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On the generic part of the cohomology of non-compact PEL-type A Shimura varieties

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Statement of Originality

I hereby declare that this thesis is my own work. Information derived from the published and unpublished work of others has been acknowledged in the text and references are given in the list of sources.

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Abstract

In this thesis, we use the new methods of Koshikawa to prove that the generic l -adic cohomology of non-compact type A Shimura varieties vanishes below middle degree, when localized at a maximal ideal of the Hecke algebra. This generalizes previously established vanishing results of Caraiani-Scholze.

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Chapter 1

Introduction

In this thesis, we extend some vanishing results of the torsion étale cohomology of unitary Shimura varieties, due to Caraiani and Scholze ([CS17], [CS19]), using some new methods by Koshikawa ([Kos21]).

Let us start by briefly recalling the general vanishing conjecture of locally symmetric spaces, as formulated for example in [Eme14] and [CG18]. Let G be a connected reductive group over $\mathbb{Q}$, and X be the associated symmetric space for the Lie group $G(\mathbb{R})$. If we fix a neat open compact subgroup $K \subset G(\mathbb{A}_f)$, we can also define a locally symmetric space

$$X_K(G) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K,$$

which is a smooth Riemannian manifold. There are two important invariants associated to this data. Let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup, and A_∞ be the identity component $T(\mathbb{R})$, where T denotes the maximal $\mathbb{Q}$ -split torus in the center of G . Define

$$l_0 := \mathrm{rk}(K_\infty) - \mathrm{rk}(A_\infty)$$

and

$$q_0 := \frac{1}{2}(d - l_0)$$

where $d = \dim_{\mathbb{R}}(X)$. The conjecture then briefly states that the l -adic cohomology of X_K is degenerate outside the range $[q_0, q_0 + l_0]$ (which is symmetric about the middle degree $\frac{1}{2}d$). To precisely state the conjecture, one needs to define non-Eisenstein maximal ideals inside the Hecke algebra, and to consider the localization of the cohomology at such an ideal (c.f. [Car22]). In the case where (G, X) give rise to a Shimura variety, we have to use the invariants $l_0 = l_0(G^{\mathrm{ad}})$ and $q_0 = q_0(G^{\mathrm{ad}})$ due to a slight difference in the definition of the symmetric spaces. In this case, we have that $l_0 = 0$, and that therefore the conjecture predicts that the non-Eisenstein part of the l -adic cohomology of $X_K(G)$ actually vanishes outside the middle degree.

Some instances of these vanishing results were originally proved for certain Shimura varieties by Lan-Suh in [LS12] and [LS13], under some very strong assumptions, and for Shimura varieties of Harris-Taylor type by Boyer in [Boy19]. But only more recently have there been significant

advances, brought up by the revolutionary work of Caraiani and Scholze. In their original paper, [CS17], they were able to prove the conjecture for unitary compact Shimura varieties of type A, and they later extended it in [CS19] for Shimura varieties associated to a quasi-split similitude unitary group. The added difficulty in [CS19] is that one needs to juggle with the (minimal and toroidal) compactifications of the Shimura variety $\mathcal{S}_K$. The Hilbert case has also since been proven by Caraiani and Tamiozzo in [CT21]. These results are useful, for example, to apply the Calegari–Geraghty method to prove automorphy lifting theorems. Some applications include, for example, the use of the vanishing results in [CS17] to prove certain instances of the Beilinson–Bloch–Kato conjecture in [LTXZZ21]. The vanishing results in [CS19] were also used in [ACC⁺18] to lift torsion classes in the cohomology of the Borel–Serre boundary of a Shimura variety to characteristic zero, resulting in local-global compatibility and modularity lifting results.

The general strategy of [CS17] consists of relating the cohomology of Shimura varieties to that of Igusa varieties, via the Hodge–Tate period morphism

$$\pi_{\text{HT}} : \mathbf{S}_{K^p} \rightarrow \mathcal{F}_{G,\mu},$$

where $\mathcal{F}_{G,\mu}$ denotes the usual flag variety attached to G and the cocharacter μ . They then prove that the geometric fibers of π_{HT} are (perfect) Igusa varieties. They also use the fact that π_{HT} is "affine" and (partially) proper, to see that the complex $R\pi_{\text{HT}*}\mathbb{F}_l$ behaves like a perverse sheaf on $\mathcal{F}_{G,\mu}$ (when localized at certain maximal ideals of the Hecke algebra). This result, together with certain trace formula methods of Shin, [Shi10], imply the l -adic cohomology of the Igusa varieties is concentrated in the middle degree, and that therefore so is $R^i\pi_{\text{HT}*}\mathbb{F}_l$. Finally, the Leray spectral sequence applied to π_{HT} implies that the torsion cohomology of the Shimura varieties, under a certain genericity assumption, vanishes outside the middle degree. [CS19] extends this picture to the respective compactifications, using a similar strategy. We remark that since the polarizations in their particular case are always principal, the moduli problems appearing e.g. in the definition of the toroidal compactifications simplifies quite a lot, when compared to more general, non-compact PEL-type Shimura varieties (c.f. [Lan13]).

More recently, [Kos21] was able to use the cutting edge work of Fargues–Scholze [FS21], to reprove, and to even strengthen, the cohomology vanishing results in [CS17] and [CS19]. In order to use his new methods to extend these vanishing results to new non-compact cases, the only missing piece is the geometry part of [CS19]. This has been done in my thesis in greater generality, as I will now describe.

In the first part of the thesis, we work in the general case of PEL-type Shimura varieties, and our strategy and result are very similar to the geometric results in [CS19]. We consider an integral PEL-datum $(\mathcal{O}_B, *, L, \langle \cdot, \cdot \rangle, h)$ as introduced in section 2.1, and we let p be an unramified prime, which we also assume to be good in the sense of [Lan13]. To such a tuple, one obtains an associated algebraic group of unitary similitudes over $\mathbb{Z}$, defined as

$$G(R) := \{(g, r) \in \text{GL}_{\mathcal{O}_B \otimes R}(L \otimes R) \times R^\times \mid \langle gx, gy \rangle = r \langle x, y \rangle \ \forall x, y \in L \otimes R\},$$

for any $\mathbb{Z}$ -algebra R . We assume that the derived group of G has type A or C . This gives rise to a

Shimura datum, and allows us therefore to define PEL-type Shimura varieties. Let $K(n)$ denote the principal level- n congruence subgroup of G for $n \geq 3$ coprime to p , and denote by $S_{K(n)}$ the corresponding Shimura variety. This is an algebraic variety defined over $\mathrm{Spec}(\mathcal{O}_{E,(p)})$, where E denotes the reflex field of the Shimura datum. Recall furthermore that these parametrize tuples $(A, \lambda_A, \iota_A, \alpha_n)$ consisting of an abelian variety A , with a prime-to- p polarization λ_A , an $\mathcal{O}_B$ -action and level structures, which are compatible (c.f. definition 2.1.3). They are often non-compact, but admit toroidal and minimal compactifications following [Lan13], denoted respectively by $\mathcal{S}_{K(n),\Sigma}^{\mathrm{tor}}$ and $\mathcal{S}_{K(n)}^*$. The toroidal compactifications depend on a choice of certain combinatorial data, namely on a family of cone decompositions Σ , but have nice local moduli descriptions, which often allows one to explicitly work with them. The minimal compactifications do not depend on any choices, but are constructed abstractly, and are therefore generally harder to work with. In the second chapter of the thesis, we recall the construction and the main properties of these compactifications due to Lan. We also explain that, as proven by Lan in [Lan16], these compactifications also exist when p divides n , albeit not satisfying all the good properties as in the case $p \nmid n$.

In the third chapter, we introduce Igusa varieties, and their compactifications. Namely, let $p \nmid n$ be an unramified prime as above, and let $K := K(n)$. Fix a p -divisible group $\mathbb{X}$ over $\bar{\mathbb{F}}_p$, with extra PEL-structures consisting of a principal polarization and an $\mathcal{O}_B$ -action satisfying certain compatibilities. We can consider the Oort central leaf $\mathcal{C}^{\mathbb{X}}$, a locally closed smooth subscheme of $S_{K,\bar{\mathbb{F}}_p} = S_K \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$, defined as

$$\mathcal{C}^{\mathbb{X}} = \{x \in \mathcal{S}_{K,\bar{\mathbb{F}}_p} \mid A[p^\infty]_{k(\bar{x})} \cong \mathbb{X} \times_{\bar{\mathbb{F}}_p} k(\bar{x})\} \subset \mathcal{S}_{K,\bar{\mathbb{F}}_p},$$

where the isomorphisms are as p -divisible groups with the extra structures, so in particular they must be compatible with the polarizations, and commute with the $\mathcal{O}_B$ -actions. Following the work of Boxer in [Box15], Lan and Stroth introduce in [LS18] the notion of a “well-positioned subset”. These are certain subsets of $\mathcal{S}_{K,\bar{\mathbb{F}}_p}$ that behave particularly well with respect to compactifications: in particular, they admit “nice” toroidal and minimal compactifications with many of the same properties as $\mathcal{S}_{K,\Sigma}^{\mathrm{tor}}$ and $\mathcal{S}_K^*$. One can prove that central leaves are an example of well-positioned subsets, and one can therefore construct compactifications $\mathcal{C}_\Sigma^{\mathbb{X},\mathrm{tor}}$ and $\mathcal{C}^{\mathbb{X},*}$.

Over such a leaf, we can consider the perfect Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}} \rightarrow \mathcal{C}^{\mathbb{X}}$ as introduced in [CS17]. These are defined as follows:

Definition 1.0.1. The Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$, is the moduli space over $\mathrm{Spec}(\bar{\mathbb{F}}_p)$ parametrizing pairs (A, ρ) , where A is an abelian variety with extra structures, and $\rho : A[p^\infty] \cong \mathbb{X}$ is an isomorphism compatible with the extra structures.

These are pro-étale covers of $\mathcal{C}_{\mathrm{perf}}^{\mathbb{X}}$. Furthermore, by [CS17], $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$ does not depend on the choice of the p -divisible group $\mathbb{X}$ within its isogeny class. The first important construction we generalize from [CS19] is that of the toroidal compactification of $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$. In particular, there exists a scheme $\mathfrak{I}\mathfrak{g}_\Sigma^{\mathbb{X},\mathrm{tor}}$ over $\mathcal{C}_\Sigma^{\mathbb{X},\mathrm{tor}}$, satisfying the following properties.

Theorem 1.0.2. *The pro-étale cover $\mathcal{I}\mathfrak{g}^{\mathbb{X}} \rightarrow \mathcal{C}_{\text{perf}}^{\mathbb{X}}$ with Galois group $\text{Aut}(\mathbb{X})(\overline{\mathbb{F}}_p)$ extends uniquely to a pro-étale cover on toroidal compactifications $\mathcal{I}\mathfrak{g}_{\Sigma}^{\mathbb{X},\text{tor}} \rightarrow \mathcal{C}_{\text{perf}}^{\mathbb{X},\text{tor}}$ with the same Galois group. Furthermore, one can describe the local structure of $\mathcal{I}\mathfrak{g}_{\Sigma}^{\mathbb{X},\text{tor}}$ using an Igusa toroidal boundary chart, just as in the case of Shimura varieties.*

The idea to prove this theorem, is to define $\mathcal{I}\mathfrak{g}_{\Sigma}^{\mathbb{X},\text{tor}}$ with a moduli description over $\mathcal{C}_{\Sigma,\text{perf}}^{\mathbb{X},\text{tor}}$, just as we did with $\mathcal{I}\mathfrak{g}^{\mathbb{X}}$. However, in this case it is a bit more subtle, since we work with a universal *semiabelian* (instead of abelian) scheme $\mathcal{G}$ over $\mathcal{S}_{K,\Sigma}^{\text{tor}}$, meaning that $\mathcal{G}[p^{\infty}]$ might not be a p -divisible group. Here, we exploit the fact that the connected part $\mathcal{G}^{\circ}[p^{\infty}]$ is a p -divisible group. This definition allows us to explicitly see that $\mathcal{I}\mathfrak{g}_{\Sigma}^{\mathbb{X},\text{tor}}$ recovers $\mathcal{I}\mathfrak{g}^{\mathbb{X}}$ on the open cell.

This theorem is quite decisive, because it replicates the classical picture for compactifications of Shimura varieties. The main complication compared to [CS19] comes from the fact that the polarizations in our case are not necessarily principal, so we have to handle carefully a lot of extra structure coming from duals. Furthermore, the description of $\mathcal{I}\mathfrak{g}_{\Sigma}^{\mathbb{X},\text{tor}}$ in terms of Igusa boundary charts also allows us to prove that if there is an isogeny between p -divisible groups $\lambda : \mathbb{X} \rightarrow \mathbb{X}'$ inducing an isomorphism on étale and connected parts, then $\mathcal{I}\mathfrak{g}_{\Sigma}^{\mathbb{X},\text{tor}} \cong \mathcal{I}\mathfrak{g}_{\Sigma}^{\mathbb{X}',\text{tor}}$. The assumption that the isogeny must be an isomorphism on étale and connected parts comes from the necessity to control the cone decompositions.

If $\mathbb{X}$ is completely slope divisible, then we show that the above picture also deperfects. Namely, we can work with Mantovan’s Igusa varieties (c.f. [Mant05]), and still define toroidal compactifications of these, which will also admit a nice local description in terms of toroidal boundary charts. These will, however, depend on the choice of $\mathbb{X}$.

One can also define the minimal compactification $\mathcal{I}\mathfrak{g}^{\mathbb{X},*}$ via normalization, and prove that these are affine, using results of Nie ([Nie15]) and Boxer ([Box15]), using the exact same strategy as in [CS19]. This uses, in particular, the invariance of $\mathcal{I}\mathfrak{g}_{\Sigma}^{\mathbb{X},\text{tor}}$ under certain isogenies.

In the third chapter, we focus on studying the Hodge–Tate period morphism. Namely, we need a description of the fibers of the Hodge–Tate period morphism in terms of Igusa varieties, just as in the compact and the quasi-split cases. The bottom-line is that this will allow us to connect the cohomology of the fibers of the Hodge–Tate period morphism to that of the Igusa varieties, which will in turn result in a semi-perversity statement.

We let $\mathcal{F}_{G,\mu}$ be the flag variety introduced above. Recall that the Hodge–Tate period morphism as constructed by Scholze and Caraiani and Scholze for Shimura varieties of Hodge type, is a morphism from the Shimura variety at infinite level (seen as an adic space, or more precisely a perfectoid space) to the flag variety. We can define compactified versions of this tower, namely by considering following diamonds over $\text{Spd}(\mathbb{Q}_p)$,

$$\mathbf{S}_{K(p^{\infty}N)}^{\text{tor}} := \varprojlim_n \mathcal{S}_{K(p^n N), \mathbb{Q}_p}^{\text{tor}}$$

and

$$\mathbf{S}_{K(p^n N)}^* := \varprojlim_n \mathcal{S}_{K(p^n N), \mathbb{Q}_p}^*.$$

One can then extend the Hodge-Tate period morphism to compactifications.

Theorem 1.0.3. *The Hodge-Tate period morphism*

$$\pi_{\text{HT}} : \mathbf{S}_{K(p^\infty N)} \rightarrow \mathcal{F}_{G,\mu},$$

extends to compactifications. In particular, there exists a morphism

$$\pi_{\text{HT}}^* : \mathbf{S}_{K(p^\infty N)}^* \rightarrow \mathcal{F}_{G,\mu}$$

which induces by projection a map

$$\pi_{\text{HT}}^{\text{tor}} : \mathbf{S}_{K(p^\infty N)}^{\text{tor}} \rightarrow \mathcal{F}_{G,\mu}.$$

We then need to analyze the fibers of these morphisms. Let C be a fixed algebraically closed, non-archimedean field over $\mathbb{Q}_p$. Denote by $\mathcal{O}_C$ its ring of integers, and let k be its residue field. Fix $x \in \mathcal{F}_{G,\mu}(C)$. By the main theorem of [SW13], one can associate to x a p -divisible group $\mathbb{X}_{\mathcal{O}_C}$ with extra structures over $\mathcal{O}_C$, together with a trivialization $\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of its Tate module. We denote by $\mathbb{X}/k$ its special fiber. In particular, we consider the Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$ associated to this p -divisible group, and its compactifications $\mathfrak{I}\mathfrak{g}_{\Sigma}^{\mathbb{X},\text{tor}}$ and $\mathfrak{I}\mathfrak{g}^{\mathbb{X},*}$. Since these schemes are perfect, they admit lifts to $\mathcal{O}_C$. We have the following result.

Theorem 1.0.4. *There is a morphism of p -adic formal schemes*

$$g_{\mathcal{O}_C}^{\text{tor}} : \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X},\text{tor}} \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C}^{\text{tor}}$$

extending the map over the special fiber, inducing an open immersion of perfectoid spaces

$$f^{\text{tor}} : \mathfrak{I}\mathfrak{g}_C^{\mathbb{X},\text{tor}} \hookrightarrow (\pi_{\text{HT}}^{\text{tor}})^{-1}(x),$$

with the same rank 1 points. Furthermore, the morphism

$$\mathfrak{I}\mathfrak{g}_C^{\mathbb{X},*} \hookrightarrow (\pi_{\text{HT}}^*)^{-1}(x)$$

induced by f^{tor} is also an open immersion of affinoid perfectoid spaces.

This theorem is proved using precisely the local description of the toroidal compactifications of Igusa varieties. Namely, one can explicitly describe the map $\pi_{\text{HT}}^{\text{tor}}$ on the boundary (so at each cusp), and then directly compare its fibers (using the local description of the toroidal compactifications of the Shimura varieties at infinite level) to the description indicated in 1.0.2.

Finally, in chapter 5, we focus on the cohomological results. We start by using the geometrical result on the fibers of the Hodge-Tate period morphism, using the same methods as in section 4.6. in [CS19], to obtain a semiperversity result on the nearby cycles of $R\pi_{\text{HT},*}\mathbb{F}_l$:

Theorem 1.0.5. *Let $\mathcal{S}_{K(p^\infty N), \mathbb{Q}_p}^\circ := \varprojlim_n \mathcal{S}_{K(p^n N), \mathbb{Q}_p}^\circ$ where $\mathcal{S}_{K(N), \mathbb{Q}_p}^\circ \subset \mathcal{S}_{K(N), \mathbb{Q}_p}$ denotes the locus of good reduction. Let π_{HT}° denote the restriction of π_{HT} to $\mathcal{S}_{K(p^\infty N), \mathbb{Q}_p}^\circ$. Let l be a*

prime different from p , and denote by d the dimension of the Shimura variety we are considering. Consider the complex

$$R\pi_{\mathrm{HT}*}\mathbb{F}_l \in D(\mathcal{F}_{G,\mu,C}, \mathbb{F}_l),$$

and denote by $R\Psi(R\pi_{\mathrm{HT}*}\mathbb{F}_l)$ its nearby cycles. Then any geometric point $\bar{x}$ of $\mathcal{F}_{G,\mu,C}$ admits a cofinal system of affinoid étale neighborhoods $U = \mathrm{Spa}(A) \rightarrow \mathcal{F}_{G,\mu,C}$, with associated formal scheme $\mathcal{U} = \mathrm{Spf}(A^\circ)$, such that the nearby cycles $R\Psi(R\pi_{\mathrm{HT}*}\mathbb{F}_l)|_U$ satisfy

$$R\Psi(R\pi_{\mathrm{HT}*}\mathbb{F}_l)|_U \in {}^pD^{\geq d}(\mathcal{U}_k, \mathbb{F}_l).$$

This critically uses that the structure of the toroidal compactifications is nice, in particular they are smooth and can be chosen such that the boundary is a normal crossings divisor.

The last piece we need in order to obtain our vanishing result, is Koshikawa's argument, which we will now explain. We need to restrict ourselves to the setting in [Kos21]. We take a CM field F , and a type A PEL-datum $(B, *, V, (\cdot, \cdot))$, where B is a central simple F -algebra. We denote by G the unitary similitudes group attached to this data. We assume that p is a prime that is unramified for the Shimura datum, and that splits completely in F . We can write $G_{\mathbb{Q}_p}$ as a product

$$G_{\mathbb{Q}_p} \cong \mathbb{G}_m^\times \times \prod G_{v_i}$$

for $p = v_1 \cdots v_m \in F$, where each G_{v_i} is a product of general linear groups over $F_{v_i} \cong \mathbb{Q}_p$. Let $K = K_p K^p \subset G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ be a sufficiently small open compact subgroup with $K_p = \mathbb{Z}_p^\times \times \prod K_{v_i} \subset G(\mathbb{Q}_p)$ an hyperspecial open compact subgroup. We once more denote by $\mathcal{S}_K$ the PEL-type Shimura variety attached to this data, and let d be its dimension. The Hecke algebra at p is defined as

$$H_{K_p} := \mathbb{Z}_l[K_p \backslash G(\mathbb{Q}_p)/K_p]$$

for some prime $l \neq p$, and it admits an action on the cohomology groups

$$H^i(\mathcal{S}_K, \bar{\mathbb{F}}_l)$$

and

$$H_c^i(\mathcal{S}_K, \bar{\mathbb{F}}_l),$$

via correspondences. Let $\mathfrak{m}_{H_{K_p}} \subset H_{K_p}$ be a maximal ideal of the local Hecke algebra in the support of $H_{(c)}^i(\mathcal{S}_K, \bar{\mathbb{F}}_l)$, and let $\lambda_{\mathfrak{m}_{H_{K_p}}}$ be the corresponding character. We have associated L-parameters, which consists of products of representations

$$\rho_{\mathfrak{m}_{v_i}} : \mathrm{Gal}(\bar{F}_{v_i}/F_{v_i}) \rightarrow \prod_{j \in J} \mathrm{GL}_{n_j}(\bar{\mathbb{F}}_l)$$

for each place v_i of F above p , as above. $\rho_{\mathfrak{m}_{v_i}}$ is generic if for every $j \in J$, the eigenvalues $\alpha_{j,1}, \dots, \alpha_{j,n_j} \in \bar{\mathbb{F}}_l$ of $\rho_{\mathfrak{m}_{v_i}}(\mathrm{Frob})$ for the Frobenius homomorphism $\mathrm{Frob} \in \mathrm{Gal}(\bar{F}_{v_i}/F_{v_i})$, regarded as elements in $\mathrm{GL}_{n_j}(\bar{\mathbb{F}}_l)$ satisfy $\alpha_{j,k}/\alpha_{j,m} \neq p$ for $k \neq m$. Our main result can be stated as follows.

Theorem 1.0.6. *Let p be a prime which splits completely in F . Let $\mathfrak{m}_{H_{K_p}} \subset H_{K_p}$ be a maximal ideal of the Hecke algebra, with associated L -parameters $\rho_{\mathfrak{m}_{v_i}}$, which we assume to be generic for every i . Then $H^i(\mathcal{S}_K, \bar{\mathbb{F}}_l)_{\mathfrak{m}_{H_{K_p}}} = 0$ for $i < d$ and $H^i(\mathcal{S}_K, \bar{\mathbb{F}}_l)_{\mathfrak{m}_{H_{K_p}}} = 0$ for $i > d$.*

Let us summarize the ideas behind Koshikawa's method. The first ingredient one needs, is a local vanishing result. We consider the local PEL-data $(G_{\mathbb{Q}_p}, b, \mu)$, associated to our global type A PEL-datum $(B, *, V, (\cdot, \cdot))$. Let $\mathcal{M}_{(G, b, \mu), K_p}$ the corresponding local Shimura variety. Then, keeping the same notation as above for the Hecke algebra, and under the same genericity assumption, we have that the cohomology groups

$$H_c^i(\mathcal{M}_{(G, b, \mu), K_p}, \mathbb{Z}_l)_{\mathfrak{m}_{H_{K_p}}}$$

vanish for any i , if we assume that J_b is not quasi-split. The main ingredients to prove this result are the results in [FS21] on the constructions of L -parameters associated to certain representations, as well as excursion operators.

The second important ingredient in the proof, is a certain type of cohomological Mantovan's product formula. This relates the cohomology of the Shimura variety to the cohomology of Igusa varieties and of the corresponding Rapoport-Zink space. Namely, using the geometric Mantovan's product formula from [CS19], Koshikawa is able to prove that $R\Gamma(\mathcal{S}_{K^p, \bar{\mathbb{Q}}}, \bar{\mathbb{F}}_l)$ admits a filtration, with graded pieces

$$\begin{aligned} R\Gamma(\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l)^{\text{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G, b, \mu), \infty}, \bar{\mathbb{F}}_l(d_b))[2d_b] \cong \\ R\Gamma_c(\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l(d_b)^*)^{\text{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G, b, \mu), \infty}, \bar{\mathbb{F}}_l(d_b)), \end{aligned}$$

for $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$, where d_b denotes the dimension of $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$.

Since, under our assumptions on p , J_b is quasi-split if and only if b is ordinary, we can use the local vanishing result to conclude that

$$R\Gamma_c(\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l(d_b)^*)_{\mathfrak{m}_{H_{K_p}}}^{\text{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G, b, \mu), \infty}, \bar{\mathbb{F}}_l(d_b))_{\mathfrak{m}_{H_{K_p}}} = 0$$

for all non-ordinary b . In particular, it suffices to work with the ordinary element. Finally, the semiperversity result then allows us to finish the proof of theorem 1.0.6. The proof uses, once more, results from [FS21].

Overview of the thesis. This thesis is structured as follows. In the second chapter, we recall the construction of toroidal and minimal PEL-type Shimura varieties due to Lan. In the third chapter, we recall the definition of Igusa varieties, and generalize the constructions of their toroidal and minimal compactifications introduced in [CS19] to general PEL-type Shimura varieties. In the forth chapter, we recall the Hodge-Tate period morphism, extend it to compactifications, and study their fibers. Finally, in chapter five, we prove the semiperversity result, and recall Koshikawa's methods introduced in [Kos21] to obtain our main result.

Chapter 2

Compactifications of Shimura varieties

The geometry of PEL-type Shimura varieties and their compactifications is in the heart of this thesis. We must therefore start by recalling the definition of these varieties, as well as the construction of their toroidal and minimal compactifications. We also setup some of the notation we will be using throughout the thesis.

2.1 PEL-type Shimura varieties

Shimura Varieties are certain smooth, quasi-projective algebraic varieties Sh_K , defined over the so called reflex field E , which is a number field. They were firstly introduced (in certain cases) by Shimura in [Shi], and later generalized by Deligne in [Del71]. They admit both Galois and Hecke actions, which has proven to be quite useful when realizing the Langlands correspondence over number fields, as conjectured by Langlands. They are associated to certain pairs (G, X) called Shimura datum, consisting of a connected reductive group G over $\mathbb{Q}$, and a conjugacy class of homomorphisms $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \rightarrow G_{\mathbb{R}}$, X , satisfying a series of axioms (c.f. [Del71], [Mil05]). One can use a Shimura data to construct a Shimura variety Sh_K as above, whose complex points are actually given by the double coset

$$X_K := [G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K)],$$

where $K \subset G(\mathbb{A}_f)$ is an open compact subgroup. This can be proven to be a locally symmetric space

For a much more complete introduction to the topic, see for example [Mil05] or [Lan18].

Let us now focus on the PEL-case, which will be a natural generalization of the modular curve.

Let B be a finite-dimensional semisimple algebra over $\mathbb{Q}$, together with a positive involution $*$. We fix a $\mathbb{Z}$ -order $\mathcal{O}_B$ such that the restriction of $*$ to $\mathcal{O}_B$ is still defined, and is therefore

also an involution. We assume that $\mathcal{O}_B$ is maximal at some unramified prime p (i.e. a prime where the semisimple algebra is split, so $B_{\mathbb{Q}_p} \cong M_n(K)$ for K some unramified extension of $\mathbb{Q}_p$). We furthermore fix a finite-dimensional B -module V , and an $\mathcal{O}_B$ -lattice L satisfying condition 1.4.3.10. in [Lan13], together with an alternating bilinear form $\langle \cdot, \cdot \rangle : L \times L \rightarrow \mathbb{Z}(1)$, which we assume to be non-degenerate, and to satisfy the equality $\langle bx, y \rangle = \langle x, b^*y \rangle$ for all $x, y \in L$ and $b \in \mathcal{O}_B$. We assume furthermore that this pairing is self-dual when localized at p . Here, $\mathbb{Z}(1) = \ker(\exp : \mathbb{C} \rightarrow \mathbb{C}^\times)$, so in particular it is a free $\mathbb{Z}$ -module of rank 1, and each choice of $\sqrt{-1}$ in $\mathbb{C}$ determines an isomorphism between $\mathbb{Z}(1)$ and $\mathbb{Z}$ (but there is no canonical one). Finally, we fix an $\mathbb{R}$ -algebra homomorphism $h : \mathbb{C} \rightarrow \text{End}_{(\mathcal{O}_B \otimes \mathbb{R})}(L \otimes \mathbb{R})$, satisfying that for all $x, y \in L \otimes \mathbb{R}$ and $c \in \mathbb{C}$, $\langle h(c)x, y \rangle = \langle x, h(\bar{c})y \rangle$, and that $\langle \cdot, \sqrt{-1} \cdot \rangle$ is symmetric and positive definite for any choice of isomorphism $\mathbb{Z}(1) \cong \mathbb{Z}$ as above. The tuple $(\mathcal{O}_B, *, L, \langle \cdot, \cdot \rangle, h)$ is called an integral PEL-datum. To such data, we can associate the unitary similitude group over $\mathbb{Z}$, defined as

$$G(R) := \{(g, r) \in \text{GL}_{\mathcal{O}_B \otimes R}(L \otimes R) \times R^\times \mid \langle gx, gy \rangle = r \langle x, y \rangle \ \forall x, y \in L \otimes R\},$$

for any $\mathbb{Z}$ -algebra R . We assume that the derived group of G has type A or C .

As explained in [Lan13], sections 1.2.1. and 1.2.5., if we tensor h by $\mathbb{C}$, we obtain an action of $\mathbb{C}$ on $L \otimes \mathbb{C}$, which induces a decomposition $L \otimes \mathbb{C} = V_0 \oplus \bar{V}_0$, where $h(z)$ acts as $1 \otimes z$ on V_0 and as $1 \otimes \bar{z}$ on $\bar{V}_0$, and both these spaces are totally isotropic with respect to the pairing $\langle \cdot, \cdot \rangle$. We let F_0 be the field of definition of V_0 as an $\mathcal{O}_B \otimes \mathbb{C}$ -module (c.f. [Lan13, 1.2.5.4]). Taking X to be the conjugacy class of h , we obtain a Shimura datum (G, X) . We denote $S_0 = \text{Spec}(\mathcal{O}_{F_0, (p)})$.

Remark 2.1.1. • Our assumptions on the PEL-datum, and namely the conditions at p , are equivalent to p being a good prime in the sense of [Lan13, Defn. 1.4.1.1.], c.f. [Box15, Defn. 2.1.6.].

- The condition we impose on the lattice L (namely condition 1.4.3.10. of [Lan13]) essentially states that the $\mathcal{O}_B$ action must extend to an action of a maximal order of in B containing $\mathcal{O}_B$. It will be important later on, to insure that a certain filtration is split.

In order to define the moduli spaces of abelian schemes associated to the data above, we need to explain Kottwitz's determinantal condition.

Definition 2.1.2. 1. Let $\alpha_1, \dots, \alpha_n$ be any basis of $\mathcal{O}_B$ as a $\mathbb{Z}$ -algebra, and $\alpha_1^\vee, \dots, \alpha_n^\vee$ the dual basis in $\mathcal{O}_B^\vee := \text{Hom}(\mathcal{O}_B, \mathbb{Z})$. There is an isomorphism of free $\mathbb{Z}$ -modules $\mathcal{O}_B^\vee \cong \mathbb{Z}X_1 \oplus \dots \oplus \mathbb{Z}X_n$ by sending $\alpha_i^\vee$ to X_i for all i .

For any finitely generated $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{C}$ -module M , we can define

$$\text{Det}_{\mathcal{O}_B|M} = \text{Det}_{\mathcal{O}_B|M}^{\alpha_1, \dots, \alpha_n}(X_1, \dots, X_n) := \text{Det}_{\mathbb{C}}(\alpha_1 X_1 + \dots + \alpha_n X_n | M) \in \mathbb{C}[\mathcal{O}_B^\vee].$$

2. Let $S \rightarrow S_0$ be a scheme. Let $\alpha_1, \dots, \alpha_n$ and $\alpha_1^\vee, \dots, \alpha_n^\vee$ be basis of $\mathcal{O}_B$ resp. $\mathcal{O}_B^\vee$, and consider $\mathcal{O}_S[\mathcal{O}_B^\vee] \cong \mathcal{O}_S[X_1, \dots, X_n]$, once more given by sending $\alpha_i^\vee$ to X_i for all i .

For any locally free $\mathcal{O}_S$ module $\mathcal{M}$ of finite rank, with an $\mathcal{O}_S$ -linear action of $\mathcal{O}_B$, define

$$\text{Det}_{\mathcal{O}_B|\mathcal{M}} = \text{Det}_{\mathcal{O}_B|\mathcal{M}}^{\alpha_1, \dots, \alpha_n}(X_1, \dots, X_n) := \text{Det}_{\mathcal{O}_S}(\alpha_1 X_1 + \dots + \alpha_n X_n | \mathcal{M}) \in \mathcal{O}_S[\mathcal{O}_B^\vee].$$

3. Let $(L \otimes \mathbb{R}, \langle, \rangle, h)$ be a triple as given in an integral PEL-datum. Recall that $L \otimes \mathbb{C} = V_0 \oplus \bar{V}_0$. Consider an abelian scheme $A \rightarrow S$ over S_0 , together with a prime-to- p polarization and an $\mathcal{O}_B$ -action, inducing an $\mathcal{O}_B$ -action on $\underline{\text{Lie}}_{A/S}$. $\underline{\text{Lie}}_{A/S}$ is said to satisfy the Kottwitz determinantal condition defined by $(L \otimes \mathbb{R}, \langle, \rangle, h)$ if there is an equality

$$\text{Det}_{\mathcal{O}_B|V_0} = \text{Det}_{\mathcal{O}_B|\underline{\text{Lie}}_{A/S}}$$

over $\mathcal{O}_S[\mathcal{O}_B^\vee]$.

We are only interested in the case of principal level- n structures, for n prime-to- p . In particular, fix such a n , we assume that $n \geq 3$, and let

$$K = K(n) = \{g \in G(\hat{\mathbb{Z}}) \mid g \equiv 1 \pmod{n}\}.$$

Definition 2.1.3. Let $\mathcal{S}_K$ be the functor over S_0 parametrizing over a test scheme S tuples $(A, \lambda, \iota, (\alpha_n, \nu_n))$ consisting of:

1. an abelian scheme A over S ;
2. a prime-to- p polarization $\lambda : A \rightarrow A^\vee$;
3. an $\mathcal{O}_B$ -action $\iota : \mathcal{O}_B \rightarrow \text{End}_S(A)$ satisfying the equality $\iota(b)^\vee \circ \lambda = \lambda \circ \iota(b^*)$ for every $b \in \mathcal{O}_B$;
4. $\underline{\text{Lie}}_{A/S}$ must satisfy the determinantal condition defined by $(L \otimes \mathbb{R}, \langle, \rangle, h)$ as in 2.1.2;
5. (α_n, ν_n) is a level- n structure: namely, $\alpha_n : (L/n)_S \xrightarrow{\sim} A[n]$ is an $\mathcal{O}_B$ -equivariant isomorphism of group schemes over S , and $\nu_n : (\mathbb{Z}/n\mathbb{Z}(1))_S \xrightarrow{\sim} (\mu_n)_S$ is an isomorphism of group schemes over S such that the diagram

$$\begin{array}{ccc} (L/n)_S \times (L/n)_S & \longrightarrow & (\mathbb{Z}/n\mathbb{Z}(1))_S \\ \downarrow & & \downarrow \\ A[n] \times A[n] & \longrightarrow & (\mu_n)_S \end{array}$$

commutes, where the top horizontal arrow is the alternating pairing on L , the bottom horizontal arrow is the Weil pairing on $A[n]$ induced from the polarization, and the two vertical arrows are the obvious ones. We furthermore require that the level- n structure is symplectic liftable in the sense of [Lan13, Defn. 1.3.6.2.]). This roughly means that α_n lifts compatibly to isomorphisms $\alpha_m : (L/m)_S \xrightarrow{\sim} A[m]$ for all m prime-to- p which are divided by n .

Two such tuples $(A, \lambda, \iota, (\alpha_n, \nu_n))$ and $(A', \lambda', \iota', (\alpha'_n, \nu'_n))$ are equivalent if there is an isomorphism $f : A \rightarrow A'$, compatible with the $\mathcal{O}_B$ -action, such that $\lambda = f^\vee \circ \lambda' \circ f$, and such that $\alpha_n = f \circ \alpha'_n$, $\nu_n = \nu'_n$.

One can show that this functor is representable by a finite type, smooth scheme over S_0 , see for example [Lan13, Section 7.2.4].

2.2 Degeneration data

Toroidal compactifications of integral models of Shimura Varieties were constructed on the Siegel case by Faltings and Chai on their book [FC90], and later generalized to the PEL-case by Lan in his thesis, [Lan13]. It has since been further generalized to Hodge-type Shimura varieties by Keerthi Madapusi Pera in [Pera11]. The main idea for this construction (both in [FC90] and [Lan13]), consists in proving that there exists an equivalence between two categories: one of them is very much similar to our abelian tuples $(A, \lambda, \iota, (\alpha_n, \nu_n))$ above, but instead of an abelian scheme we will be considering semi-abelian schemes with extra structures. The other category is what is called degeneration data, and consists of a collection of objects characterizing exactly the degeneration of abelian varieties into semi-abelian varieties. We will try to be very precise when describing this categories, and we will also explain how to construct the functor in one direction, which is relatively easy to understand. It will become clear when analyzing the construction and local behavior of the toroidal compactifications why the degeneration data is so important: one needs to construct a space parametrizing the degeneration data over a PEL-type Shimura variety, and then, using a gluing process, one is able to construct a proper scheme containing our original Shimura variety as an open dense subset having a semi-abelian universal object (depending on a choice of a cone decomposition), and being "locally" defined by these moduli spaces. Since we are only interested in the PEL-case, we will mainly follow [Lan13] in this section. There are also good summaries on the topic in [Wu19] and [IM11].

Throughout this section, we will be considering objects over an affine scheme $S = \text{Spec}(R)$, where R is a Noetherian, normal domain which is I -complete for some ideal $I \in R$, and with fraction field K . We let $R_0 = R/I$, and $R_i = R/I^{i+1}$ for $i \in \mathbb{N}$, and S_0, S_i accordingly. We also let $\eta = \text{Spec}(K)$ and $S_{\text{for}} = \text{Spf}(R, I)$. We will always work with an integer n , which we assume is prime-to- p .

2.2.1 Main theorem on Degeneration

We start by recalling the definition of a semi-abelian scheme

Definition 2.2.1. A semi-abelian scheme G over S is a separated, smooth commutative group scheme with geometrically connected fibers, which is fiberwise given by an extension of an abelian variety by a torus.

We can immediately define one of the categories we are interested in:

Definition 2.2.2. Let $\text{DEG}_{\text{PEL}, \mathcal{S}_K}(R, I)$ denote the category whose objects are given by tuples $(G, \lambda_\eta, \iota, \alpha_n)$, where:

1. G is a semi-abelian scheme over S , with abelian generic fiber G_η , and such that $G_0 = G \times_S S_0$ is globally an extension of a torus by an abelian variety, so in particular one has a short exact sequence

$$0 \rightarrow T_0 \rightarrow G_0 \rightarrow A_0 \rightarrow 0$$

where T_0 is an isotrivial torus, and A_0 an abelian variety over S_0 ;

2. $\lambda_\eta : G_\eta \rightarrow G_\eta^\vee$ is a polarization of the generic fiber G_η of G ;
3. $\iota : \mathcal{O}_B \rightarrow \text{End}_S(G)$ is a ring homomorphism, satisfying the equality $\iota_\eta(b)^\vee \circ \lambda_\eta = \lambda_\eta \circ \iota_\eta(b^*)$ on the generic fiber for every $b \in \mathcal{O}_B$;
4. $\underline{\text{Lie}}_{G_\eta/S}$ satisfies the determinantal condition defined by $(L \otimes \mathbb{R}, \langle \cdot, \cdot \rangle, h)$;
5. $\alpha_n = (\alpha_n, \nu_n)$ is a level- n structure on G_η as in definition 2.1.3. In particular, $\alpha_n : L/nL \xrightarrow{\sim} G[n]_\eta$, and $\nu_n : \mathbb{Z}/n\mathbb{Z}(1) \xrightarrow{\sim} \mu_n$ are isomorphisms of group schemes over S , such that $(G_\eta, \lambda_\eta, \iota_\eta, \alpha_n) \in \mathcal{S}_K(S)$.

Morphisms are isomorphisms respecting all the extra structure.

Remark 2.2.3. Note that in the definition of semi-abelian scheme, one asks that G is fiberwise an extension of an abelian variety by a torus, but this does not imply that this is true globally. In particular, the condition that G_0 is an extension of an abelian variety by a torus is significant.

Remark 2.2.4. Under our assumptions on S , for any polarization λ_η as in (2), one can actually prove that there exists a unique lifting to a homomorphism $\lambda : G \rightarrow G^\vee$. More explicitly, by [Ray70, IX, 1.13], since S is Noetherian and normal, G carries some ample cubical invertible sheaf $\mathcal{L}$. In particular, we know that one can also define $G_\eta^\vee$ as the quotient $G_\eta/K(\mathcal{L}_\eta)$, where $K(\mathcal{L}_\eta)$ is the kernel of the polarization $\lambda(\mathcal{L}_\eta) : G_\eta \rightarrow G_\eta^\vee$ induced by $\mathcal{L}_\eta$ (c.f. [Mum70]). $K(\mathcal{L}_\eta)$ extends to a flat quasi-finite subscheme $K(\mathcal{L})$ of G over S by [MB85, IV, 2.4.]. Define $G^\vee = G/K(\mathcal{L})$ (which is still a semi-abelian scheme by [MB85, IV, 7.1.2.]), extending $G_\eta/K(\mathcal{L}_\eta)$. The existence of the lifting λ as above is then proven in [Lan13, Theorem 3.4.3.2.], following [MB85, IV, 7.1.]. We would like to point out that by construction $(G^\vee)_\eta = (G_\eta)^\vee$. In particular, we can unambiguously consider objects $(G, \lambda, \iota, \alpha_n)$ with $\lambda : G \rightarrow G^\vee$ as above.

Remark 2.2.5. We could have also defined ι as an action on G_η , since $\text{End}_S(G) \cong \text{End}_\eta(G_\eta)$ (c.f. [FC90, Chapter 2]).

The above remarks immediately show that all the extra structure on G are completely determined by its restriction to the generic fiber.

This finishes the definition of one of the categories we mentioned in the beginning of the section. Before defining the second one, let us introduce the concept of a cusp label.

Definition 2.2.6. Consider the set of tuples $(Z_n, (\underline{X}, \underline{Y}, \phi, \varphi_{-2,n}, \varphi_{0,n}), \delta_n)$, where:

1. Z_n is a filtration

$$Z_n = \{0 \subset Z_{n,-2} \subset Z_{n,-1} \subset Z_{n,0} = L/nL\}$$

on L/nL that can be written as the reduction mod- n of a filtration

$$0 \subset Z_{-2} \subset Z_{-1} \subset Z_0 = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$$

on $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$. This filtration must lift to a filtration $Z_{\mathbb{A}^p}$ on $L \otimes_{\mathbb{Z}} \mathbb{A}^p$, that is symplectic ($Z_{\mathbb{A}^p, -1}$ and $Z_{\mathbb{A}^p, -2}$ are each others' annihilators under the alternating pairing $\langle \cdot, \cdot \rangle_{\mathbb{A}^p}$ on $L \otimes_{\mathbb{Z}} \mathbb{A}^p$), integrable (meaning that each graded piece $\text{Gr}_{\mathbb{A}^p, -i} = Z_{\mathbb{A}^p, -i} / Z_{\mathbb{A}^p, -i-1}$ is equal to $M_i \otimes_{\mathbb{Z}} \mathbb{A}^p$ for some torsion-free, finitely generated $\mathcal{O}_B$ -module M_i) and split (so, there is an isomorphism $L \otimes_{\mathbb{Z}} \mathbb{A}^p \cong \bigoplus_i \text{Gr}_{\mathbb{A}^p, -i}$);

2. $\underline{X}$ and $\underline{Y}$ are étale sheaves of free, commutative groups that have the same rank, and $\phi : \underline{Y} \rightarrow \underline{X}$ is an injective homomorphism with finite cokernel. Furthermore, we assume that both sheaves are equipped with $\mathcal{O}_B$ -module structures $\iota_X : \mathcal{O}_B \rightarrow \text{End}_S(\underline{X})$, $\iota_Y : \mathcal{O}_B \rightarrow \text{End}_S(\underline{Y})$, and ϕ must be $\mathcal{O}_B$ -equivariant. For simplicity, we will actually assume from now on that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y ;

3. $\varphi_{-2,n}$ and $\varphi_{0,n}$ are liftable isomorphisms

$$\varphi_{-2,n} : \text{Gr}_{-2,n} \xrightarrow{\sim} \text{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1))$$

and

$$\varphi_{0,n} : \text{Gr}_{0,n} \xrightarrow{\sim} Y/nY$$

, satisfying the equation

$$\langle \varphi_{-2,n}(\cdot), \phi \circ \varphi_{0,n}(\cdot) \rangle_{\text{can}} = \langle \cdot, \cdot \rangle_{20,n} \quad (2.1)$$

where $\langle \cdot, \cdot \rangle_{\text{can}} : \text{Hom}((X/n, (\mathbb{Z}/n\mathbb{Z})(1)) \times X/nX \rightarrow (\mathbb{Z}/n\mathbb{Z})(1))$ is the canonical pairing, and $\langle \cdot, \cdot \rangle_{20,n} : \text{Gr}_{-2,n} \times \text{Gr}_{0,n} \rightarrow (\mathbb{Z}/n\mathbb{Z})(1)$ is the pairing induced from the pairing in L/nL , that exists by the symplectic condition on Z_n ;

4. $\delta_n : \bigoplus_i \text{Gr}_{-i,n} \xrightarrow{\sim} L/nL$ is a liftable choice of splitting.

We then define an equivalence class on this set: two such tuples $(Z_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), \delta_n)$ and $(Z'_n, (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n}), \delta'_n)$ are equivalent if $Z_n = Z'_n$ and there exists a pair of isomorphisms $\gamma_X : X' \xrightarrow{\sim} X$ and $\gamma_Y : Y \xrightarrow{\sim} Y'$ such that $\phi = \gamma_X \circ \phi' \circ \gamma_Y$, $\varphi'_{-2,n} = \gamma_X^\vee \circ \varphi_{-2,n}$ and $\varphi'_{0,n} = \gamma_Y \circ \varphi_{0,n}$. A cusp label at level n is an equivalence class of such tuples, denoted $[(Z_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), \delta_n)]$.

Definition 2.2.7. Let $\text{DD}_{\text{PEL}, \mathcal{S}_K}(R, I)$ denote the category whose objects are given by tuples $(A, \lambda_A, \iota_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha_n^\natural])$, where:

1. (A, λ_A, ι_A) is an abelian scheme over S together with a polarization $\lambda_A : A \rightarrow A^\vee$, and an $\mathcal{O}_B$ -action $\iota_A : \mathcal{O}_B \rightarrow \text{End}_S(A)$ satisfying $\iota_A(b)^\vee \circ \lambda_A = \lambda_A \circ \iota_A(b^*)$ for every $b \in \mathcal{O}_B$;
2. $\underline{X}$ and $\underline{Y}$ are as above, and we again assume these are both constant with values X and Y ;
3. Two $\mathcal{O}_B$ -equivariant homomorphisms

$$c : X \rightarrow A^\vee$$

and

$$c^\vee : Y \rightarrow A$$

satisfying the equality $\lambda_A \circ c^\vee = c \circ \phi$;

4. τ is a trivialization

$$\tau : \mathbf{1}_{Y \times_S X, \eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{A, \eta}^{\otimes -1}$$

where $\mathcal{P}_{A, \eta}$ is the Poincaré sheaf on $A_\eta \times A_\eta^\vee$, satisfying the symmetry and the positivity conditions, which we will explain bellow. Furthermore, τ must be compatible with the $\mathcal{O}_B$ -action, i.e. it must satisfy the equality

$$(\iota_Y(b) \times \text{id}_X)^* \tau = (\text{id}_Y \times \iota_X(b^*))^* \tau$$

for all $b \in \mathcal{O}_B$.

5. The level- n data $\alpha_n^\natural = (Z_n, \varphi_{-2, n}, (\varphi_{-1, n}, \nu_{-1, n}), \varphi_{0, n}, \delta_n, c_n, c_n^\vee, \tau_n)$ is a tuple such that $(Z_n, (X, Y, \phi, \varphi_{-2, n}, \varphi_{0, n}), \delta_n)$ is a tuple as in the definition of a cusp label, so Z_n is a filtration on L/nL that lifts to a symplectic, integrable, split filtration on $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$, $\varphi_{-2, n}$ and $\varphi_{0, n}$ are liftable isomorphisms $\varphi_{-2, n} : \text{Gr}_{-2, n} \xrightarrow{\sim} \text{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1))$ and $\varphi_{0, n} : \text{Gr}_{0, n} \xrightarrow{\sim} Y/nY$, satisfying the equations above. Furthermore, $\varphi_{-1, n} : \text{Gr}_{-1, n} \xrightarrow{\sim} A[n]_\eta$ and $\nu_{-1, n} : \mathbb{Z}/n\mathbb{Z}(1) \xrightarrow{\sim} \mu_{n, \eta}$ are isomorphisms making the tuple $(A_\eta, \lambda_{A, \eta}, \iota_{A, \eta}, (\varphi_{-1, n}, \nu_{-1, n}))$ an element of $\mathcal{S}_K(\eta)$ with respect to the PEL datum given by the lattice $\text{Gr}_{-1, n}$, which has an induced alternating pairing also by the conditions we assumed on the filtration. $\delta_n : \bigoplus_i \text{Gr}_{-i, n} \xrightarrow{\sim} L/nL$ is a liftable choice of splitting. c_n and $c_n^\vee$ are two homomorphisms

$$c_n : \frac{1}{n}X_\eta \rightarrow A_\eta^\vee$$

and

$$c_n^\vee : \frac{1}{n}Y_\eta \rightarrow A_\eta$$

that lift c and $c^\vee$ respectively over the generic fiber, i.e. such that $c_\eta = c_n \circ (X_\eta \hookrightarrow \frac{1}{n}X_\eta)$ and $c_\eta^\vee = c_n^\vee \circ (Y_\eta \hookrightarrow \frac{1}{n}Y_\eta)$. These must be compatible with the splitting δ_n , that is, it must satisfy the equality,

$$\langle \varphi_{-1, n}(\cdot), (\lambda_A \circ c_n^\vee - c_n \circ \phi) \circ \varphi_{0, n}(\cdot) \rangle_{A_\eta} = \nu_{-1, n} \circ \langle \cdot, \cdot \rangle_{10, n}$$

where $\langle \cdot, \cdot \rangle_{A_\eta}$ is the Weil pairing on $A[n]_\eta$ and $\langle \cdot, \cdot \rangle_{10,n} : \text{Gr}_{-1,n} \times \text{Gr}_{0,n} \rightarrow (\mathbb{Z}/n\mathbb{Z})(1)$ is defined as $\langle \cdot, \cdot \rangle_{10,n} = \langle \delta_n(\cdot), \delta_n(\cdot) \rangle$. They also need to satisfy a liftability condition as in [Lan13, 5.2.7.8]. Finally, τ_n is a trivialization

$$\tau_n : \mathbf{1}_{\frac{1}{n}Y \times_S X, \eta} \xrightarrow{\sim} (c_n^\vee \times c)^* \mathcal{P}_{A, \eta}^{\otimes -1}$$

lifting τ , that is also required to satisfy certain compatibility relations with δ_n . Namely, let

$$\langle \cdot, \cdot \rangle_{00} : \text{Gr}_{0,n} \times \text{Gr}_{0,n} \rightarrow (\mathbb{Z}/n\mathbb{Z})(1)$$

be defined as $\langle \cdot, \cdot \rangle_{00} = \langle \delta_n(\cdot), \delta_n(\cdot) \rangle$, and

$$d_{00,n} : \frac{1}{n}Y/Y \times \frac{1}{n}Y/Y \rightarrow \mu_{n,\eta}$$

is given by $d_{00,n}(\frac{1}{n}y, \frac{1}{n}y') := \tau_n(\frac{1}{n}y, \phi(y'))\tau_n(\frac{1}{n}y', \phi(y))^{-1}$. Then, the equality

$$d_{00,n}(\varphi_{0,n}(\cdot), \varphi_{0,n}(\cdot)) = \nu_{-1,n} \circ \langle \cdot, \cdot \rangle_{00,n}$$

must hold.

One then defines an equivalence relation on these tuples, to eliminate the choice of splitting (as explained in [Lan13, Section 5.2.7]). All tuples in an equivalence class will have the same $(Z_n, \varphi_{-2,n}, (\varphi_{-1,n}, \nu_{-1,n}), \varphi_{0,n})$, so only the remaining data changes. See [Lan13, 5.2.7.11.] for the full definition.

6. $(A_\eta, \lambda_{A,\eta}, \iota_{A,\eta})$ satisfies the determinantal condition determined by $(\text{Gr}_{-1,\mathbb{R}}^Z, \langle \cdot, \cdot \rangle_{11,\mathbb{R}}, h_{-1})$, where $\langle \cdot, \cdot \rangle_{11,\mathbb{R}} : \text{Gr}_{-1,\mathbb{R}}^Z \times \text{Gr}_{-1,\mathbb{R}}^Z \rightarrow \mathbb{R}$ and $h_1 : \mathbb{C} \rightarrow \text{End}_{\mathcal{O}_B \otimes \mathbb{R}}(\text{Gr}_{-1,\mathbb{R}}^Z)$ are given by [Lan13, Prop. 5.1.1.2.] (c.f. [Lan13, Rmk. 1.4.3.14.] and Lemma 5.2.2.2.).

Morphisms are isomorphisms respecting all the extra structure.

Theorem 2.2.8. [Lan13, Thm. 5.2.7.14.] *There is an equivalence of categories between $\text{DEG}_{\text{PEL}, \mathcal{S}_K}(R, I)$ and $\text{DD}_{\text{PEL}, \mathcal{S}_K}(R, I)$.*

2.2.2 Explanation of the functor

We now wish to explain how to construct the functor from $\text{DEG}_{\text{PEL}, \mathcal{S}_K}(R, I)$ to $\text{DD}_{\text{PEL}, \mathcal{S}_K}(R, I)$. We will do this step-by-step.

