Sharp Bounds on Connectivity Functions for Random Walks and Random Interlacements



Centre for Doctoral Training
Mathematics of Random Systems

Yuriy Yuryevich Shulzhenko

Supervisors: Dr Pierre-François Rodriguez[1],
Dr Subhajit Goswami[2],
Professor Xue-Mei Li[1]

Department of Mathematics
Imperial College London

This dissertation is submitted for the degree of
Doctor of Philosophy

November 2024

[1]Imperial College London
Department of Mathematics
London SW7 2AZ
United Kingdom.
p.rodriguez@imperial.ac.uk
yuriy.shulzhenko16@imperial.ac.uk
xue-mei.li@imperial.ac.uk

[2]School of Mathematics
Tata Institute of Fundamental Research
1, Homi Bhabha Road
Colaba, Mumbai 400005, India.
goswami@math.tifr.res.in

Copyright Declaration

The copyright of this thesis rests with the author. Unless otherwise indicated, its contents are licensed under a Creative Commons Attribution-Non Commercial 4.0 International Licence (CC BY-NC).

Under this licence, you may copy and redistribute the material in any medium or format. You may also create and distribute modified versions of the work. This is on the condition that: you credit the author and do not use it, or any derivative works, for a commercial purpose.

When reusing or sharing this work, ensure you make the licence terms clear to others by naming the licence and linking to the licence text. Where a work has been adapted, you should indicate that the work has been changed and describe those changes.

Please seek permission from the copyright holder for uses of this work that are not included in this licence or permitted under UK Copyright Law.

In loving memory of my mother, Elena, who hoped for this journey but never witnessed it.

Declaration

I hereby declare that the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. Any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Yuriy Yuryevich Shulzhenko
November 2024

Acknowledgements

First and foremost, I extend my deepest gratitude to my supervisor, Pierre-François Rodriguez. Under his mentorship, I have come to truly appreciate the elegance and depth of percolation theory. What started off as a small piece of a puzzle in my eyes, has manifested into a vast and fascinating field, brimming with intricate ideas and surprising interconnections. I feel privileged to have been guided by his passion and dedication throughout the years, and am immensely grateful for his support for my development, both academically and personally.

I would also like to express my sincere thanks to my co-supervisor, Subhajit Goswami, for his selfless and unwavering support, particularly in the final months before submission. Our discussions were always enlightening, and his insights were consistently inspired and well-timed. I am truly honoured to have worked in collaboration with him and Pierre - this work would not have been possible without their shared wisdom. In this regard, I am also grateful to Xue-Mei Li for encouraging me to pursue a PhD in the first place, and for her trust in me when I needed it most. As a final thanks regarding matters directly concerning this manuscript, I would like to thank Balázs Ráth and Ajay Chandra for kindly agreeing to be the examiners.

Throughout my studies, there were two particular groups of people who were influential to my life, those in Weeks and Kemp Porter. I am grateful to all those in the former group for keeping me energized and motivated by their example - particularly to Luca and Nicola, who were always positive and supportive even in tough times. As for my office-at-home at Kemp Porter, I am honored to have worked alongside such remarkable individuals in the subwarden team during my final years at Imperial. Their friendship will be sorely missed as I embark on the next chapter of my life.

Finally, I would like to thank those people dearest to me. The first in the list is my father, who has encouraged and supported me in every aspect of my life - I attribute my dedication and passion to him. I would also like to thank my mother for

always encouraging and inspiring philosophical discussion. She has taught me academic curiosity, patience and kindness - qualities which will remain with me always. The last in this list is Cătălina. Thank you for being patient with me, I hope that I can support your greatest endeavours as you have with mine.

Abstract

Consider the vacant set $\mathcal{V}_N$ induced by a simple random walk on a d-dimensional torus with side length N. We determine the typical size of the largest vacant cluster in the subcritical regime $u > u_*$, and the second largest cluster in the supercritical regime $u < u_*$ for all dimensions $d \geq 3$. In the case $d \geq 4$, these are shown to have a diameter which is logarithmic, meanwhile when $d = 3$, an additional, multiplicative, doubly logarithmic term appears, coinciding with other long-range correlation models such as the Gaussian free field level set. Moreover, for $d = 3$, we obtain the precise polynomial divergence of cluster sizes as $u \to u_*$.

Our findings stem from matching upper and lower bounds which we derive for the truncated two-point connectivity function for the random interlacements vacant set $\mathcal{V}^u$. In the case of the lower bounds, a change-of-measure argument is used in order to apply spatial modulations to the random interlacements, rendering them locally sub- or supercritical, depending on whether we wish to insulate, or facilitate, connections. A novelty in our approach is that extra care is taken to deal with rotational invariance of the resulting connectivity function, requiring new capacity bounds on oblique tubes in the case of $d = 3$.

For the upper bounds, we invoke a localisation and bootstrapping argument, which effectively handles the obstacle of long-range correlations inherently present in our model. In order to deal with the non-monotonicity appearing in the supercritical regime, a new approach involving sub-collections of the random interlacements is presented, effectively reformulating the required events in a monotonic way in exchange for an acceptable combinatorial cost. As a final highlight, we tailor an algorithmic approach to overcome the structural rigidity enforced by the connectedness of trajectories comprising the random interlacements model.

Contents

Figures **xv**

1 Introduction **1**
1.1 Motivations of Percolation Theory 1
1.2 Illustrative Examples 4
1.2.1 Independent Percolation 4
1.2.2 Dependent Percolation 8
1.3 The Random Walk on a Torus 12
1.4 Outline of Results 19

2 Preliminaries **23**
2.1 Notation 23
2.2 Review of Potential Theory 25
2.3 Capacities of Oblique Sets 31
2.4 Random Interlacements 39
2.5 A Useful Entropy Inequality 45
2.6 Soft Local Times 47

3 Lower Bounds **53**
3.1 Generalised Lower Bound 53
3.2 Tilted Interlacements Construction 64
3.3 Estimates for the Tilted Harmonic Measure 74
3.4 Coupling Tilted Interlacements 85

4 Upper Bounds **99**
4.1 Geometric Properties 99
4.1.1 Crossings and Blocking Interfaces 99
4.1.2 Propagation of Good Events 104
4.2 Localisation 111

4.2.1 Coupling RI with Independent Excursions 111
4.2.2 Estimates on Collections of $\mathcal{F}_z^{u,v}$ 118
4.3 Renormalisation 121
4.3.1 Coarse-graining 122
4.3.2 The Event $\mathscr{G}$ 127
4.3.3 Bootstrapping 134
4.4 Upper Bounds: Collecting the Pieces 137
4.4.1 Subcritical Regime 138
4.4.2 Supercritical Regime 141
4.4.3 Torus Cluster Sizes 159
4.5 Assortment of Technical Results 164
4.5.1 Estimate on V_z^c 166
4.5.2 Conditional Finite Energy Bounds 188
4.5.3 Proofs of A Priori Estimates 194

References **205**

Figures

1.1 Bond vs site percolation 5
1.2 Simulation of clusters of the vacant set on a torus 13
1.3 Depiction of structural rigidity 18

2.1 Chaining Argument 38

3.1 Tilting Random Interlacements 70

4.1 Coarse Graining of γ 124
4.2 Intersecting Concentric Shells 125

Chapter 1

Introduction

1.1 Motivations of Percolation Theory

Percolation theory is a broad and exciting area of research which has gained significant popularity in the mid-late 20^{th} century due to its deep connections to physical systems and natural phenomena. At its core, this subset of mathematics concerns itself with the study of the properties of porous materials, which are modelled as probability distributions on the space of random 'structures' and are referred to as percolation models. By this description, 'percolation' is somewhat a misnomer, given that we are interested in a more diverse range of properties than simply whether the induced random structure allows some fluid-like substance to percolate through it. In fact, the name is rather an acknowledgement to the origins of the theory as an alternative for diffusive models when studying the spread of a fluid through a medium, i.e. whether the fluid percolates. In particular, while the diffusive approach assumes that all sources of randomness lie with the dispersion of the fluid, one can alternatively view the motion of the fluid as deterministic given the realisation of the random structure of the medium [56]. Such a reinterpretation of the problem has opened up other avenues of investigation which are tied more closely with features of the random structure such as connectivity, fractal dimension and critical behaviour. These mathematical notions establish principles by which we can interpret many surprising aspects of the physical world, from ferromagnetism to the spread of forest fires or even evaluating the effectiveness of a sample in measuring the porosity of soil (a problem important in the oil and gas industry [39]).

A common starting point for constructing percolation models is to consider an underlying space represented by a graph $\mathcal{G} = (V, E)$ comprised of a set of vertices V and

edges E, and then assigning each edge (or vertex, see bond vs site percolation in §1.2.1) a binary state, typically referred to as open or closed, according to a predetermined rule. Once the states have been determined, some natural questions arise when considering the sub-graph $\mathcal{V} \subset \mathcal{G}$ of open edges, for example concerning the connectivity of $\mathcal{V}$ or the sizes of its constituent clusters. Moreover, these geometric properties correspond to important analogies in the physical world, positioning percolation models as an effective tool for understanding natural phenomena. To effectively illustrate these analogies, let us momentarily assign a physical interpretation to our general space $\mathcal{G}$. Consider the popularised example of a porous stone immersed in water [31]; in this case, we model the stone as closed vertices, and its holes as open vertices, where we assume that the stone can be viewed as a connected subset with non-empty interior of a sufficiently fine 3-dimensional integer lattice. Thus, in light of such a model, the question of whether water passes through the porous stone is reduced to one of topological connectivity: is there a path of nearest-neighbour open vertices starting at one 'face' of the stone and ending at another? Intuitively, the answer to this question depends on the density of the stone, with a low density likely yielding a positive response. Therefore, by parameterising the density according to some $p \in (0, 1)$, one can investigate the probability of such a connection occurring, say $\theta(p)$, as a function of the parameter p.

In the language of the general model, $\theta(p)$ is closely related to both the structure of the underlying space $\mathcal{G}$ as well as the 'rule' for opening and closing vertices. Given this flexibility in terms of the structure and rule, it is conceivable that there are many possible behaviours arising for $\theta(\cdot)$, ranging from smooth, predictable transitions to abrupt, critical phenomena. For instance, consider for the moment an unspecified model on an infinite graph $\mathcal{G}$, and modify the definition of $\theta(p)$ instead to be the probability that there exists an infinite connected component $\mathscr{C}_\infty$ of open vertices. Such a definition for $\theta(p)$ forms a dichotomy where, above a certain critical parameter p_c, the cluster $\mathscr{C}_\infty$ exists with non-zero probability, meanwhile, below the threshold p_c, clusters remain finite with probability 1. In this context, we say that a *phase transition* occurs at p_c, where the regimes $p < p_c$ and $p > p_c$ are referred to as subcritical and supercritical respectively.

Determining the exact value of p_c is one of the fundamental problems in percolation theory, which is made challenging by its dependence on the underlying graph $\mathcal{G}$. Despite this difficulty, several mathematical works have rigorously determined the exact values of critical parameters for various percolation models [37], [60], [67], and some others

developed foundational inequalities that support the determination of percolation thresholds for specific lattice models [54], [27].

Near the critical point p_c, percolation models exhibit fascinating and complex behaviours, which are typically referred to as critical phenomena. In this regime, the system displays large-scale correlations, with clusters of open vertices spanning a wide range of sizes. One of the remarkable features of this critical behaviour is its universality - despite differences in the microscopic structure of the underlying graph $\mathcal{G}$, certain macroscopic properties, such as the critical exponents that describe how quantities like the cluster size or correlation length scale near p_c, remain consistent across different systems [25]. These exponents are not influenced by the specifics of the graph but rather depend only on the dimension of the space, highlighting the deep connection between percolation theory and other areas of statistical mechanics. For instance, critical percolation in two dimensions shares characteristics with other models of phase transitions, such as the Ising model, further reinforcing the notion that percolation serves as a unifying framework for studying phase transitions in disordered systems [31].

Percolation models offer rich and intriguing problems not only at criticality but also in the regime far from it. In fact, this far-from-critical regime is central to the investigations in this thesis. When the system is away from the critical threshold, its behaviour exhibits a marked contrast to the intricate, scale-invariant phenomena observed near criticality [58], [34]. Typically, the types of problems that are instead considered are those of various observables, such as the size of the largest (or second largest) connected component, two point connectivity functions and capacity bounds - these characteristically display exponential decay in the appropriate sub/supercritical regimes. Such insights have been explored in both classical and contemporary studies, which include foundational work by [38], and more recent advancements by [36], [53], [28].

While $\mathcal{G}$ can, in theory, take on a wide variety of forms, and there has been significant research aimed at maintaining the generality of these graphs in certain models [16], [65], [42], [55], in practice, one does not need to venture far to encounter interesting and non-trivial behaviour. The Bernoulli percolation model on the integer lattice, which we explore in §1.2.1, already provides a demonstrative example of such complexity and depth.

1.2 Illustrative Examples

Let us make some of the previously mentioned terminology rigorous with the help of some illustrative examples. As before, the setup will consist of a graph $\mathcal{G} = (V, E)$, where for the purposes of this section V will be restricted to the d-dimensional integer lattice $\mathbb{Z}^d$ and E is the collection of unordered nearest-neighbour pairs of the lattice. On $\mathcal{G}$ we can also define a percolation configuration, denoted by $\omega = (\omega_v)_{v \in V} \in \{0, 1\}^V$, as one realisation of assigned vertex states, where the values 0, 1 represent closed and open vertices respectively. Finally, define a percolation model as a probability distribution on the space of percolation configurations on $\mathcal{G}$. In words, the rule for opening/closing vertices is to sample a percolation configuration from the chosen percolation model.

1.2.1 Independent Percolation

The first and simplest percolation model was the Bernoulli model on the integer lattice, which was initially studied by Broadbent and Hammersley in [7], so this is a natural starting point for us also. Depending on whether we consider the random structure as a probability distribution over the vertices or edges, one arrives at two flavours of the Bernoulli model: site or bond percolation, respectively (see Figure 1.1). In the following, we shall mostly rely on site percolation for the presentation, since, in any case, any bond percolation model can be represented as a site percolation model on, potentially, a different lattice [31].

In the Bernoulli (site) percolation model, each vertex is independently open with probability $p \in (0, 1)$ and closed with probability $(1 - p)$. To this process, we associate the probability space $(\{0, 1\}^V, \mathcal{F}, \mathbb{P}_p)$, where the σ-algebra generated by events on finitely many vertices. The independence of vertices simplifies the structure of the model, allowing the probability distribution for any finite subset of the lattice to be expressed as a product measure over the vertices. This property makes many computations tractable and positions the Bernoulli model as a fundamental starting point for both theoretical exploration and practical simulations in percolation theory.

The simplicity of the Bernoulli model is further highlighted by its computational efficiency: sampling a configuration requires an algorithm with complexity linear in the volume of the system. This allows researchers to visually explore properties like the critical parameter, cluster sizes, and continuity of observables through simulations with relative ease. Despite its simplicity, the Bernoulli model serves as a fertile ground



(a) Bond Percolation **(b)** Site Percolation

Fig. 1.1 A possible realisation for each of the bond and site Bernoulli percolation model on the integer lattice (i.e. when $V = \mathbb{Z}^d$, $E = \{(x, y) : x, y \in V\}$)

for inquiry, as many open problems - such as precise estimates of critical thresholds [37] and critical exponents - remain unsolved. Interestingly, some of these questions have clearer answers in more complex models [35], [15], which we will explore later in this work.

Given the Bernoulli model on a d-dimensional integer lattice as our working example, we can now elaborate on the questions alluded to within the general overview in a more concrete setting. First, we consider the so-called vacant set arising as the collection of open vertices in the resulting configuration, denoted as $\mathcal{V}_{\text{Bernoulli}} = \{v \in \mathbb{Z}^d : \omega_v = 1\}$. In this set, we also define the connected components as the maximal connected clusters in $\mathcal{V}_{\text{Bernoulli}}$, and return to the problem of the infinite cluster. Note that, by translation invariance of the lattice and probability measure, we lose no generality by assuming that 0 belongs to the infinite cluster [31], so let us denote the connected component of $\mathcal{V}_{\text{Bernoulli}}$ containing 0 by $\mathscr{C}$. Recalling the probability space $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{F}, \mathbb{P}_p)$ from above, we define

$$\theta(p) \coloneqq \mathbb{P}_p[|\mathscr{C}| = \infty] \tag{1.2.1}$$

where $|\mathscr{C}|$ is used to denote the cardinality of $\mathscr{C}$. For a preliminary investigation of the quantity $\theta(p)$, a natural approach is to seek lower and upper bounds in hopes of determining the critical threshold. One such argument for an upper bound in this specific example can be obtained via combinatorial reasoning as in [7]. First, observe that whenever an infinite cluster exists, for any $n > 0$ there must be a (self-avoiding) path inside $\mathcal{V}_{\text{Bernoulli}}$ of length n which starts at the origin. Since the paths are self-avoiding, they must intersect exactly n vertices, each of which is open. By independence, this

occurs with probability p^n. Moreover, since the number of choices for such a path is at most $(2d)^n$, a union bound yields the estimate $\theta(p) \leq (2dp)^n$ for *any* $n > 0$. By letting $n \to \infty$, this reasoning determines a lower bound on the critical threshold $p_c \coloneqq \sup\{p : \theta(p) = 0\}$, since $\theta(p) = 0$ whenever $p \in (0, \frac{1}{2d})$.

Noticing this, a natural immediate question is whether $\theta(p)$ is in fact non-trivial, i.e. is $p_c < 1$. In this direction, the work in [33] gives an initial confirmation of non-triviality in the case $d \geq 2$ for p sufficiently close to 1 by applying a version of Peierls's argument. This is developed even further in [7], where it is stated that there exists a $p_c(\mathcal{G}) \in (0, 1)$ for any transitive graph $\mathcal{G}$ such that $\theta(p) = 0$ for $p \in (0, p_c(\mathcal{G}))$, meanwhile $\theta(p) = 1$ for $p \in (p_c(\mathcal{G}), 1)$. Aside from determining the non-triviality of the phase transition, it has also been determined that θ_p is a continuous function of p in the supercritical regime $p > p_c$ [8] for any $d \geq 2$ and that it is continuous at the critical threshold for the special case $d = 2$. However, the general case for $d \geq 3$ remains open, with only a partial solution provided by [34] for $d \geq 19$ and recently improved in [26] to $d \geq 11$ by using methods in mean-field theory.

Further investigations of various systems at criticality were undertaken by numerous individuals during the late 20^{th} century - this sustained effort illuminated the deep connections of percolation to other disciplines, including complex and Fourier analysis. One such work by Belavin, Polyakov and Zamolodchikov in [4], [5] suggested that the scaling limit of the planar Bernoulli percolation model at criticality should be conformally invariant, that is, any angle-preserving transformation would also preserve the law of local observables in the system. Mathematically, such a conformal invariance of the model is ill-posed since it is not clear what the scaling limit of the discrete percolation model should be, so typically interfaces (separating paths between 1s and 0s) of the discrete model are considered instead. The upshot of such an approach is that the limit of a discrete collection of neighbouring points can be interpreted as a continuous curve in the space, for which we can form a rigorous foundation.

Significant progress was made in this direction with the introduction of Schramm-Loewner Evolution (SLE) by Schramm in 2000 [57]. This work identified the SLE processes as the only possible conformally invariant scaling limits of cluster interfaces in discrete percolation models at criticality. Shortly after, Smirnov [58] provided a rigorous confirmation that the scaling limit of cluster interfaces in the Bernoulli percolation model on the triangular lattice does indeed exist as the lattice mesh size

approaches zero. Furthermore, he demonstrated that this limit is precisely described by the stochastic Loewner evolution process. Building on this, Smirnov applied these results in subsequent work [59] to determine the critical exponents for the triangular lattice model, which had previously been conjectured through simulations. The key result showed that, as p converges to $p_c = \frac{1}{2}$ from above, $\theta(p) \approx (p - p_c)^{\frac{5}{36}}$, marking a decisive advancement in the understanding of discrete lattice models.

Let us now revisit the definition of the critical threshold p_c. This parameter was initially defined through the function $\theta(p)$, which, while useful, focuses solely on the existence of an infinite cluster and does not provide direct information regarding the distribution or rate of decay of cluster sizes. Given this, one could explore alternative characterizations of percolation, particularly ones that provide insight into the connectivity over finite distances. One such alternative is to define a function $\chi_N(p)$, which considers the probability that a path exists from the origin to the boundary of a finite ball S_N in the lattice:

$$\chi_N(p) = \mathbb{P}_p[0 \xleftrightarrow{\mathcal{V}_{\text{Bernoulli}}} \partial S_N], \quad N \geq 1 \tag{1.2.2}$$

where ∂S_N is the set of all vertices in $\mathbb{Z}^d$ that can be reached by a nearest-neighbor path of length exactly N starting from the origin, and $\{0 \xleftrightarrow{\mathcal{V}_{\text{Bernoulli}}} \partial S_N\}$ represents the event that such a path exists within the open cluster. From this, we can define an alternative critical parameter, p_{**}, as

$$p_{**} = \sup\{p \in [0,1] : \chi_N(p) \leq e^{-cN} \text{ for some } c > 0\} \tag{1.2.3}$$

Here, p_{**} corresponds to the highest value of p for which the probability of connectivity from the origin to ∂S_N decays exponentially with N. This raises a natural question: are the critical thresholds p_c and p_{**} identical, even though they are defined through different perspectives on percolation? This question is central to the famous problem known as the *sharpness* of the phase transition, which examines whether the transition between sub- and supercritical regimes occurs abruptly at a unique threshold or if these critical values could differ depending on the chosen criterion. For Bernoulli percolation on the d-dimensional integer lattice, the sharpness of the phase transition has been rigorously established for $d \geq 2$. One such proof appears in, for example, [24, Theorem 1.1].

We conclude this subsection with a few closing remarks that will help contextualize the forthcoming material. Firstly, it is worth noting that $\chi_N(p)$ serves as an effective

proxy for the maximal cluster size in the subcritical regime of independent percolation. In fact, the relationship in the definition of p_{**} (1.2.3) suggests that, for $p < p_{**}$, the average maximal cluster size within S_N scales as $\log N$, which aligns with the behaviour of the maximal cluster size in the classical Erdös-Rényi model, also in the subcritical regime. This observation leads to the hypothesis that these cluster sizes are intricately linked to the correlation function of points in the percolation configuration. Secondly, now within the realm of the supercritical regime ($p > p_c$), one may question the co-existence of infinite clusters. Once such a possibility is excluded, as has been done for a broad range of models in [8], investigating the sizes of the remaining, finite, clusters is a canonical step. In order to observe the non-macroscopic dynamics, often the second largest cluster is considered [12], [43]. Each of these topics will be explored in greater detail as we present our results.

1.2.2 Dependent Percolation

Previously, we considered the case of a simple Bernoulli percolation model where each of the states are independent. Despite the inherent manageability afforded to us by independence, even this base-case model challenged mathematicians for decades and still presents open problems as described above. With this in mind, the complexity introduced by the presence of correlations between states is substantial, and has forced great strides in developing the necessary methodology to handle such scenarios [55]. To demonstrate some of the complexity, we introduce the Gaussian Free Field (GFF) and study its level sets as a specific example of a percolation model exhibiting correlation. In parallel, to facilitate comparability, we remark that the Bernoulli model can also be represented as a level set of a Gaussian field with the absence of any correlation between states. In this way, let $(\Omega, \mathcal{F})$ be a measurable space on which we consider the collection of random variables:

$$\phi_x : \Omega \to \mathbb{R}, \qquad x \in \mathbb{Z}^d \tag{1.2.4}$$

and define the probability measures $\mathbb{P}^B$, $\mathbb{P}^G$ in such a way that, under each of the measures, the collection $\{\phi_x\}_{x\in\mathbb{Z}^d}$ forms a centred Gaussian field with covariances $\mathbb{E}^B[\phi_x\phi_y] = \delta_x(y)$ and $\mathbb{E}^G[\phi_x\phi_y] = g(x,y)$ respectively, where $g(\cdot,\cdot)$ is the discrete Green's function. Note that the choice of correlation function in the latter case is polynomially decaying whenever the dimension satisfies $d \geq 3$ since $g(x,y) \asymp |x-y|^{2-d}$. In what follows, we denote Φ as the cumulative distribution function of the standard normal random variable. Under the field dictated by $\mathbb{P}^B$, the level set $\{x \in \mathbb{Z}^d : \phi_x \leq \Phi^{-1}(p)\}$

is equal in law to the Bernoulli percolation model described above. Meanwhile, $\{x \in \mathbb{Z}^d : \phi_x \geq h\}$ is the GFF level set percolation model at level h, a model which has recently seen significant developments in dimensions $d \geq 3$ [16], [53], [19], [63], etc.

As expected, many of the same problems related to criticality and phase transitions presented in the realm of independent percolation are still of interest in the case of the GFF. Moreover, there are instances in which this seemingly more complex field has yielded clearer and more conclusive answers - see Theorem 1.2.1 for one such example. For some time, the phenomenology of this model has been understood to have a close interplay with potential theory and electrical networks, but more recently it has also driven developments in decorrelation and renormalisation methods. Presumably, such techniques form a blueprint for dealing with a broader range of percolation models exhibiting polynomially decaying correlations: intuitively, the former allow us to isolate the dependent effects, meanwhile the latter introduces a machinery which iterates on a priori probability bounds on certain observables, often leading to optimality at a sufficiently large scale.

This said, in the GFF case, the decorrelation process depends on an important decomposition for Gaussian fields which is not available in other models. In particular, if we consider a finite set $U \subset\subset \mathbb{Z}^d$, one can decompose the GFF ϕ as the sum of a harmonic average ξ^U (the global effects) and a field ψ^U independent of $\phi|_{U^c}$ (the local fluctuations) [53, Lem. 1.2]:

$$\psi_x^U = \phi_x - \xi_x^U, \quad \xi_x^U = \mathbb{E}_x[\phi_{X_{H_{U^c}}}, H_{U^c} < \infty] \tag{1.2.5}$$

where the expectation is taken over simple random walks on the integer lattice starting at $x \in \mathbb{Z}^d$. The potency of this approach lies in the fact that one may select a collection of disjoint sets $\{U_z\}_{z\in\mathcal{I}}$, $U_z \subset\subset \mathbb{Z}^d$ and then control each of the ξ^{U_z} to have small absolute value so that we retain only the independent collections of $\{\psi^{U_z}\}_{z\in\mathcal{I}}$ at a slightly deviated height of the level set. Then, these collections can be used to transfer certain suitable properties from a small scale to a larger one with better probabilistic approximations. To be more concrete, we will view an example in the context of 1-arm and 2-arm connectivity events, which admit the described approach.

Once again, we define a critical parameter in terms of the existence of an infinite cluster

$$h_* := \inf\{h \geq 0 : \mathbb{P}[0 \overset{\{\phi \geq h\}}{\longleftrightarrow} \infty] = 0\} \tag{1.2.6}$$

The critical parameter h_* is known to define a non-trivial phase transition for $d \geq 3$ in the sense that $0 < h_* < \infty$ (see [6], [14], [53]). Recently, it has also been shown in [19] that this critical threshold is sharp for the GFF. In particular, the result therein determines the equality $h_* = h_{**} = \bar{h}$, where h_{**} and $\bar{h}$ are critical levels characterised by the regimes displaying a stretched-exponential rate of decay of connectivity and strongly percolative properties respectively. The latter levels are defined as

$$h_{**} := \inf\left\{h \geq 0 : \text{ for all } N \geq 0,\ \mathbb{P}[0 \overset{\{\phi \geq h\}}{\longleftrightarrow} \partial B_N] \leq e^{-cN^{c'}}\right\} \tag{1.2.7}$$

$$\bar{h} := \sup\{h \geq 0 : \phi \text{ strongly percolates up to level } h\} \tag{1.2.8}$$

where ∂B_N is the boundary of a radius N supremum ball centred at the origin, and ϕ is said to strongly percolate below h whenever, for any two levels h, h' with $h > h'$, the following inequalities hold for some positive constant $c > 0$:

$$\mathbb{P}\left\{\begin{matrix}\{\phi \geq h\} \cap B \text{ has a connected} \\ \text{component of diameter } \geq \dfrac{N}{5}\end{matrix}\right\} \geq 1 - e^{-cN^{c'}} \tag{1.2.9}$$

$$\mathbb{P}\left\{\begin{matrix}\text{If there are connected components of } B \cap \{\phi \geq h\} \\ \text{and } B^{'} \cap \{\phi \geq h\} \text{ of diameters } \geq \dfrac{N}{10}, \text{ then they} \\ \text{are connected in } D \cap \{\phi \geq h^{'}\}\end{matrix}\right\} \geq 1 - e^{-cN^{c'}} \tag{1.2.10}$$

where $B = [0, N)^d$, $B^{'} = Ne + B$ with $|e| = 1$ and $D = [-3N, 4N)^d$. Combining the events in (1.2.9) and (1.2.10) yields a strongly supercritical regime of the connectivity structure of the GFF level sets, since we can glue connected clusters from neighbouring boxes to create increasingly larger interfaces and therefore pass to connected clusters at higher length-scales. This remark implies that, even before the strong result of equality was proven, the strongly supercritical regime dictated by $\bar{h}$ already imposed the existence of an infinite cluster with non-zero probability, hence $\bar{h} \leq h_*$ - this is an important consequence for models other than the GFF which have not yet seen such developments.

It turns out that, for GFF percolation model, it is possible to improve upon the precision of the decay in (1.2.7). To be precise, one can extract the exact dependence of the rate of decay in relation to the proximity between h and h_*, which essentially amounts to determining the critical exponents for this model. Such a relation has been obtained by finding upper and lower bounds on the connectivity observable which match the principle rate - the state-of-the-art results can be summarised in Theorem 1.2.1 below. Before proceeding to its statement, we highlight the significance of this result by remarking that, in the case of a d-dimensional Bernoulli percolation, such a precise conclusion is still not available.

Theorem 1.2.1. *[28] For $d = 3$ and all $h \in \mathbb{R}$,*

$$\lim_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}\left[0 \overset{\phi \geq h}{\longleftrightarrow} \partial B_N, 0 \overset{\phi \geq h}{\not\longleftrightarrow} \infty\right] = -\frac{\pi}{6}(h - h_*)^2 \tag{1.2.11}$$

Theorem 1.2.2. *[28] For every $d \geq 4$ and $h \neq h_*$, there exist $C = C(d,h), c = c(d,h) \in \mathbb{R}_{>0}$ such that*

$$e^{-CN} \leq \mathbb{P}\left[0 \overset{\phi \geq h}{\longleftrightarrow} \partial B_N, 0 \overset{\phi \geq h}{\not\longleftrightarrow} \infty\right] \leq e^{-cN} \tag{1.2.12}$$

Apart from these results on the GFF, connectivity properties have been studied on more general Gaussian fields, where a sharpness result has recently been obtained in [44] and the subcritical regime for such fields has seen some developments in [45].

Returning to the theorems above, one immediately notices the significant difference in strength between the theorems when dealing with the $d = 3$ case and $d \geq 4$ case. This relates closely to our discussion of the decomposition of the field ϕ into its local and global components, as well as to properties of the capacity of sets in the integer lattice. In particular, we remark that the capacity of a 1-dimensional line takes the form:

$$\mathrm{cap}(([0,N] \cap \mathbb{Z}) \times \{0\}^{d-1}) \sim \frac{\pi}{3} \frac{N}{\log N} \text{ when } d = 3 \tag{1.2.13}$$

$$\mathrm{cap}(([0,N] \cap \mathbb{Z}) \times \{0\}^{d-1}) \asymp N \text{ when } d \geq 4 \tag{1.2.14}$$

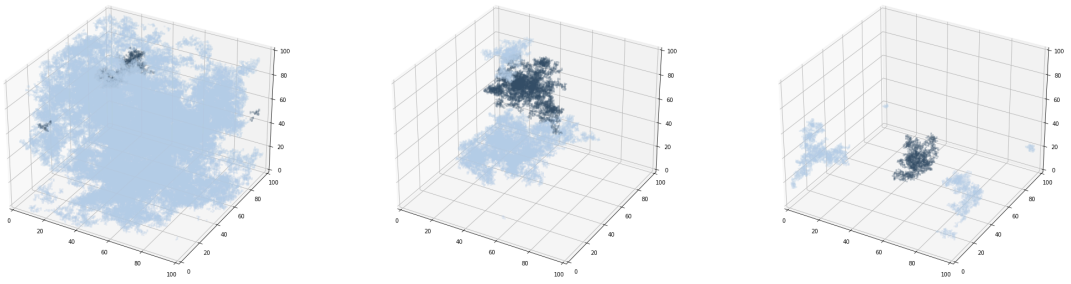
In the process of obtaining the bounds (1.2.11), (1.2.12), a coarse-graining approach is applied to pass from connections at a large scale to connections in combinations of boxes at a smaller scale, inside of which the fields are decomposed as described above. Such an approach necessitates separately controlling the local fluctuations and the

global harmonic average across the coarse-grained collections. The log-probability of the latter control failing is comparable to the capacity of a thin tube in d-dimensions, so by selecting an appropriate width for the tube, the bounds obtained asymptotically converge to the values in (1.2.13) and (1.2.14). On the other hand, the local fluctuations resulting in bad configurations have a probability which is exponentially decaying in any dimension, but with no precise constant as a prefactor. As a result, the global effects are dominating in $d = 3$, meanwhile in higher dimensions they are of the same order as the local ones, consequently requiring a more careful account of constants in order to find precise bounds depending on the level h and dimension d.

To conclude, let us more closely examine the currently available results on the correlation length for the Bernoulli model, in contrast with Theorems 1.2.1 and 1.2.2 above. To this end, we consider the (truncated) two-point connectivity event, effectively replacing $\{\phi \geq h\}$ from (1.2.11) with $\mathcal{V}_{\text{Bernoulli}}$. Although a dominant decay of $e^{-N/\xi(p)}$ for $p \neq p_c$ has been determined for this observable in the Bernoulli regime (on the d-dimensional integer lattice) [2], [32], [10], the correlation length $\xi(p)$ has not yet been given a precise form. The current best upper bound on $\xi(p)$, $p \neq p_c$ in intermediate dimensions can be found in [23], which shows $\xi(p) \leq e^{C|p-p_c|^{-2}}$. On the other hand, a sketch-proof in the same reference shows polynomial lower bounds of the form $\xi(p) \geq |p - p_c|^{-c(d)}$ which matches the expected principal rate of decay and therefore suggests there is still work to be done for the upper bounds. Together, these results depict a stark difference in comparison with Theorem 1.2.1 and even the weaker Theorem 1.2.2, where the *exact* rate of decay has been confirmed for 2-point connectivity of the GFF.

1.3 The Random Walk on a Torus

Given the current state of affairs within the realm of percolation theory, let us proceed by introducing our model of interest in this work: the simple random walk on a discrete torus. In its nascence, such a setup was an implementation of a naïve erosion model, wherein an active site is eroded, becomes inactive, and one of its neighbours is chosen uniformly at random to become the new active site. For simplicity, we assume that eroded sites are not excluded from being chosen in the final step, although the study of self-avoiding erosive models is also a widespread area of research (see [3] for an introduction). As we will come to realise, this is an algorithmically-simple, distributionally-complex example of a dependent percolation model, resulting in a



(a) u=2.5 **(b)** u=3.0 **(c)** u=3.5

Fig. 1.2 Largest and second largest connected components of the vacant set $\mathcal{V}_{100}(uN^d) \coloneqq \mathbb{T}_N^3 \setminus X_{[0,uN^d]}$.

rich list of hypotheses via numerical simulations with plenty of room for substantive rigorous conclusions - a perfect candidate for theoretical research.

Let $\mathbb{T}_N^d \coloneqq (\mathbb{Z}/N\mathbb{Z})^d$ denote a d-dimensional torus for $d \geq 3$ with side length $N \geq 1$ endowed with its usual nearest-neighbour graph structure. Also, Let Ω be the space of nearest-neighbour paths on $\mathbb{T}_N^d$, which can be viewed as functions from discrete time space to states in the torus $\Omega \ni \omega : \mathbb{Z}_{\geq 0} \to \mathbb{T}_N^d$. Then, define the projection $X_n(\omega) \coloneqq \omega(n)$ and let $X_{[0,n]}(\omega) \coloneqq \bigcup_{k=0}^n \{X_k(\omega)\}$ be the trace of the nearest neighbour path up to the n^{th} step. Finally, associate to the space of paths a canonical σ-algebra $\mathcal{F}$ (generated by all finite paths on Ω) and a probability measure P_N under which X_0 is uniformly distributed on $\mathbb{T}_N^d$ and X_n is a simple random walk (SRW) starting at X_0. With this construction, it turns out that the interesting time-scale at which to consider the trace of the SRW is when it is proportional to the volume of the torus under investigation - with such a choice, the probability of any vertex being open is bounded away from 0 and 1 uniformly in N, yielding non-trivial dynamics.

In line with our previous reasoning, the percolation model that we consider is the vacant set of the random walk, which we denote by $\mathcal{V}_N^u \coloneqq \mathbb{T}_N^d \setminus X_{[0,uN^d]}$. Compelling evidence that N^d is the appropriate time-scale can be seen visually by simulating $\mathcal{V}_N^u$ for various values of $u > 0$: a distinctive phase transition appears at a critical parameter $u_* \approx 3$ where macroscopic vacant clusters (connected subsets of $\mathcal{V}_N^u$) are present for $u < u_*$ and only tiny fragments appear for $u > u_*$, just as in the examples we have already presented theoretically (see Figure 1.2). However, since the torus is a finite space, the critical parameter cannot be rigorously defined in terms of the existence of an infinite connected cluster as before. Although the natural reaction would be to simply redefine the critical parameter, a more fruitful approach was suggested by

Sznitman in [61], wherein a framework is introduced such that a model called the random interlacements (RI) situated on the infinite lattice $\mathbb{Z}^d$ locally behaves like the projection of the SRW on a torus.

Let us briefly remark that, in the literature, there are other instances where an infinite-space model is related to a finite-space counterpart, see for example [1] for an approximation of the GFF on a torus. The percolative properties of the corresponding finite-space level set are not as well-studied as those of $\mathcal{V}^u_N$, although there has been some progress to determine cluster sizes in other finite-space models such as for certain Gaussian ensembles [46].

Intuitively, the RI are a Poissonian cloud of simple random walks where, for any finite box, the subset of the RI intersecting the box is comprised of a Poisson number of SRWs. The Poissonian parameter is selected to be $u\nu(\cdot)$, where ν is a specially-chosen intensity measure which takes into account the capacity of the finite region under consideration and u is a non-negative density parameter referred to as the level of the RI (see (2.4.9)). Since it is known that the RI are a.s. connected [61], we instead study their corresponding level u vacant set, $\mathcal{V}^u \coloneqq \mathbb{Z}^d \setminus \mathcal{I}^u$, rendering the problem meaningful with respect to connectivity. In the resulting vacant set percolation model, we define the parameters u_*, $\bar{u}$ and u_{**} as analogues of h_*, $\bar{h}$ and h_{**} respectively by replacing the level set of the GFF in equations (1.2.6), (1.2.8) and (1.2.7) by the level u vacant set of the RI. In the seminal works [20], [21], [22], the critical parameter u_*, and its alternatives $\bar{u}$, u_{**}, have since been shown to satisfy the equality:

$$u_* = \bar{u} = u_{**} \tag{1.3.1}$$

as well as to coincide with the macroscopic-fragmented transition of maximal clusters in the torus as described above.

The purpose of the present research is to conclusively quantify the size of the maximal cluster in the subcritical regime, as well as the second largest cluster in the supercritical regime - these findings are summarised in Theorem 1.3.1 and Theorem 1.3.2. To be precise, we use $\mathcal{C}^u(x) = \mathcal{C}^u_N(x)$ to denote the connected component of $x \in \mathbb{T}^d_N$, meanwhile $\mathcal{C}^u_{\max} = \mathcal{C}^u_{\max,N}$ is the cluster maximising $|\mathcal{C}^u(x)|$ over $x \in \mathbb{T}^d_N$ and $\mathcal{C}^u_{\text{sec}}$ is the cluster maximising over $x \in \mathbb{T}^d_N \setminus \mathcal{C}^u_{max}$. Such an endeavour is further justified by previously known results which ensure that $\mathcal{C}^u_{max}$ is topologically non-trivial with high probability in the supercritical case [22, (1.8)], meanwhile its diameter

has stretched exponential upper tails in the subcritical case [22, 1.7]. Up until now, the state of the art results on the observable $\mathcal{C}^u_{\max}$ could be found in [22, Thm 1.1], which itself was an improvement on [66, Thm 1.2] by virtue of the equality of critical parameters (1.3.1). Effectively, the aforementioned theorem supports the empirical view of the SRW on $\mathbb{T}^d_N$ phase transition by showing that with high P_N-probability as $N \to \infty$, and for some large $\lambda = \lambda(u, d) \in (0, \infty)$, one has

$$diam(\mathcal{C}^u_{\max}) \leq (\log N)^\lambda, \text{ for all } u > u_* \text{ and } d \geq 3 \tag{1.3.2}$$

where, given a set $\mathcal{C} \subset \mathbb{T}^d_N$ we write $diam(\mathcal{C})$ to be the diameter of $\mathcal{C}$ according to the Euclidean distance. Meanwhile, the same reference also shows that in the supercritical regime $u < u_*$, $diam(\mathcal{C}^u_{\max}) \geq N$, identifying the dichotomy between the subcritical and supercritical regimes. On the other hand, conclusions regarding the observable $\mathcal{C}^u_{\text{sec}}$ were somewhat less precise, of which the latest appears as [66, Thm 1.4], where again for high P_N-probability as $N \to \infty$ and for some $\lambda = \lambda(u, d) \in (0, \infty)$ sufficiently large,

$$diam(\mathcal{C}^u_{\text{sec}}) \leq (\log N)^\lambda, \text{ for } u \ll (1 \wedge u_*) \text{ and } d \geq 5 \tag{1.3.3}$$

Noticeably, (1.3.3) does not deal with $d = 3, 4$, nor does it guarantee the tiny-fragment behaviour in the vicinity of the critical parameter as the methodology requires a markedly lower density parameter of the random walk, i.e. the perturbative regime.

Prior to stating the theorems which develop (1.3.2) and (1.3.3), we introduce the appropriate length-scale $r_N(u)$ for consideration with regards to the diameters of the clusters $\mathcal{C}^u_{\max}$ and $\mathcal{C}^u_{\text{sec}}$:

$$r_N(u) \coloneqq \left(\sqrt{u} - \sqrt{u_*}\right)^{-2} \rho_N, \text{ for } N \geq 2, u > 0 \tag{1.3.4}$$

where

$$\rho_N = \rho_N(d) = \begin{cases} \frac{9}{\pi} \log N \cdot \log\log N, & \text{if } d = 3 \\ \log N, & \text{if } d \geq 4 \end{cases} \tag{1.3.5}$$

Theorem 1.3.1 (Lower Bounds)**.** *[29] For $d = 3$ and all $\lambda \in (0, 1)$, the following hold:*

(i) For all $u > u_*$,

$$\lim_N P_N \left[diam(\mathcal{C}^u_{max}) \geq \lambda r_N(u) \right] = 1 \tag{1.3.6}$$

(ii) For all $u < u_*$,

$$\lim_N P_N \left[diam(\mathcal{C}^u_{sec}) \geq \lambda r_N(u) \right] = 1 \tag{1.3.7}$$

Meanwhile, for the case of $d \geq 4$ the results (1.3.6), (1.3.7) remain true when replacing $r_N(u)$ with ρ_N and selecting some $\lambda \in (0, \infty)$ depending on u and d.

Theorem 1.3.2 (Upper Bounds)**.** *[30] For $d = 3$ and all $\lambda \in (1, \infty)$ one has:*

(i) For all $u > u_*$,

$$\lim_N P_N \left[diam(\mathcal{C}_{max}) \geq \lambda r_N(u) \right] = 0 \tag{1.3.8}$$

(ii) For all $u < u_*$,

$$\lim_N P_N \left[diam(\mathcal{C}^u_{sec}) \geq \lambda r_N(u) \right] = 0 \tag{1.3.9}$$

Moreover, when $d \geq 4$, the results (1.3.8), (1.3.9) still hold for some $\lambda(u, d) \in (1, \infty)$ if we replace $r_N(u)$ with ρ_N.

Clearly, the potency of the above theorems can only be appreciated in their common context. Together, they form probability bounds on the diameters of $\mathcal{C}^u_{\max}$ and $\mathcal{C}^u_{\text{sec}}$ which match in their principal rate of decay up to a multiplicative constant. Remarkably, in $d = 3$ we observe the precise relationship between the rate and the proximity of the density parameter u to the critical threshold u_*. The demonstrated relationship shows a polynomial divergence, suggesting that cluster sizes at criticality are in fact macroscopic. Moreover, again in the case $d = 3$, one observes an extra $\log \log N$ factor, which has previously appeared in other long-range correlation models such as the Gaussian free field. This additional term distinguishes the behaviour of such models from the independent model discussed above, and suggests that they form a common universality class.

Key components in showing the results (1.3.6)-(1.3.9) are obtained in the infinite volume scenario of the RI vacant set model, where we prove the decay rate of truncated

two-point connectivity events. For any $x, y \in \mathbb{Z}^d$, let us define

$$\tau_v^{tr}(x,y) := \mathbb{P}\left[x \overset{\mathcal{V}^v}{\longleftrightarrow} y, x \overset{\mathcal{V}^v}{\not\longleftrightarrow} \infty\right] \tag{1.3.10}$$

Since $\tau_v^{tr}(x,y)$ is symmetric and translation-invariant due to the properties of the vacant set, we have $\tau_v^{tr}(x,y) = \tau_v^{tr}(0,y-x)$ and write $\tau_v^{tr}(x) := \tau_v^{tr}(0,x)$. Dealing with the above quantity is more straightforward in the subcritical regime $u > u_*$ since the truncation occurs with probability 1 and has no effect on the overall decay. However, in the supercritical regime $u < u_*$, the disconnection from infinity must be handled more carefully, taking into account that the comprising events of (1.3.10) have opposite monotonicities as functions of v. Remark that, although previous works [22] could be adapted in a relatively effortless fashion in order to deduce stretched-exponential bounds on a sprinkled version of the above event, i.e. where the disconnection occurs at a strictly lower level $u < v$, the results we present hereafter crucially maintain the same level and require an alternative strategy. In the theorem below, we give precise asymptotics of the truncated two-point connectivity function at all non-critical levels.

Theorem 1.3.3. *[29], [30] For all $u \neq u_*$, denoting by $|\cdot|$ the Euclidean distance on $\mathbb{Z}^d$, there exist constants $c(u,d) \in (0,1)$, $C(u,d) \in (0,\infty)$ such that*

$$\frac{1}{|x|}\log \tau_u^{tr}(x) \in [-C(u,d), -c(u,d)], \ \textit{whenever } d \geq 4 \tag{1.3.11}$$

$$\lim_{|x|\to\infty} \frac{\log|x|}{|x|}\log \tau_u^{tr}(x) = -\frac{\pi}{3}\left(\sqrt{u} - \sqrt{u_*}\right)^2, \ \textit{whenever } d = 3. \tag{1.3.12}$$

As in the finite volume case, lines (1.3.11) and (1.3.12) are shown by proving the respective upper and lower bounds separately, where vastly different approaches are applied. Despite the differences, the obstacles encountered are thematic, two of which have already been referred to - long range correlations and opposing monotonicities.

Let us briefly illustrate how these obstacles arise with two prominent examples. The presence of the correlations encumbers coarse-graining efforts in the case of upper bounds, since when lifting from a small scale to a larger scale we can no longer directly view the small-scale events as independent copies. On the other hand, the appearing opposite monotonicities often lead to sprinkled versions of the required events since we seek to use the advantageous notions of sub-/supercriticality where relevant. However, in this work one desires a stronger statement, rendering such methods unsatisfactory. Evidently, such complications have already been encountered in the case of the GFF

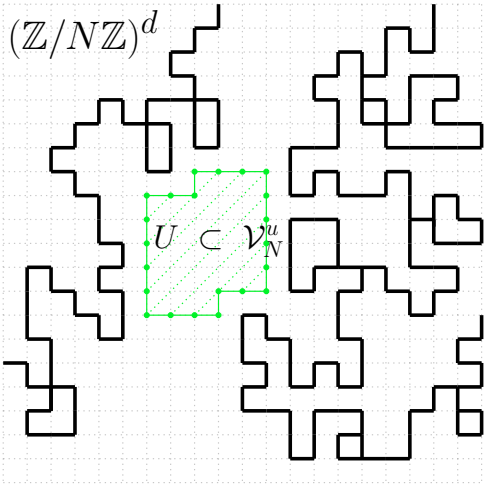


Fig. 1.3 A depiction of structural rigidity of $\mathcal{V}_N^u$. Conditional on the green vertices being open, the interior of the boundary they form must also be open with probability 1 since no random walk could have entered this region without crossing the boundary.

percolation model which we have briefly discussed and suggested solutions to. Despite these modifications, our percolative model also exhibits an additional impediment in the form of a certain structural rigidity which precludes the finite energy property. The latter property would guarantee that, conditional on surrounding states, every state of the underlying graph has a probability $p \in (0, 1)$ of being open, and is clearly present in all previously discussed examples. In the absence of this condition, probabilities of certain collections of events exude meaningless lower bounds of 0 and do not lend themselves well for our purposes.

To be more concrete, we describe how this poses a problem when dealing with the supercritical upper bound, with reference to the method applied in [28] which relies on the aforementioned finite energy condition. As alluded to, in this regime we seek to connect to a large radius via the vacant set while avoiding the unique infinite cluster, for otherwise the cluster is itself infinite. When doing so, the finite cluster is gradually explored and at each point where it comes within close proximity of the infinite cluster, we may form an upper bound on the truncated connection probability by opening every vertex inside a small box containing both the finite and infinite cluster. The lower bound on the probability of such an event occuring in the GFF case is strictly positive, whereas it may well be 0 in the presence of the structural rigidity we have described. Clearly, considering intersections of such events then does not iteratively yield an improved bound via renormalisation methods.

1.4 Outline of Results

Let us now turn to describing the format of the exposition. Chapter 2 intends to provide an introduction to all the key concepts and results invoked thereafter. In particular, in §2.2 we discuss potential theory, which deals with Green's functions, equilibrium measures and capacity in the continuous time setting. §2.3 computes some (new) results on the capacities of oblique tubes which will be relied on extensively in the proof of the lower bounds. §2.4 provides a rigorous introduction to Poisson Point Processes (PPP) and the infinite volume percolation model of interest - the Random Interlacements, including any notation pertaining to them. Then, we proceed to describing some results which, although could be included in the main body of the lower bound results, are presented in the preliminaries to improve readability. §2.5 introduces some additional concepts and identities on PPPs, together with more general ideas such as stochastic domination and the entropy computation. §2.6 discusses the soft local time method which is crucial in both the lower and upper bound results due to its mileage in constructing couplings between Markov chains (as we will see in §3.4, the RI can be reformulated to conform to the underlying assumptions). From this point onwards, the chapters are dedicated to the main results.

Chapter 3 deals with all the material used to show the lower bounds asserted by Theorem 1.3.1 and the relevant parts of Theorem 1.3.3. We commence by introducing a generalized lower bound in §3.1 in the form of Theorem 3.1.1. Moreover, in the same section we subsequently show that the general result is sufficiently flexible to handle the proofs of all the stated lower bounds by selecting the events G^u therein appropriately (see below (3.1.1)). The proof for the infinite volume percolation case is almost immediate, and requires us to use some conditional finite energy properties in the proximity of the points 0 and x which we intend to connect. On the other hand, the proof for the torus reduces to a second moment method applied to a collection of specially chosen events which imply the required conditions on $\mathcal{C}^u_{\max}$ and $\mathcal{C}^u_{\text{sec}}$ (see below 1.3.1 for their definitions). In the subcritical case, these consist of one-arm connectivity events whereas in the supercritical case we deal with the insulated one-arm connectivity events instead. As we will see, the lack of monotonicity in the latter events introduces some extra difficulties which justify the full generality of Theorem 3.1.1. With this in mind, the remaining sections are therefore devoted to proving the fully general result claimed by Theorem 3.1.1. The strategy of the proof is to spatially modulate the density of the random interlacements in order to profit from known-results contextualised by the subcritical and supercritical regimes regarding

various connectivity probabilities. Henceforth, we refer to such a modification as 'tilting' the RI, and present it rigorously in §3.2. In the subcritical regime, we seek to tilt in such a way that the interlacements have an effective level below u_* inside a narrow elongated tube containing 0 and x. With this effective level, by virtue of the strongly percolative regime, we are then in a position to iteratively apply properties akin to (1.2.9),(1.2.10) to form a connection in the vacant set between small boxes containing 0 and x. As a consequence, the problem of obtaining lower bounds is reframed as one of computing the cost of the suggested spatial modulation. Finally, in §3.4 we justify the proposed tilting scheme by proving that the effective levels post-modification are as claimed. This relies heavily on technical results in §3.3 which propose estimates on tilted and natural harmonic/equilibrium measures, as well as on the soft local time method introduced in the preliminaries.

Finally, in Chapter 4 we will wrap things up by showing the remaining upper bounds proposed by Theorem 1.3.2 and 1.3.3. In preparation for these, a substantial amount of groundwork is necessary, which we now overview and contextualise. Throughout the course of this Chapter, we repurpose renormalisation ideas previously encountered in the literature (see for example [28], [64] etc.), with the purpose of bootstrapping small-scale a priori bounds to more precise bounds at larger scales, thereby more accurately characterising the system macroscopically. Once the necessary tools are presented, the subcritical upper bounds are obtained almost immediately, whereas the supercritical case requires an even more intricate approach, both regarding the choice of a priori events as well as with respect to book-keeping of various rates of decay and sub-collections of excursions. A key theme which will play a central role in determining the 'correct' events to bootstrap in the supercritical case is that of local uniqueness, which essentially prevents the existence of any large, finite clusters in this regime. In §4.1, we collect all the geometric properties that will be necessary, with the highlights appearing in Proposition 4.1.1 and Lemma 4.1.5. The first of these describes a kind of duality between collections of paths and surrounding interfaces, meanwhile the latter guarantees connectivity of large clusters via good points under a certain 'n-good' condition (see (4.1.27)). The n-goodness property admits some amount of flexibility and has a convenient bound available in the form of Lemma 4.1.3. To describe this bound, the generic n-goodness events are associated to a sequence of increasing scales L_n obtained geometrically after choosing an initial L_0 and multiplicative factor λ_0. Then, given some estimate at the smallest scale L_0, a super-exponentially decaying bound in n is obtained at *any* scale L_n. This salient

feature allows us to boost estimates from a small, *fixed*, scale L_0 to possibly very large scales L which will subsequently be fed into the renormalisation machinery. In §4.2 we build on the soft local time material presented in §2.6 as well as applied in the context of tilted Random Interlacements in §3.4. This is, predictably, done with the aim of localising the vacant set in collections of disjoint boxes such that we can profit from the resulting independence and numerology. Naturally, we also quantify the probability-cost of such localisation methods in §4.2.2. In §4.3 we present the renormalisation techniques in three distinct steps, which will be discussed in more detail in the preamble there. First, in §4.3.1 a suitable coarse-graining of paths is identified which allows us to represent connectivity by paths instead as a collection of 'good' boxes at scale L, which themselves display likely properties. Then, in §4.3.2 an estimate is determined for the joint-occurrence of such good boxes paving any crossing path - this joint occurrence is represented by the event $\mathscr{G}$. Finally, in §4.3.3, we present how the event $\mathscr{G}$ is subsequently related to the comprising good events at a larger scale N. At this stage, note that the bootstrapping step in this Section exhibits an intimate connection between L and N, such that both will eventually be considered in the limit $L, N \to \infty$ - this is in contrast with the geometric renormalisation of §4.1.2. In §4.4, we finally provide the proofs of the main results, contingent on the previous sections as well as certain technical results discussed in §4.5. The subcritical case arises as a streamlined version of the bootstrapping method, where the 'good' a priori event takes the form of a simple disconnection event (which is indeed likely in the subcritical regime). For the supercritical case, the overarching idea is to construct very many surrounding interfaces which contain vacant clusters that are 'difficult' to avoid. The latter statement will be made precise in the steps leading to yet another local uniqueness condition, the estimate for which will be determined by the series of technical results concluding the thesis. Refer to (4.4.15) for the aforementioned event and Proposition 4.4.5 for the corresponding estimate. In order to deal with the non-monotonicity of this event, especially in the context of iterative localisation/delocalisation steps, we introduce various collections of subsequences of the RI excursions. These collections are chosen in such a way as to be sufficiently exhaustive, in the sense that we still have a plethora of Random Interlacements results at hand, applicable to the chosen subsequences, while still retaining a tenable complexity i.e. the cardinalities of the collections are not too large. Such features permit for a relatively frictionless supercritical bootstrapping step, which is initialised by the local uniqueness event occurring for *every* subsequence of the discussed collection. By additionally incorporating truncations of excursions in these collections of subsequences, these results can be transposed to the case of the SRW on a

torus by yet another coupling. The truncation is necessary to account for the start and end points of the random walk, which terminate the path abruptly - a behaviour not natively present in the cloud of bi-infinite trajectories of the Random Interlacements model. The details of this are discussed in §4.4.3. In the final section of the thesis, §4.5, we handle all the intricate details relating to the estimate on the supercritical local uniqueness event (§4.5.1) as well as various a priori bounds which are crucial for the supercritical bootstrapping (§4.5.3). At the heart of these details lies the finite energy technique, Lemma 4.5.1, which ensures that a cost is paid whenever a path comes close to a vacant surrounding interface, which is itself constructed by bootstrapping small-scale local uniqueness events via the event $\mathscr{G}$. Note that, in the Finite Energy Lemma we actually present two conditional vacancy results - this is an artefact of our choice of excursion collections, which consist of both *large* subsequences and *small* subsequences, identified according to whether they contain a large consecutive sequence of the RI excursions. For the large subsequences, we can only afford to empty a small box to ensure connections locally, whereas for the small subsequences we instead resort to an argument inspired by [18], where we create a kind of vacant lattice of edges at a sufficiently low level of the Random Interlacements, accounting for the stray excursions comprising the small subsequences. The proof of both of these results is presented in §4.5.2, and relies on excursion rerouting arguments.

Chapter 2

Preliminaries

Most of the material in this chapter, with the exception of §2.3, can be found in the literature, with the relevant references provided as the results are stated. Many of the introductory materials on the RI are sourced from [17] which provides a useful overview of some of the main properties of the percolation model, meanwhile the potential theory results can, for example, be found in [62]. §2.3 on the other hand presents some new estimates on the capacities of oblique tubes, which consequently facilitate the precision of the lower bound estimates for 2-point connectivity in the RI vacant set. Finally, in §2.6 we provide a more general version of [11, Lemma 2.1] in the form of Theorem 2.6.2, and prove the corresponding extension.

2.1 Notation

Let us now fix the notation to be used throughout this thesis. Let $d \geq 1$ and consider $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$. We shall represent the usual l^p-norm of x by $|x|_p$, recalling that

$$|x|_p := \begin{cases} \left\{ \sum_{i=1}^d |x_i|^p \right\}^{\frac{1}{p}} & p < \infty \\ \max_i(|x_i|) & p = \infty \end{cases} \tag{2.1.1}$$

In the above, we shall write $|x| = |x|_2$ to simplify notation for the Euclidean norm as it is frequently used. Moreover, we will write $B(x, r) := \{y \in \mathbb{Z}^d : |y - x|_\infty \leq r\}$ and $B^p(x, r) := \{y \in \mathbb{Z}^d : |y - x|_p \leq r\}$ as the closed supremum and p-balls respectively, centred at x with radius r. In the sequel, we will often refer to l_∞-balls simply as boxes. Moreover, given a point $x \in \mathbb{Z}^d$ and a set $A \subset \mathbb{Z}^d$ we will write $d_p(x, A) = \inf_{y \in A} |x - y|_p$ and for any point $z \in \mathbb{R}^d$ denote by $[z] \in \mathbb{Z}^d$ the point achieving $d_2(z, \mathbb{Z}^d)$, i.e. $[z]$ is the point in the integer lattice closest to z according to Euclidean distance (to decide

in the case of a draw, one can simply assign an ordering to points in the lattice). Given any two points $x, y \in \mathbb{Z}^d$, they will be called nearest-neighbours ($*$-neighbours), whenever they satisfy $|x-y| = 1$ ($|x-y|_\infty = 1$), and we will write $x \sim y$ ($x \overset{*}{\sim} y$) in this instance. Moreover, a (nearest-neighbour) path $\gamma \subset \mathbb{Z}^d$ is an ordered collection of points $\{\gamma(0), \dots, \gamma(n)\}$ satisfying $\gamma(i) \sim \gamma(i+1)\ \forall i \in \{0, n-1\}$. If the condition is weakened to $\gamma(i) \overset{*}{\sim} \gamma(i+1)\ \forall i \in \{0, n-1\}$ then we call γ a $*$-path.

Throughout, we use $\mathbb{Z}^d$ interchangeably to denote either the set of points on the integer lattice or the graph consisting of such points endowed with all edges between nearest-neighbour points. On the lattice $\mathbb{Z}^d$, we also define the following function spaces:

$$L^2(\mathbb{Z}^d) = \{f \in \mathbb{R}^{\mathbb{Z}^d} : \sum_{x \in \mathbb{Z}^d} |f(x)|^2 < \infty\} \tag{2.1.2}$$

$$H(\mathbb{Z}^d) = \left\{ f \in \mathbb{R}^{\mathbb{Z}^d} : \frac{1}{2} \sum_{x,y \in \mathbb{Z}^d} \frac{1}{2d} \Big(f(y) - f(x)\Big)^2 < \infty \right\} \tag{2.1.3}$$

Given $f, g \in \mathbb{R}^d$, we write $\langle f, g \rangle$ to denote the L^2-scalar product

$$\langle f, g \rangle = \sum_{x \in \mathbb{Z}^d} f(x) g(x) \tag{2.1.4}$$

Furthermore, ∂A denotes the boundary of A when $A \subset \mathbb{R}^d$ and $\partial A, \partial^{\text{out}} A$ denote the inner and outer boundary respectively of A when $A \subset \mathbb{Z}^d$:

$$\partial A \coloneqq \{x \in A : \exists y \notin A \text{ st } x \sim y\}, \quad \partial^{\text{out}} A \coloneqq \{x \in \mathbb{Z}^d \setminus A : \exists y \in A \text{ st } x \sim y\} \tag{2.1.5}$$

For any $U, V, W \subset \mathbb{Z}^d$, we write $U \overset{V}{\leftrightarrow} W$ whenever there exists a nearest neighbour path $\gamma = \{\gamma(0), \dots, \gamma(n)\} \subset V$ with $\gamma(0) \in U$, $\gamma(n) \in W$. If U or V are point sets, we replace the set with the element in the above notation, eg. $x \overset{V}{\leftrightarrow} W$ when $U = \{x\}$ for some $x \in \mathbb{Z}^d$. Moreover, for x as before, we use the notation $x \overset{B}{\leftrightarrow} \infty$ to mean $x \overset{B}{\leftrightarrow} \partial B(x, N)\ \forall N \in \mathbb{N}^*$. We say that U is ($*$-)connected if for any $x, y \in U$ there is a ($*$-)path $\gamma \subset U$ connecting x and y. A ($*$-)connected component of U, denoted as $\mathcal{C} \subset U$, is defined as a maximal ($*$-)connected set, in the sense that any $x \in U$ connected to $\mathcal{C}$ by a ($*$-)path in U must also satisfy $x \in \mathcal{C}$.

Finally, for any two functions $f, g : \mathbb{Z}^d \to \mathbb{R}$, we write $f \sim g$ if

$$\lim_{|x|\to\infty} \frac{f(x)}{g(x)} = 1 \tag{2.1.6}$$

Meanwhile, we write $f \asymp g$ if there are two positive constants c, C such that for any $x \in \mathbb{Z}^d$.

$$cf(x) \leq g(x) \leq Cf(x) \tag{2.1.7}$$

2.2 Review of Potential Theory

Let W^+ denote the space of $\mathbb{Z}^d \times [0, \infty)$-valued sequences satisfying the following properties:

(i) the sequence of first coordinates $\{Z_n(w)\}_{n\in\mathbb{Z}_{\geq 0}}$ is a nearest-neighbour path in $\mathbb{Z}^d$ with finitely many entries belonging to finite sets $K \subset\subset \mathbb{Z}^d$,

(ii) the sequence of second coordinates $\{\xi_n(w)\}_{n\in\mathbb{Z}_{\geq 0}}$, which represents the waiting times of the path at the corresponding state, has an unbounded sum of entries,

where we have denoted an element $w \in W^+$. We similarly define W as the space of doubly-infinite $\mathbb{Z}^d \times [0, \infty)$-valued sequences where the properties above hold for $\{Z_n(w)\}_{n\in\mathbb{Z}_{\geq 0}}$, $\{\xi_n(w)\}_{n\in\mathbb{Z}_{\geq 0}}$ as well as $\{Z_n(w)\}_{n\in\mathbb{Z}_{<0}}$, $\{\xi_n(w)\}_{n\in\mathbb{Z}_{<0}}$. Aside from the coordinate projections, we also introduce time projections on $w \in W^+$ as the collection of random variables $X_\cdot(w) : [0, \infty) \to \mathbb{Z}^d$ by writing $\Xi_0 \coloneqq 0$, $\Xi_n \coloneqq \Xi_{n-1} + \xi_{n-1}$ for $n \geq 1$ and letting for $t \geq 0$

$$X_t(w) = Z_n(w) \text{ whenever } \Xi_n(w) \leq t < \Xi_{n+1}(w) \tag{2.2.1}$$

Formally, X_t allows us to view the elements $w \in W^+$ as continuous time nearest neighbour paths in $\mathbb{Z}^d$. By a similar approach for $w \in W$, we can view $X_\cdot(w)$ as a map $X_\cdot(w) : (-\infty, \infty) \to \mathbb{Z}^d$, with the subtlety that $X_0(w)$ is defined to be first time we are at state Z_0, and for $n > 0$, $\Xi_{-n} \coloneqq \Xi_{-n+1} - \xi_{-n}$ such that for $t > 0$

$$X_{-t}(w) = Z_{-n}(w) \text{ whenever } \Xi_{-n} \leq -t < \Xi_{-n+1} \tag{2.2.2}$$

Moreover, we introduce the discrete time shift for $n \in \mathbb{Z}$ as the map $\theta_n : \mathcal{W} \to \mathcal{W}$ satisfying $\theta_n(w(\cdot)) = w(\Xi_n + \cdot)$. In particular, when viewed in terms of the discrete skeleton, we have $Z_\cdot(\theta_n(w)) = Z_{\cdot+n}(w)$.

To the path spaces W^+, W we associate the σ-algebras $\mathcal{W}^+$, $\mathcal{W}$ generated by the coordinate maps Z_n, ξ_n. Then, define P_x for any $x \in \mathbb{Z}^d$ to be the probability distribution on $(W^+, \mathcal{W}^+)$ such that $Z = (Z_n)_{n\in\mathbb{N}}$ is a $\mathbb{Z}^d$-valued process with the law of a simple random walk on $\mathbb{Z}^d$ starting at x, and $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables exponentially distributed with mean 1 and independent from Z. For some $K \subset\subset \mathbb{Z}^d$, $A \in \mathcal{W}^+$, and measure $\mu(\cdot)$ supported on K, we also write

$$P_\mu(A) = \sum_{y\in K} \mu(y) P_y(A) \tag{2.2.3}$$

Moreover, letting $U \subset \mathbb{Z}^d$ and recalling that $\xi_i(w)$ is the i^{th} exponential waiting time for some SRW $w \in W$, we denote the entrance, hitting and exit times in the following familiar fashion:

$$\begin{aligned} H_U &\coloneqq \inf\{t : X_t \in U\} \\ \tilde{H}_U &\coloneqq \inf\{t \geq \xi_0 : X_t \in U\} \\ T_U &\coloneqq \inf\{t : X_t \notin U\} \end{aligned} \tag{2.2.4}$$

For the purposes of the exposition, let us also define the sub-Markovian transition semi-group $R_{t,U}$ for $t \geq 0$, $f : \mathbb{Z}^d \to \mathbb{R}$ and some $x \in \mathbb{Z}^d$, $U \subset \mathbb{Z}^d$

$$R_{t,U} f(x) = E_x[f(X_t), t < T_U] \tag{2.2.5}$$

For the special case of $U = \mathbb{Z}^d$, we have an explicit summation expansion:

$$R_{t,\mathbb{Z}^d} f(x) = E_x[f(X_t)] = \sum_{n\geq 0} e^{-t} \frac{t^n}{n!} E_x[f(Z_n)] \tag{2.2.6}$$

We will also consider the one-step transition operator $P : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$

$$Pf(x) = E_x[f(X_{\xi_0})] = E_x[f(Z_1)] = \sum_{y\sim x} \frac{1}{2d} f(y) \tag{2.2.7}$$

Given these semigroups, we define the discrete Laplacian Δ, Dirichlet function and energy on the space of functions $f, g : \mathbb{Z}^d \to \mathbb{R}$ as:

$$\Delta f(x) = (P - I) f(x) \tag{2.2.8}$$

where I is simply the identity, $If \equiv f$.

$$E_U(\nu, \mu) \coloneqq \sum_{x,y} \nu(x) g_U(x, y) \mu(y), \tag{2.2.9}$$

$$\mathcal{E}(f, g) \coloneqq \frac{1}{2} \sum_{x \sim y} \frac{1}{2d} \Big(f(x) - f(y)\Big)\Big(g(x) - g(y)\Big). \tag{2.2.10}$$

Remark that both $E_U(\cdot, \cdot)$, $\mathcal{E}(\cdot, \cdot)$ are positive definite, symmetric bilinear forms. When dealing with the entropy calculation in the lower bounds, we will also rely on the following discrete version of the Gauss-Green theorem.

Theorem 2.2.1 (Discrete Gauss-Green)**.** *Given $f \in H$, $g \in L^2$, then*

$$\mathcal{E}(f, g) = -\langle \Delta f, g \rangle \tag{2.2.11}$$

Alternatively, (2.2.11) also holds when at least one of f or g is compactly supported.

Proof. The result follows immediately from a direct computation:

$$\begin{aligned}
\langle \Delta f, g \rangle &\overset{(2.1.4)}{=} \sum_{x \in \mathbb{Z}^d} \Delta f(x) g(x) \overset{(2.2.7),(2.2.8)}{=} \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \frac{1}{2d} \Big(f(y) - f(x)\Big) g(x) \mathbb{1}\{y \sim x\} \\
&= \frac{1}{2} \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \frac{1}{2d} \Big(f(y) - f(x)\Big) g(x) \mathbb{1}\{y \sim x\} \\
&\qquad + \frac{1}{2} \sum_{y \in \mathbb{Z}^d} \sum_{x \in \mathbb{Z}^d} \frac{1}{2d} \Big(f(x) - f(y)\Big) g(y) \mathbb{1}\{y \sim x\} \\
&= -\frac{1}{2} \sum_{\substack{x,y \in \mathbb{Z}^d \\ x \sim y}} \frac{1}{2d} \Big(f(y) - f(x)\Big)\Big(g(y) - g(x)\Big) \\
&\overset{(2.2.10)}{=} -\mathcal{E}(f, g)
\end{aligned} \tag{2.2.12}$$

□

It will often be useful to represent properties of the simple random walk in terms of the killed Green's function, defined as follows for $x, y \in \mathbb{Z}^d$ and $U \subset \mathbb{Z}^d$:

$$g_U(x, y) \coloneqq \int_0^\infty R_{t,U} \mathbb{1}_y(x) dt = E_x\Big[\int_0^{T_U} \mathbb{1}\{X_t = y\} dt\Big] \tag{2.2.13}$$

When $U = \mathbb{Z}^d$, we simply obtain the Green's function $g(x, y) = g_{\mathbb{Z}^d}(x, y)$. The killed Green's function, and therefore Green's function as a special case, satisfy symmetry

and finiteness, meanwhile $g(\cdot,\cdot)$ is also translation invariant. Namely, for all $x, y \in \mathbb{Z}^d$:

$$g(x,y) = g(0, y-x), \quad g(x,y) = g(y,x), \quad g_U(x,y) = g_U(y,x) \tag{2.2.14}$$

Furthermore, we have the following asymptotic bounds for each of $g(\cdot,\cdot)$ and $g_{B_N}(\cdot,\cdot)$ as $|x|$ tends to infinity (but $x \in B_N$ for the killed Green's case):

$$g(0,x) = \bar{c}_0|x|^{2-d} + O(|x|^{1-d}), \quad g_{B_N}(0,x) = \bar{c}_0(|x|^{2-d} - N^{2-d}) + O(|x|^{1-d}) \tag{2.2.15}$$

Given the Green's function above (which can be found, for example, in [41, (1.7),(1.10)]), we also find it instructive to define the following closely related operator:

$$G_U f(x) = \sum_{y \in \mathbb{Z}^d} g_U(x,y) f(y) \tag{2.2.16}$$

It turns out that this operator is anti-commuting with the discrete Laplacian defined in (2.2.8).

Lemma 2.2.2. *Let $U \subset \mathbb{Z}^d$ and $f \in \mathbb{R}^{\mathbb{Z}^d}$. For any $x \in U$,*

$$\Delta G_U f(x) = -f(x) \tag{2.2.17}$$

Proof. First, consider any $y \in U$, then

$$\begin{aligned}
g_U(x,y) &\overset{(2.2.13)}{=} \int_0^\infty R_{t,U}\mathbb{1}_y(x)dt \overset{(2.2.5)}{=} \int_0^\infty E_x[\mathbb{1}_y(X_t), t < T_U]dt \\
&= \int_0^\infty \sum_{n\geq 0} E_x[\Xi_n \leq t < \Xi_{n+1} \leq T_U, \mathbb{1}_y(Z_n)]dt \\
&\overset{Z_0 = X_0 \in K \subset U}{=} \int_0^\infty E_x[0 \leq t < \Xi_0, \mathbb{1}_y(Z_0)]dt \\
&\qquad + \int_0^\infty \sum_{n\geq 1} E_x[\Xi_n \leq t < \Xi_{n+1} \leq T_U, \mathbb{1}_y(X_{\Xi_n})]dt \\
&\overset{\text{Strong Markov}}{=} \delta_x(y) + \int_0^\infty \sum_{n\geq 0} E_x[E_{X_{\xi_0}}[\Xi_n \leq t < \Xi_{n+1} \leq T_U, \mathbb{1}_y(X_{\Xi_n})]]dt \\
&= \delta_x(y) + E_x\left[\int_0^\infty R_{t,U}\mathbb{1}_y(Z_1)dt\right]
\end{aligned} \tag{2.2.18}$$

Then, by applying the scalar product (2.1.4) with f over $y \in \mathbb{Z}^d$ to both the left and right hand side one obtains:

$$G_U f(x) = f(x) + E_x \left[\int_0^\infty R_{t,U} f(Z_1) dt \right] \overset{(2.2.7)}{=} f(x) + P G_U f(x) \tag{2.2.19}$$

Hence rearranging and recalling (2.2.8) yields the required result. □

Having defined the Green's functions, we must recall the closely related definitions of equilibrium, capacity and their killed counterparts. Given two sets $K \subset U \subset \mathbb{Z}^d$, we first define the equilibrium and relative equilibrium measures:

$$e_K(x) \coloneqq P_x[\tilde{H}_K = \infty] \mathbb{1}_K(x), \quad e_{K,U}(x) \coloneqq P_x[\tilde{H}_K > T_U] \mathbb{1}_K(x) \tag{2.2.20}$$

Moreover, define the relative capacity directly as a sum of the equilibrium measures:

$$\mathrm{cap}_U(K) = \sum_{x \in K} e_{K,U}(x) \tag{2.2.21}$$

For any $x \in \mathbb{Z}^d$ we will also consider the standardized equilibrium measure:

$$\bar{e}_{K,U}(x) \coloneqq \frac{e_{K,U}(x)}{\mathrm{cap}_U(K)} \tag{2.2.22}$$

With these definitions, let us take the opportunity to introduce two useful results related to equilibrium measures which we will rely on henceforth. First of all, we have the following last-exit decomposition result.

Lemma 2.2.3. *For $x \in \mathbb{Z}^d$ and $K \subset\subset U \subset \mathbb{Z}^d$,*

$$P_x[H_K < T_U] = G_U e_{K,U}(x) = \sum_{y \in K} g_U(x,y) e_{K,U}(y) \tag{2.2.23}$$

Secondly, we introduce the following sweeping identity, c.f. [63, (1.12)], which asserts that

$$e_K(y) = P_{e_{K'}}[H_K < \infty, X_{H_K} = y], \quad \forall K \subset K' \subset\subset \mathbb{Z}^d, \ y \in \mathbb{Z}^d. \tag{2.2.24}$$

Aside from the direct definition above, there are two classical variational formulations for relative capacity which we will rely on. The resulting characterisations are given by the following lemma, where we note that the first is in terms of the reciprocal of the energy, meanwhile the second relates capacity to the Dirichlet form.

Lemma 2.2.4. *Let $K \subset\subset U \subset \mathbb{Z}^d$ and recall $E_U(\cdot,\cdot)$, $\mathcal{E}(\cdot,\cdot)$ from (2.2.9), (2.2.10) respectively. We have that*

$$cap_U(K) = \frac{1}{\inf_\nu E_U(\nu,\nu)} \tag{2.2.25}$$

where the infimum runs over all probability measures ν supported on K, with its minimum attained at $\nu = \bar{e}_{K,U}$. Moreover, we have

$$cap_U(K) = \inf_{f\in\mathcal{F}} \mathcal{E}(f,f) \tag{2.2.26}$$

where now the infimum is over all functions in the set

$$\mathcal{F} \coloneqq \{f : f(x) = 1 \ \forall x \in K, f(x) = 0 \ \forall x \notin U\} \tag{2.2.27}$$

and the unique minimiser of the Dirichlet form in (2.2.26) is $f(x) = h_{K,U}(x) \coloneqq P_x[H_K < T_U]$.

Proof. Since $E_U(\cdot,\cdot)$ is a positive definite, symmetric bilinear form, for an arbitrary probability measure ν supported on K we have

$$\begin{aligned} 0 \le E_U(\nu,\nu) &= E_U(\nu - \bar{e}_{K,U} + \bar{e}_{K,U}, \nu - \bar{e}_{K,U} + \bar{e}_{K,U}) \\ &= E_U(\nu - \bar{e}_{K,U}, \nu - \bar{e}_{K,U}) + 2E_U(\nu - \bar{e}_{K,U}, \bar{e}_{K,U}) + E_U(\bar{e}_{K,U}, \bar{e}_{K,U}) \end{aligned} \tag{2.2.28}$$

but the second term disappears, since

$$\begin{aligned} E_U(\nu - \bar{e}_{K,U}, \bar{e}_{K,U}) &= \sum_{x,y\in\mathbb{Z}^d} (\nu(x) - \bar{e}_{K,U}(x)) g_U(x,y) \bar{e}_{K,U}(y) \\ &\overset{(2.2.23)}{=} \sum_{x\in K} (\nu(x) - \bar{e}_{K,U}(x)) \frac{1}{\mathrm{cap}_U(K)} = 0 \end{aligned} \tag{2.2.29}$$

Hence the unique minimiser of (2.2.28) is $\nu = \bar{e}_{K,U}$, with

$$\inf_\nu E_U(\nu,\nu) = E_U(\bar{e}_{K,U}, \bar{e}_{K,U}) = \frac{1}{\mathrm{cap}_U(K)} \tag{2.2.30}$$

as required. Next, we show (2.2.26) with a similar argument.

$$\begin{aligned} 0 \le \mathcal{E}(f,f) &= \mathcal{E}(f - h_{K,U} + h_{K,U}, f - h_{K,U} + h_{K,U}) \\ &= \mathcal{E}(f - h_{K,U}, f - h_{K,U}) + 2\mathcal{E}(f - h_{K,U}, h_{K,U}) + \mathcal{E}(h_{K,U}, h_{K,U}) \end{aligned} \tag{2.2.31}$$

In (2.2.31), we deal with each term separately, starting with the second.

$$\begin{aligned}\mathcal{E}(f-h_{K,U},h_{K,U}) &\overset{(2.2.11),(2.2.23)}{=} -\langle f-h_{K,U},\Delta G_U e_{K,U}\rangle \\ &\overset{(2.2.17)}{=} \langle f-h_{K,U},e_{K,U}\rangle = \sum_{y\in K}\Big(f(y)-h_{K,U}(y)\Big)e_{K,U}=0\end{aligned} \tag{2.2.32}$$

where we used that $f(y)=h_{K,U}(y)=1$ for $y\in K$ and $f\in\mathcal{F}$ as in (2.2.27). The last term yields

$$\mathcal{E}(h_{K,U},h_{K,U}) = -\langle \Delta G_U e_{K,U},h_{K,U}\rangle = \langle e_{K,U},h_{K,U}\rangle \overset{(2.2.21)}{=} \mathrm{cap}_U(K) \tag{2.2.33}$$

Hence the minimum of $\mathcal{E}(f,f)$ over $\mathcal{F}$ is $\mathrm{cap}_U(K)$ and it is attained at $f=h_{K,U}$, since the first term in (2.2.31) is non-negative. □

Next, we recall an important lemma which gives us upper bounds on the capacity in terms of the Green's function.

Lemma 2.2.5. *[17, (2.3.13)] For any $K\subset\subset\mathbb{Z}^d$:*

$$\frac{|K|}{\sup_{x\in K}\sum_{y\in K}g_U(x,y)} \le cap_U(K) \le \frac{|K|}{\inf_{x\in K}\sum_{y\in K}g_U(x,y)} \tag{2.2.34}$$

Proof. The result follows immediately by summing (2.2.23) over all $x\in K$. □

Finally, we state the following well-known estimate for the capacity of d-dimensional balls (see for instance [22, (2.8)-(2.9)]). For any $L>0$, we have

$$cL^{d-2}\le \mathrm{cap}(B(0,L))\le CL^{d-2} \tag{2.2.35}$$

2.3 Capacities of Oblique Sets

In the sequel, particularly in the case of the lower bounds, we will require precise capacity estimates on oblique tubes in $\mathbb{Z}^d$, arising as certain discretized cylindrical sets. With this in mind, recall the relevant notation from Section 2.1 and consider a point $x\in\mathbb{Z}^d\setminus\{0\}$ and direction $e=\frac{x}{|x|}$. With this, we introduce the following sets, referred to as the oblique line and oblique tube respectively

$$\begin{aligned}T(x)=T(x,0)&=\{[je]:0\le j\le\lceil|x|\rceil\}\\ T(x,r)&=\{y\in\mathbb{Z}^d: d_2(y,T(x))\le r\},\ r\ge 0\end{aligned} \tag{2.3.1}$$

Moreover, we now fix a collection of nested oblique tubes, parameterized by $\delta \in (0, \frac{1}{6})$ which will eventually be taken as $\delta \to 0$ after the appropriate manipulations have been carried out. Fixing some $x \in \mathbb{Z}^d$, let $T^1 \subset \ldots \subset T^6$ be defined as

$$\begin{aligned} T^i &= T(x, (|x| \vee 100)^{i\delta}),\ 1 \leq i \leq 5, \\ T^6 &= T(x, 4(|x| \vee 100)^{5\delta}) \end{aligned} \tag{2.3.2}$$

and introduce the set $U = B(0, 10^3|x|)$ such that it contains the largest of the nested tubes, i.e. $T^6 \subset U$. When the tubes are parallel to one of the axes we denote the special cases $T_N = T(Ne_1), T_N(r) = T(Ne_1, r)$ where e_1 is a standard basis vector (by symmetry, it is enough to select a single axis direction e_1 with $|e_1|_2 = 1$). Also, for $\delta \in (0, 1)$, we shall write $T_N^\delta := T_N(N^\delta)$. We now collect some crucial approximations on the capacities of the objects above in a useful lemma. This result will come into play when calculating the entropy between tilted and untilted probability measures and is indeed quite delicate in its precision in the case of $d = 3$. The entropy calculation manifests itself as the required cost of our modification procedure according to line (2.5.5), and turns out to represent the desired rate of probability decay of the connectivity studied herein. Naturally, one then expects that the choice of transformation of the space is fundamental - this is discussed further when the relevant transformations are introduced in §3.2.

Lemma 2.3.1. *($d = 3$) For all $\delta \in (0, \frac{1}{2})$, $|x| \geq C(\delta)$ and $i = 1, \ldots, 6$,*

$$cap(T^i) \leq (1 + C\delta)\frac{\pi|x|}{3\log|x|}, \tag{2.3.3}$$

$$cap_U(T^i) \leq \Big(1 + \frac{C}{\log|x|}\Big) cap(T^i), \tag{2.3.4}$$

$$cap_{T^3}(T^2) \leq C\delta^{-1} cap(T^2). \tag{2.3.5}$$

When $d \geq 4$, the capacities on the left of (2.3.3), (2.3.4) are bounded by $C|x|$.

In order to show Lemma 2.3.1 we will require an additional technical result describing the hitting probability of an oblique set, the proof of which we defer to the end of this section in order to improve the flow of the presentation.

Lemma 2.3.2. *($d = 3$) For all $\delta \in (0, \frac{1}{2})$, $|x| \geq C(\delta)$,*

$$\inf_{y \notin T^3} P_y[H_{T^2} = \infty] \geq c\delta, \tag{2.3.6}$$

Although there are already results in the literature regarding hitting times when the tubes are parallel to the axes e.g. [28, (2.25)] or when rotational invariance exists such as in the case of Brownian motion, the approaches therein are not directily applicable in our situation. To be precise, typically there is a projection argument involved to simplify the case to the well-studied one-dimensional Gambler's ruin problem, but unfortunately such a projection does not yield a simple random walk when considering integer-valued oblique sets. Thus, we will show a new martingale approach which yields approximations used in the aforementioned technical result.

Proof of Lemma 2.3.1. In order to show (2.3.3), we start by noticing

$$T^1 \subset \bigcup_{j=0}^{\lceil 2|x|^{1-\delta}\rceil} B_j, \quad B_j \coloneqq B\Big(\big[\frac{j|x|^\delta}{2}e\big], |x|^\delta\Big) \tag{2.3.7}$$

for $|x| \geq C(\delta)$, where we have effectively covered the oblique tube T^1 with a collection of intersecting l^∞ boxes. Then, due to the fact that for any $i, j \in \{1, \dots, \lceil 2|x|^{1-\delta}\rceil\}$, $i \neq j$, $x \in B_i$, $y \in B_j$ we have $d(x,y) \leq (|i-j| + d^{\frac{1}{2}})\frac{|x|^\delta}{2}$, we can apply [48, (2.24)] with $\delta < d^{-\frac{1}{2}}$, $N = |x|$, $P = \lceil 2|x|^{1-\delta}\rceil + 1$ therein. By remarking that $\nu = 1$ (see notation in the reference) on the 3-dimensional integer lattice and $\nu > 1$ in higher dimensions, we obtain the required result in both $d = 3$ and $d \geq 4$ for T^1. To prove the result for all $i \in \{1, \dots, 6\}$ simply replace δ with $i\delta$ in the steps above. Next, we proceed to the proof of (2.3.4). Recall the definition of U from below (2.3.2), and consider some $y \notin U$. Then, we have that $\min_{z \in \partial T^i} |y - z| \geq 9|x|$, and so by (2.2.15), $g(y,z) \leq C|x|^{-1}$ for $z \in T^i$. Then, applying a last-exit decomposition as in (2.2.23) for $U = \mathbb{Z}^d$, it follows that

$$P_y[H_{T^i} < \infty] \leq C|x|^{-1} cap(T^i) \overset{(2.3.3)}{\leq} C'(\log|x|)^{-1} \tag{2.3.8}$$

Then, for $z \in T^i$, by decomposing the equilibrium measure of T^i over the exit position of U and applying the bound from (2.3.8) one yields

$$P_z[\tilde{H}_{T^i} = \infty] = \sum_{y \in \partial^{\text{out}} U} P_z[\tilde{H}_{T^i} > T_U, X_{T_U} = y] P_y[H_{T^i} = \infty] \geq \Big(1 - \frac{C}{\log|x|}\Big) P_z[\tilde{H}_{T^i} > T_U] \tag{2.3.9}$$

Summing over $z \in T^i$ yields the conclusion for $d = 3$. Remark that for $d \geq 4$ we get the required result since the bound (2.3.8) instead becomes $P_y[H_{T^i} < \infty] \leq C'|x|^{-1}$ by investigating the Green's function approximation for higher dimensions. We are thus

left with showing (2.3.5). By yet another exit position decomposition, this time at the exit time of T^3, we obtain for $z \in T^2$

$$P_z[\tilde{H}_{T^2} = \infty] \geq P_z[\tilde{H}_{T^2} > T_{T^3}] \inf_{y \in \partial^{\text{out}} T^3} P_y[H_{T^2} = \infty] \tag{2.3.10}$$

By applying the technical Lemma 2.3.2 and summing over $z \in T^2$ we are done. □

Prior to delving into the proof of Lemma 2.3.2, we introduce some notation alongside the key ideas. First of all, we reduce the problem into considering $\partial^{\text{out}} T^3 = \mathbb{S} \cup \mathbb{L}$ where $\mathbb{S}$ and $\mathbb{L}$ are disjoint 'short' and 'long' boundaries of the oblique set respectively. Abbreviating the thickness of the tube T^3 by $r_3 = (|x| \vee 100)^{3\delta}$, these are defined as

$$\begin{aligned} \mathbb{L} &\coloneqq \{z \in T(x, 4r_3) \setminus T^3 : |\langle z - x, e\rangle| \wedge |\langle z, e\rangle| > 4r_3\}, \\ \mathbb{S} &\coloneqq \partial^{\text{out}} T^3 \cap \mathbb{L}^c \end{aligned} \tag{2.3.11}$$

The reason behind this distinction is that, in the process of deriving the approximations, the boundary of the oblique set is partitioned into disjoint segments where it is convenient for the segments to contain not too many points and so that the points in the segments are approximately equidistant from the innermost width 0 tube $T \equiv T(x)$. By nature of the endpoints, there are more points on $\partial^{\text{out}} T^3$ which are closest to either 0 or x so these are viewed separately as the short boundary $\mathbb{S}$. Moreover, we will show that the hitting probability problem when starting at the short boundary simplifies to the long boundary case, so it is sufficient to consider the latter scenario.

