

Diversification Benefits of Commodities for Cryptoasset Portfolios

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Abstract—The aim for balance between risk and reward in investment portfolios often requires studying the diversification contribution of its constituents. This objective requires to specify whether investors can extend their exposure in certain asset classes and benefit their portfolios in a statistically significant way. In this paper, we address this issue of diversification in the context of cryptoasset portfolios and examine whether their risk-adjusted performance can be enhanced through seeking exposure into the commodities class. For an equally-weighted portfolio of five cryptoassets, we first consider the addition of physical gold, as conceptualised by the CoinShares Gold and Cryptoassets Index, a diversified, monthly-rebalanced index that seeks exposure to both asset classes. We further consider modifying the index composition by replacing physical gold with a basket of five commodities. Mean-variance spanning tests reveal that the addition of physical gold in the original cryptoasset portfolio translates to a significant shift in the efficient frontier, both in terms of the global minimum variance and the tangency portfolios. Additionally, expanding the exposure in the commodity side confirms a statistically significant improvement, with the diversification benefit arising from a shift in the tangency portfolio. We further generate a number of price paths for the original index, the modified index and their components, according to a Dynamic Conditional Correlation GARCH specification, to assess the efficiency of the index weighted risk contribution scheme. Results demonstrate a superior performance of the two indices when compared against their constituents in terms of Omega ratio. The modified index appears more appropriate for investors that seek higher annual returns, while the original composition would be more appropriate for individuals with moderate annual return goals.

Index Terms—Cryptoassets, Gold, Commodities, Index, Mean-Variance Spanning, Diversification

I. INTRODUCTION

Cryptoassets have experienced broad market acceptance in the last few years and their inclusion in investment portfolios has become an important topic of interest. Their highly volatile nature has made the original works of Engle [1] and Bollerslev [2] around Generalised Autoregressive Conditional Heteroskedasticity (GARCH) models, a popular choice when evaluating cryptoasset price fluctuations. GARCH-type models are commonly used in financial time series modeling, in cases where time-varying volatility and volatility clustering are observed, and earlier studies have examined their application

in the cryptocurrency space as well. Chu et al. [3] examine various specifications of univariate GARCH-type models for twelve, major at the time, cryptoassets and assess them according to their ability to estimate Value-at-Risk. Multivariate settings are also typically considered, with Katsiampa [4] employing an Asymmetric Diagonal BEKK (named after Baba, Engle, Kraft and Kroner [5]) multivariate GARCH setting to examine the individual conditional variances as well as the volatility co-movements of five major cryptocurrencies. Overall, results reject normality and homoskedasticity and time-varying conditional correlations are persistent and positive. Kim et al. [6] also utilise multivariate approaches and examine the relationship between Bitcoin, Gold and the S&P 500 index using a variety of Dynamic Conditional Correlation models (DCC) GARCH specifications. Results demonstrate a positive time-varying relationship between the examined markets.

One of the main steps of portfolio optimisation is selecting a target function that needs to be optimised. Investors that are interested in mitigating risk, may choose to form a portfolio that is constructed through minimisation of the standard deviation. Another popular choice is constructing a portfolio that gives the highest expected return per unit of risk. The Sharpe ratio is a metric that is used on this occasion, and is defined as the return divided by the standard deviation of the investment. Maximising the Sharpe ratio produces a “risk-efficient” allocation that is commonly referred to as the tangency portfolio. While the Sharpe ratio is the most commonly used risk–return assessment metric by investors, its main disadvantage is that it accounts only for the first two moments of the returns distribution (average asset return and standard deviation). The Omega ratio [7], defined as the probability-weighted ratio of gains over losses given a specific level of threshold return, is an alternative metric that is used in portfolio theory to address the shortcomings of the Sharpe ratio. Asset allocation methods can also follow simplistic approaches, such as equally-weighted portfolios. Recent studies have examined the applications of the aforementioned principles of modern portfolio theory in investment strategies that include cryptoassets.

In [8], Liu considers ten major cryptoassets and examines the out-of-sample performance of commonly-used asset allocation strategies. Results demonstrate that diversifying across the crypto-market improves risk-adjusted performance, with

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the equal-weighting scheme achieving the best results in terms of Sharpe ratio. Brauneis and Mestel [9] also assess the risk-adjusted performance of cryptoasset portfolios, based on the traditional Markowitz mean-variance framework. The authors employ different mean-variance strategies to form diversified portfolios of 500 cryptoassets and compare their performance to individual cryptoasset investments. They also consider two benchmarks for performance comparison, an equally-weighted portfolio and the CRIX [10] market capitalisation index. In terms of the Sharpe ratio, they conclude that naïvely diversified portfolios outperform single cryptoasset investments and mean-variance optimised portfolios.

Castro et al. [11] suggest a framework that maximises the Omega ratio. They consider four diversified investment portfolios and conclude that, while crypto-exposure improves returns, it also increases risk. Bedi and Nashier [12] construct diversified portfolios, denominated in USD and converted to GBP, EUR, JPY and CNY and conclude that CNY, JPY and USD portfolios record significantly higher gains if they include Bitcoin. Wu and Pandley [13] also examine the impact of including Bitcoin in a portfolio of stocks, bonds, currencies, real estate and commodities. Results confirm that investors can benefit from holding a small amount of Bitcoin in their portfolios. For a comprehensive overview of cryptocurrency trading-related research, readers can consult the works of Fang et al. [14]

The inclusion of risky assets, such as cryptoassets, in investment portfolios requires careful consideration of the impact on the risk-adjusted performance. One way to account for this impact is to examine the changes in the portfolio's efficient frontier. The efficient frontier refers to a set of optimal portfolios that provide the highest level of returns at a given level of risk. Graphically, the efficient frontier is formed by plotting the portfolio's expected return against its standard deviation and is curved because each unit of risk added to a portfolio gains a diminishing amount of return. Huberman and Kandel [15] propose a multivariate test setting that examines the hypothesis that the efficient frontier of a benchmark set of K assets is the same as the efficient frontier of an augmented set of $K + N$ assets. Failure to reject the proposed hypothesis would suggest that the initial portfolio setting cannot be benefited by investing in the additional N assets under consideration. Kan and Zhou [16] conduct a comprehensive study of tests for mean-variance spanning based on the regression framework of Huberman and Kandel [15] and present in detail the Likelihood Ratio, Wald, Lagrange multiplier and step-down F tests.

