

Spectral Inequalities on Manifolds of Negative Curvature

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Declaration of Originality

I hereby confirm that this thesis and the research that led to it are the result of my own work and that ideas or quotations of others, published or unpublished, are duly referenced according to the established practices of the discipline.

Timon Ruben Weinmann

Abstract

This thesis is devoted to the study of the Laplace–Beltrami operator in hyperbolic space and the Schrödinger operators it produces. The text can be divided into four parts. In the first part comprising chapters 2 to 5 certain notions and classic theorems from the spectral theory of self-adjoint operators in general and of Schrödinger operators in particular are reviewed. Particular attention is paid to Lieb–Thirring inequalities in euclidean space. The second part, which comprises chapter 6, constitutes the core of this thesis. Here it is shown that many Lieb–Thirring inequalities that hold in euclidean space have analogues in hyperbolic space. The third part, chapter 7, uses elementary methods to prove a Weyl law for the hyperbolic Laplacian on domains. Though such a result can be obtained using different established approaches, our proof shows the utility of the method of coherent states. Finally, in the fourth part, chapter 8, a Weyl law is obtained for Schrödinger operators having a logarithmic differential part and an increasing potential that is bounded from below.

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1 Introduction

The main result of this thesis is a class of Lieb–Thirring inequalities in the hyperbolic space $\mathbf{H}^d$. Lieb–Thirring inequalities, which appeared first in the 70s in the seminal paper [LT75] by Lieb and Thirring, have played an important role in mathematical physics, in particular in the study of the stability of matter. Consider a Schrödinger operator $-\Delta + V$ on $L^2(\mathbf{R}^d)$ with $V \in L^{1+d/2}(\mathbf{R}^d)$. The classical Lieb–Thirring inequality stipulates that

$$\mathrm{tr}(-\Delta + V)_- \leq L_d \int_{\mathbf{R}^d} V(x)_-^{1+d/2} dx,$$

where $\mathrm{tr}(-\Delta + V)_-$ is the trace of the non-negative operator $(-\Delta + V)_-$. The constant L_d depends only on the dimension d . This inequality turns out to be equivalent to

$$\sum_{j=1}^N \int_{\mathbf{R}^d} |\nabla u_j(x)|^2 dx \geq K_d \int_{\mathbf{R}^d} \left(\sum_{j=1}^N |u_j(x)|^2 \right)^{1+2/d} dx,$$

for $u_1, \dots, u_N \in H^1(\mathbf{R}^d)$ that are orthonormal in $L^2(\mathbf{R}^d)$. The term on the left-hand side can be interpreted as the kinetic energy of a system of particles. The Lieb–Thirring was later generalized to

$$\mathrm{tr}(-\Delta + V)_-^\gamma \leq L_{\gamma,d} \int_{\mathbf{R}^d} V(x)_-^{\gamma+d/2} dx,$$

for certain values of γ when V is appropriately integrable. To make sense of all the notions involved, chapter 2 reviews some basic notions of spectral theory. Of particular importance here is the Courant–Fischer theorem which provides a way of computing the eigenvalues $\lambda_k(A)$ of a lower semibounded self-adjoint operator A associated with a quadratic form a (i.e. $D(A) \subseteq \mathcal{D}[a]$ and $a[u, v] = \langle u, Av \rangle$ for $u \in \mathcal{D}[a]$ and $v \in D(A)$) via

$$\lambda_k(A) = \inf_{\substack{E \subseteq \mathcal{D}[a] \\ \dim E = k}} \max_{\substack{u \in E \\ \|u\|=1}} a[u].$$

In certain situations, an equivalent formulation of this formula, known as Glazman’s lemma, is advantageous. Glazman’s lemma affirms that

$$N_A(\lambda) = \sup\{\dim E \mid E \subseteq \mathcal{D}(a) \text{ is a subspace s.t. } \forall u \in E \setminus \{0\}: a[u] < \lambda \|u\|^2\},$$

where N_A is the spectral counting function of the operator A . Chapter 3 discusses coherent states, which form a special case of overfilling families of functions. A family $e_\alpha \in \mathcal{H}$, $\alpha \in A$ of elements of a separable Hilbert space $\mathcal{H}$ indexed by a measure space (A, μ) is said to

be overfilling if the map $A \ni \alpha \mapsto e_\alpha \in \mathcal{H}$ is measurable and if for all $f \in \mathcal{H}$ one has

$$\|f\|^2 = \int_A |\langle e_\alpha, f \rangle|^2 d\mu(\alpha).$$

It turns out that overfilling families of functions provide a way of computing the traces of non-negative operators via

$$\mathrm{tr} T = \int_A \langle e_\alpha, T e_\alpha \rangle d\mu(\alpha).$$

The prime example of an overfilling family of functions in $L^2(\mathbf{R}^d)$ is given by

$$e_{\xi,y}(x) = e^{i\xi x} g(x - y)$$

where $g \in \mathcal{S}(\mathbf{R}^d)$ is real-valued and such that $g(-x) = g(x)$ and $\int g(x)^2 dx = 1$. In this case $A = \mathbf{R}^{2d}$ and $d\mu(\xi, y) = \frac{d\xi dy}{(2\pi)^d}$. Therefore

$$\mathrm{tr} K = \int_{\mathbf{R}^{2d}} \langle e_{\xi,y}, K e_{\xi,y} \rangle \frac{d\xi dy}{(2\pi)^d}$$

holds for non-negative trace class operators on $L^2(\mathbf{R}^d)$. The notions from Chapters 2 and 3 are applied to different realizations of the Laplace operator in chapter 4. In particular, the Weyl law for Dirichlet Laplacians on open sets $\Omega \subseteq \mathbf{R}^d$ of finite measure

$$N_{-\Delta_\Omega^D}(\lambda) = L_{0,d}^c |\Omega| \lambda^{d/2} + o(\lambda^{d/2})$$

is proven here to show the utility of the coherent state approach in spectral theory. Here, $N_{-\Delta_\Omega^D}(\lambda)$ is the number of eigenvalues of $-\Delta_\Omega^D$ lying strictly below λ and $L_{0,d}^c = \frac{|B_1|}{(2\pi)^d}$, where $|B_1|$ is the volume of the unit ball in $\mathbf{R}^d$.

Chapter 5 introduces Schrödinger operators and discusses the above mentioned Lieb–Thirring inequalities and the Lieb–Thirring conjecture, which deals with the optimal (smallest possible) constants in the Lieb–Thirring inequality. In Chapter 6 it is shown that for Schrödinger operators $-\Delta_h + V$ on $L^2(\mathbf{H}^d)$ (here $-\Delta_h$ denotes the Laplace–Beltrami operator on $\mathbf{H}^d$), the inequality

$$\sum_k \mu_k^\gamma \leq L_{\gamma,d} \int_{\mathbf{H}^d} V(x, y)_-^{d/2+\gamma} dV_h(x, y),$$

holds, where $(d-1)^2/4 - \mu_k$ is the k -th eigenvalue of $-\Delta_h + V$ below $(d-1)^2/4$, whenever $\gamma \geq 1/2$ and $V \in L^{d/2+\gamma}(\mathbf{H}^d, dV_h)$. The shift by $(d-1)^2/4$ turns out to be a natural

consequence of the geometry of $\mathbf{H}^d$. Here, V_h denotes the canonical Riemannian volume on the hyperbolic space $\mathbf{H}^d$. What is remarkable here is that the constants $L_{\gamma,d}$ are the same as in the euclidean case. Therefore, the optimal (i.e. smallest possible) constants in the hyperbolic case are at least as good (small) as the ones in the euclidean case. As a corollary of the Lieb–Thirring inequality, the kinetic energy inequality

$$\sum_{j=1}^N \int_{\mathbf{H}^d} y^{2-d} \left(|\tilde{\nabla} u_j(x)|^2 + |\partial_y u_j|^2 \right) dx dy \geq K_d \int_{\mathbf{H}^d} \left(\sum_{j=1}^N |u_j(x, y)|^2 \right)^{1+2/d} \frac{dx dy}{y^d} + N \frac{(d-1)^2}{4},$$

is obtained for $u_1, \dots, u_N \in H^1(\mathbf{H}^d)$ that are orthonormal in $L^2(\mathbf{H}^d)$. The proof of the Lieb–Thirring inequality in hyperbolic space relies on the unitary operator

$$W: L^2\left(\mathbf{H}^d, \frac{1}{y^d} dx dy\right) \rightarrow L^2(\mathbf{R}^d, dx dt)$$

given by

$$(Wu)(x, t) = e^{\frac{(1-d)}{2}t} u(x, e^t).$$

Conjugating the Laplace–Beltrami operator on $\mathbf{H}^d$ with the operator W yields

$$H = -\partial_t^2 - e^{2t} \tilde{\Delta},$$

where $-\tilde{\Delta}$ is the Laplacian in the x -coordinates. Thus an operator on $L^2(\mathbf{R}^d)$ is obtained. Chapter 7 studies the Dirichlet realizations of this operator H on bounded open sets $\Omega \subseteq \mathbf{R}^d$. It is proven that if Ω has Lipschitz boundary, then the eigenvalues λ_k of the operator H on Ω with Dirichlet boundary conditions satisfy

$$\sum_k (\lambda - \lambda_k)_+ = \lambda^{1+d/2} \int_{\mathbf{R}^d \times \Omega} (1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2)_+ \frac{d\xi dy}{(2\pi)^d} + O(\lambda^{d/2+2/3}).$$

Here we use the notation $\tilde{\xi} = (x_2, \dots, \xi_d)$. Finally, in Chapter 8 a Weyl law for Schrödinger operators with logarithmic differential part is proven. If $\log(1 - \Delta)$ denotes the operator with Fourier symbol $\log(1 + |\xi|^2)$, then one can construct self-adjoint realizations of $\log(1 - \Delta) + V$ when V is bounded from below. If V is increasing, then the spectrum of $\log(1 - \Delta) + V$ is discrete. We prove that if $V \in C(\mathbf{R}^d)$ such that $V(x) \geq \alpha|x| + c$ for some $\alpha > 0$ and $c \in \mathbf{R}$, then the eigenvalues λ_k of $\log(1 - \Delta) + V$ satisfy

$$\lim_{\lambda \rightarrow \infty} e^{-d\lambda/2} \sum_k (\lambda - \lambda_k)_+ = \frac{2|B_1|}{d(2\pi)^d} \int_{\mathbf{R}^d} e^{-dV(x)/2} dx.$$

2 A very Brief Introduction to Spectral Theory

In this chapter we recall some basic facts from the theory of linear operators on Hilbert spaces. We mainly follow the exposition in [Lew22], but some things are also taken from [Tes14], [FLW23], [Bor20] and [Wei00]. The chapter aims at introducing those parts of spectral theory that will be used in the research part of this thesis in as concise a manner as possible. Nothing in this chapter is original by any means.

2.1 Closed Operators

Definition 2.1. A *linear operator* on a Hilbert space $\mathcal{H}$ is linear map $A: \mathcal{D}(A) \rightarrow \mathcal{H}$ defined on a linear subspace $\mathcal{D}(A)$ of $\mathcal{H}$. The space $\mathcal{D}(A)$ is called the *domain* of A and in case $\mathcal{D}(A)$ is a dense subset of $\mathcal{H}$, the operator A is said to be *densely defined*. The set

$$\mathcal{G}(A) = \{(u, Au) \mid u \in \mathcal{D}(A)\} \subseteq \mathcal{H} \oplus \mathcal{H}$$

is called the *graph* of A . If $\mathcal{G}(A)$ is a closed subset of $\mathcal{H} \oplus \mathcal{H}$, then A is called a *closed operator*. The *range* of an operator is the set

$$\mathcal{R}(A) = \{Au \mid u \in \mathcal{D}(A)\}$$

and its *kernel* is

$$\mathcal{N}(A) = \{u \in \mathcal{D}(A) \mid Au = 0\}$$

Given two operators A and B on the same Hilbert space, we say that B is an *extension* of A or that A is a *restriction* of B , if

$$\mathcal{G}(A) \subseteq \mathcal{G}(B).$$

In this case we write $A \subseteq B$. An operator is *closable*, if it has a closed extension. Furthermore, an operator A is *bounded* if there exists a constant $C > 0$ such that $\|Au\| \leq C\|u\|$ for all $u \in \mathcal{D}(A)$. The smallest such constant is denoted $\|A\|$. The identity operator on $\mathcal{H}$, i.e. the map $u \mapsto u$ defined for all $u \in \mathcal{H}$ is denoted $I_{\mathcal{H}}$ or simply I if it is clear from the context which Hilbert space is being talked about. Finally, multiples of the identity operator are written like scalars: $zI = z$ for $z \in \mathbb{C}$.

Remark 1. When talking about linear operators we often omit the word “linear”. While the study of non-linear operators does give rise to an interesting subject, it does not concern this thesis.

We will mainly be interested in closed operators. When talking about bounded operators, we usually assume that they be defined on the entire space. This is justified by the theorem below. The space of all bounded operators defined everywhere on the Hilbert space $\mathcal{H}$ is denoted $\mathcal{L}(\mathcal{H})$. For $A \in \mathcal{L}(\mathcal{H})$, the above definition of $\|A\|$ coincides with $\|A\| = \sup\{\|Au\| \mid u \in \mathcal{H}, \|u\| = 1\}$. This defines a norm making $\mathcal{L}(\mathcal{H})$ a Banach algebra.

Proposition 2.2 (Closed Graph Theorem). *Let A be a linear operator on a Hilbert space. Then any two of the properties below imply the third one:*

(i) A is bounded.

(ii) A is closed.

(iii) $\mathcal{D}(A)$ is closed.

Proof. Only the implication (ii), (iii) $\implies$ (i) is not trivial. Its proof is due to Banach. Consult [Wei00, Theorem 4.4] or [Tes14, Theorem 2.9] for details. $\square$

Definition 2.3. Given two operators A and B on the same Hilbert space, we can define their *sum* $A + B$ as

$$(A + B)u = Au + Bu$$

on

$$\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$$

and their *product* AB as

$$(AB)u = A(Bu)$$

on $\mathcal{D}(AB) = B^{-1}(\mathcal{D}(A))$.

We are now ready to define a central notion in spectral theory: the spectrum of an operator.

Definition 2.4. Let A be an operator on a Hilbert space $\mathcal{H}$. The *resolvent set* $\rho(A)$ of A is the set of all complex numbers $z \in \mathbf{C}$ such that $A - z$ is a bijective map $\mathcal{D}(A) \rightarrow \mathcal{H}$ and such that $(A - z)^{-1} \in \mathcal{L}(\mathcal{H})$. The map $\rho(A) \ni z \mapsto (A - z)^{-1} \in \mathcal{L}(\mathcal{H})$ is called the *resolvent (map)* of A . The complement $\sigma(A) = \mathbf{C} \setminus \rho(A)$ of the resolvent set is called the *spectrum* of A .

In order to examine the topological properties of the sets $\rho(A)$ and $\sigma(A)$, we need the following easy result.

Lemma 2.5. *Let $A \in \mathcal{L}(\mathcal{H})$ such that $\|A\| < 1$. Then $I - A$ is invertible and*

$$(I - A)^{-1} = \sum_{j=0}^{\infty} A^j.$$

Proof. See any book on Banach algebras or functional analysis, for instance [Rud87]. $\square$

Theorem 2.6. *Let A be a linear operator. Then its resolvent set $\rho(A)$ is open.*

Proof. Suppose that $z \in \rho(A)$. For $z' \in \mathbf{C}$ we have

$$A - z' = (1 - (z' - z)(A - z)^{-1})(A - z).$$

By assumption $A - z$ is invertible and its inverse is bounded. If $|z' - z| < \frac{1}{\|(A - z)^{-1}\|}$, then $1 - (z' - z)(A - z)^{-1}$ is invertible by 2.5, again with bounded inverse. Consequently, $A - z'$ is invertible and $(A - z')^{-1} \in \mathcal{L}(\mathcal{H})$. Therefore, the resolvent set contains the open ball of radius $\frac{1}{\|(A - z)^{-1}\|}$ around z . $\square$

A direct consequence of this theorem is of course that the spectrum of any operator is closed.

Remark 2. Let us consider two special cases. Assume that A be an operator that is not closed and that $z \in \rho(A)$. Then $(A - z)^{-1}$ is a bounded operator. Suppose that $(u_n) \subseteq \mathcal{D}(A)$ and that $u_n \rightarrow u \in \mathcal{H}$ and $Au_n \rightarrow v \in \mathcal{H}$. Then $(A - z)u_n \rightarrow v - zu$. By continuity, $u_n \rightarrow (A - z)^{-1}(v - zu)$. Thus $u = (A - z)^{-1}(v - zu) \in \mathcal{D}(A)$ and $Au = v$. *Consequently, operators that are not closed must have empty resolvent set.* This partly justifies why we are primarily interested in closed operators. Assume now that $A \in \mathcal{L}(\mathcal{H})$. Then the map $\rho(A) \rightarrow \mathbf{C}$, $f: z \mapsto \langle u, (A - z)^{-1}u \rangle$ is holomorphic and tends to zero as $|z| \rightarrow \infty$ (see [Lew22, Remark 2.9]). Suppose that $\rho(A) = \mathbf{C}$, then $f: \mathbf{C} \rightarrow \mathbf{C}$ is holomorphic and bounded, tending to zero at infinity. Therefore $f(z) = 0$ for all $z \in \mathbf{C}$ by Liouville's theorem. Since u is arbitrary, $(A - z)^{-1} = 0$ for all $z \in \mathbf{C}$, which is clearly absurd. Thus, every bounded operator has non-empty spectrum.

One simple version of the *spectral mapping theorem* will be useful later.

Proposition 2.7. *Let A be a linear operator and let $z \in \rho(A)$. The following identity holds for the spectrum of $(A - z)^{-1}$:*

$$\sigma((A - z)^{-1}) = \{(z' - z)^{-1} \mid z' \in \sigma(A)\},$$

if $A \in \mathcal{L}(\mathcal{H})$ and

$$\sigma((A - z)^{-1}) = \{(z' - z)^{-1} \mid z' \in \sigma(A)\} \cup \{0\}$$

if A is unbounded.

Proof. By the identity

$$A - z' = (1 - (z' - z)(A - z)^{-1})(A - z),$$

we see that if $z' \neq z$, then $(z' - z)^{-1} \in \rho((A - z)^{-1})$ is equivalent to $z' \in \rho(A)$. Clearly, the map $\mathbf{C} \setminus \{z\} \rightarrow \mathbf{C} \setminus \{0\}$ given by $z' \mapsto (z' - z)^{-1}$ is a bijection. If $\lambda \in \sigma((A - z)^{-1})$, then it cannot be written as $\lambda = (z' - z)^{-1}$ for $z' \in \rho(A)$. Consequently, $\lambda = 0$ or $\lambda = (z' - z)^{-1}$ for $z' \in \sigma(A)$, proving that

$$\sigma((A - z)^{-1}) \subseteq \{(z' - z)^{-1} \mid z' \in \sigma(A)\} \cup \{0\}.$$

On the other hand, the above equivalence yields

$$\{(z' - z)^{-1} \mid z' \in \sigma(A)\} \subseteq \sigma((A - z)^{-1}).$$

Finally, $0 \in \rho((A - z)^{-1})$ if and only if $(A - z) = ((A - z)^{-1})^{-1}$ is in $\mathcal{L}(A)$ which is clearly equivalent to $A \in \mathcal{L}(\mathcal{H})$. $\square$

2.2 Symmetric Operators

Lemma 2.8. *A linear subspace $\mathcal{G} \subseteq \mathcal{H} \oplus \mathcal{H}$ is the graph of some operator if and only if $(0, v) \in \mathcal{G}$ implies $v = 0$.*

Proof. The “only if” part is obvious. Now, suppose we are given a subspace $\mathcal{G} \subseteq \mathcal{H} \oplus \mathcal{H}$ satisfying the above property. We define $\mathcal{D}(A) = \{u \in \mathcal{H} \mid \exists v \in \mathcal{H}: (u, v) \in \mathcal{G}\}$. This is obviously a linear space. Note that, by assumption, for every $u \in \mathcal{D}(A)$, the vector v such that $(u, v) \in \mathcal{G}$ is unique. Therefore $Au = v$ is well-defined and yields a linear operator A with $\mathcal{G} = \mathcal{G}(A)$. $\square$

Lemma 2.9. *Let A be a closable operator. Then it has a minimal closed extension $\overline{A}$ in the sense that all closed extensions B of A satisfy $A \subseteq \overline{A} \subseteq B$.*

Proof. Define $\mathcal{G}(\overline{A}) = \overline{\mathcal{G}(A)}$. Then for all closed extensions B of A one has $\mathcal{G}(A) \subseteq \mathcal{G}(\overline{A}) \subseteq \mathcal{G}(B)$. Therefore $(0, v) \in \mathcal{G}(\overline{A})$ implies $v = 0$. Thus $\mathcal{G}(\overline{A})$ is actually the graph of some operator $\overline{A}$ and this operator is closed. $\square$

Definition 2.10. Let A be a closable operator. Its minimal closed extension $\overline{A}$ is called its *closure*.

Definition 2.11. Let A be a densely defined operator. Its *adjoint*, denoted A^* , is the operator defined by the graph

$$\mathcal{G}(A^*) = \{(Au, -u) \mid u \in \mathcal{D}(A)\}^\perp.$$

Remark 3. The condition that A be densely defined is necessary for the set defined above to be the graph of some operator, as is easily seen.

Note that since the $\mathcal{G}(A^*)$ is the orthogonal complement of some set, it is a closed subspace. Therefore, *adjoint operators are always closed*. By definition of A^* , one has

$$\langle Au, v \rangle = \langle u, A^*v \rangle$$

for all $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}(A^*)$. An immediate consequence of this definition is the following fact: A densely defined operator A is closable if and only if $\mathcal{D}(A^*)$ is dense. In this case $(A^*)^* = \overline{A}$ and $\overline{A}^* = A^*$.

Proposition 2.12. Let A be a densely defined operator. Then

$$\mathcal{N}(A^* - \overline{z}) = \mathcal{R}(A - z)^\perp.$$

Proof. Note that $v \in \mathcal{N}(A^* - \overline{z})$ if and only if $(v, \overline{z}v) \in \mathcal{G}(A^*)$, which, by definition, means that $\langle v, Au \rangle = \langle \overline{z}v, u \rangle = \langle v, zu \rangle$ for all $u \in \mathcal{D}(A)$. This in turn is clearly equivalent to $v \in \mathcal{R}(A - z)^\perp$. $\square$

Remark 4. If A is densely defined and closable, then

$$\mathcal{R}(\overline{A} - z)^\perp = \mathcal{N}(\overline{A}^* - \overline{z}) = \mathcal{N}(A^* - \overline{z}) = \mathcal{R}(A - z)^\perp.$$

Definition 2.13. A densely defined operator A is called *symmetric* if it is a restriction of its adjoint, i.e. if $A \subseteq A^*$.

In other words, a densely defined operator A is symmetric if and only

$$\langle Au, v \rangle = \langle u, Av \rangle$$

for all $v, u \in \mathcal{D}(A)$.

Remark 5. Since adjoint operators are closed, any symmetric operator is closable.

Let us try to categorize the spectra of symmetric operators.

Theorem 2.14. *Let A be a symmetric operator. Then its spectrum is one of the following:*

- (i) the entire complex plane $\sigma(A) = \mathbf{C}$
- (ii) the upper half-plane $\sigma(A) = \overline{\mathbf{C}_+}$
- (iii) the lower half-plane $\sigma(A) = \overline{\mathbf{C}_-}$
- (iv) a subset of the real line $\sigma(A) \subseteq \mathbf{R}$

Remark 6. The first option is in particular true when A is not closed. But there are other cases when the spectrum might be the entire complex plane. The fourth option is of particular interest to us. We will see later that the condition that the spectrum be a subset of the real line is equivalent to the notion of self-adjointness. Finally, if $A \in \mathcal{L}(\mathcal{H})$, then, by 2.5, its spectrum is contained in the disk of radius $\|A\|$ around the origin. Thus, in this case, only the fourth option is possible.

Proof. The idea is to show that if $z \in \rho(A) \setminus \mathbf{R}$, then the entire open half-plane containing z is included in the resolvent set. Indeed, in the proof of 2.6, we have seen that if $z \in \rho(A)$, then the open ball of radius $\|(A - z)^{-1}\|^{-1}$ centered at z is contained in $\rho(A)$. Not that for a symmetric operator we always have

$$\|(A - z)u\| \geq |\operatorname{Im} z| \|u\|$$

and therefore $\|(A - z)^{-1}\|^{-1} \geq |\operatorname{Im} z|$. It remains to show that the spectrum of a symmetric operator is non-empty. Indeed, suppose it were empty. This would in particular imply that $0 \in \rho(A)$ and thus $A^{-1} \in \mathcal{L}(\mathcal{H})$. But then $\sigma(A^{-1}) = \{0\}$, which implies $A^{-1} = 0$, which is clearly absurd. $\square$

2.3 Self-Adjoint Operators

We now define the most important class of linear operators in mathematical quantum mechanics.

Definition 2.15. A densely defined operator A is called *self-adjoint* if it coincides with its adjoint, i.e. if $A = A^*$.

Remark 7. In practice, it is often fairly simple to verify whether an operator is symmetric. To prove self-adjointness of an operator A , however, one needs to prove its symmetry and that $\mathcal{D}(A^*) \subseteq \mathcal{D}(A)$.

The following theorem gives sufficient and necessary conditions for a symmetric operator to be self-adjoint.

Theorem 2.16. *Let A be a symmetric operator. The following assertions are equivalent:*

(i) A is self-adjoint.

(ii) $\sigma(A) \subseteq \mathbf{R}$.

(iii) $\mathcal{R}(A + i) = \mathcal{R}(A - i) = \mathcal{H}$.

Proof. (i) $\implies$ (iii) : Note that $\mathcal{N}(A \mp i) = \mathcal{R}(A \pm i)^\perp$. Clearly, $\pm i$ cannot be eigenvalues of any symmetric operator. Thus $\overline{\mathcal{R}(A \pm i)} = \mathcal{H}$. As seen in the proof of 2.14, we have $\|(A \pm i)u\| \geq \|u\|$. Thus, if $((A \pm i)u_n)$ is Cauchy sequence, then so is (u_n) . Since adjoint operators are always closed and A is its own adjoint, this now implies that $u = \lim u_n \in \mathcal{D}(A)$ and $\lim(A \pm i)u_n = (A \pm i)u \in \mathcal{R}(A \pm i)$. Consequently, $\mathcal{R}(A \pm i)$ is closed.

(iii) $\implies$ (ii) : The spectrum of A corresponds to one of the cases in 2.14. We know from $\|(A \pm i)u\| \geq \|u\|$ that $A \pm i$ has a bounded inverse and from assumption (iii), that said inverse is defined everywhere. Hence $\pm i \in \rho(A)$. Thus the only possible option in 2.14 is (iv).

(ii) $\implies$ (i) : It suffices to prove that $\mathcal{D}(A^*) \subseteq \mathcal{D}(A)$. Let $z \in \mathbf{C} \setminus \mathbf{R}$. Then $A - z$ and $A - \bar{z}$ are bijective. Let $u \in \mathcal{D}(A^*)$ and define $v = (A - z)^{-1}(A^* - z)u \in \mathcal{D}(A) \subseteq \mathcal{D}(A^*)$. Then $Av = A^*v$ and $(A - z)v = (A^* - z)u$. Therefore $(A^* - z)(u - v) = (A - z)v - (A - z)v = 0$ and $u - v \in \mathcal{N}(A^* - z) = \mathcal{R}(A - \bar{z})^\perp = \{0\}$. Consequently, $u = v \in \mathcal{D}(A)$. $\square$

Remark 8. Combining 2.14 and 2.16 we see that if A is a symmetric operator and $z \in \mathbf{C}$ is such that both z and $\bar{z}$ belong to $\rho(A)$, then A is self-adjoint, since the only possible option in 2.14 would be that $\sigma(A) \subseteq \mathbf{R}$. If $z \in \mathbf{C} \setminus \mathbf{R}$, then we would get the bounded invertibility of $A - z$ for free, meaning that we only have to require that $A - z$ and $A - \bar{z}$ be surjective to conclude the self-adjointness of A .

2.4 Semibounded Operators and Forms

In this section we examine the connection between semibounded self-adjoint operators and semibounded forms.

Definition 2.17. A sesquilinear form $a: \mathcal{D}[a] \times \mathcal{D}[a] \rightarrow \mathbf{C}$ defined on some subspace $\mathcal{D}[a]$ of a Hilbert space $\mathcal{H}$ is said to be semibounded from below if there exists a constant $c \in \mathbf{R}$ such that $a[u, u] \geq c \|u\|^2$ for all $u \in \mathcal{D}[a]$. If c can be chosen to be (strictly) positive, then the form a is said to be *coercive*.

Remark 9. For simplicity of notation we shall write $a[u] = a[u, u]$. The map $u \mapsto a[u]$ is called the *quadratic form* associated with the sesquilinear form $(u, v) \mapsto a[u, v]$. Furthermore, we will often omit the specification “from below”, speaking simply of semibounded forms. One could talk about forms that are semibounded from above, but multiplication by -1 would immediately reduce the situation to semiboundedness from below. Note that any semibounded form is automatically symmetric by polarization. Finally, any coercive sesquilinear form is an inner product.

We can associate to any linear operator A a sesquilinear form a by defining $\mathcal{D}[a] = \mathcal{D}(A)$ and $a[u, v] = \langle u, Av \rangle$. If A is symmetric, then so is a . Conversely, if a is symmetric and $\mathcal{D}(A)$ is dense, then A must be symmetric. Consequently, if a densely-defined form obtained this way is semibounded, the operator must be symmetric to begin with.

Definition 2.18. A symmetric operator A is said to be *semibounded (from below)*, if the form $\mathcal{D}(A) \times \mathcal{D}(A) \ni (u, v) \mapsto \langle u, Av \rangle \in \mathbf{C}$ is semibounded (from below).

In what follows we are exclusively interested in forms that arise in the context of symmetric operators. It is therefore natural to assume that *all sesquilinear forms be defined on a dense subspace* of some Hilbert space and we will tacitly do so from now on.

If a is a coercive form, then $\mathcal{D}[a]$ equipped with a is an inner product space. This space may or may not be complete with respect to the metric induced by this inner product. This brings us to the following definition.

Definition 2.19. A coercive sesquilinear form a is said to be *closed* if $\mathcal{D}[a]$ is a Hilbert space when equipped with a as inner product. It is called *closable* if it has a closed extension b , in the sense that $\mathcal{D}[a] \subseteq \mathcal{D}[b]$ and $b[u] = a[u]$ for $u \in \mathcal{D}[a]$.

While any inner product space can be completed to obtain a Hilbert space, in the case of a coercive form a it is not clear whether the completion of $\mathcal{D}[a]$ can still be identified with a subspace of the ambient Hilbert space. Indeed, *not all coercive forms are closable*, an important class of closable forms, however, are those that are obtained according to the above procedure starting with a symmetric operator.

Theorem 2.20. *Let A be a symmetric operator and let the form a be defined by $\mathcal{D}[a] = \mathcal{D}(A)$ and $a[u, v] = \langle u, Av \rangle$. If a is coercive, then it is closable.*

Proof. Without loss of generality, assume that $a[u] \geq \|u\|^2$. Any a -Cauchy sequence (u_n) is then also a Cauchy sequence in $\mathcal{H}$. Denote its limit by u . Define $b[u] = \lim a[u_n]$. This limit exists due to the reverse triangle inequality

$$|\sqrt{a[u_n]} - \sqrt{a[u_m]}| \leq \sqrt{a[u_n - u_m]}.$$

Furthermore, the definition of $b[u]$ does not depend on the choice of the sequence (u_n) . Indeed, if (v_n) is another a -Cauchy sequence converging to u in $\mathcal{H}$, then letting $w_n = u_n - v_n$ yields a a -Cauchy sequence converging to zero in $\mathcal{H}$. If $\lim a[w_n] = 0$, then, again by the reversed triangle inequality, we see that $b[u]$ is well-defined. By Cauchy–Schwarz we have

$$a[w_n] \leq |a[w_n, w_n - w_m]| + |a[w_n, w_m]| \leq \sqrt{a[w_n]} \sqrt{a[w_n - w_m]} + \|w_m\| \|Aw_n\|.$$

Since (w_n) is an a -Cauchy sequence, it is a -bounded and thus the first term can be made arbitrarily small by choosing n and m large enough. The second term converges to zero as $m \rightarrow \infty$. Consequently $\lim_n a[w_n] = 0$. $\square$

We need some notation: Given a sesquilinear form a and a real number $\alpha \in \mathbf{R}$, we define the form a_α by setting $\mathcal{D}[a_\alpha] = \mathcal{D}[a]$ and $a_\alpha[\cdot, \cdot] = a[\cdot, \cdot] + \alpha \langle \cdot, \cdot \rangle$. If a is a semibounded form and

$$\alpha > - \inf_{u \in \mathcal{D}[a], \|u\|=1} a[u],$$

then a_α is clearly coercive. This way, the notions of closedness and closability extend to semibounded forms by calling such a form a *closed* or *closable* if a_α is for some (and hence all) α for which a_α is coercive. Using this definition we can generalize the above theorem to assert that *any semibounded operator defines a closable semibounded sesquilinear form*. Now that we established that semibounded forms induced by symmetric operators are closable, we shall see that one can associate a semibounded self-adjoint operator to any closed semibounded form and vice versa. This way, any given semibounded (symmetric) operator has a distinguished self-adjoint extension. The correspondence between closed semibounded forms and semibounded self-adjoint operators is characterized by the following to properties:

- (i) $\mathcal{D}(A) \subseteq \mathcal{D}[a]$.
- (ii) $a[w, u] = \langle w, Au \rangle$ for all $w \in \mathcal{D}[a]$ and $u \in \mathcal{D}(A)$.

Proposition 2.21. *Let a be a closed semibounded sesquilinear form. Then there exists a unique semibounded self-adjoint operator A such that the properties (i) and (ii) from*

above are satisfied.

