

Nonlinear waves in viscous multilayer shear flows in the presence of interfacial slip

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Abstract

The stability of immiscible two-fluid Couette flows is considered when slip is present at the liquid-liquid interface. The phenomena are modelled by incorporating a Navier slip condition at the interface to replace that of no slip. A nonlinear asymptotic theory is developed when the flow geometry consists of a thin layer slipping over a thick fluid layer that scales with the channel height. A nonlocal, nonlinear evolution equation is derived that is valid at finite Reynolds numbers, slip lengths, viscosity and density ratios. The nonlocal term arises from the coupling between the phases and its Fourier symbol is calculated in closed form in terms of Airy functions, thus generalising past results by the inclusion of slip. The linear spectrum is calculated and it is shown that in geometries containing a thin layer, the role of slip is to introduce dispersion and reduce instability or enhance stability.

Keywords: interfacial waves, multilayer shear flow, Navier slip

1. Introduction

Interfacial instabilities in viscous multilayer flows are a fundamental problem in numerous technological and physical applications. The pioneering work of Yih [1] identified an unstable mode in a two-fluid Couette flow and has led to a vast theoretical and experimental literature. Briefly, it is established that interfacial instabilities in two-fluid Couette flows are possible when inertia is present and the more viscous fluid occupies the thinner of the two layers - see for example [2, 3], and the experiments in [4]. Nonlinear multiscale asymptotic theories utilising the presence of a thin layer and accounting for a coupling between the two phases were developed in [5, 6]. The model equation contains a nonlocal term that arises from the coupling between the thin and thick layers. In [7] the new equation was shown to accurately predict direct numerical simulations (DNS) and experiments. Of particular importance is the ability of the nonlocal model to predict bistability phenomena observed in the experiments of [4]. In fact the analysis and computations of [8] construct families of spatially periodic unimodal and bimodal travelling wave solutions and prove their coexistence rigorously using the theory of quasi-solutions.

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Extensions to three-dimensional flows including the effects of insoluble surfactants have been considered in [6] and [9] (the latter work proves the existence of local solutions in the general case, and global solutions in a special limit - the general case remains an open problem). We note that extensions to pressure-driven rather than wall-driven two-fluid flows, also utilising a thin layer region and also DNS, can be found in [10].

In all the above studies a no slip condition was imposed at liquid-liquid interfaces. In the present study, motivated by experiments in extrusion of liquid polymer melts, we allow for slip at the interface and develop theories analogous to those for no slip. As stated in the experiments the polymer melt liquids behave as Newtonian ones and this is the case considered here. In what follows we give a review of the experimental literature of multilayer polymer blend flows with particular interest on the interfacial slippage phenomena observed.

The existence of slippage at solid-liquid interfaces is widely accounted for in film flow models [11], and is usually modelled theoretically using a Navier-slip boundary condition [12]. High-resolution experimental observations on the micro-scale [13] identified the phenomenon and attracted interest due to the need to study, understand and model the behaviour of important polymer melt materials [14]. The presence of slip at liquid-liquid interfaces arose in the theoretical modelling of polymer blends, where a considerable decrease in measured viscosity was observed in experiments when their interfaces were undergoing shear. In order to achieve better agreement between experiments and models, slip was introduced for the first time as a parameter termed “friction factor”, regarded as the frictional force between interfaces of adjacent polymeric components [15]. The term “slip” was not used and, as pointed out later [16], the friction factor was rather serving fitting purposes and there was no attempt to explain the underlying mechanism.

The phenomenon was identified further in experiments predicting a large decrease in measured viscosity during the co-extrusion of multi-laminated polymers, and was attributed to interfacial slippage [16]. The authors observed a further decrease in measured viscosity as the number of co-extruded layers increased. In a different experiment on the behaviour of two superimposed polymer layers under the effect of an electric field, [17], [18], the presence of slippage was also identified. Employing flow velocimetry, slip was also observed during the extrusion of a polymer-fluoropolymer mixture [19] - a fluoropolymer layer coated the walls of the die and the slippage between the two fluids increased significantly the surface quality of the extrudate.

A later study [14], considered the extrusion of a polymer melt through a polymer-coated capillary die providing results for the flow rate and the slip velocity. The pair of polymers exhibited high surface tension at their interface. The authors observed no slip for low shear stresses and significant slippage, increasing with the increase of the shearing, above a threshold. The phenomenon did not arise for a non-coated die and was suppressed in the presence of an additive that reacted with the polymers and resulted in the creation of covalent bonds across their interface. It was pointed out that the selection of the pair plays a significant role, as high incompatibility of the fluids results in higher slippage. Existence of slip was also reported at the interface of two immiscible polymer layers undergoing shear in a surface-forces apparatus, [20]. It was highlighted that the phenomenon arose in a regime where the polymers exhibit Newtonian behaviour, thus justifying our modelling assumptions.

The phenomenon was also addressed in theoretical and computational studies; [21] attempted to describe microscopically the interfaces of unentangled polymer layers, and

in [22] a scaling theory for weakly entangled polymer interfaces was proposed. Of particular interest here are the molecular dynamics (MD) simulations in [23, 24]. Consideration of two immiscible liquid layers in Couette and Poiseuille flow, revealed that the immiscibility results in an almost empty interfacial region of magnitude comparable to the size of the repulsive core. The authors reported rapid changes in the density and velocity across the interface, pointing out that these would be perceived as discontinuities in experiments and continuum models where the sub-angstrom resolution provided by MD simulations is absent. To model the presence of slippage at liquid-liquid interfaces, a boundary condition similar to the solid-liquid case [12] is required. Zhao and Macosko [16] were the first to introduce a Navier-slip equation to relate the velocity jump to the shear rate. Using it, they fitted their data to estimate the slip length, which appears as the constant of proportionality. The validity of this condition was further confirmed by Koplick and Banavar [23] who characterised it as suitable and genuine thanks to the agreement exhibited with their DNS results. In a recent study [25], MD simulations are used to infer the viscosity from density and mole fraction distributions in the vicinity of the interface; this variable viscosity, if used in the Navier-Stokes equations, removes the need to interfacial conditions including a Navier-slip condition, and provides an alternative theoretical approach that will not be pursued here.

Despite the strong evidence on its existence, the impact of slip on the stability of liquid-liquid interfaces in shear flows has not been considered. In [26] a system consisting of two static superposed Newtonian polymer films on a solid substrate was studied focusing on the effect of interfacial slippage on rupture and dewetting driven by attractive van der Waals forces. Significant dependence on the magnitude of slip at both the solid-liquid and the liquid-liquid interfaces was found. To the best of our knowledge, the presence of slippage on a fluid-fluid interface and its impact on interfacial instabilities has not been studied. Therefore, we consider a two-fluid Couette flow in a channel and allow slippage at the liquid-liquid interface but not at the walls, in order to isolate the mechanism and evaluate quantitatively its effect on wave formation. Physically, we consider systems where the solid-liquid slip lengths are negligible relative to the liquid-liquid ones.

The rest of the paper is organised as follows. In Section 2 we provide the mathematical model and nonlinear boundary conditions. Section 3 develops the asymptotic theory for thin upper layers and derives the new nonlinear evolution equation that incorporates liquid-liquid interfacial slippage (mathematical details are included in an appendix); the limits of small and large slip-lengths are also carried out. Section 4 analyses the general case and in particular calculates a closed form expression for the nonlocal term in terms of Airy functions and their weighted integrals. The spectrum is also calculated for wide ranges of the parameters. Section 5 contains our conclusions and future outlook.

