

Output-to-State Stability for systems on manifolds with multiple invariant sets

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Abstract—Output-to-State Stability (OSS) is a notion of detectability for nonlinear systems that is formulated in the ISS framework. We generalize the notion of OSS for systems evolving on manifolds and having multiple invariant sets. Building upon a recent extension of the Input-to-State Stability (ISS) theory for this very class of systems [1], the paper provides equivalent characterizations of the OSS property in terms of asymptotic estimates of the state trajectories and, in particular, in terms of existence of Lyapunov-like functions.

I. INTRODUCTION

As a central notion in control theory, detectability corresponds roughly to the property of having small states whenever the outputs are small, regardless of the initial state. The state of any detectable and stabilizable system can in fact be driven to zero using only output measurements. For this reason, detectability has proved its usefulness both in state- and output-feedback design, as well as in state estimation. While the concept of detectability is well defined and fully understood in the context of linear systems, the same does not hold for nonlinear systems where several possible formulations are available. One such formulation is obtained by dualizing the notion of Input-to-State Stability (ISS) [1]. Indeed, ISS formalizes the idea of having small states whenever the inputs are small, regardless of the initial state, and thus appears as a natural candidate for detectability by simply replacing inputs with outputs. The resulting notion is called Output-to-State Stability (OSS) [2] and embeds a nonlinear gain function of the infinity norm of the output to ultimately bound the norm of the state from above. Equivalent Lyapunov-like characterizations of OSS have been provided in [3] and [4].

At the same time, the analysis of the stability and robustness properties of nonlinear systems exhibiting a variety of non-trivial dynamical behaviors such as multiple equilibria, periodicity, almost-periodicity, chaos - in short, multistable systems - is becoming more and more appealing from the perspective of systems and control theory due to its importance for several scientific disciplines ranging from mechanics and electronics to biology and neuroscience. In a recent work [5], the ISS concept has been generalized for such multistable systems. In particular, the authors in [5] have shown that the most natural way of investigating stability properties of multistable systems by means of

Lyapunov-like dissipation inequalities is to relax the Lyapunov stability requirement on the invariant set of the system, and to incorporate instead relatively mild assumptions on the decomposition of such a set.

One may wonder if it is possible to dualize ISS also in this generalized context, so as to obtain an analogous property for OSS by simply using outputs instead of inputs in the definition. Indeed, this paper provides a notion of detectability for multistable systems which portrays the behavior of ultimately having small distance of the states from the invariant set of the system whenever the outputs are small. Specifically, we characterize such generalized OSS in terms of asymptotic estimates via nonlinear gain functions and in terms of Lyapunov/LaSalle-like dissipation inequalities.

Notation: The symbol $\mathfrak{d}[x_1, x_2]$ denotes the Riemannian distance between two points x_1 and x_2 of a Riemannian manifold M . For a set $D \subset M$ define $|\cdot|_D$ as $|x|_D = \inf_{a \in D} \mathfrak{d}[x, a]$. For a compact set $A \subset M$, $\partial(A)$ and $\text{int}\{A\}$ respectively denote the boundary and the interior of A . We denote the differential [6, Definition 4.2.5] of a smooth function $V : M \rightarrow \mathbb{R}$ by the covector field $dV : M \rightarrow T^*M$, and then we denote the Lie derivative [6, Definition 4.2.6] of V along a vector field $X : M \rightarrow TM$ at a point $x \in M$ by $\mathcal{L}_X V(x) := dV(X)(x) = dV(x) \cdot X(x)$, namely the action of dV on X at x . The norm induced by the Riemannian metric g on $T_x M$ is denoted as $|\cdot|_g$. Let $x_0 \in M$ be the “origin” element of M . Notations $|v|$ and $|x|$ respectively indicate the standard Euclidean norm of vector $v \in \mathbb{R}^n$ and the Riemannian distance of $x \in M$ from x_0 , namely $|x| = \mathfrak{d}[x, x_0]$. For a measurable function $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ define its infinity norm over the time interval $[t_1, t_2]$ as $\|d\|_{[t_1, t_2]} = \text{ess sup}_{t_1 \leq t \leq t_2} |d(t)|$, and denote $\|d\| := \|d\|_{[0, +\infty)}$. Define the infinity norm of $d(\cdot)$ with respect to a compact set S as follows: $\|d\|_S = \text{ess sup}_{t_1 \leq t \leq t_2} |d(t)|_S$, and define $\|d\|_S := \|d\|_{[0, +\infty)}|_S$. We define the saturation function as $\text{sat}(x) := \text{sign}\{x\} \min\{1, |x|\}$.

II. DEFINITIONS AND MAIN RESULT

Let M be a geodesically complete, connected, n -dimensional Riemannian manifold without boundary. Let Y be a closed subset of $\mathbb{R}^l$ containing the origin. Consider the autonomous system:

$$\dot{x}(t) = f(x(t)) \quad , \quad y(t) = h(x(t)) \quad , \quad (1)$$

with state x taking values in M , output y taking values in $Y \subseteq \mathbb{R}^l$, and where $f(x) : M \rightarrow T_x M$ is a locally Lipschitz continuous vector field, and $h(x) : M \rightarrow Y$ is a locally Lipschitz continuous mapping. Denote by $X(t, x)$

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the uniquely defined solution of (1) at time t fulfilling $X(0, x) = x$. Let y denote the signal $y(\cdot) = h(X(\cdot, x))$. We then make the following Assumption.

Assumption 1: Without loss of generality, all solutions of the unperturbed system (1) are assumed to be defined for all $t \in \mathbb{R}^1$. At the same time, all α - and ω - limit sets of (1) are assumed to be compact (and possibly empty).

We then let $\mathcal{W}$ be a compact invariant sets which contains all α - and ω -limit sets of (1). Crucial in the stability analysis by means of Lyapunov-like analytical tools are the notions of decomposition for a compact invariant set and of absence of cycles in the decomposition itself, as detailed in the following definitions.