In order to handle the hitting probability when starting on $\mathbb{L}$, we will reformulate the problem with the help of a specially-chosen martingale, which will have certain uniform bounds on its expectation (see Lemma (2.3.3)). To facilitate the uniformity, it is necessary to consider the following extension of $K \subset \mathbb{Z}^d$ by two 'stings'

$$\tilde{K} \coloneqq K \cup \Big\{[je] : j \in \mathbb{Z} \cap \Big([-|x|, 0] \cup [|x|, 2|x|]\Big)\Big\} \tag{2.3.12}$$

Then, we let

$$M_t = \sum_{z \in \tilde{T}} g(X_t, z), \tag{2.3.13}$$

and define the stopping time $\tau = H_{\tilde{T}^2} \wedge T_U = T_{U \setminus \tilde{T}^2}$. Using the fact that $g(\cdot, z)$ is harmonic outside of z and that any harmonic function of X_t, we have that $(M_{t \wedge \tau})_{t \geq 0}$ is a martingale under P_y for any $y \in \partial^{\text{out}} T^3$. Denote also the extremities of the stings

$Ext(\tilde{T}) \subset \tilde{T}$ as

$$Ext(\tilde{T}) \coloneqq \{z \in \tilde{T} : d(z, x_+) \wedge d(z, x_-) \leq \tfrac{|x|}{2}\} \tag{2.3.14}$$

where $x_+ = [2|x|e] = [2x]$ and $x_- = [-|x|e] = [-x]$. Finally, we present the following estimates on the martingale from (2.3.13)

Lemma 2.3.3. *Let $d = 3$, $\delta \in (0, \frac{1}{2})$ and denote*

$$c_2 = \lim_{|x|\to\infty} |x|^{-1} g(0, x) = \frac{3}{2\pi}. \tag{2.3.15}$$

For $|x| \geq C(\delta)$, one has

$$\text{for all } z \in \mathbb{L} : E_z[M_0] \leq 2c_2(1 - 3\delta)(\log|x| + C), \tag{2.3.16}$$

$$\text{for all } z \in \tilde{T}^2 \setminus Ext(\tilde{T}) : E_z[M_0] \geq 2c_2(1 - 2\delta)(\log|x| - C). \tag{2.3.17}$$

Proof. We start by deriving a useful estimate on distance from an arbitrary point to the extended oblique line $\tilde{T}$. In this way, consider the line $l = \{te : t \in \mathbb{R}\}$ and, for $z \notin l$ define $y_z \in l$ as the point minimizing the distance to z. If we denote by $\langle \cdot, \cdot \rangle$ the standard inner product, we then have

$$\langle z - y_z, e \rangle = 0 \tag{2.3.18}$$

Next, let $j_z \in \mathbb{Z}$ be such that $|je - y_z|$ is minimized when $j = j_z$. By definition, from the triangle inequality we have

$$|j_z e - y_z| \leq 1 \tag{2.3.19}$$

Hence, letting $y \equiv [je] \in \tilde{T}$, $z \in \mathbb{Z}^d$ and abbreviating $d_z = dist_{l^2}(z, l) \equiv |z - y_z|$, we have

$$\begin{aligned} |y - z| = |[je] - z| \geq |je - z| - C &\overset{(*)}{=} \sqrt{|je - y_z|^2 + d_z^2} - C \\ &\overset{(\dagger)}{\geq} d_z \sqrt{1 + \frac{((|j - j_z| - 1) \vee 0)^2}{d_z^2}} - C \end{aligned} \tag{2.3.20}$$

where $(*)$ is simply a consequence of the fact that $\langle je - y_z, z - y_z \rangle = 0$ by collinearity, and $(\dagger)$ holds by using (2.3.19) since $|je - y_z| \geq |j - j_z| - |j_z e - y_z| \geq |j - j_z| - 1$. Next, let $z \in \mathbb{L}$ and notice that whenever $|x| \geq C(\delta)$ one has $j_z \in [-|x|/2, 3|x|/2]$. For

such z, we let $I_k \coloneqq \{j \in \mathbb{Z} : |j - j_z| - 1 \in [kd_z, (k+1)d_z)\}$ and $K = 2|x|/d_z$, so that

$$\{j \in \mathbb{Z} : [je] \in \tilde{T}\} \setminus \{j_z\} \subset \bigcup_{k=0}^{K} I_k \tag{2.3.21}$$

Hence, recalling the definition of M_t from (2.3.13) one obtains for any $z \in \mathbb{L}$ and $|x| \geq C'(\delta)$

$$\begin{aligned}
(1 + \frac{\delta}{10^3})^{-1} E_z[M_0] &\overset{(2.3.15)}{\leq} c_2 \sum_{y \in \tilde{T}} \frac{1}{|z-y|} \\
&\overset{(2.3.21)}{=} c_2(|z - j_z e|)^{-1} + c_2 \sum_{k=0}^{K} \sum_{y \in \tilde{T}} \frac{\mathbb{1}\{y = [je] \text{ for a } j \in I_k\}}{|z-y|} \\
&\overset{(2.3.20)}{\leq} C d_z^{-1} + c_2 d_z^{-1} \sum_{k=0}^{K} |I_k| \frac{1}{\sqrt{1 + k^2 - \frac{C}{d_z}}} \\
&\overset{(*)}{\leq} (1 + \frac{\delta}{10^3}) 2c_2 \Big(C + \sum_{k=1}^{K} k^{-1} \Big) \\
&\overset{(\dagger)}{\leq} (1 + \frac{\delta}{10^3}) 2c_2 (C + (1 - 3\delta) \log|x|) \\
&\leq 2c_2(1 - 2.9\delta)(C + \log|x|)
\end{aligned} \tag{2.3.22}$$

where in $(*)$ we have bounded $\frac{1}{\sqrt{1+k^2 - C/d_z}}$ by $\frac{1}{k(1-C/d_z)}$ for $k \geq 1$ and used $|I|_k \leq 2d_z$, meanwhile $(\dagger)$ holds for $|x| \geq C(\delta)$ as then we have $c|x|^{3\delta} \leq d_z \leq C|x|^{3\delta}$. A similar approach is used to show (2.3.17) which we outline precisely, paying special attention to the effect of the extension $\tilde{T}$ as opposed to T. First of all, a matching bound for (2.3.20) yields

$$|[je] - z| \leq d_z \sqrt{1 + \frac{(|j - j_z| + 1)^2}{d_z^2}} + C \tag{2.3.23}$$

Using the definition of $Ext(\tilde{T})$ from (2.3.14), when $z \in \tilde{T}^2 \setminus Ext(\tilde{T})$ we retain the condition that $j_z \in [-|x|/2, 3|x|/2]$. Then, by definition of $\tilde{T}$ from (2.3.12), by setting $K' = c|x|/d_z$ for sufficiently small c and defining $I'_k \coloneqq \{j \in \mathbb{Z} : |j - j_z| \in [kd_z, (k+1)d_z)\}$, one obtains the inclusion

$$\{j \in \mathbb{Z} : [je] \in \tilde{T}\} \supset \bigcup_{k=1}^{K'} I'_k. \tag{2.3.24}$$

where the buffer afforded to us by $Ext(\tilde{T})$ in the set on the left hand side is essential for selecting K' as above. Equipped with (2.3.23) and (2.3.24), playing the roles of (2.3.20) and (2.3.21) respectively, we perform a computation akin to (2.3.22) for $z \in \tilde{T}^2 \setminus Ext(\tilde{T})$

$$\begin{aligned}(1-\frac{\delta}{10^3})^{-1}E_z[M_0] &\overset{(2.3.15)}{\geq} c_2 \sum_{y\in\tilde{T}} \frac{1}{|z-y|} \\ &\overset{(2.3.24)}{\geq} c_2 \sum_{k=1}^{K'} \sum_{y\in\tilde{T}} \frac{\mathbb{1}\{y=[je] \text{ for a } j\in I'_k\}}{|z-y|} \\ &\overset{(2.3.23)}{\geq} c_2 d_z^{-1} \sum_{k=1}^{K'} |I'_k| \frac{1}{\sqrt{1+(k+1)^2+\frac{C}{d_z}}}\end{aligned} \tag{2.3.25}$$

Then, we deal with the final line in a similar way as before, enforcing $|x| \geq C(\delta)$. □

Let us conclude this section by fitting together all the necessary pieces to prove the technical lemma.

Proof of Lemma 2.3.2. Let us start of by dealing with the case $y \in \mathbb{L}$, deferring the short boundary case to the end of the proof. Referring to the bounds presented in Lemma 2.3.3, and observing that $|M_{t\wedge\tau}| < C$ a.s., via an application of the optional stopping theorem we see that for all $y \in \mathbb{L}$, $|x| \geq C(\delta)$

$$\begin{aligned}&2c_2(1-2.9\delta)\log|x| \overset{(2.3.16)}{\geq} E_y[M_0] = E_y[M_\tau] \geq \sum_{x\in\tilde{T}\setminus Ext(\tilde{T})} E_y[M_\tau|X_\tau=x]P_y[X_\tau=x] \\ &= \sum_{x\in\tilde{T}\setminus Ext(\tilde{T})} E_x[M_0]P_y[X_\tau=x] \overset{(2.3.17)}{\geq} P_y[H_{\tilde{T}^2}\leq T_U, X_\tau\notin Ext(\tilde{T})]\cdot 2c_2(1-2.1\delta)\log|x|\end{aligned} \tag{2.3.26}$$

Next, remarking that (2.3.3) was shown independently of Lemma 2.3.2 we immediately see that $cap(Ext(\tilde{T}) \leq \frac{C|x|}{\log|x|}$ which, in view of $d(T^3, Ext(\tilde{T})) \geq \frac{x}{4}$ whenever $|x| \geq C$, obtains

$$\sup_{|x|\to\infty} P_y[H_{Ext(\tilde{T})} < \infty] \leq C(\log|x|)^{-1} \to 0 \text{ as } |x|\to\infty \tag{2.3.27}$$

where once again we profited from the familiar decomposition (2.2.23). Therefore, by virtue of (2.3.27) we deduce that $P_y[H_{\tilde{T}^2} \leq T_U, X_\tau \in Ext(\tilde{T})] \leq 0.1\delta$ for $y \in \mathbb{L}$

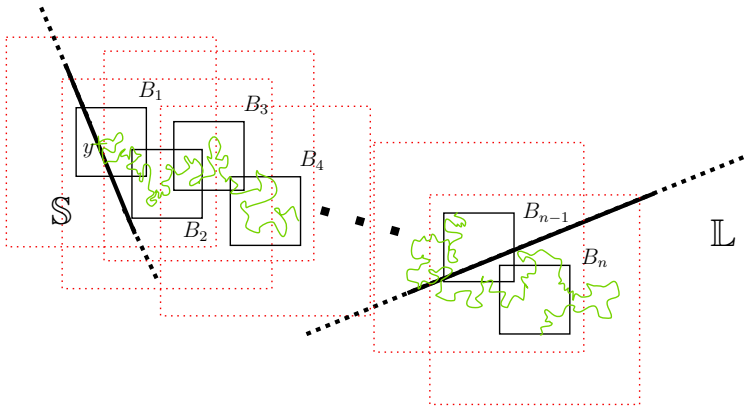


Fig. 2.1 A depiction of the chaining argument to bound the probability of hitting the long boundary uniformly from 0. Crucially, $y \in B_1$ and $B_n \subset \mathbb{L}$.

whenever $|x| \geq C(\delta)$. Combining this with (2.3.26) yields

$$P_y[H_{T^2} \leq T_U] \overset{T^2 \subset \tilde{T}^2}{\leq} P_y[H_{\tilde{T}^2} \leq T_U] \leq 1 - \left(\frac{0.8\delta}{1-2.1\delta} - 0.1\delta\right) \tag{2.3.28}$$

for all $y \in \mathbb{L}$ and $|x| \geq C(\delta)$. By applying the strong Markov property at the exit time of U alongside similar ideas to those used in (2.3.27) we also have $\sup_{z\in\mathbb{Z}^d} P_z[T_U < H_{T^2} < \infty] \to 0$ as $|x| \to \infty$. Together with (2.3.28), we have therefore deduced (2.3.6) uniformly in $y \in \mathbb{L}$. To conclude, we need only treat the case when $y \in \mathbb{S}$. In conjunction with the long boundary case, it is enough to show the following

$$\inf_{y\in\mathbb{S}} P_y[H_{\mathbb{L}} < H_{T^2}] \geq c \tag{2.3.29}$$

Once we have (2.3.29) in hand, the proof is completed by applying the strong Markov property at time $H_{\mathbb{L}}$

$$P_y[H_{T^2} = \infty] \geq P_y[H_{T^2} = \infty, H_{\mathbb{L}} < H_{T^2}] \geq \left(\inf_{z\in\mathbb{L}} P_z[H_{T^2} = \infty]\right) \cdot P_y[H_{\mathbb{L}} < H_{T^2}] \tag{2.3.30}$$

In order to show (2.3.29), we present a chaining argument (see Figure 2.1). By definition, we can find a finite number of boxes $B_i = B(x_i, \frac{r_3}{100})$, $1 \leq i \leq n$ (with n uniform in y) such that $y \in B_1$, B_i and B_{i+1} have non-empty overlap containing a translation of $B(0, \frac{r_3}{300})$ for every $i \in \{1, \ldots, n-1\}$, $B_n \subset \mathbb{L}$ and lastly $B(x_i, \frac{r_3}{10}) \cap T^2 = \emptyset$ to not have to avoid T^2 at every step while chaining. Since $B_{i+1} \subset B(x_i, \frac{3r_3}{100})$, using [49, (A.4)] with $\delta = 2$ we obtain $\inf_{z\in B_i} P_z[H_{B_{i+1}} < H_{T^2}] \geq c$ for all $i \leq 1 < n$. A

repeated application of the strong Markov property at the hitting times H_{B_i} and H_{F_n} then shows (2.3.29). □

To conclude this section, we collect a useful result on the capacity of an axis-aligned line.

Lemma 2.3.4 (Capacity of a line)**.** *[28, Lemma 2.2] For $d = 3$, and given a direction $e_1 \in \mathbb{R}^d$, $|e_1| = 1$ parallel to one of the axes, we have*

$$cap(T(Ne_1)) \sim \frac{\pi}{3} \frac{N}{\log N}, \quad \text{as } N \to \infty. \tag{2.3.31}$$

Meanwhile, for $d > 3$ there exists a $c_3(d) \in (0, 1)$ such that for all $N \geq 1$

$$c_3 N \leq cap(T_N) \leq c_3^{-1} N. \tag{2.3.32}$$

2.4 Random Interlacements

Given that the Random Interlacements (RI) model was initially introduced as a tool to study percolative properties of the SRW on the torus (TSRW), we would be remiss to not first introduce the latter model concretely. In what follows, we consider a dimension $d \geq 3$, a side length $N \geq 1$ and introduce the d-dimensional torus $\mathbb{T} = \mathbb{T}_N^d := (\mathbb{Z}/N\mathbb{Z})^d$ endowed with its natural nearest-neighbour graph structure (similarly to the integer lattice above). Then, we introduce the canonical projection $\phi : \mathbb{Z}^d \to \mathbb{T}$, fix a density $u > 0$ and define the resulting percolative model as the vacant set $\mathcal{V}_N^u = \mathbb{T} \setminus \mathbf{Z}_{[0,uN^d]}$ where $\{\mathbf{Z}_i\}_{i \geq 1} = \{\phi(Z_i)\}_{i \geq 1}$ is sampled according to the probability measure

$$P_N := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d \cap B(0, N/2)} \phi \circ P_x \tag{2.4.1}$$

which corresponds to a SRW in $\mathbb{Z}^d$ with a uniformly-sampled starting point inside $B(0, N/2)$ which is projected onto the torus $\mathbb{T}$. With these definitions, under P_N we also introduce the local occupation times of the TSRW

$$\ell_x^u = \ell_x^u(\mathbf{Z}) = \sum_{i=1}^{\lfloor uN^d \rfloor} \mathbb{1}\{\mathbf{Z}_i = x\}, \quad x \in \mathbb{T}, \ u > 0 \tag{2.4.2}$$

More generally, given any right-continuous, piecewise-constant trajectory $w : [0, \infty) \to \mathbb{Z}^d$ with $\tau_0 = 0$ and (jump) discontinuities at times $\{\tau_i\}_{i \geq 1}$ we also write

$$\ell_x(w) \coloneqq \int_0^\infty \mathbb{1}\{w_t = x\}dt, \quad \ell_{x,\text{dis}}(w) \coloneqq \sum_{i \geq 0} \mathbb{1}\{w_{\tau_i} = x\}, \quad x \in \mathbb{Z}^d \tag{2.4.3}$$

While $\ell_x(w)$ can be interpreted as the total length of time the path w stays at state x, $\ell_{x,\text{dis}}(w)$ instead counts the total number of times w intersects x. Then, Before we proceed with introducing the RI model, we briefly recall the definition of Poisson Point Processes. For the purposes of this section, let $(W_0, \mathcal{W}_0)$ denote a locally compact, Polish measurable space (not necessarily the space of paths defined earlier). On this space, we can consider an infinite point measure $\hat{\omega} = \sum_{i=1}^\infty \delta_{w_i}$, where each $w_i \in W_0$, and δ_{w_i} is the Dirac delta concentrated at w_i such that for any $A \subset W_0$, $\hat{\omega}(A) = \sum_{i=1} \mathbb{1}\{w_i \in A\}$. Then, we can denote the space of infinite point measures on W_0 by:

$$\Omega(W_0) \coloneqq \{\hat{\omega} = \sum_{i=1}^\infty \delta_{w_i} : w_i \in W_0 \ \forall i \geq 1, \ \hat{\omega}(K) < \infty \ \forall K \subset W_0 \text{ compact}\} \tag{2.4.4}$$

and associate to this space the σ-algebra $\Sigma(W_0)$ generated by the evaluation maps $\chi_A : \Omega \to \mathbb{Z}_{\geq 0}$ for any $A \in \mathcal{W}_0$ where $\chi_A(\hat{\omega}) = \hat{\omega}(A)$. Then, we can consider random variables on the space $\Omega(W_0)$ as random point measures. Given a point process $\hat{\omega} = \sum_{i \geq 0} \delta_{w_i}$ and a function $F : W_0 \to \mathbb{R}$, define $\langle \hat{\omega}, F \rangle \coloneqq \sum_{i \geq 0} F(w_i)$.

Definition 2.4.1 (Poisson Point Process)**.** *[17, Def 5.4] Given a σ-finite measure λ on $(W_0, \mathcal{W}_0)$, a random point measure μ is a Poisson Point Process (PPP) on W_0 with intensity measure λ if*

(i) for all $B \in \mathcal{W}_0$, $\mu(B) \sim Poisson(\lambda(B))$,

(ii) if $B_1, \ldots, B_n \subset W_0$ are disjoint, then $\mu(B_1), \ldots \mu(B_n)$ are independent random variables.

Remark that the PPP for a given intensity measure λ is uniquely determined by the above definition by an application of the π-λ theorem, since the distribution of μ is already given on a π-system. Additionally, we state a useful property for PPP introduced by [51, Prop. 3.6]. Given a PPP η on E with finite intensity measure μ, and $F : E \to \mathbb{R}$ a measurable function, one has

$$E[e^{\langle \eta, F \rangle}] = e^{\int_E e^F - 1 d\mu} \tag{2.4.5}$$

Next, let us introduce the underlying path spaces in which we are interested for the RI construction. Recalling the space W from the start of Section 2.2, For $w, w' \in W$, define the following maps:

$$w^+ := (w_n)_{n\geq 0}, \quad w^- := (w_n)_{n\leq 0} \tag{2.4.6}$$

as well as the useful equivalence relation:

$$w \sim w' \iff Z_\cdot(w) = Z_\cdot(\theta_k(w')) \text{ for some } k \in \mathbb{Z}$$

With the latter relation, define the space $W^* := W \backslash \sim$, by virtue of the associated projection $\pi^* : W \to W^*$, and define the σ-algebra $\mathcal{W}^*$ on W by

$$A \in \mathcal{W}^* \Leftrightarrow (\pi^*)^{-1}(A) \in \mathcal{W} \tag{2.4.7}$$

In the sequel, it will also be useful to define certain subspaces of W and W^*. To this end, let $K \subset\subset \mathbb{Z}^d$ and write $W_K \subset W$ to be the space of doubly-infinite sequences whose first coordinate Z_n is in K at least once, and similarly define $W_K^+ \subset W^+$ as the positively-indexed space of paths satisfying the same condition. Furthermore, define the projected space of equivalence classes $W_K^* := \pi^*(W_K) = \{\pi^*(w) : w \in W_K\}$. Given $\omega^* \in W_K^*$ we define the map $s_K : W_K^* \to W_K$ by the property:

$$s_K(\omega^*) = \omega \text{ where } \omega \text{ is the unique element of } W_K \text{ such that } \pi^*(\omega) = \omega^* \text{ and } H_K(\omega) = 0 \tag{2.4.8}$$

With this, we are now ready to introduce the level u random interlacements, which arise as a PPP on the product space $W^* \times \mathbb{R}_{\geq 0}$ corresponding to labelled doubly-infinite projected paths. To this end, in view of Definition 2.4.1, it is sufficient to construct a σ-finite intensity measure on $W^* \times \mathbb{R}_{\geq 0}$ to yield a unique characterization of the RI. First, set a dimension $d \geq 3$ and let ν be a σ-finite measure on the measurable space $(W^*, \mathcal{W}^*)$ such that, for any finite $K \subset\subset \mathbb{Z}^d$,

$$\mathbb{1}_{W_K^*} \cdot \nu = \pi^* \circ Q_K \tag{2.4.9}$$

where Q_K is a probability measure defined on $(W, \mathcal{W})$ satisfying the formula:

$$Q_K[(Z_{-n})_{n\geq 0} \in A, Z_0 = x, (Z_n)_{n\geq 0} \in B] = P_x[A|\tilde{H}_K = \infty]P_x[B]e_K(x) \tag{2.4.10}$$

for all $A, B \in \mathcal{W}^+$ and $x \in \mathbb{Z}^d$. For a proof that there indeed exists such a measure and that it is unique, refer to [17, Thm 6.2]. Then, recalling the definition of $\Omega(W_0)$ from (2.4.4) and introducing the notation $\Omega \coloneqq \Omega(W^* \times \mathbb{R}_{\geq 0})$, $\Sigma \coloneqq \Sigma(W^* \times \mathbb{R}_{\geq 0})$, we define the probability space $(\Omega, \Sigma, \mathbb{P})$, where $\mathbb{P}$ is the probability measure on (Ω, Σ) such that the canonical point process $\hat{\omega}$ is a Poisson Point Process (PPP) on $W^* \times \mathbb{R}_{\geq 0}$ with intensity measure $\nu(\cdot) \otimes d\lambda$ for λ the Lebesgue measure on $\mathbb{R}_{\geq 0}$. In parallel, we also consider the probability space $(\Omega^u, \Sigma^u, \mathbb{P}^u)$ corresponding to the PPP on W^* with intensity measure $u\nu(\cdot)$. The latter will be employed in cases where only one level of the RI is of interest in order to simplify certain procedures. In the sequel, each of PPPs above will be referred to as the random interlacements point process.

Next, given a finite set $K \subset\subset \mathbb{Z}^d$, we define another useful map. Let $\hat{\omega} = \sum_{i \geq 0} \delta_{(w_i^*, u_i)} \in \Omega(W^* \times \mathbb{R}_{\geq 0})$ be a point process, recalling the map s_K from line (2.4.8), define the additional maps $\mu_K^+, \mu_K^- : \Omega(W^* \times \mathbb{R}_{\geq 0}) \to \Omega(W_K^+ \times \mathbb{R}_{\geq 0})$:

$$\begin{aligned} \mu_K^+(\hat{\omega}) &\coloneqq \sum_{i \geq 0} \mathbb{1}\{w_i^* \in W_K^*\} \delta_{(s_K(w_i^*)^+, u_i)} \\ \mu_K^-(\hat{\omega}) &\coloneqq \sum_{i \geq 0} \mathbb{1}\{w_i^* \in W_K^*\} \delta_{(s_K(w_i^*)^-, u_i)} \end{aligned} \tag{2.4.11}$$

By definition of the RI, c.f. (2.4.10), we have that

$$\text{under } \mathbb{P}, \mu_K^+ \text{ is a PPP on } W^+ \times \mathbb{R}_{\geq 0} \text{ with intensity measure } P_{e_K} \otimes d\lambda. \tag{2.4.12}$$

Similarly, for a fixed level $u > 0$, we define $\mu_{K,u}^+$ and $\mu_{K,u}^-$ as in (2.4.11) except restricting the summation over $u_i \leq u$. In this way, $\mu_{k,u}^+$ is then a PPP on W^+ with intensity measure uP_{e_K}.

Bearing in mind the formulations of the RI Point Process discussed below (2.4.10), depending on whether the space we consider is of the form $W^* \times \mathbb{R}_{\geq 0}$ or W^*, we define the following random subset of $\mathbb{Z}^d$:

$$\mathcal{I}^u(\hat{\omega}) = \begin{cases} \bigcup_{u_i \leq u} \text{Range}(w_i^*) & \hat{\omega} = \sum_{i \geq 0} \delta_{(w_i^*, u_i)} \in \Omega(W^* \times \mathbb{R}_{\geq 0}) \\ \bigcup_{i \geq 0} \text{Range}(w_i^*) & \hat{\omega} = \sum_{i \geq 0} \delta_{w_i^*} \in \Omega(W^*) \end{cases} \tag{2.4.13}$$

where $\text{Range}(w^*) \coloneqq \bigcup_{n \in \mathbb{Z}} \{Z_n(w) : w \in (\pi^*)^{-1}(w^*)\}$ for $w^* \in W^*$. Then, the random interlacements at level u are defined as $\mathcal{I}^u$ under $\mathbb{P}$ when the space is (Ω, Σ) and $\mathcal{I}^u$ under $\mathbb{P}^u$ when the space under consideration is instead (Ω^u, Σ^u). A key property

of defining the random interlacements in this way is that the law of $\mathcal{I}^u$ is the same whether we are on the space Ω or Ω^u, hence it makes sense to speak of the law of the random interlacements. Moreover, this fact will be essential when transferring properties of the former to the latter in order to obtain the large deviations bounds we seek. As briefly mentioned in the introduction, we also define the vacant set at level u as $\mathcal{V}^u := \mathbb{Z}^d \setminus \mathcal{I}^u$. It turns out that considering the vacant set is essential to observe interesting behaviour, since the RI themselves do not exhibit a phase transition in the sense described due to the fact that they are a.s. connected at all levels $u > 0$. Aside from connectedness, the RI have been shown to be translation invariant and ergodic with respect to lattice shifts, and they also satisfy

$$\mathbb{P}[\mathcal{I}^u \cap K = \emptyset] = e^{-u\mathrm{cap}(K)}, \ K \subset\subset \mathbb{Z}^d \tag{2.4.14}$$

Remark that, although we could have defined $\mathcal{I}^u$ as a random set on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies the equality (2.4.14) on all cylindrical sets, in this work we will frequently rely on the intimate connection between the RI and collections of random walks. In the sequel, it will also be useful to have the definition of a related map $\mathcal{I}^u : \Omega(W^+ \times \mathbb{R}_{\geq 0}) \to \mathcal{B}(\mathbb{Z}^d)$:

$$\mathcal{I}^u(\hat{\omega}^+) := \bigcup_{u_i \leq u} \mathrm{Range}(w_i^+), \quad \hat{\omega}^+ = \sum_{i \geq 0} \delta_{(w_i^+, u_i)} \in \Omega(W^+ \times \mathbb{R}_{\geq 0}) \tag{2.4.15}$$

where we canonically define $\mathrm{Range}(w^+) := \bigcup_{n \geq 0} Z_n(w^+)$ for $w^+ \in W^+$. Evidently, by considering $\mathcal{I}$ and $\mathcal{I}^u$ without the underlying trajectories we lose some information in the form of the occupation time field, which we now introduce. Consider a level $u > 0$ and a point $x \in \mathbb{Z}^d$, then for a PPP $\hat{\omega} \in \Omega(W^* \times \mathbb{R}_{\geq 0})$, under $\mathbb{P}$ we define the continuous occupation time field

$$\ell_x^u = \ell_x^u(\hat{\omega}) := \langle \ell_x, \mu_{\{x\},u}^+ \rangle \tag{2.4.16}$$

as well as the discrete occupation time field

$$\ell_{x,dis}^u = \langle \ell_{x,\mathrm{dis}}, \mu_{\{x\},u}^+ \rangle \tag{2.4.17}$$

where we recall the general local time notation from (2.4.3). Although the inclusion of both of the above definitions may seem cumbersome, it turns out necessary in order to fully take advantage of the results in the literature. In this specific instance, the continuous version will be employed in the process of proving certain couplings between

the RI with a tilted version (see ((3.2.14))), meanwhile the discrete version is more appealing elsewhere to avoid unnecessary complication. Another benefit of discussing the discrete version is that we are afforded the following important coupling between the discrete local times of the RI (2.4.16) and the TSRW (2.4.2)

Theorem 2.4.2. *[49, Cor. 5.2] For all $\delta > 0$, there exist $c, C \in (0, \infty)$ depending only on δ and d such that the following holds. For every $u > 0$, $\epsilon \in (0,1)$ and $N \in \mathbb{N}$ there exists a coupling Q between $(l^u_x)_{x\in\mathbb{T}}$ under P_N and $(l_{x,dis})^{u(1\pm\epsilon)}_{x\in B(0,N/2)}$ under $\mathbb{P}$ such that*

$$Q[l^{u(1-\epsilon)}_{x,dis} \leq l^u_{\phi(x)} \leq l^{u(1+\epsilon)}_{x,dis}, \forall x \in B(0, N(1-\delta)/2)] \geq 1 - CN^{2d}\lceil uN^{d-2}\rceil \exp(-c\epsilon\sqrt{uN^{d-2}}) \tag{2.4.18}$$

To conclude, let us also introduce two decoupling theorems for the RI. The first is applied in the case that events measurable with respect to the RI take place in disjoint sets, meanwhile the second is a conditional decoupling, where the entrance and exit points of the RI inside a certain set are known. In view of the latter, we first present some useful notation. Given sets $A \subset\subset U \subset \mathbb{Z}^d$, a density parameter $u > 0$, and the RI PP $\hat{\omega} \in \Omega(W^* \times \mathbb{R}_{\geq 0})$, define the finite multiset $\mathcal{C}_u = \mathcal{C}^{A,U}_u(\hat{\omega})$ as the collection of all pairs of start- and endpoints of any excursion between A and $\partial_{\text{out}}U$ in the support of ω with label not exceeding u (see §3.4 for an explicit construction of such excursions). The random variable $\mathcal{C}_u$ takes values in the measure space $(\Omega_{\mathcal{C}}, \mathcal{A}_{\mathcal{C}})$, where one can simply take $\mathcal{A}_{\mathcal{C}} = 2^{\Omega_{\mathcal{C}}}$ since $\Omega_{\mathcal{C}}$ is countable. Moreover, letting $B \subset A$, define $\mathcal{D}_u$ as the collection of all the excursions of $\hat{\omega}$ visiting B, from their first entrance of B until the last time they are in A prior to leaving U. The measure space underlying the random variable $\mathcal{D}_u$ is $(\Omega_{\mathcal{D}}, 2^{\Omega_{\mathcal{D}}})$. For a precise mathematical formulation, refer to the preamble before [20, Proposition 2.3].

Theorem 2.4.3. *[47, Thm. 2.1] Let A_1, A_2 be two non-intersecting subsets of $\mathbb{Z}^d$, with at least one of them being finite. Let $s = d(A_1, A_2)$ and $r = \min(diam(A_1), diam(A_2))$. Then, there exist positive constants γ_0, γ_1 depending only on d such that for all $u > 0$, $\epsilon \in (0,1)$ there exists a coupling Q between $\mathcal{I}^u$ and two independent RI processes $(\mathcal{I}^u_1)_{u\geq 0}$ and $(\mathcal{I}^u_2)_{u\geq 0}$ such that*

$$Q[\mathcal{I}^{u(1-\epsilon)}_k \cap A_k \subset \mathcal{I}^u \cap A_k \subset \mathcal{I}^{u(1+\epsilon)}_k \cap A_k,\ k = 1,2] \geq 1 - \gamma_0(r+s)^d \exp(-\gamma_1\epsilon^2 u s^{d-2}). \tag{2.4.19}$$

Theorem 2.4.4. *[20, Remark 2.4,3] Let $k \geq 1$. There exist $c_2 \in (0, \frac{1}{2})$ and $C_3 > 0$ with the property that for all $r \geq C$ there exists sets A, U with $B = B_r \subset A \subset U \subset B_{r+r^{1-c_2}}$*

such that, for any collection $u_1, \dots, u_k > 0$*;* $\delta_1, \dots, \delta_k \in (0,1)$*, one can find an event* $\mathcal{A} \in \mathcal{A}_{\mathcal{C}}^{\otimes k}$ *with*

$$\mathbb{P}[(\mathcal{C}_{u_1}, \dots, \mathcal{C}_{u_k}) \in \mathcal{A}] \geq 1 - e^{-c_2 u \delta^2 r^{c_2}}, \tag{2.4.20}$$

where $u \coloneqq \min\{u_1, \dots, u_k\}$ *and* $\delta \coloneqq \min\{\delta_1, \dots, \delta_k\}$*, and for all fixed* $(\xi_1, \dots, \xi_k) \in \mathcal{A}$*, there is a coupling* $\hat{\mathbb{Q}} = \hat{\mathbb{Q}}_{\xi_1,\dots,\xi_k}$ *between the family of* $\{0,1\}^{\mathbb{Z}^d}$*-valued random variables* $(\hat{\mathcal{D}}_v : v \in \{u_i(1 \pm \delta_i) : 1 \leq i \leq k\}) \overset{law}{=} (\mathcal{D}_v : v \in \{u_i(1 \pm \delta_i) : 1 \leq i \leq k\})$ *and* $(\hat{\mathcal{D}}_{u_1} \dots \hat{\mathcal{D}}_{u_k})$ *having the law of* $(\mathcal{D}_{u_1} \dots \mathcal{D}_{u_k})$ *under* $\mathbb{P}[\cdot|(\mathcal{C}_{u_1}, \dots, \mathcal{C}_{u_k}) = (\xi_1, \dots, \xi_k)]$*, and*

$$\hat{\mathbb{Q}}[\hat{\mathcal{D}}_{u_i(1-\delta_i)} \subset \hat{\mathcal{D}}_i \subset \hat{\mathcal{D}}_{u_i(1+\delta_i)} \text{ for all } 1 \leq i \leq k] \geq 1 - kC_3 e^{-c_2 u \delta^2 r^{c_2}}. \tag{2.4.21}$$

2.5 A Useful Entropy Inequality

As briefly discussed in Section 1.4, the strategy for proving the lower bounds requires us to *tilt* the RI measure $\mathbb{P}$ via a Girsanov-type argument in such a way that we can take advantage of the various sub- and supercritical properties available. To be concrete, comparison with the subcritical regime is possible when the resulting modified distribution *dominates* a subcritical level of the RI, which results in good disconnection properties. Symmetrically, to have good connectivity properties one would need to dominate the resulting distribution by a supercritical level of RI. By domination, we are informally referring to a property similar to the one exhibited in the coupling (2.4.19) where, in the same vein, we would say that e.g. $\mathcal{I}_1^{u(1+\epsilon)}$ dominates $\mathcal{I}^u$ in the set A_1. In this section, we present the main tools which will subsequently quantify the cost of the tilting process exhibiting the properties described above. To this end, we introduce some elementary notions.

Remark that the space of subsets of $\mathbb{Z}^d$ can be represented as the space $\{0,1\}^{\mathbb{Z}^d}$ to which we associate the maximal σ-algebra $\sigma(\{0,1\}^{\mathbb{Z}^d})$. On this space, there exists a natural partial ordering dictated by inclusion of sets. In other words, for $\xi, \xi' \in \{0,1\}^{\mathbb{Z}^d}$,

$$\xi \leq \xi' \iff \{x \in \mathbb{Z}^d : \xi(x) = 1\} \subset \{x \in \mathbb{Z}^d : \xi'(x) = 1\} \tag{2.5.1}$$

With this natural ordering, we define an increasing event $A \in \sigma(\{0,1\}^{\mathbb{Z}^d})$ to be an event satisfying for any $\xi, \xi' \in \{0,1\}^{\mathbb{Z}^d}$

$$\xi \leq \xi', \xi \in A \Rightarrow \xi' \in A \tag{2.5.2}$$

Similarly, a decreasing event is one which satisfies

$$\xi \geq \xi', \xi \in A \Rightarrow \xi' \in A \tag{2.5.3}$$

so that, given any increasing event $A \in \sigma(\{0,1\}^{\mathbb{Z}^d})$, A^c is decreasing and vice versa. Also, the increasing events are closed under unions and intersections.

Lemma 2.5.1 (FKG Inequality)**.** *Let $d \geq 3$, $u \geq 0$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be the measure space associated to the RI. The FKG inequality holds on this probability space, namely given two increasing events $A, B \in \mathcal{F}$, we have:*

$$\mathbb{P}[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B] \tag{2.5.4}$$

Finally, we present the following pivotal inequality which will quantify the cost of tilting (see [41, (1.61)]):

$$\mathbb{P}[A] \geq \tilde{\mathbb{P}}[A] e^{-\frac{1}{\tilde{\mathbb{P}}[A]}(H(\tilde{\mathbb{P}}|\mathbb{P}) + \frac{1}{e})} \tag{2.5.5}$$

where, for $\tilde{\mathbb{P}} \ll \mathbb{P}$, we write $H(\tilde{\mathbb{P}}|\mathbb{P}) = \tilde{\mathbb{E}}\Big[\log \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\Big]$ as the entropy between the two measures. Concisely, when proving the lower bounds, a so-called tilted interlacements measure $\tilde{\mathbb{P}}^u_N \ll \mathbb{P}^u$ will be introduced which acts as a modification of $\mathbb{P}^u$ with higher/lower densities inside tubes of the form (2.3.2) (as alluded to in the introduction, the way in which this modification is done depends on the regime that we are working in). When we speak of locally increasing the density of random interlacements inside a set B, we mean to say that, inside B, the random set $\mathcal{I}^u$ under $\tilde{\mathbb{P}}^u_N$ dominates (in the sense above) the random set $\mathcal{I}^{u_1}$ under $\mathbb{P}^{u_1}$ for some $u_1 > u$. Analogously, locally decreasing the density in B suggests that $\mathcal{I}^{u_2}$ is dominated by $\mathcal{I}^u$ in the box B for some level $u_2 < u$. For example, in the subcritical case we seek a probability measure $\tilde{\mathbb{P}}^u_N \ll \mathbb{P}^u$ which takes advantage of the strongly percolative regime while maintaining a low cost of transformation according to inequality (2.5.5).

2.6 Soft Local Times

A possibly surprising property of Poisson Point Processes is that they can be used to sample collections of Markov chains with predetermined initial and transition probabilities. Furthermore, a single PPP has the flexibility to sample more than one such collection, opening up the theory as a tool to generate couplings between Markov processes. For clarity, when one speaks of a 'coupling' the intended meaning is that random variables originally defined on separate probability spaces are redefined under a common probability measure in such a way that their laws are preserved. In the sequel, we present the coupling procedure associated to PPPs which we henceforth refer to as the soft local time method. To this end, recall the notation presented around display (2.4.4) and consider the probability space $\Big(\Omega(W_0 \times \mathbb{R}_{\geq 0}), \Sigma(W_0 \times \mathbb{R}_{\geq 0}), \mathbb{Q}\Big)$ associated to a PPP η with intensity measure $\mu \times d\lambda$. For sampling an individual random variable from this PPP, one has the following proposition.

Proposition 2.6.1. *[47, Proposition 4.1] Let $\pi : W_0 \to \mathbb{R}_+$ be a measurable function with $\int_{W_0} \pi(z) d\mu(z) = 1$. For $\eta = \sum_{\lambda \in \Lambda} \delta_{(z_\lambda, u_\lambda)} \in \Omega(W_0 \times \mathbb{R}_{\geq 0})$, we define*

$$\xi = \inf\{t \geq 0 : \exists \lambda \in \Lambda \text{ such that } t\pi(z_\lambda) \geq u_\lambda\} \tag{2.6.1}$$

Then under the probability measure $\mathbb{Q}$ determining the PPP η with intensity measure $\mu \times d\lambda$, the following hold:

(i) there exists an a.s. unique $\hat{\lambda} \in \Lambda$ such that $\xi\pi(z_{\hat{\lambda}}) = u_{\hat{\lambda}}$

(ii) $(z_{\hat{\lambda}}, \xi)$ is distributed as $\pi(z)\mu(dz) \otimes Exp(1)$

(iii) $\eta' \coloneqq \sum_{\lambda \neq \hat{\lambda}} \delta_{(z_\lambda, u_\lambda - \xi\pi(z_\lambda))}$ has the same law as η and is independent of $(\xi, \hat{\lambda})$

As alluded to, the above proposition can then be applied iteratively with the goal of generating a sequence of Markov processes with known starting and transition densities. In other words, assume that we have a Markov process $(Z_k)_{k \geq 1} \subset W_0$ with the following properties:

$$\begin{aligned} &Z_1 \sim \pi_0(z)\mu(dz) \\ &\mathbb{P}[Z_{k+1} \in dz | Z_k] = \pi(Z_k, z)\mu(dz), \text{ for } k \geq 1 \end{aligned} \tag{2.6.2}$$

where $\pi_0(\cdot)$ and $\pi(\cdot, \cdot)$ are $\mathcal{W}_0$-measurable in all coordinates and $\pi_0(\cdot)$, $\pi(x, \cdot)$ (for any fixed $x \in W_0$) both integrate to 1 with respect to μ. Then, in order to sample $\{z_i\}_{i \geq 1}$ from the PPP η, we initialise the process by sampling $(z_1, \xi_1)(= (z_{\hat{\lambda}}, \xi))$ according to

the steps in Proposition 2.6.1 with $\pi = \pi_0$, then set $G_1(z) = \xi\pi_0(z)$ and iteratively define for $i \geq 2$

$$\begin{aligned}
&\xi_i \coloneqq \inf\left\{t \geq 0 : \exists(z_\lambda, u_\lambda) \notin \{(z_n, u_n)\}_{n<i} \text{ s.t. } G_{i-1}(z_\lambda) + t\pi(z_{i-1}^i, z_\lambda) \geq u_\lambda\right\} \\
&G_i(z) = G_{i-1}(z) + \xi_i\pi(z_{i-1}, z) \\
&(z_i, u_i) \notin \{(z_n, u_n)\}_{n<i} \text{ satisfies } G_i(z_i) = u_i
\end{aligned} \tag{2.6.3}$$

By the conclusions of Proposition 2.6.1, we have that

$$\{\xi_i\}_{i\geq 1} \text{ are i.i.d } \textit{exponential}(1) \text{ random variables.} \tag{2.6.4}$$

$$\{z_i\}_{i\geq 1} \overset{\text{law}}{=} \{Z_i\}_{i\geq 0} \text{ and are independent of the } \xi_i\text{'s.} \tag{2.6.5}$$

As an aside, the functions G_i are called the soft local times of the Markov process $\{Z_i\}_{i\geq 1}$ because their expectation is closely related to that of the classical local time of the same Markov process. For more details of this method and remark, one can refer to [47, Proposition 4.10], [47, Theorem 4.6] respectively.

To conclude this section, we introduce a powerful result which shows staggered inclusions of sequences sampled according to the soft local time method described by (2.6.3). Explicitly, given $m_0 \geq 1$, $\epsilon > 0$ and two sequences $Z = \{z_i\}_{i\geq 1}, \hat{Z} = \{\hat{z}_i\}_{i\geq 1} \subset W_0$ sampled from the same PPP $\eta \in \Omega(W_0 \times \mathbb{R}_{\geq 0})$, we show that the event

$$\text{Incl}^{\epsilon,m_0}(Z, \hat{Z}) = \begin{cases} \{z_1, \ldots, z_{\lfloor(1-\epsilon)m\rfloor}\} \subset \{\hat{z}_1, \ldots, \hat{z}_{\lfloor(1+3\epsilon)m\rfloor}\}, \\ \{\hat{z}_1, \ldots, \hat{z}_{\lfloor(1-\epsilon)m\rfloor}\} \subset \{z_1, \ldots, z_{\lfloor(1+3\epsilon)m\rfloor}\} \end{cases} \quad \text{holds for all } m \geq m_0. \tag{2.6.6}$$

occurs with high probability, and is closely related to a concentration event of a Poisson random variable. The utility of this result manifests itself when the transition probabilities of the sampled sequences Z, $\hat{Z}$ are approximately equal, in the sense determined by condition (2.6.10), as this permits a control on the precision of the inclusions parametrized by ϵ. An additional feature of Theorem 2.6.2 is that it considers a subset $W_0' \subset W_0$ of the underlying value space of the PPP η, allowing us to also tune the inclusion subsets of interest. In particular, assuming that the sequences above satisfy the additional condition $\hat{Z} \subset W_0'$, we can restrict Z to the subsequence

$$Z|_{W_0'} \coloneqq \{z_{\sigma_i}\}_{i\geq 1} \subset W_0' \tag{2.6.7}$$

where we recursively define

$$\sigma_1 = \min\{i \geq 1 : z_i \in W_0'\}, \quad \sigma_n = \min\{i > \sigma_{n-1} : z_i \in W_0'\}, \quad n \geq 2 \tag{2.6.8}$$

and guarantee $\text{Incl}^{\epsilon,m_0}(Z|_{W_0'}, \hat{Z})$, which markedly retains the precision of (2.6.6). In the statement of the theorem, we will also rely on the following random variables for $a, b \in [0,\infty)$, $W_0' \subset W_0$, $\hat{\pi} : W_0 \to \mathbb{R}_{\geq 0}$

$$N_{W_0'}(a,b) \coloneqq \#\{\lambda \in \Lambda : z_\lambda \in W_0',\ a\hat{\pi}(z_\lambda) < u_\lambda \leq b\hat{\pi}(z_\lambda)\}, \tag{2.6.9}$$

where we often abbreviate the special case $N(a,b) = N_{W_0}(a,b)$. Under $\mathbb{Q}$, in the proof of Theorem 2.6.2 we will see that $N_{W_0'}(a,b) \sim \textit{Poisson}(b-a)$ when $\hat{\pi}$ satisfies the assumptions of π in Proposition 2.6.1.

Theorem 2.6.2. *Let $\{(z_i,\xi_i)\}_{i\geq 1}$ be a sequence of pairs sampled from a PPP $\eta \in \Omega(W_0 \times \mathbb{R}_{\geq 0})$ with intensity measure $\mu \times d\lambda$ according to display (2.6.3) with initial and transition densities $\pi_0 : W_0 \to \mathbb{R}_{\geq 0}$, $\pi : W_0 \times W_0 \to \mathbb{R}_{\geq 0}$ respectively. Consider also $\{(\hat{z}_i,\hat{\xi}_i)\}_{i\geq 1}$, sampled from the same PPP η by setting the initial and transition densities as $\pi_0(\cdot) \equiv \pi(w,\cdot) \equiv \hat{\pi}(\cdot)$ for any choice $w \in W_0$, for some $\mathcal{W}_0$-measurable density $\hat{\pi} : W_0 \to \mathbb{R}_{\geq 0}$ supported on $W_0' \subset W_0$ with $\int_{W_0} \hat{\pi} d\mu = 1$ and $W_0' \subset W_0$ nonempty. Let $\epsilon \in (0,1)$ and assume that the following condition is satisfied*

$$\sup_{\substack{w\in W_0,\\ w'\in W_0'}} \left\{ \left| \frac{\pi_0'(w')}{\hat{\pi}(w')} - 1 \right|, \left| \frac{\pi'(w,w')}{\hat{\pi}(w')} - 1 \right| \right\} \leq \frac{\epsilon}{3}, \tag{2.6.10}$$

where $\pi_0'(w') = \pi / \int_{W_0'} \pi_0 d\mu$, $\pi'(w,w') = \pi(w,w') / \int_{W_0'} \pi(w,w') d\mu(w')$. Then

$$U^{\epsilon,m_0} \subset \textit{Incl}^{\epsilon,m_0}(Z|_{W_0'}, \hat{Z}) \tag{2.6.11}$$

for $Z = \{z_i\}_{i\geq 1}$, $\hat{Z} = \{\hat{z}_i\}_{i\geq 1}$, where we have defined

$$U^{\epsilon,m_0} \coloneqq \{N'(m,(1+\epsilon)m) < 2\epsilon m, (1-\epsilon)m < N'(0,m) < (1+\epsilon)m, \forall m \geq m_0\} \tag{2.6.12}$$

for $N'(\cdot,\cdot) = N_{W_0'}(\cdot,\cdot)$ as in (2.6.9). Moreover,

$$\mathbb{Q}[(U^{\epsilon,m_0})^c] \leq e^{-c\epsilon m_0}. \tag{2.6.13}$$

Proof. First we show (2.6.13) as it requires a standalone argument. Indeed, recalling the definition of $N_{W_0'}$ in (2.6.9) and the process described by (2.6.3), we have the following equality

$$N'(0,t) = N(0,t) = \sup\{i \geq 1 : \hat{\Xi}_{i-1} < t\}, \tag{2.6.14}$$

where $\hat{\Xi}_i \coloneqq \sum_{j=1}^{i} \hat{\xi}_j$, setting $\hat{\Xi}_0 \coloneqq 0$ and the first equality follows from the fact that $\hat{\pi}$ is supported on W_0'. However, by points *(ii)* and *(iii)* from Proposition 2.6.1, the $\hat{\xi}_i$ are i.i.d *exponential*(1) random variables, so their sum is gamma distributed and we have for any $n \in \mathbb{N}$

$$\begin{aligned} P[N(0,t) = n] &= P[\hat{\Xi}_n \geq t, \hat{\Xi}_{n-1} < t] \\ &= \int_0^t \int_{t-x}^{\infty} e^{-y} dy \frac{1}{\Gamma(n-1)} x^{n-1} e^{-x} dx = \frac{t^n e^{-t}}{n!}, \end{aligned} \tag{2.6.15}$$

hence $N(0,t)$ is *Poisson*(t) distributed. Since $N(a,b) = N(0,b) - N(0,a)$, by independence of the $\hat{\xi}_i$ and the memoryless property of exponential random variables, we also have $N(a,b) \sim N(0,b-a)$, so the bound (2.6.13) boils down to a classical large deviations result on the Poisson distribution.

To show (2.6.11), we proceed as in the proof of [11, Lemma 2.1], except in our version we define

$$y^{(k)} \coloneqq \arg\min_{y \in W_0'} \frac{G_k(y)}{\hat{\pi}(y)}, \tag{2.6.16}$$

and obtain the following controls for $y \in W_0'$

$$\begin{aligned} \frac{G_k(y)}{\hat{\pi}(y)} &= \frac{\pi_0(y)}{\hat{\pi}(y)} + \sum_{i=2}^{k} \xi_i \frac{\pi(z_i, y)}{\hat{\pi}(y)} \\ &\overset{(2.6.10)}{\leq} (1+\epsilon)\Big(\frac{\pi_0(y^{(k)})}{\hat{\pi}(y^{(k)})} + \sum_{i=2}^{k} \xi_i \frac{\pi(z_i, y^{(k)})}{\hat{\pi}(y^{(k)})} \Big) = (1+\epsilon) \frac{G_k(y^{(k)})}{\hat{\pi}(y^{(k)})}. \end{aligned} \tag{2.6.17}$$

Then, by abbreviating $k = \sigma_{(1-\epsilon)m}$, we have $\frac{G_k(y^{(k)})}{\hat{\pi}(y^{(k)})} \leq m$, otherwise

$$\begin{aligned}&\frac{G_k(y)}{\hat{\pi}(y)} \overset{(2.6.16)}{\geq} \frac{G_k(y^{(k)})}{\hat{\pi}(y^{(k)})} > m, \quad \forall y \in W_0' \\ \Rightarrow &|(z_\lambda, u_\lambda) : z_\lambda \in W_0',\ G_k(z_\lambda) \geq u_\lambda| > |(z_\lambda, u_\lambda) : m\hat{\pi}(z_\lambda) \geq u_\lambda| \overset{(2.6.12)}{\geq} (1-\epsilon)m\end{aligned} \tag{2.6.18}$$

where the first inequality is strict since the process by which we sample (z_λ, u_λ) implies that $\exists(z_{\hat{\lambda}}, y_{\hat{\lambda}})$ such that $G_k(z_{\hat{\lambda}}) = u_{\hat{\lambda}} > m\hat{\pi}(z_{\hat{\lambda}})$, therefore $(z_{\hat{\lambda}}, u_{\hat{\lambda}}) \notin \{(z_\lambda, u_\lambda) : m\hat{\pi}(z_\lambda) \geq u_\lambda\}$. (2.6.18) is a contradiction because of our choice of k. Then, by (2.6.17) we have

$$\frac{G_k(y)}{\hat{\pi}(y)} \leq (1+\epsilon)m, \quad \forall y \in W_0' \tag{2.6.19}$$

which, by virtue of (2.6.12), implies

$$\{z_{\sigma_1}, \ldots, z_{\sigma_{(1-\epsilon)m}}\} \subset \{\hat{z}_1, \ldots, \hat{z}_{(1+3\epsilon)m}\} \tag{2.6.20}$$

To show the inclusion in the other direction, we abbreviate $k' = \sigma_{(1+3\epsilon)m}$ and see that $\frac{G_k(y^{(k)})}{\hat{\pi}(y^{(k)})} \geq m$ for otherwise, by analogous reasoning to above, we have

$$|(z_\lambda, u_\lambda) : z_\lambda \in W_0',\ G_{k'}(z_\lambda) \geq u_\lambda| < |(z_\lambda, u_\lambda) : (1+\epsilon)m\hat{\pi}(z_\lambda) \geq u_\lambda| \overset{(2.6.12)}{\leq} (1+3\epsilon)m \tag{2.6.21}$$

which is a contradiction. Hence, by applying the first condition in the event (2.6.12) once more, we have

$$\{\hat{z}_1, \ldots, \hat{z}_{(1-\epsilon)m}\} \subset \{z_{\sigma_1}, \ldots, z_{\sigma_{(1+3\epsilon)m}}\} \tag{2.6.22}$$

as required. □

Chapter 3

Lower Bounds

In this chapter, we obtain the main (new) lower bounds on the two percolation models of interest within this thesis: the SRW on a torus, and the RI vacant sets. Each of these results arises as a consequence of a certain novel generalised lower bound appearing in Theorem 3.1.1, applied in tandem with couplings which previously appeared in the literature, namely in the form of Theorem 2.4.19 and Theorem 2.4.3. In order to show the generalised lower bound, we follow change-of-measure ideas introduced in [41], with a more elaborate choice of functions f_0, f_1 (see (3.2.21), (3.2.23) and their analogue [41, (2.4)]) to account for the sub- and supercritical regimes respectively, with the latter handling the non-monotonicity in the vacant field. The differing choice of functions effectively necessitates us to supplement the work in the aforementioned reference by showing the decay of return times of the SRW under our version of the tilted measure, thereby producing estimates on the tilted equilibrium measure. The resulting estimates are then used in §3.4 to obtain the coupling in Theorem 3.4.1, replacing Proposition 4.1 and 4.2 from [41]. Although the coupling is entirely novel, the soft local time method employed in its derivation has previously been presented in [47]. Crucially, our approach adapts the methodology therein to simultaneously handle multiple density levels of the RI.

3.1 Generalised Lower Bound

In this section, we state a generalised lower bound in Theorem 3.1.1 from which we obtain the lower bounds from Theorems 1.3.1 and 1.3.3 dealing with both the sizes of the clusters $\mathcal{C}^u_{max}, \mathcal{C}^u_{sec} \subset \mathbb{T}^d_N$ as well as the truncated connectivity function $\tau^u_{tr}(x, y)$. As will become clear, there are three crucial components of the generalisation which render it an effective and flexible tool. First of all, it simultaneously deals with connection

and disconnection events, such that the resulting probability bounds are informative in both the sub- and supercritical regimes. Moreover, the events admit a small amount of unfavourable 'sprinkling' parameterised by η, in the sense that the disconnection event occurs at a slightly lower level of the vacant set than the connection event, meaning that the overall event is strictly less likely. Finally, the source of the aforementioned flexibility of the theorem lies in the good event $G^u(x)$ which, in exchange for some decay and measurability conditions, allows us to e.g. perform surgery locally by connecting a vacant cluster to a specified nearby point (chosen as 0 or x in the case of the truncated connectivity bound). To introduce $G^u(x)$ precisely, we recall the notation from §2.1 as well as the oblique tubes from display (2.3.2) and the local times of the RI PP $(l^u_y)_{y\in\mathbb{Z}^d,u>0}$ from equation (2.4.16). In what follows, recall x, δ from the definitions of the oblique tubes and let

$$r_0 = r_0(x,\delta) \coloneqq 100 \vee \frac{|x|^\delta}{100},\ r_1 = r_0^{\frac{1}{4}} \tag{3.1.1}$$

Then, for $u > 0$, $x \in \mathbb{Z}^d$, $\delta \in (0,1)$ and $\eta \in [0,1)$, consider the event $G^u = \bigcap_{i\in I} G_{B_i}$, where $(B_i : i \in I)$ is a finite collection of boxes with possible duplicates (i.e. it could occur that $B_i = B_j$ for $i \neq j$) where each box has radius $10r_1$ and centre in $T(x)$. The aforementioned measurability condition is that for any box $B \in \{B_i : i \in I\}$, the event $G_B = G_B^{v,w}$ is expressible measurably as at most two occupation fields $(l^v_y)_{y\in B}$ and $(l^w_y)_{y\in B}$ where $v, w \in \{u(1 - \frac{k}{4}\eta\sqrt{\delta}) : k = 0,\dots,4\}$ with $v > w$ and that G_B is increasing in $(l^v_y)_{y\in B}$ and decreasing in $(l^w_y)_{y\in B}$.

Theorem 3.1.1. *Let G^u be of the above form and define, for $u \neq u_*$, $\eta \in [0,1)$, $x \in \mathbb{Z}^d$ and $\delta \in (0,1)$, the event:*

$$A^u(x,\delta) = \Big\{G^u, B(0,r_1) \overset{\mathcal{V}^u \cap T^1}{\longleftrightarrow} B(x,r_1), T^4 \overset{\mathcal{V}^{u(1-\eta\sqrt{\delta})}}{\not\longleftrightarrow} \partial T^5\Big\} \tag{3.1.2}$$

For $u > u_*$, we omit the disconnection event in the definition of $A^u(x,\delta)$ above. If the following decay condition holds

$$\lim_{|x|\to\infty} |I| \cdot \sup_{i\in I} \mathbb{P}[(G^{v,w}_{B_i})^c] = 0,\ \textit{for all } v > w \textit{ with } v, w \in [\frac{u_*}{2}, 2u_*], \tag{3.1.3}$$

then for all $u \neq u_*$, $0 \leq \eta \leq c_1$ and $d = 3$ one has that

$$\liminf_{\delta\downarrow 0} \liminf_{|x|\to\infty} \frac{\log|x|}{|x|} \log \mathbb{P}[A^u(x,\delta)] \geq -\frac{\pi}{3}\Big(\sqrt{u_*} - \sqrt{u}(1 - C\eta)\Big)^2 \tag{3.1.4}$$

The remaining sections in this chapter will be devoted to proving the theorem above, meanwhile we will immediately derive the main lower bound results as an application of Theorem 3.1.1. Let us commence with

Proof of Theorem 1.3.3 (Lower Bounds). We start by proving the subcritical case $u > u_*$, since the connectivity event is simpler due to the lack of truncation in this instance. For $x \in \mathbb{Z}^d$ and $\delta > 0$, abbreviating $B_x = B(x, r_1)$ with r_1 as in (3.1.1), by an application of the FKG inequality from Lemma 2.5.1

$$\begin{aligned} \log \tau_u^{tr}(x) = \log \mathbb{P}\Big[0 \overset{\mathcal{V}^u}{\longleftrightarrow} x\Big] &\geq \log \mathbb{P}\Big[B_0 \overset{\mathcal{V}^u}{\longleftrightarrow} B_x, (B_0 \cup B_x) \subset \mathcal{V}^u\Big] \\ &\overset{(2.4.14)}{\geq} \log \mathbb{P}\Big[B_0 \overset{\mathcal{V}^u}{\longleftrightarrow} B_x\Big] - u\mathrm{cap}(B_0 \cup B_x) \end{aligned} \tag{3.1.5}$$

Then, recalling r_1 from (3.1.1), by subadditivity we have $\mathrm{cap}(B_0 \cup B_x) \leq C|x|^{d\delta/4}$ so that, as soon as $\delta < \frac{4}{d}$, the second term disappears after multiplying by $\frac{\log|x|}{|x|}$ and taking the limit $|x| \to \infty$. On the other hand, the first term is dealt with via an application of Theorem 3.1.1 by letting $G^u = \Omega$, the space of the RI PPP. After letting $|x| \to \infty$ and $\delta \downarrow 0$, we therefore yield line (1.3.12). The case of $d = 4$ follows similarly by the comment below line (3.1.4), and multiplying by $\frac{1}{|x|}$ in place of $\frac{\log|x|}{|x|}$.

Now we turn to showing the supercritical case, $u < u_*$. Remark that we can no longer apply the FKG inequality as before since the events comprising τ_u^{tr} have opposite monotonicities. To overcome this hurdle, we will set the event G^u appropriately in such a way that we can perform surgery on the cloud of paths around 0 and x with an acceptable multiplicative cost. The outcome will be that r_1-radius boxes around 0 and x become completely vacant at a slightly lower level of the vacant set. To conclude, we apply Theorem 3.1.1 to an event where the initial connection between the boxes $B(0, r_1)$ and $B(x, r_1)$ occurs at a slightly higher level, i.e. it is negatively-sprinkled, and thus facilitates the suggested surgeries around 0 and x, subsequently yielding 2-point connectivity at the required, unsprinkled, level. To this end, let $y \in \mathbb{Z}^d$ and let $A_y \subset \tilde{A}_y$ denote the annuli defined as $A_y = B(x, 6r_1) \setminus B(x, 4r_1)$ and $\tilde{A}_y = B(y, 7r_1) \setminus B(x, 3r_1)$. Next, let $\eta \in (0, c_1)$ be sufficiently small such that $\frac{u}{1-\eta} < u_*$, rendering $v = \frac{u}{1-\eta\sqrt{\delta}} (< u_*)$ supercritical. By replacing u with v in Theorem 3.1.1, v will eventually become our starting level to carry out the strategy laid out above. With these notations, let us introduce the necessary good event

$$G^v(x) := G_0^v \cap G_x^v = \bigcap_{z \in \{0,x\}} G_{z,1} \cap G_{z,2} \cap G_{z,3} \tag{3.1.6}$$

where, setting $v_2 = v(1 - \frac{\eta\sqrt{\delta}}{2})$ so that $u < v_2 < v (< u_*)$, we define

$$\begin{aligned} G_{z,1} &:= \bigcap_{y,y' \in \mathcal{I}^{v_2} \cap A_z} \left\{ y \overset{\mathcal{I}^v \cap \tilde{A}_z}{\longleftrightarrow} y' \right\}, \\ G_{z,2} &:= \{ (\mathcal{I}^{v_2} \setminus \mathcal{I}^u) \cap B(z, 2r_1) \neq \emptyset \}, \text{ and} \\ G_{z,3} &:= \bigcap_{y \in B(z, 9r_1)} \{ l_y^v \leq r_1 \} \end{aligned} \tag{3.1.7}$$

Remark that the event $G^v(x)$ is of the required form for a choice of boxes $B_i, i \in I$, with $|I| = 6$ and with centres either 0 or x. Indeed, each of the $G_{z,i}$, $i \in \{1,2,3\}$, $z \in \{0, x\}$ are measurable with respect to at most two levels of the occupation field $(l_y^w)_{y \in B(z,10r_1)}$, and also have the required monotonicities. To illustrate, in the notation above, $G_{0,1}$ can be written as $G_{B_0}^{v,v_2}$, is increasing in $(l_y^v)_{y \in B_0}$ and decreasing in $(l_y^{v_2})_{y \in B_0}$. With this, we need only show that the decay condition (3.1.3) is satisfied in order to apply Theorem 3.1.1. To this end, $\mathbb{P}[(G_{z,1})^c]$ is dealt with by an application of [15, Thm 5.1], $\mathbb{P}[(G_{z,2})^c]$ is bounded by using the formula (2.4.14) in conjunction with the fact that $\mathcal{I}^{v_2} \setminus \mathcal{I}^u$ is distributed as $\mathcal{I}^{v_2 - u}$ and $\mathrm{cap}(B(x, 2r_1)) \geq c r_1^{d-2}$ and finally the required bound on $\mathbb{P}[(G_{z,3})^c]$ is a consequence of a union bound together with a large-deviation estimate to deal with the compound Poisson-distributed local occupation times. For each of the events $G_{z,i}$ we have therefore obtained exponentially decaying bounds in terms of r_1, where $r_1 \to \infty$ as $|x| \to \infty$, so we have shown condition (3.1.3). As a result, from line (3.1.4), for $d = 3$ we have

$$\begin{aligned} \liminf_{\delta \downarrow 0} \liminf_{|x| \to \infty} \frac{\log |x|}{|x|} \log \mathbb{P}[G^v(x), B(0, r_1) \overset{\mathcal{V}^v \cap T^1}{\longleftrightarrow} B(x, r_1), T^4 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \partial T^5] \\ \geq -\frac{\pi}{3} (\sqrt{u_*} - \sqrt{u}(1 - C\eta))^2 \end{aligned} \tag{3.1.8}$$

In fact, the events $G_{z,i}$ correspond to the events $\hat{F}_B^i$ of [20, Prop. 3.1] with $r = r_1$, $B = B(z, r_1)$, $\hat{B} = B(z, 8r_1)$, $u_1 = u_2 = u_3 = v$, and $2\delta_1 = \delta_2 = \eta\sqrt{\delta} v$ therein. Consequently, we conclude that the sprinkled finite energy property holds, since $\hat{F}_B \subset \tilde{F}_B$, where the latter event satisfies:

$$\mathbb{P}\Big[B \subset \mathcal{V}^u | \sigma(\omega_{\hat{B}}^-, \mathcal{I}^v \cap \hat{B}) \Big] \mathbb{1}_{\tilde{F}_B} \geq e^{-C r_1^{2d}} \tag{3.1.9}$$

for $\omega_{\hat{B}}^-$ the PP obtained from the RI PP $\omega = \sum_{i \geq 0} \delta_{(w_i^*, u_i)}$ by removing all sub-excursions (excluding their final points) of $\hat{B}$ ending at $\partial^{\mathrm{out}} \hat{B}$ from each doubly-infinite projected path w_i^*. As we will see, an iterative application of property (3.1.9) yields the following

for $d = 3$, $\delta \in (0, c)$ and $\eta \in (0, c(u))$

$$\begin{aligned}(\tau_u^{tr}(x) \geq)\mathbb{P}\Big[0 \overset{\mathcal{V}^u \cap T^1}{\longleftrightarrow} x,\, T^4 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \partial T^5\Big] \\ \geq e^{-Cr_1^{2d}}\mathbb{P}\Big[G^v(x), B(0,r_1) \overset{\mathcal{V}^v \cap T^1}{\longleftrightarrow} B(x,r_1), T^4 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \partial T^5\Big]\end{aligned} \tag{3.1.10}$$

Then, for now assuming the result above holds, by combining (3.1.8) with (3.1.10), taking logarithms, multiplying by $\frac{\log|x|}{|x|}$, taking the limit $|x| \to \infty$ for fixed $\delta < \frac{1}{2d}$ and finally letting $\delta \downarrow 0$, $\eta \downarrow 0$ we yield the lower bound for the case $d = 3$ (1.3.12). Therefore, to finish this case, it remains to show (3.1.10) from (3.1.9). Denoting by $E = \{B(0,r_1) \overset{\mathcal{V}^v \cap T^1}{\longleftrightarrow} B(x,r_1), T^4 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \partial T^5\}$, and noticing that for $(z,z') \in \{(0,x),(x,0)\}$ each of the $G_{z,i}$ and $\{B_z \subset \mathcal{V}^u\}$ are measurable with respect to $\sigma(\omega^-_{\hat{B}_{z'}})$ and E is measurable with respect to $\sigma(\omega^-_{\hat{B}_z}, \mathcal{I}^v \cap \hat{B}_z)$ for $z \in \{0,x\}$, we have by taking conditional probabilities and applying the tower rule:

$$\begin{aligned}&\mathbb{P}\Big[0 \overset{\mathcal{V}^u \cap T^1}{\longleftrightarrow} x,\ T^4 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \partial T^5\Big] \geq \mathbb{P}[B_0 \subset \mathcal{V}^u,\ B_x \subset \mathcal{V}^u,\ E] \\ &= \mathbb{E}\Big[\mathbb{1}\{E, B_x \subset \mathcal{V}^u\}\mathbb{P}[B_0 \subset \mathcal{V}^u | \sigma(\omega^-_{\hat{B}_0}, \mathcal{I}^v \cap \hat{B}_0)]\Big] \overset{(3.1.9)}{\geq} e^{-Cr_1^{2d}}\mathbb{P}[E, B_x \subset \mathcal{V}^u, G_0^v] \\ &= e^{-Cr_1^{2d}}\mathbb{E}\Big[\mathbb{1}\{E, G_0^v\}\mathbb{P}[B_x \subset \mathcal{V}^u | \sigma(\omega^-_{\hat{B}_x}, \mathcal{I}^v \cap \hat{B}_x)]\Big] \overset{(3.1.9)}{\geq} e^{-Cr_1^{2d}}\mathbb{P}[E, G^v(x)]\end{aligned} \tag{3.1.11}$$

Finally, we present an ad hoc argument which handles the remaining case $d \geq 4$. Let us define $\ell \subset \mathbb{Z}^d$ as the minimal, connected subset containing both 0 and x, and consider the set $\Sigma \coloneqq \partial B(\ell, 1)$, where $B(\ell,1) \coloneqq \bigcup_{x \in \ell} B(x,1)$. Observe that, by construction, Σ is a connected set such that any unbounded path starting from ℓ must intersect Σ. Moreover, by the monotonicity and subadditivity of capacity, we have that $c \leq \mathrm{cap}(\{0\}) \leq \mathrm{cap}(\Sigma) \leq C|x|$, where we have used that the cardinality of Σ is at most $C|x|$. Recalling the RI PP $\hat{\omega}$ defined below (2.4.10), we let $\mathcal{N}^u = \hat{\omega}(\Sigma \times [0,u])$ be the number of excursions with label at most u intersecting Σ - observe that the definition of PPPs implies $\mathcal{N}^u \sim \mathrm{Poisson}(u\mathrm{cap}(\Sigma))$. In light of these considerations, we have the following inequalities

$$\tau_u^{\mathrm{tr}}(x) \geq \mathbb{P}[\ell \subset \mathcal{V}^u, \Sigma \subset \mathcal{I}^u] \geq cue^{-Cu|x|}\mathbb{P}[\ell \subset \mathcal{V}^u, \Sigma \subset \mathcal{I}^u | \mathcal{N}^u = 1] \tag{3.1.12}$$

In order to estimate the conditional probability on the right-hand side, we remark that the probability

$$\inf_{z \in \Sigma} P_z[\Sigma \subset \mathrm{Range}(X), \ell \cap \mathrm{Range}(X) = \emptyset] \tag{3.1.13}$$

is necessarily a lower bound, and then we proceed with an explicit, deterministic, construction for a realisation of the path X which occurs with a probability exhibiting the desired rate of decay. Indeed, starting from z, we consider a (deterministic) nearest neighbour path which visits all points of Σ without intersecting ℓ in its first $C|x|$ steps, and then proceeds to exit $B(\Sigma, |x|)$ in the next $C|x|$ steps. Remark that the former attribute is indeed possible due to the connectedness of Σ together with $|\Sigma| \leq C|x|$. The probability that X agrees with this nearest neighbour path up until its exit time from $B(\Sigma, |x|)$ is $(2d)^{-C|x|}$. Moreover, for a point z' a distance of $|x|$ away from Σ, we have the following estimate on the return probability to Σ

$$P_{z'}[H_\Sigma < \infty] \overset{(2.2.23)}{\leq} C|x|^{2-d}\mathrm{cap}(\Sigma) \leq C'|x|^{3-d} \tag{3.1.14}$$

All in all, we conclude the result by applying the Markov property to X and observing that the realisation of X described above implies the event occurring in (3.1.13), therefore constituting a lower bound. □

Remark 3.1.2 (one-arm estimates)**.** *(i) By choosing $x = Ne_1$ in the lower bounds from (1.3.11) and (1.3.12), we immediately see the following one-arm probability estimates for $u \neq u_*$:*

$$\textit{for } d = 3: \quad \liminf_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}\Big[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_N,\ 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\Big] \geq -\frac{\pi}{3}(\sqrt{u} - \sqrt{u_*})^2, \tag{3.1.15}$$

$$\textit{for } d \geq 4: \quad \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\Big[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_N,\ 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\Big] \geq -C(u) \tag{3.1.16}$$

We state these results independently as they will be used in the proof of the subcritical bound of Theorem 1.3.1.

(ii) It turns out that slightly more is required than the one-arm estimates above in order to prove the supercritical bound of Theorem 1.3.1, since a sprinkling is unavoidable when coupling the RI with the SRW on the torus. Upon investigating the steps in the proof above, paying particular attention to lines (3.1.8) and (3.1.10), we see that a slight modification yields for $x = Ne_1$, $v = u(1 - \frac{1}{4}\eta\sqrt{\delta})$, $v' = u(1 - \frac{1}{2}\eta\sqrt{\delta})$:

$$\liminf_{\delta\downarrow 0} \liminf_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}\Big[0 \overset{\mathcal{V}^v \cap T^1}{\longleftrightarrow} Ne_1,\ T^4 \overset{\mathcal{V}^{v'}}{\not\longleftrightarrow} \partial T^5\Big] \geq -\frac{\pi}{3}(\sqrt{u_*} - \sqrt{u}(1 - C\eta))^2, \tag{3.1.17}$$

when $d = 3$. For $d \geq 4$, the factor $\log N$ on the left-hand side is removed and the lower bound is replaced by $-C(\eta, u)$. Crucially, notice that the disconnection can now occur at a disadvantageous lower level. To obtain this result, we follow the steps in display (3.1.11), except we start with levels v and v' for the connection and disconnection respectively, and raise the connection as in event E therein to a level u. Identical steps follow, modifying the event G^v by taking:

$$v \to u, \ v_2 \to v, \ u \to v'$$

which maintains its measurability and decay requirements as described above Theorem 3.1.1 and in (3.1.3).

Next, we recall the notation from §1.3 and proceed to the proofs for the main lower bound results for the RW on a torus. As alluded to in the introduction, the proof relies on the above results in the case of the RI vacant set, particularly on those emphasised in Remark 3.1.2. The strategy of the proof is to apply a second moment argument with suitably defined events in the sub- and supercritical scenarios respectively, where a coupling is necessary to transfer from events on $\mathcal{V}^u$ to analogous events on $\mathcal{V}^u_N$. Although the general idea is clear, the subtlety in the argument lies in effectively handling the error terms and the sprinkling which are introduced after coupling the vacant set on the torus to the RI vacant set - this is where we display the full power of the one arm estimates (3.1.15) - (3.1.17). Let us briefly introduce some necessary tools in the form of two lemmas, the first of which is the second moment method and the second provides us with an explicit formulation for the Radon-Nikodym derivative between two Poisson Point Processes:

Lemma 3.1.3 (Second Moment Method)**.** *[68] Let X be a non-negative random variable which is not identically equal to 0. Then:*

$$\mathbb{P}[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} \tag{3.1.18}$$

Lemma 3.1.4. *[52, Lem. 2.1] Let ν and ν' be two finite measures on a common measurable space $(W, \mathcal{W})$ and assume that $\nu' \ll \nu$. Then $\mathbb{P}_{\nu'} \ll \mathbb{P}_\nu$, where $\mathbb{P}_\nu$, $\mathbb{P}_{\nu'}$ denote the laws of the canonical Poisson point processes with intensity measures ν, ν' respectively. Moreover, we have the following explicit formulation of the Radon-Nikodym derivative between them:*

$$\frac{d\mathbb{P}_{\nu'}}{d\mathbb{P}_\nu}(\omega) = e^{-(\nu'(W)-\nu(W))} \prod_{i=1}^{n} \frac{d\nu'}{d\nu}(w_i) \tag{3.1.19}$$

where $\omega = \sum_{i=1}^{n} \delta_{w_i}$ is a generic finite point measure in W.

We yield an important consequence of Lemma 3.1.4 by considering the restriction of two levels of the random interlacements to a set W_K^* for some compact $K \subset\subset \mathbb{Z}^d$. Remark that the restricted process $\mu_{K,u}^+ = \sum_{i=1}^{\mathcal{N}_K^u} \delta_{S_K(w_i)^+}$ is a Poisson Point Process on $\Omega(W^+)$ with intensity measure $u\mathrm{cap}(K)P_{\bar{e}_K}$ (see around (2.4.12) for context) - we denote its associated probability measure on $\Omega(W^+)$ by $\mathbb{P}_K^u$. Recalling that the general definition of the space of Point Processes $\Omega(\mathcal{X})$ guarantees that all point processes on a compact set are finite, we can directly apply the lemma above:

$$\frac{d\mathbb{P}_K^u}{d\mathbb{P}_K^{u'}}(\omega) = e^{(u'-u)\mathrm{cap}(K)}\left(\frac{u}{u'}\right)^{n(\omega)} \tag{3.1.20}$$

Where ω is a finite point process as in the lemma above, and $n(\omega) = |\omega(K)|$.

Proof of Theorem 1.3.1. Let us commence by showing the subcritical case (1.3.6). Although we will focus on the case $d = 3$ since it is more involved, a comment will be made regarding the proof for $d \geq 4$ at the end. In the sequel, we simplify notation by writing $\mathbb{T} = \mathbb{T}_N^d$, and we shall denote by $\pi : \mathbb{Z}^d \to \mathbb{T}$ the canonical projection from the integer lattice onto the torus. Thus, let $u > u_*$, $\lambda \in (0,1)$ and for $\delta \in (0,1)$, define $\mathbb{L}_\delta = \{x : x \in \lfloor N^\delta \rfloor \mathbb{Z}^d \cap [0, (1-2\delta)N]^d\} \subset \mathbb{Z}^d$. An important property of the thinned lattice $\mathbb{L}_\delta$ is that, for any $x, y \in \mathbb{L}_\delta$ with $x \neq y$, we have $d_{\mathbb{T}}(x,y) \geq cN^\delta$, where $d_{\mathbb{T}}$ refers to the graph distance on the torus $\mathbb{T}$. Define under the RI measure $\mathbb{P}$

$$\Xi_{N,\delta}(\mathcal{V}^u) = \sum_{x \in \mathbb{L}_\delta} \mathbb{1}_{A_x(\mathcal{V}^u)} \tag{3.1.21}$$

where

$$A_x(\mathcal{V}) = \{x \overset{\mathcal{V}}{\longleftrightarrow} \partial B(x, \lambda r_N(u))\} \tag{3.1.22}$$

with $r_N(u)$ as in (1.3.4). In parallel, we introduce $\Xi_{N,\delta}^{\mathbb{T}}(\mathcal{V}_N^u)$ under the probability measure P_N as the torus equivalent of $\Xi_{N,\delta}$ by swapping the vacant set $\mathcal{V}^u$ with $\mathcal{V}_N^u$ and replacing $A_x(\mathcal{V})$ with $A_x^{\mathbb{T}}(\mathcal{V}) = \{\pi(x) \overset{\mathcal{V}}{\longleftrightarrow} \pi(\partial B(x, \lambda r_N(u))\}$. Noting that the occurrence of any $A_x^{\mathbb{T}}(\mathcal{V}_N^u)$ immediately implies $diam(\mathcal{C}_{max}^u) \geq \lambda r_n(u)$, we can apply the coupling from Theorem 2.4.2 to show that for all $\delta, \epsilon_1 \in (0,1)$ and $N \geq 1$, with $v_1 = u(1+\epsilon_1)$,

$$P_N[diam(\mathcal{C}_{max}^u) \geq \lambda r_N(u)] \geq P_N[\Xi_{N,\delta}^{\mathbb{T}}(\mathcal{V}_N^u) > 0] \overset{\text{Thm. 2.4.2}}{\geq} \mathbb{P}[\Xi_{N,\delta}(\mathcal{V}^{v_1}) > 0] - \eta_{1,N} \tag{3.1.23}$$

where $\eta_{1,N} = \eta_{1,N}(u,\lambda,\delta,\epsilon_1) = C(u,\delta,\lambda)\exp\{-c(\delta)\epsilon_1 u^{\frac{1}{2}}N^{(d-2)/2}\}$. We remark that, in order to apply the coupling we rely on the fact that $\Xi_{N,\delta}(\mathcal{V}^u)$ and $\Xi^{\mathbb{T}}_{N,\delta}(\mathcal{V}^u_N)$ are both decreasing in u and measurable with respect to the discrete local times $l^u_{x,dis}$ for $x \in B(0,(1-\delta)N/2)$ and l^u_x for $x \in \mathbb{T}$ respectively for $N \geq C(u,\delta,\lambda)$. Next, with the aim of applying Lemma 3.1.3 to the final quantity in line (3.1.23), we control $\mathbb{E}[\Xi_{N,\delta}(\mathcal{V}^{v_1})^2]$ with the help of the decoupling from Theorem 2.4.3. To this end, let $\epsilon_2 \in (0,1)$ and $v_2 = u(1+\epsilon_1)(1-\epsilon_2)$ so that $v_2 < v_1$. Then, with the abbreviation $\Xi = \Xi_{N,\delta}(\mathcal{V}^{v_1})$, from the translation invariance of $\mathcal{V}^{v_1}$ we have

$$\mathbb{E}[\Xi^2] = \mathbb{E}[\Xi] + \sum_{x\neq y}\mathbb{P}[A_x(\mathcal{V}^{v_1})\cap A_y(\mathcal{V}^{v_1})] \overset{\text{Thm. 2.4.3}}{\leq} \mathbb{E}[\Xi] + |\mathbb{L}_\delta|^2(\mathbb{P}[A_0(\mathcal{V}^{v_2})]^2 + \eta_{2,N}), \tag{3.1.24}$$

where $\eta_{2,N} = \eta_{2,N}(v_2,\epsilon_2) = CN^{d\delta}\exp\{-c\epsilon_2^2 v_2 N^{\delta(d-2)}\}$. Overall, observing that $\mathbb{E}[\Xi] = |\mathbb{L}_\delta|\cdot\mathbb{P}[A_0(\mathcal{V}^{v_1})]$ by another application of translation invariance, we have that for any $\delta,\epsilon_1,\epsilon_2 \in (0,1)$ and for suitably large $N \geq C(u,\epsilon_1,\lambda)$

$$\frac{\mathbb{E}[\Xi]^2}{\mathbb{E}[\Xi^2]} \geq \left(\mathbb{E}[\Xi]^{-1} + \left(\frac{\mathbb{P}[A_0(\mathcal{V}^{v_2})]}{\mathbb{P}[A_0(\mathcal{V}^{v_1})]}\right)^2 + \mathbb{P}[A_0(\mathcal{V}^{v_1})]^{-2}\eta_{2,N}\right)^{-1} \tag{3.1.25}$$

We deal with each term on the right-hand side of (3.1.25) separately. For a suitable selection of the parameters $\delta,\epsilon_1,\epsilon_2$, it will be shown that the first and last terms both decay to 0 in the large N limit, meanwhile the central term will tend to 1 in the same limit. Therefore, in order to handle the first term, we notice that for $d=3$, $|\mathbb{L}_\delta| \geq (1-2\delta)e^{3(1-\delta)\log N}$, so our aim is to show an upper bound on $\mathbb{P}[A_0(\mathcal{V}^{v_1})]^{-1}$ which is dominated by the rate of decay of $|\mathbb{L}_\delta|^{-1}$. As a consequence of (3.1.22), (3.1.15) and (1.3.4), we have for all $N \geq C(u,\epsilon_1,\lambda)$

$$\mathbb{P}[A_0(\mathcal{V}^{v_1})]^{-1} \leq \exp\left\{3\lambda(1+\epsilon_1)\left(\frac{\sqrt{v_1}-\sqrt{u_*}}{\sqrt{u}-\sqrt{u_*}}\right)^2\frac{(\log N)\log\log N}{\log\log N + \eta_{3,N}}\right\} \tag{3.1.26}$$

where $\eta_{3,N} = \eta_{3,N}(u,\lambda)$ is a term arising from $\log r_N$ satisfying $\lim_{N\to\infty}\frac{\eta_{3,N}}{\log\log N} = 0$. Note also that the 3 appearing in the above bound is important and precise, manifesting as $3 = \frac{9}{\pi}\frac{\pi}{3}$ where the first factor comes from the definition of $r_N(u)$ and the second from (3.1.15). Then, we choose $\epsilon_1 = \frac{\delta}{10}$ and $\delta = \delta(u,\lambda) \in (0,1)$ small enough such that for all $N \geq C(u,\lambda)$,

$$\lambda < \frac{(1-\delta)}{(1+\epsilon_1)}\left(\frac{\sqrt{v_1}-\sqrt{u_*}}{\sqrt{u}-\sqrt{u_*}}\right)^{-2}\left(1+\frac{\eta_{3,N}}{\log\log N}\right) \tag{3.1.27}$$

which, in light of (3.1.26) and the preceding paragraph implies

$$\lim_{N\to\infty} \mathbb{E}[\Xi]^{-1} = 0 \tag{3.1.28}$$

In order to deal with the second term in (3.1.25), we apply the consequence (3.1.20) of Lemma 3.1.4 with the choice $K = B(0, \lambda r_N(u))$ to get $\frac{d\mathbb{P}_K^{v_2}}{d\mathbb{P}_K^{v_1}} \leq e^{(v_1-v_2)cap(K)}$ since $v_2 < v_1$. Therefore

$$1 \overset{v_2 < v_1}{\leq} \frac{\mathbb{P}[A_0(\mathcal{V}^{v_2})]}{\mathbb{P}[A_0(\mathcal{V}^{v_1})]} \leq e^{(v_1-v_2)cap(K)} \leq e^{C\epsilon_2 v_1(\lambda r_N(u))^{d-2}} \overset{N}{\longrightarrow} 1 \tag{3.1.29}$$

where, to ensure that the last limit holds, we recall ρ_N from (1.3.5) and set

$$\lim_N \rho_N \cdot \epsilon_2^{\frac{1}{d-2}} = 0 \text{ and } \liminf_N \epsilon_2^2 N^{\delta\frac{(d-2)}{2}} > 0 \tag{3.1.30}$$

As a consequence of condition (3.1.30), we can also deal with the final term of (3.1.25) by recalling the definition of $\eta_{2,N}$ and applying the upper bound (3.1.26) on $\mathbb{P}[A_0(\mathcal{V}^{v_1})]^{-1}$ to yield $\mathbb{P}[A_0(\mathcal{V}^{v_1})]^{-2} \cdot \eta_{2,N} \to 0$ as $N \to \infty$. Combining this result with (3.1.28), (3.1.29), (3.1.25) and the second moment method from Lemma 3.1.3, we see the required result from line (3.1.23).

Let us now proceed to the proof of the supercritical case, (1.3.7). Thus, let $u < u_*$ and $\lambda \in (0, 1)$. In comparison to the subcritical case, there are now additional complications due to the fact that we deal with the second largest cluster $\mathcal{C}_{sec}^u$, thereby introducing non-monotonic effects in our calculations. In the proof, we will again rely on a second moment method argument and, to this end, we must introduce a new event to replace $A_x(\mathcal{V})$ which will eventually imply the existence of a second largest cluster with required diameter. For $v_1, v_2 > 0$, $\delta \in (0, 1)$ and $r_N = r_N(u)$ as in (1.3.4), under $\mathbb{P}$ consider the event

$$A_x^{v_1,v_2} := \{x \overset{\mathcal{V}^{v_1} \cap T_x^1}{\longleftrightarrow} x + \lambda r_N e_1,\ T_x^4 \overset{\mathcal{V}^{v_2}}{\not\longleftrightarrow} \partial T_x^5\}, \quad x \in \mathbb{Z}^d \tag{3.1.31}$$

where $T^i = T^{i\delta}_{\lambda r_N}$ and we denote by T_x^i the translation of all points in T^i by x. Similarly, under P_N we let

$$A_x^{v_1,v_2} := \{\phi(x) \overset{\mathcal{V}_N^{v_1} \cap \phi(T^1)}{\longleftrightarrow} \phi(x + \lambda r_N e_1),\ \phi(T_x^4) \overset{\mathcal{V}_N^{v_2}}{\not\longleftrightarrow} \phi(\partial T_x^5)\}, \quad x \in \mathbb{T} \tag{3.1.32}$$

Finally, for our purposes we will require the additional supercritcally-likely event

$$E_u := \{\mathbb{T} \cap \mathcal{V}_N^u \text{ contains a cluster with diameter } \geq \frac{N}{100}\} \tag{3.1.33}$$

With a similar flavour to the steps presented in the subcritical scenario, we have the following inclusion for $N \geq C(\lambda)$

$$\bigcap_{x \in \mathbb{L}_\delta} A_{\phi(x)}^{u,u} \cap E_u \subset \{\mathcal{C}_{sec}^u \geq \lambda r_N\} \tag{3.1.34}$$

Notice that the inclusion (3.1.34) must necessarily take place because the cluster guaranteed by the event E_u cannot be connected to a cluster obtained from any of the $A_{\phi(x)}^{u,u}$, since that would imply a crossing of $\phi(T^5 \setminus T^4)$ which is precluded. Then

$$P_N[\mathcal{C}_{sec}^u \geq \lambda r_N] \geq P_N\Big[\bigcup_{x \in \mathbb{L}_\delta} A_{\phi(x)}^{u,u} \cap E_u\Big] \geq P_N\Big[\bigcup_{x \in \mathbb{L}_\delta} A_{\phi(x)}^{u,u}\Big] - P_N[E_u^c] \tag{3.1.35}$$

At this stage, we show that the limit of $P_N[E_u^c]$ in N is 0, by proving a lower bound on its complement which converges to 1.

$$\begin{aligned} P_N[E_u] &\geq P_N[B(0, \frac{N}{10}) \cap \mathcal{V}_N^u \text{ contains a cluster with diameter } \geq \frac{N}{100}] \\ &\geq \mathbb{P}[B(0, \frac{N}{10}) \cap \mathcal{V}^{u_1} \text{ contains a cluster with diameter } \geq \frac{N}{100}] - \eta_{1,N} \\ &\geq 1 - e^{-cN^c} - \eta_{1,N} \end{aligned} \tag{3.1.36}$$

where we have used that $u_1 < u_*$ is a supercritical density, so that we can profit from the good connectivity properties of the strongly percolative regime. From now on, under $\mathbb{P}$ we write $\Xi_{N,\delta}^{v_1,v_2} = \sum_{x \in \mathbb{L}_\delta} \mathbb{1}_{A_x^{v_1,v_2}}$, and when instead considering $\Xi_{N,\delta}^{v_1,v_2}$ under P_N we take the sum over $\phi(\mathbb{L}_\delta)$. Letting $\epsilon_1 \in (0,1)$, $\bar{u}_1 = u(1+\epsilon_1)$, $\underline{u}_1 = u(1+\epsilon_1)$ and bearing in mind that $A_x^{v_1,v_2}$ is decreasing in v_1 and increasing in v_2, we have

$$P_N\Big[\bigcup_{x \in \mathbb{L}_\delta} A_{\phi(x)}^{u,u}\Big] = P_N[\Xi_{N,\delta}^{u,u} > 0] \overset{\text{Thm 2.4.2}}{\geq} \mathbb{P}[\Xi_{N,\delta}^{\bar{u}_1,\underline{u}_1} > 0] - \eta_{1,N} \tag{3.1.37}$$

where $\eta_{1,N}$ takes the same form as below (3.1.23). At this stage, we emphasise that $\Xi_{N,\delta}^{\bar{u}_1,\underline{u}_1}$ is a sum over non-independent functions, since the $A_x^{\bar{u}_1,\underline{u}_1}$ are functions of the vacant set. Using the conditional decoupling from Theorem 2.4.4 in the case $k=2$ (corresponding to two different values of u, here $\bar{u}_1$ and $\underline{u}_1$) for $B(x, \lfloor \frac{1}{100} N^\delta \rfloor) \subset A \subset U \subset B(x, \lfloor \frac{1}{10} N^\delta \rfloor)$ and each $x \in \mathbb{L}_\delta$, one finds with $\bar{u}_2 = \bar{u}_1(1+\epsilon_1)$, $\underline{u}_2 = \underline{u}_1(1-\epsilon_1)$, that for all

$N \geq C(\delta, \eta, u, \lambda)$

$$\mathbb{P}[\Xi_{N,\delta}^{\bar{u}_1,\underline{u}_1} > 0] \geq \hat{\mathbb{P}}[\hat{\Xi} > 0] - |\mathbb{L}_\delta|\eta_{2,N}(u_1, \epsilon_1) \tag{3.1.38}$$

with $\eta_{2,N}(\cdot,\cdot)$ as below (3.1.24) and $\hat{\Xi} = \hat{\Xi}_{N,\delta}^{\bar{u}_2,\underline{u}_2}$ under $\hat{\mathbb{P}}$ consisting of the events $\{\hat{A}_x^{\bar{u}_2,\underline{u}_2} : x \in \mathbb{L}_\delta\}$ where the vacant sets which appear in $A_x^{\bar{u}_2,\underline{u}_2}$ are replaced by independent copies $\mathcal{V}_x^u$, $u \in \{\bar{u}_2, \underline{u}_2\}$, $x \in \mathbb{L}_\delta$, rendering the $\hat{A}_x^{\bar{u}_2,\underline{u}_2}$ independent, with the same distribution as $A_x^{\bar{u}_2,\underline{u}_2}$. Therefore, applying the second moment method yields that, for $N \geq C(\delta, \eta, u, \lambda)$

$$\mathbb{P}[\Xi_{N,\delta}^{\bar{u}_1,\underline{u}_1} > 0] \geq (\hat{\mathbb{E}}[\hat{\Xi}]^{-1} + 1 - |\mathbb{L}_\delta|^{-1})^{-1} - |\mathbb{L}_\delta|\eta_{2,N}(u_1, \epsilon_1) \tag{3.1.39}$$

To conclude, we therefore need to control $\hat{\mathbb{E}}[\hat{\Xi}]^{-1} = |\mathbb{L}_\delta|^{-1}\mathbb{P}[A_0^{\bar{u}_2,\underline{u}_2}]^{-1}$. To facilitate this, let $\epsilon_1(\delta, \eta)$ be chosen sufficiently small such that $\bar{u}_2 = u(1+\epsilon_1)^2 \leq u(1+\frac{\eta}{10}\sqrt{\delta})$ and $\underline{u}_2 = u(1-\epsilon_1)^2 \geq u(1-\frac{\eta}{10}\sqrt{\delta})$. Then, in light of (3.1.17), wherein we replace u with $u' = u\big(\frac{1+\eta\sqrt{\delta}/10}{1-\eta\sqrt{\delta}/4}\big)$ and set $v = u'(1-\frac{1}{4}\eta\sqrt{\delta})$, $v' = u'(1-\frac{1}{2}\eta\sqrt{\delta})$, since $A_0^{v,v'} \subset A_0^{\bar{u}_2,\underline{u}_2}$, we have for sufficiently large $N \geq C(u, \lambda, \epsilon_1)$

$$\mathbb{P}[A_0^{\bar{u}_2,\underline{u}_2}]^{-1} \leq \exp\left\{3\lambda(1+\epsilon_1)\Big(\frac{\sqrt{u_*} - \sqrt{u'}(1-C\eta)}{\sqrt{u_*} - \sqrt{u}}\Big)^2 \frac{\log N \cdot \log\log N}{\log\log N + \eta_{3,N}}\right\} \tag{3.1.40}$$

so that, by selecting δ, η sufficiently small, by a similar argument to the one presented around (3.1.27), we obtain $\lim_{N\to\infty} \hat{\mathbb{E}}[\hat{\Xi}]^{-1} = 0$. The result follows immediately by noticing that $|\mathbb{L}_\delta|^{-1}$, $|\mathbb{L}_\delta|\eta_{2,N}(u_1, \epsilon_1)$ and $\eta_{1,N}$ all tend to 0 as $N \to \infty$.