2. Mathematical model and governing equations

Consider two viscous, immiscible stratified fluids in a channel of height D and infinite parallel walls. Fluid 1 has density ρ_1 and viscosity μ_1 and occupies the lower layer adjacent to the lower wall, while fluid 2 has density ρ_2 and viscosity μ_2 and occupies the upper layer. Using a standard Cartesian system (x, y) and restricting attention to two-dimensional flows having a velocity field $\mathbf{u} = (u, v)$, we take the lower and upper walls to be at $y = 0, D$, region 1 is described laterally by $0 < y < h(x, t)$, and region 2

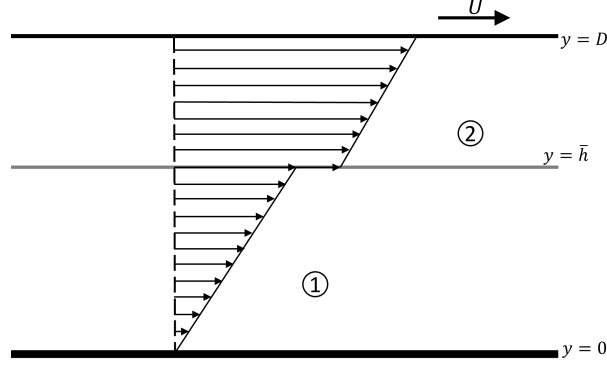


Figure 1: Schematic of the flow geometry depicting the basic flow in the presence of liquid-liquid interfacial slip.

has vertical extent $h(x, t) < y < D$. Here $y = h(x, t)$ denotes the position of the sharp interface between the two fluids, and must be determined as part of the solution; t is time. A shear is imposed to the two-fluid system by sliding the upper plate in its plane with velocity U . An exact solution exists corresponding to a flat interface $y = \bar{h}$, where $\bar{h}$ is a constant $0 < \bar{h} < D$. A schematic is given in Fig. 1.

The governing equations are the incompressible Navier-Stokes equations in each region, along with no-slip conditions at the walls and interfacial conditions at $y = h(x, t)$. The latter are modified due to the presence of slip at the liquid-liquid interface. The conditions to be imposed at $y = h(x, t)$ are kinematic conditions, an impermeability condition at the interface, normal and tangential stress balance equations, and a slip condition with slip length b , see [26] for a discussion of the model. Defining the stress tensors in each fluid (see Appendix A for array form)

$$\mathbf{\Pi}^{(1),(2)} = -p_{(1),(2)} \mathbf{I} + \mu_{(1),(2)} (\nabla \mathbf{u}_{(1),(2)} + (\nabla \mathbf{u}_{(1),(2)})^T), \quad (1)$$

where $p_{(1),(2)}$ are the pressures in the respective fluids, these conditions on $y = h(x, t)$ read (in dimensional form)

$$v_2 = h_t + u_2 h_x, \quad \text{or} \quad v_1 = h_t + u_1 h_x, \quad (2)$$

$$(\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{n} = 0 \quad \Rightarrow \quad v_2 - v_1 = h_x (u_2 - u_1), \quad (3)$$

$$\mathbf{n} \cdot (\mathbf{\Pi}^{(1)} - \mathbf{\Pi}^{(2)}) \cdot \mathbf{n} = \gamma \kappa, \quad (4)$$

$$\mathbf{n} \cdot (\mathbf{\Pi}^{(1)} - \mathbf{\Pi}^{(2)}) \cdot \mathbf{t} = 0, \quad (5)$$

$$(\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{t} = b \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \mathbf{n} \cdot \mathbf{\Pi}^{(2)} \cdot \mathbf{t}. \quad (6)$$

Here, $\mathbf{n} = \frac{(-h_x, 1)}{\sqrt{1+h_x^2}}$ and $\mathbf{t} = \frac{(1, h_x)}{\sqrt{1+h_x^2}}$ are the unit normal pointing into fluid 2, and the unit tangent at any point on the interface. In addition, $\kappa = \frac{h_{xx}}{(1+h_x^2)^{3/2}}$ is the signed curvature of the interface, γ is the coefficient of surface tension and b is the slip length. Note that the impermeability condition (3) is identical to the result of subtracting one kinematic

condition from the other in equation (2). Hence, we will use only one of the kinematic conditions along with (3). Of course (3) is trivially satisfied in the absence of slip and equations (3) and (6) are replaced by a no-slip condition at the interface, i.e. the two conditions yield $\mathbf{u}_1 = \mathbf{u}_2$ at $y = h(x, t)$. We also remark that in view of (5) we could have used $\mathbf{\Pi}^{(1)}$ in equation (6) instead of $\mathbf{\Pi}^{(2)}$.

The equations and boundary conditions are made dimensionless using the channel height D to scale lengths (including the slip length), U to scale velocities, D/U to scale time, and $\rho_2 U^2$ to scale pressures. In what follows we use the same dependent variables to denote dimensionless functions. The Navier-Stokes equations in regions 1 and 2 become

$$\mathbf{u}_{1t} + (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_1 = -r \nabla p_1 + \frac{r}{m Re} \nabla^2 \mathbf{u}_1 - G_r^2 \mathbf{e}_y, \quad \nabla \cdot \mathbf{u}_1 = 0, \quad (7)$$

$$\mathbf{u}_{2t} + (\mathbf{u}_2 \cdot \nabla) \mathbf{u}_2 = -\nabla p_2 + \frac{1}{Re} \nabla^2 \mathbf{u}_2 - G_r^2 \mathbf{e}_y, \quad \nabla \cdot \mathbf{u}_2 = 0, \quad (8)$$

and we have denoted $\nabla^2 = \partial_x^2 + \partial_y^2$. The boundary conditions at the solid walls $y = 0, 1$ become

$$y = 0, \quad u_1 = v_1 = 0, \quad (9)$$

$$y = 1, \quad u_2 - 1 = v_2 = 0. \quad (10)$$

In dimensionless form the kinematic conditions (2), (3) remain unchanged, while conditions (4)-(6) read (written out in full)

$$\begin{aligned} \frac{R_e}{2} (p_2 - p_1) (1 + h_x^2) + h_x^2 \left(\frac{u_{1x}}{m} - u_{2x} \right) - h_x \left(\frac{1}{m} (u_{1y} + v_{1x}) - u_{2y} - v_{2x} \right) \\ + \frac{1}{m} v_{1y} - v_{2y} = \frac{1}{C_a} \frac{h_{xx}}{\sqrt{1 + h_x^2}}, \end{aligned} \quad (11)$$

$$2h_x \left(\frac{1}{m} (v_{1y} - u_{1x}) + u_{2x} - v_{2y} \right) + (1 - h_x^2) \left(\frac{1}{m} (u_{1y} + v_{1x}) - u_{2y} - v_{2x} \right) = 0, \quad (12)$$

$$u_2 - u_1 + h_x (v_2 - v_1) - \frac{L(1+m)}{\sqrt{1+h_x^2}} [(1-h_x^2)(u_{2y} + v_{2x}) + 2h_x(v_{2y} - u_{2x})] = 0. \quad (13)$$

The dimensionless parameters appearing in equations (7)-(13) are

$$m = \frac{\mu_2}{\mu_1}, \quad r = \frac{\rho_2}{\rho_1}, \quad R_e = \frac{\rho_2 U D}{\mu_2}, \quad C_a = \frac{2\mu_2 U}{\gamma}, \quad G_r = \frac{1}{F_r} = \frac{\sqrt{gD}}{U}, \quad L = \frac{b}{D} \quad (14)$$

and represent fluid 2 to fluid 1 viscosity and density ratios, the Reynolds number using fluid 2 quantities, the capillary number, a Grashoff number (inverse Froude number F_r) measuring gravitational effects, and the dimensionless slip length. Throughout this work we will assume that $R_e = \mathcal{O}(1)$.