Definition 1: Let $\Lambda \subset M$ be a compact and invariant set for (1). A *decomposition* of Λ is a finite, disjoint family of compact invariant sets $\Lambda_1, \dots, \Lambda_K$ such that

$$\Lambda = \bigcup_{i=1}^K \Lambda_i.$$

For an invariant set Λ , its attracting and repulsing subsets are defined as follows:

$$\begin{aligned} \mathfrak{A}(\Lambda) &= \{x \in M : |X(t, x)|_\Lambda \rightarrow 0 \text{ as } t \rightarrow +\infty\}, \\ \mathfrak{R}(\Lambda) &= \{x \in M : |X(t, x)|_\Lambda \rightarrow 0 \text{ as } t \rightarrow -\infty\}. \end{aligned}$$

Given two invariant and disjoint subsets $\mathcal{W}_1, \mathcal{W}_2 \subset \mathcal{W}$, we define the relation $\mathcal{W}_1 \prec \mathcal{W}_2$ if $\mathfrak{A}(\mathcal{W}_1) \cap \mathfrak{R}(\mathcal{W}_2) \neq \emptyset$ (this relation implies that there is a trajectory going backward to set $\mathcal{W}_2$ and forward to set $\mathcal{W}_1$).

Definition 2: Let $\Lambda_1, \dots, \Lambda_K$ be a decomposition of Λ .

- 1) An r -cycle ($r \geq 2$) is an ordered r -tuple of distinct indices $i_1, \dots, i_r$ such that $\Lambda_{i_1} \prec \dots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
- 2) A 1-cycle is an index i such that $[\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i)] - \Lambda_i \neq \emptyset$.
- 3) A *filtration ordering* is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

Existence of an r -cycle for (1) with $r \geq 2$ is equivalent to existence of a heteroclinic cycle, and existence of a 1-cycle implies existence of a homoclinic orbit. Typically, system (1) is required to satisfy the following assumptions on $\mathcal{W}$.

Assumption 2 (No cycle condition): The set $\mathcal{W}$ admits a finite decomposition, i.e. $\mathcal{W} = \bigcup_{i=1}^K \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, which shows no cycle between the $\mathcal{W}_i$ s and which satisfies a filtration ordering, as detailed in Definitions 1 and 2. Under the specified assumptions, the set $\mathcal{W}$ is said to *satisfy the no-cycle condition under the flow of (1)*.

Assumption 3: Assume that the set $\mathcal{Z} := \{z \in M : z \in \mathfrak{A}(\mathcal{W}_i) \cap \mathfrak{R}(\mathcal{W}_j) \text{ for some } i < j\}$ is compact.

Once the set $\mathcal{W}$ has been specified and ascertained to satisfy Assumptions 1, 2, and 3, a further assumption needs to be made on the output mapping h , namely we require $h(x) = 0$ whenever $x \in \mathcal{W}$. We are now going to list several properties which we are interested in comparing throughout this work.

¹Without loss of generality, system (1) can be made backward and forward complete by slowing down the dynamics with $\dot{x} = \frac{1}{1+|x|} f(x)$.

Definition 3 (O-pAG): System (1) is said to satisfy the *output practical asymptotic gain* (O-pAG) property if, for some $\eta_y \in \mathcal{K}$ and a positive constant q , the following estimate holds for all $t \geq 0$ and all $x \in M$:

$$\limsup_{t \rightarrow +\infty} |X(t, x)|_{\mathcal{W}} \leq \eta_y(\|y\|) + q. \quad (2)$$

If $q = 0$, then we say that the *output asymptotic gain* (O-AG) property holds.

Definition 4 (O-LIM): System (1) is said to satisfy the *output limit* (O-LIM) property if, for some $\mu_y \in \mathcal{K}$, the following estimate holds for all $x \in M$:

$$\inf_{t \geq 0} |X(t, x)|_{\mathcal{W}} \leq \mu_y(\|y\|). \quad (3)$$

Definition 5 (O-pGS): System (1) is said to satisfy the *output practical global stability* (O-pGS) property if there exists class- $\mathcal{K}_\infty$ functions β_0, β_y and a constant $q \geq 0$ such that the following estimate holds for all $t \geq 0$ and all $x \in M$:

$$|X(t, x)|_{\mathcal{W}} \leq \max\{\beta_0(|x|_{\mathcal{W}}), \beta_y(\|y_{[0,t]}\|)\} + q. \quad (4)$$

If $q = 0$, then we say that the *output global stability* (O-GS) property holds.

Definition 6 (GATTMO): System (1) is said to satisfy the *global attractivity modulo output* (GATTMO) property if there exist a function $\rho \in \mathcal{K}_\infty$ such that the following property holds:

$$\begin{aligned} \forall x \in M \quad \forall \varepsilon > 0 \quad \exists T_{x,\varepsilon} > 0 : \quad & \forall T \in [T_{x,\varepsilon} + \infty) \\ \text{if } \left(\forall t \in [0, T], |X(t, x)|_{\mathcal{W}} \geq \rho(|h(X(t, x))|) \right), & \quad (5) \\ \text{then } \left(\forall t \in [T_{x,\varepsilon}, T], |X(t, x)|_{\mathcal{W}} \leq \varepsilon \right). & \quad (6) \end{aligned}$$

Definition 7 (OSS-Lyapunov function): Any function $V : M \rightarrow \mathbb{R}$ is called a *practical OSS-Lyapunov function* for system (1) if it satisfies the following two properties:

- there exist a constant $c \geq 0$, and class- $\mathcal{K}_\infty$ functions α_1, α_2 such that:

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{W}} + c) \quad \forall x \in M; \quad (7)$$

- V is differentiable along trajectories of (1), namely $V(X(t, x))$ is differentiable in $t \in \mathbb{R}$ for all $x \in M$. Furthermore, there exist class- $\mathcal{K}_\infty$ functions α, γ_y and a constant $q_0 \geq 0$ such that

$$\frac{d}{dt} V(X(t, x)) \leq -\alpha(|X(t, x)|_{\mathcal{W}}) + \gamma_y(\|h(X(t, x))\|) + q_0 \quad (8)$$

for all $t \geq 0$ and all $x \in M$.