Notwithstanding past studies on crypto-market dynamics and the diversification properties of cryptoassets, their combination with traditional asset classes remains an important topic of discussion. The primary goal of this paper is to examine the diversification benefits of commodities for the cryptoasset space and to gain insight into how their inclusion in cryptoasset portfolios affects the mean-variance frontier. We first consider the case of the CoinShares Gold and Cryptoassets Index and assess the diversification benefit of physical gold

for a portfolio of five equally-weighted crypto-constituents. We further consider a case of diversified commodity exposure by replacing physical gold with a basket of five commodities and assess whether there exist opportunities to improve the index risk-adjusted performance. Both cases reject the mean-variance spanning null hypothesis, with gold shifting both the global minimum variance and tangency portfolios. The formation of the commodity basket shifts only the tangency portfolio of the index. We also isolate the main market determinants of the formed portfolios, namely the crypto-basket, gold and the commodity basket, and study the dynamics of their conditional correlation. We produce 1000 paths for each component and report the Omega ratio following the index rebalancing scheme. We find that the original index approach is more appropriate for investors that seek moderate annual returns, while the modified index can target higher returns.

The remainder of this paper is organised as follows. Section II presents the essential background in regards to the design and aim of the index. In Section III, we elaborate on regression-based mean-variance spanning tests and DCC-GARCH approaches for price simulations. Section IV demonstrates the geometrical interpretation of the spanning test results and the derived omega ratios for a set of 1000 price path simulations, according to the dynamic conditional correlation specification. Section V concludes.

II. BACKGROUND

The CoinShares Gold and Cryptoassets Index (CGCI) [17] is an index that offers investors exposure to digital assets. The CGCI is comprised of two components; an equally-weighted basket of five cryptoassets and an investment in physical gold. The crypto-constituents are selected and rebalanced monthly and include the top five eligible cryptoassets based on the 6-month rolling mean of their free-float market capitalisation. In contrast with traditional market capitalisation index products that serve as market benchmarks (e.g. [10], [18], [19]), the CGCI is designed to achieve a better risk profile through the concepts of diversification and frequent rebalancing.

Two main characteristics of the cryptoasset space that make portfolio and index construction challenging tasks, are the high levels of volatility and high intraclass correlations. Strategies that revolve around risk distribution among constituents have been capable of improving risk-adjusted performance in traditional asset classes and are a popular choice in index design. Equal Risk Contribution (ERC) falls under this category and is a strategy that aims to distribute risk, rather than capital, equally across all constituents. The weighting is based on inverse volatility which implies that generally the most volatile components have the lowest weights. If correlations across all constituent pairs are the same in the ERC setting, then each constituent weight is given as the reciprocal of its volatility divided by the sum of the reciprocals of the volatilities of all constituents:

$$x_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^N \sigma_j^{-1}}, i = 1, \dots, N \quad (1)$$

The CGCI considers the former and combines the cryptoasset class with physical gold, an uncorrelated wealth-preserving asset class, to achieve superior risk-adjusted performance. One way to account for the highly volatile nature of the crypto-component is to consider a weighting scheme that is specifically concerned with distributing risk. The additional lack of correlation between the two asset classes indicates that the setting of Eq. 1 is appropriate for weighting the two components. However, due to the significant differences in volatility levels between cryptoassets and gold, any attempt to equally distribute risk among them leads to overconcentration in the low-volatility asset.

We consequently propose a Weighted Risk Contribution scheme (WRC), which avoids overconcentration by allowing the proportion of risk contributed by each asset class to be configurable. We follow the main ERC definitions and obtain the two-asset-case vector of risk contributions:

$$\frac{1}{\sigma(x)} \begin{pmatrix} x_1^2 \sigma_1^2 + x_1 x_2 \rho \sigma_1 \sigma_2 \\ x_2^2 \sigma_2^2 + x_1 x_2 \rho \sigma_1 \sigma_2 \end{pmatrix},$$

where $x = (x_1, x_2)$ is the weighting among two components and ρ denotes the correlation. In the case of uncorrelated assets and supposing that we desire the risk contribution of asset 1 to be α times the risk contribution of asset 2, we need to solve for x_1 in $x_1^2 \sigma_1^2 = \alpha (x_2^2 \sigma_2^2)$. Given a long-only constraint, $x_i \in [0, 1]$ and $\sum_{i=1}^2 x_i = 1$, this yields:

$$x_1 = \frac{\sqrt{\alpha} \sigma_1^{-1}}{\sqrt{\alpha} \sigma_1^{-1} + \sigma_2^{-1}} \quad (2)$$

In the case of CGCI, $x_1 = x_c$ and $x_2 = x_g$ represent the weights for a basket of equally-weighted cryptoassets and gold respectively. Additionally, $\sigma_1 = \sigma_c$ and $\sigma_2 = \sigma_g$ are the historical volatilities of the crypto-basket and gold logarithmic daily returns, and the risk contribution ratio is set as $\alpha = 4$, which indicates that 80% of the overall portfolio risk emanates from the crypto-basket. Setting $\alpha = 1$ would result in the traditional ERC scheme.

In summary, index calculation involves the following steps:

- 1) Computation of the 6-month rolling volatility of (a) the equally-weighted crypto-basket, and (b) gold;
- 2) Allocation among the crypto-basket and gold expressed as the weighted risk contribution problem presented in Eq. 2, where $\alpha = 4$.