The basic undisturbed flow is given by a flat interface $y = \Theta = \bar{h}/D$ and a unidirec-

tional parallel flow

$$\bar{u}_1(y) = \frac{my}{1 + L + (\Theta + L)m - \Theta}, \quad \bar{p}_1(y) = \frac{G_r^2}{r} [\Theta(1 - r) + r - y], \quad (15)$$

$$\bar{u}_2(y) = 1 + \frac{y - 1}{1 + L + (\Theta + L)m - \Theta}, \quad \bar{p}_2(y) = G_r^2(1 - y). \quad (16)$$

where the denominators never vanish in the range the parameters are defined ($L \geq 0$, $m \geq 0$, $0 \leq \Theta \leq 1$). Evaluating the basic velocities at the interface provides the jump in velocity

$$\bar{u}_2|_{y=\Theta} - \bar{u}_1|_{y=\Theta} = \frac{L(1 + m)}{1 + L + (\Theta + L)m - \Theta}, \quad (17)$$

which is seen to vanish when $L = 0$ as expected.

The objective of the present work is to study the nonlinear interfacial waves in canonical shear flows in the presence of slip. In its absence ($L = 0$) analogous analyses and studies of the resulting equations have been carried out in the case when one layer is thin (either the upper or lower layer) - see [27, 6, 28, 8, 9], and the asymptotic results were used in favourable comparisons with direct numerical simulations [7] and experiments [4]. The key to the derivation is the coupling between the main part of the flow with that in the thin layer. The presence of slip somewhat complicates the scalings and in what follows we present a detailed asymptotic derivation for such flows. The analysis is similar whether the thin layer is the upper or lower one, even though the resulting equations are slightly different. We proceed with a thin upper layer mostly due to experimental considerations - for example to have a thin more viscous layer in a water-oil system, the lighter oil will be at the top as seen in the experiments of [4]. Results for a thin lower layer are analogous and hence are not included here.

3. Nonlinear evolution equations for thin upper layers

Assuming that layer 2 has mean dimensionless thickness $\varepsilon \ll 1$, then the undisturbed interface is at $y = \Theta = 1 - \varepsilon$. We look for nonlinear interfacial disturbances that are (i) weakly nonlinear, (ii) have order one wavelengths that scale with the channel height. A detailed derivation of the asymptotic expansions is included in Appendix B. Briefly, the derivation proceeds by assuming an interfacial deflection that drives a perturbation capillary-induced pressure gradient via the normal stress condition (11), that in turn drives a perturbation flow in the film region 2. The tangential stress balance (12) and Navier-slip law (13) in turn provide balances that determine the perturbation velocities and pressure in the main region 1. The sizes of dimensionless parameters are selected to ensure retention of as much of the physics as possible within the expansion scheme - this leads to $m, r, Re, L = \mathcal{O}(1)$, and $Ca, F_r^2 \sim \varepsilon$, the latter scalings made precise in Appendix B. Finally, with the scalings at hand, the leading order kinematic condition (2) provides the desired evolution equation.

From Appendix B, the weakly nonlinear interfacial deflection is given by

$$y_i = h(x, t) = 1 - \varepsilon + \varepsilon^2 H(x, t). \quad (18)$$

(The subscript i is a label for the interface.) The film region 2 is described by the stretched variable

$$Y = \frac{1-y}{\varepsilon}, \quad (19)$$

and combining this with (18) provides the interfacial position in terms of the stretched variable

$$Y_i = 1 - \varepsilon H. \quad (20)$$

The expansions in the lower thick region 1, given by $0 \leq y \leq 1 - \varepsilon - \varepsilon^2 H$, are

$$\begin{aligned} u_1 &= \bar{u}_1(y) + \varepsilon^2 U_1 + \dots, & v_1 &= \varepsilon^2 V_1 + \dots, \\ p_1 &= \frac{\bar{G}_r^2}{\varepsilon r} [(1-\varepsilon)(1-r) + r - y] + \varepsilon^2 P_1 + \dots, \end{aligned} \quad (21)$$

while those in the thin upper region 2, given by $0 \leq Y \leq 1 - \varepsilon^2 H$, are

$$u_2 = \bar{u}_2(Y) + \varepsilon^3 U_2 + \dots, \quad v_2 = \varepsilon^4 V_2 + \dots, \quad p_2 = \bar{G}_r^2 Y + \varepsilon P_2 + \dots \quad (22)$$

The basic states $\bar{u}_1(y)$ and $\bar{u}_2(Y)$ are given by (15), (16), where it is understood that (19) is used in the latter.

Substitution of (22) into (8) yields, to leading order,

$$U_{2YY} = R_e P_{2x}, \quad P_{2Y} = 0, \quad U_{2x} - V_{2Y} = 0. \quad (23)$$

The second equation implies $P_2 \equiv P_2(x, t)$, hence on integration and use of the no-slip and no-penetration conditions $U_2|_{Y=0} = V_2|_{Y=0} = 0$ at the upper wall, yields

$$U_2 = \frac{1}{2} Y^2 R_e P_{2x} + AY, \quad V_2 = \frac{1}{6} Y^3 R_e P_{2xx} + \frac{1}{2} Y^2 A_x, \quad (24)$$

where $A(x, t)$ is a function that needs to be determined. The perturbation pressure P_2 follows from the leading order contributions to the tangential stress balance (11),

$$P_2 = \bar{G}_r^2 \left(1 - \frac{1}{r}\right) H + \frac{2}{R_e \bar{C}_a} H_{xx}. \quad (25)$$

where the scaled parameters $\bar{G}_r$ and $\bar{C}_a$ are defined in Appendix B.

The flow in region 1 couples to the solutions (24) and is necessary in determining the function A . Substitution of the expansions (21) into the Navier-Stokes equations (7), along with the appropriate Galilean transformation and new slow time-scale ($\partial_t \rightarrow -\bar{u}_{20}(\varepsilon)\partial_x + \varepsilon^2\partial_\tau$ - see Appendix B) gives the linear system

$$[\bar{u}_1(y) - 1]U_{1x} + \bar{u}_{1y}V_1 = -P_{1x} + \frac{r}{mR_e}\nabla^2 U_1, \quad (26)$$

$$[\bar{u}_1(y) - 1]V_{1x} = -P_{2y} + \frac{r}{mR_e}\nabla^2 V_1, \quad (27)$$

$$U_{1x} + V_{1y} = 0. \quad (28)$$

Elimination of P_2 and U_1 provides the following equation for the vertical perturbation

velocity field

$$\nabla^4 V_1 - \frac{mR_e}{r} \bar{g}(y) \nabla^2 V_{1x} = 0, \quad -\infty < x < \infty, \quad 0 < y < 1, \quad (29)$$