Let us start with pairs (G, λ) , where G is a semi-abelian scheme over S satisfying the properties described in 2.2.2, and λ is a polarization. The category of these pairs, with morphisms being isomorphisms compatible with λ , is usually denoted by $\text{DEG}_{\text{pol}}(R, I)$. Then, by assumption, we have a short exact sequence

$$0 \rightarrow T_0 \rightarrow G_0 \rightarrow A_0 \rightarrow 0$$

over S_0 . By [Lan13, Thm. 3.1.1.2.], one can lift T_0 to a multiplicative subgroup of G_i (recall that $G_i = G \times_S \text{Spec}(R/I^{i+1})$) over S_i , and its quotient is an abelian scheme (c.f. [Lan13, Section 3.3.3.3]). In particular, we obtain short exact sequences

$$0 \rightarrow T_i \rightarrow G_i \rightarrow A_i \rightarrow 0$$

for all integers i , and thus a sequence of formal schemes

$$0 \rightarrow T_{\text{for}} \rightarrow G_{\text{for}} \rightarrow A_{\text{for}} \rightarrow 0$$

where T_{for} is a formal torus and A_{for} is a formal abelian scheme S_{for} . We would now like to know that the above sequence is algebraizable, meaning that there exists a torus T , a group scheme $\tilde{G}$ and an abelian scheme over S such that $T_{\text{for}} = T \otimes S_{\text{for}}$, etc., so that

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow A \rightarrow 0$$

is still a short exact sequence. This is not true in general, but if one can prove that there exists an ample cubical invertible sheaf on G , then one can use Grothendieck's existence theorem to see that this is indeed the case, c.f. [Lan13, 3.3.3.6.], [EGA]. The existence of this cubical invertible sheaf follows by a hard theorem of Grothendieck, stating that under the normality hypothesis on S , G is actually quasi-projective (c.f. [Ray70, XI, 1.13.]).

The resulting extension

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow A \rightarrow 0$$

is called the Raynaud extension associated to G , and its construction is functorial. We denote by $\underline{X}$ the character group of T , $\underline{X} = \underline{X}(T) = \text{Hom}(T, \mathbb{G}_m)$. We assume for simplicity, up to passing to an étale extension, that T is split, so that $\underline{X}$ is a constant group with value X . One can see that giving such an extension

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow A \rightarrow 0$$

is equivalent to giving a group homomorphism $c : X \rightarrow A^\vee$ as follows: let $\chi : T \rightarrow \mathbb{G}_m$ be a character of T . Then we can consider the push-out diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T & \longrightarrow & \tilde{G} & \xrightarrow{\pi} & A \longrightarrow 0 \\ & & \downarrow \chi & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{O}^X & \longrightarrow & A \longrightarrow 0 \end{array}$$

where $\mathcal{O}^X = \tilde{G} \times^{T, X} \mathbb{G}_m$ is the push-out of the left square. $\mathcal{O}^X$ is actually an element of $\text{Pic}^0(A/S) = A^\vee$ (c.f. [Lan13, Prop. 3.1.5.1.]), holding the desired homomorphism. One can also think about this sheaf in another way: we think of $\tilde{G}$ as a T -torsor over A . Then, it is relatively affine, and one can prove that it is given by

$$\tilde{G} \cong \underline{\text{Spec}}_{\mathcal{O}_A}(\pi_* \mathcal{O}_{\tilde{G}}) \cong \underline{\text{Spec}}_{\mathcal{O}_A}(\bigoplus_{\chi \in X} \mathcal{O}_\chi),$$

where $\mathcal{O}_\chi$ is the eigensheaf of $\pi_*\mathcal{O}_{\tilde{G}}$ with weight χ for the T -action, and $\pi : \tilde{G} \rightarrow A$. If one thinks about a T -torsor "locally", as the data of local trivializations and gluing data, and one analyzes both $\mathcal{O}_\chi$ and $\mathcal{O}^\chi$, then it is possible to conclude that $\mathcal{O}_\chi = \mathcal{O}^{-\chi}$. This is done explicitly in [Lan13, 3.1.2.]. In order to avoid sign problems, we will from now on change the sign of c so that $c(\chi) = \mathcal{O}_\chi$. The fact that c is a group homomorphism, endows $\bigoplus_{\chi \in X} \mathcal{O}_\chi$ with an $\mathcal{O}_A$ -algebra structure, coming from isomorphisms $\mathcal{O}_{\chi+\chi'} \cong \mathcal{O}_\chi \otimes_{\mathcal{O}_A} \mathcal{O}_{\chi'}$.

One can construct $c^\vee$ and $\underline{Y}$ in a similar way, after defining the dual Raynaud extension. Recall that we have already seen that there exists a "dual" semiabelian scheme $G^\vee$ of G , that is identified with the actual dual on the generic fiber, so that in particular applying the same strategy to $G^\vee$ as we did with G above, we obtain the so called dual Raynaud extension

$$0 \rightarrow T^\vee \rightarrow \tilde{G}^\vee \rightarrow A^\vee \rightarrow 0,$$

where $A^\vee$ is, as one would expect, the dual of A as seen in [Lan13, Prop. 3.4.4.1.]. In particular, the notation makes sense. We denote by $\underline{Y}$ the character group of $T^\vee$, and note that $T^\vee$ must be split if T is, and thus by our assumptions, $\underline{Y}$ is constant with value Y . Using the exact same computations as above yields the construction of the group homomorphism $c^\vee : Y \rightarrow A$.

The next step will be to explain the construction of ϕ and λ_A . Note that a map $\phi : Y \rightarrow X$ is equivalent to a map between the tori $T \rightarrow T^\vee$. Both ϕ and λ_A are induced from the polarization λ on G as follows: by functoriality of Raynaud extensions, λ induces a homomorphism $\tilde{\lambda} : \tilde{G} \rightarrow \tilde{G}^\vee$ between the Raynaud extensions, inducing a polarization on A . By restriction to the torus, this induces the desired map $\tilde{\lambda}|_T : T \rightarrow T^\vee$, and a polarization on A , giving therefore rise to a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T & \longrightarrow & \tilde{G} & \xrightarrow{\pi} & A \longrightarrow 0 \\ & & \downarrow \phi & & \downarrow \tilde{\lambda} & & \downarrow \lambda_A \\ 0 & \longrightarrow & T^\vee & \longrightarrow & \tilde{G}^\vee & \longrightarrow & A^\vee \longrightarrow 0. \end{array}$$

The equality $\lambda_A \circ c^\vee = c \circ \phi$ is equivalent to the commutativity of the above diagram. Also, ϕ and λ_A uniquely determine $\tilde{\lambda}$, since there are no nontrivial homomorphisms from an abelian scheme to a torus.

This shows how one obtains a tuple $(A, \lambda_A, \iota_A, X, Y, \phi, c, c^\vee)$ from $(G, \lambda) \in \text{DEG}_{\text{pol}}(R, I)$. The only missing piece on our tuple $(A, \lambda_A, \iota_A, X, Y, \phi, c, c^\vee, \tau, [\alpha_n^h]) \in \text{DD}_{\text{PEL}, \mathcal{S}_K}(R, I)$ (forgetting the $\mathcal{O}_B$ -actions and the Lie algebra condition) that we can still construct from our pair $(G, \lambda) \in \text{DEG}_{\text{pol}}(R, I)$ is τ , which is also the most complicated part.

Let us start by making a few observations. $\tau : \mathbf{I}_{Y \times_S X, \eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{A, \eta}^{\otimes -1}$ is a trivialization of the biextension $(c^\vee \times c)^* \mathcal{P}_{A, \eta}^{\otimes -1}$. The precise definition of a biextension is given in [Lan13, Defn. 3.2.1.1.], but τ being a trivialization of a biextension essentially means that it is bilinear, and the symmetry condition says that this bilinear form is symmetric. To be more precise, since X and Y are constant we can think of τ as a set of sections $\{\tau(y, \chi)\}$ of the line bundles $\mathcal{P}_{A, \eta}^{\otimes -1}(c^\vee(y), c(\chi)) = (c^\vee(y) \times c(\chi))^* \mathcal{P}_{A, \eta}^{\otimes -1} = (c^\vee(y))^* \mathcal{O}_{\chi, \eta}^{\otimes -1} = \mathcal{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$ over S_η for $y \in Y$

and $\chi \in X$. We think of these as maps

$$\tau(y, \chi) : \mathcal{O}_\chi(c^\vee(y))_\eta \rightarrow \mathcal{O}_{S, \eta}$$

using that $\mathcal{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$ is invertible. The first partial multiplicative map of biextensions structure gives an isomorphism

$$\mathcal{P}_{A, \eta}^{\otimes -1}(c^\vee(y_1), c(\chi)) \otimes_{\mathcal{O}_{S, \eta}} \mathcal{P}_{A, \eta}^{\otimes -1}(c^\vee(y_2), c(\chi)) \xrightarrow{\sim} \mathcal{P}_{A, \eta}^{\otimes -1}(c^\vee(y_1 + y_2), c(\chi))$$

for $y_1, y_2 \in Y$ and $\chi \in X$. Using the second partial multiplicative map of biextensions, we get a similar isomorphism on the second variable

$$\mathcal{P}_{A, \eta}^{\otimes -1}(c^\vee(y), c(\chi_1)) \otimes_{\mathcal{O}_{S, \eta}} \mathcal{P}_{A, \eta}^{\otimes -1}(c^\vee(y), c(\chi_2)) \xrightarrow{\sim} \mathcal{P}_{A, \eta}^{\otimes -1}(c^\vee(y), c(\chi_1 + \chi_2))$$

for $y \in Y$ and $\chi_1, \chi_2 \in X$. Then the bilinearity condition says that the image of $\tau(y_1, \chi) \otimes \tau(y_2, \chi)$ via the first map must be $\tau(y_1 + y_2, \chi)$, and similarly the image of $\tau(y, \chi_1) \otimes \tau(y, \chi_2)$ via the second map must be $\tau(y, \chi_1 + \chi_2)$. The symmetry condition is the only reasonable one after composing τ with $\text{id}_Y \circ \phi$.

We can also now state the positivity condition for τ : it says that for every $y \in Y$,

$$\tau(y, \phi(y)) : \mathcal{O}_{\phi(y)}(c^\vee(y))_\eta \rightarrow \mathcal{O}_{S, \eta}$$

lifts to a section on S ,

$$\tau(y, \phi(y)) : \mathcal{O}_{\phi(y)}(c^\vee(y)) \rightarrow \mathcal{O}_S$$

and that if y is not zero, then the image of $\tau(y, \phi(y))$ factors through $\underline{I}$ (the ideal sheaf generated by the ideal I).

Remark 2.2.9. One can also prove that having a trivialization τ is equivalent to having a period morphism $\iota : Y \rightarrow \tilde{G}_\eta$ lifting $c_\eta^\vee$. Indeed, to construct ι from τ , recall that we can think of $\tau(y, \chi)$ as sections

$$\tau(y, \chi) : \mathcal{O}_\chi(c^\vee(y))_\eta \rightarrow \mathcal{O}_{S, \eta},$$

and thus define maps

$$c^\vee(y)^* \mathcal{O}_{\tilde{G}, \eta} \cong c^\vee(y)^* \left(\bigoplus_{\chi \in X} \mathcal{O}_\chi \right) \xrightarrow{\sum_\chi \tau(y, \chi)} \mathcal{O}_{S, \eta},$$

for each $y \in Y$. One can show that this is an $\mathcal{O}_{S, \eta}$ -algebra homomorphism, using τ 's second multiplicative law. Putting all of these morphisms together gives the desired map $\iota : Y \rightarrow \tilde{G}_\eta$. It is actually a map of group schemes, because of the first partial multiplicative law. This is all explained in detail in [FC90].

We can now start the construction of τ . Recall that we have the Raynaud extension

$$0 \rightarrow T \xrightarrow{i} \tilde{G} \xrightarrow{\pi} A \rightarrow 0$$

and we have already mentioned that we can pick an ample cubical invertible sheaf $\mathcal{L}$ on the semi-abelian scheme G . We assume that the polarization on G_η is the one induced from $\mathcal{L}$, so $\lambda_\eta = \lambda_{\mathcal{L}_\eta}$. By considering the formal completion of $\mathcal{L}$, just as we have done when constructing the Raynaud extension, one can prove that $\mathcal{L}$ actually extends to an ample cubical invertible sheaf $\tilde{\mathcal{L}}$ on $\tilde{G}$ (c.f. [Lan13, Prop. 3.3.3.11.]). We pick a cubical trivialization $i^*\tilde{\mathcal{L}} \cong \mathcal{O}_T$, so that $\tilde{\mathcal{L}}$ descends to an ample invertible sheaf $\mathcal{M}$ on A , namely we have $\pi^*\mathcal{M} \cong \tilde{\mathcal{L}}$. Recall that we can think of $\tilde{G}$ as a relative affine scheme over A and write

$$\tilde{G} \cong \underline{\mathrm{Spec}}_{\mathcal{O}_A}(\pi_*\mathcal{O}_{\tilde{G}}) \cong \underline{\mathrm{Spec}}_{\mathcal{O}_A}\left(\bigoplus_{\chi \in X} \mathcal{O}_\chi\right)$$

where $\mathcal{O}_\chi$ is the eigensheaf of $\pi_*\mathcal{O}_{\tilde{G}}$ with weight χ , so in particular we have that

$$\pi_*\tilde{\mathcal{L}} \cong \bigoplus_{\chi \in X} \mathcal{M}_\chi = \bigoplus_{\chi \in X} \mathcal{M} \otimes_{\mathcal{O}_A} \mathcal{O}_\chi,$$

yielding the equality

$$\Gamma(\tilde{G}, \tilde{\mathcal{L}}) = \Gamma(A, \pi_*\mathcal{O}_{\tilde{G}}) = \bigoplus_{\chi \in X} \Gamma(A, \mathcal{M}_\chi).$$

Also recall that we have short exact sequences

$$0 \rightarrow T_i \rightarrow G_i \xrightarrow{\pi_i} A_i \rightarrow 0$$

over $\mathrm{Spec}(R/I^i)$, holding in the limit an algebraizable short exact sequence of formal schemes

$$0 \rightarrow T_{\mathrm{for}} \rightarrow G_{\mathrm{for}} \rightarrow A_{\mathrm{for}} \rightarrow 0$$

defining the Raynaud extension. In particular, we have equalities

$$\Gamma(G_i, \mathcal{O}_{G_i}) = \Gamma(A_i, \pi_{i,*}\mathcal{O}_{G_i}) = \bigoplus_{\chi \in X} \Gamma(A_i, \mathcal{O}_\chi)$$

and thus, taking the limit, yields equalities

$$\Gamma(G_{\mathrm{for}}, \mathcal{O}_{G_{\mathrm{for}}}) = \widehat{\bigoplus_{\chi \in X} \Gamma(A_{\mathrm{for}}, \mathcal{O}_{\chi, \mathrm{for}})} = \widehat{\bigoplus_{\chi \in X} \Gamma(A, \mathcal{O}_\chi)}$$

where $\widehat{\bigoplus}$ means I -adic completion and the last equality follows from [Lan13, Prop. 2.3.1.1.]. By the same reasoning, we have

$$\Gamma(G_i, \mathcal{L}_i) = \bigoplus_{\chi \in X} \Gamma(A_i, \mathcal{M}_{\chi, i})$$

and

$$\Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) = \widehat{\bigoplus_{\chi \in X} \Gamma(A, \mathcal{M}_\chi)}.$$

These are essentially saying that we can write an element $s \in \Gamma(G_{\mathrm{for}}, \mathcal{O}_{G_{\mathrm{for}}})$ or $s \in \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ as a sum

$$s = \sum_{\chi \in X} \sigma_\chi(s)$$

with $\sigma_\chi(s) \in \Gamma(A, \mathcal{O}_\chi)$ respectively $\sigma_\chi(s) \in \Gamma(A, \mathcal{M}_\chi)$, that is I -adic convergent, so for any i , there are only finitely many coefficients in this expression that are not zero modulo I^i . This is called the Fourier expansion of theta function. Precomposing the last equality with the inclusion

$$\Gamma(G, \mathcal{L}) \hookrightarrow \Gamma(G_{\text{for}}, \mathcal{L}_{\text{for}})$$

and taking the projection to the χ -component

$$\bigoplus_{\chi \in X} \Gamma(A, \mathcal{M}_\chi) \rightarrow \Gamma(A, \mathcal{M}_\chi)$$

gives a map

$$\sigma_\chi : \Gamma(G, \mathcal{L}) \rightarrow \Gamma(A, \mathcal{M}_\chi).$$

Tensoring both sides with K , we obtain a map

$$\sigma_\chi : \Gamma(G_\eta, \mathcal{L}_\eta) \rightarrow \Gamma(A_\eta, \mathcal{M}_{\chi, \eta})$$

between K -vector spaces. The idea now is that τ comes from the functional equation of σ_χ . In particular, one needs to study the relation between σ_χ and $\sigma_{\chi+\phi(y)}$ for $y \in Y$.

Let $y \in Y$, and $c^\vee(y) \in A(S)$ its image under the morphism $c^\vee : Y \rightarrow A$. Let $T_{c^\vee(y)} : A \rightarrow A$ be the translation by $c^\vee(y)$. By assumption, $\lambda_A = \lambda_{\mathcal{M}}$, so in particular the equality $\lambda_A(x) = T_x^* \mathcal{M} \otimes \mathcal{M}^{-1} \otimes \mathcal{M}_x^{-1}$ holds. Using this, and the relation $\lambda_A \circ c^\vee = c \circ \phi$, we obtain an isomorphism

$$T_{c^\vee(y)}^* \mathcal{M}_\chi \cong \mathcal{M}_{\chi+\phi(y)} \otimes_R \mathcal{M}_\chi(c^\vee(y)).$$

Indeed, recall that $c(\phi(y)) = \mathcal{O}_{\phi(y)}$ and that by definition $\mathcal{M}_\chi = \mathcal{M} \otimes_{\mathcal{O}_A} \mathcal{O}_\chi$. Using this isomorphism, we get a natural map

$$T_{c^\vee(y)}^* \circ \sigma_\chi : \Gamma(G_\eta, \mathcal{L}_\eta) \rightarrow \Gamma(A_\eta, T_{c^\vee(y)}^* \mathcal{M}_{\chi, \eta}) = \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y), \eta}) \otimes_K \mathcal{M}_\chi(c^\vee(y))_\eta.$$

The idea is to compare this to

$$\sigma_{\chi+\phi(y)} : \Gamma(G_\eta, \mathcal{L}_\eta) \rightarrow \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y), \eta}).$$

One sees using representations of theta groups that these are actually proportional, so that there is a functional equation of the form

$$\sigma_{\chi+\phi(y)} = \psi(y, \chi) T_{c^\vee(y)}^* \circ \sigma_\chi$$

for some unique section $\psi(y, \chi) \in \mathcal{M}_\chi(c^\vee(y))_\eta^{\otimes -1}$. By definition of $\mathcal{M}$, we have that

$$\mathcal{M}_\chi(c^\vee(y))_\eta^{\otimes -1} = \mathcal{M}(c^\vee(y))_\eta^{\otimes -1} \otimes_{\mathcal{O}_{S, \eta}} \mathcal{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$$

so in particular, can write $\psi(y, \chi) = \psi(y) \tau(y, \chi)$, where $\psi(y) : \mathcal{M}(c^\vee(y))_\eta \xrightarrow{\sim} \mathcal{O}_{S, \eta}$ is a trivialization of $\mathcal{M}(c^\vee(y))_\eta^{\otimes -1}$, and $\tau(y, \chi) : \mathcal{O}_\chi(c^\vee(y))_\eta \rightarrow \mathcal{O}_{S, \eta}$ is a section of $\mathcal{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$. This is exactly what we were searching for, and it is explained in section 4.3.1. in [Lan13]. The fact that

τ is uniquely characterized by the functional equation above follows from the fact that σ_χ is not the zero map over the generic fiber (c.f [Lan13, 4.3.1.]). Furthermore, τ does not depend on the choice of $\mathcal{L}$ by [Lan13, Lemma 4.3.1.14]. All the properties of τ now follow formally from the definition, as in [Lan13, Prop. 4.3.1.8.].

Let us denote the category consisting of tuples $(A, \lambda_A, \iota_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ as we have just constructed (so, tuples in $\mathrm{DD}_{\mathrm{PEL}, \mathcal{S}_K}(R, I)$ without splitting assumptions on $\underline{X}$ and $\underline{Y}$, level-structures, $\mathcal{O}_B$ -actions, and the Lie algebra condition) by $\mathrm{DD}_{\mathrm{pol}, \mathcal{S}_K}(R, I)$, with the morphisms being isomorphisms on A , $\underline{X}$ and $\underline{Y}$, that respect all the extra structures. We obtain the following.

Theorem 2.2.10. *[FC90, Ch. II, Thm. 6.2] There is a functor*

$$F_{\mathrm{pol}} : \mathrm{DEG}_{\mathrm{pol}, \mathcal{S}_K}(R, I) \rightarrow \mathrm{DD}_{\mathrm{pol}, \mathcal{S}_K}(R, I)$$

sending $(G, \lambda) \in \mathrm{DEG}_{\mathrm{pol}, \mathcal{S}_K}(R, I)$ to a tuple $(A, \lambda_A, \iota_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau) \in \mathrm{DD}_{\mathrm{pol}, \mathcal{S}_K}(R, I)$, inducing an equivalence of categories. If we assume that the character groups of the torus parts are split on both sides, then this functor is the one constructed above.

The inverse functor was constructed in the totally degenerate case by Mumford in [Mum72], and then extended to the case in hands by Faltings-Chai in [FC90] and Lan in [Lan13].

The analysis of the $\mathcal{O}_B$ -action data is not too hard: namely, the functoriality of F_{pol} induces an equivalence of categories, once we had the $\mathcal{O}_B$ -actions on both the categories $\mathrm{DEG}_{\mathrm{pol}, \mathcal{S}_K}(R, I)$ and $\mathrm{DD}_{\mathrm{pol}, \mathcal{S}_K}(R, I)$.

The analysis of the determinantal condition is done in detail by Lan in [Lan13], see section 5.1.2. for the details.

Finally, we want to explain how we obtain the level-structure data on a tuples

$$(A, \lambda_A, \iota_A, X, Y, \phi, c, c^\vee, \tau),$$

from the level- n structure α_n on G_η . Let us start by recalling that the level- n structure on degeneration data is given by a tuple $\alpha_n^\natural = (Z_n, \varphi_{-2,n}, (\varphi_{-1,n}, \nu_{-1,n}), \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$, consisting of a filtration on L/nL , liftings of c , $c^\vee$ and τ to $\frac{1}{n}X$, $\frac{1}{n}Y$ and $\frac{1}{n}Y \times X$ respectively, and some extra isomorphisms and splittings. We want to start by building the filtration Z_n on L/nL from a tuple $(G, \lambda, \iota, \alpha_n) \in \mathrm{DEG}_{\mathrm{PEL}, \mathcal{S}_K}(R, I)$. For that, we need the following proposition:

Proposition 2.2.11. *[Lan13, Prop. 5.2.2.1.] Let n be an integer coprime to p . There is a canonical, $\mathcal{O}_B$ -equivariant short exact sequence of finite étale S -group schemes*

$$0 \rightarrow \tilde{G}_\eta[n] \rightarrow G_\eta[n] \rightarrow Y/nY \rightarrow 0,$$

which extends to a short exact sequence on Tate modules

$$0 \rightarrow \mathrm{T}^p \tilde{G}_\eta \rightarrow \mathrm{T}^p G_\eta \rightarrow Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \rightarrow 0,$$

by taking the limit over all integers n coprime to p .

Recall that we also have short exact sequences,

$$0 \rightarrow T[n] \rightarrow \tilde{G}[n] \rightarrow A[n] \rightarrow 0$$

over S , by taking the n -torsion of the Raynaud extension. Once again, this induces a short exact sequence on the Tate modules

$$0 \rightarrow T^p T_\eta \rightarrow T^p \tilde{G}_\eta \rightarrow T^p A_\eta \rightarrow 0,$$

by taking the limit over integers n which are coprime to p . This immediately induces a filtration

$$0 \subset T[n]_\eta \subset \tilde{G}[n]_\eta \subset G[n]_\eta$$

on $G[n]_\eta$. We denote its graded pieces given by $\{\text{Gr}_i\}_{-2 \leq i \leq 0}$, and we notice that these are isomorphic to $T[n]_\eta$, $A[n]_\eta$ and $(Y/nY)_\eta$ respectively. Now notice that, if we have a level structure $(\alpha_n : L/nL \xrightarrow{\sim} G[n]_\eta, \nu_n : \mathbb{Z}/n\mathbb{Z}(1) \xrightarrow{\sim} \mu_n)$ as in definition 2.2.2, α_n induces the desired filtration

$$0 \subset Z_{n,-2} \subset Z_{n,-1} \subset Z_{n,0} = L/nL$$

on L/nL given by the pullback of the filtration on $G[n]_\eta$ that we just defined. Since the filtration on $G[n]_\eta$ extends to Tate-modules, this filtration will also satisfy a similar liftability condition. Namely, it lifts to a filtration

$$0 \subset Z_{-2} \subset Z_{-1} \subset Z_0 = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p.$$

The fact that the graded pieces $\text{Gr}_{n,i}$ are isomorphic to $T[n]_\eta$, $A[n]_\eta$ and $(Y/nY)_\eta$ and the liftability of the short exact sequences in proposition 2.2.11 to the Tate modules, together with the liftability conditions imposed on the level- n structure on G , imply that we obtain isomorphisms

$$\varphi_{-2,n} : \text{Gr}_{-2,n} \xrightarrow{\sim} T[n]_\eta = \text{Hom}(X/nX, \mu_n) \cong \text{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1)),$$

$$\varphi_{0,n} : \text{Gr}_{0,n} \xrightarrow{\sim} Y/nY$$

and

$$\varphi_{-1,n} : \text{Gr}_{-1,n} \xrightarrow{\sim} A[n]_\eta,$$

which lift to isomorphisms

$$\varphi_{-2} : \text{Gr}_{-2} \xrightarrow{\sim} \text{Hom}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p, \hat{\mathbb{Z}}^p(1)),$$

and

$$\varphi_0 : \text{Gr}_0 \xrightarrow{\sim} Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p,$$

as required. We take $\nu_{-1,n} := \nu_n : \mathbb{Z}/n\mathbb{Z}(1) \xrightarrow{\sim} \mu_{n,\eta}$. By [Lan13, Prop. 5.2.2.], we know that under the Weil pairing on $T^p G_\eta$ (which is an abelian scheme by assumption), $T^p T_\eta$ and $T^p \tilde{G}_\eta$

are identified as annihilators of each other, inducing therefore the Weil pairing on $T^p A_\eta$, and thus a pairing between $\text{Gr}_{-2,n}$ and $\text{Gr}_{0,n}$

$$e^\phi : T^p T_\eta \times (Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p) \rightarrow T^p \mathbb{G}_{m,\eta}.$$

One can prove that e^ϕ is actually just the canonical pairing, i.e. we know that we have an isomorphism $T^p T_\eta \cong \text{Hom}_{\hat{\mathbb{Z}}^p}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p, T^p \mathbb{G}_{m,\eta})$, and e^ϕ is just the composition

$$\text{Hom}_{\hat{\mathbb{Z}}^p}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p, T^p \mathbb{G}_{m,\eta}) \times (Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p) \xrightarrow{\text{id} \times \phi} \text{Hom}_{\hat{\mathbb{Z}}^p}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p, T^p \mathbb{G}_{m,\eta}) \times (X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p) \xrightarrow{\text{can}} T^p \mathbb{G}_{m,\eta}.$$

By [Lan13, Lemma 5.2.2.4], under our previous assumption that condition 1.4.3.10 in [Lan13] is satisfied, any filtration on L/nL induced by a filtration on $G[n]_\eta$ as above is split. Pick a $\mathcal{O}_B$ -linear splitting $\delta_n : \bigoplus_i \text{Gr}_{-i,n}^{Z_n} \xrightarrow{\sim} L/nL$. It is not hard to see that having the isomorphisms $\varphi_{i,n}$, and splittings of both

$$0 \rightarrow \tilde{G}[n]_\eta \rightarrow G[n]_\eta \rightarrow (Y/nY)_\eta \rightarrow 0 \quad (2.2)$$

and

$$0 \rightarrow T[n]_\eta \rightarrow \tilde{G}[n]_\eta \rightarrow A[n]_\eta \rightarrow 0 \quad (2.3)$$

is equivalent to having a level structure $\alpha_n : G[n]_\eta \xrightarrow{\sim} L/nL$ inducing the given filtration Z_n (although we should notice that at this point, we still need to verify the compatibility with the pairings). Indeed, the splittings above induce an isomorphism

$$G[n]_\eta \cong^{2.2} \tilde{G}[n]_\eta \oplus (Y/nY)_\eta \cong^{2.3} T[n]_\eta \oplus A[n]_\eta \oplus (Y/nY)_\eta$$

that is required to be compatible with the Weil pairing. Therefore, if we have the data of homomorphisms $\varphi_{i,n}$ and splittings, we recover a level structure by taking the composition

$$\alpha_n : L/nL \xrightarrow{\delta_n^{-1}} \bigoplus_i \text{Gr}_{-i,n}^{Z_n} \xrightarrow{\oplus \varphi_{i,n}} T[n]_\eta \oplus A[n]_\eta \oplus (Y/nY)_\eta \cong G[n]_\eta.$$

Remark 2.2.12. This explains how to obtain a level structure on G_η from the level structure on degeneration data. We can explicitly see that having the isomorphisms on the graded pieces is not sufficient to recover the level structure, which is why we need to consider splittings δ_n . These allow us to recover isomorphisms between the whole spaces. Our equivalence relation on $\alpha_n^\#$ then allows us to identify all the splittings, so that its choice does not matter.

Similarly, if we have a level structure $\alpha_n : G[n]_\eta \xrightarrow{\sim} L/nL$, with induced filtration Z_n on L/nL , δ_n allows us to recover the splittings of the sequences 2.2 and 2.3, via the induced isomorphism

$$G[n]_\eta \xrightarrow{\alpha_n} L/nL \xrightarrow{\delta_n} \bigoplus_i \text{Gr}_{-i,n}^{Z_n} \xrightarrow{\oplus \varphi_{-i,n}} T[n]_\eta \oplus A[n]_\eta \oplus (Y/nY)_\eta.$$

We just need analyze and rewrite what it means to have such splittings, which is how we will produce the triple $(c_n, c_n^\vee, \tau_n)$.

Let us start with analyzing splittings of the sequence 2.3. Splitting 2.3 corresponds to giving a map $A[n]_\eta \rightarrow \tilde{G}[n]_\eta$, whose image gives a closed subgroup scheme of $\tilde{G}[n]_\eta$, which we denote again by $A[n]_\eta$. By taking the quotient $\tilde{G}'_\eta = \tilde{G}_\eta/A[n]_\eta$, we get an isogeny $\tilde{G}_\eta \rightarrow \tilde{G}'_\eta$ with kernel $A[n]_\eta$. Since $A[n]_\eta \cap T_\eta$ is empty, T_η still embeds in $\tilde{G}'_\eta$, and we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_\eta & \longrightarrow & \tilde{G}_\eta & \longrightarrow & A_\eta \longrightarrow 0 \\ & & \downarrow \sim & & \downarrow \text{pr} & & \downarrow \cdot n \\ 0 & \longrightarrow & T_\eta & \longrightarrow & \tilde{G}'_\eta & \longrightarrow & A_\eta \longrightarrow 0 \end{array}$$

where the map between the torus is an isomorphism, the map between the semi-abelian schemes is the projection and the map on the abelian varieties is the multiplication-by- n map. This can be completed to a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_\eta & \longrightarrow & \tilde{G}_\eta & \longrightarrow & A_\eta \longrightarrow 0 \\ & & \downarrow \sim & & \downarrow \text{pr} & & \downarrow \cdot n \\ 0 & \longrightarrow & T_\eta & \longrightarrow & \tilde{G}'_\eta & \longrightarrow & A_\eta \longrightarrow 0 \\ & & \downarrow \cdot n & & \downarrow & & \downarrow \sim \\ 0 & \longrightarrow & T_\eta & \longrightarrow & \tilde{G}_\eta & \longrightarrow & A_\eta \longrightarrow 0 \end{array}$$

where the three vertical maps are the multiplication-by- n maps. Then, the abelian scheme $\tilde{G}'_\eta$ together with the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_\eta & \longrightarrow & \tilde{G}'_\eta & \longrightarrow & A_\eta \longrightarrow 0 \\ & & \downarrow \cdot n & & \downarrow & & \downarrow \sim \\ 0 & \longrightarrow & T_\eta & \longrightarrow & \tilde{G}_\eta & \longrightarrow & A_\eta \longrightarrow 0 \end{array}$$

where the map between the torus is the multiplication-by- n map, and the map between the abelian varieties is just the identity, determines the splitting. Identifying the map $T_\eta \xrightarrow{n} T_\eta$ with the dual of the inclusion map $X \hookrightarrow \frac{1}{n}X$ between character groups, and using the same argument as in the construction of c , we see that having the above diagram is equivalent to having a map $c_n : \frac{1}{n}X \rightarrow A_\eta^\vee$ lifting $c : X \rightarrow A_\eta^\vee$ over the generic fiber. Doing a similar analysis as in [Lan13, Section 5.2.3], one can prove that giving a splitting of 2.2 is equivalent to giving a lifting of the period map $\iota : Y \rightarrow \tilde{G}_\eta$, $\iota_n : \frac{1}{n}Y \rightarrow \tilde{G}_\eta$, which is equivalent to a lift of τ by the previous remark (c.f. [Lan13, Lemma 5.2.3.2.]). Composing this with $\pi : \tilde{G}_\eta \rightarrow A_\eta$ gives the desired lifting of $c^\vee$, $c_n^\vee$. We finally take δ_n to be the splitting on L/nL induced from the one on $G[n]_\eta$, thus finishing the construction of the tuple $\alpha_n^\natural = (Z_n, \varphi_{-2,n}, (\varphi_{-1,n}, \nu_{-1,n}), \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$ from an element in $\text{DEG}_{\text{PEL}, \mathcal{S}_K}(R, I)$.

Remark 2.2.13. Heuristically, one should think of $\tilde{G}$ as a covering of G , and namely think of G as a quotient of $\tilde{G}$ by Y . Having this in account, it is intuitive to think that a splitting $Y/nY \rightarrow G[n]_\eta$ of 2.2 is given by a map ι_n as above.

The only thing remaining, is to analyze the Weil pairings, and to find conditions in order to insure that the level structure α_n is compatible with these pairings. Following the notation in [Lan13], let us denote by W_n the filtration on $G[n]_\eta$ obtained by pulling back the filtration Z_n on L/nL , so that in particular the triple $(c_n, c_n^\vee, \tau_n)$ determines a liftable splitting $\zeta_n : \oplus_i \mathrm{Gr}_i^{W_n} \xrightarrow{\sim} G[n]_\eta$. The goal is to study the pairing on $\mathrm{Gr}_i^{W_n}$ induced from the Weil pairing on $G[n]_e$. As we have already stated, $W_{n,-2}$ is the annihilator of $W_{n,-1}$, and we obtain induced pairings on $\mathrm{Gr}_{-2}^{W_n} \times \mathrm{Gr}_0^{W_n}$ and $\mathrm{Gr}_{-1}^{W_n} \times \mathrm{Gr}_{-1}^{W_n}$, which are independent of the splitting. By carefully analyzing the splittings and the induced pairings, Lan was able to write down explicit equations for the remaining pairings between the other graded pieces. Namely, if we write the induced pairing on $\oplus_i \mathrm{Gr}_i^{W_n}$ as a matrix, we will obtain a matrix of the form

$$\begin{pmatrix} 0 & 0 & e_{20,n} \\ 0 & e_{11,n} & e_{10,n} \\ e_{02,n} & e_{01,n} & e_{00,n} \end{pmatrix}$$

where $e_{ij,n} : \mathrm{Gr}_{-i}^{W_n} \times \mathrm{Gr}_{-j}^{W_n} \rightarrow \mu_{n,\eta}$. Notice that the matrix has this form, since $W_{n,-2}$ is the annihilator of $W_{n,-1}$. Furthermore, since $\langle \cdot, \cdot \rangle$ is alternating, $e_{02,n}$ and $e_{01,n}$ are already determined by $e_{20,n}$ and $e_{10,n}$ respectively, and we already know that $e_{20,n}$ corresponds to the canonical pairing, and that $e_{11,n}$ corresponds to the Weil pairing on $A[n]_\eta$.

It remains to analyze $e_{01,n}$ and $e_{00,n}$. What Lan proves in section 5.2., is that in order for α_n to be compatible with the Weil pairings, $e_{10,n} : A[n]_\eta \times \frac{1}{n}Y/Y \rightarrow \mu_{n,\eta}$ must be given by

$$e_{10,n}(a, \frac{1}{n}y) = \langle a, (\lambda_{A,\eta} \circ c_n^\vee - c_n \circ \phi)(\frac{1}{n}y) \rangle_{A[n]_\eta},$$

where $\langle \cdot, \cdot \rangle_{A[n]_\eta}$ denotes the Weil pairing on $A[n]_\eta$, and $e_{00,n} : \frac{1}{n}Y/Y \times \frac{1}{n}Y/Y \rightarrow \mu_{n,\eta}$ by

$$e_{00,n}(\frac{1}{n}y, \frac{1}{n}y') = \tau_n(\frac{1}{n}y, \phi(y')) \tau_n(\frac{1}{n}y', \phi(y))^{-1},$$

which are exactly the conditions we stated in definition 2.2.7.

2.3 Toroidal Compactifications

In this section, we want to state the main theorem on toroidal compactifications for PEL-type Shimura varieties due to Lan, and briefly explain its construction. We want to obtain a proper scheme, containing our original Shimura variety as an open dense subscheme, and having as a universal object a semiabelian variety with extra structures, whose restriction to $\mathcal{S}_K$ gives back the universal abelian scheme. The idea is to construct moduli spaces of degenerating data as we defined above, and to glue them to our original Shimura variety. A problem arises when doing this, namely that the equivalence of categories we saw only works over a complete basis. This means that we only get an isomorphism between the moduli space of degeneration data and the completion of the toroidal compactification along the boundary. One then needs to algebraize everything in order to obtain the whole toroidal compactification. However, the construction of

the toroidal compactification is not unique: it depends on some combinatorial data, namely on a choice of a compatible cone decomposition. We will start by explaining the general setup in the next section.

2.3.1 Toroidal Embeddings

We will be following [Lan13] and [FC90] in this section. Let T be a split torus, with character group X and cocharacter group $X^\vee$. Consider the $\mathbb{R}$ -vector space $X_{\mathbb{R}}^\vee = X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.3.1. A rational polyhedral cone σ in $X_{\mathbb{R}}^\vee$ is a subset of $X_{\mathbb{R}}^\vee$ of the form

$$\sigma = \mathbb{R}_{>0}v_1 + \cdots + \mathbb{R}_{>0}v_n$$

where $\{v_i\}_i \in X_{\mathbb{Q}}^\vee$. The closure of σ is given by

$$\bar{\sigma} = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_n.$$

A rational polyhedral cone σ is non-degenerate if its closure does not contain any nonzero $\mathbb{R}$ -subvector space of $X_{\mathbb{R}}^\vee$, and it is smooth if the v_i can be chosen to form a $\mathbb{Z}$ basis of $X^\vee$.

Definition 2.3.2. Let σ be a rational polyhedral cone. A supportive hyperplane H of σ is a hyperplane in $X_{\mathbb{R}}^\vee$, such that σ does not overlap with both sides of H .

Definition 2.3.3. Let σ be a rational polyhedral cone. A face τ of σ is a rational polyhedral cone such that there exists a supportive hyperplane H of σ satisfying $H \cap \bar{\sigma} = \bar{\tau}$.

Recall that we have a natural pairing

$$\langle \cdot, \cdot \rangle : X \times X^\vee \rightarrow \mathbb{Z}$$

which defines by extension of scalars a pairing

$$\langle \cdot, \cdot \rangle : X \times X_{\mathbb{R}}^\vee \rightarrow \mathbb{R}.$$

Definition 2.3.4. Let σ be a rational polyhedral cone. We define the following semi-groups of X :

$$\sigma^\vee := \{x \in X \mid \langle x, \sigma \rangle \geq 0\}$$

$$\sigma_0^\vee := \{x \in X \mid \langle x, \sigma \rangle > 0\}$$

and

$$\sigma^\perp := \{x \in X \mid \langle x, \sigma \rangle = 0\}.$$

It is clear from the definition that $\sigma^\perp$ defines a subgroup of X , and thus a quotient torus T_σ .

Let Γ be a group acting on X , and thus also on $X^\vee$. Let P be a cone in $X_{\mathbb{R}}^\vee$ (i.e. a $\mathbb{R}_{>0}^\times$ -invariant subset).

Definition 2.3.5. A Γ -admissible rational polyhedral cone decomposition of P is a collection $\Sigma = \{\sigma_i\}_{i \in I}$ of non-degenerate rational polyhedral cones such that P is the (set theoretic) disjoint union of all cones in Σ , so

$$P = \coprod_{i \in I} \sigma_i.$$

This is required to be a stratification, so the closure of any σ_i in P must be the union of σ_j with $j \in I$. We also assume that each of these σ_j appearing in the closure of σ_i must be a face of σ_i . Furthermore, Σ must be Γ -invariant, so Γ permutes the cones in Σ , and the set of Γ -orbits, Σ/Γ , must be finite.

Such a cone decomposition Σ is said to be smooth if every cone in Σ is smooth.

We now want to use these cones to obtain toroidal embeddings. So, let $\mathcal{M}$ be a T -torsor over some scheme Z , and write by abuse of notation (c.f. [Lan13, 4.2.3.])

$$\mathcal{O}_{\mathcal{M}} = \bigoplus_{\chi \in X} \mathcal{O}_{\mathcal{M}, \chi},$$

where $\mathcal{O}_{\mathcal{M}, \chi}$ is the weight- χ subsheaf under the T -action, using as usual that $\mathcal{M}$ is relatively affine over Z . As we have already seen, we have isomorphisms

$$\mathcal{O}_{\mathcal{M}, \chi} \otimes \mathcal{O}_{\mathcal{M}, \chi'} \xrightarrow{\sim} \mathcal{O}_{\mathcal{M}, \chi + \chi'}$$

inducing on $\mathcal{O}_{\mathcal{M}}$ the structure of an $\mathcal{O}_Z$ -sheaf of algebras. Fix a rational polyhedral cone σ , and define

$$\mathcal{M}(\sigma) = \underline{\text{Spec}}_{\mathcal{O}_Z}(\bigoplus_{\chi \in \sigma^\vee} \mathcal{O}_{\mathcal{M}, \chi}).$$

$\bigoplus_{\chi \in \sigma^\vee} \mathcal{O}_{\mathcal{M}, \chi}$ gets an induced structure of $\mathcal{O}_Z$ algebras from the one on $\mathcal{O}_{\mathcal{M}}$. The inclusion $\bigoplus_{\chi \in \sigma^\vee} \mathcal{O}_{\mathcal{M}, \chi} \subset \mathcal{O}_{\mathcal{M}}$ induces an open immersion

$$\mathcal{M} \hookrightarrow \mathcal{M}(\sigma).$$

The T -action on $\mathcal{M}$ extends naturally to $\mathcal{M}(\sigma)$. Furthermore, for any face τ of σ , the inclusion

$$\sigma^\vee \subset \tau^\vee$$

induces a canonical embedding

$$\mathcal{M}(\tau) \hookrightarrow \mathcal{M}(\sigma).$$

In particular, if we have a Γ -admissible rational polyhedral cone decomposition Σ , we can consider the schemes $\mathcal{M}(\sigma_i)$ for every $\sigma_i \in \Sigma$, and use these embeddings to glue them all, obtaining a scheme $\tilde{\mathcal{M}}_\Sigma$ over Z - the toroidal embedding of $\mathcal{M}$ defined by Σ , which is called the toroidal embedding of $\mathcal{M}$ defined by Σ . Before stating its properties, let us just define a subscheme of $\mathcal{M}(\sigma)$.

Definition 2.3.6. Define

$$\mathcal{M}_\sigma = \mathcal{M}(\sigma) - \bigcup_{\tau \text{ face of } \sigma} \mathcal{M}(\tau) \subset \mathcal{M}(\sigma)$$

where τ runs through the proper faces of σ . This is a closed subscheme of $\mathcal{M}(\sigma)$ given by the ideal sheaf

$$\mathcal{I}_\sigma = \sum_{\chi \in \sigma_0^\vee} \mathcal{O}_{\mathcal{M}, \chi}$$

which can therefore be identified with

$$\mathcal{M}_\sigma = \underline{\text{Spec}}_{\mathcal{O}_Z} (\oplus_{\chi \in \sigma^\perp} \mathcal{O}_{\mathcal{M}, \chi}).$$

This is called the σ -stratum of $\mathcal{M}(\sigma)$. $\mathcal{M}_\sigma$ is a T_σ -torsor over Z .

Proposition 2.3.7. *[Lan13, Thm. 6.1.2.8.], [FC90, Ch. IV, Thm. 2.5.] Let P be a cone in $X_{\mathbb{R}}^\vee$, and let $\Sigma = \{\sigma_i\}_{i \in I}$ be a Γ -admissible rational polyhedral cone decomposition of P . Let $\mathcal{M}$ be a T -torsor, and $\bar{\mathcal{M}}_\Sigma$ be the toroidal embedding of $\mathcal{M}$ defined by Σ . Then, the following properties are satisfied:*

1. $\bar{\mathcal{M}}_\Sigma$ is separated, locally of finite type scheme over Z , and it has an induced action of T . It contains both $\mathcal{M}$ and $\mathcal{M}(\sigma_i)$ for $i \in I$ as open dense subschemes, and the open immersions $\mathcal{M} \hookrightarrow \bar{\mathcal{M}}_\Sigma$ and $\mathcal{M}(\sigma_i) \hookrightarrow \bar{\mathcal{M}}_\Sigma$ are T -equivariant. Namely, $\bar{\mathcal{M}}_\Sigma$ is the union of the $\mathcal{M}(\sigma_i)$, with gluings given the the inclusions $\mathcal{M}(\tau) \hookrightarrow \mathcal{M}(\sigma)$ for faces τ of σ . In particular, if $\bar{\sigma}_l = \bar{\sigma}_j \cap \bar{\sigma}_k$ with σ_l the largest common face of σ_j and σ_k , then $\mathcal{M}(\sigma_l) = \mathcal{M}(\sigma_j) \cap \mathcal{M}(\sigma_k)$;
2. $\bar{\mathcal{M}}_\Sigma$ admits a natural stratification by locally closed subschemes

$$\bar{\mathcal{M}}_\Sigma = \coprod_{i \in I} \mathcal{M}_{\sigma_i}$$

with $\mathcal{M}_{\sigma_i}$ defined as in definition 2.3.6. Namely, by construction, if we denote by $\bar{\mathcal{M}}_{\sigma_i}$ the closure of $\mathcal{M}_{\sigma_i}$ in $\bar{\mathcal{M}}_\Sigma$, $\bar{\mathcal{M}}_{\sigma_i} \subset \bar{\mathcal{M}}_{\sigma_j}$ if and only if $\bar{\sigma}_j \subset \bar{\sigma}_i$;

3. Γ acts on $\bar{\mathcal{M}}_\Sigma$ sending $\mathcal{M}(\sigma_i)$ (resp. $\mathcal{M}_{\sigma_i}$) to $\mathcal{M}(\gamma\sigma_i)$ (resp. $\mathcal{M}_{\gamma\sigma_i}$) for each $\gamma \in \Gamma$. We will assume that if $\gamma\bar{\sigma} \cap \bar{\sigma} \neq \{0\}$ for some $\gamma \in \Gamma$, then a power of γ acts trivially on the smallest boundary of $X_{\mathbb{R}}^\vee$ containing σ ;
4. If the cone decomposition Σ is smooth, then $\bar{\mathcal{M}}_\Sigma$ is smooth over Z , and the complement $\bar{\mathcal{M}}(\sigma_i) \setminus \mathcal{M}$ has normal crossings divisors for each $\sigma_i \in \Sigma$.

2.3.2 Main Theorem on Toroidal Compactifications

In this section, we state and sketch the proof of the main theorem on toroidal compactifications. As was already mentioned, we will want to construct a space that parametrizes tuples $(A, \lambda_A, \iota_A, X, Y, \phi, c, c^\vee, \tau, [\alpha_n^h])$ as in definition 2.2.8 (where we once more assume that n is prime-to- p), not necessarily over a complete base. In fact, we first want to construct a space parametrizing tuples $(A, \lambda_A, \iota_A, X, Y, \phi, c, c^\vee, \tau, \alpha_n^h)$ without the equivalence relation on α_n^h , and then quotient it out by a certain group to identify the elements on the same equivalence class.

Furthermore, notice that there is some redundancy in the definition: namely, c_n , $c_n^\vee$ and τ_n already determine c , $c^\vee$ and τ . In particular, what we really want to parametrize are tuples

$$(Z_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, \iota_A, (\varphi_{-1,n}, \nu_{-1,n})), \delta_n, (c_n, c_n^\vee, \tau_n)),$$

with elements as described in the previous sections. We somehow want to construct this moduli space in several steps: first, we need to fix the discrete data, meaning a cusp label or a representative of one, $(Z_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), \delta_n)$ as in definition 2.2.6. We can use the filtration Z_n to define a smaller PEL-type Shimura variety, which will parametrize the abelian part of the tuples above, namely tuples $(A, \lambda_A, \iota_A, (\varphi_{-1,n}, \nu_{-1,n}))$, as follows:

Definition 2.3.8. Let $\mathcal{S}_K^{Z_n}$ be the functor over S_0 parametrizing over a test scheme S tuples $(A, \lambda, \iota, (\alpha_n, \nu_n))$ consisting of:

1. an abelian scheme A over S ;
2. a prime-to- p polarization $\lambda : A \rightarrow A^\vee$;
3. an $\mathcal{O}_B$ -action $\iota : \mathcal{O}_B \rightarrow \text{End}_S(A)$ satisfying the equality $\iota(b)^\vee \circ \lambda = \lambda \circ \iota(b^*)$ for every $b \in \mathcal{O}_B$;
4. $\text{Lie}_{A/S}$ must satisfy the determinantal condition determined by $(\text{Gr}_{-1,\mathbb{R}}^{Z_n}, \langle \cdot, \cdot \rangle_{11,\mathbb{R}}, h_{-1})$ as in definition 2.1.2, also c.f. definition 2.2.7;
5. (α_n, ν_n) is a level- n structure: namely, $\alpha_n : \text{Gr}_{-1,S}^{Z_n} \xrightarrow{\sim} A[n]$ is an $\mathcal{O}_B$ -equivariant isomorphism of group schemes over S , and $\nu_n : (\mathbb{Z}/n\mathbb{Z}(1))_S \xrightarrow{\sim} (\mu_n)_S$ is an isomorphism of group schemes over S such that the diagram

$$\begin{array}{ccc} \text{Gr}_{-1,S}^{Z_n} \times \text{Gr}_{-1,S}^{Z_n} & \longrightarrow & (\mathbb{Z}/n\mathbb{Z}(1))_S \\ \downarrow & & \downarrow \\ A[n] \times A[n] & \longrightarrow & (\mu_n)_S \end{array}$$

commutes, where the top horizontal arrow is the alternating pairing induced by the filtration Z_n on $\text{Gr}_{-1,S}^{Z_n}$, the bottom horizontal arrow is the Weil pairing on $A[n]$ induced from the polarization, and the two vertical arrows are the obvious ones. We furthermore require that the level- n structure is symplectic liftable in the sense of [Lan13, Defn. 1.3.6.2.]). This roughly means that α_n lifts compatibly to isomorphisms $\alpha_m : \text{Gr}_{-1,S}^{Z_m} \xrightarrow{\sim} A[m]$ for all m prime-to- p which are divided by n .

Two such tuples $(A, \lambda, \iota, (\alpha_n, \nu_n))$ and $(A', \lambda', \iota', (\alpha'_n, \nu'_n))$ are equivalent if there is an isomorphism $f : A \rightarrow A'$, compatible with the $\mathcal{O}_B$ -action, such that $\lambda = f^\vee \circ \lambda' \circ f$, and such that $\alpha_n = f \circ \alpha'_n$, $\nu_n = \nu'_n$.

Remark 2.3.9. • Lemma 5.2.7.3. in [Lan13] provides us with a PEL-type lattice, associated with $\text{Gr}_{-1,S}^{Z_n}$ (this uses the fact that our assumptions on Z_n imply that $\text{Gr}_{-1,S}^{Z_n}$ admits an alternating bilinear pairing $\langle \cdot, \cdot \rangle_{1,n}$). By abuse of notation, we still denote this lattice by $\text{Gr}_{-1,S}^{Z_n}$. It is then clear that $\mathcal{S}_K^{Z_n}$ is a PEL-type Shimura variety as in definition 2.1.3.

- On the construction of the category of degenerations $\text{DD}_{\text{PEL},\mathcal{S}_K}(R,I)$ parametrizing tuples of the form $(A, \lambda_A, \iota_A, X, Y, \phi, c, c^\vee, \tau, [\alpha_n^\natural])$, where $\alpha_n^\natural = (Z_n, \varphi_{-2,n}, (\varphi_{-1,n}, \nu_{-1,n}), \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$, if we pick a representative of $[\alpha_n^\natural]$, the tuple $(A, \lambda_A, \iota_A, (\varphi_{-1,n}, \nu_{-1,n}))$ is exactly a point in $\mathcal{S}_K^{Z_n}$, just as required.
- $\mathcal{S}_K^{Z_n}$ does not depend on the choice of representative of the cusp label, since it only depends on Z_n which is independent of this choice.

The next step of the construction of our moduli space, will be to construct a space C_{ϕ_n} over $\mathcal{S}_K^{Z_n}$, parametrizing the Raynaud extensions c_n and $c_n^\vee$. The naive definition for this space will not be exactly what we want, and we will need to cut out a certain geometrically connected component which will be related to the liftability conditions on c_n and $c_n^\vee$. The final step, will be to parametrize the trivializations τ_n , which is also the hardest part. We will also have to be careful to keep track of all the extra relations, such as the compatibilities with Weil pairings.

We will now make a few preparations to state the main theorem on toroidal compactifications. We will then sketch the proof of the theorem, by explaining the construction of our moduli space.

Denote by $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ the torus part of the representative of the cusp label we fixed above.

