

FUNCTIONAL INEQUALITIES FOR DUNKL OPERATORS WITH APPLICATIONS

A THESIS PRESENTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY OF IMPERIAL COLLEGE LONDON
AND THE
DIPLOMA OF IMPERIAL COLLEGE
BY
CORNEL ANDREI VELICU

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE
180 QUEEN'S GATE, LONDON SW7 2AZ

APRIL 2019

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

COPYRIGHT

The copyright of this thesis rests with the author. Unless otherwise indicated, its contents are licensed under a Creative Commons Attribution-NonCommercial 4.0 International Licence (CC BY-NC).

Under this licence, you may copy and redistribute the material in any medium or format. You may also create and distribute modified versions of the work. This is on the condition that: you credit the author and do not use it, or any derivative works, for a commercial purpose.

When reusing or sharing this work, ensure you make the licence terms clear to others by naming the licence and linking to the licence text. Where a work has been adapted, you should indicate that the work has been changed and describe those changes.

Please seek permission from the copyright holder for uses of this work that are not included in this licence or permitted under UK Copyright Law.

Functional Inequalities for Dunkl Operators with Applications

ABSTRACT

In this thesis we thoroughly study several families of classical functional inequalities in the setting of Dunkl operators. The following families are considered: Sobolev, Hardy, Gagliardo-Nirenberg, logarithmic Sobolev, Poincaré inequalities. In each case, several versions of the inequality are discussed, as well as related results. In the case of the Sobolev inequality for $p = 2$ we provide bounds on the optimal constant, and in the process we also determine the precise optimal constants for the equivalent weighted Sobolev inequality for the usual gradient. In terms of spectral inequalities, we obtain Lieb-Thirring and Cwikel-Lieb-Rozenblum inequalities for Schrödinger operators with Dunkl laplacian.

As an application of the Gagliardo-Nirenberg inequality, we prove an existence and uniqueness result for the Cauchy problem with small data of the nonlinear damped wave equation. Furthermore, we prove ergodicity for an infinite dimensional Markov semigroup defined in terms of Dunkl operators.

ACKNOWLEDGMENTS

Firstly, I wish to thank my supervisor, Bogusław Zegarliński, for all his help throughout my PhD, and for motivating me to branch out and become a well rounded mathematician. I am particularly grateful for his patience and constant encouragement.

I would also like to express my gratitude to Ari Laptev, who, without having any obligation towards my studies, took an active pastoral role and later kindly agreed to be my supervisor. I owe him my discovery of the fascinating area of spectral analysis.

A special thanks must go to Jean-Philippe Anker who introduced me to Dunkl theory before I even started my PhD, and who continued to take an interest in my work and support my career development.

I am also grateful to all those colleagues from Imperial College and from elsewhere who shared with me their insight.

Last, but not least, I would like to thank my family, my girlfriend and my friends for their unreserved support.

CONTENTS

INTRODUCTION	8
1 PRELIMINARIES ON DUNKL THEORY	14
1.1 Root systems	14
1.2 Dunkl operators	16
1.3 The Dunkl kernel	20
1.4 Dunkl transform	22
1.5 The one-dimensional case	23
1.6 The heat semigroup	24
1.7 Sobolev spaces	27
1.8 Spherical h-harmonics	28
2 SOBOLEV INEQUALITIES	30
2.1 The carré-du-champ operator	32
2.2 Nash's inequality	35
2.3 Pseudo-Poincaré inequality	39
2.4 Estimates of the best constant	48
2.4.1 Isoperimetric inequality	49
2.4.2 Sharp constants	54
3 HARDY INEQUALITIES	59
3.1 L^p Hardy inequality	60
3.2 Hardy inequality for fractional laplacian	64
3.3 Improved Hardy's inequalities	65
3.4 Other Hardy-type inequalities	69
3.4.1 The Rellich Inequality	69
3.4.2 The Caffarelli-Kohn-Nirenberg Inequality	72

4	SPECTRAL ESTIMATES	76
4.1	The Lieb-Thirring inequality	77
4.2	The CLR inequality	83
5	THE GAGLIARDO-NIRENBERG INEQUALITY AND AN APPLICATION	85
5.1	The Gagliardo-Nirenberg inequality I	86
5.2	The Gagliardo-Nirenberg inequality II	88
5.3	The nonlinear damped wave equation	93
5.3.1	The linear case	93
5.3.2	The main result	97
6	LOG-SOBOLEV INEQUALITIES	103
6.1	The classical Log-Sobolev inequality	104
6.2	U-bounds	108
6.3	Log-Sobolev inequalities for weighted measures	112
6.4	Poincaré inequalities	115
6.5	Tight log-Sobolev inequalities	120
7	INFINITE DIMENSIONAL ANALYSIS	124
7.1	The finite dimensional case	125
7.1.1	Gradient bounds	126
7.1.2	Invariant measures	130
7.2	The infinite dimensional case	136
7.2.1	Construction of the semigroup	136
7.2.2	Invariant measure	144
	CONCLUSIONS AND FUTURE WORK	149
	APPENDIX A PROOF OF THEOREM 6.4.6	151
	APPENDIX B SOME PROBABILITY THEORY	155
	REFERENCES	157

INTRODUCTION

Rational Dunkl operators were introduced by Dunkl in [31] to study special functions associated to root systems but have since been studied extensively and have found many applications outside harmonic analysis. They are differential-difference operators defined in terms of a root system. The geometric structure of the root system provides Dunkl operators with a rich theory; an example of a surprising remarkable property is the fact that Dunkl operators commute.

Another important property is the existence of an intertwining operator which intertwines between the usual partial derivatives and Dunkl operators. With the help of this operator, the theory of Dunkl transform can be established. This generalises the usual theory of Fourier transform and shares many of its powerful characteristics.

As mentioned above, Dunkl operators were first introduced by Dunkl in [31] and then subsequently developed in the papers [32], [33], [34]. Important early contributions appeared also in [28], [80] and [35]. The next important achievement was due to Rösler who showed in [86] that the intertwining operator is positive, and thus the Dunkl heat kernel (which she introduced and studied in [85]) is also positive. A more complete account of the early history of Dunkl history can be found in [87].

More recently, there has been extensive work to generalise classical harmonic analysis results to Dunkl theory. To give a few examples, maximal functions and Riesz potentials are discussed in [99], [100], [53], [1], [47] and [48], singular integrals in [2], while the Hardy space H^1 was studied in [4], [36], [5] and [37]. For a survey of the more recent directions in Dunkl theory see [3].

Rational Dunkl operators are just one class of differential-difference operators associated to reflection groups. Another important class are the trigonometric Dunkl operators of Cherednik, see [3] for an introduction to this class and for an overview of general Dunkl theories. In this thesis we only consider the rational case.

Apart from the theory of special functions and harmonic analysis associated to root systems, Dunkl operators have found applications most notably in mathematical physics, where they are used in the study of quantum many body problems associated to the Calogero-Moser-Sutherland model, see [104] for more details. In probability theory, Dunkl operators have been used to define Markov processes with remarkable properties; for an

overview of probabilistic aspects of Dunkl theory see [49].

Functional inequalities lie at the heart of Analysis, being a fundamental tool for estimating quantities that are a priori unknown. Classical inequalities such as Sobolev or Hardy inequalities are ubiquitous in Analysis and their importance cannot be overstated. These classical inequalities were later refined to deal with more precise problems, for example in the theory of partial differential equations (inequalities of Nash, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, etc.).

In this thesis we begin a study of functional inequalities for rational Dunkl operators. The special nature of Dunkl operators means that they do not fit into any prescribed category, so no “recipe” can be applied in their study. We have thus surveyed a large number of methods and ideas corresponding to various pathological cases. Harmonic analysis methods have proven very useful at times, and the theory of Dunkl transform is particularly powerful. On the other hand, to consider just one example, the fact that the Leibniz rule does not hold is problematic and has prevented us from using several very nice approaches.

In the context of Dunkl theory, functional inequalities have not been explicitly studied before. There exist a few results in the literature, usually as byproducts of a more general harmonic analysis theory. The Sobolev inequality for a restricted range of indices was obtained in [1] as a consequence of the theory of Riesz potentials for Dunkl operators. Pitt’s inequality was first proved in [95] in a special case, and then in [46] in full generality. A Hardy-Rellich inequality on a Lizorkin space was obtained in [48] using the Dunkl-Riesz potential.

We now give an outline of the thesis. More background on each family of inequalities and a literature review is given at the beginning of the relevant chapter.

Chapter 1 briefly introduces the theory of Dunkl operators and their main properties, and it also sets the notation that will be used throughout the whole thesis. This chapter was designed so that it provides a clear introduction to Dunkl theory and it covers all the material that will be necessary in the subsequent chapters. However, it is not intended to give a complete account of Dunkl theory and its modern developments; more detailed surveys can be found in [87] and [3].

In Chapter 2 we study the Sobolev inequality for Dunkl operators. Combining the results of this chapter with previous results in the literature, we can provide the Sobolev inequality in full generality

$$\|f\|_q \leq C \|\nabla_k f\|_p, \quad (1)$$

where $1 \leq p < N + 2\gamma$ and $q = \frac{p(N+2\gamma)}{N+2\gamma-p}$. An elementary proof of the important case $p = 2$

is provided as a consequence of the Nash inequality

$$\|f\|_2^{1+\frac{2}{N+2\gamma}} \leq C \|\nabla_k f\|_2 \|f\|_1^{\frac{2}{N+2\gamma}}.$$

Using a general method based on the pseudo-Poincaré inequality

$$\|f - H_t f\|_p \leq C \sqrt{t} \|\nabla_k f\|_p,$$

where H_t is the Dunkl heat semigroup, we prove a more general Besov space embedding result that implies the Sobolev inequality for $1 \leq p \leq 2$. A special case of the Gagliardo-Nirenberg inequality is also obtained from this result.

In the second part of the chapter we study the optimal constant in inequality (1) for $p = 2$. An essential ingredient here is the simple but powerful inequality

$$\int_{\mathbb{R}^N} |\nabla f|^2 d\mu_k \leq \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k, \quad (2)$$

which we obtain through carré-du-champ methods. This inequality allows us to obtain bounds for the optimal constant in (1), in terms of the optimal constant in its classical counterpart (when $k = 0$). More precisely, we determine the sharp constant in the classical weighted Sobolev inequality

$$\int_{\mathbb{R}^N} |f|^q d\mu_k \leq C \int_{\mathbb{R}^N} |\nabla f|^2 d\mu_k,$$

where $q = \frac{2(N+2\gamma)}{N+2\gamma-2}$. This is obtained by first proving the isoperimetric inequality for the weighted measure μ_k , which then implies a rearrangement inequality.

Chapter 3 contains a thorough discussion of many forms of Hardy's inequality for Dunkl operators and some applications. Firstly, the L^p Hardy inequality is proved:

$$\int_{\mathbb{R}^N} \frac{|f|^p}{|x|^p} d\mu_k \leq C \int_{\mathbb{R}^N} |\nabla_k f|^p d\mu_k, \quad (3)$$

for $1 < p < \frac{N+2\gamma}{1+2\gamma}$. In case $p = 2$ we are able to remove this restriction on γ by using the stronger inequality (2); in this case we obtain (3) for $N + 2\gamma > 2$. In case $p = 2$ we also determine the optimal constant $\frac{(N+2\gamma-2)^2}{4}$, which, as in the classical case, is not achieved. Furthermore, we also prove the improved Hardy inequality

$$\int_{\Omega} |\nabla_k f|^2 d\mu_k \geq \frac{(N + 2\gamma - 2)^2}{4} \int_{\Omega} \frac{f^2}{|x|^2} d\mu_k + C \left(\int_{\Omega} |f|^q X^{1+\frac{q}{2}} \left(\frac{|x|}{\delta} \right) d\mu_k \right)^{\frac{2}{q}},$$

where $q = \frac{2(N+2\gamma)}{N+2\gamma-2}$, $\delta = \sup_{x \in \Omega} |x|$, and $X(t) = (1 - \log t)^{-1}$.

We conclude the chapter with two Hardy-type inequalities: the Rellich inequality

$$\int_{\mathbb{R}^N} |\Delta_k f|^2 d\mu_k \geq \frac{(N+2\gamma)^2(N+2\gamma-2)^2}{16} \int_{\mathbb{R}^N} \frac{|f|^2}{|x|^4} d\mu_k,$$

with sharp constant, and the Caffarelli-Kohn-Nirenberg inequality

$$\int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} d\mu_k \leq C \left(\frac{|f|^p}{|x|^{pb}} \right)^{\frac{2}{p}},$$

where $a \leq b \leq a+1$, $a < \frac{N+2\gamma}{2} - 1$, and $p = \frac{2(N+2\gamma)}{N+2\gamma-2+2(b-a)}$.

Chapter 4 is devoted to spectral estimates for the Dunkl Schrödinger operator

$$H := -\Delta_k + V,$$

for an admissible potential V . This operator has discrete negative spectrum consisting of eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots < 0$ repeated according to multiplicity. Our first result is the Lieb-Thirring inequality

$$\sum_i |\lambda_i|^\zeta \leq C \int_{\mathbb{R}^N} V_+^{\frac{N+2\gamma}{2}+\zeta} d\mu_k,$$

for any $\zeta > 0$, where $V_+(x) = \max\{V(x), 0\}$. The case $\zeta = 0$, known as the Cwikel-Lieb-Rozenblum inequality, estimates the number $N(V)$ of negative eigenvalues of H , and is the subject of our next result:

$$N(V) \leq C \int_{\mathbb{R}^N} V_+^{\frac{N+2\gamma}{2}} d\mu_k.$$

In Chapter 5 we prove two forms of the Gagliardo-Nirenberg inequality. The first one is a simple consequence of the Sobolev inequality and it states that

$$\|f\|_q \leq C \|\nabla_k f\|_r^\theta \|f\|_p^{1-\theta},$$

where the coefficients $1 \leq p, q < \infty$, $1 \leq r < N+2\gamma$ and $0 \leq \theta \leq 1$ satisfy the relation

$$\theta \left(\frac{1}{N+2\gamma} + \frac{1}{p} - \frac{1}{r} \right) = \frac{1}{p} - \frac{1}{q}.$$

The second Gagliardo-Nirenberg inequality is more involved and it required harmonic analysis methods based on the Littlewood-Paley theory. It states that for $1 < p < \infty$, $0 \leq s \leq \frac{N+2\gamma}{p}$ and $0 < \theta < 1$ we have

$$\|(-\Delta_k)^{s(1-\theta)/2}\|_p \leq C \|(-\Delta_k)^{s/2}\|_p^{1-\theta} \|f\|_p^\theta.$$

In the second part of the chapter, we discuss the nonlinear damped wave equation for Dunkl laplacian

$$\partial_{tt}^2 u - \Delta_k u + b \partial_t u + mu = f(u), \quad (4)$$

where $b, m > 0$ and the nonlinearity f satisfies the assumptions $f(0) = 0$ and

$$|f(a) - f(b)| \leq C(|a|^{p-1} + |b|^{p-1})|a - b|.$$

The main result is a global existence and uniqueness theorem for the Cauchy problem with small data associated to equation (4).

Chapter 6 discusses logarithmic Sobolev inequalities. In the case of the measure μ_k , with the help of the Sobolev inequality we prove that

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 d\mu_k} d\mu_k \leq \epsilon \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k + C(\epsilon) \int_{\mathbb{R}^N} f^2 d\mu_k,$$

which holds for all $\epsilon > 0$, where $C(\epsilon)$ is a decreasing function. The decay of $C(\epsilon)$ allows us to deduce the ultracontractivity property of the heat semigroup.

In the case of weighted probability Dunkl measures of the form $d\mu_U := \frac{1}{Z} e^{-U} d\mu_k$, we prove several generalised log-Sobolev inequalities. This is achieved with the help of U -bounds, which are inequalities of the form

$$\int_{\mathbb{R}^N} |f|^p V d\mu_u \leq C \int_{\mathbb{R}^N} |\nabla_k f|^p d\mu_U + C_2 \int_{\mathbb{R}^N} |f|^p d\mu_U,$$

where V is a suitable function which depends on U .

Also discussed in this chapter are Poincaré inequalities for the Dunkl gradient. We obtain these results from their counterparts for the usual gradient ∇f with Dunkl weight w_k , which we prove by showing that w_k is a Muckenhoupt weight for sufficiently small γ . The resulting Poincaré inequalities are finally used to improve the log-Sobolev inequalities proved above into tight versions, with no dependence of $\|f\|_p$ on the right hand side.

In Chapter 7 we construct and study a Markov semigroup on an infinite dimensional space, defined as an infinite product of finite dimensional diffusions generated by operators of the form

$$L := \Delta_k + b \cdot \nabla_k,$$

with additional interactions. The main tool for defining and studying this semigroup are gradient bounds of the form

$$\tilde{\Gamma}(P_t f) \leq e^{-ct} P_t(\tilde{\Gamma}(f)).$$

A suitable gradient form $\tilde{\Gamma}$ is considered, which takes into account the reflections asso-

ciated to Dunkl operators. Apart from the construction of the semigroup, we prove the existence of invariant measure and smoothing properties of the semigroup. This extends results of [111] for differential-difference operators with a single reflection term, to general Dunkl operators for any root system. Markov processes associated to Dunkl operators have also been studied (in the finite dimensional case) for example in [88], [43] and [44]; see also [49] (and references therein).

Some of the results in this thesis are contained in [105], [106] and [107].

1

PRELIMINARIES ON DUNKL THEORY

This chapter serves as a self-contained introduction to the main concepts and results in Dunkl theory. While it aims to provide sufficient background and context for the subsequent chapters, it is not our intention to give a full introduction to the topic, and for more details we recommend the excellent survey papers [87] and [3].

1.1 ROOT SYSTEMS

Root systems lie at the foundation of Dunkl theory, their geometric structure being the “secret” behind some of the most striking properties of Dunkl operators. However, we should note that their history goes back a long way, especially in relation to the classification problem for Lie algebras. We give here a small summary of facts about root systems that will be relevant to us later; the reader should consult [57] for a more complete exposition.

In order to introduce the main notions, we first make some definitions.

Definition 1.1.1. A **root system** is a finite set $R \subset \mathbb{R}^N \setminus \{0\}$ such that

- (i) $R \cap \alpha\mathbb{R} = \{-\alpha, \alpha\}$ for all $\alpha \in R$, and
- (ii) $\sigma_\alpha(R) = R$ for all $\alpha \in R$, where σ_α is the reflection in the hyperplane orthogonal to α , i.e.,

$$\sigma_\alpha x = x - 2 \frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

Definition 1.1.2. The **reflection group associated to the root system** R is the

subgroup G of the orthogonal group $O(N)$ generated by all the reflections σ_α for $\alpha \in R$:

$$G := \langle \sigma_\alpha \mid \alpha \in R \rangle.$$

Definition 1.1.3. The **Weyl chambers** associated to the root system R are the connected components of

$$\mathbb{R}^N \setminus \{x \in \mathbb{R}^N : \langle \alpha, x \rangle = 0 \text{ for some } \alpha \in R\}.$$

Let E be the set of all functions $\epsilon : R_+ \rightarrow \{-1, 1\}$, and for each $\epsilon \in E$ let

$$\mathbb{R}_\epsilon^N = \{x \in \mathbb{R}^N : \text{sgn}(\langle \alpha, x \rangle) = \epsilon(\alpha) \text{ for all } \alpha \in R_+\}.$$

Weyl chambers are all of the form $\mathbb{R}_\epsilon^N$ for some $\epsilon \in E$; equivalently, if $\mathbb{R}_\epsilon^N \neq \{0\}$, then $\mathbb{R}_\epsilon^N$ is a Weyl chamber.

The following result summarises some properties of the reflection group G .

Proposition 1.1.4. (i) G is a finite group.

(ii) G acts simply transitively on the set of Weyl chambers, i.e., for any Weyl chambers H, H' there exists exactly one $g \in G$ such that $gH = H'$.

(iii) The number of Weyl chambers equals the order of G .

We can write the root system R as a disjoint union $R = R_+ \cup (-R_+)$, and we call R_+ a **positive subsystem**; this decomposition is not unique, but, as we will see, the particular choice of positive subsystem does not make a difference in our definitions below.

A root system is **irreducible** if it cannot be written as the disjoint union of two orthogonal root systems. Irreducible root systems are the building blocks of root systems, as any root system can be uniquely decomposed into irreducible components. Moreover, all irreducible root systems are fully classified using Coxeter graphs; there are four infinite families of root systems:

$$A_N, B_N, D_N \text{ and } I_2(m),$$

as well as the exceptional cases:

$$E_6, E_7, E_8, F_4, H_3 \text{ and } H_4.$$

We give details below about the infinite families and we refer the reader to [57] for the exceptional cases.

1. A_N : this is a root system in $\mathbb{R}^{N+1}$, defined for $N \geq 1$, with

$$R = \{\pm(e_i - e_j) : 1 \leq i < j \leq N+1\},$$

and reflection group the symmetric group S_{N+1} which acts on $\mathbb{R}^N$ by permuting the standard basis $e_1, \dots, e_{N+1}$.

2. B_N : this root system is defined in $\mathbb{R}^N$ for $N \geq 2$ by

$$R = \{\pm e_i : 1 \leq i \leq N\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq N\}.$$

The reflection group G is isomorphic to the semidirect product $\mathbb{Z}_2^N \rtimes S_N$, and it is generated by permutations of the standard basis as above, as well as sign changes $e_i \mapsto -e_i$.

3. D_N : in this case we have a root system defined in $\mathbb{R}^N$ for $N \geq 3$ by

$$R = \{\pm e_i \pm e_j : 1 \leq i < j \leq N\},$$

with reflection group $\mathbb{Z}_2^{N-1} \rtimes S_N$, acting on $\mathbb{R}^N$ by permutations of the standard basis, and by an even number of sign changes, i.e., by reflections $e_i + e_j \mapsto -(e_i + e_j)$.

4. $I_2(m)$: this is a root system corresponding to reflection group $D_{2m} = \mathbb{Z}_m \rtimes \mathbb{Z}_2$, the dihedral group of order $2m$. The best way to describe the root system is in $\mathbb{C}$, by

$$R = \left\{ e^{i \frac{j\pi}{m}} : 0 \leq j < 2m \right\}.$$

1.2 DUNKL OPERATORS

From now on we fix a root system R in $\mathbb{R}^N$ with positive subsystem R_+ and associated reflection group G . We also assume without loss of generality that $|\alpha|^2 = 2$ for all $\alpha \in R$.

Definition 1.2.1. A **multiplicity function** on the root system R is a map $k : R \rightarrow [0, \infty)$ which is G -invariant, i.e.,

$$k(\alpha) = k(g\alpha) \quad \forall g \in G, \quad \forall \alpha \in R.$$

We will normally write $k_\alpha = k(\alpha)$ as these will be the coefficients in our Dunkl operators.

It is possible to define Dunkl operators for general multiplicity functions $k : R \rightarrow \mathbb{R}$ but some of the results below only hold in the case $k \geq 0$. For our purpose it is enough to consider only this case.

Definition 1.2.2. Fix a multiplicity function k for the root system R . For $i = 1, \dots, N$, we define the **Dunkl operator** T_i on $C^1(\mathbb{R}^N)$ by

$$T_i f(x) = \partial_i f(x) + \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

Note that this definition does not depend on the choice of positive subsystem R_+ since, by the G -invariance of k , we can also write

$$T_i f(x) = \partial_i f(x) + \frac{1}{2} \sum_{\alpha \in R} k_\alpha \alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

Note also that in the case of $k = 0$ we recover the usual partial derivatives.

The denominator $\langle \alpha, x \rangle$ appearing in the definition of T_i does not actually produce a singularity as can be seen through the following representation formula

$$\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} = \int_0^1 \alpha \cdot \nabla f(x - t\langle \alpha, x \rangle \alpha) dt. \quad (1.1)$$

Moreover, with the help of (1.1) we can describe the action of Dunkl operators on classical spaces of functions.

Lemma 1.2.3. 1. If $f \in C^m(\mathbb{R}^N)$ with $m \geq 1$, then $T_i f \in C^{m-1}(\mathbb{R}^N)$;

2. Dunkl operators leave invariant the space of smooth compactly supported functions, $C_c^\infty(\mathbb{R}^N)$, and the space of Schwarz functions, $\mathcal{S}(\mathbb{R}^N)$.

One of the fundamental results, due to Dunkl [31] at the beginnings of the theory, is that Dunkl operators commute.

Theorem 1.2.4. For any $i, j = 1, \dots, N$, we have

$$T_i T_j = T_j T_i.$$

So we see that in some ways Dunkl operators behave just like partial derivatives. However, and this is what makes the theory interesting, some well-known properties of partial derivatives (and other useful families of operators) fail for Dunkl operators. Most notably, the Leibniz formula does not hold in general for Dunkl operators. Instead, we have the following product rule.

Lemma 1.2.5. For any $f, g \in C^1(\mathbb{R}^N)$ and for any $i = 1, \dots, N$, we have

$$T_i(fg)(x) = T_i f(x)g(x) + f(x)T_i g(x) - \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{(f(x) - f(\sigma_\alpha x))(g(x) - g(\sigma_\alpha x))}{\langle \alpha, x \rangle}.$$

In particular, taking $f = g$, we have

$$T_i(f^2)(x) = 2f(x)T_i f(x) - \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{(f(x) - f(\sigma_\alpha x))^2}{\langle \alpha, x \rangle}.$$

Proof. By definition, we have

$$\begin{aligned} T_i(fg)(x) &= \partial_i(fg)(x) + \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{fg(x) - fg(\sigma_\alpha x)}{\langle \alpha, x \rangle} \\ &= \partial_i f(x)g(x) + f(x)\partial_i g(x) + g(x) \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} \\ &\quad + f(\sigma_\alpha x) \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{g(x) - g(\sigma_\alpha x)}{\langle \alpha, x \rangle} \\ &= T_i f(x)g(x) + f(x)T_i g(x) - \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{(f(x) - f(\sigma_\alpha x))(g(x) - g(\sigma_\alpha x))}{\langle \alpha, x \rangle}. \end{aligned}$$

□

An important particular case of this is when one of the functions f and g is G -invariant, in which case the Leibniz formula does hold. This will be exploited extensively and so we record it below.

Corollary 1.2.6. *Let $f, g \in C^1(\mathbb{R}^n)$ such that one of f and g is G -invariant. Then, for any $i = 1, \dots, N$, we have*

$$T_i(fg) = T_i(f)g + fT_i(g).$$

Starting from Dunkl operators, we will consider problems similar to those of classical analysis. Two important concepts that are generalisation to Dunkl operators of their classical counterparts, and which will play a central role in the analysis below, are the Dunkl gradient and Dunkl laplacian.

Definition 1.2.7. The **Dunkl gradient** is defined by

$$\nabla_k := (T_1, \dots, T_N),$$

and the **Dunkl laplacian** is defined by

$$\Delta_k := \sum_{i=1}^N T_i^2.$$

Note that for $k = 0$ Dunkl operators reduce to partial derivatives, and $\nabla_0 = \nabla$ and $\Delta_0 = \Delta$ are the usual gradient and laplacian.

The Dunkl laplacian can be expressed in terms of the usual gradient and laplacian using the following formula.

Proposition 1.2.8. *For any $f \in C^2(\mathbb{R}^N)$ we have*

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k_\alpha \left[\frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2} \right]. \quad (1.2)$$

The natural spaces to study Dunkl operators are the weighted spaces $L^p(\mu_k)$, with

$$d\mu_k = w_k(x) dx,$$

where the **Dunkl weight** is

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k_\alpha}.$$

This is a homogeneous function of degree 2γ , where

$$\gamma := \sum_{\alpha \in R_+} k_\alpha.$$

We will mostly work in the spaces $L^p(\mu_k)$, and we will write the norm of these spaces simply $\|\cdot\|_p$. With respect to this weighted measure we have the following integration by parts formula:

Proposition 1.2.9. *For any $f \in C_b^1(\mathbb{R}^N)$ and $g \in \mathcal{S}(\mathbb{R}^N)$, we have*

$$\int_{\mathbb{R}^N} T_i(f)g d\mu_k = - \int_{\mathbb{R}^N} f T_i(g) d\mu_k.$$

An important integral in Dunkl theory is the following Macdonald-Mehta constant which is a generalisation to root systems of a previous integral by Mehta.

Definition 1.2.10. The **Macdonald-Mehta constant** associated to the system R with multiplicity function k is

$$M_k := \int_{\mathbb{R}^N} e^{-|x|^2/2} d\mu_k.$$

Macdonald conjectured in [75] that

$$M_k = (2\pi)^{N/2} \prod_{\alpha \in R_+} \frac{\Gamma(2k_\alpha + 1)}{\Gamma(k_\alpha + 1)}. \quad (1.3)$$

He also proved the result for infinite classes of root systems. Later, Opdam proved in

[79] the result in general for all crystallographic root systems. Finally, a proof in full generality was given by Etingof in [38].

One application of the Macdonald-Mehta integral that we will often use is in computing integrals of radial functions.

Proposition 1.2.11. *Let $f : \mathbb{R}^N \rightarrow \mathbb{C}$ be a radial function. Then*

$$\int_{\mathbb{R}^N} f(x) d\mu_k(x) = \frac{M_k}{2^{\frac{N+2\gamma}{2}-1} \Gamma\left(\frac{N+2\gamma}{2}\right)} \int_0^\infty f(r) r^{N+2\gamma-1} dr.$$

Proof. Using polar coordinates $x = r\theta$ for $r > 0$ and $\theta \in \mathbb{S}^{N-1}$, we have, since w_k is homogeneous of degree 2γ ,

$$\int_{\mathbb{R}^N} f(x) d\mu_k(x) = p(B_1) \int_0^\infty f(r) r^{N+2\gamma-1} dr, \quad (1.4)$$

where

$$p(B_1) = \int_{\mathbb{S}^{N-1}} w_k(\theta) d\sigma(\theta).$$

Applying this to the Macdonald-Mehta integral, we obtain

$$M_k = p(B_1) \int_0^\infty e^{-r^2/2} r^{N+2\gamma-1} dr = p(B_1) 2^{\frac{N+2\gamma}{2}-1} \Gamma\left(\frac{N+2\gamma}{2}\right),$$

so

$$p(B_1) = \frac{M_k}{2^{\frac{N+2\gamma}{2}-1} \Gamma\left(\frac{N+2\gamma}{2}\right)}. \quad (1.5)$$

The conclusion then follows from (1.4). $\square$

1.3 THE DUNKL KERNEL

An important function associated with Dunkl operators is the Dunkl kernel $E_k(x, y)$ which acts as a root systems generalisation of the exponential $e^{\langle x, y \rangle}$. Apart from being a crucial tool in the development of the Dunkl transform, the Dunkl kernel is also significant in the study of special functions.

There are two ways to define the Dunkl kernel. In order to proceed with the first route, we need to introduce the Dunkl intertwining operator, which is also of independent interest. The intertwining operator, which originates in the works of Dunkl, is a linear isomorphism on the space $\mathcal{P}$ of polynomials on $\mathbb{R}^N$, which intertwines between partial derivatives and Dunkl operators.

Proposition 1.3.1. *Let $\mathcal{P}_n$, for $n \in \mathbb{N}_0$, be the space of homogeneous polynomials on $\mathbb{R}^N$ of degree n . Then, there exists a unique linear isomorphism*

$$V_k : \mathcal{P} \rightarrow \mathcal{P}$$

such that $V_k(c) = c$ for all $c \in \mathcal{P}_0$, $V_k(\mathcal{P}_n) = \mathcal{P}_n$ for all $n \in \mathbb{N}_0$, and

$$T_i V_k = V_k \partial_i \quad \text{for all } i = 1, \dots, N. \quad (1.6)$$

*The operator V_k thus defined will be called the **Dunkl intertwining operator**.*

The proof of this Proposition, due to Dunkl [33], is an exercise in linear algebra and it is not constructive. In fact, it is in general very difficult to compute the intertwining operator explicitly. Trimèche [102] later showed that the operator V_k can be extended to the space $C^\infty(\mathbb{R}^N)$ of smooth functions, while preserving the intertwining property (1.6). This allows us to define the Dunkl kernel.

Definition 1.3.2. The **Dunkl kernel** or **Dunkl exponential** is the map $E_k : \mathbb{R}^N \times \mathbb{C}^N \rightarrow \mathbb{C}$ defined by

$$E_k(x, y) := V_k(e^{\langle \cdot, y \rangle})(x).$$

As in the classical case, the Dunkl kernel provides a class of eigenfunctions for Dunkl operators.

Proposition 1.3.3. *Let $y \in \mathbb{C}^N$. Then, the function $f = E_k(\cdot, y)$ is the unique real analytic solution of the system of equations*

$$\begin{cases} T_i f = y_i f & \forall i = 1, \dots, N \\ f(0) = 1. \end{cases} \quad (1.7)$$

Remark. This property could be used as an alternative definition of the Dunkl kernel, which avoids the intertwining operator. More precisely, we could define $E_k(\cdot, y)$ as the unique real analytic solution of the system of equations (1.7).

Below are some basic properties of the Dunkl kernel.

Proposition 1.3.4. (i) *The Dunkl kernel E_k extends to a holomorphic map on $\mathbb{C}^N \times \mathbb{C}^N$.*

(ii) *$E_k(x, y) = E_k(y, x)$ for all $x, y \in \mathbb{C}^N$.*

(iii) *$E_k(gx, gy) = E_k(x, y)$ for all $g \in G$, $x, y \in \mathbb{C}^N$.*

(iv) *$E_k(tx, y) = E_k(x, ty)$ for all $t \in \mathbb{C}$, $x, y \in \mathbb{C}^N$.*

(v) $\overline{E_k(x, y)} = E_k(\bar{x}, \bar{y})$ for all $x, y \in \mathbb{C}^N$.

As with the intertwining operator, it is in general very difficult to compute the Dunkl kernel explicitly. Some further understanding of the Dunkl exponential is provided by the results of [86]. In this paper Rösler showed that the intertwining operator is positive on the space of polynomials $\mathcal{P}$, and using this she found a representation formula in terms of a family of probability measures. The main results are summarised in the following Theorem.

Theorem 1.3.5. (i) V_k is positive on $\mathcal{P}$, i.e., $V_k p \geq 0$ for all $p \in \mathcal{P}$ with $p \geq 0$.

(ii) For each $x \in \mathbb{R}^N$ there exists a unique Borel probability measure μ_k^x such that

$$V_k f(x) = \int_{\mathbb{R}^N} f(y) d\mu_k^x(y).$$

The support of μ_k^x is contained in the convex hull of the G -orbit of x .

(iii) From the previous point it follows that for $x \in \mathbb{R}^N$ and $y \in \mathbb{C}^N$ we have the representation formula

$$E_k(x, y) = \int_{\mathbb{R}^N} e^{\langle x, y \rangle} d\mu_k^x(y).$$

There are two important consequences of this result, which we spell out in the next two Propositions. The first simply shows that the Dunkl exponential is positive on $\mathbb{R}^N \times \mathbb{R}^N$, while the second provides some very useful bounds on the Dunkl kernel.

Proposition 1.3.6. The Dunkl kernel is positive on $\mathbb{R}^N \times \mathbb{R}^N$, i.e.,

$$E_k(x, y) > 0 \quad \forall x, y \in \mathbb{R}^N.$$

Proposition 1.3.7. Let $\zeta \in \mathbb{Z}_+^N$ be a multi-index. For any $x, y \in \mathbb{C}^N$ we have the bound

$$|\partial_y^\zeta E_k(x, y)| \leq |x|^{|\zeta|} \max_{g \in G} e^{\operatorname{Re} \langle gx, y \rangle}.$$

We will mostly use the particular case

$$|E_k(ix, y)| \leq 1 \quad \forall x, y \in \mathbb{R}^N.$$

1.4 DUNKL TRANSFORM

Having defined the Dunkl exponential, we are now ready to introduce the Dunkl transform, which is a generalisation to root system of the usual Fourier transform. Indeed, in

the case $k \equiv 0$, so when Dunkl operators equal the usual partial derivatives, we recover the theory of Fourier transform. The Dunkl transform shares many important properties with the classical Fourier transform and, as we will see in the next chapters, it is a very useful tool in proving analytic results for Dunkl operators.

Definition 1.4.1. The **Dunkl transform** is the operator $\mathcal{D}_k : L^1(\mu_k) \rightarrow C_b(\mathbb{R}^N)$ defined by

$$\mathcal{D}_k(f)(\xi) := \frac{1}{M_k} \int_{\mathbb{R}^N} f(x) E_k(-i\xi, x) d\mu_k(x) \quad \forall \xi \in \mathbb{R}^N.$$

Remark. The fact that the operator $\mathcal{D}_k$ is well-defined follows from the bounds on the Dunkl kernel from Proposition 1.3.7.

The main properties of the Dunkl transform are collected in the following Proposition.

Proposition 1.4.2. (i) *The Dunkl transform is an automorphism of the Schwarz space $\mathcal{S}(\mathbb{R}^N)$, with inverse given by*

$$\mathcal{D}_k^{-1}f(\xi) = \mathcal{D}_kf(-\xi).$$

(ii) $\mathcal{D}_k(T_j f)(\xi) = i\xi_j \mathcal{D}_k(f)(\xi)$ for all $j = 1, \dots, N$.

(iii) $T_j(\mathcal{D}_k(f)) = -i\mathcal{D}_k(x_j f)$ for all $j = 1, \dots, N$.

(iv) *The Dunkl transform has a unique extension to an isometric isomorphism of $L^2(\mu_k)$. In particular, the Plancherel formula holds*

$$\|\mathcal{D}_k(f)\|_2 = \|f\|_2.$$

1.5 THE ONE-DIMENSIONAL CASE

To illustrate the concepts introduced above, in this short section we give an account of the one-dimensional case. In this case there is only one root system, namely $R = \{-\sqrt{2}, \sqrt{2}\}$ (taking into account our normalisation assumption that $|\alpha|^2 = 2$ for all $\alpha \in R$), which generates a reflection group isomorphic to $\mathbb{Z}_2$ that acts by changing the sign. The unique Dunkl operator is then

$$Tf(x) := f'(x) + k \frac{f(x) - f(-x)}{x},$$

for a constant $k \geq 0$. Using formula (1.2), one can compute the Dunkl laplacian

$$\Delta_k f(x) = f''(x) + 2k \frac{f'(x)}{x} - k \frac{f(x) - f(-x)}{x^2}.$$

As mentioned above, it is difficult in general to obtain explicit expression for the intertwining operator or the Dunkl kernel; however, the simple form of the Dunkl operator in the one-dimensional case allows one to compute the following:

$$V_k p(x) = \frac{\Gamma(k + 1/2)}{\Gamma(k)\Gamma(1/2)} \int_{-1}^1 p(xt)(1-t)^{k-1}(1+t)^k dt,$$

and also

$$E_k(x, y) = e^{xy} \cdot {}_1F_1(k, 2k + 1, -2xy),$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind.

1.6 THE HEAT SEMIGROUP

In analogy to the classical equation, the Dunkl heat equation is the initial-value problem

$$\begin{cases} (\Delta_k - \partial_t)u(x, t) = 0 \\ u(x, 0) = f(x). \end{cases} \quad (1.8)$$

As in the classical case, this can be solved using Dunkl transform. First, we need to introduce the concept of Dunkl translation.

Definition 1.6.1. For any $x \in \mathbb{R}^N$, the **generalised** or **Dunkl translation** is the operator $\tau_x : \mathcal{S}(\mathbb{R}^N) \rightarrow \mathcal{S}(\mathbb{R}^N)$ defined by

$$\tau_x f(y) = \frac{1}{M_k} \int_{\mathbb{R}^N} \mathcal{D}_k f(\xi) E_k(ix, \xi) E_k(iy, \xi) d\mu_k(\xi). \quad (1.9)$$

An extension of the Dunkl translation to the space $C^\infty(\mathbb{R}^N)$ is due to Trimèche [103], where he uses the definition

$$\tau_x f(y) = V_k^y V_k^x [(V_k^{-1} f)(x + y)],$$

where V_k^x and V_k^y are the intertwining operators acting on the x and y variables, respectively. The two definitions agree on the Schwarz space $\mathcal{S}(\mathbb{R}^N)$.

Similarly to the definition over $\mathcal{S}(\mathbb{R}^N)$, we can define the generalised translation over $L^2(\mu_k)$ via the formula (1.9). Moreover, the operator $\tau_x : L^2(\mu_k) \rightarrow L^2(\mu_k)$ is bounded, with norm $\|\tau_x\| \leq 1$. Apart from this case $p = 2$, which relies on the Plancherel formula, it remains the most important question in Dunkl theory to show that Dunkl translations are uniformly bounded on $L^p(\mu_k)$ for general $p \geq 1$. This problem is equivalent to proving Young's inequality in the Dunkl setting; before we state this, we need to define

a generalised convolution.

Definition 1.6.2. The **Dunkl convolution product** for two functions $f, g \in C_c^\infty(\mathbb{R}^N)$ is defined as

$$f *_k g(x) := \int_{\mathbb{R}^N} \tau_x f(y) g(y) d\mu_k(y) \quad \forall x \in \mathbb{R}^N.$$

The open problem mentioned above can now be stated as the Young's inequality for Dunkl convolution.

Open Problem (Young's inequality). Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then there exists a constant $C \geq 0$ such that for all $f, g \in C_c^\infty(\mathbb{R}^N)$ the following inequality holds

$$\|f *_k g\|_r \leq C \|f\|_p \|g\|_q.$$

Remark. Apart from the one-dimensional case, this problem has also been solved for the case of radial functions. Let by some abuse of notation $f(x) = f(|x|)$. Then there exists an explicit expression for its translation

$$\tau_x f(y) = \int_{\mathbb{R}^N} f\left(\sqrt{|x|^2 + |y|^2 + 2\langle y, z \rangle |x|}\right) d\mu_k^{\frac{x}{|x|}}(z),$$

where μ_k^w are the measures introduced in Theorem 1.3.5. Using this, one can see that the Young's inequality holds if one of f, g is a radial function.

Here are some properties of the generalised translation and convolution.

Proposition 1.6.3. (i) If $k \equiv 0$, then $\tau_x f(y) = f(x + y)$.

(ii) $\tau_x f(y) = \tau_y f(x)$ for all $x, y \in \mathbb{R}^N$.

(iii) $\mathcal{D}_k(\tau_x f)(\xi) = E_k(ix, \xi) \mathcal{D}_k f(\xi)$ for all $x, \xi \in \mathbb{R}^N$.

(iv) $T_j \tau_x f = \tau_x T_j f$ for all $j = 1, \dots, N, x \in \mathbb{R}^N$.

(v) $\tau_x(E_k(\cdot, \xi))(y) = E_k(x, \xi) E_k(y, \xi)$ for all $x, y, \xi \in \mathbb{R}^N$.

(vi) $\mathcal{D}_k(f *_k g)(\xi) = \mathcal{D}_k f(\xi) \mathcal{D}_k g(\xi)$ for all $\xi \in \mathbb{R}^N$.

Going back to the Dunkl heat equation, we want to obtain the solution as a convolution with a convenient “fundamental solution”. This role will be fulfilled by the function

$$g_k(x, t) := \frac{1}{(2t)^{\frac{N+2\gamma}{2}} M_k} e^{-\frac{|x|^2}{4t}}.$$

Indeed, it can be checked by a direct computation that the function g_k satisfies the heat equation $(\Delta_k - \partial_t)u = 0$, and, moreover

$$\int_{\mathbb{R}^N} g_k(x, t) d\mu_k(x) = 1 \quad \forall t > 0,$$

and

$$\mathcal{D}_k(g_k)(\xi, t) = \frac{1}{M_k} e^{-t|\xi|^2}.$$

Definition 1.6.4. The **Dunkl heat kernel** is the function

$$h_t(x, y) = \tau_{-x} g_k(y, t) = \frac{1}{M_k(2t)^{\frac{N+2\gamma}{2}}} e^{-\frac{|x|^2+|y|^2}{4t}} E_k\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right).$$

Remark. The second equality follows from the properties of g_k listed above, as well as the formula [34]

$$\int_{\mathbb{R}^N} E_k(x, z) E_k(y, z) e^{-\frac{|z|^2}{2}} d\mu_k(z) = M_k e^{\frac{|x|^2+|y|^2}{2}} E_k(x, y).$$

The following Proposition collects some properties of the Dunkl heat kernel.

Proposition 1.6.5. (i) $h_t(x, y) > 0$ for all $x, y \in \mathbb{R}^N$, $t > 0$.

(ii) For fixed $y \in \mathbb{R}^N$ and $t > 0$, the function $x \mapsto h_t(x, y)$ belongs to the Schwarz space $\mathcal{S}(\mathbb{R}^N)$.

(iii) $h_t(x, y) = \frac{1}{M_k^2} \int_{\mathbb{R}^N} e^{-t|\xi|^2} E_k(ix, \xi) E_k(-iy, \xi) d\mu_k(\xi)$.

(iv) $h_t(x, y) = h_t(y, x)$.

(v) $\int_{\mathbb{R}^N} h_t(x, y) d\mu_k(x) = 1$ for all $x \in \mathbb{R}^N$ and $t > 0$.

(vi) $h_{t+s}(x, y) = \int_{\mathbb{R}^N} h_t(x, z) h_s(y, z) d\mu_k(z)$.

(vii) $h_t(x, y) \leq \frac{1}{M_k(2t)^{\frac{N+2\gamma}{2}}} \max_{g \in G} e^{-\frac{|gx-y|^2}{4t}}$.

(viii) For any $y \in \mathbb{R}^N$, the function $(x, t) \mapsto h_t(x, y)$ is a solution to the Dunkl heat equation.

Remark. The explicit expression for the Dunkl kernel which is available in one dimension (see section 1.5) allows one to determine the global behaviour of the Dunkl heat kernel in this case; this was done in [4]. More recently, the global behaviour of the heat kernel was determined in full generality in [5] and [37].

Finally, we can define the Dunkl heat semigroup. Consider, for all $f \in C_b(\mathbb{R}^N)$, the operators

$$H_t f(x) := (g_k(\cdot, t) *_k f)(-x) = \int_{\mathbb{R}^N} h_t(x, y) f(y) d\mu_k(y),$$

for $t > 0$, and $H_0 f = f$. As in the classical case, these operators form a positivity-preserving contraction semigroup which solves the initial-value problem (1.8). This is summarised in the following result.

Theorem 1.6.6. *The operators $(H_t)_{t \geq 0}$ define a positivity-preserving contraction semigroup on $C_0(\mathbb{R}^N)$ with infinitesimal generator Δ_k . Moreover, for any $f \in C_0(\mathbb{R}^N)$, the function $(x, t) \mapsto H_t f(x)$ solves the initial-value problem (1.8).*

Definition 1.6.7. The semigroup obtained in Theorem 1.6.6 will be called the **Dunkl heat semigroup**.

1.7 SOBOLEV SPACES

Definition 1.7.1. The **Sobolev space** $H_{\mathcal{D}}^1(\mathbb{R}^N)$ consists of all $f \in L^2(\mu_k)$ such that also $\nabla_k f \in L^2(\mu_k)$, where the gradient is understood in a weak sense. More precisely, $f \in H_{\mathcal{D}}^1(\mathbb{R}^N)$ if and only if $f \in L^2(\mu_k)$ and there exist $f_1, \dots, f_N \in L^2(\mu_k)$ such that

$$\int_{\mathbb{R}^N} f T_i \varphi d\mu_k = - \int_{\mathbb{R}^N} f_i \varphi d\mu_k \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N), \quad \forall i = 1, \dots, N.$$

This defines a Hilbert space with inner product

$$\langle f, g \rangle_{H_{\mathcal{D}}^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} f \bar{g} d\mu_k + \int_{\mathbb{R}^N} \nabla_k f \cdot \nabla_k \bar{g} d\mu_k.$$

It is clear that $C_c^\infty(\mathbb{R}^N) \subset H_{\mathcal{D}}^1(\mathbb{R}^N)$. Moreover, using the same techniques as in the classical case, it can be seen that $C_c^\infty(\mathbb{R}^N)$ is in fact dense in $H_{\mathcal{D}}^1(\mathbb{R}^N)$. A bit of care is needed here because the classical proof of this fact relies on mollifiers and Young's inequality, which does not hold in general in the Dunkl setting. However, a partial result is available, when one of the functions is radial, so density of $C_c^\infty(\mathbb{R}^N)$ in $H_{\mathcal{D}}^1(\mathbb{R}^N)$ can be proved by using radial mollifiers.

As in the classical case, we can provide an alternative characterisation of the Sobolev space $H_{\mathcal{D}}^1(\mathbb{R}^N)$ in terms of the Dunkl transform as follows. Given $f \in L^2(\mu_k)$, we have $f \in H_{\mathcal{D}}^1(\mathbb{R}^N)$ if and only if the map $\xi \mapsto |\xi| \mathcal{D}_k(f)(\xi)$ is in $L^2(\mu_k)$; in this case, by Plancherel's formula we have

$$\|f\|_{H_{\mathcal{D}}^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\mathcal{D}_k(f)(\xi)|^2 (1 + |\xi|^2) d\mu_k(\xi).$$

This justifies the following more general definition.