If (8) holds for $q_0 = 0$, then V is called an *OSS-Lyapunov function*. If, in addition, the set $\bigcup_{x \in \mathcal{W}_i} \{V(x)\}$ is a singleton for all $i \in \{1, \dots, K\}$, then V is called an *OSS-Lyapunov function constant on invariant sets*.

Theorem 1: Consider system (1) and let Assumptions 1 and 2 be satisfied for the compact invariant set $\mathcal{W}$ containing all α - and ω -limit sets. Then the following properties are equivalent for system (1):

- 1) O-AG;
- 2) O-pAG;
- 3) O-LIM and O-pGS;
- 4) GATTMO;

- 5) existence of an OSS-Lyapunov function.
- 6) existence of a practical OSS-Lyapunov function.
- 7) existence of an OSS-Lyapunov function constant on invariant sets.

Proof: For implications proving sufficiency, i.e. $5 \Rightarrow 3$, $3 \Rightarrow 2$, $7 \Rightarrow 1$, $2 \Rightarrow 1$, and $6 \Rightarrow 2$, the proofs are omitted due to space constraints (however, we refer the reader to [7]). Implications $7 \Rightarrow 5$, and $5 \Rightarrow 6$ are trivial. Necessity of the existence of an OSS-Lyapunov function, i.e. implications $1 \Rightarrow 4$ and $4 \Rightarrow 7$, is proved throughout Section III. ■

Definition 8: System (1) is said to be *OSS in the multistable sense* with respect to the set $\mathcal{W}$ if it satisfies the properties in Theorem 1.

Remark 1: Suppose system (1) is such that $h(x) \equiv 0$ for all $x \in M$. Then, Theorem 1 provides necessary and sufficient conditions for the global attractivity with zero inputs (zero-GATT) property, i.e. $\lim_{t \rightarrow +\infty} |X(t, x)|_{\mathcal{W}} = 0$ for all $x \in M$. In particular, we retrieve the existence of a lower-semicontinuous proper Lyapunov function constant on invariant sets, class- $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha$ and a constant $c \geq 0$ satisfying (7) and $\dot{V}(x) \leq -\alpha(|x|_{\mathcal{W}})$ for all $x \in M$.

III. EXISTENCE OF OSS-LYAPUNOV FUNCTION

This Section addresses the construction of a lower semicontinuous OSS-Lyapunov function by using Assumptions 1 and 2 and the O-AG property. We omit the proofs of Lemmas 4, 5, 6, 7, and 8 due to space constraints.

A. Asymptotic stability in sectors

Lemma 1: O-AG $\Rightarrow$ GATTMO.

Remark 2: The underlying assumption in all subsequent results of this Section is the fulfillment of the GATTMO property for (1).

In this Subsection, we consider a particular definition of invariant sets $\bar{A}_i, A_i, B_i, C_i$, and D_i , which follows along the lines of [8] and [5]. In particular, we will show that the GATTMO property is a sufficient condition for the Global Asymptotic Stability Modulo Output (GASMO) property - that is, classical OSS - of sets A_i in C_i , and for the GASMO property modulo infinity of sets B_i in D_i . Define the sets:

$$\begin{aligned}\mathcal{D} &:= \{x \in M : |x|_{\mathcal{W}} \leq \rho(|h(x)|)\}, \\ \mathcal{E} &:= M \setminus \mathcal{D}, \\ \mathcal{E}_0 &:= \mathcal{E} \cup \partial\mathcal{D}.\end{aligned}$$

Define for all $x \in \mathcal{E}_0$ the first hitting time $\lambda(x)$ as:

$$\lambda(x) := \inf \{t \geq 0 : X(t, x) \in \mathcal{D}\}.$$

Lemma 2: $\mathcal{W} \subseteq \mathcal{D}$.

Proof: $x \in \mathcal{W} \Rightarrow |x|_{\mathcal{W}} = 0 \Rightarrow |x|_{\mathcal{W}} \leq \rho(|h(x)|) \Rightarrow x \in \mathcal{D}$. ■

Lemma 3: For all $x \in \mathcal{E}$ it holds $\lambda(x) < +\infty$ or $\lim_{t \rightarrow +\infty} |X(t, x)|_{\mathcal{W}} = 0$.

Proof: Assume $\lambda(x) = +\infty$ for some $x \in \mathcal{E}$. Then, by definition of $\lambda(x)$, it holds that $X(t, x) \notin \mathcal{D}$ for all $t \geq 0$ which implies $|X(t, x)|_{\mathcal{W}} > \rho(|h(X(t, x))|)$ for all $t \geq 0$. By GATTMO, $\lim_{t \rightarrow +\infty} |X(t, x)|_{\mathcal{W}} = 0$. ■

Corollary 1: If $\lim_{t \rightarrow +\infty} |X(t, x)|_{\mathcal{W}} = +\infty$, then $\lambda(x) < +\infty$ and there exists $\bar{t} > 0$ such that $X(\bar{t}, x) \notin \mathcal{E}_0$.

