Fluctuations of SHE and KPZ equations with non-integrable spatial correlations

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Statement of Originality

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Luca Gerolla
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To My Family

Abstract

This thesis examines the large scales fluctuations of the solution to a nonlinear stochastic heat equation (SHE) and the KPZ equation in dimensions $d \geqslant 3$, in the presence of long-range spatial correlations. Specifically, the equations are driven by Gaussian noise that is white in time but has non-integrable spatial covariance (at infinity), with polynomial decay rate $\kappa \in (2, d)$.

We prove that the fluctuations of both equations, when appropriately rescaled, converge to a Gaussian limit described by the corresponding additive stochastic heat equation. Unlike the case of compactly supported covariance, the noise in the limit equation retains spatial correlation with a Riesz kernel $|x|^{-\kappa}$ as covariance, along with a different effective variance and fluctuations scales. Additionally, by considering the spaces of locally Hölder continuous distributions $\mathcal{C}^\alpha$, we demonstrate convergence in optimal Hölder topologies as distribution-valued processes.

To establish our main results, we also prove a priori moment bounds on the solutions and their Malliavin derivatives, as well as the existence of space-time stationary solutions, which underpin the fluctuations theory of both models. Interestingly, in contrast to the case of integrable covariance, we show that the KPZ equation converges in probability to its scaling limit.

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Chapter 1

Introduction

The focus of this thesis is devoted to study of large scales statistical properties of the solutions to a stochastic (non-linear) heat equation (SHE) and the Kardar-Parisi-Zhang (KPZ) equation, driven by noise that exhibits non-integrable spatial correlations. We shall consider the following heat equation on $\mathbb{R}^d$ perturbed by a multiplicative random forcing:

$$\partial_t u = \tfrac{1}{2}\Delta u + \beta\sigma(u)\xi \ , \tag{1.0.1}$$

where σ is a real valued Lipschitz continuous function on $\mathbb{R}$ and ξ is a mean zero Gaussian noise that is white in time and coloured in space. The two equations are related via the Cole-Hopf transform. Formally, if u solves (1.0.1) with $\sigma(u) = u$, then $h = \log u$ solves the KPZ equation

$$\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + \beta\xi \ . \tag{1.0.2}$$

Throughout, our work will focus in the high-dimensional case $d \geqslant 3$. This regime has been the subject of intense study in the recent years. In a number of works [GRZ18, MU18, DGRZ20, CNN22, LZ22, GL20, GK21b, DGRZ21, CCM24], it was shown that both the SHE and KPZ equation exhibit Gaussian fluctuations at large scales in the weak disorder phase, namely when the coupling constant β is sufficiently small. In this regime, the one time distribution of the fluctuations has been shown to solve in the limit the stochastic heat

equation with additive noise - the Edwards–Wilkinson (EW) equation [EW82] - given by

$$\partial_t \mathcal{U}^{EW} = \tfrac{1}{2}\Delta \mathcal{U}^{EW} + \beta\,\nu\,\dot{W}\ , \tag{1.0.3}$$

where $\dot{W} = \dot{W}(t,x)$ denotes space-time white noise. The results hold provided that the correlations of the driving noise have *compact support*. The effective variance ν depends on the spatial covariance of ξ in the microscopic model. Dissipation of correlations and the EW fluctuations limit may not hold when the noise exhibits heavy-tailed correlations. In this chapter we concisely review the wider related literature, and look at the main results in $d \geqslant 3$ for the compactly supported case; whence we give an overview of our main contributions.

1.1 Background

The study of the above mentioned equations, besides their mathematical interest, is also motivated by Physics. Introduced in [KPZ86] as a model for random interfaces growth, since then the KPZ equation has been widely researched given its relation with many important physics and probabilistic models [HHZ95, Kru97, BG97], and in order to elucidate the KPZ universality class [Qua12, Cor12, Tak18].

Besides its relation to the KPZ (and Burgers equation) through the Cole-Hopf transform, the linear SHE provides a linearised model of chemical kinetics, or more generally models the evolution of a scalar field in a non-stationary (in time) random medium [Mol91]. Much research on the equation has also been drawn due to its connections with discrete polymer models, exclusion processes and intermittency phenomena[(1)] (see for example [CM94, BC95, BG99, Com17, Zyg24]).

[(1)] Intermittency arise with equation driven by multiplicative noise. Informally, it emerges as the random field highly concentrates in few very high peaks, leading to severe growth of its moments with time. We refer also to [HHNT15] and references therein, the notes [Gu20a] provide some intuition relevant to our work and the results outlined below.

While one can make sense of the linear SHE through Itô integration, the KPZ equation (1.0.2) is ill-posed when ξ is a space-time white noise. Since, given the regularity of the solution, the non-linearity $|\nabla h|^2$ would entail a product of distributions, which cannot be defined. A rigorous notion of solution to the KPZ equation in $d = 1$ was first obtained in [Hai13], employing the theory of controlled rough paths, then encompassed by the general framework of Regularity Structures [Hai14b].

The KPZ is, in fact, one example of so-called *singular SPDEs*. In general, these are ill-posed equations of the form

$$Lu = F(u) + G(u)\,\xi\ , \tag{1.1.1}$$

where L is a linear differential operator, $F(u)$ and $G(u)$ are nonlinear terms depending on u and its lower-order derivatives [Hai14a]. Due to the irregularity of the driving noise ξ (and thus the solution u), the terms in $F(u)$ and the pointwise product of $G(u)$ with ξ may not make formal sense. The main challenge, therefore, is to give rigorous sense to the ill-posed equation and establish suitable notions of solutions and their existence. One natural approach is to smooth out the irregular noise ξ^ε (e.g. by means of a mollifier φ_ε approximating the identity), so that one can make sense of (1.1.1) and the associated solution u_ε. The aim is to prove the existence of a limiting object u as $\varepsilon \to 0$ and investigate its behavior, working towards a canonical notion of existence and uniqueness of solutions, independent of the mollifier. As we shall see later, in *higher dimensions*, the limit behaviour of the regularised model may resamble the linearised model and is tied to the *large-scales* dynamics of the equation.

In general, to carry out the regularisation approach, appropriate renormalization procedures are needed to compensate for turning off the mollifier, typically requiring the equation to be re-centered by some diverging constant(s) $C_\varepsilon \to \infty$, as $\varepsilon \to 0$. This is because of the regularisation of singular nonlinearities, to obtain non-trivial and meaningful solutions [CW17]. The actual constant(s) and rate may depend on dimension, nonlinearities and noise in the

equation. The need of renormalisation can also be motivated by the underlying physics, given that singular SPDEs often emerge as universal limits of microscopic models for physical systems [CS20].

In the last decade, we have witnessed extraordinary progress in the field, thanks to the seminal framework of Regularity Structures [Hai14b]. Along with the paracontrolled calculus approach of [GIP15, GP17] and renormalisation group techniques [Kup16, Duc21], these theories revolutionised the field of *subcritical* singular SPDEs, allowing to tackle a variety of important equations [CHS18, BGHZ22, CCHS22a, BCCH21, CCHS22b, GH21, GKO24].

Informally, SPDEs are referred to as subcritical when, at small scales, the effect of the non-linearity $F(u)$ is negligible and the noise dominates (cf. [CW17, Defn. 1.3], [Hai14b, Assump. 8.3]). However, in higher dimensions, such theories no longer apply, as these equations typically become *supercritical*. In the case of SHE and KPZ, these equations are subcritical in $d = 1$, and supercritical when $d \geqslant 3$, which is the regime we are considering.

In one dimension, the KPZ equation interpolates between two universality classes: the (Gaussian) Edwards–Wilkinson class at small scales and the (non-Gaussian) KPZ class at large scales [Cor12]. The latter long-standing conjecture was only recently proven in [QS23, Vir20], where it was shown that, up to shifting and scaling, the rescaled solution $\varepsilon^{\frac{1}{2}} h(\varepsilon^{-\frac{3}{2}} t, \varepsilon^{-1} x)$ converges as $\varepsilon \to 0$ to a universal space-time process known as the KPZ fixed point. This was built upon extensive research and progresses over the last few decades (cf. [BDJ99, ACQ11, BQS11, AKQ14, MQR21], see also the surveys [Qua12, Cor12, QS15] and more recently [CS20, Rem22, Bai22]).

In the critical dimension $d = 2$, unlike $d = 1$, a phase transition occurs depending on the value of the coupling constant β. Much activity has focused on elucidating the behaviours of SHE and KPZ in the "weak coupling" regime (and its boundary [BC98]). There, when the rescaled β is sufficiently small ($\beta < \beta_c$), a series of works [CSZ17, Gu20b, CSZ20, CD20, NN23] showed that Gaussian fluctuations emerge at large scales under a logarithmic rescaling

of order $\sqrt{|\log \varepsilon|}$. Recent work also examined the anisotropic KPZ equation [CES21, CET21, CHST22, CET20], which (formally) admits the free field as its invariant distribution, and how the behaviour of SHE is affected in the presence of a nonlinearity [DG22, Tao22, DG24, DG23]. More recently, this line of research culminated in remarkable progresses also addressing the behaviour at critical $\beta \approx \beta_c$ [CSZ19, GQT21, CSZ23a, CSZ23b].

1.2 Related work in higher dimensions

Here we are interested in the large-scale dynamics of SHE and the KPZ equation in $d \geqslant 3$. Tipically, one would like to understand the model driven by space-time white noise $\dot{W}$, which is a Gaussian random distribution with delta distributions as covariance functions; formally

$$\mathbb{E}[\, \dot{W}(t,x)\dot{W}(s,y)\,] = \delta(t-s)\delta(x-y)\ .$$

In the sense that, for any $\psi, \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1})$, we have

$$\mathbb{E}\big[\ \langle \dot{W}, \psi\rangle\langle \dot{W}, \varphi\rangle\ \big] = \int_{\mathbb{R}}\int_{\mathbb{R}^d} \psi(s,x)\varphi(s,x)\ dx\, ds\ .$$

In higher dimensions, solutions fail to be function-valued when the driving noise is $\xi = \dot{W}$, this is due to the irregularity of $\dot{W}$, One then tries to appropriately regularise the SPDEs via mollification, in order to study the behaviour of the solutions. The starting point is to consider the solution u to the linear SHE, namely (1.0.1) with $\sigma(u) = u$, driven by a regularised Gaussian noise ξ:

$$\partial_t u(t,x) = \tfrac{1}{2}\Delta u(t,x) + \beta u(t,x)\xi(t,x)\ , \quad u(0,x) = u_0(x)\ , \tag{1.2.1}$$

with the product term intepreted as an Itô integral. Tipically, the noise is taken to be a mollification of $\dot{W}$ by a spatial mollifier $\phi : \mathbb{R}^d \to \mathbb{R}$:

$$\xi(t,x) = \int_{\mathbb{R}^d} \phi(x-y)\dot{W}(t,y)\, dy\ . \tag{1.2.2}$$

Then, the regularised noise ξ is Gaussian, white in time and coloured in space, with covariance

$$\mathbb{E}[\, \xi(t,x)\xi(s,y)\,] = \delta(t-s)R(x-y)\ , \tag{1.2.3}$$

where the homogeneous spatial covariance R is given by

$$R(x) = \int_{\mathbb{R}^d} \phi(x+y)\phi(y)dy \ .$$

Usually, the mollifier is taken to be a smooth compactly supported function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R}_+)$. Given the mollification, the equation is now well-posed and there exist a unique function valued solution [Dal99, PZ00].

To study the large scales behaviour, one then introduces a small parameter $\varepsilon > 0$, and considers the parabolic rescaled solution $u_\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon)$. Given the scale invariance of $\dot{W}$, u_ε satisfies the (regularised) equation:

$$\partial_t u_\varepsilon(t,x) = \tfrac{1}{2}\Delta u_\varepsilon(t,x) + \beta\varepsilon^{\frac{d}{2}-1} u_\varepsilon(t,x)\xi^\varepsilon(t,x) \ , \tag{1.2.4}$$

where ξ^ε is a white-in-time Gaussian noise, with spatial covariance $\varepsilon^{-d}R(\varepsilon^{-1}\cdot)$. Note that, as we let $\varepsilon \to 0$, we are effectively zooming out to consider the large scales behaviour of u. Since $d \geqslant 3$, the factor in front of the noise decays to 0, "turning off" the noise in the equation.

To build intuition and draw the differences with our work, we now briefly review the main results on the large-scales behaviour of SHE and KPZ, in the case of *compactly supported* (integrable) covariance. Namely, we shall assume in the rest of the section that $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R}_+)$ (thus also R) is nonnegative, smooth and compactly supported.

1.2.1 Law of large numbers and β-phase transition

Before investigating the fluctuations, one should understand the mean and asymptotic behaviour of u_ε. Depending on the value of the coupling constant β (also referred to as inverse temperature) there occurs a phase transition. This has been long understood in the discrete polymer literature [Bol89, IS88, CY06, Com17]. More recently, [MSZ16] considered the continous case, focusing on the linear SHE with flat initial conditions, they proved there occurs a phase transition depending on the value of the coupling constant β.

Theorem A. *[MSZ16, Thm. 2.1-2.4] Assume u is the solution of (1.2.1) with coupling constant β and with initial conditions $u_0 \equiv 1$. There exists a constant $\beta_c > 0$ such that, for any $x \in \mathbb{R}^d$, as $t \to \infty$, the following convergence in distribution holds:*

$$u(t,x) \Rightarrow Z \quad \textit{where} \quad \begin{cases} Z > 0 \ a.s. & \textit{if} \quad \beta < \beta_c \ , \\ Z = 0 \ a.s. & \textit{if} \quad \beta > \beta_c \ . \end{cases} \tag{1.2.5}$$

Moreover, for sufficiently small $\beta < \beta_* < \beta_c$, u_ε converges in probability, as $\varepsilon \to 0$, when tested against $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} u_\varepsilon(t,x) g(x) dx \quad \xrightarrow{\mathbb{P}} \quad \int_{\mathbb{R}^d} g(x) dx \ . \tag{1.2.6}$$

The regime below the critical value β_c, i.e. $\beta < \beta_c$, is known as *weak disorder*, whereas $\beta > \beta_c$ as *strong disorder*[(2)], where intermittency onset. Given that $u_0 \equiv 1$, one has $\mathbb{E}u(t,x) = 1$ for all $t \geqslant 0$ and $x \in \mathbb{R}^d$, hence (1.2.6) can be seen as a law of large numbers (LLN), where u_ε converges to its mean as a random distribution. The LLN no longer holds when $\beta > \beta_c$, where u ceases to be uniformly integrable in time. Later in [CNN22], the authors actually showed that (1.2.6) holds in the whole weak disorder regime $\beta \in (0, \beta_c)$, and for general initial conditions $u_0 \in \mathcal{C}_b(\mathbb{R}^d)$.

A crucial tool in their proof, and in the fluctuations results of the wider literature, is that the linear SHE admits a Feynman–Kac representation formula:

$$u(t,x) = \mathbf{E}_B \left[u_0(x + B_t) \exp\left(\beta \int_0^t \xi(t-s, x + B_s) ds - \frac{1}{2} \beta^2 R(0) t \right) \right] , \tag{1.2.7}$$

where $\mathbb{E}_B$ denotes the expectation with respect to a Brownian motion path $(B_s)_{s \geqslant 0}$, independent of ξ.

This expression indicates that $u(t,x)$ can be seen as the partition function of a (Brownian) directed polymer in the white noise environment ξ. The LLN

[(2)] see [BC20, BM19, Com17] and references therein concerning the strong disorder regime and some results on the localisation of the polymer model.

(1.2.6) follows from the large scales spatial de-correlation of u. Namely, by a second moment calculation, through (1.2.7), for $x \neq y$, one can show

$$\operatorname{cov}\big(u_\varepsilon(t,x), u_\varepsilon(t,0)\big) \simeq \int_{\mathbb{R}^d} \frac{R(z)}{|z - x/\varepsilon|^{d-2}}\, dz \ , \tag{1.2.8}$$

which vanishes as $\varepsilon \to 0$, when R is compactly supported and thus obtain convergence via Portenko's lemma [Por75].

However, by a simple computation, we note that the expression on the right in (1.2.8) may still vanish, as long as, for large x,

$$R(x) \approx |x|^{-\kappa} \quad \text{with} \quad \kappa > 2 \ . \tag{1.2.9}$$

This observation supported our exploration to see whether a LLN (and thus a fluctuations CLT) may hold even when the spatial covariance is no longer integrable at infinity. As we shall see in Section 2.5.3 (cf. Remark 2.5.6), analogous de-correlations hold in our settings, cf. (1.3.6), which will turn out to be crucial to prove the fluctuations limit.

1.2.2 Large scales fluctuations

There have been numerous works looking at the large scales fluctuations and stationary solutions of the two equations.

Linear SHE. The fluctuations, in the weak sense, of the *linear* stochastic heat equation in $d \geqslant 3$ have been studied in various works [GRZ18, MU18, DGRZ21, CNN22, LZ22]. Recall u_ε satisfies (1.2.4). With R compactly supported, the noise ξ^ε converges to space-time white noise $\nu_0 \dot{W}$, with variance

$$\nu_0^2 = \int_{\mathbb{R}^d} R(y)\, dy \ .$$

This can be seen by a simple computation, since ξ^ε is white in time and has spatial covariance $\varepsilon^{-d} R(\varepsilon^{-1}\cdot)$, for any $\psi, \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1})$, we have

$$\begin{aligned} \mathbb{E}\big[\langle \xi^\varepsilon, \psi\rangle \langle \xi^\varepsilon, \varphi\rangle\big] &= \int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} \psi(s,x)\varphi(s,x')\, \varepsilon^{-d} R\big(\tfrac{x-x'}{\varepsilon}\big)\, dx\, dx'\, ds \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} \psi(s,x)\varphi(s,x-\varepsilon y)\, R(y)\, dx\, dy\, ds \end{aligned}$$

$$\longrightarrow \Big(\int_{\mathbb{R}^d} R(y)dy \Big) \int_{\mathbb{R}} \int_{\mathbb{R}^d} \psi(s,x)\varphi(s,x)\, dx\, ds$$
$$= \nu_0^2\, \mathbb{E}\big[\, \langle \dot{W}, \psi \rangle \langle \dot{W}, \varphi \rangle \,\big] \,.$$

On the other hand, from the asymptotic result (1.2.5), u converges, in *microscopic time*, towards an almost sure positive random variable Z when $\beta < \beta_c$, where $\mathbb{E}Z = 1$. Thus, the product term in (1.2.4) stabilises. Rewriting the equation,

$$\partial_t \big(\tfrac{u_\varepsilon(t,x)-1}{\varepsilon^{\frac{d}{2}-1}} \big) = \tfrac{1}{2} \Delta \big(\tfrac{u_\varepsilon(t,x)-1}{\varepsilon^{\frac{d}{2}-1}} \big) + \beta u_\varepsilon(t,x) \xi^\varepsilon(t,x) \,,$$

we may expect the fluctuations to live at $\varepsilon^{\frac{d}{2}-1}$ scales, and to converge in the limit towards the EW solution $\nu_0 \mathcal{U}^{EW}$, where

$$\partial_t \mathcal{U}^{EW} = \tfrac{1}{2} \Delta \mathcal{U}^{EW} + \beta\, \dot{W} \,, \qquad \mathcal{U}^{EW}(0,\cdot) \equiv 0 \,. \tag{1.2.10}$$

While this heuristic intuition hints correctly to the EW limit, it turns out the interaction between the noise and u_ε leads to a non trivial effective variance ν_{eff}, with $\nu_{\text{eff}} > \nu_0$. In fact, rescaling appopriately the fluctuations from the mean, one has the following:

Theorem B. *[GRZ18, MU18, CNN22, LZ22] Assume u is the solution of (1.2.1) with coupling constant β and with initial conditions $u_0 \equiv 1$. There exists a constant $\beta_0 > 0$ such that, for any $\beta < \beta_0$, the following holds, as $\varepsilon \to 0$, for any $t > 0$ and test function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,*

$$\varepsilon^{1-\frac{d}{2}} \int_{\mathbb{R}^d} \big(u_\varepsilon(t,x) - \mathbb{E}u_\varepsilon(t,x) \big) g(x)\, dx \implies \nu_{\text{eff}} \int_{\mathbb{R}^d} \mathcal{U}^{EW}(t,x) g(x)\, dx \,. \tag{1.2.11}$$

Where the effective variance is given by

$$\nu_{\text{eff}}^2 = \int_{\mathbb{R}^d} R(y) \mathbb{E}_B \left[\exp\left(\beta^2 \int_0^\infty R(B_{2s} + y)\, ds \right) \right] dy \,. \tag{1.2.12}$$

The CLT result in (1.2.11) was proved in [GRZ18, MU18], allowing even for short-range time correlations. It was later extended in [CNN22, LZ22] to the whole L^2-region $\beta \in (0, \beta_{L^2})$, also with general initial conditions u_0 and

considering multi-time convergence. For simplicity, here we state the result for the single-time distribution, when SHE has flat initial conditions and driven by white in time noise. In the case of general u_0, we have $\mathbb{E}u_\varepsilon(t,x) = \bar{u}(t,x)$, where $\bar{u}$ is the solution of the heat equation

$$\partial_t \bar{u} = \tfrac{1}{2}\Delta \bar{u} \ , \qquad \bar{u}(t,0) = u_0 \ ,$$

and the fluctuation limit in (1.2.11) is then given by $\nu_{\text{eff}}\bar{u}(t,x)\mathcal{U}_t^{EW}$. In the presence of *short-range* time correlations, namely for $\rho \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$,

$$\mathbb{E}[\,\xi(t,x)\xi(s,y)\,] = \rho(t-s)R(x-y) \ , \tag{1.2.13}$$

the Laplacian is modulated by a non-trivial diffusivity in the limit.

The L^2 region represents the values of β such that the corresponding solution to (1.2.1) has uniformly bounded L^2 moments for all times, with

$$\beta_{L^2} := \sup\left\{\beta > 0 \ : \ \sup_x \sup_{t\geqslant 0} \mathbb{E}[u(t,x)^2] < \infty\right\}.$$

We note that $\beta_{L^2} \leqslant \beta_c$, with strict inequality expected from works on discrete polymers, cf. [CNN22, Sec. 1.2]. When $\beta > \beta_{L^2}$, Theorem B is no longer expected to hold, as the effective variance ν_{eff} blows up.

The fluctuations limit and identification of the variance, crucially rely on the stabilization of the solution u at large times. In fact, it was shown in [GL20, Thm. 1.1], that there exist a spatial stationary field $\big(Z(x)\big)_{x\in\mathbb{R}^d}$, such that:

$$u(t,\cdot) \implies Z \quad \text{in} \quad \mathcal{C}(\mathbb{R}^d), \quad \text{as } t\to\infty \ . \tag{1.2.14}$$

Then, we can rewrite the effective variance in (1.2.11) as

$$\nu_{\text{eff}}^2 = \int_{\mathbb{R}^d} R(y)\mathbb{E}[Z(y)Z(0)]\,dy \ . \tag{1.2.15}$$

KPZ equation and nonlinear SHE. The EW limit for the KPZ in $d \geqslant 3$ was first shown in [MU18] using perturbation techniques borrowed from constructive Quantum Field Theory, considering a finite range *space-time* mollification of space-time white noise $\dot{W}$. Later, [DGRZ20] proved analogous

result for small β employing Malliavin Calculus techniques. While [CNN22, LZ22] demostrated that the EW limit holds in the full L^2 region $\beta \in (0, \beta_{L^2})$, using different methods, both inspired by the directed polymers literature. Letting $h_\varepsilon(t,x) = h(t/\varepsilon^2, x/\varepsilon)$, we have the following.

Theorem C. *[MU18, DGRZ20, CNN22, LZ22] Assume h is the solution of the KPZ equation (1.0.2) with flat initial conditions $h(0,x) \equiv 0$. Then, for any $\beta < \beta_{L^2}$, for any $t > 0$ and test function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, as $\varepsilon \to 0$,*

$$\varepsilon^{1-\frac{d}{2}} \int_{\mathbb{R}^d} \big(h_\varepsilon(t,x) - \mathbb{E}h_\varepsilon(t,x)\big) g(x)\, dx \implies \nu_{\text{eff}} \int_{\mathbb{R}^d} \mathcal{U}^{EW}(t,x) g(x)\, dx\ . \tag{1.2.16}$$

We see that fluctuations from the mean of KPZ converge to the same limit as for the linear SHE. We refer to [CNN22, Sec. 2] for more general statements with non-flat initial conditions and about the fluctuations of the *stationary* solutions. A key property used to carry out the proofs is that, through the Cole-Hopf transform, $h = \log u$, one can examine the large scale behaviour of the KPZ leveraging the properties of SHE and the Feynman-Kac formula.

A question one may ask is whether similar fluctuations results hold for transformations of u other than the Cole-Hopf, or in the case of the nonlinear SHE (1.0.1). This is indeed the case. In fact, the proof of [DGRZ20] allows to conclude that the fluctuations of $\Phi(u_\varepsilon)$, where $\Phi : \mathbb{R}_+ \to \mathbb{R}$ and its derivatives satisfy suitable polynomial growth conditions, also converge in law to $c_\Phi\, \nu_{\text{eff}}\, \mathcal{U}_t^{EW}$, in the sense of random distributions, as in (1.2.16), where the modulation factor is given by

$$c_\Phi = \mathbb{E}[\, Z\, \Phi'(Z)\,]\ . \tag{1.2.17}$$

Note that in the Cole-Hopf case $\Phi(u) = \log(u)$, one indeed has $c_\Phi = 1$, retrieving (1.2.16). Their proof holds for $\beta < \beta_\Phi < \beta_{L^2}$, requiring uniform control of moments higher than two. Leveraging the Feynman-Kac representation (1.2.7) and Clark-Ocone formula to express the fluctuations, through a separation of timescales and Brownian paths estimates, they identify (1.2.12)

and show variance convergence. To show Gaussianity of the limit they employ the second order Poincaré inequality [NPR09, Thm. 1.1], through estimates on the Malliavin derivatives of u.

Concerning the nonlinear SHE in $d \geqslant 3$ with a Lipschitz nonlinearity, [GL20] showed that the fluctuations converge to the EW limit as in Theorem B. In this case, the effective variance is modulated by the nonlinearity of the equation, with the fluctuations now converging to $\nu_\sigma \, \mathcal{U}_t^{EW}$, where

$$\nu_\sigma^2 = \int_{\mathbb{R}^d} R(y) \mathbb{E}\big[\sigma\big(Z(y)\big)\sigma\big(Z(0)\big)\big] \, dy \, . \tag{1.2.18}$$

The works of [GL20, DGRZ20] were an important inspiration when we started our works on SHE and the KPZ equation.

In summary, we have seen that in higher dimensions $d \geqslant 3$, all these models fall in the (Gaussian) Edwards–Wilkinson class, but their limit variances crucially depend on the noise mollification, and defined in terms of the underlying stationary solutions.

1.3 Main results

We are now going to outline the main contributions of this thesis and some intuitions. We will provide more details, precise statements and prove the results in the later chapters.

Non-integrable covariance. A natural question in view of the above results is what happens if the noise covariance R changes; it is unclear if one should still observe Gaussian fluctuations under the same scaling. We note that the proofs of all the above results hold as long as R is *compactly supported* (or in principle R decays fast enough to be integrable). Whilst we commented above that one may expect the LLN (1.2.6) to hold (for linear SHE) even when $R \notin L^1(\mathbb{R}^d)$, as long as it is not too heavy tailed (1.2.9), it is not clear a priori what happens at the level of fluctuations. If we recall the effective variance of

the EW limit (1.2.15), we noted that $\nu_{\text{eff}} > \nu_0$, where

$$\nu_0^2 = \int_{\mathbb{R}^d} R(x)\,dx = +\infty\ ,$$

when $R \notin L^1(\mathbb{R}^d)$. In fact, the proofs and fluctuations limit no longer hold in this setting, given that the above effective variances blows up when R is no longer integrable. Understanding what happens in these regimes and probe possible phase transitions was one of the main motivations for our work.

Our setup. We consider the nonlinear SHE (1.0.1), with globally Lipschitz nonlinearity σ, driven by white in time Gaussian noise ξ with covariance as in (1.2.3). We assume its smooth spatial covariance R is such that $|R(x)| \lesssim (1+|x|)^{-\kappa}$ and, as $\varepsilon \to 0$, it has the following limiting profile,

$$\varepsilon^{-\kappa} R(\varepsilon^{-1}x) \to |x|^{-\kappa} \quad \text{with} \quad \kappa \in (2,d)\ . \tag{1.3.1}$$

Whilst bounded and integrable near 0, R fails to be integrable at infinity, given $\kappa < d$. We investigate the case where u is the solution with flat initial conditions $u(0,x) \equiv 1$, and consider the parabolic rescaled solution $u_\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon)$. Balancing out the scalings, we note that u_ε now satisfies the rescaled equation

$$\partial_t u_\varepsilon = \tfrac{1}{2}\Delta u_\varepsilon + \beta\varepsilon^{\frac{\kappa}{2}-1}\sigma(u_\varepsilon)\,\xi^\varepsilon\ , \qquad u_\varepsilon(0,x) \equiv 1\ , \tag{1.3.2}$$

where, in our case, ξ^ε is a white in time Gaussian noise with spatial covariance $\varepsilon^{-\kappa}R(\frac{x}{\varepsilon})$. From assumption (1.3.1), we have $\varepsilon^{-\kappa}R(\varepsilon^{-1}x) \to |x|^{-\kappa}$ as $\varepsilon \to 0$. Since the law of centred Gaussian white noise is characterised by its covariance function, ξ^ε converges in law to a spatially correlated noise ξ^0, with Riesz kernel covariance:

$$\mathbb{E}\big[\,\xi^0(t,x)\,\xi^0(s,y)\,\big] = \delta(t-s)|x-y|^{-\kappa}\ . \tag{1.3.3}$$

Moments estimates and stationary solutions. Besides uniform moments bounds, a crucial ingredient for the fluctuations results to hold in the *compactly supported covariance* case was that, asymptotically as $t \to \infty$, $u(t,x)$ converges

in law to a stationary field $Z(x)$, cf. (1.2.14). Its law characterises the effective variance in the limit, and the asymptotic field is non-trivial because the coupling constant was taken to be small enough $\beta < \beta_c$, before the onset of intermittency (1.2.5). Inspired by [GL20, Thm. 1.1], we show that in our long-range correlations setting there exists a space-time stationary field $\vec{Z}(t,x)$. Pull back and stationary fields also appeared in [DGRZ21, CCM24, CCM20]. Considering the pullback solutions $u^{(s)}$, where $u^{(s)}$ denotes the solution to SHE (1.0.1) with initial condition 1 at time s, this is defined as the limit

$$\vec{Z}(t,x) = \lim_{s\to-\infty} u^{(s)}(t,x) \ .$$

The convergence holds in $L^p(\Omega)$ (over compacts) and we quantify its rate, dependent on κ. For a random variable X, we denote with $\|X\|_p = \big(\mathbb{E}[X^p]\big)^{1/p}$ the $L^p(\Omega)$ norm with respect to the underlying probability space.

Theorem 1. *Let $d \geqslant 3$, σ be Lipschitz continuous and R satisfies $|R(x)| \lesssim (1+|x|)^{-\kappa}$, with $\kappa \in (2,d)$. Then, for any $p > 1$, there exists a positive constant $\beta_1(p)$, such that for $\beta \leqslant \beta_1$, there exists a space-time stationary field $\vec{Z}$ such that, uniformly in $t > s$:*

$$\sup_{x\in\mathbb{R}^d} \big\|u^{(s)}(t,x) - \vec{Z}(t,x)\big\|_p \lesssim 1 \wedge (t-s)^{\frac{2-\kappa}{4}} \ . \tag{1.3.4}$$

In order to show the above, we first obtain a-priori moment bounds on u and pointwise estimate on its Malliavin derivative. We show that for any $p \geqslant 2$, there exists a constant $C_p > 0$, such that, for β sufficiently small $\beta < \beta_0(p)$,

$$\sup_{s\geqslant 0, x\in\mathbb{R}^d} \|u(s,x)\|_p \leqslant C_p \ , \qquad \|D_{r,z}u(t,x)\|_p \lesssim C_p P_{t-r}(x-z) \ . \tag{1.3.5}$$

Employing some Gaussian-covariance decay estimates, we also derive spatial de-correlation bounds on u (and $\vec{Z}$), of the form

$$\operatorname{cov}\Big(u(t,x), u(t,0)\Big) \lesssim 1 \wedge |x|^{2-\kappa} \ . \tag{1.3.6}$$

The stationary field $\vec{Z}$ underpins our main result on the fluctuations of SHE, and it characterises the effective variance emerging due to the spatial de-correlation at large scales, still present in our setting. Identifying the stationary

field and obtaining a precise rate of convergence (1.3.4) will turn out to be crucial also for our work on the KPZ equation. Given (1.3.6), the law of large numbers (1.2.6) immediately follows in our settings too. These results are fully stated and proved in Sections 2.5-2.6.

Fluctuations of SHE. Given the above and looking back at the rescaled equation (3.1.1), it may come natural to conjecture that also in our settings one may see the fluctuations of u_ε converging to a Gaussian limit, solving an additive stochastic heat equation. Using (1.3.6) and the covariance assumptions (1.3.1), through some simple calculations, one observes that the variance of the fluctuations $\varepsilon^{1-\frac{\kappa}{2}} u_\varepsilon$ indeed converge. The limit is the same as the variance of $\nu_{\text{eff}}\mathcal{U}_0$, where ν_{eff} is now given by (1.3.9), and $\mathcal{U}_0$ denotes the solution to the additive stochastic heat equation

$$\partial_t \mathcal{U}_0(t,x) = \tfrac{1}{2}\Delta \mathcal{U}_0(t,x) + \beta\xi^0(t,x)\ , \qquad \mathcal{U}_0(0,\cdot) \equiv 0\ , \tag{1.3.7}$$

where ξ^0 is the Gaussian noise with covariance (1.3.3). Given $t > 0$ and a test function g, letting

$$X_t^{\varepsilon,g} = \varepsilon^{1-\frac{\kappa}{2}} \int_{\mathbb{R}^d} \big(u_\varepsilon(t,x) - \mathbb{E}u_\varepsilon(t,x)\big) g(x) dx\ ,$$

we are able to prove the following:

Theorem 2. *There exists a $\beta_0 > 0$ such that for all $\beta \leqslant \beta_0$, for any time indices $0 < t_1 \leqslant \ldots \leqslant t_n$, and for any smooth functions with compact support $\{g_i\}_{i=1,\ldots,n} \subset \mathcal{C}_c^\infty(\mathbb{R}^d)$, as $\varepsilon \to 0$, the following holds:*

$$(X_{t_1}^{\varepsilon,g_1}, \ldots, X_{t_n}^{\varepsilon,g_n}) \Rightarrow \nu_{\text{eff}} \bigg(\int_{\mathbb{R}^d} \mathcal{U}_0(t_1,x) g_1(x)\, dx, \ldots, \int_{\mathbb{R}^d} \mathcal{U}_0(t_n,x) g_n(x)\, dx \bigg). \tag{1.3.8}$$

Where the effective variance ν_{eff} is given by

$$\nu_{\text{eff}}^2 = \big| \mathbb{E}\big[\, \sigma\big(Z(0)\big)\, \big] \big|^2\ . \tag{1.3.9}$$

This is one of our main results. Compared to the compactly supported covariance case, we note three differences:

(i) The limit is no longer the solution $\mathcal{U}^{EW}$ of the EW equation (1.2.10), now the limit equation (1.3.7) is driven by coloured noise with correlations of polynomial decay.

(ii) The fluctuations, now of order $\varepsilon^{\frac{\kappa}{2}-1}$, live on a larger scale depending on the decay parameter κ.

(iii) The effective variance (1.3.9) is of different nature compared to (1.2.18). It is still modulated by the nonlinearity σ, but somewhat resambles more the behaviour observed in $d = 2$ [Tao22, DG24], where

$$\nu_{\text{eff}}^2 = \mathbb{E}\big[\,\sigma\big(Z(0)\big)^2\,\big]\ .$$

Interestingly, in the case of linear SHE ($\sigma(u) = u$) or KPZ, as we shall see later, our effective variance (1.3.9) reduces to 1, which is *independent* of β, unlike the compactly supported case or results in $d = 2$.

To show the joint convergence (1.3.8), we employ Stein's method (cf. [HNV20, Prop. 2.3]), inspired by [GL20] earlier work in the compactly supported case. To employ the method, we make use of a divergence representation of the fluctuations $X_t^{\varepsilon,g} = \delta(v_t^{\varepsilon,g})$, readily available given the mild formulation of SHE. Then, in order to show convergence to the Gaussian limit, through Stein's criterion, this reduces to show that as $\varepsilon \to 0$,

$$\langle DX_{t_i}^{\varepsilon,g_i}, v_{t_j}^{\varepsilon,g_j}\rangle_{\mathcal{H}} \xrightarrow{L^2(\Omega)} \Sigma_{i,j}\ , \tag{1.3.10}$$

where Σ denote the covariance matrix of the EW limit, on the right hand side of (1.3.8). The main technical step is to extract the Gaussian limit, which reduces to show vanishing of some higher order terms in the fluctuations. This is achieved by decomposing $\langle DX_{t_i}^{\varepsilon,g_i}, v_{t_j}^{\varepsilon,g_j}\rangle_{\mathcal{H}}$ and careful estimates, leveraging the bounds (1.3.5). Eventually, we bound the remainder terms by multiple integrals involving singular kernels. These integrals are shown to be finite by viewing them as Feynman diagrams. To carry out these arguments, we adapt the general framework of [Hai18] to our needs, allowing us to systematically handle singular kernels and their small/large scales behaviour.

Convergence in Hölder topologies. Building on the multidimensional convergence (1.3.8), we then show a stronger result: we view X^ε and $\mathcal{U}_0$ as distribution valued stochastic processes and consider the spaces of locally Hölder continuous distributions $\mathcal{C}^\alpha(\mathbb{R}^d)$, with $\alpha < 0$. These spaces, as introduced for example in [Hai14b], allow to measure and compare regularity of distributions. Using these as state space, we formulate a Kolmogorov-type criterion to show a random distibution-valued process belongs to $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$, with $\gamma > 0$. With moments control on the associated norm, we obtain a tightness criterion. Hence we are able to show that X^ε are tight and converge in law to $\mathcal{U}_0$ in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$, where the Hölder exponents depend on the covariance decay parameter κ, obtaining the following:

Theorem 3. *For any $\gamma \in (0, \frac{1}{2})$ and $\alpha < 1 - \frac{\kappa}{2} - 2\gamma$, there exists $\beta_{\gamma,\alpha} > 0$ such that, for any $\beta \leqslant \beta_{\gamma,\alpha}$ and $T > 0$, we have as $\varepsilon \to 0$,*

$$X^\varepsilon \implies \nu_{\text{eff}}\,\mathcal{U}\,, \qquad \text{in} \qquad \mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)\,.$$

The convergence in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$ spaces is novel to our knowledge and the Hölder exponents appear optimal in this setting. Some previous works in the related literature [CCM24, CD20] made use of the criterion of [FM17, Thm 1.1] (cf. also [Hai14b, Thm. 10.7]) to show tightness and convergence of fluctuations in $\mathcal{C}^\alpha$ for fixed time $t > 0$, with sub-optimal α. We hope the intuitive and simple framework, as formulated in the embeddings of Proposition 3.3.7, can be applied in different contexts too. Together with Theorems 2-3, these will be the main focus of Chapter 3.

Homogenisation and the KPZ scaling limit. Understanding the KPZ fluctuations with long range correlations was one of the main objectives since start of the PhD. The strategy is to consider the Cole-Hopf transform to study the large scales fluctuations. By setting $h_\varepsilon(t,x) = \varepsilon^{1-\frac{\kappa}{2}} \log u_\varepsilon(t,x)$, this satisfies

$$\partial_t h_\varepsilon = \tfrac{1}{2}\Delta h_\varepsilon + \tfrac{1}{2}\varepsilon^{\frac{\kappa}{2}-1}|\nabla h_\varepsilon|^2 - \tfrac{1}{2}\beta^2 R(0)\varepsilon^{-1-\frac{\kappa}{2}} + \beta\xi^\varepsilon\,, \qquad h_\varepsilon(0,x) = h_0(x)\,. \tag{1.3.11}$$

Due to the modulation through a singular mapping, it is not immediately evident whether the large scale fluctuations of the KPZ obey the same effective dynamics as SHE. For example, one difference is that, because of the nonlinearity, we no longer readily have a divergence expression for the fluctuations. This introduces additional challenges, both in demonstrating variance convergence (to identify the limit) and proving Gaussianity of the limit. Furthermore, one now requires control of the negative moments of u. Different methods have been employed to study the KPZ fluctuations in the compactly supported covariance case, all fundamentally based on the Feynman-Kac formula (1.2.7) for the linear SHE [DGRZ20, CNN22, LZ22] (with the exception of [MU18], where perturbation techniques are used).

Here we deviate from these strategies. Initially inspired by [DGRZ20, Thm. 1.2], we can consider more general transformations of u and look at

$$Y_t^{\varepsilon,g} = \varepsilon^{1-\frac{\kappa}{2}} \int_{\mathbb{R}^d} \big(\Phi(u_\varepsilon(t,x)) - \mathbb{E}[\Phi(u_\varepsilon(t,x))] \big) g(x)\, dx \ ,$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is a transformation satisfying some small and large scales polynomial growth bounds (with the Cole–Hopf transform $\phi(u) = \log u$ satisfying them too). By the Clark–Ocone formula, we retrieve a divergence representation of $Y_t^{\varepsilon,g}$,

$$Y_t^{\varepsilon,g} = \varepsilon^{1-\frac{\kappa}{2}} \int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \mathbb{E}[D_{s,y}\Phi(u_\varepsilon(t,x))|\mathcal{F}_s]\, g(x)\, dx \Big) \xi(ds,dy) \ ,$$

which we can then express in terms of the fundamental solutions of the linear SHE. We denote these by $w_{s,y}(t,x)$, where $w_{s,y}(t,x)$ is the solution at time $t > s$ of (1.2.1) initialiased with Dirac-delta initial conditions δ_y at time s. From here, the key idea is that the fundamental solution exhibits large scales homegenisation. Namely, when $t \gg s$, we observe

$$w_{s,y}(t,x) \approx P_{t-s}(x-y)\overleftarrow{Z}(s,y)\vec{Z}(t,x) \ ,$$

where $P_t(x) = (2\pi t)^{-d/2} e^{-\frac{|x|^2}{2t}}$ denotes the d-dimensional heat kernel, and $\overleftarrow{Z}(s,y)$ is another stationary process, representing $\vec{Z}(t,x)$ "running backwards

in time". More concretely, the stationary fields satisfy

$$\begin{aligned} \vec{Z}(t,x) &= 1 + \beta \int_{-\infty}^{t} \int_{\mathbb{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)\,\xi(dr,dz)\ , \\ \overleftarrow{Z}(s,y) &= 1 + \beta \int_{s}^{\infty} \int_{\mathbb{R}^d} P_{r-s}(y-z)\overleftarrow{Z}(r,z)\,\xi(dr,dz)\ , \end{aligned} \tag{1.3.12}$$

with the stochastic integral in the expression of $\overleftarrow{Z}$ being of backwards Itô type. Then, we prove the following:

Theorem 4. *Let $\mu = \frac{1}{2} - \frac{1}{\kappa}$. For any $p \geqslant 1$, for sufficiently small β, the following holds, uniformly in $t > s$ and $x, y \in \mathbb{R}^d$,*

$$\left\| \frac{w_{s,y}(t,x)}{P_{t-s}(y-x)} - \vec{Z}(t,x)\overleftarrow{Z}(s,y) \right\|_p \lesssim (1+t-s)^{-\mu}(1+(1+t-s)^{-\frac{1}{2}}|x-y|)\ . \tag{1.3.13}$$

Setting up the proof of this was one of the key technical challenges. By appropriately decomposing the difference $w_{s,y}(t,x)/P_{t-s}(y-x) - \vec{Z}(t,x)\overleftarrow{Z}(s,y)$, to exploit the characterisation of the fields (1.3.12), through careful moments and kernel estimates, we are able to derive the spatially weighted decay rate (1.3.13), tailored for our later proofs. A posteriori, we note that the result can be seen as an analogue of some recent local limit theorems for polymers in discrete [Gab23, Jun23] and continuous setting [NN23, CCM20].

Crucially, using mixing of the stationary fields and separation of scales, the homegenisation result allows us to show that the fluctuations of $\Phi(u_\varepsilon)$ are approximately equal, up to a modulation factor ν_Φ, to the ones of u_ε solving the linear SHE. Namely, we show the following convergence in $L^p(\Omega)$:

$$\lim_{\varepsilon \to 0} \|Y_t^{\varepsilon,g} - \nu_\Phi X_t^{\varepsilon,g}\|_p = 0\ . \tag{1.3.14}$$

The convergence (1.3.14) is actually uniform in $t > 0$, with a precise rate which allows us to show convergence in Hölder spaces through the Kolmogorov-like embeddings of Chapter 3.

Decomposing the fluctuations of SHE, we are then able to show *convergence in probability* to the fluctuations limit $\nu_\Phi \mathcal{U}_0$, since, under a suitable

coupling, the driving noise ξ^0 of the limit equation is itself the scaling limit of ξ^ε. This is unlike the case of integrable correlations, where only convergence in law is to be expected[(3)].

Theorem 5. *Let $\Phi \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{R})$ be a function satisfying for some $q > 0$,*

$$|\Phi(z)| + |\Phi'(z)| + |\Phi''(z)| \lesssim z^{-q} + z^q \,, \qquad \forall z \in \mathbb{R}_+ \,.$$

Set

$$u_\varepsilon^\Phi(t,x) = \varepsilon^{1-\frac{\kappa}{2}} \big(\Phi(u_\varepsilon(t,x)) - \mathbb{E}[\Phi(u_\varepsilon(t,x))]\big) \,.$$

Then, given any $p \geqslant 1$, and $\alpha < 1 - \frac{\kappa}{2}$, there exists $\beta_0^\Phi(\alpha,p) > 0$ and $\nu_\Phi \geqslant 0$ such that, for all $\beta < \beta_0^\Phi$ and any good coupling of the noise, the random processes u_ε^Φ converge in probability to $\nu_\Phi \mathcal{U}^0$ in $L^p([0,T],\mathcal{C}^\alpha(E))$ as $\varepsilon \to 0$, for any $T > 0$ and compact $E \subset \mathbb{R}^d$. The effective variance is given by

$$\nu_\Phi = \mathbb{E}[Z\,\Phi'(Z)] \,, \tag{1.3.15}$$

where Z denotes a real-valued random variable with the same law as $\vec{Z}(t,x)$.

We observe that the modulation factor ν_Φ is the same as the one appearing in [DGRZ20] (cf. also [NN23, Thm. 1.3] in $d = 2$), as we saw above (1.2.17). However, now we have different fluctuations scales and limit equation, as in SHE. In the case of the Cole–Hopf transform, $\Phi(u) = \log(u)$, we recover the KPZ fluctuations (with flat initial conditions $h_0 \equiv 0$) and we note that $\nu_\Phi = 1$, indicating that the KPZ has the same fluctuations limit as the *linear SHE*, which is also covered by the case $\Phi(u) = u$.

Furthermore, with all the above ingredients in place, we can consider the rescaled KPZ with *general initial conditions* $h_0 \in \mathcal{C}_b(\mathbb{R}^d)$ and probe its scaling limit. Namely, we are able to show that there exists a deterministic constant $c < 0$, such that $h_\varepsilon(t,x) - \varepsilon^{1-\frac{\kappa}{2}} c$ converges and identify the limit.

[(3)]We are grateful for the helpful suggestions from Alex Dunlap and Yu Gu to pursue strong convergence in our settings.

Theorem 6. *For any $p \geqslant 1$, and $\alpha < 1 - \frac{\kappa}{2}$, there exists a constant $c < 0$ and a positive value $\beta_0 = \beta_0(p, \alpha)$ such that for any $\beta < \beta_0$, for any deterministic initial condition $h_0 \in \mathcal{C}_b(\mathbb{R}^d)$, and any good coupling of the noise, one has, for any $T > T_0 > 0$,*

$$\lim_{\varepsilon \to 0} \big(h_\varepsilon(t,x) - \varepsilon^{1-\frac{\kappa}{2}} c \big) = \mathcal{U} \ ,$$

in probability in $L^p([T_0, T], \mathcal{C}^\alpha(E))$, for any compact $E \subset \mathbb{R}^d$. Here, $\mathcal{U}$ denotes the solution of (1.3.7), initialised with $\mathcal{U}(0,x) = h_0(x)$.

This is the main result of the last Chapter 4. Finally, it is also interesting to note that the Gaussianity of the KPZ scaling limit, with long-range correlations, aligns with the predictions from the physics literature [KS99, KCDW14].

1.3.1 Related directions

In view of our results, there are various open problems one may want to address. Notably, how the large-scales dynamics would change with higher values of $\beta > \beta_0$ in our setting (if intermittency onsets and observe a phase transition as in the case of compactly supported covariance). Below we briefly review other related research and interesting possible directions for further work.

Refine fluctuations description. Our study has been focused on fluctuations around the mean. Alternatively one could investigate also other kinds of fluctuations. In the case of compactly supported R, [CCM24] proved convergence results for the fluctuations *about* the stationary solutions of the KPZ (namely $h - \log \vec{Z}$ in our notation) and the fluctuations (about the mean) of the stationary solutions themselves. Their results, when combined, provide a finer description of the fluctuations $h_\varepsilon - \mathbb{E}h_\varepsilon$. One may expect analogue results also in our setting, but these remain open at present. See also [DGRZ21, CCM18] for other related (pointwise) results on SHE and its fluctuations. Given the central limit type of results, another interesting direction may be then to look at the large deviations of $\Phi(u_\varepsilon)$, cf. [MCS03, LT21, GLLT23], and explore how these depend on the correlations decay parameter κ.