Definition 1.7.2. For any $s > 0$ we define the **Sobolev space** $H_{\mathcal{D}}^s$ to be the space of all $f \in L^2(\mu_k)$ such that the map $\xi \mapsto |\xi|^s \mathcal{D}_k(f)(\xi)$ is in $L^2(\mu_k)$. Then $H_{\mathcal{D}}^s$ is a Banach space under the norm

$$\|f\|_{H_{\mathcal{D}}^s}^2 = \int_{\mathbb{R}^N} |\mathcal{D}_k(f)(\xi)|^2 (1 + |\xi|^2)^s d\mu_k(\xi).$$

1.8 SPHERICAL H-HARMONICS

In this section we briefly introduce h-harmonics; our presentation here is based on [23] and we invite the interested reader to this reference for more details.

Definition 1.8.1. An *h-harmonic polynomial* of degree n is a homogeneous polynomial p of degree n that satisfies

$$\Delta_k p = 0.$$

Spherical h -harmonics (or just *h-harmonics*) of degree n are then defined to be restrictions of h -harmonic polynomials of degree n to the sphere $\mathbb{S}^{N-1}$.

Let $\mathcal{H}_n^N$ be the space of h -harmonics of degree n ; this is a finite-dimensional space and we denote its dimension by $d(n)$. Moreover, the space $L^2(\mathbb{S}^{N-1}, w_k(\xi) d\sigma(\xi))$, where σ is the surface measure on $\mathbb{S}^{N-1}$, is the orthogonal direct sum of the spaces $\mathcal{H}_n^N$, for $n = 0, 1, 2, \dots$. Let

$$Y_i^n \quad \text{for } i = 1, \dots, d(n)$$

be an orthonormal basis of $\mathcal{H}_n^N$. In spherical polar coordinates $x = r\xi$, for $r \in [0, \infty)$ and $\xi \in \mathbb{S}^{N-1}$, we can write the Dunkl laplacian as

$$\Delta_k = \frac{\partial^2}{\partial r^2} + \frac{N + 2\gamma - 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{k,0}, \quad (1.10)$$

where $\Delta_{k,0}$ is a generalisation of the Laplace-Beltrami operator on the sphere, and it only acts on the ξ variable. Then the spherical h-harmonics are eigenvalues of $\Delta_{k,0}$, i.e.,

$$\Delta_{k,0} Y = -n(n + N + 2\gamma - 2)Y =: \lambda_n Y, \quad \text{for all } Y \in \mathcal{H}_n^N.$$

The h -harmonic expansion of a function $f \in L^2(\mu_k)$ is given by

$$f(r\xi) = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} f_{n,i}(r) Y_i^n(\xi),$$

where

$$f_{n,i}(r) = \int_{\mathbb{S}^{N-1}} f(r\xi) Y_i^n(\xi) w_k(\xi) \, d\sigma(\xi), \quad (1.11)$$

and σ is the surface measure on the sphere $\mathbb{S}^{N-1}$.

2

SOBOLEV INEQUALITIES

The classical Sobolev inequality states that

$$\|f\|_q \leq C \|\nabla f\|_p \quad \forall f \in C_c^\infty(\mathbb{R}^N) \quad (2.1)$$

where $1 \leq p < N$ and $q = \frac{Np}{N-p}$. This is a fundamental result in analysis and it has been widely studied in a variety of contexts, see e.g. the classical references [76], [92].

We will be concerned with this inequality (and related inequalities) in the context of Dunkl theory. The general approach to inequality (2.1) is to represent f as an integral expression involving ∇f . In the Dunkl setting, this type of representation is given by the Riesz transform and Riesz potential, and indeed the Sobolev inequality for $1 < p < N + 2\gamma$ was obtained as a corollary of this theory in [1] (see also [53]).

Combining our results with the existing results of [1], we have obtained the Sobolev inequality for Dunkl operators in full generality. That is, we have the following Theorem.

Theorem 2.0.1. *[Sobolev inequality] Let $1 \leq p < N + 2\gamma$ and $q = \frac{p(N+2\gamma)}{N+2\gamma-p}$. Then there exists a constant $C > 0$ such that we have the inequality*

$$\|f\|_q \leq C \|\nabla_k f\|_p \quad \forall f \in C_c^\infty(\mathbb{R}^N).$$

In this chapter we explore different approaches that provide simpler proofs, or which improve on the existing results. We will first prove the Nash inequality in the Dunkl setting and use this to obtain an elementary proof of the Sobolev inequality in the classical case $p = 2$. Nash's inequality is another important result in analysis and it was first

proved in [77] where it was used in the study of parabolic and elliptic equations. Nash's inequality can be seen as a weaker version of the Sobolev inequality as it can be deduced from the latter using only Hölder's inequality, but the two are in fact equivalent. We will prove this in our context using a nice elementary result of [8].

Using a different approach, based on a pseudo-Poincaré inequality and a method of Ledoux [64], we will prove a more general Besov space result. This implies in particular the Sobolev inequality in the case $1 \leq p \leq 2$ (the limitation $1 \leq p \leq 2$ comes from the pseudo-Poincaré inequality). Note that this includes the case $p = 1$ which was not known before. A Gagliardo-Nirenberg inequality is also obtained.

Finally, we consider the problem of estimating the best constant in the Sobolev inequality. In the classical case (inequality (2.1)), this amounts to finding the supremum

$$C = \sup_{f \in C_c^1(\mathbb{R}^N)} \frac{\|f\|_q}{\|\nabla f\|_p},$$

and the question was answered in [97] and [7]. In the paper of Talenti, it was shown that replacing f by its symmetric decreasing rearrangement f^* increases the fraction in the definition of C , so it is enough to consider the supremum over radial functions. This was done using the Pólya-Szegő inequality

$$\|\nabla f^*\|_p \leq \|\nabla f\|_p. \quad (2.2)$$

which holds for all $p > 1$. This simplifies the problem to that of maximising a functional over a space of functions defined on the real positive half-line.

Throughout this chapter we use a simple but very useful result based on the carré-du-champ operator which provides a link between the L^2 norms of the usual gradient ∇f , and the Dunkl gradient $\nabla_k f$. Namely, we have

$$\|\nabla f\|_2 \leq \|\nabla_k f\|_2.$$

This is particularly useful in this chapter in estimating the best constant in the Dunkl Sobolev inequality by linking it to its weighted counterpart involving only the usual gradient.

We deal with this question in Section 2.4. First we prove an isoperimetric inequality for the weighted Dunkl measure μ_k , using a method inspired from [15]. The isoperimetric inequality then allows us to use a general result of Talenti [98], who proves a weighted version of the Pólya-Szegő inequality. This provides us with a weighted rearrangement inequality that holds for the usual gradient ∇f , on Weyl chambers. In particular, we find

the precise best constant for the weighted Sobolev inequality in this classical case. We will then connect this inequality to Dunkl operators by exploiting properties of Dirichlet forms, as discussed above.

The structure of the chapter is as follows. Section 2.1 introduces the carré-du-champ operator and we also prove some related results that will be very important in all subsequent proofs. In section 2.2 we prove Nash's inequality and then deduce the Sobolev inequality in case $p = 2$. In Section 2.3 we prove the pseudo-Poincaré inequality and the Besov space inequality. Finally, in Section 2.4 we prove the isoperimetric inequality and the rearrangement inequality, and find exact best constants for the weighted Sobolev inequality for usual partial derivatives. This is then used to deduce important results about best constants for the Sobolev inequality in the Dunkl setting.

2.1 THE CARRÉ-DU-CHAMP OPERATOR

A key ingredient in our proofs below will be the carré-du-champ operator. This is defined as

$$\Gamma(f) = \frac{1}{2}(\Delta_k(f^2) - 2f\Delta_k f).$$

The following Lemma gives an expression of this operator.

Lemma 2.1.1. *We have*

$$\Delta_k(f^2) = 2f\Delta_k f + 2|\nabla f|^2 + 2 \sum_{\alpha \in R_+} k_\alpha \left(\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} \right)^2.$$

In particular, we obtain the following expression for the carré-du-champ operator

$$\Gamma(f) = |\nabla f|^2 + \sum_{\alpha \in R_+} k_\alpha \left(\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} \right)^2.$$

Proof. Using (1.2), we have

$$\begin{aligned}
\Delta_k(f^2) &= \Delta(f^2) + 2 \sum_{\alpha \in R_+} k_\alpha \left(\frac{\langle \nabla(f^2), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f^2(x) - f^2(\sigma_\alpha x)}{\langle \alpha, x \rangle^2} \right) \\
&= 2f\Delta f + 2|\nabla f|^2 + 4f(x) \sum_{\alpha \in R_+} k_\alpha \left(\frac{\langle \nabla f, \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2} \right) \\
&\quad + 2 \sum_{\alpha \in R_+} k_\alpha \left(\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} \right)^2 \\
&= 2f\Delta_k f + 2|\nabla f|^2 + 2 \sum_{\alpha \in R_+} k_\alpha \left(\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} \right)^2.
\end{aligned}$$

The expression for $\Gamma(f)$ then follows immediately from this and the definition. $\square$

In the Euclidean case, as well as on Riemannian manifolds with Laplace-Beltrami operator, we have $\Gamma(f) = |\nabla f|^2$. The same is not true in the Dunkl case; however, using integration by parts, we can compute the Dirichlet form to obtain

$$\int_{\mathbb{R}^N} \Gamma(f) d\mu_k = \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k.$$

This, together with Lemma 2.1.1, give the following useful relation.

Lemma 2.1.2. *For all $f \in C_0^1(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k \geq \int_{\mathbb{R}^N} |\nabla f|^2 d\mu_k.$$

A similar very useful result that will be used in the proof of Sobolev inequality is the following.

Lemma 2.1.3. *For all $f \in C_0^1(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} |\nabla_k |f||^2 d\mu_k \leq \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k.$$

Proof. We note that $(|f(x)| - |f(\sigma_\alpha x)|)^2 \leq (f(x) - f(\sigma_\alpha x))^2$ holds for all $x \in \mathbb{R}^N$ and all $\alpha \in R$. Since also $|\nabla |f|| = |\nabla f|$, then from Lemma 2.1.1 we have

$$\Gamma(|f|) \leq \Gamma(f).$$

Using Dirichlet forms as above, the conclusion follows. $\square$

We also have the following pointwise estimate of the carré-du-champ operator in terms of the Dunkl gradient.

Proposition 2.1.4. *There exists a constant $C > 0$ such that the following inequality holds for all functions f for which all terms below are well-defined*

$$\Gamma(f) \geq C |\nabla_k f|^2.$$

Proof. For $i = 1, \dots, N$, let

$$D_i f(x) = \sum_{\alpha \in R_+} \alpha_i k_\alpha \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle},$$

so

$$T_i f(x) = \partial_i f(x) + D_i f(x).$$

From Lemma 2.1.1, we have (recall that $|\alpha|^2 = 2$)

$$\Gamma(f)(x) = \sum_{i=1}^N \left(|\partial_i f(x)|^2 + \frac{1}{2} \sum_{\alpha \in R_+} \alpha_i^2 k_\alpha \left(\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} \right)^2 \right).$$

We will estimate the α -sum first. Take $\tilde{C} := \min_{\alpha \in R_+} \frac{1}{2k_\alpha}$. Here we use the convention that if $k_\alpha = 0$, then $\frac{1}{k_\alpha} = \infty$. Since k does not vanish identically and R_+ is a finite set, then $\tilde{C}$ is well-defined and finite. We then have

$$\begin{aligned} \frac{1}{2} \sum_{\alpha \in R_+} \alpha_i^2 k_\alpha \left(\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} \right)^2 &\geq \tilde{C} \sum_{\alpha \in R_+} \alpha_i^2 k_\alpha^2 \left(\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} \right)^2 \\ &\geq \frac{\tilde{C}}{\sqrt{|R_+|}} (D_i f(x))^2. \end{aligned}$$

Going back to the carré-du-champ operator, we now have

$$\Gamma(f)(x) \geq \sum_{i=1}^N \left(|\partial_i f(x)|^2 + \frac{\tilde{C}}{\sqrt{|R_+|}} (D_i f(x))^2 \right). \quad (2.3)$$

The following inequality holds for all $x, y \in \mathbb{R}$ and all $c > 0$

$$x^2 + cy^2 \geq \frac{c}{c+1} (x+y)^2. \quad (2.4)$$

Indeed, by rearranging the terms, this is equivalent to

$$(x - cy)^2 \geq 0.$$

Using this inequality separately for each $i = 1, \dots, N$, from inequality (2.3) above we obtain that

$$\Gamma(f)(x) \geq C \sum_{i=1}^N (\partial_i f(x) + D_i f(x))^2 = C |\nabla_k f(x)|^2,$$

where $C = \frac{\tilde{C}/\sqrt{|R_+|}}{1+\tilde{C}/\sqrt{|R_+|}}$ is the constant obtained from inequality (2.4). $\square$

2.2 NASH'S INEQUALITY

In this section we will prove the Nash inequality using an elementary method that exploits the Dunkl transform and its rich theory. Using a nice method of [8] we can prove that this in turn implies the Sobolev inequality.

Proposition 2.2.1 (Nash inequality). *There exists a constant $C > 0$ that depends on N and k such that the following inequality holds for all $f \in L^1(\mu_k)$ such that $\nabla_k f \in L^2(\mu_k)$:*

$$\|f\|_{L^2(\mu_k)}^{1+2/(2\gamma+N)} \leq C \|\nabla_k(f)\|_{L^2(\mu_k)} \|f\|_{L^1(\mu_k)}^{2/(2\gamma+N)}.$$

Proof. Fix an $R > 0$ and let B_R be the ball in $\mathbb{R}^N$ of radius R and centred at the origin. We have

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R} |\mathcal{D}_k(f)(\xi)|^2 d\mu_k &\leq \int_{\mathbb{R}^N \setminus B_R} \frac{|\xi|^2}{R^2} |\mathcal{D}_k(f)(\xi)|^2 d\mu_k \\ &\leq \frac{1}{R^2} \int_{\mathbb{R}^N} |\xi|^2 |\mathcal{D}_k(f)(\xi)|^2 d\mu_k. \end{aligned}$$

Using the property that

$$\mathcal{D}_k(T_j f)(\xi) = i\xi_j \mathcal{D}_k(f)(\xi)$$

for any $j = 1, \dots, N$, then the right hand side of the above double inequality becomes

$$\frac{1}{R^2} \int_{\mathbb{R}^N} |\mathcal{D}_k(\nabla_k f)|^2 d\mu_k = \frac{1}{R^2} \int_{\mathbb{R}^N} |\nabla_k(f)|^2 d\mu_k,$$

where we used Plancherel's formula.

On the other hand, looking at the integral over B_R , we have

$$\begin{aligned}
\int_{B_R} |\mathcal{D}_k(f)(\xi)|^2 d\mu_k &= \frac{1}{M_k^2} \int_{B_R} \left| \int_{\mathbb{R}^N} f(x) E_k(-i\xi, x) d\mu_k(x) \right|^2 d\mu_k(\xi) \\
&\leq \frac{1}{M_k^2} \int_{B_R} \left(\int_{\mathbb{R}^N} |f(x) E_k(-i\xi, x)| d\mu_k(x) \right)^2 d\mu_k(\xi) \\
&\leq \frac{1}{M_k^2} \int_{B_R} \left(\int_{\mathbb{R}^N} |f(x)| d\mu_k(x) \right)^2 d\mu_k(\xi) \\
&= \frac{1}{M_k^2} \mu_k(B_R) \|f\|_{L^1(\mu_k)}^2.
\end{aligned}$$

Here we used the bounds $|E_k(-i\xi, x)| \leq 1$ of the Dunkl exponential.

Using the homogeneity of the Dunkl weight, we can compute using spherical coordinates

$$\mu_k(B_R) = \int_{B_R} d\mu_k(x) = \int_{\mathbb{S}_{N-1}} \int_0^R r^{N+2\gamma-1} w_k(\theta) dr d\sigma(\theta) = \frac{p(B_1)}{N+2\gamma} R^{N+2\gamma},$$

where $p(B_1) = \int_{\mathbb{S}_{N-1}} w_k(\theta) d\theta$.

Putting the above together and using Plancherel's formula, we have obtained

$$\begin{aligned}
\int_{\mathbb{R}^N} |f(x)|^2 d\mu_k(x) &= \int_{\mathbb{R}^N} |\mathcal{D}_k(f)(\xi)|^2 d\mu_k(\xi) \\
&\leq \frac{1}{R^2} \int_{\mathbb{R}^N} |\nabla_k(f)|^2 d\mu_k + \frac{p(B_1)}{M_k^2(N+2\gamma)} R^{N+2\gamma} \|f\|_{L^1(\mu_k)}^2
\end{aligned}$$

The right hand side is optimised for

$$R = \left(\frac{2M_k^2 \|\nabla_k(f)\|_{L^2(\mu_k)}^2}{p(B_1) \|f\|_{L^1(\mu_k)}^2} \right)^{1/(N+2\gamma+2)}$$

and upon substituting this above and raising everything to the power $\frac{2\gamma+N+2}{2(2\gamma+N)}$, we obtain finally

$$\|f\|_{L^2(\mu_k)}^{1+2/(2\gamma+N)} \leq C \|\nabla_k(f)\|_{L^2(\mu_k)} \|f\|_{L^1(\mu_k)}^{2/(2\gamma+N)},$$

for a constant C which can be computed explicitly from the above. $\square$

We can now deduce the Sobolev inequality using the elementary argument from [8].

Theorem 2.2.2 (Sobolev inequality). *Suppose $N+2\gamma > 2$. Then there exists a constant $C > 0$ such that for all $f \in C_c^\infty(\mathbb{R}^N)$ we have the inequality*

$$\|f\|_{L^q(\mu_k)} \leq C \|\nabla_k f\|_{L^2(\mu_k)},$$

where $q = \frac{2(N+2\gamma)}{N+2\gamma-2}$.

As announced above, we will prove that the Sobolev inequality follows from the Nash inequality. Before giving the proof, we note that the opposite implication also holds. Indeed, this follows from Hölder's inequality $\|FG\|_{L^1(\mu_k)} \leq \|F\|_{L^P(\mu_k)} \|G\|_{L^Q(\mu_k)}$ by applying it to $F = f^{2\theta}$, $G = f^{2(1-\theta)}$, with a suitable choice of $\theta \in (0, 1)$ and P, Q conjugate. We obtain

$$\int_{\mathbb{R}^N} f^2 d\mu_k \leq \left(\int_{\mathbb{R}^N} f^{2P\theta} d\mu_k \right)^{1/P} \left(\int_{\mathbb{R}^N} f^{2(1-\theta)Q} d\mu_k \right)^{1/Q}.$$

Since we need L^1 and $L^{2(N+2\gamma)/(N+2\gamma-2)}$ norms on the right hand side, we let

$$2\theta P = 1 \quad \text{and} \quad 2(1-\theta)Q = \frac{2(N+2\gamma)}{N+2\gamma-2}.$$

These equations, together with $\frac{1}{P} + \frac{1}{Q} = 1$, have a unique solution. Making the substitution and using the Sobolev inequality, the Nash inequality follows.

Proof. Fix first a smooth function $f \geq 0$ of compact support. Let

$$f_j(x) = \begin{cases} 0, & \text{for } f(x) < 2^j \\ f(x) - 2^j, & \text{for } 2^j \leq f(x) \leq 2^{j+1} \\ 2^j, & \text{for } f(x) > 2^{j+1} \end{cases}$$

for any $j \in \mathbb{Z}$. Then $f_j \in L^1(\mathbb{R}^N)$ and $\nabla_k f \in L^2(\mu_k)$, so we can apply the Nash inequality obtained above to get

$$\left(\int_{\mathbb{R}^N} f_j^2 d\mu_k \right)^{1+2/(2\gamma+N)} \leq C^2 \int_{S_j \setminus S_{j+1}} |\nabla_k f|^2 d\mu_k \cdot \left(\int_{\mathbb{R}^N} f_j d\mu_k \right)^{4/(2\gamma+N)}, \quad (2.5)$$

where $S_j = f^{-1}([2^j, \infty))$. Using the definition of f_j , we also see that

$$\int_{\mathbb{R}^N} f_j^2 d\mu_k \geq 2^{2j} \mu_k(S_{j+1}), \quad (2.6)$$

and

$$\int_{\mathbb{R}^N} f_j d\mu_k \leq 2^j \mu_k(S_j). \quad (2.7)$$

Let $q = \frac{2(N+2\gamma)}{N+2\gamma-2}$, so we want to estimate $\|f\|_{L^q(\mu_k)}$. We have

$$\begin{aligned} \int_{\mathbb{R}^N} f^q d\mu_k &= \sum_{j \in \mathbb{Z}} \int_{S_j \setminus S_{j+1}} f^q d\mu_k \\ &\leq \sum_{j \in \mathbb{Z}} 2^{q(j+1)} (\mu_k(S_j) - \mu_k(S_{j+1})) = (2^q - 1) \sum_{j \in \mathbb{Z}} 2^{qj} \mu_k(S_j). \end{aligned}$$

This suggests introducing the notation $a_j = 2^{qj} \mu_k(S_j)$ and $b_j = \int_{S_j \setminus S_{j+1}} |\nabla_k f|^2 d\mu_k$. In light of the above, now we need to estimate $\sum_{j \in \mathbb{Z}} a_j$.

Using the bounds (2.6) and (2.7) into the inequality (2.5), we obtain after some manipulation

$$a_{j+1} \leq 2^q C^{2p} b_j^p a_j^{4/(N+2\gamma+2)},$$

where $p = \frac{N+2\gamma}{N+2\gamma+2}$. Summing up over all $j \in \mathbb{Z}$ and noting that $\frac{2}{N+2\gamma+2} = 1 - p$, we can apply Hölder's inequality to obtain

$$\begin{aligned} \sum_{j \in \mathbb{Z}} a_j &= \sum_{j \in \mathbb{Z}} a_{j+1} \leq 2^q C^{2p} \sum_{j \in \mathbb{Z}} b_j^p a_j^{4/(N+2\gamma+2)} \\ &\leq 2^q C^{2p} \left(\sum_{j \in \mathbb{Z}} b_j \right)^p \left(\sum_{j \in \mathbb{Z}} a_j^2 \right)^{1-p} \\ &\leq 2^q C^{2p} \left(\sum_{j \in \mathbb{Z}} b_j \right)^p \left(\sum_{j \in \mathbb{Z}} a_j \right)^{2(1-p)}. \end{aligned}$$

Thus, we have

$$\|f\|_q^q \leq (2^q - 1) \sum_{j \in \mathbb{Z}} a_j \leq 2^{q/(2p-1)} (2^q - 1) C^{2p/(2p-1)} \left(\sum_{j \in \mathbb{Z}} b_j \right)^{p/(2p-1)}.$$

But clearly $\sum_{j \in \mathbb{Z}} b_j \leq \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k$, and since $\frac{p}{2p-1} = \frac{q}{2}$, we obtain

$$\|f\|_{L^q(\mu_k)} \leq 2^{1/(2p-1)} (2^q - 1)^{1/q} C \|\nabla_k f\|_{L^2(\mu_k)},$$

as required.

If f is not non-negative, then applying the above result to the non-negative function $|f|$ and using Lemma 2.1.3, we have

$$\|f\|_q \leq \tilde{C} \|\nabla_k |f|\|_2 \leq \tilde{C} \|\nabla_k f\|_2.$$

This completes the proof. □

2.3 PSEUDO-POINCARÉ INEQUALITY

In this section we will prove a sharper Sobolev-type inequality, involving Besov spaces. Besov spaces B_{pq}^s are generalisations of the classical Sobolev spaces; for example, we have $B_{2,2}^s = H^s$ for any $s > 0$. They can be characterised by many equivalent definitions which are all rather technical; see [101] for more information. Here, we will be concerned only with the spaces $B_{\infty,\infty}^s$, for $s < 0$, which admit a simpler definition in terms of the heat semigroup.

From now on we will work only in the Dunkl setting and in order to simplify notation, we will also use $B_{\infty,\infty}^s$ to denote the corresponding Dunkl-Besov space. Recall that $H_t = e^{t\Delta_k}$ is the heat semigroup. For any $s < 0$, we define the Dunkl-Besov space $B_{\infty,\infty}^s$ as the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^N)$ for which the norm

$$\|f\|_{B_{\infty,\infty}^s} := \sup_{t>0} t^{-s/2} \|H_t f\|_{\infty}$$

is finite. The aim of this section is to prove the following improved Sobolev inequality.

Theorem 2.3.1. *Let $1 \leq p < q < \infty$ such that $p \leq 2$. For any f such that $\|\nabla_k f\|_p < \infty$ we have the inequality*

$$\|f\|_q \leq C \|\nabla_k f\|_p^{p/q} \|f\|_{B_{\infty,\infty}^{p/(p-q)}}^{1-p/q},$$

where $C > 0$ is a constant.

This inequality, in the classical case of simple derivatives, was first proved in [19] using wavelets. A simplified proof was given by Ledoux in [64]. Similar inequalities were also studied in [18] and [10]. Here we will follow Ledoux's method that makes use of the known bounds on the Dunkl heat kernel.

The essential ingredient in this proof is the following pseudo-Poincaré inequality.

Proposition 2.3.2. *For any $1 \leq p \leq 2$, there exists a constant $C > 0$ such that for all f we have*

$$\|f - H_t f\|_p \leq C\sqrt{t} \|\nabla_k f\|_p.$$

This type of inequality is interesting in its own right as it has proven to be a useful tool for proving Sobolev inequalities in particularly difficult geometric settings. It can be stated more generally as

$$\|f - A_r f\|_p \leq Cr \|\nabla f\|_p,$$

where A_r is some sort of averaging operator. See [92] and [93] (and references therein) for more details, as well as some historical remarks.

Before we start proving the pseudo-Poincaré inequality, we need the following two lemmas. The first of these concerns gradient bounds, while the second one provides a Poincaré-type inequality for the semigroup H_t .

Lemma 2.3.3. *The following inequality holds*

$$|\nabla_k H_t f| \leq C H_t(|\nabla_k f|)$$

for a constant $C > 0$.

Proof. We first note that since the Dunkl operators commute, then ∇_k and H_t also commute. Then, using the inequality $\sum_i a_i^2 \leq (\sum_i a_i)^2$ which holds for nonnegative real numbers a_i , and then Cauchy-Schwarz, we obtain

$$\begin{aligned} |\nabla_k H_t f|(x) &= |H_t \nabla_k f|(x) = \left(\sum_{i=1}^N \left(\int_{\mathbb{R}^N} T_i f(y) h_t(x, y) \, d\mu_k(y) \right)^2 \right)^{1/2} \\ &\leq \sum_{i=1}^N \int_{\mathbb{R}^N} |T_i f(y)| h_t(x, y) \, d\mu_k(y) \\ &\leq \sqrt{N} \int_{\mathbb{R}^N} |\nabla_k f(y)| h_t(x, y) \, d\mu_k(y) \\ &= \sqrt{N} H_t(|\nabla_k f|)(x), \end{aligned}$$

as required. □

Lemma 2.3.4. *The following inequality holds*

$$H_t(f^2) - (H_t f)^2 \geq C t |\nabla_k H_t f|^2,$$

for a constant $C > 0$.

Proof. We note that

$$\begin{aligned} H_t(f^2) - (H_t f)^2 &= \int_0^t \frac{d}{ds} (H_s((H_{t-s} f)^2)) \, ds \\ &= \int_0^t H_s (\Delta_k((H_{t-s} f)^2) - 2 H_{t-s} f \Delta_k(H_{t-s} f)) \, ds \\ &= 2 \int_0^t H_s (\Gamma(H_{t-s} f)) \, ds. \end{aligned}$$

Using first Proposition 2.1.4, and then Cauchy-Schwarz, this becomes

$$\begin{aligned} H_t(f^2) - (H_t f)^2 &\geq 2C \int_0^t H_s(|\nabla_k H_{t-s} f|^2) \, ds \\ &\geq 2C \int_0^t (H_s(|\nabla_k H_{t-s} f|))^2 \, ds. \end{aligned}$$

Finally, using Lemma 2.3.3, we obtain

$$\begin{aligned} H_t(f^2) - (H_t f)^2 &\geq \frac{2C}{\sqrt{N}} \int_0^t |\nabla_k H_t f|^2 \, ds \\ &= \frac{2C}{\sqrt{N}} t |\nabla_k H_t f|^2, \end{aligned}$$

and this completes the proof. □

We can now prove the pseudo-Poincaré inequality.

Proof of Proposition 2.3.2. We note that

$$H_t f - f = \int_0^t \frac{d}{ds} (H_s f) \, ds = \int_0^t \Delta_k H_s f \, ds.$$

Fix a smooth function g such that $\|g\|_{p^*} \leq 1$, where $\frac{1}{p} + \frac{1}{p^*} = 1$. Then

$$\begin{aligned} \int_{\mathbb{R}^N} g(f - H_t f) \, d\mu_k &= \int_{\mathbb{R}^N} (g - H_t g) f \, d\mu_k \\ &= - \int_0^t \int_{\mathbb{R}^N} f \Delta_k H_s g \, d\mu_k \, ds \\ &= \int_0^t \int_{\mathbb{R}^N} \nabla_k f \cdot \nabla_k H_s g \, d\mu_k \, ds \\ &\leq \|\nabla_k f\|_p \int_0^t \|\nabla_k H_s g\|_{p^*} \, ds. \end{aligned}$$

From Lemma 2.3.4 we have

$$H_s(g^2) \geq H_s(g^2) - (H_s g)^2 \geq Cs |\nabla_k H_s g|^2.$$

Since $p^* \geq 2$, then the function $|\cdot|^{p^*/2}$ is convex and so, using Jensen's inequality,

$$\int_{\mathbb{R}^N} |\nabla_k H_s g|^{p^*} \, d\mu_k \leq (Cs)^{-p^*/2} \int_{\mathbb{R}^N} H_s(g^2)^{p^*/2} \, d\mu_k \leq (Cs)^{-p^*/2} \int_{\mathbb{R}^N} H_s(|g|^{p^*}) \, d\mu_k,$$

and thus

$$\|\nabla_k H_s g\|_{p^*} \leq \frac{1}{\sqrt{Cs}} \left\| H_s(|g|^{p^*}) \right\|_1^{1/p^*} \leq \frac{1}{\sqrt{Cs}} \|g\|_{p^*} \leq \frac{1}{\sqrt{Cs}}.$$

Here we used the fact that H_t is a contraction on $L^1(\mu_k)$. Therefore

$$\int_{\mathbb{R}^N} g(f - H_t f) d\mu_k \leq C\sqrt{t} \|\nabla_k f\|_p.$$

Since g was arbitrary, this shows that

$$\|f - H_t f\|_p \leq C\sqrt{t} \|\nabla_k f\|_p,$$

as required. □

We are now in a position to give a proof of the improved Sobolev inequality.

Proof of Theorem 2.3.1. The proof goes in three steps: the first step is proving a weak form of the inequality, the second step is proving the inequality for f that satisfies the additional assumption $f \in L^q$, and finally, in the last step, we remove the assumption $f \in L^q$ from the previous step.

Step 1. In this step we establish the weak inequality

$$\|f\|_{q,w} \leq C \|\nabla_k f\|_p^\theta \|f\|_{B_{\infty,\infty}^{\theta/(\theta-1)}}^{1-\theta}, \quad (2.8)$$

where $\theta = \frac{p}{q} \in (0, 1)$, and the weak L^q norm is defined by

$$\|f\|_{q,w}^q := \sup_{t>0} t^q \mu_k(\{|f(x)| > t\}).$$

By homogeneity, we can assume that $\|f\|_{B_{\infty,\infty}^{\theta/(\theta-1)}} \leq 1$, i.e.,

$$|H_t f| \leq t^{\theta/2(\theta-1)} \quad \text{for all } t > 0. \quad (2.9)$$

For every $t > 0$, let $s_t = t^{2(\theta-1)/\theta}$, so, by (2.9), we have $|H_{s_t} f| \leq t$.

If $|f| \geq 2t$, then $|f - H_{s_t} f| \geq |f| - |H_{s_t} f| \geq t$, so

$$\mu_k(\{|f(x)| \geq 2t\}) \leq \mu_k(|f - H_{s_t} f| \geq t) \leq t^{-p} \int_{\mathbb{R}^N} |f - H_{s_t} f|^p d\mu_k. \quad (2.10)$$

Using the pseudo-Poincaré inequality from Proposition 2.3.2, this implies that

$$t^q \mu_k(\{|f(x)| \geq 2t\}) \leq C t^{q-p} s_t^{p/2} \|\nabla_k f\|_p^p. \quad (2.11)$$

But from the choice of s_t , and because $\theta = \frac{p}{q}$, we have $t^{q-p}s_t^{p/2} = 1$, so taking infimum over $t > 0$, by the definition of the weak L^q norm, we finally obtain (2.8).

Step 2. Here we will impose the additional assumption $f \in L^q$. As before, we can assume by homogeneity that $\|f\|_{B_{\infty,\infty}^{\theta/(\theta-1)}} = 1$, so we need to prove that

$$\int_{\mathbb{R}^N} |f|^q d\mu_k \leq C \int_{\mathbb{R}^N} |\nabla_k f|^p d\mu_k. \quad (2.12)$$

Keeping with the notation above, for any $t > 0$ we let $s_t = t^{2(\theta-1)/\theta}$ and so $|H_{s_t} f| \leq t$.

Using the cake layer representation theorem (see [68, Theorem 1.13]) we have

$$\int_{\mathbb{R}^N} |f|^q d\mu_k = 5^q \int_0^\infty \mu_k(|f| \geq 5t) d(t^q). \quad (2.13)$$

Fix a constant $c \geq 4$. Define the function

$$\tilde{f}_t = (f - t)_+ \wedge (ct) + (f + t)_- \vee (-ct).$$

If $|f(x)| \geq 5t$, then either $|\tilde{f}(x)| = |f(x)| - t$, or $|\tilde{f}| = ct$. It then follows that (recall that we required $c \geq 4$)

$$\{|f| \geq 5t\} \subset \{|\tilde{f}| \geq 4t\},$$

and thus

$$\int_0^\infty \mu_k(|f| \geq 5t) d(t^q) \leq \int_0^\infty \mu_k(|\tilde{f}| \geq 4t) d(t^q). \quad (2.14)$$

The triangle inequality implies that

$$\begin{aligned} |\tilde{f}| &\leq |\tilde{f} - H_{s_t} \tilde{f}| + |H_{s_t}(\tilde{f} - f)| + |H_{s_t} f| \\ &\leq |\tilde{f} - H_{s_t} \tilde{f}| + H_{s_t}(|\tilde{f} - f|) + t, \end{aligned}$$

where we used the convexity of the modulus function and the fact that $|H_{s_t} f| \leq t$. It follows that

$$\{|\tilde{f}| \geq 4t\} \subset \{|\tilde{f} - H_{s_t} \tilde{f}| \geq t\} \cup \{H_{s_t}(|\tilde{f} - f|) \geq 2t\}.$$

Hence

$$\begin{aligned} \int_0^\infty \mu_k(|\tilde{f}| \geq 4t) d(t^q) &\leq \int_0^\infty \mu_k(|\tilde{f} - H_{s_t} \tilde{f}| \geq t) d(t^q) \\ &\quad + \int_0^\infty \mu_k(H_{s_t}(|\tilde{f} - f|) \geq 2t) d(t^q). \end{aligned} \quad (2.15)$$

We will estimate the two terms appearing on the right hand side of this inequality sepa-

rately. Firstly,

$$\begin{aligned}\mu_k \left(|\tilde{f} - H_{s_t} \tilde{f}| \geq t \right) &\leq \int_{\mathbb{R}^N} \frac{|\tilde{f} - H_{s_t} \tilde{f}|^p}{t^p} d\mu_k \\ &\leq C t^{-p} s_t^{p/2} \int_{\mathbb{R}^N} |\nabla_k \tilde{f}|^p d\mu_k,\end{aligned}$$

where in the second line we used the pseudo-Poincaré inequality from Proposition 2.3.2. We note that by the construction of $\tilde{f}$ we have that $\nabla_k \tilde{f} = \nabla_k f$ on $t \leq |f| \leq (c+1)t$, and $\nabla_k \tilde{f} = 0$ otherwise. Consequently,

$$\begin{aligned}\int_0^\infty \mu_k \left(|\tilde{f} - H_{s_t} \tilde{f}| \geq t \right) d(t^q) &\leq C \int_0^\infty t^{-q} \int_{\mathbb{R}^N} \mathbf{1}_{\{t \leq |f| \leq (c+1)t\}} |\nabla_k f|^p d\mu_k d(t^q) \\ &= Cq \int_{\mathbb{R}^N} |\nabla_k f|^p \int_{|f|/(c+1)}^{|f|} \frac{1}{t} dt d\mu_k \\ &= Cq \log(c+1) \int_{\mathbb{R}^N} |\nabla_k f|^p d\mu_k.\end{aligned}$$

Secondly, in order to estimate the second term appearing in (2.15), we first note that

$$\begin{aligned}|f - \tilde{f}| &= |f - \tilde{f}| \mathbf{1}_{\{|f| \leq (c+1)t\}} + |f - \tilde{f}| \mathbf{1}_{\{|f| \geq (c+1)t\}} \\ &\leq t + |f| \mathbf{1}_{\{|f| \geq (c+1)t\}},\end{aligned}$$

and using this we have

$$\begin{aligned}\int_0^\infty \mu_k \left(H_{s_t}(|f - \tilde{f}|) \geq 2t \right) d(t^q) &\leq \int_0^\infty \mu_k \left(H_{s_t}(|f| \mathbf{1}_{\{|f| \geq (c+1)t\}}) \geq t \right) d(t^q) \\ &\leq \int_0^\infty \int_{\mathbb{R}^N} \frac{|f| \mathbf{1}_{\{|f| \geq (c+1)t\}}}{t} d\mu_k d(t^q) \\ &= \int_{\mathbb{R}^N} |f| \int_0^{|f|/(c+1)} q t^{q-2} dt d\mu_k \\ &= \frac{q}{q-1} \frac{1}{(c+1)^{q-1}} \int_{\mathbb{R}^N} |f|^q d\mu_k.\end{aligned}$$

Putting these back in (2.15), and also using (2.14) and (2.13), we finally have

$$\int_{\mathbb{R}^N} |f|^q d\mu_k \leq Cq5^q \log(c+1) \int_{\mathbb{R}^N} |\nabla_k f|^p d\mu_k + \frac{q}{q-1} \frac{5^q}{(c+1)^{q-1}} \int_{\mathbb{R}^N} |f|^q d\mu_k.$$

Choosing $c \geq 4$ large enough so that $\frac{q5^q}{q-1} < (c+1)^{q-1}$, we obtain (2.12) for a constant $C > 0$.

Step 3. This is a technical step in which we remove the unnecessary assumption $f \in L^q$

from the previous step. The proof also follows that of Step 2.

Let $f \in W^{1,p}(\mathbb{R}^N)$ and, as before, we can assume by homogeneity that $\|f\|_{B_{\infty,\infty}^{\theta/(\theta-1)}} = 1$. Fix also $0 < \epsilon < 1$ and define

$$N_\epsilon(f) = \int_\epsilon^{1/\epsilon} \mu_k(|f| \geq 5t) \, d(t^q).$$

By the weak inequality (2.8) this is a finite quantity. Moreover, following the proof of Step 2, we can bound this as

$$N_\epsilon(f) \leq Cq \log(c+1) \int_{\mathbb{R}^N} |\nabla_k f|^p \, d\mu_k + \int_\epsilon^{1/\epsilon} \frac{1}{t} \int_{\mathbb{R}^N} |f| \mathbb{1}_{\{|f| \geq (c+1)t\}} \, d\mu_k \, d(t^q). \quad (2.16)$$

We now estimate the second term on the right hand side of this inequality. Changing the order of integration using Fubini's theorem, we have

$$\begin{aligned} & \int_\epsilon^{1/\epsilon} \frac{1}{t} \int_{\mathbb{R}^N} |f| \mathbb{1}_{\{|f| \geq (c+1)t\}} \, d\mu_k \, d(t^q) \\ &= q \int_{\mathbb{R}^N} |f| \left[\mathbb{1}_{\{\epsilon \leq \frac{|f|}{c+1} \leq 1/\epsilon\}} \int_\epsilon^{|f|/(c+1)} t^{q-2} \, dt + \mathbb{1}_{\{\frac{|f|}{c+1} > 1/\epsilon\}} \int_\epsilon^{1/\epsilon} t^{q-2} \, dt \right] d\mu_k \\ &= \frac{q}{q-1} \int_{\mathbb{R}^N} \left[\mathbb{1}_{\{\epsilon \leq \frac{|f|}{c+1} \leq 1/\epsilon\}} \left(\frac{|f|^q}{(c+1)^{q-1}} - \epsilon^{q-1} |f| \right) + \mathbb{1}_{\{\frac{|f|}{c+1} > 1/\epsilon\}} \left(\frac{|f|}{\epsilon^{q-1}} - \epsilon^{q-1} |f| \right) \right] d\mu_k. \end{aligned}$$

Using a similar method to the above, we also note that

$$\begin{aligned} & \int_\epsilon^{1/\epsilon} \mu_k(|f| \geq (c+1)t) \, d(t^q) \\ &= \int_\epsilon^{1/\epsilon} \int_{\mathbb{R}^N} \mathbb{1}_{\{|f| \geq (c+1)t\}} \, d\mu_k \, d(t^q) \\ &= \int_{\mathbb{R}^N} \left[\mathbb{1}_{\{\epsilon \leq \frac{|f|}{c+1} \leq 1/\epsilon\}} \int_\epsilon^{|f|/(c+1)} d(t^q) + \mathbb{1}_{\{\frac{|f|}{c+1} > 1/\epsilon\}} \int_\epsilon^{1/\epsilon} d(t^q) \right] d\mu_k \\ &= \int_{\mathbb{R}^N} \left[\mathbb{1}_{\{\epsilon \leq \frac{|f|}{c+1} \leq 1/\epsilon\}} \left(\frac{|f|^q}{(c+1)^q} - \epsilon^q \right) + \mathbb{1}_{\{\frac{|f|}{c+1} > 1/\epsilon\}} \left(\frac{1}{\epsilon^q} - \epsilon^q \right) \right] d\mu_k. \end{aligned}$$

From these last two computations we can now deduce that

$$\begin{aligned} \int_\epsilon^{1/\epsilon} \frac{1}{t} \int_{\mathbb{R}^N} |f| \mathbb{1}_{\{|f| \geq (c+1)t\}} \, d\mu_k \, d(t^q) &\leq \frac{q(c+1)}{q-1} \int_\epsilon^{1/\epsilon} \mu_k(|f| \geq (c+1)t) \, d(t^q) \\ &\quad + \frac{q}{q-1} \int_{\mathbb{R}^N} \mathbb{1}_{\{\frac{|f|}{c+1} > 1/\epsilon\}} \left(\frac{|f|}{\epsilon^{q-1}} - \frac{c+1}{\epsilon^q} + (c+1)\epsilon^q - |f|\epsilon^{q-1} \right) d\mu_k. \end{aligned}$$

We note also that

$$\begin{aligned} \int_{1/\epsilon}^{\infty} \mu_k(|f| \geq (c+1)t) dt &= \int_{1/\epsilon}^{\infty} \int_{\mathbb{R}^N} \mathbb{1}_{\{|f| \geq (c+1)t\}} d\mu_k dt \\ &= \int_{\mathbb{R}^N} \mathbb{1}_{\{\frac{|f|}{c+1} > 1/\epsilon\}} \left(\frac{|f|}{c+1} - \frac{1}{\epsilon} \right) d\mu_k. \end{aligned}$$

Finally, these give

$$\begin{aligned} \int_{\epsilon}^{1/\epsilon} \frac{1}{t} \int_{\mathbb{R}^N} |f| \mathbb{1}_{\{|f| \geq (c+1)t\}} d\mu_k d(t^q) &\leq \frac{q(c+1)}{q-1} \int_{\epsilon}^{1/\epsilon} \mu_k(|f| \geq (c+1)t) d(t^q) \\ &+ \frac{q}{q-1} \frac{c+1}{\epsilon^{q-1}} \int_{1/\epsilon}^{\infty} \mu_k(|f| \geq (c+1)t) dt. \end{aligned} \quad (2.17)$$

At this point we use the definition of the weak norm $\|\cdot\|_{q,w}$ and a change of variables to obtain

$$\int_{\epsilon}^{1/\epsilon} \mu_k(|f| \geq (c+1)t) d(t^q) \leq \left(\frac{5}{c+1} \right)^q N_{\epsilon}(f) + \frac{q}{(c+1)^q} \log \frac{c+1}{5} \|f\|_{q,w}^q,$$

and also

$$\int_{1/\epsilon}^{\infty} \mu_k(|f| \geq (c+1)t) dt \leq \frac{\epsilon^{q-1}}{(c+1)^q(q-1)} \|f\|_{q,w}^q.$$

Using these in (2.17) and then (2.16), we have that

$$\begin{aligned} N_{\epsilon}(f) &\leq Cq \log(c+1) \|\nabla_k f\|_p^p + \frac{q}{q-1} \frac{5^q}{(c+1)^{q-1}} N_{\epsilon}(f) \\ &+ \frac{q}{q-1} \frac{1}{(c+1)^{q-1}} \left(q \log \frac{c+1}{5} + \frac{1}{q-1} \right) \|f\|_{q,w}^q. \end{aligned}$$

From Step 1 we know that $\|f\|_{q,w} < \infty$, and from our assumption we have $\|\nabla_k f\|_p < \infty$. Thus, by choosing c large enough (independent of ϵ), we have $N_{\epsilon}(f) < \infty$. Therefore, taking the limit $\epsilon \rightarrow 0$, this implies that $\|f\|_q < \infty$. We have now reduced the problem to Step 2, so the proof is complete. $\square$

We can show that the inequality of Theorem 2.3.1 implies the classical Sobolev and Gagliardo-Nirenberg inequalities, but first we need the following ultracontractivity result.

Proposition 2.3.5. *Let $1 \leq p \leq \infty$. Then, there exists a constant $C > 0$ such that for any $f \in L^p(\mu_k)$ and for any $t > 0$, we have*

$$\|H_t f\|_{\infty} \leq C t^{-(2\gamma+N)/2p} \|f\|_p.$$

Proof. Firstly, if $f \in L^1(\mu_k)$, then, using the bounds on the Dunkl heat kernel from Proposition 1.6.5, we deduce that

$$|H_t f(x)| \leq \int_{\mathbb{R}^N} \frac{1}{(2t)^{\gamma+N/2} M_k} |f(y)| w_k(y) dy = C t^{-\gamma-N/2} \|f\|_1,$$

for a constant $C > 0$ that does not depend on t . Thus the Proposition holds true for $p = 1$.

On the other hand, if $f \in L^\infty(\mu_k)$, then, using the fact that $\int_{\mathbb{R}^N} h_t(x, y) w_k(y) dy = 1$ for any $x \in \mathbb{R}^N$, we have

$$|H_t f(x)| \leq \|f\|_\infty,$$

so the Proposition holds true in the case $p = \infty$.

The general case then follows by interpolation, using the Riesz-Thorin Theorem (see for example Theorem 1.1.5 in [25]). $\square$

Using this result we are now ready to present the classic Sobolev and Gagliardo-Nirenberg inequalities.

Corollary 2.3.6 (Sobolev inequality). *Let $1 \leq p < \max\{2, N + 2\gamma\}$ and define $q = \frac{p(N+2\gamma)}{N+2\gamma-p}$. Then, for any $f \in C_c^\infty(\mathbb{R}^N)$, we have the inequality*

$$\|f\|_q \leq C \|\nabla_k f\|_p,$$

where $C > 0$ is a constant.

