Intertemporal Price Discrimination via Randomized Promotions

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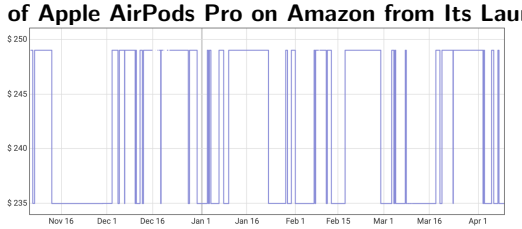
Problem definition: The undesirable but inevitable consequence of running promotions is that consumers can be trained to time their purchases strategically. In this paper, we study randomized promotions, where the firm randomly offers discounts over time, as an alternative strategy of intertemporal price discrimination. **Methodology/results:** We consider a base model where a monopolist sells a single product to a market with a constant stream of two market segments. The segments are heterogeneous in both their product valuations and patience levels. The firm pre-commits to a price distribution, and in each period, a price is randomly drawn from the chosen distribution. We characterize the optimal price distribution as a randomized promotion policy and show that it serves as an intertemporal price discrimination mechanism such that high-valuation customers would make a purchase immediately at a regular price upon arrival and low-valuation customers would wait for a random promotion. Compared against the optimal cyclic pricing policy, which is optimal within the strategy space of all deterministic pricing policies, the optimal randomized pricing policy beats the optimal cyclic pricing policy if low-valuation customers are sufficiently patient and the absolute discrepancy between high and low customer valuations is large enough. We extend the model in three directions. First, we consider the case where a portion of customers are myopic, who would never wait. We show that the existence of myopic customers is detrimental to the firm's profitability, and the expected profit from an optimal randomized pricing policy decreases as the proportion of myopic customers in the population increases. Second, we consider Markovian pricing policies where prices are allowed to be intertemporally correlated in a Markovian fashion. This additional maneuver allows the firm to reap an even higher profit when low-valuation customers are sufficiently patient, by avoiding consecutive promotions but on average running the promotion more frequently with a smaller discount size. Lastly, we consider a model with multiple customer segments and show that a two-point price distribution remains optimal and our conclusion from the two-segment base model still holds under certain conditions that are adopted in the literature. **Managerial implications:** Our results imply that the firm may want to deliberately randomize promotions in the presence of forward-looking customers.

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1. Introduction

Dynamic or contingent pricing is now widely adopted by firms in a range of sectors such as the airline and e-commerce industry and proven to be a strategy that can improve a firm's bottom line substantially. It is especially prevalent in online retail markets, even for durable goods, because retailers are no longer inhibited by physical price tags that require time and effort to change manually. Many firms manipulate prices frequently and, from the customers' point of view, in a somewhat random fashion. For example, Figure 1 displays the price trajectory of Apple AirPods Pro on Amazon from its launch in Nov 2019 to Apr 2020. The price varied between its list price of $249.00 and a discounted price of $234.99. If a price-sensitive consumer wants to wait for the discount of $14, it is not clear when the promotion might happen outside the Christmas shopping season and how long it will last.

Figure 1 Pricing Trajectory of Apple AirPods Pro on Amazon from Its Launch in Nov 2019 to Apr 2020



Note. Source: https://keepa.com/#!product/1-B07ZPC9QD4

When customers are forward-looking, a seller may benefit from deliberately varying prices over time in a *random* fashion. Consider a price-insensitive or delay-sensitive consumer who wanted to buy an AirPods Pro arrives at Amazon. Even though she is fully aware of the possibility of a promotion in the near future by checking the historical price trajectory through websites such as CamelCamelCamel or Keepa, the random occurrence of such a price discount may deter the consumer from waiting and result in her purchase at the list price upon her arrival. Moon et al. (2017) empirically show the success of an established North American specialty retailer in implementing a committed state-independent randomized markdown policy through avoiding "'conditioning' or 'training' its customers to wait for markdowns."[1]

Comparing with running promotions on predictable time instances, there are clearly pros and cons associated with randomized pricing. On one hand, under a randomized price policy, customers cannot reliably predict future prices, making it hard to time their purchases. Hence randomized pricing would be able to alleviate strategic customer behavior to some extent, compared to a deterministic markdown policy, especially when it was close to a predetermined promotion time.

[1] In an Online Appendix, Moon et al. (2017) also numerically show when randomized pricing outperforms a deterministic markdown schedule.

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However, the flip side of the coin is that, though customers cannot predict future prices accurately, they would expect some price variations at any point in time, and still have tendency to delay their purchases regardless of when their consumption need arises. Thus it is unclear under what conditions randomized pricing would dominate a deterministic pricing policy such as a static pricing policy and a cyclic pricing policy. The former completely eliminates strategic customer behavior and the latter can intertemporally price discriminate customers (see, e.g., Conlisk et al. 1984). Therefore, a full-blown theoretical investigation is needed to complement the empirical study by Moon et al. (2017).

To that end, we address a set of questions that require formal modeling and a complete scan of the parameter and strategy space. For examples, how may randomized promotions and pricing work to benefit the firm? Is it possible that randomized pricing is even worse than a static pricing policy?[2] Under what conditions does randomized pricing outperform the optimal deterministic pricing such as cyclic pricing[3] that has been well studied in the literature and vice versa? Is it beneficial to further expand the search for the optimal pricing strategy from randomized prices independently drawn from a price distribution to more sophisticated randomized pricing policies, such as Markovian pricing where the price process evolves in a Markovian fashion, e.g., whether there will be a promotion in the next period depends on whether there is one in the current period?

We answer these questions with a theoretical model in discrete time. In particular, we consider a base model where a monopolist sells a single product to two segments of customers. Customers arrive sequentially at a constant rate, and they are heterogeneous in both product valuations and patience levels. Upon arrival, a customer can choose to purchase the product immediately, delay purchases for future periods with the hope for a lower price, or leave immediately. The decision of randomized pricing for the firm is to decide on and commit to a price distribution before the sales horizon starts, from which a price is randomly drawn in each period of the sales horizon.

First, we characterize customers' best responses to randomized pricing, as well as structural properties that optimal randomized pricing policies must obey. It turns out, under any optimal randomized pricing policies, high-valuation customers would either purchase or leave immediately upon arrival, while low-valuation customers would either buy immediately if the price happens to be sufficiently low or delay the purchase. We then construct a simple two-point price distribution with the higher price to be the valuation of high-type customers and prove its optimality by

[2] In an Online Appendix, Moon et al. (2017) analytically show that deterministic markdowns are preferred over the optimal randomized price policy in the extreme case when the waiting cost of high valuation customers is zero.

[3] A cyclic pricing policy is commonly observed in practice, even for durable goods. Take games for Nintendo Switch as an example. Other than the usual seasonal promotions (such as discounts on Mario games on March 10th to celebrate "Mario Day" and promotions of horror games around Halloween), digital games would go on sale on Nintendo's digital store each Thursday, so patient gamers will be able to purchase games at a discounted price (source: https://bucketlist.games/do-nintendo-switch-games-ever-go-on-sale-how-to-find-lowest-prices/).

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showing that such a distribution can attain an upper bound of the firm's expected profit. Under this two-point price distribution (when it is non-degenerate to a single mass point), high-valuation customers would always purchase immediately upon arrival, while low valuations would always purchase with the lower discounted price either immediately or in a future period. That is, the optimal randomized pricing policy serves as an intertemporal price discrimination mechanism such that high-valuation customers would make a purchase immediately at a regular price upon arrival and low-valuation customers would wait for a promotion.

Next, we investigate the firm's profitability, consumer surplus, social welfare, and surplus allocation across the two market segments under the optimal randomized pricing policy. We show that, compared to the optimal static pricing policy, the optimal randomized pricing policy yields a higher profit when low-valuation customers are sufficiently patient. Hence, randomized pricing either dominates or reduces to static pricing. If under the optimal static pricing policy the low-valuation customers are priced out of the market, then the optimal randomized pricing policy leads to (weakly) higher surplus for both market segments and hence higher social welfare (which is the total surplus of the firm and all customers). Otherwise, the optimal randomized pricing policy results in a lower surplus difference between the two market segments. Thus, shifting from static pricing to randomized pricing leads to a (weakly) higher profit for the firm, and moreover, either (weakly) higher social welfare or a (weakly) fairer surplus allocation among customers.

We further evaluate the performance of randomized pricing policies by benchmarking against cyclic pricing policies. We first show that, in our setup, cyclic pricing policies arise to be optimal within the strategy space of all deterministic policies. Under a cyclic pricing policy, the same pricing schedule within a cycle is repeated over time. We show that an optimal cyclic pricing policy follows a (weakly) markdown pattern. Interestingly, neither randomized pricing nor cyclic pricing always dominates the other over the full parameter space. However, when low-valuation customers are sufficiently patient and the absolute discrepancy in customers' valuations is sufficiently large, an optimal randomized pricing policy is more profitable than an optimal cycle deterministic pricing policy. The underlying rationale is as follows. When the absolute discrepancy in customers' valuations is sufficiently large, it is important for the firm to capture most of the surplus from high-valuation customers. This can be achieved under the optimal randomized pricing policies by a low frequency of promotions, and thus the majority of high-valuation customers would make a purchase at the price equal to their valuation. Meanwhile, the low-valuation customers wait for the promotion as they are sufficiently patient. On the other hand, the optimal cyclic pricing policy follows a predetermined price trajectory, known to all customers. Thus, the firm needs to compensate high-valuation customers to prevent them from waiting to purchase at a discounted price, especially when it is closer to the end of a cycle. This compensation cuts into the firm's profitability, leading to underperformance of cyclic pricing policies compared to randomized pricing policies where high-valuation customers pay at their valuation most of the time.

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Lastly, we extend the base model in three directions. First, we consider the case where a portion of customers are myopic, who would never wait. We show that in the presence of myopic customers, the optimal randomized pricing policy still follows a two-point distribution, which is composed of a regular price and a discounted price that clears the market of strategic low-valuation customers from time to time. As the proportion of myopic customers in the population increases, the firm may either run promotions less frequently with a lower discounted price or run promotions more frequently with a higher discounted price. The optimal strategy depends on the ratio of valuations between the two customer segments. Overall, the existence of myopic customers is detrimental to the firm's profitability, and the expected profit from an optimal randomized pricing policy decreases as the proportion of myopic customers in the population increases. Second, we consider a Markovian pricing policy, where prices are allowed to be intertemporally correlated in a Markovian fashion. We show that Markovian pricing may not always dominate randomized pricing. However, when it generates a higher profit, it does so by avoiding consecutive promotions and extracting weakly more profit from low-valuation customers and strictly more profit from high-valuation customers, than an optimal randomized pricing policy. When low-valuation customers are sufficiently patient, the extra maneuver with Markovian pricing policies leads to a higher expected profit through, on average, running the promotion more frequently with a smaller discount size. Third, we consider a model with multiple customer segments. We show that, under certain conditions that are also used in the literature, the optimal randomized pricing policy still follows a two-point price distribution, under which there exists a cutoff on customers' valuations such that any customer with a valuation less than the cutoff would choose to wait, while any customer with a valuation greater than or equal to that cutoff would either purchase or leave immediately upon arrival. This randomized pricing policy is guaranteed to outperform an optimal static pricing policy when customers of the lowest valuation are sufficiently patient. Under the same set of conditions that guarantee the optimality of a two-point price distribution for randomized pricing, we show that, when the absolute discrepancies in customers' valuations are large enough, an optimal randomized pricing policy is always more profitable than an optimal cyclic pricing policy.

Our results imply that firms may want to deliberately randomize promotions in the presence of forward-looking customers who sequentially arrive, because doing so can lead to higher profits. As a default assumption in the literature of dynamic mechanism design, commitment to a randomized pricing policy is practically easier to enforce nowadays than ever, thanks to a provable track record displayed by price tracker websites such as CamelCamelCamel or Keepa. E-commerce websites can develop a reputation of running random promotions to train consumers to make an immediate purchase as they see an acceptable price upon arrival rather than to wait for a lower price.

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2. Literature Review

Our paper is related to the economic literature on the usage of price dispersion for the purpose of price discrimination. The origin of this stream of literature can be traced back to the 1970s. Back then, economists have come to recognize that the "law of one price" is no law at all (Varian 1980). A significant degree of price dispersion is observed across a wide range of industries. The first few papers, including but not limited to Salop and Stiglitz (1977), Salop (1977), and Reinganum (1979), seek to provide rationale regarding why multiple retailers offer an identical item at different prices contemporaneously. This type of price dispersion is also referred to as *spatial* price dispersion. Our work is more closely related to a different kind of price dispersion, namely *temporal* price dispersion, where prices vary over time, and at any moment, a cross section of the market would exhibit price dispersion. Temporal price dispersion is first studied in the seminal paper by Varian (1980), who shows that the existence of temporal price dispersion is sustained by information heterogeneity in prices. Varian (1980) assumes that there are two segments of consumers, i.e., consumers who are informed of all current prices, and consumers who are uninformed. Informed consumers would only purchase from the lowest priced seller, while uninformed consumers would visit sellers randomly. The author shows that, when there are both informed and uninformed consumers, there is a symmetric mixed equilibrium in which all sellers randomize according to the same price distribution. In that setting, the existence of temporal price dispersion is sustained by information heterogeneity in prices. Since then, temporal price dispersion has been studied with various twists from the information perspective (see, e.g., Baye and Morgan 2001 and references therein). The randomized pricing policy considered in our paper is also one type of temporal price dispersion. By contrast, we focus on one single firm, rather than multiple firms. We unravel the effectiveness of randomized pricing policies by a single firm in serving as an intertemporal price discrimination mechanism.

Along the line of intertemporal price discrimination, researchers have tried to tackle the intertemporal pricing problems with sequentially arriving strategic customers who are forward-looking and able to time their purchase. With strategic customers, the optimal class of policies within the strategy space of all deterministic policies has been shown to be *cyclic* pricing policies. For example, Conlisk et al. (1984) and Besbes and Lobel (2015) study a setting very close to ours with an infinite sales horizon and a steady stream of strategic customers of heterogeneous valuation or patience levels. A subtle difference is that Conlisk et al. (1984) adopt a discounting factor in capturing the customers' cost of waiting, while Besbes and Lobel (2015) assume an exogenously given shopping window over which a customer compares prices. By contrast to the shopping window assumption, we assume customers have waiting cost per period as an alternative assumption to a discounting factor[4] and fully endogenize strategic customer behavior as a solution to a dynamic program.

[4] We allow customers to have heterogeneous waiting costs per period, while Conlisk et al. (1984) assume the same discounting factor for all customers.

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Similar to Conlisk et al. (1984) and Besbes and Lobel (2015), we also show that the optimal deterministic pricing policy is a cyclic pricing policy among deterministic policies in our setting. Then, we identify conditions under which randomized pricing outperforms cyclic pricing.

Sobel (1984) extends Conlisk et al. (1984) to account for competition and shows that in the symmetric equilibrium, competing symmetric sellers use randomized (mixed) strategies. The equilibrium strategy has the following form: there exists a finite time length during which all sellers charge the regular price to sell to high-valuation customers and after which any seller may run a promotion. When a promotion or multiple promotions occur, all of the accumulated low-valuation customers buy from the lowest price seller, and the process repeats. Sinitsyn (2017) builds on Sobel (1984) but assumes that the sellers must schedule their price promotion in advance over a finite horizon, i.e., each seller commits to a future period in which to run a promotion to sell to low-valuation customers. The author shows that sellers often use mixed strategies in equilibrium, choosing the future promotion period according to a probability distribution function. In contrast to these papers that study competition where randomized (mixed) strategies may naturally arise in equilibrium, we study whether a monopolist seller can do better by expanding its strategy space from deterministic pricing policies to randomized pricing policies.

In a parallel setup to the line of Conlisk et al. (1984), Besanko and Winston (1990) assume a finite number of customers lining up in the beginning of the sales horizon and show the subgame perfect Nash equilibrium pricing policy involves intertemporal price discrimination through price skimming. Liu and Zhang (2013) study a competitive version of this setup for two firms offering vertically differentiated products. In contrast, we follow the line of Conlisk et al. (1984) by assuming the entry of new customers over time, which captures the sequential customer arrivals in many practical settings such as purchasing an AirPods Pro online as customers' needs arise over time.

Aviv and Pazgal (2008) show that strategic customer behavior suppresses the benefits of price segmentation. Chen et al. (2019) point out that static prices eliminate any strategic customer behavior and show its competitive ratio is $1-1/e$ for a broad class of customer utility models. In our model, we characterize sufficient conditions under which randomized pricing reduces to static pricing, or strictly outperforms static pricing. In addition, various operational-level maneuvers, such as rationing by deliberately understocking products (Liu and van Ryzin 2008), implementing a dry period with zero inventory (Chen and Chu 2020), displaying one unit of inventory at a time (Yin et al. 2009) and offering price guarantees under Markovian pricing (Wu et al. 2020[5]), have been proposed to reduce the propensity of consumers to strategically wait for discounts. Contributing

[5] There are several major differences between our paper and Wu et al. (2020). On the demand side, Wu et al. (2020) assume that customers have an exponentially distributed lifetime with price monitoring costs, while in our model, there is neither any price monitoring cost nor a shopping window, but instead, customers incur waiting costs. On the supply side, Wu et al. (2020) assume a menu cost for each price change, while we do not have such a cost. Wu et al. (2020) focus on price guarantees under Markovian pricing, while we compare randomized vs. cyclic pricing.

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to this literature, we study randomized pricing as a mechanism to alleviate strategic customer behavior. We identify under what conditions such randomized pricing policies can indeed benefit the firm over static pricing or cyclic pricing.

There are two papers that are closest to this paper: Moon et al. (2017) and Wu et al. (2014). Moon et al. (2017) introduce the randomized pricing as a mechanism of intertemporal price discrimination and emphasize the customers' active monitoring as a critical factor in determining the success of randomized pricing. Other than that Moon et al. (2017) is a structural estimation model and ours is an analytical model, there are a couple of distinct differences. First, the models have many differences. Moon et al. (2017) study a continuous-time model with an infinite sales horizon, a finite initial inventory level and a finite market size, while we consider a discrete-time model with an infinite horizon, no capacity constraint and a constant stream of new customers (essentially infinitely many customers). Ours is a common setting to study strategic consumer behavior, see, e.g., Conlisk et al. (1984) and Besbes and Lobel (2015). Moon et al. (2017) assume that the firm has the information on the remaining numbers of customers in the market, while we do not. Second, the main focuses are different. Moon et al. (2017) compare randomized pricing with contingent pricing dependent on the inventory level and the remaining numbers of strategic customers.[6] As our model has neither inventory constraint nor the information of outstanding strategic customers, state-dependent contingent pricing reduces to a static pricing policy. In addition to an analytical comparison between randomized and static pricing, we also theoretically compare randomized pricing with cyclic pricing, which is proven to be optimal among deterministic policies.[7]

Like ours, Wu et al. (2014) also build a model to study the impact of randomized pricing policies on a firm's profitability. However, our paper differs from Wu et al. (2014) in the following essential aspects. First, Wu et al. (2014) adopt an exogenously given shopping window to capture customers' strategic waiting behavior. In their paper, high-valuation customers are assumed to wait for up to one period, and low-valuation customers would wait for up to a given finite number of periods. In contrast, we assume that each segment of customers have their own cost of waiting per time period and fully endogenize strategic waiting behavior as a solution to an optimal stopping time problem for each individual customer. Second, the price distribution considered in Wu et al. (2014) is beforehand restricted to a two-point distribution. By contrast, we start with general price distributions and characterize the sufficient conditions under which a two-point distribution is optimal. Lastly, with our general model, we provide a much more comprehensive set of comparisons among randomized, static, and cyclic pricing.

[6] The state in Moon et al. (2017) is time independent.

[7] Moon et al. (2017, Appendix F) compare randomized pricing with a deterministic price schedule which may be analogous to cyclic pricing in the case of no inventory constraint. For a special case, it is shown that the deterministic price schedule dominates randomized pricing. We identify general conditions under which randomized pricing dominates cyclic pricing.

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In sum, the main contributions of this paper are threefold. First, we complement the empirical work of Moon et al. (2017) to theorize randomized pricing. Second, we show after expanding the strategy space of the firm from deterministic policies to random policies, the firm can do strictly better and identify conditions under which the firm is better off with randomized pricing, compared with cyclic pricing. Third, we show that the firm can do even better with Markovian pricing, compared with time-independent randomized pricing, and characterize the conditions under which a two-point price distribution remains optimal with multiple customer segments.

3. A Model of Randomized Pricing

We consider a monopolist who sells a single product over an infinite horizon. The seller uses a randomized pricing policy. That is, the firm chooses and announces a distribution of prices ex ante, and then a random price P is drawn from the chosen distribution for each period. In our base model, prices are assumed to be independent and identically distributed across different periods, which will be relaxed in Section 4.2 where the price process follows a Markovian process. We denote the price mass or density function by $f(p)$ and the cumulative distribution function by $F(p)$. Since no customer would make a purchase at an infinitely high price, we just consider the support of P to be finite. As general mechanism design without commitment presents well-known difficulties (Bester and Strausz 2001), commitment to a (potentially randomized) policy is the most common approach in the literature nowadays.[8]

Customers are infinitesimal and arrive sequentially over time. We consider a discrete-time model, and the customer arrival rate is normalized to one for each time period. Each customer purchases up to one unit of the product. Upon arrival, a customer decides among three options: purchase immediately in the current period, wait in the system with the hope of a more favorable price in future periods, or leave immediately. If a customer chooses to delay purchase, waiting cost is incurred for each period.[9] Customers are heterogeneous along two dimensions, namely, valuation and patience level. We assume there are two customer segments.[10] A fraction α of low-valuation customers (bargain hunters) value the product at v_L and the remaining $1-\alpha$ customers value the product at v_H, where $v_H > v_L$. The one-period waiting costs for low- and high-valuation customers are denoted by c_L and c_H, respectively. Notice that we do not require that low-valuation customers be more patient than high-valuation customers, i.e., c_L can be larger than c_H.

[8] Under the assumption of perfect commitment, a mechanism design problem reduces to an optimization problem subject to the well-known incentive compatibility constraints, thanks to the revelation principle. However, the revelation principle generally fails when the mechanism designer cannot credibly commit to an outcome induced by the mechanism.

[9] An alternative way of modeling utility loss due to customer waiting is to discount future payoffs. Charging waiting cost proportional to the wait time is more common in the operations literature. Our main results shall hold qualitatively for the discounting scheme under some conditions.

[10] An extension with multiple segments of customers is studied in Section 4.3.

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We assume that customers benefit from waiting, i.e., a customer's waiting cost per period is less than her valuation. Otherwise, it becomes trivial that customers would either purchase the product or leave immediately upon arrival.

ASSUMPTION (S) (STRATEGIC CUSTOMERS) $0 < c_L < v_L$ and $0 < c_H < v_H$.

3.1. Optimal Customer Response

We first characterize the optimal response of customers under *any* given price distribution. It suffices to consider one segment of customers with valuation of v and a one-period waiting cost of c. Suppose a customer arrives in period t and makes a purchase in period t'. Her utility is thus given by $v - p_{t'} - (t'-t)c$, $\forall t' \geq t$. Upon arrival, a customer makes a choice among three options: buy immediately (buy), delay purchase (wait), or leave immediately (quit). The customer surplus of quitting and adopting an outside option is assumed to be zero. At period t, her utility from purchasing in period $t'(> t)$ would be uncertain under a randomized pricing policy. Customers are assumed to be rational and make decisions to maximize their utilities. Denote by $V_t(p_t)$ a customer's maximum utility given price p_t in the current period t. V_t would be independent of time because we consider a stationary system with an infinite horizon. Thus, without abuse of notation, we omit the subscript and use V throughout the rest of the paper. Therefore, $V(p_t)$ is given by

$$V(p_t) = \max\{\underbrace{v - p_t}_{\text{(buy)}}, \underbrace{-c + \mathsf{E}[V(P)]}_{\text{(wait)}}, \underbrace{0}_{\text{(quit)}}\}.$$

We show in the following lemma that customers' optimal response is governed by a threshold policy. They would purchase the product immediately if the price in the current period is sufficiently low; otherwise, they would either delay purchase or leave immediately.

LEMMA 1. *There exists a threshold $\underline{p} \in (0, v]$ such that*

$$V(p_t) = \begin{cases} v - p_t & \text{if } p_t \leq \underline{p}, \\ [-c + \mathsf{E}[V(P)]]^+ & \text{otherwise.} \end{cases} \tag{1}$$

Whether a customer would wait or leave depends on her valuation v. If a customer's valuation v is sufficiently high, then she is better off waiting for a favorable price in future periods. Otherwise, she would either buy now or leave immediately. In case of ties between acting immediately and later, an agent is assumed to act immediately. We apply this tie-breaking rule throughout this paper. In particular, if a customer is otherwise indifferent between, quitting or buying immediately, and waiting, she is assumed to quit or buy immediately. We formalize this result as follows.

PROPOSITION 1. (THRESHOLD RULE OF CUSTOMER) *A customer with valuation v would be willing to wait and eventually purchase a unit of the product (either immediately or in a future period) if $v > \underline{v}^* = \max\{v' \mid \mathsf{E}[(v'-P)^+] \leq c\}$, and her purchase threshold $\underline{p} = \underline{v}^*$. Otherwise, the customer would purchase the product if $v \geq p_t$, or leave immediately if $v < p_t$.*

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Proposition 1 shows that a customer's decision, in particular whether she will wait or leave, depends critically on the firm's choice of the price distribution. This is not surprising in the sense that if a relatively low price can be drawn from the price distribution with a high probability, then a customer is better off waiting because her expected utility from waiting outweighs her waiting cost. On the other hand, if a low price is not likely to be drawn from the price distribution, i.e., a customer is expected to wait for a long time before seeing a favorable price, then the best option for the customer is not to wait. Proposition 1 implies that the firm's choice of the price distribution shapes customers' decisions in a critical way.

More precisely, valuation $\underline{v}^* = \max\{v' \mid \mathsf{E}[(v'-P)^+] \le c\}$ is the break-even point. For a customer with valuation exactly at $\underline{v}^*$, she would be indifferent between to wait and not to wait. At this break-even point $v = \underline{v}^*$, the expected benefit of waiting for one more period, $\mathsf{E}[(v-P)^+]$, obtaining a surplus $v - P$ when $P \le v$, is exactly equal to the one-period waiting cost c. For a customer whose valuation is strictly higher than $\underline{v}^*$, she would be willing to wait: if she observes a favorable low price $p_t \le \underline{v}^*$, she would buy one unit in period t and otherwise, she would wait, because her expected benefit of waiting for one more period is always strictly higher than the waiting cost c. In period $t+1$, the exactly *same* threshold rule applies because the last period cost of waiting, c, has been *sunk*. As a result, for a customer with $v > \underline{v}^*$, she would eventually purchase a unit of the product, either immediately upon arrival or in a future period. For those customers whose valuation is strictly lower than $\underline{v}^*$, they would not wait and only make purchase upon arrival when they see a price lower than their valuation. For those whose valuation at $\underline{v}^*$, our tie-breaking rule assumes them to conform to the latter.

Other than the price distribution, it is apparent that customers' own traits of valuation and waiting cost also affect their optimal responses. For a given waiting cost c, perhaps somewhat counterintuitively, a customer is more willing to wait if her valuation is higher, because the customer's valuation is more likely to exceed the threshold $\underline{v}^*$. On the other hand, more intuitively, with a given valuation v, a customer is less willing to wait with a higher waiting cost.[11] These observations are important to understand the firm's optimal choice of the price distribution, in the presence of multiple market segments with heterogeneous customer valuation and waiting cost.

Lastly, we formally summarize the firm's expected profit from one customer segment with valuation v and per-period waiting cost c in the following lemma.

[11] Customers' optimal response under randomized pricing policies closely resembles the searching behavior as commonly observed in the search literature (see, e.g., Lippman and Mccall 1976 and Weitzman 1979). In a typical search model, an agent pays a cost to sample a new item with a random reward, whereas analogously in our model, a customer is able to sample a new random price by waiting for one more time period (and pay a per-period waiting cost c). As a result, customers' optimal response as characterized in Proposition 1 can be viewed as a variation of the well-known reservation price policy. That is, $\underline{v}^*$ is the reservation price for a customer with valuation v and per-period waiting cost c. The customer will stop waiting and purchase immediately if the price in the time period is no greater than v^* (should she decide to wait in the first place).

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Lemma 2 (Firm's Expected Profit from a Single Segment). *Consider a single market segment with customer valuation v and waiting cost c. Under an arbitrary randomized price distribution P:*

(i) If $v > \underline{v}^*$, the monopolist's expected profit is $\mathsf{E}[P|P \le \underline{v}^*]$.

(ii) Otherwise, the monopolist's expected profit is $F(v)\mathsf{E}[P|P \le v]$.

For the case when customers are willing to wait (i.e., $v > \underline{v}^*$), they would wait, potentially for many periods, until buy at a price lower than $\underline{v}^*$. Hence, the monopolist's expected profit is given by $\mathsf{E}[P|P \le \underline{v}^*]$. Of course, one underlying assumption of this expression is that the firm does not discount future payoffs, which is consistent with the long-run average objective adopted in a similar setting of Besbes and Lobel (2015). For the case when customers are not willing to wait (i.e., $v \le \underline{v}^*$), they would only buy upon arrival seeing a price no higher than their valuation v and hence, the monopolist's expected profit is $F(v)\mathsf{E}[P|P \le v]$. For both cases, the expected profit is no more than v. As an immediate ramification, the optimal randomized pricing policy is trivial with only one segment of customers: it reduces to an optimal static pricing policy with a price at v.

3.2. Optimal Randomized Pricing Policy

Next we derive the optimal randomized pricing policy with two segments of customers who differ in both valuations and patience levels. The flow of this section is as follows. We first discuss structural properties that an optimal randomized pricing policy must satisfy. Then, based on those properties, we construct an optimal price distribution and prove its optimality by showing that an upper bound of the expected revenue can be attained by such a price distribution.

A benchmark that we will use throughout the rest of this section is the optimal static pricing policy, under which customers behave myopically and the expected profit is given by $\max\{v_L, (1-\alpha)v_H\}$. That is, the monopolist either sets a static price at v_L to sell to both segments or sets a static price at v_H to sell only to the high-valuation customers. When an optimal static pricing policy outperforms randomized pricing policies, the price distribution in an optimal randomized pricing policy reduces to a constant.

To facilitate the following discussion, we determine critical thresholds that govern customers' decisions as follows, based on the single-segment analysis in Section 3.1. That is, a low (resp., high) valuation customer would either purchase or wait if and only if $\underline{p}^L < v_L$ (resp., $\underline{p}^H < v_H$), where

$$\underline{p}^i = \max\{v' \mid \mathsf{E}[(v'-P)^+] \le c_i\}, \qquad i = L, H.$$

3.2.1. Structural Properties. We start by proving some key features that an optimal randomized pricing policy must satisfy. The properties shed light on market characteristics that the firm can potentially benefit from using randomized pricing policies. They also narrow down the candidate policies, and set the stage for us to derive the optimal randomized pricing policy.

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Lemma 3. (Absolute Patience Level) *An optimal randomized pricing policy reduces to a static pricing policy if* $c_L \geq c_H$.

Waiting cost parameters c_L and c_H indicate the absolute patience levels for low- and high-valuation customers, respectively. Lemma 3 shows that, under any randomized pricing policy, the monopolist cannot obtain a higher profit than that from an optimal static pricing policy when high-valuation customers are more patient. This result is not surprising in the sense that when high-valuation customers are more patient, they can wait for at least as long as low-valuation customers, and thus would make a purchase at a price no higher than what low-valuation customers would pay. The highest price that a low-valuation customer would pay for is v_L. Since the firm cannot effectively separate the two segments of customers when $c_L \geq c_H$, the optimal expected profit per period can be no higher than v_L, which is the expected profit under the static price v_L.

Lemma 4. (Relative Patience Level) *An optimal randomized pricing policy reduces to a static pricing policy if* $v_L/c_L \leq v_H/c_H$.

In Lemma 4, we show a weaker condition, $v_L/c_L \leq v_H/c_H$ (i.e., $c_L/c_H \geq v_L/v_H$), under which the firm would not be better off from a randomized pricing policy. Essentially, the ratio of a customer's valuation and her per-period waiting cost, i.e., v_i/c_i, $i = L, H$, indicates her relative patience level. It manifests the maximum number of time periods that a customer could potentially wait before her utility becomes negative. Lemma 4 shows that the firm can only effectively price discriminate two segments when low-valuation customers are relatively more patient. Moon et al. (2017) show, in an Online Appendix, a special case of Lemma 4 when the high-valuation customers have zero waiting cost, i.e., $c_H = 0$.

Furthermore, as shown in Lemma OS.1, $v_L/c_L > v_H/c_H$ is a necessary condition for low-valuation customers to wait, and for high-valuation customers to either purchase or leave immediately under any randomized pricing policy. Intuitively, this is a situation that benefits the most for the firm because the firm can charge a higher price for high-valuation customers, while at the same time set a lower price randomly once a while and clear the market of hanging around low-valuation customers. This intuition is formally confirmed in the following proposition.[12]

Proposition 2. (Customer Behavior under an Optimal Randomized Pricing Policy) *Under any optimal randomized policy that outperforms the optimal static pricing policy, low-valuation customers would wait, and high-valuation customers would either purchase or leave immediately upon arrival.*

[12] We present the result here to facilitate the flow, however the proof of this result relies on many of the following analysis, and we thus suggest readers to postpone the reading of the proof until the end of this section.

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3.2.2. Optimal Price Distribution. Next we derive the optimal randomized pricing policy. Based on Proposition 2, we only need to focus on those pricing policies under which low-valuation customers would wait, while high-valuation customers would not, i.e., $v_L > \underline{p}^L$ and $v_H \le \underline{p}^H$. Recall that $\underline{p}^i = \max\{v' \mid \mathsf{E}[(v'-P)^+] \le c_i\}$, $i = L, H$. Because $\mathsf{E}[(v'-P)^+]$ increases in v', $v_L > \underline{p}^L$ and $v_H \le \underline{p}^H$ are equivalent to $\mathsf{E}[(v_L-P)^+] > c_L$ and $\mathsf{E}[(v_H-P)^+] \le c_H$, respectively. An optimal randomized pricing policy that satisfies the preceding conditions can be derived by solving the following optimization problem, with the objective function derived from our preceding discussion and the constraints based on Proposition 1:

$$\begin{aligned} \max_{F(p)} \quad & \alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha) F(v_H) \mathsf{E}[P|P \le v_H] \\ \text{s.t.} \quad & \underline{p}^L = \max\{v' \mid \mathsf{E}[(v'-P)^+] \le c_L\}, \\ & \mathsf{E}[(v_L-P)^+] > c_L, \quad \mathsf{E}[(v_H-P)^+] \le c_H. \end{aligned} \tag{2}$$

This is a non-trivial problem because we need to optimize over a distribution function with no obvious structure to exploit. The solution approach we take here is that we first find an upper bound of the expected profit from problem (2), and then construct a price distribution achieving the upper bound.

Because $\mathsf{E}[P|P \le \underline{p}^L] \le \mathsf{E}[P|P \le v_L]$, we can remove $\underline{p}^L$ and obtain a relaxation of the above optimization problem as follows:

$$\begin{aligned} \max_{F(p)} \quad & \alpha \mathsf{E}[P|P \le v_L] + (1-\alpha) F(v_H) \mathsf{E}[P|P \le v_H] \\ \text{s.t.} \quad & \mathsf{E}[(v_L-P)^+] > c_L, \quad \mathsf{E}[(v_H-P)^+] \le c_H. \end{aligned} \tag{3}$$

Notice that, for any feasible solution of this relaxation, we have $F(v_L) < \frac{c_H - c_L}{v_H - v_L}$. This is because by the two constraints of problem (3), $c_H - c_L > \mathsf{E}[(v_H-P)^+] - \mathsf{E}[(v_L-P)^+] = F(v_H)\mathsf{E}[v_H - P|P \le v_H] - F(v_L)\mathsf{E}[v_L - P|P \le v_L] \ge F(v_L)\mathsf{E}[v_H - P|P \le v_L] - F(v_L)\mathsf{E}[v_L - P|P \le v_L] = (v_H - v_L)F(v_L)$. As such, we restrict our discussion to those distributions where this condition is satisfied in the following analysis.

Lemma 5. *Let R_P be the support of any optimal price distribution of problem (3). If there exists $p \in R_P$ such that $p > v_L$, then $p = v_H$.*

Lemma 5 shows an important structural property of an optimal price distribution for problem (3). The underlying rationale is as follows. First, it does not make sense to charge a price greater than v_H, because neither high-valuation nor low-valuation customers would make a purchase at such a price. Second, for any feasible solution of problem (3), high-valuation customers would either purchase or leave immediately upon arrival. As such, whenever a price greater than v_L but less than or equal to v_H is drawn from the price distribution, high-valuation customers would always purchase immediately, while low-valuation customers would wait because the price is greater than their valuation. Thus, the best strategy for the firm is to charge the highest possible price, i.e., v_H, such that all surplus from high-valuation customers can be extracted.

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LEMMA 6. (UPPER BOUNDS) *Let* $\Delta_R = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$.

(i) When $\Delta_R \geq 1$, *the optimal expected profit from (3) is no more than that from an optimal static pricing policy;*

(ii) When $\Delta_R < 1$,

(a) if $\Delta_R < \frac{c_H - c_L}{v_H - v_L}$, *the optimal expected profit from (3) is no more than* $U_1 \equiv \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - 2(1-\alpha)(v_H - v_L)\Delta_R$;

(b) otherwise, the optimal expected profit from (3) is no more than $U_2 \equiv \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_H - \alpha c_L \frac{v_H - v_L}{c_H - c_L}$.

Based on Lemma 5, we can derive upper bounds of the expected profit from problem (3), which is summarized in Lemma 6. It turns out that, when $\Delta_R \geq 1$, any randomized pricing policy is dominated by the optimal static pricing policy. It is easy to verify that Δ_R increases in c_L, which implies that randomized pricing policies are more likely being dominated when low-valuation customers are less patient. The underlying rationale is that the effectiveness of randomized pricing policies relies on intertemporal price discrimination of the two segments of customers by charging high and low prices for high and low-valuation customers respectively. However, when low-valuation customers are less patient, they become less likely to delay their purchases, and thus the firm would either lose more low-valuation customers or reduce prices to accommodate both low- and high-valuation customers. In the case of $c_L \geq c_H$, i.e., high-valuation customers are more patient than low-valuation customers, randomized pricing policies become completely ineffective, as shown in Lemma 3. Similarly, we notice that Δ_R decreases in $v_H - v_L$. That is, randomized pricing policies are more likely being dominated when the difference in valuations between two segments becomes smaller. When customers' valuations become closer, high-valuation customers can mimic the behavior of low-valuation customers more easily, making it more costly for the firm to price discriminate the two segments.