Towards criticality. In view of our results, effectively the decay parameter κ appears to play the role of space dimension d. At present it is unclear what would happen with $\kappa \leqslant 2$ and in the edge case $\kappa = d$. As mentioned in Section 1.1, various work has been carried in the critical dimension $d = 2$, where weak disorder regime and large scales Gaussian fluctuations emerge, if the coupling constant β is tuned with a logarithmic correction in ε [CSZ17, CSZ20, Gu20b, Tao22, DG24]. It would be interesting to understand if something similar happens when $\kappa = 2$, with spatial dimension $d \geqslant 3$. In relation to this, one may try to characterise also in higher dimensions the distribution of the limiting stationary field $\vec{Z}$, cf. the results of [DG22, DG23] in critical dimension $d = 2$. We expect the role of logarithmic tuning would be decisive to observe fluctuations in this setting. In fact, earlier work [MT04] studied the case of linear SHE in $d \geqslant 3$,

$$\partial_t u = \tfrac{1}{2}\Delta u + \beta\, u\, \xi, \tag{1.3.16}$$

with Gaussian noise ξ with singular covariance

$$\mathbb{E}[\,\xi(s,y)\,\xi(t,x)\,] = \delta(t-s)|x-y|^{-2}.$$

They showed that, for sufficiently small β, there exist martingale solutions which are *singular measures*, and studied their properties. It would be interesting to understand what happens if one introduces a nonlinearity $\sigma(u)$ in the multiplicative term, viewing $\sigma(u)$ as a perturbation of the identity. Namely, examine whether one can still obtain well-posedness results and how properties of the solutions are impacted by the nonlinearity, in relation to the study of the logarithmically tuned fluctuations above mentioned. Interestingly, very recent work [KT24] explored the time-dependent spatial averages of the linear model (1.3.16) with singular covariance, and they identified three limit regimes based on the space-time scales considered.

Alternative models and time correlations. In our work we focused on the large scale fluctuations of SHE and KPZ in dimensions $d \geqslant 3$, over the whole space.

However, there are other settings and models that could provide interesting problems to explore the impact of long range correlations on the macroscopic dynamics. For example, we mention recent works on the KPZ over the one dimensional *finite* torus [GK21b, DGK21]; Gaussian fluctuations emerging in the Schrödinger equation [GK21a], Burgers' [CGT24] and Navier–Stokes [CK24] equations. Typically, the noise considered in these studies exhibits integrable spatial correlations. It may be interesting to understand if the above, or other related systems, still fall within the (Gaussian) Edwards–Wilkison class when non-integrable (at infinity) spatial correlations are present, probing possible phase transitions due to the stronger noise correlations. We note that in lower dimensions $d \leqslant 2$, observations from physics on the KPZ equation (cf. [SC19] and references therein) suggest that universality class and scalings are indeed affected by the introduction of long-range (time and/or spatial) correlations in the model.

Slowly decaying covariances emerge, for example, when using SHE as a model for temperatures changes in a fluid flow (cf. [MT04, CM94]). Orthogonally to our setting, and actually the problem we attempted at start of the PhD, it would be interesting to understand how *time correlations* impact the large-scale fluctuations. Note that in our proofs later, we shall rely on the martingale structure of the noise, with ξ being white-in-time. Some previous works [GRZ18, DGRZ21, MU18] relaxed such assumption in $d \geqslant 3$, recall (1.2.13), allowing also for short-range (compactly supported) time covariance (see recent [Kot24] for the case of critical dimension $d = 2$), which results in a non-trivial diffusivity in the Edwards–Wilkinson limit. Interestingly, [Muk22] showed that SHE undergoes homogenisation even in presence of long-range time correlations, behaving like

$$\rho(t) \lesssim (1+t)^{-\gamma}\,, \qquad \text{with} \quad \gamma > 2\,.$$

However, homogenisation (and therefore a Gaussian CLT analogue to the above) are no longer expected with slower decaying potential, as [OS99] observed there can be non-unique Gibbs measures when $\gamma \in (1, 2]$. Therefore, at

present it remains an open question if there exists a scaling limit for SHE/KPZ with *non-compactly supported time covariance* $\rho(t)$, which may induce phase transitions based on the time-correlation decay parameter γ.

1.4 Structure of the thesis

In this thesis we only present the main two projects that led to original contributions during the PhD. The results are based on the following two articles, both joint works with Martin Hairer and Xue-Mei Li:

• [GHL23], which was accepted in the journal ***Annals of Applied Probability***.

• [GHL24], currently submitted and under review in the journal ***Probability Theory and Related Fields***.

The rest of this thesis is then structured as follows:

- ▷ In Chapter 2, after introducing notation and technical tools from Malliavin Calculus, we focus on showing estimates and properties of the stochastic heat equation (SHE), in particular the existence of underlying stationary solutions. All these will be crucially used to study the large scale fluctuations in the later chapters.
- ▷ In Chapter 3, we examine the fluctuations of SHE and set up our framework to show convergence in Hölder spaces.
- ▷ In Chapter 4, we finally show the existence of a Gaussian scaling limit for the KPZ equation, through a large scale homogenisation of the fundamental solutions of SHE.

The main results and content of Chapters 2-3 are mostly based on [GHL23], whereas Chapter 4 is based on [GHL24].

Chapter 2

Properties of SHE

In this chapter we outline preliminary notation and tools, then present key results concerning the properties of the solutions of SHE, from the first half of the article [GHL23], joint work with Martin Hairer and Xue-Mei Li. Some parts are verbatim to the article.

As mentioned in the introduction, we shall focus on studying the following nonlinear stochastic heat equation (SHE):

$$\partial_t u(t,x) = \tfrac{1}{2}\Delta u(t,x) + \beta\sigma(u(t,x))\xi(t,x)\ , \quad u(0,x) \equiv 1\ , \tag{2.0.1}$$

where σ is a global Lipschitz function, $\beta > 0$ is a coupling constant and ξ is Gaussian noise, with covariance given by

$$\mathbb{E}[\,\xi(t,x)\xi(s,y)\,] = \delta(t-s)R(x-y)\ . \tag{2.0.2}$$

In the case of $\sigma(u) = u$, we shall refer to it as *linear SHE*. The well-posedness theory of the equation follows from classical theories of [Wal86, DPZ14]. The product on the right-hand side of the equation is interpreted as an Itô–Walsh integral and the unique (mild) solution of (2.0.1) is given by the solution to the fixed point problem

$$u(t,x) = 1 + \beta\int_0^t\int_{\mathbb{R}^d} P_{t-s}(x-y)\sigma(u(s,y))\xi(ds,dy)\ , \tag{2.0.3}$$

given that we consider flat initial conditions $u(0,x) \equiv 1$. In the next sections, we make use of the following assumption on the noise and its covariance, which guarantees well-posedness of the equation with function valued solutions.

Assumption 2.0.1. The noise ξ in (2.0.1) is a mean zero space time Gaussian noise with covariance as in (2.0.2) where $R : \mathbb{R}^d \to \mathbb{R}$ is smooth and positive definite. There exists $\kappa \in (2,d)$ such that $|R(x)| \lesssim (1+|x|)^{-\kappa}$ and, for any $x \in \mathbb{R}^d \setminus \{0\}$,

$$\varepsilon^{-\kappa} R(\varepsilon^{-1}x) \to |x|^{-\kappa} \quad \text{as} \quad \varepsilon \to 0\,. \tag{2.0.4}$$

2.1 Main results and notation

In this chapter, after introducing setup and useful technical tools, we focus on proving key covariance estimates and properties of the solutions of SHE (2.0.1), which will be used in the proofs of the next sections. The structure is as follows:

- ▷ In Section 2.2, we concisely introduce our Malliavin Calculus setup and the main tools used later.
- ▷ In Section 2.3, we present an integration criterion, Lemma 2.3.2, to estimate multiple integrals of singular kernels by recasting them as Feynman diagrams. This is based on the more general framework of [Hai18], tailored to our technical needs for the fluctuations theory of SHE (Chapter 3), but also applicable in different contexts.
- ▷ After setting out preliminary technical tools, in Section 2.4 we gather elementary pointwise Gaussian bounds and covariance (decay) estimates. These are key to show the main results in the next two sections, and they will also be frequently used in the later chapters.
- ▷ In Section 2.5, the main result is Lemma 2.5.2, where we are able to show that for sufficiently small $\beta \leqslant \beta_0(p)$, the moments of u are uniformly

bounded and the moments of its Malliavin derivative are controlled (up to a constant) by the heat kernel:

$$\sup_{s\geqslant 0, x\in\mathbb{R}^d} \|u(s,x)\|_p \leqslant C_p \ , \qquad \|D_{r,z}u(t,x)\|_p \lesssim C_p P_{t-r}(x-z) \ .$$

The latter relies on a pointwise contraction Lemma 2.5.1, which allows also for negative-valued covariance functions. Moreover, we are able to show large scales de-correlation estimates (cf. Lemma 2.5.4) of the type

$$\operatorname{cov}\left(u(t,x), u(t,0)\right) \lesssim 1 \wedge |x|^{2-\kappa} \ ,$$

which imply a law of large numbers, and later allow us to identify of effective limit variance of the fluctuations of SHE, cf. Section 3.2.2.

▷ Finally, in Section 2.6, we show that u converges in the large time limit, Theorem 2.6.1. More specifically, by considering pull back solutions, we prove an L^p convergence rate towards a space-time stationary field $\vec{Z}$ which underpins our fluctations theory for SHE, and also plays a vital role when we look at the KPZ equation in Chapter 4.

We provide precise statements of the results as we go along. Note that similar results to Lemma 2.5.2 and Theorem 2.6.1 were shown in [GL20, Lem. 2.1–2.2, Thm 1.1] for the case when R is compactly supported.

Notation. In the next sections and later chapters, we shall use the following notation and conventions:

▷ Throughout, we consider $(\Omega, \mathcal{F}, \mathbb{P})$ as background probability space, with filtration $\mathcal{F}_t$ obtained by completing the σ-algebra generated by ξ, containing sets of the form

$$\{\xi\big(\mathbf{1}_{[s_0,s_1]}(\cdot)\mathbf{1}_A(\cdot)\big) \ : \ -\infty < s_0 < s_1 \leqslant t, \ \ A \in \mathcal{B}_b(\mathbb{R}^d)\} \ ,$$

where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the bonded Borel subsets of $\mathbb{R}^d$ We denote with $\mathbb{E}[\,\cdot\,]$ the expectation with respect to $\mathbb{P}$. The norm $\|\cdot\|_p$ stands for the

$L^p(\Omega)$ norm for real valued random variables. With $\mathcal{X}$ denoting any normed space, we denote with $L^p(\Omega, \mathcal{X})$ the space of $\mathcal{X}$-valued $L^p(\Omega)$ integrable random variables.

- ▷ As introduced in the next section, $\mathcal{H}$ will denote the Hilbert space underlying the isonormal Gaussian noise ξ. We denote with D the corresponding Malliavin derivative operator and δ for its adjoint.
- ▷ We denote with $|f|_{\mathrm{Lip}}$ the Lipschitz constant of a function $f : \mathbb{R}^d \to \mathbb{R}$, such that $|f(x) - f(y)| \leqslant |f|_{\mathrm{Lip}}|x - y|$ holds for all $x, y \in \mathbb{R}^d$.
- ▷ With $a \lesssim b$ we mean $a \leqslant Cb$ for some constant $C > 0$, independent of the scaling paremeter ε.
- ▷ For $a, b \in \mathbb{R}$, we denote their minimum with $a \wedge b = \min(a, b)$.
- ▷ For $E \subset \mathbb{R}^d$ open, we write $\mathcal{C}_c^\infty(E)$ for the set of smooth functions on E with compact support. We set $\mathcal{D} = \mathcal{C}_c^\infty(\mathbb{R}^d)$ and denote its dual space of distributions with $\mathcal{D}'$.
- ▷ For $E \subset \mathbb{R}^d$ open and nonnegative integer $k \in \mathbb{Z}_+$, we denote with $\mathcal{C}^k(E)$ the space of k-times continuously differentiable functions on E, and $\mathcal{C}_b^k(E)$ the subset of bounded $\mathcal{C}^k(E)$ functions with all partial derivatives up to order k bounded.
- ▷ For $r \in [0, 1)$, we denote by $\mathcal{C}^r = \mathcal{C}^r(\mathbb{R}^d, \mathbb{R})$ the space of little-Hölder continuous functions, namely the r-Hölder functions f such that $\lim_{y \to x} |y - x|^{-\alpha}|f(y) - f(x)| = 0$ locally uniformly in x. This is the completion of smooth functions under the system of seminorms

$$\|f\|_{E,\mathcal{C}^r} := \sup_{\substack{x,y \in E \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^r} ,$$

where $E \subset \mathbb{R}^d$ is open and bounded. For $r \geq 1$, we define recursively the space $\mathcal{C}^r$ as the space of continuously differentiable functions such that each directional derivative is in $\mathcal{C}^{r-1}$. We denote by $\mathcal{C}_c^r$ the subset of little-Hölder continuous compactly supported functions.

▷ For $\alpha < 0$, we denote with $\mathcal{C}^\alpha(\mathbb{R}^d) \subset \mathcal{D}'$ the spaces of locally Hölder continuous distributions. These are introduced in Section 3.3.1.

2.2 Malliavin Calculus

In the following, various tools from Malliavin calculus will turn out very useful for various steps of the proofs. The development of Malliavin calculus was initiated in the seminal work of [Mal78], as an attempt to provide a probabilistic proof of Hörmander's Theorem [Hö67]. The theorem showed that solutions to SDEs, under certain conditions on the equation coefficients (Hörmander's condition), have a density with respect to Lebesgue measure (cf. also [Hai11] for a self-contained proof and further references). Since then, it has provided numerous techniques to study stochastic evolutions and "perform calculus" with respect to Gaussian noise.

2.2.1 Basic definitions

We start by recalling fundamental definitions and basic facts, see e.g. [Nua06] for a comprehensive introduction to the subject.

White noise and the Malliavin derivative. Given a real separable Hilbert space $\mathcal{H}$, the stochastic process $\xi = \{\xi(h) : h \in \mathcal{H}\}$ is defined to be an isormal Gaussian process if ξ is a centered Gaussian family of random variables, such that for $h, g \in \mathcal{H}$:

$$\mathbb{E}[\,\xi(h)\xi(g)\,] = \langle h, g\rangle_{\mathcal{H}}\ .$$

Hence ξ defines a linear isometry between $\mathcal{H}$ and a subset of $L^2(\Omega)$ made of centred Gaussians. The idea is now to define a linear operator by perturbing functionals of the noise in $\mathcal{H}$ directions.

Let $\mathcal{C}_p^\infty(\mathbb{R}^n)$ denote the space of smooth functions $F : \mathbb{R}^n \to \mathbb{R}$ growing (with their derivatives) at most polynomially at infinity. Then consider the

space of smooth random variables $\mathcal{S} \subset L^2(\Omega)$ of the form

$$F(\xi(h_1), \dots, \xi(h_n)) \qquad \text{with} \quad F \in \mathcal{C}_p^\infty(\mathbb{R}^n) \, .$$

For $X = F(\xi(h_1), \dots, \xi(h_n)) \in \mathcal{S}$ one defines the Malliavin derivative operator D by

$$DX = \sum_{k=1}^{n} \partial_k F(\xi(h_1), \dots, \xi(h_n)) \, h_k \, .$$

We note that $D\,\xi(h) = h$ and we can interpret DX as a directional derivative, in the sense that for any element $h \in \mathcal{H}$,

$$\langle DX, h\rangle_{\mathcal{H}} = \lim_{\delta \to 0} \frac{1}{\delta} \Big(F\big(\xi(h_1) + \delta\langle h_1, h\rangle_{\mathcal{H}}, \dots, \xi(h_n) + \delta\langle h_n, h\rangle_{\mathcal{H}}\big) - F(\xi(h_1), \dots, \xi(h_n)) \Big) \, .$$

As short hand notation, we may use $D^h X := \langle DX, h\rangle_{\mathcal{H}}$. The space $\mathcal{S}$ is dense in $L^2(\Omega)$, thus the operator $D : \mathcal{S} \subset L^2(\Omega) \to L^2(\Omega, \mathcal{H})$ is unbounded densely defined. The operator satisfies the following integration by parts formula

$$\mathbb{E}\langle DX, h\rangle_{\mathcal{H}} = \mathbb{E}\,[\, X\xi(h)\,] \, . \tag{2.2.1}$$

The formula follows by properties of the Gaussian density and makes D closable, as a linear operator from $L^p(\Omega)$ to $L^p(\Omega, \mathcal{H})$ for any $p \geqslant 1$ [Nua06, Lem. 1.2.1, Prop. 1.2.1]. We shall still denote its closure by D, and consider the associate domain $\mathbb{D}^{1,p} \subset L^p(\Omega)$, given by the closure of $\mathcal{S}$ under the norm

$$\|X\|_{\mathbb{D}^{1,p}} = \big(\mathbb{E}\,[\,|X|^p\,] + \mathbb{E}\,[\,\|DX\|_{\mathcal{H}}^p\,]\big)^{\frac{1}{p}} .$$

One can consider iterated Malliavin derivatives D^k and define corresponding (Sobolev-like) spaces $\mathbb{D}^{k,p}$, we refer to [Nua06] for these. One important fact is that Malliavin differentiability is preserved under transformations, and a chain rule, analogue to the one in classical calculus, holds:

Proposition 2.2.1. *[Nua06, Prop. 1.2.3] Let $p \geqslant 1$. Assume $\psi \in \mathcal{C}_b^1(\mathbb{R}^m)$ and $X = (X^1, \dots, X^m)$ is a random vector, where each component belongs to the space $\mathbb{D}^{1,p}$. Then $\psi(X) \in \mathbb{D}^{1,p}$, and*

$$D[\psi(X)] = \sum_{i=1}^{m} (\partial_i \psi)(X) \, DX^i \, .$$

Adjoint operator and properties. We denote with δ the adjoint of the operator D. This is a closed unbounded operator $\delta : L^2(\Omega, \mathcal{H}) \to L^2(\Omega)$ such that:

(i) The domain of δ is the subset of $\mathcal{H}$-valued random variables $u \in L^2(\Omega, \mathcal{H})$ for which there exists a constant $c = c(u)$, such that for all $X \in \mathbb{D}^{1,2}$ the following holds:

$$\left|\mathbb{E}\left[\, \langle DX, u\rangle_{\mathcal{H}} \,\right]\right| \leqslant c\|X\|_2 \,.$$

(ii) If $u \in \operatorname{Dom}\delta$, then $\delta(u) \in L^2(\Omega)$ and satisfies for all $X \in \mathbb{D}^{1,2}$:

$$\mathbb{E}\left[\, X\delta(u) \,\right] = \mathbb{E}\left[\, \langle DX, u\rangle_{\mathcal{H}} \,\right] .$$

The adjoint operator is also referred to as the divergence operator or Skorokhod integral. We note that $\mathcal{H}$-valued Malliavin differentiable random variables $u \in \mathbb{D}^{1,2}(\mathcal{H})$ belong to the domain of δ. Moreover, if $u \in \mathbb{D}^{1,2}(\mathcal{H})$ and $D^h u \in \operatorname{Dom}\delta$, the following commutation relation holds [Nua06, Prop. 1.3.2]:

$$D^h\delta(u) = \langle u, h\rangle_{\mathcal{H}} + \delta(D^h u) \,. \tag{2.2.2}$$

Note that in the case that h is a deterministic element of $\mathcal{H}$, we have $D\delta(h) = h$. We now state an important property for later.

Proposition 2.2.2. *[Nua06, Prop. 1.3.3] Let $X \in \mathbb{D}^{1,2}$ and $u \in \operatorname{Dom}\delta$ such that $Xu \in L^2(\Omega, \mathcal{H})$ and $X\delta(u) - \langle DX, u\rangle_{\mathcal{H}}$ is square integrable. Then the following holds:*

$$X\delta(u) = \delta(Xu) + \langle DX, u\rangle_{\mathcal{H}} \,.$$

In the setting where δ corresponds to the Itö integral (when u is a suitable adapted process), by introducing a correction term, the proposition allows us to factor into the (Skorokhod) integral non-adapted random variables. In addition, we shall make use of the following control on the divergence, the proof is based on Meyer's inequalities, cf. [Nua06, Sec. 1.5].

Proposition 2.2.3. *[Nua06, Prop. 1.5.4] For all $p > 1$, the operator δ is continuous from $\mathbb{D}^{1,p}(\mathcal{H})$ to $L^p(\Omega)$.*

2.2.2 Our setup

We now can review our setup for stochastic integration and Malliavin calculus notation going forward.

Gaussian noise. In our settings, the driving noise has covariance given by (2.0.2). Since R is positive definite, we can view ξ as an isonormal Gaussian process on the Hilbert space $\mathcal{H}$, defined to be the closure of $\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ under the inner product

$$\langle f,g\rangle_{\mathcal{H}} = \int_{-\infty}^{\infty}\int_{\mathbb{R}^{2d}} f(s,y_1)g(s,y_2)R(y_1-y_2)dy_1dy_2ds\ , \tag{2.2.3}$$

where we quotient out elements with zero norm. Then, for any deterministic $f,g\in\mathcal{H}$, we have

$$\mathbb{E}\big[\,\xi(f)\,\xi(g)\,\big] = \langle f,g\rangle_{\mathcal{H}}$$

We denote the induced norm by

$$\|f\|_{\mathcal{H}} = \sqrt{\langle f,f\rangle_{\mathcal{H}}}\ .$$

The Itô–Walsh stochastic integral with respect to ξ, which we denote by

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} v(s,y)\,\xi(ds,dy)\ ,$$

is well defined for any jointly measurable random process $\{v(s,y):(s,y)\in\mathbb{R}\times\mathbb{R}^d\}$ that is $\mathcal{F}_s$-adapted and satisfies

$$\mathbb{E}[\,\|v\|_{\mathcal{H}}^2\,] = \mathbb{E}\Big[\int_{-\infty}^{\infty}\int_{\mathbb{R}^{2d}} v(s,y_1)v(s,y_2)R(y_1-y_2)dy_1dy_2ds\Big] < \infty\ . \tag{2.2.4}$$

The construction of the stochastic integral follows by an isometry extension, as for classical Itô calculus, see e.g. [Dal99, Sec. 2].

In particular, for an adapted intregrable process v, Itô isometry holds:

$$\mathbb{E}\Big[\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^d} v(s,y)\,\xi(ds,dy)\Big)^2\Big] = \mathbb{E}[\,\|v\|_{\mathcal{H}}^2\,]\ . \tag{2.2.5}$$

Morover, for $v\in\operatorname{Dom}\delta$ which is $\mathcal{F}_s$-adapted, the Skorokhod integral $\delta(v)$ coincides with the Itô integral $\int_{\mathbb{R}\times\mathbb{R}^d} v(s,y)\xi(ds,dy)$.

Well-posedness. Under our assumptions on σ being Lipschitz and with flat initial condition $u_0 \equiv 1$, by [Dal99, Thm. 13, Remark 14], the solution to (2.0.1) is unique and function valued, satisfying for any $T > 0$ and $p \geqslant 1$:

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \mathbb{E}\big[|u(t,x)|^p\big] < \infty \,, \tag{2.2.6}$$

as long as the spatial covariance R satisfies "Dalang's condition":

$$\int_{\mathbb{R}^d} \frac{\hat{R}(z)}{1+|z|^2} dz < \infty \,,$$

where $\hat{R}$ denotes the Fourier transform $\hat{R}(z) = \int_{\mathbb{R}^d} R(x) e^{-iz\cdot x}\, dx$ of R.

This holds from our Assumption (2.0.1), since by Plancherel's identity,

$$\int_{\mathbb{R}^d} \hat{R}(z)|z|^{-2} dz = c \int_{\mathbb{R}^d} R(y)|y|^{-d+2} dy \lesssim \int_{\mathbb{R}^d} (1+|y|)^{-\kappa}|y|^{-d+2} dy < \infty \,.$$

The proof of well-posedness and (2.2.6) in [Dal99] is based on a Picard iteration argument, cf. proof of Lemma 2.5.2 where we prove stronger moment estimates, as required in our later proofs.

We refer also to [PZ00, Thm 0.2], where they demonstrate well-posedness solutions in the weighted space $L^2(\mathbb{R}^d, e^{-|x|}dx)$, under the same hypothesis.

Notation and commutation relation. For a random variable $X \in \mathbb{D}^{1,2}$, among its equivalence class in $\mathcal{H}$, in the following we shall denote DX by $D_{\cdot,\cdot}X$ for a function valued representative of the Malliavin derivative of X. Then $D_{r,z}X$ denotes the pointwise evaluation of $D_{\cdot,\cdot}X$ at (r,z) in $\mathbb{R}\times\mathbb{R}^d$, which makes sense up to modification on zero Lebesgue measure subsets of $\mathbb{R}\times\mathbb{R}^d$. For example, if $h \in \mathcal{H} \cap L^2(\mathbb{R}^{d+1})$ is a deterministic measurable function in $\mathcal{H}$, then $D_{r,z}\xi(h) = h(r,z)$.

From (2.2.2), the following commutation relation is satisfied as elements of $\mathcal{H}$:

$$D_{r,z}\delta(v) = v(r,z) + \int_{-\infty}^{\infty}\int_{\mathbb{R}^d} D_{r,z}v(s,y)\,\xi(ds,dy) \,, \tag{2.2.7}$$

where the integral on the right-hand side is in the Itô sense, as long as $D_{r,z}v(s,y)$ is $\mathcal{F}_s$-adapted (otherwise it is a Skorokhod integral in general).

With the notation introduced, the Clark–Ocone formula will be an important tool to express in divergence form the fluctuations from the mean and derive covariance estimates.

Proposition 2.2.4. *[CKNP21, Prop 6.3] For any $X \in \mathbb{D}^{1,2}$, the following equality holds almost surely:*

$$X = \mathbb{E}[X] + \int_{\mathbb{R}\times\mathbb{R}^d} \mathbb{E}[D_{s,y}X|\mathcal{F}_s]\,\xi(ds,dy)\ . \tag{2.2.8}$$

Finally, another important tool we shall use to prove convergence in law to a Guassian limit is the following formulation of Stein's method:

Proposition 2.2.5. *[HNV20, Prop. 2.3] Let $X = (X_1, \dots, X_n)$ be a random vector such that $X_i \in \mathbb{D}^{1,2}$ and $X_i = \delta(v_i)$ for $v_i \in \mathrm{Dom}\,\delta$, $i = 1, \dots n$. Let Z be an n-dimensional centered Gaussian vector with covariance matrix $(\Sigma_{i,j})_{1\leqslant i,j\leqslant n}$. Then, for any function $h : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{C}_b^2(\mathbb{R}^n)$, letting $\|D^2h\|_\infty = \max_{1\leqslant i,j\leqslant n} \sup_{x\in\mathbb{R}^n} |\partial_{x_i}\partial_{x_j}h(x)|$, we have*

$$|\mathbb{E}h(X) - \mathbb{E}h(Z)| \leqslant \frac{1}{2}\|D^2h\|_\infty \sqrt{\sum_{i,j=1}^n \mathbb{E}[(\Sigma_{i,j} - \langle DX_i, v_j\rangle_{\mathcal{H}})^2]}\ .$$

2.3 Feynman diagrams integration

The aim of this section is to formulate a criterion (Lemma 2.3.2) to verify that integrals I_Γ involving multiple kernerls of the form (2.3.1) are finite. We start by introducing the notions needed from the general framework of [Hai18].

2.3.1 Definitions

Throughout this section, we fix a family $\{K_\mathfrak{t} : \mathfrak{t} \in \mathfrak{L}\}$, where $\mathfrak{L}$ denotes some finite index set and each $K_\mathfrak{t}$ is a function $K_\mathfrak{t} : \mathbb{R}^d \to \mathbb{R}$ (also sometimes referred to as a "kernel"). All our kernels are assumed to be smooth, except at the origin where they are allowed to have a singularity.

Consider now a directed graph $\Gamma = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a finite index set, interpreted as the vertices of Γ, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, interpreted as the set of edges, with elements denoted by $e = (e_-, e_+)$. Given a label map $\mathfrak{t}\colon \mathcal{E} \to \mathfrak{L}$, it will be convenient to have bounds on general integrals of the form

$$I_\Gamma(\varphi) = \int_{(\mathbb{R}^d)^{\mathcal{V}}} \Big(\prod_{e \in \mathcal{E}} K_{\mathfrak{t}(e)}(x_{e_+} - x_{e_-}) \Big) \varphi(x_{\mathcal{V}_\ell})\, dx\ . \tag{2.3.1}$$

Here, $\mathcal{V}_\ell \subset \mathcal{V}$ is some fixed subset of vertices, sometimes called "legged vertices" [(1)], while $x_{\mathcal{V}_\ell} \in (\mathbb{R}^d)^{\mathcal{V}_\ell}$ denotes the restriction of $x \in (\mathbb{R}^d)^{\mathcal{V}}$ to $\mathcal{V}_\ell$, and $\varphi \in \mathcal{C}_c((\mathbb{R}^d)^{\mathcal{V}_\ell})$ is a test function. We also sometimes write $\mathcal{V}_{\mathrm{int}} = \mathcal{V} \backslash \mathcal{V}_\ell$ for the "interior vertices". We also think of I_Γ as a distribution on $(\mathbb{R}^d)^{\mathcal{V}_\ell}$, in which case integration takes place over the interior vertices and the legged vertices are kept fixed. Our test functions will typically be of the form $\varphi(x_{\mathcal{V}_\ell}) = \prod_{i \in \mathcal{V}_\ell} g_i(x_i)$ for some $g_i \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

We refer to the triple $(\mathcal{V}, \mathcal{E}, \mathfrak{t})$ as a "Feynman diagram". In our applications, we indicate the edge labels by colour / degree in the diagram and we ignore the orientation of edges corresponding to symmetric kernels.

In order to bound (2.3.1), we need some quantitative control on the kernels $K_{\mathfrak{t}}$, whose behaviour is characterised by their label $\mathfrak{t}$ and associated degrees. Namely, we assume that there are "degree" maps

$$\deg_0\colon \mathfrak{L} \to (-\infty, 0), \qquad \deg_\infty\colon \mathfrak{L} \to (-\infty, 0),$$

capturing the small / large scale behaviour of the corresponding kernel:

$$|K_{\mathfrak{t}}(x)|\, \mathbf{1}_{|x| \leqslant 1} \leqslant c_1 |x|^{\deg_0 \mathfrak{t}}, \qquad |K_{\mathfrak{t}}(x)|\, \mathbf{1}_{|x| > 1} \leqslant c_2 |x|^{\deg_\infty \mathfrak{t}}. \tag{2.3.2}$$

We will also need the following two notions in order to formulate the assumptions guaranteeing integrability of (2.3.1) at small / large scales respectively.

[(1)] In the more general setting of [Hai18], these are precisely the vertices that are target of some incoming leg. In our specific case each vertex in $\mathcal{V}_\ell$ has exactly one incoming leg, and $(\mathcal{V}, \mathcal{E})$ correspond to $(\mathcal{V}_*, \mathcal{E}_*)$ in the notation used in [Hai18].

(i) A *subgraph* $\bar{\Gamma} \subset \Gamma$ consists of a subset of edges $\bar{\mathcal{E}} \subset \mathcal{E}$, as well as the subset of vertices $\bar{\mathcal{V}}$ incident to at least one edge in $\bar{\mathcal{E}}$. Note that a subgraph does not need to contain all the edges in Γ that are connected vertices in $\bar{\mathcal{V}}$.

(ii) Let $\mathcal{P}_\Gamma = \{A_i\}$ be a partition of $\mathcal{V}$. An edge (e_-, e_+) is said to be a "traversing" edge, if the two vertices belong to different elements of the partition. We denote the set of traversing edges with $\mathcal{E}(\mathcal{P}_\Gamma)$. A partition $\mathcal{P}_\Gamma$ with at least two elements is said to be *tight* if there exists $A \in \mathcal{P}_\Gamma$ with $\mathcal{V}_\ell \subset A$, containing all the legged vertices.

Before the main Lemma, we review the notions introduced in an example:

Example 2.3.1. Consider the integral I_Γ defined as:

$$I_\Gamma(\phi) = \int_{\mathbb{R}^{4d}} \int_{\mathbb{R}^{2d}} \big(|x_1 - z_1||x_3 - z_2|\big)^\gamma \big(|x_2 - z_1||x_4 - z_2|\big)^\alpha |z_1 - z_2|^\beta \phi(x)\, dx\, dz \ ,$$

where $\phi \in \mathcal{C}_c(\mathbb{R}^{4d})$, $dx = \prod_{i=1}^4 dx_i$ and $dz = \prod_{i=1}^2 dz_i$. The Feynman diagram Γ associated to the integral is the following:

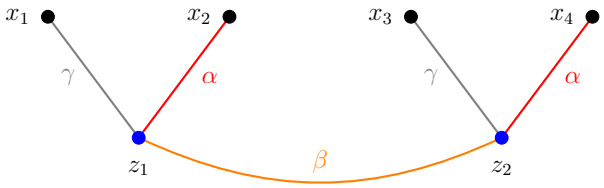


Diagram Γ

Our convention will be to denote with variables x_i the legged vertices integrated against a test function ϕ, and use different colours in the diagram to visually differentiate them from the internal vertices. In this case, we have

$$\mathcal{V}_\ell = \{\bullet_{x_i} \ : \ i = 1, 2, 3, 4\} \, , \qquad \mathcal{V}_{\text{int}} = \{\bullet_{z_i} \ : \ i = 1, 2\} \ .$$

For example, consider the partition

$$\mathcal{P} = \{\{\mathcal{V}_\ell, \bullet_{z_1}\}, \{\bullet_{z_2}\}\} \ .$$

This partition is tight, of size 2 and its traversing edges are $[z_1, z_2]$, $[z_2, x_3]$ and $[z_2, x_4]$, with degrees β, γ and α respectively. Thus, the corresponding large scale degree (defined below at (2.3.4)) is given by

$$\deg_\infty \mathcal{P} = \sum_{e \in \mathcal{E}(\mathcal{P})} \deg_\infty \mathfrak{t}(e) + d(|\mathcal{P}| - 1) = \beta + \gamma + \alpha + d \, .$$

In Lemma 3.2.5 we will determine conditions such that I_Γ is finite.

2.3.2 Integration lemma and applications

Given the definitions above, combining [Hai18, Prop. 2.3 & 4.1] (see also [Wei60] for the a formulation for the small-scale behaviour) we have the following lemma:

Lemma 2.3.2. *Let Γ be a Feynman diagram with at least one edge. Suppose that the following small and large scale conditions are satisfied.*

(i) For every non-empty subgraph $\bar{\Gamma} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$,

$$\deg_0 \bar{\Gamma} := \sum_{e \in \bar{\mathcal{E}}} \deg_0 \mathfrak{t}(e) + d(|\bar{\mathcal{V}}| - 1) \; > \; 0. \tag{2.3.3}$$

(ii) For every tight partition $\mathcal{P}_\Gamma$, denoting $\mathcal{E}(\mathcal{P}_\Gamma)$ its set of traversing edges,

$$\deg_\infty \mathcal{P}_\Gamma := \sum_{e \in \mathcal{E}(\mathcal{P}_\Gamma)} \deg_\infty \mathfrak{t}(e) + d(|\mathcal{P}_\Gamma| - 1) \; < \; 0. \tag{2.3.4}$$

Then $I_\Gamma(\varphi) < \infty$ for every $\varphi \in \mathcal{C}_c((\mathbb{R}^d)^{\mathcal{V}_\ell})$.

Remark 2.3.3. If Γ is a tree (no undirected cycle), to check (2.3.3) is sufficient to verify that for all $e \in \mathcal{E}_*$

$$\deg_0 \mathfrak{t}(e) > -d. \tag{2.3.5}$$

Below we provide an example, which will be used to determine the limit variance of the fluctuations of SHE. The lemma will also be crucially used in

Section 3.2.4 in order to show that higher order non-Gaussian terms vanish. Before the example, we gather here some useful estimates for convolutions.

Convolution estimates. For any numbers $\alpha, \beta > -d$ such that $\alpha + \beta + d < 0$, one has

$$\int_{\mathbb{R}^d} |z|^\alpha |x - z|^\beta dz = c_{\alpha,\beta} |x|^{\alpha+\beta+d} , \tag{2.3.6}$$

for some finite constant $c_{\alpha,\beta}$. This can be seen by performing a change of variables $z = |x|y$ (cf. [Hai14b, Lem. 10.14]). More generally, as in [HQ18, Sec. 6.3], one can define for $\alpha, \beta \geq 0$ the spaces $\mathcal{B}_{\alpha,\beta}$ of functions F such that

$$\|F\|_{\alpha,\beta} \stackrel{\text{def}}{=} \sup_{|x|\leqslant 1} |x|^\alpha |F(x)| + \sup_{|x|\geq 1} |x|^\beta |F(x)| < \infty , \tag{2.3.7}$$

allowing different decay/blow up coefficients at large and small scales. It is shown in [HQ18, Lem. 6.8] that[(2)], provided that $\alpha_i < d$ and $\beta_1 + \beta_2 > d$, one has

$$\|F_1 \star F_2\|_{\alpha,\beta} \lesssim \|F_1\|_{\alpha_1,\beta_1} \|F_2\|_{\alpha_2,\beta_2} , \tag{2.3.8}$$

with $\alpha = 0 \vee (\alpha_1 + \alpha_2 - d)$ and $\beta = (\beta_1 + \beta_2 - d) \wedge \beta_1 \wedge \beta_2$.

Example 2.3.4. For any $f, g \in \mathcal{C}_c(\mathbb{R}^d)$, any $\kappa < d$, and any $\lambda \in (0, \frac{\kappa}{2})$,

$$I_{\Gamma_0}(f, g; \lambda) := \int_{\mathbb{R}^{4d}} |y_1 - y_2|^{-\kappa} \prod_{i=1}^{2} |x_i - y_i|^{\lambda-d} f(x_1) g(x_2) dy dx < \infty. \tag{2.3.9}$$

Since $\kappa < d$ and $\lambda > 0$, by homogeneity $\int |x_2 - y_2|^{\lambda-d} |y_1 - y_2|^{-\kappa} dy_2$ is a constant multiple of $|y_1 - x_2|^{\lambda-\kappa}$. Then, up to constant, the integral I_{Γ_0} is controlled by the integral $I_{\Gamma_0'}$, with diagram below. This corresponds to collapsing the edge $[y_1, y_2]$ in Γ_0 and removing the node $\{\bullet_{y_2}\}$ from the diagram. Since $2\lambda < \kappa$, we then estimate as follows:

$$I_{\Gamma_0} \lesssim I_{\Gamma_0'} = \int_{\mathbb{R}^{3d}} |x_1 - y_1|^{\lambda-d} |x_2 - y_1|^{\lambda-\kappa} |f(x_1) g(x_2)| dy_1 dx$$

[(2)]The statement given there is on $\mathbb{R}^{d+1}$ with parabolic scaling. The exact same proof applies to the Euclidean scaling.

$$\lesssim \int_{\mathbb{R}^{2d}} |x_1 - x_2|^{2\lambda-\kappa} |f(x_1)g(x_2)| dx < \infty.$$

This simple computation is a good place to illustrate notation and to demonstrate how to apply Lemma 2.3.2. We draw the Feynman diagram Γ_0 associated with the integral and its reduced graph Γ_0', once we integrated out all possible edges between the internal vertices $\mathcal{V}_i$.

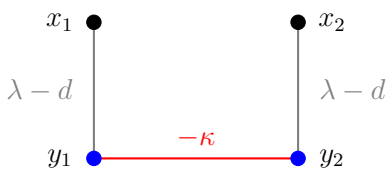


Diagram Γ_0

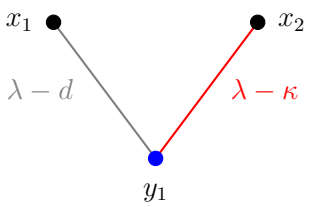


Diagram Γ_0'

The small scales $\deg_0$ and the large scales $\deg_\infty$ of each edge is the same and labelled. To explain the terminology we work with Γ_0. Note that all 3 labels are greater than $-d$, then the small scale criterion is trivially satisfied by Remark 2.3.3. Let $\mathcal{V}_\ell = \{\bullet_{x_1}, \bullet_{x_2}\}$ denote the set of single-legged vertices and the remaining $\mathcal{V}_i = \{\bullet_{y_1}, \bullet_{y_2}\}$. A tight non-trivial partition must have an element A containing $\mathcal{V}_\ell$, so they are:

$$\mathcal{P}_1 = \{\bullet_{x_1}, \bullet_{x_2}, \bullet_{y_1}\} \cup \{\bullet_{y_2}\}, \quad \mathcal{P}_2 = \{\bullet_{x_1}, \bullet_{x_2}, \bullet_{y_2}\} \cup \{\bullet_{y_1}\},$$
$$\mathcal{P}_3 = \{\bullet_{x_1}, \bullet_{x_2}\} \cup \{\bullet_{y_1}, \bullet_{y_2}\}, \quad \mathcal{P}_4 = \{\bullet_{x_1}, \bullet_{x_2}\} \cup \{\bullet_{y_1}\} \cup \{\bullet_{y_2}\}\ .$$

By Lemma 2.3.2 the integral is finite if $\deg_\infty(\mathcal{P}_i) < 0$ for each partition, which is trivial to check. The traversing edges for $\mathcal{P}_1$ are: $[y_1, y_2]$ and $[x_2, y_2]$, hence in this case,

$$\deg_\infty(\mathcal{P}_1) = \lambda - d - \kappa + d < 0\ ,$$

and similarly for $\mathcal{P}_2$. Whilst, $\deg_\infty \mathcal{P}_3 = 2(\lambda - d) + d = 2\lambda - d$, and $\deg_\infty \mathcal{P}_4 = 2(\lambda - d) - \kappa + 2d = 2\lambda - \kappa < 0$, concluding the proof of (2.3.9).

2.4 Elementary Gaussian and covariance estimates

In this subsection we gather the main estimates involving the d-dimensional heat kernel

$$P_t(x) = (2\pi t)^{-d/2} e^{-\frac{|x|^2}{2t}} ,$$

and covariance R used below. Firstly, we have the following basic estimates.

Lemma 2.4.1. *For any $T > 0$, $n \in \mathbb{N}$ such that $\frac{nd}{2} > 1$ and $x_i \in \mathbb{R}^d \setminus \{0\}$ for $i = 1, \dots, n$,*

$$\int_0^\infty \prod_{i=1}^n P_s(x_i)\, ds \leqslant c_{d,n} \left(\sum_{i=1}^n |x_i|^2 \right)^{1-\frac{nd}{2}} . \tag{2.4.1}$$

In particular, when $d \geqslant 3$,

$$\int_0^\infty P_s(x)\, ds = \frac{\Gamma(\frac{d}{2}-1)}{2(\pi)^{\frac{d}{2}}} |x|^{2-d}. \tag{2.4.2}$$

Proof. Letting $a = \frac{1}{2}\sum_{i=1}^n |x_i|^2$, through a change of variable $\tau = \frac{a}{s}$ we have

$$\begin{aligned} \int_0^\infty \prod_{i=1}^n P_s(x_i) ds &= (2\pi)^{-\frac{nd}{2}} \int_0^\infty s^{-\frac{nd}{2}} e^{-a/s} ds \\ &= (2\pi)^{-\frac{nd}{2}} a^{1-\frac{nd}{2}} \int_0^\infty \tau^{\frac{nd}{2}-2} e^{-\tau} d\tau = (2\pi)^{-\frac{nd}{2}} \Gamma(\tfrac{nd}{2}-1) a^{1-\frac{nd}{2}}, \end{aligned}$$

completing the proof. □

Remark 2.4.2. Noting $2a \geqslant |x_i|^2$ for any i, we have $a^{1-\frac{nd}{2}} \lesssim \Pi_{i=1}^n |x_i|^{\frac{2}{n}-d}$ and

$$\int_0^\infty \prod_{i=1}^n P_s(x_i) ds \leqslant c_{d,n}\, \Pi_{i=1}^n |x_i|^{\frac{2}{n}-d}. \tag{2.4.3}$$

The above can be used to average over time the product of multiple heat kernels. However, when time indices are not matching, the following pointwise estimates are useful. Moreover, we shall make use the Hölder regularity in time (2.4.5) of the heat kernel.

Lemma 2.4.3. *Given any $\lambda \in (0, d)$, the following bounds hold uniformly in $s > 0$, and $x \in \mathbb{R}^d \setminus \{0\}$:*

$$P_s(x) \lesssim s^{-\frac{\lambda}{2}} |x|^{\lambda - d} , \qquad |\partial_s P_s(x)| \lesssim s^{-\frac{\lambda}{2} - 1} |x|^{\lambda - d} . \tag{2.4.4}$$

Furthermore, for every $\gamma \in [0, 1]$ and $t > s$, one has

$$|P_t(x) - P_s(x)| \lesssim (t - s)^{\gamma} s^{-\gamma - \frac{\lambda}{2}} |x|^{\lambda - d} . \tag{2.4.5}$$

Proof. It suffices to recall that by scale invariance of the heat kernel, we have $|\partial_t^k P_t(x)| \lesssim (\sqrt{|t|} + |x|)^{-d-2k}$. Applying Young's product inequality, with $\alpha = \lambda/d$, we note

$$P_s(x) \lesssim (\sqrt{s} + |x|)^{-d} \lesssim (s^{\alpha/2} |x|^{1-\alpha})^{-d} ,$$

so that (2.4.4) follows. Similarly we obtain the bound for $\partial_s P_s$. For the last bound, it follows from (2.4.4) that

$$\begin{aligned} |P_t(x) - P_s(x)| &\leqslant \int_s^t |\partial_r P_r(x)| \, dr \lesssim |x|^{\lambda - d} \int_s^t r^{-1 - \frac{\lambda}{2}} \, dr \\ &\lesssim |x|^{\lambda - d} s^{-\frac{\lambda}{2} - 1} \big(s \wedge |t - s| \big) , \end{aligned}$$

and (2.4.5) follows, concluding the proof. □

Lemma 2.4.4. *If $\kappa \in (0, d)$, then for any $r > 0$, any $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \frac{P_r(z - x)}{|z|^{\kappa}} dz \lesssim (|x| + \sqrt{r})^{-\kappa}, \quad \int_{\mathbb{R}^d} \frac{P_r(z - x)}{1 + |z|^{\kappa}} dz \lesssim (1 + |x| + \sqrt{r})^{-\kappa}.$$

Proof. It is sufficient to show the first inequality, the second follows from the fact that $\int_{\mathbb{R}^d} P_r(z - x)(1 + |z|^{\kappa})^{-1} dz \leqslant 1$ and a bound of the order $\min(1, (|x| + \sqrt{r})^{-\kappa})$ is equivalent to a bound $C(1 + |x| + \sqrt{r})^{-\kappa}$.

By a change of variable,

$$\begin{aligned} \Lambda(r, x) &= \int_{\mathbb{R}^d} \frac{P_r(z - x)}{|z|^{\kappa}} dz = r^{-\frac{\kappa}{2}} \int \frac{P_1(z - \frac{x}{\sqrt{r}})}{|z|^{\kappa}} dz \\ &= r^{-\frac{\kappa}{2}} (P_1 \star |\cdot|^{-\kappa})(x/\sqrt{r}). \end{aligned}$$

We note that one has $P_1 \in \mathcal{B}_{0,d}$ and $z \mapsto |z|^{-\kappa} \in \mathcal{B}_{\kappa,\kappa}$, where $\mathcal{B}_{\alpha,\beta}$ denote the weighted space as defined in (2.3.7). Then, by (2.3.8), we see that $P_1 \star |\cdot|^{-\kappa}$ belongs to the space $\mathcal{B}_{0,\kappa}$, implying that:

$$\Lambda(r,x) \lesssim |x|^{-\kappa}\mathbf{1}_{\{|x|>\sqrt{r}\}} + r^{-\frac{\kappa}{2}}\mathbf{1}_{\{|x|\leqslant\sqrt{r}\}}.$$

Hence, the required bound follows, concluding the proof. □

We now gather some often used estimate about R in the following lemma. Given $\kappa \in (2,d)$, the kernel $|\cdot|^{-\kappa}$ is integrable near 0 and decays sufficiently fast so that the following estimates hold.

Lemma 2.4.5. *Let $R : \mathbb{R}^d \to \mathbb{R}^d$ be a Borel measurable function such that $|R(x)| \lesssim \frac{1}{1+|x|^\kappa}$ for a number $\kappa \in (2,d)$. Then, there exists $C_R > 0$ such that the following hold for any $t > 0$ and $x \in \mathbb{R}^d$:*

$$\int_{\mathbb{R}^d} P_t(z-x)R(z)\,dz \leqslant C_R(1+|x|+\sqrt{t})^{-\kappa}, \tag{2.4.6}$$

$$\int_0^\infty \int_{\mathbb{R}^d} P_r(z-x)R(z)\,dr\,dz \leqslant C_R(1\wedge|x|^{2-\kappa}), \tag{2.4.7}$$

$$\int_0^t \int_{\mathbb{R}^{2d}} P_r(z_1-x)P_{t-r}(z_2)R(z_1-z_2)\,dr\,dz_1\,dz_2 \leqslant C_R\,t(1+|x|+\sqrt{t})^{-\kappa}. \tag{2.4.8}$$

Proof. Given the decay assumption on R, the first inequality follows immediately from Lemma 2.4.4.