Proof. Let $\theta = \frac{p}{q} = \frac{N+2\gamma-p}{N+2\gamma}$. By Proposition 2.3.5 we have

$$\|H_t f\|_\infty \leq C t^{-(2\gamma+N)/2q} \|f\|_q,$$

so, by the definition of the Besov norm, we deduce that

$$\|f\|_{B_{\infty, \infty}^{-(2\gamma+N)/q}} \leq \|f\|_q.$$

We note that $\frac{2\gamma+N}{q} = -\frac{N+2\gamma-p}{p} = -\frac{\theta}{\theta-1}$ so Theorem 2.3.1 gives

$$\|f\|_q \leq C \|\nabla_k f\|_p^\theta \|f\|_q^{1-\theta},$$

from which the Sobolev inequality follows immediately. $\square$

Corollary 2.3.7 (Gagliardo-Nirenberg inequality). *Let $1 \leq p < q < \infty$ such that also*

$p \leq 2$, and define $\frac{r}{q(N+2\gamma)} = \frac{1}{p} - \frac{1}{q}$. Then, for any $f \in C_c^\infty(\mathbb{R}^N)$, we have the inequality

$$\|f\|_q \leq C \|\nabla_k f\|_p^{p/q} \|f\|_r^{1-p/q}.$$

Proof. This follows from Theorem 2.3.1 as above since $\theta = \frac{p}{q}$ and $\frac{\theta}{\theta-1} = -\frac{N+2\gamma}{r}$. $\square$

2.4 ESTIMATES OF THE BEST CONSTANT

In this section we provide estimates for the best constant in the Dunkl Sobolev inequality, i.e., we study

$$C_{DS} := \sup \frac{\|f\|_q}{\|\nabla_k f\|_2},$$

where the supremum is considered over all non-zero functions in $C_c^1(\mathbb{R}^N)$. From Lemma 2.1.2 we see that

$$\frac{\|f\|_q}{\|\nabla_k f\|_2} \leq \frac{\|f\|_q}{\|\nabla f\|_2},$$

so we can obtain estimates by comparison to the optimal constant in the Sobolev inequality in the weighted space $L^2(\mu_k)$ but with usual gradient ∇ . More precisely, consider the inequality

$$\|f\|_q \leq C_{CS} \|\nabla f\|_2,$$

with optimal constant, i.e.,

$$C_{CS} := \sup \frac{\|f\|_q}{\|\nabla f\|_2}.$$

From the observation above we see straight away that

$$C_{DS} \leq C_{CS}.$$

In what follows we will find the precise value of C_{CS} by first proving an isoperimetric inequality for the weighted measure on Weyl chambers, and then obtaining a rearrangement inequality using a general result of Talenti [98] for weighted measures. This will provide the upper estimate of C_{DS} . For the lower estimate, we compute the supremum in the definition of C_{DS} but over all radial functions. We end the section with a conjecture, that the supremum is actually achieved for a radial function.

The main result is the following:

Theorem 2.4.1. *With the same notation as above, we have*

$$|G|^{-\frac{1}{N+2\gamma}} C_{CS} \leq C_{DS} \leq C_{CS}.$$

More precisely, we have

$$\begin{aligned} & \left(\frac{2}{(N+2\gamma)(N+2\gamma-2)} \right)^{1/2} \cdot \left[\frac{1}{M_k} \frac{\Gamma(N+2\gamma)}{\Gamma((N+2\gamma)/2)} \right]^{\frac{1}{N+2\gamma}} \\ & \leq C_{DS} \leq \left(\frac{2}{(N+2\gamma)(N+2\gamma-2)} \right)^{1/2} \cdot \left[\frac{|G|}{M_k} \frac{\Gamma(N+2\gamma)}{\Gamma((N+2\gamma)/2)} \right]^{\frac{1}{N+2\gamma}}. \end{aligned}$$

2.4.1 ISOPERIMETRIC INEQUALITY

For a bounded Lipschitz domain $A \subset \mathbb{R}^N$ we define

$$p(A) := \int_{\partial A} w_k(x) \, d\sigma(x),$$

where σ is the surface measure on ∂A . The aim of this subsection is to prove the following isoperimetric inequality.

Theorem 2.4.2 (Isoperimetric inequality). *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then we have the inequality*

$$\mu_k(\Omega)^{1-\frac{1}{N+2\gamma}} \leq Cp(\Omega),$$

for a constant $C = \frac{\mu_k(B_1^\epsilon)^{1-\frac{1}{N+2\gamma}}}{p(B_1^\epsilon)}$, where $\epsilon \in E$ is such that $\mathbb{R}_\epsilon^N$ is a Weyl chamber, $B_1 = \{|x| < 1\}$ and $B_1^\epsilon = B_1 \cap \mathbb{R}_\epsilon^N$.

Proof. First assume that Ω is contained in a Weyl chamber $\mathbb{R}_\epsilon^N$ for some $\epsilon \in E$. Moreover, since Ω is a Lipschitz domain, it can be approximated by smooth domains that also approximate the perimeter and area (see, for example, [29]), so we can assume further that $\overline{\Omega} \subset \mathbb{R}_\epsilon^N$. We start by considering the Neumann problem

$$\begin{cases} w_k^{-1} \nabla \cdot (w_k \nabla u) = c & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega, \end{cases} \quad (2.18)$$

where ν is outward normal to the boundary of Ω , and c is a constant. If the equation has a solution $u \in H^1(\Omega)$, then, by the divergence theorem, we must have

$$\int_{\Omega} \nabla \cdot (w_k \nabla u) \, dx = \int_{\partial\Omega} w_k \frac{\partial u}{\partial \nu} \, d\sigma(x),$$

so $c = \frac{p(\Omega)}{\mu_k(\Omega)}$. Conversely, if $c = \frac{p(\Omega)}{\mu_k(\Omega)}$, then the general theory of elliptic equations tells us that the problem has a unique solution $u \in H^1(\Omega)$ (up to a constant). Moreover, by

the assumption on Ω at the beginning of the proof, the operator

$$w_k^{-1} \nabla \cdot (w_k \nabla u) = \Delta u + \sum_{\alpha \in R_+} \frac{2k_\alpha}{\langle \alpha, x \rangle} \langle \alpha, \nabla u \rangle \quad (2.19)$$

is uniformly elliptic on Ω . Thus, by the regularity theory, the solution u is smooth.

Consider the set

$$\Gamma_u := \{x \in \Omega : u(y) - u(x) \geq \nabla u(x) \cdot (y - x) \text{ for all } y \in \overline{\Omega}\}.$$

Geometrically, this is the set of all points x for which the graph of u lies entirely above the tangent hyperplane at x . Consider also the set

$$\Gamma_u^\epsilon := \{x \in \Gamma_u : \operatorname{sgn}(\langle \alpha, \nabla u(x) \rangle) = \epsilon(\alpha) \text{ for all } \alpha \in R_+\}.$$

We will show that $B_1^\epsilon \subset \nabla u(\Gamma_u^\epsilon)$. Indeed, take $\zeta \in B_1^\epsilon$, and consider the point $x \in \overline{\Omega}$ for which

$$\min_{y \in \overline{\Omega}} (u(y) - \zeta \cdot y) = u(x) - \zeta \cdot x.$$

If $x \in \partial\Omega$, then

$$\frac{\partial u(x)}{\partial \nu} = \lim_{h \downarrow 0} \frac{u(x - h\nu) - u(x)}{-h} \leq \zeta \cdot \nu \leq |\zeta| < 1,$$

which contradicts the boundary condition. So $x \in \Omega$ and so, by definition, $x \in \Gamma_u$. Also, being a minimum of the function $u(y) - \zeta \cdot y$, by taking gradient we have

$$\nabla u(x) = \zeta.$$

Since $\zeta \in B_1^\epsilon$, this shows that $x \in \Gamma_u^\epsilon$, and consequently $B_1^\epsilon \subset \nabla u(\Gamma_u^\epsilon)$.

This implies that

$$\mu_k(B_1^\epsilon) \leq \int_{\nabla u(\Gamma_u^\epsilon)} w_k(x) \, dx.$$

Using the change of variables $y = (\nabla u)^{-1}(x)$ (notice that by definition ∇u is injective on Γ_u), we have

$$\int_{\nabla u(\Gamma_u^\epsilon)} w_k(x) \, dx = \int_{\Gamma_u^\epsilon} w_k(\nabla u(y)) |\det(H(u)(y))| \, dy.$$

Here $H(u)$ is the Hessian matrix of u , i.e.,

$$H(u) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq N}.$$

Denote by $\lambda_1(y), \dots, \lambda_N(y)$ the eigenvalues of $H(u)(y)$. Then the Jacobian factor can

also be written as

$$|\det(H(u)(y))| = \lambda_1(y) \cdots \lambda_N(y).$$

Before moving to the next step, we note that the Hessian matrix is positive-semidefinite on Γ_u . Indeed, assume that for some $x \in \Gamma_u$ there exists a vector $v \in \mathbb{R}^N$ such that

$$v^T \cdot H(u)(x) \cdot v < 0.$$

By continuity of $H(u)$, there exists $r > 0$ such that the same holds over a small ball $B_{2r}(x) \subset \Omega$, i.e.,

$$v^T \cdot H(u)(y) \cdot v < 0 \quad \text{for all } y \in B_{2r}(x). \quad (2.20)$$

By the mean value theorem, there exists $\theta \in (0, 1)$ such that

$$u(x + rv) - u(x) = r \nabla u(x) \cdot v + r^2 v^T \cdot H(u)(x + \theta rv) \cdot v.$$

By (2.20) we have

$$u(x + rv) - u(x) < r \nabla u(x) \cdot v,$$

contradicting the fact that $x \in \Gamma_u$. Therefore $H(u)(x)$ is positive-semidefinite and so $\lambda_1(x), \dots, \lambda_N(x) \geq 0$.

Applying the weighted mean value inequality in the above, we have

$$\begin{aligned} \mu_k(B_1^\epsilon) &\leq \int_{\Gamma_u^\epsilon} w_k(\nabla u(y)) |\det(H(u)(y))| \, dy \\ &= \int_{\Gamma_u^\epsilon} \prod_{\alpha \in R_+} \left(\frac{\langle \alpha, \nabla u(y) \rangle}{\langle \alpha, x \rangle} \right)^{2k_\alpha} \cdot \lambda_1(y) \cdots \lambda_N(y) w_k(y) \, dy \\ &\leq \int_{\Gamma_u^\epsilon} \frac{1}{(N + 2\gamma)^{N+2\gamma}} \left[\sum_{\alpha \in R_+} 2k_\alpha \frac{\langle \alpha, \nabla u(y) \rangle}{\langle \alpha, x \rangle} + \lambda_1(y) + \cdots + \lambda_N(y) \right]^{N+2\gamma} d\mu_k(y). \end{aligned}$$

We note that

$$\lambda_1(y) + \cdots + \lambda_N(y) = \text{Tr}(H(u)(y)) = \Delta u(y),$$

so using (2.19) we have obtained

$$\begin{aligned} \mu_k(B_1^\epsilon) &\leq \int_{\Gamma_u^\epsilon} \frac{1}{(N + 2\gamma)^{N+2\gamma}} [w_k(y)^{-1} \nabla \cdot (w_k \nabla u)(y)]^{N+2\gamma} d\mu_k(y) \\ &\leq \left(\frac{1}{N + 2\gamma} \cdot \frac{p(\Omega)}{\mu_k(\Omega)^{1 - \frac{1}{N+2\gamma}}} \right)^{N+2\gamma}. \end{aligned} \quad (2.21)$$

To conclude the proof, we note that the function $u(x) = \frac{1}{2}|x|^2$ solves the Neumann

problem (2.18) in B_1^ϵ with constant $c = N + 2\gamma$. Due to our observation that the Neumann problem has a solution if and only if $c = \frac{p(\Omega)}{w_k(\Omega)}$, we have that

$$\frac{p(B_1^\epsilon)}{\mu_k(B_1^\epsilon)} = N + 2\gamma. \quad (2.22)$$

Thus, (2.21) implies that

$$\frac{p(B_1^\epsilon)}{\mu_k(B_1^\epsilon)^{1-\frac{1}{N+2\gamma}}} \leq \frac{p(\Omega)}{\mu_k(\Omega)^{1-\frac{1}{N+2\gamma}}},$$

which is what we wanted to prove.

Before we consider the case of general domain Ω , we make some observations about the constant that appears in this inequality. The reflection group associated to the root system R acts transitively on the set of Weyl chambers, i.e., for any H, H' Weyl chambers, there exists $g \in G$ such that $gH = H'$. With a change of variables $gx = y$, this shows that

$$p(H \cap B_1) = p(H' \cap B_1).$$

This, together with equation (2.22), shows that the constant

$$\frac{p(H \cap B_1)}{\mu_k(H \cap B_1)^{1-\frac{1}{N+2\gamma}}}$$

does not depend on the choice of the Weyl chamber H .

For general Ω , let $\Omega_H = \Omega \cap H$ for each Weyl chamber H . The subscript H in the sums below will mean that the sum is taken over all Weyl chambers H . Since the sets Ω_H intersect only along the boundary, and $\overline{\Omega} = \bigcup_H \overline{\Omega_H}$, it is clear that

$$\mu_k(\Omega) = \sum_H \mu_k(\Omega_H).$$

Moreover, because $w_k(x) = 0$ for any $x \in \partial\Omega_H \setminus \partial\Omega$, then we also have

$$p(\Omega) = \sum_H p(\Omega_H).$$

Consequently, we have

$$\frac{p(\Omega)}{\mu_k(\Omega)^{1-\frac{1}{N+2\gamma}}} \geq \min_H \frac{p(\Omega_H)}{\mu_k(\Omega_H)^{1-\frac{1}{N+2\gamma}}}. \quad (2.23)$$

Indeed, if this were not true, then for any Weyl chamber H' for which $p(\Omega_{H'}) \neq 0$ and

$\mu_k(\Omega_{H'}) \neq 0$, we have

$$\frac{\sum_H p(\Omega_H)}{p(\Omega_{H'})} < \left(\frac{\sum_H \mu_k(\Omega_H)}{\mu_k(\Omega_{H'})} \right)^{1 - \frac{1}{N+2\gamma}} \leq \frac{\sum_H \mu_k(\Omega_H)}{\mu_k(\Omega_{H'})}.$$

By taking the power -1 and summing over all the Weyl chambers H' , we obtain a contradiction. Therefore, it is enough to prove the inequality for Ω_H where H realises the minimum of the expression in (2.23), which was done in the first part of the proof. $\square$

In the next step we will obtain a rearrangement inequality as a consequence of the isoperimetric inequality discussed above. Before this, we need to introduce some definitions.

Fix a Weyl chamber $\mathbb{R}_\epsilon^N$. For any measurable subset $\Omega \subset \overline{\mathbb{R}_\epsilon^N}$, we define its rearrangement to be the set

$$\Omega^* = B_r(0) \cap \mathbb{R}_\epsilon^N,$$

where $B_r(0)$ is the ball of radius r centred at the origin, and $r \geq 0$ is such that

$$\mu_k(\Omega^*) = \mu_k(\Omega).$$

For any measurable function $f : \overline{\mathbb{R}_\epsilon^N} \rightarrow \mathbb{C}$, we define its symmetric decreasing rearrangement $f^* : \overline{\mathbb{R}_\epsilon^N} \rightarrow [0, \infty)$ by

$$f^*(x) = \int_0^\infty \mathbf{1}_{\{|f|>t\}^*}(x) dt.$$

The function f^* is radial and decreasing, and it satisfies the property

$$\mu_k(\{x \in \mathbb{R}_\epsilon^N : |f(x)| > t\}) = \mu_k(\{x \in \mathbb{R}_\epsilon^N : f^*(x) > t\}),$$

for every $t > 0$. As a consequence, we have

$$\int_{\mathbb{R}_\epsilon^N} |f|^p d\mu_k = \int_{\mathbb{R}_\epsilon^N} |f^*|^p d\mu_k,$$

for any $p \geq 1$.

The main result of Talenti [98] then implies the following rearrangement inequality.

Proposition 2.4.3. *Let $1 \leq p < \infty$, and let $H = \mathbb{R}_\epsilon^N$ be a Weyl chamber. For any $f \in W^{1,p}(H)$, we have the inequality*

$$\int_{\mathbb{R}_\epsilon^N} |\nabla f^*|^p d\mu_k \leq \int_{\mathbb{R}_\epsilon^N} |\nabla f|^p d\mu_k.$$

Indeed, the result in [98] proves a general rearrangement inequality on a subset of the Euclidean space with a weighted measure, as long as an isoperimetric inequality holds. Moreover, when equality in the isoperimetric inequality is given by balls, then the rearrangement function is radial.

Another consequence of Theorem 2.4.2 is the Sobolev inequality for the usual gradient ∇ on the space weighted space $L^2(\mu_k)$. It is a classical fact that the sharp isoperimetric inequality is equivalent to the sharp Sobolev inequality for $\nabla f \in L^1$. The proof of this equivalence is done through the co-area formula, see [76] for details. The general case, with $\nabla f \in L^p$, follows by taking $|f|^a$, for a suitable a , in the L^1 result.

Proposition 2.4.4. *Let $\mathbb{R}_\epsilon^N$ be a Weyl chamber. Let $1 \leq p < N + 2\gamma$ and $q = \frac{p(N+2\gamma)}{N+2\gamma-p}$. Then there exists a constant $C > 0$ such that for any $f \in C_c^1(\overline{\mathbb{R}_\epsilon^N})$ we have*

$$\left(\int_{\mathbb{R}_\epsilon^N} |f|^q d\mu_k \right)^{1/q} \leq C \left(\int_{\mathbb{R}_\epsilon^N} |\nabla f|^p d\mu_k \right)^{1/p}. \quad (2.24)$$

2.4.2 SHARP CONSTANTS

Before we prove the main result, we need some information about the best constants that appear in the Sobolev inequality from Proposition 2.4.4, so we first compute

$$C_W := \sup \frac{\left(\int_{\mathbb{R}_\epsilon^N} |f|^q d\mu_k \right)^{1/q}}{\left(\int_{\mathbb{R}_\epsilon^N} |\nabla f|^p d\mu_k \right)^{1/p}}.$$

The choice of Weyl chamber $\mathbb{R}_\epsilon^N$ in this definition does not matter because we can obtain any Weyl chamber from another one through an element g of the reflection group G . Firstly, it is clear that it is enough to consider only non-negative functions f because replacing f by $|f|$ leaves the quotient in the definition of C_W invariant. By Proposition 2.4.3, and since $\int_{\mathbb{R}_\epsilon^N} f^q d\mu_k = \int_{\mathbb{R}_\epsilon^N} (f^*)^q d\mu_k$, we have

$$\frac{\left(\int_{\mathbb{R}_\epsilon^N} |f|^q d\mu_k \right)^{1/q}}{\left(\int_{\mathbb{R}_\epsilon^N} |\nabla f|^p d\mu_k \right)^{1/p}} \leq \frac{\left(\int_{\mathbb{R}_\epsilon^N} |f^*|^q d\mu_k \right)^{1/q}}{\left(\int_{\mathbb{R}_\epsilon^N} |\nabla f^*|^p d\mu_k \right)^{1/p}}.$$

But for f radial, say $f(x) = g(|x|)$, where $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, we have

$$\nabla f(x) = \frac{x}{|x|} g'(|x|),$$

so, using polar coordinates, we can compute

$$\frac{\left(\int_{\mathbb{R}_\epsilon^N} |f|^q d\mu_k\right)^{1/q}}{\left(\int_{\mathbb{R}_\epsilon^N} |\nabla f|^p d\mu_k\right)^{1/p}} = p(B_1^\epsilon)^{\frac{1}{q}-\frac{1}{p}} \frac{\left(\int_0^\infty g(r)^q r^{N+2\gamma-1} dr\right)^{1/q}}{\left(\int_0^\infty |g'(r)|^p r^{N+2\gamma-1} dr\right)^{1/p}}. \quad (2.25)$$

We have thus reduced the problem to maximising the functional

$$J(g) = \frac{\left(\int_0^\infty g(r)^q r^{N+2\gamma-1} dr\right)^{1/q}}{\left(\int_0^\infty |g'(r)|^p r^{N+2\gamma-1} dr\right)^{1/p}}. \quad (2.26)$$

But this problem was studied by Talenti in [97], where he obtained that

$$J(g) \leq (N+2\gamma)^{-1/p} \left(\frac{p-1}{N+2\gamma-p}\right)^{1/p'} \left[\frac{1}{p'} B\left(\frac{N+2\gamma}{p}, \frac{N+2\gamma}{p'}\right)\right]^{-\frac{1}{N+2\gamma}}, \quad (2.27)$$

where B is the beta function and p' is the conjugate of p , i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Equality is achieved for functions of the form

$$\varphi(r) = (a + br^{p'})^{1-\frac{N+2\gamma}{p}}, \quad (2.28)$$

where $a, b > 0$.

Recall that we have

$$p(B_1) = \frac{M_k}{2^{\frac{N+2\gamma}{2}-1} \Gamma\left(\frac{N+2\gamma}{2}\right)},$$

where M_k is the Macdonald-Mehta constant. Moreover, by a change of variables we can see that $p(B_1^\epsilon)$ does not depend on the Weyl chamber, and since the number of Weyl chambers is equal to the order of the reflection group G , it follows that

$$p(B_1^\epsilon) = \frac{1}{|G|} p(B_1).$$

Therefore, combining (2.25) and (2.27), we obtain the value of the best constant in the Sobolev inequality (2.24); this is summarised in the following result.

Proposition 2.4.5. *Let $\mathbb{R}_\epsilon^N$ be a Weyl chamber. Let $1 < p < N + 2\gamma$, and $q = \frac{p(N+2\gamma)}{N+2\gamma-p}$.*

Then the best constant in the Sobolev inequality (2.24) is given by

$$C_W = (N + 2\gamma)^{-1/p} \left(\frac{p-1}{N + 2\gamma - p} \right)^{1/p'} \cdot \left[\frac{2^{\frac{N+2\gamma}{2}-1} p' |G|}{M_k} \frac{\Gamma(N + 2\gamma) \Gamma((N + 2\gamma)/2)}{\Gamma((N + 2\gamma)/p) \Gamma((N + 2\gamma)/p')} \right]^{\frac{1}{N+2\gamma}}.$$

Remark. This generalises the results of [15] where the authors consider the weighted Sobolev inequality with monomial weight $|x_1|^{A_1} \dots |x_l|^{A_l}$; this corresponds to the Dunkl weight with root system $R = \{e_1, \dots, e_l\}$, where $1 \leq l \leq N$ and $e_1, \dots, e_N$ is the standard basis of $\mathbb{R}^N$.

We now turn to the Sobolev inequality on the whole space $\mathbb{R}^N$, which follows from the above.

Proposition 2.4.6. *Let $1 < p < N + 2\gamma$ and $g = \frac{p(N+2\gamma)}{N+2\gamma-p}$. Then there exists a constant $C_{CS} > 0$ such that for any $f \in C_c^\infty(\mathbb{R}^N)$, we have the inequality*

$$\|f\|_q \leq C_{CS} \|\nabla f\|_p. \quad (2.29)$$

Moreover, the sharp constant C_{CS} satisfies

$$C_{CS} = C_W,$$

with equality if and only if f is supported on the closure of a Weyl chamber, where it takes the form (2.28).

Proof. Fix $f \in C_c^\infty(\mathbb{R}^N)$ and for any Weyl chamber H let $f|_H$ denote the restriction of f to H . From the Sobolev inequality applied on each Weyl chamber, we have

$$\|f\|_q = \left(\sum_H \|f|_H\|_q^q \right)^{1/q} \leq C_W \left(\sum_H \|\nabla f|_H\|_p^q \right)^{1/q}.$$

Now we apply the inequality $\|y\|_{l_q} \leq \|y\|_{l_p}$ which holds on finite dimensional spaces since $p \leq q$, with equality if and only if $y = (0, \dots, 0, 1, 0, \dots, 0)$. It follows that

$$\|f\|_q \leq C_W \left(\sum_H \|\nabla f|_H\|_p^p \right)^{1/p} = C_W \|\nabla f\|_p.$$

This proves that the Sobolev inequality (2.29) holds, and also that $C_{CS} \leq C_W$. Moreover, following the proof backwards we see that equality does hold if and only if f is supported

on $\overline{H}$ for some Weyl chamber H , and we have equality in the corresponding Weyl chamber inequality, i.e.,

$$f(x) = (a + b|x|^{p'})^{1 - \frac{N+2\gamma}{p}} \quad \forall x \in H,$$

for some $a, b > 0$. □

Remark. In view of Lemma 2.1.2, this result for $p = 2$ provides yet another proof of Theorem 2.2.2.

We now specialise to the case $p = 2$. This limitation comes from the fact that we rely on the Dirichlet form method and Lemma 2.1.2. It remains an open question to prove the inequality

$$\|\nabla f\|_p \leq \|\nabla_k f\|_p$$

for $p \neq 2$. If this holds, then the estimates below generalise immediately.

We are now ready to prove the main result.

Proof of Theorem 2.4.1. The upper bound follows immediately from the above, using Lemma 2.1.2. Indeed, we have

$$\frac{\|f\|_q}{\|\nabla_k f\|_2} \leq \frac{\|f\|_q}{\|\nabla f\|_2},$$

and taking supremum on both sides we obtain $C_{DS} \leq C_{CS}$. In the case $p = 2$ the value of C_W simplifies and we have indeed

$$C_{CS} = C_W = \left(\frac{2}{(N+2\gamma)(N+2\gamma-2)} \right)^{1/2} \cdot \left[\frac{|G|}{M_k} \frac{\Gamma(N+2\gamma)}{\Gamma((N+2\gamma)/2)} \right]^{\frac{1}{N+2\gamma}}.$$

For the lower bound, we note that

$$\sup_{f \text{ radial}} \frac{\|f\|_q}{\|\nabla_k f\|_2} \leq C_{DS}.$$

The left hand side can be computed as above, making use of the functional (2.26) and Talenti's result. We obtain

$$\sup_{f \text{ radial}} \frac{\|f\|_q}{\|\nabla_k f\|_2} = \left(\frac{2}{(N+2\gamma)(N+2\gamma-2)} \right)^{1/2} \cdot \left[\frac{1}{M_k} \frac{\Gamma(N+2\gamma)}{\Gamma((N+2\gamma)/2)} \right]^{\frac{1}{N+2\gamma}}.$$

This completes the proof. □

Remark. If we could show that a maximiser exists for the optimal constant problem,

then from the proof above we deduce that

$$C_{DS} < C_{CS}.$$

Considering the above remark, as well as computations in the one dimensional case, we propose the following conjecture.

Conjecture 2.4.7. *The optimal constant in the Sobolev inequality for Dunkl operators is achieved by radial functions, so*

$$C_{DS} = |G|^{-\frac{1}{N+2\gamma}} C_{CS}.$$

3

HARDY INEQUALITIES

The classical Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla f|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} dx$$

is one of the most important results in analysis. It has seen an incredible development from its beginnings in Hardy's papers, having been refined and extended to various settings; see [11], [81], [27] and references therein for an overview of the topic.

Here we begin a systematic study of Hardy's inequality and its variants for Dunkl operators. As far as we can see, Hardy type inequalities for Dunkl operators have not been explicitly studied. In [46], the authors proved a Pitt's inequality which implies the Hardy's inequality for fractional powers of the Dunkl laplacian. In [17], the ground state representation method was used to obtain Hardy's inequality for fractional powers of the Dunkl-Hermite operators. Finally, in [48] the authors prove a Hardy-Rellich inequality for Dunkl operators on a Lizorkin space with sharp constant, a particular case of this is the sharp L^2 Hardy inequality on the Lizorkin space.

Apart from the classical Hardy's inequality with sharp constant, we also study in this chapter L^p inequalities which hold for small coefficients γ . As in the classical case, the sharp constant in the Hardy inequality is not achieved and we prove here an improved Hardy inequality using a method based on spherical h-harmonics. Two Hardy-type inequalities are also discussed, the Rellich inequality (with sharp constant), and a special case of the Caffarelli-Kohn-Nirenberg inequality. More background on these variants is given in each corresponding section.

The chapter is organised as follows. In section 3.1 we discuss the L^p Hardy inequality; for general p we obtain this inequality for small γ , while in the case $p = 2$ we obtain the L^2 Hardy inequality for the full range of γ . In section 3.2 we prove the Hardy inequality for fractional Dunkl laplacian. In section 3.3 we prove an improved Hardy inequality and as a corollary we deduce the Poincaré inequality for Dunkl operators. Finally, section 3.4 contains two Hardy-type results: the Rellich inequality and the Caffarelli-Kohn-Nirenberg inequality.

3.1 L^p HARDY INEQUALITY

In this section we study general L^p Hardy inequalities for Dunkl operators. Firstly we prove two Hardy-type inequalities in the weighted spaces $L^p(\mu_k)$, for the usual gradient and the Dunkl gradient. Using this inequality, we obtain the L^p inequality for a restricted range of p . In the particular case $p = 2$ we are able to improve on this method by using a stronger estimate, and so we finally prove the L^2 Hardy inequality for Dunkl operators in full generality.

We begin with the following Hardy-type inequalities.

Theorem 3.1.1. *Let $1 < p < \infty$. Then, for any $f \in C_c^\infty(\mathbb{R}^N)$ we have the inequality*

$$\int_{\mathbb{R}^N} |x \cdot \nabla f|^p d\mu_k \geq \left(\frac{N + 2\gamma}{p} \right)^p \int_{\mathbb{R}^N} |f|^p d\mu_k. \quad (3.1)$$

If, in addition, we have $p < \frac{N+2\gamma}{2\gamma}$, then we also have

$$\int_{\mathbb{R}^N} |x \cdot \nabla_k f|^p d\mu_k \geq \left(\frac{N + 2\gamma - 2p\gamma}{p} \right)^p \int_{\mathbb{R}^N} |f|^p d\mu_k \quad (3.2)$$

for all $f \in C_c^\infty(\mathbb{R}^N)$.

Proof. We first note that

$$\sum_{i=1}^N T_i(x_i) = \sum_{i=1}^N \left(1 + \sum_{\alpha \in R_+} k_\alpha \alpha_i^2 \right) = N + 2\gamma.$$

Thus, we have

$$\begin{aligned}
(N + 2\gamma) \int_{\mathbb{R}^N} |f|^p d\mu_k &= \sum_{i=1}^N \int_{\mathbb{R}^N} T_i(x_i) |f|^p d\mu_k \\
&= - \sum_{i=1}^N \int_{\mathbb{R}^N} x_i T_i(|f|^p) d\mu_k \\
&= - \int_{\mathbb{R}^N} x \cdot \left(\nabla(|f|^p) + \sum_{\alpha \in R_+} k_\alpha \alpha \frac{|f(x)|^p - |f(\sigma_\alpha x)|^p}{\langle \alpha, x \rangle} \right) d\mu_k \\
&= -p \int_{\mathbb{R}^N} f |f|^{p-2} x \cdot \nabla f d\mu_k - \sum_{\alpha \in R_+} k_\alpha \int_{\mathbb{R}^N} (|f(x)|^p - |f(\sigma_\alpha x)|^p) d\mu_k
\end{aligned}$$

Note that by a change of variables $y = \sigma_\alpha x$, we have

$$\int_{\mathbb{R}^N} |f(\sigma_\alpha x)|^p d\mu_k(x) = \int_{\mathbb{R}^N} |f(y)|^p d\mu_k(y),$$

so

$$(N + 2\gamma) \int_{\mathbb{R}^N} |f|^p d\mu_k = -p \int_{\mathbb{R}^N} f |f|^{p-2} x \cdot \nabla f d\mu_k. \quad (3.3)$$

By Hölder's inequality with exponents $\frac{p}{p-1}$ and p , we have

$$- \int_{\mathbb{R}^N} f |f|^{p-2} x \cdot \nabla f d\mu_k \leq \left(\int_{\mathbb{R}^N} |f|^p d\mu_k \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |x \cdot \nabla f|^p d\mu_k \right)^{\frac{1}{p}}.$$

Combining this inequality with (3.3), we obtain (3.1).

On the other hand, going back to (3.3), we have

$$\begin{aligned}
(N + 2\gamma) \int_{\mathbb{R}^N} |f|^p d\mu_k &= -p \int_{\mathbb{R}^N} f |f|^{p-2} x \cdot \nabla_k f d\mu_k \\
&\quad + p \sum_{\alpha \in R_+} k_\alpha \int_{\mathbb{R}^N} f |f|^{p-2} (f(x) - f(\sigma_\alpha x)) d\mu_k \quad (3.4) \\
&= -p \int_{\mathbb{R}^N} f |f|^{p-2} x \cdot \nabla_k f d\mu_k + p\gamma \int_{\mathbb{R}^N} |f|^p d\mu_k \\
&\quad - p \sum_{\alpha \in R_+} k_\alpha \int_{\mathbb{R}^N} f(x) f(\sigma_\alpha x) |f(x)|^{p-2} d\mu_k.
\end{aligned}$$

By Hölder's inequality with exponents $\frac{p}{p-1}$ and p , we have

$$\begin{aligned} - \int_{\mathbb{R}^N} f(x) f(\sigma_\alpha x) |f(x)|^{p-2} d\mu_k &\leq \left(\int_{\mathbb{R}^N} |f(x)|^p d\mu_k \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |f(\sigma_\alpha x)|^p d\mu_k \right)^{\frac{1}{p}} \\ &= \int_{\mathbb{R}^N} |f|^p d\mu_k, \end{aligned}$$

where in the last step we used a change of variables. Thus, going back to the computations above, we now have

$$(N + 2\gamma - 2p\gamma) \int_{\mathbb{R}^N} |f|^p d\mu_k \leq -p \int_{\mathbb{R}^N} f |f|^{p-2} x \cdot \nabla_k f d\mu_k$$

and as before, using Hölder's inequality, we obtain (3.2). $\square$

From this inequality we easily obtain an L^p Hardy inequality for a restricted range of p .

Theorem 3.1.2. *Let $1 < p < \frac{N+2\gamma}{1+2\gamma}$. Then, for any $f \in C_c^\infty(\mathbb{R}^N)$, we have the inequality*

$$\int_{\mathbb{R}^N} |\nabla_k f|^p d\mu_k \geq \left(\frac{N + 2\gamma - 2p\gamma - p}{p} \right)^p \int_{\mathbb{R}^N} \frac{|f|^p}{|x|^p} d\mu_k.$$

Proof. This follows by taking $\frac{f}{|x|}$ in (3.2). Indeed, we have

$$\frac{N + 2\gamma - 2p\gamma}{p} \left\| \frac{f}{|x|} \right\|_p \leq \left\| x \cdot \nabla_k \left(\frac{f}{|x|} \right) \right\|_p. \quad (3.5)$$

By the Leibniz rule we have

$$\nabla_k \left(\frac{f}{|x|} \right) = \frac{1}{|x|} \nabla_k f - f \frac{x}{|x|^3},$$

so

$$\left\| x \cdot \nabla_k \left(\frac{f}{|x|} \right) \right\|_p \leq \|\nabla_k f\|_p + \left\| \frac{f}{|x|} \right\|_p. \quad (3.6)$$

Thus, combining (3.5) and (3.6), we obtain the desired L^p Hardy inequality. $\square$

We obtained this L^p Hardy inequality through the Hardy-type inequality (3.2), which in turn relies on an estimate based on the Hölder's inequality of the term containing the difference $f(x) - f(\sigma_\alpha x)$ in (3.4). A better result can be obtained in the case $p = 2$ by relying instead on the Hardy-type inequality (3.1) for the usual gradient, and the Dirichlet form estimate

$$\|\nabla f\|_2 \leq \|\nabla_k f\|_2$$

from Lemma 2.1.2. Using this approach, we can obtain the L^2 Hardy inequality without any restrictions on γ .

Theorem 3.1.3. *Assume $N + 2\gamma > 2$. Then, for any $f \in C_c^\infty(\mathbb{R}^N)$, we have the inequality*

$$\int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k \geq \left(\frac{N + 2\gamma - 2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|f|^2}{|x|^2} d\mu_k,$$

where the constant is sharp.

Proof. We proceed similarly to the proof of the previous Theorem, but now we take $\frac{f}{|x|}$ in inequality (3.1) instead. We have

$$\frac{N + 2\gamma}{2} \left\| \frac{f}{|x|} \right\|_2 \leq \left\| x \cdot \nabla \left(\frac{f}{|x|} \right) \right\|_2 \leq \|\nabla f\|_2 + \left\| \frac{f}{|x|} \right\|_2.$$

Thus

$$\frac{N + 2\gamma - 2}{2} \left\| \frac{f}{|x|} \right\|_2 \leq \|\nabla f\|_2 \leq \|\nabla_k f\|_2,$$

where we used Lemma 2.1.2.

To check that the constant $\frac{(N+2\gamma-2)^2}{4}$ is sharp we consider, for each $n = 1, 2, \dots$, the radial function $f_n(x) = h_n(|x|)$, where

$$h_n(r) = \begin{cases} \frac{1}{c_n} & \text{if } r \leq 1 \\ \frac{1}{c_n} r^{c_n} & \text{if } r > 1, \end{cases}$$

where $c_n = -\frac{1}{n} - \frac{N+2\gamma-2}{2}$. Then we have

$$\frac{\int_{\mathbb{R}^N} |\nabla_k f_n|^2 d\mu_k}{\int_{\mathbb{R}^N} \frac{f_n(x)^2}{|x|^2} d\mu_k} = \frac{\int_0^\infty (h'_n(r))^2 r^{N+2\gamma-1} dr}{\int_0^\infty h_n(r)^2 r^{N+2\gamma-3} dr} = \frac{\frac{n}{2}}{\frac{1}{c_n^2} \left(\frac{1}{N+2\gamma} + \frac{n}{2} \right)} \rightarrow \frac{(N + 2\gamma - 2)^2}{4}$$

as $n \rightarrow \infty$. □

Remark. As mentioned above, we were able to prove the Hardy inequality in case $p = 2$ for the full range $N + 2\gamma > 2$ because we replaced a Hölder inequality estimate for difference terms by the stronger estimate of Lemma 2.1.2. If we have in general

$$\|\nabla f\|_p \leq \|\nabla_k f\|_p,$$

then the same method applies to obtain L^p Hardy inequality in full generality for $1 < p < N + 2\gamma$.

3.2 HARDY INEQUALITY FOR FRACTIONAL LAPLACIAN

In this short section we prove the Hardy inequality for fractional laplacian using the Pitt's inequality. In particular, we obtain a new proof of the L^2 Hardy inequality from Theorem 3.1.3.

We can define the fractional Dunkl laplacian $(-\Delta_k)^s$ for $s > 0$ using Dunkl transform by the formula

$$\mathcal{D}_k((-\Delta_k)^s f)(\xi) = |\xi|^{2s} \mathcal{D}_k(f)(\xi).$$

The main result of this section is the following Hardy inequality for the fractional laplacian $(-\Delta_k)^s$.

Theorem 3.2.1. *For all $0 \leq s < \frac{N+2\gamma}{2}$ and all $f \in \mathcal{S}(\mathbb{R}^N)$ we have*

$$C(s)^2 \int_{\mathbb{R}^N} \frac{|f(x)|^2}{|x|^{2s}} d\mu_k \leq \int_{\mathbb{R}^N} |(-\Delta_k)^{s/2} f|^2 d\mu_k,$$

with sharp constant given by

$$C(s) = 2^s \frac{\Gamma(\frac{1}{2}(N/2 + \gamma + s))}{\Gamma(\frac{1}{2}(N/2 + \gamma - s))}. \quad (3.7)$$

This will follow easily from the following Pitt's inequality.

Proposition 3.2.2 ([46]). *Let $0 \leq s < \frac{N+2\gamma}{2}$. Then, for all $g \in \mathcal{S}(\mathbb{R}^N)$, the following inequality holds*

$$C(s) \left\| |\xi|^{-s} \mathcal{D}_k(g)(\xi) \right\|_2 \leq \left\| |x|^s g(x) \right\|_2,$$

with sharp constant $C(s)$ given by (3.7).

Proof of Theorem 3.2.1. The Hardy inequality for fractional laplacian is essentially a rewriting of Pitt's inequality. Indeed, let $f = \mathcal{D}_k(g)$, then by Proposition 3.2.2 it follows that

$$C(s) \left\| |x|^{-s} f \right\|_2 \leq \left\| |\xi|^s \mathcal{D}_k^{-1}(f)(\xi) \right\|_2 = \left\| |\xi|^s \mathcal{D}_k(f) \right\|_2.$$

Here in the last step we used the property that $\mathcal{D}_k^{-1}(f)(\xi) = \mathcal{D}_k(f)(-\xi)$ and a change of variables $\xi \mapsto -\xi$. But, by Plancherel's formula

$$\left\| |\xi|^s \mathcal{D}_k(f)(\xi) \right\|_2 = \left\| \mathcal{D}_k((-\Delta_k)^{s/2} f) \right\|_2 = \left\| (-\Delta_k)^{s/2} f \right\|_2.$$

Thus, we have obtained that

$$C(s)^2 \int_{\mathbb{R}^N} \frac{|f(x)|^2}{|x|^{2s}} d\mu_k \leq \int_{\mathbb{R}^N} |(-\Delta_k)^{s/2} f|^2 d\mu_k,$$

as required. $\square$

In the particular case $s = 1$, we obtain the classical L^2 Hardy inequality for Dunkl operators from Theorem 3.1.3. To see that this inequality does indeed follow from Theorem 3.2.1 it is enough to check that

$$\int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k = \int_{\mathbb{R}^N} |(-\Delta_k)^{1/2} f|^2 d\mu_k.$$

By Plancherel's formula and the definition of fractional laplacian, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |(-\Delta_k)^{1/2} f|^2 d\mu_k &= \int_{\mathbb{R}^N} |\mathcal{D}_k((-\Delta_k)^{1/2} f)|^2 d\mu_k(\xi) = \int_{\mathbb{R}^N} |\xi|^2 |\mathcal{D}_k(f)(\xi)|^2 d\mu_k(\xi) \\ &= \int_{\mathbb{R}^N} \mathcal{D}_k((-\Delta_k)f)(\xi) \cdot \overline{\mathcal{D}_k(f)(\xi)} d\mu_k(\xi) = - \int_{\mathbb{R}^N} \Delta_k f \cdot \bar{f} d\mu_k = \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k, \end{aligned}$$

where in the end we used integration by parts.

3.3 IMPROVED HARDY'S INEQUALITIES

It is a well known fact that the best constant in the classical Hardy inequality is not achieved, i.e., there is no $f \neq 0$ such that

$$\int_{\mathbb{R}^N} |\nabla f|^2 = \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{f^2}{|x|^2}.$$

Based on this observation, improved Hardy inequalities were proved, where the error in the classical Hardy inequality is bounded from below, usually by a term involving a suitable potential V . More precisely, inequalities of the following form are studied

$$\int_{\Omega} |\nabla f|^2 dx - \frac{(N-2)^2}{4} \int_{\Omega} \frac{f^2}{|x|^2} dx \geq \int_{\Omega} |V| f^2 dx. \quad (3.8)$$

The first such result was proved by Brezis and Vázquez in [14] where it was used in the study of singular extremal solutions of a semilinear elliptic equation. In that paper the above inequality is proved for constant potential V that depends on the domain Ω . The proof is based on a symmetrisation argument. However, the authors noticed that even in this case the best constant is not achieved, so they posed the question of whether the improvement appearing on the right hand side of inequality (3.8) is just the first term of a series. This was answered positively by Filippas and Tertikas in [40], where such a construction can be found. Similar improved inequalities have also been found for other Hardy type inequalities, for example Hardy inequality on domains, with weights

depending on the distance to the boundary.

In this section we will prove an improved Hardy's inequality for Dunkl operators using a method similar to [40] based on spherical h-harmonics.

Theorem 3.3.1. *Let $X(t) = (1 - \log t)^{-1}$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\delta = \sup_{x \in \Omega} |x|$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(\Omega)$ we have the inequality*

$$\int_{\Omega} \left[|\nabla_k f|^2 - \frac{(N + 2\gamma - 2)^2}{4} \frac{f^2}{|x|^2} \right] d\mu_k \geq C \left(\int_{\Omega} |f|^q X^{1+q/2} \left(\frac{|x|}{\delta} \right) d\mu_k \right)^{2/q},$$

where $q = \frac{2(N+2\gamma)}{N+2\gamma-2}$ is the Sobolev coefficient.

We will need the following Lemma, which is itself a weighted Hardy inequality in one dimension. For a comprehensive treatment of such inequalities, see [81] and [76]. The following Lemma is a direct consequence of Theorem 1.3.2/3 in [76].

Lemma 3.3.2. *Let $q \geq 2$ and $\delta > 0$. Then there exists a constant $C > 0$ such that the following inequality holds for all $g \in C_0^\infty(0, \delta)$*

$$\int_0^\delta t |g'(t)|^2 dt \geq C \left(\int_0^\delta \frac{|g(t)|^q}{t} X \left(\frac{t}{\delta} \right)^{1+\frac{q}{2}} dt \right)^{2/q}.$$

Proof of Theorem 3.3.1. We note first that it is enough to prove the result in the case when Ω is the ball B_δ of radius δ centred at the origin. Indeed, for a general Ω we have $\Omega \subset B_\delta$, so the result for B_δ implies in particular the inequality for Ω .

Consider the h -harmonic expansion of a function $f \in L^2(\mu_k)$

$$f(r\xi) = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} f_{n,i}(r) Y_i^n(\xi).$$

The functions $f_{n,i}$, given by (1.11), are defined on $\mathbb{R}_+$, but with a slight abuse of notation we will also see them as radial functions on $\mathbb{R}^N$ by identifying $f_{n,i}(x) = f_{n,i}(|x|)$. Using the formula (1.10) for the Dunkl laplacian, we have

$$\int_{\Omega} |\nabla_k f|^2 d\mu_k = - \int_{\Omega} f \Delta_k f d\mu_k = - \int_{\Omega} f \left[\frac{\partial^2 f}{\partial r^2} + \frac{N + 2\gamma - 1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \Delta_{k,0} f \right] d\mu_k.$$

Using the orthogonality of the h-harmonics $\{Y_i^n\}$, as well as the fact that Y_i^n are eigen-

functions of the operator $\Delta_{k,0}$ with eigenvalues λ_n , it follows that

$$\begin{aligned}
& \int_{\Omega} |\nabla_k f|^2 d\mu_k \\
&= - \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{\delta} \left[f_{n,i}(r) f_{n,i}''(r) + \frac{N+2\gamma-1}{r} f_{n,i}(r) f_{n,i}'(r) + \lambda_n \frac{1}{r^2} f_{n,i}^2(r) \right] r^{N+2\gamma-1} dr \\
&= - \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{\delta} \left[f_{n,i}(x) \Delta_k f_{n,i}(x) + \lambda_n \frac{f_{n,i}^2(x)}{|x|^2} \right] dx \\
&= \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{\delta} \left[|\nabla_k f_{n,i}(x)|^2 - \lambda_n \frac{f_{n,i}^2(x)}{|x|^2} \right] dx.
\end{aligned}$$

Let $\Lambda := \left(\frac{N+2\gamma-2}{2}\right)^2$ denote the Hardy inequality constant. From the above, we have

$$\int_{\Omega} \left[|\nabla_k f|^2 - \Lambda \frac{f^2}{|x|^2} \right] d\mu_k = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} I_{n,i},$$

where

$$I_{n,i} := \int_{\Omega} \left[|\nabla_k f_{n,i}|^2 - (\Lambda + \lambda_n) \frac{f_{n,i}^2(x)}{|x|^2} \right] d\mu_k.$$

When $n > 0$, it can be checked by rearranging the terms that Hardy's inequality implies the following

$$I_{n,i} \geq \frac{\lambda_n}{\lambda_n - \Lambda} \int_{\Omega} \left[|\nabla_k f_{n,i}|^2 - \lambda_n \frac{f_{n,i}^2}{|x|^2} \right] d\mu_k.$$

Thus, we have that

$$\sum_{n=1}^{\infty} \sum_{i=1}^{d(n)} I_{n,i} \geq C_1 \sum_{n=1}^{\infty} \sum_{i=1}^{d(n)} \int_{\Omega} \left[|\nabla_k f_{n,i}|^2 - \lambda_n \frac{f_{n,i}^2}{|x|^2} \right] d\mu_k = C_1 \int_{\Omega} |\nabla_k(f - f_{0,1})|^2 d\mu_k, \quad (3.9)$$

where $C_1 = \min_{n \geq 1} \frac{\lambda_n}{\lambda_n - \Lambda} > 0$. Using the Sobolev inequality, we have that

$$\begin{aligned}
\int_{\Omega} |\nabla_k(f - f_{0,1})|^2 d\mu_k &\geq C_2 \left(\int_{\Omega} |f - f_{0,1}|^q d\mu_k \right)^{2/q} \\
&\geq C_2 \left(\int_{\Omega} |f - f_{0,1}|^q X^{1+q/2} \left(\frac{|x|}{\delta} \right) d\mu_k \right)^{2/q}, \quad (3.10)
\end{aligned}$$

for a constant $C_2 > 0$. Here, in the second inequality, we used the fact that X is bounded above by 1.

When $n = 0$, we have that $d(0) = 1$ and $\lambda_0 = 0$, so

$$I_{0,1} = \int_{\Omega} \left[|\nabla_k f_{0,1}|^2 - \Lambda \frac{f_{0,1}^2}{|x|^2} \right] d\mu_k.$$

Let $u(r) = r^{(N+2\gamma-2)/2} f_{0,1}$ so after an easy computation we find that

$$I_{0,1} = \int_{\Omega} |x|^{-(N+2\gamma-1)} \left[-(N+2\gamma-2)u(|x|)u'(|x|) + |x|u'(|x|)^2 \right] d\mu_k(x).$$

Using polar coordinates we then have

$$I_{0,1} = p(B_1) \int_0^\delta \left[-(N+2\gamma-2)u(r)u'(r) + ru'(r)^2 \right] dr = p(B_1) \int_0^\delta ru'(r)^2 dr,$$

where $p(B_1) = \int_{\mathbb{S}^{N-1}} w_k(\theta) d\sigma(\theta)$, and in the last equality we used the fact that $u(0) = u(\delta) = 0$. Applying Lemma 3.3.2, this implies

$$\begin{aligned} I_{0,1} &\geq p(B_1) C_3 \left(\int_0^\delta \frac{|u|^q}{r} X^{1+q/2} \left(\frac{r}{\delta} \right) dr \right)^{2/q} \\ &= p(B_1)^{1-2/q} C_3 \left(\int_{\Omega} |f_{0,1}|^q X^{1+q/2} \left(\frac{|x|}{\delta} \right) d\mu_k \right)^{2/q}. \end{aligned} \quad (3.11)$$

Finally, from (3.9) we obtain

$$\begin{aligned} \int \left[|\nabla_k f|^2 - \Lambda \frac{f^2}{|x|^2} \right] d\mu_k &\geq I_{0,1} + C_1 \int |\nabla_k (f - f_{0,1})|^2 d\mu_k \\ &\geq C \left(\int |f|^q X^{1+q/2}(|x|) d\mu_k \right)^{2/q}, \end{aligned}$$

for a constant $C > 0$, where in the last line we used (3.10), (3.11), and the triangle inequality in the space $L^q(\mu_k)$. $\square$

The following Corollary, which is a Dunkl equivalent of the original result of Brezis and Vazquez, is very important because it establishes a Poincaré inequality for Dunkl operators.

