Buckling of thick cylindrical shells under external pressure: A new analytical expression for the critical load and comparison with elasticity solutions.

by Dr George Papadakis

Department of Mechanical Engineering
King's College London
Strand, WC2R 2LS, London, UK

E-mail: george.papadakis@kcl.ac.uk
Tel : (+44)-020-7848-2049
Fax (+44)-020-7848-2932

Paper accepted in Int. J. of Solids and Structures

May 2008

Abstract

In this paper a set of stability equations for thick cylindrical shells is derived and solved analytically. The set is obtained by integration of the differential stability equations across the thickness of the shell. The effects of transverse shear and the non-linear variation of the stresses and displacements are accounted for with the aid of the higher order shell theory proposed by Voyiadjis and Shi (1991). For a thick shell under external hydrostatic pressure, the stability equations are solved analytically and yield an improved expression for the buckling load. Reference solutions are also obtained by solving numerically the differential stability equations. Both the full set that contains strains and rotations as well as the simplified set that contains rotations only were solved numerically. The relative magnitude of shear strain and rotation was examined and the effect of thickness was quantified. Differences between the benchmark solutions and the analytic expressions based on the refined theory and the classical shell theory are analysed and discussed. It is shown that the new analytic expression provides significantly improved predictions compared to the formula based on thin shell theory.

Keywords: Buckling under external pressure, thick cylindrical shells, stability equations, critical load.

1. Introduction

Thick shells are widely used in various engineering applications such as cooling towers, arch dams, pressure vessels etc. Parts of the human body as well can be thought of as moderately thick shells carrying fluid, for example aorta, lung airways etc. The classical theory of thin shells was developed by Love and is based on the Kirchhoff-Love assumption for the deformation in the circumferential and radial direction but it ignores radial stress effects and the transverse shear deformation.

Based on the classical thin shell theory a simple expression for the buckling load under external pressure for two dimensional isotropic shells in plain strain can be easily derived. This expression is $p_{cr}^{tsh} = \frac{1}{4}\frac{E}{1-\nu^2}\frac{h^3}{a^3}$ where a is the radius of the midsurface of the shell, h is the thickness, E the modulus of elasticity and ν is the Poisson ratio (Timoshenko and Gere (1961)). In this expression as well as the ones that follow in the next sections, the superscript "tsh" denotes "thin shell theory". However, this expression overestimates the critical load for thick shells i.e. leads to non-conservative results. For example, for the ratio of external to internal radius $R_2 / R_1 = 0.4$ the overestimation is equal to 23.7% (Kardomateas (1993)). The reason for this overprediction can be traced to the different features that thick shells have in comparison to thin shells. For example transverse shear can no longer be neglected while the circumferential and radial stresses vary non-linearly across the thickness of the shell. For sandwich cylindrical shells that contain a low-modulus core, the effect of transverse shear can be very dramatic (Kardomateas and Simitses (2005)).

Many theories have been developed in order to account for the effect of shear deformation. A historical review of these theories as primarily applied to the buckling problem is presented by Simitses (1996). In the first order shear deformation theory, the displacement field is assumed to vary linearly with respect to thickness (measured from the midsurface) and the rotations of the normal to the midsurface are independent variables. Fu and Waas (1995) applied this theory to study the initial post-buckling behaviour of thick rings under uniform, external hydrostatic pressure. Takano (2008) used the same theory in order to extend Flügge's (1960) stability equations for moderately thick anisotropic cylindrical shells under axial loading. Higher order shear deformation theories in which the displacements fields are expressed as cubic functions of the thickness coordinate and the transverse displacement is assumed to be constant through the thickness have been proposed by Reddy and Liu (1985) as well as Simitses and Anastasiadis (1992). This approximation results in a parabolic distribution of the transverse shear strain across the thickness. Shen (2001) employed the theory of Reddy and Liu (1985) to study the post buckling of shear deformable cross-ply laminated cylindrical shells under combined external pressure and axial compression. Shariat and Eslami (2007) studied the buckling of thick plates using a third order shear deformation theory and obtained closed form solutions for the critical mechanical and thermal loads. The theory of Simitses and Anastasiadis (1992) was used by the same authors (Anastasiadis and Simitses (1993)) as well as Simitses et al (1993) for the linear buckling analysis of finite- and infinite-long laminated shells under the action of external pressure. The general conclusion from these studies is that first order shear deformation theories improve significantly the predictions of the buckling load compared to the thin shell Kirchhoff-Love assumption. However, the improvement offered by the higher order theories over the first order ones is much smaller.

Voyiadjis and Shi (1991) also proposed a refined shell theory suitable for thick cylinders that incorporates not only for the effect of transverse shear but also that of transverse strain and the non-linearity of the in-plane stresses. It differs from the previous theories in that the deformations in the circumferential and radial direction are obtained by solving analytically the ordinary differential equations obtained from the stress-strain relations and keeping only the low order terms in the Taylor series expansions of $\ln(z+a)$ and $1/(z+a)$. Incorporation of transverse shear deformation follows the work of Reissner (1945). The theory was first developed and applied to the problem of wave propagation in isotropic elastic plates by Voyiadjis and Baluch (1981) and was later extended for thick spherical shells by Voyiadjis and Woelke (2004).

All the above shell theories that are used to derive improved approximations of the buckling loads are based on assumptions on the distribution of the displacement field across the thickness of the shell. However, it is possible to obtain the critical loads exactly (i.e. without making such assumptions) by solving directly the stability equations of Novozhilov (1953). In this case, the displacement field is obtained as part of the solution. Kardomateas (1993), Kardomateas and Chung (1994) as well as Kardomateas and Simitses (2005) followed this approach and derived the buckling load for thick shells under external hydrostatic pressure. These exact solutions can then be used to check the accuracy of the developed shear deformation theories presented above.

The theory of Voyiadjis and Shi (1991) is used in the present paper for the estimation of the stress and moment resultants and the derivation of an improved analytical expression for the estimation of the buckling load for thick isotropic shells under external hydrostatic pressure. The accuracy of the derived expression is assessed against benchmark results obtained from the numerical solution of the differential stability equations of Novozhilov (1953).

For thin shells, strains are negligible compared to rotations so the differential stability equations contain the effect of rotations only. However, the validity of this assumption needs to be examined carefully in the context of thick shells. Kardomateas (2000) examined the effect of strains and found that they result in further decrease of the critical load.

The fact that strains are important for thick shells means that conjugate stress-strains pairs should be used to arrive at accurate benchmark results as demonstrated originally by Bažant (1971). The wider context as well as more details about different measures of finite strain and stress is provided in the book of Bažant and Cedolin (2003). It is possible to change form one conjugate pair to another using a transformation formula for the stiffness tensor. By the way, this is how the controversy between Engesser and Harinx formulae for the critical load in a beam with shear deformation is resolved. In the present paper we use the Green strain tensor and the 2nd Piola-Kirchhoff stress tensor pair. In order to assess the effect of thickness on the relative magnitude of the strains with respect to rotations, solutions were obtained using the full set (that contains rotations and strains) as well as the simpler set that contains rotations only. In order to simplify the algebra, only the latter set was used in order to derive the analytic solution. Of course, the analytic results are assessed against the numerical solution of the full set.

The paper is organised as follows: In section 2 the differential stability equations are presented along with the associated boundary conditions for the load case under examination (hydrostatic pressure). The various steps that lead to the stability equations that contain rotations only are examined and the underlying assumptions are highlighted. This section also includes details for the numerical solution of the differential equations. In section 3 the simplified differential equations that contain rotations only are integrated across the thickness of the shell. In the following section, the stress and moment resultants obtained from the refined shell theory are substituted to the stability equations and the resulting homogenous system is solved analytically in section 5 yielding an improved expression for the critical pressure. Section 6 presents a detailed comparison between the derived expression and benchmark solutions for a range of h/a values. Finally section 7 summarises the main findings of the paper.

2. Differential stability equations.

In the present paper only the two-dimensional, plain strain problem in the *θ, r* plane is examined. The displacements in these two directions are denoted as *v* and *w* respectively. The basic geometric variables and load condition are shown in figure 1. Details on the derivation of the differential stability equations in three dimensions can be found in Novozhilov (1953) and Kardomateas (1993). Here only the two dimensional form is examined. In this and the following sections, the superscript "0" denotes the base load state, that is the initial equilibrium position.

The initial displacements are perturbed by a small amount i.e. the new equilibrium position is described by:

$$\begin{aligned} v(r,\theta) &= v^0(r,\theta) + \varepsilon \cdot v'(r,\theta) \\ w(r,\theta) &= w^0(r,\theta) + \varepsilon \cdot w'(r,\theta) \end{aligned} \tag{1}$$

where ε is an infinitesimally small quantity and the functions $v'(r,\theta)$ and $w'(r,\theta)$ are assumed finite. The displacement perturbations lead to stress perturbations that are given by:

$$\begin{aligned} \sigma_{rr}(r,\theta) &= \sigma_{rr}^0(r,\theta) + \varepsilon \cdot \sigma_{rr}'(r,\theta) + \varepsilon^2 \cdot \sigma_{rr}''(r,\theta) \\ \sigma_{\theta\theta}(r,\theta) &= \sigma_{\theta\theta}^0(r,\theta) + \varepsilon \cdot \sigma_{\theta\theta}'(r,\theta) + \varepsilon^2 \cdot \sigma_{\theta\theta}''(r,\theta) \\ \tau_{r\theta}(r,\theta) &= \tau_{r\theta}^0(r,\theta) + \varepsilon \cdot \tau_{r\theta}'(r,\theta) + \varepsilon^2 \cdot \tau_{r\theta}''(r,\theta) \end{aligned} \tag{2}$$

The second order terms appear because the strain-displacement relations contain quadratic terms (for example Malvern (1969), Bažant and Cedolin (2003)).