When $\Delta_R < 1$, we show upper bounds of the optimal expected profit from problem (3) in Lemma 6(ii). These upper bounds can be achieved by a two-point price distribution as we will show shortly. In the first best solution, when the firm knows the private information of each individual customer's valuation, the optimal expected profit is given by $\alpha v_L + (1-\alpha)v_H$, which certainly cannot be attained by anonymous posted price mechanisms due to information asymmetry and customers' self-selection behavior. Lemma 6 shows that, under an optimal randomized pricing policy, the profit loss, compared to the first best solution, depends on the market composition α, per-period waiting costs c_L and c_H for both low- and high-valuation customers, and the difference in valuations across two market segments $v_H - v_L$.

Next we are ready to discuss optimal randomized pricing policies. Similar to Su (2007), we will focus on ϵ-optimal policies as a consequence of our restriction to anonymous posted price mechanisms. An ϵ-optimal policy is defined as follows.

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Definition 1. (ϵ-Optimal Policy) For any arbitrarily small $\epsilon > 0$, a policy is ϵ-optimal if its revenue is greater than $1-\epsilon$ fraction of the optimal revenue.[13]

For brevity, we refer to ϵ-optimal policies as "optimal" policies in the rest of the analysis.

Proposition 3. (Optimal Randomized Pricing Policy) *Suppose* $\Delta_R = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < 1$. *The following two-point price distribution is optimal:* $f(p) = \begin{cases} \beta^* & \text{if } p = \underline{p}^*, \\ 1-\beta^* & \text{if } p = v_H, \end{cases}$ *where* $\beta^* = \min\left\{\Delta_R, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$, $\underline{p}^* = v_L - \frac{c_L}{\beta^*} - \eta(\delta)$ *and* $\delta > 0$ *is sufficiently small. Moreover,* $\lim_{\delta \searrow 0} \eta(\delta) = 0$.

Proposition 3 shows that the firm can attain the optimal profit with a simple two-point distribution. Under this two-point price distribution, high-valuation customers would always purchase immediately upon arrival, while low valuations would always purchase with the lower price of $\underline{p}^*$. Notice that the lower price from the price distribution is less than v_L by an amount of c_L/β^*. This is the "price" the firm has to pay to incentivize low-valuation customers to wait (instead of leaving immediately). The compensation increases in c_L but decreases in β^*. It is intuitive that the firm needs to compensate more with larger c_L due to low-valuation customers being less patient. The reason why the compensation decreases in β^* is as follows: β^* is the probability of drawing the lower price from the optimal price distribution. When the lower price is drawn, the market will be cleared and all customers arrived so far would make a purchase immediately and then leave the market. So, if the lower price is drawn less frequently, then low-valuation customers are expected to wait for a longer time on average, rendering more compensation required to incentivize them to wait.

Proposition 4. *There exists a threshold on* c_L, *below which the optimal randomized pricing policy outperforms the optimal static pricing policy.*[14]

The optimal static pricing policy is either to set the price at v_H to only serve the high-valuation customers, often referred to as the margin strategy (if $v_L < (1-\alpha)v_H$) or to set the price at v_L to serve both segment of customers, often referred to as the volume strategy (if $v_L \geq (1-\alpha)v_H$). The optimal two-point price distribution would allow the firm to achieve an upper bound as specified in Lemma 6. Whether the optimal expected profit from randomized pricing is greater than that from static pricing depends on system parameters. When low-valuation customers are sufficiently

[13] More specifically, for any arbitrarily small $\epsilon > 0$, there exists a sufficient small $\delta > 0$ such that the revenue of a policy that depends on δ is greater than $1-\epsilon$ fraction of the optimal revenue. Because we assume that in case of ties between acting immediately and later an agent is assumed to act immediately, by offering a sufficiently small δ off a price, we could induce one segment of customers to wait rather than to buy immediately, so to achieve intertemporal price discrimination. This is why we achieve the ϵ-optimality, because we leave a small amount of surpluses to some customers. But with δ made infinitesimally small, we almost achieve the full optimality. The ϵ-optimality does not affect our comparison between two ϵ-optimal policies, because the ϵ values for both policies can be made sufficiently small, so the comparison always holds for sure.

[14] We obtain a more general set of sufficient conditions in the proof of Proposition 4.

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patient, the optimal randomized pricing policy is always better than the optimal static pricing policy, as shown in Proposition 4. In particular, when c_L converges to 0, the upper bound given in Lemma 6 converges to the expected profit of the first best solution, i.e., $\alpha v_L + (1-\alpha)v_H$. That is, when low-valuation customers are sufficiently patient, an optimal randomized pricing policy is able to almost *perfectly* price discriminate the two segments through occasional random promotions to clear low-valuation customers, allowing the firm to extract almost *all* surplus from customers.

Corollary 1. (Consumer Surplus and Social Welfare) *Compared with the optimal static pricing policy, the optimal randomized pricing policy yields:*

(i) if $v_L < (1-\alpha)v_H$, higher social welfare, and higher consumer surplus for both low valuation and high-valuation customers;

(ii) if $v_L \geq (1-\alpha)v_H$, lower social welfare, and higher consumer surplus for low-valuation customers, but lower consumer surplus for high-valuation customers when $c_H < v_H - v_L$.

(All above statements hold weakly if the optimal randomized pricing policy reduces to a static price.)

Corollary 1 illustrates the impact of the optimal randomized pricing policy on social welfare and consumer surplus. Let us first discuss social welfare, which is the total surplus of the firm and all customers. Under the optimal randomized pricing policy, each customer is guaranteed to purchase a product either immediately or in future periods. As high-valuation customers always make a purchase under both the optimal randomized and static pricing, the loss of efficiency is solely due to cost of waiting from low-valuation customers. If $v_L < (1-\alpha)v_H$, low-valuation customers are induced to wait and make a purchase under the optimal randomized pricing policy, but are excluded from the market under the optimal static pricing policy (because the optimal static price is v_H), leading to higher social welfare under the optimal randomized pricing policy. On the other hand, when $v_L \geq (1-\alpha)v_H$, the social welfare is maximized under the optimal static pricing policy (because the optimal static price is v_L) as there is no efficiency loss. Consequently, the optimal randomized pricing policy would only hurt social welfare in this case, with the efficiency loss from waiting by low-valuation customers.

On another note, low-valuation customers would always weakly benefit from randomized pricing. The underlying rationale is that consumer surplus from low-valuation customers is always zero under the optimal static pricing policy–they would either purchase at a price of v_L or leave immediately. However, under the optimal randomized pricing policy, the lower price from the price distribution, which is less than v_L, incentivizes low-valuation customers to wait rather than leaving immediately and compensates for their expected wait c_L/β^*, where $1/\beta^*$ is the expected number of periods to see the low price. In addition, the optimal policy also leaves a tiny amount of positive expected surplus, $\eta(\delta)$, for low-valuation customers. If $v_L < (1-\alpha)v_H$, consumer surplus

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from high-valuation customers would also be higher under the optimal randomized pricing policy. This is because their surplus is always equal to zero under the optimal static pricing policy with the price of v_H, however under the randomized pricing policy, some high-valuation customers can purchase at the lower price of $\underline{p}^*$, leading to a positive surplus for them. Consequently, using the optimal randomized pricing policy by the firm could result in higher social welfare as long as $v_L < (1-\alpha)v_H$, because both consumer surplus and the firm's profit can be higher. However, if $v_L \geq (1-\alpha)v_H$, consumer surplus from high-valuation customers could be higher under either the optimal randomized or static pricing policy, depending on the system parameters.

Lastly, we compare the difference in consumer surplus between low- and high-valuation customers under the two alternative pricing policies. Many researchers have shown that customers would compare with their peers as a reference to evaluate their own payoffs (see e.g., Ho and Su 2009). As a result, the difference in surplus from purchasing the product between the two segments of customers can be regarded as a measure for fairness. We show in the following corollary that the optimal randomized pricing policy would make the surplus allocation fairer if $v_L \geq (1-\alpha)v_H$.

Corollary 2. (Fairness) *Compared with the optimal static pricing policy, the discrepancy in consumer surplus between low- and high-valuation customers is smaller under the optimal randomized pricing policy if* $v_L \geq (1-\alpha)v_H$.

The reason that randomized pricing can be fairer is exactly due to the same reason why higher social welfare can be generated. If $v_L \geq (1-\alpha)v_H$, under the optimal static pricing policy of charging v_L, high-valuation customers enjoy a positive surplus by hiding their identity while low-valuation customers have zero surplus. However, under the optimal randomized pricing policy, better price discrimination reduces the surplus of high-valuation customers, while still leaving a tiny amount of surplus on the table for the low-valuation customers. As a result, the difference between the surpluses of the two segments is reduced. Combining the two corollaries, we see that implementing the optimal randomized pricing, as opposed to the optimal static pricing, either achieves higher social welfare benefiting all, or a fairer situation among all customers.

3.3. Optimal Deterministic Pricing: Cyclic Pricing

In this section, we seek to gauge the performance of randomized pricing by comparing it against deterministic pricing, which has been studied extensively in the literature. To that end, we first need to derive the optimal deterministic pricing policy in our setting, which has not been studied before when customers are forward looking and heterogeneous in both valuations and per-period waiting costs. Under a deterministic pricing policy, the firm needs to determine a sequence of prices $\mathbf{p} = \{p_t\}_{t\in\mathbb{N}}$. The sequence of prices is public information, announced to all customers. For a customer of valuation v and per-period waiting cost c, her utility from buying in period t' is given

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by $v - p_{t'} - (t' - t)c$ if she arrives in period t. She would thus compare net utilities from buying in all periods $t' \geq t$, and choose to buy in the period where her utility is maximized, conditional on it being non-negative.

We first show that one does not need to consider beyond cyclic pricing for the optimal deterministic pricing. Under a cyclic pricing policy, there exists an integer T such that $p_t = p_{t+T}$ for all t, where T is the cycle length. The firm's long-run average expected profit with a cyclic pricing policy $\mathbf{p}$ is given by

$$\Pi(\mathbf{p}) = \lim_{T' \to \infty} \frac{1}{T'} \sum_{t=1}^{T'} \pi_t(\mathbf{p}),$$

where $\pi_t(\mathbf{p})$ is the profit from customers arriving in period t. We have $\pi_t(\mathbf{p}) = \pi_{t+T}(\mathbf{p})$ due to prices being cyclic, and thus an alternative formulation of the firm's long-run average profit is given by $\Pi(\mathbf{p}) = \frac{1}{T} \sum_{t=1}^{T} \pi_t(\mathbf{p})$.

Proposition 5. (Optimality of Cyclic Pricing) *Cyclic deterministic pricing is optimal within the strategy space of deterministic policies.*[15]

Proposition 5 significantly narrows down candidates for optimal deterministic policies. Our goal is to find a cyclic pricing policy $\mathbf{p}$ such that $\Pi(\mathbf{p})$ is maximized. The flow of this section is as follows. We first discuss structural properties of optimal cyclic policies, and then construct an optimal cyclic policy based on those structural results.

3.3.1. Structural Properties. First, it is easy to verify that the firm would not charge a price higher than v_H, the highest valuation from customers, under any optimal cyclic pricing policy. That is, if $\mathbf{p} = \{p_t\}_{t \in \mathbb{N}}$ is an optimal cyclic pricing policy, we have $p_t \leq v_H$ for any t.

Lemma 7. *An optimal cyclic pricing policy reduces to a static pricing policy if $c_L \geq c_H$.*

As the counterpart of Lemma 3, Lemma 7 illustrates the impact of customers' patience levels on the structure of the optimal cyclic pricing policy. When high-valuation customers are more patient than low-valuation customers, cyclic pricing policies would not perform better than the optimal static pricing policy. This is because high-valuation customers would wait for at least as long as low-valuation customers, and pay no more than low-valuation customers, rendering any intertemporal price discrimination mechanism, including cyclic pricing policies, ineffective at all.

Next, we characterize the behavior of high-valuation customers under an optimal cyclic pricing policy in Proposition 6. Both cyclic deterministic and randomized pricing policies are variations of intertemporal price discrimination mechanisms, and thus customers' behavior under the optimal policies is similar. Proposition 6 shows that the optimal cyclic pricing policy will be designed in such a way that high-valuation customers would have no incentive to wait.

[15] We prove the result under the general case with n customer segments.

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PROPOSITION 6. *Under any optimal cyclic pricing policy, high-valuation customers would always purchase immediately upon arrival.*

3.3.2. Optimal Cyclic Deterministic Pricing Policy. Proposition 6 sets the stage for us to derive optimal cyclic pricing policies. For this purpose, we assume $c_L < c_H$ in view of Lemma 7. If this assumption fails, the optimal cyclic pricing policy reduces to a static price. We construct a cyclic pricing policy, and prove its optimality in Proposition 7. With such a policy, prices weakly decrease over time within each cycle.

PROPOSITION 7. (OPTIMAL CYCLIC DETERMINISTIC PRICING POLICY) *An optimal cyclic pricing policy is given by* $p_{t'} = v_H - [v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+$, $t' = 1, 2 \ldots, T$, *where* T *is the cyclic length and* $\delta > 0$ *is arbitrarily small. Under this policy, all low-valuation customers buy in period* T *and high-valuation customers buy immediately without delay.*

Consistent with Conlisk et al. (1984), Proposition 7 shows that the optimal cyclic pricing policy follows a markdown pattern. Within a cycle, the prices between two consecutive periods either stay constant at v_H or drop by a size no more than c_H, which eliminates the incentive for high-valuation customers to wait. More precisely, in general, within a cycle, the prices stay constant at v_H for some time (which can be zero), drops by a size no more than c_H and then finally drops by a size of exactly c_H to the end-of-cycle price $v_L - (T-1)c_L - \delta$; see Figure 2(a) for an illustration. On the other hand, low-valuation customers would wait patiently in the market upon arrival, because their waiting is more than fully compensated. Eventually, the market of low-valuation customers will be cleared at the end of each cycle. There is still a loose part in Proposition 7, which is the determination of the cycle length T. This is a common issue for characterizing the optimal cyclic pricing policy, see, e.g., Conlisk et al. (1984) and Besbes and Lobel (2015). We cannot characterize it analytically in general, however we show that it can be computed for a special case, see Proposition 8(i).

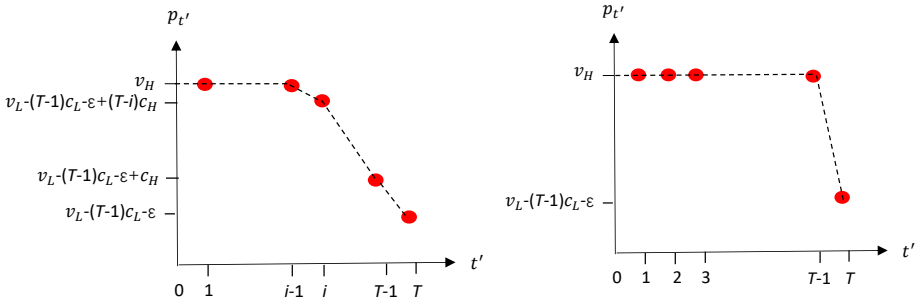
The optimal cyclic pricing policy may reduce to a static pricing policy. The following corollary identifies two sufficient conditions for such a degeneracy.

COROLLARY 3. *If (i)* $c_H/c_L \le 2/(1-\alpha)$ *or (ii)* $v_H - v_L \le c_L/(1-\alpha)$, *an optimal cyclic pricing policy reduces to a static pricing policy.*

By Lemma 7, if $c_H/c_L \le 1$, the optimal cyclic pricing policy reduces to a static price. Corollary 3(i) further shows a weaker condition for such a degeneracy. That is, when the patience level of high-valuation customers relative to low-valuation customers is below a threshold of $2/(1-\alpha)$, cyclic pricing reduces to static pricing, and that situation is more likely to occur if there is a larger fraction of low-valuation customers. In addition, Corollary 3(ii) shows that if $v_H - v_L \le c_L/(1-\alpha)$, i.e., $(1-\alpha)(v_H - v_L) \le c_L$, the optimal cyclic pricing policy reduces to a static price, in which case the benefit of intertemporal price discrimination through having low-valuation customers wait for one period is no more than the waiting cost. Corollary 3 implies that we cannot say cyclic pricing always outperforms static pricing.

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Figure 2 Optimal cyclic prices



3.3.3. Performance Comparison. Neither can we have the same ranking of the performances of randomized pricing and cyclic pricing for all scenarios. We identify the following sufficient conditions under which one does better than the other.

PROPOSITION 8. (RANDOMIZED PRICING VS. CYCLIC DETERMINISTIC PRICING) *Suppose* $v_H - v_L > c_L/(1-\alpha)$. *(i) If* $v_H - v_L$ *is below the threshold of* $c_H \Big/ \left(1+\frac{1}{2}\sqrt{\frac{1-\alpha}{\alpha}}\right)$, *the optimal cyclic pricing policy is in the form of the first* $T-1$ *periods priced at* v_H *and the last period priced at* $v_L-(T-1)c_L-\delta$ *with the optimal cycle length* $T=\lfloor T_1\rfloor$ *or* $\lfloor T_1\rfloor+1$, *and yields a* higher *profit than an optimal randomized pricing policy, where* $T_1=\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}$ *and* $\lfloor x\rfloor$ *represents the greatest integer that is no more than* x. *(ii) If* v_H-v_L *is sufficiently large, the optimal cyclic pricing policy is in the general form of a series of markdowns as illustrated in Figure 2(a) and yields a* lower *profit than an optimal randomized pricing policy.*

For the rest of this section, we assume $v_H - v_L > c_L/(1-\alpha)$. Otherwise, by Corollary 3(ii), the optimal cyclic pricing policy reduces to a static pricing policy, then we can resort to the comparisons between the optimal randomized and static pricing policies studied in Section 3.2.

Proposition 8(i) provides a sufficient condition under which cyclic pricing outperforms randomized pricing. When the valuation difference between the two segments $v_H - v_L$ is below a threshold, the optimal cyclic pricing policy is in the form of the first $T-1$ periods priced at v_H and the last period priced at $v_L-(T-1)c_L-\delta$; see Figure 2(b) for an illustration. That is, under the optimal cyclic pricing, the frequency of running a promotion is $1/T$. Recall that under the optimal randomized pricing, the frequency of running a promotion is β^*. If we equalize these two frequencies, the revenue earned from the high-valuation customers would be the same. However, the waiting cost compensation under the cyclic deterministic policy is lower than that under the randomized policy, because under the cyclic deterministic policy the wait time for a promotion is strictly less than T periods, but under the randomized policy that is expected to be $1/\beta^*$ periods. As a result, it takes less for the cyclic deterministic policy to compensate the low-valuation customers and hence retain more revenue than the randomized policy. Therefore, the optimal cyclic pricing policy has

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a lower promotion frequency, charges a higher promotion price and earns a higher revenue than the optimal randomized pricing policy. To illustrate this, we consider an example with parameters $v_H = 10$, $v_L = 5$, $c_H = 8$, $c_L = 0.3125$, and $\alpha = 0.5$. The optimal randomized pricing policy is to run a promotion with probability $\beta = 0.25$ and a discount price $\underline{p}^* = 3.75$, which achieves a profit of 6.09. On the other hand, the optimal cyclic pricing policy is $\{10, 10, 10, 4.06 - \delta\}$ with a cycle of 4 periods and achieves a profit of almost 6.29.

Proposition 8(ii) shows that randomized pricing performs better than cyclic pricing when the valuation difference between the two segments $v_H - v_L$ is sufficiently large. In this case, the optimal cyclic deterministic policy is in the general form of a series of markdowns as illustrated in Figure 2(a). The underlying rationale can be understood by considering a high-valuation customer who arrives at the second-to-last period under a cyclic deterministic policy. If she waited for a period, she would, for sure, obtain the discounted price that is targeted to the low type. However, this is not the case under randomized pricing. In any period, if a high-valuation customer decided to wait, there would be no guarantee at all she would get a discounted price in the next period. If the discrepancy between v_H and v_L is large enough, it is important for the firm to capture most of the surplus from high-valuation customers. This can be achieved under optimal randomized pricing policies by maintaining a low frequency of offering the discounted price $\underline{p}^*$, and thus the majority of high-valuation customers would make a purchase at the high price v_H upon their arrivals. Given the low-valuation customers are sufficiently patient, the firm can still capture those customers at the occasional promotions. On the other hand, the optimal cyclic pricing policy follows a predetermined price trajectory, known to all customers. Thus, the firm either gives up the low-valuation customers completely by not running promotions at all, or adopts a sequence of markdowns by capturing the low-valuation at the end of a cycle. However, neither is ideal. The former loses the low segment compared to the optimal randomized pricing, and the latter needs to compensate high-valuation customers to prevent them from waiting through a series of markdowns. This compensation cuts into the firm's profitability, again leading to a lower profit compared to the optimal randomized pricing. To illustrate this, we consider an example with parameters $v_H = 10$, $v_L = 1$, $c_H = 5$, $c_L = 0.1$, and $\alpha = 0.5$. The optimal randomized pricing policy is to run a promotion with probability $\beta = 0.1054$ and a discount price $\underline{p}^* = 0.05$, which achieves a profit of 4.50. On the other hand, the optimal cyclic pricing policy is $\{10, 10, 10, 10, 10, 10, 10, 10, 5.1 - \delta, 0.1 - \delta\}$ with a cycle of 10 periods and achieves a profit of almost 4.31.

4. Extensions

In this section, we extend the base model in three directions. We first consider the case where a portion of customers are myopic, and study the impact of myopic customers on the optimal randomized pricing policy, as well as the firm's expected profit. Then we consider a Markovian

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pricing policy, where prices are allowed to be intertemporally correlated in a Markovian fashion, and lastly consider a general model with multiple customer segments.

4.1. Myopic Customers

In practice, it may be the case that some customers are myopic in the sense that they are not patient or sophisticated enough to wait for promotions in the future. In this section, we extend our base model to account for myopic customers, and study how the optimal randomized pricing policy varies with the proportion of myopic customers in the population.

Suppose that $\gamma \in [0,1]$ fraction of customers are strategic, and the remaining customers are myopic, who will never wait. We assume that whether a customer is myopic is independent of their valuation. The rest of the settings are the same as the base model in Section 3. Thus, $\alpha\gamma$ (resp., $(1-\alpha)\gamma$) fraction of low-valuation (resp., high-valuation) customers have a per-period waiting cost c_L (resp., c_H). On the other hand, $\alpha(1-\gamma)$ (resp., $(1-\alpha)(1-\gamma)$) fraction of low-valuation (resp., high-valuation) customers would never wait, i.e., their per-period waiting cost is equal to ∞.

It is easy to show that Proposition 2 still holds for strategic customers by following a similar analysis. That is, under any optimal randomized policy that outperforms the optimal static pricing policy, strategic low-valuation customers would wait, and strategic high-valuation customers would either purchase or leave immediately upon arrival. Consequently, we can derive the optimal randomized pricing policy by solving the following optimization problem:

$$\begin{aligned} \max_{F(p)} \quad & \alpha\gamma \mathsf{E}[P|P \le \underline{p}^L] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] \\ \text{s.t.} \quad & \underline{p}^L = \max\{v' \mid \mathsf{E}[(v'-P)^+] \le c_L\}, \\ & \mathsf{E}[(v_L - P)^+] > c_L, \quad \mathsf{E}[(v_H - P)^+] \le c_H. \end{aligned} \tag{4}$$

The objective function is derived based on Lemma 2, and the constraints are given by Proposition 1. Problem (4) reduces to Problem (2) when $\gamma = 1$. Denote $\tilde{\Delta}_R = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L}}$. We show in the proof of Proposition 9 that an optimal randomized pricing policy reduces to a static pricing policy when either $\tilde{\Delta}_R \ge 1$ or $\tilde{\Delta}_R$ does not exist, i.e., $(1-\alpha)(v_H - v_L) \le \alpha(1-\gamma)v_L$.

Proposition 9. (Optimal Randomized Pricing Policy with Myopic Customers) *If $\tilde{\Delta}_R < 1$, the following two-point price distribution is optimal: $f(p) = \begin{cases} \tilde{\beta}^* & \text{if } p = \underline{\tilde{p}}^*, \\ 1-\tilde{\beta}^* & \text{if } p = v_H, \end{cases}$ where $\tilde{\beta}^* = \min\left\{\tilde{\Delta}_R, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$, $\underline{\tilde{p}}^* = v_L - \frac{c_L}{\tilde{\beta}^*} - \eta(\delta)$ and $\delta > 0$ is sufficiently small. Moreover, $\lim_{\delta \searrow 0} \eta(\delta) = 0$.*

In the presence of myopic customers, the optimal randomized pricing policy still follows a two-point distribution which is composed of a regular price equal to v_H and a discounted price that clears the market of strategic low-valuation customers from time to time. As shown in Corollary 4(i), if $v_L \le (1-\alpha)v_H$, it is optimal to run promotions less frequently with a lower discounted price as the proportion of myopic customers in the population increases; otherwise, the firm is

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better off running promotions more frequently with a higher discounted price. This is because, on the one hand, if $v_L \le (1-\alpha)v_H$, i.e., v_H is high relative to v_L, it is imperative to extract surplus from high-valuation customers. The expected price that a high-valuation customer would pay upon arrival is given by $\tilde{\beta}^*\underline{\tilde{p}}^* + (1-\tilde{\beta}^*)v_H$, which decreases in $\tilde{\beta}^*$. As the proportion of myopic customers increases, high-valuation customers become more important, and thus the firm is better off running promotions less frequently, i.e., choosing a smaller $\tilde{\beta}^*$. The distribution converges to a constant of v_H as $\gamma \to 0$. On the other hand, if $v_L > (1-\alpha)v_H$, it is important for the firm to capture surplus from low-valuation customers. As the proportion of myopic customers increases, the firm should run promotions more frequently so that it will be able to sell to more myopic low-valuation customers. Overall, the existence of myopic customers is detrimental to the firm's profitability, and the expected profit from an optimal randomized pricing policy decreases as the proportion of myopic customers in the population increases, as shown in Corollary 4(ii).

COROLLARY 4. *(i) If $v_L \le (1-\alpha)v_H$, both $\tilde{\beta}^*$ and $\underline{\tilde{p}}^*$ increase in γ; otherwise, they both decrease in γ; (ii) The expected profit of the optimal randomized pricing policy increases in γ.*

4.2. Markovian Pricing

Rather than drawing prices independently from a price distribution in the base randomized pricing policy, we consider a Markovian pricing policy, where random prices over time form a Markov process (see, e.g., Wu et al. 2014, 2020). A Markovian pricing policy gives more control to the firm as the price trajectory is state dependent, and we want to investigate whether this extra layer of control would benefit the firm.

Motivated by the optimal randomized pricing policy characterized in Proposition 3, we restrain our discussion to a Markovian pricing policy with only two values, i.e., a regular price of p and a discounted price of $p_d (\le v_L)$. We denote the probability transition matrix by $\mathbf{M} = \begin{pmatrix} q_L & 1-q_L \\ q_H & 1-q_H \end{pmatrix}$, where q_L is the transition probability from p_d to p_d, and q_H is the transition probability from p to p_d. Similar to our discussion in Section 3, under any optimal Markovian pricing policy, we can show that high-valuation customers would always purchase immediately upon arrival, while low-valuation customers would make a purchase immediately if the price in the current period is p_d and wait otherwise. Consequently, the optimal regular price is given by $p = v_H$, which is assumed to be the case in the rest of our analysis.

For a high-valuation customer arriving in period t, her utility of buying immediately is given by $v_H - p_t$. Denote by $V(p_t)$ the maximum utility for a low-valuation customer arriving in period t, conditional on the price in the previous period being v_H. Then, $V(p_t)$ is given by

$$V(p_t) = \begin{cases} v_L - p_d & \text{if } p_t = p_d, \\ -c_L + \mathsf{E}[V(p_{t+1})] & \text{if } p_t = v_H. \end{cases} \tag{5}$$

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We first study the structural property for an optimal Markovian pricing policy. Lemma 8 characterizes the necessary and sufficient conditions such that high-valuation customers would always purchase immediately upon arrival, while low-valuation customers would either purchase immediately or wait. They are also necessary conditions for a Markovian pricing policy to be optimal.

LEMMA 8. *High-valuation customers would always purchase immediately upon arrival, while low-valuation customers would either purchase immediately or wait if and only if* $q_H(v_L - p_d) > c_L$ *and* $q_H(v_H - p_d) \leq c_H$.

Based on Lemma 8, we can derive an optimal Markovian pricing policy and compare its performance with other pricing policies.

PROPOSITION 10. (OPTIMAL MARKOVIAN PRICING POLICY) *Let* $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}}$.

(i) If $(1-\alpha)(v_H - v_L) \leq c_L$ *(under which either* Δ_M *is not well-defined as a real number or* $\Delta_M \geq 1$*), an optimal Markovian pricing policy (and an optimal randomized pricing policy as well) reduces to a static pricing policy.*

(ii) If $1/2 \leq \Delta_M < 1$ *and* $\frac{c_H - c_L}{v_H - v_L} > 1$*, an optimal Markovian pricing policy is given by* $q_L^* = 0$, $q_H^* = 1$, $p_d^* = v_L - c_L - \eta(\delta)$ *for an arbitrarily small* $\delta > 0$ *and* $\lim_{\delta \searrow 0} \eta(\delta) = 0$*. Moreover, it is a cyclic pricing policy with a cyclic length of* 2*. The expected profit from this optimal Markovian pricing policy is always greater than that from an optimal randomized pricing policy.*

(iii) Otherwise, an optimal Markovian pricing policy is given by $q_L^* = 0$, $q_H^* = \min\left\{\frac{\Delta_M}{1-\Delta_M}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$, $p_d^* = v_L - \frac{c_L}{q_H^*} - \eta(\delta)$ *for an arbitrarily small* $\delta > 0$ *and* $\lim_{\delta \searrow 0} \eta(\delta) = 0$*. The expected profit from this optimal Markovian pricing policy is always greater than that from an optimal randomized pricing policy.*

There are a couple of interesting observations from Proposition 10. First, we show in Proposition 10(i) that, when $(1-\alpha)(v_H - v_L) \leq c_L$, both an optimal Markovian pricing policy and an optimal randomized pricing policy as a special case of the Markovian policy reduce to a static pricing policy. However, when $0 \leq \Delta_M < 1$, an optimal Markovian pricing policy is guaranteed to be more profitable than an optimal randomized pricing policy, though it may reduce to a cyclic pricing policy (see Proposition 10(ii)). Second, q_L^* is equal to 0 in the optimal Markovian pricing policy. That is, after running a promotion in one period by charging the lower price p_d, the firm would not want to do it again immediately in the following period. This is sensible because p_d essentially serves as a market clearing price, and all customers arrived so far would make a purchase and leave immediately with the lower price p_d. Then in the next period, the system restarts, and the firm would be better off charging the higher price to extract surplus from high-valuation customers. Such an extra tune-up of the Markovian pricing policy potentially avoids running promotions consecutively and leaving surplus on the table for high-valuation customers. It indeed allows the

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firm to gain higher profit when $\Delta_M < 1$, which requires the low-valuation customers be sufficiently patient.

Now we compare the key decision variables in an optimal Markovian pricing policy with those in an optimal randomized pricing policy when Markovian pricing dominates randomized pricing. The results are summarized in Corollary 5.

COROLLARY 5. *If c_L is sufficiently small, $p_d^* \geq \underline{p}^*$ and $\pi_L^* \geq \beta^*$, where π_L^* is the steady state probability of p_d^*. Moreover, $L_M^* = 1/\pi_L^* \leq L_F^* = 1/\beta^*$, where L_M^* and L_F^* are the expected length of time periods between two consecutive discounts under optimal Markovian pricing and randomized pricing, respectively.*

By Proposition 10, if c_L is sufficiently small, the optimal Markovian pricing policy dominates the optimal randomized pricing policy, which further dominates the optimal static pricing policy. Interestingly, the optimal Markovian pricing policy would run the promotion more often than the optimal randomized pricing policy, but with a smaller size of the discount. This is because, as explained, the optimal Markovian pricing policy avoids running promotions consecutively, and as a result, the low-valuation customers need to be compensated more for waiting. To do so, the firm can either increase the frequency of promotions, but at the same time, increase the discount price, or alternatively, decrease the frequency of promotion but offer a more appealing discount price. The former is more profitable for the firm, as it reduces the surplus left to the high-valuation customers arriving in the promotional period. Overall, such a Markovian tune-up ensures the incentive compatibility for low-valuation customers to wait, while reducing the surplus enjoyed by high-valuation customers if they happen to arrive at a promotion period.

Finally, we confirm our intuition that Markovian pricing fares better than randomized pricing because it extracts weakly more surplus from both market segments.

COROLLARY 6. *If $v_H - v_L > c_L$ and $\Delta_M < 1$, an optimal Markovian pricing policy can extract weakly more profit from low-valuation customers, and strictly more profit from high-valuation customers than an optimal randomized pricing policy.*

Under the condition that $v_H - v_L > c_L$ and $\Delta_M < 1$, by Proposition 10, Markovian pricing dominates randomized pricing. It does so by smartly running promotions and avoiding leaving surplus on the table to the high-valuation customers as in randomized pricing, while still keeping the low-valuation customers incentive compatible to wait for a promotion.

4.3. Multiple Customer Segments

Next we consider an extension of the base model with n customer segments. A fraction α_i of customers are of type-i, who value the product at v_i. Without loss of generality, we assume $v_i < v_{i+1}$,

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for $i=1,2,\ldots,n-1$. The one-period waiting cost for type-i customers is denoted by c_i. We make the following assumptions in this section.

Assumption (W) (Customers May Wait) $0<c_i<v_i$, $i=1,2,\ldots,n$.

Assumption (O) (Ordered Waiting Costs) $c_1<c_2<\cdots<c_n$.

Assumption (W) is an extension of Assumption (S) for multiple customer segments. The model becomes trivial if it is violated, because customers would either purchase the product or leave immediately upon arrival. Assumption (O) implies that the waiting cost per period is higher for customers with a higher valuation. This assumption is innocuous in the sense that, as we will show in Lemmas 9 and OA.5, if $c_1\geq c_2\geq\cdots\geq c_n$, both an optimal randomized pricing policy and an optimal deterministic pricing policy reduce to a static pricing policy. In other words, if there exists a pair of i and $j(>i)$ such that $c_i\geq c_j$, neither randomized pricing policies nor deterministic pricing policies are able to effectively price discriminate the two segments. Thus, this scenario is as if they belong to the same segment, and our analysis is still applicable with the number of customer segments reduced by one.

4.3.1. Optimal Randomized Pricing Policy. Denote the power set of the set $\{1,2,\ldots,n\}$ by $Q(n)$, and, under any randomized pricing policy, denote the set of customer segments that will wait by $\mathcal{W}\in Q(n)$. The rest of customers will either purchase or leave immediately upon arrival. The firm's problem can be formulated as the following optimization problem:

$$\begin{aligned}\max_{F(p),\mathcal{W}\in Q(n)}\quad & \sum_{i\in\mathcal{W}}\alpha_i\mathsf{E}[P|P\leq \underline{p}^i]+\sum_{i\notin\mathcal{W}}\alpha_i F(v_i)\mathsf{E}[P|P\leq v_i]\\ \text{s.t.}\quad & \underline{p}^i=\max\{v'\mid \mathsf{E}[(v'-P)^+]\leq c_i\},\quad \forall\, i\in\mathcal{W},\\ & \mathsf{E}[(v_i-P)^+]>c_i,\quad \forall\, i\in\mathcal{W},\\ & \mathsf{E}[(v_i-P)^+]\leq c_i,\quad \forall\, i\notin\mathcal{W}.\end{aligned}\tag{6}$$

The objective function is derived from our discussion in Section 3.1, and the constraints are due to Proposition 1. Potentially, we may solve problem (6) by conditioning on $\mathcal{W}$, and evaluate the corresponding profits from optimal price distributions across all possible $\mathcal{W}$. But this is a non-trivial problem due to: (i) Conditioning on $\mathcal{W}$, we still need to optimize the price distribution, where there is no obvious structure that we can exploit; (ii) More critically, the cardinality of the power set $Q(n)$ increases exponentially in n. That is, the number of optimization problems that we need to evaluate increases exponentially in the number of customer segments. Thus it demands us to start with exploring structural properties of the optimal randomized pricing policy.

Notice that a static pricing policy is a special case of randomized pricing policies, where the price distribution reduces to a constant. The expected profit from an optimal static pricing policy, under which customers behave myopically, is given by $\max\limits_{1\leq j\leq n}\left\{\sum\limits_{i=j}^{n}\alpha_i v_j\right\}$. We first show a sufficient condition under which an optimal randomized pricing policy degenerates to a static pricing policy.

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Lemma 9. (Impact of Patience Level) *If $v_1/c_1 \le v_2/c_2 \le \cdots \le v_n/c_n$, there exists $k \in \{1,2,\ldots,n\}$ for any randomized pricing policy such that (i) any customer of valuation greater than v_k will wait; (ii) any customer of valuation less than or equal to v_k will either purchase or leave immediately upon arrival. Under the above condition, an optimal randomized pricing policy reduces to a static pricing policy.*

As the counterpart of Lemma 4, Lemma 9 shows that, under the condition $v_i/c_i \le v_j/c_j$ (i.e., $c_i/c_j \ge v_i/v_j$), $\forall\, i<j$, the firm would not be better off with any price randomization. We make some observations. First, Lemma 9 provides justification for Assumption (O) because it implies that, under $c_1 \ge c_2 \ge \cdots \ge c_n$ that completely flips the order in Assumption (O), an optimal randomized pricing policy reduces to a static pricing policy. That is, under any randomized pricing policy, the monopolist cannot obtain a higher profit than that from an optimal static pricing policy when customers of higher valuations have lower waiting costs per period. Second, as shown in the proof of Lemma 9, $v_i/c_i > v_j/c_j$, $\forall\, i<j$, is a necessary condition for customers of a lower valuation to wait and customers of a higher valuation to either purchase or leave immediately under any randomized pricing policy. Intuitively, this is a situation that benefits the most for the firm because the firm can charge higher prices for customers of higher valuations, while at the same time set lower prices randomly once a while and clear the market of hanging-around customers of lower valuations. However, $v_i/c_i > v_j/c_j$, $\forall\, i<j$, is not a sufficient condition for this to happen. We are able to identify one sufficient condition.

Assumption (R) (Ranked Valuation Decay) $v_1 - c_1 \ge v_2 - c_2 \ge \cdots \ge v_n - c_n$.

