

Credit Modelling

Generating Spread Dynamics with Intensities and Creating Dependence with Copulas

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Declaration

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline

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- I dedicate this thesis to my mother.

Abstract

The thesis is an investigation into the pricing of credit risk under the intensity framework with a copula generating default dependence between obligors. The challenge of quantifying credit risk and the derivatives that are associated with the asset class has seen an explosion of mathematical research into the topic. As credit markets developed the modelling of credit risk on a portfolio level, under the intensity framework, was unsatisfactory in that either:

1. The state variables of the intensities were driven by diffusion processes and so could not generate the observed level of default correlation (see Schönbucher (2003a)) or,
2. When a jump component was added to the state variables, it solved the problem of low default correlation, but the model became intractable with a high number of parameters to calibrate to (see Chapovsky and Tevaras (2006)) or,
3. Use was made of the conditional independence framework (see Duffie and Garleanu (2001)). Here, conditional on a common factor, obligors' intensities are independent. However the framework does not produce the observed level of default correlation, especially for portfolios with obligors that are dispersed in terms of credit quality.

Practitioners seeking to have interpretable parameters, tractability and to reproduce observed default correlations shifted away from generating default dependence with intensities and applied copula technology to credit portfolio pricing. The one factor Gaussian copula and some natural extensions, all falling under the factor framework, became standard approaches. The factor framework is an efficient means of generating dependence between obligors. The problem with the factor framework is that it does not give a representation to the dynamics of credit risk, which arise because credit spreads evolve with time.

A comprehensive framework which seeks to address these issues is developed in the thesis. The framework has four stages:

1. Choose an intensity model and calibrate the initial term structure.
2. Calibrate the variance parameter of the chosen state variable of the intensity model.

3. When extended to a portfolio of obligors choose a copula and calibrate to standard market portfolio products.
4. Combine the two modelling frameworks, copula and intensity, to produce a dynamic model that generates dependence amongst obligors.

The thesis contributes to the literature in the following way:

- It finds explicit analytical formula for the pricing of credit default swaptions with an intensity process that is driven by the extended Vasicek model. From this an efficient calibration routine is developed.

Many works (Jamshidian (2002), Morini and Brigo (2007) and Schönbucher (2003b)) have focused on modelling credit swap spreads directly with modified versions of the Black and Scholes option formula. The drawback of using a modified Black and Scholes approach is that pricing of more exotic structures whose value depend on the term structure of credit spreads is not feasible. In addition, directly modelling credit spreads, which is required under these approaches, offers no explicit way of simulating default times.

In contrast, with intensity models, there is a direct mechanism to simulate default times and a representation of the term structure of credit spreads is given.

Brigo and Alfonsi (2005) and Bielecki et al. (2008) also consider intensity modelling for the purposes of pricing credit default swaptions. In their works the dynamics of the intensity process is driven by the Cox Ingersoll and Ross (CIR) model. Both works are constrained because the parameters of the CIR model they consider are constant. This means that when there is more than one tradeable credit default swaption exact calibration of the model is usually not possible. This restriction is not in place in our methodology.

- The thesis develops a new method, called the *loss algorithm*, in order to construct the loss distribution of a portfolio of obligors. The current standard approach developed by Turc et al. (2004) requires differentiation of an interpolated curve (see Hagan and West (2006) for the difficulties of such an approach) and assumes the existence of a base correlation curve. The *loss algorithm* does not require the existence of a base correlation curve or differentiation of an interpolated curve to imply the portfolio loss distribution.
- Schubert and Schönbucher (2001) show theoretically how to combine copula models and stochastic intensity models. In the thesis the Schubert and Schönbucher (2001)

framework is implemented by combining the extended Vasicek model and the Gaussian copula model. An analysis of the impact of the parameters of the combined models and how they interact is given. This is as follows:

- The analysis is performed by considering two products, securitised loans with embedded triggers and leverage credit linked notes with recourse. The two products both have dependence on two obligors, a counterparty and a reference obligor.
 - Default correlation is shown to impact significantly on pricing.
 - We establish that having large volatilities in the spread dynamics of the reference obligor or counterparty creates a de-correlating impact: the higher the volatility the lower the impact of default correlation.
 - The analysis is new because, classically, spread dynamics are not considered when modelling dependence between obligors.
- The thesis introduces a notion called the *stochastic liquidity threshold* which illustrates a new way to induce intensity dynamics into the factor framework.
 - Finally the thesis shows that the valuation results for single obligor credit default swaptions can be extended to portfolio index swaptions after assuming losses on the portfolio occur on a discretised set and independently to the index spread level.

Contents

1	Introduction	2
1.0.1	Summary of sections in this chapter	2
1.1	Credit as an asset class	3
1.1.1	Historical defaults	4
1.1.2	Credit derivatives and obligations	5
1.2	Classical Modelling Routes	6
1.2.1	Single obligor	7
1.2.2	Multiple obligor	9
1.3	Summary of chapters of this thesis	13
2	Products	17
2.0.1	Summary of sections in this chapter	17
2.1	Motivating Example	18
2.2	Credit Default Swap	19
2.2.1	Forward CDS and credit default swaptions	23
2.3	Portfolio Derivatives	23
2.3.1	NTDs	23
2.3.2	Collateralised debt obligations	27
2.3.3	Portfolio risk characteristics	28
2.3.4	Standardised portfolios	29
2.4	Credit linked notes	31
2.4.1	Leveraged credit linked notes	32
2.5	Chapter conclusion	34

3	Single Name Valuation, Modelling and Calibration	36
3.0.1	Summary of sections in this chapter	36
3.1	Preliminary results	38
3.1.1	No arbitrage pricing	38
3.1.2	Intensity modelling	40
3.1.3	Filtration switching	43
3.2	Pricing in the single obligor case	46
3.2.1	Losses	46
3.2.2	Pricelets	47
3.2.3	Valuation of a CDS	49
3.2.4	Market standard for CDS valuation	49
3.2.5	Numerical example: CDS valuation	51
3.2.6	Forwards and Swaptions	54
3.2.7	Pre-default quantities and equilibrium swap spreads	57
3.3	The extended Vasicek Model	59
3.3.1	Model setup	59
3.3.2	Analytical valuation for the extended Vasicek model	61
3.3.3	Historical credit spreads, convexity adjustments and the real world measure	64
3.3.4	Model calibration	67
3.3.5	Numerical results	69
3.3.6	Analysis of the extended Vasicek model	73
3.4	Leveraged credit linked note without recourse	74
3.4.1	Default time construction	74
3.4.2	The pricing method	75
3.4.3	Fair spread and risk analysis	76
3.4.4	Numerical results	77
3.5	Alternative models for the intensity process	92
3.5.1	Cox Ingersoll Ross model	92