For (2.4.7), we integrate $P_t(z-x)$ over time as in Lemma 2.4.1, obtaining

$$\int_0^\infty \int_{\mathbb{R}^d} P_t(z-x)\big|R(z)\big|dz = c_d \int_{\mathbb{R}^d} \frac{|R(z)|}{|z-x|^{d-2}}dz,$$

Then, the second inequality follows from

$$\int_{\mathbb{R}^d} \frac{|R(z)|}{|z-x|^{d-2}}dz \lesssim 1\wedge|x|^{2-\kappa}. \tag{2.4.9}$$

This can be seen by noting as above that $R \in \mathcal{B}_{0,\kappa}$ and $z \mapsto |z|^{2-d} \in \mathcal{B}_{d-2,d-2}$ (as defined in (2.3.7)), hence (2.4.9) holds given that $\kappa \in (2,d)$.

It remains to show (2.4.8). By a change of variables and integrating the convolution of the heat kernels, the left hand side equals

$$\int_0^t \int_{\mathbb{R}^d} P_t(y-x)R(y)\,dr\,dy \lesssim t(1+|x|+\sqrt{t})^{-\kappa}\ ,$$

where we used (2.4.6). This concludes the proof.

□

In the case of R bounded and L^1-integrable (e.g. $\kappa > d$ or compactly supported R) the estimates (2.4.6)-(2.4.8) hold with κ replaced by d.

2.5 Moments and correlation estimates

In this section we are going to prove a priori moment estimates on u and its Malliavin derivative. We shall use the setup and notation concerning Malliavin Calculus introduced in Section 2.2. Recall the underlying Hilbert space $\mathcal{H}$, with inner product (2.2.3):

$$\langle f,g\rangle_{\mathcal{H}} = \int_{-\infty}^{\infty}\int_{\mathbb{R}^{2d}} f(s,y_1)g(s,y_2)R(y_1-y_2)dy_1dy_2ds\ .$$

By estimate (2.4.7) in Lemma 2.4.5, we note that the heat kernel belongs to $\mathcal{H}$, provided $|R(x)| \lesssim (1+|x|^\kappa)^{-1}$ with $\kappa \in (2,d)$.

We start by proving below a simple criterion to establish pointwise estimates with respect to the heat kernel.

2.5.1 Basic contraction lemma

The pointwise bounds on the Malliavin derivative crucially rely on the following estimate, which is inspired by [CH19, Lemma 2.2] but drops the requirement for R to be positive. We note that the only requirement (2.5.1) on R will follow by (2.4.6). We write $\mathcal{B}_+(\mathbb{R}_+\times\mathbb{R}^d)$ for the set of measurable functions from $\mathbb{R}_+\times\mathbb{R}^d$ to $\mathbb{R}_+$ and a map $J_t(s,x) : \mathcal{B}_+(\mathbb{R}_+\times\mathbb{R}^d)\times\mathcal{B}_+(\mathbb{R}_+\times\mathbb{R}^d)\to\mathbb{R}$ defined as follows:

$$J_t(s,x)(g_1,g_2) = \int_{\mathbb{R}^{2d}} P_{t-s}(x-y_1)P_{t-s}(x-y_2)|R(y_1-y_2)|g_1(s,y_1)g_2(s,y_2)dy_1dy_2.$$

Lemma 2.5.1. *Let $\lambda > 0$ and $g_n : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ be nonnegative measurable functions. Assume the function $R : \mathbb{R}^d \to \mathbb{R}$ satisfies the following, for some $\alpha > 1$,*

$$\int_{\mathbb{R}^{2d}} P_r(z_1 - x_1)P_r(z_2 - x_2)\big|R(z_1 - z_2)\big|dz_1dz_2 \leqslant C_R(1 + r)^{-\alpha}. \quad (2.5.1)$$

If $g_0(t, x) \leqslant P_t(x)$ and g_n satisfy for all $n \geqslant 0$, $t > 0$ and $x \in \mathbb{R}^d$,

$$g_{n+1}^2(t, x) - (P_t(x))^2 \leqslant \lambda \int_0^t J_t(s, x)(g_n, g_n)ds < \infty, \quad (2.5.2)$$

and if furthermore $\lambda \in (0, \frac{1}{c_\alpha})$, where $c_\alpha := 4C_R \int_0^\infty (1 + s)^{-\alpha}ds$, then

$$\sup_{n\geqslant 0} g_n(t, x) \leqslant \frac{P_t(x)}{\sqrt{1 - \lambda c_\alpha}}. \quad (2.5.3)$$

Proof. Let $f_0(t, x) = P_t(x)$ and for $n \geqslant 0$, let f_n be the positive solution to

$$f_n^2(t, x) = P_t(x)^2 + \lambda \int_0^t J_t(s, x)(f_{n-1}, f_{n-1})ds. \quad (2.5.4)$$

We note that $g_n \leqslant f_n$ for all $n \geqslant 0$. Indeed $g_0 \leqslant f_0$ and, for all $n \geq 0$, $g_n \leqslant f_n$ implies that

$$g_{n+1}^2(t, x) \leqslant (P_t(x))^2 + \lambda \int_0^t J_t(s, x)(f_n, f_n)ds = f_{n+1}^2(t, x) \ ,$$

so that $g_n \leqslant f_n$ since both functions are positive by assumption / definition.

Letting $H_n = \sum_{j=0}^n (\lambda c_\alpha)^j$, we show by induction that $f_n(t, x) \leqslant P_t(x)\sqrt{H_n}$. The case $n = 0$ is trivial. Using the relation

$$P_{t-s}(x - y)P_s(y) = P_t(x)P_{\frac{s(t-s)}{t}}(y - \tfrac{s}{t}x)$$

and the induction hypothesis on f_n we have

$$\begin{aligned}
&\int_0^t J_t(s, x)(f_n, f_n)ds \\
&\leqslant H_n \int_0^t \int_{\mathbb{R}^{2d}} P_{t-s}(x - y_1)P_{t-s}(x - y_2)|R(y_1 - y_2)|P_s(y_1)P_s(y_2)dyds \\
&\leqslant H_n P_t(x)^2 \int_0^t \int_{\mathbb{R}^{2d}} P_{\frac{s(t-s)}{t}}(y_1 - \tfrac{s}{t}x)P_{\frac{s(t-s)}{t}}(y_2 - \tfrac{s}{t}x)|R(y_1 - y_2)|dyds
\end{aligned}$$

$$\leqslant C_R H_n P_t(x)^2 \int_0^t (1 + \tfrac{s(t-s)}{t})^{-\alpha} ds \leqslant 2C_R H_n P_t(x)^2 \int_0^{t/2} (1 + \tfrac{s}{2})^{-\alpha} ds$$
$$\leqslant 4C_R H_n P_t(x)^2 \int_0^{\infty} (1+s)^{-\alpha} ds \leqslant c_\alpha H_n P_t(x)^2 \,,$$

where we used assumption (2.5.1) to obtain the penultimate line. It then follows from (2.5.4) that

$$f_{n+1}^2(t,x) = P_t(x)^2 + \lambda \int_0^t J_t(s,x)(f_n, f_n) ds$$
$$\leqslant P_t(x)^2 (1 + \lambda c_\alpha H_n) = P_t(x)^2 H_{n+1},$$

concluding the induction step. It follows that $g_n(t,x) \leqslant P_t(x)\sqrt{H_\infty} = P_t(x)/\sqrt{1-\lambda c_\alpha}$, concluding the proof. □

2.5.2 Moments bounds

The key idea is now to combine the commutation relation (2.2.7)[(3)] with the mild formulation of the solution u, which allows for a useful characterisation of its Malliavin derivative [GL20, p. 5] and to derive the estimates below through the criterion above, see (2.5.7). Given the Lipschitz assumption on σ, by [Nua06, Prop. 1.2.4] the chain rule still holds and we have

$$D_{r,z}\sigma(u(s,y)) = \Sigma(s,y) D_{r,z} u(s,y) \,, \tag{2.5.5}$$

where Σ is an adapted process bounded by $|\sigma|_{\mathrm{Lip}}$ (if σ is continuously differentiable then $\Sigma(s,y) = \sigma'(u(s,y))$ as given by Proposition 2.2.1).

Set $\|\sigma\|^2 = |\sigma(0)|^2 + |\sigma|_{\mathrm{Lip}}^2$, to keep track of constants below. Our moment estimates hold if $\beta < \frac{c}{|\sigma|_{\mathrm{Lip}}}$, where c is a constant depending on p, κ and R.

Lemma 2.5.2. *Let $d \geqslant 3$, σ be Lipschitz continuous, and suppose that ξ satisfies Assumption 2.0.1. Then, for any $p \geq 2$, there exists positive constants $\beta_0(p)$ and C_p, such that if $\beta \leqslant \beta_0(p)$, the solution of (2.0.1) has uniform pth*

[(3)] The identity (2.2.7) will also be useful in the next chapter to express the Malliavin derivative of the fluctuations $u_\varepsilon - \mathbb{E}[u_\varepsilon]$.

moments:

$$\sup_{s\geqslant 0, x\in\mathbb{R}^d} \|u(s,x)\|_p = \sup_{s\geqslant 0} \|u(s,0)\|_p \leqslant C_p. \tag{2.5.6}$$

Furthermore, $u(t,x)$ *is Malliavin differentiable with* $\sup_{t\geqslant 0, x\in\mathbb{R}^d} \|u(t,x)\|^2_{\mathbb{D}^{1,p}} < \infty$ *and, for all* $x, z \in \mathbb{R}^d$ *and* $t > r > 0$*,*

$$\|D_{r,z}u(t,x)\|_p \lesssim C_p P_{t-r}(x-z). \tag{2.5.7}$$

Remark 2.5.3. Given the flat initial condition $u(0,\cdot) \equiv 1$ and translation invariance of ξ, the process $u(t,\cdot)$ is stationary in space. The constant C_p is monotone increasing in p. The threshold $\beta_0(p)$ is monotone decreasing in p.

Proof. The key for the proof is Lemma 2.5.1, which allows us to show pointwise moment estimate on the Malliavin derivative $D_{s,y}u(t,x)$, the approach is otherwise quite standard and inspired by [GL20, Lem. 2.1-2.2], [CKNP21, Thm 6.4].

First we recall that by the Burkholder–Davis–Gundy inequality, for any time interval I,

$$\Big\| \int_{I\times\mathbb{R}^d} f(s,y)\xi(ds,dy)\Big\|_p \leqslant c_p \,\mathbb{E}\left[\Big(\int_I \int_{\mathbb{R}^{2d}} f(s,y_1) f(s,y_2) R(y_1-y_2) dy ds \Big)^{\frac{p}{2}} \right]^{\frac{1}{p}}. \tag{2.5.8}$$

We apply it to the stochastic integral $\int_0^r \int_{\mathbb{R}^d} P_{t-s}(x-y)\sigma(u(s,y))\xi(ds,dy)$ in the mild formulation, with $0 \leqslant r \leqslant t$, obtaining the bound below where we denote $dy = dy_1 dy_2$,

$$\|u(t,0)\|_p \leqslant 1 + c_p\beta \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^{2d}} \Big(\prod_{j=1}^2 P_{t-s}(y_j)\sigma(u(s,y_j)) \Big) R(y_1-y_2) dy ds \right)^{\frac{p}{2}} \right]^{\frac{1}{p}}. \tag{2.5.9}$$

Since $u(s,\cdot)$ is stationary, by Hölder's inequality,

$$\|\sigma(u(s,y_1))\sigma(u(s,y_2))\|_{\frac{p}{2}} \leqslant \|\sigma(u(s,0))\|_p^2 \leqslant 2|\sigma(0)|^2 + 2|\sigma|^2_{\mathrm{Lip}} \|u(s,0)\|_p^2.$$

Then, letting $f(t) := \|u(t,0)\|_p$, and applying Minkowski inequality in (2.5.9),

$$f(t) \leqslant 1 + \sqrt{2} c_p \beta \Big(\int_0^t \int_{\mathbb{R}^{2d}} P_{t-s}(y_1) P_{t-s}(y_2) \times$$

$$\Big(|\sigma(0)|^2 + |\sigma|^2_{\mathrm{Lip}} f^2(s)\Big)\Big| \, R(y_1 - y_2)\Big| dyds\Bigg)^{\frac{1}{2}}.$$

By (2.4.6), it follows that there exist constants[4] $C_1, C_2 > 0$ (independent of t) such that

$$f(t) \leqslant C_2 + \beta C_1 \left(\int_0^t (1 + t - s)^{-\frac{\kappa}{2}} f^2(s) ds \right)^{\frac{1}{2}},$$

Since $\kappa > 2$, the integral $c_\kappa = \int_0^\infty (1 + r)^{-\frac{\kappa}{2}} dr$ is finite. It follows that $\sup_t f(t) \leqslant C_2 + \sqrt{c_\kappa} C_1 \beta \sup_t f(t)$. For any $\beta \leqslant (2\sqrt{c_\kappa} C_1)^{-1}$, we have that $\sup_t f(t) \leqslant 2C_2$. The bound (2.5.6) follows from the fact that the process $u(t, \cdot)$ is stationary in space.

For (2.5.7), let us consider $u_0(t, x) = 1$, and for $n \geqslant 0$ define the Picard iteration

$$u_{n+1}(t, x) = 1 + \beta \int_0^t \int_{\mathbb{R}^d} P_{t-s}(x - y) \sigma(u_n(s, y)) \xi(ds, dy).$$

A similar argument as for $f(t)$ above shows that, for any $x \in \mathbb{R}^d$,

$$\begin{aligned} &\|u_{n+1}(t, x) - u_n(t, x)\|_p \\ &\quad \leqslant \beta c_p |\sigma|_{\mathrm{Lip}} \Bigg(\int_0^t \int_{\mathbb{R}^{2d}} P_{t-s}(y_1) P_{t-s}(y_2) \\ &\qquad\qquad \|u_n(s, 0) - u_{n-1}(s, 0)\|_p^2 \big| R(y_1 - y_2) \big| dyds \Bigg)^{\frac{1}{2}} \\ &\quad \leqslant \tfrac{1}{2} \sup_s \sup_x \|u_n(s, x) - u_{n-1}(s, x)\|_p, \end{aligned}$$

for sufficiently small β as above. Thus $\{u_n(t, x)\}$ is a Cauchy sequence and converges to the unique solution u.

Moreover, it follows from (2.2.7) and (2.5.5) that

$$\begin{aligned} D_{r,z} u_{n+1}(t, x) &= P_{t-r}(x - z) \sigma(u_n(r, z)) \\ &\quad + \beta \int_r^t \int_{\mathbb{R}^d} P_{t-s}(x - y) \Sigma_n D_{r,z} u_n(s, y) \xi(ds, dy), \end{aligned}$$

[4] The constants are $C_1 = c_p \sqrt{2C_R} |\sigma|_{\mathrm{Lip}}$ and $C_2 = 1 + \beta c_p \sqrt{2C_R} |\sigma(0)|$, where C_R is from (2.4.6).

where $\Sigma_n(t,x)$ is a stochastic process bounded by $|\sigma|_{\mathrm{Lip}}$, which, intuitively speaking, represents a version of the derivative $\sigma'(u_n(s,y))$. Observe that $\sup_t \sup_x \|\sigma(u_n(t,x))\|_p$ is uniformly bounded in n, c.f. (2.5.10). We apply again BDG to the integral term, followed by Minkowski and Hölder inequalities, obtaining[5]

$$\|D_{r,z}u_{n+1}(t,x)\|_p^2 \leqslant C_2' P_{t-r}^2(x-z) + \beta^2 C_1' \int_r^t \int_{\mathbb{R}^{2d}} |R(y_1-y_2)| \Big(\prod_{i=1}^2 P_{t-s}(x-y_i) \|D_{r,z}u_n(s,y_i)\|_p \Big) dy ds.$$

The covariance bound (2.4.6) implies (2.5.1) with $\alpha = \kappa/2 > 1$, so that Lemma 2.5.1 applies[6], in which we let $t = \theta + r$, $x = \eta + z$ and $g_n(\theta,\eta) = \|D_{r,z}u_n(\theta+r,\eta+z)\|_p$, to obtain

$$\|D_{r,z}u(t,x)\|_p \leqslant \sup_{n\geqslant 0} \|D_{r,z}u_n(t,x)\|_p \lesssim C_p P_{t-r}(x-z)\ .$$

To show that $u(t,x) \in \mathbb{D}^{1,p}$, it now suffices to bound $\|Du(t,x)\|_{L^p(\Omega,\mathcal{H})}$. Using Minkowski and Hölder's inequality, followed by (2.5.7), we have

$$\begin{aligned}
\|Du(t,x)\|_{L^p(\Omega,\mathcal{H})} &= \mathbb{E}\Big[\Big(\int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} D_{r,z_1}u(t,x) D_{r,z_2}u(t,x) R(z_1-z_2)\, dr\, dz \Big)^{\frac{p}{2}} \Big]^{\frac{1}{p}} \\
&\leqslant \Big(\int_s^t \int_{\mathbb{R}^{2d}} \|D_{r,z_1}u(t,x) D_{r,z_2}u(t,x)\|_{\frac{p}{2}}\, R(z_1-z_2)\, dr\, dz \Big)^{\frac{1}{2}} \\
&\leqslant C_p \Big(\int_s^t \int_{\mathbb{R}^{2d}} \prod_{i=1}^2 P_{t-r}(x-z_i)\, R(z_1-z_2)\, dr\, dz \Big)^{\frac{1}{2}} \lesssim C_p\ ,
\end{aligned}$$

where the last inequality follows by (2.4.7). Hence we obtain $\|u(t,x)\|_{\mathbb{D}^{1,p}} < \infty$, uniformly over t and x, thus concluding the proof. □

2.5.3 Correlation estimate and the Law of Large numbers

We now proceed to obtain de-correlation estimates. Using Clark–Ocone formula and (2.4.7), we show solutions decorrelate like $|x|^{2-\kappa}$ for large x.

[5] The constants are given by $C_1' = 2c_p^2|\sigma|_{\mathrm{Lip}}^2$ and $C_2' = 4\|\sigma\|^2 C_p^2$.

[6] As long as $4\beta^2 C_1' C_R c_\kappa < 1$, therefore can take $\beta_0(p)$ such that $\beta_0^2 c_p^2 c_\kappa C_R |\sigma|_{\mathrm{Lip}}^2 < \frac{1}{8}$.

Lemma 2.5.4. *Let $d \geqslant 3$, σ be Lipschitz continuous, and suppose that ξ satisfies Assumption 2.0.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $p \geqslant 2$. For $\beta \leqslant \beta_0(p)$,*

$$\sup_{s\geqslant 0, x\in\mathbb{R}^d} \|f(u(s,x))\|_p \lesssim 1. \tag{2.5.10}$$

For any $\beta \leqslant \beta_0(2)$, the following holds uniformly in $x \in \mathbb{R}^d$,

$$\sup_{t,s>0} \operatorname{cov}\left(f\left(u(t,x)\right), f\left(u(s,0)\right)\right) \leqslant C|f|^2_{\mathrm{Lip}} \left(1 \wedge |x|^{2-\kappa}\right). \tag{2.5.11}$$

Proof. Recall that $u(t,x)$ has uniform L^p bounds by Lemma 2.5.2. Noting that $\|f(u(s,x))\|_p \leqslant |f(0)| + |f|_{\mathrm{Lip}}\|u(s,x)\|_p$, the first inequality follows. For the second bound, we note $D_{r,z}f(u(t,x)) = \Sigma(t,x)\, D_{r,z}u(t,x)$, for some $\Sigma(t,x)$ bounded by $|f|_{\mathrm{Lip}}$ (cf. (2.5.5)). Recall the Clark–Ocone formula (see Proposition 2.2.4): for any $X \in \mathbb{D}^{1,2}$,

$$X - \mathbb{E}[X] = \int_{\mathbb{R}_+\times\mathbb{R}^d} \mathbb{E}[D_{r,z}X|\mathcal{F}_r]\xi(dr,dz)\,. \tag{2.5.12}$$

Applying this to $f(u(t,x))$, we obtain

$$f(u(t,x)) - \mathbb{E}f(u(t,x)) = \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}[\Sigma(t,x)D_{r,z}u(t,x)|\mathcal{F}_r]\xi(dr,dz).$$

By Itô's isometry, followed by the Cauchy–Schwarz inequality and (conditional) Jensen's inequality, we have

$$\begin{aligned}
&\operatorname{cov}\left(f(u(t,x)), f(u(s,0))\right)\\
&\quad= \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big(\mathbb{E}[\Sigma(t,x)D_{r,z_1}u(t,x)|\mathcal{F}_r]\\
&\qquad\qquad \mathbb{E}[\Sigma(s,0)D_{r,z_2}u(s,0)|\mathcal{F}_r]\Big)R(z_1-z_2)dz_1dz_2dr\\
&\quad\leqslant |f|^2_{\mathrm{Lip}} \int_0^{t\wedge s} \int_{\mathbb{R}^d} \|D_{r,z_2}u(t,x)\|_2\|D_{r,z_2}u(s,0)\|_2\big|R(z_1-z_2)\big|dz_1dz_2dr.
\end{aligned}$$

Then (2.5.11) follows by using the $L^2(\Omega)$ estimate (2.5.7) on Du and (2.4.7),

$$\begin{aligned}
&\operatorname{cov}\big(f(u(t,x)), f(u(s,0))\big)\\
&\quad\lesssim |f|^2_{\mathrm{Lip}} \int_0^{t\wedge s} \int_{\mathbb{R}^{2d}} P_{t-r}(x-z_1)P_{s-r}(z_2)\big|R(z_1-z_2)\big|dz_1dz_2dr\\
&\quad\lesssim |f|^2_{\mathrm{Lip}}(1\wedge|x|^{2-\kappa}),
\end{aligned}$$

concluding the proof of the second statement. □

In Chapter 4, when looking at the KPZ scaling limit, we shall prove analougous estimates for more general transformations of the solution of the *linear SHE*, lifting the Lipschitz assumption and leveraging a priori bounds also on the *negative moments*. Much attention will be devoted also to the fundamental solutions of SHE and their large time homogenisation.

With the moments and decorrelation estimates above in place, the law of large number immediately follows with rate $O(\varepsilon^{\frac{\kappa}{2}-1})$ as ε converges to 0. This can also be deduced from the fluctuations result of the next chapter.

Corollary 2.5.5. *Let $d \geqslant 3$, σ be Lipschitz continuous, and suppose that ξ satisfies Assumption 2.0.1. Then, for any $\beta \leqslant \beta_0(2)$ and any test function $g \in \mathcal{C}_c(\mathbb{R}^d)$, the following holds:*

$$\left\| \int_{\mathbb{R}^d} u_\varepsilon(t,x) g(x)\, dx - \int_{\mathbb{R}^d} g(x)\, dx \right\|_2 \lesssim \varepsilon^{\frac{\kappa}{2}-1} .$$

Proof. We make use of the fact that the mean is constant $\mathbb{E}[u(t,x)] = 1$. Consider $\tilde{u}_\varepsilon = u_\varepsilon - 1$, then by computing the second moments we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} \tilde{u}_\varepsilon(t,x) g(x)\, dx\Big)^2\Big] \\
&\qquad = \mathbb{E}\Big[\prod_{i=1}^2 \int_{\mathbb{R}^d} u_\varepsilon(t,x_i) g(x_i)\, dx_i\Big] - \prod_{i=1}^2 \int_{\mathbb{R}^d} g(x_i)\, dx_i \\
&\qquad = \int_{\mathbb{R}^{2d}} \operatorname{cov}\big(u_\varepsilon(t,x_1), u_\varepsilon(t,x_2)\big) g(x_1) g(x_2)\, dx_1\, dx_2 \\
&\qquad \lesssim \varepsilon^{\kappa-2} \int_{\mathbb{R}^{2d}} |x_1 - x_2|^{2-\kappa} g(x_1) g(x_2)\, dx_1\, dx_2 \lesssim \varepsilon^{\kappa-2} ,
\end{aligned}$$

where we used (2.5.11) in the last line. The LLN then follows. □

Remark 2.5.6. (i) For the *linear* stochastic heat equation, the Feynman–Kac formula yields

$$\operatorname{cov}(u_\varepsilon(t,x), u_\varepsilon(t,0)) = \mathbb{E}_B \exp\Big(\frac{\beta^2}{2} \int_0^{\frac{2t}{\varepsilon^2}} R(\tfrac{x}{\varepsilon} + B_r) dr\Big) - 1.$$

Here $\mathbb{E}_B$ denotes expectation with respect to an independent Brownian motion B. Using Portenko's lemma [Por75], then their covariance

converges to 0 as long as the following holds

$$\mathbb{E}_B\left[\int_0^{\frac{2t}{\varepsilon^2}} R(\tfrac{x}{\varepsilon}+B_r)dr\right] \leqslant \mathbb{E}_B\left[\int_0^{\infty} \big|R(\tfrac{x}{\varepsilon}+B_r)\big|dr\right] = \int_{\mathbb{R}^d} \frac{|R(z)|}{|\frac{x}{\varepsilon}-z|^{d-2}}\,dz \to 0.$$

With this observation, and a moments computation like above, [MSZ16] derived the LLN for the compactly supported case (cf. [MSZ16, Lem. 3.1]).

(ii) The asymptotic stationary field which is subject of the next section also decorrelates spatially like u, namely $\operatorname{cov}(Z(x), Z(0)) \approx |x|^{2-\kappa}$.

2.6 Pull back solutions and asymptotics

The next result is inspired by [GL20, Thm. 1.1] and shows that the solution with constant initial condition converges in law towards some stationary field Z for large times, with an explicit convergence rate. In fact, we show the stronger convergence in probability in the pullback sense. For this, let us write $u^{(K)}(t,x)$ for the solution of the nonlinear stochastic heat equation with initial condition 1 at time $-K$, namely

$$u^{(K)}(t,x) = 1 + \beta \int_{-K}^{t}\int_{\mathbb{R}^d} P_{t-s}(x-y)\sigma(u^{(K)}(s,y))\xi(ds,dy). \tag{2.6.1}$$

These pull back solutions and stationary field(s) will play a crucial role also in the study of the KPZ equation in Chapter 4. With the notation above at hand, the claimed convergence is as follows.

Theorem 2.6.1. *Let $d \geqslant 3$, σ be Lipschitz continuous, and suppose that ξ satisfies Assumption 2.0.1. Then, for any $p > 1$, there exists a positive constant $\beta_1(p)$, such that for $\beta \leqslant \beta_1$, uniformly in $t > -(K \wedge K')$:*

$$\sup_{x\in\mathbb{R}^d} \|u^{(K')}(t,x) - u^{(K)}(t,x)\|_p \lesssim \beta(1+t+K\wedge K')^{\frac{2-\kappa}{4}}. \tag{2.6.2}$$

Furthermore, there exists a space-time stationary field $\vec{Z}$ such that, for any compact region $\mathfrak{K} \subset \mathbb{R}^d$ and every $t \in \mathbb{R}$, one has $\lim_{K\to\infty} \mathbb{E}\sup_{x\in\mathfrak{K}} |u^{(K)}(t,x) - \vec{Z}(t,x)|^p \to 0$.

Proof. Without loss of generality, in the proof we assume that $0 \leqslant K < K'$. Let us consider the following seminorm:

$$M_p := \sup_{x\in\mathbb{R}^d} \sup_{t>-K} (t+K+1)^{\frac{\kappa-2}{4}} \|u^{(K')}(t,x) - u^{(K)}(t,x)\|_p \,. \tag{2.6.3}$$

We will first show this is finite, implying (2.6.2). For $t > -K$, using the mild formulation we can rewrite the difference in two terms:

$$\begin{aligned} u^{(K')}(t,x) - u^{(K)}(t,x) \\ &= \beta \int_{-K}^{t} \int_{\mathbb{R}^d} P_{t-s}(x-y)[\sigma(u^{(K')}(s,y)) - \sigma(u^{(K)}(s,y))]\xi(ds,dy) \\ &\quad + \beta \int_{-K'}^{-K} \int_{\mathbb{R}^d} P_{t-s}(x-y)\sigma(u^{(K')}(s,y))\xi(ds,dy) \,. \end{aligned}$$

Thus, by the Burkholder–Davis–Gundy inequality (2.5.8) and the triangle inequality, for a universal constant c_p we have

$$\|u^{(K')}(t,x) - u^{(K)}(t,x)\|_p \leqslant c_p\beta\big(A_1(t,x) + A_2(t,x)\big), \tag{2.6.4}$$

where we denote with A_i the following terms:

$$\begin{aligned} A_1(t,x) = \Big\| \int_{-K}^{t} \int_{\mathbb{R}^{2d}} \Big(\prod_{j=1}^{2} P_{t-s}(x-y_j) \\ &\qquad [\sigma(u^{(K')}(s,y_j)) - \sigma(u^{(K)}(s,y_j))]\Big) R(y_1-y_2)dyds\Big\|_{\frac{p}{2}}^{\frac{1}{2}} , \\ A_2(t,x) = \Big\| \int_{-K'}^{-K} \int_{\mathbb{R}^{2d}} \Big(\prod_{j=1}^{2} P_{t-s}(x-y_j)\sigma(u^{(K')}(s,y_j))\Big) R(y_1-y_2)dyds\Big\|_{\frac{p}{2}}^{\frac{1}{2}} . \end{aligned}$$

By the Minkowski and Hölder inequalities, using (2.4.6) in the last step, we have

$$\begin{aligned} A_1(t,x) \leqslant \Big(\int_{-K}^{t} \int_{\mathbb{R}^{2d}} \Big(\prod_{j=1}^{2} P_{t-s}(y_j-x) \\ &\qquad \|\sigma(u^{(K')}(s,y_j)) - \sigma(u^{(K)}(s,y_j))\|_p\Big) R(y_1-y_2)dyds\Big)^{\frac{1}{2}} \\ &\leqslant |\sigma|_{\mathrm{Lip}} \sup_{s>-K} (s+K+1)^{\frac{\kappa-2}{4}} \sup_{y} \|u^{(K')}(s,y) - u^{(K)}(s,y)\|_p \times \end{aligned}$$

$$\Big(\int_{-K}^{t}(s+K+1)^{-\frac{\kappa-2}{2}}\int_{\mathbb{R}^{2d}}P_{t-s}(y_1)P_{t-s}(y_2)R(y_1-y_2)dyds\Big)^{\frac{1}{2}}$$
$$\leqslant |\sigma|_{\mathrm{Lip}}\,M_p\Big(C_R\int_{-K}^{t}(s+K+1)^{-\frac{\kappa-2}{2}}(1+t-s)^{-\frac{\kappa}{2}}ds\Big)^{\frac{1}{2}}.$$

We split the integration region $[-K,t]$ of the final integral into two equal parts. On the interval $[-K,\frac{t-K}{2}]$, we have $t-s\geq\frac{t+K}{2}$, so that

$$\int_{-K}^{\frac{t-K}{2}}(s+K+1)^{-\frac{\kappa-2}{2}}(1+t-s)^{-\frac{\kappa}{2}}ds\leqslant 2^{\frac{\kappa}{2}}-1(1+t+K)^{-\frac{\kappa}{2}}(t+K)$$
$$\leqslant 2^{\frac{\kappa}{2}}-1(t+K+1)^{1-\frac{\kappa}{2}},$$

while on $[\frac{t-K}{2},t]$ we have $s+K\geq\frac{t+K}{2}$, so that

$$\int_{\frac{t-K}{2}}^{t}(s+K+1)^{-\frac{\kappa-2}{2}}(1+t-s)^{-\frac{\kappa}{2}}ds\leqslant 2^{\frac{\kappa}{2}}-1(1+t+K)^{1-\frac{\kappa}{2}}\int_{-\infty}^{t}(1+t-s)^{-\frac{\kappa}{2}}ds\,.$$

Combining both bounds, we obtain[(7)]

$$A_1(t,x)\lesssim\ M_p\,(1+t+K)^{\frac{2-\kappa}{4}}\,.$$

Similarly, given that $\sup_{s,y,K'}\|\sigma(u^{(K')}(s,y))\|_p$ is uniformly bounded for $\beta<\beta_0(p)$ by a constant $c(\beta_0)$ (cf. (2.5.10)), the second term is controlled up to a constant by

$$A_2(t,x)\lesssim c(\beta_0)\Big(\int_{-K'}^{-K}(1+t-s)^{-\frac{\kappa}{2}}ds\Big)^{\frac{1}{2}}\lesssim c(\beta_0)(1+t+K)^{\frac{2-\kappa}{4}}.$$

Returning to (2.6.4) with these estimates, rearranging $(1+t+K)^{\frac{2-\kappa}{4}}$, and taking suprema in x in $t>-K$ of both sides, we obtain that for some constants C_1 and C_2[(8)],

$$M_p\leqslant\beta C_2+\beta C_1M_p.$$

For sufficiently small $\beta\leqslant\frac{1}{2C_1}$, we have $M_p\leqslant\frac{\beta C_2}{1-\beta C_1}\leqslant 2\beta C_2$. When combined with (2.6.3), this implies (2.6.2). This shows in particular that

[(7)]The implicit constant is given by $2^{\frac{\kappa-2}{4}}(1+\sqrt{c_\kappa})|\sigma|_{\mathrm{Lip}}\sqrt{C_R}$, where $c_\kappa=\int_0^\infty[1\wedge r^{-\frac{\kappa}{2}}]dr$ as before.

[(8)]Where this time $C_1=c_p2^{\frac{\kappa-2}{4}}(1+\sqrt{c_\kappa})|\sigma|_{\mathrm{Lip}}\sqrt{C_R}$, thus can fix $\beta_1\leqslant\frac{1}{2C_1}$.

there exists a space-time stationary field $\vec{Z}(t,x)$ such that, for every (t,x), $\vec{Z}(t,x) = \lim_{K\to\infty} u^{(K)}(t,x)$ in $L^p(\Omega)$.

To show that this L^p convergence is locally uniform in x, we make use of [HL22a, Prop. 3.12]: combined with Kolmogorov's criterion and a simple interpolation argument, it suffices to show that there exists some $\delta > 0$ such that, for every $p \geq 1$,

$$\|u^{(K)}(t,x_1) - u^{(K)}(t,x_2)\|_p \lesssim |x_1 - x_2|^\delta ,$$

uniformly over K, t, and $|x_1 - x_2| \leqslant 1$. Since our bounds are uniform, we can restrict ourselves to the case $K = 0$. Letting $G_{x_1,x_2}(s,y) = P_s(x_1 - y) - P_s(x_2 - y)$, as above by BDG we have

$$\begin{aligned}\|u(t,x_1) - u(t,x_2)\|_p &= \Big\| \int_0^t \int_{\mathbb{R}^d} G_{x_1,x_2}(t-s,y)\sigma(u(s,y))\xi(ds,dy)\Big\|_p \\ &\lesssim \Big(\int_0^t \int_{\mathbb{R}^{2d}} \Big(\prod_{i=1}^2 G_{x_1,x_2}(t-s,y_i)\|\sigma(u(s,y_i))\|_p\Big) R(y_1-y_2)dyds\Big)^{\frac{1}{2}}.\end{aligned}$$

By [CH19, Lem. 3.1], for $\delta \in (0,1)$, we have

$$G_{x_1,x_2}(t-s,y) \lesssim (t-s)^{-\frac{\delta}{2}}(P_{2(t-s)}(x_1-y) + P_{2(t-s)}(x_2-y))|x_1-x_2|^\delta.$$

Then, since $\|\sigma(u(s,y))\|_p$ is uniformly bounded, we obtain from (2.4.6) the bound

$$\begin{aligned}&\|u(t,x_1) - u(t,x_2)\|_p \\ &\quad\lesssim |x_1-x_2|^\delta \sum_{i,j=1}^2 \Big(\int_0^t (t-s)^{-\delta} \int_{\mathbb{R}^{2d}} P_{2(t-s)}(x_i - y_1) \\ &\qquad\qquad P_{2(t-s)}(x_j - y_2)R(y_1-y_2)dyds\Big)^{\frac{1}{2}} \\ &\quad\lesssim |x_1-x_2|^\delta \Big(\int_0^t (t-s)^{-\delta}(1 \wedge (t-s)^{-\frac{\kappa}{2}})ds\Big)^{\frac{1}{2}} \lesssim |x_1-x_2|^\delta ,\end{aligned}$$

as desired, concluding the proof. □

Remark 2.6.2. We remark that the restrictions required on β_0 and β_1 are such that $\beta_1 = a_\kappa \beta_0$, for some constant $a_\kappa > 0$. Both are inversely proportional to $|\sigma|_{\mathrm{Lip}}$.

Remark 2.6.3. Given our use and for simplicity of exposition, we considered constant initial conditions $u(0,x) \equiv 1$ in Lemma 2.5.2 and Theorem 2.6.1. However, the requirement may be relaxed and proofs adapted to more general initial conditions. For example, one may consider pertubations of a constant by an integrable function of the form $u(0,x) = \lambda + v(x)$ with $v \in L^1(R^d) \cap L^\infty(\mathbb{R}^d)$ and $\lambda > 0$, cf. [GL20, Rem. 3.6].

Chapter 3

Fluctuations of SHE

In this chapter we present the results on the fluctuations and convergence in Hölder topologies of SHE, based on the 2nd revised version of the article [GHL23], joint work with Martin Hairer and Xue-Mei Li. Note that in the 2nd version we have improved and simplified various parts of the original exposition - in particular Theorem 2.6.1, proofs concerning our tightness criterion and convergence in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$ spaces. We thank anonymous referees for the careful review and helpful suggestions. Further details are added in this chapter, whilst some parts are verbatim to the article.

In this chapter we investigate the large scales fluctuations of (2.0.1), the nonlinear stochastic heat equation (SHE) on $\mathbb{R}^d$ subject to multiplicative random forcing driven by the Gaussian noise ξ. Unlike most of previous work on the subject, our focus is on the case where the noise ξ exhibits heavy-tailed correlations that are in particular non-integrable at infinity, recall Assumption 2.0.1. We will restrict ourselves to the high-dimensional case $d \geqslant 3$. Building on the results presented in the previous chapter, we now study the large-scale fluctuations from the mean, leading to a central limit type theorem.

The SHE, while significant in its own right [Mol91, CM94], has garnered considerable interest due to its connection, via the Cole–Hopf transform [BG97], to the KPZ (Kardar–Parisi–Zhang) equation – a model for surface

growth [KPZ86]. The solution of the linear SHE where $\sigma(u) = u$ describes the partition function of a system of directed polymers in a quenched random potential, thus establishing a connection to statistical mechanics [BC95].

As motivated in the introductory chapter, the question of large-scale fluctuations is of particular interest. Recent studies, [GRZ18, MU18, DGRZ20, CNN22, LZ22, GL20, GK21b, DGRZ21, CCM18], have demonstrated that, provided the correlations of the driving noise have compact support or decay sufficiently fast, both the SHE and KPZ equation exhibit Gaussian fluctuations at large scales in the weak disorder phase, notably when the interaction strength β is small. In this case the fluctuations converge to the Edwards–Wilkinson model, with effective variance dependent on the noise's spatial mollification (cf. Remark 3.1.7).

Differently from the case of integrable correlations, we now demonstrate that when the noise correlation decays at a polynomial rate of order $\kappa \in (2, d)$, we still observe a Gaussian CLT, but spatial correlation in the noise persists in the scaling limit. This translates in fluctuations described by the stochastic heat equation with additive, spatially coloured noise $\dot{W}^{\kappa}$. Our result diverges from the compactly supported noise case in other two key aspects. First, the scaling factor is of order $\varepsilon^{2-\kappa}$, suggesting that the decay order of the correlations plays the role of an effective dimension. Second, the formula for the effective variance of the limit $\mathcal{U}$ is different from that appearing in the integrable case. (This is to be expected since the latter diverges, cf. Remark 3.1.7 and Section 1.3.)

The natural question whether our theorem could inform the fluctuations analysis of the KPZ equation, or be applicable to the critical dimension, reveals substantial challenges. The Cole-Hopf transform, $h = \log(u)$, is a singular mapping, complicating direct application of the results proved here for SHE. These issues are addressed in [GHL24], and will be the subject of Chapter 4.

We now outline the main results and organisation of the chapter.

3.1 Main results

We let u be as in (2.0.1) and, for $\varepsilon > 0$, we write $u_\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon)$, which solves

$$\partial_t u_\varepsilon(t,x) = \tfrac{1}{2}\Delta u_\varepsilon(t,x) + \beta\varepsilon^{\frac{\kappa}{2}-1}\sigma(u_\varepsilon(t,x))\xi^\varepsilon(t,x), \qquad u_\varepsilon(0,x) \equiv 1\ , \tag{3.1.1}$$

where $\xi^\varepsilon(t,x) = \varepsilon^{-\frac{\kappa}{2}-1}\xi(t/\varepsilon^2, x/\varepsilon)$ is a Gaussian noise that is white in time with spatial covariance $\varepsilon^{-\kappa}R(\frac{x}{\varepsilon})$ and the product is again interpreted as an Itô integral. Given a test function g, we furthermore set

$$X_t^{\varepsilon,g} = \varepsilon^{1-\frac{\kappa}{2}}\int_{\mathbb{R}^d}\big(u_\varepsilon(t,x) - \mathbb{E}u_\varepsilon(t,x)\big)g(x)dx. \tag{3.1.2}$$

Note that $\mathbb{E}u_\varepsilon(t,x) = 1$ by our assumption on the initial condition. Throughout the chapter, we make the following assumptions on the noise and the nonlinearity.

Assumption 3.1.1. The noise ξ in (2.0.1) is a mean zero space time Gaussian noise with $\mathbb{E}[\,\xi_t(x)\xi_s(y)\,] = \delta(t-s)R(x-y)$ satsifying Assumption 2.0.1. The nonlinearity $\sigma : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, such that

$$|\sigma(a) - \sigma(b)| \leqslant |\sigma|_{\mathrm{Lip}}|a-b|\ .$$

Example 3.1.2. The noise ξ with spatial covariance function $R(x) = (1+|x|^2)^{-\kappa/2}$ satisfies Assumption 2.0.1.

From assumption (2.0.4), it is immediate that ξ^ε converges in law to a spatially correlated noise with covariance $|x|^{-\kappa}$. In Theorem 2.6.1, we demonstrated that the solution of (2.0.1) converges uniformly as $t \to \infty$ to a stationary field $\vec{Z}$. Letting Z a random variable with distribution equal to $\vec{Z}(0,0)$, our main fluctuations result can then be formulated as follows.

Theorem 3.1.3. *Let $d \geqslant 3$. Suppose that ξ and σ, satisfy Assumption 3.1.1. Then, there exists a $\beta_0 > 0$ such that for all $\beta \leqslant \beta_0$, and for any time indices $0 < t_1 \leqslant \ldots \leqslant t_n$, and for any smooth functions with compact support*

$\{g_i\}_{i=1,\dots,n} \subset \mathcal{C}_c^\infty(\mathbb{R}^d)$, the following random variables converge, as $\varepsilon \to 0$, in distribution:

$$(X_{t_1}^{\varepsilon,g_1},\dots,X_{t_n}^{\varepsilon,g_n}) \Rightarrow \nu_{\text{eff}}\bigg(\int_{\mathbb{R}^d}\mathcal{U}(t_1,x)g_1(x)\,dx,\dots,\int_{\mathbb{R}^d}\mathcal{U}(t_n,x)g_n(x)\,dx\bigg). \tag{3.1.3}$$

Here, $\mathcal{U}$ denotes the solution to the additive stochastic heat equation

$$\partial_t\mathcal{U}(t,x) = \tfrac{1}{2}\Delta\mathcal{U}(t,x) + \beta\,\dot{W}^\kappa(t,x), \qquad \mathcal{U}(0,\cdot)\equiv 0, \tag{3.1.4}$$

with effective variance $\nu_{\text{eff}}^2 = |\mathbb{E}[\sigma(Z)]|^2$ and covariance of the Gaussian noise given by

$$\mathbb{E}[\,\dot{W}^\kappa(t,x)\dot{W}^\kappa(s,y)\,] = \delta(t-s)|x-y|^{-\kappa}. \tag{3.1.5}$$

The proof of this result will be given in the next Section 3.2.

Remark 3.1.4. The restriction to small values $\beta \leqslant \beta_0$ comes from the uniform bounds on the solution u and its Malliavin derivative derived in the previous chapter. We note that the existence of the weak disorder regime and a transition at $\kappa = 2$ was already pointed out in a similar context in [Lac11]. Whilst one may expect a phase transition as in the compactly supported covariance case [MSZ16, CNN22], where (2.0.1) fails to admit a stationary solution and the fluctuations limit cease to hold for higher values of β, it is at present unclear whether a similar behaviour is present in the long-range correlated setting.

We note that the fluctuation result validates a posteriori the LLN of Corollary 2.5.5 proved in the previous chapter. Building on the convergence of the finite dimensional marginal distributions, we considered the fluctuations X^ε in suitable Hölder topologies and examined how to show convergence in time while capturing also its distributional regularity, which depends on the correlation parameter κ.

In Section 3.3, we firstly characterise the spaces of (negative) Hölder continuous distributions $\mathcal{C}^\alpha$, with $\alpha < 0$. We then formulate Kolmogorov-type criterions (Propositions 3.3.7-3.3.9) to show tightness in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$. These may be of independent interest and are also crucially used in the next chapter to show quantitative convergence results for the KPZ scaling limit.

With this simple framework, in Section 3.4 we show that X^ε converge weakly to $\nu_{\text{eff}}\,\mathcal{U}$ as a *Hölder continuous distribution-valued* processes:

Theorem 3.1.5. *Let $d \geqslant 3$. Suppose that ξ and σ, satisfy Assumption 3.1.1. Then, for any $\gamma \in (0, \frac{1}{2})$ and $\alpha < 1 - \frac{\kappa}{2} - 2\gamma$, there exists $\beta_{\gamma,\alpha} > 0$ such that, for any $\beta \leqslant \beta_{\gamma,\alpha}$ and $T > 0$, X^ε converges to $\nu_{\text{eff}}\,\mathcal{U}$ weakly in $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha)$.*

Remark 3.1.6. The Hölder coefficients are consistent with the expected regularity gain by applying the heat semigroup to the driving noise $\dot{W}^\kappa$ of the limit equation, cf. Remark 3.4.2. In this chapter the limit noise is denoted with $\dot{W}^\kappa$, to keep clar the convergence is in law here.

Remark 3.1.7. As claimed in the introduction, Theorem 3.1.3 proves a Gaussian CLT, even in the presence of long-range correlations. However, it crucially differs from the compactly supported case given that now the limit solution is driven by coloured noise with correlations of polynomial decay. Moreover, depending on the decay parameter κ, we see that the fluctuations are of order $\varepsilon^{\frac{\kappa}{2}-1}$ (in place of $\varepsilon^{\frac{d}{2}-1}$) and the effective variance (1.3.9) is of different nature compared to (1.2.18). The new effective variance is due to the slower decorrelation, and fluctuations emerging at different scales. Since $v_\varepsilon = \varepsilon^{1-\kappa/2}(u^\varepsilon - 1)$ solves the equation

$$\partial_t v_\varepsilon(t,x) = \tfrac{1}{2}\Delta v_\varepsilon(t,x) + \beta\sigma(\varepsilon^{\kappa/2-1}v_\varepsilon(t,x)+1)\xi^\varepsilon(t,x), \qquad v_\varepsilon(0,x) \equiv 0\ ,$$

one would naïvely expect at first that in the limit (3.1.3), $\nu_{\text{eff}} = \sigma(1)$, but this is incorrect due to resonances between the fluctuations of u_ε and ξ^ε. Given large time results of the last chapter, we can give a heuristic argument leading to the correct expression $\nu^2_{\text{eff}} = \lim_{t\to\infty} |\mathbb{E}[\sigma(u(t,0))]|^2$.

For this, let us test the rescaled noise term in (3.1.1) against a test function g, thus considering the expression

$$\int_{\mathbb{R}^d}\int_0^t \sigma(u_\varepsilon(s,x))g(x)\ \xi^\varepsilon(ds,dx),$$

whose covariance is given by

$$\int_{\mathbb{R}^{2d}}\int_0^t C(\varepsilon^{-2}s, \varepsilon^{-1}(x-y))\,ds\,\varepsilon^{-\kappa}R(\tfrac{x-y}{\varepsilon})g(x)g(y)\,dx\,dy\ , \tag{3.1.6}$$

where we set

$$C(s,x) = \mathbb{E}[\sigma(u(s,0))\sigma(u(s,x))] \, .$$

Given Theorem 2.6.1, we know u converges in law to some stationary process Z as $t \to \infty$, so we expect $C(s,\cdot)$ to converge as $s \to \infty$ to some bounded function $\bar{C} : \mathbb{R}^d \to \mathbb{R}$, whose spatial correlations vanish (as long as Z is sufficiently mixing), furthermore has the asymptotic $\lim_{|x|\to\infty} \bar{C}(x) = |\mathbb{E}\sigma(Z)|^2$.

In case R is $L^1(\mathbb{R}^d)$ integrable, so is $R\bar{C}$, and we expect (3.1.6) to converge (provided $\kappa = d$) to $\tilde{\nu}^2 \|g\|_{L^2}^2 t$ with

$$\tilde{\nu}^2 = \int_{\mathbb{R}^d} \bar{C}(y) R(y) \, dy \, ,$$

thus leading to the limit $\tilde{\nu}\mathcal{U}$, solving the equation $\partial_t \mathcal{U} = \frac{1}{2}\Delta\mathcal{U} + \beta\dot{W}$, where $\dot{W}$ is standard space time white noise. For R with compact support, this heuristic agrees with the result in [MSZ16, GRZ18], see also [CNN22] and discussion in the introduction (cf. (1.2.10) and the variance expression (1.2.15)).

