ON THE DETERMINATION OF STRUCTURAL SYSTEM RELIABILITY.

A Thesis Presented for the degree of
Master of Philosophy in the
Faculty of Engineering of
the University of London

by

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November 1981

ABSTRACT

There are several reliability analysis methods which can be used to calculate the probability of a structural member failing in a single failure mode. Some of these methods use the complete set of basic variable values which will cause a local failure to obtain the failure probability (Level 3 methods), and are therefore extremely unwieldy for practical use, while others simplify that set by certain idealisations (Level 2 methods). The failure probabilities obtained by two of the Level 2 methods have been compared with those obtained by a Level 3 method in order to establish whether the inaccuracy caused by these idealisations can reasonably be neglected.

A review has been carried out of some papers which describe methods for evaluating the probability of system failure by using properties associated with each potential local failure in the structural system. These methods tend to use the structural elements resistance against the load effect associated with a failure mode as basic variables. Therefore a method has been put forward which estimates the parameters of the distribution of such a resistance from a function of the basic variables and which is consistent with the best available estimate of the probability of that local failure being realised.

Ditlevsen's (1979) method has been extended so that it can also deal with failure paths consisting of multiple non-ductile local failures. This proposed method has been developed so that bounds on the system reliability of any idealised structure can be produced, with the single provision that no load applied to the structure causes non-linear load effects on any of the elements. It has been demonstrated that, for some simple examples attempted, this method produces relatively narrow bounds which are consistent with values obtained by some of those methods which were reviewed together with a Monte Carlo simulation.

ACKNOWLEDGEMENTS

The author wishes to thank Dr F C Walley, Mr D F Evans (then both with the Directorate of Civil Engineering Services, PSA) and Mr I C Macpherson (then of the Building Regulations Professional Division, DOE) who, because of their interest in the field of structural reliability and in its application to the work of the DOE, gave the author the time and the opportunity to carry out these studies.

The work was carried out under the immediate supervision of Mr M J Baker and Dr T A Wyatt, to whom the author is deeply grateful for their guidance and advice. Thanks are also due to the author's colleagues during his work at Imperial College, Dr K Ramachandran, Mr W Manners and Mr A H S Wickham, for the useful discussions about reliability analysis and associated topics.

Finally, but in no way least, the author wishes to thank his wife for providing help and support in so many ways during the course of the study and in particular for correcting some of the mistakes in the English of the earlier drafts.

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NOTATION

$a_{i,j}$	The event of the excess resistance of that element which would be the j^{th} element to fail if the i^{th} failure path were realised being less than or equal to zero after the excess load from the first (j-1) local failures in that path have been redistributed.
A_i	The event of the i^{th} failure path being realised.
$c_{i,j,k}$	The resistance coefficient by which the resistance $r_{i,k}$ is multiplied in the form of the excess resistance for the j^{th} element which would fail if the i^{th} failure path occurs.
$\underline{C}$	The transformation matrix which transforms a set of correlated basic variables into a set of mutually uncorrelated standardised variables.
^{c}	Complement (ie event of the original event not being realised).
$d_{i,j,k}$	The loading coefficient by which the load S_k is multiplied in the form of the excess resistance for the j^{th} element which would fail if the i^{th} failure path occurs.
E(X)	The expected value (ie mean) of X.
$f_X(x)$	The probability density function of X, evaluated at X = x.
F	The event of system failure.
$F_X(x)$	The cumulative distribution function of X, equal to the probability of $X \leqslant x$.
$g(\underset{\sim}{x})$	The form of a failure criterion as a function of the basic variables, $\underset{\sim}{X}$.
$h(\underset{\sim}{u})$	The form of a failure criterion as a function of the standardised normal basic variables, $\underset{\sim}{U}$.
m	The number of failure paths in the failure set.
n	The number of basic variables.
n_i	The number of local failures which constitute the i^{th} failure path.
n_k	The number of time intervals during which the k^{th} time-varying load maintains a uniform intensity in the reference period, T.
n_s	The number of loads applied to the structure.
p(a)	The probability of event 'a' being realised.
p(a b)	The probability of event 'a' being realised conditional on event 'b' having been realised.
$r_{i,j}$	The resistance of the element which would be the i^{th} element to fail if the j^{th} failure path were realised.
$\mathcal{R}$	The reliability of a system or element = 1.0 - p(F).
S_i	The i^{th} load applied to the structure.
t_n	The difference between the maximum and the minimum values from a set of n trials.
T	The reference period of the structure, usually the design life.

u A parameter for the extreme distributions.

u_i A value of the i^{th} standardised normal basic variable

u_i^* The value of the i^{th} standardised normal basic variable at the Advanced Level 2 design point.

$\underset{\sim}{U}$ The set of random standardised normal basic variables, U_i, which have values u_i.

$\underset{\sim}{V}_x$ The co-variance matrix for the basic variables.

x_i A value of the i^{th} basic variable.

x_i^* The value of the i^{th} basic variable at the Advanced Level 2 design point.

x_{MAX}^n The maximum value from a set of n trials of X.

x_{MIN}^n The minimum value from a set of n trials of X.

$\underset{\sim}{X}$ The set of random basic variables, X_i, which have values x_i.

$Z_{i,j}$ The excess resistance of the i^{th} element which would fail if the j^{th} failure path occurs, and is equal to $g(\underset{\sim}{x})$ and $h(\underset{\sim}{u})$.

α A parameter for the extreme distributions.

α_i The sensitivity factor of the i^{th} basic variable in Level 2 reliability analysis.

$\beta_{i,j}$ The reliability index for the i^{th} element which would fail if the j^{th} failure path occurs, equal to the distance from the origin to the nearest point on the (idealised) failure boundary in standardised normal space. Usually assumed $\doteq -\Phi^{-1}(F_{Z_{i,j}}(0.0))$

$\Gamma(.)$ The gamma function.

δx_i A small increment in the value of the i^{th} basic variable, X_i.

δu_i A small increment in the value of the i^{th} standardised normal basic variable, U_i.

$\Delta_k T$ The interval during which the k^{th} time-varying load maintains a constant intensity = T/n_k

$$\Theta_{i,j,k,l} = \left[\frac{\left| \rho(Z_{i,j}, Z_{k,l}) \right| + \rho(Z_{i,j}, Z_{k,l})}{2\rho(Z_{i,j}, Z_{k,l})} \right] = \psi(-\rho(Z_{i,j}, Z_{k,l}))$$

Θ Angle between the axes of two standardised normal variables.

Π Signifies product of terms.

$\rho(X,Y)$ The correlation coefficient of X and Y.

ϵ The minimum value for the extreme type III minima distribution.

Σ Signifies summation of terms.

$\sigma(X)$ The standard deviation of X.

$\phi(u)$	The probability density function for the standardised normal distribution, evaluated at U = u.
$\Phi(u)$	The cumulative distribution function for the standardised normal distribution, equal to the probability of $U \leqslant u$.
$\chi(y)$	$= (y-\|y\|)/2y$
$\cap$	Signifies intersection of events (ie the event of all the component events being realised).
U	Signifies union of events (ie the event of at least one of the component events being realised).
$I_k \subset I_m$	Operate with I_k as each sub-set of size k from I_m in turn.
$j \in I_k$	Operate with j as each element of I_k in turn.
$I_m \setminus I_k$	The sub-set of Im containing all elements not in I_k.
$\underline{\underline{\ }}$	Signifies a matrix.
$\underset{\sim}{\ }$	Signifies a vector.

NB: In general when a capital letter is used for a random variable the same letter in lower case represents the values which it may take on.

CHAPTER 1 - INTRODUCTION

1.1 BACKGROUND

1.1.1 Limit State Design

The methods available to an engineer for the design or analysis of structures are continually being refined and new methods are regularly being developed. One result of this trend is that the overall safety margin for structures is being erroded as some of the previously intangible reserves of strength are incorporated explicitly in the calculations to help carry the applied loads.

Partly as a result of this trend and partly as a result of a more rigorous approach generally, recent structural codes of practice have been based on the limit state approach, both in this country and internationally. In the UK the first Limit state code was CP 110:1972, "The Structural Use of Concrete"[1]. In a limit state code each structural element is designed explicitly against failure by several modes rather than being designed to a general set of rules intended to cover all such modes. Those modes of failure which are chosen to be designed against are known as the limit states, and are intended to be the most probable of the known modes of failure.

1.1.2 Partial Safety Factor Format

At the same time as codes began to be written using the limit state approach they also changed to incorporate their safety element by means of partial safety factors instead of permissible stresses. With permissible stresses, the safety factor is a global one which, in practice, has been used to divide the yield stress to obtain the permissible stress. These safety factors are not given explicitly in the codes.

When the safety element is incorporated by means of partial safety factors, different partial safety factors are applied to several of the design variables which serve to increase or decrease their nominal values. The values of the partial safety factors are given explicitly in the codes rather than being already incorporated into the given values of the design variables. The values of the partial safety factors are generally regarded as reflecting the uncertainty in the variables' values and therefore should lead to the overall factor of safety reflecting the uncertainty in the design. Furthermore the use of explicitly given partial safety factors allows their revision to be carried out relatively easily if and when further knowledge makes that desirable.

1.1.3 Calibration

The overall intention of each limit state criterion is to ensure that any element designed to that criterion has an acceptable reserve of strength against that mode of failure. Since the values of the design variables are not known precisely (eg the maximum load applied to an element or an actual material strength), a designer can never be entirely certain whether there is any reserve of strength. However, code writers have to decide on values for partial safety factors so that the probability of there not being any reserve of strength is acceptably low without making any design uneconomic. This implies choosing partial safety factors values so that each design produces an element with a probability of failure equal, as near as is practicable, to a target probability whose value is based on the level considered socially acceptable.

The simplest method to do this is to assume that the existing codes produce designs having the correct degree of safety and calibrate the new code to produce similar designs. The assumption can be made if the old code was used extensively without a history of failures. However, a code which just imitates its predecessor is not likely to advance the state of the art of engineering.

Alternatively, if the probabilities of failure can be calculated explicitly for each element in each mode of failure, the past experience with the whole code can be used to produce a target probability against which each design to the new code can be calibrated.

1.1.4 Reliability Analysis

Theoretically the probability of the failure of a structural element by a particular mode of failure can be calculated from a knowledge of the failure mechanism (and hence the sets of values for the design variables which, if any of them occurred, would result in failure) and the probability distributions for the values of the design variables. However, in most cases this can only be achieved by numerical methods and as such is extremely time-consuming.

In order to reduce the extent of the calculations required, various methods have been devised which give approximations to the failure probability by making simplifications. The accuracy achieved with these methods varies depending on the extent of the simplification. In order to differentiate these methods from those which do not make such simplifications, the latter are called Level 3 methods of reliability analysis and the former Level 2 methods. The use of partial safety factors in design equations are termed Level 1 methods.

It should be noted that any failure probability will be conditional on the existing knowledge used in the calculation, and that neither the variability of the design variables nor the mechanisms causing failure are perfectly understood. Therefore any such failure probabilities can only be considered as subjective and not frequentist measures of failure occurring. This should not be too significant in calibrating codes since any errors will apply to both the old and new codes.

1.1.5 System Reliability

The Level 2 methods mentioned above were developed to analyse single failure events (eg a beam failure in shear or a slab having excessive deflection). There are, however, some modes of failure which involve system failures, where a system failure is one whose occurrence results from several individual, or local, failures occuring concurrently. An example of system failure is progressive collapse, the importance of which was demonstrated in 1968 when a corner of the Ronan Point block of flats in Canning Town collapsed killing four people, following a gas explosion on the eighteenth floor. This failure resulted in an additional requirement, clause D17, in the Building Regulations [22].

A major difference between calculating system and local failure probabilities is that while a local failure is governed by a single criterion (eg a load effect being greater than a resistance or a deflection being greater than a specified limit), a system failure can be dependent on any one of a number of combinations of events (eg any two of four supports to a simply supported slab failing). Furthermore, the failure criteria for those events will be governed by variables some of which will be common to two or more failure criteria and some of which to only one. Therefore the events of constituent local failures will be neither totally mutually independent nor mutually dependent, so that the probability of system failure cannot be obtained from just a knowledge of the probabilities of the local failures.

Also, in design or analysis of structures, elements cannot be isolated from each other except by idealisations. For example, the load effect on a specific element cannot be known precisely because of:

i. inaccurate structural analysis for anything other than a truly statically determinate structure;

ii. the load-carrying capacity of nominally non-structural elements, and,

iii. load re-distribution as the structure deforms under load.

By analysing the whole structural system together such inaccuracies can be reduced, whether for conventional or reliability analysis.

Therefore, methods of system reliability are being developed to complement those used to calculate the probability of single local failures. Because of the additional complexity of systems, these methods are not as developed and contain greater simplifications than the methods for single failure events.

1.2 OBJECT AND SCOPE OF STUDY

1.2.1 Accuracy of Level 2 methods

The principal division of the methods for obtaining the probability of an element failing is into Level 3 methods using numerical integration, Level 3 methods using Monte Carlo simulation, Mean Value Level 2 methods and Advanced Level 2 methods (see sections 2.3 and 2.4 for descriptions of these methods). Experience with different computer programs shows that reliability analysis based on an Advanced Level 2 method is usually very much faster than programs using numerical integration or even those using a simulation method, while Mean Value Level 2 methods are faster still.

The disadvantage with the Level 2 procedures is that they give only an approximation to the failure probability, and so these methods need to be checked to see if the results which they produce can be treated with the desired degree of confidence. Therefore, in Part 1, some results obtained using these methods were compared to those obtained using a Level 3 numerical integration procedure. This is the continuation of a study by Baker[2], which had shown that the errors occuring as a result of the simplifications that are implicit in an Advanced Level 2 method are fairly small for the simple structural problem examined.

For this study the same failure criterion was used, but with a wider range of coefficients of variation for the basic variables. The comparisons used both a Mean Value Level 2 and an Advanced Level 2 method to investigate whether the extra calculations involved in the latter is justified in terms of improved accuracy. A series of randomly generated failure criteria were also studied to establish whether the form of the failure criterion had any effect on the accuracy of the results. These random criteria were arbitary and were not intended to have any structural significance.

1.2.2 Structural System Reliability Analysis

Part 2 deals with structural system reliability analysis. Some earlier papers dealing with the subject are reviewed.

A method is proposed in which the resistance of the elements in a structure and the loads applied to it are treated as variables which have normal, or Gaussian, distributions. These variables are used to define idealised failure criteria for the local failures in a system which make allowance for load redistribution after each local failure. These criteria are then incorporated in a technique of system reliability analysis to obtain bounds on the probability of system failure. This method is similar to Ditlevsen's [17], but is more general since it allows for parallel combinations of failure events as well as series combinations. The principal disadvantage of this generality is the increased complexity of the method. The only limitation on the structures for which this method can be used is that no load applied to the structure should cause non-linear effects on any constituent element.

The proposed method was demonstrated in the analysis of some simple examples. The results were compared to those obtained when using some of the methods that had been reviewed, together with the system failure probability obtained from a Monte Carlo simulation.

In both the proposed method and some of those described by the reviewed papers, the resistances of the elements are used as variables. Therefore a method has been proposed which obtains the parameters for the distributions of such resistances from the basic variables and which is consistent with the best estimate of the probability of the member failing when considered in isolation.

PART 1 - LEVEL 2 RELIABILITY METHODS

CHAPTER 2 - CONSIDERATIONS WHICH AFFECT RELIABILITY

2.1 BASIC VARIABLES

2.1.1 Uncertainty in Design Variables

In a design office, the values of all design variables are generally treated as being deterministic. In practice, however, the precise value of each variable will not be that assumed by the designer, although it should bear some relationship to it. In reliability analysis this uncertainty is allowed for by assuming that all design variables (with the exception of any mathematical or physical constants) are modelled by probabilistic distributions or stochastic processes.

The uncertainty in the design variables is due to a number of causes. The values of the mechanical properties of construction materials (eg the yield strength of steel, the compression strength of concrete or its Young's modulus) will depend on a large number of factors, some of which have been investigated by Baker[1,4] and others. Similarly the variability of the structural dimensions and geometry has been studied[42].

The permanent loads (eg the self-weight or floor finishes) will be dependent on the structural dimensions together with the density of the materials that are used, and hence they can be correlated to properties of the materials and/or the structural dimensions. The variable loads (eg the occupancy load or wind load) will depend on the actual use to which the structure is put, and since this may change with time the maximum intensity of any variable load cannot be known until the end of the design life of the structure, and usually not even then.

Another uncertainty is the uncertainty of the theoretical model used to predict structure behaviour. Even in the best relationship derived to model the behaviour of a type of member, various effects may be neglected, either because they are not known or because they are too complex to be included. These give rise to errors in the theoretical relationship and the ratio of the actual to the predicted behaviour can be considered as a further variable.

2.1.2 Basic Variables

The above variables are known as the basic variables, and the parameters for their distributions may be estimated by taking a large enough sample from

their populations. For simplicity in the past they were often treated as having normal distributions, but if there is enough data it is preferable to fit the most appropriate distribution to them. In some cases this can also be done from theoretical considerations (eg the log normal distribution if the quantity is the product of a set of random events, the normal distribution if the quantity is the sum of a set of random events, or the extreme type I maxima distribution if the quantity is the maximum value from a large set of normally distributed variables).

An alternative approach is to use a knowledge of the higher moments, usually the coefficients of skewness and kurtosis, to fit a distribution from a family of similar distributions[34]. This has the advantage of modelling the distribution only using direct observations of the quantity, but has the disadvantages that:

i. any theoretical knowledge about the causes of the variability is not considered; and

ii. the higher moments are sensitive to the sample taken unless a very large number are included.

2.1.3 Correlated Variables

The basic variables are often idealised as a set of mutually independent (and therefore uncorrelated) variables. If they are not, then they can be transformed into a set of mutually uncorrelated variables. This can be demonstrated by considering the transformation of a set of correlated variables, $\underset{\sim}{X}$, into a set of mutually uncorrelated variables, $\underset{\sim}{U}$, such that $E(U_i) = 0.0$ and $\sigma(U_i) = 1.0$ for all i. If the transformation is

$$\underset{\sim}{U} = \underline{C}.(\underset{\sim}{X} + \underset{\sim}{W}) \qquad (2.1.1)$$

where $\underline{C}$ is a constant matrix and $\underset{\sim}{W}$ is a constant vector.
Then

$$E(\underset{\sim}{U}) = \underline{C}.(E(\underset{\sim}{X}) + \underset{\sim}{W}) = \underset{\sim}{0} \qquad (2.1.2)$$

Therefore $\underset{\sim}{W}$ must be $-E(\underset{\sim}{X})$ and the transformation is

$$\underset{\sim}{U} = \underline{C}.(\underset{\sim}{X} - E(\underset{\sim}{X})) \qquad (2.1.3)$$

If $\underline{V}_X$ is the covariance matrix of $\underset{\sim}{X}$, then

$$\begin{aligned} \underline{V}_X &= E[(\underset{\sim}{X} - E(\underset{\sim}{X})).(\underset{\sim}{X} - E(\underset{\sim}{X}))^T] \\ &= E[\underline{C}^{-1}.\underset{\sim}{U}.(\underline{C}^{-1}.\underset{\sim}{U})^T] \\ &= \underline{C}^{-1}.E[\underset{\sim}{U}.\underset{\sim}{U}^T].(\underline{C}^{-1})^T \\ &= (\underline{C}^T.\underline{C})^{-1} \end{aligned} \qquad (2.1.4)$$

since

$$E[\underset{\sim}{U}.\underset{\sim}{U}^T] = \underline{V}_U = \underline{I} \qquad (2.1.5)$$

It can be shown[11] that $\underline{C}$ is in fact equal to

$$\underline{C} = \underline{A}^T . \underline{V}_Y^{-1} \qquad (2.1.6)$$

where $\underline{A}$ is an orthogonal matrix with column vectors equal to the orthonormal eigenvectors of $\underline{V}_X$, and $\underline{V}_Y$ is a diagonal matrix composed of the eigenvalues of $\underline{V}_X$.

If the original variables do not have normal distributions, the transformed variables are not defined by common distributions. If all the original variables do have normal distributions, however, then the transformed variables are also normal.

2.1.4 Example of Correlation

One simple method of avoiding correlated variables is to use a combination of several mutually uncorrelated variables. When the uncorrelated variables are treated as a product, the means for all but one of the variables are taken as unity; while when they are treated as a sum, the means for all but one of them are taken as zero. For example, the yield strength of each reinforcing bar in a concrete beam[1,4] has been modelled using four basic variables. The model is

$$F_Y = F_1 \times F_2 \times F_3 \times F_4 \qquad (2.1.7)$$

where

- F_Y is the yield strength of a particular reinforcing bar in the beam;
- F_1 models the variability of all bars of the same nominal diameter, type and grade with appropriate mean;
- F_2 models the variability between different works with unit mean, and will be common to all the bars in a beam unless part of a large project which warrants more than one supplier;
- F_3 models the variability between billets with unit mean, and in many cases can be combined with F_2; and
- F_4 models the variability between bars rolled from a single billet with unit mean, and will be a different basic variable for each bar.

In general, however, this precision in modelling does not occur because it would require considerable statistical knowledge for it to make a significant

contribution to the accuracy of any analysis, and to model all parameters perfectly would require more statistical data than is at present, or ever likely to be, available[1,4,42].

When analysis is being carried out for a whole structure rather than just a single member, then the set of basic variables for each local failure will be a sub-set of the complete set of basic variables for the structure.

2.2 FAILURE CRITERIA

A limit state is defined as a particular limiting condition for an element (or number of elements) beyond which it (or they) becomes unfit for one or more of its purposes. The limit states are usually divided into ultimate and serviceability classifications, where an ultimate limit state implies some form of collapse while a serviceability limit state implies that the member can no longer fulfil its design function in some respect.

For the purposes of analysis, it is assumed that the basic variables can be used to define a function, $g(\underset{\sim}{x})$, such that if $g(\underset{\sim}{x}) > 0.0$ that limit state has not been reached, but if $g(\underset{\sim}{x}) \leqslant 0.0$ then it has been reached. This function will be referred to as a failure criterion, which is therefore a general term implying failure to comply with a specified design requirement and not necessarily collapse. The function $g(\underset{\sim}{x})$ need not necessarily be a simple equation, so that the boundary between the safe region (defined as the subset of $\underset{\sim}{X}$ in which $g(\underset{\sim}{x}) > 0.0$) and the failure region (the subset of $\underset{\sim}{X}$ in which $g(\underset{\sim}{x}) \leqslant 0.0$) is not necessarily a smooth hyper-surface. This boundary is known as the failure boundary.

The probability of failure of an element or structure with respect to a particular limit state can be defined as the probability of the limit state function being less than or equal to zero, that is

$$\begin{aligned} p(F) &= \int_{g(X)=-\infty}^{0} f_{g(\underset{\sim}{X})}(g(\underset{\sim}{x}))\, d(g(\underset{\sim}{x})) \\ &= \int_{X \in Q} f_{g(\underset{\sim}{X})}(g(\underset{\sim}{x}))\, \frac{d(g(\underset{\sim}{x}))}{d\underset{\sim}{x}}\, d\underset{\sim}{x} \\ &= \int_{X \in Q} f_{\underset{\sim}{X}}(\underset{\sim}{x})\, d\underset{\sim}{x} \end{aligned} \qquad (2.2.1)$$

where Q is the subset of $\underset{\sim}{X}$ in the failure region and $f_{\underset{\sim}{X}}(\underset{\sim}{x})$ is the joint probability density function for the variables $\underset{\sim}{X}$.

If the element or structure has to satisfy several limit states, the subset Q is different for each limit state. The total probability of failure will be defined by that subset of $\underset{\sim}{X}$ that contains all points of failure in the basic variable space. In general, it will not be either the sum or the maximum of the values for the separate limit states, though these will form bounds to it.

The reliability, $\mathcal{R}$, of an element or structure is defined as the probability that it will not attain the relevant limit state(s), and as such is given by

$$\mathcal{R} = 1.0 - p(F) \qquad (2.2.2)$$

where $p(F)$ is the probability of failure. In general the probability of failure is very small (typical values are around 10^{-6}) so that the reliability is most conveniently expressed in terms of one minus the probability of failure.

2.3 LEVEL 3 RELIABILITY ANALYSIS METHODS

2.3.1 Definitions

The definition proposed for Level 3 methods [11] is "safety checking based on an exact probabilistic analysis for whole structural systems, using a full distribution approach, with reliability levels based on agreed failure probabilities, interpreted in the sense of relative frequency". Most of the structures that have previously been analysed by these methods are single members considered to fail in only one mode. This is both because it is difficult to define the joint probability density function of the various basic variables and because accurate numerical solutions are extremely time-consuming.

The object of any analysis by this method is to measure the cumulative probability of the function $g(\underset{\sim}{x})$ being less than or equal to zero. If there are n basic variables, this requires rewriting equation (2.2.1) in the form

$$p(F) = \int_{x_1=-\infty}^{+\infty} \int_{x_2=-\infty}^{+\infty} \cdots \int_{x_n=-\infty}^{+\infty} f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) \times \chi(g(\underset{\sim}{x})) \, dx_1 \, dx_2 \ldots dx_n \qquad (2.3.1)$$

where

$$\chi(y) = \tfrac{1}{2} - |y|/2y \qquad (2.3.2)$$

2.3.2 Numerical Integration Approach

Analytical solutions to equation (2.3.1) are available for a few special cases, but in general this equation can only be solved numerically.

Numerical integration of a convolution integral is an extremely time-consuming procedure, particularly for a large number of variables, and therefore would not be recommended for general use. However, it is a suitable method for obtaining an accurate estimate of the probability of failure with which to compare the estimates obtained from other methods. If this method is used to assess the errors arising from an approximate method of reliability analysis, then the accuracy of the numerical integration procedure has to be at least an order of magnitude more accurate than the expected error of the approximate method. The size of the finite increments of the variable space added at each step and the number of such steps can be adjusted so that the appropriate degree of accuracy in the value of the failure probability is attained for the minimum amount of calculation.

The division of the variable space into finite increments for numerical integration can be carried out in a number of ways. The simplest approximate form of the failure probability, for mutually independant basic variables, is obtained by dividing the variable space into steps of size δx in each of the n dimensions. This gives

$$p(F) \doteq \sum_{j_1=-k_1}^{l_1} \sum_{j_2=-k_2}^{l_2} \cdots \sum_{j_n=-k_n}^{l_n} \left[\delta x^n \cdot f_{X_1}(j_1 \cdot \delta x) \cdot f_{X_2}(y_2 \cdot \delta x) \cdot \cdots \cdot f_{X_n}(j_n \cdot \delta x) \cdot \psi(g(\underset{\sim}{x})) \right] \qquad (2.3.3)$$

where $g(\underset{\sim}{x})$ is the numerical value of the failure criteria when $x_i = j_i \cdot \delta x$ for $1 \leqslant i \leqslant n$. Each value of l_i and k_i is chosen so as to include all of the failure region which has a relatively high probability of realisation.

2.3.3 Refinements to Numerical Integration Approach

To evaluate the failure probability by equation (2.3.3) is not sensible since it is equivalent to dividing the basic variable space into equal sized hypercubes and then summing the probability content of those in the failure region. A more efficient approach is to carry out the convolution integration as a series of stacked "loops". The inside "loop", which finds $p(F|x_2,\ldots,x_n)$ is obtained from

$$p(F|x_2,\ldots,x_n) = \begin{cases} F_{x_1}(x_1') & \text{if } g(x_1' + \delta x, x_2,\ldots,x_n) \geqslant 0.0 \\ 1.0 - F_{x_1}(x_1') & \text{if } g(x_1' + \delta x, x_2,\ldots,x_n) < 0.0 \end{cases} \quad (2.3.4)$$

where x_1' is the value of the first variable such that $g(x_1', x_2,\ldots,x_n) = 0.0$ for the conditional values of x_i, $2 \leqslant i \leqslant n$.

The second "loop" can then be carried out using the values of this conditional probability for each increment in the second basic variable, and similarly for subsequent "loops" (figure 2.3.1), using

$$p(F|x_{i+1},\ldots,x_n) = \sum_{j=-k_i}^{l_i} \left[\delta x \cdot f_{x_i}(j \cdot \delta x) \cdot p(F|x_i = j.\delta x, x_{i+1}, x_{i+2},\ldots,x_n) \right] \quad (2.3.5)$$

This is repeated until the n^{th}, and outer, "loop" has produced an unconditional failure probability. It should be noted, however that this approach is only applicable to structures in which the position of the failure surface can be uniquely defined by specifying any $(n-1)$ of the n basic variables and the condition $g(\underset{\sim}{x}) = 0$.

For both equation (2.3.3) and (2.3.5) the accuracy will be improved if Simpson's Rule is used to sum the probabilities rather than the more straightforward Trapezoidal Rule. A further refinement is to vary the values of δx such that they are smaller in regions of high probability, usually near the mean values, and larger in regions of low probability. A refined Level 3 method, using these and other modifications, is described in sub-section 3.1.3.

2.3.4 Monte Carlo Analysis

An alternative method to numerical integration, which also uses the full distribution approach, is Monte-Carlo simulation [17]. In each of a number of trials a value is randomly generated from the distribution modelling each basic variable, and the failure criterion, $g(\underset{\sim}{x})$, is evaluated for that set of values. The probability of failure is calculated as the proportion of the trials undertaken for which $g(\underset{\sim}{x}) \leqslant 0.0$. The precision is dependent on the number of trials carried out and the expected number of those trials that will result in failure.

This precision can sometimes be improved by fitting a curve, such as the Pearson Distribution curves[18], using the central moments of the sample values, $g(\underset{\sim}{x})$, and calculating the probability of $g(\underset{\sim}{x}) \leqslant 0$ directly. The main problem with curve fitting is that the tail of the parent distribution may not be the same shape as that of the fitted distribution.

Monte Carlo simulation is useful for the verification of other methods where numerical integration would prove excessively expensive in computer time. This method will become inefficient with relatively reliable structures since the number of trials required to give a meaningful result is inversely related to the element's probability of failure.

2.4 LEVEL 2 RELIABILITY ANALYSIS METHOD

2.4.1 Definition

The proposed definition of a Level 2 method[11] is "a design method incorporating safety checks only at a selected point (or points) on the failure boundary (as defined by the appropriate limit state equation), rather than as a continuous process, as at Level 3". The different Level 2 methods, therefore, are different algorithms for identifying a suitable safety checking point and the idealisation of the failure boundary in the region of that point to determine reliabilities described either by safety indices or equivalent "operational" probabilities in the same sense as for Level 3 methods.

2.4.2 Mean Value Methods

The Mean Value Level 2 methods[14,36] expand the failure criterion in terms of a Taylor series of mutually independent basic variables about the set of mean values. Using the first two terms is equivalent to linearising the failure surface at the mean point. This is given by

$$z = g(\underset{\sim}{x})$$

$$\doteq g(E(\underset{\sim}{X})) + \sum_{i=1}^{n} (x_i - E(X_i)) \frac{\partial g(\underset{\sim}{x})}{\partial x_i} \Bigg|_{\underset{\sim}{x}=E(\underset{\sim}{X})} \qquad (2.4.1)$$

where n is the number of basic variables. Hence, using the rules for expectation and variance[6],

$$E(z) = g(E(\underset{\sim}{X})) \qquad (2.4.2)$$

$$\sigma^2(z) = \sum_{i=1}^{n} \left[\sigma(X_i) \cdot \frac{\partial g(x)}{\partial x_i} \Bigg|_{\underset{\sim}{x}=E(\underset{\sim}{X})} \right]^2 \qquad (2.4.3)$$

The reliability index, β, is defined as $E(z)/\sigma(z)$, and the "operational" failure probability by $\Phi(-\beta)$. This will be exact if the failure criterion is a linear function of normal variables.

This method suffers from not being invariant [15] to any equivalent transformations of the failure criterion which produce the same failure and safe regions in basic variable space, including statistically acceptable transformations of the basic variables. However, the amount of calculation required to obtain an estimate of the failure probability by this method is relatively small since the linearisation point is coincident with the mean point.

2.4.3 Transformations of the Variables

In order to eliminate the lack of invariance to transformations of the failure criterion, the linearisation can be carried out in the region of the point of maximum probability density on the failure surface [20, 23]. This point is known as the design point and has to be obtained by iteration. The first step is to transform the basic variables.

For the particular case of normally distributed variables, the following transformation can be used

$$u_i = \frac{x_i - E(X_i)}{\sigma(X_i)} \qquad (2.4.4)$$

This makes the design point the closest point to the origin in the transformed variable space. Other transformations can be made for the cases of non-normal variables.

One approach [16] to using these transformations is to model specific basic variables with set distribution types, and then make prescribed transformations on them. This has the advantage of consistency of approach between different failure criteria, but does not allow the best available models for each basic variable to be used.

A more general method is to transform each variable from whatever distribution type has been used to model it into a standardised normal variable [3, 23]. This can be done by taking the cumulative distribution function $F_{X_i}(x)$ of X_i, as given by

$$F_{X_i}(x) = p(X_i \leqslant x) \qquad (2.4.5)$$

and then finding a value, u_i, of a standardised normal distribution which has the same cumulative probability by

$$u_i = \Phi^{-1} (F_{x_i}(x_i)) \tag{2.4.6}$$

where $\Phi(u)$ is the cumulative distribution function of the standardised normal variable u. A good polynomial approximation to $\Phi(u)$ for computational purposes has been given by Hastings [21].

This transformation may also be carried out [9] by finding the mean, m, and standard deviation, s, for a hypothetical normal distribution which has the same cumulative probability and probability density at the specific value x_i as the distribution of X_i, followed by the further transformation

$$u_i = \frac{x_i - m}{s} \tag{2.4.7}$$

Values of m and s have to be found for each value of x_i used.

The transformation of x_i into u_i for all the basic variables is referred to as the transformation from X-space (ie that of the original basic variables) into U-space (ie that of the standardised normal variables) (figure 2.4.1). Similarly u_i can be transformed into x_i by

$$x_i = F_{x_i}^{-1} (\Phi (u_i)) \tag{2.4.8}$$

and the failure criterion can be transformed into U-space by

$$g(\underset{\sim}{x}) = g(F_{\underset{\sim}{X}}^{-1} (\Phi (\underset{\sim}{u})) = h(\underset{\sim}{u}) \tag{2.4.9}$$

In practice this last transformation is carried out by transforming each value of u_i into the equivalent x_i before substituting into $g(\underset{\sim}{x})$.

The latest modification is to allow for the inclusion of correlated basic variables since it has been shown that the transformations of the variables into another, mutually independent set can be carried out within the design procedure [34].

2.4.4 Advanced Method

The algorithm which is referred to as the Advanced Level 2 method in following chapters is based on Horne's algorithm [23], but it is modified to include correlated basic variables. The failure criterion, equation (2.4.9), is transformed into U-space and then linearised at the design point u*, so that it becomes

$$Z \doteq \sum_{i=1}^{n} \left[a_i (u_i - u_i^*) \right] \tag{2.4.10}$$

where

$$a_i = \left. \frac{\partial h(\underline{u})}{\partial u_i} \right|_{u=u^*} \tag{2.4.11}$$

The mean and standard deviation of Z can be found in a similar manner to those for the Mean Value method. Since $E(U_i) = 0.0$ and $\sigma(U_i) = 1.0$, these are

$$E(Z) = \sum_{j=1}^{n} \left[-a_j u_j^* \right] \tag{2.4.12}$$

and

$$\sigma^2(Z) = \sum_{j=1}^{n} \sum_{k=1}^{n} \left[a_j a_k \rho(U_j, U_k) \right] \tag{2.4.13}$$

where $\rho(U_j, U_k)$ is the correlation coefficient for u_i and u_j. If a pair of the basic variables are either mutually independent or both have normal distributions, then $\rho(U_j, U_k) = \rho(X_j, X_k)$, but in other cases the relationship has to be established explicitly, or else one has to be assumed.

The reliability index is defined by

$$\beta = \frac{E(Z)}{\sigma(Z)} = \frac{\sum_{j=1}^{n} \left[-a_j u_j^* \right]}{\sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} \left[a_j a_k \rho(U_j, U_k) \right]}} \tag{2.4.14}$$

Therefore

$$\sum_{i=1}^{n} \left[\frac{\partial \beta}{\partial u_i^*} \delta u_i^* \right] = \sum_{i=1}^{n} \left[\left\{ \frac{\sum_{j=1}^{n} \left[-\frac{\partial a_j}{\partial u_i^*} u_j^* \right] - a_i}{\sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} \left[a_j a_k \rho(U_j, U_k) \right]}} - \frac{\sum_{j=1}^{n} \left[-a_j u_j^* \right] \times \sum_{j=1}^{n} \sum_{k=1}^{n} \left[\left(a_j \frac{\partial a_k}{\partial u_i^*} + \frac{\partial a_i}{\partial u_i^*} a_k \right) \rho(U_j, U_k) \right]}{2 \times \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \left[a_j a_k \rho(U_j, U_k) \right] \right)^{3/2}} \right\} \delta u_i^* \right] \tag{2.4.15}$$

Since $\rho(U_j,U_k) = \rho(U_k,U_j)$, this simplifies to

$$\sum_{i=1}^{n}\left[\frac{\partial\beta}{\partial u_i^*}\,\delta u_i^*\right] = \sum_{i=1}^{n}\left[\left\{\frac{\sum_{j=1}^{n}\left[-\frac{\partial a_j}{\partial u_i^*}\,u_j^*\right] - a_i}{\sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n}\left[a_j a_k \rho(U_j,U_k)\right]}} - \beta \times \frac{\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{\partial a_j}{\partial u_i^*}\,a_k\rho(U_j,U_k)\right]}{\sum_{j=1}^{n}\sum_{k=1}^{n}\left[a_j a_k \rho(U_j,U_k)\right]}\right\}\delta u_i^*\right] \qquad (2.4.16)$$

If β is to be a stationary (maximum or minimum) value, then

$$\sum_{i=1}^{n}\left[\frac{\partial\beta}{\partial u_i^*}\,\delta u_i^*\right] = 0.0 \qquad (2.4.17)$$

subject to the point remaining on the failure boundary, so that

$$\sum_{i=1}^{n}\left[a_i\,\delta u_i^*\right] = 0.0 \qquad (2.4.18)$$

Therefore, if equation (2.4.16) is to be satisfied for all possible combinations of δu_i^*

$$\sum_{j=1}^{n}\left[-\frac{\partial a_j}{\partial u_i^*}\,u_j^*\right] - \beta\,\frac{\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{\partial a_j}{\partial u_i^*}\,a_k\rho(U_j,U_k)\right]}{\sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n}\left[a_j a_k\rho(U_j,U_k)\right]}} = 0.0 \qquad (2.4.19)$$

must be true for $1 \leqslant i \leqslant n$. This gives n simultaneous equations in the n values of u_i^*, these equations being satisfied when

$$u_j^* = -\beta\,\frac{\sum_{i=1}^{n}\left[a_i\rho(U_i,U_j)\right]}{\sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n}\left[a_i a_k\rho(U_i,U_k)\right]}} \qquad (2.4.20)$$

or

$$u_i^* = \alpha_i \beta \tag{2.4.21}$$

where

$$\alpha_i = \frac{\sum_{j=1}^{n} \left[a_j \rho(U_i, U_j) \right]}{\sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} \left[a_j a_k \rho(U_j, U_k) \right]}} \tag{2.4.22}$$

2.4.5 Iterative Procedure

In general, any trial set of estimated values of u_i^* will not exactly satisfy the failure criterion as defined by the equation

$$h(\underset{\sim}{u}) = 0.0 \tag{2.4.23}$$

so therefore a correction factor δu_i^* is required to be applied to u_i^*.

If

$$h(u^*) = \Delta h \tag{2.4.24}$$

then

$$\sum_{i=1}^{n} \left[a_i \delta u_i^* \right] + \Delta h = 0.0 \tag{2.4.25}$$

If the value of β obtained from rearranging equation (2.4.21) for each value of i is β_i, so that

$$\beta_i = u_i^* / \alpha_i \tag{2.4.26}$$

then

$$\beta_i + \delta u_i^* / \alpha_i = \beta \tag{2.4.27}$$

So that

$$\sum_{i=1}^{n} \left[a_i \alpha_i (\beta - \beta_i) \right] + \Delta h = 0.0 \tag{2.4.28}$$

which can be rewritten as

$$\beta = \frac{\sum_{i=1}^{n} \left[a_i \alpha_i \beta_i \right] - \Delta h}{\sum_{i=1}^{n} \left[a_i \alpha_i \right]} \tag{2.4.29}$$

and the correction factors can be found from

$$\delta u_i^* = \alpha_i (\beta - \beta_i) \tag{2.4.30}$$

Similarly to the Mean Value Level 2 method, the "operational" failure probability is found from

$$p(F) = \Phi (-\beta) \tag{2.4.31}$$

2.4.5 Mutually Uncorrelated Basic Variables

The Advanced Level 2 method was originally developed for use with mutually uncorrelated basic variables. This simplifies some of the equations since

$$\rho(U_i, U_j) = \begin{cases} 0.0 & \text{if } i \neq j \\ 1.0 & \text{if } i = j \end{cases} \tag{2.4.32}$$

which implies that

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \left[a_j a_k \rho(U_j, U_k) \right] = \sum_{j=1}^{n} \left[a_j^2 \right] \tag{2.4.33}$$

and

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \left[\frac{\partial a_i}{\partial u_j} a_k \rho(U_j, U_k) \right] = \sum_{j=1}^{n} \left[\frac{\partial a_i}{\partial u_j} a_j \right] \tag{2.4.34}$$

An example of a failure criterion in three mutually independent variables is given in figure 2.4.2. The failure boundary is shown in one octant of the standardised normal space. The true probability of failure is the probability of any realisation $\underset{\sim}{u}$ occurring in the failure region, while the "operational" failure probability is the probability of $\underset{\sim}{u}$ being beyond the linearised failure boundary. Therefore any error will be due to the probability of $\underset{\sim}{u}$ lying between the true and linearised failure boundaries, which are tangential at the design point.

The additional error generated by using the mean point as the linearising point is illustrated in figure 2.4.3. The mean value linearisation of the failure criterion means that the approximate failure boundary may not even interesect the true failure boundary, so that the error can be several orders of magnitude larger than the Advanced Level 2 method.

The Level 2 methods can only be used to estimate the failure probability (and hence the reliability) for modes of failure whose failure surfaces do not have gross discontinuities close to the transformed origin. This limits the method to the determination of the reliability against individual local failures as a general case.

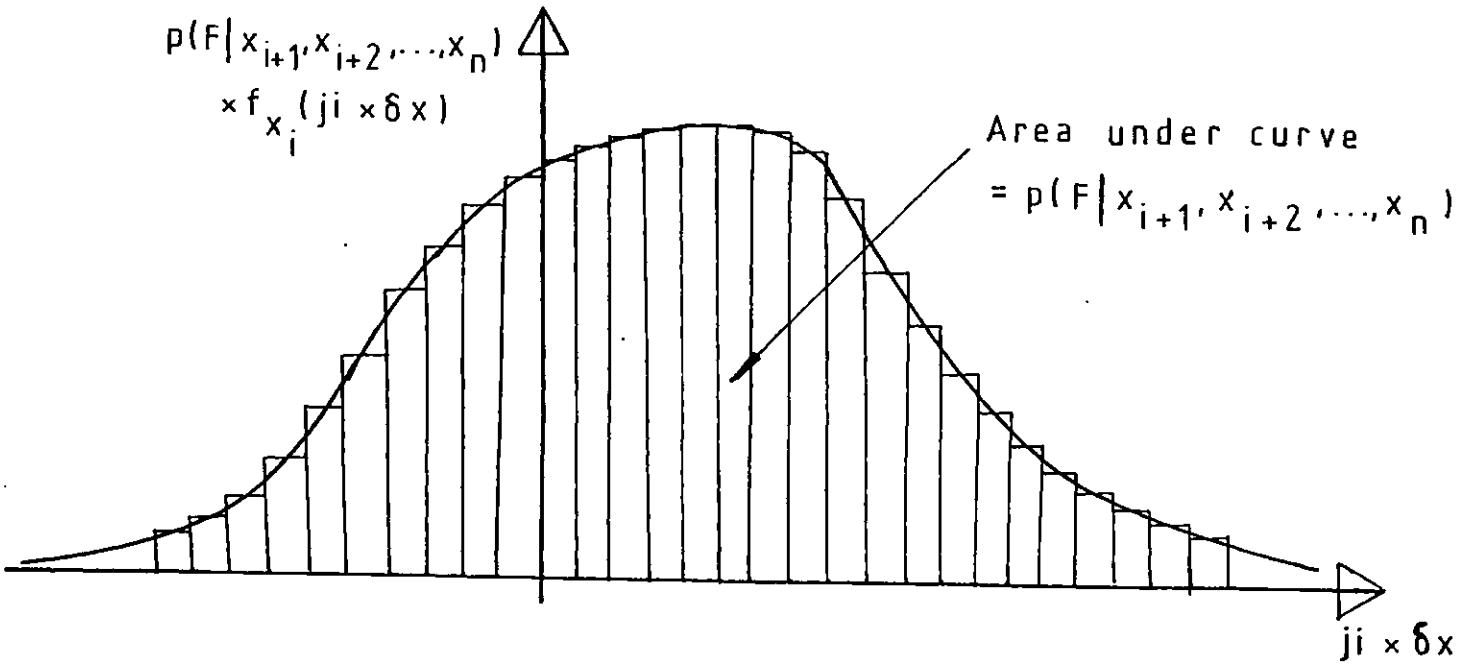


Fig. 2.3.1 - Contribution of Variable X_i to the Conditional Probability $p(F|x_{i+1},\ldots,x_n)$

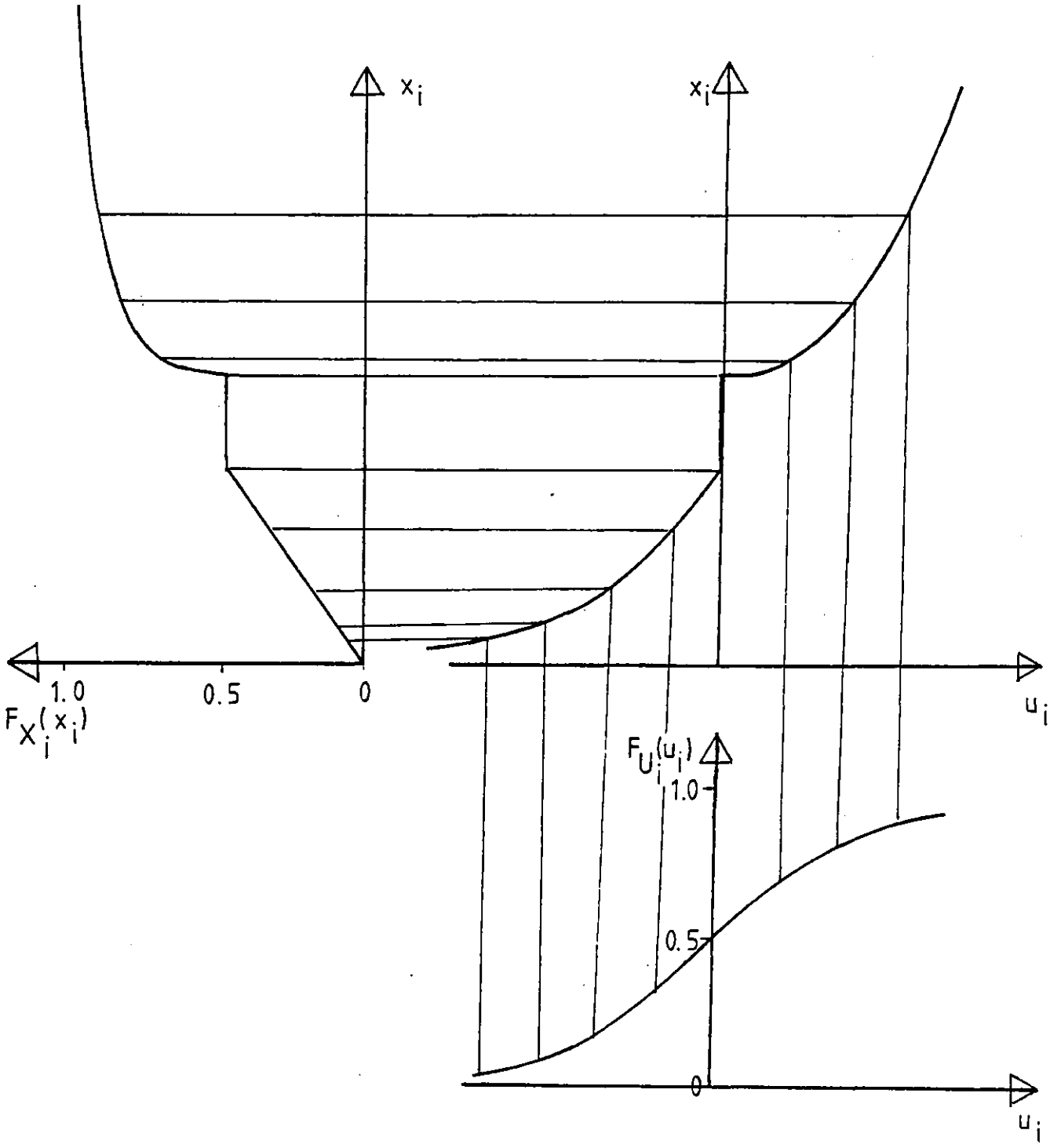


Fig. 2.4.1 - Example of Transformation of a Variable X_i into a Standardised Normal Variable U_i

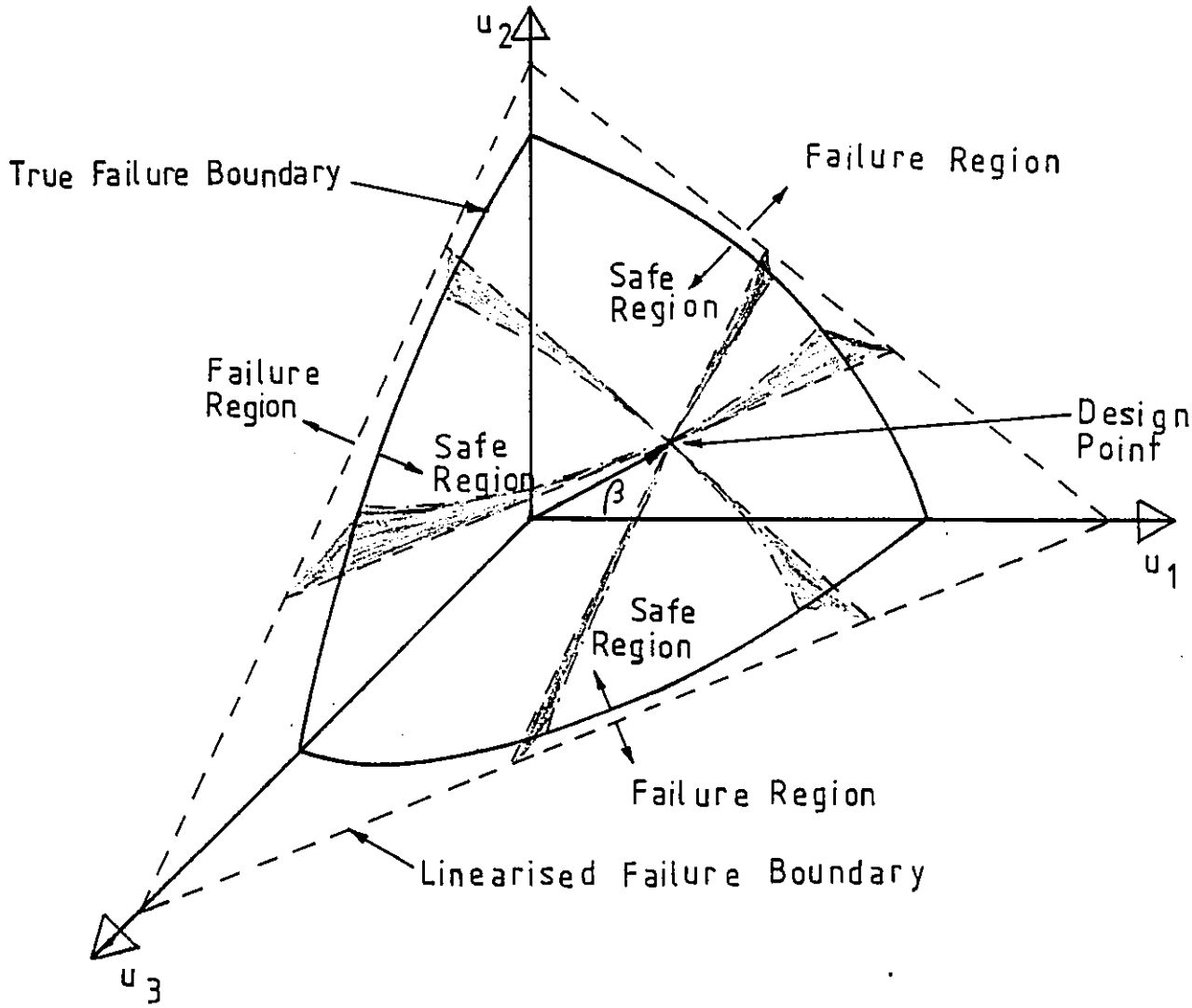


Fig. 2.4.2 - Three Variable Failure Boundary in Standardised Normal Space

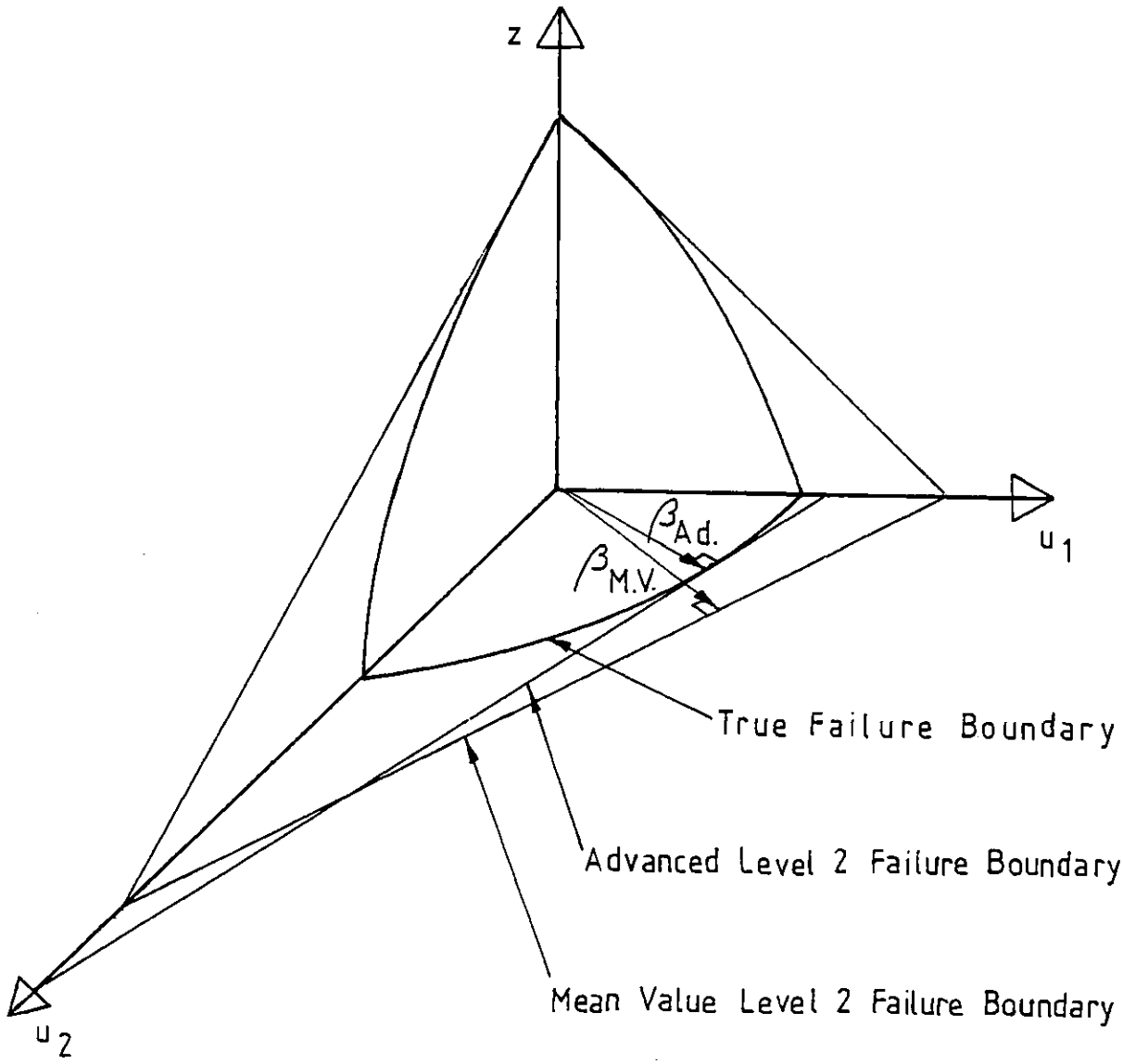


Fig.2.4.3 - Two Variable Failure Criterion in Standardised Normal Space

CHAPTER 3 - THE ACCURACY OF LEVEL 2 METHODS

3.1 EVALUATION OF LEVEL 2 ERRORS

3.1.1 Sources of Level 2 Errors

The Level 2 methods of structural reliability analysis (as described in section 2.4) involve making some approximations in order to evaluate the probability of failure. At present these techniques are widely used [5,11,14,25] without the quantitative effect of the simplifying assumptions having been fully established. Some work has been done by Baker [2] to examine the effect of these assumptions by comparing some failure probabilities obtained using the Advanced Level 2 method with those obtained using the Level 3 analysis by numerical integration. This chapter is an extension of that work.

The error in the failure probability calculated by the Advanced Level 2 method can be divided into that which:

i. is due to converging at the 'wrong' design point (ie not the closest point on the failure boundary to the origin in standardised normal space);

ii. is due to using inaccurate probabilistic models because of insufficient data; and

iii. is due to the probability content of the basic variable space between the true and linearised failure boundaries.

The error due to i. should be capable of elimination by refinement to the algorithm to ensure that the point found is not a local minimum, while any errors due to inaccurate probabilistic models are ignored since any error due to ii. will also affect the Level 3 control values. Therefore this study is primarily to examine the errors due to iii.

3.1.2 Examples Used

This study first examines the error in the relatively unimportant structural example of a rod in axial tension, both for the Advanced and Mean Value Level 2 methods, using 117 sets of probability distributions for the basic variables. Each rod was designed to one of four reliability levels (Section 3.2). Since this first study examined only one combination of untransformed basic variables, 108 arbitrary failure criteria, against each of which designs were carried out to three reliability levels, were then examined to check that the first set of results were not dependent on the form of that criterion. These arbitrary failure criteria had no structural meaning unless by pure coincidence (section 3.3). Finally a non-structural failure criterion which contained two potential design points was studied to examine the errors in a failure boundary which was grossly non-linear (sub-section 3.3.8).

Horne[23] has proposed a method of determining the error by assuming that the curvature components of the failure boundary are constant to infinity, and equal to the values at the design point. Since the important part of the failure region is that in the vicinity of the point of maximum probability density on the failure boundary, this assumption should not be too inaccurate provided that the curvature does not change significantly close to that region (for example when there is a discontinuity in the slope). However, direct comparison with the failure probability obtained by the Level 3 method is needed to find the precise error, and therefore that is used for this study.

3.1.3 Evaluation of Level 3 Probability

The method used for calculating the Level 3 failure probability was based on that used by Baker[2]. The failure criterion was transformed into standardised normal space, equation (2.4.9), and the basic variables were ranked in order of decreasing absolute values of α_i (as given by equation (2.4.22) in the Advanced Level 2 analysis). The failure probability conditional on all but two of the standardised normal variables having specified values can be calculated using the chi distribution[6]. The two non-specified variables were taken as the highest ranked pair. The chi distribution models the square root of the sum of the squares of ν independent standardised normal random variables, and therefore its cumulative distribution gives the failure probability of a hyper-spherical failure boundary at the relevant radius in ν dimensions, where ν are the number of degrees of freedom.

The chi probability density function is given by

$$f_Y(y) = \frac{y \cdot \left(\frac{y^2}{2}\right)^{\left(\frac{\nu}{2} - 1\right)} \cdot \exp\left(-\frac{y^2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \qquad (3.1.1)$$

where $\Gamma(.)$ is the gamma function. For $\nu = 2$ this simplifies to

$$f_Y(y) = \frac{y \cdot \left(\frac{y^2}{2}\right)^{0} \cdot \exp\left(-\frac{y^2}{2}\right)}{\Gamma(1)} = y \cdot \exp\left(-\frac{y^2}{2}\right) \qquad (3.1.2)$$

with the associated cumulative distribution function given by

$$F_Y(y) = \int_0^y y \cdot \exp\left(-\frac{y^2}{2}\right) dy = 1.0 - \exp\left(-\frac{y^2}{2}\right) \qquad (3.1.3)$$

Hence the probability, $p_{2\pi}$, of being outside a circle of radius y centred at the origin in two-dimensional standardised normal space is given by

$$p_{2\pi} = \exp(-\frac{y^2}{2}) \qquad (3.1.4)$$

and the probability of being outside that circle in a segment of Δ radians is given by

$$p_{\Delta} = \frac{\Delta}{2\pi} \cdot \exp(-\frac{y^2}{2}) \qquad (3.1.5)$$

Therefore the plane in standardised normal space which included the two most important variables was divided into segments of Δ radians, with the distance from the origin to the failure boundary in the i^{th} segment being y_i (figure 3.1.1). Therefore the failure probability is given by

$$p(F|u_3,u_4,\ldots,u_n) \doteq \frac{\Delta}{6\pi} \sum_{i=-k}^{l} \left[\exp(-\tfrac{1}{2} y_{(2i-1)}^2) + 4 \cdot \exp(-\tfrac{1}{2} y_{2i}^2) + \exp(-\tfrac{1}{2} y_{(2i+1)}^2) \right] \qquad (3.1.6)$$

The limits k and l correspond to those obtained by applying negative and positive incremental angles respectively to the line at an angle $\tan^{-1}(u_2^*/u_1^*)$ to the u_1 axis until the contribution to the probability from a segment in the incremental angle became insignificant, whilst ensuring that $(k+1)\Delta \leqslant \pi$ so that the contribution from a segment is not summed more than once.

The third and subsequent variables were incremented in steps and the probability, conditional on the values of the less sensitive variable, obtained using Simpson's Rule by

$$p(F|u_{i+1},u_{i+2},\ldots,u_n) = \frac{\delta u_i}{3\sqrt{2\pi}} \left[\sum_{j=-k_i}^{l_i} \left\{ \exp(-\frac{(2j-1)^2}{2} \cdot \delta u_i^2) \times p(F|u_i=(2j-1)\cdot\delta u_i,u_{i+1},\ldots,u_n) + 4 \cdot \exp(-j^2 \cdot \delta u_i^2) \times p(F|u_i=2j\cdot\delta u_i,u_{i+1},\ldots,u_n) + \exp(-\frac{(2j+1)^2}{2} \cdot \delta u_i^2) \times p(F|u_i=(2j+1)\cdot\delta u_i,u_{i+1},\ldots,u_n) \right\} \right] \qquad (3.1.7)$$

with the values of δu_i increasing with decreasing absolute value of α_i. The limits k_i and l_i correspond to these obtained by applying negative and positive increments respectively to u_i until the contribution to the probability from an increment was considered insignificant.

The unconditional probability of failure, p(F), was obtained from equation (3.1.7) when i=n, the number of basic variables in the failure criterion. The values used for the limits and increments were chosen following a set of trial calculations so as to give an accuracy of three decimal places in $\log_{10}$ (p(F)). This level of accuracy was considered sufficient for the control values.

3.1.4 Design and Analysis Procedure

The procedure used for each example was to obtain a design for which the probability of failure calculated by the Level 2 method was equal to the target, or design, probability. This was achieved by altering the mean of one of the basic variables in the failure criterion in an iterative process. The "true" failure probability was then calculated by the Level 3 analysis method described above.

For those cases in which the iterative procedure stopped before the value from the Level 2 analysis for $\log_{10}$ (failure probability) was within three decimal places of the target value, the Level 2 and 3 values were treated as being proportional to each other and adjusted accordingly. This adjustment did not change any failure probability by more than 2½%.

3.2 EXAMPLE OF ROD IN AXIAL TENSION

3.2.1 Form of Failure Criterion

The first failure criterion to be studied in order to assess the difference between the failure probability determined using the Level 2 methods and the "exact" value was the structurally simple problem of a cylindrical rod in axial tension. This has been used previously for other comparisons [2,11].
For the purposes of the present study the following assumptions were made:

i. The bar was a perfect cylinder
ii. Failure was defined by the onset of yielding
iii. The rod was perfectly homogeneous
iv. The load was applied axially
v. The diameter, yield stress, permanent load and live load were independent random variables.

Given these conditions the form of the failure criterion is given by

$$\mathcal{G} = g(D,Y,P,L) = \frac{\pi}{4} \times D^2 \times Y - P - L = 0.0 \qquad (3.2.1)$$

where

D = the diameter of the rod

Y = the yield stress of the rod

P = the permanent load applied to the rod

L = the live load applied to the rod

This was a convenient example with which to examine the errors due to the linearisation implicit in all Level 2 procedures since Z is a non-linear function of four basic variables.

The design of each rod was carried out by altering the mean diameter to give the target failure probability as found with the Advanced Level 2 method. This design was then analysed by the Level 3 method and the Mean Value Level 2 method. For the latter, the analysis was repeated with the form of the failure criterion rearranged from the load format of equation (3.2.1) to the stress format of

$$Z' = g(D,Y,P,L) = Y - \frac{4\times(P+L)}{\pi \times D^2} \qquad (3.2.2)$$

3.2.2 Advanced Level 2 Results

All four variables were treated as random variables only for the first seven cases considered (Table 3.2.1), with coefficients of variation from 0.05 to 0.25 for yield stress, 0.05 to 0.40 for both permanent and live loads and 0.242 to 0.414 for diameter; these ranges being used to cover extreme cases rather than reflect values that would normally be encountered. A target failure probability of 10^{-3} was used. The Mean Value Level 2 analysis was not applied to these particular cases.

The Advanced Level 2 method underestimated the probability of failure in each case, the furthest from the target value being $10^{-2.821}$. The failure boundary for this case was plotted in transformed space in six mutually orthogonal planes through the design point (figure 3.2.1), with all but two of the variables held constant for each plane (figure 3.2.2 in two dimensions). This shows that the failure boundary in four of these planes are concave when viewed from the safe region, one is approximately linear and one is convex. Of the four concave surfaces, the one which has the diameter and yield stress varying is the least linear and hence results in the greatest error.

For the remaining cases (Table 3.2.2), the coefficient of variation for the permanent load was kept at zero in order to economise on the computation

required. The other coefficients of variation ranged from 0.05 to 0.25 for yield stress, 0.05 to 1.00 for the live load and 0.034 to 0.606 for diameter. The target failure probabilities were set at 10^{-3}, 10^{-5}, 10^{-7} or 10^{-9}. The yield stress was modelled as having a normal distribution, so therefore its coefficient of variation had to always be less than $-1.0/\Phi^{-1}$ (target probability) in order to stop the probability of the yield stress theoretically having a negative value being greater than the target probability.

The Advanced Level 2 method generally underestimated the probability of failure for this failure criterion. The range of failure probabilities were $10^{-2.784}$ to $10^{-2.996}$, $10^{-4.696}$ to $10^{-4.996}$, $10^{-6.838}$ to $10^{-7.009}$ and $10^{-8.819}$ to $10^{-9.039}$ for the target probabilities 10^{-3}, 10^{-5}, 10^{-7} and 10^{-9} respectively. The values of the failure probability evaluated by the Level 3 method were plotted on a logarithmic scale in figure 3.2.3, demonstrating their lack of scatter from the target values. Cases from each end of the range for the target probabilities 10^{-3} and 10^{-9} were plotted in figures 3.2.4 - 3.2.7 in a similar manner to figure 3.2.1.

3.2.3 Shape of the Failure Boundaries

As might be expected, those cases which gave approximately the correct failure probability have much more linear failure boundaries than those which underestimated it, whose boundaries were significantly concave. In case 83, where the Advanced Level 2 method overestimated the failure probability, the transformed failure boundary in the plane with varying diameter and live load is convex, despite being concave in that plane for the other three cases plotted. These diagrams, therefore, demonstrate that:

i. A concave transformed failure boundary will cause the Advanced Level 2 analysis to underestimate the failure probability, and vice versa for a convex one;

ii. The shape of the failure boundary can change from convex to concave, or vice versa, with changes in the parameters of the distributions modelling the basic variables because of the transformation into standardised normal space.

The latter conclusion means that it will be difficult to generalise as to whether, or by how much, the Advanced Level 2 overestimates or underestimates the true failure probability, even for a given failure criterion with given probabilistic models for the basic variables.

The errors in $\log_{10}(p(F))$ did not alter significantly with increased reliability, so that the proportional error in Advanced Level 2 reliability analysis decreased with increasing reliability for this example. However, this is only the error conditional on the probabilistic models being correct, and the errors due to inaccuracies of these models could be expected to increase with increased reliability since the design point will be then further into the tails of the distributions modelling the basic variables. (The effect of the reliability level on the expected error is further discussed in section 3.3).

3.2.4 Mean Value Level 2 Results

The Mean Value Level 2 analysis gave two widely different failure probabilities with the two forms for the failure criterion. With the load form it tended to overestimate the true probability, while with the stress form it always underestimated it. This demonstrates the range of results that the Mean Value Level 2 method can give for the same problem due to its lack of invariance to the form of the failure criterion. Therefore great care must be taken if using this method to define a consistent form in which all failure criteria are written.

A set of rods were designed to a target probability of 10^{-5} by the Mean Value Level 2 method, but the exact failure probability was obtained by the Advanced Level 2 rather than the Level 3 method because the previous study had demonstrated that the Advanced Level 2 method is considerably more accurate than the Mean Value Level 2 method. The results are given in table 3.2.3 and figure 3.2.7. The extra calculation required to carry out the iterative procedures in the Advanced Level 2 method were shown to be fully justified for non-linear failure criteria since the error in the failure probability of these designs were several orders of magnitude, and they were not to a consistent pattern which would have been virtually cancelled out when using them for a calibration exercise. However, the Mean Value Level 2 method is used still quite widely[25] because of its simplicity.

It is suggested here that the Advanced Level 2 method should always be used in preference to the Mean Value Level 2 method. This is because, when the latter approach is adequate to give a good approximation to the failure probability, the former will only require a limited number of iterations to find the design point, and hence will not take significantly longer to evaluate the result.

3.3 EXAMPLES USING ARBITRARY FAILURE CRITERIA

3.3.1 Form of Failure Criteria

To extend the range of the comparison so as to eliminate any dependence on the form of the failure criterion, a set of failure criteria from a family of 16,731 possible arbitrary failure criteria with mutually uncorrelated variables were analysed. The general form of the failure criteria was given by

$$\mathcal{Z} = X_1^{n1} \circledast_1 X_2^{n2} \circledast_2 X_3^{n3} + k \qquad (3.3.1)$$

where

$\circledast_i$ = one of the operators +, −, x or ÷

n_i = fixed index

X_i = basic variable with fixed distribution type, mean and standard deviation

} randomly chosen within fixed bounds (Table 3.3.1)

k = constant, used to design the criterion to give failure probabilities of 10^{-3}, 10^{-5} and 10^{-7} by the Advanced Level 2 method.

These failure criteria were not intended to have any structural significance they were purely to give a wide range of possible combinations of basic variables and hence shapes of transformed failure boundaries.

The non-integer values of the indices were limited to those variables which have distributions that cannot have negative values. Any variable that is a denominator in a failure criterion was limited to the non-negative distribution types to eliminate the possibility of having an infinite as well as a zero failure boundary (as demonstrated in figure 3.3.1). The errors resulting from this type of failure criterion would have to be examined if any practical failure criterion were shown to exhibit such behaviour.

3.3.2 Design Point Error

Of the 108 different failure criteria that were analysed (Table 3.3.2), case No 26 resulted in the largest error. When the failure boundary was plotted in standardised normal space (figure 3.3.2), the curve was shown to have two local minima and hence two possible design points. The Advanced Level 2 failure probability had been obtained after iterating to the one further from the origin. When the iteration procedure was re-started near the correct, rather than the pseudo, design point each time, the iterative design procedure only succeeded for the target probability of 10^{-3}. However, even with two approximately equal possible values of β for this reliability level, the error in $\log_{10}(p(F))$ is less than 0.15 (see also Sub-sections 3.3.7 and 3.3.8).

3.3.3 Scatter of Results

The remaining trials were plotted in a histogram of failure probabilities as computed by Level 3 analysis (figure 3.3.3), for which the mean and standard deviations were calculated for the three reliability levels (table 3.3.3). These statistics demonstrate that the errors are not large for most cases, and that the tendency is for the Advanced Level 2 method to overestimate rather than underestimate the probability of failure for this family of failure criteria. (This is the opposite of what was observed with the single failure criterion of a rod in axial tension.) Further, the scatter does not appear to be dependent on the target probability of failure, although a larger sample would be required in order to confirm this.

3.3.4 Approximate Error Function

In order to examine the theoretical effect of the value of the target probability on the error, assume that the change of the constant only alters the distance between the origin and the design point in U-space (ie β). In two dimensions (which is considered for simplicity), this transformed boundary can be plotted with the line joining the origin to the design point as the x-axis and an orthogonal line to it through the origin as the y-axis (figure 3.3.4).

The true failure boundary passes through the point $(\beta, 0)$ and can be defined by

$$y = \begin{cases} f_1(x-\beta) & 0 \leqslant y \leqslant +\infty \\ f_2(x-\beta) & -\infty \leqslant y \leqslant 0 \end{cases} \qquad (3.3.2)$$

where $(x-\beta) < 0.0$ for a concave failure boundary and $(x-\beta) > 0.0$ for a convex one. The proportion of the probability content of any thin slice parallel to the y-axis which lies between the true and linearised failure boundaries is given by

$$r_e(x) = \Phi(-f_1(x-\beta)) + \Phi(f_2(x-\beta)) \qquad (3.3.3)$$

since the distribution of y is a marginal standard normal one. Therefore, for a convex failure boundary, the total error probability, p_e, is given by

$$p_e = \int_\beta^\infty r_e(x) * \phi(x)\, dx$$

$$= \int_\beta^\infty \left[\Phi(-f_1(x-\beta)) + \Phi(f_2(x-\beta)) \right] \phi(x)\, dx \qquad (3.3.4)$$

and for a concave failure boundary

$$p_e = \int_{-\infty}^\beta \left[\Phi(-f_1(\beta-x)) + \Phi(f_2(\beta-x)) \right] \phi(x)\, dx \qquad (3.3.5)$$

Assuming that the true failure boundary is approximately linear, $f_1(x)$ and $f_2(x)$ will be large unless $x \doteqdot 0.0$, so that $\Phi(-f_1(x)) + \Phi(f_2(x))$ will quickly tend to zero as x increases from zero. Therefore, since the majority of the integration will take place at $x \doteqdot \beta$, replace $\phi(x)$ by $\phi(\beta)$ so that equation (3.3.4) becomes

$$p_e \doteqdot \int_\beta^\infty \left[\Phi(-f_1(x-\beta)) + \Phi(f_2(x-\beta)) \right] \phi(\beta)\, dx$$

$$= \phi(\beta) \int_0^\infty \left[\Phi(-f_1(x-\beta)) + \Phi(f_2(x-\beta)) \right] d(x-\beta) \quad (3.3.6)$$

Moving the failure boundary relative to the origin means that β changes but $(x-\beta)$ remains the same. Therefore equation (3.3.6) implies that equation (3.3.4) (and similarly for equation (3.3.5)) can be written as

$$p_e \doteqdot \eta \cdot \phi(\beta) \quad (3.3.7)$$

where η is a constant, dependent only on the shape of the failure boundary near the design point since the integral is in terms of $(x-\beta)$. This result can similarly be obtained for more than two dimensions.

Hence the value obtained by Advanced Level 2 analysis for the probability of failure for failure boundaries of a given shape can be corrected by

$$\Phi(-\beta) = p(F) - \eta \times \phi(\beta) \quad (3.3.8)$$

Therefore

$$\log_e(\Phi(-\beta)) = \log_e\left[p(F) - \eta \times \phi(\beta)\right]$$

$$= \log_e(p(F)) + \log_e\left[1.0 - \eta \frac{\phi(\beta)}{p(F)}\right]$$

$$= \log_e(p(F)) - \eta \frac{\phi(\beta)}{p(F)} - \frac{\eta^2}{2}\left(\frac{\phi(\beta)}{p(F)}\right)^2 - \frac{\eta^3}{3}\left(\frac{\phi(\beta)}{p(F)}\right)^3 \quad (3.3.9)$$

But, since the error, $\eta \times \phi(x) \ll p(F)) \doteqdot \Phi(-\beta)$,

$$\log_e(\Phi(-\beta)) \doteqdot \log_e(p(F)) - \eta \frac{\phi(\beta)}{\Phi(-\beta)} \quad (3.3.10)$$

or

$$\log_{10}(\Phi(-\beta)) \doteqdot \log_{10}(p(F)) - \eta' \frac{\phi(\beta)}{\Phi(-\beta)} \quad (3.3.11)$$

3.3.5 Change in Expected Error with Probability Level

Equation (3.3.11) implies that, for an approximately linear failure boundary of a given shape, the error in $\log_{10}(p(F))$ as found by Advanced Level 2 analysis will be proportional to the ratio of the probability density to the cumulative probability of $-\beta$ (Table 3.3.4). This means that, if the shape of a boundary is maintained, the error increases but the proportional error decreases with increased reliability.

With the trial values obtained, dividing the mean errors in $\log_{10}(p(F))$ by $\phi(\beta)/\Phi(-\beta)$ for the three reliability levels produced very close results (0.0228, 0.0232 and 0.0229 for $p(F) = 10^{-3}$, 10^{-5} and 10^{-7} respectively). This confirms that the errors are proportional to this ratio, although the standard deviations calculated from the same results do not follow any such pattern.

3.3.6 Shape of Failure Boundaries

Two of the trials that are of particular interest are cases No 33 and 34. In case No 33 the error reduced markedly with reduced target failure probability. The failure boundary in standardised normal space (figure 3.3.5) is nearly parallel to the U_1, U_2 plane, and, since X_3 has a normal distribution, this implies that the failure criterion, $Z = X_1^3/X_2^6 + X_3 + k$, can be approximated by the linear equation $Z \doteq X_3 + k$ and that $X_1 \ll X_2^2$ near the design point. As the target failure probability is reduced, the value of u_3^* increases and the concave "warp" in the failure boundary must have a decreasing influence.