3.5.2 Pure jump model	93
3.5.3 Numerical results	94
3.6 Chapter conclusion	99
4 Multiple Defaults And Dependence	101
4.0.1 Summary of sections in this chapter	101
4.1 Portfolio pricing	103
4.2 Mutually independent defaults and the conditionally independent framework .	105
4.2.1 The case of mutually independent default times	106
4.2.2 Conditional independence	107
4.2.3 Generating default dependence by correlating intensities	108
4.3 Copula and the factor framework	110
4.3.1 Copula	110
4.3.2 Factor framework	112
4.3.3 One factor Gaussian copula	114
4.4 Index swap valuation	116
4.4.1 The standard Index and market conventions	117
4.4.2 The credit index swap basis	119
4.5 Correlation Curves	120
4.5.1 Calibrating the correlation curve	120
4.5.2 Compound correlation	121
4.5.3 Base correlation	123
4.6 Random recovery	126
4.6.1 Numerical example	129
4.6.2 Arbitrage in the random recovery	132
4.7 Generating a portfolio's loss distribution: a new loss algorithm	133
4.7.1 The current method of extracting the loss distribution	134
4.7.2 The loss algorithm	136
4.7.3 Numerical example	138

4.8	Random factor loading	139
4.9	Chapter conclusion	142
5	Spread Dynamics With Default Correlation	145
5.0.1	Summary of sections in this chapter	146
5.1	Framework	147
5.1.1	The full intensity of an obligor	148
5.1.2	The survival and threshold copula	150
5.2	Securitised loans	151
5.2.1	Collateralisation	152
5.2.2	Documentation	153
5.2.3	Risk characteristics	153
5.2.4	The pricing method and model	154
5.2.5	Numerical example	157
5.2.6	Remarks	160
5.3	Leverage with recourse credit linked notes	160
5.3.1	The pricing method and model	161
5.3.2	Numerical Example	162
5.4	Chapter conclusion	166
6	Future Research	168
6.0.1	Summary of sections in this chapter	168
6.1	Factor framework: a new way forward	169
6.1.1	Toward a model with spread evolution	171
6.2	Index swaptions with intensity models	174
6.2.1	Framework	176
6.2.2	Pricing	177
6.3	Chapter conclusion	182
7	Conclusion Of Thesis	184

A The Structural Model	187
B Mathematical Importance Of Conditional Poisson Process	189
B.1 Bremaud's characterisation	189
B.2 Girsanov's theorem for point processes	193
B.3 Correlation neutral measure	194
C Interest Rates: Theory And Application To Credit	197
C.1 Zero coupon bonds and the forward rate	197
C.2 The Heath-Jarrow-Morton framework and the T-Forward measure	198
D Affine Models	201
D.1 Diffusion models	201
D.2 Jump diffusion models	203
E Extended Vasicek Model	206
E.1 Gaussian processes	206
E.2 Model solution	207
F The Factor Framework And Copula Algorithms	210
F.1 Recursion Method	210
F.1.1 Generic method of extracting the expected loss	210
F.2 Bespoke CDO Pricing	211
F.2.1 Mixing base correlation	212
F.3 The basic computational algorithm for the Threshold and Survival copulas . . .	213
G Market Data	214
References	221

Nomenclature

Acronyms

BCF	Base correlation framework
BCLN	Basic credit linked note
CDIS	Credit default index swap
CDO	Collateralised debt obligation
CDS	Credit default swap
CIR	Cox Ingersoll and Ross
CLN	Credit linked note
CLS	Credit linked swap
CPP	Conditional Poisson process
FCDS	Forward credit default swap
FEP	Front end protection
HJM	Heath Jarrow Morton
ISDA	International Swaps and Derivatives Association
JTD	Jump to default
LAIS	Loss adjusted index spread
LCLN	Leveraged credit linked note
LGD	Loss given default
MTM	Mark to market
NTD	n^{th} to default swap

OFGC One factor Gaussian copula

PB Protection buyer

PS Protection seller

RFL Random factor loading

Credit Risky Quantities

$\mathbf{D}(t, T)$ The discounted payoff of the default leg of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\mathbf{D}^I(t, T)$ The discounted payoff of the default leg of a CDIS started at time $t \geq 0$ for a maturity of $T > t$

$\tilde{\mathbf{D}}(t, T)$ The pre-default present value of the default leg of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\tilde{\mathbf{D}}^I(t, T)$ The pre-collapse present value of the default leg of a CDIS started at time $t \geq 0$ for a maturity of $T > t$

$\bar{\mathbf{D}}_L(t, T)$ The present value of the default leg of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\bar{\mathbf{D}}_L^I(t, T)$ The present value of the default leg of a CDIS started at time $t \geq 0$ for a maturity of $T > t$

$\mathbf{val}(t, T)$ The discounted payoff of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\mathbf{val}^I(t, T)$ The discounted payoff of a CDIS started at time $t \geq 0$ for a maturity of $T > t$

$\mathbf{P}(t, T)$ The discounted payoff of the premium leg of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\mathbf{P}^I(t, T)$ The discounted payoff of the premium leg of a CDIS started at time $t \geq 0$ for a maturity of $T > t$

$\tilde{\mathbf{P}}(t, T)$ The pre-default present value of the premium leg of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\tilde{\mathbf{P}}^I(t, T)$ The pre-collapse present value of the premium leg of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\bar{\mathbf{P}}_{\mathbf{L}}(t, T)$ The present value of the premium leg of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\bar{\mathbf{P}}_{\mathbf{L}}^I(t, T)$ The present value of the premium leg of a CDIS started at time $t \geq 0$ for a maturity of $T > t$

$\overline{\mathbf{val}}_{\mathbf{L}}(t, T)$ The present value of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\overline{\mathbf{val}}_{\mathbf{L}}^I(t, T, k)$ The present value of a CDIS started at time $t \geq 0$ for a maturity of $T > t$

$\widetilde{\mathbf{val}}(t, T)$ The pre-default present value of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\widetilde{\mathbf{val}}^I(t, T)$ The pre-collapse present value of a CDIS started at time $t \geq 0$ for a maturity of $T > t$

$\tilde{s}(t, T)$ The pre-default fair swap spread for a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$\tilde{s}^I(t, T)$ The pre-collapse fair CDIS swap spread started at time $t \geq 0$ for a maturity of $T > t$

$Q(t, T)$ The survival probability of an obligor at time $t \geq 0$ for a maturity of $T > t$

R Recovery rate

$s(t, T)$ The swap spread of a credit contingent swap started at time $t \geq 0$ for a maturity of $T > t$