Proof: $\lambda(x) < +\infty$ follows trivially from Lemma 3. In order to prove the existence of such $\bar{t}$, assume by contradiction that for all $\bar{t} > 0$ we have $X(\bar{t}, x) \in \mathcal{E}_0$. Then, by GATTMO property, $\limsup_{t \rightarrow +\infty} |X(t, x)|_{\mathcal{W}} = 0$, which contradicts the hypothesis $\lim_{t \rightarrow +\infty} |X(t, x)|_{\mathcal{W}} = +\infty$. ■

We define:

$$\begin{aligned}\mathfrak{A}(\infty) &:= \left\{x \in M : \lim_{t \rightarrow +\infty} |X(t, x)| = +\infty\right\} \\ \mathfrak{R}(\infty) &:= \left\{x \in M : \lim_{t \rightarrow -\infty} |X(t, x)| = +\infty\right\}.\end{aligned}$$

Consider the following sets for any $1 \leq i \leq K$:

$$A_i := \bigcup_{j \leq i} \mathfrak{R}(\mathcal{W}_j), \quad B_i := \bigcup_{i < l} \mathfrak{A}(\mathcal{W}_l), \quad \bar{A}_i := A_i \setminus \mathfrak{A}(\infty) \quad (9)$$

$$C_i := \mathfrak{A}(\infty) \cup \bigcup_{j \leq i} \mathfrak{A}(\mathcal{W}_j), \quad D_i := \mathfrak{R}(\infty) \cup \bigcup_{i < l} \mathfrak{R}(\mathcal{W}_l). \quad (10)$$

We are now going to prove several interesting properties which hold true for the sets $A_i, B_i, C_i, D_i, \bar{A}_i$.

- 1) *Complementarity.* $C_i = M \setminus B_i$ and $D_i = M \setminus A_i$. Indeed, by Assumption 1 it holds $M = \mathfrak{A}(\infty) \cup \bigcup_{1 \leq i \leq K} \mathfrak{A}(\mathcal{W}_i)$. Applying the same arguments to the backward flow proves that $D_i = M \setminus A_i$.
- 2) *Invariance, boundedness and closedness.* By their definition, the sets $A_i, B_i, C_i, \bar{A}_i$ are invariant for the dynamics (1). Lemma 5 in [8] proves that C_i is an open set containing A_i . Being C_i the complement of B_i in M , it follows that B_i is closed. Applying the same arguments to the backward flow proves that A_i is closed and D_i is an open set containing B_i . Observe that M is the disjoint union of the $\mathfrak{R}(\mathcal{W}_j)$ and $\mathfrak{R}(\infty)$, and thus we can write $\bar{A}_i = A_i \setminus (\mathfrak{R}(\infty) \cup \mathfrak{A}(\infty))$. Moreover, the complement of $\mathfrak{R}(\infty) \cup \mathfrak{A}(\infty)$ is $\mathcal{Z}$. For this reason, we conclude that $\bar{A}_i$ is compact².
- 3) *Inclusion.* $\bar{A}_i \subseteq A_i \subseteq C_i$ and $B_i \subseteq D_i$. Indeed, the set $\mathfrak{R}(\mathcal{W}_j)$ with $1 \leq j \leq K$ consists of $\mathcal{W}_j$ together with all the initial conditions repelled by $\mathcal{W}_j$. By the filtration ordering definition, all such initial conditions are attracted either to ∞ , thus lie in $\mathfrak{A}(\infty)$, or by some $\mathcal{W}_h$ with $h < j$, thus lie in $\mathfrak{A}(\mathcal{W}_h)$. We can then write:

$$\mathfrak{R}(\mathcal{W}_j) \subseteq \mathcal{W}_j \cup \mathfrak{A}(\infty) \cup \bigcup_{h < j} \mathfrak{A}(\mathcal{W}_h) \subset \mathfrak{A}(\infty) \cup \bigcup_{h \leq j} \mathfrak{A}(\mathcal{W}_h) \quad (11)$$

By iterating (11) for all $j \leq i$, we obtain:

$$A_i = \bigcup_{j \leq i} \mathfrak{R}(\mathcal{W}_j) \subset \bigcup_{j \leq i} \left(\mathfrak{A}(\infty) \cup \bigcup_{h \leq j} \mathfrak{A}(\mathcal{W}_h) \right) = C_i. \quad (12)$$

Applying the same arguments to the backward flow proves that $B_i \subseteq D_i$.

- 4) *Attractivity.* For all $x \in C_i$, it holds $\lambda(x) < +\infty$ or $\lim_{t \rightarrow +\infty} |X(t, x)|_{\bar{A}_i} = 0$. Indeed, by definition of C_i , all initial conditions in C_i are attracted to ∞ , hence

²In this passage, we made use of Assumption 3, which we intend to relax in further works.

$\lambda(x) < +\infty$ by Lemma 1, or to some $\mathcal{W}_j$ with $j \leq i$, hence to $\bar{A}_i$.

- 5) *Repulsion.* For all $x \in D_i$, it holds $\lim_{t \rightarrow -\infty} |X(t, x)| = +\infty$ or $\lim_{t \rightarrow -\infty} |X(t, x)|_{B_i} = 0$. Indeed, by definition of D_i , all initial conditions in D_i are repelled from ∞ , or from some $\mathcal{W}_l$ with $l > i$, hence from B_i .

Lemma 4: (Existence of a $\mathcal{KL}$ bound given GASMO of $\bar{A}_i$ in a closed subset of C_i) Let Assumptions 1 and 2 hold true. Assume furthermore the GATTMO property as in Definition 6. Then, for any closed subset $\bar{C}_i \subset C_i$ there exists a class $\mathcal{KL}$ function β such that, for all x in a compact subset of $(\mathcal{E} \cap \bar{C}_i) \cup \bar{A}_i$,

$$|X(t, x)|_{\bar{A}_i} \leq \beta(|x|_{\bar{A}_i}, t) \quad \text{for all } t \in [0, \lambda(x)).$$