Assumption (R) says that though upon arrival customer valuations are ordered such that type 1 has the lowest valuation, the order is reversed if all wait. That is, type 1 customers have the highest residual valuation after waiting (for one period), so they have more incentive to wait. An analogous assumption is made in Golrezaei et al. (2020) for two customer types (see Figure 1 therein).[16]

Proposition 11. (Optimality of Two-Price Randomized Pricing Policy) *Under the additional Assumption (R), an optimal randomized pricing policy follows a two-point price distribution. In particular, under the optimal two-point price distribution: (i) There exists $k_r \in \{1,\ldots,n\}$ such that any customer with valuation less than v_{k_r} would choose to wait, while any customer with valuation greater than or equal to v_{k_r} would either purchase or leave immediately upon arrival; (ii) The lower price point is less than v_1; (iii) The higher price point is equal to v_{h_r}, where $h_r = \arg\max_{k_r \le j \le n} \left(\sum_{i=j}^{n} \alpha_i v_j \right)$.*

[16] In particular, if customers of type-i have a discount factor $\gamma_i \in (0,1]$, Assumption (R) would be equivalent to $\gamma_1 v_1 \ge \gamma_2 v_2 \ge \cdots \ge \gamma_n v_n$.

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Proposition 11 is handy when it comes to identifying the optimal randomized pricing policy. First, it shows that under the additional Assumption (R), customer behavior is sorted in the sense that there exists a cutoff k_r in type, where any customer with valuation less than v_{k_r} would choose to wait, while any customer with valuation greater than or equal to v_{k_r} would either purchase or leave immediately upon arrival. Consequently, instead of solving 2^n optimization problems as suggested in problem (6), we only need to evaluate n optimization problems in order to identify the optimal price distribution, which represents a significant improvement computationally. Second, for a given k_r, n segments of customers have 3 types of behavior under the optimal randomized pricing policy that turns out to follow a two-point distribution: those low-valuation customers from type 1 to type $k_r - 1$ would wait for the discount price (lower than v_1 so that type 1 customers feel worthwhile waiting), those high-valuation customers from type h_r and above would always buy upon arrival, and those customers with types in between only buy at the discount price or leave immediately upon arrival. The latter intermediate case would not exist when there were two customer segments.

4.3.2. Optimal Cyclic Pricing Policy. As shown in the proof of Proposition 5, the optimal customers' response under any cyclic pricing policy can be summarized as follows.

Corollary 7. (Optimal Customer Response under Cyclic Pricing) *The optimal customer response can be characterized as follows: there exist $k_c \in \{1, 2, \ldots, n\}$ and $k_0 \in \{1, 2, \ldots, k_c - 1\}$ such that, (i) any customer of valuation greater than or equal to v_{k_c} will either buy or leave immediately upon arrival; (ii) any customer of valuation less than v_{k_0} will leave immediately upon arrival; (iii) further, there exists τ_i for any type-i customer, $k_0 \le i < k_c$, such that she leaves immediately upon arrival if she arrives before the τ_i^{th} period within a cycle; otherwise, she will wait to buy at the end of a cycle.*

Following a similar approach as the case with two customer segments, we investigate structural properties, and then construct an optimal cyclic policy based on them. The optimal cyclic pricing policy is summarized in the proposition below, and results on structural properties are relegated to Online Appendix B.

Proposition 12. (Optimal Cyclic Pricing) *Under the additional Assumption (R), an optimal cyclic pricing policy has the following characterizations: (i) $\tau_i \le \tau_{i+1}$, $\forall\, i < k_c - 1$. In particular, $\tau_1 = 1$; (ii) Moreover, if $v_1 \ge (\alpha_2 + \cdots + \alpha_n)v_2 \ge \cdots \ge \alpha_n v_n$, the optimal pricing schedule is given by $p_T = \min\limits_{1 \le i < k_c} \{v_i - (T - \tau_i)c_i\} - \delta$ and $p_t = \min\{v_{k_c}, p_T + (T-t)c_{k_c}\}$, $t = 1, 2, \ldots, T-1$, where $\delta > 0$ is arbitrarily small.*

Similar to that of Proposition 11, in order to derive more insights and structural properties of the optimal cyclic pricing policy, we restrict ourselves with the additional Assumption (R). We make

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some observations from Proposition 12. First, under the stipulated condition, customers of the lowest valuation would definitely buy at the end of a cycle, while customers of a higher valuation tend to wait as it comes closer to the end of a cycle. Intuitively, this is a very favorable situation to the firm due to: (1) customers of the lowest valuation has the lowest per-period waiting cost. Thus, incentive provided for their waiting is not too costly for the firm; (2) the firm is able to extract surplus from a large fraction of customers of lower valuations at the end of a cycle.

Moreover, we are able to characterize the optimal pricing schedule if we further assume that $v_1 \geq (\alpha_2 + \cdots + \alpha_n)v_2 \geq \cdots \geq \alpha_n v_n$. In this case, the optimal static price is given by v_1, which is often referred to as volume strategy. Under the optimal cyclic pricing policy, the price level across two consecutive periods either stays constant at v_{k_c} or drops by a size no more than c_{k_c}, which eliminates the incentive for customers of valuation greater than or equal to v_{k_c} to wait. More precisely, in general, within a cycle, the prices stay constant at v_{k_c} for some time (which can be zero), drops by a size no more than c_{k_c} and then finally drops by a size of exactly c_{k_c} to the end-of-cycle price $\min_{1\leq i<k_c}\{v_i - (T-\tau_i)c_i\} - \delta$. On the other hand, customers of valuation less than v_{k_c} would wait patiently in the market upon arrival (conditional on being sufficiently close to the end of a cycle), because their waiting is more than fully compensated. Eventually, the market of customers of lower valuations will be cleared at the end of a cycle.

4.3.3. Randomized Pricing vs. Cyclic Pricing. Performance comparison between optimal randomized pricing policies and optimal cyclic pricing policies with n customer segments is summarized as follows.

Proposition 13. (Randomized Pricing vs. Cyclic Pricing) *Under the additional Assumption (R) and $v_1 \geq (\alpha_2 + \cdots + \alpha_n)v_2 \geq \cdots \geq \alpha_n v_n$, we have:*

(i) the problem with n customer segments is equivalent to the problem with two customer segments in which the $\sum_{j=1}^{k_r-1}\alpha_j$ fraction of customers have valuation v_{k_r-1} and per-period waiting cost c_{k_r-1} and the rest fraction of customers have valuation v_{k_r} and per-period waiting cost c_{k_r}.

(ii) If $v_{k_r} - v_{k_r-1} > c_{k_r-1}$ and $v_{k_r} - v_{k_r-1} < c_{k_r} \Big/ \left(1 + \frac{1}{2}\sqrt{\sum_{j=k_r}^{n}\alpha_j \Big/ \sum_{j=1}^{k_r-1}\alpha_j}\right)$, an optimal cyclic pricing policy yields a higher *profit than an optimal randomized pricing policy.*

(iii) There exists a threshold on $v_{k_c} - v_{k_c-1}$, above which an optimal cyclic pricing policy yields a lower *profit than an optimal randomized pricing policy.*

As the counterpart of Proposition 8, Proposition 13 shows that our conclusion from the two-segment case remains in the general case with n customer segments under certain conditions. In particular, Proposition 13 is derived under the extra Assumption (R) and the condition $v_1 \geq (\alpha_2 + \cdots + \alpha_n)v_2 \geq \cdots \geq \alpha_n v_n$. The former condition ensures that an optimal randomized pricing policy follows a two-point price distribution, with the lower price being less than v_1 and the higher price equal to $v_{h_r} = v_{k_r}$, with the identity ensured by the latter condition. We supplement this theoretical result with comprehensive numerical studies in Online Appendix C for additional insights.

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5. Conclusion

In this paper, we consider randomized pricing as an alternative dynamic pricing strategy. We show that, compared against the optimal static pricing policy, the optimal randomized pricing policy always benefits the firm in terms of profitability, when customers of the lowest valuation are sufficiently patient. We further gauge the performance of an optimal randomized pricing policy by comparing it against cyclic pricing policies, which arise to be optimal among deterministic policies. In this case, neither policy always dominates the other. However, when the valuation differences among customers are sufficiently large, an optimal randomized pricing policy yields higher profit than cyclic pricing policies. We expect that these results continue to hold when customers discount utilities from future periods. Furthermore, customers' patience levels can differ significantly among different types of products such as hedonic vs. utilitarian products. Our model provides useful guidance for firms regarding whether randomized pricing can be beneficial, evaluated at the estimated parameters, no matter which type of product it is.

We model randomized pricing as a price distribution that a firm commits itself to, and a price is randomly drawn from this distribution in each time period. This is, arguably, a stylized model for randomized pricing, which allows us to analyze the problem analytically. However, a randomized pricing policy may manifest itself in different ways in practice. The key essence of randomized pricing, compared to deterministic pricing, is that prices shall appear somewhat randomly from the perspective of customers, which hinders their ability to predict prices accurately. For instance, firms might well adopt deterministic pricing algorithms that depend on factors such as the marginal cost of procuring/selling a product, which are not directly observable to customers. Due to a random disruption at the supply source or a fluctuation in the inventory level, prices generated by the pricing algorithms may vary accordingly, which would seem random in the eye of consumers, as they have no visibility of these price-governing states and thus are not able to predict the future promotions. Our results show that the firm may benefit from this "randomization" without deliberation, especially when the variation in customer valuations is significant.

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Online Appendix to "Intertemporal Price Discrimination via Randomized Promotions": The Case of Multiple Customer Segments

A. Randomized Pricing

The following notation is used throughout proofs for randomized pricing policies with n customer segments.

(i) $u_i = \underline{p}^i$, $\forall\, i \in \mathcal{W}$, and $u_i = v_i$, $\forall\, i \notin \mathcal{W}$, where $\underline{p}^i$ is defined in problem (6);

(ii) $\beta_1 = F(u_1)$, and $\beta_i = F(u_i) - F(u_{i-1})$, $\forall\, i > 1$;

(iii) $x_1 = \mathsf{E}[P|P \le u_1]$, and $x_i = \mathsf{E}[P|u_{i-1} < P \le u_i]$, $\forall\, i > 1$.

With this set of notation, problem (6) can be reformulated as follows, conditional on the set of customers who will wait.

Lemma OA.1. *Consider a set of randomized pricing policies, where the set of customers that will wait under any policy in the set remains the same, which is denoted by $\mathcal{W}$, and the rest of customers will either purchase or leave immediately upon arrival. The optimal policy in this set can be derived by solving the following optimization problem.*

$$
\begin{aligned}
\max_{\beta,\mathbf{x}} \quad & \sum_{i=1}^{n} \alpha_i \pi_i \\
s.t. \quad & \beta_1 v_1 - \beta_1 x_1 > c_1, \quad \textit{if } 1 \in \mathcal{W}, \\
& \sum_{j=1}^{i} \beta_j x_i - \sum_{j=1}^{i} \beta_j x_j = c_i, \quad \forall\, i \in \mathcal{W} \setminus \{1\}, \\
& \sum_{j=1}^{i} \beta_j v_i - \sum_{j=1}^{i} \beta_j x_j \le c_i, \quad \forall\, i \notin \mathcal{W}, \\
& x_i < v_i, \quad \forall\, i \in \mathcal{W}, \\
& x_i \le v_i, \quad \forall\, i \notin \mathcal{W}, \\
& \sum_{i=1}^{n} \beta_i = 1,
\end{aligned}
\tag{OA.1}
$$

where $\pi_i = \sum_{j=1}^{i} \beta_j x_j \Big/ \sum_{j=1}^{i} \beta_j$, $\forall\, i \in \mathcal{W}$; *Otherwise,* $\pi_i = \sum_{j=1}^{i} \beta_j x_j$.

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LEMMA OA.2. *There exists a threshold on c_1, below which an optimal randomized pricing policy outperforms an optimal static pricing policy.*[OA.1]

Lemma OA.2 shows that when customers of the lowest valuation are sufficiently patient, an optimal randomized pricing policy will not reduce to a static pricing policy. The rationale is as follows. Recall that an optimal static pricing policy is to set the price at v_{i^*} such that $i^* = \arg\max_i \sum_{j=i}^{n} \alpha_j v_i$. So one simple way to construct a randomized pricing strategy is to charge v_{i^*} most of the time, but run random promotions with a discount price lower than v_1 once in a while to extract surplus from customers of the lowest valuation. Because customers of valuation v_1 are sufficiently patient, it can be guaranteed that extra surplus from type-1 customers outweighs the profit loss from customers of higher valuations, due to type-1 customers' purchases during promotion that can be run sufficiently infrequently. Thus, a randomized pricing policy constructed as mentioned above outweighs an optimal static pricing policy, and thus so does an optimal randomized pricing policy.

B. Deterministic Cyclic Pricing

Proposition 5 ensures that we do not need to consider beyond cyclic pricing for the optimal deterministic pricing. As shown in the proof of Proposition 5, the optimal customers' response under any cyclic pricing policy can be summarized as follows.

COROLLARY OA.1. (OPTIMAL CUSTOMER RESPONSE UNDER CYCLIC PRICING) *The optimal customer response can be characterized as follows: there exists $k_c \in \{1, 2, \ldots, n\}$ and $k_0 \in \{1, 2, \ldots, k_c - 1\}$ such that,*

(i) any customer of valuation greater than or equal to v_{k_c} will either buy or leave immediately upon arrival;

(ii) any customer of valuation less than v_{k_0} will leave immediately upon arrival;

(iii) there exists τ_i for any type-i customer, $k_0 \le i < k_c$, such that she leaves immediately upon arrival if she arrives before the τ_i^{th} period within a cycle; Otherwise, she will wait to buy at the end of a cycle.

[OA.1] It boils down to Proposition 4 if $n = 2$.

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Corollary OA.1(i) shows that there exists a cutoff on customer valuation, denoted by v_{k_c}, such that any customer with valuation greater than or equal to v_{k_c} will never wait. This is a consequence of Assumption (O), which assumes that per-period waiting costs follow an ascending order. Thus, with the benefit of waiting the same for all types of customers (because they see the same deterministic price path), it is more costly for customers of a higher valuation to wait. In the other extreme, Corollary OA.1(ii) says that there may exist a segment of customers of low valuations who will leave immediately upon arrival. This segment includes those customers whose valuation is less than p_t, $\forall t$. Then, we also have a segment of customers of intermediate valuations who will wait if and only if she arrives sufficiently close to the end of a cycle. Otherwise, her utility from waiting becomes negative due to the cost of waiting, and thus she is better-off leaving immediately upon arrival.

Given customers' optimal response as characterized in Corollary OA.1, the firm's long-run average expected profit with a cyclic pricing policy $\mathbf{p}$ is given by $\Pi(\mathbf{p}) = \lim_{T'\to\infty} \frac{1}{T'} \sum_{t=1}^{T'} \pi_t(\mathbf{p})$, where $\pi_t(\mathbf{p})$ is the profit from customers arriving in period t. We have $\pi_t(\mathbf{p}) = \pi_{t+T}(\mathbf{p})$ due to prices being cyclic. Our goal is to find a cyclic pricing policy $\mathbf{p}$ such that $\Pi(\mathbf{p})$ is maximized.

Next we prove some key structural properties that an optimal cyclic pricing policy must satisfy. We derive these based on the following partial characterization of an optimal cyclic pricing policy.

Lemma OA.3. (Optimal Cyclic Pricing conditional on Cutoffs) *The optimal pricing schedule is in the following form, with k_c and k_0 defined in Corollary OA.1:*

(i) At the T^{th} period, $p_T = \min_{k_0 \le i < k_c} \{v_i - (T - \tau_i)c_i\} - \delta$, where $\delta > 0$ is sufficiently small;

(ii) At the t^{th} period, $t = 1, \ldots, T-1$, we have

(a) *if $p_{t+1} + c_{k_c} \le v_{k_c}$, $p_t = p_{t+1} + c_{k_c}$;*

(b) *if $p_{t+1} + c_{k_c} > v_{k_c}$,*

$$p_t = \begin{cases} p_{t+1} + c_{k_t}, & \text{if } (\alpha_{k_t} + \cdots + \alpha_n)(p_{t+1} + c_{k_t}) > (\alpha_{l_t} + \cdots + \alpha_n)v_{l_t}, \\ v_{l_t}, & \text{if } (\alpha_{k_t} + \cdots + \alpha_n)(p_{t+1} + c_{k_t}) \le (\alpha_{l_t} + \cdots + \alpha_n)v_{l_t}, \\ v_l, & \text{if } k_t \text{ does not exist}, \end{cases}$$

where $k_t = \arg\min_{k_c \le i} \{p_{t+1} + c_i \le v_i\}$, $l_t = \arg\max_{k_c \le i < k_t} \{(\alpha_i + \cdots + \alpha_n)v_i\}$ and $l = \arg\max_{k_c \le i} \{(\alpha_i + \cdots + \alpha_n)v_i\}$.

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Lemma OA.3 sheds light on the structure of an optimal cyclic pricing policy. The price at period T is simply the highest price under which customers' purchasing decisions are consistent with that characterized in Corollary OA.1. Then conditional on p_T, we can construct the optimal pricing schedule recursively backwards in time. At time period $t<T$, p_t cannot be greater than $p_{t+1}+c_{k_t}$, where $k_t=\underset{k_c\le i}{\arg\min}\{p_{t+1}+c_i\le v_i\}$; otherwise, a customer of valuation v_{k_t} is better off waiting to buy in future periods, which contradicts to the optimal customer response as characterized in Corollary OA.1. Therefore, subject to this constraint, the optimal price at any time period $t<T$ is the one that maximizes the firm's profit in that focal period.

With Lemma OA.3 in hand, we are able to characterize some structural properties of an optimal cyclic pricing policy. First, we show that an optimal cyclic pricing policy follows a markdown pattern, which is summarized formally in the lemma below.

Lemma OA.4. (Markdown Pattern) *An optimal cyclic pricing policy follows a (weakly) markdown pattern.*

This result is not surprising in the sense that, given that customers of higher valuations would either buy or leave immediately upon arrival, and customers of lower valuations would wait, the firm has incentive to charge higher prices early on in a cycle to extract as much surplus from customers of higher valuations as possible, and then clear the market of customers of lower valuations when it approaches the end of a cycle. Having said that, this strategy is only viable when it is incentive compatible for customers of higher valuations to buy immediately, which implies that the price reduction in any two consecutive periods shall be less than or equal to the waiting cost of those customers of higher valuations.

Lemma OA.3 also implies that, under any optimal cyclic pricing policy, the firm will not charge a price higher than v_n, the highest valuation from customers. With this, we are able to identify two sufficient conditions under which an optimal cyclic pricing policy reduces to a static pricing policy.

Lemma OA.5. (Degeneration of Optimal Cyclic Pricing to Static Pricing) *An optimal cyclic pricing policy reduces to a static pricing policy, if either of the following conditions is satisfied:*

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(i) $v_1/c_1 \le v_2/c_2 \le \cdots \le v_n/c_n$;

(ii) $c_{i+1}/c_i \le \sum_{j=i}^{n} \alpha_j \Big/ \sum_{j=i+1}^{n} \alpha_j$, $\forall i < n$.

In particular, under condition (i), we have $\tau_{i+1} \le \tau_i$, *for any* $k_0 \le i < k_c - 1$, *where* τ_i *is defined in Corollary OA.1.*

As the counterpart of Lemma 9, Lemma OA.5(i) illustrates the impact of customers' patience levels on the structure of the optimal cyclic pricing policy. When customers of a higher valuation are more patient than customers of a lower valuation, i.e., $v_i/c_i \le v_j/c_j$, $\forall i \le j$, cyclic pricing policies cannot outperform the optimal static pricing policy. In particular, under the condition in Lemma OA.5(i), among customers with valuation v_i, $k_0 \le i < k_c - 1$ who might wait by Corollary OA.1, customers of a higher valuation are more patient than customers of a lower valuation in the sense that they will start to wait earlier within a cycle, i.e., $\tau_{i+1} \le \tau_i$, for any $k_0 \le i < k_c - 1$. This contributes negatively to the firm's profit due to: (i) the firm is not able to effectively price discriminate among customers who will wait as customers of a higher valuation can effectively mimic behavior of customers of a lower valuation; (2) the firm needs to charge a lower price at the end of a cycle to compensate for the waiting of customers of a higher valuation, who also have a higher per-period waiting cost.

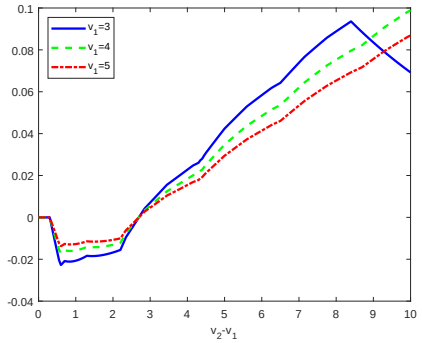
An alternative way to understand Lemma OA.5(i) is as follows. We can rewrite the condition in Lemma OA.5(i) as $c_{i+1}/c_i \le v_{i+1}/v_i$, $\forall i < n$. That is, if the ratio of per-period waiting costs of two customer segments is less than the ratio of their valuations, i.e., the discrepancy of per-period waiting costs of any two neighboring customer types is sufficiently small, any cyclic pricing policy cannot effectively price discriminate among the two customers. Lemma OA.5(ii) shows an alternative threshold for the ratio of per-period waiting costs, below which any cyclic pricing policy cannot outperform an optimal static pricing policy.

C. Performance Comparison: Numerical Results

Analytically, we are able to characterize the optimal randomized pricing policy under certain conditions with multiple customer segments, as shown in Section 4.3. In this section, we seek to complement our theoretical analysis with comprehensive numerical studies.

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Figure OA.1 Relative Profit Gap between Optimal Randomized Pricing and Optimal Cyclic Pricing



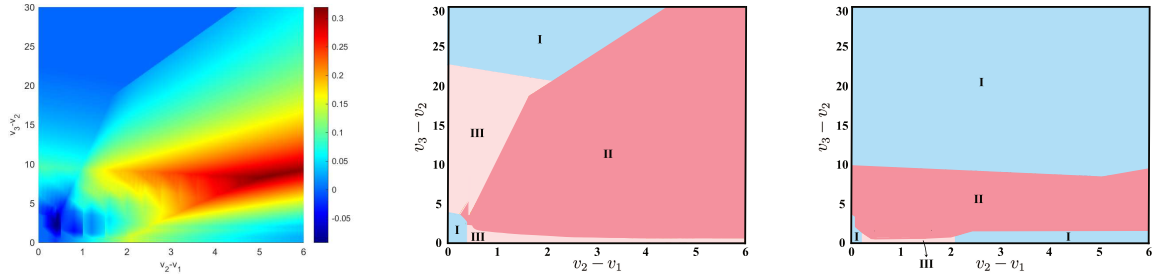
Note. Consider the case with two customer segments. Half of them value the product at v_1 with a per-period waiting cost of $c_1 = 0.1$, and the other half value the product at v_2 with a per-period waiting cost of $c_2 = 2.5$.

Our first study focuses on the case when there are two segments of customers. The parameters of the study are specified as follows: one half of customers value the product at v_1 with a per-period waiting cost of $c_1 = 0.1$, and the other half of customers value the product at v_2 with a per-period waiting cost of $c_2 = 2.5$. We consider 3 scenarios, where v_1 takes a value of 3, 4 and 5, respectively. In each scenario, $v_2 - v_1$ ranges from 0 to 10 with an incremental value of 0.05. Figure OA.1 summarizes the relative profit differences between an optimal randomized pricing policy and an optimal cyclic pricing policy. When the difference between the valuations is marginal, neither policies can effectively price discriminate the two customer segments, and thus both of them reduce to the optimal static pricing policy, leading to zero performance gap. As $v_2 - v_1$ increases, an optimal cyclic pricing policy first outperforms an optimal randomized pricing policy, but this relationship is reversed when $v_2 - v_1$ becomes sufficiently large. This result corroborates our theoretical finding in Proposition 13. In particular, the turning points are identical across the three scenarios, which suggests that whether randomized pricing can outperform cyclic pricing solely depends on the difference in valuations, rather than their absolute values. This result has empirical implications, as in many circumstances, it is easier to estimate the valuation difference than the valuation itself.

Next we move on to the case with multiple customer segments. In particular, we will focus on the case when there are three segments of customers, which proves to be a sufficiently sophisticated context that allows us to investigate the nuances properly. The setup of the study is as follows: 1/3 of customers have a low valuation of $v_1 = 3$ with a per-period waiting cost of $c_1 = 0.1$, 1/3

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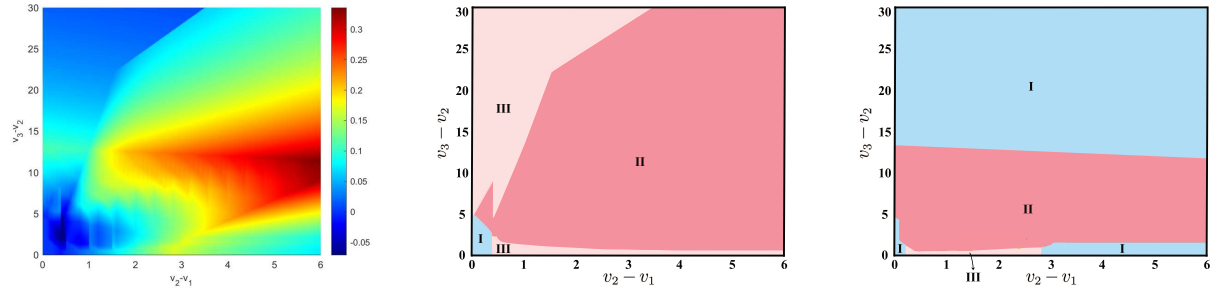
Figure OA.2 Optimal Randomized Pricing vs. Optimal Cyclic Pricing when $n = 3$ and $v_1 = 3$



(a) Relative Profit Gap (b) Optimal Randomized Pricing (c) Optimal Cyclic Pricing

Note. In Figures OA.2(b) and OA.2(c), in those regions of type (I), the corresponding pricing policy reduces to a static pricing policy; in those regions of type (II), customers of valuations v_1 and v_2 would wait, and customers of valuation v_3 would buy immediately upon arrival under the corresponding pricing policy; in those regions of type (III), customers of valuation v_1 would wait, customers of valuation v_2 would either purchase or leave immediately upon arrival, and customer of valuation v_3 would buy immediately upon arrival under the corresponding pricing policy.

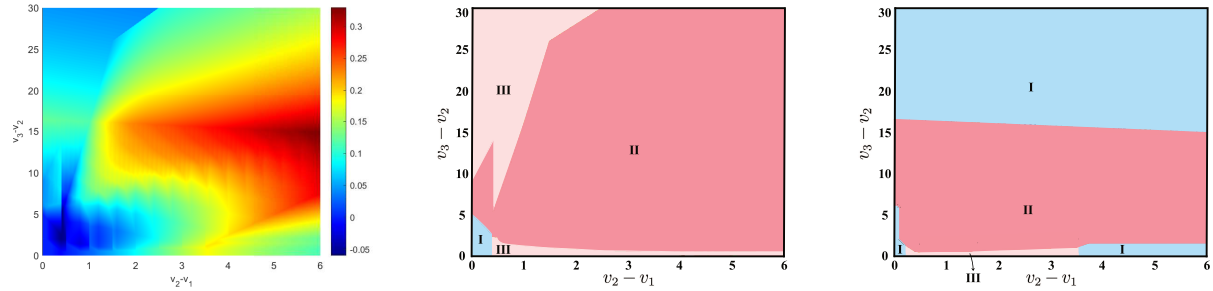
Figure OA.3 Optimal Randomized Pricing vs. Optimal Cyclic Pricing when $n = 3$ and $v_1 = 4$



(a) Relative Profit Gap (b) Optimal Randomized Pricing (c) Optimal Cyclic Pricing

Note. Parameters are the same as those in Figure OA.2, except for $v_1 = 4$.

Figure OA.4 Optimal Randomized Pricing vs. Optimal Cyclic Pricing when $n = 3$ and $v_1 = 5$



(a) Relative Profit Gap (b) Optimal Randomized Pricing (c) Optimal Cyclic Pricing

Note. Parameters are the same as those in Figure OA.2, except for $v_1 = 5$.

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of customers have a medium valuation v_2 with a per-period waiting cost of $c_2 = 0.5$, and the rest 1/3 of customers have a high valuation of v_3 with a per-period waiting cost of $c_3 = 2.5$. In this study, $v_2 - v_1$ ranges from 0 to 6, and $v_3 - v_2$ ranges from 0 to 30. Figure OA.2(a) summarizes the relative profit gap between optimal randomized pricing policies and optimal cyclic pricing policies. To facilitate the understanding of Figure OA.2(a), we also plot the customers' responses under the optimal randomized pricing and optimal cyclic pricing policies in Figures OA.2(b) and OA.2(c), respectively. We test the robustness of these observations with other parameters, and observe similar patterns, see Figures OA.3 and OA.4.

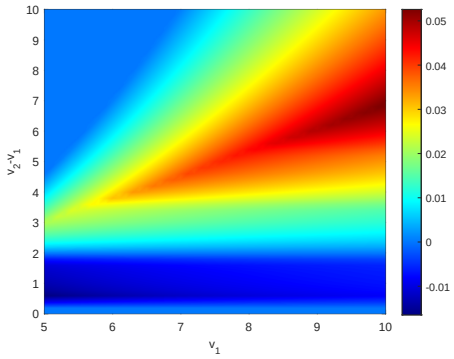
We make some observations from Figure OA.2. First, when the differences of valuations (i.e., both $v_2 - v_1$ and $v_3 - v_2$) are sufficiently small, neither randomized pricing nor cyclic pricing can effectively price discriminate customers, and thus both of them reduce to an optimal static pricing policy, leading to zero performance gap. As $v_2 - v_1$ or $v_3 - v_2$ increases, cyclic pricing first becomes effective in price discriminating customers whereas optimal randomized pricing still reduces to optimal static pricing, as suggested by the smaller region of type (I) in the bottom left corner of Figure OA.2(c) than that in OA.2(b). However, as $v_2 - v_1$ and $v_3 - v_2$ continue to increase, this relationship is reversed where optimal randomized pricing consistently outperforms optimal cyclic pricing. This observation is consistent with our analytical result in Proposition 13, even though the condition $v_1 - c_1 \geq v_2 - c_2 \geq v_3 - c_3$ is not satisfied in the majority of the parameter space in this study.

Second, the relative performance gap between optimal randomized pricing and cyclic pricing is nonlinear, which reaches its maximum when both $v_2 - v_1$ and $v_3 - v_2$ are of medium values. The rationale is as follows. When both $v_2 - v_1$ and $v_3 - v_2$ are relatively small, performance gaps are small, with cyclic pricing dominating randomized pricing under certain circumstances, as discussed above. As $v_2 - v_1$ and/or $v_3 - v_2$ increase, the effectiveness of randomized pricing becomes more salient, as it is able to extract most of surplus from customers of higher valuations while keeping customers of lower valuations waiting in the system, whereas optimal cyclic pricing policies quickly reduce to optimal static pricing policies, as suggested by the large area of type (I) in Figure OA.2(c). This is because, when the differences in customers' valuations are large, an optimal cyclic pricing

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policy needs to follow a sequence of markdowns to prevent the high valuation customers from waiting, which leaves too much surplus to those customers, rendering its degeneration to a static pricing policy. On the other end of the spectrum when the differences in valuations are extremely large, especially when the highest valuation is significantly higher than the rest, capturing the surplus from customers of the highest valuation becomes the utmost priority, and thus an optimal static pricing policy becomes more favorable, compared to those price discrimination mechanisms. Thus, the performance gap becomes smaller as a result. In fact, when v_3 dominates, no price discrimination policy would be effective, and thus both randomized pricing and cyclic pricing policies reduce to an optimal static pricing policy. This is reflected by the area of type (I) in the top left corner in both Figures OA.2(b) and OA.2(c).

Figure OA.5 Relative Profit Gap between Optimal Randomized Pricing and Optimal Cyclic Pricing



Note. Parameters of this study are consistent with estimates in Moon et al. (2017). In particular, we consider two customer segments with $\alpha_1 = 0.18$, $\alpha_2 = 0.82$, $c_1 = 0.2$, and $c_2 = 2.5$.

To validate the generality of our results, we run another numerical study by adopting parameters from the empirical context of Moon et al. (2017). We consider the case with two customer segments, where 18% of them have a low valuation of v_1 with a per-period waiting cost of $c_1 = 0.2$ and 82% of customers have a high valuation of v_2 with a per-period waiting cost of $c_2 = 2.5$.[OA.2] In this study, v_1 ranges from 5 to 10 (i.e., equivalent to \$50 to \$100 in Moon et al. 2017), and $v_2 - v_1$ varies from 0

[OA.2] Under a stationary randomized pricing policy, the monitoring cost estimated in Moon et al. (2017) is equivalent to customers' per-period waiting cost in our setting, as both measure the cost for customers to sample another price point. We also scale the estimated valuations and costs in Moon et al. (2017) down by a factor of 10 to make results comparable to the rest of our numerical analysis.

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to 10. Figure OA.5 summarizes the relative profit gap between optimal randomized pricing policies and optimal cyclic pricing policies. Consistent with our findings in Figure OA.1, we observe that an optimal cyclic pricing policy outperforms an optimal randomized pricing policy when $v_2 - v_1$ is relatively small, whereas this dominance is reversed when $v_2 - v_1$ becomes sufficiently large. Moreover, we also see the benefit of randomized pricing generally reaches its maximum when the differences in customer valuations, $v_2 - v_1$, are of medium values.

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Online Supplement to "Intertemporal Price Discrimination via Randomized Promotions": Proofs

A. Proofs of Results in the Main Body

Proof of Lemma 1. Note that $V(p_t) = \max\{v - p_t, -c + \mathsf{E}[V(P)], 0\}$, which is equivalent to $V(p_t) = \max\{v - p_t, [-c + \mathsf{E}[V(P)]]^+\}$. It is easy to see that $v - p_t$ is decreasing in p_t, and $[-c + \mathsf{E}[V(P)]]^+$ is independent of p_t. On the one hand, $v - 0 = v > [-c + \mathsf{E}[V(P)]]^+$, because $V(p_t) = \max\{v - p_t, -c + \mathsf{E}[V(P)], 0\} \leq v - p_t < v$. On the other hand, $v - v = 0 \leq [-c + \mathsf{E}[V(P)]]^+$. Thus, there exists a threshold $\underline{p} \in (0, v]$ such that $v - p_t \geq [-c + \mathsf{E}[V(P)]]^+$ if and only if $p_t \leq \underline{p}$. Consequently, $V(p_t)$ is in the form of Equation (1). $\square$

Proof of Proposition 1. Denote $\underline{v}^* = \max\{v' \mid \mathsf{E}[(v' - P)^+] \leq c\}$. Since no customer would make a purchase at an infinitely high price, we just consider the support of P as being finite. It is easy to verify that $\mathsf{E}[(v' - P)^+]$ is weakly increasing in v' and eventually strictly increasing in v' for v' higher than the upper end of P's finite support, $\mathsf{E}[(0 - P)^+] = 0 < c$ and $\lim_{v' \nearrow \infty} \mathsf{E}[(v' - P)^+] = \infty > c$. Hence, the existence of $\underline{v}^*$ is guaranteed.

Consider first when $v > \underline{v}^*$. We prove by contradiction that a customer with valuation v would be willing to wait and eventually purchase a unit of product in a future period. Suppose on the contrary, the customer would not wait by either purchasing the product now or leaving immediately. Under the assumption of zero utility from the outside option, the customer would purchase the product if and only if $v \geq p_t$. Thus, under the stipulation that the customer would not wait by either purchasing the product now or leaving immediately, we must have $\mathsf{E}[V(P)] = \mathsf{E}[(v - P)^+]$. In such a case, her expected utility from waiting, i.e., $-c + \mathsf{E}[V(P)]$, must be less than or equal to 0, which means that $\mathsf{E}[(v - P)^+] \leq c$ due to $\mathsf{E}[V(P)] = \mathsf{E}[(v - P)^+]$. On the other hand, recall that by definition, $\underline{v}^* = \max\{v' \mid \mathsf{E}[(v' - P)^+] \leq c\}$. Because $v > \underline{v}^*$, we have $\mathsf{E}[(v - P)^+] > c$, which leads to a contradiction. Hence, a customer with valuation $v > \underline{v}^*$ would be willing to wait and eventually purchase a unit of product.

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Next, we prove that the customer's purchase threshold $\underline{p}$ is equal to $\underline{v}^*$, meaning that a customer with valuation $v \geq \underline{v}^*$ would make a purchase if and only if the price in the focal period is less than or equal to $\underline{v}^*$. First, as shown above, we must have $-c + \mathsf{E}[V(P)] > 0$ when $v > \underline{v}^*$. Given a purchase theshold $\underline{p}$ (its existence is guranteed by Lemma 1), we have $\mathsf{E}[V(P)] = F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] + \bar{F}(\underline{p})[-c + \mathsf{E}[V(P)]]$, which implies that

$$\mathsf{E}[V(P)] = \frac{F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] - c}{F(\underline{p})} + c. \tag{OS.1}$$

Thus, we have $v - \underline{p} - (-c + \mathsf{E}[V(P)]) = (-F(\underline{p})\mathsf{E}[\underline{p} - P | P \leq \underline{p}] + c)/F(\underline{p}) \geq 0$, where the inequality is guaranteed by Lemma 1. This implies $\mathsf{E}[(\underline{p} - P)^+] = F(\underline{p})\mathsf{E}[\underline{p} - P | P \leq \underline{p}] \leq c$. As $\underline{v}^* = \max\{v' \mid \mathsf{E}[(v' - P)^+] \leq c\}$ by definition, we conclude that $\underline{v}^* \geq \underline{p}$. Lastly, we prove by contradiction that $\underline{p} = \underline{v}^*$. Suppose $\underline{v}^* > \underline{p}$. Because $v - p_t \geq -c + \mathsf{E}[V(P)]$ if and only if $p_t \leq \underline{p}$, we have $v - \underline{v}^* < -c + \mathsf{E}[V(P)] = (F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] - c)/F(\underline{p})$, which is equivalent to $F(\underline{p})\mathsf{E}[\underline{v}^* - P | P \leq \underline{p}] > c$. Because $\underline{v}^* > \underline{p}$, we have $F(\underline{p})\mathsf{E}[\underline{v}^* - P | P \leq \underline{p}] \leq F(\underline{v}^*)\mathsf{E}[\underline{v}^* - P | P \leq \underline{v}^*] = \mathsf{E}[(\underline{v}^* - P)^+] \leq c$, which leads to a contradiction. Hence, the behavior of a customer with valuation $v > \underline{v}^*$ should be as characterized in Proposition 1.