Definition 2.3.10. Define the group S_{Φ_n} as the quotient group of

$$\check{S}_{\Phi_n} := \left(\frac{1}{n} Y \otimes_{\mathbb{Z}} X \right) / (y \otimes \phi(y') - y' \otimes \phi(y), (b \frac{1}{n} y) \otimes \chi - \frac{1}{n} y \otimes (b^* \chi))_{y, y' \in Y, \chi \in X, b \in \mathcal{O}_B}$$

by its torsion subgroup.

In the construction of the moduli space we mentioned above, the scheme parametrizing the trivializations τ_n will give rise to a torsor for the split torus with character group S_{Φ_n} will show up. We will then want to use the previous section to produce torus embeddings of this torsor, in order to produce compactifications of this space. We remark that there is a natural isomorphism between $S_{\Phi_n, \mathbb{R}}^\vee := S_{\Phi_n}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ where $S_{\Phi_n}^\vee$ is the $\mathbb{Z}$ -linear dual of S_{Φ_n} and the space of Hermitian pairings on $Y_{\mathbb{R}}$ with values in $B \otimes_{\mathbb{Q}} \mathbb{R}$, $(|\cdot, \cdot|) : Y_{\mathbb{R}} \times Y_{\mathbb{R}} \rightarrow B \otimes_{\mathbb{Z}} \mathbb{R}$ (c.f. [Lan13, sect. 6.2.5.]).

Let $P_{\Phi_n} \subset S_{\Phi_n, \mathbb{R}}^\vee$ be the subset consisting of positive semi-definite Hermitian forms (see [Lan13, Defn. 6.2.5.3.] for definition of positivity in this setup), with admissible radicals, i.e. such that the radicals are the $\mathbb{R}$ -span of some direct summand $\mathcal{O}_B$ -submodule of Y . Let $P_{\Phi_n}^+ \subset P_{\Phi_n}$ be the subset of positive definite Hermitian forms. We will apply the constructions of the previous section, namely by considering Γ -admissible rational polyhedral cone decompositions of P , with $X := S_{\Phi_n}$, $P := P_{\Phi_n}$ and $\Gamma = \Gamma_{\Phi_n}$ (where we are using the notation introduced in section 2.3.1 for X , P and Γ), where $\Gamma_{\Phi_n} \subset \text{GL}_{\mathcal{O}_B}(X) \times \text{GL}_{\mathcal{O}_B}(Y)$ is the group of pairs of isomorphisms

$\gamma_X : X \xrightarrow{\sim} X$ and $\gamma_Y : Y \xrightarrow{\sim} Y$ satisfying $\phi = \gamma_X \circ \phi \circ \gamma_Y$, $\varphi_{-2,n} = \gamma_X^\vee \circ \varphi_{-2,n}$ and $\varphi_{0,n} = \gamma_Y \circ \varphi_{0,n}$ (recall that under definition 2.2.6 of a cusp label, these relations are exactly the ones defining equivalent classes when $X = X'$ and $Y = Y'$). It is clear that Γ_{Φ_n} acts on P_{Φ_n} . One can prove that there exists a Γ_{Φ_n} -admissible, smooth rational, polyhedral cone decomposition of P_{Φ_n} , c.f. [AMRT75, Ch. 2], so let $\Sigma_{\Phi_n} = \{\sigma_i\}_{i \in I}$ be such a cone decomposition.

In order to construct the toroidal compactification, one need to perform a gluing process: we need to pick a representative for each cusp label, and to construct polyhedral cone decompositions for each P_{Φ_n} as we have just defined. This will allow us to define torus embeddings for each cusp label, which we then want to be able to glue together to get a space. But in order to be able to do this, the choice of these cone decompositions need to be "compatible". Let us make this precise.

Definition 2.3.11. Let (Z_n, Φ_n, δ_n) and $(Z'_n, \Phi'_n, \delta'_n)$ be two representatives of cusp labels such that we have a surjection

$$(Z_n, \Phi_n, \delta_n) \twoheadrightarrow (Z'_n, \Phi'_n, \delta'_n)$$

(see [Lan13, Defn. 6.3.3.4.] for the definition), inducing an embedding $P'_{\Phi_n} \subset P_{\Phi_n}$ of cones. Choose two polyhedral cone decompositions Σ_{Φ_n} and Σ'_{Φ_n} . Then

$$(\Phi_n, \delta_n, \Sigma_{\Phi_n}) \twoheadrightarrow (\Phi'_n, \delta'_n, \Sigma'_{\Phi_n})$$

is a surjection if $\Sigma_{\Phi_n}|_{P'_{\Phi_n}} = \Sigma'_{\Phi_n}$.

So, compatible means that after picking representatives for each cusp label (Z_n, Φ_n, δ_n) , and cone decompositions Σ_{Φ_n} , for each surjection between these representatives $(Z_n, \Phi_n, \delta_n) \twoheadrightarrow (Z'_n, \Phi'_n, \delta'_n)$, $(\Phi_n, \delta_n, \Sigma_{\Phi_n}) \twoheadrightarrow (\Phi'_n, \delta'_n, \Sigma'_{\Phi_n})$ must also be a surjection.

We will also need the following definitions:

Definition 2.3.12. Let (Z_n, Φ_n, δ_n) and $(Z'_n, \Phi'_n, \delta'_n)$ be two representatives of cusp labels.

- Let σ and σ' be two nondegenerate rational polyhedral cones in P_{Φ_n} resp. P'_{Φ_n} . Two tuples $(Z_n, \Phi_n, \delta_n, \sigma)$ and $(Z'_n, \Phi'_n, \delta'_n, \sigma')$ are equivalent if there exists a pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ making (Z_n, Φ_n, δ_n) and $(Z'_n, \Phi'_n, \delta'_n)$ equivalent, and the induced isomorphism $P'_{\Phi_n} \xrightarrow{\sim} P_{\Phi_n}$ sends σ' to σ .
- Let Σ_{Φ_n} and Σ'_{Φ_n} be two choices of Γ_{Φ_n} (resp. $\Gamma_{\Phi'_n}$) polyhedral cone decompositions of P_{Φ_n} (resp. $P_{\Phi'_n}$). Two triples $(\Phi_n, \delta_n, \Sigma_{\Phi_n})$ and $(\Phi'_n, \delta'_n, \Sigma_{\Phi'_n})$ are said to be equivalent if there exists a pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ making (Z_n, Φ_n, δ_n) and $(Z'_n, \Phi'_n, \delta'_n)$ equivalent, and identifying the two polyhedral cone decompositions under the induced isomorphism $P'_{\Phi_n} \xrightarrow{\sim} P_{\Phi_n}$.

Definition 2.3.13. A compatible choice of admissible smooth rational polyhedral cone decomposition data for $\mathcal{S}_K$ is a complete set of compatible choices of polyhedral cone decompositions $\Sigma = \{\Sigma_{\Phi_n}\}_{[(\Phi_n, \delta_n)]}$, for (Φ_n, δ_n) running through all representatives of cusp labels.

Remark 2.3.14. • There always exist a compatible choice of admissible smooth rational polyhedral cone decomposition data for $\mathcal{S}_K$, c.f. [Lan13, Prop. 6.3.3.5].

- The choice of a compatible choice Σ_{Φ_n} for the chosen representative (Φ_n, δ_n) extends to a compatible choice for all representatives (Φ'_n, δ'_n) . This implies that we only have to make one choice for one such representative ([Lan13, Rmk. 6.3.3.3]).

Definition 2.3.15. Let $[(Z_n, \Phi_n, \delta_n, \sigma)]$ and $[(Z'_n, \Phi'_n, \delta'_n, \sigma')]$ be two equivalence classes of tuples as in the definition above. We say that $[(Z'_n, \Phi'_n, \delta'_n, \sigma')]$ is a face of $[(Z_n, \Phi_n, \delta_n, \sigma)]$ if there exist representatives $(Z_n, \Phi_n, \delta_n, \sigma)$ and $(Z'_n, \Phi'_n, \delta'_n, \sigma')$ such that there is a surjection

$$(Z_n, \Phi_n, \delta_n) \twoheadrightarrow (Z'_n, \Phi'_n, \delta'_n),$$

inducing an embedding $P'_{\Phi'_n} \hookrightarrow P_{\Phi_n}$ of cones, such that the image of σ' under the embedding is contained in the Γ_{Φ_n} -orbit of σ .

Theorem 2.3.16. [Lan13, Thm. 6.4.1.1.], [LS18, Prop. 2.1.3.] *Let $n \geq 3$ be an integer, and $K = K(n)$ a principal level. Let $\Sigma = \{\Sigma_{(\Phi_n, \delta_n)}\}_{(\Phi_n, \delta_n)}$ be a compatible choice of an admissible, smooth, projective rational polyhedral cone decomposition data. Then there is a projective, smooth scheme $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ over $\text{Spec}(\mathcal{O}_{F_0, (p)})$, containing $\mathcal{S}_K$ as an open, dense subscheme*

$$j_{\Sigma}^{\text{tor}} : \mathcal{S}_K \hookrightarrow \mathcal{S}_{K, \Sigma}^{\text{tor}}$$

together with a universal semi-abelian tuple

$$(G, \lambda, \iota, \alpha_n)$$

as in definition 2.2.2, such that the restriction of $(G, \lambda, \iota, \alpha_n)$ to $\mathcal{S}_K$ is the universal object over $\mathcal{S}_K$. Furthermore, $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ satisfies the following properties:

1. It admits a (set-theoretical) stratification by locally closed subschemes

$$\mathcal{S}_{K, \Sigma}^{\text{tor}} = \coprod_{[(\Phi_n, \delta_n, \sigma)]} Z_{[(\Phi_n, \delta_n, \sigma)]}$$

where the union is over a complete set of equivalence classes of tuples $(\Phi_n, \delta_n, \sigma)$, with $\sigma \in \Sigma_{\Phi_n}$ such that $\sigma \subset P_{\Phi_n}^+$. Each stratum $Z_{[(\Phi_n, \delta_n, \sigma)]}$ lies in the closure of another stratum $Z_{[(\Phi'_n, \delta'_n, \sigma')]}$ if and only if $Z_{[(\Phi_n, \delta_n, \sigma)]}$ is a face of $Z_{[(\Phi'_n, \delta'_n, \sigma')]}$. $\mathcal{S}_K$ corresponds to the class where $X = Y = 0$. Each stratum is smooth over $\text{Spec}(\mathcal{O}_{F_0, (p)})$;

2. The complement of $\mathcal{S}_K$ in $\mathcal{S}_{K, \Sigma}^{\text{tor}}$, with its reduced structure, is a relative Cartier divisor with normal crossings;
3. Let $(\mathcal{S}_{K, \Sigma}^{\text{tor}})_{Z_{[(\Phi_n, \delta_n, \sigma)]}}^{\wedge}$ denote the formal completion of $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ along the $[(\Phi_n, \delta_n, \sigma)]$ stratum. For each choice of representative (Z_n, Φ_n, δ_n) of a cusp label as above, there is a toroidal boundary chart

$$\begin{array}{ccc}
\Xi_{\Phi_n, \delta_n} & \longrightarrow & \Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}} = \coprod_{\sigma \in \Sigma_{\Phi_n, \delta_n}} \Xi_{\Phi_n, \delta_n}(\sigma) \\
\downarrow & \swarrow & \\
C_{\Phi_n, \delta_n} & & \\
\downarrow & & \\
\mathcal{S}_K^{Z_n} & &
\end{array}$$

where:

- $\mathcal{S}_K^{Z_n}$ is the Shimura variety associated to the filtration Z_n as in definition 2.3.8 (in particular, it parametrizes tuples $(A, \lambda_A, \iota_A, \alpha_n)$ of abelian schemes with extra structures);
- C_{Φ_n, δ_n} is an abelian scheme over $\mathcal{S}_K^{Z_n}$ (as we will explain below, this should parametrize the Raynaud extension: namely, the data c_n and $c_n^\vee$);
- Ξ_{Φ_n, δ_n} is a torsor under the split torus with character group S_{Φ_n} (this should parametrize τ_n). More precisely, Ξ_{Φ_n, δ_n} is defined as

$$\Xi_{\Phi_n, \delta_n} = \underline{\text{Spec}}_{\mathcal{O}_{C(\Phi_n, \delta_n)}} (\oplus_{l \in S_{\Phi_n}} \Psi_n(l))$$

where $\Psi_n(l)$ is a line bundle which we will define below. Then, since Ξ_{Φ_n, δ_n} is a torsor, we can define the scheme $\Xi_{\Phi_n, \delta_n}(\sigma) = \underline{\text{Spec}}_{\mathcal{O}_{C(\Phi_n, \delta_n)}} (\oplus_{l \in \sigma^\vee} \Psi_n(l))$ for any rational polyhedral cone $\sigma \in \Sigma_{\Phi_n}$ as in the previous section, so that $\Xi_{\Phi_n, \delta_n} \hookrightarrow \Xi_{\Phi_n, \delta_n}(\sigma)$ is a toroidal embedding.

We will see below that Ξ_{Φ_n, δ_n} will parametrize the period morphism τ_n above C_{Φ_n, δ_n} , as desired. In particular, the toroidal boundary chart yields the construction of a moduli space, parametrizing degeneration tuples just as expected. Let $\mathcal{X}_{\Phi_n, \delta_n, \sigma}$ denote the formal completion of $\Xi_{\Phi_n, \delta_n}(\sigma)$ along the boundary $\Xi_{\Phi_n, \delta_n, \sigma} \hookrightarrow \Xi_{\Phi_n, \delta_n}(\sigma)$, where $\Xi_{\Phi_n, \delta_n, \sigma}$ denotes the ideal sheaf of $\Xi_{\Phi_n, \delta_n}(\sigma)$ given by

$$\Xi_{\Phi_n, \delta_n, \sigma} = \underline{\text{Spec}}_{\mathcal{O}_{C(\Phi_n, \delta_n)}} (\oplus_{l \in \sigma^\perp} \Psi_n(l)).$$

Let furthermore $\Gamma_{\Phi_n, \sigma} \subset \Gamma_{\Phi_n}$ be the subgroup consisting of (γ_X, γ_Y) mapping σ to himself under the Γ_{Φ_n} -action on P_{Φ_n} . There is a canonical isomorphism

$$(\mathcal{S}_{K, \Sigma}^{\text{tor}})_{Z[(\Phi_n, \delta_n, \sigma)]}^\wedge \cong \mathcal{X}_{\Phi_n, \delta_n, \sigma} / \Gamma_{\Phi_n, \sigma}$$

which is independent on the choice of representative of the cusp label, respecting stratifications.

4. Let

$$\Xi_{\Phi_n, \delta_n}(\sigma)^+ := \bigcup_{\tau \in \Sigma_{\Phi_n}^+, \bar{\tau} \subset \bar{\sigma}} \Xi_{\Phi_n, \delta_n, \tau}. \quad (2.4)$$

For each $\sigma \in \Sigma_{\Phi_n}^+$, consider the formal scheme $\mathcal{X}_{\Phi_n, \delta_n, \sigma}^\circ$, given by the formal completion of $\Xi_{\Phi_n, \delta_n}(\sigma)$ along $\Xi_{\Phi_n, \delta_n}(\sigma)^+$. Then, there is a canonical isomorphism

$$(\mathcal{S}_{K, \Sigma}^{\text{tor}})_{\bigcup_{\tau \in \Sigma_{\Phi_n}^+, \tau \subset \bar{\sigma}} Z[(\Phi_n, \delta_n, \tau)]}^\wedge \cong \mathcal{X}_{\Phi_n, \delta_n, \sigma}^\circ / \Gamma_{\Phi_n, \sigma}$$

extending the canonical isomorphism

$$(\mathcal{S}_{K, \Sigma}^{\text{tor}})_{Z[(\Phi_n, \delta_n, \sigma)]}^\wedge \cong \mathcal{X}_{\Phi_n, \delta_n, \sigma}^\circ / \Gamma_{\Phi_n, \sigma}.$$

Let furthermore $\text{Spf}(R) \rightarrow \mathcal{X}_{\Phi_n, \delta_n, \sigma}^\circ$ be an affine open formal subscheme of $\mathcal{X}_{\Phi_n, \delta_n, \sigma}^\circ$. The isomorphism 2.4 induces morphisms $\text{Spec}(R) \rightarrow \Xi_{\Phi_n, \delta_n}(\sigma)$ and $\text{Spec}(R) \rightarrow \mathcal{S}_{K, \Sigma}^{\text{tor}}$. Then, the stratifications induced by $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ and $\Xi_{\Phi_n, \delta_n}(\sigma)$ via these maps on $\text{Spec}(R)$ agree. We denote by W^0 the preimage of $\mathcal{S}_K$ (or Ξ_{Φ_n, δ_n}) under this map.

5. Let S be an irreducible, Noetherian, normal scheme over $\text{Spec}(\mathcal{O}_{F_0, (p)})$, and assume that we have a degenerating family $(G^\dagger, \lambda^\dagger, \iota^\dagger, \alpha_n^\dagger)$ over S , as in definition 2.2.2. Then there exists a unique morphism

$$S \rightarrow \mathcal{S}_{K, \Sigma}^{\text{tor}}$$

over $\text{Spec}(\mathcal{O}_{F_0, (p)})$ such that $(G^\dagger, \lambda^\dagger, \iota^\dagger, \alpha_n^\dagger)$ is the pullback of the universal object $(G, \lambda, \iota, \alpha_n)$ if and only if the following condition is satisfied:

Let $\bar{s} \in S$ be a geometric point of S , and $\text{Spec}(V) \rightarrow S$ be any dominant morphism centered at $\bar{s}$, from the spectrum of a complete discrete valuation ring V with fraction field K , algebraically closed residue field k and valuation v . Let $(G^\dagger, \lambda^\dagger, \iota^\dagger, \alpha_n^\dagger) \rightarrow \text{Spec}(V)$ be the pullback of the degenerating family to V . Then $(G^\dagger, \lambda^\dagger, \iota^\dagger, \alpha_n^\dagger) \rightarrow \text{Spec}(V)$ is an element in $\text{DEG}_{\text{PEL}, \mathcal{S}_K}(V, \mathfrak{m})$, and thus, by theorem 2.2.8, it corresponds to an element

$$(A^\dagger, \lambda_A^\dagger, \iota_A^\dagger, X^\dagger, Y^\dagger, \phi^\dagger, c^\dagger, c^{\vee, \dagger}, \tau^\dagger, [\alpha_n^\dagger]^\dagger) \in \text{DD}_{\text{PEL}, \mathcal{S}_K}(V, \mathfrak{m}).$$

In particular, it determines a filtration $Z_n^\dagger$ in L/n as usual, and a torus argument $\Phi_n^\dagger$. As we already know, the trivialization

$$\tau^\dagger : \mathbf{1}_{Y^\dagger \times_S X^\dagger, \eta} \xrightarrow{\sim} (c^{\vee, \dagger} \times c^\dagger)^* \mathcal{P}_{A^\dagger, \eta}^{\otimes -1}$$

over the generic fiber $\eta = \text{Spec}(K)$, can be thought of as a set of sections $\tau^\dagger(y, \chi)$ for $y \in Y$, $\chi \in X$ which we think of as sections

$$\tau^\dagger(y, \chi) : \mathcal{O}_{\text{Spec}(V), \eta} \xrightarrow{\sim} \mathcal{O}_\chi(c^{\vee, \dagger}(y))_\eta^{\otimes -1}.$$

There is an integral structure on $\mathcal{O}_\chi(c^{\vee, \dagger}(y))_\eta^{\otimes -1}$ given by $\mathcal{O}_\chi(c^{\vee, \dagger}(y))^{\otimes -1}$, which we can pullback via this section to obtain an invertible V -submodule of K , $I_{y, \chi}$. This defines a morphism

$$\begin{aligned} Y^\dagger \times X^\dagger &\rightarrow \text{Inv}(V) \\ (y, \chi) &\mapsto I_{y, \chi} \end{aligned}$$

where $\text{Inv}(V)$ denotes the group of invertible V -submodules of K . This descends to a homomorphism

$$B^\ddagger : S_{\Phi_n} \rightarrow \text{Inv}(V)$$

and by noticing that we have a natural identification $\text{Inv}(V) \cong \mathbb{Z}$ via $\pi^m V \leftrightarrow m$ where π denotes the uniformizer of V , we obtain a homomorphism

$$v \circ B^\ddagger : S_{\Phi_n} \rightarrow \mathbb{Z}$$

which is an element of $S_{\Phi_n}^\vee$. The condition is that for some choice of splitting $\delta_n^\ddagger$, there is a cone $\sigma^\ddagger$ in the chosen cone decomposition $\Sigma_{\Phi_n^\ddagger}$ of $P_{\Phi_n^\ddagger}$ whose closure contains all $v \circ B^\ddagger$ obtained this way.

We want to sketch the proof of the theorem, as done by Lan in his thesis. Namely, we want to explain how to construct the toroidal boundary chart. It will be clear by the construction that it should parametrize the desired tuples. We also want to explain the proof of the universal property.

Proof. As we already explained, we want to start by constructing a space parametrizing degeneration data tuples

$$(Z_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, \iota_A, (\varphi_{-1,n}, \nu_{-1,n})), \delta_n, c, c^\vee, \tau, c_n, c_n^\vee, \tau_n).$$

As we have already remarked, c , $c^\vee$ and τ are determined by c_n , $c_n^\vee$ and τ_n respectively, so we really want to parametrize tuples

$$(Z_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, \iota_A, (\varphi_{-1,n}, \nu_{-1,n})), \delta_n, c_n, c_n^\vee, \tau_n).$$

Note that on the statement of the theorem, we fix a set of representatives of cusp labels (Z_n, Φ_n, δ_n) , and we want to construct a toroidal boundary chart for each of these. So let us fix a cusp label $[(Z_n, \Phi_n, \delta_n)]$, and a representative (Z_n, Φ_n, δ_n) . The abelian part,

$$(A, \lambda_A, \iota_A, (\varphi_{-1,n}, \nu_{-1,n})),$$

is a point in $\mathcal{S}_K^{Z_n}$ by definition. By abuse of notation, let A denote the universal abelian scheme over $\mathcal{S}_K^{Z_n}$. Next we want to parametrize c_n and $c_n^\vee$. For that, notice that we can just consider the schemes $\underline{\text{Hom}}_{\mathcal{O}_B}(\frac{1}{n}X, A^\vee)$ and $\underline{\text{Hom}}_{\mathcal{O}_B}(\frac{1}{n}Y, A)$ over $\mathcal{S}_K^{Z_n}$. Recall that c_n and $c_n^\vee$ are liftings of c and $c^\vee$ respectively, and that these are required to satisfy the equality $\lambda_A \circ c^\vee = c \circ \phi$. This can easily be obtained, by considering the scheme

$$\ddot{C}_{\Phi_n, \delta_n} := \underline{\text{Hom}}_{\mathcal{O}_B}(\frac{1}{n}X, A^\vee) \times_{\underline{\text{Hom}}_{\mathcal{O}_B}(Y, A^\vee)} \underline{\text{Hom}}_{\mathcal{O}_B}(\frac{1}{n}Y, A)$$

where the map $\underline{\text{Hom}}_{\mathcal{O}_B}(\frac{1}{n}X, A^\vee) \rightarrow \underline{\text{Hom}}_{\mathcal{O}_B}(Y, A^\vee)$ is $c_n \mapsto c_n \circ (X \hookrightarrow \frac{1}{n}X) \circ \phi$ and the map $\underline{\text{Hom}}_{\mathcal{O}_B}(\frac{1}{n}Y, A) \rightarrow \underline{\text{Hom}}_{\mathcal{O}_B}(Y, A^\vee)$ is $c_n^\vee \mapsto \lambda_A \circ c_n^\vee \circ (Y \hookrightarrow \frac{1}{n}Y)$.

Also recall that there is a compatibility condition with the splitting δ_n , namely the equality

$$\langle \varphi_{-1,n}(\cdot), (\lambda_A \circ c_n^\vee - c_n \circ \phi) \circ \varphi_{0,n}(\cdot) \rangle_{A_\eta} = \nu_{-1,n} \circ \langle \cdot, \cdot \rangle_{10,n}$$

must be satisfied. Let $b_{\Phi_n, \delta_n} : \frac{1}{n}Y/Y \rightarrow A^\vee[n]$ be the (liftable) homomorphism defined by the equation

$$\langle \varphi_{-1,n}(\cdot), b_{\Phi_n, \delta_n} \circ \varphi_{0,n}(\cdot) \rangle_{A_\eta} = \nu_{-1,n} \circ \langle \cdot, \cdot \rangle_{10,n}.$$

Then the equality above translates to the equality

$$b_{\Phi_n, \delta_n} = \lambda_A \circ c_n^\vee - c_n \circ \phi.$$

Now just consider the morphism

$$\partial_n : \ddot{C}_{\Phi_n, \delta_n} \rightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_B}(\frac{1}{n}Y/Y, A^\vee[n])$$

sending $(c_n, c_n^\vee)$ to $\lambda_A \circ c_n^\vee - c_n \circ \phi$, and let $\ddot{C}_{\Phi_n, b_n} = \partial_n^{-1}(b_{\Phi_n, \delta_n})$ be the fiber of ∂_n at b_{Φ_n, δ_n} . This space cuts out the relation we want. The only problem is that this is not necessarily an abelian scheme, namely it might not have geometrically connected fibers, so it is not yet the scheme C_{Φ_n, δ_n} that we are looking for. In order to obtain C_{Φ_n, δ_n} one needs to pick the "right" geometrically connected component of $\ddot{C}_{\Phi_n, b_n}$, which will actually be exactly the one that cuts out the liftability conditions on the pairs $(c_n, c_n^\vee)$. Let us recall its construction.

Construction 2.3.17. Let $\underline{\mathrm{Hom}}_{\mathcal{O}_B}(\frac{1}{n}X, A)^\circ$ be the fiberwise geometric identity component of $\underline{\mathrm{Hom}}_{\mathcal{O}_B}(\frac{1}{n}X, A)$, as in [Lan13, Prop. 5.2.3.9.]. Consider the map

$$\underline{\mathrm{Hom}}_{\mathcal{O}_B}(\frac{1}{n}X, A)^\circ \rightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_B}(\frac{1}{n}X, A^\vee) \times_{\underline{\mathrm{Hom}}_{\mathcal{O}_B}(Y, A^\vee)} \underline{\mathrm{Hom}}_{\mathcal{O}_B}(\frac{1}{n}Y, A)$$

over $\mathcal{S}_K^{Z_n}$, and denote its image (which is an abelian subscheme of $\ddot{C}_{\Phi_n, \delta_n}$) by $\ddot{C}_{\Phi_n}^\circ$. It is clear that its kernel is given by the finite étale group scheme

$$\underline{\mathrm{Hom}}_{\mathcal{O}_B}(\frac{1}{n}X/\Phi_n(\frac{1}{n}Y), \ker(\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}_B}(\frac{1}{n}X, A)^\circ.$$

Denote by $\pi_0(\ddot{C}_{\Phi_n}, \mathcal{S}_K^{Z_n})$ the group scheme of fiberwise geometric connected components of $\ddot{C}_{\Phi_n}$, as defined in [Lan13, Defn. 5.2.3.8.]. One can prove that this is a finite étale scheme over $\mathcal{S}_K^{Z_n}$, and that its rank has no prime factors other than those of the discriminant of $\mathcal{O}_B$, $[X : \phi(Y)]$, the rank of the kernel of λ_A and n (c.f. [Lan13, Prop. 6.2.3.2.]). Now, consider the scheme $\ddot{C}_{\Phi_n, b_n}$ defined above. Recall that we have a liftability condition on b_{Φ_n, δ_n} , and therefore a compatible choice of liftings $\hat{b} = \{b_m\}_{n|m, p \nmid m}$ with $b_m : \frac{1}{m}Y/Y \rightarrow A^\vee[m]$. If we let m be an integer such that $p \nmid m$ and $n|m$, then we obtain a natural morphism $\ddot{C}_{\Phi_n, b_m} \rightarrow \ddot{C}_{\Phi_n, b_n}$, and therefore we can consider the common schematic image of these morphisms, denoted $\ddot{C}_{\Phi_n, b_n}^{\mathrm{com}}$. This is already the scheme that we are looking for, but we still need to prove that it is geometrically connected.

We have the following result:

Lemma 2.3.18. [Lan13, Lemma 6.2.3.11.] One can produce a uniquely determined compatible system of elements $\{\tilde{b}_m\}_{n|m, p \nmid m}$, where each $\tilde{b}_m$ is an element of $\underline{\mathrm{Hom}}_{\mathcal{O}_B}(\frac{1}{m}X/\phi(Y), A^\vee) \times \underline{\mathrm{Hom}}_{\mathcal{O}_B}(Y, A^\vee)$ $E \subset \ddot{C}_{\Phi_m, \delta_n}$ that is sent to b_m under the map

$$\partial_m : \ddot{C}_{\Phi_m, \delta_m} \rightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_B}(\frac{1}{m}Y/Y, A^\vee[m]).$$

By compatible, we mean that for $m|l$, $\tilde{b}_l$ is sent to $\tilde{b}_m$ via the canonical map $\ddot{C}_{\Phi_l, b_l} \rightarrow \ddot{C}_{\Phi_m, b_m}$. This implies that, under the conditions above, $\tilde{b}_m$ translates $\ddot{C}_{\Phi_m, 0}$ to $\ddot{C}_{\Phi_m, b_m}$ for each $m \geq 1$ such that $n|m$, and that therefore the underlying geometric spaces of $\ddot{C}_{\Phi_n, 0}$ and $\ddot{C}_{\Phi_n, b_n}$ are isomorphic.

Proof. Just take $\tilde{b}_m$ to be $(-b_l|_{\frac{1}{m}X}, 0)$, where l is an integer satisfying $n|l$, $p \nmid l$ and $\frac{1}{m}X \subset \frac{1}{l}\Phi(Y)$. One needs to be a bit careful to make sure that this makes sense, but once one does so, the result follows trivially.

The last thing we need to prove, is the following:

Proposition 2.3.19. [Lan13, Lemma 6.2.3.13.] The common schematic image $\ddot{C}_{\Phi_n, 0}^{\mathrm{com}}$ given by the natural morphisms $\ddot{C}_{\Phi_m, 0} \rightarrow \ddot{C}_{\Phi_n, 0}$, for m such that $p \nmid m$ and $n|m$, is $\ddot{C}_{\Phi_n}^\circ$.

Proof. We start by noticing that the image of the natural map $\ddot{C}_{\Phi_m, 0} \rightarrow \ddot{C}_{\Phi_n, 0}$ given by the inclusions $\frac{1}{n}X/X \hookrightarrow \frac{1}{m}X/X$ and $\frac{1}{n}Y/Y \hookrightarrow \frac{1}{m}Y/Y$ is the same as the multiplication-by- $\frac{m}{n}$ map on $\ddot{C}_{\Phi_n, 0}$, just by definition. Furthermore, the canonical isomorphisms $\frac{1}{n}X \cong X \cong \frac{1}{m}X$ and $\frac{1}{n}Y \cong Y \cong \frac{1}{m}Y$ induce canonical isomorphisms

$$\ddot{C}_{\Phi_n, 0} \times_{\mathcal{S}_K^{Z_n}} \mathcal{S}_K^{Z_m} \xrightarrow{\sim} \ddot{C}_{\Phi_1} \times_{\mathcal{S}_K^{Z_1}} \mathcal{S}_K^{Z_m} \xrightarrow{\sim} \ddot{C}_{\Phi_m, 0}. \quad (2.5)$$

In particular, the rank of the group scheme $\pi_0(\ddot{C}_{\Phi_n, 0}, \mathcal{S}_K^{Z_n})$ of fiberwise connected components of $\ddot{C}_{\Phi_n, 0}$ over $\mathcal{S}_K^{Z_n}$ is prime-to- p , using that the rank of $\pi_0(\ddot{C}_{\Phi_1}, \mathcal{S}_K^{Z_1})$ is prime-to- p , and that the pullback of $\pi_0(\ddot{C}_{\Phi_1}, \mathcal{S}_K^{Z_1})$ under 2.5 is $\pi_0(\ddot{C}_{\Phi_n, 0}, \mathcal{S}_K^{Z_n})$. Therefore, there exists an integer m satisfying $n|m$, such that multiplication by $\frac{m}{n}$ kills $\pi_0(\ddot{C}_{\Phi_n, 0}, \mathcal{S}_K^{Z_n})$, and thus the schematic image of the map $\ddot{C}_{\Phi_m, 0} \rightarrow \ddot{C}_{\Phi_n, 0}$ for the chosen m is contained in $\ddot{C}_{\Phi_n}^\circ$ by our initial remark (c.f. [Lan13, Prop. 5.2.3.9., (4)]). But since $\ddot{C}_{\Phi_n}^\circ$ is an abelian scheme, the multiplication by integers are surjective on it, and thus it must both coincide with the schematic image of the map above, and be contained in the image of $\ddot{C}_{\Phi_l, 0} \rightarrow \ddot{C}_{\Phi_n, 0}$ for any $n|l$. The result follows.

This ends the construction, since it is now clear by the previous two results that the following corollary holds.

Corollary 2.3.20. [Lan13, Cor. 6.2.3.14.] $\ddot{C}_{\Phi_n, b_n}^{\mathrm{com}}$ is the translation of $\ddot{C}_{\Phi_n}^\circ$ by $\tilde{b}_n$, and it is therefore an abelian scheme.

We take $C_{\Phi_n, \delta_n} = \ddot{C}_{\Phi_n, b_n}^{\mathrm{com}}$.

The only thing left to parametrize is τ_n . We want once again to think τ , or more precisely τ_n as a set of sections $\{\tau(\frac{1}{n}y, \chi)\}$ of the line bundles $(c^\vee(\frac{1}{n}y) \times c(\chi))^* \mathcal{P}_{A, \eta}^{\otimes -1}$ over S_η for $y \in Y$ and

$\chi \in X$. Let $(c_n, c_n^\vee)$ be the two universal homomorphisms over C_{Φ_n, δ_n} . We can define a map

$$\frac{1}{n}Y \times X \rightarrow \text{Pic}(C_{\Phi_n, \delta_n})$$

by sending $\frac{1}{n}y \otimes \chi$ to $(c^\vee(\frac{1}{n}y) \times c(\chi))^* \mathcal{P}_{A, \eta}^{\otimes -1}$. This is well defined due to the formal properties of the pullback of the Poincaré sheaf, such as linearity in the first and second variables, and symmetry with respect to ϕ as we have already explained (c.f. [Lan13], section 6.2.2. for a detailed exposition). Recall that we have defined the abelian group

$$\ddot{S}_{\Phi_n} := (\frac{1}{n}Y \otimes_{\mathbb{Z}} X) / (y \otimes \phi(y') - y' \otimes \phi(y), (b\frac{1}{n}y) \otimes \chi - \frac{1}{n}y \otimes (b^*\chi))_{y, y' \in Y, \chi \in X, b \in \mathcal{O}_B}.$$

These formal properties actually imply that the map above descends to a map

$$\Psi_n : \ddot{S}_{\Phi_n} \rightarrow \text{Pic}(C_{\Phi_n, \delta_n}).$$

The bilinearity of these pullbacks induces canonical isomorphisms

$$\Psi_n(l) \otimes_{C_{\Phi_n, \delta_n}} \Psi_n(l') \xrightarrow{\sim} \Psi_n(l + l')$$

for any $l, l' \in \ddot{S}_{\Phi_n}$, and therefore an $\mathcal{O}_{C_{\Phi_n, \delta_n}}$ -algebra structure on

$$\oplus_{l \in \ddot{S}_{\Phi_n}} \Psi_n(l).$$

We now define

$$\ddot{\Xi}_{\Phi_n} := \underline{\text{Spec}}_{C_{\Phi_n, \delta_n}} (\oplus_{l \in \ddot{S}_{\Phi_n}} \Psi_n(l))$$

and denote by $\ddot{E}_{\Phi_n} = \underline{\text{Hom}}(\ddot{S}_{\Phi_n}, \mathbb{G}_m)$ the group of multiplicative type with character group $\ddot{S}_{\Phi_n}$. Then, $\ddot{\Xi}_{\Phi_n}$ is a $\ddot{E}_{\Phi_n}$ -torsor ([Lan13, Sect. 6.3.2.]). This space parametrizes trivializations τ_n by construction, and it thus admits a universal trivialization

$$\tau_n : \mathbf{1}_{\frac{1}{n}Y \times X} \xrightarrow{\sim} (c_n^\vee \times c)^* \mathcal{P}_A^{\otimes -1}.$$

However, note that since $\ddot{S}_{\Phi_n}$ is not necessarily torsion free, $\ddot{E}_{\Phi_n}$ might not be a torus. This is solved in a similar fashion as with $\ddot{C}_{\Phi_n, \delta_n}$, namely one needs again to choose the subspace Ξ_{Φ_n, δ_n} of $\ddot{\Xi}_{\Phi_n}$, having again to do with the liftability condition on τ_n , as well as the Weil pairing condition, and this will actually be a torsor under the torus with character group $S_{\Phi_n} = \ddot{S}_{\Phi_n, \text{free}}$, the subgroup of $\ddot{S}_{\Phi_n}$ of the torsion-free elements.

The only things left to take care of are the positivity condition of τ , and also the fact that we are not necessarily working over the spectrum of an I -complete local ring R (which is actually needed to make sense of the positivity condition), so at the moment we can not yet use the equivalence of categories as in theorem 2.2.8. Recall that the positivity condition states that if we are over $S = \text{Spec} R$ for some I -completed normal ring R as above, and denoting by $\eta = \text{Spec} K$ is generic fiber, then for every $y \in Y$,

$$\tau(y, \phi(y)) : \mathcal{O}_{\phi(y)}(c^\vee(y))_\eta \rightarrow \mathcal{O}_{S, \eta}$$

lifts to a section on S ,

$$\tau(y, \phi(y)) : \mathcal{O}_{\phi(y)}(c^\vee(y)) \rightarrow \mathcal{O}_S$$

and that if y is not zero, then the image of $\tau(y, \phi(y))$ factors through the ideal of definition of R . Using the same notation as in the statement of the theorem 2.3.16, recall that the boundary of the morphism $\Xi_{\Phi_n, \delta_n, \sigma} \hookrightarrow \Xi_{\Phi_n, \delta_n}(\sigma)$ is defined by the ideal sheaf $\mathcal{I}_{\Phi_n, \delta_n, \sigma} = (\oplus_{l \in \sigma_0^\vee} \Psi_n(l))$. Also note the following: if $l = [y \otimes \phi(y)]$ for $y \in Y$, and $\sigma \subset P_{\Phi_n}$ (resp. $\sigma \subset P_{\Phi_n}^+$) is positive semi-definite (resp. positive definite), then $\sigma(y, \Phi(y)) \geq 0$ (resp. $\sigma(y, \Phi(y)) > 0$). This is [Lan13, Lemma 6.2.5.6.], but let us explain how to prove this: recall that we have an isomorphism between the $\mathbb{R}$ -linear dual of S_{Φ_n} , $S_{\Phi_n, \mathbb{R}}^\vee$, and the space of certain Hermitian forms on $Y_{\mathbb{R}}$ with values in $B \otimes_{\mathbb{Q}} \mathbb{R}$. This identification is obtained by sending a Hermitian pairing $|(\cdot, \cdot)|$ to the function $y \otimes \Phi(y') \mapsto \text{Tr}_{B/\mathbb{Q}}(|y, y'|)$. Then, for $\sigma \in P_{\Phi_n}$ (resp. $\sigma \in P_{\Phi_n}^+$), we have $\sigma(y, \phi(y)) = \text{Tr}_{B/\mathbb{Q}}(|y, y|)$ for some positive semi-definite (resp. positive definite) Hermitian pairing $|(\cdot, \cdot)|$. In particular $\sigma(y, \Phi(y)) \geq 0$ (resp. $\sigma(y, \Phi(y)) > 0$), so $(y, \phi(y)) \in \sigma^\vee$ (resp. $(y, \phi(y)) \in \sigma_0^\vee$). This implies that if $\sigma \in P_{\Phi_n}$, the universal trivialization

$$\tau(y, \Phi(y)) : \mathcal{O}_{\Xi_{\Phi_n, \delta_n}} \otimes_{\mathcal{O}_{C_{\Phi_n, \delta_n}}} \Psi_n(y, \Phi(y)) \xrightarrow{\sim} \mathcal{O}_{\Xi_{\Phi_n, \delta_n}}$$

extends to a section

$$\tau(y, \Phi(y)) : \mathcal{O}_{\Xi_{\Phi_n, \delta_n}(\sigma)} \otimes_{\mathcal{O}_{C_{\Phi_n, \delta_n}}} \Psi_n(y, \Phi(y)) \rightarrow \mathcal{O}_{\Xi_{\Phi_n, \delta_n}(\sigma)},$$

since the image of $\mathcal{O}_{\Xi_{\Phi_n, \delta_n}(\sigma)} \otimes_{\mathcal{O}_{C_{\Phi_n, \delta_n}}} \Psi_n(y, \Phi(y))$ through the composition of

$$\mathcal{O}_{\Xi_{\Phi_n, \delta_n}(\sigma)} \otimes_{\mathcal{O}_{C_{\Phi_n, \delta_n}}} \Psi_n(y, \Phi(y)) \rightarrow \mathcal{O}_{\Xi_{\Phi_n, \delta_n}} \otimes_{\mathcal{O}_{C_{\Phi_n, \delta_n}}} \Psi_n(y, \Phi(y)) \quad (2.6)$$

induced from the toroidal embedding $\Xi_{\Phi_n, \delta_n} \hookrightarrow \Xi_{\Phi_n, \delta_n}(\sigma)$, with $\tau(y, \Phi(y))$, lies in $\mathcal{O}_{\Xi_{\Phi_n, \delta_n}(\sigma)}$ (note that $\oplus_{l \in \sigma^\vee} \Psi_n(l)$ is a subalgebra of $\oplus_{l \in S_{\Phi_n}} \Psi_n(l)$). If $\sigma \in P_{\Phi_n}^+$, then its image under 2.6 is contained in $\mathcal{I}_{\Phi_n, \delta_n, \sigma}$. This is essentially the positivity condition. Just as in proposition 2.3.7, we can glue the schemes $\Xi_{\Phi_n, \delta_n}(\sigma)$ together to obtain a toroidal compactification, which we will denote by $\bar{\Xi}_{\Phi_n, \delta_n, \Sigma_{\Phi_n}}$. By varying the cusp labels, we can consider the union of these spaces

$$\coprod_{(Z_n, \Phi_n, \delta_n)} \bar{\Xi}_{\Phi_n, \delta_n, \Sigma_{\Phi_n}}.$$

This disjoint union is almost the moduli space parametrizing degeneration tuples that we were looking for. There are only two things missing: the first one is that in order to use the theorem on the equivalence of categories between semi-abelian data and degeneration data, c.f. theorem 2.2.8, we need to work over a complete base. This is the reason we consider the completion of $\bar{\Xi}_{\Phi_n, \delta_n, \Sigma_{\Phi_n}}$ along the boundary strata $\coprod_{\sigma \in \Sigma_{(\Phi_n)}} \bar{\Xi}_{\Phi_n, \delta_n, \sigma}$, obtaining therefore a formal scheme which we denote by $\mathcal{X}_{\Phi_n, \delta_n, \Sigma_{\Phi_n}}$. It admits a universal semiabelian scheme as in the statement, by functoriality of the Mumford construction (one is able to glue the semi-abelian schemes coming from the equivalence in 2.2.8 at each I -complete affine normal base, and thus obtain a global one). The second thing missing, is to identify the different equivalence classes of $\alpha_n^\sharp$. Since our

polyhedral cone decomposition Σ_{Φ_n} is Γ_{Φ_n} -admissible, its action extends to $\bar{\Xi}_{\Phi_n, \delta_n, \Sigma_{\Phi_n}}$, and thus to $\mathcal{X}_{\Phi_n, \delta_n, \Sigma_{\Phi_n}}$. By taking the quotient

$$\mathcal{X}_{\Phi_n, \delta_n, \Sigma_{\Phi_n}} / \Gamma_{\Phi_n},$$

we are identifying equivalent cusp labels, and thus obtaining the desired moduli space of degeneration data. One then needs to algebraize all these maps to obtain gluing data, finishing the construction of the toroidal compactification. This is where the compatibility condition of the cone decomposition is needed.

We now want to explain how to prove the universal property of the toroidal compactifications. Let us start by stating the universal property for $\Xi_{\Phi_n, \delta_n}(\sigma)$:

Lemma 2.3.21. *[Lan13, Prop. 6.2.5.11.] Let R be a Noetherian normal integral domain with quotient field K , and suppose that we have a map $\text{Spec} R \rightarrow C_{\Phi_n, \delta_n}$ which is liftable to a map $\text{Spec} K \rightarrow \Xi_{\Phi_n, \delta_n}$. Recall that under these assumptions, we can construct a homomorphism*

$$v \circ B : S_{\Phi_n} \rightarrow \mathbb{Z},$$

which is an element of $S_{\Phi_n}^\vee$, by considering the sections $\tau(y, \chi) : (c^\vee(y) \times c(\chi))^ \mathcal{P}_{A_K} \xrightarrow{\sim} K$, and the integral structure $(c^\vee(y) \times c(\chi))^* \mathcal{P}_{A_R} \rightarrow (c^\vee(y) \times c(\chi))^* \mathcal{P}_{A_K}$ on $(c^\vee(y) \times c(\chi))^* \mathcal{P}_{A_K}$. Then the map $\text{Spec}(K) \rightarrow \Xi_{\Phi_n, \delta_n}$ lifts to a map $\text{Spec}(R) \rightarrow \Xi_{\Phi_n, \delta_n}(\sigma)$ if and only if for every discrete valuation $v : K^\times \rightarrow \mathbb{Z}$ with $v(R) \geq 0$, the corresponding map $v \circ B$ is in the closure of σ .*

This can be easily proven: all we really want to know is whether the ideals $I_{y, \chi}$ given by the map $B : S_{\Phi_n} \rightarrow \text{Inv}(R)$ are actually contained in R for elements $(y, \chi) \in \sigma^\vee$. But this is the same as $v \circ B(y, \chi) \geq 0$, which is by definition the same as it lying in the closure of σ .

The universal property for $\Xi_{\Phi_n, \delta_n}(\sigma)$ immediately implies a similar universal property for $\bar{\Xi}_{\Phi_n, \delta_n, \Sigma_{\Phi_n}}$ which is as follows: let $\bar{S}$ be a Noetherian normal scheme over C_{Φ_n, δ_n} , and let $S \subset \bar{S}$ be a dense open subscheme. Assume furthermore that we have a map $S \rightarrow \Xi_{\Phi_n, \delta_n}$ defined over $\bar{S} \rightarrow C_{\Phi_n, \delta_n}$. Then $S \rightarrow \Xi_{\Phi_n, \delta_n}$ extends to a morphism $\bar{S} \rightarrow \bar{\Xi}_{\Phi_n, \delta_n, \Sigma_{\Phi_n}}$ if and only if for every geometric point $x \in \bar{S}$, and every dominant morphism $\text{Spec}(V) \rightarrow \bar{S}$ centered at x from a discrete valuation ring V , the associated morphism

$$S_{\Phi_n} \rightarrow \mathbb{Z}$$

constructed as above lies in the closure of some cone σ in the cone decomposition Σ_{Φ_n} . The last part of the theorem then follows formally (one needs to be careful with the gluing).

2.4 Minimal compactifications

In this section, we quickly review the main theorem on minimal compactifications, as proven in [Lan13].

Theorem 2.4.1. [Lan13, 7.2.4.1., 7.2.4.3.] Let $n \geq 3$ be an integer prime-to- p , and $K = K(n)$ a principal level. There is a normal, projective, flat scheme $\mathcal{S}_K^*$ over $\mathrm{Spec}(\mathcal{O}_{F_0, (p)})$, containing $\mathcal{S}_K$ as an open, dense subscheme

$$j^* : \mathcal{S}_K \hookrightarrow \mathcal{S}_K^*.$$

Furthermore, $\mathcal{S}_K^*$ admits a stratification by locally closed subschemes

$$\mathcal{S}_K^* = \coprod_{[(Z_n, \Phi_n, \delta_n)]} \mathcal{S}_K^{Z_n}$$

running through a complete set of cusp labels, such that $\mathcal{S}_K^{Z'_n}$ lies in the closure of $\mathcal{S}_K^{Z_n}$ iff there is a surjection of cusp labels from $[(Z'_n, \Phi'_n, \delta'_n)]$ to $[(Z_n, \Phi_n, \delta_n)]$. In addition, it satisfies the following properties:

- Let Σ be a choice of a compatible admissible smooth rational polyhedral cone decomposition data for $\mathcal{S}_K$ as in definition 2.3.13. Let $\mathcal{S}_{K, \Sigma}^{\mathrm{tor}}$ be the toroidal compactification attached to this data. Then, there is a map

$$\pi_\Sigma : \mathcal{S}_{K, \Sigma}^{\mathrm{tor}} \rightarrow \mathcal{S}_K^*$$

between compactifications, such that the equality $j_K^* = \pi_\Sigma \circ j_{K, \Sigma}^{\mathrm{tor}}$ holds. For each cusp label $[(Z_n, \Phi_n, \delta_n)]$, the (set-theoretic) preimage of the $[(Z_n, \Phi_n, \delta_n)]$ -stratum along π_Σ agrees with $Z_{[(\Phi_n, \delta_n)]} := \coprod_{\sigma \in \Sigma_{\Phi_n}} Z_{[(\Phi_n, \delta_n, \sigma)]}$ (using the notation as in theorem 2.3.16). Furthermore,

$$\pi_{\Sigma, *} \mathcal{O}_{\mathcal{S}_{K, \Sigma}^{\mathrm{tor}}} = \mathcal{O}_{\mathcal{S}_K^*}.$$

- Let $K(n), K(m) \subset G(\hat{\mathbb{Z}}^p)$ be two principal levels, and $g \in G(\mathbb{A}^p)$ such that $g^{-1}K(n)g \subset K(m)$. Then, there exists a finite surjective morphism

$$[g] : \mathcal{S}_{K(n)}^* \rightarrow \mathcal{S}_{K(m)}^*,$$

extending the usual one on Shimura varieties.

2.5 Level- p -structures

In order to define the Hodge-Tate period morphism in precedent chapters, we will need to consider toroidal and minimal compactifications with additional level- p -structures. This was studied in [Lan16], and we just want to quickly review his results.

We work in the same setup as in the last few sections, namely we consider an integral PEL-data $(\mathcal{O}_B, *, L, \langle \cdot, \cdot \rangle, h)$, with associated unitary similitude group G over $\mathbb{Z}$, and we fix a good prime $p \nmid [L^\sharp : L]$. Let $K^p = K(n) \subset G(\mathbb{A}_f^p)$ be a principal open compact subgroup with $p \nmid n$, so that in particular we can consider the associated Shimura variety $\mathcal{S}_{K^p}$ as in definition 2.1.3, defined over $\mathrm{Spec}(\mathcal{O}_{F_0, (p)})$, where F_0 denotes the reflex field (recall in particular that $\mathcal{S}_{K^p}$ parametrizes tuples $(A, \lambda, \iota, (\alpha_n, \nu_n))$ consisting of an abelian scheme with extra structures, and that the fact that p is a good prime implies that the polarization λ is prime-to- p).

Notice that we can consider the exact same PEL-data, but with $\square = \emptyset$ (as in the notation of [Lan13], [Lan16]). In particular, we can then consider open compact subgroups $K = K_p K^p \subset G(\hat{\mathbb{Z}})$, with $K^p = K(n)$, and obtain an associated Shimura variety $\mathcal{S}_K^{\text{pre}}$ over F_0 , parametrizing tuples $(A, \lambda, \iota, \alpha_K)$ similar to the ones above, but where α_K is now a level- K structure. We still assume that p is a good prime, so that λ is still a prime-to- p polarization. We are actually only interested in the case $K = K(p^m n)$ where n is prime-to- p , so that α_K is a $p^m n$ -level structure. It will then consist of a level- n structure α_n , and of a level- p level structure α_p . We obtain an induced finite morphism

$$\pi^{\text{pre}} : \mathcal{S}_K^{\text{pre}} \rightarrow \mathcal{S}_{K^p} \otimes_{\mathbb{Z}} \mathbb{Q}$$

sending a tuple $(A, \lambda, \iota, \alpha_K)$ to $(A, \lambda, \iota, \alpha_n)$, where α_n is the associated level- n structure, such that the pullback of the universal object on $\mathcal{S}_{K^p} \otimes_{\mathbb{Z}} \mathbb{Q}$ is exactly the universal object in $\mathcal{S}_K^{\text{pre}}$.

Remark 2.5.1. This is [Lan16, Prop. 4.12.], with $j = 1$, and $(\mathcal{O}_{\text{aux}}, *, \iota_{\text{aux}}, L_{1,\text{aux}}, \langle \cdot, \cdot \rangle_{1,\text{aux}}, h_{1,\text{aux}}) = (\mathcal{O}_B, *, L, \langle \cdot, \cdot \rangle, h)$.

In this setup, $\mathcal{S}_K^{\text{pre}}$ does not necessarily admit a good integral model (as in [Box15, Defn. 3.4.1.]). We need therefore to define $\mathcal{S}_K$ as the normalization of $\mathcal{S}_{K^p}$ in $\mathcal{S}_K^{\text{pre}}$, under the morphism π_{pre} . By [Lan16, Prop. 6.1.], $\mathcal{S}_K$ is a normal, flat scheme over $\text{Spec}(\mathcal{O}_{F_0, (p)})$, with a canonical isomorphism

$$\mathcal{S}_K \times_{\text{Spec}(\mathcal{O}_{F_0, (p)})} \text{Spec}(F_0) \cong \mathcal{S}_K^{\text{pre}}$$

over $\text{Spec}(F_0)$. It still admits a finite morphism $\pi : \mathcal{S}_K \rightarrow \mathcal{S}_{K^p}$ by definition.

We can also analyze what happens on the level of toroidal and minimal compactifications. This is done on sections 5, 6 and 7 of [Lan16].

2.5.1 Toroidal compactifications at level- p

We want to briefly explain what we obtain on the level of toroidal compactifications, when we add the level- p -structures. Let us start by, just as above, considering the Shimura variety $\mathcal{S}_K^{\text{pre}}$ with $K = K(p^m n)$, with $\square = \emptyset$. We can then consider toroidal compactifications of $\mathcal{S}_K^{\text{pre}}$, just by using [Lan13]. We want to construct a morphism between such a compactification, to a toroidal compactification of $\mathcal{S}_{K^p} \otimes_{\mathbb{Z}} \mathbb{Q}$, extending the morphism π^{pre} we saw above. In order to do so, let us start by analyzing the cusp labels.