The equations of equilibrium are written in terms of the second Piola-Kirchhoff tensor $\boldsymbol{\sigma}$ as

$$div\left(\boldsymbol{\sigma} \cdot \boldsymbol{F}^T\right) = 0 \tag{3}$$

where $\boldsymbol{F}$ is the deformation gradient. The finite strain tensor conjugate with this stress tensor is the Green's Lagrangian tensor. The boundary condition is

$$(\boldsymbol{F\sigma}) \cdot \vec{n} = \vec{t} \tag{4}$$

where $\vec{t}$ is the traction and $\vec{n}$ is the outward pointing unit normal vector.

For a two dimensional, plain strain problem, the expanded form of 3 is:

$$
\begin{aligned}
&\frac{\partial}{\partial r}\left[\sigma_{rr}(1+e_{rr})+\tau_{r\theta}\left(\frac{1}{2}e_{r\theta}-\omega_x\right)\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\tau_{r\theta}(1+e_{rr})+\sigma_{\theta\theta}\left(\frac{1}{2}e_{r\theta}-\omega_x\right)\right]+\\
&\frac{1}{r}\left[\sigma_{rr}(1+e_{rr})-\sigma_{\theta\theta}(1+e_{\theta\theta})-2\tau_{r\theta}\omega_x\right]=0 \qquad (a)\\
&\frac{\partial}{\partial r}\left[\sigma_{rr}\left(\frac{1}{2}e_{r\theta}+\omega_x\right)+\tau_{r\theta}(1+e_{\theta\theta})\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\tau_{r\theta}\left(\frac{1}{2}e_{r\theta}+\omega_x\right)+\sigma_{\theta\theta}(1+e_{\theta\theta})\right]+\\
&\frac{1}{r}\left[\sigma_{rr}\left(\frac{1}{2}e_{r\theta}+\omega_x\right)+\sigma_{\theta\theta}\left(\frac{1}{2}e_{r\theta}-\omega_x\right)+\tau_{r\theta}(2+e_{rr}+e_{\theta\theta})\right]=0 \qquad (b)
\end{aligned}
\tag{5}
$$

in the radial and circumferential directions respectively. In the above equations, e_{ij} denote the linear strains

$$
e_{rr}=\frac{\partial w}{\partial r},\quad e_{\theta\theta}=\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{w}{r},\quad e_{r\theta}=\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r} \tag{6}
$$

and ω_x the linear rotation

$$
\omega_x=\frac{1}{2}\left(\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial w}{\partial\theta}\right) \tag{7}
$$

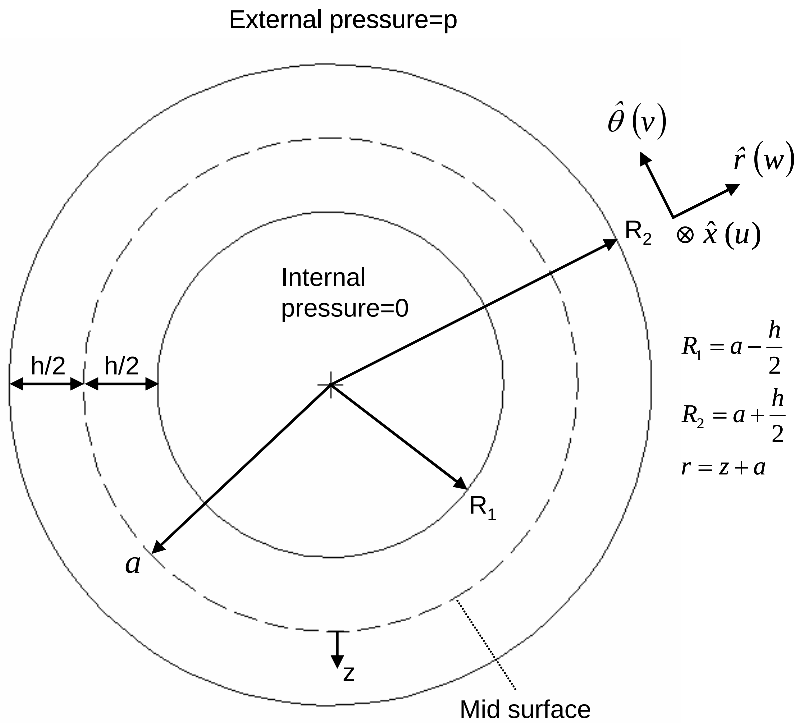

Figure 1. Basic geometric variables and load condition.

The set of equations 5 is applied for the initial equilibrium as well as the perturbed equilibrium position. For the stress tensor in the latter position only the linear terms in ε are retained (the quadratic terms are associated with the initial postbuckling behaviour). The two equations are then subtracted and the resulting expressions are shown below:

$$\begin{aligned}
&\frac{\partial}{\partial r}\left[\sigma_{rr}^{0}e'_{rr}+\sigma'_{rr}\left(1+e_{rr}^{0}\right)\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\tau'_{r\theta}\left(1+e_{rr}^{0}\right)+\sigma_{\theta\theta}^{0}\left(\frac{1}{2}e'_{r\theta}-\omega'_{x}\right)\right]+\\
&\frac{1}{r}\left[\sigma_{rr}^{0}e'_{rr}+\sigma'_{rr}\left(1+e_{rr}^{0}\right)-\sigma_{\theta\theta}^{0}e'_{\theta\theta}-\sigma'_{\theta\theta}\left(1+e_{\theta\theta}^{0}\right)\right]=0 \qquad (a)\\
&\frac{\partial}{\partial r}\left[\tau'_{r\theta}\left(1+e_{\theta\theta}^{0}\right)+\sigma_{rr}^{0}\left(\frac{1}{2}e'_{r\theta}+\omega'_{x}\right)\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\sigma'_{\theta\theta}\left(1+e_{\theta\theta}^{0}\right)+\sigma_{\theta\theta}^{0}e'_{\theta\theta}\right]+\\
&\frac{1}{r}\left[\tau'_{r\theta}\left(2+e_{rr}^{0}+e_{\theta\theta}^{0}\right)+\sigma_{rr}^{0}\left(\frac{1}{2}e'_{r\theta}+\omega'_{x}\right)+\sigma_{\theta\theta}^{0}\left(\frac{1}{2}e'_{r\theta}-\omega'_{x}\right)\right]=0 \qquad (b)
\end{aligned} \tag{8}$$

The shear strain, shear stress and rotation are all zero ($e_{r\theta}^{0}=\tau_{r\theta}^{0}=\omega_{x}^{0}=0$) under the examined base load condition and so the corresponding terms have been dropped from the above set. Assuming that the non-zero normal strains $e_{rr}^{0}, e_{\theta\theta}^{0}$ are much smaller than 1 (i.e. $1+e_{rr}^{0}\approx 1, 1+e_{\theta\theta}^{0}\approx 1$) we have:

$$\begin{aligned}
&\frac{\partial}{\partial r}\left(\sigma_{rr}^{0}e'_{rr}+\sigma'_{rr}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\tau'_{r\theta}+\sigma_{\theta\theta}^{0}\left(\frac{1}{2}e'_{r\theta}-\omega'_{x}\right)\right]+\frac{1}{r}\left(\sigma'_{rr}+\sigma_{rr}^{0}e'_{rr}-\left(\sigma'_{\theta\theta}+\sigma_{\theta\theta}^{0}e'_{\theta\theta}\right)\right)=0 \qquad (a)\\
&\frac{\partial}{\partial r}\left[\tau'_{r\theta}+\sigma_{rr}^{0}\left(\frac{1}{2}e'_{r\theta}+\omega'_{x}\right)\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\sigma'_{\theta\theta}+\sigma_{\theta\theta}^{0}e'_{\theta\theta}\right)+\frac{1}{r}\left[2\tau'_{r\theta}+\sigma_{rr}^{0}\left(\frac{1}{2}e'_{r\theta}+\omega'_{x}\right)+\sigma_{\theta\theta}^{0}\left(\frac{1}{2}e'_{r\theta}-\omega'_{x}\right)\right]=0 \quad (b)
\end{aligned} \tag{9}$$

In the above sets 8 and 9 the primed stresses $\sigma'_{rr}, \sigma'_{\theta\theta}, \tau'_{r\theta}$ are evaluated from the primed linear strains $e'_{rr}, e'_{\theta\theta}, e'_{r\theta}$ from:

$$\begin{aligned}
\sigma'_{rr}&=\left(2G+\lambda\right)e'_{rr}+\lambda e'_{\theta\theta}\\
\sigma'_{\theta\theta}&=\left(2G+\lambda\right)e'_{\theta\theta}+\lambda e'_{rr}\\
\tau'_{r\theta}&=Ge'_{r\theta}
\end{aligned} \tag{10}$$

where $\lambda=\dfrac{E\nu}{(1+\nu)(1-2\nu)}, G=\dfrac{E}{2(1+\nu)}$ are the Lamé coefficients (it has been assumed that $e'_{xx}=0$ because of plain strain conditions). This set is appropriate for the adopted stress/strain conjugate pair that we use. Using this constitutive set, $\sigma_{rr}^0 e'_{rr} + \sigma'_{rr} = \left(2G + \lambda + \sigma_{rr}^0\right)e'_{rr} + \lambda e'_{\theta\theta}$ and $\sigma'_{\theta\theta} + \sigma_{\theta\theta}^0 e'_{\theta\theta} = \left(2G + \lambda + \sigma_{\theta\theta}^0\right)e'_{\theta\theta} + \lambda e'_{rr}$ and therefore if the critical load p is such that

$$\sigma_{rr}^0, \sigma_{\theta\theta}^0 << 2G + \lambda \qquad (11)$$

the terms $\sigma_{rr}^0 e'_{rr}$ and $\sigma_{\theta\theta}^0 e'_{\theta\theta}$ can be dropped as much smaller compared to σ'_{rr} and $\sigma'_{\theta\theta}$ respectively, leading to:

$$\begin{aligned}
&\frac{\partial \sigma'_{rr}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\tau'_{r\theta} + \sigma_{\theta\theta}^0\left(\frac{1}{2}e'_{r\theta} - \omega'_x\right)\right] + \frac{1}{r}\left(\sigma'_{rr} - \sigma'_{\theta\theta}\right) = 0 \qquad (a)\\
&\frac{\partial}{\partial r}\left[\tau'_{r\theta} + \sigma_{rr}^0\left(\frac{1}{2}e'_{r\theta} + \omega'_x\right)\right] + \frac{1}{r}\frac{\partial \sigma'_{\theta\theta}}{\partial \theta} + \frac{1}{r}\left[2\tau'_{r\theta} + \sigma_{rr}^0\left(\frac{1}{2}e'_{r\theta} + \omega'_x\right) + \sigma_{\theta\theta}^0\left(\frac{1}{2}e'_{r\theta} - \omega'_x\right)\right] = 0 \quad (b)
\end{aligned} \qquad (12)$$