In case No 34 (figure 3.3.6) the error changes from an underestimation of the failure probability to an overestimation with reduced target probability. This results from the convex boundary in the U_1, U_3 plane remaining approximately the same distance from the origin in that plane, and hence contributing a similar error, while the concave boundary in the U_2, U_3 plane moves away from the origin in that plane with reduced target probability, and hence contributes a reduced error. (Both these error components would be proportional to $\phi(\sqrt{U_i^2 + U_j^2})$ in terms of equation (3.3.8).) Hence the one error is replaced by the other as the dominant error with changing reliability levels.

3.3.7 Design Point Errors

The linearisation of the failure boundary in standardised normal space at the closest, or design, point has been shown not to lead to gross errors for this family of arbitary failure criteria when estimating the probability of failure. However, when there are multiple local minima, as in case No 26, the choice of the true, rather than a pseudo, design point can be very critical. In order to reduce the probability of the wrong design point being found either:

i. the Advanced Level 2 iteration procedure can be started from several different points, and the resultant values of β compared to find the minima when they differ; or

ii. points on the hyper-sphere formed by points equidistant with the assumed design point from the origin in standardised normal space can be checked to see if they are in the failure region, and the Advanced Level 2 iteration procedure restarted from that point if it is in the failure region.

An excessive number of such checks would be required to be entirely certain that the point found is the design point. It may be possible to reduce this with experience of the particular failure criterion, so that the checks are carried out in regions of the variable space where the curvature is such that any local minima are most likely to exist. Two possible examples are:

i. When there is a discontinuity of slope due to a change in the basis of the failure criterion (eg in an RC beam when the reinforcing bars move from the plastic to the elastic portion of the stress-strain curve, and vice versa); and

ii. When the distribution modelling a basic variable, before being transformed into a standardised normal one, has a step in its probability density function (eg when the yield stress is modelled by a lognormal distribution, but below a particular value the probability density is reduced to a fraction of the normal value (with the complete distribution re-normalised) to represent quality control procedures).

3.3.8 Multiple Local Minima

An arbitrary failure criterion in two variables was therefore analysed, in which both variables had normal distributions with unit standard deviations, to examine the effect of multiple local minima when the correct design point is found. The failure criterion used was the cubic equation

$$\mathcal{Z} = g(X_1, X_2) = X_1^3 + 4.5X_1^2 + 5.3X_1 - 1.2X_2 + 5.7 \qquad (3.3.12)$$

Since both variables had normal distributions, any change in their mean values only moved the origin of the transformed surface without affecting its shape. The failure boundary (figure 3.3.7) had a definite point of inflection and two localised minimum distances to most of the points used as transformed origins.

This is again a non-structural failure criteria. Furthermore this type of curve is unlikely to occur to such a marked extent in any genuine failure criterion model since it is not monotonic. In general the significant basic variables will be either loading variables (and therefore any increase in value will cause a decrease in the failure criterion's value) or resisting variables (and therefore any increase in value will cause an increase in the failure criterion's value).

The particular computer program used for the Advanced Level 2 method failed to find the design point for a set of the transformed origins, and for three of these the program used for the Level 3 method also failed to calculate the failure probability. For the remaining transformed origins the apparent error

in $\log_{10}$ (failure probability) calculated by the Advanced Level 2 method is never more than about double or less than half that calculated by the Level 3 method.

The results showed that once the origin was a reasonable distance (about 2 standard deviations) from those parts of the failure boundary with sharp curvature, the accuracy of the Level 2 analysis was within acceptable agreement of the Level 3 results for such values as were obtained.

3.3.9 Extension of Present Work

If multiple design points are found or sharp curvature is suspected, then a possible method to find how "sharp" that curvature is near to a design point in any plane U_i, U_j is to increase u_i^* by δu_i and find the value of u_j, u_{j+}, which satisfies the failure criterion given that all other values are at their design point values. Repeating this procedure using $u_i^* - \delta u_i$ to find u_{j-} allows the curvature in that plane at the design point to be approximated by

$$\frac{\partial^2 u_j}{\partial u_i^2} \doteq \frac{(u_{j+} - 2u_j^* + u_{j-})}{\delta u_i^2} \qquad (3.3.13)$$

The limiting absolute value of $\frac{\partial^2 u_j}{\partial u_i^2}$, above which gross errors in the Level 2 reliability analysis are likely, will be a function of β, but to establish this function is outside the scope of this study. The relevant size of δu_i may not necessarily be small since the sharp curvature causing an error may not occur at the design point (eg with transformed origin (-2.0,4.0) in the example above, or if there is a complete discontinuity of slope).

In modelling real structural members, where there are likely to be a large number of basic variables, the feasibility of checking for any such marked non-linear behaviour of the transformed failure surface will be small. However, error is most likely to occur when there are multiple local minima with similar values of β but with widely differing $\underset{\sim}{\alpha}$ vectors, where the variables with the most altered α_i values should be examined first.

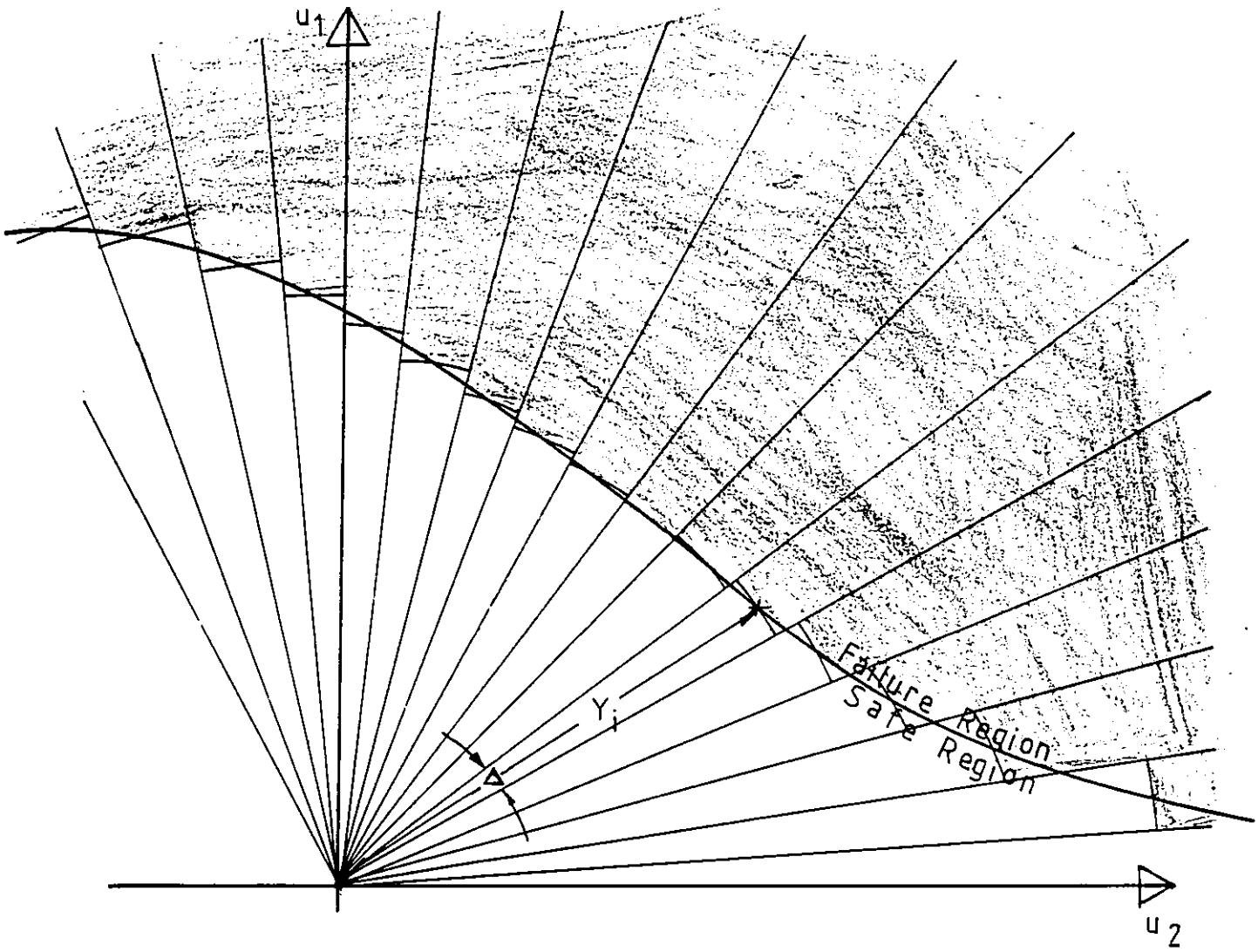


Fig. 3.1.1 - Failure Boundary in Standardised Normal Space for the First Two Variables

Variable	Mean	Distribution Type
Diameter	Varies	Lognormal
Yield Stress	290.0	Normal
Permanent Load	30000.0	Normal
Live Load	50000.0	Extreme Type I Maxima

Log_{10} (Design Failure Probability) =-3.0

Case No	Coefficients of Variation				Diameter Mean	Log_{10} (Failure Probability)
	Yield Stress	Permanent Load	Live Load	Diameter		
1	0.05	0.05	0.10	0.271	44.36	- 2.981
2	0.05	0.05	0.40	0.260	46.10	- 2.985
3	0.05	0.40	0.40	0.250	46.49	- 2.975
4	0.25	0.40	0.10	0.242	49.58	- 2.821
5	0.05	0.05	0.40	0.414	72.36	- 2.990
6	0.25	0.05	0.05	0.407	73.73	- 2.836
7	0.25	0.40	0.40	0.398	75.41	- 2.832

TABLE 3.2.1 – Advanced Level 2 Comparison with Four Variables

Case No	Coefficients of Variation			Diameter Mean	$\log_{10}$ (Failure Probability)		
					Exact	Mean Value Level 2	
	Yield Stress	Live Load	Diameter			(Load)	(Stress)
8	0.05	0.05	0.064	23.34	- 2.922	- 2.269	- 3.820
9	0.05	0.10	0.063	23.73	- 2.905	- 2.465	- 4.188
10	0.05	0.40	0.052	29.02	- 2.994	- 4.565	- 7.104
11	0.25	0.05	0.038	39.42	- 2.996	- 2.818	- 3.000
12	0.25	0.40	0.037	40.40	- 2.926	- 2.856	- 3.028
13	0.05	0.25	0.105	28.66	- 2.960	- 2.318	- 6.334
14	0.10	1.00	0.074	40.33	- 2.994	- 3.990	- 6.587
15	0.25	0.25	0.075	40.19	- 2.939	- 2.444	- 3.013
16	0.05	0.05	0.207	36.22	- 2.980	- 1.389	- 8.476
17	0.05	0.40	0.195	38.55	- 2.979	- 1.583	- 10.452
18	0.10	0.10	0.205	36.62	- 2.945	- 1.387	- 6.307
19	0.10	0.40	0.193	38.88	- 2.962	- 1.564	- 7.333
20	0.25	0.10	0.173	43.45	- 2.893	- 1.550	- 3.065
21	0.05	0.05	0.271	44.38	- 2.987	- 1.180	- 12.336
22	0.05	0.40	0.260	46.10	- 2.977	- 1.264	- 14.246
23	0.10	0.10	0.269	44.63	- 2.960	- 1.178	- 8.387
24	0.25	0.05	0.246	48.86	- 2.821	- 1.214	- 3.215
25	0.25	0.40	0.239	50.21	- 2.784	- 1.263	- 3.302
26	0.05	0.05	0.422	71.14	- 2.995	- 0.885	- 28.309
27	0.10	0.40	0.413	72.59	- 2.970	- 0.901	- 14.737
28	0.05	0.10	0.521	96.04	- 2.991	- 0.775	- 43.422
29	0.10	0.40	0.514	97.36	- 2.975	- 0.782	- 17.857
30	0.05	0.05	0.606	123.68	- 2.992	- 0.704	- 56.406
31	0.10	0.40	0.600	124.91	- 2.979	- 0.712	- 19.782

TABLE 3.2.2(i) - Rod in Axial Tension with Design $p(F) = 10^{-3}$

Case No	Coefficients of Variation			Diameter Mean	$\log_{10}$ (Failure Probability)		
					Exact	Mean Value Level 2	
	Yield Stress	Live Load	Diameter			(Load)	(Stress)
32	0.05	0.05	0.060	25.00	- 4.876	- 3.385	- 6.869
33	0.05	0.40	0.044	34.48	- 4.996	- 8.866	- 16.103
34	0.10	0.05	0.056	26.79	- 4.881	- 3.451	- 5.368
35	0.10	0.40	0.042	35.53	- 4.968	- 6.652	- 9.269
36	0.05	0.25	0.091	33.10	- 4.980	- 3.702	- 14.350
37	0.10	0.05	0.098	30.58	- 4.894	- 2.620	- 6.512
38	0.10	0.40	0.080	37.72	- 4.974	- 4.293	- 9.866
39	0.20	0.10	0.061	49.06	- 4.976	- 3.897	- 5.008
40	0.05	0.40	0.163	46.08	- 4.995	- 2.249	- 24.035
41	0.10	0.10	0.178	42.43	- 4.908	- 1.816	- 10.294
42	0.10	0.40	0.161	46.71	- 4.971	- 2.200	- 12.298
43	0.05	0.05	0.233	51.57	- 4.975	- 1.497	- 26.000
44	0.05	0.40	0.218	55.41	- 4.985	- 1.686	- 31.643
45	0.10	0.05	0.230	52.08	- 4.955	- 1.488	- 13.003
46	0.10	0.40	0.214	55.95	- 4.965	- 1.669	- 14.479
47	0.20	0.40	0.198	60.51	- 4.696	- 1.690	- 5.299
48	0.10	0.05	0.349	85.85	- 4.974	- 1.068	- 18.564
49	0.10	0.40	0.338	88.82	- 4.959	- 1.111	- 19.096
50	0.20	0.10	0.339	88.56	- 4.769	- 1.070	- 5.860
51	0.05	0.10	0.427	117.14	- 4.979	- 0.919	- 67.500
52	0.10	0.40	0.416	120.32	- 4.961	- 0.932	- 20.910
53	0.05	0.05	0.492	152.48	- 4.993	- 0.827	- 76.295
54	0.05	0.40	0.482	155.69	- 4.965	- 0.839	- 21.824
55	0.20	0.40	0.474	158.14	- 4.872	- 0.837	- 6.320

TABLE 3.2.2(ii) – Rod in Axial Tension with Design $p(F) = 10^{-5}$

Case No	Coefficients of Variation			Diameter Mean	Log_{10} (Failure Probability)		
					Exact	Mean Value Level 2	
	Yield Stress	Live Load	Diameter			(Load)	(Stress)
56	0.05	0.05	0.057	26.35	- 6.838	- 4.400	- 10.060
57	0.05	0.40	0.038	39.23	- 6.996	- 13.322	- 25.852
58	0.10	0.05	0.052	29.08	- 6.862	- 4.603	- 7.353
59	0.10	0.40	0.037	40.90	- 6.960	- 9.041	- 12.370
60	0.05	0.25	0.081	36.99	- 6.993	- 5.090	- 23.595
61	0.10	0.05	0.090	33.17	- 6.855	- 3.286	- 8.640
62	0.10	0.40	0.069	43.31	- 6.968	- 5.792	- 13.019
63	0.05	0.40	0.143	52.46	- 7.009	- 2.880	- 37.522
64	0.10	0.10	0.162	46.44	- 6.869	- 2.172	- 12.984
65	0.10	0.40	0.140	53.46	- 6.978	- 2.793	- 15.405
66	0.05	0.05	0.270	57.06	- 6.963	- 1.754	- 38.025
67	0.05	0.40	0.190	63.14	- 7.002	- 2.074	- 46.722
68	0.10	0.05	0.208	57.80	- 6.934	- 1.743	- 15.646
69	0.10	0.40	0.187	64.02	- 6.975	- 2.045	- 17.276
70	0.05	0.40	0.296	101.24	- 6.983	- 1.301	- 68.935
71	0.10	0.40	0.294	101.96	- 6.961	- 1.296	- 20.653
72	0.05	0.10	0.375	133.20	- 6.966	- 1.037	- 76.716
73	0.10	0.40	0.361	138.64	- 6.958	- 1.073	- 21.804
74	0.05	0.05	0.431	174.72	- 6.988	- 0.922	- 82.121
75	0.10	0.40	0.417	180.05	- 6.958	- 0.949	- 22.353

TABLE 3.2.2(iii) – Rod in Axial Tension with Design $p(F) = 10^{-7}$

Case No	Coefficient of Variation			Diameter Mean	$\log_{10}$ (Failure Probability)		
					Exact	Mean Value Level 2	
	Yield Stress	Live Load	Diameter			(Load)	(Stress)
76	0.05	0.05	0.054	27.55	- 8.827	- 5.377	- 13.363
77	0.05	0.40	0.034	43.51	- 8.997	- 17.633	- 34.869
78	0.10	0.05	0.048	31.49	- 8.860	- 5.814	- 9.273
79	0.10	0.40	0.033	45.97	- 8.949	- 11.020	- 14.616
80	0.05	0.25	0.074	40.53	- 9.000	- 6.466	- 32.514
81	0.10	0.05	0.084	35.59	- 8.819	- 3.926	- 10.451
82	0.10	0.40	0.062	48.53	- 8.958	- 7.160	- 15.213
83	0.05	0.40	0.129	58.16	- 9.039	- 3.491	- 48.468
84	0.10	0.10	0.150	50.16	- 8.830	- 2.491	- 14.814
85	0.10	0.40	0.126	59.60	- 8.977	- 3.359	- 17.310
86	0.05	0.05	0.195	61.56	- 8.951	- 1.977	- 47.396
87	0.05	0.40	0.171	70.05	- 9.015	- 2.447	- 57.554
88	0.10	0.05	0.192	62.61	- 8.911	- 1.965	- 17.235
89	0.10	0.40	0.168	71.26	- 8.981	- 2.404	- 18.839
90	0.05	0.05	0.285	105.17	- 8.971	- 1.354	- 72.319
91	0.10	0.40	0.264	113.51	- 8.971	- 1.467	- 21.378
92	0.05	0.10	0.341	146.53	- 8.954	- 1.140	- 80.809
93	0.10	0.40	0.324	154.56	- 8.964	- 1.198	- 22.192

TABLE 3.2.2(iv) – Rod in Axial Tension with Design $p(F) = 10^{-9}$

Case No	Coefficients of Variation		Load Format			Stress Format		
			Diameter		Log_{10} (Failure Probability)	Diameter		Log_{10} (Failure Probability)
	Yield Stress	Live Load	Mean	C of V		Mean	C of V	
94	0.05	0.05	27.07	0.055	- 8.20	24.05	0.062	- 3.79
95	0.05	0.40	29.64	0.051	- 3.22	27.26	0.055	- 2.43
96	0.10	0.05	29.84	0.050	- 7.66	26.36	0.057	- 4.64
97	0.10	0.40	32.01	0.047	- 3.79	29.26	0.051	- 2.90
98	0.05	0.25	36.73	0.082	- 6.87	27.63	0.109	- 2.59
99	0.10	0.05	39.53	0.076	- 12.07	28.76	0.104	- 3.77
100	0.10	0.40	40.33	0.074	- 5.92	30.59	0.098	- 2.76
101	0.20	0.10	69.81	0.043	- 5.76	49.38	0.061	- 5.00
102	0.05	0.40	70.58	0.106	- 14.06	33.78	0.222	- 1.98
103	0.10	0.10	76.27	0.098	- 20.15	34.74	0.216	- 2.47
104	0.10	0.40	76.25	0.098	- 15.04	35.33	0.212	- 2.24
105	0.05	0.05	107.94	0.111	- 2.50	37.96	0.316	- 1.78
106	0.05	0.40	107.87	0.111	- 23.55	38.12	0.315	- 1.71
107	0.10	0.05	116.64	0.103	- 21.92	39.44	0.304	- 2.00
108	0.10	0.40	116.55	0.103	- 21.84	39.62	0.303	- 1.89
109	0.20	0.40	202.22	0.059	- 6.45	55.49	0.216	- 4.25
110	0.10	0.05	283.80	0.106	- 22.92	53.33	0.563	- 1.38
111	0.10	0.40	283.74	0.106	- 22.92	53.03	0.566	- 1.35
112	0.20	0.10	492.55	0.061	- 6.53	68.23	0.440	- 2.56
113	0.05	0.10	436.20	0.115	- 31.25	63.34	0.789	- 1.10
114	0.10	0.40	471.34	0.106	- 23.04	64.67	0.773	- 1.15
115	0.05	0.05	653.60	0.115	- 71.31	75.62	0.992	- 0.99
116	0.10	0.40	706.27	0.106	- 23.08	76.90	0.975	- 1.04
117	0.20	0.40	1225.81	0.061	- 6.54	93.37	0.803	- 1.55

TABLE 3.2.3 - Rod in Axial Tension with Design $p(F) = 10^{-5}$ by Level 2 Mean Value Method

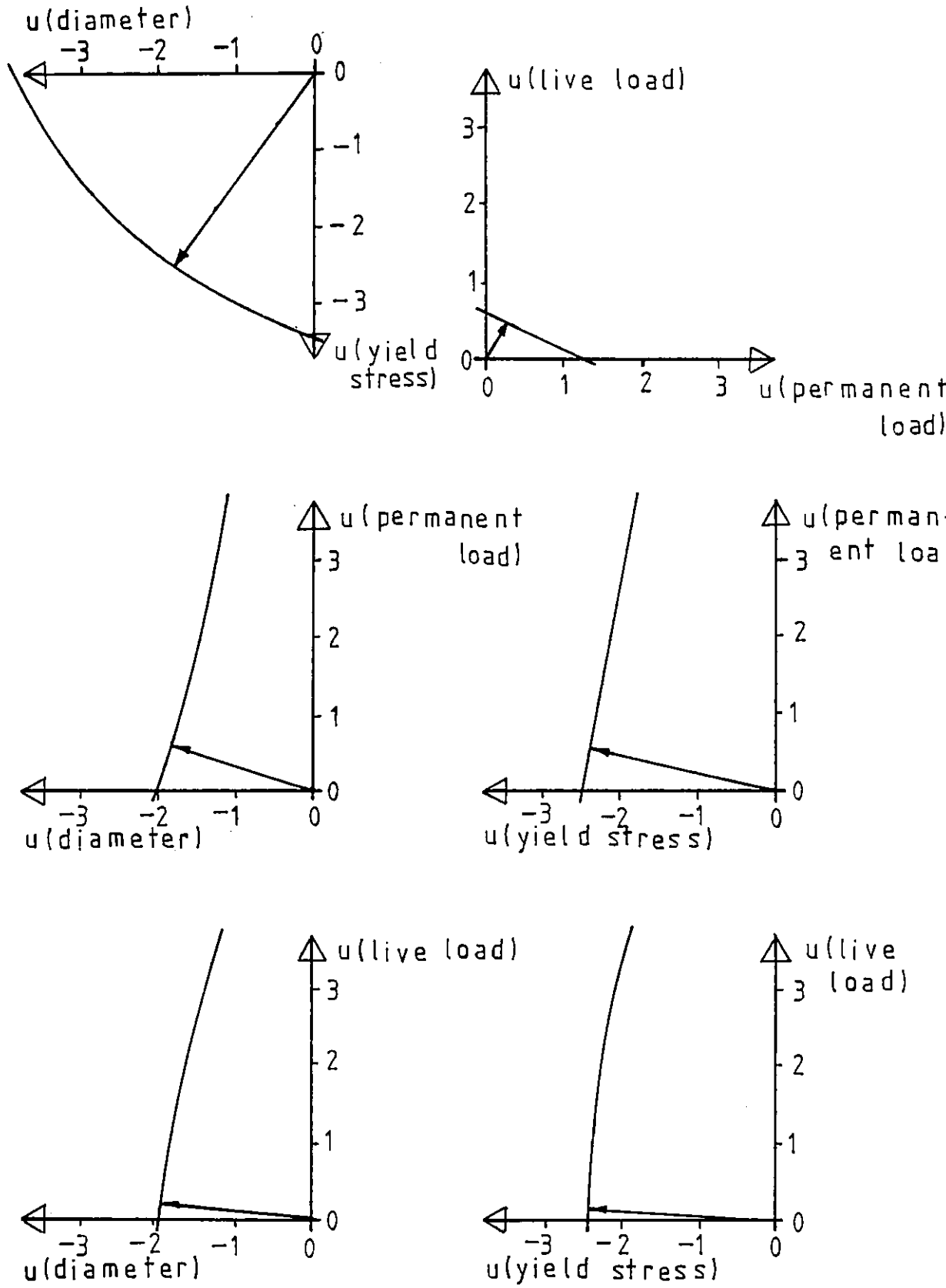


Fig. 3.2.1 - Failure Boundary for Case 4 Plotted in Standardised Normal Space

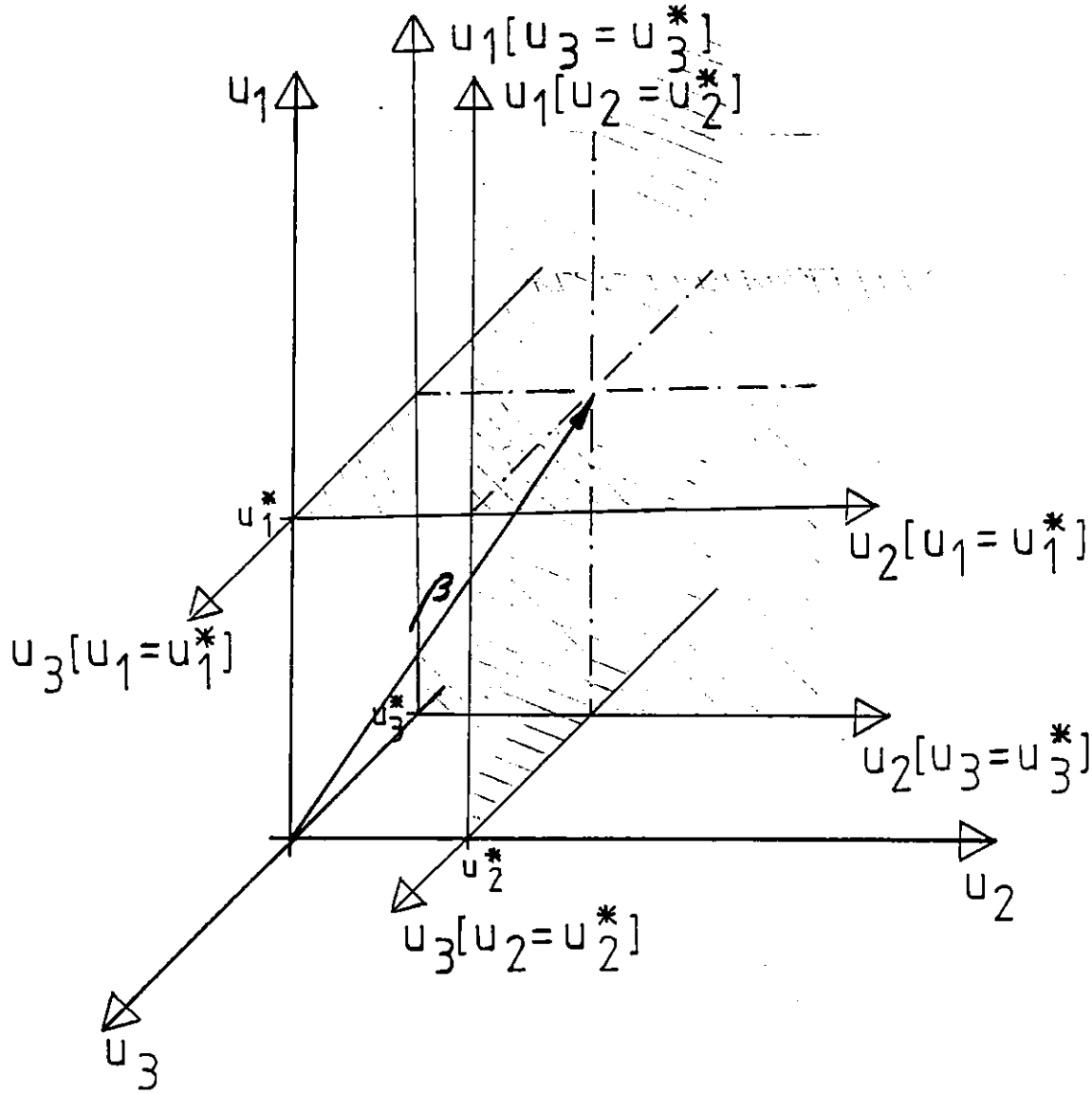


Fig. 3.2.2 - Planes Used for the Plotting a Three Dimensional Failure Boundary

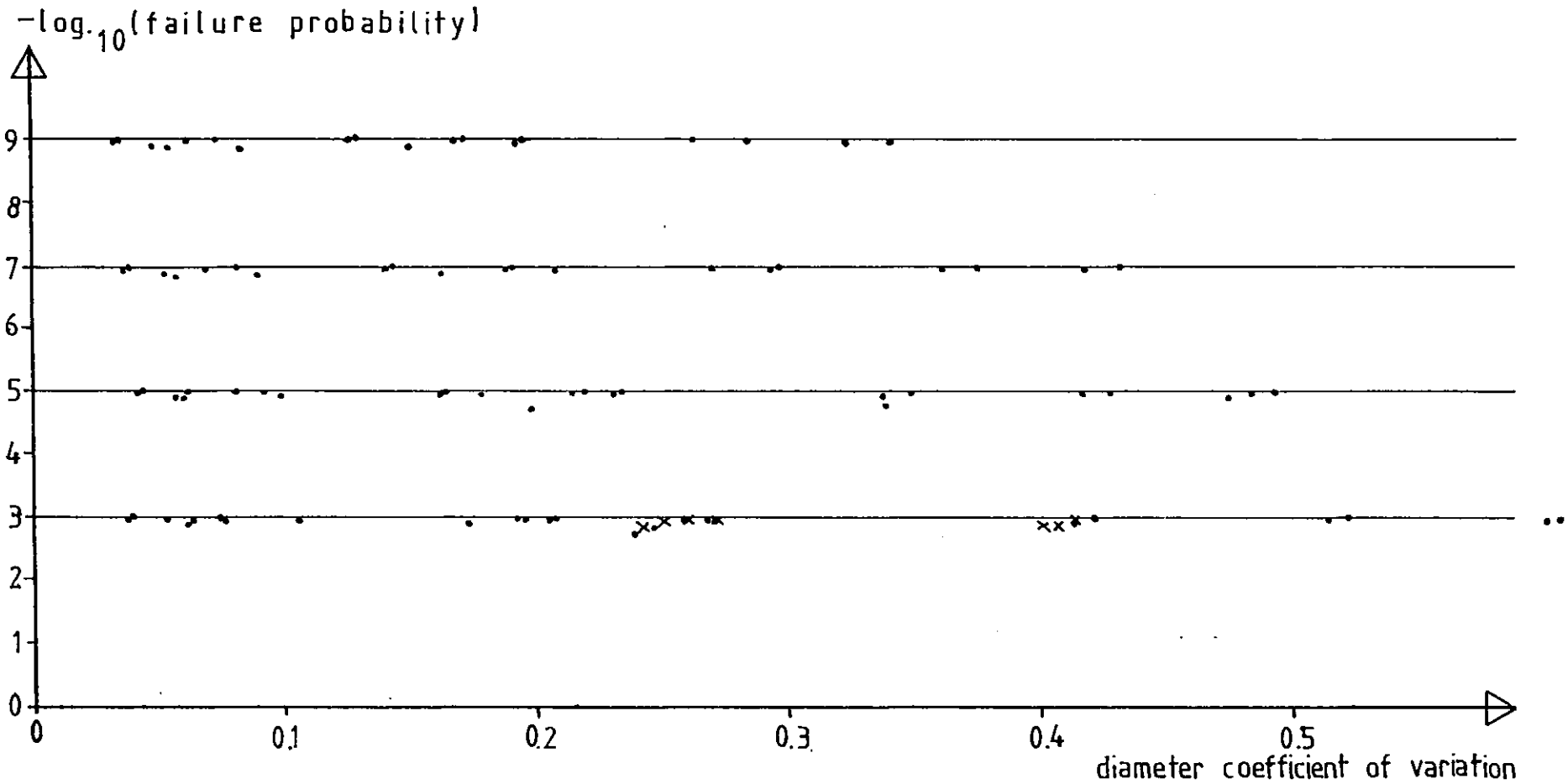


Fig. 3.2.3 - Exact Failure Probabilities for Rods Designed by Advanced Level 2 Method

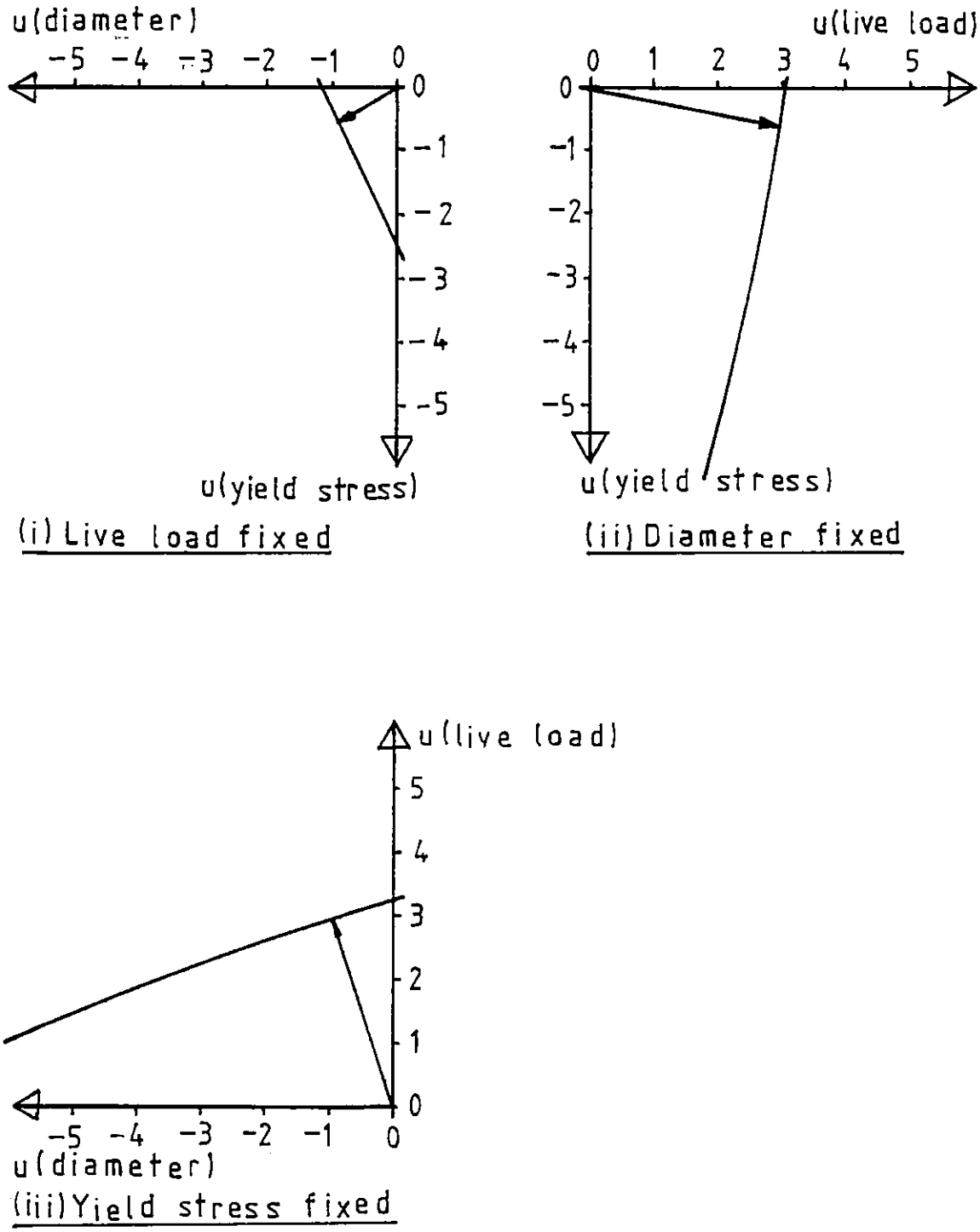
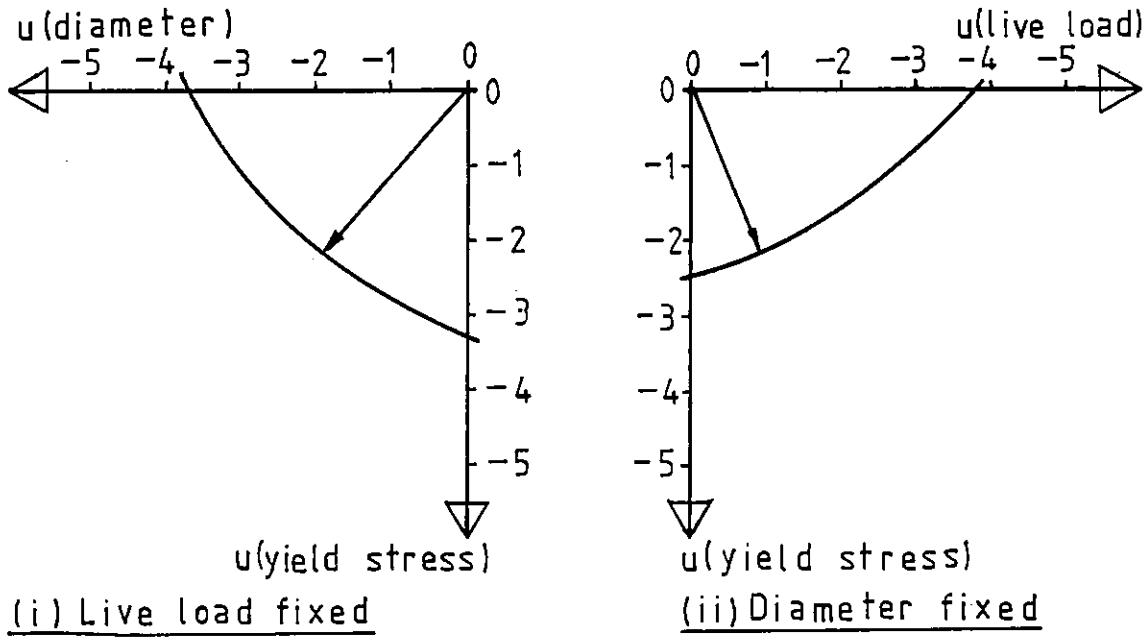


Fig. 3.2.4 - Failure Boundary for Case 10 Plotted in Standardised Normal Space



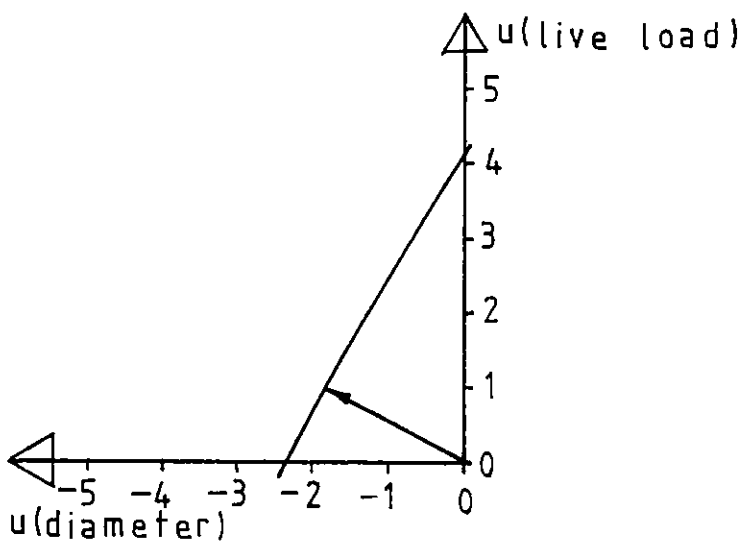
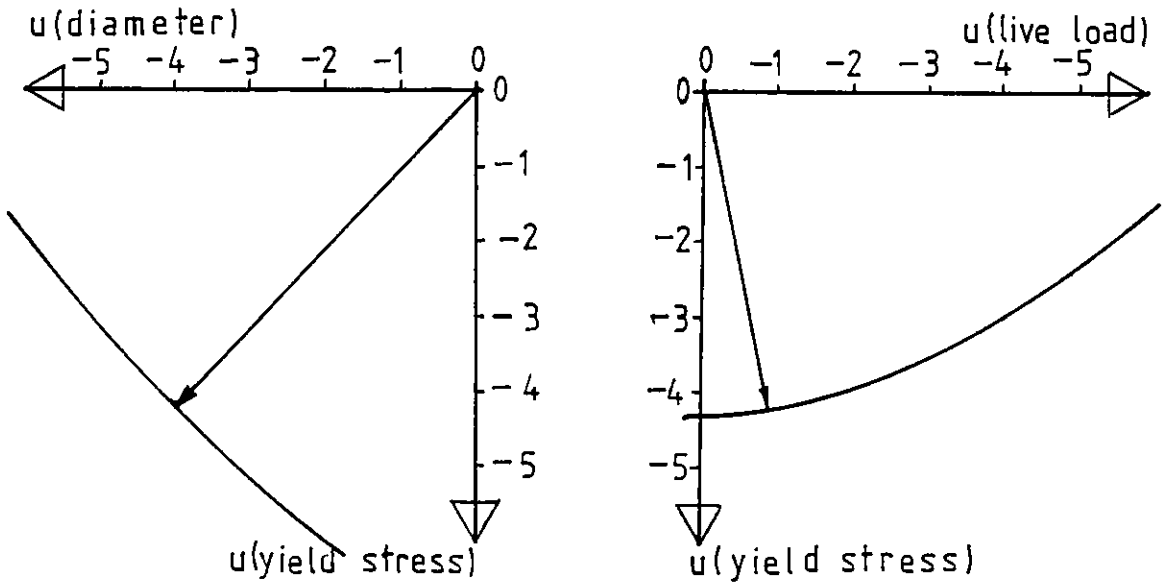
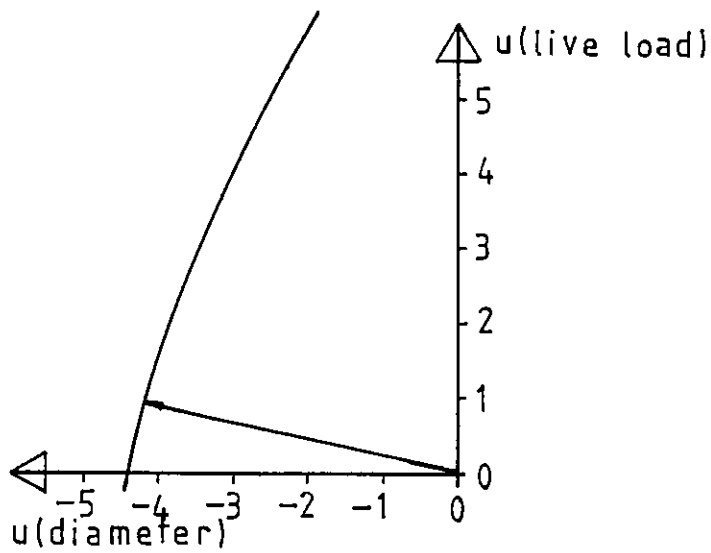


Fig.3.2.5 - Failure Boundary for Case 25 Plotted in Standardised Normal Space



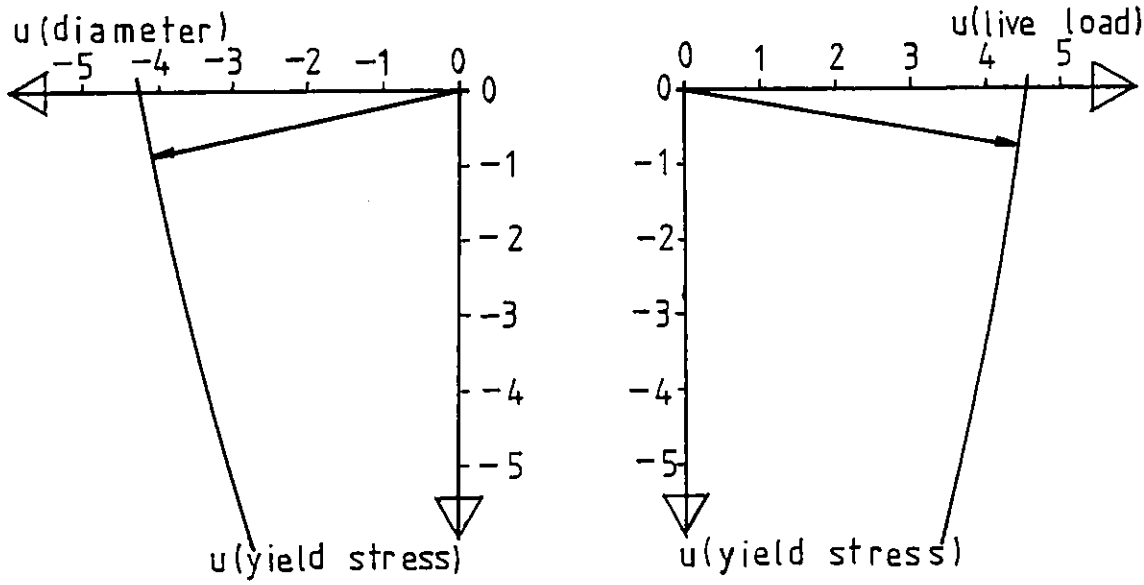
(i) Live load fixed

(ii) Diameter fixed



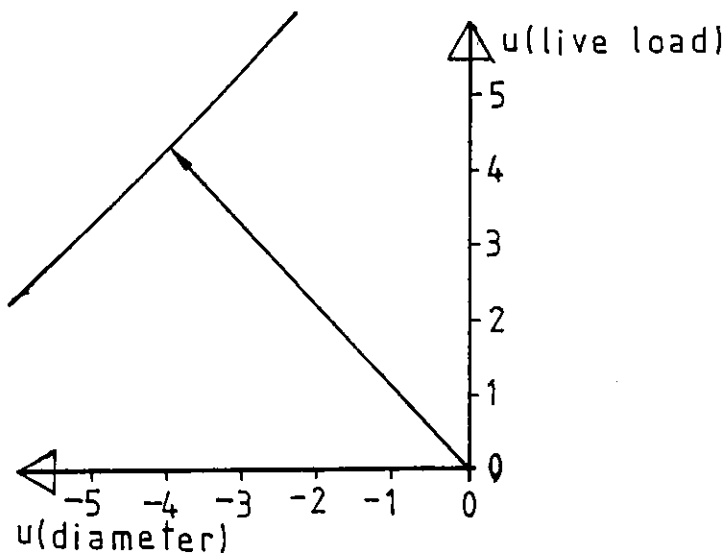
(iii) Yield stress fixed

Fig.3.2.6 - Failure Boundary for Case 81 Plotted in Standardised Normal Space



(i) Live load fixed

(ii) Diameter fixed



(iii) Yield stress fixed

Fig. 3.2.7 - Failure Boundary for Case 83 Plotted in Standardised Normal Space

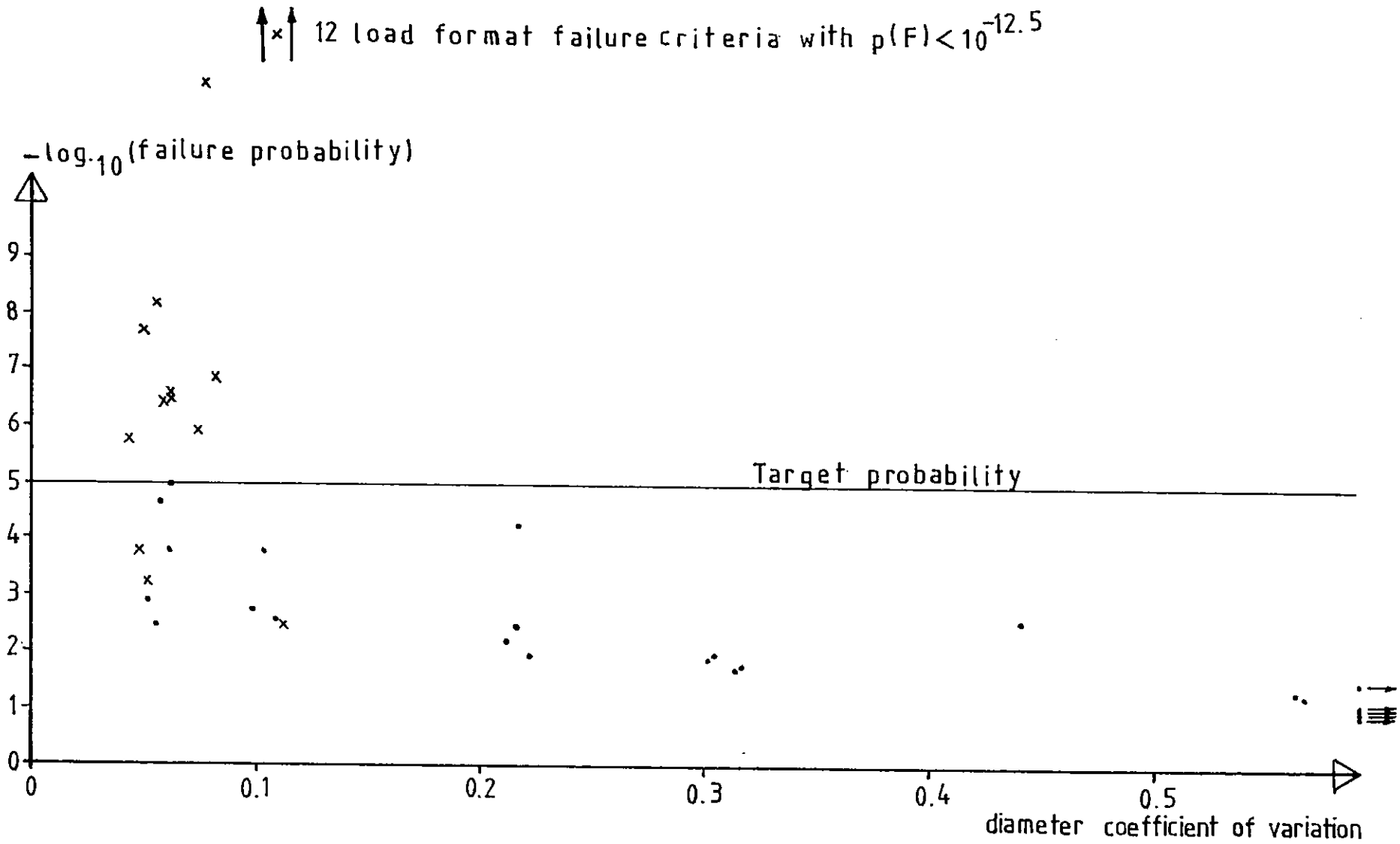


Fig. 3.2.8 - Exact Failure Probabilities for Rods Designed by Mean Value Level 2 Method

FAILURE CRITERION: $Z = X_1^{n1} \circledast_1 X_2^{n2} \circledast_2 X_3^{n3} + K$

SIGNS: $p(\circledast_i = +) = p(\circledast_i = -) = p(\circledast_i = \times) = p(\circledast_i = \div) = 0.25$

Distribution Type for X_i	p(Distribution Type for X_i) if		p(n_i = m | Distribution Type for X_i) where		
	i=1 or i=2 | $\circledast_1 \neq \div$ or i=3 | $\circledast_2 \neq \div$	i=2 | $\circledast_1 = \div$ or i=3 | $\circledast_2 = \div$	m=1/6, 1/5, 1/4, 1/3 or 1/2	m=1	m=2,3,4,5 or 6
Normal	0.20	0.0	0.0	0.1667	0.1667
Lognormal	0.20	0.40	0.0833	0.1667	0.0833
Extreme Type I Maxima	0.10	0.0	0.0	0.1667	0.1667
Extreme Type I Minima	0.10	0.0	0.0	0.1667	0.1667
Extreme Type II Maxima	0.10	0.20	0.0833	0.1667	0.0833
Extreme Type III Minima	0.10	0.20	0.0833	0.1667	0.0833
Rectangular	0.10	0.0	0.0	0.1667	0.1667
Exponential	0.10	0.20	0.0833	0.1667	0.0833

DISTRIBUTION TYPES AND INDICES

MEANS: $f_{E(X_i)}(E(X_i)) = \frac{1}{9.0}$, $1.0 \leqslant E(X_i) < 10.0$

STANDARD DEVIATIONS: $f_{\sigma(X_i)}(\sigma(X_i)) = \frac{4.0}{E(X_i)}$, $0.05E(X_i) \leqslant (X_i) < 0.30E(X_i)$

NB: If distribution type for X_i is exponential, reset $E(X_i) = \sigma(X_i)$

CONSTANT: K is set to give the target failure probability

TABLE 3.3.1 – ARBITRARY FAILURE CRITERIA

Case No	Design Failure Probability			Case No	Design Failure Probability			Case No	Design Failure Probability		
	10^{-3}	10^{-5}	10^{-7}		10^{-3}	10^{-5}	10^{-7}		10^{-3}	10^{-5}	10^{-7}
1	- 2.979	- 4.971	- 7.026	37	- 2.906	- 5.294	- 7.089	73	- 3.103	- 5.116	- 7.124
2	- 3.010	- 5.010	- 7.071	38	- 3.010	- 5.010	- 7.010	74	- 3.009	- 5.009	- 7.010
3	- 3.058	- 5.034	- 7.035	39	- 3.015	- 5.019	- 7.025	75	- 3.051	- 5.303	- 7.332
4	- 2.930	- 4.879	- 6.830	40	- 3.044	- 5.051	- 7.055	76	- 3.054	- 5.063	- 7.069
5	- 3.004	- 5.035	- 7.051	41	- 2.998	- 4.892	- 6.897	77	- 3.085	- 5.132	- 7.186
6	- 3.355	- 5.363	- 7.099	42	- 3.129	- 5.141	- 7.149	78	- 3.010	- 5.010	- 7.010
7	- 3.010	- 5.010	- 7.011	43	- 3.000	- 5.001	- 7.001	79	- 3.337	- 5.346	- 7.134
8	- 3.278	- 5.111	- 7.107	44	- 3.870	- 5.907	- 7.839	80	- 3.002	- 5.008	- 7.010
9	- 3.079	- 5.295	- 7.387	45	- 3.040	- 5.049	- 7.057	81	- 3.118	- 5.183	- 8.794
10	- 3.139	- 5.211	- 7.300	46	- 3.020	- 5.028	- 7.036	82	- 3.049	- 5.105	- 7.154
11	- 3.206	- 5.010	- 7.011	47	- 3.010	- 5.010	- 7.010	83	- 3.014	- 5.015	- 7.015
12	- 3.010	- 5.010	- 7.011	48	- 3.010	- 5.010	- 7.010	84	- 3.009	- 5.010	- 7.010
13	- 3.010	- 5.010	- 7.010	49	- 2.853	- 4.870	- 7.427	85	- 3.930	- 6.199	- 7.932
14	- 3.285	- 5.029	- 7.034	50	- 2.782	- 4.727	- 6.703	86	- 2.996	- 4.994	- 6.993
15	- 2.989	- 4.977	- 6.967	51	- 3.014	- 5.015	- 7.016	87	- 3.010	- 5.010	- 7.011
16	- 3.016	- 5.016	- 7.017	52	- 3.045	- 5.051	- 7.055	88	- 3.010	- 5.012	- 7.014
17	- 3.015	- 5.025	- 7.044	53	- 3.007	- 5.010	- 7.025	89	- 3.099	- 5.090	- 7.082
18	- 3.010	- 5.010	- 7.010	54	- 3.049	- 5.072	- 7.092	90	- 3.066	- 5.065	- 7.063
19	- 3.546	- 5.198	- 7.210	55	- 3.235	- 5.007	- 7.000	91	- 3.062	- 5.070	- 7.075
20	- 3.205	- 5.649	- 7.847	56	- 3.093	- 5.047	- 7.049	92	- 3.010	- 5.010	- 7.020
21	- 3.010	- 5.010	- 7.010	57	- 3.342	- 5.277	- 7.002	93	- 3.506	- 5.824	- 7.280
22	- 3.017	- 5.018	- 7.019	58	- 3.011	- 5.011	- 7.011	94	- 3.042	- 5.260	- 7.074
23	- 3.059	- 5.071	- 7.080	59	- 2.913	- 4.931	- 6.953	95	- 3.312	- 5.367	- 7.300
24	- 2.878	- 4.905	- 6.885	60	- 3.077	- 5.083	- 7.084	96	- 3.198	- 4.998	- 7.007
25	- 3.009	- 5.024	- 7.053	61	- 2.857	- 4.903	- 6.986	97	- 3.056	- 5.914	- 8.992
26	- 2.856	- 3.441	- 3.486	62	- 3.018	- 5.002	- 7.002	98	- 3.052	- 5.065	- 7.075
27	- 3.010	- 5.010	- 7.011	63	- 3.010	- 5.010	- 7.010	99	- 3.011	- 5.012	- 7.013
28	- 3.013	- 5.014	- 7.016	64	- 3.066	- 5.093	- 7.114	100	- 3.044	- 5.049	- 7.054
29	- 3.041	- 5.049	- 7.052	65	- 3.045	- 5.317	- 7.318	101	- 3.107	- 5.118	- 7.123
30	- 3.020	- 5.023	- 7.026	66	- 3.010	- 5.010	- 7.010	102	- 3.049	- 5.056	- 7.060
31	- 3.076	- 5.107	- 7.131	67	- 2.910	- 4.842	- 7.052	103	- 3.062	- 5.426	- 7.600
32	- 3.012	- 5.013	- 7.014	68	- 3.019	- 5.025	- 7.031	104	- 3.010	- 5.010	- 7.010
33	- 3.751	- 5.189	- 7.015	69	- 3.055	- 5.072	- 7.085	105	- 3.022	- 5.043	- 7.071
34	- 2.976	- 5.765	- 7.348	70	- 3.080	- 5.101	- 7.115	106	- 3.011	- 5.264	- 7.361
35	- 3.007	- 4.996	- 6.999	71	- 3.026	- 5.061	- 7.075	107	- 2.908	- 4.927	- 6.972
36	- 3.193	- 4.993	- 6.991	72	- 2.962	- 4.935	- 6.906	108	- 3.105	- 5.118	- 7.126

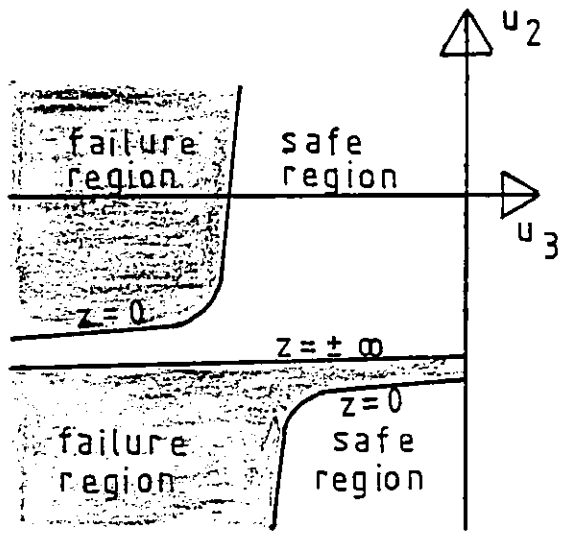
TABLE 3.3.2 - LOG_{10} (FAILURE PROBABILITY) FOR ARBITRARY FAILURE CRITERIA

Target Probability	10^{-3}	10^{-5}	10^{-7}
Sample Size	107	107	107
Mean	- 3.077	- 5.104	- 7.123
Standard Deviation	0.175	0.050	0.303
Coefficient of Skewness	2.81	2.67	4.21
Coefficient of Kurtosis	12.5	11.1	23.5

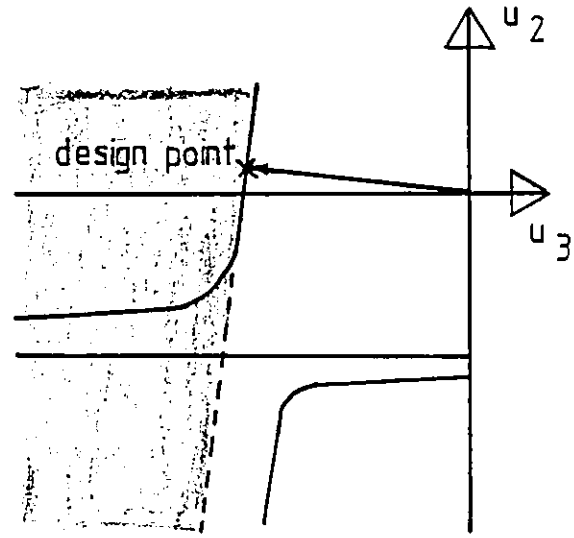
TABLE 3.3.3 - STATISTICS OF LOG_{10} (p(F)) FOR ARBITRARY FAILURE CRITERIA DESIGNED USING ADVANCED LEVEL 2 METHOD

β	Probability Density, $\phi(\beta)$	Cumulative Probability, $\Phi(-\beta)$	$\frac{\phi(\beta)}{\Phi(-\beta)}$	$\frac{\phi(\beta)/\Phi(-\beta)}{-\log_{10}\{\Phi(-\beta)\}}$
0.0	0.399	0.50	0.80	2.65
0.67	0.318	0.25	1.27	2.11
1.28	0.176	0.10	1.76	1.76
2.32	0.0267	0.01	2.67	1.33
3.09	0.00337	0.001	3.37	1.12
3.72	0.000396	0.0001	3.96	0.94
4.27	4.48×10^{-5}	10^{-5}	4.48	0.90
4.75	5.03×10^{-6}	10^{-6}	5.03	0.84
5.20	5.38×10^{-7}	10^{-7}	5.38	0.77
5.61	5.85×10^{-8}	10^{-8}	5.85	0.73
6.00	6.08×10^{-9}	10^{-9}	6.08	0.68
6.36	6.57×10^{-10}	10^{-10}	6.57	0.66
6.71	6.67×10^{-11}	10^{-11}	6.67	0.61

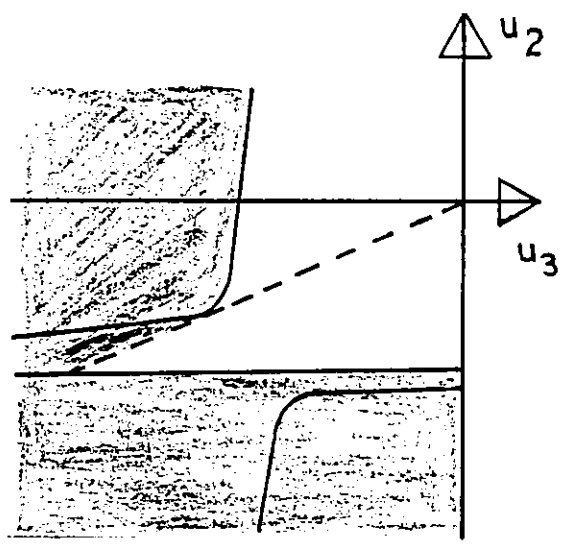
TABLE 3.3.4 - RATIO OF PROBABILITY DENSITY AND CUMULATIVE PROBABILITY FUNCTIONS FOR STANDARD NORMAL DISTRIBUTION



(i) True failure regions



(ii) Level 2 method failure region

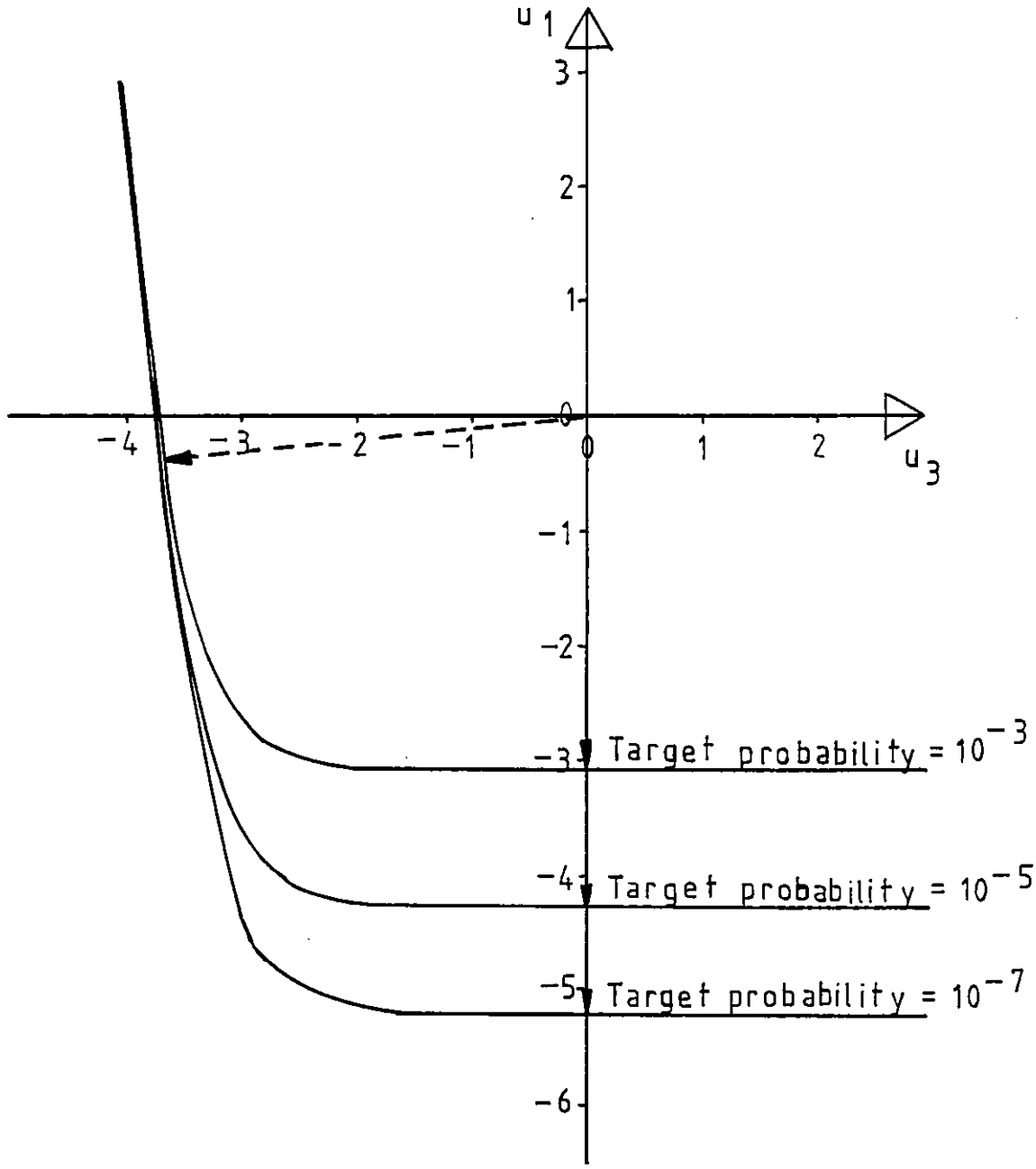


(iii) Level 3 method failure region

Failure Criterion

$$Z = X_1^3 / X_2 + X_3^4 + k$$

Fig. 3.3.1 - Example of a Denominator which can Take on Negative Values



Failure criterion: $Z = X_1^6 - X_2^4/X_3 + k$

X_2 held at design point values

Fig. 3.3.2 - Failure Boundary Plotted in Standardised Normal Space

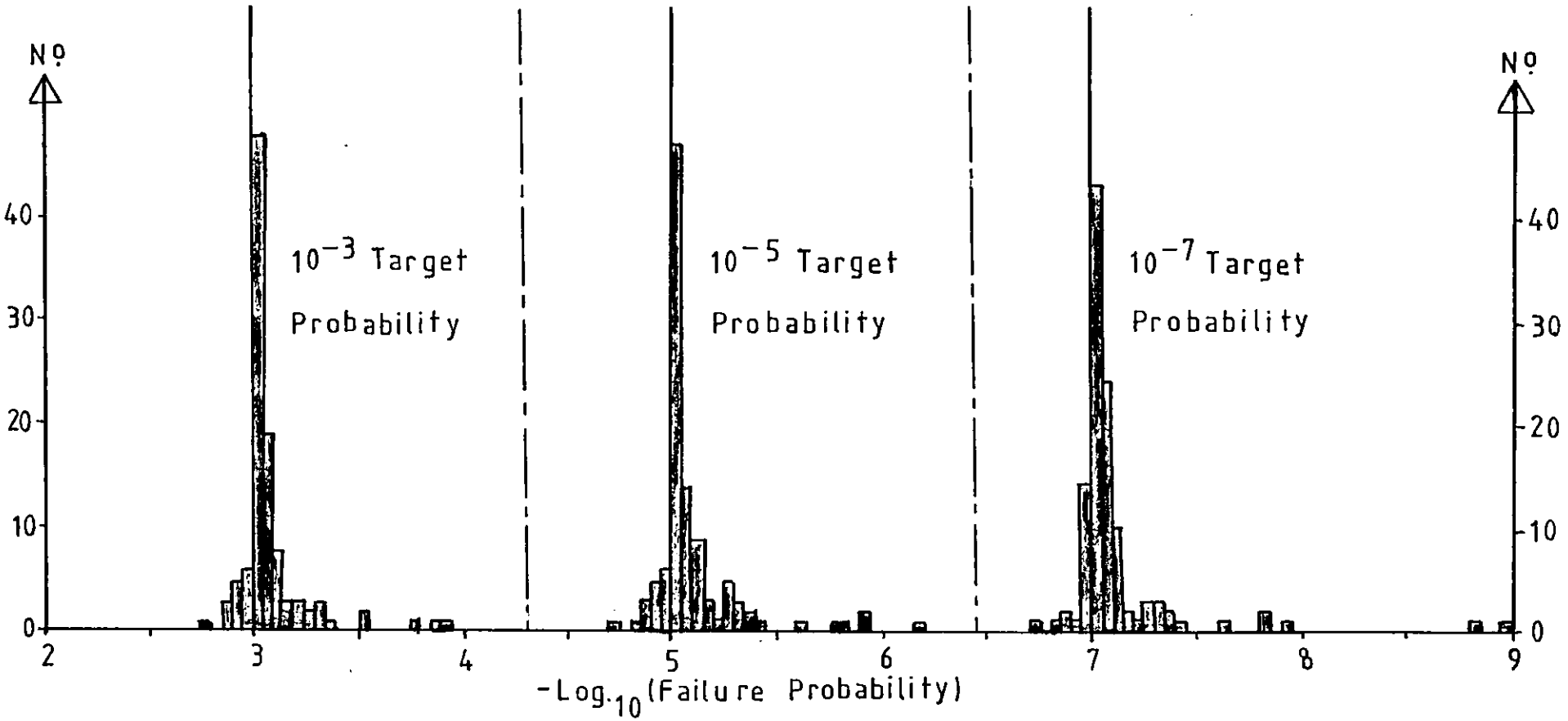


Fig. 3.3.3 - Histogram of Level 3 Failure Probabilities for Arbitary Failure Criteria Designed to Target Probabilities by Advanced Level 2 Analysis

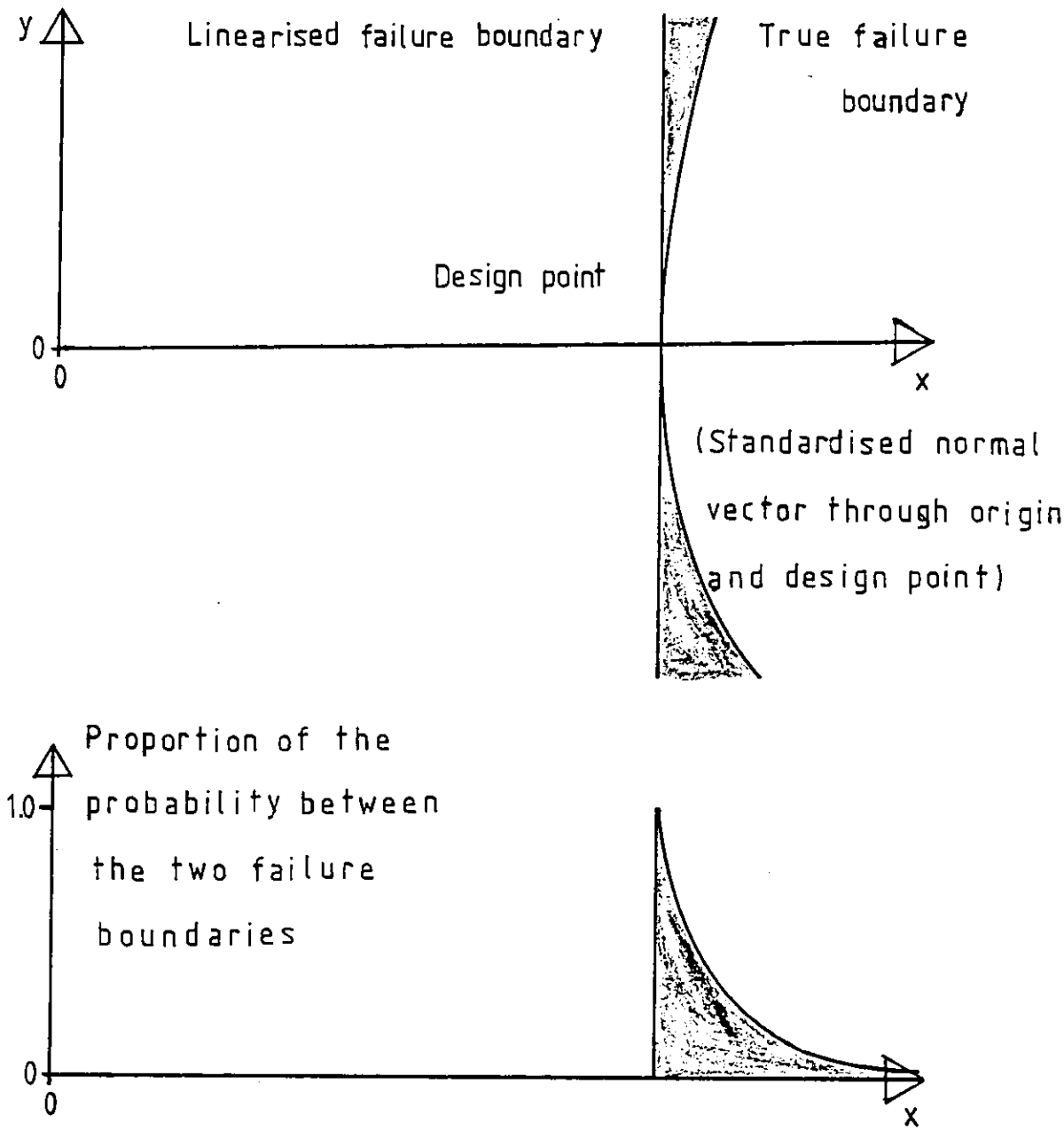
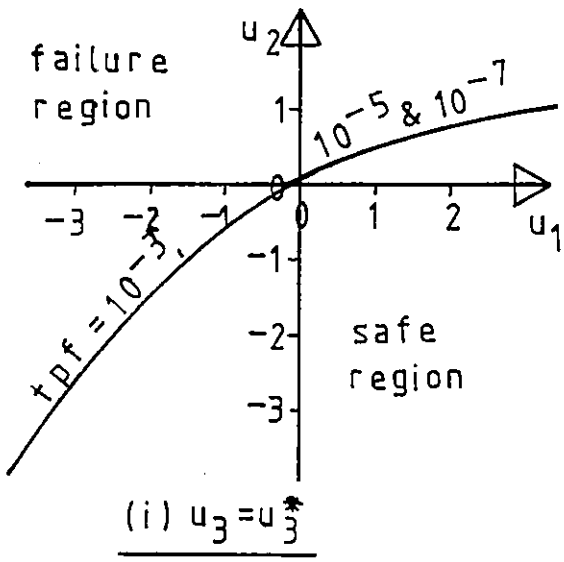
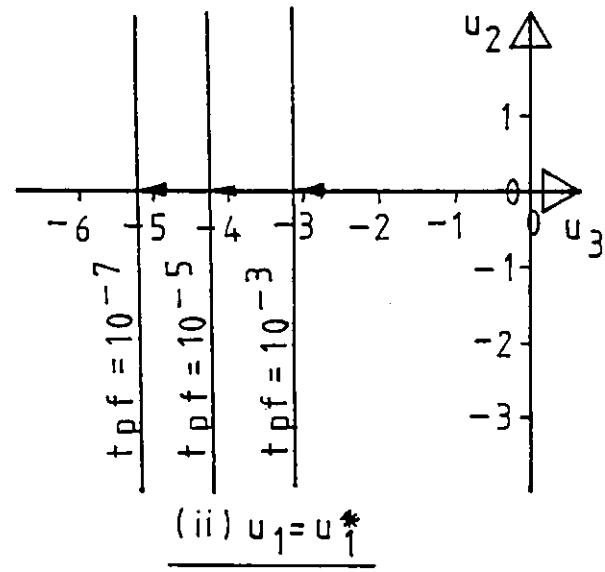
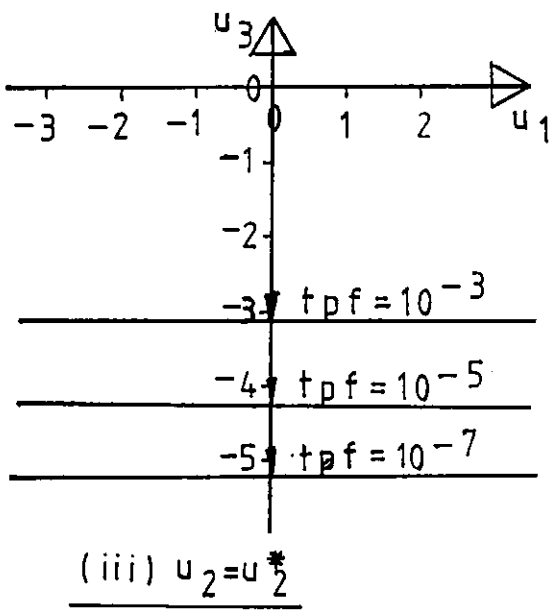


Fig. 3.3.4 - Error in Marginal Normal Distribution of a Convex Failure Criterion





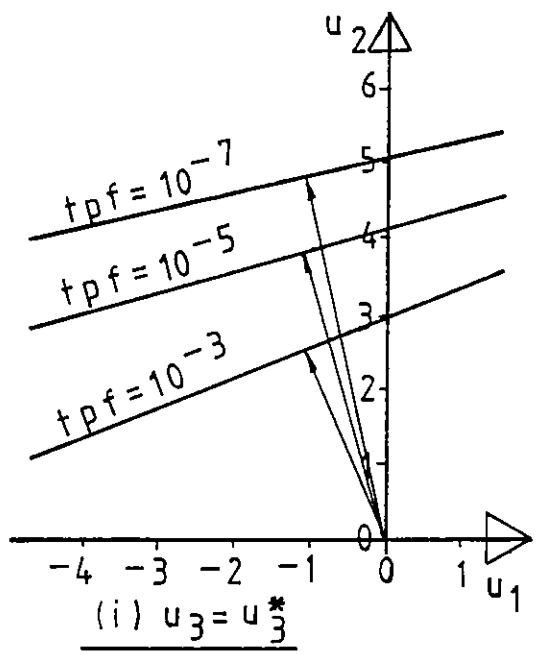


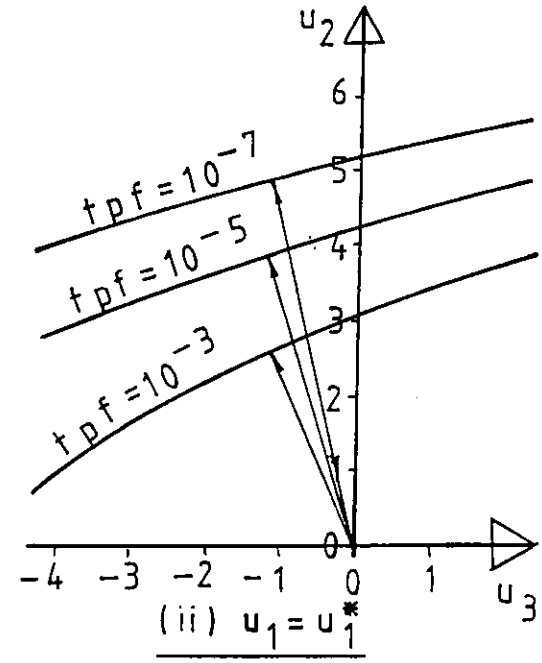
tpf = target failure probability

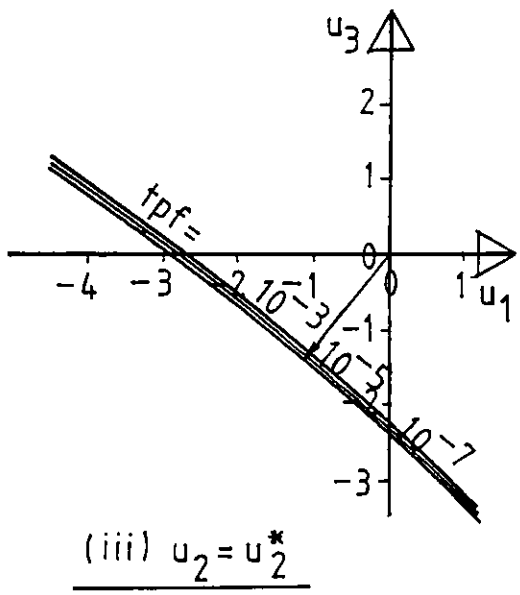
Failure criterion

$$Z = X_1^3 / X_2^6 + X_3 + k = 0$$

Fig. 3.3.5 - Failure Boundary for Case 33 Plotted in Standardised Normal Space





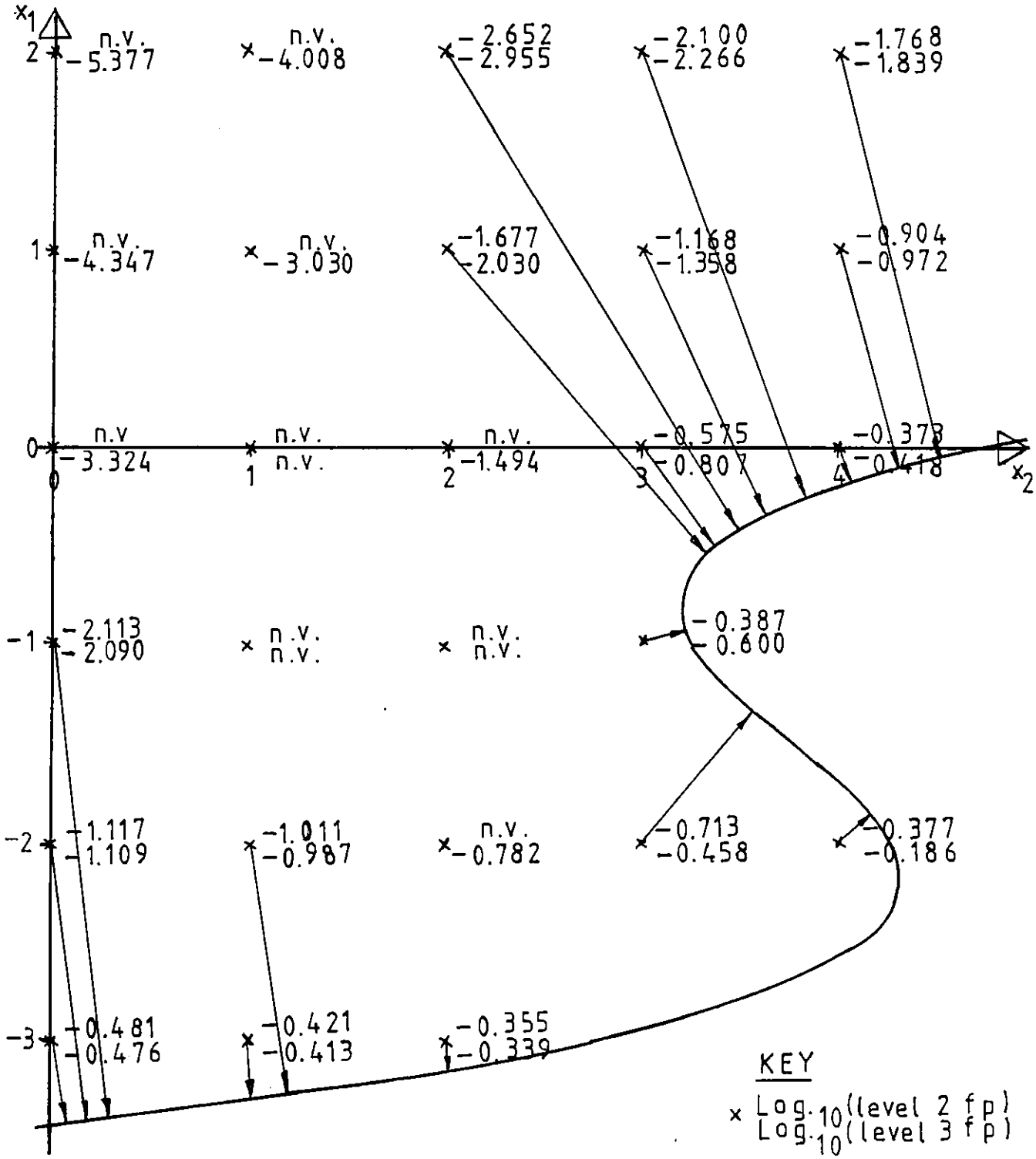


tpf = target failure probability

failure criterion

$$Z = (X_1^3 / X_2^2) \times X_3^2 = 0$$

Fig. 3.3.6 - Failure Boundary for Case 34 Plotted in Standardised Normal Space



fp = failure probability

n.v. = no value found by computer program used

Fig. 3.3.7 - Failure Boundary for two Standardised Normal Variable Failure Criterion

PART 2 - SYSTEM RELIABILITY ANALYSIS

CHAPTER 4 - PREVIOUS WORK ON SYSTEM RELIABILITY

4.1 IDEALISATIONS OF STRUCTURAL SYSTEMS

4.1.1 Binary Analogy

The reliability analysis methods considered in Part 1 are for calculating the probability of a single event being realised - usually the failure of an element to resist the load effect applied to it - where that event has been idealised mathematically as $g(\underset{\sim}{x}) \leqslant 0.0$. As discussed in section 1.1, system reliability analysis examines the probability of an event which is the combination, by union and/or by intersection, of several such events. The component events leading to system failure are referred to here as local failures.