$v(t, T)$ Defaultable zero coupon bond price at time $t \geq 0$ for a maturity of $T > t$

Functions And Distributions

$\Phi(x)$ Standard cumulative normal distribution

$E(y)$ The standard error function

$E_i(y) = E(iy)/i$, where $E(.)$ is the standard error function and i is imaginary

$(x)^+ = \max(x, 0)$

$[x]$ The integer part of a number $x \in \mathbb{R}_+$

$\mathbb{C}(X_1, X_2)$ The correlation between random variables X_1 and X_2

$\mathbb{V}(X_1, X_2)$ The variance between random variables X_1 and X_2

$\phi(x)$ The density function of a standard normal with $x \in \mathbb{R}$

$\Phi_2(z_1, z_1, \rho)$ The bivariate normal distribution function with $z_1, z_2 \in \mathbb{R}$ and correlation coefficient $\rho \in [-1, 1]$

C^S The survival copula

C^T The threshold copula

$N(b, a)$ Normal distribution with mean $b \in \mathbb{R}$ and variance $a \in \mathbb{R}_+$

$p_i(t)$ The default probability of an obligor with identity i for the period between 0 and time $t \geq 0$

$F(\mathbf{x})$ The joint distribution of a vector of random variables, where $\mathbf{x} = (x_1, \dots, x_n)$ and $n \in \mathbb{N}$

Filtrations

$(\hat{\mathcal{F}}_t)_{t \geq 0} = \hat{\mathbb{F}}$ The filtration containing the sub-filtration and information about all but the last obligor to default

$(\mathcal{F}_t)_{t \geq 0} = \mathbb{F}$ Sub-filtration

$(\mathcal{G}_t)_{t \geq 0} = \mathbb{G}$ Enlarged filtration

$(\tilde{\mathcal{G}}_t)_{t \geq 0} = \tilde{\mathbb{G}}$ Partial filtration

$(\mathcal{D}_t)_{t \geq 0} = \mathbb{D}$ The minimal filtration generated by default processes

Default-Free Interest Rate Quantities

$P(t, T)$ Risk free zero coupon bond price at time $t \geq 0$ for a maturity of $T > t$

Processes

$(\lambda_t)_{t \geq 0}$ Intensity process

$(L_t)_{t \geq 0}$ Credit portfolio loss process

$\tau_{(m)}$ The default time of the m^{th} obligor to default, with $m \in \mathbb{N}$

$(\hat{F}_t)_{t \geq 0}$ The $\hat{\mathbb{F}}$ – conditional cumulative distribution process of the collapse time $\hat{\tau}$

$(\hat{Q}_t)_{t \geq 0}$ The $\hat{\mathbb{F}}$ – conditional survival process of the collapse time, $\hat{\tau}$

$(F_t)_{t \geq 0}$ The conditional cumulative distribution process of the default time, τ , of an obligor

$(N_t)_{t \geq 0}$ Point process

$(Q_t)_{t \geq 0}$ The $\mathbb{F}$ – *conditional survival process* of the default time, τ , of an obligor

List of Figures

1.1	Observed default events in the last 100 years. Data source: <i>Moody's website</i>	5
2.1	The historical 5 year CDS swap spread of Brazil. Data from <i>MarkIt</i>	22
2.2	Sensitivities of n^{th} -to default spreads to changes in default correlation. The basket of obligors are from Table 2.1.	26
2.3	The loss of a tranche per unit of tranche notional for each percentage loss of the portfolio. Here $k_1 = 15\%$ and $k_2 = 30\%$.	28
3.1	Stripped hazard rate curve for the Federative Republic of Brazil with a piece-wise constant intensity curve.	52
3.2	Survival probability.	53
3.3	Protection leg.	53
3.4	Premium leg.	54
3.5	Forward hazard rate of Federative Republic of Brazil CDS curve and a simulated hazard rate path.	71
3.6	Integrated forward hazard rate and a simulated integrated hazard rate.	71
3.7	Surface of spot volatility against option expiry and speed of mean reversion.	72
3.8	LCLN spread premium above the bench mark interest rate, Libor, as a function of varying speed of mean reversion and extended Vasicek spot volatilities	78
3.9	LCLN $CS01$ with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities	81
3.10	LCLN hedge position of equivalent maturity CDS with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities	82
3.11	LCLN $CS01$ with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.	83

3.12	LCLN hedge position of equivalent maturity CDS with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.	84
3.13	LCLN <i>Gamma</i> with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities	87
3.14	LCLN <i>Gamma</i> with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.	88
3.15	LCLN <i>Vega</i> with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities	90
3.16	LCLN <i>Vega</i> with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.	91
3.17	The stripped forward hazard rate of the Federative Republic of Brazil along side a CIR simulated hazard rate path.	96
3.18	The integrated forward hazard rate and a simulated integrated hazard rate path for the CIR model.	97
3.19	The stripped forward hazard rate of the Federative Republic of Brazil along side a pure jump simulated hazard rate path.	97
3.20	The integrated forward hazard rate and a simulated integrated hazard rate path for the pure jump model. The simulation is done with $\mu_S = 0$, $\sigma_S = 4\%$, constant jump intensity for Γ equal to 4% and mean reversion speed of 0. . . .	98
4.1	Portfolio loss density	116
4.2	The compound correlation curve for data in Table 4.1.	123
4.3	The base correlation loss ratio curve for data in Table 4.2 with monotonic cubic interpolation and linear extrapolation.	126
4.4	The compound correlation random recovery curve for data in Table 4.3.	131
4.5	The base correlation random recovery loss ratio curve for data in Table 4.3 with monotonic cubic interpolation.	131
4.6	Tranchlet prices for Itraxx series 10.	132
4.7	Implied loss distribution using the OFGC BCF.	138
4.8	Implied loss distribution using the random recovery BCF.	139
4.9	RFL loss distributions.	142
5.1	Sensitivity of the repo spread to changes in volatility	158

5.2	Sensitivity of the repo spread to changes in the margin calling threshold. . . .	159
5.3	Sensitivity of the repo spread to changes in volatility for various default correlation values.	159
5.4	Leverage with recourse spread premium above the bench mark interest rate, Libor, as a function of varying speed of mean reversion and extended Vasicek spot volatilities with correlation fixed at 20%, 40%, 60% and 90% respectively.	165
D.1	The effects of varying the parameter a (the speed of mean reversion), defined in Definition D.1. The model used is the extended Vasicek model (see Chapter 3 (Section 3.3) for more details) with a flat credit swap spread curve of 400bps. We use a constant spot volatility, $\sigma = 15\%$	202
G.1	Part 1: obligor swap spread data for the Itraxx index on the 09 June 2009. Data source from <i>MarkIt</i>	214
G.2	Part 2: obligor swap spread data for the Itraxx index on the 09 June 2009. Data source from <i>MarkIt</i>	215