The validity of inequality 11 for the examined problem will be checked later in section 6. For thin shells the rotations substantially exceed strains (Brush and Almroth (1975), Bažant and Cendolin (2003)) so the above equations take the following simplified form:

$$\begin{aligned}
&\frac{\partial \sigma'_{rr}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\tau'_{r\theta} - \sigma_{\theta\theta}^0 \omega'_z\right) + \frac{1}{r}\left(\sigma'_{rr} - \sigma'_{\theta\theta}\right) = 0 \qquad (a)\\
&\frac{\partial}{\partial r}\left(\tau'_{r\theta} + \sigma_{rr}^0 \omega'_x\right) + \frac{1}{r}\frac{\partial \sigma'_{\theta\theta}}{\partial \theta} + \frac{1}{r}\left(2\tau'_{r\theta} + \sigma_{rr}^0 \omega'_x - \sigma_{\theta\theta}^0 \omega'_x\right) = 0 \quad (b)
\end{aligned} \qquad (13)$$

However, for thick shells this assumption needs to be re-examined (again this is deferred to section 6).

Applying the boundary condition 4 in the initial and perturbed equilibrium positions and subtracting the two equations we get for the loading case considered here:

$$\begin{aligned}
&\sigma'_{rr}\left(a + \frac{h}{2}, \theta\right) = \sigma'_{rr}\left(a - \frac{h}{2}, \theta\right) = 0 \qquad (a)\\
&\tau'_{r\theta}\left(a + \frac{h}{2}, \theta\right) = \tau'_{r\theta}\left(a - \frac{h}{2}, \theta\right) = 0 \qquad (b)
\end{aligned} \qquad (14)$$

The stresses $\sigma_{\theta\theta}^0, \sigma_{rr}^0$ are given by the well known expressions from linear elasticity (Lai et al (1996)):

$$
\begin{aligned}
\sigma^0_{\theta\theta}(r) &= -p\left[1+\left(\frac{R_1}{r}\right)^2\right]\cdot\left[1-\left(\frac{R_1}{R_2}\right)^2\right]^{-1} = f_{rr}(r)\cdot p \qquad (a)\\
\sigma^0_{rr}(r) &= -p\left[1-\left(\frac{R_1}{r}\right)^2\right]\cdot\left[1-\left(\frac{R_1}{R_2}\right)^2\right]^{-1} = f_{\theta\theta}(r)\cdot p \qquad (b)
\end{aligned}
\tag{15}
$$

In order to quantify the previous assumptions for thick shells, both the simplified set 13 (that contains only rotations) as well as the more complete set 9 (that contains rotations and strains) were solved numerically.

The solution procedure is explained below. Assuming the following distribution of perturbed displacements

$$
\begin{aligned}
v(r,\theta) &= A(r)\cdot\sin m\theta\\
w(r,\theta) &= B(r)\cdot\cos m\theta
\end{aligned}
\tag{16}
$$

the linear strains and rotation become (from definitions 6 and 7):

$$
\begin{aligned}
e'_{rr} &= \frac{dB}{dr}\cos m\theta\\
e'_{\theta\theta} &= \frac{1}{r}\left(Am+B\right)\cos m\theta\\
e'_{r\theta} &= \left(\frac{dA}{dr}-\frac{A+Bm}{r}\right)\sin m\theta\\
\omega'_x &= \frac{1}{2}\left(\frac{dA}{dr}+\frac{A+Bm}{r}\right)\sin m\theta
\end{aligned}
\tag{17}
$$

These expressions are substituted into 10 to obtain the corresponding stress expressions and these are then inserted to either 9 or 13. For example, the final expressions for set 9 are

$$
\begin{aligned}
&(2G+\lambda)\frac{d^2B}{dr^2}+\frac{m}{r}(\lambda+G)\frac{dA}{dr}+\frac{2G+\lambda}{r}\frac{dB}{dr}-\frac{1}{r^2}(\lambda+3G)mA-\frac{1}{r^2}\left(2G+\lambda+Gm^2\right)B=\\
&p\left[-\frac{d}{dr}\left(f_{rr}(r)\cdot\frac{dB}{dr}\right)-\frac{f_{rr}(r)}{r}\frac{dB}{dr}+\frac{2m}{r^2}f_{\theta\theta}(r)A+\frac{f_{\theta\theta}(r)}{r^2}\left(m^2+1\right)B\right]\\
&G\frac{d^2A}{dr^2}+\frac{G}{r}\frac{dA}{dr}-\frac{(\lambda+G)m}{r}\frac{dB}{dr}-\frac{1}{r^2}\left(G+m^2(2G+\lambda)\right)A-\frac{m}{r^2}(3G+\lambda)B=\\
&p\left[-\frac{d}{dr}\left(f_{rr}(r)\frac{dA}{dr}\right)-\frac{f_{rr}(r)}{r}\frac{dA}{dr}+\frac{f_{\theta\theta}(r)\left(m^2+1\right)}{r^2}A+\frac{2m}{r^2}f_{\theta\theta}(r)B\right]
\end{aligned}
\tag{18}
$$

for the radial and circumferential directions respectively (the factors $f_{rr}(r), f_{\theta\theta}(r)$ are defined in 15). Simpler expressions are obtained for set 13.

The finite volume method (Versteeg and Malasekera (2007)) was used to discretise the equations. For the evaluation of the derivatives, second order accurate approximations were used for the internal points and first order forward or backward expressions for the boundaries. The resulting generalised eigenvalue problem was solved using the QZ decomposition technique (Pozrikidis (1998)). This method is implemented by Argonne National Laboratory in the FORTRAN subroutines `CQZHES`, `CQZVAL` and `CQZVEC` that can be found in the netlib repository (www.netlib.org). The benchmark solutions obtained were employed to assess the accuracy of the analytic expression for the critical load derived using the refined shell theory in the sections 3, 4 and 5 below.

3. Integration of differential stability equations.

The differential stability equations presented earlier will be integrated across the thickness of the shell. In order to simplify the algebra, the simplified set that contains rotations only will be integrated. The error between the derived analytic solution and the numerical solution of the full set will be quantified in section 6.

Integrating the set of equations 13 across the thickness of the cylinder we have:

$$\begin{aligned}&\int_{-h/2}^{h/2}\left[r\frac{\partial\sigma'_{rr}}{\partial r}+\frac{\partial}{\partial\theta}\left(\tau'_{r\theta}-\sigma^0_{\theta\theta}\omega'_x\right)+\left(\sigma'_{rr}-\sigma'_{\theta\theta}\right)\right]dz=0\\&\int_{-h/2}^{h/2}\left[r\frac{\partial}{\partial r}\left(\tau'_{r\theta}+\sigma^0_{rr}\omega'_x\right)+\frac{\partial\sigma'_{\theta\theta}}{\partial\theta}+\left(2\tau'_{r\theta}+\sigma^0_{rr}\omega'_x-\sigma^0_{\theta\theta}\omega'_x\right)\right]dz=0\end{aligned}\qquad(19)$$

Integrating by parts the first term of the integrand we get:

$$
\begin{aligned}
&\left[r\sigma'_{rr}\right]_{-h/2}^{h/2} - \int_{-h/2}^{h/2}\sigma'_{rr}\,dz + \int_{-h/2}^{h/2}\frac{\partial\tau'_{r\theta}}{\partial\theta}dz - \int_{-h/2}^{h/2}\frac{\partial\left(\sigma^0_{\theta\theta}\omega'_x\right)}{\partial\theta}dz + \int_{-h/2}^{h/2}\left(\sigma'_{rr}-\sigma'_{\theta\theta}\right)dz = 0\\
&\left[r\left(\tau'_{r\theta}+\sigma^0_{rr}\omega'_x\right)\right]_{-h/2}^{h/2} - \int_{-h/2}^{h/2}\left(\tau'_{r\theta}+\sigma^0_{rr}\omega'_x\right)dz + \int_{-h/2}^{h/2}\left(\frac{\partial\sigma'_{\theta\theta}}{\partial\theta}+2\tau'_{r\theta}+\sigma^0_{rr}\omega'_x-\sigma^0_{\theta\theta}\omega'_x\right)dz = 0
\end{aligned}
\tag{20}
$$