$$\bar{g}(y) = \left(\frac{m(y-1) - L(1+m)}{m + L(1+m)} \right), \quad (30)$$

where the leading order of the expression (15) has been used. Equation (29) requires conditions as $|x| \rightarrow \infty$ or appropriate spatial periodicity conditions, and also four conditions in total at the lateral boundaries $y = 0$ and $y = 1$. The former are those of no-slip and no-penetration (9), namely,

$$V_1 = V_{1y} = 0 \quad \text{at} \quad y = 0. \quad (31)$$

The conditions at $y = 1$ come from the remaining four interfacial conditions: continuity of vertical velocity (3), tangential stress balance (12), and the Navier slip law (13), and the kinematic condition (2). Note that our solutions (24) in region 2 contain the unknowns $H(x, \tau)$, $A(x, \tau)$, confirming that we have four equations for four unknowns, enough to determine a single evolution equation for $H(x, \tau)$. To leading order these equations give (we eliminate U_1 in favour of V_1 using the continuity equation and appropriate x -derivatives)

$$V_1|_{y=1} = -\frac{L(1+m)}{m + L(1+m)} H_x, \quad (32)$$

$$\frac{1}{m} (V_{1xx} - V_{1yy})|_{y=1} + R_e P_{2xx} + A_x = 0, \quad (33)$$

$$\frac{(1-m)H_x}{m + L(1+m)} + V_{1y}|_{y=1} + L(1+m)[R_e P_{2xx} + A_x] = 0, \quad (34)$$

from which (33) and (34) are combined to find

$$\frac{(1-m)H_x}{m + L(1+m)} + V_{1y}|_{y=1} - \frac{L(1+m)}{m} (V_{1xx} - V_{1yy})|_{y=1} = 0. \quad (35)$$

We can then use equations (32) and (35) (along with (31)) as the four conditions of problem (29) to find a separable solution in Fourier space as described below.

Finally we consider the kinematic condition (2) and incorporate the Galilean transformation (B.22) to find

$$V_2|_{Y=1} = H_\tau + \frac{1}{m + L(1+m)} HH_x. \quad (36)$$

Using (24), (25) and (33) casts (36) into

$$H_\tau + \frac{1}{m + L(1+m)} HH_x + \frac{1}{3} R_e \bar{G}_r^2 \left(1 - \frac{1}{r} \right) H_{xx} + \frac{2}{3\bar{C}_a} H_{xxxx} + \frac{1}{2m} (V_{1xx} - V_{1yy})|_{y=1} = 0. \quad (37)$$

The remaining task is to determine the last term in terms of H in order to close the

system. We can anticipate, therefore, that a non-local evolution equation will emerge as we detail next.

3.1. Solution in region 1 and the final evolution equation

To find A , and hence V_2 needed in (36), we need to solve for V_1 . Transforming the problem into the Fourier domain, with hats denoting Fourier transforms, e.g. $\mathcal{F}[H] := \hat{H} = \int_{-\infty}^{\infty} H e^{-ikx} dx$ etc., and writing

$$\hat{V}_1(y) = ik\hat{H}F(y), \quad (38)$$

we find the inhomogeneous boundary value problem

$$\left(\frac{d^2}{dy^2} - k^2\right)^2 F - ik\frac{mR_e}{r}\bar{g}(y)\left(\frac{d^2}{dy^2} - k^2\right)F = 0, \quad (39)$$

$$F(0) = F'(0) = 0, \quad (40)$$

$$F(1) = -\frac{L(1+m)}{m+L(1+m)}, \quad (41)$$

$$\frac{L(1+m)}{m}[F''(1) + k^2F(1)] + F'(1) = -\frac{(1-m)}{m+L(1+m)}. \quad (42)$$

Once F is determined it is used to calculate the last term in (37). In Fourier space we have

$$\begin{aligned} \mathcal{F}[(V_{1xx} - V_{1yy})|_{y=1}] &= -ik^3\hat{H}F(1) - ik\hat{H}F''(1) \\ &= -\frac{L(1+m)}{m+L(1+m)}(ik)^3\hat{H} - ik\hat{H}F''(1), \end{aligned} \quad (43)$$

and hence the evolution equation (37) for H becomes

$$\begin{aligned} H_\tau + \frac{1}{m+L(1+m)}HH_x + \frac{1}{3}R_e\bar{G}_r^2\left(1 - \frac{1}{r}\right)H_{xx} + \frac{2}{3\bar{C}_a}H_{xxx} \\ - \frac{L(1+m)}{2m[m+L(1+m)]}H_{xxx} + \frac{1}{2m}\mathcal{F}^{-1}[-ikF''(1)\hat{H}] = 0. \end{aligned} \quad (44)$$

The general solution for F and the corresponding form of (44) is considered in Section 4. We end this section by considering the two limits $L \rightarrow 0$ and $L \rightarrow \infty$ that correspond to zero and infinite slip length respectively. The former limit recovers a subset of the equations derived and studied by [6], [8], [9], and the reader is referred to those articles for details.

3.1.1. The limit $L \rightarrow 0$

When $L \rightarrow 0$ the boundary value problem (39)-(42) becomes identical to that studied by [6], [8], [9]. Of interest here is the fact that when the thinner layer is more viscous, $m > 1$, and $R_e \neq 0$, the Fourier symbol $-ikF''(1)$ of the nonlocal term in (44) has a negative real part, implying instability for a range of wavenumbers. This crucially leads to the formation of nonlinear waves at the interface.

3.1.2. The limit $L \rightarrow \infty$

Inspection of (44) for $L \gg 1$ shows that the nonlinearity is retained if $H \sim L$, and writing $H(x, \tau) = L\eta(x, \tau)$ with $1 \ll L \ll 1/\varepsilon$ for consistency with the weakly nonlinear expansion (18), gives

$$\begin{aligned} \eta_\tau + \frac{1}{m+1}\eta\eta_x + \frac{1}{3}R_e\bar{G}_r^2\left(1 - \frac{1}{r}\right)\eta_{xx} + \frac{2}{3\bar{C}_a}\eta_{xxx} \\ - \frac{1}{2m}\eta_{xxx} + \frac{1}{2m}\mathcal{F}^{-1}[-ikG''(1)\hat{\eta}] = 0, \end{aligned} \quad (45)$$

where the function G is the leading order approximation to F for large L . It follows from (39)-(42) and (30) that

$$\left(\frac{d^2}{dy^2} - k^2\right)^2 G + ik\frac{mR_e}{r}\left(\frac{d^2}{dy^2} - k^2\right)G = 0, \quad (46)$$

$$G(0) = G'(0) = 0, \quad (47)$$

$$G(1) = -1, \quad (48)$$

$$G''(1) + k^2G(1) = 0. \quad (49)$$

The solution of (46)-(49) can be found using elementary methods, but the solution details are not necessary in determining the form of (45). Inspection of (48) and (49) implies

$$G''(1) = -k^2G(1) = k^2. \quad (50)$$

Hence $\mathcal{F}^{-1}[-ikG''(1)\hat{\eta}] = \eta_{xxx}$, thus last term in (45) exactly cancels the intrinsic dispersive effect present at all values of L . We can conclude that large L does not introduce any instabilities (or indeed any other effects such as dispersion). Nonuniform interfaces can only arise through Rayleigh-Taylor instabilities via the third term of (45) when $r > 1$.