Proof. Let $\mathcal{D}(A)$ be the subspace of all $u \in \mathcal{D}[a]$ for which there exists a $v \in \mathcal{H}$ such that

$$a[w, u] = \langle w, v \rangle$$

for all $w \in \mathcal{D}[a]$. Then $Au = v$ is well-defined, since $\mathcal{D}[a]$ is assumed to be dense. The linearity of A is straightforward. What remains to be shown is that A is densely defined and that it is self-adjoint. Let α be chosen such that a_α is an inner product. We claim that $A + \alpha$ is surjective. Indeed, the coercivity of a_α yields that for fixed $v \in \mathcal{H}$, the map $w \mapsto \langle w, v \rangle$ is a continuous anti-linear form on $(\mathcal{D}[a], a_\alpha)$. Therefore, there exists a $u \in \mathcal{D}[a]$ such that $\langle w, v \rangle = a_\alpha[w, u]$ for all $w \in \mathcal{D}[a]$. This means precisely that $u \in \mathcal{D}(A)$ and $v - \alpha u = Au$. Having established this surjectivity, the density of $\mathcal{D}(A)$ is easy to see. Let $w \in \mathcal{D}[a]$ be orthogonal to $\mathcal{D}(A)$ in the sense of a_α . Then

$$0 = a_\alpha[w, u] = a[w, u] + \alpha \langle w, u \rangle = \langle w, v + \alpha u \rangle = \langle w, (A + \alpha)u \rangle,$$

for all $u \in \mathcal{D}(A)$. This means that $w \in \mathcal{R}(A + \alpha)^\perp$. By surjectivity of $A + \alpha$, it follows that $w = 0$. By coercivity of a_α , density of $\mathcal{D}(A)$ in $\mathcal{D}[a]$ with respect to a_α implies density with respect to $\langle \cdot, \cdot \rangle$. Since $\mathcal{D}[a]$ itself was assumed to be dense in $\mathcal{H}$, we see that $\mathcal{D}(A)$ is dense in $\mathcal{H}$. We proceed to proving that A is self-adjoint. That A is symmetric is obvious. We have seen that $A + \alpha$ is surjective. Furthermore, by coercivity of a_α , the operator $A + \alpha$ is boundedly invertible. Therefore $\alpha \in \rho(A)$. In the theorem [2.14](#), the only possible option is thus (iv), i.e. that $\sigma(A) \subseteq \mathbf{R}$. By theorem [2.16](#), A is self-adjoint. Finally, uniqueness of A is seen as follows: Let B be a semibounded self-adjoint operator satisfying the properties (i) and (ii) above. In fact, these two properties alone imply that $\mathcal{D}(B)$ is dense $\mathcal{D}[a]$ in the sense of a_α . Indeed, if $v \in \mathcal{D}[a]$ such that $a_\alpha[u, v] = 0$ for all $u \in \mathcal{D}(B)$, then $v \in \mathcal{R}(B + \alpha)^\perp = \mathcal{N}(B + \alpha)$. This space is clearly trivial, since a_α is coercive. Property (ii) states that $a[w, u] = \langle w, Bu \rangle$ for all $w \in \mathcal{D}[a]$ and $u \in \mathcal{D}(B)$. By definition, this means that $\mathcal{D}(B) \subseteq \mathcal{D}(A)$ and that $Bu = Au$ on $\mathcal{D}(A)$. In other words, $B \subseteq A$. Since both operators are self-adjoint, we see that $A = B$. $\square$

Conversely, given a semibounded self-adjoint operator, we can find a unique semibounded form associated with it.

Proposition 2.22. *Let A be a semibounded self-adjoint operator. Then there exists a unique closed semibounded sesquilinear form a such that properties (i) and (ii) are satisfied.*

Proof. Let the form b be constructed by applying [2.20](#) to the operator A . Denote its

closure by a . Then, obviously, $\mathcal{D}(A) \subseteq \mathcal{D}[a]$. Furthermore, $a[w, u] = b[w, u] = \langle w, Au \rangle$ for all $w \in \mathcal{D}[b] = \mathcal{D}(A)$ and $u \in \mathcal{D}(A)$. By definition, $\mathcal{D}[b]$ is dense in $\mathcal{D}[a]$. Therefore $a[w, u] = \langle w, Au \rangle$ for all $w \in \mathcal{D}[a]$ and $u \in \mathcal{D}(A)$. This concludes the proof of existence. To prove uniqueness, assume that there exists another form c satisfying (i) and (ii). Then $\mathcal{D}(A)$ is dense in both $\mathcal{D}[a]$ and $\mathcal{D}[c]$ and

$$a[w, u] = \langle w, Au \rangle = c[w, u]$$

for all $w, u \in \mathcal{D}(A)$. By closedness of a , we see that $\mathcal{D}[c] \subseteq \mathcal{D}[a]$ and that the two forms coincide on $\mathcal{D}[c]$. Conversely, $\mathcal{D}[a] \subseteq \mathcal{D}[c]$. This concludes the argument. $\square$

We are now in a position to prove that any semibounded symmetric operator has a self-adjoint extension.

Theorem 2.23. *Let A be a semibounded symmetric operator. Then it has a lower semi-bounded self-adjoint extension.*

Proof. Define the form a via $\mathcal{D}[a] = \mathcal{D}(A)$ and $a[u, v] = \langle u, Av \rangle$. By 2.20, a has a closed extension b . Now, by 2.21 there exists a unique semibounded self-adjoint operator B associated with b . The only thing that is left to show is that B is indeed an extension of A . We have $b[w, v] = \langle w, Bv \rangle$ for all $w \in \mathcal{D}[b]$ and $v \in \mathcal{D}(B)$. Since $\mathcal{D}(A) = \mathcal{D}[a] \subseteq \mathcal{D}[b]$, this identity holds in particular for $w \in \mathcal{D}(A)$. Now, since $\mathcal{D}[a] = \mathcal{D}(A)$ is dense in $\mathcal{D}[b]$, we see that $b[w, v] = \langle Aw, v \rangle$ for all $w \in \mathcal{D}(A)$ and $v \in \mathcal{D}[b]$. Therefore $\langle Aw, v \rangle = \langle w, Bv \rangle$ for all $w \in \mathcal{D}(A)$ and $v \in \mathcal{D}(B)$. Consequently, $B \subseteq A^*$ and thus $A \subseteq \overline{A} \subseteq B^* = B$. $\square$

We conclude this section by noting that the lower bounds for an operator and its associated form coincide.

Proposition 2.24. *Let A be a semibounded self-adjoint operator and let a be a closed semibounded form associated with it. Then*

$$\inf_{u \in \mathcal{D}[a] \setminus \{0\}} \frac{a[u]}{\|u\|^2} = \inf_{u \in \mathcal{D}(A) \setminus \{0\}} \frac{\langle u, Au \rangle}{\|u\|^2}.$$

Proof. The inequality “ $\leq$ ” is obvious and “ $\geq$ ” follows from the fact that $\mathcal{D}(A)$ is dense in $\mathcal{D}[a]$. $\square$

2.5 The Spectral Theorem

We begin this section by considering a class of particularly nice operators. Let (X, μ) be a measure space and let $f: X \rightarrow \mathbf{C}$ be measurable. On the Hilbert space $L^2(X, \mu)$ we

define the *operator of multiplication by f* via

$$(M_f u)(x) = f(x)u(x), \quad \mu\text{-a.e. } x \in X$$

on the maximal domain

$$\mathcal{D}(M_f) = \{u \in L^2(X, \mu) \mid fu \in L^2(X)\}.$$

Multiplication operators have the following properties:

Proposition 2.25. *Let (X, μ) be a measure space and let $f: X \rightarrow \mathbf{C}$ be measurable.*

(i) *M_f is closed.*

(ii) *M_f is densely defined and $(M_f)^* = M_{\bar{f}}$.*

Proof. Since adjoint operators are always closed, it suffices to prove assertion (ii). Let $u \in L^2(X, \mu)$. Define $X_n = \{x \in X \mid |f(x)| \leq n\}$. Then $X = \bigcup_n X_n$. Define $u_n = \chi_{X_n} u$. Clearly $u_n \in \mathcal{D}(M_f)$ and furthermore $u_n \rightarrow u$ in $L^2(X, \mu)$ by Lebesgue's dominated convergence theorem. Obviously $M_{\bar{f}} \subseteq (M_f)^*$. Conversely, let $v \in \mathcal{D}((M_f)^*)$ and let $w = (M_f)^* v$. Then

$$\int_X \overline{w}(x)u(x)d\mu(x) = \int_X \overline{v}(x)f(x)u(x)d\mu(x),$$

for all $u \in \mathcal{D}(M_f)$ and in particular for $u \in L^2(X, \mu)$ supported in X_n . Therefore

$$\int_{X_n} \overline{(w(x) - \bar{f}(x)v(x))}u(x)d\mu(x) = 0,$$

for any $u \in L^2(X_n, \mu)$. Since f is bounded on X_n , the restriction of $w - \bar{f}v$ to X_n is in $L^2(X_n, \mu)$. Consequently, $w - \bar{f}v = 0 \in L^2(X_n, \mu)$ and thus $w(x) = \bar{f}(x)v(x)$ for almost every $x \in X_n$ and hence for almost every $x \in X$. $\square$

Remark 10. Note that if f is real-valued, then M_f is a self-adjoint operator.

We are also interested in the spectral properties of multiplication operators. Recall that the *essential image* or *essential range* of a measurable function $f: X \rightarrow \mathbf{C}$ defined on a measure space (X, μ) is the set of all $\lambda \in \mathbf{C}$ for which the preimage of any open set containing λ has positive measure. In other words

$$\text{ess-range}(f) = \{\lambda \in \mathbf{C} \mid \forall \varepsilon > 0 : \mu(\{x \in X \mid |f(x) - \lambda| < \varepsilon\}) > 0\}.$$

Obviously, the essential range of any measurable function is a closed set.

Proposition 2.26. *Let (X, μ) be a semi-finite measure space and let $f: X \rightarrow \mathbf{C}$ be a measurable function. Then the spectrum of M_f is the essential range of f :*

$$\sigma(M_f) = \text{ess-range}(f).$$

Furthermore, $\lambda \in \mathbf{C}$ is an eigenvalue of M_f if and only if $\mu(\{x \in X \mid f(x) = \lambda\}) > 0$.

To prove this assertion, we use the following lemma:

Lemma 2.27. *Let A be an operator on a Hilbert space and let $\lambda \in \mathbf{C}$. If there exists a sequence $(u_n) \subseteq \mathcal{D}(A)$ such that $\|u_n\| = 1$ and $(A - \lambda)u_n \rightarrow 0$, then $\lambda \in \sigma(A)$.*

Proof of the lemma. Suppose that $\lambda \in \rho(A)$. Then

$$1 = \|u_n\| = \|(A - \lambda)^{-1}(A - \lambda)u_n\| \leq \|(A - \lambda)^{-1}\| \|(A - \lambda)u_n\| \rightarrow 0,$$

which is clearly a contradiction. □

Remark 11. For self-adjoint operators, the inverse implication holds as well.

Proof of the theorem. Suppose that $\lambda \in \mathbf{C} \setminus \text{ess-range}(f)$. Then there exists $\varepsilon > 0$ such that $|f(x) - \lambda| \geq \varepsilon$ for a.e. $x \in X$. Then $|(f(x) - \lambda)^{-1}| \leq 1/\varepsilon$ almost everywhere and therefore the operator of multiplication by $(f - \lambda)^{-1}$ is in $\mathcal{L}(L^2(X, \mu))$. Clearly $M_{(f-\lambda)^{-1}}$ is the inverse of $M_{f-\lambda} = M_f - \lambda$, therefore $\sigma(M_f) \subseteq \text{ess-range}(f)$. Conversely, suppose that $\lambda \in \text{ess-range}(f)$. Let $X_n = \{x \in X \mid |f(x) - \lambda| < 1/n\}$. Then $\mu(X_n) > 0$ for every $n \in \mathbf{N}$. Since the measure μ is semi-finite, there exist measurable sets $Y_n \subseteq X_n$ such that $0 < \mu(Y_n) < \infty$. Define $u_n = \chi_{Y_n} / \sqrt{\mu(Y_n)}$. Then (u_n) is a sequence of normalized elements of $\mathcal{D}(M_f)$, since f is bounded on every X_n . Furthermore,

$$\|(M_f - \lambda)u_n\|^2 = \frac{1}{\mu(Y_n)} \int_{Y_n} |f(x) - \lambda|^2 d\mu(x) \leq \frac{1}{n^2} \rightarrow 0.$$

Applying the above lemma concludes the proof of the assertion about the spectrum. Now assume that $\mu(\{x \in X \mid f(x) = \lambda\}) > 0$ and let $Y \subseteq \{x \in X \mid f(x) = \lambda\}$ such that $0 < \mu(Y) < \infty$. Then χ_Y is clearly an eigenvector of M_f with eigenvalue λ . Conversely, suppose that $M_f u = \lambda u$ for some $u \in L^2(X, \mu)$. Then

$$0 = \|(M_f - \lambda)u\|^2 = \int_X |f(x) - \lambda|^2 |u(x)|^2 d\mu(x)$$

and consequently $(f(x) - \lambda)u(x) = 0$ almost everywhere. If $\{x \in X \mid f(x) = \lambda\}$ were of measure zero, u would have to vanish almost everywhere. $\square$

Operators of multiplication by essentially bounded functions are bounded.

Proposition 2.28. *Let (X, μ) be a semi-finite measure space and let $f: X \rightarrow \mathbf{C}$ be a measurable function. Then $M_f \in \mathcal{L}(L^2(X, \mu))$ if and only if f is μ -essentially bounded. In this case*

$$\|M_f\| = \text{ess-sup}|f|.$$

Proof. Suppose that f is essentially bounded. Then

$$\int_X |fu|^2 d\mu \leq (\text{ess-sup}|f|)^2 \int_X |u|^2 d\mu$$

for every $u \in L^2(X, \mu)$. Thus M_f is in $\mathcal{L}(L^2(X, \mu))$ with $\|M_f\| \leq \text{ess-sup}|f|$. For $\varepsilon > 0$ define $X_\varepsilon = \{x \in X \mid |f(x)| \geq \text{ess-sup}|f| - \varepsilon\}$. By the definition of essential suprema, $\mu(X_\varepsilon) > 0$ and by semi-finiteness of μ , there exists a set $Y \subseteq X_\varepsilon$ such that $0 < \mu(Y) < \infty$. Then $u = \chi_Y \in L^2(X, \mu)$ and $\|M_f u\| \geq (\text{ess-sup}|f| - \varepsilon) \|u\|$. Therefore $\|M_f\| = \text{ess-sup}|f|$. An analogous argument shows that M_f cannot be bounded if f is not essentially bounded. $\square$

We are now able to state the spectral theorem. We reached the point at which we need to restrict ourselves to separable Hilbert spaces. The spectral theorem stipulates that any self-adjoint operator in a separable Hilbert space is unitarily equivalent to a multiplication operator.

Theorem 2.29. *Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. There exist a σ -finite measure space (X, μ) , a unitary operator $V: \mathcal{H} \rightarrow L^2(X, \mu)$ and a measurable function $f: X \rightarrow \mathbf{R}$ such that*

$$A = V^{-1}M_f V.$$

In particular $\mathcal{D}(A) = V^{-1}\mathcal{D}(M_f) = \{u \in \mathcal{H} \mid \int_X |f|^2 |Vu|^2 d\mu < \infty\}$.

Proving this classical result would exceed the scope of this introductory chapter, which is why we refer to [\[Wei00\]](#) for a proof.

Remark 12. Since any σ -finite measure space is also semi-finite, the multiplication operators obtained by the spectral theorem satisfy the conditions of [\[2.26\]](#).

The spectral theorem is extremely useful since spectral properties are invariant under unitary equivalence in the sense that if A is an operator on a Hilbert space $\mathcal{H}$ and A' is

an operator on a Hilbert space $\mathcal{H}'$ and if $V: \mathcal{H} \rightarrow \mathcal{H}'$ is unitary such that $A = V^{-1}A'V$, then $\sigma(A) = \sigma(A')$, $\sigma_d(A) = \sigma_d(A')$, and the multiplicities of eigenvalues coincide.

Given this multiplication operator representation of self-adjoint operators, it is easy to define functions of self-adjoint operators. The idea is to define

$$\phi(A) = V^{-1}M_{\phi \circ f}V.$$

To justify the notation $\phi(A)$, this definition should not depend on the choice of the representation of A as a multiplication operator, as this choice is not unique. We first prove this assertion for bounded functions. We denote by $B(\mathbf{R})$ the space of all bounded measurable functions $\mathbf{R} \rightarrow \mathbf{C}$ equipped with the supremum norm: $\|\phi\|_\infty = \sup_{\lambda \in \mathbf{R}} |\phi(\lambda)|$. Clearly, $B(\mathbf{R})$ is also a $*$ -algebra.

Theorem 2.30. *Let A be a self-adjoint operator on a separable Hilbert space. There exists a unique $*$ -homomorphism $\Psi: B(\mathbf{R}) \rightarrow \mathcal{L}(\mathcal{H})$, $\phi \mapsto \Psi(\phi) =: \phi(A)$ such that*

- (i) $\|\phi(A)\| \leq \sup_{\lambda \in \sigma(A)} |\phi(\lambda)|$, in particular, Ψ is continuous.
- (ii) If $\phi(\lambda) = (\lambda - z)^{-1}$ for some $z \in \rho(A)$, then $\phi(A) = (A - z)^{-1}$.
- (iii) If (ϕ_n) is a (uniformly) bounded sequence in $B(\mathbf{R})$ and $\phi_n \rightarrow \phi$ pointwise, then $\phi_n(A) \rightarrow \phi(A)$ strongly.

The map Ψ is also called a functional calculus.

Remark 13. We know from the theory of C^* -algebras that any homomorphism of C^* -algebras is a contraction. In our case this is equivalent to saying that $\|\phi(A)\| \leq \|\phi\|_\infty$.

Proof. The spectral theorem provides us with a measure space (X, μ) , a function $f: X \rightarrow \mathbf{R}$ and a unitary operator $V: \mathcal{H} \rightarrow L^2(X, \mu)$ such that $A = V^{-1}M_fV$. Define $\Psi(\phi) = V^{-1}M_{\phi \circ f}V$. Then $\Psi(\phi)$ is equivalent to the operator of multiplication by a bounded function and thus a bounded operator by [2.28](#). Indeed, its operator norm is the μ -essential supremum of $\phi \circ f$, but since f is essentially supported on $\sigma(A)$, we obtain the bound claimed in (i).

The normalization, i.e. assertion (ii) has been discussed above. We proceed to the proof of (iii). Let $u \in \mathcal{H}$. We estimate

$$\|\Psi(\phi_n)u - \Psi(\phi)u\|^2 \leq \int_X |\phi_n(f(x)) - \phi(f(x))|^2 |Vu(x)|^2 d\mu(x).$$

Lebesgue's dominated convergence theorem establishes the claim. The proof of uniqueness of Ψ makes use of the density of functions of the form $\phi(\lambda) = (\lambda - z)^{-1}$ in $B(\mathbf{R})$, invoking the Stone–Weierstraß theorem. Indeed, suppose we had two functional calculi Ψ_1 and Ψ_2 . Set

$$\mathcal{A} = \{\phi \in B(\mathbf{R}) \mid \Psi_1(\phi) = \Psi_2(\phi)\}.$$

By property (ii), $\mathcal{A}$ contains functions of the form $\phi(\lambda) = (\lambda - z)^{-1}$, $z \in \mathbf{C} \setminus \mathbf{R}$. By the Stone–Weierstraß theorem, the algebra generated by functions of this form is dense in $C_0(\mathbf{R})$. Clearly, $\mathcal{A}$ is an algebra as well. By property (iii), $\mathcal{A}$ is closed under pointwise limits of uniformly bounded sequences of functions. It is thus a fortiori closed under uniform limits and hence

$$C_0(\mathbf{R}) \subseteq \mathcal{A}.$$

On the other hand, every measurable function can be realized as the point-wise limit of $C_0(\mathbf{R})$ functions. Consequently,

$$B(\mathbf{R}) = \mathcal{A}. \quad \square$$

One special case of functions of self-adjoint operators that bears particular importance is that of spectral projections. Suppose that A is a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. For a measurable set $\Omega \subseteq \mathbf{R}$, define $P(\Omega) = \chi_\Omega(A)$. The operators $P(\Omega)$ satisfy a series of important properties.

Proposition 2.31. *In the situation described above, we find that*

- (i) $P(\Omega)$ is an orthogonal projection for any $\Omega \in \mathcal{B}(\mathbf{R})$.
- (ii) $P(\bigcup_n \Omega_n)u = \sum_n P(\Omega_n)u$ for disjoint $\Omega_n \in \mathcal{B}(\mathbf{R})$ and $u \in \mathcal{H}$.
- (iii) $P(\mathbf{R}) = I$.

Proof. To see (i), we need to prove that each $P(\Omega)$ is idempotent and self-adjoint, i.e. that $P(\Omega) = P(\Omega)^* = P(\Omega)^2$. This is, however, an immediate consequence of the properties of indicator functions. Assertion (ii) is a consequence of [2.30](#) and (iii) is obvious. $\square$

A map $P: \mathcal{B}(\mathbf{R}) \rightarrow \mathcal{L}(\mathcal{H})$ satisfying the properties in the above proposition is called a *projection-valued measure*. For any projection-valued measure P one can find a self-adjoint operator A such that $P(\Omega) = \chi_\Omega(A)$ and one can define integrals with respect to projection-valued measures (see [Tes14](#) for instance). Such integrals would be linear operators. In fact, one formulation of the spectral theorem is that any self-adjoint operator can be written as an integral with respect to a projection-valued measure. Introducing

this concept would demand too much notation and technical work while it would not yield any significant advantages compared to the multiplication operator version of the spectral theorem.

It will be convenient to decompose the spectra of self-adjoint operators.

Definition 2.32. Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. The set of all isolated eigenvalues of finite multiplicity of A is called its *discrete spectrum* and it is denoted $\sigma_d(A)$. The *essential spectrum* of A is the set $\sigma_e(A) = \sigma(A) \setminus \sigma_d(A)$.

The spectrum of A and its components described above can be described in terms of P . Indeed, from [2.26](#) we immediately obtain

$$\sigma(A) = \{\lambda \in \mathbf{C} \mid P(\lambda - \varepsilon, \lambda + \varepsilon) \neq 0 \text{ for all } \varepsilon > 0\},$$

and that a number $\lambda \in \mathbf{R}$ is an eigenvalue of A if and only if $P(\{\lambda\}) \neq 0$. In this case $P(\{\lambda\})$ is the orthogonal projection onto $\ker(A - \lambda)$. The discrete spectrum is given by

$$\sigma_d(A) = \{\lambda \in \sigma(A) \mid \dim \mathcal{R}(P(\lambda - \varepsilon, \lambda + \varepsilon)) < \infty \text{ for some } \varepsilon > 0\},$$

and the essential spectrum by

$$\sigma_e(A) = \{\lambda \in \sigma(A) \mid \dim \mathcal{R}(P(\lambda - \varepsilon, \lambda + \varepsilon)) = \infty \text{ for all } \varepsilon > 0\}.$$

We can characterize the points of the spectrum and of the essential spectrum using *Weyl-sequences*.

Proposition 2.33. Let A be a self-adjoint operator in a separable Hilbert space.

- (i) $\lambda \in \sigma(A)$ if and only if there exists a sequence $(u_n) \subseteq D(A)$ of normalized elements such that $(A - \lambda)u_n \rightarrow 0$.
- (ii) $\lambda \in \sigma_e(A)$ if and only if there exists a sequence $(u_n) \subseteq D(A)$ of normalized elements such that $(A - \lambda)u_n \rightarrow 0$ and $u_n \rightharpoonup 0$.

Proof. The “if”-part of assertion (i) was already proven in [2.27](#). Suppose conversely that $\lambda \in \sigma(A)$. For every $n \in \mathbf{N}$ we have $\dim \mathcal{R}(P(\lambda - 1/n, \lambda + 1/n)) > 0$ and we can thus choose a normalized $u_n \in \mathcal{R}(P(\lambda - 1/n, \lambda + 1/n))$. Since $P(\lambda - 1/n, \lambda + 1/n)u_n = u_n$, we find $u_n \in D(A)$ and $\|(A - \lambda)u_n\| \leq \frac{1}{n}$, proving the claim. So see (ii), suppose we had a normalized sequence $(u_n) \subseteq D(A)$ such that $(A - \lambda)u_n \rightarrow 0$ and $u_n \rightharpoonup 0$. Since

$(A - \lambda)u_n \rightarrow 0$, we know $\lambda \in \sigma(A)$. If $\lambda \in \sigma_d(A)$, then $\dim \mathcal{R}(P(\lambda - \varepsilon, \lambda + \varepsilon)) < \infty$ for some $\varepsilon > 0$. Choose such an ε and let $v_1, \dots, v_N$ be an orthonormal basis of $\mathcal{R}(P(\lambda - \varepsilon, \lambda + \varepsilon))$. Since $(A - \lambda)u_n \rightarrow 0$, we find $u_n \in \mathcal{R}(P(\lambda - \varepsilon, \lambda + \varepsilon))$ for sufficiently large n . Therefore,

$$\|u_n\|^2 = \sum_{j=1}^N |\langle v_j, u_n \rangle|^2 \rightarrow 0,$$

which clearly contradicts the normalization assumption $\|u_n\| = 1$. Hence λ must lie in $\sigma_e(A)$. Finally, assume that $\lambda \in \sigma_e(A)$. Every $\mathcal{R}(P(\lambda - 1/n, \lambda + 1/n))$ is infinite-dimensional. We can therefore inductively define $u_n \in \mathcal{R}(P(\lambda - 1/n, \lambda + 1/n))$ such that $u_n \perp u_{n-1}, \dots, u_1$ and $\|u_n\| = 1$. Just like in part (i) we get $(A - \lambda)u_n \rightarrow 0$ and the orthogonality of the (u_n) implies the weak convergence $u_n \rightharpoonup 0$. $\square$

One consequence of this proposition that will turn out to be extremely useful is the following criterion of the discreteness of the spectrum of a self-adjoint operator.

Proposition 2.34. *Let A be a self-adjoint operator. The spectrum of A is purely discrete if $(A - z)^{-1}$ is a compact operator for some (and hence all) $z \in \rho(A)$.*

Proof. Let $\lambda \in \sigma(A)$. Suppose that $\lambda \in \sigma_e(A)$. Then there exists a sequence $(u_n) \subseteq \mathcal{D}(A)$ with $\|u_n\| = 1$ such that $(A - \lambda)u_n \rightarrow 0$ and $u_n \rightharpoonup 0$. Let $z \in \rho(A)$ and assume that $(A - z)^{-1}$ is compact. By definition, $(A - z)^{-1}u_n \rightarrow 0$, since $u_n \rightharpoonup 0$. Therefore

$$u_n = (A - z)^{-1}(A - z)u_n = (A - z)^{-1}(A - \lambda)u_n + (\lambda - z)(A - z)^{-1}u_n \rightarrow 0,$$

which contradicts the assumptions that $\|u_n\| = 1$. $\square$

We even have a further criterion for the discreteness of the spectrum of a semibounded self-adjoint operator in terms of its form.

Proposition 2.35. *Let A be a semibounded self-adjoint operator on the Hilbert space $\mathcal{H}$ and let a be its associated form. If the embedding $\mathcal{D}[a] \hookrightarrow \mathcal{H}$ is compact, then the operator A has discrete spectrum.*

Proof. Suppose that the embedding operator $J: \mathcal{D}[a] \hookrightarrow \mathcal{H}$ be compact. Let $\alpha \in \mathbf{R}$ be such that a_α is an inner product. For $z \in \rho(A)$, the resolvent $(A - z)^{-1}$, considered as an

operator $R: \mathcal{H} \rightarrow \mathcal{D}[a]$, is bounded, since

$$\begin{aligned}
a_\alpha[(A - z)^{-1}u] &= \langle u, (A + \alpha)(A - z)^{-1}u \rangle \\
&= \langle u, ((A - z) + z + \alpha)(A - z)^{-1}u \rangle \\
&= \langle u, (1 + (z + \alpha)(A - z)^{-1})u \rangle \\
&\leq \|u\|^2 + |z + \alpha| \|(A - z)^{-1}u\| \|u\| \\
&\leq (1 + |z + \alpha| \|(A - z)^{-1}\|) \|u\|^2.
\end{aligned}$$

Therefore, $(A - z)^{-1} = J \circ R: \mathcal{H} \rightarrow \mathcal{H}$ is compact as the composition of a compact and a bounded operator. According to the preceding proposition, $\sigma(A) = \sigma_d(A)$. $\square$

We conclude this section with a discussion of some consequences of the spectral theorem for semibounded operator. Suppose that A be a self-adjoint lower semibounded operator on a separable Hilbert space $\mathcal{H}$, and denote its associated form by a . By the spectral theorem, we know that there exist a σ -finite measure space (X, μ) , a unitary operator $V: \mathcal{H} \rightarrow L^2(X, \mu)$ and a measurable function $f: X \rightarrow \mathbf{R}$ such that $A = V^{-1}M_fV$. The domain of A is then given by

$$\mathcal{D}(A) = V^{-1}\mathcal{D}(M_f) = \{u \in \mathcal{H} \mid \int_X |f|^2 |Vu|^2 d\mu < \infty\}.$$

We have never explicitly computed the domain of the associated form. This will be necessary, however, in the next section. We will prove the following claim

$$\mathcal{D}[a] = \{u \in \mathcal{H} \mid \int_X f |Vu|^2 d\mu < \infty\}.$$

Note that we do not have to take the absolute value of f , because, by assumption, f is bounded from below. We have to prove that

1. $a[w, v] = \int_X f \overline{Vw} Vv d\mu$ defines a closed form on $\mathcal{D}[a]$,
2. $\mathcal{D}(A) \subseteq \mathcal{D}[a]$
3. $a[w, v] = \langle w, Av \rangle$ for $w \in \mathcal{D}[a]$ and $v \in \mathcal{D}(A)$.

Without loss of generality, assume that $\inf_{u \in \mathcal{D}[a]} a[u] / \|u\|^2 \geq 0$. Then $f \geq 0$ and the first assertion is equivalent to the completeness of $L^2(X, f d\mu)$, which is clear. The second assertion is an immediate consequence of $f \leq 1 + |f|^2$ and the third assertion is obvious.

2.6 Variational Principles

Consider a self-adjoint operator A and suppose that it is semibounded from below. By definition, the part of the spectrum lying to the left of $\inf \sigma_e(A)$ is discrete, i.e. it consists of isolated eigenvalues of finite multiplicity. A question one might ask is the following: Given some real number $\lambda \leq \inf \sigma_e(A)$, how many eigenvalues (counting multiplicity) are there $< \lambda$? One might also ask if there is any way one could explicitly compute the eigenvalues of A that lie below $\inf \sigma_e(A)$. Answers to these questions are given, respectively, by *Glazman's lemma* and the *Courant–Fischer formula* or *min-max formula*. It turns out that these two theorems are in fact equivalent. To formulate Glazman's lemma, we introduce the spectral counting function.

Definition 2.36. For a self-adjoint operator A , the function $N_A: \mathbf{R} \rightarrow \mathbf{N}_0 \cup \{\infty\}$ given by

$$N_A(\lambda) = \dim \mathcal{R}(P((-\infty, \lambda)))$$

is called the *spectral counting function* of the operator A .

There are several things to comment on regarding this definition. First, note that $N_A(\lambda) = \text{tr } \chi_{(-\infty, \lambda)}(A)$, since the trace of a projection is the same as its rank. Second, if A is lower semibounded and $\lambda < \inf \sigma_e(A)$, then, by the argument outlined above, the values $N_A(\lambda)$ must be finite.

Theorem 2.37 (Glazman's Lemma). *Let A be a semibounded self-adjoint operator on a separable Hilbert space with associated form a . For all $\lambda \in \mathbf{R}$ we have*

$$N_A(\lambda) = \sup\{\dim E \mid E \subseteq \mathcal{D}[a] \text{ is a subspace s.t. } \forall u \in E \setminus \{0\}: a[u] < \lambda \|u\|^2\}.$$

Proof. By the spectral theorem, there exist a σ -finite measure space (X, μ) , a unitary operator $V: \mathcal{H} \rightarrow L^2(X, d\mu)$ and a measurable function $f: X \rightarrow \mathbf{R}$ such that $A = V^{-1}M_fV$. As before, denote $P(\Omega) = \chi_\Omega(A)$ for measurable $\Omega \subseteq \mathbf{R}$. To prove the “ $\leq$ ” inequality, simply take $E = \mathcal{R}(P((-\infty, \lambda)))$. Then $\dim E = N_A(\lambda)$. Indeed, since A is semibounded from below, for any $u \in \mathcal{H}$, the projection $P((-\infty, \lambda))u$ belongs to the domain of A . Thus $E \subseteq \mathcal{D}(A) \subseteq \mathcal{D}[a]$ and for all $u \in E$ we have $P((-\infty, \lambda))u = u$ and thus $\chi_{f < \lambda}Vu = Vu$. Consequently,

$$a[u] = \langle u, Au \rangle = \int_X f|Vu|^2 d\mu = \int_X \chi_{f < \lambda} f|Vu|^2 d\mu < \lambda \int_X |Vu|^2 d\mu = \lambda \|u\|^2,$$

for $u \neq 0$. To see the “ $\geq$ ” inequality, consider an $F \subseteq \mathcal{D}[a]$ and suppose that $\dim F > N_A(\lambda)$. Then there exists a $u \in F \cap \mathcal{R}(P((-\infty, \lambda)))^\perp$. Note that $\mathcal{R}(P((-\infty, \lambda)))^\perp =$

$\mathcal{R}(P([\lambda, \infty)))$, since $P((-\infty, \lambda)) + P([\lambda, \infty)) = P(\mathbf{R}) = I$. Therefore, arguing in the same way as above, we find

$$a[u] \geq \lambda \|u\|^2. \quad \square$$

Remark 14. This proof is an adaptation of the proof in [FLW23]. The original formulation in Glazman's book ([AG66]) was slightly different and did not require the semiboundedness of the operator. Mysteriously, the theorem did not appear in the English translation ([AG93]) of the book.

A more common formulation of this theorem is known as the Courant–Fischer or min-max formula.

Theorem 2.38. *Let A be a semibounded self-adjoint operator on a separable Hilbert space with associated form a . For $k \in \mathbf{N}$, define the numbers λ_k via*

$$\lambda_k(A) = \inf_{\substack{E \subseteq \mathcal{D}[a] \\ \dim E = k}} \max_{\substack{u \in E \\ \|u\|=1}} a[u].$$

Then

- $\lambda_k(A)$ is equal to the k -th eigenvalue if $k \leq N_A(\inf \sigma_e(A))$,
- or $\lambda_k(A) = \inf \sigma_e(A)$.

We present an adaptation of the very elegant proof that can be found in [Bor20]. The proof is very similar to that of Glazman's lemma.

Proof. Let $\lambda \in \mathbf{R}$. Then $\mathcal{R}(P((-\infty, \lambda))) \subseteq \mathcal{D}(A) \subseteq \mathcal{D}[a]$, by the semiboundedness of A . Suppose that $\dim \mathcal{R}(P((-\infty, \lambda))) \geq k$. Then there exists a k -dimensional subspace $E \subseteq \mathcal{R}(P((-\infty, \lambda)))$. For this E we then clearly have

$$\max_{\substack{u \in E \\ \|u\|=1}} a[u] < \lambda,$$

and therefore, a fortiori, $\lambda_k < \lambda$. If, on the other hand, $\dim \mathcal{R}(P((-\infty, \lambda))) < k$, then for each k -dimensional $E \subseteq \mathcal{D}[a]$, there exists a normalized $u \in E \cap \mathcal{R}(P([\lambda, \infty)))$. For this u we clearly have $a[u] \geq \lambda$. Hence $\lambda_k \geq \lambda$. If we consider the contrapositions of these two assertions, we find that for every $\varepsilon > 0$ we have

$$\begin{aligned} \dim \mathcal{R}(P((-\infty, \lambda_k - \varepsilon))) &< k \\ \dim \mathcal{R}(P((-\infty, \lambda_k + \varepsilon))) &\geq k. \end{aligned}$$

Taking the difference of these two inequalities, and taking into account that ε can be chosen arbitrarily, we obtain

$$\dim \mathcal{R}(P((\lambda_k - \varepsilon, \lambda_k + \varepsilon))) \geq 1,$$

for all $\varepsilon > 0$. This implies that in any case $\lambda_k \in \sigma(A)$. If $\dim \mathcal{R}(P((\lambda_k - \varepsilon, \lambda_k + \varepsilon)))$ is finite for some ε , then $\lambda_k \in \sigma_d(A)$ and the two inequalities above imply that λ_k is the k -th eigenvalue. If on the other hand $\dim \mathcal{R}(P((\lambda_k - \varepsilon, \lambda_k + \varepsilon)))$ is infinite for all ε , then $\lambda_k \in \sigma_e(A)$. The first of the two inequalities above then guarantees that no point to the left of λ_k can belong to the essential spectrum, implying that $\lambda_k = \inf \sigma_e(A)$. $\square$

We introduce an order relation on the set of semibounded self-adjoint operators on $\mathcal{H}$.