So far, the discussion has been oriented solely around $d = 3$, so let us describe how the above steps can be adapted to handle the remaining case $d \geq 4$. In the subcritical regime, $u > u_*$, we replace the use of (3.1.15) with (3.1.16), which consequently alleviates the stringent condition on λ from (3.1.27) instead to just requiring λ to be sufficiently small. We proceed in a similar manner when $u < u_*$ - by replacing the use of (3.1.17) with its $d \geq 4$ version (presented directly beneath it), we determine that $\hat{\mathbb{E}}[\hat{\Xi}]^{-1} \to 0$ as $N \to \infty$ and conclude the proof as before. □

3.2 Tilted Interlacements Construction

The remaining sections in this chapter are devoted to the proof of Theorem 3.1.1. In this section, we go through the necessary steps to construct a probability measure $\tilde{\mathbb{P}}_f \ll \mathbb{P}$ on the space of $W^* \times \mathbb{R}_{\geq 0}$-valued point processes which have the necessary

properties for each of the sub- and supercritical lower bound problems. In particular, for the subcritical case we seek a locally decreased density inside the strip T^1 as alluded to below equation (2.5.5), meanwhile for the supercritical case we want an increased density of random interlacements inside $T^5 \setminus T^4$ in order to synchronously maintain the likelihood of connectivity within the narrower tube T^1 while, with high probability, precluding connections within the annulus $T^5 \setminus T^4$. Finally, in both cases we endeavour to show that their entropy with the original interlacements measure is sufficiently small. Note that the probability measure $\tilde{\mathbb{P}}_f$ will henceforth be referred to as the measure of the tilted interlacements and, since it will be clear from context whether we are working in the sub- or supercritical regime, we will not change the notation between the two.

To start off with, we consider a generic function $f : \mathbb{Z}^d \to \mathbb{R}_{\geq 0}$ which is non-negative, finite, and identically equal to 1 outside a finite set. Moreover, we write $V_f = -\frac{\Delta f}{f}$, where we recall that Δ denotes the discrete Laplacian c.f. (2.2.8), and introduce the stochastic process

$$M_t^f = \frac{f(X_t)}{f(X_0)} \exp\Big\{ \int_0^t V_f(X_s) ds \Big\},\ t \geq 0 \tag{3.2.1}$$

By [41, Lemma 1.2], it is known that for all $x \in \mathbb{Z}^d$, this process is an $\mathcal{F}_t$-adapted martingale under P_x. Moreover, it is uniformly integrable under P_x, and it has a well-defined limit (by the martingale convergence theorem) which satisfies the following important property:

$$E_x[M_\infty^f] = \frac{1}{f(x)} E_x[e^{\int_0^\infty V_f(X_s) ds}] = 1 \tag{3.2.2}$$

Then, we define the measure $\tilde{P}_x^f \ll P_x$ in terms of the Radon-Nikodym derivative

$$\frac{d\tilde{P}_x^f}{dP_x} = M_\infty^f = \frac{1}{f(x)} e^{\int_0^\infty V_f(X_s) ds} \tag{3.2.3}$$

As a corollary to our above remarks, $\tilde{P}_x^f$ is in fact a probability measure on W^+ for all $x \in \mathbb{Z}^d$, which we designate as the tilted path measure. Refer to [41, Corollary 1.3] for some additional properties of this path measure, which crucially determine that the Random walk under the tilted measure is still a Markov process, although it may exhibit variable holding times unless f is chosen as a constant. In the following, we will also require several definitions and results transposed from our original path probability space to our tilted one - all of these are collected from [41] for the purpose

of convenience. To start with, we shall denote the tilted versions of the Green's and killed Green's functions as:

$$\tilde{g}^f(x,y) \coloneqq \frac{1}{f(y)^2}\tilde{E}_x\Big[\int_0^\infty \mathbb{1}\{X_s = y\}ds\Big], \quad \tilde{g}^f_U(x,y) = \frac{1}{f(y)^2}\tilde{E}_x\Big[\int_0^{T_U} \mathbb{1}\{X_s = y\}ds\Big] \tag{3.2.4}$$

where $\frac{1}{f(x)^2}$ is an invariant measure such that the tilted continuous time Markov chain is reversible. From the literature, it is known that the tilted Green's and killed Green's functions defined in this way are symmetric and finite. Moreover, let us define the tilted equilibrium measure and capacity for some $K \subset\subset \mathbb{Z}^d$:

$$\tilde{e}^f_K(x) \coloneqq \tilde{P}^f_x[\tilde{H}_K = \infty]\mathbb{1}_K(x)f(x)\Big(\frac{1}{2d}\sum_{|e|=1} f(x+e)\Big), \text{ for } x \in \mathbb{Z}^d \tag{3.2.5}$$

$$\tilde{\text{cap}}^f(K) \coloneqq \sum_{x\in K} \tilde{e}^f_K(x) \tag{3.2.6}$$

Similarly to equation (2.2.22), the standardized tilted equilibrium measure is defined for any $x \in \mathbb{Z}^d$:

$$\bar{\tilde{e}}^f_K(x) \coloneqq \frac{\tilde{e}^f_K(x)}{\tilde{\text{cap}}^f(K)} \tag{3.2.7}$$

Crucially, with the above definitions we obtain a modification of Lemma 2.2.3 to the tilted SRW case

$$\sum_{y\in K} \tilde{g}^f(x,y)\tilde{e}^f_K(y) = 1, \ \forall x \in K, \tag{3.2.8}$$

and, letting $K \subset K' \subset\subset \mathbb{Z}^d$, we have the analogue of the sweeping identity (2.2.24)

$$\tilde{P}^f_{\tilde{e}_{K'}}[H_K < \infty, X_{H_K} = y] = \sum_{x\in K'} \tilde{e}^f_{K'}(x)\tilde{h}^f_K(x,y) = \tilde{e}^f_K(y), \ \forall y \in K \tag{3.2.9}$$

where, given sets $K \subset U \subset \mathbb{Z}^d$ and points $x, y \in \mathbb{Z}^d$, we define the harmonic measure as

$$h_{K,U}(x,y) \coloneqq P_x[H_K < T_U, X_{H_K} = y] \tag{3.2.10}$$

and introduce the following related quantities

$$h_{K,U}(x) \coloneqq \sum_{y \in K} h_{K,U}(x,y),\ h_K(x,y) = h_{K,\mathbb{Z}^d}(x,y) \tag{3.2.11}$$

To define the tilted versions of the above for some choice of $f \in \mathcal{F}$, namely $\tilde{h}^f_{K,U}(\cdot,\cdot)$, $\tilde{h}^f_{K,U}(\cdot)$ and $\tilde{h}^f_K(\cdot)$, all instances of P_x are replaced with $\tilde{P}^f_x$. Additionally, referring to the discussion below equation (2.4.10), on the measurable space (Ω, Σ) corresponding to the RI we endeavour to find a suitable probability measure which governs the behaviour of the tilted interlacements - this will be our tilted probability measure $\tilde{\mathbb{P}}_f$. To this end, define a function on the space of positively-indexed paths $F : W^+ \to \mathbb{R}$:

$$F_f(w) = \int_0^\infty V_f(X_s)(w)ds \tag{3.2.12}$$

Fixing a reference density $u > 0$, and recalling the notation from (2.4.6), (2.4.8), this function induces the mapping F^*_f on $W^* \times \mathbb{R}_{\geq 0}$ in the following way

$$F^*_f(w^*, v) = \begin{cases} F_f(s_U(w^*)^+) & \text{whenever } (w^*, v) \in W^*_U \times [0,u] \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.13}$$

Then, recalling the notation for the RI measure $\mathbb{P}$ again from below equation (2.4.10), we define the measure $\tilde{\mathbb{P}}_f \ll \mathbb{P}$ via the corresponding Radon-Nikodym derivative:

$$\frac{d\tilde{\mathbb{P}}_f}{d\mathbb{P}} = e^{\langle \hat{\omega}, F^*_f \rangle} \tag{3.2.14}$$

Henceforth, we shall refer to the measure constructed via the above steps as the tilted interlacements measure. Having concluded the construction, we collect its crucial properties in a convenient proposition. As part of its statement, we also highlight the resulting connection between the tilted measures on paths and interlacements.

Proposition 3.2.1. *[41, Proposition 2.1] Fix $u > 0$, let f be as above (3.2.1). The tilted random interlacements measure from (3.2.14) satisfies the following properties*

(i) *$\tilde{\mathbb{P}}_f$ defines a probability measure on Ω,*

(ii) *under $\tilde{\mathbb{P}}_f$ the point process $\hat{\omega} \coloneqq \sum_{u_i \leq u} \delta_{(w^*_i, u_i)}$ is a PPP on $W^* \times \mathbb{R}_{\geq 0}$ with intensity measure $e^{F^*_f}(\nu \otimes \lambda)$, and*

(iii) *for $K \subset\subset \mathbb{Z}^d$, μ^+_K is a PPP on $W^+ \times \mathbb{R}_{\geq 0}$ with intensity measure $\tilde{P}^f_{\tilde{e}_K} \otimes \lambda$ when viewed under the measure $\tilde{\mathbb{P}}_f$.*

Remark 3.2.2. *A consequence of this proposition is that the level u tilted interlacements also follow an interlacements-like structure in the sense that, when restricted to the set W_K^*, the intensity measure $\tilde{\nu}$ can be represented in the form (2.4.9). In particular, for any $K \subset\subset \mathbb{Z}^d$, we have:*

$$\mathbb{1}_{W_K^*} \cdot \tilde{\nu} = \pi \circ \tilde{Q}_K \tag{3.2.15}$$

where $\tilde{Q}_K$ is a probability measure on the space $(W, \mathcal{W})$ such that:

$$\tilde{Q}_K[(X_{-n})_{n\geq 0} \in A, X_0 = x, (X_n)_{n\geq 0} \in B] = \tilde{P}_x[A|\tilde{H}_K = \infty]\tilde{P}_x[B]\tilde{e}_K(x) \tag{3.2.16}$$

for $A, B \in \mathcal{W}^+$, $x \in \mathbb{Z}^d$.

By applying the above remark together with line (2.4.9), we immediately obtain the following corollary which will be useful in §3.4 when the random interlacements restricted to a local set are rewritten as the union of a random number of excursions in that set.

Corollary 3.2.2.1. *Recalling the notation from display (2.4.11), the laws of (μ_K^+, μ_K^-) under $\mathbb{P}$ and $\tilde{\mathbb{P}}_f$ are Poisson Point Processes on $\Big(\Omega(W^+ \times \mathbb{R}_{\geq 0}) \times \Omega(W^+ \times \mathbb{R}_{\geq 0})\Big)$ with intensity measures characterized by:*

$$\zeta_K(\bar{B}_1 \times \bar{B}_2) \coloneqq \lambda_\Delta(I_1 \times I_2) \sum_{x\in K} e_K(x) P_x[B_1] P_x[B_2|\tilde{H}_K = \infty] \tag{3.2.17}$$

and

$$\tilde{\zeta}_K(\bar{B}_1 \times \bar{B}_2) \coloneqq \lambda_\Delta(I_1 \times I_2) \sum_{x\in K} \tilde{e}_K(x) \tilde{P}_x[B_1] \tilde{P}_x[B_2|\tilde{H}_K = \infty] \tag{3.2.18}$$

respectively, where $\bar{B}_i = B_i \times I_i$, $B_i \in \mathcal{W}^+$, $I_i \subset \mathbb{R}_{\geq 0}$ for $i \in \{1, 2\}$, and λ_Δ is the Lebesgue measure on the diagonal divided by $\sqrt{2}$.

Let us also introduce a result which utilizes the extension of $\mathcal{I}^u$ to PPs on one-sided paths from line (2.4.15) in conjunction with the properties of μ_K^+ and μ_K^-.

Proposition 3.2.3. *Given a finite set $K \subset\subset \mathbb{Z}^d$ and a Poisson Point Process $\hat{\omega} \in \Omega(W^* \times \mathbb{R}_{\geq 0})$ dictated by either the untilted or tilted probability measures, we have the following important observation*

$$\mathcal{I}^u(\hat{\omega}) \cap K = \mathcal{I}^u(\mu_K^+(\hat{\omega})) \cap K, \qquad \mathbb{P}/\tilde{\mathbb{P}}_f \text{ a.s.} \tag{3.2.19}$$

Proof. First notice that $\mathcal{I}^u(\hat{\omega}) \cap K = \left\{\mathcal{I}^u(\mu_K^+(\hat{\omega})) \cup \mathcal{I}^u(\mu_K^-(\hat{\omega}))\right\} \cap K$ since the maps μ_K^+ and μ_K^- collect all possible paths which intersect K, so that the remaining paths in $\mathcal{I}^u(\hat{\omega})$ which have not been accounted for are a subset of $W^* \setminus W_K^*$, and hence have an empty intersection of their ranges with the set K. Mathematically, this is seen in the following way:

$$\begin{aligned}
&\mathcal{I}^u(\hat{\omega}) \cap K \\
&\quad = \mathcal{I}^u(\mathbb{1}\{W_K^*\}\hat{\omega} + \mathbb{1}\{W^* \setminus W_K^*\}\hat{\omega}) \cap K = \mathcal{I}^u(\mathbb{1}\{W_K^*\}\hat{\omega}) \cup \mathcal{I}^u(\mathbb{1}\{W^* \setminus W_K^*\}\hat{\omega}) \cap K \\
&\quad = \mathcal{I}^u(\mathbb{1}\{W_K^*\}\hat{\omega}) \cap K = \Big(\bigcup_{u_i \le u} \mathrm{Range}(\mathbb{1}\{W_K^*\}w_i^*) \Big) \cap K \\
&\quad = \Big(\bigcup_{\substack{u_i \le u \\ w_i^* \in K}} \mathrm{Range}(s_K(w_i^*)^+) \cup \mathrm{Range}(s_K(w_i^*)^-) \Big) \cap K = \left\{\mathcal{I}^u(\mu_K^+(\hat{\omega})) \cup \mathcal{I}^u(\mu_K^-(\hat{\omega}))\right\} \cap K
\end{aligned} \tag{3.2.20}$$

To conclude, we notice that $\mathcal{I}^u(\mu_K^-(\hat{\omega})) \cap K \subset \mathcal{I}^u(\mu_K^+(\hat{\omega})) \cap K$, $\mathbb{P}$ / $\tilde{\mathbb{P}}_f$-a.s., since by Corollary 3.2.2.1, $\mu_K^-(\hat{\omega})$ picks up paths w with law $P_x[\cdot|\tilde{H}_K = \infty]$ (or $\tilde{P}_x[\cdot|\tilde{H}_K = \infty]$ for the tilted case), and for each $x \in \partial_i K$, $\mathrm{Range}(w) = X_0(w)$, $P_x[\cdot|\tilde{H}_K = \infty]/\tilde{P}_x[\cdot|\tilde{H}_K = \infty]$ -a.s. □

Finally, we are ready to diverge in procedure depending on whether we are interested in the subcritical or supercritical problem. With this in mind, we commence by defining f_0 for subcritical purposes

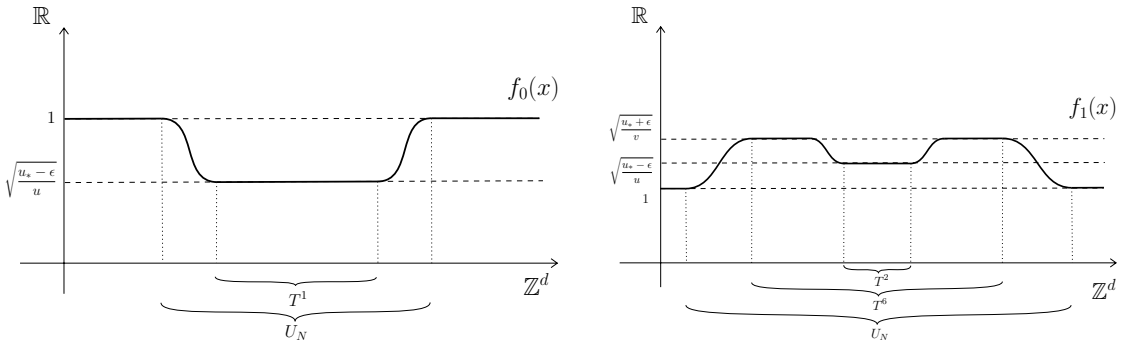
$$f_0(x) \equiv f_0^{u;\epsilon}(x) = 1 - \Big(1 - \sqrt{\frac{u_* - \epsilon}{u}}\Big) h_0(x), \quad u > u_*, \tag{3.2.21}$$

where $h_0(x) \coloneqq P_x(H_{T^1} < T_U)$. An essential property of this function is the following

$$f_0(x) = \begin{cases} \sqrt{\frac{u_* - \epsilon}{u}} & x \in T^1 \\ 1 & x \notin U \end{cases} \tag{3.2.22}$$

Symmetrically, for the supercritical problem, we consider $0 < v \le u < u_*$ and define

$$f_1(x) \equiv f_1^{v,u;\epsilon} = 1 + \Big(\sqrt{\frac{u_* + \epsilon}{v}} - 1\Big) h_2(x) - \Big(\sqrt{\frac{u_* + \epsilon}{v}} - \sqrt{\frac{u_* - \epsilon}{u}}\Big) h_1(x), \tag{3.2.23}$$



(a) f_0 Transformation **(b)** f_1 Transformation

Fig. 3.1 A depiction of the relevant tilting functions, $f_0^{u;\epsilon}$ and $f_1^{v,u;\epsilon}$ for the sub- and supercritical regimes respectively (see (3.2.21), (3.2.23)).

where $h_1(x) \coloneqq P_x[H_{T^2} < T_{T^3}]$, $h_2(x) \coloneqq P_x[H_{T^6} < T_U]$. This function now satisfies

$$f_1(x) = \begin{cases} \sqrt{\frac{u_*-\epsilon}{u}}, & x \in T^2 \\ \sqrt{\frac{u_*+\epsilon}{v}}, & x \in T^6 \setminus T^3 \\ 1 & x \notin U \end{cases} \tag{3.2.24}$$

In particular, each of the proposed functions f_0, f_1 are finite, non-negative and identically equal to 1 outside of a finite set. Given this observation, these functions are therefore both valid candidates for defining our tilted path and random interlacements measures - refer to the steps outlined in equations (3.2.1) to (3.2.14) to construct $\tilde{\mathbb{P}}_f$ for f_0, f_1. Let us systematically examine the effects of selecting such functions for the definitions of the tilted interlacements.

(i) In the subcritical regime, the effect of $\tilde{\mathbb{P}}_{f_0}$ is to decrease the density of the random interlacements inside the strip T_1 to appear supercritical, and hence to facilitate connections inside the vacant set.

(ii) In the supercritical regime, tilting introduced by $\tilde{\mathbb{P}}_{f_1}$ increases the density of the random interlacements inside the annulus $T^6 \setminus T^3$, thereby inhibiting the existence of an infinite cluster. In addition to this, inside T^2 the tilting also 'shifts' the density of the interlacements to be close to the tilted interlacements inside the surrounding annulus, while still being supercritical. Although the latter step may seem unnecessary, its exigency is clarified in the proof of Proposition 3.2.4 (the effective density levels must be sufficiently close together as a function of δ so that the resulting bound is well-behaved as $\delta \downarrow 0$).

(iii) The negative sprinkling present in the supercritical tilted interlacements deals with the complication of multiple starting densities of interest. Letting $v < u < u_*$, we aim to connect 0 to $x \in \mathbb{Z}_d \setminus B_N^2$ via a vacant cluster at level u, meanwhile disconnecting the vacant cluster from infinity at a level v, affording us some pivotal flexibility in certain scenarios.

Additionally, notice that both f_0 and f_1 return to 1 in a harmonic way outside of U. Such a harmonicity will come into play when satisfying the low-cost requirement of our tilting procedure, as this turns out to be the least disruptive way of modulating space. In the following proposition, we quantify this cost precisely with a bound on the relative entropy (recall the definition of $H(\cdot|\cdot)$ below (2.5.5)) in both the subcritical and supercritical cases. In the following, we use the abbreviation $\log^* x = \log x$ for $d = 3$ and $\log^* x = 1$ for $d \geq 4$.

Proposition 3.2.4. *Let $d \geq 3$. There exists $C_1(d) \in (0, \infty)$ such that for all $u < u_*$, the bound*

$$\limsup_{\delta \downarrow 0} \limsup_{\epsilon \downarrow 0} \limsup_{|x| \to \infty} \frac{\log^* |x|}{|x|} H(\tilde{\mathbb{P}}_f | \mathbb{P}) \leq C_1 (\sqrt{u} - \sqrt{u_*})^2 (1 + C\eta) \tag{3.2.25}$$

holds for $f = f_1^{u(1-\eta\sqrt{\delta}),u;\epsilon}$ and all $\eta \in [0, c)$. For $u > u_*$, (3.2.25) remains valid for $f = f_0^{u;\epsilon}$ with $\eta = 0$ on the right-hand side. Moreover, one can choose $C_1 = \frac{\pi}{3}$ when $d = 3$.

Another important feature of the probability measure $\tilde{\mathbb{P}}_f$ is the fact that it assigns a mass of 1 to the event $A^u(x, \delta)$ from (3.1.2) whenever $u \neq u_*$, meaning that the large deviations behaviour of the event under $\mathbb{P}$ is entirely encapsulated by the cost of the transformation (3.2.25). In the following proposition, any assumptions on the event G^u as discussed above Theorem 3.1.1 along with (3.1.3) are assumed to hold, and the dependence on the sprinkling parameter $\eta \in [0, 1)$ inherited by $A^u(x, \delta)$ is made implicit in our notation.

Proposition 3.2.5. *For all $\delta \in (0, 1)$, $u \neq u_*$, $\eta \in [0, c)$ and $\epsilon \in (0, u_*(1 \wedge \frac{\eta\sqrt{\delta}}{10}))$,*

$$\lim_{|x| \to \infty} \tilde{\mathbb{P}}_f[A^u(x, \delta)] = 1 \text{ holds for } \begin{cases} u > u_* \text{ with } f = f_0^{u;\epsilon} \\ u < u_* \text{ with } f = f_1^{u(1-\eta\sqrt{\delta}),u;\epsilon}. \end{cases} \tag{3.2.26}$$

Proof of Theorem 3.1.1. Let $u > u_*$ and select $f = f_0^{u;\epsilon}$ for $\epsilon > 0$. Since by construction $\tilde{\mathbb{P}}_f$ is absolutely continuous with respect to $\mathbb{P}$ c.f. (3.2.14), we can apply the entropy

inequality (2.5.5) to show that for all $x \in \mathbb{Z}^d$ and $\delta \in (0,1)$,

$$\log \mathbb{P}[A^u(x,\delta)] \geq \log(\tilde{\mathbb{P}}_f[A^u(x,\delta)]) - \frac{H(\tilde{\mathbb{P}}_f|\mathbb{P}) + e^{-1}}{\tilde{\mathbb{P}}_f[A^u(x,\delta)]} \tag{3.2.27}$$

Multiplying both sides by $\frac{\log^*(|x|)}{|x|}$ and taking $|x| \to \infty$, $\epsilon \downarrow 0$ and $\delta \downarrow 0$ in that order, we obtain (3.1.4) (with $\eta = 0$) by combining (3.2.27) with the bounds (3.2.25) and (3.2.26). For $u < u_*$, the proof is analogous by choosing $f = f_1^{u(1-\eta\sqrt{\delta}),u;\epsilon}$ instead, and the conclusions hold for all $0 \leq \eta < c$. □

In the remainder of this section, we aim to show Proposition 3.2.4 by collecting the capacity results on oblique tubes from §2.3 and applying the entropy inequality (2.5.5). On the other hand, the proof of Proposition 3.2.5 is deferred to §3.4. We start off by showing a fundamental connection between entropy and the Dirichlet form for generic functions taking the form described above (3.2.1).

Lemma 3.2.6. *Let $u > 0$. For the RI measure $\mathbb{P}$ as in the discussion below equation (2.4.10) and the tilted RI measure $\tilde{\mathbb{P}}_f$ as defined by virtue of equation (3.2.14), we find that:*

$$H(\tilde{\mathbb{P}}_f|\mathbb{P}) = u\mathcal{E}(f,f) \tag{3.2.28}$$

Proof. Recalling the definition of entropy from below (2.5.5) we have

$$H(\tilde{\mathbb{P}}_f|\mathbb{P}) = \tilde{\mathbb{E}}_f[\log \frac{d\tilde{\mathbb{P}}_f}{d\mathbb{P}}] \overset{(3.2.14)}{=} \tilde{\mathbb{E}}_f[\langle \hat{\omega}, F_f^* \rangle] = \tilde{\mathbb{E}}_f[\langle \mu_U^+, F_f \cdot \mathbb{1}_{[0,u]} \rangle] \tag{3.2.29}$$

where, in the final equality we recall the definition of $\langle \cdot, \cdot \rangle$ in the context of PPPs from the discussion below (2.4.4), and remark that the underlying space for the PPP μ_U^+ is $W^+ \times \mathbb{R}_{\geq 0}$, hence the introduction of $F_f \cdot \mathbb{1}_{[0,u]} : W^+ \times \mathbb{R}_{\geq 0} \to \mathbb{R}$. Then, by applying the definition of a PPP we can compute its expectation

$$(3.2.29) \overset{\text{Prop. } 3.2.1,(iii)}{=} u\langle \tilde{P}^f_{\tilde{e}_U}, F_f \rangle = u\tilde{E}^f_{\tilde{e}_U}\Big[\int_0^\infty V_f(X_s)ds\Big] \tag{3.2.30}$$

$$\overset{(3.2.4)}{=} u \sum_{x \in U} \sum_{x' \in \mathbb{Z}^d} \tilde{e}_{U,f}(x)\tilde{g}_f(x,x')f(x')^2 V_f(x') \tag{3.2.31}$$

$$\overset{(3.2.8)}{=} u \sum_{x \in \mathbb{Z}^d} V_f(x)f(x)^2 = -u \sum_{x \in \mathbb{Z}^d} f(x)\Delta(f(x)-1) \tag{3.2.32}$$

$$= \frac{u}{4d} \sum_{x \sim y} \Big(f(x) - f(y)\Big)^2 \overset{(2.2.10)}{=} u\mathcal{E}(f,f) \tag{3.2.33}$$

where for equation (3.2.32) we used that $V_f(\cdot)$ is supported in U and that the tilted Green's function is symmetric. To obtain equation (3.2.33), we used the fact that $\Delta(f+c) = \Delta f$ for any constant c and that $f-1$ is finitely supported to apply the discrete Gauss-Green theorem. □

Lemma 3.2.7. *For all $u, \epsilon > 0$ and $0 < v \leq u$, one has for f_0, f_1 as defined in (3.2.21), (3.2.23) respectively, that $f_0, f_1 \in H(\mathbb{Z}^d)$ (see 2.1.3) as well as*

$$\mathcal{E}(f_0, f_0) = u^{-1}(\sqrt{u} - \sqrt{u_* - \epsilon})^2 cap_U(T^1), \tag{3.2.34}$$

$$\mathcal{E}(f_1, f_1) = v^{-1}[(\sqrt{u_* + \epsilon} - \sqrt{v})^2 cap_U(T^6) + (\sqrt{u_* + \epsilon} - \sqrt{v/u}\sqrt{u_* - \epsilon})^2 cap_{T^3}(T^2)]. \tag{3.2.35}$$

Proof. The remark that $f_0, f_1 \in H(\mathbb{Z}^d)$ follows immediately from the fact that they are constant outside of the finite set U - in view of this, we can proceed to the computations. Let us only tackle the case for (3.2.35) since (3.2.34) is a simplification of the same steps. Introducing the shorthand notations $\alpha = \left(\frac{u_*+\epsilon}{v}\right)^{1/2} - 1$ and $\beta = \left(\frac{u_*+\epsilon}{v}\right)^{1/2} - \left(\frac{u_*-\epsilon}{u}\right)^{1/2}$, we have that:

$$\mathcal{E}(f_1, f_1) = \alpha^2 \mathcal{E}(h_2, h_2) - 2\alpha\beta\mathcal{E}(h_2, h_1) + \beta^2 \mathcal{E}(h_1, h_1) = \alpha^2 \mathcal{E}(h_2, h_2) + \beta^2 \mathcal{E}(h_1, h_1) \tag{3.2.36}$$

Where the cross-term $\mathcal{E}(h_1, h_2) = 0$ due to the fact that h_2 is constant wherever h_1 is not constant, so at least one of the gradients $h_i(x) - h_i(y)$ is 0 for any $x \sim y$. The result follows from the variational formulation of capacity presented in Lemma 2.2.4, particularly taking note of its unique minimiser $P_x[H_K < T_U]$. □

With Lemma 3.2.6 and Lemma 3.2.7 available to us, Proposition 3.2.4 becomes an easy consequence.

Proof of Proposition 3.2.4. As usual, let us focus on the case $u < u_*$ with $f = f_1^{u(1-\eta\sqrt{\delta}),u;\epsilon}$ since the reasoning for $u > u_*$ simply replaces the use of (3.2.35) with (3.2.34) in what follows. Combining (3.2.28) with (3.2.35) with the choice $v = u(1-\eta\sqrt{\delta})$ for $\eta \in [0,1]$, and subsequently feeding the bounds (2.3.3)-(2.3.5), one finds that

$$\hat{\pi}(\tilde{\mathbb{P}}|\mathbb{P}) \leq \frac{1+C\delta}{1-\eta\sqrt{\delta}} \frac{C_1|x|}{\log|x|} \Bigg[\Big(\sqrt{u_* + \epsilon} - \sqrt{u(1-\eta\sqrt{\delta})}\Big)^2 \Big(1 + \frac{C}{\log|x|}\Big) + C\delta^{-1}\Big(\sqrt{u_* + \epsilon} - \sqrt{(1-\eta\sqrt{\delta})u_* - \epsilon}\Big)^2 \Bigg] \tag{3.2.37}$$

for all $u, \epsilon > 0$, $\eta \in [0, c)$ and $|x| \geq C(\delta)$. Multiplying both sides by $\frac{\log^* |x|}{|x|}$, the desired bound follows by taking the successive limits $|x| \to \infty, \epsilon \downarrow 0$ and $\delta \downarrow 0$, using for the term in the second line that $|1 - \sqrt{1-x}| \leq Cx$ for $0 < x < c$ with $x = \eta\sqrt{\delta}$, which upon squaring precludes the explosion of the factor δ^{-1} in the limit $\delta \downarrow 0$. □

3.3 Estimates for the Tilted Harmonic Measure

The purpose of this section is to show a comparison lemma between the tilted and untilted counterparts of the harmonic measure (see (3.2.10), (3.2.11)), as well as to show the matching result for the equilibrium measure (see (2.2.20), (3.2.5)). The estimates we obtain herein will be used extensively in §3.4 when we show certain couplings of the local times of the RI with the tilted RI. With this goal in mind, let us fix a direction x for our nested oblique tubes, and define the sets $\Gamma_{f_0} \coloneqq \Gamma_{int} = T(x)$, $\Gamma_{f_1} \coloneqq \Gamma_{int} \cup \Gamma_{ext}$, $\Gamma_{ext} \coloneqq \partial T^4$. $\Gamma_f, f \in \mathcal{F}$. Next, for a choice of $f \in \mathcal{F}$, we define an accompanying collection of nested l^∞-balls which are anchored at points in the set Γ_f

$$B_i^y \coloneqq B(y, r_1^{(i+2)/8}),\ y \in \Gamma_f,\ i \in \{1, \dots, 4\} \tag{3.3.1}$$

Notice that, with this choice of radii, we have $B_4^y \subset T^1$ whenever $y \in \Gamma_{int}$ and $B_4^y \subset T^5 \setminus T^3$ whenever $y \in \Gamma_{ext}$. The advantage of such a nesting is twofold. First of all, by analysing the tilting method presented in §3.2, we observe that random walks do not experience the effect of tilting inside T^1 or $T^5 \setminus T^3$, since $V_f(x) = 0$ in these regions. As a result, an excursion of B_i for $\{i \in 1, \dots, 4\}$ which is killed upon exiting B_i has a law agnostic from the choice P_x or $\tilde{P}_x^f$ for $x \in \partial B_i$. Secondly, we will see in Lemma 3.3.2 that by considering $|x| \geq C(\epsilon)$ one can bound by ϵ the tilted probability of re-entrances of a random walk to B_i after exiting B_{i+1}. This result mirrors the situation in the untilted case which we present for completeness below, and, in combination with the first remark, implies diminishing effects of the tilting on an individual RW as $|x| \to \infty$.

Lemma 3.3.1. *For $x \in \mathbb{Z}^d \setminus \{0\}$, $\delta \in (0, 1)$, for each $i \in \{1, \dots, 3\}$, we have:*

$$\beta_0(x) \coloneqq \sup_{i=1,2,3} \sup_{z \in \mathbb{Z}^d, \nu \in \partial B_{i+1}^z} P_\nu[H_{B_i^z} < \infty] \to 0 \text{ as } |x| \to \infty \tag{3.3.2}$$

Proof. By translational invariance of the law of a SRW, we let $z \in \mathbb{Z}^d$ be arbitrary and elide dependence on z in our notation. Fix a $\nu \in \partial B_{i+1}$, and proceed in the following

way

$$P_v(H_{B_i} < \infty) \overset{\text{lem. 2.2.3}}{=} \sum_{y \in B_i} g(v,y) e_{B_i}(y) \leq C_g \sum_{y \in B_i} |v-y|^{2-d} e_{B_i}(y) \overset{\text{cap}(B(0,r)) \asymp r^{d-2}}{\leq} o(r_1^{-c}) \quad (3.3.3)$$

where the final term decays to 0 as $|x| \to \infty$. □

In this context, we synchronously produce the bound on the tilted version of the above quantity. Naturally, this is more involved and boils down to producing a control on the Radon-Nikodym derivative between the tilted and untilted measures. The argument which we invoke will hinge on Lemma 3.3.3 wherein we bound a component on which the Radon-Nikodym derivative is well-behaved and will feed directly into a Chebyshev argument.

Lemma 3.3.2. *Let $x \in \mathbb{Z}^d \setminus \{0\}$, $\delta \in (0,1)$ For $f \in \mathcal{F}$, we have:*

$$\beta_f(x) \coloneqq \sup_{i=1,2} \sup_{y \in \Gamma_f, v \in \partial B_{i+1}} \tilde{P}_v^f[H_{B_i} < \infty] \to 0 \text{ as } |x| \to \infty \quad (3.3.4)$$

Proof. For simplicity, we only deal with the case $i = 2$ since the other case is treated in the same way. To commence the proof, we select arbitrary $f \in \mathcal{F}$, $y \in \Gamma_f$ and $v \in \partial B_3$ and decompose $\tilde{P}_v^f(H_{B_2} < \infty)$ over the exit time from B_4:

$$\tilde{P}_v^f[H_{B_2} < \infty] = \tilde{P}_v^f[H_{B_2} < T_{B_4}] + \tilde{P}_v^f[T_{B_4} \leq H_{B_2} < \infty] \quad (3.3.5)$$

Since, prior to exiting B_4, the law of the tilted simple random walk is identical to that of the usual simple random walk ($B_4 \subset T^1$ whenever $y \in \Gamma_{f_0}$ and $B_4 \subset T^1 \cup T^5 \setminus T^3$ whenever $y \in \Gamma_{f_1}$, and f_0, f_1 are constant on these respective sets), we can estimate the first value on the RHS as follows:

$$\tilde{P}_v^f[H_{B_2} < T_{B_4}] = P_v[H_{B_2} < T_{B_4}] \leq P_v[H_{B_2} < \infty] \overset{(3.3.3)}{=} o(r_1^{-c}) \quad (3.3.6)$$

where the bound is uniform in the choice of f, y, v such that

$$\beta_0^f(x) \coloneqq \sup_{y \in \Gamma_f, v \in \partial B_3} \tilde{P}_v^f[H_{B_2} < T_{B_4}] = o(r_1^{-c}) \quad (3.3.7)$$

and so decays to 0 when $|x| \to \infty$. We now deal with the second component of the RHS of equation (3.3.5) by applying the strong Markov property at the exit time of

B_4 followed by the entrance time of B_3:

$$\tilde{P}_v^f[T_{B_4} \leq H_{B_2} < \infty] \leq \max_{z\in\partial B_4} \tilde{P}_z^f[\tilde{H}_{B_3} < \infty] \max_{v'\in\partial B_3} \tilde{P}_{v'}^f[H_{B_2} < \infty] \tag{3.3.8}$$

Then, by taking the maximum over $v \in \partial B_3$ on both sides of equation (3.3.5), substituting in equation (3.3.8) and rearranging we obtain:

$$\max_{v\in\partial B_3} \tilde{P}_v^f[H_{B_2} < \infty] \leq \frac{\beta_0^f(x)}{1 - \max_{z\in\partial B_4} \tilde{P}_z^f[\tilde{H}_{B_3} < \infty]} \tag{3.3.9}$$

Hence, since we know already that $\beta_0^f(N)$ decays to 0 polynomially in r_1, we need only show that $\liminf_{|x|\to\infty} \min_{z\in\partial B_4} \tilde{P}_z^f[\tilde{H}_{B_3} = \infty] > 0$. Remark that, for sufficiently large $|x|$ and for all $z \in \partial B_4$, we have $P_z[\tilde{H}_{B_3} = \infty] > c \in (0,1)$. For $\alpha > 0$, let $I_f(\alpha) \coloneqq \Big\{ \int_0^\infty V_f(X_s)ds \geq -\alpha \Big\}$. Denoting $V_f^- = (-V_f) \vee 0$, we have the following modification of Chebyshev's inequality:

$$\begin{aligned}
P_y[I_f^c] &= \int_{W^+} \mathbb{1}\Big\{ \int_0^\infty V_f(w_s^+)ds < -\alpha \Big\} dP_y(w^+) \\
&\leq \int_{W^+} \mathbb{1}\Big\{ \int_0^\infty V_f^-(w_s^+)ds > \alpha \Big\} dP_y(w^+) \\
&\leq \int_{W^+} \frac{\int_0^\infty V_f^-(w_s^+)ds}{\alpha} \mathbb{1}\Big\{ \int_0^\infty V_f^-(w_s^+)ds > \alpha \Big\} dP_y(w^+) \\
&\leq \frac{E_y\Big[\int_0^\infty V_f^-(X_s)ds\Big]}{\alpha} \overset{\text{lem. } 3.3.3}{\leq} \frac{c'(u)}{\alpha}
\end{aligned} \tag{3.3.10}$$

Then, by writing $I_f = I_f\Big(\frac{2c'(u)}{c}\Big)$ for c chosen above:

$$P_y[I_f] = 1 - P_y\Big[-\int_0^\infty V_f(X_s)ds > \frac{2c'(u)}{c} \Big] \geq 1 - \frac{c}{2} \tag{3.3.11}$$

Hence $P_z[\{\tilde{H}_{B_3} = \infty\} \cap I_f] \geq \frac{c}{2}$, and on I_f we have that:

$$\frac{d\tilde{P}_z^f}{dP_z} = \frac{1}{f(z)} e^{\int_0^\infty V_f(X_s)ds} \geq e^{-\frac{2c'(u)}{c}} = c'' > 0 \tag{3.3.12}$$

so we finally get:

$$\begin{aligned}\liminf_{|x|\to\infty} \min_{z\in\partial B_4} \tilde{P}_z^f[\tilde{H}_{B_3}=\infty] &\geq \liminf_{|x|\to\infty} \min_{z\in\partial B_4} \tilde{P}_z^f[\{\tilde{H}_{B_3}=\infty\}\cap I_f] \\ &= \liminf_{|x|\to\infty} \min_{z\in\partial B_4} E_z[\frac{d\tilde{P}_z^f}{dP_z}\mathbb{1}\{\tilde{H}_{B_3}=\infty, I_f\}] \\ &\geq \frac{c''c}{2} > 0\end{aligned} \tag{3.3.13}$$

which, together with equation (3.3.9) completes the proof by remarking that the bound we obtained is uniform in y. □

Lemma 3.3.3. *Let $x\in\mathbb{Z}^d$, $\delta\in(0,1)$. For $f\in\mathcal{F}$ there exists some constant $c'(u)>0$ such that*

$$\sup_{y\in\Gamma_f(x), z\in\partial B_4^y} E_z\Big[\int_0^\infty V_f^-(X_s)ds\Big] \leq c'(u) \tag{3.3.14}$$

where $V_f^- \coloneqq (-V_f)\vee 0$.

Proof. First we deal with the case of $f=f_0$:

$$\begin{aligned}E_z\Big[\int_0^\infty V_{f_0}^-(X_s)ds\Big] &= E_z\Big[\int_0^\infty V_{f_0}^-(X_s)\mathbb{1}\{X_s\in\partial_i T^1\cup\partial U\}ds\Big] \\ &= \sum_{z'\in\partial T^1\cup\partial^{\text{out}}U} V_{f_0}^-(z')g(z,z')\end{aligned} \tag{3.3.15}$$

where in the first line we have used the fact that $V_{f_0}(z')=0$ whenever $z'\notin\partial T^1\cup\partial^{\text{out}}U$. Next, we recall the definition of V_f from above (3.2.1) and abbreviate $\alpha=1-\sqrt{\frac{u_*-\epsilon}{u}}$ such that $f_0=1-\alpha h_0$ such that

$$V_{f_0}(z') = -\frac{\Delta f_0(z')}{f_0(z')} = \frac{\alpha\Delta h_0(z')}{1-\alpha h_0(z')} = \frac{\alpha}{2d(1-\alpha h_0(z'))}\sum_{\xi\sim z'}(h_0(\xi)-h_0(z')) \tag{3.3.16}$$

Moreover, recall that when $z'\in\partial^{\text{out}}U, h_0(z')=0$, so that $V_{f_0}(z')=\frac{\alpha}{2d}\sum_{\xi\sim z'}h_0(\xi)\geq 0$, hence the sum over $z'\in\partial^{\text{out}}U$ can be ignored as $V_{f_0}^-(z')=0$. Meanwhile, $h_0(z')=1$

whenever $z' \in \partial T^1$, hence:

$$\begin{aligned}(3.3.16) &= \frac{-\alpha}{2d(1-\alpha)} \sum_{\xi \sim z'} (1 - P_\xi[H_{T^1} < T_U]) = \frac{-\alpha}{2d(1-\alpha)} \sum_{\xi \sim z'} P_\xi[H_{T^1} \geq T_U] \\ &\geq \frac{-\alpha}{d(1-\alpha)} \sum_{\xi \sim z'} P_\xi[H_{T^1} = \infty] = \frac{-2\alpha}{1-\alpha} P_z[\tilde{H}_{T^1} = \infty] \qquad (3.3.17) \\ &= -\frac{2\alpha}{1-\alpha} e_{T^1}(z') \geq -c'(u) e_{T^1}(z')\end{aligned}$$

where we have used that $P_\xi[H_{T^1} \geq T_U] \leq 2P_\xi[H_{T^1} = \infty]$ for sufficiently large $|x|$ together with $\sum_{\xi \sim z'} P_\xi[H_{T^1} = \infty] = 2dP_{z'}[\tilde{H}_{T^1} = \infty]$ for $z' \in \partial T^1$. The first of these results is seen by applying the strong Markov property at exit times of U.

$$\begin{aligned}P_\xi[H_{T^1} = \infty] &= \sum_{\xi' \in \partial^{\text{out}} U} P_\xi[H_{T^1} \geq T_U, X_{T_U} = \xi'] P_{\xi'}[H_{T^1} = \infty] \\ &\geq \min_{\xi' \in \partial U} P_{\xi'}[H_{T^1} = \infty] P_\xi[H_{T^1} \geq T_U]\end{aligned} \qquad (3.3.18)$$

and then using that $\min_{\xi' \in \partial^{\text{out}} U} P_{\xi'}[H_{T^1} = \infty] \to 1$ by an approach similar to that in (3.3.3). To obtain the second result, simply apply the Markov property at X_1 - for z' chosen as above and $\xi \sim z'$ with $\xi \in \partial T^1$, we have:

$$P_{z'}[\tilde{H}_{T^1} = \infty] = E_{z'}[P_{X_1}[H_{T^1} = \infty]] = \frac{1}{2d} \sum_{\xi \sim z'} P_\xi[H_{T^1} = \infty] \qquad (3.3.19)$$

Then, plugging in the lower bound on $V_{f_0}(z')$ from line (3.3.17) into equation (3.3.15) yields:

$$(3.3.15) \leq c'(u) \sum_{z' \in \partial T^1} e_{T^1}(z') g(z, z') \overset{\text{lem. } 2.2.3}{=} c'(u) \qquad (3.3.20)$$

Now we deal with the case $f = f_1$. Similarly to the steps in display (3.3.15), we have:

$$E_z\Big[\int_0^\infty V_{f_1}^-(X_s) ds\Big] = \sum_{z' \in \partial T^2 \cup \partial^{\text{out}} T^3 \cup \partial T^6 \cup \partial^{\text{out}} U} V_{f_1}^-(z') g(z, z') \qquad (3.3.21)$$

Since $V_{f_1}(z) = 0$ whenever $z \notin \partial T^2 \cup \partial^{\text{out}} T^3 \cup \partial T^6 \cup \partial^{\text{out}} U$. Next, we abbreviate $\alpha = \sqrt{\frac{u_* + \epsilon}{v}} - 1$, $\beta = \sqrt{\frac{u_* + \epsilon}{v}} - \sqrt{\frac{u_* - \epsilon}{u}}$ such that $f_1 = 1 + \alpha h_2 - \beta h_1$ and evaluate $V_{f_1}(z')$ for $z' \in \partial T^2 \cup \partial^{\text{out}} T^3 \cup \partial T^6 \cup \partial^{\text{out}} U$ by considering each of the boundaries comprising

the union separately:

$$V_{f_1}(z') = -\frac{\Delta f_1(z')}{f_1(z')} = \frac{\beta\Delta h_1(z') - \alpha\Delta h_2(z')}{1 + \alpha h_2(z') - \beta h_1(z')} \tag{3.3.22}$$

For $z \in \partial T^2$, we have $h_2(z) = 1, \Delta h_2(z) = 0$ and $h_1(z) = 1$, hence:

$$V_{f_1}(z') = -\frac{\beta}{2d(1+\alpha-\beta)} \sum_{\xi \sim z'} P_\xi[H_{T^2} \geq T_{T^3}] \geq -c'(u) e_{T^2}(z') \tag{3.3.23}$$

By similar steps to those in display (3.3.17), except replacing T^1 with T^2 and adapting (3.3.18) to a decomposition over the exit times from T^3. Hence, for such z we have:

$$\sum_{z' \in \partial T^2} V_{f_1}^-(z') g(z, z') \leq c'(u) \sum_{z' \in \partial T^2} e_{T^2}(z') g(z, z') \leq c'(u) \tag{3.3.24}$$

For $z' \in \partial^{\text{out}} T^3$, we have $h_2(z') = 1, \Delta h_2(z') = 0, h_1(z') = 0$, so that:

$$V_{f_1}(z') = \frac{\beta}{2d(1+\alpha)} \sum_{\xi \sim z'} h_1(\xi) \geq 0 \tag{3.3.25}$$

For $z' \in \partial T^6$, we have $h_1(z') = 0, \Delta h_1(z') = 0, h_2(z') = 1$, so we have again that:

$$V_{f_1}(z') = \frac{\alpha}{2d(1+\alpha)} \sum_{\xi \sim z'} (1 - h_2(\xi)) \geq 0 \tag{3.3.26}$$

Hence $V_{f_1}^-(z') = 0$ for $z' \in \partial^{\text{out}} T^3 \cup \partial T^6$ and the corresponding sum is ignored in equation (3.3.21). Finally, for $z' \in \partial^{\text{out}} U$, we have $h_1(z') = 0, \Delta h_1(z') = 0, h_2(z') = 0$, such that:

$$V_{f_1}(z') = -\frac{\alpha}{2d} \sum_{\xi \sim z'} P_\xi[H_{T^6} < T_U] = -\alpha P_{z'}[H_{T^6} < \tilde{H}_{U^c}] \tag{3.3.27}$$

Where the final equality comes from similar steps to line (3.3.19), noting that $T_U = H_{U^c}$. By construction, see (2.3.2), $T^6 \subset U/2$, where $U/2$ denotes the box concentric to U with half the radius, hence we have $P_{z'}[H_{T^6} < \tilde{H}_{U^c}] \leq P_{z'}[H_{U/2} < \tilde{H}_{U^c}]$. Then, to bound the quantity $P_{z'}[H_{U/2} < \tilde{H}_{U^c}]$ we formalize a 1-dimensional gambler's ruin argument. Let $e_i \in \mathbb{Z}^d$ be the unit vector in the i'th direction, $X_t \coloneqq X_t^i \cdot e_i$ and $X^i \coloneqq \{X_t^i\}_{t \geq 0}$ the projections in the i'th dimension of the points of a discrete-timed path X. Then,

define for $I \subset \mathbb{Z}$:

$$H_I^i \coloneqq \inf\{t \geq 0 : X_t^i \in I\},\ \tilde{H}_I^i \coloneqq \inf\{t > 0 : X_t^i \in I\} \tag{3.3.28}$$

Also, since $z' \in \partial^{\text{out}}U$, it must be on a face orthogonal to direction i for some $i \in \{1, \dots, d\}$, hence $z' \cdot e_i = 10^3|x|$. With these definitions, we notice that:

$$\left\{H_{U/2} < \tilde{H}_{U^c}, X_0 = z'\right\} \subset \left\{H^i_{(-\infty, 5\cdot 10^2|x|]} < \tilde{H}^i_{[10^3|x|+1,\infty)}, X_0^i = 10^3|x| + 1\right\} \tag{3.3.29}$$

Since any path which reaches $U/2$ before it reaches U^c must necessarily reach the plane orthogonal to e_i with intercept $5 \cdot 10^2|x|$ before it reaches the plane orthogonal to e_i with intercept $10^3|x| + 1$. As a result, we have:

$$P_z[H_{U/2} < \tilde{H}_{U^c}] \leq \frac{1}{2} P^1_{10^3|x|}[H_{\{5\cdot 10^2|x|\}} < H_{\{10^3|x|+1\}}] \leq \frac{c}{|x|} \tag{3.3.30}$$

where in the last line P^1_α represents the probability measure associated to the simple random walk in 1-dimension starting at $x \in \mathbb{Z}$. Hence, returning to bounding (3.3.21), we have:

$$\begin{aligned} \sum_{z' \in \partial^{\text{out}}U} V^-_{f_1}(z') g(z, z') &\leq \frac{c}{|x|} \sum_{z' \in \partial^{\text{out}}U} g(z, z') \\ &\overset{(2.2.15)}{\leq} \frac{c}{|x|} |\partial^{\text{out}}U| |z - z'|^{2-d} \leq \frac{c}{|x|} \cdot |x|^{d-1} \cdot |x|^{d-2} = c \end{aligned} \tag{3.3.31}$$

where we have used that, for $y \in \Gamma_{f_1}$ and $z \in \partial B_4^y$, $|z - z'| \geq c|x|$. Combining each of the results from (3.3.24), (3.3.25), (3.3.26) and (3.3.31) obtains the required bound on (3.3.21) and completes the proof. □

We are now in a position to state and prove the required estimate on the harmonic measure. Part of the result is classical, as the untilted harmonic measure is well-studied in the literature, meanwhile the tilted version is tailor-made for our scenario and is an immediate consequence of Lemma 3.3.2 and Lemma 3.3.5. We postpone the statement of the latter lemma until after the estimate as it is used technically in the body of its proof.

Lemma 3.3.4. *Let $f \in \mathcal{F}$, $\epsilon_0 > 0$, $\delta \in (0, \frac{1}{6})$, $|x| \geq C(\epsilon_0, \delta)$, $y \in \Gamma_f(x)$, $z \in \partial B_2^y$ and $z' \in \mathbb{Z}^d$. Recalling the notation from (3.2.10), (3.2.11) and abbreviating $B_i = B_i^y$ we have*

$$h_{B_1,B_3}(z, z') \leq h_{B_1}(z, z') \leq (1 + \epsilon_0) \min_{\tilde{z} \in \partial B_2} h_{B_1,B_3}(\tilde{z}, z') \tag{3.3.32}$$

as well as

$$h_{B_1,B_3}(z,z') \le \tilde{h}^f_{B_1}(z,z') \le (1+\epsilon_0) \min_{\tilde{z}\in\partial B_2} h_{B_1,B_3}(\tilde{z},z') \tag{3.3.33}$$

Proof. Let us fix $f \in \mathcal{F}$ and henceforth declutter the notation by tacitly assuming that tilting is always carried out with this choice of f, writing $\tilde{P}_z = \tilde{P}^f_z$, $\tilde{h}_{K,U} = \tilde{h}^f_{K,U}$ and so on. Note that the the lower bounds in (3.3.32) and (3.3.33) are both immediate by definition of the equilibrium potential and, for (3.3.33), due to the fact $X_{\cdot\wedge T_{B_3}}$ has the same law under P_z as under $\tilde{P}^f_z$. Moreover, the upper bound in (3.3.32) is the content of [41, Lemma 3.5], so the proof is found therein. Finally, for the upper bound in (3.3.33), we take similar steps to those in equations (3.3.5) and (3.3.8). Specifically, recalling the definition of $\beta_f(x)$ from (3.3.4), we have

$$\begin{aligned}\tilde{h}_{B_1}(z,z') &= \tilde{h}_{B_1,B_3}(y,z) + \tilde{P}_z(T_{B_3} < H_{B_1} < \infty, X_{H_{B_1}} = z') \\ &\le \tilde{h}_{B_1,B_3}(z,z') + \max_{\alpha\in\partial B_2} \tilde{h}_{B_1}(\alpha,z')\beta_f(x)\end{aligned} \tag{3.3.34}$$

where to obtain the conclusion of display (3.3.34) we first decompose the equilibrium potential over the escape time from B_3 and then we apply the strong Markov property, first at T_{B_3} and then at H_{B_2}

$$\begin{aligned}&\tilde{P}_z[T_{B_3} < H_{B_1} < \infty, X_{H_{B_1}} = z'] \\ &\quad= \tilde{E}_z\Big[\tilde{E}_z[\mathbb{1}\{H_{B_1} < \infty, X_{H_{B_1}} = z'\}|\mathcal{F}_{T_{B_3}}]\mathbb{1}\{T_{B_3} < H_{B_1}\}\Big] \\ &\quad\le \max_{\tilde{z}\in\partial B_3} \tilde{P}_{\tilde{z}}[H_{B_1} < \infty, X_{H_{B_1}} = z']\tilde{P}_z[T_{B_3} < H_{B_1}] \\ &\quad= \max_{\tilde{z}\in\partial B_3} \tilde{E}_{\tilde{z}}\Big[\tilde{E}_{\tilde{z}}[\mathbb{1}\{H_{B_1} < \infty, X_{H_{B_1}} = z'\}|\mathcal{F}_{H_{B_2}}]\mathbb{1}\{H_{B_2} < \infty\}\Big]\tilde{P}_z[T_{B_3} < H_{B_1}] \\ &\quad\le \max_{\alpha\in\partial B_2} \tilde{P}_\alpha[H_{B_1} < \infty, X_{H_{B_1}} = z'] \max_{\tilde{z}\in\partial B_3} \tilde{P}_{\tilde{z}}[H_{B_2} < \infty)\tilde{P}_z[T_{B_3} < H_{B_1}] \\ &\quad\le \max_{\alpha\in\partial_i B_3} \tilde{h}_{B_1,f}(\alpha,z)\beta_f(N)\end{aligned}$$

Now, returning to display (3.3.34), we take the maximum over $z \in \partial B_2$ on both sides and rearrange to yield

$$\max_{z\in\partial B_3} \tilde{h}_{B_1}(z,z') \le \frac{1}{1-\beta_f(x)} \max_{z\in\partial B_2} \tilde{h}_{B_1,B_3}(z,z') \tag{3.3.35}$$

Then, by Lemma 3.3.2 we have $\beta_f(x) \to 0$, so we restrict x to be large enough such that $\frac{1}{1-\beta_f(x)} \le 1 + \frac{\epsilon_0}{2}$. We obtain our conclusion by the fact that $\max_{y\in\partial_i B_3} h_{B_1,B_4}(y,z) \le (1+\frac{\epsilon_0}{2}) \min_{y\in\partial_i B_3} h_{B_1,B_4}(y,z)$ for $|x| \ge C(\epsilon_0,\delta)$ by Lemma 3.3.5. □

Lemma 3.3.5. *Let $x \in \mathbb{Z}^d \setminus \{0\}$ satisfying $|x| \geq C$, $f \in \mathcal{F}$, $y \in \Gamma_f$ and $z \in \partial B_1^y$. We have*

$$\max_{\tilde{z}\in\partial B_2} h_{B_1,B_3}(\tilde{z},z) \leq (1+|x|^{-c\delta}) \min_{\tilde{z}\in\partial B_2} h_{B_1,B_3}(\tilde{z},z) \tag{3.3.36}$$

Proof. First we consider an arbitrary $\tilde{z} \in \partial B_2$ and decompose the killed equilibrium potential over the last step before entering B_1:

$$h_{B_1,B_3}(\tilde{z},z) = \frac{1}{2d} \sum_{\substack{z'\sim z \\ z'\in\partial B_1}} g_{B_3\setminus B_1}(\tilde{z},z') \tag{3.3.37}$$

Hence we see that our problem boils down to comparing $g_{B_3\setminus B_1}(\tilde{z},z')$ for different $\tilde{z} \in \partial B_2$. By taking the steps outlined in equations (3.34) to (3.36) from [41, Lemma 3.5], we have $\forall \tilde{z} \in \partial B_2$:

$$\left| \frac{g_{B_3\setminus B_1}(\tilde{z},z)}{g_{B_3}(\tilde{z},z) P_z[H_{B_1} > T_{B_3}]} - 1 \right| \leq r_1^{-c} \tag{3.3.38}$$

Therefore, assuming the following comparison holds true for any $\tilde{z}, z' \in \partial B_2$:

$$\left| \frac{g_{B_3}(\tilde{z},z)}{g_{B_3}(z',z)} - 1 \right| \leq r_1^{-c} \tag{3.3.39}$$

from equations (3.3.38) and (3.3.39) we obtain that $g_{B_3\setminus B_1}(\tilde{z},z) \geq (1-r_1^{-c}) g_{B_3\setminus B_1}(z',z)$ for any $\tilde{z}, z' \in \partial B_2$, which in turn yields the required result by substituting into equation (3.3.37). Let us then complete the proof by showing equation (3.3.39). By (2.2.15), letting $\underline{B} = B(\tilde{z}, \frac{1}{2} r_1^{\frac{5}{8}}) \subset B_3 \subset \bar{B} = B(\tilde{z}, 2r_1^{\frac{5}{8}})$ we have

$$g_{B_3}(\tilde{z},z) \leq g_{\bar{B}}(\tilde{z},z) \leq \bar{c}_0 |\tilde{z}-z|^{2-d} - c r_1^{\frac{5}{8}(2-d)} + c r_1^{\frac{1}{2}(1-d)} \tag{3.3.40}$$

$$g_{B_3}(\tilde{z},z) \geq g_{\underline{B}}(\tilde{z},z) \geq \bar{c}_0 |\tilde{z}-z|^{2-d} - c r_1^{\frac{5}{8}(2-d)} - c r_1^{\frac{1}{2}(1-d)} \tag{3.3.41}$$

such that, after some rearrangement:

$$\frac{g_{B_3}(\tilde{z},z)}{g_{B_3}(z',z)} - 1 \leq c \left\{ \left(\frac{|z'-z|}{|\tilde{z}-z|} \right)^{d-2} - 1 \right\} \leq c \left\{ \left(\frac{r_1^{\frac{1}{2}} + r_1^{\frac{3}{8}}}{r_1^{\frac{1}{2}} - r_1^{\frac{3}{8}}} \right)^{d-2} - 1 \right\} \leq r_1^{-c}$$

Where we used that the maximum distance between $\tilde{z} \in \partial B_2$ is $r_1^{\frac{1}{2}} + r_1^{\frac{3}{8}}$, while the minimum is $r_1^{\frac{1}{2}} - r_1^{\frac{3}{8}}$. A lower bound can be obtained on the above quantity in a similar fashion. □

In the remainder of this section, we concern ourselves with showing the estimates for capacity and equilibrium, which will be the content of Lemma 3.3.8. First, we introduce a lemma which provides an estimate on the asymptotic expected time spent inside a radius r ball by a simple random walk with exponential holding times which is started on the boundary of said ball.

Lemma 3.3.6. *[41, Lemma 1.1]*

$$\alpha(r) \coloneqq \sup_{y \in \partial B(0,r)} \left| \frac{E_y\left[\int_0^\infty \mathbb{1}_{B(0,r)}(X_s) ds\right]}{c_1 r^2} - 1 \right| \to 0 \text{ as } r \to \infty \tag{3.3.42}$$

Next, to pave the way towards the capacity estimates, we present some notation governing the inhomogeneous density felt by the tilted RW.

$$\begin{aligned} \tilde{u}_f^z &= \begin{cases} u \text{ if } f \in \mathcal{F} \text{ and } z \in \Gamma_{int} \\ v \text{ if } f = f_1 \text{ and } z \in \Gamma_{ext} \end{cases} \\ u_f^z = f^2(z)\tilde{u}_f^z &= \begin{cases} u_* - \epsilon \text{ if } f \in \mathcal{F} \text{ and } z \in \Gamma_{int} \\ u_* + \epsilon \text{ if } f = f_1 \text{ and } z \in \Gamma_{ext} \end{cases} \end{aligned} \tag{3.3.43}$$

Lemma 3.3.7. *Let $x \in \mathbb{Z}^d \setminus \{0\}$, $f \in \mathcal{F}$, $y \in \Gamma_f$. Recalling the tilted path measure as defined via the Radon-Nikodym derivative in line (3.2.3), and abbreviating notation $B_2 = B_2^y$, we have:*

$$\tilde{u}_f^y \tilde{E}^f_{\tilde{e}_{B_2}}\left[\int_0^\infty \mathbb{1}_{B_2}(X_s) ds\right] = u_f^y E_{e_{B_2}}\left[\int_0^\infty \mathbb{1}_{B_2}(X_s) ds\right] \tag{3.3.44}$$

Proof. The proof is almost identical to that of [41, Lemma 3.2], paying extra attention to the specific values taken by $f^2(x)$ in our case. □

Finally, with the aid of Lemma 3.3.2 and Lemma 3.3.7, we are ready to show the capacity and equilibrium measure estimate.

Lemma 3.3.8. *For all $\epsilon_0 \in (0,1)$, $f \in \mathcal{F}$, $x \in \mathbb{Z}^d$, $y \in \Gamma_f(x)$ and $|x| \geq C(\epsilon_0, \delta)$,*

$$(1-\epsilon_0) u_f^z \cdot cap(B_2^z) \leq \tilde{u}_f^z \cdot \tilde{cap}_f(B_2^z) \leq (1+\epsilon_0) u_f^z \cdot cap(B_2^z), \text{ and} \tag{3.3.45}$$

$$(1-\epsilon_0) u_f^z \cdot e_{B_1^z}(y) \leq \tilde{u}_f^z \cdot \tilde{e}_{B_1^z}(y) \leq (1+\epsilon_0) u_f^z \cdot e_{B_1^z}(y), \ y \in \mathbb{Z}^d \tag{3.3.46}$$

Proof. The lower bound in line (3.3.45) arises immediately by taking the same steps as in [41, Proposition 3.1], but replacing their lemmas 3.2 and 3.3 with our lemmas 3.3.7 and 3.3.2 respectively. On the other hand, to prove the upper bound of (3.3.45) some adaptations are necessary which we describe here in detail. The strategy of the proof is to use equation (3.3.44) alongside the now-familiar fact that the tilted random walk has the same law up to the stopping time T_{B_3} as the untilted walk. Equation (3.3.44) is applied by showing its left hand side is approximately an upper bound for $\tilde{u}_f^z$ times the tilted capacity, while the right hand side is approximately a lower bound for u_f^z times the untilted capacity, which henceforth permits us to make the required comparison for equation (3.3.45). We proceed by showing the former:

$$\begin{aligned}
\tilde{E}_{\tilde{e}_{B_2}}\Big[\int_0^\infty \mathbb{1}_{B_2}(X_s)ds\Big] &\geq \tilde{E}_{\tilde{e}_{B_2}}\Big[\int_0^{T_{B_3}} \mathbb{1}_{B_2}(X_s)ds\Big] \\
&\overset{(3.2.6)}{\geq} \tilde{\mathrm{cap}}_f(B_2) \min_{z\in\partial B_2} \tilde{E}_z\Big[\int_0^{T_{B_3}} \mathbb{1}_{B_2}(X_s)ds\Big] \\
&= \tilde{\mathrm{cap}}_f(B_2) \min_{z\in\partial B_2} E_z\Big[\int_0^{T_{B_3}} \mathbb{1}_{B_2}(X_s)ds\Big] \\
&= \tilde{\mathrm{cap}}_f(B_2) \min_{z\in\partial B_2} \Big\{E_z\Big[\int_0^\infty \mathbb{1}_{B_2}(X_s)ds\Big] - E_z\Big[\int_{T_{B_3}}^\infty \mathbb{1}_{B_2}(X_s)ds\Big]\Big\}
\end{aligned} \tag{3.3.47}$$

In equation (3.3.47), let us first focus our attention on the second expectation. Let $\in \partial B_2$, and apply the strong Markov property at the exit time from B_3 and the consecutive re-entrance time to B_2:

$$\begin{aligned}
E_z\Big[\int_{T_{B_3}}^\infty \mathbb{1}_{B_2}(X_s)ds\Big] &= E_z\Big[E\Big[\int_{T_{B_3}}^\infty \mathbb{1}_{B_2}(X_s)ds\Big|\mathcal{F}_{T_{B_3}}\Big]\Big] = E_z\Big[E\Big[\int_0^\infty \mathbb{1}_{B_2}(X_s\circ\theta_{T_{B_3}})ds\Big|\mathcal{F}_{T_{B_3}}\Big]\Big] \\
&= E_z\Big[E_{X_{T_{B_3}}}\Big[\int_0^\infty \mathbb{1}_{B_2}(X_s)ds\Big]\Big] = \sum_{z'\in\partial B_3} E_{z'}\Big[\int_{H_{B_2}}^\infty \mathbb{1}_{B_2}(X_s)ds\Big] P_z[X_{T_{B_3}} = z'] \\
&\leq \max_{z'\in\partial B_3} E_{z'}\Big[\int_{H_{B_2}}^\infty \mathbb{1}_{B_2}(X_s)ds\Big] = \max_{z'\in\partial B_3} E_{z'}\Big[E_{X_{H_{B_2}}}\Big[\int_0^\infty \mathbb{1}_{B_2}(X_s)ds\Big]\Big] \\
&\leq \max_{z'\in\partial B_3} P_{z'}[H_{B_2}<\infty] \max_{\alpha\in\partial B_2} E_\alpha\Big[\int_0^\infty \mathbb{1}_{B_2}(X_s)ds\Big]
\end{aligned}$$

By Lemma 3.3.1, we see that $\max_{z'\in\partial B_3} P_{z'}[H_{B_2}<\infty] \leq \beta_0(x) \to 0$ as $|x|\to\infty$. Additionally, by Lemma 3.3.6 we get that $c_1r_1(1+\alpha(r_1^{\frac{1}{2}})) \geq E_y\Big[\int_0^\infty \mathbb{1}_{B_2}(X_s)ds\Big] \geq c_1r_1(1-\alpha(r_1^{\frac{1}{2}}))$, so we have:

$$(3.3.47) \geq \tilde{\mathrm{cap}}_f(B_2)c_1r_1\Big\{\Big(1-\alpha(r_1^{\frac{1}{2}})\Big) - \beta_0(x)\Big(1+\alpha(r_1^{\frac{1}{2}})\Big)\Big\} = \tilde{\mathrm{cap}}_f(B_2)c_1r_1\Big\{1-\beta_2(x)\Big\}$$

Where we have defined $\beta_2(x) \coloneqq \alpha(r_1^{\frac{1}{2}}) + \beta_0(x) + \alpha(r_1^{\frac{1}{2}})\beta_0(x)$ which also tends to 0. On the other hand, also by Lemma (3.3.6), the following upper bound is immediate:

$$E_{e_{B_2}}\Big[\int_0^\infty \mathbb{1}_{B_2}(X_s)ds\Big] \leq \mathrm{cap}(B_2)c_1 r_1\Big\{1+\alpha(r_1^{\frac{1}{2}})\Big\} \tag{3.3.48}$$

In order to complete the proof we apply Lemma 3.3.7:

$$\begin{aligned}\tilde{u}_f^z \tilde{\mathrm{cap}}_f(B_2)c_1 r_1\Big\{1-\beta_2(x)\Big\} &\leq \tilde{u}_f^z \tilde{E}_{\tilde{e}_{B_2}}\Big[\int_0^\infty \mathbb{1}_{B_2}(X_s)ds\Big] \\ &= u_f^z E_{e_{B_2}}\Big[\int_0^\infty \mathbb{1}_{B_2}(X_s)ds\Big] \leq u_f^z \mathrm{cap}(B_2)c_1 r_1\Big\{1+\alpha(r_1^{\frac{1}{2}})\Big\}\end{aligned}$$

Since $\lim_{|x|\to\infty}\alpha(r_1^{\frac{1}{2}}) = 0$ and $\lim_{x\to\infty}\beta_2(x) = 0$, we have for $|x| \geq C(\epsilon_0, \delta)$:

$$\tilde{u}_f^z \tilde{\mathrm{cap}}_f(B_2) \leq (u_f^z + \epsilon_0)\mathrm{cap}(B_2) \tag{3.3.49}$$

Finally, we endeavour to show the equilibrium estimate portion of the lemma, line (3.3.46) - this is obtained as a consequence of (3.3.33) and (3.3.45), as we now explain. For the desired upper bound on $u_f^z \cdot \tilde{e}_{B_1}(y)$, applying the sweeping identity to the sets $B_1 \subset B_2$ yields that for all $|x| \geq C'(\epsilon_0, \delta)$

$$\begin{aligned}\tilde{e}_{B_1}(y) = \sum_{y'\in\partial B_2} \tilde{e}_{B_2}(y')\tilde{h}_{B_1}(y', y) &\overset{(3.3.33)}{\leq} (1+\frac{\epsilon_0}{2})\tilde{\mathrm{cap}}_f(B_2) \min_{\tilde{y}'\in\partial B_2} h_{B_1,B_3}(\tilde{y}', y) \\ &\overset{(3.3.45)}{\leq} (1+\epsilon_0)\frac{u_f^z}{\tilde{u}_f^z}\mathrm{cap}(B_2^z) \min_{\tilde{y}'\in\partial B_2} h_{B_1,B_3}(\tilde{y}', y) \\ &\leq (1+\epsilon_0)\frac{u_f^z}{\tilde{u}_f^z}\sum_{\tilde{y}'\in\partial B_2} e_{B_2}(y')h_{B_1}(\tilde{y}', y) = (1+\epsilon_0)\frac{u_f^z}{\tilde{u}_f^z}e_{B_1}(y)\end{aligned} \tag{3.3.50}$$

where we apply the sweeping identity once more in the final step. The lower bound can obtained similarly. □

3.4 Coupling Tilted Interlacements

Up to this point, our construction of the tilted interlacements has largely been motivated by intuition - in particular, recall that in the subcritical case our aim was to decrease the density of the interlacements within the strip T^1 to make connections between two distant $|x|^\delta$-radius boxes more likely, meanwhile in the supercritical case we do the opposite inside the oblique annulus $T^6 \setminus T^3$. Henceforth, we devote the present section to justifying these intuitive claims via the following coupling, which is the culmination

of the tilted estimates from §3.3 in tandem with the so-called soft local time method presented in §2.6.

Theorem 3.4.1. *For all $\delta \in (0,1)$, $\epsilon \in (0, \frac{u_*}{10} \wedge 1)$ and $\eta \in [0, c_3)$, the following holds.*

(i) If $u < u_*$ and $z \in \Gamma_{ext}$, then with $f = f_1^{v,u;\epsilon}$, $v = u(1-\eta\sqrt{\delta})$ and abbreviating $B = B_1^z$, there exists a coupling $\mathbb{Q}_B$ of $(\ell_x^{u_*+\frac{1}{2}\epsilon})_{x\in B}$, $(\ell_x^{u_*+\frac{3}{2}\epsilon})_{x\in B}$ (each under $\mathbb{P}$) and $(\tilde{\ell}_x^v)_{x\in B}$ (having the same law as $(\ell_x^v)_{x\in B}$ under $\tilde{\mathbb{P}}_f$), such that for $\tilde{c} = \tilde{c}(u,\delta,\epsilon,\eta)$,

$$\mathbb{Q}_B[\ell_x^{u_*+\frac{1}{2}\epsilon} \le \tilde{\ell}_x^v \le \ell_x^{u_*+\frac{3}{2}\epsilon}, x \in B] \ge 1 - e^{-\tilde{c}r_0^{\tilde{c}}}. \tag{3.4.1}$$

(ii) If $u < u_*$ and $z \in \Gamma_{int}$, with $s_i(t) = t(1-\gamma_i)^{\sigma_i}$ for $\gamma_i \in [0,1)$, $\sigma_i \in \{\pm 1\}$, $i = 1,2$, there exists a coupling $\mathbb{Q}_B$ of $(\ell_x^{s_i(u_*-\frac{2k-1}{2}\epsilon)})_{x\in B,\ k,i\in\{1,2\}}$ and $(\tilde{\ell}_x^{s_i(u)})_{x\in B,\ i=1,2}$, such that

$$\mathbb{Q}_B[\ell_x^{s_i(u_*-\frac{3}{2})} \le \tilde{\ell}_x^{s_i(u)} \le \ell_x^{s_i(u_*-\frac{1}{2}\epsilon)}, x \in B, i = 1,2] \ge 1 - e^{-\tilde{c}r_0^{\tilde{c}}}, \tag{3.4.2}$$

for $\tilde{c} = \tilde{c}(u,\delta,\epsilon,\eta,\gamma_i,\sigma_i)$. If $u > u_*$, then for all $z \in \Gamma_{int}$ there exists a coupling $\mathbb{Q}_B$ with the same properties, but with $f = f_0^{u;\epsilon}$ now underlying the marginal law of $(\tilde{\ell}_x^{s_i(u)})_{x\in B, i=1,2}$.

Before delving into the necessary steps required to prove Theorem 3.4.1, we show that Proposition 3.2.5 is indeed a consequence of the above coupling.

Proof of Proposition 3.2.5. We will freely (and tacitly) assume that various statements hold for $|x| \ge C(\delta)$, which is no loss of generality in view of (3.2.26) since the latter concerns the limit $|x| \to \infty$ only. Let $u < u_*$ and $f = f_1^{v,u;\epsilon}$ with $v = u(1-\eta\sqrt{\delta})$ and $\eta \in [0, c_3)$ as for the conclusions of Theorem 3.4.1 to hold. The following conclusions will always hold uniformly in η as above, $\delta \in (0,1)$ and $\epsilon \in (0, u_*(1 \wedge \frac{\eta\sqrt{\delta}}{10}))$, as postulated in the statement of Proposition 3.2.5. Using Theorem 3.4.1 we argue separately that

$$\lim_{|x|\to\infty} \tilde{\mathbb{P}}_f[T^4 \overset{\mathcal{V}^v}{\longleftrightarrow} \partial T^5] = 0, \tag{3.4.3}$$

$$\lim_{|x|\to\infty} \tilde{\mathbb{P}}_f[G^u, B(0,r_1) \overset{\mathcal{V}^u \cap T^1}{\longleftrightarrow} B(x,r_1)] = 1. \tag{3.4.4}$$

Recalling $A^u(x,\delta)$ from 3.1.2, the claim (3.2.26) for $u > u_*$ immediately follows from 3.4.3 and (3.4.4). To obtain (3.4.3) applying the relevant coupling from Theorem 3.4.1,*(i)*, one finds that for all $z \in \Gamma_{\text{ext}}$, with $B = B_1^z$,

$$\tilde{\mathbb{P}}_f[z \overset{\mathcal{V}^v}{\longleftrightarrow} \partial B] = \mathbb{Q}_B[z \overset{\{\tilde{\ell}^v_\cdot = 0\}}{\longleftrightarrow} \partial B] \overset{(3.4.1)}{\leq} \mathbb{Q}_B[z \overset{\{\tilde{\ell}^{u_*+\frac{1}{2}\epsilon}_\cdot = 0\}}{\longleftrightarrow} \partial B] + e^{-c|x|^{c'(\delta)}}$$
$$= \mathbb{P}\Big[z \overset{\mathcal{V}^{u_*+\frac{1}{2}\epsilon}}{\longleftrightarrow} \partial B\Big] + e^{-cr_0^{\tilde{c}}} \leq Ce^{-c|x|^{c'(\delta)}},$$

where the last step follows using [22, Thm. 1.2,(i)] and recalling that $r_0 \geq c|x|^\delta$. Applying a union bound over $z \in \partial T^4 (= \Gamma_{ext})$ and using the previous bound on $\tilde{\mathbb{P}}_f[z \overset{\mathcal{V}^v}{\longleftrightarrow} \partial B]$ yields (3.4.3). To deal with (3.4.4), we use part *(ii)* of Theorem 3.4.1. We consider the two events appearing in (3.4.4) separately, and start with the connection event appearing there. Throughout the rest of the proof, constants may implicitly depend on all of $u, \delta, \epsilon, \eta$. To this end, we introduce the following events inspired by the strongly percolative regime (see around (1.2.8))

$$E_{z,r}(\mathcal{V}) \coloneqq \Big\{B(z,r) \cap \mathcal{V} \text{ has a connected component of diameter } \geq \frac{r}{5}\Big\}$$
$$U_{z,r}(\mathcal{V}, \mathcal{V}') \coloneqq \begin{Bmatrix}\text{Any two connected components of } B(z,r) \cap \mathcal{V} \text{ with} \\ \text{diameters } \geq \frac{r}{10} \text{ are connected in } B(z,2r) \cap \mathcal{V}'\end{Bmatrix} \tag{3.4.5}$$

which, by [22, Thm 1.2], for any $0 < v \leq u < u_*$, satisfy

$$\mathbb{P}[E_{z,r}(\mathcal{V}^u)] \geq 1 - Ce^{-r^c}, \tag{3.4.6}$$
$$\mathbb{P}[U_{z,r}(\mathcal{V}^u, \mathcal{V}^v)] \geq 1 - Ce^{-r^c}, \tag{3.4.7}$$

Then, letting $r = \frac{1}{2}r_1^{\frac{3}{8}}$ so that $B(z, 2r_1) = B_1^z$ as in (3.3.1), we see that chaining the above events yields the following inclusion

$$\bigcap_{z \in \Gamma_{int}} E_{z,r}(\mathcal{V}^u) \cap U_{z,2r}(\mathcal{V}^u, \mathcal{V}^u) \subset \{B(0,r_1) \overset{\mathcal{V}^u \cap T^1}{\longleftrightarrow} B(x,r_1)\} \tag{3.4.8}$$

Hence, by applying a union bound together with Theorem 3.4.1 *(ii)*, keeping track of the monotonicities of $E_{z,r}$ and $U_{z,r}$ with respect to their respective parameters, we have

$$\begin{aligned}\tilde{\mathbb{P}}_f[B(0,r_1) \overset{\mathcal{V}^u \cap T^1}{\longleftrightarrow} B(x,r_1)] &\leq |\Gamma_{int}|\Big\{\tilde{\mathbb{P}}_f[E_{0,r}(\mathcal{V}^u)^c] + \tilde{\mathbb{P}}_f[U_{0,2r}(\mathcal{V}^u, \mathcal{V}^u)^c]\Big\} \\ &\leq |\Gamma_{int}|\Big\{\tilde{\mathbb{P}}_f[E_{0,r}(\mathcal{V}^{u_*-\frac{\epsilon}{2}})^c] + \tilde{\mathbb{P}}_f[U_{0,2r}(\mathcal{V}^{u_*-\frac{3\epsilon}{2}}, \mathcal{V}^{u_*-\frac{\epsilon}{2}})^c] + ce^{-\tilde{c}r_0^{\tilde{c}}}\Big\} \\ &\leq C|x|e^{-c|x|^{c\delta}} \to 0\end{aligned} \tag{3.4.9}$$

where we used that, by definition, $\mathcal{V}^u = \{\ell^u_\cdot = 0\}$. To accommodate the presence of G^u in (3.4.4), recalling its form specified above Theorem 3.1.1 and applying a union

bound, one sees that it is enough to argue that

$$\lim_{|x|\to\infty} |I| \cdot \sup_i \widetilde{\mathbb{P}}_f[(G_{B_i})^c] = 0. \tag{3.4.10}$$

This is obtained by combining (3.1.3) and Theorem 3.4.1,*(ii)*. Indeed, recall to this effect (cf. above Theorem 3.1.1) that B_i is a box of radius $10r_1$ with center $z \in \Gamma_{\text{int}}$. By choice of r_1 in (3.1.1) and in view of (3.3.1), this means that $B_i \subset B_1^z = B$. In particular, the event G_{B_i} is thus measurable relative to $(\tilde{\ell}_x^{s_i(u)})_{x\in B, i=1,2}$, where $s_i(u) = u(1-\gamma_i)$ for suitable choice of $\gamma_i \in \{\frac{k}{4}\eta\sqrt{\delta} : k = 0,\dots,4\}$. Assume for concreteness that $s_1(u) > s_2(u)$, so that G_{B_i} is increasing in $(\tilde{\ell}_x^{s_1(u)})_{x\in B}$ and decreasing in $(\tilde{\ell}_x^{s_2(u)})_{x\in B}$. With $(v,w) = (s_1(u), s_2(u))$, $(v',w') = (s_1(u_* - \frac{3}{2}\varepsilon), s_2(u_* - \frac{1}{2}\varepsilon))$, it then follows by application of (3.4.2), using the (separate) monotonicity of $G_{B_i} = G_{B_i}^{v,w}$, that

$$\widetilde{\mathbb{P}}_f[G_{B_i}^{v,w}] \geq \mathbb{P}[G_{B_i}^{v',w'}] - e^{-\tilde{c}r_0^{\tilde{c}}}.$$

The claim (3.4.10) now follows using (3.1.3), noting in particular that the assumption $\varepsilon < u_* \frac{\eta\sqrt{\delta}}{10}$ guarantees that $v' > w'$.

To deduce (3.2.26) for $u < u_*$, one shows (3.4.4) now with $f = f_0^{u;\varepsilon}$ in exactly the same manner as above but resorting to the second part of item *(ii)* in Theorem 3.4.1 instead. □

From now on, we endeavour to find a probability measure $\mathbb{Q}_B$ which exhibits the couplings asserted by Theorem 3.4.1. In view of the soft local times method presented in §2.6, we hope to represent the RI and tilted RI as Markov processes with tangible initial and transition probabilities. Such a representation will enable the iterative application of Proposition 2.6.1 which, by proxy of a single PPP, outlines the procedure to generate both the tilted and natural RI, thereby coupling them. In order to formulate such a Markov representation, the RI are effectively chopped up into excursions of a predetermined box (corresponding to B in the coupling $\mathbb{Q}_B$) and then, via the selection of an appropriate distribution, the law of the local times of the resulting excursions is equated to that of the RI/tilted RI.

To formalize the preceding discussion, we consider the finite sets $A \subset U \subset\subset \mathbb{Z}^d$, and iteratively introduce a sequence of stopping times on the filtered space of nearest-neighbour paths $(W^+, \mathcal{W}^+, \{\mathcal{W}_n^+\}_{n\geq 0})$ where $\{\mathcal{W}_n^+\}_{n\geq 0}$ is the canonical filtration. Set

$T_0^{A,U} = R_0^{A,U} = 0$, and define for $k \geq 1$

$$R_k^{A,U} = H_A \circ \theta_{T_{k-1}^{A,U}} + T_{k-1}^{A,U}, \quad T_k^{A,U} = T_U \circ \theta_{R_k^{A,U}} + R_k^{A,U} \tag{3.4.11}$$

where we refer to the $R_i^{A,U}$ and $T_i^{A,U}$ as successive return and departure times to A and from U respectively. Moreover, we shall write:

$$\tau_{A,U}(w) \coloneqq \sup\{k \geq 0 : R_k^{A,U}(w) < \infty\} \tag{3.4.12}$$

to be the number of excursions from A to U by the path $w \in W^+$. Additionally, recalling the boxes $B_1 \subset B_2 \subset\subset \mathbb{Z}^d$ from (3.3.1), we shall shorten notation by writing $R_i \coloneqq R_i^{B_1,B_2}$, $T_i \coloneqq T_i^{B_1,B_2}$ and $\tau \coloneqq \tau_{B_1,B_2}$. Occasionally, we will write $\tilde{\tau}$ to refer to τ when viewed under the tilted law, so that it makes sense to speak of the distributions of τ and $\tilde{\tau}$ without specifying the probability space each time they are mentioned. This will become useful when τ and $\tilde{\tau}$ will be considered under the same probability measure, so it will be necessary to distinguish between the two without reference to the probability space.

Next, consider a PPP $\hat{\omega} = \sum_{i \geq 0} \delta_{(w_i^*, u_i)} \in \Omega(W^* \times \mathbb{R}_{\geq 0})$ and recall its projection $\mu_{B_1}^+(\hat{\omega})$ from (2.4.11). By Proposition 3.2.3, as long as we restrict ourselves to the probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}_f$ for $f \in \mathcal{F}$ on the space $\Omega(W^* \times \mathbb{R}_{\geq 0})$, we can write the following equality $\mathbb{P}/\tilde{\mathbb{P}}_f$-almost surely:

$$B_1 \cap \mathcal{I}^u(\hat{\omega}) = B_1 \cap \mathcal{I}(\mu_{B_1}^+(\hat{\omega})) = B_1 \cap \bigcup_{u_i \leq u} \text{Range}(s_{B_1}(w_i^*)^+) \tag{3.4.13}$$

where we recall the notation from equations (2.4.13), (2.4.15). Then, by Corollary 3.2.2.1, we can simulate the RHS of equation (3.4.13) under the measure $\mathbb{P}$ by the algorithm below

(i) Recalling that under $\mathbb{P}$, $\mu_{B_1,u}^+(W_{B_1}^*) \sim \text{Poisson}(u\text{cap}(B_1))$, generate the random variable $\Theta^u \sim \text{Poisson}(u\text{cap}(B_1))$, representing the number of paths from $\mu_{B_1,u}^+$ which will intersect B_1.

(ii) For $i = 1, \ldots, \Theta^u$, let $x_i \sim \bar{e}_{B_1}$, and generate trajectories w_i with laws $P_{x_i}(\cdot)$.