The term $\int_{-h/2}^{h/2}\sigma'_{rr}dz$ cancels out in the first equation as does the term $\int_{-h/2}^{h/2}\left(\tau'_{r\theta}+\sigma^0_{rr}\omega'_x\right)dz$ in the second, so:

$$
\begin{aligned}
&\left[r\sigma'_{rr}\right]_{-h/2}^{h/2} + \frac{\partial}{\partial\theta}\int_{-h/2}^{h/2}\tau'_{r\theta}dz - \int_{-h/2}^{h/2}\frac{\partial\left(\sigma^0_{\theta\theta}\omega'_x\right)}{\partial\theta}dz - \int_{-h/2}^{h/2}\sigma'_{\theta\theta}dz = 0\\
&\left[r\left(\tau'_{r\theta}+\sigma^0_{rr}\omega'_x\right)\right]_{-h/2}^{h/2} + \int_{-h/2}^{h/2}\left(\frac{\partial\sigma'_{\theta\theta}}{\partial\theta}+\tau'_{r\theta}-\sigma^0_{\theta\theta}\omega'_x\right)dz = 0
\end{aligned}
\tag{21}
$$

Using the definitions of the stress resultants

$$
N'_\theta = \int_{-h/2}^{h/2}\sigma'_{\theta\theta}dz \qquad Q'_\theta = \int_{-h/2}^{h/2}\tau'_{r\theta}dz
\tag{22}
$$

the buckling equations become:

$$
\begin{aligned}
&\left[r\sigma'_{rr}\right]_{-h/2}^{h/2} + \frac{\partial Q'_\theta}{\partial\theta} - \int_{-h/2}^{h/2}\frac{\partial\left(\sigma^0_{\theta\theta}\omega'_x\right)}{\partial\theta}dz - N'_\theta = 0\\
&\left[r\left(\tau'_{r\theta}+\sigma^0_{rr}\omega'_x\right)\right]_{-h/2}^{h/2} + \frac{\partial N'_\theta}{\partial\theta} + Q_\theta - \int_{-h/2}^{h/2}\sigma^0_{\theta\theta}\omega'_x dz = 0
\end{aligned}
\tag{23}
$$

If we define the integral I'_θ as

$$
\mathrm{I}'_\theta = \int_{-h/2}^{h/2}\sigma^0_{\theta\theta}\omega'_x dz
\tag{24}
$$

we get:

$$
\begin{aligned}
&\left[r\sigma'_{rr}\right]_{-h/2}^{h/2} + \frac{\partial Q'_\theta}{\partial\theta} - \frac{\partial \mathrm{I}'_\theta}{\partial\theta} - N'_\theta = 0\\
&\left[r\left(\tau'_{r\theta}+\sigma^0_{rr}\omega'_x\right)\right]_{-h/2}^{h/2} + \frac{\partial N'_\theta}{\partial\theta} + Q'_\theta - \mathrm{I}'_\theta = 0
\end{aligned}
\tag{25}
$$

Using the boundary conditions 14 and the fact that $\sigma^0_{rr}\left(a+\frac{h}{2},\theta\right)=-p,\ \sigma^0_{rr}\left(a-\frac{h}{2},\theta\right)=0$, we have

$$\begin{aligned}&\frac{\partial Q'_\theta}{\partial\theta}-\frac{\partial \mathrm{I}'_\theta}{\partial\theta}-N'_\theta=0\\&-\left(a+\frac{h}{2}\right)p\omega'_{x(a+h/2)}+\frac{\partial N'_\theta}{\partial\theta}+Q'_\theta-\mathrm{I}'_\theta=0\end{aligned} \tag{26}$$

This set is complemented by the moment balance equation in the x direction (shear-moment relation)

$$\frac{1}{a}\frac{\partial M'_\theta}{\partial\theta}+Q'_\theta=0 \quad \left(where\ M'_\theta=-\int_{-h/2}^{h/2} z\sigma'_{\theta\theta}dz\right) \tag{27}$$

The above set is general i.e. valid for either thick or thin shells. For thin shells this set can be simplified and compared with the equations of Flügge (1960) (see Appendix A). In order to proceed $N'_\theta, Q'_\theta, \mathrm{I}'_\theta$ and $\omega'_{x(a+h/2)}$ must be written in terms of characteristic displacements and shear angles. For thin shells the standard shell theory of Love has been used extensively. However for thick shells a refined shell theory is more suitable and is applied in the next section to derive the shell stability equations.

4. Stability equations based on a higher order shell theory.

The theory of Voyiadjis and Shi (1991) provides expressions for the variation of the displacements fields v, w as functions of z, the transverse shear resultant Q_θ, the moment stress resultants M_θ, M_x as well as the external pressure loading. For example, for the load case examined in this paper, these expressions take the form:

$$\begin{aligned}&v(z,\theta)=\left(1+\frac{z}{a}\right)\left[\begin{array}{l}v_{ms}+\dfrac{Q_\theta}{2hG}z\left(3-\dfrac{4z^2}{h^2}+\dfrac{3z}{a}\left(-\dfrac{1}{2}+\dfrac{z^2}{h^2}\right)\right)+\dfrac{\partial w_{ms}}{\partial\theta}\dfrac{z}{a}\left(-1+\dfrac{z}{a}\right)+\\ 2\dfrac{\nu}{E}\dfrac{\partial(M_\theta+M_x)}{\partial\theta}\dfrac{z^3}{h^3}\dfrac{1}{a}\left(1-\dfrac{3z}{2a}\right)\end{array}\right]\\&w(z,\theta)=w_{ms}-6\frac{\nu}{E}\frac{z^2}{h^3}(M_\theta+M_x)-\frac{p}{E}\left[z-\frac{R_1{}^2}{a^2}\left(z-\frac{z^2}{a}\right)\right]\cdot\left[1-\left(\frac{R_1}{R_2}\right)^2\right]^{-1}\end{aligned} \tag{28}$$

where the subscript "ms" denotes the value of the displacement at the mid surface (z=0) while the moment stress resultants M_θ, M_x are defined by

$$M_\theta = -\int_{-h/2}^{h/2} z\sigma_{\theta\theta}dz, \quad M_x = -\int_{-h/2}^{h/2} \sigma_{xx}\left(1+\frac{z}{a}\right)z\, dz \tag{29}$$

These expressions are then employed to derive constitutive equations for N_θ and M_θ in terms of the values on the mid-surface or more compactly in terms of the average displacements defined as

$$\begin{aligned} \overline{w} &= w_{ms} - \frac{3\nu}{10hE}\left(M_\theta + M_x\right) - \frac{p}{E}\frac{R_1^2}{a^3}\frac{h^2}{20}\cdot\left[1-\left(\frac{R_1}{R_2}\right)^2\right]^{-1} \\ \overline{v} &= v_{ms} \end{aligned} \tag{30}$$

and the shear angle γ_θ defined as

$$\gamma_\theta = \frac{Q_\theta}{\frac{5}{6}Gh} \tag{31}$$

For comparison, the expressions of the standard thin shell theory are (Flügge (1960)):

$$\begin{aligned} v^{tst}(z,\theta) &= \left(1+\frac{z}{a}\right)v_{ms} - \frac{z}{a}\frac{\partial w_{ms}}{\partial\theta} \\ w^{tst}(z,\theta) &= w_{ms} \end{aligned} \tag{32}$$

It is clear that in the refined theory, effects that are important for thick shells are taken into account, such as the non linear variation of the displacements along z and the transverse shear. These are excluded from the much simpler kinematic expressions of the standard thin shell theory.

It is easy now to derive the corresponding expressions for the primed quantities, for example $v'(z,\theta), w'(r,\theta)$. Simply the equations 28 are applied to the initial and perturbed equilibrium positions and they are subtracted. The terms that contain the external loading p cancel out and since the equations are linear in terms of Q_θ, M_θ, M_x we get for the primed displacement field:

$$
\begin{aligned}
v'(z,\theta) &= \left(1+\frac{z}{a}\right)\left[\begin{array}{l} v'_{ms} + \dfrac{Q'_\theta}{2hG} z\left(3-\dfrac{4z^2}{h^2}+\dfrac{3z}{a}\left(-\dfrac{1}{2}+\dfrac{z^2}{h^2}\right)\right)+\dfrac{\partial w'_{ms}}{\partial\theta}\dfrac{z}{a}\left(-1+\dfrac{z}{a}\right)+ \\ 2\dfrac{\nu}{E}\dfrac{\partial\left(M'_\theta+M'_x\right)}{\partial\theta}\dfrac{z^3}{h^3}\dfrac{1}{a}\left(1-\dfrac{3z}{2a}\right) \end{array}\right] \\
w'(z,\theta) &= w'_{ms} - 6\frac{\nu}{E}\frac{z^2}{h^3}\left(M'_\theta + M'_x\right)
\end{aligned}
\tag{33}
$$

After integration the primed stress resultants can be written as:

$$
\begin{aligned}
N'_\theta &= \frac{D}{a}\left(\frac{\partial\overline{v}'}{\partial\theta}+\overline{w}'\right)+\frac{K}{a^3}\left(\frac{\partial^2\overline{w}'}{\partial\theta^2}+\overline{w}'\underline{-\frac{7}{16}a\frac{\partial\gamma'_\theta}{\partial\theta}}\right) \\
\mathrm{M}'_\theta &= \frac{K}{a^2}\left(\frac{\partial^2\overline{w}'}{\partial\theta^2}+\overline{w}'\underline{-a\frac{\partial\gamma'_\theta}{\partial\theta}}\right)
\end{aligned}
\tag{34}
$$

where

$$
D=\frac{Eh}{1-\nu^2},\ K=\frac{Eh^3}{12\left(1-\nu^2\right)}
\tag{35}
$$

The underlined terms are absent from the corresponding expressions of the standard theory. An expression for the rotation can be obtained by substituting the expressions 33 into the definition 7. If the resulting expression is evaluated at $z=\dfrac{h}{2}$ we get:

$$
\omega_{z(a+h/2)} = \frac{1}{a}\left(\overline{v}' + \underline{\frac{5}{12}\gamma'_\theta h} - \frac{\partial\overline{w}'}{\partial\theta}\right)
\tag{36}
$$

Substituting the circumferential stress distribution 15(a) and the expression for rotation in 24 the following equation for I'_θ is obtained:

$$
I'_\theta = p\left(1+\underline{\frac{h}{2a}}\right)\left(\frac{\partial\overline{w}'}{\partial\theta}-\overline{v}'\underline{-\frac{5}{12}\gamma'_\theta a}\right)
\tag{37}
$$

Again the underlined terms are omitted in the thin shell theory. Substituting equations 34 and 37 into 26 and 27 results in a homogenous system of 3 equations and 3 unknowns $\overline{v}', \overline{w}', \gamma'_\theta$.