Definition 2.39. Let A and B be semibounded self-adjoint operators on the Hilbert space $\mathcal{H}$. Let a and b be the forms corresponding to A and B respectively. We write $A \leq B$ if $\mathcal{D}[a] \supseteq \mathcal{D}[b]$ and $a[u] \leq b[u]$ for all $u \in \mathcal{D}[b]$.

Proposition 2.40. Let A and B be semibounded self-adjoint operators on a separable Hilbert space. Suppose that $A \leq B$. Then $N_B(\lambda) \leq N_A(\lambda)$ for all $\lambda \in \mathbf{R}$. In particular, $\inf \sigma_e(A) \leq \inf \sigma_e(B)$ and $\lambda_n(A) \leq \lambda_n(B)$.

Proof. This is an immediate consequence of Glazman's lemma. Suppose that $A \leq B$. If $E \subseteq \mathcal{D}[b]$ and $b[u] < \lambda \|u\|^2$ for all $u \in E \setminus \{0\}$, then we also have $E \subseteq \mathcal{D}[a]$ and $a[u] < \lambda \|u\|^2$ by definition of the relation $A \leq B$. Thus, the supremum in the formula for $N_A(\lambda)$ is taken over a larger space than in the case of $N_B(\lambda)$ and hence $N_B(\lambda) \leq N_A(\lambda)$ for all $\lambda \in \mathbf{R}$. If we chose $\lambda = \lambda_n(A)$ to be the n -th eigenvalue of A , then $N_B(\lambda_n(A)) \leq N_A(\lambda_n(A)) < n$. On the other hand, $N_B(\lambda) \geq n$ for all $\lambda > \lambda_n(B)$. Thus $\lambda_n(A) \leq \lambda_n(B)$. Finally, if $N_B(\lambda) = \infty$ for some λ , then $N_A(\lambda) = \infty$ and thus $\sigma_e(B) \subseteq \sigma_e(A)$, so $\inf \sigma_e(A) \leq \inf \sigma_e(B)$. $\square$

2.7 Some Perturbation Theory

In this section we shall examine how to obtain new self-adjoint operators from old ones and which properties might be preserved.

Theorem 2.41 (Kato–Rellich). Let A be a self-adjoint operator and let B be a symmetric operator with $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. Suppose that there exist an $\alpha \in [0, 1)$ and a $c > 0$ such that

$$\|Bu\| \leq \alpha \|Au\| + c \|u\|$$

for all $u \in \mathcal{D}(A)$. Then the operator $A + B$ (defined on $\mathcal{D}(A)$) is self-adjoint.

Proof. Clearly $A + B$ is a symmetric operator. By Remark 8 it is therefore sufficient to find some $z \in \mathbf{C} \setminus \mathbf{R}$ for which $A + B - z$ and $A + B - \bar{z}$ are surjective. Let $\mu \in \mathbf{R}$ and consider the operator $A + B - i\mu$. Note that $(A - i\mu)$ is boundedly invertible since A is self-adjoint and that

$$A + B - i\mu = (1 + B(A - i\mu)^{-1})(A - i\mu).$$

If we can prove that $B(A - i\mu)^{-1}$ is bounded with norm < 1 , we are done by 2.5. Indeed, given $u \in \mathcal{H}$ we find

$$\|B(A - i\mu)^{-1}u\| \leq \alpha \|A(A - i\mu)^{-1}u\| + c \|(A - i\mu)^{-1}u\|$$

and therefore $B(A - i\mu)^{-1}$ is bounded. Note that for $v \in \mathcal{D}(A)$ we have

$$\|(A - i\mu)v\|^2 = \|Av\|^2 + |\mu|^2 \|v\|^2,$$

and therefore

$$\|u\|^2 = \|A(A - i\mu)^{-1}u\|^2 + |\mu|^2 \|(A - i\mu)^{-1}u\|^2$$

Consequently,

$$\|A(A - i\mu)^{-1}\| \leq 1$$

and

$$\|(A - i\mu)^{-1}\| \leq \frac{1}{|\mu|}.$$

Finally,

$$\|B(A - i\mu)^{-1}\| \leq \alpha + \frac{c}{|\mu|}.$$

Since $\alpha < 1$ we find $\|B(A - i\mu)^{-1}\| < 1$ for sufficiently large $|\mu|$. □

Under certain conditions, perturbations of self-adjoint operators by symmetric ones leave the essential spectrum invariant.

Definition 2.42. Let A be a self-adjoint operator and let B be a symmetric operator with $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. We say that the operator B is A -compact, if $B(A - i)^{-1}$ is a compact operator on $\mathcal{H}$.

Remark 15. If $B(A - i)^{-1}$ is compact, then $B(A - z)^{-1}$ is compact for any $z \in \rho(A)$, as is seen from the resolvent identity.

If B is A -compact, then it satisfies the conditions of the Kato–Rellich theorem. Indeed, we have the even stronger affirmation:

Proposition 2.43. *Let A be a self-adjoint operator and let B be a symmetric operator that is A -compact. Then for every $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that*

$$\|Bu\| \leq \varepsilon \|Au\| + c_\varepsilon \|u\|.$$

Proof. Suppose $B(A - i)^{-1}$ is compact. We can write $B(A - i\mu)^{-1}$ as

$$B(A - i\mu)^{-1} = B(A - i)^{-1}(A - i)(A - i\mu)^{-1}.$$

Note that $((A - i)(A - i\mu)^{-1})^* = (A + i)(A + i\mu)^{-1} = f_\mu(A)$ for

$$f_\mu(x) = \frac{x + i}{x + i\mu}.$$

For $\mu \geq 0$, we have $|f_\mu(x)| \leq 1$. Furthermore, $f_\mu(x) \rightarrow 0$ for all $x \in \mathbf{R}$ as $\mu \rightarrow \pm\infty$. By [2.30](#), we find $f_\mu(A) \rightarrow 0$, $\mu \rightarrow \infty$ in the strong operator topology. Since $B(A - i)^{-1}$ is compact, so is its adjoint, and therefore, by [9.5](#), we find

$$B(A - i\mu)^{-1} = B(A - i)^{-1}(A - i)(A - i\mu)^{-1} = (A + i)(A + i\mu)^{-1}(B(A - i)^{-1})^* \rightarrow 0$$

in the operator norm as $\mu \rightarrow \pm\infty$. Therefore, $\|B(A - i\mu)^{-1}\|$ can be made arbitrarily small. Let $\varepsilon > 0$ and let $|\mu|$ be so large that $\|B(A - i\mu)^{-1}\| < \varepsilon$. For $u \in \mathcal{D}(A)$ we then find

$$\|Bu\| = \|B(A - i\mu)^{-1}(A - i\mu)u\| \leq \varepsilon (\|Au\| + |\mu| \|u\|),$$

and choosing $c_\varepsilon = \varepsilon|\mu|$ concludes the argument. $\square$

Theorem 2.44 (Weyl). *Let A be a self-adjoint operator and let B be a symmetric operator that is A -compact. Then $\sigma_e(A) = \sigma_e(A + B)$.*

Proof. First note that the above discussion established that A and B satisfy the conditions of the Kato–Rellich theorem and therefore $A + B$ is actually a self-adjoint operator.

Suppose that $\lambda \in \sigma_e(A)$. Then there exists a sequence of normalized elements $(u_n) \subseteq \mathcal{D}(A)$ such that $(A - \lambda)u_n \rightarrow 0$ and $u_n \rightharpoonup 0$. Clearly we have $(A + B - \lambda)u_n = (A - \lambda)u_n + B(A - i)^{-1}(A - i)u_n$. On the other hand, $(A - i)u_n = (A - \lambda)u_n + (\lambda - i)u_n$ converges weakly to zero. Hence $B(A - i)^{-1}(A - i)u_n \rightarrow 0$ by compactness of $B(A - i)^{-1}$. Finally $(A + B - \lambda)u_n \rightarrow 0$ and thus $\lambda \in \sigma_e(A + B)$. The reverse inclusion is obtained by changing

the roles of A and B by $A + B$ and $-B$ respectively. In this case $-B$ is $A + B$ -compact, since

$$A + B - i\mu = (1 + B(A - i\mu)^{-1})(A - i\mu)$$

implies

$$-B(A + B - i\mu)^{-1} = -B(A - i\mu)^{-1}(1 + B(A - i\mu)^{-1})^{-1}$$

for sufficiently large $|\mu|$. □

Let us now turn toward perturbations of quadratic forms.

Theorem 2.45 (Kato–Lax–Lions–Milgram–Nelson). *Let a be a closed coercive form and let b be a symmetric sesquilinear form with $\mathcal{D}[a] \subseteq \mathcal{D}[b]$. Suppose that there exist $\alpha \in [0, 1)$ and $c > 0$ such that*

$$|b[u]| \leq \alpha a[u] + c \|u\|^2$$

for all $u \in \mathcal{D}[a]$. Then $u \mapsto a[u] + b[u]$ defines a closed coercive form on $\mathcal{D}[a]$. In particular, this form can be associated with a semibounded self-adjoint operator C . If additionally the form b is weakly continuous on $\mathcal{D}[a]$, i.e. $u_n \rightharpoonup 0$ in the Hilbert space $\mathcal{D}[a]$ implies $b[u_n] \rightarrow 0$, then the essential spectra of the operator A corresponding to the form a and of C coincide

$$\sigma_e(C) = \sigma_e(A).$$

Proof. Note that for $u \in \mathcal{D}[a]$ we have

$$(1 - \alpha)a[u] - c \|u\|^2 \leq a[u] + b[u] \leq (1 + \alpha)a[u] + c \|u\|^2.$$

Therefore the norms induced by a and by $a + b$ are equivalent on $\mathcal{D}[a]$ and completeness with respect to one of them implies completeness with respect to the other one. For the assertion regarding the essential spectra, see [\[Lew22\]](#). □

Remark 16. If a is the form corresponding to a self-adjoint operator A and b is defined by a symmetric operator B , then the condition in this theorem reads, for $u \in \mathcal{D}(A)$ as

$$|\langle u, Bu \rangle| \leq \alpha \langle u, Au \rangle + c \|u\|^2,$$

which is of a different nature than the condition in Kato–Rellich’s theorem.

3 Coherent States

This chapter on coherent states is taken from the author's preprint [\[Wei24\]](#).

3.1 Overfilling Families of Functions

We introduce overfilling families and prove how they can be used to compute traces of operators following [\[Ber73\]](#).

Definition 3.1. Let $\mathcal{H}$ be a separable Hilbert space and let (A, μ) be a measure space. A family $e_\alpha \in \mathcal{H}$, $\alpha \in A$ of elements of $\mathcal{H}$ indexed by A is said to be overfilling if the map $A \ni \alpha \mapsto e_\alpha \in \mathcal{H}$ is measurable and if for all $f \in \mathcal{H}$ one has

$$\|f\|^2 = \int_A |\langle e_\alpha, f \rangle|^2 d\mu(\alpha).$$

Given an overfilling family, we obtain an isometric embedding $\mathcal{H} \rightarrow L^2(A)$ by setting

$$f(\alpha) = \langle e_\alpha, f \rangle$$

for $f \in \mathcal{H}$. Note in particular that $e_\alpha(\beta) = \langle e_\beta, e_\alpha \rangle = \overline{e_\beta(\alpha)}$. We can therefore use the same notation for the inner products in $\mathcal{H}$ and in $L^2(A)$. Note that

$$\langle g, f \rangle = \int_A \langle g, e_\alpha \rangle \langle e_\alpha, f \rangle d\mu(\alpha)$$

for any $g, f \in \mathcal{H}$.

Theorem 3.2. *Let T be a non-negative trace-class operator on $\mathcal{H}$. Then*

$$\mathrm{tr} T = \int_A \langle e_\alpha, T e_\alpha \rangle d\mu(\alpha).$$

Proof. Indeed let ψ_k be an orthonormal basis of $\mathcal{H}$ consisting of eigenfunctions of T ,

$$\begin{aligned}
\int_A \langle e_\alpha, T e_\alpha \rangle d\mu(\alpha) &= \int_A \sum_k \langle e_\alpha, \psi_k \rangle \langle \psi_k, T e_\alpha \rangle d\mu(\alpha) \\
&= \sum_k \int_A \langle e_\alpha, \psi_k \rangle \langle \psi_k, T e_\alpha \rangle d\mu(\alpha) \\
&= \sum_k \int_A \langle T^* \psi_k, e_\alpha \rangle \langle e_\alpha, \psi_k \rangle d\mu(\alpha) \\
&= \sum_k \langle T^* \psi_k, \psi_k \rangle \\
&= \sum_k \langle \psi_k, T \psi_k \rangle.
\end{aligned}$$

Note that interchanging the sum and the integral is justified since

$$\begin{aligned}
\sum_k \int_A |\langle e_\alpha, \psi_k \rangle \langle \psi_k, T e_\alpha \rangle| d\mu(\alpha) &\leq \sum_k \left(\int_A |\langle e_\alpha, \psi_k \rangle|^2 d\mu(\alpha) \right)^{1/2} \left(\int_A |\langle T \psi_k, e_\alpha \rangle|^2 d\mu(\alpha) \right)^{1/2} \\
&= \sum_k \|\psi_k\| \|T \psi_k\| \\
&= \sum_k \langle \psi_k, T \psi_k \rangle < \infty
\end{aligned}$$

by virtue of T being trace-class and the ψ_k being eigenfunctions. $\square$

Let us discuss two simple examples of overfilling families.

Consider the case when $A = \mathbf{N}$ and $(e_n)_{n \in \mathbf{N}}$ is an orthonormal basis of $\mathcal{H}$. In this case the above procedure yields the well-known isometric isomorphism $\mathcal{H} \rightarrow \ell^2(\mathbf{N})$ and $f(n)$ is simply the n -th coefficient c_n in the Fourier expansions $f = \sum_n c_n e_n$. Given a trace-class operator T on $\mathcal{H}$, we obtain $\text{tr } T = \sum_k T_{k,k}$, where $T_{k,l} = \langle e_k, T e_l \rangle$, as expected.

Let $\Omega \subseteq \mathbf{R}^d$ be open and bounded. Let $e_\xi(x) = e^{i\xi x}$. By extending $f \in L^2(\Omega)$ by zero outside Ω , we may compute its Fourier transform. Plancherel's theorem yields

$$\int_\Omega |f(x)|^2 dx = \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbf{R}^d} |\langle e_\xi, f \rangle|^2 \frac{d\xi}{(2\pi)^d}.$$

Thus $(e_\xi)_{\xi \in \mathbf{R}^d}$ is an overfilling family in $L^2(\Omega)$ with $A = \mathbf{R}^d$ and $d\mu(\xi) = (2\pi)^{-d} d\xi$.

3.2 Coherent States

The rest of this thesis will deal with a class of overfilling families in L^2 -spaces constructed as follows. Let $g \in \mathcal{S}(\mathbf{R}^d)$ be real-valued and such that $g(-x) = g(x)$ and $\int g(x)^2 dx = 1$. For $\xi, y \in \mathbf{R}^d$ define $e_{\xi,y}(x) = e^{i\xi x} g(x - y)$. Clearly $\|e_{\xi,y}\|_{L^2(\mathbf{R}^d)} = 1$. The space $\mathbf{R}^{2d}$ indexing the family $(e_{\xi,y})$ shall be equipped with the measure $(2\pi)^{-d} d\xi dy$. The functions $e_{\xi,y} \in L^2(\mathbf{R}^d)$ are called *coherent states*.

Proposition 3.3. *The family $e_{\xi,y}$, $(\xi, y) \in \mathbf{R}^{2d}$ is overfilling in $L^2(\mathbf{R}^d)$.*

Proof. We need to show that

$$\int_{\mathbf{R}^{2d}} |\langle e_{\xi,y}, f \rangle|^2 \frac{d\xi dy}{(2\pi)^d} = \int_{\mathbf{R}^d} |f(x)|^2 dx.$$

Note that $\langle e_{\xi,y}, f \rangle = \int e^{-i\xi x} g(x - y) f(x) dx$ is simply the Fourier transform of the function $x \mapsto (2\pi)^{d/2} g(x - y) f(x)$. Therefore, Plancherel's theorem yields

$$\int_{\mathbf{R}^d} |\langle e_{\xi,y}, f \rangle|^2 d\xi = (2\pi)^d \int_{\mathbf{R}^d} g(x - y)^2 |f(x)|^2 dx$$

Integration with respect to y concludes the proof. $\square$

Remark 17. This theorem yields in particular that

$$\text{tr } K = \int_{\mathbf{R}^{2d}} \langle e_{\xi,y}, K e_{\xi,y} \rangle \frac{d\xi dy}{(2\pi)^d}.$$

for non-negative trace-class operators K on $L^2(\mathbf{R}^d)$.

Note that we have not used the symmetry $g(-x) = g(x)$ yet. This, however, will be important in the following technical lemma.

Lemma 3.4.

$$\langle e_{\xi,y}, -\partial_{x_j}^2 e_{\xi,y} \rangle = \xi_j^2 + \int_{\mathbf{R}^d} |\partial_{z_j} g(z)|^2 dz$$

Proof. Direct computation yields

$$\partial_{x_j} e_{\xi,y}(x) = i\xi_j e^{i\xi x} g(x - y) + e^{i\xi x} \partial_{x_j} g(x - y)$$

and

$$\partial_{x_j}^2 e_{\xi,y}(x) = -\xi_j^2 e^{i\xi x} g(x - y) + 2i\xi_j e^{i\xi x} \partial_{x_j} g(x - y) + e^{i\xi x} \partial_{x_j}^2 g(x - y). \quad (1)$$

Therefore

$$\begin{aligned}\langle e_{\xi,y}, -\partial_{x_j}^2 e_{\xi,y} \rangle &= \int_{\mathbf{R}^d} \xi_j^2 (g(x-y))^2 dx + 2i\xi_j \int_{\mathbf{R}^d} \partial_{x_j} g(x-y) g(x-y) dx \\ &\quad - \int_{\mathbf{R}^d} \partial_{x_j}^2 g(x-y) g(x-y) dx\end{aligned}$$

By symmetry of g , the second integral on the right vanishes and by integration by parts, the third integral is equal to $\int_{\mathbf{R}^d} (\partial_{z_j} g(z))^2 dz$. $\square$

Lemma 3.5. *The Laplacian acts on the functions $e_{\xi,y}$ in the following way:*

$$\langle e_{\xi,y}, -\Delta e_{\xi,y} \rangle_{L^2(\mathbf{R}^d)} = |\xi|^2 + \int_{\mathbf{R}} |\nabla g(z)|^2 dz.$$

Proof. Follows immediately from Lemma 3.4. $\square$

Instead of the entire space $\mathbf{R}^d$, consider now an open set $\Omega \subseteq \mathbf{R}^d$ of finite measure. For $\varepsilon > 0$ let $\Omega^\varepsilon = \{x \in \mathbf{R}^d : \text{dist}(x, \Omega) < \varepsilon\}$. We impose on g the additional condition that $g \in C_c^\infty(\mathbf{R}^d)$ with $\text{supp } g \subseteq \{z \in \mathbf{R}^d : |z| < 1\}$. Note that $g^\varepsilon(z) = \varepsilon^{-d/2} g(z/\varepsilon)$ still satisfies the symmetry and normalization conditions demanded of g . We may therefore define $e_{\xi,y}^\varepsilon(x) = e^{i\xi x} g^\varepsilon(x-y)$.

Any function $f \in L^2(\Omega)$ can be extended by zero to a function in $L^2(\mathbf{R}^d)$. Thus

$$\langle e_{\xi,y}^\varepsilon, f \rangle_{L^2(\mathbf{R}^d)} = \langle e_{\xi,y}^\varepsilon, f \rangle_{L^2(\Omega)}.$$

The map $y \mapsto \langle e_{\xi,y}^\varepsilon, f \rangle$ vanishes outside $\Omega + \{z \in \mathbf{R}^d : |z| < \varepsilon\} = \Omega^\varepsilon$, since, by rescaling,

$$\text{supp } g^\varepsilon \subseteq \{z \in \mathbf{R}^d : |z| < \varepsilon\}.$$

Therefore

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} |\langle e_{\xi,y}^\varepsilon, f \rangle_{L^2(\mathbf{R}^d)}|^2 \frac{d\xi dy}{(2\pi)^d} = \int_{\mathbf{R}^d \times \Omega^\varepsilon} |\langle e_{\xi,y}^\varepsilon, f \rangle_{L^2(\Omega)}|^2 \frac{d\xi dy}{(2\pi)^d}$$

Finally,

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} |\langle e_{\xi,y}^\varepsilon, f \rangle_{L^2(\mathbf{R}^d)}|^2 \frac{d\xi dy}{(2\pi)^d} = \|f\|_{L^2(\mathbf{R}^d)}^2 = \|f\|_{L^2(\Omega)}^2.$$

This proves the following

Proposition 3.6. *Let $\Omega \subseteq \mathbf{R}^d$ be an open set of finite measure. Assume that g is supported in the open ball $\{z \in \mathbf{R}^d : |z| < 1\}$. Then the family $(e_{\xi,y}^\varepsilon)$, $y \in \Omega^\varepsilon$, $\xi \in \mathbf{R}^d$ of coherent states constructed using g^ε is overfilling in $L^2(\Omega)$.*

4 The Laplacian on a Domain

In this chapter we apply some of the methods discussed in Chapter 1 to certain realizations of the Laplace operator. Everything in this chapter is well-known. We mainly follow [FLW23], [Bor20], and [LMP24].

4.1 Realizations of the Laplacian

Recall that for an open set $\Omega \subseteq \mathbf{R}^d$ the Laplacian on $C^\infty(\Omega)$ is defined via the differential expression

$$-\Delta = -\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}.$$

Our goal is to obtain self-adjoint realizations of this expression in the Hilbert space $L^2(\Omega)$. The theory of quadratic form permits us to do so rather easily.

4.1.1 The Free Laplacian

For $u \in H^1(\mathbf{R}^d)$ define

$$a[u] = \int_{\mathbf{R}^d} |\nabla u|^2 dx.$$

This expression clearly defines a non-negative quadratic form on $L^2(\mathbf{R})$.

Proposition 4.1. *The quadratic form a is closed.*

Proof. By [9.12] $H^1(\mathbf{R}^d)$ is a Hilbert space. Note that for $u \in H^1(\mathbf{R}^d)$ one has

$$\|u\|_{H^1(\mathbf{R}^d)}^2 = a[u] + \|u\|_{L^2(\mathbf{R})}^2.$$

Therefore, the form a is closed, by completeness of $H^1(\mathbf{R})$. □

Definition 4.2. The unique non-negative self-adjoint operator corresponding to the form a is called the *Laplacian* on $\mathbf{R}^d$ and it is denoted by $-\Delta$.

The operator $-\Delta$ is particularly easy to describe because it can be explicitly diagonalized. Note that $\mathcal{D}(-\Delta)$ is the subspace of all $u \in \mathcal{D}[a] = H^1(\mathbf{R}^d)$ for which there exists a $v \in L^2(\mathbf{R}^d)$ such that $a[w, u] = \langle w, v \rangle$ which, using integration by parts, is exactly $H^2(\mathbf{R}^d)$. Using the Fourier transform we find, for $u \in H^2(\mathbf{R}^d)$, that

$$-\Delta u = \mathcal{F}^{-1} |\xi|^2 \mathcal{F} u.$$

This means that $-\Delta$ is unitarily equivalent to $M_{|\xi|^2}$. Therefore we have

$$\sigma(-\Delta) = \sigma(M_{|\xi|^2}) = \text{ess-range}|\xi|^2 = [0, \infty)$$

and the parts of the spectrum (discrete spectrum, essential spectrum) of $-\Delta$ also coincide with those of the spectrum of $M_{|\xi|^2}$.

Theorem 4.3. *Let $-\Delta$ be the Laplacian on $\mathbf{R}^d$. Then*

$$\sigma_d(-\Delta) = \emptyset, \quad \sigma_e(-\Delta) = [0, \infty).$$

One could have simply defined $-\Delta$ via $\mathcal{F}^{-1}|\xi|^2\mathcal{F}$ thus immediately obtaining the self-adjointness and the spectral properties of this operator. This approach, however, only works when considering the entire space $\mathbf{R}^d$. The approach we chose on the other hand can easily be extended to open domains $\Omega \subseteq \mathbf{R}^d$.

4.1.2 The Dirichlet Laplacian

Let Ω be an open subset of $\mathbf{R}^d$. Similar to the argument above we define

$$a^D[u] = \int_{\Omega} |\nabla u|^2 dx.$$

for $u \in H_0^1(\Omega)$. This is again a non-negative quadratic form. Since $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ and $H^1(\Omega)$ is complete, $H_0^1(\Omega)$ is itself complete, and therefore, arguing in the same way as for the free Laplacian, the form a^D is closed.

Definition 4.4. The unique non-negative self-adjoint operator corresponding to the form a^D is called the *Dirichlet Laplacian* on Ω and it is denoted by $-\Delta_{\Omega}^D$.

We are interested in the cases when $-\Delta_{\Omega}^D$ has discrete spectrum.

Theorem 4.5. *Let Ω be an open subset of $\mathbf{R}^d$ of finite measure. Then the spectrum of $-\Delta_{\Omega}^D$ is discrete.*

Proof. By [2.35], it suffices to prove that $\mathcal{D}[a^D] \hookrightarrow L^2(\Omega)$ is compact. Since $\mathcal{D}[a^D] = H_0^1(\Omega)$ and since $a^D[u] + \|u\|_{L^2(\Omega)}^2 = \|u\|_{H_0^1(\Omega)}^2$, we just need to show that $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. This is precisely the assertion of [9.18]. $\square$

Having established the discreteness of $\sigma(-\Delta^D)$, we proceed to the study of the eigenvalues of $-\Delta^D$, which by [2.38] can be written as an increasing sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

Since $\sigma_e(-\Delta^D)$ is empty, these eigenvalues can only accumulate at infinity. Consequently, $\lambda_n \rightarrow \infty$.

We are interested in studying the properties of the sequence (λ_n) .

Proposition 4.6. *Let Ω be an open subset of $\mathbf{R}^d$ of finite measure. The first eigenvalue λ_1 of the Dirichlet Laplacian on Ω is strictly positive.*

Proof. We will prove something even stronger and show that $\lambda_1 \geq \frac{C_d}{|\Omega|^{2/d}}$, where C_d is the constant from Poincaré's inequality [9.21](#). Recall that by [2.38](#), we have

$$\lambda_1 = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx}.$$

For any $u \in H_0^1(\Omega)$, Poincaré's inequality [9.21](#) holds:

$$\frac{C_d}{|\Omega|^{2/d}} \int_{\Omega} |u|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx.$$

Consequently

$$\lambda_1 \geq \frac{C_d}{|\Omega|^{2/d}}. \quad \square$$

It is worth noting that this lower bound only depends on the volume of Ω . Among all open set $\Omega \subseteq \mathbf{R}^d$ of a fixed volume, it turns out that balls minimize the first Dirichlet eigenvalue. This circumstance is known as the *Faber–Krahn inequality*. Since we will now consider different domains at the same time, we shall introduce the notation $\lambda_n(\Omega)$ for the n -th eigenvalue of the Dirichlet Laplacian on Ω to avoid confusion. Given a bounded open set $\Omega \subseteq \mathbf{R}^d$, we define its *symmetrization* Ω^* to be the unique open ball centered around zero having the same measure as Ω .

Proposition 4.7 (Faber–Krahn). *Let $\Omega \subseteq \mathbf{R}^d$ be a bounded open set and let Ω^* be its symmetrization. Then*

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*).$$

Remark 18. The theorem in this general form (in arbitrary dimension) was first proven by Krahn in [\[Kra26\]](#). See [\[LL01\]](#) for a modern proof.

The Dirichlet eigenvalues satisfy an interesting property which is known as domain monotonicity and which is a consequence of the Courant–Fischer theorem.

Theorem 4.8. *Let $\Omega_1, \Omega_2 \subseteq \mathbf{R}^d$ be bounded open sets such that $\Omega_1 \subseteq \Omega_2$. Then*

$$\lambda_n(\Omega_1) \geq \lambda_n(\Omega_2).$$

Remark 19. A musician might paraphrase this property as “The larger the drum, the lower the pitch”.

Proof. One can extend any function in $H_0^1(\Omega_1)$ by zero to a function in $H_0^1(\Omega_2)$. Therefore, the assertion is a direct consequence of [2.38](#) $\square$

Remark 20. The domain monotonicity for Dirichlet eigenvalues is even strict in the sense that if $\Omega_2 \setminus \Omega_1$ contains a non-empty open set, then $\lambda_n(\Omega_1) > \lambda_n(\Omega_2)$. See [\[Wel72\]](#) for a proof of this fact.

4.1.3 The Neumann Laplacian

We proceed in the same way as for the Dirichlet Laplacian. Consider an open subset Ω of $\mathbf{R}^d$. Define

$$a^N[u] = \int_{\Omega} |\nabla u|^2 dx.$$

for $u \in H^1(\Omega)$. Since $H^1(\Omega)$ is complete, the form a^N is closed. The proof is analogous to the proof of the corresponding assertion about the Dirichlet Laplacian.

Definition 4.9. The unique non-negative self-adjoint operator corresponding to the form a^N is called the *Neumann Laplacian* on Ω and it is denoted by $-\Delta_{\Omega}^N$.

Theorem 4.10. *Let Ω be an open subset of $\mathbf{R}^d$ with finite measure and the extension property. Then the spectrum of $-\Delta_{\Omega}^N$ is discrete.*

Proof. Arguing in a similar way as for the Dirichlet Laplacian, we can use theorem [9.19](#) to obtain the assertion. $\square$

Unlike in the Dirichlet case, the first eigenvalue of the Neumann Laplacian is not positive. Indeed it is always zero, since constant functions belong to $H^1(\Omega)$ if Ω is of finite measure. We denote the sequence of Neumann eigenvalues by

$$0 = \mu_1 \leq \mu_2 \leq \dots$$

As in the Dirichlet case, we have $\mu_n \rightarrow \infty$. There is an analogue of the Faber–Krahn inequality for the *second* Neumann eigenvalue. The inequality in this case is reversed,

that is

$$\mu_2(\Omega) \leq \mu_2(\Omega^*).$$

See [Wei56] for a proof of this fact, which is also known as the *Szegő–Weinberger inequality*. Contrary to the Dirichlet case, in the Neumann case we do not have domain monotonicity. See [Bor20] for a simple counterexample.

4.2 Eigenvalue Comparisons

Consider an open set $\Omega \subseteq \mathbf{R}^d$ with the extension property and finite measure. According to the preceding discussions, the spectra of $-\Delta_\Omega^D$ and $-\Delta_\Omega^N$ are discrete. As before, we write the spectra as increasing sequences $\sigma(-\Delta_\Omega^D) = \{\lambda_n\}_{n=1}^\infty$ and $\sigma(-\Delta_\Omega^N) = \{\mu_n\}_{n=1}^\infty$. An immediate consequence of [2.40] is that

$$\lambda_n \geq \mu_n$$

for all $n \in \mathbf{N}$, since $\mathcal{D}[a^D] = H_0^1(\Omega) \subseteq H^1(\Omega) = \mathcal{D}[a^N]$ and therefore $-\Delta_\Omega^D \geq -\Delta_\Omega^N$. It turns out that we can make an even stronger assertion.

Theorem 4.11. *Let Ω be an open subset of $\mathbf{R}^d$ with the extension property and finite measure. Denote by λ_k , respectively μ_k , the k -th eigenvalue of $-\Delta_\Omega^D$, respectively $-\Delta_\Omega^N$. The following inequality holds:*

$$\lambda_n \geq \mu_{n+1}.$$

Remark 21. Friedlander proved this assertion for domains with C^1 -boundary in [Fri91]. Filonov's ([Fil05]) proof is shorter and does not demand any regularity of the boundary other than the requirement that Ω have the extension property.

4.3 Eigenvalue Asymptotics

Given an open subset Ω of $\mathbf{R}^d$ with finite measure, what can one say about the asymptotic behavior of the Dirichlet eigenvalues λ_n as $n \rightarrow \infty$? In 1911, Hermann Weyl proved ([Wey11]) that in the case $d = 2$, the Dirichlet eigenvalues satisfy

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = \frac{4\pi}{|\Omega|}.$$

In terms of the counting function $N(\lambda) = N_{-\Delta_\Omega^D}(\lambda)$, this is equivalent to saying that

$$\lim_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\lambda} = \frac{|\Omega|}{4\pi}.$$

Soon afterwards, in 1912, Weyl proved ([Wey12]) a similar result for the case $d = 3$. Generally, that is in arbitrary dimension d , we find

$$\lim_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{|\Omega| |B_1|}{(2\pi)^d}.$$

According to [Ivr16], this generalization was first proven in [Car35]. Generally, one can consult [ANPS09] and [Ivr16] for a more detailed history of Weyl's law.

Our goal is to prove the asymptotic formula above. The usual approach of doing this involves a technique called *Dirichlet–Neumann–bracketing*.

Our procedure is the following:

1. Using the theory of coherent states we prove the asymptotic behavior of the Riesz-means $\sum_n (\lambda - \lambda_n)_+$ as $\lambda \rightarrow \infty$.
2. From this we derive the behavior of $\sum_n e^{-t\lambda_n}$ as $t \rightarrow 0+$.
3. Applying Karamata's Tauberian theorem yields Weyl's law for $N(\lambda)$.

4.3.1 Weyl's Law for the Dirichlet Laplacian

Our proof of the Weyl asymptotic formula for the eigenvalues of the Dirichlet Laplacian is somewhat non-standard. However, no originality is claimed whatsoever, since the components of the proof are all well-established methods in the field. The first part of our proof, i.e. the asymptotic formula for $\sum_n (\lambda - \lambda_n)_+$ is taken from the author's preprint [Wei24]. Our approach yields a proof of Weyl's law that might be considered less technical than Dirichlet–Neumann bracketing, thus illustrating the utility of coherent state methods.

Lemma 4.12. *Let Ω be an open subset of $\mathbf{R}^d$ of finite measure and denote by λ_k , $k \in \mathbf{N}$ the eigenvalues of the Dirichlet Laplacian in Ω counted with multiplicity. Then*

$$\liminf_{\lambda \rightarrow \infty} \lambda^{-1-d/2} \sum_k (\lambda - \lambda_k)_+ \geq |\Omega| \int_{\mathbf{R}^d} (1 - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d}.$$

Proof. Let $\varepsilon > 0$ and define $\Omega^\varepsilon = \{x \in \mathbf{R}^d : \text{dist}(x, \Omega) < \varepsilon\}$. Let $g \in C_c^\infty(\mathbf{R}^d)$ be real-valued and such that $g(-x) = g(x)$ and $\int g(x)^2 dx = 1$. Furthermore, assume that $\text{supp } g \subseteq \{z \in \mathbf{R}^d : |z| < 1\}$. Then, by [3.6], we know that $e_{\xi, y}^\varepsilon(x) = e^{i\xi x} g^\varepsilon(x - y)$, $y \in \Omega^\varepsilon$, $\xi \in \mathbf{R}^d$ defines an overfilling family in $L^2(\Omega)$.