The occurrence of events unique to the cryptoasset space, such as hard forks and airdrops, poses additional challenges. A hard fork, for example, may be problematic when calculating index price levels, particularly if there is significant disagreement about the ticker symbols that both the old and new cryptoassets should trade under. In the case of CGCI, exchange decisions about the ticker symbols used to describe the resulting cryptoassets and the markets that they sustain, determine how hard forks are handled. For details around the full pricing methodology of the crypto-constituents, the crypto-basket and the CGCI price level computations as well

as treatment of hard forks and airdrops, readers can consult the official methodology document¹.

III. METHODOLOGY

A. Regression-Based Tests of Mean-Variance Spanning

The process of constituent selection in portfolio construction often raises the question of whether a set of additional assets can improve the risk-adjusted returns of a base asset set. Huberman and Kandel [15] address this issue and, given a set of K assets (benchmark assets) and a set of N additional assets (test assets), they test the hypothesis that the efficient frontier of K is the same as the at of $K + N$. The concept of mean-variance spanning therefore states that a set of K assets spans a larger set of $N + K$ assets if the minimum-variance frontier of K assets is identical to the minimum-variance frontier $K + N$.

Relevant works in financial literature [16], [20], [21] analyse whether investors, given a set of K assets, can benefit through investing in an additional set of N assets. Following the definitions of Kan and Zhou [16], we denote with $R_t = [R'_{1t}, R'_{2t}]$ the returns of the assets under consideration. Here R'_{1t} is a vector of length K , with the returns of the K benchmark assets at time t and R'_{2t} is a vector of length N , with the returns of the N test assets at time t . The expected returns and the covariance matrix of the $N + K$ assets are given by:

$$\mu = E[R_t] = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad V = \text{Var}[R_t] = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \quad (3)$$

where $E[R_{1t}] = \mu_1$ and $E[R_{2t}] = \mu_2$, V_{11} is the covariance matrix of the K benchmark assets, V_{22} is the covariance matrix of the N test assets, V_{12} , V_{21} represent the covariances of the benchmark assets with the test assets. For the mean-variance spanning test setting, R_{2t} is projected on R_{1t} :

$$R_{2t} = \alpha + \beta R_{1t} + \epsilon_t, \quad (4)$$

where $E[\epsilon_t] = 0_N$ and $E[\epsilon_t R'_{1t}] = 0_{N \times K}$. In Eq. 4, α and β are given by $\alpha = \mu_2 - \beta \mu_1$ and $\beta = V_{21} V_{11}^{-1}$.

In matrix notation, assuming length T for the assets' time-series, we have:

$$R = XB + E, \quad (5)$$

where R is the $T \times N$ matrix of the test asset returns R_{2t} , X is the $T \times (K + 1)$ matrix where each row is given by $[1, R'_{1t}]$, $t \in [1, T]$, $B = [\alpha, \beta]'$ and E is a $T \times N$ matrix where each row is given by ϵ'_t . The residuals ϵ_t are assumed to be independent and identically distributed as multivariate normal with zero mean and variance Σ .

Setting $\delta = 1_N - \beta 1_K$, the null hypothesis for spanning, provided by Huberman and Kandel [15], is given by:

$$H_0 : \alpha = 0_N, \delta = 0_N \quad (6)$$

The null hypothesis highlights the fact that mean-variance spanning tests are joint tests. Condition $\alpha = 0$ tests whether the test assets improve the tangency portfolio and $\delta = 0$ tests

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whether they improve the global minimum variance portfolio. The two conditions together imply that adding the N test assets to the base portfolio of K assets does not shift the efficient frontier significantly.

The null hypothesis in Eq. 6 can be rewritten as $H_0 : \Theta = 0_{2 \times N}$, where $\Theta = [\alpha, \delta]'$. In this case, $\Theta = AB - C$, where

$$A = \begin{bmatrix} 1 & 0'_K \\ 0 & -1'_K \end{bmatrix}, \quad C = \begin{bmatrix} 0'_N \\ -1'_N \end{bmatrix}, \quad (7)$$

Mean-Variance spanning is presented in [16] as a likelihood ratio test (LR), based on the regression given by Eq. 4, and can accordingly be supplemented with the Wald (W) and the Lagrange multiplier (LM) tests. We define the estimation matrices

$$\hat{G} = TA(X'X)^{-1}A', \quad \hat{H} = \hat{\Theta}\hat{\Sigma}^{-1}\hat{\Theta}' \quad (8)$$

$\hat{\Sigma}$ in Eq. 8 is the unconstrained maximum likelihood estimator of the covariance matrix of error terms Σ . Denoting by $\hat{B}$ the estimator of B , we have:

$$\hat{B} = [\hat{\alpha}, \hat{\beta}]' = (X'X)^{-1}(X'R), \quad (9)$$

$$\hat{\Sigma} = \frac{1}{T} (R - X\hat{B})' (R - X\hat{B}). \quad (10)$$

Further denoting by λ_1 and λ_2 the eigenvalues of $\hat{H}\hat{G}^{-1}$, with $\lambda_1 \geq \lambda_2 \geq 0$, we obtain the three test statistics:

$$\text{LR} = T \sum_{i=1}^2 \ln(1 + \lambda_i) \stackrel{A}{\sim} \chi_{2N}^2, \quad (11)$$

$$\text{W} = T(\lambda_1 + \lambda_2) \stackrel{A}{\sim} \chi_{2N}^2, \quad (12)$$

$$\text{LM} = T \sum_{i=1}^2 \frac{\lambda_i}{1 + \lambda_i} \stackrel{A}{\sim} \chi_{2N}^2. \quad (13)$$

All three tests have an asymptotic χ_{2N}^2 distribution, although we must have $\text{W} \geq \text{LR} \geq \text{LM}$ in finite samples [22], [23]. The three tests might produce conflicting results, with W favoring rejection and LM favoring acceptance. It is therefore important to perform all three tests when deciding upon the mean-variance spanning outcome.