Corollary 3.3.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then there exists a constant $C(\Omega) > 0$ such that for any $f \in C_0^\infty(\Omega)$ we have the inequality*

$$\int_{\Omega} |\nabla_k f|^2 d\mu_k \geq \frac{(N+2\gamma-2)^2}{4} \int_{\Omega} \frac{f^2}{|x|^2} d\mu_k + C(\Omega) \int_{\Omega} f^2 d\mu_k.$$

Proof. This follows from the previous Theorem and Hölder's inequality applied to the

function $f^2 X^{1+\frac{2}{q}} \in L^{q/2}$ and $X^{-1-\frac{2}{q}} \in L^{q/(q-2)}$. □

Corollary 3.3.4 (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then there exists a constant $C(\Omega) > 0$ such that for any $f \in C_0^\infty(\Omega)$ we have the inequality*

$$\int_{\Omega} |\nabla_k f|^2 d\mu_k \geq C(\Omega) \int_{\Omega} f^2 d\mu_k.$$

3.4 OTHER HARDY-TYPE INEQUALITIES

In this section we present two results closely related to the Hardy inequality: the Rellich inequality and a Caffarelli-Kohn-Nirenberg inequality.

3.4.1 THE RELICH INEQUALITY

This classical inequality, first proved by Rellich in [82] (see also [83]), states that for all $f \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ we have

$$\int_{\mathbb{R}^N} |\Delta f|^2 dx \geq \frac{N^2(N-4)^2}{16} \int_{\mathbb{R}^N} \frac{f^2}{|x|^4} d\mu_k,$$

where the constant is sharp.

Here we prove the Dunkl analogue of this inequality. Our proof below uses the method of spherical h-harmonics already employed above to obtain an improved Hardy inequality, and it is similar in style to the original proof of Rellich.

Theorem 3.4.1 (Rellich inequality). *Suppose that $N + 2\gamma \neq 2$. Then, for any $f \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, we have the inequality*

$$\int_{\mathbb{R}^N} |\Delta_k f|^2 d\mu_k \geq \frac{(N + 2\gamma)^2(N + 2\gamma - 4)^2}{16} \int_{\mathbb{R}^N} \frac{f^2}{|x|^4} d\mu_k.$$

The constant in this inequality is sharp.

Proof. Consider the expansion of f in terms of spherical h-harmonics

$$f(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} f_{n,i}(r) Y_i^n(\xi).$$

We then have

$$\Delta_k f(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \left[f''_{n,i}(r) + \frac{N + 2\gamma - 1}{r} f'_{n,i}(r) + \frac{\lambda_n}{r^2} f_{n,i}(r) \right] Y_i^n(\xi).$$

In order to simplify the computations below, we introduce the notation $\overline{N} := N + 2\gamma$. Thus, from the orthogonality properties of $\{Y_{n,i}\}$ we have

$$\int_{\mathbb{R}^N} (\Delta_k f)^2 d\mu_k = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{\infty} \left[f''_{n,i}(r) + \frac{\overline{N} - 1}{r} f'_{n,i}(r) + \frac{\lambda_n}{r^2} f_{n,i}(r) \right]^2 r^{\overline{N}-1} dr.$$

Expanding the brackets and computing the terms containing products of mixed derivatives using integration by parts (recall that $f_{n,i}$ has compact support away from 0), we have

$$\begin{aligned} \int_{\mathbb{R}^N} (\Delta_k f)^2 d\mu_k &= \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{\infty} \left[(f''_{n,i})^2 r^{\overline{N}-1} + (\overline{N} - 2\lambda_n - 1)(f'_{n,i})^2 r^{\overline{N}-3} \right. \\ &\quad \left. + \lambda_n(\lambda_n - 2(\overline{N} - 4))f_{n,i}^2 r^{\overline{N}-5} \right] dr. \end{aligned}$$

Fix now some $n = 0, 1, \dots$, and some $i = 1, \dots, d(n)$, and define

$$g_{n,i} = r^{(\overline{N}-4)/2} f_{n,i}.$$

To simplify notation, let $u = f_{n,i}$ and $v = g_{n,i}$; let also $c = \overline{N} - 2\lambda_n - 1$. We then have

$$\begin{aligned} u'(r) &= \frac{4 - \overline{N}}{2} r^{(2-\overline{N})/2} v + r^{(4-\overline{N})/2} v' \\ u''(r) &= \frac{(4 - \overline{N})(2 - \overline{N})}{4} r^{-\overline{N}/2} v + (4 - \overline{N}) r^{(2-\overline{N})/2} v' + r^{(4-\overline{N})/2} v''. \end{aligned} \tag{3.12}$$

Thus, using these relations and integrating by parts the terms involving mixed products

of derivatives of v , we have

$$\begin{aligned}
& \int_0^\infty \left[(f''_{n,i})^2 r^{\bar{N}-1} + (\bar{N} - 2\lambda_n - 1)(f'_{n,i})^2 r^{\bar{N}-3} \right] dr = \int_0^\infty \left[(u'')^2 r^{\bar{N}-1} + c(u')^2 r^{\bar{N}-3} \right]^2 dr \\
&= \int_0^\infty \left[(v'')^2 r^3 + ((4 - \bar{N})^2 + c)(v')^2 r + \frac{(4 - \bar{N})^2}{4} \left(\frac{(2 - \bar{N})^2}{4} + c \right) v^2 r^{-1} \right. \\
&\quad \left. + 2 \frac{4 - \bar{N}}{2} \left(\frac{(4 - \bar{N})(2 - \bar{N})}{2} + c \right) vv' \right. \\
&\quad \left. + 2 \frac{(4 - \bar{N})(2 - \bar{N})}{4} vv'' r + 2(4 - \bar{N})v'v'' r^2 \right] dr \\
&= \int \left[(v'')^2 r^3 + \left(c + \frac{(4 - \bar{N})(2 - \bar{N})}{2} \right) (v')^2 r + \frac{(4 - \bar{N})^2}{4} \left(\frac{(2 - \bar{N})^2}{4} + c \right) v^2 r^{-1} \right] dr \\
&= \frac{\bar{N}^2(4 - \bar{N})^2}{16} \int_0^\infty v^2 r^{-1} dr \\
&\quad + \int_0^\infty \left[(v'')^2 r^3 + \left(\frac{1}{2}(\bar{N} - 2)^2 + 1 - 2\lambda_n \right) (v')^2 r - \lambda_n \frac{(4 - \bar{N})^2}{2} v^2 r^{-1} \right] dr.
\end{aligned}$$

It then follows that

$$\begin{aligned}
& \int_0^\infty ((f''_{n,i})^2 r^{\bar{N}-1} + (\bar{N} - 2\lambda_n - 1)(f'_{n,i})^2 r^{\bar{N}-3} + \lambda_n(\lambda_n - 2(\bar{N} - 4))f_{n,i}^2 r^{\bar{N}-5}) dr \\
&\geq \frac{\bar{N}^2(4 - \bar{N})^2}{16} \int_0^\infty v^2 r^{-1} dr + \int_0^\infty [(v'')^2 r^3 + A(v')^2 r + Bv^2 r^{-1}] dr,
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{1}{2}(\bar{N} - 2)^2 + 1 - 2\lambda_n = 2n(n + \bar{N} - 2) + \frac{1}{2}(\bar{N} - 2)^2 + 1 \\
B &= \lambda_n(\lambda_n - 2(\bar{N} - 4)) - \lambda_n \frac{(4 - \bar{N})^2}{2} = n(n + \bar{N} - 2) \left[n(n + \bar{N} - 2) + \frac{\bar{N}}{2}(\bar{N} - 4) \right].
\end{aligned}$$

It is then clear that $A \geq 0$ for all $n = 0, 1, \dots$ without any restrictions on $\bar{N}$, whilst $B \geq 0$ for all n as long as $\bar{N} \neq 2$ (which is why we made this assumption). Finally, we have obtained

$$\begin{aligned}
& \int_0^\infty ((f''_{n,i})^2 r^{\bar{N}-1} + (\bar{N} - 2\lambda_n - 1)(f'_{n,i})^2 r^{\bar{N}-3} + \lambda_n(\lambda_n - 2(\bar{N} - 4))f_{n,i}^2 r^{\bar{N}-5}) dr \\
&\geq \frac{\bar{N}^2(4 - \bar{N})^2}{16} \int_0^\infty v^2 r^{-1} dr = \frac{\bar{N}^2(4 - \bar{N})^2}{16} \int_0^\infty f_{n,i}^2 r^{\bar{N}-5} dr.
\end{aligned}$$

Adding these up for all $n = 0, 1, \dots$ and $i = 1, \dots, d(n)$, from the above, and reconstruct-

ing f back from its spherical h-harmonics components, we have obtained that

$$\int_{\mathbb{R}^N} (\Delta_k f)^2 d\mu_k \geq \frac{(N+2\gamma)^2(N+2\gamma-4)^2}{16} \int_{\mathbb{R}^N} \frac{f^2}{|x|^4} d\mu_k,$$

as required.

To check that the constant is sharp we can use a similar example as in the classical case. More precisely, for $n = 3, 4, \dots$ let $f_n(x) = |x|^{2-\frac{N+2\gamma}{2}} h_n(|x|)$, where

$$h_n : [0, \infty) \rightarrow [0, 1] \quad \text{is such that} \quad h_n(r) = \begin{cases} 0 & \text{if } r \leq 1 \\ 1 & \text{if } 2 \leq r \leq n \\ 0 & \text{if } r \geq 2n, \end{cases}$$

with derivatives satisfying

$$|h'_n| \leq \frac{c_1}{n} \quad \text{and} \quad |h''_n| \leq \frac{c_2}{n^2}$$

for some constants $c_1, c_2 > 0$ (an explicit such h_n can be found in [83]). Then we can compute

$$\int_{\mathbb{R}^N} |\nabla_k f_n|^2 d\mu_k = C_1 + p(B_1) \frac{(N+2\gamma)^2(N+2\gamma-4)^2}{16} \int_2^{2n} \frac{1}{r} dr$$

and

$$\int_{\mathbb{R}^N} \frac{f_n^2}{|x|^4} d\mu_k = C_2 + p(B_1) \int_2^{2n} \frac{1}{r} dr,$$

where C_1 and C_2 can be bounded by constants that do not depend on n . Thus

$$\lim_{n \rightarrow \infty} \frac{\int_{\mathbb{R}^N} |\nabla_k f_n|^2 d\mu_k}{\int_{\mathbb{R}^N} \frac{f_n^2}{|x|^4} d\mu_k} = \frac{(N+2\gamma)^2(N+2\gamma-4)^2}{16}.$$

□

3.4.2 THE CAFFARELLI-KOHN-NIRENBERG INEQUALITY

The Caffarelli-Kohn-Nirenberg inequality [16] is

$$\left(\int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pb}} dx \right)^{1/p} \leq C \left(\int_{\mathbb{R}^N} \frac{|\nabla f|^r}{|x|^{ra}} dx \right)^{\theta/r} \left(\int_{\mathbb{R}^N} \frac{|f|^q}{|x|^{qc}} dx \right)^{(1-\theta)/q}.$$

In [16] necessary and sufficient conditions on the parameters p, q, r, a, b, c, θ are given for which the above inequality holds for all $f \in C_c^\infty(\mathbb{R}^N)$. We prove here the Dunkl analogue of a particular case of this inequality, corresponding to the values $r = 2$ and $\theta = 1$. In this special case the inequality was known before the work of Caffarelli, Kohn and Nirenberg, see for example [59], and it is sometimes known as the Hardy-Sobolev inequality as it generalises both these results (which correspond to the values $a = 0, b = 0$, and $a = 0, b = 1$, respectively). Best constants are also known, see [76, Corollary 4.8]. Our proof below is inspired by the method of [45].

Theorem 3.4.2. *Let $a \leq b \leq a + 1$ and $p = \frac{2(N+2\gamma)}{N+2\gamma-2+2(b-a)}$, and suppose that $a < \frac{N+2\gamma-2}{2}$. Then, for any $f \in C_c^\infty(\mathbb{R}^N)$ we have the inequality*

$$\int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} d\mu_k \geq C_{a,b} \left(\int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pb}} d\mu_k \right)^{2/p},$$

where $C_{a,b} > 0$ is a constant.

Proof. The strategy of the proof is to establish the inequality in the end cases $b = a + 1$ and $b = a$ separately, and then to interpolate between these two cases to obtain the result in full generality.

Step 1. Suppose $b = a + 1$, so $p = 2$. We begin by considering the function $u = \frac{f}{|x|^a}$, so

$$T_i u = \frac{T_i f}{|x|^a} - a \frac{x_i}{|x|^{a+2}} f. \quad (3.13)$$

Then we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} d\mu_k &= \int_{\mathbb{R}^N} \left| \nabla_k u + a \frac{x}{|x|^{a+2}} f \right|^2 d\mu_k \\ &= \int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k + a^2 \int_{\mathbb{R}^N} \frac{f^2}{|x|^{2a+2}} d\mu_k + 2a \int_{\mathbb{R}^N} \frac{f}{|x|^{a+2}} x \cdot \nabla_k u d\mu_k. \end{aligned} \quad (3.14)$$

Let $\epsilon > 1$. Applying the inequality $2xy \geq -\epsilon x^2 - \frac{1}{\epsilon} y^2$, we can estimate the last term on the right hand side of the previous equality

$$2a \int_{\mathbb{R}^N} \frac{f}{|x|^{a+2}} x \cdot \nabla_k u d\mu_k \geq -\epsilon a^2 \int_{\mathbb{R}^N} \frac{f^2}{|x|^{2a+2}} d\mu_k - \frac{1}{\epsilon} \int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k.$$

Plugging this in (3.14), we have obtained

$$\int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} d\mu_k \geq a^2(1 - \epsilon) \int_{\mathbb{R}^N} \frac{f^2}{|x|^{2a+2}} d\mu_k + (1 - \frac{1}{\epsilon}) \int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k.$$

Applying Hardy's inequality to the last term on the right hand side of the above inequality, we have

$$\int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} d\mu_k \geq \left(a^2(1 - \epsilon) + (1 - \frac{1}{\epsilon}) \frac{(N + 2\gamma - 2)^2}{4} \right) \int_{\mathbb{R}^N} \frac{f^2}{|x|^{2a+2}} d\mu_k.$$

This holds for all $\epsilon > 1$ and since $a < \frac{N+2\gamma-2}{2}$, we can choose for example $\epsilon = \frac{N+2\gamma-2}{2a}$ to obtain a positive constant.

Step 2. Suppose now that $a = b$. In this case $p = q := \frac{2(N+2\gamma)}{N+2\gamma-2}$, the Sobolev coefficient. Using the Sobolev inequality we have

$$\left(\int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pa}} d\mu_k \right)^{2/p} \leq C \int_{\mathbb{R}^N} \left| \nabla_k \left(\frac{f}{|x|^a} \right) \right|^2 d\mu_k.$$

Using (3.13), we obtain

$$\left| \nabla_k \left(\frac{f}{|x|^a} \right) \right|^2 = \left| \frac{\nabla_k f}{|x|^a} - a \frac{x}{|x|^{a+2}} f \right|^2 \leq 2 \frac{|\nabla_k f|^2}{|x|^{2a}} + 2a^2 \frac{f^2}{|x|^{2a+2}}.$$

Thus, from the last two relations it follows that

$$\begin{aligned} \left(\int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pa}} d\mu_k \right)^{2/p} &\leq 2Ca^2 \int_{\mathbb{R}^N} \frac{|f|^2}{|x|^{2a+2}} d\mu_k + 2C \int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} d\mu_k \\ &\leq 2C(a^2 C_{a,a+1}^{-1} + 1) \int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} d\mu_k, \end{aligned}$$

where we used the previous step.

Step 3. We now look at the case $a < b < a + 1$. As above, let $q = \frac{2(N+2\gamma)}{N+2\gamma-2}$ be the Sobolev coefficient. We have $2 < p < q$, so there exists $\theta \in (0, 1)$ such that

$$p = 2\theta + q(1 - \theta),$$

so

$$b = a + \frac{\theta(N + 2\gamma - 2)}{N + 2\gamma - 2\theta},$$

and also

$$pb = 2(a + 1)\theta + qa(1 - \theta).$$

Then, using Hölder's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pb}} d\mu_k &= \int_{\mathbb{R}^N} \frac{|f|^{2\theta+q(1-\theta)}}{|x|^{2(a+1)\theta+qa(1-\theta)}} d\mu_k \\ &\leq \left(\int_{\mathbb{R}^N} \frac{|f|^2}{|x|^{2(a+1)}} d\mu_k \right)^\theta \left(\int_{\mathbb{R}^N} \frac{|f|^q}{|x|^{qa}} d\mu_k \right)^{1-\theta}. \end{aligned}$$

Using the two steps above, this implies

$$\int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pb}} d\mu_k \leq C_{a,a+1}^{-\theta} C_{a,a}^{-q(1-\theta)/2} \left(\int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} d\mu_k \right)^{p/2},$$

as required. This completes the proof. $\square$

Remark. One could prove a more general Caffarelli-Kohn-Nirenberg inequality of the form

$$\left(\int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pb}} d\mu_k \right)^{1/p} \leq C \left(\int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} d\mu_k \right)^{\theta/2} \left(\int_{\mathbb{R}^N} \frac{|f|^q}{|x|^{qc}} d\mu_k \right)^{(1-\theta)/q},$$

which holds for all $f \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, subject to the assumption that

$$\frac{1}{p} - \frac{b}{N+2\gamma} = \theta \left(\frac{1}{2} - \frac{a+1}{N+2\gamma} \right) + (1-\theta) \left(\frac{1}{q} - \frac{c}{N+2\gamma} \right),$$

where $b = (1-\theta)c + \theta d$, for parameters $p, q, a, c, d, \theta \in \mathbb{R}$ such that $p > 0$, $q \geq 1$, $\theta \in [0, 1]$, and such that all integrals above are finite. This could be achieved by interpolating using Hölder's inequality between the case $\theta = 0$ (which is trivial as $p = q$, $b = c$, and both sides reduce to $\| |x|^{-b} f \|_p$), and the case $\theta = 1$, which was done in the previous Theorem. However, this only works for a more restricted and rather complicated range of θ depending on p and q .

4

SPECTRAL ESTIMATES

Consider the Schrödinger operator on $L^2(\mathbb{R}^N)$

$$H := -\Delta - V,$$

where $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the usual Laplacian on $\mathbb{R}^N$ and V is a potential. Under certain conditions on the potential V , it can be shown that the negative spectrum of H is discrete; in this case the negative eigenvalues of H can be defined using the variational principle (see for example [68, Chapter 11] or [26] for a comprehensive discussion of this). When this holds, we denote the negative eigenvalues of H repeated according to multiplicity by $\lambda_1 \leq \lambda_2 \leq \dots < 0$. The Lieb-Thirring inequality [70] gives estimates on the sum of powers of the negative eigenvalues, namely

$$S_\zeta(V) := \sum_i |\lambda_i|^\zeta \leq L_{N,\zeta} \int_{\mathbb{R}^N} V_+(x)^{N/2+\zeta} dx, \quad (4.1)$$

where $L_{N,\zeta}$ is a constant, and we use the notation $f_+(x) = \max\{f(x), 0\}$. This inequality holds for $\zeta > 0$ if $N \geq 2$ and for $\zeta > \frac{1}{2}$ if $N = 1$. After it was first proved, this inequality has been studied extensively and generalised to different settings, for example to Riemannian manifolds. For an account of the Lieb-Thirring inequality and its applications to stability of matter, see [69]. The value of the optimal constant $L_{n,\zeta}$ is in general not known. By comparison to the semi-classical approximation, we have

$$L_{N,\zeta} \geq L_{N,\zeta}^{\text{cl}} := \frac{1}{(4\pi)^{N/2}} \frac{\Gamma(\zeta + 1)}{\Gamma\left(\frac{N}{2} + \zeta + 1\right)}.$$

Lieb and Thirring proved in [70] that for $N = 1$ and $\zeta \geq \frac{3}{2}$ we have in fact equality

$$L_{1,\zeta} = L_{1,\zeta}^{\text{cl}}.$$

In [63], Laptev and Weidl showed that this equality holds more generally for all $N \geq 1$ and $\zeta \geq \frac{3}{2}$. Finally, in the case $N = 1$, $\zeta = \frac{1}{2}$ the optimal value $L_{1,1/2} = 2L_{1,1/2}^{\text{cl}} = \frac{1}{2}$ was computed in [58]. See [62] for a survey of the best known values of the constants in the inequality (4.1).

The case $\zeta = 0$ was proved individually by Rozenblum [90], Lieb [67] and Cwikel [20], and it is usually referred to as the Cwikel-Lieb-Rozenblum inequality, or, in short, the CLR inequality. A later proof was given by Li and Yau in [66].

Here we will prove these inequalities for Schrödinger operators involving the Dunkl laplacian. Namely, we consider

$$H := -\Delta_k - V,$$

for a potential V . As above, when the negative spectrum of this operator is discrete, we denote by $\lambda_1 \leq \lambda_2 \leq \dots < 0$ its negative eigenvalues repeated according to multiplicity, and for $\zeta > 0$ we denote

$$S_\zeta(V) := \sum_i |\lambda_i|^\zeta.$$

We first prove the Lieb-Thirring inequality for Dunkl operators using the method of the original proof [70]. For the CLR inequality, we obtain this as a consequence of the Sobolev inequality of Theorem 2.2.2 from a general result of Levin and Solomyak [65] for Markov semigroups.

4.1 THE LIEB-THIRRING INEQUALITY

The main result is the following.

Theorem 4.1.1. *Assume that $N + 2\gamma > 2$. Fix $\zeta > 0$ and let V be a potential such that $V_+ \in L^{N/2+\gamma+\zeta}(\mu_k)$. Then we have*

$$S_\zeta(V) \leq L_\zeta \int_{\mathbb{R}^N} V_+^{N/2+\gamma+\zeta} d\mu_k,$$

with a constant

$$L_\zeta = \frac{2^{-\frac{N+2\gamma}{2}+1} M_k^{-1} \zeta^{\zeta+1}}{\Gamma(N/2 + \gamma + \zeta + 1)} \inf_{\frac{N+2\gamma}{2} < m < \frac{N+2\gamma}{2} + \zeta} [mF(m - N/2 - \gamma)F(N/2 + \gamma + \zeta - m)],$$

for $F(x) := \Gamma(x)x^{-x}$.

Remark. As will be clear from the discussion below, the condition on V implies that the Schrödinger operator H is semi-bounded self-adjoint, so we can define the negative eigenvalues using the usual variational approach.

The proof will be based on the Birman-Schwinger principle which was first proved in [13] and [94]. We will give here a proof of this principle adapted to our case.

Proposition 4.1.2 (Birman-Schwinger principle). *Let $c > 0$ and assume that the potential $V \in L^{N/2+\gamma+\zeta}(\mu_k)$ is non-negative. Consider the operator*

$$T_c := V^{1/2}(-\Delta_k + c)^{-1}V^{1/2}.$$

Denote by $n(c)$ the number of eigenvalues greater or equal to 1 of T_c . Then the number of eigenvalues at most $-c$ of H is $n(c)$.

Proof. To simplify notation, let $\tilde{N} = \frac{N}{2} + \gamma + \zeta$.

Firstly we note that $(-\Delta_k + c)^{-1/2}$ is a bounded operator from $L^2(\mu_k)$ to $H_D^1(\mathbb{R}^N)$. Indeed, for any $f \in L^2(\mu_k)$ we have

$$\mathcal{D}_k((-\Delta_k + c)^{-1/2}f)(\xi) = (|\xi|^2 + c)^{-1/2}\mathcal{D}_k(f)(\xi).$$

Since $\frac{|\xi|}{(|\xi|^2 + c)^{1/2}} \leq 1$ and $f \in L^2(\mu_k)$, then the map $\xi \mapsto |\xi|\mathcal{D}_k((-\Delta_k + c)^{-1/2}f)(\xi)$ is in $L^2(\mu_k)$, so by the Fourier characterisation of the Sobolev space we have $(-\Delta_k + c)^{-1/2}f \in H_D^1(\mathbb{R}^N)$. Moreover,

$$\|(-\Delta_k + c)^{-1/2}f\|_{H_D^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} \frac{1 + |\xi|^2}{c + |\xi|^2} |\mathcal{D}_k(f)(\xi)|^2 d\mu_k(\xi) \leq \sup_{\xi \in \mathbb{R}^N} \frac{1 + |\xi|^2}{c + |\xi|^2} \|f\|_2.$$

On the other hand, $V^{1/2}$ is a bounded operator from $H_D^1(\mathbb{R}^N)$ to $L^2(\mu_k)$. To see this, we first use the Hölder inequality with exponents $\tilde{N}$ and $\frac{\tilde{N}}{\tilde{N}-1}$ to obtain

$$\left(\int_{\mathbb{R}^N} V(x)|f(x)|^2 d\mu_k(x) \right)^{1/2} \leq \|V\|_{N/2+\gamma+\zeta}^{1/2} \left(\int_{\mathbb{R}^N} |f(x)|^{\frac{2\tilde{N}}{\tilde{N}-1}} d\mu_k(x) \right)^{\frac{\tilde{N}-1}{2\tilde{N}}}. \quad (4.2)$$

Let $q = \frac{2(N+2\gamma)}{N+2\gamma-2}$. Since $2 < \frac{2\tilde{N}}{\tilde{N}-1} < q$, there exists $\theta \in (0, 1)$ such that $\frac{2\tilde{N}}{\tilde{N}-1} = 2\theta + q(1-\theta)$. Thus, by Hölder's inequality again, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |f(x)|^{\frac{2\tilde{N}}{\tilde{N}-1}} d\mu_k(x) \right)^{\frac{\tilde{N}-1}{2\tilde{N}}} &\leq \|f\|_2^{2\theta \frac{\tilde{N}-1}{2\tilde{N}}} \|f\|_q^{(1-\theta)q \frac{\tilde{N}-1}{2\tilde{N}}} \\ &\leq 2\theta \frac{\tilde{N}-1}{2\tilde{N}} \|f\|_2 + (1-\theta)q \frac{\tilde{N}-1}{2\tilde{N}} \|f\|_q, \end{aligned} \quad (4.3)$$

where in the last step we used Young's inequality. Finally, combining (4.2) and (4.3), and using Sobolev's inequality, we have

$$\left(\int_{\mathbb{R}^N} V(x) |f(x)|^2 d\mu_k(x) \right)^{1/2} \leq C_1 \|f\|_2 + C_2 \|\nabla_k f\|_2,$$

for some constants $C_1, C_2 > 0$.

Therefore, the operator $U := V^{1/2}(-\Delta_k + c)^{-1/2}$ is bounded from $L^2(\mu_k)$ to $L^2(\mu_k)$. Since $T_c = UU^*$, this shows that T_c is also bounded and self-adjoint. Moreover, we will see below that for large enough $m > 0$ we have $\text{Tr}(T_c^m) < \infty$, so T_c is a compact operator.

The connection between the operator T_c and our Schrödinger operator H is given by the fact that there is a one-to-one correspondence between eigenfunctions with eigenvalue 1 of T_c , and eigenfunctions with eigenvalue $-c$ of H .

Indeed, let first $f \in L^2(\mu_k)$ be such that $T_c f = f$, and define $g = (-\Delta_k + c)^{-1} V^{1/2} f$. If we can prove that $g \in H_{\mathcal{D}}^1(\mathbb{R}^N)$, then we have

$$(-\Delta_k + c)g = V^{1/2} f = Vg,$$

so g is an eigenfunction of H with eigenvalue $-c$. To prove that $g \in H_{\mathcal{D}}^1(\mathbb{R}^N)$, we use first Hölder's inequality to obtain

$$\int_{\mathbb{R}^N} (\sqrt{V} f)^{\frac{2\tilde{N}}{\tilde{N}+1}} d\mu_k \leq \|V\|_{\frac{\tilde{N}}{\tilde{N}+1}} \|f\|_2^{\frac{\tilde{N}}{\tilde{N}+1}},$$

so $\sqrt{V} f \in L^{\frac{2\tilde{N}}{\tilde{N}+1}}(\mu_k)$. But $1 < \frac{2\tilde{N}}{\tilde{N}+1} < 2$ and the Dunkl transform $\mathcal{D}_k$ is a bounded operator from $L^1(\mu_k)$ to $L^\infty(\mu_k)$ and also from $L^2(\mu_k)$ to $L^2(\mu_k)$, so by the Riesz-Thorin interpolation theorem we have $\mathcal{D}_k(\sqrt{V} f) \in L^{\frac{2\tilde{N}}{\tilde{N}-1}}(\mu_k)$. Since $\mathcal{D}_k((- \Delta_k + c)^{-1} \sqrt{V} f)(\xi) = \frac{1}{|\xi|^2 + c} \mathcal{D}_k(\sqrt{V} f)(\xi)$, by Hölder's inequality we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|\xi|^2}{(|\xi|^2 + c)^2} \mathcal{D}_k(\sqrt{V} f)(\xi)^2 d\mu_k(\xi) &\leq \int_{\mathbb{R}^N} \frac{\mathcal{D}_k(\sqrt{V} f)(\xi)^2}{|\xi|^2 + c} d\mu_k(\xi) \\ &\leq \left\| \mathcal{D}_k(\sqrt{V} f) \right\|_{\frac{\tilde{N}}{\tilde{N}-1}}^{\frac{\tilde{N}}{\tilde{N}-1}} \left(\int_{\mathbb{R}^N} \frac{1}{(|\xi|^2 + c)^{\tilde{N}}} d\mu_k(\xi) \right)^{1/\tilde{N}}. \end{aligned}$$

As we will see below in the proof of the Lieb-Thirring inequality, the integral on the right hand side is finite since $\tilde{N} > N/2 + \gamma$. Thus, we proved that the map $\xi \mapsto |\xi| \mathcal{D}_k((- \Delta_k + c)^{-1} \sqrt{V} f)(\xi)$ is in $L^2(\mu_k)$, which means that $g \in H_{\mathcal{D}}^1(\mathbb{R}^N)$.

On the other hand, if $g \in H_{\mathcal{D}}^1(\mathbb{R}^N)$ is such that $Hg = -cg$, then take $f = V^{1/2}g$. By the above, then $f \in L^2(\mu_k)$ and clearly $T_c f = f$.

To finish the proof, we note that by the min-max principle, if $c < c'$, we have $T_c \geq T_{c'}$. Thus, if we denote the eigenvalues of T_c and $T_{c'}$ by $\lambda_1(c) \geq \lambda_2(c) \geq \dots \geq 0$ and $\lambda_1(c') \geq \lambda_2(c') \geq \dots \geq 0$, respectively, then for any $i = 1, 2, \dots$, we have $\lambda_i(c) \geq \lambda_i(c')$. Moreover, $\lambda_i(c)$ depends continuously on c . Given all this, it is then not difficult to conclude the proof. $\square$

Before we begin the proof of the Lieb-Thirring inequality, we will also need the following result also due to Lieb and Thirring, which we state here without proof (see [70, Appendix B], or [6] for a different proof).

Proposition 4.1.3. *Let A, B be positive, self-adjoint operators on a separable Hilbert space $\mathcal{H}$. Then, for each real $m \geq 1$, we have*

$$\mathrm{Tr}(B^{1/2}AB^{1/2})^m \leq \mathrm{Tr}(B^{m/2}A^mB^{m/2}).$$

Proof of Theorem 4.1.1. Let N_c be the number of eigenvalues of H that are at most c (counted with multiplicity). From this definition of N_c we obtain straight away that

$$S_\zeta(V) = \sum_j |\lambda_j|^\zeta = \zeta \int_0^\infty c^{\zeta-1} N_{-c} dc. \quad (4.4)$$

Take $c > 0$ and $0 \leq t \leq 1$. It can then be checked directly that N_{-c} is equal to the number of eigenvalues that are at most $-tc$ of the operator

$$-\Delta_k - (V - (1-t)c).$$

By the min-max principle, since

$$-\Delta_k - (V - (1-t)c) \geq -\Delta_k - (V - (1-t)c)_+,$$

we obtain that N_{-c} is bounded above by the number of eigenvalues at most $-tc$ of the operator

$$-\Delta_k - (V - (1-t)c)_+.$$

However, by the Birman-Schwinger principle, the number of eigenvalues that are smaller or equal to $-tc$ of this latter operator equals the number of eigenvalues that are at least 1 of the operator

$$T_c^{(t)} := (V - (1-t)c)_+^{1/2} (-\Delta_k + tc)^{-1} (V - (1-t)c)_+^{1/2}.$$

Therefore, we have that, for any $m \geq 1$,

$$N_{-c} \leq \text{Tr}((T_c^{(t)})^m).$$

In order to estimate the right hand side of this last relation, we note that in light of Proposition 4.1.3, it is enough to estimate the trace of the operator

$$T_c^{m,t} := (V - (1-t)c)_+^{m/2} (-\Delta_k + tc)^{-m} (V - (1-t)c)_+^{m/2}.$$

Using the following property of the Dunkl transform

$$\mathcal{D}_k(-\Delta_k f)(\xi) = |\xi|^2 \mathcal{D}_k(f)(\xi),$$

as well as the inversion formula, it follows that

$$\begin{aligned} -\Delta_k f(x) &= \mathcal{D}_k(|\xi|^2 \mathcal{D}_k(f)(\xi))(-x) = M_k^{-1} \int_{\mathbb{R}^N} |y|^2 \mathcal{D}_k(f)(y) E_k(ix, y) d\mu_k(y) \\ &= M_k^{-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(z) |y|^2 E_k(-iz, y) E_k(ix, y) d\mu_k(y) d\mu_k(z). \end{aligned}$$

Thus, for a Borel measurable function φ , we can define

$$\varphi(-\Delta_k) f(x) = M_k^{-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(z) \varphi(|y|^2) E_k(-iz, y) E_k(ix, y) d\mu_k(y) d\mu_k(z),$$

so $\varphi(-\Delta_k)$ is an integral operator with kernel

$$K_\varphi(x, z) = M_k^{-2} w_k(z) \int_{\mathbb{R}^N} \varphi(|y|^2) E_k(-iz, y) E_k(ix, y) d\mu_k(y).$$

Going back to our proof, we have that $T_c^{m,t}$ is an integral operator with kernel given by

$$(V(x) - (1-t)c)_+^{m/2} (V(z) - (1-t)c)_+^{m/2} K_\varphi(x, z),$$

where now $\varphi : u \mapsto (u + tc)^{-m}$. Then we can compute

$$\text{Tr}(T_c^{m,t}) = \int_{\mathbb{R}^N} (V(x) - (1-t)c)_+^m K_\varphi(x, x) dx.$$

We have

$$K_\varphi(x, x) = M_k^{-2} w_k(x) \int_{\mathbb{R}^N} (|y|^2 + tc)^{-m} |E_k(ix, y)|^2 d\mu_k(y),$$

which we can estimate using the inequality $|E_k(ix, y)| \leq 1$. We obtain

$$K_\varphi(x, x) \leq M_k^{-2} w_k(x) \int_{\mathbb{R}^N} \frac{1}{(|y|^2 + tc)^m} d\mu_k(y).$$

In order to compute this integral we use polar coordinates and obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{1}{(|y|^2 + tc)^m} d\mu_k(y) &= \frac{M_k}{2^{\frac{N+2\gamma}{2}-1} \Gamma\left(\frac{N+2\gamma}{2}\right)} \int_0^\infty \frac{r^{N+2\gamma-1}}{(r^2 + tc)^m} dr \\ &= \frac{M_k}{2^{\frac{N+2\gamma}{2}-1} \Gamma\left(\frac{N+2\gamma}{2}\right)} (tc)^{\gamma+N/2-m} B\left(\frac{N}{2} + \gamma, m - \frac{N}{2} - \gamma\right), \end{aligned}$$

where we require $m > N/2 + \gamma$. Here B is the usual beta function. To summarise, we have obtained that

$$N_{-c} \leq \text{Tr}(T_c^{m,t}) \leq C(tc)^{N/2+\gamma-m} \int_{\mathbb{R}^N} (V(x) - (1-t)c)_+^m d\mu_k(x),$$

where

$$C = \frac{1}{M_k 2^{\frac{N+2\gamma}{2}-1} \Gamma\left(\frac{N+2\gamma}{2}\right)} B\left(\frac{N}{2} + \gamma, m - \frac{N}{2} - \gamma\right).$$

As in the previous proof, let $\tilde{N} = \frac{N}{2} + \gamma + \zeta$. Replacing the result above in (4.4), we obtain that

$$S_\zeta(V) \leq \zeta C t^{N/2+\gamma-m} \int_{\mathbb{R}^N} \int_0^\infty c^{\tilde{N}-m-1} (V(x) - (1-t)c)_+^m d\mu_k(x) dc.$$

By swapping the order of integration in the integral on the right hand side using the Fubini-Tonelli theorem, and using the change of variables $V_+(x)b = (1-t)c$, we have

$$\begin{aligned} S_\zeta(V) &\leq \zeta C t^{N/2+\gamma-m} (1-t)^{m-\tilde{N}} \int_{\mathbb{R}^N} \int_0^1 V_+(x)^{\tilde{N}} b^{\tilde{N}-m-1} (1-b)^m db d\mu_k(x) \\ &= \zeta C t^{N/2+\gamma-m} (1-t)^{m-\tilde{N}} B(\tilde{N}-m, m+1) \int_{\mathbb{R}^N} V_+(x)^{\tilde{N}} d\mu_k(x) \end{aligned}$$

We note here that in the last equality we require $\zeta + \gamma + N/2 - m = \tilde{N} - m > 0$. The constant in this inequality is optimised for

$$t = \frac{m - \gamma - N/2}{\zeta},$$

so we obtain finally

$$S_\zeta(V) \leq 2^{-\frac{N+2\gamma}{2}+1} M_k^{-1} m \zeta^{\zeta+1} F(m-\gamma-N/2) F(\tilde{N}-m) \frac{1}{\Gamma(\tilde{N}+1)} \int_{\mathbb{R}^N} V_+(x)^{\tilde{N}} d\mu_k(x),$$

where $F(x) := \Gamma(x)x^{-x}$. The Theorem follows immediately by taking infimum over all possible m . $\square$

4.2 THE CLR INEQUALITY

As mentioned above, the CLR inequality was proved independently by Rozenblum, Lieb and Cwikel. Later, a different proof was given by Li and Yau in [66] which also produced the best known constant. This simple proof pertained to further generalisation and in [65] it was proved in the general setting of a Markov generator which satisfies the Sobolev inequality.

Theorem 4.2.1 ([65]). *Let (Ω, μ) be a σ -finite measure space and let A be a Markov generator on $L^2(\Omega)$ with quadratic form Q . Suppose that for some $q > 0$, A satisfies a Sobolev inequality of the form*

$$\|f\|_{L^q(\mu)}^2 \leq CQ(f) \quad (4.5)$$

for any $f \in \text{Dom}(Q)$. Take $V \in L^p(\mu)$, $V \geq 0$, where $p = \frac{q}{q-2}$ and let $N(V)$ be the number of negative eigenvalues of the operator

$$-A - V.$$

Then we have

$$N(V) \leq e^p C^p \int_{\Omega} V^p d\mu,$$

where C is the constant in (4.5).

By Rösler's positivity results for the Dunkl intertwining operator (which, in particular, show that the Dunkl heat semigroup is Markov), together with the Sobolev inequality we obtained above (Theorem 2.2.2), the conditions of Theorem 4.2.1 are satisfied for $A = \Delta_k$ and therefore we finally obtain the desired inequality.

Corollary 4.2.2 (CLR inequality). *Suppose $N + 2\gamma > 2$. Let $V \geq 0$ be in $L^{N/2+\gamma}(\mathbb{R}^N)$. Then the number $N(V)$ of negative eigenvalues of the operator H satisfies the bound*

$$N(V) \leq e^{N/2+\gamma} C_{DS}^{N+2\gamma} \int_{\mathbb{R}^N} V^{N/2+\gamma}(x) d\mu_k,$$

where C_{DS} is the optimal constant in the Sobolev inequality.

We will not give a proof of Theorem 4.2.1 here, but very briefly sketch the proof in our particular case. Firstly, we can assume that $V \in L^1 \cap L^\infty$ with $V > 0$ by a density argument in a suitable space of compact operators. We will consider the quadratic form

$$Q(f) := \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k$$

with domain $H_{\mathcal{D}}^1(\mathbb{R}^N)$. Notice that Q in fact defines a norm on the Hilbert space $H_{\mathcal{D}}^1(\mathbb{R}^N)$ and let X be the completion of $H_{\mathcal{D}}^1(\mathbb{R}^N)$ under this norm. Consider now

$$Q'(f) = \int_{\mathbb{R}^N} V|f|^2 d\mu_k.$$

By the Sobolev inequality, the quadratic form Q' defines a bounded self-adjoint operator on X , say T_V . If we see Q' as a quadratic form in the weighted space $L^2(V d\mu_k)$, it can be checked that this also defines a self-adjoint operator, say K , which is related to T_V via the relation

$$K^{-1} = T_V.$$

The operator K generates a semigroup which is ultracontractive, so it has an integral kernel $k_t(x, y)$. Moreover, K can be seen to have a discrete spectrum. By proving bounds on k_t we can then obtain bounds on the eigenvalues of K , and so, also on the eigenvalues of T_V . Finally, a general version of the Birman-Schwinger principle allows us to compare the negative eigenvalues of H with the eigenvalues of T_V .

5

THE GAGLIARDO-NIRENBERG INEQUALITY AND AN APPLICATION

The Gagliardo-Nirenberg inequality was proved independently by Gagliardo [42] and Nirenberg [78] and it states that

$$\|\nabla^j f\|_q \leq C \|\nabla^l f\|_r^\theta \|f\|_p^{1-\theta}, \quad (5.1)$$

for all $f \in W^{l,r}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, where $1 \leq p \leq \infty$, $j, l \in \mathbb{N}$ and $\frac{j}{l} \leq \theta \leq 1$ are such that

$$\frac{1}{q} = \frac{j}{N} + \theta \left(\frac{1}{r} - \frac{l}{N} \right) + \frac{1-\theta}{p}.$$

There is an exceptional case when $r > 1$ and $l - j - \frac{N}{r}$ is a non-negative integer, in which case we must also require $\theta < 1$. See [41] for historical remarks on this result and a complete proof, and also the references therein for an account of the many applications of the Gagliardo-Nirenberg inequality to PDEs.

This chapter will consist of two parts. In the first part we will prove the Gagliardo-Nirenberg inequality for Dunkl operators in two different particular cases. Firstly, in section 5.1, we prove the case $j = 0$, $l = 1$, which follows from the Sobolev inequality, and which is the main tool in the analysis of the nonlinear damped wave equation in later sections. Secondly, in section 5.2 we prove the more involved case $p = q = r$, which relies on Littlewood-Paley theory.

In the second part of the chapter we use the obtained Gagliardo-Nirenberg inequality to

study the nonlinear damped wave equation

$$\begin{cases} \partial_{tt}^2 u(t, x) - \Delta_k u(t, x) + b \partial_t u(t, x) + m u(t, x) = f(u)(t, x) \\ u(0, x) = \epsilon u_1(x) \\ \partial_t u(0, x) = \epsilon u_2(x), \end{cases}$$

with nonlinearity satisfying the assumptions $f(0) = 0$ and

$$|f(a) - f(b)| \leq C(|a|^{p-1} + |b|^{p-1})|a - b|,$$

for some $p > 1$. To begin with, in section 5.3.1 we study the linear version of this equation (corresponding to $f = 0$), and in section 5.3.2 we prove a global existence and uniqueness result for the general Cauchy problem with small data. This follows the treatment of [91], where similar problems are considered for Rockland operators on Lie groups.

5.1 THE GAGLIARDO-NIRENBERG INEQUALITY I

In this section we prove the Dunkl analogue of the case $j = 0$, $l = 1$ of the classical Gagliardo-Nirenberg inequality (5.1).

Theorem 5.1.1. *Let $1 \leq p, q < \infty$, $1 \leq r < N + 2\gamma$, and $0 \leq \theta \leq 1$ satisfying the assumption*

$$\theta \left(\frac{1}{N + 2\gamma} + \frac{1}{p} - \frac{1}{r} \right) = \frac{1}{p} - \frac{1}{q}. \quad (5.2)$$

Then the following inequality holds for all $f \in H_D^1(\mathbb{R}^N) \cap L^p(\mu_k)$

$$\|f\|_q \leq C \|\nabla_k f\|_r^\theta \|f\|_p^{1-\theta},$$

for some constant $C > 0$.

Proof. We note first that it is enough to prove the result for $f \in C_c^\infty(\mathbb{R}^N)$ and then use a density argument. If $\theta = 0$, then $p = q$ and the inequality reduces to the trivial case $\|f\|_p \leq C \|f\|_p$. For the rest of the proof we will assume that $\theta > 0$.

Since $r < N + 2\gamma$, from (5.2) we obtain $(1 - \theta)q < p$. We can then define $s > 0$ by the equation

$$\frac{q(1 - \theta)}{p} + \frac{1}{s} = 1. \quad (5.3)$$

Then, by Hölder's inequality we have

$$\begin{aligned} \int_{\mathbb{R}^N} |f|^q d\mu_k &\leq \left(\int_{\mathbb{R}^N} |f|^{qs\theta} d\mu_k \right)^{1/s} \left(\int_{\mathbb{R}^N} |f|^p d\mu_k \right)^{\frac{q(1-\theta)}{p}} \\ &= \|f\|_{qs\theta}^{q\theta} \|f\|_p^{q(1-\theta)}. \end{aligned} \quad (5.4)$$

At this point, we would like to bound $\|f\|_{qs\theta}$ from above by $\|\nabla_k f\|_r$, which, together with the above inequality, concludes the Theorem. This can be achieved by the Sobolev inequality of Theorem 2.0.1 as long as $1 \leq r < N + 2\gamma$ and

$$\frac{1}{qs\theta} = \frac{1}{r} - \frac{1}{N + 2\gamma}.$$

This follows from the assumption (5.2) and the definition of s in equation (5.3). Thus we can indeed apply the Sobolev inequality to obtain

$$\|f\|_{qs\theta} \leq C \|\nabla_k f\|_r. \quad (5.5)$$

Therefore, from (5.4) and (5.5) we can conclude that

$$\|f\|_q \leq C \|\nabla_k f\|_r^\theta \|f\|_p^{1-\theta},$$

which is what we needed to prove. □

We record in the next Corollary the very useful particular case when $p = r = 2$, while in the second Corollary we prove a consequence of Gagliardo-Nirenberg involving the Sobolev norm that will be essential later in the study of the wave equation.

Corollary 5.1.2. *Suppose $N + 2\gamma > 2$. Let $2 \leq q \leq \frac{2(N+2\gamma)}{N+2\gamma-2}$ and*

$$\theta = (N + 2\gamma) \frac{q - 2}{2q}. \quad (5.6)$$

Then there exists a constant $C > 0$ such that the inequality

$$\|f\|_q \leq C \|\nabla_k f\|_2^\theta \|f\|_2^{1-\theta}$$

holds for any $f \in H_D^1(\mathbb{R}^N)$.

Corollary 5.1.3. *Suppose $N + 2\gamma > 2$. Then for any $2 \leq q \leq \frac{2(N+2\gamma)}{N+2\gamma-2}$ there exists a constant $C > 0$ such that the inequality*

$$\|f\|_q \leq C \|f\|_{H_D^1(\mathbb{R}^N)}$$

holds for any $f \in H_{\mathcal{D}}^1(\mathbb{R}^N)$.

Proof. We use the Young inequality

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b,$$

which holds for all $a, b \geq 0$ and $0 \leq \theta \leq 1$. From the previous Corollary, with θ as in (5.6), we have

$$\|f\|_q^2 \leq C \max\{\theta, 1-\theta\} \cdot (\|\nabla_k f\|_2^2 + \|f\|_2^2) \leq C \|f\|_{H_{\mathcal{D}}^1(\mathbb{R}^N)}^2,$$

and the result follows immediately. $\square$

5.2 THE GAGLIARDO-NIRENBERG INEQUALITY II

In this section we prove the case $p = q = r$ of the Gagliardo-Nirenberg inequality. Note that in this case the assumptions in (5.1) imply that $\theta = \frac{j}{l}$. However, we adopt a slightly different notation in this section that will be more convenient for the present case. The main result is the following.