Let $[(Z, \Phi, \delta)]$ be a cusp label at level $K(p^m n)$. In particular, if we pick a representative (Z, Φ, δ) , we can think of $Z = \{Z_{-1}\}$ as a symplectic, admissible filtration of $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. By considering the base change

$$Z_{n, -i} := Z_{-i} \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}^p,$$

we obtain a symplectic, admissible filtration Z_n of $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$. For the torus argument, if we write $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ where $\varphi_{-2} : \text{Gr}_{-2}^Z \xrightarrow{\sim} \text{Hom}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1))$ and $\varphi_0 : \text{Gr}_0^Z \xrightarrow{\sim} Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, then we can consider the induced isomorphisms over $\hat{\mathbb{Z}}^p$, just by tensoring φ_{-2} and φ_0 with $\hat{\mathbb{Z}}^p$. We

obtain a torus argument at level- n , $\Phi_n = (X, Y, \phi, \varphi_{n,-2}, \varphi_{n,0})$. We also let δ_n be any splitting of Z_n .

This construction yields a well-defined assignment

$$[(Z, \Phi, \delta)] \rightarrow [(Z_n, \Phi_n, \delta_n)]$$

of cusp labels. In particular, we obtain induced isomorphisms

$$(S_\Phi)_\mathbb{Q}^\vee \cong (S_{\Phi_n})_\mathbb{Q}^\vee,$$

and

$$P_\Phi^{(+)} \cong P_{\Phi_n}^{(+)},$$

and therefore the choices of admissible smooth rational cone decompositions for $\mathcal{S}_K^{\text{pre}}$ and $\mathcal{S}_{K^p}$ are the same. By [Lan16, Prop. 5.24.], this induces morphisms

$$\mathcal{S}_{K,\Sigma}^{\text{tor,pre}} \rightarrow \mathcal{S}_{K^p,\Sigma}^{\text{tor}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

between toroidal compactifications (where $\mathcal{S}_{K,\Sigma}^{\text{tor,pre}}$ denotes the compactification constructed in [Lan13]), for each compatible choice of an admissible rational polyhedral cone decomposition data for $\mathcal{S}_K^{\text{pre}}$, extending the morphism π^{pre} . Each strata $Z_{[(\Phi,\delta,\sigma)]}$ is sent to the corresponding strata $Z_{[(\Phi_n,\delta_n,\sigma)]}$ (with the same notation as in theorem 2.3.16).

Using the analysis of principal level structures in [Lan13], one can analyze what the extra structures coming from p -level structures are on the degeneration data, and thus on toroidal boundary charts. In particular, the extra structures on toroidal boundary charts on $\mathcal{S}_{K,\Sigma}^{\text{tor,pre}}$, are given by a splitting of the short exact sequence

$$0 \rightarrow T[p^m]_\eta \rightarrow \tilde{G}[p^m]_\eta \rightarrow A[p^m]_\eta \rightarrow 0 \quad (2.7)$$

which is given, respectively, by a lift $c_{p^m} : \frac{1}{p^m}X \rightarrow A^\vee$ of the Raynaud extension $c : X \rightarrow A^\vee$, and by a lift $\iota_{p^m} : \frac{1}{p^m}Y \rightarrow \tilde{G}$ of the period morphism $\iota : Y \rightarrow \tilde{G}$.

In order to obtain integral models of these compactifications, we need to consider compactifications defined by normalization (the same is already true for the associated Shimura varieties with p -level structures, as we saw above). Namely, we make the following definition.

Definition 2.5.2. We define $\mathcal{S}_{K,\Sigma}^{\text{tor}}$ as the normalization of $\mathcal{S}_{K^p,\Sigma}^{\text{tor}}$ in $\mathcal{S}_K$, under the morphism

$$\mathcal{S}_K \rightarrow \mathcal{S}_{K^p} \rightarrow \mathcal{S}_{K^p,\Sigma}^{\text{tor}}.$$

Even though this is a quite abstract definition, Lan was able to prove that one can describe $\mathcal{S}_{K,\Sigma}^{\text{tor}}$ in a similar way as $\mathcal{S}_{K^p,\Sigma}^{\text{tor}}$ was described in theorem 2.3.16.

Theorem 2.5.3. [Lan16, Lemma 7.9., Sect. 8, Thm. 9.13., Thm. 10.13.]

Let $n \geq 3$ be an integer, and $K = K(np^m)$ a principal level. Let $\Sigma = \{\Sigma_{(\Phi, \delta)}\}_{(\Phi, \delta)}$ be a compatible choice of an admissible, smooth rational polyhedral cone decomposition data. Then $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ is a proper, flat scheme over $\text{Spec}(\mathcal{O}_{F_0, (p)})$, containing $\mathcal{S}_K$ as an open, dense subscheme

$$j_{\Sigma}^{\text{tor}} : \mathcal{S}_K \hookrightarrow \mathcal{S}_{K, \Sigma}^{\text{tor}},$$

together with a universal semi-abelian tuple

$$(G, \lambda, \iota, \alpha).$$

Furthermore, $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ satisfies the following properties:

1. It admits a (set-theoretical) stratification by locally closed subschemes

$$\mathcal{S}_{K, \Sigma}^{\text{tor}} = \coprod_{[(\Phi, \delta, \sigma)]} Z_{[(\Phi, \delta, \sigma)]}$$

where the union is over a complete set of equivalence classes of tuples (Φ, δ, σ) , with $\sigma \in \Sigma_{\Phi}$ such that $\sigma \subset P_{\Phi}^+$. Each stratum $Z_{[(\Phi, \delta, \sigma)]}$ lies in the closure of another stratum $Z_{[(\Phi', \delta', \sigma')]}$ if and only if $Z_{[(\Phi, \delta, \sigma)]}$ is a face of $Z_{[(\Phi', \delta', \sigma')]}$. $\mathcal{S}_K$ corresponds to the class where $X = Y = 0$;

2. Let $(\mathcal{S}_{K, \Sigma}^{\text{tor}})_{Z_{[(\Phi, \delta, \sigma)]}}^{\wedge}$ denote the formal completion of $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ along the $[(\Phi, \delta, \sigma)]$ stratum. For each choice of representative (Z, Φ, δ) of a cusp label as above, there is a toroidal boundary chart similar to the one in theorem 2.3.16,

$$\begin{array}{ccc} \Xi_{\Phi, \delta} & \longrightarrow & \Xi_{\Phi, \delta, \Sigma_{\Phi, \delta}} = \coprod_{\sigma \in \Sigma_{\Phi, \delta}} \Xi_{\Phi, \delta}(\sigma) \\ \downarrow & \swarrow & \\ C_{\Phi, \delta} & & \\ \downarrow & & \\ \mathcal{S}_K^Z & & \end{array}$$

defined by normalizations (c.f. [Lan16], section 7). There is a canonical isomorphism

$$(\mathcal{S}_{K, \Sigma}^{\text{tor}})_{Z_{[(\Phi, \delta, \sigma)]}}^{\wedge} \cong \mathcal{X}_{\Phi, \delta, \sigma}.$$

3. $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ satisfies a universal property similar to the one in theorem 2.3.16.

Remark 2.5.4. As seen in chapter 10 in [Lan16], the universal level structure α in the previous statement is only required to be defined in characteristic 0.

2.5.2 Minimal Compactifications at level- p

Just as in the case of toroidal compactifications, we can define the minimal compactification as follows.

Definition 2.5.5. Define $\mathcal{S}_K^*$ as the normalization of $\mathcal{S}_{K^p}^*$ in $\mathcal{S}_K$ under the morphism

$$\mathcal{S}_K \rightarrow \mathcal{S}_{K^p} \rightarrow \mathcal{S}_{K^p}^*.$$

Definition 2.5.6. Let $\mathcal{S}_{K^p}^{Z_n}$ be the scheme defined in 2.3.8, for any level- n filtration Z_n . We can make a similar definition of $\mathcal{S}_K^{Z, \text{pre}}$ with level $p^m n$ -structures (c.f. [Lan13, Defn. 5.4.2.6.]), and we obtain a morphism

$$\mathcal{S}_K^{Z, \text{pre}} \rightarrow \mathcal{S}_{K^p}^{Z_n} \otimes_{\mathbb{Z}} \mathbb{Q}$$

if Z_n is obtained from Z just by forgetting the p^m -structure, which is defined as the morphism π^{pre} we saw above. Define $\mathcal{S}_K^Z$ as the normalization of $\mathcal{S}_{K^p}^{Z_n}$ in $\mathcal{S}_K^{Z, \text{pre}}$ under the morphism

$$\mathcal{S}_K^{Z, \text{pre}} \rightarrow \mathcal{S}_{K^p}^{Z_n} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathcal{S}_{K^p}^{Z_n}.$$

We obtain the following generalization of theorem 2.4.1

Theorem 2.5.7. [Lan16, Prop. 6.4., Prop. 7.11., Thm. 12.1., Thm. 12.16.] *Let $n \geq 3$ be an integer prime-to- p , and $K = K(p^m n)$ a principal level. There is a normal, projective, flat scheme $\mathcal{S}_K^*$ over $\text{Spec}(\mathcal{O}_{F_0, (p)})$, containing $\mathcal{S}_K$ as an open, dense subscheme*

$$j^* : \mathcal{S}_K \hookrightarrow \mathcal{S}_K^*.$$

Furthermore, $\mathcal{S}_K^*$ admits a stratification by locally closed subschemes

$$\mathcal{S}_K^* = \coprod_{[(Z, \Phi, \delta)]} \mathcal{S}_K^Z$$

running through a complete set of cusp labels, such that $\mathcal{S}_{K'}^{Z'}$ lies in the closure of $\mathcal{S}_K^Z$ iff there is a surjection of cusp labels from $[(Z', \Phi', \delta')]$ to $[(Z, \Phi, \delta)]$. In addition, it satisfies the following property:

Let Σ be a choice of a compatible admissible smooth rational polyhedral cone decomposition data for $\mathcal{S}_K$ as in definition 2.3.13. Let $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ be the toroidal compactification attached to this data. Then, there is a map

$$\pi_{\Sigma} : \mathcal{S}_{K, \Sigma}^{\text{tor}} \rightarrow \mathcal{S}_K^*$$

between compactifications, such that the equality $j_K^* = \pi_{\Sigma} \circ j_{K, \Sigma}^{\text{tor}}$ holds. For each cusp label $[(Z, \Phi, \delta)]$, the (set-theoretic) preimage of the $[(Z, \Phi, \delta)]$ -stratum along π_{Σ} agrees with $Z_{[(\Phi, \delta)]} := \coprod_{\sigma \in \Sigma_{\Phi}} Z_{[(\Phi, \delta, \sigma)]}$ (using the notation as in theorem 2.3.16).

Chapter 3

Compactifications of Igusa varieties

3.1 Igusa varieties

In this section, we want to review the definition and main properties of central leaves and Igusa varieties. We will keep the notation from the previous chapter. Our main sources are going to be [Mant05], [CS17] and [CS19].

3.1.1 The Kottwitz set

We need to recall the definition of the Kottwitz set. Let G be an unramified reductive group over $\mathbb{Q}_p$. Fix a maximal torus and a Borel $T \subset B \subset G$, and the associated root system $(X^*(T), \Delta, X_*(T), \Delta^*)$. Let $\mathcal{O}_L = W(\bar{\mathbb{F}}_p)$ and $L = W(\bar{\mathbb{F}}_p)[\frac{1}{p}]$, and σ be the automorphism induced from the Frobenius automorphism on $\bar{\mathbb{F}}_p$. It induces an action by conjugation on $G(L)$, given by $g \mapsto hg\sigma(h)^{-1}$.

Definition 3.1.1. The Kottwitz set $B(G)$ is the set of σ -conjugacy classes of elements $b \in G(L)$.

One also need to define a subset of $B(G, \mu) \subset B(G)$ associated to a cocharacter $\mu \in X_*(T)$. In order to do so, we need to recall the existence of a couple of maps, which will allow us to define a poset structure in $B(G)$.

We let $\mathbb{D}$ be a pro-algebraic torus with character group $\mathbb{Q}$. Define the following group:

$$\mathcal{N}(G) := (\text{Inn}(G) \backslash \text{Hom}_L(\mathbb{D}, G))^{\langle \sigma \rangle},$$

which is the set of $\langle \sigma \rangle$ -invariants in the set of conjugacy classes of homomorphisms $\mathbb{D}_L \rightarrow G_L$. There is a σ -equivariant, well defined map constructed by Kottwitz in [Kot85]

$$\nu : B(G) \rightarrow \mathcal{N}(G),$$

which is called the Newton map.

Remark 3.1.2. There is another way of interpreting the set $B(G)$. Recall that an isocrystal over $\bar{\mathbb{F}}_p$ is a pair (V, ϕ) , where V is a finite dimensional L -vector space together with a σ -equivariant automorphism $\phi : V \rightarrow V$. More generally, isocrystal with G -structure is a tensor functor

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \rightarrow \{\mathrm{Isoc}/\bar{\mathbb{F}}_p\}.$$

One can prove that if $G = \mathrm{GL}_n$, then there is a bijection between $B(G)$ and the set of isomorphism classes of isocrystals of height n over $\bar{\mathbb{F}}_p$, sending $[b] \in B(G)$ to $(L^n, b\sigma)$ (c.f. [Mazur73]). More generally, there is a bijection between $B(G)$ and the set of isomorphism classes of isocrystals with G -structure.

Example 3.1.3. If $G = \mathrm{GL}_n$, then $T = \mathbb{G}_m^n$ and $X^*(T) \cong \mathbb{Q}^n$, and $[b] \in B(G)$ corresponds to the isocrystal $(L^n, b\sigma)$ by the remark above. In this case, the Newton map coincides with the usual Newton polygon of $(L^n, b\sigma)$. This explains the origin of the name of ν .

There is still another map we need to define.

Definition 3.1.4. Let

$$\pi_1(G, T) := \pi_1(G) = X_*(T) / \sum_{\alpha^\vee \in \phi^\vee} \mathbb{Z}\alpha^\vee$$

where $\phi^\vee$ is the set of cocharacters of T .

Remark 3.1.5. Originally in [Kot85], $\pi_1(G)$ was actually defined as the character group of the center of the Langlands dual, $X^*(Z(\hat{G}))$. We use the definition as in [RR96]. One can prove that the definitions are equivalent c.f. [RR96]. Let $\pi_1(G, T)_\sigma := X^*(Z(\hat{G})^\sigma)$.

Consider the Cartan decomposition

$$G(L) = \bigcup_{\mu \in X_*(T)^+} G(\mathcal{O}_L)\mu(\varpi)G(\mathcal{O}_L)$$

where $X_*(T)^+ \subset X_*(T)$ is the set of dominant coroots (c.f. [SW20, Sect. 19.2.]).

Define a map (the Kottwitz map)

$$K_G : B(G) \rightarrow \pi_1(G)_\sigma$$

sending $[b]$ to the image of μ (labeling the Cartan stratum of b) under the composition $X_*(T) \twoheadrightarrow \pi_1(G) \twoheadrightarrow \pi_1(G)_\sigma$.

The Kottwitz map together with the Newton map allows us to define an injective map (c.f. [Kot85, Sect. 3])

$$\begin{aligned} B(G) &\rightarrow X_*(T)_{\mathbb{Q}}^{+, \langle \sigma \rangle} \times \pi_1(G)_\sigma \\ [b] &\mapsto (\nu_b, K_G([b])). \end{aligned}$$

Definition 3.1.6 (Bruhat order). Let x_1, x_2 be two elements in $X_*(T)_{\mathbb{Q}}$. Then $x_1 \leq x_2$ if $x_1 - x_2$ is a non-negative, rational, linear combination of simple coroots of G .

The map above endows $B(G)$ with a poset structure, by setting

$$[b] \leq [b'] \iff (K_G([b]) = K_G([b']) \text{ and } \nu_b \leq \nu_{b'})$$

We can finally define the set $B(G, \mu)$.

Definition 3.1.7. Let $\mu \in X_*(T)^+ \subset X_*(T)$ be a dominant cocharacter. Let

$$\bar{\mu} := \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\mu)$$

where m is the degree of the splitting field of G/F over F . An element $[b] \in B(G)$ is in $B(G, \mu)$ if

$$\nu_b \leq \bar{\mu} \text{ and } K_G([b]) = [\mu]$$

with $[\mu] \in \pi_1(G)_\sigma$ being the image of μ under the composition $X_*(T) \twoheadrightarrow \pi_1(G) \twoheadrightarrow \pi_1(G)_\sigma$.

Remark 3.1.8. Assume that $F = \mathbb{Q}_p$. There is a bijection between the set of dominant cocharacters $X_*(T)^+$ and the set of conjugacy classes of cocharacters $\mathbb{G}_m \rightarrow G_{\mathbb{Q}_p}$ (c.f. [SW20, Sect. 20.2.]).

3.1.2 p -divisible groups

Let us start by recalling the definition of the universal cover of a p -divisible group, following [SW13]. Let R be a p -adic complete $\mathbb{Z}_p$ -algebra, and denote by Nilp_R the category of R algebras where p is nilpotent. Let $\text{Nilp}_R^{\text{opp}}$ be the opposite category.

We will think of a p -divisible group $\mathcal{G}$ as a sheaf on $\text{Nilp}_R^{\text{opp}}$, sending an R -algebra S to $\varinjlim \mathcal{G}[p^n](S)$ (c.f. [SW13, Sect. 3.1.]).

Definition 3.1.9. Let $\mathcal{G}$ be a p -divisible group over some p -adic complete $\mathbb{Z}_p$ -algebra R . The universal cover $\tilde{\mathcal{G}}$ of $\mathcal{G}$ is the sheaf in $\text{Nilp}_R^{\text{opp}}$ sending an R -algebra S to

$$\tilde{\mathcal{G}}(S) := \varprojlim_{p:\mathcal{G} \rightarrow \mathcal{G}} \mathcal{G}(S).$$

Proposition 3.1.10. [SW13, Prop. 3.1.3.] *If $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is an isogeny between p -divisible groups over R , then the induced morphism $\tilde{f} : \tilde{\mathcal{G}}_1 \rightarrow \tilde{\mathcal{G}}_2$ between the universal covers is an isomorphism.*

Example 3.1.11. If R is perfect, and we consider $\mathcal{G} = \text{Spf}(R[[x_1, \dots, x_m]])$, then $\tilde{\mathcal{G}}$ is given by $\text{Spf}(R[[x_1^{1/p^\infty}, \dots, x_m^{1/p^\infty}]])$ (c.f. [SW13, Prop. 3.1.3.] and [CS17, Prop. 4.1.2.]).

Also recall that to any p -divisible group $\mathcal{G}$ over $\mathbb{F}_p$, one can associate an isocrystal by considering its Dieudonné-module:

Let $\mathcal{G}$ be a p -divisible group over $\mathbb{F}_p$, and denote by $\text{E}\mathcal{G}$ its universal vector extension

$$0 \rightarrow V \rightarrow \text{E}\mathcal{G} \rightarrow \mathcal{G} \rightarrow 0$$

Example 3.1.12. If $\mathcal{G} = \mathbb{Q}_p/\mathbb{Z}_p$, then its universal vector extension is given by the short exact sequence

$$0 \rightarrow \mathbb{G}_a \rightarrow \frac{\mathbb{G}_a \oplus \mathbb{Q}_p}{\mathbb{Z}_p} \rightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

Consider its Lie algebra $\text{Lie}(\mathcal{E}\mathcal{G})$, seen as a crystal on the crystalline site of $\bar{\mathbb{F}}_p$ (c.f. [BBM82], [Mes72]).

Definition 3.1.13. The rational Dieudonné module $M_{\mathcal{G}}$ associated to $\mathcal{G}$ is the isocrystal over $W(\bar{\mathbb{F}}_p)[p^{-1}]$ obtained as follows: we evaluate the crystal $\text{Lie}(\mathcal{E}\mathcal{G})$ on the PD-thickening $W(\bar{\mathbb{F}}_p) \rightarrow \bar{\mathbb{F}}_p$, and then localize at p .

By the Dieudonné-Manin classification (c.f. [DO00], [Manin63]), every isocrystal over $W(\bar{\mathbb{F}}_p)[p^{-1}]$ can be written as a direct sum of simple isocrystals, i.e. isocrystals of the form

$$(V, \phi) = (W(\bar{\mathbb{F}}_p)[p^{-1}]^n, \phi_{\frac{m}{n}})$$

where

$$\phi_{\frac{m}{n}}(x_1, \dots, x_n) = (\sigma(x_2), \dots, \sigma(x_n), p^m \sigma(x_1))$$

with σ is the Frobenius on $W(\bar{\mathbb{F}}_p)[p^{-1}]$, and n coprime to m . We denote this isocrystal by $V_{\frac{m}{n}}$. $\frac{m}{n}$ is called the slope of $V_{\frac{m}{n}}$.

By the Dieudonné-Manin classification, one can talk about the slopes of any isocrystal. Indeed, write $V = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_r}$ for $\lambda_i = \frac{m_i}{n_i}$ and $\lambda_i \leq \lambda_{i+1}$ for all i . We write the slopes of V as the tuple $(\lambda_1, \dots, \lambda_r)$. By our construction of the Dieudonné module, we can also define the slopes of a p -divisible group.

Definition 3.1.14. A p -divisible group $\mathcal{G}$ over $\bar{\mathbb{F}}_p$ is isoclinic if its associated isocrystal $M_{\mathcal{G}}$ is isoclinic (i.e. it only has one slope). If the slope of $M_{\mathcal{G}}$ is λ , set the slope of $\mathcal{G}$ to be $-\lambda$.

Example 3.1.15. 1. If $\mathcal{G} = \mathbb{Q}_p/\mathbb{Z}_p$, then $M_{\mathcal{G}} = V_0$ has slope 0.

2. If $\mathcal{G} = \mu_{p^\infty}$, then $M_{\mathcal{G}} = V_{-1}$ has slope 1.

Theorem 3.1.16. [FC90], [Hao09, Thm. 1.3.5.], [Dem72] *The functor from p -divisible groups over $\bar{\mathbb{F}}_p$ up to isogeny to isocrystals over $W(\bar{\mathbb{F}}_p)[p^{-1}]$ of slopes between 0 and 1 is fully faithful. In particular, an isocrystal over $W(\bar{\mathbb{F}}_p)[p^{-1}]$ uniquely determines an isogeny class of p -divisible groups, and uniquely determines their universal cover.*

The second part of the theorem follows immediately from 3.1.10.

Let us go back to a more general setting. Namely, let us consider a p -divisible group over some scheme S over $\text{Spec}(\mathbb{F}_p)$, and let $\text{Frob}_{\mathcal{G}}$ be the relative Frobenius. Consider the quasi-isogeny

$$p^{-m} \text{Frob}_{\mathcal{G}}^n : \mathcal{G} \rightarrow \mathcal{G}^{(p^n)}$$

for m, n coprime integers.

Definition 3.1.17. Let $\mathcal{G}/S$ as above. $\mathcal{G}$ is said to be

- **Isoclinic** if there exists some $\lambda = \frac{m}{n} \in \mathbb{Q}_{\geq 0}$ such that $p^{-m}\text{Frob}_{\mathcal{G}}^n$ is an isomorphism.
- **Slope divisible** with respect to $\lambda = \frac{m}{n} \in \mathbb{Q}_{\geq 0}$ if $p^{-m}\text{Frob}_{\mathcal{G}}^n$ is an isogeny.
- **Completely slope divisible** if there is a filtration

$$0 = \mathcal{G}_{\geq 0} \subset \mathcal{G}_{\geq 1} \subset \cdots \subset \mathcal{G}_{\geq r} = \mathcal{G}$$

such that $\mathcal{G}_i := \mathcal{G}_{\geq i} / \mathcal{G}_{\geq i-1}$ is isoclinic of slope λ_i , and each $\mathcal{G}_{\geq i}$ is slope divisible with respect to λ_i , with $\lambda_i \leq \lambda_{i+1}$ for all i .

Remark 3.1.18. If $\mathcal{G}$ is a p -divisible group over $\bar{\mathbb{F}}_p$, then both definitions of isoclinic coincide, and being completely slope divisible is equivalent to

$$\mathcal{G} = \bigoplus_{i=1}^r \mathcal{G}_i$$

for $\mathcal{G}_i$ isoclinic p -divisible groups, as we will see bellow. Also note that by all we have seen in this section, there always exists a completely slope divisible p -divisible group in any given isogeny class of p -divisible groups over $\bar{\mathbb{F}}_p$.

Proposition 3.1.19. [Zink01, Corollary 11] *Let $\mathcal{G}$ be a completely slope divisible p -divisible group over a perfect field K , so in particular it admits a slope filtration*

$$0 = \mathcal{G}_{\geq 0} \subset \mathcal{G}_{\geq 1} \subset \cdots \subset \mathcal{G}_{\geq r} = \mathcal{G}$$

where each quotient $\mathcal{G}_i$ is isotypical of strictly increasing slopes λ_i . Then the slope filtration splits canonically.

Proof. By induction, we can assume that $r = 2$, so that the slope filtration is

$$0 \subset \mathcal{G}_{\geq 1} \subset \mathcal{G}_{\geq 2} = \mathcal{G},$$

where $\mathcal{G}_1 = \mathcal{G}_{\geq 1}$ is isoclinic of slope $\lambda_1 = \frac{t_1}{s}$ and $\mathcal{G}_2 = \mathcal{G}_{\geq 2} / \mathcal{G}_{\geq 1}$ is isoclinic of slope $\lambda_2 = \frac{t_2}{s}$ with $t_1 > t_2$. We get a short exact sequence

$$0 \rightarrow \mathcal{G}_{\geq 1} \rightarrow \mathcal{G} \rightarrow \mathcal{G}_2 \rightarrow 0,$$

and splitting the slope filtration is equivalent to splitting this sequence. Passing to Dieudonné modules, we are reduced to proving the following claim: Assume that we have a short exact sequence of finitely generated free W -modules,

$$0 \rightarrow M^1 \rightarrow M \rightarrow M^2 \rightarrow 0,$$

over $W(K)$, together with an F -linear map $V : M \rightarrow M$ (corresponding to $p^{-t_2}V^s$), which acts topologically nilpotently on M^1 , and is a bijection on M^2 . Then the sequence splits canonically.

This follows from Lemma 6.16 of [Zink84], but let us explain the proof. Since M is a finitely generated free W -module, $M_n := M/p^n$ is of finite length for all n , so in particular it follows from our hypothesis that $M_n^1 := M^1/p^n = \ker(V_n^r)$ and $M_n^2 := M^2/p^n = \operatorname{Im}(V_n^r)$ for r large enough. In particular, we think of M_n^2 as being contained in M_n . Now, let $m \in M_n$. There exists some element m' such that $V^r m = V^{2r} m'$, and thus $V^r m' - m \in \ker(V^r) = M_n^1$, giving a splitting of

$$0 \rightarrow M_n^1 \rightarrow M_n \rightarrow M_n^2 \rightarrow 0.$$

The result follows by taking inverse limits.

Finally, we recall the following lemma, which will be quite useful in the remaining sections of this chapter.

Lemma 3.1.20. *[Zink84, Thm. 7] Let X be a connected, regular scheme over $\operatorname{Spec}(\mathbb{F}_p)$, and let $\mathcal{G}$ be a p -divisible group over X . Let η denote the generic point of X , and $\bar{\eta}$ be a geometric point above it. If $\mathcal{G}_{\bar{\eta}}$ is completely slope divisible, then so is $\mathcal{G}$.*

3.1.3 Local PEL-data

In order to define Oort central leaves, we need to introduce the notion of a PEL-type p -divisible group, so let us start by recalling the local PEL-data as in [RZ96].

Let B be a finite-dimensional, semisimple algebra over $\mathbb{Q}_p$, together with a positive involution $*$. We fix a finite-dimensional B -module V , endowed with an alternating bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{Q}_p,$$

satisfying the equality $\langle bx, y \rangle = \langle x, b^* y \rangle$ for all $x, y \in V$ and $b \in B$. To such data, we can associate the unitary similitude group over $\mathbb{Q}_p$, by setting

$$G(R) := \{(g, r) \in \operatorname{GL}_{B \otimes R}(V \otimes R) \times R^\times \mid \langle gx, gy \rangle = r \langle x, y \rangle \ \forall x, y \in V \otimes R\},$$

for any $\mathbb{Q}_p$ -algebra R . We assume as in the global case that the derived group of G has type A or C , and that the data is unramified. Fix a maximal $\mathbb{Z}_p$ -order $\mathcal{O}_B$ such that the restriction of $*$ to $\mathcal{O}_B$ in B is defined, and is therefore an involution. We assume that there exists a self-dual $\mathcal{O}_B$ -lattice Λ . In particular, we can define a reductive group $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$, given by

$$G_{\mathbb{Z}_p}(R) := \{(g, c) \in \operatorname{GL}_{\mathcal{O}_B \otimes R}(\Lambda \otimes R) \times R^\times \mid \langle gx, gy \rangle = c \langle x, y \rangle \ \forall x, y \in \Lambda \otimes R\},$$

for any $\mathbb{Z}_p$ -algebra R .

Fix a conjugacy class of cocharacters $\mu : \mathbb{G}_m \rightarrow G_{\bar{\mathbb{Q}}_p}$, such that only weights 0 and 1 appearing in the induced weight decomposition in $V_{\bar{\mathbb{Q}}_p}$. In particular, we can write

$$V_{\bar{\mathbb{Q}}_p} = V_0 \oplus V_1,$$

and the composition $c \circ \mu : \mathbb{G}_m \rightarrow \mathbb{G}_m$ (by thinking of c as a map from $G_{\bar{\mathbb{Q}}_p}$ to $\mathbb{G}_m$), is the identity, so in particular V_0 and V_1 are totally isotropic. Denote by E the field of definition of μ .

Fix an element $b \in B(G, \mu^{-1})$, where μ is the cocharacter in the conjugacy class above such that μ^{-1} is dominant (which exist and is unique by Remark 3.1.8). A local rational PEL-data is defined as a tuple $(\mathcal{O}_B, *, \Lambda, (\cdot, \cdot), \mu, b)$ with elements as explained above.

To the element $b \in B(G, \mu^{-1})$, we can associate an isocrystal

$$(V \otimes_{\mathbb{Q}_p} L, b(\text{id} \otimes \sigma)).$$

Our conditions on b and on the weights of μ insure that $(V \otimes_{\mathbb{Q}_p} L, b(\text{id} \otimes \sigma))$ has weights between -1 and 0 . In particular, there is an isogeny class of p -divisible groups $\mathbb{X}_b$ over $\bar{\mathbb{F}}_p$, whose p -divisible groups have rational Dieudonné module given by the isocrystal above. Their universal cover is uniquely determined (c.f. 3.1.16). By functoriality, $\mathbb{X}_b$ is equipped with a B -action, $\iota_b : B \hookrightarrow \text{End}(\tilde{\mathbb{X}}_b)$, and a symmetric polarization $\lambda_b : \mathbb{X}_b \rightarrow \mathbb{X}_b^\vee$ compatible with the Rosatti involution. Let $\check{E} = E \cdot L$, and let $\text{Nilp}_{\mathcal{O}_{\check{E}}}$ denote the category of $\mathcal{O}_{\check{E}}$ -algebras on which p is nilpotent.

Let us also recall the definition of the Rapoport-Zink space attached to a rational PEL-data

Definition 3.1.21. Let $(\mathcal{O}_B, *, \Lambda, (\cdot, \cdot), \mu, b)$ be a local rational PEL-data. The Rapoport-Zink space $\mathcal{M}_{(G, b, \mu)}$ is the functor on $\text{Nilp}_{\mathcal{O}_{\check{E}}}^{\text{op}}$ sending $R \in \text{Nilp}_{\mathcal{O}_{\check{E}}}^{\text{op}}$ to the set of isomorphism classes of tuples $(\mathcal{G}, \iota, \lambda, \rho)$ where:

1. $\mathcal{G}$ is a p -divisible group over R ;
2. $\iota : \mathcal{O}_B \rightarrow \text{End}(\mathcal{G})$ is a $\mathcal{O}_B$ -action on $\mathcal{G}$;
3. $\lambda : \mathcal{G} \rightarrow \mathcal{G}^\vee$ is a principal polarization, compatible with the Rosatti involution and satisfying the determinantal condition;
- 4.

$$\rho : \mathbb{X}_b \otimes_{\bar{\mathbb{F}}_p} R/p \rightarrow \mathcal{G} \otimes_R R/p$$

is a quasi-isogeny compatible with the extra structures.

We have the following representability result.

Theorem 3.1.22. [RZ96, Thm. 3.25., Sect. 3.82] $\mathcal{M}_{(G, b, \mu)}$ is representable by a formally smooth, formally locally of finite type formal scheme over $\text{Spf}(\mathcal{O}_{\check{E}})$.

3.1.4 The Newton Stratification and Oort's Central leaves

Let $(\mathcal{O}_B, *, L, \langle \cdot, \cdot \rangle, h)$ be an integral PEL-datum as defined in the first chapter. Let $\mathcal{S}_K$ be the associated Shimura variety at principal level $K = K(n)$ for $n \geq 3$, and consider the base change $\mathcal{S}_{K, \bar{\mathbb{F}}_p}$ to $\bar{\mathbb{F}}_p$. Let p be a prime which is unramified in B and prime-to n .

It is possible to define certain subsets of $\mathcal{S}_{K, \bar{\mathbb{F}}_p}$, related to the isogeny classes of the p -divisible groups associated to the Shimura varieties it parametrizes. Namely, let $\mathcal{A}$ denote the universal abelian variety with extra structures over $\mathcal{S}_{K, \bar{\mathbb{F}}_p}$. Let $\mathcal{G} := \mathcal{A}[p^\infty]$ be the associated

p -divisible group with extra structures induced from the ones on $\mathcal{A}$. For any point $x \in \mathcal{S}_{K, \mathbb{F}_p}$, we can consider the associated p -divisible group with extra structures over $k(\bar{x})$, $\mathcal{G}_{k(\bar{x})}$, for any geometric point $\bar{x}$ over x , whose isogeny class corresponds to some $b \in B(G, \mu^{-1})$ by [RR96, Sect. 3.5.], c.f. [Vie15, Sect. 3.3]. Denote it by $b(\mathcal{G}_{k(\bar{x})})$. Then, for $b \in B(G)$, define the subsets

$$\mathcal{S}_{K, \mathbb{F}_p}^{\leq b} := \{x \in \mathcal{S}_{K, \mathbb{F}_p} \mid b(\mathcal{G}_{k(\bar{x})}) \leq b\}$$

and

$$\mathcal{S}_{K, \mathbb{F}_p}^b := \{x \in \mathcal{S}_{K, \mathbb{F}_p} \mid b(\mathcal{G}_{k(\bar{x})}) = b\}$$

of $\mathcal{S}_{K, \mathbb{F}_p}$. We have the following result.

Theorem 3.1.23. [RR96, Thm. 3.6.] $\mathcal{S}_{K, \mathbb{F}_p}^{\leq b}$ defines a closed subscheme of $\mathcal{S}_{K, \mathbb{F}_p}$. Furthermore, $\mathcal{S}_{K, \mathbb{F}_p}^b$ is an open subscheme of $\mathcal{S}_{K, \mathbb{F}_p}^{\leq b}$.

We call $\mathcal{S}_{K, \mathbb{F}_p}^b$ the Newton strata associated to b . These define a well-defined stratification called the Newton stratification on $\mathcal{S}_{K, \mathbb{F}_p}$ (c.f. [Vie15], [Lov17]). Furthermore, $\mathcal{S}_{K, \mathbb{F}_p}^b$ is non-empty for any $[b] \in B(G, \mu^{-1})$ by [VW13, Thm. 1.6.].

Let us now recall the definition of the Oort's central leaves, as introduced in [Mant05]. By definition, our integral PEL-data $(\mathcal{O}_B, *, L, \langle \cdot, \cdot \rangle, h)$ gives rise to a local integral PEL-datum $(\mathcal{O}_{\mathbb{Z}_p}, *, L_{\mathbb{Z}_p}, \langle \cdot, \cdot \rangle_{\mathbb{Q}_p}, \mu, b)$. Fix a p -divisible group with extra structures $\mathbb{X}_b$ (not just an isogeny class) over $\mathbb{F}_p$ of PEL- type as in the previous section. Recall that $\mathcal{G} := \mathcal{A}[p^\infty]$, where $\mathcal{A}$ is the universal abelian variety over $\mathcal{S}_{K, \mathbb{F}_p}$.

Definition 3.1.24. The central leaf attached to the p -divisible group $\mathbb{X}_b$ is the subset of $\mathcal{S}_{K, \mathbb{F}_p} = \mathcal{S}_K \otimes_{\mathbb{F}_p} \mathbb{F}_p$ given by

$$\mathcal{C}^{\mathbb{X}_b} = \{x \in \mathcal{S}_{K, \mathbb{F}_p} \mid \mathcal{G}_{k(\bar{x})} \cong \mathbb{X}_b \times_{\mathbb{F}_p} k(\bar{x})\} \subset \mathcal{S}_{K, \mathbb{F}_p}^b,$$

where the isomorphisms are as p -divisible groups with the extra structures, so in particular they must preserve the polarizations up to an element in $\mathbb{Z}_p^\times$, and commute with the $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -actions.

Proposition 3.1.25. [Mant05, Prop. 1] $\mathcal{C}^{\mathbb{X}_b}$ is a closed subset of $\mathcal{S}_{K, \mathbb{F}_p}^b$. Once we endow it with the induced reduced subscheme structure, it actually becomes a smooth subscheme.

3.1.5 Igusa varieties

We can now define Igusa varieties. We will define two different Igusa varieties: the ones introduced in [Mant05] for completely slope divisible p -divisible groups, and the perfect Igusa varieties as defined in [CS17]. We fix a p -divisible group with extra structures $\mathbb{X}_b$ as above.

Definition 3.1.26. The Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$, is defined as the moduli space over $\text{Spec}(\mathbb{F}_p)$, sending an $\text{Spec}(\mathbb{F}_p)$ -algebra R to the set of isomorphism classes of pairs

$$\{(A, \rho) \mid A \in \mathcal{S}_K(R), \rho : A[p^\infty] \rightarrow \mathbb{X}_b \times_{\mathbb{F}_p} R\}$$

where ρ is an isomorphism of p -divisible groups satisfying the following:

1. ρ is compatible with the $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -actions on both sides.
2. the isomorphisms commute with the isomorphisms induced by the polarization on $A[p^\infty]$ and $\mathbb{X}_b$ up to an element $\mathbb{Z}_p^\times$. In particular, there exists an element in $\mathbb{Z}_p^\times$ such that the diagram

$$\begin{array}{ccc} A[p^\infty] & \xrightarrow{\rho} & \mathbb{X}_b \times_{\mathbb{F}_p} R \\ \downarrow \lambda_A & & \downarrow \lambda_b \\ A^\vee[p^\infty] & \xleftarrow[\rho^\vee]{} & (\mathbb{X}_b)^\vee \times_{\mathbb{F}_p} R \end{array}$$

commute up to that element.

This space is representable by a scheme by [CS17, Prop. 4.3.3.]. Furthermore, since $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$ is reduced, the natural map

$$\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b} \rightarrow \mathcal{S}_K$$

factors through the Oort central leaf $\mathcal{C}^{\mathbb{X}_b}$ (c.f. [CS17]).

One can actually think about these Igusa varieties as the moduli space of abelian varieties up to p -power isogenies by the lemma bellow.

Lemma 3.1.27. [CS17, Lemma 4.3.4.], see also [CT21, Lemma 4.2.2]

Let R be a $\mathbb{F}_p$ -algebra. The set $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}(R)$ is in bijection with the set of isomorphism classes of tuples $(A, \lambda, \iota, (\alpha_n, \nu_n), \rho)$ consisting of:

1. an abelian scheme A over $S := \text{Spec}(R)$;
2. a prime-to- p polarization $\lambda : A \rightarrow A^\vee$;
3. an $\mathcal{O}_B$ -action $\iota : \mathcal{O}_B \rightarrow \text{End}_S(A)$ satisfying the equality $\iota(b)^\vee \circ \lambda = \lambda \circ \iota(b^*)$ for every $b \in \mathcal{O}_B$;
4. (α_n, ν_n) is a level- n structure as in definition 2.1.3;
5. $\rho : A[p^\infty] \rightarrow \mathbb{X}_b \times_{\mathbb{F}_p} R$ is a quasi-isogeny, commuting with the $\mathcal{O}_B$ -actions, and respecting the polarizations up to a $\mathbb{Q}_p^\times$ -factor.

Two such tuples $(A, \lambda, \iota, (\alpha_n, \nu_n), \rho)$ and $(A', \lambda', \iota', (\alpha'_n, \nu'_n), \rho')$ over R are isomorphic if there is a p -power quasi-isogeny $A \rightarrow A'$ commuting with $\mathcal{O}_B$ -actions, level-structures and ρ , and respecting the polarizations up to a p -power.

Let us give a brief idea of the proof.

Proof. Clearly, if we have a point $(A, \rho) \in \mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}(R)$, we obtain a tuple as the one described above.

Conversely, assume that we are given a tuple $(A, \lambda, \iota, (\alpha_n, \nu_n), \rho)$ as above, and assume for simplicity that $\rho : A[p^\infty] \rightarrow \mathbb{X}_b \times_{\mathbb{F}_p} R$ is an isogeny. Let $K = \ker(\rho)$, and consider that abelian

scheme $A' := A/K$. It is p -isogenous to A by definition, and its associated p -divisible group $A'[p^\infty]$ is isomorphic to $\mathbb{X}_b$, just by construction. In particular, we obtain an isomorphism $\rho' : A'[p^\infty] \rightarrow \mathbb{X}_b \times_{\mathbb{F}_p} R$, since ρ factors through $A'[p^\infty]$, so in particular $\rho = \rho' \circ \pi$ where $\pi : A \rightarrow A'$ denotes the projection. This will provide us with the desired point in $\mathfrak{Ig}^{\mathbb{X}_b}$, once we can define the extra structures in the statement. For the action, one just takes $\iota'(x) = \pi \circ \iota(x) \circ \pi^{-1}$, and the level- n -structure is just the induced one (recall that n is prime-to- p). Finally, we can define a polarization on A' as $\lambda_{A'} = (\pi^\vee)^{-1} \circ \lambda \circ \pi^{-1}$. However, recall that ρ must respect the polarizations up to a factor $c_p \in \mathbb{Q}_p^\times$, so it must satisfy the equality $c_p \lambda_b = (\rho^\vee)^{-1} \circ \lambda \circ \rho^{-1}$. This implies in particular that $c_p \lambda_b = (\rho'^\vee)^{-1} \circ \lambda_{A'} \circ \rho'^{-1}$, just by definition of all the maps involved. But notice that for an element (A, ρ) to be in $\mathfrak{Ig}^{\mathbb{X}_b}(R)$, ρ must commute with polarizations up to an element in $\mathbb{Z}_p^\times$. In order to fix this, one just takes $\lambda' = p^{-v_p(c_p)} \lambda_{A'}$. This finishes the proof.

This shows several things. For starters, we already knew that the group of automorphisms of $\mathbb{X}_b$ respecting the extra structures acted on $\mathfrak{Ig}^{\mathbb{X}_b}$, but Lemma 3.1.27 actually shows that the group of quasi-isogenies of $\mathbb{X}_b$ respecting the extra structures also acts on $\mathfrak{Ig}^{\mathbb{X}_b}$. This implies that $\mathfrak{Ig}^{\mathbb{X}_b}$ is independent of the choice of $\mathbb{X}_b$ in a given isogeny class. Furthermore, one can prove that $\mathfrak{Ig}^{\mathbb{X}_b}$ is perfect (c.f. [CS17, Cor. 4.3.5.]).

In order to define Mantovan's Igusa varieties, let us now assume that $\mathbb{X}_b$ is the completely slope divisible p -divisible group in the given isogeny class. In particular, we have a decomposition

$$\mathbb{X}_b \cong \bigoplus_{i=1}^r \mathbb{X}_{b,i}$$

of $\mathbb{X}_b$, where each $\mathbb{X}_{b,i}$ is isoclinic of strictly increasing slopes $\lambda_i \in [0, 1]$.

The quasi-polarization on $\mathbb{X}_b$ induces isomorphisms $\lambda_{b,i} : \mathbb{X}_{b,i} \cong (\mathbb{X}_{b,j})^\vee$, for all i, j with $\lambda_{b,i} + \lambda_{b,j} = 1$, with the property that $\lambda_{b,i} = c(\lambda_b) \lambda_{b,j}^\vee$ for some constant $c(\lambda_b) \in \mathbb{Z}_{(p)}^\times$. Denote by $\mathcal{G}_b := \mathcal{G}_{\mathbb{X}_b}$, the p -divisible group associated to the restriction of the universal abelian variety $\mathcal{A}/\mathcal{S}_K$ to $\mathcal{C}^{\mathbb{X}_b}$, with induced polarization $\lambda_{\mathcal{G}_b}$. It is completely slope divisible by Lemma 3.1.20, so in particular it admits a slope filtration

$$0 = \mathcal{G}_{b, \geq 0} \subset \mathcal{G}_{b, \geq 1} \subset \cdots \subset \mathcal{G}_{b, \geq r} = \mathcal{G}_b$$

where each quotient $\mathcal{G}_{b,i}$ is isotypical of strictly increasing slopes λ_i , and just as above the polarization on $\mathcal{G}_b$ induces $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -equivariant isomorphisms $\lambda_{\mathcal{G}_b}^i : \mathcal{G}_{b,i} \cong (\mathcal{G}_{b,j})^\vee$ for all i, j with $\lambda_i + \lambda_j = 1$, such that $(\lambda_{\mathcal{G}_b}^i)^\vee = c(\lambda_{\mathcal{G}_b}) \lambda_{\mathcal{G}_b}^j$ for some constant $c(\lambda_{\mathcal{G}_b}) \in \mathbb{Z}_{(p)}^\times$ as in [Mant05].

Definition 3.1.28. Fix an integer m . The finite Igusa variety $\text{Ig}_m^{\mathbb{X}_b}$ associated to $\mathbb{X}_b$ is the map

$$\text{Ig}_m^{\mathbb{X}_b} \rightarrow \mathcal{C}^{\mathbb{X}_b},$$

parametrizing over each $\mathcal{C}^{\mathbb{X}_b}$ -scheme S , tuples $(\rho_i^m)_{i=1}^r$ consisting of isomorphisms

$$\rho_i^m : \mathcal{G}_{b,i}[p^m] \times_{\mathcal{C}^{\mathbb{X}_b}} S \xrightarrow{\sim} \mathbb{X}_{b,i}[p^m] \times_{\mathbb{F}_p} S$$

satisfying the following:

1. each ρ_i^m lifts fppf locally for all $m' \geq m$.
2. ρ_i^m are compatible with the $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -actions on both sides.
3. the isomorphisms commute with the isomorphisms induced by the polarization on $\mathcal{G}[p^m]$ and $\mathbb{X}_b[p^m]$ up to an element $(\mathbb{Z}/p^m\mathbb{Z})^\times(S)$. In particular, there exists an element in $(\mathbb{Z}/p^m\mathbb{Z})^\times(S)$ such that all diagrams

$$\begin{array}{ccc}
\mathcal{G}_{b,i}[p^m] \times_{\mathcal{C}^{\mathbb{X}_b}} S & \longrightarrow & \mathbb{X}_{b,j}[p^m] \times_{\mathbb{F}_p} S \\
\downarrow & & \downarrow \\
(\mathcal{G}_{b,j})^\vee[p^m] \times_{\mathcal{C}^{\mathbb{X}_b}} S & \longleftarrow & (\mathbb{X}_{b,j})^\vee[p^m] \times_{\mathbb{F}_p} S
\end{array}$$

commute up to that element.

Passing to the inverse limit, we obtain:

Definition 3.1.29. The pro-Igusa variety $\mathrm{Ig}^{\mathbb{X}_b}$ associated to $\mathbb{X}_b$ is the map

$$\mathrm{Ig}^{\mathbb{X}_b} \rightarrow \mathcal{C}^{\mathbb{X}_b},$$

parametrizing over each $\mathcal{C}^{\mathbb{X}_b}$ -scheme S , tuples $(\rho_i)_{i=1}^r$ of isomorphisms

$$\rho_i : \mathcal{G}_i \times_{\mathcal{C}^{\mathbb{X}_b}} S \xrightarrow{\sim} \mathbb{X}_{b,i} \times_k S$$

satisfying the following:

1. ρ_i are compatible with the $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -actions on $\mathcal{G}_i$ and $\mathbb{X}_{b,i}$,
2. the isomorphisms ρ_i commute with the isomorphisms induced by the polarization on $\mathcal{G}_i$ and $\mathbb{X}_{b,i}$ up to an element of $\mathbb{Z}_{(p)}^\times(S)$, which is independent of i .

We have the following results on the geometry of Igusa varieties.

Definition 3.1.30. Let $\mathbb{X}_b/\bar{\mathbb{F}}_p$ be a PEL-type p -divisible group, not necessarily completely slope divisible,. We define:

1. For $m \in \mathbb{Z}_{\geq 1}$, let $\Gamma_{b,m}$ be the group of automorphisms of $\mathbb{X}_b[p^m]$, compatible with $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -actions and polarization, that lift to automorphisms of $\mathbb{X}_p[p^{m'}]$ for all $m' > m$.
2. Define $\Gamma_b = \varprojlim_m \Gamma_{b,m}$, as the profinite group of automorphisms of $\mathbb{X}_b$ compatible with $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -actions and polarization.
3. Let $\underline{\mathrm{Aut}}(\mathbb{X}_b)$ be the group of automorphisms of $\mathbb{X}_b$ compatible with the $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -action and polarization.

Proposition 3.1.31. [Mant05], [CS17], [CS19] Let $\mathbb{X}_b/\bar{\mathbb{F}}_p$ be a PEL-type p -divisible group. Then:

1. The perfect Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$ over $\mathcal{C}^{\mathbb{X}_b}$ is representable by an $\underline{\text{Aut}}(\mathbb{X}_b)$ -torsor over $\mathcal{C}^{\mathbb{X}_b}$.
2. Assume that $\mathbb{X}_b$ is completely slope divisible. Then Mantovan's finite Igusa variety $\text{Ig}_m^{\mathbb{X}_b}$ is representable by a $\Gamma_{b,m}$ -torsor over $\mathcal{C}^{\mathbb{X}_b}$. In particular, the pro-Igusa variety $\text{Ig}^{\mathbb{X}_b}$ is representable by a Γ_b -torsor over $\mathcal{C}^{\mathbb{X}_b}$.

Proof. All the statements are proven in [CS19, Sect. 2.2.], namely the first part is Proposition 2.2.7., and the second one in Theorem 2.2.4. and Proposition 2.2.6..

Assume now that $\mathbb{X}_b$ is completely slope divisible, and consider the perfect Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$, and the pro-Igusa variety $\text{Ig}^{\mathbb{X}_b}$. These are not necessarily the same. Indeed, the slope filtration does not necessarily split over a non-perfect base, and therefore the isomorphisms ρ_i do not induce isomorphisms on the whole p -divisible groups. However, over a perfect base the slope filtration does split uniquely by Proposition 3.1.19. This allows one to prove the following result.

Proposition 3.1.32. [CS17, Prop. 4.3.8.] *There is a natural map $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b} \rightarrow \text{Ig}^{\mathbb{X}_b}$ over $\mathcal{C}^{\mathbb{X}_b}$, since an isomorphism $\mathcal{G}_b \cong \mathbb{X}_b$ over $\mathcal{C}^{\mathbb{X}_b}$ induces isomorphisms $\mathcal{G}_{b,i} \cong \mathbb{X}_{b,i}$ between the isoclinic pieces of both $\mathcal{G}_b$ and $\mathbb{X}_b$ by [CS17, Cor. 4.1.10.]. It induces an isomorphism between $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$ and the perfection of $\text{Ig}^{\mathbb{X}_b}$.*

3.2 Toroidal compactifications of Igusa varieties

In this section, we will construct toroidal compactifications of Igusa varieties. This was already done for the quasi-split unitary case in [CS19], and we want to extend their construction for any Shimura variety of PEL-type as considered in the previous chapter. We start by reviewing the construction of the toroidal compactification of the Oort central leaves $\mathcal{C}^{\mathbb{X}_b}$. This is done by considering the notion of a well positioned subset, as introduced in [Box15] and [LS18].

3.2.1 Partial toroidal compactification of Oort central leaves

Let us fix a prime p unramified in B , a principal level $K = K(n)$ for some positive integer $n \geq 3$, which we assume to be prime-to- p , and a compatible choice of a polyhedral cone decomposition Σ for choices of representatives of all cusp labels, as in the first chapter. Also fix a p -divisible group of PEL-type $\mathbb{X}_b$ over $\bar{\mathbb{F}}_p$. Let us start by defining well-positioned subsets, following [LS18]. These will be subsets of the special fiber of the Shimura variety, which allows one to define toroidal and minimal compactifications on them. In particular, their compactifications will admit a local structure very similar to the one we have on compactifications of the Shimura varieties.

Definition 3.2.1. Let $\text{Spf}(R) \rightarrow \mathcal{X}_{\Phi_n, \delta_n, \sigma}^\circ$ be an affine open formal subscheme of $\mathcal{X}_{\Phi_n, \delta_n, \sigma}^\circ$ for some representative of a cusp label $[Z_n, \Phi_n, \delta_n]$ as in theorem 2.3.16, and recall that the stratifications induced on $\text{Spec}(R)$ by the induced morphisms $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ and $\Xi_{\Phi_n, \delta_n}(\sigma)$ coincide, and that W^0 denotes the preimage of $\mathcal{S}_K$ under this map. Let $Y \subset \mathcal{S}_{K, \bar{\mathbb{F}}_p}$ be a locally closed subset.

We say that Y is a well-positioned subset if there exists a collection

$$Y^\natural = \{Y_{Z_n}^\natural\}_{Z_n}$$

of locally closed subsets of $\mathcal{S}_{K, \mathbb{F}_p}^{Z_n}$, running through a set of representatives of cusp labels at level n , such that

$$Y \times_{\mathcal{S}_{K, \mathbb{F}_p}} W^0 = Y_{Z_n}^\natural \times_{\mathcal{S}_{K, \mathbb{F}_p}^{Z_n}} W^0$$

under the morphisms $W^0 \rightarrow \mathcal{S}_{K, \mathbb{F}_p}$ and $W^0 \rightarrow \Xi_{\Phi_n, \delta_n} \rightarrow \mathcal{S}_{K, \mathbb{F}_p}^{Z_n}$ respectively.

This condition can be checked on points. Following [CS19, Section 3.1.], let us explain exactly how to translate this condition on points.