By theorem [3.2](#), which allows for a computation of traces using coherent states, we have

$$\sum_k (\lambda - \lambda_k)_+ = \text{tr}(\lambda + \Delta_\Omega^D)_+ = \int_{\mathbf{R}^d \times \Omega^\varepsilon} \langle e_{\xi,y}^\varepsilon, (\lambda + \Delta_\Omega^D)_+ e_{\xi,y}^\varepsilon \rangle_{L^2(\Omega)} \frac{d\xi dy}{(2\pi)^d}.$$

Since $\mu \mapsto \mu_+$ is convex, applying Jensen's inequality to the spectral measure of $-\Delta_\Omega^D$, we obtain

$$\langle e_{\xi,y}^\varepsilon, (\lambda + \Delta_\Omega^D)_+ e_{\xi,y}^\varepsilon \rangle_{L^2(\Omega)} \geq (\langle e_{\xi,y}^\varepsilon, (\lambda + \Delta_\Omega^D) e_{\xi,y}^\varepsilon \rangle_{L^2(\Omega)})_+.$$

Clearly

$$\int_{\mathbf{R}^d \times \Omega^\varepsilon} (\langle e_{\xi,y}^\varepsilon, (\lambda + \Delta_\Omega^D) e_{\xi,y}^\varepsilon \rangle_{L^2(\Omega)})_+ \frac{d\xi dy}{(2\pi)^d} \geq \int_{\mathbf{R}^d \times \Omega_\varepsilon} (\langle e_{\xi,y}^\varepsilon, (\lambda + \Delta_\Omega^D) e_{\xi,y}^\varepsilon \rangle_{L^2(\Omega)})_+ \frac{d\xi dy}{(2\pi)^d},$$

where $\Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \Omega^c) > \varepsilon\}$. For $y \in \Omega_\varepsilon$ the function $e_{\xi,y}^\varepsilon$ is supported in $\Omega_\varepsilon + B_\varepsilon = \Omega$ and consequently

$$\langle e_{\xi,y}^\varepsilon, (\lambda + \Delta_\Omega^D) e_{\xi,y}^\varepsilon \rangle_{L^2(\Omega)} = \langle e_{\xi,y}^\varepsilon, (\lambda + \Delta) e_{\xi,y}^\varepsilon \rangle_{L^2(\mathbf{R}^d)} = \lambda - |\xi|^2 - \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz.$$

The last equality is due to [3.5](#). Consequently,

$$\begin{aligned} \sum_k (\lambda - \lambda_k)_+ &\geq \int_{\mathbf{R}^d \times \Omega_\varepsilon} \left(\lambda - |\xi|^2 - \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz \right)_+ \frac{d\xi dy}{(2\pi)^d} \\ &= \lambda^{1+d/2} \int_{\mathbf{R}^d \times \Omega_\varepsilon} \left(1 - |\xi|^2 - \frac{1}{\lambda} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz \right)_+ \frac{d\xi dy}{(2\pi)^d} \\ &\geq \lambda^{1+d/2} |\Omega_\varepsilon| \int_{\mathbf{R}^d} \left(1 - |\xi|^2 - \frac{1}{\lambda} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz \right)_+ \frac{d\xi}{(2\pi)^d}. \end{aligned} \quad (2)$$

Finally we obtain

$$\liminf_{\lambda \rightarrow \infty} \lambda^{-1-d/2} \sum_k (\lambda - \lambda_k)_+ \geq |\Omega_\varepsilon| \int_{\mathbf{R}^d} (1 - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d}$$

by monotone convergence. Letting $\varepsilon \rightarrow 0$ concludes the proof, since $\chi_{\Omega_\varepsilon}(x) \rightarrow \chi_\Omega(x)$ for all $x \in \mathbf{R}^d$ and thus $|\Omega_\varepsilon| \rightarrow |\Omega|$ by dominated convergence. $\square$

We now seek to prove an upper bound on $\sum_k (\lambda - \lambda_k)_+$.

Proposition 4.13. *Let $\Omega \subseteq \mathbf{R}^d$ be an open set of finite measure and let $\lambda_k, k \in \mathbf{N}$ be the*

eigenvalues of the Dirichlet Laplacian on Ω . Then

$$\sum_k (\lambda - \lambda_k)_+ \leq |\Omega| \int_{\mathbf{R}^d} (\lambda - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d}.$$

Remark 22. This result is equivalent to one obtained by Li and Yau [LY83] and our proof follows [Lap97]. An analogous assertion for Neumann eigenvalues was proven in [Krö92].

Proof. Let $(u_k)_k \subseteq L^2(\Omega)$ be an orthonormal system of eigenfunctions corresponding to $(\lambda_k)_k$. In particular, $u_k \in H_0^1(\Omega)$. Then, since every u_k can be extended by zero to a function $H^1(\mathbf{R}^d)$, we have

$$\lambda_k = \int_{\Omega} |\nabla u_k(x)|^2 dx = \int_{\mathbf{R}^d} |\xi|^2 |\hat{u}_k(\xi)|^2 d\xi.$$

Therefore

$$(\lambda - \lambda_k)_+ = \left(\int_{\mathbf{R}^d} (\lambda - |\xi|^2) |\hat{u}_k(\xi)|^2 d\xi \right)_+ \leq \int_{\mathbf{R}^d} (\lambda - |\xi|^2)_+ |\hat{u}_k(\xi)|^2 d\xi,$$

by Jensen's inequality, since $|\hat{u}_k(\xi)|^2 d\xi$ is a probability measure. Summing over k we find

$$\begin{aligned} \sum_k (\lambda - \lambda_k)_+ &\leq \sum_k \int_{\mathbf{R}^d} (\lambda - |\xi|^2)_+ |\hat{u}_k(\xi)|^2 d\xi \\ &= \int_{\mathbf{R}^d} (\lambda - |\xi|^2)_+ \sum_k |\hat{u}_k(\xi)|^2 d\xi. \end{aligned}$$

Note that

$$\hat{u}_k(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{-ix\xi} u_k(x) dx = \frac{1}{(2\pi)^{d/2}} \int_{\Omega} e^{-ix\xi} u_k(x) dx = \langle (2\pi)^{-d/2} e^{i\xi \cdot}, u_k \rangle_{L^2(\Omega)}.$$

Since the u_k form an orthonormal basis of $L^2(\Omega)$, we obtain

$$\sum_k |\hat{u}_k(\xi)|^2 = \sum_k |\langle (2\pi)^{-d/2} e^{i\xi \cdot}, u_k \rangle_{L^2(\Omega)}|^2 = \left\| (2\pi)^{-d/2} e^{i\xi \cdot} \right\|_{L^2(\Omega)}^2 = (2\pi)^{-d} |\Omega|.$$

This concludes the argument. □

The inequality thus obtained can be rewritten as

$$\lambda^{-d/2-1} \sum_k (\lambda - \lambda_k)_+ \leq |\Omega| \int_{\mathbf{R}^d} (1 - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d}.$$

Combining the two inequalities finally yields the following result.

Proposition 4.14. *For the eigenvalues (λ_k) of the Dirichlet Laplacian on an open set Ω of finite measure, the following asymptotic formula holds*

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1-d/2} \sum_k (\lambda - \lambda_k)_+ = |\Omega| \int_{\mathbf{R}^d} (1 - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d}.$$

4.3.2 The Error Term

This subsection is also taken from [Wei24]. Recall that in the proof of the lower bound on the Riesz mean we obtained:

$$\sum_k (\lambda - \lambda_k)_+ \geq \lambda^{1+d/2} |\Omega_\varepsilon| \int_{\mathbf{R}^d} \left(1 - |\xi|^2 - \frac{1}{\lambda} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz \right)_+ \frac{d\xi}{(2\pi)^d}.$$

Denoting $B = \{\xi \in \mathbf{R}^d \mid |\xi|^2 \leq 1\}$, we can make the estimate

$$\begin{aligned} \sum_k (\lambda - \lambda_k)_+ &\geq \lambda^{1+d/2} |\Omega_\varepsilon| \int_B \left(1 - |\xi|^2 - \frac{1}{\lambda} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz \right)_+ \frac{d\xi}{(2\pi)^d} \\ &\geq \lambda^{1+d/2} |\Omega_\varepsilon| \int_B (1 - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d} - \lambda^{d/2} \frac{|\Omega_\varepsilon| |B|}{(2\pi)^d} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz \\ &\geq \lambda^{1+d/2} |\Omega_\varepsilon| \int_B (1 - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d} - \lambda^{d/2} \frac{|\Omega| |B|}{(2\pi)^d} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz \\ &\geq \lambda^{1+d/2} |\Omega| \int_B (1 - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d} - R(\lambda, \varepsilon) \end{aligned}$$

where

$$R(\lambda, \varepsilon) = \lambda^{1+d/2} (|\Omega| - |\Omega_\varepsilon|) \int_B (1 - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d} + \lambda^{d/2} \frac{|\Omega| |B|}{(2\pi)^d} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz$$

Note that

$$\int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz = \varepsilon^{-2} \int_{\mathbf{R}^d} |\nabla g(z)|^2 dz.$$

Let $\varepsilon = \lambda^{-\alpha}$ with $\alpha > 0$. Then there are constants $L, C \geq 0$ such that

$$R(\lambda, \varepsilon) = L \lambda^{1+d/2} (|\Omega| - |\Omega_\varepsilon|) + C \lambda^{d/2+2\alpha}.$$

Suppose that $|\Omega| - |\Omega_\varepsilon| = O(\varepsilon)$. This is in particular the case when the inner Minkowski content of Ω exists which was proven to be the case when the boundary of Ω is Lipschitz-

regular and Ω itself is bounded in [ACV08] by Ambrosio, Colesanti, and Villa. Then,

$$R(\lambda, \lambda^{-\alpha}) = C\lambda^{d/2+2\alpha} + O(\lambda^{1+d/2-\alpha}).$$

Minimizing $\max\{d/2 + 2\alpha, 1 + d/2 - \alpha\}$, we find $\alpha = 1/3$. Consequently

$$\sum_k (\lambda - \lambda_k)_+ \geq \lambda^{1+d/2} |\Omega| \int_B (1 - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d} + O(\lambda^{d/2+2/3}).$$

Combining this with the upper estimate 4.13 we obtain the following theorem.

Theorem 4.15. *Let Ω be a bounded domain whose boundary is Lipschitz-regular. Then the eigenvalues (λ_k) of the Dirichlet Laplacian on Ω satisfy*

$$\sum_k (\lambda - \lambda_k)_+ = \lambda^{1+d/2} |\Omega| \int_B (1 - |\xi|^2)_+ \frac{d\xi}{(2\pi)^d} + O(\lambda^{d/2+2/3}).$$

This is a significantly weaker result than the one obtained in [FL20] by Frank and Larson, where the following two-term asymptotic formula was proven for bounded Lipschitz domains:

$$\sum_k (\lambda - \lambda_k)_+ = L_d |\Omega| \lambda^{d/2+1} - \frac{L_{d-1}}{4} \mathcal{H}^{d-1}(\partial\Omega) \lambda^{(d-1)/2+1} + o(\lambda^{(d-1)/2+1}),$$

for constants $L_d, L_{d-1} \geq 0$.

Remark 23. Geisinger considered similar coherent state methods to prove Weyl laws for fractional Laplacians in the article [Gei14].

4.3.3 Asymptotics of the Counting Function

In the previous subsection we discussed the asymptotic behavior of $\sum_k (\lambda - \lambda_k)_+$ as $\lambda \rightarrow \infty$, where (λ_k) are the eigenvalues of the Dirichlet Laplacian on a domain Ω . Now, given $\lambda > 0$, we would like to know the number of k 's for which $\lambda_k < \lambda$. By definition, this number is equal to the value of spectral counting function of the Dirichlet Laplacian in λ , i.e.

$$\#\{k \in \mathbf{N} \mid \lambda_k < \lambda\} = N_{-\Delta_\Omega^D}(\lambda).$$

We will use the classic Tauberian theorem of Karamata.

Lemma 4.16 (Karamata). *Let μ be a (positive) measure on $[0, \infty)$ such that*

$$\int_0^\infty e^{-tx} d\mu(x) < \infty$$

for all $t > 0$. Suppose that

$$\lim_{t \rightarrow 0^+} t^\alpha \int_0^\infty e^{-tx} d\mu(x) = A$$

for some $\alpha > 0$, $A \geq 0$. Then, for the measure μ , we find

$$\lim_{s \rightarrow \infty} s^{-\alpha} \mu([0, s]) = \frac{A}{\Gamma(\alpha + 1)}.$$

Remark 24. This theorem was first proven in [Kar31]. A modern proof can be found in [Bor20].

For the sake of brevity, we introduce the following notation

$$L_{\gamma, d}^c = \int_{\mathbf{R}^d} (1 - |\xi|^2)_+^\gamma \frac{d\xi}{(2\pi)} = \frac{\Gamma(\gamma + 1)}{(4\pi)^{d/2} \Gamma(\gamma + d/2 + 1)}$$

for $d \in \mathbf{N}$, $\gamma \geq 0$. The second identity can be seen by utilizing the properties of the beta function. The Weyl law we have proven in this section thus reads

$$\sum_k (\lambda - \lambda_k)_+ = L_{1, d}^c |\Omega| \lambda^{1+d/2} + O(\lambda^{d/2+2/3}).$$

The crucial ingredient in our proof of the Weyl law for the spectral counting function is the following lemma.

Lemma 4.17. *Let Ω be an open domain of finite measure. Denote by (λ_k) the eigenvalues of the Dirichlet Laplacian on Ω . Then*

$$\lim_{t \rightarrow 0^+} t^{d/2} \sum_{k \geq 0} e^{-t\lambda_k} = \frac{|\Omega|}{(4\pi)^{d/2}}.$$

Proof. Note that for $t > 0$ and $\lambda \in \mathbf{R}$, we find

$$e^{-t\lambda} = t^2 \int_{\mathbf{R}} (x - \lambda)_+ e^{-tx} dx,$$

which can easily be seen by performing integration by parts. Consequently,

$$\sum_k e^{-t\lambda_k} = \sum_k t^2 \int_{\mathbf{R}} (x - \lambda_k)_+ e^{-tx} dx = t^2 \int_0^\infty \sum_k (x - \lambda_k)_+ e^{-tx} dx,$$

by monotone convergence. By [4.14](#) we know that for $x \geq 0$ we have

$$\sum_k (x - \lambda_k)_+ = L_{1,d}^c |\Omega| x^{1+d/2} - r(x),$$

where r is a non-negative (by [4.13](#)) measurable function satisfying $\frac{r(x)}{x^{d/2+1}} \rightarrow 0$ as $x \rightarrow \infty$. Note that

$$t^2 \int_0^\infty x^{1+d/2} e^{-tx} dx = t^{-d/2} \int_0^\infty y^{d/2+1} e^{-y} dy = t^{-d/2} \Gamma(d/2 + 2).$$

On the other hand, $\lim_{x \rightarrow \infty} \frac{r(x)}{x^{d/2+1}} = 0$ implies

$$t^{d/2+1} r\left(\frac{y}{t}\right) = y^{d/2+1} \left(\frac{t}{y}\right)^{d/2+1} r\left(\frac{y}{t}\right) \rightarrow 0$$

for fixed $y > 0$ and $t \rightarrow 0+$. Since

$$r(x) = L_{1,d}^c |\Omega| x^{1+d/2} - \sum_k (x - \lambda_k)_+ \leq 2L_{1,d}^c |\Omega| x^{1+d/2},$$

we also find

$$t^{d/2+1} r\left(\frac{y}{t}\right) \leq 2L_{1,d}^c |\Omega| y^{1+d/2},$$

where the right-hand side of this inequality is clearly integrable on $[0, \infty)$ with respect to $e^{-y} dy$. Dominated convergence thus yields

$$t^{d/2+2} \int_{\mathbf{R}} r(x) e^{-tx} dx = t^{d/2+1} \int_{\mathbf{R}} r\left(\frac{y}{t}\right) e^{-y} dy \rightarrow 0$$

as $t \rightarrow 0+$. To conclude, we find

$$\lim_{t \rightarrow 0+} t^{d/2} \sum_{k \geq 0} e^{-t\lambda_k} = L_{1,d}^c \Gamma(d/2 + 2) |\Omega| = \frac{|\Omega|}{(4\pi)^{d/2}},$$

by the definition of the constant $L_{1,d}^c$. □

We can now state the Weyl law for Dirichlet Laplacians in its classical form in terms of the spectral counting function.

Theorem 4.18. *Let Ω be an open domain of finite measure. The spectral counting func-*

tion of the Dirichlet Laplacian on Ω exhibits the following asymptotic behavior

$$N_{-\Delta_{\Omega}^D}(\lambda) = L_{0,d}^c |\Omega| \lambda^{d/2} + o(\lambda^{d/2}),$$

as $\lambda \rightarrow \infty$.

Proof. Define $\mu([0, \lambda)) = N_{-\Delta_{\Omega}^D}(\lambda)$. Clearly, μ extends to a measure on $[0, \infty)$, since $N_{-\Delta_{\Omega}^D}$ is increasing, continuous from the left and $\lim_{\lambda \rightarrow -\infty} N_{-\Delta_{\Omega}^D} = 0$. See [LG06] for a proof of this classical result from measure theory.

If we again denote by λ_k the k -th eigenvalue (counting multiplicity) of the Dirichlet Laplacian on Ω , then integrals of non-negative measurable functions f on $[0, \infty)$ are given by

$$\int_0^{\infty} f d\mu = \sum_k f(\lambda_k).$$

Thus

$$\sum_{k \geq 0} e^{-t\lambda_k} = \int_0^{\infty} e^{-tx} d\mu(x),$$

and we can apply Karamata's Tauberian theorem with $\alpha = d/2$ and $A = \frac{|\Omega|}{(4\pi)^{d/2}}$ to obtain

$$\lim_{\lambda \rightarrow \infty} \frac{N_{-\Delta_{\Omega}^D}(\lambda)}{\lambda^{d/2}} = \lim_{\lambda \rightarrow \infty} \frac{\mu([0, \lambda))}{\lambda^{d/2}} = \lim_{\lambda \rightarrow \infty} \frac{\mu([0, \lambda])}{\lambda^{d/2}} = \frac{|\Omega|}{(4\pi)^{d/2} \Gamma(d/2 + 1)} = L_{0,d}^c |\Omega|. \quad \square$$

We can formulate a similar theorem for the Neumann Laplacian (see [FLW23] for a proof).

Theorem 4.19. *Let Ω be an open, bounded domain with the extension property. The spectral counting function of the Neumann Laplacian on Ω satisfies*

$$N_{-\Delta_{\Omega}^N}(\lambda) = L_{0,d}^c |\Omega| \lambda^{d/2} + o(\lambda^{d/2}),$$

as $\lambda \rightarrow \infty$.

What is remarkable about 4.18 is that the asymptotic behavior of $\lambda^{-d/2} N_{-\Delta_{\Omega}^D}(\lambda)$ as $\lambda \rightarrow 0$ depends only on the measure of Ω and not on other possible properties it might have. On the other hand, given the eigenvalues of the Dirichlet Laplacian on some domain, 4.18 gives us a way of computing the measure of the domain. Two domains are said to be *isospectral*, if they produce the same Dirichlet eigenvalues. Weyl's law thus implies that isospectral domains must have the same measure. For bounded domains Ω whose boundary is smooth and that satisfy a further technical condition, one can even prove (see [Ivr80]) a two term

asymptotic formula

$$N_{-\Delta_\Omega^D}(\lambda) = L_{0,d}^c |\Omega| \lambda^{d/2} + \frac{1}{4} L_{0,d-1}^c \mathcal{H}^{d-1}(\partial\Omega) \lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}),$$

as $\lambda \rightarrow \infty$. This means that in certain cases, we can not only obtain the d -dimensional measure of Ω but also the $d-1$ -dimensional measure of $\partial\Omega$ from the asymptotic behavior of $N_{-\Delta_\Omega^D}$. One might ask what further characteristics of a domain might be determined knowing only the corresponding Dirichlet eigenvalues, or even if the eigenvalues determine the domain completely (up to isometries). More concisely: does isospectrality imply isometry? Mark Kac formulated this question in [Kac66] very poetically as “*Can one hear the shape of a drum?*”. For some decades this question remained unanswered, until Gordon, Webb, and Wolpert produced a counterexample in [GWW92a]. See also [GWW92b] for further discussion.

4.3.4 Asymptotics of Riesz Means

Let us now discuss Weyl’s law for Riesz means, i.e. for expressions of the form

$$\mathrm{tr}(\lambda - (-\Delta_\Omega^D))_+^\gamma = \sum_k (\lambda - \lambda_k)_+^\gamma$$

for $\gamma \geq 0$. We use the convention

$$\sum_k (\lambda - \lambda_k)_+^0 = N_{-\Delta_\Omega^D}(\lambda).$$

Theorem 4.20 (Weyl’s law for Riesz means). *Let $\Omega \subseteq \mathbf{R}^d$ be an open domain of finite measure and let $\gamma \geq 0$. The Dirichlet eigenvalues (λ_k) for Ω satisfy*

$$\sum_k (\lambda - \lambda_k)_+^\gamma = L_{\gamma,d}^c |\Omega| \lambda^{d/2+\gamma} + o(\lambda^{d/2+\gamma}),$$

as $\lambda \rightarrow \infty$.

Proof. First note that for any $\lambda \geq 0$ we have

$$\lambda^\gamma = \gamma \int_0^\infty \chi_{\{\tau < \lambda\}} \tau^{\gamma-1} d\tau.$$

Therefore, by monotone convergence, we find

$$\sum_k (\lambda - \lambda_k)_+^\gamma = \gamma \int_0^\infty N_{-\Delta_\Omega^D}(\lambda - \tau) \tau^{\gamma-1} d\tau.$$

We are going to need one identity that can easily be checked using the definition of the Γ -function:

$$L_{0,d}^c \gamma \int_0^\lambda (\lambda - \kappa)^{d/2} \kappa^{\gamma-1} d\kappa = L_{\gamma,d}^c \lambda^{d/2+\gamma}.$$

The Weyl law for the spectral counting function can be formulated as follows: for every ε , there exists a λ_ε such that

$$\lambda^{-d/2} |N_{-\Delta_\Omega^D}(\lambda) - L_{0,d}^c |\Omega| \lambda^{d/2}| \leq \varepsilon$$

for all $\lambda \geq \lambda_\varepsilon$. Given $\varepsilon > 0$, we can therefore make the following estimate

$$|\sum_k (\lambda - \lambda_k)_+^\gamma - L_{\gamma,d}^c |\Omega| \lambda^{d/2+\gamma}| \leq \gamma \int_0^\lambda |N_{-\Delta_\Omega^D}(\lambda - \kappa) - L_{0,d}^c |\Omega| (\lambda - \kappa)^{d/2}| \kappa^{\gamma-1} d\kappa.$$

We split this into two parts. First, when $\kappa \leq \lambda - \lambda_\varepsilon$, then

$$|N_{-\Delta_\Omega^D}(\lambda - \kappa) - L_{0,d}^c |\Omega| (\lambda - \kappa)^{d/2}| \leq \varepsilon (\lambda - \kappa)^{d/2},$$

by Weyl's law. Thus

$$\begin{aligned} & \gamma \int_0^{\lambda - \lambda_\varepsilon} |N_{-\Delta_\Omega^D}(\lambda - \kappa) - L_{0,d}^c |\Omega| (\lambda - \kappa)^{d/2}| \kappa^{\gamma-1} d\kappa \\ & \leq \varepsilon \gamma \int_0^{\lambda - \lambda_\varepsilon} (\lambda - \kappa)^{d/2} \kappa^{\gamma-1} d\kappa \\ & \leq \varepsilon \gamma \int_0^\lambda (\lambda - \kappa)^{d/2} \kappa^{\gamma-1} d\kappa \\ & = \varepsilon \frac{L_{\gamma,d}^c}{L_{0,d}^c} \lambda^{d/2+\gamma}. \end{aligned}$$

When $\kappa > \lambda - \lambda_\varepsilon$, we make the estimate

$$\begin{aligned}
& \gamma \int_{\lambda - \lambda_\varepsilon}^{\lambda} |N_{-\Delta_\Omega^D}(\lambda - \kappa) - L_{0,d}^c |\Omega| (\lambda - \kappa)^{d/2}| \kappa^{\gamma-1} d\kappa \\
& \leq \gamma \int_{\lambda - \lambda_\varepsilon}^{\lambda} \left(N_{-\Delta_\Omega^D}(\lambda_\varepsilon) + L_{0,d}^c |\Omega| \lambda_\varepsilon^{d/2} \right) \kappa^{\gamma-1} d\kappa \\
& \leq \gamma \left(N_{-\Delta_\Omega^D}(\lambda_\varepsilon) + L_{0,d}^c |\Omega| \lambda_\varepsilon^{d/2} \right) \int_0^{\lambda} \kappa^{\gamma-1} d\kappa \\
& = \left(N_{-\Delta_\Omega^D}(\lambda_\varepsilon) + L_{0,d}^c |\Omega| \lambda_\varepsilon^{d/2} \right) \lambda^\gamma.
\end{aligned}$$

We therefore find

$$\limsup_{\lambda \rightarrow \infty} \lambda^{-d/2-\gamma} \left| \sum_k (\lambda - \lambda_k)_+^\gamma - L_{\gamma,d}^c |\Omega| \lambda^{d/2+\gamma} \right| = \varepsilon \frac{L_{\gamma,d}^c}{L_{0,d}^c}.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the claim follows. $\square$

Remark 25. This proof was taken from [\[FLW23\]](#).

4.4 Pólya Inequalities

We conclude this chapter on Laplacians with a discussion of Pólya type inequalities. Recall that by [4.13](#), for an open set of finite measure $\Omega \subseteq \mathbf{R}^d$ we have

$$\sum_k (\lambda - \lambda_k)_+ \leq L_{1,d}^c |\Omega| \lambda^{d/2+1}$$

for the Dirichlet eigenvalues (λ_k) on Ω . This means that the leading order term in the asymptotic formula for $\sum_k (\lambda - \lambda_k)_+$ actually acts as global upper bound. The question arises whether this is still the case for $\gamma = 0$, i.e. if

$$N_{-\Delta_\Omega^D}(\lambda) \leq L_{0,d}^c |\Omega| \lambda^{d/2} \tag{3}$$

for all $\lambda \geq 0$.

We actually obtain a bound for $N_{-\Delta_\Omega^D}$ from [4.13](#). Indeed, let $\kappa > \lambda \geq 0$. For the eigenvalues $\lambda_k < \lambda$ we find $(\kappa - \lambda) \leq (\kappa - \lambda_k)$ and summing over all k such that $\lambda_k < \lambda$, yields

$$N_{-\Delta_\Omega^D}(\lambda)(\kappa - \lambda) \leq \sum_{\lambda_k < \lambda} (\kappa - \lambda_k)_+ \leq \sum_k (\kappa - \lambda_k)_+.$$

Therefore,

$$N_{-\Delta_{\Omega}^D}(\lambda) \leq \frac{1}{(\kappa - \lambda)} \sum_k (\kappa - \lambda_k)_+ \leq \frac{1}{(\kappa - \lambda)} L_{1,d}^c |\Omega| \kappa^{d/2+1},$$

and choosing $\kappa = (1 + \frac{2}{d}) \lambda$ yields

$$N_{-\Delta_{\Omega}^D}(\lambda) \leq \frac{d}{2} \left(1 + \frac{2}{d}\right)^{d/2+1} L_{1,d}^c |\Omega| \lambda^{d/2}.$$

Bearing in mind that $(\frac{d}{2} + 1) = \frac{L_{0,d}^c}{L_{1,d}^c}$ we thus obtain

$$N_{-\Delta_{\Omega}^D}(\lambda) \leq \left(1 + \frac{2}{d}\right)^{d/2} L_{0,d}^c |\Omega| \lambda^{d/2},$$

which is clearly a worse estimate than [\[3\]](#). Direct computation shows that [\[3\]](#) holds for cubes. It was proven by Pólya in [\[Pól61\]](#) that [\[3\]](#) holds for any bounded domain that is tiling the entire space $\mathbf{R}^d$. By this we mean the following.

Definition 4.21. A bounded open set $\Omega \subseteq \mathbf{R}^d$ is said to be *tiling* $\mathbf{R}^d$ if there exists a sequence (F_n) of isometries of $\mathbf{R}^d$ such that for $\Omega_n = F_n \Omega$ we have

1. $\Omega_n \cap \Omega_m = \emptyset$ for $n \neq m$,
2. $|\mathbf{R}^d \setminus \bigcup_n \Omega_n| = 0$.

In other words, we want $\mathbf{R}^d$ to be (almost) filled by congruent copies of Ω that do not intersect. Note that when $F: \Omega \rightarrow \tilde{\Omega}$ is an isometry, then $-\Delta_{\Omega}^D$ and $-\Delta_{\tilde{\Omega}}^D$ are unitarily equivalent. In particular, their eigenvalues coincide.

Theorem 4.22 (Pólya). *Let $\Omega \subseteq \mathbf{R}^d$ be an open tiling domain. Then*

$$N_{-\Delta_{\Omega}^D}(\lambda) \leq L_{0,d}^c |\Omega| \lambda^{d/2}.$$

That [\[3\]](#) should hold for all bounded open domains is known as *Pólya's conjecture*. Indeed, no counterexample to the estimate could be constructed so far. For a long time it was an open question whether the unit ball in $\mathbf{R}^d$ satisfies Pólya's conjecture. Quite recently, this was proven to be the case by Levitin, Polterovich, Sher, and Filonov in [\[FLPS23\]](#).

5 Schrödinger Operators

In this chapter we are interested in Schrödinger operator, i.e. operators of the form

$$-\Delta + V,$$

where $V \in L^1_{\text{loc}}(\mathbf{R}^d)$ is a real-valued function. While this expression makes perfect sense when applied to test functions, we would like to obtain self-adjoint realizations. There are two principle approaches for defining $-\Delta + V$ as a self-adjoint operator. The obvious approach is to simply consider $-\Delta + V$ as a sum of two operators. Kato–Rellich’s theorem provides a criterion for this sum to be self-adjoint. The second approach is to define a quadratic form via

$$u \mapsto \int_{\mathbf{R}^d} (|\nabla u(x)|^2 + V(x)|u(x)|^2) dx,$$

and by abuse of notation denote the operator corresponding to this form by $-\Delta + V$. In both cases, if we impose certain conditions on V , we find that

$$\sigma_e(-\Delta + V) = \sigma_e(-\Delta) = [0, \infty),$$

meaning that the discrete spectrum of $-\Delta + V$ must lie strictly below zero.

5.1 The Kato–Rellich Approach

The integrability of V is of great importance for the possible self-adjointness of $-\Delta + V$. We will focus on the case $V \in L^p(\mathbf{R}^d)$, where p will depend on d in the following way

$$\begin{cases} p \geq 2, & d = 1, 2, 3 \\ p > 2, & d = 4 \\ p \geq d/2, & d \geq 5 \end{cases} \quad (4)$$

Theorem 5.1. *Let $V \in L^p(\mathbf{R}^d)$ be a real-valued function with p satisfying (4). For every $\varepsilon > 0$ there exists a constant $C_\varepsilon \geq 0$ such that*

$$\|Vu\|_{L^2(\mathbf{R}^d)} \leq \varepsilon \|-\Delta u\|_{L^2(\mathbf{R}^d)} + C_\varepsilon \|u\|_{L^2(\mathbf{R}^d)},$$

for all $u \in H^2(\mathbf{R}^d)$.

Remark 26. The inequality in this theorem implies that $\mathcal{D}(-\Delta) = H^2(\mathbf{R}^d) \subseteq \mathcal{D}(M_V)$. Furthermore, since ε is arbitrary, we can choose $\varepsilon < 1$ thus satisfying the conditions of the

Kato–Rellich theorem [2.41](#).

A proof of this assertion can be found in [\[Nam21\]](#). The conditions on p depend on the existence of a continuous Sobolev embedding $H^2(\mathbf{R}^d) \subseteq L^q(\mathbf{R}^d)$, where $q = \frac{2p}{p-2}$.

5.2 The Form Approach

Consider now the case that $V \in L^p(\mathbf{R}^d)$ with

$$\begin{cases} p \geq 1, & d = 1 \\ p > 1, & d = 2 \\ p \geq d/2, & d \geq 3 \end{cases} \quad (5)$$

Theorem 5.2. *Let $V \in L^p(\mathbf{R}^d)$ be a real-valued function with p satisfying [\(5\)](#). For every $\varepsilon > 0$ there exists a constant $C_\varepsilon \geq 0$ such that*

$$\int_{L^2(\mathbf{R}^d)} |V(x)| |u(x)|^2 dx \leq \varepsilon \int_{L^2(\mathbf{R}^d)} |\nabla u(x)|^2 dx + C_\varepsilon \int_{L^2(\mathbf{R}^d)} |u(x)|^2 dx,$$

for all $u \in H^1(\mathbf{R}^d)$.

Remark 27. Define

$$q[u] = \int_{\mathbf{R}^d} (|\nabla u(x)|^2 + V(x)|u(x)|^2) dx,$$

for $u \in H^1(\mathbf{R}^d)$. Then q is a well-defined lower semibounded closed quadratic form on $\mathcal{D}[q] = H^1(\mathbf{R}^d)$. The corresponding self-adjoint operator, denoted $-\Delta + V$, has domain

$$\mathcal{D}(-\Delta + V) = \{u \in H^1(\mathbf{R}^d) \mid -\Delta u + Vu \in L^2(\mathbf{R}^d)\},$$

where $-\Delta u + Vu$ is understood in the sense of distributions.

Remark 28. Note that [\(5\)](#) comprises a larger class of admissible values for p than [\(4\)](#).

Remark 29. More generally, one could consider potentials in $L^p(\mathbf{R}^d) + L^\infty(\mathbf{R}^d)$ with p as in [\(5\)](#). One would also obtain self-adjoint Schrödinger operators this way using the form approach. In order to guarantee that $\sigma_e(-\Delta + V) = [0, \infty)$ one could add the additional requirement that the potential belong to $L^p(\mathbf{R}^d) + L^\infty_\varepsilon(\mathbf{R}^d)$ (See Lewin’s book [\[Lew22\]](#) for a definition). In the rest of this book we are primarily interested in Lieb–Thirring inequalities. These inequalities demand integrability of the potential and since $L^q(\mathbf{R}^d) \subseteq L^p(\mathbf{R}^d) + L^\infty_\varepsilon(\mathbf{R}^d)$ for $q \geq p$, we can thus use our more restrictive approach without any loss of significant generality.

Proof. Let us sketch the idea of the proof. A complete proof can be found in [Lew22]. For simplicity, we restrict ourselves to the case $d \geq 3$. The other cases can be proved in a similar way.