Theorem 5.2.1 (Gagliardo-Nirenberg inequality). *Let $1 < p < \infty$, $0 \leq s \leq \frac{N+2\gamma}{p}$ and $0 < \theta < 1$. Then, for any $f \in \mathcal{S}(\mathbb{R}^N)$, we have the inequality*

$$\|(-\Delta_k)^{s(1-\theta)/2} f\|_p \leq C \|(-\Delta_k)^{s/2} f\|_p^{1-\theta} \|f\|_p^\theta,$$

where $C > 0$ is a constant.

In order to prove this result, we need to introduce some Littlewood-Paley theory. Let $\psi \in C_c^\infty(\mathbb{R})$ with support contained in the annulus $\frac{1}{2} \leq |x| \leq 2$ and such that

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j}x) = 1 \quad \text{for all } x \in \mathbb{R} \setminus \{0\}. \quad (5.7)$$

For convenience denote $\psi_j(x) = \psi(2^{-j}x)$ and note that ψ_j has support contained in the annulus $2^{j-1} \leq |x| \leq 2^{j+1}$. Define the following operators on $L^2(\mu_k)$

$$P_j(f) = \mathcal{D}_k^{-1}(\psi_j \mathcal{D}_k(f)).$$

Then (5.7) implies that

$$\sum_{j \in \mathbb{Z}} P_j = I, \quad (5.8)$$

where I is the identity operator on $L^2(\mu_k)$. The following result will be essential in the proof of Theorem 5.2.1.

Proposition 5.2.2. *For any $s \geq 0$ and $1 < p < \infty$ there exist two constants $C_1, C_2 > 0$ such that*

$$C_1 \|(-\Delta_k)^{s/2} f\|_p \leq \left\| \left(\sum_{j \in \mathbb{Z}} |2^{js} P_j f|^2 \right)^{1/2} \right\|_p \leq C_2 \|(-\Delta_k)^{s/2} f\|_p$$

holds for all $f \in \mathcal{S}(\mathbb{R}^N)$.

Before we prove the Proposition, we need the following two results. The first is the Hörmander-Mikhlin multiplier theorem which was proved in the case of Dunkl operators in [22] (see also [23, Chapter 7]). The second result is a Lemma which corresponds to the particular case $s = 0$ in Proposition 5.2.2, but where the conditions on the function ψ are relaxed.

Theorem 5.2.3 (Hörmander-Mikhlin multiplier theorem - Theorem 4.1, [22]). *Let $m : (0, \infty) \rightarrow \mathbb{R}$ be a function satisfying $\|m\|_\infty \leq A$ and the Hörmander's condition*

$$\frac{1}{R} \int_R^{2R} |m^{(r)}(x)| dx \leq AR^{-r} \quad \forall R > 0,$$

where r is the smallest integer greater than or equal to $\frac{N+1}{2} + \gamma$. Let $\mathcal{T}_m$ be the operator on $L^2(\mu_k)$ defined by

$$\mathcal{D}_k(\mathcal{T}_m f)(\xi) = m(|\xi|) \mathcal{D}_k(f)(\xi) \quad \forall \xi \in \mathbb{R}^N.$$

Then for all $1 < p < \infty$ there exists a constant $C_p > 0$ such that

$$\|\mathcal{T}_m f\|_p \leq C_p A \|f\|_p$$

for all $f \in \mathcal{S}(\mathbb{R}^N)$.

Lemma 5.2.4. *Let $\varphi \in C_c^\infty(\mathbb{R})$ with support contained in the annulus $\frac{1}{2} \leq |x| \leq 2$. As in the above, let $\varphi_j(x) = \varphi(2^{-j}x)$ and consider the operator on $L^2(\mu_k)$ given by*

$$P_j^\varphi f = \mathcal{D}_k^{-1}(\varphi_j \mathcal{D}_k(f)).$$

Then, for any $1 < p < \infty$ there exists a constant $C > 0$ such that

$$\left\| \left(\sum_{j \in \mathbb{Z}} |P_j^\varphi f|^2 \right)^{1/2} \right\|_p \leq C \|f\|_p. \quad (5.9)$$

Proof. Let $\{\epsilon_j\}_{j \in \mathbb{Z}}$ be a sequence of independent identically distributed random variables with $\mathbb{P}(\epsilon_j = \pm 1) = \frac{1}{2}$. Recall Khinchin's inequality: if $1 < p < \infty$, then there exist two constants $C_1, C_2 > 0$ such that for any $y_1, \dots, y_n \in \mathbb{C}$ we have

$$C_1 \left(\sum_{j=1}^n |y_j|^2 \right)^{1/2} \leq \left(\mathbb{E} \left| \sum_{j=1}^n \epsilon_j y_j \right|^p \right)^{1/p} \leq C_2 \left(\sum_{j=1}^n |y_j|^2 \right)^{1/2}.$$

Consider then the operator

$$\mathcal{T}f = \sum_{j \in \mathbb{Z}} \epsilon_j P_j^\varphi f.$$

Then

$$\mathcal{D}_k(\mathcal{T}f) = \sum_{j \in \mathbb{Z}} \epsilon_j \varphi_j \mathcal{D}_k(f),$$

so the operator $\mathcal{T}$ is given by the multiplier

$$m(x) = \sum_{j \in \mathbb{Z}} \epsilon_j \varphi_j(x).$$

We wish to apply the Hörmander-Mikhlin multiplier theorem. Let r be the smallest integer greater than or equal to $\frac{N+1}{2} + \gamma$. We have

$$m^{(r)}(x) = \sum_{j \in \mathbb{Z}} 2^{-jr} \epsilon_j \varphi^{(r)}(2^{-j}x).$$

By considering the support of φ , at most two terms in this sum are non-zero, corresponding to the $j \in \mathbb{Z}$ such that

$$\log_2 x - 1 \leq j \leq \log_2 x + 1.$$

Thus

$$|m^{(r)}(x)| \leq C |2^{-jr}| \leq C x^{-r},$$

so

$$\int_R^{2R} |m^{(r)}(x)| dx \leq C R^{-r+1},$$

so m does indeed satisfy the requirements of the Hörmander-Mikhlin theorem. Therefore

$$\|\mathcal{T}f\|_p \leq C \|f\|_p. \quad (5.10)$$

On the other hand, by Khinchin's inequality we have

$$\mathbb{E}(|\mathcal{T}f(x)|^p) = \mathbb{E} \left(\left| \sum_{j \in \mathbb{Z}} \epsilon_j P_j^\varphi f(x) \right|^p \right) \geq C \left(\sum_{j \in \mathbb{Z}} |P_j^\varphi f(x)|^2 \right)^{p/2}.$$

But (5.10) implies that

$$\mathbb{E}(\|\mathcal{T}f\|_p^p) \leq C \|f\|_p^p.$$

The last two inequalities finally imply (5.9). $\square$

Proof of Proposition 5.2.2. We first prove the inequality on the right hand side. Since $\mathcal{D}_k((-\Delta_k)^{s/2}f)(\xi) = |\xi|^s \mathcal{D}_k(f)(\xi)$, we have

$$2^{js} P_j f(x) = \mathcal{D}_k^{-1}(2^{js} |\xi|^{-s} \psi_j \mathcal{D}_k((-\Delta_k)^{s/2} f)). \quad (5.11)$$

Let $\varphi(\xi) = \frac{\psi(\xi)}{|\xi|^s}$ and let P_j^φ be defined as in the previous Lemma. Also let $g = (-\Delta_k)^{s/2} f$. Then from (5.11) it follows that

$$\sum_{j \in \mathbb{Z}} |2^{js} P_j f|^2 = \sum_{j \in \mathbb{Z}} |P_j^\varphi g|^2,$$

so by Lemma 5.2.4 we have

$$\left\| \left(\sum_{j \in \mathbb{Z}} |2^{js} P_j f|^2 \right)^{1/2} \right\|_p = \left\| \left(\sum_{j \in \mathbb{Z}} |P_j^\varphi g|^2 \right)^{1/2} \right\|_p \leq C \|g\|_p = C \|(-\Delta_k)^{s/2} f\|_p. \quad (5.12)$$

We now turn to the left hand side inequality. Let $g \in \mathcal{S}(\mathbb{R}^N)$ be arbitrary. By considering the support of ψ we can see that

$$\int_{\mathbb{R}^N} P_j f \cdot P_i g \, d\mu_k = 0$$

for all $|j - i| > 2$. Thus, by (5.8), we have

$$\begin{aligned} \int_{\mathbb{R}^N} (-\Delta_k)^{s/2} f \bar{g} \, d\mu_k &= \sum_{i=-2}^2 \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^N} P_j ((-\Delta_k)^{s/2} f) P_{j+i} \bar{g} \, d\mu_k \\ &= \sum_{i=-2}^2 \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^N} \psi_j(\xi) \psi_{j+i}(\xi) |\xi|^s \mathcal{D}_k(f)(\xi) \mathcal{D}_k(\bar{g})(\xi) \, d\mu_k(\xi) \\ &= \sum_{i=-2}^2 2^{is} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^N} 2^{js} P_j(f) P_{j+i}^\varphi(\bar{g}) \, d\mu_k, \end{aligned}$$

where $\varphi(\xi) = |\xi|^s \psi(\xi)$. Here we used the Plancherel identity twice. Thus, by the Cauchy-

Schwarz inequality we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (-\Delta_k)^{s/2} f \bar{g} \, d\mu_k \right| &\leq \sum_{i=-2}^2 2^{is} \int_{\mathbb{R}^N} \left(\sum_{j \in \mathbb{Z}} |2^{js} P_j f|^2 \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} |P_{j+i}^\varphi \bar{g}|^2 \right)^{1/2} d\mu_k \\ &\leq 5 \cdot 4^s \int_{\mathbb{R}^N} \left(\sum_{j \in \mathbb{Z}} |2^{js} P_j f|^2 \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} |P_j^\varphi \bar{g}|^2 \right)^{1/2} d\mu_k. \end{aligned}$$

By Hölder's inequality and then Lemma 5.2.4, this implies that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (-\Delta_k)^{s/2} f \bar{g} \, d\mu_k \right| &\leq 5 \cdot 4^s \left\| \left(\sum_{j \in \mathbb{Z}} |2^{js} P_j f|^2 \right)^{1/2} \right\|_p \cdot \left\| \left(\sum_{j \in \mathbb{Z}} |P_j^\varphi \bar{g}|^2 \right)^{1/2} \right\|_{p'} \\ &\leq C \|g\|_{p'} \cdot \left\| \left(\sum_{j \in \mathbb{Z}} |2^{js} P_j f|^2 \right)^{1/2} \right\|_p. \end{aligned}$$

Finally, since $g \in \mathcal{S}(\mathbb{R}^N)$ was arbitrary, it follows that

$$\|(-\Delta_k)^{s/2} f\|_p \leq C \left\| \left(\sum_{j \in \mathbb{Z}} |2^{js} P_j f|^2 \right)^{1/2} \right\|_p.$$

This completes the proof of Proposition 5.2.2. □

We are now ready to prove the main result of this section.

Proof of Theorem 5.2.1. By Proposition 5.2.2, we have

$$\|(-\Delta_k)^{s(1-\theta)/2} f\|_p \leq C \left\| \left(\sum_{j \in \mathbb{Z}} |2^{js(1-\theta)} P_j f|^2 \right)^{1/2} \right\|_p.$$

But by Hölder's inequality we have

$$\sum_{j \in \mathbb{Z}} |2^{js(1-\theta)} P_j f|^2 \leq \left(\sum_{j \in \mathbb{Z}} |2^{js} P_j f|^2 \right)^{1-\theta} \cdot \left(\sum_{j \in \mathbb{Z}} |P_j f|^2 \right)^\theta.$$

Applying Hölder's inequality again, the last two results imply

$$\begin{aligned} \|(-\Delta_k)^{s(1-\theta)/s} f\|_p &\leq C \left\| \left(\sum_{j \in \mathbb{Z}} |2^{js} P_j f|^2 \right)^{1/2} \right\|_p^{1-\theta} \cdot \left\| \left(\sum_{j \in \mathbb{Z}} |P_j f|^2 \right)^{1/2} \right\|_p^\theta \\ &\leq C \|(-\Delta_k)^{s/2} f\|_p^{1-\theta} \|f\|_p^\theta, \end{aligned}$$

where in the last step we used Proposition 5.2.2. $\square$

5.3 THE NONLINEAR DAMPED WAVE EQUATION

5.3.1 THE LINEAR CASE

Let us first consider the following linear Cauchy problem on $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$

$$\begin{cases} \partial_{tt}^2 u(t, x) - \Delta_k u(t, x) + b \partial_t u(t, x) + m u(t, x) = 0 \\ u(0, x) = u_0(x) \\ \partial_t u(0, x) = u_1(x), \end{cases} \quad (5.13)$$

where $b, m > 0$. We take Dunkl transform with respect to the variable x in this equation to obtain

$$\begin{cases} \partial_{tt}^2 \mathcal{D}_k(u)(t, \xi) + |\xi|^2 \mathcal{D}_k(u)(t, \xi) + b \partial_t \mathcal{D}_k(u)(t, \xi) + m \mathcal{D}_k(u)(t, \xi) = 0 \\ \mathcal{D}_k(u)(0, \xi) = \mathcal{D}_k(u_0)(\xi) \\ \partial_t \mathcal{D}_k(u)(0, \xi) = \mathcal{D}_k(u_1)(\xi). \end{cases} \quad (5.14)$$

To simplify notation, let $U := \mathcal{D}_k(u)$, $U_0 := \mathcal{D}_k(u_0)$ and $U_1 := \mathcal{D}_k(u_1)$. In this notation, we have to solve the second order linear homogeneous ordinary differential equation in t

$$\partial_{tt}^2 U + b \partial_t U + (m + |\xi|^2) U = 0, \quad (5.15)$$

with initial conditions $U(0, \xi) = U_0(\xi)$ and $\partial_t U(0, \xi) = U_1(\xi)$. As is well known, the solution of this equation depends on the sign of

$$D := b^2 - 4(m + |\xi|^2)$$

and we treat the three different cases separately below.

Case 1: $D > 0$. The solution of (5.15) is then

$$U(t, \xi) = e^{-\frac{b}{2}t} \left[U_0(\xi) \left(\cosh \frac{t\sqrt{D}}{2} + \frac{b}{\sqrt{D}} \sinh \frac{t\sqrt{D}}{2} \right) + U_1(\xi) \frac{2}{\sqrt{D}} \sinh \frac{t\sqrt{D}}{2} \right]. \quad (5.16)$$

This implies that

$$|U(t, \xi)| \leq e^{-\frac{b}{2}t + \frac{\sqrt{D}}{2}t} \left[|U_0(\xi)| \left(1 + \frac{bt}{2} \right) + |U_1(\xi)| \frac{1}{\sqrt{D}} \right], \quad (5.17)$$

where we used the fact that $\frac{1-e^{-z}}{z} \leq 1$ for all $z \geq 0$.

For any $c > 0$ we have $\frac{bct}{2} \leq e^{bct/2}$, so

$$\frac{bt}{2} e^{-\frac{b}{2}t + \frac{\sqrt{D}}{2}t} \leq \frac{1}{c} e^{-\frac{b}{2}(1-c)t + \frac{\sqrt{D}}{2}t}.$$

We know that for any $\xi \in \mathbb{R}^N$ we have $D \leq b^2 - 4m$, so

$$-\frac{b}{2}(1-c) + \frac{\sqrt{D}}{2} \leq -\frac{b}{2}(1-c) + \frac{\sqrt{b^2 - 4m}}{2} < 0$$

if we choose $0 < c < 1 - \frac{\sqrt{b^2 - 4m}}{b}$. In this case, we obtain $\delta > 0$ independent of ξ or t such that

$$\frac{bt}{2} e^{-\frac{b}{2}t + \frac{\sqrt{D}}{2}t} \leq \frac{1}{c} e^{-\delta t}$$

and

$$e^{-\frac{b}{2}t + \frac{\sqrt{D}}{2}t} \leq e^{-\delta t}$$

for all t and ξ . Consequently, (5.17) becomes

$$|U(t, \xi)| \leq C e^{-\delta t} \left[|U_0(\xi)| + \frac{1}{\sqrt{D}} |U_1(\xi)| \right]. \quad (5.18)$$

Case 2: $D < 0$. The solution of equation (5.15) in this case is

$$U(t, \xi) = e^{-\frac{bt}{2}} \left[U_0(\xi) \left(\cos \frac{t\sqrt{|D|}}{2} + \frac{b}{\sqrt{|D|}} \sin \frac{t\sqrt{|D|}}{2} \right) + U_1(\xi) \frac{2}{\sqrt{|D|}} \sin \frac{t\sqrt{|D|}}{2} \right]. \quad (5.19)$$

We then have that

$$|U(t, \xi)| \leq e^{-\frac{bt}{2}} \left[|U_0(\xi)| \left(1 + \frac{bt}{2} \right) + \frac{2}{\sqrt{|D|}} |U_1(\xi)| \right],$$

and using, for example, the inequality $\frac{bt}{2} e^{-\frac{bt}{2}} \leq e^{-\frac{bt}{4}}$, we obtain the same estimate (5.18)

as above for a constant $\delta > 0$ independent of t or ξ .

Case 3: $D = 0$. The solution of (5.15) in this case is

$$U(t, \xi) = e^{-\frac{bt}{2}} \left[U_0(\xi) \left(1 + \frac{bt}{2} \right) + tU_1(\xi) \right]. \quad (5.20)$$

A similar reasoning to above shows that in this case there exists a positive constant δ such that we have the estimate

$$|U(t, \xi)| \leq Ce^{-\delta t} [|U_0(\xi)| + |U_1(\xi)|]. \quad (5.21)$$

Proposition 5.3.1. *Fix $s \in \mathbb{R}$. Let u be a solution of the Cauchy problem (5.13) with initial data $u_0 \in H_{\mathcal{D}}^s$ and $u_1 \in H_{\mathcal{D}}^{s-1}$. Then there exists a constant $\delta > 0$ (which does not depend on the initial data) such that, for all $t > 0$ we have the estimates*

$$\|u(t, \cdot)\|_{H_{\mathcal{D}}^s}^2 \leq Ce^{-2\delta t} \left(\|u_0\|_{H_{\mathcal{D}}^s}^2 + \|u_1\|_{H_{\mathcal{D}}^{s-1}}^2 \right), \quad (5.22)$$

Moreover, if $l \in \mathbb{N}_0$ and $u_0 \in H_{\mathcal{D}}^{s+l}$ and $u_1 \in H_{\mathcal{D}}^{s+l-1}$, then we have the estimate

$$\|\partial_t^l u(t, \cdot)\|_{H_{\mathcal{D}}^s}^2 \leq Ce^{-2\delta t} \left(\|u_0\|_{H_{\mathcal{D}}^{s+l}}^2 + \|u_1\|_{H_{\mathcal{D}}^{s+l-1}}^2 \right). \quad (5.23)$$

Proof. We first consider the case when $4m > b^2$. Then $D < 0$ for all $\xi \in \mathbb{R}^N$, so we are always in the Case 2 above. We have, keeping the same notation as above and using estimate (5.18), that

$$\begin{aligned} \|u(t, \cdot)\|_{H_{\mathcal{D}}^s}^2 &= \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |U(t, \xi)|^2 d\xi \\ &\leq Ce^{-2\delta t} \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |U_0(t, \xi)|^2 d\xi + Ce^{-2\delta t} \int_{\mathbb{R}^N} \frac{(1 + |\xi|^2)^s}{4m - b^2 + 4|\xi|^2} |U_1(t, \xi)|^2 d\xi. \end{aligned}$$

Due to the condition $4m > b^2$, the function $\xi \mapsto \frac{1+|\xi|^2}{4m-b^2+4|\xi|^2}$ is bounded on $\mathbb{R}^N$, so the above estimates implies

$$\begin{aligned} \|u(t, \cdot)\|_{H_{\mathcal{D}}^s}^2 &\leq Ce^{-2\delta t} \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |U_0(t, \xi)|^2 d\xi + Ce^{-2\delta t} \int_{\mathbb{R}^N} (1 + |\xi|^2)^{s-1} |U_1(t, \xi)|^2 d\xi \\ &= Ce^{-2\delta t} \left(\|u_0\|_{H_{\mathcal{D}}^s}^2 + \|u_1\|_{H_{\mathcal{D}}^{s-1}}^2 \right), \end{aligned}$$

as required.

The remaining case $4m \leq b^2$ is a little more involved. The issue in the above computations is that now we have a singularity at $|\xi| = \frac{\sqrt{b^2-4m}}{2}$. We will now split our integral over $B := \{\xi \in \mathbb{R}^N : |D| < 1\}$ and its complement, $B^C = \mathbb{R}^N \setminus B$; more precisely, we use the

following

$$\|u(t, \cdot)\|_{H_{\mathcal{D}}^s}^2 = \int_B (1 + |\xi|^2)^s |U(t, \xi)|^2 d\xi + \int_{B^C} (1 + |\xi|^2)^s |U(t, \xi)|^2 d\xi. \quad (5.24)$$

On B^C we are always in Cases 1 and 2, and in both cases we have the same estimate (5.18). Using the same reasoning as above, because on B^C the function $\xi \mapsto \frac{1+|\xi|^2}{|b^2-4m-4|\xi|^2|}$ is bounded, we deduce that

$$\begin{aligned} & \int_{B^C} (1 + |\xi|^2)^s |U(t, \xi)|^2 d\xi \\ & \leq C e^{-2\delta t} \left[\int_{B^C} (1 + |\xi|^2)^s |U_0(\xi)|^2 d\xi + \int_{B^C} (1 + |\xi|^2)^{s-1} |U_1(\xi)|^2 d\xi \right]. \end{aligned} \quad (5.25)$$

We now turn to the integral over B . Starting from the explicit expressions we obtained in (5.16) and (5.19), using the fact that $|\frac{\sinh z}{z}| \leq 1$ and $|\frac{\sin z}{z}| \leq 1$ for all $z \neq 0$, we find the following estimate which holds in all cases

$$|U(t, \xi)| \leq e^{-\tilde{\delta}t} \left[|U_0(\xi)| \left(1 + \frac{bt}{2}\right) + t |U_1(\xi)| \right],$$

for some $\tilde{\delta} > 0$. As in Case 3, this implies that the estimate (5.21) holds in general. Thus, we have the inequality

$$\int_B (1 + |\xi|^2)^s |U(t, \xi)|^2 d\xi \leq C e^{-2\delta t} \left[\int_B (1 + |\xi|^2)^s |U_0(\xi)|^2 d\xi + \int_B (1 + |\xi|^2)^s |U_1(\xi)|^2 d\xi \right].$$

But on B we have $|D| < 1$, so

$$4|\xi|^2 \leq |4|\xi|^2 + 4m - b^2| + |b^2 - 4m| < |b^2 - 4m| + 1.$$

In other words, on B the quantity $1 + |\xi|^2$ is bounded above by a constant independent of t , so the estimate above becomes

$$\begin{aligned} & \int_B (1 + |\xi|^2)^s |U(t, \xi)|^2 d\xi \\ & \leq C e^{-2\delta t} \left[\int_B (1 + |\xi|^2)^s |U_0(\xi)|^2 d\xi + \int_B (1 + |\xi|^2)^{s-1} |U_1(\xi)|^2 d\xi \right]. \end{aligned} \quad (5.26)$$

Putting together (5.24), (5.25) and (5.26), this completes the proof of estimate (5.22).

For the second estimate (5.23), we differentiate the explicit formulas (5.16), (5.19) and (5.20) with respect to time, and then perform similar estimates as above. The proof of the Proposition is thus complete. $\square$

5.3.2 THE MAIN RESULT

In this section we consider the nonlinear wave equation

$$\begin{cases} \partial_{tt}^2 u(t, x) - \Delta_k u(t, x) + b \partial_t u(t, x) + m u(t, x) = f(u)(t, x) \\ u(0, x) = \epsilon u_1(x) \\ \partial_t u(0, x) = \epsilon u_2(x), \end{cases} \quad (5.27)$$

where $b, m, \epsilon > 0$, and the nonlinearity is a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that will satisfy the assumptions

$$f(0) = 0 \quad (5.28)$$

and

$$|f(a) - f(b)| \leq C(|a|^{p-1} + |b|^{p-1})|a - b|, \quad (5.29)$$

for some $p > 1$.

Our main result is the following global existence result.

Theorem 5.3.2. *Suppose that $N + 2\gamma > 2$ and let $1 \leq p \leq \frac{N+2\gamma}{N+2\gamma-2}$. Let $u_0 \in H_{\mathcal{D}}^1(\mathbb{R}^N)$ and $u_1 \in L^2(\mu_k)$. Assume that f is a function that satisfies the assumption (5.28-5.29). Then there exists a constant $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ the Cauchy problem (5.27) has a unique global solution*

$$u \in C(\mathbb{R}_+; H_{\mathcal{D}}^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}_+; L^2(\mu_k)).$$

Moreover, there exists a constant $\delta_0 > 0$ such that the solution satisfies the estimate

$$\|u(t, \cdot)\|_{H_{\mathcal{D}}^1} + \|\partial_t u(t, \cdot)\|_2 \leq C e^{-\delta_0 t}$$

for all $t > 0$.

Proof. Part 1: Existence. We will prove existence of the solution using the contraction theorem. Consider the Banach space

$$X := C(\mathbb{R}_+; H_{\mathcal{D}}^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}_+; L^2(\mu_k)),$$

with the norm

$$\|u\|_X := \sup_{t \geq 0} \left[(1+t)^{-1/2} e^{\delta t} \left(\|u(t, \cdot)\|_{H_{\mathcal{D}}^1} + \|\partial_t u(t, \cdot)\|_2 \right) \right],$$

where $\delta > 0$ is that obtained in Proposition 5.3.1. For $M > 0$ which will be conveniently

chosen later, consider

$$Y := \{u \in X : \|u\|_X \leq M\},$$

which is a closed subset of X and thus a complete metric space in the metric induced from X .

Define the map

$$S(u)(t, x) := \varphi(t, x) + \int_0^t T(f(u(s, \cdot)))(t - s, x) \, ds,$$

where φ is the solution of the linear problem (the Cauchy problem (5.27) with $f \equiv 0$), and $T(w)$ is defined generally, for $w : \mathbb{R}^N \rightarrow \mathbb{R}$, as the solution of the linear Cauchy problem

$$\begin{cases} \partial_{tt}^2 u - \Delta_k u + b \partial_t u + m u = 0 \\ u(0, x) = 0 \\ \partial_t u(0, x) = w(x). \end{cases}$$

The strategy of our existence proof is to show that S is a contraction on Y , i.e., to show that for all $u, v \in Y$ we have

- (i) $\|S(u)\|_X \leq M$ (in other words, S maps Y into itself), and
- (ii) $\|S(u) - S(v)\|_X \leq c \|u - v\|_X$ for some constant $0 < c < 1$.

Assuming these hold, then the Banach fixed point theorem guarantees the existence of a fixed point of S in Y , which in turn is a solution to our Cauchy problem.

We will now focus on proving point (ii), and (i) will follow as a special case. To simplify the computations below, we introduce the notation

$$\tilde{T}_u(t, x) := \int_0^t T(f(u(s, \cdot)))(t - s, x) \, ds.$$

We have, for any $t > 0$,

$$\begin{aligned} & \| (S(u) - S(v))(t, \cdot) \|_{H_D^1} + \| \partial_t (S(u) - S(v))(t, \cdot) \|_2 \\ &= \| (\tilde{T}_u - \tilde{T}_v)(t, \cdot) \|_{H_D^1} + \| \partial_t (\tilde{T}_u - \tilde{T}_v)(t, \cdot) \|_2. \end{aligned} \tag{5.30}$$

Noting that $T(f(u(s, \cdot))) - T(f(v(s, \cdot))) = T(f(u(s, \cdot)) - f(v(s, \cdot)))$, we first estimate

$$\begin{aligned}
\|(\tilde{T}_u - \tilde{T}_v)(t, \cdot)\|_{H_{\mathcal{D}}^1}^2 &= \|\nabla_k(\tilde{T}_u - \tilde{T}_v)(t, \cdot)\|_2^2 + \|(\tilde{T}_u - \tilde{T}_v)(t, \cdot)\|_2^2 \\
&= \int_{\mathbb{R}^N} \left| \int_0^t \nabla_k \left(T(f(u(s, \cdot)) - f(v(s, \cdot))) \right)(t-s, x) ds \right|^2 d\mu_k(x) \\
&\quad + \int_{\mathbb{R}^N} \left| \int_0^t T(f(u(s, \cdot)) - f(v(s, \cdot)))(t-s, x) ds \right|^2 d\mu_k(x) \\
&\leq t \int_{\mathbb{R}^N} \int_0^t \left| \nabla_k \left(T(f(u(s, \cdot)) - f(v(s, \cdot))) \right)(t-s, x) \right|^2 ds d\mu_k(x) \\
&\quad + t \int_{\mathbb{R}^N} \int_0^t |T(f(u(s, \cdot)) - f(v(s, \cdot)))(t-s, x)|^2 ds d\mu_k(x) \\
&= t \int_0^t \|T(f(u(s, \cdot)) - f(v(s, \cdot)))(t-s, \cdot)\|_{H_{\mathcal{D}}^1}^2 ds.
\end{aligned}$$

Here in the last line we simply interchanged the order of integration. Using the estimates from Proposition 5.3.1, we obtain

$$\|(\tilde{T}_u - \tilde{T}_v)(t, \cdot)\|_{H_{\mathcal{D}}^1}^2 \leq Cte^{-2\delta t} \int_0^t e^{2\delta s} \|f(u(s, \cdot)) - f(v(s, \cdot))\|_2^2 ds. \quad (5.31)$$

Using Leibniz's formula for differentiation of an integral we have, since $T(w)(0, x) = 0$ for any function w , that

$$\frac{\partial}{\partial t} \left(\int_0^t T(f(w))(t-s, x) ds \right) = \int_0^t \partial_t(T(f(w)))(t-s, x) ds.$$

Using this, we estimate

$$\begin{aligned}
\|\partial_t(\tilde{T}_u - \tilde{T}_v)(t, \cdot)\|_2^2 &= \int_{\mathbb{R}^N} \left| \int_0^t \partial_t(T(f(u(s, \cdot)) - f(v(s, \cdot)))(t-s, x) ds \right|^2 d\mu_k(x) \\
&\leq t \int_0^t \|\partial_t(T(f(u(s, \cdot)) - f(v(s, \cdot)))(t-s, \cdot)\|_2^2 ds \\
&\leq Cte^{-2\delta t} \int_0^t e^{2\delta s} \|f(u(s, \cdot)) - f(v(s, \cdot))\|_2^2 ds.
\end{aligned} \quad (5.32)$$

We are left to estimate

$$\|f(u(s, \cdot)) - f(v(s, \cdot))\|_2.$$

To this end, we use the assumption (5.29) and Hölder's inequality with exponents $\frac{p}{p-1}$

and p to obtain

$$\begin{aligned}
& \int_{\mathbb{R}^N} |f(u(s, x)) - f(v(s, x))|^2 d\mu_k(x) \\
& \leq C \int_{\mathbb{R}^N} [|u(s, x)|^{2(p-1)} + |v(s, x)|^{2(p-1)}] |u(s, x) - v(s, x)|^2 d\mu_k(x) \\
& \leq C \left[\|u(s, \cdot)\|_{2p}^{2(p-1)} + \|v(s, \cdot)\|_{2p}^{2(p-1)} \right] \|u(s, \cdot) - v(s, \cdot)\|_{2p}^2.
\end{aligned}$$

It is at this point that we use the Gagliardo-Nirenberg inequality in the form of Corollary 5.1.3. It is this result that imposes the restriction $1 \leq p \leq \frac{N+2\gamma}{N+2\gamma-2}$. Thus, using this result, we have

$$\begin{aligned}
& \int_{\mathbb{R}^N} |f(u(s, x)) - f(v(s, x))|^2 d\mu_k(x) \\
& \leq C \|u(s, \cdot) - v(s, \cdot)\|_{H_D^1}^2 \left[\|u(s, \cdot)\|_{H_D^1}^{2(p-1)} + \|v(s, \cdot)\|_{H_D^1}^{2(p-1)} \right].
\end{aligned} \tag{5.33}$$

For the next step, we note that for any $w \in X$ we have

$$\|w(s, \cdot)\|_{H_D^1} \leq (1+s)^{1/2} e^{-\delta s} \|w\|_X.$$

In light of this, and recalling that $\|u\|_X \leq M$ and $\|v\|_X \leq M$, (5.33) yields

$$\|f(u(s, \cdot)) - f(v(s, \cdot))\|_2^2 \leq C(1+s)^p e^{-2p\delta s} M^{2p-2} \|u - v\|_X^2.$$

Replacing this in (5.31) and (5.32), we have obtained that

$$\begin{aligned}
& \left\| (\tilde{T}_u - \tilde{T}_v)(t, \cdot) \right\|_{H_D^1} + \left\| \partial_t (\tilde{T}_u - \tilde{T}_v)(t, \cdot) \right\|_2 \\
& \leq C \sqrt{t} e^{-\delta t} M^{p-1} \|u - v\|_X \left(\int_0^t (1+s)^p e^{-2(p-1)\delta s} ds \right)^{1/2}.
\end{aligned}$$

The integral on the right hand side of this inequality can be bounded by a constant independently of t :

$$\begin{aligned}
\int_0^t (1+s)^p e^{-2(p-1)\delta s} ds & \leq \int_0^\infty (1+s)^p e^{-2(p-1)\delta s} ds \\
& = \frac{e^{2(p-1)\delta}}{(2(p-1)\delta)^{p+1}} \int_{2(p-1)\delta}^\infty e^{-s} s^p ds < \infty.
\end{aligned}$$

Consequently, going back to (5.30), we now have

$$\begin{aligned} (1+t)^{-1/2} e^{\delta t} \left(\|S(u) - S(v)(t, \cdot)\|_{H_{\mathcal{D}}^1} + \|\partial_t(S(u) - S(v))(t, \cdot)\|_2 \right) \\ \leq C_1 \left(\frac{t}{1+t} \right)^{1/2} M^{p-1} \|u - v\|_X, \end{aligned}$$

for a constant $C_1 > 0$. Taking supremum over all $t > 0$, we finally obtain

$$\|S(u) - S(v)\|_X \leq C_1 M^{p-1} \|u - v\|_X. \quad (5.34)$$

We now want to bound $\|S(u)\|_X$. On the one hand, putting $v = 0$ in (5.34), since $f(0) = 0$ and so $S(0) = \varphi$, we obtain

$$\|S(u) - \varphi\|_X \leq C_1 M^p.$$

On the other hand, from the our linear case estimates in Proposition 5.3.1, we obtain

$$\|\varphi\|_X \leq C_2 \epsilon (\|u_0\|_{H_{\mathcal{D}}^1} + \|u_1\|_2),$$

for a constant $C_2 > 0$, and thus

$$\|S(u)\|_X \leq \|\varphi\|_X + \|S(u) - \varphi\|_X \leq C_1 M^p + C_2 \epsilon (\|u_0\|_{H_{\mathcal{D}}^1} + \|u_1\|_2). \quad (5.35)$$

Fixing $c \in (0, 1)$, let $M = \left(\frac{c}{C_1}\right)^{\frac{1}{p-1}}$, so $C_1 M^{p-1} = c$, and $\epsilon_0 = \frac{M(1-c)}{C_2(\|u_0\|_{H_{\mathcal{D}}^1} + \|u_1\|_2)}$. Then, for $0 < \epsilon < \epsilon_0$, from (5.34) and (5.35) we have

$$\|S(u) - S(v)\|_X \leq c \|u - v\|_X$$

and

$$\|S(u)\|_X \leq M,$$

i.e., points (i) and (ii) above. The proof of existence is thus complete.

Part 2: Uniqueness. Suppose that there exist $u, v \in X$ that both solve the Cauchy problem (5.27). Let $\psi := u - v$ and fix $t^* > 0$. We will show that $\psi = 0$ in $[0, t^*] \times \mathbb{R}^N$ and since t^* was arbitrary, this implies that $\psi \equiv 0$ and so $u = v$.

Both u and v are fixed points of the map S , and using the computations from the previous

part we obtain that

$$\begin{aligned}
\|\psi(t, \cdot)\|_{H_D^1}^2 + \|\partial_t \psi(t, \cdot)\|_2^2 &= \left\| \tilde{T}_u(t, \cdot) - \tilde{T}_v(t, \cdot) \right\|_{H_D^1}^2 + \left\| \partial_t(\tilde{T}_u - \tilde{T}_v)(t, \cdot) \right\|_2^2 \\
&\leq Cte^{-2\delta t} \int_0^t e^{2\delta s} \|f(u(s, \cdot)) - f(v(s, \cdot))\|_2^2 ds \\
&\leq Cte^{-2\delta t} \int_0^t \left[\|u(s, \cdot)\|_{H_D^1}^{2(p-1)} + \|v(s, \cdot)\|_{H_D^1}^{2(p-1)} \right] \|\psi(s, \cdot)\|_{H_D^1}^2 ds.
\end{aligned}$$

From the definition of the Banach space X , the map $t \mapsto \|u(t, \cdot)\|_{H_D^1}^{2(p-1)} + \|v(t, \cdot)\|_{H_D^1}^{2(p-1)}$ is continuous, and so it is bounded on the compact set $[0, t^*]$. Therefore, the inequality above becomes

$$\|\psi(t, \cdot)\|_{H_D^1}^2 + \|\partial_t \psi(t, \cdot)\|_2^2 \leq C \int_0^t [\|\psi(s, \cdot)\|_{H_D^1}^2 + \|\partial_t \psi(s, \cdot)\|_2^2] ds,$$

and so, by Gronwall's lemma, we finally obtain

$$\|\psi(t, \cdot)\|_{H_D^1}^2 + \|\partial_t \psi(t, \cdot)\|_2^2 = 0$$

for all $0 \leq t \leq t^*$. This proves uniqueness.

Part 3: Estimates. In the first two parts we have proved that our Cauchy problem has a unique solution, say $u \in Y$. By the definition of the X -norm, we have

$$\|u(t, \cdot)\|_{H_D^1} + \|\partial_t u(t, \cdot)\|_2 \leq (1+t)^{1/2} e^{-\delta t} M \leq Ce^{-\frac{\delta}{2}t},$$

for any $t > 0$. □

6

LOG-SOBOLEV INEQUALITIES

The logarithmic-Sobolev inequality (or log-Sobolev inequality, in short), on a general measure space $(\Omega, \mathcal{F}, \mu)$ with a quadratic form Q defined on a suitable space of functions on Ω , states that

$$\int_{\Omega} f^2 \log \frac{f^2}{\int_{\Omega} f^2 d\mu} d\mu \leq CQ(f) + D \int_{\Omega} f^2 d\mu, \quad (6.1)$$

for some constants C and D . If $D = 0$, we say that (6.1) is a tight log-Sobolev inequality. Although this inequality was used before, it was first explicitly recognised in Gross's seminal paper [51]. His main result was the equivalence of log-Sobolev inequalities to hypercontractivity. More precisely, if e^{-tH} is a Markov semigroup on $L^2(\mu)$, then the log-Sobolev inequality (6.1) is equivalent to the property that the semigroup e^{-tH} is hypercontractive, i.e.,

$$\|e^{-tH} f\|_{L^{q(t)}(\mu)} \leq e^{M(t)} \|f\|_{L^p(\mu)},$$

where $q(t) = 1 + (p-1)e^{\frac{2t}{C}}$ and $M(t) = D \left(\frac{1}{p} - \frac{1}{q(t)} \right)$.

The log-Sobolev inequality in a rather general setting was the crucial tool used by Davies in [25] to obtain bounds on the heat kernel associated to a positive semigroup. More precisely, he showed that the log-Sobolev inequality (6.1) with sufficiently decaying constants is equivalent to the ultracontractivity property. This equivalence then allows one to deduce lower and upper bounds for the heat kernel. Recall that a semigroup $(P_t)_{t>0}$ on $L^2(\mu)$ is called ultracontractive if for each $t > 0$ the operator P_t is bounded from $L^2(\mu)$ to $L^\infty(\mu)$.

Among the many applications of the log-Sobolev inequality we mention its use in statistical mechanics by providing bounds on the entropy. Indeed, the relative entropy functional

is defined as

$$\text{Ent}(f) := \int_{\Omega} f \log f \, d\mu - \int_{\Omega} f \, d\mu \log \int_{\Omega} f \, d\mu,$$

for $f \geq 0$, so the log-Sobolev inequality can be rephrased as

$$\text{Ent}(f^2) \leq CQ(f) + D \|f\|_{L^2(\mu)}^2.$$

An important feature of the log-Sobolev inequality is the product property: if (6.1) holds on two measure spaces $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$, then a form of log-Sobolev also holds on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \times \mu_2)$. For more information about this and the use in statistical mechanics see the lecture notes [52].

Another important property of the log-Sobolev inequality which we exploit below is its connection with the Poincaré inequality. In general, the tight log-Sobolev inequality implies the Poincaré inequality. On the other hand, a non-tight log-Sobolev inequality, in the presence of a Poincaré inequality, can be improved to obtain a tight log-Sobolev inequality. See [9] for more information on this, and also for an excellent account of the log-Sobolev inequality.

In this chapter we start out by proving the log-Sobolev inequality for the Dunkl measure μ_k with Dirichlet form $\int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_k$ in section 6.1. We then study the same problem for weighted Dunkl measure $e^{-U} \, d\mu_k$ for suitable G -invariant functions U (usually of the form $|x|^p$). These inequalities are of interest, for example, because of their association with semigroups generated by operators of the form $\Delta_k + b \cdot \nabla_k$ (see Example 7.1.9). The main method for studying these weighted inequalities is provided by [54] where powerful machinery based on U -bounds was introduced. Thus we discuss U -bounds in section 6.2, which we then apply in section 6.3 to obtain the desired weighted log-Sobolev inequality. Finally, in section 6.4 we prove Poincaré inequalities for the Dunkl measure μ_k which holds for small γ and then use this result, in section 6.5, to obtain tight log-Sobolev inequalities.

6.1 THE CLASSICAL LOG-SOBOLEV INEQUALITY

To begin with, we have the following simple log-Sobolev inequality.

Theorem 6.1.1. *There exists a constant $c \in \mathbb{R}$ such that for any $\epsilon > 0$ and for any $f \in C_c^\infty(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int_{\mathbb{R}^N} f^2 \, d\mu_k} \, d\mu_k \leq \epsilon \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_k + C(\epsilon) \int_{\mathbb{R}^N} f^2 \, d\mu_k,$$

where $C(\epsilon) = \frac{N+2\gamma}{2}(\log \frac{1}{\epsilon} - c)$.

Proof. Fix $f \in C_c^\infty(\mathbb{R}^N)$, $f \neq 0$. Then $\frac{f^2}{\int_{\mathbb{R}^N} f^2 d\mu_k} d\mu_k$ is a probability measure, and so by Jensen's inequality we have, for any $\delta > 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int_{\mathbb{R}^N} f^2 d\mu_k} d\mu_k &= \frac{1}{\delta} \int_{\mathbb{R}^N} f^2 d\mu_k \cdot \int_{\mathbb{R}^N} \frac{f^2}{\int_{\mathbb{R}^N} f^2 d\mu_k} \log \left(\frac{f^2}{\int_{\mathbb{R}^N} f^2 d\mu_k} \right)^\delta d\mu_k \\ &\leq \frac{1}{\delta} \int_{\mathbb{R}^N} f^2 d\mu_k \cdot \log \int_{\mathbb{R}^N} \left(\frac{f^2}{\int_{\mathbb{R}^N} f^2 d\mu_k} \right)^{1+\delta} d\mu_k \\ &= \frac{\delta+1}{\delta} \|f\|_2^2 \log \frac{\|f\|_{2+2\delta}^2}{\|f\|_2^2}. \end{aligned}$$

We then use the elementary inequality

$$\log x \leq \epsilon x + \log \frac{1}{\epsilon} - 1,$$

which holds for all $x, \epsilon > 0$. Thus

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int_{\mathbb{R}^N} f^2 d\mu_k} d\mu_k \leq \frac{\delta+1}{\delta} \left[\epsilon \|f\|_{2+2\delta}^2 + (\log \frac{1}{\epsilon} - 1) \|f\|_2^2 \right].$$

Finally, by choosing $\delta > 0$ such that $2 + 2\delta = \frac{2(N+2\gamma)}{N+2\gamma-2}$, we can apply Sobolev's inequality to deduce

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int_{\mathbb{R}^N} f^2 d\mu_k} d\mu_k \leq \frac{\delta+1}{\delta} C_{DS}^2 \epsilon \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_k + \frac{\delta+1}{\delta} (\log \frac{1}{\epsilon} - 1) \int_{\mathbb{R}^N} f^2 d\mu_k,$$

where C_{DS} is the Sobolev constant. A simple relabelling of the constants finishes the proof. $\square$

Using the general results of [25], from this inequality we can deduce a more general L^p result, as well as the ultracontractivity property.

Corollary 6.1.2. *Let $2 < p < \infty$. Then, for any $\epsilon > 0$ and for any $f \in C_c^\infty(\mathbb{R}^N)$ such that $f \geq 0$, we have*

$$\int_{\mathbb{R}^N} f^p \log \frac{f^p}{\int_{\mathbb{R}^N} f^p d\mu_k} d\mu_k \leq \epsilon \int_{\mathbb{R}^N} \nabla_k f \cdot \nabla_k (f^{p-1}) d\mu_k + C \left(\frac{2\epsilon}{p} \right) \int_{\mathbb{R}^N} f^p d\mu_k,$$

where $C(\epsilon)$ is as in the previous Theorem.

Proof. This follows from the previous Theorem and [25, Lemma 2.2.6]. $\square$

Finally, we recover the ultracontractivity property for the Dunkl heat semigroup. This was already established above in Proposition 2.3.5 using properties of the heat kernel; using the new log-Sobolev approach, no a priori bounds on the heat kernel are necessary.

Corollary 6.1.3. *The Dunkl heat semigroup $(H_t)_{t \geq 0}$ on $L^2(\mu_k)$ with generator Δ_k is ultracontractive. More precisely, for all $t > 0$ we have*

$$\|H_t f\|_\infty \leq C t^{-\frac{N+2\gamma}{4}} \|f\|_2,$$

for a constant $C > 0$.

Proof. This follows from the previous Corollary and [25, Theorem 2.2.7]. $\square$

More generally, we want to prove similar results for probability measures

$$d\mu_U := \frac{1}{Z} e^{-U} d\mu_k,$$

where $U \geq 0$ is a G -invariant function and $Z = \int_{\mathbb{R}^N} e^{-U} d\mu_k$. To illustrate the method and to motivate the study of U -bounds in the next section, we first consider Gaussian weight in the following Theorem. This result will be further refined and generalised in the next sections, but the main lines of the proof will remain the same.