An identical approach with all natural quantities replaced by their tilted counterparts (such as for capacity and equilibrium measure) is used to simulate $\widetilde{\Theta}^u$, $\tilde{x}_i$ and $\tilde{w}_i$. In this way, the steps above yield the following equalities, on which the soft local time

method will at last be tenable.

$$B_1 \cap \mathcal{I}^u \overset{\text{law}}{=} B_1 \cap \bigcup_{j=1}^{\Theta^u} \bigcup_{k=1}^{\tau(w_j)} X_{[R_k,T_k]}(w_j), \tag{3.4.14}$$

$$B_1 \cap \tilde{\mathcal{I}}_f^u \overset{\text{law}}{=} B_1 \cap \bigcup_{j=1}^{\tilde{\Theta}_f^u} \bigcup_{k=1}^{\tau(\tilde{w}_j)} X_{[R_k,T_k]}(\tilde{w}_j). \tag{3.4.15}$$

where $X_{[u,v]}(w)$ is the trajectory given by $X_{[u,v]}(w)(s) = X_{(u+s)\wedge v}(w)$. We note for clarity that equations (3.4.14) and (3.4.15) hold under any probability measure such that $\mathcal{I}^u$ and $\tilde{\mathcal{I}}_f^u$ are distributed as $\mathcal{I}^u$ under $\mathbb{P}$ and $\tilde{\mathbb{P}}_{N,f}$ respectively, meanwhile the random variables present on the RHS are distributed in the manner described above. With this remark, our problem becomes analogous to simulating excursions of both of the Markov processes $\{X_t(w_j)\}_{t\geq 0}$ and $\{X_t(\tilde{w}_j)\}_{t\geq 0}$ for $j \geq 1$ from a single Poisson Point Process with a suitable intensity measure. The resulting sequences of excursions will from now on be denoted as:

$$Z_k^j \coloneqq X_{[R_k,T_k]}(w_j), \quad \tilde{Z}_k^j \coloneqq X_{[R_k,T_k]}(\tilde{w}_j) \tag{3.4.16}$$

for $k \geq 1$. Denote the space comprising all such excursions from B_1 to B_2 by

$$W_{B_1,B_2}^+ \coloneqq \left\{ \begin{matrix} w(\cdot \wedge T_{B_2}) : w \text{ is a right-continuous, finite, nearest} \\ \text{neighbour } \mathbb{Z}^d\text{-valued path with } w(0) \in \partial B_1 \end{matrix} \right\} \tag{3.4.17}$$

Notice the crucial fact that the sequences $\{Z_k^j\}_{k\geq 1}$ and $\{\tilde{Z}_k^j\}_{k\geq 1}$ inherit the Markov property from the underlying processes $\{X_t(w_j)\}_{t\geq 0}$ and $\{X_t(\tilde{w}_j)\}_{t\geq 0}$. In fact, we may view the collection of independent Markov processes $\{Z_k^j\}_{k\geq 1}$ as a single Markov process under a specially-chosen probability measure Q. We let Q be the probability measure on the augmented space $\mathbb{N}^{\mathbb{R}} \times (W^+)^\infty$ on which $\{\Theta^u\}_{u\geq 0}$ is a Poisson Process with rate $\text{cap}(B_1)$, and which prescribes an infinite sequence of independent $w_i \in W^+$ for $i \geq 0$ with laws $P_{\tilde{e}_{B_1}}$. Then, define the process $\{Z_j^{B_1,B_2}\}_{j\geq 1} \subset W_{B_1,B_2}^+$ as follows:

$$\begin{aligned} Z_1^{B_1,B_2} &\coloneqq Z_1^1 \\ Z_2^{B_1,B_2} &\coloneqq \begin{cases} Z_2^1 & \text{if } R_2^{B_1,B_2}(w_1) < \infty \\ Z_1^2 & \text{if } R_2^{B_1,B_2}(w_1) = \infty \end{cases} \end{aligned} \tag{3.4.18}$$

and, if $Z_{n-1}^{B_1,B_2} = Z_j^k$ has already been chosen for some $j, k \in \mathbb{N}$, then:

$$Z_n^{B_1,B_2} := \begin{cases} Z_{j+1}^k & \text{if } R_{j+1}^{B_1,B_2}(w_k) < \infty \\ Z_0^{k+1} & \text{if } R_{j+1}^{B_1,B_2}(w_k) = \infty \end{cases} \tag{3.4.19}$$

and, for the purposes of this chapter, we use the abbreviation $Z_j = Z_j^{B_1,B_2}$. The key properties of this new sequence $\{Z_n\}_{n\geq 0}$ are that it is still a Markov process and that it is equivalent in law to the collection $\{Z_k^j\}_{j,k\geq 0}$ after relabelling. On this space, let $\tau_i := \tau(w_i)$. We use an identical method to deal with $\{\tilde{Z}_k^j\}_{k\geq 1}$: define $\tilde{Q}_f$ for $f \in \mathcal{F}$ on the same augmented space $\mathbb{R}^{\mathbb{N}} \times (W^+)^\infty$ such that $\{\tilde{\Theta}_f^u\}_{u\geq 0}$ is a Poisson process with rate $\tilde{\text{cap}}_f(B_1)$ and we have an infinite sequence of independent $\tilde{w}_i$, $i \geq 0$ now with laws $\tilde{P}^f_{\tilde{e}_{B_1}}$. Completing the steps with these tilted variants, we obtain the relabelled version of $\{\tilde{Z}_k^j\}_{j,k\geq 0}$, namely the sequence $\{\tilde{Z}_n\}_{n\geq 0}$. Now, we let $\tilde{\tau}_i := \tau(\tilde{w}_j)$, where dependence on f is implicit and made clear due to context. Although this extra work of relabelling seems superfluous (since by the discussion presented in [47, Section 4] we can synchronously construct a collection of Markov chains), it allows us to avoid introducing a cemetery state to represent the case where a SRW does not return to B_1 and hence does not produce any more excursions in $W^+_{B_1,B_2}$. Finally, let us derive the transition probabilities of Z_n for $z, z' \in W^+_{B_1,B_2}$ by applying the strong Markov property:

$$\begin{aligned} Q[Z_n = z' | Z_{n-1} = z] &\overset{(3.4.19)}{=} Q[Z_n = z', R_j < \infty | Z_{n-1} = z] \\ &\qquad + Q[Z_n = z', R_j = \infty | Z_{n-1} = z] \\ &= P_{X_{T_{B_2}}(z)}[X_{[H_{B_1}, T_{B_2}]} = z', H_{B_1} < \infty] \\ &\qquad + P_{\bar{e}_{B_1}}[X_{[0,T_{B_2}]} = z'] P_{X_{T_{B_2}}(z)}[H_{B_1} = \infty] \end{aligned} \tag{3.4.20}$$

We now turn to finding a suitable PPP from which we can sample both Z_n as well as $\tilde{Z}_n$. Define the PPP $\eta := \sum_{n\geq 0} \delta_{(z_n,u_n)}$ governed by the probability measure $\mathbb{Q}_{B_1}$ on the space of locally finite point processes on the underlying space $W^+_{B_1,B_2} \times \mathbb{R}_+$ such that its intensity measure is $\mu_{B_1} \otimes d\lambda$, where:

$$\mu_{B_1}(\cdot) := \sum_{x \in \partial B_1} P_x[X_{[0,T_{B_2}]} \in \cdot] \tag{3.4.21}$$

and $d\lambda$ is the Lebesgue measure as before. Since the change of measure M_∞ which defines our tilted probability space on paths $\tilde{P}_x^f$ is equal to 1 on paths which are wholly

contained in the strip T^1 (or on paths wholly contained in $T^6 \setminus T^3$ in the supercritical case, which is important when B_1 and B_2 are centred at $x \in \Gamma_{f_1}$), we have the useful property that $P_x[X_{[0,T_{B_2}]} \in \cdot\,] = \tilde{P}^f_x[X_{[0,T_{B_2}]} \in \cdot\,]$, so we can write

$$\tilde{\mu}_{B_1}(\cdot) \coloneqq \sum_{x \in \partial B_1} \tilde{P}^f_x[X_{[0,T_{B_2}]} \in \cdot] = \mu_{B_1}(\cdot) \tag{3.4.22}$$

By recalling the form of our transition probabilities in (3.4.20), we then have the following two Radon Nikodym derivatives for $x = (x_0, \dots, x_m)$, $y = (y_0, \dots, y_n) \in W^+_{B_1,B_2}$:

$$\frac{dQ[Z_k \in \cdot\,|Z_{k-1} = x]}{d\mu_{B_1}}(y) = h_{B_1}(x_m, y_0) + \bar{e}_{B_1}(y_0) P_{x_m}(H_{B_1} = \infty) \tag{3.4.23}$$

$$\frac{d\tilde{Q}_f[\tilde{Z}_k \in \cdot\,|\tilde{Z}_{k-1} = x]}{d\tilde{\mu}_{B_1}}(y) = \tilde{h}^f_{B_1}(x_m, y_0) + \bar{\tilde{e}}^f_{B_1}(y_0) \tilde{P}^f_{x_m}(H_{B_1} = \infty) \tag{3.4.24}$$

So, to sample the random interlacements and tilted random interlacements from the same PPP η, we have the following respective transition functions:

$$\pi(x,y) = \frac{dQ[Z_k \in \cdot\,|Z_{k-1} = x]}{d\mu_{B_1}}(y), \qquad \tilde{\pi}^f(x,y) = \frac{d\tilde{Q}_f[\tilde{Z}_k \in \cdot\,|\tilde{Z}_{k-1} = x]}{d\tilde{\mu}_{B_1}}(y) \tag{3.4.25}$$

The final requirement is to determine the initial distributions as set out by equation (2.6.2). These are determined as:

$$\pi_0(x) = \bar{e}_{B_1}(x_0), \qquad \tilde{\pi}^f_0(x) = \bar{\tilde{e}}^f_{B_1}(x_0) \tag{3.4.26}$$

More details for obtaining line (3.4.26) are included in [47, Proposition 5.1], where the case for the tilted interlacements is symmetrical by replacing P with $\tilde{P}^f$. For convenience, from now on we will tacitly associate the excursions in $W^+_{B_1,B_2}$ with their starting and ending values so that we consider the densities $\pi(\cdot,\cdot) : W^+_{B_1,B_2} \times W^+_{B_1,B_2} \to \mathbb{R}^+$ instead as $\pi(\cdot,\cdot) : \partial^{\text{out}}B_2 \times \partial B_1 \to \mathbb{R}^+$ (recall that the soft local times in our construction only depend on the entrance and exit points in ∂B_1 and $\partial^{\text{out}}B_2$ respectively). With the representation in equation (3.4.14), we therefore have all the tools necessary to construct both the standard and tilted interlacements by following the procedure in display (2.6.3). To be precise, using the soft local time method above, we sample sequences of excursions $\{z_i\}_{i\geq 0}, \{\tilde{z}_i\}_{i\geq 0} \subset \{z_\lambda\}_{\lambda \in \Lambda}$ from η

which, under the probability measure of the PPP $\mathbb{Q}_{B_1}$ have the same laws as the $\{Z_i\}_{i\geq 0}, \{\tilde{Z}_i\}_{i\geq 0} \subset W^+_{B_1,B_2}$ generated by the path probability measures Q and $\tilde{Q}_f$.

One final construction which we will find useful is a sequence of independent excursions in $W^+_{B_1,B_2}$ from the same PPP η. To construct this sequence we set $\pi(x,y) = \pi_0(y) = \hat{\pi}(y)$, where we define:

$$\hat{\pi}(y) \coloneqq \bar{e}_{B_1}(y) \tag{3.4.27}$$

We shall label the sequence of excursions obtained via this procedure by $\{\hat{z}_n\}_{n\geq 1}$. With all of the above, we are almost in a position to apply Theorem 2.6.2 in order to couple a deterministic number of untilted excursions with their tilted counterparts - our last ingredient is the following lemma comparing the PPP sampling densities of the independent and interlacement excursions. In particular, the result guarantees that condition (2.6.10) is satisfied for $|x|$ sufficiently large.

Lemma 3.4.2. *For all $f \in \mathcal{F}$, $x \in \mathbb{Z}^d$, $z \in \Gamma_f(x)$, we have the following relation on the initial and transition densities (see* (3.4.25)-(3.4.27)*)*

$$\sup_{\xi,\xi' \in \Xi_{B_1}} \left\{ \left| \frac{\pi_0(\xi)}{\hat{\pi}(\xi)} - 1 \right|, \left| \frac{\pi(\xi,\xi')}{\hat{\pi}(\xi')} - 1 \right| \right\} \leq c r_0^{-c} \tag{3.4.28}$$

Moreover, (3.4.28) *holds when replacing π_0, π with $\tilde{\pi}_0, \tilde{\pi}$ respectively.*

Proof. To deal with the initial distribution comparison, $\frac{\pi_0}{\hat{\pi}}$, $\frac{\tilde{\pi}_0}{\hat{\pi}}$, in inequality (3.3.46) sum over B_1^z, and subsequently divide the same inequality by the result to yield a normalised analogue of (3.3.46). In view of the definitions of $\pi_0(\cdot)$ and $\hat{\pi}(\cdot)$, and remarking that ϵ_0 from Lemma 3.3.8 can be chosen polynomially-small in r_0, we have the required conclusion.

Next, we handle the case $\frac{\pi}{\hat{\pi}}$, $\frac{\tilde{\pi}}{\hat{\pi}}$. Observe that, for all ϵ_0, $\delta \in (0, \frac{1}{6})$ and $|x| \geq C(\epsilon_0, \delta)$, uniformly in $f \in \mathcal{F}$, $z \in \Gamma_f(x)$, $y \in \partial^{\text{out}} B_2^z$ and $y' \in \partial B_1^z$,

$$(1-\epsilon_0)\bar{e}_{B_1}(y') \leq \frac{\tilde{h}^f_{B_1}(y,y')}{\tilde{P}^f_y[H_{B_1} < \infty]} \leq (1+\epsilon_0)\bar{e}_{B_1}(y') \tag{3.4.29}$$

Indeed, the untilted version of (3.4.29) is a classical result (see for example [40, Theorem 2.1.3]), therefore, to obtain (3.4.29) as shown, it is sufficient to compare $\frac{\tilde{h}^f_{B_1}}{\tilde{P}^f}$ to $\frac{h^f_{B_1}}{P^f}$. To this effect, we use Lemma 3.3.4 twice: once directly for the numerator and a second time

on the denominator by first summing (3.3.33) over z'. Hence, recalling the definition of $\tilde{\pi}$ from (3.4.24), (3.4.25), we have for $\xi, \xi' \in W^+_{B_1,B_2}$

$$\begin{aligned}\frac{\tilde{\pi}(\xi,\xi')}{\hat{\pi}(\xi')} &= \frac{\tilde{h}_{B_1}(\xi_{-1},\xi'_0)}{\tilde{\bar{e}}_{B_1}(\xi'_0)} + \tilde{P}_{\xi_{-1}}(H_{B_1}=\infty) \\ &\overset{(3.4.29)}{\leq} (1+\epsilon_0)P_{\xi_{-1}}[H_{B_1}<\infty] + P_{\xi_{-1}}[H_{B_1}=\infty] \\ &\leq 1+\epsilon_0 P_{\xi_{-1}}[H_{B_1}<\infty]\end{aligned} \tag{3.4.30}$$

where ξ_0 and ξ_{-1} denote the starting and ending points of $\xi \in W^+_{B_1,B_2}$ respectively. By deriving a similar lower bound, overall we have

$$\left|\frac{\tilde{\pi}(\xi,\xi')}{\hat{\pi}(\xi')} - 1\right| \leq \epsilon_0 h_{B_1}(\xi_{-1}) \tag{3.4.31}$$

The same bound as (3.4.31) can be obtained for $\frac{\pi}{\hat{\pi}}$ by using the untilted version of (3.4.29) instead. □

By feeding in Lemma 3.4.2 into a twofold application of Theorem 2.6.2 (with $W'_0 = W_0 = W^+_{B_1,B_2}$), we obtain $U^v_{m_0} \subset \mathrm{Incl}^{v,m_0}(Z,\hat{Z}) \cap \mathrm{Incl}^{v,m_0}(\tilde{Z},\hat{Z})$ for RI trajectories $Z = \{z_i\}_{i\geq 1}$ sampled according to π_0, π, tilted trajectories $\tilde{Z} = \{z_i\}_{i\geq 1}$ with $\tilde{\pi}_0$, $\tilde{\pi}$ and finally independent trajectories $\hat{Z} = \{\hat{z}_i\}_{i\geq 1}$ with $\hat{\pi}$. In particular, at this stage we have couplings of deterministic counts of (tilted) RI excursions with independent excursions sampled according the scheme in display (2.6.3). After combining these pairs in order to have inclusions directly between the tilted and untilted RI (without the intermediary independent excursions $\hat{Z}$), one sees that for any $v \in \left(0, \frac{1}{8}\right)$, whenever $|x| \geq C(v,\delta)$ on the event $U^v_{m_0}$ (see (2.6.9), (2.6.12)) we have the following inclusions for all $m \geq m_0$

$$\begin{aligned}\{z_1,\ldots,z_{(1-v)m}\} &\subset \{\tilde{z}_1,\ldots,\tilde{z}_{(1+8v)m}\} \\ \{\tilde{z}_1,\ldots,\tilde{z}_{(1-v)m}\} &\subset \{z_1,\ldots,z_{(1+8v)m}\}\end{aligned} \tag{3.4.32}$$

where $\{z_i\}_{i\geq 0} \overset{law}{=} \{Z_i\}_{i\geq 0}$ and $\{\tilde{z}_i\}_{i\geq 0} \overset{law}{=} \{\tilde{Z}_i\}_{i\geq 0}$ by nature of the soft local time sampling procedure. Recall that, in this instance, the excursions $Z_i, \tilde{Z}_i$ correspond to just one box $B_1 = B^y_1$, $y \in \Gamma_f$, $f \in \mathcal{F}$, however, Theorem 2.6.2 can also guarantee inclusions of excursions across multiple boxes simultaneously under the same coupling measure $\mathbb{Q}$ by tuning $W'_0 = W^+_{B^z_1,B^z_2}$ for several versions of $\hat{\pi}^z$ corresponding to a well-separated collection of boxes $\{B^z_1\}_{z\in\mathcal{C}}$ (see (4.2.2)).

As we now explain, comparisons between a deterministic number of excursions as given

by (3.4.32) do not directly attain the coupling we seek on their own. Let

$$\mathcal{N}^u = \mathcal{N}^u_{B_1,B_2} = \langle \mu^+_{B_1,u}, \tau_{B_1,B_2} \rangle, \text{ under } \mathbb{P} \tag{3.4.33}$$

represent the total number of RI excursions of the box B_1 and relate them in law to the procedure described below (3.4.13). To define $\tilde{\mathcal{N}}^u$, we simply replace $\mathbb{P}$ with $\tilde{\mathbb{P}}^f$ in (3.4.33). Explicitly, recalling the notation in (3.4.12) and below (3.4.13), we have

$$\mathcal{N}^u \overset{\text{law}}{=} \sum_{i=1}^{\Theta^u} \tau_i, \quad \widetilde{\mathcal{N}}^u \overset{\text{law}}{=} \sum_{i=1}^{\tilde{\Theta}^u_f} \tilde{\tau}_i. \tag{3.4.34}$$

Therefore, in order to couple the local times $\{\ell^u_z\}_{z\in B}$ and $\{\tilde{\ell}^{u'}_z\}_{z\in B}$, concentrations of the excursion counts (3.4.34) need to be taken into account. In light of this, the effective levels appearing in (3.4.1), (3.4.2) boil down to an estimate on the compound Poisson processes $\mathcal{N}^u$, $\widetilde{\mathcal{N}}^u$ presented in the following lemma.

Lemma 3.4.3. *For all $\delta \in (0,1)$, $\varepsilon \in (0, \frac{u_*}{10} \wedge 1)$ and $\eta \in [0, c_3)$, the following hold.*

i) If $u < u_*$ and $z \in \Gamma_{\text{ext}}$, then with $f = f_1^{v,u;\varepsilon}$, $v = u(1-\eta\sqrt{\delta})$,

$$\mathbb{Q}_B\Big[(1+c\varepsilon)\mathcal{N}^{u_*+\frac{1}{2}\varepsilon} \le \widetilde{\mathcal{N}}^v \le (1+c\varepsilon)^{-1}\mathcal{N}^{u_*+\frac{3}{2}\varepsilon}\Big] \ge 1 - e^{-\tilde{c}r_0^{\tilde{c}}}. \tag{3.4.35}$$

ii) If $u \neq u_*$ and $z \in \Gamma_{int}$, then with $s_i(t) = t(1-\gamma_i)^{\sigma_i}$ for $\gamma_i \in [0,1)$, $\sigma_i \in \{\pm 1\}$, $i = 1,2$, one has for f as above when $u < u_*$ and $f = f_0^{u;\varepsilon}$ when $u > u_*$ that

$$Q_B\Big[(1+\varepsilon)\mathcal{N}^{s_i(u_*-\frac{3}{2}\varepsilon)} \le \widetilde{\mathcal{N}}^{s_i(u)} \le (1+\varepsilon)^{-1}\mathcal{N}^{s_i(u_*-\frac{1}{2}\varepsilon)}\Big] \ge 1 - e^{-\tilde{c}r_0^{\tilde{c}}}, \ i = 1,2, \tag{3.4.36}$$

for suitable $\tilde{c}$ depending on $u, \delta, \varepsilon, \eta, \gamma_i, \sigma_i$.

Prior to proving Lemma 3.4.3, we introduce an important inequality on the moment generating function of $\tilde{\tau}$.

Lemma 3.4.4. *For all $\lambda \in (-\infty, c)$, $f \in \mathcal{F}$, $x \in \mathbb{Z}^d$, $z \in \Gamma_f(x)$, with $\tau = \tau_{B_1^z, B_2^z}$,*

$$e^\lambda(1-\beta^f(x)) \le \tilde{E}^f_{\bar{e}_{B_1}}\Big[e^{\lambda\tau}\Big] \le \frac{e^\lambda(1-\beta)}{1-e^\lambda\beta}, \tag{3.4.37}$$

where $\beta \le \beta^f(x)$ (see (3.3.4) for $\beta^f(x)$). Moreover, (3.4.37) continues to hold for $E_{\bar{e}_{B_1}}[e^{\lambda\tau}]$ if we replace β directly with $\beta_0(x)$ from (3.3.2).

Proof. The lower bound is immediate by bounding $\tilde{E}^f_{\tilde{\epsilon}_{B_1}}\left[e^{\lambda\tau}\right] \geq e^{\lambda}\tilde{P}^f_{\tilde{\epsilon}_{B_1}}[\tau = 1]$, so we are left with showing the upper bound. Remark that, for $y \in \partial B_1, \tilde{P}^f_y$ a.s. we have $\tau = 1 + (\tau \circ \theta_{H_{B_1}} \mathbb{1}\{H_{B_1} < \infty\}) \circ \theta_{T_{B_2}}$. Hence, as in [64, Lemma 1.7], by applying the strong Markov property first at time T_{B_2} and then at H_{B_1}, we have

$$\begin{aligned} \tilde{E}^f_y[e^{\lambda\tau}] &= e^{\lambda}\tilde{E}^f_y[e^{\lambda(\tau\circ\theta_{H_{B_1}}\mathbb{1}\{H_{B_1}<\infty\})\circ\theta_{T_{B_2}}}] \\ &= e^{\lambda}\tilde{E}^f_y[\tilde{E}^f_{X_{T_{B_2}}}[\mathbb{1}\{H_{B_1} < \infty\}E_{H_{B_1}}[e^{\lambda\tau}] + \{H_{B_1} = \infty\}]] \end{aligned} \tag{3.4.38}$$

Hence, by setting $\phi = \sup_{y\in\partial B_1} \tilde{E}^f_y[e^{\lambda\tau}]$, $\beta = \sup_{y\in\partial B_1} \tilde{E}^f_y[\tilde{P}_{T_{B_2}}[H_{B_1} < \infty]] \leq \beta^f(x)$, we have

$$\phi \leq e^{\lambda}(\beta\phi + (1-\beta)) \tag{3.4.39}$$

from which the claim follows directly. Notice that, in the above, the choice of β circumvents the requirement $\tilde{E}^f_y[e^{\lambda\tau}] \geq 1$, which is not necessarily true for negative values of λ. The closing remark at the end of the lemma's statement follows from identical steps with obvious replacements. □

Now we are ready for the

Proof of Lemma 3.4.3. Let us focus on (3.4.35) as the remaining bounds are obtained similarly. Thus let $f = f_1^{v,u;\varepsilon}$, $v = u(1-\eta\sqrt{\delta})$ for some $u < u_*$ and $B = B_1^z$ for some $z \in \Gamma_{\text{ext}}$. From (3.4.34), Lemma 3.4.4 and Lemma 3.3.8, we infer that for $\lambda \in (-\infty, c)$, $\epsilon_0 \in (0,1)$ and $|x| \geq C(\epsilon_0)$,

$$\begin{aligned} \log \widetilde{\mathbb{E}}^f\left[e^{\lambda\cdot\widetilde{\mathcal{N}}^v}\right] &\overset{(2.4.5)}{=} v\cdot\widetilde{\text{cap}}(B)\Big(\widetilde{E}^f_{\tilde{\epsilon}_B}\left[e^{\lambda\tau}\right] - 1\Big) \overset{(3.4.37)}{\leq} v\cdot\widetilde{\text{cap}}(B)\frac{e^{\lambda}-1}{1-e^{\lambda}\beta^f(x)} \\ &\overset{(3.3.46)}{\leq} (u_*+\varepsilon)\text{cap}(B)\frac{(1+\epsilon_0)(e^{\lambda}-1)}{1-e^{\lambda}\beta^f(x)}, \end{aligned} \tag{3.4.40}$$

where to apply (2.4.5) we rewrote the excursion counting number as a function of the PPP $\mu^+_{B,u}$ (c.f. below (2.4.12)), namely $\tilde{\mathcal{N}}^v = \langle \tau, \mu^+_{B,u}\rangle$. Next, in applying (3.3.46), we have summed over $y \in \mathbb{Z}^d$ and used that $v \cdot \frac{u_f^z}{\tilde{u}_f^z} = u_* + \varepsilon$ by (3.3.43) (and using that $z \in \Gamma_{\text{ext}}$ with $f = f_1$). A lower bound corresponding to (3.4.40) can be derived similarly, along with similar estimates for $\log \mathbb{E}[e^{\lambda\cdot\mathcal{N}^w}]$, $w > 0$, obtained by means of the final remark in Lemma 3.4.4 (but no longer requiring Lemma 3.3.8). Using that $e^{\lambda} - 1 \leq \lambda(1 + C\lambda)$ and $1 - e^{-\lambda} \geq \lambda(1 - C\lambda)$for $0 \leq \lambda \leq 1$ and applying Chebyshev's inequality separately to $\mathcal{N}^{u_*+\frac{1}{2}\varepsilon}$, $\widetilde{\mathcal{N}}^v$ and $\mathcal{N}^{u_*+\frac{3}{2}\varepsilon}$, selecting in each case $\lambda = \epsilon_0 = c\varepsilon$

in (3.4.40), while using Lemma 3.3.2 to control β^f in (3.4.40), one ensures that with probability at least $1 - e^{-\tilde{c}r_0^{\tilde{c}}}$ and for $|x| \geq C(\varepsilon, \delta)$, the inequalities

$$\begin{aligned}
&\mathcal{N}^{u_*+\frac{1}{2}\varepsilon} \leq \mathrm{cap}(B)(u_* + \tfrac{5}{8}\varepsilon),\\
&\mathrm{cap}(B)(u_* + \tfrac{7}{8}\varepsilon) \leq \widetilde{\mathcal{N}}^v \leq \mathrm{cap}(B)(u_* + \tfrac{9}{8}\varepsilon),\\
&\mathcal{N}^{u_*+\frac{3}{2}\varepsilon} \geq \mathrm{cap}(B)(u_* + \tfrac{11}{8}\varepsilon)
\end{aligned}$$

all hold. From this (3.4.35) immediately follows for sufficiently small choice of $c \in (0, 1)$. □

Finally, armed with (3.4.32) and Lemma 3.4.3, we are ready to prove the raison d'être of this section.

Proof of Theorem 3.4.1. Let us define $\mathbb{Q}_B$ as the measure introduced below (3.4.20) which drives the PPP on $W^+_{B_1,B_2}$ together with the random variables Θ and τ_i, $i \geq 0$ along with their tilted counterparts. Recall that, in line with the discussion around (3.4.13), under $\mathbb{Q}_B$ the random measures

$$(\xi^u, \tilde{\xi}^v)_{u,v>0} := \Big(\sum_{1\leq k\leq \mathcal{N}^u} \delta_{Z_k}, \sum_{1\leq k\leq \widetilde{\mathcal{N}}^v} \delta_{\tilde{Z}_k} \Big)_{u,v>0} \tag{3.4.41}$$

have the same law as the excursions between $B = B_1^z$ and B_2^z induced by the trajectories in the support of μ_B^+ with label at most u and v under $\mathbb{P}$ and $\tilde{\mathbb{P}}^f$ respectively. As can be seen from (2.4.16), the occupation time field $(\ell_x^u)_{x\in B,u>0}$ is clearly a measurable function of the excursions induced by $\mu_B(\omega)$ under $\mathbb{P}$, and similarly for $(\tilde{\ell}_x^u)_{x\in B,u>0}$ under $\widetilde{\mathbb{P}}_f$. Hence, $\ell_x^u, \tilde{\ell}_x^v$, $x \in B, u, v > 0$ can be viewed (in law) as functionals of the point measures in (3.4.41), as

$$\ell_x^u \overset{\text{law}}{=} \ell_x(\xi^u) \overset{\text{def.}}{=} \sum_{1\leq k\leq \mathcal{N}^u} \int_0^{\mathrm{len}(Z_k)} 1\{Z_k(t) = x\}\mathrm{d}t, \quad x \in B,\, u > 0, \tag{3.4.42}$$

where $\mathrm{len}(Z_k)$ refers to the length (duration) of the excursion Z_k, and similarly for $\tilde{\ell}_x^u$. In particular, $\mathbb{Q}_B$ thus furnishes a coupling of the occupation time fields $(\ell_x^u, \tilde{\ell}_x^v : x \in B, u, v > 0)$.

We now argue that (3.4.1) holds (which implicitly entails that z, f and $B = B_1^z$ have been chosen accordingly). For two point measures ξ, ξ' on the excursion space Ξ, as in (3.4.41) for instance, we write $\xi \leq \xi'$ if $\mathrm{supp}(\xi) \subset \mathrm{supp}(\xi')$. As follows plainly from (3.4.42), the occupation time field is clearly monotone with respect to this order,

i.e. $\xi \leq \xi'$ implies $\ell(\xi) \leq \ell(\xi')$. Fix $v = c\varepsilon$ with c small enough so that the conclusions of (3.4.32) hold and $\frac{1+8v}{1-v} \leq 1 + c\varepsilon$. Let $m_0 = \mathcal{N}^{u_*}$. With these choices for v and m_0, if the event $U^v_{m_0}$ (see (2.6.12)) appearing in the prelude to (3.4.32) and the event on the left-hand side of (3.4.35) jointly occur (under $\mathbb{Q}_B$), one deduces using $\mathcal{N}^{u_*+\frac{1}{2}\varepsilon} \geq \mathcal{N}^{u_*}$ when applying (3.4.32) that

$$\{Z_1, \dots, Z_{\mathcal{N}^{u_*+\frac{1}{2}\varepsilon}}\} \overset{(3.4.32)}{\subset} \{\widetilde{Z}_1, \dots, \widetilde{Z}_{(1+c\varepsilon)\mathcal{N}^{u_*+\frac{1}{2}\varepsilon}}\} \overset{(3.4.35)}{\subset} \{\widetilde{Z}_1, \dots, \widetilde{Z}_{\widetilde{\mathcal{N}}^v}\}.$$

With a view towards (3.4.41), this yields that $\xi^{u_*+\frac{1}{2}\varepsilon} \leq \tilde{\xi}^v$ and hence $\ell_x^{u_*+\frac{1}{2}\varepsilon} = \ell_x(\xi^{u_*+\frac{1}{2}\varepsilon}) \leq \ell_x(\tilde{\xi}^v) = \tilde{\ell}^v_x$ for all $x \in B$ by monotonicity. The other inequality $\tilde{\ell}^v_x \leq \ell_x^{u_*+\frac{3}{2}\varepsilon}$ inherent to the event in (3.4.1) is obtained similarly, now exploiting (3.4.32) together with the second inequality in (3.4.35). To conclude the proof of (3.4.1) it thus suffices to combine the bound on the event in (3.4.32), together with a suitable estimate on $\mathbb{Q}_B[U^{c\varepsilon}_{n=\mathcal{N}^{u_*}}]$. The latter is obtained by combining (3.4.32), first for $|x| \geq C(\varepsilon)$ but eventually for all x by possibly adapting the constant $\tilde{c}$, and the fact (see for instance the steps in the proof of Lemma 3.4.3) that $Q_B[\mathcal{N}^{u_*} \geq \frac{1}{2}u_*\mathrm{cap}(B)] \geq 1 - e^{-c\cdot\mathrm{cap}(B)}$.

The proof of item *(ii)* in Theorem 3.4.1 follows a similar reasoning as that of item *(i)*, now combining (3.4.32) and (3.4.36) to deduce (3.4.2). □

Chapter 4

Upper Bounds

4.1 Geometric Properties

The material in §4.1.1 is an extension of a similar result appearing in [28, Lemma 2.1]. In particular, we additionally show the maximality of the arising surrounded connected set $\mathcal{O}$ in part (iii) of Proposition 4.1.1. On the other hand, in §4.1.2 we present a streamlined version of the material appearing in [16]. To be precise, rather than using a general graph structure as in the case of the reference, we restrict to the case of an integer lattice, as this is always our underlying state space. As a remark, the results in §4.1.2 will only be required to show certain a priori bounds in §4.5.3 when the renormalisation procedure derived in §4.3 proves insufficient. As we will see, this is necessary when the smallest scale of the renormalisation is required to stay constant.

4.1.1 Crossings and Blocking Interfaces

In contrast with the lower bounds presented in Chapter 3, the upper bounds no longer rely on a change-of-measure strategy as the inherent goal of such a scheme is to select a constituent event and boost its probability. Naturally, in the present case this is insufficient as all possible events need to be accounted for in order to determine an upper bound. To this end, in this section, we provide a construction in Proposition 4.1.1 which is guaranteed to occur when certain, purely topological, conditions are met concerning any possible path. Prior to doing so, we build on the definitions introduced in Section 2.1 to satisfy our notational requirements for this section. As usual, we consider the integer lattice $\mathbb{Z}^d$ with the graph structure induced by l^2 nearest neighbours, and consider two sets $U \subset V \subset \mathbb{Z}^d$. Next, define $\partial_V U$, $\partial_V^{\text{out}} U$ as the inner

and outer boundaries of U with respect to V,

$$\partial_V(U) \coloneqq \{x \in U : \exists y \in V \setminus U \text{ such that } x \sim y\}, \quad \partial_V^{\text{out}} \coloneqq \partial_V(V \setminus U) \tag{4.1.1}$$

Moreover, in the case that U is finite, there is at most one infinite component in the complement of U, which we denote by U_∞^c. Then, we also define the filled-in version of U as $\text{Fill}(U) \coloneqq \mathbb{Z}^d \setminus U_\infty^c$, which simply removes any holes of U. With this, the exterior boundary of U is defined as $\partial^{ext}U = \partial U_\infty^c = \partial^{\text{out}}\text{Fill}(U)$. Finally, now letting both U, V be finite sets, we say that V surrounds U, denoted as

$$U \preceq V, \text{ whenever any infinite connected set } \gamma \text{ intersecting } U \text{ also intersects } V \tag{4.1.2}$$

where we remark that this defines a reflexive and transitive relation between finite sets. As an example, we notice that the finite components of U^c are surrounded by U which is in turn surrounded by $\partial^{ext}U$. Indeed, if there is an infinite connected set γ intersecting the holes of U which does not intersect U itself, then since U is finite, γ must also intersect U_c^∞ so then U_c^∞ is connected to the holes of U in U^c (via γ) which is a contradiction by the maximality assumption of components. A similar argument handles U and $\partial^{ext}U$, using that $U \subset \text{Fill}(U)$.

Proposition 4.1.1 (Existence of Blocking Interfaces)**.** *Let $U \subset\subset \mathbb{Z}^d$ be a $*$-connected set and $V \subset\subset \mathbb{Z}^d$ be a box such that $U \subset V$. Suppose that $\Sigma \subset V \setminus U$ is a set such that any path γ connecting U and ∂V intersects Σ in at least $k \geq 1$ points. Then, there exists an integer $l \geq 1$ and $*$-connected sets $O_1, \dots, O_l \subset \Sigma$ satisfying the following three properties:*

(i) $U \preceq O_1 \preceq \dots \preceq O_l \preceq \partial V$,

(ii) any path γ connecting U to ∂V intersects $O \coloneqq \bigcup_{1 \leq i \leq l} O_i$ in at least k points, and

(iii) given any pair of indices $j \in \{1, \dots, l\}$, $j' \in \{j, (j+1) \wedge l\}$, then whenever two points $x \in \bar{O}_j$, $x' \in \bar{O}_{j'}$ are connected by a path $\gamma \subset V \setminus O$, they are also connected by a path $\gamma' \subset V \setminus \Sigma$. Moreover, if O_l is connected to ∂V outside O, it is connected to ∂V outside Σ.

Before turning to the proof of this proposition, we introduce an auxiliary lemma which will aid in showing the maximality criterion posited by item *(iii)*. To this end, for a path $\gamma = \{\gamma(0), \dots, \gamma(n)\}$ we denote by $\mathring{\gamma} = \bigcup_{1 \leq i < n} \gamma(i)$ as the path with its endpoints removed.

Lemma 4.1.2. *Let $\Sigma \subset\subset \mathbb{Z}^d$, $U \subset \Sigma$ a $*$-component of Σ and W either (i) a connected set or (ii) also a $*$-component of Σ. If the following conditions hold*

$$U \preceq W, \tag{4.1.3}$$

$$\bar{U} \text{ is connected to } \bar{W} \text{ in } \Sigma^c, \tag{4.1.4}$$

then whenever a point in U is connected to W by a path γ satisfying $\mathring{\gamma} \cap U = \emptyset$, then the same point is connected to W by a path γ' with $\mathring{\gamma}' \cap \Sigma = \emptyset$.

Proof. Define the set

$$NC := \{x \in \mathbb{Z}^d : \nexists \text{ path } \gamma \text{ connecting } x \text{ to } W \text{ with } \mathring{\gamma} \cap \Sigma = \emptyset\} \tag{4.1.5}$$

For a contradiction, assume that there exists a point $x \in NC$ with a path γ connecting x to W with $\mathring{\gamma} \cap U = \emptyset$. Remark that $\mathring{\gamma} \neq \emptyset$, since otherwise $\mathring{\gamma} \cap \Sigma = \emptyset$ and therefore $x \notin NC$, contradicting our assumption. In the sequel, we will repeatedly rely on the fact that

$$\text{for } U \subset\subset \mathbb{Z}^d, x \in \mathrm{Fill}(U), y \in \mathrm{Fill}(U)^c, \forall \gamma \text{ such that } x \overset{\gamma}{\leftrightarrow} y, \gamma \cap \partial^{ext} U \neq \emptyset \tag{4.1.6}$$

Then, let $\mathcal{C}_x$ be the $*$-connected component of $\Sigma^c \cup \{x\}$ containing x. By definition, $\partial^{ext}\mathcal{C}_x \subset \partial^{\mathrm{out}}\mathcal{C}_x \subset \Sigma$. Moreover, since $x \in NC$ and $\mathcal{C}_x$ is connected, we must have *(i)* $\bar{\mathcal{C}_x} \cap W = \emptyset$, as well as *(ii)* $\mathcal{C}_x \subset NC$. In fact, we claim that

$$\mathrm{Fill}(\mathcal{C}_x) \subset NC \tag{4.1.7}$$

which, as we now describe, leads to a contradiction. By (4.1.4), there is a path $\gamma' \subset \Sigma^c$ from $\bar{U}$ to $\bar{W}$. By definition of closure of sets, there exists a point $y \in U$ neighbouring γ' which, by the preceding statement, satisfies $y \notin NC$. Furthermore, since U is $*$-connected, there is a $*$-path $\gamma \subset U$ from x to y. Since $x \in \mathrm{Fill}(\mathcal{C}_x)$, $y \in NC^c \subset \mathrm{Fill}(\mathcal{C}_x)^c$ are endpoints of the path γ, there must be a point on γ which is either contained in, or a neighbour of, $\partial^{ext}\mathcal{C}_x$. Since $\mathcal{C}_x$ is connected then, by virtue of [13, Lemma 2.1*(i)*], $\partial^{ext}\mathcal{C}_x$ is $*$-connected. Using that $\partial^{ext}\mathcal{C}_x \subset \Sigma$ and $\gamma \subset U$ are neighbours, one obtains by maximality that they must be part of the same component U of Σ, such that $\partial^{ext}\mathcal{C}_x \subset U$. On the other hand, by assumption we had that x is connected to W by a path γ with $\mathring{\gamma}$ with $\mathring{\gamma} \cap U = \emptyset$. Since $W \cap \mathrm{Fill}(\mathcal{C}_x) = \emptyset$, the endpoint of γ is in $W \subset \mathrm{Fill}(\mathcal{C}_x)^c$, so that $\mathring{\gamma}$ must contain a point in $\partial^{ext}\mathcal{C}_x$ such that $\partial^{ext}\mathcal{C}_x \cap U^c \neq \emptyset$ which is a contradiction.

Hence, to conclude, we need only to show (4.1.7). Let $w \in \mathrm{Fill}(\mathcal{C}_x)$ and assume that $w \notin NC$. First, we show that this implies that

$$W \text{ has a non-empty intersection with a finite component of } \mathcal{C}_x^c. \tag{4.1.8}$$

Indeed, if (4.1.8) does not hold, then by property *(i)* from above (4.1.7), we have $W \cap (\mathrm{Fill}(\mathcal{C}_x) \cup \partial^{ext}\mathcal{C}_x) = \emptyset$, hence for any path γ from $w \in \mathrm{Fill}(\mathcal{C}_x)$ to $W \subset (\mathrm{Fill}(\mathcal{C}_x) \cup \partial^{ext}\mathcal{C}_x)^c$, $\mathring{\gamma}$ must contain a point of $\partial^{ext}\mathcal{C}_x \subset \Sigma$, which is a contradiction as this implies $w \in NC$. Finally, we use the assumptions on W in full force. By (4.1.3), we have that $x \preceq W$. Therefore, on account of *(i)* above, $\bar{\mathcal{C}_x} \preceq W$, and in turn

$$\mathrm{Fill}(\mathcal{C}_x) \cup \partial^{ext}\mathcal{C}_x \preceq W \tag{4.1.9}$$

by transitivity of the surrounding relation. Property *(i)* combined with (4.1.9) imply

$$(\mathrm{Fill}(\mathcal{C}_x) \cup \partial^{ext}\mathcal{C}_x)^c \cap W \neq \emptyset \tag{4.1.10}$$

where we remark that property *(i)* was necessary for this conclusion due to the reflexivity of the surrounding relation. To conclude, we distinguish between the two possibilities of W. In the case when W is a connected set, by using (4.1.8) and (4.1.10) we construct a path γ inside W passing from $\mathrm{Fill}(\mathcal{C}_x)$ to $(\mathrm{Fill}(\mathcal{C}_x) \cup \partial^{ext}\mathcal{C}_x)^c$, so that by (4.1.6) $\mathring{\gamma}$ must intersect $\partial^{ext}\mathcal{C}_x$, which contradicts property *(i)*. On the other hand, if W was a $*$-connected component of Σ, (4.1.8) and (4.1.10) imply, by similar reasoning, that there exists a $*$-path $\gamma \subset W$ starting in $Fill(\mathcal{C}_x)$ and ending in $(\mathrm{Fill}(\mathcal{C}_x) \cup \partial^{ext}\mathcal{C}_x)^c$ which must contain a point either inside or neighbouring $\partial^{ext}\mathcal{C}_x$. Since $\partial^{ext}\mathcal{C}_x$ is itself a $*$-connected set of Σ, by maximality we have $\partial^{ext}\mathcal{C}_x \subset W$, which is again a contradiction by property *(i)*. □

Proof of Proposition 4.1.1. First we show part *(i)*. Since any path from U to ∂V crosses Σ in $k \geq 1$ points, U is not connected to ∂V in $\mathbb{Z}^d \setminus \Sigma$. Let us set up an iterative procedure to obtain the $\{O_i\}_{1 \leq i \leq l}$ postulated by the proposition. In what follows, denote by $\mathcal{C}_U$ the $*$-connected component of $\mathbb{Z}^d \setminus \Sigma$ containing U, and define O_1 as the $*$-connected component of $\partial^{ext}\mathcal{C}_U$ in Σ. Indeed, this is possible since the exterior boundary of a finite $*$-connected set is itself $*$-connected by [13, Lemma 2.1*(i)*]. Remark that $U \subset \mathcal{C}_U \preceq \partial^{ext}\mathcal{C}_U \subset O_1$, so in particular $U \preceq O_1$ as required. Next, set

$U_0 = U$, $\Sigma_0 = \Sigma$ and define

$$U_n \coloneqq \mathcal{C}_{U_{n-1}} \cup O_n, \quad \Sigma_n \coloneqq \Sigma_{n-1} \setminus O_{n-1}, \tag{4.1.11}$$

for $n \geq 1$, $k_{U_{n-1},\Sigma_{n-1}} \geq 1$, where O_n is the $*$-component of $\partial^{ext}\mathcal{C}_{U_{n-1}}$ in Σ, and

$$k_{U,\Sigma} \coloneqq \min\{|\gamma \cap \Sigma| : \gamma \text{ is any path s.t. } U \overset{\gamma}{\leftrightarrow} \partial V\}. \tag{4.1.12}$$

In this way, according to identical reasoning as above, we have $U \preceq O_1 \preceq \ldots \preceq O_l \preceq V$ for $l = \inf\{n : k_{U_n,\Sigma_n} = 0\}$ with each O_i $*$-connected. Note that the last surrounding relation is immediate since $O_i \subset \Sigma \subset V \preceq \partial V$ $\forall i \in \{0, \ldots, l\}$ as $V \subset \mathbb{Z}^d$ is a box.

Next, under the assumption of property *(iii)* which we defer to the end of the proof, we show property *(ii)*. Thus, consider any path γ connecting U to ∂V with at most one point in ∂V. Let $\gamma_1, \ldots, \gamma_m$ denote the maximal non-trivial segments of γ whose interiors lie outside O. By property *(i)* of the Proposition, each segment must either have both its endpoints in $\{O_j, O_{j+1}\}$ or one endpoint in O_l and the other in ∂V. Moreover, by property *(iii)*, each of the segments γ_i of γ can be replaced by γ_i' with their interior in Σ^c to construct a new path γ' wielding the property $\gamma' \cap \Sigma = \gamma' \cap O = \gamma \cap O$. Consequently, we have $|\gamma \cap O| \geq k$, due to the assumption that any path γ' from U to ∂V must necessarily satisfy $|\gamma' \cap \Sigma| \geq k$.

Finally, to prove property *(iii)*, let $x \in O_j$ and $y \in \{O_j, O_{j+1}\}$ (or $y \in \{O_l, \partial V\}$ when $j = l$) be such that x is connected to y by a path γ with its interior in O^c, and assume that there are no paths from x to y with their interior wholly in Σ^c.

Let us distinguish between the different possibilities for y, starting with the case $y \in O_j$. Define $\mathcal{C}_x$ as the connected component of x in $V \setminus \Sigma$ and, using the assumption that x is not connected to y in Σ^c, let $\mathcal{C}_y^x$ be the connected component of y in $V \setminus \mathcal{C}_x$. In fact, the underlying assumption additionally implies $y \in \mathcal{C}_y^x \setminus \partial_V \mathcal{C}_y^x$ as well as

$$\partial_V \mathcal{C}_y^x \subset \Sigma \tag{4.1.13}$$

for otherwise, in the absence of the first condition, there would be a path from x to y with interior in $\mathcal{C}_x \setminus \{x\} \subset \Sigma^c$ and, in the absence of (4.1.13), maximality of the component $\mathcal{C}_x \subset \Sigma^c \cup \{x\}$ is violated. In fact, since x is connected to y by a path

whose interior is in $V \setminus O$, and $y \in \mathcal{C}_y^x \setminus \partial_V \mathcal{C}_y^x$,

$$\partial_V \mathcal{C}_y^x \cap O^c \neq \emptyset \tag{4.1.14}$$

On the other hand, since $x, y \in O_j$ which is a $*$-connected component of Σ, there is a $*$-path $\gamma \in \Sigma$ from x to y. Any such $*$-path must either intersect or be a neighbour of $\partial_V \mathcal{C}_y^x$, which is itself $*$-connected by [13, Lemma 2.1*(ii)*] since $\mathcal{C}_x$ is a connected subset of V. Thus, since $\partial_V \mathcal{C}_y^x \cup \gamma \subset \Sigma$ by (4.1.13), $\partial_V \mathcal{C}_y^x$ and γ belong to the same $*$-connected component of Σ. Since $O_j \cap (\partial_V \mathcal{C}_y^x \cup \gamma) \neq \emptyset$ (e.g. x, y both belong to this intersection), $\partial_V \mathcal{C}_y^x \subset O_j$ by maximality of the $*$-connected component O_j. Evidently, this contradicts (4.1.14).

Next, we deal with the case $y \in O_{j+1}$, $j \neq l$. By constructing $\mathcal{C}_x$ and $\mathcal{C}_y^x$ in the same way as before, we see that the conclusions (4.1.13), (4.1.14) still hold. However, the remainder of the argument does not follow in the same way as x and y now no longer belong to the same $*$-component of Σ. We remedy this with an application of Lemma 4.1.2. Since $O_j \subset \Sigma$, $x \in O_j \preceq O_{j+1}$ and $\bar{O}_j$ is connected to $\bar{O}_{j+1}$ in Σ^c by construction, letting (U, Σ, W) in Lemma 4.1.2 we can find a point $z \in \partial_V^{\text{out}} \mathcal{C}_x \cap O_{j+1}$. This is due to the fact that x was assumed to be connected to $y \in O_{j+1}$ by a path with its interior in O_j^c, so there must be a path connecting x to O_{j+1} with its interior in Σ^c. z is taken to be the endpoint of this other path. Noticing that $\partial_V^{\text{out}} \mathcal{C}_x = \partial_V \mathcal{C}_y^x$, together with the fact that O_{j+1} is $*$-connected, then there exists a $*$-path from $z \in \partial_V \mathcal{C}_y^x$ to y. Once again using the maximality of the $*$-component O_{j+1}, by (4.1.13) and the fact that $y \in O_{j+1}$, $\partial_V \mathcal{C}_y^x \subset O_{j+1}$. This contradicts (4.1.14).

Finally the case where $x \in O_l$ and $y \in \partial V$ follows almost immediately by setting $(U, \Sigma, W) = (O_l, \Sigma, \partial V')$ in Lemma 4.1.2 for any box V' whose interior contains V. □

4.1.2 Propagation of Good Events

In this subsection, we present some standalone results which, in the context of this work, will be useful in proving certain a priori estimates and will subsequently be fed into our renormalisation scheme (see the discussion in the introduction to §4.3). The material we present here amounts to a simplification of Proposition 7.1 and Lemma 8.6 from [16] to the case of the integer lattice $\mathbb{Z}^d$, and can be viewed as a kind of geometric renormalisation scheme for good events. In practice, this approach enables us to preserve an ensemble of certain local properties, typically defined at a fixed

scale L_0 (which is kept general here and will be specified in (4.3.35)), at larger scales. Importantly, this is done without incurring an excessively high cost to maintain these properties, which may otherwise arise due to the increasing cardinality of the ensemble. In the following, we consider the triple of integers $\mathcal{L} = (L_0, \bar{\lambda}, \lambda_0)$ such that

$$L_0 \geq 1, \bar{\lambda} \geq 2 \text{ and } \lambda_0 \geq 16(\sqrt{d} + \gamma_0)\bar{\lambda} \tag{4.1.15}$$

where we recall γ_0 as the constant from the decoupling Theorem 2.4.3. Next, denoting $L_n = \lambda_0^n L_0$ and the corresponding sub-lattices $\mathbb{L}_n := L_n\mathbb{Z}^d$, for a family of events $\mathcal{E} := \{\mathcal{E}_x : x \in \mathbb{L}_0\}$ we describe an associated collection of n-good events $G^{\mathcal{L}}_{x,n}(\mathcal{E})$ according to the following iterative procedure:

$$\begin{aligned} G^{\mathcal{L}}_{x,0}(\mathcal{E}) &= \mathcal{E}_x, \quad x \in \mathbb{L}_0, \\ G^{\mathcal{L}}_{x,n}(\mathcal{E}) &= \bigcap_{\substack{y,y' \in \mathbb{L}_{n-1} \cap B(x, \bar{\lambda}L_n) \\ |y-y'| \geq L_n}} G^{\mathcal{L}}_{y,n-1}(\mathcal{E}) \cup G^{\mathcal{L}}_{y',n-1}(\mathcal{E}), \quad x \in \mathbb{L}_n, n \geq 1. \end{aligned} \tag{4.1.16}$$

In fact, as we will see shortly, this is a special case of n-goodness which we will generalise. We are now ready to present our first result, which propagates a bound on a 0-bad event to an improved bound on an n-bad event.

Lemma 4.1.3 (Bound on n-bad events)**.** *There exist constants $c_7 > 0$, $C_7 \geq 1$ such that for all choices of L_0, $\bar{\lambda}$ and λ_0 as in* (4.1.15)*, all $\epsilon > 0$ and $u_1, u_2 > 0$ satisfying the condition*

$$\frac{u\epsilon^2(\sqrt{\lambda_0}L_0/2)^{d-2}}{\log(L_0+1)} \geq C_7 \tag{4.1.17}$$

and all families $\mathcal{E} = \{\mathcal{E}_x(u_1, u_2) : x \in \mathbb{L}_0\}$ such that the events $\mathcal{E}_x(u_1, u_2)$ are decreasing in u_1, increasing in u_2 (according to inclusion), are measurable with respect to $\sigma\big(\mathcal{I}^{u_1}|_{B(x,\bar{\lambda}L_0)}, \mathcal{I}^{u_2}|_{B(x,\bar{\lambda}L_0)}\big)$, and satisfy

$$\mathbb{P}[\mathcal{E}_x(u_1, u_2)^c] \leq \frac{c_7}{\lambda_0^{4d}} \quad \forall x \in \mathbb{L}_0 \tag{4.1.18}$$

Then one has for all $n \in \mathbb{N}$, $x \in \mathbb{L}_n$

$$\mathbb{P}[G^{\mathcal{L}}_{x,n}(\mathcal{E}_x(u_1 - \epsilon, u_2 + \epsilon))^c] \leq 2^{-2^n} \tag{4.1.19}$$

Proof. Fix some $\epsilon > 0$, $u_1, u_2 > 0$, $\bar{\lambda}$, λ_0 as in (4.1.15) and $\mathcal{E}_x(u_1, u_2)$ satisfying the conditions in the statement of the Lemma for any $x \in \mathbb{L}_0$. Define the sequences

$(u_{i,n})_{n\geq 0}$ for $i = 1, 2$ by $u_{i,0} = u_i$ and $u_{1,n} = u_1 - \sum_{k=1}^{n} \frac{\epsilon \wedge 1}{2^k}$, $u_{2,n} = u_2 + \sum_{k=1}^{n} \frac{\epsilon \wedge 1}{2^k}$, and argue that if (4.1.17) holds, then

$$\mathbb{P}\Big[G^{\mathcal{L}}_{x,n}(\mathcal{E}_x(u_{1,n}, u_{2,n})^c\Big] \leq \frac{2^{-2^n}}{2^{2d+1} \cdot \lambda_0^{4d}}, \qquad \text{for all } n \in \mathbb{N} \tag{4.1.20}$$

from which (4.1.19) follows immediately since, by monotonicity in each of its parameters, $\mathcal{E}_x(u_{1,n}, u_{2,n}) \subset \mathcal{E}_x(u_1 - \epsilon, u_2 + \epsilon)$. We prove inequality (4.1.20) by induction on n. The basis case, $n = 0$, is given immediately by selecting $c_7 = \frac{2^{-1}}{2^{2d+1}}$. Thus, we assume that (4.1.20) holds for $n - 1$ for some $n \geq 1$, and aim to show the case n. From the definition of $G^{\mathcal{L}}_{x,n-1}$ in (4.1.16) and the measurability conditions imposed on $\mathcal{E}_x(u_1, u_2)$, we have that $G^{\mathcal{L}}_{x,n-1}(\mathcal{E}_x(u_{1,n}, u_{2,n}))$ are measurable relative to $\sigma\Big(\mathcal{I}^{u_{1,n-1}}|_{B(x,2\bar{\lambda}L_{n-1})}, \mathcal{I}^{u_{2,n-1}}|_{B(x,2\bar{\lambda}L_{n-1})}\Big)$. Then, we let $r_n = \operatorname{diam}(B(x, 2\bar{\lambda}L_{n-1})) = 4\bar{\lambda}L_{n-1}$ and, for any $x \in \mathbb{L}_n, y, y' \in \mathbb{L}_{n-1} \cap B(x, \bar{\lambda}L_n)$ such that $|y - y'| \geq L_n$, we observe that

$$\frac{3}{2}L_n \overset{(4.1.15)}{\geq} s_n \coloneqq d\Big(B(y, r_n/2), B(y', r_n/2)\Big) \overset{(4.1.15)}{\geq} \frac{1}{2}L_N. \tag{4.1.21}$$

Note that for the choices $A_1 = B(y, \bar{\lambda}L_{n-1})$ and $A_2 = B(y', \bar{\lambda}L_{n-1})$ in Theorem 2.4.3, for any pair y, y' as described, we now have bounds on the diameters $r(= r_n)$ of the sets and on the separation distance $s(= s_n)$, based on the conditions outlined in the theorem. In particular, we are therefore in a position to apply the decoupling therein. Overall, by applying a union bound, these considerations yield

$$\begin{aligned}
&\mathbb{P}\Big[G^{\mathcal{L}}_{x,n}(\mathcal{E}(u_{1,n}, u_{2,n}))^c\Big] \\
&\leq \Big(\frac{2\bar{\lambda}L_n}{L_{n-1}}\Big)^{2d} \sup_{y,y'} \mathbb{P}\Big[G^{\mathcal{L}}_{y,n-1}(\mathcal{E}(u_{1,n}, u_{2,n}))^c \cap G^{\mathcal{L}}_{y',n-1}(\mathcal{E}(u_{1,n}, u_{2,n}))^c\Big] \\
&\overset{\text{Thm. 2.4.3}}{\leq} (2\lambda_0^2)^{2d}\Big\{ \sup_{y \in B(x,\bar{\lambda}L_n) \cap \mathbb{L}_{n-1}} \mathbb{P}\Big[G^{\mathcal{L}}_{y,n-1}(\mathcal{E}(u_{1,n-1}, u_{2,n-1}))^c\Big]^2 \\
&\qquad\qquad + \gamma_0(r_n + s_n)^d \exp(-\gamma_1 2^{-2n}\epsilon^2 u s_n^{d-2}\Big\} = (\dagger)
\end{aligned} \tag{4.1.22}$$

Hence, by applying the induction hypothesis together with our estimates on r_n, s_n and the assumption (4.1.17), we have that

$$\begin{aligned}(\dagger) &\overset{(4.1.20)}{\leq} \frac{1}{2}\Big(\frac{2^{-2^n}}{2^{2d+1}\cdot\lambda_0^{4d}}\Big) + \gamma_0 2^{2d}\lambda_0^{4d}(2L_n)^d \exp(-\gamma_1 2^{-2n}\epsilon^2 u(L_n/2)^{d-2}) \\ &\overset{(4.1.17)}{\leq} \frac{1}{2}\Big(\frac{2^{-2^n}}{2^{2d+1}\cdot\lambda_0^{4d}}\Big) + \gamma_0 2^{3d}\lambda_0^{(4+n)d} L_0^d \exp\Big(-\gamma_1 2^{-2n} C_7 \log(1+L_0)(\lambda_0^{n-\frac{1}{2}})^{d-2}\Big)\end{aligned} \tag{4.1.23}$$

Finally, for a sufficiently large constant C_7 and with our assumptions on λ_0 in (4.1.15), we have that

$$\gamma_0 2^{3d}\lambda_0^{(4+n)d} L_0^d \exp\Big(-\gamma_1 2^{-2n} C_7 \log(1+L_0)(\lambda_0^{n-\frac{1}{2}})^{d-2}\Big) \leq \frac{1}{2}\Big(\frac{2^{-2^n}}{2^{2d+1}\cdot\lambda_0^{4d}}\Big), \tag{4.1.24}$$

which yields the required result. □

Remark 4.1.4. *Although the events belonging to the family $\mathcal{E}(u_1, u_2)$ are considered to be measurable with respect to the trace of the random interlacements, one may also weaken this measurability condition by instead strengthening the decoupling of Theorem 2.4.3 to deal with local times. Since the proof of such a decoupling works direct with inclusions of excursions (see for example the event* (2.6.6)*), such an approach is easily available.*

In the remainder of this subsection, we lay the groundwork to show a certain connectivity result related to events of the form (4.1.16). As a consequence, Lemma 4.1.3 becomes an effective tool for constructing estimates on local-uniqueness type events. To this end, let us consider a family of events $\mathcal{E} := \big\{\mathcal{E}_x^k, x \in \mathbb{Z}^d, k \in \{1,\dots,m\}\big\}$ for some finite $m \in \mathbb{N}^*$, and associate to it the collection

$$\boldsymbol{\mathcal{E}}_x^k = \bigcap_{y \in B(x, \bar{\lambda} L_0)} \mathcal{E}_y^k, \quad x \in \mathbb{L}_0, \quad k = 1,\dots,m \tag{4.1.25}$$

effectively adding an extra 'ground' level to the sequence of n-good events in (4.1.16). For a point $y \in \mathbb{Z}^d$, we say that

$$y \text{ is } \mathcal{E}\text{-good if } \bigcap_{k=1}^{m} \mathcal{E}_y^k \text{ holds.} \tag{4.1.26}$$

Moreover, finally arriving at the generalisation referred to below (4.1.16), given a point $y \in \mathbb{L}_n$, we say that

$$y \text{ is } (n,\mathcal{E})\text{-good if } \bigcap_{k=1}^{m} G^{\mathcal{L}}_{y,n}(\boldsymbol{\mathcal{E}}^k) \text{ holds.} \tag{4.1.27}$$

In the case that a family $\mathcal{E}$ is fixed and known, we often abbreviate by simply writing 'y is good', or 'y is n-good' respectively.

Lemma 4.1.5 (Local Uniqueness from n-goodness)**.** *Fix $m \in \mathbb{N}^*$, $c_m = 5(m+1)^2$, $\bar{\lambda} = 22c_m$ and consider a family of events $\mathcal{E} := \left\{\mathcal{E}^k_x, x \in \mathbb{Z}^d, k \in \{1,\dots,m\}\right\}$. For any $L_0 \geq 1$, $n \in \mathbb{N}$, if there exists a vertex $x \in \mathbb{L}_n$ which is n-good, then every two connected components of $B(x, 20c_mL_n)$ with diameter at least c_mL_n are connected via a path of good vertices in $B(x, 30c_mL_n)$.*

Before proceeding to the proof of the result above, we first present the following Lemma which will be applied therein.

Lemma 4.1.6. *Consider two finite sets $A, B \subset \mathbb{Z}^d$, and recall the definitions of Fill$(\cdot)$, $\partial^{ext}(\cdot)$ and '$\preceq$' from the preamble in §4.1.1. We have that*

$$A \preceq B \text{ and } B \preceq A \iff \partial Fill(A) = \partial Fill(B) \tag{4.1.28}$$

Proof. The direction ($\Leftarrow$) is immediate, so we are left with showing ($\Rightarrow$). Let $x \in \partial\mathrm{Fill}(A)$. By definition of ∂^{ext}, there exists an unbounded path γ starting from x such that no point of γ apart from its starting point is in A. Since $A \preceq B$, γ must intersect B at some point x' (wlog, we can assume that x' is the final point in γ which is also in B, otherwise finiteness of B is contradicted). Since we also have $B \preceq A$, and γ *only* intersects A at the starting point x, it must be that $x = x'$ and therefore $x \in B$. Since $x \in \partial\mathrm{Fill}(A)$ was arbitrary, we yield $\partial\mathrm{Fill}(A) \subset \partial\mathrm{Fill}(B)$. By symmetry, we also have $\partial\mathrm{Fill}(B) \subset \partial\mathrm{Fill}(A)$, concluding the result. □

Proof of Lemma 4.1.5. Let us proceed by induction. For $n = 0$, if x is 0-good, then in view of (4.1.25)-(4.1.27) and the definition (4.1.16), we have

$$\bigcap_{k=1}^{m} G_{x,0}(\boldsymbol{\mathcal{E}}^k) \overset{(4.1.16)}{=} \bigcap_{k=1}^{m} \boldsymbol{\mathcal{E}}^k_x \overset{(4.1.25),(4.1.26)}{=} \bigcap_{y \in B(x,\bar{\lambda}L_0)} \{y \text{ is good}\} \tag{4.1.29}$$

such that all vertices in $B(x, 22c_mL_0)$ are good, and therefore any two clusters in $B(x, 20c_mL_0)$ are connected by a nearest neighbour path of good vertices.

Now, assume that the conclusion of the lemma holds for the case $n-1$ for some $n \geq 1$, and let $x \in \mathbb{L}_n$ be an n-good vertex. Let $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{Z}^d$ be any two connected components of $B(x, 20c_m L_n)$ with diameter at least $c_m L_n$. First, we aim to show that

$$\begin{aligned}&\text{there exists a finite collection } \gamma = \{\gamma(i)\}_{i=1}^{n_\gamma} \subset \mathbb{L}_{n-1} \cap B(x, 22c_m L_n) \text{ of} \\ &(n-1)\text{-good vertices such that } B(\gamma(1), L_{n-1}) \cap \mathcal{U}_1 \neq \emptyset,\ B(\gamma(n_\gamma), L_{n-1}) \cap \mathcal{U}_2 \neq \emptyset \\ &\text{and } B(\gamma(i), L_{n-1}) \cap B(\gamma(i+1), L_{n-1}) \neq \emptyset\ \forall i \in \{1, \dots n_\gamma - 1\}.\end{aligned} \tag{4.1.30}$$

To show (4.1.30), we first aim to find connected sub-components $\mathcal{U}_i' \subset \mathcal{U}_i$ for each $i = 1, 2$ such that the collections $\{y \in \mathbb{L}_{n-1} : B(y, L_{n-1}) \cap \mathcal{U}_i' \neq \emptyset\}$ consist entirely of $(n-1)$-good vertices. To this end, for each $i = 1, 2$, we select $m+1$ connected subsets $(\mathcal{U}_i^k)_{k \in \{1,\dots,m+1\}}$ of $\mathcal{U}_i$ such that for any $k, k' \in \{1, \dots, m+1\}$ with $k \neq k'$

$$d(\mathcal{U}_i^k, \mathcal{U}_i^{k'}) \geq L_n + 2\sqrt{d}L_{n-1}, \text{ and } \operatorname{diam}(\mathcal{U}_i^k) \geq 4(m+1)L_n \tag{4.1.31}$$

To construct these sub-components, we take the following approach. By assumption, since $\operatorname{diam}(\mathcal{U}_i) \geq c_m L_n$, there exist two points $y_0^s, y_0^e \in \mathcal{U}_i$ such that $|y_0^s - y_0^e| \geq c_m L_n$. Thus, we let

$$\begin{aligned}&\mathcal{U}_i^1 \text{ be the connected component of } y_0^s \text{ in } \mathcal{U}_i \cap B^2(y_0^s, 4(m+1)L_n), \text{ and} \\ &\mathcal{U}_i^{-1} \text{ be the connected component of } y_0^e \text{ in } \mathcal{U}_i \cap B^2(y_0^s, 5(m+1)L_n)^c.\end{aligned} \tag{4.1.32}$$

Now, since $\operatorname{diam}(\mathcal{U}_i^{-1}) \geq m \cdot 5(m+1)L_n$, we can iterate the process above for $k = 1, \dots, m-1$ by finding $y_k^s, y_k^e \in \mathcal{U}_i^{-k}$ satisfying $|y_k^s - y_k^e| \geq (m+1-k) \cdot 5(m+1)L_n$, and obtaining $\mathcal{U}_i^{k+1}$, $\mathcal{U}_i^{-(k+1)}$ by replacing the pair (y_0^s, y_0^e) with (y_k^s, y_k^e) in (4.1.32). Crucially, at each step we have $\operatorname{diam}(\mathcal{U}_i^{-k}) \geq (m+1-k) \cdot 5(m+1)L_n$ for $k = 1, \dots, m$, permitting the selection of the points y_k^s, y_k^e. To conclude, we set $\mathcal{U}_i^{m+1} = \mathcal{U}_i^{-m}$. Inspecting the procedure above shows that $(\mathcal{U}_i^k)_{k \in \{1,\dots,m+1\}}$ satisfies (4.1.31).

Thus, we assume that for each $k \in \{1, \dots, m+1\}$, there exists an $(n-1)$-bad vertex $y_k \in \mathbb{L}_{n-1}$ such that $B(y_k, L_{n-1}) \cap \mathcal{U}_i^k \neq \emptyset$ (therefore $y_k \in B(x, 20c_m L_n + L_{n-1})$). By the Dirichlet principle, at least two of these bad points, say y_k and $y_{k'}$, must satisfy $G_{y,n-1}^{\mathcal{L}}(\boldsymbol{\mathcal{E}}^j)^c$, $y \in \{y_k, y_k'\}$ for some $j \in \{1, \dots, m\}$. By (4.1.31), y_k and $y_{k'}$ are separated by at least L_n, so we conclude that x cannot be n-good, which is a contradiction. Thus, for each $i = 1, 2$ we let $\mathcal{U}_i^{k_i}$ be the connected component such that any point $y \in \mathbb{L}_{n-1}$ with $B(y, L_{n-1}) \cap \mathcal{U}_i^{k_i} \neq \emptyset$ is $(n-1)$-good. Furthermore, we define the set

$\mathrm{comp}_{n-1}(\mathcal{U}_i^{k_i}) \subset \mathbb{Z}^d$ as the connected component in

$$\bigcup_{\substack{y \in \mathbb{L}_{n-1} \cap B(x, \bar{\lambda} L_n): \\ y \text{ is } (n-1)\text{-good}}} B(y, L_{n-1})$$

containing $\mathcal{U}_i^{k_i}$. Then, the claim (4.1.30) can be reformulated as showing $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) = \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$. For a contradiction, assume that $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) \neq \mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$. By definition of connectedness, this occurs if and only if $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) \cap \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2}) = \emptyset$. Since $\mathcal{U}_1^{k_1}, \mathcal{U}_2^{k_2} \subset B(x, 20c_m L_n)$, there exists a nearest neighbour path $\gamma = (\gamma(i))_{i=1}^{n_\gamma} \subset B(x, 20c_m L_n)$ connecting $\mathcal{U}_1^{k_1}$ and $\mathcal{U}_2^{k_2}$. With this, and the definition of ∂^{ext}, we introduce the following dichotomy for such paths: since $\gamma(1) \in \mathcal{U}_1^{k_1}$, either there exists $n'_\gamma \in \{1, \dots, n_\gamma\}$ such that $\gamma(n'_\gamma) \in \partial^{\mathrm{ext}} \mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$, or every unbounded nearest neighbour path starting from $\gamma(n_\gamma)$ intersects $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$. A similar dichotomy is obtained for the reversed path starting in $\mathcal{U}_2^{k_2}$. Observe that, if any unbounded nearest neighbour path starting from $\gamma(n_\gamma) \in \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$ intersects $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$ *and* any unbounded nearest neighbour path from $\gamma(1) \in \mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$ intersects $\mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$, then by connectivity any unbounded path beginning in $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$ intersects $\mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$ and vice versa. Overall, we have shown

$$\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) \preceq \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2}), \text{ and } \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2}) \preceq \mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) \tag{4.1.33}$$

which, by an application of Lemma 4.1.6, yields that $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) \cap \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2}) \neq \emptyset$ - this is a contradiction. Therefore, returning to the dichotomy, we have that either there exists n'_γ such that $\gamma(n'_\gamma) \in \partial^{\mathrm{ext}} \mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$, or $n'_\gamma \in \partial^{\mathrm{ext}} \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$. Assume that we are in the first scenario since the second follows similarly. Due to the fact that $\gamma \subset B(x, 20c_m, L_n)$, we have that $\partial^{\mathrm{ext}} \mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) \cap B(x, 20c_m, L_n) \neq \emptyset$. From the condition on the diameter on $\mathcal{U}_1^{k_1}$ and the $*$-connectedness of the exterior boundary, we have that there exists a $*$-path in $\partial^{\mathrm{ext}} \mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$ starting from $B(x, 20c_m L_n)$ with diameter between $(m+1)L_n$ and $(m+2)L_n$. Moreover, any point on this path must lie inside $B(y, L_{n-1})$ for some $(n-1)$-bad $y \in \mathbb{L}_{n-1}$ and therefore there must be at least $m+1$ bad points in $B(x, (20c_m + m + 1 + \lambda_0^{-1})L_n) \subset B(x, 22c_m L_n)$ with mutual separation at least L_n. By applying a Dirichlet argument in the same spirit as before, we once again yield that x cannot be n-good, which is a contradiction. At this point, we have therefore shown $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) = \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$ and, in turn, (4.1.30).

Now, we present an iterative procedure to produce the desired connection between $\mathcal{U}_1^{k_1}$ and $\mathcal{U}_2^{k_2}$ via good vertices. Let $y_0 \in \mathcal{U}_1^{k_1}$ and $y_{N+1} \in \mathcal{U}_2^{k_2}$, and apply (4.1.30) to

obtain a sequence of $(n-1)$-good vertices $y_1, \dots, y_N \in \mathbb{L}_{n-1} \cap B(x, 22c_m L_n)$ such that

$$\begin{aligned} 5c_m L_{n-1} \leq & |y_j - y_{j+1}| \leq 6c_m L_{n-1} \text{ for } j \in \{1, \dots, N\}, \text{ and} \\ & |y_j - y_{j+1}| \leq 6c_m L_{n-1} \text{ for } j \in \{0, N+1\}. \end{aligned} \tag{4.1.34}$$

Then, for each $j \in \{0, \dots, N+1\}$ construct a nearest neighbour path π_j starting at y_j and satisfying $c_m L_{n-1} \leq \text{diam}(\pi_j) \leq 2c_m L_{n-1}$, with π_0, π_{N+1} having the additional requirement that $\pi_0 \subset \mathcal{U}_1^{k_1}$, $\pi_{N+1} \subset \mathcal{U}_2^{k_2}$. Remark that, for each consecutive pair π_j, π_{j+1}, we have that $\pi_j, \pi_{j+1} \subset B(y_{j+1}, 10c_m L_{n-1})(\subset B(y_{j+1}, 20c_m L_{n-1}))$, so by the inductive step there exists a nearest neighbour path of good vertices $\bar{\pi}_{j+1} \subset B(y_{j+1}, 30c_m L_{n-1})(\subset B(x, 30c_m L_{n-1}))$ starting in π_j and ending in π_{j+1}. From $\bar{\pi}_{j+1}$ we extract a sub-path $\hat{\pi}_{j+1} \subset B(y_{j+1}, 10c_m L_{n-1})$ with $\hat{\pi}_{j+1}(0) \in \pi_j$ and $\text{diam}(\hat{\pi}_{j+1}) \geq c_m L_{n-1}$. Thus, we apply this step first on the pair π_0, π_1 to produce $\hat{\pi}_1$. Then, we replace π_1 with $\hat{\pi}_1$, and observe that $\hat{\pi}_1, \pi_2 \subset B(y_2, 20c_m L_{n-1})$ so that the above process still follows to produce $\hat{\pi}_2$ with the required conditions. By repeating these construction and replacement steps sequentially for $j = 0, \dots, N$, we obtain a sequence of nearest neighbour paths $\hat{pi}_j$, for $j = 1, \dots, N-1$, consisting entirely of good vertices. Moreover, since $\hat{\pi}_1(0) \in \pi_0$, $\hat{\pi}_{j+1}(0) \in \hat{\pi}_j$ for every $j \in \{1, \dots, m-1\}$ and $\bar{\pi}_{N+1}(0)$ connects $\hat{\pi}_N$ to π_{N+1}, we can concatenate $(\hat{\pi}_j)_{j=1}^N$ and $\bar{\pi}_{N+1}$ to obtain the required nearest neighbour path of good vertices between $\mathcal{U}_1^{k_1}$ and $\mathcal{U}_2^{k_2}$. □

4.2 Localisation

Although the content in §4.2.1 appears in various forms throughout the literature (see for example [47]), by now turning to our Theorem 2.6.2 we render the material more concise and directly applicable for our purposes. §4.2.2 on the other hand is an extension of the results in [64] where, for example, only one direction of Lemma 4.2.3 and Proposition 4.2.5 can be found (for their analogues in [64], see Proposition 3.1 and Theorem 3.2).

4.2.1 Coupling RI with Independent Excursions

One of the difficulties in proving various connectivity results for the vacant set of the RI percolation model is the inherent long-range correlation structure. Although this issue was largely circumvented in the previous chapter simply by applying union bounds, we have observed the necessity of decouplings when applying Theorem 2.4.3 in the proof of Theorem 1.3.1 - our main lower bound for the SRW on a torus. Such

complications are inevitable when the events we consider depend on the configuration of the RI synchronously across a collection of boxes, which is precisely the situation we face for the upper bounds. In particular, to form the upper bounds we will consider coarse-grained versions of arbitrary paths (see §4.3.1) which will consist of many disjoint boxes. The purpose of this section is therefore to apply the soft local time method in conjunction with Theorem 2.6.2 in order to simultaneously localise the RI in each of the constituent boxes of a large collection. As observed in Section 3.4, this amounts to proving that condition (2.6.10) is satisfied for an appropriate choice of underlying spaces $W_0' \subset W_0$, initial density π_0, and transition densities $\pi, \hat{\pi}$ (see the setup of Theorem 2.6.2). Without further delay, we shall now precisely formulate the relevant framework. In the sequel, we will consider a scale $L \geq 1$ and a separation parameter $K \geq 100$ which both take integer values. Then, we associate a lattice $\mathbb{L} \coloneqq L\mathbb{Z}^d$ to our choice of scale, and consider boxes anchored at points $z \in \mathbb{L}$

$$\begin{aligned} &C_z \coloneqq z + [0, L)^d, \quad \tilde{C}_z \coloneqq z + [-L, 2L)^d, \\ &\tilde{D}_z \coloneqq z + [-2L, 3L)^d, \quad D_z \coloneqq z + [-3L, 4L)^d, \quad \breve{D}_z \coloneqq z + [-6L, 7L)^d \\ &U_z \coloneqq z + [-KL, (K+1)L)^d, \quad \breve{U}_z \coloneqq z + [-(K+3)L, (K+4)L)^d \end{aligned} \tag{4.2.1}$$

where, if necessary for clarity, we include the dependence on the scale by writing $C_{z,L}, D_{z,L}$ etc. Moreover, we say that

$$\text{two points } z, z' \in \mathbb{L} \text{ are well-separated (at scale } L) \text{ whenever } \breve{U}_z \cap \breve{U}_{z'} = \emptyset, \tag{4.2.2}$$

and, for a collection of pairwise well-separated points $\mathcal{C} \subset \mathbb{L}$ with $\breve{U}_z \cap \breve{U}_{z'} = \emptyset$, we define

$$\Sigma_D(\mathcal{C}) = \bigcup_{z \in \mathcal{C}} \breve{D}_z, \quad \Sigma_U(\mathcal{C}) = \bigcup_{z \in \mathcal{C}} \breve{U}_z \tag{4.2.3}$$

and abbreviate $\Sigma_U = \Sigma_U(\mathcal{C})$, $\Sigma = \Sigma_D(\mathcal{C})$ when the context is clear. Next, employing a similar approach as expressed around (3.4.13), we decompose the RI corresponding to the PPP $\hat{\omega} = \sum_{i \geq 0} \delta_{w_i^*, u_i}$ into a collection of excursions

$$\Sigma \cap \mathcal{I}^u(\hat{\omega}) = \Sigma \cap \bigcup_{\substack{u_i \leq u \\ w_i^* \in W_\Sigma^*}} \bigcup_{j=1}^{\tau(s_\Sigma^+(w_i^*))} X_{[R_j, T_j]}(s_\Sigma^+(w_i^*)) \tag{4.2.4}$$

where we abbreviate $\tau = \tau_{\Sigma,\Sigma_U}$, $R_j = R_j^{\Sigma,\Sigma_U}$, $T_j = T_j^{\Sigma,\Sigma_U}$ and recall all the relevant notation from (2.4.6), (2.4.8), (2.4.13) and (3.4.11). Then, we consider a probability measure Q on under which $w_i \sim P_{\bar{e}_{\Sigma(\mathcal{C})}}$ for $i \geq 1$ so that $W_\Sigma^+ \ni w_i \overset{\text{law}}{=} s_\Sigma^+(w_i^*)$ whenever $w_i^* \in W_\Sigma^*$. With this, we define the following Markov chain on W_{Σ,Σ_U}^+

$$\begin{aligned} Z_1 &= X_{[R_1,T_1]}(w_1) \\ Z_n &= \begin{cases} X_{[R_{j+1},T_{j+1}]}(w^k) & R_{j+1} < \infty \\ X_{[R_j,T_j]}(w^{k+1}) & R_{j+1} = \infty \end{cases} \quad \text{for } n \geq 2 \text{ when } Z_{n-1} = X_{[R_j,T_j]}(w^k) \end{aligned} \tag{4.2.5}$$

with $j, k \in \mathbb{N}$. Furthermore, if we endow Q with the additional random variable $\Theta^u \sim \mathit{Poisson}(u\mathrm{cap}(\Sigma))$, we have

$$\Sigma \cap \mathcal{I}^u(\hat{\omega}) \overset{\text{law}}{=} \Sigma \cap \bigcup_{i=1}^{\Theta^u} Z_i \tag{4.2.6}$$

Hence, as before, to sample excursions in W_{Σ,Σ_U} with law dictated by the RI, it suffices to consider the Markov chain $\{Z_i\}_{i\geq 1}$ from (4.2.5). This places us within the scope of the soft local time method (2.6.3), meaning we only need to identify the initial and transition densities for this Markov chain with respect to the following intensity measure

$$\mu(\cdot) = \sum_{x \in \partial\Sigma} P_x[X_{[R_1,T_1]} \in \cdot] \tag{4.2.7}$$

By similar working to (3.4.20), we see that by setting

$$\pi_0(y) = \bar{e}_\Sigma(y_0), \quad \pi(x,y) = h_\Sigma(x_{-1}, y_0) + \bar{e}_\Sigma(y_0) P_{x_{-1}}[H_\Sigma = \infty] \tag{4.2.8}$$

with $x, y \in W_{\Sigma,\Sigma_U}^+$, and x_0, x_{-1} denoting the first and last points of x respectively, we have

$$Q[Z_1 \in \cdot] = \int_\cdot \pi_0 d\mu, \quad Q[Z_k \in \cdot | Z_{k-1} = x] = \int_\cdot \pi(x,y) d\mu(y), \ k \geq 2 \tag{4.2.9}$$

With these considerations, let $\mathbb{Q}_\mathcal{C}$ be the probability measure associated to a PPP η with intensity measure $\mu \times d\lambda$ on $W_{\Sigma,\Sigma_U}^+ \times \mathbb{R}_{\geq 0}$, and apply the steps described around (2.6.3) with π_0 and π from (4.2.8) to sample $\{(z_i, \xi_i)\}_{i\geq 1}$ from η. Proposition 2.6.1 then asserts that $Z \coloneqq \{z_i\}_{i\geq 1} \overset{\text{law}}{=} \{Z_i\}_{i\geq 1}$ and $\{\xi_i\}_{i\geq 1}$ i.i.d *exponential*(1) random variables.

Thereafter, we also consider a probability distribution $\hat{\mathbb{P}}_{\mathcal{C}}$ governing collections of independent excursions $\{\hat{Z}_k^{\check{D}_z,\check{U}_z}\}_{k\geq 1} \subset W^+_{\check{D}_z,\check{U}_z}$ for each $z \in \mathcal{C}$, and aim to sample $\hat{Z}_z = \{\hat{z}_k^z\}_{k\geq 1} \overset{\text{law}}{=} \{\hat{Z}_k^{\check{D}_z,\check{U}_z}\}_{k\geq 1}$ from the same PPP η under $\mathbb{Q}_{\mathcal{C}}$. In view of Theorem 2.6.2, this approach will yield the localisation we require in every box of $\Sigma(\mathcal{C})$ as long as condition (2.6.10) is satisfied. To elaborate on the necessary steps, fix $z \in \mathcal{C}$ and in the Theorem set $W_0 = W^+_{\Sigma,\Sigma_U}$, $W_0' = W^+_{\check{D}_z,\check{U}_z}$ as well as

$$\hat{\pi}^z(y) = \bar{e}_{\check{D}_z}(y_0),\ y \in W^+_{\Sigma,\Sigma_U} \tag{4.2.10}$$

so that $\hat{\pi}^z$ is supported on $W^+_{\check{D}_z,\check{U}_z}$ and $\int_{W^+_{\Sigma,\Sigma_U}} \hat{\pi}^z d\mu = 1$. In order to show that condition (2.6.10) is indeed satisfied for π_0, π, $\hat{\pi}^z$, we first introduce a mixing result for the entrance distribution of a random walk starting far from the set in question.

Proposition 4.2.1. *[64, Proposition 1.5] There exist $C_4 \in [1,\infty)$, $C_5 \in (0,\infty)$ such that for all $K \geq C_4$, $L \geq 1$, non-empty $A \subset B_L$ and $B \subset\subset \mathbb{Z}^d$ such that $B \cap B_{KL} = A$ with $\mathbb{Z}^d \setminus B$ connected, one has for any $y \in A$ and $x \in \mathbb{Z}^d \setminus (B \cup B_{KL})$,*

$$(1 - C_5K^{-1})\bar{e}_A(y) \leq P_x[X_{H_B} = y | H_B < \infty, X_{H_B} \in A] \leq (1 + C_5K^{-1})\bar{e}_A(y) \tag{4.2.11}$$

and

$$(1 - C_5K^{-1})\bar{e}_A(y) \leq \frac{\bar{e}_B(y)}{\bar{e}_B(A)} \leq (1 + C_5K^{-1})\bar{e}_A(y) \tag{4.2.12}$$

where $\bar{e}_B(A) = \sum_{x\in A} \bar{e}_B(x)$.

With this result in our arsenal, we are now ready to show estimates on $\frac{\pi_0^z}{\hat{\pi}^z}$, $\frac{\pi^z}{\hat{\pi}^z}$, where for $x, y \in W_{\Sigma,\Sigma_U}$

$$\pi_0^z(y) = \pi_0(y) / \int_{W^+_{\check{D}_z,\check{U}_z}} \pi_0 d\mu, \quad \pi^z(x,y) = \pi(x,y) / \int_{W^+_{\check{D}_z,\check{U}_z}} \pi(x,y) d\mu(y) \tag{4.2.13}$$

First, we evaluate the denominators of π_0^z and π^z

$$\begin{aligned}
&\int_{W^+_{\check{D}_z,\check{U}_z}} \pi_0 d\mu = Q[Z_1 \in W^+_{\check{D}_z,\check{U}_z}] = \bar{e}_\Sigma(\check{D}_z) \\
&\int_{W^+_{\check{D}_z,\check{U}_z}} \pi(x,y) d\mu(y) = Q[Z_2 \in W^+_{\check{D}_z,\check{U}_z} | Z_1 = x] \\
&\qquad\qquad = P_{x_{-1}}[X_{H_\Sigma} \in \check{D}_z, H_\Sigma < \infty] + \bar{e}_\Sigma(\check{D}_z) P_{x_{-1}}[H_\Sigma = \infty]
\end{aligned} \tag{4.2.14}$$

Then, in view of (4.2.8), (4.2.10) and (4.2.14), we observe that for any $\epsilon \in (0,1)$ and $K > 3C_5/\epsilon$

$$\frac{\pi_0^z(y)}{\hat{\pi}^z(y)} \overset{(4.2.12)}{\leq} (1+\frac{\epsilon}{3})\frac{\pi_0(y)}{\bar{e}_\Sigma(y_0)}\Big/\frac{\int_{W^+_{\breve{D}_z,\breve{U}_z}}\pi_0 d\mu}{\bar{e}_\Sigma(\breve{D}_z)} = 1+\frac{\epsilon}{3} \tag{4.2.15}$$

as well as

$$\pi(x,y) \overset{(4.2.11),(4.2.12)}{\leq} (1+\frac{\epsilon}{3})\bar{e}_{\breve{D}_z}(y_0)\Big\{P_{x_{-1}}[X_{H_\Sigma}\in\breve{D}_z, H_\Sigma<\infty] + \bar{e}_\Sigma(\breve{D}_z)P_{x_{-1}}[H_\Sigma=\infty]\Big\} \tag{4.2.16}$$

which, recalling the computations (4.2.14), implies $\pi^z(x,y) \leq (1+\frac{\epsilon}{3})\bar{e}_{\breve{D}_z}(y_0)$. All in all, we have

$$\frac{\pi^z(x,y)}{\hat{\pi}^z} \leq 1+\frac{\epsilon}{3} \tag{4.2.17}$$

Analogous reasoning to (4.2.15)-(4.2.17) also yields the lower bounds $\frac{\pi_0^z(y)}{\hat{\pi}^z}, \frac{\pi^z(x,y)}{\hat{\pi}^z} \geq 1-\frac{\epsilon}{3}$, so we have therefore satisfied condition (2.6.10) for any $z\in\mathcal{C}$. We conclude, by defining U_z^{ϵ,m_0} as in (2.6.12) with $N'(\cdot,\cdot) = N_{W^+_{\breve{D}_z,\breve{U}_z}}(\cdot,\cdot)$ and applying Theorem 2.6.2, we have for $K > 3C_5/\epsilon$

$$U_z^{\epsilon,m_0} \subset \mathrm{Incl}_z^{\epsilon,m_0} := \mathrm{Incl}^{\epsilon,m_0}(Z|_{W^+_{\breve{D}_z,\breve{U}_z}}, \hat{Z}_z),\ \forall z\in\mathcal{C} \tag{4.2.18}$$

Since the inclusions on the RHS of the display (4.2.18) are agnostic to the true number of excursions of the RI which visit each box $\breve{D}_z$, one final step is necessary in order to complete the localisation of the RI. In the tilted interlacements, the analogue of this is Lemma 3.4.3 which estimated the true, random, excursion counts $\mathcal{N}^u$, $\tilde{\mathcal{N}}^u$ of natural and tilted RI respectively (see (3.4.33)). Correspondingly, we need to simultaneously estimate the excursion counts of the RI in each $\breve{D}_z$ in order to determine an appropriate choice of m_0 to guarantee inclusions of the form (4.2.18) for the RI. To this end, for any $u,v\geq 0$, $z\in\mathbb{Z}^d$ we define the events

$$\mathcal{F}_z^{u,v} = \mathcal{F}_{z,L}^{u,v} := \begin{cases} \{\mathcal{N}^u_{z,L} \leq v\mathrm{cap}(\breve{D}_{0,L})\}, & \text{if } u\leq v \\ \{\mathcal{N}^u_{z,L} \geq v\mathrm{cap}(\breve{D}_{0,L})\}, & \text{if } u>v \end{cases} \tag{4.2.19}$$

where with hopefully obvious notation

$$\mathcal{N}^u_{z,L} \coloneqq \langle \mu^+_{\breve{D}_{z,L},u}, \tau_{\breve{D}_{z,L},\breve{U}_{z,L}} \rangle \tag{4.2.20}$$

To ensure that these events are well-defined under $\mathbb{Q}_{\mathcal{C}}$, we simply augment the probability space corresponding to $\mathbb{Q}_{\mathcal{C}}$ of the PPP η with $\{\mathcal{N}^u_{z,L}\}_{z\in\mathcal{C}}$ (this is akin to augmenting by a collection of correlated compound Poisson RVs which are independent from the samples $\{(z_i,\xi_i)\}_{i\geq 1}$,$\{(\hat{z}_i,\hat{\xi}_i)\}_{i\geq 1}$). Let us briefly detour to show some concentration bounds on a single occurrence of $\mathcal{F}^{u,v}_{z,L}$.