Next consider the case when $v \leq \underline{v}^*$. We first show by contradiction that the customer would either purchase the product now or leave immediately. Suppose the customer would be willing to wait, which means that her expected utility from waiting must be greater than zero, i.e., $-c + \mathsf{E}[V(P)] > 0$. Based on Lemma 1, there exists a threshold $\underline{p}$ such that the customer would be willing to wait if and only if the price in the current period is greater than $\underline{p}$. Thus Equation (OS.1) still holds, which implies that $-c + \mathsf{E}[V(P)] = (F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] - c)/F(\underline{p}) > 0$. This is equivalent to $F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] > c$. Based on Lemma 1, we also know that $v - \underline{p} \geq -c + \mathsf{E}[V(P)] > 0$, i.e., $v > \underline{p}$. Coupling with the increasing monotonicity of $F(v')\mathsf{E}[v - P | P \leq v']$ in v' for any $v' \leq v$, we have $\mathsf{E}[(v - P)^+] = F(v)\mathsf{E}[v - P | P \leq v] \geq F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] > c$, which implies that $v > \underline{v}^*$ because $\mathsf{E}[(v' - P)^+]$ is increasing in v' and $\underline{v}^* = \max\{v' \mid \mathsf{E}[(v' - P)^+] \leq c\}$. This contracts to the aforementioned condition that $v \leq \underline{v}^*$. Hence, a customer with valuation $v \leq \underline{v}^*$ would either purchase the product now or leave immediately. Under the assumption of zero utility from the outside option, her purchase threshold is simply her valuation v, meaning that the customer would purchase the product if and only if $v \geq p_t$. We thus obtain the announced results. □

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Proof of Lemma 2. (i) If $v > \underline{v}^*$, customers will wait until the price is no more than $\underline{p} = \underline{v}^*$, as shown in Proposition 1. The probability that customers purchase in the nth period after arrival is given by $\left(\bar{F}(\underline{p})\right)^{n-1} F(\underline{p})$, and if they indeed make a purchase in a period, the firm's expected profit is $\mathsf{E}[P|P \le \underline{p}]$. Consequently, the monopolist's expected profit ex ante is

$$\sum_{n=1}^{\infty} \left(\bar{F}(\underline{p})\right)^{n-1} F(\underline{p})\mathsf{E}[P|P \le \underline{p}] = \lim_{n\to\infty} F(\underline{p})\mathsf{E}[P|P \le \underline{p}] \frac{1-\left(\bar{F}(\underline{p})\right)^{n}}{1-\bar{F}(\underline{p})} = \mathsf{E}[P|P \le \underline{p}] = \mathsf{E}[P|P \le \underline{v}^*],$$

where the second equality is due to $\bar{F}(\underline{p}) < 1$.

(ii) If $v \le \underline{v}^*$, customers will either buy with a price no more than v or quit immediately. Consequently, the expected profit is $F(v)\mathsf{E}[P|P \le v]$. □

Proof of Lemma 3. Note that $\underline{p}^i = \max\{v' | \mathsf{E}[(v'-P)^+] \le c_i\}$, $i = L, H$. Because $\mathsf{E}[(v'-P)^+]$ is increasing in v' and $c_L \ge c_H$, we have $\underline{p}^L \ge \underline{p}^H$. Next, we consider three mutually exclusive but collectively exhaustive cases: (a1) $\underline{p}^L \ge \underline{p}^H \ge v_H > v_L$; (b1) $v_H > \underline{p}^H$, $v_L \le \underline{p}^L$; (c1) $v_H > v_L > \underline{p}^L \ge \underline{p}^H$.

(a1) $\underline{p}^L \ge \underline{p}^H \ge v_H > v_L$. Due to Proposition 1, neither customers with a high-valuation nor customers with a low valuation would wait. Consequently, the firm's expected profit, as shown in Lemma 2, is given by

$$\begin{aligned}
&\alpha F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] \\
&= \alpha F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)[F(v_H) - F(v_L)]\mathsf{E}[P|v_L < P \le v_H] + (1-\alpha)F(v_L)\mathsf{E}[P|P \le v_L] \\
&= F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)[F(v_H) - F(v_L)]\mathsf{E}[P|v_L < P \le v_H] \\
&\le v_L F(v_L) + (1-\alpha)v_H[F(v_H) - F(v_L)] \le \max\{v_L, (1-\alpha)v_H\}F(v_H) \le \max\{v_L, (1-\alpha)v_H\},
\end{aligned}$$

where the first inequality is due to $\mathsf{E}[P|P \le v_L] \le \mathsf{E}[v_L|P \le v_L] \le v_L$ and $\mathsf{E}[P|v_L \le P \le v_H] \le \mathsf{E}[v_H|v_L \le P \le v_H] = v_H$. Thus, in the case when $\underline{p}^L \ge \underline{p}^H \ge v_H > v_L$, the expected profit from the optimal randomized pricing is no greater than that from the optimal static pricing policy.

(b1) $v_H > \underline{p}^H$, $v_L \le \underline{p}^L$. In this case, customers with a high-valuation will wait until the price is no higher than $\underline{p}^H$, but customers with a low valuation will quit the market immediately if the price is higher than their valuation v_L. According to Lemma 2, the monopolist's expected profit is given by

$$\alpha F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H].$$

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Consider first when $\underline{p}^H \le v_L$. Then, we have

$$\alpha F(v_L)\mathsf{E}[P|P\le v_L]+(1-\alpha)\mathsf{E}[P|P\le \underline{p}^H]\le \alpha v_L F(v_L)+(1-\alpha)\underline{p}^H\le \alpha v_L F(v_L)+(1-\alpha)v_L\le v_L,$$

where the first inequality is due to $\mathsf{E}[P|P\le v_L]\le \mathsf{E}[v_L|P\le v_L]\le v_L$ and $\mathsf{E}[P|P\le \underline{p}^H]\le \mathsf{E}[\underline{p}^H|P\le v_H]=\underline{p}^H$.

Next consider the circumstance when $v_H>\underline{p}^H>v_L$. The condition implies that $\mathsf{E}[P|P\le \underline{p}^H]\le \mathsf{E}[P|P\le v_H]$. Then, we have

$$\begin{aligned}
&\alpha F(v_L)\mathsf{E}[P|P\le v_L]+(1-\alpha)\mathsf{E}[P|P\le \underline{p}^H]\\
\le&\alpha F(v_L)\mathsf{E}[P|P\le v_L]+(1-\alpha)\mathsf{E}[P|P\le v_H]\le \alpha\frac{F(v_L)}{F(v_H)}\mathsf{E}[P|P\le v_L]+(1-\alpha)\mathsf{E}[P|P\le v_H]\\
=&\frac{\alpha F(v_L)\mathsf{E}[P|P\le v_L]+(1-\alpha)F(v_H)\mathsf{E}[P|P\le v_H]}{F(v_H)}\le \frac{\max\{v_L,(1-\alpha)v_H\}F(v_H)}{F(v_H)}=\max\{v_L,(1-\alpha)v_H\},
\end{aligned}$$

where the second inequality is due to $0<F(v_H)\le 1$, and the last inequality is based on the proof in (a1).

(c1) $v_H>v_L>\underline{p}^L\ge \underline{p}^H$. In this case, based on Proposition 1, both customers with a high-valuation and customers with a low valuation would be willing to wait. As a result, the expected profit, as shown in Lemma 2, is given by

$$\alpha\mathsf{E}[P|P\le \underline{p}^L]+(1-\alpha)\mathsf{E}[P|P\le \underline{p}^H]\le \alpha\underline{p}^L+(1-\alpha)\underline{p}^H<v_L,$$

where the first inequality is due to $\mathsf{E}[P|P\le \underline{p}^L]\le \underline{p}^L$ and $\mathsf{E}[P|P\le \underline{p}^H]\le \underline{p}^H$, the second inequality is due to $v_L>\underline{p}^L\ge \underline{p}^H$.

We thus obtained the announced results. □

Proof of Lemma 4. Based on Lemma 3, we can restrict our discussion to the situation when $c_L<c_H$. Note that $\underline{p}^i=\max\{v'|\mathsf{E}[(v'-P)^+]\le c_i\}$, $i=L,H$. Because $c_L<c_H$ implies that $\underline{p}^L<\underline{p}^H$, we need to consider four mutually exclusive but collective exhaustive cases: (a2) $v_L\le \underline{p}^L, v_H\le \underline{p}^H$; (b2) $v_L\le \underline{p}^L, v_H>\underline{p}^H$; (c2) $v_L>\underline{p}^L, v_H\le \underline{p}^H$; (d2) $v_L>\underline{p}^L, v_H>\underline{p}^H$.

Under the case (a2), based on Proposition 1, we know that neither high-valuation customers nor low-valuation customers would wait. Then following a similar approach as in the proof of case (a1) in Lemma 3, we can show that an optimal static pricing policy dominates randomized pricing

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policies. Under the case (b2), due to Proposition 1, low-valuation customers would never wait but high-valuation customers would wait in the system with the hope of a favorable price in future periods. The case is similar to the case (b1) in the proof of Lemma 3, and we can show that an optimal static pricing policy dominates randomized pricing policies following a similar approach.

Next we show in an auxiliary lemma below that case (c2) cannot happen under the condition that $\frac{v_L}{c_L} \le \frac{v_H}{c_H}$.

LEMMA OS.1. *$\frac{v_L}{c_L} > \frac{v_H}{c_H}$ is a necessary condition for (c2).*

Proof of Lemma OS.1. By Proposition 1, (c2) implies that $F(v_L)\mathsf{E}[v_L - P|P \le v_L] = \mathsf{E}[(v_L - P)^+] > c_L$ and $F(v_H)\mathsf{E}[v_H - P|P \le v_H] = \mathsf{E}[(v_H - P)^+] \le c_H$. Then, we have

$$c_H \ge F(v_H)\mathsf{E}[v_H - P|P \le v_H] > F(v_L)\mathsf{E}[v_H - P|P \le v_L] = v_H F(v_L) - F(v_L)\mathsf{E}[P|P \le v_L],$$

and

$$c_L < F(v_L)\mathsf{E}[v_L - P|P \le v_L] = v_L F(v_L) - F(v_L)\mathsf{E}[P|P \le v_L].$$

The above two inequalities suggest that

$$\frac{c_H}{c_L} > \frac{v_H - \mathsf{E}[P|P \le v_L]}{v_L - \mathsf{E}[P|P \le v_L]} > \frac{v_H}{v_L},$$

where the second inequality is due to $(v_H - \mathsf{E}[P|P \le v_L])\, v_L - (v_L - \mathsf{E}[P|P \le v_L])\, v_H = (v_H - v_L)\mathsf{E}[P|P \le v_L] > 0$. We thus obtain the announced result. □

Lastly, we show that the optimal static pricing policy outperforms randomized pricing policies under (d2), i.e., $v_L > \underline{p}^L, v_H > \underline{p}^H$, and $\frac{c_H}{c_L} \le \frac{v_H}{v_L}$. By Propositions 1, we know that customers would either purchase immediately or wait. Due to Lemma 2, the corresponding expected profit is given by

$$\alpha\mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H].$$

On the one hand, $\underline{p}^L = \max\{v'|\mathsf{E}[(v'-P)^+] \le c_L\}$, and then for any $v > \underline{p}^L$, we have $\mathsf{E}[(v-P)^+] > c_L$. Consequently, $F(v_L)\mathsf{E}[v_L - P|P \le v_L] = \mathsf{E}[(v_L - P)^+] > c_L$ due to $v_L > \underline{p}^L$. On the other hand, $\underline{p}^H = \max\{v'|\mathsf{E}[(v'-P)^+] \le c_H\}$, which implies $F(\underline{p}^H)\mathsf{E}[\underline{p}^H - P|P \le \underline{p}^H] = \mathsf{E}[(\underline{p}^H - P)^+] \le c_H$. We

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also observe that $\frac{c_H}{c_L} > \frac{\underline{p}^H}{v_L}$, which can be shown with the same approach as that in the proof of Lemma OS.1.

Consider first when $\underline{p}^H \le v_L$. We have $\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \le \alpha \underline{p}^L + (1-\alpha)\underline{p}^H < v_L \le \max\{v_L, (1-\alpha)v_H\}$. That is, the firm is worse off under a randomized pricing policy.

Next consider the scenario when $\underline{p}^H > v_L$. We start by showing that $F(v_L) < \frac{c_H - c_L}{\underline{p}^H - v_L}$. This is true because

$$\begin{aligned} c_H - c_L &> F(\underline{p}^H)\mathsf{E}[\underline{p}^H - P|P \le \underline{p}^H] - F(v_L)\mathsf{E}[v_L - P|P \le v_L] \\ &> F(v_L)\mathsf{E}[\underline{p}^H - P|P \le v_L] - F(v_L)\mathsf{E}[v_L - P|P \le v_L] = (\underline{p}^H - v_L)F(v_L), \end{aligned}$$

where the first inequality is due to $F(v_L)\mathsf{E}[v_L - P|P \le v_L] > c_L$ and $F(\underline{p}^H)\mathsf{E}[\underline{p}^H - P|P \le \underline{p}^H] \le c_H$, and the second inequality is due to $\underline{p}^H > v_L$. As a result, we have

$$\begin{aligned} &\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \le \alpha \mathsf{E}[P|P \le v_L] + (1-\alpha)\underline{p}^H \\ <&\alpha\left(v_L - \frac{c_L}{F(v_L)}\right) + (1-\alpha)\underline{p}^H < \alpha\left(v_L - \frac{c_L}{c_H - c_L}(\underline{p}^H - v_L)\right) + (1-\alpha)\underline{p}^H \\ =&\alpha\frac{c_H}{c_H - c_L}v_L + \left(1 - \alpha - \frac{\alpha c_L}{c_H - c_L}\right)\underline{p}^H. \end{aligned}$$

The first inequality is due to $\underline{p}^L < v_L$, the second inequality is due to $v_L F(v_L) - F(v_L)\mathsf{E}[P|P \le v_L] = F(v_L)\mathsf{E}[v_L - P|P \le v_L] > c_L$, and the third inequality is due to $F(v_L) < \frac{c_H - c_L}{\underline{p}^H - v_L}$.

If $(1-\alpha) - \alpha\frac{c_L}{c_H - c_L} \le 0$, then we have

$$\alpha\frac{c_H}{c_H - c_L}v_L + \left(1 - \alpha - \frac{\alpha c_L}{c_H - c_L}\right)\underline{p}^H < \alpha\frac{c_H}{c_H - c_L}v_L + \left(1 - \alpha - \frac{\alpha c_L}{c_H - c_L}\right)v_L = v_L \le \max\{v_L, (1-\alpha)v_H\},$$

where the first inequality is due to $v_L < \underline{p}^H$. Otherwise, i.e., if $(1-\alpha) - \alpha\frac{c_L}{c_H - c_L} > 0$, we have

$$\begin{aligned} &\alpha\frac{c_H}{c_H - c_L}v_L + \left(1 - \alpha - \frac{\alpha c_L}{c_H - c_L}\right)\underline{p}^H < \alpha\frac{c_H}{c_H - c_L}v_L + \left(1 - \alpha - \frac{\alpha c_L}{c_H - c_L}\right)\frac{c_H}{c_L}v_L \\ =&(1-\alpha)\frac{c_H}{c_L}v_L \le (1-\alpha)v_H \le \max\{v_L, (1-\alpha)v_H\}. \end{aligned}$$

The first inequality is due to $\frac{c_H}{c_L} > \frac{\underline{p}^H}{v_L}$, and the second inequality is due to $\frac{c_H}{c_L} \le \frac{v_H}{v_L}$. □

Proof of Proposition 2. Recall the four mutually exclusive but collective exhaustive cases considered in the proof of Lemma 4: (a2) $v_L \le \underline{p}^L, v_H \le \underline{p}^H$; (b2) $v_L \le \underline{p}^L, v_H > \underline{p}^H$; (c2) $v_L > \underline{p}^L, v_H \le \underline{p}^H$; (d2) $v_L > \underline{p}^L, v_H > \underline{p}^H$. We already show that a randomized pricing policy cannot outperform an optimal static pricing policy under (a2) and (b2). Under the case (c2), low-valuation customers

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would make a purchase immediately or wait in the system, while high-valuation customers would either purchase the product upon arrival or leave immediately. Thus, all we need to show is that case (d2) is dominated by either (c2) or an optimal static pricing policy.

Based on Proposition 1 and Lemma 2, we can derive the optimal expected profit under case (d2) by solving the following optimization problem.

$$\begin{aligned} \max_{F(p)} \quad & \alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H]. \\ \text{s.t.} \quad & \underline{p}^L = \max\{v' | \mathsf{E}[(v'-P)^+] \le c_L\}, \\ & \underline{p}^H = \max\{v' | \mathsf{E}[(v'-P)^+] \le c_H\}, \\ & \mathsf{E}[(v_L - P)^+] > c_L, \quad \mathsf{E}[(v_H - P)^+] > c_H. \end{aligned} \tag{OS.2}$$

If $\underline{p}^H \le v_L$, we have $\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \le \alpha \underline{p}^L + (1-\alpha)\underline{p}^H < v_L$, which suggests that the optimal randomized pricing policy under case (d2) reduces to a static pricing policy. Thus we only need to restrict our attention to the case when $\underline{p}^H > v_L$. Suppose $F(p)$ is an optimal solution to problem (OS.2). Because no customer would buy at a price greater than v_H, the upper bound of an optimal distribution will be no more than v_H, i.e., $F(v_H) = 1$.

Based on Proposition 1 and Lemma 2, the optimal expected profit under case (c2) can be derived by solving problem (2). We construct a price distribution $G(p)$ as follows: $G(p)$ is the same as $F(p)$ for any $p \le \underline{p}^H$, and it takes one single value of v_H for any $p > \underline{p}^H$, with a probability mass equal to $1 - F(\underline{p}^H)$. We first show that $G(p)$ is a feasible solution of problem (2). Because $G(p)$ is equal to $F(p)$ for any $p \le v_L < \underline{p}^H$, we have $\mathsf{E}_G[(v_L - P)^+] > c_L$ and $\underline{p}^L = \max\{v' | \mathsf{E}[(v'-P)^+] \le c_L\} = \max\{v' | \mathsf{E}_G[(v'-P)^+] \le c_L\}$. On the other hand, $\mathsf{E}_G[v_H - P | \underline{p}^H < P \le v_H] = 0$ as $G(p)$ takes only one value of v_H for any $p > \underline{p}^H$. Consequently, $G(v_H)\mathsf{E}_G[v_H - P|P \le v_H] = G(\underline{p}^H)\mathsf{E}_G[v_H - P|P \le \underline{p}^H] = F(\underline{p}^H)\mathsf{E}[v_H - P|P \le \underline{p}^H] \le c_H$, where the last equality is due to the definition of $\underline{p}^H$. Thus, $G(p)$ is a feasible solution of problem (2).

Lastly, we prove that case (d2) is dominated by case (c2) by showing that the expected profit from $F(p)$ is lower than that from $G(p)$:

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$$
\begin{aligned}
&\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \\
=&\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)F(\underline{p}^H)\mathsf{E}[P|P \le \underline{p}^H] + (1-\alpha)(1-F(\underline{p}^H))\mathsf{E}[P|P \le \underline{p}^H] \\
<&\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)F(\underline{p}^H)\mathsf{E}[P|P \le \underline{p}^H] + (1-\alpha)(1-F(\underline{p}^H))v_H \\
=&\alpha \mathsf{E}_G[P|P \le \underline{p}^L] + (1-\alpha)G(\underline{p}^H)\mathsf{E}_G[P|P \le \underline{p}^H] + (1-\alpha)(1-G(\underline{p}^H))\mathsf{E}_G[P|\underline{p}^H < P \le v_H] \\
=&\alpha \mathsf{E}_G[P|P \le \underline{p}^L] + (1-\alpha)G(v_H)\mathsf{E}_G[P|P \le v_H],
\end{aligned}
$$

where the inequality is due to $\underline{p}^H < v_H$, and the last equality is due to $G(v_H) = F(v_H) = 1$. We thus obtain the announced result. □

Proof of Lemma 5. Suppose $F(p)$ is a feasible solution for the optimization problem (3). We can construct another distribution, say $G(p)$, as follows. $G(p)$ is equal to $F(p)$ for any $p \le v_L$. However, under $G(p)$, the distribution has only one value, namely v_H, above v_L, and its corresponding probability mass is equal to $1 - F(v_L)$. Because $G(p)$ is equal to $F(p)$ for any $p \le v_L$, it is easy to verify that $G(v_L)\mathsf{E}_G[v_L - P|P \le v_L] > c_L$ holds. On the other hand, $\mathsf{E}_G[v_H - P|v_L < P \le v_H] = 0$ as $G(p)$ has only one value of v_H for any $p > v_L$. Consequently, $G(v_H)\mathsf{E}_G[v_H - P|P \le v_H] = G(v_L)\mathsf{E}_G[v_H - P|P \le v_L] + [G(v_H) - G(v_L)]\mathsf{E}_G[v_H - P|v_L < P \le v_H] = G(v_L)\mathsf{E}_G[v_H - P|P \le v_L] = F(v_L)\mathsf{E}[v_H - P|P \le v_L] \le F(v_H)\mathsf{E}[v_H - P|P \le v_H] \le c_H$. Thus, $G(p)$ is also a feasible solution. To establish the announced result, we next show that the expected profit under $G(p)$ always dominates that under $F(p)$. That is,

$$
\begin{aligned}
&\alpha \mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] \\
=&\alpha \mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)[F(v_H) - F(v_L)]\mathsf{E}[P|v_L < P \le v_H] \\
\le&\alpha \mathsf{E}_G[P|P \le v_L] + (1-\alpha)G(v_L)\mathsf{E}_G[P|P \le v_L] + (1-\alpha)v_H(1-F(v_L)) \\
=&\alpha \mathsf{E}_G[P|P \le v_L] + (1-\alpha)G(v_H)\mathsf{E}_G[P|P \le v_H],
\end{aligned}
$$

where the inequality is due to $F(v_H) \le 1$ and $\mathsf{E}[P|v_L < P \le v_H] \le \mathsf{E}[v_H|v_L < P \le v_H] = v_H$, and the second equality is due to $v_H[1 - F(v_L)] = v_H[1 - G(v_L)] = [G(v_H) - G(v_L)]\mathsf{E}_G[P|v_L < P \le v_H]$. The inequality is strict if $F(p)$ takes any value other than v_H within the interval $(v_L, v_H]$. We thus obtain the announced result. □

Proof of Lemma 6. Denote $\beta = F(v_L)$ and $U(\beta) = \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \frac{\alpha c_L}{\beta} - (1 -$

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$\alpha)\beta(v_H - v_L)$. We first show that $U(\beta)$ is an upper bound for the expected profit from the optimization problem (3). Suppose $F(p)$ is an optimal solution for (3). Then, the expected profit under $F(p)$ satisfies

$$\begin{aligned}
&\alpha \mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] \\
=&\alpha \mathsf{E}[P|P \le v_L] + (1-\alpha)[F(v_L)\mathsf{E}[P|P \le v_L] + (1-F(v_L))v_H] \\
<&\alpha v_L - \frac{\alpha c_L}{F(v_L)} + (1-\alpha)[v_L F(v_L) - c_L + (1-F(v_L))v_H] \\
=&\alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \frac{\alpha c_L}{\beta} - (1-\alpha)\beta(v_H - v_L),
\end{aligned}$$

where $\beta = F(v_L)$. The first equality is due to Lemma 5, and the inequality is due to $c_L < F(v_L)\mathsf{E}[v_L - P|P \le v_L] = v_L F(v_L) - F(v_L)\mathsf{E}[P|P \le v_L]$.

(i) $\frac{\partial U(\beta)}{\partial \beta} = \frac{\alpha c_L}{\beta^2} - (1-\alpha)(v_H - v_L)$. If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} \ge 1$, $\frac{\partial U(\beta)}{\partial \beta}$ is guaranteed to be greater than or equal to 0, because $\beta \le 1$. That is, $U(\beta)$ is increasing in β when $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} \ge 1$. Consequently, $U(\beta) \le \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \alpha c_L - (1-\alpha)(v_H - v_L) = v_L - c_L \le \max\{v_L, (1-\alpha)v_H\}$.

(ii) Because $\frac{\partial U(\beta)}{\partial \beta} = \frac{\alpha c_L}{\beta^2} - (1-\alpha)(v_H - v_L)$ is decreasing in β, $U(\beta)$ is concave in β. $\frac{\partial U(\beta)}{\partial \beta} = 0$ is realized at $\beta = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$. Recall that any feasible solution to Problem (3) satisfies $\beta < \frac{c_H - c_L}{v_H - v_L}$. Consequently, if $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L}$, the maximum of $U(\beta)$ is realized at $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$, and thus $U(\beta) \le U_1 = U(\beta^*)$. Otherwise, $U(\beta) \le U(\frac{c_H - c_L}{v_H - v_L} - \delta) = \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_H - \alpha c_L \frac{v_H - v_L}{c_H - c_L} - \eta(\delta)$, where $\lim_{\delta \searrow 0} \eta(\delta) = 0$. Thus, we obtain the announced results. □

Proof of Proposition 3. Recall that the optimal profit given by Problem (2) is bounded from above by U_1 if $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L}$, or by U_2 if $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} \ge \frac{c_H - c_L}{v_H - v_L}$. Next we show that the two-point distribution shown in the theorem is a feasible solution to Problem (2), and the corresponding profit converges to the upper bounds when δ converges to 0.

For the feasibility, we notice that $\mathsf{E}[(v_L - P)^+] = \beta^*(v_L - \underline{p}^*) = c_L + \beta^*\eta(\delta) > c_L$, and $\mathsf{E}[(v_H - P)^+] = \beta^*(v_H - \underline{p}^*) = \beta^*(v_H - v_L) + c_L + \beta^*\eta(\delta) \le c_H$, which is due to $0 < \beta^* < \frac{c_H - c_L}{v_H - v_L}$ and $\eta(\delta) \searrow 0$. Thus the two-point distribution is a feasible solution to Problem (2).

Next we prove the optimality of the two-point distribution. Consider first when $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L}$. The expected profit is given by

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$$\begin{aligned}
&\alpha \mathsf{E}[P|P \le \underline{p}^*] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] \\
=&\alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \frac{\alpha c_L}{\beta^*} - (1-\alpha)\beta^*(v_H - v_L) - [\alpha + (1-\alpha)\beta^*]\eta(\delta) \\
=&U_1 - (\alpha + (1-\alpha)\beta^*)\eta(\delta).
\end{aligned}$$

Next when $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} \ge \frac{c_H - c_L}{v_H - v_L}$, the expected profit is given by

$$\begin{aligned}
&\alpha \mathsf{E}[P|P \le \underline{p}^*] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] \\
=&\alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \frac{\alpha c_L}{\beta^*} - (1-\alpha)\beta^*(v_H - v_L) - [\alpha + (1-\alpha)\beta^*]\eta(\delta) \\
=&U_2 - \left[\alpha v_L \frac{v_H - v_L}{c_H - c_L} \frac{1}{\frac{c_H - c_L}{v_H - v_L} - \delta} - (1-\alpha)(v_H - v_L)\right]\delta - [\alpha + (1-\alpha)\beta^*]\eta(\delta).
\end{aligned}$$

We thus obtain the announced result. □

Proof of Proposition 4. To facilitate the discussion, we first prove an auxiliary lemma as described below.

LEMMA OS.2. *(i) The necessary and sufficient conditions for $U_1 > v_L$ and $U_1 > (1-\alpha)v_H$ are given by $\sqrt{\frac{c_L}{v_H - v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$ and $\sqrt{\frac{c_L}{v_H - v_L}} < \sqrt{\frac{\alpha}{1-\alpha}}\left(\sqrt{\frac{v_H}{v_H - v_L}} - 1\right)$, respectively;*

(ii) The necessary and sufficient conditions for $U_2 > v_L$ and $U_2 > (1-\alpha)v_H$ are given by $\frac{c_H}{v_H - v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)} < 1$ and $\alpha \frac{v_L c_H - c_L v_H}{c_H - c_L} > (1-\alpha)c_H$, respectively.

Proof of Lemma OS.2. (i) $U_1 - v_L = (1-\alpha)(v_H - v_L) - (1-\alpha)c_L - 2\sqrt{\alpha(1-\alpha)c_L(v_H - v_L)} = (1-\alpha)(v_H - v_L)\left[1 - \frac{c_L}{v_H - v_L} - 2\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}\right]$. Thus, $U_1 > v_L$ if and only if $\sqrt{\frac{c_L}{v_H - v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$. On the other hand, $U_1 - (1-\alpha)v_H = \alpha v_L - (1-\alpha)c_L - 2\sqrt{\alpha(1-\alpha)c_L(v_H - v_L)} = (1-\alpha)(v_H - v_L)\left[\frac{\alpha v_L}{(1-\alpha)(v_H - v_L)} - \frac{c_L}{v_H - v_L} - 2\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}\right]$. Thus, $U_1 > (1-\alpha)v_H$ if and only if $\sqrt{\frac{c_L}{v_H - v_L}} < \sqrt{\frac{\alpha}{1-\alpha}}\left(\sqrt{\frac{v_H}{v_H - v_L}} - 1\right)$.

(ii) $U_2 - v_L = (1-\alpha)(v_H - v_L) - (1-\alpha)c_H - \alpha c_L \frac{v_H - v_L}{c_H - c_L} = (1-\alpha)(v_H - v_L)\left[1 - \frac{c_H}{v_H - v_L} - \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)}\right]$. Thus, $U_2 > v_L$ if and only if $\frac{c_H}{v_H - v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)} < 1$. On the other hand, $U_2 - (1-\alpha)v_H = \alpha v_L - (1-\alpha)c_H - \alpha c_L \frac{v_H - v_L}{c_H - c_L} = \alpha \frac{v_L c_H - c_L v_H}{c_H - c_L} - (1-\alpha)c_H$. Thus, $U_2 > (1-\alpha)v_H$ if and only if $\alpha \frac{v_L c_H - c_L v_H}{c_H - c_L} > (1-\alpha)c_H$. □

Now we are ready to prove the main results. Consider first when $v_L \ge (1-\alpha)v_H$. Recall that, if $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L}$ and $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < 1$, the profit with an optimal randomized pricing policy is U_1, based on Lemma 6. The profit with an optimal static pricing policy is v_L when

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$v_L \geq (1-\alpha)v_H$. To this end, we only need to compare U_1 and v_L. Based on Lemma OS.2(i), $U_1 > v_L$ if and only if $\sqrt{\frac{c_L}{v_H - v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$. Moreover, $\frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}} < \sqrt{\frac{1-\alpha}{\alpha}}$, which is due to $\sqrt{\alpha}(1-\sqrt{\alpha}) < 1-\alpha = (1+\sqrt{\alpha})(1-\sqrt{\alpha})$. As such, $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < 1$ is guaranteed by the condition $\sqrt{\frac{c_L}{v_H - v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$.

If $1 > \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} \geq \frac{c_H - c_L}{v_H - v_L}$, the profit with an optimal randomized pricing policy is approximately U_2, by Lemma 6. Thus, we only need to compare U_2 and v_L. According to Lemma OS.2(ii), $U_2 > v_L$ if and only if $\frac{c_H}{v_H - v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)} < 1$. Moreover, we can show that $\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)} < \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)}$ because $\frac{c_H - c_L}{v_H - v_L} < \frac{c_H}{v_H - v_L} < 1$, where the second inequality is due to $\frac{c_H}{v_H - v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)} < 1$. Consequently, $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < 1$ is guaranteed by the condition $\frac{c_H}{v_H - v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)} < 1$.

Consider next when $v_L \leq (1-\alpha)v_H$, i.e., $\alpha v_L \leq (1-\alpha)(v_H - v_L)$. The profit with an optimal static pricing policy is $(1-\alpha)v_H$. Based on Assumption (S), we have $c_L < v_L$, and thus $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \sqrt{\frac{\alpha v_L}{(1-\alpha)(v_H - v_L)}} \leq 1$. Then following a similar approach as described in the preceding paragraphs, we can obtain the conditions that an optimal randomized pricing policy dominates an optimal static pricing policy. That is, if $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L}$, the condition is given by $\sqrt{\frac{c_L}{v_H - v_L}} < \sqrt{\frac{\alpha}{1-\alpha}}\left(\sqrt{\frac{v_H}{v_H - v_L}} - 1\right)$; Otherwise, the condition is given by $\alpha \frac{v_L c_H - c_L v_H}{c_H - c_L} > (1-\alpha)c_H$.

To sum up, an optimal randomized pricing policy outperforms an optimal static pricing policy if $\frac{v_L}{c_L} > \frac{v_H}{c_H}$ and the following conditions hold.

(i) $v_L \geq (1-\alpha)v_H$.

  (a) If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L}$, $\sqrt{\frac{c_L}{v_H - v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$;

  (b) Otherwise, $\frac{c_H}{v_H - v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)} < 1$.

(ii) $v_L < (1-\alpha)v_H$.

  (a) If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L}$, $\sqrt{\frac{c_L}{v_H - v_L}} < \sqrt{\frac{\alpha}{1-\alpha}}\left(\sqrt{\frac{v_H}{v_H - v_L}} - 1\right)$;

  (b) Otherwise, $\alpha \frac{v_L c_H - c_L v_H}{c_H - c_L} > (1-\alpha)c_H$.

When c_L is sufficiently small, either the set of conditions (i)-(a) or (ii)-(a) is satisfied, and we thus obtain the desired result. □

Proof of Corollary 1. Under the optimal randomized pricing policy, the expected profit from low valuation and high-valuation customers are given by $\mathsf{E}[P|P \leq \underline{p}^*]$ and $F(v_H)\mathsf{E}[P|P \leq v_H]$,

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respectively. The surplus of low-valuation customers is $\frac{F(\underline{p}^*)\mathsf{E}[v_L-P|P\le\underline{p}^*]-c_L}{F(\underline{p}^*)}+c_L$, according to Equation (OS.1), and the surplus of high-valuation customers is $\mathsf{E}[(v_H-P)^+]$. Based on Lemma 5, $F(v_H)=1$, and thus $F(v_H)\mathsf{E}[P|P\le v_H]+\mathsf{E}[(v_H-P)^+]=v_H$. Consequently, the total social welfare under an optimal randomized pricing policy is given by

$$\begin{aligned}&\alpha\left(\mathsf{E}[P|P\le\underline{p}^*]+\frac{F(\underline{p}^*)\mathsf{E}[v_L-P|P\le\underline{p}^*]-c_L}{F(\underline{p}^*)}+c_L\right)+(1-\alpha)v_H\\ =&\alpha\left(v_L-\frac{c_L}{F(\underline{p}^*)}+c_L\right)+(1-\alpha)v_H.\end{aligned}$$

When $v_L<(1-\alpha)v_H$, the optimal static price is v_H and the corresponding social welfare is $(1-\alpha)v_H$, which is less than $\alpha\left(v_L-\frac{c_L}{F(\underline{p}^*)}+c_L\right)+(1-\alpha)v_H$. In this case, the surplus of all customers is equal to zero under the optimal static pricing policy, which is thus less than that under an optimal randomized pricing policy. On the other hand, when $v_L\ge(1-\alpha)v_H$, the optimal static price is v_L and the corresponding social welfare is $\alpha v_L+(1-\alpha)v_H$, which is greater than $\alpha\left(v_L-\frac{c_L}{F(\underline{p}^*)}+c_L\right)+(1-\alpha)v_H$. In this case, the surplus for low-valuation customers is also zero under the optimal static pricing policy, while customer surplus for high-valuation customers is given by v_H-v_L. Recall that, under the optimal randomized pricing policy, the surplus of high-valuation customers is $\mathsf{E}[(v_H-P)^+]$, which is less than or equal to c_H according to the constraint in the optimization problem (2). Thus, the surplus of high-valuation customers under the optimal randomized pricing policy would be lower when $c_H<v_H-v_L$, and we obtain the announced results. □

Proof of Corollary 2. Under the optimal randomized pricing policy, the surplus of low-valuation customers is $\frac{F(\underline{p}^*)\mathsf{E}[v_L-P|P\le\underline{p}^*]-c_L}{F(\underline{p}^*)}+c_L$, according to Equation (OS.1), and the surplus of high-valuation customers is $\mathsf{E}[(v_H-P)^+]$. Due to Proposition 3, $\frac{F(\underline{p}^*)\mathsf{E}[v_L-P|P\le\underline{p}^*]-c_L}{F(\underline{p}^*)}+c_L=v_L-\underline{p}^*-\frac{\bar{F}(\underline{p}^*)}{F(\underline{p}^*)}c_L$, and $\mathsf{E}[(v_H-P)^+]=F(\underline{p}^*)(v_H-\underline{p}^*)$. Therefore, the difference of the two is given by $(v_H-v_L)-\bar{F}(\underline{p}^*)\left(v_H-\underline{p}^*-\frac{c_L}{F(\underline{p}^*)}\right)$. Because $v_H>v_L\ge\underline{p}^*+\frac{c_L}{F(\underline{p}^*)}$, the difference in consumer surplus between high-valuation and low-valuation customers would be less than v_H-v_L. When $v_L\ge(1-\alpha)v_H$, the optimal static pricing policy is to charge a fixed price of v_L, and thus the difference in consumer surplus would be exactly v_H-v_L. We thus obtain the announced result. □

Proof of Proposition 5. We prove the result for the general case with n customer segments. A fraction α_i of customers are of type-i, who value the product at v_i and incur per-period waiting time

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c_i. Without loss of generality, we assume that $c_1 < c_2 < \cdots < c_n$. This is an innocuous assumption as we do not require $v_i < v_j$ for any $i < j$. We first prove an auxiliary lemma.

LEMMA OS.3. *Without loss of generality, we assume that $c_1 < c_2 < \cdots < c_n$. Consider any deterministic pricing policy. If a customer with valuation v_i, for any $i \in \{1, \ldots, n\}$, arriving in period 1 purchases in period j, then*

(i) any customer with valuation v_i arriving in period $t \in \{2, \ldots, j\}$ will purchase in period j;

(ii) any customer with valuation $v_{i'}$, for any $i' < i$, arriving in period $t \in \{1, \ldots, j\}$ will not purchase earlier than period j, should she ever purchase.

Proof of Lemma OS.3. First, we prove Lemma OS.3(i). A customer with valuation v_i arriving in period 1 purchases in period j implies that $p_j + (j-1)c_i = \min_{t \geq 1}\{p_t + (t-1)c_i\}$ and $p_j + (j-1)c_i < v_i$. As a direct consequence, we have $p_j + (j-1)c_i = \min_{t \geq t'}\{p_t + (t-1)c_i\}$, for any $t' = 2, 3 \ldots, j$. Thus, all type-i customers arriving in period $t \in [2, j]$ will purchase in period j.