The concept of system failure can be demonstrated easily for simple systems that consist of idealised components which may be in one of two possible (binary) states, S, (ie $0 \equiv$ safe and $1 \equiv$ failed) as in some electrical circuits. Let an open switch represent the local failure state, a closed switch the local safe state, and a broken circuit represent the system failure state. The simplest combinations of components are:

i. Parallel systems (fig. 4.1.1(i)), for which system failure occurs only when <u>all</u> the switches are open; and

ii. Series systems (fig. 4.1.1(ii)), for which system failure occurs when <u>any</u> of the switches are open.

These two types of systems can be combined into more complex arrangements, which can be shown schematically as a failure network (fig 4.1.2). Each "switch" is shown as a box, and system failure may be defined as the condition when "current" can no longer flow between A and B.

The probability of the event of local failure, a_i, in such a binary system is equal to the expected value of that component's state, S_i. Similarly, the probability of system failure is equal to the expected value of the system's state. The state of any such system can be obtained from the states of its components by the rules

$$\text{i.} \quad S_{system} = \prod_{i=1}^{n} (S_i) \qquad \text{for parallel systems}$$

and

$$\text{ii.} \quad S_{system} = 1.0 - \prod_{i=1}^{n} (1.0 - S_i) \qquad \text{for series systems} \qquad (4.1.1)$$

where S_{System} is the state of a system with n components. These rules do not, however, apply to the expected values of the states unless the component states are mutually independent.

4.1.2 Real Behaviour of Materials

The local failure states are usually far from binary in nature when considering real structural systems. For example, all real structural materials can be in states other than failure and non-failure if linear elastic behaviour is regarded as non-failure and an increasing deflection without any change of load effect as failure. The proportion of yield stress at which failure is deemed to occur when calculating the reliability of a single element can be chosen purely by considering when that element will enter a particular limit state. However, the effect of the local failure on the response of the structure to the applied load must also be considered when analysing the system reliability. The load effect must be redistributed to allow for the element which has failed having a reduced capacity to carry load if the local failure does not result in system failure.

The two most common idealisations for the failed states are for:

i. perfectly ductile elements, which will continue to carry the load-effect which causes local failure, but no more; and

ii. perfectly brittle elements, which carry no load-effect after local failure.

These are shown as idealisations of some typical material properties in fig. 4.1.3.

4.1.3 Local Failure Criteria

In structural engineering, each local event, a_i, is associated with several basic variables, $\underset{\sim}{X}$, which are combined to define the binery state, S_i, by the failure criterion, $g_i(\underset{\sim}{X})$ so that

$$S_i = \begin{cases} 1 \text{ if } g_i(\underset{\sim}{x}) \leqslant 0.0 \\ 0 \text{ if } g_i(\underset{\sim}{x}) > 0.0 \end{cases} \qquad (4.1.2)$$

However, since many of the basic variables are used to define more than one state, in general, the failure criteria are not mutually uncorrelated, ie

$$\rho(g_i(\underset{\sim}{X}), g_j(\underset{\sim}{X})) \neq 0.0 \qquad (4.1.3)$$

This implies that neither are the states mutually uncorrelated, ie

$$\rho(S_i, S_j) \neq 0.0 \qquad \ldots\ldots (4.1.4)$$

and therefore, as stated in section 1.1, the probability of the intersection of the events is not the product of the probabilities of those events, ie

$$P(a_i \cap a_j) \neq P(a_i) * P(a_j) \qquad \ldots\ldots (4.1.5)$$

Hence more complex methods have to be used to determine the structural system reliability (which is defined as unity minus the probability of system failure). However, a failure criterion has to be determined, which defines when the structure no longer fulfils the design purpose for which the probability of system failure is required, before any analysis can be carried out. This criterion does not have to be related to the structural system problem of when a mechanism is formed. It can be:

i. when the first local failure occurs (even if the structure is highly redundant and therefore would remain standing);

ii. when overall failure (ie collapse) occurs; or

iii. at any stage between i. and ii. when the structure is deemed as unfit for its purpose by some other criterion.

The precise nature of the system failure criterion is also a consideration in deciding upon the appropriate idealisation for the local failure criteria as a binary choice of states (ie failed and non-failed).

4.1.4 System Failure Criteria

An idealised two storey frame is shown in fig. 4.1.4, together with some possible mechanisms caused by the formation of plastic hinges. The system failure criterion could be defined so that system failure occurs when any deflection is greater than a specified value, or when neither bay of the lower storey is standing (vi). As well as these extreme examples, there are many more possible failure criteria, including those for which system failure is the formation of any hinge, the formation of any mechanism (ii. or iii.) or the collapse of part of the structure (iv. or v.).

Once the failure criterion has been established for the required degree of system failure, the possible failure groups can be found. A failure group here refers to a group of local failures which, if they were all realised, would cause system failure. A failure group can be sub-divided into failure paths

(where each failure path is a sequence of local failures which result in system failure) when the order in which the local failures are realised is significant. The exhaustive set of failure groups (or failure paths) for the structure failing according to that system failure criterion is referred to here as the failure set.

The failure paths can be shown schematically in a failure tree. Fig. 4.1.6 shows the failure tree for a simple system for which the failure network is shown in fig. 4.1.5. The main disadvantage with using a failure tree to describe a failure set is its size for complex structures. There are a maximum of $\frac{n!}{(n-m)!}$ failure paths for a structure with n potential local failures, any m of which will cause system failure. However, an advantage that a failure tree has over a failure network as a diagramatic aid to understanding a failure set is that the failure network cannot show the relationship between elements which are in parallel for some failure paths and in series for others without introducing additional symbols (fig. 4.1.7).

4.2 EXACT APPROACHES TO SYSTEM RELIABILITY ANALYSIS

4.2.1 Level 3 Methods

The definition of a Level 3 method (section 2.3) is as applicable to system reliability analysis as it is to elemental reliability analysis. However, the number of basic variables that have to be considered is increased, so that the complexity, and hence the computing requirements, will be significantly greater. Also the failure criterion will no longer necessarily reduce to a single convolution integral, so that the function(s) that has to be integrated, usually by numerical methods, and the limits of the integrations will be arrived at only after considerable manipulation. This latter point should not affect a Monte Carlo simulation method to the same extent since, for any set of basic variable values, the set of local events which would result can be determined from which the safe/failed condition of the structure can be found.

Therefore the Level 3 methods are usually considered to require too much computing effort to be practical for analysing most structures. However, several methods of system reliability analysis have been proposed which, while not being truly exact or Level 3 methods, do make their assumptions and simplifications in the manner of solving equations, particularly for the cumulative probability of multivariate distributions, rather than in the approach to obtaining the mathematical model describing the event of system failure.

4.2.2 Stevenson and Moses' Method

Stevenson and Moses [39] have proposed a method of setting up a series of convolution integrals to obtain the probability of system failure. The excess resistance of each of the modes of failure has to be of the form

$$Z_i = \sum_{j=1}^{n_r} (c_{i,j}.R_j) - \sum_{j=1}^{n_s} (d_{i,j}.S_j) \qquad (4.2.1)$$

where R_j and S_j are independent continuous random variables modelling the resistances of the members and the applied loads respectively, and $c_{i,j}$ and $d_{i,j}$ are constants. This can be expressed in a more compact form as

$$Z_i = \sum_{j=1}^{n_t} (b_{i,j}.X_i) \qquad (4.2.2)$$

A function $\phi_k(x)$ is then defined as

$$\phi_k(x) = p\left(\sum_{j=1}^{k}(b_{i,j}.X_i) \leqslant x\right)$$

$$= \int_{-\infty}^{x} \phi_{(k-1)}(x-t).\ p(b_{i,k}.X_i = t)\ dt \qquad (4.2.3)$$

Hence $\phi_k(x)$ can be found by numerical integration of the convolution integral, and the probability of that mode of failure occurring is

$$p(Z_i \leqslant 0.0) = \phi_i\ (0.0) \qquad (4.2.4)$$

The probability of system failure for a structure with m possible modes of failure is

$$p(F) = \sum_{i=1}^{m} \left[\ p\left(\bigcap_{j=1}^{(i-1)} \left\{Z_j > 0.0\right\} \cap \left\{Z_i \leqslant 0.0\right\}\right)\right] \qquad (4.2.5)$$

where

$$p\left(\bigcap_{j=1}^{(i-1)} \left\{Z_j > 0.0\right\} \cap \left\{Z_i \leqslant 0.0\right\}\right) = \prod_{j=1}^{i} \left[p\left(Z_j > 0.0 \,\middle|\, \bigcap_{k=(j+1)}^{i} \left\{Z_k \leqslant 0.0\right\}\right)\right]$$

$$= \int_{t_i=-\infty}^{t_i=0.0} \int_{t_{(i-1)}=0.0}^{t_{(i-j)}=\infty} \cdots \int_{t_i=0.0}^{t_i=\infty} \prod_{j=1}^{i} \left[p\left(Z_j = t_j \bigcap_{k=(j+1)}^{i} \left\{Z_k = t_k\right\}\right)\right] dt_1 \ldots dt_i \qquad (4.2.6)$$

However, the evaluation of equation (4.2.6) is extremely difficult to carry out, particularly if the multivariate form of the distribution of Z_1 through Z_m is unknown. Therefore Stevenson and Moses suggest some approximations to allow the analysis to be carried out.

4.2.3 Approximations to Permit Analysis

The modes of failure are ranked in descending order of their probability of occurring, so that

$$p(z_i \leqslant 0.0) \geqslant p(z_{(1+i)} \leqslant 0.0) \tag{4.2.7}$$

for all i. This will minimise the errors when the first three terms only are considered, so that

$$P(F) \doteq \sum_{i=1}^{3} \left[p\left(\bigcap_{j=1}^{(i-1)} \{z_j > 0.0\} \cap \{z_i \leqslant 0.0\} \right) \right]$$

$$\doteq \sum_{i=1}^{3} \left[\int_{-\infty}^{0} \prod_{j=1}^{i} \left[p\left(z_j > 0.0 \bigcap_{k=(j+1)}^{i} \{z_k = z_k^*\} \right) \right] dt_i \right] \tag{4.2.8}$$

where

$$z_k^* = \int_0^{\infty} \frac{t_k \times p(z_k = t_k)}{p(z_k > 0.0)} \, dt_k \tag{4.2.9}$$

In order to carry out the numerical integrations involved, the final approximation is that the multivariate distribution of the excess resistances, z_i, is taken as being normal.

This method was proposed before the Advanced Level 2 method of structural reliability analysis had been developed to obtain close estimates of the probability of single elements failing. The use of Advanced Level 2 analysis would dispense with the need to restrict the form of the failure criteria for local events to equation (4.2.2), the probability of which is then evaluated by equation (4.2.3) and (4.2.4) requiring the numerical evaluation of a convolution integral, although it would also invalidate the rigour of the approach. However, equation (4.2.6) requires approximation for its solution, and even then will require a considerable numerical effort, so the method will not necessarily give the exact probability of system failure despite the computation involved.

4.2.4 Rackwitz and Krzykacz's Method

Rackwitz and Krzykacz [35], in a paper on Nuclear Reactor plants produced a method which diverges only slightly from a theoretically exact method. They started by defining the failure probability of each component local failure, a_j, as

$$p(a_j) = 1.0 - \int_{\{\underset{\sim}{x} : g_j(\underset{\sim}{x}) > 0\}} df_{\underset{\sim}{X}}(\underset{\sim}{x}) \tag{4.2.10}$$

where the intended failure criterion is

$$Z_j = g_j(\underset{\sim}{X}) = h_j(\underset{\sim}{U}) \qquad (4.2.11)$$

and $f_{\underset{\sim}{X}}(\underset{\sim}{x})$ is the joint density function of $\underset{\sim}{X}$. Z_j is transformed into a standardised normal variable Y_j by

$$y_j = \Phi^{-1}\left(F_{Z_j}(z_j)\right) \doteq \frac{z_j - E(Z_j)}{\sigma(Z_j)} \qquad (4.2.12)$$

where $E(Z_j)$ and $\sigma(Z_j)$ are as defined in equations (2.4.12) and (2.4.13) respectively, and β_j is $E(Z_j)/\sigma(Z_j)$. This approximation makes the analysis a Level 2 one, but if $F_{Z_j}(z_j)$ could be defined analytically without integrals for any specific case it would be a true Level 3 method, and as such is included in this section.

These univariate standardised normal variables can be regarded as the marginal distributions of a multi variate standardised normal distribution, for which the correlation coefficients are obtained from

$$\rho(Y_i, Y_j) = \underset{\sim}{\alpha}_i^T \cdot \underline{V}_u \cdot \underset{\sim}{\alpha}_j \qquad (4.2.12)$$

where $\underset{\sim}{\alpha}_i$ and $\underset{\sim}{\alpha}_j$ are the vectors of alpha values found by equation (2.4.22) for the ith and jth local failure respectively, and $\underline{V}_u$ is the correlation coefficient matrix of the transformed basic variables, $\underset{\sim}{U}$.

This allows the probability of system failures to be given as a function of the probability density functions of multivariate normal distributions. Three simple examples given are

i. Series:

$$p(F) = 1.0 - p\left(\bigcap_{j=1}^{m} (Y_j > -\beta_j)\right)$$

$$= 1.0 - \int_{-\underset{\sim}{\beta}}^{\infty} \phi(\underset{\sim}{Y} ; \underline{V}_y)\, d\underset{\sim}{Y} \qquad (4.2.13)$$

ii. Parallel:

$$p(F) = p\left(\bigcap_{j=1}^{m} (Y_j \leqslant -\beta_j)\right)$$

$$= \int_{-\infty}^{-\underset{\sim}{\beta}} \phi(\underset{\sim}{Y} ; \underline{V}_y)\, d\underset{\sim}{Y}$$

$$= \Phi(-\underset{\sim}{\beta} ; \underline{V}_y) \qquad (4.2.14)$$

iii. k out of m (where system failure is any k local failures out of m possible local failures, each numbered from set I_m):

$$P(F) = \sum_{I_k \subset I_m} P\Big[\bigcap_{j \in I_k} \{ u_j > -\beta_j \} \cap \bigcap_{j \in I_m \setminus I_k} \{ u_j \leq -\beta_j \} \Big]$$

$$= \sum_{I_k \subset I_m} \int_{-\beta(I_m)}^{\infty} \Big[\int_{-\infty}^{-\underset{\sim}{\beta}(I_m \setminus I_k)} \underset{\sim}{\phi}(\underset{\sim}{Y} ; \underset{\sim}{V}_y) dy_{(I_m \setminus I_k)} \Big] dy_{(I_k)} \qquad (4.2.15)$$

where $\underset{\sim}{V}_y$ is the correlation coefficient with elements $\rho(Y_i, Y_j)$.

The principal difficulties of this method are to provide a simple algorithm with which to set up the integral for the general case, and the mathematics of evaluating that integral over a large number oi possible local failures. The latter problem is discussed in an appendix to their paper. A method is demonstrated to solve several examples by using Markov chains.

4.2.5 Casciati's Method

Casciati [8] has proposed a method for studying elastic-plastic frames by using a parametric Monte Carlo simulation technique. This method obtains the deflections at incipient collapse as well as the probability of system failure due to a load carrying capacity criterion. One advantage of this approach is that each simulated structure is statically analysed, so allowing for the effect of the variability of stiffnesses, etc, on the global distribution of load effects throughout the structure. The use of Monte Carlo simulation and repeated statical analyses, however, makes it inefficient in most cases for obtaining system reliability when there are other methods available.

4.3 APPROXIMATIONS TO THE SYSTEM FAILURE PROBABILITY

4.3.1 General

The methods outlined in section 4.2 are not practical for many of the possible applications of system reliability analysis since they would involve a very considerable amount of calculation. Therefore other methods of structural system reliability analysis are required which can obtain a reasonably close estimate of the probability of system failure without involving an excessive amount of calculating effort. This requires the structures to become more idealised, which in turn limits the structures that can be analysed by such methods to those structures for which the idealisations are appropriate.

4.3.2 Vanmarke's Method

Vanmarke [41] limits the problem to one of series combinations of local failures only by assuming that any local failure will cause system failure. He produces a method for this system failure criterion which obtains an upper bound on the probability of system failure that he assumes is close enough to be a good approximation. The value of the probability of each local failure made is found by treating the resistance and load as uncorrelated normal variables, so that

$$\beta_i = \frac{E(R_i) - E(S_i)}{\sigma^2(R_i) + \sigma^2(S_i)} = \frac{E(Z_i)}{\sigma(Z_i)} \qquad (4.3.1)$$

and

$$p(A_i) = \Phi(-\beta_i) \qquad (4.3.2)$$

where R_i is the element's resistance and S_i the load effect applied to it. System failure occurs simultaneously with any local failure, so that simple bounds on the probability of system failure are

$$\max_{i=j}^{m} (p(A_i)) \leq p(F) \leq \sum_{i=j}^{m} (p(A_i)) \qquad (4.3.3)$$

To get a better approximation to the probability of system failure, the local failure modes are ranked so that

$$\beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots \leq \beta_m \qquad (4.3.4)$$

and constants, c_i, are defined so that

$$c_1 = 1.0$$

and

$$c_i = \min_{j=1}^{(i-1)} (p(A_j^c \mid A_i))$$

$$\doteq 1.0 - \frac{\Phi\left(-\text{Max}\left\{\min_{j=1}^{(i-1)}\left(\frac{\beta_j}{|\rho(Z_i, Z_j)|}, \beta_i\right)\right\}\right)}{\Phi(-\beta_i)} \qquad \text{for } i > 1 \qquad (4.3.5)$$

Then the probability of system failure is approximated to by

$$p(F) = \sum_{i=1}^{m} (c_i \Phi(-\beta_i)) \qquad (4.3.6)$$

Although Vanmarke's method is relatively simple, it is essentially only an upper bound to the value, and therefore is superceded by methods such as Ditlevsen's (sub-sections 4.4.4 to 4.4.7).

4.3.3 Moses's Unzipping Technique

Moses[27-30] does not limit the system failure to the first local failure when he proposes his "unzipping" technique to approximate the probability of system failure. His method consists of assuming that there is an incremental load that increases until a local failure takes place. The load effects on the elements are then amended to allow for the modified structural behaviour and the load increases again until the next local failure occurs. This method is limited to structures with only one loading variable acting on them because of the use of an incremental load.

The load increment necessary to cause the jth local failure mode in failure path i after the (j-1)th local failure has occurred is defined as $\delta R_{i,j}$. It is assumed that the load effect of the imposed load is known deterministically at every stage of the failure path, so that the load effect on the element due to the jth local failure, $r_{i,j}$, can be found by

$$r_{i,j} = \sum_{k=1}^{j} (c_{i,j,k} \times \delta R_{i,k}) \qquad (4.3.7)$$

where $c_{i,j,k}$ are constants. At this value of load effect the failure only just occurs, so that $r_{i,j}$ also represents that element's resistance. (Note that in this method the lower and upper case r will represent elemental and structural resistances respectively, rather than an individual value and the population of a variable).

4.3.4 Example

An example of this technique for a pin-jointed truss is given in Appendix B. In the first failure path (elements EF, CD and DF failing in that order) the first element fails under the load effect from the initial load increment (fig. B.2), so that

$$r_{1,1} = r_{EF} = 7/27\ \delta R_{1,1} \qquad (4.3.8)$$

The second element fails under the combined load effect from the first two load increments (Figs. B.2 and B.3), so that, if element EF is perfectly ductile,

$$r_{1,2} = r_{CD} = 10/27\ \delta R_{1,1} + 1/2\ \delta R_{1,2} \qquad (4.3.9)$$

However, if element EF is perfectly brittle,

$$r_{1,2} = r_{CD} = 1/2\ (\delta R_{1,1} + \delta R_{1,2}) \qquad (4.3.10)$$

The elements can also be modelled by other material behaviours, such as only being able to resist half the load effect causing failure which would give

$$r_{1,2} = r_{CD} = 10/27 \;\; 1/2\delta R_{1,1} + 1/2(1/2\delta R_{1,1} + \delta R_{1,2})$$

$$= 47/108 \; \delta R_{1,1} + 1/2\delta R_{1,2} \qquad (4.3.11)$$

The third element fails under the combined load effect from the first three load increments (Figs B.2, B.3 and B.7), so that, if elements EF and CD are perfectly ductile,

$$r_{1,3} = r_{DF} = 20/27 \; \delta R_{1,1} + \delta R_{1,2} + \delta R_{1,3} \qquad (4.3.12)$$

The value of the load increments can be found in terms of the elements' resistances by inverting equation (4.3.7) into

$$\delta R_{i,k} = \sum_{j=1}^{k} (b_{i,j,k} \cdot r_{i,j}) \qquad (4.3.13)$$

Inverting equations (4.3.8), (4.3.9) and (4.3.12) gives

$$\left.\begin{aligned} \delta R_{1,1} &= 27/7 \; r_{EF} \\ \delta R_{1,2} &= -20/7 \; r_{EF} + 2 \; r_{CD} \\ \delta R_{1,3} &= \qquad\quad - 2 \; r_{CD} + r_{DF} \end{aligned}\right\} \qquad (4.3.14)$$

The resistance of the structure against failure by that failure path is the sum of the n_i load increments since system failure will be realised simultaneously with the n_ith local failure. Therefore

$$R_i = \sum_{j=1}^{n_i} (\delta R_{ij}) = \sum_{j=1}^{n_i} \left(\sum_{k=1}^{n_i} (b_{i,j,k}) * r_{i,j}\right)$$

$$= \sum_{j=1}^{n_i} (d_{i,j} \; r_{i,j}) \qquad (4.3.15)$$

Substituting in the values for the example given above from equation (4.3.14) gives

$$R_1 = r_{EF} + r_{DF} \qquad (4.3.16)$$

4.3.5 Resistance of the Structure

By using the first two terms of the Taylor Series expansion of the structure's resistance to failure by that failure path as a function of the individual element's resistances, the first two moments of R_i are

$$E(R_i) \doteq \sum_{j=1}^{n_i} (d_{i,j} \times E(r_{i,j})) \qquad (4.3.17)$$

and

$$\sigma^2(R_i) \doteq \sum_{j=1}^{n_i} \left(\sum_{k=1}^{n_i} (d_{i,j} \times d_{i,k} \times \rho(r_{i,j}, r_{i,k}) \times \sigma(r_{i,j}) \times \sigma(r_{i,k}))\right) \qquad (4.3.18)$$

Moses treated the element resistances as basic variables. If they are not, then the values of $E(r_{i,j})$ and $\sigma(r_{i,j})$ can be obtained in a similar manner to equations (4.3.17) and (4.3.18) from the first two terms of the Taylor Series expansion of the individual element's resistance written as a function of the basic variables (see also section 5.5).

The structure's resistence is used in the simple R-S format in order to obtain the probability of failure by that failure path as

$$p(A_i) \doteq \Phi(-\beta_i) = \Phi\left(-\frac{E(R_i) - E(S)}{\sqrt{\sigma^2(R_i) + \sigma^2(S)}}\right) \qquad (4.3.19)$$

where S is the loading variable. This is repeated for the m possible failure paths, and the probability of system failure is approximated to by

$$p(F) \gtrsim \max_{i=1}^{m} (p(A_i)) \qquad (4.3.20)$$

4.3.6 Limitations of Moses' Unzipping Method

The mean and standard deviation of the structure's resistance against the fourteen failure paths of the examples, together with the values of β_i, are given in table B.2 using arbitary statistics for the elements' resistances and applied load (table B.1). These give a failure probability of 0.22×10^{-3} for failure path No 1, which is the maximum value for any path, but it is equalled by two other failure paths out of those 14 considered.

This method, therefore, requires three assumptions to be made. The first is that the distributions of the elements' resistances and the load can be described by the normal distribution. When each resistance can be modelled by a linear combination of basic variables with normal distributions, equations (4.3.17) and (4.3.18) will give results which are exact rather than approximate, and then this assumption will be correct provided that the load is also normally distributed.

Care must be taken when carrying out this analysis that negative load increments are only included if subsequent increments decrease the probability of failure.

Negative values are likely to occur in the less probable failure paths or with brittle elements, when the jth local failure may be almost certain to occur if the load is large enough to cause the (j-1)th local failure. The reason that negative load increments occur is that the value given for the structure's resistance against failure by a particular failure path, R_i, is in fact only the resistance against the applied load for the final mode of local failure in the failure path, but with the structure distributing the load effect as if the other local failures had occurred. Therefore the true value of the probability of that failure path occurring is the intersection of the probabilities of all the local failure modes, which is given by

$$p(A_i) = p\left(\bigcap_{j=1}^{n_i} a_{i,j}\right) \leqslant \operatorname*{Min}_{j=1}^{n_i} \left(p(a_{i,j})\right) \qquad (4.3.21)$$

where, for variables with normal distributions

$$p(a_{i,j}) = \Phi\left(- \frac{\sum_{k=1}^{j} \left(d_{i,k} * E(r_{i,k})\right) - E(S)}{\sqrt{\sum_{k=1}^{j} \left(\sum_{l=1}^{j} \left(d_{i,k} d_{i,l} \rho(r_{i,k}, r_{i,l}) \sigma(r_{i,k}) \sigma(r_{i,l})\right)\right) + \sigma^2(S)}} \right) \qquad (4.3.22)$$

Hence the second assumption is that the resistances, $\sum_{k=1}^{j} (\delta R_{i,k})$, for any failure path have strong positive correlation, so that the inequality in equation (4.3.21) is a close approximation to the correct value.

The decrease in failure probability with more load increments is demonstrated in several of the failure paths in the example described in Appendix B. In some of the paths the final value of β is the largest (eg failure path No. 6, when β for the first local mode of failure is 2.99, for the first two is 2.20 and for the system failure is 3.52) but not for all of them (eg failure paths No. 4, where the β values are 1.74, 4.22 and 3.65, and No 13. where the β values are 4.74, 3.96 and 3.65).

The final assumption in this method is to use equation (4.3.20) to obtain the probability of system failure. This produces a lower bound on the true value of that probability, provided that equation (4.3.19) gives the true value for the probability of the realisation of the failure paths. This will not result in a significant error if a few of the values of $p(A_i)$ are much larger than the others (unlike the example in Appendix B), and that those few have strong mutual correlation. When this is not the case, this method could be used to estimate the parameters defining the structure's resistance against the various failure paths for use with another method, such as Vanmarke's method.

The correlation coefficients for the structure's resistances, which would be required, can be obtained similarly to equation (4.3.18) as

$$\rho(R_i,R_j) = \frac{\left[\sum_{k=1}^{n_i} \left(\sum_{l=1}^{n_j} (b_{i,k}.b_{j,l}.\rho(r_{i,k},r_{j,l}).\sigma(r_{i,k}).\sigma(r_{j,l}))\right)\right]}{\sigma(R_i).\sigma(R_j)} \qquad (4.3.23)$$

However, this would mean considerable calculations which would still be liable to gross errors if:

i. the resistances had highly skewed or other non-normal distributions; or

ii. the load increments for each path did not have strong positive mutual correlation.

4.3.7 Marshall and Bea's Method

Marshall and Bea [26] propose a method of structural system reliability analysis for structures with specific correlations of the elements' resistances (depending on their physical position in the structure) and which resist hind-cast loads. They assume that the load effect, S_i, on each element can be found deterministically for a given load, S, on the structure. The element has a nominal resistance to resist this load effect, and the ratio of this nominal resistance to the load effect is the safety factor, γ_i. Marshall and Bea give graphs for the elements, but if the resistances have normal distributions the nominal resistance will be $(E(R_i) + k \times \sigma(R_i))$, where k is a constant. Therefore

$$\beta_i = E(R_i) - s_i / \sigma(R_i)$$

$$= \left[\gamma_i - 1.0\right] \frac{s_i}{\sigma(R_i)} - k \qquad (4.3.24)$$

They propose two methods in order to model brittle-redundant and ductile redundant structures. These methods can be used as limiting cases for those structures which contain:

i. both brittle and ductile elements; and/or

ii. elements which cannot be idealised as either perfectly brittle or perfectly brittle.

The limiting cases cannot be regarded as bounds on the true probability of system failure since the two methods contain conflicting idealisations.

4.3.8 Brittle Redundant Model

The probability of any initial local failure in their brittle-redundant model is assumed to be

$$p(A_1|s) = 1.0 - \prod_{i=1}^{m} (1.0 - \Phi(-\beta_i)) \qquad (4.3.25)$$

where m is the number of local failure modes which can cause an initial local failure leading to collapse. The values of s_i are then recalculated after removal of one of the elements, the choice of which element to remove being dependent on maximising the probability of future local failures. The values of β_i are then recalculated and $p(A_2|A_1,s)$ is obtained as for equation (4.3.25). This is repeated until the removal of an element causes system failure. Then

$$p(F|s) = \prod_{i=1}^{n} (p(A_i|A_1, \ldots, A_{(i-j)}, s)) \qquad (4.3.26)$$

where n is the number of elements that have to be removed to cause system failure.

This method assumes that all the resistances are mutually uncorrelated even when a local failure mode can occur at several stages in the failure sequence. This will tend to make the final probability of system failure conservative, but it is balanced, hopefully, by the multiplication of each component of the probability of failure at stage i by the probability of failure at stage (i + 1) after the most disadvantageous (in terms of high reliability) element has been removed.

Applying this model to the example described in Appendix B gives

$$p(A_1|s = 2.0) = 0.151 \times 10^{-5} \qquad (4.3.27)$$

Element DF is removed in order to obtain the maximum probability of subsequent failures, which results in

$$p(A_2|A_1, s=2.0) = 1.0 \qquad (4.3.28)$$

This gives

$$p(F\ S=2.0) = p(A_1|s=2.0) \times p(A_2|A_1, s=2.0) = 0.151 \times 10^{-5} \qquad (4.3.29)$$

However, the probability of elements EF and FG failing provide the vast majority of the numerical value of $p(A_1|s=2.0)$ and $p(A_2|A_1, s=2.0)$. This is despite the fact that only one of the two elements is physically able to fail in a mode system failure consisting of two modes of local failure. This shows that the balancing referred to above does not always succeed in producing a reasonable probability of system failure.

Element EF can be removed after the first stage, instead of element DF, which produces a second stage of the local failure of element DF (since it is the only element where local failure produces system failure), or

$$p(A_2|A_1,\ s_1 = 2.0) = \Phi(-\frac{5.0-2.0}{0.5}) = 1.0 \times 10^{-9} \qquad (4.3.30)$$

Alternatively, a stage 1.5 can be included where any of the other five elements can fail, so allowing stage 2 to include another failure from those elements, or

$$p(A_2|A_1,\ s=2.0) = 1.0 - \left[1.0 - 1.0 \times 10^{-9}\right] \times \left[1.0 - p(A_{1.5}|A_1,\ s=2.0) \times p(A_{2'}|A_1, A_{1.5},\ s=2.0)\right] \qquad (4.3.31)$$

where

$$p(A_{1.5}|A_1, s=2.0) = 1.0 - \left[1.0 - \Phi(-\frac{3.0-1.0}{0.3})\right]^3 \times \left[1.0 - \Phi(-\frac{2.0-1.0}{0.2})\right]^2 = 0.57 \times 10^{-6} \qquad (4.3.32)$$

Event $A_{1.5}$ is the removal of element CD rather than element AB for similar reasons to the removal of element EF being event A_1, so that

$$p(A_{2'}|A_1, A_{1.5},\ s=2.0) = 1.0 - \left[1.0 - 1.0 \times 10^{-9}\right] \times \left[1.0 - 0.57 \times 10^{-6} \times 0.13 \times 10^{-3}\right] = 0.11 \times 10^{-8} \qquad (4.3.34)$$

and

$$p(F|s=2.0) = 0.151 \times 10^{-5} \times 0.11 \times 10^{-8} = 0.166 \times 10^{-14} \qquad (4.3.35)$$

In contrast to the result given by equation (4.3.29), the result given by equation (4.3.35) is likely to be too small. This is because only a proportion of the possible failure paths have now been included in this modified method. This demonstrates that this model is only really suitable for purely series or purely parallel systems, since with mixed systems it must include some impossible events (eg system failure resulting from only the local failures of elements EF and FG in the example) in order to include all the failure paths.

4.3.9 Ductile Redundant Model

The failure network is used for Marshall and Bee's ductile redundant model, so the component elements have to be arranged together in a set of series and parallel combinations. This method then combines the safety indices of each element in stages to produce a single safety index for the whole structure.

Elements with the same resistance characteristics which act in parallel are assumed to be perfectly correlated. Therefore the safety factor on the assemblage will be the mean of the elements' safety factors weighted with respect to their load effects, S_i, if the resistance of the assemblage is taken as the sum of the elements' resistances. This will also mean that the coefficient of variation of the assemblage will equal that of the elements if equation (4.3.24) holds. Therefore

$$E(R_{ass}) = \sum_{i=1}^{n} E(R_i) \tag{4.3.36}$$

$$\sigma(R_{ass}) = \sum_{i=j}^{n} \sigma(R_i) \tag{4.3.37}$$

$$s_{ass} = \sum_{i=1}^{n} s_i \tag{4.3.38}$$

where n is the number of members to be combined. Hence

$$\begin{aligned} \beta_{ass} &= (E(R_{ass}) - s_{ass})/\ (R_{ass}) \\ &= (\sum_{i=1}^{n} (E(R_i)) - \sum_{i=1}^{n} (s_i))/ \sum_{i=1}^{n} (\sigma(R_i)) \\ &= \frac{\sum_{i=1}^{n} (\beta_i * \sigma(R_i))}{\sum_{i=1}^{n} (\sigma(R_i))} \end{aligned} \tag{4.3.39}$$

Therefore this weighted mean of β_i can be used when the elements do not have the same safety index/safety factor relationship.

Elements or assemblages that are combined in series are treated as if their resistances are uncorrelated, so that

$$\beta_{ass} = -\Phi^{-1} \left[1.0 - \prod_{i=1}^{n} (1.0 - \Phi(\beta_i)) \right] \tag{4.3.40}$$

where n is the number of elements or assemblages being combined. In the example described in Appendix B with s = 4.5, the safety index for assemblage a (elements AB, BC and BD in series) is found as

$$\beta_a = -\Phi^{-1} \left[1.0 - (1.0 - \Phi(- \frac{3.0-1.667}{0.3}))^3 \right] = 4.20 \tag{4.3.41}$$

The combination of a set of series assemblages in parallel is not clearly defined in their paper, but equations (4.3.36) to (4.3.39) could be used if the means and standard deviations of the series assemblages can be obtained. The mean resistance of a series assemblage will be approximately the mean resistance of its weakest component since the series assemblage will fail once the weakest of its components has failed. The load effect on the assemblage will be equal to the load effect on each of its components so that the standard deviation can be approximated to by

$$\sigma(R_{ass}) \doteq \frac{\left[\underset{i=1}{\overset{n}{\text{Min}}} (E(R_i)) - S_{ass}\right]}{\beta_{ass}} \qquad (4.3.42)$$

4.3.10 Example

For assemblage a in Appendix B, this approach gives

$$E(R_a) = \text{Min } (3.0, 3.0, 3.0) = 3.0 \qquad (4.3.43)$$

and, since $S_a = 1.667$,

$$(R_a) = (3.0 - 1.667)/4.20 = 0.317 \qquad (4.3.44)$$

Using these values, together with similar values for assemblage b, the values for assemblage d, the parallel combination of assemblages a and b, become

$$S_d = S_a + S_b = 3.333 \qquad (4.3.45)$$

$$E(R_d) = E(R_a) + E(R_b) = 5.0 \qquad (4.3.46)$$

$$\sigma(R_d) = \sigma(R_a) + E(R_b) = 0.571 \qquad (4.3.47)$$

and hence

$$\beta_d = \frac{E(R_d) - S_d}{(R_d)} = 2.91 \qquad (4.3.48)$$

Equation (4.3.39) need only be used to find β_{ass} directly when the other parameter are not required, for example when obtaining β_{system}. In this example

$$\beta_{system} = \frac{(-2.83 \times 0.589 + 2.84 \times 0.587)}{(0.589 + 0.587)} = 0.0 \qquad (4.3.49)$$

Once the safety indices have been combined into a single index, the probability of failure being realised when load S is applied is found by

$$p(F|S) = \Phi(-\beta_{system}) \qquad (4.3.50)$$

which, for this example, gives

$$p(F|s = 4.5) = \Phi(-0.0) = 0.5 \qquad (4.3.51)$$

4.3.11 Comparison of Ductile- and Brittle- Redundant Models

The brittle-redundant model does not always produce a higher probability of system failure when compared to the ductile-redundant model. This is because resistances in parallel are treated as mutually uncorrelated in the former and as mutually perfectly positively correlated in the latter. The simplest example of the result of these conflicting assumptions is two identical elements in parallel, both with resistances R and with load effect s_1 applied to them. The brittle-reundant model gives

$$p(F|s) = \left[2\,\Phi\left(-\frac{E(R)-s_1}{\sigma(R)}\right) - \Phi^2\left(-\frac{E(R)-s_1}{\sigma(R)}\right)\right] \times \Phi\left(-\frac{E(R)-2s_1}{\sigma(R)}\right) \qquad (4.3.52)$$

while the ductile-redundant model gives

$$p(F|s) = \Phi\left(-\frac{E(R)-s_1}{\sigma(R)}\right) \qquad (4.3.53)$$

Hence the brittle-redundant model will give a smaller probability of system failure when equation (4.3.52) gives a lower value than equation (4.3.53), which occurs when

$$\Phi\left(-\frac{E(R)-2s_1}{\sigma(R)}\right) < \frac{1.0}{2.0 - \Phi\left(\frac{E(R)-s_1}{(R)}\right)} \lesssim \frac{1}{2} \qquad (4.3.54)$$

This includes all cases where $E(R) > 2s_1$. Therefore the use of the two models to produce bounds will be highly suspect, particularly for simple structures.

These calculations are relatively easy to perform by hand, but if the probability of failure is required for a variable load, the probability of system failure is given by

$$p(F) = \int_{-\infty}^{+\infty} p(F|s) \times p(s)\, ds \qquad (4.3.55)$$

This equation requires the use of numerical integration, which will have to be carried out over each loading variable if there is more than one. Therefore this method can become laborious as a hand method for the unconditional probability of system failure, and it is better suited to check the failure probability for structures that have experienced a known maximum load in order to update the estimate of its resistance. Such Bayesian calibration [26,31] is used with offshore structures, for which Marshall and Bea's paper is intended.

4.4 BOUNDS ON THE SYSTEM FAILURE PROBABILITY

4.4.1 Need for Bounds

The methods described in section 4.3 demonstrate that gross assumptions about and/or severe restraints on the idealised system have to be made in order to obtain a value for the probability of system failure without excessive calculations. These assumptions and restraints limit the confidence that can be placed in the value of any results obtained by these methods and reduce the opportunities for the use of these methods respectively. An alternative approach is to make complementary assumptions, when assumptions have to be made, so as to obtain an upper and a lower bound on the probability of system failure. These should then lead to a measure of the reliability of the structural system (the mean of the two bounds) together with a measure of the confidence that can be placed in that reliability (the difference between the two bounds) for a more general set of structures.

The methods which employ bounds tend to make idealisations of the correct expression for the probability of system failure (as Vanmarke's method) rather than to make idealisations of the system, which then leads directly to simpler expressions for the probability of system failure (as in Moses' or Marshall and Bea's methods). The full expression for the probability of system failure, in terms of the component events of local failures, can be obtained by considering a failure tree. Failure trees describe the event of system failure as the union of a set of failure paths, each failure path being the intersections of a series of modes of local failure. Therefore the probability of the event of system failure can be written as

$$p(F) = p\left(\bigcup_{i=1}^{m} \left\{ \bigcap_{j=1}^{n_i} \left[a_{i,j} \right] \right\}\right) \qquad (4.4.1)$$

where $a_{i,j}$ is the event of the j^{th} local failure in the i^{th} failure path, n_i is the number of local failure modes in the i^{th} failure path and m is the number of failure paths in the failure set.

The previous works about bounds on the probability of system failures that are studied here limit the structures that can be analysed by them to those which only fail by series combinations of local failures. This implies that the event of system failure consists only of the union of the events of local failures (ie $n_i = 1$ for all i), so that equation (4.4.1) reduces to

$$p(F) = p\left(\bigcup_{i=1}^{m} \left\{ A_i \right\}\right) \qquad (4.4.2)$$

Therefore these methods can be used only with structures where system failure is defined as the occurrance of the first local failure.

It should be noted here that local failure modes can include the formation of mechanisms if the elements are perfectly ductile. This is because the failure event in that case can be described explicitly by a single criterion (eg the sum of the moments induced by the applied load being greater than or equal to the sum of the ultimate moments of the elements) rather than as the event of a series of failure events, each described by a different criteria (eg each moment induced by the applied load being greater than or equal to the ultimate moment of the appropriate element).

4.4.2 Cornell's Bounds

Cornell[13] defines his system as structures with n_m modes of local failure acted upon by a set of loads which are applied n_s times. If $A_{i,j}$ is the event of failure by the i^{th} mode of failure during the j^{th} load application, then bounds can be built up for the union of such events. These bounds are based on the relationships:

i. for mutually perfectly negatively correlated excess resistances (fig 4.4.1) (ie $\rho(A,B) = -1.0$) of

$$\left.\begin{aligned} p(a \cap b) &= \text{Max } (0.0,\ p(a) + p(b) - 1.0) \\ p(a \cup b) &= \text{Min } (p(a) + p(b),\ 1.0) \end{aligned}\right\} \qquad (4.4.3)$$

ii. for mutually uncorrelated excess resistances (ie $\rho(A,B) = 0.0$) of

$$\left.\begin{aligned} p(a \cap b) &= p(a) \times p(b) \\ p(a \cup b) &= p(a) + p(b) - p(a) \times p(b) \end{aligned}\right\} \qquad (4.4.4)$$

and iii. for mutually perfectly positively correlated excess resistances (fig 4.4.2) (ie $\rho(A,B) = +1.0$) of

$$\left.\begin{aligned} p(a \cap b) &= \text{Min } (p(a),\ p(b)) \\ p(a \cup b) &= \text{Max } (p(a),\ p(b)) \end{aligned}\right\} \qquad (4.4.5)$$

where $\rho(A,B)$ is the correlation coefficient of the excess resistances, A and B, for the events of local failure a and b respectively. The excess resistance for a failure event is a numerical value which is positive if the failure does not occur and negative if it does.

The various bounds (Table 4.4.1) are given for the cases of:

i. independent (ie $\rho(A,B) = 0.0$);

ii. perfectly (functionally) dependent (ie $\rho(A,B) = +1.0$); and

iii. "imperfect" or unknown degree of dependence (ie $-1.0 \leqslant \rho(A,B) \leqslant +1.0$).

for both the applied loads and the structure's resistance to each mode of failure.

The lower bound is equal to the maximum probability of any of the local failures in every case, while the upper bound can be many times this value. The most typical case is likely to be independent load applications (ie assuming that the peaks of the load intensity are independent) with an imperfect degree of dependence between the resistance to the modes of failure (eg in a portal frame where some plastic hinges will be coincident for different mechanisms, or modes of failure). The upper bound in that case is the sum of the probability of each of the local modes of failure, so that the bounds could differ by several orders of magnitude if there are a large number of approximately equal component failure probabilities. Hence these bounds are too general for most applications.

4.4.3 Moses and Kinser's Bounds

Moses and Kinser[27] assume that there is no negative correlation of the excess resistances for their bounds. Using the same system that Cornell defined, this gives the bounds for the most general case as

$$\underset{i=1}{\overset{n_m}{\text{Max}}} \left[\underset{j=1}{\overset{n_s}{\text{Max}}} \left\{ p(A_{i,j}) \right\} \right] \leqslant p(F) \leqslant 1.0 - \prod_{i=1}^{n_m} \left[\prod_{j=1}^{n_s} \left\{ 1.0 - p(A_{i,j}) \right\} \right]$$

$$\doteq \sum_{i=1}^{n_m} \left[\sum_{j=1}^{n_s} \left\{ p(A_{i,j}) \right\} \right] \qquad (4.4.6)$$

However, this inequality can include negative correlation of the excess resistances, since, although the upper bound gives the correct value for mutually uncorrelated excess resistances, the approximation to the upper bound (for cases when $p(A_{i,j}) \ll 1.0$) gives the correct value for perfectly negatively correlated excess resistances (provided the sum does not exceed unity). The lower bound gives the correct value for fully perfectly positively correlated excess resistances, as in Cornell's bounds.

In order to obtain a closer upper bound, the event of a local failure is defined for a structure in which the load effects are linearly and deterministically related to the applied loads, as when

$$r_i - c_{i,j} s_j \leqslant 0.0$$

where r_i is the structure's resistance to the i^{th} mode of failure, s_j is the j^{th} load application and $c_{i,j}$ is a constant. Then

$$p(F) \leqslant \sum_{i=j}^{n_m} \left[\sum_{j=1}^{n_s} \left\{ M_{i,j} \right\} \right] \tag{4.4.8}$$

where $M_{i,j}$ is the probability of failure by mode i under load j, conditional on the survival of the structure:

i. under modes 1 through to (i-1) against load j; and

ii. under mode i against load 1 through to (j-1).

Moses and Kinser consider the case where all the resistances and all the loads are mutually uncorrelated, so that $M_{i,j}$ can be described mathematically as

$$M_{i,j} = \int_0^{\infty} \left\{ \int_0^{y} \left[\prod_{k=1}^{(j=1)} \left\{ F_{s_k} \left(\frac{C_{n_m,n_s}}{C_{n_m,k}} x \right) \times f_{r_i} \left(C_{k,n_s} x \right) \right\} \right] ds \right.$$

$$\left. \times \left[\prod_{k=1}^{(i-1)} \left\{ 1.0 - F_{r_k} \left(C_{k,n_s} \; y \right) \right\} \right] \times F_{s_k} \left(\frac{C_{n_m,n_s}}{C_{n_m,k}} \; y \right) \right\} dy \tag{4.4.9}$$

Equation (4.4.8) gives the exact probability of system failure when there is only one load application, but the resistances will still need to be mutually independent for the values of $M_{i,1}$ from equation (4.4.9) for use in equation (4.4.8) to be correct.

Equation (4.4.9) is a double convolution integral which will usually have to be solved by numerical integration for each of the $n_m \times n_s$ values of $M_{i,j}$ required for a complete, but approximate, upper bound. Therefore this method is too complex to justify the assumptions which are required to be made about the system. Since this method was put forward, improved reliability analysis techniques for evaluating local failures have been developed (section 2.4) which make this method relatively inefficient.

4.4.4 Ditlevsen's Bounds on the Probability of System Failure

Ditlevsen [17] also considers structural systems which are deemed to fail with the occurrance of any local failure, but he does not consider different load applications separately. This is common practice since the load variables can usually be defined by the probability distribution of the maximum load intensity in the reference period (usually the design life of the structure) rather than the instantaneous intensity of the load.

A_i is used to represent the event of the i^{th} local failure mode, which has a probability of occurring of $p(A_i)$, and F is used to represent the event of system failure, which has a probability of occurring of $p(F)$. Ditlevsen gives as bounds on the probability of system failure

$$P(A_1) + \sum_{i=2}^{n} \lfloor \text{Max} \left\{ P(A_i) - \sum_{j=1}^{i-1} \left[P(A_i \cap A_j) \right], 0.0 \right\} \rfloor \leqslant P(F) \leqslant$$

$$P(A_1) + \sum_{i=2}^{n} \lfloor P(A_i) - \underset{j=1}{\overset{i-1}{\text{Max}}} \left\{ P(A_i \cap A_j) \right\} \rfloor \qquad (4.4.10)$$

To demonstrate these bounds, consider the most general failure set with four possible local failure modes (A_1, A_2, A_3 and A_4). This is shown in fig 4.4.3, where 'a' represents $P(A_1 \cap A_2^c \cap A_3^c \cap A_4^c)$, ab represents $P(A_1 \cap A_2 \cap A_3^c \cap A_4^c)$, abc represents $P(A_1 \cap A_2 \cap A_3 \cap A_4^c)$, etc. Hence $P(A_1)$ is represented by the total area covered by regions with a in their titles, $P(A_2)$ by the total area with b titles, $P(A_3)$ by the total area with c titles and $P(A_4)$ by the total area with d titles. The bounds on the probability of system failure for this method when considering the local failure modes in ascending order is therefore

$$\begin{aligned} P(F) \geqslant & \lfloor (a + ab + ac + ad + abc + abd + acd + abcd) \rfloor \\ & + \lfloor (b + ab + bc + bd + abc + abd + bcd + abcd) - (ab + abc + abd + abcd) \rfloor \\ & + \text{Max} \lfloor 0, (c + ac + bc + cd + abc + acd + bcd + abcd) \\ & - (ac + abc + acd + abcd) - (bc + abc + bcd + abcd) \rfloor \\ & + \text{Max} \lfloor 0, (d + ad + bd + cd + abd + acd + bcd + abcd) \\ & - (ad + abd + acd + abcd) - (bd + abd + bcd + abcd) - (cd + acd + bcd + abcd) \rfloor \\ & = P(F) - \text{Min} (c + cd, abc + adcd) - \text{Min} (d, abd + acd + bcd + 2 \times abcd) \end{aligned}$$

(4.4.11)

and

$$\begin{aligned} P(F) \leqslant & \lfloor (a + ab + ac + ad + abc + abd + acd + abcd) \rfloor \\ & + \lfloor (b + ab + bc + bd + abc + abd + bcd + abcd) - (ab + abc + abd + abcd) \rfloor \\ & + \lfloor (c + ac + bc + cd + abc + acd + bcd + abcd) - \text{Max} (ac + abc + acd + \\ & \quad abcd, bc + abc + bcd + abcd) \rfloor \\ & + \lfloor (d + ad + bd + cd + abd + acd + bcd + abcd) - \text{Max} (ad + abd + acd + \\ & \quad abcd, bd + abd + bcd + abcd, cd + acd + bcd + abcd) \rfloor \\ & = P(F) + \text{Min} (bc + bcd, ac + acd) + \text{Min} (bd + cd + bcd, ad + cd + acd, \\ & \quad ad + bd + abd) \end{aligned}$$

(4.4.12)

The order in which the local failure modes are considered may affect the magnitude of the computed bounds, so the procedure can be repeated with a different order so as to improve either bound if the computed values seem too wide. However, there are n! sets of bounds if all the permutations in the general case are considered.

4.4.5 The width of the Bounds on the Probability of System Failure

It can be seen from equation (4.4.11) that the lower bound will be exact if no more than two events can occur simultaneously, ie if most events are mutually exclusive (eg A_1 and A_2 plus A_3 and A_4). Similarly in equation (4.4.12) the upper bound will be exact if every fresh mode local failure considered is mutually exclusive of all but one of the modes of local failure already included. This means that certain failure systems, such as those shown as Venn diagrams in fig 4.4.4, will produce equal bounds giving the exact value of $p(F)$. Such systems will be rare in practice, since most local failure modes will not be mutually exclusive, although the bounds can appear to be identical when the probability of two or more local failures occurring simultaneously is several orders of magnitude less than the probability of a single local failure occurring.

The probability of failure has to be several orders of magnitude smaller than the probability of safety in nearly all practical situations. Provided that the excess resistances are not strongly positively mutually correlated, this means that in most cases a (which represents $p(A_1 \cap A_2^c \cap A_3^c \cap A_4^c)$, b, c and d will be very much larger than ab (which represents $p(A_1 \cap A_2 \cap A_3^c \cap A_4^c)$), ac, ad, bc, bd and cd; which in turn will be very much larger than abc (which represents $p(A_1 \cap A_2 \cap A_3 \cap A_4^c)$, abd, acd and bcd; which in turn will be very much larger than abcd (which represents $p(A_1 \cap A_2 \cap A_3 \cap A_4)$). Therefore the error terms in equations (4.4.11) and (4.4.12) will often be negligible, particularly the lower bound which has third order terms rather than the second order terms of the upper bound. Only if the excess resistances of the local failure modes are strongly positively correlated (ie when $p(A_i \cap A_j) \doteq$ Min $(p(A_i), p(A_j)$, etc) can these error terms become significant, and then not if Max $(p(A_i), p(A_j)) \gg$ Min $(p(A_i), p(A_j))$. The strong correlation of the excess resistances is not an uncommon case, however, since the resistances of the elements in a structure can be strongly correlated when they come from a common source, and the load effect on each element is liable to be a proportion of an overall load.

Four cases of the system described by fig 4.4.3 are given in table 4.4.2. The values of the probabilities of failure were chosen arbitarily so that the four cases illustrate respectively:

i. Strong positive mutual correlation with approximately equal probability of each local failure;

ii. Weak positive or negative mutual correlation with approximately equal probability of each local failure;

iii. Strong positive mutual correlation with unequal probability of each local failure; and

iv. Weak positive or negative mutual correlation with unequal probability of each local failure,

where correlation refers to their excess resistances. All the errors in the bounds are small, but the largest is for case i, as anticipated, in which the lower bound error is smaller than the upper, again as anticipated.

4.4.6 Ditlevsen's Bounds on the Probability of Two Local Failures

Ditlevsen considered the specific case where each mode of failure in the failure set is defined by an inhomogeneous linear combination of mutually uncorrelated standard normal variables. For this case the excess resistance for each mode of local failure can be written as

$$Z_i = c_i + \sum_{k=1}^{n} (d_{i,k} U_k) \tag{4.4.13}$$

where U_k are the variables and c_i and $d_{i,k}$ are constants. Hence the mean, standard deviation and correlation coefficients of Z_i can be found by

$$E(Z_i) = c_i \tag{4.4.14}$$

$$\sigma^2(Z_i) = \sum_{k=1}^{n} (d_{i,k}^2) \tag{4.4.15}$$

and

$$\rho(Z_i, Z_j) = \left\{ E(Z_i \times Z_j) - E(Z_i) \times E(Z_j) \right\} / \left\{ \sigma(Z_i) \times \sigma(Z_j) \right\}$$

$$= \frac{\sum_{k=j}^{n} (d_{i,k} \times d_{j,k})}{\sqrt{\sum_{k=1}^{n} (d_{i,k}^2) \times \sum_{k=1}^{n} (d_{j,k}^2)}} \tag{4.4.16}$$

Since the basic variables have normal distributions, the excess resistance also has a normal distribution and

$$p(A_i) = \Phi(-E(Z_i) / \sigma(Z_i)) = \Phi(-\beta_i) \tag{4.4.17}$$

A knowledge of the physical significance in the variable space of the value of $\rho(Z_1, Z_2)$, as evaluated from equation (4.4.16) is required in order to establish narrow bounds on the value of $p(A_i \cap A_j)$. This relationship is that $\rho(Z_1, Z_2)$ is equal to the cosine of the angle, Θ, between the normals to the hyper plane failure boundaries which pass through the origin. This can be demonstrated by considering the example of two local failure modes, with excess resistances Z_i and Z_j, such that

$$Z_i = c_i + \sum_{k=1}^{n} (d_{i,k} \; U_k) \tag{4.4.18}$$

and

$$Z_j = c_j + \sum_{k=1}^{n} (d_{j,k} \; U_k) \tag{4.4.19}$$

The direction cosines of the normals to these boundaries are given by

$$\alpha_{i,k} = \frac{d_{i,k}}{\sqrt{\sum_{l=1}^{n} (d_{i,l}^2)}} \quad ; \quad \alpha_{j,k} = \frac{d_{i,k}}{\sqrt{\sum_{l=1}^{n} (d_{j,l}^2)}} \tag{4.4.20}$$

and, from n-dimensional geometry,

$$\cos\Theta = \sum_{k=1}^{n} (\alpha_{i,k} \; \alpha_{j,k}) \tag{4.4.21}$$

Hence substituting equation (4.4.20) in equation (4.4.21) and comparing with equation (4.4.21) gives

$$\cos\Theta = \frac{\sum_{k=1}^{n} (d_{i,k} \; d_{j,k})}{\sqrt{\sum_{l=1}^{n} (d_{i,l}^2) \times \sum_{l=1}^{n} (d_{j,l}^2)}} = \rho(Z_i, Z_j) \tag{4.4.22}$$

This relationship is used by Ditlevsen when he considers the plane in the n-dimensional U-space which contains both these normals, and hence also the origin (figs 4.4.5 and 4.4.6). He establishes, by simple geometry, his bounds on $P(A_i \cap A_j)$ as

$$\text{Max} \left[\Phi(-\beta_i) \times \Phi(-\beta_j^*), \; \Phi(-\beta_j) \times \Phi(-\beta_i^*) \right] \leqslant P(A_i \cap A_j) \leqslant$$

$$\Phi(-\beta_i) \times \Phi(-\beta_j^*) + \Phi(-\beta_j) \times \Phi(-\beta_i^*) \Big]$$

$$\text{for} \; \rho(Z_i, Z_j) > 0.0 \tag{4.4.23}$$

and

$$\text{Max} \left[0.0, \; \Phi(-\beta_i) + \Phi(-\beta_j) - 1.0 \right] \leqslant P(A_i \cap A_j) \leqslant$$

$$\text{Min} \left[\Phi(-\beta_i) \times \Phi(-\beta_j^*), \; \Phi(-\beta_j) \times \Phi(-\beta_i^*) \right]$$

$$\text{for} \; \rho(\beta_i, \beta_j) < 0.0 \tag{4.4.24}$$

where

$$\beta_i^* = \frac{\beta_i - \beta_j \cos\Theta}{\sqrt{1.0 - \cos^2\Theta}} = \frac{\beta_i - \beta_j \rho(Z_i, Z_j)}{\sqrt{1.0 - \rho^2(Z_i, Z_j)}} \tag{4.4.25}$$

and similarly for β_j^*

4.4.7 The Width of the Bounds on the Probability of Two Local Failures

The physical significance of β_j^* can be found by considering the distribution of Z_j conditional on $z_i = 0.0$. Since

$$E(Z_j | z_i = 0.0) = E(Z_j) + \rho(Z_i, Z_j) \times \frac{\sigma(Z_j)}{\sigma(Z_i)} \times (0.0 - E(Z_i))$$

$$= E(Z_j) - E(Z_i) \times \frac{\sigma(Z_j)}{\sigma(Z_i)} \times \rho(Z_i, Z_j) \qquad (4.4.26)$$

and

$$\sigma^2(Z_j | z_i = 0.0) = \sigma^2(Z_j) \times (1.0 - \rho^2(Z_i, Z_j)) \qquad (4.4.27)$$

Then

$$p(A_j | z_i = 0.0) = \Phi\left[- \frac{E(Z_j) - E(Z_i) \times \frac{\sigma(Z_j)}{\sigma(Z_i)} \times \rho(Z_i, Z_j)}{\sigma(Z_j) \sqrt{1.0 - \rho^2(Z_i, Z_j)}} \right]$$

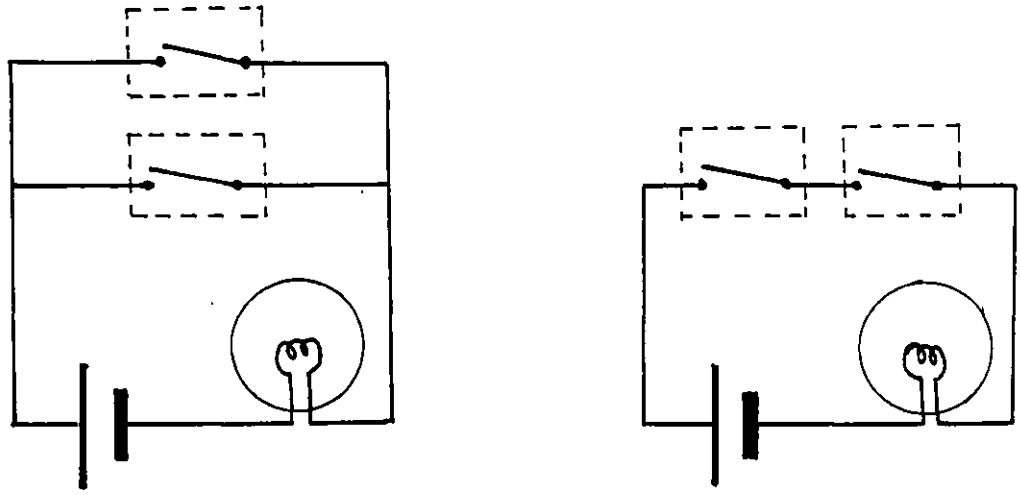
$$= \Phi(-\beta_j^*) \qquad (4.4.28)$$

Therefore, $p(A_j | z_i = 0.0) \leqslant p(A_j | A_i)$ if $\rho(Z_i, Z_j) \geqslant 0.0$ and $p(A_j | z_i = 0.0) \geqslant p(A_j | A_i)$ if $\rho(Z_i, Z_j) \leqslant 0.0$. When considering fig 4.4.6, the probability density contours will be circles centred at the origin, which is the point of maximum density. Therefore, the value of $p(A_i) \times p(A_j | z_i = 0.0)$ and $p(A_j) \times p(A_i | z_j = 0.0)$ which differs least from $p(A_i \cap A_j)$ will be the one whose quadrant is closest to the origin, since that is the one whose error segment is farthest from the origin. This will be $p(A_i) \times p(A_j | z_i = 0.0)$ when $\beta_i < \beta_j$, and similarly vice versa. This error segment will become a smaller proportion of the probability as the difference between β_i and β_j increases since the contours of probability density will tend towards being parallel with the failure boundary of local failure mode J (fig 4.4.7).

The bounds on $p(A_i \cap A_j)$ found in equations(4.4.23) and (4.4.24) are used to find the bounds on $p(F)$ by equation (4.4.10). The lower bound on $p(A_i \cap A_j)$ has to be used for the upper bound on $p(F)$, and similarly the upper bound for the lower bound, in order that the bounds on $p(F)$ are accurate since only negative values of $p(A_i \cap A_j)$ are used in equation (4.4.10).

The ratio of the lower bound on $p(A_i \cap A_j)$ to its upper bound can be as low as a half when the excess resistances are positively correlated (this value occurs

when $\rho(Z_i, Z_j)$ tends towards zero, so that $\beta_i \doteq \beta_i^*$ and $\beta_j \doteq \beta_j^*$, but equation (4.4.23) still applies), and zero when negatively correlated (since the lower bound is usually zero in that case). Therefore any difference in the system bounds in the bounds on the probability of system failure can usually be attributed principally to the bounds on the intersection of pairs of local failure modes since the inherent errors in equations (4.4.11) and (4.4.12) are small, except in the case of equally probable local failure modes whose excess resistances are strongly mutually positively correlated.



(i) Parallel (ii) Series

Fig.4.1.1 - Combination of Binary Systems

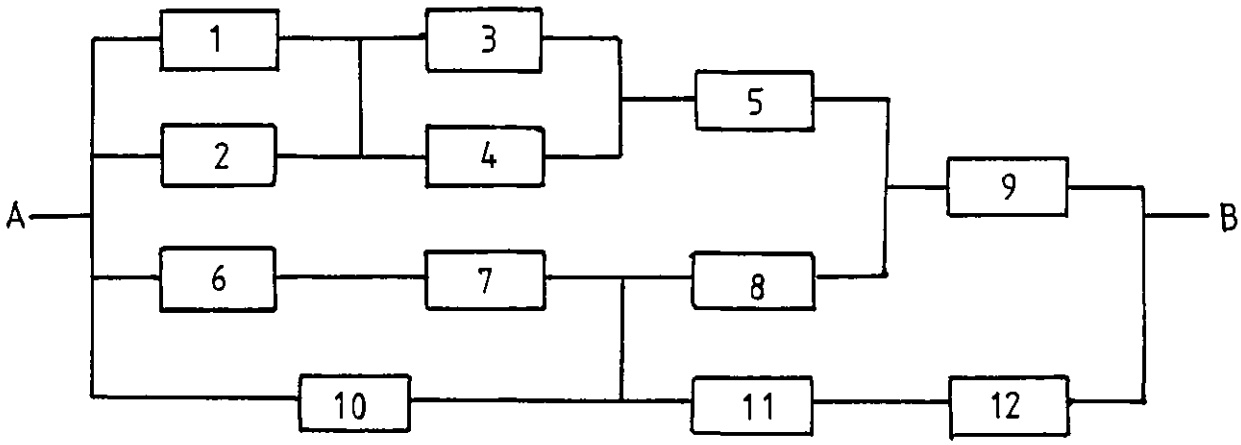
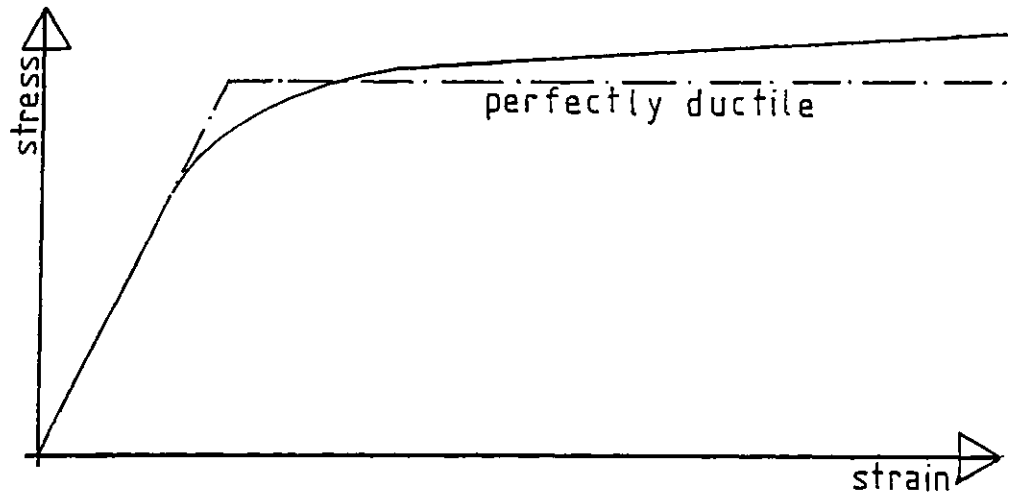
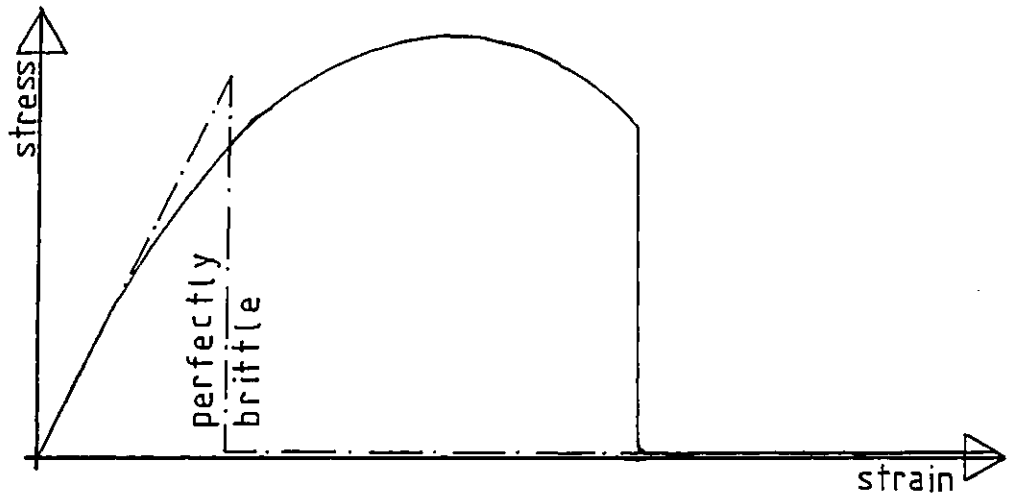


Fig.4.1.2 - Example of a Failure Network



(i) Example of behaviour typically assumed ductile



(ii) Example of behaviour typically assumed brittle

Fig.4.1.3 - Behaviour of Materials under Load

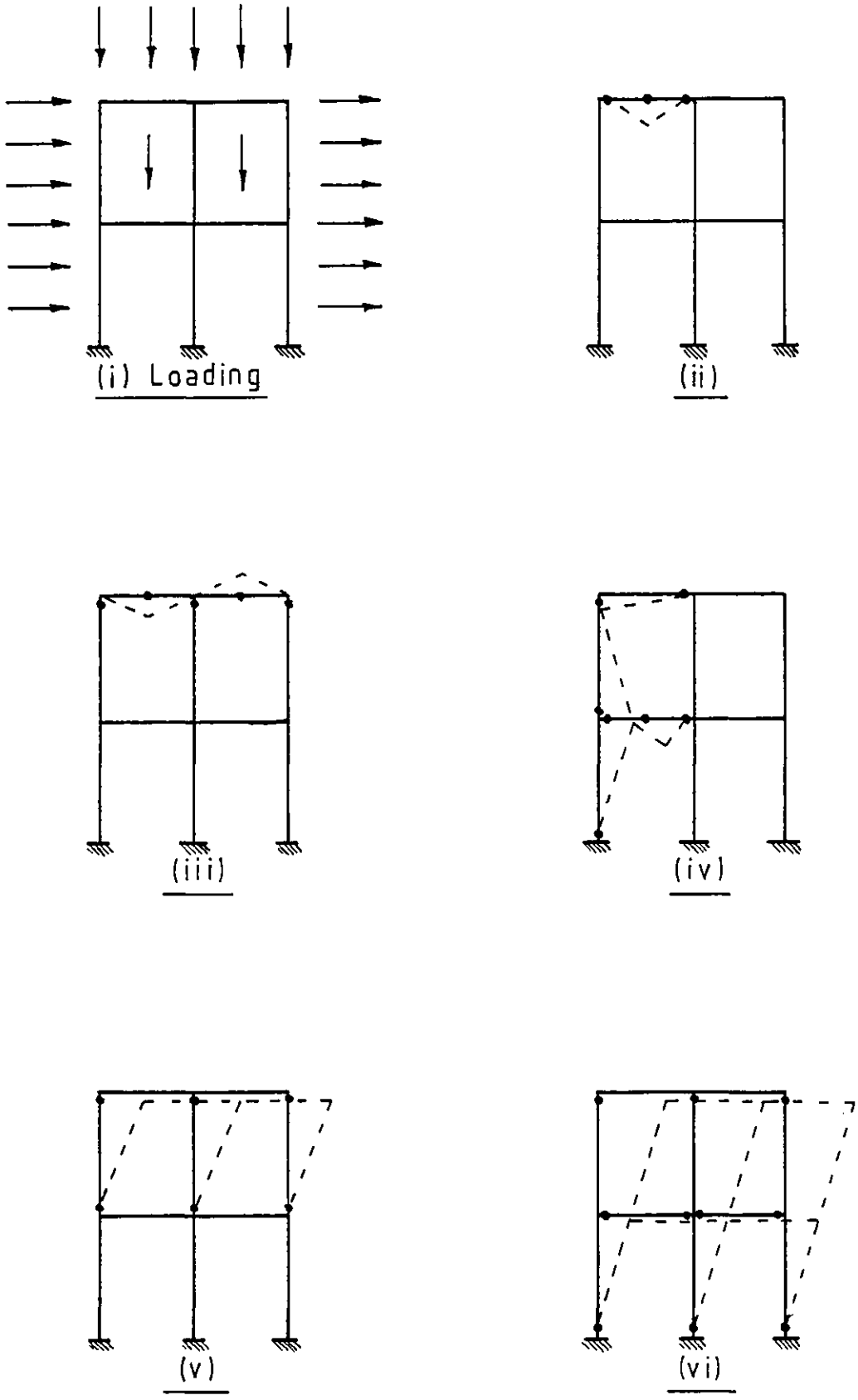


Fig.4.1.4 - Loaded Structure with some Possible Failures of Sub-Systems

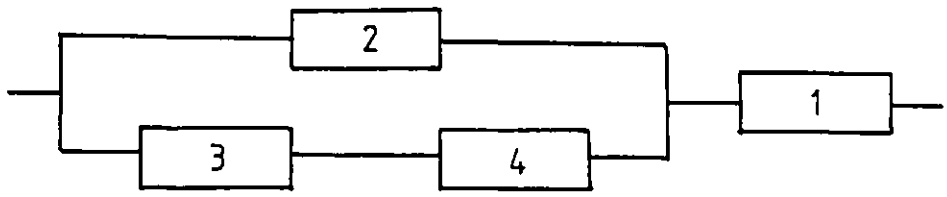


Fig. 4.1.5 - Failure Network for a Simple System

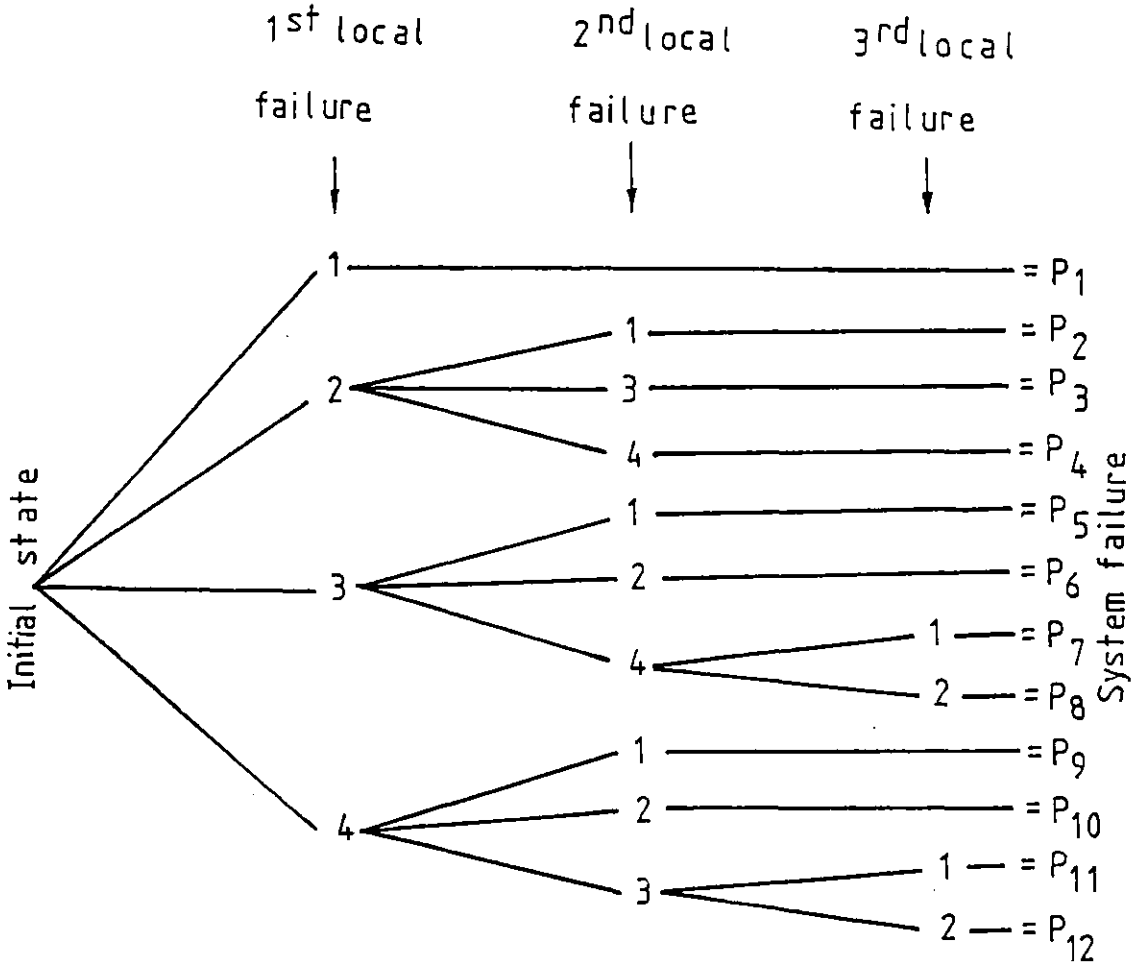
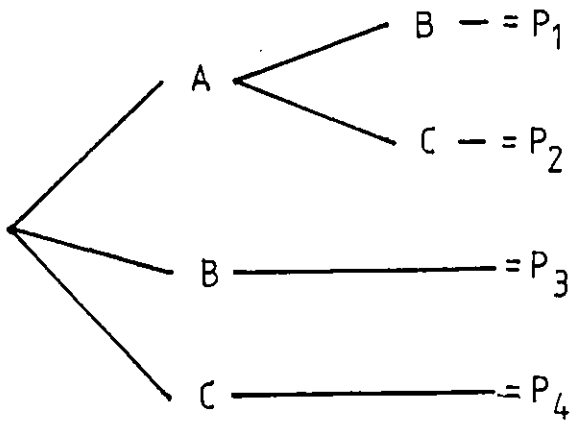
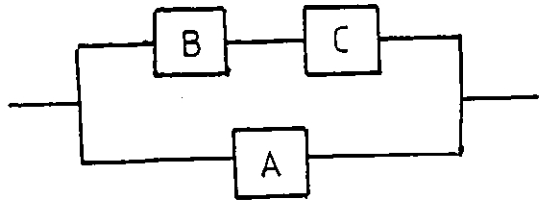


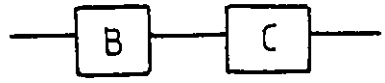
Fig. 4.1.6 - Failure Tree for a Simple System



(i) Failure tree

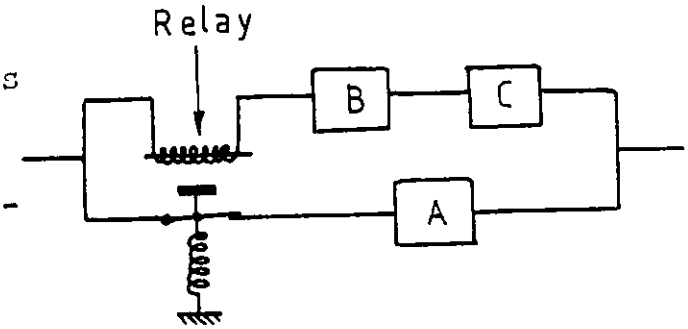


(ii) Failure network for paths P_1 and P_2



(iii) Failure network for paths P_3 and P_4

(The structure constists of three brittle members (A, B and C). The local failure of B or C results in system failure, but not of A since the structure is assumed to be able to redistribute all the load effect from A to B and C.)