These limits indicate that instabilities that are supported in the absence of slip appear to get stabilised as L becomes asymptotically large. In what follows we analyse the effect of arbitrary L on the spectrum.

4. Solutions for general values of the slip length

To study (44) for general values of L we need to determine $F''(1)$. Writing

$$\omega(y; k) = \left(\frac{d^2}{dy^2} - k^2\right)F, \quad (51)$$

casts (39) into an Airy equation for ω (given the linearity of $\bar{g}(y)$). Solving for ω and in turn (51) for F using variation of parameters, provides the general solution

$$F(y; k) = c_1 \sinh(ky) + c_2 \cosh(ky) + C_1 A_1(y) + C_2 A_2(y), \quad (52)$$

where

$$A_1(y) = \frac{1}{k} \int_0^y \sinh(k(y-y')) A_i(z') dy', \quad A_2(y) = \frac{1}{k} \int_0^y \sinh(k(y-y')) A_i(z' e^{\frac{2i\pi}{3}}) dy' \\ z' = \beta^{1/3} \left[y' + \beta^{-1} \left(k^2 - \frac{ikmR_e}{r} \right) \right], \quad \beta = \frac{ikm^2 R_e}{r[m + L(1+m)]}. \quad (53)$$

Conditions (40) imply that $c_1 = c_2 = 0$, and conditions (41)-(42) give a linear system for C_1, C_2 , readily solved (and checked with a symbolic manipulator) to find

$$C_1 = - \frac{A_2(1) [m(m-1) + 2k^2 L^2(1+m)^2] + mL(1+m)A_2'(1) + L^2(1+m)^2 A_i(e^{\frac{2i\pi}{3}} z_1)}{m(m + L(1+m))K}, \quad (54)$$

$$C_2 = \frac{A_1(1) [m(m-1) + 2k^2 L^2(1+m)^2] + mL(1+m)A_1'(1) + L^2(1+m)^2 A_i(z_1)}{m(m + L(1+m))K}, \quad (55)$$

where

$$z_1 = \beta^{1/3} \left[1 + \beta^{-1} \left(k^2 - \frac{ikmR_e}{r} \right) \right], \quad (56)$$

$$K = A_1(1)A_2'(1) - A_1'(1)A_2(1) + \frac{L(1+m)}{m} \left(A_1(1)A_i(e^{\frac{2i\pi}{3}} z_1) - A_2(1)A_i(z_1) \right). \quad (57)$$

The following formulas that follow from (53) will also be useful

$$A_1'(1) = \int_0^1 \cosh(k(y-y')) A_i(z') dy', \quad A_2'(1) = \int_0^1 \cosh(k(y-y')) A_i(e^{\frac{2i\pi}{3}} z') dy', \quad (58)$$

$$A_1''(1) = k^2 A_1(1) + A_i(z_1), \quad A_2''(1) = k^2 A_2(1) + A_i(e^{\frac{2i\pi}{3}} z_1), \quad (59)$$

$$F''(1) = C_1 A_1''(1) + C_2 A_2''(1). \quad (60)$$

4.1. Linear stability properties

The stability of (44) follows by linearising with respect to η and looking for normal mode solutions proportional to $e^{ikx + \sigma\tau}$. The eigenvalue $\sigma = \sigma_R + i\sigma_I$ is given by

$$\sigma = \frac{1}{3} R_e \overline{G}_r^2 \left(1 - \frac{1}{r} \right) k^2 - \frac{2}{3\overline{C}_a} k^4 - \frac{L(1+m)}{2m[m + L(1+m)]} ik^3 + \frac{ik}{2m} F''(1). \quad (61)$$

The physical origin of the different terms in equation (61) are an instability/stability due to density differences, short-wave damping due to surface tension, a dispersive term due to the presence of slip, and a complex term $F''(1)$ accounting for the dynamics in region 1 and inducing both dispersive effects as well as growth or damping, depending on parameters. There are six physical parameters appearing in (61), namely $R_e, \overline{G}_r, r, \overline{C}_a, m, L$. In the results that follow we assume equal densities, $r = 1$, thus switching off the effect of density stabilisation or destabilisation. In addition we take $\overline{C}_a = 1$ in order to illustrate the stability properties for the three crucial parameters R_e, m, L .

We present results for the variation of the growth rate σ_R with wavenumber k , as well as the wavespeed $c = -\frac{\sigma_R}{k}$. It is known, e.g. [1], that long-wave instability is possible if (i) inertia is present, and (ii) the more viscous fluid is in the thinner of the two layers. In our formulation this translates to $R_e \neq 0$ and $m > 1$. In Figure 2 we fix the slip length parameter to $L = 0.1$ and present the growth rates for $m = 2$ in panel 2(a), and $m = 0.5$ in panel 2(b), for a range of Reynolds numbers. It can be seen that for $m = 2$ there is long-wave instability with the band of unstable waves and the maximum growth rate increasing as R_e increases. For $m = 0.5$, however, the flow is stable and indeed becomes more stable as R_e increases.

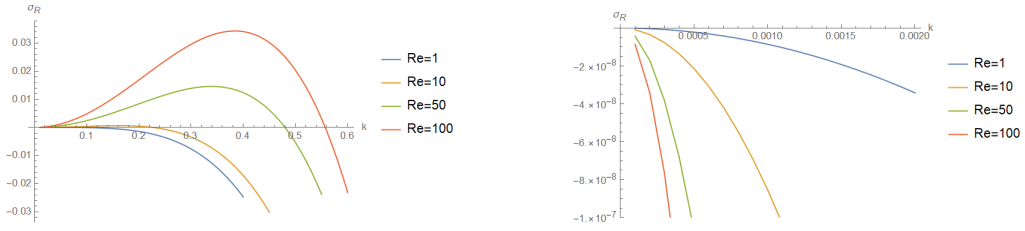


Figure 2: The growth rate σ_R for $L = 0.1$ and $R_e = 1, 10, 50, 100$: (a) more viscous film $m = 2$; (b) less viscous film $m = 0.5$.

In Figure 3 we evaluate the effect of L on the stability. It was shown analytically in section 3.1.2 that for $L \gg 1$ and $r = 1$ the flow is damped for all wavenumbers. Figure 3 starts with the unstable configuration $R_e = 50$ and $m = 2$ at $L = 0$, and shows σ_R for $L \leq 1$. It can be seen that the presence of slip has a dramatic effect on decreasing the instability and indeed makes it disappear completely for $L = 1$. In fact, the long wave analysis carried out in Appendix C shows that the transition to stability occurs at $L = 1/\sqrt{3} \approx 0.58$ - the neutral stability curves in the $L - m$ plane are shown in figure C.5.

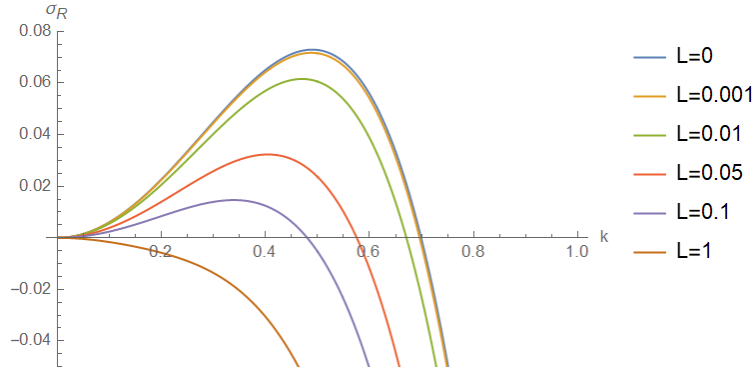


Figure 3: The growth rate σ_R for $m = 2$, $R_e = 50$ with L varying between 0 and 1.