The final expressions are:

$$\begin{aligned}
&\frac{5}{6}Gh\frac{\partial\gamma'_\theta}{\partial\theta}-p\left(1+\frac{h}{2a}\right)\left(\frac{\partial^2\overline{w}'}{\partial\theta^2}-\frac{\partial\overline{v}'}{\partial\theta}-\frac{5}{12}a\frac{\partial\gamma_\theta}{\partial\theta}\right)-\frac{D}{a}\left(\frac{\partial\overline{v}'}{\partial\theta}+\overline{w}'\right)-\frac{K}{a^3}\left(\frac{\partial^2\overline{w}'}{\partial\theta^2}+\overline{w}'-\frac{7}{16}a\frac{\partial\gamma'_\theta}{\partial\theta}\right)=0\\
&-p(a+\frac{h}{2})\frac{1}{a}\left(\overline{v}'+\frac{5}{12}\gamma'_\theta h-\frac{\partial\overline{w}'}{\partial\theta}\right)+\frac{D}{a}\left(\frac{\partial^2\overline{v}'}{\partial\theta^2}+\frac{\partial\overline{w}'}{\partial\theta}\right)+\frac{K}{a^2}\frac{9}{16}\frac{\partial^2\gamma'_\theta}{\partial\theta^2}-p\left(1+\frac{h}{2a}\right)\left(\frac{\partial\overline{w}'}{\partial\theta}-\overline{v}'-\frac{5}{12}\gamma'_\theta a\right)=0\\
&\frac{K}{a^3}\left(\frac{\partial^3\overline{w}'}{\partial\theta^3}+\frac{\partial\overline{w}'}{\partial\theta}-a\frac{\partial^2\gamma'_\theta}{\partial\theta^2}\right)+\gamma_\theta\frac{5}{6}Gh=0
\end{aligned}\tag{38}$$

The pressure terms in the circumferential equation can be simplified and finally we get:

$$\begin{aligned}
&\frac{5}{6}Gh\frac{\partial\gamma'_\theta}{\partial\theta}-p\left(1+\frac{h}{2a}\right)\left(\frac{\partial^2\overline{w}'}{\partial\theta^2}-\frac{\partial\overline{v}'}{\partial\theta}-\frac{5}{12}a\frac{\partial\gamma'_\theta}{\partial\theta}\right)-\frac{D}{a}\left(\frac{\partial\overline{v}'}{\partial\theta}+\overline{w}'\right)-\frac{K}{a^3}\left(\frac{\partial^2\overline{w}'}{\partial\theta^2}+\overline{w}'-\frac{7}{16}a\frac{\partial\gamma'_\theta}{\partial\theta}\right)=0\\
&p\left(1-\frac{h}{2a}\right)\frac{5}{12}\gamma'_\theta a+\frac{D}{a}\left(\frac{\partial^2\overline{v}'}{\partial\theta^2}+\frac{\partial\overline{w}'}{\partial\theta}\right)+\frac{K}{a^2}\frac{9}{16}\frac{\partial^2\gamma'_\theta}{\partial\theta^2}=0\\
&\frac{K}{a^3}\left(\frac{\partial^3\overline{w}'}{\partial\theta^3}+\frac{\partial\overline{w}'}{\partial\theta}-a\frac{\partial^2\gamma'_\theta}{\partial\theta^2}\right)+\gamma_\theta\frac{5}{6}Gh=0
\end{aligned}\tag{39}$$

This is the final set of stability equations for thick shells that contains only the effect of rotations. An analytic solution for the critical load is derived in the next section.

5. Analytic solution of the stability system.

We assume the following variation with respect to the angle θ:

$$\begin{aligned}
\overline{v}' &= A\sin m\theta\\
\overline{w}' &= B\cos m\theta\\
\gamma'_\theta &= C\sin m\theta
\end{aligned}\tag{40}$$

where m is an integer. Introducing this to 39 and cancelling out the trigonometric functions we get the homogenous linear system:

$$\begin{aligned}
&A\cdot\left[p\left(1+\frac{h}{2a}\right)-\frac{D}{a}\right]m+B\cdot\left[p\left(1+\frac{h}{2a}\right)m^2-\frac{D}{a}+\frac{K}{a^3}\left(m^2-1\right)\right]+C\cdot\left[\frac{5}{6}Gh+\frac{K}{a^2}\frac{7}{16}+p\left(1+\frac{h}{2a}\right)\frac{5}{12}a\right]m=0\\
&A\cdot\left(\frac{D}{a}m^2\right)+B\cdot\left(\frac{D}{a}m\right)+C\cdot\left[\frac{K}{a^2}\frac{9}{16}m^2-pa\left(1-\frac{h}{2a}\right)\frac{5}{12}\right]=0\\
&B\cdot\left[\frac{K}{a}m\left(m^2-1\right)\right]+C\cdot\left(Km^2+a^2\frac{5}{6}Gh\right)=0
\end{aligned}\tag{41}$$

The trivial solution is A=B=C=0. For a non-trivial solution the determinant of the system should be equal to 0. This condition leads to the following quadratic equation for the critical pressure p:

$$\frac{K}{a^3}(m^2-1)(1-\xi)+p^2\frac{\xi}{m^2}\frac{5}{12}\frac{a}{D}\left(1-\frac{h^2}{4a^2}\right)-p\left\{\left(1+\frac{h}{2a}\right)-\xi\frac{5}{12}\left[\left(1+\frac{h}{2a}\right)-\frac{1}{m^2}\left(1-\frac{h}{2a}\right)\right]\right\}=0 \tag{42}$$

where ξ is a function of m, $\frac{h}{a}$ and ν

$$\xi=\frac{Km^2}{Km^2+a^2\frac{5}{6}Gh}=\frac{\frac{h^2}{a^2}m^2}{\frac{h^2}{a^2}m^2+5(1-\nu)} \tag{43}$$

If we ignore the quadratic term as being very small, we get the following approximate analytic expression:

$$p_{cr}=\frac{\frac{K}{a^3}(m^2-1)(1-\xi)}{1+\frac{h}{2a}-\xi\frac{5}{12}\left[\left(1+\frac{h}{2a}\right)-\frac{1}{m^2}\left(1-\frac{h}{2a}\right)\right]} \tag{44}$$

To the best of the author's knowledge, this expression has not appeared before in the literature. The minimum value of p_{cr} is for m=2 and reads

$$p_{cr(m=2)}=\frac{1}{4}\frac{E}{1-\nu^2}\left(\frac{h}{a}\right)^3\cdot\frac{\left(1-\xi_{(m=2)}\right)}{1+\frac{h}{2a}-\xi_{(m=2)}\left(\frac{5}{16}+\frac{25}{96}\frac{h}{a}\right)} \tag{45}$$

For very thin shells ($\frac{h}{a}<<1$) and small values of m ($\xi<<1$), equation 44 simplifies to the familiar expression from thin shell theory:

$$p_{cr}^{tst}=\frac{K}{a^3}(m^2-1)=\frac{1}{12}\frac{E}{1-\nu^2}\frac{h^3}{a^3}(m^2-1) \tag{46}$$

It's interesting also to note that while the simplified expression gives an ever increasing load as m increases i.e. $\lim_{m\to\infty}p_{cr}^{tst}\to\infty$, the new expression tends to the asymptotic value:

$$\lim_{m\to\infty} p_{cr} = \frac{\frac{E}{1-\nu^2}\left(\frac{h}{12a}\right)5(1-\nu)}{\left(1+\frac{h}{2a}\right)\left(\frac{7}{12}\right)} \tag{47}$$

The reason is the competing behaviour of the two terms in the nominator of equation 44: as m increases (m^2-1) increases quadratically, $(1-\xi)$ tends to 0 (also quadratically), but their product is finite. This asymptotic behaviour agrees with the benchmark solution as will be shown later in section 6.

It should be mentioned at this point that the expression for buckling pressure for thick rings can be easily obtained from 45. The derived formula suitable for thick rings is

$$p_{cr(ring)} = \frac{E}{4}\left(\frac{h}{a}\right)^3 \cdot \frac{\left(1-\xi_{(ring)}\right)}{1+\frac{h}{2a}-\xi_{(ring)}\left(\frac{5}{16}+\frac{25}{96}\frac{h}{a}\right)}, \quad \xi_{(ring)} = \frac{4\frac{h^2}{a^2}}{4\frac{h^2}{a^2}+\frac{5}{1+\nu}} \tag{48}$$

It can be easily seen that if E is replaced by $\frac{E}{1-\nu^2}$ and ν by $\frac{\nu}{1-\nu}$ equation 45 is obtained.