Consider the backward flow:

$$\dot{x}(t) = f^*(x(t), 0) = -f(x(t), 0), \quad (13)$$

with f as in (1). Let $X^*(t, x)$ and $\varphi_t^*(x)$ equivalently denote the uniquely defined solution of (13) for all $t \in \mathbb{R}$ with initial condition $x \in M$. Clearly, it holds $X^*(t, x) = X(-t, x)$ and $\varphi_t^*(x) = \varphi_{-t}(x)$. Define the following norm:

$$\mathbb{D}(x) := \min \left\{ |x|_{B_i}, \frac{1}{|x|} \right\}. \quad (14)$$

Furthermore, define for all $x \in \mathcal{E}_0$ the first hitting time $\lambda^*(x)$:

$$\lambda^*(x) := \inf_{t \geq 0} \{t \geq 0 : X^*(t, x) \in \mathcal{D}\}. \quad (15)$$

Lemma 5: (Existence of a $\mathcal{KL}$ bound given GASMO modulo infinity of B_i in a closed subset of D_i) Let Assumptions 1 and 2 hold true. Assume furthermore the GATTMO property as in Definition 6. Then, for any closed subset $\bar{D}_i \subset D_i$ there exists a class $\mathcal{KL}$ function β such that, for all x in a compact subset of $(\mathcal{E} \cap \bar{D}_i) \cup B_i$,

$$\mathbb{D}(X^*(t, x)) \leq \beta(\mathbb{D}(x), t) \quad \text{for all } t \in [0, \lambda^*(x)).$$

B. OSS-Lyapunov functions in sectors

We bring to the attention of the reader that the GASMO property being satisfied in C_i and D_i (the latter in the modulo-infinity sense) respectively allows the construction of classical OSS-Lyapunov functions in C_i and D_i (in the modulo-infinity sense) which follows along the lines of [4]. The existence of such OSS-Lyapunov functions is proved by Lemmas 6 and 7. Let $\mathcal{E}_3 := \{x \in M : |x|_{\mathcal{W}} \geq 3\rho(|h(x)|)\}$.

Lemma 6: (Existence of an OSS-Lyapunov function in a closed subset of C_i) Let the assumptions of Lemma 4 hold true. Then, for any closed subset $\bar{C}_i \subset C_i$, there exists a lower-semicontinuous function $V_1 : \bar{C}_i \rightarrow \mathbb{R}_{\geq 0}$ and class- $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \beta_1$ such that the following two properties hold:

$$\alpha_1(|x|_{\bar{A}_i}) \leq V_1(x) \leq \alpha_2(|x|_{\bar{A}_i}) \quad \forall x \in \bar{C}_i \quad (16)$$

$$\left. \frac{d}{dt} V_1(X(t, x)) \right|_{t=0} \leq -\beta_1(|x|_{\bar{A}_i}) \quad \forall x \in (\mathcal{E}_3 \cap \bar{C}_i) \cup \bar{A}_i. \quad (17)$$

In particular, $\left. \frac{d}{dt} V_1(X(t, x)) \right|_{t=0} = 0$ whenever $x \in \bar{A}_i$.

Lemma 7: (Existence of an OSS-Lyapunov function in a closed subset of D_i) Let the assumptions of Lemma 5 hold

true. Then, for any closed subset $\bar{D}_i \subset D_i$, there exists a lower-semicontinuous function $V_2 : \bar{D}_i \rightarrow \mathbb{R}_{\geq 0}$ and of class- $\mathcal{K}_\infty$ functions $\alpha_3, \alpha_4, \beta_2$ such that the following two properties hold:

$$\alpha_3(\mathbb{D}(x)) \leq V_2(x) \leq \alpha_4(\mathbb{D}(x)) \quad \forall x \in \bar{D}_i \quad (18)$$

$$\left. \frac{d}{dt} V_2(X^*(t, x)) \right|_{t=0} \leq -\beta_2(\mathbb{D}(x)) \quad \forall x \in (\mathcal{E}_3 \cap \bar{D}_i) \cup B_i. \quad (19)$$

In particular, $\left. \frac{d}{dt} V_2(X^*(t, x)) \right|_{t=0} = 0$ whenever $x \in B_i$.

We are now going to verify that the contributions of functions V_1 and V_2 can be combined to form an OSS-Lyapunov function L_i which is defined over the whole manifold M and vanishes on $\bar{A}_i$. Indeed, recall that $B_i = M \setminus C_i$ and that $A_i = M \setminus D_i$. Furthermore, $\text{int}\{C_i \setminus \bar{A}_i\} \neq \emptyset$ and $\text{int}\{D_i \setminus B_i\} \neq \emptyset$. Therefore, we can define $\rho_{max}^i := \inf_{a \in A_i, b \in B_i} \delta[a, b]$ with the property $\rho_{max}^i > 0$. Choose $\rho_1, \rho_2 \in \mathbb{R}$ so that $0 < \rho_1 < \rho_2 < \rho_{max}^i$. Select the following closed set:

$$\begin{aligned} \bar{C}_i &:= \{x \in C_i : |x|_{A_i} \leq \rho_2\} \\ &\supseteq \{x \in C_i : |x|_{\bar{A}_i} \leq \rho_2\} \end{aligned} \quad (20)$$

$$\bar{D}_i := D_i \cap \{x \in M : |x|_{\bar{A}_i} > (\alpha_2^{-1} \circ \alpha_1)(\rho_1)\}. \quad (21)$$

By making use of the distributivity property of operations $\cup$ and $\cap$, it is straightforward to prove that:

$$\bar{C}_i \cup \bar{D}_i = M, \quad \forall i \in \{1, \dots, K\}.$$

Lemma 8: Let the assumptions of Lemmas 6 and 7 hold true. Then, for all $i \in \{1, \dots, K\}$, there exists a lower-semicontinuous function $L_i : M \rightarrow \mathbb{R}_{\geq 0}$ which satisfies the following two properties:

- 1) there exists a class- $\mathcal{K}$ function α_1^i such that:

$$\alpha_1^i(|x|_{\bar{A}_i}) \leq L_i(x) \quad \forall x \in M.$$

In particular, $L_i(x) = 0$ whenever $x \in \bar{A}_i$ and there exists a constant $b^i > 0$ such that $L_i(x) = b^i$ whenever $x \in B_i$.