The main determinant of the power of the spanning tests is the difference between the risk levels in the two minimum variance portfolios [16]. That is because, although spanning tests are joint tests, they weigh the estimates of the two constants according to their accuracy. In this case, the estimation of δ is statistically more accurate since it does not involve any estimations of expected returns. Additionally, a statistically significant shift in the minimum variance portfolio might not be as important to an investor as an improvement to the tangency portfolio. It is therefore suggested to examine the two parts of the hypothesis in Eq. 6 separately. This involves a step-down procedure which first examines whether $\alpha = 0_N$ through an F-test (F_1) and secondly the condition $\delta = 0_N$, also through an F-test (F_2). A rejection of the F_1 test will hint that the two tangency portfolios are different while the rejection of F_2 will indicate a significant difference in the

minimum variance portfolios. The original hypothesis will be accepted only if both individual tests are accepted. F_1 has a central F-distribution with N and $T - K - N$ degrees of freedom, given by:

$$F_1 = \left(\frac{T - K - N}{N} \right) \left(\frac{|\hat{\Sigma}|}{|\hat{\Sigma}|} - 1 \right) = \left(\frac{T - K - N}{N} \right) \left(\frac{\hat{\alpha} - \hat{\alpha}_1}{1 + \hat{\alpha}_1} \right), \quad (14)$$

where $\hat{\Sigma}$ is an unconstrained estimate of the covariance matrix Σ and $\hat{\Sigma}$ is an estimate of Σ , conditional on $\alpha = 0_N$. Additionally, $\hat{\alpha} = \hat{\mu}'\hat{V}^{-1}\hat{\mu}$, $\hat{\mu} = \frac{1}{T} \sum_{t=1}^T R_t$, $\hat{V} = \frac{1}{T} \sum_{t=1}^T (R_t - \hat{\mu})(R_t - \hat{\mu})'$, $\hat{\alpha}_1 = \hat{\mu}_1'\hat{V}_{11}^{-1}\hat{\mu}_1$. Here $\hat{\mu}$ and $\hat{V}$ represent the maximum likelihood estimates of the expected return and covariance matrix of the augmented portfolio, respectively.

Similarly, F_2 has a central F-distribution with N and $T - K - N + 1$ degrees of freedom, it is independent of F_1 and is given by:

$$F_2 = \left(\frac{T - K - N + 1}{N} \right) \left(\frac{|\hat{\Sigma}|}{|\hat{\Sigma}|} - 1 \right) = \left(\frac{T - K - N + 1}{N} \right) \left[\left(\frac{\hat{c} + \hat{d}}{\hat{c}_1 + \hat{d}_1} \right) \left(\frac{1 + \hat{\alpha}_1}{1 + \hat{\alpha}} \right) - 1 \right], \quad (15)$$

where $\hat{\Sigma}$ is an estimate of the residual covariance matrix Σ , conditional on $\alpha = 0_N$ and $\delta = 0_N$, $\hat{b}_1 = \hat{\mu}_1'\hat{V}_{11}^{-1}1_K$, $\hat{c}_1 = 1'_K\hat{V}_{11}^{-1}1_K$, $\hat{d}_1 = \hat{\alpha}\hat{c} - \hat{b}^2$, $\hat{b} = \hat{\mu}'\hat{V}^{-1}1_{N+K}$, $\hat{c} = 1'_{N+K}\hat{V}^{-1}1_{N+K}$ and $\hat{d} = \hat{\alpha}\hat{c} - \hat{b}^2$. The mentioned constants will later also assist the geometrical interpretation of the performed spanning tests.

B. Conditional Variances and Correlations

In financial econometrics, it is important to consider the dependence structure in the second-order moments of asset returns. Multivariate GARCH models deal with the issue of correlation and their main application is studying the relationship between the volatilities of different markets. Multivariate frameworks are typically more relevant than independent univariate models and improve the decision-making process in volatility prediction, portfolio selection and risk management.

Multivariate GARCH models can generally be expressed as:

$$\mathbf{r}_t = \boldsymbol{\mu}_t + \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t = \mathbf{H}_t^{1/2} \mathbf{z}_t, \quad (16)$$

where $\boldsymbol{\epsilon}_t$ is the error term at time t , expressed through the mean-corrected time-series of $\mathbf{r}_t$, $E[\boldsymbol{\epsilon}_t] = 0$, $\boldsymbol{\mu}_t$ is the $n \times 1$ vector of the expected values of the asset returns $\mathbf{r}_t$, $\mathbf{H}_t$ is the $n \times n$ matrix of conditional variances of $\boldsymbol{\epsilon}_t$ at time t and $\mathbf{z}_t$ is the $n \times 1$ vector of the independent and identically distributed residuals. $\mathbf{H}_t^{1/2}$ can be obtained through the Cholesky factorisation of $\mathbf{H}_t$.

There exist different approaches of the multivariate GARCH setting that allow to specify the conditional covariance matrix $\mathbf{H}_t$. One method models the conditional variances and correlations rather than straightforwardly modelling the conditional

covariance matrix. In this case, the conditional covariance matrix $\mathbf{H}_t$ is decomposed as follows:

$$\mathbf{H}_t = \mathbf{D}_t \mathbf{R}_t \mathbf{D}_t, \quad (17)$$

where $\mathbf{D}_t$ is the diagonal matrix of conditional volatilities $\mathbf{D}_t = \text{diag}(h_{1t}^{1/2}, \dots, h_{nt}^{1/2})$, and $\mathbf{R}_t$ denotes the conditional correlation matrix of the mean-corrected time-series, ϵ_t .