Let C be a complete, algebraically closed, non-archimedean field with ring of integers $\mathcal{O}_C$. Consider a point $x = (A, \lambda, \iota, \alpha_n) \in \mathcal{S}_K(C)$. By the degeneration theorem, A admits a model over $\mathcal{O}_C$ (i.e. there exists a semi-abelian scheme over $\mathcal{O}_C$ with generic fiber A). We can consider its Raynaud extension (c.f. as defined in section 2.5.1. of [CS19]),

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow B \rightarrow 0$$

which is defined over $\mathcal{O}_C$. The extra structures on A induce a polarization, $\mathcal{O}_B$ -action and level n -structure on B , which therefore defines a point in $\pi(x) \in \mathcal{S}_Z(C)$, where Z is the cusp into which x degenerates.

Let Y be a locally closed subset of $\mathcal{S}_{K, \mathbb{F}_p}$. Then the condition of Y being well-positioned, translates to the existence of a collection $Y^\natural = \{Y_{Z_n}^\natural\}_{Z_n}$ of locally closed subsets of $\mathcal{S}_{K, \mathbb{F}_p}^{Z_n}$, such that for any complete, algebraically closed, non-archimedean field C above $\mathbb{F}_p$, and any point $x = (A, \lambda, \iota, \alpha_n) \in \mathcal{S}_K(C)$ that degenerates to a cusp Z_n , $y \in Y$ if and only if $\pi(y) \in Y_{Z_n}^\natural$.

Let us define the partial toroidal and minimal compactifications associated to a well positioned subset Y , following the work of Lan and Stroh in [LS18].

Definition 3.2.2. Let $Y \subset \mathcal{S}_{K, \mathbb{F}_p}$ be a well-positioned subset. Let $\bar{Y}$ denote the closure of Y in $\mathcal{S}_{K, \mathbb{F}_p}$, and let $Y_0 := \bar{Y} - Y$ denote its complement.

1. Consider the closure of $\bar{Y}$ (resp. Y_0) in $\mathcal{S}_{K, \Sigma, \mathbb{F}_p}^{\text{tor}}$, and denote it by $\bar{Y}_\Sigma^{\text{tor}}$ (resp. $Y_{0, \Sigma}^{\text{tor}}$). The partial toroidal compactification of Y is defined as

$$Y_\Sigma^{\text{tor}} := \bar{Y}_\Sigma^{\text{tor}} - Y_{0, \Sigma}^{\text{tor}}.$$

There is an induced closed immersion

$$j_\Sigma^{\text{tor}} : Y \hookrightarrow Y_\Sigma^{\text{tor}}.$$

2. Consider the closure of $\bar{Y}$ (resp. Y_0) in $\mathcal{S}_{K, \mathbb{F}_p}^*$, and denote it by $\bar{Y}^*$ (resp. Y_0^*). The partial minimal compactification of Y is defined as

$$Y^* := \bar{Y}^* - Y_0^*.$$

There is an induced closed immersion

$$j^* : Y \hookrightarrow Y^*.$$

We have the following result on the structure of these schemes.

Theorem 3.2.3. *[LS18, Thm. 2.3.2.], [LS(NC)18, Prop. 2.2.]*

Let Y be a well-positioned subset, with a family $Y^\natural = \{Y_{Z_n}^\natural\}_{Z_n}$ of locally closed subsets of $\mathcal{S}_{K, \mathbb{F}_p}^{Z_n}$ as in definition 3.2.1. The proper, surjective morphism $\mathcal{S}_{K, \Sigma}^{\text{tor}} \rightarrow \mathcal{S}_K^*$ induces a morphism

$$f : Y^{\text{tor}} \rightarrow Y^*$$

satisfying the equality $j^* = f \circ j_\Sigma^{\text{tor}}$. The stratification of $\mathcal{S}_{K, \Sigma}^{\text{tor}}$, resp. $\mathcal{S}_K^*$, induces a similar stratification on Y_Σ^{tor} , resp. Y^* , given by

$$Y_\Sigma^{\text{tor}} = \coprod_{[(\Phi_n, \delta_n, \sigma)]} Z_{[(\Phi_n, \delta_n, \sigma)]} \times_{\mathcal{S}_{K, \Sigma}^{\text{tor}}} Y_\Sigma^{\text{tor}} =: Y_{Z_{[(\Phi_n, \delta_n, \sigma)]}}^\natural$$

with notation as in 2.3.16, resp.

$$Y^* = \coprod_{Z_n} \mathcal{S}_{Z_n} \times_{\mathcal{S}_K^*} Y^* = \coprod_{Z_n} Y_{Z_n}$$

inducing a bijection between $Y_{Z_n}^\natural$ and $Y_{Z_n, \mathbb{F}_p}$. Furthermore, consider the toroidal boundary chart associated to $\mathcal{S}_{K, \Sigma}^{\text{tor}}$

$$\begin{array}{ccc} \Xi_{\Phi_n, \delta_n} & \longrightarrow & \Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}} \\ \downarrow & \swarrow & \\ C_{\Phi_n, \delta_n} & & \\ \downarrow & & \\ \mathcal{S}_K^{Z_n} & & \end{array}$$

as in 2.3.16. By taking pullbacks, we get a similar toroidal boundary chart for Y

$$\begin{array}{ccc} Y_{\Xi_{\Phi_n, \delta_n}}^\natural & \longrightarrow & Y_{\Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}}}^\natural \\ \downarrow & \swarrow & \\ Y_{C_{\Phi_n, \delta_n}}^\natural & & \\ \downarrow & & \\ Y_{Z_n}^\natural & & \end{array}$$

There is a canonical isomorphism,

$$(Y_\Sigma^{\text{tor}})_{Y_{Z_{[(\Phi_n, \delta_n, \sigma)]}}^\natural}^\wedge \cong Y_{\mathcal{X}_{\Phi_n, \delta_n, \sigma}}^\natural / \Gamma_{\Phi_n, \sigma}$$

extending the canonical isomorphism

$$(\mathcal{S}_{K, \Sigma}^{\text{tor}})_{Z_{[(\Phi_n, \delta_n, \sigma)]}}^\wedge \cong \mathcal{X}_{\Phi_n, \delta_n, \sigma} / \Gamma_{\Phi_n, \sigma}$$

seen in theorem 2.3.16.

The reason we need these results, is that we will have to consider partial compactifications on Oort-central leaves. We have the following proposition.

Proposition 3.2.4. *[LS18, Prop. 3.4.2.] The Oort central leaf $\mathcal{C}^{\mathbb{X}_b}$ associated to the p -divisible group $\mathbb{X}_b$ is well positioned. Furthermore, we can describe the locally closed subsets $\mathcal{C}_Z^{\mathbb{X}_b, \mathfrak{h}}$ in the following way: for any cusp label $Z_n = (Z_n, (\underline{X}, \underline{Y}, \phi, \varphi_{-2,n}, \varphi_{0,n}), \delta_n)$, $\mathcal{C}_{Z_n}^{\mathbb{X}_b, \mathfrak{h}}$ is either empty, or the central leaf $\mathcal{C}_{Z_n}^{\mathbb{X}_b, Z_n} \subset \mathcal{S}_{K, \mathbb{F}_p}^{Z_n}$ associated to a certain p -divisible group $\mathbb{X}_{b, Z_n}$ to be described below.*

Proof. We want to use the equivalent definition of well-positioned subsets using the description on points in order to prove the statement. In particular, we want to deduce what $\mathbb{X}_{b, Z}$ should be by describing what happens to a point in $\mathcal{C}^{\mathbb{X}_b, Z}$ when we analyze its degeneration data. Let $x = (A, \lambda, \iota, \alpha_n) \in \mathcal{S}_{\mathbb{F}_p}(C)$ be a point, giving rise to a Raynaud extension

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow B \rightarrow 0,$$

defining a point $\pi(x) \in \mathcal{S}_{Z, k}(C)$ for a certain cusp Z as above. Recall that for any $m \in \mathbb{Z}_{\geq 1}$, we have two short exact sequences

$$0 \rightarrow \tilde{G}[p^m]_C \rightarrow A[p^m]_C \rightarrow (Y/p^m Y)_C \rightarrow 0$$

and

$$0 \rightarrow T[p^m]_C \rightarrow \tilde{G}[p^m]_C \rightarrow B[p^m]_C \rightarrow 0.$$

Taking the inverse limit over m , we obtain a symplectic, $\mathcal{O}_B$ -linear filtration on $A[p^\infty]_C$,

$$0 \subset T[p^\infty]_C \subset \tilde{G}[p^\infty]_C \subset A[p^\infty]_C,$$

with graded pieces $\text{Gr}_{-2} = T[p^\infty]_C$, $\text{Gr}_{-1} = B[p^\infty]_C$ and $\text{Gr}_0 = Y \otimes \mathbb{Q}_p / \mathbb{Z}_p$. Since Gr_{-2} is multiplicative, and Gr_0 is étale [CS19, Prop. 2.2.1.] tells us that the filtration splits, inducing an isomorphism

$$A[p^\infty]_C \cong \text{Hom}(X, \mu_{p^\infty}) \oplus B[p^\infty]_C \oplus Y \otimes \mathbb{Q}_p / \mathbb{Z}_p.$$

Now the result follows if we just take $(\mathbb{X}_{b, Z_n})_C := B[p^\infty]_C$.

In particular, we can now consider partial toroidal and minimal compactifications of $\mathcal{C}^{\mathbb{X}_b}$, which we denote by $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$ and $\mathcal{C}^{\mathbb{X}_b, *}$ respectively.

3.2.2 Toroidal compactifications of Igusa varieties

In this section, we aim to define toroidal compactifications of the Igusa varieties we introduced in the previous section, and therefore generalize section 3.2. of [CS19]. The idea will be to impose a condition on semi-abelian tuples $(G, \lambda, \iota, \alpha_n)$ over the partial toroidal compactification of the Oort central leaf, $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$, very much similar to the ones above defining the Igusa varieties above. However, the argument is more subtle. Namely, note that if $(\mathcal{G}, \lambda, \iota, \alpha_n)$ denotes the restriction

of the universal object of $\mathcal{S}_{K,\Sigma}^{\text{tor}}$ to $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$, then $\mathcal{G}[p^\infty]$ is not a p -divisible group ($\mathcal{G}[p^m]$ are quasi-finite, but not finite group schemes), so we can not just ask there to be an isomorphism between $\mathcal{G}[p^\infty]$ and some PEL-type p -divisible group. The solution for this, as seen in [CS19], is as follows: even though $\mathcal{G}[p^\infty]$ is not a p -divisible group, its connected part $\mathcal{G}[p^\infty]^\circ$ is, as we will see below. In [CS19], they are working in the case of the split unitary group and the polarizations on their moduli spaces are principal. In particular, on the open cell, one can recover the isomorphism between the étale parts of the p -divisible group associated to the universal abelian variety and the fixed PEL-type p -divisible group, just from the isomorphism on the multiplicative parts. The fact that the polarizations showing up on our moduli problems are not principal could potentially complicate our constructions. Fortunately, we have the following (known) result.

Lemma 3.2.5. *Let A be an abelian variety over some scheme S , and $\lambda : A \rightarrow A$ be a prime-to- p polarization. Then the induced polarization on associated p -divisible groups $\lambda : A[p^\infty] \rightarrow A[p^\infty]$ is principal*

Proof. Let $K \subset A$ denote the kernel of λ . Since λ is an isogeny, K is finite, and λ being prime-to- p means that K has order n , where n is some integer which is coprime to p . By Deligne's theorem (c.f. [Gal16, Thm. 2.1.1.]), this implies that all points in K are actually n -torsion points, so $K = K[n]$. But the intersection of $K[n]$ and $K[p]$ is trivial, since p and n are coprime, and therefore so is $K[p]$.

With this in mind, the strategy for the proof of proposition 3.2.1. in [CS19] carries over to our setup:

Proposition 3.2.6. *Let $\mathcal{G}^\circ := \widehat{\mathcal{G}}$ be defined as the formal completion of $\mathcal{G}$ along its identity section. Then $\mathcal{G}[p^\infty]^\circ$ is a p -divisible group over $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$.*

Proof. By the construction of the toroidal compactifications (c.f. [LS18, Sect. 3.3.1]), we can prove the claim over some strict complete local ring $x = \text{Spec}(R) \rightarrow \mathcal{C}^{\mathbb{X}_b, \text{tor}}$. Let Z_n be the cusp at which x lies in, and consider the associated Raynaud extension

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow A \rightarrow 0$$

over $\text{Spec}(R)$. By the construction of the Raynaud extension, we know that there is an isomorphism of formal schemes $\widehat{\tilde{G}} \cong \widehat{\mathcal{G}}$ over $\text{Spf}(R)$, which is actually defined over $\text{Spec}(R)$ by formal GAGA. In particular, it induces an isomorphism $\mathcal{G}[p^\infty]^\circ \cong \tilde{G}[p^\infty]^\circ$. It remains to prove that $\tilde{G}[p^\infty]^\circ$ is a p -divisible group. Since $\tilde{G}[p^\infty]$ is a p -divisible group, the result follows by noticing that its connected part has constant rank ($A[p^\infty]$ is determined by a map from $\text{Spec}(R)$ to $\mathcal{C}_Z^{\mathbb{X}_b, Z}$).

Recall as in Prop. 3.2.4, that over any geometric point $\bar{x}$ of $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$, there is an $\mathcal{O}_B$ -linear isomorphism of p -divisible groups

$$\mathbb{X}_b^\circ \cong \text{Hom}(X, \mu_{p^\infty}) \oplus \mathbb{X}_{b, Z_n}^\circ$$

where Z_n is the cusp into which $\bar{x}$ degenerates into. This immediately implies by the proof of the proposition we just saw that there is an $\mathcal{O}_B$ -linear isomorphism

$$\mathcal{G}[p^\infty]^\circ \times_{\mathcal{C}^{\mathbb{X}_b, \text{tor}}} k(\bar{x}) \cong \mathbb{X}_b^\circ \times_{\mathbb{F}_p} k(\bar{x}),$$

and thus the multiplicative part $\mathcal{G}[p^\infty]^\mu$ has constant rank, and is also a p -divisible group. We let

$$\mathcal{G}[p^\infty]^{(0,1)} = \mathcal{G}[p^\infty]^\circ / \mathcal{G}[p^\infty]^\mu$$

denote the biconnected part of $\mathcal{G}[p^\infty]^\circ$. From the discussion above, one can prove the following.

Proposition 3.2.7. *$\mathcal{G}[p^\infty]^{(0,1)}$ admits a polarization, extending the existent one over $\mathcal{C}^{\mathbb{X}_b}$.*

Proof. This can once more be checked over some strict complete local ring R of $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$, where the isomorphism $\mathcal{G}[p^\infty]^\circ \cong \tilde{G}[p^\infty]^\circ$ over $\text{Spec}(R)$ induces an isomorphism $\mathcal{G}[p^\infty]^{(0,1)} \cong A[p^\infty]^{(0,1)}$, and $A[p^\infty]^{(0,1)}$ carries the desired polarization.

The completely slope divisible case

Let start by constructing compactifications of Mantovan's Igusa varieties. In particular, we need to assume that our fixed PEL-type p -divisible group $\mathbb{X}_b$ is completely slope divisible, and write $\mathbb{X}_b \cong \bigoplus_{i=1}^r \mathbb{X}_{b,i}$, with $\mathbb{X}_{b,i}$ its isotypical components. Recall that we defined in 3.1.29 a pro-Igusa variety $\text{Ig}^{\mathbb{X}_b}$ over $\mathcal{C}^{\mathbb{X}_b}$, which by proposition 3.1.31 is a pro-finite étale cover of $\mathcal{C}^{\mathbb{X}_b}$ with Galois group Γ_b . Denote once more by $(\mathcal{G}, \lambda, \iota, \alpha_n)$ the restriction of the universal object of $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ to $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$. By lemma 3.1.20, and the existence of isomorphisms

$$\mathcal{G}[p^\infty]^\circ \times_{\mathcal{C}^{\mathbb{X}_b, \text{tor}}} k(\bar{x}) \cong \mathbb{X}_b^\circ \times_{\mathbb{F}_p} k(\bar{x}),$$

for any geometric point in $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$, the assumption that $\mathbb{X}_b$ is completely slope divisible implies that so is $\mathcal{G}^\circ$, and therefore there exists a filtration

$$0 = \mathcal{G}_{\geq 0} \subset \mathcal{G}_{\geq 1} \subset \cdots \subset \mathcal{G}_{\geq r} = \mathcal{G}^\circ$$

with isotypical quotients $\mathcal{G}_i$ of slope $\lambda_i < 1$, as in definition 3.1.17. In particular, we can make the following definition.

Definition 3.2.8. Let $\mathbb{X}_b \cong \bigoplus_{i=1}^r \mathbb{X}_{b,i}$ be our fixed PEL-type p -divisible group, with isotypical components $\mathbb{X}_{b,i}$ of slopes λ_i . Let S be a $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$ -scheme.

1. For a positive integer m , a finite Igusa level p^m -structure on S is the data $\{\rho_i\}_i$, for i such that $\lambda_i < 1$ consisting of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -compatible isomorphisms

$$\rho_i : \mathcal{G}_i[p^m] \times_{\mathcal{C}^{\mathbb{X}_b, \text{tor}}} S \xrightarrow{\sim} \mathbb{X}_{b,i}[p^m] \times_{\mathbb{F}_p} S,$$

lifting fppf locally for all $m' \geq m$, such that for i, j with $\lambda_i, \lambda_j > 0$ $\lambda_i + \lambda_j = 1$, the induced diagram

$$\begin{array}{ccc}
\mathcal{G}_{b,i}[p^m] \times_{\mathcal{C}^{\mathbb{X}_b}} S & \longrightarrow & \mathbb{X}_{b,i}[p^m] \times_{\mathbb{F}_p} S \\
\downarrow & & \downarrow \\
(\mathcal{G}_{b,j})^\vee[p^m] \times_{\mathcal{C}^{\mathbb{X}_b}} S & \longleftarrow & (\mathbb{X}_{b,j})^\vee[p^m] \times_{\mathbb{F}_p} S
\end{array}$$

commutes up to an element in $(\mathbb{Z}/p^m\mathbb{Z})^\times(S)$ independent of i .

2. A pro-Igusa level structure on S is the data $\{\rho_i\}_i$, for i such that $\lambda_i < 1$, consisting of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -compatible isomorphisms

$$\rho_i : \mathcal{G}_i[p^\infty] \times_{\mathcal{C}^{\mathbb{X}_b, \text{tor}}} S \xrightarrow{\sim} \mathbb{X}_{b,i} \times_{\mathbb{F}_p} S,$$

such that for i, j with $\lambda_i, \lambda_j > 0$ $\lambda_i + \lambda_j = 1$, the induced diagram

$$\begin{array}{ccc}
\mathcal{G}_{b,i}[p^\infty] \times_{\mathcal{C}^{\mathbb{X}_b}} S & \longrightarrow & \mathbb{X}_{b,i} \times_{\mathbb{F}_p} S \\
\downarrow & & \downarrow \\
(\mathcal{G}_{b,j})^\vee[p^\infty] \times_{\mathcal{C}^{\mathbb{X}_b}} S & \longleftarrow & (\mathbb{X}_{b,j})^\vee \times_{\mathbb{F}_p} S
\end{array}$$

commutes up to an element in $\mathbb{Z}_p^\times(S)$, independent of i .

Definition 3.2.9. 1. We define the toroidal compactification of the finite Igusa variety $\text{Ig}_m^{\mathbb{X}_b}$ as the moduli space over $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$ parametrizing finite Igusa p^m -structures up to isomorphism, and denote it by $\text{Ig}_m^{\mathbb{X}_b, \text{tor}}$.

2. We define the toroidal compactification of the pro-Igusa variety $\text{Ig}^{\mathbb{X}_b}$ as the moduli space over $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$ parametrizing pro-Igusa structures up to isomorphism, and denote it by $\text{Ig}^{\mathbb{X}_b, \text{tor}}$.

We obtain the following generalization of [CS19, Thm. 3.2.4.].

Theorem 3.2.10. 1. The finite-étale cover $\text{Ig}_m^{\mathbb{X}_b} \rightarrow \mathcal{C}^{\mathbb{X}_b}$, with Galois group $\Gamma_{b,m}$, extends uniquely to a finite-étale cover $\text{Ig}_m^{\mathbb{X}_b, \text{tor}} \rightarrow \mathcal{C}^{\mathbb{X}_b, \text{tor}}$, with the same Galois group.

2. The pro-étale cover $\text{Ig}^{\mathbb{X}_b} \rightarrow \mathcal{C}^{\mathbb{X}_b}$, with Galois group Γ_b , extends uniquely to a pro-étale cover $\text{Ig}^{\mathbb{X}_b, \text{tor}} \rightarrow \mathcal{C}^{\mathbb{X}_b, \text{tor}}$, with the same Galois group.

Proof. For the first statement, one once more applies [CS19, Thm 2.2.4.] in the following way: one considers the moduli spaces over $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$ parametrizing:

- An $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -compatible isomorphism

$$\rho_i : \mathcal{G}_i[p^\infty] \times_{\mathcal{C}^{\mathbb{X}_b, \text{tor}}} S \xrightarrow{\sim} \mathbb{X}_{b,i} \times_{\mathbb{F}_p} S,$$

for i such that $\lambda_i < \frac{1}{2}$. We do this for all such i .

- An $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -compatible isomorphism

$$\rho_i : \mathcal{G}_i[p^\infty] \times_{\mathcal{C}^{\mathbb{X}_b, \text{tor}}} S \xrightarrow{\sim} \mathbb{X}_{b,i} \times_{\mathbb{F}_p} S,$$

for i such that $\lambda_i = \frac{1}{2}$, commuting with polarizations up to an element in $\mathbb{Z}_p^\times(S)$.

[CS19, Theorem 2.2.4.] implies that these are all finite étale covers of $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$, with Galois group $\Gamma_{b,m}$. The isomorphisms

$$\rho_i : \mathcal{G}_i[p^\infty] \times_{\mathcal{C}^{\mathbb{X}_b, \text{tor}}} S \xrightarrow{\sim} \mathbb{X}_{b,i} \times_{\mathbb{F}_p} S,$$

for i with $\frac{1}{2} < \lambda_i < 1$ are then formally determined by the ones above up to our element in $\mathbb{Z}_p^\times(S)$, by taking polarizations. The result follows by taking fiber products of these spaces. One knows that $\text{Ig}_m^{\mathbb{X}, \text{tor}}$ is then representable by a finite étale scheme over $\mathcal{C}^{\mathbb{X}, \text{tor}}$. Taking inverse limits yields the claim.

On the open cell, we obtain the usual Igusa variety by lemma 3.2.5, which implies that the isomorphism on the multiplicative part induces, by taking polarizations, one on the étale part.

The second statements follow from taking inverse limits.

Even though it is already good to know that toroidal compactifications of Igusa varieties exist, we would like to have a more explicit description of them, just as in theorems 2.3.16 and 3.2.3. In particular, we would like to understand the formal completions of $\text{Ig}_m^{\mathbb{X}_b, \text{tor}}$ and $\text{Ig}_m^{\mathbb{X}_b, \text{tor}}$ along certain boundary strata.

In order to do that, recall that by theorem 3.2.3, $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$ admits a boundary toroidal chart

$$\begin{array}{ccc} \mathcal{C}_{\Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & \longrightarrow & \mathcal{C}_{\Xi_{\Phi_n, \delta_n}, \Sigma_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \\ \downarrow & \swarrow & \\ \mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & & \\ \downarrow & & \\ \mathcal{C}_{Z_n}^{\mathbb{X}_b} & & \end{array}$$

where $\mathcal{C}_{Z_n}^{\mathbb{X}_b} = \mathcal{C}^{\mathbb{X}, *}_K \times_{\mathcal{S}_K} \mathcal{S}_{Z_n}$, $\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} = \mathcal{C}_{Z_n}^{\mathbb{X}} \times_{\mathcal{S}_{Z_n}} C_{\Phi_n, \delta_n}$, $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b} = \mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \times_{\mathcal{S}_{Z_n}} \Xi_{\Phi_n, \delta_n}$ and $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}, \Sigma_{\Phi_n, \delta_n}}^{\mathbb{X}_b} = \mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \times_{\mathcal{S}_{Z_n}} \Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}}$. Furthermore, one can identify the formal completion of $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$ along the boundary stratum $\mathcal{C}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b}$ (with notation as in theorem 3.2.3), denoted $\widehat{\mathcal{C}}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}}$, with the quotient

$$\mathcal{C}_{\mathcal{X}_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b} / \Gamma_{\Phi_n, \sigma}$$

between the formal completion of $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}(\sigma)$ along $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}, \sigma}^{\mathbb{X}_b}$ and $\Gamma_{\Phi_n, \sigma}$.

We can now define make the following definition.

Definition 3.2.11. 1. Define the boundary stratum $\text{Ig}_{m, Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}} \subset \text{Ig}_m^{\mathbb{X}_b, \text{tor}}$ as the pre-image of $\mathcal{C}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b}$ along the finite-étale cover $\text{Ig}_m^{\mathbb{X}_b, \text{tor}} \rightarrow \mathcal{C}^{\mathbb{X}_b, \text{tor}}$. Denote by $\widehat{\text{Ig}}_{m, Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}}$ the formal completion of $\text{Ig}_m^{\mathbb{X}_b, \text{tor}}$ along the boundary stratum $\text{Ig}_{m, Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}} \subset \text{Ig}_m^{\mathbb{X}_b, \text{tor}}$.

2. Define the boundary stratum $\text{Ig}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}} \subset \text{Ig}^{\mathbb{X}_b, \text{tor}}$ as the pre-image of $\mathcal{C}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b}$ along the pro-étale cover $\text{Ig}^{\mathbb{X}_b, \text{tor}} \rightarrow \mathcal{C}^{\mathbb{X}_b, \text{tor}}$. Denote by $\widehat{\text{Ig}}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}}$ the formal completion of $\text{Ig}^{\mathbb{X}_b, \text{tor}}$ along the boundary stratum $\text{Ig}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}} \subset \text{Ig}^{\mathbb{X}_b, \text{tor}}$.

The question now is whether we have a descriptions of $\widehat{\text{Ig}}_{m,Z[(\Phi_n,\delta_n,\sigma)]}^{\mathbb{X}_b,\text{tor}}$ and $\widehat{\text{Ig}}_{Z[(\Phi_n,\delta_n,\sigma)]}^{\mathbb{X}_b,\text{tor}}$ as a quotient associated to a toroidal boundary chart as in the case of the Shimura varieties and well-positioned subsets.

Recall that the main trick to construct toroidal compactification of Shimura varieties and to prove this type of statement is to construct a moduli space that parametrizes certain tuples, the degeneration data, that are over certain basis equivalent to a semi-abelian tuple, as the universal semi-abelian scheme over $\mathcal{S}_{K,\Sigma}^{\text{tor}}$. In the present case, we added the datum of certain trivializations of the $[p^m]$ -torsion resp. $[p^\infty]$ -torsion of the isoclinic pieces of the p -divisible group obtained from the connected part of this semi-abelian tuple, so in principle, what we should try do is to add a similar data to the toroidal boundary chart associated to $\mathcal{C}^{\mathbb{X}_b,\text{tor}}$, and thus to our degeneration data.

As in theorem 2.3.16, over C_{Φ_n,δ_n} we are parametrizing Raynaud extensions and their duals (satisfying some extra compatibility conditions). In particular, we have a universal semi-abelian scheme $\tilde{G}_{\Phi_n,\delta_n}$, lying in a short exact sequence

$$0 \rightarrow T \rightarrow \tilde{G}_{\Phi_n,\delta_n} \rightarrow A_{\Phi_n,\delta_n} \rightarrow 0,$$

where T is the torus with character group X given by the cusp label (Z_n, Φ_n, δ_n) , and A_{Φ_n,δ_n} is the pullback of the universal abelian scheme on $\mathcal{S}_K^{Z_n}$ to C_{Φ_n,δ_n} . Recall that by the proof of proposition 3.2.6, over $\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}$, the connected part of the p -divisible group attached to $\tilde{G}_{\Phi_n,\delta_n}$ is still a p -divisible group. Let us we denote it by $\tilde{\mathcal{G}}_{\Phi_n,\delta_n}$. The fact that we are assuming that $\mathbb{X}_b$ is completely slope divisible, together with the definition of the central leaf and lemma 3.1.20, imply that $\tilde{\mathcal{G}}_{\Phi_n,\delta_n}$ is completely slope divisible. Denote its isoclinic pieces by $\tilde{\mathcal{G}}_{\Phi_n,\delta_n,i}$. Also, write $\mathbb{X}_b = \oplus_i \mathbb{X}_{b,i}$ with $\mathbb{X}_i$ isoclinic of slope λ_i .

Definition 3.2.12. Let (Z_n, Φ_n, δ_n) be a cusp label.

1. A degeneration Igusa p^m -level structure attached to (Z_n, Φ_n, δ_n) on a $\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}$ -scheme S consists of the data $\{\rho_i\}_i$ for i such that $\lambda_i < 1$, consisting of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -compatible isomorphisms

$$\rho_i : \tilde{\mathcal{G}}_{\Phi_n,\delta_n,i}[p^m] \times_{\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}} S \xrightarrow{\sim} \mathbb{X}_i[p^m] \times_{\mathbb{F}_p} S,$$

lifting fppf locally for all $m' \geq m$, such that for i, j with $\lambda_i, \lambda_j > 0$, $\lambda_i + \lambda_j = 1$, the induced diagram

$$\begin{array}{ccc} \tilde{\mathcal{G}}_{\Phi_n,\delta_n,i}[p^m] \times_{\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}} S & \longrightarrow & \mathbb{X}_{b,i}[p^m] \times_{\mathbb{F}_p} S \\ \downarrow & & \downarrow \\ (\tilde{\mathcal{G}}_{\Phi_n,\delta_n,j})^\vee[p^m] \times_{\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}} S & \longleftarrow & (\mathbb{X}_{b,j})^\vee[p^m] \times_{\mathbb{F}_p} S \end{array}$$

commutes up to an element in $(\mathbb{Z}/p^m\mathbb{Z})^\times(S)$, independent of i .

2. A degeneration pro-Igusa level structure attached to (Z_n, Φ_n, δ_n) on a $\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ -scheme S consists of the data $\{\rho_i\}_i$ for i such that $\lambda_i < 1$, consisting of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -compatible isomorphisms

$$\rho_i : \tilde{\mathcal{G}}_{\Phi_n, \delta_n, i} \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}} S \xrightarrow{\sim} \mathbb{X}_i \times_{\mathbb{F}_p} S,$$

such that for i, j with $\lambda_i, \lambda_j > 0$, $\lambda_i + \lambda_j = 1$, the induced diagram

$$\begin{array}{ccc} \tilde{\mathcal{G}}_{\Phi_n, \delta_n, i} \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}} S & \longrightarrow & \mathbb{X}_{b, i} \times_{\mathbb{F}_p} S \\ \downarrow & & \downarrow \\ (\tilde{\mathcal{G}}_{\Phi_n, \delta_n, j})^\vee \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}} S & \longleftarrow & (\mathbb{X}_{b, j})^\vee \times_{\mathbb{F}_p} S \end{array}$$

commutes up to an element in $(\mathbb{Z}_p)^\times(S)$, independent of i .

If we had this data to our toroidal boundary chart, the result will follow almost immediately.

- Definition 3.2.13.** 1. We define $\text{Ig}_{m, Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ as the moduli space over $\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ parametrizing degeneration Igusa p^m -level structures attached to (Z_n, Φ_n, δ_n) .
2. We define $\text{Ig}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ as the moduli space over $\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ parametrizing degeneration pro-Igusa level structures attached to (Z_n, Φ_n, δ_n) .

By the same argument as in theorem 3.2.10, $\text{Ig}_{m, Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ defines a finite-étale cover with Galois group $\Gamma_{b, m}$, and $\text{Ig}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ a pro-étale cover with Galois group Γ_b . We denote their pullbacks to $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ by $\text{Ig}_{m, Z_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ resp. $\text{Ig}_{Z_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$, and similarly for $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}, \Sigma_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$, $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}(\sigma)}^{\mathbb{X}_b}$ and $\mathcal{C}_{\Xi_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b}$. Denote by $\text{Ig}_{m, \mathcal{X}_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b}$ the formal completion of $\text{Ig}_{m, Z_n, \Xi_{\Phi_n, \delta_n}(\sigma)}^{\mathbb{X}_b}$ along $\text{Ig}_{m, Z_n, \Xi_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b}$, and by $\text{Ig}_{\mathcal{X}_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b}$ the formal completion of $\text{Ig}_{Z_n, \Xi_{\Phi_n, \delta_n}(\sigma)}^{\mathbb{X}_b}$ along $\text{Ig}_{Z_n, \Xi_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b}$. We obtain the following generalization of [CS19, Theorem 3.2.6].

Theorem 3.2.14. *There are canonical isomorphisms between $\widehat{\text{Ig}}_{m, Z_{[(\Phi_n, \delta_n, \sigma)]}}^{\mathbb{X}_b, \text{tor}}$ and the quotient*

$$\text{Ig}_{m, \mathcal{X}_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b} / \Gamma_{\Phi_n, \sigma},$$

and between $\widehat{\text{Ig}}_{Z_{[(\Phi_n, \delta_n, \sigma)]}}^{\mathbb{X}_b, \text{tor}}$ and the quotient

$$\text{Ig}_{\mathcal{X}_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b} / \Gamma_{\Phi_n, \sigma}.$$

Proof. We only have to prove that the pullbacks of $\mathcal{G}[p^\infty]^\circ$ and $\tilde{\mathcal{G}}_{\Phi_n, \delta_n}$ to $\widehat{\mathcal{C}}_{Z_{[(\Phi_n, \delta_n, \sigma)]}}^{\mathbb{X}_b, \text{tor}}$ are isomorphic. This comes down to proving that if we cover $\widehat{\mathcal{C}}_{Z_{[(\Phi_n, \delta_n, \sigma)]}}^{\mathbb{X}_b, \text{tor}}$ by affine open formal subschemes $\text{Spf}(R)$, that arise from formal completion of affine opens in $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}, \Sigma_{\Phi_n, \delta_n}}^{\mathbb{X}_b, \text{tor}}$ and $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}(\sigma)}^{\mathbb{X}_b}$, and that lift to affine opens in the formal completion of $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}(\sigma)}^{\mathbb{X}_b}$ along the σ -stratum, then the pullbacks of $\mathcal{G}[p^\infty]^\circ$ and $\tilde{\mathcal{G}}_{\Phi_n, \delta_n}$ through the induced morphisms

$$f_1 : \text{Spec}(R) \rightarrow \mathcal{C}_{\Xi_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b, \text{tor}}$$

and

$$f_2 : \text{Spec}(R) \rightarrow \mathcal{C}_{\Xi_{\Phi_n, \delta_n}(\sigma)}^{\mathbb{X}_b}$$

are isomorphic. But this follows easily, since the completions along the identity of the pullbacks of $\mathcal{G}$ and of the Raynaud extension $\tilde{G}_{\Phi_n, \delta_n}$ along f_1 resp. f_2 agree (c.f. proof of proposition 3.2.6).

This result allows us to generalize the main theorem on toroidal compactifications. However, we can refine this result a bit further. Note that on the main theorem on toroidal compactifications for Shimura varieties, all the discrete data (the data related to the tori T and $T^\vee$), is given by the cusp label. However, this is not the case if we consider the toroidal compactifications of Igusa varieties, since the isomorphisms between $T[p^\infty] \subset \mathcal{G}[p^\infty]^\circ$ and a certain multiplicative part of $\mathbb{X}_b$ are part of the data given by the degeneration Igusa p^m -level structure. In order to somehow fix this, we want to add the data related to $T[p^\infty]$ to the cusp label as follows. For the following definition, we do not need to assume that $\mathbb{X}_b$ is completely slope divisible.

Definition 3.2.15. Consider the set of tuples $(Z^{\mathbb{X}_b}, Z_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), \psi, \delta^{\mathbb{X}_b}, \delta_n)$, where:

1. $(Z_n, (X, Y, \varphi_{-2,n}, \varphi_{0,n}), \delta_n)$ is a tuple as in the definition of a cusp label;
2. $Z^{\mathbb{X}_b}$ is a filtration on $\mathbb{X}_b$

$$Z^{\mathbb{X}_b} = \{0 \subset Z_{-2}^{\mathbb{X}_b} \subset Z_{-1}^{\mathbb{X}_b} \subset Z_0^{\mathbb{X}_b} = \mathbb{X}_b\}$$

such that $\text{Gr}_{-2}^{\mathbb{X}_b} = Z_{-2}^{\mathbb{X}_b}$ is multiplicative, and $\text{Gr}_0^{\mathbb{X}_b}$ is identified with its Cartier dual via the principal polarization on $\mathbb{X}_b$ (in particular, $\text{Gr}_0^{\mathbb{X}_b}$ is étale). It thus induces a principal polarization on $\text{Gr}_{-1}^{\mathbb{X}_b}$;

3. ψ is an isomorphism

$$\psi : \text{Gr}_0^{\mathbb{X}_b} \xrightarrow{\sim} Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p);$$

4. $\delta^{\mathbb{X}_b}$ is an $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -splitting of $Z^{\mathbb{X}_b}$, so an isomorphism

$$\delta^{\mathbb{X}_b} : \mathbb{X}_b \cong \bigoplus_i \text{Gr}_{-i}^{\mathbb{X}_b}.$$

We define an equivalence class on this set as follows: two such tuples $(Z^{\mathbb{X}_b}, Z_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), \psi, \delta^{\mathbb{X}_b}, \delta_n)$ and $(Z'^{\mathbb{X}_b}, Z'_n, (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n}), \psi', \delta'^{\mathbb{X}_b}, \delta'_n)$ are equivalent if $Z_n = Z'_n$, $Z_m^{\mathbb{X}_b} = Z'_m{}^{\mathbb{X}_b}$, and there exists a pair of isomorphisms $\gamma_X : X' \xrightarrow{\sim} X$ and $\gamma_Y : Y \xrightarrow{\sim} Y'$ such that $\phi = \gamma_X \circ \phi' \circ \gamma_Y$, $\varphi'_{-2,n} = \gamma_X^\vee \circ \varphi_{-2,n}$, $\varphi'_{0,n} = \gamma_Y \circ \varphi_{0,n}$ and $\psi' = \gamma_Y \circ \psi$. An Igusa cusp label is an equivalence class of such tuples, denoted $[(Z^{\mathbb{X}_b}, Z_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), \psi, \delta^{\mathbb{X}_b}, \delta_n)]$.

For simplicity, we will simply denote an Igusa cusp label by $\tilde{Z}_n$.

Let us go back to the assumption that $\mathbb{X}_b$ is completely slope divisible. We now need to figure out what the "missing data" is. By this, we mean the following: if we are working over $\mathcal{C}_{\Phi_n, \delta_n}^{\mathbb{X}_b}$, we have a universal Raynaud extension

$$0 \rightarrow T \rightarrow \tilde{G}_{b, \Phi_n, \delta_n} \rightarrow A_{b, \Phi_n, \delta_n} \rightarrow 0,$$

and we fix an Igusa cusp label over some cusp label (Z_n, Φ_n, δ_n) , what is the data we still need in order to identify the isoclinic pieces of $\tilde{\mathcal{G}}_{\Phi_n, \delta_n}$ (which we recall it denotes the connected part of the p -divisible group attached to $\tilde{G}_{b, \Phi_n, \delta_n}$), which are not given by the the Igusa cusp label, with certain isoclinic pieces of $\mathbb{X}_b^\circ$ (or their p^m -torsion, in the finite case). The answer is the following.

Definition 3.2.16. Let (Z_n, Φ_n, δ_n) be a cusp label. A pro-Igusa level structure on a $\mathcal{C}_{\Phi_n, \delta_n}^{\mathbb{X}_b}$ -scheme S , compatible with an Igusa cusp label $\tilde{Z}_n$ over (Z_n, Φ_n, δ_n) , consists of the data of:

1. An isomorphism

$$\rho_{A_{b, \Phi_n, \delta_n}^\mu} : A_{b, \Phi_n, \delta_n}[p^\infty]^\mu \times_{\mathcal{C}_{\Phi_n, \delta_n}^{\mathbb{X}_b}} S \xrightarrow{\sim} (\mathrm{Gr}_{-1}^{\mathbb{X}_b})^\mu \times_{\mathbb{F}_p} S$$

compatible with the $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -compatible actions;

2. A splitting

$$\delta : A_{b, \Phi_n, \delta_n}[p^\infty]^\mu \times_{\mathcal{C}_{\Phi_n, \delta_n}^{\mathbb{X}_b}} S \rightarrow \tilde{\mathcal{G}}_{\Phi_n, \delta_n}^\mu \times_{\mathcal{C}_{\Phi_n, \delta_n}^{\mathbb{X}_b}} S$$

compatible with the $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -actions;

3. The data $\{\rho_i\}_i$ for all i with $\lambda_i \in (0, 1)$, of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -isomorphisms

$$\rho_i : A_{b, \Phi_n, \delta_n}[p^\infty]_i \times_{\mathcal{C}_{\Phi_n, \delta_n}^{\mathbb{X}_b}} S \xrightarrow{\sim} \mathbb{X}_{b, i} \times_{\mathbb{F}_p} S$$

such that for i, j with $\lambda_i + \lambda_j = 1$, the induced diagrams

$$\begin{array}{ccc} A_{b, \Phi_n, \delta_n}[p^\infty]_i \times_{\mathcal{C}_{\Phi_n, \delta_n}^{\mathbb{X}_b}} S & \longrightarrow & \mathbb{X}_{b, i} \times_{\mathbb{F}_p} S \\ \downarrow & & \downarrow \\ (A_{b, \Phi_n, \delta_n}[p^\infty]_j)^\vee \times_{\mathcal{C}_{\Phi_n, \delta_n}^{\mathbb{X}_b}} S & \longleftarrow & (\mathbb{X}_{b, j})^\vee \times_{\mathbb{F}_p} S \end{array}$$

commute up to an element in $\mathbb{Z}_p^\times(S)$, independent of i .

The upshot is that if we fix a cusp label (Z_n, Φ_n, δ_n) , an Igusa cusp label above it $\tilde{Z}_n$, and a pro-Igusa level structure compatible with $\tilde{Z}_n$, we can recover a degeneration pro-Igusa level structure attached to (Z_n, Φ_n, δ_n) . Indeed, over a $\mathcal{C}_{\Phi_n, \delta_n}^{\mathbb{X}_b}$ -scheme S , the splitting $\delta^{\mathbb{X}_b}$, is an isomorphism $\delta^{\mathbb{X}_b} : \mathbb{X}_b \cong \bigoplus_i \mathrm{Gr}_{-i}^{\mathbb{X}_b}$. By duality, the isomorphism ψ induces an identification between $\mathrm{Gr}_{-2}^{\mathbb{X}_b}$ and $T[p^\infty]$. The splitting δ of the short exact sequence

$$0 \rightarrow T[p^\infty] \rightarrow \tilde{\mathcal{G}}_{\Phi_n, \delta_n}^\mu \rightarrow A_{b, \Phi_n, \delta_n}[p^\infty]^\mu \rightarrow 0$$

over S , gives us an isomorphism $\tilde{\mathcal{G}}_{\Phi_n, \delta_n}^\mu \cong T[p^\infty] \oplus A_{b, \Phi_n, \delta_n}[p^\infty]^\mu$, and since $\tilde{\mathcal{G}}_{\Phi_n, \delta_n}^{(0,1)} \cong A_{b, \Phi_n, \delta_n}[p^\infty]^{(0,1)}$, the isomorphisms ρ_{A^μ} and ρ_i give us the rest of the data. Similarly, given a cusp label (Z_n, Φ_n, δ_n) and a degeneration pro-Igusa level structure attached to (Z_n, Φ_n, δ_n) , we can construct a filtration $Z^{\mathbb{X}_b}$ simply by taking $Z_{-2}^{\mathbb{X}_b}$ to be $T[p^\infty] \subset \tilde{\mathcal{G}}_{\Phi_n, \delta_n}^\mu$ and the rest of the filtration being defined by its dual. Picking any splitting $\delta^{\mathbb{X}_b}$ (they are all identified by the equivalence relation) immediately gives us a pro-Igusa cusp label. The pro-Igusa level structure compatible with $\tilde{Z}_n$ is already given in the definition of the degeneration pro-Igusa level structure attached to (Z_n, Φ_n, δ_n) .

Definition 3.2.17. Let (Z_n, Φ_n, δ_n) be a cusp label, and $\tilde{Z}_n$ be an Igusa cusp label over it. We define $\text{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ as the moduli space over $\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ parametrizing pro-Igusa level structures compatible with $\tilde{Z}_n$.

Putting all of this together, we obtain the following result.

Theorem 3.2.18. Let (Z_n, Φ_n, δ_n) be a cusp label, and $\hat{\text{Ig}}_{Z_n[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}}$ be as defined in 3.2.11.

1. $\hat{\text{Ig}}_{Z_n[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}}$ admits a decomposition into open and closed formal subschemes

$$\hat{\text{Ig}}_{Z_n[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}} = \sqcup_{\tilde{Z}_n} \hat{\text{Ig}}_{\tilde{Z}_n, \sigma}^{\mathbb{X}_b, \text{tor}}$$

where $\tilde{Z}_n$ runs through the Igusa cusp labels above (Z_n, Φ_n, δ_n) .

2. Fix an Igusa cusp label $\tilde{Z}_n$. Recall the toroidal boundary chart

$$\begin{array}{ccc} \Xi_{\Phi_n, \delta_n} & \longrightarrow & \Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}} \\ \downarrow & \swarrow & \\ C_{\Phi_n, \delta_n} & & \\ \downarrow & & \\ \mathcal{S}_K^{Z_n} & & \end{array}$$

in theorem 2.3.16, describing $(\mathcal{S}_{K, \Sigma}^{\text{tor}})_{Z_n[(\Phi_n, \delta_n, \sigma)]}^\wedge$. The moduli space $\text{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ in definition 3.2.17 admits a morphism to the Igusa variety $\text{Ig}^{\mathbb{X}_b, Z_n}$ over the central leaf $\mathcal{C}^{\mathbb{X}_b, Z_n}$, and it fits in a commutative diagram

$$\begin{array}{ccc} \text{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & \longrightarrow & C_{\Phi_n, \delta_n} \\ \downarrow & & \downarrow \\ \text{Ig}^{\mathbb{X}_b, Z_n} & \longrightarrow & \mathcal{S}_K^{Z_n}. \end{array}$$

Let $\text{Ig}_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \rightarrow \text{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ resp. $\text{Ig}_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}}}^{\mathbb{X}_b} \rightarrow \text{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ denote the pullback of $\text{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \rightarrow C_{\Phi_n, \delta_n}$ along $\Xi_{\Phi_n, \delta_n} \rightarrow C_{\Phi_n, \delta_n}$ resp. $\Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}} \rightarrow C_{\Phi_n, \delta_n}$. These define an Igusa toroidal boundary chart

$$\begin{array}{ccc}
\mathrm{Ig}_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & \longrightarrow & \mathrm{Ig}_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}, \Sigma_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \\
\downarrow & \swarrow & \\
\mathrm{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & & \\
\downarrow & & \\
\mathrm{Ig}_{Z_n}^{\mathbb{X}} & &
\end{array}$$

For our given rational polyhedral cone $\sigma \in \Sigma_{\Phi_n}$, consider the schemes

$$\mathrm{Ig}_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}(\sigma) := \mathrm{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \times_{C_{\Phi_n, \delta_n}} \Xi_{\Phi_n, \delta_n}(\sigma)$$

and

$$\mathrm{Ig}_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}, \sigma}^{\mathbb{X}_b} := \mathrm{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \times_{C_{\Phi_n, \delta_n}} \Xi_{\Phi_n, \delta_n, \sigma}.$$

Let $\mathrm{Ig}_{\tilde{Z}_n, \mathcal{X}_{\Sigma_{\Phi_n, \delta_n}, \sigma}}^{\mathbb{X}_b}$ denote the formal completion of $\mathrm{Ig}_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}(\sigma)$ along $\mathrm{Ig}_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}, \sigma}^{\mathbb{X}_b}$. Then, there is an isomorphism

$$\widehat{\mathrm{Ig}}_{\tilde{Z}_n, \sigma}^{\mathbb{X}, \mathrm{tor}} \cong \mathrm{Ig}_{\tilde{Z}_n, \mathcal{X}_{\Sigma_{\Phi_n, \delta_n}, \sigma}}^{\mathbb{X}_b} / \Gamma_{\tilde{Z}_n, \sigma},$$

where $\Gamma_{\tilde{Z}_n, \sigma}$ is the subgroup of $\Gamma_{\Phi_n, \sigma}$ such that a pair $(\gamma_X, \gamma_Y) \in \Gamma_{\Phi_n, \sigma}$ is in $\Gamma_{\tilde{Z}_n, \sigma}$ if $\psi = \gamma_Y \circ \psi$.

Proof. Our discussion above already implies most of the parts of the theorem. Namely, we have seen that given a cusp label (Z_n, Φ_n, δ_n) and a degeneration pro-Igusa level structure attached to (Z_n, Φ_n, δ_n) , one obtains a uniquely determined pro-Igusa cusp label $\tilde{Z}_n$ over (Z_n, Φ_n, δ_n) . Then, the moduli space $\mathrm{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \rightarrow \mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ parametrizes pro-Igusa level structures compatible with $\tilde{Z}_n$, giving us the rest of the data parametrized by $\mathrm{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$. This induces a decomposition on $\mathrm{Ig}_{\tilde{Z}_n, \mathcal{X}_{\Sigma_{\Phi_n, \delta_n}, \sigma}}^{\mathbb{X}_b}$, and thus one on $\widehat{\mathrm{Ig}}_{\tilde{Z}_n, \sigma}^{\mathbb{X}, \mathrm{tor}}$ by Theorem 3.2.14. It also immediately implies that we obtain an isomorphism on completions

$$\widehat{\mathrm{Ig}}_{\tilde{Z}_n, \sigma}^{\mathbb{X}, \mathrm{tor}} \cong \mathrm{Ig}_{\tilde{Z}_n, \mathcal{X}_{\Sigma_{\Phi_n, \delta_n}, \sigma}}^{\mathbb{X}_b} / \Gamma_{\tilde{Z}_n, \sigma},$$

where we consider the group $\Gamma_{\tilde{Z}_n, \sigma}$ instead of $\Gamma_{\Phi_n, \sigma}$, since we have to also identify all the splittings $\delta^{\mathbb{X}_b}$.

Note that even though in the case of toroidal compactifications of Shimura varieties one can prove that C_{Φ_n, δ_n} is an abelian scheme, this is not the case for $\mathrm{Ig}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ in the theorem above, since we are working at an infinite level at p . However, one can wonder whether that holds true at p^m -level. For example, in [CS19, Thm. 3.3.11.], it is proved that, in their notation, the scheme $C_{\tilde{Z}_n}$ defining a toroidal boundary chart of the toroidal compactification of the Igusa variety at level p^m is indeed an abelian scheme. We were not able to prove that the same holds true in our setup, but we can still try to study the corresponding scheme $\mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ as introduced below. We need a few definitions. Recall that we have been working at principal

level $K = K(n)$ since the beginning. But now we will need to be careful with the level at p . Notice that $J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ acts on Igusa cusp labels. Consider the open compact subgroup $K_m = \Gamma_b(p^m)K^p(n) \subset J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$, where $\Gamma_b(p^m) = \ker(\Gamma_b \rightarrow \Gamma_{b,m})$. Consider K_m -orbits of Igusa cusp labels. We can identify these as follows.

Definition 3.2.19. Consider the set of tuples $(Z_m^{\mathbb{X}_b}, Z_n, X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), \psi_m, \delta_m^{\mathbb{X}_b}, \delta_n)$, where:

1. $(Z_n, (X, Y, \varphi_{-2,n}, \varphi_{0,n}), \delta_n)$ is a tuple as in the definition of a cusp label;
2. $Z_m^{\mathbb{X}_b}$ is a filtration on $\mathbb{X}_b[p^m]$

$$Z_m^{\mathbb{X}_b} = \{0 \subset Z_{m,-2}^{\mathbb{X}_b} \subset Z_{m,-1}^{\mathbb{X}_b} \subset Z_{m,0}^{\mathbb{X}_b} = \mathbb{X}_b[p^m]\}$$

such that $\mathrm{Gr}_{m,-2}^{\mathbb{X}_b} = Z_{m,-2}^{\mathbb{X}_b}$ is multiplicative, and $\mathrm{Gr}_{m,0}^{\mathbb{X}_b}$ is identified with its Cartier dual via the principal polarization on $\mathbb{X}_b$ (in particular, $\mathrm{Gr}_0^{\mathbb{X}_b}$ is étale), which will automatically lift to $p^{m'}$ for $m' > m$;

3. ψ_m is an isomorphism

$$\psi_m : \mathrm{Gr}_{m,0}^{\mathbb{X}_b} \xrightarrow{\sim} Y/p^m Y;$$

4. $\delta_m^{\mathbb{X}}$ is an $\mathcal{O}_B$ -splitting of $Z_m^{\mathbb{X}_b}$, so an isomorphism

$$\delta_m^{\mathbb{X}_b} : \mathbb{X}_b[p^m] \cong \bigoplus_i \mathrm{Gr}_{m,-i}^{\mathbb{X}_b}.$$

We then define an equivalence class on this set: two such tuples $(Z_m^{\mathbb{X}}, Z_n, X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), \psi_m, \delta_m^{\mathbb{X}}, \delta_n)$ and $(Z_m^{\mathbb{X}'}, Z'_n, X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n}), \psi'_m, \delta_m^{\mathbb{X}'}, \delta'_n)$ are equivalent if $Z_n = Z'_n$, $Z_m^{\mathbb{X}} = Z_m^{\mathbb{X}'}$, and there exists a pair of isomorphisms $\gamma_X : X' \xrightarrow{\sim} X$ and $\gamma_Y : Y \xrightarrow{\sim} Y'$ such that $\phi = \gamma_X \circ \phi' \circ \gamma_Y$, $\varphi'_{-2,n} = \gamma_X^\vee \circ \varphi_{-2,n}$, $\varphi'_{0,n} = \gamma_Y \circ \varphi_{0,n}$ and $\psi'_m = \gamma_Y \circ \psi_m$. An Igusa cusp label at level K_m is an equivalence class of such tuples, denoted $[(Z_m^{\mathbb{X}}, Z_n, X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), \psi_m, \delta_m^{\mathbb{X}}, \delta_n)]$.

Notice that once more, this equivalence class identifies all the splittings of both filtrations $Z_m^{\mathbb{X}_b}$ and Z_n . For simplicity, we will denote an Igusa cusp label at level K_m by $\tilde{Z}_n^m$. Let us also define Igusa level structures at finite level.