(iv) Failure network including relay

Fig.4.1.7 - Failure Tree Requiring a Non-Standard Failure Network Symbol

LOADS	SYSTEM FAILURE MODES		
	PERFECT (FUNCTIONALLY) DEPENDENT RESISTANCES	"IMPERFECT" OR UNKNOWN DEGREE OF DEPENDENCE	INDEPENDENT RESISTANCES
PERFECT (FUNCTIONALLY) DEPENDENT	$p(F) = \overset{n_m}{\underset{i=1}{\text{Max}}}\left(\overset{n_s}{\underset{j=1}{\text{Max}}}\left(p(A_{i,j})\right)\right)$	$\overset{n_s}{\underset{j=1}{\text{Max}}}\left(\sum_{i=1}^{n_m}\left(p(A_{i,j})\right)\right) \geqslant p(F) \geqslant \overset{n_m}{\underset{i=1}{\text{Max}}}\left(\overset{n_s}{\underset{j=1}{\text{Max}}}\left(p(A_{i,j})\right)\right)$	
"IMPERFECT" OR UNKNOWN DEGREE OF DEPENDENCE	$\sum_{j=1}^{n_s}\left(\overset{n_m}{\underset{i=1}{\text{Max}}}\left(p(A_{i,j})\right)\right) \geqslant p(F) \geqslant$	$\sum_{i=1}^{n_m}\left(\sum_{j=1}^{n_s}\left(p(A_{i,j})\right)\right) \geqslant p(F) \geqslant \overset{n_m}{\underset{i=1}{\text{Max}}}\left(\overset{n_s}{\underset{j=1}{\text{Max}}}\left(p(A_{i,j})\right)\right)$	$\left.\begin{array}{c}\sum_{i=1}^{n_m}\left(\sum_{j=1}^{n_s}\left(p(A_{i,j})\right)\right) \\ \text{to} \\ \overset{n_s}{\underset{j=1}{\text{Max}}}\left(\sum_{i=1}^{n_m}\left(p(A_{i,j})\right)\right)\end{array}\right\} \geqslant p(F) \geqslant \overset{n_m}{\underset{i=1}{\text{Max}}}\left(\overset{n_s}{\underset{j=1}{\text{Max}}}\left(p(A_{i,j})\right)\right)$
INDEPENDENT	$\overset{n_m}{\underset{i=1}{\text{Max}}}\left(\overset{n_s}{\underset{j=1}{\text{Max}}}\left(p(A_{i,j})\right)\right)$	$\left.\begin{array}{c}\sum_{i=1}^{n_m}\left(\sum_{j=1}^{n_s}\left(p(A_{i,j})\right)\right) \\ \text{to} \\ \sum_{j=1}^{n_s}\left(\overset{n_m}{\underset{i=1}{\text{Max}}}\left(p(A_{i,j})\right)\right)\end{array}\right\} p(F) \geqslant \overset{n_m}{\underset{i=1}{\text{Max}}}\left(\overset{n_s}{\underset{j=1}{\text{Max}}}\left(p(A_{i,j})\right)\right)$	$\sum_{i=1}^{n_m}\left(\sum_{j=1}^{n_s}\left(p(A_{i,j})\right)\right) \geqslant p(F) \geqslant \overset{n_m}{\underset{i=1}{\text{Max}}}\left(\overset{n_s}{\underset{j=1}{\text{Max}}}\left(p(A_{i,j})\right)\right)$

$A_{i,j}$ is the event of failure by the i^{th} mode of failure during the j^{th} load application

n_m = number of system modes of failure

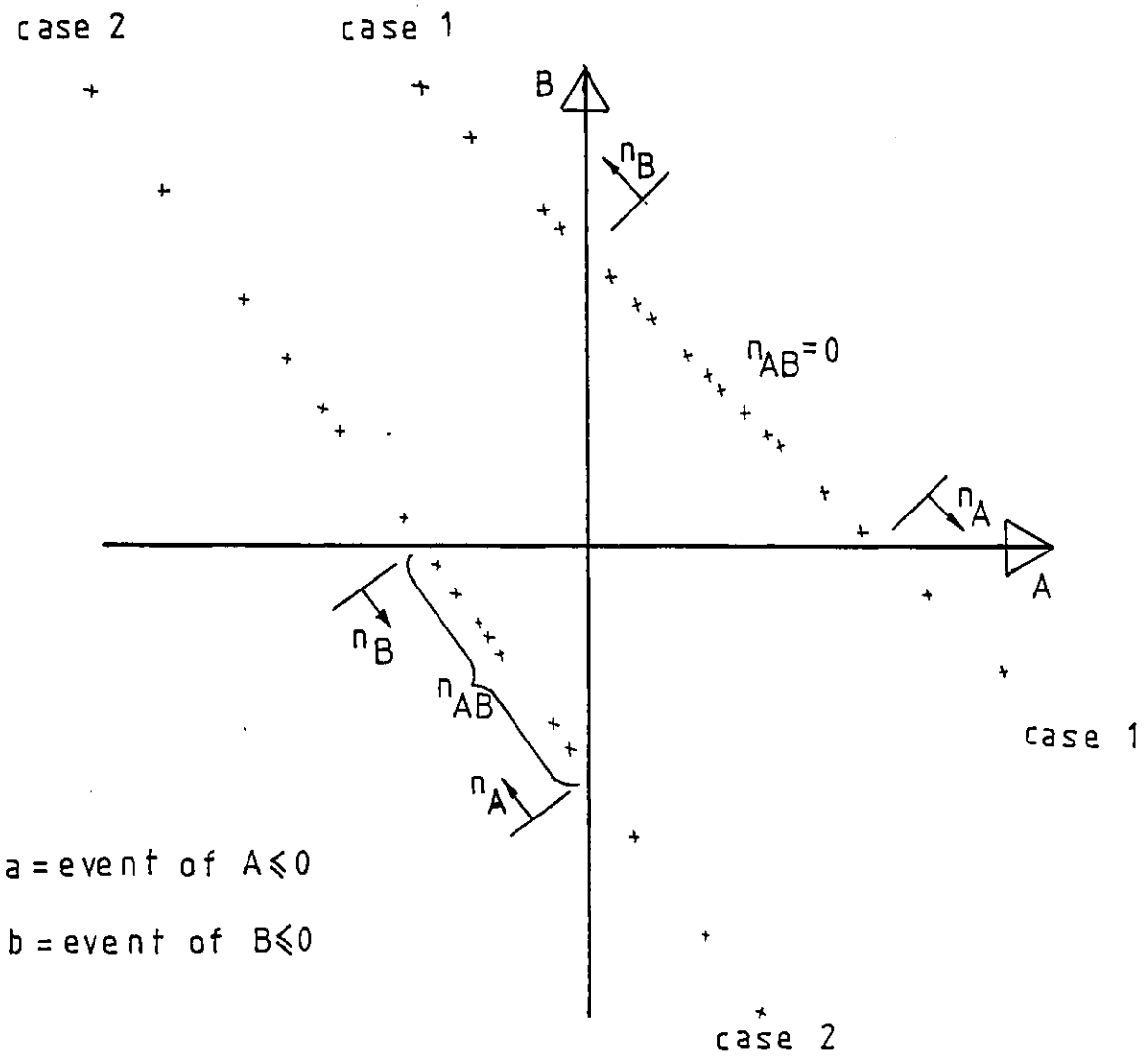
n_s = number of load application

TABLE 4.4.1 – BOUNDS ON THE FAILURE PROBABILITY OF A STRUCTURE SUBJECTED TO SUCCESIVE LOADS (FROM CORNELL [13])

	CASE (i)	CASE (ii)	CASE (iii)	CASE (iv)
a	0.005×10^{-4}	0.250×10^{-4}	0.005×10^{-4}	0.085×10^{-4}
b	0.005×10^{-4}	0.300×10^{-4}	0.900×10^{-4}	0.900×10^{-4}
c	0.005×10^{-4}	0.200×10^{-4}	0.001×10^{-4}	0.009×10^{-4}
d	0.005×10^{-4}	0.230×10^{-4}	0.000×10^{-4}	0.001×10^{-4}
ab	0.016×10^{-4}	0.003×10^{-4}	0.080×10^{-4}	0.004×10^{-4}
ac	0.014×10^{-4}	0.002×10^{-4}	0.001×10^{-4}	0.000×10^{-4}
ad	0.012×10^{-4}	0.003×10^{-4}	0.000×10^{-4}	0.000×10^{-4}
bc	0.010×10^{-4}	0.004×10^{-4}	0.001×10^{-4}	0.000×10^{-4}
bd	0.010×10^{-4}	0.003×10^{-4}	0.001×10^{-4}	0.000×10^{-4}
cd	0.008×10^{-4}	0.002×10^{-4}	0.000×10^{-4}	0.000×10^{-4}
abc	0.040×10^{-4}	0.001×10^{-4}	0.008×10^{-4}	0.001×10^{-4}
abd	0.020×10^{-4}	0.000×10^{-4}	0.000×10^{-4}	0.000×10^{-4}
acd	0.030×10^{-4}	0.000×10^{-4}	0.000×10^{-4}	0.000×10^{-4}
bcd	0.020×10^{-4}	0.001×10^{-4}	0.001×10^{-4}	0.000×10^{-4}
abcd	0.800×10^{-4}	0.001×10^{-4}	0.002×10^{-4}	0.000×10^{-4}
$p(A_1)$	0.937×10^{-4}	0.261×10^{-4}	0.096×10^{-4}	0.090×10^{-4}
$p(A_2)$	0.921×10^{-4}	0.313×10^{-4}	0.993×10^{-4}	0.905×10^{-4}
$p(A_3)$	0.927×10^{-4}	0.210×10^{-4}	0.014×10^{-4}	0.010×10^{-4}
$p(A_4)$	0.905×10^{-4}	0.240×10^{-4}	0.004×10^{-4}	0.001×10^{-4}
$p(A_1 \cap A_2)$	0.876×10^{-4}	0.005×10^{-4}	0.089×10^{-4}	0.005×10^{-4}
$p(A_1 \cap A_3)$	0.884×10^{-4}	0.004×10^{-4}	0.010×10^{-4}	0.001×10^{-4}
$p(A_1\ A_4)$	0.862×10^{-4}	0.004×10^{-4}	0.001×10^{-4}	0.000×10^{-4}
$p(A_2 \cap A_3)$	0.870×10^{-4}	0.007×10^{-4}	0.011×10^{-4}	0.001×10^{-4}
$p(A_2 \cap A_4)$	0.850×10^{-4}	0.005×10^{-4}	0.003×10^{-4}	0.000×10^{-4}
$p(A_3 \cap A_4)$	0.858×10^{-4}	0.004×10^{-4}	0.002×10^{-4}	0.000×10^{-4}
p(F)	1.0×10^{-4}	1.0×10^{-4}	1.0×10^{-4}	1.0×10^{-4}
Lower Bound Error	0.018×10^{-4}	0.005×10^{-4}	0.001×10^{-4}	0.001×10^{-4}
Upper Bound Error	0.068×10^{-4}	0.007×10^{-4}	0.001×10^{-4}	0.000×10^{-4}

TABLE 4.4.2 - ARBITARY FAILURE PROBABILITIES FOR THE SYSTEM IN FIG. 4.4.3

Total population of each case = N



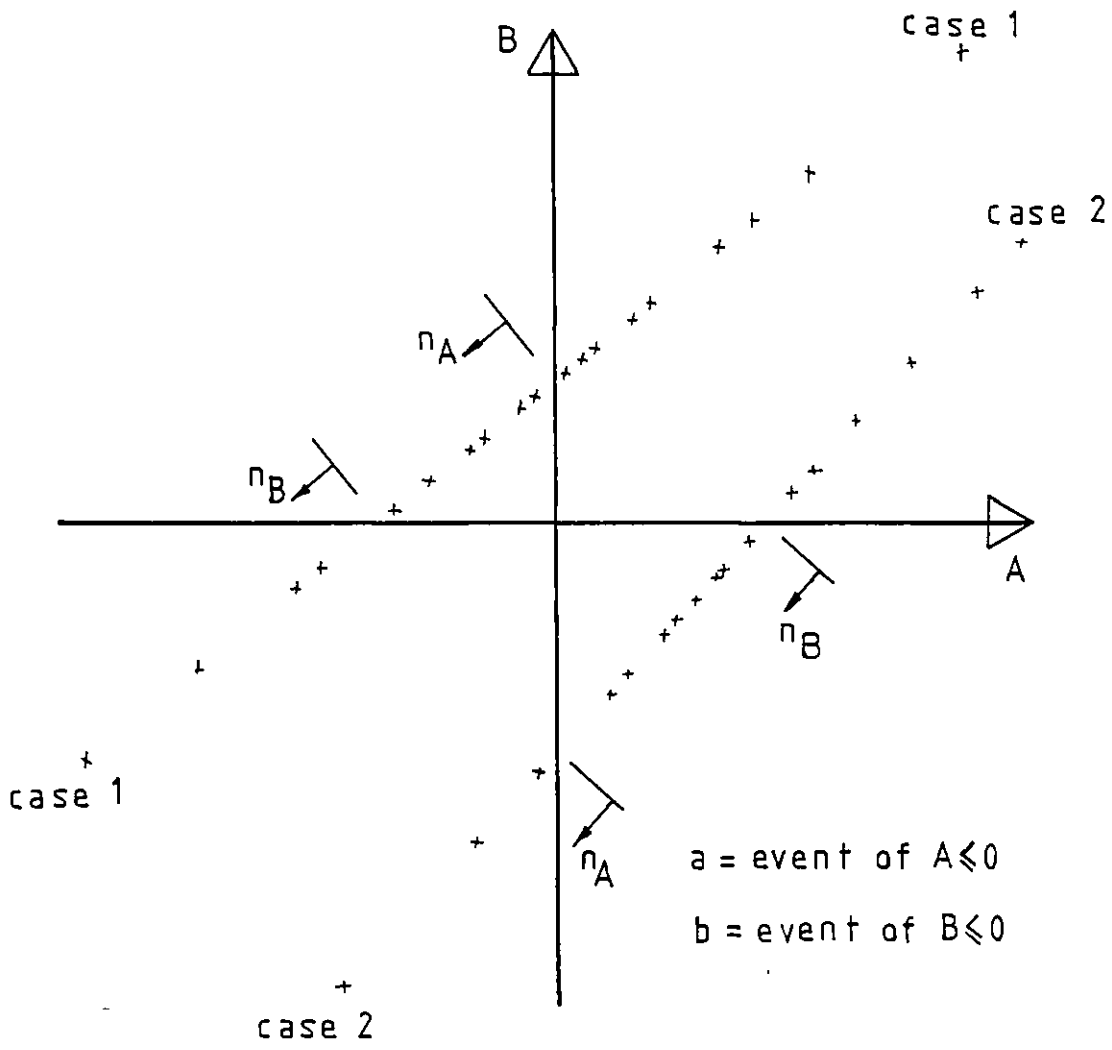
$$p(a) = n_A/N \qquad p(b) = n_B/N$$

$$p(a \cap b) = n_{AB}/N = \frac{Max[\,n_A + n_B - N\,,\,0.0\,]}{N} = Max[\,p(a)+p(b)-1.0\,,\,0.0\,]$$

$$p(a \cup b) = \frac{[\,n_A + n_B - n_{AB}\,]}{N} = \frac{Min[\,n_A + n_B\,,\,N\,]}{N} = Min[\,p(a)+p(b)\,,\,1.0\,]$$

Fig.4.4.1 - Probabilities Associated with two Mutually Perfectly Negatively Correlated Variables

Total population of each case = N



$$p(a) = {}^{n_A}/_{N} \qquad p(b) = {}^{n_B}/_{N}$$

$$p(a\ b) = \frac{\text{Min}[n_A, n_B]}{N} = \text{Min}[p(a), p(b)]$$

$$p(a\ b) = \frac{\text{Max}[n_A, n_B]}{N} = \text{Max}[p(a), p(b)]$$

Fig.4.4.2 - Probabilities Associated with two Mutually Perfectly Positively Correlated Variables

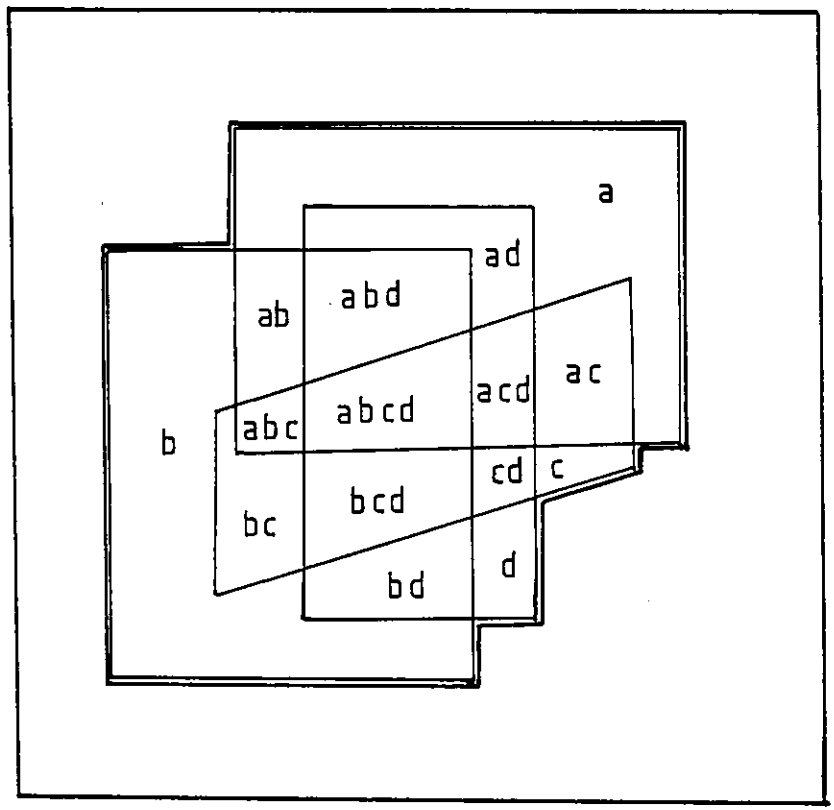


Fig.4.4.3 - Venn Diagram of Four Failure Events

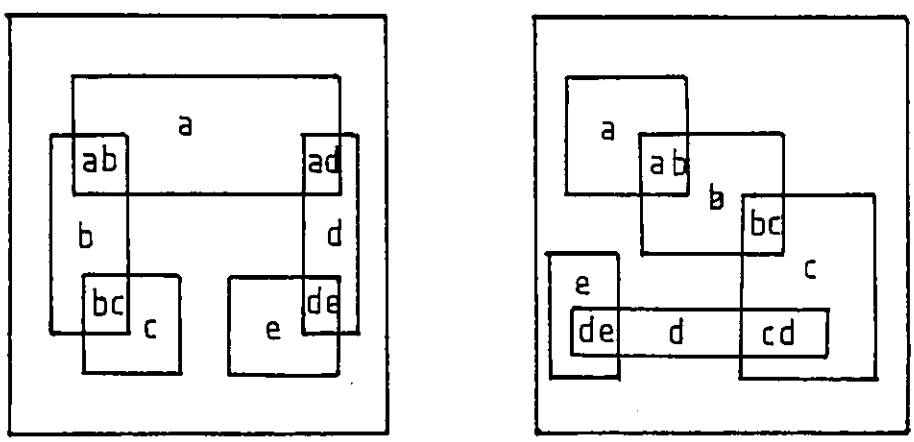


Fig.4.4.4 - Venn Diagrams for Possible Failure Systems with Exact Bounds

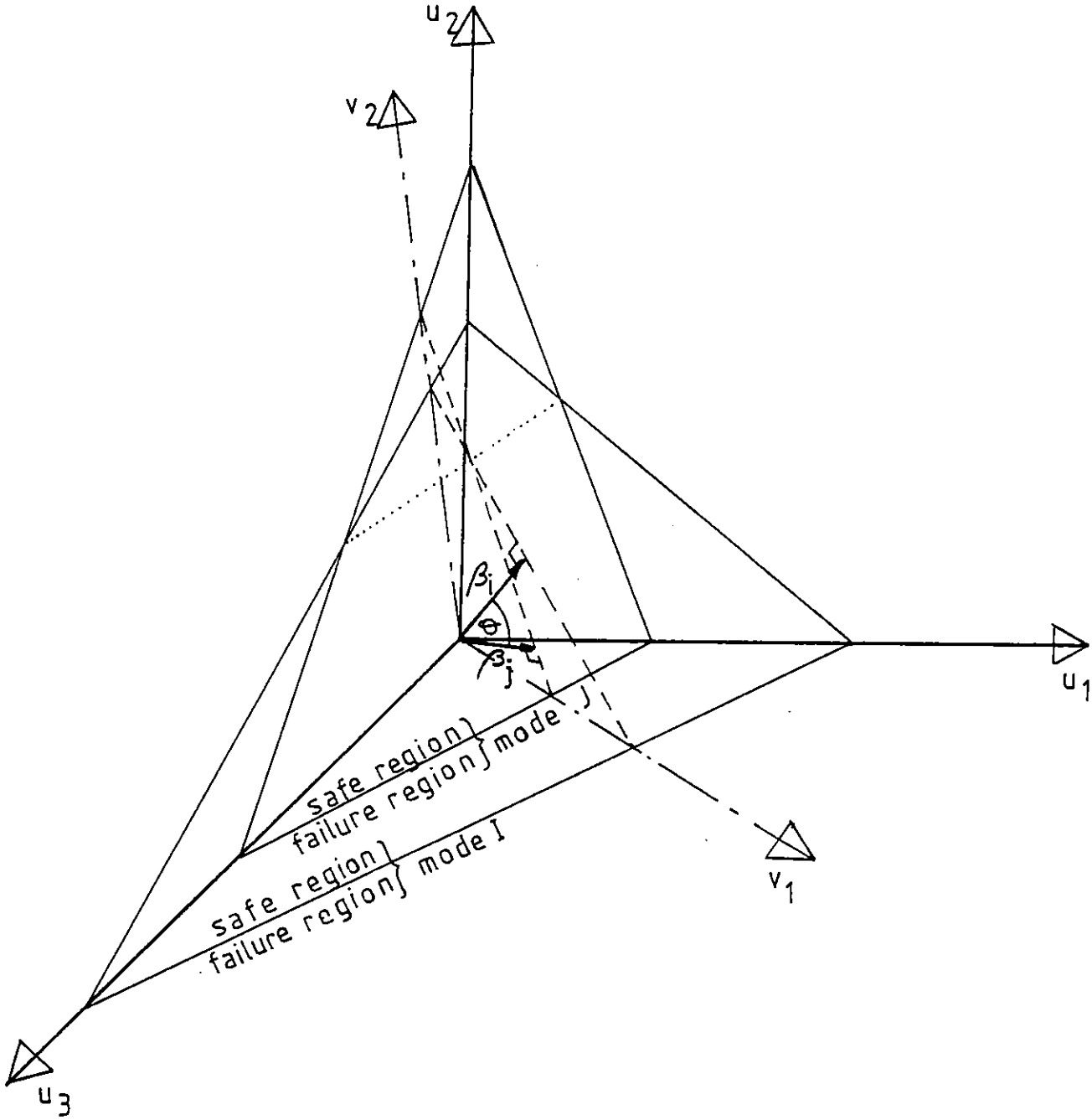
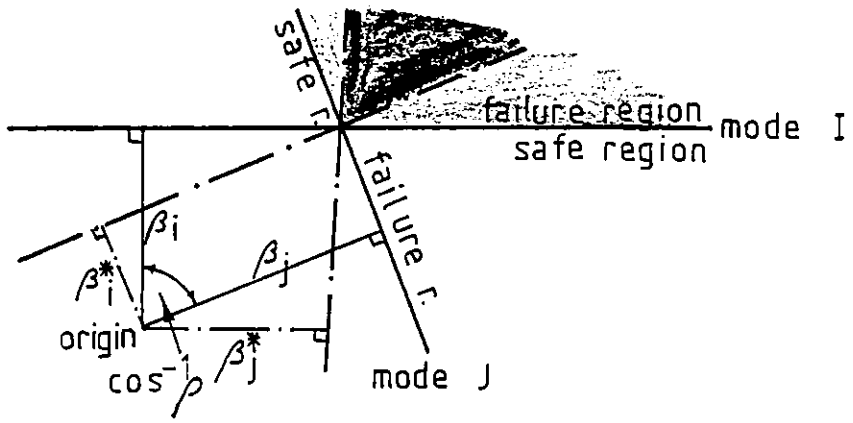
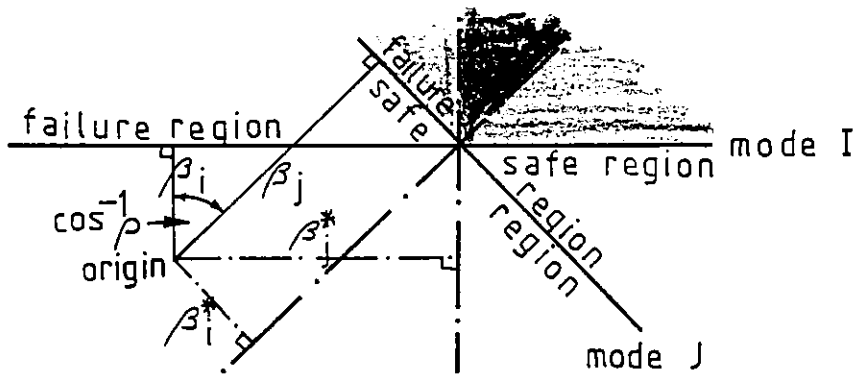


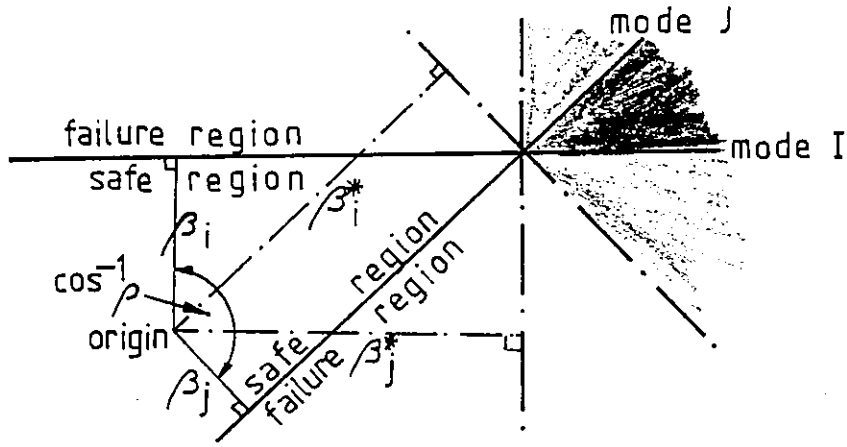
Fig.4.4.5 - Two Linear Failure Boundaries in 3 Dimensions Showing Plane V_1, V_2 which Contains Both Normals Passing through the Origin



(i) Positive correlation



(ii) Positive correlation



(iii) Negative correlation

Fig. 4.4.6 - Linear Failure Boundaries in the Standised Normal Plane V_1, V_2 Shown in Fig. 4.4.5

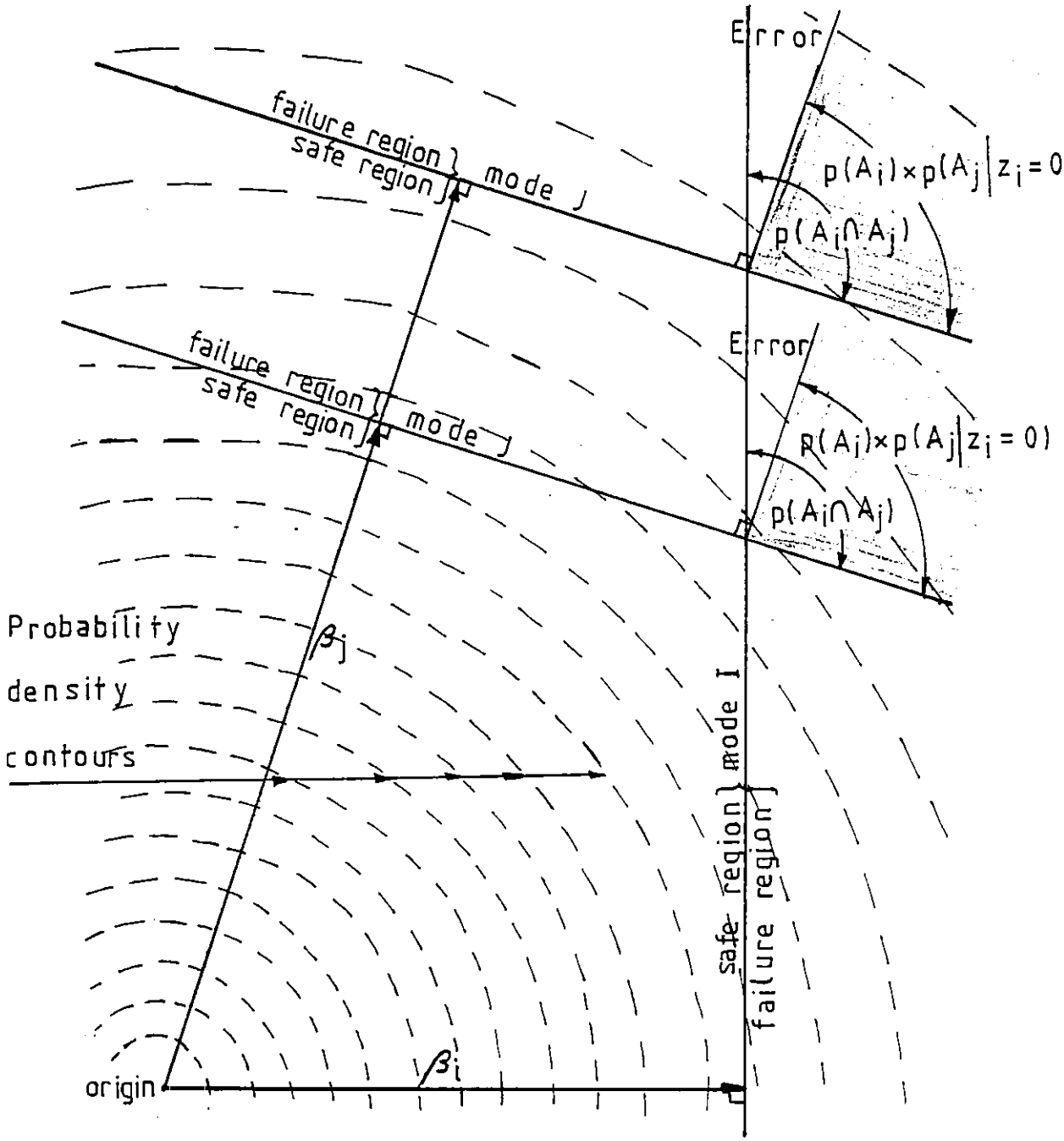


Fig. 4.4.7 - Approximation to $p(A_i \cap A_j)$ as $[\beta_j - \beta_i]$ Increases for Positive $\rho(Z_i, Z_j)$

CHAPTER 5 - PROPOSED APPROACH TO SYSTEM RELIABILITY ANALYSIS

5.1 THE BASIS OF THE PROPOSED METHOD

5.1.1 The Need for an Improved Method

The intention in this chapter is to propose an approach for obtaining bounds on the probability of structural system failure that can include those structures for which system failure occurs as the result of a sequence of local failure events. This approach is a development of the method put forward by Ditlevsen [17] (sub-sections 4.4.4 to 4.4.7), with the inclusion of an idealised redistribution of load-effect following the failure of each element based on Moses' unzipping technique [28-30] (sub-sections 4.3.3 to 4.3.6).

The main disadvantage of Ditlevsen's approach is that it considers explicitly only the union of failure events. This limits the method for use with structural systems where:

i. system failure occurs with the first local failure; or

ii. all the elements can be idealised as behaving in a ductile manner so that plastic analysis can be used to define each mode of failure.

In practice there are structural systems with modes of failure which do not fall into these categories, but for which the ability to calculate the probability of failure is important. One obvious example is tower blocks and the probability of their progressive collapse, as at Ronan Point, Canning Town, in 1968.

5.1.2 Failure Boundaries for Structural Systems

Consider the failure boundary, separating the safe region from the failure region in the n-dimensional basic variable space, for a structural system which conforms to the above limits and which can fail by m different modes of system failure. This boundary will consist of m intersecting hyper-surfaces. Starting from any point in the safe region, the failure region will be reached once any of those hyper-surfaces has been crossed. Therefore the events of being in the failure region for that system is the union of the events of being in the failure regions for each mode of system failure. If the failure boundaries for the modes of system failure are, or can be idealised as, hyper-planes in standardised normal space (as in Ditlevsen's examples), then the failure boundary for the system can be shown as in fig 5.1.1 for any plane through that space.

In the more general case, however, each mode of system failure is itself made up from n_i modes of local failure, each of which has to occur before the system will fail. The failure boundary corresponding to such a mode of system failure will

consist of n_i intersecting hyper-surfaces. In contrast to the above, starting from any point in the failure region, the safe region for that mode of system failure will be reached once any of those hyper-surfaces has been crossed. Therefore the event of being in the failure region for that mode of system failure is the intersection of the events of being in the failure region for each constituent mode of local failure. If the failure boundaries for the modes of local failure are, or can be idealised as, hyper-planes in standardised normal space, then the failure boundary for the system can be shown as in fig 5.1.2 for any plane through that space. This idealisation for the modes of local failure simplifies the calculation of the probability of their occurance, both singly and jointly, since these values are then the cumulative probabilities from marginal distributions of a multi-variate normal distribution. This simplification is employed in the proposed method.

5.1.3 Constituent Events

The event of being in the failure region for a mode of local failure implies that the set of basic variable values are such that the imposed load-effect is greater than the resistance of that element. In many structural systems a number of local failures have to occur successively for system failure to be realised. After each local failure there will be a redistribution of load-effect which can modify the form of the failure criteria for subsequent modes of local failure in the failure path. This leads to an important, but not obvious, consideration. The sets of basic variable values, from those in the total variable space, which correspond to a negative excess resistance for a mode of local failure will not necessarily correspond to a negative excess resistance for each of the previous modes of local failure in that failure path. For this reason, the event of being in the failure region for a mode of local failure does not necessarily imply that local failure has physically occurred. (fig 5.1.3 and table 5.1.1).

5.1.4 Definitions

The event of being in the failure region for the $_j$th mode of local failure of the $_i$th failure path will be referred to as the event a_{ij}, and similarly the event of the $_i$th failure path occurring will be the event A_i. The reason for working in terms of the events $a_{i,j}$ is because of the difficulty in modifying the distributions of the basic variables to take account of any previous local failures when calculating the probability of local failures.

The physical local failure associated with $a_{i,j}$ can be represented as $(\bigcap_{k=1}^{j} a_{i,k})$, since all the events $a_{i,1}$ through $a_{i,j}$ have to occur before the jth local failure in the ith failure path can take place. Note that if j is the last mode of local

failure in the failure path, the occurrence of that physical local failure is identical with the occurrence of the failure path, A_i, since failure of the structural system will occur with that local failure./ Also the occurrence of the jth local failure given the (j-1)th local failure has taken place in the i^{th} failure path is represented as $(a_{i,j} \mid \bigcap_{k=1}^{(j-1)} a_{i,k})$ (fig 5.1.3 and table 5.1.1).

It is helpful to summarise the preceding concepts as follows:

i. the form of the failure criterion defining the event $a_{i,j}$ is based on events $a_{i,1}$ through $a_{i,(j-1)}$ having occurred ; but

ii. the magnitude of the variables used to evaluate whether that failure criterion is satisfied are not modifed to allow for events $a_{i,1}$ through $a_{i,(j-1)}$ having occurred.

For further clarification these concepts can be demonstrated in a simple example.

5.1.5 Example

Consider three beams with a point load applied to the top beam (fig 5.1.4). With the assumptions that:

i. self-weight can be ignored;

ii. the beams will only fail in flexure;

iii. the beams are perfectly brittle at failure;

iv. impact effects can be ignored; and

v. each beam has uniform geometrical and mechanical properties,

the failure criteria for the three events $a_{i,1}$, $a_{i,2}$ and $a_{i,3}$ are

$$\left.\begin{aligned} a_{i,1}&: Z_{i,1} = R_1 - \tfrac{1}{4}WL \leqslant 0.0 \\ a_{i,2}&: Z_{i,2} = R_2 - \tfrac{1}{4}WL \leqslant 0.0 \\ a_{i,3}&: Z_{i,3} = R_3 - \tfrac{1}{4}WL \leqslant 0.0 \end{aligned}\right\} \qquad (5.1.1)$$

where R_1 , R_2 and R_3 are the resistances of the three beams to any applied bending moment.

The load-effect on the second beam, $\frac{1}{4}WL$, will only be applied to it after failure of the first beam, but the form of the failure criterion for $a_{1,2}$ does not include any limit on the value of the load-effect to ensure that it is greater than the resistance of the first beam, R_1 - despite $\frac{1}{4}WL$ having to be greater than R1 for $a_{1,1}$ to occur, and hence physically for the load-effect to be applied to the second beam. If $R_1 = \frac{3}{8}WL$ and $R_2 = \frac{1}{8}WL$, $a_{1,2}$ will occur but the second beam will not physically fail since no load will ever be applied to it as $a_{1,1}$ will not occur. Similar considerations apply to the event $a_{1,3}$ in terms of both $a_{1,1}$ and $a_{1,2}$.

5.1.6 The Approach Used to Obtain the Bounds

The events $a_{i,j}$ and A_i can be combined into the event of system failure, F, by

$$F = \bigcup_{i=1}^{m} (A_i) = \bigcup_{i=1}^{m} \left(\bigcap_{j=1}^{n_i} [a_{i,j}] \right) \qquad (5.1.2)$$

where m is the total number of failure paths by which the structural system can fail and n_i is the number of local failures which have to occur to cause the i^{th} failure path to be realised. Therefore the probability of system failure can be given by

$$P(F) = P\left(\bigcup_{i=1}^{m} (A_i)\right) = P\left(\bigcup_{i=1}^{m} \left\{ \bigcap_{j=1}^{n_i} [a_{i,j}] \right\}\right) \qquad (5.1.3)$$

The proposed method of structural system reliability analysis obtains bounds on the probability of system failure in terms of the probabilities of the various failure paths occurring, which are similar to those obtained by Ditlevsen in terms of the probabilities of modes of system failure occurring. However, it is first necessary to obtain bounds on the probability of each failure path occurring in terms of the probability of local events $a_{i,j}$.

5.1.7 The Failure of an Element

Most structural systems could conceivably fail by many failure paths, each of which require several local failures to occur in succession. In most cases, the potential failure of each element will be included in a sizeable sub-set of the failure paths. For some of those failure paths a particular element will be the first to fail, in some the second, etc. Therefore there are usually several events $a_{i,j}$ associated with the failure of any particular element. The form of the failure criteria defining this sub-set of the events $a_{i,j}$ will often be based on different events $a_{i,1}$ through $a_{i,(j-1)}$ having occured, and so the expression for the excess resistance in each case can also differ.

5.1.8 Assumptions

In order to make the manipulation of the variables more tractable, the expression for each excess resistance should be written as a linear equation of normal variables. Therefore:

i. the structures to be analysed are limited to those which can be modelled as behaving in a linear elastic manner between the events of local failure;

ii. the variables considered are limited to the resistances of the elements, $R_{i,j}$ and the applied loads, S_K, all of which are treated as normally distributed (see also section 5.5);

iii. each load is independant of each resistance; and

iv. the load-effects are assumed to be equal to a deterministic linear function of the applied loads.

These assumptions allow the failure boundary for each mode of local failure to be idealised as a hyper-plane in standardised normal space.

5.1.9 Mathematical Form of the Local Failure Criteria

Following from these assumptions the form of the failure criterion for the event $a_{i,1}$ is

$$Z_{i,1} = R_{i,1} - \sum_{k=1}^{n_s} \left[d_{i,1,k} * S_K \right] \leqslant 0.0 \qquad (5.1.4)$$

where n_s is the number of applied loads and $d_{i,1,k}$ are appropriate constants.

After an element has failed there will be redistribution of load-effect away from that element. If the element is perfectly ductile, the residual resistance is equal to the initial resistance, $R_{i,1}$, and the excess load-effect, which is to be restributed, is equal to $-Z_{i,1}$. If, on the other hand, theelement is perfectly brittle, the residual resistance is zero and the excess load-effect is equal to $\sum_{k=1}^{n_s} \left[d_{i,1,k} S_k \right]$, the initial load-effect.

The actual load-carrying capacity of an element after the load-effect applied has exceeded its initial resistance will depend on the mechanical properties of its constituent materials. In practice these matrial properties may be time- or strain-dependents such as strain hardening or load-shedding. These forms of behaviour can be approximately modelled by assuming that the residual resistance can be calculated deterministically when each subsequent local failure occurs. More sophisticated modelling is not possible if the form of the failure criteria are to remain linear.

The form of the failure criterion for the second mode of local failure in the i^{th} failure path will include a proportion of the excess load-effect redistributed from the first element. Therefore

$$Z_{i,2} = R_{i,2} - \sum_{k=1}^{n_s} \left[d'_{i,2,k} \times S_k \right] - \gamma_1 \left\{ \sum_{k=1}^{n_s} \left[d_{i,1,k} \times S_k \right] - \gamma_2 R_{i,1} \right\}$$

$$= \gamma_1 \times \gamma_2 \times R_{i,1} + R_{i,2} - \sum_{k=1}^{n_s} \left[\left\{ d'_{i,2,k} + \gamma_1 \times d_{i,1,k} \right\} \times S_k \right]$$

$$= c_{1,2,1} \times R_{i,1} + R_{i,2} - \sum_{k=1}^{n_s} \left[d_{i,2,k} \times S_k \right] \leqslant 0.0 \qquad (5.1.5)$$

where $d'_{i,2,k}$ are the influence coefficients for the load-effect on the element before the first failure occurs; γ_1 is the proportion of the excess load-effect redistributed from the first element to the second; and γ_2 is the ratio of the residual to the initial resistance for the first element ($\gamma_2 = 1.0$ if perfectly ductile, $\gamma_2 = 0.0$ if perfectly brittle, $\gamma_2 \geqslant 1.0$ if strain hardening occurs, and $1.0 \geqslant \gamma_2 \geqslant 0.0$ if load shedding occurs, fig 5.1.5). Similarly, the form of the failure criterion for any mode of local failure in a failure path can be written as

$$Z_{i,j} = \sum_{k=1}^{j} \left[c_{i,j,k} \times R_{i,k} \right] - \sum_{k=1}^{n_s} \left[d_{i,j,k} \times S_k \right] \leqslant 0.0 \qquad (5.1.6)$$

where $c_{i,j,j} = 1.0$

5.1.10 Similarity with Moses' Approach

If the elements' resistances are written as $r_{i,j}$ instead of $R_{i,j}$, for a single load variable, S_1, eqn.(5.1.6) becomes

$$Z_{i,j} = \sum_{k=1}^{j} \left[c_{i,j,k} - r_{i,k} \right] - d_{i,j,1} \times S_1 \leqslant 0.0 \qquad (5.1.7)$$

and hence, by rearranging this inequality,

$$S_1 \leqslant \sum_{k=1}^{j} \left[\frac{c_{i,j,k}}{d_{i,j,1}} \times r_{i,k} \right] \qquad (5.1.8)$$

The RHS of this inequality can be considered as the resistance of the structural system to the single applied load, S_1, and is of the form used by Moses, eqn. (4.3.15).

5.1.11 The Failure Set

Before any structure can be analysed, the exhaustive set of possible failure paths has to be identified. These constitute the failure set for that structural system.

In practice, a local failure could occur at any point along a structural member, but it is assumed that local failures are confined to specific cross-sectional points on each member for the idealised system. The position of such points, which are referred to as elements, will usually be where either there is step change in the value of the resistance (eg. due to the depth of a beam being reduced), or the load-effect is expected to be maximum (eg. at a support for a shear mode of local failure).

The influence coefficients, $d_{i,j,k}$ and $c_{i,j,k}$ in eqn. (5.1.6) have to be calculated for each of the modes of local failure in each failure path. For statically inderminant structures, the elastic properties of the members must be assumed deterministic, even though their ultimate resistances are treated as statistical variables.

5.1.12 Further Example

The simple example of a pinned portal frame, to which a single horizontal load is applied, (fig 5.1.5) can be used to demonstrate the intentions of the proposed method of structural system reliability analysis. The idealisations of this system are that:

i. the structural properties of each member (AB,BC and CD) are constant along its length;

ii. the formation of plastic hinges are the only local failure modes considered; and

iii. the material from which the frame is constructed behaves in a ductile manner.

In this structural system there are 4 possible modes of local failure, being the hinges at B in members AB and BC and at C in members BC and CD. The complete failure tree is as shown in fig 5.1.7.

The event $a_{1,1}$ is defined as the excess resistance of member AB at B being negative, that is

$$Z_{i,1} = R_i - \tfrac{1}{2}LS \leqslant 0.0 \qquad (5.1.9)$$

The event $a_{1,2}$ is defined as the excess resistance of member BC at C being negative when the structure is treated as if there is a plastic hinge at B, that is

$$Z_{1,2} = R_2 - \tfrac{1}{2}LS - (\tfrac{1}{2}LS - R_1)$$

$$= R_1 + R_2 - LS \leqslant 0.0 \qquad (5.1.10)$$

It should be noted that the event $a_{1,2}$ does not imply a plastic hinge occurs at C on BC unless a plastic hinge occurs first a B on AB. This can be shown by taking $R_1 = \frac{5}{8}LS$ and $R_2 = \frac{1}{4}LS$, so that $Z_{1,1} = \frac{1}{8}LS \not\leqslant 0.0$ while $Z_{1,2} = -\frac{1}{8}LS \leqslant 0.0$. Hence the occurance of a failure path is dependent on <u>all</u> the modes of local failure within it being realised.

5.2 THE PROBABILITY OF LOCAL EVENTS

5.2.1 The Probability of $a_{i,j}$ Occurring

The event $a_{i,j}$ occurs when the associated excess resistance $Z_{i,j}$ is less than or equal to zero. The assumptions set out in sub-section 5.1.8 include the variables, $R_{i,k}$ and S_k, in the linear expression for $Z_{i,j}$, eqn. (5.1.6), being treated as having normal distributions. This assumption results in the excess resistances, $Z_{i,j}$, also having normal distributions. Hence the probability of the event $a_{i,j}$ occurring is given by

$$P(a_{i,j}) = \Phi(-\beta_{i,j}) = \Phi(-E(Z_{i,j}) / \sigma(Z_{i,j})) \qquad (5.2.1)$$

where, provided $c_{i,j,k}$ and $d_{i,j,k}$ are known constants,

$$E(Z_{ij}) = E\left(\sum_{k=1}^{j} \left[c_{i,j,k} \times R_{i,k} \right] - \sum_{k=1}^{n_s} \left[d_{i,j,k} \times S_k \right] \right)$$

$$= \sum_{k=1}^{j} \left[c_{i,j,k} \times E(R_{i,k}) \right] - \sum_{k=1}^{n_s} \left[d_{i,j,k} \times E(S_k) \right]$$

(5.2.2)

and

$$\sigma^2(Z_{i,j}) = \sigma^2 \left(\sum_{k=1}^{j} \left[c_{i,j,k} \times R_{i,k} \right] - \sum_{k=1}^{n_s} \left[d_{i,j,k} \times S_k \right] \right)$$

$$= \sum_{k=1}^{j} \sum_{l=1}^{j} \left[c_{i,j,k} \times c_{i,j,l} \times \rho(R_{i,k}, R_{i,l}) \times \sigma(R_{i,k}) \times \sigma(R_{i,l}) \right]$$

$$+ \sum_{k=1}^{n_s} \sum_{L=1}^{n_s} \left[d_{i,j,k} \times d_{i,j,l} \times \rho(S_k, S_l) \times \sigma(S_k) \times \sigma(S_l) \right]$$

(5.2.3)

since $\rho(R_{i,k}, S_l)$ is assumed to be zero for all i, k and l.

5.2.2 Bounds Using The Probability of Single Local Events

Bounds on the probability of system failure can be obtained using direct knowledge of only the probabilities of single local events. These bounds are calculated using the relationships

$$0.0 \leqslant P\left(\bigcap_{j=1}^{n_i} (a_{i,j})\right) = P(A_i) \leqslant \min_{j=1}^{n_i} \left[P(a_{i,j}) \right] \qquad (5.2.4)$$

and

$$\max_{i=1}^{m} \left[P(A_i) \right] \leqslant P\left(\bigcap_{i=1}^{m} (A_i)\right) = P(F) \leqslant \sum_{i=1}^{m} \left[P(A_i) \right] \qquad (5.2.5)$$

Eqn (5.2.4) can be improved when there is no mutually negative correlation of the excess resistances to

$$\prod_{j=1}^{n_i} \left[P(a_{i,j}) \right] \leqslant P(A_i) \min_{j=1}^{n_i} \left[P(a_{i,j}) \right] \qquad (5.2.6)$$

Hence

$$\max_{i=1}^{m} \left\{ \Theta_i \times \prod_{j=1}^{n_i} \left[P(a_{i,j}) \right] \right\} \leqslant P(F) \leqslant \sum_{i=1}^{m} \left\{ \min_{j=1}^{n_i} \left[P(a_{i,j}) \right] \right\} \qquad (5.2.7)$$

where Θ_i = 1.0 unless there is negative correlation of two excess resistance in the i^{th} failure path, when Θ_i = 0.0. These bounds on the probability of system failure are wide in most cases.

5.2.3 Level of Sophistication

Closer bounds on the probability of system failure can be obtained if direct knowledge of the probability of some compound events are known. The type of compound events whose probabilities are required can be either the union or the intersection of local events, since these are connected by the relationship [67]

$$P(a_{i,j} \cup a_{k,l}) = P(a_{i,j}) + P(a_{k,l}) - P(a_{i,j} \cap a_{k,l}) \qquad (5.2.8)$$

For this work, as in most others, the intersection of local events is the compound event whose probability will be found directly.

The level of sophistication of a method to find the probability of a compound event can be regarded as the maximum number of local events in any compound event whose probability is obtained directly rather than by combinations of the probabilities of lower ordered compound events. If the level of sophistication were able to be increased to the number of local events in the failure set, $\sum_{i=1}^{m} (n_i)$, the actual probability of system failure would be obtained rather than just bounds on it. In this method, however, the second level of sophistication is employed, so that the equations have to be written in terms of the probabilities of single local events, $P(a_{i,j})$, and the probabilities of the intersection of two local events, $P(a_{i,j} \cap a_{k,l})$.

The probability of two local events, $a_{i,j}$ and $a_{k,l}$, both occurring is the probability of two normal variables, $Z_{i,j}$ and $Z_{k,l}$, each being less than or equal to zero. Since these variables are not necessarily independent, the evaluation of this probability can be approached either by:

i. constructing bounds on this probability (sub-section 5.2.4 below); or

ii. treating the two variables as having marginal distributions of a bivariate normal distribution and obtaining the cumulative probability of that bivariate distribution (sub-section 5.2.5 to 5.2.8 below).

5.2.4 Bounds on the Probability of Joint Events

The first approach is used by Ditlevsen (section 4.4). Combining eqns. (4.4.23) and (4.4.24), his bounds on the probability of $a_{i,j}$ and $a_{k,l}$ both occurring are

$$\text{Max}\left\{ \left[1.0 - \Theta_{i,j,k,l}\right] \times \left[\Phi(-\beta_{i,j}) + \Phi(-\beta_{k,l}) - 1.0 \right], \Theta_{i,j,k,l} \times \text{Max}\left[\Phi(-\beta_{i,j}) \times \Phi(-\beta^*_{k,l}), \Phi(-\beta_{k,l}) \times \Phi(\beta_{i,j}^*) \right] \right\} \leq P(a_{i,j} \cap a_{k,l}) \leq$$

$$\Theta_{i,j,k,l} \times \left\{ \Phi(-\beta_{i,j}) \times \Phi(-\beta^*_{k,l}) + \Phi(-\beta_{k,l}) \times \Phi(-\beta^*_{i,j}) \right\}$$

$$+ \left\{ 1.0 - \Theta_{i,k,k,l} \right\} \times \text{Min}\left[\Phi(-\beta_{i,j}) \times \Phi(-\beta^*_{k,l}), \Phi(-\beta_{k,l}) \times \Phi(-\beta^*_{i,j}) \right] \qquad (5.2.9)$$

where $$\beta_{i,j} = \frac{E(z_{i,j})}{\sigma(z_{i,j})} \quad ; \quad \beta_{k,l} = \frac{E(z_{k,l})}{\sigma(z_{k,l})} \qquad (5.2.10)$$

$$\beta_{i,j}^* = \frac{\beta_{i,j} - \rho(z_{i,j}, z_{k,l}) \times \beta_{k,l}}{\sqrt{1.0 - \rho^2(z_{i,j}, z_{k,l})}} \quad ; \quad \beta_{k,l}^* = \frac{\beta_{k,l} - \rho(z_{i,j}, z_{k,l}) \times \beta_{i,j}}{\sqrt{1.0 - \rho^2(z_{i,j}, z_{k,l})}} \qquad (5.2.11)$$

and $$\Theta_{i,j,k,l} = \frac{\left|\rho(z_{i,j}, z_{k,l})\right| + \rho(z_{i,j}, z_{k,l})}{2 \times \rho(z_{i,j}, z_{k,l})} \qquad (5.2.12)$$

The correlation coefficient, $\rho\,(Z_{i,j},\ Z_{k,l})$, can be found from

$$\rho(Z_{i,j},Z_{k,l}) = \frac{E(Z_{i,j} \times Z_{k,l}) - E(Z_{i,j}) \times E(Z_{k,l})}{\sigma(Z_{i,j}) \times \sigma(Z_{k,l})}$$

$$= \Big\{ E\Big(\Big[\sum_{m=1}^{j} \Big\{ c_{i,j,m} \times R_{\cdot,m} \Big\} - \sum_{m=1}^{n_s} \Big\{ d_{i,j,m} \times S_m \Big\} \Big] \times \sum_{n=1}^{l} \Big\{ c_{k,l,m} \times R_{k,n} \Big\} - \sum_{n=1}^{n_s} \Big\{ d_{k,l,n} \times S_n \Big\} \Big] \Big) - E(Z_{i,j}) \times E(Z_{k,l}) \Big\} \Big/ \Big\{ \sigma(Z_{i,j}) \times \sigma(Z_{k,l}) \Big\}$$

(5.2.13)

After manipulation, this yields

$$\rho(Z_{i,j},Z_{k,l}) = \Big\{ \sum_{m=1}^{j} \sum_{n=1}^{l} \Big[c_{i,j,m} \times c_{k,l,n} \times \rho(R_{l,m},R_{k,n}) \times \sigma(R_{i,m}) \times \sigma(R_{k,n}) \Big] + \sum_{m=2}^{n_s} \sum_{n=1}^{n_s} \Big[d_{i,j,m} \times d_{k,l,m} \times \rho(S_m,S_n) \times \sigma(S_m) \times \sigma(S_n) \Big] \Big\} \Big/ \Big\{ \sigma(Z_{i,j}) \times \sigma(Z_{k,l}) \Big\} \qquad (5.2.14)$$

5.2.5 Bivariate Normal Distribution

The alternative to using Ditlevsen's bounds is to solve the bivariate normal integral, as given by [6]

$$(a_{i,j} \cap a_{k,l}) = \Big[\int_{-\infty}^{0} \int_{-\infty}^{0} \exp\Big(\frac{\rho(Z_{i,j},Z_{k,l}) \times \{ E(Z_{i,j}) - z_{i,j} \} \times \{ E(Z_{k,l}) - z_{k,l} \}}{\{ 1.0 - \rho^2(Z_{i,j},Z_{k,l}) \} \times \sigma(Z_{i,j}) \times \sigma(Z_{k,l})} - \tfrac{1}{2} \Big\{ \frac{E(Z_{i,j}) - z_{i,j}}{\sqrt{1.0 - \rho^2(Z_{i,j},Z_{k,l}) \times \sigma(Z_{i,j})}} \Big\}^2 - \tfrac{1}{2} \Big\{ \frac{E(Z_{k,l}) - z_{k,l}}{\sqrt{1.0 - \rho^2(Z_{i,j},Z_{k,l}) \times \sigma(Z_{k,l})}} \Big\}^2 \Big) dz_{i,j} dz_{k,l} \Big] \Big/ \Big[2\pi \sqrt{1.0 - \rho^2(Z_{i,j},\ Z_{i,j})} \times \sigma(Z_{i,j}) \times \sigma(Z_{k,l}) \Big] \qquad (5.2.15)$$

where $\rho\,(Z_{i,j},Z_{k,l})$ can again be found from eqn. (5.2.14)

The cumulative probability of any bivariate distribution is a double convolution integral. This is time consuming to solve accurately by numerical integration, but that is the method that must be employed unless the equation can be:

i. solved analytically;

ii. approximated to closely; or

iii. reduced to a single integral.

There is no analytical solution for the normal distribution, but for the univariant case Hastings has produced a good approximation to $\Phi(u)$ [21]. No similar approximation has been devised for the bivariate case. Three approaches to achieve a suitable simplification of eqn. (5.2.15) are Owen's methods, Hermite-Gauss quedrative and Simpson's rule approximation [24, 38]. These methods employ Hastings' approximation.

5.2.6 Owen's Method

Owen's method reduces eqn. (5.2.15) to

$$\Phi(-\beta_{i,j},-\beta_{k,l},\rho(z_{i,j},z_{k,l})) = \tfrac{1}{2}\left\{\Phi(-\beta_{i,j}) + \Phi(-\beta_{k,l}) - \tfrac{1}{4}\left|\frac{\beta_{i,j}}{|\beta_{i,j}|} - \frac{\beta_{k,l}}{|\beta_{k,l}|}\right|\right.$$

$$\left. - T\left(\beta_{i,j}, \frac{\beta_{k,l} - \rho(z_{i,j}, z_{k,l})}{\beta_{i,j} \times \sqrt{1.0 - \rho(z_{i,j}, z_{k,l})^2}}\right) - T\left(\beta_{k,l}, \frac{\beta_{i,j} - \rho(z_{i,j},z_{k,l}) \times \beta_{k,l}}{\beta_{k,l} \times \sqrt{1.0 - \rho^2(z_{i,j},z_{k,l})}}\right)\right. \tag{5.2.16}$$

where

$$T(h,a) = \frac{\tan^{-1}(a) \times \sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(2m+1)} \times \left\{1.0 - \exp\left(-\frac{h^2}{2}\right) \times \sum_{n=0}^{m}\left[\frac{\left(\frac{1}{2}h^2\right)^n}{n!}\right]\right\} \times a^{(2m+1)}\right]}{2\pi}$$

$$\text{for } h > 0.0 \text{ and } 0.0 < a \leqslant 1.0 \tag{5.2.17}$$

$$T(ah, 1/a) = \tfrac{1}{2}\left\{\Phi(-h) + \Phi(-ah)\right\} - \Phi(-h) \times \Phi(-ah) - T(a,h)$$

$$\text{for } h > 0.0 \text{ and } a > 0.0 \tag{5.2.18}$$

and

$$T(h,a) = -T(h,-a) = T(-h,a) \tag{5.2.19}$$

5.2.7 Hermite-Gauss Quadrature

The double convolution integral of eqn. (5.2.15) is replaced, in the Hermite-Gauss quadrature, by the series

$$\Phi(-\beta_{i,j}, -\beta_{k,l}, \rho(z_{i,j}, z_{k,l})) = \sum_{l=1}^{m} H_l \times \Phi\left(\frac{\sqrt{2}\, c_1 \cdot x_l - \beta_{i,j}}{\sqrt{1.0 - c_1^2}}\right) \Phi\left(\frac{\sqrt{2}\, c_2 \cdot x_l - \beta_{k,l}}{\sqrt{1.0 - c_2^2}}\right)$$

$$+ E \qquad (5.2.20)$$

where m is the number of points used for the quadrature;

$$c_1 = + \sqrt{|\rho(z_{i,j}, z_{k,l})|} \quad ; \quad c_2 = \frac{\rho(z_{i,j}, z_{k,l})}{c_1} \qquad (5.2.21)$$

$$H_l = \frac{2^{(m+1)} \times m!}{\{H_{(m+1)}(x_l)\}^2} \quad , \quad l = 1, \ldots\ldots, m \qquad (5.2.22)$$

$H_n(x)$ is the Hermite polynominal of order n;

$x_1, \ldots\ldots, x_n$ are the zeros of $H_n(x)$ [19,37]; and

E is the error function given by

$$E = \frac{m!}{(2m)!} \times \Phi^{2m}\left(\frac{c_1 \eta - \beta_{i,j}}{\sqrt{1.0 - c_1^2}}\right) \times \Phi^{2m}\left(\frac{c_2 \eta - \beta_{k,l}}{\sqrt{1.0 - c_2^2}}\right) \qquad (5.2.19)$$

for some value η .

5.2.8 Simpson's Rule

The double convolution integral in eqn. (5.2.15) is reduced to a single convolution for the approximation by Simpson's Rule. The integrals used are

$$\Phi(-\beta_{i,j},\beta_{k,l},\rho(z_{i,j},z_{k,l})) = \begin{cases} \Phi(-\beta_{k,l}) - \int_{-\infty}^{-\beta_{k,l}} \frac{\exp(-\frac{x^2}{2})}{\sqrt{2\pi}} \Phi\left\{\frac{\beta_{i,j}+\rho(z_{i,j},z_{k,l})x}{\sqrt{1.0-\rho^2(z_{i,j},z_{k,l})}}\right\} dx \\ \qquad \text{for} \quad \rho(z_{i,j},z_{k,l}) \geqslant 0.0 \text{, and} \\ \int_{-\infty}^{-\beta_{k,l}} \frac{\exp(-\frac{x^2}{2})}{\sqrt{2\pi}} \Phi\left\{\frac{-\beta_{i,j}+\rho(z_{i,j},z_{k,l})x}{1.0-\rho^2(z_{i,j},z_{k,l})}\right\} dx \quad \text{for} \\ \qquad \rho(z_{i,j},z_{k,l}) > 0.0 \end{cases} \tag{5.2.24}$$

5.2.9 Comparison of the Methods

Sowden and Ashford[38] suggest that Owen's method is the quickest one of the three for computer use in calculating the cumulative probability to a particular degree of accuracy for the majority of cases. However Owen's method includes expressions of the form $\left[\frac{1}{2}\Phi(-\beta_1) - T\left(-\beta_1, \frac{\beta_2-\rho(z_1,z_2)\beta_1}{\beta_1\sqrt{1.0-\rho^2(z_1,z_2)}}\right)\right]$ in which difference between the two terms can be less than the accuracy of Hastings' approximation for $\Phi(u)$, which is about 10^{-7}, for relatively large values of β encountered in structural reliability analysis. Therefore the table given by Sowden and Ashford has been amended (table 5.2.1) by replacing Owen's method with the Simpson's Rule approximation when:

(i) $\beta_{i,j}+\beta_{k,l} \geqslant 3.0 \times [1.0+\rho(z_{i,j},z_{k,l})]$; or (5.2.25)

(ii) Max $(\beta_{i,j},\beta_{k,l}) \geqslant 6.0$ (5.2.26)

Hence the best way of obtaining a value for the probability of joint events, $P(a_{i,j} \cap a_{k,l})$, is to use the appropriate method from table 5.2.1. The use of these methods rather than the bounds of eqn. (5.2.9) increases the computer time necessary to find the bounds on the system reliability, but it also narrows the gap between these bounds. The use of a function to find the cumulative frequency for the standardised trivariate (or other multivariate) normal distribution would reduce that gap still further, but the additional computing required would be daunting - especially considering the number of times such a function would be used.

5.3 THE PROBABILITY OF THE REALISATION OF FAILURE PATHS

5.3.1 Lower Bound

The probability of the i^{th} failure path occurring is the probability of the occurrance of all the events $a_{i,j}$ which constitute that failure path. Therefore

$$P(A_i) = P\left(\bigcap_{j=1}^{n_i} [a_{i,j}]\right)$$

$$= P(a_{i,1}) \times \prod_{j=2}^{n_i} \left\{ P\left(a_{i,j} \,\middle|\, \bigcap_{2=1}^{(j-1)} [a_{i,1}]\right)\right\} \qquad (5.3.1)$$

where A_i is the event of the i^{th} failure path occurring and $a_{i,j}$ is the event of $Z_{i,j} \leqslant 0.0$. Unlike the occurrence of an event $a_{i,j}$ (sub-section 5.1.4), the occurrence of an event A_i implies the physical failure of the structural system.

When the excess resistance, $Z_{i,j}$, for the local failure mode $a_{i,j}$ is not negatively correlated with any of the excess resistances, $Z_{i,l}$, for the local failure modes $a_{i,l}$ which occur before $a_{i,j}$ in that failure path, then

$$P\left(a_{i,j} \,\middle|\, \bigcap_{l=1}^{(j-1)} [a_{i,l}]\right) \geqslant P(a_{i,j} \mid a_{i,k}) = \frac{P(a_{i,j} \cap a_{i,k})}{P(a_{i,k})} \geqslant P(a_{i,j}) \qquad (5.3.2)$$

is true for all values of k less than j. Choosing k to maximise the value of $P(a_{i,j} \cap a_{i,k})$ and substituting eqn. (5.3.2) into eqn. (5.3.1) gives a lower bound on the probability of that failure path occurring. This bound is

$$P(A_i) \geqslant P(a_{i,1}) \times \prod_{j=1}^{n_i} \left\{ \max_{k=1}^{(j-1)} \left[\frac{P(a_{i,j} \cap a_{i,k})}{P(a_{i,k})} \right] \right. \qquad (5.3.3)$$

When there is negative correlation of the excess resistance of local failure mode $a_{i,j}$ with the excess resistance of at least one of preceeding local failure modes in the failure path, then eqn. (5.3.2) is not necessarily true for all values of k and should be replaced by

$$P\left(a_{i,j} \,\middle|\, \bigcap_{2=1}^{(j-1)} \left[a_{i,1}\right]\right) \geqslant P\left(a_{i,j} \,\middle|\, \bigcap_{1=1}^{n} \left[a_{j,k_1}\right]\right) \qquad (5.3.4)$$

where a_{i,k_1} is one of the n local failure modes whose excess resistance is negatively correlated with that of local failure mode $a_{i,j}$. A lower bound on this probability can be found by

$$P\left(a_{i,j} \,\middle|\, \bigcap_{1=1}^{n} \left[a_{i,k_1}\right]\right) = \frac{P\left(\bigcap_{1=1}^{n} \left[a_{i,k_1}\right] \,\middle|\, a_{i,j}\right) * P(a_{i,j})}{P\left(\bigcap_{1=1}^{n} \left[a_{i,k_1}\right]\right)}$$

$$= \frac{\left\{1.0 - P\left(\bigcup_{1=1}^{n} \left[a^{c}_{i,k_1}\right] \,\middle|\, a_{i,j}\right)\right\} * P(a_{i,j})}{P\left(\bigcap_{1=1}^{n} \left[a_{i,k_1}\right]\right)}$$

$$\geqslant \frac{P(a_{i,j}) - \sum_{1=1}^{n} \left[P\left(a^{c}_{i,k_1} \cap a_{i,j}\right)\right]}{P(a_{i,k_m})}$$

$$\frac{P(a_{i,j}) - \sum_{1=1}^{n} \left[P(a_{i,j}) - P\left(a_{i,k_1} \cap a_{i,j}\right)\right]}{P(a_{i,k_m})} \qquad (5.3.5)$$

which is true for any value of $m \leqslant n$, where E^{c} represents the complimentary event to E. The inclusion of eqn. (5.3.5) produces a more complete lower bound on the probability of the failure path occurring since it allows for negative correlation. This lower bound is

$$P(A_i) \geq p(a_{i,1}) \times \prod_{j=2}^{n_i} \left\{ \prod_{k=1}^{(j-1)} \left[\Theta_{i,j,i,k} \right] \times \max_{k=1}^{(j-1)} \left[\frac{P(a_{i,j} \cap a_{i,k})}{p(a_{i,k})} \right] + \left\{ 1.0 - \prod_{k=1}^{(j-1)} \left[\Theta_{i,j,i,k} \right] \right\} \times \frac{\max \left[0.0, p(a_{i,j}) - \sum_{k=1}^{(j-1)} \left[\left\{ 1.0 - \Theta_{i,k,i,k} \right\} \times \right. \right.}{\min_{k=1}^{(j-1)} \left[\left\{ 1.0 - \Theta_{i,k,1,k} \right\} \times p(a_{i,k}) + \Theta_{i,j,i,k} \right]} \right.$$

$$\frac{p(a_{i,j}) - P(a_{i,j} \cap a_{i,k})}{} = P_1(A_i) \qquad (5.3.6)$$

where $\Theta_{i,j,i,k}$ is as defined in eqn. (5.2.12).

5.3.2 Upper Bound

To find an upper bound on $P(A_i)$, the probability of the occurrence of two of the local events $a_{i,j}$ are considered as being unity, their maximum possible value. This simplification has the effect of reducing eqn. (5.3.1) to the inequality

$$P(A_i) \leq P(a_{i,j} \cap a_{i,k}) \qquad (5.3.7)$$

Minimising the right hand side of eqn. (5.3.7) to produce the closest upper bound gives

$$P(A_i) \leq \min_{j=1}^{(n_i-1)} \left[\min_{k=(j+1)}^{n_i} P(a_{i,j} \cap a_{i,k}) \right]$$

$$= P_u(A_i) \qquad (5.3.8)$$

Hence the complete bounds on the probability of the i^{th} failure path occurring, $P(A_i)$, are

$$P_l(A_i) \leq P(A_i) \leq P_u(A_i) \qquad (5.3.9)$$

The bounds become identical when $n_i \leq 2$ ($p(A_i) = P(a_{i,1} \cap a_{i,2})$ if $n_i = 2$ and $P(A_i) = P(a_{i,1})$ if $n_i = 1$).