Bearing in mind that $L^q(\mathbf{R}^d) \subseteq L^p(\mathbf{R}^d) + L^\infty_\varepsilon(\mathbf{R}^d)$ for $q \geq p$, we can assume without loss of generality that $V \in L^{d/2}(\mathbf{R}^d)$. By dominated convergence, we have

$$\lim_{M \rightarrow \infty} \int_{|V| > M} |V(x)|^{d/2} dx = 0.$$

Thus, for $\varepsilon > 0$ there exists an M such that

$$\int_{|V| > M} |V(x)|^{d/2} dx \leq \varepsilon.$$

Write $V_1 = V\chi_{\{|V| > M\}}$ and $V_2 = V\chi_{\{|V| \leq M\}}$. Then $V = V_1 + V_2$ and $V_2 \in L^\infty(\mathbf{R}^d)$. Recall the Sobolev embedding theorem for $d \geq 3$: there exists a constant $C > 0$ such that

$$\|u\|_{L^{2d/(d-1)}(\mathbf{R}^d)} \leq C \|u\|_{H^1(\mathbf{R}^d)}$$

for all $u \in H^1(\mathbf{R}^d)$. Applying Hölder's inequality, we thus find

$$\begin{aligned} \int_{L^2(\mathbf{R}^d)} |V(x)| |u(x)|^2 dx &\leq \int_{L^2(\mathbf{R}^d)} |V_1(x)| |u(x)|^2 dx + \|V_2\|_{L^\infty(\mathbf{R}^d)} \|u\|_{L^2(\mathbf{R}^d)}^2 \\ &\leq \|V_1\|_{L^{d/2}(\mathbf{R}^d)} \|u\|_{L^{2d/(d-2)}(\mathbf{R}^d)}^2 + \|V_2\|_{L^\infty(\mathbf{R}^d)} \|u\|_{L^2(\mathbf{R}^d)}^2 \\ &\leq C^2 \varepsilon^{2/d} \|u\|_{H^1(\mathbf{R}^d)}^2 + \|V_2\|_{L^\infty(\mathbf{R}^d)} \|u\|_{L^2(\mathbf{R}^d)}^2 \\ &\leq C^2 \varepsilon^{2/d} \int_{\mathbf{R}^d} |\nabla u(x)|^2 dx + (C^2 \varepsilon^{2/d} + \|V_2\|_{L^\infty(\mathbf{R}^d)}) \|u\|_{L^2(\mathbf{R}^d)}^2 \end{aligned}$$

Clearly, we can make $C^2 \varepsilon^{2/d}$ arbitrarily small. This concludes the proof. $\square$

Remark 30. One might even consider potentials $V = V_+ - V_-$ where only the negative part V_- is in $L^p(\mathbf{R}^d)$ and the positive part V_+ is merely in $L^1_{loc}(\mathbf{R}^d)$. On $C_c^\infty(\mathbf{R}^d)$ one would still obtain a lower semibounded symmetric sesquilinear form. The domain of its closure, however, would generally not coincide with $H^1(\mathbf{R}^d)$. We restrict ourselves here to the case outlined in the theorem above. This is justified by the circumstance that we are mostly interested in Lieb–Thirring inequalities and can thus assume that the potential is purely negative by remark [34].

It turns out that the operator $-\Delta + V$ thus constructed satisfies the conditions for the assertion about essential spectra in the KLMN theorem [2.45] (See [Lew22] for a proof).

Theorem 5.3. *Let V be as in the theorem above. The operator $-\Delta + V$ satisfies*

$$\sigma_e(-\Delta + V) = [0, \infty).$$

Since we are primarily interested in the discrete eigenvalues of finite multiplicity, we will focus on the negative part of the spectra of Schrödinger operators. In our case the spectrum thus looks as follows: the non-negative half-line is the essential spectrum. If there are any negative eigenvalues, then there is a lowest eigenvalue, called the ground state energy, because $-\Delta + V$ is lower semibounded. Other negative eigenvalues are called excited state energies and they form a sequence of negative numbers that, if they are infinitely many, accumulate at zero. They could not possibly accumulate anywhere else. The first question we would like to ask is whether there are infinitely many negative eigenvalues. The next section gives an answer to this question.

5.3 The Cwikel–Lieb–Rozenblum Inequality

As discussed in the last section, given a Schrödinger operator as defined in [5], the discrete spectrum is negative. We now ask ourselves the question whether the discrete spectrum is finite. Generally, it is not. Consider the hydrogen atom as an example. Under certain integrability conditions for V we can, however, enforce finiteness of the discrete spectrum. The following result was proven independently from one another by Cwikel ([Cwi77]), Lieb ([Lie80]), and Rozenblum ([Roz72]) in the 70s.

For a real-valued function V satisfying the conditions of Theorem [5], we abbreviate

$$N(V) = N_{-\Delta+V}(0),$$

where $-\Delta + V$ is Schrödinger operator constructed in Theorem [5]. It turns out that for the following estimate, only the negative part of V is of relevance. We use the convention $V = V_+ - V_-$, $V_{\pm} \geq 0$ for positive and negative parts of functions.

Theorem 5.4 (Cwikel–Lieb–Rozenblum, '70s). *Let $d \geq 3$. There exists a constant $L_{0,d}$ such that for all $V \in L^{d/2}(\mathbf{R}^d)$, we have*

$$N(V) \leq L_{0,d} \int_{\mathbf{R}^d} V_-(x)^{d/2} dx.$$

Remark 31. A short proof of this inequality can be found in [Fra14]. It is based on methods developed by Rumin in [Rum10]. The CLR inequality can be significantly generalized, see

the paper [LS97] by Levin and Solomyak for details.

In dimensions one and two, no such bound can exist. Indeed, if $d \leq 2$ and $V \in C_c^\infty(\mathbf{R}^d)$ is non-positive and not identically zero, then $-\Delta + V$ has at least one negative eigenvalue. We refer to [RS78] for a proof. Nonetheless, there are estimates for Schrödinger operators on $(0, \infty)$ (Calogero's bound, [RS78]) and on $\mathbf{R} \times (0, \infty)$ ([FLR24]) when Dirichlet boundary conditions are imposed.

5.4 Lieb–Thirring Inequalities

We now turn toward a class of inequalities that is of central importance in the spectral theory of Schrödinger operators. Lieb–Thirring inequalities have been studied intensively throughout the past decades. For an overview of the subject see [Sch22] or [Fra21] and for a comprehensive treatment see [FLW23]. The situation is the same as before: we have a real-valued potential V that is integrable to a sufficiently high degree and we consider the Schrödinger operator constructed using the form method. We denote the negative eigenvalues of $-\Delta + V$ by

$$\lambda_1(V) \leq \lambda_2(V) \leq \dots \leq 0.$$

If there are only n negative eigenvalues, we set $\lambda_{n+1}(V) = \lambda_{n+2}(V) = \dots = 0$. We are interested in *Riesz means*, i.e. in expressions of the form

$$\sum_j |\lambda_j(V)|^\gamma,$$

where $\gamma \geq 0$. We use the convention

$$\sum_j |\lambda_j(V)|^0 = N(V).$$

It is often convenient to use the notation

$$\mathrm{tr}(-\Delta + V)_-^\gamma = \sum_j |\lambda_j(V)|^\gamma.$$

The following theorem, known as the *Lieb–Thirring inequality* was first proven in a special case in [LT75]. Lieb and Thirring used the eigenvalue bound in the case $\gamma = 1$ to derive a lower bound on the kinetic energy of a system of fermions thus providing a simple proof of the stability of matter of second kind, proven earlier by Dyson and Lenard in [DL67], [LD68]. For an overview of the topic of stability of matter, see also the book by Lieb and

Seiringer [LS09].

Theorem 5.5. *Let $d \geq 0$ and let $\gamma \geq 0$ be such that*

$$\begin{cases} \gamma \geq \frac{1}{2}, & d = 1 \\ \gamma > 0, & d = 2 \\ \gamma \geq 0, & d \geq 3. \end{cases} \quad (6)$$

There exists a constant $L_{\gamma,d} > 0$ such that for all real-valued functions $V \in L^{d/2+\gamma}(\mathbf{R}^d)$, the negative eigenvalues $\lambda_j(V)$ of the Schrödinger operator $-\Delta + V$ satisfy

$$\sum_j |\lambda_j(V)|^\gamma \leq L_{\gamma,d} \int_{\mathbf{R}^d} V_-(x)^{d/2+\gamma} dx.$$

Remark 32. Note that the conditions on γ in [6] imply that V satisfies the conditions in [5] ensuring that we can construct $-\Delta + V$ using the form approach.

Remark 33. The case $d = 1$, $\gamma = \frac{1}{2}$ was proven by Weidl in [Wei96]. The case $d \geq 3$, $\gamma = 0$ is precisely the Cwikel–Lieb–Rozenblum inequality discussed in the previous section. All the remaining cases were proven by Lieb and Thirring in their paper [LT76]. Lieb and Thirring used a version of the *Birman–Schwinger principle* to prove the inequalities they considered. Of fundamental importance for their proof is the general trace inequality

$$\mathrm{tr}(B^{1/2}AB^{1/2})^m \leq \mathrm{tr}(B^{m/2}A^mB^{m/2})$$

for positive bounded operators A and B on Hilbert spaces.

Remark 34. In the form sense, we have

$$-\Delta - V_- \leq -\Delta + V.$$

Consequently, $\lambda_j(-V_-) \leq \lambda_j(V)$ by proposition [2.40]. Note that these numbers are negative. Thus $|\lambda_j(V)| \leq |\lambda_j(-V_-)|$ and

$$\mathrm{tr}(-\Delta + V)_-^\gamma \leq \mathrm{tr}(-\Delta - V_-)_-^\gamma.$$

In the right-hand side of the Lieb–Thirring inequality, only the negative part of V contributes. When proving Lieb–Thirring inequalities, one can therefore restrict oneself to the Schrödinger operators $-\Delta - V$ with $V \geq 0$.

Lieb and Thirring proved the stability of matter using a lower bound on the kinetic energy of particle system that they derived from the Lieb–Thirring inequality in the case $\gamma = 1$.

Theorem 5.6. *Let $d \geq 1$. There exists a constant $K_d > 0$ such that for all $N \in \mathbf{N}$ and $u_1, \dots, u_N \in H^1(\mathbf{R}^d)$ that are orthonormal in $L^2(\mathbf{R}^d)$*

$$\sum_{j=1}^N \int_{\mathbf{R}^d} |\nabla u_j(x)|^2 dx \geq K_d \int_{\mathbf{R}^d} \left(\sum_{j=1}^N |u_j(x)|^2 \right)^{1+2/d} dx.$$

Remark 35. The optimal (largest possible) constant K_d in the above inequality is related to the optimal (smallest possible) constant $L_{1,d}$ in the Lieb–Thirring inequality via

$$((1 + d/2)L_{1,d})^{1+2/d} ((1 + 2/d)K_d)^{1+d/2} = 1.$$

Remark 36. This kinetic energy inequality can be proven directly, without using the Lieb–Thirring inequality. See [Rum10] or [Sab16] for instance.

5.4.1 Optimal Constants

The Lieb–Thirring inequality should be compared to the *Weyl asymptotic formula* for the eigenvalues of Schrödinger operators which reads

$$\lim_{\alpha \rightarrow \infty} \alpha^{-d/2-\gamma} \sum_j |\lambda_j(\alpha V)|^\gamma = L_{\gamma,d}^c \int_{\mathbf{R}^d} V_-(x)^{d/2+\gamma} dx,$$

where

$$L_{\gamma,d}^c = \int_{\mathbf{R}^d} (1 - |\xi|^2)_+^\gamma \frac{d\xi}{(2\pi)} = \frac{\Gamma(\gamma + 1)}{(4\pi)^{d/2} \Gamma(\gamma + d/2 + 1)},$$

for potentials satisfying $V \in L^{d/2+\gamma}(\mathbf{R}^d)$. One consequence of the Weyl asymptotic formula is that the constant $L_{\gamma,d}$ in the Lieb–Thirring inequality must satisfy

$$L_{\gamma,d} \geq L_{\gamma,d}^c.$$

One can try to find the lowest possible constant $L_{\gamma,d}$ for which the Lieb–Thirring inequality holds, but it cannot be strictly smaller than the semi-classical constant $L_{\gamma,d}^c$. See [Fra21] for an overview over the current best constants. Clearly, if

$$\sum_j |\lambda_j(V)|^\gamma \leq L \int_{\mathbf{R}^d} V_-(x)^{d/2+\gamma} dx,$$

then

$$|\lambda_1(V)|^\gamma \leq L \int_{\mathbf{R}^d} V_-(x)^{d/2+\gamma} dx.$$

If we denote the smallest possible constant for which the above inequality holds by $L_{\gamma,d}^{(1)}$, then we see that

$$L_{\gamma,d} \geq L_{\gamma,d}^{(1)}.$$

Lieb and Thirring conjectured in [LT76] that the optimal (smallest) constant in the Lieb–Thirring inequality is given by

$$\max\{L_{\gamma,d}^{(1)}, L_{\gamma,d}^c\}.$$

This conjecture has come to be known as the *Lieb–Thirring conjecture* and it is known to hold in some cases and to fail in others. In particular, the Lieb–Thirring conjecture is known to hold in all dimensions for $\gamma \geq 3/2$. In this case, $\max\{L_{\gamma,d}^{(1)}, L_{\gamma,d}^c\}$ coincides with $L_{\gamma,d}^c$. In [LW00], Laptev and Weidl proved the Lieb–Thirring inequality for $d = 1$, $\gamma = 3/2$ in the case of operator-valued potentials with the semi-classical constant $L_{3/2,1}^c$. A simpler proof for this inequality was later found by Benguria and Loss in [BL00]. Laptev and Weidl then proceeded, by using a lifting argument, to prove the Lieb–Thirring inequality (5.5) for $\gamma = 3/2$ in any dimension with the constant $L_{3/2,d}^c$. An argument by Aizenman [AL78] and Lieb extends the Lieb–Thirring inequality with semi-classical constants $L_{\gamma,d}^c$ to the case $\gamma \geq 3/2$. The Lieb–Thirring conjecture also holds for $d = 1$, $\gamma = 1/2$ as proven by Hundertmark, Lieb, and Thomas in [HLT98] using a modified Birman–Schwinger principle. In this case, the sharp constant is $\frac{1}{2} = L_{1/2,1}^{(1)} = 2L_{1/2,1}^c$.

In all other cases, the conjecture either fails or its validity is unknown. Schimmer’s paper [Sch22] discusses the current (2022) state of the Lieb–Thirring conjecture. Even in the cases in which the conjecture is known to fail, there might be sharp constants. On the other hand, in the cases in which there is no known sharp constant, one might still be interested in the smallest currently known constant. In the following cases, we have an upper bound on the lowest constant in the Lieb–Thirring inequality:

$$\begin{cases} 2L_{\gamma,d}^c, & 1/2 \leq \gamma < 1, \ d = 1 \\ 2R_{1,1}L_{\gamma,d}^c, & 1/2 \leq \gamma < 1, \ d \geq 2 \\ R_{1,1}L_{\gamma,d}^c, & 1 \leq \gamma < 3/2, \ d \geq 1 \end{cases}$$

Here $R_{1,1} \leq 1.456\dots$. In the first case, the assertion is a consequence of the Aizenman–Lieb argument, while the remaining two cases were proven by Frank, Hundertmark, Jax, and Nam in [FHJN21].

6 Lieb–Thirring Inequalities in Hyperbolic Spaces

This is the main chapter of the thesis. Its results have been published in [ILW24].

6.1 Hyperbolic Space

Riemannian manifolds can be equipped with two objects that allow us to consider spectral theoretic problems on said manifolds. These two objects are the *Riemannian measure* and the *Laplace–Beltrami operator*. Usually, they are defined locally, i.e. in local coordinates. It is then shown that the definitions do not depend on the choice of coordinates. Since smooth manifolds are paracompact, they admit smooth partitions of unity. The local definitions of our two objects can thus be “glued” together to define global objects. For details on Riemannian manifolds, see for instance the classic book [GHL04] by Gallot, Hulin, and Lafontaine.

Let us turn toward the local definitions of the two objects we are interested in.

Definition 6.1. Consider a Riemannian manifold (M, g) of dimension d . In local coordinates $x_1, \dots, x_d$, we define the *Riemannian measure* as

$$dV_g(x) = \sqrt{\det(g(x))} dx_1 \dots dx_d.$$

By $\det(g(x))$ we mean the determinant of the matrix $(g_{ij}(x))$ formed by the entries of the Riemannian metric tensor in the coordinates $x_1, \dots, x_d$:

$$g(x) = \sum_{i,j=1}^d g_{ij}(x) dx_i \otimes dx_j$$

As mentioned earlier, one would have to show that this definition does not depend on the choice of coordinates whenever two coordinate charts intersect. We refer to Grigoryan’s book [Gri09] for a proof.

Definition 6.2. The Laplace–Beltrami operator on a Riemannian manifold is given by

$$-\Delta_g = -\frac{1}{\sqrt{\det(g(x))}} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(\sqrt{\det(g(x))} g^{ij}(x) \frac{\partial}{\partial x_j} \right)$$

in local coordinates $x_1, \dots, x_d$. Here, $g^{ij}(x)$ is the ij -th entry of the inverse matrix of the matrix formed by the $g_{ij}(x)$.

Given a measure and a differential expression, we can define $-\Delta_g$ in the weak (distributional) sense just like we would in the euclidean case. Gradients are even easier to define. For a smooth function u on a Riemannian manifold (M, g) , there exists unique smooth vector field $\text{grad } u$ such that for all smooth vector fields X on M

$$g(\text{grad } u, X) = du(X) = Xu.$$

The vector field $\text{grad } u$ is called the *gradient* of u . One can then define the Sobolev norm

$$\|u\|_{H^1(M)}^2 = \int_M (|u|_g^2 + |\text{grad } u|_g^2) dV_g.$$

The Sobolev space $H^1(M)$ is then the closure of $C^\infty(M)$ in this norm. We can thus construct a self-adjoint realization of $-\Delta_g$ by simply associating a semibounded self-adjoint operator to the quadratic form

$$q[u] = \int_M |\text{grad } u|_g^2 dV_g$$

defined on $H^1(M)$.

We are interested in a particular Riemannian manifold – the hyperbolic space. One nice property of the hyperbolic space is that it can be covered by a single chart. In this work, we use the half-space model: The d -dimensional hyperbolic space $\mathbf{H}^d$ is the set

$$\mathbf{H}^d = \{(x_1, \dots, x_{d-1}, y) \in \mathbf{R}^d \mid y > 0\},$$

equipped with the metric

$$h = \frac{1}{y^2}(dx_1^2 + \dots + dx_{d-1}^2 + dy^2).$$

For convenience, we shall write $x = (x_1, \dots, x_{d-1})$. The metric tensor is given by a diagonal matrix in these coordinates

$$h_{ij}(x, y) = \frac{1}{y^2} \delta_{ij}.$$

Therefore,

$$\sqrt{\det(h(x, y))} = \sqrt{\frac{1}{y^{2d}}} = \frac{1}{y^d},$$

and the volume is given by

$$dV_h(x, y) = \frac{1}{y^d} dx_1 \dots dx_{d-1} dy.$$

The inverse of matrix of the metric tensor is clearly given by

$$h^{ij}(x, y) = y^2 \delta_{ij}.$$

For the Laplace–Beltrami operator, we thus find

$$-\Delta_h = -y^2 \tilde{\Delta} - y^d \partial_y (y^{2-d} \partial_y),$$

where $\tilde{\Delta} = \sum_{j=1}^{d-1} \partial_{x_j}^2$. Recall that the Laplacian in (the entire) euclidean space has the spectrum

$$\sigma(-\Delta) = [0, \infty)$$

and that its spectrum is purely essential. For the Laplace–Beltrami operator in hyperbolic space we find

$$\sigma(-\Delta_h) = \left[\frac{(d-1)^2}{4}, \infty \right),$$

and again this spectrum is purely essential. The papers by McKean ([\[McK70\]](#)) and Perry ([\[Per87\]](#)) deal with the Laplace–Beltrami operator in hyperbolic space. We will prove these assertions about the spectrum of $-\Delta_h$ in the next section.

6.2 The Laplace–Beltrami Operator

Recall that the hyperbolic space $\mathbf{H}^d$ is equipped with the measure $dV_h(x, y) = \frac{1}{y^d} dx dy$, where $dx = dx_1 \dots dx_{d-1}$. We are thus interested in the function space $L^2\left(\mathbf{H}^d, \frac{1}{y^d} dx dy\right)$. The core idea in our computations is to replace y by e^t where $t \in \mathbf{R}$. We thus work on $\mathbf{R}^d$, rather than a half-space, again. Consider the operator

$$W: L^2\left(\mathbf{H}^d, \frac{1}{y^d} dx dy\right) \rightarrow L^2(\mathbf{R}^d, dx dt)$$

given by

$$(Wu)(x, t) = e^{\frac{(1-d)}{2}t} u(x, e^t).$$

To check that Wu is actually square integrable, we compute

$$\begin{aligned}\int_{\mathbf{R}^d} |Wu(x, t)|^2 dx dt &= \int_{\mathbf{R}^d} e^{(1-d)t} |u(x, e^t)|^2 dx dt \\ &= \int_{\mathbf{H}^d} |u(x, y)|^2 \frac{dx dy}{y^d}.\end{aligned}$$

The operator W is therefore even an isometry. Clearly, the change of variables can be reversed and W is thus seen to be a unitary operator. Our approach to studying the operator $-\Delta_h$ is now to consider the operator $W(-\Delta_h)W^*$ on $L^2(\mathbf{R}^d, dx dt)$ which is by definition unitarily equivalent to $-\Delta_h$. Spectral properties thus carry over.

For simplicity, write $v(x, t) = Wu(x, t)$ for $u \in L^2(\mathbf{H}^d, y^{-d} dx dy)$. For $u \in \mathcal{D}(-\Delta_h)$ we find

$$\begin{aligned}\langle u, -\Delta_h u \rangle_{L^2(\mathbf{H}^d, y^{-d} dx dy)} &= \int_{\mathbf{H}^d} y^2 \left(|\tilde{\nabla} u(x, y)|^2 + |\partial_y^2 u(x, y)|^2 \right) \frac{dx dy}{y^d} \\ &= \int_{\mathbf{R}^d} \left(e^{2t} |\tilde{\nabla} v(x, t)|^2 + |\partial_t^2 v(x, t)|^2 + \frac{(d-1)^2}{4} |v(x, t)|^2 \right) dx dt \\ &= \langle v, (H + (d-1)^2/4) v \rangle_{L^2(\mathbf{R}^d, dx dt)},\end{aligned}$$

where the operator H is given by the differential expression

$$H = -\partial_t^2 - e^{2t} \tilde{\Delta}.$$

Just like before, $-\tilde{\Delta}$ is the Laplacian in the x -coordinates and $\tilde{\nabla}$ is the gradient in the x -coordinates. Clearly, the spectrum of H is essential and equals $[0, \infty)$. Shifting H by $(d-1)^2/4$ yields our claim about the spectrum of $-\Delta_h$.

6.3 Schrödinger Operators in Hyperbolic Space

We would like to define Schrödinger operators on $\mathbf{H}^d$. The crucial ingredient in the proof of Theorem 5.2 is the Sobolev embedding theorem. In order to use the form approach to define Schrödinger operators on $\mathbf{H}^d$, we need appropriate embeddings $H^1(\mathbf{H}^d) \hookrightarrow L^p(\mathbf{H}^d)$.

Theorem 6.3. *Let $d \geq 3$. Then the Sobolev space $H^1(\mathbf{H}^d)$ embeds continuously into $L^p(\mathbf{H}^d)$ with $p = \frac{2d}{d-2}$.*

Remark 37. See Hebey's book [Heb96b] for a proof. In this source, the author proves a more general case. Specifically, the validity of the Sobolev embedding is proven for complete Riemannian manifolds whose Ricci curvature is bounded from below and that

admit a uniform lower bound on the volumes of balls independent of their centers. Since the curvature of $\mathbf{H}^d$ is constant, so is its Ricci curvature. Furthermore, the volumes of balls in hyperbolic space do not even depend on the center. Sobolev embedding theorems for complete manifolds are also discussed in Aubin's article [Aub76]. Hebey's paper [Heb96a] deals with optimal constants in Sobolev inequalities on complete manifolds.

This theorem covers the case $d \geq 3$. For $d = 2$ we have the following assertion.

Theorem 6.4. *For any $p \geq 2$, the Sobolev space $H^1(\mathbf{H}^2)$ embeds continuously into $L^p(\mathbf{H}^2)$.*

Remark 38. See the paper [MT97] by Mugelli and Talent for a proof.

We can thus define Schrödinger operators in the same way as in the euclidean case. Specifically, if V is a real-valued function on $\mathbf{H}^d$ such that $V \in L^p(\mathbf{H}^d)$, where

$$\begin{cases} p \geq 1, & d = 1 \\ p > 1, & d = 2 \\ p \geq d/2, & d \geq 3 \end{cases}$$

then $-\Delta_h + V$ can be defined as a lower semibounded self-adjoint operator whose essential spectrum coincides with

$$\sigma_e(-\Delta_h + V) = \sigma(-\Delta_h) = \left[\frac{(d-1)^2}{4}, \infty\right).$$

When talking about the discrete spectrum of $-\Delta_h + V$ we are consequently interested in the part of the spectrum lying below $\frac{(d-1)^2}{4}$.

We have an analogue of the CLR inequality for Schrödinger operators on $\mathbf{H}^d$, $d \geq 3$. Rather than the number of negative eigenvalues, one now estimates the number $N_{-\Delta_h + V}(\frac{(d-1)^2}{4})$ of eigenvalues lying below $\frac{(d-1)^2}{4}$.

Theorem 6.5. *Let $d \geq 3$. There exists a constant $C > 0$ such that for all real-valued functions $V \in L^{d/2}(\mathbf{H}^d)$*

$$N_{-\Delta_h + V}\left(\frac{(d-1)^2}{4}\right) \leq C \int_{\mathbf{H}^d} V_-(x, y)^{d/2} \frac{dx dy}{y^d}.$$

A proof of this inequality can be found in the article [LS97] by Levin and Solomyak. Here the authors also prove a Weyl-type asymptotic formula

Theorem 6.6. *If $d \geq 3$ and $V \in L^{d/2}(\mathbf{H}^d)$, then*

$$\lim_{\alpha \rightarrow \infty} \alpha^{-d/2} N_{-\Delta_h + \alpha V} \left(\frac{(d-1)^2}{4} \right) = L_{0,d}^c \int_{\mathbf{H}^d} V_-(x, y)^{d/2} \frac{dx dy}{y^d}.$$

Remark 39. It is remarkable, that the same constant, $L_{0,d}^c$, appears as in the euclidean case.

6.4 Lieb–Thirring Inequalities in Hyperbolic Space

We have now reached the core subject of this chapter. Given a potential $V \in L^{d/2+\gamma}(\mathbf{H}^d)$, we would like to estimate sums of the form

$$\sum_j (\lambda_j(V) - (d-1)^2/4)_-^\gamma,$$

where $\lambda_j(V)$ are the eigenvalues of the Schrödinger operator $-\Delta_h + V$. Note that we have to shift the eigenvalues by the constant $(d-1)^2/4$ before we consider their powers and sums thereof. It turns out that these shifts of eigenvalues, let us denote them by

$$\mu_j(V) = \frac{(d-1)^2}{4} - \lambda_j(V),$$

satisfy an inequality that is strikingly similar to the euclidean Lieb–Thirring inequality. The following theorem is the main assertion of the article [\[ILW24\]](#).

Theorem 6.7. *Let $d \geq 2$, $\gamma \geq 1/2$ and let $V \in L^{d/2+\gamma}(\mathbf{H}^d, dV_h)$ be real-valued. The numbers $\mu_k(V)$ of $-\Delta_h + V$ satisfy*

$$\sum_k \mu_k(V)^\gamma \leq L_{\gamma,d} \int_{\mathbf{H}^d} V(x, y)_-^{d/2+\gamma} dV_h(x, y),$$

where the constant $L_{\gamma,d}$ is the same as in the euclidean case.

For reasons that will become clear later, we can only consider the case $\gamma \geq 1/2$. Before proving this result, let us discuss some of its consequences.

Studying Lieb–Thirring inequalities on Riemannian manifolds that are not the euclidean space is not a new idea. Indeed, Lieb–Thirring inequalities on cylinders, spheres, and tori have been studied by Laptev, Loss, Ilyin, and Zelik in the articles [\[ILLZ16\]](#) and [\[ILZ20\]](#). In these cases, the constant in the Lieb–Thirring inequality is generally larger (worse) than

in the euclidean case. Remarkably, in our situation the constants are the same as in the euclidean case. The optimal constants are thus at least as good as in the euclidean case.

6.5 Operator-Valued Potentials

We would like to use a lifting argument similar to the one used by Laptev and Weidl to lift results from $\mathbf{R}^{d-1}$ to $\mathbf{R}^d$ in [LW00]. For this, we need some basic notions about vector-valued functions.

Let $\mathcal{H}$ be a separable Hilbert space. As a topological space, $\mathcal{H}$ can be equipped with the Borel σ -algebra. One can thus consider measurable functions taking values in $\mathcal{H}$. Generally, when considering functions taking values in Banach spaces, one uses a weaker definition of measurability called *Bochner measurability*. Since $\mathcal{H}$ is separable, however, these two notions coincide. The book by Hytönen, van Neerven, Veraar, and Weis [HvNVW16] deals with the analysis of functions taking values in more general Banach spaces.

Let $u: \mathbf{R} \rightarrow \mathcal{H}$ be a measurable function. The map $t \mapsto \langle v, u(t) \rangle_{\mathcal{H}}$ is measurable for all $v \in \mathcal{H}$. Indeed, any $\langle v, \cdot \rangle$ is a bounded (continuous) linear form on $\mathcal{H}$ and therefore $t \mapsto \langle v, u(t) \rangle_{\mathcal{H}}$ is measurable as a composition of a measurable and a continuous function. Given two measurable functions $u, v: \mathbf{R} \rightarrow \mathcal{H}$, the map $t \mapsto \langle v(t), u(t) \rangle_{\mathcal{H}}$ is measurable, since $\langle \cdot, \cdot \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathbf{C}$ is continuous.

Those measurable functions $u: \mathbf{R} \rightarrow \mathcal{H}$ for which

$$\int_{\mathbf{R}} \|u(t)\|_{\mathcal{H}}^2 dt < \infty,$$

form a Hilbert space $L^2(\mathbf{R}, \mathcal{H})$ (after identifying functions agreeing almost everywhere, just like in the scalar case), when equipped with the inner product

$$\langle u, v \rangle_{L^2(\mathbf{R}, \mathcal{H})} = \int_{\mathbf{R}} \langle u(t), v(t) \rangle_{\mathcal{H}} dt.$$

The proof of this assertion carries over verbatim from the corresponding result about $L^2(\mathbf{R})$. Note that by the Fubini–Tonelli theorem we have

$$L^2(\mathbf{R}, L^2(\mathbf{R}^{d-1})) = L^2(\mathbf{R}^d).$$

Similarly, one defines $L^1(\mathbf{R}, \mathcal{H})$ as the space of those measurable functions $u: \mathbf{R} \rightarrow \mathcal{H}$ for

which the norm (after identifying functions agreeing almost everywhere)

$$\int_{\mathbf{R}} \|u(t)\|_{\mathcal{H}} dt$$

is finite. In this case we can make sense of the integral

$$\int_{\mathbf{R}} u(t) dt.$$

Indeed, the map

$$\mathcal{H} \ni v \mapsto \int_{\mathbf{R}} \langle u(t), v \rangle_{\mathcal{H}} dt \in \mathbf{C}$$

is a bounded linear functional on $\mathcal{H}$. Riesz' representation theorem now allows us to define

$$\langle \int_{\mathbf{R}} u(t) dt, v \rangle_{\mathcal{H}} = \int_{\mathbf{R}} \langle u(t), v \rangle_{\mathcal{H}} dt.$$

We write $u \in L^1_{loc}(\mathbf{R}, \mathcal{H})$ if $\chi_K u \in L^1(\mathbf{R}, \mathcal{H})$ for all compact $K \subseteq \mathbf{R}$. Obviously, $L^2(\mathbf{R}, \mathcal{H}) \subseteq L^1_{loc}(\mathbf{R}, \mathcal{H})$. We say that $w \in L^1_{loc}(\mathbf{R}, \mathcal{H})$ is the *weak derivative* of a function $u \in L^1_{loc}(\mathbf{R}, \mathcal{H})$ if

$$\int_{\mathbf{R}} \phi(t) w(t) dt = - \int_{\mathbf{R}} \phi'(t) u(t) dt,$$

for all $\phi \in C_c^\infty(\mathbf{R})$. If for given u such a w exists, then it is unique and we denote it by $w = u'$. We can now define Sobolev spaces of vector-valued functions. We define

$$H^1(\mathbf{R}, \mathcal{H}) = \{u \in L^2(\mathbf{R}, \mathcal{H}) \mid u' \in L^2(\mathbf{R}, \mathcal{H})\}$$

and

$$\langle u, v \rangle_{H^1(\mathbf{R}, \mathcal{H})} = \langle u, v \rangle_{L^2(\mathbf{R}, \mathcal{H})} + \langle u', v' \rangle_{L^2(\mathbf{R}, \mathcal{H})}.$$

Just like in the scalar case, $H^1(\mathbf{R}, \mathcal{H})$ is a Hilbert space. On $H^1(\mathbf{R}, \mathcal{H})$, one may define the quadratic form

$$q[u] = \int_{\mathbf{R}} \|u'(t)\|_{\mathcal{H}}^2 dt.$$

Like in the scalar case, q is a closed form. The corresponding self-adjoint operator on $L^2(\mathbf{R}, \mathcal{H})$ shall be denoted by $-\frac{d^2}{dt^2}$.

We would like to study operator-valued functions. A map $V: \mathbf{R} \rightarrow \mathcal{L}(\mathcal{H})$ is said to be measurable, if for all $u \in \mathcal{H}$, the map $t \mapsto V(t)u$ is measurable. If $V: \mathbf{R} \rightarrow \mathcal{L}(\mathcal{H})$ and $u: \mathbf{R} \rightarrow \mathcal{H}$ are measurable, then so is $t \mapsto V(t)u(t)$. Indeed, expanding $u(t)$ with respect

to an orthonormal basis (e_n) of $\mathcal{H}$ we find

$$V(t)u(t) = \sum_n \langle e_n, u(t) \rangle V(t)e_n$$

and hence $V(t)u(t)$ is the pointwise limit of measurable functions. Consequently, if $u, v: \mathbf{R} \rightarrow \mathcal{H}$ are measurable functions and if $V: \mathbf{R} \rightarrow \mathcal{L}(\mathcal{H})$ is measurable, then $t \mapsto \langle v(t), V(t)u(t) \rangle_{\mathcal{H}}$ is measurable.

Suppose that $V(t)$ is a non-negative operator for all t . We find

$$\begin{aligned} |\langle v(t), V(t)u(t) \rangle_{\mathcal{H}}| &\leq \|v(t)\|_{\mathcal{H}} \|V(t)u(t)\|_{\mathcal{H}} \\ &\leq \|V(t)\|_{\mathcal{L}(\mathcal{H})} \|v(t)\|_{\mathcal{H}} \|u(t)\|_{\mathcal{H}} \\ &\leq \|V(t)\|_{S^p(\mathcal{H})} \|v(t)\|_{\mathcal{H}} \|u(t)\|_{\mathcal{H}}, \end{aligned}$$

where $\|V(t)\|_{S^p(\mathcal{H})} = (\text{tr}_{\mathcal{H}} V(t)^p)^{1/p}$ is the p -Schatten norm of $V(t)$ for $p \geq 1$. Note the following result whose proof can be found in Weidmann's book [Wei00]

$$\|V(t)\|_{S^p(\mathcal{H})} = \sup\{(\sum_n |\langle u_n, V(t)v_n \rangle_{\mathcal{H}}|^p)^{1/p} \mid (u_n), (v_n) \text{ orthonormal bases of } \mathcal{H}\}.$$

The function $t \mapsto \|V(t)\|_{S^p(\mathcal{H})}$ is therefore the pointwise supremum of measurable functions and thus measurable itself, as is $\|V(\cdot)\|_{\mathcal{L}(\mathcal{H})}$, since

$$\|V(t)\|_{\mathcal{L}(\mathcal{H})} = \sup_{\|u\|_{\mathcal{H}}=1} \|V(t)u\|_{\mathcal{H}}$$

is the pointwise supremum of measurable functions.