Theorem 6.1.4. *Let $U = |x|^2$ and consider the probability measure $\mu_U := \frac{1}{Z} e^{-U} \mu_k$, where $Z = \int_{\mathbb{R}^N} e^{-U} d\mu_k$. Then, there exist constants $C_1, C_2 > 0$ such that the following inequality holds for all $f \in C_c^\infty(\mathbb{R}^N)$:*

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 d\mu_U} d\mu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + C_2 \int_{\mathbb{R}^N} f^2 d\mu_U.$$

Proof. Plugging $\frac{1}{\sqrt{Z}} f e^{-U/2}$ in Theorem 6.1.1, we have

$$\begin{aligned} \int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 d\mu_U} d\mu_U &\leq \frac{1}{Z} \epsilon \int_{\mathbb{R}^N} |\nabla_k (f e^{-U/2})|^2 d\mu_k \\ &\quad + (C(\epsilon) + \log Z) \int_{\mathbb{R}^N} f^2 d\mu_U + \int_{\mathbb{R}^N} f^2 U d\mu_U. \end{aligned}$$

Since U is G -invariant, we can use the Leibniz rule to compute

$$\begin{aligned} \frac{1}{Z} \int_{\mathbb{R}^N} |\nabla_k(fe^{-U/2})|^2 d\mu_k &= \int_{\mathbb{R}^N} |\nabla_k f - \frac{1}{2}f\nabla U|^2 d\mu_U \\ &\leq 2 \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + \frac{1}{2} \int_{\mathbb{R}^N} f^2 |\nabla U|^2 d\mu_U \\ &= 2 \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + 2 \int_{\mathbb{R}^N} f^2 U d\mu_U, \end{aligned}$$

where, in the last line, we used the fact that $\nabla U = 2x$, so $|\nabla U|^2 = 4U$. Replacing this inequality in the above, we have

$$\begin{aligned} \int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 d\mu_U} d\mu_U &\leq 2\epsilon \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U \\ &\quad + (C(\epsilon) + \log Z) \int_{\mathbb{R}^N} f^2 + (1 + 2\epsilon) \int_{\mathbb{R}^N} f^2 U d\mu_U. \end{aligned} \quad (6.2)$$

We now use the identity

$$\nabla_k(fe^{-U/2}) = e^{-U/2}\nabla_k f - \frac{1}{2}fe^{-U/2}\nabla U \quad (6.3)$$

to deduce

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U &= \frac{1}{Z} \int_{\mathbb{R}^N} |\nabla_k(fe^{-U/2})|^2 d\mu_k + \frac{1}{4} \int_{\mathbb{R}^N} f^2 |\nabla U|^2 d\mu_U \\ &\quad + \frac{1}{Z} \int_{\mathbb{R}^N} f\nabla U \cdot \nabla_k(fe^{-U/2})e^{-U/2} d\mu_k. \end{aligned} \quad (6.4)$$

Keeping in mind that $\nabla U = 2x$, this equality implies that

$$\int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U \geq \int_{\mathbb{R}^N} f^2 U d\mu_U + A, \quad (6.5)$$

where

$$A := \frac{1}{Z} \int_{\mathbb{R}^N} f\nabla U \cdot \nabla_k(fe^{-U/2})e^{-U/2} d\mu_k$$

We now compute the quantity A . Firstly, by integration by parts we have

$$A = - \sum_{i=1}^N \frac{1}{Z} \int_{\mathbb{R}^N} fe^{-U/2} T_i(2x_i fe^{-U/2}) d\mu_k(x).$$

Using Lemma 1.2.5, we have

$$\begin{aligned}
T_i(x_i f e^{-U/2}) &= x_i T_i(f e^{-U/2}) + f e^{-U/2} T_i(x_i) - \sum_{\alpha \in R_+} k_\alpha \alpha_i e^{-U(x)/2} \frac{(f(x) - f(\sigma_\alpha x))(x_i - (\sigma_\alpha x)_i)}{\langle \alpha, x \rangle} \\
&= x_i T_i(f e^{-U/2}) + f e^{-U/2} (1 + \sum_{\alpha \in R_+} k_\alpha \alpha_i^2) - \sum_{\alpha \in R_+} k_\alpha \alpha_i^2 e^{-U(x)/2} (f(x) - f(\sigma_\alpha x))
\end{aligned} \tag{6.6}$$

Thus

$$A = -A - 2(N + 2\gamma) \int_{\mathbb{R}^N} f^2 d\mu_U + 4 \sum_{\alpha \in R_+} k_\alpha \int_{\mathbb{R}^N} f(x)(f(x) - f(\sigma_\alpha x)) d\mu_U(x),$$

and so

$$A = -N \int_{\mathbb{R}^N} f^2 d\mu_U - 2 \sum_{\alpha \in R_+} k_\alpha \int_{\mathbb{R}^N} f(x)f(\sigma_\alpha x) d\mu_U(x).$$

Using the elementary inequality $2XY \leq X^2 + Y^2$, we obtain

$$A \geq -(N + 2\gamma) \int_{\mathbb{R}^N} f^2 d\mu_U.$$

Replacing this in equation (6.5), we obtain

$$\int_{\mathbb{R}^N} f^2 U d\mu_U \leq \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + (N + 2\gamma) \int_{\mathbb{R}^N} f^2 d\mu_U. \tag{6.7}$$

Finally, using this in (6.2), we have

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 d\mu_U} d\mu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + C_2 \int_{\mathbb{R}^N} f^2 d\mu_U,$$

for some constants $C_1, C_2 > 0$, as required. $\square$

6.2 U-BOUNDS

Looking back at the proof of the weighted log-Sobolev inequality in Theorem 6.1.4, we can see that inequality (6.7) was the key element. Inequalities of this form are called U-bounds (cf. [54]). In this section we will prove more general U-bounds by adapting our proof slightly, and these will later be used to deduce log-Sobolev inequalities.

Proposition 6.2.1. *Let $p > 1$ and consider the function $U(x) = |x|^p$ and the probability*

measure $\mu_U := \frac{1}{Z} e^{-U} \mu_k$, where $Z = \int_{\mathbb{R}^N} e^{-U} d\mu_k$. We then have

$$\int_{\mathbb{R}^N} |f| \cdot |x|^{p-1} d\mu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f| d\mu_U + D \int_{\mathbb{R}^N} |f| d\mu_U,$$

for some constants $C, D > 0$.

Proof. In order to avoid a singularity that will arise at the origin, we first consider a function f that vanishes on the unit ball. As before, we start with identity (6.3). Noting that in this case we have

$$\nabla U(x) = p|x|^{p-1} \nabla(|x|),$$

the identity above now reads

$$\nabla_k(f e^{-U}) = e^{-U} \nabla_k f - p f |x|^{p-1} e^{-U} \nabla(|x|).$$

Taking inner product with $\nabla(|x|)$ and integrating on both sides, we have

$$\begin{aligned} \frac{1}{Z} \int_{\mathbb{R}^N} \nabla(|x|) \cdot \nabla_k(f e^{-U}) d\mu_k \\ = \int_{\mathbb{R}^N} \nabla_k(|x|) \cdot \nabla_k f d\mu_U - p \int_{\mathbb{R}^N} |\nabla(|x|)|^2 f |x|^{p-1} d\mu_U. \end{aligned} \quad (6.8)$$

We can use integration by parts on the left hand side to obtain

$$\frac{1}{Z} \int_{\mathbb{R}^N} \nabla(|x|) \cdot \nabla_k(f e^{-U}) d\mu_k = - \int_{\mathbb{R}^N} \Delta_k(|x|) f d\mu_U.$$

Replacing this in (6.8), and using also the fact that $|\nabla(|x|)| = 1$ for $x \neq 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} f |x|^{p-1} d\mu_U &= \frac{1}{p} \int_{\mathbb{R}^N} \nabla(|x|) \cdot \nabla_k f d\mu_U + \frac{1}{p} \int_{\mathbb{R}^N} \Delta_k(|x|) f d\mu_U \\ &\leq \frac{1}{p} \int_{\mathbb{R}^N} |\nabla_k f| d\mu_U + \frac{1}{p} \int_{\mathbb{R}^N} \Delta_k(|x|) f d\mu_U. \end{aligned}$$

Finally, we have

$$T_i^2(|x|) = T_i \left(\frac{x_i}{|x|} \right) = \frac{1}{|x|} - \frac{x_i^2}{|x|^3} + \sum_{\alpha \in R_+} k_\alpha \frac{\alpha_i^2}{|x|},$$

so

$$\Delta_k(|x|) = (N + 2\gamma - 1) \frac{1}{|x|}.$$

Therefore, from the above we deduce that (recall that f vanishes on the unit ball)

$$\int_{\mathbb{R}^N} f|x|^{p-1} d\mu_U \leq \frac{1}{p} \int_{\mathbb{R}^N} |\nabla_k f| d\mu_U + \frac{N+2\gamma-1}{p} \int_{\mathbb{R}^N} |f| d\mu_U.$$

Writing $f = f_+ - f_-$, where $f_+(x) = \max(f(x), 0)$ and $f_-(x) = -\min(f(x), 0)$, we can apply this inequality to f_+ and f_- separately. Adding the two resulting inequalities, we have

$$\int_{\mathbb{R}^N} |f| \cdot |x|^{p-1} d\mu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f| d\mu_U + D_1 \int_{\mathbb{R}^N} |f| d\mu_U,$$

where $C_1 = \frac{1}{p}$ and $D_1 = \frac{N+2\gamma-1}{p}$.

Consider now a function $f \in L^1(d\mu_U)$. Let $\varphi : \mathbb{R}_+ \rightarrow [0, 1]$ be a smooth function such that $\varphi = 0$ on $[0, 1]$ and $\varphi = 1$ on $[2, \infty)$. Assume also that $|\varphi'| \leq 1$ on $\mathbb{R}_+$. Let $\phi(x) = \varphi(|x|)$. The strategy now will be to split $f = \phi f + (1 - \phi)f$; the first term vanishes on the unit ball so the above can be applied to it, while the second term has compact support and it will be easy to bound. We have

$$\begin{aligned} \int_{\mathbb{R}^N} |f| \cdot |x|^{p-1} d\mu_U &\leq \int_{\mathbb{R}^N} |\phi f| \cdot |x|^{p-1} d\mu_U + \int_{\mathbb{R}^N} |(1 - \phi)f| \cdot |x|^{p-1} d\mu_U \\ &\leq C_1 \int_{\mathbb{R}^N} |\nabla_k(\phi f)| d\mu_U + D_1 \int_{\mathbb{R}^N} |\phi f| d\mu_U + 2^{p-1} \int_{\mathbb{R}^N} |f| d\mu_U \\ &\leq C_1 \int_{\mathbb{R}^N} |\nabla_k f| d\mu_U + (C_1 + D_1 + 2^{p-1}) \int_{\mathbb{R}^N} |f| d\mu_U. \end{aligned}$$

Here, in the last step, we used the fact that

$$\nabla_k(\phi f) = f \nabla_k \phi + \phi \nabla_k f$$

and

$$|\nabla \phi(x)| = \left| \frac{x}{|x|} \varphi'(|x|) \right| \leq 1.$$

This completes the proof of the Proposition. $\square$

Proposition 6.2.2. *Let $p > 1$ and consider the function $U(x) = |x|^p$ and the probability measure $\mu_U := \frac{1}{Z} e^{-U} \mu_k$, where $Z = \int_{\mathbb{R}^N} e^{-U} d\mu_k$. We then have*

$$\int_{\mathbb{R}^N} |f|^2 \cdot |x|^{2(p-1)} d\mu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + D \int_{\mathbb{R}^N} |f|^2 d\mu_U, \quad (6.9)$$

for some constants $C, D > 0$.

Proof. We will follow more closely the proof of (6.7). We have

$$\nabla U(x) = p|x|^{p-2}x.$$

From (6.4), we have

$$\frac{p^2}{4} \int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} d\mu_U \leq \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U - A,$$

where

$$A = \frac{1}{Z} \int_{\mathbb{R}^N} f \nabla U \cdot \nabla_k (f e^{-U/2}) e^{-U/2} d\mu_k.$$

As before

$$\begin{aligned} 2A &= -[p(p-2) + p(N+2\gamma) - 2\gamma p] \int_{\mathbb{R}^N} f^2 |x|^{p-2} d\mu_U \\ &\quad - 2p \sum_{\alpha \in R_+} k_\alpha \int_{\mathbb{R}^N} |x|^{p-2} f(x) f(\sigma_\alpha x) d\mu_U(x) \\ &\geq -[p^2 + p(N+2\gamma-2)] \int_{\mathbb{R}^N} f^2 |x|^{p-2} d\mu_U. \end{aligned}$$

Thus

$$\int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} d\mu_U \leq \frac{4}{p^2} \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + \frac{2[p^2 + p(N+2\gamma-2)]}{p^2} \int_{\mathbb{R}^N} f^2 |x|^{p-2} d\mu_U. \quad (6.10)$$

Assume first that $p > 2$ and let $\epsilon > 0$. Then, using Hölder's inequality with coefficients $\frac{p}{2(p-1)}$ and $\frac{p-2}{2(p-1)}$, and then Young's inequality with the same coefficients, we have

$$\begin{aligned} \int_{\mathbb{R}^N} f^2 |x|^{p-2} d\mu_U &\leq \left(\int_{\mathbb{R}^N} f^2 d\mu_U \right)^{\frac{p}{2(p-1)}} \left(\int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} d\mu_U \right)^{\frac{p-2}{2(p-1)}} \\ &\leq \frac{p}{2(p-1)} \epsilon^{-\frac{p-2}{p}} \int_{\mathbb{R}^N} f^2 d\mu_U + \frac{p-2}{2(p-1)} \epsilon \int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} d\mu_U. \end{aligned}$$

Thus, by choosing $\epsilon > 0$ small enough such that

$$1 > \frac{(p-2)[p^2 + p(N+2\gamma-2)]}{p^2(p-1)} \epsilon,$$

we obtain inequality (6.9) for some constants $C, D > 0$.

The case $1 < p < 2$ requires more care. The inequality (6.10) holds for all $p > 1$, and so

taking $\mathbf{1}_{B_1^C} f$ in this result, where $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$ and $B_1^C = \mathbb{R}^N \setminus B_1$, we obtain

$$\int_{B_1^C} f^2 |x|^{2(p-1)} d\mu_U \leq \frac{4}{p^2} \int_{B_1^C} |\nabla_k f|^2 d\mu_U + \frac{2[p^2 + p(N + 2\gamma - 2)]}{p^2} \int_{B_1^C} f^2 |x|^{p-2} d\mu_U.$$

Note that, since $p < 2$,

$$\int_{B_1^C} f^2 |x|^{p-2} d\mu_U \leq \int_{B_1^C} f^2 d\mu_U,$$

and so

$$\int_{B_1^C} f^2 |x|^{2(p-1)} d\mu_U \leq \frac{4}{p^2} \int_{B_1^C} |\nabla_k f|^2 d\mu_U + \frac{2[p^2 + p(N + 2\gamma - 2)]}{p^2} \int_{B_1^C} f^2 d\mu_U.$$

On the other hand, since $p > 1$, we have

$$\int_{B_1} f^2 |x|^{2(p-1)} d\mu_U \leq \int_{B_1} f^2 d\mu_U.$$

Adding the last two inequalities, we obtain

$$\int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} d\mu_U \leq \frac{4}{p^2} \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + \frac{2[p^2 + p(N + 2\gamma - 2)]}{p^2} \int_{\mathbb{R}^N} f^2 d\mu_U,$$

which concludes the proof of the Proposition. □

6.3 LOG-SOBOLEV INEQUALITIES FOR WEIGHTED MEASURES

In this section we use the U -bounds obtained above to deduce log-Sobolev inequalities. To begin with, from Proposition 6.2.2 we obtain a generalised $L^2(\mu_U)$ log-Sobolev inequality for the weighted probability measure $d\mu_U = \frac{1}{Z} e^{-U(x)} d\mu_k$, for $U(x) = |x|^p$ for $1 \leq p \leq 2$. Similarly, from Proposition 6.2.1 we obtain a log-Sobolev inequality in $L^1(\mu_U)$ for general $1 < p < \infty$. The approach in both these results is similar to that of Theorem 6.1.1: first employing Jensen's inequality to take the logarithm outside the integral, and then using the classical Sobolev inequality. U -bounds will be used to control residual terms arising from the introduction of a weight.

Proposition 6.3.1. *Let $U(x) = |x|^p$ for $1 < p \leq 2$ and consider the probability measure $d\mu_U = \frac{1}{Z} e^{-U} d\mu_k$, where $Z = \int_{\mathbb{R}^N} e^{-U} d\mu_k$. Let $s = 2\frac{p-1}{p}$. Then we have the inequality*

$$\int_{\mathbb{R}^N} f^2 \left| \log \frac{f^2}{\int f^2 d\mu_U} \right|^s d\mu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + C_2 \int_{\mathbb{R}^N} f^2 d\mu_U,$$

for some constants $C_1, C_2 > 0$.

Proof. Consider the function $h = \frac{1}{\sqrt{Z}} f e^{-U/2}$. Then $\int_{\mathbb{R}^N} f^2 d\mu_U = \int_{\mathbb{R}^N} h^2 d\mu_k$, and

$$\begin{aligned} \int_{\mathbb{R}^N} f^2 \left| \log \frac{f^2}{\int f^2 d\mu_U} \right|^s d\mu_U &= \int_{\mathbb{R}^N} h^2 \left| \log \frac{h^2}{\int h^2 d\mu_k} + U + \log Z \right|^s d\mu_k \\ &\leq \int_{\mathbb{R}^N} h^2 \left| \log \frac{h^2}{\int h^2 d\mu_k} \right|^s d\mu_k + \int_{\mathbb{R}^N} h^2 U^s d\mu_k + |\log Z|^s \int_{\mathbb{R}^N} h^2 d\mu_k, \end{aligned} \quad (6.11)$$

where in the last inequality we used the fact that since $s \in (0, 1]$ we have $(X + Y)^s \leq X^s + Y^s$ for $X, Y > 0$.

Before we start the usual procedure of applying Jensen's inequality, we note that the function $x |\log x|^s$ is bounded on $(0, 1)$, and let

$$D = \sup_{x \in (0, 1)} x |\log x|^s < \infty.$$

Consider now the function $\log_+ x := \max\{0, \log x\}$. Then the above observation implies that

$$x |\log x|^s \leq x (\log_+ x)^s + D \quad \text{for all } x > 0.$$

With this in mind, we have

$$\begin{aligned} \int_{\mathbb{R}^N} h^2 \left| \log \frac{h^2}{\int h^2 d\mu_k} \right|^s d\mu_k &= \int_{\mathbb{R}^N} h^2 d\mu_k \cdot \int_{\mathbb{R}^N} \frac{h^2}{\int h^2 d\mu_k} \left| \log \frac{h^2}{\int h^2 d\mu_k} \right|^s d\mu_k \\ &\leq \int_{\mathbb{R}^N} h^2 d\mu_k \cdot \left[\int_{\mathbb{R}^N} \frac{h^2}{\int h^2 d\mu_k} \left(\log_+ \frac{h^2}{\int h^2 d\mu_k} \right)^s d\mu_k + D \right]. \end{aligned} \quad (6.12)$$

For the fixed function h the measure $\frac{h^2}{\int h^2 d\mu_k} d\mu_k$ is a probability measure and so we can apply Jensen's inequality to the concave function $(\log_+ t)^s$ in the below as follows

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{h^2}{\int h^2 d\mu_k} \left(\log_+ \frac{h^2}{\int h^2 d\mu_k} \right)^s d\mu_k &= \frac{1}{\delta^s} \int_{\mathbb{R}^N} \frac{h^2}{\int h^2 d\mu_k} \left(\log_+ \left(\frac{h^2}{\int h^2 d\mu_k} \right)^\delta \right)^s d\mu_k \\ &\leq \frac{1}{\delta^s} \left(\log_+ \int_{\mathbb{R}^N} \left(\frac{h^2}{\int h^2 d\mu_k} \right)^{1+\delta} d\mu_k \right)^s \\ &= \frac{(1+\delta)^s}{\delta^s} \left(\log_+ \frac{\|h\|_{2+2\delta}^2}{\|h\|_2^2} \right)^s. \end{aligned}$$

By standard elementary means we can show that there exists a decreasing function $\overline{C}(\epsilon)$

defined for all $\epsilon > 0$ such that the inequality

$$|\log_+ x|^s \leq \epsilon x + \overline{C}(\epsilon)$$

holds for all $x > 0$. Applying this in the previous inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{h^2}{\int h^2 d\mu_k} \left(\log_+ \frac{h^2}{\int h^2 d\mu_k} \right)^s d\mu_k &\leq \frac{(1+\delta)^s}{\delta^s} \left(\epsilon \frac{\|h\|_{2+2\delta}^2}{\|h\|_2^2} + \overline{C}(\epsilon) \right) \\ &\leq \frac{(1+\delta)^s}{\delta^s} \frac{1}{\int h^2 d\mu_k} [\epsilon C_{DS}^2 \|\nabla_k h\|_2^2 + \overline{C}(\epsilon) \|h\|_2^2]. \end{aligned} \quad (6.13)$$

Here in the last step we used the Sobolev inequality which holds if we choose $\delta > 0$ such that $2 + 2\delta = \frac{2(N+2\gamma)}{N+2\gamma-2}$.

Next, we have

$$\|\nabla_k h\|_2^2 = \frac{1}{Z} \int_{\mathbb{R}^N} |\nabla_k (f e^{-U/2})|^2 d\mu_k \leq 2 \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + \frac{1}{2} \int_{\mathbb{R}^N} f^2 |\nabla U|^2 d\mu_U. \quad (6.14)$$

Combining (6.11), (6.12), (6.13) and (6.14), we have

$$\begin{aligned} \int_{\mathbb{R}^N} f^2 \left| \log \frac{f^2}{\int f^2 d\mu_U} \right|^s d\mu_U &\leq \left(\frac{N+2\gamma}{2} \right)^s 2C_{DS}^2 \epsilon \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U \\ &\quad + \left[\left(\frac{N+2\gamma}{2} \right)^s \overline{C}(\epsilon) + |\log Z|^s \right] \int_{\mathbb{R}^N} f^2 d\mu_U \\ &\quad + \left(\frac{N+2\gamma}{2} \right)^s \frac{C_{DS}^2}{2} \epsilon \int_{\mathbb{R}^N} f^2 |\nabla U|^2 d\mu_U + \int_{\mathbb{R}^N} f^2 U^s d\mu_U. \end{aligned}$$

But $|\nabla U| = p|x|^{p-1}$ and $U^s = |x|^{2(p-1)s}$, so the last two terms can be computed as follows

$$\int_{\mathbb{R}^N} f^2 U^s d\mu_U = \int_{\mathbb{R}^N} f^2 |x|^{2(p-1)s} d\mu_U = \frac{1}{p^2} \int_{\mathbb{R}^N} f^2 |\nabla U|^2 d\mu_U.$$

Finally, from Proposition 6.2.2 and a relabelling of the constants, we obtain

$$\int_{\mathbb{R}^N} f^2 \left| \log \frac{f^2}{\int f^2 d\mu_U} \right|^s d\mu_U \leq \epsilon \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + C(\epsilon) \int_{\mathbb{R}^N} f^2 d\mu_U,$$

where $C(\epsilon) = A_1 \overline{C}(a\epsilon) + A_2 \epsilon + A_3$, for some constants $A_1, A_2, A_3, a > 0$. Choosing small $\epsilon > 0$ we have $C(\epsilon) > 0$. This concludes the proof. $\square$

Proposition 6.3.2. *Let $U(x) = |x|^p$ with $1 \leq p < \infty$ and consider the finite measure*

$d\mu_U = e^{-U} d\mu_k$. Let $s = \frac{p-1}{p}$. Then we have the inequality

$$\int_{\mathbb{R}^N} f \left| \log \frac{|f|}{\int |f| d\mu_U} \right|^s d\mu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f| d\mu_U + C_2 \int_{\mathbb{R}^N} |f| d\mu_U,$$

for some constants $C_1, C_2 > 0$.

Proof. The proof is similar to that of the previous result except for in this case we rely on the Sobolev inequality

$$\|h\|_q \leq C \|\nabla_k h\|_1$$

where $q = \frac{N+2\gamma}{N+2\gamma-1}$, and the U -bound of Proposition 6.2.1. $\square$

6.4 POINCARÉ INEQUALITIES

In this section we discuss Poincaré inequalities for Dunkl operators. These will be used in the next section to improve some of our previous log-Sobolev inequalities, but are also of independent interest.

In order to prove Poincaré inequalities, we first introduce Muckenhoupt weights and show that the Dunkl weight is a Muckenhoupt weight for small coefficients k_α . This implies a weighted Poincaré inequality for the usual gradient ∇f , via a classic result of [39]. Then, using the usual technique based on Dirichlet forms, we show that this implies a Poincaré inequality for the Dunkl gradient.

Definition 6.4.1 (Muckenhoupt weights). Let w be a locally integrable function on $\mathbb{R}^N$, and assume also that $w(x) > 0$ for almost every $x \in \mathbb{R}^N$. We say that $w \in A_p$ for some $p > 1$ if there exists a constant $A < \infty$ such that

$$\frac{1}{|B|} \int_B w(x) dx \left(\frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq A \quad (6.15)$$

holds for all balls $B \subset \mathbb{R}^N$. Similarly, we say that $w \in A_1$ if there exists a constant $A > 0$ such that

$$\frac{1}{|B|} \int_B w(y) dy \leq Aw(x) \quad \text{for a.e. } x \in B$$

holds for all balls $B \subset \mathbb{R}^N$.

The weights satisfying property A_p have been studied extensively in harmonic analysis, particularly because of their connection with maximal operators. The maximal operator on $\mathbb{R}^N$ is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy,$$

where $B_r(x) = \{y \in \mathbb{R}^N : |y - x| < r\}$. We then have the following fundamental result due to Muckenhoupt.

Theorem 6.4.2 (Muckenhoupt's theorem). *Let $1 < p < \infty$ and let w be a locally integrable function on $\mathbb{R}^N$. Then $w \in A_p$ if and only if there exists a constant $A > 0$ such that*

$$\int_{\mathbb{R}^N} |Mf(x)|^p w(x) \, dx \leq A \int_{\mathbb{R}^N} |f(x)|^p w(x) \, dx$$

holds for all $f \in L^p(w(x) \, dx)$.

We will only present here some basic properties of these weights, for proofs and a more detailed introduction see [96] or [50].

Proposition 6.4.3. *We have the following properties of the classes A_p :*

- (i) *The constant functions belong to all A_p classes for $p \geq 1$.*
- (ii) *The classes A_p increase with p , i.e., if $w \in A_p$ and $1 \leq p < q < \infty$, then $w \in A_q$.*
- (iii) *Let $w_1 \in A_{p_1}$ and $w_2 \in A_{p_2}$ for some $1 \leq p_1, p_2 < \infty$. Let also $\theta \in [0, 1]$ and $1 \leq p < \infty$ such that $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then, if $w^{\frac{1}{p}} = w_1^{\frac{\theta}{p_1}} w_2^{\frac{1-\theta}{p_2}}$, we have $w \in A_p$.*
- (iv) *Let $1 \leq p_1, \dots, p_n < \infty$ and $w_j \in A_{p_j}$ for $j = 1, \dots, n$. Let also $0 < \theta_1, \dots, \theta_n < 1$ such that $\theta_1 + \dots + \theta_n = 1$. Then*

$$w_1^{\theta_1} \cdot \dots \cdot w_n^{\theta_n} \in A_{\max\{p_1, \dots, p_n\}}.$$

A typical example of an A_p weight is the function $w(x) = |x|^a$ for which we have $w \in A_p$ if and only if $-N < a < N(p - 1)$. Our aim is to establish for which values of p the Dunkl weight w_k is of class A_p . The proof we provide below is based on the following crucial result from [84].

Lemma 6.4.4. *Let $P(x)$ be a polynomial of degree d in $\mathbb{R}^N$. Let B_1 be the unit open ball of radius 1, and let $\epsilon < \frac{1}{d}$. Then we have*

$$\int_{B_1} |P(x)|^{-\epsilon} \, dx \leq C_\epsilon \left(\int_{B_1} |P(x)| \, dx \right)^{-\epsilon},$$

where the constant C_ϵ does not depend on the polynomial P .

We then have the following result.

Proposition 6.4.5. *Let $1 < p < \infty$. If $\gamma < \frac{p-1}{2}$, then the Dunkl weight w_k is of class A_p .*

Proof. Fix $\alpha \in R_+$. We will consider the polynomial of degree 1 given by $P(x) = \langle \alpha, x \rangle$. By Lemma 6.4.4, we have

$$\int_{B_1} |\langle \alpha, x \rangle|^{-\epsilon} dx \leq C_\epsilon \left(\int_{B_1} |\langle \alpha, x \rangle| dx \right)^{-\epsilon}, \quad (6.16)$$

which holds for $\epsilon < 1$.

We note that we can replace the unit ball B_1 in Lemma 6.4.4 with any ball $B_R(z) = \{x : |x - z| < R\}$ since composing the polynomial $P(x)$ with any dilation or translation results also in a polynomial of degree d in x . More precisely, we consider the polynomial $Q(x) = P(\frac{x-z}{R})$ and apply Lemma 6.4.4 to Q to obtain the general result.

Thus we also have (6.16) for any ball $B \subset \mathbb{R}^N$. Taking $\epsilon = \frac{1}{q-1}$ for some $q > 2$, this gives (6.15) with p replaced by q , i.e.,

$$|\langle \alpha, x \rangle| \in A_q \quad \text{for any } q > 2.$$

We now want to show that $|\langle \alpha, x \rangle|^a \in A_p$ if $0 < a < p - 1$. We distinguish two cases:

Case 1: $0 < a < \min(1, p - 1)$. Take $p_1 = \frac{p-1}{a} + 1$ and $p_2 = 1$. Also, let

$$w_1(x) = |\langle \alpha, x \rangle| \quad \text{and} \quad w_2(x) = 1.$$

From our assumption, we have $p_1 > 2$, and so, by the observation above, $w_1 \in A_{p_1}$. By Proposition 6.4.3(i), we also have $w_2 \in A_{p_2}$. Take

$$\theta = \frac{p_1(p-1)}{p(p_1-1)} = \frac{p-1+a}{p} \in (0, 1),$$

so we have $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. By Proposition 6.4.3(iii) we have $w := w_1^{\frac{p\theta}{p_1}} = |\langle \alpha, x \rangle|^a \in A_p$.

Case 2: $1 \leq a < p - 1$. Consider the space Π_1 of polynomials on $\mathbb{R}^N$ of degree at most 1. As this is a finite vector space, all norms on Π_1 are equivalent, so there exists a constant $\tilde{C}$ such that

$$\int_{B_1} |P(x)| dx \geq \tilde{C} \left(\int_{B_1} |P(x)|^a dx \right)^{1/a}$$

holds for all $P \in \Pi_1$. From this, together with (6.16) applied to $\epsilon = \frac{a}{p-1}$, we have

$$\int_{B_1} |\langle \alpha, x \rangle|^{-\frac{a}{p-1}} dx \leq C_\epsilon \tilde{C}^{-\frac{a}{p-1}} \left(\int_{B_1} |\langle \alpha, x \rangle|^a dx \right)^{-\frac{1}{p-1}}.$$

Using the observation after inequality (6.16), we can replace B_1 by any ball $B \subset \mathbb{R}^N$, and so this implies that $|\langle \alpha, x \rangle|^a \in A_p$.

Finally, assume that $\gamma < \frac{p-1}{2}$. Let $\delta = \frac{1}{|R_+|} \left(1 - \frac{2\gamma}{p-1}\right)$ and define $\theta_\alpha = \frac{2k_\alpha}{p-1} + \delta$. Assuming that $|R_+| > 1$ ($|R_+| = 1$ corresponds to the case proved above), we have $0 < \theta_\alpha < 1$ for all $\alpha \in R_+$, and $\sum_{\alpha \in R_+} \theta_\alpha = 1$. Since $\frac{2k_\alpha}{\theta_\alpha} < p-1$, by the above we have $|\langle \alpha, x \rangle|^{\frac{2k_\alpha}{\theta_\alpha}} \in A_p$ for all $\alpha \in R_+$. By Proposition 6.4.3(iv), we conclude that

$$w_k(x) = \prod_{\alpha \in R_+} \left(|\langle \alpha, x \rangle|^{\frac{2k_\alpha}{\theta_\alpha}} \right)^{\theta_\alpha} \in A_p,$$

as required. $\square$

The following important result from [39] shows that the weighted Poincaré inequality holds for Muckenhoupt weights.

Theorem 6.4.6 ([39]). *Let $1 < p < \infty$ and $w \in A_p$. Then there exists a constant $C > 0$ such that for all balls $B_r(x)$ of centre x and radius $r > 0$, and for all functions f defined on $B_r(x)$ that are Lipschitz continuous, we have the inequality*

$$\int_{B_r(x)} |f(y) - f_{B_r(x)}^w|^p w(y) dy \leq Cr^p \int_{B_r(x)} |\nabla f(y)|^p w(y) dy,$$

where $f_{B_r(x)}^w = \frac{1}{w(B_r(x))} \int_{B_r(x)} f(z) w(z) dz$ is the weighted average of f over the ball $B_r(x)$.

A proof of this Theorem is given in Appendix A.

With the help of this result we obtain the following Poincaré inequality for Dunkl operators.

Theorem 6.4.7. *Assume that $\gamma < \frac{1}{2}$. Then there exists a constant $C > 0$ such that for any function f Lipschitz continuous on $B_R(0)$ we have the inequality*

$$\int_{B_R(0)} \left| f - \frac{1}{\mu_k(B_R(0))} \int_{B_R(0)} f d\mu_k \right|^2 d\mu_k \leq CR^2 \int_{B_R(0)} |\nabla_k f|^2 d\mu_k.$$

Proof. Since $\gamma < \frac{1}{2}$, by Proposition 6.4.5 we have $w_k \in A_2$. Thus, by Theorem 6.4.6 we have the Poincaré inequality

$$\int_{B_R(0)} |f - f_{B_R(0)}^{w_k}|^2 d\mu_k \leq CR^2 \int_{B_R(0)} |\nabla f|^2 d\mu_k.$$

But, since $B_R(0)$ is G -invariant, by Lemma 2.1.2 we have

$$\int_{B_R(0)} |\nabla f|^2 d\mu_k \leq \int_{B_R(0)} |\nabla_k f|^2 d\mu_k.$$

This, together with the previous inequality, prove the Theorem. $\square$

We can now use this result together with the U -bounds proved above to obtain a Poincaré inequality for the weighted measure μ_U .

Proposition 6.4.8. *Let $p > 1$ and consider the weighted probability measure $d\mu_U = \frac{1}{Z} e^{-U} d\mu_k$, where $U(x) = |x|^p$ and $Z = \int_{\mathbb{R}^N} e^{-U} d\mu_k$. Assume, in addition, that $\gamma < \frac{1}{2}$. We then have the Poincaré inequality*

$$\int_{\mathbb{R}^N} \left| f - \int_{\mathbb{R}^N} f d\mu_U \right|^2 d\mu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U$$

which holds for all functions f locally Lipschitz continuous on $\mathbb{R}^N$ and such that $f \in L^1(\mu_U)$, with a constant $C > 0$ independent of f .

Proof. We first note that for any constant $\zeta \in \mathbb{R}$ we have

$$\left| \int_{\mathbb{R}^N} f d\mu_U - \zeta \right| \leq \int_{\mathbb{R}^N} |f - \zeta| d\mu_U \leq \left(\int_{\mathbb{R}^N} |f - \zeta|^2 d\mu_U \right)^{\frac{1}{2}},$$

so by the triangle inequality for the $L^2(\mu_U)$ norm

$$\int_{\mathbb{R}^N} \left| f - \int_{\mathbb{R}^N} f d\mu_U \right|^2 d\mu_U \leq 4 \int_{\mathbb{R}^N} |f - \zeta|^2 d\mu_U.$$

Thus, it is enough to prove the inequality

$$\int_{\mathbb{R}^N} |f - \zeta|^2 d\mu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U \quad (6.17)$$

for some $\zeta \in \mathbb{R}$.

Let $R > 0$ and let $B_R(0) = \{|x| \leq R\}$. We will prove (6.17) with $\zeta = \frac{1}{\mu_k(B_R(0))} \int_{B_R(0)} f d\mu_k$

for large enough R . Firstly, we have

$$\begin{aligned}
\int_{B_R(0)} |f - \zeta|^2 d\mu_U &\leq \frac{1}{Z} \int_{B_R(0)} \left| f - \frac{1}{\mu_k(B_R(0))} \int_{B_R(0)} f d\mu_k \right|^2 d\mu_k \\
&\leq \frac{C}{Z} R^2 \int_{B_R(0)} |\nabla_k f|^2 d\mu_k \\
&\leq CR^2 e^{Rp} \int_{B_R(0)} |\nabla_k f|^2 d\mu_U.
\end{aligned} \tag{6.18}$$

Here we used the Poincaré inequality of Theorem 6.4.7 and the bounds $e^{-Rp} \leq e^{-U} \leq 1$ on $B_R(0)$.

On the other hand, we can use the U -bounds of Proposition 6.2.2 to the remaining integral as follows

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B_R(0)} |f - \zeta|^2 d\mu_U &\leq R^{-2(p-1)} \int_{|x| \geq R} |f(x) - \zeta|^2 |x|^{2(p-1)} d\mu_U \\
&\leq CR^{-2(p-1)} \int_{|x| \geq R} |\nabla_k f|^2 d\mu_U + DR^{-2(p-1)} \int_{|x| \geq R} |f - \zeta|^2 d\mu_U.
\end{aligned}$$

But R was an arbitrary positive number so we are free to choose it such that $DR^{-2(p-1)} < 1$. Then we have

$$\int_{\mathbb{R}^N \setminus B_R(0)} |f - \zeta|^2 d\mu_U \leq \frac{CR^{-2(p-1)}}{1 - DR^{-2(p-1)}} \int_{|x| \geq R} |\nabla_k f|^2 d\mu_U. \tag{6.19}$$

Adding the inequalities (6.18) and (6.19), we obtain (6.17), and therefore, by the observation above, the Proposition is proved. $\square$

6.5 TIGHT LOG-SOBOLEV INEQUALITIES

We now have all the ingredients to obtain tight log-Sobolev inequalities. The first is a tight version of the log-Sobolev inequality for Gaussian measure from Theorem 6.1.4.

Theorem 6.5.1. *Assume that $\gamma < \frac{1}{2}$. Let $U(x) = |x|^2$ and consider the probability measure $d\mu_U = \frac{1}{Z} e^{-U} d\mu_k$, where $Z = \int_{\mathbb{R}^N} e^{-U} d\mu_k$. Then there exists a constant $C > 0$ such that we have the inequality*

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 d\mu_U} d\mu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U.$$

In order to prove this result we will need the following inequality, known as Rothaus's

lemma.

Lemma 6.5.2 ([89]). *Recall that*

$$\text{Ent}(g) := \int_{\mathbb{R}^N} g \log g \, d\mu_U - \int_{\mathbb{R}^N} g \, d\mu_U \log \int_{\mathbb{R}^N} g \, d\mu_U,$$

for $g \geq 0$. Then, for all f measurable with $\int_{\mathbb{R}^N} f \, d\mu_U = 0$, we have the inequality

$$\text{Ent}((f + c)^2) \leq \text{Ent}(f^2) + 2 \int_{\mathbb{R}^N} f^2 \, d\mu_U,$$

for all $c \in \mathbb{R}$.

Proof of Theorem 6.5.1. By Rothaus's lemma we have

$$\text{Ent}(f^2) \leq \text{Ent} \left(\left(f - \int f \, d\mu_U \right)^2 \right) + 2 \int_{\mathbb{R}^N} \left(f - \int f \, d\mu_U \right)^2 \, d\mu_U.$$

Furthermore, from Theorem 6.1.4, we have

$$\text{Ent}(f^2) \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U + (2 + C_2) \int_{\mathbb{R}^N} \left(f - \int f \, d\mu_U \right)^2 \, d\mu_U.$$

Finally, using the Poincaré inequality of Proposition 6.4.8, we obtain

$$\text{Ent}(f^2) \leq (C_1 + C(2 + C_2)) \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U,$$

which is exactly what we wanted to prove. $\square$

The following inequality is also a tight version of the generalised log-Sobolev inequality of Proposition 6.3.1, and it is obtained from this result in a manner very similar to the proof that we have just seen.

Theorem 6.5.3. *Assume that $\gamma < \frac{1}{2}$. Let $1 \leq p \leq 2$ and $s = 2\frac{p-1}{p}$. Let $U(x) = |x|^p$ and consider the probability measure $d\mu_U = \frac{1}{Z} e^{-U} \, d\mu_k$, where $Z = \int_{\mathbb{R}^N} e^{-U} \, d\mu_k$. Let also*

$$\Phi(x) = x(\log(x + 1))^s.$$

Then we have the inequality

$$\int_{\mathbb{R}^N} \Phi(f^2) \, d\mu_U - \Phi \left(\int_{\mathbb{R}^N} f^2 \, d\mu_U \right) \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U,$$

for a constant $C > 0$.

As before, we need the following generalisation of Rothaus's lemma.

Lemma 6.5.4 ([72]). *Let Φ be as in the statement of the Theorem and define, for $g \geq 0$,*

$$\text{Ent}_\Phi(g) := \int_{\mathbb{R}^N} \Phi(g) \, d\mu_U - \Phi \left(\int_{\mathbb{R}^N} g \, d\mu_U \right).$$

Then there exist constants $A_1, B_1 > 0$ such that for any f with $\int_{\mathbb{R}^N} f \, d\mu_U = 0$ we have

$$\text{Ent}_\Phi((f + c)^2) \leq A_1 \text{Ent}_\Phi(f^2) + B_1 \int_{\mathbb{R}^N} f^2 \, d\mu_U$$

for all $c \in \mathbb{R}$.

Proof of Theorem 6.5.3. The proof of this goes along the same lines as the proof of Theorem 6.5.1 although in this case we cannot apply Proposition 6.3.1 directly. Instead, we note that

$$\begin{aligned} \text{Ent}_\Phi(g) &= \int_{\mathbb{R}^N} g \left[(\log(1 + g))^s - \left(\log \left(1 + \int_{\mathbb{R}^N} g \right) \right)^s \right] d\mu_U \\ &\leq \int_{\mathbb{R}^N} g \left| \log \frac{g + 1}{\int g \, d\mu_U + 1} \right|^s d\mu_U, \end{aligned}$$

where we used the inequality $(X + Y)^s \leq X^s + Y^s$ which holds for all $X, Y \geq 0$ since $s \in [0, 1]$. We compute the integral on the right hand side separately over $X = \left\{ x : g(x) \geq \int_{\mathbb{R}^N} g \, d\mu_U \right\}$ and $\bar{X} = \mathbb{R}^N \setminus X$. On X we have

$$1 \leq \frac{g + 1}{\int g \, d\mu_U + 1} \leq \frac{g}{\int g \, d\mu_U},$$

so

$$\int_X g \left| \log \frac{g + 1}{\int g \, d\mu_U + 1} \right|^s d\mu_U \leq \int_{\mathbb{R}^N} g \left| \log \frac{g}{\int g \, d\mu_U} \right|^s d\mu_U.$$

On the other hand, on $\bar{X}$ we have

$$1 \leq \frac{\int g \, d\mu_U + 1}{g + 1} \leq 1 + \frac{\int g \, d\mu_U}{g},$$

so

$$\begin{aligned} \int_{\overline{X}} g \left| \log \frac{g+1}{\int g d\mu_U + 1} \right|^s d\mu_U &= \int_{\overline{X}} g \left(\log \frac{\int g d\mu_U + 1}{g+1} \right)^s d\mu_U \\ &\leq \int_{\overline{X}} g \left(\frac{\int g d\mu_U}{g} \right)^s d\mu_U \leq \int_{\mathbb{R}^N} g d\mu_U, \end{aligned}$$

where we first used the inequality $\log(1+x) \leq x$ for all $x \geq 0$, and then the fact that $s \leq 1$, so $\left(\frac{\int g d\mu_U}{g}\right)^s \leq \frac{\int g d\mu_U}{g}$. Thus

$$\text{Ent}_{\Phi}(g) \leq \int_{\mathbb{R}^N} g \left| \log \frac{g}{\int g d\mu_U} \right|^s d\mu_U + \int_{\mathbb{R}^N} g d\mu_U.$$

We can now apply the same strategy as before. First, by Proposition 6.3.1, we have

$$\text{Ent}_{\Phi}(g^2) \leq C_1 \int_{\mathbb{R}^N} |\nabla_k g|^2 d\mu_U + (C_2 + 1) \int_{\mathbb{R}^N} g^2 d\mu_U.$$

Taking $g = f - \int_{\mathbb{R}^N} f d\mu_U$ in this inequality and applying the previous Lemma, we have

$$\text{Ent}_{\Phi}(f^2) \leq A_1 C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 d\mu_U + (A_1(C_2 + 1) + B_1) \int_{\mathbb{R}^N} \left(f - \int_{\mathbb{R}^N} f d\mu_U \right)^2 d\mu_U.$$

Finally, using the Poincaré inequality of Proposition 6.4.8, the proof is complete. □

7

INFINITE DIMENSIONAL ANALYSIS

In this chapter we study infinite dimensional models of interacting dissipative systems generated by Dunkl operators. More precisely, we study the Markov semigroup on the infinite dimensional space $(\mathbb{R}^N)^{\mathbb{Z}^d}$ generated by

$$\mathcal{L} := \sum_{l \in \mathbb{Z}^d} (L^{(l)} + L_1^{(l)}).$$

The operators $L^{(l)}$ and $L_1^{(l)}$ are defined by

$$L^{(l)} = \Delta_k^{(l)} + b^{(l)} \cdot \nabla_k^{(l)},$$

and

$$L_1^{(l)} = e^{(l)} \cdot \nabla_k^{(l)},$$

where $\Delta_k^{(l)}$ and $\nabla_k^{(l)}$ are the usual Dunkl laplacian and gradient on $\mathbb{R}^N$ which act only on the l component of $\omega = (\omega_l)_{l \in \mathbb{Z}^d} \in (\mathbb{R}^N)^{\mathbb{Z}^d}$. The functions $b^{(l)} : (\mathbb{R}^N)^{\mathbb{Z}^d} \rightarrow \mathbb{R}^N$ depend only on the ω_l component, while $e^{(l)} : (\mathbb{R}^N)^{\mathbb{Z}^d} \rightarrow \mathbb{R}^N$ depend on the components ω_j with $|j-l| < R$ for a finite R called the range of interactions. In other words, the operators $L^{(l)}$ are operators that only depend on the ω_l component and so they define finite dimensional models. In the absence of the interaction terms $e^{(l)}$, these evolve independently of each other.

We first study the finite dimensional case of a single operator $L^{(l)}$ acting on $\mathbb{R}^N$, for which a priori the existence of an invariant measure is unknown. To begin with, we prove gradient bounds for the resulting semigroup. An essential trick here is to consider

a symmetrised gradient which takes into account the reflections associated to the root system defining Dunkl operators. This allows us to deal with the resulting reflection terms and to obtain appropriate bounds. Using these bounds we then prove the existence of an invariant measure for the semigroup in the case of small coefficients k_α .

This analysis is then extended to an infinite dimensional setting, by introducing interaction terms. In this case, corresponding to the description above, the construction of the semigroup is not obvious and it requires an approximation procedure. This requires gradient bounds similar to the finite dimensional case, but which are technically more involved. The bounds obtained allow us to prove that the semigroup has an invariant measure and study its ergodicity properties.

Dissipative dynamics for infinite systems of interacting particles has been studied a long time; see the classic references [71] and [52] for more details. For a study of Markov semigroups in infinite dimensional settings see [110] in the case of elliptic operators, [30] for subelliptic operators, [108, 109] for Levy-type operators and [61] for operators of hypocoercive type. Similar analysis for Dunkl-type operators in the infinite dimensional setting but in a simplified context corresponding to the very particular case of a root system A_1 , was studied in [111]; we extend these results to the case of general root systems in this chapter.

7.1 THE FINITE DIMENSIONAL CASE

We consider the operator

$$L := \sum_{i=1}^N (T_i^2 + b_i T_i) = \Delta_k + b \cdot \nabla_k, \quad (7.1)$$

where each $b_i \in C^1(\mathbb{R}^N)$ for $i = 1, \dots, N$. We will assume throughout this section that L generates a Markov semigroup on $C_b(\mathbb{R}^N)$, the space of bounded continuous functions on $\mathbb{R}^N$. Let $P_t := e^{tL}$ be the semigroup generated by L and denote for brevity $f_t = P_t f$. The carré-du-champ operator associated with the operator L is given by

$$\Gamma_L(f) := \frac{1}{2} (L(f^2) - 2fLf).$$

This should not be confused with the carré-du-champ operator associated with the Dunkl laplacian

$$\Gamma(f) := \frac{1}{2} (\Delta_k(f^2) - 2f\Delta_k f),$$

which will also be used below.

In order to avoid long expressions, in this chapter we use the notation

$$A_\alpha f(x) := \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}$$

for the α -dependent difference part in the definition of the Dunkl operators.

Example 7.1.1. The main example throughout this section is the linear case corresponding to $b(x) = -cx$ for some $c > 0$. This case is related to the generalised Ornstein-Uhlenbeck semigroup studied in [88], where it was shown that the operator

$$L_{OU} := \Delta_k - cx \cdot \nabla f$$

generates a Markov semigroup on $C_b(\mathbb{R}^N)$.

We note that our operator is related to L_{OU} via

$$Lf = L_{OU}f - c \sum_{\alpha \in R_+} k_\alpha (f - f \circ \sigma_\alpha).$$

Since the linear operator $f \mapsto -c \sum_{\alpha \in R_+} k_\alpha (f - f \circ \sigma_\alpha)$ is clearly bounded on $C_b(\mathbb{R}^N)$, standard results from general semigroup theory imply that L generates a contraction semigroup $(P_t)_{t \geq 0}$ (the perturbation of a contraction semigroup generator by a bounded operator generates a contraction semigroup, see for example [24, Theorem 3.1]).

It is easy to see that the assumption (7.2) that we impose below on the drifts b will assure the positive maximum principle holds and thus $(P_t)_{t \geq 0}$ is indeed positivity-preserving, so a Markov semigroup (see [60, Theorem 19.11]).

Moreover, adding suitably small perturbations to the linear drifts $b(x) = -cx$ should produce further Markov semigroups with generators of the desired form (7.1).

Remark. We expect that the class of functions b for which the operator L generates a Markov semigroup is much larger but this is the subject of a different investigation.

7.1.1 GRADIENT BOUNDS

The main result of this section is a bound on the symmetrised gradient form

$$\tilde{\Gamma}(f)(x) := \sum_{g \in G} |\nabla_k f(gx)|^2.$$

This form of the gradient has the advantage that it is G -invariant, so we do not have to deal with the reflected terms that will appear in the computations below.