In our case R is not integrable, $\tilde{\nu}^2 = \infty$ for the SHE, but we have instead the large scale structure $\varepsilon^{-\kappa} R(\frac{x-y}{\varepsilon}) \approx |x-y|^{-\kappa}$. Since this is integrable near 0, we expect (3.1.6) to behave like

$$t \int_{\mathbb{R}^{2d}} \bar{C}(\varepsilon^{-1}(x-y)) \frac{g(x)g(y)}{|x-y|^\kappa} \, dx \, dy \approx t|\mathbb{E}\sigma(Z)|^2 \int_{\mathbb{R}^{2d}} \frac{g(x)g(y)}{|x-y|^\kappa} \, dx \, dy \, ,$$

which is precisely what is suggested by Theorem 3.1.5.

Remark 3.1.8. The isotropic convergence assumption (2.0.4) can be relaxed to

$$\varepsilon^{-\kappa} R(\varepsilon^{-1}x) \to \theta\big(\tfrac{x}{|x|}\big)|x|^{-\kappa} \quad \text{as} \quad \varepsilon \to 0,$$

for some positive continuous function $\theta : \partial B(0,1) \to \mathbb{R}_+$. This results in an Edwards–Wilkinson limit driven by Gaussian white noise with covariance

$$\mathbb{E}[\, \dot{W}^\kappa(t,x) \dot{W}^\kappa(s,y) \,] = \delta(t-s) \, \theta\big(\tfrac{x-y}{|x-y|}\big)|x-y|^{-\kappa},$$

in place of (3.1.5).

Proofs ideas and organisation of the chapter

- ▷ Section 3.2 is devoted to the proof of Theorem 3.1.3. Our basic strategy is to apply Stein's method, as in Proposition 2.2.5. For this, let $Z_i := \nu_{\text{eff}} \int_{\mathbb{R}^d} \mathcal{U}(t_i, x) g_i(x) dx$, for a collection of test functions g_i and times t_i and $\mathcal{U}$ the solution to the stochastic heat equation (3.1.4). Let Σ denote the covariance matrix of $\{Z_i\}$, and write $X^{\varepsilon,g}$ as the divergence of $v^{\varepsilon,g}$ through the mild formulation of the equation. It is sufficient to show that the quantity, $\langle DX^{\varepsilon,g_i}_{t_i}, v^{\varepsilon,g_j}_{t_j} \rangle_{\mathcal{H}} - \Sigma_{i,j}$, converges to zero in $L^2(\Omega)$. We decompose this quantity into the sum of the difference of the covariances between $X^{\varepsilon,g_i}_{t_i}$ and Z_j – which converges to zero by direct computation – and a remainder term. The main technical hurdle is to control the $L^2(\Omega)$ norm of this residual term, which we bound by a multiple integral involving singular kernels. The latter are then shown to be bounded by recasting them as Feynman diagrams, see Lemma 2.3.2. This is carried in Section 3.2.

- ▷ In Section 3.3, we introduce the spaces of (negative) Hölder continuous distributions $\mathcal{C}^\alpha$, with $\alpha < 0$, and a useful characterisation in terms of a single test function from [CZ20]. We then define spaces satisfying Kolmogorov-like conditions, and show these embed continuously into Hölder spaces with control on moments and regularities. The conditions are simple to verify and allow us to formulate a tightness criterion for $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha)$ spaces.

- ▷ In Section 3.4, leveraging moments bounds from the previous chapter and the Kolmorogov-type embeddings of Section 3.3, we show tightness of X^ε in $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha)$ for suitable $\gamma < \frac{1}{2}$ and $\alpha < 0$, concluding the proof of Theorem 3.1.5. In contrast with the first version of our article, besides simplifying our proofs and framework, we prove convergence over the whole optimal range of Hölder exponents[(1)].

[(1)] Namely, by careful estimates, we were able to lift the additional requirement of $\alpha < -1$.

3.2 Convergence of marginals

Throughout this section, we assume $\beta \leqslant \beta_0$, where $\beta_0 > 0$ is taken sufficiently small so that Theorem 2.6.1 and Lemma 2.5.4 hold, and we can apply Lemma 2.5.2 in the arguments below to estimate moments.[(2)]

Our main tool is Stein's method to show convergence to the Gaussian limit. We start with the proof that marginals converge in distribution, Theorem 3.1.3, relying on the results shown in the following subsections. There, we show the variance converges to the identified limit and prove that higher orderh (non-Gaussian) terms vanish as ε converge to 0, through moments estimates and Feynman diagrams analysis.

3.2.1 Proof of Theorem 3.1.3

Proof of Theorem 3.1.3. We follow the general strategy of [GL20]. Recall Proposition 2.2.5 and suppose Z is an n-dimensional centered Gaussian vector with covariance matrix $(\Sigma_{i,j})_{1\leqslant i,j\leqslant n}$. Since $\mathcal{C}^2$ functions with compact support are weak convergence determining over $\mathbb{R}^n$ (c.f. [EK86, Prop III.4.4]), it follows immediately that the convergence

$$\langle DX_i^\varepsilon, v_j^\varepsilon\rangle_{\mathcal{H}} \longrightarrow \Sigma_{i,j} \quad \text{in } L^2(\Omega)\ , \tag{3.2.1}$$

for a sequence $X_i^\varepsilon = \delta(v_i^\varepsilon)$ implies that X^ε converges in law to Z.

In order to prove Theorem 3.1.3, we then apply this result to $X_i^\varepsilon := X_{t_i}^{\varepsilon,g_i}$, defined by (3.1.2). We also set

$$Z_i \overset{\text{def}}{=} Z_{t_i}(g_i) = \nu_{\text{eff}} \int_{\mathbb{R}^d} \mathcal{U}(t_i,x) g_i(x) dx\ ,$$

where $\mathcal{U}$ is as in Theorem 3.1.3. By the definition of the mild solution, we

Now we only require $\alpha + 2\gamma < 1 - \frac{\kappa}{2}$, which is the space-time regularity constraint expected.

[(2)] Keeping track of the various constraints, one finds that it suffices to set $\beta_0 = \beta_0(4) \wedge \beta_1(2)$ with $\beta_0(p)$ as in Lemma 2.5.2 and $\beta_1(2)$ as in Theorem 2.6.1.

have $X_i^\varepsilon = \delta(v_i^\varepsilon)$ with

$$v_i^\varepsilon(s,y) := v_{t_i}^{\varepsilon,g_i}(s,y) = \beta\varepsilon^{1-\kappa/2}\mathbf{1}_{[0,\frac{t_i}{\varepsilon^2}]}(s)\sigma(u(s,y))\int_{\mathbb{R}^d} P_{\frac{t_i}{\varepsilon^2}-s}(\tfrac{x}{\varepsilon}-y)g_i(x)dx\ , \tag{3.2.2}$$

so it remains to show that (3.2.1) holds with $\Sigma_{i,j} = \mathbb{E}(Z_iZ_j)$.

Applying the commutation relation (2.2.7) to $D\delta(v_i^\varepsilon)$, we obtain the decomposition $\langle DX_i^\varepsilon, v_j^\varepsilon\rangle_{\mathcal{H}} = A_{1,\varepsilon}^{i,j} + A_{2,\varepsilon}^{i,j}$, where

$$A_{1,\varepsilon}^{i,j} = \langle v_i^\varepsilon, v_j^\varepsilon\rangle_{\mathcal{H}}\ , \qquad A_{2,\varepsilon}^{i,j} = \int_{-\infty}^{\infty}\int_{\mathbb{R}^d}\langle Dv_i^\varepsilon(r,z), v_j^\varepsilon\rangle_{\mathcal{H}}\,\xi(dr,dz)\ .$$

With a simple calculation we can rewrite the first term as follows:

$$\begin{aligned}A_{1,\varepsilon}^{i,j} &= \int_{\mathbb{R}^{1+2d}} R(y_1-y_2)v_i^\varepsilon(s,y_1)v_j^\varepsilon(s,y_2)dyds\\ &= \frac{\beta^2}{\varepsilon^{\kappa-2}}\int_0^{\frac{t_i\wedge t_j}{\varepsilon^2}}\int_{\mathbb{R}^{2d}} R(y_1-y_2)\Big(\sigma(u(s,y_1))\int_{\mathbb{R}^d} P_{\frac{t_i}{\varepsilon^2}-s}(\tfrac{x_1}{\varepsilon}-y_1)g_i(x_1)dx_1\Big)\\ &\qquad\Big(\sigma(u(s,y_2))\int_{\mathbb{R}^d} P_{\frac{t_j}{\varepsilon^2}-s}(\tfrac{x_2}{\varepsilon}-y_2)g_j(x_2)dx_2\Big)\ dyds.\end{aligned}$$

Through the change of variables $s\mapsto\frac{s}{\varepsilon^2}$, $y_i\mapsto\frac{y_i}{\varepsilon}$ and the fact that $P_{\frac{r}{\varepsilon^2}}(\frac{x}{\varepsilon}) = \varepsilon^d P_r(x)$, we then have

$$\begin{aligned}A_{1,\varepsilon}^{i,j} &= \frac{\beta^2}{\varepsilon^{\kappa+2d}}\int_0^{t_i\wedge t_j}\int_{\mathbb{R}^{2d}} R(\tfrac{y_1-y_2}{\varepsilon})\,\sigma(u(\tfrac{s}{\varepsilon^2},\tfrac{y_1}{\varepsilon}))\sigma(u(\tfrac{s}{\varepsilon^2},\tfrac{y_2}{\varepsilon}))\\ &\qquad\Big(\int_{\mathbb{R}^{2d}} P_{\frac{t_i-s}{\varepsilon^2}}(\tfrac{x_1-y_1}{\varepsilon})P_{\frac{t_j-s}{\varepsilon^2}}(\tfrac{x_2-y_2}{\varepsilon})g_i(x_1)g_j(x_2)dx\Big)dyds\\ &= \beta^2\int_0^{t_i\wedge t_j}\int_{\mathbb{R}^{4d}}\varepsilon^{-\kappa}R(\tfrac{y_1-y_2}{\varepsilon})\Big(\prod_{m=1}^2\sigma(u(\tfrac{s}{\varepsilon^2},\tfrac{y_m}{\varepsilon}))\Big)\\ &\qquad P_{t_i-s}(x_1-y_1)P_{t_j-s}(x_2-y_2)g_i(x_1)g_j(x_2)\,dxdyds\ .\end{aligned}$$

Regarding the term $A_{2,\varepsilon}^{i,j}$, it follows from (3.2.2) and the identity (cf. (2.5.5)) $D_{s,y}\sigma(u(r,z)) = \Sigma(r,z)D_{s,y}u(r,z)$ for a bounded, adapted function-valued process $\Sigma(r,\cdot)$, that, setting

$$\tilde{A}_{2,\varepsilon}^{\tau}(x,s,y) := \int_s^{\frac{\tau}{\varepsilon^2}}\int_{\mathbb{R}^d} P_{\frac{\tau}{\varepsilon^2}-r}(\tfrac{x}{\varepsilon}-z)\Sigma(r,z)D_{s,y}u(r,z)\xi(dr,dz)\ , \tag{3.2.3}$$

by a similar calculation we have

$$A_{2,\varepsilon}^{i,j} = \frac{\beta^2}{\varepsilon^{\kappa+d}}\int_0^{t_i\wedge t_j}\int_{\mathbb{R}^{4d}} g_i(x_1)g_j(x_2)\tilde{A}_{2,\varepsilon}^{t_i}(x_1,\tfrac{s}{\varepsilon},\tfrac{y_1}{\varepsilon})\times$$

$$\sigma(u_\varepsilon(s,y_2))P_{t_j-s}(x_2-y_2)R(\tfrac{y_1-y_2}{\varepsilon})dydxds.$$

With the above decomposition in place, we now write

$$\|\langle DX_i^\varepsilon, v_j^\varepsilon\rangle - \Sigma_{i,j}\|_2 \leqslant \left\|\mathbb{E}A_{1,\varepsilon}^{i,j} - \Sigma_{i,j}\right\|_2 + \mathrm{var}(A_{1,\varepsilon}^{i,j})^{\frac{1}{2}} + \|A_{2,\varepsilon}^{i,j}\|_2.$$

We note that $\mathbb{E}[X_i^\varepsilon X_j^\varepsilon] = \mathbb{E}A_{1,\varepsilon}^{i,j}$ so that $\left|\mathbb{E}A_{1,\varepsilon}^{i,j} - \Sigma_{i,j}\right|$ is the difference between covariances and converges to 0 by Lemma 3.2.2 below.

On the other hand, by Proposition 3.2.8 the terms $\mathrm{var}(A_{1,\varepsilon}^{i,j})$ and $\|A_{2,\varepsilon}^{i,j}\|_2$ vanish as $\varepsilon \to 0$, concluding the proof of Theorem 3.1.3. □

It remains to prove Lemma 3.2.2 and Proposition 3.2.8, which is done in sections 3.2.2 and 3.2.5 respectively.

3.2.2 Convergence of covariances

We drop superscripts, fix $f, g \in \mathcal{C}_c(\mathbb{R}^d)$ and fix $\tau, v > 0$. The covariance $\Sigma_{\tau,v}^{f,g}$ of the limiting terms $Z_\tau(f)$ and $Z_v(g)$ is

$$\Sigma_{\tau,v}^{f,g} := \beta^2\nu_{\text{eff}}^2 J_{\tau,v}(f,g),$$
$$J_{\tau,v}(f,g) := \int_0^{\tau\wedge v}\int_{\mathbb{R}^{4d}} |y_1-y_2|^{-\kappa} P_{\tau-s}(x_1-y_1) P_{v-s}(x_2-y_2) f(x_1)g(x_2)dxdyds.$$

Recall I_{Γ_0}, defined at (2.3.9) and finite for $\kappa \in (0,d)$ and $\lambda < \frac{\kappa}{2}$,

$$I_{\Gamma_0}(f,g;\lambda) = \int_{\mathbb{R}^{4d}} |y_1-y_2|^{-\kappa} \prod_{i=1}^2 |x_i-y_i|^{\lambda-d} f(x_1)g(x_2)dydx.$$

We first observe the following bound on $J_{\tau,v}(f,g)$ in terms of I_{Γ_0}:

Lemma 3.2.1. *Let $f, g \in \mathcal{C}_c(\mathbb{R}^d, \mathbb{R}_+)$, $\tau, v > 0$, $d \geqslant 3$ and $\lambda \in (0,1)$. Then,*

$$J_{\tau,v}(f,g) \lesssim (\tau\wedge v)^{1-\lambda} I_{\Gamma_0}(f,g;\lambda). \tag{3.2.4}$$

Proof. Let $t = \tau \wedge v$. For any fixed $\lambda \in (0,1)$, with Lemma 2.4.3, we can bound the Gaussian kernels as follows:

$$P_{\tau-s}(x_1-y_1)P_{v-s}(x_2-y_2) \lesssim (t-s)^{-\lambda} \prod_{i=1}^2 |x_i-y_i|^{\lambda-d},$$

and the claim follows by inserting this into the expression for $J_{\tau,v}(f,g)$ and performing the time integral. □

We now present one of the lemmas needed for proving Theorem 3.1.3.

Lemma 3.2.2. *Assume the assumptions of Theorem 3.1.3 on κ, R, σ and $\beta < \beta_0$. Let $f, g \in \mathcal{C}_c(\mathbb{R}^d)$ and for $\tau, v > 0$ let*

$$A_{1,\varepsilon} = \beta^2 \int_0^{\tau\wedge v} \int_{\mathbb{R}^{4d}} \varepsilon^{-\kappa} R(\tfrac{y_1-y_2}{\varepsilon}) \Big(\prod_{m=1}^{2} \sigma(u(\tfrac{s}{\varepsilon^2}, \tfrac{y_m}{\varepsilon})) \Big) \times P_{\tau-s}(x_1-y_1) P_{v-s}(x_2-y_2) f(x_1) g(x_2)\, dxdyds. \tag{3.2.5}$$

Then the following statements hold.

▷ *For any $\lambda \in (0,1)$, the following bound holds uniform in $\varepsilon \in (0,1)$,*

$$\mathbb{E}|A_{1,\varepsilon}| \lesssim (\tau \vee v)^{1-\lambda} I_{\Gamma_0}(|f|,|g|;\lambda)\ . \tag{3.2.6}$$

▷ *Furthermore,*

$$\lim_{\varepsilon\to 0} \mathbb{E}\, A_{1,\varepsilon} = \beta^2 \nu_{\text{eff}}^2\, J_{\tau,v}(f,g) = \Sigma^{f,g}_{\tau,v}\ . \tag{3.2.7}$$

Proof. Note that under our assumptions I_{Γ_0} is finite by Example 2.3.4. By the definition of $A_{1,\varepsilon}$, one has

$$\mathbb{E}|A_{1,\varepsilon}| \leqslant \beta^2 \int_0^{\tau\wedge v} \int_{\mathbb{R}^{4d}} \varepsilon^{-\kappa} \big|R(\tfrac{y_1-y_2}{\varepsilon})\big|\, \mathbb{E}\big[\sigma(u(\tfrac{s}{\varepsilon^2}, \tfrac{y_1}{\varepsilon}))\sigma(u(\tfrac{s}{\varepsilon^2}, \tfrac{y_2}{\varepsilon}))\big] P_{\tau-s}(x_1-y_1) P_{v-s}(x_2-y_2) |f(x_1)g(x_2)|\, dxdyds. \tag{3.2.8}$$

By the uniform L^p bounds in (2.5.10),

$$\sup_{\varepsilon,s,y_1,y_2} \mathbb{E}\big[\sigma(u(\tfrac{s}{\varepsilon^2}, \tfrac{y_1}{\varepsilon}))\sigma(u(\tfrac{s}{\varepsilon^2}, \tfrac{y_2}{\varepsilon}))\big] < \infty.$$

Since $\varepsilon^{-\kappa}|R(\frac{x}{\varepsilon})| \lesssim |x|^{-\kappa}$ uniformly in ε, it follows that $\mathbb{E}|A_{1,\varepsilon}| \lesssim \beta^2 J_{\tau,v}(|f|,|g|)$, so that (3.2.6) follows from Lemma 3.2.1.

To show (3.2.7), we use the stationarity in space of $\{u(t,x)\}_{x\in\mathbb{R}^d}$ to write

$$\mathbb{E}\big[\sigma(u(\tfrac{s}{\varepsilon^2}, \tfrac{y_1}{\varepsilon}))\sigma(u(\tfrac{s}{\varepsilon^2}, \tfrac{y_2}{\varepsilon}))\big] =$$

$$\mathbb{E}\big[\sigma(u(\tfrac{s}{\varepsilon^2},0))\big]^2 + \operatorname{cov}\big(\sigma(u(\tfrac{s}{\varepsilon^2},\tfrac{y_1-y_2}{\varepsilon})),\sigma(u(\tfrac{s}{\varepsilon^2},0))\big)\,.$$

Thus, we have the decomposition $\mathbb{E}A_{1,\varepsilon} = \Sigma^\varepsilon + B^\varepsilon$, where

$$\Sigma^\varepsilon := \beta^2 \int_0^{\tau\wedge v}\int_{\mathbb{R}^{4d}} \varepsilon^{-\kappa} R(\tfrac{y_1-y_2}{\varepsilon})\,\mathbb{E}\big[\sigma(u(\tfrac{s}{\varepsilon^2},0))\big]^2 \\ P_{\tau-s}(x_1-y_1)P_{v-s}(x_2-y_2)f(x_1)g(x_2)\,dxdyds, \tag{3.2.9}$$

$$B^\varepsilon := \beta^2 \int_0^{\tau\wedge v}\int_{\mathbb{R}^{4d}} \varepsilon^{-\kappa} R(\tfrac{y_1-y_2}{\varepsilon})\,\operatorname{cov}(\sigma(u(\tfrac{s}{\varepsilon^2},\tfrac{y_1-y_2}{\varepsilon})),\sigma(u(\tfrac{s}{\varepsilon^2},0))) \\ P_{\tau-s}(x_1-y_1)P_{v-s}(x_2-y_2)f(x_1)g(x_2)\,dxdyds. \tag{3.2.10}$$

By (2.5.6), we have $\mathbb{E}\big[\sigma(u(\frac{s}{\varepsilon^2},0))\big] \leqslant C$ uniformly over $s \geqslant 0$ and $\varepsilon > 0$. Furthermore by Theorem 2.6.1, $u(t,x)$ converges in law as $t\to\infty$, so $\nu_{\text{eff}} := \lim_{t\to\infty}|\mathbb{E}[\sigma(u(t,0))]|$ is well defined and for every $s>0$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\big[\sigma(u(\tfrac{s}{\varepsilon^2},0))\big]^2 = \nu_{\text{eff}}^2\,. \tag{3.2.11}$$

Similarly, using (2.5.11), it follows that for any $y_1 \neq y_2$,

$$\operatorname{cov}\Big(\sigma(u(\tfrac{s}{\varepsilon^2},\tfrac{y_1-y_2}{\varepsilon})),\sigma(u(\tfrac{s}{\varepsilon^2},0))\Big) \leqslant C(1\wedge\tfrac{\varepsilon^{\kappa-2}}{|y_1-y_2|^{2-\kappa}}) \longrightarrow 0, \quad \text{as } \varepsilon\to 0\,, \tag{3.2.12}$$

and this quantity is uniformly bounded, again by (2.5.6). Since $\varepsilon^{-\kappa}|R(\frac{x}{\varepsilon})| \lesssim |x|^{-\kappa}$, it follows from (3.2.11)-(3.2.12) that the integrands of both (3.2.9) and (3.2.10) are uniformly bounded by an integrable function. It then follows from Lebesgue's dominated convergence theorem that

$$B^\varepsilon \longrightarrow 0, \qquad \Sigma^\varepsilon \longrightarrow \beta^2\nu_{\text{eff}}^2 J_{\tau,v} = \Sigma^{f,g}_{\tau,v}\,,$$

thus concluding the proof. □

3.2.3 Estimates on residual terms

The focus of this subsection is to obtain preliminary estimates on the residual terms $\operatorname{var}(A_{1,\varepsilon})$ and $\|A_{2,\varepsilon}\|_2$, where $A_{1,\varepsilon}$ is given in (3.2.5) and

$$A_{2,\varepsilon} = \varepsilon^{-d}\beta^2 \int_0^{\tau\wedge v}\int_{\mathbb{R}^{4d}} \varepsilon^{-\kappa} R(\tfrac{y_1-y_2}{\varepsilon})\tilde{A}^\tau_{2,\varepsilon}(x_1,\tfrac{s}{\varepsilon},\tfrac{y_1}{\varepsilon})\sigma(u_\varepsilon(s,y_2)) \\ P_{v-s}(x_2-y_2)f(x_1)g(x_2)dydxds, \tag{3.2.13}$$

with $\tilde{A}^{\tau}_{2,\varepsilon}$ given by (3.2.3).

We use the short-hand notation $dx = \prod_{i=1}^4 dx_i$, $dy = \prod_{i=1}^4 dy_i$, $dz = \prod_{i=1}^2 dz_i$. Moreover, for $f, g : \mathbb{R}^d \to \mathbb{R}$, we set

$$\psi_{f,g}(x) := f(x_1)g(x_2)g(x_3)f(x_4), \tag{3.2.14}$$

$$\varphi_{f,g}(x) := f(x_1)g(x_2)f(x_3)g(x_4), \tag{3.2.15}$$

$$Q(x) := \int_0^\infty \int_{\mathbb{R}^{2d}} P_r(x - z_1)P_r(z_2)|R(z_1 - z_2)|\, dz\, dr. \tag{3.2.16}$$

Letting $t = \tau \wedge v$, we have the following preliminary estimates:

Lemma 3.2.3. *For any test functions $f, g : \mathbb{R}^d \to \mathbb{R}$, any $\lambda \in (0, \frac{1}{2})$, one has*

$$\mathrm{var}(A_{1,\varepsilon}) \lesssim \varepsilon^{\kappa-2} t^{1-2\lambda} \sum_{\substack{i=1,2\\ j=3,4}} \int_{\mathbb{R}^{8d}} \Big(\prod_{k=1}^4 |y_k - x_k|^{\lambda-d}\Big) |y_i - y_j|^{2-\kappa} \\ |y_1 - y_2|^{-\kappa} |y_3 - y_4|^{-\kappa} \, |\varphi_{f,g}(x)| \, dxdyds.$$

Proof. We follow [GL20] and begin with

$$\mathrm{var}(A_{1,\varepsilon}) = \beta^2 \int_0^{\tau\wedge v} \int_{\mathbb{R}^{8d}} \varepsilon^{-\kappa} R(\tfrac{y_1-y_2}{\varepsilon}) \varepsilon^{-\kappa} R(\tfrac{y_3-y_4}{\varepsilon}) \operatorname{cov}\Big(\Lambda_\varepsilon(s, y_1, y_2), \Lambda_\varepsilon(s, y_3, y_4)\Big) \\ \Big(\prod_{k=1}^2 P_{\tau-s}(x_k - y_k) P_{v-s}(x_{k+2} - y_{k+2})\Big) f(x_1)g(x_2)f(x_3)g(x_4)\, dxdyds$$

where $\Lambda_\varepsilon(s, y_1, y_2) := \sigma(u(\frac{s}{\varepsilon^2}, \frac{y_1}{\varepsilon}))\sigma(u(\frac{s}{\varepsilon^2}, \frac{y_2}{\varepsilon}))$. By the Clark–Ocone formula,

$$\operatorname{cov}\Big(\Lambda_\varepsilon(s, y_1, y_2), \Lambda_\varepsilon(s, y_3, y_4)\Big) = \\ \int_0^{\frac{s}{\varepsilon^2}} \int_{\mathbb{R}^{2d}} R(z_1 - z_2)\, \mathbb{E}\Big[\mathbb{E}[D_{r,z_1}\Lambda_\varepsilon(s, y_1, y_2)|\mathcal{F}_r|]\mathbb{E}[D_{r,z_2}\Lambda_\varepsilon(s, y_3, y_4)|\mathcal{F}_r|]\Big] dzdr.$$

Using the chain rule to expand $D_{r,z}\Lambda_\varepsilon(s, y_1, y_2)$ and the bounds (2.5.7)-(2.5.10),

$$\|D_{r,z_1}\Lambda_\varepsilon(s, y_1, y_2)\|_2 \lesssim P_{\frac{s}{\varepsilon^2}-r}(\tfrac{y_1}{\varepsilon} - z) + P_{\frac{s}{\varepsilon^2}-r}(\tfrac{y_2}{\varepsilon} - z). \tag{3.2.17}$$

Then, we have

$$\Big|\operatorname{cov}\Big(\Lambda_\varepsilon(s, y_1, y_2), \Lambda_\varepsilon(s, y_3, y_4)\Big)\Big| \lesssim \sum_{\substack{i=1,2\\ j=3,4}} Q(\tfrac{y_i-y_j}{\varepsilon}), \tag{3.2.18}$$

thus concluding

$$\mathrm{var}(A_{1,\varepsilon}) \lesssim \sum_{\substack{i=1,2\\ j=3,4}} \int_0^{\tau\wedge v} \int_{\mathbb{R}^{8d}} \Big(\prod_{k=1}^{2} P_{\tau-s}(y_k - x_k) P_{v-s}(y_{2+k} - x_{2+k}) \Big) Q\big(\tfrac{y_i - y_j}{\varepsilon}\big) \\ \varepsilon^{-\kappa} \big| R\big(\tfrac{y_1 - y_2}{\varepsilon}\big) \big| \varepsilon^{-\kappa} \big| R\big(\tfrac{y_3 - y_4}{\varepsilon}\big) \big| \, |\varphi_{f,g}(x)| \, dx dy dy ds. \tag{3.2.19}$$

Since $\varepsilon^{-\kappa}|R(\frac{x}{\varepsilon})| \lesssim |x|^{-\kappa}$, and by Lemma 2.4.5, we see that

$$Q\big(\tfrac{x}{\varepsilon}\big) \lesssim 1 \wedge \varepsilon^{\kappa-2}|x|^{2-\kappa} \lesssim \varepsilon^{\kappa-2}|x|^{2-\kappa}.$$

Plugging these estimates into (3.2.19) and bringing the time integral inside, we have

$$\mathrm{var}(A_{1,\varepsilon}) \lesssim \varepsilon^{\kappa-2} \sum_{\substack{i=1,2\\ j=3,4}} \int_{\mathbb{R}^{8d}} \int_0^{\tau\wedge v} \Big(\prod_{k=1}^{2} P_{\tau-s}(y_k - x_k) P_{v-s}(y_{2+k} - x_{2+k}) \Big) ds \\ |y_i - y_j|^{2-\kappa} |y_1 - y_2|^{-\kappa} |y_3 - y_4|^{-\kappa} \, |\varphi_{f,g}(x)| \, dx dy.$$

Let $t = \tau \wedge v$ and $\lambda \in (0, \frac{1}{2})$. We use again the pointwise estimate (2.4.4) from Lemma 2.4.3. For $r \in \{\tau, v\}$, we have $r \geqslant t$, so that $P_{r-s}(x) \lesssim (t-s)^{-\frac{\lambda}{2}} |x|^{\lambda-d}$ uniformly in t, s and $x \neq 0$. Hence, we estimate as follows

$$\begin{aligned} \mathrm{var}(A_{1,\varepsilon}) &\lesssim \varepsilon^{\kappa-2} \sum_{\substack{i=1,2\\ j=3,4}} \int_{\mathbb{R}^{8d}} \Big(\int_0^t (t-s)^{-2\lambda} \, ds \Big) \Big(\prod_{k=1}^{4} |y_k - x_k|^{\lambda-d} \Big) \\ &\qquad |y_i - y_j|^{2-\kappa} \, |y_1 - y_2|^{-\kappa} |y_3 - y_4|^{-\kappa} \, |\varphi_{f,g}(x)| \, dx dy ds \\ &= \varepsilon^{\kappa-2} t^{1-2\lambda} \sum_{\substack{i=1,2\\ j=3,4}} \int_{\mathbb{R}^{8d}} \Big(\prod_{k=1}^{4} |y_k - x_k|^{\lambda-d} \Big) |y_i - y_j|^{2-\kappa} \\ &\qquad |y_1 - y_2|^{-\kappa} |y_3 - y_4|^{-\kappa} \, |\varphi_{f,g}(x)| \, dx dy dy, \end{aligned}$$

completing the proof. □

Lemma 3.2.4. *For any test functions $f, g : \mathbb{R}^d \to \mathbb{R}$, one has*

$$\|A_{2,\varepsilon}\|_2 \lesssim \varepsilon^{\frac{\kappa}{2}-1} \sqrt{t} \Big[\int_{\mathbb{R}^{10d}} |\psi_{f,g}(x)| \, |y_1 - y_4|^{-\kappa} |z_1 - y_2|^{-\kappa} |z_2 - y_3|^{-\kappa} \\ \Big(\int_0^t P_{v-s}(y_2 - x_2) P_{v-s}(y_3 - x_3) \int_s^{\tau} P_{\tau-r}(x_1 - y_1) P_{\tau-r}(x_4 - y_4)$$

$$P_{r-s}(y_1 - z_1)P_{r-s}(y_4 - z_2)drds\Big)dydzdx\Big]^{\frac{1}{2}}.$$

Proof. Setting

$$B^\tau_\varepsilon(s, x_1, x_1', y_1, y_1', y_2, y_2') = \mathbb{E}\Big[\tilde{A}^\tau_{2,\varepsilon}(x_1, \tfrac{s}{\varepsilon}, \tfrac{y_1}{\varepsilon})\tilde{A}^\tau_{2,\varepsilon}(x_1', \tfrac{s}{\varepsilon}, \tfrac{y_1'}{\varepsilon})\sigma(u_\varepsilon(s, y_2))\sigma(u_\varepsilon(s, y_2'))\Big],$$

and using Minkowski inequality, we obtain:

$$\|A_{2,\varepsilon}\|_2 \leqslant \varepsilon^{-d}\int_0^{\tau\wedge v}\Big(\int_{\mathbb{R}^{8d}} B^\tau_\varepsilon(s, x_1, x_1', y_1, y_1', y_2, y_2')P_{v-s}(x_2 - y_2)P_{v-s}(x_2' - y_2')$$
$$\varepsilon^{-\kappa}\big|R(\tfrac{y_1-y_2}{\varepsilon})\big|\varepsilon^{-\kappa}\big|R(\tfrac{y_1'-y_2'}{\varepsilon})\big|f(x_1)g(x_2)f(x_1')g(x_2')dxdx'dydy'\Big)^{\frac{1}{2}}ds.$$

Recalling the definition (3.2.3) of $\tilde{A}^\tau_{2,\varepsilon}$, we first apply Itô isometry, then estimate via Hölder inequality and the moment bounds on Du, $\sigma(u)$ from Lemma 2.5.2, and finally perform a change of variables,

$$\begin{aligned}
&|B^\tau_\varepsilon(s, x_1, x_1', y_1, y_1', y_2, y_2')| \\
&= \Big|\int_{\frac{s}{\varepsilon^2}}^{\frac{\tau}{\varepsilon^2}}\int_{\mathbb{R}^{2d}}\mathbb{E}\big[\Sigma(r, z_1)\Sigma(r, z_2)D_{\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}}u(r, z_1)D_{\frac{s}{\varepsilon^2},\frac{y_1'}{\varepsilon}}u(r, z_2)\sigma(u(\tfrac{s}{\varepsilon^2}, \tfrac{y_2}{\varepsilon}))\sigma(u(\tfrac{s}{\varepsilon^2}, \tfrac{y_2'}{\varepsilon}))\big] \\
&\qquad P_{\frac{\tau}{\varepsilon^2}-r}(\tfrac{x_1}{\varepsilon} - z_1)P_{\frac{\tau}{\varepsilon^2}-r}(\tfrac{x_1'}{\varepsilon} - z_2)R(z_1 - z_2)\,dzdr\Big| \\
&\lesssim \int_{\frac{s}{\varepsilon^2}}^{\frac{\tau}{\varepsilon^2}}\int_{\mathbb{R}^{2d}} P_{r-\frac{s}{\varepsilon^2}}(z_1 - \tfrac{y_1}{\varepsilon})P_{r-\frac{s}{\varepsilon^2}}(z_2 - \tfrac{y_1'}{\varepsilon}) \\
&\qquad P_{\frac{\tau}{\varepsilon^2}-r}(\tfrac{x_1}{\varepsilon} - z_1)P_{\frac{\tau}{\varepsilon^2}-r}(\tfrac{x_1'}{\varepsilon} - z_2)\big|R(z_1 - z_2)\big|\,dzdr \\
&= \varepsilon^{\kappa-2+2d}\int_s^\tau\int_{\mathbb{R}^{2d}} P_{r-s}(z_1 - y_1)P_{r-s}(z_2 - y_1') \\
&\qquad P_{\tau-r}(x_1 - z_1)P_{\tau-r}(x_1' - z_2)\varepsilon^{-\kappa}\big|R(\tfrac{z_1-z_2}{\varepsilon})\big|\,dzdr.
\end{aligned}$$

Returning to $\|A_{2,\varepsilon}\|_2$ and relabelling integrating variables[(3)], we conclude

$$\|A_{2,\varepsilon}\|_2 \lesssim \varepsilon^{\frac{\kappa}{2}-1}\int_0^{\tau\wedge v}\Big[\int_s^\tau\int_{\mathbb{R}^{10d}} P_{\tau-r}(x_1 - y_1)P_{\tau-r}(x_4 - y_4)$$

[(3)]Namely, we relabel (x_1, x_1', x_2, x_2') with (x_1, x_4, x_2, x_3), and $(z_1, z_2, y_1, y_1', y_2, y_2')$ with $(y_1, y_4, z_1, z_2, y_2, y_3)$.

$$P_{r-s}(y_1-z_1)P_{r-s}(y_4-z_2)\,\varepsilon^{-\kappa}\left|R(\tfrac{y_1-y_4}{\varepsilon})\right|\varepsilon^{-\kappa}\left|R(\tfrac{z_1-y_2}{\varepsilon})\right|\varepsilon^{-\kappa}\left|R(\tfrac{z_2-y_3}{\varepsilon})\right|$$
$$P_{v-s}(y_2-x_2)P_{v-s}(y_3-x_3)\left|\psi_{f,g}(x)\right|dydzdxdr\Big]^{\frac{1}{2}}ds. \tag{3.2.20}$$

Since $\varepsilon^{-\kappa}|R(\frac{x}{\varepsilon})|\lesssim|x|^{-\kappa}$, letting $t=v\wedge\tau$, we have

$$\begin{aligned}\|A_{2,\varepsilon}\|_2\lesssim\varepsilon^{\frac{\kappa}{2}-1}\int_0^t\Big[\int_s^\tau\int_{\mathbb{R}^{10d}}&|\psi_{f,g}(x)|\,P_{\tau-r}(x_1-y_1)P_{\tau-r}(x_4-y_4)\\&P_{r-s}(y_1-z_1)P_{r-s}(y_4-z_2)\,|y_1-y_4|^{-\kappa}|z_1-y_2|^{-\kappa}|z_2-y_3|^{-\kappa}\\&P_{v-s}(y_2-x_2)P_{v-s}(y_3-x_3)dydzdxdr\Big]^{\frac{1}{2}}ds.\end{aligned}$$

Applying Jensen's inequality (with respect to the normalised measure $\frac{ds}{t}$) and bringing the time integrals inside, we have

$$\begin{aligned}\|A_{2,\varepsilon}\|_2\lesssim\varepsilon^{\frac{\kappa}{2}-1}\sqrt{t}\Big[\int_{\mathbb{R}^{10d}}&|\psi_{f,g}(x)|\,|y_1-y_4|^{-\kappa}|z_1-y_2|^{-\kappa}|z_2-y_3|^{-\kappa}\\&\Big(\int_0^t P_{v-s}(y_2-x_2)P_{v-s}(y_3-x_3)\int_s^\tau P_{\tau-r}(x_1-y_1)P_{\tau-r}(x_4-y_4)\\&P_{r-s}(y_1-z_1)P_{r-s}(y_4-z_2)drds\Big)dydzdx\Big]^{\frac{1}{2}},\end{aligned}$$

obtaining the required inequality. □

3.2.4 Feynman diagrams integrals

We now bound the integrals appearing in the estimates of Lemmas 3.2.3-3.2.4, for this we employ the Feynman diagrams analysis introduced in Section 2.3.

Lemma 3.2.5. *Let $F\in\mathcal{C}_c(\mathbb{R}^{4d})$. Let $\lambda>0$, $\alpha>-d$, and $\beta>-d$ satisfy $\lambda+\alpha<\min(-\frac{1}{2}\beta,\frac{1}{2}d,-\beta)$. Then the following integral is finite,*

$$I_{\tilde{\Gamma}}(F;\lambda)=\int_{\mathbb{R}^{6d}}(|z_1-x_1||z_2-x_3|)^{\lambda-d}\,(|z_1-x_2||z_2-x_4|)^{\alpha}\,|z_1-z_2|^{\beta}\,|F(x)|\,dxdz. \tag{3.2.21}$$

Proof. The graph associated with the integral is

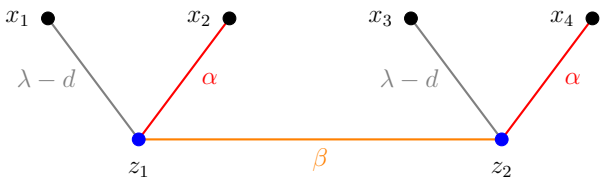


Diagram $\tilde{\Gamma}$

The small scale degree on each edge of the tree graph is clearly greater than the dimension $-d$, since there are no cycle to apply Lemma 2.3.2 we only need to check the large scale condition is satisfied (cf. Remark 2.3.3). Namely, for any tight partition $\mathcal{P}$,

$$\deg_\infty \mathcal{P} := \sum_{e\in\mathcal{E}(\mathcal{P}_\Gamma)} \deg_\infty \mathfrak{t}(e) + d(|\mathcal{P}_\Gamma| - 1) \ < \ 0,$$

where $\mathcal{E}(\mathcal{P})$ is the set of traversing edges associated to $\mathcal{P}$.

Since $\mathcal{V}_\ell = \{\bullet_{x_1}, \bullet_{x_2}, \bullet_{x_3}, \bullet_{x_4}\}$, the tight partitions are given by:

$$\mathcal{P}_1 = \{\mathcal{V}_\ell, \{\bullet_{z_1}\}, \{\bullet_{z_2}\}\}, \qquad \mathcal{P}_2 = \{\mathcal{V}_\ell, \{\bullet_{z_1}, \bullet_{z_2}\}\},$$
$$\mathcal{P}_3 = \{\{\mathcal{V}_\ell, \bullet_{z_1}\}, \{\bullet_{z_2}\}\}, \qquad \mathcal{P}_4 = \{\{\mathcal{V}_\ell, \bullet_{z_2}\}, \{\bullet_{z_1}\}\}.$$

Identifying the traversing edges corresponding to each partition, we compute

$$\deg_\infty \mathcal{P}_1 = 2\lambda - 2d + 2\alpha + \beta + 2d, \qquad \deg_\infty \mathcal{P}_2 = 2\lambda - 2d + 2\alpha + d,$$
$$\deg_\infty \mathcal{P}_3 = \deg_\infty \mathcal{P}_4 = \lambda - d + \alpha + \beta + d,$$

concluding the proof. □

Lemma 3.2.6. *For $F \in \mathcal{C}_c(\mathbb{R}^{4d})$, let*

$$I_{\Gamma_1}(F;\lambda) = \int_{\mathbb{R}^{8d}} \Big(\prod_{k=1}^{4} |y_k - x_k|^{\lambda-d}\Big) \tag{3.2.22}$$
$$|y_1 - y_4|^{2-\kappa}|y_1 - y_2|^{-\kappa}|y_3 - y_4|^{-\kappa}\,|F(x)|\,dxdy,$$

$$I_{\Gamma_2}(F;\lambda) = \int_{\mathbb{R}^{10d}} |y_1 - y_4|^{-\kappa}|z_1 - y_2|^{-\kappa}|z_2 - y_3|^{-\kappa} \tag{3.2.23}$$
$$|y_1 - z_1|^{\lambda-d}|y_4 - z_2|^{\lambda-d}\Big(\prod_{i=1}^{4} |x_i - y_i|^{\lambda-d}\Big)|F(x)|dydzdx.$$

Then $I_{\Gamma_1}(F;\lambda)$ is finite for any $\lambda \in (0, \frac{3\kappa-2}{4})$; and $I_{\Gamma_2}(F;\lambda)$ is finite for any $\lambda \in (0, \frac{\kappa}{2})$.

Proof. The diagram associated to the first integral is Γ_1 below, with the labels indicating the edge kernel degrees $\deg_0$ (which equals also $\deg_\infty$).

The integral $I_{\Gamma_1}(F;\lambda)$ simplifies after convolution / integrating out y_2 and y_3. For example, under our conditions, by (2.3.6) we have

$$\int |y_2 - x_2|^{\lambda-d}|y_1 - y_2|^{-\kappa}\, dy_2 = C|y_1 - x_2|^{\lambda-\kappa}.$$

As briefly explained in Section 2.3, this procedure collapses diagram Γ_1 in the sense of replacing $|y_2 - x_2|^{\lambda-d}|y_1 - y_2|^{-\kappa}$ with $|y_1 - x_2|^{\lambda-\kappa}$ corresponds to removing the node y_2 in the diagram. Similarly, node y_3 is removed.

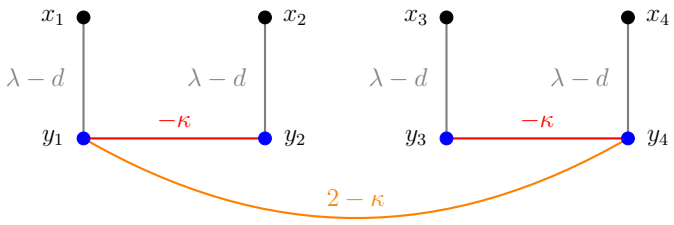


Diagram Γ_1

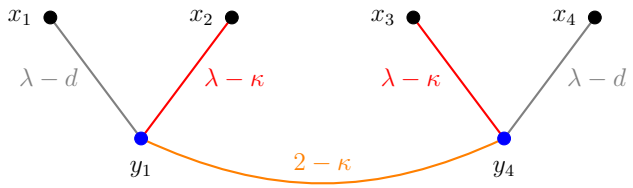


Diagram Γ_1 collapsed

This corresponds to reduce the integral $I_{\Gamma_1}(F;\lambda)$ as follows

$$I_{\Gamma_1}(F;\lambda) \lesssim \int_{\mathbb{R}^{6d}} (|y_1 - x_1||y_4 - x_4|)^{\lambda-d}\, (|y_1 - x_2||y_4 - x_3|)^{\lambda-\kappa} \\ |y_1 - y_4|^{2-\kappa}\, |F(x)|\, dy_1 dy_4 dx.$$

Hence, I_{Γ_1} is controlled by the integral represented in graph $\tilde{\Gamma}$ in the proof of Lemma 3.2.5, where we take $\alpha = \lambda - \kappa$ and $\beta = 2 - \kappa$. In other words, I_{Γ_1} is finite whenever $I_{\tilde{\Gamma}}$ is, and we conclude the proof by Lemma 3.2.5.

The diagram associated to I_{Γ_2} is as follows:

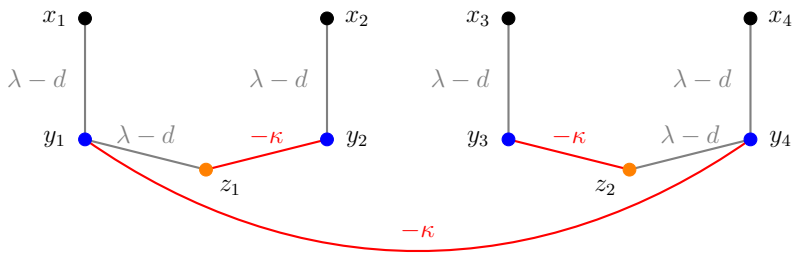


Diagram Γ_2

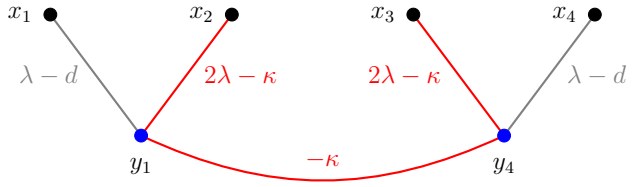


Diagram Γ_2 collapsed

We note that, integrating out y_2 , y_3 , z_1 and z_2, removing the associated nodes in Γ_2 reduces the graph to $\tilde{\Gamma}$ where $\alpha = 2\lambda - \kappa$ and $\beta = -\kappa$. Applying Lemma 3.2.5, we see that also I_{Γ_2} is finite and we conclude the proof. □

With the estimates of Lemmas 3.2.3-3.2.4 in place, we are going to recast the bounds for $\mathrm{var}(A_{1,\varepsilon})$ and for $\|A_{2,\varepsilon}\|_2$ in terms of the form $o(\varepsilon)I_{\Gamma_i}$, where I_{Γ_i} are finite by the Lemma just proved.

3.2.5 Vanishing of residual terms

We begin with the following basic estimate.

Lemma 3.2.7. *Let $w_1, w_2, x_1, x_2, y_1, y_2 \in \mathbb{R}^d \setminus \{0\}$. For any $\tau, v \geqslant t > 0$ and $\frac{1}{2} \leqslant \lambda < \frac{2}{3}$,*

$$J = \int_0^t P_{v-s}(w_1)P_{v-s}(w_2)\int_s^\tau P_{\tau-r}(x_1)P_{\tau-r}(x_2)P_{r-s}(y_1)P_{r-s}(y_2)drds$$
$$\lesssim t^{2-3\lambda}\Big(\prod_{i=1}^2 |w_i|^{\lambda-d}|x_i|^{\lambda-d}|y_i|^{\lambda-d}\Big) .$$

Proof. Since $\tau, v \geqslant t$, we can use the pointwise bound (2.4.4), yielding

$$J \lesssim \Big(\prod_{i=1}^{2} |w_i|^{\lambda-d}|x_i|^{\lambda-d}|y_i|^{\lambda-d}\Big) \int_0^t (v-s)^{-\lambda} \int_s^\tau (\tau - r)^{-\lambda}(r-s)^{-\lambda} dr ds. \tag{3.2.24}$$

For a constant c,

$$\begin{aligned} &\int_0^t (v-s)^{-\lambda} \int_s^\tau (\tau - r)^{-\lambda}(r-s)^{-\lambda} dr ds \\ &= c \int_0^t (v-s)^{-\lambda}(\tau - s)^{1-2\lambda} ds \leqslant \int_0^t (t-s)^{1-3\lambda} ds \lesssim t^{2-3\lambda}\,, \end{aligned}$$

this concludes the proof. We need $1 - 2\lambda \leqslant 0$ to replace $\tau - s$ by $t - s$. □

Collecting the estimates from the Lemmas above, we can now control the vanishing residual terms as required in the proof of Theorem 3.1.3.

Proposition 3.2.8. *Let $f, g \in \mathcal{C}_c(\mathbb{R}^d)$ and $\tau, v > 0$. Then, for any given λ in the region stated below, the following bounds hold uniformly in $\varepsilon \in (0,1)$:*

$$\operatorname{var}(A_{1,\varepsilon}) \lesssim \varepsilon^{\kappa-2}(\tau \wedge v)^{1-2\lambda} I_{\Gamma_1}(F;\lambda), \qquad \lambda \in (0, \tfrac{1}{2})\,, \tag{3.2.25a}$$

$$\|A_{2,\varepsilon}\|_2 \lesssim \varepsilon^{\frac{\kappa}{2}-1}(\tau \wedge v)^{\frac{3}{2}(1-\lambda)} [I_{\Gamma_2}(F;\lambda)]^{\frac{1}{2}}, \quad \lambda \in [\tfrac{1}{2}, \tfrac{2}{3}); \tag{3.2.25b}$$

where $F = \prod_{i=1}^4 F_i$, with $F_i \in \{f, g\}$.