We would like to make sense of Schrödinger operators with operator-valued potentials. In order to use the form method of defining Schrödinger operators, the last ingredient we need is some sort of Sobolev embedding $H^1(\mathbf{R}, \mathcal{H}) \rightarrow L^\infty(\mathbf{R}, \mathcal{H})$, where $L^\infty(\mathbf{R}, \mathcal{H})$ is the space of all measurable $u: \mathbf{R} \rightarrow \mathcal{H}$ for which $t \mapsto \|u(t)\|_{\mathcal{H}}$ is essentially bounded, equipped with the essential supremum norm: $\|u\|_{L^\infty(\mathbf{R}, \mathcal{H})} = \text{ess-sup}_t \|u(t)\|_{\mathcal{H}}$. Suppose that $u \in H^1(\mathbf{R}, \mathcal{H})$. By definition

$$\int_{\mathbf{R}} \phi(t) \langle v, u'(t) \rangle_{\mathcal{H}} dt = - \int_{\mathbf{R}} \phi'(t) \langle v, u(t) \rangle_{\mathcal{H}} dt,$$

for all $\phi \in C_c^\infty(\mathbf{R})$ and $v \in \mathcal{H}$. Therefore, $\langle v, u(\cdot) \rangle_{\mathcal{H}}$ is weakly differentiable for all $v \in \mathcal{H}$ with $\langle v, u(\cdot) \rangle'_{\mathcal{H}} = \langle v, u'(\cdot) \rangle_{\mathcal{H}}$. Furthermore, $\langle v, u(\cdot) \rangle_{\mathcal{H}}, \langle v, u'(\cdot) \rangle_{\mathcal{H}} \in L^2(\mathbf{R})$ by the Cauchy–

Schwarz inequality. In summary, $\langle v, u'(\cdot) \rangle_{\mathcal{H}} \in H^1(\mathbf{R})$. The classical Sobolev embedding theorem in one dimension states that there exists a constant $C > 0$ such that for all $\psi \in H^1(\mathbf{R})$

$$\|\psi\|_{L^\infty(\mathbf{R}^d)} \leq C \|\psi\|_{H^1(\mathbf{R})}.$$

Note that for $\psi = \langle v, u(\cdot) \rangle_{\mathcal{H}}$

$$\begin{aligned} \|\psi\|_{H^1(\mathbf{R})}^2 &= \int_{\mathbf{R}} |\psi(t)|^2 dt + \int_{\mathbf{R}} |\psi'(t)|^2 dt \\ &= \int_{\mathbf{R}} |\langle v, u(t) \rangle_{\mathcal{H}}|^2 dt + \int_{\mathbf{R}} |\langle v, u'(t) \rangle_{\mathcal{H}}|^2 dt \\ &\leq \int_{\mathbf{R}} \|u(t)\|_{\mathcal{H}}^2 dt + \int_{\mathbf{R}} \|u'(t)\|_{\mathcal{H}}^2 dt. \end{aligned}$$

Therefore

$$\|\psi\|_{H^1(\mathbf{R})} \leq \|u\|_{H^1(\mathbf{R}, \mathcal{H})}.$$

On the other hand,

$$\|u(t)\|_{\mathcal{H}} = \sup_{\|v\|_{\mathcal{H}}=1} |\langle v, u(t) \rangle_{\mathcal{H}}|$$

and thus

$$\begin{aligned} \text{ess-sup}_t \|u(t)\|_{\mathcal{H}} &= \text{ess-sup}_t \sup_{\|v\|_{\mathcal{H}}=1} |\langle v, u(t) \rangle_{\mathcal{H}}| \\ &= \sup_{\|v\|_{\mathcal{H}}=1} \text{ess-sup}_t |\langle v, u(t) \rangle_{\mathcal{H}}| \\ &\leq C \|u\|_{H^1(\mathbf{R}, \mathcal{H})}. \end{aligned}$$

In conclusion,

$$\|u\|_{L^\infty(\mathbf{R}, \mathcal{H})} \leq C \|u\|_{H^1(\mathbf{R}, \mathcal{H})}$$

for all $u \in H^1(\mathbf{R}, \mathcal{H})$.

Suppose now that $V: \mathbf{R} \rightarrow \mathcal{L}(\mathcal{H})$ is measurable and such that $\|V(\cdot)\|_{\mathcal{L}(\mathcal{H})} \in L^p(\mathbf{R})$ for some $p \geq 1$ and such that $V(t)$ is a non-negative operator almost everywhere. For every $\varepsilon > 0$ there exists a constant $C_\varepsilon \geq 0$ such that

$$\left| \int_{L^2(\mathbf{R})} \langle u(t), V(t)u(t) \rangle_{\mathcal{H}} dt \right| \leq \varepsilon \int_{L^2(\mathbf{R})} \|u'(t)\|_{\mathcal{H}}^2 dt + C_\varepsilon \int_{L^2(\mathbf{R})} \|u(t)\|_{\mathcal{H}}^2 dt,$$

for all $u \in H^1(\mathbf{R}, \mathcal{H})$. Note that the condition $\|V(\cdot)\|_{\mathcal{L}(\mathcal{H})} \in L^p(\mathbf{R})$ is satisfied whenever

$\mathrm{tr}_{\mathcal{H}} V(\cdot)^p \in L^1(\mathbf{R})$. Just like in the scalar-valued case, we find that

$$q[u] = \int_{L^2(\mathbf{R})} \|u'(t)\|_{\mathcal{H}}^2 dt - \int_{L^2(\mathbf{R})} \langle u(t), V(t)u(t) \rangle_{\mathcal{H}} dt$$

defined on $H^1(\mathbf{R}, \mathcal{H})$ is a closed quadratic form. The unique lower-semibounded self-adjoint operator associated with q shall be denoted $-\frac{d^2}{dt^2} - V(t)$.

We are now able to formulate a Lieb–Thirring inequality for Schrödinger operators with operator-valued potentials. It was proven by Laptev and Weidl in [LW00].

Lemma 6.8. *Let $\gamma \geq 1/2$. There exists a constant $L_{\gamma,1}$ such that for operator-valued measurable functions $V: \mathbf{R} \rightarrow \mathcal{L}(\mathcal{H})$ for which $V(t) \geq 0$ is a non-negative operator and $\mathrm{tr}_{\mathcal{H}} V(\cdot)^{1/2+\gamma} \in L^1(\mathbf{R})$ we have*

$$\mathrm{tr}_{L^2(\mathbf{R}, \mathcal{H})} \left(-\frac{d^2}{dt^2} - V(t) \right)_{-}^{\gamma} \leq L_{\gamma,1} \int_{\mathbf{R}} \mathrm{tr}_{\mathcal{H}} (V(t))^{1/2+\gamma} dt.$$

Remark 40. Just like in the scalar case, for $\gamma \geq 3/2$ the optimal constant equals $L_{3/2,1}^c$ and is sharp. This is proven by Laptev and Weidl in [LW00]. For $\gamma = 1/2$, the best constant is $2L_{1/2,1}^c$ and it is sharp, which was proven by Hundertmark, Laptev, and Weidl in [HLW00]. The Aizenman–Lieb argument shows that the optimal constant in the case $1/2 < \gamma < 1$ is at most $2L_{1/2,1}^c$. If $1 \leq \gamma < 3/2$, the optimal constant is at most $R_{1,1}L_{\gamma,1}^c$.

6.6 Proof of the Lieb–Thirring Inequality in Hyperbolic Space

We are finally in a position to prove the main theorem of this chapter, the Lieb–Thirring inequality in hyperbolic space [6.7]. Suppose that $d \geq 2$ and that $\gamma \geq 1/2$. Suppose that $V \in L^{d/2+\gamma}(\mathbf{H}^d)$ be a real-valued function and let $\lambda_1 \leq \lambda_2 \leq \dots \leq \frac{(d-1)^2}{4}$ be the eigenvalues of the operator $-\Delta_h + V$ below $(d-1)^2/4$.

If $\mu_j = \frac{(d-1)^2}{4} - \lambda_j$, then the numbers $-\mu_1 \leq -\mu_2 \leq \dots \leq 0$ are the negative eigenvalues of the operator

$$-\partial_t^2 - e^{2t}\tilde{\Delta} + V(x, e^t).$$

Consequently the Riesz means satisfy

$$\begin{aligned}
\sum_j \mu_j^\gamma &= \operatorname{tr}_{L^2(\mathbf{R}^d)} \left(-\partial_t^2 - e^{2t} \tilde{\Delta} + V(x, e^t) \right)_-^\gamma \\
&\leq \operatorname{tr}_{L^2(\mathbf{R}^d)} \left(-\partial_t^2 - (e^{2t} \tilde{\Delta} - V(x, e^t))_+ \right)_-^\gamma \\
&= \operatorname{tr}_{L^2(\mathbf{R}, L^2(\mathbf{R}^{d-1}))} \left(-\partial_t^2 - (e^{2t} \tilde{\Delta} - V(x, e^t))_+ \right)_-^\gamma \\
&\leq L_{\gamma,1} \int_{\mathbf{R}} \operatorname{tr}_{L^2(\mathbf{R}^{d-1})} (e^{2t} \tilde{\Delta} - V(x, e^t))_+^{1/2+\gamma} dt.
\end{aligned}$$

The first inequality here is due to the variational principle and the second inequality is precisely the Lieb–Thirring inequality for Schrödinger operators with vector-valued potentials. For the integrand in the last term we find

$$\begin{aligned}
\operatorname{tr}_{L^2(\mathbf{R}^{d-1})} (e^{2t} \tilde{\Delta} - V(x, e^t))_+^{1/2+\gamma} &= e^{2t(\gamma+1/2)} \operatorname{tr}_{L^2(\mathbf{R}^{d-1})} (-\tilde{\Delta} + e^{-2t} V(x, e^t))_-^{1/2+\gamma} \\
&\leq e^{2t(\gamma+1/2)} L_{\gamma+1/2, d-1} \int_{\mathbf{R}^{d-1}} e^{-2t(\gamma+d/2)} V(x, e^t)_-^{\gamma+d/2} dx \\
&= e^{t(1-d)} L_{\gamma+1/2, d-1} \int_{\mathbf{R}^{d-1}} V(x, e^t)_-^{\gamma+d/2} dx
\end{aligned}$$

due to the standard Lieb–Thirring inequality in $\mathbf{R}^{d-1}$. Finally, we obtain

$$\begin{aligned}
\sum_j \mu_j^\gamma &\leq L_{\gamma,1} \int_{\mathbf{R}} e^{t(1-d)} L_{\gamma+1/2, d-1} \int_{\mathbf{R}^{d-1}} V(x, e^t)_-^{\gamma+d/2} dx dt \\
&= L_{\gamma,1} L_{\gamma+1/2, d-1} \int_{\mathbf{R}^d} e^{t(1-d)} V(x, e^t)_-^{\gamma+d/2} dx dt \\
&= L_{\gamma,1} L_{\gamma+1/2, d-1} \int_{\mathbf{R}^d} V(x, y)_-^{\gamma+d/2} \frac{dx dy}{y^d}.
\end{aligned}$$

Recall that

$$L_{\gamma, d}^c = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2} \Gamma(\gamma + d/2 + 1)}$$

and hence

$$L_{\gamma,1}^c L_{\gamma+1/2, d-1}^c = L_{\gamma, d}^c.$$

For $\gamma \geq 3/2$, the optimal constant in the Lieb–Thirring inequality in hyperbolic space is therefore simply the semiclassical constant and it is sharp. In the other cases, the best

known constants for Lieb–Thirring inequalities in euclidean space are

$$\begin{cases} 2L_{\gamma,d}^c, & 1/2 \leq \gamma < 1, \ d = 1 \\ 2R_{1,1}L_{\gamma,d}^c, & 1/2 \leq \gamma < 1, \ d \geq 2 \\ R_{1,1}L_{\gamma,d}^c, & 1 \leq \gamma < 3/2, \ d \geq 1. \end{cases}$$

Consequently, the optimal constant in the hyperbolic case for $1 \leq \gamma < 3/2$ is at most

$$R_{1,1}L_{\gamma,1}^c \cdot L_{\gamma+1/2,d-1}^c = R_{1,1}L_{\gamma,d}^c$$

and for $1/2 \leq \gamma < 1$ it is at most

$$2L_{\gamma,1}^c \cdot R_{1,1}L_{\gamma+1/2,d-1}^c = 2R_{1,1}L_{\gamma,d}^c.$$

The optimal constants in the hyperbolic case are therefore at least as good as in the euclidean case. This concludes the proof of theorem [6.7](#). $\square$

Remark 41. Because we used a lifting argument based on Lieb–Thirring inequalities on $\mathbf{R}^1$, we could only consider the case $\gamma \geq 1/2$ even on $\mathbf{H}^d$ with $d \geq 2$.

6.7 A Kinetic Inequality

Just like in the euclidean case, the Lieb–Thirring inequality in hyperbolic space in the case $\gamma = 1$ is equivalent to an inequality for the kinetic energy. The proof follows the same strategy as in the euclidean case (See [\[FLW23\]](#) for a discussion of the euclidean case).

Theorem 6.9. *For all $d \geq 2$, there exists a constant K_d such that for all $N \in \mathbf{N}$ and $u_1, \dots, u_N \in H^1(\mathbf{H}^d)$ that are orthonormal in $L^2(\mathbf{H}^d)$*

$$\sum_{j=1}^N \int_{\mathbf{H}^d} y^{2-d} \left(|\tilde{\nabla} u_j(x)|^2 + |\partial_y u_j|^2 \right) dx dy \geq K_d \int_{\mathbf{H}^d} \rho^{1+2/d}(x, y) \frac{dx dy}{y^d} + N \frac{(d-1)^2}{4},$$

where

$$\rho(x, y) = \sum_{j=1}^N |u_j(x, y)|^2.$$

Proof. Let $u_1, \dots, u_N \in H^1(\mathbf{H}^d)$ be orthonormal in $L^2(\mathbf{H}^d)$. Let $0 \leq V \in L^{d/2+1}(\mathbf{H}^d)$ and let $\lambda_1, \dots, \lambda_N$ be its first N eigenvalues of the Schrödinger operator $-\Delta_h - V$ below

$(d-1)^2/4$. By the variational principle, we have

$$\begin{aligned} \sum_{j=1}^N \lambda_j &\leq \sum_{j=1}^N \int_{\mathbf{H}^d} (|\text{grad } u_j(x, y)|_h^2 - V(x, y)|u_j(x, y)|^2) dV_h(x, y) \\ &= \sum_{j=1}^N \int_{\mathbf{H}^d} \left(y^2 (|\tilde{\nabla} u_j(x)|^2 + |\partial_y u_j|^2) - V(x, y)|u_j(x, y)|^2 \right) \frac{dx dy}{y^d}. \end{aligned}$$

On the other hand, by the Lieb–Thirring inequality we have

$$\sum_{j=1}^N \left(\frac{(d-1)^2}{4} - \lambda_j \right) \leq L_{1,d} \int_{\mathbf{H}^d} V(x, y)^{d/2+1} \frac{dx dy}{y^d}.$$

Combining these two inequalities, we obtain

$$\begin{aligned} &\sum_{j=1}^N \int_{\mathbf{H}^d} \left(y^2 (|\tilde{\nabla} u_j(x)|^2 + |\partial_y u_j|^2) - V(x, y)|u_j(x, y)|^2 \right) \frac{dx dy}{y^d} \\ &\geq N \frac{(d-1)^2}{4} - L_{1,d} \int_{\mathbf{H}^d} V(x, y)^{d/2+1} \frac{dx dy}{y^d}, \end{aligned}$$

and thus

$$\begin{aligned} &\sum_{j=1}^N \int_{\mathbf{H}^d} y^2 (|\tilde{\nabla} u_j(x)|^2 + |\partial_y u_j|^2) \frac{dx dy}{y^d} \\ &\geq N \frac{(d-1)^2}{4} + \int_{\mathbf{H}^d} \left(V(x, y) \rho(x) - L_{1,d} V(x, y)^{d/2+1} \right) \frac{dx dy}{y^d}, \end{aligned}$$

where

$$\rho(x, y) = \sum_{j=1}^N |u_j(x, y)|^2.$$

Choosing the potential V to be given by

$$V(x, y) = \left(\frac{\rho(x, y)}{L_{1,d}(1 + d/2)} \right)^{1+2/d},$$

we find

$$\begin{aligned} & \sum_{j=1}^N \int_{\mathbf{H}^d} y^2 \left(|\tilde{\nabla} u_j(x)|^2 + |\partial_y u_j|^2 \right) \frac{dx dy}{y^d} \\ & \geq N \frac{(d-1)^2}{4} + \frac{d/2}{(1+d/2)^{2/d+1}} L_{1,d}^{-2/d} \int_{\mathbf{H}^d} \rho(x, y)^{1+2/d} \frac{dx dy}{y^d}. \end{aligned}$$

Setting

$$K_d = \frac{d/2}{(1+d/2)^{2/d+1}} L_{1,d}^{-2/d}$$

yields the claim. $\square$

One simple consequence of this inequality is that if we set $N = 1$, we obtain the following Sobolev-type inequality: If $u \in H^1(\mathbf{H}^d)$, $u \neq 0$, then

$$\begin{aligned} & \int_{\mathbf{H}^d} y^2 \left(|\tilde{\nabla} u(x)|^2 + |\partial_y u|^2 \right) \frac{dx dy}{y^d} \\ & \geq \frac{(d-1)^2}{4} \int_{\mathbf{H}^d} |u(x, y)|^2 \frac{dx dy}{y^d} + K_d \int_{\mathbf{H}^d} |u(x, y)|^{2+4/d} \frac{dx dy}{y^d} \left(\int_{\mathbf{H}^d} |u(x, y)|^2 \frac{dx dy}{y^d} \right)^{-2/d}. \end{aligned}$$

This inequality is not new. Indeed, in the case $d \geq 3$, this inequality can also easily be obtained from the standard Sobolev inequality on $\mathbf{H}^d$ (see [\[Aub76\]](#)) using Hölder's inequality.

7 Weyl's Law for the Hyperbolic Laplacian

This chapter is taken from the preprint [\[Wei24\]](#).

Motivated by the preceding chapter, we are interested in the operator

$$H = -\partial_{x_1}^2 - e^{2x_1} \Delta_{\tilde{x}}$$

where $\Delta_{\tilde{x}} = \sum_{j=2}^d \partial_{x_j}^2$ on the space $L^2(\mathbf{R}^d)$. Just like for the usual Laplacian in chapter 4, we intend to prove a Weyl law for the Dirichlet realization of H on a bounded domain $\Omega \subseteq \mathbf{R}^d$.

To this end we will, once again, employ coherent state methods. Consider an open bounded set $\Omega \subseteq \mathbf{R}^d$. The operator H is clearly elliptic on any bounded Ω . A classical theorem from the theory of elliptic differential operators states that the spectrum of the Dirichlet realization of an elliptic differential operator on a bounded domain is discrete. See Evans' classic book [\[Eva10\]](#) for a proof of this fact. In our case, this can easily be seen in the following way: for any $u \in C_c^\infty(\mathbf{R}^d)$ one has

$$\langle u, Hu \rangle_{L^2(\mathbf{R}^d)} = \int_{\mathbf{R}^d} \left(|\partial_{x_1} u(x)|^2 + e^{2x_1} \sum_{j=2}^d |\partial_{x_j} u(x)|^2 \right) dx.$$

Suppose now that $\Omega \subseteq \mathbf{R}^d$ is a *bounded* domain. Then there exist numbers $r, R \in \mathbf{R}$ such that $r < x_1 < R$ for all $x \in \Omega$. Consequently, if $u \in C_c^\infty(\Omega)$, then

$$\frac{e^r}{e^r + 1} \int_{\Omega} |\nabla u(x)|^2 dx \leq \int_{\Omega} \left(|\partial_{x_1} u(x)|^2 + e^{2x_1} \sum_{j=2}^d |\partial_{x_j} u(x)|^2 \right) dx \leq (1 + e^R) \int_{\Omega} |\nabla u(x)|^2 dx.$$

Therefore,

$$u \mapsto \int_{\Omega} \left(|\partial_{x_1} u(x)|^2 + e^{2x_1} \sum_{j=2}^d |\partial_{x_j} u(x)|^2 \right) dx$$

defines a closed quadratic form on $H_0^1(\Omega)$. Furthermore, the spectrum of the corresponding self-adjoint operator is discrete, since $H_0^1(\Omega)$ embeds compactly into $L^2(\Omega)$ and the quadratic form discussed here is equivalent to the (square of the) Sobolev norm. Clearly, the operator is non-negative. The spectrum thus consists of numbers $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ tending to $+\infty$. In the rest of this chapter we will be interested in the asymptotic behavior

of the Riesz means

$$\sum_{k=1}^{\infty} (\lambda - \lambda_k)_+$$

as $\lambda \rightarrow +\infty$.

The main theorem of this chapter is the following.

Theorem 7.1. *Let $\Omega \subseteq \mathbf{R}^d$ be a bounded open set and let λ_k , $k \in \mathbf{N}$ be the eigenvalues of the operator H on Ω with Dirichlet boundary conditions. Then*

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1-d/2} \sum_k (\lambda - \lambda_k)_+ = \int_{\mathbf{R}^d \times \Omega} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2\right)_+ \frac{d\xi dy}{(2\pi)^d}.$$

We prove this in two parts. First we prove the lower bound

$$\liminf_{\lambda \rightarrow \infty} \lambda^{-1-d/2} \sum_k (\lambda - \lambda_k)_+ \geq \int_{\mathbf{R}^d \times \Omega} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2\right)_+ \frac{d\xi dy}{(2\pi)^d}$$

and then the upper bound

$$\limsup_{\lambda \rightarrow \infty} \lambda^{-1-d/2} \sum_k (\lambda - \lambda_k)_+ \leq \int_{\mathbf{R}^d \times \Omega} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2\right)_+ \frac{d\xi dy}{(2\pi)^d}.$$

The integral term can be rewritten, performing the change of variables $\eta_1 = \xi_1$, $\tilde{\eta} = e^{y_1} \tilde{\xi}$, as

$$\begin{aligned} \int_{\mathbf{R}^d \times \Omega} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2\right)_+ \frac{d\xi dy}{(2\pi)^d} &= \int_{\Omega} \int_{\mathbf{R}^d} \left(1 - |\eta|^2\right)_+ \frac{d\eta}{(2\pi)^d} e^{(1-d)y_1} dy \\ &= L_{1,d}^c \int_{\Omega} e^{(1-d)y_1} dy \\ &= L_{1,d}^c V_h(\Omega'), \end{aligned}$$

where $V_h(\Omega')$ is the Riemannian volume in the hyperbolic space $\mathbf{H}^d$ of the set $\Omega' = \{(x_1, \tilde{x}) \in \mathbf{R}^d \mid (\ln(x_1), \tilde{x}) \in \Omega\}$. It is easy to see that Ω' is precisely the preimage of Ω under the coordinate transform discussed in the last chapter.

Remark 42. Note that this formula also follows from a more general discussion ([\[Brü74\]](#)) of the asymptotic behavior of the spectral functions of elliptic differential operators with smooth coefficients. Our approach, however, exemplifies the utility of the coherent state approach, a theory that has classically been applied to powers of euclidean Laplacians and Schrödinger operators.

In the final section of this chapter we give an estimate on the error term in the asymptotic formula given above.

7.1 Lower Trace Bound

We are interested in the action of the operator H on a coherent state of the form $e_{\xi,y}(x) = e^{i\xi x}g(x-y)$, where $g \in C_c^\infty(\mathbf{R}^d)$ with $\|g\|_{L^2(\mathbf{R}^d)} = 1$.

Lemma 7.2.

$$\begin{aligned} & \langle e_{\xi,y}, H e_{\xi,y} \rangle_{L^2(\mathbf{R}^d)} \\ &= \xi_1^2 + e^{2y_1} |\tilde{\xi}|^2 \int_{\mathbf{R}^d} e^{2z_1} g(z)^2 dz + e^{2y_1} \int_{\mathbf{R}^d} e^{2z_1} |\nabla_{\bar{z}} g(z)|^2 dz + \int_{\mathbf{R}^d} (\partial_{z_1} g(z))^2 dz. \end{aligned}$$

Proof. The contribution of $-\partial_{x_1}^2$ was already determined in [3.4]. We compute the expression $\langle e_{\xi,y}, -e^{2x_1} \partial_{x_j}^2 e_{\xi,y} \rangle_{L^2(\mathbf{R}^d)}$ for $j > 1$. Equation [1] in the proof of Lemma [3.4] implies

$$\begin{aligned} \langle e_{\xi,y}, -e^{2x_1} \partial_{x_j}^2 e_{\xi,y} \rangle &= \xi_j^2 \int_{\mathbf{R}^d} e^{2x_1} (g(x-y))^2 dx + 2i\xi_j \int_{\mathbf{R}^d} e^{2x_1} \partial_{x_j} g(x-y) g(x-y) dx \\ &\quad - \int_{\mathbf{R}^d} e^{2x_1} \partial_{x_j}^2 g(x-y) g(x-y) dx. \end{aligned}$$

The first term on the right equals

$$e^{2y_1} \xi_j^2 \int_{\mathbf{R}^d} e^{2(x_1-y_1)} (g(x-y))^2 dx = e^{2y_1} \xi_j^2 \int_{\mathbf{R}^d} e^{2z_1} (g(z))^2 dz.$$

For the second integral we find

$$\int_{\mathbf{R}^d} e^{2x_1} \partial_{x_j} g(x-y) g(x-y) dx = \int_{\mathbf{R}} e^{2x_1} \int_{\mathbf{R}^{d-1}} \partial_{x_j} g(x-y) g(x-y) d\tilde{x} dx_1,$$

where the inner integral vanishes due to symmetry of g . Finally, for the third integral we find, again by splitting the integral, that

$$\begin{aligned} \int_{\mathbf{R}^d} e^{2x_1} \partial_{x_j}^2 g(x-y) g(x-y) dx &= -e^{2y_1} \int_{\mathbf{R}} e^{2(x_1-y_1)} \int_{\mathbf{R}^{d-1}} \partial_{x_j}^2 g(x-y) g(x-y) d\tilde{x} dx_1 \\ &= e^{2y_1} \int_{\mathbf{R}} e^{2(x_1-y_1)} \int_{\mathbf{R}^{d-1}} (\partial_{x_j} g(x-y))^2 d\tilde{x} dx_1 \\ &= e^{2y_1} \int_{\mathbf{R}^d} e^{2z_1} (\partial_{z_j} g(z))^2 dz \end{aligned}$$

Summing over $j = 2, \dots, d$ completes the argument. $\square$

Theorem 7.3. *Let $\Omega \subseteq \mathbf{R}^d$ be a bounded open set and let $\lambda_k, k \in \mathbf{N}$ be the eigenvalues of the operator H on Ω with Dirichlet boundary conditions. Then*

$$\liminf_{\lambda \rightarrow \infty} \lambda^{-1-d/2} \sum_k (\lambda - \lambda_k)_+ \geq \int_{\mathbf{R}^d \times \Omega} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2\right)_+ \frac{d\xi dy}{(2\pi)^d}.$$

Proof. Let us again consider a g supported in the unit ball and construct $e_{\xi,y}^\varepsilon$ accordingly. By the same argument as for the Dirichlet Laplacian in the proof of Proposition 4.12, we have

$$\sum_k (\lambda - \lambda_k)_+ \geq \int_{\mathbf{R}^d \times \Omega_\varepsilon} \left(\langle e_{\xi,y}^\varepsilon, (\lambda - H)e_{\xi,y}^\varepsilon \rangle_{L^2(\mathbf{R}^d)} \right)_+ \frac{d\xi dy}{(2\pi)^d}.$$

Using the expression obtained for the symbol of H in 7.2 then gives

$$\sum_k (\lambda - \lambda_k)_+ \geq \int_{\mathbf{R}^d \times \Omega_\varepsilon} \left(\lambda - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 c_3^\varepsilon - e^{2y_1} c_2^\varepsilon - c_1^\varepsilon \right)_+ \frac{d\xi dy}{(2\pi)^d}, \quad (7)$$

where

$$\begin{aligned} c_1^\varepsilon &= \int_{\mathbf{R}^d} (\partial_{z_1} g^\varepsilon(z))^2 dz \\ c_2^\varepsilon &= \int_{\mathbf{R}^d} e^{2z_1} |\nabla_{\bar{z}} g^\varepsilon(z)|^2 dz \\ c_3^\varepsilon &= \int_{\mathbf{R}^d} e^{2z_1} g^\varepsilon(z)^2 dz. \end{aligned}$$

Note that since $g(z)^2$ is a mollifier, we have $c_3^\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$. After a change of coordinates, we obtain

$$\sum_k (\lambda - \lambda_k)_+ \geq \lambda^{1+d/2} \int_{\mathbf{R}^d \times \Omega_\varepsilon} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 c_3^\varepsilon - \frac{1}{\lambda} e^{2y_1} c_2^\varepsilon - \frac{1}{\lambda} c_1^\varepsilon \right)_+ \frac{d\xi dy}{(2\pi)^d}. \quad (8)$$

Dominated convergence yields

$$\liminf_{\lambda \rightarrow \infty} \lambda^{-1-d/2} \sum_k (\lambda - \lambda_k)_+ \geq \int_{\mathbf{R}^d \times \Omega_\varepsilon} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 c_3^\varepsilon \right)_+ \frac{d\xi dy}{(2\pi)^d}.$$

Letting $\varepsilon \rightarrow 0$ and using dominated convergence concludes the argument. $\square$

7.2 Upper Trace Bound

Before proving an upper bound on the trace of the hyperbolic Laplacian, we need to consider some further technicalities concerning coherent states. Consider again the general situation where $g \in C_c^\infty(\mathbf{R}^d)$ is real-valued and such that $g(-x) = g(x)$ and $\int g(x)^2 dx = 1$. As before, define $e_{\xi,y}(x) = e^{i\xi x} g(x - y)$. The operator

$$\Phi: L^2(\mathbf{R}^d, dx) \rightarrow L^2(\mathbf{R}^{2d}, (2\pi)^{-d} dy d\xi)$$

given by

$$(\Phi\psi)(\xi, y) = \langle e_{\xi,y}, \psi \rangle_{L^2(\mathbf{R}^d)}$$

is called the coherent state transform. Since $(e_{\xi,y})$ is overfilling, Φ is an isometry, i.e. $\Phi^* \Phi = I_{L^2(\mathbf{R}^d)}$.

In this section's theorem we shall make use of the two following lemmas (See also [\[LL01\]](#)).

Lemma 7.4. *Let $\psi \in H^1(\mathbf{R}^d)$. Then*

$$\langle \psi, \Phi^* \xi_1^2 \Phi \psi \rangle = \|\partial_{x_1} \psi\|^2 + \int_{\mathbf{R}^d} (\partial_{z_1} g(z))^2 dz \|\psi\|^2.$$

Proof. Let us introduce some notation. We write $g_y(x) = g(x - y)$. Recall that for the Fourier transform $\mathcal{F}: L^2(\mathbf{R}^d) \rightarrow L^2(\mathbf{R}^d)$ we used the convention

$$\mathcal{F}f(\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{-ix\xi} f(x) dx,$$

for $f \in L^2(\mathbf{R}^d) \cap L^1(\mathbf{R}^d)$. Finally, we write $X = L^2(\mathbf{R}^d, dx)$ and $Y = L^2(\mathbf{R}^{2d}, (2\pi)^{-d} dy d\xi)$ for brevity.