Definition 3.2.20. Let (Z_n, Φ_n, δ_n) be a cusp label, and $\tilde{Z}_n^m$ be an Igusa cusp label at level K_m over it. An Igusa-level p^m -structure on a $\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ -scheme S , compatible with $\tilde{Z}_n^m$ over (Z_n, Φ_n, δ_n) , consists of the data of:

1. An isomorphism

$$\rho_{A_{b, \Phi_n, \delta_n}^{\mu}, m} : A_{b, \Phi_n, \delta_n}^{\mu}[p^m]^{\mu} \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}} S \xrightarrow{\sim} (\mathrm{Gr}_{-1}^{\mathbb{X}_b})[p^m]^{\mu} \times_{\mathbb{F}_p} S$$

compatible with the $\mathcal{O}_B$ -actions;

2. A splitting

$$\delta_m : A_{b,\Phi_n,\delta_n}[p^m]^\mu \times_{\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}} S \rightarrow \tilde{\mathcal{G}}_{\Phi_n,\delta_n}^\mu[p^m] \times_{\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}} S$$

compatible with the $\mathcal{O}_B$ -actions;

3. The data $\{\rho_{i,m}\}_i$ for all i with $\lambda_i \in (0, 1)$, of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -compatible isomorphisms

$$\rho_{i,m} : A_{b,\Phi_n,\delta_n}[p^m]_i \times_{\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}} S \xrightarrow{\sim} \mathbb{X}_{b,i}[p^m] \times_{\mathbb{F}_p} S$$

that lift fppf locally to $p^{m'}$ -torsion for all $m' > m$, such that for i, j with $\lambda_i + \lambda_j = 1$, the induced diagrams

$$\begin{array}{ccc} A_{b,\Phi_n,\delta_n}[p^m]_i \times_{\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}} S & \longrightarrow & \mathbb{X}_{b,i}[p^m] \times_{\mathbb{F}_p} S \\ \downarrow & & \downarrow \\ (A_{b,\Phi_n,\delta_n}[p^m])_j^\vee \times_{\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}} S & \longleftarrow & (\mathbb{X}_{b,j}[p^m])^\vee \times_{\mathbb{F}_p} S \end{array}$$

commute up to an element in $\mathbb{Z}/p^m\mathbb{Z}^\times(S)$, independent of i .

Definition 3.2.21. Let (Z_n, Φ_n, δ_n) be a cusp label, and $\tilde{Z}_n^m$ be an Igusa cusp label at level K_m over it. We define $\mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}$ as the moduli space over $\mathcal{C}_{C_{\Phi_n,\delta_n}}^{\mathbb{X}_b}$ parametrizing Igusa level p^m -structures compatible with $\tilde{Z}_n^m$.

Theorem 3.2.22. Let (Z_n, Φ_n, δ_n) be a cusp label, and $\hat{\mathrm{Ig}}_{m, Z_{[(\Phi_n, \delta_n, \sigma)]}}^{\mathbb{X}_b, \mathrm{tor}}$ be as defined in 3.2.11.

1. $\hat{\mathrm{Ig}}_{m, Z_{[(\Phi_n, \delta_n, \sigma)]}}^{\mathbb{X}_b, \mathrm{tor}}$ admits a decomposition into open and closed formal subschemes

$$\hat{\mathrm{Ig}}_{m, Z_{[(\Phi_n, \delta_n, \sigma)]}}^{\mathbb{X}_b, \mathrm{tor}} = \sqcup_{\tilde{Z}_n^m} \hat{\mathrm{Ig}}_{\tilde{Z}_n^m, \sigma}^{\mathbb{X}_b, \mathrm{tor}}$$

where $\tilde{Z}_n^m$ runs through the Igusa cusp labels at level K_m above (Z_n, Φ_n, δ_n) .

2. Fix an Igusa cusp label at level K_m , $\tilde{Z}_n^m$. Recall the toroidal boundary chart

$$\begin{array}{ccc} \Xi_{\Phi_n, \delta_n} & \longrightarrow & \Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}} \\ \downarrow & \swarrow & \\ C_{\Phi_n, \delta_n} & & \\ \downarrow & & \\ \mathcal{S}_K^{Z_n} & & \end{array}$$

in theorem 2.3.16, describing $(\mathcal{S}_{K, \Sigma}^{\mathrm{tor}})_{\tilde{Z}_n^m}^{\wedge_{[(\Phi_n, \delta_n, \sigma)]}}$. The moduli space $\mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ in definition 3.2.21 is smooth, and it fits in a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & \longrightarrow & C_{\Phi_n, \delta_n} \\
\downarrow & & \downarrow \\
\mathrm{Ig}_m^{\mathbb{X}_{Z_n}} & \longrightarrow & \mathcal{S}_K^{Z_n}.
\end{array}$$

Let $\mathrm{Ig}_{\tilde{Z}_n^m, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \rightarrow \mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ resp. $\mathrm{Ig}_{\tilde{Z}_n^m, \Xi_{\Phi_n, \delta_n}, \Sigma_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \rightarrow \mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ denote the pullback of $\mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \rightarrow C_{\Phi_n, \delta_n}$ along $\Xi_{\Phi_n, \delta_n} \rightarrow C_{\Phi_n, \delta_n}$ resp. $\Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}} \rightarrow C_{\Phi_n, \delta_n}$. These define an Igusa toroidal boundary chart

$$\begin{array}{ccc}
\mathrm{Ig}_{\tilde{Z}_n^m, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & \longrightarrow & \mathrm{Ig}_{\tilde{Z}_n^m, \Xi_{\Phi_n, \delta_n}, \Sigma_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \\
\downarrow & \swarrow & \\
\mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & & \\
\downarrow & & \\
\mathrm{Ig}_m^{\mathbb{X}_{Z_n}} & &
\end{array}$$

where $\mathrm{Ig}_{\tilde{Z}_n^m, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ is also smooth. For our given smooth rational polyhedral cone $\sigma \in \Sigma_{\Phi_n}$, consider the schemes

$$\mathrm{Ig}_{\tilde{Z}_n^m, \Xi_{\Phi_n, \delta_n}(\sigma)}^{\mathbb{X}_b} := \mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \times_{C_{\Phi_n, \delta_n}} \Xi_{\Phi_n, \delta_n}(\sigma)$$

and

$$\mathrm{Ig}_{\tilde{Z}_n^m, \Xi_{\Phi_n, \delta_n}, \sigma}^{\mathbb{X}_b} := \mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \times_{C_{\Phi_n, \delta_n}} \Xi_{\Phi_n, \delta_n, \sigma}.$$

Let $\mathrm{Ig}_{\tilde{Z}_n^m, \mathcal{X}_{\Sigma_{\Phi_n, \delta_n}, \sigma}}^{\mathbb{X}_b}$ denote the formal completion of $\mathrm{Ig}_{\tilde{Z}_n^m, \Xi_{\Phi_n, \delta_n}(\sigma)}^{\mathbb{X}_b}$ along $\mathrm{Ig}_{\tilde{Z}_n^m, \Xi_{\Phi_n, \delta_n}, \sigma}^{\mathbb{X}_b}$. Then, there is an isomorphism

$$\widehat{\mathrm{Ig}}_{\tilde{Z}_n^m, \sigma}^{\mathbb{X}_b, \mathrm{tor}} \cong \mathrm{Ig}_{\tilde{Z}_n^m, \mathcal{X}_{\Sigma_{\Phi_n, \delta_n}, \sigma}}^{\mathbb{X}_b} / \Gamma_{\tilde{Z}_n^m, \sigma},$$

where $\Gamma_{\tilde{Z}_n^m, \sigma}$ is the subgroup of $\Gamma_{\Phi_n, \sigma}$ such that a pair $(\gamma_X, \gamma_Y) \in \Gamma_{\Phi_n, \sigma}$ is in $\Gamma_{\tilde{Z}_n^m, \sigma}$ if $\psi_m = \gamma_Y \circ \psi_m$.

Proof. This follows exactly as in 3.2.18, except for the smoothness statements that will follow from the discussion below.

Let us now try to identify the scheme $\mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ in a different way.

Note that the Igusa level structure at level K_m compatible with $\tilde{Z}_n^m$ immediately gives us the map to $\mathrm{Ig}_m^{\mathbb{X}_{Z_n}}$, and $\mathrm{Ig}_m^{\mathbb{X}_{Z_n}}$ parametrizes the data of type (1) and (3) in definition 3.2.2. Furthermore, the abelian scheme C_{Φ_n, δ_n} parametrizes the Raynaud extension $0 \rightarrow T \rightarrow \tilde{G} \rightarrow A \rightarrow 0$ and its dual $c^\vee$, together with splittings of its $[n]$ -torsion, which are given by a pair $(c_n, c_n^\vee)$ as in the proof of 2.3.16. We additionally only have to parametrize the splitting in the

second part of definition 3.2.2, so just as in theorem 3.3.12. of [CS19], if we forget the relation b_{Φ_n, δ_n} and the lifting conditions as we did in the beginning of the proof of theorem 2.3.16, we obtain the scheme

$$\ddot{C}_{\tilde{Z}_n^m} := \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, (A/A[p^m]^\mu)^\vee) \times_{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)} \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A).$$

Since $c_n^m \in \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, (A/A[p^m]^\mu)^\vee)$ is a lifting of c_n , it is clear that the equality $\lambda_A \circ c^\vee = c \circ \phi$ is still satisfied. It admits a map

$$\ddot{C}_{\tilde{Z}_n^m} \rightarrow \ddot{C}_{\Phi_n, \delta_n}$$

induced from the projection $\mathrm{pr} : A \rightarrow A/A[p^m]^\mu$ (or rather its dual isogeny $\mathrm{pr}^\vee : (A/A[p^m]^\mu)^\vee \rightarrow A^\vee$). Since the kernel $A[p^m]^\mu$ of the projection pr is of multiplicative type, the dual isogeny $\mathrm{pr}^\vee$ is étale, and therefore the map $\ddot{C}_{\tilde{Z}_n^m} \rightarrow \ddot{C}_{\Phi_n, \delta_n}$ is finite étale. In particular, since $\ddot{C}_{\Phi_n, \delta_n}$ is smooth, so is $\ddot{C}_{\tilde{Z}_n^m}$. We can now see that we are in the same situation as [Lan13]: we have a scheme $\ddot{C}_{\tilde{Z}_n^m}$ which is representable by a proper smooth scheme, but it is not necessarily abelian because its geometric fibers might not be connected.

We now need to imitate Lan's construction of C_{Φ_n, δ_n} , that is, we need to add the relation b_{Φ_n, δ_n} and the lifting conditions. Indeed, recall that the strategy to define C_{Φ_n, δ_n} was to take a system of elements $b_{mn|m, p \nmid m}$ lifting our fixed element b_n , and to consider the common schematic image of the natural morphisms $\ddot{C}_{\Phi_m, b_m} \rightarrow \ddot{C}_{\Phi_n, b_n}$. This is what we will attempt to generalize bellow.

We define a morphism

$$\partial_n^m : \ddot{C}_{\tilde{Z}_n^m} \rightarrow \underline{\mathrm{Hom}}(\frac{1}{n}Y/Y, A^\vee[n])$$

just by precomposing ∂_n with $\ddot{C}_{\tilde{Z}_n^m} \rightarrow \ddot{C}_{\Phi_n, \delta_n}$, and thus we can still consider the fiber $\partial_n^{m, -1}(b_n)$, which we will denote by $\ddot{C}_{\tilde{Z}_n^m, b_n}$. Choosing once more a system $\{b_l\}_{n|l, p \nmid l}$ lifting b_n , we have natural morphisms

$$\ddot{C}_{\tilde{Z}_l^m, b_l} \rightarrow \ddot{C}_{\tilde{Z}_n^m, b_n}.$$

We let $\ddot{C}_{\tilde{Z}_n^m, b_n}^{\mathrm{com}}$ denote the common schematic image of these morphisms. This is exactly the scheme we are looking for, since by definition it is the moduli space of pairs $(c_n, c_n^\vee) \in C_{\Phi_n, \delta_n}$ over $\mathrm{Ig}_m^{\mathbb{Z}_n}$, admitting the desired splitting as in the second part of definition 3.2.2.

We can also generalize [Lan13, Lemma 6.2.3.11.]:

Lemma 3.2.23. *One can produce a uniquely determined compatible system of elements $\{\tilde{b}_l\}_{n|l, p \nmid l}$, where each $\tilde{b}_l$ is an element of $\underline{\mathrm{Hom}}(\frac{1}{l}X/\phi(Y), (A/A[p^m]^\mu)^\vee) \times E \subset \ddot{C}_{\tilde{Z}_n^m}$ that is sent to b_l under the map*

$$\partial_{l, m} : \ddot{C}_{\tilde{Z}_l^m} \rightarrow \underline{\mathrm{Hom}}(\frac{1}{l}Y/Y, A^\vee[l]).$$

By compatible, we mean that for $k|l$, $\tilde{b}_k$ is sent to $\tilde{b}_l$ via the canonical map $\ddot{C}_{\Phi_k, b_k} \rightarrow \ddot{C}_{\Phi_l, b_l}$. In particular, $\tilde{b}_l$ translates $\ddot{C}_{\tilde{Z}_l^m, 0}$ to $\ddot{C}_{\tilde{Z}_l^m, b_l}$ under the conditions above.

Proof. Since l is prime-to- p , we have that $A^\vee[l] = (A/A[p^m]^\mu)^\vee[l]$. In particular, we can think of b_n as an element in $\underline{\mathrm{Hom}}(\frac{1}{n}Y/Y, (A/A[p^m]^\mu)^\vee[n])$, and of ∂_l^m as a map $\partial_l^m : \ddot{C}_{\tilde{Z}_l, m} \rightarrow \underline{\mathrm{Hom}}(\frac{1}{l}Y/Y, (A/A[p^m]^\mu)^\vee[l])$. Then the exact same construction as in [Lan13, Lemma 6.2.3.11.] carries over: namely, let k be an integer satisfying $n|k$, $p \nmid k$ and $\frac{1}{l}X \subset \frac{1}{k}\Phi(Y)$. Then we take $\tilde{b}_l = (-b_k|_{\frac{1}{l}X}, 0)$.

In particular, if we were able to prove that $\ddot{C}_{\tilde{Z}_n, 0}^{\mathrm{com}}$ is geometrically connected, we would conclude that $\mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ is an abelian scheme. However, we were not able to generalize Lan's proof to our situation, since we are not able to prove that $\pi_0(\ddot{C}_{\tilde{Z}_n^m}, \mathrm{Ig}_{\tilde{Z}_n^m}^{\mathbb{X}_b})$ has no p -factors, which is crucial in Lan's proof (c.f. in the proof of proposition 2.3.19, one really needs that there exists an l such that $n|l$ and $p \nmid l$ that kills $\pi_0(\ddot{C}_{\Phi_n, 0}, \mathcal{S}_K^{\mathbb{Z}_n})$).

Remark 3.2.24. It follows just as in the case of toroidal compactifications of Shimura varieties, from the fact that $\mathrm{Ig}_{\tilde{Z}_n^m, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$, and more specifically $\mathrm{Ig}_{\tilde{Z}_n^m, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ is smooth, that the toroidal compactification of the finite Igusa variety $\mathrm{Ig}_m^{\mathbb{X}_b, \mathrm{tor}}$ is also smooth, with our choice of smooth rational polyhedral cone decomposition.

The case of perfect Igusa varieties

In this section, we want to define toroidal compactifications of the perfect Igusa varieties, and describe their boundary strata as in the completely slope divisible case. So we fix a PEL-type p -divisible group $\mathbb{X}_b$ which is not necessarily completely slope divisible. Recall that we defined our perfect Igusa variety $\mathfrak{Ig}^{\mathbb{X}_b}$ in 3.1.26, and we saw it admits a map $\mathfrak{Ig}^{\mathbb{X}_b} \rightarrow \mathcal{C}^{\mathbb{X}_b}$, which by proposition 3.1.31 is a pro-finite étale cover of $\mathcal{C}_{\mathrm{perf}}^{\mathbb{X}_b}$, with Galois group $\Gamma_{\mathbb{X}_b} = \underline{\mathrm{Aut}}(\mathbb{X}_b)$. Once again, let $(\mathcal{G}, \lambda, \iota, \alpha_n)$ denote the restriction of the universal object of $\mathcal{S}_{K, \Sigma}^{\mathrm{tor}}$ to $\mathcal{C}^{\mathbb{X}_b, \mathrm{tor}}$. Let us start with an analogous of Theorem 3.2.10.

Definition 3.2.25. A perfect Igusa-level structure on a perfect $\mathcal{C}^{\mathbb{X}_b, \mathrm{tor}}$ -scheme S consists of the data of an $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -equivariant isomorphism

$$\rho : \mathcal{G}[p^\infty]^\circ \times_{\mathcal{C}^{\mathbb{X}_b, \mathrm{tor}}} S \xrightarrow{\sim} \mathbb{X}_b^\circ \times_{\mathbb{F}_p} S,$$

such that the induced isomorphism on biconnected parts

$$\rho_{(0,1)} : \mathcal{G}[p^\infty]^{(0,1)} \times_{\mathcal{C}^{\mathbb{X}_b, \mathrm{tor}}} S \xrightarrow{\sim} \mathbb{X}_b^{(0,1)} \times_{\mathbb{F}_p} S$$

commutes with the polarizations up to an element in $\mathbb{Z}_p^\times(S)$.

Definition 3.2.26. We define the perfect toroidal compactification of the perfect Igusa variety $\mathfrak{Ig}^{\mathbb{X}_b}$ as the moduli space over $\mathcal{C}_{\mathrm{perf}}^{\mathbb{X}_b, \mathrm{tor}}$ parametrizing perfect Igusa-level structures up to isomorphism, and denote it by $\mathfrak{Ig}^{\mathbb{X}_b, \mathrm{tor}}$.

Theorem 3.2.27. *The pro-étale cover $\mathfrak{Ig}^{\mathbb{X}_b} \rightarrow \mathcal{C}_{\mathrm{perf}}^{\mathbb{X}_b}$, with Galois group $\Gamma_{\mathbb{X}_b}$, extends uniquely to a pro-étale cover $\mathfrak{Ig}^{\mathbb{X}_b, \mathrm{tor}} \rightarrow \mathcal{C}_{\mathrm{perf}}^{\mathbb{X}_b, \mathrm{tor}}$, with the same Galois group.*

Proof. This follows exactly by the same arguments as in the proof of Theorem 3.2.10, except one uses [CS19, Prop. 2.2.6.] instead of [CS19, Thm 2.2.4.].

Our aim is now once again to better understand the geometry of $\mathcal{I}\mathfrak{g}^{\mathbb{X}_b, \text{tor}}$. Namely, we want to describe the formal completion of $\mathcal{I}\mathfrak{g}^{\mathbb{X}_b, \text{tor}}$ along boundary strata.

Definition 3.2.28. Define the boundary stratum $\mathcal{I}\mathfrak{g}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}} \subset \mathcal{I}\mathfrak{g}^{\mathbb{X}_b, \text{tor}}$ as the pre-image of $\mathcal{C}_{Z[(\Phi_n, \delta_n, \sigma)], \text{perf}}^{\mathbb{X}_b}$ along the pro-étale cover $\mathcal{I}\mathfrak{g}^{\mathbb{X}_b, \text{tor}} \rightarrow \mathcal{C}_{\text{perf}}^{\mathbb{X}_b, \text{tor}}$. Denote by $\widehat{\mathcal{I}\mathfrak{g}}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}}$ the formal completion of $\mathcal{I}\mathfrak{g}^{\mathbb{X}_b, \text{tor}}$ along the boundary stratum $\mathcal{I}\mathfrak{g}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}} \subset \mathcal{I}\mathfrak{g}^{\mathbb{X}_b, \text{tor}}$.

Recall that by theorem 3.2.3, $\mathcal{C}^{\mathbb{X}_b, \text{tor}}$ admits a boundary toroidal chart that allows one to identify $\mathcal{C}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}}$, with the quotient

$$\mathcal{C}_{\mathcal{Z}_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b} / \Gamma_{\Phi_n, \sigma}$$

between the formal completion of $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}(\sigma)}^{\mathbb{X}_b}$ along $\mathcal{C}_{\Xi_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b}$ and $\Gamma_{\Phi_n, \sigma}$. In particular, taking perfections induces an identification between $\widehat{\mathcal{C}}_{Z[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}}$ and a similar quotient, between the formal completion of $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}(\sigma), \text{perf}}^{\mathbb{X}_b}$ along $\mathcal{C}_{\Xi_{\Phi_n, \delta_n, \sigma}, \text{perf}}^{\mathbb{X}_b}$ and $\Gamma_{\Phi_n, \sigma}$.

Recall once more that if we fix a cusp label (Z_n, Φ_n, δ_n) , by theorem 2.3.16, over C_{Φ_n, δ_n} we have a universal Raynaud extension

$$0 \rightarrow T \rightarrow \tilde{G}_{\Phi_n, \delta_n} \rightarrow A_{\Phi_n, \delta_n} \rightarrow 0,$$

where T is the torus with character group X given by the cusp label, and A_{Φ_n, δ_n} the pullback of the universal abelian scheme on $\mathcal{S}_K^{Z_n}$ to C_{Φ_n, δ_n} . Over $\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$, the connected part of the p -divisible group attached to $\tilde{G}_{\Phi_n, \delta_n}$ is still a p -divisible group, which we once more denote by $\tilde{\mathcal{G}}_{\Phi_n, \delta_n}$.

Definition 3.2.29. Let (Z_n, Φ_n, δ_n) be a cusp label. A degeneration perfect Igusa-level structure attached to (Z_n, Φ_n, δ_n) , on a perfect $\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ -scheme S , consists of an $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -isomorphism

$$\rho : \tilde{\mathcal{G}}_{\Phi_n, \delta_n} \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}} S \xrightarrow{\sim} \mathbb{X}_b^\circ \times_{\mathbb{F}_p} S,$$

such that the restriction to the biconnected parts

$$\rho^{(0,1)} : \tilde{\mathcal{G}}_{\Phi_n, \delta_n}^{(0,1)} \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}} S \xrightarrow{\sim} \mathbb{X}_b^{(0,1)} \times_{\mathbb{F}_p} S,$$

is compatible with the polarizations up to a scalar in $\mathbb{Z}_p^\times(S)$.

Definition 3.2.30. We define $\mathcal{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ as the moduli space over $\mathcal{C}_{C_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}$, parametrizing degeneration perfect Igusa-level structures attached to (Z_n, Φ_n, δ_n) .

Once again by [CS19, Prop. 2.2.6.], $\mathcal{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ defines a pro-finite étale cover over $\mathcal{C}_{C_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}$ with Galois group $\underline{\text{Aut}}(\mathbb{X}_b)$. We denote its pullback to $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}$ by $\mathcal{I}\mathfrak{g}_{Z_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$, and similarly for $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}, \Sigma_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}$, $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}(\sigma), \text{perf}}^{\mathbb{X}_b}$ and $\mathcal{C}_{\Xi_{\Phi_n, \delta_n, \sigma}, \text{perf}}^{\mathbb{X}_b}$. Denote by $\mathcal{I}\mathfrak{g}_{\mathcal{Z}_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b}$ the formal completion of $\mathcal{I}\mathfrak{g}_{Z_n, \Xi_{\Phi_n, \delta_n}(\sigma)}^{\mathbb{X}_b}$ along $\mathcal{I}\mathfrak{g}_{Z_n, \Xi_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b}$. We obtain the following generalization of [CS19, Thm. 3.2.13.].

Theorem 3.2.31. *There is a canonical isomorphism between $\widehat{\mathfrak{I}\mathfrak{g}}_{Z_{[(\Phi_n, \delta_n, \sigma)]}}^{\mathbb{X}_b, \text{tor}}$ and the quotient*

$$\mathfrak{I}\mathfrak{g}_{\mathcal{Z}_{\Phi_n, \delta_n, \sigma}}^{\mathbb{X}_b} / \Gamma_{\Phi_n, \sigma}.$$

Proof. This follows exactly as in theorem 3.2.14.

Let us give a better description of a few objects appearing in the Igusa toroidal boundary chart. In particular, we can prove the following generalization of [CS19, Prop. 3.3.11.].

Lemma 3.2.32. *Fix a cusp label (Z_n, Φ_n, δ_n) , and a point $(A, \lambda_A, \iota_A, (\varphi_{-1, n}, \nu_{-1, n}), c_n, c_n^\vee, \rho) \in \mathfrak{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}(S)$. In particular, recall that $(A, \lambda_A, \iota_A, (\varphi_{-1, n}, \nu_{-1, n}))$ is a point in the smaller Shimura variety $\mathcal{S}_K^{Z_n}$, consisting of an abelian scheme over S with extra structures, c_n and $c_n^\vee$ denote the Raynaud extension and its dual. The Raynaud extension is given by a short-exact sequence*

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow A \rightarrow 0.$$

Finally, $\rho : \tilde{\mathcal{G}}_{\Phi_n, \delta_n} \xrightarrow{\sim} \mathbb{X}_b^\circ$ is a degeneration perfect Igusa-level structure attached to (Z_n, Φ_n, δ_n) . Then, there is an $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -equivariant embedding

$$\varrho : \tilde{G}[p^\infty] \hookrightarrow \mathbb{X}_b$$

extending ρ , which is compatible with the polarizations. The induced filtration $Z_{\mathbb{X}_b}$

$$T[p^\infty] \subset \tilde{G}[p^\infty] \subset \mathbb{X}_b$$

is symplectic, and the induced isomorphism

$$\text{Gr}_{-1}^{\mathbb{X}_b} \cong \tilde{G}[p^\infty] / T[p^\infty] = A[p^\infty]$$

is compatible with the polarizations.

Proof. Over a perfect base the étale-connected sequence splits (c.f. [CS19, Sect. 2.2.]), so $\mathbb{X}_b \cong \mathbb{X}_b^\mu \oplus \mathbb{X}_b^{(0,1)} \oplus \mathbb{X}_b^{\text{ét}}$. Furthermore, we have that $\tilde{G}[p^\infty] \cong T[p^\infty] \oplus A[p^\infty]$, and therefore $\tilde{G}[p^\infty]^\mu \cong T[p^\infty] \oplus A[p^\infty]^\mu$, and $\tilde{G}[p^\infty]^{\text{ét}} \cong A[p^\infty]^{\text{ét}}$. Since $\tilde{\mathcal{G}}_{\Phi_n, \delta_n} \cong \tilde{G}[p^\infty]^\circ$, ρ induces an isomorphism between $\tilde{G}[p^\infty]^\circ$ and $\mathbb{X}_b^\circ$. By Lemma 3.2.5, the induced polarization on $A[p^\infty]$ is an isomorphism, so $A[p^\infty]^{\text{ét}}$ is the dual of $A[p^\infty]^\mu$, giving us the desired embedding. The isomorphism between the first graded piece of the filtration $Z_{\mathbb{X}_b}$ and $A[p^\infty]$, and the compatibility with polarizations follows immediately by construction and by the general theory of degeneration.

We can also describe $\mathfrak{I}\mathfrak{g}_{Z_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ more explicitly, as follows.

Proposition 3.2.33. *Let us denote the torus with character group S_{Φ_n} by E_{Φ_n} , and let $S_{\Phi_n, \text{perf}}$ resp. $E_{\Phi_n, \text{perf}}$ denote their perfections. The $S_{\Phi_n, \text{perf}}$ -torsor $\mathfrak{I}\mathfrak{g}_{Z_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \rightarrow \mathfrak{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ is the perfection of the natural E_{Φ_n} -torsor over $\mathfrak{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$, that parametrizes lifts of the dual Raynaud extension $c^\vee : Y \rightarrow A$ to $\iota : Y \rightarrow \tilde{G}$. In particular, it parametrizes symmetric lifts $\tilde{\iota} : Y[\frac{1}{p}] \rightarrow \tilde{G}$ of the period morphism $\iota : Y \rightarrow \tilde{G}$ (c.f. remark 2.2.9).*

Proof. That $\mathfrak{I}\mathfrak{g}_{Z_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \rightarrow \mathfrak{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ is the perfection of the described S_{Φ_n} -torsor over $\mathfrak{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ is clear by definition. Furthermore, a symplectic splitting of the filtration $Z_{\mathbb{X}_b}$ of $\mathbb{X}_b$ over C_{Φ_n, δ_n} , induces a splitting of the short exact sequence

$$0 \rightarrow T[p^\infty] \rightarrow \tilde{G}[p^\infty] \rightarrow A[p^\infty] \rightarrow 0$$

and dually of the sequence

$$0 \rightarrow T^\vee[p^\infty] \rightarrow \tilde{G}^\vee[p^\infty] \rightarrow A^\vee[p^\infty] \rightarrow 0$$

via the embedding

$$\varrho : \tilde{G}[p^\infty] \hookrightarrow \mathbb{X}_b.$$

Splitting the second sequence is equivalent to giving a lift $\tilde{c}_n^\vee : Y[\frac{1}{p}] \rightarrow A$ of the dual Raynaud extension $c^\vee : Y \rightarrow A$ (c.f. [Lan13, Lemma 5.2.3.1.]). The description in the statement follows by uniqueness (of such a perfect torus lift of the described E_{Φ_n} -torsor), since this indeed defines a lift of the E_{Φ_n} -torsor $\Xi_{\Phi_n, \delta_n} \rightarrow C_{\Phi_n, \delta_n}$ to a $E_{\Phi_n, \text{perf}}$ -torsor.

Once again, we want to try to refine this using Igusa cusp labels. We need to define perfect Igusa level structures.

Definition 3.2.34. Let (Z_n, Φ_n, δ_n) be a cusp label. A perfect Igusa-level structure on a perfect $\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ -scheme S , compatible with an Igusa cusp label $\tilde{Z}_n$ over (Z_n, Φ_n, δ_n) , consists of the data of:

1. An isomorphism

$$\rho_{A_{b, \Phi_n, \delta_n}^\mu} : A_{b, \Phi_n, \delta_n}^\mu[p^\infty] \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}} S \xrightarrow{\sim} (\text{Gr}_{-1}^{\mathbb{X}_b})^\mu \times_{\mathbb{F}_p} S$$

compatible with the $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -actions.

2. A splitting

$$\delta : A_{b, \Phi_n, \delta_n}[p^\infty]^\mu \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}} S \rightarrow \tilde{\mathcal{G}}_{\Phi_n, \delta_n}^\mu \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}} S$$

compatible with the $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -actions.

3. An isomorphism

$$\rho_{A_{b, \Phi_n, \delta_n}^{(0,1)}} : A_{b, \Phi_n, \delta_n}[p^\infty]^{(0,1)} \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}} S \xrightarrow{\sim} \mathbb{X}_b^{(0,1)} \times_{\mathbb{F}_p} S$$

commuting with the polarizations up to an element in $\mathbb{Z}_p^\times(S)$.

We can do the same analysis as above. Namely, if we fix a cusp label together with the Igusa data associated to an Igusa cusp label $\tilde{Z}_n$, and a perfect Igusa level structure compatible with $\tilde{Z}_n$, we can recover the perfect Igusa level structure: the splitting $\delta^\mathbb{X}$, gives us an isomorphism $\delta^\mathbb{X} : \mathbb{X} \cong \bigoplus_i \text{Gr}_{-i}^\mathbb{X}$. By duality, ψ gives us an identification of $\text{Gr}_{-2}^\mathbb{X}$ with $T[p^\infty]$. The splitting

δ gives us an isomorphism $G[p^\infty]^\mu \cong T[p^\infty] \oplus A[p^\infty]^\mu$, and since $\mathcal{G}_{\Phi_n, \delta_n}^{(0,1)} \cong A_{b, \Phi_n, \delta_n}[p^\infty]^{(0,1)}$, the isomorphisms $\rho_{A_{b, \Phi_n, \delta_n}^{(0,1)}}$ and $\rho_{A_{b, \Phi_n, \delta_n}^\mu}$ give us the rest of the data. Similarly, given a cusp label (Z_n, Φ_n, δ_n) and a perfect degeneration Igusa level structure attached to (Z_n, Φ_n, δ_n) , we can construct a filtration $Z_n^{\mathbb{X}_b}$ as in the previous lemma 3.2.32. Picking any splitting $\delta^{\mathbb{X}_b}$ (they are all identified by the equivalence relation) immediately gives us an Igusa cusp label. The perfect Igusa level structure compatible with $\tilde{Z}_n$ is already given in the definition of the perfect degeneration Igusa level structure attached to (Z_n, Φ_n, δ_n) . We obtain the following result.

Definition 3.2.35. Let (Z_n, Φ_n, δ_n) be a cusp label, and $\tilde{Z}_n$ be an Igusa cusp label over it. We define $\mathcal{I}\mathfrak{g}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ as the moduli space over $\mathcal{C}_{C_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}$ parametrizing perfect Igusa-level structures compatible with $\tilde{Z}_n$.

Theorem 3.2.36. Let (Z_n, Φ_n, δ_n) be a cusp label, and $\widehat{\mathcal{I}\mathfrak{g}}_{Z_n[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}}$ be as defined in 3.2.28.

1. $\widehat{\mathcal{I}\mathfrak{g}}_{Z_n[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}}$ admits a decomposition into open and closed formal subschemes

$$\widehat{\mathcal{I}\mathfrak{g}}_{Z_n[(\Phi_n, \delta_n, \sigma)]}^{\mathbb{X}_b, \text{tor}} = \sqcup_{\tilde{Z}_n} \widehat{\mathcal{I}\mathfrak{g}}_{\tilde{Z}_n, \sigma}^{\mathbb{X}_b, \text{tor}}$$

where $\tilde{Z}_n$ runs through the Igusa cusp labels above (Z_n, Φ_n, δ_n) .

2. Fix an Igusa cusp label $\tilde{Z}_n$. Recall the toroidal boundary chart

$$\begin{array}{ccc} \Xi_{\Phi_n, \delta_n} & \longrightarrow & \Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}} \\ \downarrow & \swarrow & \\ C_{\Phi_n, \delta_n} & & \\ \downarrow & & \\ \mathcal{S}_K^{Z_n} & & \end{array}$$

in theorem 2.3.16, describing $(\mathcal{S}_{K, \Sigma}^{\text{tor}})^{\wedge}_{Z_n[(\Phi_n, \delta_n, \sigma)]}$. The moduli space $\mathcal{I}\mathfrak{g}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ in definition 3.2.35 admits a morphism to the Igusa variety $\mathcal{I}\mathfrak{g}^{\mathbb{X}_b, Z_n}$ over the perfection of the central leaf $\mathcal{C}_{\text{perf}}^{\mathbb{X}_{Z_n}}$, and it fits in a commutative diagram

$$\begin{array}{ccc} \mathcal{I}\mathfrak{g}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & \longrightarrow & \mathcal{C}_{C_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b} \\ \downarrow & & \downarrow \\ \mathcal{I}\mathfrak{g}^{\mathbb{X}_b, Z_n} & \longrightarrow & \mathcal{C}_{Z_n, \text{perf}}^{\mathbb{X}_b} \end{array}$$

Let $\mathcal{I}\mathfrak{g}_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \rightarrow \mathcal{I}\mathfrak{g}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ resp. $\mathcal{I}\mathfrak{g}_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}}}^{\mathbb{X}_b} \rightarrow \mathcal{I}\mathfrak{g}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b}$ denote the pullback of $\mathcal{I}\mathfrak{g}_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \rightarrow \mathcal{C}_{C_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}$ along $\mathcal{C}_{\Xi_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b} \rightarrow \mathcal{C}_{C_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}$ resp. $\mathcal{C}_{\Xi_{\Phi_n, \delta_n, \Sigma_{\Phi_n, \delta_n}}, \text{perf}}^{\mathbb{X}_b} \rightarrow \mathcal{C}_{C_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}$. These define an Igusa toroidal boundary chart

$$\begin{array}{ccc}
\mathcal{I}g_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & \longrightarrow & \mathcal{I}g_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}, \Sigma_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \\
\downarrow & \swarrow & \\
\mathcal{I}g_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} & & \\
\downarrow & & \\
\mathcal{I}g_{\tilde{Z}_n, Z_n}^{\mathbb{X}_b} & &
\end{array}$$

For our given rational polyhedral cone $\sigma \in \Sigma_{\Phi_n}$, consider the schemes

$$\mathcal{I}g_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}(\sigma) := \mathcal{I}g_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}} \mathcal{C}_{\Xi_{\Phi_n, \delta_n}(\sigma), \text{perf}}^{\mathbb{X}_b}$$

and

$$\mathcal{I}g_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}, \sigma}^{\mathbb{X}_b} := \mathcal{I}g_{\tilde{Z}_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_b} \times_{\mathcal{C}_{C_{\Phi_n, \delta_n}, \text{perf}}^{\mathbb{X}_b}} \mathcal{C}_{\Xi_{\Phi_n, \delta_n}, \sigma, \text{perf}}^{\mathbb{X}_b}.$$

Let $\mathcal{I}g_{\tilde{Z}_n, \mathcal{X}, \Sigma_{\Phi_n, \delta_n}, \sigma}^{\mathbb{X}_b}$ denote the formal completion of $\mathcal{I}g_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_b}(\sigma)$ along $\mathcal{I}g_{\tilde{Z}_n, \Xi_{\Phi_n, \delta_n}, \sigma}^{\mathbb{X}_b}$. Then, there is an isomorphism

$$\widehat{\mathcal{I}g}_{\tilde{Z}_n, \sigma}^{\mathbb{X}_b, \text{tor}} \cong \mathcal{I}g_{\tilde{Z}_n, \mathcal{X}, \Sigma_{\Phi_n, \delta_n}, \sigma}^{\mathbb{X}_b} / \Gamma_{\tilde{Z}_n, \sigma},$$

where $\Gamma_{\tilde{Z}_n, \sigma}$ is the subgroup of $\Gamma_{\Phi_n, \sigma}$ such that a pair $(\gamma_X, \gamma_Y) \in \Gamma_{\Phi_n, \sigma}$ is in $\Gamma_{\tilde{Z}_n, \sigma}$ if $\psi = \gamma_Y \circ \psi$.

Proof. This just follows once more by our discussion above.

3.3 Minimal compactifications of Igusa varieties

In this section, we want to recall the definition of the minimal compactifications of Igusa varieties, and prove its affineness. To do so, we need the following result, which is a generalization of [CS19, Cor. 3.2.14.].

Proposition 3.3.1. *Let $\lambda : \mathbb{X}_1 \rightarrow \mathbb{X}_2$ be an isogeny of p -divisible groups with extra structures, such that the isogeny is an isomorphism on the multiplicative and étale parts. Then this induces an isomorphism*

$$\mathcal{I}g^{\mathbb{X}_1, \text{tor}} \cong \mathcal{I}g^{\mathbb{X}_2, \text{tor}}$$

extending the isomorphism

$$\mathcal{I}g^{\mathbb{X}_1} \cong \mathcal{I}g^{\mathbb{X}_2}.$$

We want to use the explicit moduli description of the formal completions at the boundary, and the universal property of toroidal compactifications to prove this.

Proof. The idea to prove this result is the usual: we start by fixing a cusp label (Z_n, Φ_n, δ_n) , for which we have a perfect Igusa toroidal boundary chart. Then, using the moduli description of $\widehat{\text{Ig}}_{\Sigma(\Phi_n, \delta_n), Z_n}^{\mathbb{X}_b, \text{tor}}$, we will produce an isomorphism

$$\widehat{\text{Ig}}_{\Sigma(\Phi_n, \delta_n), Z_n}^{\mathbb{X}_1, \text{tor}} \cong \widehat{\text{Ig}}_{\Sigma(\Phi_n, \delta_n), Z_n}^{\mathbb{X}_2, \text{tor}}$$

between the formal completions at the boundaries. Gluing everything together will then yield the desired isomorphism between toroidal compactifications.

Let us thus fix a cusp label (Z_n, Φ_n, δ_n) . Consider the Igusa toroidal boundary chart

$$\begin{array}{ccc} \mathcal{I}\mathfrak{g}_{Z_n, \Xi_{\Phi_n, \delta_n}}^{\mathbb{X}_1} & \longrightarrow & \mathcal{I}\mathfrak{g}_{Z_n, \Xi_{\Phi_n, \delta_n}, \Sigma_{\Phi_n, \delta_n}}^{\mathbb{X}_1} \\ \downarrow & \swarrow & \\ \mathcal{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_1} & & \\ \downarrow & & \\ \mathcal{I}\mathfrak{g}^{\mathbb{X}_{Z_n}} & & \end{array}$$

as defined in the previews section. Recall that by lemma 3.2.32, $\mathcal{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_1}$ parametrizes points $(A, \lambda_A, \iota_A, (\alpha_{-1, n})) \in \mathcal{S}_K^{Z_n}$, together with triples $(c_n, c_n^\vee, \varrho)$, consisting of the Raynaud extension $c_n : X \rightarrow A^\vee$ and its dual $c_n^\vee : Y \rightarrow A$, given as usual respectively by the short exact sequences

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow A \rightarrow 0$$

and

$$0 \rightarrow T^\vee \rightarrow \tilde{G}^\vee \rightarrow A^\vee \rightarrow 0,$$

and where $\varrho : \tilde{G}[p^\infty] \hookrightarrow \mathbb{X}_1$ is an $\mathcal{O}_B$ -linear embedding. Let $\lambda : \mathbb{X}_1 \rightarrow \mathbb{X}_2$ be an isogeny as in the statement, so that in particular it is an isomorphism on the multiplicative and étale parts. Denote by $K \subset \mathbb{X}_1$ the kernel of λ . By our assumptions, K is actually contained in $\mathbb{X}_1^{(0,1)}$, since it does not meet either the multiplicative, or the étale part of $\mathbb{X}_1$. By abuse of notation, let us also denote by K its preimage under the embedding $\varrho : \tilde{G}[p^\infty] \hookrightarrow \mathbb{X}_1$. Since λ is an isogeny, K is actually contained in $\tilde{G}[p^m]$ for some integer m called the degree of λ , and we can consider the quotient $\tilde{G}' := \tilde{G}/K$. This defines a new semi-abelian scheme, fitting in the short-exact sequence

$$0 \rightarrow T \rightarrow \tilde{G}' \rightarrow A' \rightarrow 0,$$

where $A' := A/K_A$, where K_A is the image of K through the surjection $\tilde{G} \rightarrow A$. It admits an injection

$$\varrho' : \tilde{G}'[p^\infty] \hookrightarrow \mathbb{X}_2$$

by noticing that the composition

$$\tilde{G}[p^\infty] \xrightarrow{\varrho} \mathbb{X}_1 \xrightarrow{\lambda} \mathbb{X}_2$$

factors through $\tilde{G}'[p^\infty]$, inducing an isomorphism on connected pieces, just by definition of $\tilde{G}'$.

Let furthermore $(c')^\vee : Y \rightarrow (A')^\vee$ be defined by the equality $(c')^\vee = \pi_A \circ c^\vee$, where $\pi_A : A \rightarrow A'$ denotes the projection. $(c')^\vee$ defined the dual Raynaud extension $(\tilde{G}')^\vee$, and is equivalent to a short exact sequence

$$0 \rightarrow T^\vee \rightarrow (\tilde{G}')^\vee \rightarrow (A')^\vee \rightarrow 0.$$

In particular, $Y' = Y$, since K does not intersect the étale part of $\tilde{G}$. This already induces an isomorphism

$$\lambda_* : \mathfrak{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_1} \cong \mathfrak{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_2}.$$

By analyzing the construction of the equivalence between c' and the short exact sequence

$$0 \rightarrow T \rightarrow \tilde{G}' \rightarrow A' \rightarrow 0,$$

we also see that $(\pi_A^\vee)^{-1} \circ c = p^m \circ c'$, where $\pi_A^\vee : (A')^\vee \rightarrow A^\vee$ is the isogeny dual to π_A , and $(\pi_A^\vee)^{-1} : A^\vee \rightarrow (A')^\vee$ is the isogeny whose composition with $\pi_A^\vee$ is p^m . This implies that, before perfection, the pullback along λ_* of the natural S_{Φ_n} -torsor over $\mathfrak{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_2}$ is the pushout of the natural S_{Φ_n} -torsor over $\mathfrak{I}\mathfrak{g}_{Z_n, C_{\Phi_n, \delta_n}}^{\mathbb{X}_1}$ under $p^m \times \text{id}_X : S_{\Phi_n} \rightarrow S_{\Phi_n}$. In particular, we obtain an isomorphism after perfection. It is also clear that this induces the same action as in lemma 3.1.27 on the open cell, implying the result. The glueing follows from the lemma bellow.

Lemma 3.3.2. [CS19, Lemma 4.3.3.] *Let X and Y be qcqs schemes, with Y separated, and $U \subset X$ be a quasicompact open schematically dense subscheme. Let furthermore $f : U \rightarrow Y$ be a map of schemes. If there is an extension of f to a map $\bar{f} : X \rightarrow Y$, then this extension is unique, and to check whether this extension exists, it suffices to find an fpqc cover $X' \rightarrow X$ such that $f' = f|_{X' \times_X U} : X' \times_X U \rightarrow Y$ extends to $\bar{f}' : X' \rightarrow Y$.*

This allows us to generalize theorem 3.3.2. in [CS19] to our setup, to obtain the following result.

Theorem 3.3.3. $\mathcal{C}^{\mathbb{X},*}$ is affine.

This result follows verbatim as in the proof of theorem 3.3.2. in [CS19]. For completion, let us briefly recall the idea.

Proof. The first step of the prove, is to see that in any given isogeny class, there exists a p -divisible group $\mathbb{X}$ such that $\mathcal{C}^{\mathbb{X},*}$ is affine. This follows by picking a certain p -divisible group $\mathbb{X}$, whose p -torsion is an Ekedahl–Oort stratum which is contained in the given Newton stratum (the existence of such an $\mathbb{X}$ is proven in [Nie15, Cor. 1.6.] for general PEL-type Shimura varieties). The minimal compactification of this strata is then affine by [Box15, Thm. C], and the claim follows by [CS19, Prop. 3.1.3], using that $\mathcal{C}^{\mathbb{X}}$ is a closed subset of the corresponding Ekedahl–Oort stratum. It then suffices to see that if $\mathbb{X}'$ is such a p -divisible group, and there is an isogeny $\lambda : \mathbb{X}' \rightarrow \mathbb{X}$, then $\mathcal{C}^{\mathbb{X},*}$ is also affine. Assuming that λ is an isomorphism on the étale and multiplicative parts, and using proposition 3.3.1, yields a correspondence

$$\mathcal{C}^{\mathbb{X}',*} \leftarrow \mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}} \rightarrow \mathcal{C}^{\mathbb{X},*}$$

extending the diagram

$$\mathcal{C}^{\mathbb{X}'} \leftarrow \mathfrak{I}\mathfrak{g}^{\mathbb{X}} \rightarrow \mathcal{C}^{\mathbb{X}}.$$

It now follows formally that $\mathcal{C}^{\mathbb{X},*}$ is affine, by lemma 3.3.3 in [CS19]. Namelly, by considering the diagram

$$\begin{array}{ccccc} \mathcal{C}^{\mathbb{X}',*,\text{bdry}} & \longleftarrow & \mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor},\text{bdry}} & \longrightarrow & \mathcal{C}^{\mathbb{X},*,\text{bdry}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}^{\mathbb{X}',*} & \longleftarrow & \mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}} & \longrightarrow & \mathcal{C}^{\mathbb{X},*} \end{array}$$

where $\mathcal{C}^{\mathbb{X},*,\text{bdry}}$, $\mathcal{C}^{\mathbb{X}',*,\text{bdry}}$, and $\mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor},\text{bdry}}$ denote the boundaries of $\mathcal{C}^{\mathbb{X},*}$, $\mathcal{C}^{\mathbb{X}',*}$, and $\mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}}$ respectively.

Definition 3.3.4. We define $\mathfrak{I}\mathfrak{g}^{\mathbb{X},*}$ as the normalization of $\mathcal{C}^{\mathbb{X},*}$ in $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$.

We obtain the following generalization of [CS19, Prop. 3.3.4.].

Proposition 3.3.5. $\mathfrak{I}\mathfrak{g}^{\mathbb{X},*} \rightarrow \mathcal{C}^{\mathbb{X},*}$ is integral, and thus $\mathfrak{I}\mathfrak{g}^{\mathbb{X},*}$ is affine by the above theorem. Furthermore, it agrees with the Stein factorization of

$$\mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}} \rightarrow \mathcal{C}^{\mathbb{X},*},$$

and therefore any isogeny $\lambda : \mathbb{X} \rightarrow \mathbb{X}'$ inducing an isomorphism on multiplicative and étale parts induces an isomorphism

$$\mathfrak{I}\mathfrak{g}^{\mathbb{X},*} \cong \mathfrak{I}\mathfrak{g}^{\mathbb{X}',*}$$

by Proposition 3.3.1.

One can do something similar in the completely slope divisible case. Namely, we fix a completely slope divisible p -divisible group $\mathbb{X}_b$ for some b in the Kottwitz set, and we consider the non-perfect, pro-Igusa variety $\text{Ig}^b \rightarrow \mathcal{C}^b$. We define the minimal compactification as above:

Definition 3.3.6. Define $\text{Ig}^{b,*}$ as the normalization of $\mathcal{C}^{b,*}$ in Ig^b .

The following statements follow from the definitions, just as in [CS19, Lemma 3.3.8.].

Lemma 3.3.7. *The morphism*

$$\text{Ig}^{b,*} \rightarrow \mathcal{C}^{b,*}$$

is finite surjective, and hence $\text{Ig}^{b,}$ is affine. Furthermore, $\text{Ig}^b \hookrightarrow \text{Ig}^{b,*}$ is a dense open immersion, and $\text{Ig}^{b,*}$ is normal.*

Chapter 4

The Hodge-Tate Period morphism

In this chapter, we recall the construction of the Hodge-Tate period morphism on Shimura varieties. We then generalize section 4 of [CS19], extending the Hodge-Tate period morphism to the toroidal and the minimal compactifications, and analyze its fibers.

4.1 Hodge-Tate Period Morphism on Shimura Varieties

Let us start by recalling one of the major results in [CS17], namely the existence of a Hodge-Tate period morphism for Shimura varieties of Hodge type.

Let (G, X) be a Shimura datum, namely G is a reductive group and X a $G(\mathbb{R})$ -conjugacy class of homomorphisms $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \rightarrow G_{\mathbb{R}}$ satisfying certain axioms. We have a Shimura variety associated to this data. Furthermore, h determines a Hodge cocharacter

$$\mu : \mathbb{G}_{m, \mathbb{C}} \xrightarrow{z \mapsto (z, 1)} \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}} \xrightarrow{h_{\mathbb{C}}} G_{\mathbb{C}},$$

which in turn allows one to define a parabolic subgroup P_{μ} . The associated flag variety $Fl_{G, \mu}$ parametrizes parabolic subgroups in the conjugacy class of P_{μ} .

Recall that a Shimura datum is of Hodge type if it admits an embedding $(G, X) \hookrightarrow (\tilde{G}, \tilde{X})$ for some Siegel Shimura datum $(\tilde{G}, \tilde{X})$. In particular, the Shimura variety associated to (G, X) has a universal abelian variety, which is the restriction of the universal abelian variety associated to the Siegel Shimura variety one obtains from $(\tilde{G}, \tilde{X})$.

Let $Fl_{(G, \mu)}$ (resp. $Fl_{(\tilde{G}, \tilde{\mu})}$) denote the flag variety associated to (G, X) (resp. $(\tilde{G}, \tilde{X})$). Fixing an embedding $(G, X) \hookrightarrow (\tilde{G}, \tilde{X})$ as above, induces an embedding of flag varieties $Fl_{(G, \mu)} \hookrightarrow Fl_{(\tilde{G}, \tilde{\mu})}$. Fix open compact subgroups $K \subset G(\mathbb{A}_f)$ and $\tilde{K} \subset \tilde{G}(\mathbb{A}_f)$ such that there is an embedding of Shimura varieties

$$\mathcal{S}_K \hookrightarrow \tilde{\mathcal{S}}_{\tilde{K}} \otimes E.$$

Fix a prime p , write $K = K^p K_p \subset G(\mathbb{A}_f^p) \times G(\mathbb{Q}_p)$ and fix a place $\mathfrak{p}$ of E above p . Consider the adic spaces $\mathcal{F}_{G, \mu}$, $\mathcal{F}_{\tilde{G}, \tilde{\mu}}$ over $\mathrm{Spa}(E_{\mathfrak{p}})$ associated to $Fl_{(G, X)}$, resp. $Fl_{(\tilde{G}, \tilde{\mu})}$. We have the following result by Scholze and Caraiani and Scholze.

Theorem 4.1.1. [Sch15, Thm. 4.1.1.], [CS17, Thm. 2.1.3.]

1. If K^p is sufficiently small, there exists a perfectoid space $\mathbf{S}_{K^p}$ over $E_{\mathfrak{p}}$ such that

$$\mathbf{S}_{K^p} \sim \varprojlim_{K_p} (\mathcal{S}_{K_p K^p} \otimes_E E_{\mathfrak{p}})^{\text{ad}};$$

2. There is a $G(\mathbb{Q}_p)$ -equivariant Hodge-Tate period morphism

$$\pi_{\text{HT}} : \mathbf{S}_{K^p} \rightarrow \mathcal{F}_{\tilde{G}, \tilde{\mu}},$$

that factors through $\mathcal{F}_{G, \mu}$. In particular, we obtain a morphism

$$\pi_{\text{HT}} : \mathbf{S}_{K^p} \rightarrow \mathcal{F}_{G, \mu}. \quad (4.1)$$

The construction of π_{HT} is quite complicated in the Hodge case, so we do not want to get into it. We do recall however that in the Siegel case, π_{HT} is given on C -points, for C a complete algebraically closed field over $\mathbb{Q}_p$, by the Hodge-Tate decomposition

$$0 \rightarrow (\text{Lie})(A)(1) \rightarrow T_p A \otimes C \rightarrow (\text{Lie} A^{\vee})^{\vee} \rightarrow 0$$

for $x \in \mathbf{S}_{K^p}(C, C^+)$ corresponding to the abelian variety A of dimension g over C , since $T_p A \otimes C \cong C^{2g}$, and $(\text{Lie})(A)(1) \hookrightarrow T_p A \otimes C$ is a totally isotropic subspace.

4.2 Mantovan's product formula

Before starting to study the Hodge-Tate period morphism on compactifications, we want to recall Mantovan's product formula for PEL-type Shimura varieties.

A product formula was originally constructed by Mantovan in [Mant05]. She worked with proper Shimura varieties of PEL-type, and considered a completely slope divisible p -divisible group $\mathbb{X}_b$, and the associated finite Igusa varieties $\text{Ig}_m^{\mathbb{X}_b}$. She was then able to use vanishing cycles to write a formula describing the l -adic cohomology of the Shimura varieties in terms of the l -adic cohomology of the corresponding Rapoport-Zink spaces and Igusa varieties (c.f. [Mant05, Sect. 8]).