List of Tables

2.1	Federative Republic of Brazil, Federation of Russia, Republic of Hungary and Indonesia CDS swap spreads as of February 2009. Data source from <i>MarkIt</i> .	26
2.2	Sensitivities of n^{th} - to default spreads to changes in the OFGC default correlation. The basket of obligors are from Table 2.1.	26
2.3	The tranches available on Itraxx and CDX	30
3.1	Federative Republic of Brazil CDS spreads as of February 2009. Data source from <i>MarkIt</i>	52
3.2	Hazard rate node values, survival probability values, protection leg values and premium leg values. Computed using the Federative Republic of Brazil CDS curve.	52
3.3	Calibrated ATM Brazil spot volatility curve with an inferred speed of mean reversion of 5.04.	70
3.4	Simulation of a 3m5y payer swaption valuation with ATM forward strike.	70
3.5	ATM spot volatilities matrix for different speed of mean reversion levels	72
3.6	LCLN spread premium above the bench mark interest rate, Libor, as a function of varying speed of mean reversion and extended Vasicek spot volatilities.	78
3.7	LCLN <i>CS01</i> with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities.	81
3.8	LCLN hedge position of equivalent maturity CDS with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities.	82
3.9	LCLN <i>CS01</i> with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.	83
3.10	LCLN hedge position of equivalent maturity CDS with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.	84

3.11 LCLN <i>Gamma</i> with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities.	87
3.12 LCLN <i>Gamma</i> with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.	88
3.13 LCLN <i>Vega</i> with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities.	89
3.14 LCLN <i>Vega</i> with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.	90
3.15 Calibrated parameters for the CIR, pure jump and extended Vasicek models respectively. The models have been calibrated to a 2y3y ATM payer swaption with 60% log-normal volatility.	98
3.16 Risk metrics for calibrated CIR, pure jump and extended Vasicek models respectively. The models have been calibrated to a 2y3y ATM payer swaption with 60% log-normal volatility. The risks are produced for an LCLN trade. . . .	99
4.1 Itraxx series 10 compound correlation values for 09 June 2009.	122
4.2 Itraxx series 10 loss ratios and base correlations for 09 June 2009. 5y index level at 120bps	126
4.3 Itraxx series 10 base correlations and compound correlations for 09 June 2009. With random recovery parameters $\alpha = 0$ and $\beta = 50\%$. Also displayed are the loss ratios.	130
5.1 Goldman Sachs and Gazprom CDS swap spreads. Data source from <i>MarkIt</i> , February 2009	158
5.2 Leverage with recourse spread premium above the bench mark interest rate, Libor, as a function of varying correlation and extended Vasicek spot volatilities with speed of mean reversion fixed at 0. We use a hypothetical flat CDS spread of 400bps for both counterparty and underlying.	164

Chapter 1

Introduction

In this thesis we consider the modelling of credit as an asset class. Credit refers to the capacity of obligors to service their debts. By obligors we mean entities whether governments or firms who have payment obligations to creditors. There are two basic challenges with credit modelling:

- Quantify the probability that an obligor no longer meets its payment obligations i.e. goes into default.
- Develop a representation to the timing of default.

Over recent decades derivatives have been written with payoffs that are contingent on the credit performance of obligors. These derivatives are called *credit derivatives* and have helped bring transparency to the level of risk premia required to assume the credit risk of obligors (see Mingle (2007)). The complex structure of some credit derivatives, that may depend on a portfolio of obligors, presents a modelling challenge.

This thesis is concerned with the modelling of credit derivatives. In this chapter we will provide the reader with an overview of the asset class, its complexities and the possible modelling routes. Precise mathematical definitions are subsequently provided in later chapters.

1.0.1 Summary of sections in this chapter

This chapter is split into three sections:

- In Section 1.1 we provide a definition of credit risk and credit derivatives. In the same section we proceed to discuss aspects of credit risk which are challenging from a modelling perspective e.g. representing default times and quantifying default losses. We

conclude Section 1.1 by commenting on the remarkable growth of credit derivatives over the last decade, the impact of the credit crisis in 2008 and the continued economic rationale for credit derivatives.

- In Section 1.2 we outline classic modelling routes for credit derivatives. The section considers single obligor modelling approaches and multi-obligor modelling approaches.
- In Section 1.3 we provide a summary of the chapters of this thesis.

1.1 Credit as an asset class

Credit is unlike other asset classes (such as stocks, interest rates or commodities) because it refers to the capacity of an obligor to service its debt and not on a tradeable asset. The debts of an obligor are called *underlyings* and are tradeable. What we are seeking to do is model the diminished capacity or willingness to service debts. Underlyings, through their prices, help us infer the markets perception of this capacity.

Definition 1.1 (Credit Risk and the Credit Spread). *By credit risk we mean the risk of realising financial losses due to unexpected changes in the credit quality of an obligor in a financial agreement. It can occur because of a default, changes in credit spreads or changes in the rating of the obligor. By credit spread we mean the premium above the risk free rate charged to an obligor to provide it with financing.*

There does not need to be a default in order to have financial losses due to an obligor's credit worthiness. Creditors may find an obligor's debt cheapened in value because of a perception of the obligor having lower credit quality. However, the financial loss in this case will only be realised if the creditor sells the obligor's debt and locks in the drop in value of the debt. It is only when there is a default that the creditor cannot avoid realising a loss regardless of whether he sells the debt or not; this is because the obligor cannot or will not service the debt.

Defaults include bankruptcy, failure to pay interest or principal in accordance with an agreed schedule, moratorium and restructuring of the terms of the debts of an obligor. The degree of losses when there is a default is usually difficult to predict, especially for an obligor that is large with complex interests and varied assets on its balance sheet. There are multiple components determining the loss a creditor incurs given a default, for example:

- Liquidation of the assets of an obligor in default may be inefficiently executed by administrators in bankruptcy proceedings, leading to a lower than expected realised value for the assets.
- The assets of an obligor may be correlated with the obligor's default, affecting the market unexpectedly. This may cause the value of these assets to fall.
- Alternatively, and perhaps more positively, an agreement between creditors could be reached, leading to a restructuring of the terms of the debt obligations. This may mean creditors do not incur significant losses as the obligor continues as a going concern.

1.1.1 Historical defaults

In this section we consider observed historical default events.

Figure 1.1 shows the observed default events in the last 100 years.