Next, we show that any customer with valuation v_i', for any $i' < i$, arriving in period $t \in [1, j]$ will only purchase in period j or afterwards, should she ever purchase. Because $p_j + (j-1)c_i \leq p_t + (t-1)c_i$ for any $t \leq j$, we have $p_j - p_t \leq (t-j)c_i \leq (t-j)c_{i'}$ for any $i' < i$, which implies that $p_j + (j-1)c_{i'} \leq p_t + (t-1)c_{i'}$. We thus obtained the announced result. □

Note that under any deterministic pricing policy, a customer upon arrival would either leave immediately without purchasing or choose to purchase a unit of the product (either immediately or in a future period). We first show that under any deterministic pricing policy, there exists a cutoff period, say period T, such that all customers arriving before or during period T would have left by the end of period T. Without loss of generality, suppose type-$\underline{i}$ is the lowest type of customers who would make a purchase at some point under such a pricing policy. That is, any customer of type-i, $i < \underline{i}$, always leaves immediately. Then our preceding statement that any customer arriving before or during period T would have left by the end of the period holds automatically for customers of type-i, for any $i < \underline{i}$.

Consider a customer with valuation $v_{\underline{i}}$. Suppose that $t_{\underline{i}} \geq 1$ is the first period that a customer with valuation $v_{\underline{i}}$ would choose to purchase the product. That is, all customers with valuation $v_{\underline{i}}$ arriving

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before period $t_{\underline{i}}$ would leave immediately without purchasing. Denote the period in which the type-$\underline{i}$ customer arriving in period $t_{\underline{i}}$ makes the purchase as T. Based on Lemma OS.3(i), any customer with valuation $v_{\underline{i}}$ arriving in period $t \in \{t_{\underline{i}}+1, \ldots, T\}$ will make a purchase in period T. We first prove by contradiction that any customer with valuation v_i, $i \in \{\underline{i}+1, \ldots, n\}$, arriving in period $t \in \{t_{\underline{i}}, \ldots, T\}$ would make a purchase no later than period T, should she ever purchase. Suppose on the contrary, a customer with valuation $v_{i'}$, $i' \in \{\underline{i}+1, \ldots, n\}$, arriving in period $t' \in \{t_{\underline{i}}, \ldots, T\}$, would make a purchase in period $T' > T$. Then Lemma OS.3(ii) would imply that any customer with valuation $v_{\underline{i}}$ arriving in period t' will not purchase earlier than period T', which contradicts with the aforementioned result. Thus, any type-i customer, $i \geq \underline{i}$, arriving between period $t_{\underline{i}}$ and period T would have left (with or without purchasing) by the end of period T.

Next we show by contradiction that any customer with valuation v_i, $i \in \{\underline{i}+1, \ldots, n\}$, arriving in period $t \in \{1, \ldots, t_{\underline{i}}-1\}$ would make a purchase no later than period T, should she ever purchase. Suppose a customer with valuation $v_{i'}$, $i' \in \{\underline{i}+1, \ldots, n\}$, arriving in period $t' \in \{1, \ldots, t_{\underline{i}}-1\}$ would make a purchase in period $T' > T$. By Lemma OS.3(i), any customer with valuation $v_{i'}$ arriving in period $t \in \{t_{\underline{i}}, \ldots, T\}$ would make a purchase in period T', which however contradicts with the result in the preceding paragraph. Thus, any type-i customer, $i \geq \underline{i}$, arriving between period 1 and period $t_{\underline{i}}-1$ would have left (with or without purchasing) by the end of period T.

Therefore, under any deterministic pricing policy, there exists a period T such that all customers arriving between period 1 and period T would have left by the end of period T. This statement also holds for, a fortiori, any optimal deterministic pricing policy, say $\mathbf{p}^* = \{p_t^*\}_{t\in\mathbb{N}}$. Following a similar approach, we can show that starting from period $T+1$, there exists a T' such that all customers arriving between period $T+1$ and period $T+T'$ would have left by the end of period $T+T'$. Due to the optimality of the pricing policy, we must have $T = T'$ and $p_t = p_{T+t}$, $t \in \{1, \ldots, T\}$, and thus we obtain the announced result. $\square$

Proof of Lemma 7. Consider the scenario when $c_L \geq c_H$. If low-valuation customers arriving in period t would make a purchase with price $p_{t'} \leq v_L$, then high-valuation customers arriving in the same period would also pay no more than $p_{t'}$ because $p_{t'} + (t'-t)c_H \leq p_{t'} + (t'-t)c_L \leq v_L < v_H$. In this case, $\pi_t(\mathbf{p}) \leq v_L$. On the other hand, if low-valuation customers in period t choose to leave

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without purchasing, then the firm's profit is bounded below by $(1-\alpha)v_H$, i.e., $\pi_t(\mathbf{p}) \le (1-\alpha)v_H$. Therefore, $\Pi(\mathbf{p}) = \frac{1}{T}\sum_{t=1}^{T}\pi_t(\mathbf{p}) \le \max\{v_L, (1-\alpha)v_H\}$. $\square$

Proof of Proposition 6. Suppose $\mathbf{p} = \{p_1, p_2, \ldots, p_T\}$ is an optimal cyclic pricing policy where some high-valuation customers would wait. Without loss of generality, we assume that high-valuation customers arriving in period 1 would wait and purchase in period $j > 1$. That is, $p_j + (j-1)c_H = \min\{p_1, p_2 + c_H, \ldots, p_T + (T-1)c_H\}$ and $p_j + (j-1)c_H < v_H$. As a direct consequence, we have $p_j + (j-t)c_H = \min\{p_t, p_{t+1} + c_H, \ldots, p_T + (T-t)c_H\}$, for any $t = 2, 3 \ldots, j$. Next we show that any high-valuation customers arriving in period t, where $1 \le t \le j$, would always purchase in period j. Clearly, for a customer arriving in period t, purchasing in period j dominates the option of purchasing in any period between t and T. To this end, we need to show that the customer is worse off if she purchases in any period between $T+1$ and $T+t-1$. Due to $p_j + (j-1)c_H = \min\{p_1, p_2 + c_H, \ldots, p_T + (T-1)c_H\}$, we have $p_j + (j-1)c_H \le p_{t'} + (t'-1)c_H$, $t' = 1, 2, \ldots, t-1$. Coupling with the fact that $p_{t'} = p_{T+t'}$, we have $p_j + (j-t)c_H \le p_{t'} + (t'-t)c_H = p_{T+t'} + (t'-t)c_H < p_{T+t'} + (T+t'-t)c_H$. As a result, any high-valuation customer arriving in period $t \in [1, j]$ would make a purchase in period j.

Next we show that any low-valuation customer arriving in period $t \in [1, j]$ would also purchase in period j. We restrict our discussion to the case when $c_H > c_L$ because the optimal cyclic pricing policy degenerates into a static pricing policy when $c_H \le c_L$, based on Lemma 7. Because $p_j + (j-1)c_H \le p_t + (t-1)c_H$ for any $t \le j$, we have $p_j - p_t \le (t-j)c_H < (t-j)c_L$, which implies that $p_j + (j-1)c_L \le p_t + (t-1)c_L$. That is, low-valuation customers arriving in period $t \in [1, j]$ would make a purchase in period j and afterwards if they would ever purchase. Now we show that low-valuation customers arriving in period $t \in [1, j]$ will never purchase after period j by contradiction. Suppose low-valuation customers purchase in period $j' > j$. Let us denote $s = \arg\min_{1 \le i \le j-1}\{p_i + (i-1)c_H\}$. Recall that $p_j + (j-1)c_H = \min\{p_1, p_2 + c_H, \ldots, p_T + (T-1)c_H\}$, then $p_j + (j-1)c_H \le p_s + (s-1)c_H$. However, if $p_j + (j-1)c_H = p_s + (s-1)c_H$, note $s < j$, with tie-breaking rule, the high-valuation customers arriving in period $t \in [1, s]$ would make a purchase in period s, which contradicts with the result that any high-valuation customer arriving in period $t \in [1, j]$ would make a purchase in period j from the previous paragraph. Therefore, we have $p_j + (j-1)c_H < p_s + (s-1)c_H$. Then, we can

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design a new pricing policy $\mathbf{p}'$, which is exactly the same as $\mathbf{p}$ except for p'_s. We let $p'_s+(s-1)c_H = p_j+(j-1)c_H$, and then $p'_s > p_j$ because $s<j$. Now we will show that the behavior of low-valuation customers under the new policy is exactly the same as before, however, the firm's profit from high-valuation customers would be higher with $\mathbf{p}'$. Consider the low-valuation customers' behavior, since low-valuation customers purchase in period $j'>j$ under policy $\mathbf{p}$, thus $p_{j'}+(j'-1)c_L = \min\{p_1, p_2+c_L, \ldots, p_T+(T-1)c_L\}$. Then, note $s<j$, we have $p_{j'}+(j'-1)c_L \le p_j+(j-1)c_L = p_j+(j-1)c_H-(j-1)(c_H-c_L) = p'_s+(s-1)c_H-(j-1)(c_H-c_L) < p'_s+(s-1)c_H-(s-1)(c_H-c_L) = p'_s+(s-1)c_L$. As a result, we have $p'_{j'}+(j'-1)c_L = \min\{p'_1, p'_2+c_L, \ldots, p'_T+(T-1)c_L\}$ because $\mathbf{p}'$ is exactly the same as $\mathbf{p}$ except for p'_s. Then low-valuation customers arriving in period $t\in[1,j]$ will also purchase in period j' under the new policy. It is easy to see $p'_{j'+T}+(j'+T-t)c_L = \min\{p'_{T+1}+(T+1-t)c_L, p'_{T+2}+(T+2-t)c_L, \ldots, p'_{2T}+(2T-t)c_L\}$ for $t=j+1, j+2\ldots, T$. Then low-valuation customers arriving in period $t\in[j+1,T]$ will purchase in a period between period t and T or in period $j'+T$ under policy $\mathbf{p}'$, which is as same as that under policy $\mathbf{p}$. Consider the high-valuation customers, recall that $p'_s+(s-1)c_H = p_j+(j-1)c_H = \min\{p_1, p_2+c_H, \ldots, p_T+(T-1)c_H\}$, then the high-valuation customers arriving in period $t\in[1,s]$ purchase in period s with price $p'_s > p_j$ by tie-breaking rule under policy $\mathbf{p}'$. And the high-valuation customers arriving in period $t\in[s+1,j]$ still purchase in period j. It is easy to see $p'_{s+T}+(s+T-t)c_H = p'_{j+T}+(j+T-t)c_H = \min\{p'_{T+1}+(T+1-t)c_H, p'_{T+2}+(T+2-t)c_H, \ldots, p'_{2T}+(2T-t)c_H\}$ for $t=j+1, j+2\ldots, T$. Then high-valuation customers arriving in period $t\in[j+1,T]$ will purchase in a period between period t and T or in period $s+T$ under policy $\mathbf{p}'$. For the first case, the behavior of high-valuation customer is as same as that under policy $\mathbf{p}$. For the second case, high-valuation customers arriving in period $t\in[j+1,T]$ will purchase in period $j+T$ with price $p_{j+T} = p_j < p'_s = p'_{s+T}$ under policy $\mathbf{p}$. Thus the firm's profit from high-valuation customers would be higher or equal to that under policy $\mathbf{p}$. The result contradicts with the assumption that $\mathbf{p}$ is an optimal cyclic pricing policy. Thus we conclude that low-valuation customers arriving in period $t\in[1,j]$ also purchase in period j.

Following the same approach, we can show that if high-valuation customers arriving in period $j+1$ choose to purchase in period $j+k$, then all customers arriving in period $t\in[j+1,j+k]$ will purchase in period $j+k$. That is, the cyclic policy $\mathbf{p}$ can be decomposed into many mini cycles. Without

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of loss generality, we assume that there are m mini cycles, where the length of each mini cycle is denoted by n_l, $l=1,\ldots,m$ and $\sum_{l=1}^{m} n_l = T$. Denote the profit from customers who made a purchase in period t by $\tilde{\pi}_t(\mathbf{p})$. Then, $\Pi(\mathbf{p}) = [n_1\tilde{\pi}_{n_1}(\mathbf{p}) + n_2\tilde{\pi}_{n_1+n_2}(\mathbf{p}) + \cdots + n_m\tilde{\pi}_{n_1+n_2+\cdots+n_m}(\mathbf{p})]/T$. Denote $\tilde{\pi}_{n_1+n_2+\cdots+n_{l^*}}(\mathbf{p}) = \max\{\tilde{\pi}_{n_1+n_2+\cdots+n_l}(\mathbf{p}), l=1,2,\ldots,m\}$. Then, we have $\Pi(\mathbf{p}) \le \tilde{\pi}_{n_1+n_2+\cdots+n_{l^*}}(\mathbf{p})$.

Last, we show that $\mathbf{p}$ cannot be an optimal cyclic pricing policy by contradiction. If $\mathbf{p}$ is an optimal cyclic pricing policy, then $\Pi(\mathbf{p}) \ge \max\{v_L, (1-\alpha)v_H\}$ because a static pricing policy is a special case of cyclic pricing policies. The analysis in preceding paragraphs shows that all customers arriving in period $t \in [1,j]$ would make a purchase in period j under policy $\mathbf{p}$. Thus, for any $t \le j$, if low-valuation customers purchase in period j, $\tilde{\pi}_t(\mathbf{p}) = p_j < v_L$; otherwise, only high-valuation customers would make a purchase in period j, and thus $\tilde{\pi}_t(\mathbf{p}) = (1-\alpha)p_j \le (1-\alpha)v_H$. Therefore, the average expected profit from this mini cycle cannot exceed $\max\{v_L, (1-\alpha)v_H\}$, and thus $\Pi(\mathbf{p}) \le \tilde{\pi}_{n_1+n_2+\cdots+n_{l^*}}(\mathbf{p}) < \max\{v_L, (1-\alpha)v_H\}$, which contradicts to the optimality of $\mathbf{p}$. □

Proof of Proposition 7. Suppose $\mathbf{p} = \{p_1, p_2, \ldots, p_T\}$ is an optimal cyclic pricing policy. Let us denote by $p^L_{\min} = \min\{p_1, p_2 + c_L, \ldots, p_T + (T-1)c_L\}$, and $t^L_{\min}$ the earliest time period such that $p_{t^L_{\min}} + (t^L_{\min} - 1)c_L = p^L_{\min}$. Following the same approach as that in the proof of Proposition 6, we can show that any low-valuation customer arriving in period $t \in \{1,2,\ldots,t^L_{\min}\}$ will purchase in period $t^L_{\min}$ with price $p_{t^L_{\min}}$. Thus, $p_{t^L_{\min}} < v_L - (t^L_{\min} - 1)c_L$ because any low-valuation customer arriving in period 1 purchases in period $t^L_{\min}$. On the other hand, high-valuation customers will not wait under an optimal cyclic pricing policy as shown in Proposition 6. Thus, $p_{t'} \le p_{t^L_{\min}} + (t^L_{\min} - t')c_H$, where $t' \in \{1,2\ldots,t^L_{\min}\}$. Otherwise, high-valuation customers would be better off postponing purchase until period $t^L_{\min}$. Because $p_{t'} \le v_H$, we have

$$\begin{aligned} p_{t'} &\le \min\{v_H,\, p_{t^L_{\min}} + (t^L_{\min} - t')c_H\} < \min\{v_H,\, v_L - (t^L_{\min} - 1)c_L + (t^L_{\min} - t')c_H\} \\ &= v_H - [v_H - v_L + (t^L_{\min} - 1)c_L - (t^L_{\min} - t')c_H]^+. \end{aligned}$$

Thus, the profit in the first $t^L_{\min}$ is bounded from above by

$$\begin{aligned} \sum_{t=1}^{t^L_{\min}} \pi_t(\mathbf{p}) &= (1-\alpha)(p_1 + p_2 + \cdots + p_{t^L_{\min}}) + \alpha t^L_{\min} p_{t^L_{\min}} \\ &< (1-\alpha)\sum_{i=1}^{t^L_{\min}} (v_H - [v_H - v_L + (t^L_{\min} - 1)c_L - (t^L_{\min} - t')c_H]^+) + \alpha t^L_{\min}(v_L - (t^L_{\min} - 1)c_L). \end{aligned}$$

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The upper bound can be approximated by $p_{t'} = v_H - [v_H - v_L + (t^L_{\min} - 1)c_L + \delta - (t^L_{\min} - t')c_H]^+$ for $t' = 1, 2 \ldots, t^L_{\min}$. It is easy to verify that $p_1 \geq p_2 \geq \cdots \geq p_{t^L_{\min}}$. Furthermore, we have $p_{t^L_{\min}} = v_H - [v_H - v_L + (t^L_{\min} - 1)c_L + \delta - (t^L_{\min} - t^L_{\min})c_H] = v_L - (t^L_{\min} - 1)c_L - \delta$ and thus $p_{t^L_{\min}} + (t^L_{\min} - 1)c_L = v_L - \delta$. For any $t' \in \{1, 2 \ldots, t^L_{\min} - 1\}$, $p_{t'}$ is equal to either v_H or $v_H - [v_H - v_L + (t^L_{\min} - 1)c_L + \delta - (t^L_{\min} - t')c_H] = v_L - (t^L_{\min} - 1)c_L + (t^L_{\min} - t')c_H - \delta$. If $p_{t'} = v_H$, then $p_{t'} + (t' - 1)c_L = v_H + (t' - 1)c_L > v_L - \delta$. On the other hand, if $p_{t'} = v_L - (t^L_{\min} - 1)c_L + (t^L_{\min} - t')c_H - \delta$, then $p_{t'} + (t' - 1)c_L = v_L + (t^L_{\min} - t')(c_H - c_L) - \delta$, which is again greater than $v_L - \delta$ because $t' < t^L_{\min}$ and $c_H > c_L$. Thus all low type customers will wait until period $t^L_{\min}$ to purchase. We can then follow the same approach and show that optimal prices from period $t^L_{\min} + 1$ onward are simply replications of $\{p_1, p_2, \ldots, p_{t^L_{\min}}\}$. Thus, $T = t^L_{\min}$, and the optimal expected profit under policy $\mathbf{p}$ is given by $\Pi(\mathbf{p}) = \frac{1}{T} \sum_{t=1}^{T} \pi_t(\mathbf{p})$. $\square$

Proof of Corollary 3. (i) Based on Proposition 7, the profit from an optimal cyclic pricing policy is given by

$$\begin{aligned}
\Pi(T) &= (1-\alpha)\frac{\sum_{t'=1}^{T} v_H - [v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+}{T} + \alpha(v_L - (T-1)c_L - \delta) \\
&< (1-\alpha)\frac{\sum_{t'=1}^{T} [v_L - (T-1)c_L + (T-t')c_H]}{T} + \alpha(v_L - (T-1)c_L) \\
&= (1-\alpha)\frac{T-1}{2}c_H + (v_L - (T-1)c_L) \leq v_L \leq \max\{v_L, (1-\alpha)v_H\},
\end{aligned}$$

where the first inequality is due to $\delta > 0$ and $[v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+ \geq v_H - v_L + (T-1)c_L + \delta - (T-t')c_H$, and the second inequality is due to $c_H/c_L \leq 2/(1-\alpha)$. A static pricing policy can achieve a profit of $\max\{v_L, (1-\alpha)v_H\}$, with either the volume strategy of pricing at v_L or the margin strategy of pricing at v_H. Thus we obtain the announced result.

(ii) Based on Proposition 7, the profit from an optimal cyclic pricing policy is given by

$$\begin{aligned}
\Pi(T) &= (1-\alpha)\frac{\sum_{t'=1}^{T} v_H - [v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+}{T} + \alpha(v_L - (T-1)c_L - \delta) \\
&< (1-\alpha)\left[\frac{T-1}{T}v_H + \frac{1}{T}(v_L - (T-1)c_L)\right] + \alpha(v_L - (T-1)c_L) \equiv R(T),
\end{aligned}$$

where the first inequality is due to $\delta > 0$ and $v_H - [v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+ \leq v_H$ for $t' \leq$

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$T-1$. Taking the first order derivative with respect to T, we have $R'(T)=(1-\alpha)\frac{v_H-v_L-c_L}{T^2}-\alpha c_L$, which is no more than 0 when $v_H-v_L\le c_L$. Now we consider $c_L<v_H-v_L\le c_L/(1-\alpha)$, then

$$R'(T)=(1-\alpha)\frac{v_H-v_L-c_L}{T^2}-\alpha c_L\le(1-\alpha)(v_H-v_L-c_L)-\alpha c_L\le 0,$$

where the first inequality is due to $T\ge 1$, the second inequality is due to $v_H-v_L\le c_L/(1-\alpha)$. Hence, $R(T)$ is decreasing in T. Consequently, $\Pi(T)<R(T)\le R(1)=v_L\le\max\{v_L,(1-\alpha)v_H\}$. Thus we obtain the announced result. □

Proof of Proposition 8. First, we show that if an optimal cyclic pricing policy is in the form of the first $T-1$ periods priced at v_H and the last period priced at $v_L-(T-1)c_L-\delta$, then it is always better than an optimal randomized pricing policy. Moreover, under the conditions that $v_H-v_L>c_L/(1-\alpha)$ and $v_H-v_L<c_H\Big/\left(1+\frac{1}{2}\sqrt{\frac{1-\alpha}{\alpha}}\right)$, the optimal cyclic pricing policy is in this form. Suppose a cyclic pricing policy is in the form of the first $T-1$ periods priced at v_H and the last period priced at $v_L-(T-1)c_L-\delta$, by the proof of Corollary 3 (ii), then the profit of this policy is $\Pi(T)=R(T)-(1-\alpha)\frac{1}{T}\delta-\alpha\delta$. It is easy to verify that $R'(T)$ is decreasing in T when $v_H-v_L>c_L/(1-\alpha)>c_L$, and thus we conclude that $R(T)$ is concave, with its maximum achieved at $T_1=\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}$. Since a cyclic length should be an integer, the maximum of $R(T)$ is thus given by $\max\{R(\lfloor T_1\rfloor),R(\lfloor T_1\rfloor+1)\}$, where $\lfloor x\rfloor$ represents the greatest integer that is no more than x. Note that $\max\{R(\lfloor T_1\rfloor),R(\lfloor T_1\rfloor+1)\}$ is an upper bound of the profit of an optimal cyclic pricing policy. Let $T=\lfloor T_1\rfloor$ or $T=\lfloor T_1\rfloor+1$, then this cyclic pricing policy is asymptotically optimal. Note $R(T)$ is concave and decreasing when $T\ge T_1$, then $R(\lfloor T_1\rfloor+1)\ge R(T_1+1)$ since $T_1+1\ge\lfloor T_1\rfloor+1$. Consequently, $\max\{R(\lfloor T_1\rfloor),R(\lfloor T_1\rfloor+1)\}\ge R(T_1+1)$, i.e., $R(T_1+1)$ is a lower bound of the profit of an optimal cyclic pricing policy. Hence, we have

$$\begin{aligned}
R(T_1+1)&=(1-\alpha)\left[\frac{T_1}{T_1+1}v_H+\frac{1}{T_1+1}(v_L-T_1c_L)\right]+\alpha(v_L-T_1c_L)\\
&=\alpha v_L+(1-\alpha)v_H-(1-\alpha)c_L-\alpha T_1c_L-(1-\alpha)\frac{1}{T_1+1}(v_H-v_L-c_L)\\
&>\alpha v_L+(1-\alpha)v_H-(1-\alpha)c_L-\alpha T_1c_L-(1-\alpha)\frac{1}{T_1}(v_H-v_L-c_L)\\
&=\alpha v_L+(1-\alpha)v_H-(1-\alpha)c_L-2\sqrt{\alpha(1-\alpha)c_L(v_H-v_L-c_L)}\\
&>\alpha v_L+(1-\alpha)v_H-(1-\alpha)c_L-2\sqrt{\alpha(1-\alpha)c_L(v_H-v_L)}=U_1,
\end{aligned}$$

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where the first inequality is due to $T_1 > 0$ and $v_H - v_L > c_L/(1-\alpha) > c_L$, the second inequality is due to $c_L > 0$, the second equality is due to $T_1 = \sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}}$. Based on Lemma 6, the expected profit of an optimal randomized pricing policy is either U_1 or U_2. Note that $U_1 \geq U_2$. Now we will prove $v_H - v_L + (T-1)c_L + \delta - (T-t')c_H \leq 0$ for $t' = 1, 2, \ldots, T-1$ when $T = \lfloor T_1 \rfloor$ or $T = \lfloor T_1 \rfloor + 1$ under the condition $v_H - v_L < c_H \Big/ \left(1 + \frac{1}{2}\sqrt{\frac{1-\alpha}{\alpha}}\right)$, which induces the cyclic pricing policy is in the form of the first $T-1$ periods priced at v_H and the last period priced at $v_L - (T-1)c_L - \delta$. Thus, we have

$$v_H - v_L + (T-1)c_L + \delta - (T-t')c_H < v_H - v_L + T_1 c_L - c_H \leq v_H - v_L + \frac{1}{2}\sqrt{\frac{1-\alpha}{\alpha}}(v_H - v_L) - c_H < 0,$$

where the first inequality is due to $\delta > 0$, $T = \lfloor T_1 \rfloor$ or $T = \lfloor T_1 \rfloor + 1$, $t' \leq T-1$, the second inequality is due to $T_1 c_L = \sqrt{\frac{(1-\alpha)c_L(v_H - v_L - c_L)}{\alpha}}$ is maximized at $c_L = (v_H - v_L)/2$, the third inequality is due to $v_H - v_L < c_H \Big/ \left(1 + \frac{1}{2}\sqrt{\frac{1-\alpha}{\alpha}}\right)$. Hence, we get the announced result of Proposition 8(i).

Next, we show under the conditions that $v_H - v_L > c_L/(1-\alpha)$ and $v_H - v_L$ is higher than a threshold, an optimal randomized pricing policy is always better. Since the optimal cyclic pricing policy is in the form of the prices stay constant at v_H for some time, drops by a size no more than c_H and then drops by a size of exactly c_H to the end-of-cycle price $v_L - (T-1)c_L - \delta$ by Proposition 7. By the previous paragraph, we know that if an optimal cyclic pricing policy is in the form of the first $T-1$ periods priced at v_H and the last period priced at $v_L - (T-1)c_L - \delta$, then it is always better than an optimal randomized pricing policy. Therefore, if an optimal randomized pricing policy is always better, then the optimal cyclic pricing policy should be in the general form of a series of markdowns as illustrated in Figure 2(a). By Corollary 3(i), we just need to consider the case $c_H/c_L > 2/(1-\alpha)$. Based on Proposition 7, the profit from an optimal cyclic pricing policy is given by

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$$\begin{aligned}\Pi(T) &= (1-\alpha)\frac{\sum_{t'=1}^{T} v_H - [v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+}{T} + \alpha(v_L - (T-1)c_L - \delta)\\ &< (1-\alpha)\frac{\sum_{t'=1}^{T} v_H - [v_H - v_L + (T-1)c_L - (T-t')c_H]^+}{T} + \alpha(v_L - (T-1)c_L)\\ &< (1-\alpha)\frac{\int_0^T v_H - [v_H - v_L + (T-1)c_L - (T-t)c_H]^+ \mathrm{d}t}{T} + \alpha(v_L - (T-1)c_L)\\ &= (1-\alpha)v_H + \alpha(v_L - (T-1)c_L) - (1-\alpha)\frac{[v_H - v_L + (T-1)c_L]^2}{2Tc_H} \equiv H(T),\end{aligned}$$

where the first inequality is due to $\delta > 0$, and the second is due to $v_H - [v_H - v_L + (T-1)c_L - (T-t)c_H]^+$ being monotonically decreasing in t. Taking the first order derivative with respect to T, we have $H'(T) = -\alpha c_L + (1-\alpha)\frac{(v_H - v_L - c_L)^2}{2c_H T^2} - (1-\alpha)\frac{c_L^2}{2c_H}$. It is easy to verify that $H'(T)$ is decreasing in T, and thus we conclude that $H(T)$ is concave, with its maximum achieved at $T_2 = \sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)^2}{2\alpha c_L c_H + (1-\alpha)c_L^2}}$. The upper bound for the profit of a cyclic pricing policy is thus given by

$$H(T_2) = \alpha v_L + (1-\alpha)v_H - \alpha(T_2 - 1)c_L - \alpha T_2 c_L - (1-\alpha)\frac{T_2 c_L^2}{c_H} - (1-\alpha)\frac{c_L}{c_H}(v_H - v_L - c_L).$$

Based on Lemma 6, the expected profit of an optimal randomized pricing policy is either U_1 or U_2. Note that $U_1 \geq U_2$ and $U_2 = \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_H - \alpha c_L \frac{v_H - v_L}{c_H - c_L}$. Now we show $H(T_2) < U_2$ if $v_H - v_L$ is higher than a threshold. First, we show $T_2 > \frac{v_H - v_L}{c_H - c_L}$ if $v_H - v_L$ is higher than a threshold. Since $T_2 - \frac{v_H - v_L}{c_H - c_L} = \sqrt{\frac{1-\alpha}{2\alpha c_L c_H + (1-\alpha)c_L^2}}(v_H - v_L) - \frac{v_H - v_L}{c_H - c_L} - \sqrt{\frac{1-\alpha}{2\alpha c_L c_H + (1-\alpha)c_L^2}}c_L$, it is sufficiently to show $T_2 - \frac{v_H - v_L}{c_H - c_L}$ is increasing in $v_H - v_L$. In other words, $\frac{(1-\alpha)}{2\alpha c_L c_H + (1-\alpha)c_L^2} > \frac{1}{(c_H - c_L)^2}$, which is equivalent to $(1-\alpha)(c_H^2 - 2c_L c_H) > 2\alpha c_L c_H$. It is easy to verify $(1-\alpha)(c_H^2 - 2c_L c_H) > 2\alpha c_L c_H$ when $c_H/c_L > 2/(1-\alpha)$. Consequently, we have $-\alpha T_2 c_L < -\alpha c_L \frac{v_H - v_L}{c_H - c_L}$. Hence, a sufficient condition for $H(T_2) < U_2$ is given by

$$-\alpha(T_2 - 1)c_L - (1-\alpha)\frac{T_2 c_L^2}{c_H} - (1-\alpha)\frac{c_L}{c_H}(v_H - v_L - c_L) < -(1-\alpha)c_H.$$

Note the left hand of the inequality is decreasing in $v_H - v_L$ when c_H and c_L are given. In other words, there is a threshold on $v_H - v_L$ above which the inequality holds. We thus obtain the announced result of Proposition 8(ii). □

Proof of Proposition 9. We first prove an auxiliary lemma below, which gives an upper bound for Problem (4).

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LEMMA OS.4. *Let* $\tilde{\Delta}_R = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}}$.

(i) When $(1-\alpha)(v_H-v_L) \le \alpha(1-\gamma)v_L + \alpha\gamma c_L$, *the optimal expected profit from* (4) *is no more than that from an optimal static pricing policy;*

(ii) When $(1-\alpha)(v_H-v_L) > \alpha(1-\gamma)v_L + \alpha\gamma c_L$, *i.e.,* $\tilde{\Delta}_R < 1$,

(a) if $\tilde{\Delta}_R < \frac{c_H-c_L}{v_H-v_L}$, *the optimal expected profit from* (4) *is no more than* $\tilde{U}_1 \equiv \alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - 2[(1-\alpha)(v_H-v_L) - \alpha(1-\gamma)v_L]\tilde{\Delta}_R$;

(b) otherwise, the optimal expected profit from (4) *is no more than* $\tilde{U}_2 \equiv \alpha\gamma v_L + (1-\alpha)v_H - \alpha(1-\gamma)c_L - (1-\alpha)c_H - \alpha\gamma c_L\frac{v_H-v_L}{c_H-c_L} + \alpha(1-\gamma)v_L\frac{c_H-c_L}{v_H-v_L}$.

Proof of Lemma OS.4. Denote $\beta = F(v_L)$ and $\tilde{U}(\beta) = \alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - \frac{\alpha\gamma c_L}{\beta} - (1-\alpha)\beta(v_H-v_L) + \alpha(1-\gamma)\beta v_L$. We first show that $\tilde{U}(\beta)$ is an upper bound for the expected profit from the optimization problem (4). Suppose $F(p)$ is an optimal solution for (4). Then, the expected profit under $F(p)$ satisfies

$$
\begin{aligned}
&\alpha\gamma \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P \le v_L] \\
\le&\alpha\gamma \mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P \le v_L] \\
=&\alpha\gamma \mathsf{E}[P|P \le v_L] + (1-\alpha)[F(v_L)\mathsf{E}[P|P \le v_L] + (1-F(v_L))v_H] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P \le v_L] \\
<&\alpha\gamma v_L - \frac{\alpha\gamma c_L}{F(v_L)} + (1-\alpha)[v_L F(v_L) - c_L + (1-F(v_L))v_H] + \alpha(1-\gamma)[v_L F(v_L) - c_L] \\
=&\alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - \frac{\alpha\gamma c_L}{\beta} - (1-\alpha)\beta(v_H-v_L) + \alpha(1-\gamma)\beta v_L,
\end{aligned}
$$

where $\beta = F(v_L)$. The first inequality is due to $\underline{p}^L \le v_L$, the second inequality is due to $c_L < \mathsf{E}[(v_L-P)^+] = v_L F(v_L) - F(v_L)\mathsf{E}[P|P \le v_L]$, and the first equality is due to Lemma 5, which also holds for problem (4) by following a similar proof. Taking the derivative of $\tilde{U}(\beta)$ with respect to β, we have $\frac{\partial\tilde{U}(\beta)}{\partial\beta} = \frac{\alpha\gamma c_L}{\beta^2} - (1-\alpha)(v_H-v_L) + \alpha(1-\gamma)v_L$.

(i) If $(1-\alpha)(v_H-v_L) \le \alpha(1-\gamma)v_L + \alpha\gamma c_L$, $\frac{\partial\tilde{U}(\beta)}{\partial\beta}$ is guaranteed to be greater than or equal to 0 because $\beta \le 1$. That is, $\tilde{U}(\beta)$ is increasing in β when $(1-\alpha)(v_H-v_L) \le \alpha(1-\gamma)v_L + \alpha\gamma c_L$. Consequently, $\tilde{U}(\beta) \le \tilde{U}(1) = \alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - \alpha\gamma c_L - (1-\alpha)(v_H-v_L) + \alpha(1-\gamma)v_L = v_L - c_L \le \max\{v_L, (1-\alpha)v_H\}$.

(ii) If $(1-\alpha)(v_H-v_L) > \alpha(1-\gamma)v_L + \alpha\gamma c_L$, $\tilde{U}(\beta)$ is concave in β. $\frac{\partial\tilde{U}(\beta)}{\partial\beta} = 0$ is realized at $\beta = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}}$. Recall that any feasible solution to problem (4) satisfies $\beta <$

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$\frac{c_H-c_L}{v_H-v_L}$. Consequently, if $\sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}} < \frac{c_H-c_L}{v_H-v_L}$, the maximum of $\tilde{U}(\beta)$ is realized at $\tilde{\beta}^* = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}}$, and thus $\tilde{U}(\beta) \le \tilde{U}_1 = U(\tilde{\beta}^*)$. Otherwise, $\tilde{U}(\beta) \le \tilde{U}(\frac{c_H-c_L}{v_H-v_L} - \delta) = \alpha\gamma v_L + (1-\alpha)v_H - \alpha(1-\gamma)c_L - (1-\alpha)c_H - \alpha\gamma c_L\frac{v_H-v_L}{c_H-c_L} + \alpha(1-\gamma)v_L\frac{c_H-c_L}{v_H-v_L} - \eta(\delta)$, where $\lim_{\delta\searrow 0}\eta(\delta) = 0$. Thus, we obtain the announced results. □

Lemma OS.4 shows that the optimal profit given by problem (4) is bounded below by $\tilde{U}_1$ if $\sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}} < \frac{c_H-c_L}{v_H-v_L}$, or by $\tilde{U}_2$ if $\sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}} \ge \frac{c_H-c_L}{v_H-v_L}$. Next we show that the two-point distribution shown in the theorem is a feasible solution to problem (4), and the corresponding profit converges to the upper bounds when δ converges to 0.

For the feasibility, we notice that $\mathsf{E}[(v_L-P)^+] = \tilde{\beta}^*(v_L - \underline{\tilde{p}}^*) = c_L + \tilde{\beta}^*\eta(\delta) > c_L$, and $\mathsf{E}[(v_L-P)^+] = \tilde{\beta}^*(v_H - \underline{\tilde{p}}^*) = \tilde{\beta}^*(v_H - v_L) + c_L + \tilde{\beta}^*\eta(\delta) \le c_H$, which is due to $\tilde{\beta}^* < \frac{c_H-c_L}{v_H-v_L}$ and $\eta(\delta) \searrow 0$. Thus the two-point distribution is a feasible solution to problem (4).

Next we prove the optimality of the two-point distribution. Consider first when $\sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}} < \frac{c_H-c_L}{v_H-v_L}$. The expected profit is given by

$$
\begin{aligned}
&\alpha\gamma\mathsf{E}[P|P\le\underline{\tilde{p}}^*] + (1-\alpha)F(v_H)\mathsf{E}[P|P\le v_H] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P\le v_L] \\
=&\alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - \frac{\alpha\gamma c_L}{\tilde{\beta}^*} - (1-\alpha)\tilde{\beta}^* v_H + (1-\alpha\gamma)\tilde{\beta}^* v_L - [\alpha\gamma + (1-\alpha\gamma)\tilde{\beta}^*]\eta(\delta) \\
=&\tilde{U}_1 - [\alpha\gamma + (1-\alpha\gamma)\tilde{\beta}^*]\eta(\delta).
\end{aligned}
$$

Next when $\sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}} \ge \frac{c_H-c_L}{v_H-v_L}$, the expected profit is given by

$$
\begin{aligned}
&\alpha\gamma\mathsf{E}[P|P\le\underline{\tilde{p}}^*] + (1-\alpha)F(v_H)\mathsf{E}[P|P\le v_H] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P\le v_L] \\
=&\alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - \frac{\alpha\gamma c_L}{\tilde{\beta}^*} - (1-\alpha)\tilde{\beta}^* v_H + (1-\alpha\gamma)\tilde{\beta}^* v_L - [\alpha\gamma + (1-\alpha\gamma)\tilde{\beta}^*]\eta(\delta) \\
=&\tilde{U}_2 - \left[\alpha\gamma v_L\frac{v_H-v_L}{c_H-c_L}\frac{1}{\frac{c_H-c_L}{v_H-v_L}-\delta} - (1-\alpha)(v_H-v_L) + \alpha(1-\gamma)\right]\delta - [\alpha\gamma + (1-\alpha\gamma)\tilde{\beta}^*]\eta(\delta).
\end{aligned}
$$

We thus obtain the announced result. □

Proof of Corollary 4. We first consider the monotonicity of $\tilde{\beta}^*$ and $\underline{\tilde{p}}^*$ with respect to γ. Note that

$$\frac{\partial\tilde{\Delta}_R^2}{\partial\gamma} = \frac{\alpha c_L[(1-\alpha)v_H - v_L]}{[(1-\alpha)(v_H-v_L) - \alpha(1-\gamma)v_L]^2}.$$

Thus, $\tilde{\Delta}_R$ is increasing in γ if $v_L \le (1-\alpha)v_H$, and decreasing in γ otherwise. As a direct consequence, both $\tilde{\beta}^* = \min\left\{\tilde{\Delta}_R, \frac{c_H-c_L}{v_H-v_L} - \delta\right\}$ and $\underline{\tilde{p}}^* = v_L - \frac{c_L}{\tilde{\beta}^*} - \eta(\delta)$ are increasing in γ if $v_L \le (1-\alpha)v_H$, and decreasing in γ otherwise.