More details of the effect of the viscosity ratio m on the stability characteristics are considered in Figures 4(a)-(b). Here, we fix $R_e = 1$, consider a relatively small value

$L = 0.1$ and vary m from 0.2 to 5. Panel 4(a) shows the development of the growth rate σ_R as m increases, while panel 4(b) depicts the corresponding wave speed $c = -\sigma_R/k$. It can be seen from 4(b) that stable modes, e.g. $m = 0.2$ and $m = 0.5$, have rather large and positive wave speeds, whereas unstable modes have negative wave speeds. Note that in the derivation we have introduced a Galilean transformation of speed $\bar{u}_{20} = \frac{m(1+L)}{m+L(1+m)}$ (see equation (B.21)), to leading order, and so the speed in a fixed frame is $c + \bar{u}_{20}$. As m increases the waves are slower than the Galilean frame speed.

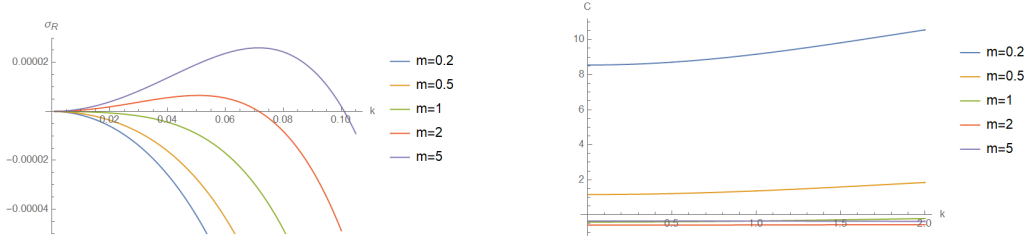


Figure 4: (a) The growth rate σ_R and (b) the wavespeed c for $Re = 1$, $L = 0.1$ and varying viscosity ratio $m = 0.2, 0.5, 1, 2, 5$.

5. Conclusions

In this study we carried out a theoretical investigation of the effect of interfacial slip on the stability of viscous multilayer shear flows. We introduced a Navier slip law at the fluid-fluid interface and considered its effects on wave formation. The general mathematical model was posed and analytical progress was made in the limit when the upper layer (near the moving plate) is thin. Using asymptotic long-wave expansions that couple thin-film dynamics with the thick lower layer dynamics, we derived a novel class of nonlocal equations that incorporate the physics of interfacial slip with those of shear, surface tension, and Rayleigh-Taylor instability. The nonlocal term is computed exactly in the sense that its Fourier symbol is expressed explicitly in terms of Airy functions and their weighted integrals. This lends itself to both numerical and careful asymptotic treatments as in [8].

The main finding regarding the solution properties of the new equation, is that the presence of slip tends to stabilise the flow and indeed if L is sufficiently large, then unstable modes can become completely damped - see Figure 3 for $L = 1$ for example. Linear stability results by the authors for the general problem when the two layers are of comparable thickness, indicate that in other regions of the parameter space the slip can act to destabilise otherwise stable long waves. There are two future directions currently under consideration: (i) the analysis and computations of the solutions of the nonlinear equation (44), and (ii) the investigation of the linear stability problem in the most general case. These will be reported elsewhere.

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Appendix A. Array form of the tensors in (1)

In each one of the layers, let $\ell = 1, 2$ indicate which one, the tensor equation (1) may be written as

$$\mathbf{\Pi}^\ell = -p_\ell \mathbf{I} + \mu_\ell (\nabla \mathbf{u}^\ell + (\nabla \mathbf{u}^\ell)^T)$$

or, using the Einstein notation for the 2D case

$$\Pi_{ij}^\ell = -p_\ell \delta_{ij} + \mu_\ell \left(\frac{\partial u_i^\ell}{\partial x_j} + \frac{\partial u_j^\ell}{\partial x_i} \right) \quad i, j = 1, 2$$

where δ_{ij} is the Kronecker delta function. Recalling that in the present model the system of coordinates is $(x_i, x_j) = (x, y)$ and the velocity field $\mathbf{u}^\ell = (u_i^\ell, u_j^\ell) = (u^\ell, v^\ell)$, the stress tensors may be written in an array form as:

$$[\Pi_{ij}^\ell] = -p_\ell \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu_\ell \begin{pmatrix} 2 \frac{\partial u^\ell}{\partial x} & \frac{\partial u^\ell}{\partial y} + \frac{\partial v^\ell}{\partial x} \\ \frac{\partial u^\ell}{\partial y} + \frac{\partial v^\ell}{\partial x} & 2 \frac{\partial v^\ell}{\partial y} \end{pmatrix}$$

Appendix B. Derivation of the asymptotic expansions

Here we give details of the derivation of the asymptotic scalings that lead to our nonlinear evolution equations. Weakly nonlinear wavy disturbances are written as

$$y_i = h(x, t) = 1 - \varepsilon + \delta H(x, t), \quad (\text{B.1})$$

where $\delta \ll \varepsilon$ is to be determined. A key observation is the fact that due to viscosity differences between the two phases, there results a local velocity difference, that scales with the interfacial amplitude, when the base flow is evaluated at the disturbed interface. To see this we recall that $\Theta = 1 - \varepsilon$, so that the difference between the velocities (15)-(16) at y_i given by (B.1), reads

$$\bar{u}_2|_{y_i} - \bar{u}_1|_{y_i} = \frac{L(1+m)}{m + \varepsilon(1-m) + L(1+m)} + \frac{\delta(1-m)H(x, t)}{m + \varepsilon(1-m) + L(1+m)}. \quad (\text{B.2})$$

As expected, the $\mathcal{O}(\delta)$ velocity jump vanishes when the viscosities are equal, $m = 1$, but when $m \neq 1$ a velocity perturbation is induced that contributes to setting the canonical scales in the asymptotic solutions that follow.

Since $\varepsilon \ll 1$ we introduced a stretched variable in the y -direction in region 2:

$$y = 1 - \varepsilon Y, \quad \partial_x \rightarrow \partial_x, \quad \partial_y \rightarrow -\frac{1}{\varepsilon} \partial_Y, \quad Y = \mathcal{O}(1). \quad (\text{B.3})$$