Lemma 4.2.2. *Let $L \geq 1$, $z \in \mathbb{L}$, and $u, v > 0$ such that $\frac{u}{v} \vee \frac{v}{u} < \frac{3}{2}$. Then for some constant $c > 0$ we have*

$$\mathbb{P}[(\mathcal{F}^{u,v}_{z,L})^c] \leq e^{-c\max(u^{-1},v^{-1})(u-v)^2 cap(\breve{D}_z)} \tag{4.2.21}$$

Proof. We simplify our proof by showing only the case $u > v$, as the other case is almost identical, and abbreviating $\tau = \tau_{\breve{D}_z,\breve{U}_z}$. In the usual manner, by an application of Chebyshev's inequality and recalling the definition of $\mu^+_{K,u}$ below (2.4.12), we have for any $\delta > 0$, and $K \geq C(\delta)$ (recall $\mathcal{N}^u_z$ tacitly depends on K)

$$\begin{aligned}
\mathbb{P}[(\mathcal{F}^{u,v}_{z,L})^c] &= \inf_{\lambda>0} \mathbb{P}[e^{-\lambda\mathcal{N}^u_z} \geq e^{-\lambda v \mathrm{cap}(\breve{D}_z)}] \leq \inf_{\lambda>0} e^{\lambda v \mathrm{cap}(\breve{D}_z)} \mathbb{E}[e^{-\lambda\mathcal{N}^u_z}] \\
&= \inf_{\lambda>0} e^{\lambda v \mathrm{cap}(\breve{D}_z)} \mathbb{E}[e^{\langle \mu^+_{\breve{D}_z,u}, -\lambda\tau\rangle}] \overset{(2.4.5)}{=} \inf_{\lambda>0} e^{\lambda v \mathrm{cap}(\breve{D}_z)} e^{u\mathrm{cap}(\breve{D}_z) E_{\bar{e}_{\breve{D}_z}}[e^{-\lambda\tau}-1]} \\
&\overset{\text{Lem. 3.4.4}}{\leq} \inf_{\lambda>0} \exp\left\{ \mathrm{cap}(\breve{D}_z)\left[\lambda v - \frac{u(1-e^{-\lambda})}{1-\delta e^{-\lambda}}\right]\right\} \qquad (4.2.22) \\
&\leq \inf_{\lambda>0} \exp\left\{ \mathrm{cap}(\breve{D}_z)\left[\lambda v - u(1-e^{-\lambda})\right]\right\} \qquad (4.2.23)
\end{aligned}$$

Then, by setting $\lambda = \log(\frac{u}{v})$ and applying a Taylor series, we have that $\lambda v - u(1-e^{-\lambda}) \leq -cv^{-1}(u-v)^2$, which yields the required result. □

When viewed together, the events $\mathrm{Incl}^{\epsilon,m_0}_z$ and $\mathcal{F}^{u,v}_z$ will facilitate switching back and forth between interlacement sets comprising different sets of excursions with varying degrees of localisation. In particular, let us introduce

$$\begin{aligned}
Z^u_z &= \{Z^{\breve{D}_z,\breve{U}_z}_k\}_{1\leq k\leq \mathcal{N}^u_z}, \\
\bar{Z}^u_z &= \{Z^{\breve{D}_z,\breve{U}_z}_k\}_{1\leq k\leq \lfloor u\mathrm{cap}(\breve{D}_z)\rfloor}, \\
\hat{Z}^u_z &= \{\hat{Z}^{\breve{D}_z,\breve{U}_z}_k\}_{1\leq k\leq \lfloor u\mathrm{cap}(\breve{D}_z)\rfloor}
\end{aligned} \tag{4.2.24}$$

where the $\{Z_k^{\breve{D}_z,\breve{U}_z}\}_{k\geq 1}$ are the interlacement excursions in $W^+_{\breve{D}_z,\breve{U}_z}$, and $\{\hat{Z}_i^{\breve{D}_z,\breve{U}_z}\}_{i\geq 1}$ are independent excursions as sampled from η according to $\hat{\pi}^z$. From now on, let us also denote $\bar{\mathcal{N}}_z^u = \hat{\mathcal{N}}_z^u = u\mathrm{cap}(\breve{D}_z)$ so that we can uniformly refer to the excursion counts via $\tilde{\mathcal{N}}_z^u$ depending on the choice $\tilde{Z}_z^u \in \{Z_z^u, \bar{Z}_z^u, \hat{Z}_z^u\}$. Then, for some excursion set $Z = \{Z_k\}_{1\leq k\leq n_Z}$ (i.e. $Z_k \in W^+_{D,U}$ for some $D \subset U$) where n_Z may or may not be random, we write

$$\mathcal{I}(Z) \coloneqq \bigcup_{1\leq k\leq n_Z} \mathrm{range}(Z_k), \quad \mathcal{V}(Z) \coloneqq \mathbb{Z}^d \setminus \mathcal{I}(Z) \tag{4.2.25}$$

to denote the interlacement set and vacant set of Z respectively. Moreover, we often rely on the following abbreviations

$$\mathcal{V}_z^u = \mathcal{V}(Z_z^u),\ \bar{\mathcal{V}}_z^u = \mathcal{V}(\bar{Z}_z^u),\ \hat{\mathcal{V}}_z^u = \mathcal{V}(\hat{Z}_z^u) \text{ etc.} \tag{4.2.26}$$

and remark that for any $u > v$, $\epsilon \in (0,1)$ and $\lfloor m_0(1+3\epsilon)\rfloor = \lfloor v\mathrm{cap}(D_0)\rfloor$, under the event $\mathcal{F}_z^{u,v} \cap \mathrm{Incl}_z^{\epsilon,m_0}$ we have the following set of inclusions

$$\mathcal{V}^u \cap \breve{D}_z = \mathcal{V}_z^u \subset \bar{\mathcal{V}}_z^v \subset \hat{\mathcal{V}}_z^{v'}, \text{ for all } v' \text{ such that } \left\lfloor v'\mathrm{cap}(\breve{D}_0)\right\rfloor \leq \lfloor (1-\epsilon)m_0\rfloor \tag{4.2.27}$$

Analogous inclusions in the opposite direction are naturally also true.

To complete this section, we introduce some noised versions of the above fields which will facilitate certain finite-energy techniques by reducing the inherent rigidity of the interlacements structure. Let us therefore augment our probability measure $\mathbb{Q}$ corresponding to the PPP η with a collection of independent i.i.d uniform random variables $\mathrm{U} = \{\mathrm{U}_x : x \in \mathbb{Z}^d\}$. For $\delta \in [0,1)$ and $\mathcal{V} \subset \mathbb{Z}^d$, let $N_\delta(\mathcal{V}) \subset \mathbb{Z}^d$ denote the set with the occupation variables

$$\mathbb{1}_{\{x\in N_\delta(\mathcal{V})\}} = \mathbb{1}_{\{x\in\mathcal{V}\}}\mathbb{1}_{\{\mathrm{U}_x\geq\delta\}} \tag{4.2.28}$$

An immediate consequence of this definition is that

$$N_\delta(\mathcal{V}) \text{ is increasing in } \mathcal{V} \text{ and decreasing in } \delta \text{ w.r.t. set inclusion.} \tag{4.2.29}$$

In the sequel, with reference to the notation in (4.2.26), we will abbreviate $(\mathcal{V})_\delta = N_\delta(\mathcal{V})$ for $\mathcal{V} \in \{\mathcal{V}_z^u, \bar{\mathcal{V}}_z^u, \hat{\mathcal{V}}_z^u\}$.

4.2.2 Estimates on Collections of $\mathcal{F}_z^{u,v}$

To ensure the meaningfulness of a transfer between vacant sets with varying degrees of localization (see (4.2.24), (4.2.26)), as described in (4.2.27), it must occur without incurring an excessively high cost. For now, we refer to this cost in an abstract sense, intending to clarify its precise meaning when dependencies arise later, and simply state that it is encapsulated by collections of the events $(U^{\epsilon}_{z,m_0})^c$ and $(\mathcal{F}_z^{u,v})^c$ with $z \in \mathcal{C} \subset \mathbb{L}$. Since the events $\{U^{\epsilon}_{z,m_0}\}_{z\in\mathcal{C}}$ are independent for well-separated collections, we can control the former using the product of the bounds on $\mathbb{Q}_{\mathcal{C}}[(U^{\epsilon}_{z,m_0})^c]$ as given in (2.6.13). However, the events $\mathcal{F}_z^{u,v}$ are not independent due to the highly interconnected behaviour of the constituent random walks of the RI, requiring us to develop a more nuanced approach. Crucially, we emphasise that computing the cost using a simple union bound on the probabilities of events of the form $\bigcup_{z\in\mathcal{C}}(U^{\epsilon}_{z,m_0})^c$ is insufficient since the necessary exponential decay rate is unattainable in this context, namely one with exponent proportional to $\mathrm{cap}(\Sigma(\mathcal{C}))$. It is important to note that, due to the subadditivity of capacity, the product bound obtained for $\mathbb{Q}_{\mathcal{C}}\Big[\bigcap_{z\in\mathcal{C}}(U^{\epsilon}_{z,m_0})^c\Big]$ indeed surpasses this requirement. To this end, we commence by introducing a computationally-tractable observable $h^u = h^u(\mathcal{C})$ which we will later relate to collections of $\mathcal{F}_z^{u,v}$. Consider the function V on $\mathbb{Z}^d$ defined as

$$V(x) = \sum_{\breve{D}\in\mathcal{C}} e_\Sigma(\breve{D})\bar{e}_{\breve{D}}(x), \quad x \in \mathbb{Z}^d, \tag{4.2.30}$$

where $e_\Sigma(\breve{D}) = \sum_{y\in\breve{D}} e_\Sigma(y)$ and the sum ranges over all $\breve{D}$ such that $\breve{D} = \breve{D}_z$ for some $z \in \mathcal{C}$. First, we present a result which assures that V is well-approximated by e_Σ when K becomes large.

Lemma 4.2.3. *For all $\epsilon \in (0,1)$, $L \geq 1$, $K \geq \frac{C_5}{\epsilon}$ and $\mathcal{C} \subset \mathbb{L}$ a well-separated collection of points (see (4.2.2)), one has*

$$(1-\epsilon)e_\Sigma(x) \leq V(x) \leq (1+\epsilon)e_\Sigma(x), \quad x \in \mathbb{Z}^d \tag{4.2.31}$$

Proof. From the definition of V, we see that it has the same support as $\bar{e}_\Sigma$, hence by applying (4.2.12) with $A = \breve{D}$, $B = \Sigma$ one immediately yields the required result. □

Now, recalling the notation for $\mu^+_{K,u}$ from below (2.4.12), define

$$h^u = h^u(\mathcal{C}) := \langle \mu^+_{\Sigma,u}, \int_0^\infty V(X_s)ds \rangle \tag{4.2.32}$$

where $\int_0^\infty V(X_s)ds$ represents the map $W^+ \ni w \to \int_0^\infty V(X_s(w))ds \in \mathbb{R}$. Before showing the main deviation estimates on h^u, we introduce an important result which, in the context of Lemma 4.2.3, gives an approximation to the Lagrangian of h^u.

Lemma 4.2.4. *Let $u > 0$, $L \geq 1$, and $\mathcal{C} \subset \mathbb{L}$ well-separated. For any $a < 1$, we have*

$$\mathbb{E}\Big[\exp\Big(a\langle \mu^+_{\Sigma,u}, \int_0^\infty e_\Sigma(X_s)ds\rangle\Big)\Big] = \exp\left(\frac{ua\,cap(\Sigma)}{1-a}\right) \tag{4.2.33}$$

Proof. First by recalling the discussion below (2.4.12), we see that $\mu^+_{\Sigma,u}$ is a PPP with intensity measure uP_{e_Σ}. Hence, we can apply the identity (2.4.5) for Poisson functionals to obtain

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(\langle \mu^+_{\Sigma,u}, a\int_0^\infty e_\Sigma(X_s)ds\rangle\Big)\Big] \\ = \exp\left\{u\mathrm{cap}(\Sigma)E_{\bar{e}_\Sigma}\Big[\exp\Big(a\int_0^\infty e_\Sigma(X_s)ds\Big) - 1\Big]\right\}\end{aligned} \tag{4.2.34}$$

Where we apply [64, Lemma 2.1] to determine that, under $P_{\bar{e}_{\Sigma(\mathcal{C})}}$, the quantity $\int_0^\infty e_{\Sigma(\mathcal{C})}(X_s)ds$ has the same law as an *exponential*(1) random variable. Applying this fact to the RHS of (4.2.34) and evaluating the final expression yields the required result. □

Proposition 4.2.5. *Let $\epsilon \in (0,1)$ and $0 < \frac{u_-}{1-\epsilon} < u < \frac{u_+}{1+\epsilon}$. Then, for any $K \geq \frac{C_5}{\epsilon}$, $L \geq 1$, and $\mathcal{C} \subset \mathbb{L}$ well-separated, one has that*

$$\mathbb{P}[h^u \geq u_+\,cap(\Sigma)] \leq \exp\left(-\left(\sqrt{\frac{u_+}{1+\epsilon}} - \sqrt{u}\right)^2 cap(\Sigma)\right) \tag{4.2.35}$$

and

$$\mathbb{P}[h^u \leq u_-\,cap(\Sigma)] \leq \exp\left(-\left(\sqrt{u} - \sqrt{\frac{u_-}{1-\epsilon}}\right)^2 cap(\Sigma)\right) \tag{4.2.36}$$

Proof. Using the approximation on V from Lemma 4.2.3 and subsequently applying Chebyshev's inequality, we can find bounds on the tails of h^u in terms of the Lagrangian presented on the LHS of (4.2.33), which is dealt with by the corresponding Lemma 4.2.4. Let us present the steps for each direction separately, starting with the upper tail (4.2.35).

$$\mathbb{P}[h^u \geq u_+\mathrm{cap}(\Sigma)] \leq \mathbb{P}\Big[(1+\epsilon)\langle \mu^+_{\Sigma,u}, \int_0^\infty e_\Sigma(X_s)ds\rangle \geq u_+\mathrm{cap}(\Sigma)\Big]$$

$$\leq \exp(-au_+\mathrm{cap}(\Sigma))\exp\left(\frac{ua(1+\epsilon)\mathrm{cap}(\Sigma)}{1-a(1+\epsilon)}\right)$$

whenever $a(1+\epsilon) \in (0,1)$ as per the conditions of Lemma 4.2.4. Hence, recalling that $u < \frac{u_+}{1+\epsilon}$, we can set $a = \frac{1}{1+\epsilon}\left(1 - \sqrt{\frac{(1+\epsilon)u}{u_+}}\right) \in (0,1)$ to yield the claim. The second inequality (4.2.36) follows similarly

$$\mathbb{P}[h^u \leq u_-\mathrm{cap}(\Sigma)] \leq \mathbb{P}\Big[-a(1-\epsilon)\langle \mu^+_{\Sigma,u}, \int_0^\infty e_\Sigma(X_s)ds\rangle \geq -au_-\mathrm{cap}(\Sigma)\Big]$$
$$\leq \exp(au_-\mathrm{cap}(\Sigma))\exp\left(\frac{-ua(1-\epsilon)\mathrm{cap}(\Sigma)}{1+a(1-\epsilon)}\right)$$

where $a > 0$. Hence, we deduce the required result by recalling our assumption $u > \frac{u_-}{1-\epsilon}$ and substituting $a = \frac{1}{1-\epsilon}\left(\sqrt{\frac{u(1-\epsilon)}{u_-}} - 1\right) > 0$. □

To wrap up this section, we elucidate the connection between the observable h^u and the collections $\{\mathcal{F}^{u,v}_z\}_{z\in\mathcal{C}}$, which has been the primary objective of our discussion. With this relationship established, the tail bounds we derived in Proposition 4.2.5 will now prove to be useful. To this effect, for $u, v \geq 0$, and for a sequence of excursions $\tilde{Z}^u_z = \{\tilde{Z}_k\}_{1\leq k\leq \tilde{\mathcal{N}}^u_z} \in \{Z^u_z, \bar{Z}^u_z, \hat{Z}^u_z\}$ (see (4.2.24) to recap the various degrees of localisation), we introduce the following event

$$\mathcal{E}^v(\tilde{Z}^u_z) \coloneqq \begin{cases} \sum_{1\leq l\leq \tilde{\mathcal{N}}^u_z} \int_0^{T_{\breve{U}_z}} e_{\breve{D}_z}(\tilde{Z}_k)ds \leq v\mathrm{cap}(\breve{D}_0), & u \leq v \\ \sum_{1\leq l\leq \tilde{\mathcal{N}}^u_z} \int_0^{T_{\breve{U}_z}} e_{\breve{D}_z}(\tilde{Z}_k)ds \geq v\mathrm{cap}(\breve{D}_0), & u > v \end{cases} \tag{4.2.37}$$

where we remark that $\mathcal{E}^v(Z^u_z) = \{h^u(\{z\}) \leq v\mathrm{cap}(\breve{D}_z)\}$ when $u \leq v$, with the inequality reversed when $u > v$. Moreover, as a corollary of Lemma 4.2.2, we have

Lemma 4.2.6. *Let $L \geq 1$, $z \in \mathbb{L}$ $u > 0$ and $\epsilon \in (0, \frac{1}{2})$. Then, letting $v = u(1 \pm \epsilon)$ and recalling the probability measure $\mathbb{P}_z = \mathbb{P}_{\{z\}}$ from the paragraph preceding* (4.2.10)*, for some constant $c > 0$ we have*

$$\mathbb{P}_z[\mathcal{E}^v(\hat{Z}^u_z)^c] \leq e^{-c\epsilon^2 cap(\breve{D}_{z,L})} \tag{4.2.38}$$

Proof. Let us assume for the proof that $v < u$. We reverse the localisation of the excursions $\hat{Z}^u_z$ by noticing the following inclusion for $u' = u(1-\frac{\epsilon}{3})$ and $u'' = u'(1-\frac{\epsilon}{3})$

$$\mathcal{E}^v(\hat{Z}^u_z)^c \subset \mathcal{E}^v(Z^{u''}_z)^c \cup (U_z^{\frac{\epsilon}{10}, \frac{1}{2}\min(u,v)\mathrm{cap}(\breve{D}_0)})^c \cup (\mathcal{F}^{u'',u'}_z)^c \tag{4.2.39}$$

Then, recalling the remark directly preceding this proof and combining the bounds from Proposition 4.2.5 (for $\mathcal{C} = \{z\}$, and remarking that $(\sqrt{u(1+\epsilon)} - \sqrt{u})^2 = \frac{u}{4}\epsilon^2 + \mathcal{O}(\epsilon^4)$), Lemma 4.2.2 and display (2.6.13), we see the required result. □

Finally, we set the groundwork to extending the bound on $\mathcal{F}^{u,v}_{z,L}$ introduced in 4.2.2 from just one point $z \in \mathbb{L}$ to collections $\mathcal{C} \subset \mathbb{L}$.

Lemma 4.2.7. *Let $u, v, v' \geq 0, L \geq 1$ and consider a well-separated collection $\mathcal{C} \subset \mathbb{L}$. One has the following inclusion*

$$\bigcap_{z\in\mathcal{C}} \mathcal{E}^v(\bar{Z}^{v'}_z) \cap (\mathcal{F}^{u,v'}_z)^c \subset \begin{cases} \left\{h^u \leq v\,cap(\Sigma)\right\}, & v' < v < u \\ \left\{h^u \geq v\,cap(\Sigma)\right\}, & u < v < v' \end{cases} \tag{4.2.40}$$

Proof. First we show the result for $u < v < v'$, which essentially consists of the same steps as in [64, (4.9),(4.10)]. Assuming the LHS of (4.2.40), then for any $z \in \mathcal{C}$

$$\begin{aligned} \langle \mu^+_{\Sigma,u}, \int_0^\infty e_{\breve{D}_z}(X_s)\rangle &\overset{\breve{D}_z\subset\Sigma}{=} \langle \mu^+_{\breve{D}_z,u}, \int_0^\infty e_{\breve{D}_z}(X_s)\rangle \\ &\overset{(\mathcal{F}^{u,v'}_z)^c}{\geq} \sum_{1\leq l\leq v'\mathrm{cap}(D_0)} \int_0^{T_{\breve{U}_z}} e_{\breve{D}_z}(Z^l_s)ds \overset{\mathcal{E}^v(\bar{Z}^{v'}_z)}{\geq} v\mathrm{cap}(D_0) \end{aligned} \tag{4.2.41}$$

Thus, recalling the definition of h^u from (4.2.32) we obtain

$$h^u = \sum_{z\in\mathcal{C}} \frac{e_\Sigma(\breve{D}_z)}{\mathrm{cap}(\breve{D}_z)} \langle \mu^+_{\Sigma,u}, \int_0^\infty e_{\breve{D}_z}(X_s)\rangle \geq v\mathrm{cap}(\Sigma) \tag{4.2.42}$$

The case $v' < v < u$ follows identically, remarking that the limit of integration in the second line of (4.2.41) is $T_{\breve{U}_z}$ due to the fact that we are considering excursions $\bar{Z}_k \in W^+_{\breve{D}_z,\breve{U}_z}$ rather than a full path corresponding to the RI. □

4.3 Renormalisation

In essence, the concept behind renormalisation methods is to identify an event that, through iterative aggregation and rescaling, progressively enhances the accuracy of probabilistic estimates predicated by some initial approximation. Given this fundamental idea, we give a high-level overview of the specific approach herein via the example of a 'good' connectivity event $\mathcal{G}_{z,L}$ which takes place inside a box $D_{z,L}$. Starting with a path $\gamma \subset \mathcal{V}^u \cap B_{0,N}$ passing from 0 to ∂B_N, assuming a proper definition of $\mathcal{G}_{z,L}$, one can reformulate the occurrence of such a path as the event where a collection of

disjoint scale-$L \ll N$ boxes 'crossed' by the path and anchored at points in $\mathcal{C} \subset \mathbb{L}$ all exhibit the property $\mathcal{G}_{z,L}$. Such a paving of the path γ is called a coarse-graining, and allows us to find a representative set of boxes for any path, thereby reducing the complexity of the problem. According to the previously outlined blueprint, one then introduces a renormalization event $\mathscr{G}$, which guarantees that any coarse-graining has a minimum number of its constituent boxes satisfying $\mathcal{G}_{z,L}$. In essence, $\mathscr{G}$ facilitates a bootstrapping argument whereby the combination of good connectivity events at scale L implies the same criterion is also observed at the large scale N, i.e. $\mathcal{G}_{0,N}$ holds, but now with a more favorable probabilistic estimate. This improvement is achieved through a combinatorial approach, where the number of possible pavings competes against the probability that the criterion holds for any choice of coarse-graining.

In the following, the coarse-graining procedure that we present is an extension of one which has already been used for similar problems for GFF level set percolation in [28]. Indeed, we allow for coarsenings of crossings of Euclidean annuli, where the original source only necessitated crossings of l^∞ annuli, amounting to nested boxes. This modification requires an alternative choice of projection in the proof of Proposition 4.3.2, where we rely on energy arguments linked to the variational formulation of the capacity. Moreover, in the same proposition, we generalise the requirements on the scale L and separating distance K. The remaining subsections 4.3.2 and 4.3.3 take inspiration from [28], although require substantial modification due to differences between the RI and GFF in localising procedure. The manner in which the results are presented are also more general, albeit in the context of the RI, such that they are easily applied in a broad range of renormalising scenarios.

4.3.1 Coarse-graining

In light of the renormalisation description above, our first goal is to construct a class of admissible coarse-grainings where any individual paving will exceed a required capacity threshold while the entire collection maintains a good control on the total number of possibilities. To this effect, we recall the notation from Section 2.1 and for $x \in \mathbb{Z}^d$, $r \geq 0$ abbreviate the closed l^2 ball $B_r^2 = B^2(0,r) \subset \mathbb{Z}^d$. We remark that B_r^2 is simply connected, i.e. both B_r^2 and its complement are connected in $\mathbb{Z}^d$ for any $r \geq 0$, hence $\partial^{\text{ext}} B_r^2 = \partial^{\text{out}} B_r^2$. This can be seen by noticing that for any $x \in \mathbb{Z}^d \setminus \{0\}$ there are nearest neighbours y, z of x with $|y| < |x| < |z|$, so by starting with $x \in B_r^2$, we can connect to any other point $x' \in B_r^2$ via a nearest neighbour path by first visiting 0. Similarly, for $x \notin B_r^2$ we can first visit any point of $\partial^{\text{out}} B_r \subset (B_r^2)^c$ which is itself

connected. For $U \subset V \subset\subset \mathbb{Z}^d$ where V is simply connected, we say that

$$\text{a path } \gamma \text{ crosses } V \setminus U \text{ if it contains a point in both } U \text{ and } \partial V, \tag{4.3.1}$$

and we omit reference to U when $U = \{0\}$, i.e γ crosses $V \ni 0$ if it intersects 0 and ∂V. In the sequel, we consider coarse-grainings of paths crossing sets $\Lambda_N \subset \mathbb{Z}^d$ of the following form

$$\Lambda_N \in S_N := \{B_N^2, B_N^2 \setminus B_{\sigma N}^2, \sigma \in \Big(0, \frac{1}{3}\Big), B_{2N}^2 \setminus B_N^2, \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}\} \tag{4.3.2}$$

To allow for a uniform presentation it will be convenient to define $\sigma = \sigma(\Lambda_N)$ for all choices in (4.3.2) by setting $\sigma(B_N^2 \setminus B_{\sigma N}^2) = \sigma$ and $\sigma(\Lambda_N) = 0$ otherwise. Note that $\sigma \in [0, \frac{1}{3})$ and that for any choice of Λ_N, the quantity $(1-\sigma)N$ frequently encountered below roughly corresponds to the l^2 diameter (l^∞ diameter in the case of $\Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$) of Λ_N. With this groundwork, we are now ready to introduce the category of admissible coarse-grainings, which arise as a special case of well-separated subsets of $\mathbb{L}$ with additional useful properties.

Definition 4.3.1 (Γ-Admissibility)**.** *[28, Def. 4.2] Let $K \geq 100\sqrt{d}, L \geq 1, N \geq 10KL$ and consider an increasing entropy rate $\Gamma : [1, \infty) \to [0, \infty)$. Given $\Lambda_N \in \mathcal{S}_N$ and $a \in (0,1)$, a family $\mathcal{A} := \mathcal{A}_L^K(\Lambda_N)$ of collections $\mathcal{C} \subset \mathbb{L}$ is called (a, Γ)-admissible if*

(i) all $\mathcal{C} \in \mathcal{A}$ have equal cardinality $n := |\mathcal{C}|$ satisfying $n \in [\frac{a(1-\sigma)N}{u(KL)}, \frac{(1-\sigma)N}{u(KL)}]$, where $u(x) = x$ if $d = 3$ and $u(x) = x(\log x)^2$ if $d \geq 4$,

(ii) each of the collections $\mathcal{C} \in \mathcal{A}$ satisfy the mutual separation distance requirement described by (4.2.2), as well as the condition that $D_{z,L} \subset \Lambda_N$ for all $z \in \mathcal{C}$,

(iii) for any $*$-path γ crossing Λ_N, there exists $\mathcal{C} \in \mathcal{A}$ such that γ crosses $\tilde{D}_{z,L} \setminus C_{z,L}$ for all $z \in \mathcal{C}$,

(iv) $\log|\mathcal{A}| \leq \Gamma(N/L)$.

At this point, we can clearly specify the postulated claim regarding the existence of an admissible family of coarsenings $\mathcal{A}$ where for each $\mathcal{C} \in \mathcal{A}$, $\Sigma(\mathcal{C})$ exhibits a sufficiently large capacity without surpassing an entropy specified by some choice of Γ. The resulting lower bound on the aforementioned quantity turns out to be intimately related to the capacity of a tube T_N (see (2.3.1) together with the abbreviation below (2.3.2)), and thus requires extra care in order to preserve the correct constants in $d = 3$.

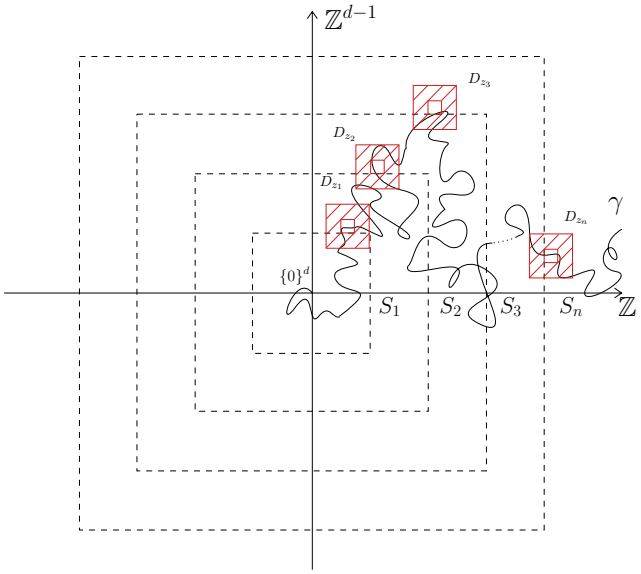


Fig. 4.1 For one possible crossing path γ of $B_n \setminus \{0\}$, the boxes D_{z_i}, $1 \le i \le n$ represent a coarse-graining belonging to a Γ-admissible family. See the proof of Proposition 4.3.2 for the S_i.

Proposition 4.3.2 (Admissible Coarsenings)**.** *There exist $C_2 \in [1, \infty)$ and $c_3 \in (0, \frac{1}{100d})$ such that, for all $K \ge 100$, $L \ge 1$, $N \ge c_3^{-1} h(KL)$ and $\Lambda_N \in S_N$ (see (4.3.2)), there exists a (c_3, Γ)-admissible collection $\mathcal{A} = \mathcal{A}_L^K(\Lambda_N)$ with the following properties.*

(i) If $d = 3$, one has for all $\rho \in (0,1)$ and with $\Gamma(x) = C_2 K^{-1} x \log ex$,

$$\liminf_{N\to\infty} \inf_{\substack{K\in[K_-,K_+],\\ L\in[L_-,L_+]}} \inf_{\mathcal{C}\in\mathcal{A}} \inf_{\substack{\tilde{\mathcal{C}}\subset\mathcal{C},\\ |\tilde{\mathcal{C}}|\ge(1-\rho)|\mathcal{C}|}} \frac{cap(\Sigma(\tilde{C}))}{(1-\frac{C_3}{K})\, cap(T_{(1-\sigma)N})} \ge 1-\rho, \tag{4.3.3}$$

where $C_3 \in [200, \infty)$, $\Sigma(\tilde{C})$ is as in (4.2.3) and $K_\pm = K_\pm(N)$, $L_\pm(N)$ satisfy

$$\begin{aligned} &K_-(N) = 100, \lim_N L_-(N) = \infty, \text{ and} \\ &\lim_N \left(\frac{\log(K_+(N)L_+(N))}{\log N} \right)^{\frac{1}{K_+(N)}} = 0, \quad L_+(N) \le c_3 N/K_+(N) \end{aligned} \tag{4.3.4}$$

(ii) If $d \ge 4$, choosing instead $\Gamma(x) = C_2 x$, the bound (4.3.3) remains valid with $L_-(N) = 1$, any fixed $K \ge 100$, $L_+(N) = c_3 N/K$ and $(1 - \frac{c_3}{K})$ replaced by some $c(K) \in (0, 1]$.

Proof. This proposition is based on [28, Proposition 4.3], except that we extend the set S_N to include l^2 balls. As such, for the proof we will only elaborate on the necessary modifications to the existing result. For example, the case for $d \ge 4$ is immediate by

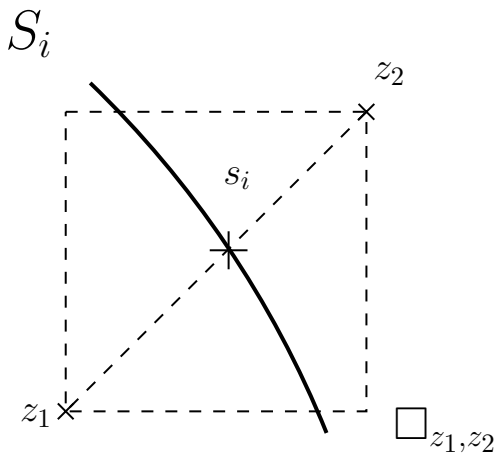


Fig. 4.2 A visualisation of $\square_{z_1,z_2}$ for $z_1, z_2 \in \gamma$.

using $B_{r/\sqrt{d}} \subset B_r^2 \subset B_r$, since in this instance the referenced work replaces $\lambda(K)$ with $\lambda(K)/\sqrt{d} \in (0,1]$ which simply corresponds to decreasing our choice of $c(K) \in (0,1]$. However, in the case of $d = 3$ we cannot afford to lose any precision in this way, so we must introduce a modification to the construction of our coarse graining. As in the citation, we restrict our proof to the case of $\Lambda_N = B_N^2$, although the other cases can be handled similarly to they way they were dealt with at the end of Section 4.1 in [28]. In the following, for a set $A \subset \mathbb{R}^d$, we shall denote the trace of A at scale L by

$$\mathbb{L}(A) \coloneqq \{x \in \mathbb{L} : C_{x,L} \cap A \neq \emptyset\} \tag{4.3.5}$$

Then, we proceed similarly to the referenced work by defining the concentric shells (now l^2 rather than l^∞) $S_i \coloneqq \partial_{\mathbb{R}^d} B^2_{3\sqrt{d}KLi}$ for $i = 1, \ldots, n \coloneqq \left\lfloor N/3\sqrt{d}KL \right\rfloor - 1$ ($n \geq 1$ by the assumptions) where $\partial_{\mathbb{R}^d}$ is the topological boundary and $B^2_{3\sqrt{d}KLi}$ is temporarily viewed as the Euclidean ball in $\mathbb{R}^d$. Remark that, for any $i \in \{1, \ldots n\}$, any $*$-path γ crossing Λ_N must intersect $C_{z,L}$ for some $z \in \mathbb{L}(S_i)$. Indeed, there must be two points $z_1 \overset{*}{\sim} z_2 \in \gamma$ whose interpolation (in $\mathbb{R}^d$) intersects S_i at some point s_i. Let $\square_{z_1,z_2}$ be the unique box $C_{x,1}$ such that $s_i \in \square_{z_1,z_2}$. By assumption, $s_i \in \square_{z_1,z_2} \subset \bigcup_{z \in \mathbb{L}(S_i)} C_{z,L}$, therefore at least one of z_1,z_2 must be in $\bigcup_{z \in \mathbb{L}(S_i)} C_{z,L}$. In light of the above guarantee, given any crossing γ of B_N^2, we can construct a sequence of points

$$\mathcal{C} = \{z_i\}_{1 \leq i \leq n} \subset \mathbb{L} \text{ where } C_{z_i} \cap S_i \neq \emptyset \text{ and } \gamma \text{ crosses } \tilde{D}_{z_i,L} \setminus C_{z_i,L}. \tag{4.3.6}$$

For concreteness, by considering the first exit times of γ from each of the sets $S_i + C_{0,L}$, we obtain z_i as the anchor point of the first such box $C_{z_i,L}$ which is crossed. Finally,

$\mathcal{A} = \mathcal{A}_L^K(B_N^2)$ is defined as the family of all possible collections $\mathcal{C}$ as described in (4.3.6) for any path γ crossing B_N^2. Any such $\mathcal{C}$ is well-separated by an application of the triangle inequality since $d(S_i, S_{i+1}) \geq 3KL$, and immediately satisfies conditions (i) and (iii) from the definition of Γ-admissibility by construction. Hence, we are left with showing point (iv) and the lower bound (4.3.3). An estimate for the cardinality of $\mathcal{A}$ can be computed by noticing that any C_z intesecting S_i is necessarily a subset of $A_i \coloneqq B^2_{4\sqrt{d}KLi} \setminus B^2_{2\sqrt{d}KLi}$. Hence, the number of coices for z_i is bounded by

$$\frac{\mathrm{Vol}_{\mathbb{R}^d}(A_i)}{\mathrm{Vol}_{\mathbb{R}^d}(C_{0,L})} \leq C\frac{(4\sqrt{d}KLi)^d - (2\sqrt{d}KLi)^d}{L^d} \leq C\Big(\frac{N}{L}\Big)^{d-1} \tag{4.3.7}$$

which immediately implies that $|\mathcal{A}| \leq \left(C\frac{N}{L}\right)^{n(d-1)}$, thereby proving point (iv) of Definition 4.3.1 with $\Gamma(x) = C_2K^{-1}x\log ex$ for a suitable choice of C_2. In order to complete the lower bound (4.3.3), we recall that the proof in [28] relied on a two-step approach. First, a comparison between $\mathrm{cap}(\Sigma(\tilde{C}))$ and a porus line $\tilde{T}_N$ is shown in Lemma 4.5, which relies on the variational formulation of capacity (2.2.25) and a projection argument. Then, the bound is completed by comparing the capacities of a porus and solid line in Lemma 4.6. From this description, it is clear that the only impact from our alternative construction is on the inequalities arising from projecting points of $\tilde{C}$ onto a specified collection of points on the horizontal axis. To be precise, this is the result of the more complicated geometry of $\mathbb{L}(S_i)$, where the constituent points can deviate from ∂S_i by a distance of up to $\sqrt{d}L$ (this does not occur in the case of the l^∞ norm as the constituent points of ∂B_{3KLi} are axis-aligned columns). As a result, the inequality (2.21) from [28] is replaced by

$$|\tau(z_i) - \tau(z_j)| \leq |z_i - z_j| + 2\sqrt{d}L\mathbb{1}_{i\neq j} \text{ for all } z_i, z_j \in \mathcal{C} \in \mathcal{A} \tag{4.3.8}$$

which does not affect the subsequent computations, as a similar discrepancy is already accounted for through the upper bound on κ in the explanation presented directly after (4.23).

Finally, we explain the new addition of the factor $(1 - \frac{C_3}{K})$ and the first condition in the second line of (4.3.4). The former follows from display (4.24) in [28] and the latter is a result of the joint-limit over K and L by inspecting the proof of Lemma 4.6 therein. In particular, we notice that the inequality in the last display of the proof of Lemma 4.6 is obtained via an application of Lemma 2.4, which presents an upper bound of the form $C(\log(1 + d(x, \tilde{T}_N))/\log N)^{c(\gamma)}$, and requires that this bound converges to 0 as $N \to \infty$.

By delving into the proof of Lemma 2.4, we see that we can replace $c(\gamma) = c'\gamma$ for some $c' \in (0,1)$. Moreover, in the context of [28], $\tilde{T}_N \subset T_N$ is a perforated line and x is a point a l^∞ distance of at least CKL away from $\tilde{T}_N$, so we yield the required condition from (4.3.4). □

4.3.2 The Event $\mathscr{G}$

With the foundations of admissible coarse-grainings established, we are now prepared to introduce the key event $\mathscr{G}$ that will play a crucial role in refining our subsequent problems through rescaling. This event will guarantee the simultaneous occurrence of two additional families of events, $\mathcal{F} \coloneqq \{\mathcal{F}_{z,L} : z \in \mathbb{L}\}$ and $\mathcal{G} \coloneqq \{\mathcal{G}_{z,L} : z \in \mathbb{L}\}$ which will be specified shortly. The families $\mathcal{F}$ and $\mathcal{G}$ loosely correspond to groups of excursion counting events and 'good' connectivity events respectively, where 'good' indicates a high likelihood in the regime of choice. In fact, the specific choice for the family $\mathcal{G}$ is flexible, and will allow us to tune our approach to tackle the various problems of interest. Now, letting Λ_N be one of the choices from (4.3.2), for any $\rho \in (0,1)$ we define

$$\mathscr{G} = \mathscr{G}(\Lambda_N, \mathcal{G}, \mathcal{F}; \rho) \coloneqq \bigcap_{\mathcal{C} \in \mathcal{A}} \bigcup_{\substack{\tilde{\mathcal{C}} \subset \mathcal{C}, \\ |\tilde{\mathcal{C}}| \geq \rho |\mathcal{C}|}} \bigcap_{z \in \tilde{\mathcal{C}}} (\mathcal{F}_{z,L} \cap \mathcal{G}_{z,L}), \tag{4.3.9}$$

under $\mathbb{P}$, where $\mathcal{A} = \mathcal{A}_L^K(\Lambda_N)$ is a family of coarse-grainings as ensured by Proposition 4.3.2, implicitly requiring $K \geq 100$, $L \geq 1$ and $N \geq c_3^{-1} h(KL)$. In this section, we aim to show that $\mathscr{G}$ is a 'good' event, i.e. it takes place with high probability. To this end, we first specify the family $\mathcal{F} = \{\mathcal{F}_{z,L}\}_{z \in \mathbb{L}}$ which effectively extends the definition of $\mathcal{F}_{z,L}^{u,v}$ to incorporate a *set* of excursion level pairs $(u_i, v_i)_{1 \leq i \leq k}$, $u_i, v_i > 0$. Given $L \geq 1$, $z \in \mathbb{L}$, $k \in \mathbb{N}$ and denoting $\boldsymbol{u} = \{u_i\}_{1 \leq i \leq k}$, $\boldsymbol{v} = \{v_i\}_{1 \leq i \leq k}$, we let

$$\mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}} \coloneqq \bigcap_{i=1}^{k} \mathcal{F}_{z,L}^{u_i,v_i}. \tag{4.3.10}$$

With these events at hand, they become the constituents $\mathcal{F}_{z,L}$ of the family $\mathcal{F} = \mathcal{F}_L^{\boldsymbol{u},\boldsymbol{v}} = \{\mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}} : z \in \mathbb{L}\}$. Next, recall the i.i.d excursions $\hat{Z}^{\breve{D}_z, \breve{U}_z} = \{\hat{Z}_k^{\breve{D}_z, \breve{U}_z}\}_{k \geq 1}$ governed by the probability distribution $\hat{\mathbb{P}}_z = \hat{\mathbb{P}}_{\{z\}}$ introduced above (4.2.10) and endow this probability space with an independent collection of i.i.d. uniform random variables $\mathsf{U} = \{\mathsf{U}_x : x \in \mathbb{Z}^d\}$. Regarding the events forming the family $\mathcal{G} = \mathcal{G}_L = \{\mathcal{G}_{z,L} : z \in \mathbb{L}\}$, we assume that there exists an event $\hat{G}_{z,L}$ for each $z \in \mathbb{L}$ which is measurable relative

to $\hat{Z}^{\check{D}_z,\check{U}_z}$, U, and, for some $\epsilon_L \in (0,1)$ and integer $m_L > 0$, satisfies the inclusion

$$(\hat{\mathcal{G}}_{z,L} \cap \mathrm{Incl}_z^{\epsilon_L,m_L}) \subset \mathcal{G}_{z,L} \text{ for any coupling } \mathbb{Q} \text{ of } \mathbb{P} \text{ and } \hat{\mathbb{P}}_z, \tag{4.3.11}$$

recalling that the event $\mathrm{Incl}_z^{\epsilon_L,m_L}$ from (2.6.6), (4.2.18) couples deterministic counts of excursions in $W^+_{D_z,\check{U}_z}$ with precision ϵ_L and a minimum threshold m_L. With this groundwork in place, we are now ready for the

Proposition 4.3.3 (Estimate for $\mathscr{G}^c$)**.** *For any choice of $\Lambda_N \in S_N$ (see (4.3.2)), $\rho \in (0, \frac{1}{2}]$, an integer $k \geq 1$ and families $\mathcal{F} = \mathcal{F}_L^{\boldsymbol{u},\boldsymbol{v}}$, $\mathcal{G} = \mathcal{G}_L$ as above, the following hold. If, for some $K_0 \geq 1$, $\beta' \in (0,1)$ and $\theta \in (0,\infty)$, $\sup_{z\in\mathbb{L}} \hat{\mathbb{P}}_z[\hat{G}^c_{z,L}] \leq p_L$ for all $K \geq K_0$ and $L \geq 1$ with*

$$\limsup_{L\to\infty} L^{-\beta'} \log p_L < 0 \text{ and } \limsup_{L\to\infty} \frac{(\log L)^\theta}{L} \log \mathbb{Q}_0[(U_0^{\epsilon_L,m_L})^c] = -\infty \tag{4.3.12}$$

(see (2.6.12) and the remark above (4.2.18) for the definition of $U_z^{\epsilon_L,m_L}$, as well as the discussion below (4.2.9) for $\mathbb{Q}_0 = \mathbb{Q}_{\{0\}}$), then

(i) for $d = 3$, there exists $\alpha = \alpha(\beta') \in (0,\infty)$ such that with $L(N) = \lfloor (\log N)^\alpha \rfloor$, one has for any $N \geq C(\boldsymbol{u},\boldsymbol{v},\rho,K_0,\{p_L,\epsilon_L,m_L\})$,

$$\begin{aligned}\sup_{K\in[K_-,K_+]} \Big(1-\frac{C_3}{K}\Big)^{-1} \log \mathbb{P}[\mathscr{G}^c(\Lambda_N,\mathcal{G}_{L(N)},\mathcal{F}^{\boldsymbol{u},\boldsymbol{v}}_{L(N)};\rho)] \\ \leq -(1-\sigma)(1-\rho)\frac{\pi}{3k}\Big[\min_{1\leq i\leq k}(\sqrt{u_i}-\sqrt{v_i})^2\Big]\frac{N}{(\log N)^\beta}\end{aligned} \tag{4.3.13}$$

for some $\beta = \beta(\beta') \in (1,\infty)$ if $\beta' \leq \frac{1}{2}$ and $\beta = 1$ otherwise, where $K_- = \frac{3C_5}{\epsilon_{L(N)}} \vee C(\boldsymbol{u},\boldsymbol{v},\beta') \vee K_0$, $K_+ = \sqrt{\log\log e^2 N}$ and the constants C_3, C_5 are from Proposition 4.3.2 and 4.2.1 respectively.

(ii) For $d \geq 4$ and $\epsilon_L = \epsilon \in (0,1)$ uniformly in L, we have for any fixed $L \geq C(\boldsymbol{u},\boldsymbol{v},\epsilon,L_0)$, $K = C(\boldsymbol{u},\boldsymbol{v},\epsilon) \vee K_0$ and $N \geq C(\boldsymbol{u},\boldsymbol{v},\epsilon,K_0,L_0)$,

$$N^{-1} \log \mathbb{P}[\mathscr{G}^c(\Lambda_N,\mathcal{G}_L,\mathcal{F}_L^{\boldsymbol{u},\boldsymbol{v}};\frac{1}{2})] \leq -c(\boldsymbol{u},\boldsymbol{v},\epsilon,K_0,L_0)(<0). \tag{4.3.14}$$

Proof. Let us deal with the cases $d = 3$ and $d = 4$ simultaneously. Throughout the proof, assume that $K \geq 100$, $L \geq 1$ and $N \geq c_3^{-1} h(KL)$ so as to satisfy the conditions of Proposition 4.3.2, thereby guaranteeing the existence of a (c_3,Γ)-admissible family of coarse-grainings $\mathcal{A}_L^K(\Lambda_N)$ such that the event $\mathscr{G}(\Lambda_N,\mathcal{G},\mathcal{F};\rho)$ is well-defined. Then,

by proceeding with a union bound, condition *(iv)* of Γ-admissibility yields

$$\log \mathbb{P}[\mathscr{G}^c(\Lambda_N, \mathcal{G}_L, \mathcal{F}_L^{\boldsymbol{u},\boldsymbol{v}}; \rho)] \leq \Gamma(N/L) + \sup_{\mathcal{C}\in\mathcal{A}} \log \mathbb{P}\Big[\Big(\bigcup_{\substack{\tilde{\mathcal{C}}\subset\mathcal{C}:\\ |\tilde{\mathcal{C}}|\geq\rho|\mathcal{C}|}} \bigcap_{z\in\tilde{\mathcal{C}}} \mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}} \cap \mathcal{G}_{z,L} \Big)^c \Big], \quad (4.3.15)$$

thus, deferring the complexity factor to the end of the proof, we first deal with the second quantity on the RHS of (4.3.15). To this end, keeping in mind the definition of $\mathcal{F}_{z,L}$ from (4.3.10), we justify the following inclusion

$$\bigcap_{\substack{\tilde{\mathcal{C}}\subset\mathcal{C}:\\ |\tilde{\mathcal{C}}|\geq\rho|\mathcal{C}|}} \bigcup_{z\in\tilde{\mathcal{C}}} ((\bar{\mathcal{F}}_{z,L}^{\boldsymbol{u},\boldsymbol{v}})^c \cap \mathcal{G}_{z,L}^c) \subset E_{\mathcal{G}} \cup \bigcup_{1\leq i\leq k} E_{\mathcal{F}}^i \quad (4.3.16)$$

where we denote the events

$$\begin{aligned} E_{\mathcal{G}} &:= \{\exists \mathcal{C}_{\mathcal{G}} \subset \mathcal{C} : |\mathcal{C}_{\mathcal{G}}| = \lfloor \rho|\mathcal{C}| \rfloor \text{ , and } \forall z \in \mathcal{C}_{\mathcal{G}}, \mathcal{G}_{z,L}^c \text{ holds}\}, \\ E_{\mathcal{F}}^i &:= \left\{ \exists \mathcal{C}_{\mathcal{F}}^i \subset \mathcal{C} : |\mathcal{C}_{\mathcal{F}}^i| = \left\lfloor \frac{1-2\rho}{k} |\mathcal{C}| \right\rfloor \text{ st } \forall z \in \mathcal{C}_{\mathcal{F}}^i, (\mathcal{F}_{z,L}^{u_i,v_i})^c \text{ holds} \right\}, \quad 1 \leq i \leq k \end{aligned} \quad (4.3.17)$$

Indeed, the inclusion (4.3.16) follows by a Dirichlet-type argument. By contradiction, if the right-hand side were false, then $\mathcal{G}_{z,L}^c$ must hold for fewer than $\lfloor \rho|\mathcal{C}| \rfloor$ points of $\mathcal{C}$ and each $(\mathcal{F}_{z,L}^{u_i,v_i})^c$ must hold for fewer than $\left\lfloor \frac{1-2\rho}{k}|\mathcal{C}| \right\rfloor$ points of $\mathcal{C}$. In total, $(\mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}})^c \cup \mathcal{G}_{z,L}^c$ must hold for strictly fewer than $\lfloor \rho|\mathcal{C}| \rfloor + k \left\lfloor \frac{1-2\rho}{k}|\mathcal{C}| \right\rfloor \leq \lfloor (1-\rho)|C| \rfloor$ points of $\mathcal{C}$, meaning that there exist at least $\rho|\mathcal{C}| + 1$ points of $\mathcal{C}$ satisfying $\mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}} \cap \mathcal{G}_{z,L}$, contradicting the LHS of (4.3.16). Such a decomposition will be used repeatedly throughout this proof; therefore, the details will be omitted from this point onward.

Let us first control $\mathbb{P}[E_{\mathcal{G}}]$, as the approach required for $E_{\mathcal{F}}^i$, $1 \leq i \leq k$ is more intricate and requires localisation ideas developed in Section 4.2.1, together with the estimates on $\mathcal{F}_{z,L}^{u,v}$ via the observable h^u from Section 4.2.2. On the other hand, for $E_{\mathcal{G}}$ the comparatively broad properties of $\mathcal{G}_{z,L}$ imposed around (4.3.11) in conjunction with the assumption (4.3.12) yield the required implications in a relatively straightforward manner. In particular, line (4.3.11) itself implies $\mathcal{G}_{z,L}^c \subset \hat{\mathcal{G}}_{z,L}^c \cup (\mathrm{Incl}_{z,L}^{\epsilon_L,m_L})^c$ under $\mathbb{Q}_{\mathcal{C}}$. Moreover, as soon as $K \geq 3C_5/\epsilon_L \vee K_0$, (4.2.18) implies that $U_{z,L}^{\epsilon_L,m_L} \subset \mathrm{Incl}_{z,L}^{\epsilon_L,m_L}$ for any $z \in \mathcal{C}$ under $\mathbb{Q}_{\mathcal{C}}$, which collectively shows

$$\mathcal{G}_{z,L}^c \subset \hat{\mathcal{G}}_{z,L}^c \cup (U_{z,L}^{\epsilon_L,m_L})^c, \ \forall z \in \mathcal{C}, \text{ under } \mathbb{Q}_{\mathcal{C}}. \quad (4.3.18)$$

Thus, by applying a union bound over the choices $\mathcal{C}_{\mathcal{G}} \subset \mathcal{C}$ indicated in the event $E_{\mathcal{G}}$ followed by a similar Dirichlet argument as above, one sees

$$\begin{aligned}\mathbb{P}[E_{\mathcal{G}}] &\leq \binom{|\mathcal{C}|}{\lfloor \rho|\mathcal{C}|\rfloor} \sup_{\substack{\mathcal{C}_{\mathcal{G}}\subset\mathcal{C}:\\ |\mathcal{C}_{\mathcal{G}}|=\lfloor\rho|\mathcal{C}|\rfloor}} \mathbb{Q}_{\mathcal{C}}\Big[\bigcap_{z\in\mathcal{C}_{\mathcal{G}}} \hat{\mathcal{G}}^c_{z,L} \cup (U^{\epsilon_L,m_L}_{z,L})^c\Big] \\ &\leq \binom{|\mathcal{C}|}{\lfloor \rho|\mathcal{C}|\rfloor}\binom{\lfloor\rho|\mathcal{C}|\rfloor}{\lfloor\rho|\mathcal{C}|/2\rfloor}\Bigg\{\sup_{\substack{\mathcal{C}'\subset\mathcal{C}:\\ |\mathcal{C}'|=\lfloor\rho|\mathcal{C}|/2\rfloor}} \mathbb{Q}_{\mathcal{C}}\Big[\bigcap_{z\in\mathcal{C}'}\hat{\mathcal{G}}^c_{z,L}\Big] + \sup_{\substack{\mathcal{C}'\subset\mathcal{C}:\\ |\mathcal{C}'|=\lfloor\rho|\mathcal{C}|/2\rfloor}} \mathbb{Q}_{\mathcal{C}}\Big[\bigcap_{z\in\mathcal{C}'}(U^{\epsilon_L,m_L}_{z,L})^c\Big]\Bigg\}\end{aligned} \tag{4.3.19}$$

Then, owing to the classical bounds on combinations $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ together with the independence of the $U^{\epsilon_L,m_L}_{z,L}$ across $z \in \mathbb{L}$ and the measurability of $\hat{G}_{z,L}$ with respect to independent excursions, the above can be evaluated as

$$(4.3.19) \leq e^{C\rho|\mathcal{C}||\log\rho|}\Big\{p_L^{\lfloor\rho|\mathcal{C}|/2\rfloor} + \mathbb{Q}_z[(U^{\epsilon_L,m_L}_{z,L})^c]^{\lfloor\rho|\mathcal{C}|/2\rfloor}\Big\} \tag{4.3.20}$$

Hence, all in all, by applying assumption (4.3.12) in line (4.3.20), keeping in mind that $\beta' \in (0,1)$, we have for $L \geq L_0 \vee C(\beta')$ and $K \geq K_0 \vee 3C_5/\epsilon_L$,

$$\mathbb{Q}_{\mathcal{C}}[E_{\mathcal{G}}] \leq \exp\{-\rho|\mathcal{C}|(cL^{\beta'} - C|\log\rho|)\} \tag{4.3.21}$$

Now, we turn to handling the events $E^i_{\mathcal{F}}$. By applying a union bound and taking into consideration the combinatorial complexity of choosing $\mathcal{C}^i_{\mathcal{F}}$ from $\mathcal{C}$, we have

$$\mathbb{P}[E^i_{\mathcal{F}}] \leq e^{C|\mathcal{C}|}\sup_{\mathcal{C}^i_{\mathcal{F}}\subset\mathcal{C}} \mathbb{P}\Big[\bigcap_{z\in\mathcal{C}^i_{\mathcal{F}}}(\mathcal{F}^{u_i,v_i}_{z,L})^c\Big] \tag{4.3.22}$$

To handle the probability in (4.3.22), we let $v' \in (u,v)(=(u\wedge v, u\vee v))$ and decompose $(\mathcal{F}^{u_i,v_i}_{z,L})^c = \{(\mathcal{F}^{u_i,v_i}_{z,L})^c \cap \mathcal{E}^{v'_i}(\bar{Z}^{v_i}_z)\} \cup \mathcal{E}^{v'_i}(\bar{Z}^{v_i}_z)^c$, attributing $2\rho|\mathcal{C}|/k$ points of $\mathcal{C}^i_{\mathcal{F}}$ to the second event in the union in a similar manner to before. Then, we use the relation (4.2.40) on the remaining points (of which there are $(1-4\rho)|\mathcal{C}|/k$) satisfying $(\mathcal{F}^{u_i,v_i}_{z,L})^c \cap$

$\mathcal{E}^{v_i'}(\bar{Z}_z^{v_i})$. Uniformly over choices of $\mathcal{C}' \subset \mathbb{L}$ as above, we have for $u_i < v_i' < v_i$

$$\begin{aligned}\mathbb{P}\Big[\bigcap_{z\in\mathcal{C}_{\mathcal{F}}^i}(\mathcal{F}_{z,L}^{u_i,v_i})^c\Big] \\ \leq (c\rho^{-1})^{2\rho|\mathcal{C}|/k}\Big\{\sup_{\substack{\mathcal{C}_1\subset\mathcal{C}_{\mathcal{F}}^i:\\ |\mathcal{C}_1|=\lfloor(1-4\rho)|\mathcal{C}|/k\rfloor}}\mathbb{Q}_{\mathcal{C}_{\mathcal{F}}^i}[h^{u_i}(\mathcal{C}_1)\geq v_i'\mathrm{cap}(\Sigma(\mathcal{C}_1))] \\ + \sup_{\substack{\mathcal{C}_2\subset\mathcal{C}_{\mathcal{F}}^i:\\ |\mathcal{C}_2|=\lfloor 2\rho|\mathcal{C}|/k\rfloor}}\mathbb{Q}_{\mathcal{C}_{\mathcal{F}}^i}\Big[\bigcap_{z\in\mathcal{C}_2}\mathcal{E}^{v_i'}(\bar{Z}_z^{v_i})^c\Big]\Big\}\end{aligned} \tag{4.3.23}$$

and an identical approach can be carried out for $v_i < v_i' < u_i$ by swapping the direction of the inequality in the first event on the RHS. Continuing with the case $u_i < v_i' < v_i$, the first probability on the RHS of (4.3.23) is immediately bounded by an application of Proposition 4.2.5 to get,

$$\mathbb{Q}_{\mathcal{C}'}[h^{u_i}(\mathcal{C}_1)\geq v_i'\mathrm{cap}(\Sigma(\mathcal{C}_1))] \leq \exp\Big\{-\Big(\sqrt{\frac{v_i'}{1+\tilde{\epsilon}}}-\sqrt{u_i}\Big)^2\mathrm{cap}(\Sigma(\mathcal{C}_1))\Big\}, \tag{4.3.24}$$

where a similar bound is obtained by using (4.2.36) instead of (4.2.35), and $\tilde{\epsilon}\in(0,1)$ will be determined shortly. Therefore, we endeavour to describe the necessary steps for the second probability. We remark that the set $\bar{Z}_z^{v_i}$ (see (4.2.24)) consists of a deterministic number of RI excursions, which we aim to decouple. This requires the introduction of localising events $U_{z,L}^{\tilde{\epsilon},\tilde{m}}$ with a careful choice for the precision $\tilde{\epsilon}$ depending on u_i, v_i for each $1\leq i\leq k$ (to ensure uniformity across the $E_{\mathcal{F}}^i$). In particular, for a given $\delta\in(0,1)$ we define the parameters

$$\tilde{\epsilon}=\frac{\delta}{100}\min_{1\leq i\leq k}|u_i-v_i|, \quad \tilde{m}=2^{-1}\min\{u_i,v_i : 1\leq i\leq k\}\mathrm{cap}(D_0) \tag{4.3.25}$$

and, since $v_i'\in(u_i,v_i)$ was arbitrary, select

$$v_i'=v_i(1+10\tilde{\epsilon}(\mathbb{1}_{u_i>v_i}-\mathbb{1}_{u_i<v_i})) \tag{4.3.26}$$

Let us proceed with the remaining steps with the assumption that $u_i<v_i$, as the other case is identical. Then, using the Dirichlet approach on the decomposition $(\mathcal{E}^{v_i'}(\bar{Z}_z^{v_i})^c\cap\mathrm{Incl}_{z,L}^{\tilde{\epsilon},\tilde{m}})\cup(\mathrm{Incl}_{z,L}^{\tilde{\epsilon},\tilde{m}})^c$ with $\rho|\mathcal{C}|/k$ points designated to each constituent event of the union and relying on the inclusions

$$\mathcal{E}^{v_i'}(\bar{Z}_z^{v_i})^c\cap\mathrm{Incl}_{z,L}^{\tilde{\epsilon},\tilde{m}}\subset\mathcal{E}^{v_i'}(\hat{Z}_z^{(1-5\tilde{\epsilon})v_i})^c \tag{4.3.27}$$

and (4.2.18) for $K \geq 3C_5/\tilde{\epsilon}$, we have by independence and translation invariance of the collections $U_{z,L}^{\tilde{\epsilon},\tilde{m}}$ and $\hat{Z}_z^{(1-5\tilde{\epsilon})v_i}$ across well-separated z

$$\begin{aligned}\mathbb{Q}_{\mathcal{C}'}\Big[\bigcap_{z\in\mathcal{C}_2}\mathcal{E}^{v_i'}(\bar{Z}_z^{v_i})^c\Big] &\leq e^{c\rho|\mathcal{C}|/k}\Big\{\mathbb{Q}_0[\mathcal{E}^{v_i'}(\hat{Z}_0^{(1-5\tilde{\epsilon})v_i})^c]^{\rho|\mathcal{C}|/k}+\mathbb{Q}_0[(U_{0,L}^{\tilde{\epsilon},\tilde{m}})^c]^{\rho|\mathcal{C}|/k}\Big\} \\ &\leq \exp\{-ck^{-1}\rho|C|v_i\tilde{\epsilon}^2\mathrm{cap}(\breve{D}_{0,L})\}\end{aligned} \tag{4.3.28}$$

where we have used Lemma 4.2.6, the bound (2.6.13) and the independent choice of $\tilde{\epsilon}$ from L to determine the leading order term. Altogether, combining the bounds (4.3.28) and (4.3.24) subsequently feeding them into (4.3.23) yields for all $L \geq C(u_i, v_i, \delta)$ (recall $\delta \in (0,1)$ is implicit in $\tilde{\epsilon}$) and $K \geq C\tilde{\epsilon}^{-1}$

$$\begin{aligned}&\mathbb{P}\Big[\bigcap_{z\in\mathcal{C}_{\mathcal{F}}^i}(\mathcal{F}_{z,L}^{u_i,v_i})^c\Big] \\ &\leq (C\rho^{-1})^{k^{-1}\rho|\mathcal{C}|}\Bigg(\exp\{-ck^{-1}\rho|\mathcal{C}|v_i\tilde{\epsilon}^2\mathrm{cap}(\breve{D}_{0,L})\} \\ &\qquad + \sup_{\substack{\mathcal{C}_1\subset\mathcal{C}_{\mathcal{F}}^i:\\ |\mathcal{C}_1|=\lfloor(1-4\rho)|\mathcal{C}|/k\rfloor}}\exp\Big\{-\Big(\sqrt{(1-11\tilde{\epsilon})v_i}-\sqrt{u_i}\Big)^2\mathrm{cap}(\Sigma(\mathcal{C}_1))\Big\}\Bigg)\end{aligned} \tag{4.3.29}$$

Now, we lay out the steps required to conclude the proof in the case $d = 3$ - the remaining case of $d \geq 4$ is less intricate and will be discussed immediately after. We collect the intermediary steps presented above with an aim to apply Proposition 4.3.2 to observe the required asymptotic behaviour. Our discussion starts off once again with $\mathbb{P}[E_{\mathcal{G}}]$. By setting $L(N) = \lfloor(\log N)^\alpha\rfloor$ as in the assumption of part *(i)* of the Proposition, and letting $N \geq C(\rho, \beta')$ so as to ensure $cL^{\beta'} - C|\log\rho| \geq c'L^{\beta'}$, then the fact that $\mathcal{C}$ satisfies condition *(i)* of Definition 4.3.1 passed into (4.3.21) implies

$$\log\mathbb{Q}_{\mathcal{C}}[E_{\mathcal{G}}] \leq -\frac{c\rho(1-\sigma)N}{K(\log N)^{\alpha(1-\beta')}}, \tag{4.3.30}$$

where $\alpha > 0$ is yet to be determined. Next, we handle the $\mathbb{P}[E_{\mathcal{F}}^i]$, which we will see to comprise the leading asymptotic rate of decay. To start off with, by examining the first term in the parentheses of (4.3.29), and relying on the fact $\mathrm{cap}(\breve{D}_{0,L}) \geq cL^{d-2} \overset{d=3}{=} cL$, we see that it decays faster than the bound on $\mathbb{Q}_{\mathcal{C}}[E_{\mathcal{G}}]$ derived in (4.3.30) (by using the same steps therein) and so becomes negligible for any $i \in \{1,\dots,k\}$ after assembling as soon as $N \geq C(\beta', \rho, k, \delta, \boldsymbol{u}, \boldsymbol{v}, L_0)$ and $K \geq K_0 \vee C\tilde{\epsilon}^{-1}$. Thus, the second term appearing on the right-hand side of (4.3.29) becomes the principal term, which we

deal with by remarking that (4.3.4) is satisfied for $L = L(N)(= L_- = L_+)$ with $K_+ = \sqrt{\log\log e^2 N}$ and applying (4.3.3). Overall, this yields, for all $\delta \in (0,1)$, $N \geq C(\beta', \rho, k, \delta, \boldsymbol{u}, \boldsymbol{v}, L_0)$ and $K_0 \vee C\tilde{\epsilon}^{-1} \leq K \leq K_+$

$$\log \mathbb{Q}_{\mathcal{C}}[E^i_{\mathcal{F}}] \leq -\frac{(1-\frac{\delta}{2})\gamma N}{\log N}, \text{ for each } 1 \leq i \leq k, \tag{4.3.31}$$

where

$$\gamma = \frac{\pi}{3k}(1-C\rho)(1-\sigma)\min_{1\leq i\leq k}(\sqrt{v_i}-\sqrt{u_i})^2. \tag{4.3.32}$$

Returning to (4.3.15) in light of the inclusion (4.3.16), we now handle $\Gamma(N/L(N))$ by recalling property *(i)* of the admissible coarsenings Proposition 4.3.2 which implies $\Gamma(N/L) \leq C_2 N(\log N)/KL$. Together with the bounds (4.3.30), (4.3.31), this argument shows that for all $N \geq C(\beta', \rho, k, \delta, \boldsymbol{u}, \boldsymbol{v}, L_0)$ and $K \geq K_0 \vee C\tilde{\epsilon}^{-1} \vee 3C_5(\epsilon_{L(N)})^{-1}$,

$$N^{-1}\log \mathbb{P}[\mathscr{G}^c] \leq \frac{C_2}{K_-(N)(\log N)^{\alpha-1}} - \left(\frac{c}{K_+(N)(\log N)^{\alpha(1-\beta')}} \wedge \frac{(1-\frac{\delta}{2})\gamma}{\log N}\right) \tag{4.3.33}$$

In order for the term in parenthesis to be larger than the first term on the right of (4.3.33), and because $K_- \geq 1$ whereas $K_+ \leq C \vee \sqrt{\log\log N}$, it is sufficient that $\alpha - 1 > \alpha(1-\beta') \vee 1$. In particular this requires $\alpha > 2$. There are now two cases to consider, depending on the value of $\beta' \in (0,1)$. If $0 < \beta' \leq \frac{1}{2}$, one simply picks any $\alpha > \frac{1}{\beta'} (> 2)$, for instance $\alpha = \frac{1}{\beta'} + 1$. Since for such β', one has $\alpha(1-\beta') \geq 1$, the right-hand side of (4.3.33) is thus bounded by $-c/(2K_+(N)(\log N)^{\alpha(1-\beta')})$ for sufficiently large N. By choosing any β ever so slightly larger than $\alpha(1-\beta')$, one can easily absorb for sufficiently large N the factor $1/2K_+(N)$ and produce instead the desired pre-factor, leading to (4.3.13) in this case. If instead $\frac{1}{2} < \beta' < 1$, one picks a value of α satisfying $2 < \alpha < (1-\beta')^{-1}$, for instance, $\alpha = 1 + \frac{1}{2}(1-\beta')^{-1}$. The decay in (4.3.33) is then governed by the second term in parentheses (since now $\alpha(1-\beta') < 1$), and for suitably large N one ensures that the right-hand side of (4.3.33) is at most $\frac{(1-\delta)\gamma}{\log N}$, yielding (4.3.13).

The case $d \geq 4$ is simpler, notably because the complexity $\Gamma(N/L) \leq CN/L$ never requires fine-tuning of L beyond choosing L large (in a manner depending on the various parameters). For instance, in the case of $\mathbb{Q}_{\mathcal{C}}[E_{\mathcal{G}}]$, recalling that $|\mathcal{C}| \geq cN/L\log(KL)^2$ from ((i)), one obtains a bound on $\log \mathbb{Q}_{\mathcal{C}}[E_{\mathcal{G}}]$ effectively of the form $c\frac{N}{L}\frac{L^{\beta'}}{\log(KL)^2}$ and the second fraction is more than enough for large L to produce a decay of exponential order

N/L with arbitrary large pre-factor. The case of $\mathbb{Q}_{\mathcal{C}}[E^i_{\mathcal{F}}]$ is handled similarly to before, using that the capacity of a box of side-length L grows at least quadratically when $d > 3$ to handle the first term in (4.3.29), and for the second appealing to item *(ii)* of Proposition 4.3.2. Notice in particular that Proposition 4.3.2 yields an exponential decay in N rather than N/L in this case. These considerations yield (4.3.14). □

4.3.3 Bootstrapping

Previously, we introduced the family $\mathcal{G} = \mathcal{G}_L = \{\mathcal{G}_{z,L} : z \in \mathbb{L}\}$ in rather abstract terms according to certain inclusion and measurability conditions (see (4.3.11) and the preceding paragraph). In this section, we elaborate on the kind of events that we hope to bootstrap, keeping in mind that these will be sufficiently flexible for use both in the sub- and supercritical regimes. Moreover, we introduce a new 'base' scale $L_0 \geq 1$ with $(N \gg)L \gg L_0$ which will remain unchanged throughout the bootstrapping procedure. For the remainder of this section, the three concurrent scales L_0, L, N and the scaling factor K are tacitly assumed to satisfy

$$K \geq 100, N \geq C_4 c_3^{-1} 10^d h(KL) \text{ and } L \geq 100 L_0, \tag{4.3.34}$$

so as to keep in line with the conditions for the various results from Sections §4.3.1 and §4.3.2. Consistently with the scale L, we denote $\mathbb{L}_0 = L_0\mathbb{Z}^d$, and for a set $A \subset \mathbb{Z}^d$ we define

$$\mathbb{L}_0(A) = \{z \in \mathbb{L}_0 : C_{z,L_0} \cap A \neq \emptyset\} \tag{4.3.35}$$

which, for a path γ will be abbreviated as $\mathbb{L}_0(\gamma) = \mathbb{L}_0(\text{Range}(\gamma))$. In what follows, the adaptability of $\mathcal{G}$ will be articulated through the families of events $V = \{V_z : z \in \mathbb{L}\}$, $W = \{W_{z,y} : z \in \mathbb{L}, y \in \mathbb{L}_0\}$ and the family of random finite subsets $\mathscr{C} = \{\mathscr{C}_z \subset \mathbb{Z}^d : z \in \mathbb{L}$, each of which will be selected according to the specific regime we are working in to achieve the desired outcome.

Definition 4.3.4 (The events $\mathcal{G}_{z,L}$)**.** *For $a \geq 0$ and $(V, W, \mathscr{C})$ as above, let $\mathcal{G}_z = \mathcal{G}_{z,L}(= \mathcal{G}_{z,L,L_0})$ be given by*

$$\mathcal{G}_z(V, W, \mathscr{C}; a) = G_z(W, \mathscr{C}; a) \cap V_z, \tag{4.3.36}$$

where the event $G_z = G_{z,L}$ is defined as

$$G_{z,L} \coloneqq \left\{ \begin{array}{r} \text{for any crossing } \gamma \text{ of } \tilde{D}_z \setminus \tilde{C}_z, \text{ there exists a collection} \\ \text{of points } S_\gamma \subset \mathbb{L}_0(\gamma) \text{ such that } |S_\gamma| \geq a \text{ and for each} \\ y \in S_\gamma, W_{z,y} \text{ occurs and } C_{y,L_0} \cap \mathscr{C}_z \neq \emptyset \end{array} \right\} \tag{4.3.37}$$

Remark 4.3.5. *In the above definition, we remark that $\mathcal{G}_z$ only depends on $V, W, \mathscr{C}$ through $V_z, W_{z,\cdot}$ and $\mathscr{C}_z$. Moreover, when $a = 0$ each of $W, \mathscr{C}$ and G_z become superfluous in view of* (4.3.37) *and $\mathcal{G}_z$ coincides with V_z. This setup is already non-trivial and will be pertinent in the (simpler) subcritical regime.*

To conclude the renormalisation process detailed in the preamble of §4.3, let us present the result that, modulo some re-localisation steps, demonstrates how the event $\mathscr{G}$ introduced in §4.3.2 enables us to transition from collections of $\mathcal{G}_{z,L}$ at a scale L to the event $\mathcal{G}_{0,N}$ at scale N. In the body of the result, we use the notation

$$\Lambda_{N,-L} \coloneqq V_{N,-L} \setminus U_N \text{ with } \Lambda_N = V_N \setminus U_N \text{ and } V_{N,-L} \coloneqq \{z \in \mathbb{Z}^d : D_{z,L} \subset V_N\}, \tag{4.3.38}$$

and insist that, although in practice the events $\mathcal{F}_{z,L}$ will be of the form (4.3.10), they can in fact be chosen more generally.

Proposition 4.3.6 (Bootstrap of $\mathcal{G}_{z,L}$)**.** *Under the conditions posited in* (4.3.34) *and for any choice of $(V^1, W^1, \mathscr{C})$, $(\tilde{V}^1, \tilde{W}^1, \tilde{\mathscr{C}})$ as described below* (4.3.35)*, all $\Lambda_N \in S_N$, $a \geq 0$ and $\rho \in (0, \frac{1}{2}]$, the following hold. If*

$$(\mathcal{G}_{z,L}(V^1, W^1, \mathscr{C}^1; a) \cap \mathcal{F}_{z,L}) \subset \mathcal{G}_{z,L}(\tilde{V}^1, \tilde{W}^1, \tilde{\mathscr{C}}^1; a), \text{ for each } z \in \Lambda_N \cap \mathbb{L}, \tag{4.3.39}$$

then there exists a non-empty set $\mathcal{O} \subset \Lambda_{N,-L}$ such that, writing $\Lambda_N = V_N \setminus U_N$, each $*$-component $\mathcal{O}'$ of $\mathcal{O}$ satisfies $U_N \cap \mathbb{L} \preceq \mathcal{O}' \preceq \partial_{\mathbb{L}}(V_N \cap \mathbb{L})$ as subsets of the coarse-grained lattice $\mathbb{L}$ (see §4.1.1 for notation) and, abbreviating $\mathcal{G}^1 = \{\mathcal{G}_{z,L}(V^1, W^1, \mathscr{C}^1; a) : z \in \mathbb{L}\}$, one has the inclusion

$$\mathscr{G}(\Lambda_N, \mathcal{G}^1, \mathcal{F}; \rho) \subset \bigcap_{z \in \mathcal{O}} \mathcal{G}_{z,L}(\tilde{V}^1, \tilde{W}^1, \tilde{\mathscr{C}}^1; a) \Big(\subset \bigcap_{z \in \mathcal{O}} \tilde{V}^1_z \Big) \tag{4.3.40}$$

Furthermore, if $\tilde{W}^1_{z,\cdot} = \tilde{W}^1_{0,\cdot}$ for all $z \in \mathbb{L}$, then

$$\mathscr{G}(\Lambda_N, \mathcal{G}^1, \mathcal{F}; \rho) \subset \mathcal{G}_{0,N}(V^2, W^2, \mathscr{C}^2; b), \text{ when } \Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}, \tag{4.3.41}$$

where $b = 10^{-d} \cdot \left\lfloor \frac{(1-\rho)c_3 N}{h(KL)} \right\rfloor \cdot a$, $V_0^2 := \bigcap_{z\in\mathcal{O}} \tilde{V}_z^1$, $W_{0,y}^2 = \tilde{W}_{0,y}^1$ *for all* $y \in \mathbb{L}_0$ *and* $\mathscr{C}_0^2 := \bigcup_{z\in\mathcal{O}} \mathscr{C}_z^1$ *(note that, on account of Remark 4.3.5, only* V_0^2, $W_{0,\cdot}^2$ *and* $\mathscr{C}_0^2$ *are required to define* $\mathcal{G}_{0,N}$ *in* (4.3.41)*).*

Proof. The statement in (4.3.40) will follow from a correct interpretation of the problem in the form of the blocking surfaces presented in §4.1.1. To this end, we write $\Lambda_N = V_N \setminus U_N$, where Λ_N ranges over S_N defined in (4.3.2). In the notation of Proposition 4.1.1, we let $U = U_N \cap \mathbb{L}$, $V = V_N \cap \mathbb{L}$ and consider the set $\Sigma \subset (\mathbb{L} \cap \Lambda_N)$ defined as

$$\Sigma = \{z \in (\Lambda_{N,-L} \cap \mathbb{L}) : \mathcal{G}_{z,L}(V^1, W^1, \mathscr{C}^1; a) \cap \mathcal{F}_{z,L} \text{ occurs}\}, \tag{4.3.42}$$

and aim to apply the result on the coarse-grained lattice $\mathbb{L}$. Under the assumption that $\mathscr{G} = \mathscr{G}(\Lambda_N, \mathcal{G}^1, \mathcal{F}; \rho)$ holds, we will show that the conditions laid out in Proposition 4.1.1 are satisfied, namely that any path $\gamma \subset \mathbb{L}$ passing from U to $\partial_{\mathbb{L}} V$ must intersect Σ in at least

$$k := \left\lfloor \frac{(1-\rho)c_3 N}{h(KL)} \right\rfloor \tag{4.3.43}$$

points; the fact that U and V are $*$-connected in $\mathbb{L}$ is immediate. Note that, with a slight abuse of notation, $\partial_{\mathbb{L}}$ refers to the inner boundary consisting of nearest-neighbours in $\mathbb{L}$. Thus, let us consider an arbitrary path $\gamma \subset \mathbb{L}$ which crosses $V \setminus U$ in $\mathbb{L}$, and define the $*$-path $\bar{\gamma} \subset \mathbb{Z}^d$ as an extension of γ crossing Λ_N which satisfies $\mathrm{Range}(\bar{\gamma}) \subset \bigcup_{z\in\gamma} C_{z,L}$, e.g. by extending along the box edges between neighbours in $\mathbb{L}$. Since $\bar{\gamma}$ crosses Λ_N, $\mathscr{G}$ implies the existence of a coarsening $\mathcal{C} = \mathcal{C}(\bar{\gamma}) \in \mathcal{A}_K^L(\Lambda_N)$ such that $\bar{\gamma}$ crosses $\tilde{D}_z \setminus C_z$ for each $z \in \mathcal{C}$ and such that $\mathcal{G}_{z,L}(V^1, W^1, \mathscr{C}^1; a) \cap \mathcal{F}_{z,L}$ holds for every $z \in \tilde{\mathcal{C}} \subset \mathcal{C}$ with $|\tilde{\mathcal{C}}| = k$ from (4.3.43). Note that the cardinality of $\tilde{\mathcal{C}}$ is a consequence of property *(i)* of the Γ-admissibility of $\mathcal{A}$. Note that, by construction, γ must also intersect each $z \in \tilde{\mathcal{C}} \subset \Sigma$, and hence the final assumption of Proposition 4.1.1 is satisfied on the coarse-grained lattice $\mathbb{L}$ for $U, V, \Sigma \subset \mathbb{L}$ as above, yielding $*$-connected sets $O_1, \dots O_l \subset \mathbb{L}$ satisfying properties *(i)-(iii)* from the Proposition. Letting $\mathcal{O} := \bigcup_i O_i$, item *(i)* then implies that any $*$-component $\mathcal{O}' \subset \mathcal{O}$ satisfies $U_N \cap \mathbb{L} \preceq \mathcal{O}' \preceq \partial_{\mathbb{L}}(V_N \cap \mathbb{L})$. Moreover, since $\mathcal{O} \subset \Sigma$, by using the assumption (4.3.39) we have the first inclusion in (4.3.40). The second inclusion (in brackets) in (4.3.40) is simply a consequence of the definition of $\mathcal{G}_{z,L}$ in (4.3.36). All that is left to show is (4.3.41). Recalling the definition (4.3.36) and applying the second inclusion in (4.3.40), one plainly sees that V_0^2 holds, reducing our problem to showing $G_{0,N}(W^2, \mathscr{C}^2; b)$ instead. To this effect, our approach is to

amalgamate collections of points S_γ across boxes obtained from our coarse-graining in order to verify the event (4.3.37) is satisfied at a larger scale for W^2, $\mathscr{C}^2$, b as chosen beneath (4.3.41). Let γ be a path crossing $\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$, and define $\gamma' \subset \mathbb{L}$ as the ordered collection of $z \in \mathbb{L}$ such that γ intersects $C_{z,L}$. By construction of $\mathcal{O}$, we see that γ' must intersect at least k points of $\mathcal{O}$ (with k as in (4.3.43)), each of which satisfies $G_{z,L}(\tilde{W}^1, \tilde{\mathscr{C}}^1; a)$. Note that, in particular, γ must cross each $\tilde{D}_{z,L} \setminus \tilde{C}_{z,L}$, so by our previous remark there must be a collection of points $S_\gamma(z)$ with $|S_\gamma(z)| \geq a$ and for each $y \in S_\gamma(z), \tilde{W}^1_{z,y} = \tilde{W}^1_{0,y} = W^2_{0,y}$ holds and $\emptyset \neq C_{y,L_0} \cap \mathscr{C}_z \subset C_{y,L_0} \cap \bigcup_{z \in \mathcal{O}} \mathscr{C}^1_z = C_{y,L_0} \cap \mathscr{C}^2_0$. The claim (4.3.41) now holds by extracting $S_\gamma \subset \bigcup_{z \in \mathcal{O} \cap \mathrm{Range}(\gamma')} S_\gamma(z)$ by only retaining disjoint boxes $\tilde{D}_{z,L}$ (e.g. a fraction 10^{-d} of the original boxes) to ensure that the sets $S_\gamma(z)$ are also disjoint, and their cardinalities are additive. As a result, we have sourced a set $S_\gamma \subset \gamma$ with cardinality at least $10^{-d} \cdot k \cdot a = b$ such that all its constituent points satisfy the required conditions for $G_{0,N}(W^2, \mathscr{C}^2; b)$ to hold. □

4.4 Upper Bounds: Collecting the Pieces

In the previous sections, we have introduced a plethora of relevant machinery which will be directed at proving Theorems 1.3.2 and 1.3.3. Of these, Proposition 4.3.6 be furnished with an appropriate choice of data $(V, W, \mathscr{C})$ dictated by the specific problem and regime under consideration. Over the course of the subsequent sections, we will show in dimensions at least four

Theorem 4.4.1. *For every $d \geq 4$ and $u \in (0, \infty) \setminus \{u_*\}$, there exists $c = c(u, d) \in (0, \infty)$ such that*

$$\frac{1}{N} \log \mathbb{P}\Big[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B^2_N, 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\Big] \leq -c \tag{4.4.1}$$

for all $N \geq 1$.

Recalling the notation below (2.1.1) and abbreviating $B^2_N = B^2(0, N)$, $B_N = B(0, N)$, the inclusions $B^2_N \subset B_N \subset B^2_{C'(d)N}$ imply that the above theorem also holds for B_N in place of B^2_N, possibly with $c(u, d)$. Moreover, we have the somewhat more precise result in $d = 3$, which is a consequence of the care taken to retain parameters corresponding to leading-order rates of decay in Proposition 4.3.3.

Theorem 4.4.2. *For $d = 3$ and all $u \in (0, \infty) \setminus \{u_*\}$, we have*

$$\limsup_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}\Big[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B^2_N, 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\Big] \leq -\frac{\pi}{3}(\sqrt{u} - \sqrt{u_*})^2, \tag{4.4.2}$$

and the same limit also holds for B_N.

In conjunction, the theorems above yield the desired upper bounds on the two-point connectivity function immediately, by virtue of the inclusion $\tau_u^{tr}(x) \subset \{0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B^2_{|x|}, 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\}$ for any $x \in \mathbb{Z}^d$. This simple remark permits us to avoid a lot of the work presented in the lower bound case to facilitate point-point connection (see §3.1 for an in-depth discussion of the necessary finite-energy techniques).

Although the material comprising the following sections is inspired by various sources in the literature, it is entirely novel - particularly the concluding results described in the preamble above. Some inspirations include the algorithmic procedure described in the proof of [28, Lemma 5.11], the finite energy techniques used in [20, Prop. 3.1] which we modify for our purposes, and the small-density considerations in [18] which we subsequently use to distinguish between 'small' and 'large' collections of excursions (this will appear in §4.5.1). In this section, we also produce an original approach based on collections of sub-excursions in order to monotonize local uniqueness events.

4.4.1 Subcritical Regime

To provide a clear and straightforward introduction, we first present the solution of the subcritical two-point connectivity problem. This choice is motivated by its comparative simplicity stemming from the degeneracy of the corresponding event $\mathcal{G}_{z,L}(V, W, \mathscr{C}; a)$. Specifically, in this scenario, it turns out that we only need to consider $a = 0$ contact points, which leads to the streamlined version $\mathcal{G}_{z,L}(V, W, \mathscr{C}; 0) = V_z$, and henceforth allows us to ignore W and $\mathscr{C}$ (see Remark 4.3.5). In the sequel, we often refer to the vacant fields with varying degrees of localisation as introduced in (4.2.26). Then, for a subcritical level $u > u_*$, $z \in \mathbb{L}$ and $\tilde{\mathcal{V}}_{\mathbb{L}} \in \{\mathcal{V}_{\mathbb{L}}, \bar{\mathcal{V}}_{\mathbb{L}}, \hat{\mathcal{V}}_{\mathbb{L}}\}$ as one of the aforementioned fields, we consider the event

$$\mathrm{Dis}_z(\tilde{\mathcal{V}}_{\mathbb{L}}, u) := \{\tilde{C}_z \overset{\tilde{\mathcal{V}}^u_z}{\not\longleftrightarrow} \partial \tilde{D}_z\}, \tag{4.4.3}$$

which will henceforth supply the various forms of the data V. In typical fashion, we will use the contents of §4.2 to pass between the various forms of the event (4.4.3) with

the help of the following inclusions for any $u_1 < u_2$

$$\begin{aligned} \mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_1) \cap \mathcal{F}_{z,L}^{u_1,u_2} &\subset \mathrm{Dis}_z(\bar{\mathcal{V}}_{\mathbb{L}}, u_2), \\ \mathrm{Dis}_z(\bar{\mathcal{V}}_{\mathbb{L}}, u_1) \cap \mathcal{F}_{z,L}^{u_2,u_1} &\subset \mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_2), \end{aligned} \tag{4.4.4}$$

as well as, for any $u > u_*$ and $\epsilon \in (0,1)$

$$\begin{aligned} \mathrm{Dis}_z(\bar{\mathcal{V}}_{\mathbb{L}}, u) \cap \mathrm{Incl}_z^{\frac{\epsilon}{6}, u_* \mathrm{cap}(\breve{D}_0)} &\subset \mathrm{Dis}_z(\hat{\mathcal{V}}_{\mathbb{L}}, u(1+\epsilon)), \\ \mathrm{Dis}_z(\hat{\mathcal{V}}_{\mathbb{L}}, u) \cap \mathrm{Incl}_z^{\frac{\epsilon}{6}, u_* \mathrm{cap}(\breve{D}_0)} &\subset \mathrm{Dis}_z(\bar{\mathcal{V}}_{\mathbb{L}}, u(1+\epsilon)), \end{aligned} \tag{4.4.5}$$

where we recall the definitions of $\mathcal{F}_z^{u,v}$ and $\mathrm{Incl}_z^{\epsilon,m}$ from (4.2.19) and (4.2.18) respectively. From now on, we tacitly assume that all the scale and separation conditions stated in (4.3.34) are satisfied for the purposes of successfully carrying out the bootstrapping approach, and turn to the proof of the subcritical case of Theorem 4.4.2.

Proof of Theorem 4.4.2 ($u > u_*$). Let $d = 3$ and consider $u_* < u_1 < u_2$. From the second line of (4.4.4), we see that the assumption (4.3.39) of Proposition (4.3.6) is satisfied for the choices $V_z = \mathrm{Dis}_z(\bar{\mathcal{V}}_{\mathbb{L}}, u_1)$, $\tilde{V}_z = \mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_2)$, $a = 0$ and $\mathcal{F}_{z,L} = \mathcal{F}_{z,L}^{u_2,u_1}$. In this way, fixing some $S_N \ni \Lambda_N = V_N \setminus U_N$ and $\rho \in (0, \frac{1}{2})$, an application of (4.3.40) shows that the occurrence of $\mathscr{G}(\Lambda_N, \mathrm{Dis}_\cdot(\bar{\mathcal{V}}_{\mathbb{L}}, u_1), \mathcal{F}_\cdot^{u_2,u_1}; \rho)$ yields a $*$-connected component $\mathcal{O}' \subset \mathbb{L}$ satisfying $U_N \cap \mathbb{L} \preceq \mathcal{O}' \preceq \partial_{\mathbb{L}}(V_N \cap \mathbb{L})$, that $\mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_2)$ holds for each $z \in \mathcal{O}'$, and that any path crossing Λ_N (in $\mathbb{L}$) must intersect $\mathcal{O}'$. In turn, this implies that any path $\gamma \subset \mathbb{Z}^d$ crossing Λ_N must cross $\tilde{D}_z \setminus \tilde{C}_z$ for some $z \in \mathcal{O}'$, which is endowed with the disconnection property $\mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_2)$. Consequently, a connection inside $\mathcal{V}^{u_2}$ between U_N and ∂V_N is precluded, i.e.

$$\mathscr{G}(\Lambda_N, \mathrm{Dis}_\cdot(\bar{\mathcal{V}}_{\mathbb{L}}, u_1), \mathcal{F}_\cdot^{u_2,u_1}; \rho) \subset \{U_N \overset{\mathcal{V}^{u_2}}{\not\longleftrightarrow} \partial V_N\} \tag{4.4.6}$$

Thus, we observe that an upper bound on connectivity from U_N to ∂V_N in the vacant set can be obtained by instead bounding the complement of the left-hand side of (4.4.6) - this is firmly within the domain of Proposition 4.3.3. To this end, we endeavour to show the underlying probabilistic assumptions therein.