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We next consider the monotonicity of the expected profit from the optimal randomized pricing policy with respect to γ. Based on Lemma OS.4, we know that the optimal expected profit is bounded above by either $\tilde{U}_1$ or $\tilde{U}_2$. First taking the derivative of $\tilde{U}_1$ with respect to γ, we have

$$\frac{\partial \tilde{U}_1}{\partial \gamma} = \alpha v_L + \alpha c_L - \frac{\alpha c_L[(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L] + \alpha\gamma c_L \alpha v_L}{\sqrt{\alpha\gamma c_L[(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L]}} = \alpha v_L + \alpha c_L - \frac{\alpha c_L}{\tilde{\Delta}_R} - \alpha v_L \tilde{\Delta}_R.$$

As $\frac{\alpha c_L}{\tilde{\Delta}_R} + \alpha v_L \tilde{\Delta}_R$ is convex in $\tilde{\Delta}_R$, we have $\frac{\alpha c_L}{\tilde{\Delta}_R} + \alpha v_L \tilde{\Delta}_R < \alpha v_L + \alpha c_L$ for any $c_L/v_L < \tilde{\Delta}_R < 1$. Next we show that $\tilde{\Delta}_R > c_L/v_L$. If $(1-\alpha)v_H < v_L$, $\tilde{\Delta}_R = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L}} > \sqrt{\frac{\alpha\gamma c_L}{\alpha v_L - \alpha(1-\gamma)v_L}} = \sqrt{\frac{c_L}{v_L}} > \frac{c_L}{v_L}$. If $(1-\alpha)v_H \geq v_L$, the profit of the optimal static pricing policy is given by $(1-\alpha)v_H$. Thus, we have $\tilde{U}_1 \geq (1-\alpha)v_H$, which implies $\alpha\gamma v_L > (1-\alpha\gamma)c_L + 2[(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L]\tilde{\Delta}_R > [(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L]\tilde{\Delta}_R$. This is equivalent to $\sqrt{(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L} < \frac{\alpha\gamma v_L}{\sqrt{\alpha\gamma c_L}}$. Consequently, $\tilde{\Delta}_R = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L}} > \frac{\alpha\gamma c_L}{\alpha\gamma v_L} = \frac{c_L}{v_L}$, and thus we conclude $\frac{\partial \tilde{U}_1}{\partial \gamma} > 0$. Next we show that $\tilde{U}_2$ increases in γ. Recall that $\tilde{U} = \tilde{U}_2$ when $\frac{c_H - c_L}{v_H - v_L} < \tilde{\Delta}_R < 1$. As shown in Lemma OS.1, $\frac{v_L}{c_L} > \frac{v_H}{c_H}$ is a necessary condition for low-valuation customers to wait, and for high-valuation customers to either purchase or leave immediately under any randomized pricing policy. It is easy to verify $\frac{c_L}{v_L} < \frac{c_H - c_L}{v_H - v_L}$ due to $\frac{v_L}{c_L} > \frac{v_H}{c_H}$, and thus

$$\frac{\partial \tilde{U}_2}{\partial \gamma} = \alpha v_L + \alpha c_L - \alpha c_L \frac{v_H - v_L}{c_H - c_L} - \alpha v_L \frac{c_H - c_L}{v_H - v_L} = \alpha \left(1 - \frac{c_H - c_L}{v_H - v_L}\right)\left(v_L - c_L \frac{v_H - v_L}{c_H - c_L}\right) > 0.$$

The expected profit from the optimal randomized pricing policy differs from the upper bounds $\tilde{U}_1$ and $\tilde{U}_2$ only by an infinitesimal term, and therefore we obtain the announced results. □

Proof of Lemma 8. Consider the low-valuation customers first. Based on the transition matrix $\mathbf{M}$ and Equation (5), we have $\mathsf{E}[V(P)] = q_H(v_L - p_d) + (1-q_H)[-c_L + \mathsf{E}[V(P)]]$, and thus $\mathsf{E}[V(P)] = v_L - p_d - c_L\frac{1-q_H}{q_H}$. Low valuation customers would wait for a price of p_d if and only if $-c_L + \mathsf{E}[V(P)] > 0$, i.e., $q_H(v_L - p_d) > c_L$. Similarly, we can show that high-valuation customers always purchase immediately if and only if $q_H(v_H - p_d) \leq c_H$. □

Proof of Proposition 10. Denote by π_L and π_H the steady state probabilities for p_d and v_H, respectively. Solving the equations $(\pi_L, \pi_H)\mathbf{M} = (\pi_L, \pi_H)$ and $\pi_L + \pi_H = 1$, we have $\pi_L = \frac{q_H}{1-q_L+q_H}$, and $\pi_H = \frac{1-q_L}{1-q_L+q_H}$. Under an optimal Markovian pricing policy, low-valuation customers always

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purchase with price p_d, while high-valuation customers purchase immediately upon arrival. Thus, the average expected profit under an optimal Markovian pricing policy is given by

$$\begin{aligned}\Pi(\mathbf{M}) =&\alpha p_d + (1-\alpha)(\pi_L p_d + (1-\pi_L)v_H)\\ \le&\alpha p_d + (1-\alpha)\left(\frac{q_H}{1+q_H}p_d + \frac{1}{1+q_H}v_H\right)\\ <&\alpha(v_L - \frac{c_L}{q_H}) + (1-\alpha)\left[\frac{q_H}{1+q_H}(v_L - \frac{c_L}{q_H}) + \frac{1}{1+q_H}v_H\right] \equiv U^M(q_H).\end{aligned}$$

The first inequality holds because $\pi_L = \frac{q_H}{1-q_L+q_H}$ increases in q_L and $\Pi(\mathbf{M})$ decreases in π_L, and thus its maximum is realized when $q_L = 0$. The second inequality is due to $q_H(v_L - p_d) > c_L$. Let $q_L = 0$ and $q_d = v_L - \frac{c_L}{q_H} - \eta(\delta)$ for an arbitrarily small $\delta > 0$ and $\lim_{\delta \searrow 0}\eta(\delta) = 0$. Then $\Pi(\mathbf{M})$ converges to the upper bound $U^M(q_H)$ when δ converges to 0. Therefore, the optimal Markovian pricing policy is letting $q_L^* = 0$ and $q_d^* = v_L - \frac{c_L}{q_H} - \eta(\delta)$. Now we turn to the optimal q_H. Taking the derivative of $U^M(q_H)$ with respect to q_H, we have

$$\frac{\partial U^M(q_H)}{\partial q_H} = \alpha\frac{c_L}{q_H^2} - (1-\alpha)\frac{v_H - v_L - c_L}{(1+q_H)^2} = \frac{\alpha c_L}{(1+q_H)^2}\left[\left(\frac{1+q_H}{q_H}\right)^2 - \frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}\right].$$

(i) If $v_H - v_L \le c_L$ or $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} \ge 1$.

When $v_H - v_L \le c_L$, we have

$$\begin{aligned}U^M(q_H) =&\alpha(v_L - \frac{c_L}{q_H}) + (1-\alpha)\left[\frac{q_H}{1+q_H}(v_L - \frac{c_L}{q_H}) + \frac{1}{1+q_H}v_H\right]\\ \le&\alpha(v_L - \frac{c_L}{q_H}) + (1-\alpha)\left[\frac{q_H}{1+q_H}(v_L - \frac{c_L}{q_H}) + \frac{1}{1+q_H}(v_L + c_L)\right]\\ =&v_L - \alpha\frac{c_L}{q_H} < v_L \le \max\{v_L, (1-\alpha)v_H\}.\end{aligned}$$

Therefore, when $v_H - v_L \le c_L$, a Markovian pricing policy is strictly dominated by an optimal static pricing policy. Consider $v_H - v_L > c_L$, then Δ_M is well-defined as a real number. Because $0 \le q_H \le 1$, we have $(\frac{1+q_H}{q_H})^2 = (1 + \frac{1}{q_H})^2 \ge 4$. Thus if $\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L} \le 4$, i.e., $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} \ge \frac{1}{2}$, then $\frac{\partial U^M(q_H)}{\partial q_H} \ge 0$. Coupling with $q_H < \frac{c_H - c_L}{v_H - v_L}$ as shown in Lemma 8, $U^M(q_H)$ is maximized at $q_H = \min\left\{1, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$. When $0 < \frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L} \le 1$, i.e., $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} \ge 1$, we have $c_L \ge (1-\alpha)(v_H - v_L)$. Consequently,

$$\begin{aligned}U^M(q_H) \le U^M(1) &= \alpha(v_L - c_L) + (1-\alpha)(\frac{1}{2}(v_L - c_L) + \frac{1}{2}v_H) = v_L - c_L + \frac{1}{2}(1-\alpha)(v_H - v_L + c_L)\\ &\le v_L - c_L + \frac{1}{2}(1-\alpha)c_L + \frac{1}{2}c_L = v_L - \frac{1}{2}\alpha c_L \le \max\{v_L, (1-\alpha)v_H\}.\end{aligned}$$

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That is, if $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} \geq 1$, an optimal Markovian pricing policy reduces to a static pricing policy.

Next we show that an optimal randomized pricing policy also reduces to a static pricing policy when $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} \geq 1$ or when Δ_M is not well-defined as a real number. Note that the preceding conditions is equivalent to $c_L \geq (1-\alpha)(v_H - v_L)$. Lemma OS.2 shows that the necessary and sufficient conditions for $U_1 > v_L$ and $U_2 > v_L$ are given by $\sqrt{\frac{c_L}{v_H - v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$ and $\frac{c_H}{v_H - v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)} < 1$, respectively. Now we show that $U_1 \leq v_L$ or $U_2 \leq v_L$ when $c_L \geq (1-\alpha)(v_H - v_L)$. When $\Delta_R < 1$ and $\Delta_R < \frac{c_H - c_L}{v_H - v_L}$, Lemma 6 shows that the expected profit from an optimal randomized pricing policy is no more than U_1. In this case, as $c_L \geq (1-\alpha)(v_H - v_L)$, we have $\sqrt{\frac{c_L}{v_H - v_L}} \geq \sqrt{1-\alpha} > \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$ since $0 < \alpha < 1$, and thus the expected profit from an optimal randomized pricing policy cannot be greater than v_L. On the other hand, when $\frac{c_H - c_L}{v_H - v_L} \leq \Delta_R < 1$, Lemma 6 shows that the expected profit from an optimal randomized pricing policy is no more than U_2. In this case, we have $\frac{c_H}{v_H - v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)} > \frac{c_L}{v_H - v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H - c_L)} \geq 1 - \alpha + \alpha \frac{v_H - v_L}{c_H - c_L} > 1$, where the second inequality is due to the condition $c_L \geq (1-\alpha)(v_H - v_L)$, and the last inequality is due to $0 < \frac{c_H - c_L}{v_H - v_L} < 1$. Therefore, an optimal randomized pricing policy also reduces to a static pricing policy when $\frac{c_H - c_L}{v_H - v_L} \leq \Delta_R < 1$.

(ii) If $1/2 \leq \Delta_M < 1$ and $\frac{c_H - c_L}{v_H - v_L} > 1$.

As shown in the previous paragraph, $U^M(q_H)$ is maximized at $q_H = \min\left\{1, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$ when $1/2 \leq \Delta_M < 1$. Coupling with $\frac{c_H - c_L}{v_H - v_L} > 1$, we have $q_H^* = 1$ and $q_d^* = v_L - \frac{c_L}{q_H^*} - \eta(\delta) = v_L - c_L - \eta(\delta)$. Since we have shown $q_L^* = 0$, then $\pi_L^* = \frac{q_H^*}{1 - q_L^* + q_H^*} = 1/2$ and $\pi_H = \frac{1 - q_L^*}{1 - q_L^* + q_H^*} = 1/2$. Then the Markovian pricing policy reduces to a cyclic pricing policy with a cyclic length of 2. Next we show that $U^M(q_H^*)$ is greater than $U(\beta^*)$, which is the expected profit from an optimal randomized pricing policy. We have

$$\begin{aligned}
U^M(q_H^*) =& \alpha\left(v_L - \frac{c_L}{q_H^*}\right) + (1-\alpha)\left[\frac{q_H^*}{1+q_H^*}\left(v_L - \frac{c_L}{q_H^*}\right) + \frac{1}{1+q_H^*} v_H\right] \\
=& \alpha v_L + (1-\alpha) v_H - (1-\alpha)\frac{c_L}{1+q_H^*} - \alpha\frac{c_L}{q_H^*} - (1-\alpha)\frac{q_H^*}{1+q_H^*}(v_H - v_L) \\
\geq& \alpha v_L + (1-\alpha) v_H - (1-\alpha)\frac{c_L}{1+\beta^*} - \alpha\frac{c_L}{\beta^*} - (1-\alpha)\frac{\beta^*}{1+\beta^*}(v_H - v_L) \\
>& \alpha v_L + (1-\alpha) v_H - (1-\alpha) c_L - \alpha\frac{c_L}{\beta^*} - (1-\alpha)\beta^*(v_H - v_L) = U(\beta^*),
\end{aligned}$$

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which the first inequality is due to $U^M(q_H^*) \geq U^M(q_H)$, for any $q_H < \frac{c_H - c_L}{v_H - v_L}$ and $\beta^* < \frac{c_H - c_L}{v_H - v_L}$, and the second inequality is due to $\beta^* > 0$.

(iii) Otherwise.

Consider the case that $1/2 \leq \Delta_M < 1$ and $\frac{c_H - c_L}{v_H - v_L} \leq 1$. From the analysis in the previous paragraph, note $\frac{\Delta_M}{1-\Delta_M} \geq 1$, we know that $q_H^* = \min\left\{1, \frac{c_H - c_L}{v_H - v_L} - \delta\right\} = \frac{c_H - c_L}{v_H - v_L} - \delta = \min\left\{\frac{\Delta_M}{1-\Delta_M}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$. Consider the case when $\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L} > 4$, i.e., $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} < \frac{1}{2}$. Solving $\frac{\partial U^M(q_H)}{\partial q_H} = 0$, we have $q_H = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1} = \frac{\Delta_M}{1-\Delta_M}$. It is easy to verify that $\frac{\partial U^M(q_H)}{\partial q_H} \geq 0$ for any $q_H \leq \frac{\Delta_M}{1-\Delta_M}$; Otherwise $\frac{\partial U^M(q_H)}{\partial q_H} \leq 0$. Coupling with $q_H < \frac{c_H - c_L}{v_H - v_L}$ as shown in Lemma 8, the maximum of $U^M(q_H)$ is realized when $q_H = \min\left\{\frac{\Delta_M}{1-\Delta_M}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$ in this case. At the optimality, $q_L^* = 0$, and $p_d^* = v_L - \frac{c_L}{q_H^*} - \eta(\delta)$. With the same approach in the proof of case (ii), we have the expected profit from this optimal Markovian pricing policy is greater than that from an optimal randomized pricing policy. □

Proof of Corollary 5. According to Lemma 6(i) and Proposition 10(i), if $\Delta_R \geq 1$ or $\Delta_M \geq 1$, randomized or Markovian pricing policy is dominated by the optimal static pricing policy. Note $\Delta_R < \Delta_M$, then we only consider the case $\Delta_R < \Delta_M < 1$. Based on Proposition 3, $\beta^* = \min\left\{\Delta_R, \frac{c_H - c_L}{v_H - v_L} - \delta\right\} = \min\left\{\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\} < 1$, and $\underline{p}^* = v_L - \frac{c_L}{\beta^*} - \eta(\delta)$. Because $\left(\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1\right)\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} = \sqrt{\frac{v_H - v_L - c_L}{v_H - v_L}} - \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \sqrt{\frac{v_H - v_L - c_L}{v_H - v_L}} < 1$, we have $\frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1} > \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$ when $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} < 1$. Based on Proposition 10, $q_H^* = \min\left\{\frac{\Delta_M}{1-\Delta_M}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\} = \min\left\{\frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$ or $q_H^* = 1$. As a result, $q_H^* \geq \beta^*$, and thus $p_d^* = v_L - \frac{c_L}{q_H^*} - \eta(\delta) \geq \underline{p}^* = v_L - \frac{c_L}{\beta^*} - \eta(\delta)$.

If $\frac{c_H - c_L}{v_H - v_L} \leq \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}$, then $\beta^* = \frac{c_H - c_L}{v_H - v_L} - \delta$. Note $\frac{c_H - c_L}{v_H - v_L} - \delta \leq \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < 1$, by Proposition 10(iii), we have $q_H^* = \frac{c_H - c_L}{v_H - v_L} - \delta$. Consequently, we have $\pi_L^* = \frac{q_H^*}{1 - q_L^* + q_H^*} = \frac{q_H^*}{1 + q_H^*} < q_H^* = \beta^*$.

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L} \leq \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}$ and $\frac{c_H - c_L}{v_H - v_L} > 1$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$, and $q_H^* = 1$, based on Proposition 10(ii). As a result, we have $\pi_L^* = \frac{q_H^*}{1 - q_L^* + q_H^*} = \frac{q_H^*}{1 + q_H^*} = \frac{1}{2}$. Because we restrict our discussion to the case when an optimal randomized pricing policy dominates an optimal static pricing policy, all conditions in Lemma OS.2 shall be satisfied. That is, when $v_L \geq (1 -$

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$\alpha)v_H$, we have $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L}$ and $\sqrt{\frac{c_L}{v_H-v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$. Consequently, $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} = \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{c_L}{v_H-v_L}} < \sqrt{\frac{\alpha}{1-\alpha}}\frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}} = \frac{\sqrt{\alpha}}{1+\sqrt{\alpha}} < \frac{1}{2} = \pi_L^*$. On the other hand, when $v_L < (1-\alpha)v_H$, we have $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L}$ and $\sqrt{\frac{c_L}{v_H-v_L}} < \sqrt{\frac{\alpha}{1-\alpha}}\left(\sqrt{\frac{v_H}{v_H-v_L}}-1\right)$. Consequently, $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} = \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{c_L}{v_H-v_L}} < \frac{\alpha}{1-\alpha}\left(\sqrt{\frac{1}{\alpha}}-1\right) = \frac{\sqrt{\alpha}}{1+\sqrt{\alpha}} < \frac{1}{2} = \pi_L^*$, where the first inequality is due to $v_L < (1-\alpha)v_H$, i.e., $\frac{v_H}{v_H-v_L} < \frac{1}{\alpha}$.

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L} \le \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$ and $\frac{c_H-c_L}{v_H-v_L} \le 1$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}$, and $q_H^* = \frac{c_H-c_L}{v_H-v_L} - \delta$, based on Proposition 10(iii). Recall that $\pi_H^* = \frac{q_H^*}{1+q_H^*}$ is increasing in q_H^*, then $\frac{\beta^*}{1+\beta^*} < \pi_H^* < \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}}$, which is due to $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < q_H^* = \frac{c_H-c_L}{v_H-v_L} - \delta < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$. It is easy to see $\frac{\beta^*}{1+\beta^*} < \beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}}$. Therefore, we can't say $\pi_L^* \ge \beta^*$ or $\pi_L^* \le \beta^*$ in this case.

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1} < \frac{c_H-c_L}{v_H-v_L}$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}$, and $q_H^* = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$ or 1. If $q_H^* = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$, we have $\pi_L^* = \frac{q_H^*}{1+q_H^*} = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}} = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}} \ge \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} = \beta^*$. If $q_H^* = 1$, we have shown in the preceding paragraph that $\pi_L^* \ge \beta^*$ in this case.

To sum up, comparing an optimal Markovian pricing policy with an optimal randomized pricing policy, we have

(i) If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}} < 1$, then $q_H^* \ge \beta^*$ and $p_d^* \ge \underline{p}^*$, where β^* and $\underline{p}^*$ are given by Proposition 3;

(ii) If $\frac{c_H-c_L}{v_H-v_L} \le \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$, then $\pi_L^* < \beta^*$;

(iii) If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L} \le \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$ and $\frac{c_H-c_L}{v_H-v_L} > 1$, or $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1} < \frac{c_H-c_L}{v_H-v_L}$, then $\pi_L^* \ge \beta^*$.

When c_L is sufficiently small, the sets of conditions (i) and (iii) are satisfied, and we thus obtain the desired result.

Lastly, we show $L_M^* = 1/\pi_L^*$ and $L_F^* = 1/\beta^*$. For an optimal Markovian pricing policy, when the price of current period i is p_d^*, then the probability of next period's price is also p_d^* is q_L^*. Moreover, for $j \ge 2$, the discount price appears in period $i+j$ for the first time, then its probability is $(1-q_L^*)(1-q_H^*)^{j-2}q_H^*$. Thus $L_M^* = q_L^* + \sum_{j=2}^{\infty} j(1-q_L^*)(1-q_H^*)^{j-2}q_H^*$. Let

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$$S=\sum_{j=2}^{\infty} j(1-q_L^*)(1-q_H^*)^{j-2}q_H^* = 2(1-q_L^*)q_H^* + \sum_{j=3}^{\infty} j(1-q_L^*)(1-q_H^*)^{j-2}q_H^*$$
$$= 2(1-q_L^*)q_H^* + \sum_{j=2}^{\infty}(j+1)(1-q_L^*)(1-q_H^*)^{j-1}q_H^*,$$

then $(1-q_H^*)S=\sum_{j=2}^{\infty} j(1-q_L^*)(1-q_H^*)^{j-1}q_H^*$. Therefore,

$$q_H^*S = 2(1-q_L^*)q_H^* + \sum_{j=2}^{\infty}(j+1)(1-q_L^*)(1-q_H^*)^{j-1}q_H^* - \sum_{j=2}^{\infty} j(1-q_L^*)(1-q_H^*)^{j-1}q_H^*$$
$$= (1-q_L^*)q_H^*\left[2+\sum_{j=2}^{\infty}(1-q_H^*)^{j-1}\right] = (1-q_L^*)q_H^*\left[2+\lim_{n\to\infty}(1-q_H^*)\frac{1-(1-q_H^*)^n}{1-(1-q_H^*)}\right] = (1-q_L^*)(1+q_H^*).$$

Note $q_L^*=0$ and $\pi_L^*=\frac{q_H^*}{1-q_L^*+q_H^*}=\frac{q_H^*}{1+q_H^*}$, then $S=(1-q_L^*)(1+1/q_H^*)$, and $L_M^*=q_L^*+S=1+1/q_H^*=1/\pi_L^*$. For an optimal randomized pricing policy, when the price of current period i is $\underline{p}^*$, then the probability of discount price $\underline{p}^*$ appears in period $i+j$ for the first time is $\beta^*(1-\beta^*)^{j-1}$. Then $L_F^*=\sum_{j=1}^{\infty} j\beta^*(1-\beta^*)^{j-1}$. Consequently,

$$\beta^*L_F^* = L_F^* - (1-\beta^*)L_F^* = \sum_{j=1}^{\infty} j\beta^*(1-\beta^*)^{j-1} - \sum_{j=1}^{\infty} j\beta^*(1-\beta^*)^{j}$$
$$= \beta^* + \sum_{j=2}^{\infty} j\beta^*(1-\beta^*)^{j-1} - \sum_{j=1}^{\infty} j\beta^*(1-\beta^*)^{j}$$
$$= \beta^* + \sum_{j=1}^{\infty}(j+1)\beta^*(1-\beta^*)^{j} - \sum_{j=1}^{\infty} j\beta^*(1-\beta^*)^{j}$$
$$= \sum_{j=1}^{\infty}\beta^*(1-\beta^*)^{j-1} = \beta^*\lim_{n\to\infty}\frac{1-(1-\beta^*)^n}{1-(1-\beta^*)} = 1.$$

Therefore, $L_F^*=1/\beta^*$. □

Proof of Corollary 6. First we show it is true for low-valuation customers. By Corollary 5, we know $p_d^*\geq\underline{p}^*$. Recall the low-valuation customers always buy with the price p_d^* and $\underline{p}^*$ under an optimal Markovian and randomized pricing policy, respectively.

Next we consider high-valuation customers. Since high-valuation customers will buy immediately under both polices, we have

$$U_H^M(q_H^*)=\pi_L^*p_d^*+(1-\pi_L^*)v_H, \quad U_H(\beta^*)=\beta^*\underline{p}^*+(1-\beta^*)v_H,$$

where $U_H^M(q_H^*)$ and $U_H(\beta^*)$ are the expected profit earned from high-valuation customers under an optimal Markovian and randomized pricing policy, respectively. Now we

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will show $U_H^M(q_H^*) > U_H(\beta^*)$. According to Proposition 3, $\beta^* = \min\left\{\Delta_R, \frac{c_H-c_L}{v_H-v_L} - \delta\right\} = \min\left\{\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}, \frac{c_H-c_L}{v_H-v_L} - \delta\right\} < 1$, and $\underline{p}^* = v_L - \frac{c_L}{\beta^*} - \eta(\delta)$. Based on Proposition 10, $q_H^* = \min\left\{\frac{\Delta_M}{1-\Delta_M}, \frac{c_H-c_L}{v_H-v_L} - \delta\right\} = \min\left\{\frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}, \frac{c_H-c_L}{v_H-v_L} - \delta\right\}$ or $q_H^* = 1$. Note $\pi_L^* = \frac{q_H^*}{1-q_L^*+q_H^*} = \frac{q_H^*}{1+q_H^*}$, thus,

$$U_H^M(q_H^*) = \pi_L^* p_d^* + (1-\pi_L^*)v_H = v_H - c_L - \frac{q_H^*}{1+q_H^*}\eta(\delta) - \frac{q_H^*}{1+q_H^*}(v_H - v_L - c_L),$$

$$U_H(\beta^*) = \pi_L^* p_d^* + (1-\pi_L^*)v_H = v_H - c_L - \beta^*\eta(\delta) - \beta^*(v_H - v_L).$$

Note $\lim_{\delta \searrow 0} \eta(\delta) = 0$. Therefore, if we want to show $U_H^M(q_H^*) > U_H(\beta^*)$, it is equivalent to verify $\frac{q_H^*}{1+q_H^*}(v_H - v_L - c_L) < \beta^*(v_H - v_L)$.

If $\frac{c_H-c_L}{v_H-v_L} \le \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$, then $\beta^* = \frac{c_H-c_L}{v_H-v_L} - \delta$. Note $\frac{c_H-c_L}{v_H-v_L} \le \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < 1$, by Proposition 10(iii), we have $q_H^* = \frac{c_H-c_L}{v_H-v_L} - \delta$. Hence, we have $\frac{q_H^*}{1+q_H^*}(v_H - v_L - c_L) < q_H^*(v_H - v_L - c_L) < \beta^*(v_H - v_L)$, where the first inequality is due to $0 < q_H^* \le 1$, and the second is due to $c_L > 0$.

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L} \le \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$ and $\frac{c_H-c_L}{v_H-v_L} > 1$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}$. Note $1 < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$, i.e., $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}} > \frac{1}{2}$, then $q_H^* = 1$, based on Proposition 10(ii). Hence, we have

$$\begin{aligned}\frac{q_H^*}{1+q_H^*}(v_H - v_L - c_L) &= \frac{1}{2}(v_H - v_L - c_L) < \frac{1}{2}\sqrt{v_H - v_L - c_L}\sqrt{v_H - v_L} \\ &< \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}}\sqrt{v_H - v_L - c_L}\sqrt{v_H - v_L} \\ &= \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}(v_H - v_L) = \beta^*(v_H - v_L).\end{aligned}$$

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L} \le \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$ and $\frac{c_H-c_L}{v_H-v_L} \le 1$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}$, and $q_H^* = \frac{c_H-c_L}{v_H-v_L} - \delta$, based on Proposition 10(iii). Hence, we have

$$\begin{aligned}\frac{q_H^*}{1+q_H^*}(v_H - v_L - c_L) &< \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}}(v_H - v_L - c_L) \\ &< \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}}\sqrt{v_H - v_L - c_L}\sqrt{v_H - v_L} \\ &= \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}(v_H - v_L) = \beta^*(v_H - v_L),\end{aligned}$$

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where the first inequality is due to $\frac{q_H^*}{1+q_H^*}$ is increasing in q_H^* and $q_H^* = \frac{c_H - c_L}{v_H - v_L} - \delta < \frac{c_H - c_L}{v_H - v_L} \leq \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}$, and the second inequality is due to $c_L > 0$.

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1} < \frac{c_H - c_L}{v_H - v_L}$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$ and $q_H^* = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}$ or 1. If $q_H^* = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}$, we have $\frac{q_H^*}{1+q_H^*} = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}}} = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}}$. Hence,

$$\begin{aligned}\frac{q_H^*}{1+q_H^*}(v_H - v_L - c_L) &= \sqrt{\frac{\alpha c_L}{1-\alpha}}\sqrt{v_H - v_L - c_L} < \sqrt{\frac{\alpha c_L}{1-\alpha}}\sqrt{v_H - v_L} \\ &= \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}(v_H - v_L) = \beta^*(v_H - v_L).\end{aligned}$$

If $q_H^* = 1$, we have shown in the preceding paragraph that $\frac{q_H^*}{1+q_H^*}(v_H - v_L - c_L) < \beta^*(v_H - v_L)$ in this case. □

Proof of Lemma 9. In order to show the result, we first prove an auxiliary lemma.

LEMMA OS.5. *$\frac{c_{i_1}}{v_{i_1}} < \frac{c_{i_2}}{v_{i_2}}$ is a necessary condition that type-i_1 customers would wait and type-i_2 customers would either purchase or leave immediately, for any $i_1 < i_2$.*

Proof of Lemma OS.5. According to Lemma OA.1, type-i_1 customers wait and type-i_2 customers either purchase or leave immediately imply that $\sum_{j=1}^{i_1} \beta_j v_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j > \sum_{j=1}^{i_1} \beta_j x_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j = c_{i_1}$ and $\sum_{j=1}^{i_2} \beta_j v_{i_2} - \sum_{j=1}^{i_2} \beta_j x_j \leq c_{i_2}$, respectively. Thus, we have

$$c_{i_2} \geq \sum_{j=1}^{i_2} \beta_j v_{i_2} - \sum_{j=1}^{i_2} \beta_j x_j > \sum_{j=1}^{i_1} \beta_j v_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j,$$

and

$$c_{i_1} < \sum_{j=1}^{i_1} \beta_j v_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j.$$

Rearranging the above two inequalities, we have

$$\frac{c_{i_2}}{c_{i_1}} > \frac{\sum_{j=1}^{i_2} \beta_j v_{i_2} - \sum_{j=1}^{i_2} \beta_j x_j}{\sum_{j=1}^{i_1} \beta_j v_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j} > \frac{v_{i_2}}{v_{i_1}},$$

where the second inequality is due to $\left(\sum_{j=1}^{i_2} \beta_j v_{i_2} - \sum_{j=1}^{i_2} \beta_j x_j\right) v_{i_1} - \left(\sum_{j=1}^{i_1} \beta_j v_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j\right) v_{i_2} = v_{i_1}\left(\sum_{j=i_1+1}^{i_2} \beta_j v_{i2} - \sum_{j=i_1+1}^{i_2} \beta_j x_j\right) + (v_{i_2} - v_{i_1}) \sum_{j=1}^{i_1} \beta_j x_j > 0$. We thus obtain the announced result. □

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Lemma OS.5 and the condition $\frac{c_1}{v_1} \geq \cdots \geq \frac{c_n}{v_n}$ imply that there exists a k such that any customer with valuation greater than v_k will wait, and any customer with valuation less than or equal to v_k will either purchase or leave immediately upon arrival, under any randomized pricing policy. Thus, based on Lemma OA.1, its optimal expected profit can be derived by solving the following problem.

$$\begin{aligned} \max_{\beta,\mathbf{x}} \quad & \sum_{i=1}^{n} \alpha_i \pi_i \\ \text{s.t.} \quad & \sum_{j=1}^{i} \beta_j v_i - \sum_{j=1}^{i} \beta_j x_j \leq c_i, \quad \forall\, i \leq k, \\ & \sum_{j=1}^{i} \beta_j x_i - \sum_{j=1}^{i} \beta_j x_j = c_i, \quad \forall\, i > k, \\ & x_i \leq v_i, \forall\, i \leq k, \text{ and } x_i < v_i, \forall\, i > k, \text{ and } \sum_{i=1}^{n} \beta_i = 1, \end{aligned} \tag{OS.3}$$

where $\pi_i = \sum_{j=1}^{i} \beta_j x_j$, $\forall\, i \leq k$, and $\pi_i = \sum_{j=1}^{i} \beta_j x_j \Big/ \sum_{j=1}^{i} \beta_j$ otherwise.

First consider the case when $k = n-1$ or n. In this case, we have

$$\sum_{i=1}^{n} \alpha_i \pi_i = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{i} \beta_j x_j = \beta_1 x_1 + \sum_{i=2}^{n} \alpha_i \beta_2 x_2 + \cdots + \alpha_n \beta_n x_n \leq \max_j \left\{ \sum_{i=j}^{n} \alpha_i x_j \right\} \leq \max_j \left\{ \sum_{i=j}^{n} \alpha_i v_j \right\},$$

where the last term indicates the expected profit under an optimal static pricing policy.

Next consider the case when $k = 0$. We will prove the dominance of static pricing by induction. When $n = 2$, the profit function is given by $\alpha_1 x_1 + \alpha_2[\beta_1 x_1 + (1-\beta_1)x_2]$, and constraints are $\beta_1(v_1 - x_1) > c_1$ and $\beta_1(x_2 - x_1) = c_2$. That is, $x_1 < v_1 - \frac{c_1}{\beta_1}$ and $x_2 = x_1 + \frac{c_2}{\beta_1}$. Thus, we have

$$\alpha_1 x_1 + \alpha_2[\beta_1 x_1 + (1-\beta_1)x_2] = x_1 + \alpha_2 \left(\frac{1}{\beta_1} - 1 \right) c_2 < v_1 - \frac{c_1}{\beta_1} + \alpha_2 \left(\frac{1}{\beta_1} - 1 \right) c_2 \equiv h(\beta_1),$$

where the equality is due to $x_2 = x_1 + \frac{c_2}{\beta_1}$ and $\alpha_1 + \alpha_2 = 1$, and the inequality is due to $x_1 < v_1 - \frac{c_1}{\beta_1}$. Taking the derivative of $h(\beta_1)$ with respect to β_1, we have $\frac{\partial h(\beta_1)}{\partial \beta_1} = \frac{c_1 - \alpha_2 c_2}{\beta_1^2}$. If $c_1 \geq \alpha_2 c_2$, $h(\beta_1)$ is increasing in β_1 and thus $h(\beta_1) \leq h(1) = v_1 - c_1$. If $c_1 < \alpha_2 c_2$, $h(\beta_1)$ is decreasing in β_1 and thus $h(\beta_1) \leq h(c_1/v_1) = \alpha_2 \left(\frac{v_1}{c_1} - 1 \right) c_2 < \alpha_2 \frac{v_1}{c_1} c_2 \leq \alpha_2 v_2$, where the first inequality is due to $v_1 - \frac{c_1}{\beta_1} > 0$, and the last inequality is due to $\frac{c_1}{v_1} \geq \frac{c_2}{v_2}$. So the expected profit from any randomized pricing policy is less than or equal to the expected profit from the optimal static pricing policy, which is given by $\max\{v_1, \alpha_2 v_2\}$.

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Now suppose that, under the case when $k=0$, an optimal randomized pricing policy reduces to the optimal static pricing policy for $n-1$, $\forall\, n \geq 3$. We next show that this statement also holds for n. Based on Problem (OS.3), we can rewrite the expected profit function as:

$$\sum_{i=1}^{n} \alpha_i \pi_i = \alpha_1 x_1 + \cdots + \alpha_{n-1}\left(x_{n-1} - \frac{c_{n-1}}{\beta_1 + \cdots + \beta_{n-1}}\right) + \alpha_n \left(x_n - c_n\right),$$

where $x_n = x_{n-1} + \frac{c_n - c_{n-1}}{\beta_1 + \cdots + \beta_{n-1}}$. Taking the derivative of $\sum_{i=1}^{n} \alpha_i \pi_i$ with respect to $(\beta_1 + \cdots + \beta_{n-1})$, we have $\frac{\partial\left(\sum_{i=1}^{n} \alpha_i \pi_i\right)}{\partial(\beta_1 + \cdots + \beta_{n-1})} = \frac{(\alpha_{n-1}+\alpha_n)c_{n-1} - \alpha_n c_n}{(\beta_1 + \cdots + \beta_{n-1})^2}$. If $(\alpha_{n-1} + \alpha_n)c_{n-1} \geq \alpha_n c_n$, $\sum_{i=1}^{n} \alpha_i \pi_i$ is increasing in $\beta_1 + \cdots + \beta_{n-1}$. Thus, it is optimal to let $\beta_1 + \cdots + \beta_{n-1} = 1$ and $\beta_n = 0$, which reduces to the case with $n-1$ types of customers. On the other hand, if $(\alpha_{n-1} + \alpha_n)c_{n-1} < \alpha_n c_n$, $\sum_{i=1}^{n} \alpha_i \pi_i$ is decreasing in $\beta_1 + \cdots + \beta_{n-1}$. In this case, it is optimal to let $\beta_1 + \cdots + \beta_{n-1} = \beta_1 + \cdots + \beta_{n-2}$, i.e., $\beta_{n-1} = 0$, which again reduces to the case with $n-1$ types of customers. We thus establish the result for n.