The Navier-Stokes equations (7) in region 2 transform to

$$u_{2t} + u_2 u_{2x} - \frac{1}{\varepsilon} v_2 u_{2Y} = -p_{2x} + \frac{1}{R_e} \left(u_{2xx} + \frac{1}{\varepsilon^2} u_{2YY} \right), \quad (\text{B.4})$$

$$v_{2t} + u_2 v_{2x} - \frac{1}{\varepsilon} v_2 v_{2Y} = \frac{1}{\varepsilon} p_{2Y} + \frac{1}{R_e} \left(v_{2xx} + \frac{1}{\varepsilon^2} v_{2YY} \right) - G_r^2, \quad (\text{B.5})$$

$$u_{2x} - \frac{1}{\varepsilon} v_{2Y} = 0. \quad (\text{B.6})$$

We seek an asymptotic solution of (B.4)-(B.6) by writing flow variables as a basic part plus perturbations

$$u_2 = \bar{u}_2(Y) + \delta_u U_2 + \text{h.o.t.}, \quad v_2 = \varepsilon \delta_u V_2 + \text{h.o.t.}, \quad p_2 = \varepsilon G_r^2 Y + \delta_p P_2 + \text{h.o.t.}, \quad (\text{B.7})$$

where h.o.t. denotes higher order terms, and δ_u, δ_p must be determined. The size of v_2 in (B.7) follows from the continuity equation (B.6). Note that in terms of Y the interface position is given by

$$Y_i = 1 - \frac{\delta}{\varepsilon} H(x, t). \quad (\text{B.8})$$

In region 1 we introduce the expansions

$$u_1 = \bar{u}_1(y) + \Delta_u U_1 + \text{h.o.t.}, \quad v_1 = \Delta_u V_1 + \text{h.o.t.}, \\ p_1 = \frac{G_r^2}{\rho} (1 - y - \varepsilon(1 - \rho)) + \Delta_u P_1 + \text{h.o.t.}, \quad (\text{B.9})$$

where Δ_u also needs to be determined. The perturbations in region 1 are all of the same order Δ_u , due to the rectangular domain and lack of separation of spatial scales.

Substitution of (B.7) into (B.4)-(B.6) yields, to leading order,

$$-\delta_p P_{2x} + \frac{\delta_u}{\varepsilon^2 R_e} U_{2YY} = 0, \quad (\text{B.10})$$

$$\frac{1}{\varepsilon} P_{2Y} = 0, \quad (\text{B.11})$$

$$U_{2x} - V_{2Y} = 0. \quad (\text{B.12})$$

Hence $P_2 \equiv P_2(x, t)$, and a balance of terms in (B.10) requires

$$\delta_p \sim \frac{\delta_u}{\varepsilon^2}, \quad (\text{B.13})$$

noting that we assume $R_e = \mathcal{O}(1)$. The pressure perturbation that drives the flow is obtained from the normal stress balance (11); retention of gravitational and surface tension forces also, yields

$$R_e \delta_p P_2 + \delta R_e G_r^2 \left(\frac{1}{\rho} - 1 \right) H - \frac{\delta}{C_a} H_{xx} = \mathcal{O}(\Delta_u, \frac{\delta \delta_u}{\varepsilon}, \dots). \quad (\text{B.14})$$

Balancing terms we find

$$\delta_p \sim \delta G_r^2 \sim \frac{\delta}{C_a}, \quad (\text{B.15})$$

and combining this with (B.13) provides the perturbation velocity scale

$$\delta_u = \frac{\varepsilon^2 \delta}{C_a}. \quad (\text{B.16})$$

The terms of order Δ_u on the right hand side of (B.14) include the pressure gradient and stresses in region 1, and we have anticipated $\Delta_u \ll \delta_p$ as we demonstrate next. To do this, consider leading terms in the tangential stress balance equation (12), and the Navier slip condition (13) that yield, respectively,

$$\frac{\Delta_u}{m}(U_{1y} + V_{1x})|_{y=1} + \frac{\delta_u}{\varepsilon}U_{2Y}|_{Y=1} = \mathcal{O}(\delta\Delta_u, \varepsilon\delta_u, \dots), \quad (\text{B.17})$$

$$\frac{\delta(1-m)H(x,t)}{m+L(1+m)+\varepsilon(1-m)} - \Delta_u U_1 + \frac{\delta_u}{\varepsilon}L(1+m)U_{2Y} = \mathcal{O}(\delta_u, \delta\Delta_u, \dots). \quad (\text{B.18})$$

To retain coupling between the two phases and keep as many physical mechanisms as possible, it follows from (B.17)-(B.18) that

$$\Delta_u \sim \frac{\delta_u}{\varepsilon} \sim \delta, \quad (\text{B.19})$$

which in view of (B.13) immediately gives $\Delta_u \ll \delta_p$ as anticipated in (B.14). Furthermore, equation (B.19) gives $\delta_u \sim \varepsilon\delta$, and combining this with (B.16) implies strong surface tension according to

$$C_a = \varepsilon \bar{C}_a, \quad \bar{C}_a = \mathcal{O}(1). \quad (\text{B.20})$$

It remains to determine δ as a function of ε . This comes from the kinematic conditions, and without loss of generality we work with the film-sided equation, along with the normal velocity continuity condition (3). Evaluating (2) at y_i (equivalently at Y_i given by (B.8)), and Taylor expanding about the undisturbed interface, yields

$$\varepsilon\delta_u V_2|_{Y=1} = \delta(H_t + \bar{u}_{20}H_x) + \frac{\delta^2}{m+L(1+m)}HH_x + \dots, \quad (\text{B.21})$$

where $\bar{u}_{20} = \frac{m(1-\varepsilon)+Lm}{m(1-\varepsilon)+\varepsilon+L(1+m)}$ is the constant basic velocity at the undisturbed interface from the film side. Applying a Galilean transformation and a slow time scale through the change of variables

$$(x, t) = (X = x - \bar{u}_{20}t, \tau/\delta), \quad \partial_t \rightarrow -\bar{u}_{20}\partial_X + \delta\partial_\tau, \quad (\text{B.22})$$

casts (B.21) into

$$\varepsilon \delta_u V_2|_{Y=1} = \delta^2 \left(H_\tau + \frac{1}{m + L(1 + m)} H H_X \right) + \dots, \quad (\text{B.23})$$

implying that $\varepsilon \delta_u \sim \delta^2$ which on use (B.19) yields $\delta \sim \varepsilon^2$. Hence,

$$\delta = \varepsilon^2, \quad \delta_u = \varepsilon^3, \quad \delta_p \sim \varepsilon, \quad \Delta_u = \varepsilon^2, \quad (\text{B.24})$$

and

$$V_2|_{Y=1} = H_\tau + \frac{1}{m + L(1 + m)} H H_X. \quad (\text{B.25})$$

Equation (B.25) is the desired evolution equation for the wavy weakly nonlinear dynamics. It remains to determine $V_2|_{Y=1}$ as described next.

Solving (B.10) subject to (10) gives

$$U_2 = \frac{R_e}{2} Y^2 P_{2xx} + A Y, \quad (\text{B.26})$$

with the function A to be determined. Integration of the continuity equation (B.12) subject to (10) yields

$$V_2 = \frac{R_e}{6} Y^3 P_{2xx} + \frac{1}{2} Y^2 A_x. \quad (\text{B.27})$$

The pressure P_2 follows from (B.14),

$$P_2 = \frac{1}{R_e \overline{C}_a} H_{xx} + \overline{G}_r^2 \left(1 - \frac{1}{r} \right), \quad (\text{B.28})$$

where the scaled Grashof number $G_r^2 = \overline{G}_r / \varepsilon$ has been introduced to retain density stratification effects. Hence, use of (B.27)-(B.28) in (B.25), completes the derivation once $A(x, t)$ is expressed in terms of $H(x, t)$, as shown in section 3.