Proof. Note that the integrals $I_{\Gamma_i}(F;\lambda)$ are finite by Lemmas 3.2.6. The proof of the proposition relies crucially on the bounds collected in Lemma 3.2.3-3.2.4. Firstly, we note

$$\begin{aligned} \operatorname{var}(A_{1,\varepsilon}) &\lesssim \varepsilon^{\kappa-2} t^{1-2\lambda} \sum_{\substack{i=1,2 \\ j=3,4}} \int_{\mathbb{R}^{8d}} \Big(\prod_{k=1}^4 |y_k - x_k|^{\lambda-d}\Big) |y_i - y_j|^{2-\kappa} \times \\ &\qquad |y_1 - y_2|^{-\kappa} |y_3 - y_4|^{-\kappa} \left|\varphi_{f,g}(x)\right| dx dy ds \\ &\lesssim \varepsilon^{\kappa-2}(\tau \wedge v)^{1-2\lambda} I_{\Gamma_1}(F;\lambda), \end{aligned}$$

where $F = \prod_{i=1}^4 F_i$, with $F_i \in \{f, g\}$ depending on the (i,j)-pair in the summation, with the integral I_{Γ_1} finite by Lemma 3.2.6.

For the second term, we use estimate from Lemma 3.2.4, to which we apply Lemma 3.2.7 with some fixed $\lambda \in [\frac{1}{2}, \frac{2}{3})$ to obtain with $F = \psi_{f,g}$,

$$\begin{aligned}\|A_{2,\varepsilon}\|_2 \lesssim \varepsilon^{\frac{\kappa}{2}-1} t^{\frac{3}{2}(1-\lambda)} \Bigg[\int_{\mathbb{R}^{10d}} |\psi_{f,g}(x)||y_1 - y_4|^{-\kappa} |z_1 - y_2|^{-\kappa} |z_2 - y_3|^{-\kappa} \\ |y_1 - z_1|^{\lambda-d} |y_4 - z_2|^{\lambda-d} \Big(\prod_{i=1}^{4} |x_i - y_i|^{\lambda-d} \Big) dy dz dx \Bigg]^{\frac{1}{2}} \\ = \varepsilon^{\frac{\kappa}{2}-1} (\tau \wedge v)^{\frac{3}{2}(1-\lambda)} \big[I_{\Gamma_2}(F, \lambda) \big]^{\frac{1}{2}}.\end{aligned}$$

This concludes the proof of Proposition 3.2.8. □

3.3 Tigthness criterion in $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha)$

We start by introducing notation and definitions of locally Hölder continuous distributions $\mathcal{C}^\alpha$ of negative exponent. Intuitively, a distribution ζ is said to be of Hölder regularity $\alpha < 0$ if, when tested against a rescaled test function g^λ (approximating unity as $\lambda \to 0$), it blows up like λ^α:

$$|\langle \zeta, g^\lambda \rangle| \approx \lambda^\alpha \, .$$

The formal definition is given below, and the corresponding seminorms are finite if the blow-up rate holds uniformly over *all* suitable Hölder continuous functions. The characterisation of $\mathcal{C}^\alpha$ spaces in terms of a *single* test function is key to control $L^p(\Omega)$ moments of the seminorms, and then prove our Kolmogorov-type embeddings in the following subsection.

3.3.1 Definitions and characterisation of negative Hölder spaces

Let $\mathcal{D}$ denote the space of smooth functions with compact support, and $\mathcal{D}'$ the dual space of all distributions. For a function $g \in \mathcal{C}(\mathbb{R}^d)$, $z \in \mathbb{R}^d$ and $\lambda > 0$, with g_x^λ we denote its transformation by scaling and translation as follows:

$$g_x^\lambda(y) := \lambda^{-d} g(\lambda^{-1}(y - x)) \, . \tag{3.3.1}$$

If $\lambda = 1$ or $x = 0$, the corresponding index will be dropped. Note that these operations are L^1 invariant.

For a given $\alpha < 0$, denote $\lceil -\alpha \rceil \geqslant -\alpha$ the smallest (positive) integer bigger than $-\alpha$. We shall use the following subset of $\mathcal{C}^r(\mathbb{R}^d)$ test functions:

$$\mathcal{B}_r := \{g \in \mathcal{D} : \mathrm{supp}(g) \subset B_1, \|g\|_{\mathcal{C}^r} \leqslant 1\} ,$$

where B_a denote the open ball of radius a centred at 0.

Let $\mathcal{C}^\alpha \subset \mathcal{D}'$ denote the subset of locally Hölder continuous distributions defined as follows. Note that they are not necessarily tempered distributions.

Definition 3.3.1. [Hai14b, Def. 3.7] A distribution ζ is said to be in $\mathcal{C}^\alpha$, if for every compact set $E \subset \mathbb{R}^d$, the following

$$\|\zeta\|_{\alpha;E} := \sup_{\eta \in \mathcal{B}_r} \sup_{\lambda \in (0,1)} \sup_{x \in E} \lambda^{-\alpha} |\langle \zeta, \eta_x^\lambda \rangle|, \tag{3.3.2}$$

is finite, where $r = \lceil -\alpha \rceil$.

For any compact $E \subset \mathbb{R}^d$, we denote by $\mathcal{C}^\alpha(E)$ the space of distributions such that the seminorm in (3.3.2) is finite.

Given the family of semi-norms, the space $\mathcal{C}^\alpha$ is metrisable, with

$$d_\alpha(\zeta, \zeta') \stackrel{\text{def}}{=} \sum_{m \geqslant 1} 2^{-m} \big(1 \wedge \|\zeta - \zeta'\|_{\alpha;\bar{B}_m}\big) , \tag{3.3.3}$$

making it a Fréchet space. As metric spaces, $\mathcal{C}^{\alpha'}$ is compactly embedded in $\mathcal{C}^\alpha$ for any $\alpha' > \alpha$.

There are a number of useful characterizations for the spaces $\mathcal{C}^\alpha$. They can be characterised in terms of wavelets [Hai14b, Sec 3.1-2], but we choose here to use the characterisation from [CZ20] which only requires us to test against translations and rescalings of one single essentially arbitrary test function.

For any function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we introduce the notation for its rescaled version at scale 2^{-n}:

$$\varphi^{(n)} = 2^{nd} \varphi(2^n \cdot) , \qquad \varphi_x^{(n)} = 2^{nd} \varphi\big(2^n (\cdot - x)\big) .$$

We also write $\bar{\mathcal{D}} \subset \mathcal{D}$ for the set of test functions φ such that $\operatorname{supp}\varphi \subset B_1$ and $\int \varphi = 1$. We now fix such function $\varphi \in \bar{\mathcal{D}}$ once and for all for the remaining chapters. Our key tool is the following analytic criterion to characterise $\mathcal{C}^\alpha$ spaces. Although φ is *fixed*, this shows that control on $\langle \zeta, \varphi_x^{(k)} \rangle$ allows to get a similar bound for *all* test functions and control the seminorm (3.3.2).

Theorem 3.3.2 ([CZ20, Theorem 12.4])**.** *Let $\zeta \in \mathcal{D}'$ be a distribution and let $\alpha \in (-\infty, 0]$. Then, one has $\zeta \in \mathcal{C}^\alpha$ if and only if, for any compact set E and uniformly over $n \in \mathbb{N}$,*

$$\sup_{x \in E} |\langle \zeta, \varphi_x^{(n)} \rangle| \lesssim 2^{-n\alpha}.$$

Furthermore, the semi-norms of ζ can be estimated by

$$\|\zeta\|_{\alpha;E} \leqslant C \sup_{x \in E_2} \sup_{n \in \mathbb{N}} 2^{n\alpha} |\langle \zeta, \varphi_x^{(n)} \rangle| \tag{3.3.4}$$

where $C = C(\varphi, \alpha, d)$ is an explicit constant and $E_2 = \{z : d(z, E) \leqslant 2\}$.

The above criterion immediately implies the following control in $\mathcal{C}^\alpha$ spaces.

Lemma 3.3.3. *Let $p \geqslant 1$ and $\alpha < 0$. Let ζ be a random distribution. If for any compact E there exists a constant C_E such that*

$$\sup_n 2^{\alpha n} \left\| \sup_{x \in E} |\langle \zeta, \varphi_x^{(n)} \rangle| \right\|_p \leqslant C_E, \tag{3.3.5}$$

then, for any $\alpha' < \alpha$, $\zeta \in \mathcal{C}^{\alpha'}$ almost surely and

$$\|\zeta\|_{L^p(\Omega, \mathcal{C}^{\alpha'}(E))} \leqslant C_{\alpha', E_2},$$

for an explicit constant $C_{\alpha', E_2} = C(\varphi, \alpha, d, \alpha', E_2)$.

Proof. Setting $\zeta_n(x) = \langle \zeta, \varphi_x^{(n)} \rangle$, it follows from (3.3.4) that

$$\left\| \|\zeta\|_{\alpha', E} \right\|_p \leqslant C \left\| \sup_{n \geqslant 1} \sup_{x \in E_2} 2^{\alpha' n} |\zeta_n(x)| \right\|_p \leqslant C \sum_{n \geqslant 1} 2^{(\alpha' - \alpha)n} \left\| \sup_{x \in E_2} 2^{\alpha n} |\zeta_n(x)| \right\|_p.$$

By (3.3.5), we conclude letting $C_{\alpha', E_2} = C(\varphi, \alpha, d) C_{E_2} \sum_{n=1}^\infty 2^{(\alpha' - \alpha)n}$. □

3.3.2 Embeddings and tightness criterion

Let us now define a space of distribution-valued processes with spatial Hölder regularity when tested against translates of the functions $\varphi^{(n)}$.

Definition 3.3.4. A random distribution ζ is said to belong to $\mathcal{C}_p^{\alpha_0}$ if there exists $C > 0$ such that the following estimates hold for any $n \geqslant 1$:

$$\sup_{x\in\mathbb{R}^d} \left\| \langle \zeta, \varphi_x^{(n)} \rangle \right\|_p \leqslant C 2^{-n\alpha_0}, \tag{3.3.6}$$

$$\sup_{\substack{x,y\in\mathbb{R}^d, \\ 0<|x-y|\leqslant 2^{-n}}} \left\| \langle \zeta, \varphi_x^{(n)} - \varphi_y^{(n)} \rangle \right\|_p \leqslant C 2^{-n(\alpha_0-1)} |x-y| \,, \tag{3.3.7}$$

where $|\cdot|$ denotes the sup norm on $\mathbb{R}^d$. The smallest such constant C is denoted by $\|\zeta\|_{\mathcal{C}_p^{\alpha_0}}$. We write $\mathcal{C}_p^{\gamma_0,\alpha_0}$, as a shorthand, for the space $\mathcal{C}^{\gamma_0}(\mathbb{R}_+, \mathcal{C}_p^{\alpha_0})$ and, with $T > 0$ finite constant, $\mathcal{C}_{p,T}^{\gamma_0,\alpha_0}$ for the space $\mathcal{C}^{\gamma_0}([0,T], \mathcal{C}_p^{\alpha_0})$.

Remark 3.3.5. Note that if (3.3.6) holds uniformly over the class of test functions $\bar{\mathcal{D}}$, then (3.3.7) follows immediately just like in the proof of Lemma 3.4.6.

Remark 3.3.6. For simplicity and our applications, in the definition we consider global spatial estimates in (3.3.6)-(3.3.7), which are tailored to the embeddings in Proposition 3.3.7 below. One could easily remodulate proofs to show analogous embeddings imposing weaker local conditions (with constant(s) dependent on the compact set of reference), tuning the weights of the d_α metric (3.3.3) based on how the seminorms grow.

The following allows for redistribution of regularity across time and space, leading to the tightness criterion (Proposition 3.3.9) for distribution valued processes with positive Hölder continuous trajectory in time. This is very similar to other Kolmogorov-type criteria as for example in [Hai14b, Thm 10.7], [CCM24, Prop. 2.4], or [FM17, Thm 1.1], but given the range of relation between the spatial and temporal regularity exponents, it doesn't seem to be an immediate corollary of one of these results.

Proposition 3.3.7. *Let $\alpha_0 < 0$, $p > d$, and $\gamma_0 > \frac{1}{p}$. Then, for any $\gamma \in (0, \gamma_0 - \frac{1}{p})$, $\alpha < \alpha_0 - \frac{d}{p}$ and any compact $E \subset \mathbb{R}^d$, one has the continuous embeddings*

$$\mathcal{C}_p^{\alpha_0} \subset L^p(\Omega, \mathcal{C}^\alpha(E)) \;, \qquad \mathcal{C}_p^{\gamma_0,\alpha_0} \subset L^p(\Omega, \mathcal{C}^\gamma(\mathbb{R}_+, \mathcal{C}^\alpha(E))) \;.$$

Viewing $\mathcal{C}^\alpha$ as the metric space equipped with the distance d_α as in (3.3.3), it follows that

$$\mathcal{C}_p^{\gamma_0,\alpha_0} \subset L^p(\Omega, \mathcal{C}^\gamma(\mathbb{R}_+, \mathcal{C}^\alpha)) \;.$$

Remark 3.3.8. If we consider a process on a finite interval $[0, T]$, the same proof clearly holds and $\mathcal{C}_{p,T}^{\gamma_0,\alpha_0}$ is continuously embedded in $L^p(\Omega, \mathcal{C}^\gamma([0, T], \mathcal{C}^\alpha(E)))$.

Proof. We will show that, given any compact E,

$$\big\| \|\zeta\|_{\alpha;E} \big\|_p \lesssim \|\zeta\|_{\mathcal{C}_p^{\alpha_0}} (1 + \mathrm{diam}(E_2))^{\frac{d}{p}} \;. \tag{3.3.8}$$

This implies the first embedding, but also

$$\big\| \|\zeta_r - \zeta_t\|_{\alpha;E} \|_p \lesssim \|\zeta_r - \zeta_t\|_{\mathcal{C}_p^{\alpha_0}} \leqslant |t - r|^{\gamma_0} \|\zeta\|_{\mathcal{C}_p^{\gamma_0,\alpha_0}} \;,$$

where we dropped constant dependent on E. Then, we see that the second embedding follows from the usual Kolmogorov continuity theorem.

Let us now focus on (3.3.8) and set $C = \|\zeta\|_{\mathcal{C}_p^{\alpha_0}}$ going forward. We denote with $C_x^{(n)}$ the cube of sidelength 2^{1-n} centred at x and set $\zeta^n(x) = \langle \zeta, \varphi_x^{(n)} \rangle$. We note that by (3.3.7) the rescaled process $\tilde{\zeta}_x^n(z) := \zeta^n(2^{-n}(z - x))$ satisfies $\|\tilde{\zeta}_x^n(y) - \tilde{\zeta}_x^n(z)\|_p \leqslant C 2^{-n\alpha_0} |y - z|$ for $|y - z| \leqslant 1$. By Kolmogorov's criterion, for any $x \in \mathbb{R}^d$ we obtain

$$\Big\| \sup_{y \in C_0^{(0)}} |\tilde{\zeta}_x^n(y) - \tilde{\zeta}_x^n(0)| \Big\|_p \lesssim C 2^{-n\alpha_0} \;.$$

When combined with (3.3.6), this in turn yields

$$\big\| \sup_{y \in C_x^{(n)}} |\zeta^n(y)| \big\|_p \lesssim C 2^{-n\alpha_0} \;.$$

Given a compact set $E \subset \mathbb{R}^d$, write $E_n \subset 2^{-n}\mathbb{Z}^d$ for the smallest set so that $E \subset \bigcup_{x\in E_n} C_x^{(n)}$. It follows that

$$\Big\|\sup_{y\in E}|\zeta^n(y)|\Big\|_p \leqslant (\#E_n)^{\frac{1}{p}} \sup_{x\in E_n}\Big\|\sup_{y\in C_x^{(n)}}|\zeta^n(y)|\Big\|_p .$$

To summarise, we have the estimate:

$$\Big\|\sup_{y\in E}|\langle\zeta,\varphi_y^{(n)}\rangle|\Big\|_p \lesssim C(1+2^n\operatorname{diam}(E))^{\frac{d}{p}}2^{-n\alpha_0} . \tag{3.3.9}$$

In particular, we can estimate as in Lemma 3.3.3 to obtain

$$\big\|\|\zeta\|_{\alpha;E}\big\|_p \lesssim \sum_{n\geqslant 1} 2^{(\alpha-\alpha_0)n}\Big\|\sup_{y\in E_2}2^{n\alpha}|\zeta^n(y)|\Big\|_p \lesssim C(1+\operatorname{diam}(E_2))^{\frac{d}{p}} . \tag{3.3.10}$$

Thus, (3.3.8) follows, concluding the proof of the first two embeddings.

Consider the metric space $\mathcal{C}^\alpha$, given the above we have

$$\begin{aligned}\|d_\alpha(\zeta_r,\zeta_t)\|_p &\leqslant \sum_{m\geqslant 1}2^{-m}\big(1\wedge\big\|\|\zeta_r-\zeta_t\|_{\alpha;B_{m+2}}\big\|_p\big)\\ &\lesssim \|\zeta_r-\zeta_t\|_{\mathcal{C}_p^{\alpha_0}}\sum_{m\geqslant 1}2^{-m}(1+\operatorname{diam}(B_{m+2})^{\frac{d}{p}})\\ &\lesssim |t-r|^{\gamma_0}\|\zeta\|_{\mathcal{C}_p^{\gamma_0,\alpha_0}} .\end{aligned}$$

Then last embedding immediately follows again by Kolmogorov's continuity. □

As a corollary, we obtain the following tightness criterion.

Proposition 3.3.9 (Tightness criterion)**.** *Let $\{\zeta^k, k\in\mathcal{I}\}$ be a family of distribution-valued stochastic processes such that $\sup_k\|\zeta^k\|_{\mathcal{C}_p^{\gamma_0,\alpha_0}}<\infty$ for some $\alpha_0<0$, $p>d$, $\gamma_0>\frac{1}{p}$. Then, for any $T>0$, any $\alpha<\alpha_0-\frac{d}{p}$, and any $\gamma<\gamma_0-\frac{1}{p}$, $\{\zeta^k, k\in\mathcal{I}\}$ is tight in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$.*

Proof. For $\gamma<\gamma_0-\frac{1}{p}$, by Proposition 3.3.7, $\big\|\|\zeta\|_{\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)}\big\|_p$ is uniformly bounded in k. Tightness follows from the fact that $\mathcal{C}^{\gamma'}([0,T],\mathcal{C}^{\alpha'})$ is compactly embedded in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$ for any $0<\gamma'<\gamma$ and $\alpha'<\alpha$. □

3.4 Convergence in Hölder topologies

With the setup and embedding(s) of the previous section, we can now prove weak convergence with respect to Hölder topologies. We start with the proof of the second main Theorem 3.1.5. We shall prove the required estimates and tightness in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha(\mathbb{R}^d))$ in the following sections.

3.4.1 Proof of Theorem 3.1.5

We recall $\kappa \in (2,d)$ is the decay exponent of the spatial covariance R, $\mathcal{U}_t(g)$ denotes the solution of (3.1.4) at time $t>0$ tested against a test function g. Let $\dot{W}^\kappa$ denotes the centred Gaussian noise with spatial covariance $|x|^{-\kappa}$,

$$\mathcal{U}_t(g) = \beta \int_0^t \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} P_{t-s}(x-y)g(x)dx \Big) dW^\kappa(s,y). \tag{3.4.1}$$

Let us also recall

$$X_t^{\varepsilon,g} = \varepsilon^{1-\frac{\kappa}{2}} \int_{\mathbb{R}^d} \big(u_\varepsilon(t,x)-1\big)g(x)dx.$$

In this section we show that X^ε converges weakly to $\nu_{\text{eff}}\,\mathcal{U}$, as distribution valued stochastic processes with Hölder continuous in time trajectories. Throughout this section, $\gamma \in (0,\frac{1}{2})$ will parametrise Hölder continuity in time and $\alpha < 1-\frac{\kappa}{2}-2\gamma$ the distributional regularity.

We now prove Theorem 3.1.5, which we restate here for the convenience of the reader.

Theorem 3.4.1. *Let $d \geqslant 3$. Suppose that ξ and σ, satisfy Assumption 3.1.1. Then, for any $\gamma \in (0,\frac{1}{2})$ and $\alpha < 1-\frac{\kappa}{2}-2\gamma$, there exists $\beta_{\gamma,\alpha}>0$ such that, for any $\beta \leqslant \beta_{\gamma,\alpha}$ and $T>0$, X^ε converges to $\nu_{\text{eff}}\,\mathcal{U}$ weakly in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$.*

Proof. The law of a distribution-valued random variable $\zeta \in \mathcal{D}'$ is uniquely determined by the family of probability distributions of random vectors for the form $(\zeta(g_1),\dots,\zeta(g_n))$, where $n \in \mathbb{N}$, and $\{g_i\}_{i=1,\dots,n} \subset \mathcal{D}$ are smooth compactly supported functions (cf. e.g. [Fer67, Prop. III.4.2b, Thm. III.6.5]).

Observe that $\mathcal{C}^\alpha \subset \mathcal{D}'$, then the law of $\zeta \in \mathcal{C}^\alpha$ is uniquely determined by such distributions. In Theorem 3.1.3, we have shown that for $\beta < \beta_0(4) \wedge \beta_1(2)$, for any $0 < t_1 \leqslant t_2 \leqslant \dots t_n \leqslant T$, any $\{g_i\}_{i=1,\dots,n} \subset \mathcal{D}$, as $\varepsilon \to 0$, the following convergence holds in distribution:

$$(X_{t_1}^{\varepsilon,g_1}, \dots, X_{t_n}^{\varepsilon,g_n}) \Rightarrow \nu_{\text{eff}}(\mathcal{U}_{t_1}(g_1), \dots, \mathcal{U}_{t_n}(g_n)).$$

This identifies the law of the accumulation points of X^ε, as a random variable in $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha)$.

It remains to show tightness. In the previous section we characterised $\mathcal{C}^\alpha$ spaces and showed Proposition 3.3.7, a Kolmogorov-type criterion for $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha(\mathbb{R}^d))$ processes. To apply this, in section 3.4.2, we prove some elementary estimates to be used in Lemmas 3.4.6-3.4.7. With these, we identify a range of exponents α and γ (cf. Remark 3.4.2 below for some heuristics on these) so that $\{X^\varepsilon, \varepsilon \in (0,1]\}$ is tight for sufficiently small $\beta < \beta_{\gamma,\alpha}$. Hence, in Proposition 3.4.8, items ((i)) and ((ii)a), we control in L^p the $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha(\mathbb{R}^d))$ norms of $\mathcal{U}$ and X^ε, uniformly in ε. Finally, in ((ii)b) of Proposition 3.4.8, we show $\{X^\varepsilon, \varepsilon \in (0,1]\}$ is tight. This concludes the proof of Theorem 3.1.5. ≣

Remark 3.4.2. The admissible range of Hölder exponents can be derived as follows. One can verify that $\dot{W}^\kappa$ satisfies the bound

$$\|\langle \dot{W}^\kappa, \lambda^{-1}\eta^{-d}\psi(\lambda^{-1}\cdot, \eta^{-1}\cdot)\rangle\|_p \lesssim \lambda^{-\frac{1}{2}}\eta^{-\frac{\kappa}{2}}, \qquad \lambda, \eta \in (0,1],$$

uniformly over all test functions ψ that are compactly supported in some ball of unit radius and bounded by 1. Since this appears on the right-hand side of a heat equation, one would expect the regularity of solutions to be at best improved by one time derivative, two space derivatives, or a mixture of both. In other words, one would expect solutions u to be such that, for every $\theta \in [0,1]$ and test function ψ as before

$$\|\langle u, \lambda^{-1}\eta^{-d}\psi(\lambda^{-1}\cdot, \eta^{-1}\cdot)\rangle\|_p \lesssim \lambda^{\theta-\frac{1}{2}}\eta^{2-2\theta-\frac{\kappa}{2}},$$

thus suggesting that the range of exponents in Theorem 3.4.1 is sharp, except that one could probably allow for $\gamma \in (-\frac{1}{2}, \frac{1}{2})$.

3.4.2 Some elementary estimates

We collect the following elementary bounds. The estimates of Lemma 3.4.5 will determine the (sharp) range of Hölder exponents.

Lemma 3.4.3. *For any compactly supported bounded $f : \mathbb{R}^d \to \mathbb{R}^m$, define*

$$J_{\kappa,\delta}(f) := \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^{2} |f(x_i)||x_i - y_i|^{\delta - d} \Big) |y_1 - y_2|^{-\kappa} dy dx. \tag{3.4.2}$$

Denoting M the diameter of the support f, then for $\delta \in (0, \frac{\kappa}{2})$,

$$J_{\kappa,\delta}(f) \lesssim M^{2d+2\delta-\kappa} \|f\|_\infty^2 .$$

Furthermore, with $f_x^\lambda(y) = \lambda^{-d} f(\frac{y-x}{\lambda})$, one has the scaling relation

$$J_{\kappa,\delta}(f_x^\lambda) = \lambda^{2\delta-\kappa} J_{\kappa,\delta}(f) . \tag{3.4.3}$$

Proof. Recall (2.3.6), we use the fact that

$$\int_{\mathbb{R}^d} |x-y|^\alpha |y-w|^\beta dy = c_{\alpha,\beta} |w-x|^{\alpha+\beta+d},$$

provided $\alpha + \beta + d < 0$ and $\alpha < -d$, $\beta < -d$. Then, for a universal constant c, we have

$$J_{\kappa,\delta}(f) = c \int_{\mathbb{R}^{2d}} |f(x_1)||f(x_2)||x_1 - x_2|^{2\delta-\kappa} \, dx.$$

From this, the inequality and scaling relation follow. □

Definition 3.4.4. For $g : \mathbb{R}^d \to \mathbb{R}$, set $g^\lambda(y) := \lambda^{-d} g(\frac{y}{\lambda})$. Define

$$\begin{aligned} I_A(r,t) = \int_r^t \int_{\mathbb{R}^{2d}} |y_1 - y_2|^{-\kappa} \\ \Big(\prod_{i=1}^{2} \int_{\mathbb{R}^d} P_{t-s}(x_i - y_i) |g^\lambda(x_i)| dx_i \Big) \, dy ds, \end{aligned} \tag{3.4.4}$$

$$\begin{aligned} I_B(r,t) = \int_0^r \int_{\mathbb{R}^{2d}} |y_1 - y_2|^{-\kappa} \\ \Big(\prod_{i=1}^{2} \int_{\mathbb{R}^d} \big| P_{t-s}(x_i - y_i) - P_{r-s}(x_i - y_i) \big| |g^\lambda(x_i)| dx_i \Big) \, dy ds. \end{aligned}$$

Lemma 3.4.5. *For any $\delta \in (0,1)$, $\gamma \in (0, 1-\delta)$, $\lambda \in (0,1]$, $0 < r \leqslant t$, and $g \in \mathcal{C}_c^1$, we have:*

$$I_A(r,t) \lesssim M^{2d+2\delta-\kappa} \ |t-r|^{1-\delta} \lambda^{2\delta-\kappa} \|g\|_\infty^2, \tag{3.4.5}$$

$$I_B(r,t) \lesssim M^{2d+2\delta-\kappa} \, r^{1-\delta-\gamma} |t-r|^\gamma \lambda^{2\delta-\kappa} \|g\|_\infty^2, \tag{3.4.6}$$

where M denotes the diameter of the support of g.

Proof. By change of time variable and the heat kernel estimate (2.4.4) with λ there taken to be δ, we have

$$\begin{aligned} I_A(r,t) &= \int_{\mathbb{R}^{4d}} \int_0^{t-r} \Big(\prod_{i=1}^2 P_s(x_i - y_i) \Big) ds \, \Big(\prod_{i=1}^2 |g^\lambda(x_i)| \Big) |y_1 - y_2|^{-\kappa} dy dx \\ &\lesssim \int_0^{t-r} s^{-\delta} ds \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^2 |g^\lambda(x_i)| |x_i - y_i|^{\delta - d} \Big) |y_1 - y_2|^{-\kappa} dy dx \\ &\lesssim (t-r)^{1-\delta} J_{\kappa,\delta}(g^\lambda) = (t-r)^{1-\delta} \lambda^{2\delta-\kappa} J_{\kappa,\delta}(g). \end{aligned}$$

Since $\delta < 1 < \frac{\kappa}{2}$, (3.4.5) holds by Lemma 3.4.3.

To estimate I_B, we use Hölder continuity in time of the heat kernel. For any $t \geqslant r > 0$ we obtain from the second part of Lemma 2.4.3:

$$|P_t(x) - P_r(x)| \lesssim (t-r)^{\frac{\gamma}{2}} r^{-\frac{\gamma+\delta}{2}} |x|^{\delta - d} \, .$$

Inserting this bound into (3.4.4), we obtain

$$\begin{aligned} &I_B(r,t) \\ &\lesssim (t-r)^\gamma \int_0^r (r-s)^{-\gamma-\delta} \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^2 |x_i - y_i|^{\delta-d} |g^\lambda(x_i)| \Big) |y_1 - y_2|^{-\kappa} dy dx ds \\ &\lesssim (t-r)^\gamma r^{1-\gamma-\delta} J_{\kappa,\delta}(g^\lambda) \, , \end{aligned}$$

so that (3.4.6) follows. □

3.4.3 Tightness in $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha)$

In the next two lemmas we gather the time increments and spatial regularity bounds for $\mathcal{U}$ and X^ε, allowing us to identify γ and α such that the required tightness holds in $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha)$. Recall the notation $g_x^\lambda(y) := \lambda^{-d} g(\frac{y-x}{\lambda})$.

Lemma 3.4.6. *Let $p \geqslant 2, \kappa \in (2,d)$, $\gamma \in (0,\frac{1}{2})$, $\alpha + 2\gamma < 1 - \frac{\kappa}{2}$. Set $\mathcal{U}_{r,t} = \mathcal{U}_t - \mathcal{U}_r$. Then, the following holds, for any $0 < r < t \leqslant T$, $x, y \in \mathbb{R}^d$ and $\lambda > 0$, and any $g \in \mathcal{C}_c^1$,*

$$\|\mathcal{U}_{r,t}(g_x^\lambda)\|_p \lesssim \beta M^{d+\alpha}|t-r|^\gamma \lambda^\alpha \|g\|_\infty \,, \tag{3.4.7}$$

$$\|\mathcal{U}_{r,t}(g_x^\lambda - g_y^\lambda)\|_p \lesssim \beta M^{d+\alpha}|t-r|^\gamma \lambda^{\alpha-1}|g|_{\mathrm{Lip}}|x-y| \,, \tag{3.4.8}$$

where M is the diameter of the support of g.

Proof. By Gaussianity, it is sufficient to work with $p = 2$. By the invariance in law of $\mathcal{U}$ under spatial translation, we have $\mathcal{U}_t(g_x^\lambda) \overset{\text{law}}{=} \mathcal{U}_t(g^\lambda)$, so we may take $x = 0$ for the first inequality. Dividing the integration region $[0,t] = [0,r] \cup [r,t]$, using Itô isometry, we compute the second moment of $\mathcal{U}_{r,t}(g^\lambda)$. Firstly, recall (3.4.1),

$$\mathcal{U}_t(g) = \beta \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} P_{t-s}(x-y) g(x) dx \right) dW^\kappa(s,y).$$

Then,

$$\begin{aligned}
&\|\mathcal{U}_t(g^\lambda) - \mathcal{U}_r(g^\lambda)\|_2^2 \\
&\quad = \beta^2 \mathbb{E}\bigg(\int_{\mathbb{R}^{1+d}} \int_{\mathbb{R}^d} \Big(\mathbf{1}_{[0,t]}(s) P_{t-s}(z-y) - \mathbf{1}_{[0,r]}(s) P_{r-s}(z-y) \Big) \\
&\qquad\qquad \times g^\lambda(z) dz \; dW^\kappa(s,y) \bigg)^2 \\
&\quad \leqslant \beta^2 \big(I_A(r,t) + I_B(r,t) \big),
\end{aligned}$$

where I_A and I_B are defined in (3.4.4) above. According to Lemma 3.4.5, for $\tilde{\gamma} \in (0, 1-\delta)$ and $\delta, \lambda \in (0,1)$, we have:

$$\begin{aligned}
\|\mathcal{U}_{r,t}(g^\lambda)\|_2 &\leqslant \beta (|t-r|^{1-\delta} + r^{1-\delta-\tilde{\gamma}}|t-r|^{\tilde{\gamma}})^{\frac{1}{2}} M^{d+\delta-\frac{\kappa}{2}} \lambda^{\delta-\frac{\kappa}{2}} \|g\|_\infty \\
&\lesssim \beta \; M^{d+\delta-\frac{\kappa}{2}} \; |t-r|^{\frac{\tilde{\gamma}}{2}} \lambda^{\delta-\frac{\kappa}{2}} \|g\|_\infty.
\end{aligned}$$

Inequality (3.4.7) follows from setting $\gamma = \tilde{\gamma}/2$ and $\alpha := \delta - \frac{\kappa}{2}$. The constraint becomes $\alpha + 2\gamma < 1 - \frac{\kappa}{2}$.

In order to obtain the bound on $\|\mathcal{U}_{r,t}(g_x^\lambda - g_y^\lambda)\|_p$, we first assume that $|x-y| \geq \lambda M$. It then trivially follows from applying (3.4.7) separately to

g_x^λ and g_y^λ, whose diameter of support is bounded by M, and observing that $\lambda^\alpha \leqslant \lambda^{\alpha-1}|x-y|M^{-1}$, as well as $\|g\|_\infty \lesssim M|g|_{\mathrm{Lip}}$.

In the case $|x-y| \leqslant \lambda M$, it then suffices to note that

$$g_x^\lambda - g_y^\lambda = \tilde{g}_x^\lambda\,, \qquad \tilde{g} = g - g_{\frac{y-x}{\lambda}}\,,$$

and that $\|\tilde{g}\|_\infty \leqslant \lambda^{-1}|g|_{\mathrm{Lip}}|x-y|$. Furthermore, the diameter of the support of $\tilde{g}$ is bounded by twice that of g, so the bound follows again from (3.4.7). □

Lemma 3.4.7. *Let $d \geqslant 3$, $p \geqslant 2$, σ be Lipschitz continuous, $\beta < \beta_0(p)$, suppose that ξ satisfies Assumption 2.0.1, and set $X_{r,t}^\varepsilon = X_t^\varepsilon - X_r^\varepsilon$. For any $\gamma \in (0, \frac{1}{2})$, any $\alpha < 1 - \frac{\kappa}{2} - 2\gamma$ and any $g \in \mathcal{C}_c^1$ with diameter of its support equal to M, the following holds for any $0 \leqslant r < t \leqslant T$, $x, y \in \mathbb{R}^d$ and $\lambda > 0$,*

$$\sup_{\varepsilon\in(0,1)} \|X_{r,t}^{\varepsilon,g_x^\lambda}\|_p \lesssim \beta M^{d+\alpha}\,|t-r|^\gamma \lambda^\alpha\,\|g\|_\infty\,, \tag{3.4.9}$$

$$\sup_{\varepsilon\in(0,1)} \|X_{r,t}^{\varepsilon,g_x^\lambda - g_y^\lambda}\|_p \lesssim \beta M^{d+\alpha}|t-r|^\gamma \lambda^{\alpha-1}|g|_{\mathrm{Lip}}|x-y|\,, \tag{3.4.10}$$

uniformly over $\varepsilon \in (0,1]$.

Proof. Given spatial stationarity of $u_\varepsilon(t,x)$, we only need to work with g^λ, without the shifts. Note that

$$X_t^{\varepsilon,g^\lambda} - X_r^{\varepsilon,g^\lambda} = \varepsilon^{1-\frac{\kappa}{2}} \int_{\mathbb{R}^d} \big(u_\varepsilon(t,x) - u_\varepsilon(r,x)\big) g^\lambda(x)dx = \beta\varepsilon^{1-\frac{\kappa}{2}}\big(\mathcal{A}_{r,t}^\varepsilon + \mathcal{B}_{r,t}^\varepsilon\big),$$

where

$$\mathcal{A}_{r,t}^\varepsilon = \int_{\frac{r}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d} \sigma(u(s,y)) \int_{\mathbb{R}^d} P_{\frac{t}{\varepsilon^2}-s}(\tfrac{x}{\varepsilon} - y) g^\lambda(x)dx\,\xi(ds,dy),$$

$$\mathcal{B}_{r,t}^\varepsilon = \int_0^{\frac{r}{\varepsilon^2}} \int_{\mathbb{R}^d} \sigma(u(s,y)) \int_{\mathbb{R}^d} \big(P_{\frac{t}{\varepsilon^2}-s}(\tfrac{x}{\varepsilon} - y) - P_{\frac{r}{\varepsilon^2}-s}(\tfrac{x}{\varepsilon} - y)\big) g^\lambda(x)dx\,\xi(ds,dy).$$

In contrast to $\mathcal{U}_t(g)$, these processes are not Gaussian, but we can estimate their pth moments via the BDG and Minkowski inequalities, followed by a

change of variables:

$$\begin{aligned}
&\mathbb{E}[(\mathcal{A}^{\varepsilon}_{r,t})^p] \lesssim \mathbb{E}[\langle \mathcal{A}^{\varepsilon}_{r,\cdot}\rangle_t^{\frac{p}{2}}] \\
&= \mathbb{E}\Big[\Big(\int_{\frac{r}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\int_{\mathbb{R}^{4d}}\Big(\prod_{i=1}^{2}\sigma(u(s,y_i))P_{\frac{t}{\varepsilon^2}-s}(\tfrac{x_i}{\varepsilon}-y_i)g^{\lambda}(x_i)\Big)R(y_1-y_2)dxdyds\Big)^{\frac{p}{2}}\Big] \\
&\leqslant \Big(\int_{\frac{r}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\int_{\mathbb{R}^{4d}}\Big\|\prod_{i=1}^{2}\sigma(u(s,y_i))\Big\|_{\frac{p}{2}}\Big(\prod_{i=1}^{2}P_{\frac{t}{\varepsilon^2}-s}(\tfrac{x_i}{\varepsilon}-y_i)|g^{\lambda}(x_i)|\Big)|R(y_1-y_2)|dxdyds\Big)^{\frac{p}{2}} \\
&\leqslant \Big(\varepsilon^{\kappa-2}\int_{r}^{t}\int_{\mathbb{R}^{4d}}\Big\|\prod_{i=1}^{2}\sigma(u_\varepsilon(s,y_i))\Big\|_{\frac{p}{2}}\Big(\prod_{i=1}^{2}P_{t-s}(x_i-y_i)|g^{\lambda}(x_i)|\Big)\varepsilon^{-\kappa}\Big|R(\tfrac{y_1-y_2}{\varepsilon})\Big|dxdyds\Big)^{\frac{p}{2}}.
\end{aligned}$$

Recalling $\varepsilon^{-\kappa}\big|R(\frac{y_1-y_2}{\varepsilon})\big| \lesssim |y_1-y_2|^{-\kappa}$ and the moment bounds (2.5.10), we obtain

$$\begin{aligned}
\|\mathcal{A}^{\varepsilon}_{r,t}\|_p &\lesssim \varepsilon^{\frac{\kappa}{2}-1}\Big(\int_r^t\int_{\mathbb{R}^{4d}}\Big(\prod_{i=1}^{2}P_{t-s}(x_i-y_i)|g^{\lambda}(x_i)|\Big)|y_1-y_2|^{-\kappa}dxdyds\Big)^{\frac{1}{2}} \\
&= \varepsilon^{\frac{\kappa}{2}-1}I_A(r,t)^{\frac{1}{2}}.
\end{aligned}$$

Following the same steps, we also have $\|\mathcal{B}^{\varepsilon}_{r,t}\|_p \leqslant \|\langle \mathcal{B}^{\varepsilon}_{\cdot,t}\rangle_r^{\frac{1}{2}}\|_p \lesssim \varepsilon^{\frac{\kappa}{2}-1}I_B(r,t)^{\frac{1}{2}}$. Combining the two bounds, we have

$$\|X_t^{\varepsilon,g^{\lambda}} - X_r^{\varepsilon,g^{\lambda}}\|_p \leqslant \beta\varepsilon^{1-\frac{\kappa}{2}}\big(\|\mathcal{A}^{\varepsilon}_{r,t}\|_p + \|\mathcal{B}^{\varepsilon}_{r,t}\|_p\big) \lesssim I_A(r,t)^{\frac{1}{2}} + I_B(r,t)^{\frac{1}{2}}.$$

We can now apply Lemma 3.4.5 as in the previous proof to obtain (3.4.9). The bound (3.4.10) then follows in the same way as in Lemma 3.4.6. □

We can now use the estimates of Lemmas 3.4.6 and 3.4.7 to control $\mathcal{U}_t(\varphi_x^{(n)})$, $X_t^{\varepsilon,\varphi_x^{(n)}}$, which allows us to conclude that both $\mathcal{U}_t$ and X_t^{ε} belong to $\mathcal{C}^{\gamma}([0,T],\mathcal{C}^{\alpha})$.

Proposition 3.4.8. *Let $d \geqslant 3$, $\gamma \in (0,\frac{1}{2})$ and $\alpha < 1-\frac{\kappa}{2}-2\gamma$.*

(i) Then, there exists a version of $\mathcal{U}$ in $\mathcal{C}^{\gamma}([0,T],\mathcal{C}^{\alpha})$.

(ii) Let σ be Lipschitz continuous, and let ξ satisfies Assumption 2.0.1.

(a) Then, there exist $p = p_{\gamma,\alpha} > d$, such that for any $\beta < \beta_0(p_{\gamma,\alpha})$ the following holds

$$\sup_{\varepsilon\in(0,1)}\Big\|\|X^{\varepsilon}\|_{\mathcal{C}^{\gamma}([0,T],\mathcal{C}^{\alpha})}\Big\|_p < \infty. \tag{3.4.11}$$

(b) For sufficiently small $\beta < \beta_{\gamma,\alpha}$, $\{X^\varepsilon\}_{\varepsilon\in(0,1)}$ is tight in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$.

Proof. Let us denote with ζ_t either $\mathcal{U}_t$ or X^ε_t. We want to verify that $\zeta \in \mathcal{C}^{\gamma_0,\alpha_0}_{p,T}$ and apply Proposition 3.3.7. Recal $\varphi^{(n)} = 2^{nd}\varphi(2^n\cdot)$ and let us apply Lemmas 3.4.6-3.4.7 with test functions $\varphi^{(n)}_x$. For any $\gamma_0 \in (0, \frac{1}{2})$ and $\alpha_0 < 1 - \frac{\kappa}{2} - 2\gamma_0$, we obtain the estimates

$$\sup_{x\in\mathbb{R}^d} \big\|\langle \zeta_{r,t}, \varphi^{(n)}_x\rangle\big\|_p \lesssim 2^{-n\alpha_0}|t-r|^{\gamma_0}\|\varphi\|_\infty,$$
$$\sup_{x\neq y\in\mathbb{R}^d} \big\|\langle \zeta_{r,t}, \varphi^{(n)}_x - \varphi^{(n)}_y\rangle\big\|_p \lesssim 2^{-n(\alpha_0-1)}|t-r|^{\gamma_0}|\varphi|_{\mathrm{Lip}}.$$

Hence (3.3.6)–(3.3.7) are satisfied and both $\mathcal{U}_t$ and X^ε_t belong to $\mathcal{C}^{\gamma_0,\alpha_0}_{p,T}$ with $\|X^\varepsilon\|_{\mathcal{C}^{\gamma_0,\alpha_0}_{p,T}}$ uniformly bounded in ε. Choosing p sufficiently large and $\beta < \beta_0(p)$ small enough so that Lemma 3.4.7 holds, we can then apply Proposition 3.3.7 which yields (3.4.11) as claimed.

We proceed to prove the last statement ((ii)b). Given γ and α, we can pick real numbers γ' and α' such that

$$\gamma < \gamma' < \tfrac{1}{2}, \qquad \alpha < \alpha' < 1 - \tfrac{\kappa}{2} - 2\gamma'.$$

Since $\mathcal{C}^{\alpha'}$ is compactly embedded in $\mathcal{C}^\alpha$, $\mathcal{C}^{\gamma'}([0,T],\mathcal{C}^{\alpha'})$ is compactly embedded in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$. Thus $K_\Lambda := \{X \in \mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha) \,:\, \|X\|_{\mathcal{C}^{\gamma'}([0,T],\mathcal{C}^{\alpha'})} \leqslant \Lambda\}$ is compact in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$, for any $\Lambda > 0$. Given the uniform moment bound (3.4.11), which holds for sufficiently small $\beta < \beta_0(p_{\gamma',\alpha'})$, we have

$$\mathbb{P}(X^\varepsilon \notin K_\Lambda) = \mathbb{P}(\|X^\varepsilon\|_{\mathcal{C}^{\gamma'}([0,T],\mathcal{C}^{\alpha'})} > \Lambda) \leqslant \frac{\mathbb{E}\big[\,\|X^\varepsilon\|^p_{\mathcal{C}^{\gamma'}([0,T],\mathcal{C}^{\alpha'})}\,\big]}{\Lambda^p} \lesssim \frac{1}{\Lambda^p}.$$

Hence X^ε is tight in $\mathcal{C}^\gamma([0,T],\mathcal{C}^\alpha)$. This completes the proof of the proposition. □

Chapter 4

Homogenisation and the KPZ scaling limit

In this chapter we present our results on the KPZ scaling limit, based on [GHL24], joint work with Martin Hairer (MH) and Xue-Mei Li (XML). We are grateful to Alex Dunlap and Yu Gu for discussions on this topic and for suggesting that our proof may yield convergence in probability. The original attempt on this project was based on alternative strategy, partly inspired by [DGRZ20]. However, a key step of the proof was affected by a mistake - which I am very thankful to my supervisors (XML and MH) for spotting and their crucial help on restructuring our proof(s). The new strategy here presented is based on a homegenisation idea due to MH, which lifted our proof and unveiled a richer underlying structure, allowing us also to prove a sharper scaling limit result. Most parts of the proofs and exposition are verbatim to the article.

In this chapter we investigate the large-scale dynamics of the Kardar–Parisi–Zhang (KPZ) equation on $\mathbb{R}^d$ in dimension $d \geqslant 3$. The KPZ equation

$$\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + \beta\xi \,, \tag{4.0.1}$$

was originally introduced in [KPZ86] to model random interface growth. As

mentioned in Chapter 1, the KPZ equation has been extensively studied given its ability to capture the universal behaviour of many probabilistic models and the mathematical challenges presented for a well-posed solution theory.

Typically, the mean zero Gaussian noise ξ in the model has short range correlations. In that case, we reviewed in the introduction that, in a number of recent works [MU18, CCM24, DGRZ20, CNN22, LZ22], it was shown that the large-scale behaviour of the KPZ equation in dimensions $d \geqslant 3$, in the weak coupling regime, is described by the Edwards–Wilkinson model.

Whereas here, we still consider spatially coloured noise with long range correlations of polynomial decay $|x|^{-\kappa}$, with the ultimate objective to explore how these may affect the KPZ scaling limit in the supercritical regime $d \geqslant 3$. Using the Cole-Hopf transform $h = \log u$ to relate the KPZ to the stochastic heat equation (SHE), we build on the properties and fluctuations theory of SHE laid out in the previous chapters. However, a new crucial tool will be a homogenisation result for the fundamental solution of the *linear SHE* (4.1.7), which also allows us to consider fluctuations of general transformations $\Phi(u)$.

Moreover, we are able obtain convergence in probability, unlike the case of integrable correlations. Namely, under a suitable coupling, the noise driving the limiting equation turns out to be the scaling limit of the noise driving the KPZ equation. It is also interesting to note that our result on the KPZ scaling limit aligns with the physics literature [KS99, KCDW14], which predicts a Gaussian limit when $\kappa > 2$. In the following section we review setup, main results and structure of the chapter.

4.1 Main results

We begin by describing the assumptions on the noise ξ considered here.

Assumption 4.1.1. The Gaussian noise ξ is white in time and coloured in space, obtained by convolving (in space) a space-time white noise $\dot{W}$ with a

smooth function $\phi : \mathbb{R}^d \to \mathbb{R}_+$:

$$\xi(t,x) = \int_{\mathbb{R}^d} \phi(x-y)\dot{W}(t,y)\,dy\ , \tag{4.1.1}$$

such that its spatial covariance $R(x) = \int_{\mathbb{R}^d} \phi(x+y)\phi(y)dy$ is nonnegative and satifies the long range conditions as in Assumption 2.0.1.

Remark 4.1.2. For example, taking $\phi : \mathbb{R}^d \to \mathbb{R}_+$ behaving at infinity like $\phi(x) \sim |x|^{-\frac{d+\kappa}{2}}$ leads to covariance R which satisfies Assumption 2.0.1. The additional structural requirements of mollification and non-negativity of the spatial covariance R are technical: they are required only in Lemma 4.2.2 to prove bounds on the negative moments of the solution of the linear SHE.