5.3.3 Probability of Two Failure Paths

Consider a system in which the local events $a_{i,j}$ are such that two potential failure paths, A_i and A_k, exist. The mathematical event $A_i \cap A_k$ can occur if all the events $a_{i,j}$ and $a_{k,l}$ are realised, although the physical failure of that system can only occur along one failure path, the choice being governed by the loading history. The probability of this event can be written as

$$(A_i \cap A_k) = P\left(\left[\bigcap_{j=1}^{n_i} \{a_{i,j}\}\right] \cap \left[\bigcap_{l=1}^{n_k} \{a_{k,l}\}\right]\right)$$

$$= P(a_{i,1}) \cdot \prod_{j=2}^{n_i} \left\{P\left(a_{i,j} \,\middle|\, \bigcap_{m=1}^{(j-1)} [a_{i,m}]\right)\right\} \cdot P\left(a_{k,1} \,\middle|\, \bigcap_{m=1}^{n_i} [a_{i,m}]\right) \times$$

$$\prod_{l=2}^{n_k} \left\{ P\left(a_{k,l} \,\middle|\, \bigcup_{m=1}^{n_i} [a_{i,m}], \bigcap_{m=1}^{l} [a_{k,m}]\right)\right\} \qquad (5.3.10)$$

Eqn. (5.3.10) is similar to eqn. (5.3.1), so that a set of bounds on the probability of both failure paths being able to occur can be found using the same arguments as those used to obtain eqn. (5.3.9). These bounds are

$$P(a_{i,1}) \times \prod_{j=2}^{n_i} \left\{ \prod_{l=1}^{(j-1)} \left[\Theta_{i,j,i,l} \times \operatorname*{Max}_{l=1}^{(j-1)} \left[\frac{P(a_{i,j} a_{i,l})}{P(a_{i,l})}\right]\right] + \left\{1.0 - \prod_{l=1}^{(j-1)} \left[\Theta_{i,j,i,l}\right]\right\} \right.$$

$$\left. \times \operatorname{Max}\left[0.0, P(a_{i,j}) - \frac{\sum_{l=1}^{(j-1)} \left[\left\{1.0 - \Theta_{i,j,i,l}\right\} \times \left\{P(a_{i,j}) - P(a_{i,j} \cap a_{i,l})\right\}\right]}{\operatorname*{Min}_{l=1}^{(j-1)} \left[\left\{1.0 - \Theta_{i,j,i,l}\right\} \times P(a_{i,l}) + \Theta_{i,j,i,l}\right]}\right] \right\} \times$$

$$\prod_{l=1}^{n_i} \left[\Theta_{k,l,i,l} \operatorname*{Max}_{l=1}^{n_i} \left[\frac{P(a_{k,l} \cap a_{i,l})}{P(a_{i,l})}\right] + \left\{1.0 - \prod_{l=1}^{n_i} \left[\Theta_{k,l,i,l}\right]\right\} \times \right.$$

$$\left. \operatorname{Max}\left[0.0, P(a_{k,1}) - \frac{\sum_{l=1}^{n_i} \left[\left\{1.0 - \Theta_{k,1,i,l}\right\} \times \left\{P(a_{k,1}) - P(a_{k,1} a_{i,l})\right\}\right]}{\operatorname*{Min}_{l=1}^{n_i} \left[\left\{1.0 - \Theta_{k,1,i,l}\right\} \cdot P(a_{k,1}) + \Theta_{k,1,i,l}\right]}\right] \right] \times$$

$$\prod_{j=2}^{n_k}\left\{\prod_{l=1}^{n_i}\left[\Theta_{k,j,i,l}\right]\times\prod_{l=1}^{(j-1)}\left[\Theta_{k,j,k,l}\right]\times\operatorname{Max}\left\{\operatorname{Max}_{l=1}^{n_i}\left[\frac{P(a_{k,j}\cap a_{i,l})}{P(a_{i,l})}\right],\right.\right.$$

$$\left.\left.\operatorname{Max}_{l=1}^{(j-1)}\left[\frac{P(a_{k,j}\cap a_{k,l})}{P(a_{k,l})}\right]\right\}+\left\{1.0-\prod_{l=1}^{n_i}\left[\Theta_{k,j,i,l}\right]\times\prod_{l=1}^{(j-1)}\left[\Theta_{k,j,k,l}\right]\right\}\times$$

$$\operatorname{Max}\left[0.0,\ P(a_{k,j})-\sum_{l=1}^{n_i}\left\{1.0-\Theta_{k,j,i,l}\right\}\times\left\{P(a_{k,j})-P(a_{i,j}\cap a_{i,l})\right\}-\sum_{l=1}^{(j-1)}\left\{1.0-\right.\right.$$

$$\left.\left.\Theta_{k,j,k,l}\right\}\times\left\{P(a_{k,j})-P(a_{k,j}\cap a_{k,l})\right\}\right]\Big/$$

$$\operatorname{Min}\left[\operatorname{Min}_{l=1}^{n_i}\left(\left\{1.0-\Theta_{k,j,i,l}\right\}\cdot P(a_{i,l})+\Theta_{k,j,l,l}\right),\operatorname{Min}_{l=1}^{(j-1)}\left(\left\{1.0-\Theta_{k,j,k,l}\right\}\cdot P(a_{k,l})+\right.\right.$$

$$\left.\left.\Theta_{k,k,k,l}\right)\right]$$

$$\leqslant P(A_i\cap A_k)\leqslant$$

$$\operatorname{Min}\left\{\operatorname{Min}_{j=1}^{(n_i-1)}\left[\operatorname{Min}_{l=(j+1)}^{n_i}\left\{P(a_{i,j}\cap a_{i,l})\right\}\right],\ \operatorname{Min}_{j=1}^{(n_k-1)}\left[\operatorname{Min}_{l=(j+1)}^{n_k}\left\{P(a_{k,j}\cap a_{k,l})\right\}\right]\ \operatorname{Min}_{j=1}^{n_i}\left[\right.\right.$$

$$\left.\left.\operatorname{Min}_{l=1}^{n_k}\left\{P(a_{i,j}\cap a_{k,l})\right\}\right]\right\} \qquad (5.3.11)$$

5.4 THE FAILURE PROBABILITY OF THE SYSTEM

The probability that the system will fail is the probability of at least one of the failure paths occurring. Therefore

$$\begin{aligned}P(F) &= P\left(\bigcup_{i=1}^{m}\left\{A_i\right\}\right)\\ &= P(A_i)+\sum_{i=2}^{m}\left[P\left(A_i\ \bigcap_{j=1}^{(i-1)}\left\{A_j^c\right\}\right)\right.\\ &= P(A_i)+\sum_{i=2}^{m}\left[P(A_i)\times P\left(\bigcap_{j+1}^{(i-j)}\left\{A_j^c\right\}\,\middle|\,A_i\right)\right] \qquad (5.4.1)\end{aligned}$$

However, since

$$P\left(\bigcap_{j=1}^{(i-1)}\left\{A_j^c\right\}\,\middle|\,A_i\right)\leqslant P\left(A_k^c\,\middle|\,A_i\right)=1.0-\frac{P(A_i\cap A_k)}{P(A_i)} \qquad (5.4.2)$$

is true for all values of k less than i, choosing k to maximise the value of $P(A_i \cap A_k)$ and substituting eqn. (5.4.2) into eqn. (5.4.1) gives an upper bound for the probability of system failure. This bound is

$$P(F) \geq P(A_i) + \sum_{i=2}^{n} \left[P(A_i) - \max_{k=1}^{(i-2)} \left\{ P(A_i \cap A_k) \right\} \right] \qquad (5.4.3)$$

Eqn. (5.4.2) can be rearranged into

$$P\left(\bigcap_{j=1}^{(i-1)} \left\{ A_j^c \right\} \,\middle|\, A_i \right) = 1.0 - P\left(\bigcup_{j=1}^{(i-2)} \left\{ A_j^c \right\} \,\middle|\, A_i \right)$$

$$\geq 1.0 - \frac{\sum_{j=1}^{(i-1)} \left[P(A_i \cap A_j) \right]}{P(A_i)} \qquad (5.4.4)$$

Hence a lower bound can be put on the probability of system failure as

$$P(F) \geq P(A_i) + \sum_{i=2}^{m} \left[\max \left\{ P(A_i) - \sum_{j=1}^{(i-1)} \left[P(A_i \cap A_j) \right], 0.0 \right] \right] \qquad (5.4.5)$$

The true probability of the intersection of the complement of the failure paths is less than the probability of the events when they are treated as being mutually independant, provided that there is no mutual negative correlation among the failure paths. Mutual negative correlation of the failure paths is assumed not to occur when there is no mutual negative correlation of any pair of excess resistances for local failure modes from different failure paths. Therefore, given that condition,

$$P\left(\bigcap_{j=2}^{(i-1)} \left\{ A_j^c \right\} \,\middle|\, A_i \right) \geq \prod_{k=1}^{(i-1)} \left[P(A_k^c \mid A_i) \right]$$

$$= \prod_{k=1}^{(i-1)} \left[1.0 - \frac{P(A_i \cap A_k)}{P(A_i)} \right] \qquad (5.4.6)$$

a set of bounds on the failure probability of the system can be given using the maximum from eqns. (5.4.4) and (5.4.6) when there is no mutual negative correlation of the failure paths. These bounds are

$$P(A_i)+\sum_{i=2}^{m}\left[\ \text{Max}\left\{P(A_i)-\sum_{j=1}^{(i-1)}\left[P(A_i\cdot A_j)\right],\ \prod_{j=1}^{(i-1}\left[\prod_{k=1}^{n_i}\left\{\prod_{l=1}^{n_j}\left[\ominus_{i,k,j,l}\right]\right\}\right.\right.\right.$$

$$\times\ P(A_i)\ \prod_{k=1}^{(i-1)}\left[1.0-\text{Min}\left\{1.0,\ \frac{P(A_i\cap A_k)}{P(A_i)}\right\}\right]\Bigg]\Bigg\}\ \leqslant\ P(F)\ \leqslant$$

$$P\ (A_i)+\sum_{i=2}^{m}\left[P(A_i)-\max_{k=1}^{(i-1)}\left\{P\ (A_i\cap A_k)\right\}\right] \qquad (5.4.7)$$

These bounds are identical if $m \leqslant 2$ ($P(F)= P(A_i)+ P(A_2)- P(A_1\cap A_2)$ if m=2 and $P(F) = P(A_1)$ if m = 1), but their values will be identical only if those failure paths have equal values for the upper and lower bounds on their probabilities of occurring from eqns. (5.3.9) and (5.3.11).

The values of $P(A_i)$ for use in eqn. (5.4.7) are the lower bound of eqn. (5.3.9) with the lower bound, and the upper bound with the upper bound. The values of $P(A_i\cap A_k)$, however, are the upper bound of eqn. (5.3.11)with the lower bound, and vice versa, since that term is used with a negative sign. The ratio of $P(A_i\cap A_k)$ and $P(A_i)$ is used with an upper limit of unity since the upper bound on $P(A_i\cap A_k)$ is divided by the lower bound on $P(A_i)$ when evaluating the lower bound of eqn. (5.4.7).

5.5 PARAMETERS USED TO MODEL THE VARIABLES

5.5.1 Element Resistances as Functions of the Basic Variables

In order to be able to use the methods described in sections 5.1 to 5.4 (as well as some of those in sections 4.3 and 4.4) it is necessary to be able to estimate the means,standard deviations and correlation coefficients for the distributions of the element resistances and of the loads. However, it is impossible to establish these parameters for the element resistances directly by sampling, and hence they must be estimated from a knowledge of the distributions of the basic variables (section 2.1). The basic variables have distribution parameters which are generally more easily established since the same materials, etc, are used for many designs. It should be noted that the equations in sections 5.1 and 5.2 cannot easily be redefined in terms of these basic variables.

The failure criterion for the initial local failure in any failure path is eqn (5.1.4), ie

$$g_i(\underset{\sim}{X}) = Z_{i,1} = R_{i,1} - \sum_{k=1}^{n_s}\left[d_{i,1,k}\ S_k\right] \leqslant 0.0 \qquad (5.5.1)$$

Therefore the element resistance, $R_{i,1}$, can be defined as a function of the basic variables, $\underset{\sim}{X}$, by

$$R_{i,1} = R_i(\underset{\sim}{X}) = Z_{i,1} + \sum_{k=1}^{n_s} \left[d_{i,1,k} \, S_k \right] \tag{5.5.2}$$

since the loads, $\underset{\sim}{S}$, are assumed to be a sub-set of the basic variables, $\underset{\sim}{X}$.

5.5.2 Means and Standard Deviations

Provided the basic variables in $R_i(\underset{\sim}{X})$ can be assumed to be mutually independent, one approach to the estimation of the parameters for the distribution of $R_{i,1}$ is to use the first two terms of Taylor's expansion for $R_i(\underset{\sim}{X})$ [28]. This leads to the approximations for the mean and variance of $R_{i,1}$ of

$$E(R_{i,1}) \doteq R_i(\underset{\sim}{X}) \Big|_{\underset{\sim}{x} = E(\underset{\sim}{X})} \tag{5.5.3}$$

and

$$\sigma^2(R_{i,1}) \doteq \sum_{j=1}^{n} \left[\left(\frac{\partial R_i(X)}{\partial x_j} \Big|_{\underset{\sim}{x} = E(\underset{\sim}{X})} \right)^2 \sigma^2(X_j) \right] \tag{5.5.4}$$

These values can then be used, with the parameters of the applied loads, to give the probability of an initial failure as

$$p(a_{i,1}) \doteq \Phi\left(- \frac{E(R_{i,1}) - \sum_{k=1}^{n_s} \left[d_{i,1,k} \; E(S_k) \right]}{\sqrt{\sigma^2(R_{i,1}) + \sum_{j=1}^{n_s} \sum_{k=1}^{n_s} \left[d_{i,1,j} \; d_{i,1,k} \rho(S_j, S_k) \, \sigma(S_j) \, \sigma(S_k) \right]}} \right) \tag{5.5.5}$$

This value of $p(a_{i,1})$ is the probability found by the Mean Value Level 2 method, which is demonstrated to be inaccurate in Part 1. A more accurate estimate of $p(a_{i,1}) = p(Z_{i,1} = g_i(\underset{\sim}{X}) \leqslant 0.0)$ can usually be obtained by using other methods, such as:

i. the Advanced Level 2 method;

ii. Monte Carlo simulation; or

iii. numerical integration.

The standard deviation for the equivalent normal distribution can therefore be found by re-arranging eqn (5.5.5) treating $\sigma(R_{i,1})$ rather than $p(a_{i,1})$ as the unknown. This approach will produce distribution parameters for the element resistances which give the best estimates for the probability of the initial local failures in each failure path, although not necessarily for the subsequent failures. It should be noted that there is no need to make the assumption that the basic variable governing $R_{i,1}$ are mutually independent in eqn (5.5.5).

Re-arranging eqn (5.5.5) gives

$$\sigma^2(R_{i,1}) \doteq \left\{ \frac{E(R_{i,1}) - \sum_{k=1}^{n_s} \left[d_{i,1,k}\, E(S_k) \right]}{^1(p(a_{i,1}))} \right\}^2 - \sum_{j=1}^{n_s} \sum_{k=1}^{n_s} \left\{ d_{i,1,j}\, d_{i,1,k}\, \rho(S_j, S_k) \times \sigma(S_j)\, \sigma(S_k) \right\} \qquad (5.5.6)$$

The mean value of the element resistance, $E(R_{i,1})$, can still be estimated from eqn (5.5.3) provided the basic variables are mutually independent. If the variables are partially correlated, the approximation for the mean becomes[6]

$$E(R_{i,1}) \doteq \left[R_i(\underset{\sim}{X}) \Big|_{\underset{\sim}{x} = E(\underset{\sim}{X})} \right] + \tfrac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\left[\frac{\partial^2 R_i(X)}{\partial X_j \partial X_k} \Big|_{\underset{\sim}{x} = E(\underset{\sim}{X})} \right] \rho(X_j, X_k)\, \sigma(X_j)\, \sigma(X_k) \right) \qquad (5.5.7)$$

The second term, which is not included in eqn (5.5.3), is negligible provided the standard deviations of the basic variables and the non-linearity of the function are not large. If eqn (5.5.7) is used with mutually independent basic variables this term reduces to half the sum of the variances of the variables times the second derivatives of the resistance.

As shown in Appendix C, there can be a considerable difference between the standard deviation obtained by eqn (5.5.6) and that obtained by eqn (5.5.4). It should be noted that part of this difference will be due to treating the loading variables, $\underset{\sim}{S}$, as having normal distributions in eqn (5.5.6), while using the appropriate distribution to model them when calculating $p(a_{i,1})$ (extreme type I maxima for the example in Appendix C).

5.5.3 Correlation Coefficients

The correlation coefficients have to be obtained indirectly by first finding the correlation coefficients of the excess resistances. The value of these can be established by using the equality between the correlation coefficient of two excess resistances and the cosine of the angle between the normals to the failure boundaries in standardised normal space (sub-section 4.4.6). For mutually independent basic variables, eqn (4.4.21) gives the correlation coefficient of the excess resistances, or

$$\rho(Z_{i,1}, Z_{j,1}) = \cos\Theta = \sum_{k=1}^{n} (\alpha_{i,k}\, \alpha_{j,k}) \qquad (5.5.8)$$

where $\alpha_{i,k}$ and $\alpha_{j,k}$ are the values from eqn (2.4.22) used in the Advanced Level 2 analysis (sub-sections 2.4.4 & 2.4.5) of $Z_{i,1}$ and $Z_{j,1}$ respectively for the k^{th} basic variable. It should be noted that the value of $\alpha_{i,k}$ is zero for any basic

variable which governs $Z_{j,1}$ but not $Z_{i,1}$, and vice versa, so that $\rho(Z_{i,1}, Z_{j,1})$ is zero if no basic variables are common to both excess resistances, as would be expected.

For the more general case where basic variables can be correlated, the standardised normal variables, $\underset{\sim}{U}$, used in the Advanced Level 2 analysis first have to be transformed into a set of mutually independent standardised normal variables, $\underset{\sim}{Y}$ (sub-section 2.1.3). From eqn.(2.1.3), this transformation is

$$\underset{\sim}{Y} = \underline{C}(\underset{\sim}{U} - E(\underset{\sim}{U})) = \underline{C}.\underset{\sim}{U} \qquad (5.5.9)$$

since $E(\underset{\sim}{U}) = 0.0$. Transforming the $\underset{\sim}{\alpha}$ vectors into $\underset{\sim}{a}$ vectors gives

$$\underset{\sim}{a}_i = \underline{C}.\underset{\sim}{\alpha}_i \quad ; \quad \underset{\sim}{a}_j = \underline{C}.\underset{\sim}{\alpha}_j \qquad (5.5.10)$$

Therefore

$$\begin{aligned}\rho(Z_{i,1}, Z_{j,1}) = \cos\Theta &= \underset{\sim}{a}_i^T . \underset{\sim}{a}_j \\ &= (\underline{C}.\underset{\sim}{\alpha}_i)^T . (\underline{C}.\underset{\sim}{\alpha}_j) \\ &= \underset{\sim}{\alpha}_i^T . \underline{C}^T . \underline{C} . \underset{\sim}{\alpha}_j \end{aligned} \qquad (5.5.11)$$

Substituting in eqn (2.1.4) gives

$$\rho(Z_{i,1}, Z_{j,1}) = \underset{\sim}{\alpha}_i^T . \underline{V}_u^{-1} . \underset{\sim}{\alpha}_j \qquad (5.5.12)$$

where $\underline{V}_u$ is the symmetric matrix of the correlation coefficients of the standardised normal basic variables. It should be noted that the inversion of the matrix can be carried out in parts if only sub-sets of the variables are mutually correlated.

The correlation coefficient of the resistances can be found in terms of correlation coefficient of the excess resistances by re-arranging eqn (5.2.14). Since $c_{i,1,1} = c_{k,1,1} = 1.0$, putting $j = 1 = 1$ and writing j for k in eqn (5.2.14) gives

$$\rho(Z_{i,1},Z_{j,1}) = \Big\{\rho(R_{i,1},R_{j,1})\sigma(R_{i,1})\sigma(R_{j,1}) + \sum_{k=}^{n_s}\sum_{l=1}^{n_s}\big[d_{i,1,k}\, d_{j,1,1}\rho(S_k,S_1)\times \sigma(S_k)\sigma(S_1)\big\}\Big/\big[\sigma(Z_{i,1})\,\sigma(Z_{j,1})\big] \qquad (5.5.13)$$

Therefore, by re-arranging eqn (5.5.13)

$$\rho(R_{i,1},R_{j,1}) = \Big\{\rho(Z_{i,1},Z_{j,1})\,\sigma(Z_{i,1})\,\sigma(Z_{j,1}) - \sum_{k=1}^{n_s}\sum_{l=1}^{n_s} d_{i,1,k}\, d_{j,1,1}\,\rho(S_k,S_1)\times \sigma(S_k)\,\sigma(S_1)\Big\}\Big/\big[\sigma(R_{i,1})\,\sigma(R_{j,1})\big] \qquad (5.5.14)$$

The correlation coefficients for the example in Appendix C are shown in table C.5.

5.6 TIME-VARYING LOADS

5.6.1 Introduction

So far in this chapter each variable which governs the behaviour of a structure is assumed to have a single value. However, the values of some variables do not remain constant with time, such as wind pressure or the strength of high alumina

cement concrete. When there is only one such variable this variability can be included in the model by describing that variable as the extreme value, either maximum or minimum as appropriate, that occurs within the reference period. However, when there are more than one time dependent variable they cannot be modelled simply by using extreme value distributions since those extremes are unlikely to occur at the same time. Therefore more complex modelling is required to describe such a case, and several models have been developed by others to achieve this [40].

This section is not intended as a detailed study or development of these methods, but contains an extension of the proposed method of system reliability to incorporate one particular load combination model. This model for treating load combinations is due to Ferry Borges and Castanheta, and is described in CIRIA report 63 [11] from which part of the description is reproduced, with minor modifications, in sub-section 5.6.2 below. It should be noted, however, that:

i. the model requires several assumptions which restricts the applicability of it to certain design situations; and

ii. the model only allows for time-varying loads, not time-varying resistances.

5.6.2 Load Combination Model

It is assumed that for each time-varying load, the reference period, T, is subdivided into a number of elementary time intervals of equal length, $\Delta_i T$, such that the peak values of load in consecutive time intervals can be considered to be independent. It is then further assumed that each (possibly continuously varying) load, S_i, remains constant at its peak value throughout its own elementary time interval. This latter assumption is clearly conservative, but it is not unduly so if the time intervals are relatively short. This model is represented in figure 5.6.1 for three time-varying loads.

The load combination model then requires the ordering of loads according to the individual number of possible repetitions such that

$$n_{(i-1)} \leqslant n_i \leqslant n_{(i+1)} \qquad (5.6.1)$$

for $1 < i < k$, where there are k time-varying loads and where

$$n_i = T/\Delta_i T \qquad (5.6.2)$$

An approximation to the maximum combined load is then obtained by considering the load combinations in table 5.6.1. This shows, for each combination j, the number of repetitions $N_{i,j}$ for each time-varying load, S_i, to be used in determining its

modified probability distribution for combination with the other (k-1) loads. When a reference is made to a load combination it will imply one of those given in the table, for which the principal load is defined in this study as the load which is modelled as having its maximum intensity in the reference period, even when it is not the largest numerically. The reliability index, β_i, is determined (using the Advanced Level 2 method given in sub-sections 2.4.4 to 2.4.6) for each of the k combinations and the results may then be combined to estimate the total failure probability using the relationship

$$p(F) = 1.0 - \sum_{i=1}^{k} [1.0 - \Phi(-\beta_i)]$$

$$\doteq \sum_{i=1}^{k} \Phi(-\beta_i) \qquad (5.6.3)$$

5.6.3 Accuracy of the Model

It should be noted that eqn (5.6.3) assumes that the excess resistances for the element under each load combination are mutually independent, which is only likely to occur if both the load intensities from each load combination and the resistances to them are also mutually independent. Clearly the resistances will be mutually fully dependent unless the resistance of the element is also time-dependent, and even then it is not likely to have independent intensities during each load's elementary time interval. However, the variability of the load is usually greater than that of the resistance (but see Appendix C, table C.4) so that the excess resistances are not often strongly correlated.

A second possible source of error is that the load combination method does not necessarily include all the load combinations that could produce the maximum load intensity (eg if the second largest intensity of each load occurred simultaneously). This error could lead to under-estimation of the failure probability, while the previous one could lead to over-estimation. However, despite these possible inaccuracies, Turkstra and Madsen [40] have shown that this method produces reasonable results for those cases it can model.

5.6.4 Extension to System Reliability Analysis

In element reliability analysis each element is analysed for failure to resist each of the load combinations, and similarly in system reliability analysis each failure path and local event, $a_{i,j}$, has to be analysed for failure under each load combination. Therefore it is simpler to consider any failure path which has a local event $a_{i,j}$ analysed with more than one load combination applied to the structural system as the appropriate number of different failure paths within the

failure set. However, from practical considerations, those load combinations for which an increase in the principal load would not increase the probability of failure need not be considered for that local event, $a_{i,j}$.

The condition where repairs are assumed to be carried out at the end of each time interval if any local failure(s) has occurred can be modelled by applying the same load combination to the system for every local event $a_{i,j}$ with a failure path. The condition of no repairs being carried out, however, cannot be modelled accurately by applying different load combinations to the structural system for each local failure modes in any failure path. This inefficiency of the latter model results from the load combination for the second local event $a_{i,j}$ in a failure path modelling the maximum intensity of load effect on that element during the reference period and not the maximum intensity of load effect during that part of the reference period remaining after the initial local failure occurs.

It should be noted that if the loads are applied at the same point on a structure, then maximum intensity of load-effect will occur on each element at the same time and the suggested model of the structure being regularly maintained will be appropriate for structures which are not maintained. However, the more general case of different loads being applied at different points implies that the maximum intensity of load-effect is likely to occur with different load combinations for different elements, when the maintenance procedure becomes a relevant consideration in the applicability of the model.

5.6.5 Single Elements with Multiple Time-Varying Loads

If the load combination model can be applied to the system reliability analysis technique, it can also be applied to the system of a single element acted upon by several time-varying loads. This system has a probability of failure which is the probability of failure for that element and which has made due allowance for the correlation between the resistances to which the load-effects from each load combination are applied. This approach reduces one of the possible errors in this load combination method discussed in sub-section 5.6.3.

5.6.6 Parameters of the Loads

In order to use this load combination model the means, standard deviations & correlation coefficients of the loads are required. It is assumed that these parameters are known for the underlying distribution, that is the distribution of the mutually independent values during each elementary time interval. The different loads have to be able to be modelled as being independent since the correlation coefficient of the underlying variables will not be the same as the correlation coefficients of the extreme values.

The probability of an extreme value from n time intervals can be found since the load intensities are assumed to have independent values in different time intervals. The cumulative probabilities are given by [6]

$$F_{S^n_{MAX}}(s) = [F_S(s)]^n$$
$$F_{S^n_{MIN}}(s) = 1.0 - [1.0 - F_S(s)]^n \qquad (5.6.4)$$

where $F_S(s)$ is the cumulative probability function of the load intensity, S, in any time interval and $F_{S^n_{MAX}}(s)$ and $F_{S^n_{MIN}}(s)$ are the cumulative distribution functions of the maximum and minimum load intensities in n time intervals respectively. Therefore the probability density functions and the central moments of the extreme values can be found from

$$f_{S^n_{MAX}}(s) = n\,[F_S(s)]^{(n-1)} f_S(s)$$
$$f_{S^n_{MIN}}(s) = n\,[1.0 - F_S(s)]^{(n-1)} f_S(s) \qquad (5.6.5)$$

$$E((S^n_{MAX})^r) = \int_{-\infty}^{+\infty} s^r\, n\,[F_S(s)]^{(n-1)} f_S(s)\, ds$$
$$E((S^n_{MIN})^r) = \int_{-\infty}^{+\infty} s^r\, n\,[1.0 - F_S(s)]^{(n-1)} f_S(s)\, ds \qquad (5.6.6)$$

Eqns.(5.6.5) and (5.6.6) are examined for some common distributions of S in Appendix D.

The load intensity for any load will have to be modelled by different variables for each load combination, using the central moments from eqn.(5.6.5) with the appropriate value of n from table 5.6.1. The correlation coefficient for two variables modelling the same load in different load combinations will be derived from the
probability that both load combinations will include in their summation the intensity of load effect from that load which occurs during the same time interval.

5.6.7 Correlation of Extreme Values

The probability that two variables modelling the load intensity in a single time interval from the same load will use the same time interval is $1/n_k$. Therefore since the two variables will have the same distribution parameters,

$$\rho(S_i,S_j) = [E(S_i S_j) - E(S_i)\,E(S_j)] / [\sigma(S_i)\,\sigma(S_j)] = \left[\left\{\frac{1}{n_k} E(S_i^2) + \frac{(n_k - 1.0)}{n_k} \times E^2(S_i)\right\} - E^2(S_i)\right] / \sigma^2(S_i) = 1/n_k \qquad (5.6.7)$$

An example of this is the correlation coefficient for the variables modelling the intensities of load S_1 in load combinations 2 and 3 (fig 5.6.1). The probability that the maximum intensities of loads S_2 and S_3 occur in the same time interval $\Delta_1 T$ is $1/n_1 = 0.5$, which is therefore also the correlation coefficient for the two variables modelling S_1 for those load combinations.

Similarly, if two variables model the maximum (or minimum) extreme of $n_k/n_{(k-1)}$ values, then the probability of those two sets of time intervals being identical is $1/n_{(k-1)}$, so that

$$\rho(S_i, S_j) = 1/n_{(k-1)} \qquad (5.6.8)$$

The probability that a variable which models the extreme form n_k values has that extreme value in a time interval which is the set of $n_k/n_{(k-1)}$ time intervals, where a second variable models the extreme from the intensities during those time intervals, is equal to $1/n_{(k-1)}$, so that eqn. (5.6.8) applies here also.

When two variables model the maximum and minimum extreme load intensities for different sets of time intervals, then

$$\rho((S_{MAX}^{n(k-1)})_i, (S_{MIN}^{n(k-1)})_j) = \frac{E((S_{MAX}^{n(k-1)})_i (S_{MIN}^{n(k-1)})_j) - E((S_{MAX}^{n(k-1)})_i) E((S_{MAX}^{n(k-1)})_j)}{((S_{MAX}^{n(k-1)})_i) \qquad ((S_{MIN}^{n(k-1)})_j)}$$

$$= \left\{ \frac{1}{n_{(k-1)}} \left[\rho(S_{MAX}^{n(k-1)}, S_{MIN}^{n(k-1)}) \cdot \sigma(S_{MAX}^{n(k-1)}) \, \sigma(S_{MIN}^{n(k-1)}) + E(S_{MAX}^{n(k-1)}) E(S_{MIN}^{n(k-1)}) \right] + \left(\frac{n_{(k-1)} - 1.0}{n_{(k-1)}} - 1.0 \right) E(S_{MAX}^{n(k-1)}) E(S_{MIN}^{n(k-1)}) \right\} \Big/ \left\{ \sigma(S_{MAX}^{n(k-1)}) \sigma(S_{MIN}^{n(k-1)}) \right\}$$

$$= \frac{\rho(S_{MAX}^{n(k-1)}, S_{MIN}^{n(k-1)})}{n_{(k-1)}} \qquad (5.6.9)$$

where $\rho(S_{MAX}^{n(k-1)}, S_{MIN}^{n(k-1)})$ is the correlation coefficient of the maximum and minimum values from a single set of $n_{(k-1)}$ time intervals. The correlation coefficients for the variables modelling a time-variant load are tabulated in table 5.6.2, and the values of $\rho(S_{MAX}^{n(k-1)}, S_{MIN}^{n(k-1)})$ are discussed in Appendix E, with particular reference to S having a normal distribution.

5.6.8 Estimation of Element Resistence Parameters

Eqn. (5.5.6) was obtained with the assumption that there was only one load combination to consider. There will be a different failure probability for each load combination if the loading variables model more than one time-varying load. The mean, as well as the standard deviation, can be used as a variable when there are two load combinations. Given the event of an element failing by two load combinations are $a_{i,1}$ and $b_{i,1}$, then the failure criteria are given, respectively, by

$$\left.\begin{aligned} Z_{i,1} &= R_{i,1} - \sum_{k=1}^{n_s} \left[d_{i,1,k} \, (S_{MAX}^{n_a})_k \right] \\ \text{and} \quad Z_{i,1} &= R_{i,1} - \sum_{k=1}^{n_s} \left[d_{i,1,k} \, (S_{MAX}^{n_b})_k \right] \end{aligned}\right\} \qquad (5.6.10)$$

where $(S_{MAX}^{n_a})_k$ is the maximum (or minimum if appropriate) intensity of the k^{th} load in n_a time intervals, and n_a is the number of time intervals for the k^{th} load in load combination a according to table 5.6.1. (Similarly for $(S_{MAX}^{n_b})_k$ and n_b). It can be shown (Appendix F) that the required failure probabilities for both load combinations will be obtained if

$$E(R_{i,1}) = \left\{ \left[\Phi^{-1}(p(a_{i,1}))\right]^2 \sum_{k=1}^{n_s} \left[d_{i,1,k} \, E((S_{MAX}^{n_b})_k) \right] - \left[\Phi^{-1}(p(b_{i,1}))\right]^2 \sum_{k=1}^{n_s} \left[d_{i,1,k} \times E((S_{MAX}^{n_a})_k) \right] \right\} \Big/ \left\{ \left[\Phi^{-1}(p(a_{i,1}))\right]^2 - \left[\Phi^{-1}(p(b_{i,1}))\right] \right\}$$

$$\pm \Phi^{-1}(p(a_{i,1})) \, \Phi^{-1}(p(b_{i,1})) \left\{ \left[\frac{\sum_{k=1}^{n_s} \left[d_{i,1,k} \left\{ E((S_{MAX}^{n_a})_k) - E((S_{MAX}^{n_b})_k) \right\} \right]}{\left[\Phi^{-1}(p(a_{i,1}))\right]^2 - \left[\Phi^{-1}(p(b_{i,1}))\right]^2} \right]^2 - \frac{\sum_{k=1}^{n_s} \left[d_{i,1,k}^2 \left\{ \sigma^2((S_{MAX}^{n_a})_k) - \sigma^2((S_{MAX}^{n_b})_k) \right\} \right]}{\left[\Phi^{-1}(p(a_{i,1}))\right]^2 - \left[\Phi^{-1}(p(b_{i,1}))\right]^2} \right\}^{\frac{1}{2}} \qquad (5.6.11)$$

and

$$\sigma^2(R_{i,1}) = \left[\frac{E(R_{i,1}) - \sum_{k=1}^{n_s} \left[d_{i,1,k} \, E((S_{MAX}^{n_a})_k) \right]}{\Phi^{-1}(p(a_{i,1}))} \right]^2 - \sum_{k=1}^{n_s} \left[d_{i,1,k} \, \sigma^2((S_{MAX}^{n_a})_k) \right] \qquad (5.6.12)$$

(Note that the loads have been assumed mutually independant).

Some form of numerical "best fit" analysis is required when there are more than two load combinations. One approach is to minimise the value of α_a summed over each load combination by varying the values of $E(R_{i,1})$ and $\sigma(R_{i,1})$, where

$$\alpha_a = \left[\Phi^{-1}(a_{i,1}) + \frac{E(R_{i,1}) - \sum_{k=1}^{n_s} \left[d_{i,1,k} E((S^{n_a}_{MAX})_k) \right]}{\sqrt{\sigma^2(R_{i,1}) + \sum_{k=1}^{n_s} \left[d^2_{i,1,k} \sigma^2((S^{n_a}_{MAX})_k) \right]}} \right]^2 \qquad (5.6.13)$$

Having to use a "best fit" approach makes the system analyses of single elements under multiple time-variant loads less rigorous, but it will still be more theoretically correct than treating failures under different load combinations as independent events.

The correlation coefficient for pairs of resistances will have to be obtained from a "best fit" analysis whenever there is more than one time-varying load. If $\rho_a(R_{i,1},R_{j,1})$ is the correlation coefficient with load combination a by using the method described in section 5.5, $\sum(\alpha_a)$ can be minimised by varying $\rho(R_{i,1},R_{j,1})$ where

$$\alpha_a = \left[\rho(R_{i,1},R_{j,1}) - \rho_a(R_{i,1},R_{j,1}) \right]^2 \qquad (5.6.14)$$

POINT IN VARIABLE SPACE	NEGATIVE EXCESS RESISTANCE			PHYSICAL FAILURE			SYSTEM FAILURE
	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	1	2	3	A_i
I	N	N	N	S	S	S	S&N
II	O	N	N	F	S	S	S&N
III	N	O	N	S	S	S	S&N
IV	N	N	O	S	S	S	S&N
V	O	O	N	F	F	S	S&N
VI	O	N	O	F	S	S	S&N
VII	N	O	O	S	S	S	S&N
VIII	O	O	O	F	F	F	F&O

O = Occurrence of Mathematical Event

N = Non-Occurrence of Mathematical Event

S = Physical Safety of Element

F = Physical Failure of Element

TABLE 5.1.1 - IMPLICATION OF THE VALUES OF THE BASIC VARIABLES FOR THE MODES OF SYSTEM FAILURE IN FIG 5.1.3

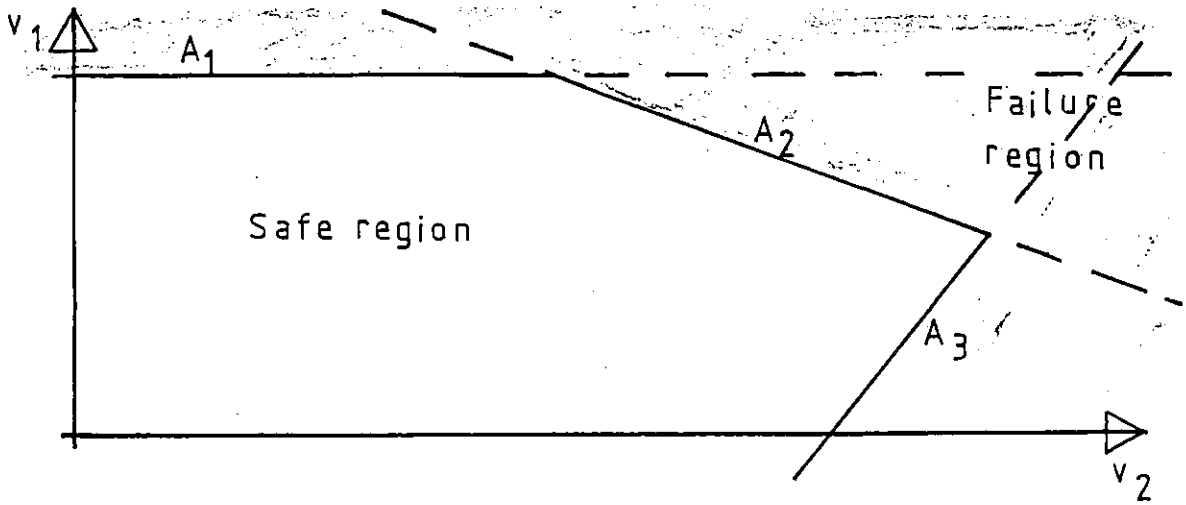


Fig.5.1.1 - Overall Failure Boundary for a System Consisting of Linear Boundaries, each Representing a Mode of System Failure

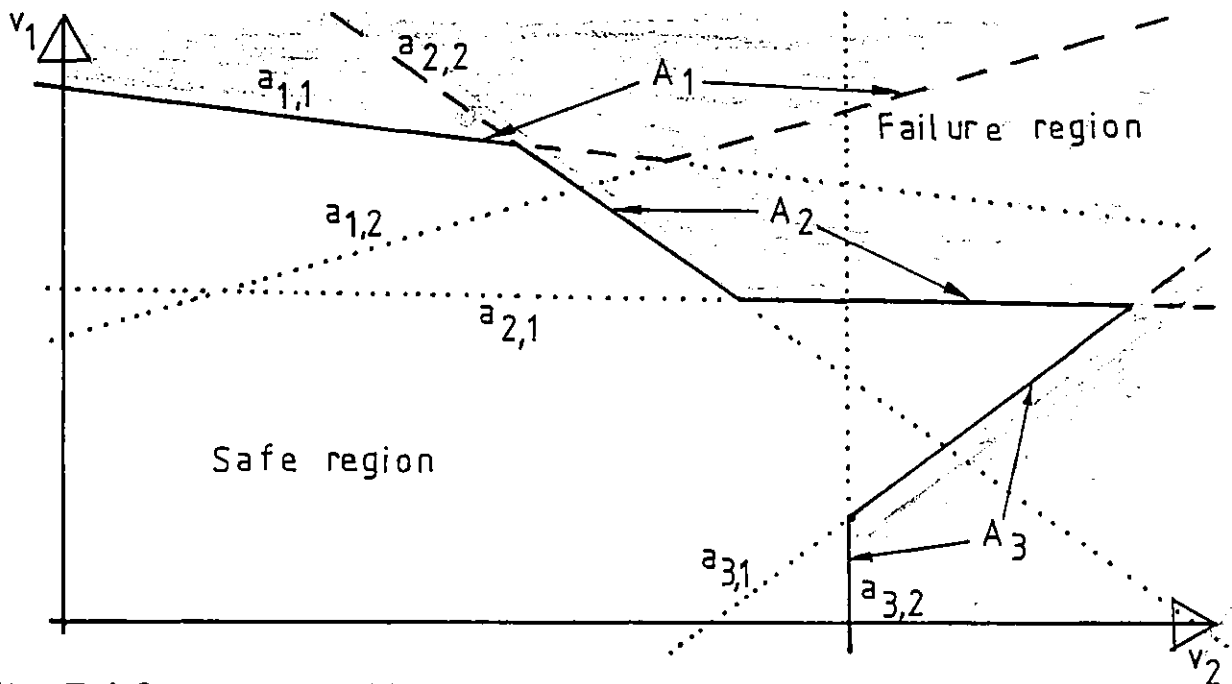


Fig.5.1.2 - Overall Failure Boundary for a System Consisting of Linear Boundaries, each Representing a Mode of Local Failure

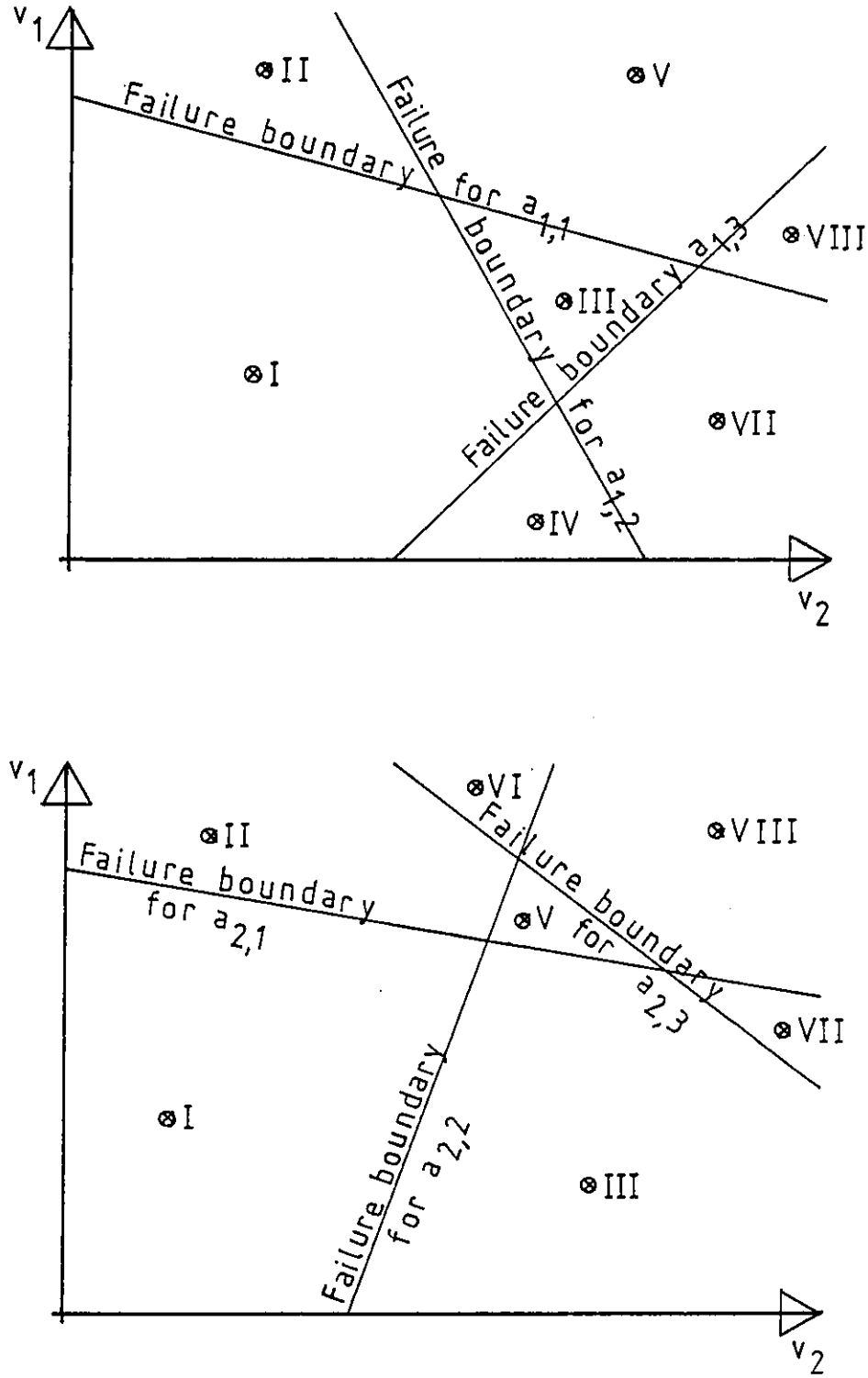


Fig. 5.1.3 - Failure Boundaries for Single Modes of System Failure in a Plane Through a Multi-Variable Space [See table 5.1.1]

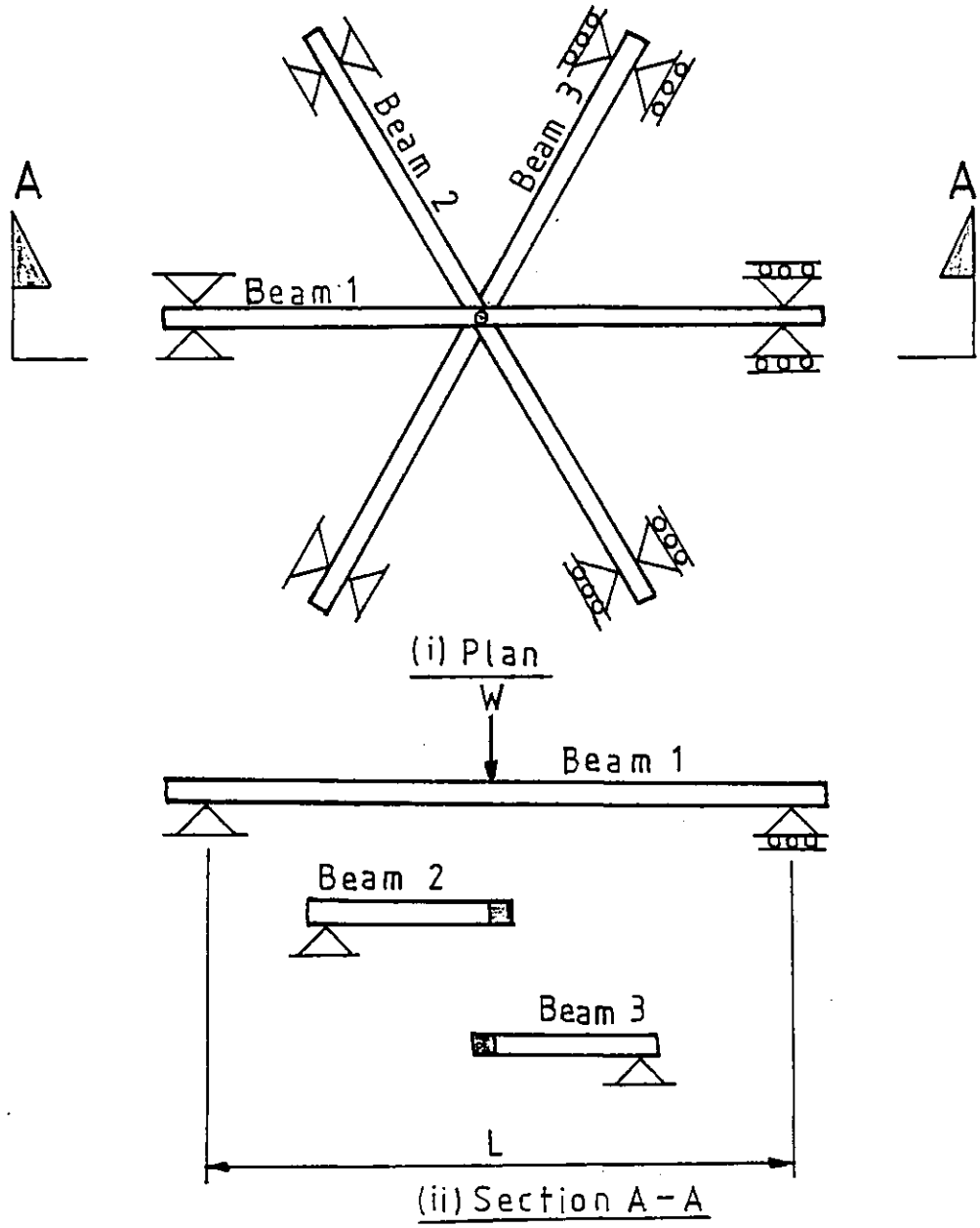
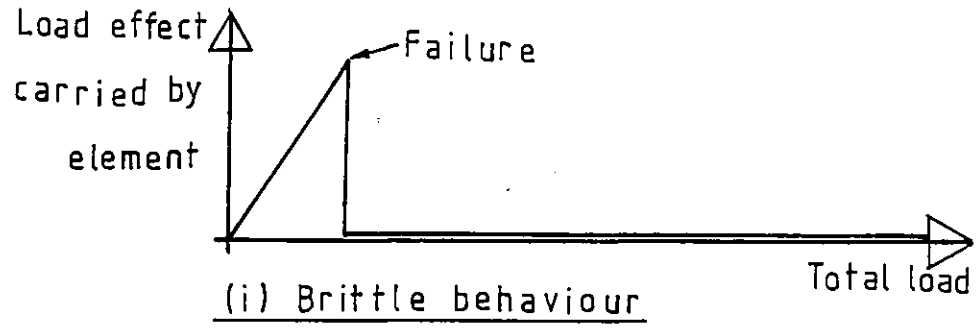
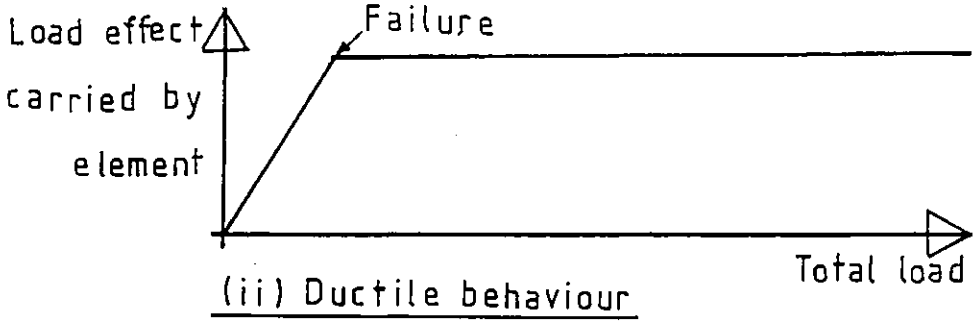


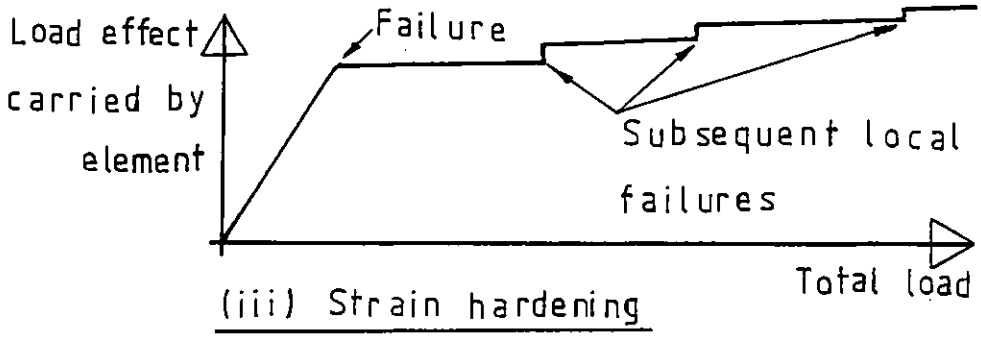
Fig.5.1.4 - Three Beams for the Support of a Central Load



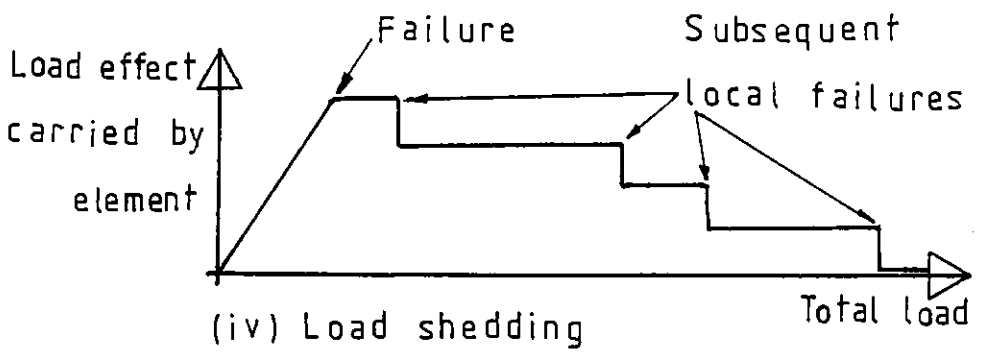
(i) Brittle behaviour



(ii) Ductile behaviour



(iii) Strain hardening



(iv) Load shedding

Fig. 5.1.5 - Idealised Element Behaviour Models

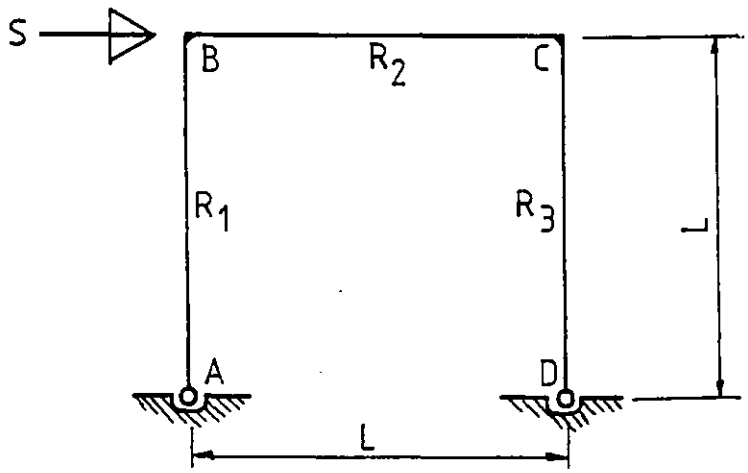


Fig.5.1.6 - Pinned Portal Frame

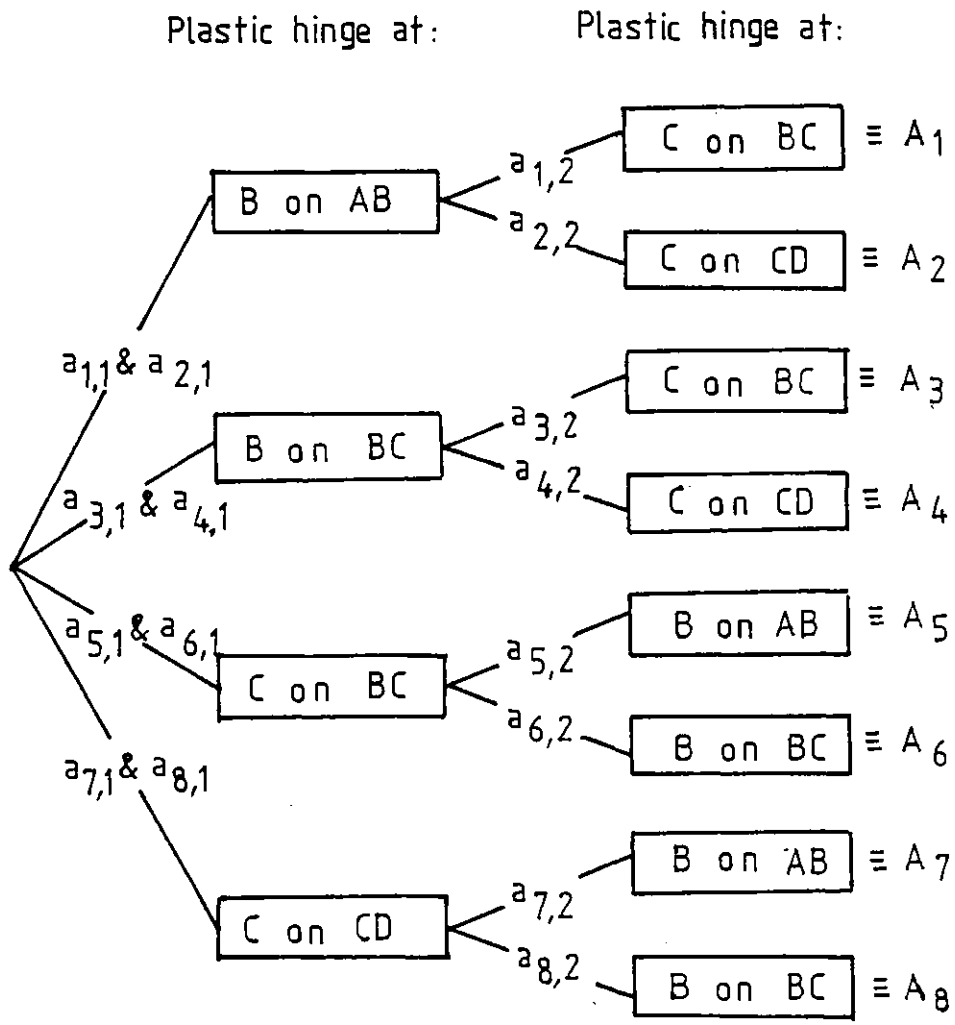


Fig.5.1.7 - Failure Tree for Pinned Portal Frame

	Simple Solution	Hermite-Gauss Quadrature	Simpson's Rule Approximation	Owen's Method
$1.0 \geq \rho > 0.99$	$\Phi(-\text{Max}(\beta 1, \beta 2))$			
$0.99 \geq \rho > 0.95$			$\lvert\beta 1 - \beta 2\rvert \geq 1$ or $\beta 1$ or $\beta 2 \geq 6$ or $\beta 1 + \beta 2 \geq 3(1+\rho)$	All other values of $\beta 1, \beta 2$
$0.95 \geq \rho > 0.20$			$\beta 1$ or $\beta 2 \geq 6$ or $\beta 1 + \beta 2 \geq 3(1+\rho)$	All other values of $\beta 1, \beta 2$
$0.20 \geq \rho > 0.001$		All $\beta 1, \beta 2$		
$0.001 \geq \rho \geq -0.001$	$\Phi(-\beta 1) \times \Phi(-\beta 2)$			
$0.001 > \rho \geq -0.40$		All $\beta 1, \beta 2$		
$-0.40 > \rho \geq -0.70$		$\beta 1$ and $\beta 2 < -1$ and $\lvert\beta 1 - \beta 2\rvert > 1$	$\beta 1$ or $\beta 2 \geq 6$ or $\beta 1 + \beta 2 \geq 3 \times (1+\rho)$	All other values of $\beta 1, \beta 2$
$-0.70 > \rho \geq -0.99$			$\beta 1$ and $\beta 2 < -1$ or $\lvert\beta 1 - \beta 2\rvert > 2$ or $\beta 1$ or $\beta 2 \geq 6$ or $\beta 1 + \beta 2 \geq 3(1+\rho)$	All other values of $\beta 1, \beta 2$
$-0.99 > \rho \geq -1.0$	$\text{Max}(0, 1 - \Phi(\beta 1) - \Phi(\beta 2))$			

TABLE 5.2.1 - METHODS TO BE USED TO OBTAIN $\Phi(-\beta 1, -\beta 2, \rho)$ [38]

LOAD	NUMBER OF TIME INTERVALS IN THE REFERENCE PERIOD	LOAD COMBINATION			
		1	2	3	... k
1	n_1	$N_{11} = n_1$	$N_{21} = 1$	$N_{31} = 1$	... $N_{k1} = 1$
2	n_2	$N_{12} = {}^{n_2}/_{n_1}$	$N_{22} = n_2$	$N_{32} = 1$	... $N_{k2} = 1$
3	n_3	$N_{13} = {}^{n_3}/_{n_2}$	$N_{23} = {}^{n_3}/_{n_2}$	$N_{33} = n_3$	... $N_{k3} = 1$
⋮	⋮	⋮	⋮	⋮	⋮
k	n_k	$N_{1k} = {}^{n_k}/_{n_{(k-1)}}$	$N_{2k} = {}^{n_k}/_{n_{(k-1)}}$	$N_{3k} = {}^{n_k}/_{n_{(k-1)}}$	... $N_{kk} = n_k$

TABLE 5.6.1 - NUMBER OF LOAD REPETIONS FOR EACH COMBINATION OF TIME - VARYING LOADS [11]

NUMBER OF TIME INTERVALS IN THE OBSERVATION PERIOD		1	n_k	${}^{n_k}/_{n_{(k-1)}}$	
			MAX^m	MIN^m	MAX^m
1		1.0 or ${}^1/_{n_k}$	${}^1/_{n_k}$	${}^1/_{n_k}$	${}^1/_{n_k}$
n_k	Max^m	${}^1/_{n_k}$	1.0	$\rho/n_{(k-1)}$	${}^1/_{n_{(k-1)}}$
${}^{n_k}/_{n_{(k-1)}}$	Min^m	${}^1/_{n_k}$	$\rho/(\ -1)$	1.0 or ${}^1/_{n_{(k-1)}}$	ρ or $\rho/n_{(k-1)}$
	Max^m	${}^1/_{n_k}$	${}^1/_{n_{(k-1)}}$	ρ or $\rho/n_{(k-1)}$	1.0 or ${}^1/_{n_{(k-1)}}$

where $\rho = \rho(S_{MAX}, S_{MIN})$ from $({}^{n_k}/_{n_{(k-1)}})$ values of S.

TABLE 5.6.2 - CORRELATION COEFFICIENTS, K^{TH} TIME-VARYING LOAD

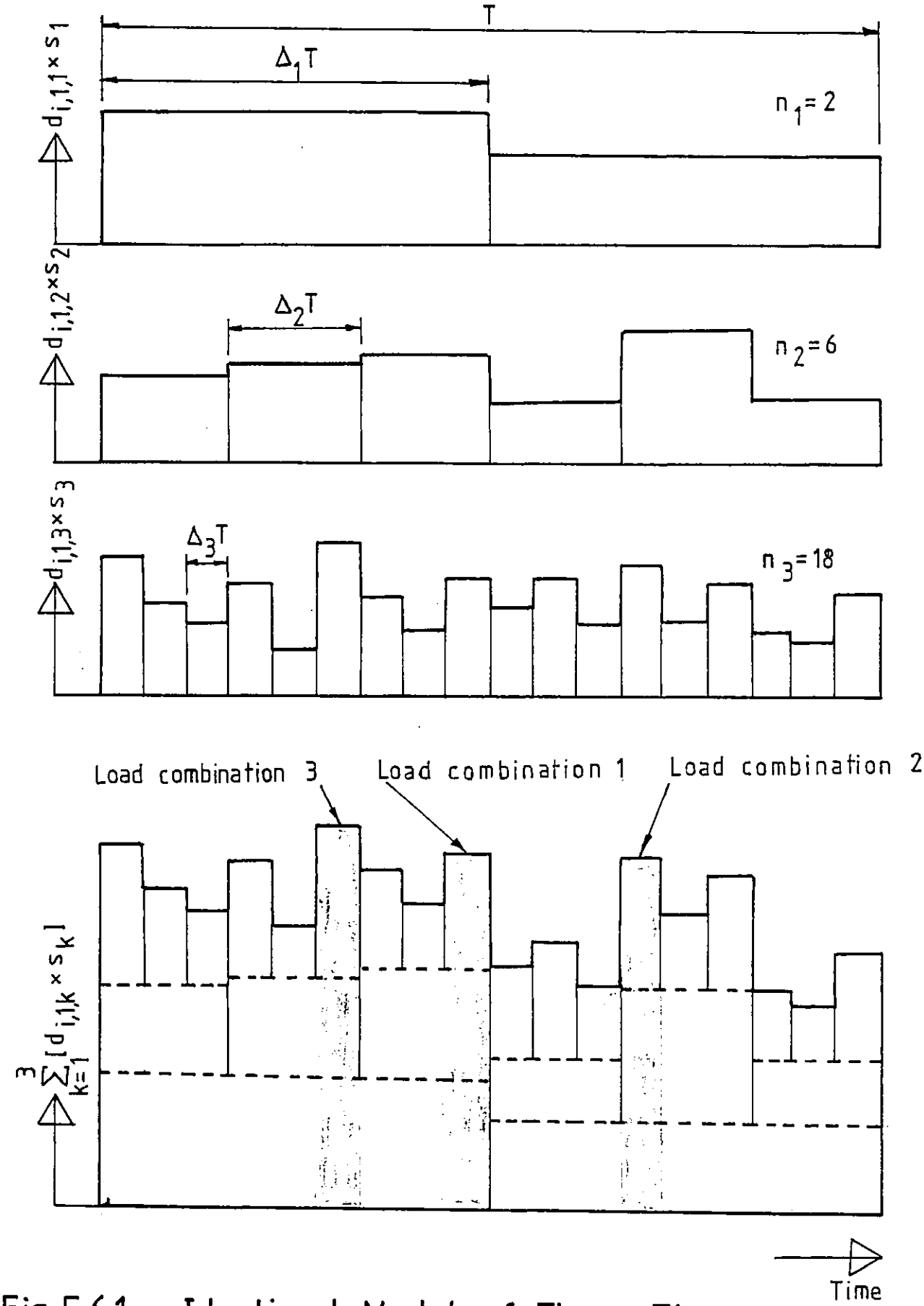


Fig. 5.6.1 - Idealised Model of Three Time-Varying Loads

CHAPTER 6 - APPLICATION OF SYSTEM RELIABILITY METHODS

6.1 SIMPLE EXAMPLE

6.1.1 Structure Under Consideration

Consider the portal frame encastre at D while remaining pinned at A. The members AB, BC and CD are numbered 1 to 3 respectively for use as subscripts.

The frame was designed using plastic analysis for a load factor of 1.6 (Appendix G, Section G.1). The load and plastic moments of resistance were assumed to have coefficients of variation of 0.20 and 0.05 respectively. For system reliability analysis three different systems were considered, in which the resistances of the three members were considered to be mutually correlated with coefficients of 0.0, 0.5 and 1.0. In each case the plastic moment of resistance along each member was assumed to be constant. These assumptions result in 24 possible failure paths, each with three modes of local failure (fig. 6.1.1).

The results of the analyses (table 6.1.1) show that the failure set is dominated by a small sub-set of the failure paths - viz. those paths consisting of local failures D → C → B. This domination occurs because all three members have the same mean resistance, which results in a high probability of the first of any local failure occurring at the point of maximum load effect (ie D).

In order to reduce this domination, a second frame was designed in which the flexural strength of the members were made proportional to the elastic bending moment in the members prior to any hinges forming (Appendix G, Section G.3). This structure was analysed as before with three sets of resistance correlation coefficients (table 6.1.2). These systems were also dominated by a sub-set of the failure paths, although not to the same extent as for the plastically designed systems. This domination was by the failure paths consisting of local failures at B_1, C_2 and D_3, since a plastic hinge forming at the junction of two members is more likely to occur on the member with the smaller mean resistance (ie B_1 rather than B_2 and C_2 rather than C_3).

6.1.2 Study of Particular Failure Paths

The method used to analyse the structures can be demonstrated by examining the probability of one failure path in one system in detail. The failure path to be examined will be P_{11}, consisting of $C_2 \rightarrow D_3 \rightarrow B_1$ in the elastically designed structure with partial correlation of the resistances. Since this was the third

most probable failure path, after P_{17} ($D_3 \rightarrow B_1 \rightarrow C_2$) and P_{21} ($D_3 \rightarrow C_2 \rightarrow B_1$), in terms of the mean of the bounds, it is referred to as the third failure path, or A_3, subsequently.

6.1.3 Failure Criteria

The failure criteria for this failure path are obtained by summing the load-effects found in Appendix G, Section G.2, and summarised in figs. G.2.3 to G.2.7. This gives

$$Z_{3,1} = R_2 - 0.3182 \times S \leqslant 0.0 \qquad (6.1.1)$$

$$\begin{aligned} Z_{3,2} &= R_3 - 0.4545 \times S - 0.6667(0.3182 \times S - R_2) \\ &= 0.667 \times R_2 + R_3 - 0.6667 \times S \leqslant 0.0 \end{aligned} \qquad (6.1.2)$$

and

$$\begin{aligned} Z_{3,3} &= R_1 - 0.2273 \times S - 0.3333(0.3182 \times S - R_2) \\ &\quad - 1.0 \times (0.6667 \times S - 0.667 \times R_2 - R_3) \\ &= R_1 + R_2 + R_3 - S \leqslant 0.0 \end{aligned} \qquad (6.1.3)$$

The coefficients for the failure criteria of the first three failure paths are given in table 6.1.3 for information.

6.1.4 Parameters of the Excess Resistances

Inserting these coefficients and the means, standard deviations and correlation coefficients of the load and member resistances (table 6.1.4) into eqns. (5.2.2), (5.2.3) and (5.2.14) gives the means, standard deviations and correlation coefficients, respectively, for the excess resistances (table 6.1.5). For example

$$\begin{aligned} E(Z_{3,3}) &= 1.0E(R_1) + 1.0E(R_2) + 1.0E(R_3) - 1.0E \\ &= 0.5266 + 0.7372 + 1.0532 - 1.0000 \\ &= 1.3169 \end{aligned} \qquad (6.1.4)$$

$$\begin{aligned} \sigma(Z_{3,3}) = \Big\{ & 1.0\sigma(R_1)\left[1.0\,\sigma(R_1)\rho(R_1,R_1) + 1.0\,\sigma(R_2)\rho(R_1,R_2) + 1.0\,\sigma(R_3)\rho(R_1,R_3)\right] \\ & + 1.0\,\sigma(R_2)\left[1.0\,\sigma(R_1)\rho(R_2,R_1) + 1.0\,\sigma(R_2)\rho(R_2,R_2) + 1.0\,\sigma(R_3)\rho(R_2,R_3)\right] \\ & + 1.0\,\sigma(R_3)\times\left[1.0\,\sigma(R_1)\rho(R_3,R_1) + 1.0\,\sigma(R_2)\rho(R_3,R_2) + 1.0\,\sigma(R_3)\rho(R_3,R_3)\right] \\ & + 1.0^2 \times \sigma^2(S)\Big\}^{\frac{1}{2}} \\ = \Big\{ & 0.02633\left[0.2633 + 0.03686 \times 0.5 + 0.05266\ 0.5\right] \\ & + 0.03686\left[0.02633 \times 0.5 + 0.03686 + 0.05266 \times 0.5\right] \\ & + 0.05266\left[0.02633 \times 0.5 + 0.03686 \times 0.5 + 0.05266\right] + 0.2^2\Big\}^{\frac{1}{2}} \\ = \ & 0.2216 \end{aligned} \qquad (6.1.5)$$

$$\text{and } \rho(Z_{3,1}, Z_{3,3}) = \{1.0\,\sigma(R_2)\,1.0\,\sigma(R_1)\,\rho(R_2, R_1) + 1.0\,\sigma(R_2)\,\rho(R_2, R_2) + 1.0\,\sigma(R_3)\,\rho(R_2, R_3) + 0.3182 \times 1.0\,\sigma^2(S)\} / \{\sigma(Z_{3,1})\,\sigma(Z_{3,3})\}$$

$$= 0.03686[0.02633 \times 0.5 + 0.03686 + 0.05266 \times 0.5] + 0.3182 \times 0.2^2\} / \{0.0735 \times 0.2216\}$$

$$= 0.954 \qquad (6.1.6)$$

It will be noted that the correlation coefficients are near unity because the principal component of the uncertainty in each excess resistance is the load, which is common to each failure criterion.

6.1.5 Probability of Local Events

The probability of the events $a_{i,j}$ being realised, singly or in pairs, can be obtained by substituting these values into the cumulative uni-variate and bi-variate normal probability distributions respectively (table 6.1.6). For example

$$p(a_{3,1}) = \Phi(-E(Z_{3,1})/\sigma(Z_{3,1})) = \Phi(-\beta_{3,1})$$

$$= \Phi(-0.4190/0.07354) = \beta(-5.698)$$

$$= 0.609 \times 10^{-8} \qquad (6.1.7)$$

and

$$p(a_{3,1} \cap a_{3,2}) = \Phi(-5.698, -5.860; 0.940)$$

$$= 0.1055 \times 10^{-8} \qquad (6.1.8)$$

It will be noted that all the values of $\beta_{i,j}$ are the same order of magnitude, varying from 5.698 to 5.942. This lack of divergence occurs because all members are assumed to be perfectly ductile after failure and each excess resistance can be regarded as a factored sum of three excess resistances. These "component" excess resistances are the initial excess resistances at the points B_1, C_2 and D_3, which results in identical values of $\beta_{i,j}$ because their resistances have been chosen to be proportional to the initial elastic load-effect applied to them. Hence any range in the values of $\beta_{i,j}$ can only occur from the lack of perfect mutual correlation between these initial excess resistances. The $\beta_{i,j}$ values for failure paths including modes of local failure other than at B_1, C_2 and D_3 will, therefore, be more divergent.

6.1.6 Probability of Failure Paths

The bounds on the probability of the third failure path occuring, $p(A_3)$, can be obtained by substituting the values from table 6.1.6 into eqn. (5.3.9). All values of $\Theta_{i,k,j,l}$ are 1.0 since there are no negative correlation coefficients, so that

any terms including $(1.0 - \Theta_{i,k,j,l})$ can be ignored. Therefore, the lower bound is

$$p(A_3) \geqslant \cancel{p(a_{3,1})} \times \frac{p(a_{3,1} \cap a_{3,2})}{\cancel{p(a_{3,1})}} \times \text{Max}\left[\frac{p(a_{3,1} \cap a_{3,3})}{p(a_{3,1})}, \frac{p(a_{3,2} \cap a_{3,3})}{p(a_{3,2})}\right]$$

$$= 1.055 \times 10^{-9} \times \text{Max}\left[\frac{0.903 \times 10^{-9}}{6.079 \times 10^{-9}}, \frac{1.233 \times 10^{-9}}{2.322 \times 10^{-9}}\right]$$

$$= 0.560 \times 10^{-9} \qquad (6.1.9)$$

and the upper bound is

$$p(A_3) \leqslant \text{Min}\left[p(a_{3,1} \cap a_{3,2}), p(a_{3,1} \cap a_{3,3}), p(a_{3,2} \cap a_{3,3})\right]$$

$$= \text{Min}\left[1.055 \times 10^{-9}, 0.903 \times 10^{-9}, 1.233 \times 10^{-9}\right]$$

$$= 0.903 \times 10^{-9} \qquad (6.1.10)$$

The analysis was carried out considering the modes of local failure in the order in which they would occur in the failure path. If this order is disregarded when manipulating the mathematical probabilities then a closer lower bound may be found. In this case, considering all possible orderings within eqn. (5.3.6) produces

$$p(A_3) \geqslant \text{Max}\ (0.560 \times 10^{-9},\ 0.903.10^{-9} \times \text{Max}\left[\frac{1.055 \times 10^{-9}}{6.079 \times 10^{-9}}, \frac{1.233 \times 10^{-9}}{1.413 \times 10^{-9}}\right]$$

$$1.233 \times 10^{-9} \times \text{Max}\left[\frac{0.903 \times 10^{-9}}{1.413 \times 10^{-9}}, \frac{1.055 \times 10^{-9}}{2,322 \times 10^{-9}}\right])$$

$$= 0.788 \times 10^{-9} \qquad (6.1.11)$$

This re-ordering of the events $a_{i,j}$ was not carried out in the actual program because of the escalation in the computation involved, particularly for the lower bound on the joint probability of two failure paths. For information, the computed bounds on the probabilities of the first two failure paths are

$$0.812 \times 10^{-9} \leqslant p(A_1) \leqslant 1.075 \times 10^{-9} \qquad (6.1.12)$$

and

$$0.572 \times 10^{-9} \leqslant p(A_2) \leqslant 1.104 \times 10^{-9} \qquad (6.1.13)$$

6.1.7 Probability of Two Failure Paths

The bounds on the probability both the first and third failure paths being able to occur, $p(A_1 \cap A_3)$, are found using eqn. (5.3.11). The lower bound, using only the ordering of events that was used for the complete analysis, is

$$p(A_1 \cap A_3) \geqslant \cancel{p(a_{1,1})} \times \frac{p(a_{1,1} \cap a_{1,2})}{\cancel{p(a_{1,1})}} \times \text{Max}\left[\frac{p(a_{1,1} \cap a_{1,3})}{p(a_{1,1})}, \frac{p(a_{1,2} \cap a_{1,3})}{p(a_{1,2})}\right]$$

$$\times \text{Max}\left[\frac{p(a_{1,1}\cap a_{3,1})}{p(a_{1,1})}, \frac{p(a_{1,2}\cap a_{3,1})}{p(a_{1,2})}, \frac{p(a_{1,3}\cap a_{3,1})}{p(a_{1,3})}\right]$$

$$\times \text{Max}\left[\frac{p(a_{1,1}\cap a_{3,2})}{p(a_{1,1})}, \frac{p(a_{1,2}\cap a_{3,2})}{p(a_{1,2})}, \frac{p(a_{1,3}\cap a_{3,2})}{p(a_{1,3})}, \frac{p(a_{3,1}\cap a_{3,2})}{p(a_{3,1})}\right]$$

$$\times \text{Max}\left[\frac{p(a_{1,1}\cap a_{2,3})}{p(a_{1,1})}, \frac{p(a_{1,2}\cap a_{2,3})}{p(a_{1,2})}, \frac{p(a_{1,3}\cap a_{3,3})}{p(a_{1,3})}, \frac{p(a_{2,1}\cap a_{3,3})}{p(a_{3,1})}, \frac{p(a_{3,2}\cap a_{3,3})}{p(a_{3,2})}\right]$$

$$= 1.519 \times 10^{-9} \times \text{Max}\left[\frac{1.117 \times 10^{-9}}{6.079 \times 10^{-9}}, \frac{1.075 \times 10^{-9}}{2.009 \times 10^{-9}}\right] \times \text{Max}\left[\frac{0.797 \times 10^{-9}}{6.079 \times 10^{-9}}, \frac{0.598 \times 10^{-9}}{2.009 \times 10^{-9}}, \frac{0.903 \times 10^{-9}}{1,413 \times 10^{-9}}\right] \times \text{Max}\left[\frac{2.045 \times 10^{-9}}{6.079 \times 10^{-9}}, \frac{1.133 \times 10^{-9}}{2.009 \times 10^{-9}}, \frac{1.233 \times 10^{-9}}{1.413 \times 10^{-9}}, \frac{1.055 \times 10^{-9}}{6.079 \times 10^{-9}}\right] \times \text{Max}\left[\frac{1.117 \times 10^{-9}}{6.079 \times 10^{-9}}, \frac{1.075 \times 10^{-9}}{2.009 \times 10^{-9}}, \frac{1.413 \times 10^{-9}}{1.413 \times 10^{-9}}, \frac{0.903 \times 10^{-9}}{6.079 \times 10^{-9}}, \frac{1.233 \times 10^{-9}}{2.322 \times 10^{-9}}\right]$$

$$= 0.453 \times 10^{-9} \qquad (6.1.14)$$

and the upper bound is

$$p(A_1 \cap A_2) \leqslant \text{Min}\left[p(a_{1,1}\cap a_{1,2}), p(a_{1,1}\cap a_{1,3}), p(a_{1,1}\cap a_{3,1}), p(a_{1,1}\cap a_{3,2}), p(a_{1,1}\cap a_{3,3}), p(a_{1,2}\cap a_{1,3}), p(a_{1,2}\cap a_{3,1}), p(a_{1,2}\cap a_{3,2}), p(a_{3,2}\cap a_{3,3}), p(a_{1,3}\cap a_{3,1}), p(a_{2,3}\cap a_{3,2}), p(a_{1,3}\cap a_{3,3}), p(a_{3,1}\cap a_{3,2}), p(a_{3,1}\cap a_{3,3}), p(a_{3,2}\cap a_{3,3})\right]$$

$$= \text{Min}\left[1.519 \times 10^{-9}, 1.117 \times 10^{-9}, 0.797 \times 10^{-9}, 2.045 \times 10^{-9}, 1.177 \times 10^{-9}, 1.075 \times 10^{-9}, 0.598 \times 10^{-9}, 1.133 \times 10^{-9}, 1.075 \times 10^{-9}, 0.903 \times 10^{-9}, 1.233 \times 10^{-9}, 1.413 \times 10^{-9}, 1.055 \times 10^{-9}, 0.903 \times 10^{-9}, 1.233 \times 10^{-9}\right]$$

$$= 0.797 \times 10^{-9} \qquad (6.1.15)$$

Similarly

$$0.635 \times 10^{-9} \leqslant p(A_1 \cap A_2) \leqslant 0.776 \times 10^{-9} \qquad (6.1.16)$$

and

$$0.421 \times 10^{-9} \leqslant p(A_2 \cap A_3) \leqslant 0.797 \times 10^{-9} \qquad (6.1.17)$$

6.1.8 Probability of System Failure

The amount by which the bounds on the probability of system failure must be increased so as to include the probability of this third failure path, $p(A_3 \cap A_1^c \cap A_2^c)$, is obtained using the appropriate terms from eqn. (5.4.7). Again all the values of $\Theta_{i,k,j,l}$ are unity, so that

$$\text{Max}\left[p_l(A_3) - p_u(A_1 \cap A_3) - p_l(A_2 \cap A_3),\ p_u(A_3)\left\{1 - \text{Min}\left[1,\ \frac{p_u(A_1 \cap A_3)}{p_l(A_3)}\right]\right\} \times \right.$$

$$\left.\left\{1 - \text{Min}\left[1,\ \frac{p_u(A_2 \cap A_3)}{p_l(A_3)}\right]\right\}\right] \leqslant p(A_3 \cap A_1^c \cap A_2^c) \leqslant$$

$$p_u(A_3) - \text{Max}\left[p_l(A_1 \cap A_3),\ p_l(A_2 \cap A_3)\right] \qquad (6.1.18)$$

where $p_u(A)$ and $P_l(A)$ indicate the upper and lower bounds, respectively, on the probability of A occurring. Therefore, the lower bound is

$$p(A_3 \cap A_1^c \cap A_2^c) \geqslant \text{Max}\left[0.560 \times 10^{-9} - 0.797 \times 10^{-9} - 0.797 \times 10^{-9},\right.$$

$$\left.0.560 \times 10^{-9} \times \left\{1 - \text{Min}\left[1,\ \frac{0.797 \times 10^{-9}}{0.560 \times 10^{-9}}\right]\right\} \times \left\{1 - \text{Min}\left[1, \frac{0.797 \times 10^{-9}}{0.560 \times 10^{-9}}\right]\right\}\right]$$

$$= 0.0 \qquad (6.1.19)$$

and the upper bound is

$$p(A_3 \cap A_1^c \cap A_2^c) \leqslant 0.903 \times 10^{-9} - \text{Max}\left[0.453 \times 10^{-9},\ 0.421 \times 10^{-9}\right]$$

$$= 0.450 \times 10^{-9} \qquad (6.1.20)$$

Note that, if the closer lower bound on $p(A_3)$ from eqn. (6.1.11) is used, the lower bound on $p(A_3 \cap A_1^c \cap A_2^c)$ is still zero for this case. A closer lower bound on $p(A_1 \cap A_3)$ or $p(A_2 \cap A_3)$, obtained by considering the modes of local failure in a different order, may be able to produce a closer upper bound on $p(A_3 \cap A_1^c \cap A_2^c)$.