- We see peaks around the time of the Great Depression in the 1930s, around the Dot Com Bubble from 1999 to 2003 and the credit crisis in 2008.
- The bar chart indicates that there is a general tendency for defaults to cluster i.e. obligors tend to default together due to systemic risk factors. In 2008 it was clearly because of the banking crisis and the ensuing liquidity drain that so many obligors defaulted.

In relative terms we observe from Figure 1.1 that there are not many defaults, in fact defaults are rare events. In 1999 during the Dot Com Bubble Moody's gave ratings to 4900 obligors; of these 106 defaulted in that year (see Keenan et al. (2000)).

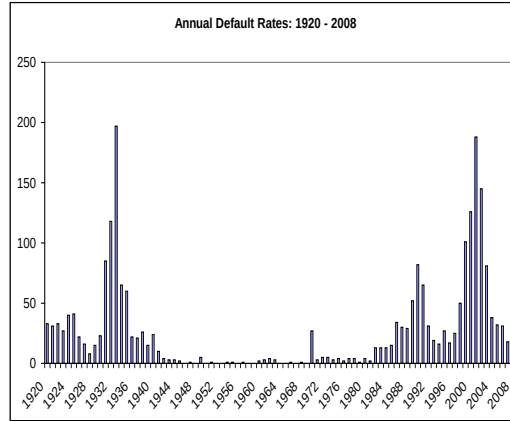


Figure 1.1: Observed default events in the last 100 years. Data source: *Moody's website*

From the discussion so far we can observe qualitatively that there are unique problems with modelling credit, which can be summarised as follows:

- Defaults are rare events.
- The credit quality of an obligor evolves with time.
- Losses can be large.
- The size of losses are unknown before a default occurs.
- Defaults are unexpected.
- Defaults cluster in time.

1.1.2 Credit derivatives and obligations

In this sub-section we discuss credit derivatives which are the main interest of this thesis:

Definition 1.2 (Credit Derivatives And Obligations). *Credit derivatives are contingent liability claims that are linked to the credit risk of obligations. Obligations may be the debt of an obligor or portfolio of obligors or even other credit derivatives.*

Every credit derivative depends on the credit performance of at least one obligor; the obligors that a credit derivative depends on are called *reference obligors*. The payoff of a credit derivative contract depends on the losses incurred due to either a default or worsening credit performance of the reference obligors.

Credit derivatives allow market participants to hedge, trade and manage the risk of portfolios of obligations. In terms of notional value outstanding credit derivatives reached a peak in January 2008, where it is estimated that outstanding notional stood at \$62 trillion, roughly three times the size of the equity markets (see Giesecke (2009)). The strong dislocation in the market in 2008 has seen a subsequent contraction in the size of credit derivatives. The bankruptcy of Lehman Brothers, the failure to pay of the Republic of Ecuador and other defaults in the period led to a flight by investors from credit derivatives. von Kleist and Mallo (2010) show that the notional outstanding of credit default swaps (defined in Chapter 2 (Section 2.2)) declined from \$57 trillion at the beginning of 2008 to \$32 trillion in mid 2009.

The economic rationale for the use of credit derivatives still holds. Financial institutions continue to make loans and give lines of credit and therefore need a mechanism to hedge credit risk. Credit derivatives perform this function (see Mengle (2007) who considers the economic importance of credit derivatives). We therefore believe the study of credit derivatives is relevant and may take on greater relevance in a regulatory environment where reducing the risk of financial losses is placed highly.

Our main objective in this thesis is to present a comprehensive modelling framework for pricing credit derivatives in the single obligor and multi-obligor setting. A significant aspect of our work is in connecting the single obligor and multi-obligor modelling frameworks. In the former the focus in the literature has been to give a representation to credit spread dynamics, while in the latter the focus has been to give a representation to default dependence between obligors. We build on the work of Schubert and Schönbucher (2001) to bridge this gap by introducing spread dynamics into the multi-obligor framework in Chapters 5 and 6.

1.2 Classical Modelling Routes

In this section we broadly discuss the common modelling approaches to familiarise the reader with the general literature. We split the discussion into two sub-sections:

- The first, Sub-section 1.2.1, deals with single obligor modelling.
- The second, Sub-section 1.2.2, deals with multi-obligor modelling.

We find it necessary to split the analysis in this way because in the literature the modelling routes for single obligor and multi-obligor products have generally had separate developments. The concern in the single obligor case has been to produce appropriate dynamics to describe credit spread evolution. In the multi-obligor case the concern has been to describe the observed dependence structure.

In general, a survey of the literature (Andersen et al. (2003), Chapovsky and Tevaras (2006), Mortensen (2006), Li (2000) and others) suggests that the features a good model should exhibit, whether modelling single obligor or multi-obligor products, are as follows:

- The model will be dynamic and able to account for the number as well as the timing of defaults.
- The model can be calibrated efficiently.
- The parameters are easily interpreted.
- The model can generate the appropriate levels of default correlation which has either been observed or is implied by current market prices of portfolio products.
- The model parameters at most grow linearly with the inclusion of obligors into a credit portfolio. By this we mean that a model should not become over parameterised due to having a large number of obligors in a credit portfolio.

1.2.1 Single obligor

There are essentially two frameworks from which to analyse credit risk in the single obligor setting:

1. The structural approach.
2. The survival analysis (or reduced form) approach.

The *structural approach* models the evolution of the value of an obligor's assets. Default is defined as the time when a structural variable, representing the value of the assets of an obligor, hits a barrier (see Appendix A for more details). The barrier is often interpreted to be a liquidity threshold and whenever the barrier is reached an obligor is said to be insolvent (see Merton (1974) where the asset values are modelled as diffusion processes).

In contrast to the structural approach the *survival analysis approach* models defaults exogenously, with parameter estimates being inferred from the market. In this thesis we focus on the survival analysis approach.

Under the survival analysis approach one typically adopts a point process with stochastic intensities. The defaults of an obligor are defined as the jump times of the point process. This form of survival analysis is called *intensity modelling*. In the context of credit risk modelling we are generally only interested in the first instance the point process, $(N_t)_{t \geq 0}$, is positive i.e. $\inf\{t \in \mathbb{R}_+ : N_t > 0\}$. This instance is called the first *default arrival time*.

In this thesis we focus on intensity models. The use of intensity models for credit pricing has been broadly studied; key works include Jarrow et al. (1997), Lando (1998), Duffie and Singleton (1999), Schubert and Schönbucher (2001), Guo et al. (2007) and Guo and Zeng (2006).