More recently, Caraiani and Scholze proved in [CS17] that if we consider the perfect Igusa varieties, we can still relate Shimura varieties to the corresponding Rapoport-Zink spaces and Igusa varieties. Let us describe their results. We keep the notation from the section 4.1 above, but we restrict ourselves to the PEL-type case. In particular, we fix some PEL-datum $(B, *, V, (\cdot, \cdot))$ over $\mathbb{Q}$ of type A or C . We obtain an associated local PEL-datum $(B_{\mathbb{Q}_p}, *, V_{\mathbb{Q}_p}, (\cdot, \cdot))$, and fix a conjugacy class of characters μ as in 3.1.3, and a $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$. We can consider the associated isogeny class of p -divisible groups, and fix one which we denote by $\mathbb{X}_b$, and the associated perfect Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$ over $\bar{\mathbb{F}}_p$, as in definition 3.1.26. Recall that $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$ is independent of the choice of p -divisible group $\mathbb{X}_b$ within an isogeny class, so, in particular, we may assume that $\mathbb{X}_b$ is

completely slope divisible. Notice that just as explained in [CS17, Sect. 4.3.], since $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$ is perfect, it admits a unique lift to a p -adic formal scheme over $W(\overline{\mathbb{F}}_p)$. If we fix a complete algebraically closed, non-archimedean extension C of $\mathbb{Q}_p$, and consider its ring of integers $\mathcal{O}_C$, then $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$ will also admit a lift to $\mathcal{O}_C$, just by base-changing $\mathfrak{I}\mathfrak{g}_{W(\overline{\mathbb{F}}_p)}^{\mathbb{X}_b}$ to $\mathcal{O}_C$, which we denote by $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}}$. Recall furthermore that we have defined the Rapoport-Zink space associated to a local PEL-datum in 3.1.21, which we denoted by $\mathcal{M}_{(G_{\mathbb{Q}_p}, b, \mu)}$, parametrizing p -divisible groups with extra structures, and that $\mathcal{M}_{(G, b, \mu)}$ is representable by a formally smooth formal scheme over $\mathrm{Spf}(\mathcal{O}_{\check{E}})$. In particular, by formal smoothness of $\mathcal{M}_{(G_{\mathbb{Q}_p}, b, \mu)}$, $\mathbb{X}_b$ also admits a lift $\mathbb{X}_{\mathcal{O}_C} \in \mathcal{M}_{(G_{\mathbb{Q}_p}, b, \mu)}(\mathcal{O}_C)$ to $\mathcal{O}_C$. By [CS17, Lemma 4.3.10.], $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}}$ will then parametrize abelian varieties with extra structures in $\mathcal{S}_{K(N)}$ together with an isomorphism $A[p^\infty] \cong \mathbb{X}_{\mathcal{O}_C} \times R$, as a moduli problem over $\mathrm{Nilp}_{\mathcal{O}_C}^{\mathrm{op}}$.

Just as in the case of Shimura varieties, we can consider the adic generic fiber of $\mathcal{M}_{(G_{\mathbb{Q}_p}, b, \mu)}$, and add level- p^∞ structures to its moduli problem (c.f. [CS17, Defn. 4.2.3.]), denoted $\mathcal{M}_{(G_{\mathbb{Q}_p}, b, \mu), \infty}$. This functor is representable by a preperfectoid adic space over $\mathrm{Spd}(\check{E}(\zeta_{p^\infty}), \mathcal{O}_{\check{E}(\zeta_{p^\infty})})$ (c.f. [CS17, Thm. 4.2.4.]). It admits a local Hodge-Tate period morphism on diamonds

$$\pi_{\mathrm{HT}}^b : \mathcal{M}_{(G_{\mathbb{Q}_p}, b, \mu), \infty} \rightarrow \mathcal{F}_{G, \mu, C}.$$

Furthermore, $\mathcal{F}_{G, \mu, C}$ admits a Newton stratification

$$\mathcal{F}_{G, \mu, C} = \coprod_{b' \in B(G_{\mathbb{Q}_p}, \mu^{-1})} \mathcal{F}_{G, \mu, C}^{b'}$$

just as in the case of the Shimura variety $\mathcal{S}_K$, where each strata $\mathcal{F}_{G_{\mathbb{Q}_p}, \mu, C}^{b'}$ is locally closed in $\mathcal{F}_{G_{\mathbb{Q}_p}, \mu, C}$. This is constructed in [CS17, Sect. 3], using a map from the flag variety to the stack of vector bundles over the Fargues-Fontaine curve. By [CS17, Prop. 4.2.6.], π_{HT}^b actually factors through $\mathcal{F}_{G_{\mathbb{Q}_p}, \mu, C}^b$. Finally, denote by $\mathbf{S}_{K^p}^\circ \subset \mathbf{S}_{K^p}$ the good reduction locus, and let $\mathbf{S}_{K^p, C}^{\mathrm{ob}}$ be the inverse image of the Newton strata in $\mathcal{S}_{K, \overline{\mathbb{F}}_p}$ along the specialization maps.

We can now state Mantovan's product formula.

Proposition 4.2.1. *[CS17, Lemma 4.3.20.] There is an isomorphism of diamonds*

$$(\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}_b})_{\eta}^{\mathrm{ad}} \times_{\mathrm{Spa}(C, \mathcal{O}_C)} \mathcal{M}_{(G, b, \mu), \infty} \cong \mathbf{S}_{K^p, C}^{\mathrm{ob}} \times_{\mathcal{F}_{G, \mu, C}^b} \mathcal{M}_{(G, b, \mu), \infty}$$

This result is quite important, as it allows us to obtain a similar product formula result on the cohomology of these spaces, thus allowing us to relate the cohomology of the Shimura variety to that of the Igusa variety and the corresponding Rapoport-Zink space, as we will see in proceeding sections.

4.3 Hodge-Tate Period Morphism on compactifications of Shimura Varieties of PEL-type

The question is now whether we can extend the Hodge-Tate period morphism 4.1 to the minimal and toroidal compactifications in our setup. Namely, we fix an integral PEL-datum $(\mathcal{O}_B, *, L, \langle \cdot, \cdot \rangle, h)$,

of type A or C as in the previous chapters, with its associated Shimura datum (G, X) of PEL-type, and consider its associated (normalized) Shimura variety at principal level $K = K(p^m N)$ for some good prime p , as in section 2.5, denoted $\mathcal{S}_K$, as usual. We consider the perfectoid space $\mathbf{S}_{K(p^\infty N)}$ obtained from $\mathbf{S}_K$ as in theorem 4.1.1. We also need to define compactifications at p^∞ -level.

Definition 4.3.1. Fix a cone decomposition Σ . Let $\mathcal{S}_{K(p^m N), \Sigma}^{\text{tor}}$ and $\mathcal{S}_{K(p^m N)}^*$ denote the toroidal and minimal compactifications of $\mathcal{S}_{K(p^m N)}$, as defined in sections 2.5.1 and 2.5.2 respectively. Consider furthermore the associated adic spaces over E_p , which we denote by $\mathcal{S}_{K(p^m N), \Sigma, E_p}^{\text{tor}}$ and $\mathcal{S}_{K(p^m N), E_p}^*$ respectively. We define the following diamonds over $\text{Spd}(E_p)$

$$\mathbf{S}_{K(p^\infty N)}^{\text{tor}} := \varprojlim_m \mathcal{S}_{K(p^m N), \Sigma, E_p}^{\text{tor}, \diamond}$$

and

$$\mathbf{S}_{K(p^\infty N)}^* := \varprojlim_m \mathcal{S}_{K(p^m N), E_p}^{*, \diamond}.$$

Just as in [CS19, Thm. 2.6.2.], one can use [BS19, Thm. 1.17.(1)] and [PS16, Thm. 0.4.] to conclude the following. A more detailed argument in the case of toroidal compactifications can also be found in [Lan22, Prop. 6.1.].

Theorem 4.3.2. *There exists a cofinal system of choices of cone decompositions such that for any cone decomposition Σ in this system the diamonds $\mathbf{S}_{K(p^\infty N)}^{\text{tor}}$ and $\mathbf{S}_{K(p^\infty N)}^*$ are both representable by perfectoid spaces.*

We get the following generalization of the second part of theorem 4.1.1, and of [CS19, Thm. 2.7.1.].

Theorem 4.3.3. *The Hodge-Tate period morphism*

$$\pi_{\text{HT}} : \mathbf{S}_{K(p^\infty N)} \rightarrow \mathcal{F}_{G, \mu},$$

extends to compactifications. Namely, there exists a morphism

$$\pi_{\text{HT}}^* : \mathbf{S}_{K(p^\infty N)}^* \rightarrow \mathcal{F}_{G, \mu}$$

fitting in the diagram

$$\begin{array}{ccc} \mathbf{S}_{K(p^\infty N)}^\circ & \longrightarrow & \mathbf{S}_{K(p^\infty N)}^* \\ & \searrow & \downarrow \\ & & \mathcal{F}_{G, \mu}. \end{array}$$

where $\mathbf{S}_{K(p^\infty N)}^\circ := \varprojlim_m \mathcal{S}_{K(p^m N), E_p}^{\circ, \diamond}$ (recall that $\mathcal{S}_{K(p^m N), E_p}^\circ \subset \mathcal{S}_{K(p^m N), E_p}$ denotes the good reduction locus). This induces, by projection, a map

$$\pi_{\text{HT}}^{\text{tor}} : \mathbf{S}_{K(p^\infty N)}^{\text{tor}} \rightarrow \mathcal{F}_{G, \mu}.$$

Proof. By [Sch15, Theorem 4.1.1.], the morphism $\pi_{\text{HT}} : \mathbf{S}_{K^p} \rightarrow \mathcal{F}_{\tilde{G}, \tilde{\mu}}$ in 4.1 factors through the minimal compactification. The argument in [CGH+20, Thm. 3.3.1.] carries on to our setup, implying the result.

4.3.1 Hodge-Tate period morphism on boundary strata

Now that we have established that the Hodge-Tate period morphism extends to toroidal and minimal compactifications, let us try to describe how it should look like on boundary strata for the toroidal compactification. This will be particularly important to understand how to describe the map on points, by using the local description of toroidal compactifications. This way, we will be able to relate its fibers to toroidal compactifications of (perfect) Igusa varieties. This section is a generalizations of [CS19, Sect. 4.4.].

For simplicity of notation, let us assume in this section that $E_p = \mathbb{Q}_p$. This is enough for the proof of our main theorem, but the whole section should easily work in more generality. Let furthermore (Z_N, Φ_N, δ_N) denote a level- N cusp label, for N coprime to p . Our main goal is to construct a map

$$\pi_{\text{HT}, Z(\Phi_N, \delta_N)} : ((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})_{Z, \mathbb{Q}_p}^\wedge)^\diamond \rightarrow \mathcal{F}_{G, \mu}$$

from a certain open subdiamond $((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})_{Z, \mathbb{Q}_p}^\wedge)^\diamond$ of $\mathbf{S}_{K(p^\infty N), \Sigma}^{\text{tor}}$ to be defined bellow, whose points should be exactly those that specialize into the cusp (Z_N, Φ_N, δ_N) . In order to use $\pi_{\text{HT}, Z(\Phi_N, \delta_N)}$ to study the fibers of $\pi_{\text{HT}}^{\text{tor}}$, it should agree with the composition

$$((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})_{Z, \mathbb{Q}_p}^\wedge)^\diamond \hookrightarrow (\mathcal{S}_{K(p^\infty N), \Sigma, \mathbb{Q}_p}^{\text{tor}})^\diamond \xrightarrow{\pi_{\text{HT}}^{\text{tor}}} \mathcal{F}_{G, \mu},$$

and we want be able to describe $\pi_{\text{HT}, Z(\Phi_N, \delta_N)}$ in terms of degeneration data.

Let us then start by defining the adic space $((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})_{Z, \mathbb{Q}_p}^\wedge)$ over $\text{Spa}(\mathbb{Q}_p)$. Consider the formal integral model

$$\mathcal{S}_{K(p^m N), \Sigma, \mathbb{Z}_p}^{\text{tor}}$$

of the adic space $\mathcal{S}_{K(p^m N), \Sigma, \mathbb{Q}_p}^{\text{tor}}$ over $\text{Spa}(\mathbb{Q}_p)$, as constructed in theorem 2.5.3, and consider the inverse limit

$$\mathcal{S}_{K(p^\infty N), \Sigma, \mathbb{Z}_p}^{\text{tor}} = \varprojlim_m \mathcal{S}_{K(p^m N), \Sigma, \mathbb{Z}_p}^{\text{tor}}$$

which we think of as a p -adic formal scheme. Recall that by definition, (Z_N, Φ_N, δ_N) is an equivalence class of tuples $(Z_N, (X, Y, \phi, \varphi_{-2, N}, \varphi_{0, N}), \delta_N)$, where Z_N is a symplectic filtration of L/NL , X and Y are $\mathcal{O}_B$ -lattices of the same rank r , $\phi : Y \rightarrow X$ is an injective homomorphism with finite cokernel, $(\varphi_{-2, N}, \varphi_{0, N})$ are certain isomorphisms relating the graded pieces of our filtration with X and Y , and δ_N is a splitting of the filtration Z_N . Consider the formal scheme

$$(\mathcal{S}_{K(N), \Sigma, Z(\Phi_N, \delta_N)}^{\text{tor}})^\wedge,$$

from theorem 2.3.16, and define

$$((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})_{Z(\Phi_N, \delta_N), \mathbb{Z}_p}^\wedge) := \mathcal{S}_{K(p^\infty N), \Sigma, \mathbb{Z}_p}^{\text{tor}} \times_{\mathcal{S}_{K(N), \Sigma, \mathbb{Z}_p}^{\text{tor}}} (\mathcal{S}_{K(N), \Sigma, Z(\Phi_N, \delta_N)}^{\text{tor}})^\wedge,$$

with the product being taken in the category of formal schemes.

To each such cusp label $[(Z_N, \Phi_N, \delta_N)]$ we have an associated parabolic subgroup $P_{[(Z_N, \Phi_N, \delta_N)]}$ of $G_{\mathbb{Q}_p}$, depending uniquely on the $\mathcal{O}_B$ -rank, r , of the lattices X and Y . In particular, the quotient $(G_{\mathbb{Q}_p}/P_{[(Z_N, \Phi_N, \delta_N)]})(\mathbb{Q}_p)$ parametrizes $\mathcal{O}_B$ -linear filtrations

$$Z_{p^\infty, -2} \subset Z_{p^\infty, -1} \subset L \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

where the first piece of the filtration $Z_{p^\infty, -2}$ has $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -rank r . This yields a natural map

$$\pi_1 : ((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^{\wedge}_{Z_{(\Phi_N, \delta_N), \mathbb{Z}_p}}) \rightarrow (G/P_{[(Z_N, \Phi_N, \delta_N)]})(\mathbb{Q}_p),$$

by remembering only the filtration given by the level p -structures. This allows us to conclude that, just as in [CS19, Sect. 4.2.], there is a natural system of $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -lattices $L_{[(Z_N, \Phi_N, \delta_N)]}$ over $((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^{\wedge}_{Z_{(\Phi_N, \delta_N), \mathbb{Z}_p}})$, given by the first graded piece $\text{Gr}_{-1}^{Z_{p^\infty}}$ of these filtrations, with a perfect alternating form. There is also a map

$$((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^{\wedge}_{Z_{(\Phi_N, \delta_N), \mathbb{Z}_p}}) \rightarrow \mathcal{S}_{K(N), \mathbb{Z}_p}^{Z_N},$$

where we recall that $\mathcal{S}_{K(N), \mathbb{Z}_p}^{Z_N}$ denotes the smaller Shimura variety associated to Z_N from definition 2.3.8. This map is just the projection map one obtains by remembering only the base (i.e. the abelian part) of the Raynaud extension A with its extra structures, which include, due to the p^∞ -levels, isomorphisms

$$\text{Gr}_{-1}^{Z_{p^\infty}} \xrightarrow{\sim} A[p^\infty],$$

over the generic fiber, or, more explicitly, compatible maps

$$\text{Gr}_{-1}^{Z_{p^\infty}}/p^m \rightarrow A[p^m]$$

for all m . These two maps immediately allow us to define the desired map: recall that our goal was to have a map

$$\pi_{\text{HT}, Z_{(\Phi_N, \delta_N)}} : ((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^{\wedge}_{Z_{(\Phi_N, \delta_N), \mathbb{Q}_p}})^{\diamond} \rightarrow \mathcal{F}_{G, \mu}.$$

Consider the generic fiber

$$((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^{\wedge}_{Z_{(\Phi_N, \delta_N), \mathbb{Q}_p}}) := ((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^{\wedge}_{Z_{(\Phi_N, \delta_N), \mathbb{Z}_p}}) \times_{\text{Spa}(\mathbb{Z}_p)} \text{Spa}(\mathbb{Q}_p)$$

of $((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^{\wedge}_{Z_{(\Phi_N, \delta_N), \mathbb{Z}_p}})$, and its associated diamond

$$((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^{\wedge}_{Z_{(\Phi_N, \delta_N), \mathbb{Q}_p}})^{\diamond}.$$

It is an open subspace of $\mathbf{S}_{K(p^\infty N), \Sigma}^{\text{tor}}$, with C -valued points being exactly those that specialize to the cusp $[(Z_N, \Phi_N, \delta_N)]$, by definition. Consider the Hodge-Tate filtration on the base of the Raynaud extension A : we obtain a totally isotropic subspace

$$\text{Lie}(A(1)) \otimes \mathcal{O}_{((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^{\wedge}_{Z_{(\Phi_N, \delta_N), \mathbb{Q}_p}})^{\diamond}} \hookrightarrow \text{Gr}_{-1}^{Z_{p^\infty}} \otimes \mathcal{O}_{((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^{\wedge}_{Z_{(\Phi_N, \delta_N), \mathbb{Q}_p}})^{\diamond}},$$

and pulling this back to $Z_{p^\infty, -1} \otimes \mathcal{O}_{((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^\wedge_{Z_{(\Phi_N, \delta_N), \mathbb{Q}_p}})^\diamond}$, yields a totally isotropic subspace of

$$L \otimes \mathcal{O}_{((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^\wedge_{Z_{(\Phi_N, \delta_N), \mathbb{Q}_p}})^\diamond},$$

defining the desired morphism $\pi_{\text{HT}, Z_{(\Phi_N, \delta_N)}}$. This map agrees with the composition

$$((\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}})^\wedge_{Z, \mathbb{Q}_p})^\diamond \hookrightarrow (\mathcal{S}_{K(p^\infty N), \Sigma, \mathbb{Q}_p}^{\text{tor}})^\diamond \xrightarrow{\pi_{\text{HT}}^{\text{tor}}} \mathcal{F}_{G, \mu},$$

by the same argument as in [CS19, Thm. 4.2.1.] (one just argues by continuity).

4.3.2 Construction of a map over $\mathcal{O}_C$ from Igusa variety to Shimura variety with p^∞ -level structures

In order to prove our main theorem on the description of the fibers of the Hodge-Tate period morphism on compactifications, we need to understand exactly how to compare certain compactifications of Igusa varieties over some algebraically closed, non-archimedean field over $\mathbb{Q}_p$, which we will from now fix and denote by C , with fibers of the compactified Hodge Tate period morphism. Recall that our Igusa varieties are defined depending on a certain choice of a p -divisible group $\mathbb{X}_b$ with extra structures. We have the following result by Scholze and Weinstein.

Theorem 4.3.4. [SW13, Thm. B] *Let C be an algebraically closed, non-archimedean field over $\mathbb{Q}_p$, and denote by $\mathcal{O}_C$ its ring of integers. There is an equivalence of categories between p -divisible groups over $\mathcal{O}_C$ and the category of pairs (T, W) consisting of a free $\mathbb{Z}_p$ -module T of finite rank, and a C -subvector space W of $T \otimes C(-1)$ (here $C(-1)$ denotes a Tate twist), via the functor sending a p -divisible group G to the pair $(T_p(G), \text{Lie}(G) \otimes C)$, where we recall that $T_p(G)$ denotes the Tate-module of G , and the fact that this pair is in the desired category comes from the Hodge-Tate sequence*

$$0 \rightarrow \text{Lie}(G) \otimes C(1) \rightarrow T_p(G) \otimes C \rightarrow \text{Lie}(G^\vee)^\vee \otimes C \rightarrow 0.$$

In particular, by the description of the flag variety in our case and this result, we see that for any point $x \in \mathcal{F}_{G, \mu}(C)$ we have an associated p -divisible group $\mathbb{X}_{\mathcal{O}_C}$ with extra structures over $\mathcal{O}_C$, together with a trivialization

$$\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

of its Tate module. Considering the base change of $\mathbb{X}_{\mathcal{O}_C}$ to the residue field k of $\mathcal{O}_C$, which we denote by $\mathbb{X}$, we can consider the perfect Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$ attached to $\mathbb{X}$, and its toroidal and minimal compactifications as usual. The problem is that, apriori, these are only defined over k .

Recall once more that since $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$ is perfect, it admits a unique lift to a p -adic formal scheme over $W(k)$, and thus to $\mathcal{O}_C$, which we already denoted by $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}}$, and that by [CS17, Lemma 4.3.10.], this parametrizes as a moduli problem over $\text{Nilp}_{\mathcal{O}_C}^{\text{op}}$, abelian varieties with extra structures in

$\mathcal{S}_{K(N)}$, together with an isomorphism $A[p^\infty] \cong \mathbb{X}_{\mathcal{O}_C} \times R$. We need to similarly define $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}, \text{tor}}$. We also need to be able to somehow extend the map

$$\mathfrak{I}\mathfrak{g}^{\mathbb{X}, \text{tor}} \rightarrow \mathcal{S}_{K(N), \Sigma, k}^{\text{tor}}$$

to $\mathcal{O}_C$. Furthermore, notice that our map is defined on level- N structures, but we really need to have a map to $\mathcal{S}_{K(p^\infty N), \Sigma, \mathcal{O}_C}^{\text{tor}}$, since $\pi_{\text{HT}}^{\text{tor}}$ is only defined from $\mathcal{S}_{K(p^\infty N), \Sigma}^{\text{tor}}$.

The fact that we are considering p -divisible groups $\mathbb{X}$ admitting a trivialization of their Tate module $\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ gives us some hope of succeeding to do this, since this trivialization together with the Igusa structures should allow us to obtain p^∞ level structures (which is exactly what happens in the open cell, as we will analyze below). Also, as is usual when working with toroidal compactifications, the idea to construct this map will be to construct maps on the boundary strata, where we use the explicit descriptions given by toroidal boundary charts, and then glue to obtain a global map. This is the same strategy as in [CS19, Section 4.3].

We will need the following technical result.

Proposition 4.3.5. *[CS19, Prop. 4.3.1.] Consider the usual natural filtration*

$$\mathbb{X}_{\mathcal{O}_C}^\mu \subset \mathbb{X}_{\mathcal{O}_C}^\circ \subset \mathbb{X}_{\mathcal{O}_C},$$

inducing a similar filtration on $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ via the trivialization $\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. This filtration is symplectic and $\mathcal{O}$ -linear, and thus admits a splitting δ , which in turn yields a splitting

$$\delta_{\mathbb{X}_{\mathcal{O}_C}} : \mathbb{X}_{\mathcal{O}_C} \simeq \mathbb{X}_{\mathcal{O}_C}^{\text{ét}} \oplus \mathbb{X}_{\mathcal{O}_C}^{(0,1)} \oplus \mathbb{X}_{\mathcal{O}_C}^\mu.$$

Let $k \rightarrow \mathcal{O}_C/p$ be a fixed section.

Then, there exists a rational number $\epsilon \in]0, 1]$, such that there is an isomorphism of p -divisible groups with extra structures

$$\rho : \mathbb{X} \times_k \mathcal{O}_C/p^\epsilon \cong \mathbb{X}_{\mathcal{O}_C} \times_{\mathcal{O}_C} \mathcal{O}_C/p^\epsilon$$

lifting the identity, so that the equality

$$\delta_{\mathbb{X}_{\mathcal{O}_C}} \times_{\mathcal{O}_C} \mathcal{O}_C/p^\epsilon = \delta_{\mathbb{X}} \times \mathcal{O}_C/p^\epsilon$$

holds.

We now want to start explaining the construction of our map. Let us then start by recalling the construction of the map

$$\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}} \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C}$$

on the open cell. We have already defined $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}}$. Its moduli description immediately gives us the desired map, since we can obtain a p^∞ -level structure by considering the composition of $\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ with T_p of the isomorphism $A[p^\infty] \cong \mathbb{X}_{\mathcal{O}_C}$. Alternatively, fix an ϵ as in the proposition above. Notice that we can base change the natural map $\mathfrak{I}\mathfrak{g}^{\mathbb{X}} \rightarrow \mathcal{S}_{K(N), \Sigma, k}$ to $\mathcal{O}_C/p^\epsilon$

via the section $k \rightarrow \mathcal{O}_C/p$. $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p}^{\mathbb{X}}$ admits an universal abelian scheme with extra structures $\mathcal{A}_{\mathcal{O}_C/p}$, which also comes with an isomorphism of p -divisible groups with extra structures

$$\mathcal{A}_{\mathcal{O}_C/p}[p^\infty] \cong \mathbb{X}_{\mathcal{O}_C} \times_{\mathcal{O}_C} \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p}^{\mathbb{X}}$$

due to the definition of the Igusa varieties. Since the deformation of an abelian scheme is determined by the deformation of the associated p -divisible group, $\mathcal{A}_{\mathcal{O}_C/p}$ lifts uniquely to $\mathcal{O}_C$ (since we already fixed a lifting $\mathbb{X}_{\mathcal{O}_C}$), and it comes with an isomorphism of p -divisible groups with extra structures

$$\mathcal{A}_{\mathcal{O}_C}[p^\infty] \cong \mathbb{X}_{\mathcal{O}_C} \times_{\mathcal{O}_C} \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}}.$$

Furthermore, once again, this isomorphism, together with the trivialization $\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ given by theorem 4.3.4, induces a p^∞ -level structure, and thus a map

$$\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}} \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C},$$

as desired.

Just as in the open cell, if we fix an ϵ and a section $k \rightarrow \mathcal{O}_C/p$ as in the proposition, we can base change our original map

$$\mathfrak{I}\mathfrak{g}^{\mathbb{X}, \text{tor}} \rightarrow \mathcal{S}_{K(N), \Sigma, k}^{\text{tor}}$$

to $\mathcal{O}_C/p$, and we obtain a map

$$\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p}^{\mathbb{X}, \text{tor}} \rightarrow \mathcal{S}_{K(N), \Sigma, \mathcal{O}_C/p}^{\text{tor}}$$

which we need to lift to $\mathcal{O}_C$ and to $\mathcal{S}_{K(p^\infty N), \Sigma, \mathcal{O}_C/p}^{\text{tor}}$. Let us finally define $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}, \text{tor}}$

Definition 4.3.6. We define $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}, \text{tor}}$ as the base change

$$\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}, \text{tor}} := W(\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p}^{\mathbb{X}, \text{tor}}) \times_{W(k)} \mathcal{O}_C,$$

where $W(\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p}^{\mathbb{X}, \text{tor}})$ is the p -adic formal scheme over $\text{Spf}(W(k))$ one obtains by taking an affine cover of $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p}^{\mathbb{X}, \text{tor}}$, applying Witt vectors and gluing.

We want to somehow use a similar strategy as in the open cell in order to lift our map on boundary strata, and in particular we will need to lift some structures such as the Raynaud extensions from $\mathcal{O}_C/p^\epsilon$ to $\mathcal{O}_C$. In order to do this, we need to use Serre-Tate theory for these semi-abelian schemes, which we shall now recall.

Theorem 4.3.7. [CS19, Thms. 2.4.1., 2.4.2.] *Let R and R' be two rings in which p is nilpotent, and let $R' \rightarrow R$ be a surjection with nilpotent kernel.*

1. Denote by $\mathbf{GSA}_R$ the category of semi-abelian schemes $\tilde{G}_R$ over R which are globally given by an extension

$$0 \rightarrow T_R \rightarrow \tilde{G}_R \rightarrow A_R \rightarrow 0$$

of an abelian scheme A_R by a split torus T_R , and whose morphisms are given by $\text{Hom}(\tilde{G}_R, \tilde{G}'_R)[\frac{1}{p}]$. Then there is an equivalence of categories between $\mathbf{GSA}_R$ and $\mathbf{GSA}_{R'}$, with functor

$$\mathbf{GSA}_{R'} \rightarrow \mathbf{GSA}_R$$

sending $\tilde{G}_{R'}$ to $\tilde{G}_{R'} \otimes_{R'} R$.

2. Consider the category of triples $(\tilde{G}_R, \mathcal{G}_{R'}, \rho)$, consisting of a semi-abelian scheme $\tilde{G}_R$ over R , which is an extension

$$0 \rightarrow T_R \rightarrow \tilde{G}_R \rightarrow A_R \rightarrow 0$$

of an abelian scheme A_R by a split torus T_R , a p -divisible group $\mathcal{G}_{R'}$ over R' and an isomorphism $\rho : \tilde{G}_R[p^\infty] \xrightarrow{\sim} \mathcal{G}_{R'} \times_{R'} R$, with morphisms given by compatible morphisms of triples. There is an equivalence of categories between the category of semi-abelian schemes $\tilde{G}_{R'}$ as above (up to isomorphism), and the category of such triples, induced by the functor sending $\tilde{G}_{R'}$ to $(\tilde{G}_{R'} \times_{R'} R, \tilde{G}_{R'}[p^\infty], \text{id})$.

We are now in good conditions to explain the proof of the following:

Theorem 4.3.8. *The morphism*

$$g : \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}} \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C}$$

extends to a morphism of p -adic formal schemes

$$g_{\mathcal{O}_C}^{\text{tor}} : \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}, \text{tor}} \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C}^{\text{tor}}.$$

As we already explained above, we will start by constructing the map on toroidal boundary charts, and then glue. In particular, we want to use the moduli description of the completions along the boundary, and lift all the structures to $\mathcal{O}_C$ as in the open cell. This will define a map without p^∞ -level structures over $\mathcal{O}_C$ on boundary strata. Then, we need to see that the trivialization coming from the Igusa variety, together with the trivialization α define a p^∞ -level structure. Finally, we glue everything together, to obtain the desired map.

Proof. We start by fixing a cusp label (Z_N, Φ_N, δ_N) , and we consider the formal scheme

$$\widehat{\mathfrak{I}\mathfrak{g}}_{Z[(\Phi_N, \delta_N, \sigma)]}^{\mathbb{X}, \text{tor}},$$

which is described by a perfect Igusa toroidal boundary chart

$$\begin{array}{ccc} \mathfrak{I}\mathfrak{g}_{Z_N, \Xi_{\Phi_N, \delta_N}}^{\mathbb{X}} & \longrightarrow & \mathcal{C}_{\Xi_{\Phi_N, \delta_N}, \text{perf}}^{\mathbb{X}} \\ \downarrow & & \downarrow \\ \mathfrak{I}\mathfrak{g}_{Z_N, C_{\Phi_N, \delta_N}}^{\mathbb{X}} & \longrightarrow & \mathcal{C}_{C_{\Phi_N, \delta_N}, \text{perf}}^{\mathbb{X}} \\ \downarrow & & \downarrow \\ \mathfrak{I}\mathfrak{g}^{\mathbb{X}}_{Z_N} & \longrightarrow & \mathcal{C}_{Z_N}^{\mathbb{X}} \end{array}$$

and that $\mathfrak{I}g_{Z_N, C_{\Phi_N, \delta_N}}^{\mathbb{X}}$ parametrizes degeneration data as for toroidal compactifications of Shimura varieties, together with degeneration perfect Igusa level structures attached to (Z_N, Φ_N, δ_N) , so isomorphisms

$$\rho : \tilde{\mathcal{G}}_{\Phi_N, \delta_N} \times_{\mathcal{C}_{C_{\Phi_N, \delta_N}}^{\mathbb{X}}} S \xrightarrow{\sim} \mathbb{X}^{\circ} \times_{\mathbb{F}_p} S,$$

compatible with polarizations on the biconnected part for any perfect $\mathcal{C}_{C_{\Phi_N, \delta_N}}^{\mathbb{X}}$ -scheme S , where we recall that $\tilde{\mathcal{G}}_{\Phi_N, \delta_N}$ denoted the connected part of the p -divisible group attached to the universal Raynaud extension over $\mathcal{C}_{C_{\Phi_N, \delta_N}}^{\mathbb{X}}$. By Lemma 3.2.32, we know that this is actually equivalent to having an $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -linear embedding

$$\varrho : \tilde{G}[p^{\infty}] \hookrightarrow \mathbb{X},$$

inducing an isomorphism on biconnected parts, with étale quotient (this just follows from the proof of the lemma). In particular, over $\mathfrak{I}g_{Z_N, C_{\Phi_N, \delta_N}}^{\mathbb{X}}$, we have a universal Raynaud extension

$$0 \rightarrow T \rightarrow \tilde{G}_{\Phi_N, \delta_N} \rightarrow A_{\Phi_N, \delta_N} \rightarrow 0,$$

and an $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -linear embedding

$$\varrho : \tilde{G}_{\Phi_N, \delta_N}[p^{\infty}] \hookrightarrow \mathbb{X},$$

with étale quotient. We base change everything to $\mathcal{O}_C/p^{\epsilon}$, using the section $k \rightarrow \mathcal{O}_C/p^{\epsilon}$ as usual. We want to lift all our datum to $\mathcal{O}_C$. Since the embedding

$$\varrho : \tilde{G}_{\Phi_N, \delta_N}[p^{\infty}]_{\mathcal{O}_C/p^{\epsilon}} \hookrightarrow \mathbb{X}_{\mathcal{O}_C/p^{\epsilon}}$$

has étale quotient, it lifts to as injection $\varrho : \tilde{G}_{\Phi_N, \delta_N}[p^{\infty}]_{\mathcal{O}_C} \hookrightarrow \mathbb{X}_{\mathcal{O}_C}$ over $\mathcal{O}_C$. We can now use theorem 4.3.7, in particular the second part, to lift the Raynaud extension

$$0 \rightarrow T_{\mathcal{O}_C/p^{\epsilon}} \rightarrow \tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C/p^{\epsilon}} \rightarrow A_{\Phi_N, \delta_N, \mathcal{O}_C/p^{\epsilon}} \rightarrow 0,$$

to $\mathcal{O}_C$. Since all the polarizations are prime-to- p , and thus induce isomorphisms of p -divisible groups on the Raynaud extension, we also get a lift of $\tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C/p^{\epsilon}}^{\vee}[p^{\infty}]$ to $\mathcal{O}_C$, and we can once more use Serre-Tate theory to lift the dual Raynaud extension to $\mathcal{O}_C$. In particular, we obtain two short-exact sequences

$$0 \rightarrow T_{\mathcal{O}_C} \rightarrow \tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C} \rightarrow A_{\Phi_N, \delta_N, \mathcal{O}_C} \rightarrow 0,$$

and

$$0 \rightarrow T_{\mathcal{O}_C}^{\vee} \rightarrow \tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C}^{\vee} \rightarrow A_{\Phi_N, \delta_N, \mathcal{O}_C}^{\vee} \rightarrow 0,$$

over $\mathfrak{I}g_{Z_N, C_{\Phi_N, \delta_N}, \mathcal{O}_C}^{\mathbb{X}}$, and an $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -linear embedding $\varrho : \tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C}[p^{\infty}] \hookrightarrow \mathbb{X}_{\mathcal{O}_C}$ satisfying the required properties. We still have to lift the torsor

$$\mathfrak{I}g_{Z_N, \Xi_{\Phi_N, \delta_N}, \mathcal{O}_C/p^{\epsilon}}^{\mathbb{X}} \rightarrow \mathfrak{I}g_{Z_N, C_{\Phi_N, \delta_N}, \mathcal{O}_C/p^{\epsilon}}^{\mathbb{X}}.$$

Recall that by proposition 3.2.33, it parametrizes symmetric lifts $\tilde{\iota} : Y[\frac{1}{p}] \rightarrow \tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C}$ of the period morphism $\iota : Y \rightarrow \tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C}$. Consider the filtration on $\mathbb{X}_{\mathcal{O}_C}$ over $\mathfrak{I}g_{Z_N, C_{\Phi_N, \delta_N}, \mathcal{O}_C/p^{\epsilon}}^{\mathbb{X}}$ given by

$$T_{\mathcal{O}_C}[p^{\infty}] \subset \tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C}[p^{\infty}] \subset \mathbb{X}_{\mathcal{O}_C},$$

and fix a splitting of it, $\delta_{\mathbb{X}_{\mathcal{O}_C}, Z_N}$, compatible with the splitting $\delta_{\mathbb{X}_{\mathcal{O}_C}}$ of $\mathbb{X}_{\mathcal{O}_C}$ we fixed in the beginning of the section, so that we have an equality

$$\delta_{\mathbb{X}_{\mathcal{O}_C}, Z_N} \times_{\mathcal{O}_C} \mathcal{O}_C/p^\epsilon = (\delta_{\mathbb{X}_{\mathcal{O}_C}, Z_N} \times_{\mathcal{O}_C} k) \times_k \mathcal{O}_C/p^\epsilon.$$

The splitting we just fixed, together with our embedding $\varrho_{\mathcal{O}_C}$, induce a splitting of the short-exact sequence

$$0 \rightarrow T_{\mathcal{O}_C}[p^\infty] \rightarrow \tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C}[p^\infty] \rightarrow A_{\Phi_N, \delta_N, \mathcal{O}_C}[p^\infty] \rightarrow 0,$$

which just as in Proposition 3.2.33 induces a splitting of the dual Raynaud extension sequence

$$0 \rightarrow T^\vee[p^\infty] \rightarrow \tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C}^\vee[p^\infty] \rightarrow A_{\Phi_N, \delta_N, \mathcal{O}_C}^\vee[p^\infty] \rightarrow 0,$$

and thus a map $\tilde{c}_{\mathcal{O}_C}^\vee : Y[\frac{1}{p}] \rightarrow A_{\Phi_N, \delta_N, \mathcal{O}_C}$. We just define $\mathfrak{I}\mathfrak{g}_{Z_N, \Xi_{\Phi_N, \delta_N}, \mathcal{O}_C}^\mathbb{X}$ as the functor parametrizing liftings

$$Y[\frac{1}{p}] \rightarrow \tilde{G}_{\Phi_N, \delta_N, \mathcal{O}_C}$$

of the map above, satisfying the symmetric condition. Just as in the proof of [CS19, Thm. 4.3.2.], this defines the desired lift.

So far, we have essentially proven that we indeed have a lift

$$\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}, \text{tor}} \rightarrow \mathcal{S}_{K(N), \mathcal{O}_C}^{\text{tor}},$$

to the Shimura variety at principal level N , up to proving that our maps on boundary charts glue. We are left to see whether we actually have a map to $\mathcal{S}_{K(p^\infty N), \mathcal{O}_C}^{\text{tor}}$, meaning that we have to analyze the level- p structures.

In order to do so, let us recall that a Igusa cusp label $\tilde{Z}_N^m$ at level $p^m N$ consists of an equivalence class of tuples

$$(Z_m^\mathbb{X}, (Z_N, X, Y, \phi, \varphi_{-2, N}, \varphi_{0, N}), \psi_m, \delta_m^\mathbb{X}, \delta_N),$$

where $((Z_N, X, Y, \phi, \varphi_{-2, N}, \varphi_{0, N}), \delta_N)$ is a usual cusp label at level N , and $Z_m^\mathbb{X}$ is a filtration on $\mathbb{X}[p^m]$, with a $\mathcal{O}_{B_{\mathbb{Q}_p}}$ -splitting $\delta_m^\mathbb{X}$ of $Z_m^\mathbb{X}$, and an isomorphism

$$\psi_m : \text{Gr}_{m,0}^\mathbb{X} \xrightarrow{\sim} Y/p^m.$$

By theorem 3.2.36, we know that $\widehat{\mathfrak{I}\mathfrak{g}}_{Z[(\Phi_N, \delta_N, \sigma)]}^{\mathbb{X}, \text{tor}}$ admits a decomposition with elements being the open and closed formal subschemes $\widehat{\mathfrak{I}\mathfrak{g}}_{\tilde{Z}_N^m, \sigma}^{\mathbb{X}, \text{tor}}$, where $\tilde{Z}_N^m$ runs through all the Igusa cusp labels above our fixed one, and that we obtain the decomposition as follows: if we have a point in $\widehat{\mathfrak{I}\mathfrak{g}}_{Z[(\Phi_N, \delta_N, \sigma)]}^{\mathbb{X}, \text{tor}}$, we just consider the induced filtration

$$0 \subset T[p^m] \subset \tilde{G}[p^m] \subset \mathbb{X}[p^m].$$

Also, notice that if we have an Igusa cusp label, we can obtain not only a cusp labels at level N (just by forgetting the extra structures), but also a cusp label at level $p^\infty N$: indeed, the

isomorphism $\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ induces an isomorphism $\mathbb{X}_{\mathcal{O}_C}[p^m](C) \cong L/p^m$ for all m , and thus the above filtration on $\mathbb{X}_{\mathcal{O}_C}[p^m]$ induces a similar one on L/p^m , giving a p^m -level structure on the cusp label, just by the assumptions on $Z_m^{\mathbb{X}}$. By abuse of notation, let $\tilde{Z}_N^m$ denote both the Igusa cusp label and the cusp label at level $p^m N$, and we fix one. We need to construct a map

$$\widehat{\mathfrak{I}\mathfrak{g}}_{\tilde{Z}_N^m, \sigma}^{\mathbb{X}, \text{tor}} \rightarrow \widehat{\mathcal{S}}_{K(p^m N), \tilde{Z}_N^m, \sigma}^{\text{tor}}$$

and we want to use the moduli description on both sides to do so. Away from p , we already know that everything works out, so we need to analyze the extra structures at p . Choose a splitting $L/p^m \cong \oplus \text{Gr}_{-i}^{Z_{p^m}}$ of the filtration we obtain on L/p^m using the trivialization α and the filtration $Z_m^{\mathbb{X}}$, and we assume it is compatible with the splitting δ chosen in proposition 4.3.5. Via the isomorphism $\mathbb{X}_{\mathcal{O}_C}[p^m](C) \cong L/p^m$, this induces a splitting of the filtration on $\mathbb{X}_{\mathcal{O}_C}[p^m]$. Since the Igusa cusp label equivalence relation identifies all splittings, we may assume that $\delta_m^{\mathbb{X}}$ is this splitting we just fixed. In particular, by definition of the filtration, this gives a splitting of the short-exact sequence

$$0 \rightarrow T[p^m] \rightarrow \tilde{G}_{\mathcal{O}_C}[p^m] \rightarrow A_{\mathcal{O}_C}[p^m] \rightarrow 0.$$

By duality, this once more induces a splitting of the dual sequence

$$0 \rightarrow T^{\vee}[p^m] \rightarrow \tilde{G}_{\mathcal{O}_C}^{\vee}[p^m] \rightarrow A_{\mathcal{O}_C}^{\vee}[p^m] \rightarrow 0.$$

We obtain lifts of the maps defining the Raynaud extension and its dual, c and $c^{\vee}$ to maps $c_m : \frac{1}{p^m} X \rightarrow A_{\mathcal{O}_C}^{\vee}$ and $c_m^{\vee} : \frac{1}{p^m} Y \rightarrow A_{\mathcal{O}_C}$. This immediately gives a map to $C_{\tilde{Z}_N^m}$ (which denotes the abelian scheme on the toroidal boundary chart defining $\widehat{\mathcal{S}}_{K(p^m N), \tilde{Z}_N^m, \sigma}^{\text{tor}}$), just by the analysis of the level structures on toroidal boundary charts. It follows from the explicit description of the torsor,

$$\mathfrak{I}\mathfrak{g}_{\tilde{Z}_N, \Xi_{\Phi_N, \delta_N}, \mathcal{O}_C}^{\mathbb{X}} \rightarrow \mathfrak{I}\mathfrak{g}_{\tilde{Z}_N, C_{\Phi_N, \delta_N}, \mathcal{O}_C}^{\mathbb{X}},$$

as the moduli space parametrizing symmetric lifts $\frac{1}{p} Y \rightarrow \tilde{G}_{\mathcal{O}_C}$ that we saw above, we obtain a map from $\mathfrak{I}\mathfrak{g}_{\tilde{Z}_N, \Xi_{\Phi_N, \delta_N}, \mathcal{O}_C}^{\mathbb{X}}$ to the torsor of lifts of $c_m^{\vee}$ to $\tilde{G}_{\mathcal{O}_C}$ over $C_{\tilde{Z}_N^m}$, that we denote by $\Xi_{\tilde{Z}_N^m, \mathcal{O}_C}$, yielding the desired map on completions along boundary charts. Taking the inverse limit over m , gives us the map

$$\widehat{\mathfrak{I}\mathfrak{g}}_{\tilde{Z}_N^m, \sigma}^{\mathbb{X}, \text{tor}} \rightarrow \widehat{\mathcal{S}}_{K(p^{\infty} N), \tilde{Z}_N^m, \sigma}^{\text{tor}}$$

as predicted. The argument for gluing follows exactly as in [CS19, Thm. 4.3.2.], using lemma 3.3.2.

4.3.3 The Fibers of the Toroidal Compactification

We are now ready to identify the fibers of the compactified Hodge-Tate period morphism with toroidal compactifications of Igusa varieties. We will see that this is just a consequence of the results we have seen in the previous two sections, and of the (local) moduli description of the toroidal compactifications. Let us start by recalling the following lemma.

Lemma 4.3.9. *[CS19, Lemma 4.4.2.] Let C be a complete, algebraically closed extension of $\mathbb{Q}_p$. Let X be a quasicompact, quasiseparated perfectoid space over C , and denote by $\bar{X}$ its canonical compactification as defined in [Sch17, Prop. 18.6.]. Let furthermore Y be a proper diamond over $\mathrm{Spd}(C)$, and $f : X \rightarrow Y$ be a map inducing isomorphisms $X(C', \mathcal{O}_{C'}) \rightarrow Y(C', \mathcal{O}_{C'})$ for all complete, algebraically closed C' over C . Then, there is an induced isomorphism $\bar{X} \rightarrow Y$, and if X over $\mathrm{Spd}(C)$ is compactifiable, f is an open immersion.*

We can now prove the following theorem.

Theorem 4.3.10. *Let $x \in \mathcal{F}_{G,\mu}(C)$ be a point in the flag variety, and let $\mathbb{X}$ denote the associated p -divisible group. The morphism*

$$g_{\mathcal{O}_C}^{\mathrm{tor}} : \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X},\mathrm{tor}} \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C}^{\mathrm{tor}}$$

constructed in theorem 4.3.8, induces an open immersion on diamonds

$$f^{\mathrm{tor}} : \mathfrak{I}\mathfrak{g}_C^{\mathbb{X},\mathrm{tor}} \hookrightarrow (\pi_{\mathrm{HT}}^{\mathrm{tor}})^{-1}(x)$$

with the same rank 1 points. In particular, $(\pi_{\mathrm{HT}}^{\mathrm{tor}})^{-1}(x)$ is the canonical compactification of $\mathfrak{I}\mathfrak{g}_C^{\mathbb{X},\mathrm{tor}}$, and thus affinoid perfectoid.

Proof. By lemma 4.3.9, one only need to check that the map induces a bijection on $(C, \mathcal{O}_C)$ -points. The second part will then follow from the fact that the target is partially proper.

Now we need to check that we indeed have a bijection on $(C, \mathcal{O}_C)$ -points. Using section 4.3.1, we see that the C -valued points on the right-hand side decompose into Igusa cusp labels $\tilde{Z}_N$, and we have already seen that the same holds true for the left-hand side. The map above is compatible with this decomposition, just by construction. In particular, we can fix such an Igusa cusp label (at infinite level). This is given by an equivalence class of tuples, consisting of a cusp label at level N , a filtration

$$Z^{\mathbb{X}} = \{0 \subset Z_{-2}^{\mathbb{X}} \subset Z_{-1}^{\mathbb{X}} \subset Z_0^{\mathbb{X}} = \mathbb{X}\}$$

on $\mathbb{X}$, together with an isomorphism

$$\psi : \mathrm{Gr}_0^{\mathbb{X}} \xrightarrow{\sim} Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p),$$

which is by duality equivalent to an isomorphism

$$\mathrm{Gr}_{-2}^{\mathbb{X}} = Z_{-2}^{\mathbb{X}} \xrightarrow{\sim} \mathrm{Hom}(X, \mu_{p^\infty}),$$

and an $\mathcal{O}_B$ -splitting of $Z^{\mathbb{X}}$, so an isomorphism

$$\delta^{\mathbb{X}} : \mathbb{X} \cong \bigoplus_i \mathrm{Gr}_{-i}^{\mathbb{X}}.$$

Using once again the trivialization $\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$, this induces a filtration

$$0 \subset Z_{p^\infty, -2} \subset Z_{p^\infty, -1} \subset L \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

with the corresponding splitting. We will ignore the structures away from p , because as usual we already know they match on both sides.

We have already analyzed what data we are parametrizing on the left-hand side. Namely, the Igusa toroidal boundary chart tells us that we have an abelian variety A over $\mathcal{O}_C$, together with a prime-to- p polarization, an $\mathcal{O}_B$ -action, and a level- N structure, belonging to a smaller Shimura variety, and fitting into the Raynaud extension

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow A \rightarrow 0$$

where T is a torus with character group X given in the Igusa cusp label. Furthermore, we have a dual Raynaud extension

$$0 \rightarrow T^\vee \rightarrow \tilde{G}^\vee \rightarrow A^\vee \rightarrow 0,$$

where $A^\vee$ is the dual abelian variety of A , and $T^\vee$ is the torus with character group Y , also given by the Igusa cusp label. These extensions are defined by maps $c : X \rightarrow A^\vee$ and $c^\vee : Y \rightarrow A$ respectively. Additionally, the Igusa structures provide us with an isomorphism

$$\tilde{G}[p^\infty] \cong Z_{-1}^\mathbb{X},$$

compatible with the isomorphism $Z_{-2}^\mathbb{X} \cong \text{Hom}(X, \mu_{p^\infty})$ coming from the Igusa cusp label, which induces, together with $\delta^\mathbb{X}$, a splitting of the short-exact sequence

$$0 \rightarrow T[p^\infty] \rightarrow \tilde{G}[p^\infty] \rightarrow A[p^\infty] \rightarrow 0.$$

Using that the polarizations on A are prime-to- p , this induces a splitting of the dual sequence

$$0 \rightarrow T^\vee[p^\infty] \rightarrow \tilde{G}^\vee[p^\infty] \rightarrow A^\vee[p^\infty] \rightarrow 0,$$

and thus liftings $\tilde{c} : X[\frac{1}{p}] \rightarrow A^\vee$ and $\tilde{c}^\vee : Y[\frac{1}{p}] \rightarrow A$ of the maps c and $c^\vee$ respectively. Finally, the data given by the torus torsors was already analyzed in previous proofs, such as in the proof of theorem 4.3.8, and we already know it matches the data parametrized by torus torsors in the Shimura variety.

Let us now analyze the data we are parametrizing on the right-hand side. We already saw in the proof of theorem 4.3.8 that we have a map from Igusa cusp labels to cusp labels at level $p^\infty N$, and thus we can use this to analyze toroidal boundary charts of the Shimura varieties using Igusa cusp labels. In particular, on the right-hand side we are also parametrizing abelian varieties A over $\mathcal{O}_C$, together with a prime-to- p polarization, an $\mathcal{O}_B$ -action, and a level- N structure. By section 4.3.1, the fact that we are trying to describe a point in $(\pi_{\text{HT}, \tilde{Z}_N})^{-1}(x)$, implies that A comes equipped with a symplectic isomorphism

$$T_p(A) \cong \text{Gr}_{-1}^{Z_{p^\infty}},$$

which in turn induces an isomorphism

$$A[p^\infty] \cong \text{Gr}_{-1}^\mathbb{X}.$$

Furthermore, A fits into a Raynaud extension

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow A \rightarrow 0$$

where T is a torus with character group X coming from the Igusa cusp label, and once more we have a dual Raynaud extension

$$0 \rightarrow T^\vee \rightarrow \tilde{G}^\vee \rightarrow A^\vee \rightarrow 0$$

where $A^\vee$ is the dual abelian variety of A , and $T^\vee$ is the torus with character group Y , also given by the Igusa cusp label. Once more, these extensions give maps $c : X \rightarrow A^\vee$ and $c^\vee : Y \rightarrow A$ respectively. We also have a splitting of the sequences

$$0 \rightarrow T[p^\infty] \rightarrow \tilde{G}[p^\infty] \rightarrow A[p^\infty] \rightarrow 0,$$

and

$$0 \rightarrow T^\vee[p^\infty] \rightarrow \tilde{G}^\vee[p^\infty] \rightarrow A^\vee[p^\infty] \rightarrow 0,$$

coming from the level- p^∞ structure, and hence once more extensions $\tilde{c} : X[\frac{1}{p}] \rightarrow A^\vee$ and $\tilde{c}^\vee : Y[\frac{1}{p}] \rightarrow A$ of c and $c^\vee$. Finally the torus torsors parametrize the usual data, and we have no extra conditions coming from the fact that we are considering the pre-image of the Hodge-Tate period morphism.

The result now follows almost trivially. Just note that the isomorphism $A[p^\infty] \cong \text{Gr}_{-1}^{\mathbb{X}}$ on the first part of the description of the right-hand side together with the splitting of

$$0 \rightarrow T[p^\infty] \rightarrow \tilde{G}[p^\infty] \rightarrow A[p^\infty] \rightarrow 0,$$

corresponds exactly to the isomorphism

$$\tilde{G}[p^\infty] \cong Z_{b,-1}$$

on the left-hand side.

We also obtain the following corollary.

Corollary 4.3.11. *There is a Hecke-equivariant isomorphism*

$$R\Gamma(\mathfrak{Jg}^{\mathbb{X},\text{tor}}, \mathbb{F}_l) \cong (R(\pi_{\text{HT}}^{\text{tor}})_* \mathbb{F}_l)_x,$$

and the stalks at higher rank points agree with the corresponding rank 1 stacks.

Proof. By [CS17, Lemma 4.4.1.], we see that there is an equality

$$(R^i(\pi_{\text{HT}}^{\text{tor}})_* \mathbb{F}_l)_x = H^i((\pi_{\text{HT}}^{\text{tor}})^{-1}(x), \mathbb{F}_l),$$

for any point $x \in \mathcal{F}_{G,\mu}(C)$. Then, theorem 4.3.10 implies the result, c.f. [CS17, Sect. 4.4.] for more details.

4.3.4 The Fibers of the Minimal Compactification

Here, we just want to record the following result.