We now start by introducing the main scaling limit result. We write h for the solution to (4.0.1), but with a slowly varying initial condition of the form $\varepsilon^{\frac{\kappa}{2}-1}h_0(\varepsilon x)$, we rescale it by setting, for $\varepsilon \in (0,1]$,

$$h_\varepsilon(t,x) = \varepsilon^{1-\frac{\kappa}{2}}h(t/\varepsilon^2, x/\varepsilon) - \tfrac{t}{2}\beta^2 R(0)\varepsilon^{-1-\frac{\kappa}{2}}\ .$$

We note that h_ε then solves the rescaled KPZ equation

$$\partial_t h_\varepsilon = \tfrac{1}{2}\Delta h_\varepsilon + \tfrac{1}{2}\varepsilon^{\frac{\kappa}{2}-1}|\nabla h_\varepsilon|^2 - \tfrac{1}{2}\beta^2 R(0)\varepsilon^{-1-\frac{\kappa}{2}} + \beta\xi^\varepsilon\ , \tag{4.1.2}$$

with initial condition h_0. Here ξ^ε is the rescaling of ξ such that

$$\mathbb{E}[\xi^\varepsilon(t,x)\xi^\varepsilon(s,y)] = \delta(t-s)\varepsilon^{-\kappa}R(\tfrac{x-y}{\varepsilon})\ . \tag{4.1.3}$$

As noted before, under this scaling ξ^ε converges in law, as $\varepsilon \to 0$, to a non-trivial Gaussian limit ξ^0 with covariance

$$\mathbb{E}[\xi^0(t,x)\xi^0(s,y)] = \delta(t-s)|x-y|^{-\kappa}\ . \tag{4.1.4}$$

In order to formulate our stronger mode of convergence, it will be convenient to upgrade this convergence to *convergence in probability*, which can be achieved by Skorokhod's representation theorem:

Definition 4.1.3. A coupling between the ξ^ε is *good* if, for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1})$, there exists a random variable $\xi^0(\psi)$ such that $\xi^\varepsilon(\psi) \Rightarrow \xi^0(\psi)$ in probability.

With ξ defined as in (4.1.1), a natural good coupling is given by

$$\xi^\varepsilon(t,x) = \varepsilon^{-(d+\kappa)/2} \int_{\mathbb{R}^d} \phi((x-y)/\varepsilon)\dot{W}(t,y)\,dy\ . \tag{4.1.5}$$

Given the limiting noise ξ^0, we write $\mathcal{U}$ for the solution to the corresponding additive stochastic heat equation:

$$\partial_t \mathcal{U} = \tfrac{1}{2}\Delta \mathcal{U} + \beta\xi^0\ , \qquad \mathcal{U}(0,\cdot) = h_0\ . \tag{4.1.6}$$

Our main result is then the following.

Theorem 4.1.4. *Let $d \geqslant 3$ and let ξ satisfy Assumption 4.1.1. For any $p \geqslant 1$, and $\alpha < 1 - \frac{\kappa}{2}$, there exists a constant $c < 0$ and a positive value $\beta_0 = \beta_0(p,\alpha)$ such that for any $\beta < \beta_0$, for any deterministic initial condition $h_0 \in \mathcal{C}_b(\mathbb{R}^d)$, and any good coupling of the noise, one has, for any $T > T_0 > 0$*

$$\lim_{\varepsilon\to 0}\big(h_\varepsilon(t,x) - \varepsilon^{1-\frac{\kappa}{2}}c\big) = \mathcal{U}\ ,$$

in probability in $L^p([T_0,T],\mathcal{C}^\alpha(E))$, for any compact $E \subset \mathbb{R}^d$.

We provide the definition and comment on the constant c in Remark 4.1.9 below, whilst the proof of this result will be given in the very last Section 4.4.3, building on the results stated next. In fact, along the way we will show a fluctuations limit theorem for general functions $\Phi(u)$ of solutions to the multiplicative stochastic heat equation (*linear SHE*)

$$\partial_t u(t,x) = \tfrac{1}{2}\Delta u(t,x) + \beta u(t,x)\xi(t,x)\ , \qquad u(0,\cdot) = 1\ . \tag{4.1.7}$$

Since we are interested in its large-scale fluctuations, as before we write $u_\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon)$ for the diffusively rescaled solution, which solves

$$\partial_t u_\varepsilon(t,x) = \tfrac{1}{2}\Delta u_\varepsilon(t,x) + \beta\varepsilon^{\frac{\kappa}{2}-1}u_\varepsilon(t,x)\xi^\varepsilon(t,x)\ , \qquad u_\varepsilon(0,\cdot) = 1, \tag{4.1.8}$$

with ξ^ε as before. We then have the following functional central limit theorem:

Theorem 4.1.5. *Consider a map $\Phi : \mathbb{R}_+ \to \mathbb{R}$ satisfying Assumption 4.4.1 and let*

$$u_\varepsilon^\Phi(t,x) = \varepsilon^{1-\frac{\kappa}{2}}\big(\Phi(u_\varepsilon(t,x)) - \mathbb{E}[\Phi(u_\varepsilon(t,x))]\big)\ .$$

With assumptions on d, R, p and α as in Theorem 4.1.4, there exists $\beta_0^\Phi = \beta_0^\Phi(\alpha, p) > 0$ and $\nu_\Phi \geqslant 0$ such that, for all $\beta < \beta_0^\Phi$, for any $T > 0$, and any good coupling, u_ε^Φ converges in probability to $\nu_\Phi \mathcal{U}^0$ in the space $L^p([0,T], \mathcal{C}^\alpha(E))$ for any compact $E \subset \mathbb{R}^d$. Here, $\mathcal{U}^0$ denotes the solution to the additive stochastic heat equation

$$\partial_t \mathcal{U}^0 = \tfrac{1}{2}\Delta \mathcal{U}^0 + \beta \xi^0 \,, \qquad \mathcal{U}^0(0,\cdot) = 0 \,. \tag{4.1.9}$$

Remark 4.1.6. Our class of possible transformations includes the Cole–Hopf transform $\Phi(u) = \log u$, for which one has $\nu_\Phi = 1$. In this case we recover the fluctuations of KPZ since $h_\varepsilon(t,x) = \varepsilon^{1-\frac{\kappa}{2}} \log u_\varepsilon(t,x)$ solves (4.1.2) with *flat initial conditions* $h_0 \equiv 0$. Hence, Theorem 4.1.4 essentially follows as a special case, but we will require additional work to handle the general initial conditions considered in Theorem 4.1.4. In fact, observe that Theorem 4.1.5 holds for $u_0 \equiv 1$, and this is also the reason why we can consider the full time interval $[0,T]$ in the convergence. We also note that, despite different attempts, here we are only able to show convergence in weaker L^p based spaces rather than $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha)$ as obtained in Theorem 3.1.5 for SHE. Given the presence of the nonlinear transformation $\Phi(u)$, is unclear whether such convergence should hold. Concerning the effective variance, we discuss and provide the explicit expression of ν_Φ in the general case in Remark 4.1.11 below. Finally, we also note that, in the case of identity $\Phi(u) = u$, we recover *convergence in probability* for the fluctuations of the *linear* SHE. Its unclear whether the stronger convergence may hold even in presence of a nonlinearity in SHE (cf. X^ε decomposition in proof of Proposition 4.4.5).

The main tool to prove the above is a homogenisation result for the fundamental solution of (4.1.7). In order to formulate it, write $w_{s,y}(t,x)$ for the solution to (4.1.7) at times $t \geqslant s$ with initial condition at time s given by $w_{s,y}(s,\cdot) = \delta_y$ which solves:

$$w_{s,y}(t,x) = P_{t-s}(x-y) + \beta \int_s^t \int_{\mathbb{R}^d} P_{t-r}(x-z) w_{s,y}(r,z)\, \xi(dr,dz) \,. \tag{4.1.10}$$

Then, we have the following:

Theorem 4.1.7. *Let $d \geqslant 3$ and ξ satisfies Assumption 4.1.1. Then for any $p \geqslant 1$, for sufficiently small β, there exist stationary random fields $\vec{Z}$ and $\overleftarrow{Z}$ such that, with $\mu = \frac{1}{2} - \frac{1}{\kappa}$,*

$$\Big\| \frac{w_{s,y}(t,x)}{P_{t-s}(y-x)} - \vec{Z}(t,x)\overleftarrow{Z}(s,y) \Big\|_p \lesssim (1+t-s)^{-\mu}(1+(1+t-s)^{-\frac{1}{2}}|x-y|) \, . \tag{4.1.11}$$

A slightly more detailed version of this statement will be formulated as Proposition 4.3.1 below. The field $\vec{Z}$ is the analogue of the pullback field already appearing in [DGRZ21, CCM24, CCM20], which we obtained in Theorem 2.6.1 as the limit $\vec{Z}(t,x) = \lim_{s\to-\infty} u^{(s)}(t,x)$, where $u^{(s)}$ denotes the solution to (4.1.7) with initial condition 1 at time s. Our homegenisation result seems related to analogue local limit theorems for polymers in $d \geqslant 2$, recently obtained in discrete and continuous settings, cf. [Gab23, Prop. 2.1], [Jun23, Cor. 1.6], [NN23, Thm. 2.10] and [CCM20, Lem. 2.3].

Remark 4.1.8. It turns out that the field $\overleftarrow{Z}$ is related to the mass process of $w_{s,y}(t,x)$ and given by "$\vec{Z}$ running backwards in time" (which motivates our notations) in a sense that will be made precise below, see (4.2.3), Theorem 2.6.1 and equation (4.2.24). A shadow of the field $\overleftarrow{Z}$ can be seen in the constant $\bar{c}$ appearing in [DGRZ21, Eq. 1.6], whose inverse would correspond to the expectation of $\overleftarrow{Z}$ (which in our case equals 1 as a consequence of the fact that we consider noise that is white in time). More explicitly, these fields satisfy the equations

$$\vec{Z}(t,x) = 1 + \beta \int_{-\infty}^{t} \int_{\mathbb{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)\,\xi(dr,dz) \, , \tag{4.1.12}$$

$$\overleftarrow{Z}(s,y) = 1 + \beta \int_{s}^{\infty} \int_{\mathbb{R}^d} P_{r-s}(y-z)\overleftarrow{Z}(r,z)\,\xi(dr,dz) \, , \tag{4.1.13}$$

with the stochastic integral appearing in the second expression being of backwards Itô type.

Returning to the Theorem 4.1.5, using the Clark–Ocone formula we recover a divergence representation for the fluctuations of $\Phi(u)$, which can be

expressed in terms of the fundamental solution of (4.1.7). Relying on the large time homogenisation of $w_{s,y}$ from Theorem 4.1.7 and mixing of the stationary fields, we are able to approximate the fluctuations of $\Phi(u)$ by the fluctuations of the linear SHE (Proposition 4.4.4), which in turn converge to $\mathcal{U}^0$. Having quantitative estimates in $\mathcal{C}^\alpha$ spaces (using the embeddings from the previous chapter), we then prove Theorem 4.1.5.

We shall explain more in details the arguments in the relevant sections, the rest of the chapters is organised as follows:

- ▷ In Section 4.2, we introduce the random fields $\vec{Z}$ and $\overleftarrow{Z}$, we collect for later sections some useful a priori bounds on these fields as well as on transformations of the solutions to the linear SHE (4.1.7) and their Malliavin derivatives.
- ▷ In Section 4.3, we prove the large-scales homogenisation Theorem 4.1.7 and the kernel/Skorokhod correction terms estimates required.
- ▷ In Section 4.4, we use the above to approximate the fluctuations of $\Phi(u)$, collect quantitative estimates for convergence in Hölder spaces and then use all of these ingredients to provide a proof of Theorems 4.1.4-4.1.5.

We conclude this section with some final remarks on the results.

Remark 4.1.9. Unlike what happens in the case of short-range correlations, the constant c in Theorem 4.1.4 is all that is left from the KPZ nonlinearity since neither the covariance of the noise nor the coefficient in front of the Laplacian end up being affected by the scaling limit. The constant is given by $c = \mathbb{E}[\log Z]$ with $Z \overset{\text{law}}{=} \vec{Z}(0,0)$. Since $\mathbb{E}Z = 1$ and Z has non-zero variance by (4.1.12), c is strictly negative. The way one can intuitively understand the presence of c is the following: since we start with a fixed ε-independent bounded continuous initial condition h_0 which looks quite different from the stationary solutions (which exhibit oscillations of size about $\varepsilon^{1-\frac{\kappa}{2}}$), the KPZ nonlinearity and the renormalisation constant appearing in (4.1.2) don't quite

balance out initially. By the time the process has become locally sufficiently close to stationarity, this imbalance has already caused it to shoot down by a substantial amount.

Remark 4.1.10. It is also important to observe, while not the case in Theorem 4.4.2 for general transformation $\Phi(u)$, the centering $\mathbb{E}[\log u_\varepsilon]$ for the KPZ can be replaced by the *fixed* limiting constant $c = \mathbb{E}[\log Z]$ in Theorem 4.1.4. This is because we are able to show $\mathbb{E}[\log u(\varepsilon^{-2}t, 0)]$ converges to c faster than $\varepsilon^{\frac{\kappa}{2}-1}$, the fluctuations scale, leveraging properties of the log transform (cf. vanishing rate (4.4.24) in the final proof of the main theorem).

Remark 4.1.11. With the notations above at hand, one has $\nu_\Phi = \mathbb{E}[Z\,\Phi'(Z)]$ in Theorem 4.1.5 where $Z = \vec{Z}(0,0)$. We will see in Lemma 4.2.4 that our assumptions on Φ guarantee that this quantity is finite. Given the dependence of the pullback stationary solution $\vec{Z}$ on β, we note that ν_Φ actually depends on β in general and that, as $\beta \to 0$, ν_Φ converges to $\Phi'(1)$. In the special case $\Phi(x) = x$, one has $\nu_\Phi = 1$ and we recover the result of the previous Chapter 3, at least in the case $\sigma(u) = u$ (linear SHE).
Note that the effective variance again differs from that obtained for the case of integrable R in [GRZ18, MU18, DGRZ20, CNN22, LZ22, GL20, DGRZ21, CCM24]. There, the effective variance ν_{eff} is given by the expression

$$\nu_{\text{eff}}^2 = \nu_\Phi^2 \int_{\mathbb{R}^d} R(x)\mathbb{E}\big[\vec{Z}(0,x)\vec{Z}(0,0)\big]\,dx\ .$$

As observed for SHE, we note that in their works the fluctuation scale depends on the dimension, while in our long-range correlations setting the fluctuations scale $\varepsilon^{\kappa-2}$ depends on the decay coefficient κ of the noise covariance. Finally, as noted in the introduction, observe that in the case of linear SHE and KPZ we have $\nu_\Phi = 1$, which is *independent* of β.

4.2 A priori bounds and stationary solutions

In this chapter we denote with u the solution of the *linear SHE* (4.1.7) and $u^{(s)}$ denotes the solution to (4.1.7) with initial condition 1 at time $s \in \mathbb{R}$:

$$u^{(s)}(t,x) = 1 + \beta \int_s^t \int_{\mathbb{R}^d} P_{t-r}(x-y) u^{(s)}(r,y)\, \xi(dr,dy)\ . \tag{4.2.1}$$

These are the same pullback solutions (2.6.1) considered in Section 2.6 (here we are in the case of linear $\sigma(u) = u$), where we denoted with $u^{(K)}(t,x)$ the solution that starts from time $-K$. We then showed in Theorem 2.6.1 that these converge to the stationary field $\vec{Z}(t,x)$ with rate

$$\big\| u^{(s)}(t,x) - \vec{Z}(t,x) \big\|_p \lesssim 1 \wedge (t-s)^{\frac{2-\kappa}{4}}\ . \tag{4.2.2}$$

We also consider the fundamental solutions $w_{s,y}(t,x)$ solving (4.1.10) and associated the mass process:

$$C_{s,y}(t) = \int_{\mathbb{R}^d} w_{s,y}(t,x)\, dx\ , \tag{4.2.3}$$

which converges similarly to the backward stationary field $\overleftarrow{Z}(s,y)$. In Section 4.2.2 we recognise an underlying cocycle structure and formalise this duality relationship. But first, in the following section, we establish useful a priori L^p bounds and correlation estimates to be used later.

Remark 4.2.1. In the following sections we shall make repeated use of the following estimate based on Burkholder–Davis–Gundy (BDG) inequality which we gather here for convenience. Given a time interval $I = [a,b]$ and an adapted random process $f(s,y)$, by BDG inequality, followed by Minkowski and Hölder inequalities, for a universal constant $c_p > 0$, we have the following

$$\begin{aligned}
&\Big\| \int_{I\times\mathbb{R}^d} f(s,y)\xi(ds,dy) \Big\|_p \\
&\quad \leqslant c_p \mathbb{E}\left[\Big(\int_I \int_{\mathbb{R}^{2d}} f(s,y_1) f(s,y_2) R(y_1-y_2)\, dy\, ds \Big)^{\frac{p}{2}} \right]^{\frac{1}{p}} \\
&\quad \leqslant c_p \Big(\int_I \int_{\mathbb{R}^{2d}} \|f(s,y_1)\|_p \|f(s,y_2)\|_p R(y_1-y_2)\, dy\, ds \Big)^{\frac{1}{2}}\ .
\end{aligned} \tag{4.2.4}$$

4.2.1 A priori moments bounds

In the next lemma we gather moment estimates on u, $\vec{Z}$ and the fundamental solution $w_{s,y}$ appearing in Theorem 4.1.7. Some statements follow by Lemma 2.5.2, but we reformulate them here for later use and consistency. Crucially, now that we are considering the *linear SHE*, for sufficiently small β, the negative moments of u are also uniformly bounded. To follows by the approach of [CCM20], which generalises to when the mollifier ϕ is not compactly supported. Control of negative moments has also been showed in [HL22b, DGRZ20, CSZ20] in the case of compactly supported covariance.

Lemma 4.2.2. *Assume that ξ satisfies Assumption 4.1.1. For any $p \geqslant 2$, there exist constants $\beta_0(p) > 0$ and C_p, such that if $\beta \leqslant \beta_0$, the following holds for the solution of (4.1.7).*

(i) The solution of (4.1.7) has positive and negative moments of all orders:

$$\sup_{s\geqslant 0, x\in\mathbb{R}^d} \|u(s,x)\|_p \leqslant C_p \ , \qquad \sup_{s\geqslant 0, x\in\mathbb{R}^d} \|u(s,x)^{-1}\|_p \leqslant C_p \ . \tag{4.2.5}$$

(ii) For all $x \in \mathbb{R}^d$ and $t > 0$, $u(t,x) \in \mathbb{D}^{1,p}$. The Malliavin derivative of u is given by

$$D_{s,y}u(t,x) = \beta u(s,y) w_{s,y}(t,x) \mathbf{1}_{[0,t]}(s) \ , \tag{4.2.6}$$

and the process w satisfies the pointwise estimate

$$\|w_{s,y}(t,x)\|_p \lesssim P_{t-s}(x-y) \ . \tag{4.2.7}$$

(iii) The process $C_{s,y}(t)$ defined in (4.2.3) satisfies $\sup_y \sup_{t>s\geqslant 0} \|C_{s,y}(t)\|_p \lesssim 1$ and

$$\|C_{s,y}(t) - C_{s,y}(\tau)\|_p \lesssim \beta(1 + (t\wedge\tau) - s)^{\frac{2-\kappa}{4}} \ . \tag{4.2.8}$$

Assertions (i)-(ii) also hold if we replace u by $\vec{Z}$, and one has

$$D_{r,z}\vec{Z}(t,x) = \beta \vec{Z}(r,z) w_{r,z}(t,x) \mathbf{1}_{(-\infty,t]}(r) \ . \tag{4.2.9}$$

Proof. The positive moment bounds in (4.2.5) follow from Lemma 2.5.2. Concerning the Malliavin derivative, we note that by the mild solution formulation we have $u(t,x) = 1 + \beta\delta(v_{t,x})$, where

$$v_{t,x}(r,z) = P_{t-r}(x-z)u(r,z)\mathbf{1}_{[0,t]}(r) \ .$$

Differentiating on both sides and using the commutation relation (2.2.7), it follows that for $s \in [0,t]$ one has

$$D_{s,y}u(t,x) = \beta P_{t-s}(x-y)u(s,y) + \beta \int_s^t \int_{\mathbb{R}^d} P_{t-r}(x-z) D_{s,y}u(r,z)\, \xi(dr,dz) \ .$$

This shows that $D_{s,y}u$ solves the same equation as u itself, but with initial condition $\beta u(s,\cdot)\delta_y$ at time s, so that (4.2.6) follows by linearity.

The moment bound on $w_{s,y}$ follows by the same steps as in the proof of Lemma 2.5.2 (we omit tracking the same constants here). Consider $w^{(0)}_{s,y}(t,x) := P_{t-s}(x-y)$, and

$$w^{(n+1)}_{s,y}(t,x) = P_{t-s}(x-y) + \beta \int_s^t \int_{\mathbb{R}^d} P_{t-r}(x-z) w^{(n)}_{s,y}(r,z)\, \xi(dr,dz) \ .$$

Letting $\tilde{w}^{(n)}_{s,y}(t,x) = w^{(n+1)}_{s,y}(t,x) - w^{(n)}_{s,y}(t,x)$, by (4.2.4) one has

$$\begin{aligned}
&\|\tilde{w}^{(n)}_{s,y}(t,x)\|_p \\
&\leqslant \beta C_p \bigg(\int_s^t \int_{\mathbb{R}^{2d}} \Big(\prod_{i=1}^2 P_{t-r}(x-z_i) \|\tilde{w}^{(n-1)}_{s,y}(r,z_i)\|_p \Big) \big| R(z_1-z_2) \big| \, dz\, dr \bigg)^{\frac{1}{2}} \\
&\lesssim \beta C_p \sup_{r>s} \sup_z \|w^{(n-1)}_{s,y}(r,z)\|_p \ .
\end{aligned}$$

Thus, for sufficiently small $\beta \leqslant \beta_0(p)$, $w^{(n)}_{s,y}$ converges to $w_{s,y}$ as $n \to \infty$. Moreover, again by triangle inequality and (4.2.4), we have

$$\begin{aligned}
\|w^{(n+1)}_{s,y}(t,x)\|_p^2 \lesssim P_{t-s}(x-y) + \\
\beta^2 \int_s^t \int_{\mathbb{R}^{2d}} R(z_1-z_2) \Big(\prod_{i=1}^2 P_{t-r}(x-z_i) \|w^{(n)}_{s,y}(s,z_i)\|_p \Big) \, dz\, dr \ .
\end{aligned}$$

Given these bounds and Lemma 2.4.5, (4.2.7) follows by Lemma 2.5.1.

The uniform control on the negative moments of u in (4.2.5) is a consequence of the fact that $\log u(t,x)$ has sub-Gaussian left tails, namely for $\theta > 0$,

$$\mathbb{P}\big(\log u(t,x) \leqslant -\theta\big) \leqslant Ce^{-\frac{\theta^2}{2}} \,. \tag{4.2.10}$$

The proof of (4.2.10) follows by [CCM20, Thm. 1.3], given that u is uniformly L^2 bounded for any $\beta \leqslant \beta_0(2)$. Their strategy to discretize and cut-off of the mollifier ϕ does not require ϕ to be compactly supported. Then, all steps carry through once noted that even with long range covariance R, uniformly in $t > 0$,

$$\mathbf{E}_{B^{(1)},B^{(2)}}\Big[\int_0^t R(B_s^{(1)} - B^{(2)})\, ds\; e^{\beta^2 \int_0^t R(B_s^{(1)}-B^{(2)})\, ds}\Big] \lesssim \sup_{\substack{\beta\leqslant\beta_0(2),\\ t>0}} \mathbb{E}[u(t,0)^2] \lesssim 1,$$

where $\mathbf{E}_{B^{(1)},B^{(2)}}$ denotes the expectation with respect to Brownian motions $B^{(i)}$ independent of ξ. The first inequality above follows by noting that, via Feynman-Kac formula, one has an explicit expression for the second moment of u (cf. [MSZ16, Lem. 3.1]):

$$\mathbb{E}[u(t,0)^2] = \mathbf{E}_{B^{(1)},B^{(2)}}\Big[e^{\beta^2 \int_0^t R(B_s^{(1)}-B^{(2)})\, ds}\Big].$$

Regarding the mass process C, given (4.2.7), the uniform bound follows by noting

$$\|C_{s,y}(t)\|_p \leqslant \int_{\mathbb{R}^d} \|w_{s,y}(t,x)\|_p\, dx \lesssim \int_{\mathbb{R}^d} P_{t-s}(x-y)\, dx = 1 \,.$$

For (4.2.8), by definition (4.3.6) we have

$$C_{s,y}(t) - C_{s,y}(\tau) = \beta \int_\tau^t w_{s,y}(r,z)\xi(dr,dz) \,.$$

By BDG estimate (4.2.4), followed by (4.2.7) and (2.4.6) we can bound the L^p norm as follows:

$$\begin{aligned}\|C_{s,y}(t) - C_{s,y}(\tau)\|_p^2 &\lesssim \int_\tau^t \int_{\mathbb{R}^{2d}} \prod_{i=1}^2 \|w_{s,y}(r,z_i)\|_p R(z_1-z_2)\, dr\, dz \\ &\lesssim \int_\tau^t \int_{\mathbb{R}^{2d}} \prod_{i=1}^2 P_{r-s}(z_i-y) R(z_1-z_2)\, dr\, dz\end{aligned}$$

$$\lesssim \int_\tau^t (1+r-s)^{-\frac{\kappa}{2}} dr \lesssim (1+\tau-s)^{1-\frac{\kappa}{2}} .$$

This implies (4.2.8), concluding the proof of (iii).

Regarding $\vec{Z}$, we note that it solves the fixed point problem

$$\vec{Z}(t,x) = 1 + \beta \int_{-\infty}^t \int_{\mathbb{R}^d} P_{t-s}(x-y)\vec{Z}(s,y)\,\xi(ds,dy) , \tag{4.2.11}$$

whence we conclude that

$$D_{r,z}\vec{Z}(t,x) = \beta P_{t-r}(x-z)\vec{Z}(r,z) + \beta \int_{-\infty}^t \int_{\mathbb{R}^d} P_{t-s}(x-y) D_{r,z}\vec{Z}(s,y)\,\xi(ds,dy).$$

Since $D_{r,z}\vec{Z}(t,x) = 0$ for $r > t$, assertion (4.2.9) holds. Since the bounds in (i) are uniform in time, their extension to $\vec{Z}$ follows. □

Remark 4.2.3. Here we proved assertion (iii) by first principles, which is required later for showing homogenisation. In the next section we explore the duality relation between the stationary fields, eventually recognising $\overleftarrow{Z}(s,y)$ as the large time L^p limit of the mass process $C_{s,y}$. In view of this, we note that (iii) can also be deduced instead using (4.2.2) from Theorem 2.6.1 and Proposition 4.2.6. For example (4.2.8) can be seen from the fact that $C_{s,y}(t) = \overleftarrow{Z}_{s-t}(s,y)$ by Proposition 4.2.6, the law invariance of ξ and (4.2.2), obtaining

$$\begin{aligned} \|C_{s,y}(t) - C_{s,y}(\tau)\|_p = \|\overleftarrow{Z}_{s-t}(s,y) - \overleftarrow{Z}_{s-\tau}(s,y)\|_p = \|Z_{s-t} - Z_{s-\tau}\|_p \\ \lesssim (1 + (t \wedge \tau) - s)^{\frac{2-\kappa}{4}} . \end{aligned} \tag{4.2.12}$$

With positive and negative moment bounds in place, we can Malliavin differentiate transformations of u and show that large scales de-correlations (4.2.17) are preserved. These will later allow us to approximate fluctuations through the homogenisation result.

Lemma 4.2.4. *Assume $\Psi \in \mathcal{C}^1(\mathbb{R}_+)$ and that, for some $M, q \geqslant 1$, satisfies for all $z \in \mathbb{R}_+$,*

$$|\Psi(z)| + |\Psi'(z)| \leqslant M(z^{-q} + z^q). \tag{4.2.13}$$

Then, for any $p > 1$ there exists $\beta_0^{M,q}(p) = \beta_0(p, d, R, q, M)$ such that for any $\beta \leqslant \beta_0^{M,q}(p)$, we have $\Psi(u(t,x)) \in \mathbb{D}^{1,p}$ with

$$D_{s,y}[\Psi(u(t,x))] = \beta \Psi'(u(t,x)) u(s,y) w_{s,y}(t,x) \mathbf{1}_{[0,t]}(s), \tag{4.2.14}$$

and the following bounds are satisfied:

$$\|\Psi(u(t,x))\|_p \lesssim M\ , \qquad \|\Psi'(u(t,x))\|_p \lesssim M\ , \tag{4.2.15}$$

$$\|D_{s,y}\Psi(u(t,x))\|_p \lesssim M P_{t-s}(x-y)\ , \quad \|\Psi(u(t,x))\|_{\mathbb{D}^{1,p}} \lesssim M\ , \tag{4.2.16}$$

uniformly over s, t, x, y. Moreover, uniformly over $x \in \mathbb{R}^d$ we have

$$\sup_{t,s>0} \left|\mathrm{cov}\left(\Psi(u(t,x)), \Psi(u(s,0))\right)\right| \lesssim M\left(1 \wedge |x|^{2-\kappa}\right)\ . \tag{4.2.17}$$

All the estimates above also hold with $\vec{Z}(t,x)$ in place of $u(t,x)$. Finally, we also have the following convergence rate

$$\|\Psi(u^{(s)}(t,x)) - \Psi(\vec{Z}(t,x))\|_p \lesssim 1 \wedge (t-s)^{\frac{2-\kappa}{4}}\ . \tag{4.2.18}$$

Proof. For $\beta \leqslant \beta_0(pq)$, bounds (4.2.15) follow immediately from (4.2.5) and the assumptions on Ψ. Observe that $\beta_0(p)$ is non-increasing in p.

Regarding the Malliavin derivative, to leverage Lemma 2.2.1, let us introduce an approximation by bounded functions $\Psi_n \in \mathcal{C}_b^1(\mathbb{R}_+)$ such that $\Psi_n = \Psi$ on $[\frac{1}{n}, n]$. Consider $\varrho_n : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\varrho_n(x) := \begin{cases} \frac{1}{2n} + \frac{n}{2}x^2 & \text{for } x \in [0, \frac{1}{n}], \\ x & \text{for } x \in [\frac{1}{n}, n], \\ \frac{3}{2}n - \frac{1}{2n}(x-2n)^2 & \text{for } x \in [n, 2n], \\ \frac{3}{2}n & \text{for } x \in [2n, +\infty). \end{cases}$$

Then, we let $\Psi_n := \Psi \circ \varrho_n$. Note that $\varrho_n \in \mathcal{C}_b^1(\mathbb{R}_+)$ approximates the identity with $\sup_x |\varrho_n'(x)| \leqslant 1$ and compact range $\varrho_n(x) \in [\frac{1}{2n}, \frac{3}{2}n]$. Thus, we have $\Psi_n \in \mathcal{C}_b^1(\mathbb{R}_+)$ for all n. Moreover, we note that uniformly in n:

$$|\varrho_n(x)| \leqslant |x| + 1, \qquad |\varrho_n(x)|^{-1} \leqslant \tfrac{1}{|x|} + 1.$$

Then, given growth control (4.2.13) of Ψ, for $\beta \leqslant \beta_0(pq)$ we can also infer

$$\sup_{n>0,s\geqslant 0,x\in\mathbb{R}^d} \|\Psi_n(u(s,x))\|_p \lesssim 1, \qquad \sup_{n>0,s\geqslant 0,x\in\mathbb{R}^d} \|\Psi_n'(u(s,x))\|_p \lesssim 1.$$

Since $u(t,x) \in \mathbb{D}^{1,p}$ by Lemma 2.5.2, we have $\Psi_n(u(t,x)) \in \mathbb{D}^{1,p}$ with $D\Psi_n(u(t,x)) = \Psi_n'(u(t,x))Du(t,x)$ by Lemma 2.2.1. We let

$$\begin{aligned} F_{r,z}^{(n)}(t,x) &= D_{r,z}\big(\Psi_n(u(t,x))\big) - \Psi'(u(t,x))D_{r,z}u(t,z) \\ &= \mathbf{1}_{\{\frac{1}{n}\leqslant u(t,x)\leqslant n\}^c}(\Psi_n'(u(t,x)) - \Psi'(u(t,x)))D_{r,z}u(t,x). \end{aligned}$$

Then, for sufficiently small β we use the moments estimates on u, Ψ', Ψ_n' and Du, to obtain

$$\begin{aligned} &\|F_{r,z}^{(n)}(t,x)\|_p \\ &\quad\leqslant \mathbb{P}(\{u<\tfrac{1}{n}\}\cup\{u>n\})^{\frac{1}{3p}}\|\Psi_n'(u(t,x)) - \Psi'(u(t,x))\|_{3p}\|D_{r,z}u(t,x)\|_{3p} \\ &\quad\lesssim n^{-\frac{1}{p}} M P_{t-r}(x-z). \end{aligned}$$

By Minkowski, Hölder's inequalities and (2.4.7),

$$\begin{aligned} \mathbb{E}\big[\|F^{(n)}(t,x)\|_{\mathcal{H}}^p\big] &= \mathbb{E}\Big[\Big(\int_0^t\int_{\mathbb{R}^{2d}}\prod_{i=1}^2 F_{r,z_i}^{(n)}(t,x)R(z_1-z_2)\,dr\,dz\Big)^{\frac{p}{2}}\Big] \\ &\leqslant \Big(\int_0^t\int_{\mathbb{R}^{2d}}\|\prod_{i=1}^2 F_{r,z_i}^{(n)}(t,x)\|_{\frac{p}{2}}R(z_1-z_2)\,dr\,dz\Big)^{\frac{p}{2}} \\ &\leqslant n^{-1}M^p\Big(\int_0^t\int_{\mathbb{R}^{2d}}\prod_{i=1}^2 P_{t-r}(x-z_i)R(z_1-z_2)\,dr\,dz\Big)^{\frac{p}{2}} \\ &\leqslant C_R M^p n^{-1}. \end{aligned}$$

Thus, $F^{(n)}(t,x) \to 0$ in $L^p(\Omega,\mathcal{H})$ as $n\to\infty$. On the other hand, $\Psi_n(u(t,x)) \to \Psi(u(t,x))$ in $L^p(\Omega)$. Since D is closed, we conclude $\Psi(u(t,x)) \in \mathrm{Dom}(D)$ and $D_{s,y}\Psi(u(t,x)) = \Psi'(u(t,x))D_{s,y}u(t,x)$. Then, given (4.2.6), representation (4.2.14) holds. Moreover, for sufficiently small β, given (4.2.7) and the uniform moments bounds on $\Psi'(u)$ and u, we have $\|D_{r,z}\Psi(u(t,x))\|_p \lesssim MP_{t-r}(x-z)$. Since the inequality constant is independent of (t,x), we estimate similarly to above

$$\mathbb{E}\big[\|D\Psi(u(t,x))(t,x)\|_{\mathcal{H}}^p\big] \leqslant \Big(\int_0^t\int_{\mathbb{R}^{2d}}\prod_{i=1}^2\|D_{r,z_i}\Psi(u(t,x))\|_p\,R(z_1-z_2)\,dr\,dz\Big)^{\frac{p}{2}}$$

$$\leqslant \Big(\int_0^\infty \int_{\mathbb{R}^{2d}} \prod_{i=1}^{2} P_{t-r}(x - z_i) R(z_1 - z_2)\, dr\, dz \Big)^{\frac{p}{2}} \lesssim 1,$$

where we used (2.4.7). Combining this with (4.2.15), we have

$$\|\Psi(u(t,x))\|_{\mathbb{D}^{1,p}} \leqslant \|\Psi(u(s,x))\|_p + \|D\Psi(u(s,x))\|_{L^p(\Omega,\mathcal{H})} \lesssim M\ ,$$

uniformly in s and x. This concludes the proof of (4.2.16).

Recall the Clark–Ocone formula (2.5.12). We apply it to $\Psi(u(s,x)) \in \mathbb{D}^{1,2}$ and estimate by the Cauchy–Schwarz inequality in the same way as in Lemma 2.5.4, obtaining

$$\begin{aligned}
&\big|\mathrm{cov}\,(\Psi(u(t,x)), \Psi(u(s,0)))\,\big| \\
&\qquad \leqslant \int_0^{t\wedge s} \int_{\mathbb{R}^{2d}} \|D_{r,z_1}\Psi(u(t,x))\|_2 \|D_{r,z_2}\Psi(u(s,0))\|_2 R(z_1 - z_2)\, dr\, dz \\
&\qquad \lesssim M \int_0^{t\wedge s} \int_{\mathbb{R}^{2d}} P_{t-r}(x - z_1) P_{s-r}(z_2) R(z_1 - z_2)\, dr\, dz\ ,
\end{aligned}$$

so that (4.2.17) follows by (2.4.7).

It remains to show (4.2.18), for which we make use of the identity

$$\begin{aligned}
&\Psi(u^{(s)}(t,x)) \;-\; \Psi(\vec{Z}(t,x)) \\
&\qquad = \big(u^{(s)}(t,x) - \vec{Z}(t,x)\big) \int_0^1 \Psi'\big(\theta u^{(s)}(t,x) + (1-\theta)\vec{Z}(t,x)\big)\, d\theta\ .
\end{aligned}$$

Using the fact that u is non-negative and the uniform a priori bounds on positive and negative moments of u and $\vec{Z}$, one obtains a uniform L^{2p} bound on $\int_0^1 \Psi'\big(\theta u(t,x) + (1-\theta)Z(t,x)\big)d\theta$ for sufficiently small β. Thus, (4.2.2) implies

$$\|\Psi(u^{(s)}(t,x)) - \Psi(\vec{Z}(t,x))\|_p \lesssim M \big\|u^{(s)}(t,x) - \vec{Z}(t,x)\big\|_{2p} \lesssim 1 \wedge (t-s)^{\frac{2-\kappa}{4}}\ ,$$

which is precisely (4.2.18), concluding the proof. □

4.2.2 Forward and backward stationary solutions

For the purpose of this section, it will be convenient to assume that the underlying probability space Ω is the space of space-time distributions $\mathcal{D}'(\mathbb{R}^{d+1})$ and that the noise ξ is simply the canonical process.

Recall $u^{(s)}(t,x)$ denotes the pullback solution (4.2.1) and the convergence (4.2.2), which implies that $\vec{Z}$ is a random fixed point for the random dynamical system generated by the stochastic heat equation (4.1.7). In fact, we can recognize a cocycle structure underpinning our main homogenisation result, as we will describe now. Our probability space admits a natural "time reversal" involution $\mathcal{R}\colon \Omega \to \Omega$ and "space-time translations" $\mathcal{T}_{t,x}\colon \Omega \to \Omega$ such that, formally

$$\big(\mathcal{R}\xi\big)(t,x) = \xi(-t,x)\;, \qquad \big(\mathcal{T}_{t,x}\xi\big)(s,y) = \xi(s+t,x+y)\;,$$

with the obvious rigorous interpretations in terms of test functions. These operations naturally yield linear operators acting on the space of random variables $A\colon \Omega \to \mathbb{R}$ by

$$(\mathcal{R}A)(\xi) \stackrel{\text{def}}{=} A(\mathcal{R}\xi)\;, \qquad (\mathcal{T}_{t,x}A)(\xi) \stackrel{\text{def}}{=} A(\mathcal{T}_{t,x}\xi)\;. \tag{4.2.19}$$

Since the law of ξ is invariant under $\mathcal{R}$ and $\mathcal{T}_{t,x}$, the operations in (4.2.19) are isometries for every $L^p(\Omega,\mathbb{P})$-space.

We will repeatedly use the fact that, by uniqueness of solutions to (4.1.7), one has the following.

Lemma 4.2.5. *For any $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, one has $\mathcal{T}_{t,x}\mathcal{R} = \mathcal{R}\mathcal{T}_{-t,x}$ and the following holds almost surely:*

$$u^{(s)}(t,x)(\mathcal{T}_{0,y}\xi) = u^{(s)}(t,x+y)(\xi)\;,$$
$$u^{(s)}(t,x)(\mathcal{T}_{r,0}\xi) = u^{(s+r)}(t+r,x)(\xi)\;. \qquad \square$$

Given any random variable A as above, we then define the stationary random fields

$$\vec{A}(t,x) = \mathcal{T}_{t,x}A\;, \quad \overleftarrow{A}(t,x) = \mathcal{T}_{t,x}\mathcal{R}A\;. \tag{4.2.20}$$

If we now define

$$Z = \lim_{s\to-\infty} Z_s\;, \qquad Z_s = u^{(s)}(0,0)\;, \tag{4.2.21}$$

then we see that this definition is consistent with our notation since by Lemma 4.2.5 $\vec{Z}(t,x) := \mathcal{T}_{t,x}Z$, defined as in (4.2.20), satisfies

$$\begin{aligned}\vec{Z}(t,x) &= \lim_{s\to-\infty} u^{(s)}(0,0)(\mathcal{T}_{t,x}\xi) = \lim_{s\to-\infty} u^{(t+s)}(t,x)(\xi) \\ &= \lim_{s\to-\infty} u^{(s)}(t,x)\ ,\end{aligned} \tag{4.2.22}$$

which is indeed the same quantity appearing in (4.2.2), as well as in (4.1.11), by virtue of (4.1.12). With the notation above

$$\overleftarrow{Z}(s,y) = \mathcal{T}_{s,y}\mathcal{R}Z = \mathcal{R}\mathcal{T}_{-s,y}Z = \mathcal{R}\,\vec{Z}(-s,y)\ , \tag{4.2.23}$$

so that

$$\overleftarrow{Z}(s,y)(\xi) = \lim_{r\to-\infty} u^{(r)}(-s,y)(\mathcal{R}\xi)\ ,$$

which is indeed consistent with (4.1.13) via the change of variables $r \mapsto -r$ (and noting that such a change of variables does indeed turn Itô integrals into backwards Itô integrals by simple Riemann sum approximations).

Let us now motivate why $\overleftarrow{Z}$ appears in (4.1.11). If it were the case that

$$w_{s,y}(t,x) = P_{t-s}(y-x)\vec{Z}(t,x)\overleftarrow{Z}(s,y)\ ,$$

then, as a consequence of the fact that $\mathbb{E}\vec{Z} = 1$ and the law of large number we could recover $\overleftarrow{Z}$ by

$$\overleftarrow{Z}(s,y) = \lim_{t\to\infty}\int_{\mathbb{R}^d} w_{s,y}(t,x)\,dx\ .$$

We observe on the right hand side is the large time limit of the mass process $C_{s,y}(t)$, defined earlier (4.2.3).

We are going to see below that indeed one has $\overleftarrow{Z}(s,y) = \lim_{t\to\infty} C_{s,y}(t)$, indentifying the limit as previously claimed. In fact we will show more, namely that with Z_s as in (4.2.21), one has the exact identity $C_{s,y}(t) = \overleftarrow{Z}_{s-t}(s,y)$.

Proposition 4.2.6. *Under the conditions of Theorem 2.6.1, one has* $C_{s,y}(t) = \overleftarrow{Z}_{s-t}(s,y)$, *and*

$$\|C_{s,y}(t) - \overleftarrow{Z}(s,y)\|_p \lesssim 1 \wedge (t-s)^{\frac{2-\kappa}{4}}.$$

Proof. By the definition (4.2.20), $\overleftarrow{Z}_{s-t}(s,y) := \mathcal{T}_{s,y}\mathcal{R}Z_{s-t} = \mathcal{R}\mathcal{T}_{-s,y}u^{(s-t)}(0,0)$. By Lemma 4.2.5,

$$\overleftarrow{Z}_{s-t}(s,y) = \mathcal{R}u^{(-t)}(-s,y), \qquad \lim_{t\to\infty} \overleftarrow{Z}_{s-t}(s,y) = \overleftarrow{Z}(s,y). \tag{4.2.24}$$

We shall use the method of duality. The basic observation is that if u denotes the SHE running forward in time with initial condition u_s at some time s and v denotes the one running backwards in time with terminal condition v_t at some time $t > s$ then $\langle u_r, v_r\rangle$, where the bracket denotes the inner product on $L^2(\mathbb{R}^d)$, is independent of $r \in [s,t]$. Formally, this is because

$$\partial_r\langle u_r, v_r\rangle = \langle \Delta u_r + \xi u_r, v_r\rangle - \langle u_r, \Delta v_r + \xi v_r\rangle = 0\ . \tag{4.2.25}$$

One can make this precise in the following way. Fix a mollifier $\rho \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ such that $\int\rho = 1$ and $\rho(t,x) = \rho(-t,x) = \rho(t,-x)$. For any $\delta > 0$, we write $\rho_\delta(t,x) = \delta^{-2-d}\rho(t/\delta^2, x/\delta)$ and we set $\xi_\delta(t,x) = (\mathcal{T}_{t,x}\xi)(\rho_\delta)$.

Fix then some initial time s and constant K_δ and consider the solution $u^\delta(r,x)$ to the random PDE

$$\partial_r u^\delta = \tfrac{1}{2}\Delta u^\delta + \beta u^\delta\xi_\delta - K_\delta u^\delta\ , \quad u^\delta(s,\cdot) = \delta_y\ . \tag{4.2.26}$$

We also fix some final time t and consider the solution v^δ to the random *backwards* PDE

$$\partial_r v^\delta = -\tfrac{1}{2}\Delta v^\delta - \beta v^\delta\xi_\delta + K_\delta v^\delta\ , \quad v^\delta(t,\cdot) = 1\ . \tag{4.2.27}$$

At this stage we note that if we set $\tilde{v}^\delta(r,x) = \mathcal{R}v^\delta(-r,x)$, then $\tilde{v}^\delta$ satisfies the *forwards* PDE

$$\partial_r \tilde{v}^\delta = \tfrac{1}{2}\Delta\tilde{v}^\delta + \beta\tilde{v}^\delta\xi_\delta - K_\delta\tilde{v}^\delta\ , \quad \tilde{v}^\delta(-t,\cdot) = 1\ . \tag{4.2.28}$$

At this point we note that (4.2.26) fits into the setting of [HL18, Thm 5.2] (in fact, the regularity of our limiting noise ξ is better than what is considered there so some of the arguments could be simplified). Combining this with [HS23] which shows that the BPHZ lift of ξ^δ to the regularity structure of [HL18,

Sec. 2.1] converges as $\delta \to 0$, we conclude that, provided that the constants K_δ are chosen appropriately, u^δ converges as $\delta \to 0$ to the fundamental solution $w_{s,y}$. This convergence takes place in an exponentially weighted space that enforces rapid decay of the solution.

By the exact same argument, one finds that $\tilde{v}^\delta$ converges to $u^{(-t)}$ as $\delta \to 0$, this time in a weighted space that allows for some slow growth of the solutions. In particular, this shows that v_r^δ converges to

$$v_r(x) = \mathcal{R}u^{(-t)}(-r, x) = \overleftarrow{Z}_{r-t}(r, x) \, .$$

We conclude that, for any $r \in [s, t]$, one has

$$\lim_{\delta \to 0} \langle u_r^\delta, v_r^\delta \rangle = \int_{\mathbb{R}^d} w_{s,y}(r, x) \overleftarrow{Z}_{r-t}(r, x) \, dx \, .$$

On the other hand, it follows immediately from (4.2.26) and (4.2.27) that $\langle u_r^\delta, v_r^\delta \rangle$ is independent of r, as in the formal calculation (4.2.25). Since $Z_0 = 1$ and $\overleftarrow{Z}_0(t, \cdot) = 1$, equating the inner product at time s and t, it follows that

$$\overleftarrow{Z}_{s-t}(s, y) = \int_{\mathbb{R}^d} w_{s,y}(t, x) \overleftarrow{Z}_0(t, x) \, dx = C_{s,y}(t) \, ,$$

which is precisely as claimed. The rate of convergence follows from (4.2.24) and (4.2.2), as remarked earlier (4.2.12).

□

4.3 Homogenisation

The focus of this section is to prove Theorem 4.1.7. We are going to show this by appropriately decomposing and estimating the quantity

$$\rho_{s,y}(t, x) = \tfrac{w_{s,y}(t,x)}{P_{t-s}(x-y)} - C_{s,y}(t) \vec{Z}(t, x) \, . \tag{4.3.1}$$

The proof crucially relies on a kernel estimate and moment bounds on Skorokhod correction terms that will be proved in Section 4.3.2.

4.3.1 Proof of Theorem 4.1.7

By Proposition 4.2.6, we know that

$$\|C_{s,y}(t) - \overleftarrow{Z}(s,y)\|_p \lesssim \beta(1+t-s)^{\frac{2-\kappa}{4}} .$$

Hence, since $\kappa > 2$, Theorem 4.1.7 follows from the moment estimate on $\rho_{s,y}(t,x)$ of the following proposition.