Lastly, consider the case when $1 \leq k \leq n-2$. Again, we prove the dominance of static pricing by induction. When $n=3$ and $k=1$, the profit function is given by $\alpha_1\beta_1 x_1 + \alpha_2 \frac{\beta_1 x_1 + \beta_2 x_2}{\beta_1 + \beta_2} + \alpha_3[\beta_1 x_1 + \beta_2 x_2 + (1 - \beta_1 - \beta_2)x_3]$, and constraints are $\beta_1(v_1 - x_1) \leq c_1$, $\beta_1(x_2 - x_1) = c_2$, and $(\beta_1 + \beta_2)x_3 - \beta_1 x_1 - \beta_2 x_2 = c_3$. Thus, we have $x_3 = x_2 + \frac{c_3 - c_2}{\beta_1 + \beta_2}$, and we can rewrite the profit function as:

$$\sum_{i=1}^{3} \alpha_i \pi_i = \alpha_1\beta_1 x_1 + \alpha_2\left(x_2 - \frac{c_2}{\beta_1 + \beta_2}\right) + \alpha_3\left(x_2 + \frac{c_3 - c_2}{\beta_1 + \beta_2} - c_3\right).$$

Taking the derivative of the profit function with respect to $(\beta_1 + \beta_2)$, we have $\frac{\partial\left(\sum_{i=1}^{3} \alpha_i \pi_i\right)}{\partial(\beta_1 + \beta_2)} = \frac{(\alpha_2 + \alpha_3)c_2 - \alpha_3 c_3}{(\beta_1 + \beta_2)^2}$. If $(\alpha_2 + \alpha_3)c_2 \geq \alpha_3 c_3$, profit is maximized by letting $\beta_1 + \beta_2 = 1$. Plugging $\beta_1 + \beta_2 = 1$ into the profit function, we have

$$\begin{aligned}\sum_{i=1}^{3} \alpha_i \pi_i &= \alpha_1\beta_1 x_1 + (\alpha_2 + \alpha_3)(x_2 - c_2) = \alpha_1\beta_1 x_1 + (\alpha_2 + \alpha_3)(\beta_1 x_1 + \beta_2 x_2) \\ &\leq \max\{x_1, (\alpha_2 + \alpha_3)x_2\} \leq \max\{v_1, (\alpha_2 + \alpha_3)v_2\},\end{aligned}$$

where the second equality is due to $\beta_1(x_2 - x_1) = c_2$. On the other hand, if $(\alpha_2 + \alpha_3)c_2 < \alpha_3 c_3$, profit is maximized by letting $\beta_2 = 0$. Plugging $\beta_2 = 0$ into the profit function, we have

$$\sum_{i=1}^{3} \alpha_i \pi_i = \alpha_1\beta_1 x_1 + \alpha_2\left(x_2 - \frac{c_2}{\beta_1}\right) + \alpha_3\left(x_2 + \frac{c_3 - c_2}{\beta_1} - c_3\right) = \alpha_1\beta_1 x_1 + \alpha_2 x_1 + \alpha_3\left[x_1 + \left(\frac{1}{\beta_1} - 1\right)c_3\right],$$

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where the second equality is due to $x_2 = x_1 + \frac{c_2}{\beta_1}$. Taking the derivative of the profit function with respect to β_1, we have $\frac{\partial\left(\sum_{i=1}^{3}\alpha_i\pi_i\right)}{\partial\beta_1} = \alpha_1 x_1 - \frac{\alpha_3 c_3}{\beta_1^2}$, which is increasing in β_1. Thus, the profit function $\sum_{i=1}^{3}\alpha_i\pi_i$ is convex in β_1. Here we need to consider two cases separately: $\beta_1 \le \frac{c_2}{v_2 - v_1}$ and $\beta_1 > \frac{c_2}{v_2 - v_1}$. When $\beta_1 > \frac{c_2}{v_2 - v_1}$, we have $x_2 = x_1 + \frac{c_2}{\beta_1} < v_2$ due to $x_1 \le v_1$. On the other hand, when $\beta_1 \le \frac{c_2}{v_2 - v_1}$, $x_1 = x_2 - \frac{c_2}{\beta_1} < v_1$ due to $x_2 \le v_2$.

Consider first when $\beta_1 \le \frac{c_2}{v_2 - v_1}$. In this case, the optimal x_2 is given by $x_2 = v_2 - \delta$, for any small $\delta > 0$. Combining with the condition $x_1 = x_2 - \frac{c_2}{\beta_1} \ge 0$, β_1 must be greater than $\frac{c_2}{v_2}$. Thus, in order to show the dominance of static pricing policy, we only need to prove that the expected profit evaluated at $\beta_1 = \frac{c_2}{v_2}$ and $\beta_1 = \frac{c_2}{v_2 - v_1}$ are less than the expected profit from an optimal static pricing policy. Plugging $\beta_1 = \frac{c_2}{v_2}$ and $x_2 = v_2 - \delta$ into the profit function, we have

$$\sum_{i=1}^{3}\alpha_i\pi_i = -(\alpha_1\beta_1 + \alpha_2 + \alpha_3)\delta + \alpha_3\left(\frac{v_2}{c_2} - 1\right)c_3 < \alpha_3\left(\frac{v_2}{c_2} - 1\right)c_3 \le \alpha_3\left(\frac{v_3}{c_3} - 1\right)c_3 < \alpha_3 v_3,$$

where the first equality is due to $x_1 = x_2 - \frac{c_2}{\beta_1} = -\delta$, and the second inequality is due to $\frac{c_2}{v_2} \ge \frac{c_3}{v_3}$. Next plugging $\beta_1 = \frac{c_2}{v_2 - v_1}$ and $x_2 = v_2 - \delta$ into the profit function, we have

$$\sum_{i=1}^{3}\alpha_i\pi_i < \alpha_1\frac{c_2}{v_2 - v_1}v_1 + \alpha_2 v_1 + \alpha_3\left[v_1 + \left(\frac{v_2 - v_1}{c_2} - 1\right)c_3\right].$$

Consider three sub-cases.

(a) $v_1 \ge \alpha_3 v_3$. Because $\frac{c_1}{v_1} \ge \frac{c_3}{v_3} = \frac{\alpha_3 c_3}{\alpha_3 v_3}$, we have $c_1 \ge \alpha_3 c_3$. Furthermore, because $(\alpha_2 + \alpha_3)c_2 < \alpha_3 c_3$ and $\frac{(\alpha_2+\alpha_3)c_2}{(\alpha_2+\alpha_3)v_2} = \frac{c_2}{v_2} \ge \frac{c_3}{v_3} = \frac{\alpha_3 c_3}{\alpha_3 v_3}$, we have $\alpha_3 v_3 > (\alpha_2 + \alpha_3)v_2$. Combining the preceding conditions, we have $v_1 > (\alpha_2 + \alpha_3)v_2$. That is, $\frac{\alpha_1 v_2}{v_2 - v_1} > 1$. Therefore,

$$\begin{aligned}
&\alpha_1\frac{c_2}{v_2 - v_1}v_1 + \alpha_2 v_1 + \alpha_3\left[v_1 + \left(\frac{v_2 - v_1}{c_2} - 1\right)c_3\right] \\
=&v_1 - \alpha_1\left(1 - \frac{c_2}{v_2 - v_1}\right)v_1 + \alpha_3\left(\frac{v_2 - v_1}{c_2} - 1\right)c_3 \\
=&v_1 - \left(\frac{v_2 - v_1}{c_2} - 1\right)\left(\alpha_1\frac{c_2}{v_2 - v_1}v_1 - \alpha_3 c_3\right) < v_1 - \left(\frac{v_2 - v_1}{c_2} - 1\right)\left(\frac{v_1}{v_2}c_2 - c_1\right) \le v_1,
\end{aligned}$$

where the first inequality is due to $\frac{\alpha_1 v_2}{v_2 - v_1} > 1$ and $c_1 \ge \alpha_3 c_3$, and the second inequality is due to $\frac{c_1}{v_1} \ge \frac{c_2}{v_2}$.

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(b) $v_1 \le (\alpha_2+\alpha_3)v_2 < \alpha_3 v_3$, i.e., $\frac{\alpha_1 v_2}{v_2-v_1} \le 1$. In this case, we have

$$\begin{aligned}
&\alpha_1 \frac{c_2}{v_2-v_1} v_1 + \alpha_2 v_1 + \alpha_3 \left[v_1 + \left(\frac{v_2-v_1}{c_2} - 1 \right) c_3 \right] \\
\le &\alpha_1 \frac{c_2}{v_2-v_1} v_1 + \alpha_2 v_1 + \alpha_3 \left[v_1 + \left(\frac{v_2-v_1}{c_2} - 1 \right) \frac{v_3}{v_2} c_2 \right] \\
= &\alpha_1 \frac{c_2}{v_2-v_1} v_1 + \alpha_2 v_1 + \alpha_3 v_1 + \alpha_3 v_3 - \alpha_3 v_3 \left(\frac{v_1}{v_2} + \frac{c_2}{v_2} \right) \\
= &\alpha_1 \frac{c_2}{v_2-v_1} v_1 - \alpha_3 v_3 \frac{c_2}{v_2} + \alpha_3 v_3 + (\alpha_2+\alpha_3) v_2 \frac{v_1}{v_2} - \alpha_3 v_3 \frac{v_1}{v_2} \\
< &\alpha_1 \frac{c_2}{v_2-v_1} v_1 - \alpha_3 v_3 \frac{c_2}{v_2} + \alpha_3 v_3 < \frac{v_1}{v_2} c_2 - v_1 \frac{c_2}{v_2} + \alpha_3 v_3 = \alpha_3 v_3,
\end{aligned}$$

where the second inequality is due to $(\alpha_2+\alpha_3)v_2 < \alpha_3 v_3$, and the third inequality is due to $\frac{\alpha_1 v_2}{v_2-v_1} \le 1$ and $v_1 < \alpha_3 v_3$.

(c) $(\alpha_2+\alpha_3)v_2 \le v_1 < \alpha_3 v_3$. If $\alpha_1 \frac{c_2}{v_2-v_1} v_1 \ge \alpha_3 c_3$, we have

$$\begin{aligned}
&\alpha_1 \frac{c_2}{v_2-v_1} v_1 + \alpha_2 v_1 + \alpha_3 \left[v_1 + \left(\frac{v_2-v_1}{c_2} - 1 \right) c_3 \right] \\
= &v_1 - \left(\frac{v_2-v_1}{c_2} - 1 \right) \left(\alpha_1 \frac{c_2}{v_2-v_1} v_1 - \alpha_3 c_3 \right) \le v_1.
\end{aligned}$$

On the other hand, if $\alpha_1 \frac{c_2}{v_2-v_1} v_1 < \alpha_3 c_3$, we have

$$\begin{aligned}
&\alpha_1 \frac{c_2}{v_2-v_1} v_1 + \alpha_2 v_1 + \alpha_3 \left[v_1 + \left(\frac{v_2-v_1}{c_2} - 1 \right) c_3 \right] \\
< &\alpha_2 v_1 + \alpha_3 v_1 + \alpha_3 c_3 \frac{v_2-v_1}{c_2} \\
< &\alpha_2 v_1 + \alpha_3 v_1 + \alpha_3 v_3 \frac{v_2-v_1}{v_2} < \alpha_3 v_3,
\end{aligned}$$

where the second inequality is due to $c_2 \le v_2$, and the third inequality is due to $(\alpha_2+\alpha_3)v_2 < \alpha_3 v_3$. Now we prove that under the condition $\beta_1 \le \frac{c_2}{v_2-v_1}$, an optimal randomized pricing policy reduces to the optimal static pricing policy. The case when $\beta_1 > \frac{c_2}{v_2-v_1}$ is trivial, as, when plugging in $\beta_1 = 1$, the profit function reduces to x_1, which is less than or equal to v_1.

Now suppose that, under the case when $1 \le k \le n-2$, an optimal randomized pricing policy reduces to an optimal static pricing policy for $n-1$, $\forall\, n \ge 3$. We next show that this statement also holds for n. Based on Problem (OS.3), we can rewrite the expected profit function as:

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i \pi_i = &\alpha_1 \beta_1 x_1 + \cdots + \alpha_k (\beta_1 x_1 + \cdots + \beta_k x_k) + \alpha_{k+1} \left(x_{k+1} - \frac{c_{k+1}}{\beta_1 + \cdots + \beta_{k+1}} \right) \\
&+ \cdots + \alpha_{n-1} \left(x_{n-1} - \frac{c_{n-1}}{\beta_1 + \cdots + \beta_{n-1}} \right) + \alpha_n \left(x_n - c_n \right),
\end{aligned}$$

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where $x_n = x_{n-1} + \frac{c_n - c_{n-1}}{\beta_1 + \cdots + \beta_{n-1}}$. Taking the derivative of $\sum_{i=1}^{n} \alpha_i \pi_i$ with respect to $(\beta_1 + \cdots + \beta_{n-1})$, we have $\frac{\partial\left(\sum_{i=1}^{n} \alpha_i \pi_i\right)}{\partial(\beta_1 + \cdots + \beta_{n-1})} = \frac{(\alpha_{n-1} + \alpha_n)c_{n-1} - \alpha_n c_n}{(\beta_1 + \cdots + \beta_{n-1})^2}$. If $(\alpha_{n-1} + \alpha_n)c_{n-1} \geq \alpha_n c_n$, $\sum_{i=1}^{n} \alpha_i \pi_i$ is increasing in $\beta_1 + \cdots + \beta_{n-1}$. Thus, it is optimal to let $\beta_1 + \cdots + \beta_{n-1} = 1$ and $\beta_n = 0$, which reduces to the case with $n-1$ types of customers. On the other hand, if $(\alpha_{n-1} + \alpha_n)c_{n-1} < \alpha_n c_n$, $\sum_{i=1}^{n} \alpha_i \pi_i$ is decreasing in $\beta_1 + \cdots + \beta_{n-1}$. In this case, it is optimal to let $\beta_1 + \cdots + \beta_{n-1} = \beta_1 + \cdots + \beta_{n-2}$, i.e., $\beta_{n-1} = 0$, which again reduces to the case with $n-1$ types of customers. We thus obtain the announced result. □

Proof of Proposition 11. Consider any randomized pricing policy. First we prove by contradiction that, under the condition $v_1 - c_1 \geq v_2 - c_2 \geq \cdots \geq v_n - c_n$, a customer with valuation v_{i_2} will not wait, if type-i_1 customers do not wait, $\forall\, i_1 < i_2$. Suppose there exists $i_2 > i_1$, where type-i_1 customers do not wait but type-i_2 customers wait. By Proposition 1, we have

$$\sum_{i=1}^{i_1} \beta_i v_{i_1} - \sum_{i=1}^{i_1} \beta_i x_i \leq c_{i_1}, \text{ and } \sum_{i=1}^{i_2} \beta_i v_{i_2} - \sum_{i=1}^{i_2} \beta_i x_i > c_{i_2}.$$

Because $x_i > v_{i_1}$, $\forall\, i > i_1$, we have $\sum_{i=1}^{i_2} \beta_i v_{i_1} - \sum_{i=1}^{i_2} \beta_i x_i < \sum_{i=1}^{i_1} \beta_i v_{i_1} - \sum_{i=1}^{i_1} \beta_i x_i \leq c_{i_1}$. Combining with the inequality $\sum_{i=1}^{i_2} \beta_i v_{i_2} - \sum_{i=1}^{i_2} \beta_i x_i > c_{i_2}$, we have

$$\sum_{i=1}^{i_2} \beta_i (v_{i_2} - v_{i_1}) > c_{i_2} - c_{i_1} \geq v_{i_2} - v_{i_1},$$

where the second inequality is due to $v_{i_1} - c_{i_1} \geq v_{i_2} - c_{i_2}$. However, the above inequality contradicts to $\sum_{i=1}^{i_2} \beta_i \leq 1$. Thus, we obtain the announced result.

The implication of this result is that there always exists $1 \leq k_r \leq n$, such that any customer with valuation greater than or equal to v_{k_r} does not wait, and any customer with valuation smaller than v_{k_r} will wait. Hence, we get Proposition 11(i). Consequently, based on Lemma OA.1, the optimal policy with such induced customer behavior can be derived by solving:

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$$\begin{aligned}
\max_{\beta,\mathbf{x}} \quad & \sum_{j=1}^{n} \alpha_j \pi_j \\
\text{s.t.} \quad & \sum_{i=1}^{j} \beta_i x_j - \sum_{i=1}^{j} \beta_i x_i = c_j, \quad \forall\, j < k_r, \\
& \sum_{i=1}^{j} \beta_i v_j - \sum_{i=1}^{j} \beta_i x_i \le c_j, \quad \forall\, j \ge k_r, \\
& x_j < v_j, j < k_r, \text{ and } x_j \le v_j, j \ge k_r, \text{ and } \sum_{j=1}^{n} \beta_j = 1,
\end{aligned} \tag{OS.4}$$

where $\pi_j = \sum_{i=1}^{j} \beta_i x_i \Big/ \sum_{i=1}^{j} \beta_i$, if $j < k_r$; Otherwise, $\pi_j = \sum_{i=1}^{j} \beta_i x_i$. The set of conditions $\sum_{i=1}^{j} \beta_i x_j - \sum_{i=1}^{j} \beta_i x_i = c_j, \forall\, j < k_r$ imply that

$$\begin{aligned}
x_1 &= x_2 - c_2/\beta_1 = x_3 - c_2/\beta_1 - (c_3 - c_2)/(\beta_1 + \beta_2) \\
&= \cdots = x_{k_r-1} - c_2/\beta_1 - (c_3 - c_2)/(\beta_1 + \beta_2) \cdots - (c_{k_r-1} - c_{k_r-2})/(\beta_1 + \beta_2 + \cdots + \beta_{k_r-2}).
\end{aligned}$$

Due to the assumption $c_1 < c_2 < \cdots < c_n$, we have

$$\begin{aligned}
v_1 - c_1/\beta_1 &> v_2 - c_2/\beta_1 > v_3 - c_2/\beta_1 - (c_3 - c_2)/(\beta_1 + \beta_2) > \cdots \\
&> v_{k_r-1} - c_2/\beta_1 - (c_3 - c_2)/(\beta_1 + \beta_2) \cdots - (c_{k_r-1} - c_{k_r-2})/(\beta_1 + \beta_2 + \cdots + \beta_{k_r-2}) > x_1,
\end{aligned}$$

where the last inequality is due to $x_{k_r-1} \le v_{k_r-1}$. As the profit function is increasing in x_j, $\forall\, j$, the optimal x_1^* is given by $x_1^* = v_{k_r-1} - c_2/\beta_1 - (c_3 - c_2)/(\beta_1 + \beta_2) \cdots - (c_{k_r-1} - c_{k_r-2})/(\beta_1 + \beta_2 + \cdots + \beta_{k_r-2}) - \delta$. The optimal x_j^*, $\forall j \in \{2, \ldots, k_r\}$ can be derived by solving the system of equations $\sum_{i=1}^{j} \beta_i x_j^* - \sum_{i=1}^{j} \beta_i x_i^* = c_j, \forall\, j < k_r$, and the optimal x_j^*, $\forall\, j \ge k_r$ is given by $x_j^* = v_j$. Plugging the optimal x_i^* into the objective function, and ignoring the sufficiently small δ, we have

$$\begin{aligned}
\sum_{j=1}^{n} \alpha_j \pi_j =& \alpha_1 \left(v_{k_r-1} - \frac{c_2}{\beta_1} - \frac{c_3 - c_2}{\beta_1 + \beta_2} \cdots - \frac{c_{k_r-1} - c_{k_r}}{\beta_1 + \beta_2 + \cdots + \beta_{k_r-2}} \right) + \alpha_2 \left(v_{k_r-1} - \frac{c_{k_r-1}}{\beta_1 + \beta_2} \right) + \cdots \\
& + \alpha_{k_r-1} \left(v_{k_r-1} - \frac{c_{k_r-1}}{\beta_1 + \beta_2 + \cdots + \beta_{k_r-1}} \right) + \alpha_{k_r} \left((\beta_1 + \beta_2 + \cdots + \beta_{k_r-1}) v_{k_r-1} - c_{k_r-1} + \beta_{k_r} v_{k_r} \right) + \cdots \\
& + \alpha_n \left((\beta_1 + \beta_2 + \cdots + \beta_{k_r-1}) v_{k_r-1} - c_{k_r-1} + \beta_{k_r} v_{k_r} + \cdots + \beta_n v_n \right).
\end{aligned}$$

It is easy to verify that the profit function is increasing in β_1, $(\beta_1 + \beta_2)$, $\ldots$, $(\beta_1 + \beta_2 + \cdots + \beta_{k_r-1})$. Thus, the profit is maximized when $\beta_2 = \beta_3 = \cdots = \beta_{k_r-2} = \beta_{k_r-1} = 0$. Next consider β_j, $\forall\, j \ge k_r$. The partial derivative of the profit function with respect to β_j is given by $\frac{\partial \left(\sum_{i=1}^{n} \alpha_i \pi_i \right)}{\partial \beta_j} = \sum_{i=j}^{n} \alpha_i v_j - \alpha_n v_n$. Denote $h_r = \arg\max_{k_r \le j \le n} \sum_{i=j}^{n} (\alpha_i v_j)$. The profit is maximized when $\beta_{h_r} = 1 - \beta_1$, and the rest are all equal to 0. We thus obtain the announced result. $\square$

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Proof of Proposition 12. The existence of k_c is guaranteed by Corollary OA.1. Next we show that $\tau_{i+1} \geq \tau_i$, for any $1 \leq i < k_c - 1$, and $\tau_1 = 1$. Because $v_{i+1} - c_{i+1} \leq v_i - c_i$, we have $v_{i+1} - (T - \tau_{i+1})c_{i+1} \leq v_i - (T - \tau_{i+1})c_i$. Thus, we know that $\tau_i \leq \tau_{i+1}$ based on the definition of τ_i, for any $1 \leq i < k_c - 1$. We show $\tau_1 = 1$ by contradiction. Suppose $\tau_1 > 1$. Then, the optimal expected profit per period from period 1 to period $\tau_1 - 1$ is given by $\sum_{j=k_c}^{n} \alpha_j v_{k_c}$, which is less than the expected profit from an optimal static pricing policy. Thus, we can increase profit by getting rid of these time periods, which contradicts to the optimality of the policy. The optimal pricing schedule can be constructed according to Lemma OA.3. In particular, $p_T = \min_{1 \leq i < k_c} \{v_i - (T - \tau_i)c_i\} - \delta$, and the price in the t^{th} period of a cycle is given by $p_t = \min\{v_{k_c}, p_T + (T - t)c_{k_c}\}$. □

Proof of Proposition 13. Proposition 13(i) can be shown as follows. By Proposition 11, under the condition $v_1 - c_1 \geq v_2 - c_2 \geq \cdots \geq v_n - c_n$, an optimal randomized pricing policy follows a two-point price distribution, with the lower price being less than v_1 and the higher price equal to v_{h_r}. Under the condition $v_1 \geq (\alpha_2 + \cdots + \alpha_n)v_2 \geq \cdots \geq \alpha_n v_n$, $h_r = k_r$. Then again by Proposition 11, customers of types from 1 to $k_r - 1$ behave the same as type $k_r - 1$ and customers of types from k_r to n behave the same as type k_r. Hence, the problem with n customer segments is equivalent to the case with two customer segments in which the $\sum_{j=1}^{k_r-1} \alpha_j$ fraction of customers have valuation v_{k_r-1} and per-period waiting cost c_{k_r-1} and the rest fraction of customers have valuation v_{k_r} and per-period waiting cost c_{k_r}.

To prove Proposition 13(ii), we first prove an auxiliary lemma.

LEMMA OS.6. *Under the conditions* $v_1 - c_1 \geq v_2 - c_2 \geq \cdots > v_{n-1} - c_{n-1} \geq v_n - c_n$ *and* $v_1 \geq (\alpha_2 + \cdots + \alpha_n)v_2 \geq \cdots \geq \alpha_n v_n$, *and consider a cyclic pricing policy with a price of* v_k *in the first* $T - 1$ *periods, a price of* $v_{k-1} - (T-1)c_{k-1} - \delta$ *in the last period. If* $v_k - v_{k-1} > c_{k-1}$ *and* $v_k - v_{k-1}$ *is below the threshold* $c_k \Big/ \left(1 + \frac{1}{2}\sqrt{\sum_{j=k}^{n} \alpha_j \Big/ \sum_{j=1}^{k-1} \alpha_j}\right)$,

(i) The optimal cycle length for this policy is either $\lfloor T_1 \rfloor$ *or* $\lfloor T_1 \rfloor + 1$, *where* $T_1 = \sqrt{\sum_{j=k}^{n} \alpha_j (v_k - v_{k-1} - c_{k-1}) \Big/ \sum_{j=1}^{k-1} \alpha_j c_{k-1}}$ *and* $\lfloor x \rfloor$ *represents the greatest integer that is no more than* x;

(ii) Any customer with valuation greater than or equal to v_k *will buy immediately upon arrival, and customer with valuation smaller than* v_k *will wait to purchase in the last period of a cycle.*

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Proof of Lemma OS.6. Consider a cyclic pricing policy where the firms charges v_k in the first $T-1$ periods, and $v_{k-1}-(T-1)c_{k-1}-\delta$ in the T^{th} period within a cycle. We first derive the optimal cycle length assuming that any customer with valuation greater than or equal to v_k will buy immediately upon arrival, and any customer with valuation smaller than v_k will wait to purchase in period T. Based on Proposition 12, the expected profit from this policy is given by $\Pi(T)=\sum_{j=1}^{k-1}\alpha_j(v_{k-1}-(T-1)c_{k-1}-\delta)+\sum_{j=k}^{n}\alpha_j\left[(T-1)v_k+v_{k-1}-(T-1)c_{k-1}-\delta\right]/T$. Denote $R(T)=\sum_{j=1}^{k-1}\alpha_j(v_{k-1}-(T-1)c_{k-1})+\sum_{j=k}^{n}\alpha_j\left[(T-1)v_k+v_{k-1}-(T-1)c_{k-1}\right]/T$ by ignoring the small δ in $\Pi(T)$. Taking the derivative of $R(T)$ with respect to T, we have

$$R'(T)=\sum_{j=k}^{n}\alpha_j\frac{v_k-v_{k-1}-c_{k-1}}{T^2}-\sum_{j=1}^{k-1}\alpha_j c_{k-1},$$

which is decreasing in T under the condition $v_k-v_{k-1}>c_{k-1}$. Thus, we conclude that $R(T)$ is concave, and it is maximized with $T_1=\sqrt{\sum_{j=k}^{n}\alpha_j(v_k-v_{k-1}-c_{k-1})\Big/\sum_{j=1}^{k-1}\alpha_j c_{k-1}}$. Thus, the optimal cycle length T^* is given by either $\lfloor T_1\rfloor$ or $\lfloor T_1\rfloor+1$, where $\lfloor x\rfloor$ represents the greatest integer that is less than or equal to x. Note that $\max\{R(\lfloor T_1\rfloor),R(\lfloor T_1\rfloor+1)\}$ is an upper bound of the expected profit from this cyclic pricing policy.

Next we show that under the conditions specified in the Lemma and the cycle length T^*, any customer with valuation greater than or equal to v_k will buy immediately upon arrival, and any customer with valuation smaller than v_k will wait to purchase in period T^*. Because $v_{i+1}-v_i<c_{i+1}-c_i\le(T^*-1)(c_{i+1}-c_i)$, we have $v_{k-1}-(T^*-1)c_{k-1}=\min_{1\le i\le k-1}(v_i-(T^*-1)c_i)<v_1$. Thus, in order to show that any customer with valuation smaller than or equal to v_{k-1} will wait to purchase in period T^*, it suffices to prove $v_{k-1}-(T^*-1)c_{k-1}\ge 0$. Note that T^* is equal to either $\lfloor T_1\rfloor$ or $\lfloor T_1\rfloor+1$. Thus, one sufficient condition is given by $\frac{v_{k-1}}{c_{k-1}}\ge T_1$. This condition holds as:

$$\frac{v_{k-1}^2}{c_{k-1}^2}>\frac{v_{k-1}}{c_{k-1}}=\sum_{j=1}^{k-1}\alpha_j v_{k-1}\Big/\sum_{j=1}^{k-1}\alpha_j c_{k-1}\ge\sum_{j=k}^{n}\alpha_j v_k\Big/\sum_{j=1}^{k-1}\alpha_j c_{k-1}>\sum_{j=k}^{n}\alpha_j(v_k-v_{k-1}-c_{k-1})\Big/\sum_{j=1}^{k-1}\alpha_j c_{k-1},$$

where the first inequality is due to $v_{k-1}>c_{k-1}$ and the second inequality is due to $\sum_{j=1}^{k-1}\alpha_j v_{k-1}\ge\sum_{j=k}^{n}\alpha_j v_k$.

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Lastly, we show that any customer with valuation greater than v_{k-1} will buy immediately upon arrival. As $v_i - c_i \geq v_{i+1} - c_{i+1}$, for any $k \leq i \leq n-1$, it suffices to show that $p_{T^*} + c_k = v_{k-1} - (T^* - 1)c_{k-1} - \delta + c_k \geq v_k$. We have

$$\begin{aligned} & v_k - v_{k-1} + (T^* - 1)c_{k-1} + \delta - c_k \\ <& v_k - v_{k-1} + T_1 c_{k-1} - c_k \\ \leq& v_k - v_{k-1} + \frac{(v_k - v_{k-1})}{2} \sqrt{\sum_{j=k}^{n} \alpha_j \Big/ \sum_{j=1}^{k-1} \alpha_j} - c_k < 0, \end{aligned}$$

where the first inequality is due to $T^* = \lfloor T_1 \rfloor$ or $\lfloor T_1 \rfloor + 1$, the second inequality is due to $T_1 c_{k-1} = \sqrt{\sum_{j=k}^{n} \alpha_j c_{k-1}(v_k - v_{k-1} - c_{k-1}) \Big/ \sum_{j=1}^{k-1} \alpha_j}$ is maximized with $c_{k-1} = (v_k - v_{k-1})/2$, and the third inequality is due to $v_k - v_{k-1} < c_k \Big/ \left(1 + \frac{1}{2}\sqrt{\sum_{j=k}^{n} \alpha_j \Big/ \sum_{j=1}^{k-1} \alpha_j}\right)$. We thus obtain the announced result. □

Now we are ready to prove Proposition 13(ii). Based on Proposition 11 and conditions specified in the proposition, we know that an optimal randomized pricing policy follows a two-point price distribution. In particular, under the optimal two-point price distribution, there exists k_r such that any customer with valuation less than v_{k_r} would wait to purchase, while any customer with valuation greater than or equal to v_{k_r} would buy immediately upon arrival. Based on the proof of Proposition 11, the expected profit from an optimal randomized pricing policy is given by $\sum_{j=1}^{k_r-1} \alpha_j (v_{k_r-1} - c_{k_r-1}/\beta_1) + \sum_{j=k_r}^{n} \alpha_j (\beta_1 v_{k_r-1} - c_{k_r-1} + (1-\beta_1) v_{k_r})$. The function is maximized when $\beta_1 = \sqrt{\sum_{j=1}^{k_r-1} \alpha_j c_{k_r-1} \Big/ \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - v_{k_r-1})}$, with a value of $\sum_{j=1}^{k_r-1} \alpha_j v_{k_r-1} + \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - c_{k_r-1}) - 2\sqrt{\left(\sum_{j=1}^{k_r-1} \alpha_j\right) \cdot \left(\sum_{j=k_r}^{n} \alpha_j c_{k_r-1}(v_{k_r} - v_{k_r-1})\right)}$.

We next show that this optimal randomized pricing policy is dominated by a cyclic pricing policy. Consider a cyclic pricing policy where the price is equal to v_{k_r} in the first $T-1$ periods, and $v_{k_r-1} - (T-1)c_{k_r-1} - \delta$ in the last period. Lemma OS.6 shows that any customer with valuation greater than or equal to v_{k_r} buy immediately upon arrival, while any customer with valuation less than v_{k_r} would wait to purchase at period T. Based on the proof of Lemma OS.6, its optimal profit is given by $\max\{R(\lfloor T_1 \rfloor), R(\lfloor T_1 \rfloor + 1)\}$. Also, we know from the proof of Lemma OS.6 that $R(T)$

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is concave and decreasing, for any $T \geq T_1$, and thus we have $R(\lfloor T_1 \rfloor + 1) \geq R(T_1 + 1)$. That is, the expected profit from this cyclic pricing is bounded below by $R(T_1+1)$. We have

$$\begin{aligned}
R(T_1+1) &= \sum_{j=k_r}^{n} \alpha_j \left[\frac{T_1}{T_1+1} v_{k_r} + \frac{1}{T_1+1}(v_{k_r-1} - T_1 c_{k_r-1}) \right] + \sum_{j=1}^{k_r-1} \alpha_j (v_{k_r-1} - T_1 c_{k_r-1}) \\
&= \sum_{j=1}^{k_r-1} \alpha_j v_{k_r-1} + \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - c_{k_r-1}) - \sum_{j=1}^{k_r-1} \alpha_j T_1 c_{k_r-1} - \sum_{j=k_r}^{n} \alpha_j \frac{1}{T_1+1}(v_{k_r} - v_{k_r-1} - c_{k_r-1}) \\
&> \sum_{j=1}^{k_r-1} \alpha_j v_{k_r-1} + \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - c_{k_r-1}) - \sum_{j=1}^{k_r-1} \alpha_j T_1 c_{k_r-1} - \sum_{j=k_r}^{n} \alpha_j \frac{1}{T_1}(v_{k_r} - v_{k_r-1} - c_{k_r-1}) \\
&= \sum_{j=1}^{k_r-1} \alpha_j v_k + \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - c_{k_r-1}) - 2\sqrt{\left(\sum_{j=1}^{k_r-1} \alpha_j \right) \left(\sum_{j=k_r}^{n} \alpha_j c_{k_r-1}(v_{k_r} - v_{k_r-1} - c_{k_r-1}) \right)} \\
&> \sum_{j=1}^{k_r-1} \alpha_j v_{k_r-1} + \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - c_{k_r-1}) - 2\sqrt{\left(\sum_{j=1}^{k_r-1} \alpha_j \right) \left(\sum_{j=k_r}^{n} \alpha_j c_{k_r-1}(v_{k_r} - v_{k_r-1}) \right)},
\end{aligned}$$

where the first inequality is due to $v_{k_r} - v_{k_r-1} > c_{k_r-1}$, and the second equality is due to $T_1 = \sqrt{\sum_{j=k_r}^{n} \alpha_j (v_{k_r} - v_{k_r-1} - c_{k_r-1}) \Big/ \sum_{j=1}^{k_r-1} \alpha_j c_{k_r-1}}$. We thus obtain Proposition 13(ii).

Next we prove Proposition 13(iii). Based on Proposition 12, we know that, under an optimal cyclic pricing policy, there exists a $k_c \in \{1, \ldots, n\}$ such that any customer with valuation greater than or equal to v_{k_c} would either purchase or leave immediately upon arrival. On the other hand, a type-i, $1 \leq i < k_c$, customer would wait upon arrival if she arrives no earlier than the τ_i^{th} period with a cycle. In particular, $\tau_1 = 1 \leq \tau_2 \leq \ldots \leq \tau_{k_c-1}$. Thus, the expected profit from an optimal cyclic pricing policy is:

$$\begin{aligned}
\Pi(T) &\leq \sum_{j=k_c}^{n} \alpha_i \frac{\sum_{t=1}^{T} v_{k_c} - [v_{k_c} - p_T - (T-t)c_{k_c}]^+}{T} + \sum_{j=1}^{k_c-1} \alpha_j p_T \\
&< \sum_{j=k_c}^{n} \alpha_j \frac{\sum_{t=1}^{T} v_{k_c} - [v_{k_c} - v_1 + (T-1)c_1 - (T-t)c_{k_c}]^+}{T} + \sum_{j=1}^{k_c-1} \alpha_j (v_1 - (T-1)c_1) \\
&< \sum_{j=k_c}^{n} \alpha_j \frac{\int_0^T v_{k_c} - [v_{k_c} - v_1 + (T-1)c_1 - (T-t)c_{k_c}]^+ \mathrm{d}t}{T} + \sum_{j=1}^{k_c-1} \alpha_j (v_1 - (T-1)c_1) \\
&= \sum_{j=k_c}^{n} \alpha_j v_{k_c} + \sum_{j=1}^{k_c-1} \alpha_i (v_1 - (T-1)c_1) - \sum_{j=k_c}^{n} \alpha_j \frac{[v_{k_c} - v_1 + (T-1)c_1]^2}{2Tc_{k_c}} \equiv H(T),
\end{aligned}$$

where the second inequality is due to $p_T < v_1 - (T-1)c_1$, and the third inequality is due to $v_{k_c} - [v_{k_c} - v_1 + (T-1)c_1 - (T-t)c_{k_c}]^+$ being monotonically decreasing in t. Taking the first order

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derivative of $H(T)$ with respect to T, we have $H'(T) = -\sum_{j=1}^{k_c-1} \alpha_j c_1 + \sum_{j=k_c}^{n} \alpha_j \frac{(v_{k_c}-v_1-c_1)^2}{2c_{k_c}T^2} - \sum_{j=k_c}^{n} \alpha_j \frac{c_1^2}{2c_{k_c}}$, which is decreasing in T. Thus, we conclude that $H(T)$ is concave in T, and its maximum is realized when $T = T_2 \equiv \sqrt{\frac{\sum_{j=k_c}^{n} \alpha_j (v_{k_c}-v_1-c_1)^2}{2\sum_{j=1}^{k_c-1} \alpha_j c_1 c_{k_c} + \sum_{j=k_c}^{n} \alpha_j c_1^2}}$. Consequently, the expected profit from an optimal cyclic pricing policy is bounded above by

$$\begin{aligned}
H(T_2) &= \sum_{j=1}^{k_c-1} \alpha_j v_1 + \sum_{j=k_c}^{n} \alpha_j v_{k_c} - \sum_{j=1}^{k_c-1} \alpha_j (T_2-1)c_1 - \sum_{j=k_c}^{n} \alpha_j \frac{(v_{k_c}-v_1-c_1)^2}{2T_2 c_{k_c}} \\
&\quad - \sum_{j=k_c}^{n} \alpha_j \frac{T_2 c_1^2}{2c_{k_c}} - \sum_{j=k_c}^{n} \alpha_j \frac{c_1}{c_{k_c}}(v_{k_c}-v_1-c_1) \\
&< \sum_{j=1}^{k_c-1} \alpha_j v_1 + \sum_{j=k_c}^{n} \alpha_j v_{k_c} - \sum_{j=1}^{k_c-1} \alpha_j (T_2-1)c_1 - \sum_{j=1}^{k_c-1} \alpha_j T_2 c_1 - \sum_{j=k_c}^{n} \alpha_j \frac{T_2 c_1^2}{2c_{k_c}} \\
&\quad - \sum_{j=k_c}^{n} \alpha_j \frac{c_1}{c_{k_c}}(v_{k_c}-v_1-c_1),
\end{aligned}$$

where the inequality is due to $T_2 = \sqrt{\frac{\sum_{j=k_c}^{n} \alpha_j (v_{k_c}-v_1-c_1)^2}{2\sum_{j=1}^{k_c-1} \alpha_j c_1 c_{k_c} + \sum_{j=k_c}^{n} \alpha_j c_1^2}} > \sqrt{\frac{\sum_{j=k_c}^{n} \alpha_j (v_{k_c}-v_1-c_1)^2}{2\sum_{j=1}^{k_c-1} \alpha_j c_1 c_{k_c}}}$.