The use of intensities, in this thesis, is motivated by a need to account for spread dynamics. Therefore intensities are stochastic in the models we consider. There are two reasons why stochastic intensities work well:

1. If one has a relatively well behaved intensity process, $(\lambda_t)_{t \geq 0}$, associated with a point process, $(N_t)_{t \geq 0}$, each adapted to filtrations $(\mathcal{F}_t)_{t \geq 0}$ and $(\mathcal{G}_t)_{t \geq 0}$ respectively, then we have that:

$$N_t - \int_0^t \lambda_s ds, \quad (1.2.1)$$

is a martingale (see Revuz and Yor (2001) for a definition of a martingale and see Bremaud (1981) for more details on Equation (1.2.1)).

2. Intensity models can be put in the context of martingale theory as they “fully acknowledge the existence of information patterns that increase with time” i.e. filtrations (see Bremaud (1981)). This enables martingale calculus to be applied, see Chapter 3 (Section 3.1).

By making use of martingale calculus we can derive tractable results when we model stochastic intensities directly rather than point processes. Quantifications such as default probabilities and the timing of defaults can be derived efficiently under this setting.

Intensity models have been generally accepted as a good modelling route in the literature for the reasons outlined above. However, intensity models are in some instances criticised for having little economic connection with the real world. The structural approach, which models real world variables such as asset values are argued to be better. We believe this argument misses the fact that information relating to the solvency of an obligor is usually incomplete. Duffie and Singleton (2003) find that trading of credit products does not take place in a liquid and transparent market and that the distribution of information is asymmetric. Therefore information flow to market participants often does not include the full

picture of any internal dysfunction that may be affecting an obligor, in particular when an obligor is close to default.

We believe that the intensity approach holds more economic relevance than the (diffusion-based)¹ structural approach because it models default as a sudden event with some pre-assigned chance of occurring over a period of time. Qualitatively this is what has been observed, see Duffie and Singleton (2003). Mathematically this means we work in a framework where, under a suitable filtration, default times are totally inaccessible stopping times (meaning default times occur unannounced, see Guo and Zeng (2006)).

Our contribution to the literature in the single obligor case is:

- To provide a tractable calibration routine for an intensity process which is driven by the extended Vasicek model. The components of this calibration routine involves:
 1. The development of an analytical formula for the valuation of credit default swap-tions.
 2. Demonstrating a relationship between the credit swap spread and the intensity process. This enables one to apply regression analysis using historical credit swap spreads in order to estimate parameters of the extended Vasicek model.
- From this a rigorous analysis of the behaviour of the leveraged credit linked note without recourse product (defined in Chapter 2 (Section 2.4.1)) is performed. The analysis establishes the significant impact of spread dynamics.

1.2.2 Multiple obligor

One way to correlate default times of multi-obligors is to correlate the intensity of each obligor. For example, if intensities follow a diffusion process, one might correlate the Brownian motions driving them, see Duffie and Singleton (2003). In Chapter 4 (Sub-section 4.2.1) we show that correlating intensities in this way is inadequate, leads to little control over the dynamics of the individual intensities and does not always guarantee one can match the observed default correlation.

Another method of correlating default times with intensity models is suggested by Duffie and Garleanu (2001). They suggest a setting with common and idiosyncratic factors. We outline the method: assume the intensity of an obligor j is λ_j , then the form of the factor approach is:

$$\lambda_j = \lambda_C + \lambda_j^{(I)}, \quad (1.2.2)$$

¹Jump versions of the structural approach have been considered, see Zhou (1997).

where λ_C is the common factor and is present for all obligors and $\lambda_j^{(I)}$ is the idiosyncratic factor unique to obligor j . The common factor adds correlation between default times of different obligors. Moreover the structure means that conditional on the common factor the obligors have independent defaults, this lends the model tractability (see Chapter 4 (Sub-section 4.2.1) where the benefits of conditional independence are discussed).

However, setups such as Equation (1.2.2) are restricted in being able to generate observed levels of default correlation whenever credit quality amongst the reference obligors are dispersed. By this we mean that if a basket of obligors contains a very risky obligor and a very low risk obligor from a credit perspective then the basket is said to be dispersed. Since the maximum level the common factor, λ_C , can take is that of the lower risk intensity, assuming the idiosyncratic part of the intensity is positive, the effects of λ_C on the higher risk intensity is minimal². Therefore the common factor setup cannot generate the appropriate range of default correlations as the common factor, λ_C , does not significantly affect the more risky obligors in dispersed portfolios.

An alternative way to extend intensity modelling to multi-obligor pricing, see Duffie (1998), is to add a common jump component to the intensities of the obligors. Duffie (1998) shows this allows one to achieve the required range of default correlations. However, practitioners found this model setting computationally slow and intractable. The tractability issue is raised by Duffie (1998).

Academic research steered away from incorporating default dependence via correlating intensities. A move was made toward copula technology and the *factor framework* (see Chapter 4). The works of Li (2000), McGinty and Beinstein (2004), Andersen and Sidenius (2004), Andersen and Sidenius (2005), Andersen (2006) were significant in the development of this new multi-obligor modelling approach.

The factor framework in its classic setup considers a vector of market drivers called *static factors* (because they have no dependence on time). For this reason, models under the framework are called *one period*. The static factors generate default dependence which gives rise to a copula.

Various forms of factor framework models have been proposed and typically with these models a low dimensional factor setting is considered (in this thesis we consider factor models that are one dimensional). The classic factor framework model is the one factor

²Mortensen (2006) has suggested a fit where he takes $\lambda_j = \alpha_j \lambda_C + \lambda_j^{(I)}$. This means that the common intensity is no longer restricted to be less than the smallest individual intensity in the basket, yet it also means there are more parameters to calibrate.

Gaussian copula (OFGC) pioneered by Li (2000)³. The OFGC model remains the market standard (see Andersen and Sidenius (2005)). The OFGC has a one dimensional Gaussian factor with one (correlation) parameter describing the dependence between obligors.

Along with the factor framework an efficient method of building portfolio loss distributions, called *the recursion method*, was developed by Andersen et al. (2003) (see Appendix F). The development means portfolio products can be evaluated semi-analytically.

The problem with the OFGC model is that there exist portfolio products called index tranches (defined in Chapter 2 (Sub-section 2.3.2)) that require a more complex dependence structure than one (correlation) parameter can describe. The OFGC correlation parameter cannot be calibrated to re-price all the index tranches simultaneously (see Andersen and Sidenius (2005)). This has led to the construction of *correlation curves* based on the implied correlation values from the OFGC model for each traded index tranche.