Proposition 4.3.1. *Let $\mu = \frac{1}{2} - \frac{1}{\kappa}$. Then, given $p \geqslant 1$, there exists $\tilde{\beta}(p) > 0$, such that for $\beta \leqslant \tilde{\beta}$, the following holds*

$$\sup_{x,y} \sup_{t>s\geqslant 0} \frac{\|\rho_{s,y}(t,x)\|_p}{(1+t-s)^{-\mu}(1+(1+t-s)^{-\frac{1}{2}}|x-y|)} \lesssim \beta . \tag{4.3.2}$$

Proof. Let us consider $\tilde{w}_{s,y}(t,x) = w_{s,y}(t,x)/P_{t-s}(x-y)$. Since $w_{s,y}(t,x)$ solves

$$w_{s,y}(t,x) = P_{t-s}(x-y) + \beta \int_s^t \int_{\mathbb{R}^d} P_{t-r}(x-z) w_{s,y}(r,z)\, \xi(dr,dz) , \tag{4.3.3}$$

one has the identity

$$\tilde{w}_{s,y}(t,x) = 1 + \beta \int_s^t \int_{\mathbb{R}^d} q^{x,y}_{s,t}(r,z) \tilde{w}_{s,y}(r,z)\, \xi(dr,dz) , \tag{4.3.4}$$

where $q^{x,y}_{s,t}$ is kernel of the Brownian bridge starting from s at x and ending at y at time t,

$$q^{x,y}_{s,t}(r,z) = \tfrac{P_{r-s}(z-y)P_{t-r}(x-z)}{P_{t-s}(x-y)} = P_{(r-s)(t-r)/(t-s)}(z - y - \tfrac{r-s}{t-s}(x-y)) . \tag{4.3.5}$$

On the other hand, integrating (4.3.3) over x, we obtain

$$C_{s,y}(t) = 1 + \beta \int_s^t \int_{\mathbb{R}^d} P_{r-s}(z-y) \tilde{w}_{s,y}(r,z) \xi(dr,dz) . \tag{4.3.6}$$

As a consequence, we can wrte $\rho_{s,y}$ as

$$\begin{aligned}\rho_{s,y}(t,x) &= \tilde{w}_{s,y}(t,x) - C_{s,y}(t)\vec{Z}(t,x)\\ &= 1 + \beta\int_s^t\int_{\mathbb{R}^d} q_{s,t}^{x,y}(r,z)\tilde{w}_{s,y}(r,z)\,\xi(dr,dz)\\ &\quad - \Big(1+\beta\int_{-\infty}^t\int_{\mathbb{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)\,\xi(dr,dz)\Big)C_{s,y}(t)\\ &= \beta\int_s^t\int_{\mathbb{R}^d}\big(q_{s,t}^{x,y}(r,z) - P_{r-s}(y-z)\big)\tilde{w}_{s,y}(r,z)\,\xi(dr,dz)\\ &\quad - \beta\int_{-\infty}^t\int_{\mathbb{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)\,\xi(dr,dz)\,C_{s,y}(t)\;.\end{aligned} \tag{4.3.7}$$

To treat the term $C_{s,y}(t)$, we split the last integral into two parts, integrating over $r\in(-\infty,s)$ leads to a contribution proportional to

$$E_1 := \int_{-\infty}^s\int_{\mathbb{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)\,\xi(dr,dz)\,C_{s,y}(t)\;. \tag{4.3.8}$$

For the other region $r\in[s,t]$, we rewrite it in terms of a Skorokhod integral using the fact (cf. Proposition 2.2.2) that $F\int G\,d\xi = \delta(FG) + \langle G, DF\rangle_{\mathcal{H}}$ with $F = C_{s,y}(t)$ and $G(r,z) = P_{t-r}(x-z)\vec{Z}(r,z)\mathbf{1}_{[s,t]}(r)$. Moreover, $C_{s,y}(r)$ is $\mathcal{F}_r$-adapted and we obtain, writing $\delta\xi$ for the Skorokhod integration:

$$\begin{aligned}&\int_s^t\int_{\mathbb{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)\,\xi(dr,dz)\,C_{s,y}(t)\\ &\quad= \int_s^t\int_{\mathbb{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)C_{s,y}(t)\,\delta\xi(dr,dz)\\ &\qquad+\int_s^t\int_{\mathbb{R}^{2d}} P_{t-r}(x-z_1)\vec{Z}(r,z_1)\,D_{r,z_2}C_{s,y}(t)R(z_1-z_2)\,dr\,dz\\ &\quad= \int_s^t\int_{\mathbb{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)C_{s,y}(r)\,\xi(dr,dz)\\ &\qquad+\int_s^t\int_{\mathbb{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)\big(C_{s,y}(t)-C_{s,y}(r)\big)\,\delta\xi(dr,dz)\\ &\qquad+\int_s^t\int_{\mathbb{R}^{2d}} P_{t-r}(x-z_1)\vec{Z}(r,z_1)\,D_{r,z_2}C_{s,y}(t)R(z_1-z_2)\,dr\,dz\;.\end{aligned}$$

Bringing all these back to (4.3.7) and given the definition (4.3.1) of $\rho_{s,y}$, we obtain

$$\begin{aligned}&\rho_{s,y}(t,x) =\\ &\quad\beta\int_s^t\int_{\mathbb{R}^d} P_{t-r}(x-z)\rho_{s,y}(r,z)\,\xi(dr,dz) + \beta E_0 - \beta\sum_{i=1}^3 E_i\;,\end{aligned} \tag{4.3.9}$$

where E_1 was defined in (4.3.8) and the other terms are given by

$$E_0 := \int_s^t \int_{\mathbb{R}^d} \big(q_{s,t}^{x,y}(r,z) - P_{r-s}(y-z) - P_{t-r}(x-z)\big)\tilde{w}_{s,y}(r,z)\,\xi(dr,dz)\;,$$
$$E_2 := \int_s^t \int_{\mathbb{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)\big(C_{s,y}(t) - C_{s,y}(r)\big)\,\delta\xi(dr,dz)\;,$$
$$E_3 := \int_s^t \int_{\mathbb{R}^{2d}} P_{t-r}(x-z_1)\vec{Z}(r,z_1)\,D_{r,z_2}C_{s,y}(t)R(z_1-z_2)\,dr\,dz\;.$$

We can now estimate their L^p norms.

Given the uniform moments bounds, on $C_{s,y}$ and $\vec{Z}$, from Lemma 4.2.2 and the covariance estimate (2.4.6), for $\beta \leqslant \beta_0(2p)$ we have

$$\begin{aligned}\|E_1\|_p &\lesssim \Big\| \int_{-\infty}^s P_{t-r}(x-z)\vec{Z}(r,z)\,\xi(dr,dz)\Big\|_{2p} \|C_{s,y}(t)\|_{2p} \\ &\lesssim \Big(\int_{-\infty}^s \int_{\mathbb{R}^{2d}} \prod_{j=1}^2 \Big(P_{t-r}(x-z_j)\|\vec{Z}(r,z_j)\|_{2p}\Big) R(z_1-z_2)\,dr\,dz\Big)^{\frac{1}{2}} \\ &\lesssim \Big(\int_{-\infty}^s (1+t-r)^{-\frac{\kappa}{2}}dr\Big)^{\frac{1}{2}} \lesssim (1+t-s)^{\frac{2-\kappa}{4}}\;,\end{aligned}$$

where we used (4.2.4) to estimate the stochastic integral of the first line.

For E_0, we note that by (4.2.7),

$$\sup_{x,y}\sup_{t>s\geqslant 0} \|\tilde{w}_{s,y}(t,x)\|_p \lesssim 1\;.$$

Then, estimating moments via BDG inequality (4.2.4) as just done above, followed by a change of variables $z_i \mapsto z_i - y$, $r \mapsto r-s$, and Lemma 4.3.3 below, we can bound E_0 as follows for $\beta \leqslant \beta_0(p)$:

$$\begin{aligned}&\|E_0\|_p \\ &\quad\leqslant C_p\Big(\int_s^t \int_{\mathbb{R}^{2d}} \prod_{i=1}^2 \big(q_{s,t}^{x,y}(r,z_i) - P_{r-s}(y-z_i) - P_{t-r}(x-z_i)\big)\times \\ &\qquad\qquad R(z_1-z_2)\,dr\,dz\Big)^{\frac{1}{2}} \\ &\quad= C_p\Big(\int_0^{t-s} \int_{\mathbb{R}^{2d}} \prod_{i=1}^2 \big(q_{0,t-s}^{x-y,0}(r,z_i) - P_{r-s}(z_i) - P_{t-r}(x-y-z_i)\big)\times \\ &\qquad\qquad R(z_1-z_2)\,dr\,dz\Big)^{\frac{1}{2}} \\ &\quad\lesssim (1+t-s)^{-\mu}(1+(1+t-s)^{-\frac{1}{2}}|x-y|)\;.\end{aligned}$$

By Lemma 4.3.2 below, for $\beta \leqslant \beta_1(3p) \wedge \beta_0(3p)$ also the terms E_2 and E_3 satisfy

$$\|E_i\|_p \lesssim (1+t-s)^{-\mu} , \qquad i = 2, 3 .$$

Among the estimates, the worst decay rate is the bound on E_0, and we obtain

$$\sum_{i=0}^{3} \|E_i\|_p \lesssim m(t-s, x-y) ,$$

where we denote the space-time weight

$$m(r, z) = (1+r)^{-\mu}(1 + (1+r)^{-\frac{1}{2}}|z|) .$$

Finally, let us consider the weighted norm

$$M := \sup_{x,y} \sup_{t>s\geqslant 0} \frac{\|\rho_{s,y}(t,x)\|_p}{m(t-s, x-y)} .$$

We estimate the first term in (4.3.9) as follows:

$$\begin{aligned}
&\Big\| \int_s^t \int_{\mathbb{R}^d} P_{t-r}(x-z)\rho_{s,y}(r,z)\, \xi(dr, dz)\Big\|_p \\
&\quad \leqslant C_p \Big(\int_s^t \int_{\mathbb{R}^{2d}} \prod_{i=1}^{2} \big(P_{t-r}(x-z_i)\|\rho_{s,y}(r,z_i)\|_p\big) R(z_1-z_2)\, dr\, dz \Big)^{\frac{1}{2}} \\
&\quad \lesssim M \Big(\int_s^t (1+r-s)^{-2\mu} \int_{\mathbb{R}^{2d}} \Big(\prod_{i=1}^{2} P_{t-r}(x-z_i)\Big) R(z_1-z_2)\, dr\, dz \Big)^{\frac{1}{2}} \\
&\qquad + M \Big(\sum_{j=1}^{2} \int_s^t (1+r-s)^{-1-2\mu} \ \times \\
&\qquad\qquad \int_{\mathbb{R}^{2d}} \Big(\prod_{i=1}^{2} |z_j - y|^2 P_{t-r}(x-z_i)\Big) R(z_1-z_2)\, dr\, dz \Big)^{\frac{1}{2}} \\
&\quad \lesssim M \Big(\int_s^t (1+r-s)^{-2\mu}(1+t-r)^{-\frac{\kappa}{2}}\, dr \Big)^{\frac{1}{2}} \\
&\qquad + M \Big(\int_s^t (1+r-s)^{-1-2\mu}(1+t-r)^{-\frac{\kappa}{2}} \big(|x-y|^2 + (t-r)\big)\, dr dr \Big)^{\frac{1}{2}} ,
\end{aligned} \tag{4.3.10}$$

where we used (2.4.6) and the fact that $\int_{\mathbb{R}^d} |z|^2 P_r(z-x)dz \lesssim r + |x|^2$ for the second integral. For $\alpha, \beta > 0$, we note

$$\int_0^T (1+\tau)^{-\alpha}(1+T-\tau)^{-1-\beta}\, d\tau \lesssim (1+T)^{-\min(\alpha,\beta)} . \tag{4.3.11}$$

This follows by splitting the integration region. On the interval $[0, T/2]$, we have $T - \tau \geqslant T/2$ and

$$\int_0^{\frac{T}{2}} (1+\tau)^{-\alpha}(1+T-\tau)^{-1-\beta}\, d\tau \lesssim (1+T)^{-1-\beta}T \lesssim (1+T)^{-\beta} ,$$

while on $[T/2, T]$, we have

$$\int_{\frac{T}{2}}^{T} (1+\tau)^{-\alpha}(1+T-\tau)^{-1-\beta}\, d\tau \lesssim (1+T)^{-\alpha} \int_0^{\infty} (1+r)^{-1-\beta}\, dr \lesssim (1+T)^{-\alpha} .$$

Then, going back to (4.3.10), we apply (4.3.11) to both terms and obtain

$$\Big\| \int_s^t \int_{\mathbb{R}^d} P_{t-r}(x-z)\rho_{s,y}(r,z)\, \xi(dr,dz) \Big\|_p \lesssim M\ m(t-s, x-y) .$$

Hence, by the estimates above, for some $C > 0$ we have

$$\|\rho_{s,y}(t,x)\|_p \leqslant C\beta M\, m(t-s,x-y) + C\beta\, m(t-s,x-y) .$$

Taking $\tilde{\beta} > 0$ such that $C\tilde{\beta} < 1$ and $\tilde{\beta} \leqslant \beta_1(3p) \wedge \beta_0(3p)$, we deduce (4.3.2) and conclude the proof. □

4.3.2 Kernel estimate and Skorokhod correction terms

We are now going to prove the Lemmas employed in the proof of our main homogenisation result. Note that the kernel estimate of Lemma 4.3.3 determines the L^p homogenisation rate.

Lemma 4.3.2. *Let $p \geqslant 1$. With notation as in the proof of Proposition 4.3.1, for any $\beta \leqslant \beta_1(3p) \wedge \beta_0(3p)$, for $i = 2, 3$, we have the following uniformly in $x, y \in \mathbb{R}^d$ and $t > s \geqslant 0$:*

$$\|E_2(s,t,y,x)\|_p \lesssim \beta\, (1+t-s)^{\frac{2-\kappa}{4}} ,$$
$$\|E_3(s,t,y,x)\|_p \lesssim \beta\, (1+t-s)^{1-\frac{\kappa}{2}} .$$

Proof. Let us start with the Skorokhod correction term E_3. Similarly to Lemma 4.2.2, where we obtained $D_{s,y}u(t,x) = \beta u(s,y) w_{s,y}(t,x)\mathbf{1}_{[0,t]}(s)$, one has

$$D_{r,z}w_{s,y}(t,x) = \beta w_{s,y}(r,z) w_{r,z}(t,x)\mathbf{1}_{[s,t]}(r) ,$$

so that, integrating over x,

$$D_{r,z}C_{s,y}(t) = \beta w_{s,y}(r,z)C_{r,z}(t)\mathbf{1}_{[s,t]}(r)\ . \tag{4.3.12}$$

Therefore, the term E_3 is given by

$$\begin{aligned} E_3 &= \int_s^t \int_{\mathbb{R}^{2d}} P_{t-r}(x-z_1)\vec{Z}(r,z_1)\, D_{r,z_2}C_{s,y}(t)R(z_1-z_2)\,dr\,dz \\ &= \beta \int_s^t \int_{\mathbb{R}^{2d}} P_{t-r}(x-z_1)\vec{Z}(r,z_1)\, w_{s,y}(r,z_2)C_{r,z_2}(t)R(z_1-z_2)\,dr\,dz\ . \end{aligned}$$

Using the moments estimates on $\vec{Z}$, w and C from Lemma 4.2.2, by Minkowski and Hölder's inequalities, followed by (2.4.8), we obtain for $\beta \leqslant \beta_0(3p)$:

$$\begin{aligned} \|E_3\|_p &\leqslant \beta \int_s^t \int_{\mathbb{R}^{2d}} P_{t-r}(x-z_1)\|\vec{Z}(r,z_1)w_{s,y}(r,z_2)C_{r,z_2}(t)\|_p R(z_1-z_2)\,dr\,dz \\ &\lesssim \beta \int_s^t \int_{\mathbb{R}^{2d}} P_{t-r}(x-z_1)P_{r-s}(z_2-y)R(z_1-z_2)\,dr\,dz \\ &\lesssim \beta(1+t-s)^{1-\frac{\kappa}{2}}\ . \end{aligned}$$

To estimate the L^p norm of E_2, we use the continuity of the Skorokhod integral $\delta : \mathbb{D}^{1,p}(\mathcal{H}) \to L^p(\Omega)$, cf. Proposition 2.2.3. Letting

$$F_{s,t,x,y}(r,z) = P_{t-r}(x-z)\vec{Z}(r,z)\big(C_{s,y}(t) - C_{s,y}(r)\big)\mathbf{1}_{[s,t]}(r)\ ,$$

we have $E_2 = \delta(F_{s,t,x,y})$. Therefore, dropping subscripts of F going forward, we have

$$\|E_2\|_p \lesssim \|F\|_{L^p(\Omega,\mathcal{H})} + \|DF\|_{L^p(\Omega,\mathcal{H}\otimes\mathcal{H})}\ . \tag{4.3.13}$$

We see that F is a pointwise product of a Malliavin differentiable random variable M and a deterministic element $h \in \mathcal{H}$. Namely, $F(r,z) = M(r,z)h(r,z)$, where

$$M(r,z) = \vec{Z}(r,z)\big(C_{s,y}(t) - C_{s,y}(r)\big)\ , \qquad h(r,z) = P_{t-r}(x-z)\mathbf{1}_{[s,t]}(r)\ .$$

Given (4.2.8), we note that

$$\|M(r,z)\|_p \lesssim \|\vec{Z}(r,z)\|_{2p}\|C_{s,y}(t) - C_{s,y}(r)\|_{2p} \lesssim \beta(1+r-s)^{\frac{2-\kappa}{4}}\ .$$

With this, using Minkowski inequality, followed by (2.4.6) and (4.3.11), we can bound the first term as follows:

$$\begin{aligned}
\|F\|_{L^p(\Omega,\mathcal{H})} &= \mathbb{E}\Big[\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^{2d}} F(r,z_1)F(r,z_2)R(z_1-z_2)\,dr\,dz\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\
&\leqslant \Big(\int_s^t \int_{\mathbb{R}^{2d}} \|M(r,z_1)M(r,z_2)\|_{\frac{p}{2}}\, h(r,z_1)h(r,z_2)R(z_1-z_2)\,dr\,dz\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\beta^2\int_s^t (1+r-s)^{1-\frac{\kappa}{2}} \int_{\mathbb{R}^{2d}} P_{t-r}(x-z_1)P_{t-r}(x-z_2)R(z_1-z_2)\,dr\,dz\Big)^{\frac{1}{2}} \\
&\lesssim \beta\Big(\int_s^t (1+r-s)^{1-\frac{\kappa}{2}}(1+t-r)^{-\frac{\kappa}{2}}\,dr\Big)^{\frac{1}{2}} \lesssim \beta(1+t-s)^{\frac{2-\kappa}{4}}\,.
\end{aligned}$$

For the derivative term, by Minkowski and Hölder's inequalities we have

$$\begin{aligned}
&\|DF\|_{L^p(\Omega,\mathcal{H}\otimes\mathcal{H})} \\
&= \mathbb{E}\Big[\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^{2d}} \langle DM(r,z_1), DM(r,z_2)\rangle_{\mathcal{H}}\; h(r,z_1)h(r,z_2)R(z_1-z_2)\,dz\,dr\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\
&\leqslant \Big(\int_s^t \int_{\mathbb{R}^{2d}} \|\langle DM(r,z_1), DM(r,z_2)\rangle_{\mathcal{H}}\|_{\frac{p}{2}}\, h(r,z_1)h(r,z_2)R(z_1-z_2)\,dz\,dr\Big)^{\frac{1}{2}} \\
&\leqslant \Big(\int_s^t \int_{\mathbb{R}^{2d}} \Big(\prod_{i=1}^{2} \|DM(r,z_i)\|_{L^p(\Omega,\mathcal{H})}\, h(r,z_i)\Big)R(z_1-z_2)\,dz\,dr\Big)^{\frac{1}{2}}\,.
\end{aligned}$$

We note that $\vec{Z}$ and C's are adapted, satisfying identities (4.2.9) and (4.3.12) respectively. Then, by the chain rule,

$$\begin{aligned}
&D_{u,\bar{z}}M(r,z) \\
&\quad= D_{u,\bar{z}}\vec{Z}(r,z)\,\big(C_{s,y}(t)-C_{s,y}(r)\big) + \vec{Z}(r,z)\,D_{u,\bar{z}}\big(C_{s,y}(t)-C_{s,y}(r)\big) \\
&\quad= \beta w_{u,\bar{z}}(r,z)\vec{Z}(u,\bar{z})\big(C_{s,y}(t)-C_{s,y}(r)\big)\mathbf{1}_{[s,r]}(u) \\
&\qquad+ \beta\vec{Z}(r,z)w_{s,y}(u,\bar{z})\big(C_{u,\bar{z}}(t)-C_{u,\bar{z}}(r)\big)\mathbf{1}_{[s,r]}(u) \\
&\qquad+ \beta\vec{Z}(r,z)w_{s,y}(u,\bar{z})C_{u,\bar{z}}(t)\mathbf{1}_{[r,t]}(u) =: \beta\sum_{j=1}^{3} m^{(j)}_{r,z}(u,\bar{z})\,.
\end{aligned}$$

By Minkowski and Hölder inequalities

$$\begin{aligned}
&\|DM(r,z)\|_{L^p(\Omega,\mathcal{H})} \lesssim \\
&\quad\sum_{j,k=1}^{3}\Big(\int_s^t\int_{\mathbb{R}^{2d}} \|m^{(j)}_{r,z}(u,\bar{z}_1)\|_p\|m^{(k)}_{r,z}(u,\bar{z}_2)\|_p\, R(\bar{z}_1-\bar{z}_2)\,d\bar{z}\,du\Big)^{\frac{1}{2}}. \qquad (4.3.14)
\end{aligned}$$

Given the a priori estimates from Lemma 4.2.2, for $\beta \leqslant \beta_0(3p)$ we have:

$$\begin{aligned}
\|m^{(1)}_{r,z}(u,\bar{z})\|_p &\lesssim P_{r-u}(z-\bar{z})(1+r-s)^{\frac{2-\kappa}{4}}\mathbf{1}_{[s,r]}(u)\ ,\\
\|m^{(2)}_{r,z}(u,\bar{z})\|_p &\lesssim P_{u-s}(\bar{z}-y)(1+r-u)^{\frac{2-\kappa}{4}}\mathbf{1}_{[s,r]}(u)\ ,\\
\|m^{(3)}_{r,z}(u,\bar{z})\|_p &\lesssim P_{u-s}(\bar{z}-y)\mathbf{1}_{[r,t]}(u)\ .
\end{aligned}$$

Hence, by Lemma 2.4.5, for $j=k=1$ we obtain

$$\begin{aligned}
&\int_s^t\int_{\mathbb{R}^{2d}}\Big(\prod_{i=1}^2\|m^{(1)}_{r,z}(u,\bar{z}_i)\|_p\Big)R(\bar{z}_1-\bar{z}_2)\,d\bar{z}\,du\\
&\quad\lesssim (1+r-s)^{1-\frac{\kappa}{2}}\int_s^r\int_{\mathbb{R}^{2d}}\Big(\prod_{i=1}^2 P_{r-u}(z-\bar{z}_i)\Big)R(\bar{z}_1-\bar{z}_2)\,d\bar{z}\,du\\
&\quad\lesssim (1+r-s)^{1-\frac{\kappa}{2}}\ .
\end{aligned}$$

On the other hand, by (2.4.6) and (4.3.11),

$$\begin{aligned}
&\int_s^t\int_{\mathbb{R}^{2d}}\|m^{(1)}_{r,z}(u,\bar{z}_1)\|_p\|m^{(2)}_{r,z}(u,\bar{z}_2)\|_p\,R(\bar{z}_1-\bar{z}_2)\,d\bar{z}\,du\\
&\quad\lesssim (1+r-s)^{\frac{2-\kappa}{4}}\int_s^r(1+r-u)^{\frac{2-\kappa}{4}}\int_{\mathbb{R}^{2d}}P_{r-u}(z-\bar{z}_1)P_{u-s}(\bar{z}_2-y)\,R(\bar{z}_1-\bar{z}_2)\,d\bar{z}\,du\\
&\quad\lesssim (1+r-s)^{\frac{2-\kappa}{4}}\int_s^r(1+r-u)^{\frac{2-\kappa}{4}}(1+u-s)^{-\frac{\kappa}{2}}\,du\lesssim(1+r-s)^{1-\frac{\kappa}{2}}\ .
\end{aligned}$$

Estimating in a similar way the other terms in (4.3.14) involving $m^{(2)}_{r,z}$ and $m^{(3)}_{r,z}$, we obtain

$$\|DM(r,z)\|_{L^p(\Omega,\mathcal{H})}\lesssim\beta(1+r-s)^{\frac{2-\kappa}{4}}\ .$$

With this, using again (2.4.6) and (4.3.11), we return to estimate DF:

$$\begin{aligned}
&\|DF\|_{L^p(\Omega,\mathcal{H}\otimes\mathcal{H})}\\
&\quad\lesssim\beta\Big(\int_s^t(1+r-s)^{1-\frac{\kappa}{2}}\int_{\mathbb{R}^{2d}}P_{t-r}(x-z_1)P_{t-r}(x-z_2)R(z_1-z_2)\,dr\,dz\,dz\,dr\Big)^{\frac12}\\
&\quad\lesssim\beta\Big(\int_s^t(1+r-s)^{1-\frac{\kappa}{2}}(1+t-r)^{-\frac{\kappa}{2}}\,dr\Big)^{\frac12}\lesssim\beta(1+t-s)^{\frac{2-\kappa}{4}}\ .
\end{aligned}$$

Gathering bounds back to (4.3.13), we obtain $\|E_2(s,t,y,x)\|_p\lesssim\beta\,(1+t-s)^{\frac{2-\kappa}{4}}$ and conclude the proof. □

Lemma 4.3.3. *Let $x \in \mathbb{R}^d$, $t > 0$, $P_r(x)$ the heat kernel, and let*

$$H_{t,x}(r,z) = P_{r(t-r)/t}(z + \tfrac{r}{t}x) - P_r(z) - P_{t-r}(z+x) \ .$$

Let $\tilde{\mu} = 1 - \frac{2}{\kappa}$, then the following holds:

$$J_t(x) := \int_0^t \int_{\mathbb{R}^{2d}} \Big(\prod_{i=1}^{2} |H_{t,x}(r,z_i)| \Big) R(z_1 - z_2)\, dz\, dr \lesssim (1+t)^{-\tilde{\mu}} (1 + (1+t)^{-1}|x|^2) \ .$$

Remark 4.3.4. Since $P_{r(t-r)/t}(z + \frac{r}{t}x) = q^{x,0}_{0,t-s}(r,z)$, this is really just a quantitative way of stating that, when considered over a very large time interval, a Brownian bridge is close to a Brownian motion near either end of the interval. Furthermore, the integral J_t is such that the contribution coming from the "bulk" is small.

Proof. Let $\alpha = \frac{2}{\kappa}$ such that $\tilde{\mu} = \alpha(\frac{\kappa}{2} - 1)$. By Lemma 2.4.5, we have $J_{t,x} \lesssim 1$ uniformly in t. Then, in the following we can assume $t > 2^{\frac{1}{1-\alpha}}$. We consider three regions of integration,

$$J_t(x) = J_{0,t^\alpha}(x) + J_{t^\alpha, t-t^\alpha}(x) + J_{t-t^\alpha,t}(x) \ ,$$

where

$$J_{t_1,t_2}(x) = \int_{t_1}^{t_2} \int_{\mathbb{R}^{2d}} \Big(\prod_{i=1}^{2} |H_{t,x}(r,z_i)| \Big) R(z_1 - z_2)\, dz\, dr \ .$$

By changing variables $z \mapsto z + x$ and $r \mapsto t - r$, we have $J_{0,t^\alpha}(x) = J_{t-t^\alpha,t}(-x)$. Hence, it suffices to estimate $J_{0,t^\alpha}(x)$ and $J_{t^\alpha,t-t^\alpha}(x)$.

For $J_{t^\alpha,t-t^\alpha}(x)$, since $r \in [t^\alpha, t - t^\alpha]$, using (2.4.6) for each Gaussian kernel:

$$\begin{aligned} J_{t^\alpha,t-t^\alpha}(x) &= \int_{t^\alpha}^{t-t^\alpha} \int_{\mathbb{R}^d} |H_{t,x}(r,z_1)| \Big(\int_{\mathbb{R}^d} |H_{t,x}(r,z_2)|\, R(z_1 - z_2)\, dz_2 \Big)\, dz_1\, dr \\ &\lesssim \int_{t^\alpha}^{t-t^\alpha} r^{-\frac{\kappa}{2}} \int_{\mathbb{R}^d} |H_{t,x}(r,z_1)|\, dz_1\, dr \lesssim t^{\alpha(1-\frac{\kappa}{2})} = t^{-\mu} \ . \end{aligned}$$

In the remainder we estimate $J_{0,t^\alpha}(x)$. For this, we split $H_{t,x}$ into two terms:

$$|H_{t,x}(r,z)| = H^1_{t,x}(r,z) + H^2_{t,x}(r,z) \ ,$$

where

$$H^{(1)}_{t,x}(r,z) = |P_{r(t-r)/t}(z+\tfrac{r}{t}x) - P_r(z)|\;, \qquad H^{(2)}_{t,x}(r,z) = P_{t-r}(z+x)\;.$$

So that,

$$J_{0,t^\alpha}(x) \leqslant \sum_{i,j=1}^{2} \int_0^{t^\alpha} \int_{\mathbb{R}^{2d}} H^{(i)}_{t,x}(r,z_1) H^{(j)}_{t,x}(r,z_2)\, R(z_1-z_2)\, dz\, dr =: \sum_{i,j=1}^{2} J^{(i,j)}_{0,t^\alpha}(x)\;.$$

By (2.4.6) from Lemma 2.4.5, we have uniformly in z_1:

$$\int_{\mathbb{R}^d} H^{(2)}_{t,x}(r,z_2)\, R(z_1-z_2)\, dz_2 \lesssim (1+t-r)^{-\frac{\kappa}{2}}\;.$$

Hence, recalling $t > 2^{\frac{1}{1-\alpha}}$, for any $i = 1, 2$, we have

$$J^{(i,2)}_{0,t^\alpha}(x) \lesssim \int_0^{t^\alpha} (t-r)^{-\frac{\kappa}{2}} \int_{\mathbb{R}^d} H^{(i)}_{t,x}(r,z_1)\, dz_1\, dr \lesssim (t-t^\alpha)^{1-\frac{\kappa}{2}} \lesssim t^{1-\frac{\kappa}{2}}\;.$$

We now need to estimate $J^{(1,1)}_{0,t^\alpha}(x)$. We rewrite

$$H^{(1)}_{t,x}(r,z) = \big|P_{r(t-r)/t}(z+\tfrac{r}{t}x) - P_r(z)\big| \leqslant \int_0^1 \big|\partial_\theta\, P_{r-\theta\frac{r^2}{t}}(z+\theta\tfrac{r}{t}x)\big|\, d\theta\;.$$

Letting $v = r^2/t$ and $w = rx/t$, with $\theta \in [0,1]$ we can compute

$$\begin{aligned}\big|\partial_\theta\, P_{r-\theta v}(z+\theta w)\big| &= \Big|\frac{dv}{2(r-\theta v)} + \frac{v|z+\theta w|^2}{2(r-\theta v)^2} - \frac{\langle z,w\rangle + \theta|w|^2}{r-\theta v}\Big| P_{r-\theta v}(z+\theta w) \\ &\lesssim \Big(\frac{v}{r-v} + \frac{v(|z|^2+|w|^2)}{(r-v)^2} + \frac{|z||w|+|w|^2}{r-v}\Big) P_{r-\theta v}(z+\theta w)\;.\end{aligned}$$

Given $r < t^\alpha$ and $t > 2^{\frac{1}{1-\alpha}}$, we have

$$\begin{aligned}\frac{v}{r-v} &= \frac{r/t}{1-r/t} \lesssim t^{\alpha-1}\;, \qquad |w| \leqslant t^{\alpha-1}|x|\;, \\ \frac{v}{(r-v)^2} &\leqslant \frac{v}{r^2-2rv} \lesssim t^{-1}\;, \qquad \frac{|w|}{r-v} = \frac{r|x|/t}{r-r^2/t} \lesssim t^{-1}|x|\;.\end{aligned}$$

Therefore, letting $\mu = \alpha(\kappa/2-1)/2 = \tilde{\mu}/2$, we have the following bound

$$\begin{aligned}&\big|\partial_\theta\, P_{r-\theta r^2/t}(z+\theta\tfrac{r}{t}x)\big| / P_{r-\theta r^2/t}(z+\theta\tfrac{r}{t}x) \\ &\quad\lesssim t^{\alpha-1} + t^{-1}\big(|z|^2 + t^{2(\alpha-1)}|x|^2\big) + t^{-1}\big(|x||z| + t^{\alpha-1}|x|^2\big)\end{aligned}$$

$$\begin{aligned}
&\lesssim t^{\alpha-1} + t^{-1}\big(|z|^2 + |x||z| + t^{\alpha-1}|x|^2\big)\\
&\lesssim t^{\alpha-1} + (t^{-1-2\mu} + t^{\alpha-2})|x|^2 + (t^{-1} + t^{-1+2\mu})|z|^2\\
&\lesssim t^{\alpha-1} + t^{-1-2\mu}|x|^2 + t^{-1+2\mu}|z|^2\,,
\end{aligned}$$

where we used the fact that $|x||z| \lesssim t^{-2\mu}|x|^2 + t^{2\mu}|z|^2$ and $\alpha - 1 \leqslant -2\mu$ when $\alpha \leqslant 2/\kappa$. Thus,

$$H^{(1)}_{t,x}(r,z) \lesssim \big(t^{\alpha-1} + t^{-1-2\mu}|x|^2 + t^{-1+2\mu}|z|^2\big)\int_0^1 P_{r-\theta r^2/t}(z + \theta\tfrac{r}{t}x)\,d\theta\,.$$

In view of this, we have two contributions:

$$\begin{aligned}
&J^{(1,1)}_{0,t^\alpha}(x) \lesssim \qquad\qquad (4.3.15)\\
&\big(t^{\alpha-1} + t^{-1-2\mu}|x|^2\big)\int_0^1\int_0^{t^\alpha}\int_{\mathbb{R}^{2d}} P_{r-\theta r^2/t}(z_1 + \theta\tfrac{r}{t}x)\,H^{(1)}_{t,x}(r,z_2)\,R(z_1-z_2)\,d\theta\,dz\,dr\\
&+ t^{-1+2\mu}\int_0^1\int_0^{t^\alpha}\int_{\mathbb{R}^{2d}} P_{r-\theta r^2/t}(z_1 + \theta\tfrac{r}{t}x)|z_1|^2\,H^{(1)}_{t,x}(r,z_2)\,R(z_1-z_2)\,d\theta\,dz\,dr
\end{aligned}$$

Employing Lemma 2.4.5, we now estimate the inner integrals uniformly in $\theta \in [0,1]$. By (2.4.7), it immediately follows that

$$\int_0^{t^\alpha}\int_{\mathbb{R}^{2d}} P_{r-\theta r^2/t}(z_1 + \theta\tfrac{r}{t}x)H^{(1)}_{t,x}(r,z_2)\,R(z_1-z_2)\,dr\,dz \lesssim 1\,.$$

Since $\int_{\mathbb{R}^d}|z|^2P_r(z+x)dz \lesssim r + |x|^2$, expanding $H^{(1)}_{t,x}$ via triangle inequality and using (2.4.6), we have

$$\begin{aligned}
&\int_0^{t^\alpha}\int_{\mathbb{R}^{2d}} |z_1|^2 P_{r-\theta r^2/t}(z_1 + \theta\tfrac{r}{t}x)\,H^{(1)}_{t,x}(r,z_2)\,R(z_1-z_2)\,dr\,dz\\
&\quad= \int_0^{t^\alpha}\int_{\mathbb{R}^d} |z_1|^2 P_{r-\theta r^2/t}(z_1 + \theta\tfrac{r}{t}x)\Big(\int_{\mathbb{R}^d} H^{(1)}_{t,x}(r,z_2)R(z_1-z_2)\,dz_2\Big)\,dz_1\,dr\\
&\quad\lesssim \int_0^{t^\alpha}\big((r-\theta\tfrac{r^2}{t}) + \theta^2\tfrac{r^2}{t^2}|x|^2\big)\big((r(t-r)/t)^{-\frac{\kappa}{2}} + r^{-\frac{\kappa}{2}}\big)\big)\,dr\\
&\quad\lesssim \int_0^{t^\alpha} r^{1-\frac{\kappa}{2}} + r^{2-\frac{\kappa}{2}}t^{-2}|x|^2\,dr \lesssim t^{\alpha(2-\frac{\kappa}{2})} + t^{\alpha(3-\frac{\kappa}{2})-2}|x|^2\\
&\quad= t^{\alpha-2\mu} + t^{2\alpha-2-2\mu}|x|^2\,.
\end{aligned}$$

Bringing these estimates back to (4.3.15), we obtain

$$J^{(1,1)}_{0,t^\alpha}(x) \lesssim t^{\alpha-1} + t^{-1-2\mu}|x|^2 + t^{2\alpha-3}|x|^2 \lesssim t^{\alpha-1} + t^{-1-2\mu}|x|^2\,.$$

Therefore, since $\max(1-\frac{\kappa}{2}, \alpha-1) \leqslant -2\mu$, we obtain

$$J_{0,t^\alpha}(x) \lesssim t^{1-\frac{\kappa}{2}} + t^{\alpha-1} + t^{-1-2\mu}|x|^2 \lesssim t^{-2\mu} + t^{-1-2\mu}|x|^2 \,.$$

Combining the estimates of $J_{0,t^\alpha}(x)$ and $J_{t^\alpha,t-t^\alpha}(x)$, we conclude the proof. □

4.4 The KPZ scaling limit

In order to prove our KPZ scaling limit, Theorem 4.1.4, we are now going to consider general transformations of u (still as in (4.1.8), i.e. with initial condition equal to 1) and prove the following (restatement of Theorem 4.1.5) for the class of functions Φ below.

Assumption 4.4.1. Let $\Phi \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{R})$ be a function satisfying for some $q > 0$,

$$|\Phi(z)| + |\Phi'(z)| + |\Phi''(z)| \lesssim z^{-q} + z^q \,, \qquad \forall z \in \mathbb{R}_+ \,. \tag{4.4.1}$$

Theorem 4.4.2. *Let $d \geqslant 3$, ξ and Φ satisfy Assumptions 4.1.1 and 4.4.1 respectively and set*

$$u_\varepsilon^\Phi(t,x) = \varepsilon^{1-\frac{\kappa}{2}}\big(\Phi(u_\varepsilon(t,x)) - \mathbb{E}[\Phi(u_\varepsilon(t,x))]\big) \,.$$

Then, given any $p \geqslant 1$, and $\alpha < 1 - \frac{\kappa}{2}$, there exists $\beta_0^\Phi(\alpha,p) > 0$ and $\nu_\Phi \geqslant 0$ such that, for all $\beta < \beta_0^\Phi$ and any good coupling of the noise, the random processes u_ε^Φ converge in probability to $\nu_\Phi \mathcal{U}^0$ in $L^p([0,T],\mathcal{C}^\alpha(E))$ as $\varepsilon \to 0$, for any $T > 0$ and compact $E \subset \mathbb{R}^d$. Here, $\mathcal{U}^0$ denotes the solution to the additive stochastic heat equation

$$\partial_t \mathcal{U}^0 = \tfrac{1}{2}\Delta \mathcal{U}^0 + \beta\xi^0 \,, \qquad \mathcal{U}^0(0,\cdot) \equiv 0 \,.$$

The effective variance is given by $\nu_\Phi = \mathbb{E}[Z\,\Phi'(Z)]$, where Z denotes a real-valued random variable with the same law as $\vec{Z}(t,x)$.

The proof of this theorem will be given in Section 4.4.3 below. Given a test function $g : \mathbb{R}^d \to \mathbb{R}$ and $\Phi : \mathbb{R}_+ \to \mathbb{R}$, in the following we write

$$Y_t^{\varepsilon,g} = \varepsilon^{1-\frac{\kappa}{2}} \int_{\mathbb{R}^d} \big(\Phi(u_\varepsilon(t,x)) - \mathbb{E}[\Phi(u_\varepsilon(t,x))]\big) g(x) dx.$$

In order to prove Theorem 4.4.2, we first derive a divergence representation for $Y_t^{\varepsilon,g}$ by using the Clark–Ocone formula.

Lemma 4.4.3. *Let $\Phi \in \mathcal{C}^2(\mathbb{R}_+)$ satisfy (4.4.1), then for β sufficiently small,*

$$Y_t^{\varepsilon,g} = \varepsilon^{1-\frac{\kappa}{2}} \int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E}[\Phi'(u_\varepsilon(t,x)) D_{s,y} u_\varepsilon(t,x) | \mathcal{F}_s]\, g(x) dx\, \xi(ds,dy). \tag{4.4.2}$$

Proof. By Lemma 4.2.4 and Clark–Ocone formula

$$\begin{aligned} Y_t^{\varepsilon,g} &= \int_{\mathbb{R}\times\mathbb{R}^d} \mathbb{E}[D_{s,y} Y_t^{\varepsilon,g} | \mathcal{F}_s]\, \xi(ds,dy) \\ &= \varepsilon^{1-\frac{\kappa}{2}} \int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E}[D_{s,y} \Phi(u_\varepsilon(t,x)) | \mathcal{F}_s] g(x)\, dx\, \xi(ds,dy)\ , \end{aligned}$$

and the claim follows from the chain rule (4.2.14). □

The strategy of proof of Theorem 4.4.2 is then as follows. We use the expression for the Malliavin derivative of the solution combined with the homogenisation result to argue that, for $\tau \gg s$,

$$\begin{aligned} \mathbb{E}[D_{s,y}\Phi(u(\tau,x))|\mathcal{F}_s] &= \beta \mathbb{E}[\Phi'(u(\tau,x)) u(s,y) w_{s,y}(\tau,x) | \mathcal{F}_s] \\ &\approx \beta \mathbb{E}[\Phi'(\vec{Z}(\tau,x)) \overleftarrow{Z}(s,y) u(s,y) P_{\tau-s}(x-y) \vec{Z}(\tau,x) | \mathcal{F}_s]\ . \end{aligned}$$

At that point, we note that the process $\vec{Z}$ is expected to mix relatively well and that furthermore for $|\tau - s| \gg 1$, $\overleftarrow{Z}(s,y)$ is $\mathcal{F}_s$-independent, of expectation 1, and roughly independent of $\vec{Z}(\tau,x)$ so that one gets

$$\begin{aligned} \mathbb{E}[D_{s,y}\Phi(u(\tau,x))|\mathcal{F}_s] &\approx \beta u(s,y) P_{\tau-s}(x-y) \mathbb{E}[\Phi'(\vec{Z}(\tau,x)) \vec{Z}(\tau,x)] \\ &\stackrel{\text{def}}{=} \beta u(s,y) P_{\tau-s}(x-y) \nu_\Phi\ . \end{aligned} \tag{4.4.3}$$

The approximations are proved in the next subsection. With precise quantitative estimates in place, in Section 4.4.2 we can show $Y_t^{\varepsilon,g}$ converges towards $\mathcal{U}^0$ in Hölder spaces. This allows us to conclude the proof of Theorem 4.4.2 in Section 4.4.3 and consider the KPZ equation with non-flat initial conditions to finally prove our main scaling limit result, Theorem 4.1.4 .

4.4.1 Quantitative approximations

The main result of this section is Proposition 4.4.4 below, which shows that we can approximate $Y_t^{\varepsilon,g}$ by ν_Φ times the large-scale fluctuations of the linear SHE which, by the mild formulation, are given by

$$X_t^{\varepsilon,g} = \beta\varepsilon^{1-\frac{\kappa}{2}} \int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P_{t/\varepsilon^2-s}(x-y)u(s,y)\,\varepsilon^d g(\varepsilon x)\,dx\,\xi(ds,dy)\ , \tag{4.4.4}$$

using the same notation as in the previous chapter. Then, in Proposition 4.4.5, we show that as $\varepsilon \to 0$ these are close to the solution of the stochastic heat equation (4.1.9) driven by noise ξ^ε, which can be written as

$$\mathcal{U}_t^\varepsilon(g) := \beta\varepsilon^{1-\frac{\kappa}{2}} \int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P_{t/\varepsilon^2-s}(x-y)\,\varepsilon^d g(\varepsilon x)\,dx\,\xi(ds,dy)\ . \tag{4.4.5}$$

Since $\mathcal{U}^\varepsilon$ converges to the limit $\mathcal{U}^0$, we conclude that $Y_t^{\varepsilon,g}$ converges to $\nu_\Phi\mathcal{U}^0$ as in Theorem 4.4.2. We note that the strong convergence can be achieved in our settings thanks to the decomposition of X^ε and estimates of Proposition 4.4.5.

Let us now recall the integral defined in the previous chapter

$$J_{\kappa,\delta}(f) \stackrel{\text{def}}{=} \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^2 |f(x_i)||x_i-y_i|^{\delta-d}\Big)|y_1-y_2|^{-\kappa}\,dy\,dx\ , \tag{4.4.6}$$

which satisfies the scaling relations and bound of Lemma 3.4.3.

Proposition 4.4.4. *Let $p \geqslant 1$. For β sufficiently small, for any test function $g \in \mathcal{C}_c(\mathbb{R}^d)$ and $t > 0$, the following holds for any $\delta_0 \in (0,1)$:*

$$\|Y_t^{\varepsilon,g} - \nu_\Phi X_t^{\varepsilon,g}\|_p \lesssim \varepsilon^{\frac{1}{2}\min(1-\delta_0,\,1-\frac{2}{\kappa})}\ , \tag{4.4.7}$$

with precise rate given by (4.4.12).

Proof. Recall (4.2.1), we denote with $u^{(s)}(t,x)$ the solution of linear SHE for $t \geqslant s$ with initial condition 1 at time s. Without loss of generality, assume $\varepsilon < t$ and let $\tau = t/\varepsilon^2$. By Lemma 4.4.3 and the expression for the Malliavin derivative (4.2.14), we can write

$$Y_t^{\varepsilon,g} = \beta\varepsilon^{1-\frac{\kappa}{2}}\int_0^\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{E}[\Phi'(u(\tau,x))u(s,y)w_{s,y}(\tau,x)|\mathcal{F}_s]\,\varepsilon^d g(\varepsilon x)\,dx\,\xi(ds,dy)\ . \tag{4.4.8}$$

Given the large times asymptotics, the homogenisation of $w_{s,y}(\tau, x)$ and mixing of the stationary process, the proof consists in approximating this fluctuations expression through separation of timescales. Namely, given moments control, the contribution coming from $\tau - s < \varepsilon^{-1}$ is negligeable. On the other hand, we let $\Psi(u) = \Phi'(u)u$ and for $\tau - s > \varepsilon^{-1}$ we show

$$\begin{aligned}
&\mathbb{E}[\Phi'(u(\tau,x))u(s,y)w_{s,y}(\tau,x)|\mathcal{F}_s] \\
&\quad\approx \mathbb{E}[\Phi'(\vec{Z}(\tau,x))u(s,y)w_{s,y}(\tau,x)|\mathcal{F}_s] \\
&\quad\approx \mathbb{E}[\Phi'(\vec{Z}(\tau,x))u(s,y)P_{\tau-s}(x-y)C_{s,y}(\tau)\vec{Z}(\tau,x)|\mathcal{F}_s] \\
&\quad\approx \mathbb{E}[u(s,y)P_{\tau-s}(x-y)C_{s,y}(\tfrac{s+\tau}{2})\Psi(u^{(\frac{s+\tau}{2})}(\tau,x))|\mathcal{F}_s] \\
&\quad= P_{\tau-s}(x-y)u(s,y)\mathbb{E}[\Psi(u^{(\frac{s+\tau}{2})}(\tau,x))] \approx P_{\tau-s}(x-y)u(s,y)\nu_\Phi\ .
\end{aligned}$$

Define the following indicator functions to separate timescales:

$$\underline{\chi}^{\varepsilon}_{\tau}(s) = \mathbf{1}_{[\tau-\varepsilon^{-1},\tau]}(s)\ , \qquad \overline{\chi}^{\varepsilon}_{\tau}(s) = \mathbf{1}_{[0,\tau-\varepsilon^{-1}]}(s)\ .$$

Then, to show (4.4.7), we write the conditional projection in (4.4.8) as

$$\mathbb{E}[\Phi'(u(\tau,x))u(s,y)w_{s,y}(\tau,x)|\mathcal{F}_s] - \nu_\Phi P_{\tau-s}(x-y)u(s,y) = \sum_{i=1}^{5} S_i(\tau,x,s,y)\ ,$$

where

$$S_1 = \Big(\mathbb{E}[\Phi'(u(\tau,x))u(s,y)w_{s,y}(\tau,x)|\mathcal{F}_s] - \nu_\Phi P_{\tau-s}(x-y)u(s,y)\Big)\,\underline{\chi}^{\varepsilon}_{\tau}(s)$$

denotes the integrand on $[0, \tau - \varepsilon^{-1}]$ and, on the remaining interval, we have 4 terms:

$$S_2 = \mathbb{E}\big[\big(\Phi'(u(\tau,x)) - \Phi'(\vec{Z}(\tau,x)\big)u(s,y)w_{s,y}(\tau,x)|\mathcal{F}_s\big]\,\overline{\chi}^{\varepsilon}_{\tau}(s)\ ,$$

$$S_3 = \mathbb{E}\big[\Phi'(\vec{Z}(\tau,x))u(s,y)\big(w_{s,y}(\tau,x) - P_{\tau-s}(x-y)C_{s,y}(\tau)\vec{Z}(\tau,x)\big)|\mathcal{F}_s\big]\,\overline{\chi}^{\varepsilon}_{\tau}(s)\ ,$$

$$\begin{aligned}
S_4 = P_{\tau-s}(x-y)u(s,y)\ \times \\
\mathbb{E}\big[C_{s,y}(\tau)\Psi(\vec{Z}(\tau,x)) - C_{s,y}(\tfrac{s+\tau}{2})\Psi(u^{(\frac{s+\tau}{2})}(\tau,x))|\mathcal{F}_s\big]\,\overline{\chi}^{\varepsilon}_{\tau}(s)\ ,
\end{aligned}$$

$$S_5 = P_{\tau-s}(x-y)u(s,y)\big(\mathbb{E}[C_{s,y}(\tfrac{s+\tau}{2})\Psi(u^{(\frac{s+\tau}{2})}(\tau,x))|\mathcal{F}_s] - \nu_\Phi\big)\,\overline{\chi}^{\varepsilon}_{\tau}(s)\ .$$

With this decomposition, we can express the difference as follows:

$$Y_t^{\varepsilon,g} - \nu_\Phi X_t^{\varepsilon,g} = \beta\varepsilon^{1-\frac{\kappa}{2}} \sum_{k=1}^{5} \mathcal{I}_{k,\varepsilon} \ ,$$

where

$$\mathcal{I}_{k,\varepsilon} := \int_0^\tau \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S_k(\tau, x, s, y)\, \varepsilon^d g(\varepsilon x)\, dx\, \xi(ds, dy) \ .$$

Estimating the L^p norms of S_j, we show that each term $\mathcal{I}_{j,\varepsilon}$ in the decomposition converges to 0. Since, through BDG inequality as in (4.2.4), we have for $k = 1, \dots, 5$,

$$\|\mathcal{I}_{k,\varepsilon}\|_p^2 \leqslant \int_0^\tau \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^{2} \|S_k(\tau, x_i, s, y_i)\|_p |g_\varepsilon(x_i)| \Big)\, R(y_1 - y_2)\, ds\, dx\, dy \ , \tag{4.4.9}$$

where $g_\varepsilon(x) = \varepsilon^d g(\varepsilon x)$.