It is now easy to derive analytic expressions for the eigenfunctions $w(z), v(z)$. The final equations are:

$$\begin{aligned} w(z) &= B+\frac{1}{4}\frac{\nu}{1-\nu^2}\frac{h^2}{a^2}\left(2\frac{z^2}{h^2}-\frac{1}{10}\right)\left(-m^2B(1+\nu)+B-\nu mA-amC(1+\nu)\right) \\ v(z) &= \left(1+\frac{z}{a}\right)\cdot\left[\begin{aligned} &A+C\frac{5}{12}z\left(3-\frac{4z^2}{h^2}+\frac{3z}{a}\left(-\frac{1}{2}+\frac{z^2}{h^2}\right)\right)-mB\frac{z}{a}\left(-1+\frac{z}{a}\right)- \\ &\frac{1}{12}\frac{\nu}{1-\nu^2}\frac{h^2}{a^2}\frac{z}{a}\left(\frac{3}{10}\left(-1+\frac{z}{a}\right)+2\frac{z^2}{h^2}\left(1-\frac{3z}{2a}\right)\right)m\left(m^2B(1+\nu)-B+\nu mA+amC(1+\nu)\right)\end{aligned}\right] \end{aligned} \tag{49}$$

where A and C are given in terms of B as:

$$\begin{aligned} A &= -\left[\frac{\left(m^2-1\right)}{m^2}\xi\left(-\frac{K}{a^3}\frac{9}{16}m+p\left(1-\frac{h}{2a}\right)\frac{5}{12m}\right)\frac{a}{D}+\frac{1}{m}\right]\cdot B \\ C &= -\frac{\left(m^2-1\right)}{am}\xi\cdot B \end{aligned} \tag{50}$$

For comparison, the eigenfunctions for the thin shell theory are

$$
\begin{aligned}
w^{tst}(z) &= B \\
v^{tst}(z) &= B\left(m\frac{z}{a} - (1+\frac{z}{a})\frac{1}{m} \right)
\end{aligned}
\tag{51}
$$

The free parameter B in 49 and 51 is evaluated so that $w\left(a-\frac{h}{2}\right)=1$. In the following section the quantitative accuracy of the new formula for the critical load will be assessed against benchmark elasticity solutions. The analytic eigenfunctions will be also validated.

6. Comparison with elasticity solutions.

The developed code for the numerical solution of the differential stability equations was first validated against the results of Kardomateas (1993 and 2000) that account for rotations and rotations/strains respectively. The value of Poisson ratio ν is equal to 0.3 (the value of E is irrelevant as only normalised results are presented below). In order to examine the effect of cell density, computations were carried out with 20, 40 and 60 cells. The 2 finer meshes produced almost indistinguishable results.

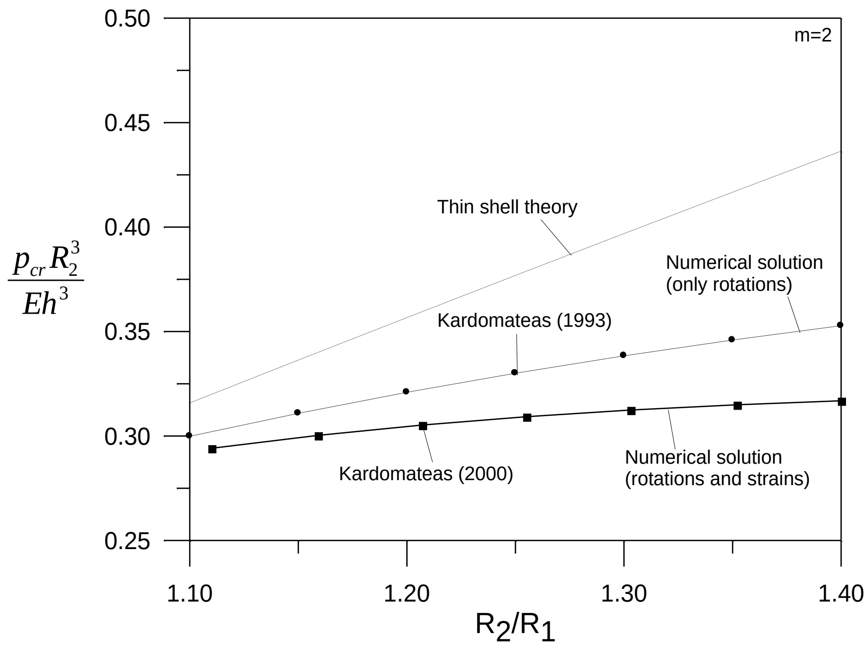

Figure 2. Critical pressure against R_2 / R_1; comparison of various approaches.

As can be seen from figure 2 the present computations match perfectly with these of Kardomateas (1993 and 2000). When strains are also included, the evaluated critical pressure is further reduced. On the other hand, the thin shell theory significantly overpredicts the critical pressure and the discrepancy increases with the thickness of the shell. It is exactly this discrepancy that the refined formula aims to correct.

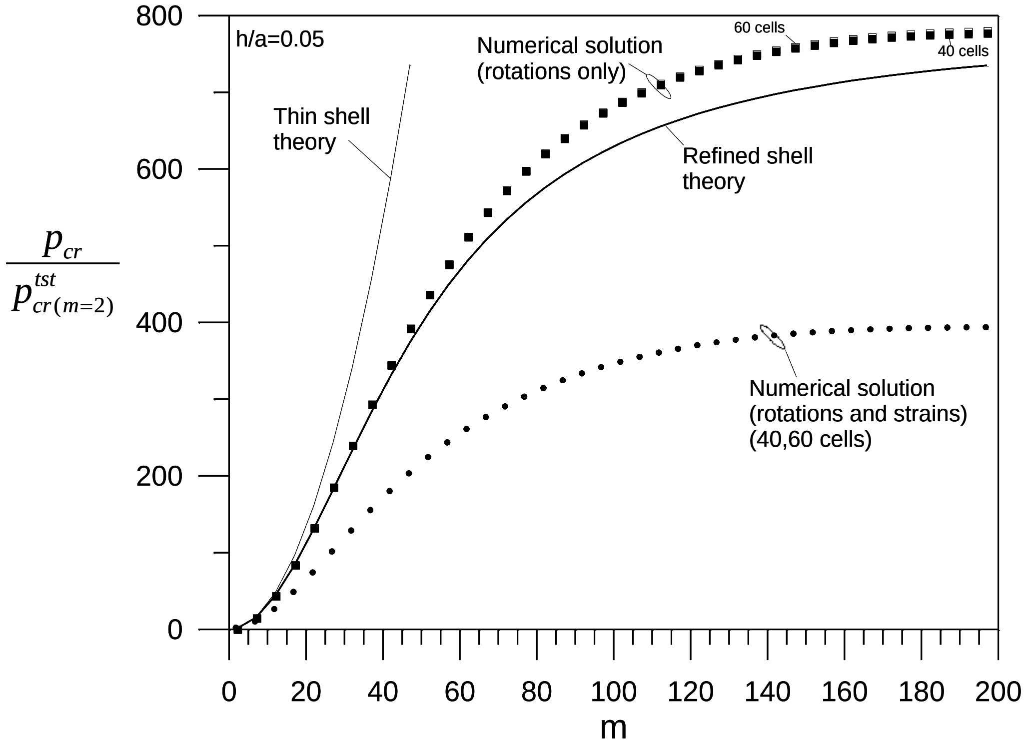

Figure 3. Variation of critical pressure with m.

Having validated the numerical code, attention is now focused on the variation of critical pressure with m. Figure 3 shows the variation of critical pressure with m for $h/a = 0.05$ (or $R_2/R_1 = 1.051$). The results are normalised with the critical pressure as predicted by the thin shell theory for m=2 $\left(p^{tst}_{cr(m=2)} = \frac{1}{4}\frac{E}{1-\nu^2}\frac{h^3}{a^3} \right)$. It is clear that the formula 44 matches closely the numerical results until about m=50. The critical load for large values of m approaches an asymptotic value and the trend is captured very well by the analytical formula. For small values of m (less than about 10) the strains have small effect on the solution but for larger values the results deviate. Finally it can be clearly seen that that thin shell solution is a good approximation to the numerical results only for small values of m but it rapidly deviates from the benchmark solution, failing to capture the asymptotic behaviour. For higher values of h/a, the behaviour is similar but the asymptotic value is reached for smaller values of m.

The critical pressure against h/a for m=2,3 and 4 is shown in figure 4. Although the smallest critical load for the case considered is obtained for m=2, it was decided to examine two more modes as there are practical problems (for example shells in elastic foundation) for which the minimum load is obtained for higher values of m (Brush and Almroth (1975)). The results are again normalised with the value from the thin shell theory. In order to facilitate the comparison the same scale is used in the vertical axis. It is clear that the novel formula does offer a significant improvement in accuracy with respect to the thin shell expression even for values of thickness to mid radius ratio (h/a) as large as 0.5. As expected, the predictions are closer to the benchmark results obtained by solving the system that contains rotations only. The effect of strains increases with the ratio h/a and the value of m. However for m=2, even for the highest value h/a=0.5, the predicted critical pressure differs from the most accurate benchmark solution (the one that includes rotations and strains) by less than 15%. This is a significant improvement compared to the 67% error from the buckling expression based on the thin shell theory.

For the largest value of $\frac{h}{a} = 0.5$ examined, the ratio $\frac{p_{cr(m=2)}}{E}$ was found to be 0.023 and the corresponding maximum stresses $\sigma_{rr}^0, \sigma_{\theta\theta}^0$ (absolute values) are equal to $p_{cr(m=2)}$ and $3.125 p_{cr(m=2)}$ respectively. The small ratios $\sigma_{rr}^0 /(2G+\lambda) = 0.017$, $\sigma_{\theta\theta}^0 /(2G+\lambda) = 0.053$ show that inequality 11 is indeed satisfied. Care however should be exercised for other types of structures, for example composite shells with soft core.