- 2) $L_i(X(t, x))$ is continuously differentiable in $t \geq 0$ for all $x \in \mathcal{E}_3 \cup \bar{A}_i \cup B_i$. Furthermore, there exists a class- $\mathcal{KL}$ function β^i such that:

$$\left. \frac{d}{dt} L_i(X(t, x)) \right|_{t=0} \leq -\beta^i(|x|_{G_i}, |x|), \quad (22)$$

for all $x \in \mathcal{E}_3 \cup \bar{A}_i \cup B_i$, with $G_i := \bar{A}_i \cup B_i$. In particular, $\left. \frac{d}{dt} L_i(X(t, x)) \right|_{t=0} = 0$ whenever $x \in G_i := \bar{A}_i \cup B_i$.

C. An OSS-Lyapunov function wrt $\mathcal{W}$ on $\mathcal{E}_3$

In this Subsection, we are going to aggregate the contributions of the L_i s in Lemma 8 to construct an OSS-Lyapunov function with respect to the set $\mathcal{W}$ which decreases along the trajectories staying in $\mathcal{E}_3 \cup \mathcal{W}$.

Instrumental in such construction will be the case $i = K$. Indeed, we have:

$$\bar{A}_K = (\mathfrak{R}(\mathcal{W}_1) \cup \dots \cup \mathfrak{R}(\mathcal{W}_K)) \setminus \mathfrak{A}(\infty)$$

$$C_K = \mathfrak{A}(\infty) \cup \mathfrak{A}(\mathcal{W}_1) \cup \dots \cup \mathfrak{A}(\mathcal{W}_K) = M$$

$$D_K = B_K = \emptyset.$$

Remark 3: The set $\bar{A}_K$ comprises all orbits of system (1) which go to some $\mathcal{W}_j$ backward in time, and to some $\mathcal{W}_i$ forward in time, with $1 \leq i < j \leq K$. In particular, if $\mathcal{W}$ is globally attractive on the whole M , it could be shown that $\bar{A}_K$ coincides with the Milnor attractor [5, equation (14)].

Select the closed set $\bar{C}_K = C_K = M$. Lemma 6 proves the existence of a lower semicontinuous Lyapunov function $V_K : M \rightarrow \mathbb{R}_{\geq 0}$ and of class- $\mathcal{K}_\infty$ functions $\alpha_1^K, \alpha_2^K, \hat{\beta}^K$ such that $\alpha_1^K(|x|_{\bar{A}_K}) \leq V_K(x) \leq \alpha_2^K(|x|_{\bar{A}_K})$ for all $x \in M$ and

$$\left. \frac{d}{dt} V_K(X(t, x)) \right|_{t=0} \leq -\hat{\beta}^K(|x|_{\bar{A}_K}) \quad (23)$$

for all $x \in \mathcal{E}_3 \cup \bar{A}_i$.

Let $L : M \rightarrow \mathbb{R}_{\geq 0}$ be the Lyapunov function defined as:

$$L(x) := V_K(x) + \sum_{i=1}^{K-1} L_i(x). \quad (24)$$

Note that $\bar{A}_1 = \mathfrak{R}(\mathcal{W}_1) \setminus \mathfrak{A}(\infty)$ and thus, by the definition of filtration ordering, it holds $\bar{A}_1 = \mathcal{W}_1$. In terms of distance norms, we will then have $|x|_{\mathcal{W}} \leq |x|_{\bar{A}_1}$ for all $x \in M$, and therefore:

$$L(x) \geq \sum_{i=1}^{K-1} \alpha_1^i(|x|_{\bar{A}_i}) + \alpha_1^K(|x|_{\bar{A}_K}) \geq \alpha_1^1(|x|_{\mathcal{W}}) + \alpha_1^K(|x|_{\bar{A}_K})$$

Since $\alpha_1^1(|x|_{\mathcal{W}})$ is a class- $\mathcal{K}$ function while α_1^K is a class- $\mathcal{K}_\infty$ and, by compactness of $\mathcal{K}$ and $\bar{A}_K$, it holds that $|x|_{\mathcal{W}} \leq |x|_{\bar{A}_K} + \nu_K$, we can define the class- $\mathcal{K}_\infty$ function $\alpha_1(s) := \alpha_1^1(s)$ for $s \leq \nu_K$ and $\alpha_1(s) := \alpha_1^1(s) + \alpha_1^K(s - \nu_K)$ for $s > \nu_K$. It follows that α_1 is a class- $\mathcal{K}_\infty$ function so that

$$\alpha_1(|x|_{\mathcal{W}}) \leq L(x) \quad \forall x \in M. \quad (25)$$

Pick any $\mathcal{W}_j$ with $j \in \{1, \dots, K\}$. Then, for any $i \in \{1, \dots, K\}$, it holds $\mathcal{W}_j \in \bar{A}_i$ or $\mathcal{W}_j \in B_i$. By virtue of property 1) in Lemma 8 and inequality (16), the L_i s and V_K vanish on the $\bar{A}_i$ s and $\bar{A}_K$ respectively, and, moreover, the L_i s take constant values on the B_i s. It then follows that the L_i s and V_K take constant values on $\mathcal{W}_j$, and we thus conclude that L is constant on any $\mathcal{W}_j$.