C. Dynamic Conditional Correlation GARCH

This paper is concerned with the issue of modeling the dependence structure of the time-varying volatilities and correlations of two asset classes through a Dynamic Conditional Correlation (DCC) GARCH model, as introduced by Engle and Sheppard in 2001 [24]. In this setting, both the correlation matrix $\mathbf{R}_t$ and the conditional variances $\mathbf{D}_t$ are time-varying and the DCC-GARCH is defined as:

$$r_t = \mu_t + \epsilon_t, \quad \epsilon_t = \mathbf{H}_t^{1/2} \mathbf{z}_t, \quad \mathbf{H}_t = \mathbf{D}_t \mathbf{R}_t \mathbf{D}_t, \quad (18)$$

The diagonal matrix $\mathbf{D}_t$ contains the time-varying volatilities, which can be expressed as univariate GARCH processes:

$$\begin{pmatrix} \sqrt{h_{1t}} & 0 & \dots & 0 \\ 0 & \sqrt{h_{2t}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{h_{nt}} \end{pmatrix} \quad (19)$$

where

$$h_{it} = \alpha_{i0} + \sum_{q=1}^{Q_i} \alpha_q \epsilon_{t-q}^2 + \sum_{p=1}^{P_i} \beta_p h_{t-p}. \quad (20)$$

The given GARCH model is a symmetric one where positive and negative returns influence volatility in the same way. However, positive innovations to volatility practically appear to correlate with negative market information (and vice versa), while positive returns should cause less uncertainty. Glosten, Jagannathan, and Runkle [25] have proposed a way to account for this asymmetry (GJR-GARCH) through modelling the positive and negative shocks on the conditional variance asymmetrically via the use of an indicator function I_t .

There is no need for the univariate GARCH models to be of the same order and μ_t in Eq. 18 can be expressed either as a constant vector or as a time-varying process. For the purposes of this paper we choose a first order ARMA model for the conditional mean and a first order GJR-GARCH for each one of the univariate GARCH models:

$$\begin{aligned} r_t &= \mu + \phi_1(X_{t-1} - \mu) + \theta_1 \epsilon_{t-1} + \epsilon_t \\ \epsilon_t &= \sigma_t Z_t \\ \sigma_t^2 &= \omega + \zeta_1 v_{1t} + (\alpha_1 \epsilon_{t-1}^2 + \gamma_1 I_{t-1} \epsilon_{t-1}^2) + \beta_1 \sigma_{t-1}^2, \end{aligned} \quad (21)$$

where ω is a volatility offset term and γ_1 represents the leverage term. The indicator function I_t takes on value of 1 for $\epsilon_t \leq 0$ and 0 otherwise.

IV. ANALYSIS AND RESULTS

A. Datasets

Both the estimation of the DCC-GARCH parameters and the mean-variance tests are performed on a historical dataset of 1308 price observations ranging from July 2nd, 2015 to November 2nd, 2020. The dataset contains the price time-series of the crypto-basket component of the CGCI rather than the individual prices of its crypto-constituents because (i) this study is not examining any alterations in the crypto-basket composition and (ii) the CGCI selection rules around the crypto-basket composition would make the simulation process unnecessarily complex. The crypto-basket historical prices are calculated using tick-by-tick trade data provided by Kaiko. The dataset also contains the time-series of five commodities. Gold prices, under the original CGCI calculations, correspond to the LBMA Gold Price PM data provided by ICE Benchmark Administration (IBA). The rest correspond to the respective price observations of iShares Silver Trust (SLV), Aberdeen Standard Physical Platinum Shares ETF (PPLT), Aberdeen Standard Physical Palladium Shares ETF (PALL) and West Texas Intermediate Crude Oil (WTI).

B. Regression-based Spanning Hypothesis Testing

The CGCI is designed based on the premise that the addition of physical gold in a cryptoasset portfolio improves the risk-return profile of the initial crypto-investment. We will seek to confirm this diversification benefit as statistically significant through the spanning tests of Section III-A. Since spanning tests are not explicitly concerned with a specific allocation style, but rather examine the shift of the frontier hyperbola as a whole, we define the benchmark investment universe as the collection of any cryptoasset c that has historically been a constituent of the CGCI. We denote their daily logarithmic returns, R_c , R_{GLD} , R_{SLV} , R_{PPLT} , R_{PALL} and R_{WTI} . The regression setting for the tests is given by:

$$\mathbf{R}_i = \boldsymbol{\alpha} + \sum_{j=1}^K \beta_j \mathbf{R}_j + \boldsymbol{\epsilon}, \quad (22)$$

where $\mathbf{R}_j$ is the set of constituents comprising the cryptoasset-only benchmark portfolio and $\mathbf{R}_i$ denotes the test asset under consideration for addition. In this first case, the test asset $\mathbf{R}_i$ corresponds to physical gold. All three tests of spanning reject the null hypothesis, with $LR = 3160$, $p < 0.001$, $W = 27723$, $p < 0.001$ and $LM = 891$, $p < 0.001$, $W \geq LR \geq LM$. We conclude that the benchmark portfolio does not span the augmented portfolio and the diversification benefit is statistically significant.

The constants that describe the efficient frontier hyperbola on the $(\hat{\sigma}, \hat{\mu})$ space of the benchmark portfolio of K assets are $\hat{\alpha}_1 = \hat{\mu}_1' \hat{V}_{11}^{-1} \hat{\mu}_1$, $\hat{b}_1 = \hat{\mu}_1' \hat{V}_{11}^{-1} \mathbf{1}_K$, $\hat{c}_1 = \mathbf{1}_K' \hat{V}_{11}^{-1} \mathbf{1}_K$ and $\hat{d}_1 = \hat{\alpha} \hat{c} - \hat{b}^2$. Accordingly, the constants that define the location of the efficient frontier of the augmented portfolio of $N + K$ assets are, $\hat{\alpha} = \hat{\mu}' \hat{V}^{-1} \hat{\mu}$, $\hat{b} = \hat{\mu}' \hat{V}^{-1} \mathbf{1}_{N+K}$, $\hat{c} = \mathbf{1}_{N+K}' \hat{V}^{-1} \mathbf{1}_{N+K}$ and $\hat{d} = \hat{\alpha} \hat{c} - \hat{b}^2$, where $\hat{\mu} = \frac{1}{T} \sum_{t=1}^T R_t$ and $\hat{V} = \frac{1}{T} \sum_{t=1}^T (R_t - \hat{\mu})(R_t - \hat{\mu})'$. Here $\hat{\mu}$ and $\hat{V}$ represent