By the moment estimates from Lemmas 4.2.2–4.2.4 (in particular (4.2.7) to bound $w_{s,y}$), we have $\|S_1(\tau, x, s, y)\|_p \lesssim P_{\tau-s}(x - y)$ provided that β is sufficiently small. By Lemma 2.4.3, with $\delta \in (0, d)$ in (2.4.4), we have

$$P_s(x) \lesssim s^{-\frac{\delta}{2}} |x|^{\delta - d} \ . \tag{4.4.10}$$

Inserting this bound into (4.4.9) yields

$$\|\mathcal{I}_{1,\varepsilon}\|_p^2 \lesssim \int_{\tau-\varepsilon^{-1}}^\tau \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^{2} P_{\tau-s}(x_i - y_i) |g_\varepsilon(x_i)| \Big) R(y_1 - y_2)\, ds\, dx\, dy \ .$$

Recall that $t = \varepsilon^2 \tau$. By the change of variables $x_i \mapsto \varepsilon x_i$, $y_i \mapsto \varepsilon y_i$, $s \mapsto \varepsilon^2 s$, and the fact that $P_{r/\varepsilon^2}(z/\varepsilon) = \varepsilon^d P_r(z)$, we have

$$\|\mathcal{I}_{1,\varepsilon}\|_p^2 \lesssim \varepsilon^{\kappa-2} \int_{t-\varepsilon}^t \int_{\mathbb{R}^{4d}} \prod_{i=1}^{2} \Big(P_{t-s}(x_i - y_i) |g(x_i)| \Big) \varepsilon^{-\kappa} R\big(\tfrac{y_1 - y_2}{\varepsilon}\big)\, ds\, dx\, dy \ .$$

Since $\varepsilon^{-\kappa} R(\frac{y_1 - y_2}{\varepsilon}) \lesssim |y_1 - y_2|^{-\kappa}$, using (4.4.10) with $\delta_0 < 1$, we estimate

$$\begin{aligned} \|\mathcal{I}_{1,\varepsilon}\|_p^2 &\lesssim \varepsilon^{\kappa-2} \int_{t-\varepsilon}^t \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^{2} P_{t-s}(x_i - y_i)\, |g(x_i)| \Big) |y_1 - y_2|^{-\kappa}\, ds\, dx\, dy \\ &\lesssim \varepsilon^{\kappa-2} \int_0^\varepsilon s^{-\delta_0}\, ds \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^{2} |x_i - y_i|^{\delta_0 - d}\, |g(x_i)| \Big) |y_1 - y_2|^{-\kappa}\, dx\, dy \end{aligned}$$

$$\lesssim \varepsilon^{\kappa-2}\varepsilon^{1-\delta_0} J_{\kappa,\delta_0}(g) \ ,$$

where $J_{\kappa,\delta}$ is defined as in (4.4.6).

For $\mathcal{I}_{2,\varepsilon}$, first apply Cauchy-Schwartz inequality, then the moment bounds (4.2.7) and (4.2.18), we have

$$\begin{aligned}\|S_2(\tau,x,s,y)\|_p &\lesssim \|\Phi'(u(\tau,x)) - \Phi'(\vec{Z}(\tau,x)\|_{3p}\|w_{s,y}(\tau,x)\|_{3p} \\ &\lesssim (1+\tau)^{\frac{2-\kappa}{4}} P_{\tau-s}(x-y) \lesssim \varepsilon^{\frac{\kappa-2}{4}} P_{\tau-s}(x-y) \ .\end{aligned}$$

By a change of variables and (4.4.10) with $\delta_1 < 1$ as above, from (4.4.9) we obtain

$$\begin{aligned}\|\mathcal{I}_{2,\varepsilon}\|_p^2 &\lesssim \varepsilon^{\frac{\kappa}{2}-1} \int_0^{\tau-\varepsilon^{-1}} \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^2 P_{\tau-s}(x_i-y_i)\,|g_\varepsilon(x_i)|\Big) R(y_1-y_2)\,ds\,dx\,dy \\ &\lesssim \varepsilon^{\kappa-2}\,\varepsilon^{\frac{\kappa}{2}-1} \int_0^{t-\varepsilon} \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^2 P_{t-s}(x_i-y_i)\,|g(x_i)|\Big) |y_1-y_2|^{-\kappa}\,ds\,dx\,dy \\ &\lesssim \varepsilon^{\kappa-2}\,\varepsilon^{\frac{\kappa}{2}-1}\,t^{1-\delta_1}\,J_{\kappa,\delta_1}(g) \ . \qquad (4.4.11)\end{aligned}$$

For the remaining $\mathcal{I}_{j,\varepsilon}$, we shall make use of the fact that $s < \tau - \varepsilon^{-1}$. Using the uniform moment bounds on $\Psi(u) = \Phi'(u)u$ and on u, and the homogenisation result Proposition 4.3.1, we have

$$\begin{aligned}\|S_3\|_p &\lesssim P_{\tau-s}(x-y)\,\|\rho_{s,y}(\tau,x)\|_{2p} \\ &\lesssim P_{\tau-s}(x-y)\,(1+\tau-s)^{-\mu}\big(1+(1+\tau-s)^{-\frac{1}{2}}|x-y|\big) \\ &\lesssim \varepsilon^\mu P_{\tau-s}(x-y) + \varepsilon^\mu |x-y| P_{\tau-s}(x-y) \ .\end{aligned}$$

Here $\mu = \frac{1}{2} - \frac{1}{\kappa}$ and $\rho_{s,y}$ is given by (4.3.1).

As for $\mathcal{I}_{2,\varepsilon}$, the contribution from $\varepsilon^\mu P_{\tau-s}(x-y)$ can be bounded by $\varepsilon^{\kappa-2}\,\varepsilon^{2\mu}\,t^{1-\delta_1}\,J_{\kappa,\delta_1}(g)$. The contribution from $\varepsilon^\mu|x-y|P_{\tau-s}(x-y)$ can be estimated as follows:

$$\begin{aligned}&\varepsilon^{2\mu}\int_0^{\tau-\varepsilon^{-1}} \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^2 |x_i-y_i| P_{\tau-s}(x_i-y_i)|g_\varepsilon(x_i)|\Big) R(y_1-y_2)\,ds\,dx\,dy \\ &\lesssim \varepsilon^{2\mu}\varepsilon^{\kappa-2}\int_0^{t-\varepsilon}(t-s)^{-\delta_2}\,ds \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^2 |x_i-y_i|^{1+\delta_2-d}\,|g(x_i)|\Big)|y_1-y_2|^{-\kappa}\,dx\,dy\end{aligned}$$

$$\lesssim \varepsilon^{2\mu}\varepsilon^{\kappa-2}\, t^{1-\delta_2}\, J_{\kappa,1+\delta_2}(g)\ .$$

We have employed again the heat kernel bound (4.4.10), with $\delta_2 < \frac{\kappa}{2} - 1$. Summing these up, we have:

$$\|\mathcal{I}_{3,\varepsilon}\|_p^2 \lesssim \varepsilon^{2\mu}\varepsilon^{\kappa-2}\big(t^{1-\delta_1} J_{\kappa,\delta_1}(g) + t^{1-\delta_2} J_{\kappa,1+\delta_2}(g)\big)\ .$$

For $\mathcal{I}_{4,\varepsilon}$ and $\mathcal{I}_{5,\varepsilon}$, we make use of mixing, separating $C_{s,y}$ and $\vec{Z}$ in independent parts. We first apply the convergence estimate (4.2.18). For $s < \tau - \varepsilon^{-1}$ and $q > 1$ we have

$$\|\Psi(\vec{Z}(\tau,x)) - \Psi(u^{(\frac{s+\tau}{2})}(\tau,x))\|_q \lesssim \beta(1+\tfrac{1}{2}(\tau-s))^{\frac{2-\kappa}{4}} \lesssim \varepsilon^{\frac{\kappa-2}{4}}\ ,$$

On the other hand, by (4.2.8) we also have

$$\|C_{s,y}(\tau) - C_{s,y}(\tfrac{s+\tau}{2})\|_q \lesssim \beta(1+\tfrac{1}{2}(\tau-s))^{\frac{2-\kappa}{4}} \lesssim \varepsilon^{\frac{\kappa-2}{4}}\ .$$

Therefore, for sufficiently small β,

$$\|C_{s,y}(\tau)\Psi(\vec{Z}(\tau,x)) - C_{s,y}(\tfrac{s+t/\varepsilon^2}{2})\Psi(u^{(\frac{s+\tau}{2})}(\tau,x))\|_{2p} \lesssim \varepsilon^{\frac{\kappa-2}{4}}\ ,$$

We have used trinagle inequality and Lemma 4.2.2. Consequently,

$$\|S_4\|_p \lesssim \varepsilon^{\frac{\kappa-2}{4}} P_{\tau-s}(x-y)\ .$$

This is the same estimates as for $\mathcal{I}_{2,\varepsilon}$, therefore, $\|\mathcal{I}_{4,\varepsilon}\|_p^2 \lesssim \varepsilon^{\kappa-2}\varepsilon^{\frac{\kappa}{2}-1}t^{1-\delta_1}J_{\kappa,\delta_1}(g)$.

By the integral expression for $C_{s,y}$ in (4.3.6), $C_{s,y}(\frac{s+\tau}{2})$ is of expectation 1 and independent of $u^{(\frac{s+\tau}{2})}(\tau,x)$. In addition, both terms are $\mathcal{F}_s$-independent, therefore we have

$$\begin{aligned}\mathbb{E}[C_{s,y}(\tfrac{s+\tau}{2})\Psi(u^{(\frac{s+\tau}{2})}(\tau,x))|\mathcal{F}_s] &= \mathbb{E}[C_{s,y}(\tfrac{s+\tau}{2})]\mathbb{E}[\Psi(u^{(\frac{s+\tau}{2})}(\tau,x))]\\ &= \mathbb{E}[\Psi(u^{(\frac{s+\tau}{2})}(\tau,x)) - \Psi(\vec{Z}(\tau,x))] + \nu_\Phi\ .\end{aligned}$$

Using again the estimate (4.2.18), we obtain

$$\begin{aligned}\|S_5\|_p &\lesssim P_{\tau-s}(x-y)\|\Psi(u^{(\frac{s+\tau}{2})}(\tau,x)) - \Psi(\vec{Z}(\tau,x))\|_{2p}\\ &\lesssim P_{\tau-s}(x-y)(1+\tfrac{1}{2}(\tau-s))^{\frac{2-\kappa}{4}} \lesssim \varepsilon^{\frac{\kappa-2}{4}} P_{\tau-s}(x-y)\ .\end{aligned}$$

This is the same bound for S_4, hence $\|\mathcal{I}_{5,\varepsilon}\|_p^2 \lesssim \varepsilon^{\kappa-2}\varepsilon^{\frac{\kappa}{2}-1}t^{1-\delta_1}J_{\kappa,\delta_1}(g)$. Putting all estimates together, observing that $\mu = \frac{1}{2} - \frac{1}{\kappa} < \frac{\kappa-2}{4}$, we obtain

$$\|Y_t^{\varepsilon,g} - \nu_\Phi X_t^{\varepsilon,g}\|_p \lesssim \varepsilon^{\frac{1-\delta_0}{2}} J_{\kappa,\delta_0}(g)^{\frac{1}{2}} + \varepsilon^{\mu}\Big(t^{\frac{1-\delta_1}{2}} J_{\kappa,\delta_1}(g)^{\frac{1}{2}} + t^{\frac{1-\delta_2}{2}} J_{\kappa,1+\delta_2}(g)^{\frac{1}{2}}\Big)\,, \tag{4.4.12}$$

where $\delta_0, \delta_1 \in (0,1)$ and $\delta_2 \in (0, \frac{\kappa}{2} - 1)$. The implicit constants are uniform in $t, \varepsilon > 0$. We conclude the proof of (4.4.7). □

Proposition 4.4.5. *Let $p \geqslant 1$ and β sufficiently small. The following holds for any $g \in \mathcal{C}_c(\mathbb{R}^d)$, $t > 0$, and $\theta \in (2, \kappa \wedge (d+2-\kappa))$:*

$$\|X_t^{\varepsilon,g} - \mathcal{U}_t^\varepsilon(g)\|_p \lesssim \varepsilon^{\frac{\theta-2}{2p}} t^{\frac{1-\delta}{2}(1+\frac{1}{p})} J_{\kappa+\theta-2\delta,\delta}(g)^{\frac{1}{2p}} J_{\kappa,\delta}(g)^{\frac{1}{2}(1-\frac{1}{p})}\,. \tag{4.4.13}$$

Proof. Recall that $X_t^{\varepsilon,g}$ is defined in (4.4.4) and $\mathcal{U}_t^\varepsilon$ is defined in (4.4.5). In the former, we substitute $u(s,y)$ by $1 + \int_0^s \int_{\mathbb{R}^d} P_{s-r}(y-z)u(r,z)\xi(dr,dz)$ allowing to recover $\mathcal{U}_t^\varepsilon(g)$ and an additional term:

$$X_t^{\varepsilon,g} = \mathcal{U}_t^\varepsilon(g) + B_\varepsilon(g)\,, \tag{4.4.14}$$

where, with $\tau = t/\varepsilon^2$,

$$\begin{aligned} B_\varepsilon = \beta^2\varepsilon^{1-\frac{\kappa}{2}} \int_0^\tau \int_{\mathbb{R}^d} \int_0^s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P_{\tau-s}(x-y) \times \\ P_{s-r}(y-z)u(r,z)\,\varepsilon^d g(\varepsilon x)dx\,\xi(dr,dz)\,\xi(ds,dy)\,. \end{aligned} \tag{4.4.15}$$

We first show that B_ε converges to 0 in L^2. By Itô's isometry,

$$\begin{aligned} \|B_\varepsilon\|_2^2 = \beta^4\varepsilon^{2-\kappa} \int_0^\tau \int_0^s \int_{\mathbb{R}^{6d}} \Big(\prod_{i=1}^2 P_{\tau-s}(x_i-y_i)P_{s-r}(y_i-z_i)\Big) R(y_1-y_2) \\ R(z_1-z_2)\mathbb{E}[u(r,z_1)u(r,z_2)]\,\varepsilon^{2d} g(\varepsilon x_1)g(\varepsilon x_2)\,ds\,dr\,dx\,dy\,dz\,. \end{aligned}$$

Using the uniform L^2 bound on u and a change of variables as before[(1)], we obtain

$$\|B_\varepsilon\|_2^2 \lesssim \int_0^t \int_0^s \int_{\mathbb{R}^{6d}} \Big(\prod_{i=1}^2 P_{t-s}(x_i-y_i)P_{s-r}(y_i-z_i)\Big)\,\varepsilon^{-2} R\big(\tfrac{y_1-y_2}{\varepsilon}\big)$$

[(1)] Namely, $x_i \mapsto \varepsilon x_i$, $y_i \mapsto \varepsilon y_i$, $z_i \mapsto \varepsilon z_i$, $s \mapsto \varepsilon^2 s$, $r \mapsto \varepsilon^2 r$, combined with $P_{r/\varepsilon^2}(z/\varepsilon) = \varepsilon^d P_r(z)$.

$$\varepsilon^{-\kappa} R(\tfrac{z_1-z_2}{\varepsilon}) |g(x_1)g(x_2)| \, ds \, dr \, dx \, dy \, dz \ .$$

Using $\varepsilon^{-\kappa} R(z/\varepsilon) \lesssim |z|^{-\kappa}$ and the kernel bound (4.4.10) with $\delta \in (0,1)$, we obtain

$$\|B_\varepsilon\|_2^2 \lesssim \int_0^t \int_0^s (t-s)^{-\delta} (s-r)^{-\delta} \, dr \, ds \, I^\varepsilon_{\delta,\kappa}(g) \lesssim t^{2-2\delta} \, I^\varepsilon_{\delta,\kappa}(g) \ ,$$

where, with $G(x) = g(x_1)g(x_2)$,

$$I^\varepsilon_{\delta,\kappa}(g) = \int_{\mathbb{R}^{6d}} \Big(\prod_{i=1}^2 |x_i - y_i|^{\delta-d} |y_i - z_i|^{\delta-d} \Big) \varepsilon^{-2} R(\tfrac{y_1-y_2}{\varepsilon}) |z_1 - z_2|^{-\kappa} |G(x)| \, dx \, dy \, dz \ .$$

Using (2.3.6), for $\beta_1 + \beta_2 + d < 0$ and $\beta_i < -d$,

$$\int_{\mathbb{R}^d} |y-z|^{\beta_1} |z-w|^{\beta_2} dz = c_{\beta_1,\beta_2} |y-w|^{\beta_1+\beta_2+d} \ ,$$

we can integrate out the z_i variables and obtain

$$I^\varepsilon_{\delta,\kappa}(g) = c \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^2 |x_i - y_i|^{\delta-d} \Big) \varepsilon^{-2} R(\tfrac{y_1-y_2}{\varepsilon}) |y_1 - y_2|^{2\delta-\kappa} |G(x)| \, dx \, dy \ .$$

Let us denote

$$K_\varepsilon(x) = \varepsilon^{-2} R(x/\varepsilon) |x|^{2\delta-\kappa} \ .$$

We know that $R(x) \lesssim |x|^{-\theta}$, for any $\theta \in [0,\kappa]$. Let us pick $\theta \in (2, \kappa \wedge (d+2-\kappa))$, so that $\theta + \kappa - d < 2$ and we fix $\delta \in (\frac{1}{2}(\theta+\kappa-d), 1)$. Then, uniformly in $\varepsilon \in (0,1)$,

$$K_\varepsilon(x) \lesssim \varepsilon^{\theta-2} |x|^{2\delta-\kappa-\theta} \ . \tag{4.4.16}$$

Note that, by the choice of θ and δ, $K_\varepsilon(x)$ is integrable at 0 and decays fast enough at infinity to integrate it over the other kernels. In fact, by (4.4.16) we obtain

$$\begin{aligned} I^\varepsilon_{\delta,\kappa}(g) &\lesssim \varepsilon^{\theta-2} \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^2 |x_i - y_i|^{\delta-d} \Big) |y_1 - y_2|^{2\delta-\kappa-\theta} |G(x)| \, dx \, dy \\ &= \varepsilon^{\theta-2} J_{\kappa+\theta-2\delta,\delta}(g) \ , \end{aligned}$$

where J is defined as in (4.4.6) and bounded by Lemma **??**, as long as $\delta < \frac{1}{2}(\kappa+\theta-2\delta)$. This is the case, given that $\delta < 1$ (thus $\delta < \frac{1}{4}(\kappa+\theta)$ as required). Then, we have obtained

$$\|B_\varepsilon\|_2 \lesssim \varepsilon^{\frac{\theta-2}{2}} t^{1-\delta} J_{\kappa+\theta-2\delta,\delta}(g)^{\frac{1}{2}} .$$

On the other hand, estimating the L^p norms of X^ε and $\mathcal{U}^\varepsilon$ by BDG estimate (4.2.4) and a change of variables as in (4.4.11) in the previous proof, we have

$$\begin{aligned}
\|B_\varepsilon\|_p &\lesssim \|X_t^{\varepsilon,g}\|_p + \|\mathcal{U}_t^\varepsilon(g)\|_p \\
&\lesssim \varepsilon^{1-\frac{\kappa}{2}} \Big(\int_0^\tau \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1} P_{\tau-s}(x_i - y_i) \varepsilon^d g(\varepsilon x_i) \Big) R(y_1 - y_2)\, dx\, dy\, ds \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\int_0^t \int_{\mathbb{R}^{4d}} \Big(\prod_{i=1}^2 P_{t-s}(x_i - y_i) |g(x_i)| \Big) |y_1 - y_2|^{-\kappa}\, ds\, dx\, dy \Big)^{\frac{1}{2}} \\
&\lesssim t^{\frac{1-\delta}{2}} J_{\kappa,\delta}(g)^{\frac{1}{2}} ,
\end{aligned}$$

By interpolation, for any $\theta \in (2, \kappa \wedge (d+2-\kappa))$ and $\delta \in (\frac{1}{2}(\theta+\kappa-d), 1)$,

$$\begin{aligned}
\|X_t^{\varepsilon,g} - \mathcal{U}_t^\varepsilon(g)\|_p = \|B_\varepsilon\|_p &\leqslant \|B_\varepsilon\|_2^{\frac{1}{p}} \|B_\varepsilon\|_{2(p-1)}^{1-\frac{1}{p}} \\
&\lesssim \varepsilon^{\frac{\theta-2}{2p}} t^{\frac{1-\delta}{2}(1+\frac{1}{p})} J_{\kappa+\theta-2\delta,\delta}(g)^{\frac{1}{2p}} J_{\kappa,\delta}(g)^{\frac{1}{2}(1-\frac{1}{p})} ,
\end{aligned} \tag{4.4.17}$$

with implicit constants uniform in $t, \varepsilon > 0$, concluding the proof. □

Remark 4.4.6. To obtain decomposition (4.4.14) we crucially made use of the mild formulation of the *linear SHE*. The vanishing of B_ε in the limit is key and it is what allows us to show strong convergence next. The decay, which we could derive by trading some powers of ε to gain integrability of $R(\cdot/\varepsilon)$, no longer holds in the case of *integrable* covariance.

4.4.2 Convergence in Hölder spaces

We are now going to combine the approximation estimates of the previous subsection, which enable us to show convergence in Hölder spaces through the embeddings of Proposition 3.3.7. Recall Definition 3.3.4 of the pre-embeddings spaces, we are going to obtain convergence rate and uniform bounds in this spaces, to be used in the next subsection for the proof of the main results.

We follow notation as in Sections 3.3.1-3.3.2, with $\bar{\mathcal{D}} \subset \mathcal{D}$ denoting the set of test functions g such that $\operatorname{supp} g \subset B_1$ and $\int g = 1$, and their rescalings:

$$g_x^\lambda(y) := \lambda^{-d} g(\lambda^{-1}(y-x)) \, .$$

We have the following:

Lemma 4.4.7. *Given any $\alpha_0 < 1 - \frac{\kappa}{2}$, $p \geqslant 1$, for sufficiently small β, there exist $a = a(\alpha_0, p, \kappa) > 0$, such that the following holds for any $T > 0$:*

$$\sup_{0<t\leqslant T} \|Y_t^\varepsilon - \nu_\Phi \mathcal{U}_t^\varepsilon\|_{\mathcal{C}_p^{\alpha_0}} \lesssim \varepsilon^a \, . \tag{4.4.18}$$

In particular, for every $\alpha < \alpha_0 - \frac{d}{p}$, there exists $a > 0$ such that

$$\| \sup_{0<t\leqslant T} \|Y_t^\varepsilon - \nu_\Phi \mathcal{U}_t^\varepsilon\|_{\alpha,E} \|_p \lesssim \varepsilon^a, \tag{4.4.19}$$

for any compact $E \subset \mathbb{R}^d$. Furthermore, for any $\gamma_0 \in (0, \frac{1}{2})$, $\alpha_0 < 1 - \frac{\kappa}{2} - 2\gamma_0$, and $p \geq 1$, one has $\sup_{\varepsilon\in(0,1]} \|\mathcal{U}^\varepsilon\|_{\mathcal{C}_{p,T}^{\gamma_0,\alpha_0}} < \infty$.

Proof. By the approximation rate (4.4.12), for $\delta_0, \delta_1 \in (0,1)$, $\delta_2 \in (0, \frac{\kappa}{2} - 1)$, and $\mu = \frac{1}{2} - \frac{1}{\kappa}$, one has

$$\sup_{t\in[0,T]} \|Y_t^{\varepsilon,g} - \nu_\Phi X_t^{\varepsilon,g}\|_p \lesssim \varepsilon^{\frac{1-\delta_0}{2}} J_{\kappa,\delta_0}(g)^{\frac{1}{2}} + \varepsilon^\mu \Big(J_{\kappa,\delta_1}(g)^{\frac{1}{2}} + J_{\kappa,1+\delta_2}(g)^{\frac{1}{2}} \Big).$$

Recall from Lemma 3.4.3, we have $J_{\kappa,\delta}(g) \lesssim M^{2d+2\delta-\kappa} \|g\|_\infty^2$, where M is the diameter of the support of g, and the scaling property (3.4.3):

$$J_{\kappa,\delta}(g_x^\lambda) = \lambda^{2\delta-\kappa} J_{\kappa,\delta}(g) \, . \tag{4.4.20}$$

Let $g \in \bar{\mathcal{D}}$ and $\delta \in (0,1)$. Then the following holds uniformly in $\lambda \in (0,1)$:

$$\begin{aligned} \sup_{t>0} \sup_{x\in\mathbb{R}^d} \|Y_t^{\varepsilon,g_x^\lambda} - \nu_\Phi X_t^{\varepsilon,g_x^\lambda}\|_p &\lesssim \varepsilon^{(\frac{1-\delta}{2})\wedge\mu} \lambda^{\delta-\frac{\kappa}{2}} \big(J_{\kappa,\delta}(g)^{\frac{1}{2}} + J_{\kappa,1+\delta_2}(g)^{\frac{1}{2}} \big) \\ &\lesssim \varepsilon^{(\frac{1-\delta}{2})\wedge\mu} \lambda^{\delta-\frac{\kappa}{2}} \|g\|_\infty \, . \end{aligned}$$

Hence, for any $\alpha_0 < 1 - \frac{\kappa}{2}$, by picking[(2)] $\delta = \delta(\alpha_0)$ sufficiently close to 1 and applying this to the test function $\varphi_x^{(n)}$ (as fixed in Section 3.3.1), we have

$$\sup_{t\in(0,T]} \sup_{n\geqslant 1} \sup_{x\in\mathbb{R}^d} 2^{n\alpha_0} \big\| \langle Y_t^\varepsilon - \nu_\Phi X_t^\varepsilon, \varphi_x^{(n)} \rangle \big\|_p \lesssim \varepsilon^{(\frac{1-\delta}{2})\wedge\mu} ,$$

which implies (3.3.6). Since the bound holds for any $g \in \bar{\mathcal{D}}$, (3.3.7) is also satisfied (cf. Remark 3.3.5). We obtain

$$\sup_{t\in(0,T]} \|Y_t^\varepsilon - \nu_\Phi X_t^\varepsilon\|_{\mathcal{C}_p^{\alpha_0}} \lesssim \varepsilon^{(\frac{1-\delta}{2})\wedge\mu} .$$

It remains to control $\|X_t^\varepsilon - \mathcal{U}_t^\varepsilon\|_{\mathcal{C}_p^{\alpha_0}}$. Following the same steps as above, for any $\theta \in (2, \kappa \wedge (d+2-\kappa))$ and $\delta \in (\frac{1}{2}(\theta+\kappa-d), 1)$, by Proposition 4.4.5 and the scaling relation (4.4.20), we have uniformly in $\lambda \in (0,1)$:

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} \sup_{t\in(0,T]} \|X_t^{\varepsilon,g_x^\lambda} - \mathcal{U}_t^\varepsilon(g_x^\lambda)\|_p &\lesssim \varepsilon^{\frac{\theta-2}{2p}} T^{\frac{1-\delta}{2}(1+\frac{1}{p})} \lambda^{(4\delta-\kappa-\theta)\frac{1}{2p}} \lambda^{(2\delta-\kappa)\frac{p-1}{2p}} \|g\|_\infty \\ &\lesssim \varepsilon^{\frac{\theta-2}{2p}} \lambda^{(\delta-\frac{\kappa}{2})(1+\frac{1}{p})} \|g\|_\infty ,\end{aligned}$$

where we used the fact that $\theta < \kappa$.

For any $\alpha_0 < 1 - \frac{\kappa}{2}$, picking sufficiently high $p(\alpha_0) \geqslant 1$ and $\delta(\alpha_0) < 1$, we have $(\delta - \frac{\kappa}{2})(1+\frac{1}{p}) > \alpha_0$. Then, by the same reasoning as above, we obtain

$$\sup_{t\in(0,T]} \|X_t^\varepsilon - \mathcal{U}_t^\varepsilon\|_{\mathcal{C}_p^{\alpha_0}} \lesssim \varepsilon^{\frac{\theta-2}{2p}} .$$

Letting $a = \frac{1}{2}\min(1-\delta, 1-\frac{2}{\kappa}, \frac{\theta-2}{p})$, by triangle inequality we conclude that (4.4.18) holds. The bound (4.4.19) follows by Proposition 3.3.7, as long as $\alpha < \alpha_0 - \frac{d}{p}$.

Regarding the bound on $\mathcal{U}_t^\varepsilon$, we note that the definition of ξ^ε implies the scaling relation

$$\mathcal{U}_t^\varepsilon(g_x^\lambda) = \varepsilon^{1-\frac{\kappa}{2}} \mathcal{U}_{t/\varepsilon^2}(g_{x/\varepsilon}^{\lambda/\varepsilon}) ,$$

so that the required bound follows immediately from Lemma 3.4.6. □

[(2)] If $\alpha_0 > -\frac{\kappa}{2}$, can set $\delta = \alpha_0 + \frac{\kappa}{2}$.

4.4.3 Proof of the main results

Lemma 4.4.8. *Consider a good coupling for the ξ^ε satisfying Assumption 4.1.1. Given a test function $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1})$ and $t \in \mathbb{R}$, write $\tilde{\psi}_t(s,x) = \psi(s,x)\mathbf{1}_{s\geq t}$. Then, one has $\lim_{\varepsilon\to 0} \xi^\varepsilon(\tilde{\psi}_t) = \xi^0(\tilde{\psi}_t)$ in probability.*

Proof. Let $\chi\colon \mathbb{R} \to \mathbb{R}_+$ be a smooth increasing function such that $\chi(s) = 0$ for $s \leqslant 0$ and $\chi(s) = 1$ for $s \geq 1$, and set $\chi_\delta(t) = \chi(t/\delta)$. Setting $\tilde{\psi}_t^\delta(s,x) = \chi_\delta(s-t)\psi(s,x)$, we then note that, as a consequence of Itô's isometry, one can find a constant C (depending on ψ) such that

$$\mathbb{E}\xi^\varepsilon(\tilde{\psi}_t^\delta - \tilde{\psi}_t)^2 \leqslant C\delta \;, \tag{4.4.21}$$

uniformly over all $\varepsilon \in [0,1]$, $\delta \leqslant 1$. Since $\tilde{\psi}_t^\delta$ is a smooth function, by coupling assumption one furthermore has $\xi^\varepsilon(\tilde{\psi}_t^\delta) \to \xi^0(\tilde{\psi}_t^\delta)$ in probability as $\varepsilon \to 0$ for any fixed δ. Combining this with (4.4.21), by rewriting

$$\xi^\varepsilon(\tilde{\psi}_t) - \xi^0(\tilde{\psi}_t) = \xi^\varepsilon(\tilde{\psi}_t - \tilde{\psi}_t^\delta) + \big(\xi^\varepsilon(\tilde{\psi}_t^\delta) - \xi^0(\tilde{\psi}_t^\delta)\big) + \xi^\varepsilon(\tilde{\psi}_t^\delta - \tilde{\psi}_t) \;,$$

the claim follows at once. □

With the Lemmas above we are now in the position to prove Theorem 4.4.2. Write $L^p([0,T], \mathcal{C}^\alpha(E))$ for the space with seminorms,

$$\|\zeta\|_{\alpha,E,p,T} := \Big(\int_0^T \|\zeta_t\|_{\alpha,E}^p \, dt\Big)^{\frac{1}{p}} \;,$$

where the semi-norms of $\mathcal{C}^\alpha(E)$ are as in Section 3.3.1, cf. (3.3.2). We see that for any $\gamma > 0$, the space $\mathcal{C}^\gamma([0,T], \mathcal{C}^\alpha(E))$ is continuously embedded in $L^p([0,T], \mathcal{C}^\alpha(E))$.

Proof of Theorem 4.4.2. From Lemma 4.4.7, given $\alpha < 1 - \frac{\kappa}{2}$, we pick $p = p(\alpha)$ suffiently high such that, as $\varepsilon \to 0$,

$$\Big\| \sup_{t\in[0,T]} \|Y_t^\varepsilon - \nu_\Phi \mathcal{U}_t^\varepsilon\|_{\alpha,E} \Big\|_p \longrightarrow 0 \;.$$

This implies that $Y^\varepsilon - \nu_\Phi \mathcal{U}^\varepsilon$ converges in probability to zero in $L^p([0,T], \mathcal{C}^\alpha(E))$.

It therefore remains to show that, for any good coupling, one has $\mathcal{U}^\varepsilon \Rightarrow \mathcal{U}^0$ in probability in $L^p([0,T], \mathcal{C}^\alpha(E))$. For this, we make use of [HL22a, Prop. 3.12] which implies that it suffices to show tightness of the laws of $\mathcal{U}^\varepsilon$ and convergence in probability of $\mathcal{U}^\varepsilon(\psi)$ for any test function $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1})$.

In Lemma 4.4.7 we showed $\sup_{\varepsilon\in(0,1]} \|\mathcal{U}^\varepsilon\|_{\mathcal{C}^{\gamma_0,\alpha_0}_{p,T}} < \infty$. By Proposition 3.3.7, we obtain for any $\gamma' \in (0, \frac{1}{2})$ and any $\alpha' < 1 - \frac{\kappa}{2} - 2\gamma'$,

$$\sup_{\varepsilon\in(0,1]} \Big\| \|\mathcal{U}^\varepsilon\|_{\mathcal{C}^{\gamma'}([0,T],\mathcal{C}^{\alpha'}(E))} \Big\|_p \lesssim 1 \,.$$

Since $\mathcal{C}^{\gamma'}([0,T], \mathcal{C}^{\alpha'}(E))$ is compactly embedded in $\mathcal{C}^{\gamma}([0,T], \mathcal{C}^{\alpha}(E))$ for any $\gamma \in (0, \gamma')$ and $\alpha < \alpha'$, we conclude that the required tightness holds.

To get convergence of $\mathcal{U}^\varepsilon(\psi)$, we note that $\mathcal{U}^\varepsilon(\psi) = \xi^\varepsilon(P\psi)$ with

$$(P\psi)(s,y) = \beta \mathbf{1}_{s\geq 0} \int_0^\infty \int_{\mathbb{R}^d} \psi(t,x) P_{t-s}(x-y)\, dx\, dt \,,$$

so that the desired convergence follows from Lemma 4.4.8. □

We now have all the ingredients in place to prove the identification of the large-scale limit of the KPZ equation given in Theorem 4.1.4.

Proof of Theorem 4.1.4. Given the initial condition h_0, let

$$\psi_\varepsilon(y) = \exp(\varepsilon^{\frac{\kappa}{2}-1} h_0(y)) - 1$$

and denote by $w^\varepsilon_{0,z}(t,x) = \varepsilon^{-d} w_{0,z/\varepsilon}(t/\varepsilon^2, x/\varepsilon)$ the rescaled fundamental solution of the linear stochastic heat equation, recall (4.1.10). Since the initial condition for u_ε is 1, it follows from the Cole–Hopf transform that

$$h_\varepsilon(t,x) = \varepsilon^{1-\frac{\kappa}{2}} \log\Big(u_\varepsilon(t,x) + \int_{\mathbb{R}^d} w^\varepsilon_{0,y}(t,x) \psi_\varepsilon(y)\, dy \Big) \,.$$

Since $2|\log(x+y) - \log(x) - y/x| \leqslant (y/x)^2$ for $x, y > 0$, it follows that

$$\begin{aligned} \Big| h_\varepsilon(t,x) - \varepsilon^{1-\frac{\kappa}{2}} \log\big(u_\varepsilon(t,x)\big) - \varepsilon^{1-\frac{\kappa}{2}} (u_\varepsilon(t,x))^{-1} \int_{\mathbb{R}^d} w^\varepsilon_{0,y}(t,x)\psi_\varepsilon(y)\, dy \Big| \\ \leqslant \frac{1}{2} \varepsilon^{1-\frac{\kappa}{2}} \Big((u_\varepsilon(t,x))^{-1} \int_{\mathbb{R}^d} w^\varepsilon_{0,y}(t,x)\psi_\varepsilon(y)\, dy \Big)^2 \,. \end{aligned}$$

By the a priori bounds for $u_\varepsilon(t,x)^{-1}$ from Lemma 4.2.2 and (4.2.7) and since $\|\psi_\varepsilon\|_\infty \lesssim \varepsilon^{\frac{\kappa}{2}-1}$, we obtain for any $p \geqslant 1$

$$\Big\| (u_\varepsilon(t,x))^{-1} \int_{\mathbb{R}^d} w^\varepsilon_{0,y}(t,x)\psi_\varepsilon(y)\,dy \Big\|_p \lesssim \varepsilon^{\frac{\kappa}{2}-1} \int_{\mathbb{R}^d} P_t(x-y)\,dy = \varepsilon^{\frac{\kappa}{2}-1}\,.$$

On the other hand, by Taylor approximating $\psi_\varepsilon(y)$, we have

$$\|\psi_\varepsilon(y) - \varepsilon^{\frac{\kappa}{2}-1} h_0\|_\infty \lesssim \varepsilon^{\kappa-2}\,,$$

consequently,

$$|\varepsilon^{1-\frac{\kappa}{2}} (u_\varepsilon(t,x))^{-1} \Big(\int_{\mathbb{R}^d} w^\varepsilon_{0,y}(t,x)\psi_\varepsilon(y)\,dy - \int_{\mathbb{R}^d} w^\varepsilon_{0,y}(t,x) h_0(y)\,dy \Big)| \lesssim \varepsilon^{\frac{\kappa}{2}-1}\,.$$

Combining these bounds, we conclude that

$$\Big\| h_\varepsilon(t,x) - \varepsilon^{1-\frac{\kappa}{2}} \log\big(u_\varepsilon(t,x)\big) - (u_\varepsilon(t,x))^{-1} \int_{\mathbb{R}^d} w^\varepsilon_{0,y}(t,x) h_0(y)\,dy \Big\|_p \lesssim \varepsilon^{\frac{\kappa}{2}-1}\,.$$

As a consequence of the homogenisation Theorem 4.1.7, with $\mu = \frac{1}{2} - \frac{1}{\kappa}$, for any $p \geqslant 1$ we have

$$\begin{aligned}
&\|w^\varepsilon_{0,y}(t,x) - \vec{Z}(t/\varepsilon^2, x/\varepsilon)\overleftarrow{Z}(0, y/\varepsilon)\varepsilon^{-d} P_{t/\varepsilon^2}(\tfrac{x-y}{\varepsilon})\|_p \\
&\quad \lesssim (t/\varepsilon^2)^{-\mu}(1 + (t/\varepsilon^2)^{-\frac{1}{2}}\varepsilon^{-1}|x-y|)\varepsilon^{-d} P_{t/\varepsilon^2}(\tfrac{x-y}{\varepsilon}) \\
&\quad \lesssim \varepsilon^{2\mu} t^{-\mu}(1 + t^{-\frac{1}{2}}|x-y|) P_t(x-y)\,,
\end{aligned}$$

where we used the fact that $\varepsilon^{-d} P_{t/\varepsilon^2}(\frac{x}{\varepsilon}) = P_t(x)$.

Given that h_0 is bounded and the uniform control of negative moments of u_ε from Lemma 4.2.2(i), we obtain

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^d} (u_\varepsilon(t,x))^{-1} \Big(w^\varepsilon_{0,y}(t,x) - \vec{Z}(t/\varepsilon^2, x/\varepsilon)\overleftarrow{Z}(0, y/\varepsilon)\varepsilon^{-d} P_{t/\varepsilon^2}(\tfrac{x-y}{\varepsilon}) \Big) h_0(y)\,dy \Big\|_p \\
&\qquad \lesssim \varepsilon^{2\mu} t^{-\mu}\, \|u_\varepsilon^{-1}(t,x)\|_{2p} \int_{\mathbb{R}^d} (1 + t^{-\frac{1}{2}}|x-y|) P_t(x-y)\,dy \lesssim \varepsilon^{2\mu} t^{-\mu}\,.
\end{aligned}$$

On the other hand, by Theorem 2.6.1 we also have

$$\Big\| \frac{\vec{Z}(t/\varepsilon^2, x/\varepsilon)}{u_\varepsilon(t,x)} - 1 \Big\|_p \lesssim \|u_\varepsilon^{-1}(t,x)\|_{2p} \|\vec{Z}(t/\varepsilon^2, x/\varepsilon) - u_\varepsilon(t,x)\|_{2p} \lesssim t^{\frac{2-\kappa}{4}} \varepsilon^{\frac{\kappa}{2}-1}\,.$$

Combining the estimates, we obtain

$$\Big\| \int_{\mathbb{R}^d} \Big(\frac{w^\varepsilon_{0,y}(t,x)}{u_\varepsilon(t,x)} - \check{Z}(0,y/\varepsilon) P_t(x-y) \Big) h_0(y)\, dy \Big\|_p \lesssim \varepsilon^{\frac{\kappa}{2}-1} t^{\frac{2-\kappa}{4}} + \varepsilon^{2\mu} t^{-\mu} \,.$$

Hence, noting that $2\mu < \frac{\kappa}{2} - 1$ and using the fact that $t \geqslant T_0 > 0$, we have

$$\Big\| h_\varepsilon(t,x) - \varepsilon^{1-\frac{\kappa}{2}} \log\big(u_\varepsilon(t,x)\big) - \int_{\mathbb{R}^d} \check{Z}(0,y/\varepsilon) P_t(x-y) h_0(y)\, dy \Big\|_p \lesssim \varepsilon^{2\mu} \,. \tag{4.4.22}$$

Let $c = \mathbb{E}[\log \vec{Z}(t,x)]$, $F(t) = \mathbb{E}[\log\big(u(t,x)\big)]$ and set $F_\varepsilon(t) = F(t/\varepsilon^2)$. Then, we decompose h_ε as follows:

$$h_\varepsilon(t,x) - \varepsilon^{1-\frac{\kappa}{2}} c = u^{\log}_\varepsilon(t,x) + \int_{\mathbb{R}^d} P_t(x-y) h_0(y)\, dy + R_{0,\varepsilon}(t) + \sum_{i=1}^{2} R_{i,\varepsilon}(t,x) \,, \tag{4.4.23}$$

where

$$\begin{aligned}
u^{\log}_\varepsilon(t,x) &= \varepsilon^{1-\frac{\kappa}{2}} \big(\log\big(u_\varepsilon(t,x)\big) - \mathbb{E}[\log\big(u_\varepsilon(t,x)\big)] \big) \,, \\
R_{0,\varepsilon}(t) &= \varepsilon^{1-\frac{\kappa}{2}} (F_\varepsilon(t) - c) \,, \\
R_{1,\varepsilon}(t,x) &= h_\varepsilon(t,x) - \varepsilon^{1-\frac{\kappa}{2}} \log\big(u_\varepsilon(t,x)\big) - \int_{\mathbb{R}^d} \check{Z}(0,y/\varepsilon) P_t(x-y) h_0(y)\, dy \,, \\
R_{2,\varepsilon}(t,x) &= \int_{\mathbb{R}^d} (\check{Z}(0,y/\varepsilon) - 1) P_t(x-y) h_0(y)\, dy \,.
\end{aligned}$$

By Theorem 4.4.2 with $\Phi = \log$, we know that $\nu_\Phi = 1$ and $u^{\log}_\varepsilon$ converge in probability to $\mathcal{U}^0$ in $L^p([0,T], \mathcal{C}^\alpha(E))$ for a good coupling of ξ^ε, where $\mathcal{U}^0$ is the solution of the stochastic heat equation (4.1.9) with zero initial conditions. Hence, the first two terms in (4.4.23) converge in law to the claimed limit $\mathcal{U}$, the solution of (4.1.6) with initial conditions h_0. It remains to show that the remainder terms $R_{i,\varepsilon}$ vanish in $L^p([T_0,T], \mathcal{C}^\alpha(E))$.

Firstly, we show that for $t > 0$:

$$|F(t) - c| \lesssim 1 \wedge |t|^{(2-\kappa)/2} \,. \tag{4.4.24}$$

Then, this implies $|R_{0,\varepsilon}(t)| \lesssim \varepsilon^{\frac{\kappa}{2}-1} t^{1-\frac{\kappa}{2}}$, which vanishes uniformly in $t \geqslant T_0$ as $\varepsilon \to 0$. Recall that

$$F(t) - c = \mathbb{E}\big(\log u(t,x) - \log \vec{Z}(t,x)\big) \,.$$

It follows that, by splitting into the $u < Z$ and $u \geq Z$ case,

$$\Big|F(t) - c - \mathbb{E}\Big(\frac{u(t,x) - \vec{Z}(t,x)}{u(t,x)}\Big)\Big| \lesssim \mathbb{E}\Big(\frac{u(t,x) - \vec{Z}(t,x)}{u(t,x) \wedge \vec{Z}(t,x)}\Big)^2 ,$$

which in turn is bounded by $1 \wedge |t|^{(2-\kappa)/2}$ by combining Theorem 2.6.1 and Lemma 4.2.2(i). The trick now is to write

$$\frac{u(t,x) - \vec{Z}(t,x)}{u(t,x)} = \int \frac{1 - \vec{Z}(0,x)}{u(t,x)} w_{0,y}(t,x)\, dy$$

and to note that $\vec{Z}(0,x)$ is independent of both $u(t,x)$ and $w_{0,y}(t,x)$, so that the expectation of this expression vanishes, and concluding the claim on $|F(t) - c|$.

From (4.4.22), we see that $\|R_{1,\varepsilon}(t,x)\|_p \lesssim \varepsilon^{2\mu}$ uniformly in x and $t \geqslant T_0$. Hence, given that the bounds are uniform in space, integrating against any spatial test function $g \in \bar{\mathcal{D}}$ implies that $R_{0,\varepsilon}$ and $R_{1,\varepsilon}$ converge to zero in $L^p([T_0, T], \mathcal{C}^\alpha(E))$.

It remains to estimate $R_{2,\varepsilon}$. We claim, for any $\delta \in (0,1)$, uniformly in $\lambda \in (0,1)$,

$$\Big\| \int_{\mathbb{R}^d} R_{2,\varepsilon}(t,x) g_z^\lambda(x)\, dx \Big\|_2 \lesssim \varepsilon^{\frac{\kappa}{2}-1} t^{-\frac{\delta}{2}} \big(J_{\kappa-2,\delta}(g_z^\lambda)\big)^{\frac{1}{2}} . \tag{4.4.25}$$

By Lemma 3.4.3, for any $g \in \bar{\mathcal{D}}$ we obtain

$$\sup_{t \geqslant T_0} \Big\| \int_{\mathbb{R}^d} R_{2,\varepsilon}(t,x) g_z^\lambda(x)\, dx \Big\|_2 \lesssim \varepsilon^{\frac{\kappa}{2}-1} \lambda^{\delta+1-\frac{\kappa}{2}} \lesssim \varepsilon^{\frac{\kappa}{2}-1} \lambda^{1-\frac{\kappa}{2}} .$$

As in the proof of Lemma 4.4.7, this implies that $\|R_{2,\varepsilon}(t,\cdot)\|_{\mathcal{C}_2^{\alpha_0}} \lesssim \varepsilon^{\frac{\kappa}{2}-1}$ for any $\alpha_0 \leqslant 1 - \frac{\kappa}{2}$, uniformly in $t \geqslant T_0$. On the other hand, by the uniform moments bounds, we know

$$\sup_{t \geqslant T_0} \Big\| \int_{\mathbb{R}^d} R_{2,\varepsilon}(t,x) g_z^\lambda(x)\, dx \Big\|_p \lesssim \sup_{t,x} \|R_{2,\varepsilon}(t,x)\|_p \lesssim 1 .$$

Then, given $\alpha < 1 - \frac{\kappa}{2}$, picking $p \geqslant 1$ sufficiently large such that $\alpha < \alpha_0 - \frac{d}{p}$, by Proposition 3.3.7 and interpolation we conclude that, as $\varepsilon \to 0$, for $p \in [2, \infty)$,

$$\sup_{t \geqslant T_0} \big\| \|R_{2,\varepsilon}(t,\cdot)\|_{\alpha,E} \big\|_p \to 0 .$$

Therefore, $R_{2,\varepsilon}$ vanishes in $L^p([T_0, T], \mathcal{C}^\alpha(E))$.

It remains to show (4.4.25), namely that $\overleftarrow{Z}(0, y/\varepsilon)$ satisfies a law of large numbers. We know that $\overleftarrow{Z}$ has constant expectation 1 and decorrelates in space by (4.2.17):

$$\Lambda(0, y/\varepsilon) := \operatorname{cov}(\overleftarrow{Z}(0,0), \overleftarrow{Z}(0, y/\varepsilon)) \lesssim 1 \wedge \varepsilon^{\kappa-2}|y|^{2-\kappa} .$$

Then, we compute the second moments and use (4.4.10) with $\delta \in (0,1)$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} R_{2,\varepsilon}(t,x) g_z^\lambda(x)\, dx\Big)^2\Big] \\
&\qquad = \int_{\mathbb{R}^{4d}} \Lambda(y_1/\varepsilon, y_2/\varepsilon) \prod_{i=1}^2 \big(P_t(x_i - y_i) h_0(y_i) g_z^\lambda(x_i)\big)\, dx\, dy \\
&\qquad \lesssim \varepsilon^{\kappa-2} t^{-\delta} \int_{\mathbb{R}^{4d}} |y_1 - y_2|^{2-\kappa} \prod_{i=1}^2 \big(|x_i - y_i|^{\delta-d} |g_z^\lambda(x_i)|\big)\, dx\, dy ,
\end{aligned}$$

which implies (4.4.25), given definition (4.4.6) of $J_{\kappa-2,\delta}$. This concludes the proof of Theorem 4.1.4.



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