Similarly

$$0.0 \leqslant p(A_2 \cap A_1^c) \leqslant 0.469 \times 10^{-9} \qquad (6.1.21)$$

Therefore, summing the first three terms in eqn. (5.4.7), $p(A_1)$, $p(A_2 \cap A_1^c)$ and $p(A_3 \cap A_1^c \cap A_2^c)$, gives the contribution for the first three failure paths to the bounds on the probability of system failure as

$$0.812 \times 10^{-9} + 0.0 + 0.0 \leqslant p(A_1 \cap A_2 \cap A_3) \leqslant 1.075 \times 10^{-9} + 0.469 \times 10^{-9}$$

$$+ 0.450 \times 10^{-9}$$

$$\therefore \quad 0.812 \times 10^{-9} \leqslant p(A_1 \cap A_2 \cap A_3) \leqslant 1.994 \times 10^{-9} \qquad (6.1.22)$$

It should be noted that the lower bound tends to increase only for failure paths containing the failure of elements which have not been included in previous failure paths.

6.1.9 Width of Bounds for Different Structures

The bounds obtained by this method (tables 6.1.1 and 6.1.2) display some interesting features. The structures have identical upper and lower bounds on their probability of system failure for the case of fully correlated resistances. This exact result is obtained because, under these conditions, each failure criterion reduces to a linear equation in two variables R and S. Therefore, each of then can be written as

$$Z_{i,j} = k_{i,j} * R - S \leqslant 0.0 \qquad (6.1.23)$$

Hence each failure path will occur when the ratio S/R is greater than or equal to the maximum value of $k_{i,j}$ from the three modes of local failure, and system failure will occur when

$$S/R \geqslant \underset{i=1}{\overset{24}{\text{Max}}} \left[\underset{j=1}{\overset{3}{\text{Min}}} (k_{i,j}) \right] = k_{i^*,j^*} \qquad (6.1.24)$$

Hence the probability of system failure will be exactly $\Phi(-\beta_{i^*,j^*})$.
As an example, P_{11} will occur when

$$S/R \geqslant \text{Min}\left(\frac{1.0}{0.3182} = 3.143, \frac{0.6667 + 1.0}{0.6667} = 2.5, \frac{1.0 + 1.0 + 1.0}{1.0} = 3.0 \right)$$

$$= 3.143 \qquad (6.1.25)$$

for the plastically designed structure, and when

$$S/R \geqslant \text{Min}\left[\frac{0.7372}{0.3182} = 2.317, \frac{0.6667 \times 0.7372 + 1.0533}{0.6667} = 2.317, \right.$$

$$\left. \frac{0.5266 + 0.7372 + 1.0533}{1.0} = 2.317 \right] = 2.317 \qquad (6.1.26)$$

for the elastically designed structure, where $R = \frac{R_1}{0.5266} = \frac{R_2}{0.7372} = \frac{R_3}{1.0532}$.

A second interesting feature is that the probability of system failure for both the plastically and elastically designed structures with perfectly correlated resistances are identical. This occurs because the load does the same amount of virtual work for system failure.

For correlation coefficients less than unity, the increase in the width of the bounds on the probability of system failure for the elastically, as compared with

the plastically, designed structures results from the failure set having a larger number of failure paths with probabilities of occurance that are of the same order of magnitude. Therefore the actual failure probability, if it were to be calculated, is likely to be greater for the elastically designed structure. However, if system failure were to be redefined as the formation of any plastic hinge, the probability of system failure would then be greater for the plastically designed structure.

6.1.10 Width of Bounds on a Failure Paths

For this example, the bounds on the probability of any particular failure path occurring can be simplified from eqn. (5.3.9) by putting $\Theta_{i,j,i,k} = 1.0$ (sub-section 6.1.6) and $n_i = 3$. Therefore

$$p(a_{i,1}) \times \frac{p(a_{i,1} \cap a_{i,2})}{p(a_{i,1})} \times \text{Max}\left[\frac{p(a_{i,1} \cap a_{i,3})}{p(a_{i,1})} , \frac{p(a_{i,2} \cap a_{i,3})}{p(a_{i,2})} \right]$$

$$= p(a_{i,1} \cap a_{i,2}) \times \text{Max}\left[\frac{p(a_{i,1} \cap a_{i,3})}{p(a_{i,1})} , \frac{p(a_{i,2} \cap a_{i,3})}{p(a_{i,2})} \right] \leqslant p(A_i)$$

$$\leqslant \text{Min}\left[p(a_{i,1} \cap a_{i,2}),\ p(a_{i,1} \cap a_{i,3}),\ p(a_{i,2} \cap a_{i,3}) \right] \qquad (6.1.27)$$

When $\rho(Z_{i,j}, Z_{i,k}) = 1.0$, the bounds are identical since $p(a_{i,j} \cap a_{i,k})$ $= \text{Min}\left[p(a_{i,j}),\ p(a_{i,k}) \right]$ (for $j = 1$ or 2 and $k = 2$ or 3, but $j \neq k$), so that

$$\text{Min}\left[p(a_{i,1}),\ p(a_{i,2}) \right] \times \text{Max}\left\{ \text{Min}\left[1.0, \frac{p(a_{i,3})}{p(a_{i,1})} \right] \right\}, \text{Max}\left\{ \text{Min}\left[1.0, \frac{p(a_{i,3})}{p(a_{i,2})} \right] \right\}$$

$$= \text{Min}\left\{ p(a_{i,1}),\ p(a_{i,2}),\ p(a_{i,3}) \right\} \leqslant p(A_i) \leqslant \text{Min}\left\{ p(a_{i,1}),\ p(a_{i,2})\ p(a_{i,3}) \right\} \qquad (6.1.28)$$

When $\rho(Z_{i,j}, Z_{i,k}) = 0.0$, the ratio of the bounds in eqn. (6.1.3) is $\text{Max}\left\{ p(a_{i,j}),\ p(a_{i,2}),\ p(a_{i,3}) \right\}$ since $p(a_{i,j} \cap a_{i,k}) = p(a_{i,j})\ p(a_{i,k})$, so that

$$p(a_{i,1})\ p(a_{i,2})\ p(a_{i,3}) \leqslant p(A_i) \leqslant \text{Min}\left\{ p(a_{i,1})\ p(a_{i,2}),\ p(a_{i,2})\ p(a_{i,3}),\ p(a_{i1})\ p(a_{i,3}) \right\} \qquad (6.1.29)$$

In general, therefore, the bounds on the probability of the occurance of a failure path, with three modes of local failure having excess resistances which are not mutually negatively correlated, will be narrow. The only exceptions will be if both:

i. the excess resistances are not strongly positively correlated; and

ii. the probability of each of the local failure events $a_{i,j}$ in the failure path is considerably less than unity.

As discussed earlier (Sub-Section 6.1.5), the excess resistances are strongly positively correlated in this example because:

i. there are only four basic variables considered (R_1, R_2, R_3 and S); and

ii. the members are assumed to behave in a ductile manner, so that the excess resistance of the second local event will include a proportion of the first, and the third will include a proportion of the first and the second.

6.1.11 Reduced Failure Set

There are some failure paths in the failure set which cannot physically occur because at least one of the local failure events not included in that failure path is certain to occur before one of those that is included (eg any failure path which has a hinge forming at B_2 as the first local failure since the load-effect at C_2 is greater and the resistances are identical). These failure paths are shown with a dotted line in the failure tree (fig. 6.1.1) and marked with an asterisk in tables 6.1.1 and 6.1.2. When the analysis was re-worked, without these failure paths being included, the bounds on the probability of system failure were close to the previous ones for the failure set which did include those failure paths. Therefore, the ommission of such failure paths from the failure set for the analysis does not invalidate the analysis, at least for this example.

6.2 COMPARISON OF PROPOSED METHOD WITH REVIEWED METHODS

6.2.1 Ditlevsen's Method

The bounds on the system failure probability for the example in section 6.1 can be found using Ditlevsen's method (section 4.4) since the failure set has failure paths consisting of the failures of ductile elements. The four variables (the three resistances, R_1, R_2 and R_3, and the load, S) can be used to form the equations describing the four modes of system failure. These equations are

$$B_1, C_2, D_3 : Z_1 = R_1 + R_2 + R_3 - S \qquad (6.2.1)$$

$$B_1, C_3, D_3 : Z_2 = R_1 + 2R_3 - S \qquad (6.2.2)$$

$$B_2, C_2, D_3 : Z_3 = 2R_2 + R_3 - S \qquad (6.2.3)$$

$$B_2, C_3, D_3 : Z_4 = R_2 + 2R_3 - S \qquad (6.2.3)$$

An exhaustive examination of the bounds on the probability of system failure was carried out since there are only $4! = 24$ possible orders for the four system failure modes, and the maximum lower and minimum upper bounds were found (Table 6.2.1). The time taken to do the calculations was considerably less

than the equivalent time for the proposed method.

The bounds on the probability of system failure are closer than those obtained with the proposed method for the elastically designed structure (Ditlevsen's bounds being identical to the quoted accuracy in each case), while for the plastically designed structure the bounds are wider. In all cases the bounds obtained by the different methods overlapped so that they do not contradict each other. Therefore, the closest bounds on the probability of system failure can be obtained by using both sets of results (Table 6.2.2).

Therefore this example shows that, for the systems that can be analysed by both methods:

i. Ditlevsen's method, including re-analysing the system for each order of the modes of system failure, can require less computation than the proposed method, with a single order of the failure paths; and

ii. which of the methods will produce the closest bounds on the probability of system failure will depend on the idealised structure.

This latter point occurs because:

i. Ditlevsen's method uses bounds on the joint probability of two modes of failure while the proposed method uses an appropriate method to calculate the cumulative probability of the bi-variate normal distribution; but

ii. the failure criterion of each of the final modes of local failure in a failure path for the proposed method are identical to the failure criterion of one of the modes of system failure for Ditlevsen's method, but, being conditional on two other modes of local failure occurring for system failure to occur in any failure path, the same mode of local failure has to be repeated until the union of those failure paths includes the entire variable space where that mode of failure will occur; and

iii. Ditlevsen's method used the closest bound obtained from each order of the modes of system failure while the proposed method used a single order both of the modes of local failure in a failure path and of the failure paths.

If the cumulative probability of the bi-variate normal distribution was used with Ditlevsen's method it would produce the closest bounds on any ductile system as can be found from analysis by these two methods.

6.2.2 Moses' Unzipping Technique

Consider a "safe" slab supported by four reinforced concrete beams (fig. 6.2.1) as an example of Moses' unzipping technique (section 4.3). The beams are the same as the one described as beam 1 in Appendix C, but with different correlation coefficients of the basic variables (Table 6.2.3) and using a more complex live load distribution model (Appendix H) in order to ensure that simple statics apply.

The modes of local failure were assumed to be bending moment at mid-span for which the members were treated as perfectly ductile, and shear force failures at each support, for which the members were treated as perfectly brittle. The parameters for the distributions of the beams' resistances to these failure modes were calculated using the methods put forward in section 5.5 (Table 6.2.4).

The values obtained for the probability of failure by each failure path are given in table 6.2.5, together with the bounds on that probability found by the proposed method. The values for the probability of system failure compare well with these bounds since the values lie between them, although it is closer to the lower bound (as would be expected since this method includes no series combinations). Again the calculation required is considerably less than for the proposed method.

6.2.3 Marshall and Bea's Methods

The structure in the above example was also analysed by Marshall and Bea's methods (section 4.3). The results from these analyses demonstrate that their brittle redundant model can give a lower probability of failure than their ductile redundant model. The failure network (fig. 6.2.2) shows two assemblages in parallel, each assemblage consisting of six local failure modes in series. The results are given in table 6.2.6, which show that the ductile redundant model did not give a lower value for the probability of failure with a fixed load than the brittle redundant model until the load was nine standard deviations from the mean.

The values of $p(F|S = s)$ were integrated from $E(S) - 6 \times \sigma(S)$ to $E(S) + 9 \times \sigma(S)$ in $0.05\,\sigma(S)$ increments, with the load, S, assumed to have an extreme type I maxima distribution. Both the values of the probability of system failure, p(F), are larger than the bounds on p(F) that were calculated previously (0.521×10^{-6} and 1.730×10^{-6} compared with $0.037 \times 10^{-6} \leqslant p(F) \leqslant 0.229 \times 10^{-6}$). Therefore, this method is not recommended for calculating the probability of system failure, except, perhaps, as a hand method to update the resistance of a structure against a hindcast load.

6.3 EXPECTED COST OF FAILURE

The structural example of a slab supported on beams (section 6.2) was also analysed by the proposed method for the system of any local failure resulting in system failure. The results imply that the upper bound on the probability of overall failure is greater than the upper bound on the probability of any local failure. This relationship obviously is not true. Therefore, the upper bound on the probability of overall failure can be reduced pragmatically to the upper bound on the probability of any local failure. In general, the upper bound on the probability of system failure can be reduced to any, smaller upper bound on the probability of system failure for a system which must fail before the initial system can fail. Similarly, the lower bound on the probability of system failure can be increased to a larger lower bound on the probability of system failure for a system which cannot fail until after the initial system has failed.

The rationalised set of bounds, on the probability of the failure of various systems for a structure, can be used to give bounds on the expected cost of failure. A simple model for the cost of failure for the slab example is:

i. a cost of 10^9 units for any local failure, and
ii. a cost of 10^{10} units for overall failure.

The bounds on the expected cost for this model are:

$10^9 \times (1887.01-364.65) \times 10^{-10} + 10^{10} \times 364.65 \times 10^{-10} = 517$ units (lower bound);

and

$10^9 \times (2265.33-2265.33) \times 10^{-10} + 10^{10} \times 2265.33 \times 10^{-10} = 2265$ Units (upper bound).

Hence the expected cost can be given as 1391 ± 874 units.

This type of calculation can be carried out even when a complex set of costings (the cost being altered for each different degree or type of failure) are required. The use of a set of different criteria to describe system failure will allow the bounds on the expected cost of failure to be found. The bounds on the expected cost of failure can be used to check the overall cost effectiveness of adjusting any of the safety factors in a Code of Practice.

6.4 COMPARISON WITH MONTE CARLO SIMULATION

6.4.1 Monte Carlo Simulation

The results obtained by any new method of analysis can be verified by comparison with those calculated by a different technique that is already established. Therefore, a Level 3 Monte Carlo Simulation (sub-section 2.3) was carried out with sufficient trials to obtain confident results to compare with the results

obtained by the proposed method. For this Monte Carlo simulation, the value of each mutually uncorrelated basic variable for each trial was produced by generating two independant random numbers, r_1 and r_2, taken from a rectangularly distributed population between 0.0 and 1.0. These numbers were combined to obtain a standardised normal random number, u_i, using the relationship

$$u_i = \text{Sin}\ (2\pi\ r_1)\ \sqrt{-2\ \log_e\ (r_2)} \qquad (6.4.1)$$

The value of u_i was transformed into the equivalent value, x_i, for the distribution describing the basic variable so that x_i had the same cumulative probability as u_i (eqn. (2.4.8)). The same value of x_i was used for each variable within each set of mutually perfectly positively correlated basic variables in a trial.

The confidence in the results of a Monte Carlo simulation can be improved by fitting curves, such as the Pearson distributions [18], to the histogram obtained by plotting the value of each trial. For system reliability analysis this value would have to be the excess resistance of the structure. The most logical idealisation for the whole structure's excess resistance is the minimum value of the excess resistances of the failure paths, each of which is, similarly, the maximum value of the excess resistance of the local failure modes in the failure path. Hence the required value would be from a "minima of maxima" type of distribution and the probability density function of the theoretical population could easily have an unusual tail, which would render the curve fitting highly inaccurate. Therefore the slab example was re-worked with the live load increased by a factor of two and a half in order to increase the probability of failure, and so reduce the number of trials necessary for reasonable accuracy.

6.4.2 Slab Example

Each variable had a different value at the design point of the Advanced Level 2 analysis for the same element when the increased load was used compared to the original load. Hence the standard deviations and correlation coefficients for the elements' resistances in this example (Table 6.4.1) are different from those used in section 6.2. This is principally because of the inaccuracies from:

i. the assumption that each element's resistance has a normal distribution; and

ii. the treatment of the live load as having an extreme type I maxima distribution while finding the design points and subsequently a normal distribution.

Both of the moments for both modes of failure are larger for this example than the example in section 6.2.

A total of 30,000 trials were carried out in six runs (Table 6.4.2), of which 219 resulted in overall failure and a further 11 resulted in at least one local failure, although not overall failure. Most of the trials which resulted in system failure could have failed by more than one failure path (the mean number of failure paths by which a trial which failed could have failed was 16.5). This fact, together with the relatively few partial failures, demonstrates that the failure paths in this example are strongly positively correlated.

The bounds on the probabilities of failure give reasonable agreement with the values obtained by the Monte Carlo simulation (Table 6.4.3). Some of the failure paths (types 2, 5 and 6 which are particularly strongly mutually correlated), however, do not give this agreement. The lack of competability could be due to the non-normal distributions of the elements' resistances and of the load being combined to produce the estimates of the central moments of the excess resistances which are incompatible with the assumption of a normal distribution for at least one local failure mode in each of these failure paths.

6.5 EXAMPLE WITH MULTIPLE LOADS

6.5.1 Portal Frame

A second single bay portal frame, with several loads applied to it, was analysed in order to investigate the effect on the results, obtained by the proposed method of system reliability analysis, of some of the excess resistances being mutually negatively correlated (Negative correlation being more liable to occur in practice due to load effects acting in opposite senses on the members than from mutually negatively correlated resistances). The frame was assumed to have pinned feet and rigid connections elsewhere. The dimensions of the frame and the applied loads are shown in fig. 6.5.1, and the bending moment distribution in Appendix J.

The structure was idealised as having the bending moment mode of failure as the only mode of local failure. The parameters of the distribution for the elements' resistances to this mode, together with those for the loads, were chosen arbitrarily to give initial failure probabilities of approximately 10^{-6} for each mode of local failure. This was done because the procedure for obtaining such parameters from the basic variables is demonstrated in the previous examples.

The mode of system failure was the frame becoming a mechanism which occurs when any pair of non-coincident plastic hinges have formed. The local failure modes were limited to occuring at three cross-sections on the horizontal member and two

cross-sections on both of the vertical members. At each of these cross-sections, failure can occur due to either a positive or negative applied bending moment being greater than the resistance so that the mode of local failure at each cross-section was described by two coincident excess resistances. The two excess resistances for the same cross-section are

$$Z_{i,1} = R_{i,1} - \sum_{k=1}^{n_s} [d_{i,1,k} S_k] \qquad (6.5.1)$$

and

$$Z_{j,1} = R_{j,1} - \sum_{k=1}^{n_s} [d_{j,1,k} S_k] \qquad (6.5.2)$$

where

$$d_{i,1,k} = - d_{j,1,k} \qquad (6.5.3)$$

and

$$R_{i,1} \leq R_{j,1} \qquad (6.5.4)$$

and any redistribution of load effect to these excess loads will be of opposite sign.

Care was taken that no failure paths included modes of local failure which had negative values of $d_{i,j,k}$ for all k since the use of two excess resistances at each cross-section involved considering both positive and negative signs with each constant. A local failure mode with all negative values of $d_{i,j,k}$ will still have a discrete theoretical probability of occurring because the loads and resistances are treated by the proposed method as having normal distributions, but such a failure would have to result from some variables having values in the negative tails of their distributions.

Modes of local failure were also ignored if the bending moment applied at the cross-section was certain to be less than that applied at another cross-section of the same member. This situation occurred when $d_{i,j,k} < d_{l,j,k}$ for all k, where the i^{th} and l^{th} failure paths have the same first $(j-1)$ local failure modes and their j^{th} local failure modes are at different cross-sections of the same member. Only nine of the fourteen local failure modes were used because of the omission of such failure paths, and of the nine there were three which were never the first failure modes in any failure path.

6.5.2 Width of Bounds

The width, between the lower and upper bounds on the probability of system failure for this example (Tables 6.5.1 and 6.5.2), compare well with those of the previous examples. The probability of failure appears to increase with increased correlation

coefficients of both the resistances and the loads, although the individual bounds do not always follow this pattern due to the varying width between the bounds. The most pronounced example of this occurs when the resistance correlation coefficients were increased from 0.95 to unity and the upper bound increased from 6.144×10^{-6} to 3.182×10^{-6} (Table 6.5.1).

This change in the upper bound could be due to the decreased uncertainty, which would imply that all the structures in table 6.5.1 should have an additional upper bound of 3.182×10^{-6}, and similarly an additional lower bound of 3.059×10^{-6}. These overall bounds cannot be used, however, since increased correlation coefficients will result in a lower probability of system failure, for any system consisting of failure paths which have the same probabilities of occurring, as well as giving a higher probability of a failure path occurring, for any failure path which has the same probabilities of its local failures. Therefore, it cannot be assumed that increasing correlation coefficients will necessarily produce a less conservative system, and hence another upper bound, nor decreasing correlation coefficients will produce another lower bound. This is demonstrated in table 6.5.1 by two lower bounds being below these 'overall' bounds of $(3.056 - 3.182) \times 10^{-6}$. If such bounds were true, they would imply that the value of the correlation coefficients have a negigible effect on the probability of system failure for this example.

6.5.3 Time-Variant Loads

The example was repeated with the three loads being treated as time-varying loads, assuming 2, 6 and 24 time intervals in the reference period. This was to demonstrate the effect of multiple load combinations on the proposed method and to examine the change in the system reliability with time-varying loads having relatively long time intervals (if the design life is 50 years, the shortest time interval in this example would be just over 2 years). The parameters for the extreme distributions of the loading variables were calculated assuming that the instantaneous loads have normal distributions with those parameters that had been used earlier in this example.

The number of failure paths that were considered increased from 25 to 45 rather than $3 \times 25 = 75$ because the criterion for ignoring local failure modes was that the bending moment at the cross-section applied by the principal load only, rather than all the loads, was either negative or certain to be less than the bending moment at another cross-section on the same member.

The probability of system failure increased:

i. with the use of the time-varying loads when compared to the instantaneous failure system using the parent distributions of the loads (the mean probability increased by 37% from table 6.5.2 to table 6.5.3);

and again

ii. with the use of the maximum design life loads as non-time-varying when compared to the failure system with time-varying loads (the mean probability increased by 16% from table 6.5.3 to table 6.5.4).

The second increase was not as large as might have been expected since an increase in one of the loads will improve the reliability of that element in some of the local failure modes, and so the extreme minima intensity in the appropriate time interval would be used for those systems which include time varying loads. The increase in a load will reduce the probability of a local failure when $d_{i,j,k}$ is negative.

The width of the bounds on the probability of system failure increased with the use of time-varying loads (the means of the difference divided by the average of the two bounds are 37%, 56% and 33% for tables 6.5.2, 5.6.3 and 5.6.4 respectively). This increase, although significant, still leaves the two bounds close enough for practical purposes. The principal reason for the increase is due to the additional number of failure paths that have to be considered.

RESISTANCE CORRELATION COEFFICIENTS	0.0		0.5		1.0
FAILURE PATHS	LOWER BOUND	UPPER BOUND	LOWER BOUND	UPPER BOUND	IDENTICAL LOWER AND UPPER BOUNDS
P_1	$.153 \times 10^{-19}$	$.153 \times 10^{-19}$	$.320 \times 10^{-19}$	$.320 \times 10^{-19}$	$.322 \times 10^{-19}$
P_2^*	$.153 \times 10^{-19}$	$.153 \times 10^{-19}$	$.320 \times 10^{-19}$	$.320 \times 10^{-19}$	$.322 \times 10^{-19}$
P_3	$.321 \times 10^{-19}$	$.321 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$
P_4	$.320 \times 10^{-19}$	$.320 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$
P_5^*	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$
P_6^*	$.153 \times 10^{-19}$	$.153 \times 10^{-19}$	$.320 \times 10^{-19}$	$.320 \times 10^{-19}$	$.322 \times 10^{-19}$
P_7^*	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$
P_8^*	$.320 \times 10^{-19}$	$.320 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$	$.322 \times 10^{-19}$
P_9	$.856 \times 10^{-20}$	$.856 \times 10^{-20}$	$.621 \times 10^{-20}$	$.621 \times 10^{-20}$	$.494 \times 10^{-21}$
P_{10}	$.494 \times 10^{-21}$	$.494 \times 10^{-21}$	$.494 \times 10^{-21}$	$.494 \times 10^{-21}$	$.494 \times 10^{-21}$
P_{11}	$.522 \times 10^{-10}$	$.523 \times 10^{-10}$	$.271 \times 10^{-9}$	$.271 \times 10^{-9}$	$.529 \times 10^{-9}$
P_{12}	$.269 \times 10^{-9}$	$.270 \times 10^{-9}$	$.483 \times 10^{-9}$	$.483 \times 10^{-9}$	$.529 \times 10^{-9}$
P_{13}^*	$.809 \times 10^{-20}$	$.914 \times 10^{-20}$	$.621 \times 10^{-20}$	$.621 \times 10^{-20}$	$.325 \times 10^{-12}$
P_{14}^*	$.809 \times 10^{-20}$	$.914 \times 10^{-20}$	$.621 \times 10^{-20}$	$.621 \times 10^{-20}$	$.325 \times 10^{-12}$
P_{15}^*	$.270 \times 10^{-9}$	$.270 \times 10^{-9}$	$.483 \times 10^{-9}$	$.483 \times 10^{-9}$	$.325 \times 10^{-12}$
P_{16}^*	$.270 \times 10^{-9}$	$.270 \times 10^{-9}$	$.483 \times 10^{-9}$	$.483 \times 10^{-9}$	$.325 \times 10^{-12}$
P_{17}	$.102 \times 10^{-13}$	$.102 \times 10^{-13}$	$.641 \times 10^{-13}$	$.641 \times 10^{-13}$	$.325 \times 10^{-12}$
P_{18}	$.102 \times 10^{-13}$	$.102 \times 10^{-13}$	$.641 \times 10^{-13}$	$.641 \times 10^{-13}$	$.325 \times 10^{-12}$
P_{19}^*	$.102 \times 10^{-13}$	$.102 \times 10^{-13}$	$.641 \times 10^{-13}$	$.641 \times 10^{-13}$	$.325 \times 10^{-12}$
P_{20}^*	$.102 \times 10^{-13}$	$.102 \times 10^{-13}$	$.641 \times 10^{-13}$	$.641 \times 10^{-13}$	$.325 \times 10^{-12}$
P_{21}	$.213 \times 10^{-9}$	$.213 \times 10^{-9}$	$.133 \times 10^{-8}$	$.133 \times 10^{-8}$	$.609 \times 10^{-8}$
P_{22}	$.749 \times 10^{-9}$	$.749 \times 10^{-9}$	$.227 \times 10^{-8}$	$.227 \times 10^{-8}$	$.609 \times 10^{-8}$
P_{23}	$.749 \times 10^{-9}$	$.749 \times 10^{-9}$	$.227 \times 10^{-8}$	$.227 \times 10^{-8}$	$.609 \times 10^{-8}$
P_{24}	$.749 \times 10^{-9}$	$.749 \times 10^{-9}$	$.227 \times 10^{-8}$	$.227 \times 10^{-8}$	$.609 \times 10^{-8}$
SYSTEM	$.1612 \times 10^{-8}$	$.1863 \times 10^{-8}$	$.4085 \times 10^{-8}$	$.4939 \times 10^{-8}$	$.6086 \times 10^{-8}$
SYSTEM EXCLUDING *	$.1612 \times 10^{-8}$	$.1863 \times 10^{-8}$	$.4085 \times 10^{-8}$	$.4637 \times 10^{-8}$	$.6086 \times 10^{-8}$

TABLE 6.1.1 - FAILURE PROBABILITIES FOR PLASTICALLY DESIGNED STRUCTURES

RESISTANCE CORRELATION COEFFICIENTS	0.0		0.5		1.0
FAILURE PATHS	LOWER BOUND	UPPER BOUND	LOWER BOUND	UPPER BOUND	IDENTICAL LOWER AND UPPER BOUNDS
P_1	$.134 \times 10^{-10}$	$.106 \times 10^{-9}$	$.231 \times 10^{-9}$	$.772 \times 10^{-9}$	$.609 \times 10^{-8}$
P_2^*	$.560 \times 10^{-18}$	$.560 \times 10^{-18}$	$.289 \times 10^{-17}$	$.289 \times 10^{-17}$	$.121 \times 10^{-16}$
P_3	$.450 \times 10^{-10}$	$.450 \times 10^{-10}$	$.453 \times 10^{-9}$	$.772 \times 10^{-9}$	$.609 \times 10^{-8}$
P_4	$.835 \times 10^{-13}$	$.250 \times 10^{-12}$	$.554 \times 10^{-12}$	$.133 \times 10^{-11}$	$.459 \times 10^{-11}$
P_5^*	$.149 \times 10^{-17}$	$.149 \times 10^{-17}$	$.149 \times 10^{-17}$	$.149 \times 10^{-17}$	$.149 \times 10^{-17}$
P_6^*	$.489 \times 10^{-22}$	$.489 \times 10^{-22}$	$.263 \times 10^{-20}$	$.263 \times 10^{-20}$	$.369 \times 10^{-17}$
P_7^*	$.964 \times 10^{-18}$	$.966 \times 10^{-18}$	$.149 \times 10^{-17}$	$.149 \times 10^{-17}$	$.149 \times 10^{-17}$
P_8^*	$.398 \times 10^{-19}$	$.614 \times 10^{-19}$	$.881 \times 10^{-18}$	$.883 \times 10^{-18}$	$.149 \times 10^{-17}$
P_9	$.357 \times 10^{-11}$	$.105 \times 10^{-9}$	$.106 \times 10^{-9}$	$.721 \times 10^{-9}$	$.609 \times 10^{-8}$
P_{10}	$.271 \times 10^{-19}$	$.271 \times 10^{-19}$	$.271 \times 10^{-19}$	$.271 \times 10^{-19}$	$.271 \times 10^{-19}$
P_{11}	$.112 \times 10^{-9}$	$.139 \times 10^{-9}$	$.560 \times 10^{-9}$	$.903 \times 10^{-9}$	$.609 \times 10^{-8}$
P_{12}	$.910 \times 10^{-12}$	$.175 \times 10^{-11}$	$.676 \times 10^{-11}$	$.110 \times 10^{-10}$	$.537 \times 10^{-10}$
P_{13}^*	$.188 \times 10^{-18}$	$.188 \times 10^{-18}$	$.288 \times 10^{-18}$	$.288 \times 10^{-18}$	$.288 \times 10^{-18}$
P_{14}^*	$.277 \times 10^{-20}$	$.279 \times 10^{-20}$	$.442 \times 10^{-20}$	$.442 \times 10^{-20}$	$.288 \times 10^{-18}$
P_{15}^*	$.288 \times 10^{-18}$	$.288 \times 10^{-18}$	$.288 \times 10^{-18}$	$.288 \times 10^{-18}$	$.288 \times 10^{-18}$
P_{16}^*	$.285 \times 10^{-18}$	$.285 \times 10^{-18}$	$.288 \times 10^{-18}$	$.288 \times 10^{-18}$	$.288 \times 10^{-18}$
P_{17}	$.606 \times 10^{-10}$	$.144 \times 10^{-9}$	$.812 \times 10^{-9}$	$.108 \times 10^{-8}$	$.609 \times 10^{-8}$
P_{18}	$.151 \times 10^{-12}$	$.360 \times 10^{-12}$	$.751 \times 10^{-12}$	$.138 \times 10^{-11}$	$.459 \times 10^{-11}$
P_{19}^*	$.625 \times 10^{-16}$	$.625 \times 10^{-16}$	$.972 \times 10^{-15}$	$.972 \times 10^{-15}$	$.101 \times 10^{-13}$
P_{20}^*	$.157 \times 10^{-16}$	$.157 \times 10^{-16}$	$.518 \times 10^{-15}$	$.518 \times 10^{-15}$	$.101 \times 10^{-13}$
P_{21}	$.132 \times 10^{-9}$	$.200 \times 10^{-9}$	$.572 \times 10^{-9}$	$.110 \times 10^{-8}$	$.609 \times 10^{-8}$
P_{22}	$.111 \times 10^{-11}$	$.167 \times 10^{-11}$	$.760 \times 10^{-11}$	$.109 \times 10^{-10}$	$.537 \times 10^{-10}$
P_{23}	$.492 \times 10^{-13}$	$.492 \times 10^{-13}$	$.542 \times 10^{-13}$	$.542 \times 10^{-13}$	$.542 \times 10^{-13}$
P_{24}	$.417 \times 10^{-15}$	$.417 \times 10^{-15}$	$.398 \times 10^{-14}$	$.398 \times 10^{-14}$	$.292 \times 10^{-13}$
SYSTEM	$.0132 \times 10^{-8}$	$.0587 \times 10^{-8}$	$.1051 \times 10^{-8}$	$.3232 \times 10^{-8}$	$.6086 \times 10^{-8}$
SYSTEM EXCLUDING P_2^*	$.0132 \times 10^{-8}$	$.0587 \times 10^{-8}$	$.1051 \times 10^{-8}$	$.3232 \times 10^{-8}$	$.6086 \times 10^{-8}$

TABLE 6.1.2 - FAILURE PROBABILITIES FOR ELASTICALLY DESIGNED STRUCTURES

VARIABLE	R_1	R_2	R_3	S
$z_{1,1}$	0	0	1.0	- 0.4545
$z_{1,2}$	1.0	0	0.60	- 0.50
$z_{1,3}$	1.0	1.0	1.0	- 1.0
$z_{2,1}$	0	0	1.0	- 0.4545
$z_{2,2}$	0	1.0	0.40	- 0.50
$z_{2,3}$	1.0	1.0	1.0	- 1.0
$z_{3,1}$	0	1.0	0	- 0.3182
$z_{3,2}$	0	0.6667	1.0	- 0.6667
$z_{3,3}$	1.0	1.0	1.0	- 1.0

TABLE 6.1.3 - COEFFICIENTS IN THE FAILURE CRITERIA

X	R_1	R_2	R_3	S
E(X)	0.5266	0.7372	1.0532	1.00
$\sigma(X)$	0.02633	0.03686	0.05266	0.20
$\rho(R_1,X)$	1.0	0.5	0.5	0.0
$\rho(R_2,X)$		1.0	0.5	0.0
$\rho(R_3,X)$			1.0	0.0
$\rho(S,X)$		Symmetrical		1.0

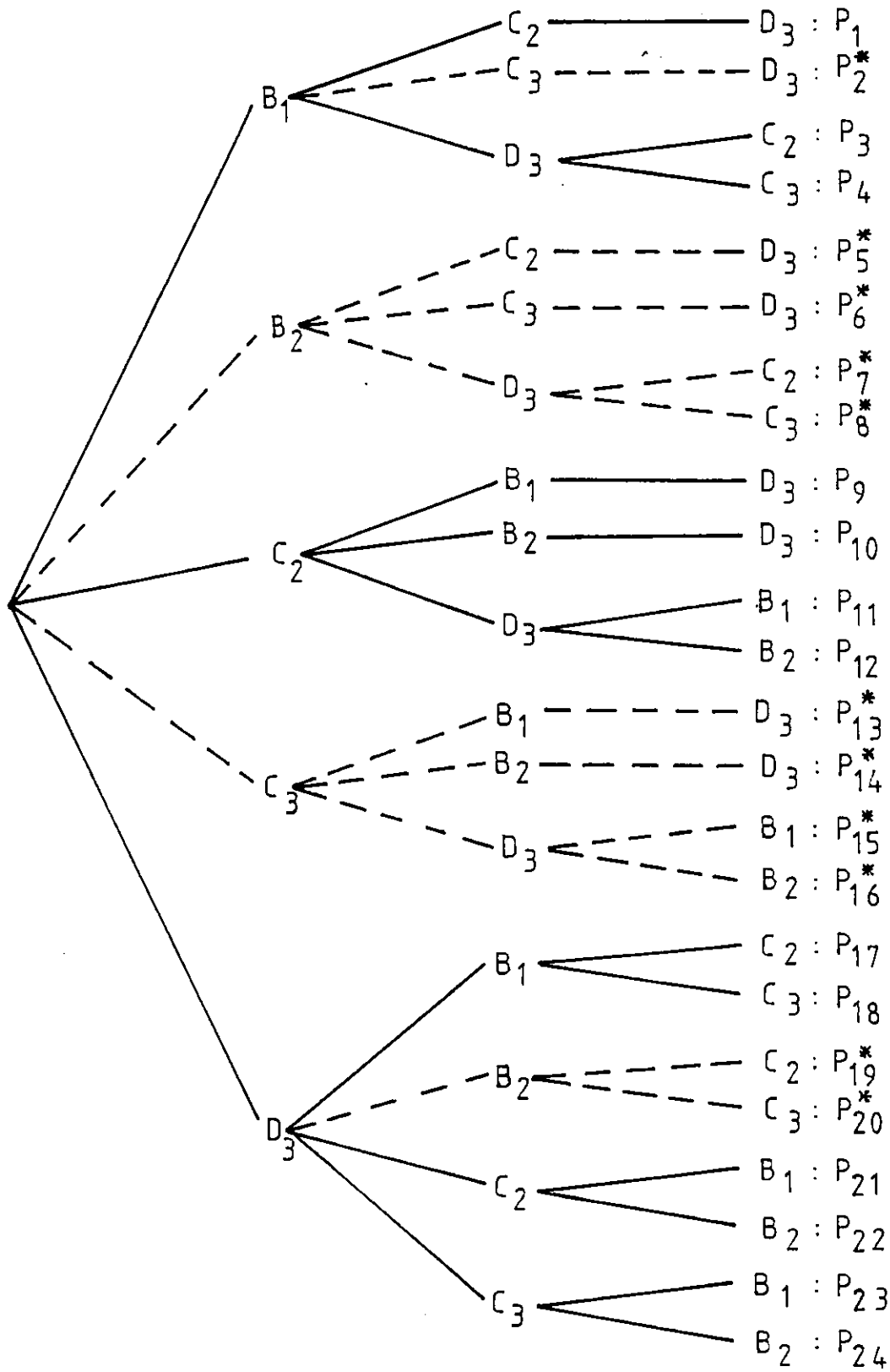
TABLE 6.1.4 - PARAMETERS OF THE DISTRIBUTIONS OF THE RESISTANCES AND LOADS

I,J	1,1	1,2	1,3	2,1	2,2	2,3	3,1	3,2	3,3
$E(z_{i,j})$	0.5986	0.6585	1.3169	0.5986	0.6585	1.3169	0.4190	0.8780	0.5986
$\sigma(z_{i,j})$	0.10506	0.11191	0.22164	0.10506	0.11215	0.22164	0.07354	0.14983	0.22164
$\beta_{i,j}$	5.698	5.884	5.942	5.698	5.872	5.942	5.698	5.860	5.942
$\rho(z_{i,j},z_{1,1})$	1.0	.974	.971	1.0	.948	.971	.874	.987	.971
$\rho(z_{i,j},z_{1,2})$		1.0	.989	.974	.957	.989	.903	.978	.989
$\rho(z_{i,j},z_{1,3})$			1.0	.971	.989	1.0	.954	.993	1.0
$\rho(z_{i,j},z_{2,1})$				1.0	.948	.971	.874	.987	.971
$\rho(z_{i,j},z_{2,2})$					1.0	.989	.983	.987	.989
$\rho(z_{i,j},z_{2,3})$						1.0	.954	.993	1.0
$\rho(z_{i,j},z_{3,1})$							1.0	.940	.954
$\rho(z_{i,j},z_{3,2})$				Symmetrical				1.0	.993
$\rho(z_{i,j},z_{3,3})$									1.0

TABLE 6.1.5 - PARAMETERS OF THE PROBABILITY DISTRIBUTIONS FOR THE EXCESS RESISTANCES OF THE FIRST THREE FAILURE PATHS

I,J	1,1	1,2	1,3	2,1	2,2	2,3	3,1	3,2	3,3
$p(a_{i,j})$	6.079	2.009	1.413	6.079	2.160	1.413	6.079	2.322	1.413
$p(a_{i,j} \cap a_{1,1})$	6.079	1.519	1.117	6.079	1.120	1.117	0.797	2.045	1.117
$p(a_{i,j} \cap a_{1,2})$		2.009	1.075	1.519	0.776	1.075	0.598	1.133	1.075
$p(a_{i,j} \cap a_{1,3})$			1.413	1.117	1.104	1.413	0.903	1.233	1.413
$p(a_{i,j} \cap a_{2,1})$				6.079	1.120	1.117	0.797	2.045	1.117
$p(a_{i,j} \cap a_{2,2})$					2.160	1.104	1.822	1.403	1.104
$p(a_{i,j} \cap a_{2,3})$						1.413	0.903	1.233	1.413
$p(a_{i,j} \cap a_{3,1})$				Symmetrical			6.079	1.055	0.903
$p(a_{i,j} \cap a_{3,2})$								2.322	1.233
$p(a_{i,j} \cap a_{3,3})$									1.413

TABLE 6.1.6 - PROBABILITIES x 10^9 OF THE EXCESS RESISTANCES BEING NEGATIVE FOR THE FIRST THREE FAILURE PATHS



P_i^* = failure path which can be ignored for structural reasons

Fig. 6.1.1 - Failure Tree for Portal Frame Example

RESISTANCE CORRELATION COEFFICIENTS	0.0	0.5	1.0	
PLASTICALLY DESIGNED STRUCTURE	0.1811×10^{-8} 0.2079×10^{-8}	0.3931×10^{-8} 0.6111×10^{-8}	0.6086×10^{-8} 0.6086×10^{-8}	LOWER UPPER } BOUND
ELASTICALLY DESIGNED STRUCTURE	0.0249×10^{-8} 0.0249×10^{-8}	0.1414×10^{-8} 0.1414×10^{-8}	0.6085×10^{-8} 0.6085×10^{-8}	LOWER UPPER } BOUND

TABLE 6.2.1 - PROBABILITY OF SYSTEM FAILURE USING DITLEVSEN'S METHOD FOR PORTAL FRAME EXAMPLE

RESISTANCE CORRELATION COEFFICIENTS	0.0	0.5	1.0
PLASTICALLY DESIGNED STRUCTURE	0.1839×10^{-8} $\pm 0.0028 \times 10^{-8}$	0.4361×10^{-8} $\pm 0.0276 \times 10^{-8}$	0.6086×10^{-8}
ELASTICALLY DESIGNED STRUCTURE	0.0249×10^{-8}	0.1414×10^{-8}	0.6085×10^{-8}

TABLE 6.2.2 - NARROWEST BOUNDS ON THE PROBABILITY OF SYSTEM FAILURE FOR PORTAL FRAME EXAMPLE

Failure Criteria Alterations to Appendix C

f_{cu} replaced by $f_{cu} \times C_u$

u_n defined by a_i, i = 1 to 9 (see Appendix H)

VARIABLE	SYMBOL	DISTRIBUTION TYPE	MEAN	STANDARD DEVIATION
Concrete Corr.	C_u	Lognormal	1.00	0.07
Load Intensity	A_i	Lognormal	1.00	0.10

Variables having unit correlation coefficients with complementary variable in other beams or at opposite support of same beams if shear failure made:

d_i , i = 1 to 5 C_u

$f_{YS,i}$, i = 1 to 5 W

Variables having zero correlation coefficients with complementary variable in other beams or at opposite support of same beam if shear failure mode:

w_b L μ_t

d_b Γ μ_c

S_v d_s f_{in}

P f_{cu} S_f b_m

TABLE 6.2.3 - VARIABLE CORRELATION FOR SLAB EXAMPLE

MODE OF FAILURE FOR MEMBER	MEAN	STANDARD DEVIATION	PROBABILITY OF INITIAL FAILURE
BENDING MOMENT	3.041	0.449	6.295×10^{-8}
SHEAR FORCE	16.793	2.663	0.847×10^{-8}

CORRELATION COEFFICIENTS		BEAM 1		
		SHEAR AT A	B M	SHEAR AT B
BEAM 1	SHEAR AT A	1.0	0.672	0.998
	B M	0.672	1.0	0.672
	SHEAR AT B	0.998	0.672	1.0
BEAM 2	SHEAR AT B	0.506	0.624	0.507
	B M	0.623	0.919	0.624
	SHEAR AT C	0.505	0.623	0.506
BEAM 3	SHEAR AT C	0.505	0.624	0.512
	B M	0.624	0.922	0.624
	SHEAR AT D	0.513	0.624	0.505

TABLE 6.2.4 - DISTRIBUTION PARAMETERS FOR BEAM RESISTANCE IN SLAB EXAMPLE

FAILURE PATH	LOWER BOUND	MOSES'S ALGORITHM	UPPER BOUND	FAILURE PATH	LOWER BOUND	MOSES'S ALGORITHM	UPPER BOUND
1	$.227 \times 10^{-7}$	$.416 \times 10^{-7}$	$.227 \times 10^{-7}$	19	$.204 \times 10^{-9}$	$.847 \times 10^{-8}$	$.205 \times 10^{-9}$
2	$.232 \times 10^{-9}$	$.822 \times 10^{-9}$	$.232 \times 10^{-9}$	20	$.132 \times 10^{-10}$	$.847 \times 10^{-8}$	$.195 \times 10^{-9}$
3		$.847 \times 10^{-8}$		21		$.822 \times 10^{-9}$	
4	$.644 \times 10^{-8}$	$.416 \times 10^{-7}$	$.194 \times 10^{-7}$	22	$.204 \times 10^{-9}$	$.847 \times 10^{-8}$	$.205 \times 10^{-9}$
5	$.716 \times 10^{-10}$	$.156 \times 10^{-7}$	$.204 \times 10^{-9}$	23		$.822 \times 10^{-9}$	
6	$.716 \times 10^{-10}$	$.822 \times 10^{-9}$	$.204 \times 10^{-9}$	24	$.132 \times 10^{-10}$	$.847 \times 10^{-8}$	$.198 \times 10^{-10}$
7	$.204 \times 10^{-9}$	$.847 \times 10^{-8}$	$.205 \times 10^{-9}$	25	$.250 \times 10^{-10}$	$.847 \times 10^{-8}$	$.275 \times 10^{-10}$
8		$.847 \times 10^{-8}$		26	$.634 \times 10^{-12}$	$.847 \times 10^{-8}$	$.275 \times 10^{-10}$
9	$.132 \times 10^{-10}$	$.847 \times 10^{-8}$	$.198 \times 10^{-9}$	27	$.653 \times 10^{-12}$	$.847 \times 10^{-8}$	$.275 \times 10^{-10}$
10	$.204 \times 10^{-9}$	$.847 \times 10^{-8}$	$.205 \times 10^{-9}$	28	$.250 \times 10^{-10}$	$.847 \times 10^{-8}$	$.275 \times 10^{-10}$
11	$.132 \times 10^{-10}$	$.847 \times 10^{-8}$	$.195 \times 10^{-9}$	29	$.634 \times 10^{-12}$	$.847 \times 10^{-8}$	$.275 \times 10^{-10}$
12		$.847 \times 10^{-8}$		30	$.643 \times 10^{-12}$	$.847 \times 10^{-8}$	$.275 \times 10^{-10}$
13	$.451 \times 10^{-8}$	$.847 \times 10^{-8}$	$.451 \times 10^{-8}$	31	$.260 \times 10^{-10}$	$.847 \times 10^{-8}$	$.287 \times 10^{-10}$
14	$.195 \times 10^{-9}$	$.847 \times 10^{-8}$	$.195 \times 10^{-9}$	32	$.662 \times 10^{-12}$	$.847 \times 10^{-8}$	$.287 \times 10^{-10}$
15		$.847 \times 10^{-8}$		33	$.662 \times 10^{-12}$	$.847 \times 10^{-8}$	$.287 \times 10^{-10}$
16	$.452 \times 10^{-8}$	$.847 \times 10^{-8}$	$.452 \times 10^{-8}$	34	$.260 \times 10^{-10}$	$.847 \times 10^{-8}$	$.287 \times 10^{-10}$
17		$.847 \times 10^{-8}$		35	$.671 \times 10^{-12}$	$.847 \times 10^{-8}$	$.287 \times 10^{-10}$
18	$.198 \times 10^{-9}$	$.847 \times 10^{-8}$	$.198 \times 10^{-9}$	36	$.671 \times 10^{-12}$	$.847 \times 10^{-8}$	$.287 \times 10^{-10}$
			SYSTEM FAILURE		$.365 \times 10^{-7}$	$.416 \times 10^{-7}$	$.229 \times 10^{-6}$
			FIRST LOCAL FAILURE		$.189 \times 10^{-6}$	$.629 \times 10^{-7}$	$.227 \times 10^{-6}$

TABLE 6.2.5 - FAILURE PROBABILITIES FOR SLAB EXAMPLE USING MOSES'S "UNZIPPING" METHOD AND THE BOUNDS ON IT BY THE PROPOSED METHOD

LOAD		BRITILE REDUNDANT MODEL	DUCTILE REDUNDANT MODEL
STANDARD DEVIATIONS	VALUE		
$E(S) - 6 \times \sigma(S)$	-3.458	$.509 \times 10^{-22}$	$.711 \times 10^{-11}$
$E(S) - 5 \times \sigma(S)$	-2.392	$.865 \times 10^{-21}$	$.288 \times 10^{-10}$
$E(S) - 4 \times \sigma(S)$	-1.325	$.150 \times 10^{-19}$	$.133 \times 10^{-9}$
$E(S) - 3 \times \sigma(S)$	-0.258	$.316 \times 10^{-18}$	$.428 \times 10^{-9}$
$E(S) - 2 \times \sigma(S)$	0.808	$.940 \times 10^{-17}$	$.161 \times 10^{-8}$
$E(S) - \sigma(S)$	1.875	$.388 \times 10^{-15}$	$.646 \times 10^{-8}$
$E(S)$	2.942	$.212 \times 10^{-13}$	$.307 \times 10^{-7}$
$E(S) + \sigma(S)$	4.008	$.142 \times 10^{-11}$	$.180 \times 10^{-6}$
$E(S) + 2 \times \sigma(S)$	5.075	$.962 \times 10^{-10}$	$.117 \times 10^{-5}$
$E(S) + 3 \times \sigma(S)$	6.142	$.525 \times 10^{-8}$	$.744 \times 10^{-5}$
$E(S) + 4 \times \sigma(S)$	7.208	$.206 \times 10^{-6}$	$.428 \times 10^{-4}$
$E(S) + 5 \times \sigma(S)$	8.275	$.552 \times 10^{-5}$	$.217 \times 10^{-3}$
$E(S) + 6 \times \sigma(S)$	9.342	$.986 \times 10^{-4}$	$.952 \times 10^{-3}$
$E(S) + 7 \times \sigma(S)$	10.408	$.116 \times 10^{-2}$	$.362 \times 10^{-2}$
$E(S) + 8 \times \sigma(S)$	11.475	$.878 \times 10^{-2}$	$.119 \times 10^{-1}$
$E(S) + 9 \times \sigma(S)$	12.542	$.429 \times 10^{-1}$	$.340 \times 10^{-1}$
	INTEGRATED	$.521 \times 10^{-6}$	$.173 \times 10^{-5}$

TABLE 6.2.6 - PROBABILITY OF SYSTEM FAILURE USING MARSHAL AND BEA'S METHODS FOR SLAB EXAMPLE

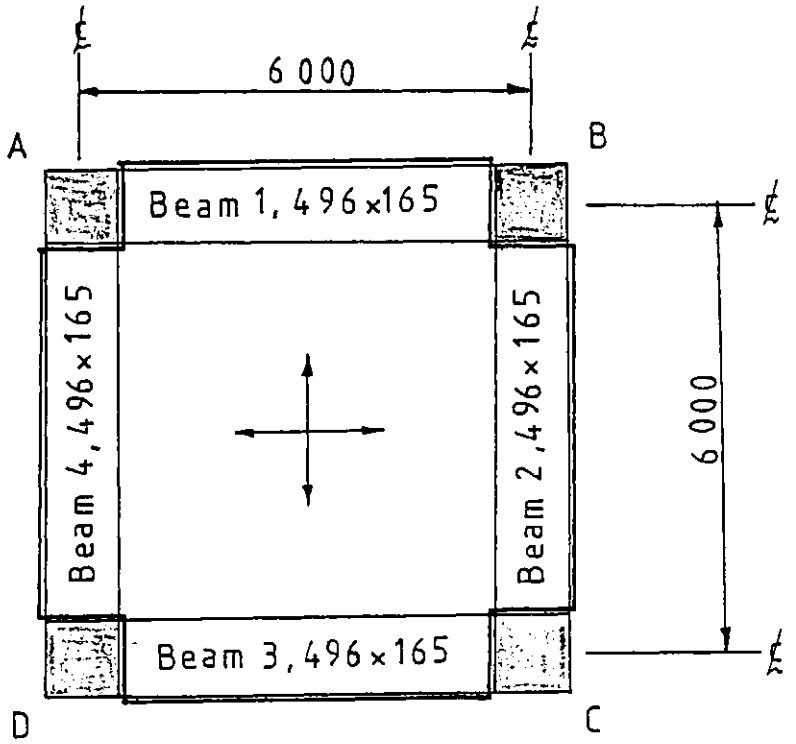


Fig.6.2.1 - Layout of Structure for Slab Example

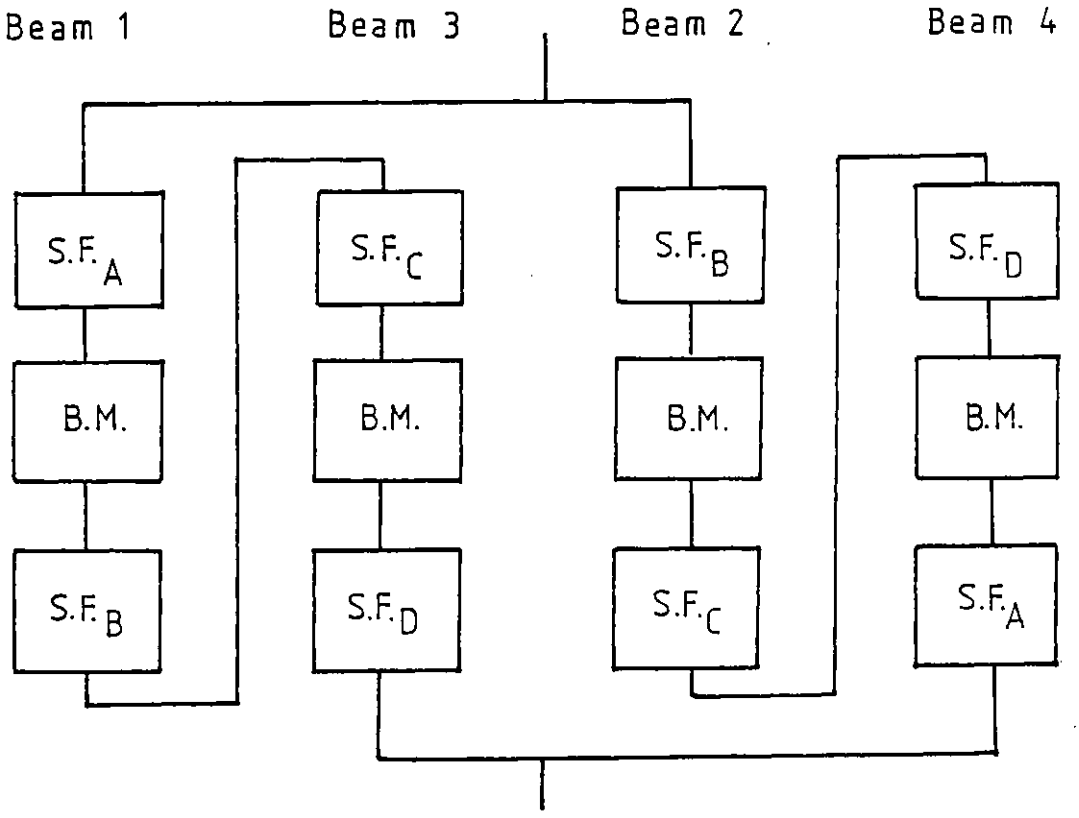


Fig. 6.2.2 - Failure Network for Slab Example

MODE OF LOCAL FAILURE	MEAN	STANDARD DEVIATION	PROBABILITY OF INITIAL FAILURE
BENDING MOMENT	3.041	0.535	0.004543
SHEAR FORCE	16.793	3.444	0.000191

CORRELATION COEFFICIENTS		BEAM 1		
		SHEAR AT A	B M	SHEAR AT B
BEAM 1	SHEAR AT A	1.0	0.702	0.998
	B M	0.702	1.0	0.702
	SHEAR AT B	0.998	0.702	1.0
BEAM 2	SHEAR AT B	0.579	0.704	0.580
	B M	0.703	0.921	0.704
	SHEAR AT C	0.578	0.703	0.579
BEAM 3	SHEAR AT C	0.578	0.705	0.585
	B M	0.705	0.923	0.705
	SHEAR AT D	0.585	0.705	0.578

TABLE 6.4.1 - DISTRIBUTION PARAMETERS FOR BEAM RESISTANCES IN SLAB EXAMPLE WITH INCREASED LOAD

	NUMBER OF FAILURES					
	RUN 1	RUN 2	RUN 3	RUN 4	RUN 5	RUN 6
TOTAL COLLAPSE	35	44	36	27	39	38
ANY FAILURE	38	45	38	29	41	39
Failure Path Type 1	94	124	124	90	116	140
Failure Path Type 2	100	146	139	104	130	142
Failure Path Type 3	16	17	27	13	23	34
Failure Path Type 4	82	102	108	80	78	120
Failure Path Type 5	75	95	113	86	84	115
Failure Path Type 6	75	95	113	86	84	115
Failure Path Type 7	2	2	4	4	4	9
Failure Path Type 8	2	2	4	3	1	5
Failure Path Type 9	2	2	4	3	3	5
Failure Path Type 10	2	2	4	4	4	9
Failure Path Type 11	2	0	3	3	2	5
Failure Path Type 12	2	0	3	3	1	4
Failure Path Type 13	2	2	6	4	6	5
Failure Path Type 14	4	0	7	4	2	5
Failure Path Type 15	0	0	1	0	0	0
Failure Path Type 16	2	2	6	4	6	9
Failure Path Type 17	0	0	1	0	0	0
Failure Path Type 18	3	2	8	4	5	8
Failure Path Type 19	2	2	4	4	4	9
Failure Path Type 20	2	0	3	3	2	5
Failure Path Type 21	2	0	3	3	1	4
Failure Path Type 22	2	2	4	4	4	9
Failure Path Type 23	2	2	4	3	1	5
Failure Path Type 24	2	2	4	3	3	6
Failure Path Type 25	0	0	0	0	0	0
Failure Path Type 26	0	0	0	0	0	0
Failure Path Type 27	0	0	0	0	0	0
Failure Path Type 28	0	0	0	0	0	0
Failure Path Type 29	0	0	0	0	0	0
Failure Path Type 30	0	0	0	0	0	0
Failure Path Type 31	0	0	0	0	0	0
Failure Path Type 32	0	0	0	0	0	0
Failure Path Type 33	0	0	0	0	0	0
Failure Path Type 34	0	0	0	0	0	0
Failure Path Type 35	0	0	0	0	0	0
Failure Path Type 36	0	0	0	0	0	0

	RUN 1	RUN 2	RUN 3	RUN 4	RUN 5	RUN 6
Element 1	17	26	21	22	22	25
Element 2	20	23	27	15	20	25
Element 3	23	32	21	18	27	20
Element 4	18	18	25	14	23	28
Element 5	2	0	2	1	0	0
Element 6	2	0	1	1	0	0
Element 7	0	0	2	1	2	1
Element 8	0	0	2	1	2	1
Element 9	0	1	0	0	2	1
Element 10	0	1	1	0	2	1
Element 11	0	0	0	2	0	4
Element 12	1	0	0	2	0	3

Number of trials per run = 5000

Number of failure paths per type = 8

TABLE 6.4.2 - RESULTS FROM THE MONTE CARLO SIMULATION

	FAILURE PROBABILITIES				
	ORIGINAL EXAMPLE		INCREASED LOAD		
	LOWER BOUND	UPPER BOUND	LOWER BOUND	MONTE CARLO	UPPER BOUND
TOTAL COLLAPSE	$.365 \times 10^{-7}$	$.227 \times 10^{-6}$	$.338 \times 10^{-2}$	$.730 \times 10^{-2}$	$.153 \times 10^{-1}$
ANY FAILURE	$.189 \times 10^{-6}$	$.227 \times 10^{-6}$	$.688 \times 10^{-2}$	$.767 \times 10^{-2}$	$.955 \times 10^{-2}$
FAILURE PATH TYPE 1	$.227 \times 10^{-7}$	$.227 \times 10^{-7}$	$.325 \times 10^{-2}$	$.287 \times 10^{-2}$	$.325 \times 10^{-2}$
FAILURE PATH TYPE 2	$.232 \times 10^{-9}$	$.232 \times 10^{-9}$	$.158 \times 10^{-3}$	$.317 \times 10^{-2}$	$.158 \times 10^{-3}$
FAILURE PATH TYPE 3	*	*	*	$.542 \times 10^{-3}$	*
FAILURE PATH TYPE 4	$.644 \times 10^{-8}$	$.194 \times 10^{-7}$	$.212 \times 10^{-2}$	$.238 \times 10^{-2}$	$.298 \times 10^{-2}$
FAILURE PATH TYPE 5	$.716 \times 10^{-10}$	$.204 \times 10^{-9}$	$.104 \times 10^{-3}$	$.238 \times 10^{-2}$	$.154 \times 10^{-3}$
FAILURE PATH TYPE 6	$.716 \times 10^{-10}$	$.204 \times 10^{-9}$	$.104 \times 10^{-3}$	$.238 \times 10^{-2}$	$.154 \times 10^{-3}$
FAILURE PATH TYPE 7	$.204 \times 10^{-9}$	$.205 \times 10^{-9}$	$.116 \times 10^{-3}$	$.104 \times 10^{-3}$	$.116 \times 10^{-3}$
FAILURE PATH TYPE 8	*	*	*	$.71 \times 10^{-4}$	*
FAILURE PATH TYPE 9	$.132 \times 10^{-10}$	$.198 \times 10^{-9}$	$.751 \times 10^{-4}$	$.79 \times 10^{-4}$	$.116 \times 10^{-3}$
FAILURE PATH TYPE 10	$.204 \times 10^{-9}$	$.205 \times 10^{-9}$	$.116 \times 10^{-3}$	$.104 \times 10^{-3}$	$.116 \times 10^{-3}$
FAILURE PATH TYPE 11	$.132 \times 10^{-10}$	$.195 \times 10^{-9}$	$.749 \times 10^{-4}$	$.63 \times 10^{-4}$	$.116 \times 10^{-3}$
FAILURE PATH TYPE 12	*	*	*	$.54 \times 10^{-4}$	*
FAILURE PATH TYPE 13	$.451 \times 10^{-8}$	$.451 \times 10^{-8}$	$.190 \times 10^{-3}$	$.121 \times 10^{-3}$	$.190 \times 10^{-3}$
FAILURE PATH TYPE 14	$.195 \times 10^{-9}$	$.195 \times 10^{-9}$	$.124 \times 10^{-3}$	$.124 \times 10^{-3}$	$.124 \times 10^{-3}$
FAILURE PATH TYPE 15	*	*	*	$.4 \times 10^{-5}$	*
FAILURE PATH TYPE 16	$.452 \times 10^{-8}$	$.452 \times 10^{-8}$	$.190 \times 10^{-3}$	$.121 \times 10^{-3}$	$.190 \times 10^{-3}$
FAILURE PATH TYPE 17	*	*	*	$.4 \times 10^{-5}$	*
FAILURE PATH TYPE 18	$.198 \times 10^{-9}$	$.198 \times 10^{-9}$	$.124 \times 10^{-3}$	$.125 \times 10^{-3}$	$.124 \times 10^{-3}$
FAILURE PATH TYPE 19	$.204 \times 10^{-9}$	$.205 \times 10^{-9}$	$.751 \times 10^{-4}$	$.121 \times 10^{-3}$	$.116 \times 10^{-3}$
FAILURE PATH TYPE 20	$.132 \times 10^{-10}$	$.195 \times 10^{-9}$	$.749 \times 10^{-4}$	$.63 \times 10^{-4}$	$.116 \times 10^{-3}$
FAILURE PATH TYPE 21	*	*	*	$.54 \times 10^{-4}$	*

	EXAMPLE 1		EXAMPLE 2		
	LOWER BOUND	UPPER BOUND	LOWER BOUND	MONTE CARLO	UPPER BOUND
FAILURE PATH TYPE 22	$.204 \times 10^{-9}$	$.205 \times 10^{-9}$	$.116 \times 10^{-3}$	$.104 \times 10^{-4}$	$.116 \times 10^{-3}$
FAILURE PATH TYPE 23	*	*	*	$.71 \times 10^{-4}$	*
FAILURE PATH TYPE 24	$.132 \times 10^{-10}$	$.198 \times 10^{-10}$	$.751 \times 10^{-4}$	$.83 \times 10^{-4}$	$.116 \times 10^{-3}$
FAILURE PATH TYPE 25	$.250 \times 10^{-10}$	$.275 \times 10^{-10}$	$.140 \times 10^{-4}$	0	$.141 \times 10^{-4}$
FAILURE PATH TYPE 26	$.634 \times 10^{-12}$	$.275 \times 10^{-10}$	$.909 \times 10^{-5}$	0	$.141 \times 10^{-4}$
FAILURE PATH TYPE 27	$.643 \times 10^{-12}$	$.275 \times 10^{-10}$	$.912 \times 10^{-5}$	0	$.141 \times 10^{-4}$
FAILURE PATH TYPE 28	$.250 \times 10^{-10}$	$.275 \times 10^{-10}$	$.140 \times 10^{-4}$	0	$.141 \times 10^{-4}$
FAILURE PATH TYPE 29	$.634 \times 10^{-12}$	$.275 \times 10^{-10}$	$.909 \times 10^{-5}$	0	$.141 \times 10^{-4}$
FAILURE PATH TYPE 30	$.643 \times 10^{-12}$	$.275 \times 10^{-10}$	$.912 \times 10^{-5}$	0	$.141 \times 10^{-4}$
FAILURE PATH TYPE 31	$.260 \times 10^{-10}$	$.287 \times 10^{-10}$	$.146 \times 10^{-4}$	0	$.146 \times 10^{-4}$
FAILURE PATH TYPE 32	$.662 \times 10^{-12}$	$.287 \times 10^{-10}$	$.943 \times 10^{-5}$	0	$.146 \times 10^{-4}$
FAILURE PATH TYPE 33	$.662 \times 10^{-12}$	$.287 \times 10^{-10}$	$.943 \times 10^{-5}$	0	$.146 \times 10^{-4}$
FAILURE PATH TYPE 34	$.260 \times 10^{-10}$	$.287 \times 10^{-10}$	$.146 \times 10^{-4}$	0	$.146 \times 10^{-4}$
FAILURE PATH TYPE 35	$.671 \times 10^{-12}$	$.287 \times 10^{-10}$	$.946 \times 10^{-5}$	0	$.146 \times 10^{-4}$
FAILURE PATH TYPE 36	$.671 \times 10^{-12}$	$.287 \times 10^{-10}$	$.946 \times 10^{-5}$	0	$.146 \times 10^{-4}$
INITIAL BENDING	$.629 \times 10^{-7}$	$.629 \times 10^{-7}$	$.454 \times 10^{-2}$	$.443 \times 10^{-2}$	$.454 \times 10^{-2}$
INITIAL SHEAR	$.847 \times 10^{-8}$	$.847 \times 10^{-8}$	$.191 \times 10^{-3}$	$.175 \times 10^{-3}$	$.191 \times 10^{-3}$

TABLE 6.4.3 - BOUNDS ON THE FAILURE PROBABILITIES FOR THE SLAB EXAMPLE, INCLUDING MONTE CARLO SIMULATION VALUES

[* = Failure Path omitted as one of the elements is due to fail in the shear force failure mode when the element at the other end of the member will automatically carry a greater load, and hence fail first]

CORRELATION COEFFICIENT OF CROSS-SECTIONS ON SAME MEMBER	CORRELATION COEFFICIENT OF THE TWO VERTICAL MEMBERS	CORRELATION COEFFICIENT OF THE HORIZONTAL AND EACH OF THE VERTICAL MEMBERS			
		0.0	0.50	0.95	1.00
0.95	0.0	$3.056\text{-}3.513 \times 10^{-6}$		$3.880\text{-}4.842 \times 10^{-6}$	
	0.50		$2.983\text{-}3.660 \times 10^{-6}$	$3.850\text{-}4.913 \times 10^{-6}$	
	0.95			$2.129\text{-}6.144 \times 10^{-6}$	
1.00	0.50		$3.370\text{-}4.270 \times 10^{-6}$		
	1.00				$2.197\text{-}3.182 \times 10^{-6}$

Table 6.5.1 - BOUNDS ON THE FAILURE PROBABILITIES FOR THE PORTAL FRAME EXAMPLE WITH 0.50 CORRELATION COEFFICIENT BETWEEN LOADS

Correlation Coefficient of Cross-Sections on same Member	Correlation Coefficient of the two Vertical Members	Correlation Coefficient of the Horizontal and Each of the Vertical Members		
		0.0	0.50	0.95
0.95	0.0	$2.989\text{-}3.448 \times 10^{-6}$		$3.808\text{-}4.768 \times 10^{-6}$
	0.50		$2.918\text{-}3.591 \times 10^{-6}$	$3.780\text{-}4.832 \times 10^{-6}$
	0.95			$2.078\text{-}6.069 \times 10^{-6}$

TABLE 6.5.2 BOUNDS ON THE FAILURE PROBABILITIES FOR THE PORTAL FRAME EXAMPLE WITH UNCORRELATED LOADS

Correlation Coefficient of Cross-Sections on Same Member	Correlation Coefficient of the two Vertical Members	Correlation Coefficient of the Horizontal and Each of the Vertical Members		
		0.0	0.50	0.95
0.95	0.0	$3.725\text{-}5.129 \times 10^{-6}$		$4.522\text{-}7.037 \times 10^{-6}$
	0.50		$3.541\text{-}5.397 \times 10^{-6}$	$4.441\text{-}7.180 \times 10^{-6}$
	0.95			$2.682\text{-}8.893 \times 10^{-6}$

TABLE 6.5.3 BOUNDS ON THE FAILURE PROBABILITIES FOR THE PORTEL FRAME EXAMPLE WITH TIME-VARYING LOADS

Correlation Coefficient of Cross-Sections on Same Member	Correlation Coefficient of the two Vertical Members	Correlation Coefficient of the Horizontal and Each of the Vertical Members		
		0.0	0.50	0.95
0.95	0.0	$4.854\text{-}5.433 \times 10^{-6}$		$6.211\text{-}7.468 \times 10^{-6}$
	0.50		$4.757\text{-}5.693 \times 10^{-6}$	$6.134\text{-}7.592 \times 10^{-6}$
	0.95			$3.467\text{-}9.292 \times 10^{-6}$

TABLE 6.5.4 BOUNDS ON THE FAILURE PROBABILITIES FOR THE PORTAL FRAME EXAMPLE WITH UNCORRELATED LOADS FOR THEIR MAXIMUM DISTRIBUTIONS

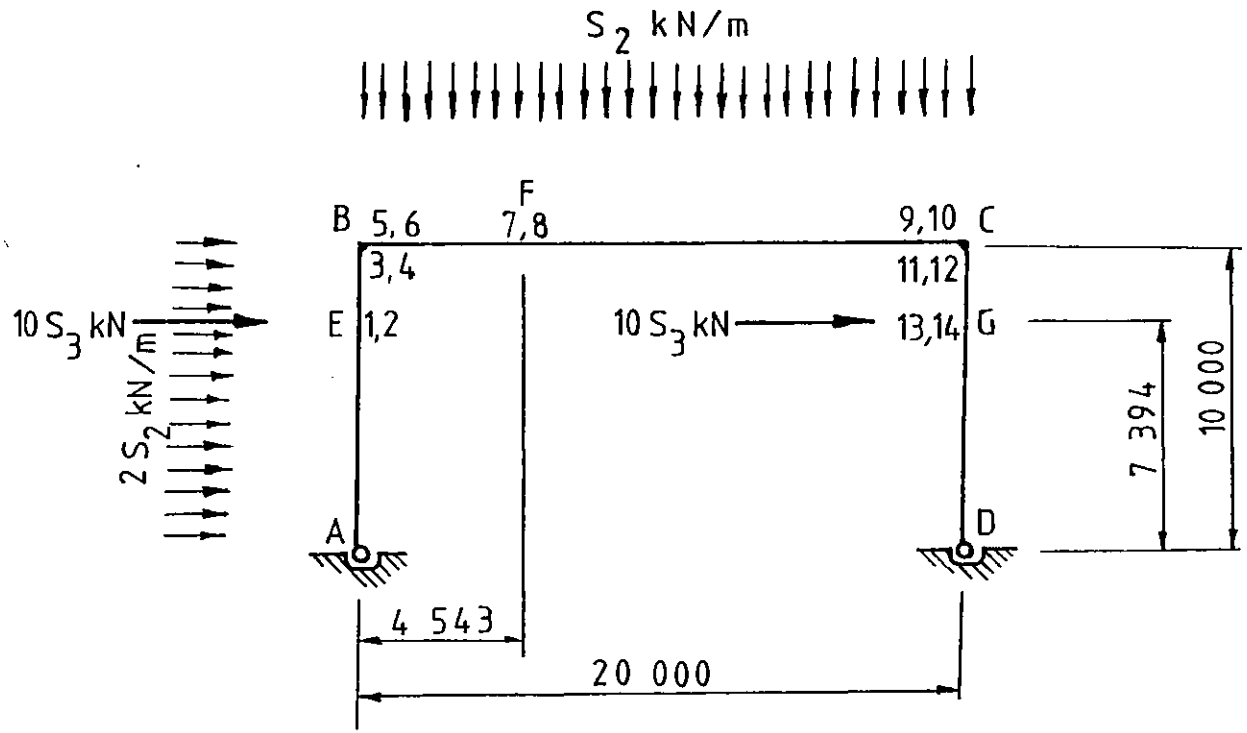


Fig.6.5.1 - Layout of Structure and Loads for Example with Multiple Loads

CHAPTER 7 - CONCLUSIONS

7.1 LEVEL 2 RELIABILITY METHODS

7.1.1 Magnitude of Errors

An error in calculating a failure probability of less than half an order of magnitude is liable to be negligible when compared to the effect of statistical uncertainties in the frequency distributions of the basic variables for most practical situations. Almost all of the Advanced Level 2 method's errors that have been calculated were negligible when assessed on that basis, while 95% of the Mean Value Level 2 method's errors were significant. Therefore the results of the comparison between the failure probabilities, as obtained from the Level 2 and Level 3 methods, that were calculated for this study have demonstrated:

i. that the errors occurring from the idealisations and approximations inherent in the Advanced Level 2 method of reliability analysis are relatively small and can be ignored; but

ii. that the same cannot be said for the Mean Value 2 method of reliability analysis.

The errors produced by the Mean Value Level 2 method were not consistent enough to be used even for calibration purposes, either:

i. with the same format of the failure criterion; or

ii. with the same elements but different formats of the failure criterion.

Therefore the additional calculation required for the iterative procedures of the Advanced method are necessary in order to obtain a result in which confidence can be placed.

7.1.2 Extension of Work

The failure criteria that were examined in the comparisons had no discontinuities in the slope of their failure boundaries. This study could be extended to examine the position and degree of any discontinuities of slope that would be necessary in order to invalidate the accuracy of the Advanced Level 2 method. Such an investigation would be useful since the failure criteria for relatively complex modes of failure are liable to include discontinuities of slope if some mode of the element's behaviour alters at discrete values of one of the parameters (eg the elastic-plastic behaviour of many materials).