the maximum likelihood estimates of the expected return and covariance matrix of the augmented portfolio and $\hat{\mu}_1$ and $\hat{V}_{11}$ represent the maximum likelihood estimates of the expected return and covariance matrix of the benchmark assets. We denote the global minimum variance and tangency benchmark portfolios by GMVP_1 and TP_1 and the global minimum variance and tangency augmented portfolios by GMVP and TP . The slopes of the asymptotes to the efficient hyperbolae of the benchmark and augmented portfolios are given by $\sqrt{\hat{d}_1/\hat{c}_1}$ and $\sqrt{\hat{d}/\hat{c}}$ respectively. Figure 1 displays the two frontier hyperbolae for the first round of spanning tests, where the benchmark portfolio only contains cryptoassets, while the augmented portfolio corresponds to the CGCI. The diversification benefit of gold is mainly attributed to a significant reduction of risk. The augmented global minimum variance portfolio corresponds to a risk level of $0.85 * \sqrt{252} = 13.5\%$ annualised volatility, compared against $4.73 * \sqrt{252} = 75.1\%$ for the cryptoasset-only benchmark. In terms of Sharpe ratio, the benchmark tangency portfolio is characterised by an annualised Sharpe value of 1.83, while the diversified portfolio presents an average simulated annualised value of 2.0. The efficient hyperbolae are characterised by asymptotes with slopes $\sqrt{\hat{d}_1/\hat{c}_1} = 0.0051$ and $\sqrt{\hat{d}/\hat{c}} = 0.0115$.

Following the rejection of the joint spanning hypothesis and the geometrical interpretation of the two portfolios, we follow the step-down procedure described in Section III-A and derive the F_1 , F_2 statistics and their p-values. The separation of the two tests therefore allows us to isolate the reasons for rejection if the original null hypothesis is rejected. If the rejection derives from F_1 , we can conclude that there is a statistically significant difference between the two tangency portfolios, while a rejection of F_2 will indicate that a significant difference is observed between the two global minimum variance portfolios. Another benefit of the step-down procedure is that it allows us to adjust the significance levels of the two tests, according to the perceived economical significance of each. Given that investors are typically interested in risk-adjusted returns rather than solely volatility, and since a statistically significant shift in the tangency portfolio is much more difficult to detect [16], it is generally advisable to set the significance level of F_1 higher and that of F_2 lower. Nevertheless, the interpretation of the p-values in the separated tests is not a trivial task. For the purposes of this study, where the tangency portfolio shift is of great economical significance, we will accept the F_1 test for p-values that exceed a value of 0.15 and reject the F_2 null for p-values falling below 0.0005. This difficulty in interpreting the F_1 test results becomes specifically relevant in this case where, despite the improvement of the annualised Sharpe ratio from 1.83 to 2, we would fail to reject the null given a 5% significance level. For this case, we derive $F_1 = 2.12$, $p = 0.146$ and $F_2 = 27432$, $p < 0.0001$ and recognise a significant statistical shift both in the tangency and global minimum variance portfolio.

An additional interest of this study is to examine whether further diversification on the commodity side can benefit the

CGCI. Here we denote by R_j the set of all the CGCI constituents (both cryptoassets and physical gold), while R_i denotes the four additional commodities, SLV, PPLT, PALL and WTI. All three tests of spanning reject the null hypothesis again at a significance level of 0.05, with $\text{LR} = 27.3$, $p = 0.0006$, $W = 27.5$, $p = 0.0006$ and $\text{LM} = 27$, $p = 0.0006$, $W \geq \text{LR} \geq \text{LM}$. We conclude that the benchmark portfolio does not span the augmented portfolio and is significantly benefited by the four additional commodities. Figure 2 displays the two frontier hyperbolae, where the benchmark portfolio corresponds to the CGCI setting, while the augmented portfolio corresponds to a CGCI portfolio that uses a commodity-basket rather than physical gold only. In this case, the diversification benefit of the commodity basket is attributable to an increase in expected returns. The augmented global minimum variance portfolio corresponds to a risk level of 13.3% volatility and does not drift away from the benchmark, which lies around the same level. In terms of the Sharpe ratio, the augmented tangency portfolio has managed to drive the risk-adjusted performance up to 2.5, compared to the previous value of 2. The slope of the hyperbola of the augmented portfolio increases to $\sqrt{\hat{d}/\hat{c}} = 0.0207$ compared to its previous benchmark of $\sqrt{\hat{d}_1/\hat{c}_1} = 0.0115$.

We follow up the joint hypothesis testing with the step-down approach. For this case, $F_1 = 2.03$, $p = 0.0889$ and $F_2 = 4.78$, $p = 0.0008$ and recognise a significant statistical shift only in the tangency portfolio. The p-value of the F_2 test in this case similarly highlights the importance of adjusting the significance levels of the step-down approach. A significance level of 0.05 would in this case indicate a significant shift in the global minimum variance portfolio, even though the test assets merely reduce its annualised standard deviation from 13.5% to 13.3%.