Theorem 4.3.12. *Let $x \in \mathcal{F}_{G,\mu}(C)$ be a point in the flag variety. The morphisms*

$$\mathfrak{Jg}_C^{\mathbb{X},\text{tor}} \rightarrow \mathfrak{Jg}_C^{\mathbb{X},*}$$

and

$$(\pi_{\text{HT}}^{\text{tor}})^{-1}(x) \rightarrow (\pi_{\text{HT}}^*)^{-1}(x)$$

induce isomorphisms on global sections. In particular, the morphism of diamonds

$$\mathfrak{Jg}_C^{\mathbb{X},*} \hookrightarrow (\pi_{\text{HT}}^*)^{-1}(x)$$

induced by f^{tor} , is an open immersion of affinoid perfectoid spaces with the same rank 1 points.

Proof. Both of the statements are purely formal. The first statement follows exactly as in [CS19, Lemma 4.5.2.], using results from [Sch17] and [Sch13]. For the second part of the theorem, we notice that $\mathfrak{Jg}_C^{\mathbb{X},*}$ is affinoid perfectoid by lemma 3.3.7, and that the same holds true for $(\pi_{\text{HT}}^*)^{-1}(x)$ by [BS19, Thm. 1.17.]. The statement then follows from the first part after taking global sections.

Corollary 4.3.13. *There is a Hecke-equivariant isomorphism*

$$R\Gamma(\mathfrak{Jg}_C^{\mathbb{X},*}, \mathbb{F}_l) \cong (R\Gamma(\pi_{\text{HT}}^*)_* \mathbb{F}_l)_x,$$

and the stalks at higher rank points agree with the corresponding rank 1 stalks.

Proof. This follows as in corollary 4.3.11.

Chapter 5

Vanishing of the cohomology of non-compact PEL-type A Shimura varieties

5.1 Semiperversity

We finally want to take advantage of all the geometric results on compactifications and the description of the fibers of the Hodge-Tate period morphism, to prove the following semiperversity result, which is a generalization of [CS19, Thm. 4.6.1.].

Theorem 5.1.1. *Let C be a complete, non-archimedean, algebraically closed extension of $\mathbb{Q}_p$, and denote by $\mathcal{O}_C$, resp. k , its ring of integers, resp. its residue field. Let l be a prime different from p . Recall that we denoted by d the dimension of the Shimura variety we are considering. Consider the complex*

$$R\pi_{\mathrm{HT}*}^{\circ} \mathbb{F}_l \in D(\mathcal{F}_{G,\mu,C}, \mathbb{F}_l),$$

and denote by $R\Psi(R\pi_{\mathrm{HT}}^{\circ} \mathbb{F}_l)$ its nearby cycles. Let $\bar{x}$ be a geometric point of $\mathcal{F}_{G,\mu,C}$. Then, $\bar{x}$ admits a cofinal system of affinoid étale neighborhoods $U = \mathrm{Spa}(A) \rightarrow \mathcal{F}_{G,\mu,C}$, with associated formal scheme $\mathcal{U} = \mathrm{Spd}(A^{\circ})$, such that the nearby cycles $R\Psi(R\pi_{\mathrm{HT}*}^{\circ} \mathbb{F}_l)|_U$ are semiperverse, ie*

$$R\Psi(R\pi_{\mathrm{HT}*}^{\circ} \mathbb{F}_l)|_U \in {}^p D^{\geq d}(\mathcal{U}_k, \mathbb{F}_l).$$

The idea to prove this is the same as in [CS19], namely, we use a result by Pink to conclude that at l^{∞} -level, the cohomology of the Shimura varieties we are considering and that of its toroidal compactifications agree. This is quite helpful, since $\pi_{\mathrm{HT}}^{\circ}$ is not proper in our non-compact case. One then uses a series of reduction steps to conclude that the resulting complex is something we already know is semiperverse, yielding the result.

Pink's Lemma

We want to start by recalling Pink's Lemma, which was proven by Pink in section 2 of [Pink92]. We also explain its proof.

1. To start, let X be an abelian variety or a torus over some separably closed field K , and let l be a rational prime, invertible in K .

Proposition 5.1.2. *[Pink92, 2.1.] The adjunction morphism $\mathbb{Z}/l^m\mathbb{Z} \rightarrow H^0(X, \mathbb{Z}/l^m\mathbb{Z})$ is an isomorphism, and the multiplication-by- n map, denoted by $[n]$, induces the identity on $H^0(X, \mathbb{Z}/l^m\mathbb{Z})$. Furthermore, for all $i \neq 0$, $[l^m]$ induces the zero morphism on $H^i(X, \mathbb{Z}/l^m\mathbb{Z})$.*

Proof. The first part follows from connectivity of X . For the second one, just recall that $[n]$ induces the multiplication-by- n map on $H^1(X, \mathbb{Z}/l^m\mathbb{Z})$, and thus the result follows from the equality

$$H^i(X, \mathbb{Z}/l^m\mathbb{Z}) = \wedge^i H^1(X, \mathbb{Z}/l^m\mathbb{Z}).$$

2. We now consider the case of a torus embedding. Let T be a torus, and Σ a smooth rational polyhedral cone decomposition of $Y \otimes \mathbb{R}$, where Y denotes the cocharacter group of T , and $j : T \hookrightarrow T_\Sigma$ the associated torus embedding. Let $[n] : T_\Sigma \rightarrow T_\Sigma$ be the extension of the multiplication-by- n map on T . Then we can extend the result above as follows.

Proposition 5.1.3. *[Pink92, 2.2.]*

- (a) *The adjunction morphism $\mathbb{Z}/l^m\mathbb{Z} \rightarrow R^0j_*\mathbb{Z}/l^m\mathbb{Z}$ is an isomorphism. Furthermore, the base change homomorphism $[l^m]^*R^i j_*\mathbb{Z}/l^m\mathbb{Z} \rightarrow R^i j_*\mathbb{Z}/l^m\mathbb{Z}$ is an isomorphism for $i = 0$ and zero otherwise.*
- (b) *The adjunction morphism $\mathbb{Z}/l^m\mathbb{Z} \rightarrow H^0(T_\Sigma, \mathbb{Z}/l^m\mathbb{Z})$ is an isomorphism, and the multiplication-by- n map, $[n]$, induces the identity on $H^0(T_\Sigma, \mathbb{Z}/l^m\mathbb{Z})$. Furthermore, for all $i \neq 0$, $[l^m]$ induces the zero morphism on $H^i(T_\Sigma, \mathbb{Z}/l^m\mathbb{Z})$.*

Proof. (a) The statement is local, so we may assume that T_Σ is affine and that our torus embedding is of the form

$$j : \mathbb{G}_m^d \hookrightarrow \mathbb{A}^e \times \mathbb{G}_m^{d-e}$$

for some integers $0 \leq e \leq d$. By the Künneth formula, we only need to consider the case $d = 1$, and thus the case $d = 1, e = 1$ since the other case is trivial and thus we have $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$. Then, the first statement follows from smoothness of $\mathbb{A}^1$. The second statement is a consequence of the fact that the stalk of $R^i j_*\mathbb{Z}/l^m\mathbb{Z}$ at 0 is isomorphic to the continuous cohomology of the tame inertia group with coefficients in $\mathbb{Z}/l^m\mathbb{Z}$, and the base change homomorphism corresponds to the restriction to the subgroup of index l^m , which immediately implies the second part of the statement.

(b) Since T_Σ is connected, the first claim follows. The Leray spectral sequence is as follows

$$E_2^{p,q} = H^p(T_\Sigma, R^q j_* \mathbb{Z}/l^m \mathbb{Z}) \Rightarrow H^{p+q}(T, \mathbb{Z}/l^m \mathbb{Z}).$$

In particular, we only need to prove that $[l^m]^*$ is zero on $E_2^{i,0}$ for all $i > 0$. By the first part of the proposition, we already know that $[l^m]^*$ is zero on $E_2^{p,q}$ for all p , and $q > 0$, and proposition 5.1.2 applied to a torus also tells us that $[l^m]^*$ is zero on $E_\infty^{p,q}$ except for $p = q = 0$. There are exact sequences

$$E_n^{p-n,n-1} \rightarrow E_n^{p,0} \rightarrow E_{n+1}^{p,0} \rightarrow 0$$

for $n \geq 2$, since $E_n^{p,q} = 0$ if $q < 0$. The fact that $[l^m]^*$ is zero on $E_2^{p-n,n-1}$ implies that the same holds for its subquotient $E_n^{p-n,n-1}$, and hence by the exact sequence above the endomorphism induced on $E_n^{p,0}$ factors through $E_{n+1}^{p,0}$. By induction, it follows that $[l^{mp}]^*$ on $E_2^{p,0}$ factors through $E_{n+1}^{p,0}$. But since $E_{p+1}^{p,0} = E_\infty^{p,0}$ (seen as the spectral sequence lives only on the first quadrant), and as we know that $[l^m]^*$ is 0 on this term, and the result follows.

3. We now want to apply the previous results to the relative case. Let us assume we have an inverse system of smooth families of tori or abelian varieties $X_\alpha \rightarrow S$ over some scheme S , so that we have commutative diagrams

$$\begin{array}{ccccc} X_\beta & \xrightarrow{\varphi_{\alpha,\beta}} & X_\alpha & \xrightarrow{\varphi_\alpha} & X \\ & \searrow f_\beta & \downarrow f_\alpha & \swarrow f & \\ & & S & & \end{array}$$

Fix a prime l invertible in S . We will assume that all $\varphi_{\alpha,\beta}$ are isogenies, and that for all α , there exists some $\beta \succ \alpha$ such that $\varphi_{\alpha,\beta}$ factors through $[l]$. We get the following result.

Proposition 5.1.4. *[Pink92, 2.6.] Under the above setup, the homomorphism*

$$\mathbb{Z}/l^m \mathbb{Z} \rightarrow \varinjlim_\alpha Rf_{\alpha*} \mathbb{Z}/l^m \mathbb{Z} \rightarrow Rf_* \varinjlim_\alpha \varphi_{\alpha*} \mathbb{Z}/l^m \mathbb{Z}$$

is an isomorphism.

Proof. The second map to $Rf_* \varinjlim_\alpha \varphi_{\alpha*} \mathbb{Z}/l^m \mathbb{Z}$ is an isomorphism since f is of finite type. In particular, we only need to prove that

$$\mathbb{Z}/l^m \mathbb{Z} \rightarrow \varinjlim_\alpha Rf_{\alpha*} \mathbb{Z}/l^m \mathbb{Z}$$

is an isomorphism. By proper base change in the case X is an abelian scheme resp. [Pink92, 2.6.6.] in the case X is a torus, every stalk of $Rf_{\alpha*} \mathbb{Z}/l^m \mathbb{Z}$ is isomorphic to the cohomology of the corresponding geometric fiber of f_α . The result now follows easily: the map $\mathbb{Z}/l^m \mathbb{Z} \rightarrow R^0 f_{\alpha*} \mathbb{Z}/l^m \mathbb{Z}$ is an isomorphism by proposition 5.1.2. Furthermore, fixing

α and choosing a β such that $\varphi_{\alpha,\beta}$ factors through $[l]$, we know again by proposition 5.1.2 that the map

$$R^p f_{\alpha*} \mathbb{Z}/l^m \mathbb{Z} \rightarrow R^p f_{\beta*} \mathbb{Z}/l^m \mathbb{Z}$$

is an isomorphism for $p = 0$ and zero otherwise, yielding the result.

4. Finally, let us assume that an inverse system as above is embedded in a compatible inverse system:

$$\begin{array}{ccccc} X_\beta & \xrightarrow{\varphi_{\alpha,\beta}} & X_\alpha & \xrightarrow{\varphi_\alpha} & X \\ \downarrow j_\beta & & \downarrow j_\alpha & & \downarrow j \\ \bar{X}_\beta & \xrightarrow{\bar{\varphi}_{\alpha,\beta}} & \bar{X}_\alpha & \xrightarrow{\bar{\varphi}_\alpha} & \bar{X} \\ & \searrow \bar{f}_\beta & \downarrow \bar{f}_\alpha & \swarrow \bar{f} & \\ & & S & & \end{array}$$

such that all horizontal maps are finite flat. We furthermore assume the following:

- $\bar{f}_\alpha$ are smooth;
- j_α are open embeddings, and that the complements are divisors with normal crossings over S (this tells us that locally for the étale topology, these look like maps $\mathbb{G}_m^d \hookrightarrow \mathbb{A}_1^d$);
- for all α , and every irreducible component of the complement $\bar{X}_\alpha \setminus X_\alpha$, there exists some $\beta \succ \alpha$ such that the ramification degree of $\bar{\varphi}_{\alpha,\beta}$ in the component is divisible by l .

Proposition 5.1.5. *[Pink92, 2.7.] Under the above setup, the homomorphism*

$$\varinjlim_\alpha \bar{\varphi}_{\alpha,*} \mathbb{Z}/l^m \mathbb{Z} \rightarrow \varinjlim_\alpha \bar{\varphi}_{\alpha,*} Rj_{\alpha*} \mathbb{Z}/l^m \mathbb{Z} \cong Rj_* \varinjlim_\alpha \varphi_{\alpha*} \mathbb{Z}/l^m \mathbb{Z}$$

is an isomorphism.

Proof. The proof goes exactly as in the previous case. We need to prove that

$$\varinjlim_\alpha \bar{\varphi}_{\alpha,*} \mathbb{Z}/l^m \mathbb{Z} \rightarrow \varinjlim_\alpha \bar{\varphi}_{\alpha,*} Rj_{\alpha*} \mathbb{Z}/l^m \mathbb{Z}$$

is an isomorphism. By the same base change result we used above for tori, every stalk of $Rj_{\alpha*} \mathbb{Z}/l^m \mathbb{Z}$ is isomorphic to the cohomology of the corresponding geometric fiber of j_α . The result then follows by proposition 5.1.3 by the same argument as above.

Now we just apply the above result to our situation.

Lemma 5.1.6. *Let $l \neq p$ a prime, and consider the Shimura variety $\mathcal{S}_{K(Nl^\infty), \mathbb{Q}}$ resp. one of its toroidal compactifications $\mathcal{S}_{K(Nl^\infty), \mathbb{Q}}^{\text{tor}}$ with l^∞ -level structures over the rationals, ie the ones we*

obtain taking the inverse limit of $\mathcal{S}_{K(Nl^m),\mathbb{Q}}$ resp. $\mathcal{S}_{K(Nl^m),\mathbb{Q}}^{\text{tor}}$ for varying m , in the category of schemes. Let

$$j_{Nl^\infty} : \mathcal{S}_{K(Nl^\infty),\mathbb{Q}} \rightarrow \mathcal{S}_{K(Nl^\infty),\mathbb{Q}}^{\text{tor}}$$

be the inclusion of the open cell. The natural map

$$\mathbb{F}_l \rightarrow Rj_{Nl^\infty,*}\mathbb{F}_l$$

is an isomorphism. In particular,

$$R\Gamma(\mathcal{S}_{K(Nl^\infty),\mathbb{Q}}^{\text{tor}}, \mathbb{F}_l) \cong R\Gamma(\mathcal{S}_{K(Nl^\infty),\mathbb{Q}}, \mathbb{F}_l).$$

Proof. The second part follows immediately from the first one, since

$$R\Gamma(\mathcal{S}_{K(Nl^\infty),\mathbb{Q}}^{\text{tor}}, \mathbb{F}_l) \cong R\Gamma(\mathcal{S}_{K(Nl^\infty),\mathbb{Q}}^{\text{tor}}, Rj_{Nl^\infty,*}\mathbb{F}_l) \cong R\Gamma(\mathcal{S}_{K(Nl^\infty),\mathbb{Q}}, \mathbb{F}_l).$$

For the first part, we just notice that Pink's Lemma (namely, proposition 5.1.5) can be applied to our case. In particular, we consider the inverse system

$$\begin{array}{ccccc} \mathcal{S}_{K(Nl^m),\mathbb{Q}} & \xrightarrow{\varphi_{n,m}} & \mathcal{S}_{K(Nl^n),\mathbb{Q}} & \xrightarrow{\varphi_n} & \mathcal{S}_{K(N),\mathbb{Q}} \\ \downarrow j_m & & \downarrow j_n & & \downarrow j \\ \mathcal{S}_{K(Nl^m),\mathbb{Q}}^{\text{tor}} & \xrightarrow{\bar{\varphi}_{n,m}} & \mathcal{S}_{K(Nl^n),\mathbb{Q}}^{\text{tor}} & \xrightarrow{\bar{\varphi}_n} & \mathcal{S}_{K(N),\mathbb{Q}}^{\text{tor}} \\ & \searrow \bar{f}_m & \downarrow \bar{f}_n & \swarrow \bar{f} & \\ & & \text{Spec}(\mathbb{Q}) & & \end{array}$$

for $m > n$. Then proposition 5.1.5 exactly tells us that $\mathbb{F}_l \rightarrow Rj_{Nl^\infty,*}\mathbb{F}_l$ is an isomorphism.

Similarly, we can prove the following.

Lemma 5.1.7. *Consider the perfect Igusa variety $\mathfrak{I}\mathfrak{g}_{K(Nl^\infty)}^{\mathbb{X}}$ at Nl^∞ -level. Then the restriction map*

$$H^i(\mathfrak{I}\mathfrak{g}_{K(Nl^\infty),\mathbb{Q}}^{\mathbb{X},\text{tor}}, \mathbb{F}_l) \rightarrow H^i(\mathfrak{I}\mathfrak{g}_{K(Nl^\infty),\mathbb{Q}}^{\mathbb{X}}, \mathbb{F}_l)$$

is an isomorphism.

Proof. By proposition 3.3.1, we may assume that $\mathbb{X}$ is completely slope divisible. The statement is clear for the non-perfect Igusa varieties at finite level by theorem 3.2.22 (see also remark 3.2.24), and thus the result follows.

Proof of semiperversity

Before starting the proof, we will need the following result.

Theorem 5.1.8. *[CS17, Thm. 4.4.4.], [CS19, Thm. 2.7.2.] Assume that p is unramified. Let $x \in \mathcal{F}_{G,\mu}(C)$ be a point in the flag variety, corresponding to a p -divisible group $\mathbb{X}_{\mathcal{O}_C}$ by theorem*

4.3.4, and let $\mathbb{X}$ denote its special fiber. Let $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$ be the associated perfect Igusa variety, $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}}$ its lift to $\mathcal{O}_C$, and $\mathfrak{I}\mathfrak{g}_C^{\mathbb{X}}$ its adic generic fiber. Then, there is an open immersion

$$\mathfrak{I}\mathfrak{g}_C^{\mathbb{X}} \hookrightarrow (\pi_{\mathrm{HT}}^{\circ})^{-1}(x)$$

with image containing all rank 1-points, inducing an isomorphism

$$(R(\pi_{\mathrm{HT}}^{\circ})_* \mathbb{F}_l)_x \cong R\Gamma(\mathfrak{I}\mathfrak{g}^{\mathbb{X}}, \mathbb{F}_l)$$

for all prime $l \neq p$.

Proof of Theorem 5.1.1 We finally give the proof of the semiperversity result. We want to prove that

$$R\Psi(R\pi_{\mathrm{HT},*}^{\circ} \mathbb{F}_l)|_U \in {}^p D^{\geq d}(\mathcal{U}_k, \mathbb{F}_l).$$

By the Hochschild-Serre spectral sequence, it actually suffices to prove that

$$R\Psi(R\pi_{\mathrm{HT},l^{\infty},*}^{\circ} \mathbb{F}_l)|_U \in {}^p D^{\geq d}(\mathcal{U}_k, \mathbb{F}_l),$$

where

$$\pi_{\mathrm{HT},l^{\infty}}^{\circ} : S_{K(p^{\infty}Nl^{\infty}),C}^{\circ} \rightarrow \mathcal{F}_{G,\mu,C}.$$

The natural map

$$R\pi_{\mathrm{HT},l^{\infty},*}^{\mathrm{tor}} \mathbb{F}_l \rightarrow R\pi_{\mathrm{HT},l^{\infty},*}^{\circ} \mathbb{F}_l$$

is an isomorphism. Indeed, by theorem 5.1.8, there is an isomorphism

$$R\pi_{\mathrm{HT},l^{\infty},*}^{\circ} \mathbb{F}_l \cong R\Gamma(\mathfrak{I}\mathfrak{g}_{K(Nl^{\infty})}^{\mathbb{X}}, \mathbb{F}_l),$$

and by lemma 5.1.7, one can identify $R\Gamma(\mathfrak{I}\mathfrak{g}_{K(Nl^{\infty})}^{\mathbb{X}}, \mathbb{F}_l)$ with $R\Gamma(\mathfrak{I}\mathfrak{g}_{K(Nl^{\infty})}^{\mathbb{X},\mathrm{tor}}, \mathbb{F}_l)$. Then corollary 4.3.11, tell us that

$$R\Gamma(\mathfrak{I}\mathfrak{g}_{K(Nl^{\infty})}^{\mathbb{X},\mathrm{tor}}, \mathbb{F}_l) \cong R\pi_{\mathrm{HT},l^{\infty},*}^{\mathrm{tor}} \mathbb{F}_l,$$

yielding the isomorphism above. We will from now on work with $R\pi_{\mathrm{HT},l^{\infty},*}^{\mathrm{tor}} \mathbb{F}_l$. Let U be a cofinal system as in the statement of theorem 5.1.1, and consider the minimal compactification $\mathcal{S}_{K(p^{\infty}N),C}^*$, and the universal fiber map $\mathcal{S}_{K(p^{\infty}N),C}^* \rightarrow \mathcal{S}_{K(p^{\infty}N),C}^*$ as in [Sch15, Section 4.1.]. By [Sch15, Thm. 4.1.1.] (see also proof of proposition 6.1.3. in [CS19]), the fiber product

$$\mathcal{S}_{K(p^{\infty}N),U}^* := \mathcal{S}_{K(p^{\infty}N),C}^* \times_{\mathcal{F}_{G,\mu,C}} U$$

is affinoid perfectoid, and furthermore for m large enough, there actually exists some affinoid $U' = \mathcal{S}_{K(p^m N),U}^*$, étale over $\mathcal{S}_{K(p^m N),C}^*$, such that

$$\mathcal{S}_{K(p^{\infty}N),U}^* = \mathcal{S}_{K(p^{\infty}N),C}^* \times_{\mathcal{S}_{K(p^m N),C}^*} U'.$$

Let $\mathcal{S}_{K(p^{\infty}N),U}^* = \mathrm{Spa}(R_{K(p^{\infty}N),U})$, and consider the induced map on formal schemes

$$\pi_{\mathrm{HT},U}^{\circ} : \mathrm{Spf}(R_{K(p^{\infty}N),U}^{\circ}) \rightarrow \mathrm{Spf}(A^{\circ}).$$

Since A°/p is of finite presentation, if we take reduction mod p of this map, the resulting map of schemes

$$\pi_{\text{HT}, \mathcal{U}_k}^\circ : \text{Spec}(R_{K(p^\infty N), U}^\circ/p) \rightarrow \text{Spec}(A^\circ/p) = \mathcal{U}_k$$

must factor over $\text{Spec}(R_{K(p^m N), U}^\circ/p)$ for m large enough, and thus we actually obtain a map

$$\text{Spec}(R_{K(p^m N), U}^\circ/p) \rightarrow \mathcal{U}_k.$$

Just as in [CS17, Prop. 6.1.3.], this satisfies the valuative criterium for properness, and thus preserves ${}^p D^{\geq d}$. Let

$$g_m^{\infty, \infty} : \mathcal{S}_{K(p^\infty Nl^\infty), U}^{\text{tor}} \rightarrow \mathcal{S}_{K(p^\infty Nl^\infty), U}^* \rightarrow U'$$

denote the projection. The fact that $\pi_{\text{HT}, \mathcal{U}_k}^\circ$ preserves ${}^p D^{\geq d}$ (and that taking nearby cycles commutes with pushforward), implies that it is enough to prove that

$$R\Psi(Rg_{m,*}^{\infty, \infty} \mathbb{F}_l) \in {}^p D^{\geq d}(\text{Spec}(R_{K(p^m N), U}^\circ/p), \mathbb{F}_l).$$

We can now replace the étale map $U \rightarrow \mathcal{F}_{G, \mu, C}$ by any étale map $U' = \text{Spa}(B) \rightarrow \mathcal{S}_{K(p^m N), C}^*$ for simplicity. We have

$$g_m^{\infty, \infty} : \mathcal{S}_{K(p^\infty Nl^\infty), U'}^{\text{tor}} \rightarrow U'$$

and we want to show that

$$R\Psi(Rg_{m,*}^{\infty, \infty} \mathbb{F}_l) \in {}^p D^{\geq d}(\text{Spec}(B^\circ/p), \mathbb{F}_l).$$

It suffices to prove a similar statement at finite levels. Indeed, if we consider the projections,

$$g_m^{m_1, m_2} : \mathcal{S}_{K(p^{m_1} Nl^{m_2}), U'}^{\text{tor}} \rightarrow U',$$

for $m_1 \geq m$ and $m_2 \geq 0$, and the open immersions

$$j_m^{m_1, m_2} : \mathcal{S}_{K(p^{m_1} Nl^{m_2}), U'} \rightarrow \mathcal{S}_{K(p^{m_1} Nl^{m_2}), U'}^{\text{tor}},$$

then there is an isomorphism

$$Rg_{m,*}^{\infty, \infty} \mathbb{F}_l = \varinjlim_{m_1, m_2} Rg_{m,*}^{m_1, m_2} \mathbb{F}_l.$$

Furthermore, if we fix m_1 , we obtain a natural map

$$\varinjlim_{m_2} Rg_{m,*}^{m_1, m_2} \mathbb{F}_l \rightarrow \varinjlim_{m_2} Rg_{m,*}^{m_1, m_2} Rj_*^{m_1, m_2} \mathbb{F}_l,$$

which is an isomorphism by Pink's lemma. The upshot of this, is that it suffices to prove that for all $m_1, m_2 > 0$,

$$R\Psi(Rg_{m,*}^{m_1, m_2} j_*^{m_1, m_2} \mathbb{F}_l) \in {}^p D^{\geq d}(\text{Spec}(B^\circ/p), \mathbb{F}_l).$$

But $Rg_{m,*}^{m_1, m_2} j_*^{m_1, m_2} \mathbb{F}_l$ lies in ${}^p D^{\geq d}$ (since it is just the composition of the open immersion $\mathcal{S}_{K(p^{m_1} Nl^{m_2}), U'} \hookrightarrow \mathcal{S}_{K(p^{m_1} Nl^{m_2}), U'}^*$, and the finite map $\mathcal{S}_{K(p^{m_1} Nl^{m_2}), U'}^* \rightarrow U'$, which lies in ${}^p D^{\geq d}$ in the scheme setting), so the result follows by t-exactness of the nearby cycles for the perverse t-structure, c.f. [Ill94]. This finishes the proof of the semiperversity.

Just as in [Kos21], we obtain the following corollary.

Corollary 5.1.9. *Let $i_{b_0} : \mathcal{F}_{G,\mu}(\mathbb{Q}_p) \hookrightarrow \mathcal{F}_{G,\mu,C}$ denote the closed immersion of the ordinary locus of $\mathcal{F}_{G,\mu,C}$. Then both local cohomology groups*

$$R\Gamma_c(\mathcal{F}_{G,\mu}(\mathbb{Q}_p), Ri_{b_0}^! R\pi_{HT*}^\circ \mathbb{F}_l)$$

and

$$R\Gamma_c([\mathcal{F}_{G,\mu}(\mathbb{Q}_p)/K_p], Ri_{b_0}^! R\pi_{HT*}^\circ \mathbb{F}_l)$$

are in ${}^p D^{\geq d}(\mathbb{F}_l)$.

Proof. The second part follows immediately from the first, so we only need to prove that

$$R\Gamma_c(\mathcal{F}_{G,\mu}(\mathbb{Q}_p), Ri_{b_0}^! R\pi_{HT*}^\circ \mathbb{F}_l) \in {}^p D^{\geq d}(\mathbb{F}_l).$$

By taking the open immersion

$$j^{b_0} : \mathcal{F}_{G,\mu,C} \setminus \mathcal{F}_{G,\mu}(\mathbb{Q}_p) \hookrightarrow \mathcal{F}_{G,\mu,C}$$

given by the inclusion of the complement of the ordinary locus in $\mathcal{F}_{G,\mu,C}$, we know by [Kosh, Lemma 8.3.] that there is a canonical fiber sequence

$$i_*^{b_0} Ri^{b_0!} R\pi_{HT*}^\circ \mathbb{F}_l \rightarrow R\pi_{HT*}^\circ \mathbb{F}_l \rightarrow Rj_*^{b_0} j^{b_0*} R\pi_{HT*}^\circ \mathbb{F}_l.$$

In particular, this gives rise to a canonical fiber sequence

$$R\Gamma_c(\mathcal{F}_{G,\mu}(\mathbb{Q}_p), Ri_{b_0}^! R\pi_{HT*}^\circ \mathbb{F}_l) \rightarrow R\Gamma(\mathcal{F}_{G,\mu}(\mathbb{Q}_p), R\pi_{HT*}^\circ \mathbb{F}_l) \rightarrow R\Gamma(\mathcal{F}_{G,\mu,C} \setminus \mathcal{F}_{G,\mu}(\mathbb{Q}_p), R\pi_{HT*}^\circ \mathbb{F}_l).$$

Just as in the previous theorem, we consider a cofinal system of affinoid étale neighborhoods $U \rightarrow \mathcal{F}_{G,\mu,C}$, and denote by $\mathcal{U}$ the associated formal scheme, so that we have a map $\mathcal{U}_k \rightarrow \mathcal{F}_{G,\mu,k}$. One can consider the image of the ordinary locus under the specialization map

$$i_Z : Z \hookrightarrow \mathcal{F}_{G,\mu,k},$$

which is a closed set consisting of finitely many closed points. By the previous theorem,

$$R\Psi(R\pi_{HT*}^\circ \mathbb{F}_l)|_U \in {}^p D^{\geq d}(\mathcal{F}_{G,\mu,k}, \mathbb{F}_l)|_U,$$

and since $Ri_Z^!$ is left t-exact, this implies that

$$Ri_Z^! R\Psi(R\pi_{HT*}^\circ \mathbb{F}_l) \in {}^p D^{\geq d}(Z, \mathbb{F}_l).$$

We can also use the above strategy to obtain a canonical fiber sequence in cohomology: we consider the open immersion

$$j_Z : \mathcal{F}_{G,\mu,k} \setminus Z \hookrightarrow \mathcal{F}_{G,\mu,k}$$

and the canonical fiber sequence

$$i_{Z*} Ri_Z^! R\Psi R\pi_{HT*}^\circ \mathbb{F}_l \rightarrow R\Psi R\pi_{HT*}^\circ \mathbb{F}_l \rightarrow Rj_{Z*} j_Z^* R\Psi R\pi_{HT*}^\circ \mathbb{F}_l$$

and thus a sequence

$$R\Gamma_Z(\mathcal{F}_{G,\mu,k}, R\Psi(R\pi_{HT*}^\circ \mathbb{F}_l)) \rightarrow R\Gamma(\mathcal{F}_{G,\mu,C}, R\pi_{HT*}^\circ \mathbb{F}_l) \rightarrow R\Gamma(\mathcal{F}_{G,\mu,C} \setminus sp^{-1}(Z), R\pi_{HT*}^\circ \mathbb{F}_l),$$

where $R\Gamma_Z(\mathcal{F}_{G,\mu,k}, R\Psi(R\pi_{HT*}^\circ \mathbb{F}_l)) := R\Gamma_c(Z, Ri_Z^! R\Psi(R\pi_{HT*}^\circ \mathbb{F}_l)) \in {}^p D^{\geq d}(\mathbb{F}_l)$ as we just saw. We can now take the limit by varying our étale system of neighborhoods. In particular, we obtain a canonical fiber sequence

$$R\varinjlim_Z R\Gamma_Z(\mathcal{F}_{G,\mu,k}, R\Psi(R\pi_{HT*}^\circ \mathbb{F}_l)) \rightarrow R\Gamma(\mathcal{F}_{G,\mu,C}, R\pi_{HT*}^\circ \mathbb{F}_l) \rightarrow R\Gamma(Fl_C \setminus \mathcal{F}_{G,\mu}(\mathbb{Q}_p), R\pi_{HT*}^\circ \mathbb{F}_l)$$

and since fiber sequences commute with limits, and we have the sequence above, we conclude that

$$R\Gamma_c(\mathcal{F}_{G,\mu}(\mathbb{Q}_p), Ri_{b_0}^! R\pi_{HT*}^\circ \mathbb{F}_l) \cong R\varinjlim_Z R\Gamma_Z(\mathcal{F}_{G,\mu,k}, R\Psi(R\pi_{HT*}^\circ \mathbb{F}_l)) \in {}^p D^{\geq d}(\mathbb{F}_l),$$

concluding the proof of the corollary.

5.2 Vanishing of Cohomology

In this section, we want to use the techniques and results of [Kos21] to prove the vanishing cohomology results. In particular, we need to restrict ourselves to the following setting: we take a CM field F , and a type A PEL-datum $(B, *, V, (\cdot, \cdot))$, where B is a central simple F -algebra. We use the same notation as in the previous chapters, so in particular we denote by G the unitary similitudes group attached to this data. We assume that p is a prime that is unramified for the Shimura datum. If we assume furthermore that p splits completely over F , we can write $G_{\mathbb{Q}_p}$ as a product

$$G_{\mathbb{Q}_p} \cong \mathbb{G}_m^\times \times \prod G_{v_i}$$

for $p = v_1 \cdots v_m \in F$, where each G_{v_i} is a product of general linear groups over $F_{v_i} \cong \mathbb{Q}_p$. Let $K = K_p K^p \subset G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ be a sufficiently small open compact subgroup with $K_p = \mathbb{Z}_p^\times \times \prod K_{v_i} \subset G(\mathbb{Q}_p)$ an hyperspecial open compact subgroup. Let $\mathcal{S}_K$ be the PEL-type Shimura variety attached to this data, and let d be its dimension. Consider the Hecke algebra at p

$$H_{K_p} := \mathbb{Z}_l[K_p \backslash G(\mathbb{Q}_p)/K_p]$$

for some prime $l \neq p$.

The local Hecke algebra H_{K_p} admits a right action on the cohomology groups

$$H^i(\mathcal{S}_K, \bar{\mathbb{F}}_l), H_c^i(\mathcal{S}_K, \bar{\mathbb{F}}_l)$$

via correspondences. Let $\mathfrak{m}_{H_{K_p}} \subset H_{K_p}$ be a maximal ideal of the local Hecke algebra in the support of $H_{(c)}^i(\mathcal{S}_K, \bar{\mathbb{F}}_l)$, or equivalently a system of Hecke eigenvalues appearing in $H_{(c)}^i(\mathcal{S}_K, \bar{\mathbb{F}}_l)$. By considering the quotient $H_{K_p}/\mathfrak{m}_{H_{K_p}}$, we obtain a corresponding character

$$\lambda_{\mathfrak{m}_{H_{K_p}}} : H_{K_p} \rightarrow \bar{\mathbb{F}}_l.$$

Furthermore, we also have associated L-parameters, which consists of products of representation

$$\rho_{\mathfrak{m}_{v_i}} : \text{Gal}(\bar{F}_{v_i}/F_{v_i}) \rightarrow \prod_j \text{GL}_{n_j}(\bar{\mathbb{F}}_l)$$

for each place v_i of F above p , as above.

Our vanishing results concern the generic part of the cohomology of these Shimura varieties. In particular, we need to recall the following genericity concept.

Definition 5.2.1. Let $\mathbb{F}$ be a finite field of characteristic l , $p \neq l$ a prime and K a finite extension of $\mathbb{Q}_p$. Consider a continuous representation $\rho : \text{Gal}(\bar{K}/K) \rightarrow \text{GL}_n(\mathbb{F})$. ρ is said to be generic if it is unramified, and if the eigenvalues $\alpha_1, \dots, \alpha_n \in \bar{\mathbb{F}}_l$ of $\rho(\text{Frob}_K)$ for an arithmetic Frobenius $\text{Frob}_K \in \text{Gal}(\bar{K}/K)$, satisfy $\alpha_i/\alpha_j \neq |\mathcal{O}_K/\mathfrak{m}_K|$ for $i \neq j$.

Note that we can apply this concept to an L-parameter $\rho_{\mathfrak{m}_{v_i}}$ as defined above: namely, $\rho_{\mathfrak{m}_{v_i}}$ is generic if $\rho_{\mathfrak{m}_{v_i},j} : \text{Gal}(\bar{F}_{v_i}/F_{v_i}) \rightarrow \text{GL}_{n_j}(\bar{\mathbb{F}}_l)$ is generic for each i and j . The main goal of this section is to prove the following:

Theorem 5.2.2. *Let p be a prime which splits completely in F . Let $\mathfrak{m}_{H_{K_p}} \subset H_{K_p}$ be a maximal ideal of the Hecke algebra, with associated L-parameters $\rho_{\mathfrak{m}_{v_i}}$, which we assume to be generic for every i . Then $H^i(\mathcal{S}_K, \bar{\mathbb{F}}_l)_{\mathfrak{m}_{H_{K_p}}} = 0$ for $i \leq d$ and $H_c^i(\mathcal{S}_K, \bar{\mathbb{F}}_l)_{\mathfrak{m}_{H_{K_p}}} = 0$ for $i \geq d$.*

We will essentially apply the results in [Kos21] to obtain the desired result. Let us start by recalling the local vanishing theorem in his paper.

We fix a finite extension K of $\mathbb{Q}_p$, with residue field $\mathbb{F}_q$. Let $(B, *, V, (\cdot, \cdot), b, \mu)$ be a local PEL-type A datum over K , obtained from a type A PEL-datum $(B, *, V, (\cdot, \cdot))$ with the same assumptions on p as above (in particular, the data must be unramified at p , and p must be completely split in the CM field F) and G/K be the corresponding local unitary similitudes group. Denote by J_b the usual inner form of G . Just as above, this implies that G is a product of general linear groups GL_{n_i} for some finite set of integers n_i . Let $K_p = \prod \text{GL}_{n_i}(\mathcal{O}_K) \subset G(K)$ again an hyperspecial open compact subgroup, and $\mathcal{M}_{(G,b,\mu),K_p}$ the corresponding local Shimura variety (c.f. [SW13, 24.1.3.]). Let H_{K_p} be the Hecke algebra defined exactly as above, $\mathfrak{m} \subset H_{K_p}$ a maximal ideal, and $\lambda_{\mathfrak{m}}, \rho_{\mathfrak{m}}$ the corresponding character resp. L-parameter (in this case, we have a single L-parameter showing up).

Theorem 5.2.3. [Kos21, Thm. 1.1] *Assume that the L-parameter $\rho_{\mathfrak{m}}$ is generic, and that furthermore J_b is not quasi-split. Then*

$$H_c^i(\mathcal{M}_{(G,b,\mu),K_p}, \mathbb{Z}_l)_{\mathfrak{m}} = 0$$

for every i .

Remark 5.2.4. We want to explain the importance of the genericity assumption in the proof of the theorem. The point is that if $\rho : \text{Gal}(\bar{K}/K) \rightarrow \text{GL}_n(\mathbb{F})$ is generic, and we consider any lift $\bar{\rho}$ to characteristic 0, it will correspond to a generic principal series representation of $\text{GL}_n(K)$ via

the local Langlands correspondence. But if J_b is not quasi-split, one can see that such a $\bar{\rho}$ can not correspond to an inner form of GL_n/K under the Jacquet–Langlands correspondence, and in particular ρ can not be the L -parameter associated to an irreducible smooth L -representation of $J_b(K)$ under the constructions in [FS21, IX 4.1, 7.1] (c.f. [Kos21, Lemma 3.1.], [Car22, Rem. 4.2.]).

The proof of the theorem consists in considering an irreducible subquotient π of the cohomology group $H_c^i(\mathcal{M}_{(G,b,\mu),K_p}, \bar{\mathbb{F}}_l)_{\mathfrak{m}}$ (seen as a $J_b(K)$ representation), and to use excursion data and the Bernstein center introduced in [FS21] to obtain an equality of L -parameters $\varphi_\pi = \rho_{\mathfrak{m}}$, where φ_π is the L -parameter associated to π as constructed in [FS21, IX.4.1, IX.7.1]. This is a contradiction by what we just discussed.

We now consider once more the global situation, and we go back to the more general setup as introduced in the first chapter. In particular, we fix a type A or C PEL-datum, which we assume to be unramified at p , and a sufficiently small compact open subgroup $K \subset G(\mathbb{A}_f)$, with $K = K_p K^p$, $K_p = G(\mathbb{Z}_p)$, and we let $\mathcal{S}_K$ be the associated PEL-type Shimura variety of type A or C defined over the reflex field E . Let $\mathfrak{p}$ be a place of E above p , and recall that $\mathcal{S}_K$ admits a canonical integral model over $\mathcal{O}_{E,\mathfrak{p}}$, which we also denote $\mathcal{S}_K$ by abuse of notation. Consider its associated adic space, and let $\mathcal{S}_K^\circ$ denote the locus of good reduction of $\mathcal{S}_K$, and the induced open immersion

$$\mathcal{S}_K^\circ \hookrightarrow \mathcal{S}_K$$

of adic spaces over $E_{\mathfrak{p}}$. Then, by [Kos21, Lemma 5.1], it follows from [LS(NC)18, Cor. 5.20.] and [Hub96, Thm. 3.5.13.] that the cohomology of the Shimura variety is completely captured by the cohomology of the good reduction locus, so in particular the natural map

$$R\Gamma(\mathcal{S}_{K,\mathbb{Q}}, \bar{\mathbb{F}}_l) \rightarrow R\Gamma(\mathcal{S}_{K,C}^\circ, \bar{\mathbb{F}}_l) \quad (5.1)$$

is an isomorphism for $l \neq p$, for C a complete, algebraically closed non-archimedean field over $\mathbb{Q}_p$.

Recall that we have a Hodge-Tate period morphism

$$\pi_{\mathrm{HT}} : \mathcal{S}_{K^p} \rightarrow \mathcal{F}_{G,\mu,C},$$

and we denote by π_{HT}° its restriction to $\mathcal{S}_{K^p}^\circ$. Also recall that we have a Newton stratification, both on the flag and the Shimura varieties, inducing a map

$$\pi_{\mathrm{HT}}^b : \mathcal{S}_{K^p,C}^b \rightarrow \mathcal{F}_{G,\mu,C}^b,$$

and allowing us therefore to define a map

$$\pi_{\mathrm{HT}}^{\circ,b} : \mathcal{S}_{K^p,C}^{\circ,b} := \mathcal{S}_{K^p,C}^\circ \times_{\mathcal{F}_{G,\mu,C}} \mathcal{F}_{G,\mu,C}^b \rightarrow \mathcal{F}_{G,\mu,C}^b,$$

which is quasicompact and quasiseparated. Koshikawa provides us with the following lemma:

Lemma 5.2.5. *[Kos21, Lemma 6.1] The natural map*

$$(R\pi_{\mathrm{HT},*}^{\circ} \bar{\mathbb{F}}_l)|_{\mathcal{F}_{G,\mu,C}^b} \rightarrow R\pi_{\mathrm{HT},*}^{\circ,b} \bar{\mathbb{F}}_l$$

is an isomorphism in $D_{\mathrm{et}}(\mathcal{F}_{G,\mu,C}^b, \bar{\mathbb{F}}_l)$.

On the other hand, Mantovan's product formula (c.f. section 4.2) allows us to prove a product formula in cohomology. Following section 7 in [Kos21], we define $R\Gamma_c(\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l(d_b))$ as the colimit of compactly supported cohomology of $\bar{\mathbb{F}}_l(d_b)$ on the finite level Igusa varieties.

Theorem 5.2.6. *[Kos21, Thm. 7.1.] There is a filtration on $R\Gamma(\mathcal{S}_{K^p,\mathbb{Q}}, \bar{\mathbb{F}}_l)$ by complexes of smooth representations of $G(\mathbb{Q}_p) \times W_{E_p}$, where W_{E_p} denotes the Weil group, with graded pieces*

$$\begin{aligned} R\Gamma(\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l)^{\mathrm{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \bar{\mathbb{F}}_l(d_b))[2d_b] \cong \\ R\Gamma_c(\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l(d_b)^*)^{\mathrm{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \bar{\mathbb{F}}_l(d_b)), \end{aligned}$$

where d_b denotes the dimension of $\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}$, and everything is equipped with a $J_b(\mathbb{Q}_p)$ -structure.

Sketch of Proof. The isomorphism from the statement follows from Poincaré duality at finite level, which implies that there is an isomorphism

$$R\Gamma_c(\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l(d_b))^* \cong R\Gamma(\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l)[2d_b].$$

By the isomorphism 5.1 we saw above, there are isomorphisms

$$R\Gamma(\mathcal{S}_{K^p,\mathbb{Q}}, \bar{\mathbb{F}}_l) \cong R\Gamma(\mathcal{S}_{K^p,C}^{\circ}, \bar{\mathbb{F}}_l) \cong R\Gamma(\mathcal{F}_{G,\mu,C}, R\pi_{\mathrm{HT}*} \bar{\mathbb{F}}_l),$$

and using excision sequences, we obtain a filtration on $R\Gamma(\mathcal{F}_{G,\mu,C}, R\pi_{\mathrm{HT}*} \bar{\mathbb{F}}_l)$, with graded pieces

$$R\Gamma_c(\mathcal{F}_{G,\mu,C}^b, R\pi_{\mathrm{HT}*} \bar{\mathbb{F}}_l)$$

which, for each b , are isomorphic to

$$R\Gamma_c(\mathcal{F}_{G,\mu,C}^b, R\pi_{\mathrm{HT}*}^{\circ,b} \bar{\mathbb{F}}_l)$$

by lemma 5.2.5. If we identify this complex with

$$R\Gamma_c(\mathfrak{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l(d_b)^*)^{\mathrm{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \bar{\mathbb{F}}_l(d_b)),$$

the result will follow. In order to do so, one starts by relating the cohomology of the flag variety with that of the Rapoport-Zink space using the local Hodge-Tate period morphism π_{HT}^b that we introduced in section 4.2. The result will then follow by Mantovan's product formula, as we will explain bellow.

More precisely, as in [Kos21, Sect. 7], we can factorize π_{HT}^b as a composition

$$\pi_{\mathrm{HT}}^b : \mathcal{M}_{(G,b,\mu),\infty} \xrightarrow{\pi_{\mathrm{unip}}^b} \mathcal{M}_{(G,b,\mu),\infty} / \tilde{J}_{b,C}^0 \xrightarrow{\pi_{\mathrm{ss}}^b} \mathcal{F}_{G,\mu,C}^b,$$

where, $\tilde{J}_{b,C}^0$ is a unipotent group of dimension d_b (c.f. [FS21, III]), and as v-sheaves, π_{unip}^b is a $\tilde{J}_{b,C}^0$ -torsor and π_{ss}^b is a $\underline{J}_b(\mathbb{Q}_p)$ -torsor. Then, by [Kos21, Lemma 7.3.], there is a $G(\mathbb{Q}_p)$ -equivariant isomorphism

$$R\Gamma_c(\mathcal{F}_{G,\mu,C}^b, R\pi_{\text{HT}*}^{\circ b} \bar{\mathbb{F}}_l) \cong R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty} / \tilde{J}_{b,C}^0, R\pi_{\text{ss}}^{b*} R\pi_{\text{HT}*}^{\circ b} \bar{\mathbb{F}}_l) \otimes_{C_c(J_b(\mathbb{Q}_p))}^L \bar{\mathbb{F}}_l. \quad (5.2)$$

In particular, we only need to identify the complex $R\Gamma_c(\mathcal{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l(d)^*)^{\text{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \bar{\mathbb{F}}_l(d))$ introduced above with the complex in 5.2.

On the other hand, Mantovan's product formula, as introduced in section 4.2, implies (c.f. [Kos21, Lemma 7.5.]) that there is an isomorphism

$$R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \pi_{\text{HT}}^{b*} R\pi_{\text{HT}*}^{\circ b} \bar{\mathbb{F}}_l \otimes_{\bar{\mathbb{F}}_l}^L R\pi_{\text{unip}}^{b!} \bar{\mathbb{F}}_l) \cong R\Gamma(\mathcal{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l) \otimes_{\bar{\mathbb{F}}_l}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, R\pi_{\text{unip}}^{b!} \bar{\mathbb{F}}_l), \quad (5.3)$$

and using the isomorphism

$$R\pi_{\text{unip}}^{b!} \bar{\mathbb{F}}_l \cong \bar{\mathbb{F}}_l(d_b)[2d_b]$$

as proved in [Kos21, Lemma 7.4.], we know that

$$R\Gamma(\mathcal{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l) \otimes_{\bar{\mathbb{F}}_l}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, R\pi_{\text{unip}}^{b!} \bar{\mathbb{F}}_l) \cong R\Gamma(\mathcal{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l) \otimes_{\bar{\mathbb{F}}_l}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \bar{\mathbb{F}}_l(d_b))[2d_b],$$

so by tensoring both sides in 5.3 by $\otimes_{C_c(J_b(\mathbb{Q}_p))}^L \bar{\mathbb{F}}_l$, and using the isomorphism

$$\begin{aligned} R\Gamma(\mathcal{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l)^{\text{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \bar{\mathbb{F}}_l) &\cong \\ (R\Gamma(\mathcal{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l) \otimes_{\bar{\mathbb{F}}_l}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \bar{\mathbb{F}}_l)) \otimes_{C_c(J_b(\mathbb{Q}_p))}^L \bar{\mathbb{F}}_l \end{aligned}$$

in [Mant04, Prop. 5.12.], one concludes that

$$\begin{aligned} R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \pi_{\text{HT}}^{b*} R\pi_{\text{HT}*}^{\circ b} \bar{\mathbb{F}}_l \otimes_{\bar{\mathbb{F}}_l}^L R\pi_{\text{unip}}^{b!} \bar{\mathbb{F}}_l) \otimes_{C_c(J_b(\mathbb{Q}_p))}^L \bar{\mathbb{F}}_l &\cong \\ (R\Gamma(\mathcal{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l)^{\text{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \bar{\mathbb{F}}_l(d_b))) [2d_b]. \end{aligned}$$

Note that the complex $(R\Gamma(\mathcal{I}\mathfrak{g}^{\mathbb{X}_b}, \bar{\mathbb{F}}_l)^{\text{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \bar{\mathbb{F}}_l(d_b))) [2d_b]$ above is exactly the one showing up in the statement of the theorem. In particular, the only think left to do is to identify

$$R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \pi_{\text{HT}}^{b*} R\pi_{\text{HT}*}^{\circ b} \bar{\mathbb{F}}_l \otimes_{\bar{\mathbb{F}}_l}^L R\pi_{\text{unip}}^{b!} \bar{\mathbb{F}}_l) \otimes_{C_c(J_b(\mathbb{Q}_p))}^L \bar{\mathbb{F}}_l$$

with

$$R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty} / \tilde{J}_{b,C}^0, R\pi_{\text{ss}}^{b*} R\pi_{\text{HT}*}^{\circ b} \bar{\mathbb{F}}_l).$$

This follows from the equivalence

$$R\pi_{\text{unip}}^{b!} R\pi_{\text{unip}}^{b!} \rightarrow \text{id}$$

in [Kos21, Lemma 7.4.], as explained in [Kos21, Section 7] (in particular, very similar arguments were given in the proof of Lemma 7.3.).

We can finally use the semiperversity and the previous results to prove theorem 5.2.2.

Proof of Main Theorem 5.2.2. By Poincaré duality, we only have to prove that $H_c^i(\mathcal{S}_K, \bar{\mathbb{F}}_l)_{\mathfrak{m}_p}$ vanishes for all $i < d$.

By [CS17, 5.5.4., 5.5.5.], J_b is quasi-split iff b is ordinary, so that in particular by theorem 5.2.3,

$$R\Gamma_c(\mathcal{M}_{(G,b,\mu),\infty}, \bar{\mathbb{F}}_l(d_b))_{\mathfrak{m}_p} = 0$$

for all non-ordinary $b \in B(G, \mu^{-1})$. With this in mind, theorem 5.2.6 immediately implies that

$$R\Gamma_c(\mathcal{S}_K, \bar{\mathbb{F}}_l)_{\mathfrak{m}_p} \cong R\Gamma(\mathfrak{Jg}^{\mathbb{X}_{b_0}}, \bar{\mathbb{F}}_l)^{\text{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G,b_0,\mu),\infty}, \bar{\mathbb{F}}_l(d))_{\mathfrak{m}_p} [2d]$$

for b_0 the unique ordinary element in $B(G, \mu^{-1})$. Let $i^{b_0} : \mathcal{F}_{G,\mu}(\mathbb{Q}_p) \rightarrow \mathcal{F}_{G,\mu,C}$ be the closed immersion corresponding to the ordinary stratum. The proof of 5.2.6 also implies that

$$R\Gamma(\mathfrak{Jg}^{\mathbb{X}_{b_0}}, \bar{\mathbb{F}}_l)^{\text{op}} \otimes_{C_c(J_b(\mathbb{Q}_p))}^L R\Gamma_c(\mathcal{M}_{(G,b_0,\mu),\infty}, \bar{\mathbb{F}}_l(d))_{\mathfrak{m}_p} [2d] \cong R\Gamma_c([\mathcal{F}_{G,\mu}(\mathbb{Q}_p)/K_p], i^{b_0*} R\pi_{\text{HT}}^\circ \bar{\mathbb{F}}_l)_{\mathfrak{m}_p}.$$

Indeed, one just has to notice that since b_0 is ordinary,

$$R\Gamma_c([\mathcal{F}_{G,\mu,C}^{b_0}/K_p], R\pi_{\text{HT}*}^{\circ b_0} \bar{\mathbb{F}}_l)_{\mathfrak{m}_p} = R\Gamma_c([\mathcal{F}_{G,\mu,C}(\mathbb{Q}_p)/K_p], i^{b_0*} R\pi_{\text{HT}}^\circ \bar{\mathbb{F}}_l)_{\mathfrak{m}_p}.$$

By corollary 5.1.9,

$$R\Gamma_c([\mathcal{F}_{G,\mu,C}(\mathbb{Q}_p)/K_p], Ri^{b_0!} R\pi_{\text{HT}*}^\circ \bar{\mathbb{F}}_l)_{\mathfrak{m}_p} \in {}^p D^{\geq d}(\bar{\mathbb{F}}_l).$$

The result follows if we identify this complex with

$$R\Gamma_c([\mathcal{F}_{G,\mu,C}(\mathbb{Q}_p)/K_p], i^{b_0*} R\pi_{\text{HT}}^\circ \bar{\mathbb{F}}_l)_{\mathfrak{m}_p}.$$

This is done in [Kos21, Prop. 1.7.], finishing therefore the proof of the main theorem.

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