Let $\epsilon \in (0, ((\frac{u}{u_*})^{\frac{1}{10}} - 1) \wedge \frac{1}{10})$. By the second localisation step (4.4.5), we see that the inclusion (4.3.11) holds for $\hat{\mathcal{G}}_{z,L} = \mathrm{Dis}_z(\hat{\mathcal{V}}_{\mathbb{L}}, u')$, $\mathcal{G}_{z,L} = \mathrm{Dis}_z(\bar{\mathcal{V}}_{\mathbb{L}}, u'(1+\epsilon))$, $\epsilon_L = \frac{\epsilon}{6}$ and $m_L = u_* \mathrm{cap}(\breve{D}_0)$ for any $u' > u_*$, so we are left with showing condition (4.3.12). In $d = 3$, the second part of this condition is true for any $\theta > 0$ by (2.6.13). To see the

first part, we must use known results on connectivity of the global vacant field and then compute the associated localisation cost to get a bound on $\hat{\mathcal{G}}_{z,L}$. In particular, from [22, Thm. 1.2. *i)*] by application of a union bound we have that for any $z \in \mathbb{L}$, there exist $c = c(d) \in (0, \infty)$ such that

$$\mathbb{P}[\mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_*(1+\epsilon))] \geq 1 - C(\epsilon)e^{-L^c} \tag{4.4.7}$$

Therefore, in combination with the first inclusion in (4.4.4), the a-priori bound (4.4.7) implies

$$\begin{aligned} &\mathbb{P}[\mathrm{Dis}_z(\bar{\mathcal{V}}_{\mathbb{L}}, u_*(1+\epsilon)^2)] \\ &\overset{(4.4.4)}{\geq} \mathbb{P}[\mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_*(1+\epsilon))] - \mathbb{P}[(\mathcal{F}^{u_*(1+\epsilon), u_*(1+\epsilon)^2})^c] \overset{\text{Lem. } 4.2.2}{\geq} 1 - C(\epsilon)e^{-L^c} \end{aligned} \tag{4.4.8}$$

Moreover, with the first inclusion of (4.4.5), the above partially-localised bound yields an estimate on the fully local event as long as $K \geq 18C_5\epsilon^{-1}$

$$\begin{aligned} &\mathbb{P}[\mathrm{Dis}_z(\hat{\mathcal{V}}_{\mathbb{L}}, u_*(1+\epsilon)^3)] \\ &\geq \mathbb{P}[\mathrm{Dis}_z(\bar{\mathcal{V}}_{\mathbb{L}}, u_*(1+\epsilon)^2)] - \mathbb{P}[(\mathrm{Incl}^{\frac{\epsilon}{6}, \frac{u}{2}\mathrm{cap}(D_0)})^c] \overset{(2.6.11),(2.6.13)}{\geq} 1 - C(\epsilon)e^{-L^c} \end{aligned} \tag{4.4.9}$$

Thus, we have shown that all the conditions of Proposition 4.3.3 are satisfied for $\beta' = c$ with the selections made above (4.4.7) for $u' = u_*(1+\epsilon)^3$. Hence, by the conclusions presented in (4.3.13) of that Proposition combined with (4.4.6) for the choice $\Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$, we obtain an improved bound at the higher scale N of the form

$$\begin{aligned} &\mathbb{P}[\tilde{C}_{0,N} \overset{\mathcal{V}^{u(1+\epsilon)^5}}{\longleftrightarrow} \partial\tilde{D}_{0,N}] \\ &\overset{(4.4.6)}{\geq} \mathbb{P}\Big[\mathscr{G}(\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}, \mathrm{Dis.}(\bar{\mathcal{V}}_{\mathbb{L}}, u(1+\epsilon)^4), \mathcal{F}_{\cdot}^{u(1+\epsilon)^5, u(1+\epsilon)^4}); \frac{1}{2})\Big] \overset{(4.3.13)}{\geq} 1 - C(\epsilon)e^{\frac{N}{(\log N)^\beta}} \end{aligned} \tag{4.4.10}$$

for some $\beta \in (1, \infty)$. Now, we apply a second bootstrapping step, replacing the a priori bound (4.4.7) with the improved estimate (4.4.10) instead. By applying the same procedure as in (4.4.8)-(4.4.9), we see that we now fulfill the assumptions of Proposition 4.3.3 with the choices for $\hat{G}_{z,L}$, $\mathcal{G}_{z,L}$, ϵ_L, m_L set above (4.4.7) for $u' = u_*(1+\epsilon)^8$ with the improved bound $p_L = C(\epsilon)e^{-\frac{L}{(\log L)^\beta}}$ in the condition (4.3.12). Hence, once more relying on the inclusion (4.4.6) for $\Lambda_N = B_N^2$ and remarking that $u < u_*(1+\epsilon)^{10}$ by

assumption, we see for $K = \sqrt{\log\log e^2 N}$, any $\rho \in (0, \frac{1}{2})$ and any $\delta \in (0,1)$

$$\limsup_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_N^2] \leq -(1-\delta)(1-\rho)\frac{\pi}{3}(\sqrt{u} - \sqrt{u_*(1+\epsilon)^9})^2. \quad (4.4.11)$$

Since δ, ρ and ϵ can be taken to be arbitrarily small (but positive), we conclude with the desired inequality

$$\limsup_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_N^2] \leq -\frac{\pi}{3}(\sqrt{u} - \sqrt{u_*})^2 \quad (4.4.12)$$

□

Proof of Theorem 4.4.1 ($u > u_*$). The proof in the case of higher dimensions follows similarly to the steps presented in (4.4.6)-(4.4.10), except using the conclusions in (4.3.14) from Proposition 4.3.3, remarking that only one bootstrapping step is necessary to achieve the required precision of the upper bound (hence the first step uses $\Lambda_N = B_N^2$ immediately). With this in mind, letting $u > u_*$ and $\epsilon \in (0, ((\frac{u}{u_*})^{\frac{1}{5}} - 1) \wedge \frac{1}{10})$ we have

$$\frac{1}{N} \log \mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_N^2] \leq -C(u,d) \text{ for all } N \geq 1 \quad (4.4.13)$$

□

4.4.2 Supercritical Regime

In the previous subsection, we already observed the fundamental strategy in a minimalistic setting. Now, we are ready to harness the full power of the renormalisation tools from §4.3 by additionally selecting an appropriate family of events $W = \{W_{z,y} : z \in \mathbb{L}, y \in \mathbb{L}_0\}$ as well as a family of random subsets $\mathcal{C} = \{\mathcal{C}_z : z \in \mathbb{L}\}$ indexed by $\mathbb{L} \times \mathbb{L}_0$ and $\mathbb{L}$ respectively. To start off with, we let W be any such family, $\tilde{\mathcal{V}} \in \{\mathcal{V}, \bar{\mathcal{V}}, \hat{\mathcal{V}}\}$ be any choice of localisation, and in view of Definition 4.3.4 we introduce the following event

$$\mathcal{G}_z(\tilde{\mathcal{V}}, a, W, \delta, u_1, u_2, u_3) \coloneqq \mathcal{G}_z(V_z(\tilde{\mathcal{V}}_{\mathbb{L}}, \delta, u_1, u_2, u_3), W, \mathscr{C}(\tilde{\mathcal{V}}_{\mathbb{L}}, \delta, u_1, u_2); a) \quad (4.4.14)$$

where $\mathscr{C}_z(\tilde{\mathcal{V}}_{\mathbb{L}}, u_1, u_2)$ is the set comprised of all clusters in $N_\delta(\tilde{\mathcal{V}}_z^{u_1}) \cap D_z$ which contain a crossing of $\tilde{D}_z \setminus \tilde{C}_z$ in $N_{2\delta}(\tilde{\mathcal{V}}_z^{u_2}) \cap D_z$, and $V_z(\tilde{\mathcal{V}}_{\mathbb{L}}, \delta, u_1, u_2, u_3)$ denotes the local

uniqueness event

$$\left\{\begin{array}{r} N_{2\delta}(\tilde{\mathcal{V}}^{u_3}) \cap D_z \text{ contains a cluster crossing } D_z \setminus C_z, \text{ and any two} \\ \text{clusters } \mathcal{C}', \mathcal{C}'' \subset N_{2\delta}(\tilde{\mathcal{V}}^{u_2}) \cap D_z \text{ crossing } \tilde{D}_{z'} \setminus \tilde{C}_{z'} \text{ and } \tilde{D}_{z''} \setminus \tilde{C}_{z''} \\ \text{respectively, for some } z', z'' \in \mathbb{L} \text{ satisfying } \max_{z_0 \in \{z', z''\}} |z_0 - z| \leq L \\ \text{are connected in } N_\delta(\tilde{\mathcal{V}}^{u_1}) \cap D_z \end{array}\right\} \tag{4.4.15}$$

One consequence of the above definition is that in the presence of the event $V(\tilde{\mathcal{V}}_{\mathbb{L}}, \delta, u_1, u_2, u_3)$, the set $\mathscr{C}_z(\tilde{\mathcal{V}}_{\mathbb{L}}, \delta, u_1, u_2)$ is in fact a connected component of $N_\delta(\tilde{\mathcal{V}}_z^{u_1})$. In the case that $u_1 = u_2 = u_3 = u$, we shall say that there is no sprinkling, and in (4.4.14) we will replace the set of parameters u_1, u_2, u_3 simply by u. Moreover, we drop the dependence on the thinning parameter δ in the notation when $\delta = 0$. In fact, in the unsprinkled scenario with no thinning, we can generalise the above definition to the case of any (random) finite sequence of $\mathbb{Z}^d$-valued nearest-neighbour piecewise constant right-continuous paths. Denoting any such path as Z, we write $V(Z)$ to signify the event in (4.4.15) with all instances of vacant sets $N_\cdot(\tilde{\mathcal{V}}^\cdot)$ replaced by $\mathcal{V}(Z)$ (see (4.2.25)).

Next, let use introduce a certain modification of an excursion, called a truncation, which will be necessary in §4.4.3 in order to deal with the abrupt start and end of a random walk on a torus. Given any excursion $Z_\circ$ between two sets $\emptyset \neq D \subset U \subset\subset \mathbb{Z}^d$,

> a truncation of $Z_\circ$ is a trajectory $Z_\circ[\tau_1, \tau_2]$ (see below (3.4.15) for definition) where $0 \leq \tau_1 \leq \tau_2 < \infty$ and each $\tau \in \{\tau_1, \tau_2\}$ is either 0 or a point of (jump) discontinuity of $Z_\circ$ i.e. $Z_\circ(\tau-) \neq Z_\circ(\tau)$.

(4.4.16)

Note that $Z_\circ[0, T_U] = Z_\circ$ where $T_U = T_U(Z_\circ)$ is relative to the excursion $Z_\circ$, and call $Z_\circ[\tau_1, \tau_2]$ a one-sided truncation of $Z_\circ$ in the special cases where $\tau_1 = 0$ or $\tau_2 = T_U(Z_\circ)$.

Observe that $V(Z)$ does not depend on the particular order in which the trajectories occur in the sequence Z and this will remain true for all the events introduced in this section having Z as one of its parameters. Henceforth, we use the phrase 'a sequence of truncated excursions' to refer to any sequence $\{Z_j[\tau_{j,1}, \tau_{j,2}]\}_{j \in J}$ where $\{Z_k\}_{k \geq 1}$ is some sequence of random excursions (e.g. $\{Z_k^{D_z, U_z}\}_{k \geq 1}$ or $\{\hat{Z}_k^{D_z, U_z}\}_{k \geq 1}$), $J \subset \mathbb{N}^*$ is finite and random and $(\tau_{k,1}, \tau_{k,2})$ is a $\mathbb{R}^2_{\geq 0}$-valued random variable for each $k \geq 1$. We

drop the qualifier 'truncated' from the phrase if $\tau_{k,1} = 0$ and $\tau_{k,2} = T_U(Z_k)$ i.e. there is no truncation involved.

Now, given a sequence of excursions $Z = \{Z_k\}_{k=1}^{n_Z}$, and recalling the definition of discrete local times $\ell_{x,\mathrm{dis}}$ from (2.4.3), we denote

$$\ell_x(Z) = \ell_{x,dis}(Z) \coloneqq \sum_{1\le k\le n_z} \ell_{x,\mathrm{dis}}(Z_k), \quad x \in \mathbb{Z}^d \tag{4.4.17}$$

where, since the continuous-time version will not be required, we have shortened the notation to $\ell_x(Z)$. Moreover, when $Z = Z_z^u$ as in (4.2.24), we shorten notation further to ℓ_x^u. Let us now lay the groundwork in order to appropriately select the data W. As it will turn out, these events shadow the strategy already presented in the lower bound regime (see the path rerouting argument in the proof of [20, Prop. 3.1]) as they facilitate performing local surgery arguments when exploring a vacant cluster of $\mathcal{V}(Z)$. First, for any $L \in \mathbb{N}^*$, let $\square(0, L)$ denote the set of all points in $C_{0,L}$ such that at least two of their coordinates lie in the set $\{0, 1, 2, L-3, L-2, L-1\}$ and let $\square(z, L) = z + \square(0, L)$ for any $z \in \mathbb{Z}^d$. Simply put, $\square(z, L)$ represents the surface of $C_{z,L}$ excluding the interior portion of each of its faces.

Definition 4.4.3. *($y \in \mathbb{L}_0$, $L_{-1} \in \mathbb{N}^*$, $L_0 > 100L_{-1}$). For any sequence of truncated excursions Z, we define the event $FE_y(Z)$ as follows:*

$$FE_y(Z) \coloneqq \bigcap_{\substack{y'\in\mathbb{L}_0\\ |y'-y|\le 3L_0}} (LU_{y'}(Z) \cap O_{y'}(Z)) \tag{4.4.18}$$

where

$$\begin{aligned} LU_y(Z) &\coloneqq \bigcap_{x,x'\in\mathcal{I}(Z)\cap(\tilde{D}_y\setminus\tilde{C}_y)} \{x \overset{\mathcal{I}(Z)\cap(\mathring{D}_y\setminus C_y)}{\longleftrightarrow} x'\}, \text{ and} \\ O_y(Z) &\coloneqq \bigcap_{x\in\partial D_y} \{\ell_x(Z) \le (\log L_0)^2\}. \end{aligned} \tag{4.4.19}$$

Moreover, recalling (4.3.36)*, we introduce the event $\mathcal{G}_{y,L_{-1}}(Z) \coloneqq \mathcal{G}_{y,L_0,L_{-1}}(V = \{\Omega : y \in \mathbb{L}_0\}, W = G(Z, L_{-1}), \mathcal{C} = \mathbb{Z}^d; a = 1)$ where $G(Z, L_{-1}) = \{G_{\mathrm{x}}(Z, L_{-1}) : y \in \mathbb{L}_0, \mathrm{x} \in \mathbb{L}_{-1} = L_{-1}\mathbb{Z}^d\}$ with*

$$G_{\mathrm{x}}(Z, L_{-1}) \coloneqq \left\{ \begin{array}{c} \square(\mathrm{x}', L_{-1}) \subset \mathcal{V}(Z) \text{ and } \sum_{x\in\partial C_{\mathrm{x}',L_{-1}}} \ell_x(Z) \le (L_{-1})^{d-1} \\ \text{for all } \mathrm{x}' \in \mathbb{L}_{-1} \text{ satisfying } |\mathrm{x} - \mathrm{x}'| \le L_{-1} \end{array} \right\} \tag{4.4.20}$$

Remark 4.4.4. *As with L_0, the scale L_{-1} will remain fixed throughout the entire bootstrapping procedure. With this in mind, we will drop it from all notations except $\mathcal{G}_{y,L_{-1}}$ to distinguish it explicitly from the overloaded definition of $\mathcal{G}_z$.*

So far, we have only ever required events to hold on a fixed chosen, albeit random, sequences of excursions. From this point onwards, we introduce a technique which provides another way of transferring between the various sequences Z_z^u, $\bar{Z}_z^u$ and $\hat{Z}_z^u$ in a more frictionless manner, with some acceptable entropy cost. In particular, consider a generic event $G_\cdot(Z,\ldots)$ which has some finite (truncated) sequence of excursions Z as one of its parameters, and introduce the family

$$\boldsymbol{Z} = \Big(\{Z_j[\tau_{j,1}^{i_j}, \tau_{j,2}^{i_j}]\}_{j\in J}\Big)_{(J,(i_j)_{j\in J})\in \boldsymbol{J}} \tag{4.4.21}$$

representing a finite collection of finite subsequences of $Z = \{Z_k\}_{k\geq 1}$. Similarly to above, $\{(\tau_{j,1}^{i_j}, \tau_{j,2}^{i_j})\}_{i\geq 1, j\geq 1}$ is a family of $\mathbb{R}^2_{\geq 0}$-valued random variable where $(\tau_{j,1}^{i_j}, \tau_{j,2}^{i_j})$ satisfies (4.4.16) with respect to Z_j for each $i, j \in \mathbb{N}^*$. Also, $\boldsymbol{J}$ is a random element from the collection of all finite sets of pairs $(J, (i_j)_{j\in J})$ where $J \subset \mathbb{N}^*$ is finite and $(i_j)_{j\in J} \in (\mathbb{N}^*)^{|J|}$. The format given in (4.4.21) is sufficiently general to represent all such collections which we refer to from here on. Now, we introduce the $\boldsymbol{Z}$ version of the event $G_\cdot(Z,\ldots)$ as

$$G_\cdot(\boldsymbol{Z},\ldots) \coloneqq \bigcap_{Z\in\boldsymbol{Z}} G_\cdot(Z,\ldots) \tag{4.4.22}$$

Let us give an important example of collection $\boldsymbol{Z}$ as in (4.4.21) which we will rely on. Given any sequence of excursions $Z = \{Z_k\}_{1\leq k\leq n_Z}$, non-negative real values $\nu_1, \nu_2 \in [0,\infty)$ and $n \in \mathbb{N}$, we let

$$\begin{aligned}&\boldsymbol{Z}(Z,\nu_1,\nu_2;n) \text{ denote the collection of (all) subsequences } \{Z_i\}_{i\in J} \text{ for}\\ &J \subset \{1,\ldots,n_Z\} \text{ such that either } J \text{ contains } \{j+1,\ldots,j+\nu_1\} \text{ for some}\\ &j \in \mathbb{N}, \text{ or is a subset of } \cup_{1\leq k\leq n} J_k \text{ where each } J_k \text{ is an interval with}\\ &\sum_{1\leq k\leq n} |J_k| \leq \nu_2\end{aligned} \tag{4.4.23}$$

where intervals are subsets of $\mathbb{Z}$ which are permitted to be empty, n will typically be fixed (but small) and ν_1, ν_2 can be random. Keeping in line with our previous notation, we use $\boldsymbol{Z}(\tilde{V}_z^u, \nu_1, \nu_2; n)$ to denote $\boldsymbol{Z}(\tilde{Z}_z^u, \nu_1, \nu_2; n)$ for $\tilde{Z}_z^u \in \{Z_z^u, \bar{Z}_z^u, \hat{Z}_z^u\}$. If we drop the first or the second condition in (4.4.23), we denote the resulting collection

as $\boldsymbol{Z}(Z,\nu_1,\star)$ or $\boldsymbol{Z}(Z,\star,\nu_2;n)$ respectively. Consequently, we see that

$$\boldsymbol{Z}(Z,\nu_1,\nu_2;n)=\boldsymbol{Z}(Z,\nu_1,\star)\cup\boldsymbol{Z}(Z,\star,\nu_2;n) \tag{4.4.24}$$

With this, we are now ready to introduce the specific choice of our bootstrapping event $\mathscr{G}$ for this subsection, to which end we recall the notation (4.3.9) for the generic data $(V,W,\mathscr{C})$. Given the parameters

$$\begin{aligned}&u_0\in(0,\infty),u<u_1<u_2<u_3<u_4\in(0,u_*)\text{ and }u_5\in[0,\infty),a,n\in\mathbb{N}^*,\\&\nu>0,\delta\in(0,\frac{1}{2}),\text{ scales }L_0,L,N\text{ satisfying (4.3.34) for some }K\geq 100,\text{ a}\\&\text{scale }L_{-1}\in\mathbb{N}^*\text{ satisfying }L_0>100L_{-1},\text{ and }z\in N\mathbb{Z}^d,\end{aligned} \tag{4.4.25}$$

let us consider the event $\mathscr{G}_z=\mathscr{G}_{z,N}(a,n;\frac{u_0}{8},u_1,u_2,u_4,2u_5)$ defined as

$$\mathscr{G}_z=\mathscr{G}(\tilde{D}_z\setminus\tilde{C}_z,\{\mathcal{G}_{\mathrm{y}}(\bar{\mathcal{V}}_{\mathbb{L}},a,\overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}},\frac{u_0}{8},u_1,2u_5;n),\delta,u_1,u_2,u_4):\mathrm{y}\in\mathbb{L}\},\mathcal{F}_L^{\boldsymbol{u}_1,\boldsymbol{u}_2};\rho=\frac{1}{2}) \tag{4.4.26}$$

(recall (4.3.9) and (4.4.14)) where the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and the family of events $\overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}},v_0,v_1,v_2;n)$ are given as follows. For $n_0\coloneqq\lceil\frac{u}{u_0}\rceil$,

$$\begin{aligned}\boldsymbol{u}_1&=(u,u_3,u_3,\boldsymbol{u}_{1,0},\dots,\boldsymbol{u}_{1,2n_0})\text{ and }\boldsymbol{u}_2=(u_1,u_2,u_4,\boldsymbol{u}_{2,0},\dots,\boldsymbol{u}_{2,2n_0}),\text{ with}\\\boldsymbol{u}_{1,k}&=\tfrac{ku_0}{2}+(0,\tfrac{u_0}{8},\tfrac{3u_0}{8},\tfrac{u_0}{2},\tfrac{u_0}{2},\tfrac{3u_0}{4},\tfrac{3u_0}{4},u_0),\text{ and}\\\boldsymbol{u}_{2,k}&=\tfrac{ku_0}{2}+(\tfrac{u_0}{8},\tfrac{u_0}{4},\tfrac{u_0}{4},\tfrac{u_0}{4},\tfrac{9u_0}{16},\tfrac{11u_0}{16},\tfrac{13u_0}{16},\tfrac{15u_0}{16})\text{ for each }k\in\{0,\dots,2n_0\}.\end{aligned} \tag{4.4.27}$$

On the other hand, for any $\tilde{\mathcal{V}}_{\mathbb{L}}\in\{\mathcal{V}_{\mathbb{L}},\bar{\mathcal{V}}_{\mathbb{L}},\hat{\mathcal{V}}_{\mathbb{L}})$ and $v_0,v_1,v_2\in(0,\infty)$,

$$\begin{aligned}\overline{\mathrm{FE}}(\tilde{\mathcal{V}},v_0,v_1,v_2;n)&=\overline{\mathrm{FE}}(\tilde{\mathcal{V}}_{\mathbb{L}},L_{-1};v_0,v_1,v_2;n)\\&=\{\overline{\mathrm{FE}}_{\mathrm{y},y}(\tilde{\mathcal{V}}_{\mathbb{L}},v_0,v_1,v_2;n):\mathrm{y}\in\mathbb{L},y\in\mathbb{L}_0\},\text{ with}\\\overline{\mathrm{FE}}_{\mathrm{y},y}(\tilde{\mathcal{V}}_{\mathbb{L}},v_0,v_1,v_2;n)&=\overline{\mathrm{FE}}_{\mathrm{y},y}(\tilde{\mathcal{V}}_{\mathbb{L}},L_{-1};v_0,v_1,v_2;n)\\&\coloneqq\mathrm{FE}_y\Big(\boldsymbol{Z}(\tilde{\mathcal{V}}_{\mathrm{y}}^{v_1},\tilde{\mathcal{N}}_{\mathrm{y}}^{v_0},\star)\Big)\cap\mathcal{G}_{y,L_{-1}}(\boldsymbol{Z}(\tilde{V}_{\mathrm{y}}^{v_1},\star,\tilde{\mathcal{N}}_{\mathrm{y}}^{v_2};n)\end{aligned} \tag{4.4.28}$$

where we recall the convention for $\tilde{\mathcal{N}}_z^u$ below (4.2.24). We are now ready to state the first major result of this section, namely Proposition 4.4.5, which effectively acts as a more sophisticated analogue of (4.4.6), now for local-uniqueness events in place of simple disconnection events. A key facet of this proposition is that the event $\mathscr{G}$,

the local uniqueness part of which exhibits sprinkling, is connected to a new local uniqueness event with no sprinkling. The latter event will be expressed in terms of the following collection of subsequences. For any sequence $Z = \{Z_k\}_{1\le k\le n_z}$, $\nu \in [0,\infty]$ and $n \in \mathbb{N}$, we let

$$\begin{aligned}&\boldsymbol{Z}_\bullet(Z,\nu;n) \text{ denote the collection of all (sub)sequences } \{Z_j\}_{j\in J} \text{ where}\\ &J \subset \{1,\dots,n_Z\} \text{ is of the form } \cup_{1\le k\le n} J_k \text{ with } J_k' \subset J_k \subset J_k' + \{0,\dots,\pm|\nu|\}\\ &\text{for some interval } J_k' \text{ and each } 1 \le k \le n\end{aligned} \tag{4.4.29}$$

Moreover, we follow the notational convention as with $\boldsymbol{Z}$ with respect to $\tilde{\mathcal{V}} \in \{\mathcal{V}, \bar{\mathcal{V}}, \hat{\mathcal{V}}\}$ (see below (4.4.23)). Additionally, given any random finite collection of sequences of excursions $\boldsymbol{Z}$, we let

$$\begin{aligned}&\mathrm{tr}(\boldsymbol{Z}) \text{ denote the collection comprising any sequence } Z' \text{ that can}\\ &\text{be obtained from some } Z \in \boldsymbol{Z} \text{ by truncating at most two of its}\\ &\text{elements as in (4.4.16)}\end{aligned} \tag{4.4.30}$$

Proposition 4.4.5. *For any (random) sequence of finite $\mathbb{Z}^d$-valued nearest-neighbour piecewise constant right-continuous trajectories Z, let $\mathcal{S}(Z) = \mathcal{S}(Z, L_{-1}) \coloneqq \{\mathrm{x} \in \mathbb{L}_{-1} : G_{\mathrm{x}}(Z) \text{ occurs}\}$. Then, with the choice of parameters as in (4.4.25) as well as $\nu \ge 1$, $3nu_0 < u_5 < u_*$, $K \ge C$ and $L \ge C(u_0)$, there exists $c = c(\delta, u_0, L_0, L_{-1}) > 0$ such that*

$$\begin{aligned}&\mathbb{P}\Big[V_z^c\Big(tr(\boldsymbol{Z}_\bullet(\mathcal{V}_z^u,\nu;n))\Big)\Big]\\ &\le \mathbb{P}[\mathscr{G}_z^c] + \mathbb{P}\Big[C_z \overset{N_{2\delta}(\tilde{\mathcal{V}}^{u_3})}{\not\longleftrightarrow} \partial D_z\Big] + C(u_0,u)^n \mathbb{P}\Big[(C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}(Z_z^{u_5})}{\not\longleftrightarrow} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1})\Big]\\ &+ C(u_0,u)\mathbb{P}\Big[\Big(\mathcal{F}_z^{\frac{u_0}{8},\frac{\nu}{cap(D_z)}}\Big)^c\Big] + C(u_0,u)\exp\{-c(am\wedge N) + C(\nu + n\log N)\}\end{aligned} \tag{4.4.31}$$

where m is the cardinality of coarsenings in $\mathcal{A}_L^K(\tilde{D}_z \setminus \tilde{C}_z)$ (see Proposition 4.3.2) and the connectivity in $\mathcal{S}(Z_z^v)$ is w.r.t. the nearest-neighbour graph structure inherited from the coarse-grained lattice $\mathbb{L}_{-1}$.

For clarity of the exposition, we postpone the proof of Proposition 4.4.5 as it requires several intermediate results and is therefore quite involved. Evidently, at some point it will be necessary to show estimates on the probabilities appearing on the RHS

of (4.4.31). It turns out that each of these can be dealt with by the bootstrapping argument along with suitable a priori estimates, one of which we now present.

Lemma 4.4.6 (Supercritical a priori estimate)**.** *$u_0 \in (0, u_*)$, $u_1 < u_2 < u_3 \in (0, u_*)$, $n \in \mathbb{N}^*$ and $K \geq 100$. There exist scales L_{-1} and $L_0 = L_0(u_0, u_1, u_2, u_3, n) \in \mathbb{N}^*$ satisfying $L_0 > 100L_{-1}$, a level $u' \in (0, u_*)$ and $\delta = \delta(L_0) \in (0, \frac{1}{2})$ such that*

$$\mathbb{P}[\mathcal{G}_0(\mathcal{V}_{\mathbb{L}}, a = 1, \overline{FE}(\mathcal{V}_{\mathbb{L}}, u_0, u_1, u'; n), \delta', u_1, u_2, u_3)] \geq 1 - Ce^{-L^c} \tag{4.4.32}$$

for any $\delta' \in [0, \delta]$, some $C = C(u_0, u_1, u_2, u_3, n) \in (0, \infty)$ and all $L > 100L_0$.

Once again, we defer the proof of this result until later (see §4.5.3) although we do briefly remark that now we intend to bootstrap not only the principal rate of decay, but also the number of 'contact points' a. Now, we proceed with the overarching strategy already sketched out through the medium of the subcritical regime. In particular, we demonstrate the localisation tools which are analogues of the inclusions (4.4.4), (4.4.5) and will consequently feed directly into Propositions 4.3.3 and 4.3.6 via the conditions (4.3.11) and (4.3.39) therein, respectively. To this end, let $u_0, u_4, v_0, v_4 \in [0, \infty]$, $u_1 < u_2 < u_3 \in (0, \infty)$ and $v_1 < v_2 < v_3$ be such that $u_0 < v_0$, $u_4 > v_4$, $u_1 > v_1$, $u_2 < v_2$ and $u_3 > v_3$. Also let N, L, K, L_0 satisfy (4.3.34). Then for any $z \in (\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}) \cap \mathbb{L}$, we have

$$\begin{aligned}
&\mathcal{G}_z(\bar{\mathcal{V}}_{\mathbb{L}}, a, \overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}}, u_0, u_1, u_4; n), \delta, u_1, u_2, u_3) \cap \mathcal{F}_z^{\boldsymbol{v},\boldsymbol{u}} \\
\subset &\mathcal{G}_z(\mathcal{V}_{\mathbb{L}}, a, \overline{\mathrm{FE}}(\mathcal{V}_{N\mathbb{Z}^d}, v_0, v_1, v_4; n), \delta, v_1, v_2, v_3), \text{ and} \\
&\mathcal{G}_z(\mathcal{V}_{\mathbb{L}}, a, \overline{\mathrm{FE}}(\mathcal{V}_{\mathbb{L}}, u_0, u_1, u_4; n), \delta, u_1, u_2, u_3) \cap \mathcal{F}_z^{\boldsymbol{u},\boldsymbol{v}} \\
\subset &\mathcal{G}_z(\bar{\mathcal{V}}_{\mathbb{L}}, a, \overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}}, v_0, v_1, v_4; n), \delta, v_1, v_2, v_3),
\end{aligned} \tag{4.4.33}$$

where $\boldsymbol{u} = (u_0, \ldots, u_4)$, $\boldsymbol{v} = (v_0, \ldots, v_4)$ and $\mathcal{F}_z^{\boldsymbol{u},\boldsymbol{v}}$ is defined in (4.2.19) and (4.3.10). Indeed, we need only compare the monotonicities of the levels in $V_z(\tilde{\mathcal{V}}_{\mathbb{L}}, u_1, u_2, u_3)$ as well as the subsequences which occur in $\boldsymbol{Z}(\tilde{\mathcal{V}}_{\mathbb{L}}^{u_1}, u_0, u_4; n)$. A similar inspection also yields

$$\mathscr{C}_z(\bar{\mathcal{V}}_{\mathbb{L}}, \delta, u_1, u_2) \cap \mathcal{F}_z^{\boldsymbol{v},\boldsymbol{u}} \subset \mathscr{C}_z(\mathcal{V}_{\mathbb{L}}, \delta, v_1, v_2) \tag{4.4.34}$$

Moreover, by recalling the definition of $\mathrm{Incl}_z^{\epsilon,m}$ from (2.6.6) and (4.2.18), we observe that under any coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\hat{\mathbb{P}}_{\{z\}}$,

$$\begin{aligned}&\mathcal{G}_z(\hat{\mathcal{V}}_{\mathbb{L}}, a, \overline{\mathrm{FE}}(\hat{\mathcal{V}}_{\mathbb{L}}, u_0, u_1, u_4; n), \delta, u_1, u_2, u_3) \cap \mathrm{Incl}_z^{\frac{c(u_0\wedge u_1)\epsilon}{nu_1}, \lfloor v\mathrm{cap}(D_z)\rfloor}\\ \subset&\mathcal{G}_z(\bar{\mathcal{V}}_{\mathbb{L}}, a, \overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}}, u_{0,\epsilon}, u_{1,-\epsilon}, u_{4,-\epsilon}; n), \delta, u_{1,-\epsilon}, u_{2,\epsilon}, u_{3,-\epsilon}),\ \text{and}\\ &\mathcal{G}_z(\bar{\mathcal{V}}_{\mathbb{L}}, a, \overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}}, u_0, u_1, u_4; n), \delta, u_1, u_2, u_3) \cap \mathrm{Incl}_z^{\frac{c(u_0\wedge u_1)\epsilon}{nu_1}, \lfloor v\mathrm{cap}(D_z)\rfloor}\\ \subset&\mathcal{G}_z(\hat{\mathcal{V}}_{\mathbb{L}}, a, \overline{\mathrm{FE}}(\hat{\mathcal{V}}_{\mathbb{L}}, u_{0,\epsilon}, u_{1,-\epsilon}, u_{4,-\epsilon}; n), \delta, u_{1,-\epsilon}, u_{2,\epsilon}, u_{3,-\epsilon}),\end{aligned} \tag{4.4.35}$$

for $v = c\epsilon\min(u_0, u_1, u_4)$, $\epsilon \in (0, \frac{1}{10})$, $L \geq C(v, \epsilon)$ and $u_{k,\epsilon'} \coloneqq (1+\epsilon')u_k$ for any $k \in \{0, \dots, 4\}$ and $\epsilon' \in \{-1, 1\}$. Next, we claim that for any $\rho \in (0, \frac{1}{2}]$,

$$\begin{aligned}&\mathscr{G}(\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}, \{\mathcal{G}_z(\bar{\mathcal{V}}_{\mathbb{L}}, a, \overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}}, u_0, u_1, u_4; n), \delta, u_1, u_2, u_3) : z \in \mathbb{L}\}, \mathcal{F}_z^{\boldsymbol{v},\boldsymbol{u}}; \rho)\\ &\cap \{C_{0,N} \overset{N_{2\delta}(\mathcal{V}^{v_3})}{\longleftrightarrow} \partial D_{0,N}\}\\ \subset&\mathcal{G}_0(\mathcal{V}_{N\mathbb{Z}^d}, a', \overline{\mathrm{FE}}(\mathcal{V}_{N\mathbb{Z}^d}, v_0, v_1, v_4; n), \delta, v_1, v_2, v_3)\end{aligned} \tag{4.4.36}$$

where $a' = \left\lfloor \frac{(1-\rho)c_3 N}{h(KL)} \right\rfloor \cdot a$ (c.f. (4.3.41)) and $\boldsymbol{u}$, $\boldsymbol{v}$ are as above. To see this, we need only remark that condition (4.3.39) from Proposition 4.3.6 is satisfied with the choices $V_z^1 = V_z(\bar{\mathcal{V}}_z, \delta, u_1, u_2, u_3)$, $\tilde{V}_z^1 = V_z(\mathcal{V}_z, \delta, v_1, v_2, v_3)$, $W^1 = \{\overline{\mathrm{FE}}_{z,y}(\bar{\mathcal{V}}_{\mathbb{L}}, u_0, u_1, u_4; n) : z \in \mathbb{L}, y \in \mathbb{L}_0\}$, $\tilde{W}^1 = \{\overline{\mathrm{FE}}_{0,y}(\mathcal{V}_{N\mathbb{Z}^d}, u_0, u_1, u_4; n) : z \in \mathbb{L}, y \in \mathbb{L}_0\}$, $\mathscr{C}_z^1 = \mathscr{C}_z(\bar{\mathcal{V}}_{\mathbb{L}}, \delta, u_1, u_2)$, $\tilde{\mathscr{C}}_z^1 = \mathscr{C}_z(\mathcal{V}_{\mathbb{L}}, \delta, v_1, v_2)$, $a \in \mathbb{N}$ and $\mathcal{F}_{z,L} = \mathcal{F}_z^{\boldsymbol{v},\boldsymbol{u}}$ - this claim is immediate by applying the first part of (4.4.33). Hence, by Proposition 4.3.6 one guarantees the existence of a (random) non-empty $\mathcal{O} \subset \tilde{D}_{0,N} \cap \mathbb{L}$ such that every $*$-component $\mathcal{O}'$ of $\mathcal{O}$ satisfies $\tilde{C}_{0,N} \cap \mathbb{L} \preceq \mathcal{O}' \preceq \partial_{\mathbb{L}}(\tilde{D}_{0,N} \cap \mathbb{L})$ in the graph induced by the coarse-grained lattice $\mathbb{L}$. Moreover, the following holds (c.f. (4.3.40))

$$\begin{aligned}&\text{on the event } \mathscr{G}(\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}, \{\mathcal{G}_z(\bar{\mathcal{V}}_{\mathbb{L}}, u_0, u_1, u_4; n), \delta, u_1, u_2, u_3) : z \in \mathbb{L}\}, \mathcal{F}_z^{\boldsymbol{v},\boldsymbol{u}}; \rho)\\ &\text{for any } N, L, K \text{ satisfying (4.3.34) and } \rho \in (0, \frac{1}{2}],\ \text{the event } \bigcap_{z\in\mathcal{O}} V_z(\mathcal{V}_{\mathbb{L}}, \delta, v_1, v_2, v_3)\\ &\text{occurs.}\end{aligned} \tag{4.4.37}$$

In turn, recalling the definition of $V_z(\mathcal{V}_{\mathbb{L}}, \delta, u_1, u_2, u_3)$ from (4.4.15), we see that on the event $\bigcap_{z\in\mathcal{O}} V_z(\mathcal{V}_{\mathbb{L}}, \delta, v_1, v_2, v_3)$, we can glue clusters of the $*$-neighbours comprising any

$*$-connected component $\mathcal{O}'$ of $\mathcal{O}$ to get

$$\begin{aligned}&\text{for any } *\text{-connected component } \mathcal{O}' \subset \mathcal{O}, \text{ there exists a component}\\ &\mathscr{C}_{\mathcal{O}'} \text{ of } D_{0,N} \cap N_\delta(\mathcal{V}^{v_1}) \text{ which contains all crossing clusters of } \tilde{D}_z \setminus \tilde{C}_z\\ &\text{in } D_z \cap N_{2\delta}(\mathcal{V}^{u_2}) \text{ for each } z \in \mathcal{O}'.\end{aligned} \tag{4.4.38}$$

By now familiar arguments, the surrounding property of $\mathcal{O}'$ implies that any path γ crossing $\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$ must also cross $\tilde{D}_z \setminus \tilde{C}_z$ for some $z \in \mathcal{O}'$. Therefore, if γ lies in $N_{2\delta}(\mathcal{V}^{v_3})(\subset N_{2\delta}(\mathcal{V}^{v_2}))$, we see that the event $\bigcap_{z\in\mathcal{O}} V_z(\mathcal{V}_{\mathbb{L}}, \delta, v_1, v_2, v_3)$ further implies

$$\text{the component of } \gamma \text{ inside } D_z \cap N_\delta(\mathcal{V}^{v_1}) \text{ is contained in } \mathscr{C}_{\mathcal{O}'}. \tag{4.4.39}$$

Thus, recalling the definition of $V_z(\tilde{\mathcal{V}}_{\mathbb{L}}, \delta, v_1, v_2, v_3)$ from (4.4.15), we get

$$\bigcap_{z\in\mathcal{O}} V_z(\mathcal{V}_{\mathbb{L}}, \delta, v_1, v_2, v_3) \cap \{C_{0,N} \overset{N_{2\delta}(\mathcal{V}^{v_3})}{\longleftrightarrow} \partial D_{0,N}\} \subset V_0(\mathcal{V}_{N\mathbb{Z}^d}, \delta, v_1, v_2, v_3). \tag{4.4.40}$$

Remark that the extra steps in contrast with Proposition 4.3.6 are indeed necessary since, without the second event in the intersection in (4.4.40), we cannot imply the right-hand side therein. To conclude, in light of (4.3.41) we need only observe that $\mathscr{C}(\mathcal{V}_{N\mathbb{Z}^d}, \delta, v_1, v_2)$ contains $\bigcup_{z\in\mathcal{O}} \mathscr{C}_z(\mathcal{V}_{\mathbb{L}}, \delta, v_1, v_2)$ (see the definition of $\mathscr{C}^2$ below (4.3.41)). However, this is immediate since every $\mathscr{C}_{\mathcal{O}'}$ is a cluster of $D_{0,N} \cap N_\delta(\mathcal{V}^{v_1})$ and, by the reasoning above, we have

$$\begin{aligned}&\bigcap_{z\in\mathcal{O}} V_z(\mathcal{V}_{\mathbb{L}}, \delta, v_1, v_2, v_3) \cap \{C_{0,N} \overset{N_{2\delta}(\mathcal{V}^{v_3})}{\longleftrightarrow} \partial D_{0,N}\}\\ &\subset \left\{\begin{array}{c}\text{there exists a component } \mathscr{C}_{\mathcal{O}} \supset \cup_{z\in\mathcal{O}} \mathscr{C}_z(\mathcal{V}_{\mathbb{L}}, \delta, v_1, v_2) \text{ of } D_{0,N} \cap N_\delta(\mathcal{V}^{v_1}) \text{ which}\\ \text{contains all crossing clusters of } \tilde{D}_z \setminus \tilde{C}_z \text{ in } D_z \cap N_{2\delta}(\mathcal{V}^{v_2}) \text{ for each } z \in \mathcal{O}\end{array}\right\}.\end{aligned} \tag{4.4.41}$$

which is obtained simply by observing that $\gamma \subset N_{2\delta}(\mathcal{V}^{v_3})$ above belongs to any of the $\mathscr{C}_{\mathcal{O}'}$ simultaneously, hence the constituent clusters $\mathscr{C}_{\mathcal{O}'}$ can be glued together to form $\mathscr{C}_{\mathcal{O}}$ with the property above. As in the subcritical case, at this point we diverge in approach depending on whether we work in $d = 3$ or $d \geq 4$.

Supercritical Upper Bounds, $d = 3$. Essentially, what ensues is a more detailed iteration of the steps outlined in §4.4.1. The core structure, which serves as a reference, is delineated in steps (4.4.7)-(4.4.12) without introducing additional complexities regard-

ing the events under consideration. To this end, let $u < u_*$ and consider any

$$\epsilon \in \Big(0, (1-(\tfrac{u}{u_*})^{\frac{1}{30}}) \wedge (1-(\tfrac{u'}{u_*})^{\frac{1}{50}}) \wedge (20^{\frac{1}{50}}-1)\Big) \tag{4.4.42}$$

where u' is from Lemma 4.4.6. Now, consider L_{-1}, L_0 and $\delta_1 = \delta(L_0) \in (0, \frac{1}{2})$ as given by Lemma 4.4.6 corresponding to the parameters $u_0 = \frac{u'}{100n}$, $u_1 = u_*(1-\epsilon)^{21}$, $u_2 = u_*(1-\epsilon)^{20}$ and $u_3 = u_*(1-\epsilon)$ therein. Then, the conclusion of the Lemma (i.e. display (4.4.32)) yields the following estimate, which we use as a starting point in our operations.

$$\mathbb{P}\Big[\mathcal{G}_z(\mathcal{V}_{\mathbb{L}}, a=1, \overline{\mathrm{FE}}(\mathcal{V}_{\mathbb{L}}, \tfrac{u'}{100n}, u_{*,21}, u'; n), \delta, u_{*,21}, u_{*,20}, u_{*,1})\Big] \geq 1 - C(\epsilon, n)e^{-L^c} \tag{4.4.43}$$

where we denote $u_{*,k} = u_{*,\epsilon,k} = u_*(1-\epsilon)^k$ for any $k \in \mathbb{N}$. From now on, we aim to progressively localize the event above in order to obtain the estimate which will satisfy condition (4.3.12) of Proposition 4.3.3. Indeed, the first step is to apply the second inclusion in (4.4.33) for $K \geq C(\epsilon)$ together with the triangle inequality to see

$$\begin{aligned}
&\mathbb{P}\Big[\mathcal{G}_z(\bar{\mathcal{V}}_{\mathbb{L}}, a=1, \overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}}, (1+\epsilon)\tfrac{u'}{100n}, u_{*,22}, (1-\epsilon)u'; n), \delta, u_{*,22}, u_{*,19}, u_{*,2})\Big] \\
\geq &\mathbb{P}\Big[\mathcal{G}_z(\mathcal{V}_{\mathbb{L}}, a=1, \overline{\mathrm{FE}}(\mathcal{V}_{\mathbb{L}}, \tfrac{u'}{100n}, u_{*,21}, u'; n), \delta, u_{*,21}, u_{*,20}, u_{*,1})\Big] - \mathbb{P}[(\mathcal{F}_z^{\boldsymbol{u}_1, \boldsymbol{u}_2})^c] \\
\overset{(4.2.21),(4.4.43)}{\geq} &\ 1 - C(\epsilon, n)e^{-L^c}
\end{aligned} \tag{4.4.44}$$

where $\boldsymbol{u}_1 = (\frac{u'}{100n}, u_{*,21}, u_{*,20}, u_{*,1}, u')$ and $\boldsymbol{u}_2 = ((1+\epsilon)\frac{u'}{100n}, u_{*,22}, u_{*,19}, u_{*,2}, (1-\epsilon)u')$. Next, recalling the coupling $\mathbb{Q}_{\{z\}}$ introduced in §4.2.1 and keeping in mind its key properties (2.6.13), (4.2.18), we can apply the second inclusion in (4.4.35) to yield

$$\begin{aligned}
&\mathbb{P}\Big[\mathcal{G}_z(\hat{\mathcal{V}}_{\mathbb{L}}, a=1, \overline{\mathrm{FE}}(\hat{\mathcal{V}}_{\mathbb{L}}, (1+\epsilon)^2\tfrac{u'}{100n}, u_{*,23}, (1-\epsilon)^2u'; n), \delta, u_{*,23}, u_{*,18}, u_{*,3})\Big] \\
\geq &\mathbb{P}\Big[\mathcal{G}_z(\bar{\mathcal{V}}_{\mathbb{L}}, a=1, \overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}}, (1+\epsilon)\tfrac{u'}{100n}, u_{*,22}, (1-\epsilon)u'; n), \delta, u_{*,22}, u_{*,19}, u_{*,2})\Big] \\
&- \mathbb{P}\Big[\Big(U_z^{\frac{c\epsilon}{n}, \lfloor \frac{c\epsilon}{n}\mathrm{cap}(\check{D}_z)\rfloor}\Big)^c\Big] \overset{(2.6.13),(4.4.44)}{\geq} 1 - C(\epsilon, n)e^{-L^c}
\end{aligned} \tag{4.4.45}$$

for all $L \geq C(\epsilon)$, $K \geq \frac{C_5}{\epsilon}$. Overall, the first inclusion of (4.4.35) together with the estimate (4.4.45) show that (4.3.11) and (4.3.12) are satisfied, respectively, for the

choice of events

$$\begin{aligned}\hat{\mathcal{G}}_{z,L} &= \mathcal{G}_z(\hat{\mathcal{V}}_{\mathbb{L}}, a=1, \overline{\mathrm{FE}}(\hat{\mathcal{V}}_{\mathbb{L}}, (1+\epsilon)^2\tfrac{u'}{100n}, u_{*,23}, (1-\epsilon)^2u'; n), \delta, u_{*,23}, u_{*,18}, u_{*,3}), \text{ and}\\ \mathcal{G}_{z,L} &= \mathcal{G}_z(\bar{\mathcal{V}}_{\mathbb{L}}, a=1, \overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}}, (1+\epsilon)^3\tfrac{u'}{100n}, u_{*,24}, (1-\epsilon)^3u'; n), \delta, u_{*,24}, u_{*,17}, u_{*,4})\end{aligned} \tag{4.4.46}$$

and for $\epsilon_L = \frac{c\epsilon}{n}$, $m_L = \left\lfloor \frac{c\epsilon}{n}\mathrm{cap}(\breve{D}_z)\right\rfloor$, $p_L = C(\epsilon, n)e^{-L^c}$, any $\theta > 0$ and $K_0 = C(\epsilon)$. Therefore, the bound (4.3.13) from Proposition 4.3.3 in combination with the inclusion (4.4.36) implies, with the choices $a = 1$,

$$\begin{aligned}\boldsymbol{v} &= \Big((1+\epsilon)^4\tfrac{u'}{100n}, u_{*,25}, u_{*,16}, u_{*,5}, (1-\epsilon)^4u'\Big), \text{ and}\\ \boldsymbol{u} &= \Big((1+\epsilon)^3\tfrac{u'}{100n}, u_{*,24}, u_{*,17}, u_{*,4}, (1-\epsilon)^3u'),\end{aligned}$$

$\delta = \frac{1}{2}$, $\rho = \frac{1}{2}$, $K = C(\epsilon)$, $L = \lfloor(\log N)^\alpha\rfloor$ for some constant $\alpha \in (0,\infty)$ and $\Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$, that

$$\begin{aligned}&\mathbb{P}\Big[\mathcal{G}_0(\mathcal{V}_{N\mathbb{Z}^d}, a = 10^{-d}\left\lfloor\tfrac{c(\epsilon)N}{(\log N)^\alpha}\right\rfloor, \overline{\mathrm{FE}}(\mathcal{V}_{N\mathbb{Z}^d}, (1+\epsilon)^4\tfrac{u'}{200n}, u_{*,25}, (1-\epsilon)^4u'; n), \delta, u_{*,25}, u_{*,16}, u_{*,5}\Big]\\ &\geq 1 - C(\epsilon,n)e^{-\frac{N}{(\log N)^\beta}} - \mathbb{P}\Big[C_{0,N} \overset{N_{2\delta}(\mathcal{V}^{u_*,5})}{\longleftrightarrow} \partial D_{0,N}\Big]\end{aligned} \tag{4.4.47}$$

for some constant $\beta \in (0,\infty)$. Then, to ensure the utility of this estimate, we introduce the following Lemma, the proof of which we defer to §4.5.3.

Lemma 4.4.7. *With ϵ and δ chosen as chosen above, we have*

$$\mathbb{P}[C_{0,N} \overset{N_{2\delta}(\mathcal{V}^{u_*,5})}{\longleftrightarrow} \partial D_{0,N}] \geq \begin{cases} 1 - C(\epsilon)e^{-\frac{N}{(\log N)^\theta}}, & d = 3\\ 1 - C(\epsilon)e^{-cN}, & d \geq 4\end{cases} \tag{4.4.48}$$

where $\theta \in (0,\infty)$ is an absolute constant.

By applying Lemma 4.4.7 in the context of (4.4.47), we yield the following improvement over the a-priori bound suggested by (4.4.43), which we will consequently feed into a second iteration of the bootstrapping argument.

$$\begin{aligned}&\mathbb{P}\Big[\mathcal{G}_0(\mathcal{V}_{N\mathbb{Z}^d}, a = 10^{-d}\left\lfloor\tfrac{c(\epsilon)N}{(\log N)^\alpha}\right\rfloor, \overline{\mathrm{FE}}(\mathcal{V}_{N\mathbb{Z}^d}, (1+\epsilon)^4\tfrac{u'}{200n}, u_{*,25}, (1-\epsilon)^4u'; n), \delta, u_{*,25}, u_{*,16}, u_{*,5}\Big]\\ &\geq 1 - C(\epsilon,n)e^{-\frac{N}{(\log N)^\beta}}\end{aligned} \tag{4.4.49}$$

Thus, resetting the scale to L and rerunning the steps (4.4.44) through (4.4.45) with (4.4.49) in place of (4.4.43), we obtain

$$\begin{aligned}&\mathbb{P}\Big[\mathcal{G}_0(\hat{\mathcal{V}}_{\mathbb{L}}, a = 10^{-d}\left\lfloor \tfrac{c(\epsilon)N}{(\log N)^{\alpha}}\right\rfloor, \overline{\mathrm{FE}}(\hat{\mathcal{V}}_{\mathbb{L}}, (1+\epsilon)^6\tfrac{u'}{200n}, u_{*,27}, (1-\epsilon)^6u'; n), \delta, u_{*,27}, u_{*,14}, u_{*,7}\Big]\\ &\geq 1 - C(\epsilon, n)e^{-\frac{L}{(\log L)^{\beta}}}\end{aligned} \tag{4.4.50}$$

thereby satisfying the inclusion (4.3.11) and the estimate (4.3.12) as per the requirements of Proposition 4.3.3 with an improved choice of P_L and corresponding $\beta'(> \frac{1}{2})$ therein, for the new selection of the events

$$\begin{aligned}\hat{\mathcal{G}}_{z,L} &= \mathcal{G}_z(\hat{\mathcal{V}}_{\mathbb{L}}, a = 10^{-d}\left\lfloor \tfrac{c(\epsilon)L}{(\log L)^{\alpha}}\right\rfloor, \overline{\mathrm{FE}}(\hat{\mathcal{V}}_{\mathbb{L}}, (1+\epsilon)^6\tfrac{u'}{100n}, u_{*,27}, (1-\epsilon)^6u'; n), \delta, u_{*,27}, u_{*,14}, u_{*,7}), \text{ and}\\ \mathcal{G}_{z,L} &= \mathcal{G}_z(\bar{\mathcal{V}}_{\mathbb{L}}, a = 10^{-d}\left\lfloor \tfrac{c(\epsilon)L}{(\log L)^{\alpha}}\right\rfloor, \overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}}, (1+\epsilon)^7\tfrac{u'}{100n}, u_{*,28}, (1-\epsilon)^7u'; n), \delta, u_{*,28}, u_{*,13}, u_{*,8}).\end{aligned} \tag{4.4.51}$$

Then, with $L = \left\lfloor (\log N)^{\alpha'}\right\rfloor$ for some absolute constant $\alpha' \in (0, \infty)$, and localisation levels $(\boldsymbol{v}, \boldsymbol{u}) = (\boldsymbol{u}_2, \boldsymbol{u}_1)$ for $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ as in (4.4.27), with $\frac{u_0}{8} = (1+\epsilon)^7\frac{u'}{100n}$, $u = u_{*,29}$, $u_1 = u_{*,28}$, $u_2 = u_{*,13}$, $u_3 = u_{*,10}$, $u_4 = u_{*,8}$ and $2u_5 = (1-\epsilon)^7u'$, display (4.3.13) yields, in the notation of (4.4.26), that for any $z \in N\mathbb{Z}^d$,

$$\mathbb{P}\Big[\mathscr{G}_{0,N}(a = 10^{-d}\left\lfloor \tfrac{c(\epsilon)L}{(\log L)^{\alpha}}\right\rfloor, n, (1+\epsilon)^7\tfrac{u'}{100n}, u_{*,28}, u_{*,13}, u_{*,8}, (1-\epsilon)^7u')\Big] \geq 1 - C(\epsilon, n)e^{-\frac{c(\epsilon,n)N}{\log N}} \tag{4.4.52}$$

Although the number of localization levels under consideration may appear excessive, since five comparisons should suffice to obtain the above inequality, this preliminary step will prove valuable when used in conjunction with Proposition 4.4.5. Indeed, an application of the aforementioned proposition with $\nu = \frac{c(\epsilon)N}{(\log\log N)^{\alpha}}$ and the remaining parameters as above (remarking that $3nu_0 < u_5$ from the last condition on ϵ in (4.4.42)), together with the estimates in (4.4.52), Lemma 4.4.7 and the lower bound on the common cardinality m of any coarsening in $\mathcal{A}_L^K(\tilde{D}_z \setminus \tilde{C}_z)$ from (4.3.1) (of at least $\frac{cN}{KL}$), gets

$$\begin{aligned}&\mathbb{P}\Big[V_z(\mathrm{tr}(\boldsymbol{Z}_{\bullet}(\mathcal{V}_z^{u_{*,29}}; \nu = \tfrac{c(\epsilon)N}{(\log\log N)^{\alpha}}; n)))\Big]\\ \geq &1 - C(\epsilon, n)e^{-\frac{N}{(\log N)^{\beta}}} - C(u_0, u)^n\mathbb{P}\Big[(C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}(Z_z^{u_5})}{\longleftrightarrow} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1})\Big]\end{aligned} \tag{4.4.53}$$

for some absolute constant $\beta \in (0,\infty)$ arising from Lemma 4.4.7. The remaining probability left to bound in the above display is the content of the following Lemma, which we also defer to §4.5.3.

Lemma 4.4.8. *With L_{-1} as chosen above, we have*

$$\mathbb{P}\Big[(C_{0,N}\cap\mathbb{L}_{-1}) \xleftarrow{\mathcal{S}(Z_z^{\frac{1}{2}(1-\epsilon)^7u'})} \partial_{\mathbb{L}_{-1}}(D_{0,N}\cap\mathbb{L}_{-1})\Big] \geq \begin{cases} 1-Ce^{-\frac{N}{(\log N)^\theta}}, & d=3 \\ 1-Ce^{-cN}, & d\geq 4 \end{cases} \tag{4.4.54}$$

for some absolute constant $\theta \in (0,\infty)$.

Then, substituting this bound into (4.4.53), we obtain

$$\mathbb{P}\Big[V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\mathcal{V}_z^{u_*,29},\nu=\tfrac{c(\epsilon)N}{(\log\log N)^\alpha};n))\Big)\Big] \geq 1-C(\epsilon,n)e^{-\frac{N}{(\log N)^\beta}} \tag{4.4.55}$$

for some absolute constant $\beta \in (0,\infty)$. Now, in order to address the $u<u_*$ case of Theorems 4.4.1 and 4.4.2, we will apply one last bootstrapping step, but now with the events $\mathcal{G}_{z,L}$ and $\hat{\mathcal{G}}_{z,L}$ being some natural modifications of the event $V_z(\cdots)$ above. Before we proceed to their definitions, we first introduce some essential inclusions which will mirror the roles played by (4.4.33), (4.4.35), and will thus facilitate the application of Propositions 4.3.3 and 4.3.6 in usual fashion. For any $0<v<u$ and $z\in\mathbb{L}$, we have

$$V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\mathcal{V}_z^u,\nu;n))\Big)\cap\mathcal{F}_z^{u,v} \subset V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}_z^v,\nu;n))\Big) \tag{4.4.56}$$

Indeed, this is immediate by noticing that, under $\mathcal{F}_z^{u,v}$, any element of the the collection $\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}^v,\nu;n)$ is contained in the collection $\boldsymbol{Z}_\bullet(\mathcal{V}^u,\nu;n)$. Moreover, under any coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\hat{\mathbb{P}}_z$, one has

$$\begin{aligned} &V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\hat{\mathcal{V}}_z^u,\nu;n))\Big)\cap\mathrm{Incl}_z^{c\epsilon,\lfloor c\epsilon L\rfloor} \subset V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}_z^{u(1-\epsilon)},\nu-c\epsilon L;n))\Big), \text{ as well as} \\ &V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}_z^u,\nu;n))\Big)\cap\mathrm{Incl}_z^{c\epsilon,\lfloor c\epsilon L\rfloor} \subset V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\hat{\mathcal{V}}_z^{u(1-\epsilon)},\nu-c\epsilon L;n))\Big) \end{aligned} \tag{4.4.57}$$

whenever $\nu\geq c\epsilon L$, $u\in(0,u_*)$ and $L\geq\frac{C}{u\epsilon}$. The proof for this set of inclusions is somewhat more involved than the previous one (hence we postpone it), and the condition on ν is evidently non-trivial in the context of (4.4.55). In fact, this is precisely where the full generality for ϵ_L from Proposition 4.3.3 will be invoked, as $\epsilon=\epsilon_L$ should be a decreasing function of L (to be identified later) in order to satisfy such an assumption. Prior to introducing the final necessary inclusion, we briefly

present some notation. For any $u > 0$, $n \in \mathbb{N}^*$, $\tilde{\Lambda}_N = W_N \setminus V'_N$ with $\Lambda_N = V_N \setminus U_N \in \{B^2_{2N} \setminus B^2_N, \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}, B^2_N \setminus B^2_{\sigma N} : \sigma \in [0, \frac{1}{3})\}$ and $V_N \subset V'_N \subset W_N \subset\subset \mathbb{Z}^d$, let

$$\begin{aligned}&\bar{\boldsymbol{Z}}(\tilde{\Lambda}_N, \mathcal{V}^u; n) \text{ denote the collection of all sequences } \{Z_j^{V'_N, W_N}\}_{j \in J} \subset W^+_{V'_N, W_N} \\ &\text{(see (3.4.17)) where } J \subset \{1, \dots, \mathcal{N}^u_{V'_N, W_N}\} \text{ is a union of at most } n \text{ disjoint} \\ &\text{intervals.}\end{aligned} \tag{4.4.58}$$

and consider the following slight modification of $\mathrm{tr}(\cdot)$ from (4.4.30). Given any random finite collection $\boldsymbol{Z}$ of sequences of excursions,

$$\begin{aligned}&\mathrm{tr}^*(\boldsymbol{Z}) \text{ denotes the collection of all sequences that are obtained from} \\ &\text{some } Z \in \boldsymbol{Z} \text{ either by replacing one element in } Z \text{ with its truncation or} \\ &\text{by replacing at most two elements of } \boldsymbol{Z} \text{ with their one-sided excursions.}\end{aligned} \tag{4.4.59}$$

In particular, note that $\mathrm{tr}^*(\boldsymbol{Z}) \subset \mathrm{tr}(\boldsymbol{Z})$. Finally, we define the event which we will consequently bound with our bootstrapping approach, and which absorbs the probability decay corresponding to (4.4.2). Recall that, in the absence of the disconnection portion of the event, the probability would otherwise converge to 1. For any $\mathcal{V} \subset \mathbb{Z}^d$ random and $\Lambda_N = V_N \setminus U_N$ as above, define

$$\text{2-arms}(\mathcal{V}, \Lambda_N) = \{\text{there exists at most one component of } \mathcal{V} \cap V_N \text{ crossing } \Lambda_N\} \tag{4.4.60}$$

and, when $\mathcal{V} = \mathcal{V}(Z)$ for some sequence of truncated excursions Z, abbreviate $\text{2-arms}(\mathcal{V}(Z), \Lambda_N)$ by $\text{2-arms}(Z, \Lambda_N)$. We now claim that for any $\rho \in (0, \frac{1}{2}]$ and $0 < u < v$,

$$\mathscr{G}\Big(\Lambda_N, \Big\{V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}^v_z, \nu; n+2))\Big) : z \in \mathbb{L}\Big\}, \mathcal{F}^{u,v}_L; \rho\Big) \subset \text{2-arms}\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_N, \mathcal{V}^u, n)), \Lambda_N\Big) \tag{4.4.61}$$

where we abbreviate

$$\mathcal{F}^{u,v}_L = \{\mathcal{F}^{u,v}_z\}_{z \in \mathbb{L}} \tag{4.4.62}$$

and remark that the $n+2$ term arises due to the fact that, when excursions of $\tilde{\Lambda}_N$ are truncated on the right-hand-side, there could be sub-excursions of Λ_N which were in the tails pre-truncation and are consequently lost. In this way, there could be at least two gaps formed in intervals of excursions (one for each lost tail) which need to

be accounted for by an extra two intervals on the left-hand-side. Indeed, by setting $V_z^1 = V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}_z^v, \nu; n+2))\Big)$ and $\tilde{V}_z^1 = V_z\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_N, \mathcal{V}^u, n))\Big)$, $a = 0$ (see remark (4.3.5)) and $\mathcal{F}_{z,L} = \mathcal{F}_z^{u,v}$, observe that condition (4.3.39) is satisfied, therefore display (4.3.40) of Proposition 4.3.6 guarantees the existence of a (random) $*$-connected set $\mathcal{O}' \subset \Lambda_{N,-L} \cap \mathbb{L}$ such that $U_N \cap \mathbb{L} \preceq \mathcal{O}' \preceq \partial_{\mathbb{L}}(V_N \cap \mathbb{L})$ and the following holds

$$\begin{aligned}&\text{on the event } \mathscr{G}\Big(\Lambda_N, \Big\{V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}_z^v, \nu; n+2))\Big) : z \in \mathbb{L}\Big\}, \mathcal{F}_L^{u,v}; \rho\Big) \text{ for any}\\ &N, L, K \text{ satisfying the separation and scaling conditions from (4.3.34)},\\ &\rho \in (0, \tfrac{1}{2}] \text{ and } V_N \subset W_N \subset\subset \mathbb{Z}^d, \text{ the event } \bigcap_{z\in\mathcal{O}'} V_z\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_N, \mathcal{V}^u, n))\Big) \text{ occurs.}\end{aligned} \tag{4.4.63}$$

Since $V_z(\cdots)$ itself is both an existence and uniqueness event (see (4.4.15) for the definition), the $*$-connectedness of $\mathcal{O}'$ permits us to glue the clusters of $\mathcal{V}(Z)$ constructed in $*$-neighbouring boxes comprising $\mathcal{O}'$. All in all, remarking that $\mathcal{O}' \subset \Lambda_{N,-L} \cap \mathbb{L}$, on the event $\bigcap_{z\in\mathcal{O}'} V_z\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_N, \mathcal{V}^u, n))\Big)$, for any sequence of truncated excursions $Z \in \mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_N, \mathcal{V}^u, n))$, one has that

$$\begin{aligned}&\text{there exists a component } \mathscr{C}_{\mathcal{O}'}(Z) \subset V_N \cap \mathcal{V}(Z) \text{ which contains}\\ &\text{all crossing clusters of } \tilde{D}_z \setminus \tilde{C}_z \text{ in } D_z \cap \mathcal{V}^u \text{ for each } z \in \mathcal{O}'.\end{aligned} \tag{4.4.64}$$

Since $U_N \cap \mathbb{L} \preceq \mathcal{O}' \preceq \partial_{\mathbb{L}}(V_N \cap \mathbb{L})$, any crossing γ of $V_N \setminus U_N$ must also cross $\tilde{D}_z \setminus \tilde{C}_z$ for some $z \in \mathcal{O}'$, hence, if such a crossing additionally satisfies $\gamma \subset \mathcal{V}(Z)$ for some $Z \in \mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_N, \mathcal{V}^u, n))$, it must hold that $\gamma \subset \mathscr{C}_{\mathcal{O}'}$ by the uniqueness property of (4.4.15). Thus, for any $Z \in \mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_N, \mathcal{V}^u, n))$, given any $\gamma \subset \mathcal{V}(Z)$,

$$\text{the component of } \gamma \text{ inside } \mathcal{V}(Z) \cap V_N \text{ is contained in } \mathscr{C}_{\mathcal{O}'}(Z) \tag{4.4.65}$$

Therefore, we obtain the following inclusion

$$\bigcap_{z\in\mathcal{O}'} V_z\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_N, \mathcal{V}^u, n))\Big) \subset \text{2-arms}\Big(\mathrm{tr}^*(\bar{Z}(\tilde{\Lambda}_N, \mathcal{V}^u, n)), \Lambda_N\Big) \tag{4.4.66}$$

which, in light of (4.4.64), shows (4.4.61). To then return to our desired result (4.4.2), we observe that for any $\sigma \in (0, \frac{1}{2}]$,

$$\text{2-arms}(\mathcal{V}^u, B_N^2 \setminus B_{\sigma N}^2) \cap \{B_{\sigma N} \overset{\mathcal{V}^u}{\longleftrightarrow} \infty\} \cap \{0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_N^2\} \subset \{0 \overset{\mathcal{V}^u}{\longleftrightarrow} \infty\} \tag{4.4.67}$$

Hence, since 2-arms$\big(\mathrm{tr}^*(\bar{Z}(\tilde{\Lambda}_N, \mathcal{V}^u, n)), \Lambda_N\big) \subset$ 2-arms$(\mathcal{V}^u, \Lambda_N)$, by (4.4.61) one sees

$$\begin{aligned}&\{0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_N^2,\ 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\}\\ &\quad\subset \mathscr{G}\Big(\Lambda_N, \Big\{V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}_z^v, \nu; n+2))\Big) : z \in \mathbb{L}\Big\}, \mathcal{F}_L^{u,v}; \rho\Big)^c \cup \{B_{\sigma N} \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\}\end{aligned} \tag{4.4.68}$$

with $\Lambda_N = B_N^2 \setminus B_{\sigma N}^2$, $\tilde{\lambda}_N = B_{2N}^2 \setminus B_N$. Now, we once more turn to satisfying the various conditions of Proposition 4.3.3 in order to handle the first event on the right-hand side of (4.4.68). In view of the first inclusion in (4.4.57), condition (4.3.11) is satisfied immediately for the choice of events

$$\begin{aligned}\hat{\mathcal{G}}_{z,L} &= V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\hat{\mathcal{V}}^{u_{*,31}}, \nu = \tfrac{c(\epsilon)L}{(\log\log L)^\alpha} - c\epsilon_L L; n))\Big)\\ \mathcal{G}_{z,L} &= V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}^{(1-\epsilon_L)u_{*,31}}, \nu = \tfrac{c(\epsilon)L}{(\log\log L)^\alpha} - 2c\epsilon_L L; n))\Big)\end{aligned} \tag{4.4.69}$$

as soon as $\epsilon_L = c(\epsilon)(\log\log L)^{-\alpha}$. Moreover, one can see that (4.3.12) is satisfied for $\hat{G}_{z,L}$ as above (with $p_L = C(\epsilon, n)e^{-\frac{N}{(\log N)^\beta}}$) by taking the same localisation steps as in (4.4.44) through (4.4.45), except starting with the event in (4.4.55) in place of (4.4.43) and using identities (4.4.56) and (4.4.57) with ϵ_L as above. Crucially, this choice of ϵ_L relies on us taking $K \geq C(\log\log L)^\alpha$ to satisfy (4.2.18), which is within the threshold of K permitted by Proposition 4.3.3 (see below (4.3.13) for K_- and K_+). Then, by letting $\mathcal{F}_L^{\boldsymbol{u},\boldsymbol{v}} = \mathcal{F}_L^{u,(1-\epsilon_L)u_{*,31}}$ and applying the Proposition, there exists an absolute constant α' such that for $L(N) = \big\lfloor (\log N)^{\alpha'} \big\rfloor$, one has for any $\delta \in (0,1)$, $\rho, \sigma \in (0, \frac{1}{2}]$, $K = C(\log\log L)^\alpha$, and for $N \geq C(\delta, \rho, K_0, L_0)$

$$\begin{aligned}\log \mathbb{P}\Big[\mathscr{G}\Big(\Lambda_N, \Big\{V_z\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}_z^{(1-\epsilon_L)u_{*,31}}, \nu; n+2))\Big) : z \in \mathbb{L}\Big\}, \mathcal{F}_L^{u,(1-\epsilon_L)u_{*,31}}; \rho\Big)^c\Big]\\ \leq -(1-\delta)(1-\sigma)(1-\rho)\frac{\pi}{3}\Big(\sqrt{(1-\epsilon_L)u_{*,31}} - \sqrt{u}\Big)^2 \frac{N}{\log N}\end{aligned} \tag{4.4.70}$$

Next, we handle the second event on the right-hand side of (4.4.68) by applying the same ideas as below display (5.72) of [28]. Indeed, let $M > 1$, and define the events

$$\begin{aligned}\mathcal{E}_1 &\coloneqq \{B_N \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_{4MN}\},\\ \mathcal{E}_{2k} &\coloneqq \text{2-arms}(\mathcal{V}^u, \Lambda_{M,N}^k), \quad \mathcal{E}_{2k+1} \coloneqq \{B_{2^k MN} \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_{2^{k+2}MN}\}, \quad k \geq 1\end{aligned} \tag{4.4.71}$$

where $\Lambda^k_{M,N} \coloneqq B_{2^{k+1}MN} \setminus B_{2^kMN}$. Then, recalling the definition of 2-arms$(\mathcal{V}, \Lambda)$, we observe that

$$\bigcap_{j\geq 0} \mathcal{E}_j \subset \{B_N \overset{\mathcal{V}^u}{\longleftrightarrow} \infty\} \tag{4.4.72}$$

Hence, from the bound on 2-arms$(\mathcal{V}^u, \Lambda^k_{M,N})$ (obtained in (4.4.70) in light of the inclusion (4.4.61) and the comment above (4.4.68)) combined with the supercritical bound on disconnection from [64, (0.6)], we see that

$$\mathbb{P}[B_N \overset{\mathcal{V}^u}{\longleftrightarrow} \infty] \leq e^{-cN} \sum_{k\geq 0} e^{-c(u)2^kMN} + \sum_{k\geq 1} e^{-c(u,u_*)\frac{2^kMN}{\log(2^kMN)}} \leq e^{-c(u,u_*)\frac{MN}{\log N}} \tag{4.4.73}$$

Hence, since M was arbitrary, we obtain that

$$\lim_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[B_N \overset{\mathcal{V}^u}{\longleftrightarrow} \infty] = 0 \tag{4.4.74}$$

Overall, by applying (4.4.70) and (4.4.74) in (4.4.68), and sequentially taking $\sigma, \rho, \delta, \epsilon \downarrow 0$, we obtain

$$\limsup_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B^2_N,\ 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty] \leq -\frac{\pi}{3}\Big(\sqrt{u_*} - \sqrt{u}\Big)^2 \tag{4.4.75}$$

□

Therefore, to complete the proof, all that is left is to show (4.4.57).

Proof of Inclusion (4.4.57). The justification for (4.4.57) proceeds in the following way, where we only show the first line as the steps for the second are identical. Similarly to the approach discussed for inclusion (4.4.56), we aim to show that, whenever $\mathrm{Incl}^{c\epsilon,\lfloor c\epsilon L\rfloor}_z$ occurs, then

$$\boldsymbol{Z}_\bullet(\hat{\mathcal{V}}^{u(1-\epsilon)}_z, \nu - c\epsilon L; n) \subset \boldsymbol{Z}_\bullet(\bar{\mathcal{V}}^u_z, \nu; n) \tag{4.4.76}$$

To this end, we temporarily denote by $[a, b]$, $\widehat{[a, b]} \subset \mathbb{Z}^d$ intervals of indices of the interlacement excursions $\{Z_i^{D_z,U_z}\}_{i\geq 1}$ and independent excursions $\{\hat{Z}_i^{D_z,U_z}\}_{i\geq 1}$ respectively. In a similar vein, when writing inclusions between such intervals, $[a, b] \subset \widehat{[c, d]}$ is used to mean $\bigcup_{j\in[a,b]} Z_j \subset \bigcup_{j\in\widehat{[c,d]}} \hat{Z}_j$. We also abbreviate $\nu' = \nu - c\epsilon L$, simplify to the case where $n = 1$ (the general case follows similarly), and consider an arbitrary element of the left-hand-side of (4.4.76), which is a subsequence of independent excursions with

index set $\hat{J} \subset \{1, \hat{\mathcal{N}}_z^{u(1-\epsilon)}\}$ satisfying (by definition of $\boldsymbol{Z}_\bullet$, see (4.4.29))

$$J \subset \hat{J} \subset J + \{0, \dots \pm |\nu'|\} \tag{4.4.77}$$

for an interval $J = \widehat{[s, e]}$, with some $s, e \in \mathbb{N}^*$, and $J + \{0, \dots \pm |\nu'|\} = \widehat{[s_0, e_0]}$, with $s_0 = s - \nu'$, $e_0 = e + \nu'$. Also, recalling the nature of our coupling $\mathbb{Q}_{\{z\}}$, denote by $\bar{J}$ the corresponding index set of RI excursions such that

$$\bigcup_{j \in \bar{J}} Z_j = \bigcup_{j \in \hat{J}} \hat{Z}_j \tag{4.4.78}$$

Let us split the approach into cases.

Case i): $(s > c\epsilon L + \nu')$ One can directly apply the coupling prescribed by $\mathrm{Incl}_z^{c\epsilon, \lfloor c\epsilon L \rfloor}$ to see

$$\begin{aligned} \big[\ (1+\epsilon)s,\ (1-\epsilon)e\big] \subset \widehat{[s, e]}\ \subset \big[\ (1-\epsilon)s,\ (1+\epsilon)e\big], \text{ as well as} \\ \big[\ (1+\epsilon)s_0,\ (1-\epsilon)e_0\big] \subset \widehat{[s_0, e_0]} \subset \big[\ (1-\epsilon)s_0,\ (1+\epsilon)e_0\big], \end{aligned} \tag{4.4.79}$$

Thus, we have that

$$\big[\ (1+\epsilon)s,\ (1-\epsilon)e\big] \subset \bar{J} \subset \big[\ (1-\epsilon)s_0,\ (1+\epsilon)e_0\big], \tag{4.4.80}$$

which, assuming the interval on the left-hand-side is non-empty, yields the required form for $\bar{J} \in \boldsymbol{Z}_\bullet(\bar{\mathcal{V}}_z^u, \nu; n)$ as soon as

$$(1+\epsilon)s - (1-\epsilon)s_0 < \nu, \text{ and } (1+\epsilon)e_0 - (1-\epsilon)e < \nu \tag{4.4.81}$$

which indeed holds for the choice of ν' above.

Case ii): $(s > c\epsilon L,\ s \leq c\epsilon L + \nu')$ Since s_0 is smaller than the coupling threshold, we cannot obtain the first line of display (4.4.79), however a similar approach to *Case i)* yields

$$\big[\ (1+\epsilon)s,\ (1-\epsilon)e\big] \subset \bar{J} \subset [0,\ (1+\epsilon)e_0], \tag{4.4.82}$$

which still satisfies a comparison as in (4.4.80) to ensure $\bar{J} \in \boldsymbol{Z}_\bullet(\bar{\mathcal{V}}_z^u, \nu; n)$.

Case iii): $(s \leq c\epsilon L,\ e > \frac{1+\epsilon}{1-\epsilon} c\epsilon L)$ Using that $\widehat{[c\epsilon L, e]} \subset \widehat{[s, e]}$ and proceeding as in *Case i)* with the intervals $\widehat{[c\epsilon L, e]}$ and $\widehat{[0, e_0]}$ yields

$$[(1+\epsilon)c\epsilon L, (1-\epsilon)e] \subset \bar{J} \subset [0, (1+\epsilon)e_0]. \tag{4.4.83}$$

We conclude as in *Case ii)*.

Case iv): ($s \leq c\epsilon L$, $e \leq \frac{1+\epsilon}{1-\epsilon} c\epsilon L$) We need only to consider the interval $\widehat{[0, e_0]}$, and any other element in $\hat{J}$, since we have

$$\bar{J} \subset [0, \tfrac{1+\epsilon}{1-\epsilon} c\epsilon L + \nu'] \tag{4.4.84}$$

and the latter interval is sufficiently small to be dealt with by the deviation ν from the chosen point.

Finally, remark that when $e - s \geq c\epsilon L$, the first interval in (4.4.80) cannot be nonempty, and otherwise we can simplify the above steps in the same spirit as *Case iv)*. □

Supercritical Upper Bounds, $d \geq 4$. To obtain (4.4.1), a largely streamlined argument is required in comparison to the case of $d = 3$. In particular, we trace the same steps from (4.4.42) up to (4.4.55), except using the $d \geq 4$ versions of Lemma 4.4.7 and Lemma 4.4.8 to obtain the bound

$$\mathbb{P}\Big[V_z\Big(\mathrm{tr}(Z_\bullet(\mathcal{V}_z^{u_{*,29}}, \nu = \tfrac{c(\epsilon)N}{(\log\log N)^\alpha}; n))\Big)\Big] \geq 1 - C(\epsilon, n)e^{-cN} \tag{4.4.85}$$

At this point we diverge in strategy, since we observe that the above bound is already sufficient for our requirements. In this way, we notice that

$$V_z\Big(\mathrm{tr}(Z_\bullet(\mathcal{V}_z^{u_{*,29}}, \nu = \tfrac{c(\epsilon)N}{(\log\log N)^\alpha}; n))\Big) \subset V_z(\mathcal{V}^u) \subset 2\text{-arms}(\mathcal{V}^u, \tilde{D}_N \setminus \tilde{C}_N) \tag{4.4.86}$$

such that, by similar steps to the conclusion of the $d = 3$ case, we have

$$\{0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial\tilde{D}_N, 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\} \subset V_z\Big(\mathrm{tr}(Z_\bullet(\mathcal{V}_z^{u_{*,29}}, \nu = \tfrac{c(\epsilon)N}{(\log\log N)^\alpha}; n))\Big)^c \cup \{\tilde{C}_N \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\} \tag{4.4.87}$$

Taking probabilities, logs, and dividing through by N yields the required result. □

4.4.3 Torus Cluster Sizes

Proof of Theorem 1.3.2 for $d = 3$: It turns out that, in light of Theorem 4.4.2, the steps necessary to yield an upper bound on cluster sizes in the subcritical regime become straightforward. First, let us revisit the salient quantity $r_N = r_N(u)$ from

(1.3.4), and observe the following relation

$$\{\mathrm{diam}(\mathcal{C}^u_{\max}) \geq \lambda r_N\} = \Big\{ \bigcup_{x\in\mathbb{T}} \{\mathrm{diam}(\mathcal{C}^u(x)) \geq \lambda r_N\} \Big\} = \Big\{ \bigcup_{x\in\mathbb{T}} \{x \overset{\mathcal{V}^u_N}{\longleftrightarrow} \partial B^2_{\lambda r_N}(x)\} \Big\} \tag{4.4.88}$$

where we recall the notation $\mathcal{C}^u_{\max}$ and $\mathcal{C}^u(x)$ from the paragraph preceding (1.3.2), as well as the vacant set $\mathcal{V}^u_N$ corresponding to the SRW on a torus. Then, we simply apply a union bound (the complexity of which is well-behaved) followed by the coupling of the torus random walk with the RI from Theorem 2.4.2, as well as our result from Theorem 4.4.2 to bound the comprising probabilities.

$$\begin{aligned}
\lim_{N\to\infty} P_N[\mathrm{diam}(\mathcal{C}^u_{\max}) \geq \lambda r_N] &\overset{(4.4.88)}{\leq} \lim_{N\to\infty} N^3 P_N\Big[0 \overset{\mathcal{V}^u_N}{\longleftrightarrow} \partial B^2_{\lambda r_N}\Big] \\
&\overset{\text{Thm. 2.4.2}}{\leq} \lim_{N\to\infty} \Big\{ e^{3\log N} \mathbb{P}\Big[0 \overset{\mathcal{V}^{u(1-\epsilon)}}{\longleftrightarrow} \partial B^2_{\lambda r_N}\Big] + e^{c\log N - c\epsilon\sqrt{uN}} \Big\} \\
&\overset{\text{Thm. 4.4.2}}{\leq} \lim_{N\to\infty} e^{3\log N - \frac{\pi}{3}(\sqrt{u(1-\epsilon)}-\sqrt{u_*})^2 \frac{\lambda r_N}{\log \lambda r_N}} \\
&\overset{(1.3.4),\,(1.3.5)}{=} \lim_{N\to\infty} \exp\left\{ \left[3 - 3\lambda \left(\frac{\sqrt{u(1-\epsilon)} - \sqrt{u_*}}{\sqrt{u} - \sqrt{u_*}} \right)^2 \right] \log N \right\}
\end{aligned} \tag{4.4.89}$$

Hence, since the left-hand-side is independent of ϵ, we have that the above quantity decays to 0 for any $\lambda \in (1,\infty)$ by letting $\epsilon \downarrow 0$.

Thus, we have shown the result for the subcritical case, and now endeavour to also prove the supercritical case. To this end, given a vacant set $\mathcal{V} \subset \mathbb{T}$ (or $\mathcal{V} \subset \mathbb{Z}^d$) and some finite subsets $A, U \subset\subset \mathbb{T}$ (or $\subset\subset \mathbb{Z}^d$), we define the event $\mathrm{LU}(\mathcal{V}, A, U)$ as

$$\left\{ \begin{matrix} \text{There exists at least one cluster of } \mathcal{V} \text{ crossing } A, \text{ and} \\ \text{there exists at most one cluster of } \mathcal{V} \text{ crossing } U. \end{matrix} \right\} \tag{4.4.90}$$

Moreover, for an anchor point $x \in \mathbb{T}$ (or $x \in \mathbb{Z}^d$) and some real positive value η, we define the annuli $A_\eta(x) = B_{(2+\eta)\lambda r_N} \setminus B_{(1-\eta)\lambda r_N}(x)$, and introduce the shortened notation

$$\mathrm{LU}_x(\mathcal{V}, \eta) = \mathrm{LU}_{x,\lambda,N}(\mathcal{V}, \eta) = \mathrm{LU}(\mathcal{V}, A_\eta(x), A_0(x)) \tag{4.4.91}$$

which, given a sequence of (truncated) excursions Z either belonging to the torus or the integer lattice, is further abbreviated as $\mathrm{LU}_x(\mathcal{V}(Z),\eta) = \mathrm{LU}_x(Z,\eta)$. For example the familiar abbreviations for $\tilde{Z}^u$ chosen from (4.2.24) yield $\mathrm{LU}_x(\tilde{\mathcal{V}}^u,\eta) = \mathrm{LU}_x(\tilde{Z}^u,\eta)$. Observe that an intersection of local uniqueness events as in (4.4.90) with anchor points x spanning the entire torus $\mathbb{T}$ is seen to imply the crucial inclusion

$$\bigcap_{x\in\mathbb{T}} \mathrm{LU}_x(\mathcal{V}_N^u,\eta) \subset \{\mathrm{diam}(\mathcal{C}_{\mathrm{sec}}^u) < \lambda r_N\} \tag{4.4.92}$$

Indeed, consider two neighbouring points $x, y \in \mathbb{T}$. Under the assumption of the event on the left-hand-side of (4.4.92), we have that there exist clusters of $\mathcal{V}_N^u$ crossing both $A_\eta(x)$ and $A_\eta(y)$, which in particular must simultaneously cross both $A_0(x)$ and $A_0(y)$. By the uniqueness part of (4.4.90), the two aforementioned clusters must comprise the same cluster in $\mathcal{V}_N^u$, so by iterating this procedure across all $x \in \mathbb{T}$ we obtain a cluster of $\mathcal{V}_N^u$ which crosses $A_0(x)$ for every $x \in \mathbb{T}$. The required inclusion follows by remarking that the diameter of the resulting cluster is at least (say) $\frac{N}{10} > \lambda r_N$, and any cluster diameter $\geq \lambda r_N$ must itself be connected to this macroscopic cluster, again by the uniqueness property stated within (4.4.90).

The remainder of the proof concerns itself with obtaining the estimate

$$P_N[\mathrm{LU}_0(\mathcal{V}_N^u,\eta)] \geq 1 - C(u,\lambda)N^{-(3+c(u,\lambda))} \tag{4.4.93}$$

which immediately yields the required result via a simple union bound and translational invariance, in light of the inclusion (4.4.92). The principal difficulty in obtaining this bound is to determine a suitable coupling between random walk excursions on the torus and RI excursions on the integer lattice - once this matter is resolved, the problem is dealt with by combining known bounds on disconnection events with the result (4.4.61) concerning the 2-arms event and the discussion below (4.4.68). To this extent, we present the pivotal inequality which aids in overcoming the stated difficulty and arises as a consequence of displays (10.3), (10.11) and Propositions 9.1 and 9.3 in [9]. Let $u > 0$ and $\epsilon_N = N^{-\frac{1}{4}}$. There exist two sets $\{\lceil \frac{N}{4} \rceil, \dots, N - \lceil \frac{N}{4} \rceil\}^d \subset D_N \subset U_N \subset \{0,\dots,N-1\}^d$ and a coupling $\mathbb{Q}_N$ of $\mathbb{P}$ and P_N such that

$$\begin{aligned} &\mathbb{Q}_N\Big[\mathcal{V}_N^u \cap D_N = \mathcal{V}(Z) \cap D_N \text{ for some } Z \in \mathrm{tr}^*\Big(\bar{\boldsymbol{Z}}(U_N \setminus D_N, u(1\pm\epsilon_N))\Big)\Big] \\ \geq &1 - C(u)e^{-c(u)N^{c(u)}} \end{aligned} \tag{4.4.94}$$

where D_N is either viewed as a subset of the integer lattice or its canonical projection on the torus, depending on context, and

$$\begin{aligned}&\bar{\boldsymbol{Z}}(U_N \setminus D_N, u(1\pm\epsilon_N)) \text{ denotes the collection of all sequences}\\&\{Z_j^{D_N,U_N}\}_{j\in J} \text{ where } J \text{ satisfies } \{1,\dots,\mathcal{N}_{D_N,U_N}^{u(1-\epsilon_N)}\} \subset J \subset \{1,\dots,\mathcal{N}_{D_N,U_N}^{u(1+\epsilon_N)}\}.\end{aligned} \tag{4.4.95}$$

Recalling the definition of the operator $\mathrm{tr}^*(\cdot)$ from (4.4.59), observe that given any $S_N \subset D_N \subset U_N \subset\subset \mathbb{Z}^d$, then for any $Z \in \mathrm{tr}^*\Big(\bar{\boldsymbol{Z}}(U_N \setminus D_N, u(1\pm\epsilon_N))\Big)$ there exists some

$$Z' \in \mathrm{tr}^*\Big(\bar{\boldsymbol{Z}}(U_N \setminus S_N, u(1\pm\epsilon_N))\Big) \text{ satisfying } \mathcal{V}(Z)\cap S_N = \mathcal{V}(Z')\cap S_N. \tag{4.4.96}$$

Display (4.4.96) provides us with sufficient flexibility to assert that the coupling $\mathbb{Q}_N$ from (4.4.94) also satisfies

$$\begin{aligned}&\mathbb{Q}_N\Big[\mathcal{V}_N^u \cap B_{(2+\eta)\lambda r_N} = \mathcal{V}(Z)\cap B_{(2+\eta)\lambda r_N} \text{ for some } Z \in \mathrm{tr}^*\Big(\bar{\boldsymbol{Z}}(U_N \setminus B_{(2+\eta)\lambda r_N}, u(1\pm\epsilon_N))\Big)\Big]\\&\geq 1 - C(u)e^{-c(u)N^{c(u)}}\end{aligned} \tag{4.4.97}$$

for any $\lambda \in (1,\infty)$ such that $B_{(2+\eta)\lambda r_N} \subset D_N$ (λ can be taken arbitrarily large as we work in the limit $N\to\infty$). Hence, remarking that

$$\mathrm{LU}_0(\mathcal{V},\eta) = \text{2-arms}(\mathcal{V}, \Lambda_{\lambda r_N}) \cap \{B_{(1-\eta)\lambda r_N} \overset{\mathcal{V}}{\longleftrightarrow} \partial B_{(2+\eta)\lambda r_N}\}, \tag{4.4.98}$$

by letting $\Lambda_{\lambda r_N(u)} = A_0(0)$ and $\tilde{\Lambda}_{\lambda r_N(u)} = U_N \setminus B_{(2+\eta)\lambda r_N(u)}$ in (4.4.58) and with any $n\in\mathbb{N}$, one obtains by virtue of (4.4.97)

$$\begin{aligned}&P_N[\mathrm{LU}_0(\mathcal{V}_N^u,\eta)]\\&\geq \mathbb{P}\Big[\text{2-arms}\Big(\mathrm{tr}^*\Big(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_{\lambda r_N}, \mathcal{V}^{u(1+\epsilon_N)}, n)\Big), \Lambda_{\lambda r_N}\Big)\Big] - \mathbb{P}[B_{(1-\eta)\lambda r_N} \overset{\mathcal{V}^{u(1+\epsilon_N)}}{\longleftrightarrow} \partial B_{(2+\eta)\lambda r_N}]\\&\qquad - \mathbb{P}[\mathcal{N}^{2u\epsilon_N}_{B_{(2+\eta)\lambda r_N},U_N} \geq n] - C(u)e^{-c(u)N^{c(u)}}\end{aligned} \tag{4.4.99}$$

where we have used monotinicity of the disconnection event appearing on the right-hand-side, and the penultimate term there arises as a consequence of definition (4.4.95), since $\bar{\boldsymbol{Z}}(U_N \setminus D_N, u(1\pm\epsilon_N))$ can be represented as at most n disjoint intervals of excursions comprising $\mathcal{V}^{u(1+\epsilon_N)}$ when the excursion number $\mathcal{N}_{D_N,U_N}^{u(1+\epsilon_N)} - \mathcal{N}_{D_N,U_N}^{u(1-\epsilon_N)} \overset{\text{law}}{=} \mathcal{N}_{D_N,U_N}^{2u\epsilon_N}$ is bounded above by n. To conclude, we deal with each of the terms on the right-hand-side

of (4.4.99), starting with the first by letting $N = \lambda r_N$ in (4.4.61) for any $n \in \mathbb{N}^*$ and referring to the discussion below (4.4.68) to obtain for all $\lambda \in (1, \infty)$ and $\eta = \eta(\lambda) > 0$ sufficiently small

$$\mathbb{P}\Big[\text{2-arms}\Big(\text{tr}^*\Big(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_{\lambda r_N}, \mathcal{V}^{u(1+\epsilon_N)}, n)\Big), \Lambda_{\lambda r_N}\Big)\Big] \geq 1 - C(u,\lambda) N^{-(3+c(u,\lambda,\eta))} \tag{4.4.100}$$

Next, remark that the disconnection event is dealt with immediately by display (0.6) of [64], which asserts that for $d = 3$

$$\mathbb{P}[B_{(1-\eta)\lambda r_N} \overset{\mathcal{V}^{u(1+\epsilon_N)}}{\longleftrightarrow} \partial B_{(2+\eta)\lambda r_N}] \leq e^{-c(u)(1-\eta)\lambda r_N} \tag{4.4.101}$$

thus, since this term has a faster rate of decay than the error in (4.4.100), it becomes inconsequential. Finally, we turn to the term concerning the excursion number, which we deal with in a similar way to the steps taken in the proof of Lemma 4.2.2, namely by applying a Chernoff bound together with Lemma 3.4.4.

$$\begin{aligned}\mathbb{P}[\mathcal{N}^{2u\epsilon_N}_{B_{(2+\eta)\lambda r_N(u)}, U_N} \geq n] &\leq \inf_{\mu>0} e^{-\mu n} \mathbb{E}\Big[e^{\mu \mathcal{N}^{2u\epsilon_N}_{B_{(2+\eta)\lambda r_N(u)}, U_N}}\Big] \\ &\overset{\text{Lem. 3.4.4}}{\leq} \inf_{\mu>0} e^{-\mu n + 2u\epsilon_N \text{cap}(B_{(2+\eta)\lambda r_N}) \frac{e^\mu}{(1-ce^\mu \text{cap}(B_{(2+\eta)\lambda r_N(u)}) N^{-1})_+}}\end{aligned} \tag{4.4.102}$$

where in the last line we used that $\sup_{x \in U_N} P_x[H_{B_{(2+\eta)\lambda r_N}} < \infty] \leq c\text{cap}(B_{(2+\eta)\lambda r_N}) N^{-1}$ by an application of (2.2.23). Then, by remarking that $\text{cap}(B_{(2+\eta)\lambda r_N(u)}) \leq C r_N(u)$, setting $\mu = \frac{1}{8} \log N$ and recalling our choice $\epsilon_N = N^{-\frac{1}{4}}$, overall one observes

$$\mathbb{P}[\mathcal{N}^{2u\epsilon_N}_{B_{(2+\eta)\lambda r_N(u)}, U_N} \geq n] \leq e^{-\frac{n}{8}\log N + C(u)}, \tag{4.4.103}$$

such that we deduce the correct rate of decay in (4.4.93) by substituting the estimates from (4.4.100), (4.4.101) and (4.4.103) into (4.4.99) and fixing some $n \geq 25$. □

Proof of Theorem 1.3.2 for $d \geq 4$: The subcritical case follows almost identically, replacing the application of Theorem 4.4.2 with Theorem 4.4.1. For the supercritical case, we first remark that the inclusion (4.4.92) still holds when replacing $\text{LU}_x(\mathcal{V}^u_N, 1)$ with $\text{LU}(\mathcal{V}^u_N, D_{x,\lambda r_N} \setminus C_{x,\lambda r_N}, \tilde{D}_{x,\lambda r_N} \setminus \tilde{C}_{x,\lambda r_N})$ and then proceed in a more direct manner i.e. without any implicit bootstrapping step as required to bound 2-arms$(\cdots)$, similar in spirit to the simplifications made in the proof of Theorem 4.4.1. In particular, by setting $\tilde{\Lambda}_{\lambda r_N(u)} = U_N \setminus D_{\lambda r_N(u)}$ and $\Lambda_{\lambda r_N(u)} = \tilde{D}_{\lambda r_N(u)} \setminus \tilde{C}_{\lambda r_N(u)}$, our analogue of (4.4.99)

becomes

$$\begin{aligned} &P_N[\mathrm{LU}(\mathcal{V}_N^u, D_{\lambda r_N(u)} \setminus C_{\lambda r_N(u)}, \tilde{D}_{\lambda r_N(u)} \setminus \tilde{C}_{\lambda r_N(u)})] \\ \geq &\mathbb{P}\Big[\mathrm{LU}\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_{\lambda r_N(u)}, \mathcal{V}^{u(1+\epsilon_N)})), D_{\lambda r_N} \setminus C_{\lambda r_N}, \tilde{D}_{\lambda r_N} \setminus \tilde{C}_{\lambda r_N}\Big)\Big] \\ &\qquad - \mathbb{P}[\mathcal{N}^{2u\epsilon_N}_{D_{\lambda r_N(u)}, U_N} \geq n] - C(u)e^{-c(u)N^{c(u)}} \end{aligned} \tag{4.4.104}$$

where we deal with the first probability on the right-hand-side by observing the inclusion

$$V_0\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_{\lambda r_N}, \mathcal{V}^u, n))\Big) \subset \mathrm{LU}\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_{\lambda r_N}, \mathcal{V}^u, n), D_{\lambda r_N} \setminus C_{\lambda r_N}, \tilde{D}_{\lambda r_N} \setminus \tilde{C}_{\lambda r_N})\Big), \tag{4.4.105}$$

and proceeding by estimating the probability of $V_0\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_{\lambda r_N}, \mathcal{V}^u, n))\Big)$. To this end, by the discussion below (4.4.62), notice that for any $u < v < u_*$

$$V_0\Big(\mathrm{tr}(\boldsymbol{Z}_\bullet(\bar{\mathcal{V}}^v, \nu, n+2))\Big) \cap \mathcal{F}_z^{u,v} \subset V_0\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(\tilde{\Lambda}_N, \mathcal{V}^u, n))\Big), \tag{4.4.106}$$

which implies, by substituting $N = \lambda r_N$ into the bound (4.4.85) (with one additional localisation step), and applying Lemma 4.2.2,

$$\mathbb{P}[V_0\Big(\mathrm{tr}^*(\bar{\boldsymbol{Z}}(U_N \setminus D_{\lambda r_N}, \mathcal{V}^u, n))\Big)] \geq 1 - C(u,n)e^{-c\lambda r_N} \tag{4.4.107}$$

In light of this, the remainder of the proof follows identically to the case of $d = 3$ by instead selecting $\epsilon_N = N^{-\frac{d-2}{4}}$ to satisfy the assumption required for inequality (4.4.94). □

4.5 Assortment of Technical Results

In the previous section, we crucially relied on several technical results in order to complete the overall bootstrapping strategy which have not yet been proven, namely Proposition 4.4.5 and Lemma 4.4.6. Now we finally delve into the details, which will crucially hinge on a certain finite energy property introduced in Lemma 4.5.1 below. The technical results that are presented here are all new, and are purposefully catered for our purposes in the preceding section. Any intermediary results that are used in their proofs are clearly stated when invoked. To start off with, we restrict our definition of a truncated excursion $Z_\circ[\tau_1, \tau_2]$ (see (4.4.16)) requiring that τ_1, τ_2 are each stopping times with respect to the filtration generated by $Z_\circ$. In the context of Proposition 4.4.5, given an excursion $Z_j = Z_j^{\check{D}_z, \check{U}_z}$ as in (3.4.18)-(3.4.19) and an $x \in U_z$, we consider

the following specific collection of stopping times, effectively corresponding to the successive hitting times of the point x by the excursion Z_j

$$H_{j,x}^{(n)} = H_x^{(n)}(Z_j) = H_{\{x\}} \circ \theta_{H_{j,x}^{(n-1)}} \text{ for } n \geq 1 \text{ with } H_{j,x}^{(0)} = 0, \tag{4.5.1}$$

with the convention that $H_{\{x\}} = \infty$ in the case that the excursion does not intersect x. In this instance, we have the following explicit formulation of the set of truncations introduced in (4.4.30)

$$\mathrm{tr}(\boldsymbol{Z}_\bullet(\mathcal{V}_z^u, \nu, n)) = \bigcup_{Z \in \boldsymbol{Z}_\bullet(\mathcal{V}_z^u,\nu,n)} \bigcup_{\substack{j_1,j_2 \in J(Z),\\ j_1 \leq j_2}} \bigcup_{\boldsymbol{x} \in U_z^4, \boldsymbol{n} \in \mathbb{N}^4} Z(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}) \tag{4.5.2}$$

where $Z = \{Z_j\}_{j \in J(Z)}$, $\boldsymbol{j} = (j_1, j_2)$, $\boldsymbol{x} = (x_1, x_1', x_2, x_2')$, $\boldsymbol{n} = (n_1, n_1', n_2, n_2')$ and $Z(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}) = \{\bar{Z}_j\}_{j \in J(Z)}$ is such that

$$\bar{Z}_j = \begin{cases} Z_{j_i}[H_{j_i,x_i}^{(n_i)}, H_{j_i,x_i'}^{(n_i')}], & \text{if } j = j_i \text{ for some } i \in \{1,2\}, \\ Z_j, & \text{otherwise.} \end{cases} \tag{4.5.3}$$

With this notation, we can finally state a Lemma which will provide uniform controls over a specified set on vacating a box in its entirety. In other words, we ensure a conditional finite energy property by restricting to an event reminiscent of $\mathrm{FE}(\cdot)$ defined in (4.4.18).