The coherent state transform of ψ is given by

$$\Phi\psi(\xi, y) = \langle e_{\xi,y}, \psi \rangle_X = \int_{\mathbf{R}^d} e^{-ix\xi} g_y(x) \psi(x) dx = (2\pi)^{d/2} \mathcal{F}(g_y \psi)(\xi).$$

Consequently,

$$\begin{aligned}
\langle \psi, \Phi^* \xi_1^2 \Phi \psi \rangle_X &= \langle \xi_1 \Phi \psi, \xi_1 \Phi \psi \rangle_Y \\
&= (2\pi)^d \|\xi_1 \mathcal{F}(g_y \psi)\|_Y^2 \\
&= (2\pi)^d \|\mathcal{F}(\partial_{x_1}(g_y \psi))\|_Y^2 \\
&= \int_{\mathbf{R}^{2d}} |\mathcal{F}(\partial_{x_1}(g_y \psi))(\xi)|^2 d\xi dy \\
&= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} |\partial_{x_1}(g_y \psi)(x)|^2 dx dy \\
&= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} |\partial_{x_1}(g_y \psi)(x)|^2 dy dx.
\end{aligned}$$

For the integrand we find

$$\begin{aligned}
|\partial_{x_1}(g(x-y)\psi(x))|^2 &= |\partial_{x_1}g(x-y)\psi(x)|^2 + |g(x-y)\partial_{x_1}\psi(x)|^2 \\
&\quad + 2 \operatorname{Re} \overline{\partial_{x_1}g(x-y)\psi(x)} g(x-y)\partial_{x_1}\psi(x).
\end{aligned}$$

Since $g(-z) = g(z)$, it is $(\partial_{z_1}g)(-z) = -(\partial_{z_1}g)(z)$ and thus

$$\int_{\mathbf{R}^d} \partial_{x_1}g(x-y)g(x-y)dy = \int_{\mathbf{R}^d} \partial_{z_1}g(z)g(z)dz = 0.$$

Finally,

$$\begin{aligned}
\langle \psi, \Phi^* \xi_1^2 \Phi \psi \rangle &= \int_{\mathbf{R}^{2d}} (|\partial_{x_1}g(x-y)\psi(x)|^2 + |g(x-y)\partial_{x_1}\psi(x)|^2) dy dx \\
&= \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |\partial_{x_1}g(x-y)|^2 dy \right) |\psi(x)|^2 dx \\
&\quad + \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |g(x-y)|^2 dy \right) |\partial_{x_1}\psi(x)|^2 dx \\
&= \int_{\mathbf{R}^d} |\partial_{z_1}g(z)|^2 dz \int_{\mathbf{R}^d} |\psi(x)|^2 dx + \int_{\mathbf{R}^d} |g(z)|^2 dz \int_{\mathbf{R}^d} |\partial_{x_1}\psi(x)|^2 dx \\
&= \int_{\mathbf{R}^d} (\partial_{z_1}g(z))^2 dz \|\psi\|^2 + \|\partial_{x_1}\psi\|^2.
\end{aligned}$$

□

Lemma 7.5. *Assume that the function in the definition of coherent states can be decomposed as $g(z) = g_1(z_1)\tilde{g}(\tilde{z})$ where g_1 and $\tilde{g}$ are symmetric, i.e. $g_1(-x_1) = g_1(x_1)$ and*

$\tilde{g}(-\tilde{x}) = \tilde{g}(\tilde{x})$, compactly supported, and satisfy the normalization conditions

$$\int_{\mathbf{R}} g_1(z_1)^2 dz_1 = \int_{\mathbf{R}^{d-1}} \tilde{g}(\tilde{z})^2 d\tilde{z} = 1.$$

Let $\psi \in H^1(\mathbf{R}^d)$ have compact support. Then

$$\langle \psi, \Phi^* e^{2y_1} |\tilde{\xi}|^2 \Phi \psi \rangle = \int_{\mathbf{R}^d} e^{2z_1} |\nabla_{\tilde{z}} g(z)|^2 dz \|e^{x_1} \psi\|^2 + \int_{\mathbf{R}^d} e^{2z_1} g(z)^2 dz \|e^{x_1} \nabla_{\tilde{x}} \psi\|^2.$$

Proof. We proceed in a similar way as in the proof of the lemma above. Adopting the same notation as before, we find for $j = 2, \dots, d$ that

$$\begin{aligned} \langle \psi, \Phi^* e^{2y_1} \xi_j^2 \Phi \psi \rangle_X &= \langle e^{y_1} \xi_1 \Phi \psi, e^{y_1} \xi_j \Phi \psi \rangle_Y \\ &= (2\pi)^d \|e^{y_1} \xi_j \mathcal{F}(g_y \psi)\|_Y^2 \\ &= (2\pi)^d \|e^{y_1} \mathcal{F}(\partial_{x_j}(g_y \psi))\|_Y^2 \\ &= \int_{\mathbf{R}^{2d}} e^{2y_1} |\mathcal{F}(\partial_{x_j}(g_y \psi))(\xi)|^2 d\xi dy \\ &= \int_{\mathbf{R}^d} e^{2y_1} \int_{\mathbf{R}^d} |\mathcal{F}(\partial_{x_j}(g_y \psi))(\xi)|^2 d\xi dy \\ &= \int_{\mathbf{R}^d} e^{2y_1} \int_{\mathbf{R}^d} |(\partial_{x_j}(g_y \psi))(\xi)|^2 dx dy \\ &= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} e^{2y_1} |\partial_{x_j}(g_y \psi)(x)|^2 dy dx. \end{aligned}$$

Summing over j we obtain

$$\langle \psi, \Phi^* e^{2y_1} |\tilde{\xi}|^2 \Phi \psi \rangle = \int_{\mathbf{R}^{2d}} e^{2y_1} |\nabla_{\tilde{x}}(g(x-y)\psi(x))|^2 dy dx.$$

Note that

$$\begin{aligned} |\nabla_{\tilde{x}}(g(x-y)\psi(x))|^2 &= |\nabla_{\tilde{x}} g(x-y)|^2 |\psi(x)|^2 + |g(x-y)|^2 |\nabla_{\tilde{x}} \psi(x)|^2 \\ &\quad + \sum_{j=2}^d 2 \operatorname{Re} \overline{\partial_{x_j} g(x-y) \psi(x)} g(x-y) \partial_{x_j} \psi(x), \end{aligned}$$

and that

$$\int_{\mathbf{R}^d} e^{2y_1} \partial_{x_j} g(x-y) g(x-y) dy = \int_{\mathbf{R}} e^{2y_1} g_1(x_1 - y_1)^2 dy_1 \int_{\mathbf{R}^{d-1}} \partial_{x_j} \tilde{g}(\tilde{x} - \tilde{y}) \tilde{g}(\tilde{x} - \tilde{y}) d\tilde{y},$$

where the second integral on the right vanishes due to the symmetry of $\tilde{g}$. Finally

$$\begin{aligned} \int_{\mathbf{R}^{2d}} e^{2y_1} |\nabla_{\tilde{x}} g(x-y)|^2 |\psi(x)|^2 dx dy &= \int_{\mathbf{R}^{2d}} e^{2(y_1-x)} |\nabla_{\tilde{x}} g(x-y)|^2 e^{2x_1} |\psi(x)|^2 dx dy \\ &= \int_{\mathbf{R}^d} e^{2z} |\nabla_{\tilde{z}} g(z)|^2 dz \int_{\mathbf{R}^d} e^{2x_1} |\psi(x)|^2 dx, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbf{R}^{2d}} e^{2y_1} |g(x-y)|^2 |\nabla_{\tilde{x}} \psi(x)|^2 dx dy &= \int_{\mathbf{R}^{2d}} e^{2(y_1-x_1)} |g(x-y)|^2 e^{2x_1} |\nabla_{\tilde{x}} \psi(x)|^2 dx dy \\ &= \int_{\mathbf{R}^d} e^{2z_1} |g(z)|^2 dz \int_{\mathbf{R}^d} e^{2x_1} |\nabla_{\tilde{x}} \psi(x)|^2 dx. \end{aligned}$$

This concludes the proof. $\square$

We are now ready to state the main theorem of this subsection.

Theorem 7.6. *Let $\Omega \subseteq \mathbf{R}^d$ be a bounded open set and let λ_k , $k \in \mathbf{N}$ be the eigenvalues of the operator H on Ω with Dirichlet boundary conditions. Then*

$$\limsup_{\lambda \rightarrow \infty} \lambda^{-1-d/2} \sum_k (\lambda - \lambda_k)_+ \leq \int_{\mathbf{R}^d \times \Omega} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 \right)_+ \frac{d\xi dy}{(2\pi)^d}$$

Proof. Assume that g is supported in the unit ball $B_1(0)$ and that it can be decomposed as $g(z) = g_1(z_1)\tilde{g}(\tilde{z})$, where $g_1, \tilde{g} \geq 0$ are symmetric in the sense that $g_1(-x_1) = g_1(x_1)$ and $\tilde{g}(-\tilde{x}) = \tilde{g}(\tilde{x})$, compactly supported and satisfy $\int_{\mathbf{R}} g_1(z_1)^2 dz_1 = \int_{\mathbf{R}^{d-1}} \tilde{g}(\tilde{z})^2 d\tilde{z} = 1$. Define $g^\varepsilon(z) = \varepsilon^{-d/2} g(z/\varepsilon)$. Denote by Ω^ε the set $\{x \in \mathbf{R}^d : \text{dist}(x, \Omega) < \varepsilon\}$. As before, let λ_k , $k \in \mathbf{N}$ be the eigenvalues of H . Since H has discrete spectrum, we can find corresponding eigenfunctions $\psi_k \in H_0^1(\Omega)$ such that $(\psi_k)_k$ is an orthonormal basis of $L^2(\Omega)$. Extending these eigenfunctions by zero outside Ω , we obtain functions in $H^1(\mathbf{R}^d)$ that are compactly supported. We can therefore apply the preceding two lemmas to the functions ψ_k , $k \in \mathbf{N}$. For the eigenvalues we find

$$\lambda_k = \langle \psi_k, H\psi_k \rangle_{L^2(\Omega)} = \langle \psi_k, H\psi_k \rangle_{L^2(\mathbf{R}^d)} = \|\partial_{x_1} \psi_k\|^2 + \|e^{x_1} \nabla_{\tilde{x}} \psi_k\|^2.$$

For brevity we denote, again,

$$\begin{aligned} c_1^\varepsilon &= \int_{\mathbf{R}^d} (\partial_{z_1} g^\varepsilon(z))^2 dz \\ c_2^\varepsilon &= \int_{\mathbf{R}^d} e^{2z_1} |\nabla_{\tilde{z}} g^\varepsilon(z)|^2 dz \\ c_3^\varepsilon &= \int_{\mathbf{R}^d} e^{2z_1} g^\varepsilon(z)^2 dz. \end{aligned}$$

Denoting $R = \sup\{y_1 \mid y \in \Omega\}$, we find

$$\begin{aligned} (\lambda - \lambda_k)_+ &= \left(\lambda - \langle \psi_k, \Phi^* \xi_1^2 \Phi \psi_k \rangle + c_1^\varepsilon - (c_3^\varepsilon)^{-1} \langle \psi_k, \Phi^* e^{2y_1} |\tilde{\xi}|^2 \Phi \psi_k \rangle + (c_3^\varepsilon)^{-1} c_2^\varepsilon \|e^{x_1} \psi_k\|^2 \right)_+ \\ &\leq \left(\lambda - \langle \psi_k, \Phi^* \xi_1^2 \Phi \psi_k \rangle + c_1^\varepsilon - (c_3^\varepsilon)^{-1} \langle \psi_k, \Phi^* e^{2y_1} |\tilde{\xi}|^2 \Phi \psi_k \rangle + (c_3^\varepsilon)^{-1} c_2^\varepsilon e^{2R} \right)_+ \\ &= \left(\langle \Phi \psi_k, (\lambda - \xi_1^2 - (c_3^\varepsilon)^{-1} e^{2y_1} |\tilde{\xi}|^2 + c_1^\varepsilon + (c_3^\varepsilon)^{-1} c_2^\varepsilon e^{2R}) \Phi \psi_k \rangle \right)_+ \\ &\leq \langle \Phi \psi_k, (\lambda - \xi_1^2 - (c_3^\varepsilon)^{-1} e^{2y_1} |\tilde{\xi}|^2 + c_1^\varepsilon + (c_3^\varepsilon)^{-1} c_2^\varepsilon e^{2R})_+ \Phi \psi_k \rangle \end{aligned}$$

where the last inequality is due to Jensen's inequality. Summing over k then yields

$$\sum_k (\lambda - \lambda_k)_+ \leq \int_{\mathbf{R}^{2d}} (\lambda - \xi_1^2 - (c_3^\varepsilon)^{-1} e^{2y_1} |\tilde{\xi}|^2 + c_1^\varepsilon + (c_3^\varepsilon)^{-1} c_2^\varepsilon e^{2R})_+ |\Phi \psi_k(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d}.$$

Since (ψ_k) is an orthonormal basis, we obtain

$$\sum_k |\Phi \psi_k(\xi, y)|^2 = \sum_k |\langle e_{\xi, y}, \psi_k \rangle|^2 = \|e_{\xi, y}\|_{L^2(\Omega)}^2 \leq \|e_{\xi, y}\|_{L^2(\mathbf{R}^d)}^2 = 1.$$

Since the map $y \mapsto \Phi \psi(\xi, y)$ is supported in $\text{supp } \psi + \text{supp } g^\varepsilon \subseteq \bar{\Omega} + B_\varepsilon(0) = \Omega^\varepsilon$ for $\psi \in H_0^1(\Omega)$, we find

$$\sum_k |\Phi \psi_k(\xi, y)|^2 \leq \chi_{\Omega^\varepsilon}(y).$$

Hence

$$\begin{aligned} \sum_k (\lambda - \lambda_k)_+ &\leq \int_{\mathbf{R}^d \times \Omega^\varepsilon} (\lambda - \xi_1^2 - (c_3^\varepsilon)^{-1} e^{2y_1} |\tilde{\xi}|^2 + c_1^\varepsilon + (c_3^\varepsilon)^{-1} c_2^\varepsilon e^{2R})_+ \frac{d\xi dy}{(2\pi)^d} \\ &= \lambda^{1+d/2} \int_{\mathbf{R}^d \times \Omega^\varepsilon} (1 - \xi_1^2 - (c_3^\varepsilon)^{-1} e^{2y_1} |\tilde{\xi}|^2 + c_1^\varepsilon \lambda^{-1} + (c_3^\varepsilon)^{-1} c_2^\varepsilon e^{2R} \lambda^{-1})_+ \frac{d\xi dy}{(2\pi)^d}. \end{aligned}$$

Consequently

$$\limsup_{\lambda \rightarrow \infty} \lambda^{-1-d/2} \sum_k (\lambda - \lambda_k)_+ \leq \int_{\mathbf{R}^d \times \Omega^\varepsilon} (1 - \xi_1^2 - (c_3^\varepsilon)^{-1} e^{2y_1} |\tilde{\xi}|^2)_+ \frac{d\xi dy}{(2\pi)^d}.$$

Letting $\varepsilon \rightarrow 0$ finishes the proof by dominated convergence and since $c_3^\varepsilon \rightarrow 1$. $\square$

7.3 The Error Term

Lemma 7.7. *Assume that $\Omega \subseteq \mathbf{R}^d$ is open and bounded and that $\partial\Omega$ is Lipschitz-regular. Then the trace of the hyperbolic Laplacian on Ω with Dirichlet boundary conditions satisfies*

$$\sum_k (\lambda - \lambda_k)_+ \geq \lambda^{1+d/2} \int_{\mathbf{R}^d \times \Omega} \left(\lambda - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 \right)_+ \frac{d\xi dy}{(2\pi)^d} + O(\lambda^{d/2+2/3}).$$

Proof. Recall that in [7](#) we found

$$\sum_k (\lambda - \lambda_k)_+ \geq \int_{\mathbf{R}^d \times \Omega_\varepsilon} \left(\lambda - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 c_3^\varepsilon - e^{2y_1} c_2^\varepsilon - c_1^\varepsilon \right)_+ \frac{d\xi dy}{(2\pi)^d}.$$

Defining $B_\varepsilon = \{(\xi, y) \in \mathbf{R}^d \times \Omega_\varepsilon : \xi_1^2 + e^{2y_1} |\tilde{\xi}|^2 \leq 1\}$, one evidently has

$$\begin{aligned} \sum_k (\lambda - \lambda_k)_+ &\geq \int_{B_\varepsilon} \left(\lambda - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 c_3^\varepsilon - e^{2y_1} c_2^\varepsilon - c_1^\varepsilon \right)_+ \frac{d\xi dy}{(2\pi)^d} \\ &= \lambda^{1+d/2} \int_{B_\varepsilon} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 c_3^\varepsilon - \frac{1}{\lambda} e^{2y_1} c_2^\varepsilon - \frac{1}{\lambda} c_1^\varepsilon \right)_+ \frac{d\xi dy}{(2\pi)^d}. \end{aligned}$$

For the integrand we find

$$\begin{aligned} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 c_3^\varepsilon - \frac{1}{\lambda} e^{2y_1} c_2^\varepsilon - \frac{1}{\lambda} c_1^\varepsilon \right)_+ &\geq \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 \right)_+ \\ &\quad - e^{2y_1} |\tilde{\xi}|^2 (c_3^\varepsilon - 1)_+ - \frac{1}{\lambda} e^{2y_1} c_2^\varepsilon - \frac{1}{\lambda} c_1^\varepsilon. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_k (\lambda - \lambda_k)_+ &\geq \lambda^{1+d/2} \int_B \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 \right)_+ \frac{d\xi dy}{(2\pi)^d} \\ &\quad - \lambda^{1+d/2} \int_{B \setminus B_\varepsilon} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 \right)_+ \frac{d\xi dy}{(2\pi)^d} \\ &\quad - \lambda^{1+d/2} (c_3^\varepsilon - 1)_+ \frac{|B|}{(2\pi)^d} - \lambda^{d/2} c_2^\varepsilon e^{2R} \frac{|B|}{(2\pi)^d} - \lambda^{d/2} c_1^\varepsilon \frac{|B|}{(2\pi)^d}, \end{aligned}$$

where $B = \{(\xi, y) \in \mathbf{R}^d \times \Omega : \xi_1^2 + e^{2y_1} |\tilde{\xi}|^2 \leq 1\}$ and $R = \sup\{y_1 : y \in \Omega\}$. Note that

$$\int_{B \setminus B_\varepsilon} \left(1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2\right)_+ \frac{d\xi dy}{(2\pi)^d} \leq \frac{|B \setminus B_\varepsilon|}{(2\pi)^d},$$

and that

$$\begin{aligned} B \setminus B_\varepsilon &= \{(\xi, y) \in \mathbf{R}^d \times (\Omega \setminus \Omega_\varepsilon) : \xi_1^2 + e^{2y_1} |\tilde{\xi}|^2 \leq 1\} \\ &\subseteq \{(\xi, y) \in \mathbf{R}^d \times (\Omega \setminus \Omega_\varepsilon) : \xi_1^2 + e^{2r} |\tilde{\xi}|^2 \leq 1\} \\ &= \{\xi \in \mathbf{R}^d : \xi_1^2 + e^{2r} |\tilde{\xi}|^2 \leq 1\} \times (\Omega \setminus \Omega_\varepsilon), \end{aligned}$$

where $r = \inf\{y_1 : y \in \Omega\}$. Clearly, the set $\{\xi \in \mathbf{R}^d : \xi_1^2 + e^{2r} |\tilde{\xi}|^2 \leq 1\}$ is bounded and thus of finite measure. Therefore, assuming that $\partial\Omega$ is Lipschitz-regular, we can infer ([ACV08]) that

$$\frac{|B \setminus B_\varepsilon|}{(2\pi)^d} = O(\varepsilon), \quad \varepsilon \rightarrow 0.$$

Consider now the term $c_3^\varepsilon - 1$. We know that

$$c_3^\varepsilon = \int_{\mathbf{R}^d} e^{2z_1} g^\varepsilon(z)^2 dz = \int_{\mathbf{R}} e^{2z_1} g_1^\varepsilon(z_1)^2 dz_1 = \int_{\mathbf{R}} e^{2z_1 \varepsilon} g_1(z_1)^2 dz_1.$$

Hence

$$c_3^\varepsilon - 1 = \int_{\mathbf{R}} (e^{2z_1 \varepsilon} - 1) g_1(z_1)^2 dz_1 = \int_{\mathbf{R}} \sum_{k=1}^{\infty} \frac{(2z_1)^k}{k!} \varepsilon^k g_1(z_1)^2 dz_1 = O(\varepsilon), \quad \varepsilon \rightarrow 0.$$

For c_2^ε we find

$$c_2^\varepsilon = \int_{\mathbf{R}^d} e^{2z_1} |\nabla_{\tilde{z}} g^\varepsilon(z)|^2 dz = \varepsilon^{-2} \int_{\mathbf{R}^d} e^{2z_1 \varepsilon} g(z_1)^2 dz_1 \int_{\mathbf{R}^d} e^{2z_1} |\nabla_{\tilde{z}} \tilde{g}(\tilde{z})|^2 d\tilde{z} = O(\varepsilon^{-2}), \quad \varepsilon \rightarrow 0,$$

and finally

$$\begin{aligned} c_1^\varepsilon &= \int_{\mathbf{R}^d} (\partial_{z_1} g^\varepsilon(z))^2 dz = \int_{\mathbf{R}} (\partial_{z_1} g_1^\varepsilon(z_1))^2 dz_1 \int_{\mathbf{R}^{d-1}} \tilde{g}(\tilde{z})^2 d\tilde{z} \\ &= \varepsilon^{-2} \int_{\mathbf{R}} (\partial_{z_1} g_1(z_1))^2 dz_1 = O(\varepsilon^{-2}), \quad \varepsilon \rightarrow 0. \end{aligned}$$

Linking ε to λ , again, via $\varepsilon = \lambda^{-\alpha}$ we obtain

$$\sum_k (\lambda - \lambda_k)_+ \geq \lambda^{1+d/2} \int_{\mathbf{R}^d \times \Omega} \left(\lambda - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 \right)_+ \frac{d\xi dy}{(2\pi)^d} - R(\lambda),$$

where $R(\lambda)$ is a linear combination of terms that are in $O(\lambda^{d/2+1-\alpha})$ and $O(\lambda^{d/2-2\alpha})$ respectively. Again, minimizing $\max\{d/2 + 2\alpha, 1 + d/2 - \alpha\}$ yields $\alpha = 1/3$ and thus

$$\sum_k (\lambda - \lambda_k)_+ \geq \lambda^{1+d/2} \int_{\mathbf{R}^d \times \Omega} \left(\lambda - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2 \right)_+ \frac{d\xi dy}{(2\pi)^d} + O(\lambda^{d/2+2/3}). \quad \square$$

In contrast to the case of the euclidean Laplacian, we need to prove an asymptotic upper bound for the hyperbolic Laplacian as well.

Lemma 7.8. *Let the same conditions as in the preceding lemma be satisfied. Then*

$$\sum_k (\lambda - \lambda_k)_+ \leq \lambda^{1+d/2} \int_{\mathbf{R}^d \times \Omega} (1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2)_+ \frac{d\xi dy}{(2\pi)^d} + O(\lambda^{d/2+2/3})$$

Proof. Recall that

$$\sum_k (\lambda - \lambda_k)_+ \leq \lambda^{1+d/2} \int_{\mathbf{R}^d \times \Omega^\varepsilon} (1 - \xi_1^2 - (c_3^\varepsilon)^{-1} e^{2y_1} |\tilde{\xi}|^2 + \lambda^{-1} c_1^\varepsilon + \lambda^{-1} (c_3^\varepsilon)^{-1} c_2^\varepsilon e^{2R})_+ \frac{d\xi dy}{(2\pi)^d}.$$

Let $\varepsilon = \lambda^{-\alpha}$. Then $((1 - (c_3^\varepsilon)^{-1}) = O(\lambda^{-\alpha})$, $\lambda^{-1} c_1^\varepsilon = O(\lambda^{-1+2\alpha})$ and $\lambda^{-1} c_2^\varepsilon = O(\lambda^{-1+2\alpha})$. If $\alpha < 1/2$, then all these terms converge to 0. In this case, for sufficiently large λ , the integrand is supported in $B^\varepsilon = \{(\xi, y) \in \mathbf{R}^d \times \Omega^\varepsilon : \xi_1^2 + e^{2y_1} |\tilde{\xi}|^2 \leq 2\}$. Furthermore, the integrand can be estimated from above in the following way

$$\begin{aligned} & (1 - \xi_1^2 - (c_3^\varepsilon)^{-1} e^{2y_1} |\tilde{\xi}|^2 + c_1^\varepsilon \lambda^{-1} + (c_3^\varepsilon)^{-1} c_2^\varepsilon e^{2R} \lambda^{-1})_+ \\ & \leq (1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2)_+ + (1 - (c_3^\varepsilon)^{-1})_+ e^{2R} |\tilde{\xi}|^2 + \lambda^{-1} c_1^\varepsilon + \lambda^{-1} (c_3^\varepsilon)^{-1} c_2^\varepsilon e^{2R}. \end{aligned}$$

Define $B = \{(\xi, y) \in \mathbf{R}^d \times \Omega : \xi_1^2 + e^{2y_1} |\tilde{\xi}|^2 \leq 2\}$. Then

$$\begin{aligned} \sum_k (\lambda - \lambda_k)_+ & \leq \lambda^{1+d/2} \int_B (1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2)_+ \frac{d\xi dy}{(2\pi)^d} \\ & \quad + \lambda^{1+d/2} \int_{B^\varepsilon \setminus B} (1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2)_+ \frac{d\xi dy}{(2\pi)^d} \\ & \quad + 2e^R \lambda^{d/2+1} ((1 - (c_3^\varepsilon)^{-1})_+ \frac{|B^\varepsilon|}{(2\pi)^d} + \lambda^{d/2} c_1^\varepsilon \frac{|B^\varepsilon|}{(2\pi)^d} + \lambda^{d/2} (c_3^\varepsilon)^{-1} c_2^\varepsilon e^{2R} \frac{|B^\varepsilon|}{(2\pi)^d}). \end{aligned}$$

Note that

$$\int_{B^\varepsilon \setminus B} (1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2)_+ \frac{d\xi dy}{(2\pi)^d} \leq \frac{|B^\varepsilon \setminus B|}{(2\pi)^d} = O(\varepsilon) = O(\lambda^{-\alpha}).$$

Therefore, the error term is of the order $O(\lambda^{\max\{d/2+2\alpha, 1+d/2-\alpha\}})$. As for the lower bound, the exponent is minimized for $\alpha = 1/3$. This concludes the argument. $\square$

Combining the two preceding lemmas, we obtain

Theorem 7.9. *Let $\Omega \subseteq \mathbf{R}^d$ be a bounded open set with Lipschitz boundary and let λ_k , $k \in \mathbf{N}$ be the eigenvalues of the operator H on Ω with Dirichlet boundary conditions. Then*

$$\sum_k (\lambda - \lambda_k)_+ = \lambda^{1+d/2} \int_{\mathbf{R}^d \times \Omega} (1 - \xi_1^2 - e^{2y_1} |\tilde{\xi}|^2)_+ \frac{d\xi dy}{(2\pi)^d} + O(\lambda^{d/2+2/3}).$$

8 The Logarithmic Laplacian

We consider operators on $L^2(\mathbf{R}^d)$ of the form $\log(1 - \Delta) + V$ where V is a non-negative, increasing potential. Here, the differential part $\log(1 - \Delta)$ is defined via

$$\langle u, \log(1 - \Delta)u \rangle = \int_{\mathbf{R}^d} \log(1 + |\xi|^2) |\hat{u}(\xi)|^2 d\xi,$$

for $u \in C_c^\infty(\mathbf{R}^d)$. The Dirichlet realizations of the operator $\log(1 - \Delta)$ or very similar ones were studied by Laptev and Weth in [LW21] and Feulefack in [Feu23].

If $V \in L_{loc}^1(\mathbf{R}^d)$ is bounded from below, then $-\log(1 - \Delta) + V$ is a lower semibounded symmetric operator on $C_c^\infty(\mathbf{R}^d)$ and it therefore admits a Friedrichs extension. The thus obtained Schrödinger operator shall be denoted by $H = \log(1 - \Delta) + V$.

Lemma 8.1. *If $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, then the spectrum of H is discrete.*

Proof. The proof is almost identical to proof of the corresponding result for Schrödinger operators $-\Delta + V$. See [RS78] for instance for a proof. $\square$

In this case, since H is lower semibounded, the spectrum consists of an increasing sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$ accumulating at infinity.

We are interested in the asymptotic behavior of these eigenvalues. We shall prove that if $V \in C(\mathbf{R}^d)$ satisfies $V(x) \geq \alpha|x| + c$ for some $\alpha > 0$ and $c \in \mathbf{R}$, then

$$\lim_{\lambda \rightarrow \infty} e^{-d\lambda/2} \sum_k (\lambda - \lambda_k)_+ = \frac{2|B_1|}{d(2\pi)^d} \int_{\mathbf{R}^d} e^{-dV(x)/2} dx.$$

Like in the preceding chapter, we proceed in two parts.

8.1 Lower Trace Bound

Theorem 8.2. *Let $V \in C(\mathbf{R}^d)$ satisfy $V(x) \geq \alpha|x| + c$ for some $\alpha > 0$ and $c \in \mathbf{R}$. The eigenvalues of the operator $H = \log(1 - \Delta) + V$ satisfy*

$$\liminf_{\lambda \rightarrow \infty} e^{-d\lambda/2} \sum_k (\lambda - \lambda_k)_+ \geq \frac{2|B_1|}{d(2\pi)^d} \int_{\mathbf{R}^d} e^{-dV(x)/2} dx$$

Remark 43. The growth condition on V enforces the finiteness of the integral $\int_{\mathbf{R}^d} e^{-dV(x)/2} dx$. For this lower trace bound, a weaker growth condition would have sufficed.

In order to prove this bound, we once again employ coherent state methods. Let $e_{\xi,y}(x) = e^{i\xi x} g(x - y)$, where $g \in C_c^\infty(\mathbf{R}^d)$ with $\|g\|_{L^2(\mathbf{R}^d)} = 1$ and $g(-x) = g(x)$.

Lemma 8.3.

$$\langle e_{\xi,y}, \log(1 - \Delta)e_{\xi,y} \rangle_{L^2(\mathbf{R}^d)} \leq \log(1 + |\xi|^2) + \frac{1}{1 + |\xi|^2} \|\nabla g\|_{L^2(\mathbf{R}^d)}^2.$$

Proof. We start by computing the Fourier transform of $e_{\xi,y}$:

$$\begin{aligned} \mathcal{F}e_{\xi,y}(p) &= (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{-ixp} e^{ix\xi} g(x-y) dx \\ &= (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{-ix(p-\xi)} g(x-y) dx \\ &= (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{-i(x+y)(p-\xi)} g(x) dx \\ &= e^{-iy(p-\xi)} (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{-ix(p-\xi)} g(x) dx \\ &= e^{-iy(p-\xi)} \hat{g}(p-\xi). \end{aligned}$$

Now

$$\begin{aligned} \langle e_{\xi,y}, \log(1 - \Delta)e_{\xi,y} \rangle_{L^2(\mathbf{R}^d)} &= \langle \mathcal{F}e_{\xi,y}, \log(1 + |\cdot|^2) \mathcal{F}e_{\xi,y} \rangle_{L^2(\mathbf{R}^d)} \\ &= \int_{\mathbf{R}^d} \log(1 + |p|^2) |\mathcal{F}e_{\xi,y}(p)|^2 dp \\ &= \int_{\mathbf{R}^d} \log(1 + |p|^2) |e^{-iy(p-\xi)} \hat{g}(p-\xi)|^2 dp \\ &= \int_{\mathbf{R}^d} \log(1 + |p|^2) |\hat{g}(p-\xi)|^2 dp \\ &= \int_{\mathbf{R}^d} \log(1 + |p + \xi|^2) |\hat{g}(p)|^2 dp \end{aligned}$$

Note that

$$\begin{aligned} \log(1 + |p + \xi|^2) &= \log \left((1 + |\xi|^2) \frac{1 + |p + \xi|^2}{1 + |\xi|^2} \right) \\ &= \log(1 + |\xi|^2) + \log \left(\frac{1 + |p + \xi|^2}{1 + |\xi|^2} \right), \end{aligned}$$

and that

$$\begin{aligned} \log \left(\frac{1 + |p + \xi|^2}{1 + |\xi|^2} \right) &= \log \left(\frac{1 + |\xi|^2 + |p|^2 + 2\xi \cdot p}{1 + |\xi|^2} \right) \\ &= \log \left(1 + \frac{|p|^2 + 2\xi \cdot p}{1 + |\xi|^2} \right). \end{aligned}$$

Recall that $\log(1 + \alpha) \leq \alpha$ for $\alpha > -1$ and that

$$\frac{|p|^2 + 2\xi \cdot p}{1 + |\xi|^2} \geq -1.$$

Thus

$$\log \left(1 + \frac{|p|^2 + 2\xi \cdot p}{1 + |\xi|^2} \right) \leq \frac{|p|^2 + 2\xi \cdot p}{1 + |\xi|^2}.$$

Consequently,

$$\begin{aligned} \langle e_{\xi,y}, \log(1 - \Delta)e_{\xi,y} \rangle_{L^2(\mathbf{R}^d)} &\leq \int_{\mathbf{R}^d} \left(\log(1 + |\xi|^2) + \frac{|p|^2 + 2\xi \cdot p}{1 + |\xi|^2} \right) |\hat{g}(p)|^2 dp \\ &= \log(1 + |\xi|^2) \int_{\mathbf{R}^d} |\hat{g}(p)|^2 dp \\ &\quad + \frac{1}{1 + |\xi|^2} \int_{\mathbf{R}^d} |p|^2 |\hat{g}(p)|^2 dp \\ &\quad + \frac{2}{1 + |\xi|^2} \int_{\mathbf{R}^d} \xi \cdot p |\hat{g}(p)|^2 dp \end{aligned}$$

Note that

$$\int_{\mathbf{R}^d} |\hat{g}(p)|^2 dp = \int_{\mathbf{R}^d} |g(x)|^2 dx = 1$$

by Plancherel's theorem, that

$$\int_{\mathbf{R}^d} \xi \cdot p |\hat{g}(p)|^2 dp = 0$$

since g is real-valued and $|\hat{g}|^2$ hence symmetric and that

$$\frac{1}{1 + |\xi|^2} \int_{\mathbf{R}^d} |p|^2 |\hat{g}(p)|^2 dp = \frac{1}{1 + |\xi|^2} \int_{\mathbf{R}^d} |\nabla g(x)|^2 dx.$$

In summary, we found

$$\langle e_{\xi,y}, \log(1 - \Delta)e_{\xi,y} \rangle_{L^2(\mathbf{R}^d)} \leq \log(1 + |\xi|^2) + \frac{1}{1 + |\xi|^2} \int_{\mathbf{R}^d} |\nabla g(x)|^2 dx$$

as claimed. □

Note also that

$$\langle e_{\xi,y}, V e_{\xi,y} \rangle = \int_{\mathbf{R}^d} V(x) g(x - y)^2 dx = \int_{\mathbf{R}^d} V(x) g(y - x)^2 dx = V * g^2(y).$$

The last ingredient we need for the proof of the theorem is the computation of the following integral.

$$\begin{aligned}\int_{\mathbf{R}^d} (-\log(|\eta|^2))_+ d\eta &= \int_{B_1} (-\log(|\eta|^2))_+ d\eta \\ &= -d|B_1| \int_0^1 \log(r^2) r^{d-1} dr\end{aligned}$$

is obtained by passing to polar coordinates. Now

$$\begin{aligned}\int_0^1 \log(r^2) r^{d-1} dr &= 2 \int_0^1 \log r r^{d-1} dr \\ &= 2 \frac{r^d}{d} \log r \Big|_0^1 - \frac{2}{d} \int_0^1 r^{d-1} dr \\ &= -\frac{2}{d^2} r^d \Big|_0^1 = -\frac{2}{d^2},\end{aligned}$$

and thus

$$\int_{\mathbf{R}^d} (-\log(|\eta|^2))_+ d\eta = \frac{2|B_1|}{d}.$$

Proof of the theorem. Let $g \in C_c^\infty(\mathbf{R}^d)$ with $\|g\|_{L^2(\mathbf{R}^d)} = 1$ and $g(-x) = g(x)$. Define $g^\varepsilon(x) = \varepsilon^{-d/2} g(x/\varepsilon)$ and $e_{\xi,y}^\varepsilon(x) = e^{i\xi x} g^\varepsilon(x-y)$. Recall that by proposition [3.3](#), the family $(e_{\xi,y}^\varepsilon)_{y,\xi}$ is overfilling in $L^2(\mathbf{R}^d)$. Note that $\sum_j (\lambda - \lambda_k)_+ = \text{tr}(\lambda - H)_+$. We compute

$$\begin{aligned}\text{tr}(\lambda - H)_+ &= \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle e_{\xi,y}^\varepsilon, (\lambda - H)_+ e_{\xi,y}^\varepsilon \rangle \frac{d\xi dy}{(2\pi)^d} \geq \int_{\mathbf{R}^d \times \mathbf{R}^d} (\langle e_{\xi,y}^\varepsilon, (\lambda - H) e_{\xi,y}^\varepsilon \rangle)_+ \frac{d\xi dy}{(2\pi)^d} \\ &\geq \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(\lambda - \log(1 + |\xi|^2) - \frac{1}{1 + |\xi|^2} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz - V * (g^\varepsilon)^2(y) \right)_+ \frac{d\xi dy}{(2\pi)^d} \\ &= \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(-\log(e^{-\lambda}(1 + |\xi|^2)) - \frac{1}{1 + |\xi|^2} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz - V * (g^\varepsilon)^2(y) \right)_+ \frac{d\xi dy}{(2\pi)^d} \\ &= \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(-\log(e^{-\lambda} + |e^{-\lambda/2} \xi|^2) - \frac{1}{1 + |\xi|^2} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz - V * (g^\varepsilon)^2(y) \right)_+ \frac{d\xi dy}{(2\pi)^d} \\ &= e^{d\lambda/2} \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(-\log(e^{-\lambda} + |\eta|^2) - \frac{1}{1 + e^\lambda |\eta|^2} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz - V * (g^\varepsilon)^2(y) \right)_+ \frac{d\eta dy}{(2\pi)^d} \\ &= \frac{e^{d\lambda/2}}{(2\pi)^d} \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(-\log(e^{-\lambda} + |\eta|^2) - \frac{1}{1 + e^\lambda |\eta|^2} \int_{\mathbf{R}^d} |\nabla g^\varepsilon(z)|^2 dz - V * (g^\varepsilon)^2(y) \right)_+ d\eta dy.\end{aligned}$$

Monotone convergence yields

$$\liminf_{\lambda \rightarrow \infty} e^{-d\lambda/2} \operatorname{tr}(\lambda - H)_+ \geq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\log(|\eta|^2) - V * (g^\varepsilon)^2(y))_+ d\eta dy.$$

Since V is continuous and $(g^\varepsilon)_\varepsilon$ is a mollifier, we have $V * (g^\varepsilon)^2(y) \rightarrow V(y)$ as $\varepsilon \rightarrow 0$ for every y . Note that $V * (g^\varepsilon)^2(y) \geq c$ for all $y \in \mathbf{R}^d$, $\varepsilon > 0$ and hence

$$(-\log(|\eta|^2) - V * (g^\varepsilon)^2(y))_+ \leq (-\log(|\eta|^2) - c)_+.$$

Letting $\varepsilon \rightarrow 0$ and using dominated convergence, we thus find

$$\begin{aligned} \liminf_{\lambda \rightarrow \infty} e^{-d\lambda/2} \operatorname{tr}(\lambda - H)_+ &\geq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\log(|\eta|^2) - V(y))_+ d\eta dy \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\log(|\eta|^2 e^{V(y)}))_+ d\eta dy \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\log(|\zeta|^2))_+ e^{-dV(y)/2} d\zeta dy \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} (-\log(|\zeta|^2))_+ d\zeta \int_{\mathbf{R}^d} e^{-dV(y)/2} dy \\ &= \frac{2|B_1|}{d(2\pi)^d} \int_{\mathbf{R}^d} e^{-dV(y)/2} dy. \end{aligned}$$

□

8.2 Upper Trace Bound

We proceed to proving the following theorem.