Lastly, based on the proof of Proposition 11, we know that the expected profit from a randomized pricing policy can be derived by maximizing $\sum_{j=1}^{k_c-1} \alpha_j (v_{k_c-1} - c_{k_c-1}/\beta_1) + \sum_{j=k_c}^{n} \alpha_j (\beta_1 v_{k_c-1} - c_{k_c-1} + (1-\beta_1)v_{k_c})$ over $\beta_1 \in [0,1]$. Plugging $\beta_1 = c_{k_c-1}/[v_{k_c-1} - v_1 + (T_2-1)c_1]$ (in which case, $v_{k_c-1} - c_{k_c-1}/\beta_1 = v_1 - (T_2-1)c_1$) into the profit function, we have

$$\sum_{j=1}^{k_c-1} \alpha_j v_1 + \sum_{j=k_c}^{n} \alpha_j v_{k_c} - \sum_{j=1}^{k_c-1} \alpha_j (T_2-1)c_1 - \sum_{j=k_c}^{n} \alpha_j \frac{v_{k_c}-v_{k_c-1}}{v_{k_c-1}-v_1+(T_2-1)c_1} c_{k_c-1} - \sum_{j=1}^{k_c-1} \alpha_j c_k \equiv U.$$

So in order to show that an optimal randomized pricing policy outperforms an optimal cyclic pricing policy, it suffices to show $H(T_2) < U$. It is easy to verify that $T_2(T_2-1)/(v_{k_c}-v_{k_c-1})$ is increasing in $v_{k_c}-v_{k_c-1}$ when $v_{k_c}-v_{k_c-1}$ is sufficiently large. Consequently, we have $-\sum_{j=k_c}^{n} \alpha_j \frac{T_2 c_1^2}{2c_{k_c}} \le -\sum_{j=k_c}^{n} \alpha_j \frac{v_{k_c}-v_{k_c-1}}{(T_2-1)c_1} c_{k_c-1} < -\sum_{j=k_c}^{n} \alpha_j \frac{v_{k_c}-v_{k_c-1}}{v_{k_c-1}-v_1+(T_2-1)c_1} c_{k_c-1}$. Comparing against the upper bound of $H(T_2)$, we know that one sufficient condition for $H(T_2) < U$ is given by

$$-\sum_{j=1}^{k_c-1} \alpha_j T_2 c_1 - \sum_{j=k_c}^{n} \alpha_j \frac{c_{k_c-1}}{c_{k_c}}(v_{k_c}-v_{k_c-1}-c_{k_c-1}) < -\sum_{j=k_c}^{n} \alpha_j c_{k_c-1}.$$

Because the left hand of the inequality is decreasing in $v_{k_c}-v_{k_c-1}$, and thus the inequality is guaranteed when $v_{k_c}-v_{k_c-1}$ is sufficiently large. We thus obtain Proposition 13(iii). □

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B. Proofs of Results in the Online Appendix

Proof of Lemma OA.1. Consider a randomized pricing policy, where the set of types of customers who will wait is denoted by $\mathcal{W}$. First we prove that β_i and x_i, $\forall i$, are properly defined by showing that $u_{i-1} < u_i$. If $i \notin \mathcal{W}$, we have $u_i = v_i > v_{i-1} \geq u_{i-1}$. If $i \in \mathcal{W}$ and $i-1 \in \mathcal{W}$, then we know that $u_i = \underline{p}^i > u_{i-1} = \underline{p}^{i-1}$ because $\underline{p}$ increases in c. The only non-trivial case is when $i \in \mathcal{W}$ and $i-1 \notin \mathcal{W}$. In this case, we have $\mathsf{E}[(v_{i-1}-P)^+] \leq c_{i-1} < c_i$. So based on the definition of $\underline{p}^i = \max\{v' \mid \mathsf{E}[(v'-P)^+] \leq c\}$, we have $u_i = \underline{p}^i > v_{i-1} = u_{i-1}$. Based on the definition of x_i, it is easy to verify that $x_i \leq \underline{p}^i < v_i$, $\forall i \in \mathcal{W}$, and $x_i \leq v_i$, $\forall i \notin \mathcal{W}$. Furthermore, the support of the price distribution of an optimal randomized pricing policy will never include any price greater than v_n, as no customers will purchase at those price points. Thus, an optimal policy must satisfy $\sum_{i=1}^{n} \beta_i = 1$.

For any $i \in \mathcal{W}$, we can rewrite the condition as $\mathsf{E}[(v_i-P)^+] = \sum_{j=1}^{i} \beta_j v_i - \sum_{j=1}^{i} \beta_j x_j \leq c_i$, $\forall i \notin \mathcal{W}$, based on the definitions of β_i and x_i. Similarly, for any $i \in \mathcal{W}$, we have $F(\underline{p}^i)\mathsf{E}[\underline{p}^i - P | P \leq \underline{p}^i] = \sum_{j=1}^{i} \beta_j \underline{p}^i - \sum_{j=1}^{i} \beta_j x_j \leq c_i$. Because $x_i \leq \underline{p}^i$, $\forall i \in \mathcal{W}$, we have $\sum_{j=1}^{i} \beta_j x_i - \sum_{j=1}^{i} \beta_j x_j \leq \sum_{j=1}^{i} \beta_j \underline{p}^i - \sum_{j=1}^{i} \beta_j x_j \leq c_i$. Notice that the profit function $\sum_{i=1}^{n} \alpha_i \pi_i$ is increasing in x_i because both $\mathsf{E}[P | P \leq \underline{p}^i] = \sum_{j=1}^{i} \beta_j x_j \Big/ \sum_{j=1}^{i} \beta_j$ and $F(v_i)\mathsf{E}[P | P \leq v_i] = \sum_{j=1}^{i} \beta_j x_j$ increase in x_i, $\forall i$. Thus, the following condition $\sum_{j=1}^{i} \beta_j x_i - \sum_{j=1}^{i} \beta_j x_j = c_i$ must be satisfied for an optimal policy; Otherwise, we can always improve profit by increasing x_i. Note that when $i = 1$, this constraint is guaranteed to be satisfied as the left hand side of the inequality is zero. So we resort to the initial condition derived in Proposition 1, i.e., $\beta_1 v_1 - \beta_1 x_1 > c_1$, should type-1 customers wait. Consequently, we can find the optimal policy, in the space of all policies where type-i customers, $\forall i \in \mathcal{W}$, wait and the rest purchase or leave immediately, by solving Problem (OA.1). □

Proof of Lemma OA.2. Consider a set of randomized pricing policies, which includes all policies under which type-1 customers would wait, and the rest of customers either purchase or leave immediately upon arrival. Based on Lemma OA.1, the optimal policy in the set can be derived by solving the following optimization problem:

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$$\begin{aligned}\max_{\beta,\mathbf{x}} \quad & \alpha_1 x_1 + \sum_{j=2}^{n} \alpha_j \sum_{i=1}^{j} \beta_i x_i \\ \text{s.t.} \quad & \beta_1 v_1 - \beta_1 x_1 > c_1, \\ & \sum_{i=1}^{j} \beta_i v_j - \sum_{i=1}^{j} \beta_i x_i \le c_j, \quad j = 2,3,\ldots,n, \\ & x_1 < v_1, \text{ and } x_j \le v_j, j = 2,3,\ldots,n, \text{ and } \sum_{j=1}^{n} \beta_j = 1.\end{aligned} \tag{OS.5}$$

Next we show that, when c_1 is sufficiently small, the optimal value of the objective function in Problem (OS.5) is greater than the expected profit from an optimal static pricing policy, which is given by $\max_{j\le n}\left\{\sum_{i=j}^{n}\alpha_i v_j\right\}$, by induction. The case when $n=2$ is proved in Proposition 4. Now suppose that the result holds when $n=k$, and we next show that it also holds when $n=k+1$.

Consider first the trivial case when $\max_{j\le k+1}\left\{\sum_{i=j}^{k+1}\alpha_i v_j\right\} = \max_{j\le k}\left\{\sum_{i=j}^{k+1}\alpha_i v_j\right\}$. In this case, we can simply let $\beta_{k+1}=0$, and adopt the optimal randomized pricing policy when $n=k$. Its expected profit is guaranteed to be greater than $\max_{j\le k}\left\{\sum_{i=j}^{k+1}\alpha_i v_j\right\}$.

Next consider the case when $\max_{j\le k+1}\left\{\sum_{i=j}^{k+1}\alpha_i v_j\right\} = \alpha_{k+1}v_{k+1}$. Conditional on β_j, $\forall\, j\in\{1,\ldots,k+1\}$, the objective function of Problem (OS.5) is increasing in x_j, $\forall\, j\in\{1,\ldots,k+1\}$. Therefore, Problem (OS.5) is maximized when $x_1 = v_1 - \frac{c_1}{\beta_1} - \delta$, for any sufficiently small δ, and $x_j = v_j$, $\forall\, j\in\{2,\ldots,k+1\}$. Thus, we can reformulate Problem (OS.5) as:

$$\begin{aligned}\max_{\beta} \quad & \alpha_1\left(v_1 - \frac{c_1}{\beta_1} - \delta\right) + \sum_{j=2}^{k+1}\alpha_j\left(\sum_{i=1}^{j}\beta_i v_i - c_1 - \beta_1\delta\right) \\ \text{s.t.} \quad & \sum_{i=1}^{j}\beta_i(v_j - v_i) < c_j - c_1, \forall\, j\in\{2,\ldots,k+1\}, \text{ and } \sum_{j=1}^{k+1}\beta_j = 1.\end{aligned} \tag{OS.6}$$

Denote $U(\beta) = \alpha_1\left(v_1 - \frac{c_1}{\beta_1}\right) + \sum_{j=2}^{k+1}\alpha_j\left(\sum_{i=1}^{j}\beta_i v_i - c_1\right)$. Thus, the derivatives of $U(\beta)$ with respect to β_j, $j=2,\ldots,k$, are given by

$$\frac{\partial U(\beta)}{\partial \beta_1} = \frac{\alpha_1 c_1}{\beta_1^2} + \sum_{i=2}^{k+1}\alpha_i v_1 - \alpha_{k+1}v_{k+1}, \text{ and } \frac{\partial U(\beta)}{\partial \beta_j} = \sum_{i=j}^{k+1}\alpha_i v_j - \alpha_{k+1}v_{k+1}, \quad j=2,3,\ldots,k.$$

Recall that $\max_{j\le k+1}\left\{\sum_{i=j}^{k+1}\alpha_i v_j\right\} = \alpha_{k+1}v_{k+1} \ge \sum_{i=j}^{k+1}\alpha_i v_j$. Thus, we can conclude that $U(\beta)$ is decreasing in β_j and the profit is maximized when $\beta_j = 0$, $\forall\, j\in\{2,\ldots,k\}$. Ignoring δ, Problem (OS.6) can thus be simplified as a single variable optimization problem below.

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$$\begin{aligned} \max_{\beta_1} \quad & \alpha_1\left(v_1-\frac{c_1}{\beta_1}\right)+\sum_{j=2}^{k+1}\alpha_j(\beta_1 v_1-c_1)+(1-\beta_1)\alpha_{k+1}v_{k+1} \\ \text{s.t.} \quad & \beta_1(v_j-v_1)<c_j-c_1, \quad \forall j\in\{2,\ldots,k+1\}. \end{aligned} \tag{OS.7}$$

The solution to the first order condition $\frac{\partial U(\beta)}{\partial \beta_1}=\frac{\alpha_1 c_1}{\beta_1^2}+\sum_{i=2}^{k+1}\alpha_i v_1-\alpha_{k+1}v_{k+1}=0$ is given by $\beta_1^*=\sqrt{\frac{\alpha_1 c_1}{\alpha_{k+1}v_{k+1}-\sum_{i=2}^{k+1}\alpha_i v_1}}$. When $\beta_1^*(v_j-v_i)<c_j-c_1, \forall j\in\{2,\ldots,k+1\}$, which holds when c_1 is sufficiently small, the expected profit with $\beta_1=\beta_1^*$ is given by:

$$\begin{aligned} &\alpha_1(v_1-\frac{c_1}{\beta_1})+\sum_{j=2}^{k+1}\alpha_j(\beta_1 v_1-c_1)+(1-\beta_1)\alpha_{k+1}v_{k+1} \\ =&\alpha_1 v_1+\alpha_{k+1}v_{k+1}-2\sqrt{\alpha_1 c_1\left(\alpha_{k+1}v_{k+1}-\sum_{i=2}^{k+1}\alpha_i v_1\right)}-\sum_{j=2}^{k+1}\alpha_j c_1, \end{aligned}$$

which is greater than $\alpha_{k+1}v_{k+1}$ when c_1 is sufficiently small. This completes the proof. □

Proof of Lemma OA.3. We first show that consumer behavior is consistent with the definition of k_0 and k_c under the price schedule as constructed in Lemma OA.3. For any $k_0\le i<k_c$, it suffices to show that either $p_T+(T-t)c_i<p_t$ or $p_t>v_i$, for any $t\ge\tau_i$. If $p_s+c_{k_c}>v_{k_c}$, for any $t<s\le T$, then we have $p_t\ge v_{k_c}>v_i$. On the other hand, if $p_s+c_{k_c}\le v_{k_c}$, for all $t<s\le T$, then we have $p_t=p_{t+1}+c_{k_c}=p_{t+2}+2c_{k_c}=\cdots=p_T+(T-t)c_{k_c}>p_T+(T-t)c_i$ due to $c_i<c_{k_c}$ for any $k_0\le i<k_c$. Next consider any $i\ge k_c$. In this case, we need to show that either $p_t\le p_{t+1}+c_i$ or $p_{t+1}+c_i>v_i$ for any $t<T$. If $p_{t+1}+c_{k_c}\le v_{k_c}$, we have $p_t=p_{t+1}+c_{k_c}\le p_{t+1}+c_i$, for any $i\ge k_c$. On the other hand, if $p_{t+1}+c_{k_c}>v_{k_c}$, we have $p_{t+1}>v_i-c_i$ for any $i<k_t$, and $p_t\le p_{t+1}+c_{k_t}\le p_{t+1}+c_i$ for any $i\ge k_t$, if k_t exists. If k_t does not exist, simply we have $p_{t+1}>v_i-c_i$ for any $i\ge k_c$.

We next show that this price policy is optimal. First consider period T. Based on the definition of τ_i, we have $v_i-(T-\tau_i)c_i\ge p_T$, for any $k_0\le i<k_c$. Thus, $p_T\le\min_{k_0\le i<k_c}\{v_i-(T-\tau_i)c_i\}$ and the optimal price for period T is given by $p_T=\min_{k_0\le i<k_c}\{v_i-(T-\tau_i)c_i\}-\delta$. Next we work backwards and consider period $t<T$. Consider first the case when $p_{t+1}+c_{k_c}\le v_{k_c}$. In this case, we prove $p_t\le p_{t+1}+c_{k_c}$ by contradiction. Suppose $p_t>p_{t+1}+c_{k_c}$. Then, we have $v_{k_c}-p_t<v_{k_c}-p_{t+1}-c_{k_c}$. Combining with the condition $p_{t+1}+c_{k_c}\le v_{k_c}$, we have $v_{k_c}-p_{t+1}-c_{k_c}\ge\max\{0,v_{k_c}-p_t\}$. That is, a customer with valuation v_{k_c} is better off purchasing at period $t+1$, which contradicts to the fact that a customer with valuation v_{k_c} will not wait. Consequently, the optimal price is $p_t=p_{t+1}+c_{k_c}$

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if $p_{t+1}+c_{k_c} \le v_{k_c}$. Last consider the case when $p_{t+1}+c_{k_c} > v_{k_c}$. Following a similar approach as described above, we can show that $p_t \le p_{t+1}+c_{k_t}$, if k_t exists. Thus, if $(\alpha_{k_t}+\cdots+\alpha_n)(p_{t+1}+c_{k_t}) > \max_{k_c \le i < k_t}\{(\alpha_i+\cdots+\alpha_n)v_i\}$, the profit maximizing price is given by $p_t = p_{t+1}+c_{k_t}$. On the other hand, if $(\alpha_{k_t}+\cdots+\alpha_n)(p_{t+1}+c_{k_t}) \le \max_{k_c \le i < k_t}\{(\alpha_i+\cdots+\alpha_n)v_i\}$, the profit maximizing price is given by v_{l_t}, where $l_t = \arg\max_{k_c \le i < k_t}\{(\alpha_i+\cdots+\alpha_n)v_i\}$. If k_t does not exist, $p_{t+1}+c_i > v_i$ for $k_c \le i$, then the firm could charge the optimal price $p_t = v_l$, without worrying about customers' strategic waiting behavior. We thus obtain the announced result. □

Proof of Lemma OA.4. We prove the result by showing that any policy as characterised in Lemma OA.3 follows a (weakly) markdown pattern. For any given cyclic pricing policy $\mathbf{p} = \{p_1, p_2, \ldots, p_T\}$, we show $p_t \ge p_{t+1}$, $\forall\, t \le T-1$ by induction. Because a customer with valuation v_i, for any $k_0 \le i < k_c$, will wait to purchase in period T, we have $p_T < p_t$, for any $t \le T-1$. Hence, we have $p_T > p_{T-1}$. Now suppose that $p_{t+1} \ge p_{t+2}$, and next we prove $p_t \ge p_{t+1}$. Let us consider three cases.

(a) $p_{t+2}+c_{k_c} \le p_{t+1}+c_{k_c} \le v_{k_c}$. Based on Lemma OA.3, we have $p_t = p_{t+1}+c_{k_c} > p_{t+2}+c_{k_c} = p_{t+1}$.

(b) $p_{t+2}+c_{k_c} \le v_{k_c} < p_{t+1}+c_{k_c}$. Based on Lemma OA.3, we have $p_{t+1} = p_{t+2}+c_{k_c} < v_{k_c}$, and $p_t > v_{k_c}$. Hence, we can conclude that $p_t > p_{t+1}$.

(c) $v_{k_c} < p_{t+2}+c_{k_c} < p_{t+1}+c_{k_c}$. It is easy to verify that $k_t \ge k_{t+1}$ due to $p_{t+1} \ge p_{t+2}$. To complete the proof, we need to further consider three sub-cases.

(c1) k_t and k_{t+1} do not exist. In this case, we have $p_t = p_{t+1} = v_l$.

(c2) k_t does not exist but k_{t+1} exists. If $p_{t+1} = v_{l_{t+1}}$, then we have $p_t = v_l \ge v_{l_{t+1}} = p_{t+1}$ due to the definitions of l and l_t. If $p_{t+1} = p_{t+2}+c_{k_{t+1}}$, which happens under the condition that $(\alpha_{k_{t+1}}+\cdots+\alpha_n)(p_{t+2}+c_{k_{t+1}}) > \max_{k_c \le i < k_{t+1}}\{(\alpha_i+\cdots+\alpha_n)v_i\}$, then we have $\max_{k_c \le i}\{(\alpha_i+\cdots+\alpha_n)v_i\} \ge (\alpha_{k_{t+1}}+\cdots+\alpha_n)v_{k_{t+1}} \ge (\alpha_{k_{t+1}}+\cdots+\alpha_n)(p_{t+2}+c_{k_{t+1}}) > \max_{k_c \le i < k_{t+1}}\{(\alpha_i+\cdots+\alpha_n)v_i\}$, where the second inequality is due to $p_{t+2}+c_{k_{t+1}} \le v_{k_{t+1}}$.The condition $\max_{k_c \le i}\{(\alpha_i+\cdots+\alpha_n)v_i\} > \max_{k_c \le i < k_{t+1}}\{(\alpha_i+\cdots+\alpha_n)v_i\}$ implies that $l \ge k_{t+1}$. Thus, we have $p_t = v_l \ge v_{k_{t+1}} \ge p_{t+2}+c_{k_{t+1}} = p_{t+1}$.

(c3) k_t and k_{t+1} exist. If $p_{t+1} = v_{l_{t+1}}$, then either $p_t = p_{t+1}+c_{k_t}$, which is greater than p_{t+1}, or $p_t = v_{l_t}$, which again is greater than $p_{t+1} = v_{l_{t+1}}$ due to $k_t \ge k_{t+1}$. Next consider the case when $p_{t+1} = p_{t+2}+c_{k_{t+1}}$. In this case, either $p_t = p_{t+1}+c_{k_t}$, which is greater than p_{t+1}, or $p_t = v_{l_t}$. To

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prove $p_t \geq p_{t+1}$ when $p_t = v_{l_t}$, we first prove k_t must be greater than k_{t+1} by contradiction. Suppose $k_t = k_{t+1}$, then we have $v_{l_t} = v_{l_{t+1}}$. Combining with the condition that $p_{t+1} = p_{t+2} + c_{k_{t+1}}$, we have $(\alpha_{k_{t+1}} + \cdots + \alpha_n)(p_{t+2} + c_{k_{t+1}}) > (\alpha_{l_{t+1}} + \cdots + \alpha_n)v_{l_{t+1}} = \max_{k_c \leq i < k_{t+1}} \{(\alpha_i + \cdots + \alpha_n)v_i\} \geq (\alpha_{k_{t+1}} + \cdots + \alpha_n)(p_{t+1} + c_{k_{t+1}})$, which contradicts to $p_{t+1} \geq p_{t+2}$. Thus, we have $k_t > k_{t+1}$. Under this condition, we have $\max_{k_c \leq i < k_t} \{(\alpha_i + \cdots + \alpha_n)v_i\} \geq (\alpha_{k_{t+1}} + \cdots + \alpha_n)v_{k_{t+1}} \geq (\alpha_{k_{t+1}} + \cdots + \alpha_n)(p_{t+2} + c_{k_{t+1}}) > \max_{k_c \leq i < k_{t+1}} \{(\alpha_i + \cdots + \alpha_n)v_i\}$, where the second inequality is due to $p_{t+2} + c_{k_{t+1}} \leq v_{k_{t+1}}$ and the last inequality is due to $p_{t+1} = p_{t+2} + c_{k_{t+1}}$. The condition $\max_{k_c \leq i < k_t} \{(\alpha_i + \cdots + \alpha_n)v_i\} > \max_{k_c \leq i < k_{t+1}} \{(\alpha_i + \cdots + \alpha_n)v_i\}$ implies that $l_t > k_{t+1}$, leading to $p_t = v_{l_t} > v_{k_{t+1}} \geq p_{t+2} + c_{k_{t+1}} = p_{t+1}$. We thus obtain the announced results. □

Proof of Lemma OA.5. We first prove Lemma OA.5(i). With a bit abuse of notation, we denote the total profit generated by customers with valuation greater than or equal to v_{k_c} in period t by π_t. We first show that, for any t, there exists $k_c \leq i_t$ such that $\pi_t \leq \sum_{i=i_t}^{n} \alpha_i(p_T + (T-t)c_{i_t})$. Consider first when $p_{t+1} + c_{k_c} \leq v_{k_c}$. In this case, we have $p_t = p_{t+1} + c_{k_c} \leq v_{k_c}$ according to Lemma OA.3. As any customer with valuation greater than or equal to v_{k_c} will not wait, the incentive compatibility constraint is given by $p_t \leq p_T + (T-t)c_{k_c}$. Consequently, we have $\pi_t = \sum_{i=k_c}^{n} \alpha_i p_t \leq \sum_{i=k_c}^{n} \alpha_i(p_T + (T-t)c_{k_c})$. Second, we consider the case when $p_{t+1} + c_{k_c} > v_{k_c}$ and $p_t = p_{t+1} + c_{k_t} \leq v_{k_t}$. In this case, any customer with valuation greater than or equal to v_{k_t} will buy immediately in period t, and thus it must satisfy that $p_t \leq p_T + (T-t)c_{k_t}$. Consequently, we have $\pi_t = \sum_{i=k_t}^{n} \alpha_i p_t \leq \sum_{i=k_t}^{n} \alpha_i(p_T + (T-t)c_{k_t})$. The third case is $p_{t+1} + c_{k_c} > v_{k_c}$ and $p_t = v_{l_t}$. In this case, any customer with valuation greater than or equal to v_{l_t} will buy immediately in period t, and thus it must satisfy that $p_t = v_{l_t} \leq p_T + (T-t)c_{l_t}$. Consequently, we have $\pi_t = \sum_{i=l_t}^{n} \alpha_i p_t = \sum_{i=l_t}^{n} \alpha_i v_{l_t} \leq \sum_{i=l_t}^{n} \alpha_i(p_T + (T-t)c_{l_t})$. Last consider the case $p_{t+1} + c_{k_c} > v_{k_c}$ and $p_t = v_l$. In this case, any customer with valuation greater than or equal to v_l will buy immediately in period t, and thus it must satisfy that $p_t = v_l \leq p_T + (T-t)c_l$. Consequently, we have $\pi_t = \sum_{i=l}^{n} \alpha_i p_t = \sum_{i=l}^{n} \alpha_i v_l \leq \sum_{i=l}^{n} \alpha_i(p_T + (T-t)c_l)$.

Recall that τ_i is defined as follows. Any type-i customer will leave immediately upon arrival if she arrives before the τ_i^{th} period within a cycle; Otherwise, she will wait till the end of the cycle to purchase. Next we show that $\tau_{i+1} \leq \tau_i$, for any $k_0 \leq i < k_c - 1$. Due to $c_1/v_1 \geq c_2/v_2 \geq \cdots \geq c_n/v_n$ and $v_i - (T - \tau_i)c_i \geq 0$, it is easy to verify that $\frac{v_{i+1} - v_i}{c_{i+1} - c_i} \geq \frac{v_{i+1}}{c_{i+1}} \geq \frac{v_i}{c_i} = T - \tau_i$, which implies that

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$v_{i+1} - (T-\tau_i)c_{i+1} \geq v_i - (T-\tau_i)c_i \geq p_T$. That is, any customer with valuation v_{i+1} arriving in period τ_i will wait to purchase in period T, and thus we have $\tau_{i+1} \leq \tau_i$, $\forall\, i < k_c - 1$.

Now we are ready to show that the expected profit from an optimal cyclic pricing is bounded above by the expected profit from an optimal static pricing policy. We first consider the case when $\tau_{k_0} = 1$. As $\tau_{i+1} \leq \tau_i$, we have $\tau_i = 1$, for any $k_0 \leq i < k_c$. Hence, the expected profit from an optimal cyclic pricing policy is bounded above by:

$$\begin{aligned}
\Pi(T) &= \sum_{t=1}^{T} \pi_t/T + \sum_{i=k_0}^{k_c-1} \alpha_i p_T \\
&\leq \sum_{t=1}^{T} \sum_{i=i_t}^{n} \alpha_i (p_T + (T-t)c_{i_t})/T + \sum_{i=k_0}^{k_c-1} \alpha_i p_T \\
&\leq \sum_{t=1}^{T} \sum_{i=i_t}^{n} \alpha_i \left[(T-t)\frac{c_{k_0}}{v_{k_0}} v_{i_t} \right]/T + \sum_{i=k_0}^{n} \alpha_i p_T \\
&\leq \max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{k_0}}{v_{k_0}} \frac{T-1}{2} + \sum_{i=k_0}^{n} \alpha_i p_T \\
&\leq \max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{k_0}}{v_{k_0}} \frac{T-1}{2} + \sum_{i=k_0}^{n} \alpha_i (v_{k_0} - (T-1)c_{k_0}),
\end{aligned}$$

where the second inequality is due to $i_t \geq k_c$ and $\frac{c_{i_t}}{v_{i_t}} \leq \frac{c_{k_0}}{v_{k_0}}$, the third inequality is due to $i_t \geq k_c$ and $\sum_{i=i_t}^{n} \alpha_i v_{i_t} \leq \max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$, and the last inequality is due to $p_T \leq v_{k_0} - (T-1)c_{k_0}$. If $\max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{k_0}}{2v_{k_0}} \leq \sum_{i=k_0}^{n} \alpha_i c_{k_0}$, then $\max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{k_0}}{v_{k_0}} \frac{T-1}{2} + \sum_{i=k_0}^{n} \alpha_i (v_{k_0} - (T-1)c_{k_0}) \leq \sum_{i=k_0}^{n} \alpha_i v_{k_0} \leq \max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$; Otherwise, it is easy to verify that $\max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{k_0}}{v_{k_0}} \frac{T-1}{2} + \sum_{i=k_0}^{n} \alpha_i (v_{k_0} - (T-1)c_{k_0})$ is increasing in $T-1$. Because of $T - 1 \leq \frac{v_{k_0}}{c_{k_0}}$, we have

$$\max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{k_0}}{v_{k_0}} \frac{T-1}{2} + \sum_{i=k_0}^{n} \alpha_i (v_{k_0} - (T-1)c_{k_0}) \leq \frac{1}{2} \max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} < \max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}.$$

Next consider the case when $\tau_{k_0} > 1$. In this case, each cycle can be decomposed into a couple of separable mini cycles, namely, $[1, \tau_{k_c-1} - 1]$, $[\tau_{k_c-1}, \tau_{k_c-2} - 1]$, $\cdots$, $[\tau_{k_0}, T]$. We prove an optimal cyclic pricing policy reducing to a static pricing policy by showing that the average profit per period within each mini cycle is no more than the expected profit from an optimal static pricing policy, i.e., $\max_{1\leq k\leq n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$. Consider the first mini cycle $[1, \tau_{k_c-1}]$. No customers arriving in these time periods will wait. Customers with valuation greater than or equal to v_{k_c} will purchase or leave

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immediately upon arrival, and thus the average profit per period is no more than $\max_{k_c \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$, which is less than or equal to $\max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$. Next consider the last mini cycle $[\tau_{k_0}, T]$. Within this mini cycle, it is as if $\tau_{k_0} = 1$, and the cycle length is $T - \tau_{k_0} + 1$. Based on our preceding discussion, we know that the expected profit per period is bounded above by the expected profit from the optimal static pricing policy.

To that end, we only need to consider the mini cycles $[\tau_{j+1}, \tau_j - 1]$, for any $k_0 \le j \le k_c - 2$. We show below that the average profit per period within any mini cycle $[\tau_{j+1}, \tau_j - 1]$ is bounded above by:

$$
\begin{aligned}
&\sum_{t=\tau_{j+1}}^{\tau_j - 1} \pi_t / (\tau_j - \tau_{j+1}) + \sum_{i=j+1}^{k_c - 1} \alpha_i p_T \\
&\le \sum_{t=\tau_{j+1}}^{\tau_j - 1} \sum_{i=i_t}^{n} \alpha_i (p_T + (T-t)c_{i_t}) / (\tau_j - \tau_{j+1}) + \sum_{i=j+1}^{k_c - 1} \alpha_i p_T \\
&\le \sum_{t=\tau_{j+1}}^{\tau_j - 1} \sum_{i=i_t}^{n} \alpha_i \left[(T-t) \frac{c_{j+1}}{v_{j+1}} v_{i_t} \right] / (\tau_j - \tau_{j+1}) + \sum_{i=j+1}^{n} \alpha_i p_T \\
&\le \max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} \frac{2T - \tau_j - \tau_{j+1} + 1}{2} + \sum_{i=j+1}^{n} \alpha_i p_T \\
&< \max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} \frac{2T - \tau_j - \tau_{j+1} + 1}{2} + \sum_{i=j+1}^{n} \alpha_i (v_{j+1} - (T - \tau_{j+1}) c_{j+1}) \\
&< \max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} (T - \tau_{j+1}) + \sum_{i=j+1}^{n} \alpha_i (v_{j+1} - (T - \tau_{j+1}) c_{j+1}),
\end{aligned}
$$

where the second inequality is due to $i_t \ge k_c$ and $\frac{c_{i_t}}{v_{i_t}} \le \frac{c_{j+1}}{v_{j+1}}$, the third inequality is due to $i_t \ge k_c > j+1$ and $\sum_{i=i_t}^{n} \alpha_i v_{i_t} \le \max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$, the fourth inequality is due to $v_{j+1} - (T - \tau_{j+1}) c_{j+1} \ge p_T$, and the last inequality is due to $\tau_{j+1} < \tau_j - 1$. Because $\max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} \ge \sum_{i=j+1}^{n} \alpha_i v_{j+1} \frac{c_{j+1}}{v_{j+1}} = \sum_{i=j+1}^{n} \alpha_i c_{j+1}$, we know that $\max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} (T - \tau_{j+1}) + \sum_{i=j+1}^{n} \alpha_i (v_{j+1} - (T - \tau_{j+1}) c_{j+1})$ is increasing in $T - \tau_{j+1}$. Combining with the condition that $T - \tau_{j+1} \le \frac{v_{j+1}}{c_{j+1}}$, we have

$$
\max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} (T - \tau_{j+1}) + \sum_{i=j+1}^{n} \alpha_i (v_{j+1} - (T - \tau_{j+1}) c_{j+1}) \le \max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}.
$$

Thus, we obtain Lemma OA.5(i).

Next we prove Lemma OA.5(ii). Consider first the case when $\tau_{k_0} = 1$. Following a similar approach

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as that of the proof of Lemma OA.5(i), we show that the expected profit from an optimal cyclic pricing policy is bounded above by:

$$\begin{aligned}
\Pi(T) &\le \sum_{t=1}^{T} \pi_t/T + \sum_{i=k_0}^{k_c-1} \alpha_i p_T \\
&\le \sum_{t=1}^{T} \sum_{i=i_t}^{n} \alpha_i (p_T + (T-t)c_{i_t})/T + \sum_{i=k_0}^{k_c-1} \alpha_i p_T \\
&\le \sum_{t=1}^{T} \sum_{i=i_t}^{n} \alpha_i ((T-t)c_{i_t})/T + \sum_{i=k_0}^{n} \alpha_i p_T \\
&\le \sum_{t=1}^{T} \sum_{i=i_t}^{n} \alpha_i ((T-t)c_{i_t})/T + \sum_{i=k_0}^{n} \alpha_i (v_{k_0} - (T-1)c_{k_0}) \\
&\le 2\sum_{t=1}^{T} \sum_{i=k_0}^{n} \alpha_i ((T-t)c_{k_0})/T + \sum_{i=k_0}^{n} \alpha_i (v_{k_0} - (T-1)c_{k_0}) \\
&= \sum_{i=k_0}^{n} \alpha_i v_{k_0} \le \max_{1\le k\le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\},
\end{aligned}$$

where the third inequality is due to $i_t \ge k_c$, the fourth inequality is due to $p_T \le v_{k_0} - (T-1)c_{k_0}$, and the fifth inequality is due to $c_j/c_i \le \sum_{k=i}^{n} \alpha_k \Big/ \sum_{k=j}^{n} \alpha_k \le 2\sum_{k=i}^{n} \alpha_k \Big/ \sum_{k=j}^{n} \alpha_k$, for any $1 \le i < j \le n$. Thus, an optimal cyclic pricing policy reduces to a static pricing policy in this case.

Next consider the case when $\tau_{k_0} > 1$. Let $l_1 = \arg\min_{k_0<i<k_c} \{\tau_i < \tau_{k_0}\}$, and we denote recursively $l_{m+1} = \arg\min_{l_m<i<k_c} \{\tau_i < \tau_{l_m}\}$. The largest properly defined m is denoted by M. As a result, any cycle can be decomposed into many separable mini cycles, namely, $[1, \tau_{l_M} - 1]$, $[\tau_{l_M}, \tau_{l_{M-1}} - 1]$, $\cdots$, $[\tau_{l_1}, \tau_{k_0} - 1]$, $[\tau_{k_0}, T]$. We prove an optimal cyclic pricing policy reducing to a static pricing policy by showing that the average profit per period within each mini cycle is no more than the expected profit per period from an optimal static pricing, namely, $\max_{1\le k\le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$. Consider the first mini cycle $[1, \tau_{l_M} - 1]$. Because only customers with valuation greater than or equal to v_{k_c} will either purchase or leave immediately upon arrival, and no customers arriving in these periods will wait, the average profit per period is thus no more than $\max_{k_c\le k\le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$, which is no more than $\max_{1\le k\le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$. Next consider the last mini cycle $[\tau_{k_0}, T]$. This mini cycle effectively reduces to the case as if $\tau_{k_0} = 1$ with a cycle length of $T - \tau_{k_0} + 1$, and thus we know that the expected profit per period is bounded above by the expected profit from an optimal static pricing policy based on the preceding discussions.

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Thus, we just need to consider $[\tau_{l_{m+1}}, \tau_{l_m} - 1]$ for any $k_0 < l_{m+1} < l_M$. The average profit per period within this mini cycle is bounded above by:

$$
\begin{aligned}
&\sum_{t=\tau_{l_{m+1}}}^{\tau_{l_m}-1} \pi_t/(\tau_{l_m} - \tau_{l_{m+1}}) + \sum_{i=l_{m+1}}^{k_c-1} \alpha_i p_T \\
\leq &\sum_{t=\tau_{l_{m+1}}}^{\tau_{l_m}-1} \sum_{i=i_t}^{n} \alpha_i (p_T + (T-t)c_{i_t})/(\tau_{l_m} - \tau_{l_{m+1}}) + \sum_{i=l_{m+1}}^{k_c-1} \alpha_i p_T \\
\leq &\sum_{t=\tau_{l_{m+1}}}^{\tau_{l_m}-1} \sum_{i=i_t}^{n} \alpha_i ((T-t)c_{i_t})/(\tau_{l_m} - \tau_{l_{m+1}}) + \sum_{i=l_{m+1}}^{n} \alpha_i p_T \\
\leq &\sum_{t=\tau_{l_{m+1}}}^{\tau_{l_m}-1} \sum_{i=i_t}^{n} \alpha_i ((T-t)c_{i_t})/(\tau_{l_m} - \tau_{l_{m+1}}) + \sum_{i=l_{m+1}}^{n} \alpha_i (v_{l_{m+1}} - (T - \tau_{l_{m+1}})c_{l_{m+1}}) \\
\leq &\sum_{t=\tau_{l_{m+1}}}^{\tau_{l_m}-1} (T-t) \sum_{i=l_{m+1}}^{n} \alpha_i c_{l_{m+1}}/(\tau_{l_m} - \tau_{l_{m+1}}) + \sum_{i=l_{m+1}}^{n} \alpha_i (v_{l_{m+1}} - (T - \tau_{l_{m+1}})c_{l_{m+1}}) \\
= &\frac{(2T - \tau_{l_{m+1}} - \tau_{l_m} + 1)}{2} \sum_{i=l_{m+1}}^{n} \alpha_i c_{l_{m+1}} + \sum_{i=l_{m+1}}^{n} \alpha_i (v_{l_{m+1}} - (T - \tau_{l_{m+1}})c_{l_{m+1}}) < \sum_{i=l_{m+1}}^{n} \alpha_i v_{l_{m+1}},
\end{aligned}
$$

where the second inequality is due to $i_t \geq k_c$, the third inequality is due to $p_T \leq v_{l_{m+1}} - (T - \tau_{l_{m+1}})c_{l_{m+1}}$, the fourth inequality is due to $c_j/c_i \leq \sum_{k=i}^{n} \alpha_k \Big/ \sum_{k=j}^{n} \alpha_k$, for any $1 \leq i < j \leq n$, and the last inequality is due to $\tau_{l_{m+1}} < \tau_{l_m} - 1$. Thus, we obtain the announced result. $\square$

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