The first correlation curve construction was *compound correlation*. The method involves implying the single OFGC correlation parameter for each traded index tranche. Index tranches are characterised by a subordination and detachment point. In order to construct the curve, the index tranche detachment points are made abscissas and the implied correlation parameters are ordinate values (this method is discussed in Chapter 4). A specifically chosen interpolation method such as piecewise linear is then applied to build up the curve. The objective with the compound correlation approach was to have a pricing curve construction similar to the construction of the volatility skew in equities (where the model is the Black and Scholes (1973) model). The problem with this curve construction method is that at times, on the one hand, we can have more than one correlation value implied by the price of an index tranche and at other times, on the other hand, no solution is possible (see O’Kane and Livesey (2004)).

McGinty and Beinstein (2004) devised the alternative curve construction method called *the base correlation approach*. The method still uses the OFGC model, but no longer suffers from lack of uniqueness and (mostly) the existence problems were fixed (this method is discussed in Chapter 4). The base correlation approach still produces a curve shape (the *base correlation skew*), signifying that the OFGC model is an inadequate means of describing the observed dependence structure.

Second generation copulas pioneered by Andersen and Sidenius (2004) tried to account for the base correlation skew by extending the OFGC model. Two extensions to the OFGC

³There is an equivalence between the structural model of Merton (1974) and the construction of the marginal default probabilities of the one factor Gaussian copula, Mortensen (2006) shows that it is inconsistent to price a collateralised debt obligation which has a smooth integral as protection leg with a structural model.

model are considered in Chapter 4 of this thesis:

- The random recovery OFGC model, where the recovery on default is random.
- The random factor loading model (RFL), where the correlation parameter changes with the level of the factor.

Both models were first developed by Andersen and Sidenius (2004). Although these models help to provide a better description of the observed dependence structure, they are still one period factor models and do not produce spread dynamics. This means, under this framework, it is impossible to price products which are sensitive to spread dynamics as well as default dependence e.g. leveraged credit linked notes with recourse (the product is discussed in Chapters 2 and 5).

The key objective of the factor models and all portfolio credit models is to provide credit portfolio loss distributions and account for the joint default distribution of obligors. Our contribution to the literature in the multi-obligor case is:

- To develop a new method called the *loss algorithm* for generating portfolio loss distributions. Our algorithm works by iteratively implying a portfolio's loss distribution from the expected loss values of a series of tranches of a portfolio. The loss algorithm is better than the standard method, developed by Turc et al. (2004), because it does not require differentiation of an interpolated curve or rely on the existence of a base correlation curve. In Turc et al. (2004) differentiation of an interpolated base correlation curve is required (this compounds interpolation error, see Hagan and West (2006)) and existence of a base correlation curve is assumed.
- To make use of the dynamics found in single obligor intensity modelling and create default dependence with a copula. Due to the work done in the single obligor case we will have a tractable calibration routine for single obligor pricing. The standard copula approach separates marginal distributions and dependence and it is this idea that is extended to give a dynamic multi-obligor credit modelling framework. The dynamic framework is developed in line with the work of Schubert and Schönbucher (2001). In their work Schubert and Schönbucher (2001) demonstrate that one can separate single obligor calibration and dependence calibration. Using this framework we develop:

1. The evaluation of counterparty risk. This is done by considering products which have embedded counterparty risk. We find new results in the behavior of such

- products. In particular, we find that volatility de-correlates obligors and, keeping all else equal, higher volatility leads to less counterparty risk.
2. A dynamic framework called the *stochastic liquidity threshold approach*. The method involves combining the developments of the factor framework with the development of the Schubert and Schönbucher (2001) approach. The idea provides a novel way to introduce local spread dynamics into the factor framework.
 3. A method to price portfolio credit default swaptions in line with the work done in the single obligor case. The result requires constructing the loss distribution of a portfolio. This is possible because of the loss algorithm we have developed.

1.3 Summary of chapters of this thesis

In this section we provide an outline of the chapters of this thesis.

In Chapter 2 we provide a detailed description of credit derivative products:

- Section 2.1 establishes a core relationship between default probabilities and credit spreads.
- In Section 2.2 we discuss credit default swaps, their structure, economic rationale and the risks they introduce.
- In Section 2.3 the importance of capturing default dependence is highlighted.
- In Section 2.4 we define leveraged credit linked notes, both recourse and non-recourse versions. In this thesis we often analyse the impact on valuation of changing model parameters with leveraged credit linked notes as the example product.

In Chapter 3 we consider the valuation of single obligor products. We distinguish between **Category 1** and **Category 2** products. **Category 1** products have no dependence on the stochasticity of credit spreads. **Category 2** products depend on the stochasticity of credit spreads:

- In Section 3.1 we present the modelling framework which is utilised in the chapter. The main result in the section is that:
 - By switching between two filtrations: the enlarged filtration, $\mathbb{G}$, and the sub-filtration, $\mathbb{F}$, we are able to increase the tractability of computations for credit pricing.

- This is primarily achieved by Proposition 5 and Corollary 6.
- Section 3.2 demonstrates how to price **Category 1** products. We also establish the market standard method for pricing the basic **Category 2** product: the credit default swaption. Finally we introduce the notion of pre-default quantities. Pre-default quantities are important because one removes arbitrage from the definition of the credit spread process.
- Section 3.3 provides a detailed analysis of the use of the extended Vasicek model as a stochastic model for the intensity process. Two developments are made here:
 1. A new analytical method for pricing credit default swaptions is provided.
 2. Under an assumption we demonstrate a relationship between credit spreads and intensities. This enables the use of historical credit spreads as a proxy for the intensity process.

These developments enable us to construct a fast and efficient calibration routine for the extended Vasicek model.

- Section 3.4 considers the valuation of leverage credit linked notes without recourse and provides numerical results in order to test the significance of spread dynamics.
- Section 3.5 examines the CIR and pure jump models as alternatives to the extended Vasicek model.

In Chapter 4 we provide an overview of multi-obligor modelling in addition to developing a new method of generating the loss distribution of a credit portfolio. In the chapter we:

- Demonstrate that attempting to generate default dependence by correlating stochastic intensities of obligors is typically inadequate. In Section 4.3 we then introduce copula methods and detail the factor framework as an alternative way of generating default dependence.
- The OFGC model is the canonical factor framework model and does not re-price all the traded tranches simultaneously. In Section 4.5 we detail the construction of correlation curves which account for the weaknesses of the model by ensuring traded tranches can be re-priced using the correlation curves in conjunction with the OFGC model.
- In Section 4.6 we consider the random recovery model which is an extension of the OFGC model. The model accounts for the empirical observation that average recovery

rates tend to be lower in bad economic times. We demonstrate that the model reduces instances of arbitrage found when using the OFGC model to generate a correlation curve.