C. DCC-GARCH Estimations

The spanning test results from the previous section confirm that the addition of commodities in cryptoasset portfolios significantly shifts the efficient frontier. In this section, we examine how this diversification benefit is reflected in the WRC weighting scheme. We compare the risk-performance profiles of a cryptoasset-only portfolio, a portfolio that adds gold, following the CGCI weighting scheme, and a portfolio that replaces gold with five commodities, also following the CGCI weighting scheme. We denote the daily value of a given financial instrument i at time t by $V_{i,t}$ and its daily logarithmic returns by $r_{i,t}$. We first form a monthly-rebalanced commodity-basket, similarly to the crypto-basket component of the CGCI. The commodity-basket base level is set on 100 on July 1st, 2015 while the price level on day t from July 2nd, 2015 onwards is calculated as:

$$\text{CoB}_t = \left(1 + \sum_{i \in N_{c,t}} x_{i,R(t)} \times \left(\frac{V_{i,t}}{V_{i,R(t)}} - 1 \right) \right) \times \text{CoB}_{R(t)}, \quad (23)$$

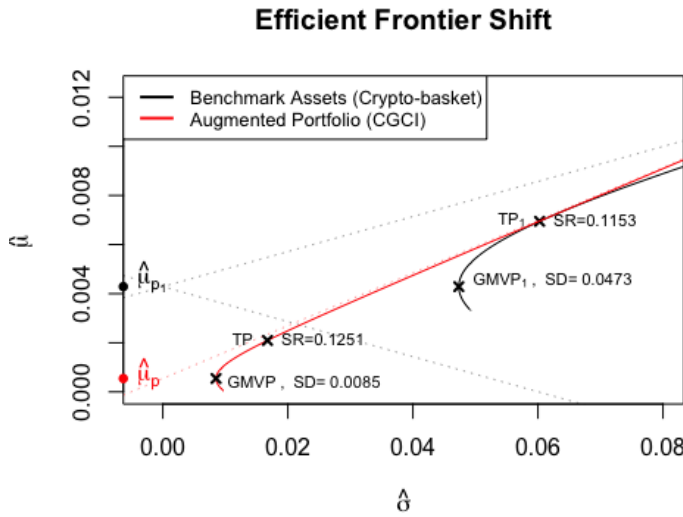


Fig. 1. Spanning Test 1: Efficient Frontiers

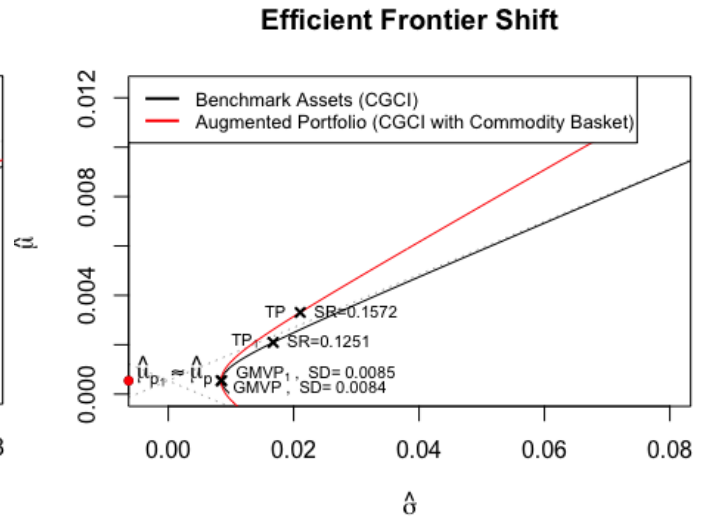


Fig. 2. Spanning Test 2: Efficient Frontiers

In Eq. 23, $N_{c,t}$ is the set of 5 commodities constituents on day t , $R(t)$ is the most recent rebalancing date preceding t , and similarly $V_{i,R(t)}$ is the value of commodity i , $x_{i,R(t)}$ is the weight of commodity i , equal to 0.2, and CoB_t is the commodity-basket price level on the last rebalancing date preceding t . The crypto-basket daily price observations are formed using the same formula and are denoted with CrB . All prices are expressed in USD. The DCC-GARCH is estimated on the three main components of interest, CrB , CoB and Gold. Table I contains the fitted model parameters.

D. Omega Ratio Comparison

Given the fitted model and a 5-year simulation horizon, we produce $N = 1000$ price paths for the CrB , CoB and gold, with the initial prices set to the last recorded prices of the historical dataset on November 2nd, 2020. We use the WRC allocation scheme (cf. Eq. 2) and produce $N = 1000$ paths for the CGCI, and $N = 1000$ paths for the modified CGCI, where the CoB replaces physical gold. We are ultimately interested in examining the individual baskets' risk-adjusted return profiles compared against the two diversified strategies.

Following the examination of the tangency portfolio shift, we also examine the risk-adjusted performance of the different portfolio specifications. We let X represent the returns of an asset and F its cumulative probability distribution function of returns. Given a selected target return threshold θ , the Omega ratio is given by:

$$\Omega(\theta) = \frac{\int_{\theta}^{\infty} [1 - F(r)] dr}{\int_{-\infty}^{\theta} F(r) dr} \quad (24)$$

The Omega ratio can be used as a way to rank investments similar to the Sharpe ratio. The threshold is first chosen to a desired target level, and investments are ranked accordingly, with higher values of preferred to lower.

DCC-GARCH Fitting					
	Crypto-basket		Commodity basket		Gold
ARMA-GJR-GARCH Parameters (Eq. 21)	ϕ_1 : 0.9669	ϕ_1 : 0.5044		ϕ_1 : -0.7792	
	θ_1 : -0.9521	θ_1 : -0.4797		θ_1 : 0.7960	
	ω : 0.00003	ω : 0.00002		ω : 0.000001	
	α_1 : 0.1168	α_1 : 0.1925		α_1 : 0.0669	
	γ_1 : -0.06346	γ_1 : -0.0397		γ_1 : -0.0259	
	β_1 : 0.9140	β_1 : 0.7345		β_1 : 0.9323	
Residual Distribution (KS*Test p-value)	Shape	4.5797		5.3186	
	(KS*Test p-value)	(0.8000)		(0.7170)	
Information Criteria Akaike Bayes	-14.784 -14.670				
Spanning Tests Results					
	LR	W	LM	F ₁	F ₂
CGCI Scenario Benchmark Assets (K): Crypto-basket Test Assets (N): Physical Gold	3160 (≤ 0.001)	27723 (≤ 0.001)	891 (≤ 0.001)	2.12 (0.1461)	27432 (≤ 0.0005)
Modified CGCI Scenario Benchmark Assets (K): Crypto-basket Test Assets (N): Commodity-basket	27.3 (0.0006)	27.5 (0.0006)	27 (0.0006)	2.03 (0.0889)	4.78 (0.0008)