Appendix C. Asymptotic results of (39)-(42) in the long wave limit

When long waves are considered it is possible to obtain the non-local term and hence the stability parameters in closed form. As $k \rightarrow 0$, $F(y)$ is expressed in terms of ascending powers of k

$$F(y) = F_0(y) + k F_1(y) + k^2 F_2(y) + k^3 F_3(y) + \dots \quad (\text{C.1})$$

and by substitution into (39)-(42), the boundary value problem at leading order is

$$F_{0yyy} = 0 \quad (\text{C.2})$$

$$\textcircled{\text{a}} \quad y = 0 : \quad F_0 = 0 \quad (\text{C.3})$$

$$F_{0y} = 0 \quad (\text{C.4})$$

$$\textcircled{\text{a}} \quad y = 1 : \quad F_0 = -\frac{L(1+m)}{m+L(1+m)} \quad (\text{C.5})$$

$$\frac{1-m}{m+L(1+m)} + F_{0y} + \frac{L(1+m)}{m} F_{0yy} = 0 \quad (\text{C.6})$$

Using a symbolic manipulator $F_{0yy}(1)$ is obtained to be real and therefore, it contributes only to the phase speed of the disturbance. The latter, at leading order becomes:

$$c = -\frac{\sigma_I}{k} = -\frac{3\bar{C}_a}{4m} F_{0yy}(1) \Rightarrow c = -\frac{3\bar{C}_a}{2} \frac{2(m-1) + 3L(1+m)}{(m+4L(1+m))(m+L(1+m))} \quad (\text{C.7})$$

The dispersive term of (61) is $\mathcal{O}(k^2)$ and thus neglected.

At the following order, $\mathcal{O}(k)$:

$$F_{1yyyy} = \frac{m}{r} R_e i \frac{m(y-1) - L(1+m)}{m+L(1+m)} F_{0yy} \quad (\text{C.8})$$

$$\textcircled{\text{a}} \quad y = 0 : \quad F_1 = 0 \quad (\text{C.9})$$

$$F_{1y} = 0 \quad (\text{C.10})$$

$$\textcircled{\text{a}} \quad y = 1 : \quad F_1 = 0 \quad (\text{C.11})$$

$$F_{1y} + \frac{L(1+m)}{m} F_{1yy} = 0 \quad (\text{C.12})$$

where the solution $F_1(y)$ is a purely imaginary 6th degree polynomial. The quantity of interest is:

$$F_{1yy}(1) = i R_e \frac{m^2}{30r} \frac{m(1-m) + 3L^2(1+m)^2}{(m+4L(1+m))(m+L(1+m))^2} \quad (\text{C.13})$$

and only affects the growth rate of the disturbance. Actually, $F_{1yy}(1)$ along with the gravity term determine the stability at leading order, since, as seen in (61), surface

tension is of $\mathcal{O}(k^4)$. Calculating the growth rate yields

$$\begin{aligned}
\sigma_R &= \frac{1}{3} R_e \bar{G}_r^2 \left(1 - \frac{1}{r}\right) k^2 - \frac{3 \bar{C}_a}{4m} \text{Im} [F_{yy}(1)] \\
&= \frac{1}{3} R_e \bar{G}_r^2 \left(1 - \frac{1}{r}\right) k^2 - \frac{3 \bar{C}_a}{4m} k \text{Im} [F_{0yy}(1) + k F_{1yy}(1)] \\
&= \frac{1}{3} R_e \bar{G}_r^2 \left(1 - \frac{1}{r}\right) k^2 - \frac{3 \bar{C}_a}{4m} k^2 \text{Im} [F_{1yy}(1)] \\
&= \frac{1}{3} R_e \bar{G}_r^2 \left(1 - \frac{1}{r}\right) k^2 - \frac{3 \bar{C}_a}{4m} \frac{m}{r} R_e \frac{m}{30} \frac{m(1-m) + 3L^2(1+m)^2}{(m+4L(1+m))(m+L(1+m))^2} k^2 \\
\sigma_R &= \frac{1}{3} R_e \bar{G}_r^2 \left(1 - \frac{1}{r}\right) k^2 + \frac{m R_e \bar{C}_a}{40r} \frac{m(m-1) - 3L^2(1+m)^2}{(m+4L(1+m))(m+L(1+m))^2} k^2 \quad (\text{C.14})
\end{aligned}$$

It is noteworthy that the Reynolds number appears only as a factor, hence it is not capable of rendering the system unstable. It may only amplify the existing regime by either further stabilising the system when it is already stable or further destabilising it, if already unstable.

In the case of equal densities the first term vanishes and the criterion for instability becomes:

$$\sigma_R > 0 \quad \Rightarrow \quad m(m-1) - 3L^2(1+m)^2 > 0,$$

or equivalently

$$(1 - 3L^2)m^2 - (6L^2 + 1)m - 3L^2 > 0. \quad (\text{C.15})$$

Neutral stability ensues along the two branches

$$m_{1,2} = \frac{1 + 6L^2 \pm \sqrt{1 + 24L^2}}{2(1 - 3L^2)}.$$

The m_2 branch is always negative hence unphysical ($m > 0$ is necessary). The other, restricted for $L > 0$, diverges to infinity for $L = 1/\sqrt{3}$, $+\infty$ for $L \rightarrow (1/\sqrt{3})^-$ and $-\infty$ otherwise. The negative branch is ignored as $m > 0$ and the behaviour is presented in Figure C.5 (a) as a neutral curve. The curve starts from $m = 1$ for $L = 0$, the neutrally stable viscosity stratification mentioned in [1].

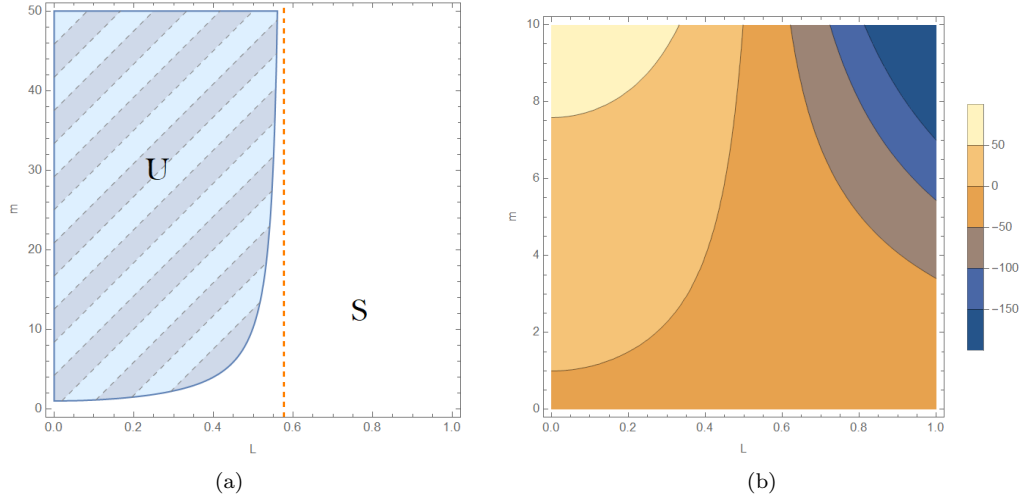


Figure C.5: Neutral curve $L - m$ in the long wave limit and region of instability (a) and contours of (C.15) in (b)

In Figure C.5 (a) the region of instability can be seen as the shadowed part of the plain. This graph explains the behaviour observed in the previous figures, namely the fact that for $L = 0.1$, $m > 1$ is unstable whilst $m < 1$ appears to be linearly stable as well as the observation that in Figure 3 the line for $L = 1$ is stable. As the above analysis indicates, the critical slip length for stabilisation is $L = 1/\sqrt{3} \sim 0.57735$ and the contours in C.5 (b) show clearly the stabilising effect of increasing L .

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