Theorem 7.1.2. *Let $b_i \in C^1(\mathbb{R}^N)$ such that $\nabla(b_i)$ is bounded for all $i = 1, \dots, N$ and which satisfy*

$$\frac{2}{\langle \alpha, x \rangle^2} - \langle b(x), \alpha \rangle \frac{1}{\langle \alpha, x \rangle} \geq 0 \quad \forall x \in \mathbb{R}^N, \forall \alpha \in R_+. \quad (7.2)$$

Consider the operator

$$L = \Delta_k + b(x) \cdot \nabla_k$$

which is assumed to generate a Markov semigroup $(P_t)_{t \geq 0}$ on $C_b(\mathbb{R}^N)$.

We then have the bound

$$\tilde{\Gamma}(P_t f) \leq e^{2\eta t} P_t(\tilde{\Gamma}(f)) \quad \forall t > 0, \quad (7.3)$$

where

$$\eta := \max_i \sup_{x \in \mathbb{R}^N} \partial_i(b_i)(x) + (N-1) \max_{i \neq j} \|\partial_j(b_i)\|_\infty + \sqrt{2}\gamma \max_{\alpha \in R_+} \|A_\alpha(b)\|_\infty.$$

Proof. Firstly, we compute

$$\begin{aligned} \frac{d}{ds} P_{t-s}(|\nabla_k f_s|^2) &= P_{t-s}(-L(|\nabla_k f_s|^2) + 2\nabla_k f_s \cdot \nabla_k L f_s) \\ &= P_{t-s}(-L(|\nabla_k f_s|^2) + 2\nabla_k f_s \cdot L \nabla_k f_s + 2\nabla_k f_s \cdot [\nabla_k, L] f_s) \\ &= P_{t-s} \left(-2 \sum_{j=1}^N \Gamma_L(T_j f_s) + I(f_s) \right), \end{aligned}$$

where

$$I(h) := 2\nabla_k h \cdot [\nabla_k, L]h = 2 \sum_{i=1}^N \nabla_k h \cdot [\nabla_k, b_i T_i]h.$$

Here and everywhere in the below the notation $[A, B]$ stands for the commutator of the operators A and B , i.e., $[A, B] := AB - BA$.

Step 1: compute $\Gamma_L(h)$. We have

$$\begin{aligned} \Gamma_L(h) &= \frac{1}{2} \left(\Delta_k(h^2) - 2h \Delta_k h + \sum_{i=1}^N b_i [T_i(h^2) - 2h T_i h] \right) \\ &= \Gamma(h) + \frac{1}{2} \sum_{i=1}^N b_i [T_i(h^2) - 2h T_i h]. \end{aligned}$$

Using Lemmas 2.1.1 and 1.2.5, we have

$$\begin{aligned}
\Gamma_L(h)(x) &= |\nabla h(x)|^2 + \sum_{\alpha \in R_+} k_\alpha \left(\frac{h(x) - h(\sigma_\alpha x)}{\langle \alpha, x \rangle} \right)^2 - \frac{1}{2} \sum_{i=1}^N \sum_{\alpha \in R_+} k_\alpha \alpha_i b_i(x) \frac{(h(x) - h(\sigma_\alpha x))^2}{\langle \alpha, x \rangle} \\
&= |\nabla h(x)|^2 + \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha (h(x) - h(\sigma_\alpha x))^2 \left(\frac{2}{\langle \alpha, x \rangle^2} - \langle \alpha, b(x) \rangle \frac{1}{\langle \alpha, x \rangle} \right).
\end{aligned}$$

Thus, assumption (7.2) assures that $\Gamma_L(h) \geq 0$.

Step 2: compute $I(h)$. We first note that, using the commutativity of Dunkl operators,

$$[T_j, b_i T_i] = [T_j, b_i] T_i + b_i [T_j, T_i] = [T_j, b_i] T_i,$$

and

$$\begin{aligned}
[T_j, b_i] T_i h(x) &= T_j(b_i T_i h)(x) - b_i(x) T_j T_i h(x) \\
&= T_j(b_i)(x) T_i h(x) - \sum_{\alpha \in R_+} k_\alpha \alpha_j \frac{(b_i(x) - b_i(\sigma_\alpha x))(T_i h(x) - T_i h(\sigma_\alpha x))}{\langle \alpha, x \rangle}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
I(h)(x) &= 2 \sum_{i,j=1}^N T_j h(x) [T_j, b_i] T_i h(x) \\
&= 2 \sum_{i=1}^N \left(\langle \nabla_k h, \nabla_k b_i \rangle(x) T_i h(x) \right. \\
&\quad \left. - \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \nabla_k h(x) \rangle \frac{(b_i(x) - b_i(\sigma_\alpha x))(T_i h(x) - T_i h(\sigma_\alpha x))}{\langle \alpha, x \rangle} \right).
\end{aligned}$$

Step 3: gradient bounds. From the previous two steps we have

$$\begin{aligned}
\frac{d}{ds} P_{t-s}(|\nabla_k f_s|^2) &\leq P_{t-s}(I(f_s)) \\
&= 2P_{t-s} \left(\sum_{i=1}^N T_i(b_i) |T_i f_s|^2 + \sum_{j \neq i} T_j(b_i) T_i f_s T_j f_s \right. \\
&\quad \left. - \sum_{i,j=1}^N \sum_{\alpha \in R_+} k_\alpha \alpha_j A_\alpha(b_i) T_j f_s (T_i f_s - (T_i f_s) \circ \sigma_\alpha) \right) \\
&= 2P_{t-s} \left(\sum_{i=1}^N \partial_i(b_i) |T_i f_s|^2 + \sum_{j \neq i} \partial_j(b_i) T_i f_s T_j f_s \right. \\
&\quad \left. + \sum_{i,j=1}^N \sum_{\alpha \in R_+} k_\alpha \alpha_j A_\alpha(b_i) T_j f_s (T_i f_s) \circ \sigma_\alpha \right).
\end{aligned}$$

Let us estimate each sum separately, using the elementary inequality $2ab \leq a^2 + b^2$ to separate products and obtain the second powers we need in the expression of the gradient. The first sum is easy to estimate. For the second sum we have

$$\begin{aligned}
2 \sum_{j \neq i} \partial_j(b_i)(x) T_i f_s(x) T_j f_s(x) &\leq \sum_{j \neq i} \|\partial_j(b_i)\|_\infty [(T_i f_s(x))^2 + (T_j f_s(x))^2] \\
&\leq 2(N-1) \max_{j \neq i} \|\partial_j(b_i)\|_\infty |\nabla_k f_s(x)|^2.
\end{aligned}$$

Finally, for the last sum we have

$$\begin{aligned}
2 \sum_{i,j=1}^N \sum_{\alpha \in R_+} k_\alpha \alpha_j A_\alpha(b_i)(x) T_j f_s(x) T_i f_s(\sigma_\alpha x) \\
&= 2 \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \nabla_k f_s(x) \rangle \langle A_\alpha(b)(x), \nabla_k f_s(\sigma_\alpha x) \rangle \\
&\leq \sqrt{2} \max_{\alpha \in R_+} \|A_\alpha(b)\|_\infty \sum_{\alpha \in R_+} k_\alpha [|\nabla_k f_s(x)|^2 + |\nabla_k f_s(\sigma_\alpha x)|^2].
\end{aligned}$$

Summing the above computations over all $g \in G$ we obtain

$$\begin{aligned}
\frac{d}{ds} P_{t-s}(\tilde{\Gamma}(f_s)) &\leq \sum_{g \in G} P_{t-s}(I(f_s) \circ g) \\
&\leq 2 \left(\max_i \sup_{x \in \mathbb{R}^N} \partial_i(b_i)(x) + (N-1) \max_{i \neq j} \|\partial_j(b_i)\|_\infty + \sqrt{2} \gamma \max_{\alpha \in R_+} \|A_\alpha(b)\|_\infty \right) P_{t-s}(\tilde{\Gamma}(f_s)).
\end{aligned}$$

So we have proved that

$$\frac{d}{ds} P_{t-s}(\tilde{\Gamma}(f_s)) \leq 2\eta P_{t-s}(\tilde{\Gamma}(f_s)),$$

where

$$\eta := \max_i \sup_{x \in \mathbb{R}^N} \partial_i(b_i)(x) + (N-1) \max_{i \neq j} \|\partial_j(b_i)\|_\infty + \sqrt{2}\gamma \max_{\alpha \in R_+} \|A_\alpha(b)\|_\infty.$$

Integrating this inequality, we obtain

$$\tilde{\Gamma}(P_t f) \leq e^{2\eta t} P_t(\tilde{\Gamma}(f)),$$

as required. $\square$

Example 7.1.3. If $b(x) = -cx$, for a constant $c > 0$, then it is clear that it satisfies the assumption (7.2) of the Theorem. Moreover, $\partial_i(b_j) = -c\delta_{ij}$ and

$$A_\alpha(b_i)(x) = c \frac{-x_i + (x - \langle \alpha, x \rangle \alpha)_i}{\langle \alpha, x \rangle} = -c\alpha_i,$$

so

$$\eta = -c + 0 + \sqrt{2}\gamma \max_\alpha |c\alpha| = -c + 2c\gamma.$$

Thus, if $\gamma < \frac{1}{2}$, the inequality (7.3) proved above is in fact a coercive bound.

Remark. It is possible to extend the previous example of linear drifts by adding small perturbations and such that the conditions of the previous Theorem still hold.

7.1.2 INVARIANT MEASURES

The strategy for proving existence of invariant measure is to use general results from probability theory such as the Prokhorov and Krylov-Bogoliubov theorems. For the convenience of the reader, in Appendix B we include a short introduction to the probability theory necessary in this chapter. A useful tool in proving the tightness conditions in these theorems is that of a Lyapunov function.

Definition 7.1.4. We say that a smooth function $\rho : \mathbb{R}^N \rightarrow [0, \infty]$ is a **Lyapunov function** for L if it satisfies the conditions:

- (i) $\rho^{-1}([0, \infty)) \neq \emptyset$;
- (ii) for any $M > 0$, the level set $\{x \in \mathbb{R}^N : \rho(x) \leq M\}$ is compact;
- (iii) there exist positive constants C_1, C_2 , with $C_2 \neq 0$, such that

$$L\rho(x) \leq C_1 - C_2\rho(x) \tag{7.4}$$

holds for all $x \in \mathbb{R}^N$ for which $\rho(x) \neq \infty$.

We then have the following easy consequence.

Lemma 7.1.5. *Let ρ be a Lyapunov function. Then, for any $x \in \mathbb{R}^N$ for which $\rho(x) \neq \infty$, $P_t \rho(x)$ is bounded in t , i.e., there exists a constant $C > 0$ such that*

$$P_t \rho(x) \leq C \quad \text{for all } t > 0.$$

Here, $P_t \rho(x)$ should be understood as the limit

$$P_t \rho(x) = \lim_{a \rightarrow \infty} P_t(\rho_a)(x),$$

where $\rho_a(x) = \min\{\rho(x), a\}$.

Proof. Note that by (7.4) we have

$$L\rho_a(y) \leq C_1 - C_2\rho_a(y)$$

holds for all $y \in \mathbb{R}^N$ if $a > \frac{C_1}{C_2}$. Since $\rho_a \in C_b(\mathbb{R}^N)$, then $P_t(\rho_a)$ is well defined and, moreover, we have

$$\frac{d}{dt} P_t(\rho_a) = P_t(L\rho_a) \leq C_1 - C_2 P_t(\rho_a).$$

Integrating this inequality, we obtain

$$P_t(\rho_a)(x) \leq e^{-C_2 t} \rho_a(x) + \frac{C_1}{C_2} (1 - e^{-C_2 t}) \leq \rho_a(x) + \frac{C_1}{C_2}.$$

Taking $a \rightarrow \infty$ we have

$$P_t(\rho) \leq \rho(x) + \frac{C_1}{C_2}.$$

Thus, the conclusion follows by taking $C = \rho(x) + \frac{C_1}{C_2}$ (recall that x is fixed). $\square$

Let us now construct a Lyapunov function as a cutoff of the euclidean distance. To this end, let $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a smooth function such that

$$\chi(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ 1 & \text{if } t \geq 2. \end{cases}$$

Consider then the function

$$\rho(x) = |x| \chi(|x|). \tag{7.5}$$

Then clearly ρ satisfies the conditions (i) and (ii) of Definition 7.1.4. In order to satisfy assumption (7.4) as well, we need to find suitable bounds on ρ . We first compute, using (1.2),

$$\begin{aligned}
\Delta_k \rho(x) &= \Delta \rho(x) + 2 \sum_{\alpha \in R_+} k_\alpha \frac{\langle \nabla \rho(x), \alpha \rangle}{\langle \alpha, x \rangle} \\
&= \sum_{i=1}^N \partial_i \left(\frac{x_i}{|x|} \chi(|x|) + x_i \chi'(|x|) \right) + 2\gamma \left(\frac{1}{|x|} \chi(|x|) + \chi'(|x|) \right) \\
&= \sum_{i=1}^N \left(\frac{1}{|x|} \chi(|x|) - \frac{x_i^2}{|x|^3} \chi(|x|) + \frac{x_i^2}{|x|^2} \chi'(|x|) + \chi'(|x|) + \frac{x_i^2}{|x|} \chi''(|x|) \right) \\
&\quad + 2\gamma \left(\frac{1}{|x|} \chi(|x|) + \chi'(|x|) \right) \\
&= (N + 2\gamma - 1) \frac{1}{|x|} \chi(|x|) + (N + 2\gamma + 1) \chi'(|x|) + |x| \chi''(|x|),
\end{aligned}$$

and we notice that, from the choice of χ , there exists a constant $C_1 \geq 0$ such that

$$\Delta_k \rho \leq C_1.$$

Thus, we have

$$\begin{aligned}
L\rho(x) &\leq C_1 + \sum_{i=1}^N b_i(x) \partial_i \rho(x) \\
&= C_1 + \langle x, b(x) \rangle \left(\frac{1}{|x|} \chi(|x|) + \chi'(|x|) \right) \\
&\leq C_2 + \frac{\langle x, b(x) \rangle}{|x|^2} \rho(x),
\end{aligned}$$

for a constant $C_2 \geq 0$. To summarise, we have proved the following Lemma.

Lemma 7.1.6. *Assume that there exists a constant $C > 0$ such that*

$$\frac{\langle x, b(x) \rangle}{|x|^2} \leq -C \quad \text{for all } x \in \mathbb{R}^N.$$

Then, the function ρ defined by (7.5) is a Lyapunov function.

Example 7.1.7. In the case of linear drifts $b(x) = -cx$ with $c > 0$, we have

$$\frac{\langle x, b(x) \rangle}{|x|^2} = -c,$$

so the function ρ does satisfy the assumption of the Lemma, so it is a Lyapunov function.

We are now ready to prove the main result of this section.

Theorem 7.1.8. *Let $\gamma < \frac{1}{2}$. Assume that there exists $\eta < 0$ such that the following inequality holds*

$$\tilde{\Gamma}(P_t f) \leq e^{2\eta t} P_t(\tilde{\Gamma}(f)).$$

Moreover, assume that there exists a Lyapunov function ρ .

Then, there exists a sequence $(P_{t_l})_{l \geq 0}$ and a probability measure ν such that for all f bounded and continuous, and for all $x \in \mathbb{R}^N$ such that $\rho(x) \neq \infty$, we have

$$P_{t_l} f(x) \rightarrow \int f d\nu \quad \text{as } l \rightarrow \infty.$$

Additionally, the measure ν is invariant for the Markov semigroup P_t .

Proof. Fix $x \in \rho^{-1}([0, \infty))$ and define the probability measures

$$p_t^x(A) := P_t(\mathbf{1}_A)(x).$$

Step 1: $(p_t^x)_{t \geq 0}$ **is a tight family.** By Markov's inequality we have, for any $M > 0$,

$$p_t^x(\{\rho \geq M\}) \leq \frac{1}{M} \int \rho dp_t^x = \frac{1}{M} P_t \rho(x) \leq \frac{C}{M},$$

where in the last step we used Lemma 7.1.5. This implies that

$$p_t^x(\{\rho \leq M\}) \geq 1 - \frac{C}{M}. \tag{7.6}$$

Since the set $\{\rho \leq M\}$ is compact, and $M > 0$ is arbitrary, this shows that the family $(p_t^x)_{t \geq 0}$ is tight. By Prokhorov's theorem, there exists a sequence $(t_l)_{l \geq 0}$ such that $(p_{t_l}^x)_{l \geq 0}$ converges weakly to a probability measure, say ν .

Step 2: $p_t^y \Rightarrow \nu$ **for all** $y \in \rho^{-1}([0, \infty))$.

Let $\gamma_{x,y}(s) = x + s(y - x)$. We first compute

$$\begin{aligned}
|P_{t_l}f(y) - P_{t_l}f(x)| &= \left| \int_0^1 \nabla(P_{t_l}f)(\gamma_{x,y}(s)) \cdot (y - x) \, ds \right| \\
&\leq |x - y| \int_0^1 |\nabla(P_{t_l}f)(\gamma_{x,y}(s))| \, ds \\
&= |x - y| \int_0^1 \left| \nabla_k(P_{t_l}f)(\gamma_{x,y}(s)) - \sum_{\alpha \in R_+} k_\alpha \alpha \frac{P_{t_l}f(\gamma_{x,y}(s)) - P_{t_l}f(\sigma_\alpha(\gamma_{x,y}(s)))}{\langle \alpha, \gamma_{x,y}(s) \rangle} \right| \, ds \\
&\leq |x - y| \int_0^1 |\nabla_k(P_{t_l}f)(\gamma_{x,y}(s))| \, ds \\
&\quad + \sqrt{2}|x - y| \sum_{\alpha \in R_+} k_\alpha \int_0^1 \left| \frac{P_{t_l}f(\gamma_{x,y}(s)) - P_{t_l}f(\sigma_\alpha(\gamma_{x,y}(s)))}{\langle \alpha, \gamma_{x,y}(s) \rangle} \right| \, ds.
\end{aligned}$$

From our assumption, we have

$$|\nabla_k(P_{t_l}f)| \leq \sqrt{\tilde{\Gamma}(P_{t_l}f)} \leq e^{\eta t} \left\| \tilde{\Gamma}(f) \right\|_\infty^{1/2}, \quad (7.7)$$

so the above becomes

$$\begin{aligned}
|P_{t_l}f(y) - P_{t_l}f(x)| &\leq |x - y| e^{\eta t_l} \left\| \tilde{\Gamma}(f) \right\|_\infty^{1/2} \\
&\quad + \sqrt{2}|x - y| \sum_{\alpha \in R_+} k_\alpha \int_0^1 \left| \frac{P_{t_l}f(\gamma_{x,y}(s)) - P_{t_l}f(\sigma_\alpha(\gamma_{x,y}(s)))}{\langle \alpha, \gamma_{x,y}(s) \rangle} \right| \, ds.
\end{aligned} \quad (7.8)$$

Without loss of generality we can assume that $x = 0$ and we will use an iteration argument applying (7.8) repeatedly. We first have

$$\begin{aligned}
|P_{t_l}f(y) - P_{t_l}f(0)| &\leq |y| e^{\eta t_l} \left\| \tilde{\Gamma}(f) \right\|_\infty^{1/2} \\
&\quad + \sqrt{2}|y| \sum_{\alpha \in R_+} \frac{k_\alpha}{|\langle \alpha, y \rangle|} \int_0^1 \frac{1}{s} |P_{t_l}f(sy) - P_{t_l}f(\sigma_\alpha(sy))| \, ds.
\end{aligned} \quad (7.9)$$

Applying (7.8) again, noting that $|sy - \sigma_\alpha(sy)| = s\sqrt{2}|\langle \alpha, y \rangle|$, we have

$$\begin{aligned}
|P_{t_l}f(y) - P_{t_l}f(0)| &\leq |y| e^{\eta t_l} \left\| \tilde{\Gamma}(f) \right\|_\infty^{1/2} \left(1 + 2 \sum_{\alpha \in R_+} k_\alpha \right) \\
&\quad + 2^{3/2}|y| \sum_{\alpha \in R_+} \sum_{\beta \in R_+} k_\alpha k_\beta \int_0^1 \int_0^1 \left| \frac{P_{t_l}f(\gamma_{\sigma_\alpha(sy), sy}(u)) - P_{t_l}f(\sigma_\beta(\gamma_{\sigma_\alpha(sy), sy}(u)))}{\langle \beta, \gamma_{\sigma_\alpha(sy), sy}(u) \rangle} \right| \, du \, ds.
\end{aligned}$$

The next step is already too difficult to write down, but we can see that we obtain

$$\begin{aligned} |P_{t_l}f(y) - P_{t_l}f(0)| &\leq |y|e^{\eta t_l} \left\| \tilde{\Gamma}(f) \right\|_{\infty}^{1/2} \sum_{i=0}^{\infty} \left[2^i \sum_{\alpha_1 \in R_+} \sum_{\alpha_2 \in R_+} \dots \sum_{\alpha_i \in R_+} k_{\alpha_1} k_{\alpha_2} \dots k_{\alpha_i} \right] \\ &= |y|e^{\eta t_l} \left\| \tilde{\Gamma}(f) \right\|_{\infty}^{1/2} \sum_{i=0}^{\infty} (2\gamma)^i. \end{aligned}$$

Therefore, if $\gamma < \frac{1}{2}$, the infinite sum converges and we have

$$|P_{t_l}f(y) - P_{t_l}f(0)| \leq C|y|e^{\eta t_l} \left\| \tilde{\Gamma}(f) \right\|_{\infty}^{1/2}. \quad (7.10)$$

Since $\eta < 0$, this shows that $p_{t_l}^y \Rightarrow \nu$, as required.

Step 3: ν is an invariant measure for P_t . We will now apply the Krylov-Bogoliubov theorem. We have, using (7.6),

$$\mu_T^x(\{\rho \leq M\}) = \frac{1}{T} \int_0^T p_t^x(\{\rho \leq M\}) dt \geq 1 - \frac{C}{M},$$

so the family $(\mu_T^x)_{T>0}$ is tight. By the Krylov-Bogoliubov theorem, there exists an invariant measure for P_t , say μ .

Fix a bounded and continuous function f and let $\epsilon > 0$. Then, since $p_{t_l}^x \Rightarrow \nu$, there exists $l_1 > 0$ such that

$$|p_{t_l}^x(f) - \nu(f)| < \frac{\epsilon}{2} \quad \text{for all } l \geq l_1.$$

On the other hand, since μ is an invariant measure, we have

$$|p_{t_l}^x(f) - \mu(f)| \leq \int |P_{t_l}f(x) - P_{t_l}f(y)| d\mu(y) \leq C \left\| \tilde{\Gamma}(f) \right\|_{\infty}^{1/2} e^{\eta t_l} \int |x - y| d\mu(y),$$

where, in the second step, we used (7.10). Since $\eta < 0$, there exists $k_2 > 0$ such that

$$|p_{t_l}^x(f) - \mu(f)| < \frac{\epsilon}{2} \quad \text{for all } l > l_2.$$

Thus, for all $l > \max\{l_1, l_2\}$, we have

$$|\mu(f) - \nu(f)| \leq |p_{t_l}^x(f) - \mu(f)| + |p_{t_l}^x(f) - \nu(f)| < \epsilon,$$

which shows that $\mu = \nu$. This finishes the proof. $\square$

Remark. We note that we needed to impose conditions for the coefficients k_{α} to be small both in order to obtain $\eta < 0$, as well as in the proof of existence of invariant measure. It would be interesting to find out whether this is just a shortfall of our chosen method,

or whether this is a deeper characteristic of Dunkl operators.

Example 7.1.9. If $b(x) = -cx$, for some $c > 0$, then an invariant measure for the resulting semigroup is given by $d\nu(x) = e^{-c|x|^2/2} d\mu_k(x)$. Indeed, we have

$$\begin{aligned} \int_{\mathbb{R}^N} Lf \, d\nu &= \int_{\mathbb{R}^N} (\Delta_k f - cx \cdot \nabla_k f) e^{-c|x|^2/2} d\mu_k \\ &= - \int_{\mathbb{R}^N} \left(\nabla \left(-\frac{c|x|^2}{2} \right) - cx \right) \cdot \nabla_k f \, d\nu = 0. \end{aligned}$$

7.2 THE INFINITE DIMENSIONAL CASE

In this section we work over the infinite dimensional space $\Omega = (\mathbb{R}^N)^{\mathbb{Z}^d}$. An element of this set will be denoted by $\omega = (\omega_l)_{l \in \mathbb{Z}^d}$, where each $\omega_l \in \mathbb{R}^N$. On the lattice $\mathbb{Z}^d$ we define the distance $|l - j| = d(l, j) := \sum_{i=1}^d |l_i - j_i|$. A cylinder function on Ω is a smooth function $f : \Omega \rightarrow \mathbb{R}$ that only depends the components ω_l for $l \in \Lambda$ and $\Lambda \subset \mathbb{Z}^d$ is a finite set. The smallest such subset $\Lambda \subset \mathbb{Z}^d$ will be denoted $\Lambda(f)$. It is known that the set of cylinder functions on Ω is dense in $\mathcal{C}(\Omega)$, see for example [52].

7.2.1 CONSTRUCTION OF THE SEMIGROUP

We want to define an infinite dimensional semigroup on $(\mathbb{R}^N)^{\mathbb{Z}^d}$ with generator

$$\mathcal{L} = \sum_{l \in \mathbb{Z}^d} L^{(l)} + \sum_{l \in \mathbb{Z}^d} e^{(l)} \cdot \nabla_k^{(l)},$$

where

$$L^{(l)} = \Delta_k^{(l)} + b^{(l)} \cdot \nabla_k^{(l)}$$

is a copy of the operator studied in the previous section, acting only on the l component of $\omega \in (\mathbb{R}^N)^{\mathbb{Z}^d}$. Here $b^{(l)} : \Omega \rightarrow \mathbb{R}^N$ is a function that only depends on the ω_l component. In addition, $e^{(l)} : \Omega \rightarrow \mathbb{R}^N$ is a C^1 function which depends only on ω_j for $|j - l| < R$ for some fixed $R > 0$ (we say that $e^{(l)}$ has finite range of interaction), and such that $e^{(l)}$ and all its first degree derivatives are uniformly bounded. As in the previous section, we assume that for any $l \in \mathbb{Z}^d$ we have

$$\frac{2}{\langle \alpha, \omega_l \rangle^2} - \langle \alpha, b^{(l)}(\omega) \rangle \frac{1}{\langle \alpha, \omega_l \rangle} \geq 0 \quad \forall \omega \in \Omega, \forall \alpha \in R_+ \quad (7.11)$$

and

$$\frac{2}{\langle \alpha, \omega_l \rangle^2} - \langle \alpha, b^{(l)}(\omega) + e^{(l)}(\omega) \rangle \frac{1}{\langle \alpha, \omega_l \rangle} \geq 0 \quad \forall \omega \in \Omega, \forall \alpha \in R_+. \quad (7.12)$$

The operators $L^{(l)}$ are commutative and, as in the previous section, each generates a finite dimensional semigroup. Thus, their sum $\sum_{l \in \mathbb{Z}^d} L^{(l)}$ also generates a semigroup. In this case, the dynamics is given by infinitely many copies of the same model acting independently of each other. Introducing the terms corresponding to the functions $e^{(l)}$ makes the dynamics more interesting as it allows for interactions between the individual diffusions.

In order to define this semigroup formally, we first consider the truncated operator for some finite $\Lambda \subset \mathbb{Z}^d$,

$$\mathcal{L}_\Lambda := \sum_{l \in \mathbb{Z}^d} L^{(l)} + \sum_{l \in \Lambda} e^{(l)} \cdot \nabla_k^{(l)}.$$

This operator is well defined on the space of cylinder functions on Ω and we assume that it generates a Markov semigroup $(P_t^\Lambda)_{t \geq 0}$. Since Λ is finite and by the finite range of interactions assumption, then in fact only the diffusions corresponding to the points of $\Lambda^R := \{l \in \mathbb{Z}^d : d(l, \Lambda) < R\}$ interact with each other. Thus, we can split $\mathcal{L}_\Lambda$ into two components

$$\mathcal{L}_\Lambda = \sum_{l \in \mathbb{Z}^d \setminus \Lambda^R} L^{(l)} + \left(\sum_{l \in \Lambda^R} L^{(l)} + \sum_{l \in \Lambda} e^{(l)} \cdot \nabla_k^{(l)} \right).$$

As discussed above, the operators $L^{(l)}$ are commutative, so the first part generates a semigroup of diffusions that do not interact with each other, while the second part is essentially finite dimensional and it can be dealt with as in the previous section.

Define the carré-du-champ operator

$$\Gamma_\Lambda(f) := \frac{1}{2} (\mathcal{L}_\Lambda(f^2) - 2f\mathcal{L}_\Lambda f).$$

Using computations similar to the finite dimensional case, we have

$$\begin{aligned} \Gamma_\Lambda(f)(\omega) &= \sum_{l \in \mathbb{Z}^d} |\nabla^{(l)} f(\omega)|^2 + \frac{1}{2} \sum_{l \in \mathbb{Z}^d \setminus \Lambda} k_\alpha A_\alpha^{(l)}(f)^2(\omega) [2 - \langle \alpha, b^{(l)}(\omega) \rangle \langle \alpha, \omega \rangle] \\ &\quad + \frac{1}{2} \sum_{l \in \Lambda} k_\alpha A_\alpha^{(l)}(f)^2(\omega) [2 - \langle \alpha, b^{(l)}(\omega) + e^{(l)}(\omega) \rangle \langle \alpha, \omega \rangle]. \end{aligned}$$

Here, in keeping with the notation above, $\nabla^{(l)}$ is the usual gradient acting only on the ω_l component of $\omega \in \Omega$, and $A_\alpha^{(l)}$ is the operator A_α acting on the same component ω_l . From the assumptions (7.11) and (7.12) we have

$$\Gamma_\Lambda(f) \geq 0.$$

We want to define the infinite semigroup for any cylindrical function f as the limit in the

uniform norm

$$P_t f = \lim_{\Lambda \rightarrow \mathbb{Z}^d} P_t^\Lambda f.$$

In order to prove that this limit exists, it is enough to show that for each increasing sequence $(\Lambda_n)_{n \in \mathbb{N}}$ of finite subsets of $\mathbb{Z}^d$, the sequence $(P_t^{\Lambda_n} f)_{n \in \mathbb{N}}$ is Cauchy. In other words, fixing a cylindrical function f , for finite $\Lambda \subset \Lambda' \subset \mathbb{Z}^d$, we want to estimate the norm

$$\left\| P_t^{\Lambda'} f - P_t^\Lambda f \right\|_\infty.$$

We assume also that Λ is large enough so that $\Lambda(f) \subset \Lambda$. We construct a sequence $\Lambda_0, \dots, \Lambda_n$, where $n = |\Lambda'| - |\Lambda|$, such that $\Lambda_0 = \Lambda$, $\Lambda_n = \Lambda'$ and for each $j = 0, \dots, n-1$, $\Lambda_{j+1} \setminus \Lambda_j = \{h_j\}$. Then, we have

$$\left\| P_t^{\Lambda'} f - P_t^\Lambda f \right\|_\infty \leq \sum_{j=0}^{n-1} \left\| P_t^{\Lambda_{j+1}} f - P_t^{\Lambda_j} f \right\|_\infty. \quad (7.13)$$

But we can compute

$$P_t^{\Lambda_{j+1}} f - P_t^{\Lambda_j} f = \int_0^t \frac{d}{ds} \left[P_{t-s}^{\Lambda_j} (P_s^{\Lambda_{j+1}} f) \right] ds = \int_0^t P_{t-s}^{\Lambda_j} (\mathcal{L}_{\Lambda_{j+1}} - \mathcal{L}_{\Lambda_j}) P_s^{\Lambda_{j+1}} f ds,$$

and since $P_t^{\Lambda_j}$ is a Markov semigroup we can estimate

$$\left| P_t^{\Lambda_{j+1}} f - P_t^{\Lambda_j} f \right| \leq \|e^{(h_j)}\|_\infty \int_0^t P_{t-s}^{\Lambda_j} \left(\left| \nabla_k^{(h_j)} (P_s^{\Lambda_{j+1}} f) \right| \right) ds$$

Similarly to the previous section, we now introduce the symmetrised gradient form

$$\tilde{\Gamma}^{(l)}(h)(\omega) = \sum_{g \in G} |\nabla_k^{(l)} h(g^{(l)} \omega)|^2, \quad (7.14)$$

defined for all $l \in \mathbb{Z}^d$, and where $g^{(l)}$ is the element g of the reflection group G acting on the $\omega_l \in \mathbb{R}^N$ coordinate of $\omega \in \Omega$. Using this definition, we have

$$\left| \nabla_k^{(h_j)} (P_s^{\Lambda_{j+1}} f) \right|^2 \leq \tilde{\Gamma}^{(h_j)}(P_s^{\Lambda_{j+1}} f),$$

and hence, from the above we deduce

$$\left\| P_t^{\Lambda_{j+1}} f - P_t^{\Lambda_j} f \right\|_\infty \leq \|e^{(h_j)}\|_\infty \int_0^t \left\| \tilde{\Gamma}^{(h_j)}(P_s^{\Lambda_{j+1}} f) \right\|_\infty^{1/2} ds. \quad (7.15)$$

Thus, we need to obtain bounds on the quantity $\left\| \tilde{\Gamma}^{(h_j)}(P_s^{\Lambda_{j+1}} f) \right\|_\infty$. This is done in the following Theorem.

Theorem 7.2.1. *Let Λ be a finite subset of $\mathbb{Z}^d$ and consider the Markov semigroup $(P_t^\Lambda)_{t \geq 0}$ generated by the operator*

$$\mathcal{L}_\Lambda := \sum_{l \in \mathbb{Z}^d} L^{(l)} + \sum_{l \in \Lambda} e^{(l)} \cdot \nabla_k^{(l)}$$

and denote $f_t := P_t^\Lambda f$. Assume that

$$\eta_l := \max_i \sup_{\omega \in \Omega} \partial_i^{(l)}(b_i^{(l)})(\omega) + (N-1) \max_{m \neq i} \|\partial_m^{(l)}(b_i)\|_\infty + \sqrt{2}\gamma \max_{\alpha \in R_+} \|A_\alpha^{(l)}(b^{(l)})\|_\infty < 0$$

and

$$\zeta := \sum_{l \in \mathbb{Z}^d} \|e^{(l)}\|_\infty < \infty.$$

Let f be a cylinder function such that $\Lambda(f) \subset \Lambda$ and let $l \notin \Lambda(f)$. Let also

$$N_l = \left\lfloor \frac{d(l, \Lambda(f))}{R} \right\rfloor + 1,$$

where the square brackets indicate integer part. Then, for any $\sigma > 0$ there exists $\tau \geq 1$ large enough such that if $N_l \geq \tau s$, and we have the bound

$$\left\| \tilde{\Gamma}^{(l)}(f_s) \right\|_\infty \leq e^{-2N_l\sigma - 2s\sigma} \sum_{j \in \mathbb{Z}^d} \left\| \tilde{\Gamma}^{(j)}(f) \right\|_\infty. \quad (7.16)$$

Proof. Fix some $i = 1, \dots, N$. We first look at

$$\begin{aligned} \partial_u P_{s-u}^\Lambda (|T_i^{(l)} f_u|^2) &= P_{s-u}^\Lambda (-\mathcal{L}_\Lambda (|T_i^{(l)} f_u|^2) + 2T_i^{(l)} f_u \cdot T_i^{(l)} \mathcal{L}_\Lambda f_u) \\ &= P_{s-u}^\Lambda (-2\Gamma_\Lambda(T_i^{(l)} f_u) + 2T_i^{(l)} f_u \cdot [T_i^{(l)}, \mathcal{L}_\Lambda] f_u) \\ &\leq P_{s-u}^\Lambda (2T_i^{(l)} f_u \cdot [T_i^{(l)}, \mathcal{L}_\Lambda] f_u), \end{aligned} \quad (7.17)$$

where we used the positivity of Γ_Λ .

We can check directly that for distinct $l, l' \in \mathbb{Z}^d$, the operators $T_i^{(l)}$ and $T_j^{(l')}$ commute for any $i, j = 1, \dots, N$. The same holds for $l = l'$, but here we need to refer to Dunkl's commutativity result (Theorem 1.2.4). As a consequence (recall also that $b^{(j)}$ depends only on the ω_j component), we have $[T_i^{(l)}, L^{(j)}] = 0$ for all $j \neq l$. Thus, we can simplify the commutator

$$\begin{aligned} [T_i^{(l)}, \mathcal{L}_\Lambda] &= [T_i^{(l)}, L^{(l)}] + \sum_{j \in \Lambda} [T_i^{(l)}, e^{(j)} \cdot \nabla_k^{(j)}] \\ &= [T_i^{(l)}, L^{(l)}] + \sum_{j \in \Lambda} [T_i^{(l)}, e^{(j)}] \cdot \nabla_k^{(j)}, \end{aligned}$$

where in the last step we used again the commutativity of $T_i^{(l)}$ with $T_u^{(j)}$. So we next need to compute $[T_i^{(l)}, e^{(j)}]$. We have

$$\begin{aligned} [T_i^{(l)}, e^{(j)}]h &= \hat{\partial}_i^{(l)}(he^{(j)}) + \sum_{\alpha \in R_+} k_\alpha \alpha_i A_\alpha^{(l)}(he^{(j)}) - e^{(j)} \hat{\partial}_i^{(l)}h - \sum_{\alpha \in R_+} k_\alpha \alpha_i A_\alpha^{(l)}(h)e^{(j)} \\ &= h \hat{\partial}_i^{(l)}(e^{(j)}) - \sum_{\alpha \in R_+} k_\alpha \alpha_i [A_\alpha^{(l)}(h)e^{(j)} - A_\alpha^{(l)}(he^{(j)})]. \end{aligned}$$

But

$$\begin{aligned} A_\alpha^{(l)}(h)e^{(j)} - A_\alpha^{(l)}(he^{(j)}) &= \frac{1}{\langle \alpha, \omega_l \rangle} [e^{(j)}(\omega)(h(\omega) - h(\sigma_\alpha^{(l)}\omega)) - (h(\omega)e^{(j)}(\omega) - h(\sigma_\alpha^{(l)}\omega)e^{(j)}(\sigma_\alpha^{(l)}\omega))] \\ &= -h(\sigma_\alpha^{(l)}\omega)A_\alpha^{(l)}(e^{(j)}). \end{aligned}$$

Thus, we have

$$[T_i^{(l)}, e^{(j)}]h = h \hat{\partial}_i^{(l)}(e^{(j)}) + \sum_{\alpha \in R_+} k_\alpha \alpha_i h(\sigma_\alpha^{(l)}\omega)A_\alpha^{(l)}(e^{(j)}).$$

Therefore, we have obtained

$$[T_i^{(l)}, \mathcal{L}_\Lambda]h = [T_i^{(l)}, L^{(l)}] + \sum_{j \in \Lambda} \left[\hat{\partial}_i^{(l)}(e^{(j)}) \cdot \nabla_k^{(j)}h + \sum_{\alpha \in R_+} k_\alpha \alpha_i A_\alpha^{(l)}(e^{(j)}) \cdot \nabla_k^{(j)}h(\sigma_\alpha^{(l)}\omega) \right]. \quad (7.18)$$

Recall the definition of the symmetrised gradient form given in (7.14). From (7.17) and (7.18) we obtain

$$\begin{aligned} \frac{d}{du} P_{s-u}^\Lambda(\tilde{\Gamma}^{(l)}(f_u)) &\leq \sum_{g \in G} \sum_{i=1}^N P_{s-u}^\Lambda \left(2T_i^{(l)} f_u(g^{(l)}\omega) [T_i^{(l)}, L^{(l)}] f_u(g^{(l)}\omega) \right) \end{aligned} \quad (7.19)$$

$$+ \sum_{g \in G} \sum_{i=1}^N \sum_{m=1}^N \sum_{j \in \Lambda} 2P_{s-u}^\Lambda \left[\left(\hat{\partial}_i^{(l)}(e_m^{(j)}) T_m^{(j)} f_u T_i^{(l)} f_u \right) \circ g^{(l)} \right] \quad (7.20)$$

$$+ \sum_{g \in G} \sum_{i=1}^N \sum_{m=1}^N \sum_{j \in \Lambda} \sum_{\alpha \in R_+} 2k_\alpha \alpha_i P_{s-u}^\Lambda \left[\left(A_\alpha^{(l)}(e_m^{(j)}) (T_m^{(j)} f_u) \circ \sigma_\alpha^{(l)} T_i^{(l)} f_u \right) \circ g^{(l)} \right]. \quad (7.21)$$

The term in (7.19) is essentially the finite dimensional case discussed in Theorem 7.1.2

since all operators involved only act on the l component. We thus have

$$\sum_{g \in G} \sum_{i=1}^N P_{s-u}^\Lambda \left(2T_i^{(l)} f_u(g^{(l)} \omega) [T_i^{(l)}, L^{(l)}] f_u(g^{(l)} \omega) \right) \leq 2\eta_l P_{s-u}^\Lambda(\tilde{\Gamma}^{(l)}(f_u)),$$

where

$$\eta_l := \max_i \sup_{\omega \in \Omega} \partial_i^{(l)}(b_i^{(l)})(\omega) + (N-1) \max_{m \neq i} \|\partial_m^{(l)}(b_i)\|_\infty + \sqrt{2}\gamma \max_{\alpha \in R_+} \|A_\alpha^{(l)}(b^{(l)})\|_\infty.$$

In order to estimate the next two terms, let

$$E_{l,j} := \max \left\{ \max_{i,m} \left\| \partial_i^{(l)}(e_m^{(j)}) \right\|_\infty, \max_{i,m,\alpha} \|A_\alpha^{(l)}(e_m^{(j)})\|_\infty \right\} \leq \sqrt{2} \max_{i,m} \left\| \partial_i^{(l)}(e_m^{(j)}) \right\|_\infty,$$

where the second inequality follows from (1.1). Clearly $E_{l,j} \geq 0$ for all $l, j \in \mathbb{Z}^d$, but note also that $E_{l,j} = 0$ if $d(l, j) \geq R$ since $e^{(l)}$ only depends on the components ω_j with $d(i, j) < R$. The strategy in estimating the remaining terms (7.20) and (7.21) is to use the inequality $2ab \leq \epsilon x^2 + \frac{1}{\epsilon} y^2$ to separate the products into sums of squares. We first estimate

$$\begin{aligned} & \sum_{g \in G} \sum_{i=1}^N \sum_{m=1}^N \sum_{j \in \Lambda} 2P_{s-u}^\Lambda \left[\left(\partial_i^{(l)}(e_m^{(j)}) T_m^{(j)} f_u T_i^{(l)} f_u \right) \circ g^{(l)} \right] \\ & \leq N \sum_{g \in G} \sum_{j \in \Lambda} E_{l,j} P_{s-u}^\Lambda \left[\epsilon |\nabla_k^{(l)} f_u|^2 \circ g^{(l)} + \frac{1}{\epsilon} |\nabla_k^{(j)} f_u|^2 \circ g^{(l)} \right] \\ & = \epsilon N \left(\sum_{j \in \Lambda} E_{l,j} \right) P_{s-u}^\Lambda(\tilde{\Gamma}^{(l)}(f_u)) + \frac{1}{\epsilon} N \sum_{g \in G} \sum_{j \in \Lambda} E_{l,j} P_{s-u}^\Lambda \left[|\nabla_k^{(j)} f_u|^2 \circ g^{(l)} \right] \\ & \leq \epsilon N \left(\sum_{j \in \Lambda} E_{l,j} \right) P_{s-u}^\Lambda(\tilde{\Gamma}^{(l)}(f_u)) + \frac{1}{\epsilon} N \sum_{g \in G} \sum_{j \in \Lambda} E_{l,j} P_{s-u}^\Lambda \left[(\tilde{\Gamma}^{(j)} f_u) \circ g^{(l)} \right] \end{aligned}$$

and similarly

$$\begin{aligned} & \sum_{g \in G} \sum_{i=1}^N \sum_{m=1}^N \sum_{j \in \Lambda} \sum_{\alpha \in R_+} 2k_\alpha \alpha_i P_{s-u}^\Lambda \left[\left(A_\alpha^{(l)}(e_m^{(j)}) (T_m^{(j)} f_u) \circ \sigma_\alpha^{(l)} T_i^{(l)} f_u \right) \circ g^{(l)} \right] \\ & \leq \sum_{g \in G} \sum_{j \in \Lambda} \sum_{\alpha \in R_+} \sqrt{2} N k_\alpha E_{l,j} P_{s-u}^\Lambda \left[\epsilon |\nabla_k^{(l)} f_u|^2 \circ g^{(l)} + \frac{1}{\epsilon} |\nabla_k^{(j)} f_u|^2 \circ (\sigma_\alpha^{(l)} \circ g^{(l)}) \right] \\ & = \sqrt{2} N \gamma \epsilon \left(\sum_{j \in \Lambda} E_{l,j} \right) P_{s-u}^\Lambda(\tilde{\Gamma}^{(l)}(f_u)) + \sqrt{2} N \gamma \frac{1}{\epsilon} \sum_{g \in G} \sum_{j \in \Lambda} E_{l,j} P_{s-u}^\Lambda \left[|\nabla_k^{(j)} f_u|^2 \circ g^{(l)} \right] \\ & \leq \sqrt{2} N \gamma \epsilon \left(\sum_{j \in \Lambda} E_{l,j} \right) P_{s-u}^\Lambda(\tilde{\Gamma}^{(l)}(f_u)) + \sqrt{2} N \gamma \frac{1}{\epsilon} \sum_{g \in G} \sum_{j \in \Lambda} E_{l,j} P_{s-u}^\Lambda \left[\tilde{\Gamma}^{(j)}(f_u) \circ g^{(l)} \right]. \end{aligned}$$

Combining all these results, we have obtained that

$$\frac{d}{du} P_{s-u}^\Lambda(\tilde{\Gamma}^{(l)}(f_u)) \leq 2\tilde{\eta}_l P_{s-u}^\Lambda(\tilde{\Gamma}^{(l)}(f_u)) + \sum_{g \in G} \sum_{j \in \Lambda} \tilde{E}_{l,j} P_{s-u}^\Lambda \left[\tilde{\Gamma}^{(j)}(f_u) \circ g^{(l)} \right], \quad (7.22)$$

where

$$\tilde{\eta}_l := \eta_l + \epsilon \frac{N}{2} (1 + \sqrt{2}\gamma) \left(\sum_{j \in \Lambda} E_{l,j} \right), \quad (7.23)$$

and

$$\tilde{E}_{l,j} := \frac{1}{\epsilon} N (1 + \sqrt{2}\gamma) E_{l,j}.$$

We choose $\epsilon > 0$ such that $\tilde{\eta}_l < 0$; this is possible because by our assumption $\eta_l < 0$. Multiplying inequality (7.22) by $e^{-2\tilde{\eta}_l u}$, and integrating with respect to u from 0 to s , we obtain

$$\tilde{\Gamma}^{(l)}(f_s) \leq e^{2\tilde{\eta}_l s} P_s^\Lambda(\tilde{\Gamma}^{(l)} f) + \sum_{g \in G} \sum_{j \in \Lambda} \tilde{E}_{l,j} \int_0^s e^{2\tilde{\eta}_l(s-u)} P_{s-u}^\Lambda \left[\tilde{\Gamma}^{(j)}(f_u) \circ g^{(l)} \right] du.$$

Taking supremum norm over this inequality, and since $\tilde{\eta}_l < 0$, we have

$$\left\| \tilde{\Gamma}^{(l)}(f_s) \right\|_\infty \leq \left\| \tilde{\Gamma}^{(l)} f \right\|_\infty + |G| \sum_{j \in \Lambda} \tilde{E}_{l,j} \int_0^s \left\| \tilde{\Gamma}^{(j)}(f_u) \right\|_\infty du. \quad (7.24)$$

Up until this point we have not used the assumption that $l \notin \Lambda(f)$ so this inequality holds for general $l \in \mathbb{Z}^d$. Since $l \notin \Lambda(f)$, then f does not depend on the ω_l coordinate and so the first term on the right hand side of inequality (7.24) vanishes.

We can improve inequality (7.24) by applying itself iteratively to the term under the integral. At the first step we obtain

$$\begin{aligned} \left\| \tilde{\Gamma}^{(l)}(f_s) \right\|_\infty &\leq |G| \sum_{j \in \Lambda} \tilde{E}_{l,j} \int_0^s \left[\left\| \tilde{\Gamma}^{(j)} f \right\|_\infty + |G| \sum_{j' \in \Lambda} \tilde{E}_{j,j'} \int_0^u \left\| \tilde{\Gamma}^{(j')} (f_v) \right\|_\infty dv \right] du \\ &\leq s |G| \sum_{j \in \Lambda} \tilde{E}_{l,j} \left\| \tilde{\Gamma}^{(j)} f \right\|_\infty + |G|^2 \sum_{j \in \Lambda} \sum_{j' \in \Lambda} \tilde{E}_{l,j} \tilde{E}_{j,j'} \int_0^s \int_0^u \left\| \tilde{\Gamma}^{(j')} (f_v) \right\|_\infty dv du. \end{aligned}$$

Note that if $d(l, \Lambda(f)) \geq R$ and $d(j, l) < R$, then $j \notin \Lambda(f)$, so the first sum vanishes again. In fact, we need to apply this method iteratively $N_l = \left\lceil \frac{d(l, \Lambda(f))}{R} \right\rceil + 1$ times in order

to get a non-trivial contribution. After infinitely many iterations, we obtain

$$\begin{aligned}\left\|\tilde{\Gamma}^{(l)}(f_s)\right\|_{\infty} &\leq \left(\frac{(sC_l)^{N_l}}{N_l!} + \frac{(sC_l)^{N_l+1}}{(N_l+1)!} + \dots\right) \sum_{j \in \mathbb{Z}^d} \left\|\tilde{\Gamma}^{(j)}(f)\right\|_{\infty} \\ &\leq \frac{(sC_l)^{N_l}}{(N_l)!} e^{sC_l} \sum_{j \in \mathbb{Z}^d} \left\|\tilde{\Gamma}^{(j)}(f)\right\|_{\infty},\end{aligned}$$

where

$$C_l = |G| \sum_{j \in \Lambda} \tilde{E}_{l,j}. \quad (7.25)$$

Note that the factorials appearing in the denominators above are a result of computing nested integrals.