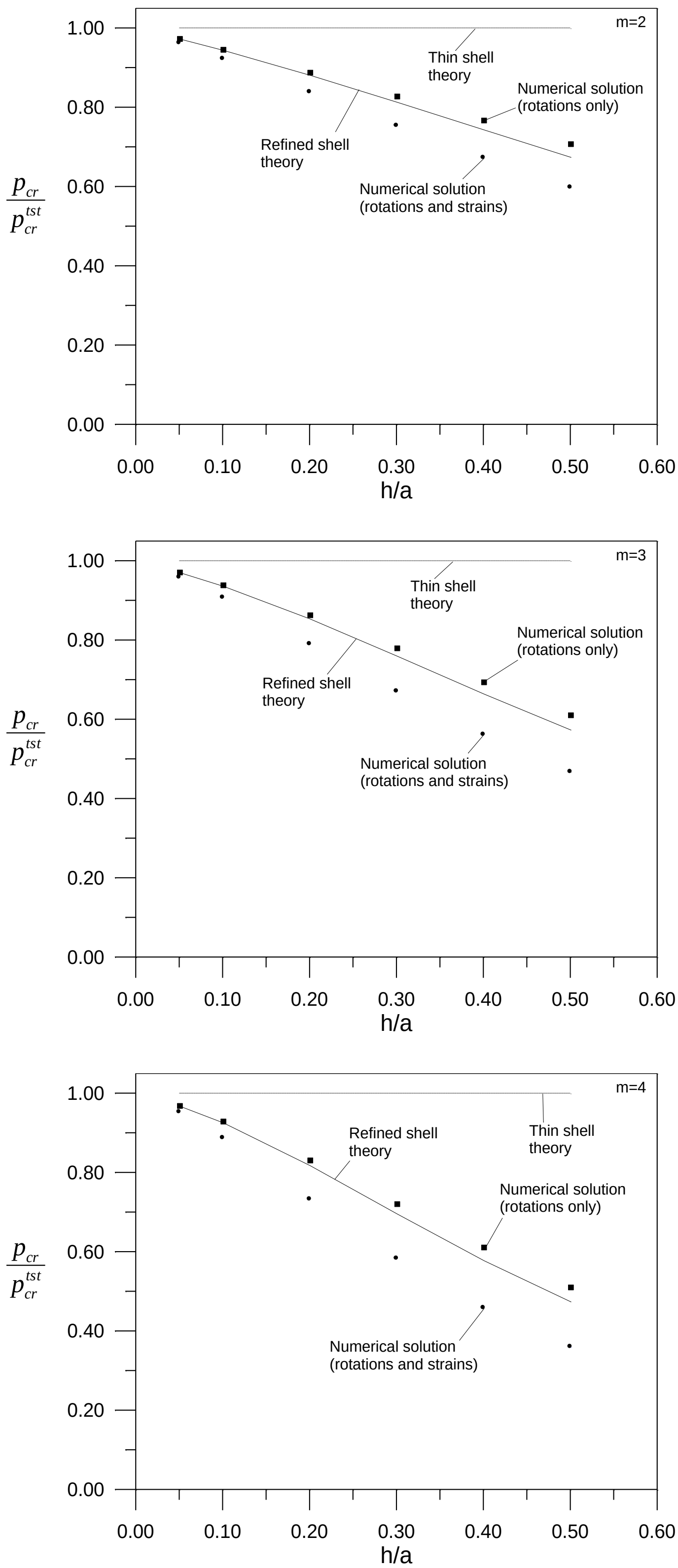

Figure 4 Variation of critical pressure against h/a for m=2, 3 and 4.

In order to further check the effect of thickness, figure 5 shows the variation of the ratio

$\frac{0.5e'_{r\theta}}{\omega'_x}$ across the thickness of the shell for various values of h/a. The ratio was evaluated from equations 17 after the eigen-solution was obtained. It is clear that for thin shells the shear strain can be neglected compared to rotation so the set of equations 13 is an accurate approximation of the full set. However as h/a increases the ratio also increases making this approximation less and less accurate.

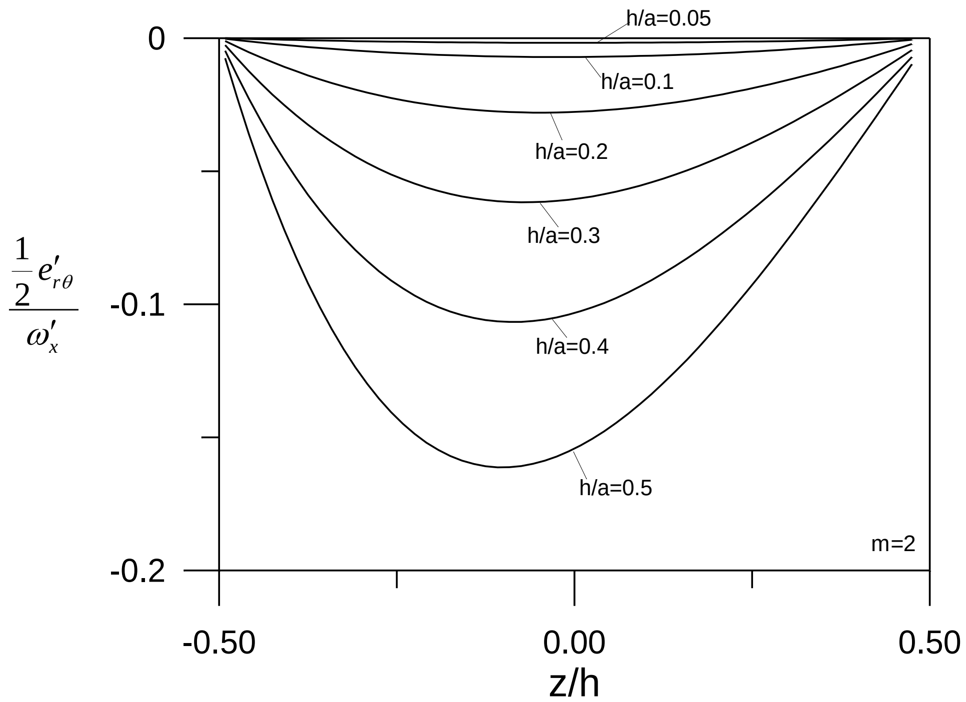

Figure 5 Variation of the ratio $\frac{0.5e'_{r\theta}}{\omega'_x}$ across the thickness for various values of h/a.

Attention is now turned to the eigenfunctions. For small values of ratio h/a the v(z) eigenfunction is a straight line and w(z) has constant value. Both the thin as well as the refined theory match very well with the benchmark solution as expected. For larger values of h/a non-linearities appear in the v(z) eigenfunction as can be seen in figure 6. This is more evident for m=3 and 4. It can be seen that the refined theory can capture very well both qualitatively and quantitatively the shape of the eigenfunction. For all values of m the standard shell theory predicts a straight line (see equation 51) with a larger slope compared to the average slope of the benchmark solution.

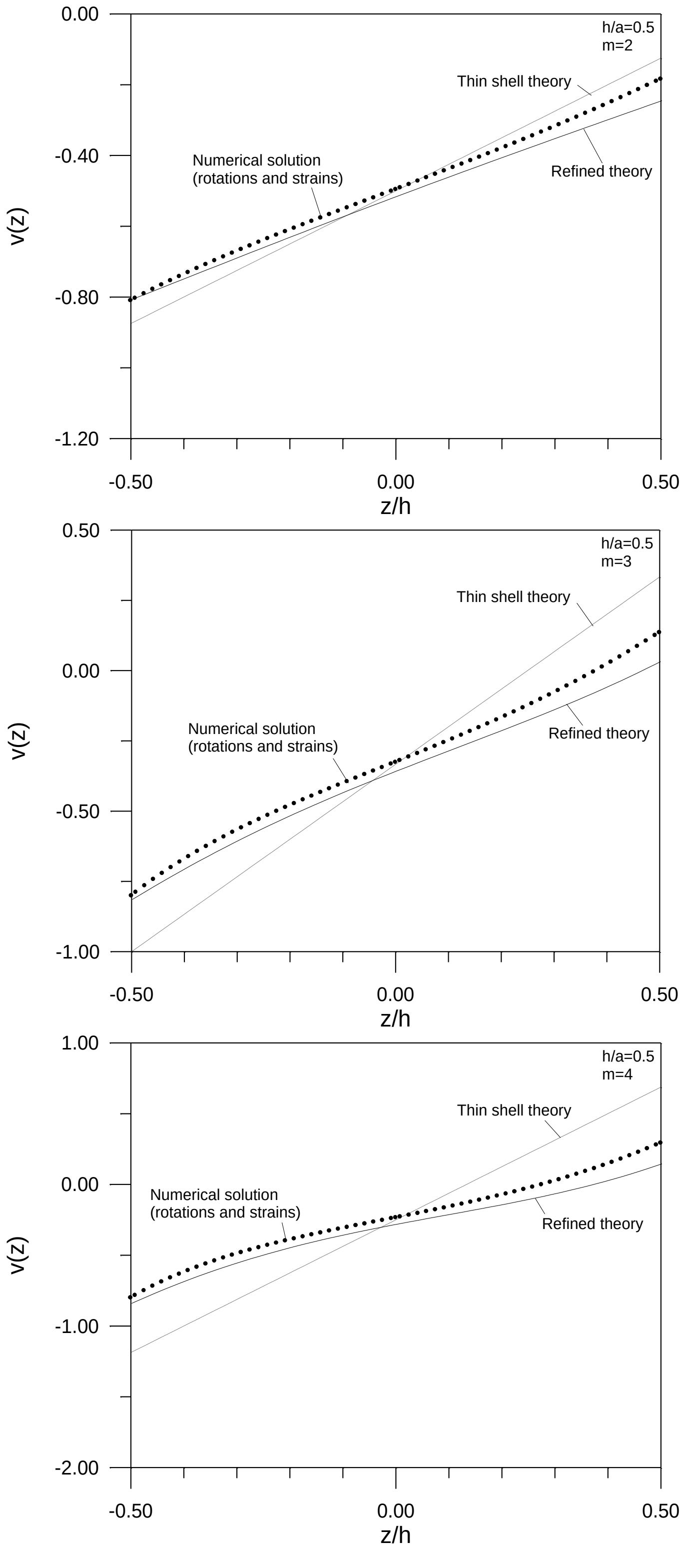

Figure 6 Eigenfunction v(z) for m=2, 3 and 4 (h/a=0.5).