Lemma 4.5.1 (Finite Energy on Good Events)**.** *Let $z \in N\mathbb{Z}^d$, $\delta \in (0, \frac{1}{2})$ and $\boldsymbol{x} = (x_1, x_1', x_2, x_2') \in U_z^4$. Also let $y \in \mathbb{L}_0 \cap \tilde{D}_z$ be such that $\{x_1, x_1', x_2, x_2'\} \cap D_{y,L_0} = \emptyset$, $J \subset \mathbb{N}^*$ be fixed, finite and $Z_J := \{Z_j^{\check{D}_z, \check{U}_z}\}_{j \in J}$. Finally we define the random set $\mathscr{C}_{\partial D_{y,L_0}} = \mathscr{C}_{\partial D_{y,L_0}}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}))$ and the event $\widetilde{LU} = \widetilde{LU}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}))$ as follows:*

$$\mathscr{C}_{\partial D_{y,L_0}} = \text{ the union of components of points on } \mathcal{I} \cap \partial D_{y,L_0} \text{ inside } \mathcal{I} \cap (D_{y,L_0} \setminus C_{y,L_0}) \tag{4.5.4}$$

where $\mathcal{I} = \mathcal{I}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}))$ and

$$\widetilde{LU}_y = \bigcap_{x,x' \in (\tilde{D}_{y,L_0} \setminus \tilde{C}_{y,L_0}) \cap \mathscr{C}_{\partial D_{y,L_0}}} \{x \overset{\mathscr{C}_{\partial D_{y,L_0}} \cap \mathring{D}_{y,L_0} \setminus C_{y,L_0}}{\longleftrightarrow} x'\} \tag{4.5.5}$$

(compare the event $\widetilde{LU}_y$ with the event LU_y defined in (4.4.19)). Then there exists $c = c(\delta) \in (0,1)$ such that for any $u' \geq u > 0$, $\boldsymbol{n} = (n_1, n_1', n_2, n_2') \in \mathbb{N}^4$ and $j_1, j_2 \in \mathbb{N}^*$,

we have

$$\mathbb{P}[C_{y,L_0} \subset \mathcal{V}|\mathcal{F}] \geq c\mathbb{1}_{\widetilde{LU}_y(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))\cap O_y(\mathcal{V}^u)\cap\{J\subset[1,\mathcal{N}_z^u]\}}, \tag{4.5.6}$$

where $\mathcal{V} = \mathcal{V}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$ *and* $\mathcal{F} = \mathcal{F}_{J,y}(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n};\delta,u,u')$ *denotes the* σ*-algebra*

$$\sigma(N_\delta(\mathcal{V}^u), N_{2\delta}(\mathcal{V}^{u'}), \mathcal{N}_z^u, \mathcal{I}_{|\mathring{D}^c_{y,L_0}}, \mathscr{C}_{\partial D_{y,L_0}}, \{\ell_x^u : x \in \partial\mathring{D}_{y,L_0}\}) \tag{4.5.7}$$

On the other hand, if $\mathrm{x} \in \mathbb{L}_{-1} \cap \tilde{D}_z$ *is such that* $\{x_1, x_1', x_2, x_2'\} \cap D_{\mathrm{x},L_{-1}} = \emptyset$ *and* $\mathrm{x}' \in \partial^{out}C_{\mathrm{x},L_{-1}}$*, then we have*

$$\mathbb{P}[\mathrm{x}' \overset{\mathcal{V}}{\longleftrightarrow} \square(\mathrm{x}, L_{-1})|\bar{\mathcal{F}}] \geq c'\mathbb{1}_{\{\mathrm{x}'\in\mathcal{V}\}\cap G_{\mathrm{x}}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))} \tag{4.5.8}$$

for some $c' = c'(L_{-1}) \in (0,1)$ *where*

$$\bar{\mathcal{F}} = \bar{\mathcal{F}}_{J,\mathrm{x}}(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}) = \sigma(\mathcal{S}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})), \mathcal{I}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))_{|C^c_{\mathrm{x},L_{-1}}}) \tag{4.5.9}$$

and $\square(x,L)$*,* $\mathcal{S}(\cdot)$ *are defined in* (4.4.17) *and Proposition 4.4.5 respectively.*

Remark 4.5.2. *Note that, by definition, the events* $\widetilde{LU}_y(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$*,* $O_y(\mathcal{V}^u)$ *and* $\{J \subset [1,\mathcal{N}_z^u]\}$ *(see Definition 4.4.3) are all measurable with respect to* $\mathcal{F}_{J,y}$ *presented in* (4.5.7)*.*

4.5.1 Estimate on V_z^c

For the purposes of this subsection, we will assume Lemma 4.5.1 is true and revisit it after completing the following proof. Thus, we proceed with the

Proof of Proposition 4.4.5. To start off with, we decompose the collection of truncated excursions $\mathrm{tr}(\boldsymbol{Z}_\bullet(\mathcal{V}^u,\nu,n))$ into two families which will eventually be handled by the two separate provisions introduced in Lemma 4.5.1 (i.e. (4.5.6) and (4.5.8)). Abbreviating $a^+ = \max(0,a)$ for any $a \in \mathbb{R}$, let

$$\begin{aligned}&\boldsymbol{Z}_{\mathrm{small}}(u_0,u;\nu;n) \text{ denote the subcollection of all sequences } \{Z_j^{\breve{D}_z,\breve{U}_z}\}_{j\in J} \text{ in}\\ &\boldsymbol{Z}_\bullet(\mathcal{V}_z^u,\nu;n) \text{ where } J \subset \{1,\dots,\mathcal{N}_z^u\} \text{ is of the form } \cup_{1\leq k\leq n} J_k \text{ with each } J_k\\ &\text{a subset of } \Big\{\mathcal{N}_z^{(\frac{i}{2}-\frac{1}{8})^+u_0}+1,\dots,\mathcal{N}_z^{(\frac{i}{2}+\frac{17}{8})u_0}\Big\} \text{ for some } 0\leq i\leq (2\left\lceil\tfrac{u}{u_0}\right\rceil-4)^+.\end{aligned} \tag{4.5.10}$$

On the other hand, we let

$$\begin{aligned}&\boldsymbol{Z}_{\mathrm{large}}(u_0,u;\nu;n) \text{ denote the subcollection of all sequences } \{Z_j^{\breve{D}_z,\breve{U}_z}\}_{j\in J} \text{ in}\\ &\boldsymbol{Z}_\bullet(\mathcal{V}^u,\nu,n) \text{ where } \{\mathcal{N}_z^{\frac{i}{2}u_0}+1,\dots,\ \mathcal{N}_z^{(\frac{i}{2}+1)u_0}\}\subset J\subset\{1,\dots,\ \mathcal{N}_z^u\} \text{ for some}\\ &0\le i\le 2\left\lceil\tfrac{u}{u_0}\right\rceil.\end{aligned} \tag{4.5.11}$$

As we will see, on the following (likely) event

$$F_z(\nu;u_0,u)\coloneqq \bigcap_{1\le i\le 2\left\lceil\frac{u}{u_0}\right\rceil}\Big\{\mathcal{N}_z^{\frac{iu_0}{2}}-\mathcal{N}_z^{\frac{iu_0}{2}-\frac{u_0}{8}}>\nu\Big\}\cap\bigcap_{0\le i\le 2\left\lceil\frac{u}{u_0}\right\rceil}\Big\{\mathcal{N}_z^{\frac{iu_0}{2}+\frac{u_0}{8}}-\mathcal{N}_z^{\frac{iu_0}{2}}>\nu\Big\} \tag{4.5.12}$$

we observe the inclusion

$$\mathrm{tr}(\boldsymbol{Z}_\bullet(\mathcal{V}_z^u,\nu;n))\subset\mathrm{tr}(\boldsymbol{Z}_{\mathrm{small}}(u_0,u;\nu;n))\cup\mathrm{tr}(\boldsymbol{Z}_{\mathrm{large}}(u_0,u;\nu;n)) \tag{4.5.13}$$

This is seen most directly by considering an arbitrary $Z=\{Z_j\}_{j\in J(Z)}\in\boldsymbol{Z}_\bullet(\mathcal{V}_z^u,\nu;n)\cap\boldsymbol{Z}_{\mathrm{large}}(u_0,u;\nu;n)^c$ and showing that $Z\in\boldsymbol{Z}_{\mathrm{small}}(u_0,u;\nu;n)$. To this end, we see that for such a choice of Z, the index set $J(Z)$ takes the form $J(Z)=\cup_{k=1}^n J_k$ for some intervals J_k' with $J_k'\subset J_k\subset J_k'\pm\{0,\dots,\nu\}$ and also contains no interval $\{\mathcal{N}_z^{\frac{i}{2}u_0}+1,\dots,\ \mathcal{N}_z^{(\frac{i}{2}+1)u_0}\}$ for any $0\le i\le 2\left\lceil\frac{u}{u_0}\right\rceil$. In particular, this means that $J(Z)$ can only contain intervals of the form $J_k'=\{\mathcal{N}_z^w+1,\dots,\mathcal{N}_z^{(w+su_0)\wedge u}\}$ for any $w\in[0,u]$, $s<\frac{3}{2}$. For such intervals, there always exists an $i\in\{0,\dots,(2\left\lceil\frac{u}{u_0}\right\rceil-4)^+\}$ such that

$$J_k'\subset\{\mathcal{N}_z^{\frac{i}{2}u_0}+1,\dots,\mathcal{N}_z^{(\frac{i}{2}+2)u_0}\}$$

hence, by combining the occurrence of $F_z(\nu;u_0,u)$ with the fact that any J_k comprising $J(Z)$ only deviates from the interval by ν steps, one finally yields

$$J_k\subset\{\mathcal{N}_z^{(\frac{i}{2}-\frac{1}{8})^+u_0}+1,\dots,\mathcal{N}_z^{(\frac{i}{2}+\frac{17}{8})u_0}\} \tag{4.5.14}$$

as required for $Z\in\boldsymbol{Z}_{\mathrm{small}}(u_0,u;\nu,n)$. Moreover, returning to the facilitating event $F_z(\nu;u_0,u)$ from (4.5.12), we observe that the stationarity of excursion count increments implies a bound of the form

$$\mathbb{P}[F_z(\nu;u_0,u)^c]\le C(u_0,u)\mathbb{P}\Big[\Big(\mathcal{F}_z^{\frac{u_0}{8},\frac{\nu}{\mathrm{cap}(\breve{D}_z)}}\Big)^c\Big] \tag{4.5.15}$$

From now on, we distinguish between our treatment of $\boldsymbol{Z}_{\text{large}}(\cdots)$ and $\boldsymbol{Z}_{\text{small}}(\cdots)$ in attaining bounds for $V_z(\text{tr}(\boldsymbol{Z}_\cdot(\cdots)))^c$. Despite the distinct approaches for these collections, there is some common resemblance, in that they both depend on an estimate derived by exploration of a vacant cluster in D_z and iterative application of (4.5.6) or (4.5.8), in the case of $\boldsymbol{Z}_{\text{large}}(\cdots)$ and $\boldsymbol{Z}_{\text{small}}(\cdots)$ respectively. Let us commence the case of $\boldsymbol{Z}_{\text{small}}(\cdots)$ by introducing some notation inspired by Proposition 4.1.1. Given any two boxes $U \subset V \subset \mathbb{Z}^d$ and $\Sigma \subset \mathbb{L} = L\mathbb{Z}^d$ for some scale $L \in \mathbb{N}^*$, the notation $k(V \setminus U, \Sigma)$ will be used to denote the maximum integer k' for which the conditions of Proposition 4.1.1 are satisfied with the above choices for U, V, Σ on the underlying coarse lattice $\mathbb{L}$. With this, fixing some finite $J \subset \mathbb{N}^*$, $\boldsymbol{j} = (j_1, j_2) \in (\mathbb{N}^*)^2$, $\boldsymbol{x} \in U_z^4$ and $\boldsymbol{n} \in \mathbb{N}^4$, the aforementioned pivotal estimate for the 'small' excursions is formulated as

$$\begin{aligned}&\mathbb{P}\Big[(C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))}{\longleftrightarrow} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1}), k(\tilde{D}_z \setminus \tilde{C}_z, \mathcal{S}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))) \geq k, V_z^c(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))\Big] \\ \leq & CN^{d-1}e^{-c(L_{-1})k}\end{aligned} \tag{4.5.16}$$

for any $k \in \mathbb{N}$. We proceed for now under the assumption that (4.5.16) holds, deferring the exploration argument to a later stage (refer to the proof of (4.5.33) below). Thus, we recall the convenient notation for $\boldsymbol{Z}(Z, \star, \nu; n)$ introduced below (4.4.23) as well as the multi-level localising event $\mathcal{F}_z^{\boldsymbol{u},\boldsymbol{v}}$ and the good event $\mathcal{G}_{y,L_{-1}}$ from (4.3.10), (4.4.20) respectively. Indeed, this is the point in the proof where we rely on the intricate choices of levels $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ detailed in (4.4.27), which are now clearly seen to arise as a consequence of the interweaving intervals described in the definition of $\boldsymbol{Z}_{\text{small}}(\cdots)$, each of which need to be coupled to the partially-local field levels playing a role in $\mathcal{G}_{y,L_{-1}}(\cdots)$. In the presence of the assumption that $3nu_0 < u_5$ (see Proposition 4.4.5), this coupling yields the following inclusion for any $\mathrm{y} \in \mathbb{L} \cap D_z$ and $y \in \mathbb{L}_0$

$$\begin{aligned}&\mathcal{G}_{y,L_{-1}}(\boldsymbol{Z}(\bar{\mathcal{V}}_{\mathrm{y}}^{u_1}, \star, 2u_5\text{cap}(\breve{D}_{\mathrm{y}}); n)) \cap \mathcal{F}_z^{\boldsymbol{u}_1,\boldsymbol{u}_2} \\ \subset &\mathcal{G}_{y,L_{-1}}(\boldsymbol{Z}(\bar{\mathcal{V}}_{\mathrm{y}}^{u_1}, \star, 2u_5\text{cap}(\breve{D}_{\mathrm{y}}); n)) \cap \mathcal{F}_z^{\boldsymbol{u}_1',\boldsymbol{u}_2'} \subset \mathcal{G}_{y,L_{-1}}\Big(\text{tr}(\boldsymbol{Z}_{\text{small}}(u_0, u; \nu; n))\Big)\end{aligned} \tag{4.5.17}$$

where $\boldsymbol{u}_1' = (u, \boldsymbol{u}_{1,0}', \ldots, \boldsymbol{u}_{1,2n_0}')$, $\boldsymbol{u}_2' = (u_1, \boldsymbol{u}_{2,0}', \ldots, \boldsymbol{u}_{2,2n_0}')$, with $\boldsymbol{u}_{1,i}' = \frac{iu_0}{2} + (\frac{u_0}{8}, \frac{3u_0}{8})$ and $\boldsymbol{u}_{2,i}' = \frac{iu_0}{2} + (\frac{u_0}{4}, \frac{u_0}{4})$ for each $i \in \{0, \ldots, 2n_0\}$ and $\lceil \frac{u}{u_0} \rceil$. Evidently, $\boldsymbol{u}_1'$, $\boldsymbol{u}_2'$ describe subsets of those levels considered by $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, where the levels of $\boldsymbol{u}_1'$ are seen to correspond to the endpoints of the intervals in (4.5.10), meanwhile $\boldsymbol{u}_2'$ contains the resulting partially-local field levels of the $\mathcal{G}_{y,L_{-1}}(\cdots)$. The inclusion (4.5.17) is immediate by observing that any sequence in $\boldsymbol{Z}_{\text{small}}(u_0, u; \nu; n)$ consists of a subset of

n intervals of length at most $\frac{5}{2}u_0$, resulting in a total length of less than $3nu_0 < 2u_5$, hence the inclusion holds by definition of $\boldsymbol{Z}(\bar{\mathcal{V}}_{\mathrm{y}}^{u_1}, \star, 2u_5\mathrm{cap}(\breve{D}_{\mathrm{y}}); n))$ (see (4.4.23)). In view of the definition of $\mathscr{G}_z$ from (4.4.26), a direct consequence of inclusion (4.5.17) becomes

$$\mathscr{G}_z \subset \Big\{k(\tilde{D} \setminus \tilde{C}_z, \mathcal{S}(Z)) \geq am \text{ for each } Z \in \mathrm{tr}(\boldsymbol{Z}_{\mathrm{small}}(u_0, u; \nu, n))\Big\} \tag{4.5.18}$$

where m is the common cardinality of each coarsening of $\tilde{D}_z \setminus \tilde{C}_z$, and a is the number of contact points prescribed by each of the constituent $\mathcal{G}_{\mathrm{y}}(\bar{\mathcal{V}}_{\mathbb{L}}, a, \overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}}, \frac{u_0}{8}, u_1, 2u_5; n))$ (see (4.4.26). This fact, combined with the estimate (4.5.16), the monotonicity of $\mathcal{S}(Z)$ in Z w.r.t inclusion as subsequences and the definition of $\boldsymbol{Z}_{\mathrm{small}}(u_0, u; \nu; n)$, implies

$$\begin{aligned}
&\mathbb{P}\Big[V_z^c\Big(\mathrm{tr}(\boldsymbol{Z}_{\mathrm{small}}(u_0, u; \nu; n))\Big)\Big] \\
\leq&\mathbb{P}\Big[\bigcup_{Z \in \mathrm{tr}(\boldsymbol{Z}_{\mathrm{small}}(u_0,u;\nu,n))} (V_z^c(Z) \cap (C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}(Z)}{\longleftrightarrow} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1}), k(\tilde{D}_z \setminus \tilde{C}_z, \mathcal{S}(Z)) \geq am\Big] \\
&+ \mathbb{P}\Big[\bigcup_{J \in J(u,u_0;n)} (C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}(\{Z_j^{\breve{D}_z, \breve{U}_z}\}_{j \in J})}{\longleftrightarrow} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1})\Big] + \mathbb{P}[\mathscr{G}_z^c]
\end{aligned} \tag{4.5.19}$$

$$\begin{aligned}
\leq&\mathbb{P}[\mathscr{G}_z^c] + \sum_{\substack{\{\hat{j}_i, \ldots, \hat{j}_i'\} \subset \hat{J}_i \subset \{(\hat{j}_i - \nu)^+, \ldots, \hat{j}_i' + \nu\}, \\ \boldsymbol{x} \in U_z^4, \hat{j}_i \leq \hat{j}_i', n_1, n_1', n_2, n_2' \leq N^3, \\ j_1, j_2, \hat{j}_i' + \nu \leq 2u_* \mathrm{cap}(D_z), 1 \leq i \leq n}} CN^{d-1} e^{-c(L_{-1})am} \\
&+ |J(u, u_0; n)| \sup_{J \in J(u,u_0;n)} \mathbb{P}\Big[(C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}(\{Z_j^{\breve{D}_z, \breve{U}_z}\}_{j \in J})}{\longleftrightarrow} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1})\Big] \\
&+ \mathbb{P}\Big[(\mathcal{F}_z^{u, 2u_*})^c \cup \big\{\max_{1 \leq j \leq 2u_* \mathrm{cap}(D_z)} |Z_j^{\breve{D}_z, \breve{U}_z}|_{\mathbb{Z}^d} > N^3\big\}\Big]
\end{aligned} \tag{4.5.20}$$

where

$$\begin{aligned}
J(u_0, u; n) &\coloneqq \{J(i_i, \ldots, i_n) : 0 \leq i_1 \leq \ldots \leq i_n \leq (2\left\lceil \tfrac{u}{u_0} \right\rceil - 4)^+\} \text{ with} \\
J(i_1, \ldots, i_n) &\coloneqq \bigcup_{1 \leq l \leq n} \{\mathcal{N}_z^{(\frac{i_l}{2} - \frac{1}{8})^+ u_0} + 1, \ldots, \mathcal{N}_z^{(\frac{i_l}{2} + \frac{17}{8}) u_0}\}
\end{aligned}$$

and $|Z_j^{\breve{D}_z, \breve{U}_z}|_{\mathbb{Z}^d}$ denotes the length of the lattice path corresponding to the excursion $Z_j^{\breve{D}_z, \breve{U}_z}$. We now proceed by bounding each of the terms in display (4.5.20) separately. First, we observe that in the summation there are at most $2^{2\nu} \cdot 2u_* \mathrm{cap}(\breve{D}_z)^2$ choices for

the subsets J_i for each i and, for any such choice, there are a further $(2u_*\mathrm{cap}(\breve{D}_z))^2 \cdot |U_z|^4 \cdot N^1 2$ choices for the truncation. Overall,

$$\begin{aligned}&\text{the number of terms under the only summation in display (4.5.20)}\\&\text{is bounded by } CN^{cn}e^{C\nu},\end{aligned} \tag{4.5.21}$$

meanwhile, we immediately see that

$$|J(u_0,u;n)| \le C\Big(\tfrac{u}{u_0}\Big)^n =: C(u,u_0)^n. \tag{4.5.22}$$

Next, we deal with the disconnection probability appearing in (4.5.20) by a series of localisations, keeping in mind the effect of the monotonicity of the event $\mathcal{S}(Z)$ in Z w.r.t inclusion as subsequences. First, we have that for any $J = J(i_1,\dots,i_n) \in J(u_0,u;n)$ and $J^* = J^*(i_1,\dots,i_n) \coloneqq \bigcup_{1\le l\le n}\{\Big\lfloor(\frac{i_l}{2}-\frac{5}{32})^+u_0\mathrm{cap}(\breve{D}_z)\Big\rfloor+1,\dots,\Big\lfloor(\frac{i_l}{2}+\frac{69}{32})u_0\mathrm{cap}(\breve{D}_z)\Big\rfloor\}$,

$$\begin{aligned}&\{(C_z\cap\mathbb{L}_{-1}) \xleftarrow{\mathcal{S}((Z_j^{\breve{D}_z,\breve{U}_z})_{j\in J^*})} \partial_{\mathbb{L}_{-1}}(D_z\cap\mathbb{L}_{-1})\}\cap\mathcal{F}_z^{\boldsymbol{u}_1,\boldsymbol{u}_2}\\ \subset&\{(C_z\cap\mathbb{L}_{-1}) \xleftarrow{\mathcal{S}((Z_j^{\breve{D}_z,\breve{U}_z})_{j\in J})} \partial_{\mathbb{L}_{-1}}(D_z\cap\mathbb{L}_{-1})\}\end{aligned} \tag{4.5.23}$$

where $(\boldsymbol{u}_1(2i),\boldsymbol{u}_1(2i+1)) = ((\frac{i}{2}-\frac{1}{8})^+u_0,(\frac{i}{2}+\frac{17}{8})u_0)$ and $(\boldsymbol{u}_2(2i),\boldsymbol{u}_2(2i+1)) = ((\frac{i}{2}-\frac{5}{32})^+u_0,(\frac{i}{2}+\frac{69}{32})u_0)$ for each $0\le i\le(2\left\lceil\frac{u}{u_0}\right\rceil-4)^+$. Moreover, under any coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\hat{\mathbb{P}}_z$, we perform a further localisation to obtain

$$\begin{aligned}&\{(C_z\cap\mathbb{L}_{-1}) \xleftarrow{\mathcal{S}((\hat{Z}_j^{\breve{D}_z,\breve{U}_z})_{j\in\hat{J}})} \partial_{\mathbb{L}_{-1}}(D_z\cap\mathbb{L}_{-1})\}\cap\mathrm{Incl}_z^{\frac{u_0}{32},\frac{u_0}{4}\mathrm{cap}(\breve{D}_z)}\\ \subset&\{(C_z\cap\mathbb{L}_{-1}) \xleftarrow{\mathcal{S}((Z_j^{\breve{D}_z,\breve{U}_z})_{j\in J^*})} \partial_{\mathbb{L}_{-1}}(D_z\cap\mathbb{L}_{-1})\}\end{aligned} \tag{4.5.24}$$

where $\hat{J} = \hat{J}(i_1,\dots,i_n) \coloneqq \bigcup_{1\le l\le n}\{\Big\lfloor(\frac{i_l}{2}-\frac{3}{16})^+u_0\mathrm{cap}(\breve{D}_z)\Big\rfloor+1,\dots,\Big\lfloor(\frac{i_l}{2}+\frac{35}{16})u_0\mathrm{cap}(\breve{D}_z)\Big\rfloor\}$. Combined with the familiar bounds from (2.6.13) (in view of (4.2.18)) and (4.2.21), the inclusions (4.5.23), (4.5.24) derived above together imply that for any $K \ge C(u_0)$,

$$\begin{aligned}&\mathbb{P}\Big[(C_z\cap\mathbb{L}_{-1}) \overset{\mathcal{S}((Z_j^{\breve{D}_z,\breve{U}_z})_{j\in J})}{\not\longleftrightarrow} \partial_{\mathbb{L}_{-1}}(D_z\cap\mathbb{L}_{-1})\Big]\\ \le&\mathbb{P}\Big[(C_z\cap\mathbb{L}_{-1}) \overset{\mathcal{S}((\hat{Z}_j^{\breve{D}_z,\breve{U}_z})_{j\in\hat{J}})}{\not\longleftrightarrow} \partial_{\mathbb{L}_{-1}}(D_z\cap\mathbb{L}_{-1})\Big] + C(u,u_0)e^{-cu_0N^{d-2}}.\end{aligned} \tag{4.5.25}$$

As a result, since the excursions $\{\hat{Z}_j^{\breve{D}_z,\breve{U}_z}\}_{j\in\hat{J}}$ are i.i.d, one can 'reset' the order of these excursions and construct another coupling to delocalise the events, in essence

retracing a simpler version of the steps above. Since $|\hat{J}| \leq \frac{5}{2} n u_0 \mathrm{cap}(D_z)$, performing this delocalisation with a precision of $\frac{u_0}{8}$ is sufficient to obtain for any $K \geq C(u_0)$,

$$\begin{aligned} &\mathbb{P}\Big[(C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}((\hat{Z}_j^{\check{D}_z,\check{U}_z})_{j\in\hat{J}})}{\longleftrightarrow\!\!\!\!\!\!/\;\;\;} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1})\Big] \\ \leq &\mathbb{P}\Big[(C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}(Z_z^{3nu_0})}{\longleftrightarrow\!\!\!\!\!\!/\;\;\;} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1})\Big] + C(u,u_0)e^{-cu_0N^{d-2}}. \end{aligned} \tag{4.5.26}$$

Hence, by recalling the assumption that $3nu_0 < u_5$ and once again employing the monotonicity of $\mathcal{S}(Z)$, the above two inequalities finally obtain

$$\begin{aligned} &\mathbb{P}\Big[(C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}((Z_j^{\check{D}_z,\check{U}_z})_{j\in J})}{\longleftrightarrow\!\!\!\!\!\!/\;\;\;} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1})\Big] \\ \leq &\mathbb{P}\Big[(C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}(Z_z^{u_5})}{\longleftrightarrow\!\!\!\!\!\!/\;\;\;} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1})\Big] + C(u,u_0)e^{-cu_0N^{d-2}}. \end{aligned} \tag{4.5.27}$$

All that is left is to deal with the final probability appearing in (4.5.20). An upper bound on $\mathbb{P}[(\mathcal{F}_z^{u,2u_*})^c]$ is given by (4.2.21) as before, meanwhile the remaining probability $\mathbb{P}\Big[|Z_j^{\check{D}_z,\check{U}_z}|_{\mathbb{Z}^d} > N^3\Big]$ is readily handled by an application of Donsker's invariance principle and the Strong Markov Property. Indeed, by letting $\{Y_n\}_{n\geq 0}$ under P_x be a discrete-time SRW on $\mathbb{Z}^d$ starting from x, the former guarantees the convergence in law of $\Big\{\frac{Y_{[nt]}}{\sqrt{n}}\Big\}_{n\geq 1}$ on $\mathbb{Z}^d$ to a d-dimensional Brownian B_t, i.e. for Y_n under P_x and B_t under P,

$$\frac{Y_{[nt]} - x}{\sqrt{n}} \xrightarrow{d} B_t \sim N(0, tI_{d\times d}). \tag{4.5.28}$$

Therefore, by fixing $t \in (0,\infty)$ such that $P\Big[|B_t| \leq 1\Big] \leq \frac{1}{4}$ (for example), we have for any $N > C$

$$P_x\Big[|Y_{[tN^2]} - x| \leq 2\sqrt{d}KN\Big] \leq \frac{1}{2} \tag{4.5.29}$$

Then, by temporarily writing $Z_j^{\check{D}_z,\check{U}_z} = \{Y_i\}_{i\geq 0}$, we rely on the following inclusion to apply the strong Markov property

$$\{|Z_j^{\check{D}_z,\check{U}_z}|_{\mathbb{Z}^d} > N^3\} \subset \bigcap_{i=0}^{\lfloor \frac{N}{t} \rfloor} \{Y_{i[tN^2]} \in U_z\} \subset \bigcap_{i=0}^{\lfloor \frac{N}{t} \rfloor} \{|Y_{(i+1)[tN^2]} - Y_{i[tN^2]}| \leq 2\sqrt{d}KN\} \tag{4.5.30}$$

Hence, by iteratively conditioning and evaluating probabilities at the times $i[tN^2]$, $i = 0, \dots, \left\lfloor \frac{N}{t} \right\rfloor$, we have by translational invariance of the SRW

$$\mathbb{P}\Big[|Z_j^{\check{D}_z, \check{U}_z}|_{\mathbb{Z}^d} > N^3\Big] \leq P_0\Big[|Y_{[tN^2]}| \leq 2\sqrt{d}KN\Big]^{\left\lfloor \frac{N}{t} \right\rfloor} \leq e^{-cN} \tag{4.5.31}$$

Now, by substituting the estimates obtained in (4.5.21), (4.5.22), (4.5.27), (4.5.31) and (4.2.21) into (4.5.20), we yield

$$\begin{aligned} &\mathbb{P}\Big[V_z^c\Big(\mathrm{tr}(\boldsymbol{Z}_{\mathrm{small}}(u_0, u; \nu; n))\Big)\Big] \\ &\leq \mathbb{P}[\mathscr{G}_z^c] + C(u_0, u)^n \mathbb{P}\Big[(C_z \cap \mathbb{L}_{-1}) \overset{\mathcal{S}(Z_z^{u_5})}{\longleftrightarrow} \partial_{\mathbb{L}_{-1}}(D_z \cap \mathbb{L}_{-1})\Big] \\ &+ C(u_0, u) \exp\{-c(u_0, L_{-1})(am \wedge N) + C(\nu + N \log N)\} \end{aligned} \tag{4.5.32}$$

In light of the inclusion (4.5.13), we have then dealt with half of the problem of showing (4.4.31) - thus we proceed to handling the excursion sets belonging to $\mathrm{tr}(\boldsymbol{Z}_{\mathrm{large}}(u, u_0; \nu; n))$. In this setting, we now introduce a crucial estimate which is the analogue of (4.5.16) and will be shown via similar, albeit more sophisticated, arguments. Consider any fixed and finite $J \subset \mathbb{N}^*$, $\boldsymbol{j} = (j_1, j_2) \in (\mathbb{N}^*)^2$, $\boldsymbol{x} \in U_z^4$ and $\boldsymbol{n} \in \mathbb{N}^4$. Then, for any $0 \leq i \leq 2\left\lfloor \frac{u}{u_0} \right\rfloor$ we have

$$\begin{aligned} &\mathbb{P}\Big[C_z \overset{N_{2\delta}(\mathcal{V}^{u_3})}{\longleftrightarrow} \partial D_z, \mathscr{G}_z, V_z^c(Z_J(\boldsymbol{J}, \boldsymbol{x}, \boldsymbol{n})), [\mathcal{N}_z^{\frac{i}{2}u_0} + 1, \mathcal{N}_z^{(\frac{i}{2}+1)u_0}] \subset J \subset [1, \mathcal{N}_z^u]\Big] \\ &\leq C N^{d-1} e^{-c(\delta, L_0)am} \end{aligned} \tag{4.5.33}$$

Assuming that (4.5.33) holds, by invoking similar arguments to those presented in the deduction of (4.5.20) one obtains

$$\begin{aligned} &\mathbb{P}\Big[V_z^c\Big(\mathrm{tr}(\boldsymbol{Z}_{\mathrm{large}}(u_0, u; \nu; n))\Big)\Big]\Big] \\ &\leq \mathbb{P}[\mathscr{G}^c] + \sum_{0 \leq i \leq 2\left\lfloor \frac{u}{u_0} \right\rfloor} \sum_{\substack{\{\hat{j}_i, \dots, \hat{j}_i'\} \subset \hat{J}_i \subset \{(\hat{j}_i - \nu)^+, \dots, \hat{j}_i' + \nu\}, \\ \boldsymbol{x} \in U_z^4, \hat{j}_i \leq \hat{j}_i', n_1, n_1', n_2, n_2' \leq N^3, \\ j_1, j_2, \hat{j}_i' + \nu \leq 2u_* \mathrm{cap}(D_z), 1 \leq i \leq n}} C N^{d-1} e^{-c(\delta, L_0)am} \\ &+ \mathbb{P}\Big[C_z \overset{N_{2\delta}(\mathcal{V}^{u_3})}{\longleftrightarrow} \partial D_z\Big] + \mathbb{P}\Big[(\mathcal{F}_z^{u, 2u_*})^c \cup \big\{\max_{1 \leq j \leq 2u_* \mathrm{cap}(D_z)} |Z_j^{\check{D}_z, \check{U}_z}|_{\mathbb{Z}^d} > N^3\big\}\Big] \end{aligned} \tag{4.5.34}$$

Therefore, by once more inserting the estimates from (4.5.21), (4.5.31) and (4.2.21) into (4.5.34) we obtain

$$\begin{aligned}&\mathbb{P}\Big[V_z^c\Big(\mathrm{tr}(\boldsymbol{Z}_{\mathrm{large}}(u_0,u;\nu;n))\Big)\Big]\\&\leq \mathbb{P}[\mathscr{G}_z^c]+\mathbb{P}\Big[C_z \overset{N_{2\delta}(\mathcal{V}^{u_3})}{\longleftrightarrow} \partial D_z\Big]+\exp\Big\{-c(\delta,u_0,L_0)(am\wedge N)+C(\nu+n\log N)\Big\}.\end{aligned} \tag{4.5.35}$$

Overall, by combining the inequalities (4.5.32), (4.5.35) and (4.5.15) as well as recalling the inclusion (4.5.13) which holds under the event $F_z(\nu;u_0,u)$ (see (4.5.12)), we obtain (4.4.31) as required.

We now focus the remainder of the proof on presenting the exploration argument required to establish (4.5.33) and, with some simplifications, (4.5.16) (refer to Remark 4.5.5 for the details of the latter). The overarching idea is that, under the event in the probability of the left-hand side of (4.5.33), we are guaranteed a path γ in $N_{2\delta}(\mathcal{V}^{u_3})$ that comes close to 'very many' surrounding interfaces of the vacant set, constructed by virtue of $\mathscr{G}_z$. Therefore, in order to ensure the non-local uniqueness appearing in the same event, one must pay a cost for each 'near miss', since the path must be disconnected from these vacant interfaces. This results in the exponential decay appearing in the right-hand side of (4.5.33) with exponent determined by the number of close encounters between γ and the vacant interfaces. Without further delay, we now devise a random algorithm by which we explore the component $\mathscr{C}_J(x;\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})$ of a point $x\in\partial\tilde{C}_z$ inside $D_z\cap\mathcal{V}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$ (see (4.5.2) for notation). To commence, for each $y\in\mathbb{L}\cap D_{0,L_0}$ we define the shifted sub-lattice $\mathbb{L}_{0,y}=y+7L_0\mathbb{Z}^d$, and remark that the sets $\mathbb{L}_{0,y}$ form a partition of $\mathbb{L}_0$ as y ranges over $\mathbb{L}_0\cap D_{0,L_0}$. Additionally, for a fixed choice of y, the collection of boxes $\{D_{y',L_0}\}_{y'\in\mathbb{L}_{0,y}}$ forms a partition of $\mathbb{Z}^d$. For the purposes of the exploration, let us also introduce the set $\hat{\mathbb{L}}_{0,y}$ obtained from $\mathbb{L}_{0,y}$ by removing all points y' with $D_{y',L_0}\cap\tilde{C}_z\neq\emptyset$, and fix our choice of $y\in\mathbb{L}_0\cap D_{0,L_0}$ in the upcoming description of the algorithm. Next, consider the (random) set

$$\mathscr{S}\coloneqq\Big\{\mathrm{y}\in\mathbb{L}\cap(\tilde{D}_z\setminus\tilde{C}_z):D_{\mathrm{y},L}\subset\tilde{D}_z\setminus\tilde{C}_z\text{ and }V_{\mathrm{y}}(\mathcal{V}_{\mathbb{L}},\delta,u,u_3,u_3)\text{ occurs}\Big\} \tag{4.5.36}$$

and recall the definition of $k(\tilde{D}_z\setminus\tilde{C}_z,\mathscr{S})\in\mathbb{N}$ above (4.5.16). Whenever $k(\tilde{D}_z\setminus\tilde{C}_z,\mathscr{S})>0$, then, with $\mathbb{L}$ as the underlying lattice, Proposition 4.1.1 assures the existence of $*$-connected sets $\mathcal{O}_1,\dots\mathcal{O}_l\subset\mathscr{S}$ satisfying properties (i) - (iii) therein with the selections $V=\mathbb{L}(\tilde{D}_z)$, $U=\mathbb{L}(\tilde{C}_z)$, $\Sigma=\mathscr{S}$ and $k=k(\tilde{D}_z\setminus\tilde{C}_z,\mathscr{S})$. To contextualize this

within the discussion above, since the $*$-connected sets $\mathcal{O}_i$ surround $\mathbb{L}(\tilde{C}_z)$ (as noted in property (i) of Proposition 4.1.1), the occurrence of $V_{\mathrm{y}}(\mathcal{V}_{\mathbb{L}}, \delta, u, u_3, u_3)$ at each point in $\mathcal{O}_i$ ensures the existence of a vacant set component of $N_\delta(\mathcal{V}^u)$ that satisfies local uniqueness properties in every box $D_{\mathrm{y},L}$, $\mathrm{y} \in \mathcal{O}_i$, for each $1 \leq i \leq l$. These are precisely the vacant components we seek to avoid in our exploration, and which will be ensured by $\mathscr{G}_z$ with certain additional, yet essential, properties. One aspect of these additional properties will be encompassed by the event

$$\widetilde{\mathrm{FE}}_{y'}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})) \coloneqq \widetilde{\mathrm{LU}}_{y'}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})) \cap O_{y'}(\mathcal{V}_z^u) \tag{4.5.37}$$

(recall (4.5.5) and Definition 4.4.3) and will henceforth facilitate the finite energy methods presented in Lemma 4.5.1. Thus, given $\mathscr{S}$ and the collection of indicator functions $\{\mathbb{1}_{\widetilde{\mathrm{FE}}_{y'}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))} : y' \in \hat{\mathbb{L}}_{0,y}\}$ from (4.5.36), (4.5.37), the proposed algorithm to explore $\mathscr{C}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ is split up into two regimes, generic and special. During periods of the generic regime, the algorithm iteratively reveals the state of the vertices of $\mathbb{Z}^d$ by starting at $x \in \tilde{C}_z$ and, at each step, inspecting the next vertex in the outer boundary of the currently explored portion of $\mathscr{C}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ according to some fixed deterministic ordering of $\mathbb{Z}^d$. On the other hand, the special regime encompasses certain, local, explorations which will determine whether or not to terminate the algorithm. Effectively, the special regime is initiated when the exploration of the vacant component is in close proximity to the aforementioned vacant interfaces, meanwhile the generic regime signifies the opposite.

In order to be precise, some further notation is necessary. Let $(w_n)_{n\geq 1}$ denote the sequence of vertices tied to the exploration, where at step n, w_n is referred to as the current vertex, such that $w_1 = x$ and, if the algorithm has exhausted all points of $\mathscr{C}_J(x, \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ by step n, $w_m = w_n$ for all $m > n$. Accordingly, we say that a vertex v has been explored by time n if $v = w_i$ for some $i < n$. For convenience, in the sequel we will abbreviate $\mathcal{I} = \mathcal{I}(Z_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}))$, $\mathcal{V} = \mathcal{V}(Z_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}))$ as in Lemma 4.5.1. Define $\tau_k, k \geq 1$ as the successive times the exploration of $\mathscr{C}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ visits

$$\partial \coloneqq \bigcup_{y' \in \hat{\mathbb{L}}_{0,y}} \partial D_{y',L_0}, \tag{4.5.38}$$

and certain additional (good) properties are satisfied. Formally, with $\tau_0 = 0$, for $k \geq 1$ we let

$$\tau_k = \inf\{n > \tau_{k-1} : w_n \in \partial, w_n \in \mathcal{V} \text{ and } (\star) \text{ holds}\}, \quad \tilde{x}_k = w_{\tau_k} \tag{4.5.39}$$

(with the convention that $\inf \emptyset = \infty$), where $(\star)$ refers to the property that if $y_k \in \hat{\mathbb{L}}_{0,y}$ denotes the unique point such that $\tilde{x}_k \in D_{y_k,L_0}$, then y_k is such that

$$\begin{aligned} &C_{y_k,L_0} \text{ intersects } \mathscr{C}_{\mathrm{y}} = \mathscr{C}_{\mathrm{y}}(\mathcal{V}_{\mathbb{L}}, \delta, u, u_3) \text{ (see (4.4.15)) for some } \mathrm{y} \in \bigcup_{1 \le j \le l} \mathcal{O}_j \\ &\text{satisfying } \{x_1, x_1', x_2, x_2'\} \cap U_{\mathrm{y},L} = \emptyset \text{ and } \widetilde{\mathrm{FE}}_{y'}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})) \text{ occurs.} \end{aligned} \tag{4.5.40}$$

keeping in mind that, by assumption, $\mathscr{S}$ (hence $\mathcal{O}_i$, $1 \le i \le l$) and the indicators of $\tilde{\mathrm{FE}}_{y'}(Z_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}))$ are known prior to starting the exploration. The times τ_k will be precisely those preceding the start of a special regime, in the sense as described above. With the notation settled, we now describe the algorithm, denoted as $\mathcal{A}_{J,y}(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$, which describes the evolution of the sequence $(w_n)_{n \ge 1}$ and, for each time step n, defines three sets $\mathscr{B}_n$, $\mathscr{W}_n$ and $\mathscr{G}_n$ of so called black, white and grey vertices respectively, whose key features are described in Lemma 4.5.3 below. Also, we let $\mathscr{C}_{J,n}(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}) \subset \mathscr{C}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ denote the component of x in $\mathscr{W}_n$. The algorithm is initialised by setting

$$\mathscr{B}_0 = \mathscr{W}_0 = \emptyset, \mathscr{G}_0 = \mathbb{Z}^d \tag{4.5.41}$$

and, in the language of the discussion below (4.5.37), a vertex v is said to be revealed when the value of $\mathbb{1}\{v \in \mathcal{I}\}$ is disclosed (remark that, a vertex may be *revealed* but not *explored* - see the special step below). Once revealed, v is attributed to either the black or white vertices, depending on whether it is in $\mathcal{I}$ or $\mathcal{V}$ respectively. Thus, at each step of the algorithm, we define the triplet $(\mathscr{B}_n, \mathscr{W}_n, \mathscr{G}_n)$ inductively from $(\mathscr{B}_{n-1}, \mathscr{W}_{n-1}, \mathscr{G}_{n-1})$ in the following way. First, if $n = 1$, let $w_n = x$, and otherwise select the 'smallest' *unexplored* w_n from $\partial^{\mathrm{out}} \mathscr{C}_{J,n-1}(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$, according to the predetermined ordering of $\mathbb{Z}^d$ - the algorithm is completed at this step if the latter is an empty set. If $n \neq \tau_k$ for any $k \ge 1$, then we perform a 'generic' step, which consists of checking whether w_n has already been revealed, and, if it has not, removing w_n from $\mathscr{G}_{n-1}$, revealing w_n, and adding w_n to $\mathscr{W}_{n-1}$ or $\mathscr{B}_{n-1}$ if it is white or black respectively, thus yielding $\mathscr{B}_n$, $\mathscr{W}_n$ and $\mathscr{G}_n$. If instead w_n was already revealed, we effectively do nothing and set $(\mathscr{B}_n, \mathscr{W}_n, \mathscr{G}_n) = (\mathscr{B}_{n-1}, \mathscr{W}_{n-1}, \mathscr{G}_{n-1})$. On the other hand, if $n = \tau_k$ for some $k \ge 1$, we split into two scenarios. Recall the definition of $\tilde{x}_k$ from (4.5.39), and denote by $y_k \in \hat{\mathbb{L}}_{0,y}$ the unique point such that $\tilde{x}_k \in \partial D_{y_k,L_0}$. If D_{y_k,L_0} is fully revealed, we simply perform a generic step. Otherwise, we perform a 'special' step, described by the procedure *(i)* -*(v)* below.

(i) Let $\left(\mathscr{B}_n|_{D^c_{y_k,L_0}}, \mathscr{W}_n|_{D^c_{y_k,L_0}}, \mathscr{G}_n|_{D^c_{y_k,L_0}}\right) = \left(\mathscr{B}_{n-1}|_{D^c_{y_k,L_0}}, \mathscr{W}_{n-1}|_{D^c_{y_k,L_0}}, \mathscr{G}_{n-1}|_{D^c_{y_k,L_0}}\right)$.

(ii) Reveal all the points in $\partial D_{y_k,L_0} \cap \mathscr{G}_{n-1}$, as well as the clusters of the points of $\mathcal{I} \cap \partial D_{y_k,L_0}$ inside $D_{y_k,L_0} \setminus C_{y_k,L_0}$ together with their outer boundaries in $D_{y_k,L_0} \setminus C_{y_k,L_0}$.

(iii) All of the revealed points in $D_{y_k,L_0} \setminus C_{y_k,L_0}$ are assigned to the intermediary collections $\mathscr{B}'_n$ or $\mathscr{W}'_n$, depending on whether they are black or white respectively, and the set of remaining points in D_{y_k,L_0} is denoted by $\mathscr{G}'_n$.

(iv) Define

$$\begin{aligned} \mathscr{G}''_n &= \text{ the points in the component of } C_{y_k,L_0} \text{ in } \mathscr{G}'_n, \\ \mathscr{B}''_n &= \mathscr{B}'_n \\ \mathscr{W}''_n &= \mathscr{W}'_n \cup (\mathscr{G}'_n \setminus \mathscr{G}''_n) \end{aligned} \tag{4.5.42}$$

and reveal all the points in $\mathscr{G}'_n \setminus \mathscr{G}''_n$ such that $\mathscr{W}''_n$ is fully revealed (indeed, $\mathscr{G}'_n \setminus \mathscr{G}''_n \subset \mathcal{V}$, see Lemma 4.5.3*(a)*).

(v) (Case I) If $\tilde{x}_k \in \mathcal{V}$ is not connected to C_{y_k,L_0} by a path in $(\mathscr{G}''_n \cup \mathscr{W}''_n) \cap D_{y_k,L_0}$, set $\mathscr{G}_n|_{D_{y_k,L_0}} = \mathscr{G}''_n, \mathscr{B}_n|_{D_{y_k,L_0}} = \mathscr{B}''_n$, and $\mathscr{W}_n|_{D_{y_k,L_0}} = \mathscr{W}''_n$.

(Case II) Otherwise, reveal all vertices in $\mathscr{G}''_n$, add them to $\mathscr{B}''_n$ or $\mathscr{W}''_n$ depending on their state, and set $\mathscr{B}_n|_{D_{y_k,L_0}}$, $\mathscr{W}_n|_{D_{y_k,L_0}}$ as in *(Case I)*, as well as $\mathscr{G}_n|_{D_{y_k,L_0}} = \emptyset$. If $\mathscr{G}''_n \subset \mathcal{V}$, *terminate* the algorithm.

Thus we have completed the specification of the algorithm $\mathcal{A}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$. Let us collect some of its essential properties in the following useful Lemma.

Lemma 4.5.3. *For all $n \geq 1$, the following hold:*

(a) $\mathscr{B}_n \subset \mathcal{I}, \mathscr{W}_n \subset \mathcal{V}$,

(b) for $S_N \in \{\mathscr{W}_n, \mathscr{B}_n, \mathscr{G}_n\}$ one has $S_N \cap D_{y_k,L_0} = S_{\tau_k} \cap D_{y_k,L_0}$ for all $k \geq 1$ and $n \geq \tau_k$ such that $\mathscr{G}_n \cap D_{y_k,L_0} \neq \emptyset$,

(c) if $n = \tau_k$ for some $k \geq 1$ such that $\mathscr{G}_{n-1} \cap D_{y_k,L_0} \neq \emptyset$ and $C_{y_k,L_0} \subset \mathcal{V}$, then $\mathscr{G}''_n \subset \mathcal{V}$ (see (4.5.42)*).*

Proof. Part *(a)*, regarding $\mathscr{B}_n \subset \mathcal{V}$, is immediate simply by inspecting the algorithm. On the other hand, for the remaining claim $\mathscr{W}_n \subset \mathcal{V}$ we rely on the observation that, for any $k \geq 1$, $\{x_1, x'_1, x_2, x'_2\} \cap D_{y_k,L_0} = \emptyset$ (see (4.5.40)) and therefore $\mathscr{G}'_{\tau_k} \setminus \mathscr{G}''_{\tau_k} \subset \mathcal{V}$. Indeed, if this were false, any points in $\mathcal{I} \cap \mathscr{G}'_{\tau_k} \setminus \mathscr{G}''_{\tau_k}$ must already have been revealed in stage *(ii)* and removed from $\mathscr{G}'_n$ in stage *(iii)* as random walk paths exit any finite set,

yielding a contradiction. Overall, we have the required claim since generic steps only add vacant points to $\mathscr{W}_n$. Part *(c)* is obtained by similar reasoning, since any points of $\mathcal{I} \cap \mathscr{G}_n''$ cannot be connected to $\partial D_{y_k,L_0}$ by a path in $\mathcal{I}$ passing through $C_{y_k,L_0} (\subset \mathcal{V})$ and must therefore have already been revealed, yielding a contradiction.

Part *(b)* holds since, in the stages following *(iv)* in the special step, either no points are revealed in D_{y_k,L_0} or all the remaining points are revealed, in which case $\mathscr{G}_n \cap D_{y_k,L_0} \neq \emptyset \ \forall n \geq \tau_k$. Also, a generic step cannot alter any of the $S_n \cap D_{y_k,L_0}$. $\square$

In the sequel, we refer to $y_1, y_2, \ldots \in \mathbb{L}_0$ (i.e. $\tilde{x}_k \in D_{y_k,L_0}$) as the sequence of points *discovered* by $\mathcal{A}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ before it stops (completes or terminates early). Note that there may not be any discovered points (in the case where $\tau_1 = \infty$) whereas the same point can be discovered multiple times. If the number of discovered points is strictly less than l, we set $y_l = \Delta$. Moreover, we refer to a point $y_l \in \hat{\mathbb{L}}_{0,y}$ discovered by $\mathcal{A}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ as *bad* if $\mathscr{G}_{\tau_l - 1} \cap D_{y_l,L_0} \neq \emptyset$, $\tilde{x}_l$ is connected to C_{y_l,L_0} by a path in $(\mathscr{G}_{\tau_l}'' \cup \mathscr{W}_{\tau_l}'') \cap D_{y_l,L_0}$, but $\mathscr{G}_n'' \cap \mathcal{I} \neq \emptyset$ (hence $C_{y_l,L_0} \not\subset \mathcal{V}$ by Lemma 4.5.3*(c)*). For any $b \in \{0, 1, 2, \ldots\}$, we denote by $k_{J,b}^y = k_{J,b}^y(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ the 'index' of the b^{th} bad point discovered by the algorithm; that is

$$\begin{aligned} k_{J,0}^y &= 0 \\ k_{J,b}^y &= \inf\{k > k_{J,b-1}^y : y_k \text{ is bad}\}, \quad b \geq 1 \end{aligned} \tag{4.5.43}$$

At this stage, we remark that a duplicate $y_l = y_k$ for $l > k$ may indeed be bad in the case that the first $\tilde{x}_k$ corresponding to y_l was not connected to C_{y_l,L_0} in $\mathscr{W}_{\tau_k}'' \cup \mathscr{G}_{\tau_k}''$. Conversely, once a point is found to be bad, it is never discovered again, implying that a point can only be bad once. Moreover, for any label $l \geq 1$ we consider the 'next' label

$$N(\ell) = \inf \left\{ l \geq \ell : \begin{array}{c} \mathscr{G}_{\tau_l - 1} \cap D_{y_l,L_0} \neq \emptyset, \ \tilde{x}_l \text{ is connected to} \\ C_{y_l,L_0} \text{ by a path in } (\mathscr{G}_{\tau_l}'' \cup \mathscr{W}_{\tau_l}'') \cap D_{y_l,L_0} \end{array} \right\} \tag{4.5.44}$$

(in view of the above, being bad requires in addition that $\mathscr{G}_{\tau_l}'' \cap \mathcal{I} \neq \emptyset$). As a consequence of the features stated in Lemma 4.5.3, we deduce three key properties of $\mathcal{A}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ which will shortly become essential in the proof of a series of claims, culminating in the inequality (4.5.33).

P1 If the explored part of the component $\mathscr{C}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ (i.e. before $\mathcal{A}_{J,y}(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ terminates) intersects a box C_{y',L_0} for some $y' \in \hat{\mathbb{L}}_{0,y}$ such that (4.5.40) holds

with y' in place of y_k, then $y' = y_l$ for some $l \geq 1$ such that $\tilde{x}_l$ is connected to C_{y_l,L_0} by a path in $(\mathscr{G}''_{\tau_l} \cup \mathscr{W}''_{\tau_l}) \cap D_{y_l,L_0}$ (see *(v)*, *(Case II)* from the special step).

P2 If $\mathcal{A}_{J,y}(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ stops prior to discovering $\mathscr{C}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$, then x is connected to $\mathscr{C}_{\mathrm{y}}$ in $\mathcal{V}$ for some $\mathrm{y} \in \bigcup_{1 \leq j \leq l} \mathcal{O}_j$ (see below (4.5.36) for their construction).

P3 For any index $b \geq 1$ and $y' \in \hat{\mathbb{L}}_{0,y}$, the event

$$\mathcal{E}_b(y') \coloneqq \{k^y_{J,b} < \infty, y_{N(k^y_{J,b})} = y'\} \tag{4.5.45}$$

is measurable with respect to $\mathcal{F} = \mathcal{F}_{J,y'}(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}; \delta, u, u_3)$ from (4.5.7).

Remark 4.5.4. *The measurability condition presented in **P3** is precisely where the choice of $\mathcal{F}_{J,y'}(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}; \delta, u, u')$ selected in Lemma 4.5.1 originates. Of course, the same estimate found therein will also hold for less restrictive σ-algebras, but these choices are insufficient for our purposes.*

From this point onwards, we present the aforementioned series of claims. The first of these harnesses the full strength of Proposition 4.1.1, particularly item *(iii)*, and introduces a crucial event, A_y, which extracts the key features of the coarse-graining.

Claim I. Fix a finite set $J \subset \mathbb{N}^*$, specify the truncations according to $\boldsymbol{j} = (j_1, j_2) \in (\mathbb{N}^*)^2$, $\boldsymbol{x} \in U_z^4$, $\boldsymbol{n} \in \mathbb{N}^4$x, select a starting point for the algorithm $x \in \partial \tilde{C}_z$, and let $0 \leq i \leq 2 \left\lfloor \frac{u}{u_0} \right\rfloor$ and $y \in \mathbb{L}_0 \cap D_{0,L_0}$. Then there exists $c = c(d) \in (0, \infty)$ such that with

$$A_y = \left\{ \sum_{y' \in \hat{\mathbb{L}}_{0,y}} \mathbb{1} \left\{ \begin{array}{c} \text{(4.5.40) holds with } y' \text{ in place of } y_k \\ \text{and } C_{y',L_0} \text{ intersects } \mathscr{C}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}) \end{array} \right\} \geq cam \right\} \tag{4.5.46}$$

we have

$$\mathscr{G}_z \cap \{x \overset{\mathcal{V}}{\leftrightarrow} \partial \tilde{D}_z\} \cap \left\{ [\mathcal{N}_z^{\frac{i}{2}u_0} + 1, \mathcal{N}_z^{(\frac{i}{2}+1)u_0}] \subset J \subset [1, \mathcal{N}_z^u] \right\} \subset \bigcup_{y \in \mathbb{L}_0 \cap D_{0,L_0}} A_y. \tag{4.5.47}$$

The next claim connects the event A_y to the number of bad points appearing during the exploration of $\mathscr{C}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$.

Claim II. Under the same set-up as in Claim I, we have

$$A_y \cap \tilde{V}^c_{z,x}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})) \subset \{k^y_{J,\lfloor cam \rfloor}(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}) < \infty\} \tag{4.5.48}$$

where we let,

$$\tilde{V}_{z,x}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \coloneqq \{x \overset{\mathcal{V}}{\leftrightarrow} \partial\tilde{D}_z, x \overset{\mathcal{V}}{\not\leftrightarrow} \mathscr{C}_{\mathrm{y}} \text{ for any y} \in \bigcup_{1\le j\le l} \mathcal{O}_j\}^c \tag{4.5.49}$$

Assuming Claim II, we then prove the following, which arises by iterated application of Lemma 4.5.1 at the bad points.

Claim III. Under the same set-up as in Claim I, one has the bound:

$$\mathbb{P}\Big[A_y, \tilde{V}^c_{z,x}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})), J \subset [1,\mathcal{N}^u_z]\Big] \le e^{-c(\delta,L_0)am}. \tag{4.5.50}$$

In our final Claim IV, we connect the pseudo-local uniqueness event $\tilde{V}_{z,x}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$ from (4.5.49) to the event of interest, $V_z(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$.

Claim IV. Under the same assumptions on J, $\boldsymbol{j}$, $\boldsymbol{x}$, $\boldsymbol{n}$ and i as above, we have

$$\mathscr{G}_z \cap \tilde{V}_z(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \cap \{C_z \overset{N_{2\delta}(\mathcal{V}^{u_3})}{\longleftrightarrow} \partial D_z\} \cap \{J \subset [1,\mathcal{N}^u_z]\} \subset V_z(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \tag{4.5.51}$$

where the event $\tilde{V}_z(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \coloneqq \bigcap_{x\in\partial\tilde{C}_z} \tilde{V}_{z,x}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$.

Overall, by combining these claims and applying a union bound, first over all $y \in \mathbb{L}_0 \cap D_{0,L_0}$ for any given $x \in \partial\tilde{C}_z$ and then over all $x \in \partial\tilde{D}_z$, we obtain (4.5.33). Indeed, starting with an application of the contra-positive of Claim IV, we observe

$$\begin{aligned} &V^c_z(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \cap \mathscr{G}_z \cap \{\tilde{C}_z \overset{N_{2\delta}(\mathcal{V}^{u_3})}{\longleftrightarrow} \partial\tilde{D}_z\} \cap \{[\mathcal{N}_z^{\frac{i}{2}u_0}+1, \mathcal{N}_z^{(\frac{i}{2}+1)u_0}] \subset J \subset [1,\mathcal{N}^u_z]\} \\ &\subset \tilde{V}^c_z(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \end{aligned} \tag{4.5.52}$$

As well as, from Claim I,

$$\begin{aligned} &\tilde{V}^c_z(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \cap \mathscr{G}_z \cap \{[\mathcal{N}_z^{\frac{i}{2}u_0}+1, \mathcal{N}_z^{(\frac{i}{2}+1)u_0}] \subset J \subset [1,\mathcal{N}^u_z]\} \\ &\subset \bigcup_{x\in\partial\tilde{C}_z} \tilde{V}^c_{z,x}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \cap \mathscr{G}_z \cap \{[\mathcal{N}_z^{\frac{i}{2}u_0}+1, \mathcal{N}_z^{(\frac{i}{2}+1)u_0}] \subset J \subset [1,\mathcal{N}^u_z]\} \\ &\subset \bigcup_{x\in\partial\tilde{C}_z} \bigcup_{y\in\mathbb{L}_0\cap D_{0,L_0}} \Big[A_y \cap \tilde{V}^c_{z,x}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \cap \{[\mathcal{N}_z^{\frac{i}{2}u_0}+1, \mathcal{N}_z^{(\frac{i}{2}+1)u_0}] \subset J \subset [1,\mathcal{N}^u_z]\}\Big] \end{aligned} \tag{4.5.53}$$

Hence, we conclude with an application of Claim III and the aforementioned union bound. Therefore, all that is left is to prove the claims. To this end, we proceed sequentially, starting from Claim I.

Proof of Claim I. In order to extract events of the form given inside the summation of (4.5.46) from the event $\mathscr{G}_z$, the following inclusion will play a pivotal role, since it delocalises the constituent events of the coarse-grainings while producing the correct collection of finite energy events. For any $\mathrm{y} \in D_z \cap \mathbb{L}$ such that $\{x_1, x_1', x_2, x_2'\} \cap U_{\mathrm{y},\mathrm{L}} = \emptyset$ and J, $\boldsymbol{j}$, $\boldsymbol{x}$, $\boldsymbol{n}$, x and i as in Claim I, we have

$$\begin{aligned}&\Big(\mathcal{G}_{\mathrm{y}}(\bar{\mathcal{V}}_{\mathbb{L}}, a, \overline{\mathrm{FE}}_{\mathrm{y}}(\bar{\mathcal{V}}_{\mathbb{L}}, L_{-1}, \tfrac{u_0}{8}, u_1, u_5; n), \delta, u_1, u_2, u_4) \cap \mathcal{F}_{\mathrm{y}}^{\boldsymbol{u}_1, \boldsymbol{u}_2} \\ &\cap \{[\mathcal{N}_z^{\frac{i}{2}u_0} + 1, \mathcal{N}_z^{(\frac{i}{2}+1)u_0}] \subset J \subset [1, \mathcal{N}_z^u]\}\Big) \subset \mathcal{G}_{\mathrm{y}}(\mathcal{V}_{\mathbb{L}}, a, \widetilde{\mathrm{FE}}_{\mathrm{y}}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})), \delta, u, u_3, u_3),\end{aligned} \tag{4.5.54}$$

where $\widetilde{\mathrm{FE}}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}))$ denotes the family of events

$$\widetilde{\mathrm{FE}}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})) \coloneqq \{\widetilde{\mathrm{FE}}_{y'}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})) : \mathrm{y} \in \mathbb{L}, y' \in \mathbb{L}_0\} \tag{4.5.55}$$

and we recall the definition of (4.5.37) as well as (4.4.26) - (4.4.27). For now, let us assume inclusion (4.5.54) holds and continue with the remainder of the proof. By using the properties of an admissible family of coarsenings, in light of the inclusion (4.5.54), the event $\mathscr{G}_z \cap \{x \overset{\mathcal{V}}{\leftrightarrow} \partial \tilde{D}_z\}$ naively seems to yield exactly what we desire simply by considering any path γ in $\mathcal{V}$ crossing $\tilde{D}_z \setminus \tilde{C}_z$, and then applying the event $\mathscr{G}_z$ to this path (i.e. to find an admissible coarsening $\tilde{\mathcal{C}}$ of γ where each point in $\tilde{\mathcal{C}}$ satisfies (4.5.40)). However, there is a subtlety: by applying a direct construction in this manner, we obtain some collection of $*$-connected sets $\bigcup_{1 \le j \le l'} \mathcal{O}_j'$ which are not guaranteed to correspond to those $\mathcal{O}_j$ obtained from $\mathscr{S}$ which is fixed at the start of the algorithm (see (4.5.36)). To resolve this issue, we call on the maximality property *(iii)* from Proposition 4.1.1 while carefully specifying the steps of our construction. To this extent, on the event $\{x \overset{\mathcal{V}}{\leftrightarrow} \partial \tilde{D}_z\}$ we observe that the component $\mathscr{C}_J(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ contains a path γ which crosses $\tilde{D}_z \setminus \tilde{C}_z$. Then, we denote by $\gamma_{\mathbb{L}}$ as the sequence of points $\{\mathrm{y}_1, \mathrm{y}_2, \ldots\} \subset \mathbb{L}$ such that the path γ visits $C_{y_1,L}, C_{y_2,L}$, etc. in that order. Notice that, whenever γ crosses $\tilde{D}_z \setminus \tilde{C}_z$, the path $\gamma_{\mathbb{L}}$ must cross $\mathbb{L}(\tilde{D}_z) \setminus \mathbb{L}(\tilde{C}_z)$ in the coarse lattice $\mathbb{L}$. Conversely, given any crossing $\boldsymbol{\gamma}'$ of $\mathbb{L}(\tilde{D}_z) \setminus \mathbb{L}(\tilde{C}_z)$ in the coarse lattice $\mathbb{L}$, there exists a path γ' crossing $\tilde{D}_z \setminus \tilde{C}_z$ in $\mathbb{Z}^d$ such that $\gamma_{\mathbb{L}}' = \boldsymbol{\gamma}'$. This can be seen by, for example, connecting each consecutive pair of points in $\boldsymbol{\gamma}'$ by a straight line,

and possibly extending the endpoints to the boundaries of their corresponding boxes.

Now, we return to the $*$-connected sets $\mathcal{O}_j$, $1 \le j \le l$ constructed from $\mathscr{S}$ according to Proposition 4.1.1. By property *(iii)* in that Proposition, recalling that it was applied on the coarse lattice $\mathbb{L}$ with $V = \mathbb{L}(\tilde{D}_z)$, $U = \mathbb{L}(\tilde{C}_z)$ and $\Sigma = \mathscr{S}$, we deduce that there exists a crossing γ' of $\tilde{D}_z \setminus \tilde{C}_z$ satisfying

$$\gamma'_{\mathbb{L}} \cap \mathscr{S} = \gamma_{\mathbb{L}} \cap \bigcup_{1 \le j \le l} \mathcal{O}_j \tag{4.5.56}$$

where $\gamma \subset \mathcal{V}$ was the path selected in the previous paragraph, and we remark that γ' need not be fully contained in $\mathcal{V}$. Indeed, starting with γ, property *(iii)* guarantees that any segment of γ which is outside $\bigcup_{1\le j\le l} \mathcal{O}_j$ can be replaced by a segment with the same endpoints which is outside $\mathscr{S}$. The required choice γ' is therefore the concatenation of the points $\gamma_{\mathbb{L}} \cap \bigcup_{1\le j\le l} \mathcal{O}_j$ with the interiors of the segments obtained in this way. Furthermore, since any coarsening in $\mathcal{A}_L^K(\tilde{D}_z \setminus \tilde{C}_z)$ which satisfies property *(iii)* of Γ-admissibility (see Definition 4.3.1) for γ' is necessarily a subset of $\gamma'_{\mathbb{L}}$, it follows that any such coarsening $\mathcal{C}$ (say) intersects $\mathscr{S}$ only at $\gamma_{\mathbb{L}} \cap \bigcup_{1\le j\le l} \mathcal{O}_j$. Moreover, since the admissible coarsening $\mathcal{C}$ is well-separated, (see (4.2.2)) it must satisfy $U_{\mathrm{y},L} \cap U_{\mathrm{y}',L} = \emptyset$ for any pair of distinct points $\mathrm{y}, \mathrm{y}' \in \mathcal{C}$. At this stage of the proof, we notice that the event $\mathscr{G}_z$ now lends itself to obtaining points with property (4.5.40) in the same 'naive' way that we alluded to before with the crucial exception that we start the process with the path γ' (instead of γ). The upshot is that, since *any* coarsening of γ' intersects $\mathscr{S}$ only at $\gamma_{\mathbb{L}} \cap \bigcup_{1\le j\le l}$, then the coarsenings $\tilde{\mathcal{C}} \subset \mathcal{C}$ of γ constructed by $\mathscr{G}_z$ must have all their points in $\mathscr{S}$ (see the definition of $\mathcal{G}_{\mathrm{y}}(\mathcal{V}_{\mathbb{L}}, a, \cdot, \delta, u, u_3, u_3)$ in (4.4.14)). Overall, this observation implies that, on the event $\mathscr{G}_z \cap \{x \overset{\mathcal{V}}{\hookleftarrow} \partial \tilde{D}_z\} \cap \Big\{[\mathcal{N}_z^{\frac{i}{2}u_0} + 1, \mathcal{N}_z^{(\frac{i}{2}+1)u_0}] \subset J \subset [1, \mathcal{N}_z^u]\Big\}$,

$$\begin{aligned}&\text{there exists } \mathscr{S}_\gamma \subset \gamma_{\mathbb{L}} \cap (\cup_{1\le j\le l} \mathcal{O}_j) \text{ such that } |\mathscr{S}_\gamma| \ge \tfrac{m}{2C_4} - 4 \ge \tfrac{m}{3C_4}\\ &\text{(recall (4.3.34)) and for each } \mathrm{y} \in \mathscr{S}_\gamma, \{x_1, x_1', x_2, x_2'\} \cap U_{\mathrm{y},L} = \emptyset, \text{ the event}\\ &\mathcal{G}_{\mathrm{y}}(\mathcal{V}, a, \widetilde{\mathrm{FE}}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})), \delta, u, u_3, u_3) \text{ occurs, and } \gamma \text{ crosses } \tilde{D}_{\mathrm{y}} \setminus C_{\mathrm{y}}.\end{aligned} \tag{4.5.57}$$

Thus, recalling the definition of $\mathcal{G}_{\mathrm{y}}(\mathcal{V}, a, \widetilde{\mathrm{FE}}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})), \delta, u, u_3, u_3)$ from Definition 4.3.4 (see also (4.4.14)), and using that the choice of $\gamma \subset \mathcal{V}$ crosses $\tilde{D}_{\mathrm{y}} \setminus C_{\mathrm{y}}$ for every $\mathrm{y} \in \mathscr{S}_\gamma$, we produce a contact points for every $\mathrm{y} \in \mathscr{S}_\gamma$ satisfying the condition in the indicator function of (4.5.46). This, together with the fact that any C_{y',L_0} intersecting $\tilde{D}_{\mathrm{y},L}$ must lie within $D_{\mathrm{y},L} \subset (\tilde{C}_z)^c$ (see (4.5.36)), implies that on the same event as

above (4.5.57),

$$\sum_{y'\in\hat{\mathbb{L}}_0}\mathbb{1}\left\{\begin{matrix}\text{(4.5.40) holds with } y \text{ in place of } y_k\\ \text{and } C_{y',L_0} \text{ intersects } \mathscr{C}_J(x;\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})\end{matrix}\right\}\geq cam, \tag{4.5.58}$$

where $\hat{\mathbb{L}}_0=\bigcup_{y\in\mathbb{L}_0\cap D_{0,L_0}}\hat{\mathbb{L}}_{0,y}$. Hence, recalling the definition of $\hat{\mathbb{L}}_{0,y}$ from the paragraph directly preceding (4.5.36), we finally obtain Claim I by an application of the Dirichlet principle.

Let us now return to the proof of inclusion (4.5.54). From one localisation step, we observe that

$$\begin{aligned}&\mathcal{G}_{\mathrm{y}}(\bar{\mathcal{V}}_{\mathbb{L}},a,\overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}},\tfrac{u_0}{8},u_1,2u_5;n),\delta,u_1,u_2,u_4)\cap\mathcal{F}_{\mathrm{y}}^{u,u_1}\cap\mathcal{F}_{\mathrm{y}}^{u_3,u_2}\cap\mathcal{F}_{\mathrm{y}}^{u_3,u_4}\\ \subset&\mathcal{G}_{\mathrm{y}}(\mathcal{V}_{\mathbb{L}},a,\overline{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}},\tfrac{u_0}{8},u_1,2u_5;n),\delta,u,u_3,u_3).\end{aligned} \tag{4.5.59}$$

Meanwhile, the definitions (4.4.19) and (4.5.5) imply

$$\mathrm{LU}_{y'}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))\subset\widetilde{\mathrm{LU}}_{y'}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \tag{4.5.60}$$

for any $y'\in\mathbb{L}_0$. To see this, observe that whenever two points $x,x'\in\mathscr{C}_{\partial D_{y,L_0}}\cap(\tilde{D}_{y,L_0}\setminus\tilde{C}_{y,L_0})$ are connected in $\mathcal{I}\cap\mathring{D}_{y,L_0}\setminus C_{y,L_0}$ then their connecting path must also belong to $\mathscr{C}_{\partial D_{y,L_0}}\setminus\partial D_{y,L_0}$ (under the assumption that $\{x_1,x_1',x_2,x_2'\}\cap D_{y,L_0}=\emptyset$). Consequently, by Definition 4.4.3 and (4.5.37), we have

$$\mathrm{FE}_{y'}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))\subset\widetilde{\mathrm{FE}}_{y'}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \tag{4.5.61}$$

for any $y'\in\mathbb{L}_0$. By recalling the definitions of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\bar{\mathrm{FE}}(\bar{\mathcal{V}}_{\mathbb{L}},\frac{u_0}{8},u_1,2u_5;n)$ from (4.4.27) and (4.4.28) respectively, taking into consideration the displays (4.5.59) and (4.5.61) leaves us with showing

$$\begin{aligned}&\mathrm{FE}_{y'}\Big(\boldsymbol{Z}(\bar{\mathcal{V}}_{\mathrm{y}}^{u_1},\tfrac{u_0}{8}\mathrm{cap}(\breve{D}_{0,L}),\star)\Big)\cap\mathcal{F}_{\mathrm{y}}^{\boldsymbol{u}''_{1,i},\boldsymbol{u}''_{2,i}}\cap\mathcal{F}_{\mathrm{y}}^{u,u_1}\\ &\cap\Big\{[\mathcal{N}_z^{\frac{i}{2}u_0}+1,\mathcal{N}_z^{(\frac{i}{2}+1)u_0}]\subset J\subset[1,\mathcal{N}_z^u]\Big\}\subset\mathrm{FE}_{y'}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))\end{aligned} \tag{4.5.62}$$

for $\boldsymbol{u}''_{1,i}=\frac{i}{2}u_0+(0,\frac{u_0}{2},\frac{u_0}{2},\frac{3u_0}{4},\frac{3u_0}{4},u_0)$ and $\boldsymbol{u}''_{2,i}=\frac{i}{2}u_0+(\frac{u_0}{8},\frac{u_0}{4},\frac{9u_0}{16},\frac{11u_0}{16},\frac{13u_0}{16},\frac{15u_0}{16})$, which are subsets of the levels prescribed by (4.4.27). This comes down to showing that, under the assumption of the additional events on the left-hand-side of (4.5.62), the set of excursions $Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})$ between $\breve{D}_z$ and $\breve{U}_z$ *induces* a set of sub-excursions between

$\breve{D}_{\mathrm{y},L}$ and $\breve{U}_{\mathrm{y},L}$ which is contained in the collection of sets of excursions described by $\boldsymbol{Z}(\bar{\mathcal{V}}_{\mathrm{y}}^{u_1}, \frac{u_0}{8}\mathrm{cap}(\breve{D}_{0,L}), \star)$. In the sequel, we denote the aforementioned induced set of sub-excursions as

$$Z_{I,L} = (Z_i^{\breve{D}_{\mathrm{y},L},\breve{U}_{\mathrm{y},L}})_{i\in I} \tag{4.5.63}$$

such that we seek to show $Z_{I,L} \in \boldsymbol{Z}(\bar{\mathcal{V}}_{\mathrm{y}}^{u_1}, \frac{u_0}{8}\mathrm{cap}(\breve{D}_{0,L}), \star)$. It is important to bear in mind that, given the condition $\{x_1, x_1', x_2, x_2'\} \cap U_{\mathrm{y},L} = \emptyset$, the (truncated) excursions of $Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ can only produce (possibly empty) sequences of full excursions between $\breve{D}_{\mathrm{y},L}$ and $\breve{U}_{\mathrm{y},L}$. In particular, no truncation is necessary in the left-hand-side of (4.5.62) to account for the excursions $Z_{I,L}$.

In this way, we first observe that $J \subset [1, \mathcal{N}_z^u]$ implies $I \subset [1, \mathcal{N}_{\mathrm{y},L}^u]$ by property of the excursion count, and, by referring to the event $\mathcal{F}_{\mathrm{y}}^{u,u_1}$, a simple localisation step yields $I \subset [1, u_1\mathrm{cap}(\breve{D}_{0,L})]$. Next, we need to verify that I contains $\{j+1, \dots, j + \left\lfloor \frac{u_0}{8}\mathrm{cap}(\breve{D}_{0,L})\right\rfloor\}$ for some $j \in \mathbb{N}$ (see (4.4.23)). In brief, the sole challenge to this claim lies in the truncations within $Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$. Consequently, we develop an argument that focuses exclusively on the segments of excursions that remain unaltered, handling each case for $\boldsymbol{j}$ individually. In particular, recalling from the paragraph below (4.5.2) that $\boldsymbol{j} = (j_1, j_2)$ with $j_1 \le j_2$, we consider three cases, namely:

(i) $\mathcal{N}_z^{\frac{i+1}{2}u_0} < j_1$,

(ii) $\mathcal{N}_z^{\frac{i+1}{2}u_0} \ge j_1$ and $\mathcal{N}_z^{(\frac{i}{2}+\frac{3}{4})u_0} < j_2$,

(iii) $\mathcal{N}_z^{\frac{i+1}{2}u_0} \ge j_1$ and $\mathcal{N}_z^{(\frac{i}{2}+\frac{3}{4})u_0} \ge j_2$.

In the following, recall the assumption $[\mathcal{N}_z^{\frac{i}{2}u_0} + 1, \mathcal{N}_z^{(\frac{i}{2}+1)u_0}] \subset J$ and observe that, for $u', v' \in [\frac{i}{2}u_0, (\frac{i}{2}+1)u_0]$ such that no excursions with labels between u' and v' are truncated, one has the inclusion $[\mathcal{N}_{\mathrm{y}}^{u'} + 1, \mathcal{N}_{\mathrm{y}}^{v'}] \subset I$. In case *(i)*, since we are on the event $\mathcal{F}_{\mathrm{y}}^{\frac{i+1}{2}u_0,(\frac{i}{2}+\frac{1}{4})u_0} \cap \mathcal{F}_{\mathrm{y}}^{\frac{i}{2}u_0,(\frac{i}{2}+\frac{1}{8})u_0}$ (see the definitions of $\boldsymbol{u}''_{1,i}$ and $\boldsymbol{u}''_{2,i}$ above), we have

$$\left\{ \left\lfloor (\tfrac{i}{2}+\tfrac{1}{8})u_0\mathrm{cap}(\breve{D}_{0,L})\right\rfloor + 1, \dots, \left\lfloor (\tfrac{i}{2}+\tfrac{1}{4})u_0\mathrm{cap}(\breve{D}_{0,L})\right\rfloor \right\} \subset I. \tag{4.5.64}$$

which leads to the required condition on I as $\lfloor x\rfloor + \lfloor y\rfloor \le \lfloor x+y\rfloor$. In case *(ii)*, we now remark that we are also on the event $\mathcal{F}_{\mathrm{y}}^{\frac{i+1}{2}u_0,(\frac{i}{2}+\frac{9}{16})u_0} \cap \mathcal{F}_{\mathrm{y}}^{(\frac{i}{2}+\frac{3}{4})u_0,(\frac{i}{2}+\frac{11}{16})u_0}$, which implies

$$\left\{ \left\lfloor (\tfrac{i}{2}+\tfrac{9}{16})u_0\mathrm{cap}(\breve{D}_{0,L})\right\rfloor + 1, \dots, \left\lfloor (\tfrac{i}{2}+\tfrac{11}{16})u_0\mathrm{cap}(\breve{D}_{0,L})\right\rfloor \right\} \subset I. \tag{4.5.65}$$

and finally in case *(iii)* we obtain a similar conclusion, since on the event $\mathcal{F}_{\mathrm{y}}^{(\frac{i}{2}+\frac{3}{4})u_0,(\frac{i}{2}+\frac{13}{16})u_0} \cap \mathcal{F}_{\mathrm{y}}^{(\frac{i}{2}+1)u_0,(\frac{i}{2}+\frac{15}{16})u_0}$ we observe

$$\Big\{ \Big\lfloor (\tfrac{i}{2}+\tfrac{13}{16})u_0 \mathrm{cap}(\breve{D}_{0,L}) \Big\rfloor + 1, \ldots, \Big\lfloor (\tfrac{i}{2}+\tfrac{15}{16})u_0 \mathrm{cap}(\breve{D}_{0,L}) \Big\rfloor \Big\} \subset I. \tag{4.5.66}$$

Hence, in every scenario we obtain the required result, and the proof of Claim I is complete. □

Proof of Claim II. Let us denote the event on the left-hand side of (4.5.48) as $\mathscr{H}_{z;y}$. From the definition of $\tilde{V}_{z,c}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$ in (4.5.49) comprising $\mathscr{H}_{z;y}$, we have that

$$\mathscr{H}_{z;y} \subset \bigcap_{\mathrm{y} \in \cup_{1 \le j \le l} \mathcal{O}_j} \{ x \overset{\mathcal{V}}{\longleftrightarrow} \mathscr{C}_{\mathrm{y}} \} \tag{4.5.67}$$

which means, by relying on **P2** above, that the algorithm $\mathcal{A}_{J,y}(x;\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})$ fully explores the component $\mathscr{C}_J(x;\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})$, which in turn must be disjoint from $\bigcup_{\mathrm{y} \in \cup_{1 \le j \le l} \mathcal{O}_j} \mathscr{C}_{\mathrm{y}}$. Next, recalling the definition of a bad point from above (4.5.43), the event $A_y \subset \mathscr{H}_{z;y}$ in conjunction with **P2** implies that the algorithm $\mathcal{A}_{J,y}(x;\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})$ discovers at least *cam* bad points (observe that the points y' with property described in the indicator functions comprising A_y are precisely those handled by **P2**). □

Proof of Claim III. Recalling the event $\mathcal{E}_b(y')$ from (4.5.45) and the definition of $k^y_{J,b}$ from the paragraph preceding (4.5.43), we perform the following decomposition

$$\begin{aligned}
\mathbb{P}\Big[k^y_{J,b+1} < \infty, J \subset [1,\mathcal{N}^u_z]\Big] &\overset{\substack{Lem.4.5.3(c),\\(4.5.45)}}{=} \sum_{y' \in \hat{\mathbb{L}}_{0,y}} \mathbb{P}\Big[\mathcal{E}_b(y') \cap \{J \subset [1,\mathcal{N}^u_z]\} \cap \{C_{y',L_0} \not\subset \mathcal{V}\}\Big] \\
&\overset{(4.5.7),\ \mathbf{P3}}{\le} \sum_{y' \in \hat{\mathbb{L}}_{0,y}} \mathbb{E}\Big[\mathbb{P}[C_{y',L_0} \not\subset \mathcal{V} | \mathcal{F}_{J,y'}] \mathbb{1}_{\mathcal{E}_b(y') \cap \{J \subset [1,\mathcal{N}^u_z]\}}\Big] \\
&\overset{(4.5.6),\ (4.5.45)}{\le} (1-c') \sum_{y' \in \hat{\mathbb{L}}_{0,y}} \mathbb{P}\Big[\mathcal{E}_b(y') \cap \{J \subset [1,\mathcal{N}^u_z]\}\Big] \\
&\le (1-c') \mathbb{P}\Big[k^y_{J,b} < \infty, J \subset [1,\mathcal{N}^u_z]\Big]
\end{aligned} \tag{4.5.68}$$

where $c' = c'(\delta, L_0) \in (0,1)$. In the third step of display (4.5.68), in order to apply the result from (4.5.6) we relied on the fact

$$\{ y_{N(k^y_{J,b})} = y' \} \subset O_{y'}(\mathcal{V}^u_z) \cap \widetilde{\mathrm{LU}}_{y'}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})) \tag{4.5.69}$$

as well as $\{x_1, x_1', x_2, x_2'\} \cap D_{y_{N(k^y_{J,b})}, L_0} = \emptyset$, both of which arise as a consequence of (4.5.40). Iterating the inequality from display (4.5.68) yields

$$\mathbb{P}\Big[k^y_{J,\lceil cam\rceil}(x;\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}) < \infty, J \subset [1,\mathcal{N}^u_z]\Big] \leq (1-c')^{cam}. \tag{4.5.70}$$

Together with Claim II this implies

$$\begin{aligned}&\mathbb{P}\Big[A_y, \tilde{V}^c_{z,x}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})), [\mathcal{N}_z^{\frac{i}{2}u_0}+1, \mathcal{N}_z^{(\frac{i}{2}+1)u_0}] \subset J \subset [1,\mathcal{N}^u_z]\Big]\\ &\leq \mathbb{P}\Big[k^y_{J,\lceil cam\rceil}(x;\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}) < \infty, J \subset [1,\mathcal{N}^u_z]\Big] \leq (1-c')^{cam}\end{aligned} \tag{4.5.71}$$

which is precisely Claim III. □

Proof of Claim IV. Recalling the definition of $V_z(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$ from (4.4.15), observe that the presence of the event $\{C_z \xleftarrow{N_{2\delta}(\mathcal{V}^{u_3})} \partial D_z\}$ on the left-hand side of (4.5.51) only leaves us with showing that any two clusters of $\mathcal{V}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$ crossing $\tilde{D}_z \setminus \tilde{C}_z$ are connected inside $D_z \cap \mathcal{V}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$. Indeed, the existence of a crossing cluster of $D_z \setminus C_z$ is evident since $N_{2\delta}(\mathcal{V}^{u_3}) \subset \mathcal{V}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$. To show this claim, the overarching principle is to construct a vacant cluster which surrounds $\tilde{C}_z$ and cannot be avoided by any path of the vacant set $\mathcal{V}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$ crossing $\tilde{D}_z \setminus \tilde{C}_z$ - by this point, the idea should be reminiscent of Lemma 4.1.1. To this end, since each $\mathcal{O}_j$ is a $*$-connected subset of $\mathscr{S}$ and satisfies property *(i)* of Lemma 4.1.1 with $\mathbb{L}$ as the underlying lattice, it follows from the definition of the event $V_{\mathrm{y}}(\mathcal{V},\delta,u,u_3,u_3)$ (see (4.4.15)) and also the definition of $\mathscr{C}_{\mathrm{y}}(\mathcal{V}_{\mathbb{L}},\delta,u,u_3)$ (see below (4.4.14)) that

$$\begin{aligned}&\text{the set } \cup_{1\leq j\leq l}\cup_{\mathrm{y}\in\mathcal{O}_j}\mathscr{C}_{\mathrm{y}} = \cup_{1\leq j\leq l}\cup_{\mathrm{y}\in\mathcal{O}_j}\mathscr{C}_{\mathrm{y}}(\mathcal{V}_{\mathbb{L}},\delta,u,u_3) \text{ is}\\ &\text{connected in } N_\delta(\mathcal{V}^u) \subset \mathcal{V}^u \text{ on the event } \{C_z \xleftarrow{N_{2\delta}(\mathcal{V}^{u_3})} \partial D_z\}.\end{aligned} \tag{4.5.72}$$

This is seen by first gluing the $\mathscr{C}_{\mathrm{y}}$ for each $y \in \mathcal{O}_i$ for a fixed $1 \leq i \leq l$ (which is possible because $\mathcal{O}_i$ is a $*$-component), followed by the fact that a cluster of $N_{2\delta}(\mathcal{V}^{u_3})$ crossing $D_z \setminus C_z$ must cross $\tilde{D}_{\mathrm{y}_i,L} \setminus \tilde{C}_{\mathrm{y}_i,L}$ for some $\mathrm{y}_i \in \mathcal{O}_i$ for each $1 \leq i \leq l$ (so must in turn be connected to $\mathscr{C}_{\mathrm{y}_i}$). Crucially, observe that on the event $\mathscr{G}_z$ (recall (4.4.26) for its definition) one has

$$\begin{aligned}&k(\tilde{D}_z \setminus \tilde{C}_z, \mathscr{S}) > 0 \text{ (see above (4.5.16)) and hence the}\\ &\text{set } \cup_{1\leq j\leq l}\cup_{\mathrm{y}\in\mathcal{O}_j}\mathscr{C}_{\mathrm{y}} \text{ is non-empty on the event } \mathscr{G}_z.\end{aligned} \tag{4.5.73}$$

Then, if there is a cluster of $\mathcal{V}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$ crossing $\tilde{D}_z \setminus \tilde{C}_z$, we have $x \xleftrightarrow{\mathcal{V}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))} \partial\tilde{D}_z$ for some $x \in \partial\tilde{C}_z$. Hence, by the definition of $\tilde{V}_z(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$, we deduce that $x \xleftrightarrow{\mathcal{V}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))} \mathscr{C}_{\mathrm{y}}$ for some $\mathrm{y} \in \cup_{1\le j\le l}\mathcal{O}_j$ (which is nonempty by (4.5.73)). By (4.5.72), any two clusters of $\mathcal{V}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))$ crossing $\tilde{D}_z \setminus \tilde{C}_z$ must therefore be connected in $\cup_{1\le j\le l}\cup_{\mathrm{y}\in\mathcal{O}_j}\mathscr{C}_{\mathrm{y}}$, which immediately yields the required result. □

All in all, we have shown all the necessary steps for (4.5.33), and therefore the proof of the Proposition is complete, conditional on Lemma 4.5.1. □

Remark 4.5.5. *Having described the sophisticated algorithm used in the proof of* (4.5.33)*, we now describe the (greatly streamlined) details required to show* (4.5.16)*. In the following, we abbreviate* $Z_J \coloneqq Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})$ *and recall* $\mathcal{I}$, $\mathcal{V}$ *from the statement of Lemma 4.5.1. Once again, we shall present an algorithm* $\bar{\mathcal{A}}_J(x;\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})$ *tied to the exploration of the cluster* $\mathscr{C}_J(x;\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})$*. Recalling the definition of* $\mathcal{S}(Z_J)$ *from* (4.4.5)*, we construct* $*$*-connected components* $\mathcal{O}_j$, $1 \le j \le l$ *of* $\mathcal{S}(Z_J)$ *by applying Proposition 4.1.1 on the underlying lattice* $\mathbb{L}_{-1}$ *with* $U = \mathbb{L}_{-1}(C_z)$, $V = \mathbb{L}_{-1}(D_z)$ *and* $\Sigma = \mathcal{S}(Z_J)$ *therein. As discussed previously,* l *could be equal to* 0 *if* $k(\mathbb{L}_{-1}(D_z)\setminus\mathbb{L}_{-1}(C_z), \mathcal{S}(Z_J)) = 0$*. Then, we consider the exploration process* $(w_n)_{n\ge1}$ *as before, and associate to it a sequence of times* τ_k, $k \ge 0$ *defined iteratively by setting* $\tau_0 = 0$ *and*

$$\tau_k = \inf\{n > \tau_{k-1} : w_n \in \mathcal{V} \text{ and } (\star)' \text{ holds.}\}, \quad w_{\tau_k} = \tilde{x}_k, \quad k \ge 1 \tag{4.5.74}$$

where, this time, $(\star)'$ *refers to the property that*

$$\begin{aligned} &\text{there exists a } \mathrm{x}_k \in \bigcup_{1\le j\le l}\mathcal{O}_j \text{ such that } \tilde{x}_k \in \partial^{out}C_{\mathrm{x}_k,L_{-1}}, \text{ as well as} \\ &|\mathrm{x}_k - \tilde{x}_j| \ge 100L_{-1} \text{ for any } 1 \le j < k, \text{ and } U_{\mathrm{x}_k,L_{-1}} \cap \{x_1, x_1', x_2, x_2'\} = \emptyset. \end{aligned} \tag{4.5.75}$$

we define the sequence of discovered points $\{\mathrm{x}_k\}_{k\ge1}$ *by letting* x_k *be the smallest point in* $\bigcup_{1\le j\le l}\mathcal{O}_j$ *(according to some predetermined ordering of* $\mathbb{Z}^d$*) such that* $\tilde{x}_k \in \partial^{out}C_{\mathrm{x},L_{-1}}$*. Then, the algorithm* $\bar{A}_J(x;\boldsymbol{j},\boldsymbol{x},\boldsymbol{n})$ *once more consists of generic steps (which remain unaltered) and special steps which occur at the times* τ_k, $k \ge 1$*. During this version of a special step, we fully reveal* $C_{\mathrm{x}_k,L_{-1}}$*, and terminate the algorithm if* $\tilde{x}_k \xleftrightarrow{\mathcal{V}(Z_J)} \square(\mathrm{x}_k, L_{-1})$*. Thus, we define the 'bad' points of the algorithm as those points* x_l *which are discovered that satisfy*

$$\{\tilde{x}_l \overset{\mathcal{V}(Z_J)}{\longleftarrow\!\!\!\!\!\!\not\longrightarrow} \square(\mathrm{x}_l, L_{-1})\},$$

and, for any $b \in \mathbb{N}^*$, we denote the 'index' of the b^{th} bad point as $k_{J,b} = k_{J,b}(x; \boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$. Concretely,

$$\begin{aligned} &k_{J,0} = 0, \\ &k_{J,b} = \inf\{l > k_{J,b-1} : \mathrm{x}_l \text{ is bad}\}, \end{aligned} \tag{4.5.76}$$

and by convention we write $\inf(\emptyset) = \infty$. In this instance, the 'next' label is just $N(l) = l + 1$, so we define

$$\mathcal{E}_b(\tilde{x}', \mathrm{x}') \coloneqq \{k_{J,b} < \infty, (\tilde{x}_{b+1}, \mathrm{x}_{b+1}) = (\tilde{x}', \mathrm{x}')\} \tag{4.5.77}$$

which is measurable with respect to $\bar{\mathcal{F}}$ from (4.5.9)*. All in all, we can show by analogous steps as in the proof of Claim II:*

Claim II' *Under the same set-up as in Claim I, we have*

$$\tilde{V}_{z,x}^c \cap \{k(\mathbb{L}_{-1}(D_z) \setminus \mathbb{L}_{-1}(C_z), \mathcal{S}(Z_J)) \geq b\} \subset \{k_{J,b} < \infty\} \tag{4.5.78}$$

where

$$\tilde{V}_{z,x} \coloneqq \{x \overset{\mathcal{V}}{\longleftrightarrow} \partial \tilde{D}_z, x \overset{\mathcal{V}}{\longleftarrow\!\!\!\!\!\!\!\!/\;\;} \square(\mathrm{x}, L_{-1}) \text{ for any } \mathrm{x} \in \bigcup_{1 \leq j \leq l} \mathcal{O}_j\} \tag{4.5.79}$$

Then, by retracing the steps in the proof of Claim III, and replacing the application of (4.5.6) *with* (4.5.8) *instead, noting that all the measurability conditions are still satisfied:*

Claim III' *Under the same set-up as in Claim I, one has the bound*

$$\mathbb{P}[k_{J,b} < \infty] \leq e^{-cb} \tag{4.5.80}$$

and finally, by remarking that $\{\mathbb{L}_{-1}(C_z) \overset{\mathcal{S}(Z_J)}{\longleftrightarrow} \mathbb{L}_{-1}(D_z)\} \subset \{C_z \overset{\mathcal{V}}{\longleftrightarrow} D_z\}$ (crucially relying on the definition of $\mathcal{S}(Z_J)$ from the statement of Proposition 4.4.5), we have

Claim IV' *With the same assumptions on the set J and the truncation parameters as in Claim I, we obtain*

$$\begin{aligned} &\tilde{V}_z \cap \{\mathbb{L}_{-1}(C_z) \overset{\mathcal{S}(Z_J)}{\longleftrightarrow} \mathbb{L}_{-1}(D_z)\} \cap \{k(\mathbb{L}_{-1}(D_z) \setminus \mathbb{L}_{-1}(C_z), \mathcal{S}(Z_J)) \geq b\} \\ &\subset V_z(Z_J) \end{aligned} \tag{4.5.81}$$

where $\tilde{V}_z = \bigcap_{x \in \partial \tilde{C}_z} \tilde{V}_{z,x}$.