Let us estimate the constant appearing in this inequality. Using the fact that for any natural number n we have $n! > \left(\frac{n}{e}\right)^n$ (which follows from the expansion of e^n), we obtain

$$\left\|\tilde{\Gamma}^{(l)}(f_s)\right\|_{\infty} \leq e^{sC_l + N_l(\log sC_l - \log N_l + 1)} \sum_{j \in \mathbb{Z}^d} \left\|\tilde{\Gamma}^{(j)}(f)\right\|_{\infty}.$$

Now fix $\sigma > 0$. There exists $\tau \geq 0$ large enough such that

$$\log \frac{C_l}{\tau} + \frac{C_l}{\tau} + 1 \leq -4\sigma. \quad (7.26)$$

If, in addition, $N_l \geq \tau s$, then

$$sC_l + N_l(\log sC_l - \log N_l + 1) \leq N_l \left(\frac{C_l}{\tau} + \log \frac{C_l}{\tau} + 1 \right) \leq -4\sigma N_l \leq -2N_l\sigma - 2s\sigma.$$

Finally, this gives

$$\left\|\tilde{\Gamma}^{(l)}(f_s)\right\|_{\infty} \leq e^{-2N_l\sigma - 2s\sigma} \sum_{j \in \mathbb{Z}^d} \left\|\tilde{\Gamma}^{(j)}(f)\right\|_{\infty}, \quad (7.27)$$

as required. $\square$

Now that we have proved the required bounds on the symmetrised gradient, we can go back to the definition of the infinite dimensional semigroup. Fix $\sigma > 0$ and let $\tilde{N} = \left\lceil \frac{d(\mathbb{Z}^d \setminus \Lambda, \Lambda(f))}{R} \right\rceil + 1$. By definition, we have $N_{h_j} \geq \tilde{N}$ for all $j = 0, \dots, n-1$. From

(7.13) and (7.15), with the help of Theorem 7.2.1, we obtain

$$\begin{aligned} \|P_t^\Lambda f - P_t^\Lambda f\|_\infty &\leq \left(\sum_{l \in \mathbb{Z}^d} \|\tilde{\Gamma}^{(l)}(f)\|_\infty \right)^{1/2} \sum_{j=0}^{n-1} \|e^{(h_j)}\|_\infty \int_0^t e^{-\sigma N_{h_j} - \sigma s} ds \\ &\leq \zeta e^{-\sigma \tilde{N}} \frac{1 - e^{\sigma t}}{\sigma} \left(\sum_{l \in \mathbb{Z}^d} \|\tilde{\Gamma}^{(l)}(f)\|_\infty \right)^{1/2}, \end{aligned} \quad (7.28)$$

which holds for all $\tilde{N} \geq \tau t$, where $\tau \geq 1$ is fixed. Recall that $\zeta = \sum_{l \in \mathbb{Z}^d} \|e^{(l)}\|_\infty$ was assumed to be finite.

This concludes the proof that for each increasing sequence $(\Lambda_n)_{n \geq 1}$ of finite subsets of $\mathbb{Z}^d$, and for each cylindrical function f , the sequence $(P_t^{\Lambda_n} f)_n$ is Cauchy. Therefore, we have proved the following Theorem.

Theorem 7.2.2. *Consider the operator*

$$\mathcal{L} := \sum_{l \in \mathbb{Z}^d} (\Delta_k^{(l)} + b^{(l)} \cdot \nabla_k^{(l)}) + \sum_{l \in \mathbb{Z}^d} e^{(l)} \cdot \nabla_k^{(l)}.$$

Assume that the coefficients $b^{(l)}$ and $e^{(l)}$ are uniformly bounded such that

$$\eta_l := \max_i \sup_{\omega \in \Omega} \partial_i^{(l)}(b_i^{(l)})(\omega) + (N-1) \max_{m \neq i} \|\partial_m^{(l)}(b_i)\|_\infty + \sqrt{2}\gamma \max_{\alpha \in R_+} \|A_\alpha^{(l)}(b^{(l)})\|_\infty < 0$$

and

$$\zeta := \sum_{l \in \mathbb{Z}^d} \|e^{(l)}\|_\infty < \infty.$$

Then, for any cylinder function f , the following limit exists in the uniform norm

$$P_t f := \lim_{\Lambda \rightarrow \mathbb{Z}^d} P_t^\Lambda f,$$

and it defines a Markov semigroup $(P_t)_{t \geq 0}$ on $C_b(\Omega)$, with infinitesimal generator given by $\mathcal{L}$.

7.2.2 INVARIANT MEASURE

In this section we discuss the existence of an invariant measure for the infinite dimensional semigroup that we have just defined. The strategy is the same as in section 7.1.2, and we start by defining a Lyapunov function ρ that is suitable to our new setting.

Let $(a_l)_{l \in \mathbb{Z}^d} \subset (0, \infty)$ be an absolutely convergent series, i.e.,

$$\sum_{l \in \mathbb{Z}^d} a_l < \infty.$$

We assume that for each $l \in \mathbb{Z}^d$ there exists a function $\rho_l : \Omega \rightarrow (0, \infty)$, which depends only on the ω_l coordinate, and such that $\omega_l \mapsto \rho_l(\omega)$ is a Lyapunov function for the operator $L^{(l)}$ on $\mathbb{R}^N$. Moreover, assume that there exist constants $C_1 \geq 0$ and $C_2 > 0$ such that

$$L^{(l)} \rho_l + e^{(l)} \cdot \nabla_k^{(l)} \rho_l \leq C_1 - C_2 \rho_l \quad \forall l \in \mathbb{Z}^d. \quad (7.29)$$

For each $0 < r \leq \infty$, define the set

$$\Omega_r := \left\{ \omega \in \Omega : \sum_{l \in \mathbb{Z}^d} a_l \rho_l(\omega) < r \right\},$$

and consider the function $\rho : \Omega_\infty \rightarrow (0, \infty)$ given by

$$\rho(\omega) := \sum_{l \in \mathbb{Z}^d} a_l \rho_l(\omega).$$

Using condition (7.29) and the fact that ρ_l only depends on the ω_l coordinate, we have

$$\mathcal{L}\rho = \sum_{l \in \mathbb{Z}^d} a_l (L^{(l)} \rho_l + e^{(l)} \cdot \nabla_k^{(l)} \rho_l) \leq \sum_{l \in \mathbb{Z}^d} a_l (C_1 - C_2 \rho_l) = C_1 \sum_{l \in \mathbb{Z}^d} a_l - C_2 \rho. \quad (7.30)$$

Then ρ is a Lyapunov function for $\mathcal{L}$.

Remark. We could take, as in section 7.1.2, $\rho_l(\omega) = |\omega_l| \chi(|\omega_l|)$ for a cut-off function χ . If $b^{(l)}$ satisfies

$$\frac{\langle \omega_l, b^{(l)}(\omega) \rangle}{|\omega_l|^2} \leq -C \quad \forall \omega_l \in \mathbb{R}^N,$$

and if the interaction coefficients $e^{(l)}$ are sufficiently small, then more generally ρ_l satisfies the assumption (7.29).

Remark. We can relax assumption (7.29) to

$$L^{(l)} \rho_l \leq C_1 - C_2 \rho_l,$$

and

$$e^{(l)} \cdot \nabla_l^{(l)} \rho_l \leq C_3 - \sum_{j \in \mathbb{Z}^d} \epsilon_{l,j} \rho_j,$$

for suitably small positive numbers $\epsilon_{l,j}$. With a little more effort, the proof below can be changed to accommodate this case.

We are ready to prove the main result of this section.

Theorem 7.2.3. *Let ρ be defined as above, and let $(P_t)_{t \geq 0}$ be the Markov semigroup with generator $\mathcal{L}$ defined in Theorem 7.2.2. Then, for any $\omega \in \Omega_\infty$ there exists a subsequence $(P_{t_j})_{j \geq 0}$, and a probability measure ν_ω such that for all $f \in C(\Omega)$ bounded cylinder functions, we have*

$$P_{t_j} f(\omega) \rightarrow \int f d\nu_\omega \quad \text{as } j \rightarrow \infty.$$

Furthermore, $\nu_\omega(\Omega_\infty) = 1$.

Proof. Fix $\omega \in \Omega_\infty$ and define

$$p_t^\omega(A) := P_t(\mathbf{1}_A)(\omega).$$

Using the same approach as in Lemma 7.1.5, from (7.30) we can show that there exists a $C > 0$ such that

$$P_t \rho(\omega) \leq C \quad \text{for all } t > 0. \quad (7.31)$$

As in the proof of Theorem 7.1.8, using (7.31) and Prokhorov's theorem, we deduce that there exists a subsequence $(p_{t_j}^\omega)_{j \geq 0}$ which converges weakly to some probability measure ν_ω .

Moreover, using Markov's inequality we have for any $r > 0$ that

$$0 \leq 1 - \nu_\omega(\Omega_r) \leq \frac{1}{r} \int \rho d\nu_\omega = \frac{1}{r} \sup_{l \in \mathbb{N}} P_{t_l} \rho(\omega) \leq \frac{C}{r},$$

where in the last step we used (7.31). Taking $r \rightarrow \infty$, we obtain $\nu_\omega(\Omega_\infty) = 1$. $\square$

For any $\epsilon > 0$ for which we have $\sum_{l \in \mathbb{Z}^d} (1 + |l|)^{-\epsilon} < \infty$, define also the set

$$\tilde{\Omega}_\epsilon := \left\{ \omega \in \Omega : \sum_{l \in \mathbb{Z}^d} \frac{|\omega_l|}{(1 + |l|)^\epsilon} < \infty \right\}.$$

Note that for ρ_l defined as in the previous remark, and with $a_l = (1 + |l|)^{-\epsilon}$, then $\tilde{\Omega}_\epsilon = \Omega_\infty$.

Remark. From the proof below it will be clear that instead of the weights $(1 + |l|)^{-\epsilon}$ we could take more general weights in the definition of $\tilde{\Omega}_\epsilon$, even of exponential decay.

Theorem 7.2.4. *Assume that $\gamma < \frac{1}{2}$. Let $(P_t)_{t \geq 0}$ be the Markov semigroup defined in Theorem 7.2.2 and assume in addition that*

$$\tilde{\eta} := \sup_{l \in \mathbb{Z}^d} \tilde{\eta}_l < 0,$$

and

$$\tilde{C} := \sup_{l \in \mathbb{Z}^d} C_l \leq -2\tilde{\eta},$$

where the constants $\tilde{\eta}_l$ and C_l were defined in (7.23) and (7.25), respectively.

For any bounded cylinder function f and any $\omega, \omega' \in \tilde{\Omega}_\epsilon$, there exists $t_0 > 0$ and a constant $C(f, \omega, \omega') < \infty$, such that

$$|P_t f(\omega) - P_t f(\omega')| \leq C(f, \omega, \omega') e^{-ct},$$

where $c > 0$ is independent of f, t, ω, ω' .

Proof. Fix a cylinder function f and $\omega, \omega' \in \tilde{\Omega}_\epsilon$. Fix $\Lambda \subset \mathbb{Z}^d$ finite and such that $\Lambda(f) \subset \Lambda$. We then have

$$\begin{aligned} |P_t f(\omega) - P_t f(\omega')| \\ \leq |P_t f(\omega) - P_t^\Lambda f(\omega)| + |P_t^\Lambda f(\omega) - P_t^\Lambda f(\omega')| + |P_t^\Lambda f(\omega') - P_t f(\omega')|. \end{aligned} \quad (7.32)$$

Using the computations in (7.28), there exists $r > 0$ such that if $\Lambda = B_{rt}(0) = \{l \in \mathbb{Z}^d : |l| < rt\}$, and letting $\Lambda' \rightarrow \infty$, then we have

$$\|P_t f - P_t^\Lambda f\|_\infty \leq C_1(f) e^{-c_1 t}, \quad (7.33)$$

which holds for all $t \geq t_0$, where t_0 is chosen such that $\Lambda(f) \subset \Lambda$. This deals with the first and the third terms on the right hand side of the inequality (7.32) above. We are thus only left to study the middle term.

If ω and ω' differ by just one coordinate, say $\omega_j = \omega'_j$ for all $j \neq l$, then let $\gamma_{\omega_l, \omega'_l}(s) = \omega_l + s(\omega'_l - \omega_l)$. We can then compute

$$\begin{aligned} |P_t^\Lambda f(\omega) - P_t^\Lambda f(\omega')| &= \left| \int_0^1 \nabla^{(l)}(P_t^\Lambda f)(\gamma_{\omega_l, \omega'_l}(s)) \cdot (\omega'_l - \omega_l) ds \right| \\ &\leq |\omega'_l - \omega_l| \int_0^1 |\nabla^{(l)}(P_t^\Lambda f)(\gamma_{\omega_l, \omega'_l}(s))| ds \\ &\leq |\omega'_l - \omega_l| \left\| \tilde{\Gamma}^{(l)}(P_t^\Lambda f) \right\|_\infty^{1/2} \\ &\quad + \sqrt{2} |\omega'_l - \omega_l| \sum_{\alpha \in R_+} k_\alpha \int_0^1 \left| \frac{P_t^\Lambda f(\gamma_{\omega_l, \omega'_l}(s)) - P_t^\Lambda f(\sigma_\alpha(\gamma_{\omega_l, \omega'_l}(s)))}{\langle \alpha, \gamma_{\omega_l, \omega'_l}(s) \rangle} \right|. \end{aligned}$$

Using the same method as in Step 2 in the proof of Theorem 7.1.8, since $\gamma < \frac{1}{2}$, then we

obtain

$$|P_t^\Lambda f(\omega) - P_t^\Lambda f(\omega')| \leq C_2 |\omega_l - \omega'_l| \left\| \tilde{\Gamma}^{(l)}(P_t^\Lambda f) \right\|_\infty^{1/2}. \quad (7.34)$$

Going back to the proof of (7.24), note that we have in fact

$$\left\| \tilde{\Gamma}^{(l)}(f_t) \right\|_\infty \leq e^{2\tilde{\eta}t} \left\| \tilde{\Gamma}^{(l)} f \right\|_\infty + |G| \sum_{j \in \Lambda} \tilde{E}_{l,j} \int_0^t \left\| \tilde{\Gamma}^{(j)}(f_u) \right\|_\infty du.$$

Using the same iteration procedure as above, this implies that

$$\left\| \tilde{\Gamma}^{(l)}(f_t) \right\|_\infty \leq e^{2\tilde{\eta}t + tC_l} \sum_{j \in \mathbb{Z}^d} \left\| \tilde{\Gamma}^{(j)}(f) \right\|_\infty. \quad (7.35)$$

By our assumption $2\tilde{\eta} + C_l \leq 2\tilde{\eta} + \tilde{C} < 0$, so this inequality provides the sufficient decay required in (7.34).

We now turn to the case of general $\omega, \omega' \in \tilde{\Omega}_\epsilon$. First, we note that if $\Lambda(f) \subset \Lambda$, then by (7.24) we have $\Lambda(P_t^\Lambda f) \subset \Lambda^R := \{l \in \mathbb{Z}^d : d(l, \Lambda) < R\}$, where R is the range of interaction. But Λ^R is a finite set, say $|\Lambda^R| = n$, and let $l_1, \dots, l_n$ be its distinct elements. We construct the sequence $\omega^0 = \omega$, $\omega_j^{i+1} = \omega_j^i$ for all $j \neq l_i$, and $\omega_{l_i}^{i+1} = \omega'_{l_i}$. In other words, each two consecutive terms of the sequence differ in just one coordinate, l_i , and $\omega_j^n = \omega'_j$ for all $j \in \Lambda^R$. Thus, by the observation above, $P_t^\Lambda(\omega^n) = P_t^\Lambda(\omega')$, and so we have

$$|P_t^\Lambda f(\omega) - P_t^\Lambda f(\omega')| \leq \sum_{i=0}^{n-1} |P_t^\Lambda f(\omega^{i+1}) - P_t^\Lambda f(\omega^i)|.$$

We estimate each term in this sum using the same method as above. More precisely, using (7.34) and (7.35), we have

$$\begin{aligned} \sum_{i=0}^{n-1} |P_t^\Lambda f(\omega^{i+1}) - P_t^\Lambda f(\omega^i)| &\leq C_2 \sum_{i=0}^{n-1} |\omega_{l_i} - \omega'_{l_i}| e^{(\tilde{\eta} + \tilde{C}/2)t} \left(\sum_{j \in \mathbb{Z}^d} \left\| \tilde{\Gamma}^{(j)}(f) \right\|_\infty \right)^{1/2} \\ &\leq C_3(f) e^{(\tilde{\eta} + \tilde{C}/2)t} \sum_{l \in B_{R+tR}(0)} \frac{|\omega_l| + |\omega'_l|}{(1+|l|)^\epsilon} (1+|l|)^\epsilon \\ &\leq C_3(f) (C_\omega + C_{\omega'}) e^{(\tilde{\eta} + \tilde{C}/2)t} (1+|R+rt|)^\epsilon \end{aligned} \quad (7.36)$$

where $C_\omega = \sum_{l \in \mathbb{Z}^d} \frac{|\omega_l|}{(1+|l|)^\epsilon}$ and $C_{\omega'} = \sum_{l \in \mathbb{Z}^d} \frac{|\omega'_l|}{(1+|l|)^\epsilon}$ which are both finite since $\omega, \omega' \in \tilde{\Omega}_\epsilon$.

From (7.33) and (7.36) we obtain the decay we require. $\square$

CONCLUSIONS AND FUTURE WORK

When we started the work contained in this thesis, not much was available in the literature in terms of functional inequalities for Dunkl operators. The few available results were usually obtained as consequences of work of a different nature. Our intention was to begin a systematic study of functional inequalities that would emphasise each inequality separately as an abstract result of independent interest rather than as a tool for solving the problem at hand. We believe we have fulfilled this aim by studying several families of important inequalities.

However, the work is far from being complete and several questions remain. Of course, there exist practically unlimited inequalities that could still be studied, but let us highlight a few questions and improvements that stem directly from the work we presented in this thesis.

The Sobolev inequality was established in full generality, but the study of optimal constants is still very difficult. Even in the case $p = 2$ considered in this thesis, the exact value of the optimal constant is still not known, although we have obtained some bounds. In more general cases $p \neq 2$ there is not information at all.

In fact, several results in this thesis relied on the very useful inequality of Lemma 2.1.2 which we obtained using carré-du-champ methods suitable only in the case $p = 2$. A more general result of the form

$$\|\nabla f\|_p \leq \|\nabla_k f\|_p,$$

apart from its intrinsic interest, would allow us to generalise some results presented here to cases other than $p = 2$.

In studying the Poincaré inequality for Dunkl operators in section 6.4, we relied on Muckenhoupt weights and an additional assumption that $\gamma < \frac{1}{2}$ was necessary. It would be interesting to determine if this assumption is necessary, or it is simply a limitation due to our chosen method. A similar issue appears in the study of existence of invariant measure in section 7.1.2, when we also required the assumption $\gamma < \frac{1}{2}$, and it is not clear if this is in fact necessary.

The important example throughout the whole Chapter 7 was that of linear drifts and small perturbations thereof, but we believe other interesting examples could be considered. It

is important first to check for which drifts b the operator $L := \Delta_k + b \cdot \nabla_k$ generates indeed a Markov semigroup. Once this is established, our assumptions for obtaining gradient bounds are quite general and they could accommodate some more of these examples.

Finally, now that we have significant progress in the topic of functional inequalities, the next important step would be applying these results to various problems in Dunkl theory. For example, methods based on the inequalities we have proved in this thesis could be employed to study the Dunkl heat kernel.

Another direction in which the results presented here could be useful is that of partial differential equations containing Dunkl operators. We have already included such an example in Chapter 5 where we studied a nonlinear damped wave equation, but much more than that should be possible to achieve.

As highlighted in the introduction, there is an important connection between Dunkl operators and mathematical physics, via the Calogero-Moser-Sutherland model, and it would be interesting to see how the results presented above could be applied in this setting. For example, multi-particle Hardy inequalities have received a lot of interest lately (see, for example, [56], [74] and [73]) and it might be possible to obtain these from the general Hardy inequalities discussed in this thesis.



PROOF OF THEOREM 6.4.6

This appendix complements the presentation of Muckenhoupt weights in chapter 6.4 by providing a proof of Theorem 6.4.6. We will follow the exposition in [55, Chapter 15].

Theorem A.1 ([39]). *Let $1 < p < \infty$ and $w \in A_p$. Then there exists a constant $C > 0$ such that for all balls $B_r(x)$ of centre x and radius $r > 0$, and for all functions f defined on $B_r(x)$ that are Lipschitz continuous, we have the inequality*

$$\int_{B_r(x)} |f(y) - f_{B_r(x)}^w|^p w(y) \, dy \leq Cr^p \int_{B_r(x)} |\nabla f(y)|^p w(y) \, dy,$$

where $f_{B_r(x)}^w = \frac{1}{w(B_r(x))} \int_{B_r(x)} f(z)w(z) \, dz$ is the weighted average of f over the ball $B_r(x)$.

Before we prove this result, we recall Muckenhoupt's Theorem (Theorem 6.4.2) in a slightly different form.

Theorem A.2. *Let $1 < p < \infty$ and let w be a locally integrable function on $\mathbb{R}^N$. Then $w \in A_p$ if and only if there exists a constant $A > 0$ such that for any ball $B \subset \mathbb{R}^N$ we have*

$$\int_B |Mf(x)|^p w(x) \, dx \leq A \int_B |f(x)|^p w(x) \, dx$$

holds for all $f \in L^p(w(x) \, dx)$.

Proof of Theorem A.1. First we show that

$$\int_{B_r(x)} |f(y) - f_{B_r(x)}|^p w(y) \, dy \leq Cr^p \int_{B_r(x)} |\nabla f(y)|^p w(y) \, dy, \quad (\text{A.1})$$

where $f_{B_r(x)} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) dy$ is the average of f with respect to the Lebesgue measure.

It is convenient to see f as function defined over the whole space $\mathbb{R}^N$ so we extend it by 0 outside $B_r(x)$. It is known (see, for example, [92, Lemma 1.4.3]) that

$$|f(y) - f_{B_r(x)}| \leq C \int_{B_r(x)} \frac{|\nabla f(z)|}{|y - z|^{N-1}} dz = CI(|\nabla f|)(y), \quad (\text{A.2})$$

where

$$Iu(y) = \int_{\mathbb{R}^N} \frac{|u(z)|}{|z - y|^{N-1}} dz$$

is the Riesz potential.

Choose any $q > 1$ such that $\inf\{s > 1 : w \in A_s\} < q < p < Nq$ and let $\kappa = \frac{N}{N-q} > 1$. We will show that

$$\left(\frac{1}{w(B_r(x))} \int_{B_r(x)} |Iu(y)|^{\kappa p} w(y) dy \right)^{\frac{1}{\kappa p}} \leq Cr \left(\frac{1}{w(B_r(x))} \int_{B_r(x)} |u(y)|^p w(y) dy \right)^{\frac{1}{p}} \quad (\text{A.3})$$

holds for any $u \in C(B_r(x))$.

For any $\epsilon > 0$ and $y \in \mathbb{R}^N$ we have, expanding u by 0 outside $B_r(x)$, that

$$\begin{aligned} \int_{|y-z| \leq \epsilon} \frac{|u(z)|}{|z - y|^{N-1}} dz &= \sum_{j=1}^{\infty} \int_{\{2^{-j}\epsilon \leq |y-z| \leq 2^{-j+1}\epsilon\}} \frac{|u(z)|}{|z - y|^{N-1}} dz \\ &\leq \sum_{j=1}^{\infty} (2^{-j}\epsilon)^{-N+1} \int_{\{2^{-j}\epsilon \leq |y-z| \leq 2^{-j+1}\epsilon\}} |u(z)| dz \\ &\leq \sum_{j=1}^{\infty} (2^{-j}\epsilon)^{-N+1} \int_{B_{2^{-j+1}\epsilon}(y)} |u(z)| dz \\ &= C\epsilon \sum_{j=1}^{\infty} 2^{-j} \frac{1}{|B_{2^{-j+1}\epsilon}(y)|} \int_{B_{2^{-j+1}\epsilon}(y)} |u(z)| dz \\ &\leq C\epsilon M u(y) \end{aligned}$$

On the other hand, by applying Hölder's inequality first with coefficients p and $\frac{p}{p-1}$, and

then $\frac{p-1}{p-q}$ and $\frac{p-1}{q-1}$, we have

$$\begin{aligned}
& \int_{|y-z|>\epsilon} \frac{|u(z)|}{|z-y|^{N-1}} dz \\
& \leq \left(\int_{B_r(x)} |u(z)|^p w(z) dz \right)^{1/p} \left(\int_{\{|y-z|>\epsilon\} \cap B_r(x)} |z-y|^{-\frac{p(N-1)}{p-1}} w(z)^{-\frac{1}{p-1}} dz \right)^{\frac{p-1}{p}} \\
& \leq \left(\int_{B_r(x)} |u(z)|^p w(z) dz \right)^{1/p} \left(\int_{|y-z|>\epsilon} |z-y|^{-\frac{p(N-1)}{p-q}} dz \right)^{\frac{p-q}{p}} \left(\int_{B_r(x)} w(z)^{-\frac{1}{q-1}} dz \right)^{\frac{q-1}{p}} \\
& = C_2 \epsilon^{\frac{p-Nq}{p}} \left(\int_{B_r(x)} |u(z)|^p w(z) dz \right)^{1/p} \left(\int_{B_r(x)} w(z)^{-\frac{1}{q-1}} dz \right)^{\frac{q-1}{p}}.
\end{aligned}$$

At this point, ϵ is an arbitrary positive number and we are free to choose it conveniently. Thus, putting these two computations together, and choosing

$$\epsilon = \left(\frac{\left(\int_{B_r(x)} |u(z)|^p w(z) dz \right)^{1/p} \left(\int_{B_r(x)} w(z)^{-\frac{1}{q-1}} dz \right)^{\frac{q-1}{p}}}{Mu(y)} \right)^{\frac{p}{Nq}},$$

we obtain

$$Iu(y) \leq C(Mu(y))^{1-\frac{p}{Nq}} \left(\int_{B_r(x)} |u(z)|^p w(z) dz \right)^{\frac{1}{Nq}} \left(\int_{B_r(x)} w(z)^{-\frac{1}{q-1}} dz \right)^{\frac{q-1}{Nq}}.$$

Thus, it follows that

$$\begin{aligned}
& \left(\frac{1}{w(B_r(x))} \int_{B_r(x)} |Iu(y)|^{\kappa p} w(y) dy \right)^{\frac{1}{\kappa p}} \\
& \leq C w(B_r(x))^{-\frac{1}{\kappa p}} \left(\int_{B_r(x)} |Mu(y)|^p w(y) dy \right)^{\frac{1}{\kappa p}} \left(\int_{B_r(x)} |u(y)|^p w(y) dy \right)^{\frac{1}{Nq}} \\
& \quad \cdot \left(\int_{B_r(x)} w(z)^{-\frac{1}{q-1}} dz \right)^{\frac{q-1}{Nq}} \\
& \leq Cr \left(\frac{1}{w(B_r(x))} \int_{B_r(x)} |u(y)|^p w(y) dy \right)^{\frac{1}{p}},
\end{aligned}$$

where we used Theorem A.2 and the definition of $w \in A_q$. This completes the proof of (A.3).

Going back to the proof of the Theorem, from (A.3) and (A.2) we have

$$\begin{aligned} \left(\frac{1}{w(B_r(x))} \int_{B_r(x)} |f(y) - f_{B_r(x)}|^{\kappa p} w(y) \, dy \right)^{\frac{1}{\kappa p}} \\ \leq Cr \left(\frac{1}{w(B_r(x))} \int_{B_r(x)} |\nabla f(y)|^p w(y) \, dy \right)^{\frac{1}{p}}. \end{aligned}$$

This proves (A.1) by noting that Hölder's inequality implies

$$\begin{aligned} \left(\frac{1}{w(B_r(x))} \int_{B_r(x)} |f(y) - f_{B_r(x)}|^p w(y) \, dy \right)^{\frac{1}{p}} \\ \leq \left(\frac{1}{w(B_r(x))} \int_{B_r(x)} |f(y) - f_{B_r(x)}|^{\kappa p} w(y) \, dy \right)^{\frac{1}{\kappa p}} \end{aligned}$$

since $\kappa > 1$.

Finally, for any constant $\zeta \in \mathbb{R}$ we have

$$\begin{aligned} |f_{B_r(x)}^w - \zeta| &= \left| \frac{1}{w(B_r(x))} \int_{B_r(x)} (f(z) - \zeta) w(z) \, dz \right| \\ &\leq \frac{1}{w(B_r(x))} \int_{B_r(x)} |f(z) - \zeta| w(z) \, dz \\ &\leq \left(\frac{1}{w(B_r(x))} \int_{B_r(x)} |f(z) - \zeta|^p w(z) \, dz \right)^{\frac{1}{p}}, \end{aligned}$$

so, taking $\zeta = f_{B_r(x)}$, this implies

$$\int_{B_r(x)} |f(y) - f_{B_r(x)}^w|^p w(y) \, dy \leq 2^p \int_{B_r(x)} |f(y) - f_{B_r(x)}|^p w(y) \, dy.$$

This last inequality together with (A.1) conclude the proof of the Theorem. $\square$

B

SOME PROBABILITY THEORY

We record here, for completeness, a few classical probability theory definitions and results that are necessary in Chapter 7 for proving existence of invariant measures.

Definition B.1. Let (X, d) be a metric space, and let $\mathcal{X}$ be the associated Borel σ -algebra, i.e., the σ -algebra generated by the open sets of (X, d) .

Let $(p_n)_{n \geq 0}$ and p be probability measures on $(X, \mathcal{X})$. We say that p_n **converges weakly** to p , and write $p_n \Rightarrow p$, if $p_n(f) \rightarrow p(f)$ for any bounded continuous function $f : X \rightarrow \mathbb{R}$.

Let Π be a family of probability measures on (X, d) . We say that Π is **relatively compact** if every sequence of elements of Π has a convergent subsequence. We say that Π is **tight** if for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset X$ such that $p(K_\epsilon) > 1 - \epsilon$ for any $p \in \Pi$.

Theorem B.2 (Prokhorov's Theorem). *Let Π be a tight family of measures on $(X, \mathcal{X})$. Then Π is relatively compact.*

This simple form of Prokhorov's theorem is all that we need here. The theorem admits a converse in the case that X is separable and complete, see [12] for more details.

The following Theorem establishes sufficient conditions for the existence of an invariant measure for a Markov semigroup. Its proof is not difficult and it goes via the Prokhorov theorem; see [21] for details. We recall that ν is an invariant measure for the Markov semigroup P_t if for all f bounded and continuous, and for all $t \geq 0$, we have

$$\int P_t(f) d\nu = \int f d\nu.$$

Theorem B.3 (The Krylov-Bogoliubov Theorem). *Let P_t be a Markov Feller semigroup on a Hilbert space H . For fixed $x \in H$ and $t > 0$, define the probability measures*

$$p_t^x(A) := P_t(\mathbf{1}_A)(x).$$

Define also the family of measures $(\mu_T^x)_{T>0}$ by

$$\mu_T^x(E) := \frac{1}{T} \int_0^T p_t^x(E) \, dt.$$

If, for some $x \in H$, the family of probability measures $(\mu_T^x)_{T>0}$ is tight, then there exists an invariant measure for P_t .

REFERENCES

- [1] B. Amri and M. Sifi. Riesz transforms for Dunkl transform. *Ann. Math. Blaise Pascal*, 19(1):247–262, 2012.
- [2] B. Amri and M. Sifi. Singular integral operators in Dunkl setting. *J. Lie Theory*, 22(3):723–739, 2012.
- [3] J.-Ph. Anker. An introduction to Dunkl theory and its analytic aspects. In *Analytic, algebraic and geometric aspects of differential equations*, Trends Math., pages 3–58. Birkhäuser/Springer, Cham, 2017.
- [4] J.-Ph. Anker, N. Ben Salem, J. Dziubański, and N. Hamda. The Hardy space H^1 in the rational Dunkl setting. *Constr. Approx.*, 42(1):93–128, 2015.
- [5] J.-Ph. Anker, J. Dziubański, and A. Hejna. Harmonic functions, conjugate harmonic functions and the Hardy space H^1 in the rational Dunkl setting. *arXiv preprint arXiv:1802.06607*, 2018.
- [6] H. Araki. On an inequality of Lieb and Thirring. *Lett. Math. Phys.*, 19(2):167–170, 1990.
- [7] T. Aubin. Meilleures constantes dans le théorème d’inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire. *J. Funct. Anal.*, 32(2):148–174, 1979.
- [8] D. Bakry, T. Coulhon, M. Ledoux, and L. Saloff-Coste. Sobolev inequalities in disguise. *Indiana Univ. Math. J.*, 44(4):1033–1074, 1995.
- [9] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.
- [10] A. Balinsky, W. D. Evans, D. Hundertmark, and R. T. Lewis. On inequalities of Hardy-Sobolev type. *Banach J. Math. Anal.*, 2(2):94–106, 2008.
- [11] A. A. Balinsky, W. D. Evans, and R. T. Lewis. *The analysis and geometry of Hardy’s inequality*. Universitext. Springer, Cham, 2015.

- [12] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [13] M. Sh. Birman. On the spectrum of singular boundary-value problems. *Mat. Sb. (N.S.)*, 55 (97):125–174, 1961.
- [14] H. Brezis and J. L. Vázquez. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Complut. Madrid*, 10(2):443–469, 1997.
- [15] X. Cabré and X. Ros-Oton. Sobolev and isoperimetric inequalities with monomial weights. *J. Differential Equations*, 255(11):4312–4336, 2013.
- [16] L. Caffarelli, R. Kohn, and L. Nirenberg. First order interpolation inequalities with weights. *Compositio Math.*, 53(3):259–275, 1984.
- [17] Ó. Ciaurri, L. Roncal, and S. Thangavelu. Hardy-type inequalities for fractional powers of the Dunkl–Hermite operator. *arXiv preprint arXiv:1602.04997*, 2016.
- [18] A. Cohen, W. Dahmen, I. Daubechies, and R. DeVore. Harmonic analysis of the space BV. *Rev. Mat. Iberoamericana*, 19(1):235–263, 2003.
- [19] A. Cohen, R. DeVore, P. Petrushev, and H. Xu. Nonlinear approximation and the space BV($\mathbf{R}^2$). *Amer. J. Math.*, 121(3):587–628, 1999.
- [20] M. Cwikel. Weak type estimates for singular values and the number of bound states of Schrödinger operators. *Ann. of Math. (2)*, 106(1):93–100, 1977.
- [21] G. Da Prato. *An introduction to infinite-dimensional analysis*. Universitext. Springer-Verlag, Berlin, 2006. Revised and extended from the 2001 original by Da Prato.
- [22] F. Dai and H. Wang. A transference theorem for the Dunkl transform and its applications. *J. Funct. Anal.*, 258(12):4052–4074, 2010.
- [23] F. Dai and Y. Xu. *Analysis on h -harmonics and Dunkl transforms*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer, Basel, 2015. Edited by Sergey Tikhonov.
- [24] E. B. Davies. *One-parameter semigroups*, volume 15 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1980.
- [25] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

- [26] E. B. Davies. *Spectral theory and differential operators*, volume 42 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.
- [27] E. B. Davies. A review of Hardy inequalities. In *The Maz'ya anniversary collection, Vol. 2 (Rostock, 1998)*, volume 110 of *Oper. Theory Adv. Appl.*, pages 55–67. Birkhäuser, Basel, 1999.
- [28] M. F. E. de Jeu. The Dunkl transform. *Invent. Math.*, 113(1):147–162, 1993.
- [29] P. Doktor. Approximation of domains with Lipschitzian boundary. *Časopis Pěst. Mat.*, 101(3):237–255, 1976.
- [30] F. Dragoni, V. Kontis, and B. Zegarliński. Ergodicity of Markov semigroups with Hörmander type generators in infinite dimensions. *Potential Anal.*, 37(3):199–227, 2012.
- [31] C. F. Dunkl. Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.*, 311(1):167–183, 1989.
- [32] C. F. Dunkl. Operators commuting with Coxeter group actions on polynomials. In *Invariant theory and tableaux (Minneapolis, MN, 1988)*, volume 19 of *IMA Vol. Math. Appl.*, pages 107–117. Springer, New York, 1990.
- [33] C. F. Dunkl. Integral kernels with reflection group invariance. *Canad. J. Math.*, 43(6):1213–1227, 1991.
- [34] C. F. Dunkl. Hankel transforms associated to finite reflection groups. In *Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991)*, volume 138 of *Contemp. Math.*, pages 123–138. Amer. Math. Soc., Providence, RI, 1992.
- [35] C. F. Dunkl, M. F. E. de Jeu, and E. M. Opdam. Singular polynomials for finite reflection groups. *Trans. Amer. Math. Soc.*, 346(1):237–256, 1994.
- [36] J. Dziubański. Riesz transforms characterizations of Hardy spaces H^1 for the rational Dunkl setting and multidimensional Bessel operators. *J. Geom. Anal.*, 26(4):2639–2663, 2016.
- [37] J. Dziubański and A. Hejna. Remark on atomic decompositions for Hardy space H^1 in the rational Dunkl setting. *arXiv preprint arXiv:1803.10302*, 2018.
- [38] P. Etingof. A uniform proof of the Macdonald-Mehta-Opdam identity for finite Coxeter groups. *Math. Res. Lett.*, 17(2):275–282, 2010.

- [39] E. B. Fabes, C. E. Kenig, and R. P. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1):77–116, 1982.
- [40] S. Filippas and A. Tertikas. Optimizing improved Hardy inequalities. *J. Funct. Anal.*, 192(1):186–233, 2002.
- [41] A. Fiorenza, M. R. Formica, T. Roskovec, and F. Soudsky. Detailed proof of classical Gagliardo-Nirenberg interpolation inequality with historical remarks. *arXiv preprint arXiv:1812.04281*, 2018.
- [42] E. Gagliardo. Ulteriori proprietà di alcune classi di funzioni in più variabili. *Ricerche Mat.*, 8:24–51, 1959.
- [43] L. Gallardo and M. Yor. Some new examples of Markov processes which enjoy the time-inversion property. *Probab. Theory Related Fields*, 132(1):150–162, 2005.
- [44] L. Gallardo and M. Yor. Some remarkable properties of the Dunkl martingales. In *In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*, volume 1874 of *Lecture Notes in Math.*, pages 337–356. Springer, Berlin, 2006.
- [45] M. Ghergu and V. D. Rădulescu. *Nonlinear PDEs*. Springer Monographs in Mathematics. Springer, Heidelberg, 2012. Mathematical models in biology, chemistry and population genetics, With a foreword by Viorel Barbu.
- [46] D. V. Gorbachev, V. I. Ivanov, and S. Yu. Tikhonov. Sharp Pitt inequality and logarithmic uncertainty principle for Dunkl transform in L^2 . *J. Approx. Theory*, 202:109–118, 2016.
- [47] D. V. Gorbachev, V. I. Ivanov, and S. Yu. Tikhonov. Positive L^p -bounded Dunkl-type generalized translation operator and its applications. *arXiv preprint arXiv:1703.06830*, 2017.
- [48] D. V. Gorbachev, V. I. Ivanov, and S. Yu. Tikhonov. Riesz potential and maximal function for Dunkl transform. *arXiv preprint arXiv:1708.09733*, 2017.
- [49] P. Graczyk, M. Rösler, and M. Yor, editors. *Harmonic and stochastic analysis of Dunkl processes*. Travaux en cour. Herman, Paris, 2008.
- [50] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.
- [51] L. Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97(4):1061–1083, 1975.

- [52] A. Guionnet and B. Zegarliński. Lectures on logarithmic Sobolev inequalities. In *Séminaire de Probabilités, XXXVI*, volume 1801 of *Lecture Notes in Math.*, pages 1–134. Springer, Berlin, 2003.
- [53] S. Hassani, S. Mustapha, and M. Sifi. Riesz potentials and fractional maximal function for the Dunkl transform. *J. Lie Theory*, 19(4):725–734, 2009.
- [54] W. Hebisch and B. Zegarliński. Coercive inequalities on metric measure spaces. *J. Funct. Anal.*, 258(3):814–851, 2010.
- [55] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original.
- [56] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev, and J. Tidblom. Many-particle Hardy inequalities. *J. Lond. Math. Soc. (2)*, 77(1):99–114, 2008.
- [57] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
- [58] D. Hundertmark, E. H. Lieb, and L. E. Thomas. A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator. *Adv. Theor. Math. Phys.*, 2(4):719–731, 1998.
- [59] V. P. Il’in. Some integral inequalities and their applications in the theory of differentiable functions of several variables. *Mat. Sb. (N.S.)*, 54 (96):331–380, 1961.
- [60] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.
- [61] V. Kontis, M. Ottobre, and B. Zegarliński. Markov semigroups with hypocoercive-type generator in infinite dimensions: ergodicity and smoothing. *J. Funct. Anal.*, 270(9):3173–3223, 2016.
- [62] A. Laptev. Spectral inequalities for partial differential equations and their applications. In *Fifth International Congress of Chinese Mathematicians. Part 1, 2*, volume 2 of *AMS/IP Stud. Adv. Math.*, 51, pt. 1, pages 629–643. Amer. Math. Soc., Providence, RI, 2012.
- [63] A. Laptev and T. Weidl. Sharp Lieb-Thirring inequalities in high dimensions. *Acta Math.*, 184(1):87–111, 2000.
- [64] M. Ledoux. On improved Sobolev embedding theorems. *Math. Res. Lett.*, 10(5-6):659–669, 2003.

- [65] D. Levin and M. Solomyak. The Rozenblum-Lieb-Cwikel inequality for Markov generators. *J. Anal. Math.*, 71:173–193, 1997.
- [66] P. Li and S. T. Yau. On the Schrödinger equation and the eigenvalue problem. *Comm. Math. Phys.*, 88(3):309–318, 1983.
- [67] E. Lieb. Bounds on the eigenvalues of the Laplace and Schroedinger operators. *Bull. Amer. Math. Soc.*, 82(5):751–753, 1976.
- [68] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.
- [69] E. H. Lieb and R. Seiringer. *The stability of matter in quantum mechanics*. Cambridge University Press, Cambridge, 2010.
- [70] E. H. Lieb and W. Thirring. Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. In B. Simon E. Lieb and A. Wightman, editors, *Studies in Mathematical Physics*, pages 269–303. Princeton University Press, 1976.
- [71] T. M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.
- [72] P. Ługiewicz and B. Zegarliński. Coercive inequalities for Hörmander type generators in infinite dimensions. *J. Funct. Anal.*, 247(2):438–476, 2007.
- [73] D. Lundholm. Geometric extensions of many-particle Hardy inequalities. *J. Phys. A*, 48(17):175203, 25, 2015.
- [74] D. Lundholm and J. P. Solovej. Hardy and Lieb-Thirring inequalities for anyons. *Comm. Math. Phys.*, 322(3):883–908, 2013.
- [75] I. G. Macdonald. Some conjectures for root systems. *SIAM J. Math. Anal.*, 13(6):988–1007, 1982.
- [76] V. Maz’ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, augmented edition, 2011.
- [77] J. Nash. Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.*, 80:931–954, 1958.
- [78] L. Nirenberg. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 13:115–162, 1959.

- [79] E. M. Opdam. Some applications of hypergeometric shift operators. *Invent. Math.*, 98(1):1–18, 1989.
- [80] E. M. Opdam. Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group. *Compositio Math.*, 85(3):333–373, 1993.
- [81] B. Opic and A. Kufner. *Hardy-type inequalities*, volume 219 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1990.
- [82] F. Rellich. Halbbeschränkte Differentialoperatoren höherer Ordnung. In *Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III*, pages 243–250. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956.
- [83] F. Rellich. *Perturbation theory of eigenvalue problems*. Assisted by J. Berkowitz. With a preface by Jacob T. Schwartz. Gordon and Breach Science Publishers, New York-London-Paris, 1969.
- [84] F. Ricci and E. M. Stein. Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals. *J. Funct. Anal.*, 73(1):179–194, 1987.
- [85] M. Rösler. Generalized Hermite polynomials and the heat equation for Dunkl operators. *Comm. Math. Phys.*, 192(3):519–542, 1998.
- [86] M. Rösler. Positivity of Dunkl’s intertwining operator. *Duke Math. J.*, 98(3):445–463, 1999.
- [87] M. Rösler. Dunkl operators: theory and applications. In *Orthogonal polynomials and special functions (Leuven, 2002)*, volume 1817 of *Lecture Notes in Math.*, pages 93–135. Springer, Berlin, 2003.
- [88] M. Rösler and M. Voit. Markov processes related with Dunkl operators. *Adv. in Appl. Math.*, 21(4):575–643, 1998.
- [89] O. S. Rothaus. Analytic inequalities, isoperimetric inequalities and logarithmic Sobolev inequalities. *J. Funct. Anal.*, 64(2):296–313, 1985.
- [90] G. V. Rozenbljum. Distribution of the discrete spectrum of singular differential operators. *Izv. Vysš. Učebn. Zaved. Matematika*, 164(1):75–86, 1976.
- [91] M. Ruzhansky and N. Tokmagambetov. Nonlinear damped wave equations for the sub-Laplacian on the Heisenberg group and for Rockland operators on graded Lie groups. *J. Differential Equations*, 265(10):5212–5236, 2018.

- [92] L. Saloff-Coste. *Aspects of Sobolev-type inequalities*, volume 289 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.
- [93] L. Saloff-Coste. Pseudo-Poincaré inequalities and applications to Sobolev inequalities. In *Around the research of Vladimir Maz'ya. I*, volume 11 of *Int. Math. Ser. (N. Y.)*, pages 349–372. Springer, New York, 2010.
- [94] J. Schwinger. On the bound states of a given potential. *Proc. Nat. Acad. Sci. U.S.A.*, 47:122–129, 1961.
- [95] F. Soltani. Pitt's inequalities for the Dunkl transform on $\mathbb{R}^d$. *Integral Transforms Spec. Funct.*, 25(9):686–696, 2014.
- [96] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
- [97] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976.
- [98] G. Talenti. A weighted version of a rearrangement inequality. *Ann. Univ. Ferrara Sez. VII (N.S.)*, 43:121–133 (1998), 1997.
- [99] S. Thangavelu and Y. Xu. Convolution operator and maximal function for the Dunkl transform. *J. Anal. Math.*, 97:25–55, 2005.
- [100] S. Thangavelu and Y. Xu. Riesz transform and Riesz potentials for Dunkl transform. *J. Comput. Appl. Math.*, 199(1):181–195, 2007.
- [101] H. Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.
- [102] K. Trimèche. The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual. *Integral Transform. Spec. Funct.*, 12(4):349–374, 2001.
- [103] K. Trimèche. Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators. *Integral Transforms Spec. Funct.*, 13(1):17–38, 2002.
- [104] J. F. van Diejen and L. Vinet, editors. *Calogero-Moser-Sutherland models*, CRM Series in Mathematical Physics. Springer-Verlag, New York, 2000.
- [105] A. Velicu. Sobolev-type inequalities for Dunkl operators. *arXiv preprint arXiv:1811.11118*, 2018.

- [106] A. Velicu. Hardy-type inequalities for Dunkl operators. *arXiv preprint arXiv:1901.08866*, 2019.
- [107] A. Velicu and N. Yessirkegenov. Rellich, Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg inequalities for Dunkl operators and applications. *arXiv preprint arXiv:1904.08725*, 2019.
- [108] L. Xu and B. Zegarliński. Ergodicity of the finite and infinite dimensional α -stable systems. *Stoch. Anal. Appl.*, 27(4):797–824, 2009.
- [109] L. Xu and B. Zegarliński. Existence and exponential mixing of infinite white α -stable systems with unbounded interactions. *Electron. J. Probab.*, 15:no. 65, 1994–2018, 2010.
- [110] B. Zegarliński. The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice. *Comm. Math. Phys.*, 175(2):401–432, 1996.
- [111] B. Zegarliński. Linear and nonlinear dissipative dynamics. *Rep. Math. Phys.*, 77(3):377–397, 2016.