Moreover, the definition of $G_i := \bar{A}_i \cup B_i$ implies that $G_K = \bar{A}_K$ and, in particular, that $G_i \supseteq \mathcal{W}$ for any $i \in \{1, \dots, K\}$. Then, by making use of property 2) in Lemma 8 and dissipation (23), we conclude that

$$\left. \frac{d}{dt} L(X(t, x)) \right|_{t=0} = 0 \quad \forall x \in \mathcal{W}. \quad (26)$$

Observe that, whenever $x \notin \mathcal{W}$, there exists some $i \in \{1, \dots, K\}$ such that $|x|_{G_i} > 0$. Viceversa, since $|x|_{G_i} \leq |x|_{\mathcal{W}}$ for any $i \in \{1, \dots, K\}$, it holds that $x \notin \mathcal{W}$ whenever $x \notin G_i$. Therefore, there exist a $\mathcal{KL}$ function ϖ' and a continuous positive definite function ϖ such that

$$\hat{\beta}^K(|x|_{G_K}) + \sum_{i=1}^{K-1} \beta^i(|x|_{G_i}, |x|) \geq \varpi'(|x|_{\mathcal{W}}, |x|) \geq \varpi(|x|_{\mathcal{W}})$$

for all $x \in M$, where last inequality follows by $|x| \leq |x|_{\mathcal{W}} + \nu_W$ with $\nu_W \geq 0$. Consider the following $\mathcal{K}_\infty$ function:

$$\beta(s) := \begin{cases} \frac{s}{1+s} \inf_{s \leq r \leq \nu_K} \varpi(r) & \text{if } s \leq \nu_K \\ \frac{\nu_K}{1+\nu_K} \varpi(c) + \hat{\beta}^K(s - \nu_K) & \text{if } s > \nu_K. \end{cases}$$

Then, it holds that:

$$\left. \frac{d}{dt} L(X(t, x)) \right|_{t=0} \leq -\beta(|x|_{\mathcal{W}}) \quad , \quad \forall x \in \mathcal{E}_3 \cup \mathcal{W}. \quad (27)$$

D. Extending the dissipation to the rest of the state space

Observe that $L(x)$ is the composition of functions which are locally Lipschitz continuous along the trajectories of system (1), namely $\frac{d}{dt} L(X(t, x))$ is locally Lipschitz continuous in t . For this reason and due to invariance of $\mathcal{W}$, we can always find a class- $\mathcal{K}_\infty$ function σ_0 such that:

$$\left. \frac{d}{dt} L(X(t, x)) \right|_{t=0} \leq \sigma_0(|x|_{\mathcal{W}}) \quad (28)$$

for all x such that $|x|_{\mathcal{W}} \leq 3\rho(|h(x)|)$. Let $\sigma(r) := \sigma_0(3r) + \beta(3r)$ for all $r \geq 0$. Observe that $\sigma_0(|x|_{\mathcal{W}}) \leq -\beta(|x|_{\mathcal{W}}) + \sigma(\rho(|h(x)|))$. It then follows from (27) and (28) that:

$$\left. \dot{L}(X(t, x)) \right|_{t=0} \leq -\beta(|x|_{\mathcal{W}}) + \sigma(\rho(|h(x)|)) \quad , \quad \forall x \in M. \quad (29)$$

We have thus proved that L is a lower-semicontinuous OSS-Lyapunov function constant on invariant sets.

IV. EXAMPLES

A. Example 1

Consider the following illustrative nonlinear system:

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_1 \\ \dot{x}_2 &= -|1 - x_1| x_2 + |x_1| x_2 \\ y &= x_2, \end{aligned} \quad (30)$$

whose phase portrait is depicted in Figure 1. The $\mathcal{W}$ -limit set for (30) is given by $\mathcal{W} := \{(-1, 0), (0, 0), (1, 0)\}$. Thus, the distance function will be analytically given as

$$|x|_{\mathcal{W}} = \sqrt{\min\{|x_1 - 1|, |x_1 + 1|, |x_1|\}^2 + x_2^2}.$$

The x_1 subsystem evolves on the real line, therefore cannot admit cycles, thus $\mathcal{W}$ satisfies Assumption 2. We are now going to select a OSS-Lyapunov function for 30. Typically, in the context of multistable systems, one selects Lyapunov functions having local minima at sinks and local maxima at sources. In our case, let:

$$V(x_1, x_2) := \frac{1}{4}(x_1 - 1)^2(x_1 + 1)^2 + x_2^2,$$

and we observe that $\alpha_1(|x|_{\mathcal{W}}) \leq V(x)$ with $\alpha_1(r) := \frac{1}{2}r^2$. Taking the time derivative of V along the trajectories of system (30) yields:

$$\begin{aligned} DV(x)f(x) &= -x_1^2(x_1 - 1)^2(x_1 + 1)^2 + x_2^2(|1 - x_1| - |x_1|) \\ &\leq -x_1^2(x_1 - 1)^2(x_1 + 1)^2 - x_2^2 + 2x_2^2 \\ &\leq \alpha(|x|_{\mathcal{W}}) + \gamma_y(|y|). \end{aligned} \quad (31)$$

with $\alpha(r) := \frac{1}{2}r^2$ and $\gamma_y(r) := 2r^2$. By virtue of Theorem 1, inequality (31) proves that the system is OSS with respect to the set $\mathcal{W}$ and output y . Figure 1 shows that any trajectory of the phase portrait of (30) with $y(t) = x_2(t) \equiv 0$ for all $t \geq 0$, that is any trajectory with $\|y\| = 0$, indeed converges to $\mathcal{W}$. In particular, Figure 1 shows that any trajectory satisfies the GATTMO property by eventually entering the set $\mathcal{D} = \{x \in M : |x|_{\mathcal{W}} \leq \rho(|h(x)|)\}$, with $\rho(s) := 3s^2$ for all $s \geq 0$.