Claim II', III' and IV' combine together to yield the required result.

4.5.2 Conditional Finite Energy Bounds

Having already witnessed its potency throughout the course of the previous subsection, we now arrive at the proof of Lemma 4.5.1.

Proof of Lemma 4.5.1. In order to prove (4.5.8), we apply some small modifications to the approach presented in Lemmas 5.10, 5.11 and 5.13 from [18]. In the flavour of this reference, we introduce two important σ-algebras which will represent the information yielded by excursions inside and outside $C_{\mathrm{x},L_{-1}}$ respectively. First, recalling the notation introduced in §2.4, define for $K \subset\subset \mathbb{Z}^d$

$$\omega_K \coloneqq \sum_{i \geq 0} \delta_{w_i^*, u_i} \mathbb{1}\{w_i^* \in W_K^*\} \text{ and } \omega - \omega_K \coloneqq \sum_{i \geq 0} \delta_{w_i^*, u_i} \mathbb{1}\{w_i^* \notin W_K^*\} \tag{4.5.82}$$

Then, we define $X_{i,j}^{\mathrm{in}}$ and $X_{i,j}^{\mathrm{out}}$ as in Section 4.2 of [18] except for any $i \geq 1$, and correspondingly let $\mathcal{F}_K^{\mathrm{in}}$, $\mathcal{F}_K^{\mathrm{out}}$ and $\mathcal{F}_K^{AB}$ be as the σ-algebras $\mathcal{F}_{K,u}^{\mathrm{in}}$, $\mathcal{F}_{K,u}^{\mathrm{out}}$ and $\mathcal{F}_{K,u}^{AB}$ as in Section 4.3 therein, except replacing $\omega_{K,u}$ with ω_K, and using the entire set of excursions $X_{i,j}$, $i \geq 0$ (i.e. excursions for any level $u > 0$). The reason for such an alteration is so that we can account for *any* fixed index-set of excursions $J \subset \mathbb{N}^*$, which a priori do not have a control on their level (although this comes into play as a probability bound via the occupation time condition present in $G_{\mathrm{x},L_{-1}}$, see (4.4.20)). Remark that, by a simple extension of [18, Lemma 4.4], $\mathcal{F}_K^{\mathrm{in}}$ and $\mathcal{F}_K^{\mathrm{out}}$ are conditionally independent on $\mathcal{F}_K^{AB}$. Finally, we let $X_J^{\mathrm{in}} = (X_j^{\mathrm{in}})_{j \in J_{\mathrm{x}}}$, $X_J^{AB} = ((X_j^s, X_j^e))_{j \in J_{\mathrm{x}}}$ denote the set of excursions of $C_{\mathrm{x},L_{-1}}$ induced by $Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$ and their endpoints respectively, and define $\mathcal{T}_{\mathrm{x},J}^{\mathrm{in}} = \mathcal{T}_{\mathrm{x},J}^{\mathrm{in}}(X_J^{AB})$ as the set of all vectors $v^{\mathrm{in}} = (v_j^{\mathrm{in}})_{j \in J_{\mathrm{x}}}$ of finite nearest-neighbour trajectories from X_j^s to X_j^e such that

i) $v_j^{\mathrm{in}} \cap \square(\mathrm{x}, L_{-1}) = \emptyset$,

ii) the total number of visits to $\partial C_{\mathrm{x},L_{-1}}$ by all the v_j^{in} is at most L_{-1}^{d-1}, and

iii) $\{X_J^{\mathrm{in}} \in \mathcal{T}_{\mathrm{x},J}^{\mathrm{in}}\} \subset G_{\mathrm{x},L_{-1}}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}))$.

Crucially, the set $\mathcal{T}_{\mathrm{x},J}^{\mathrm{in}}$ is measurable with respect to $\mathcal{F}_{C_{\mathrm{x},L_{-1}}}^{\mathrm{out}}$. Next, we observe

$$\bar{\mathcal{F}} = \sigma(\mathcal{S}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}), \mathcal{I}|_{C_{\mathrm{x},L_{-1}}^c}) \subset \sigma(\mathcal{S}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})), \mathcal{F}_{C_{\mathrm{x},L_{-1}}}^{\mathrm{out}}) =: \hat{\mathcal{F}} \tag{4.5.83}$$

such that our original problem can instead be reduced to showing, analogously to Lemma 5.10 in [18], that there exists $c = c(d, L_{-1}) > 0$ such that for any $\mathrm{x} \in \mathbb{L}_{-1} \cap (\tilde{D}_z \setminus \tilde{C}_z)$ satisfying $\{x_1, x_1', x_2, x_2'\} \cap D_{\mathrm{x},L_{-1}} = \emptyset$, $x \in \partial^{\mathrm{out}} C_{\mathrm{x},L_{-1}}$ and $\mathbb{P}$-almost surely for every realisation $\omega_{\mathcal{S}}$, $\omega_{\mathcal{F}}$ of $\mathcal{S}(Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n}))$ and $\mathcal{F}_{C_{\mathrm{x},L_{-1}}}$, there exists

$$\rho^{\mathrm{in}} = \rho^{\mathrm{in}}(\omega_{\mathcal{S}}, \omega_{\mathcal{F}}) \in \mathcal{T}^{\mathrm{in}}_{\mathrm{x},J} \tag{4.5.84}$$

such that

$$\mathbb{P}[X^{\mathrm{in}}_J = \rho^{\mathrm{in}} | \hat{\mathcal{F}}] \geq c \mathbb{1}_{\{x \in \mathcal{V}, G_{\mathrm{x},L_{-1}}(Z_J(\boldsymbol{j},\boldsymbol{x},\boldsymbol{n}))\}} \tag{4.5.85}$$

and $\{x \in \mathcal{V}\} \cap \{X^{\mathrm{in}}_J = \rho^{\mathrm{in}}\} \subset \{x \overset{\mathcal{V}}{\leftrightarrow} \square(x, L_{-1})\}$. Given this reformulation, the remaining steps follow similarly to the reference.

Now, we proceed to the proof of (4.5.6). We begin with a brief comment to provide some intuition for what follows. The main challenge we aim to address is the rigidity imposed by the sigma-algebra $\mathcal{F}_{J,y}$ on the behavior of random walk trajectories within Dy, L_0. A quick inspection of (4.5.7) reveals that $\mathcal{F}_{J,y}$ encodes information derived from RI paths both inside D_{y,L_0} and in its complement. We address the restrictions imposed by each in slightly different ways. For the latter, we introduce an alternative σ-algebra (see (4.5.87)) that 'summarizes' the information outside D_{y,L_0}. While this new σ-algebra is somewhat more restrictive, the Markov property of excursions ensures that its influence is confined to the local times of excursions on the boundary $\partial D_{y,L_0}$ - see (4.5.89). Conversely, to accommodate the constraints imposed by the information inside D_{y,L_0}, a rerouting argument becomes necessary. To make these observations rigorous, we proceed with the necessary notation.

For any nearest-neighbour transient trajectory $w = (w_n)_{n\geq 0} = ((w(n))_{n\geq 0}$ in $\mathbb{Z}^d$ and a pair of sets $\emptyset \neq D \subset U \subset\subset \mathbb{Z}^d$, let $0 \leq R_1(w) < T_1(w) < R_2(w) < \dots$ denote the successive return and departure times of w between D and $\partial^{\mathrm{out}} U$. Observe that these simply correspond to the discrete analogue of $R_k^{D,U}$, $T_{k-1}^{D,U}$ defined in (3.4.11) for continuous time-indexed trajectories. Let $w^-_{D,U}$ denote the sequence of segments $(w[0, R_1 - 1], w[T_1, R_2 - 1], \dots)$ where $w[n_1, n_2](n) = w(n_1 + n)$ for $n \in [0, n_2 - n_1] \cap \mathbb{N}$ and $w[0, R_1 - 1]$ is an 'empty sequence' when $R_1 = 0$. In words, $w^-_{D,U}$ is the sequence of segments of w obtained after removing all its excursions between D and $\partial^{\mathrm{out}} U$ minus their final points, i.e. the paths $w[R_k, T_k - 1]$ for $k \geq 1$. If $D = U = \mathring{D}_{y,L_0}$, we

abbreviate $w^-_{D,U}$ as w^-_y. With this, we consider the point process

$$\omega^-_y = \sum_{(w,v)\in\mu^+_{D_z}(\omega)} \delta_{(w^-_y,v)} \tag{4.5.86}$$

(see (2.4.11) for the definition of $\mu^+_{D_z}$). From now on, we enforce the assumption that $y \in \mathbb{L}_0 \cap \tilde{D}_z$ and $\{x_1, x_1', x_2, x_2'\}$ as introduced in the statement of Lemma 4.5.1. Given this condition, we claim that for any $u' \geq u \geq 0$, $\delta \in [0, \frac{1}{2})$,

$$\begin{aligned}&\Big(N_\delta(\mathcal{V}^u)_{|\mathring{D}^c_{y,L_0}}, N_{2\delta}(\mathcal{V}^{u'})_{|\mathring{D}^c_{y,L_0}}, \mathcal{N}^u_z, \mathcal{I}_{|\mathring{D}^c_{y,L_0}}, \{\ell^u_x : x \in \mathring{D}^c_{y,L_0}\}\Big)\\ &\text{are measurable relative to } \Big(\omega^-_y, \{\mathrm{U}_x : x \in \mathring{D}^c_{y,L_0}\}\Big)\end{aligned} \tag{4.5.87}$$

At this stage, we define $\mathcal{F}_y := \sigma(\omega^-_y, \{\mathrm{U}_x : x \in \mathring{D}^c_{y,L_0}\}$ and observe that, by definition of $\mathcal{F}$ from (4.5.7) together with the assertion above, we have

$$\mathcal{F} = \sigma(\mathcal{F}_y, N_\delta(\mathcal{V}^u) \cap \mathring{D}_{y,L_0}, N_{2\delta}(\mathcal{V}^{u'}) \cap \mathring{D}_{y,L_0}, \mathscr{C}_{\partial D_{y,L_0}}). \tag{4.5.88}$$

Thus, to show the claimed measurability in (4.5.87), we handle each of the terms on the left-hand side separately. The occupation times ℓ^u_x are immediately seen to be measurable with respect to ω^-_y whenever $x \in \mathring{D}_{y,L_0}$ since the excursions which are removed are confined to $\mathring{D}_{y,L_0}$. Furthermore, since $y \in \mathbb{L}_0 \cap \tilde{D}_z$ and $K \geq 100$, $N \geq 1000L_0$ (recall (4.3.34)), we also see that $\mathring{D}_{y,L_0} \cap \mathring{D}^c_z = \emptyset$. As a result, ω^-_y retains information about all the entrance and exit times of D_z by RI trajectories up to any level $v \geq 0$, such that $\mathcal{N}^u_z$ is measurable relative to ω^-_y. By similar reasoning, $N_{\delta'}(\mathcal{V}^v)_{|\mathring{D}^c_{y,L_0}}$ is measurable with respect to $(\omega^-_y, \{\mathrm{U}_x : x \in \mathring{D}^c_{y,L_0}\})$ for any level $v \geq 0$ and thinning parameter $\delta' \in (0,1)$. Finally, in order to deduce the measurability of $\mathcal{I}_{|\mathring{D}^c_{y,L_0}}$, we remark that $\{x_1, x_1', x_2, x_2'\} \cap D_{y,L_0} = \emptyset$ implies that the consecutive hitting times of (x_i, x_i') for $i \in \{1,2\}$ (see (4.5.1)) are measurable with respect to ω^-_y. In particular, so is the range (restricted to $\mathring{D}^c_{y,L_0}$) of the truncated trajectories comprising $Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$, such that (see above (4.5.5)) $\mathcal{I}_{|\mathring{D}^c_{y,L_0}}$ is indeed measurable relative to ω^-_y.

While on the theme of measurability, we take a moment to introduce two more constructions which are also measurable with respect to ω^-_y. The first of these is $S_v(\omega^-_y)$ for any $v \geq 0$, which denotes the multiset of all pairs of points $\Big(w(R_k(w) - 1), w(T_k(w))\Big) \in (\partial D_{y,L_0})^2$; $k \geq 1$ (provided $R_1(w) < \infty$) such that $(w, v') \in \mu^+_{D_z}(w)$ for some $v' \leq v$. The second (similar) construction is $S_J(\omega^-_y)$ for some finite set $J \subset \mathbb{N}^*$, which is again a multiset of elements in $\partial D^2_{y,L_0}$, but this time including

$\Big(Z_j(R_k(Z_j)-1, Z_j(T_k(Z_j))\Big)$ for all $k \geq 1$ with $Z_j \in Z_J(\boldsymbol{j}, \boldsymbol{x}, \boldsymbol{n})$. Notice that, for J $\subset [1, \mathcal{N}_z^u]$, we have $S_J(\omega_y^-) \subset S_u(\omega_y^-)$. Keeping these definitions in mind, we present a precise description of the law of excursions of trajectories underlying $\mu^+_{D_z}(\omega)$ as well as the variables $\{\mathrm{U}_x : x \in \mathring{D}_{y,L_0}\}$ *conditionally* on $\mathcal{F}_y$, which arises as a consequence of the Markov property for random walks. In particular,

$$\begin{aligned}&\text{the excursions } \{\boldsymbol{\gamma}_{x,x'} : (x,x') \in S_{u'(\omega_y^-)}\} \text{ (accounting for multiplicities)}\\ &\text{under the (regular) conditional law } \mathbb{P}[\cdot|\mathcal{F}_y] \text{ are distributed as independent}\\ &\text{random walk bridges where } \boldsymbol{\gamma}_{x,x'} \text{ is conditioned to start at } x, \text{ end at } x' \text{ and}\\ &\text{lie inside } \mathring{D}_{y,L_0} \text{ except at the final point independently of the i.i.d uniform}\\ &\text{random variables } \{\mathrm{U}_x : x \in \mathring{D}_{y,L_0}\}.\end{aligned} \tag{4.5.89}$$

Up until now, the discussion has concerned itself with discovering the 'correct' sigma algebra to describe $\mathcal{F}_{J,y}$ in the complement of $\mathring{D}_{y,L}$, thereby making an approach via the Markov property tenable. Let us now proceed to estimating the conditional probability in (4.5.6), given the groundwork laid out in (4.5.88) and (4.5.89). In light of the alternative representation for $\mathcal{F}$ derived in (4.5.88), given any realisation ζ_y of $(\omega_y^-, \{\mathrm{U}_x : x \in \mathring{D}^c_{y,L_0}\})$ as well as any (finite) combination of possible realisations η_1, η_1' and η_2 of $N_\delta(\mathcal{V}^u) \cap \mathring{D}_{y,L_0}$, $N_{2\delta}(\mathcal{V}^u) \cap \mathring{D}_{y,L_0}$ and $\mathscr{C}_{\partial D_{y,L_0}}$ respectively, we can write a regular conditional law $\mathbb{P}[\cdot|\mathcal{F}](\zeta_y, \eta_1, \eta_1', \eta_2)$ as follows:

$$\mathbb{P}[\cdot|\mathcal{F}](\zeta_y, \eta_1, \eta_1', \eta_2) = \mathbb{Q}_{\zeta_y}\Big[\,\cdot\,|\mathrm{N}(\eta_1, \eta_1') \cap \mathrm{C}(\eta_2)\Big] \tag{4.5.90}$$

where $\frac{0}{0}$ is interpreted as 0, $\mathbb{Q}_{\zeta_y}$ is the conditional law $\mathbb{P}[\cdot|\mathcal{F}_y]$ described in (4.5.89) evaluated at ζ_y,

$$\begin{aligned}&\mathrm{N}(\eta_1, \eta_1') \coloneqq \{N_\delta(\mathcal{V}^u) \cap \mathring{D}_{y,L_0} = \eta_1, N_{2\delta}(\mathcal{V}^{u'}) \cap \mathring{D}_{y,L_0} = \eta_1'\}, \text{ and}\\ &\mathrm{C}(\eta_2) \coloneqq \{\mathscr{C}_{\partial D_{y,L_0}} = \eta_2\}\end{aligned}$$

Thus, in the remainder of the proof, we endeavour to show

$$\mathbb{Q}_{\zeta_y}\Big[C_{y,L_0} \subset \mathcal{V}|\mathrm{N}(\eta_1, \eta_1') \cap \mathrm{C}(\eta_2)\Big] \geq c(\delta, L_0) > 0 \tag{4.5.91}$$

for each $(\eta_1, \eta_1', \eta_2)$, almost every ζ_y such that $\mathbb{Q}_{\zeta_y}[\mathrm{N}(\eta_1, \eta_1') \cap \mathrm{C}(\eta_2)] > 0$ and the parameters $(\zeta_y, \eta_1, \eta_1', \eta_2)$ satisfy the event $\widetilde{\mathrm{LU}}_y \cap O_y(\mathcal{V}_z^u) \cap \{J \subset [1, \mathcal{N}_z^u]\}$ (see (4.5.6)). Since ζ_y satisfies $\{J \subset [1, \mathcal{N}_z^u]\}$ and $u \leq u'$, one has that the event $\mathrm{N}(\eta_1, \eta_1') \cap \mathrm{C}(\eta_2)$ is measurable with respect to the excursions $\{\boldsymbol{\gamma}_{x,x'} : (x,x') \in S_{u'}(\omega_y^-)\}$ and the

variables $\{\mathrm{U}_x : x \in \mathring{D}_{y,L_0}\}$ *conditional* on ζ_y. Thus, by definition of conditional probability and measurability, we have that for almost all realisations of ζ_y such that $\mathbb{Q}_{\zeta_y}[\mathrm{N}(\eta_1, \eta_1') \cap \mathrm{C}(\eta_2)] > 0$, there exist choices of excursions $\{\gamma_{x,x'} : (x,x') \in S_{u'}(\omega_y^-)\}$ and binary values $\{b_x : x \in \mathring{D}_{y,L_0}\} \in \{0,1\}^{\mathring{D}_{y,L_0}}$, $\{b_x' : x \in \mathring{D}_{y,L_0}\} \in \{0,1\}^{\mathring{D}_{y,L_0}}$ depending only on η_1, η_1', η_2 and ζ_y such that

$$\bigcap_{(x,x')\in S_{u'}(\omega_y^-)} \{\boldsymbol{\gamma}_{x,x'} = \gamma_{x,x'}\} \cap \bigcap_{x\in\mathring{D}_{y,L_0}} \{\mathbb{1}_{\{\mathrm{U}_x \geq \delta\}} = b_x\} \cap \bigcap_{x\in\mathring{D}_{y,L_0}} \{\mathbb{1}_{\{\mathrm{U}_x \geq 2\delta\}} = b_x'\} \subset \mathrm{N}(\eta_1, \eta_1') \cap \mathrm{C}(\eta_2) \tag{4.5.92}$$

given ζ_y, and $|\gamma_{x,x'}| \leq (10L_0)^d$ for each pair of endpoints in the multiset $(x,x') \in S_u(\omega_y^-)$. Now, we aim to find collections of excursions (with probability of their occurrence bounded away from 0, conditional on ζ_y) such that the right-hand side is still preserved, meanwhile additionally satisfying $C_{y,L_0} \subset \mathcal{V}$. To this end, we give two possible classifications for the excursions of $\mathring{D}_{y,L_0}$. An excursion $\gamma_{x,x'}$ is said to be

$$\textit{type-I} \text{ if it intersects } C_{y,L_0}, \text{ and } \textit{type-II} \text{ if it does not intersect } C_{y,L_0}. \tag{4.5.93}$$

Recalling the definition of $\mathcal{V}$ from below (4.5.6), it becomes evident that we need only modify those excursions with endpoints in $S_J(\omega_y^-)$ (which is fixed, since we are conditioning on ζ_y) and which are type-I. Indeed, if none of the excursions in $\{\gamma_{x,x'} : (x,x') \in S_J(\omega_y^-\} \subset \{\gamma_{x,x'} : (x,x') \in S_{u'}(\omega_y^-\}$ from the choice in (4.5.92) are type-I, we immediately have

$$\bigcap_{(x,x')\in S_{u'}(\omega_y^-)} \{\boldsymbol{\gamma}_{x,x'} = \gamma_{x,x'}\} \subset \{C_{y,L_0} \subset \mathcal{V}\} \tag{4.5.94}$$

given ζ_y. Otherwise, if at least one of the $\gamma_{x,x'}$ with $(x,x') \in S_J(\omega_y^-)$ is type-I, we proceed via a 'rerouting' argument which preserves both the inclusion (4.5.92) (up to the choice of these type-I excursions) and the configuration of ζ_y. To proceed, we first observe that η_2 is a disjoint union of clusters of $\mathcal{I} \cap (D_{y,L_0} \setminus C_{y,L_0})$, each of which intersects $\partial D_{y,L_0}$. By assumption, $(\zeta_y, \eta_1, \eta_1', \eta_2)$ satisfies $\widetilde{\mathrm{LU}}_y$, so there exists a unique component, call it $\mathcal{C}(\eta_2)$, of $\eta_2 \setminus \partial D_{y,L_0}$ intersecting $\tilde{D}_{y,L_0}$, and thus containing $\eta_2 \cap (\tilde{D}_{y,L_0} \setminus \tilde{C}_{y,L_0})$. Building on this observation, we can infer the following:

1) Given a type-I excursion $\gamma_{x,x'}$ with $(x,x') \in S_J(\omega_y^-)$, there exists a type-II excursion $\bar{\gamma}_{x,x'}$ sharing its endpoints, and satisfying $|\bar{\gamma}_{x,x'}| \leq (10L_0)^d$ and $\mathcal{C}(\eta_2) = \mathrm{range}(\bar{\gamma}_{x,x'}) \cap \mathring{D}_{y,L_0}$.

2) For any other type-I excursion $\gamma_{\hat{x},\hat{x}'}$ with $(\hat{x},\hat{x}') \in S_J(\omega_y^-)$, there exists a type-II excursion $\bar{\gamma}_{\hat{x},\hat{x}'}$ with the same endpoints as $\gamma_{\hat{x},\hat{x}'}$ which satisfies $\mathrm{range}(\bar{\gamma}_{\hat{x},\hat{x}'}) \cap \mathring{D}_{y,L_0} \subset \mathcal{C}(\eta_2)$ as well as $|\bar{\gamma}_{\hat{x},\hat{x}'}| \leq (10L_0)^d$.

Moreover, any component of $\eta \setminus \partial D_{y,L_0}$ other than $\mathcal{C}(\eta)$ must not intersect $\tilde{D}_{y,L_0}$. Denoting such a component by $\mathcal{C}$, it therefore follows that

$$\mathcal{C} \subset (\mathring{D}_{y,L_0} \setminus \tilde{D}_{y,L_0}) \cap \bigcup_{\substack{(x,x')\in S_J(\omega_y^-):\\ \gamma_{x,x'} \text{ is type-II}}} \mathrm{range}(\gamma_{x,x'}) \tag{4.5.95}$$

for otherwise, a type-I excursion would necessarily intersect $\tilde{D}_{y,L_0}$, and therefore so would the cluster $\mathcal{C}$, implying a contradiction. Altogether, with the help of the thinned vacant sets introduced in (4.2.28), these observations imply

$$\begin{aligned} &\bigcap_{\substack{(x,x')\in S_J(\omega_y^-):\\ \gamma_{x,x'} \text{ is type-I}}} \{\boldsymbol{\gamma}_{x,x'} = \bar{\gamma}_{x,x'}\} \cap \bigcap_{\substack{(x,x')\in S_J(\omega_y^-):\\ \gamma_{x,x'} \text{ is type-II}}} \{\boldsymbol{\gamma}_{x,x'} = \gamma_{x,x'}\} \cap \\ \bigcap_{\substack{(x,x')\in\\ S_{u'}(\omega_y^-)\setminus S_J(\omega_y^-)}} &\{\boldsymbol{\gamma}_{x,x'} = \gamma_{x,x'}\} \cap \bigcap_{x\in \mathring{D}_{y,L_0}\setminus(\eta_1\cup\eta_2)} \{\mathrm{U}_x < \delta\} \cap \bigcap_{x\in\eta_1\setminus\eta_1'} \{\mathrm{U}_x \in [\delta,2\delta]\} \cap \\ &\bigcap_{x\in\eta_1'} \{\mathrm{U}_x \geq 2\delta\} \subset \mathrm{N}(\eta_1,\eta_1') \cap \mathrm{C}(\eta_2) \cap \{C_{y,L_0} \subset \mathcal{V}\} \end{aligned} \tag{4.5.96}$$

given ζ_y and that, for any type-I $\gamma_{x,x'}$ with $(x,x') \in S_J(\omega_y^-)$, the rerouted path satisfies

$$|\bar{\gamma}_{x,x'}| \leq CL_0^d \tag{4.5.97}$$

and finally, since ζ_y also satisfies the occupation time event $O_y(\mathcal{V}_z^u)$ (recall (4.4.19)), we have

$$|S_J(\omega_y^-)| \leq |S_u(\omega_y^-)| \leq CL_0^{d-1}(\log L_0)^2. \tag{4.5.98}$$

Together, (4.5.96), (4.5.97) and (4.5.98) will imply the sought-after estimate. In order to streamline the notation, for any $(x,x') \in S_J(\omega_y^-)$, we write $\tilde{\gamma}_{x,x'} = \bar{\gamma}_{x,x'}$ if $\gamma_{x,x'}$ is type-I and write $\tilde{\gamma}_{x,x'} = \gamma_{x,x'}$ otherwise. Also, we denote

$$\mathcal{U} := \bigcap_{x\in \mathring{D}_{y,L_0}\setminus(\eta_1\cup\eta_2)} \{\mathrm{U}_x < \delta\} \cap \bigcap_{x\in\eta_1\setminus\eta_1'} \{\mathrm{U}_x \in [\delta,2\delta]\} \cap \bigcap_{x\in\eta_1'} \{\mathrm{U}_x \geq 2\delta\}. \tag{4.5.99}$$

Finally, by summing over all the possibilities for $\{\boldsymbol{\gamma}_{x,x'} : (x,x') \in S_{u'}(\omega_y^-) \setminus S_J(\omega_y^-)\}$ satisfying (4.5.92), we deduce

$$\begin{aligned}\mathbb{Q}_{\zeta_y}\Big[C_{y,L_0} \subset \mathcal{V}|\mathrm{N}(\eta_1,\eta_1') \cap \mathrm{C}(\eta_2)\Big] &\overset{(4.5.96)}{\geq} \mathbb{Q}_{\zeta_y}\Big[\bigcap_{(x,x')\in S_J(\omega_y^-)}\{\boldsymbol{\gamma}_{x,x'} = \tilde{\gamma}_{x,x'}\} \cap \mathcal{U}\Big] \\ &\overset{(4.5.89)}{\geq} \mathbb{Q}_{\zeta_y}\Big[\bigcap_{(x,x')\in S_J(\omega_y^-)}\{\boldsymbol{\gamma}_{x,x'} = \tilde{\gamma}_{x,x'}\}\Big](\delta(1-2\delta))^{|\mathring{D}_{y,L_0}|} \quad (4.5.100) \\ &\overset{(4.5.89),(4.5.97),(4.5.98)}{\geq} \Big(\frac{1}{2d}\Big)^{CL_0^{2d}}(\delta(1-2\delta))^{CL_0^d}\end{aligned}$$

which yields (4.5.91) and therefore concludes the proof. □

4.5.3 Proofs of A Priori Estimates

Finally, we return to showing the estimates which set the course in motion for obtaining more refined bounds via our bootstrapping machinery. Let us start with the proof of Lemma 4.4.7, since the ideas used therein will provide some guidance with regards to the other a priori bounds, especially the one in Lemma 4.4.6. To be precise, the geometric renormalisation strategy described in §4.1.2 via Lemmas 4.1.3 and 4.1.5 will play a pivotal role in preserving a stretched exponential decay of estimates on local-uniqueness-type events when increasing the scale.

Proof of Lemma 4.4.7. Once again, as in the subcritical case presented in §4.4.1, we apply a streamlined version of the bootstrapping technique by setting $a = 0$ such that $\mathcal{G}_{z,L}(V,W,\mathscr{C};0) = V_z$. Then, for any $\delta > 0$ and $0 < u_1 \leq u_2 \leq u_3 < u_*$ we let the event $\tilde{V}_{z,L} = V_{z,L}(\tilde{\mathcal{V}}_{\mathbb{L}},\delta,u_1,u_2,u_3)$ denote

$$\{C_{z,L} \xleftrightarrow{N_\delta(\tilde{\mathcal{V}}_z^{u_3})} \partial D_{z,L}\} \cap \left\{\begin{array}{c}\text{any two clusters } \mathscr{C}_1,\mathscr{C}_2 \subset N_\delta(\tilde{\mathcal{V}}_z^{u_2}) \text{ crossing} \\ \tilde{D}_{z,L} \setminus \tilde{C}_{z,L} \text{ are connected in } N_\delta(\tilde{\mathcal{V}}_z^{u_1}) \cap D_{z,L}\end{array}\right\} \quad (4.5.101)$$

where $\tilde{\mathcal{V}} \in \{\mathcal{V},\bar{\mathcal{V}},\hat{\mathcal{V}}\}$ as usual. First, we find an a priori bound analogous to (4.4.7) for this scenario. To do this, we will make full use of Lemmas 4.1.3 and 4.1.5, and define the following events for any $x \in \mathbb{Z}^d$, $x' \in \mathbb{L}_0$ such that $x \in C_{x',L_0}$, $\delta \in (0,1)$ and $u,v > 0$

$$\mathcal{E}_x^1(u) = \{C_{x',L_0} \xleftrightarrow{\mathcal{V}^u} \partial D_{x',L_0}\} \quad (4.5.102)$$

$$\mathcal{E}_x^2(u,v) = \left\{\begin{array}{c}\text{any two clusters } \mathscr{C}_1,\mathscr{C}_2 \subset \mathcal{V}^v \text{ crossing} \\ \tilde{D}_{x',L_0} \setminus \tilde{C}_{x',L_0} \text{ are connected in } \mathcal{V}^u \cap D_{x',L_0}\end{array}\right\} \quad (4.5.103)$$

$$\mathcal{E}_x^3(\delta) = \bigcap_{y \in D_{x',L_0}} \{\mathrm{U}_y > \delta\} \tag{4.5.104}$$

and abbreviate $\mathcal{E}_x^1 = \mathcal{E}_x^1(u_3)$, $\mathcal{E}_x^2 = \mathcal{E}_x^2(u_1, u_2)$, $\mathcal{E}_x^3 = \mathcal{E}_x^3(\delta)$. Notice that on the event $\mathcal{E}_x^3$ we have $\mathcal{V}|_{D_{z,L_0}} = N_\delta(\mathcal{V})|_{D_{z,L_0}}$ for any vacant configuration $\mathcal{V}$, so that

$$\bigcap_{k=1}^{3} \mathcal{E}_x^k \subset V_{z,L_0}(\mathcal{V}_{\mathbb{L}_0}, \delta, u_1, u_2, u_3) \tag{4.5.105}$$

We aim to show that, for any $u_1 < u_2 < u_3 < u_*$, with $\boldsymbol{u} = (u_1, u_2, u_3)$, given a fixed λ_0 satisfying (4.1.15), $\exists L_0 = L_0(\boldsymbol{u})$, $c = c_0(\boldsymbol{u}) > 0$ and $\delta = \delta(L_0, \boldsymbol{u}) \in (0,1)$ such that for $L_n = \lambda_0^n L_0$, we have

$$\lim_{n\to\infty} \frac{1}{L_n^c} \log \mathbb{P}[V_{0,L_n}^c] < 0 \tag{4.5.106}$$

To this end, we recall the definitions of $\boldsymbol{\mathcal{E}}_x^k$ and $G_{x,n}^{\mathcal{L}}(\boldsymbol{\mathcal{E}}^k)$ from (4.1.25), (4.1.16) respectively, feeding in our events (4.5.102)-(4.5.104) above. Let $\epsilon \in (0,1)$ be such that $\epsilon < \min(u_* - u_3, u_2 - u_1)/2$, fix $\bar{\lambda}$ as in Lemma 4.1.5 for $m = 3$ and, in the sequel, restrict $L_0 > C(\boldsymbol{u})$ such that (4.1.17) holds for the above choices of ϵ, λ_0. First, we aim to satisfy assumption (4.1.18) of Lemma 4.1.3 for sprinkled versions of $\boldsymbol{\mathcal{E}}_x^k$, for each $x \in \mathbb{L}_0$ and $k = 1, 2, 3$. In particular, we claim that

$$\lim_{L_0\to\infty} \mathbb{P}[\hat{\boldsymbol{\mathcal{E}}}_x^k] = 1, \quad x \in \mathbb{L}_0, \quad k = 1, 2, 3 \tag{4.5.107}$$

where $\hat{\boldsymbol{\mathcal{E}}}_x^1 = \boldsymbol{\mathcal{E}}_x^1(u_3 + \epsilon)$, $\hat{\boldsymbol{\mathcal{E}}}_x^2 = \boldsymbol{\mathcal{E}}_x^2(u_1 + \epsilon, u_2 - \epsilon)$ and $\hat{\boldsymbol{\mathcal{E}}}_x^3 = \boldsymbol{\mathcal{E}}_x^3(\delta)$. The case for $k = 1, 2$ is immediate by applying a union bound together with a standard gluing argument relying on estimates (1.13), (1.14) from Theorem 1.2 in [22]. On the other hand, for $k = 3$ we use independence of the U_x combined with the choice $\delta < L_0^{-C(\bar{\lambda})}$. Observe that, since λ_0 is fixed, (4.5.107) immediately ensures condition (4.1.18) for $k = 1, 2, 3$ by fixing $L_0 > C(\boldsymbol{u}, \lambda_0)$ sufficiently large. Thus, since all the required measurability and monotonicity conditions are satisfied - measurability by making a minor modification to account for the independent random variables appearing in $\hat{\boldsymbol{\mathcal{E}}}_x^3$, and monotonicity by introducing dummy variables in $\hat{\boldsymbol{\mathcal{E}}}_x^2$ and $\hat{\boldsymbol{\mathcal{E}}}_x^3$ with respect to which the functions remain constant - we obtain, as a consequence of Lemma 4.1.3, that for all $n \in \mathbb{N}^*$

$$\mathbb{P}[G_{x,n}^{\mathcal{L}}(\boldsymbol{\mathcal{E}}^k)^c] \leq 2^{-2^n}, \quad x \in \mathbb{L}_n, \quad k = 1, 2, 3, \tag{4.5.108}$$

Then, we apply Lemma 4.1.5 to see that, when $x \in \mathbb{L}_n$ is $(n, \mathcal{E})$-good (recall (4.1.27)) any two clusters of $B(x, \frac{10}{11}\bar{\lambda}L_n)$ with diameter at least $\frac{1}{22}\bar{\lambda}L_n$ are connected via a path of $\mathcal{E}$-good vertices in $B(x, \frac{15}{11}\bar{\lambda}L_n)$. By (4.5.105), these connecting paths in fact satisfy $V_{z,L_0}(\mathcal{V}_{\mathbb{L}_0}, \delta, u_1, u_2, u_3)$ for their constituent vertices, which paves the way for another gluing argument, implying $V_{x,L}$ for $L = (\bar{\lambda}/11)L_n$. All in all, in combination with the assertion (4.5.108), this implies the required a priori bound (4.5.106). In fact, inspecting the proof shows that this a priori bound can be further generalised for any $L \in [L_n, L_{n+1})$ by applying a union bound on a bounded number of spatially-shifted copies of V^c_{0,L_n} - this yields

$$\mathbb{P}[V_{0,L}] \geq 1 - Ce^{-L^c} \tag{4.5.109}$$

for some constants $c, C > 0$ depending on $\boldsymbol{u}$. In the sequel, we will also denote

$$\boldsymbol{V}_{z,L} := \bigcap_{\substack{z' \in \mathbb{L}: \\ |z'-z| \leq L}} V_{z',L}(\mathcal{V}_{\mathbb{L}}, \delta, \boldsymbol{u}_{5,\epsilon}) \tag{4.5.110}$$

which exhibits, via a simple union bound, the same probability estimate as in (4.5.109).

Equipped with the a priori estimate (4.5.109), we now return to describing the bootstrapping argument. Consider $u_1 < u_2 < u_3 < u_*$, $\epsilon \in \left(0, (1-(\frac{u_3}{u_*})^{\frac{1}{10}}) \wedge (1-(\frac{u_2}{u_3})^{\frac{1}{20}})\right)$ and denote $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{u}_{k,\epsilon} = \left(u_1(1-\epsilon)^k, u_2(1+\epsilon)^k, u_3(1-\epsilon)^k\right)$. Then, by retracing the steps in (4.4.8)-(4.4.10), with (4.5.109) instead of (4.4.7) and using the obvious analogues of (4.4.4), (4.4.5), we yield

$$\mathbb{P}\Big[\mathscr{G}\Big(\Lambda_N, \boldsymbol{V}_z(\bar{\mathcal{V}}_{\mathbb{L}}, \delta, \boldsymbol{u}_{4,\epsilon}), \mathcal{F}_{\cdot}^{\boldsymbol{u}_{5,\epsilon}, \boldsymbol{u}_{4,\epsilon}}; \tfrac{1}{2}\Big)\Big] \geq \begin{cases} 1 - C(\epsilon)e^{-\frac{N}{(\log N)^\beta}}, & d = 3, \\ 1 - C(\epsilon)e^{-cN}, & d \geq 4. \end{cases} \tag{4.5.111}$$

for $\Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$. To conclude the proof, we make the following series of observations, referring to (4.3.34) for the restrictions on the scales L, N and the separating distance K. On the event

$$\{C_{0,N} \overset{N_\delta(\mathcal{V}^{u_1(1-\epsilon)^5})}{\longleftrightarrow} \partial D_{0,N}\},$$

there cannot be any nearest neighbour paths $\gamma \in \mathbb{L}$ crossing $\mathbb{L}(D_{0,N}) \setminus \mathbb{L}(C_{0,N})$ and satisfying $V_{z,L}$ for each $z \in \gamma$ (otherwise a gluing argument yields a contradiction). By a standard duality argument, this means that every such path $\gamma \subset \mathbb{L}$ from $\mathbb{L}(C_{0,N})$ to

$\partial_{\mathbb{L}}\mathbb{L}(D_{0,N})$ must intersect at least one $x \in \mathbb{L}$ such that $V^c_{x,L}$ holds. By Proposition 4.1.1 applied to the case $V = \mathbb{L}(D_{0,N})$, $U = \mathbb{L}(C_{0,N})$ and $\Sigma = \{x \in V \setminus U : V^c_{x,L}\text{holds}\}$ with $\mathbb{L}$ as the underlying lattice, this guarantees the existence of a $*$-connected component $\mathcal{C} \subset \Sigma$ such that $C_{0,N} \cap \mathbb{L} \preceq \mathcal{C}$. However, since the component $\mathcal{C}$ surrounds $C_{0,N}$, this implies the existence of a $*$-path in Σ crossing $\mathbb{L}(C_{z,N}) \setminus \{z\}$ for some $z \in [N, 5N] \times \{0\}^{d-1} \cap \mathbb{L}$. Moreover, such a $*$-path then implies the existence of a *path* $\gamma' \subset \mathbb{L}$ crossing $\mathbb{L}(C_{z,N}) \setminus \{z'\}$ (for some $z' \in \mathbb{L}$ with $|z' - z| \leq L$) such that $\boldsymbol{V}^c_{x,L}$ holds for all $x \in \gamma'$. In turn, this implies $\mathscr{G}(\tilde{D}_{z,\frac{N}{5}} \setminus \tilde{C}_{z,\frac{N}{5}}, \boldsymbol{V}_\cdot(\bar{\mathcal{V}}_{\mathbb{L}}, \delta, \boldsymbol{u}_{4,\epsilon}), \mathcal{F}^{\boldsymbol{u}_{5,\epsilon},\boldsymbol{u}_{4,\epsilon}}; \frac{1}{2})^c$, since otherwise there would exist a surrounding $*$-connected component of $\mathbb{L}(\tilde{C}_{z,\frac{N}{5}})$ in $\mathbb{L}(\tilde{D}_{z,\frac{N}{5}}) \setminus \mathbb{L}(\tilde{C}_{z,\frac{N}{5}})$ consisting entirely of points in $\{x \in \mathbb{L} : \boldsymbol{V}_{x,L}(\mathcal{V}_{\mathbb{L}}, \delta, \boldsymbol{u}_{5,\epsilon})\}$. This is a contradiction, since the path determined above must intersect any such component. By translation invariance and an application of the bound obtained in (4.5.111), we have the required result. □

Before we delve into the proof of Lemma 4.4.6, we present two additional lemmas which, together, deal with the probability of the event $\overline{\text{FE}}(\mathcal{V}_{\mathbb{L}}, u_0, u_1, u'; n)$ appearing in (4.4.32). The resulting probability will subsequently feed into an argument similar in spirit to the one described above as an estimate on a modified version of the family $\left\{\mathcal{E}^k_x : x \in \mathbb{Z}^d, k \in \{1, \dots, m\}\right\}$. Throughout these lemmas, we recall the notation $\text{FE}_\cdot$ and $\mathcal{G}_{\cdot,L_{-1}}$ from Definition 4.4.3, as well as $\boldsymbol{Z}(\cdots)$ from (4.4.23).

Lemma 4.5.6 ($\text{FE}_{\cdot,L_0}$ is likely)**.** *For any $u_0, u \in (0,\infty)$, $K \geq 100$ and $L_0 \in \mathbb{N}^*$, we have*

$$\mathbb{P}\Big[FE_{0,L_0}(\boldsymbol{Z}(\mathcal{V}^u_{0,L_0}, \mathcal{N}^{u_0}_{0,L_0}, \star))\Big] \geq 1 - C(u,u_0)e^{-c(u)(\log L_0)^2}. \tag{4.5.112}$$

Proof. By applying a partial-localisation step similar to (4.4.36), we obtain

$$\text{FE}_{0,L_0}(\boldsymbol{Z}(\bar{\mathcal{V}}^{2u}_{0,L_0}, \tfrac{u_0}{2}\text{cap}(\breve{D}_{0,L_0}), \star)) \cap \mathcal{F}^{u_0,\frac{u_0}{2}}_{0,L_0} \cap \mathcal{F}^{u,2u}_{0,L_0} \subset \text{FE}_{0,L_0}(\boldsymbol{Z}(\mathcal{V}^u_{0,L_0}, \mathcal{N}^{u_0}_{0,L_0}, \star)) \tag{4.5.113}$$

In the rest of the proof, we will omit the subscript L_0 for the events $\text{FE}_{y,L_0}(\cdot)$, $O_{y,L_0}(\cdot)$ etc. (see Definition 4.4.3). Since the event $O_0(Z)$ is monotonically decreasing in Z w.r.t. inclusion as subsequences and any $Z \in \boldsymbol{Z}(\bar{\mathcal{V}}^{2u}_0, \frac{u_0}{2}\text{cap}(\breve{D}_0), \star)$ is a subsequence of $\bar{Z}^{2u}_0$ (see (4.2.24) and (4.4.23)), we can deduce

$$\bigcap_{y \in \mathbb{L}_0 : |y|_\infty \leq 3L_0} O_y(\boldsymbol{Z}(\bar{\mathcal{V}}^{2u}_0, \tfrac{u_0}{2}\text{cap}(\breve{D}_0), \star)) = \bigcap_{y \in \mathbb{L}_0 : |y|_\infty \leq 3L_0} O_y(\bar{\mathcal{V}}^{2u}_0) \tag{4.5.114}$$

Next, we deal with the $\mathrm{LU}_y(Z)$ event comprising $\mathrm{FE}_y(Z)$, which incurs an additional difficulty by virtue of its non-monotonicity. First, recalling the definitions of $\boldsymbol{Z}(\bar{\mathcal{V}}_0^{2u}, u_0\mathrm{cap}(\breve{D}_0), \star)$ and the event $\mathrm{Incl}_z^{\epsilon,m_0}$ (see (4.4.23) and (4.2.18)), we obtain that under any coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\hat{\mathbb{P}}_0$

$$\mathrm{LU}_y\Big(\boldsymbol{Z}(\hat{\mathcal{V}}_0^{4u}, \tfrac{u_0}{4}\mathrm{cap}(\breve{D}_0), \star)\Big) \cap \mathrm{Incl}_0^{\frac{1}{10}, \frac{u_0}{4}\mathrm{cap}(\breve{D}_0)} \subset \mathrm{LU}_y\Big(\boldsymbol{Z}(\bar{\mathcal{V}}_0^{2u}, \tfrac{u_0}{2}\mathrm{cap}(\breve{D}_0), \star)\Big) \quad (4.5.115)$$

for any $y \in \mathbb{L}_0$. The main advantage of working with the $\mathrm{LU}_y(\cdots)$ event on the left-hand side of (4.5.115) is that the comprising excursion sets consist of i.i.d excursions. We now proceed to simplifying the combinatorial complexity of this event.

Let J, J' be two (fixed) finite, non-empty subsets of $\mathbb{N}^*$. Then, for any $y \in \mathbb{L}_0$ with $|y|_\infty \le 3L_0$ and on the event $\mathcal{I}((Z_j^{\breve{D}_0,\breve{U}_0})_{j\in J}) \cap (\tilde{D}_{y,L_0} \setminus \tilde{C}_{y,L_0}) \neq \emptyset$, we have that

$$\bigcap_{j'\in J'} \mathrm{LU}_y((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup\{j'\}}) \subset \mathrm{LU}_y((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup J'}) \quad (4.5.116)$$

Assume that we are on the event on the left-hand side of (4.5.116) and consider $x', x'' \in \mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup J'}) \cap (\tilde{D}_{y,L_0} \setminus \tilde{C}_{y,L_0})$ (if there are no such x, x' then the inclusion holds trivially). By our assumption, we also consider some $x \in \mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J}) \cap (\tilde{D}_{y,L_0} \setminus \tilde{C}_{y,L_0})$. In the case that neither $x', x'' \in \mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J})$, there exist $j' \in J'$ such that $x' \in \mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup\{j'\}})$, and the same conclusion holds for x'' with some $j'' \in J'$ instead of j'. By the left-hand side of (4.5.116), since it implies the event $\cap_{\hat{j}=j',j''}\mathrm{LU}_y((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup\{\hat{j}\}})$, we therefore have that x is connected to x' in $\mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup\{j'\}}) \cap (\mathring{D}_{y,L_0} \setminus C_{y,L_0})$, and x is also connected to x'' in $\mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup\{j''\}}) \cap (\mathring{D}_{y,L_0} \setminus C_{y,L_0})$. Overall, we obtain that x' is connected to x'' in

$$\mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup\{j',j''\}}) \cap (\mathring{D}_{y,L_0} \setminus C_{y,L_0}) \subset \mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup J'}) \cap (\mathring{D}_{y,L_0} \setminus C_{y,L_0})$$

If instead $x' \in \mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J})$, then we have that x' is directly connected to x'' in

$$\mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup\{j''\}}) \subset \mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in J\cup J'})$$

By noticing that that the argument for x'' is symmetrical, we have thus shown (4.5.116). Then in view of (4.4.18), remarking that any $\hat{Z} \in \boldsymbol{Z}(\hat{\mathcal{V}}_0^{4u}, \frac{u_0}{4}\mathrm{cap}(\breve{D}_0), \star)$ must necessarily contain $(\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j\in\{j'+1,\ldots,j'+\lfloor \frac{u_0}{4}\mathrm{cap}(\breve{D}_0)\rfloor\}}$ for some $j' \in \mathbb{N}$, by applying the inclusions (4.5.114) and (4.5.116) (with $J = \{j'+1, \ldots, j' + \left\lfloor \frac{u_0}{4}\mathrm{cap}(\breve{D}_0)\right\rfloor\} \cup \{j''\}$ for some j', j'')

together with (4.5.115), we deduce

$$\begin{aligned}\bigcap_{\substack{y\in\mathbb{L}_0:\\ |y|_\infty\le 3L_0}}\Big(O_y(\bar{\mathcal{V}}_0^{2u})\cap\bigcap_{j\in\boldsymbol{J}_{u_0,u}}\mathrm{LU}_y(\hat{Z}_J^{\breve{D}_0,\breve{U}_0})\Big)\cap\mathrm{Incl}_0^{\frac{1}{10},\frac{u_0}{4}\mathrm{cap}(\breve{D}_0)}\\ \subset\mathrm{FE}_{0,L_0}(\boldsymbol{Z}(\bar{\mathcal{V}}_0^{2u},\tfrac{u_0}{2}\mathrm{cap}(\breve{D}_0),\star))\end{aligned}\tag{4.5.117}$$

Finally, plugging this into the left-hand side of (4.5.113) yields

$$\begin{aligned}\bigcap_{\substack{y\in\mathbb{L}_0:\\ |y|_\infty\le 3L_0}}\Big(O_y(\bar{\mathcal{V}}_0^{2u})\cap\bigcap_{j\in\boldsymbol{J}_{u_0,u}}\mathrm{LU}_y(Z_J^{\breve{D}_0,\breve{U}_0})\Big)\cap\mathcal{F}_0^{u_0,\frac{u_0}{2}}\cap\mathcal{F}_0^{u,2u}\cap\mathrm{Incl}_0^{\frac{1}{10},\frac{u_0}{4}\mathrm{cap}(\breve{D}_0)}\\ \subset\mathrm{FE}_{0,L_0}(\boldsymbol{Z}(\mathcal{V}_0^{u},\mathcal{N}_0^{u_0},\star))\end{aligned}\tag{4.5.118}$$

for any L_0 such that $\frac{u_0}{4}\mathrm{cap}(D_0)\ge 1$ where $\boldsymbol{J}_{u_0,u}=\bigcup_{j''\in[1,\lfloor 4u\mathrm{cap}(\breve{D}_0)\rfloor]}\boldsymbol{J}_{u_0,u,j''}$ and

$$\boldsymbol{J}_{u_0,u,j''}\coloneqq\begin{array}{c}\text{the collection of all subsets of } \{1,\dots,\lfloor 4u\mathrm{cap}(\breve{D}_0)\rfloor\} \text{ of the form}\\ \{j+1,\dots,j+\lfloor\tfrac{u_0}{2}\mathrm{cap}(\breve{D}_0)\rfloor\}\cup\{j'\}\cup\{j''\} \text{ for some } j,j'\in\mathbb{N}\end{array}\tag{4.5.119}$$

Note that, the partition over $j''\in[1,\lfloor 4u\mathrm{cap}(\breve{D}_0)\rfloor]$ is used to guarantee the non-emptiness condition above (4.5.116) since, for any $\hat{Z}_{j''}^{\breve{D}_0,\breve{U}_0}$ intersecting $\tilde{D}_{y,L_0},\boldsymbol{J}_{u_0,u,j''}$ contains all the subsets containing this 'non-empty' excursion, and (4.5.116) applies. If none of the trajectories $\hat{Z}_{j''}^{\breve{D}_0,\breve{U}_0}$ intersect $\tilde{D}_{y,L_0}$, the inclusion holds trivially. The upshot of the inclusion (4.5.118) is that the left-hand side therein now has an improved combinatorial complexity when applying a union bound argument, since $|\boldsymbol{J}_{u_0,u}|$ is at most a power of L_0.

Let us first bound the probability of $O_y(\bar{\mathcal{V}}_0)$. To do this, we recall its definition in (4.4.19) and deduce the inclusion

$$O_y(\bar{\mathcal{V}}_0)^c\subset\bigcup_{x\in\partial D_y}\bigcup_{j\in[1,\lfloor 2u\mathrm{cap}(\breve{D}_0)\rfloor]}\Big\{\ell_x(Z_j^{\breve{D}_0,\breve{U}_0})>(\log L_0)^2\Big\}\tag{4.5.120}$$

Then, we have

$$\begin{aligned}\mathbb{P}[O_y(\bar{\mathcal{V}}_0)^c]&\le 2u\mathrm{cap}(\breve{D}_0)|\partial D_y|\sup_{z\in\partial\breve{D}_0}P_z[\ell_x(Z)>(\log L_0)^2]\\ &\le Ce^{-c(u)(\log L_0)^2}\end{aligned}\tag{4.5.121}$$

where Z under P_z is used to denote a simple random walk in $\mathbb{Z}^d$ starting from z. In the last line of display (4.5.121), we used the exponential decay of the tail of occupation times for transient random walks. Next, we want to prove that for any $K \geq 100$, $y \in \mathbb{L}_0$ such that $|y|_\infty \leq 3L_0$ and $J \in \boldsymbol{J}_{u_0,u}$, one has

$$\mathbb{P}[\mathrm{LU}_y(\hat{Z}_J^{\breve{D}_0,\breve{U}_0})] \geq 1 - C(u_0)e^{-c(u_0)L_0^c}. \tag{4.5.122}$$

To this end, we apply the following modifications to the proof of Proposition 1 in [50]. For any $R' > R \geq 1$ and $x \in \mathbb{Z}^d$ and a collection of excursions $\tilde{Z} \subset \{\tilde{Z}_j^{\breve{D}_0,\breve{U}_0}\}_{j\geq 1}$ (see discussion below (4.2.24)), start by considering the following alternative local uniqueness event

$$\widehat{\mathrm{LU}}_{x,R,R'}(\tilde{Z}) := \bigcap_{x,x' \in B_R(x)\cap\mathcal{I}(\tilde{Z})} \{x \xleftrightarrow{B_{R'}(x)\cap\mathcal{I}(\tilde{Z})} x'\}. \tag{4.5.123}$$

We will show that there exists $C > 0$ such that for any $R \geq 1$ and $x \in \mathbb{Z}^d$ satisfying $B_{CR}(x) \subset \breve{D}_0$, we have

$$\mathbb{P}[\widehat{\mathrm{LU}}_{x,R,CR}(\hat{Z}_J^{\breve{D}_0,\breve{U}_0})] \geq 1 - C(u_0)e^{-L_0^c} \tag{4.5.124}$$

for any $K \geq 100$,$y \in \mathbb{L}_0$,$J \in \boldsymbol{J}_{u_0,u}$ as before. The estimate (4.5.122) follows from this by applying a standard covering argument since $D_y \subset \breve{D}_0$ for all $y \in \mathbb{L}_0$ with $|y|_\infty \leq 3L_0$. Refer, for an example of such an argument, to the proof of Proposition 1 at the end of page 11 in [50].

Thus, we turn to the proof of (4.5.124). Since the sequence $\{\hat{Z}_j^{\breve{D}_0,\breve{U}_0}\}_{j\geq 1}$ consists of i.i.d excursions, by reshuffling it is sufficient to consider the case when $J = \{1, \ldots, \lfloor \frac{u_0}{4}\mathrm{cap}(\breve{D}_0)\rfloor + k\}$ for $k = 0, 1, 2$. Each of these cases adhere to a similar approach, so we will assume $k = 0$ in the sequel. Then, we assume without loss of generality that $u_0\mathrm{cap}(\breve{D}_0) \geq 100d$ (see (2.2.35)) and apply a delocalisation step according to (2.6.13), (4.2.18) and (4.2.21) to determine a coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\hat{\mathbb{P}}_0$ such that, for a given $v \in (0, \infty)$, $j' \in \mathbb{N}$, one has

$$\mathbb{Q}\big[\mathcal{I}^v \cap \breve{D}_0 \subset \mathcal{I}((\hat{Z}_j^{\breve{D}_0,\breve{U}_0})_{j'+1\leq j\leq j'+2v\mathrm{cap}(\breve{D}_0)}) \cap \breve{D}_0\big] \geq 1 - Ce^{-cvL_0^{d-2}} \tag{4.5.125}$$

as soon as $K \geq 100$. Invoking the independence of excursions in $\{\hat{Z}_j^{\breve{D}_0}\}_{j\geq 1}$ once more, it follows from the proofs of Lemma 9 and 10 in [50] together with (4.5.125) that for all $\epsilon \in (0, \frac{2}{3})$, $x \in \mathbb{Z}^d$ and $R > 0$ such that $B_R(x) \subset \breve{D}_0$, we have the following analogue

of Lemma 10 in [50]. For any $v \in (0,\infty)$ and $j \in \mathbb{N}$,

$$\mathbb{P}\Big[x \in \hat{\mathcal{I}}_j^v, \mathrm{cap}(\mathcal{C}(x,R;j,v)) < c(v)R^{(d-2)(1-\epsilon)}\Big] \leq C(v)e^{-c(v)R^{\frac{\epsilon}{2}}} \tag{4.5.126}$$

where $\hat{\mathcal{I}}_j^v = \mathcal{I}((\hat{Z}_{j'}^{\breve{D}_0,\breve{U}_0})_{j+1\leq j'\leq j+v\mathrm{cap}(\breve{D}_0)})$ and $\mathcal{C}(x,R;j,v)$ is the component of $\hat{\mathcal{I}}_j^v \cap B_R(x)$ containing x (by convention, $\mathcal{C}(x,R;j,v)$ if $x \notin \hat{\mathcal{I}}_j^v$). Also, from Lemma 12 in [50] and (4.5.125), we can deduce

$$\mathbb{P}\Big[U \overset{\hat{\mathcal{I}}_j^v \cap B_{CR}(x)}{\longleftrightarrow} V\Big] \geq 1 - C(v)\exp\Big(-c(v)R^{2-d}\mathrm{cap}(U)\mathrm{cap}(V)\Big) \tag{4.5.127}$$

for some $C \in (0,\infty)$, any x and R such that $B_{CR}(x) \subset \breve{D}_0$, any $j \in \mathbb{N}$ and $v \in (0,\infty)$ and all subsets $U, V \subset B_R(x)$. One final call to the independence of the excursions $\{\hat{Z}_j^{\breve{D}_0}\}_{j\geq 1}$ means that (4.5.124) can be derived from the two previous displays in exactly the same way as Lemma 13 was obtained from Lemma 10 and Lemma 12 in [50].

Finally, from (4.2.21) we have

$$\mathbb{P}[(\mathcal{F}_0^{u_0,\frac{u_0}{2}} \cap \mathcal{F}_0^{u,2u})^c] \leq 2e^{cu_0L_0^{d-2}} \tag{4.5.128}$$

Thus, by combining the estimates from (4.5.121), (4.5.122) and (4.5.128) together with the bound $|\boldsymbol{J}_{u_0,u}| \leq Cu^3L_0^C$, the inclusion (4.5.118) gives the required probability estimate (4.5.112) via a union bound. □

Lemma 4.5.7 ($\mathcal{G}_{\cdot,L_{-1}}$ is likely)**.** *There exists a scale L_{-1} and a level $u' \in (0,u_*)$ satisfying*

$$\mathbb{P}\Big[\mathcal{G}_{0,L_{-1}}\Big(\boldsymbol{Z}(\mathcal{V}_{0,L_0}^{u_*},\star,\mathcal{N}_{0,L_0}^{u'};n)\Big)\Big] \geq 1 - C(n)e^{-L_0^c} \tag{4.5.129}$$

for all $L_0 > 100L_{-1}$.

Proof. This is a consequence of Lemma 3.6 [18] after accounting for the combinatorial complexity induced by the collection $\boldsymbol{Z}(\mathcal{V}_{0,L_0}^{u_*},\star,\mathcal{N}_{0,L_0}^{u'};n)$. In particular, we use the monotonicity of the event $\mathcal{G}_{0,L_{-1}}$, and then proceed with an argument similar to that used for $\boldsymbol{Z}_{\mathrm{small}}(\cdots)$ (see (4.5.10) and (4.5.21)-(4.5.27)). □

Proof of Lemma 4.4.6. As in the statement of the lemma, we fix $u_0 \in (0,u_*)$, $u_1 < u_2 < u_3 \in (0,u_*)$, $n \in \mathbb{N}^*$ and $K \geq 100$. Then, we choose L_{-1} and $u' \in (0,\infty)$ according to Lemma 4.5.7, such that the result therein holds for any $L_0 > 100L_{-1}$. Recalling the events in (4.5.102)-(4.5.104), we define $\mathcal{E}_x^1$ and $\mathcal{E}_x^2$ as before, let $\mathcal{E}_x^3 = \mathcal{E}_x^3(2\delta)$ for

$\delta \in (0, \frac{1}{2})$ and introduce the following additional events for any $x \in \mathbb{Z}^d$, $x' \in \mathbb{L}_0$ such that $x \in C_{x',L_0}$ and $u, v > 0$:

$$\mathcal{E}^4_x(u,v) = \mathrm{FE}_{x',L_0}(\boldsymbol{Z}(\mathcal{V}^v_{x',4L_0}, \mathcal{N}^u_{x',4L_0}, \star)), \tag{4.5.130}$$

$$\mathcal{E}^5_x(u,v) = \mathcal{G}_{x',L-1}\Big(\boldsymbol{Z}(\mathcal{V}^u_{x',L_0}, \star, \mathcal{N}^v_{0,L_0}; n)\Big), \tag{4.5.131}$$

abbreviating $\mathcal{E}^4_x = \mathcal{E}^4_x(u_0, u_1)$, $\mathcal{E}^5_x = \mathcal{E}^5_x(u_1, u')$. In the sequel, we set λ_0 as in (4.1.15) and $\bar{\lambda}$ satisfying the assumptions of Lemma 4.1.5 for $m = 5$. By repeating the same steps as in the proof of Lemma 4.4.7 for $\boldsymbol{\mathcal{E}}^k_x$, $k = 1, 2, 3$ and applying a union bound together with Lemma 4.5.6 for $k = 4$ and Lemma 4.5.7 for $k = 5$, we obtain

$$\lim_{L_0 \to \infty} \mathbb{P}[\hat{\boldsymbol{\mathcal{E}}}^k_x] = 1, \quad x \in \mathbb{L}_0, \quad k \in \{1, \dots, 5\} \tag{4.5.132}$$

where $\hat{\boldsymbol{\mathcal{E}}}^1_x$, $\hat{\boldsymbol{\mathcal{E}}}^2_x$ are as below (4.5.107), $\hat{\boldsymbol{\mathcal{E}}}^3_x = \hat{\boldsymbol{\mathcal{E}}}^3_x(2\delta(L_0))$, (where $\delta(L_0) < L_0^{-C(\bar{\lambda})}$) $\hat{\boldsymbol{\mathcal{E}}}^4_x = \hat{\boldsymbol{\mathcal{E}}}^4_x(u_0 - \epsilon, u_1 + \epsilon)$, and $\hat{\boldsymbol{\mathcal{E}}}^5_x = \hat{\boldsymbol{\mathcal{E}}}^5_x(u_1 + \epsilon, u' - \epsilon)$ for a choice of $\epsilon \in \Big(0, \min(u', u_* - u_3, u_2 - u_1)/2\Big)$. Thus, we fix $L_0 > 100_{L_{-1}} \vee C(\boldsymbol{u}, \lambda_0)$ such that (4.1.17) and (4.1.18) of Lemma 4.1.3 are satisfied, from which we obtain

$$\mathbb{P}[G^{\mathcal{L}}_{x,n}(\boldsymbol{\mathcal{E}}^k)^c] \le 2^{-2^n}, \quad x \in \mathbb{L}_n, \quad k \in \{1, \dots, 5\}, \tag{4.5.133}$$

In order to complete the proof, we claim that, for $L = \bar{\lambda} L_n$, we have

$$\bigcap_{k=1}^{5} G^{\mathcal{L}}_{0,n}(\boldsymbol{\mathcal{E}}^k) \subset \mathcal{G}_0\Big(\mathcal{V}_{\mathbb{L}}, a = 1, \overline{\mathrm{FE}}(\mathcal{V}_{\mathbb{L}}, u_0, u_1, u'; n), \delta, u_1, u_2, u_3\Big) \tag{4.5.134}$$

which, in combination with the estimate (4.5.133), is immediately seen to yield (4.4.32). In order to show (4.5.134), we recall from (4.3.36) and (4.4.14) that

$$\mathcal{G}_0(\mathcal{V}_{\mathbb{L}}, a, W, \delta, u_1, u_2, u_3) = G_0(W, \mathscr{C}(\mathcal{V}_{\mathbb{L}}, \delta, u_1, u_2); a) \cap V_0(\mathcal{V}_{\mathbb{L}}, \delta, u_1, u_2, u_3) \tag{4.5.135}$$

In light of this, we reduce the problem into showing that the left-hand side of (4.5.134) (i.e. that 0 is $(n, \mathcal{E})$-good) implies each of the events on the right-hand side of (4.5.135) when $W = \overline{\mathrm{FE}}(\mathcal{V}_{\mathbb{L}}, u_0, u_1, u'; n)$ and $a = 1$. In the sequel, we assume that 0 is $(n, \mathcal{E})$-good and aim to deduce the desired conclusions, starting with the event $V_0(\cdots)$. Observe that, on the event $\mathcal{E}^3_x$, $N_{2\delta}(\mathcal{V}) = N_\delta(\mathcal{V}) = \mathcal{V}$ for any vacant configuration in D_{x,L_0}, therefore the existence part follows by Lemma 4.1.5 (see the proof of Lemma 4.4.7 for a similar argument). The uniqueness part follows from similar ideas: any cluster of $N_{2\delta}(\mathcal{V}^{u_2})$ crossing $\tilde{D}_{0,L} \setminus \tilde{C}_{0,L}$ must have diameter at least $L = \bar{\lambda} L_n > \frac{1}{22} \bar{\lambda} L_n$,

so by Lemma 4.1.5 any two such clusters must be connected by a nearest neighbour path γ of $\mathcal{E}$-good vertices. Since every constituent vertex $x \in \gamma$ satisfies $\cap_{k=1,2,3}\mathcal{E}^k_x$, we can apply a gluing argument to connect the two clusters in $N_\delta(\mathcal{V}^{u_1})$. All in all, this shows the occurrence of $V_0(\mathcal{V}_\mathbb{L}, \delta, u_1, u_2, u_3)$, and indeed also the connectedness of the component described by $\mathscr{C}(\mathcal{V}_\mathbb{L}, \delta, u_1, u_2)$. Finally, we want to show that $G_0(\cdots)$ with the parameters as above holds. Recalling (4.3.37), let γ be a crossing of $\tilde{D}_{0,L} \setminus \tilde{C}_{0,L}$, and consider any other (yet to be determined) set $\mathcal{C} \subset D_{0,L}$ with diameter at least $\frac{1}{22}L = \frac{1}{22}\bar{\lambda}L_n$. Since the diameter of γ is also at least $\frac{1}{22}\bar{\lambda}L_n$, Lemma 4.1.5 asserts the existence of an $\mathcal{E}$-good path between γ and $\mathcal{C}$ and, in particular, the existence of an $\mathcal{E}$-good vertex $x \in \gamma$. Thus, there exists an $x' \in \mathbb{L}_0$ such that $x \in C_{x',L_0}$ (and in particular, $x' \in \mathbb{L}_0(\gamma)$) such that $\overline{\mathrm{FE}}_{x',L_0}(\boldsymbol{Z}(\mathcal{V}^{u_1}_{x',4L_0}, \mathcal{N}^{u_0}_{x',4L_0}, \star)) \cap \mathcal{G}_{x',L_{-1}}(\boldsymbol{Z}(\mathcal{V}^{u_1}_{x',L_0}, \star, \mathcal{N}^{u'}_{x',L_0}; n) \subset \overline{\mathrm{FE}}_{0,x'}(\mathcal{V}, u_0, u_1, u'; n)$ holds. Moreover, by choosing $\mathcal{C}$ to be the cluster crossing $D_{0,L} \setminus C_{0,L}$ (which is guaranteed to exist by $V_0(\mathcal{V}_\mathbb{L}, \delta, u_1, u_2, u_3)$), since the path γ consists entirely of points satisfying $\cap_{k=1,2,3}\mathcal{E}^k_\cdot$, we once again apply a gluing argument to ensure that $\mathscr{C}(\mathcal{V}_\mathbb{L}, \delta, u_1, u_2)$ intersects C_{x',L_0}, and therefore conclude the proof. □

Proof of Lemma 4.4.8. Let us consider scales $N \gg L \geq 1$ as in (4.3.34), and let the scale $L_{-1} \geq 1$ and level $u' > 0$ be as chosen in Lemma 4.5.7 (as a consequence of Lemma 3.6 in [18]). For $x \in \mathbb{L}$ and $u > 0$, define the event

$$\mathrm{Loop}_{x,L}(\mathcal{V}^u) := \left\{ \begin{array}{l} (\tilde{D}_{x,L} \cap \mathbb{L}_{-1}) \setminus (\tilde{C}_{x,L} \cap \mathbb{L}_{-1}) \text{ contains a set } \mathcal{S} \subset \mathbb{L}_{-1} \\ \text{such that } \mathcal{S} \text{ is connected in } \mathbb{L}_{-1}, \text{ each x} \in \mathcal{S} \text{ satisfies} \\ G_{\mathrm{x},L_{-1}}(\mathcal{V}^u) \text{ and every } *\text{-path in } \mathbb{L}_{-1} \text{ from } \tilde{C}_{x,L} \cap \mathbb{L}_{-1} \\ \text{to } \partial_{\mathbb{L}_{-1}}(D_{x,L} \cap \mathbb{L}_{-1}) \text{ intersects } \mathcal{S} \end{array} \right\} \tag{4.5.136}$$

and let $\mathbf{Loop}_{x,L}(\mathcal{V}^u) = \cap_{x' \in \mathbb{L}: |x'-x| \leq L} \mathrm{Loop}_{x,L}(\mathcal{V}^u)$. By Corollary 3.7 of [18] and a simple union bound, we have that there exist constants $c = c(d) > 0$, $C = C(d) < \infty$ such that for all $u < u'$,

$$\mathbb{P}[\mathbf{Loop}_{x,L}(\mathcal{V}^u)] \geq 1 - Ce^{-L^c} \tag{4.5.137}$$

Thus, in the remainder of the proof, we rely on ideas presented in the concluding remarks of the proof of Lemma 4.4.7, except incorporating the a priori bound (4.5.137) in place of (4.5.109). Indeed, observe the following sequence of inclusions

$$\left\{ C_{0,N} \cap \mathbb{L}_{-1} \overset{\mathcal{S}(Z^u)}{\longleftrightarrow\!\!\!\!\!\!/} \partial_{\mathbb{L}_{-1}}(D_{0,N} \cap \mathbb{L}_{-1}) \right\}$$

$$
\begin{aligned}
&\subset \left\{ \begin{array}{l} \text{for any path } \gamma \subset \mathbb{L} \text{ crossing } D_{0,N} \cap \mathbb{L} \setminus C_{0,N} \cap \mathbb{L}, \\ \text{there exists } x \in \gamma \text{ such that } (\mathrm{Loop}_{x,L}(\mathcal{V}^u))^c \text{ holds.} \end{array} \right\} \\
&\overset{\text{Prop. 4.1.1}}{\subset} \left\{ \begin{array}{l} \exists\ *\text{-connected component } \mathcal{O} \subset \mathbb{L} \text{ in } D_{0,N} \cap \mathbb{L} \setminus C_{0,N} \cap \mathbb{L} \\ \text{such that } C_{0,N} \preceq \mathcal{O} \text{ and } \forall x \in \mathcal{O},\ (\mathrm{Loop}_{x,L}(\mathcal{V}^u))^c \text{ holds.} \end{array} \right\} \\
&\subset \left\{ \begin{array}{l} \text{For some } z \in ([N, 5N] \cap \mathbb{N}) \times \{0\}^{d-1}, \exists \text{ path crossing} \\ \tilde{D}_{z,\frac{N}{5}} \setminus \tilde{C}_{z,\frac{N}{5}} \text{ such that } \forall x \in \gamma,\ (\mathbf{Loop}_{x,L}(\mathcal{V}^u))^c \text{ holds.} \end{array} \right\} \\
&\subset \bigcup_{z \in ([N,5N] \cap \mathbb{N}) \times \{0\}^{d-1}} \mathscr{G}\Big(\tilde{D}_{z,\frac{N}{5}} \setminus \tilde{C}_{z,\frac{N}{5}}, \mathbf{Loop}_{\cdot}(\bar{\mathcal{V}}^{u/(1-\epsilon)}), \mathcal{F}^{u,u/(1-\epsilon)}, \tfrac{1}{2} \Big)^c
\end{aligned}
$$

where, by setting $u = \frac{1}{2}u'(1-\epsilon)^7$, and applying the same bootstrapping steps as used to obtain the bound (4.5.111), we yield the required result. □

References

[1] Abächerli, A. (2019). Local picture and level-set percolation of the gaussian free field on a large discrete torus. *Stochastic Processes and their Applications*, 129(9):3527–3546.

[2] Aizenman, M., Delyon, F., and Souillard, B. (1980). Lower bounds on the cluster size distribution. *Journal of Statistical Physics*, 23:267–280.

[3] Bauerschmidt, R., Duminil-Copin, H., Goodman, J., and Slade, G. (2012). Lectures on self-avoiding walks. *Probability and Statistical Physics in Two and More Dimensions (D. Ellwood, CM Newman, V. Sidoravicius, and W. Werner, eds.), Clay Mathematics Institute Proceedings*, 15:395–476.

[4] Belavin, A. A., Polyakov, A. M., and Zamolodchikov, A. B. (1984a). Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics B*, 241(2):333–380.

[5] Belavin, A. A., Polyakov, A. M., and Zamolodchikov, A. B. (1984b). Infinite conformal symmetry of critical fluctuations in two dimensions. *Journal of Statistical Physics*, 34:763–774.

[6] Bricmont, J., Lebowitz, J. L., and Maes, C. (1987). Percolation in strongly correlated systems: the massless gaussian field. *Journal of statistical physics*, 48:1249–1268.

[7] Broadbent, S. R. and Hammersley, J. M. (1957). Percolation processes: I. crystals and mazes. In *Mathematical proceedings of the Cambridge philosophical society*, volume 53, pages 629–641. Cambridge University Press.

[8] Burton, R. M. and Keane, M. (1989). Density and uniqueness in percolation. *Communications in mathematical physics*, 121:501–505.

[9] Černỳ, J. and Teixeira, A. (2016). Random walks on torus and random interlacements: macroscopic coupling and phase transition. *Ann. Appl. Probab.*

[10] Chayes, J. T., Chayes, L., and Newman, C. M. (1987). Bernoulli percolation above threshold: an invasion percolation analysis. *The Annals of Probability*, pages 1272–1287.

[11] Comets, F., Gallesco, C., Popov, S., and Vachkovskaia, M. (2013). On large deviations for the cover time of two-dimensional torus. *Electron. J. Probab.*

[12] da Silva, C., Lyra, M., and Viswanathan, G. (2002). Largest and second largest cluster statistics at the percolation threshold of hypercubic lattices. *Physical Review E*, 66(5):056107.

[13] Deuschel, J.-D. and Pisztora, A. (1996). Surface order large deviations for high-density percolation. *Probability Theory and Related Fields*, 104:467–482.

[14] Drewitz, A., Prévost, A., and Rodriguez, P.-F. (2018). The sign clusters of the massless gaussian free field percolate on $\mathbb{Z}^d$, $d \geq 3$ (and more). *Communications in Mathematical Physics*, 362:513–546.

[15] Drewitz, A., Prévost, A., and Rodriguez, P.-F. (2023). Critical exponents for a percolation model on transient graphs. *Inventiones mathematicae*, 232(1):229–299.

[16] Drewitz, A., Prévost, A., and Rodriguez, P.-F. (2024). Geometry of gaussian free field sign clusters and random interlacements. *Probability Theory and Related Fields*, pages 1–96.

[17] Drewitz, A., Ráth, B., and Sapozhnikov, A. (2014a). *An introduction to random interlacements*. Springer.

[18] Drewitz, A., Ráth, B., and Sapozhnikov, A. (2014b). Local percolative properties of the vacant set of random interlacements with small intensity. In *Annales de l'IHP Probabilités et statistiques*, volume 50, pages 1165–1197.

[19] Duminil-Copin, H., Goswami, S., Rodriguez, P.-F., and Severo, F. (2023a). Equality of critical parameters for percolation of gaussian free field level sets. *Duke Mathematical Journal*, 172(5):839–913.

[20] Duminil-Copin, H., Goswami, S., Rodriguez, P.-F., Severo, F., and Teixeira, A. (2023b). A characterization of strong percolation via disconnection. *arXiv preprint arXiv:2308.07920*.

[21] Duminil-Copin, H., Goswami, S., Rodriguez, P.-F., Severo, F., and Teixeira, A. (2023c). Finite range interlacements and couplings. *arXiv preprint arXiv:2308.07303*.

[22] Duminil-Copin, H., Goswami, S., Rodriguez, P.-F., Severo, F., and Teixeira, A. (2023d). Phase transition for the vacant set of random walk and random interlacements. *arXiv preprint arXiv:2308.07919*.

[23] Duminil-Copin, H., Kozma, G., and Tassion, V. (2021). Upper bounds on the percolation correlation length. *In and Out of Equilibrium 3: Celebrating Vladas Sidoravicius*, pages 347–369.

[24] Duminil-Copin, H. and Tassion, V. (2017). A new proof of the sharpness of the phase transition for bernoulli percolation on $\mathbb{Z}^d$. *L'Enseignement mathématique*, 62(1):199–206.

[25] Ellis, T., Kenna, R., and Berche, B. (2023). The fifty-year quest for universality in percolation theory in high dimensions. *arXiv preprint arXiv:2306.13409*.

[26] Fitzner, R. and van der Hofstad, R. (2017). Mean-field behavior for nearest-neighbor percolation in d>10. *Electron. J. Probab.*

[27] Fortuin, C. M., Kasteleyn, P. W., and Ginibre, J. (1971). Correlation inequalities on some partially ordered sets. *Communications in Mathematical Physics*, 22:89–103.

[28] Goswami, S., Rodriguez, P.-F., and Severo, F. (2022). On the radius of gaussian free field excursion clusters. *The Annals of Probability*, 50(5):1675–1724.

[29] Goswami, S., Rodriguez, P.-F., and Shulzhenko, Y. (2024a). Cluster sizes for the vacant set of the random walk and random interlacments i. *In Preparation.*

[30] Goswami, S., Rodriguez, P.-F., and Shulzhenko, Y. (2024b). Cluster sizes for the vacant set of the random walk and random interlacments ii. *In Preparation.*

[31] Grimmett, G. and Grimmett, G. (1999). *What is percolation?* Springer.

[32] Grimmett, G. R. and Marstrand, J. M. (1990). The supercritical phase of percolation is well behaved. *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, 430(1879):439–457.

[33] Hammersley, J. (1959). Bornes supérieures de la probabilité critique dans un processus de filtration. *Le Calcul des Probabilités et ses Applications, CNRS, Paris*, pages 17–37.

[34] Hara, T. and Slade, G. (1990). Mean-field critical behaviour for percolation in high dimensions. *Communications in Mathematical Physics*, 128(2):333–391.

[35] Hunt, A., Ewing, R., and Ghanbarian, B. (2014). *Percolation theory for flow in porous media*, volume 880. Springer.

[36] Jorritsma, J., Komjáthy, J., and Mitsche, D. (2024). Cluster-size decay in supercritical long-range percolation. *Electronic Journal of Probability*, 29:1–36.

[37] Kesten, H. et al. (1980). The critical probability of bond percolation on the square lattice equals 1/2. *Communications in mathematical physics*, 74(1):41–59.

[38] Kesten, H. and Zhang, Y. (1990). The probability of a large finite cluster in supercritical bernoulli percolation. *The Annals of Probability*, pages 537–555.

[39] King, P. R. and Masihi, M. (2019). *Percolation theory in reservoir engineering.* World Scientific.

[40] Lawler, G. F. (2013). *Intersections of random walks.* Springer Science & Business Media.

[41] Li, X. and Sznitman, A.-S. (2014). A lower bound for disconnection by random interlacements. *Electron. J. Probab.*

[42] Lyons, R. and Peres, Y. (2017). *Probability on trees and networks*, volume 42. Cambridge University Press.

[43] Margolina, A., Herrmann, H., and Stauffer, D. (1982). Size of largest and second largest cluster in random percolation. *Physics Letters A*, 93(2):73–75.

[44] Muirhead, S. and Severo, F. (2022). Percolation of strongly correlated gaussian fields ii. *Sharpness of the phase transition*.

[45] Muirhead, S. and Severo, F. (2024). Percolation of strongly correlated gaussian fields, i: Decay of subcritical connection probabilities. *Probability and Mathematical Physics*, 5(2):357–412.

[46] Muirhead, S. and Wigman, I. (2024). The giant component of excursions sets of spherical gaussian ensembles: existence, uniqueness, and volume concentration. *arXiv preprint arXiv:2406.09659*.

[47] Popov, S. and Teixeira, A. (2015). Soft local times and decoupling of random interlacements. *Journal of the European Mathematical Society*, 17(10):2545–2593.

[48] Prévost, A. (2023). First passage percolation, local uniqueness for interlacements and capacity of random walk. *arXiv preprint arXiv:2309.03880*.

[49] Prévost, A., Rodriguez, P.-F., and Sousi, P. (2023). Phase transition for the late points of random walk. *arXiv preprint arXiv:2309.03192*.

[50] Ráth, B. and Sapozhnikov, A. (2011). On the transience of random interlacements. *Elect. Comm. in Probab.*

[51] Resnick, S. I. (2008). *Extreme values, regular variation, and point processes*, volume 4. Springer Science & Business Media.

[52] Rodriguez, P.-F. (2014). Level set percolation for random interlacements and the gaussian free field. *Stochastic Processes and their Applications*, 124(4):1469–1502.

[53] Rodriguez, P.-F. and Sznitman, A.-S. (2013). Phase transition and level-set percolation for the gaussian free field. *Communications in Mathematical Physics*, 320:571–601.

[54] Russo, L. (1981). On the critical percolation probabilities. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 56(2):229–237.

[55] Saberi, A. A. (2015). Recent advances in percolation theory and its applications. *Physics Reports*, 578:1–32.

[56] Sahimi, M. (1994). *Applications of percolation theory*. CRC Press.

[57] Schramm, O. (2000). Scaling limits of loop-erased random walks and uniform spanning trees. *Israel Journal of Mathematics*, 118(1):221–288.

[58] Smirnov, S. (2001). Critical percolation in the plane: conformal invariance, cardy's formula, scaling limits. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 333(3):239–244.

[59] Smirnov, S. and Werner, W. (2001). Critical exponents for two-dimensional percolation. *arXiv preprint math/0109120*.

[60] Sykes, M. F. and Essam, J. W. (1964). Exact critical percolation probabilities for site and bond problems in two dimensions. *Journal of Mathematical Physics*, 5(8):1117–1127.

[61] Sznitman, A.-S. (2010). Vacant set of random interlacements and percolation. *Annals of mathematics*, pages 2039–2087.

[62] Sznitman, A.-S. (2012). *Topics in occupation times and Gaussian free fields*, volume 16. European mathematical society.

[63] Sznitman, A.-S. (2015). Disconnection and level-set percolation for the gaussian free field. *Journal of the Mathematical Society of Japan*, 67(4):1801–1843.

[64] Sznitman, A.-S. (2017). Disconnection, random walks, and random interlacements. *Probability theory and related fields*, 167:1–44.

[65] Teixeira, A. and Tykesson, J. (2013). Random interlacements and amenability. *Annals of Applied Probability*.

[66] Teixeira, A. and Windisch, D. (2011). On the fragmentation of a torus by random walk. *Communications on pure and applied mathematics*, 64(12):1599–1646.

[67] Wierman, J. C. (1981). Bond percolation on honeycomb and triangular lattices. *Advances in Applied Probability*, 13(2):298–313.

[68] Zhao, Y. (2022). 18.226 probabilistic methods in combinatorics.