A further study could also include failure boundaries with "sharp" curvature "close" to the design point in standardised normal variable space. Such a study could try to quantify "sharp" and "close", perhaps in terms of

$$\xi(\beta) = f\left(\max_{\substack{i=1,j=1 \\ i\neq j}}^{n,n}\left[\frac{\partial^2 u_i}{\partial u_j^2}\right|_{\underset{\sim}{u}=\underset{\sim}{u}^{**} \text{ where } h(\underset{\sim}{u}^{**})=0.0}\Big], \beta, \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left[u_i^{**}u_j^{**}\rho(U_i,U_j)\right]}\right) \qquad (7.1.1)$$

where $\xi(\beta)$ is a measure of the confidence in the value of $\Phi(-\beta)$ as the probability of failure. For failure boundaries with two local minima β_1 and β_2 from the origin in standardised normal variable space, the function could be in terms of

$$\xi(\text{Min}[\beta_1,\beta_2]) = f\left(\beta_1,\beta_2,\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(u_{i,1}^{*}-u_{i,2}^{*})\times(u_{j,1}^{*}-u_{j,2}^{*})\times\rho(U_i,U_j)\right]\right\}\right) \qquad (7.1.2)$$

where $u_{i,1}^{*}$ is the value of the i^{th} basic variable at the local minima β_1 from the origin, etc.

7.1.3 Limitations

The use of the Advanced Level 2 method on its own for system reliability analysis implies the linearisation of the union and/or intersection of several failure boundaries, (ie there are several discontinuities of slope in the overall systems failure boundary). This linearisation will lead to the invalidation of the accuracy of the method, as can be demonstrated by comparing the bounds on the probability of a series system failure, which have been calculated in Part 2, with the maximum probability of a local failure, which is the value that the Advanced Level 2 method should give.

In these conclusions the assumption is made that the correct design point (ie the closest point on the failure boundary to the origin in standardised normal variable space) has been found by the iterative procedure in the Advanced Level 2 method. There must be some check to confirm that the point found is the correct one, given that there is a possibility of multiple local minima on the failure boundary. This check is required because the use of the wrong linearisation point will invalidate the advantages of the Advanced over the Mean Value Level 2 method of reliability analysis.

7.2 SYSTEM RELIABILITY ANALYSIS

7.2.1 Use of System Reliability Analysis

It is now widely accepted that improved structural analysis and design techniques should be accompanied by improved methods of predicting structural safety, where the rational criterion for the safety of a structure is its reliability. This reliability is usually provided by partial safety factors since carrying out design or analysis by using reliability analysis techniques is computationally extravagant. The partial safety factors will have a rational basis if they have been obtained by calibration using the reliability of typical structural elements. This calibration is being carried out for some of the more recent Codes of Practice. The partial safety factors, however, relate to the reliability of a single element against its failure by a single mode of failure. Therefore, a set of partial safety factors cannot be used to ensure that the designed structures will have a constant level of reliability.

System reliability analysis is needed when it is required to adjust any target reliabilities used to calculate the partial safety factors so that the system reliability is maintained at a constant level. Public opinion would not tolerate the idea of a reduction of target reliabilities by such means in most cases, but to increase them when progressive collapse may occur would be generally welcomed.

Structures may have to be divided into compartments (in a similar manner to the fire restrictions in the Building Regulations [22]) in order to maintain the target system reliability level. Each compartment would have to be designed to give a system reliability level not less than a target system reliability in a similar manner to elements being designed to target element reliabilities. A failure in any compartment would have to lead to no collapse outside that compartment, so that some compartments might have to overlap.

A second use of system reliability analysis is to produce bounds on the expected cost of failure for a structure. Such bounds would be a measure of the financial implications of making any alterations to the partial safety factors. The use of different modes of system failure will allow relatively complex costing models to be used for such a comparison.

7.2.2 Methods of Analysis

Each of the methods of system reliability analysis that are revised (sections 4.3 and 4.4) can be used to analyse a limited sub-set of the possible structural systems (table 7.2.1 and fig. 7.2.1). Some of the methods also have disadvantages

where analysing the structural systems for which they were devised. The principal disadvantages of these methods are:

Rackwitz and Kzykaczi's Method:

i. The complexity of setting up the multivariant integral; and

ii. The calculating effort required to evaluate a multi-variant normal integral.

Vanmarke's Method:

The evaluated probability of system failure is only an upper bound.

Moses Unzipping Technique:

i. The assumption that the probability of one of the failure paths occurring is considerably larger than the probabilities of any other failure path occurring given that the first failure path has not occurred; and

ii. The assumption that for each failure path all but one of the probabilities of a local failure mode occurring given that the remaining local failure mode has occurred is approximately unity.

Marshall and Bea's Brittle-and Ductile-Redundant Models:

i. The requirement for either a hindcast load or numerical integration with respect to the load;

ii. The inconsistancy of the two models for some systems;

iii. The disagreement between the failure probabilities produced by the models and the method put forward in Chapter 5 (all other methods examined produced compatible results).

Cornell's bounds:

The bounds are too wide for most practical uses.

Moses and Kinser's bounds:

i. The bounds are too wide for most practical uses; and

ii. The upper bound requires solving a double convolution integral.

The examples carried out demonstrate that reasonably narrow bounds on the probability of system failure can be obtained for a potentially wide range of structures by

the structural system reliability analysis method that is proposed in Chapter 5. The principal disadvantages of this method are:

i. the load effects due to each load applied to the structure have to be able to be combined linearly for the total load effect on any element; and

ii. the amount of computation required is considerable more than is required by the other methods for which examples have been carried out in Chapter 6.

The excessive calculation required results from:

i. the set of possible failure paths having to be comprehensive for the bounds to be rigorous; and

ii. the comparison of each failure path with every other path, so that the computational effect required is approximately proportionate to the square of the product of the number of failure paths and the mean number of local failure modes in each path.

7.2.3 Recommended Methods

Therefore, it is recommended that if the system reliability of a structure is required that:

i. Ditlevsen's bounds should be used when each failure path can be described by a single failure criterion;

ii. the method proposed in Chapter 5 of this paper should be used if Ditlevsen's bounds are inapplicable; and

iii. Moses' unzipping technique may be used if an approximate value of the probability of system failure only is required, provided there is only a single loading variable.

Nearly all of the methods for system reliability analysis use the simplified resistance minus load format in order to describe the failure criteria of the individual modes of local failure. It is recommended that the parameters for the frequency distribution of the resistances are evaluated so as to give the best available estimate of the initial probability of local failure. This preference is due to the Mean Value Level 2 method of reliability analysis for elements having been demonstrated to be inaccurate in Part 1, and the evaluation of the parameters for the frequency distribution of the resistances by more traditional methods leads to evaluating the probabilities of local failures by the Mean Value Level 2 method.

7.3 POSSIBLE EXTENSIONS OF THE PRESENT STUDY

7.3.1 Possible Aims of Modification

Any improvement in the proposed method of structural system reliability analysis will need to revise it in one of two ways, the two ways tending to be mutually incompatible. An improved method can either;

i. introduce further refinements into the analysis so that the bounds on the probability of system failure are narrower and/or more rigorous; or

ii. simplify the analysis so that it takes less computational time and/or storage facilities.

The first way that the method might be improved can be sub-divided into four different objectives. These objectives are:

i. to find improved bounds in terms of being closer to the correct value;

ii. to extend the method to include the uncertainty in the distribution of the load effect for statically indeterminant structures;

iii. to remove the simplification of treating the resistances and loads as having normal frequency distributions; and

iv. to make this method into an algorithm for approximating to the multivariat distribution of Rackwitz and Kzykaczi's method.

7.3.2 Refining the Analysis

The bounds, as proposed, are not necessarily the closest that can be obtained. It is more probable that improved bounds can be found for the parallel combination of local failure modes into failure paths since this combination has received less attention previously - the methods involving bounds that were reviewed are concerned only with series combinations. Close examination of the examples in Chapter 6 demonstrates that the gap between the bounds on the probability of system failure is influenced more by extending the length of the failure paths than the number of the failure paths. Therefore, the parallel combination of events also needs more attention than the series combination.

The constants $c_{i,j,k}$ and $d_{i,j,k}$ will become variables if the method is to make allowance for the uncertainty in the load effect on each element. The use of additional variables will expand the size of the equations in sections 5.1, 5.2, 5.5 and 5.6. The values of $c_{i,j,k}$ and $d_{i,j,k}$ will be dependent on each other in order to satisfy simple statics.

The equations will have to be modified also if the loads and resistances are treated as having frequency distributions other than normal. More knowledge than the equivalent normal distribution's mean and standard deviation at one point (the Advanced Level 2 design point) will be required in order to fit a distribution to the resistance. This knowledge could be obtained by carrying out Advanced Level 2 analysis with different fixed values of the applied loads. The cumulative probability function could be built up from a set of values obtained from

$$F_{R_{i,1}}\left(r_{i,1} \leq \sum_{k=1}^{n_s} \left(d_{i,1,k} s'_k\right)\right) \doteq \Phi\left(-\beta \middle| \underset{\sim}{S} = \underset{\sim}{s}'\right) \tag{7.3.1}$$

If the proposed method was modified to obtain the multi-variate distribution of Rackwitz and Krzykacz' method, the probability of system failure that would be obtained would be accurate provided the loads and resistances can be described by the normal distribution. Such an approach could be built up from the following relationships. Given that

$$P(A_i) = \Phi(-\beta_i) \tag{7.3.2}$$

then

$$P(A_i \cap A_j) = \Phi(-\beta_i, -\beta_j, \rho(Z_i, Z_j)) \tag{7.3.3}$$

and

$$P(A_i \cup A_j) = 1.0 - \Phi(\beta_i, \beta_j, \rho(Z_i, Z_j)) \tag{7.3.4}$$

An approximation to the multi-variate normal integral, similar to Hasting's for the univariate case, would have to be devised in order to make such a method useful. Such an approximation would have wide applications beyond system reliability analysis.

7.3.3 Simplifying the Analysis

The second way that the proposed method might be improved can be sub-divided into two different objectives. These objectives are:

i. to rank the failure paths in two orders so that the maximum lower bound and the minimum upper bounds can be found without repeating the calculations for each order; and

ii. to reduce the number of possible failure paths.

To rank the failure paths would be useful since both Ditlevsen's and the proposed method are dependent on the order in which the failure paths are considered, although the ranking would be more beneficial for the proposed method since it requires considerably more calculation.

The reduction in the number of possible failure paths is required when the storage capacity of the computing facilities is exceeded by the exhaustive set of failure paths. The lower bound on the probability of system failure obtained from a reduced set of failure paths will still be rigourous, though further from the actual value, while the upper bound need not be rigourous. Therefore, the choice of which failure paths to ignore will have to be based on the upper bound. The order for the omission of failure paths will not necessarily be simply the order of increasing probability of failure, as can be shown by using a trivial example of three failure paths, where

$$p(A_1) = p(A_2) = p(A_1 \cup A_2) \tag{7.3.5}$$

and

$$p(A_3) < p(A_1) < p(A_1 \cup A_3) = p(A_2 \cup A_3) \tag{7.3.6}$$

Ignoring A_1 or A_2 will produce an accurate upper bound on the probability of system failure, while ignoring A_3 will produce an inaccurate upper bound.

TYPE OF MEMBERS	DUCTILE				MIXED				BRITTLE			
NO. OF LOCAL FAILURE MODES/FAILURE PATH	1		>1		1		>1		1		>1	
NO. OF INDEPENDENT LOADS	1	>1	1	>1	1	>1	1	>1	1	>1	1	>1
STRUCTURE TYPE	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
Cornell	B	B	B	B	B	B			B	B		
Moses and Kinser	B	A	B	A	B	A			B	A		
Vanmarke	A	A	A	A	A	A			A	A		
Marshall and Bea	(A)	(A)	A	A	(A)	(A)	(B)	(B)	(A)	(A)	A	A
Moses	(A)		A		(A)		A		(A)		A	
Ditlevsen	B	B	B	B	B	B			B	B		
Present Study	B	B	B	B	B	B	B	B	B	B	B	B

A = Approximation to the probability of system failure given certain assumptions.

B = Bounds on the probability of system failure.

( ) = Method not designed for case, but will give a result.

Table 7.2.1 - STRUCTURAL SYSTEMS FOR WHICH THE METHODS ARE APPLICABLE

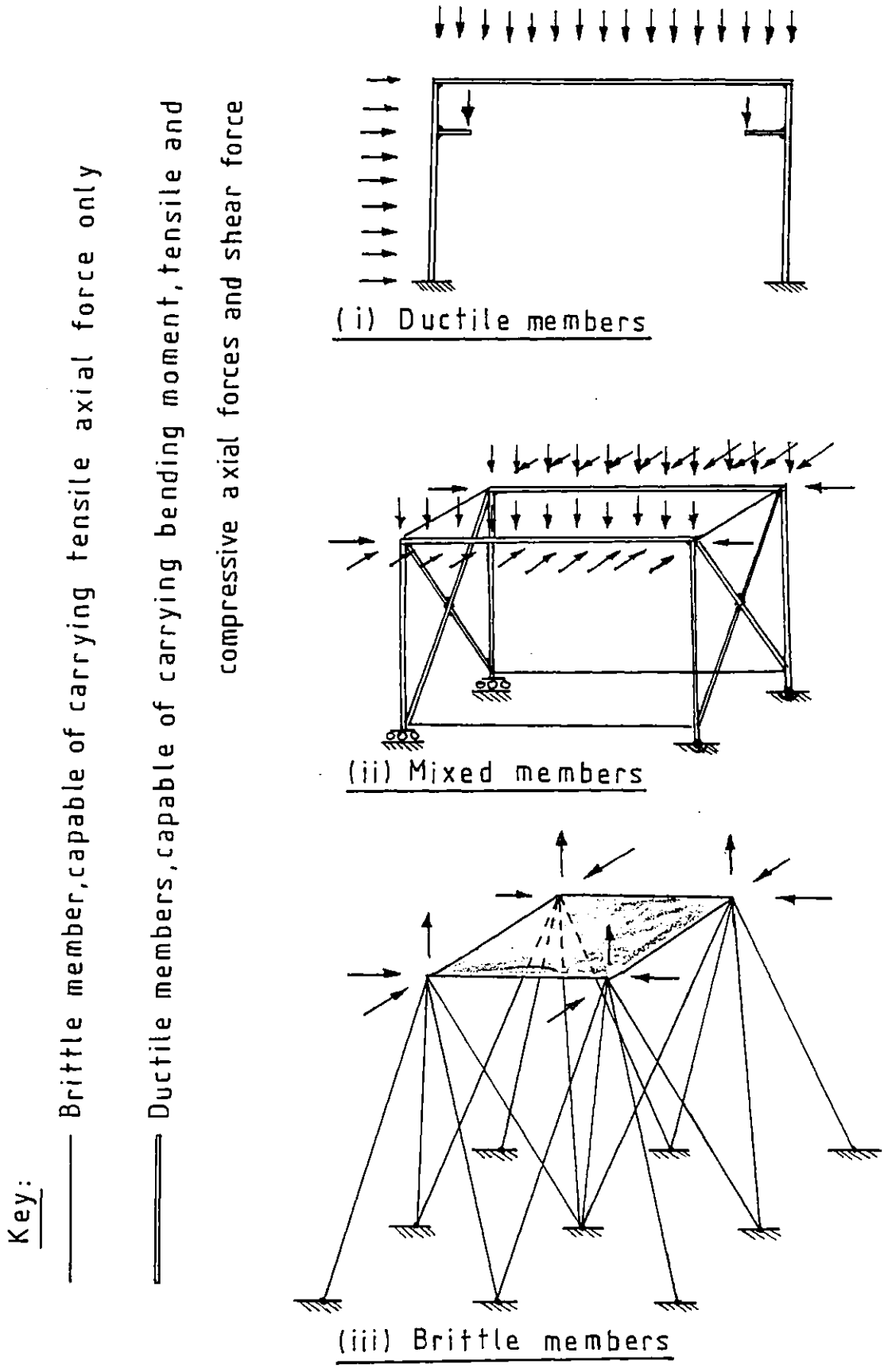


Fig.7.2.1 - Examples of Structural Systems

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APPENDIX A - GLOSSARY

(See also Appendix 8, CIRIA report 63[11])

Basic Variable: One of a set of design variables which govern the loading on and the resistance of an element and which can influence the reliability of the element. Each basic variable is modelled by a statistical distribution since the design variables cannot be defined exactly by a deterministic value.

Brittle Element: An element which is unable to carry any load effect after the local failure of that element.

Design Variable: A quantity whose value is required in order to be able to decide whether an element or structure is in a failed or non-failed state, other than mathematical or physical constants.

Ductile Element: An element which continues to carry the load effect which just causes failure after the local failure of that element.

Element: The sub-division of a structural system for which a single failure criterion is defined such that there are only failed and non-failed states (ie knowledge of its failure cannot be supplemented by further knowledge such as the degree or position of that failure). The value of each design variable for that failure criterion are regarded as being dependent on the value of a single basic variable (eg if a beam subject bending is defined as an element, then the design variables defining the load effect and the resistance at 7/16 span are treated as dependent on those at mid-span).

Excess Load Effect: The load effect which has to be redistributed from an element after local failure of that element. This is equal to the applied load effect if the element is perfectly brittle; and is equal to minus one times the excess resistance if the element is perfectly ductile.

Excess Resistance: The difference between the load carrying capacity of an element and the load effect applied to it. Local failure occurs when this is less than or equal to zero. (This has also been called the safety margin[17] and the failure indicator[5].)

Failure: The occurrence of a set of values of the basic variables so that the requirements defined by (or for) the user are not fulfilled.

Failure Criterion: Mathematical idealisation of a mode of failure in terms of basic variables such that a negative value represents the failed state and a positive value the non-failed, or safe, state.

Failure Group: A set of elements whose combined local failures would result in system failure.

Failure Network: The schematic representation of the possible local failures connected together in parallel and/or in series.

Failure Path: A sequence of local failure modes whose occurrence in that order will result in system failure.

Failure Set: The exhaustive set of the possible permutations of local failure modeswhose combined occurrence will result in system failure.

Failure Tree: The schematic representation of the possible failure paths in the failure set.

Instantaneous Failure: The failure associated with the parent distributions of any time - varying loads. This represents the instant when the loads are initially applied since initial survival theoretically then allows Bayesian updating of the non-time-varying variables.

Limit State: A mode of failure which the structure or element is designed to resist explicitly.

Load Combination: One of a set of combinations of the maxima and minima distributions of the loads acting on the structure which model the maximum load intensity during the reference period.

Local Failure: A failure which can be modelled by a single failure criterion.

Mode of Local Failure: A mode of failure whose occurrence results in a local failure.

Mode of System Failure: A mode of failure whose occurrence results in system failure.

Overall Failure: The realisation of sufficient local failures for the structure to become a mechanism.

Parallel: The intersection of two or more events so that the combined event is realised if, and only if, all of the component events are realised.

Parent Distribution: The distribution of the instantaneous values of a time-varying variable (as opposed to the distribution of the maximum or minimum value in a specified time period).

Principal Load: The time-varying load which is modelled as having its maximum intensity in the structure's reference period in a load combination, even if other loads in that load combination have larger values.

Resistance: The combination of all basic variables not designated as loading variables into a single variable in the failure criterion of an element.

Series: The union of two or more events so that the combined event is not realised if, and only if, all of the component events are not realised.

Structure: An arrangement of beams, columns, slabs and other units which are connected together in a particular manner in order to satisfy the user's requirements, and which are acted upon by a set of loads.

System Failure: The realisation of sufficient local failures to cause the structure not to fulfil its required function(s). This can be, but need not be, identical to either overall failure or the first local failure.

System Reliability: The probability that system failure will not be realised.

Time Interval: The period during which a time-varying load is maintained at a constant intensity.

Time-Varying Loads: Loads which do not maintain a constant intensity throughout the reference period (normally the design life of the structure).

APPENDIX B - PIN-JOINTED TRUSS EXAMPLE

B.1 DESCRIPTION OF SYSTEM

Structure:

Elastic pin-jointed truss with perfectly ductile elements of nominal cross-sectional area A, 2A and 3A (figure B.1.1).

Modes of Failure:

i. Local: Axial compression or tension (which are not differentiated between numerically in the value of stress at which failure is deemed to occur). Therefore failure occurs when the axial force is equal to or greater than the element's resistance (table B.1.1). The elements are assumed to be stocky, so that buckling is ignored.

ii. System: Node F becoming part of a mechanism, so that the applied load, S, cannot be resisted by the structure.

Failure Paths Considered:

The elements can be grouped in four sets where the local failure of one element in a set will result in no further load effect increments being applied to any element in that set. All elements in a set have the same nominal cross-sectional area and the same absolute value of axial load applied them, so that the failure paths consisting only of elements AB, CD, DF and EF (one from each set) will include all the possible types.

Axial Forces in Elements:

i. Initially: Let the force in BD be P and the force in DF be Q. If the horizontal displacements are given by U, then by considering the axial strain as relatively small geometrically,

$$U_B = \tfrac{2}{3}P\frac{L}{AE} \quad : U_D = (Q-P)\frac{L}{AE} \quad : U_F = 2(R-Q)\frac{L}{AE}$$

However, the strain in elements BD and DF respectively give

$$\frac{P}{3} \cdot \frac{L}{AE} = U_D - U_B = (Q - \tfrac{5}{3}P)\frac{L}{AE}$$

and $$\frac{Q}{5} \cdot \frac{L}{AE} = U_F - U_D = (2R-3Q+P)\frac{L}{AE}$$

$$\therefore \begin{cases} Q - 2P = 0 \\ 2R - \frac{16}{5}Q + P = 0 \end{cases}$$

$$\therefore P = \frac{10}{27}R \quad : Q = \frac{20}{27}R$$

See figure B.1.2 for axial force in other elements.

ii. Element EF Removed: Let the force in BD be P

$U_B = \frac{2}{3}P \times \frac{L}{AE}$: $U_D = (R-P) \times \frac{L}{AE}$

$\therefore \frac{P}{3} \cdot \frac{L}{AE} = U_D - U_B = (R - \frac{5}{3}P) \frac{L}{AE}$

$\therefore P = \frac{1}{2}R$

See figure B.1.3 for axial force in other elements.

iii. Element CD Removed: Let the force in BF be P

$U_B = \frac{2}{3}P \frac{L}{AE}$: $U_F = 2(R-P)\frac{L}{AE}$

$\therefore P (\frac{L}{3AE} + \frac{L}{5AE}) = U_F - U_B = (2R - \frac{8}{3}P)\frac{L}{AE}$

$\therefore P = \frac{5}{8}R$

See figure B.1.4 for axial force in other elements.

iv. Element AB Removed: Let the force in DF be Q

$U_D = Q \times \frac{L}{AE}$: $U_F = 2(R - Q) \frac{L}{AE}$

$\therefore \frac{Q}{5} \cdot \frac{L}{AE} = U_F - U_D = (2R - 3Q) \frac{L}{AE}$

$\therefore Q = \frac{5}{8}R$

See figure B.1.6 for axial force in other elements.

v. Other Elements Removed: Either produce statically determinant structures (figures B.1.5, B.1.7, B.1.8 and B.1.9) or else are identical to one of the cases above.

B.2 MOSES' UNZIPPING TECHNIQUE

Load Increments:

Path 1 - EF,CD,DF:

$$\begin{cases} r_{1,1} = r_{EF} = {}^{7}/_{27}\ \delta R_{1,1} \\ r_{1,2} = r_{CD} = {}^{10}/_{27}\ \delta R_{1,1} + \frac{1}{2}\delta R_{1,2} \\ r_{1,3} = r_{DF} = {}^{20}/_{27}\ \delta R_{1,1} + \delta R_{1,2} + \delta R_{1,3} \end{cases}$$

$$\therefore \begin{cases} \delta R_{1,1} = {}^{27}/_{7}\ r_{EF} \\ \delta R_{1,2} = -{}^{20}/_{7}\ r_{EF} + 2r_{CD} \\ \delta R_{1,3} + \quad - 2r_{CD} + r_{DF} \end{cases}$$

Path 2 - EF,CD,AB: $\delta R_{2,1} = \delta R_{1,1}$: $\delta R_{2,2} = \delta R_{1,2}$

$$r_{2,3} = r_{AB} = {}^{10}/_{27}\ \delta R_{2,1} + \frac{1}{2}\delta R_{2,2} + \delta R_{2,3}$$

$$\therefore \delta R_{2,3} = \quad - r_{CD} + r_{AB}$$

Path 3 - EF,DF: $SR_{3,1} = SR_{1,1}$

$$r_{3,2} = r_{DF} = {}^{20}/_{27}\, SR_{3,1} + SR_{3,2}$$

$$\therefore SR_{3,2} = -{}^{20}/_{7}\, r_{EF} + r_{DF}$$

Path 4 - EF,AB,CD: $SR_{4,1} = SR_{1,1}$

$$\begin{cases} r_{4,2} = r_{AB} = {}^{10}/_{27}\, SR_{4,1} + \frac{1}{2}\, SR_{4,2} \\ r_{4,3} = r_{CD} = {}^{10}/_{27}\, SR_{4,1} + \frac{1}{2}\, SR_{4,2} + SR_{4,3} \end{cases}$$

$$\therefore \begin{cases} SR_{4,2} = -{}^{20}/_{7}\, r_{EF} + 2r_{AB} \\ SR_{4,3} = \quad - r_{AB} + r_{CD} \end{cases}$$

Path 5 - EF,AB,DF: $SR_{5,1} = SR_{1,1}$: $SR_{5,2} = SR_{4,2}$

$$r_{5,3} = r_{DF} = {}^{20}/_{27}\, SR_{5,1} + SR_{5,2} + SR_{5,3}$$

$$\therefore SR_{5,3} = \quad - 2r_{AB} + r_{DF}$$

Path 6 - CD,EF,DF:

$$\begin{cases} r_{6,1} = r_{CD} = {}^{10}/_{27}\, SR_{6,1} \\ r_{6.2} = r_{EF} = {}^{7}/_{27}\, SR_{6,1} + {}^{3}/_{8}\, SR_{6,2} \\ r_{6,3} = r_{DF} = {}^{20}/_{27}\, SR_{6,1} + \frac{5}{8}\, SR_{6,2} + SR_{6,3} \end{cases}$$

$$\therefore \begin{cases} SR_{6,1} = {}^{27}/_{10}\, r_{CD} \\ SR_{6,2} = -{}^{28}/_{15}\, r_{CD} + {}^{8}/_{3}\, r_{EF} \\ SR_{6,3} = -{}^{5}/_{6}\, r_{CD} - {}^{5}/_{3}\, r_{EF} + r_{DF} \end{cases}$$

Path 7 - CD,EF,AB: $SR_{7,1} = SR_{6,1}$: $SR_{7,2} = SR_{6,2}$

$$r_{7,3} = r_{AB} = {}^{10}/_{27}\, SR_{7,1} + \frac{5}{8}\, SR_{7,2} + SR_{7,3}$$

$$\therefore SR_{7,3} = {}^{1}/_{6}\, r_{CD} - {}^{5}/_{3}\, r_{EF} + r_{AB}$$

Path 8 - CD,DF,EF: $SR_{8,1} = SR_{6,1}$

$$\begin{cases} r_{8,2} = r_{DF} = {}^{20}/_{27}\, SR_{8,1} + {}^{5}/_{8}\, SR_{8,2} \\ r_{8,3} = r_{EF} = {}^{7}/_{27}\, SR_{8,1} + \frac{3}{8}\, SR_{8,2} + SR_{8,3} \end{cases}$$

$$\therefore \begin{cases} SR_{8,2} = -{}^{16}/_{5}\, r_{CD} + {}^{8}/_{5}\, r_{DF} \\ SR_{8,3} = \frac{1}{2}\, r_{CD} - {}^{3}/_{5}\, r_{DF} + r_{EF} \end{cases}$$

Path 9 – CD, AB, EF: $SR_{9,1} = SR_{6,1}$

$$\begin{cases} r_{9,2} = r_{AB} = {}^{10}/_{27}\ SR_{9,1} + {}^{5}/_{8}\ SR_{9,2} \\ r_{9,3} = r_{EF} = {}^{7}/_{27}\ SR_{9,1} + {}^{3}/_{8}\ SR_{9,2} + SR_{9,3} \end{cases}$$

$$\therefore \begin{cases} SR_{9,2} = -{}^{8}/_{5}\ r_{CD} + {}^{8}/_{5}\ r_{AB} \\ SR_{9,3} = -{}^{1}/_{10}\ r_{CD} - {}^{3}/_{5}\ r_{AB} + r_{EF} \end{cases}$$

Path 10 – DF, EF:

$$\begin{cases} r_{10,1} = r_{DF} = {}^{20}/_{27}\ SR_{10,1} \\ r_{10,2} = r_{EF} = {}^{7}/_{27}\ SR_{10,1} + SR_{10,2} \end{cases}$$

$$\therefore \begin{cases} SR_{10,1} = {}^{27}/_{20}\ r_{DF} \\ SR_{10,2} = -{}^{7}/_{20}\ r_{DF} + r_{EF} \end{cases}$$

Path 11 – AB, EF, CD:

$$\begin{cases} r_{11,1} = r_{AB} = {}^{10}/_{27}\ SR_{11,1} \\ r_{11,2} = r_{EF} = {}^{7}/_{27}\ SR_{11,1} + {}^{3}/_{8}\ SR_{11,2} \\ r_{11,3} = r_{CD} = {}^{10}/_{27}\ SR_{11,1} + {}^{5}/_{8}\ SR_{11,2} + SR_{11,3} \end{cases}$$

$$\therefore \begin{cases} SR_{11,1} = {}^{27}/_{10}\ r_{AB} \\ SR_{11,2} = -{}^{28}/_{15}\ r_{AB} + {}^{8}/_{3}\ r_{EF} \\ SR_{11,3} = {}^{1}/_{6}\ r_{AB} - {}^{5}/_{3}\ r_{EF} + r_{CD} \end{cases}$$

Path 12 – AB, EF, DF: $SR_{12,1} = SR_{11,1}$: $SR_{12,2} = SR_{11,2}$

$$r_{12,3} = r_{DF} = {}^{20}/_{27}\ SR_{12,2} + {}^{5}/_{8}\ SR_{12,2} + SR_{12,3}$$

$$\therefore SR_{12,3} = -{}^{5}/_{6}\ r_{AB} - {}^{5}/_{3}\ r_{EF} + r_{DF}$$

Path 13 – AB, CD, EF: $SR_{13,1} = SR_{11,1}$

$$\begin{cases} r_{13,2} = r_{CD} = {}^{10}/_{27}\ SR_{13,1} + {}^{5}/_{8}\ SR_{13,2} \\ r_{13,3} = r_{EF} = {}^{7}/_{27}\ SR_{13,1} + {}^{3}/_{8}\ SR_{13,2} + SR_{13,3} \end{cases}$$

$$\therefore \begin{cases} SR_{13,2} = -{}^{8}/_{5}\ r_{AB} + {}^{8}/_{5}\ r_{CD} \\ SR_{13,3} = -{}^{1}/_{10}\ r_{AB} - {}^{3}/_{5}\ r_{CD} + r_{EF} \end{cases}$$

Path 14 = AB, DF, EF: $SR_{14,1} = SR_{11,1}$

$$\begin{cases} r_{14,2} = r_{DF} = {}^{20}/_{27}\, SR_{14,1} + {}^{5}/_{8}\, SR_{14,2} \\ r_{14,3} = r_{EF} = {}^{7}/_{27}\, SR_{14,1} + {}^{3}/_{8}\, SR_{14,2} + SR_{14,3} \end{cases}$$

$$\therefore \begin{cases} SR_{14,2} = -{}^{16}/_{5}\, r_{AB} + {}^{8}/_{5}\, r_{DF} \\ SR_{14,3} = \tfrac{1}{2}\, r_{AB} - {}^{3}/_{5}\, r_{DF} + r_{EF} \end{cases}$$

Probability of Failure Paths: (See Table B.2.1)

p(Failure Path 1) $\doteq \Phi(-3.52) = 0.22 \times 10^{-3}$

p(Failure Path 2) $\doteq \Phi(-3.65) = 0.13 \times 10^{-3}$

p(Failure Path 3) $\doteq \Phi(-3.52) = 0.22 \times 10^{-3}$

p(Failure Path 4) $\doteq \Phi(-4.22) = 0.13 \times 10^{-4}$

p(Failure Path 5) $\doteq \Phi(-4.22) = 0.13 \times 10^{-4}$

p(Failure Path 6) $\doteq \Phi(-3.52) = 0.22 \times 10^{-3}$

p(Failure Path 7) $\doteq \Phi(-3.65) = 0.13 \times 10^{-3}$

p(Failure Path 8) $\doteq \Phi(-3.95) = 0.39 \times 10^{-4}$

p(Failure Path 9) $\doteq \Phi(-4.32) = 0.80 \times 10^{-5}$

p(Failure Path 10) $\doteq \Phi(-3.94) = 0.42 \times 10^{-4}$

p(Failure Path 11) $\doteq \Phi(-4.74) = 0.11 \times 10^{-5}$

p(Failure Path 12) $\doteq \Phi(-4.74) = 0.11 \times 10^{-5}$

p(Failure Path 13) $\doteq \Phi(-4.74) = 0.11 \times 10^{-5}$

p(Failure Path 14) $\doteq \Phi(-4.74) = 0.11 \times 10^{-5}$

Probability of System Failure: $\doteq \Phi(-3.52) = 0.22 \times 10^{-3}$

B.3 MARSHALL AND BEA's BRITTLE REDUNDANT MODEL

i. S = 2.0

$$p(A_1\ s) = 1.0 - \prod_{\text{each element}} [1.0 - p(\text{element fails})]$$

$$= 1.0 - \left[1.0 - \Phi\left(-\frac{3.0 - 2.0 \times \frac{10}{27}}{0.3}\right)\right]^3 \times \left[1.0 - \Phi\left(-\frac{2.0 - 2.0 \times \frac{10}{27}}{0.2}\right)\right]^2 \times$$

$$\times \left[1.0 - \Phi\left(-\frac{5.0 - 2.0 \times \frac{20}{27}}{0.5}\right)\right] \times \left[1.0 - \Phi\left(-\frac{1.0 - 2.0 \times \frac{7}{27}}{0.1}\right)\right]^2$$

$$= 1.0 - \left[1.0 - \Phi(-7.53)\right]^3 \times \left[1.0 - \Phi(-6.30)\right]^2 \times \left[1.0 - \Phi(-7.04)\right] \times \left[1.0 - \Phi(-4.81)\right]^2$$

$$\doteq 3 \times 0.15 \times 10^{-13} + 2 \times 0.15 \times 10^{-9} - 0.8 \times 10^{-12} + 2 \times 0.755 \times 10^{-6}$$

$$\doteq 0.151 \times 10^{-5}$$

To maximise further failure probabilities, remove element DF

$$p(A_2 | A_1, s) = 1.0 - \left[1.0 - p(\text{EF fails})\right] \times \left[1.0 - p(\text{FG fails})\right]$$

$$= 1.0 - \left[1.0 - \Phi\left(-\frac{1.0 - 2.0}{0.1}\right)\right]^2 = 1.0 - \left[1.0 - \Phi(10.0)\right]^2$$

$$\doteq 1.0$$

$$p(F | s=2.0) = p(A_1 | s) \times p(A_2 | A_1, s) = 0.151 \times 10^{-5}$$

ii. s = 4.5

$$p(A_1 | s) = 1.0 - \left[1.0 - \Phi(-4.44)\right]^3 \times \left[1.0 - \Phi(-1.67)\right]^2 \times \left[1.0 - \Phi(-3.33)\right] \times \left[1.0 - \Phi(+1.67)\right]^2$$

$$= 1.0 - \left[1.0 - 0.45 \times 10^{-5}\right]^3 \times \left[1.0 - 0.048\right]^2 \times \left[1.0 - 0.43 \times 10^{-3}\right] \times \left[0.0479\right]^2$$

$$= 0.9979$$

To maximise further failure probabilities, remove element DF

$$p(A_2 | A_1, s) = 1.0 - \left[1.0 - \Phi(-10.0)\right]^2 = 1.0$$

$$p(F | s = 4.5) = p(A_1 | s)\, p(A_2 | A_1, s) = 0.998$$

B.4 MARSHALL AND BEA's DUCTILE REDUNDANT MODEL

Failure network - figure B.4.1

Assemblages a-e as in figure B.4.1

i. S = 2.0

$$\begin{cases} S_a = 2.0 \times \frac{10}{27} = 0.741 \\ \beta_a = -\Phi^{-1}\left(1.0 - \left[1.0 - \Phi\left(-\frac{3.0 - 0.741}{0.3}\right)\right]^3\right) = 7.3 \\ E(R_a) = E(R_{AB}) = 3.0 \\ \sigma(R_a) = (3.0 - 0.741)/7.3 = 0.31 \end{cases}$$

$$\begin{cases} S_b = 2.0 \times \frac{10}{27} = 0.741 \\ \beta_b = -\Phi^{-1}\left(1.0 - \left[1.0 - \Phi\left(-\frac{2.0 - 0.741}{0.2}\right)\right]^2\right) = 6.2 \\ E(R_b) = E(R_{CD}) = 2.0 \\ \sigma(R_b) = (2.0 - 0.741)/6.2 = 0.20 \end{cases}$$

$$\begin{cases} S_c = 2.0 \times \frac{7}{27} = 0.519 \\ \beta_c = -\Phi^{-1}\left(1.0 - \left[1.0 - \Phi\left(-\frac{1.0 - 0.519}{0.1}\right)^2\right]\right) = 4.67 \\ E(R_c) = E(R_{EF}) = 1.0 \\ \sigma(R_c) = (1.0 - 0.519)/4.67 = 0.103 \end{cases}$$

$$\begin{cases} S_d = S_a + S_b = 2 \times 0.741 = 1.482 \\ E(R_d) = E(R_a) + E(R_b) = 3.0 + 2.0 = 5.0 \\ \sigma(R_d) = \sigma(R_a) + \sigma(R_b) = 0.31 + 0.20 = 0.51 \\ \beta_d = \frac{E(R_d) - S_d}{\sigma(R_d)} = \frac{(5.0 - 1.482)}{0.51} = 6.9 \end{cases}$$

$$\begin{cases} S_e = S_d = S_{DF} = 1.482 \\ \beta_e = -\Phi^{-1}\left(1.0 - \left[1.0 - (-6.9) \times \left[1.0 - \left(-\frac{5.0 - 1.482}{0.5}\right)\right]\right]\right) = 6.85 \\ E(R_e) = \text{Min}\,(E(R_d), E(R_{DF})) = \text{Min}\,(5.0, 5.0) = 5.0 \\ \sigma(R_e) = (5.0 - 1.482)/6.85 = 0.514 \end{cases}$$

$$\beta_{sys} = \frac{\beta_c\,\sigma(R_c) + \beta_e\,\sigma(R_e)}{\sigma(R_c) + \sigma(R_e)} = \frac{4.67 \times 0.103 + 6.85 \times 0.514}{0.103 + 0.514}$$

$$= 6.49$$

$$p(F|s = 2.0) = \Phi(-\beta_{sys}) = 0.45 \times 10^{-10}$$

ii. S = 4.5

$$\begin{cases} S_a = 4.5 \times \frac{10}{27} = 1.67 \\ \beta_a = -\Phi^{-1}\left(1.0 - \left[1.0 - \Phi\left(-\frac{3.0 - 1.67}{0.3}\right)\right]^3\right) = 4.20 \\ E(R_a) = E(R_{AB}) = 3.0 \\ \sigma(R_a) = (3.0 - 1.67)/4.20 = 0.317 \end{cases}$$

$$\begin{cases} S_b = 4.5 \times \frac{10}{27} = 1.67 \\ \beta_b = -\Phi^{-1}\left(1.0 - \left[1.0 - \Phi\left(-\frac{2.0 - 1.67}{0.2}\right)\right]^2\right) = 1.31 \\ E(R_b) = E(R_{CD}) = 2.0 \\ \sigma(R_b) = (2.0 - 1.67)/1.31 = 0.254 \end{cases}$$

$$\begin{cases} S_c = 4.5 \times \frac{7}{27} = 1.17 \\ \beta_c = -\Phi^{-1}\left(1.0 - \left[1.0 - \Phi\left(-\frac{1.0 - 1.17}{0.1}\right)\right]^2\right) = -2.83 \\ E(R_c) = E(R_{EF}) = 1.0 \\ \sigma(R_c) = (1.0 - 1.17)/(-2.83) = 0.589 \end{cases}$$

$$\begin{cases} S_d = S_a + S_b = 2 \times 1.67 = 3.33 \\ E(R_d) = E(R_a) + E(R_b) = 3.0 + 2.0 = 5.0 \\ \sigma(R_d) = \sigma(R_a) + \sigma(R_b) = 0.317 + 0.254 = 0.571 \\ \beta_d = \frac{E(R_d) - S_d}{\sigma(R_d)} = \frac{5.0 - 3.33}{0.571} = 2.91 \end{cases}$$

$$\begin{cases} S_e = 3.33 \\ \beta_e = -\Phi^{-1}\left(1.0 - \left[1.0 - \Phi(-2.91)\right] \times \left[1.0 - \Phi\left(-\frac{5.0 - 3.33}{0.5}\right)\right]\right) = 2.84 \\ E(R_e) = \text{Min}\,(E(R_d), E(R_{DF})) = 5.0 \\ \sigma(R_e) = (5.0 - 3.33)/2.84 = 0.587 \end{cases}$$

$$\beta_{sys} = \frac{\beta_c\ \sigma(R_c) + \beta_e\ \sigma(R_e)}{\sigma(R_c) + \sigma(R_e)} = \frac{-2.83 \times 0.589 + 2.84 \times 0.587}{0.589 + 0.587}$$

$$= 0.0$$

$$p(F \mid s = 4.5) = \Phi(-\beta_{sys}) = 0.50$$

			CORRELATION COEFFICIENTS			
	$E(\ .\ )$	$\sigma(\ .\ )$	r_{AB}	r_{CD}	r_{DF}	r_{EF}
r_{AB}	3.0	0.30	1.0	0.3	0.3	0.3
r_{CD}	2.0	0.20	0.3	1.0	0.3	0.3
r_{DF}	5.0	0.50	0.3	0.3	1.0	0.3
r_{EF}	1.0	0.10	0.3	0.3	0.3	1.0
S	2.0	1.00	0.0	0.0	0.0	0.0

TABLE B.1.1 – ELEMENT PROPERTIES

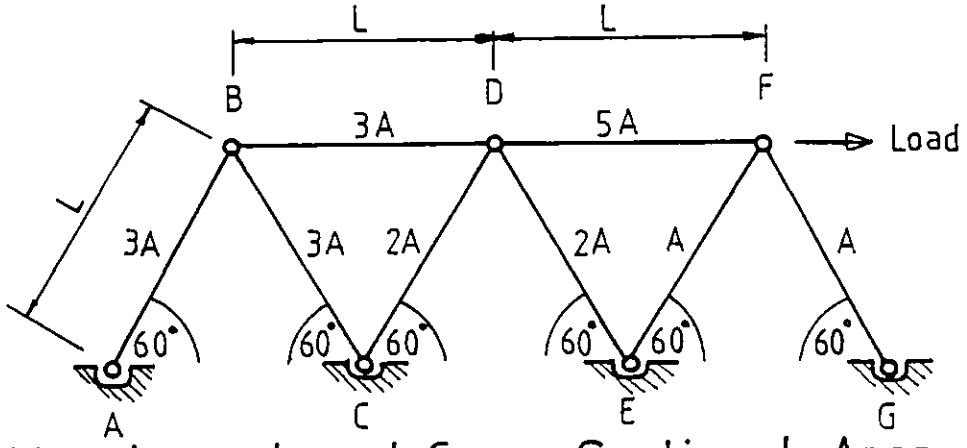


Fig.B.1.1 - Layout and Cross-Sectional Area of the Elements

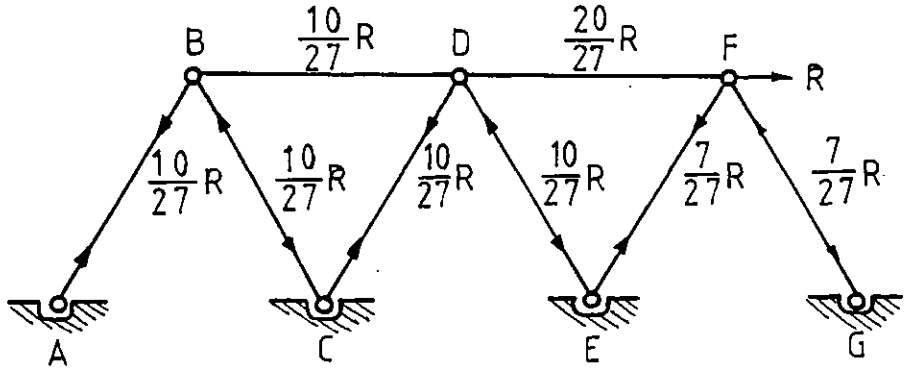


Fig. B.1.2 - Axial Forces in Elements due to the Initial Load

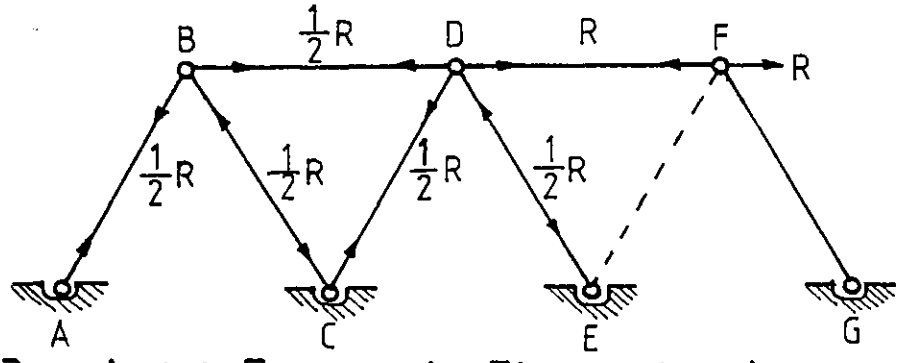


Fig.B.1.3 - Axial Forces in Elements due to the Load Increment after EF Fails

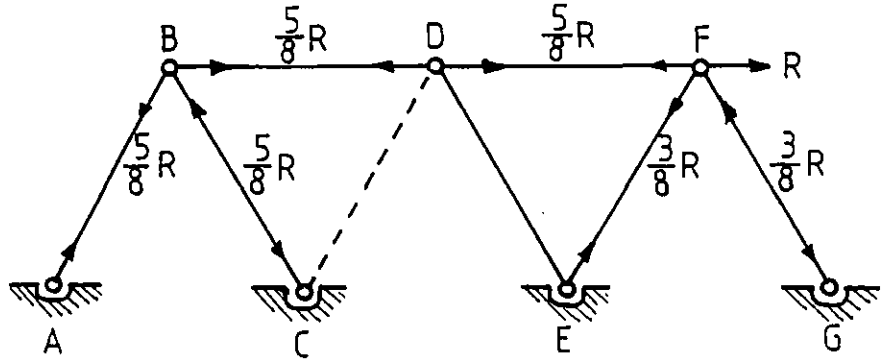


Fig.B.1.4 - Axial Forces in Elements due to the Load Increments after CD Fails

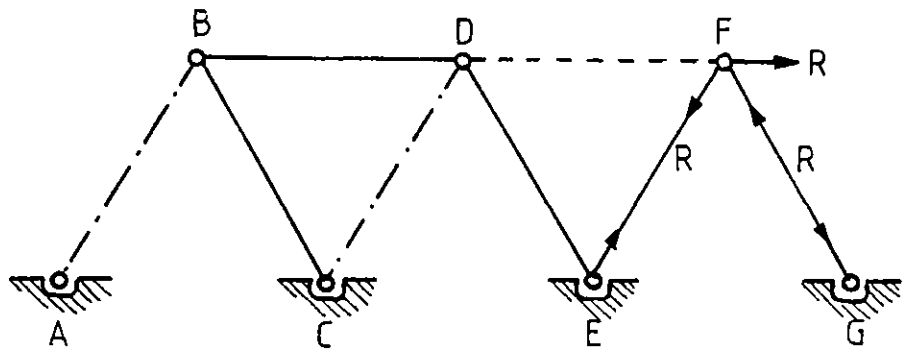


Fig.B.1.5 - Axial Forces in Elements due to the Load Increment after DF (& AB or CD) Fails

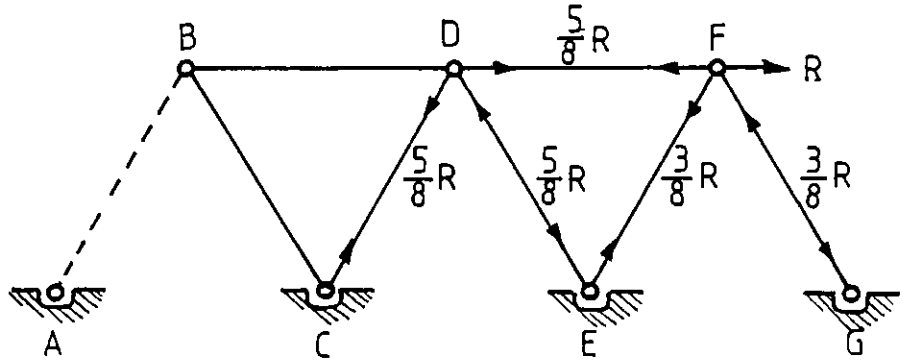


Fig.B.1.6 - Axial Forces in Elements due to the Load Increment after AB Fails

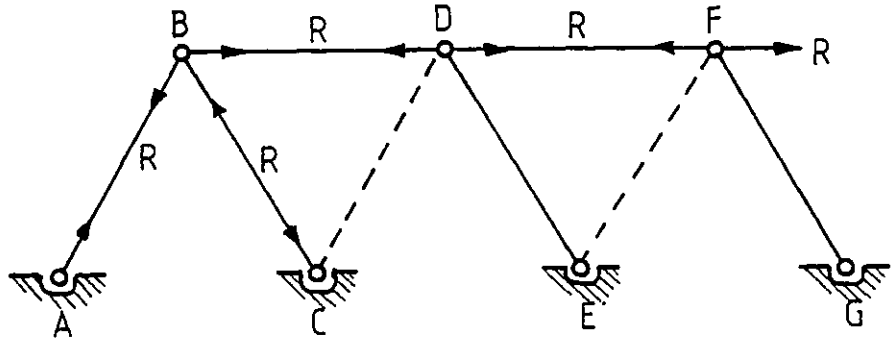


Fig.B.1.7 - Axial Forces in Elements due to the Load Increment after CD & EF Fail

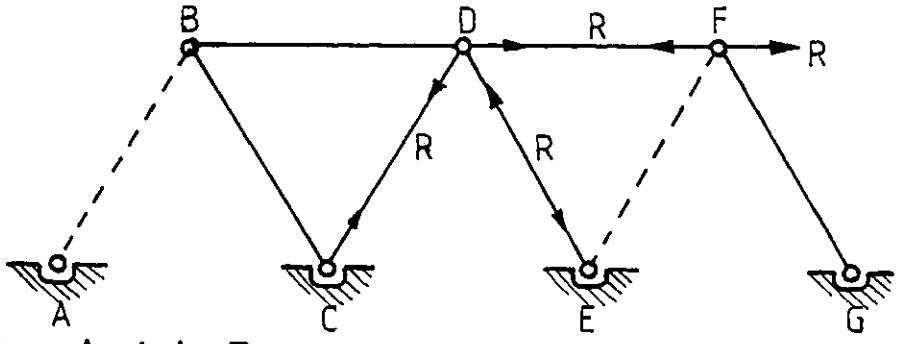


Fig.B.1.8 - Axial Forces in Elements due to the Load Increment after AB & EF Fail

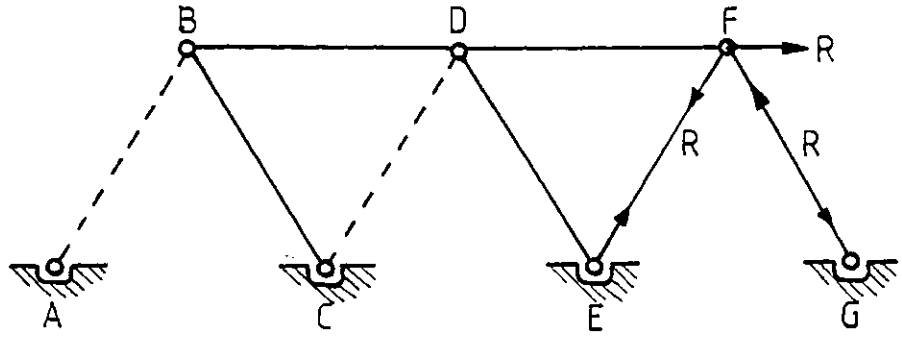


Fig.B.1.9 - Axial Forces in Elements due to the Load Increment after AB & CD Fail

TABLE B.2.1 – RESISTANCE PARAMETERS AND SAFETY INDICES FOR FAILURE PATHS

FAILURE PATH NO.	FAILURE NO.	FAILURE ELEMENT	$R_i = \sum \delta R_{i,j}$ r_{AB}	r_{CD}	r_{DF}	r_{EF}	MEAN RESISTANCE $E(R_i)$	STANDARD DEVIATION $\sigma(R_i)$	SAFETY INDEX β_i
1	1	EF	–	–	–	$^{27}/_{7}$	3.86	0.386	1.74
	2	CD	–	2	–	1	5.00	0.440	2.75
	3	DF	–	0	1	1	6.00	0.538	3.52
2	1	EF	–	–	–	$^{27}/_{7}$	3.86	0.386	1.74
	2	CD	–	2	–	1	5.00	0.440	2.75
	3	AB	1	1	–	1	6.00	0.454	3.65
3	1	EF	–	–	–	$^{27}/_{7}$	3.86	0.386	1.74
	2	DF	–	–	–	1	6.00	0.538	3.52
4	1	EF	–	–	–	$^{27}/_{7}$	3.86	0.386	1.74
	2	AB	2	–	–	1	7.00	0.638	4.22
	3	CD	1	1	–	1	6.00	0.454	3.65
5	1	EF	–	–	–	$^{27}/_{7}$	3.86	0.386	1.74
	2	AB	2	–	–	1	7.00	0.638	4.22
	3	DF	0	–	1	1	6.00	0.538	3.52
6	1	CD	–	$^{27}/_{10}$	–	$^{8}/_{3}$	5.40	0.540	2.99
	2	EF	–	$^{5}/_{6}$	–		4.33	0.354	2.20
	3	DF	–	0	1	1	6.00	0.538	3.52
7	1	CD	–	$^{27}/_{10}$	–	$^{8}/_{3}$	5.40	0.540	2.99
	2	EF	–	$^{5}/_{6}$	–		4.33	0.354	2.20
	3	AB	1	1	–	1	6.00	0.454	3.65
8	1	CD	–	$^{27}/_{10}$	$^{8}/_{5}$	–	5.40	0.540	2.99
	2	DF	–	$-^{1}/_{2}$		–	7.00	0.776	3.95
	3	EF	–	0	1	1	6.00	0.538	3.52
9	1	CD	$^{8}/_{5}$	$^{27}/_{10}$	–	–	5.40	0.540	2.99
	2	AB		$^{11}/_{10}$	–	–	7.00	0.584	4.32
	3	EF	1	1	–	1	6.00	0.454	3.65
10	1	DF	–	–	$^{27}/_{20}$	–	6.75	0.676	3.94
	2	EF	–	–	1	1	6.00	0.538	3.52
11	1	AB	$^{27}/_{10}$	–	–	$^{8}/_{3}$	8.10	0.810	4.74
	2	EF	$^{5}/_{6}$	–	–		5.17	0.842	2.43
	3	CD	1	1	–	1	6.00	0.454	3.65
12	1	AB	$^{27}/_{10}$	–	–	$^{8}/_{3}$	8.10	0.810	4.74
	2	EF	$^{5}/_{6}$	–	–		5.17	0.842	2.43
	3	DF	0	–	1	1	6.00	0.538	3.52
13	1	AB	$^{27}/_{10}$	$^{8}/_{5}$	–	–	8.10	0.810	4.74
	2	CD	$^{11}/_{10}$		–	–	6.50	0.524	3.96
	3	EF	1	1	–	1	6.00	0.454	3.65
14	1	AB	$^{27}/_{10}$	–	$^{8}/_{5}$	–	8.10	0.810	4.74
	2	DF	$-^{1}/_{2}$	–		–	6.50	0.768	3.57
	3	EF	0	–	1	1	6.00	0.538	3.52

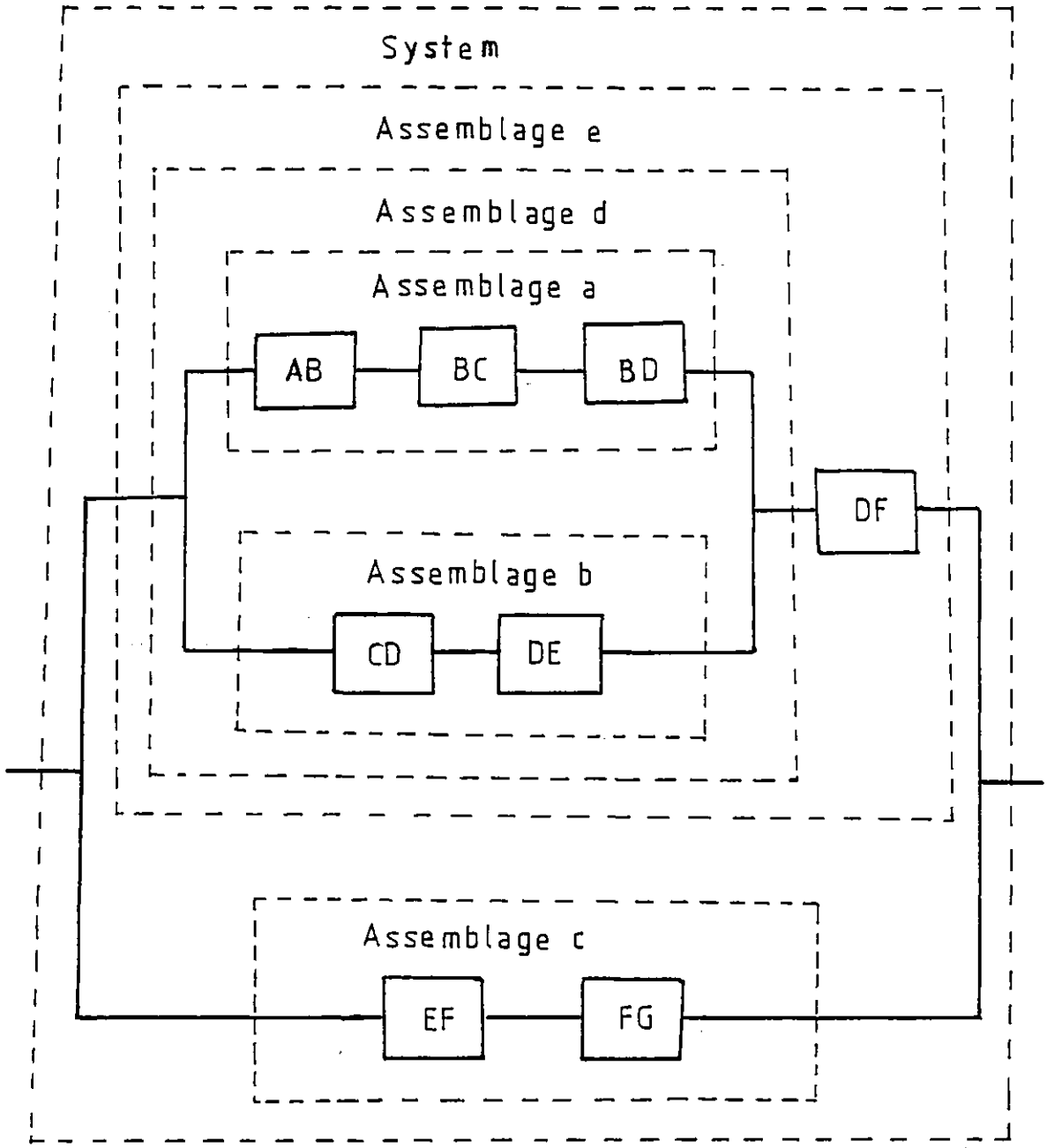


Fig.B.4.1 - Failure Network for Pin-Jointed Truss Showing Assemblages for Marshall and Bea's Ductile-Redundant Model

APPENDIX C - ESTIMATION OF RESISTANCE DISTRIBUTION PARAMETERS EXAMPLES

Design: Two 6.0m beams designed for office loading of 2.5 kNm^{-2} with width to depth ratio of $\frac{1}{3}$ and with 0% and 2% compression reinforcement [11].

Basic Variables: Tables C.1 and C.2. The basic variables which model the strength of each beam are assumed mutually uncorrelated. Some variables from each beam, however, are cross-correlated with variables from other beams to create correlation between the member resistances (Table C.1).

Local Failures Considered:

1. Beam 1 (0% compression reinforcement) failing in bending at mid-span.
2. Beam 1 (0% compression reinforcement) failing in shear at a support.
3. Beam 2 (2% compression reinforcement) failing in bending at mid-span.
4. Beam 2 (2% compression reinforcement) failing in shear at $\frac{1}{8}$ span from a support.

Bending Failure Criterion [7]:

$$D_i = \begin{cases} \mu_t + (d_b - E(d_b)) \times 10^3 + C_1 - C_i \text{ mm} & \text{for } i = 1, 2 \\ \mu_c - C_3 + C_i \text{ mm} & \text{for } i = 3, 4 \end{cases}$$

γ_i is found from fig C.1 in Nmm^{-2} where:

$$\epsilon_i = 0.0035 \, (D_i - D_{NA})/D_{NA}$$

and where D_{NA} in mm is found iteratively such that

$$600 \, W \times D_{NA} \times f_{CU} \, (1 - \sqrt{f_{CU}}/43.05) \quad \text{N}$$

$$= \sum_{i=1}^{4} \left\{ \frac{\pi}{4} N_i \times d_i^2 \times \gamma_i \right\} \quad \text{N}$$

$$D_c = D_{NA} \left[\frac{(2.0 - \sqrt{f_{CU}}/14.35)^2 + 2.0}{12.0 - \sqrt{f_{CU}}/3.5875} \right] \text{ mm}$$

$$M_{ULT} = \sum_{i=j}^{4} \left\{ (D_i - D_C) \frac{\pi}{4} N_i \times d_i^2 \times \gamma_i \times 10^{-6} \right\} \text{ kNm}$$

$$M_{APP} = \tfrac{1}{8} L^2 \left[\Gamma \left(W \times u_n + d_s \times p \times 10^{-3} + f_{in} \right) + w_v \, d_b \, p \right] \text{ kNm}$$

$$Z = b_m \, M_{ULT} - M_{APP} \quad \text{kNm}$$

Shear Failure Criterion [7]:

$$A_{ST} = \sum_{i=1}^{2} \left\{ \frac{\pi}{4} N_i \times d_i^2 \right\} \quad mm^2$$

$$D_{ST} = \frac{\sum_{i=1}^{2} \left\{ \frac{\pi}{4} N_i \times d_i^2 \times [\mu_t + (d_b - E(d_b)) \times 10^3 + C - C_i] \right\}}{A_{ST}} \quad mm$$

$$A_{ST} = 0.1 \left[\frac{A_{ST}}{D_{ST} w_b} \right] \%$$

V_c in Nmm^{-2} is interpolated from table C.3 in terms of A_{ST} and f_{cu}

$$S_{ULT} = D_{ST} \left(500 \frac{\pi}{4} d_5^2 \times f_{YS,5}/S_V + 1.5 V_C \times w_b \right) \quad kN$$

$$S_{APP} = \tfrac{1}{2} L [\Gamma W u_n + r (\Gamma (d_5 p \times 10^{-3} + f_{in}) + w_b d_b p)] \quad kN$$

$$= s_f S_{ULT} - S_{APP} \quad kN$$

Analysis:

1. Mean resistance by equation (5.5.3)
2. Standard Deviations by equation (5.5.4) } Table C.4
3. Standard Deviations by equation (5.5.6)
4. Correlation Coefficients Table C.5

VARIABLE	SYMBOL	DISTRIBUTION TYPE	UNITS	BEAM 1			BEAM 2	
				MEAN	STANDARD DEVIATION	CORRELATION COEFFICIENT	MEAN	STANDARD DEVIATION
Beam Width	w_b	Normal	m	0.165	0.005	0.0	0.126	0.005
Beam Depth	d_b	Normal	m	0.496	0.0075	0.0	0.379	0.0075
Beam Span	L	Normal	m	6.000	0.010	0.5	6.000	0.010
Tributary Width Tension	Γ	Normal	m	1.500	0.010	0.5	1.500	0.010
Centroid Depth Compression	μ_t	Lognormal	mm	450.0	4.1	0.0	333.0	4.1
Centroid Depth	μ_c	Lognormal	mm	58.3	8.0	0.0	58.3	8.0
Link Spacing	S_v	Lognormal	mm	285.0	14.25	0.0	194.0	9.7
Slab Depth	d_s	Normal	mm	106.0	4.0	1.0	106.0	4.0
Floor Finishes	f_{in}	Normal	kN/m^2	1.0	0.15	1.0	1.0	0.15
Tension Bars, Diameter 1	d_1	Lognormal	mm	32.0	0.25	0.0	25.0	0.25
Tension Bars, Diameter 2	d_2	Lognormal	mm	20.0	0.25	0.0	25.0	0.25
Compression Bars, Diameter 1	d_3	Lognormal	mm	10.0	0.25	0.0	25.0	0.25
Compression Bars, Diameter 2	d_4	Lognormal	mm	8.0	0.25	0.0	20.0	0.25
Link Bar, Diameter	d_5	Lognormal	mm	6.0	0.25	0.0	5.0	0.21
Tension Bars, Yield Stress 1	$f_{ys,1}$	Lognormal	N/mm^2	477.4	41.0	0.0	477.4	41.0
Tension Bars, Yield Stress 2	$f_{ys,2}$	Lognormal	N/mm^2	477.4	41.0	0.0	477.4	41.0
Compression Bars, Yield Stress 1	$f_{ys,3}$	Lognormal	N/mm^2	477.4	41.0	0.0	477.4	41.0
Compression Bars, Yield Stress 2	$f_{ys,4}$	Lognormal	N/mm^2	477.4	41.0	0.0	477.4	41.0
Link Bars, Yield Stress	$f_{ys,5}$	Lognormal	N/mm^2	477.4	41.0	0.0	477.4	41.0
Concrete Density	ρ	Normal	kN/m^3	23.6	0.4	1.0	23.6	0.4
Concrete Strength	f_{cu}	Lognormal	N/mm^2	35.70	2.475	1.0	35.70	2.475
Bending Moment Model Uncertainty	b_m	Lognormal		1.00	0.05	0.0	1.00	0.05
Shear Force Model Uncertainty	s_f	Lognormal		1.279	0.270	0.0	1.343	0.284
Live Load Distribution Uncertainty	u_n	Normal		1.10	0.15	0.0	1.10	0.15
Live Load	W	Extreme Type I Maxima	kN/m^2	1.471	0.533	1.0	1.471	0.533

TABLE C.1 – BASIC VARIABLES FOR EXAMPLE

	SYMBOL	BEAM 1	BEAM 2
Number of Tension Bars 1	N_1	1	2
Number of Tension Bars 2	N_2	1	1
Number of Compression Bars 1	N_3	1	1
Number of Compression Bars 2	N_4	1	1
Cover to Tension Bars 1	C_1	48.0mm	48.0mm
Cover to Tension Bars 2	C_2	48.0mm	93.0mm
Cover to Compression Bars 1	C_3	48.0mm	48.0mm
Cover to Compression Bars 2	C_4	48.0mm	48.0mm
Shear Force Node Position	r	1.0	0.75

TABLE C.2 - CONSTANT FOR EXAMPLE

A_{ST} in %	CHARACTERISTIC CONCRETE STRENGTH, f_{cu} N/mm^2			
	20	25	30	40
0.25	0.35	0.35	0.35	0.35
0.50	0.45	0.50	0.55	0.55
1.00	0.60	0.65	0.70	0.75
2.00	0.80	0.85	0.90	0.95
3.00	0.85	0.90	0.95	1.00

TABLE C.3 - ULTIMATE SHEAR STRENGTH OF CONCRETE, V_c [7] IN N mm^{-2}

MEMBER RESISTANCE NUMBER	MEAN (Eqn (5.5.3))	STANDARD DEVIATIONS	
		TRADITIONAL (Eqn (5.5.4))	PROPOSED (Eqn (5.5.6))
1	22.11 kN/m^2	3.74 kN/m^2	2.86 kN/m^2
2	30.53 kN/m^2	8.78 kN/m^2	4.27 kN/m^2
3	18.66 kN/m^2	3.28 kN/m^2	2.56 kN/m^2
4	27.83 kN/m^2	8.19 kN/m^2	4.05 kN/m^2

TABLE C.4 - STANDARD DEVIATIONS OF THE RESISTANCE

RESISTANCE	1	2	3
2	0.6557		
3	0.7238	0.5483	
4	0.5504	0.4158	0.6956

TABLE C.5 - CORRELATION COEFFICIENTS OF THE RESISTANCES

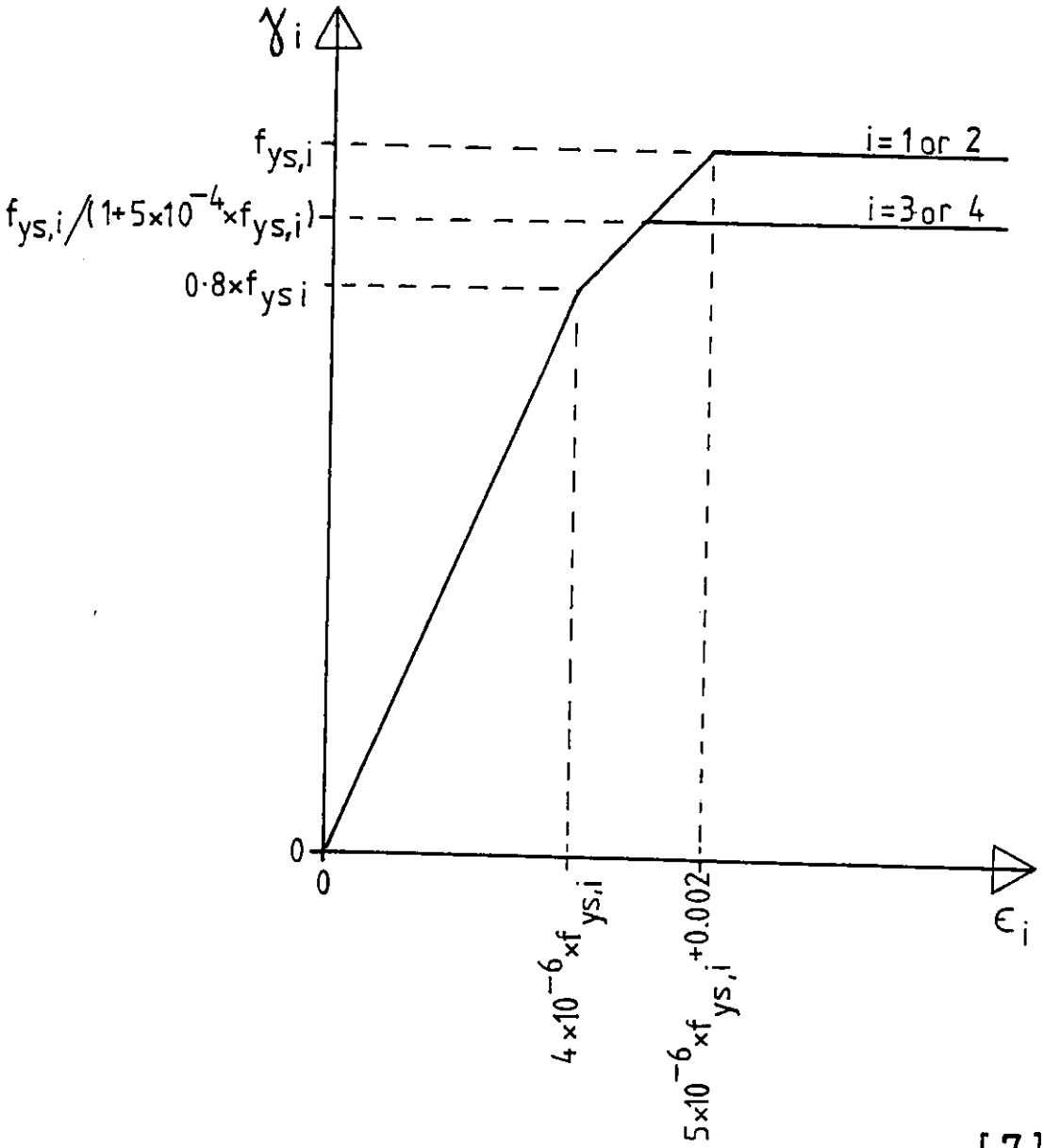


Fig.C.1 - Stress-Strain Curve for Steel [7]

APPENDIX D - DISTRIBUTION PARAMETERS FOR THE EXTREME VALUES OF LOADS

D.1 Normal Distribution

$$\begin{cases} F_{S^n_{max}}(s) = \Phi^n\left(\dfrac{s - E(S)}{\sigma(S)}\right) & \text{(D.1.1)} \\ f_{S^n_{max}}(s) = \left\{\dfrac{n}{\sqrt{2\pi}\,\sigma(S)}\right\} \exp\left(-\tfrac{1}{2}\left(\dfrac{s-E(S)}{(s)}\right)^2\right) \Phi^{(n-1)}\left(\dfrac{s-E(S)}{(S)}\right) & \text{(D.1.2)} \end{cases}$$

$$\begin{cases} F_{S^n_{max}}(s) = 1.0 - \left[1.0 - \Phi\left(\dfrac{s-E(S)}{\sigma(s)}\right)\right]^n & \text{(D.1.3)} \\ f_{S^n_{max}}(s) = \left\{\dfrac{n}{\sqrt{2\pi}\,\sigma(S)}\right\} \exp\left(-\tfrac{1}{2}\left(\dfrac{s-E(S)}{(S)}\right)^2\right)\left[1.0 - \Phi\left(\dfrac{s-E(S)}{(S)}\right)\right]^{(n-1)} & \text{(D.1.4)} \end{cases}$$

None of these functions can be solved algebraically, but it is known that as n tends to infinity these distributions will tend to the extreme type I maxima and minima respectively [6]. Therefore, if a normal and an extreme type I distribution are obtained which have the correct cumulative distributions at two points their values of mean and standard deviation should act as bounds on the true values.

Fitting a normal distribution at $s = E(S)$ and $s = E(S^n_{max})$ or $E(S^n_{max})$,

$$\begin{cases} \Phi\left(\dfrac{E(S) - E(S^n_{max})}{\sigma(S^n_{max})}\right) = \Phi^n\left(\dfrac{E(S) - E(S)}{\sigma(S)}\right) = 0.5^n & \text{(D.1.5)} \\ \Phi\left(\dfrac{E(S^n_{max}) - E(S^n_{max})}{\sigma(S^n_{max})}\right) = 0.5 = \Phi^n\left(\dfrac{E(S^n_{max})-E(S)}{\sigma(S)}\right) & \text{(D.1.6)} \end{cases}$$

Therefore

$$\begin{cases} E(S^n_{max}) = E(S) + \sigma(S) \times \Phi^{-1}(0.5^{1/n}) & \text{(D.1.7)} \\ \sigma(S^n_{max}) = -\sigma(S) \times \dfrac{\Phi^{-1}(0.5^{1/n})}{\Phi^{-1}(0.5^n)} & \text{(D.1.8)} \end{cases}$$

and

$$\begin{cases} \Phi\left(\dfrac{E(S) - E(S^n_{min})}{\sigma(S^n_{min})}\right) = 1.0 - \left[1.0 - \Phi\left(\dfrac{E(S)-E(S)}{\sigma(S)}\right)\right]^n = 1.0-0.5^n & \text{(D.1.9)} \\ \Phi\left(\dfrac{E(S^n_{min})-E(S^n_{min})}{\sigma(S^n_{min})}\right) = 0.5 = 1.0 - \left[1.0 - \Phi\left(\dfrac{E(S^n_{min}) - E(S)}{\sigma(S)}\right)\right]^n & \text{(D.1.10)} \end{cases}$$

$$\text{Therefore} \begin{cases} E(S^n_{min}) = E(S) - \sigma(S)\,\Phi^{-1}(0.5^{1/n}) & \text{(D.1.11)} \\ \sigma(S^n_{min}) = -\sigma(S)\,\dfrac{\Phi^{-1}(0.5^{1/n})}{\Phi^{-1}(0.5^n)} & \text{(D.1.12)} \end{cases}$$

Fitting an extreme type I maxima or minima distribution at s = E(S) and S = u = $E(S^n_{max}) - (\sqrt{2}\gamma/\pi)\sigma(S^n_{max})$ or $E(s^n_{min}) + (\sqrt{2}\gamma/\pi)\,\sigma(S^n_{min})$ gives

$$\begin{cases} \exp[-\exp[-\alpha(E(S) - u)]] = \Phi^n\left(\dfrac{E(S) - E(S)}{\sigma(S)}\right) = 0.5^n & \text{(D.1.13)} \\ \exp[-\exp[-\alpha(u-u)]] = e^{-1} = \Phi^n\left(\dfrac{u-E(S)}{\sigma(S)}\right) & \text{(D.1.14)} \end{cases}$$

$$\text{Therefore} \begin{cases} u = E(S) + \sigma(S)\,\Phi^{-1}(e^{-1/n}) & \text{(D.1.15)} \\ \alpha = \dfrac{\ln(-\ln(0.5^n))}{\sigma(S)\,\Phi^{-1}(e^{-1/n})} & \text{(D.1.16)} \end{cases}$$

$$\text{and hence} \begin{cases} E(S^n_{max}) = E(s) + \sigma(S)\,\Phi^{-1}(e^{-1/n})\left[1.0 + \dfrac{\gamma}{\ln(-\ln(0.5^n))}\right] & \text{(D.1.17)} \\ \sigma(S^n_{max}) = \sigma(S)\,\dfrac{\pi\,\Phi^{-1}(e^{-1/n})}{\sqrt{6}\ln(-\ln(0.5^n))} & \text{(D.1.18)} \end{cases}$$

$$\text{and} \begin{cases} 1.0-\exp[-\exp[-\alpha(E(S) - u)]] = 1.0-\left[1.0-\Phi\left(\dfrac{E(S) - E(S)}{(S)}\right)\right]^n = 1.0-0.5^n & \text{(D.1.19)} \\ 1.0-\exp[-\exp[-\alpha(u-u)]] = 1.0 - e^{-1} = 1.0 -\left[1.0-\Phi\left(\dfrac{u-E(S)}{(S)}\right)\right]^n & \text{(D.1.20)} \end{cases}$$

$$\text{Therefore} \begin{cases} u = E(S) - \sigma(S)\,\Phi^{-1}(e^{-1/n}) & \text{(D.1.21)} \\ \alpha = \dfrac{\ln(-\ln(0.5^n))}{\sigma(S)\,\Phi^{-1}(e^{-1/n})} & \text{(D.1.22)} \end{cases}$$

$$\text{and hence} \begin{cases} E(S^n_{min}) = E(S) - \sigma(S)\,\Phi^{-1}(e^{-1/n})\left(1.0 + \dfrac{\gamma}{\ln(-\ln(0.5^n))}\right) & \text{(D.1.23)} \\ \sigma(S^n_{max}) = \sigma(S)\,\dfrac{\pi\,\Phi^{-1}(e^{-1/n})}{\sqrt{6}\ln(-\ln(0.5^n))} & \text{(D.1.24)} \end{cases}$$

From consideration of eqns. (D.1.7), (D.1.8), (D.1.11), (D.1.12), D.1.17) (D.1.18), (D.1.23) and (D.1.24), the mean and standard deviation of the extreme values can be put into the form

$$\left.\begin{aligned} E(S^n_{max}) &= E(S) + k_1\,\sigma(S) \\ E(S^n_{min}) &= E(S) - k_1\,\sigma(S) \\ \sigma(S^n_{max}) &= \sigma(S^n_{min}) = k_2\,\sigma(S) \end{aligned}\right\} \qquad (D.1.25)$$

where k_1 and k_2 are independent of E(S) and $\sigma(S)$. Therefore, k_1 and k_2 will be the mean and standard deviation respectively of n standardised normal distributions (where E(S) = 0.0 and $\sigma(S) = 1.0$). Hence these values can be obtained by numerical integration from

$$\left.\begin{aligned} k_1 &= \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u \exp(-\tfrac{1}{2} u^2)\ \Phi^{(n-1)}(u)\,du \\ k_2 &= \left[\frac{n}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u^2 \exp(-\tfrac{1}{2} u^2)\ \Phi^{(n-1)}(u)\,du - k_1^2 \right]^{\frac{1}{2}} \end{aligned}\right\} \qquad (D.1.26)$$

These values, together with the bounds, are plotted in figs. D.1.1. and D.1.2.