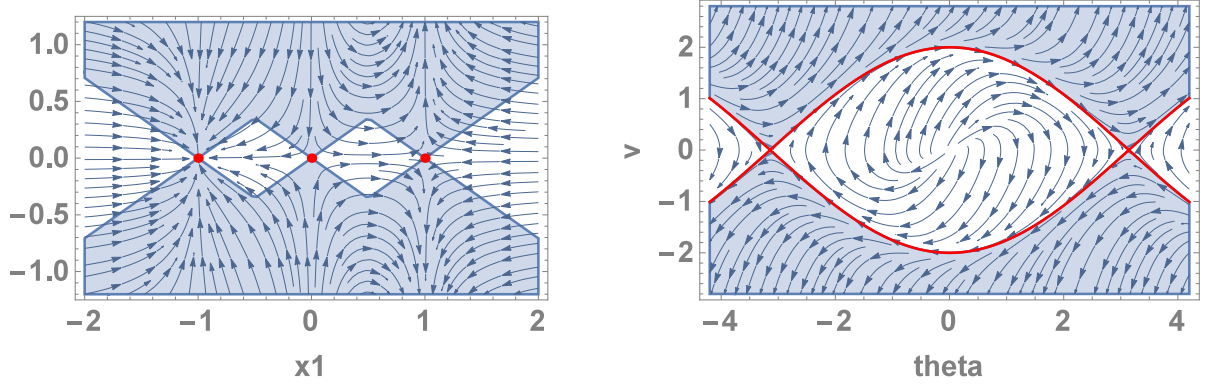


Fig. 1. Phase plot of the dynamics (30) and (32) (left and right, respectively). The sets $\mathcal{D}$ and $\mathcal{W}$ are respectively depicted in light shaded blue and red.

B. Example 2

Consider the following pendulum-like nonlinear system whose state space $x = (\theta, v)$ evolves on $M = \mathbb{S} \times \mathbb{R}$:

$$\begin{aligned}\dot{\theta} &= v \\ \dot{v} &= -\sin \theta + v|L(x)| \\ y &= h(x) = \sqrt{(\text{dist } \theta)^2 + v^2} R(-L(x)),\end{aligned}\quad (32)$$

where $L(x) := 1 - v^2/2 + \cos \theta$, $\text{dist}(\theta) := \arccos \cos \theta$ is the distance on $\mathbb{S}$, and R is the unitary ramp function, i.e. $R(s) := s$ for $s \geq 0$ and $R(s) := 0$ for $s < 0$. The phase portrait of system (32) is depicted in Figure 1. Note that the system comprises a fixed point at the origin $x = 0$ and an homoclinic trajectory to $(\pi, 0)$, that is $\Gamma := \{x \in M : L(x) = 0\}$. For ease of presentation, we denote $\Gamma_{<0} := \{x \in M : L(x) < 0\}$.

Our OSS-Lyapunov candidate is $V(x) := L(x)^2/2$. Note that $L(\Gamma) = 0$, $|L(x)| > 0$ for all $x \in M \setminus \Gamma$, and the sublevel sets $\{x \in M : |L(x)| \leq a\}$ are bounded for every $a \geq 0$, therefore there exists a class- $\mathcal{K}$ function α_Γ such that $|L(x)| \geq \alpha_\Gamma(|x|_\Gamma)$ and $V(x) \geq \alpha_\Gamma(|x|_\Gamma)^2/2$ for all $x \in M$. Taking the time derivative of $V(x)$ along the trajectories of (32) yields: $\dot{V}(x) = -v^2 L(x)|L(x)|$. Now observe that there exists no trajectory with initial condition on $\theta \neq 0, \pi$ such that $v(t) \equiv 0$ for all $t \geq 0$. In fact, trajectories of (32) are perpendicular to the horizontal axes, as proven by:

$$\left. \frac{\partial \theta}{\partial v} \right|_{v=0} = \frac{v}{-\sin \theta + v|L(x)|} \Big|_{v=0} = 0.$$

The latter fact, together with properness of V and the decrease of V on $\Gamma_{<0}$ due to $L(\Gamma_{<0}) > 0$, implies that there is no homoclinic/heteroclinic orbit among Γ and the origin. It then follows that the invariant set $\mathcal{W} := \Gamma \cup \{(0, 0)\}$ satisfies Assumption 2. By making use of property $s|s| = s^2 - 2R(-s)^2$ for all $s \in \mathbb{R}$, it is possible to show that

$$\begin{aligned}\dot{V} &= -v^2 L(x)^2 + 2R(-L(x))^2 v^2 \\ &\leq -(v^2 + 1 - \cos \theta) L(x)^2 \\ &\quad + (1 - \cos \theta + 2v^2) R(-L(x))^2 + q,\end{aligned}\quad (33)$$

where $q := 2 \max_{x \in \Gamma_{<0}} L(x)^2$. Note that, since $1 - \cos \theta \geq (\text{dist } \theta)^2/8$, there exists a class- $\mathcal{K}_\infty$ function α_0 such that

$2v^2 + 1 - \cos \theta \geq v^2 + 1 - \cos \theta \geq \alpha_0(|x|_{\{(0,0)\}})$, with $|x|_{\{(0,0)\}} = \sqrt{(\text{dist } \theta)^2 + v^2}$. It then follows from (33) that $\dot{V} \leq -\alpha(|x|_\mathcal{W}) + \gamma_y(|y|) + q$ for some class- $\mathcal{K}_\infty$ functions α and γ_y . We conclude that V is a practical OSS-Lyapunov function for system (32) and thus, by virtue of Theorem 1, (32) satisfies the O-AG and GATTMO properties with respect to the set $\mathcal{D} := \{w \in M : |x|_\mathcal{W} \leq 3|h(x)|\}$.

V. CONCLUSIONS

Dualizing the generalized ISS notion for multistable systems [5] by simply replacing inputs with outputs in the definition results into a generalized OSS property which portrays the behavior of ultimately having small distance of the states from the invariant set of the system whenever the outputs are small. This paper has provided characterizations of such property in terms of asymptotic estimates via nonlinear gain functions and in terms of Lyapunov/LaSalle-like dissipation inequalities. Our results encapsulates several characterizations of the global attractivity property ($\lim_{t \rightarrow +\infty} |X(t, x)|_\mathcal{W} = 0$ for all $x \in M$), if an everywhere-vanishing output mapping ($h(x) \equiv 0$ for all $x \in M$) is used.

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