* Kolmogorov-Smirnov

TABLE I: DCC-GARCH Parameters and diagnostics

In this study we inspect the average annual return of each component. Given a simulated price path P_i , where $i \in 1, \dots, 1000$, the annualised return is obtained through $R_i = (P_{i,1+l}/P_{i,1})^{\frac{365}{l}} - 1$ where l denotes the length of the simulated prices expressed in number of days. This yields 1000 values of annualised returns for the CGCI, crypto-basket and gold respectively.

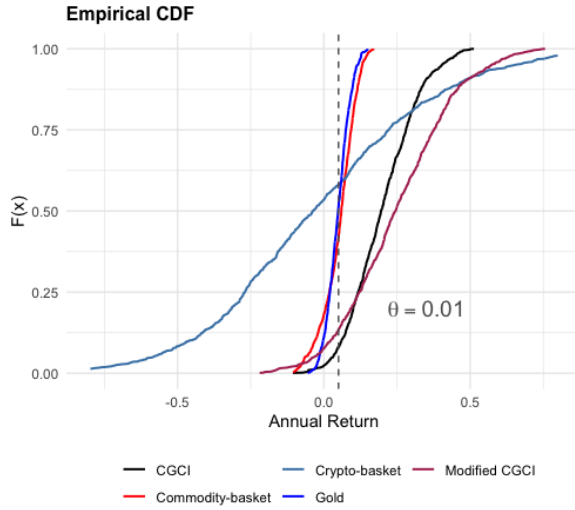


Fig. 3. Empirical Cumulative Distribution Functions

In Fig. 3 the slope of the crypto-basket cumulative distribution function appears visibly less steep compared to the rest of the components, with significantly heavier tails. Both physical gold and the formed commodity basket display a reduced profit and loss range, with the commodity basket presenting signs of slightly heavier tails. The two mixed portfolios that follow the CGCI weighting schemes appear to be relatively robust against the crypto-market's potential severe losses, while at the same time they display much higher chances of profit when compared against the commodity market.

We also plot the Omega ratio as a function of the expected profitability parameter θ , to investigate the ranking among the five portfolios given different levels of target returns. For large negative values of θ , all three Omega ratios tend to infinity, while for large positive values of θ , they tend to zero. Fig. 4 presents on a log scale the Omega ratio of the CGCI, the modified CGCI, the crypto-basket, the commodity-basket and gold as a function of θ and allows us to rank the portfolios according to the intersection points of the five curves. The intersection point of the CGCI and gold Omega lines reveals that the original index strategy is more appropriate for investors aiming for moderate positive returns up to 29.6% annually. Accordingly, gold is more effective in providing protection against negative market developments. The intersection point of the crypto-basket and original CGCI curves confirms that complete exposure to the crypto-space is appropriate for risk-tolerant investors who target annual returns that exceed 29.6%. When it comes to the modified CGCI setting, it manages to extend the range of overperformance. Specifically, the modified CGCI appears more appropriate for investors aiming up to 50.7% annual returns. When it comes to selecting between the two index methodologies, the original setting is more appropriate for investors who are interested in wealth preservation and moderate annual returns up to 12.3%, while the modified methodology is better suited for less risk-

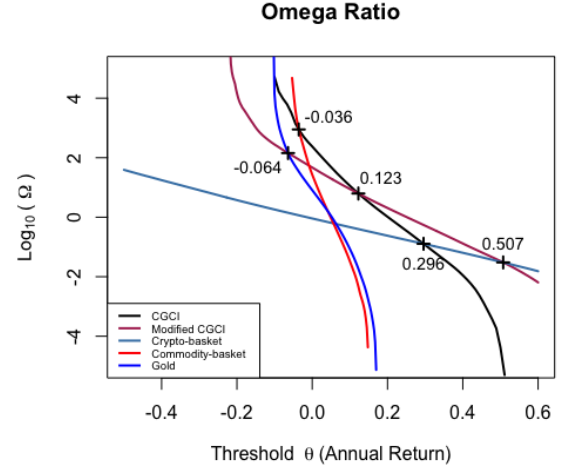


Fig. 4. Omega Ratios - Log Scale

averse individuals that set a target annual return at up to 50.7%. Both physical gold and the commodity-basket are wealth-preservation tools, with the commodity basket displaying a more volatile profile.

V. CONCLUSION

In this paper we have examined the diversification properties of the commodity class from the perspective of a cryptoasset investor. We utilise the concept of mean-variance spanning tests and examine how the addition of commodities in cryptoasset portfolios shifts the efficient frontier in a statistically significant way. A step-down approach using two separate F tests further allows us to examine changes in the global minimum variance and tangency portfolios. We first take into consideration the setting of the CoinShares Gold and Cryptoassets Index and confirm that the addition of physical gold to a basket of five equally-weighted cryptoassets translates to a significant shift in the mean-variance frontier, in both of the efficient portfolios. We also consider a second setting where gold is replaced by a basket of five commodities. The spanning tests reject the null hypothesis and the rejection is found to be due to a significant shift in the tangency portfolio.

We have additionally taken into consideration the two portfolio compositions and compared the two approaches in terms of their suitability for investors. The evolution of the portfolio risk factors have been described using a dynamic conditional correlation model, which is used to generate 1 000 different price paths. Computations of the Omega ratio for different values of the target return threshold reveal that the original index composition is more suitable for wealth-preserving investors and investors who target moderate returns. For investors with higher return expectations, up to 50.7%, the modified index approach is more appropriate.

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