D.2 Extreme Type 1 Distribution

$$\begin{aligned} F_{S^n_{max}}(s) &= (\exp(-\exp(-\alpha(s-u))))^n \\ &= \exp(-n \exp(-\alpha(s-u))) \\ &= \exp\left(-\exp\left(-\alpha\left(s - \left\{u + \frac{\ln(n)}{\alpha}\right\}\right)\right)\right) \end{aligned} \qquad (D.2.1)$$

Therefore this distribution of extreme values is also extreme type I maxima with

$$\left\{\begin{aligned} \alpha^n_{max} &= \alpha \qquad (D.2.2.) \\ u^n_{max} &= u + \frac{\ln(n)}{\alpha} \qquad (D.2.3) \end{aligned}\right.$$

Therefore

$$\left\{\begin{aligned} \sigma(S^n_{max}) &= \sigma(S) \qquad (D.2.4) \\ E(S^n_{max}) &= E(S) + \sigma(S)\ \frac{\sqrt{6}\ \ln(n)}{\pi} \qquad (D.2.5) \end{aligned}\right.$$

$$\begin{aligned} F_{S^n_{min}}(s) &= 1.0 - (1.0 - [1.0 - \exp(-\exp(\alpha(s-u))]\,)^n \\ &= 1.0 - \exp(-n \exp(\alpha(s-u))) \\ &= 1.0 - \exp\left(-\exp\left(\alpha\left(s - \left\{u - \frac{\ln(n)}{\alpha}\right\}\right)\right)\right) \end{aligned} \qquad (D.2.6)$$

Therefore this distribution of extreme values is also extreme Type I minima with

$$\begin{cases} \alpha^{n}_{min} = \alpha & (D.2.7) \\ u^{n}_{min} = u - \frac{\ln(n)}{\alpha} & (D.2.8) \end{cases}$$

Therefore

$$\begin{cases} \sigma(S^{n}_{min}) = \sigma(S) & (D.2.9) \\ E(S^{n}_{min}) = E(S) - \sigma(S) \times \frac{\sqrt{6}\ln(n)}{\pi} & (D.2.10) \end{cases}$$

NB: Only the maximum of n samples from the maxima distribution and the minimum of n samples from the minima are considered here.

D.3 Extreme Type II and III Distributions

$$F_{S^{n}_{max}}(s) = \left[\exp\left(-\left(\frac{u}{s}\right)^{k}\right)\right]^{n}$$

$$= \exp\left(-n\left(\frac{u}{s}\right)^{k}\right)$$

$$= \exp\left(-\left(\frac{n^{1/k}u}{s}\right)^{k}\right) \qquad (D.3.1)$$

Therefore this distribution of extreme values is also extreme type II maxima with

$$\begin{cases} E(S^{n}_{max}) = n^{1/k} E(S) & (D.3.2) \\ \sigma(S^{n}_{max}) = n^{1/k}\sigma(S) & (D.3.3) \end{cases}$$

where k is found from solving the equation

$$\left(\frac{\sigma(S)}{E(S)}\right)^{2} = \frac{\Gamma\left(1.0 - \frac{2.0}{k}\right)}{\Gamma^{2}\left(1.0 - \frac{1.0}{k}\right)} - 1.0 \qquad \text{for } k > 2 \qquad (D.3.4)$$

$$F_{S^{n}_{min}}(s) = 1.0 - \left[1.0 - \left\{1.0 - \exp\left(-\frac{s-\epsilon}{u-\epsilon}\right)^{k}\right\}\right]^{n}$$

$$= 1.0 - \exp\left(-n\left(\frac{s-\epsilon}{u-\epsilon}\right)^{k}\right)$$

$$= 1.0 - \exp\left(-\left[\frac{s-\epsilon}{\left\{\frac{u-\epsilon}{n^{1/k}} + \epsilon\right\}}\right]^{k}\right) \qquad (D.3.5)$$

Therefore, this distribution of extreme values is also extreme type III minima with

$$E(S^n_{min}) = \frac{E(S) - \epsilon}{n^{1/k}} + \epsilon \qquad (D.3.6)$$

$$\sigma(S^n_{min}) = \frac{\sigma(S)}{n^{1/k}} \qquad (D.3.7)$$

where ϵ is the minimum value that S can take and k is found from solving the equation

$$\left(\frac{\sigma(S)}{E(S) - \epsilon}\right)^2 = \frac{\Gamma\left(1.0 + \frac{2.0}{k}\right)}{\Gamma^2\left(1.0 + \frac{1.0}{k}\right)} - 1.0 \qquad (D.3.8)$$

NB: Only the maximum of n samples from the maxima distribution and the minimum of n samples from the minima distribution are considered here.

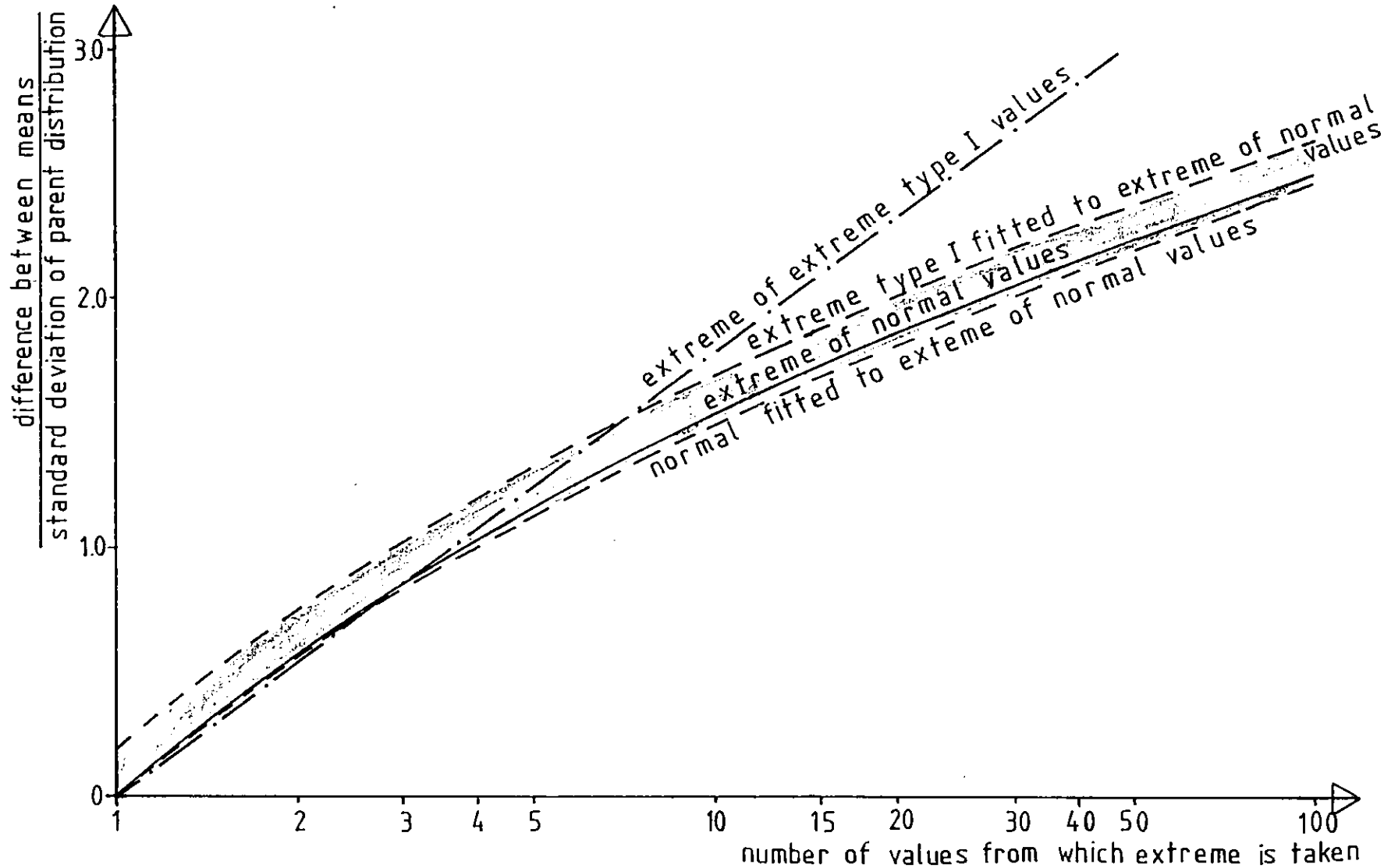


Fig. D.1.1 - Difference between Means of Extreme and Parent Distributions

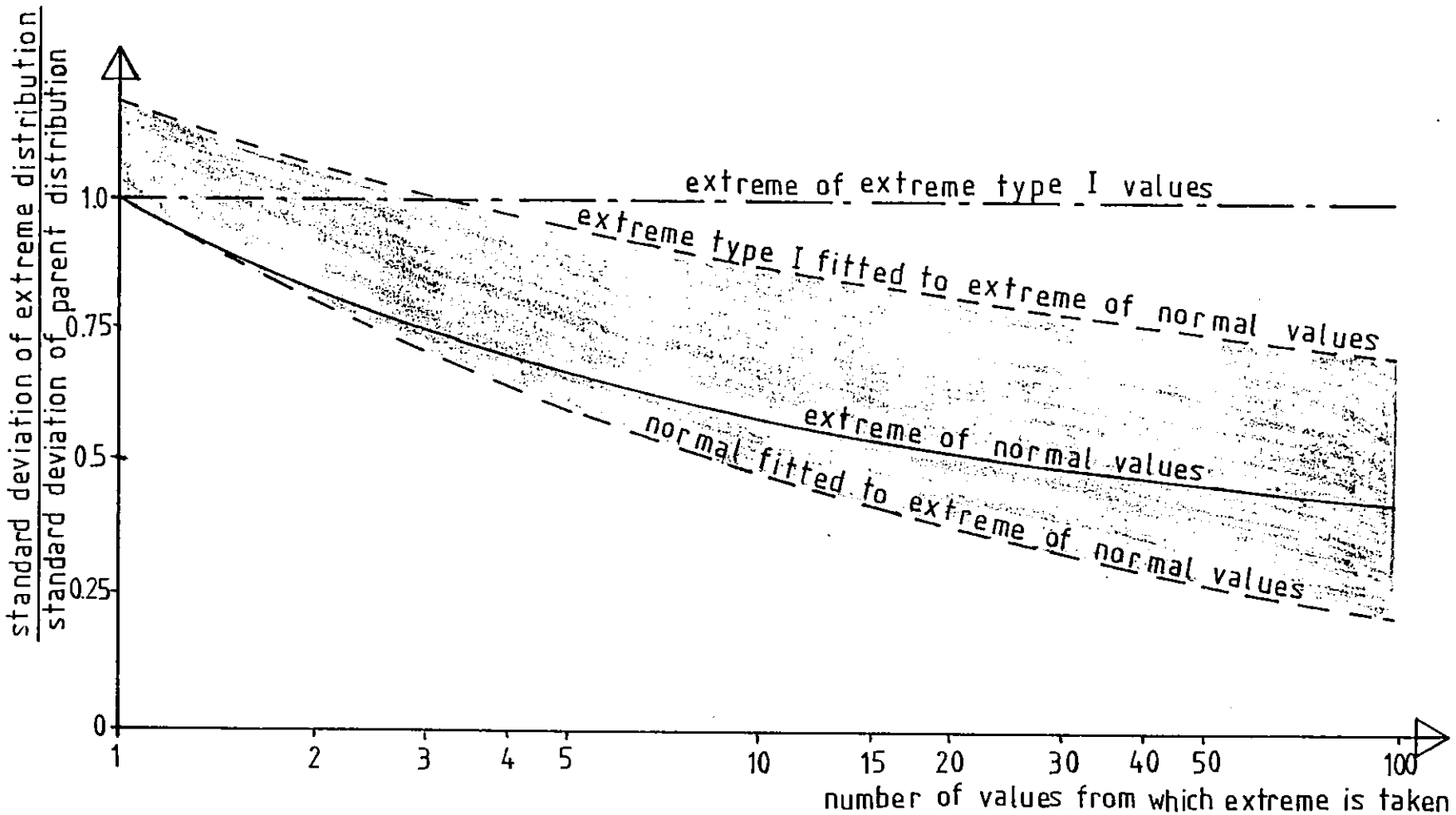


Fig. D.1.2 - Ratio between Standard Deviations of Extreme and Parent Distributions

APPENDIX E - CORRELATION COEFFICIENT FOR THE MAXIMUM AND MINIMUM LOAD INTENSITIES FROM 'N' TIME INTERVALS

$$\rho(S^n_{MAX}, S^n_{MIN}) = \frac{E(S^n_{MAX} \cdot S^n_{MIN}) - E(S^n_{MAX})\, E(S^n_{MIN})}{\sigma(S^n_{MAX})\, \sigma(S^n_{MIN})}$$

$$= \frac{\frac{1}{2}E((S^n_{MAX})^2 + (S^n_{MIN})^2 - (S^n_{MAX} - S^n_{MIN})^2) - E(S^n_{MAX})\, E(S^n_{MIN})}{\sigma(S^n_{MAX})\, \sigma(S^n_{MIN})}$$

$$= \frac{\sigma^2(S^n_{MAX}) + \sigma^2(S^n_{MIN}) - \sigma^2(T_n)}{2\sigma(S^n_{MAX})\, \sigma(S^n_{MIN})} \qquad \ldots\ldots \text{(E.1)}$$

where $T_n = S^n_{MAX} - S^n_{MIN}$. For values of $\sigma(S^n_{MAX})$ and $\sigma(S^n_{MIN})$ see Appendix D. To obtain $\sigma(T_n)$, consider the distribution of t_n conditional on S^n_{MAX}, where

$$F_{T_n | S^n_{MAX}}(t_n) = 1.0 - \left[1.0 - \frac{F_S(s-t_n)}{F_S(s)}\right]^{(n-1)} \Bigg|_{s\, =\, s^n_{MAX}} \qquad \ldots\ldots \text{(E.2)}$$

Therefore

$$f_{T_n | S^n_{MAX}}(t_n) = \frac{\partial F_{T_n | S^n_{MAX}}(t_n)}{\partial t} = (n-1)\left[1.0 - \frac{F_S(s-t)}{F_S(s)}\right]^{(n-2)} f_S(s-t_n) \Bigg|_{s\, =\, s^n_{MAX}}$$

$\ldots\ldots$ (E.3)

but

$$f_{S^n_{MAX}}(s) = n\left[F_S(s)\right]^{(n-1)} f_S(s) \qquad \ldots\ldots \text{(E.4)}$$

Therefore

$$f_{S^n_{MAX}, T_n}(s, t_n) = f_{S^n_{MAX}}(s)\, F_{T_n | S^n_{MAX}}(t_n)$$

$$= n\left[F_S(s)\right]^{(n-1)} f_S(s)\, (n-1)\left[1.0 - \frac{F_S(s-t_n)}{F_S(s)}\right]^{(n-2)} f_S(s-t_n)$$

$$= n(n-1) F_S(s) \left[F_S(s) - F_S(s-t_n)\right]^{(n-2)} f_S(s)\, f_S(s-t_n) \qquad \ldots\ldots \text{(E.5)}$$

Integrating $f_{S^n_{MAX},\, T_n}(s, t_n)$ by s to obtain the marginal distribution of T_n, $f_{T_n}(t_n)$, gives

$$f_{T_n}(t_n) = \int_{-\infty}^{+\infty} f_{S^n_{MAX},\, T_n}(s, t_n)\, ds$$

$$= n(n-1) \int_{-\infty}^{+\infty} F_S(s)\ \ F_S(s) - F_S(s-t_n)\ \ ^{(n-2)} f_S(s)\, f_S(s-t)\, ds \qquad \ldots\ldots \text{(E.6)}$$

Therefore

$$\begin{cases} E(T_n) = n(n-1) \int_{-\infty}^{+\infty} t_n F_S(s) \left[F_S(s) - F_S(s-t_n) \right]^{(n-2)} f_S(s) f_S(s-t_n) \, ds \, dt_n \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots \text{(E.7)} \\ E(T_n^2) = n(n-1) \int_{-\infty}^{+\infty} t_n^2 F_S(s) \left[F_S(s) - F_S(s-t_n) \right]^{(n-2)} f_S(s) f_S(s-t_n) \, ds \, dt_n \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots \text{(E.8)} \end{cases}$$

and

$$\sigma(T_n) = \sqrt{E(T_n^2) - E^2(T_n)} \qquad \ldots\ldots \text{(E.9)}$$

Therefore the values for $\sigma(T_n)$ will usually have to be obtained by numerical integration, and the general case for most distribution will not be able to be expressed as a simple function of $E(S)$, $\sigma(S)$ and n. Therefore consider the simplified case of eqn (E.1) where S is symmetrically distributed about the mean, so that

$$\sigma(S_{MAX}^n) = \sigma(S_{MIN}^n) \qquad \ldots\ldots \text{(E.10)}$$

and therefore

$$\rho(S_{MAX}^n, S_{MIN}^n) = 1.0 - \frac{\sigma^2(T_n)}{2\sigma^2(S_{MAX}^n)} \qquad \ldots\ldots \text{(E.11)}$$

Further, if S has a normal distribution, the distribution of T will also be normal, where t^* is the difference between two samples of s, with

$$\begin{cases} E(T^*) = E(S) - E(S) = 0.0 \qquad \ldots\ldots \text{(E.12)} \\ \sigma(T^*) = \sqrt{\sigma^2(S) + \sigma^2(S)} = \sqrt{2}\,\sigma(S) \qquad \ldots\ldots \text{(E.13)} \end{cases}$$

but

$$T_2 = |T^*| \qquad \ldots\ldots \text{(E.14)}$$

so that

$$f_{T_2}(t_2) = \frac{1.0}{\sqrt{\pi}\,\sigma(S)} \exp\left(-\left(\frac{t_2}{2\sigma(S)}\right)^2\right)$$
$$\text{for } 0 \leqslant t_2 \leqslant \infty \qquad \ldots\ldots \text{(E.15)}$$

Hence

$$E(T_2) = \int_0^{\infty} \frac{t_2}{\sqrt{\pi}\,\sigma(S)} \exp\left(-\left(\frac{t_2}{2\sigma(S)}\right)^2\right) dt_2$$

$$\ldots\ldots \text{(E.16)}$$

and

$$E(T_2^2) = E(T^{*2}) = 2\sigma^2(S) \qquad \ldots\ldots \text{(E.17)}$$

Therefore

$$\sigma(T_2) = \sqrt{2\sigma^2(S) - \left(\frac{2}{\sqrt{\pi}}\,\sigma(S)\right)^2} = \sigma(S)\sqrt{2 - \frac{4}{\pi}} \qquad \ldots\ldots \text{(E.18)}$$

Subsituting eqn (C.1.25) with n = 2 and eqn (E.18) into eqn (E.11) gives

$$\rho(S^2_{MAX},\ S^2_{MIN}) = 1.0 - \frac{\sigma^2(S)\ (2 - \frac{4}{\pi})}{2\left[0.8256\,\sigma(S)\right]^2} = 0.4669 \qquad \ldots\ldots \text{(E.19)}$$

A second value of $(S^n_{MAX},\ S^n_{MIN})$ can be found by simple logic as

$$\rho(S^1_{MAX},\ S^1_{MIN}) = 1.0 \qquad \ldots\ldots \text{(E.20)}$$

since the maximum from a sample of unit population will also be its minimum. The correlation coefficient represents the effect that the upper tail of the minima distribution has upon the lower tail of the maxima distribution, and vice versa. This effect results from the maxima distribution not being able to have a lower value than the minima distribution, so that if the minima is known, the maxima's lower tail will be truncated at that point. As n tends to infinity, the overlap of the two distributions will disappear and $(S^n_{MAX},\ S^n_{MIN})$ will tend to zero. Therefore

$$\rho(S^\infty_{MAX},\ S^\infty_{MIN}) = 0.0 \qquad \ldots\ldots \text{(E.21)}$$

Using these three values to give a simple expression with which to approximate $\rho(S^n_{MAX},\ S^n_{MIN})$ for variables having normal distributions gives

$$\rho(S^n_{MAX},\ S^n_{MIN}) \doteq \frac{0.8678}{n} + \frac{0.1322}{n^2} \qquad \ldots\ldots \text{(E.22)}$$

This should be adequate since it will quickly tail off to zero as n increases.

APPENDIX F - DISTRIBUTION PARAMETERS WITH TWO TIME-VARYING LOADS

For an element which has a failure criterion

$$Z_{i,1} = R_{i,1} - \sum_{k=1}^{n_s} \left[d_{i,1,k} \times S_k \right] \tag{F.1}$$

let the event of failure under the two load combinations, resulting from two of the loads being time-variant, be $a_{i,1}$ and $b_{i,1}$. Since the loads are assumed to be mutually independent

$$p(a_{i,1}) = \Phi \left[- \frac{E(R_{i,1}) - \sum_{k=1}^{n_s} \left[d_{i,1,k} \, E((S^{n_a}_{MAX})_k) \right]}{\sqrt{\sigma^2(R_{i,1}) + \sum_{k=1}^{n_s} \left[d^2_{i,1,k} \, \sigma^2((S^{n_a}_{MAX})_k) \right]}} \right]$$

$$p(b_{i,1}) = \Phi \left[- \frac{E(R_{i,1}) - \sum_{k=1}^{n_s} \left[d_{i,1,k} \, E((S^{n_b}_{MAX})_k) \right]}{\sqrt{\sigma^2(R_{i,1}) + \sum_{k=1}^{n_s} \left[d_{i,1,k} \, \sigma^2((S^{n_b}_{MAX})_k) \right]}} \right] \tag{F.2}$$

Rearranging eqn (F.2) in terms of $\sigma^2(R_{i,1})$ gives

$$\sigma^2(R_{i,1}) = \left[\frac{E(R_{i,1}) - \sum_{k=1}^{n_s} \left[d_{i,1,k} E((S^{n_a}_{MAX})_k) \right]}{\Phi^{-1}(p(a_{i,1}))} \right]^2 - \sum_{k=1}^{n_s} \left[d^2_{i,1,k} \, \sigma^2((S^{n_a}_{MAX})_k) \right]$$

$$= \left[\frac{E(R_{i,1}) - \sum_{k=1}^{n_s} \left[d_{i,1,k} E((S^{n_b}_{MAX})_k) \right]}{\Phi^{-1}(p(b_{i,1}))} \right]^2 - \sum_{k=1}^{n_s} \left[d^2_{i,1,k} \, \sigma^2((S^{n_b}_{MAX})_k) \right] \tag{F.3}$$

Rearranging the last equality of eqn (F.2) into a quadratic in $E(R_{i,1})$ gives

$$\left(\left[\Phi^{-1}(p(a_{i,1}))\right]^2-\left[\Phi^{-1}(p(b_{i,1}))\right]^2\right)\left[E(R_{i,1})\right]^2-2\left(\left[\Phi^{-1}(p(a_{i,1}))\right]^2\sum_{k=1}^{n_s}\left[d_{i,1,k}E((S^{n_b}_{MAX})_k)\right]\right.$$
$$\left.-\left[\Phi^{-1}(p(b_{i,1}))\right]^2\sum_{k=1}^{n_s}\left[d_{i,1,k}E((S^{n_a}_{MAX})_k)\right]\right)E(R_{i,1})+\left(\left\{\Phi^{-1}(p(a_{i,1}))\sum_{k=1}^{n_s}\left[d_{i,1,k}E((S^{n_b}_{MAX})_k)\right]\right.\right.$$
$$\left.\left.-\Phi^{-1}(p(b_{i,1}))\sum_{k=1}^{n_s}\left[d_{i,1,k}E((S^{n_a}_{MAX})_k)\right]\right\}^2+\left[\Phi^{-1}(p(a_{i,1}))\,\Phi^{-1}(p(b_{i,1}))\right]^2\sum_{k=1}^{n_s}\left[d_{i,1,k}\times\right.\right.$$
$$\left.\left.\left\{\sigma^2((S^{n_b}_{MAX})_k)-\sigma^2((S^{n_a}_{MAX})_k)\right\}\right]\right)\quad=\quad 0 \qquad (F.4)$$

Simplifying eqn. (F.4) gives

$$\left\{\left[\Phi^{-1}(p(a_{i,1}))\right]^2-\left[\Phi^{-1}(p(b_{i,1}))\right]^2\right\}\left[E(R_{i,1})\right]^2-2\sum_{k=1}^{n_s}\left[\left\{\left[\Phi^{-1}(p(a_{i,1}))\right]^2E((S^{n_b}_{MAX})_k)\right.\right.$$
$$\left.\left.-\left[\Phi^{-1}(p(b_{i,1}))\right]^2E((S^{n_a}_{MAX})_k)\right\}d_{i,1,k}\right]E(R_{i,1})+\sum_{k=1}^{n_s}\left[\left\{\left[\Phi^{-1}(p(a_{i,1}))\,E((S^{n_b}_{MAX})_k)\right]^2\right.\right.$$
$$\left.\left.-\left[\Phi^{-1}(p(b_{i,1}))E((S^{n_a}_{MAX})_k)\right]^2+\left[\Phi^{-1}(p(a_{i,1}))\,\Phi^{-1}(p(b_{i,1}))\right]^2\left[\sigma^2((S^{n_b}_{MAX})_k)-\sigma^2((S^{n_a}_{MAX})_k)\right]\right\}\times\right.$$
$$\left.d^2_{i,1,k}\right]=0$$

Therefore

$$E(R_{i,1})=\left[\sum_{k=1}^{n_s}\left[\left\{\left[\Phi^{-1}(p(a_{i,1}))\right]^2E((S^{n_b}_{MAX})_k)-\left[\Phi^{-1}(p(b_{i,1}))\right]^2E((S^{n_a}_{MAX})_k)\right\}d_{i,1,k}\right]\pm\right.$$
$$\left\{\left[\sum_{k=1}^{n_s}\left[\left\{\left[\Phi^{-1}(p(a_{i,1}))\right]^2E((S^{n_b}_{MAX})_k)-\left[\Phi^{-1}(p(b_{i,1}))\right]^2E((S^{n_a}_{MAX})_k)\right\}d_{i,1,k}\right]^2-\left\{\left[\Phi^{-1}(p(a_{i,1}))\right]^2\right.\right.$$
$$\left.\left.-\left[\Phi^{-1}(p(b_{i,1}))\right]^2\right\}\sum_{k=1}^{n_s}\left[\left\{\left[\Phi^{-1}(p(a_{i,1}))\,E((S^{n_b}_{MAX})_k)\right]^2-\left[\Phi^{-1}(p(b_{i,1}))E((S^{n_a}_{MAX})_k)\right]^2+\right.\right.$$
$$\left.\left.\left[\Phi^{-1}(p(a_{i,1}))\,\Phi^{-1}(p(b_{i,1}))\right]^2\left[\sigma^2((S^{n_b}_{MAX})_k)-\sigma^2((S^{n_a}_{MAX})_k)\right]\right\}d^2_{i,1,k}\right]^{\frac{1}{2}}\Big]\Big/\left\{\left[\Phi^{-1}(p(a_{i,1}))\right]^2\right.$$
$$\left.-\left[\Phi^{-1}(p(b_{i,1}))\right]^2\right. \qquad (F.6)$$

Simplifying eqn (F.6) gives

$$E(R_{i,1})=\frac{\sum_{k=1}^{n_s}\left[\left\{\left[\Phi^{-1}(p(a_{i,1}))\right]^2E((S^{nb}_{MAX})_k)-\left[\Phi^{-1}(p(b_{i,1}))\right]^2E((S^{na}_{MAX})_k)\right\}d_{i,1,k}\right]}{\left\{\left[\Phi^{-1}(p(a_{i,1}))\right]^2-\left[\Phi^{-1}(p(b_{i,1}))\right]^2\right\}}$$
$$\pm\Phi^{-1}(p(a_{i,1}))\Phi^{-1}(p(b_{i,1}))\left\{\left[\frac{\sum_{k=1}^{n_s}\left[d_{i,1,k}\left\{E((S^{na}_{MAX})_K)-E((S^{nb}_{MAX})_K)\right\}\right]}{\left[\Phi^{-1}(p(a_{i,1}))\right]^2-\left[\Phi^{-1}(p(b_{i,1}))\right]^2}\right]^2\right.$$

$$- \left\{ \frac{\sum_{k=1}^{n_s} \left[d_{i,1,k}^2 \left\{ \sigma^2\left(\left(S_{MAX}^{n_a}\right)_k\right) - \sigma^2\left(\left(S_{MAX}^{n_b}\right)_k\right) \right\} \right]}{\left[\Phi^{-1}(p(a_{i,1}))\right]^2 - \left[\Phi^{-1}(p(b_{i,1}))\right]^2} \right\}^{\frac{1}{2}} \qquad \text{(F.7)}$$

$\sigma(R_{i,1})$ can be found by substituting the value of $E(R_{i,1})$ from eqn. (F.7) into eqn (F.3).

APPENDIX G - DESIGN OF SIMPLE PORTAL FRAME

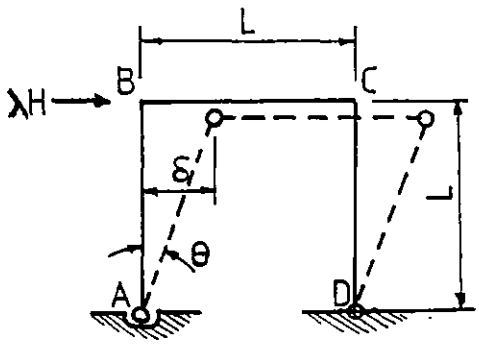
G.1 PLASTIC DESIGN

Use a load factor, λ , = 1.6, with 95% and 5% characteristic values of the resistance and load respectively. Therefore

$$M_p = E(R) - 1.645\ \sigma(R) \qquad (G.1.1)$$

$$H = E(S) + 1.645\ \sigma(S) \qquad (G.1.2)$$

where M_p is the characteristic plastic moment of the members, treated as uniform throughout, and H is the characteristic horizontal load applied.



Using plastic analysis and assuming plastic hinges at B, C and D

$$\lambda H\delta = \lambda HL\,\theta = 3M_p\,\theta \qquad (G.1.3)$$

Therefore

$$M_p = \lambda HL/3 \qquad (G.1.4)$$

Hence

$$E(R) - 1.645\ \sigma(R) = 1.6\ [E(S) + 1.645\ \sigma(S)]\ L/3 \qquad (G.1.5)$$

Therefore

$$E(R) = \frac{1.6\ [1.0 + 1.645\ \sigma(S)/E(S)]}{3\ [1.0 - 1.645\ \sigma(R)/E(R)]}\ L.E(S) \qquad (G.1.6)$$

Let L = 1.0, E(S) = 1.0, $\sigma(S)$ = 0.20 and $\frac{\sigma(R)}{E(R)} = 0.05$

Therefore

$$E(R) = 0.7723 \text{ and } \sigma(R) = 0.0386 \qquad (G.1.7)$$

G.2 ELASTIC ANALYSIS

Assume AE = ∞ , EI/L is equal for all members, and no initial lack of fit. See fig G.2.1 for internal moments sign convention and notation, and fig G.2.2 for small deflections sign convention and notation.

$$M_{AB} = \frac{2EI}{L}(2\phi_A + \phi_{B1}) - \frac{6EI}{L^2}u \qquad (G.2.1)$$

$$M_{BA} = \frac{2EI}{L}(\phi_A + 2\phi_{B1}) + \frac{6EI}{L^2}u \qquad (G.2.2)$$

$$M_{BC} = -\frac{2EI}{L}(2\phi_{B2} + \phi_{C1}) \qquad (G.2.3)$$

$$M_{CB} = \frac{2EI}{L}(\phi_{B2} + 2\phi_{C1}) \qquad (G.2.4)$$

$$M_{CD} = -\frac{2EI}{L}(2\phi_{C2} + \phi_D) + \frac{6EI}{L^2}u \qquad (G.2.5)$$

$$M_{DC} = \frac{2EI}{L}(\phi_{C2} + 2\phi_D) - \frac{6EI}{L^2}u \qquad (G.2.6)$$

Initial Distribution of Bending Moment

$$\phi_{B1} = \phi_{B2} = \phi \quad ; \quad \phi_{C1} = \phi_{C2} = \phi_C \quad ; \quad \phi_D = 0$$

$$M_{AB} = 0 \quad ; \quad M_{BA} + M_{BC} = 0 \quad ; \quad M_{CD} - M_{CD} = 0 \quad ; \quad M_{BC} - M_{CB} + M_{DC} = 1$$

$\therefore$

$$\begin{cases} 2\phi_A + \phi_B - 3u/L = 0 \\ \phi_A + 4\phi_B + \phi_C - 3u/L = 0 \\ \phi_B + 4\phi_C - 3u/L = 0 \\ 3\phi_B + 2\phi_C + 3u/L - \frac{L}{2EI} = 0 \end{cases} \qquad (G.2.7)$$

$\therefore \quad \phi_A = \frac{3L}{22EI} \quad ; \quad \phi_B = \frac{L}{44EI} \quad ; \quad \phi_C = \frac{3L}{22EI} \quad ; \quad u = \frac{13L^2}{132EI}$

$\therefore \quad M_{BC} = 0.227273 \quad ; \quad M_{CD} = -0.318182 \quad ; \quad M_{DC} = 0.454545$

(fig G.2.3)

Redistribution of Bending Moment from B

$$\phi_{C1} = \phi_{C2} = \phi_C \quad ; \quad \phi_D = 0$$

$$M_{AB} = 0 \quad ; \quad M_{BA} = 1 \quad ; \quad M_{BC} = -1 \quad ; \quad M_{CB} - M_{CD} = 0 \quad ; \quad M_{BC} - M_{CB} + M_{DC} = 0$$

$$\therefore \begin{cases} 2\phi_A + \phi_{B1} - 3u/L = 0 \\ \phi_A + 2\phi_{B1} - 3u/L + \frac{L}{2EI} = 0 \\ 2\phi_{B2} + \phi_C - \frac{L}{2EI} = 0 \\ \phi_{B2} + 4\phi_C - 3u/L = 0 \\ \phi_{B2} + \phi_C + 3u/L + \frac{L}{2EI} = 0 \end{cases} \qquad \text{(G.2.8)}$$

$$\therefore \quad \phi_A = -\frac{L}{24EI} \; ; \; \phi_{B1} = -\frac{13L}{24EI} \; ; \; \phi_{B2} = \frac{3L}{8EI} \; ; \; \phi_C = -\frac{L}{4EI} \; ; \; u = -\frac{5L^2}{24EI}$$

$$\therefore \quad M_{BC} = -1.00 \; ; \; M_{CD} = -0.25 \; ; \; M_{DC} = +0.75$$

(fig G.2.4)

Redistribution of Bending Moment from C

$$\phi_{B1} = \phi_{B2} = \phi_B \; ; \; \phi_D = 0$$

$$M_{AB} = 0 \; ; \; M_{BA} + M_{BC} = 0 \; ; \; M_{CB} = -1 \; ; \; M_{CD} = -1 \; ; \; M_{BC} - M_{CD} + M_{DC} = 0$$

$$\therefore \begin{cases} 2\phi_A + \phi_B - 3u/L \\ \phi_A + 4\phi_B + \phi_{C1} - 3u/L \\ \phi_B + 2\phi_{C1} + \frac{L}{2EI} = 0 \\ 2\phi_{C2} - 3u/L - \frac{L}{2EI} = 0 \\ -2\phi_B - \phi_{C1} + \phi_{C2} - 3u/L + \frac{L}{2EI} = 0 \end{cases} \qquad \text{(G.2.9)}$$

$$\therefore \quad \phi_A = \frac{4L}{9EI} \; ; \; \phi_B = \frac{5L}{18EI} \; ; \; \phi_{C1} = -\frac{7L}{18EI} \; ; \; \phi_{C2} = \frac{5L}{6ES} \; ; \; u = \frac{7L^2}{18EI}$$

$$\therefore \quad M_{BC} = -0.333 \; ; \; M_{CD} = -1.00 \; ; \; M_{DC} = -0.667$$

(fig G.2.5)

Redistribution of Bending Moment from D

$$\phi_{B1} = \phi_{B2} = \phi_{C1} = \phi_{C2} = \phi_C$$

$$M_{AB} = 0 \; ; \; M_{BA} + M_{BC} = 0 \; ; \; M_{CB} - M_{CD} = 0 \; ; \; M_{DC} = -1 \; ; \; M_{BC} - M_{CB} + M_{DC} = 0$$

$$\therefore \begin{cases} 2\phi_A + \phi_B - 3u/L = 0 \\ \phi_A + 4\phi_B + \phi_C - 3u/L = 0 \\ \phi_B + 4\phi_C + \phi_D - 3u/L = 0 \\ \phi_C + 2\phi_D - 3u/L + \frac{L}{2EI} = 0 \\ 3\phi_B + 3\phi_C + \frac{L}{2EI} = 0 \end{cases} \qquad \text{(G.2.10)}$$

$$\therefore \quad \phi_A = -\frac{13L}{30EI} \; ; \; \phi_B = -\frac{2L}{15EI} \; ; \; \phi_C = -\frac{L}{30EI} \; ; \; \phi_D = -\frac{11L}{15EI} \; ; \; u = -\frac{L^2}{3EI}$$

$$\therefore \quad M_{BC} = +0.60 \; ; \; M_{CB} = -0.40 \; ; \; M_{DC} = -1.00$$

(fig G.2.6)

Redistribution from second element to fail

If first failure at B, $M_{AB} = M_{BA} = M_{BC} = 0$; $M_{CB} - M_{CD} = 0$; $M_{BC} - M_{CB} + M_{DC} = 0$

$$\therefore \quad M_{CB} = M_{CD} = M_{DC}$$

If first failure at C, $M_{AB} = 0$; $M_{BA} + M_{BC} = 0$; $M_{CB} = M_{CD} = 0$;

$$M_{BC} - M_{CB} + M_{DC} = 0$$

$$\therefore \quad M_{BA} = -M_{BC} = M_{DC}$$

If first failure at D, $M_{AB} = 0$; $M_{BA} + M_{BC} = 0$; $M_{CB} - M_{CD} = 0$;

$$M_{BC} - M_{CB} + M_{BC} = 0$$

$$\therefore \quad -M_{BA} = M_{BC} = M_{CB} = M_{CD}$$

(fig G.2.7)

G.3 ELASTIC DESIGN

Assume that the frame is designed to have a constant EI throughout but with an ultimate moment of resistance of a cross-section proportional to the maximum elastic bending moment in each member before any hinge forms . Using the bending moments calculated in section G.2, this gives

$$E(R_1) - 1.645\,\sigma(R_1) = 0.2273 \times 1.6 \left[E(S) + 1.645\,\sigma(S) \right] \qquad \text{(G.3.1)}$$

$$E(R_2) - 1.645\,\sigma(R_2) = 0.3182 \times 1.6 \left[E(S) + 1.645\,\sigma(S) \right] \qquad \text{(G.3.2)}$$

$$E(R_3) - 1.645\,\sigma(R_3) = 0.4545 \times 1.6 \left[E(S) + 1.645\,\sigma(S) \right] \qquad \text{(G.3.3)}$$

Using the same parameters for the load and the same coefficient of variation for the resistances as in G.1, this gives

$$E(R_1) = 0.5266 \quad \text{and} \quad \sigma(R_1) = 0.0263 \tag{G.3.4}$$

$$E(R_2) = 0.7372 \quad \text{and} \quad \sigma(R_2) = 0.0369 \tag{G.3.5}$$

$$E(R_3) = 1.0531 \quad \text{and} \quad \sigma(R_3) = 0.0527 \tag{G.3.6}$$

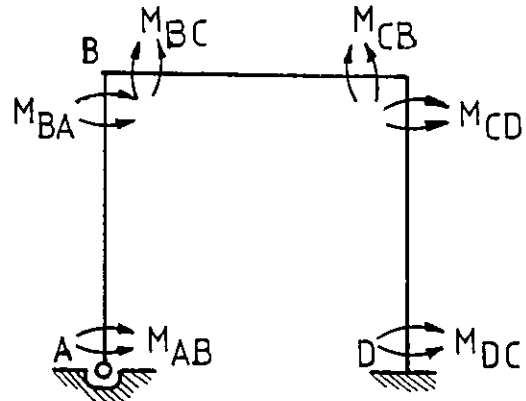


Fig.G.2.1 - Convention for Positive Internal Moments

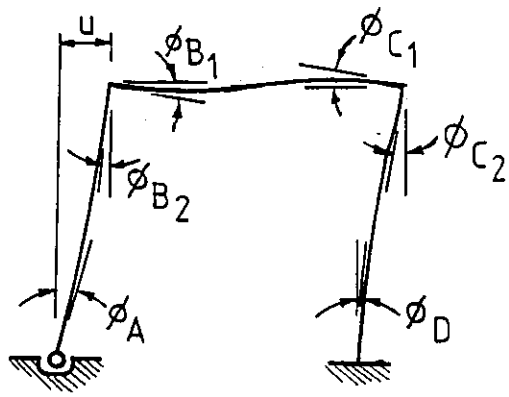


Fig.G.2.2 - Convention for Positive Deflections

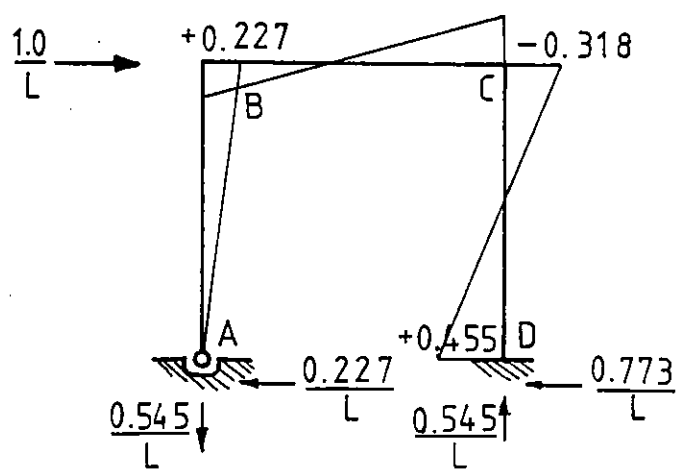


Fig.G.2.3 - Initial Bending Moment Diagram

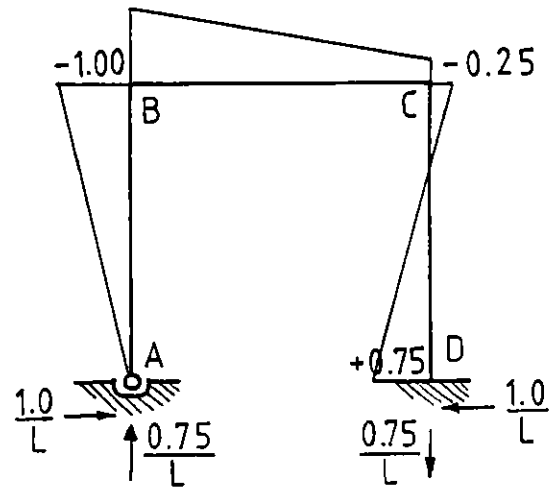


Fig.G.2.4 - Redistributed Moment from B

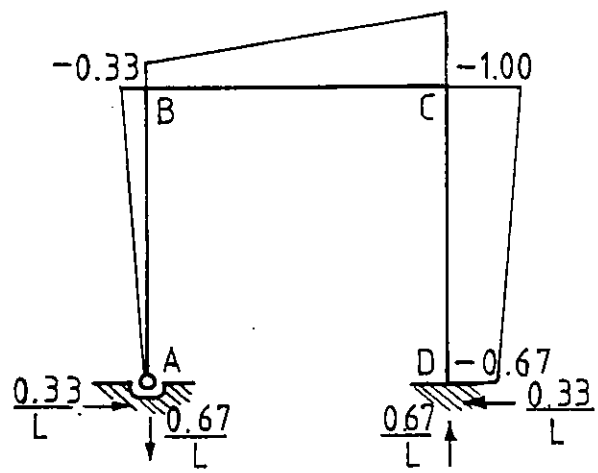


Fig.G.2.5 - Redistributed Moment from C

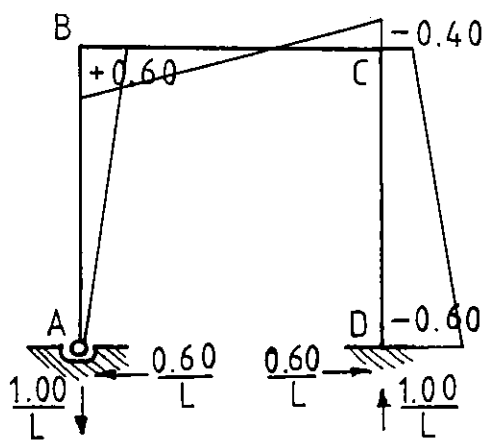
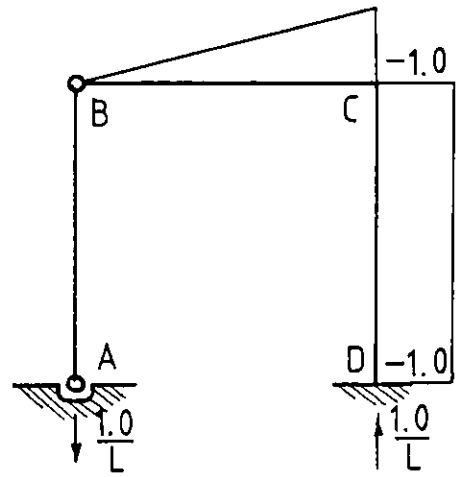
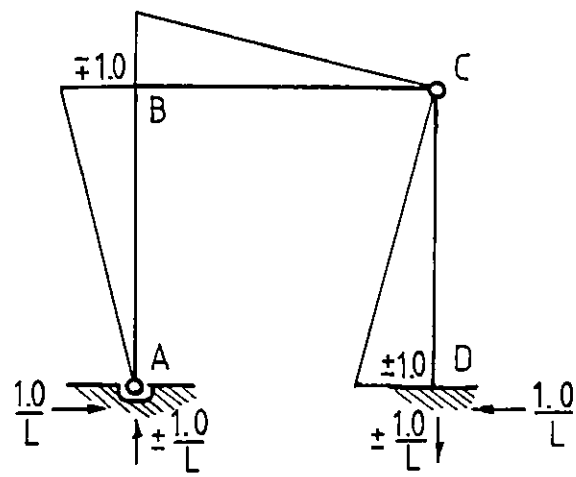


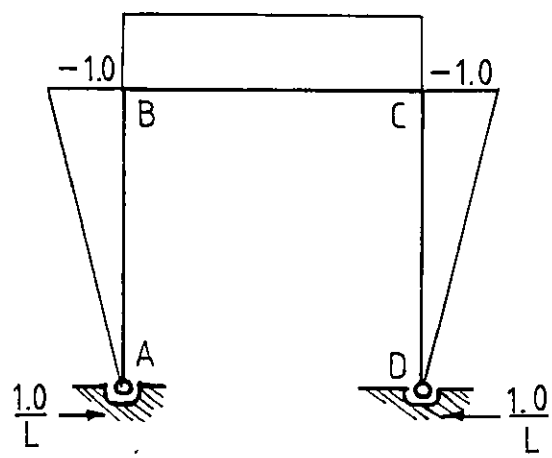
Fig.G.2.6 - Redistributed Moment from D



(i) First hinge formed at B



(ii) First hinge formed at C



(iii) First hinge formed at D

Fig.G.2.7 - Second Stage Redistribution of Moment

APPENDIX H - LIVE LOAD DISTRIBUTION FOR SLAB EXAMPLE

H.1 INITIAL DISTRIBUTION

Assume that the live load on the slab can be defined by the local intensity at nine points - the corners, mid-point of each side and the centre. If the model uses nine lognormal variables, a_i, with unit means, and a further variable, W, representing the mean live load intensity, then the live load intensity at each of the nine points, w_i, can be given by

$$w_i = 16a_i W \Big/ \left(\sum_{j=1}^{4} \left\{ a_j + 2a_{(4+j)} \right\} + 4a_9 \right) \qquad (H.1)$$

The load intensity can then be modelled as planar between the centre and each half side (figure F.1). The load at each point is assumed to be supported as spanning between the beam nearest to that point and the beam parellel to it, producing a pair of triangular tributary areas for each pair of beams (figure H.2).

$$w(x,o) = w_1 + \frac{2x}{l}(w_5 - w_1) \qquad (H.2)$$

$$w(x,x) = w_1 + \frac{2x}{l}(w_9 - w_1) \qquad (H.3)$$

$$w(x,l) = w_4 + \frac{2x}{l}(w_7 - w_4) \qquad (H.4)$$

$$w(x,l-x) = w_4 + \frac{2x}{l}(w_9 - w_4) \qquad (H.5)$$

$$w(x,x) - w(x,o) = \frac{2x}{l}(w_9 - w_5) \qquad (H.6)$$

$$w(x,l-x) - w(x,l) = \frac{2x}{l}(w_9 - w_7) \qquad (H.7)$$

$$r_x = w(x,o)\ x\ \frac{(1 - x/2)}{l} + \left[w(x,x) - w(x,o) \right] \frac{x}{2}\ \frac{(1 - 2x/3)}{l}$$

$$+ w(x,l)\ x\ \frac{x/2}{l} + \left[w(x,l-x) - w(x,l) \right] \frac{x}{2}\ \frac{2x/3}{l}$$

$$\therefore\ 6r_x/x = \left[w_1 + 2\frac{x}{l}(w_5 - w_1) \right] (6 - 3\tfrac{x}{6}) + 2\tfrac{x}{l}\ (w_9 - w_5)(3 - 2\tfrac{x}{l})$$

$$+ \left[w_4 + 2\tfrac{x}{l}\ (w_7 - w_4) \right] 3\tfrac{x}{l} + 2\tfrac{x}{l}\ (w_9 - w_7)\ 2\tfrac{x}{l}$$

$$\therefore\ r_x = w_1 x + \left[-\frac{5}{2} w_1 + \frac{1}{2} w_4 + w_5 + w_9 \right] \frac{x^2}{l} + \left[w_1 - w_4 + \frac{1}{3}\ (-w_5 + w_7) \right] \frac{x^3}{l^2} \qquad (H.8)$$

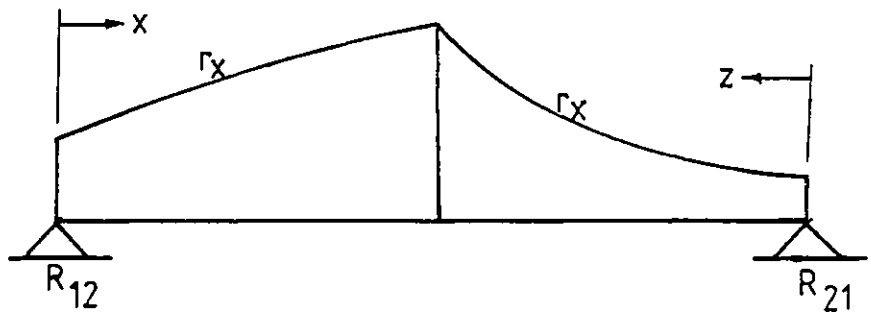
At $x = \frac{l}{2}$

$$r_x = l\left[\frac{w_1}{2} - \frac{5}{8} w_1 + \frac{1}{8} w_4 + \frac{1}{4} w_5 + \frac{1}{4} w_9 + \frac{w_1}{8} - \frac{w_4}{8} - \frac{w_5}{24} + \frac{w_7}{24} \right]$$

$$= l\left[\frac{5}{24} w_5 + \frac{1}{24} w_7 + \frac{1}{4} w_9 \right]$$

Hence $r_{x=\frac{1}{2}}$ is independent of w_1 and w_4 and no discontimuity of load intensity will occur at node 5 whether approached from points 1 or 2,

r_x is the live load intensity on the beam between points 1 and 5.



Let $r_x = c_{11}x + c_{12}x^2 + c_{13}x^3$

$$+ c_{21}\left\{x - \tfrac{1}{2}\right\} + c_{22}\left\{x - \tfrac{1}{2}\right\}^2 + c_{23}\left\{x - \tfrac{1}{2}\right\}^3 \qquad \text{(H.10)}$$

where $\left\{x - \tfrac{1}{2}\right\} = 0.0$ if $x - \tfrac{1}{2} \leqslant 0.0$

then
$$\begin{cases} c_{11} = w_1 & \text{(H.11)} \\ c_{12} = \left[-\frac{5}{2}w_1 + \frac{1}{2}w_4 + w_5 + w_9\right]/l & \text{(H.12)} \\ c_{13} = \left[w_1 - w_4 + \frac{1}{3}(-w_5 + w_7)\right]/l^2 & \text{(H.13)} \end{cases}$$

Put $z = l - x$, $x = l - z$, then

$$r_x = c_{11}(l - z) + c_{12}(l - z)^2 + c_{13}(l - z)^3 + c_{21}\left(\tfrac{l}{2} - z\right) + c_{22}\left(\tfrac{l}{2} - z\right)^2 + c_{23}\left(\tfrac{l}{2} - z\right)^3 \qquad \text{(H.14)}$$

$$\therefore \left[c_{11}\,l + c_{12}\,l^2 + c_{13}\,l^3 + c_{21}\tfrac{l}{2} + c_{22}\tfrac{l^2}{4} + c_{23}\tfrac{l^3}{8}\right] + \left[-c_{11} - 2l\,c_{12} - 3l^2\,c_{13} - c_{21} - \tfrac{2l}{2}c_{22} - \tfrac{3}{4}l^2\,c_{23}\right]z + \left[c_{12} + 3lc_{13} + c_{22} + \tfrac{3}{2}lc_{23}\right]z^2 + \left[-c_{13} - c_{23}\right]z^3$$

$$\equiv w_2 z + \left[-\tfrac{5}{2}w_2 + \tfrac{1}{2}w_3 + w_5 + w_9\right]\frac{z^2}{l} + \left[w_2 - w_3 + \tfrac{1}{3}(-w_5 + w_7)\right]\frac{z^3}{l^2} \qquad \text{(H.15)}$$

$$\therefore c_{23} = -c_{13} - (w_2 - w_3 + \tfrac{1}{3}(-w_5 + w_7))/l^2$$

$$= -\left[w_1 - w_4 + \tfrac{1}{3}(-w_5 + w_7)\right]/l^2 - \left[w_2 - w_3 + \tfrac{1}{3}(-w_5 + w_7)\right]/l^2$$

$$= \left[-w_1 - w_2 + w_3 + w_4 + \tfrac{2}{3}(w_5 - w_7)\right]/l^2 \qquad \text{(H.16)}$$

$$c_{22} = \left[-\tfrac{5}{2} w_2 + \tfrac{1}{2} w_3 + w_5 + w_9\right]/l - \left[c_{12} + 3lc_{13} + \tfrac{3}{2} l\, c_{23}\right]$$

$$= \left[-\tfrac{5}{2} w_2 + \tfrac{1}{2} w_3 + w_5 + w_9\right]/l - \left[-\tfrac{5}{2} w_1 + \tfrac{1}{2} w_4 + w_5 + w_9\right]/l - 3\left[w_1 - w_4 + \tfrac{1}{3}(-w_5 + w_7)\right]/l - \tfrac{3}{2}\left[-w_1 - w_2 + w_3 + w_4 + \tfrac{2}{3}(w_5 - w_7)\right]/l$$

$$= \left[w_1 - w_2 - w_3 + w_4\right]/l \qquad \text{(H.17)}$$

$$c_{21} = -w_2 - c_{11} - 2lc_{12} - 3l^2 c_{13} - lc_{22} - \tfrac{3}{4} l^2 c_{23}$$

$$= -w_2 - w_1 - 2\left[-\tfrac{5}{2} w_1 + \tfrac{1}{2} w_4 + w_5 + w_9\right] - 3\left[w_1 - w_4 + \tfrac{1}{3}(-w_5 + w_7)\right] - \left[w_1 - w_2 - w_3 + w_4\right] - \tfrac{3}{4}\left[-w_1 - w_2 + w_3 + w_4 + \tfrac{2}{3}(w_5 - w_7)\right]$$

$$= \tfrac{3}{4}(w_1 + w_2) + \tfrac{1}{4}(w_3 + w_4) - \tfrac{3}{2} w_5 - \tfrac{1}{2} w_7 - 2w_9 \qquad \text{(H.18)}$$

Check that the constant term is zero

$$c_{11} l + c_{12} l^2 + c_{13} l^3 + c_{21} \tfrac{l}{2} + c_{22} \tfrac{l^2}{4} + c_{23} \tfrac{l^3}{8}$$

$$= l\left[w_1 - \tfrac{5}{2} w_1 + \tfrac{1}{2} w_4 + w_5 + w_9 + w_1 - w_4 - \tfrac{1}{3} w_5 + \tfrac{1}{3} w_7 + \tfrac{3}{8} w_1 + \tfrac{3}{8} w_2 + \tfrac{1}{8} w_3 + \tfrac{1}{8} w_4 - \tfrac{3}{4} w_5 - \tfrac{1}{4} w_7 - w_9 + \tfrac{1}{4} w_1 - \tfrac{1}{4} w_2 - \tfrac{1}{4} w_3 + \tfrac{1}{4} w_4 - \tfrac{1}{8} w_1 - \tfrac{1}{8} w_2 + \tfrac{1}{8} w_3 + \tfrac{1}{8} w_4 + \tfrac{1}{12} w_5 - \tfrac{1}{12} w_7\right]$$

$$= 0$$

Total load on the beam (1,5,2) = L_{12}

$$= \int_0^1 r_x\, dx = \int_0^1 c_{11}x + c_{12}x^2 + c_{13}x^3 + c_{21}\left\{x - \tfrac{1}{2}\right\} + c_{22}\left\{x - \tfrac{1}{2}\right\}^2 + c_{23}\left\{x - \tfrac{1}{2}\right\}^3 dx$$

$$= \tfrac{1}{2} c_{11} l^2 + \tfrac{1}{3} c_{12} l^3 + \tfrac{1}{4} c_{13} l^4 + \tfrac{1}{2} c_{21} \tfrac{l^2}{4} + \tfrac{1}{3} c_{22} \tfrac{l^3}{8} + \tfrac{1}{4} c_{23} \tfrac{l^4}{16}$$

$$\therefore L_{12} = \tfrac{1}{2} c_{11} l^2 + \tfrac{1}{3} c_{12} l^3 + \tfrac{1}{4} c_{13} l^4 + \tfrac{1}{8} c_{21} l^2 + \tfrac{1}{24} c_{22} l^3 + \tfrac{1}{64} c_{23} l^4 \qquad \text{(H.19)}$$

Taking moments about point 1 to find x_c, the centre of gravity, gives

$$x_c L_{12} = \int_0^1 x r_x\, dx = \int_0^1 c_{11}x^2 + c_{12}x^3 + c_{13}x^4 + c_{21}x\left\{x - \tfrac{1}{2}\right\} + c_{22}x\left\{x - \tfrac{1}{2}\right\}^2 + c_{23}x\left\{x - \tfrac{1}{2}\right\}^3 dx$$

$$= \tfrac{1}{3} c_{11} 1^3 + \tfrac{1}{4} c_{12} 1^4 + \tfrac{1}{5} c_{13} 1^5 + \int_o^1 c_{21} \tfrac{1}{2} \left\{x - \tfrac{1}{2}\right\} + (c_{21} + c_{22} \tfrac{1}{2})\left\{x - \tfrac{1}{2}\right\}$$

$$+ (c_{22} + c_{23} \tfrac{1}{2}) \left\{x - \tfrac{1}{2}\right\}^3 + c_{32} \left\{x - \tfrac{1}{2}\right\}^4 \; dx$$

$$= \tfrac{1}{3} c_{11} 1^3 + \tfrac{1}{4} c_{12} 1^4 + \tfrac{1}{5} c_{13} 1^5 + \tfrac{1}{2} c_{21} \tfrac{1}{2} (\tfrac{1}{2})^2 + \tfrac{1}{3} (c_{21} + c_{22} \tfrac{1}{2}) (\tfrac{1}{2})^3$$

$$+ \tfrac{1}{4} (c_{22} + c_{23} \tfrac{1}{2}) (\tfrac{1}{2})^4 + \tfrac{1}{5} c_{32} (\tfrac{1}{2})^5$$

$$= \tfrac{1}{3} c_{11} 1^3 + \tfrac{1}{4} c_{12} 1^4 + \tfrac{1}{5} c_{13} 1^5 + \tfrac{5}{48} c_{21} 1^3 + \tfrac{7}{192} c_{22} 1^4 + \tfrac{9}{640} c_{23} 1^4 \quad \text{(H.20)}$$

$$\therefore x_c = \left[\frac{\tfrac{1}{3} c_{11} 1^3 + \tfrac{1}{4} c_{12} 1^4 + \tfrac{1}{5} c_{13} 1^5 + \tfrac{5}{48} c_{21} 1^3 + \tfrac{7}{192} c_{22} 1^4 + \tfrac{9}{640} c_{23} 1^4}{\tfrac{1}{2} c_{11} 1^2 + \tfrac{1}{3} c_{12} 1^3 + \tfrac{1}{4} c_{13} 1^4 + \tfrac{1}{8} c_{21} 1^2 + \tfrac{1}{24} c_{22} 1^3 + \tfrac{1}{64} c_{23} 1^4}\right]$$

$$= \left[\frac{640c_{11} + 480c_{12}\,1 + 384c_{13}\,1^2 + 200c_{21} + 70c_{22}\,1 + 27c_{23}\,1^2}{960c_{11} + 640c_{12}\,1 + 480c_{13}\,1^2 + 240c_{21} + 80c_{22}\,1 + 30c_{23}\,1^2}\right] 1 \quad \text{(H.21)}$$

$$R_{12} = L_{12}\left[1 - x_c\right]/1 \quad \text{(H.22)}$$

$$M_5 = R_{12} \tfrac{1}{2} - \int_o^{\frac{1}{2}} r_x \left(\tfrac{1}{2} - x\right) dx$$

$$= R_{12} \tfrac{1}{2} - \int_o^{\frac{1}{2}} \left[c_{11}x + c_{12}x^2 + c_{13}x^3\right]\left[\tfrac{1}{2} - x\right] \; dx$$

$$= R_{12} \tfrac{1}{2} - \int_o^{\frac{1}{2}} \tfrac{1}{2} c_{11} 1\, x + (\tfrac{1}{2} c_{12} 1 - c_{11})x^2 + (\tfrac{1}{2} c_{13} 1 - c_{12})x^3 - c_{13}x^4 \; dx$$

$$= \tfrac{1}{2} R_{12} 1 - \left[\tfrac{1}{8} c_{11} 1^3 + \tfrac{1}{24} (\tfrac{1}{2} c_{12} 1 - c_{11}) 1^3 + 64(\tfrac{1}{2} c_{13} 1 - c_{12}) 1^4 - 160\, c_{12} 1^5\right]$$

$$= \tfrac{1}{2} R_{12} 1 - \tfrac{1}{48} c_{11} 1^3 - \tfrac{1}{192} c_{12} 1^4 - \tfrac{1}{640} c_{13}\, 1^5 \quad \text{(H.23)}$$

Check by putting all values of $w_i = W$, the load on the beam will then have a triangular distribution with a maximum intensity of $\frac{1}{2}W1$ at mid-span.

$$c_{11} = W$$

$$c_{12} = c_{13} = c_{22} = c_{23} = 0$$

$$c_{21} = -2W$$

$$L_{12} = \left(\frac{1}{2} 1^2 + \frac{1}{8} (-2) 1^2\right) W = \frac{1}{4} 1^2 W$$

$$x_c = \left[(640 - 2 \times 200)/(960 - 2 \times 240)\right] 1 = \frac{1}{2}$$

$$R_{12} = \frac{1}{4} 1^2 W \left[1 - \frac{1}{2}\right]/1 = \frac{1}{8} W 1^2 \qquad \text{– o.k.}$$

$$M_5 = \frac{1}{2} \times \frac{1}{8} W 1^2 1 - \frac{1}{48} W 1^3 = \frac{1}{24} W 1^3 \text{ – o.k.}$$

Load Distribution Factor for Bending Mount at point 5

$$= M_5 / \left(\frac{W1^3}{24}\right)$$

$$= \frac{12R_{12}}{W1^2} - \frac{c_{11}}{2W} - \frac{c_{12}1}{8W} - \frac{3c_{13}1^2}{80W} \qquad \text{(H.24)}$$

Load Distribution Factor for Shear Force at point 1 on beam 12

$$= R_{12} / \left(\frac{W1^2}{8}\right)$$

$$= \frac{8R_{12}}{W1^2} \qquad \text{(H.25)}$$

H.2 REDISTRIBUTION AFTER FAILURE

If one beam fails, let the additional load on the adjacent beam be R. Treat this as a distributed load which, for reasons of statics, must have the same total value and moment as a point load at the end adjacent to the failed beam. For simplicity, assume a Triangular distribution

$$\therefore \begin{cases} (w_1 + w_2) 1/2 = W \\ w_1 \, 1 \, \frac{1}{2} + \frac{(w_2 - w_1)}{2} 1 \, \frac{21}{3} = 0 \end{cases}$$

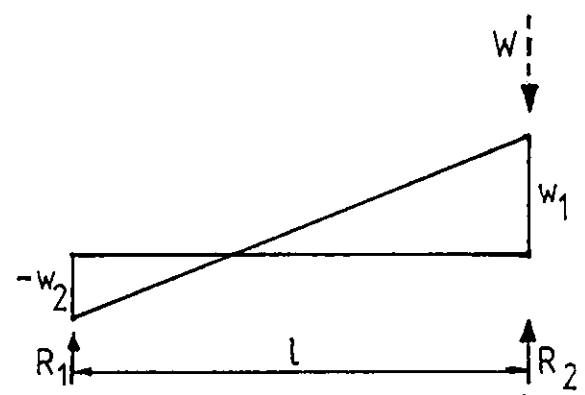
$$\therefore \quad w_1 = 2W/1 - w_2$$

$$3w_1 - 2w_1 + 2w_2 = 0$$

$$\therefore \quad w_1 = -2w_2 = \frac{2W}{1} - w_2$$

$$\therefore \quad w_2 = -\frac{2W}{1}, \quad w_1 = \frac{4W}{1}$$

$$R_1 = W \qquad R_2 = 0$$



(H.26)

$$M = \frac{1}{2} \frac{2W}{1} \frac{1}{3} \left(\frac{1}{2} - \frac{1}{9}\right) - \frac{1}{2} \frac{W}{1} \frac{1}{6} \frac{1}{18} = \frac{55}{432} W1 \qquad \text{(H.27)}$$

Moment on "failed" beam = $W\frac{1}{2} - W\frac{1}{6} = \frac{Wl}{3}$

$\therefore$ Unit bending moment redistribution produces extra

$$\begin{cases} \text{bending moment of } \frac{55}{432}/\frac{1}{3} = 0.3819444 \text{ on adjacent beams} \\ \text{shear force of } \frac{\frac{1}{24}Wl^3}{\frac{1}{8}Wl^2 \times \frac{Wl}{3}} \alpha = 1.0\,\alpha \text{ on adjacent elements} \end{cases}$$

where α is the ratio of shear force/bending moment

Unit shear force redistributed produces extra

$$\begin{cases} \text{bending moment of } \frac{55.1}{432}\frac{\frac{1}{8}Wl^2}{\frac{1}{24}Wl^3}\frac{1}{\alpha} = 0.3819444/\alpha \text{ on adjacent beams} \\ \text{shear force of 1.0 on adjacent element.} \end{cases}$$

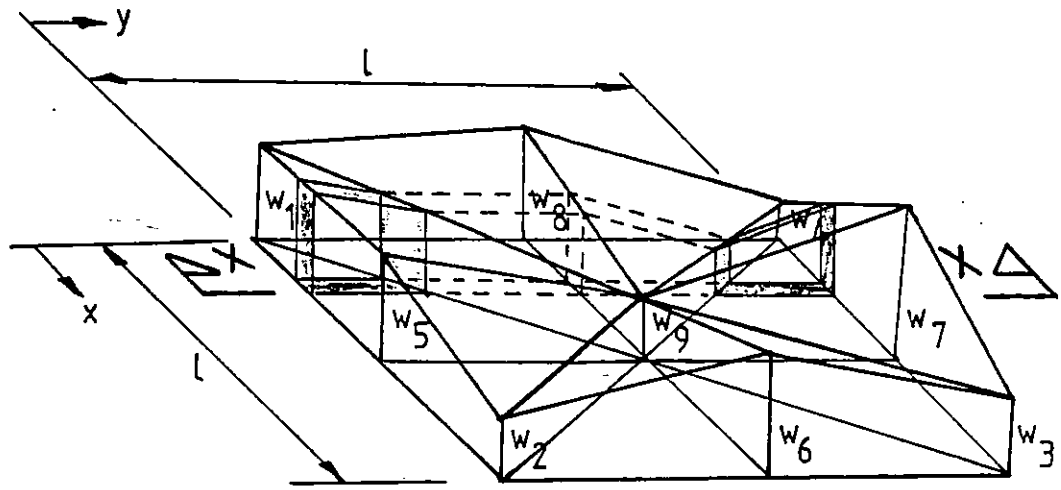


Fig.H.1 - Live Load Distribution Model

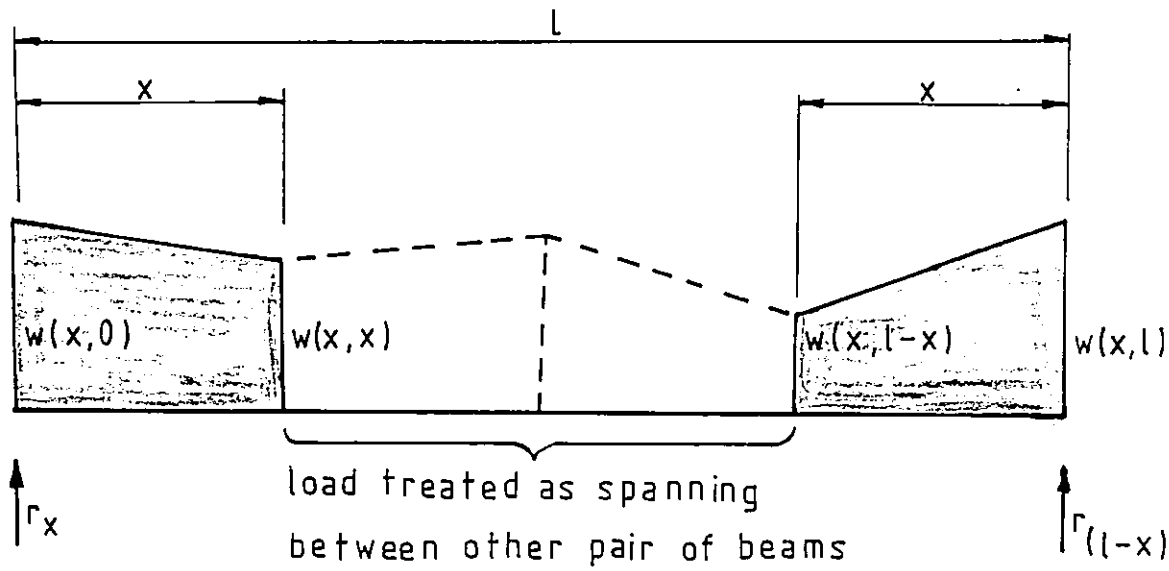


Fig.H.2 - Live Load Intensity on the Slab at X-X

APPENDIX J - LOADING ON PORTAL FRAME WITH PINNED FEET

INITIAL LOADING: Analysis is based on Frame II in Chapter 15 of "Steel Designers' Manual" [12] with the loading and dimensions as given in fig. 6.5.1.

$$k = {}^{I_2}/_{I_1} \times {}^{h}/_{l} = {}^{10.0}/_{20.0} = 0.5$$

$$N = 2k + 3.0 = 4.0$$

HORIZONTAL U.D.L. (S_1):

$$M_B = (2.0\ S_1\ 10.0^2/4)\ (-0.5/(2x4.0) + 1) = 46.8755_1 \text{ kNm}$$

$$M_C = (2.0\ S_1\ 10.0^2/4)\ (-0.5/(2x4.0 - 1) = -53.125\ S_1 \text{ kNm}$$

$$H_A = -(2.0\ S_1\ 10.0 - M_C/10.0) = -14.7875\ S_1 \text{ kN}$$

$$M_E = -H_A\ a - 2.0\ S_1\ a^2/2 = (14.7975 - a)a\ S_1 \text{ kNm}$$

Let $a = H_A / (2.0\ S_1) = 7.39375$ m for max^{m} moment

$$M_E = +54.66754\ S_1 \text{ kNm}$$

$$M_F = (+46.875 - 5.0\ x)\ S_1 \text{ kNm}$$

$$M_G = ({}^{a}/_{10.0})\ M_C = -39.2793\ S_1 \text{ kNm}$$

VERTICAL U.D.L. (S_2):

$$M_B = M_C = -(S_2\ 20.0^2)/(4x4.0) = -25.0\ S_2 \text{ kNm}$$

$$M_E = M_G = (a/10.0)\ M_B = -18.48438\ S_2 \text{ kNm}$$

$$M_F = (+10.0\ x - 25.0 - 0.50\ x^2)\ S_2 \text{ kNm}$$

HORIZONTAL POINT LOAD (S_3):

$$M_B = -M_C = M_E = -M_G = 10.0\ S_3\ (a/10.0) = 7.39375\ S_3 \text{ kNm}$$

$$M_F = (+7.39375 - 0.739375\ x)\ S_3 \text{ kNm}$$

POSITION OF F

Let F be x m from B

$$M_F = [46.875 - 5.0\ x]\ S_1 + [10.0x - 25.0 - 0.5\ x^2]\ S_2 + [7.39375 - 0.739375\ x]\ S_3 \text{ kNm}$$

$$= [46.875\ S_1 - 25.0\ S_2 + 7.39375\ S_3]$$

$$+ [5.0\ S_1 + 10.0\ S_2 - 0.739375\ S_3]x$$

$$- 0.5\ S_2\ x^2$$

Let F be at the point of max^{m} moment,

$$\therefore \quad \frac{dM_F}{dx} = \left[5.0\ S_1 + 10.0\ S_2 - 0.739375\ S_3 \right] - 2(0.5\ S_2 x) = 0.0$$

$$\therefore \quad X = 10.0 - \left[5.0\ \frac{S_1}{S_2} + 0.739375\ \frac{S_3}{S_2} \right]$$

Replace $\frac{S_1}{S_2}$ by $\frac{E(S_1)}{E(S_2)}$ and $\frac{S_3}{S_2}$ by $\frac{E(S_3)}{E(S_2)}$ since otherwise

M_F would be non-linear in S_1, S_2 and S_3,

$$\therefore \quad x = 10.0 - 5.0\ \frac{1.0}{2.0} + 0.739375\ \frac{8.0}{2.0} = 4.54.25 \text{ m}$$

$$M_F = 24.1625\ S_1 + 10.10785\ S_2 + 4.035131\ S_3$$

SUBSEQUENT LOADING:

All members are assumed perfectly ductile at failure.

Let the initial bending moment applied to the first cross-section at which failure occurs by forming a hinge be $(d_{11}\ S_1 + d_{12}\ S_2 + d_{13}\ S_3)$ and the local resistance be R_1, ∴ bending moment to be redistributed = $(d_{11}\ S_1 + d_{12}\ S_2 + d_{13} * S_3 - R_1)$ kNm.

If the initial bending moment applied to a cross-section on another member is $(d_{21}\ S_1 + d_{22}\ S_2 + d_{23}\ S_3\ )$ and its resistance is R_2, then the excess resistance after the first hinge has formed will be

$$Z = R_2 - (d_{21}\ S_1 + d_{22}\ S_2 + d_{23}\ S_3\ ) - a_{21}\ (d_{11}\ S_1 + d_{12}\ S_2 + d_{13}\ S_3 - R_1)$$

$$= a_{21}\ R_1 + R_2 - ((d_{21} + a_{21}\ d_{11})\ S_1 + (d_{22} + a_{21}\ d_{12})\ S_2 + (d_{23} + a_{21}\ d_{13})\ S_3)$$

when a_{21} is the extra bending moment applied by the redistribution of a unit bending moment from the first hinge at the second cross-section.

VALUES OF a_{21}:

Bending Moment applied at E (or G)

$$M_A = M_B = M_F = (10.0/a)\ 1.0 = 1.3617 \text{ kNm}$$

$$M_E = M_G = 1.0 \text{ kNm}$$

Bending Moment applied at A (or B or F)

$$M_A = M_B = M_F = 1.0 \text{ kNm}$$

$$M_E = M_G = (a/10.0)\ 1.0 = 0.7344 \text{ kNm}$$

Sign to be the opposite sign to the initial bending moment at the first local failure mode.