It would be very interesting to apply the approach developed in the paper to investigate theoretically the critical load under different loading conditions. For example, it is known that asthmatic lung airways (that can be thought of as moderately thick shells) collapse under the action of smooth muscle cells that impose circumferential strain in the outer surface, see Hrousis (1998). Also coupling fluid flow and tube buckling opens new possibilities to study the dynamic behaviour of shells and to this end a fluid-structure-interaction methodology developed recently by the author (Papadakis (2008)) can be used.

7. Conclusions

The buckling equations for thick cylindrical shells were derived by integrating the differential stability equations across the thickness of the shell and a higher order shell theory was employed for the estimation of the stress and moment resultants. A formula was then derived that can provide an improved prediction of the critical load under external pressure. The results were compared against benchmark solutions of the stability equations and showed that it can predict much more accurately the critical load for thick shells compared to the expression due to standard shell theory. The effect of thickness on the relative magnitude of shear strain and rotation was also quantified. It was found that the shear strain/rotation ratio increases with thickness and that the inclusion of strains leads to a further reduction of the critical pressure for the isotropic case examined in this paper.

Acknowledgement

The author would like to thank the anonymous reviewers whose substantial and constructive comments significantly improved the paper.

8. References

Anastasiadis, J.S. and Simitses, G.J., 1993, Buckling of pressure-loaded, long, shear deformable, cylindrical laminated shells. Composite Structures, 23, 221-231.

Bažant, Z.P. and Cedolin, L.,2003, Stability of Structures; elastic, inelastic, fracture and damage theories. Dover Publications, Inc.

Bažant, Z.P., 1971, A correlation study of formulations of incremental deformation and stability of continuous bodies, Journal of Applied Mechanics-Transactions of ASME, pp. 919-928.

Brush, D.O. and Almroth, B.O., 1975, Buckling of bars, plates and shells, McGraw-Hill Inc.

Flügge, W., 1960, Stresses in shells, Springer-Verlag, Berlin, Heidelberg.

Fu L. and Waas, A.M., 1995, Initial post-buckling behaviour of thick rings under uniform external hydrostatic pressure. Journal of Applied Mechanics-Transactions of ASME, 62, 338-345.

Hrousis, C.A., 1998, Computational modelling of asthmatic airway collapse. PhD thesis, Dept of Mechanical Engineering, Massachusetts Institute of Technology.

Kardomateas, G.A., 1993, Buckling of thick orthotropic cylindrical shells under external pressure, Journal of Applied Mechanics-Transactions of ASME, 60, 195-202.

Kardomateas, G.A., 2000, Effect of normal strains in buckling of thick orthotropic shells. Journal of Aerospace Engineering, 13(3), 85-91.

Kardomateas, G.A. and Chung, C.B., 1994, Buckling of thick anisotropic cylindrical shells under external pressure based on non-planar equilibrium modes. Int. J. Solids Structures, 31(16), 2195-2210.

Kardomateas, G.A. and Simitses, G.J., 2005, Buckling of long sandwich cylindrical shells under external pressure. Journal of Applied Mechanics-Transactions of ASME, 72, 493-499.

Lai, W.M., Rubin, D. and Krempl, E., 1996, Introduction to continuum mechanics, 3rd edition, Butterworth-Heinemann, Massachusetts.

Malvern, L.E., 1969, Introduction to the mechanics of a continuous medium, Prentice-Hall Inc, New Jersey.

Novozhilov, V.V., 1953, Foundations of the nonlinear elasticity theory. Dover Publications, Inc.

Papadakis, G., 2008, A novel pressure-velocity formulation and solution method for fluid-structure-interaction problems, Journal of Computational Physics, 227, 3383–3404.

Pozrikidis, C., 1998, Numerical Computation in Science and Engineering, Oxford University Press.

Reddy, J.N. and Liu, C.F., 1985, A higher-order shear deformation theory of laminated elastic shells. International Journal of Engineering Science, 23(3), 319-330.

Reissner, E., 1945, The effect of transverse shear deformation on the bending of elastic plates, Journal of Applied Mechanics-Transactions of ASME, 12, A69-A77.

Shariat, B.A.S. and Eslami, M.R., 2007, Buckling of thick functionally graded plates under mechanical and thermal loads. Composite Structures, 78, 433-439.

Shen, H.S., 2001, Postbuckling of shear deformable cross-ply laminated cylindrical shells under combined external pressure and axial compression. International Journal of Mechanical Sciences, 43, 2493-2523.

Simitses, G.J., 1996, Buckling of moderately thick laminated cylindrical shells: a review. Composites, Part B, 27B, 581-587.

Simitses, G.J. and Anastasiadis, J.S., 1992, Shear deformable theories for cylindrical laminates-equilibrium and buckling with applications. AIAA Journal, 30(3), 826-834.

Simitses, G.J., Tabiei, A. and Anastasiadis, J.S., 1993, Buckling of moderately thick, laminated cylindrical shells under lateral pressure. Composites Engineering, 3(5), 409-417.

Takano, A., 2008, Improvement of Flügge's equations for buckling of moderately thick anisotropic cylindrical shells. AIAA Journal, 46(4), 903-911.

Timoshenko, S.P. and Gere, J.M., 1961, Theory of elastic stability, McGraw Hill Book Company.

Versteeg, H.K. and Malalasekera, W., 2007, An Introduction to Computational Fluid Dynamics; The finite Volume method, 2nd edition, Pearson Education Limited, Harlow England.

Voyiadjis, G.Z. and Baluch, M.H., 1981, Refined theory for flexural motions of isotropic elastic plates, Journal of Sound and Vibration, 76(1), 57-64.

Voyiadjis, G.Z. and Shi, G., 1991, A refined two-dimensional theory for thick cylindrical shells, Int. J. Solids Structures, 27(3), 261-282.

Voyiadjis, G.Z. and Woelke, P., 2004, A refined theory for thick spherical shells, Int. J. Solids Structures, 41, 3747-3769.

Appendix A: Comparison with the stability equations of Flügge (1960) for thin shells.

For thin shells $a+\frac{h}{2}\cong a$ and the circumferential stress is $\sigma_{\theta\theta}^{0}=-p\frac{a}{h}$. Substituting also the expressions 32 into the definition of rotation 7, we find that it is independent of z $\omega_x'=\frac{1}{a}\left(v_{ms}-\frac{\partial w_{ms}}{\partial\theta}\right)$. Therefore the integral I_θ' is equal to

$$\mathrm{I}_\theta' = \int_{-h/2}^{h/2}\sigma_{\theta\theta}^{0}\omega_x' dz = -p\left(v_{ms}'-\frac{\partial w_{ms}'}{\partial\theta}\right) \tag{A1}$$

Substituting these values to the set 26 we have

$$\begin{gathered}\frac{\partial Q_\theta'}{\partial\theta}+p\left(\frac{\partial v_{ms}'}{\partial\theta}-\frac{\partial^2 w_{ms}'}{\partial\theta^2}\right)-N_\theta'=0\\ -p\left(v_{ms}'-\frac{\partial w_{ms}'}{\partial\theta}\right)+\frac{\partial N_\theta'}{\partial\theta}+Q_\theta'+p\left(v_{ms}'-\frac{\partial w_{ms}'}{\partial\theta}\right)=0\end{gathered} \tag{A2}$$

or

$$\begin{aligned}&N_\theta'-\frac{\partial Q_\theta'}{\partial\theta}+p\left(\frac{\partial^2 w_{ms}'}{\partial\theta^2}-\frac{\partial v_{ms}'}{\partial\theta}\right)=0\\ &\frac{\partial N_\theta'}{\partial\theta}+Q_\theta'=0\end{aligned} \tag{A3}$$

because the pressure terms cancel out. The set of stability equations according to Flügge (1960) is:

$$\begin{aligned}&N_\theta'-\frac{\partial Q_\theta'}{\partial\theta}+p\left(\frac{\partial^2 w_{ms}'}{\partial\theta^2}-\frac{\partial v_{ms}'}{\partial\theta}\right)=0\\ &\frac{\partial N_\theta'}{\partial\theta}+Q_\theta'-p\frac{\partial}{\partial\theta}\left(\frac{\partial v_{ms}'}{\partial\theta}+w_{ms}'\right)=0\end{aligned} \tag{A4}$$

It can seen that first (i.e. radial) stability equations are identical. However, in the second equation (theta direction) the term $p\frac{\partial}{\partial\theta}\left(\frac{\partial v_{ms}'}{\partial\theta}+w_{ms}'\right)$ is missing from the set derived in this paper i.e. it is assumed that $\frac{\partial v'_{ms}}{\partial \theta} + w'_{ms} = 0$. This is the condition of inextensional buckling (Brush and Almroth (1975)). It is not surprising that this term is missing as the circumferential strain is given by $e'_{\theta\theta} = \frac{1}{a}\left(\frac{\partial v'_{ms}}{\partial \theta} + w'_{ms}\right)$ and it was neglected from the second term in equation 9(b). The analytical expression for the buckling load derived from set A3 is of course $p_{cr} = \frac{K}{a^3}(m^2 - 1)$.