Theorem 8.4. *Let $V \in C(\mathbf{R}^d)$ be such that there exist $\alpha > 0$ and $c \in \mathbf{R}$ with*

$$V(x) \geq \alpha|x| + c$$

The eigenvalues of the operator $H = \log(1 - \Delta) + V$ satisfy

$$\limsup_{\lambda \rightarrow \infty} e^{-d\lambda/2} \sum_j (\lambda - \lambda_k)_+ \leq \frac{2|B_1|}{d(2\pi)^d} \int_{\mathbf{R}^d} e^{-dV(x)/2} dx$$

Again, we need some preliminary results.

Let $g \in C_c^\infty(\mathbf{R}^d)$ be real-valued and such that $g(-x) = g(x)$ and $\int g(x)^2 dx = 1$. Define $e_{\xi,y}(x) = e^{i\xi x} g(x - y)$. Recall that the coherent state transform is the operator

$\Phi: L^2(\mathbf{R}^d, dx) \rightarrow L^2(\mathbf{R}^{2d}, (2\pi)^{-d} dy d\xi)$ given by $(\Phi\psi)(y, \xi) = \langle e_{\xi, y}, \psi \rangle_{L^2(\mathbf{R}^d)}$ and that $\Phi^* \Phi = I_{L^2(\mathbf{R}^d)}$.

The idea now is the same as for the hyperbolic Laplacian, i.e. we want to represent $(\lambda - \lambda_k)_+$ in terms of the coherent state transform of the corresponding eigenfunction of the operator H .

Lemma 8.5. *Let $\psi \in L^2(\mathbf{R}^d)$ be such that $\int_{\mathbf{R}^d} \log(1 + |\xi|^2) |\hat{u}(\xi)|^2 d\xi < \infty$. Then*

$$\int_{\mathbf{R}^d} \log(1 + |\xi|^2) |\hat{\psi}(\xi)|^2 d\xi \geq \int_{\mathbf{R}^d \times \mathbf{R}^d} \log(1 + |\xi|^2) |\Phi\psi(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d} - \int_{\mathbf{R}^d} |\nabla g(z)|^2 dz \|\psi\|^2.$$

Proof. Recall that

$$\Phi\psi(\xi, y) = \langle e_{\xi, y}, \psi \rangle_X = \int_{\mathbf{R}^d} e^{-ix\xi} g_y(x) \psi(x) dx = (2\pi)^{d/2} \mathcal{F}(g_y \psi).$$

Note that by the convolution theorem,

$$\mathcal{F}(g_y \psi) = (2\pi)^{-d/2} \hat{g}_y * \hat{\psi}(\xi),$$

and thus

$$\Phi\psi(\xi, y) = \hat{g}_y * \hat{\psi}(\xi).$$

Note that

$$\hat{g}_y(\eta) = e^{-iy\eta} \hat{g}(\eta),$$

and thus

$$\hat{g}_y * \hat{\psi}(\xi) = \int_{\mathbf{R}^d} e^{-iy\eta} \hat{g}(\eta) \hat{\psi}(\xi - \eta) d\eta = \mathcal{F}(\Psi_\xi)(y),$$

where

$$\Psi_\xi(\eta) = (2\pi)^{d/2} \hat{g}(\eta) \hat{\psi}(\xi - \eta).$$

Thus

$$\begin{aligned} \int_{\mathbf{R}^d} |\Phi\psi(\xi, y)|^2 dy &= \int_{\mathbf{R}^d} |\mathcal{F}(\Psi_\xi)(y)|^2 dy \\ &= \int_{\mathbf{R}^d} |(\Psi_\xi)(\eta)|^2 d\eta \\ &= (2\pi)^d \int_{\mathbf{R}^d} |\hat{g}(\eta)|^2 |\hat{\psi}(\xi - \eta)|^2 d\eta. \end{aligned}$$

We can now compute

$$\begin{aligned}
\int_{\mathbf{R}^d \times \mathbf{R}^d} \log(1 + |\xi|^2) |\Phi\psi(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d} &= \int_{\mathbf{R}^d} \log(1 + |\xi|^2) \int_{\mathbf{R}^d} |\hat{g}(\eta)|^2 |\hat{\psi}(\xi - \eta)|^2 d\eta d\xi \\
&= \int_{\mathbf{R}^d} |\hat{g}(\eta)|^2 \int_{\mathbf{R}^d} \log(1 + |\xi|^2) |\hat{\psi}(\xi - \eta)|^2 d\xi d\eta \\
&= \int_{\mathbf{R}^d} |\hat{g}(\eta)|^2 \int_{\mathbf{R}^d} \log(1 + |\rho + \eta|^2) |\hat{\psi}(\rho)|^2 d\rho d\eta
\end{aligned}$$

In the previous section, we proved

$$\log(1 + |\rho + \eta|^2) \leq \log(1 + |\rho|^2) + \frac{|\eta|^2 + 2\eta \cdot \rho}{1 + |\rho|^2} \leq \log(1 + |\rho|^2) + |\eta|^2 + \frac{2\eta \cdot \rho}{1 + |\rho|^2}.$$

We therefore get

$$\begin{aligned}
\int_{\mathbf{R}^d} |\hat{g}(\eta)|^2 \int_{\mathbf{R}^d} \log(1 + |\rho + \eta|^2) |\hat{\psi}(\rho)|^2 d\rho d\eta &\leq \int_{\mathbf{R}^d} |\hat{g}(\eta)|^2 d\eta \int_{\mathbf{R}^d} \log(1 + |\rho|^2) |\hat{\psi}(\rho)|^2 d\rho \\
&\quad + \int_{\mathbf{R}^d} |\eta|^2 |\hat{g}(\eta)|^2 d\eta \int_{\mathbf{R}^d} |\hat{\psi}(\rho)|^2 d\rho \\
&\quad + 2 \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \eta \cdot \rho |\hat{g}(\eta)|^2 d\eta \frac{|\hat{\psi}(\rho)|^2}{1 + |\rho|^2} d\rho.
\end{aligned}$$

Note that

$$\int_{\mathbf{R}^d} |\hat{g}(\eta)|^2 d\eta = \int_{\mathbf{R}^d} |g(z)|^2 dz = 1$$

and

$$\int_{\mathbf{R}^d} |\eta|^2 |\hat{g}(\eta)|^2 d\eta = \int_{\mathbf{R}^d} |\nabla g(z)|^2 dz.$$

Furthermore,

$$\int_{\mathbf{R}^d} \eta \cdot \rho |\hat{g}(\eta)|^2 d\eta = 0,$$

due to the fact that g is real-valued. Finally

$$\int_{\mathbf{R}^d} |\hat{\psi}(\rho)|^2 d\rho = \|\psi\|^2.$$

This concludes the proof. □

We need to know how multiplication operators interact with the coherent state transform (See also [\[LL01\]](#)).

Lemma 8.6.

$$\int_{\mathbf{R}^d} V * g^2(x) |\psi(x)|^2 dx = \int_{\mathbf{R}^d \times \mathbf{R}^d} V(y) |\Phi\psi(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d}$$

Proof. Since $\Phi\psi(\xi, y) = (2\pi)^{d/2} \mathcal{F}(g_y\psi)$, we find by Plancherel's theorem that

$$\int_{\mathbf{R}^d} |\Phi\psi(\xi, y)|^2 \frac{d\xi}{(2\pi)^d} = \int_{\mathbf{R}^d} g(x-y)^2 |\psi(x)|^2 dx.$$

Multiplying by $V(y)$ and integrating with respect to y concludes the proof. $\square$

Finally, a generalized form of Fatou's lemma, which was proven by Kamihigashi in [\[Kam17\]](#), is needed.

Lemma 8.7. *Let $f_n \in \ell^1(\mathbf{N})$ be a sequence of non-negative summable sequences. Assume that $\limsup_n f_n \in \ell^1(\mathbf{N})$ and that*

$$\liminf_N \limsup_n \sum_{k=N} f_n(k) = 0.$$

Then Fatou's inequality

$$\limsup_n \sum_{k=1}^{\infty} f_n(k) \leq \sum_{k=1}^{\infty} \limsup_n f_n(k)$$

holds.

Proof of the theorem. Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of the operator $H = \log(1 - \Delta) + V$ and denote by $\psi_1, \psi_2, \dots \in L^2(\mathbf{R}^d)$ an orthonormal basis of $L^2(\mathbf{R}^d)$ consisting of eigenfunctions of H

$$H\psi_k = \lambda_k \psi_k.$$

In particular,

$$\lambda_k = \langle \psi_k, H\psi_k \rangle = \int_{\mathbf{R}^d} \log(1 + |\xi|^2) |\hat{\psi}_k(\xi)|^2 d\xi + \int_{\mathbf{R}^d} V(x) |\psi_k(x)|^2 dx.$$

Combining this section's lemmas we find

$$\begin{aligned} \lambda_k &\geq \int_{\mathbf{R}^d \times \mathbf{R}^d} (\log(1 + |\xi|^2) + V(y)) |\Phi\psi_k(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d} \\ &\quad - \int_{\mathbf{R}^d} |\nabla g(z)|^2 dz - \int_{\mathbf{R}^d} (V * g^2(x) - V(x)) |\psi(x)|^2 dx. \end{aligned}$$

Therefore

$$(\lambda - \lambda_k)_+ \leq \left(\int_{\mathbf{R}^d \times \mathbf{R}^d} (\lambda - \log(1 + |\xi|^2) - V(y)) |\Phi \psi_k(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d} \right. \\ \left. + \int_{\mathbf{R}^d} |\nabla g(z)|^2 dz + \int_{\mathbf{R}^d} (V * g^2(x) - V(x)) |\psi_k(x)|^2 dx \right)_+.$$

We want to estimate

$$\limsup_{\lambda \rightarrow \infty} e^{-d\lambda/2} \sum_k (\lambda - \lambda_k)_+.$$

We denote

$$\phi_\lambda(k) = e^{-d\lambda/2} (\lambda - \lambda_k)_+.$$

Then

$$\limsup_{\lambda \rightarrow \infty} e^{-d\lambda/2} \sum_k (\lambda - \lambda_k)_+ = \limsup_{\lambda \rightarrow \infty} \sum_k \phi_\lambda(k).$$

We proceed in three steps:

1. We estimate $\sum_k \limsup_{\lambda \rightarrow \infty} \phi_\lambda(k)$.
2. We show that $\liminf_N \limsup_{\lambda} \sum_{k=N}^{\infty} \phi_\lambda(k) = 0$.
3. We apply Kamihigashi's lemma to obtain:

$$\limsup_{\lambda \rightarrow \infty} \sum_k \phi_\lambda(k) \leq \sum_k \limsup_{\lambda \rightarrow \infty} \phi_\lambda(k).$$

1. Note that

$$\sum_k \limsup_{\lambda \rightarrow \infty} \phi_\lambda(k) = \lim_N \limsup_{\lambda \rightarrow \infty} \sum_{k=1}^N \phi_\lambda(k)$$

We compute

$$\sum_{k=1}^N \phi_\lambda(k) \leq \sum_{k=1}^N e^{-d\lambda/2} \int_{\mathbf{R}^d \times \mathbf{R}^d} (\lambda - \log(1 + |\xi|^2) - V(y))_+ |\Phi \psi_k(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d} \\ + \sum_{k=1}^N e^{-d\lambda/2} \left(\int_{\mathbf{R}^d} |\nabla g(z)|^2 dz + \int_{\mathbf{R}^d} (V * g^2(x) - V(x)) |\psi_k(x)|^2 dx \right)_+ \\ \leq e^{-d\lambda/2} \int_{\mathbf{R}^d \times \mathbf{R}^d} (\lambda - \log(1 + |\xi|^2) - V(y))_+ \frac{d\xi dy}{(2\pi)^d} \\ + \sum_{k=1}^N e^{-d\lambda/2} \left(\int_{\mathbf{R}^d} |\nabla g(z)|^2 dz + \int_{\mathbf{R}^d} (V * g^2(x) - V(x)) |\psi_k(x)|^2 dx \right)_+.$$

Here we used Jensen's inequality

$$\begin{aligned} & \left(\int_{\mathbf{R}^d \times \mathbf{R}^d} (\lambda - \log(1 + |\xi|^2) - V(y)) |\Phi\psi_k(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d} \right)_+ \\ & \leq \int_{\mathbf{R}^d \times \mathbf{R}^d} (\lambda - \log(1 + |\xi|^2) - V(y))_+ |\Phi\psi_k(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d} \end{aligned}$$

and

$$\sum_{k=1}^N |\Phi\psi_k(\xi, y)|^2 \leq \sum_{k=1}^{\infty} |\Phi\psi_k(\xi, y)|^2 = 1.$$

For fixed N we clearly have

$$\limsup_{\lambda \rightarrow \infty} \sum_{k=1}^N e^{-d\lambda/2} \left(\int_{\mathbf{R}^d} |\nabla g(z)|^2 dz + \int_{\mathbf{R}^d} (V * g^2(x) - V(x)) |\psi_k(x)|^2 dx \right)_+ = 0.$$

On the other hand, performing the change of variables $\eta = e^{-\lambda/2}\xi$ like in the proof for the lower bound, we find

$$\begin{aligned} & \limsup_{\lambda \rightarrow \infty} e^{-d\lambda/2} \int_{\mathbf{R}^d \times \mathbf{R}^d} (\lambda - \log(1 + |\xi|^2) - V(y))_+ \frac{d\xi dy}{(2\pi)^d} \\ & = \limsup_{\lambda \rightarrow \infty} \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(-\log(e^{-\lambda} + |\eta|^2) - V(y) \right)_+ \frac{d\eta dy}{(2\pi)^d} \\ & = \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(-\log(|\eta|^2) - V(y) \right)_+ \frac{d\eta dy}{(2\pi)^d} \\ & = \frac{2|B_1|}{d(2\pi)^d} \int_{\mathbf{R}^d} e^{-dV(y)/2} dy. \end{aligned}$$

Therefore,

$$\sum_k \limsup_{\lambda \rightarrow \infty} \phi_\lambda(k) \leq \frac{2|B_1|}{d(2\pi)^d} \int_{\mathbf{R}^d} e^{-dV(y)/2} dy.$$

2. We assumed that

$$V(x) \geq \alpha|x| + c.$$

Consequently,

$$e^{-d\lambda/2}(\lambda - \lambda_k)_+ \leq e^{-d\lambda/2}(\lambda - \tilde{\lambda}_k)_+$$

where $\tilde{\lambda}_k$ is the k -th eigenvalue of the operator $\log(1 - \Delta) + \alpha|\cdot| + c$. We can therefore assume without loss of generality that

$$V(x) = \alpha|x| + c.$$

Note that in this case

$$\begin{aligned}
V * g^2(x) &= \int_{\mathbf{R}^d} (\alpha|y| + c)g(x-y)^2 dy \\
&= \alpha \int_{\mathbf{R}^d} |y|g(x-y)^2 dy + c \\
&= \alpha \int_{\mathbf{R}^d} |x+z|g(z)^2 dz + c \\
&\leq \alpha \int_{\mathbf{R}^d} (|x| + |z|)g(z)^2 dz + c \\
&= V(x) + \alpha \int_{\mathbf{R}^d} |z|g(z)^2 dz.
\end{aligned}$$

Consequently,

$$\int_{\mathbf{R}^d} (V * g^2(x) - V(x)) |\psi_k(x)|^2 dx \leq \alpha \int_{\mathbf{R}^d} |z|g(z)^2 dz,$$

since ψ_k is normalized. For brevity, denote

$$I = \int_{\mathbf{R}^d} |\nabla g(z)|^2 dz$$

and

$$J = \alpha \int_{\mathbf{R}^d} |z|g(z)^2 dz.$$

Just like before, we can estimate

$$\begin{aligned}
(\lambda - \lambda_k)_+ &\leq \left(\int_{\mathbf{R}^d \times \mathbf{R}^d} (\lambda - \log(1 + |\xi|^2) - V(y)) |\Phi\psi_k(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d} + I + J \right)_+ \\
&= \left(\int_{\mathbf{R}^d \times \mathbf{R}^d} (\lambda - \log(1 + |\xi|^2) - V(y) + I + J) |\Phi\psi_k(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d} \right)_+ \\
&\leq \int_{\mathbf{R}^d \times \mathbf{R}^d} (\lambda - \log(1 + |\xi|^2) - V(y) + I + J)_+ |\Phi\psi_k(\xi, y)|^2 \frac{d\xi dy}{(2\pi)^d}
\end{aligned}$$

Recall that $\sum_{k=1}^{\infty} |\Phi\psi_k(\xi, y)|^2 = 1$ and thus for every $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbf{N}$ with

$$\sum_{k=N_\varepsilon}^{\infty} |\Phi\psi_k(\xi, y)|^2 \leq \varepsilon.$$

Let $\varepsilon > 0$. Then

$$\begin{aligned}
\sum_{k=N_\varepsilon}^{\infty} (\lambda - \lambda_k)_+ &\leq \varepsilon \int_{\mathbf{R}^d \times \mathbf{R}^d} (\lambda - \log(1 + |\xi|^2) - V(y) + I + J)_+ \frac{d\xi dy}{(2\pi)^d} \\
&= \varepsilon e^{d\lambda/2} \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(-\log(e^{-\lambda} + |\eta|^2) - V(y) + I + J \right)_+ \frac{d\eta dy}{(2\pi)^d} \\
&\leq \varepsilon e^{d\lambda/2} \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(-\log(|\eta|^2) - V(y) + I + J \right)_+ \frac{d\eta dy}{(2\pi)^d} \\
&= \varepsilon e^{d\lambda/2} \frac{2|B_1|}{d(2\pi)^d} \int_{\mathbf{R}^d} e^{-d(V(y)-I-J)/2} dy \\
&= \varepsilon e^{d\lambda/2} \frac{2|B_1|}{d(2\pi)^d} e^{d(I+J)/2} \int_{\mathbf{R}^d} e^{-dV(y)/2} dy.
\end{aligned}$$

Write $A = \frac{2|B_1|}{d(2\pi)^d} e^{d(I+J)/2} \int_{\mathbf{R}^d} e^{-dV(y)/2} dy$. Then

$$\sum_{k=N_\varepsilon}^{\infty} \phi_\lambda(k) \leq \varepsilon A$$

and thus

$$\limsup_{\lambda} \sum_{k=N_\varepsilon}^{\infty} \phi_\lambda(k) \leq \varepsilon A.$$

Therefore

$$\lim_N \limsup_{\lambda} \sum_{k=N}^{\infty} \phi_\lambda(k) = 0.$$

3. We just need to combine points 1. and 2. and find

$$\limsup_{\lambda} e^{-d\lambda/2} \sum_k (\lambda - \lambda_k)_+ \leq \sum_k \limsup_{\lambda} e^{-d\lambda/2} (\lambda - \lambda_k)_+ \leq \frac{2|B_1|}{d(2\pi)^d} \int_{\mathbf{R}^d} e^{-dV(y)/2} dy. \quad \square$$

9 Appendix

9.1 Compact Operators

We recall basic properties of compact operators and the spectral theorem for compact self-adjoint operators. We follow [Wei00] and [Bre83].

9.1.1 Basic Properties

Definition 9.1. A bounded linear operator $K \in \mathcal{L}(\mathcal{H})$ is said to be *compact* if it maps weakly convergent sequences to (strongly) convergent one, i.e. if $Ku_n \rightarrow 0$ for any sequence $(u_n) \subseteq \mathcal{H}$ such that $u_n \rightharpoonup 0$. We denote the set of all compact operators on $\mathcal{H}$ by $\mathcal{K}(\mathcal{H})$.

From the definition it is obvious that $\mathcal{K}(\mathcal{H})$ is a linear subspace of $\mathcal{L}(\mathcal{H})$. It turns out that $\mathcal{K}(\mathcal{H})$ is even a closed in the operator norm topology and that it is a two-sided $*$ -ideal in the algebra $\mathcal{L}(\mathcal{H})$. See [Wei00] for proofs of these assertions.

Proposition 9.2. *The space $\mathcal{K}(\mathcal{H})$ satisfies the following properties:*

1. *If $K \in \mathcal{K}(\mathcal{H})$ and $A, B \in \mathcal{L}(\mathcal{H})$, then $AKB \in \mathcal{K}(\mathcal{H})$.*
2. *If $(K_n) \subseteq \mathcal{K}(\mathcal{H})$ and $\|K - K_n\| \rightarrow 0$, then $K \in \mathcal{K}(\mathcal{H})$.*
3. *If $K \in \mathcal{K}(\mathcal{H})$, then $K^* \in \mathcal{K}(\mathcal{H})$.*
4. *For every $K \in \mathcal{K}(\mathcal{H})$, there is a sequence of finite rank operator (K_n) such that $\|K - K_n\| \rightarrow 0$.*

The spectra of compact operators exhibit nice properties.

Proposition 9.3. *Let $K \in \mathcal{K}(\mathcal{H})$ with $\dim \mathcal{H} = \infty$.*

1. $0 \in \sigma(K)$
2. $\sigma(K) \setminus \{0\}$ *is either empty, finite, or consists of a sequence converging to 0*

In other words, $\sigma(K) \setminus \{0\}$ is discrete and if $\mathcal{H}$ is infinite-dimensional, $\sigma_e(K) = \{0\}$.

9.1.2 The Spectral Theorem for Compact Self-Adjoint Operators

We state the most important theorem of this section: the spectral theorem for self-adjoint compact operators.

Theorem 9.4. *Let K be a compact self-adjoint operator in the Hilbert space $\mathcal{H}$. There exist countably many real numbers λ_n with $|\lambda_n| \geq |\lambda_{n+1}| > 0$ and orthonormal elements $u_n \in \mathcal{H}$ such that*

$$Ku = \sum_n \lambda_n \langle u_n, u \rangle u_n.$$

If $\mathcal{H}$ is infinite-dimensional, then $\lambda_n \rightarrow 0$.

Remark 44. Note that, in contrast to the spectral theorem for general self-adjoint operators, we do not require that $\mathcal{H}$ be countable. The kernel $\ker(K)$ might well be non-separable and the above theorem would still work. The separability of $\ker(K)^\perp$ on the other hand is an easy consequence of the compactness of K .

We need the following technical fact.

Proposition 9.5. *Let $K \in \mathcal{K}(\mathcal{H})$ be a compact operator and let $(A_n) \subseteq \mathcal{L}(\mathcal{H})$ be a sequence of operators converging strongly to an operator A . Then $(A_n K)$ converges in operator norm to AK .*

Proof. Without loss of generality, assume that $A = 0$. First note that $\sup \|A_n\| < \infty$ by the Banach–Steinhaus theorem. Let $\varepsilon > 0$. Since K is the limit of finite rank operators, we can find a finite rank operator L such that $\|K - L\| \leq \varepsilon$. Since L has finite rank, one can see that $A_n L \rightarrow 0$. Finally

$$\|A_n K\| \leq \|A_n L\| + \sup_k \|A_k\| \varepsilon,$$

and the right-hand side can be made arbitrarily small. □

9.1.3 Traces

An important class of compact operators is formed by the *trace class operators*.

Definition 9.6. A positive bounded operator A on a Hilbert space $\mathcal{H}$ is said to be of *trace class*, if for some orthonormal basis $(u_j)_{j \in J}$ of $\mathcal{H}$ the sum $\sum_{j \in J} \langle u_j, Au_j \rangle$ converges. The *trace* of A is then defined as

$$\operatorname{tr} A = \sum_{j \in J} \langle u_j, Au_j \rangle.$$

Remark 45. We need to prove that $\operatorname{tr} A$ is actually well-defined. Note that for the sum in the definition to converge, there can at most be countably many terms that do not vanish.

Proposition 9.7. *Let A be a positive bounded operator on a Hilbert space $\mathcal{H}$. If the sum $\sum_{j \in J} \langle u_j, Au_j \rangle$ converges for some orthonormal basis of $\mathcal{H}$, then it does so for all of them. Furthermore, the sums obtained this way coincide.*

Proposition 9.8. *Trace class operators are compact.*

A self-adjoint trace-class operator can thus be diagonalized. Its trace coincides with the sum of its eigenvalues.

Proposition 9.9. *Let A be a positive self-adjoint operator of trace-class. Then*

$$\operatorname{tr} A = \sum_n \lambda_n,$$

where λ_n are the eigenvalues of A .

9.1.4 Hilbert–Schmidt Operators

Another class of compact operators that is of particular importance is that of Hilbert–Schmidt operators. Let (X, μ) be a measure space and let $k \in L^2(X \times X, \mu \times \mu)$. Then one can define an operator $K: L^2(X, \mu) \rightarrow L^2(X, \mu)$ via

$$Ku(x) = \int_X k(x, y)u(y)d\mu(y).$$

For almost every $x \in X$, the function $k(x, \cdot)$ is square integrable by Tonelli’s theorem and therefore the integral on the right-hand side converges for almost every x . Furthermore, applying Hölder’s inequality we find that

$$|Ku(x)|^2 \leq \int_X |u(y)|^2 d\mu(y) \int_X |k(x, y)|^2 d\mu(y),$$

and integrating with respect to x yields

$$\int |Ku(x)|^2 d\mu(x) \leq \int_X |u(y)|^2 d\mu(y) \int_{X \times X} |k(x, y)|^2 d\mu(y) d\mu(x).$$

Therefore Ku is indeed an element of $L^2(X, \mu)$ and K is bounded operator on $L^2(X, \mu)$.

Proposition 9.10. *The operator K defined above is compact.*

Proof. Suppose that $(u_n) \subseteq L^2(X, \mu)$ converge weakly to zero. As noted above, $k(x, \cdot)$ is

in $L^2(X, \mu)$ for almost every $x \in X$. Therefore,

$$Ku_n(x) = \int_X k(x, y)u_n(y)d\mu(y) = \langle k(x, \cdot), u_n \rangle \rightarrow 0$$

for almost every $x \in X$. Since (u_n) converges weakly, $(\|u_n\|)$ is bounded by the Banach–Steinhaus theorem and since

$$|Ku_n(x)|^2 \leq \|u_n\|^2 \int_X |k(x, y)|^2 d\mu(y),$$

we may apply Lebesgue’s dominated convergence theorem and find that $Ku_n \rightarrow 0$. $\square$

9.2 Sobolev Spaces

This section on Sobolev spaces mainly follows the textbook [\[Bre83\]](#) by Brezis.

9.2.1 Definitions

Consider an open subset Ω of $\mathbf{R}^d$. Any function $u \in L^1_{\text{loc}}(\mathbf{R}^d)$ can be viewed as a distribution and can be differentiated as such. Its derivatives $\partial^\alpha u$, $\alpha \in \mathbf{N}_0^d$ will again be distributions, which one might or might not be able to identify with functions in $L^1_{\text{loc}}(\mathbf{R}^d)$. Since $L^p(\mathbf{R}^d) \subseteq L^1_{\text{loc}}(\mathbf{R}^d)$ for any $p \geq 1$, we can define the derivatives of arbitrary L^p -functions. The idea of Sobolev spaces is to consider only those L^p -functions whose derivatives up to a certain order are again L^p -function. We will focus on the case $p = 2$, since this is the only one we will need.

Definition 9.11. For $k \in \mathbf{N}$ the Sobolev space $H^k(\Omega)$ is defined as the space

$$H^k(\Omega) = \{u \in L^2(\Omega) \mid \forall \alpha \in \mathbf{N}_0^d: \partial^\alpha u \in L^2(\Omega)\}$$

equipped with the inner product

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}.$$

Remark 46. That $H^k(\Omega)$ is indeed a vector space is quite obvious. To see that $\langle \cdot, \cdot \rangle_{H^k(\Omega)}$ is indeed an inner product, note that

$$\langle u, u \rangle_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \langle \partial^\alpha u, \partial^\alpha u \rangle_{L^2(\Omega)} \geq \sum_{|\alpha|=0} \langle \partial^\alpha u, \partial^\alpha u \rangle_{L^2(\Omega)} = \langle u, u \rangle_{L^2(\Omega)},$$

which proves the non-degeneracy of the map $\langle \cdot, \cdot \rangle_{H^k(\Omega)}$. Symmetry and sesquilinearity are obvious.

It turns out that the Sobolev spaces $H^k(\Omega)$ are Hilbert spaces.

Theorem 9.12. *For $k \in \mathbf{N}$ the Sobolev space $H^k(\Omega)$ is complete with respect to the norm induced by the inner product $\langle \cdot, \cdot \rangle_{H^k(\Omega)}$.*

Proof. See [Bre83]. □

Sobolev spaces are extremely important in the study of partial differential equations. Spaces of differentiable functions in the classical sense, like $C^\infty(\Omega)$ or $C_c^\infty(\Omega)$ cannot be viewed as Hilbert spaces in any reasonable way. Sobolev's theory now provides us with Hilbert spaces of functions that are differentiable in some sense. This way one is enabled to use the spectral theory of operators on Hilbert spaces to study partial differential equations.

While $H^k(\Omega)$ contains functions that not differentiable in the classical sense, it is true that the $C^\infty(\Omega)$ are dense in $H^k(\Omega)$.

Theorem 9.13 (Meyers–Serrin). *Let $\Omega \subseteq \mathbf{R}^d$ be an open set. Then $C^\infty(\Omega) \cap H^k(\Omega)$ is a dense subspace of $H^k(\Omega)$.*

Proof. See [MS64]. □

This theorem states that the closure of $C^\infty(\Omega)$ in $H^k(\Omega)$ is $H^k(\Omega)$ itself. Another important class of differentiable functions on Ω is that of those that are smooth and compactly supported, i.e. $C_c^\infty(\Omega)$. Its closure in $H^k(\Omega)$ is denoted by $H_0^k(\Omega)$.

Definition 9.14. The Sobolev space $H_0^k(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in the space $H^k(\Omega)$.

Remark 47. Note that by definition, $H_0^k(\Omega)$ is a closed subspace of $H^k(\Omega)$

Remark 48. In the case $\Omega = \mathbf{R}^d$, we have $H_0^k(\mathbf{R}^d) = H^k(\mathbf{R}^d)$.

We are primarily interested in the case $k = 1$, since the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ are the ones we use to define the Neumann and Dirichlet Laplacians on Ω .

Remark 49. In the case $\Omega = \mathbf{R}^d$, we can characterize $H^1(\mathbf{R}^d)$ using the Fourier transform. Indeed,

$$H^1(\mathbf{R}^d) = \{u \in L^2(\mathbf{R}^d) \mid (1 + |\xi|^2)^{1/2} \hat{u}(\xi) \in L^2(\mathbf{R}^d)\}.$$

In this case $\|u\|_{H^1(\mathbf{R}^d)} = \|(1 + |\xi|^2)^{1/2} \hat{u}(\xi)\|_{L^2(\mathbf{R}^d)}$. This can easily be seen from Plancherel's theorem.

9.2.2 Extensions

By extending a function in $H_0^1(\Omega)$ by zero outside of Ω , we clearly obtain an element of $H^1(\mathbf{R}^d)$. Whether or not functions from $H^k(\Omega)$ can be extended to function in $H^1(\Omega)$ is the content of *extension theorems*.

Definition 9.15. Let $\Omega \subseteq \mathbf{R}^d$ be an open set. A bounded operator $E: H^1(\Omega) \rightarrow H^1(\mathbf{R}^d)$ is said to be an *extension operator* if for every $u \in H^1(\Omega)$ one has $(Eu)(x) = u(x)$ for almost every $x \in \Omega$. If Ω admits an extension operator, then it is said to have the *extension property*.

One important example for domains that have the extension property is given by Stein's theorem.

Theorem 9.16. *Any bounded Lipschitz domain $\Omega \subseteq \mathbf{R}^d$ has the extension property.*

Proof. See [Ste70]. □

9.2.3 Embedding Theorems

Keeping in mind [2.35], we might ask ourselves whether $H^1(\Omega)$ and $H_0^1(\Omega)$ can be embedded compactly into $L^2(\Omega)$. The idea is the following: $H_0^1(\Omega)$ functions can be extended to $H^1(\mathbf{R}^d)$ functions and so do $H^1(\Omega)$ functions if Ω has the extension property. The extension in both cases is a bounded linear operator. If can show that restricting $H^1(\mathbf{R}^d)$ functions to Ω yields a compact operator $H^1(\mathbf{R}^d) \rightarrow L^2(\Omega)$, then the embeddings $H_0^1(\Omega) \rightarrow L^2(\Omega)$ and $H^1(\Omega) \rightarrow L^2(\Omega)$ would be compact as compositions of bounded operators with a compact operator.

Theorem 9.17. *Let $\Omega \subseteq \mathbf{R}^d$ be an open set of finite measure. Then the map $H^1(\mathbf{R}^d) \rightarrow L^2(\Omega)$ given by restricting functions to Ω is a compact operator.*

Proof. See [Lew22]. □

As a consequence we obtain the following two embedding theorem.

Theorem 9.18. *Let Ω be an open set of finite measure. Then the embedding $H_0^1(\Omega) \rightarrow L^2(\Omega)$ is compact.*

Theorem 9.19. *Let Ω be an open set of finite measure with the extension property. Then the embedding $H^1(\Omega) \rightarrow L^2(\Omega)$ is compact.*

9.2.4 Sobolev Inequalities

One important question in Sobolev space theory is whether Sobolev spaces embed into L^p -spaces other than L^2 . More vaguely put, does integrability of the derivatives of a function imply integrability of the function to a higher degree?

Theorem 9.20. *Let $k \geq 1$, $d \geq 1$. We have the following continuous embeddings*

1. *If $k < \frac{d}{2}$, then $H^k(\mathbf{R}^d) \hookrightarrow L^p(\mathbf{R}^d)$ for $p = \frac{2d}{d-2k}$.*
2. *If $k = \frac{d}{2}$, then $H^k(\mathbf{R}^d) \hookrightarrow L^p(\mathbf{R}^d)$ for $p \in [2, \infty)$.*
3. *If $k > \frac{d}{2}$, then $H^k(\mathbf{R}^d) \hookrightarrow L^\infty(\mathbf{R}^d)$.*

Proof. See [Bre83]. □

The inequalities used to prove the embeddings above also imply the following useful inequality, known as *Poincaré's inequality*.

Theorem 9.21. *Let $d \geq 1$. There exists a constant $C_{>0}$ such that for all open sets of finite measure $\Omega \subseteq \mathbf{R}^d$ and $u \in H_0^1(\Omega)$ one has*

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{C_d}{|\Omega|^{2/d}} \int_{\Omega} |u|^2 dx.$$

Proof. See [FLW23]. □

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