- In Section 4.7 we consider how to generate loss distributions. The curve shape of a correlation curve contains information about the loss distribution of a portfolio since it enables one to re-price traded tranches. We detail the current method (Turc et al. (2004)) of recovering the loss distribution of a credit portfolio. The weakness of the approach is that it requires a very specific form of a correlation curve in order to work and requires differentiation of an interpolated curve. We provide a new method, *the loss algorithm*. The only requirements on the new method is that we must be able to extract the expected loss of an equity tranche (which the Turc et al. (2004) method also requires) and the loss density has finite support (which will always be the case).
- In Section 4.8 we review the random factor loading model. The model is an extension of the OFGC model. The model produces consistent loss distributions with those implied from traded tranches.

In Chapter 5 we:

- Detail how a dynamic multi-obligor modelling framework can be constructed when we have multiple obligors.
- In Sections 5.2 and 5.3 we consider the pricing of securitised loans and leverage credit linked notes. These products are influenced by counterparty risk.
- Some significant conclusions are reached in the chapter. In particular we find that when volatility is high it acts like a dampener on default correlation reducing instances of joint defaults.

In Chapter 6 we:

- Detail (in Section 6.1) the framework developed by Andersen (2006), which introduces the notion of *inter-temporal* dynamics. This enables one to correlate macro factors that have a common influence on obligors over multiple time horizons. From this we introduce the notion of the stochastic liquidity threshold approach. This is a method of introducing spread dynamics into the factor framework.
- In Section 6.2 we consider the valuation of credit default index swaptions. We develop, using intensity dynamics, a method of getting semi-analytical valuation for credit default index swaptions. By introducing intensity dynamics we will be able to account

for the term structure of credit spreads and develop a model which can be calibrated to multiple credit default index swaptions.

In Chapter 7 we provide a conclusion.

Chapter 2

Products

In this chapter we provide a discussion of products directly priced and used in this thesis. The complex risks borne by credit derivative products make it difficult to give a general classification of the available instruments. However there is a general structure to traded contracts. One party will be a buyer of protection against default losses and the other party will be a seller of protection. We will use the phrase protection buyer (PB) and protection seller (PS) to mean the party buying protection (gaining on default) and selling protection (losing on default) respectively. The instruments can be in unfunded format, called *credit linked swaps* (CLS) or in funded format, called *credit linked notes* (CLNs). The products considered in this chapter are:

- Credit default swaps (CDS).
- Forward credit default swaps (FCDS).
- Credit default swaptions.
- Basket default swaps (NTDs); in particular first to default swaps (FTDs).
- Credit default index swaps (CDIS).
- Collateralised debt obligations (CDOs).
- Basic credit linked notes (BCLNs).
- Leveraged credit linked notes-both *recourse* and *non-recourse* versions (LCLNs).

2.0.1 Summary of sections in this chapter

This chapter is split into five sections:

- In Section 2.1 we establish a relationship between credit spreads and default probabilities. This is done by considering a basic example of a zero coupon bond issued by an obligor that carries credit risk.
- In Section 2.2 we consider CDS, outline their important economic features, discuss the implications of having different contracts and detail the difference between physically settled and cash settled CDS. In addition, we provide an analysis that demonstrates that CDS have risk features that are similar to deep out of the money put options and that a CDS protection buyer carries counterparty risk. The section also contains a sub-section, Sub-section 2.2.1, that defines FCDS and credit default swaptions.
- In Section 2.3 we consider the pricing of credit portfolio derivatives. The section contains four sub-sections:
 - Sub-section 2.3.1 defines and considers the pricing of basket credit derivatives. The valuation of FTDs are discussed. The importance of default correlation is examined using stylised credit portfolios; we demonstrate how an NTD spread premia changes with different dependence assumptions.
 - Sub-section 2.3.2 defines CDOs and discusses the key risks of the product.
 - Sub-section 2.3.3 connects the risks assumed by an NTD investor and a CDO investor, providing a general classification of investors of these products.
 - Sub-section 2.3.4 discusses standardised credit portfolio indices. These indices are actively traded in the market either in CDO format or as CDIS.
- In Section 2.4 we discuss how credit derivatives can be constructed in note format as credit linked notes. We show that the rationale for CLNs are primarily to mitigate counterparty risk and for the CLN issuer to acquire liquidity. The section provides a definition of BCLNs. In Sub-section 2.4.1 a detailed definition of LCLNs is provided (both recourse and non-recourse versions). In the same sub-section we give an analysis of the risks of LCLNs and their relationship to BCLNs.
- In Section 2.5 we provide a conclusion.

2.1 Motivating Example

In this section we consider a simple credit instrument in order to demonstrate the relationship between credit spreads and default probabilities. *Credit spreads* are, informally, the spread above the risk free rate (taken to be *Libor* in this thesis) that an investor receives

in order to assume the credit risk of an obligor (see Sub-section 3.2.6 in Chapter 3 for a detailed definition of credit spreads). Assume we have one reference obligor and that:

- The risk free zero coupon bond compounds continuously at a constant rate of $r > 0$ for some maturity $T > 0$. The risk free zero coupon bond is therefore worth e^{-rT} today if it pays 1 at T .
- The reference obligor has zero recovery on default.

The zero coupon bond obligation of the reference obligor will be worth less than the risk free zero coupon bond i.e. $e^{-(r+s)T}$ where s is strictly positive and represents the credit spread of the obligor.

From another perspective let the probability of default on payments by the obligor within time T be $p > 0$, where the event of default is assumed independent of rates. We expect with probability $1 - p$ to be paid 1 at time T , hence the bond value is $(1 - p) \times e^{-rT}$ today. This implies that:

$$e^{-(r+s)T} = (1 - p) \times e^{-rT},$$

which gives:

$$1 - p = e^{-sT}. \quad (2.1.1)$$

From Equation (2.1.1) we see that credit spreads contain information about the default probabilities of an obligor. We have established that direct modelling of default events of an obligor is difficult due to the fact that historically we observe few defaults, see Figure 1.1 in Chapter 1. The relationship between credit spreads and default probabilities, given in Equation (2.1.1), provides a consistent modelling route because we can tackle the problem of understanding default probabilities via credit spreads. We have abundant data for this through CDS.

2.2 Credit Default Swap

A CDS protection buyer pays to protect against the default of a reference obligor. The protection buyer pays periodic (usually quarterly or semi-annual) coupons of $N \times s \times \Delta t$ where s is an annualized spread, N is the notional of the contract and Δt is the accrual period for payments¹. Coupon payments are made to the protection seller until the maturity of

¹Whilst writing this thesis, the “Big Bang” on CDS contracts has occurred meaning that instead of CDS trading with spreads, they now trade as a mix of spreads and upfront payments. In this new convention the protection buyer partially compensates the protection seller for assuming credit risk with a cash payment up-

the contract or until a default occurs. CDS enable speculative trading on the credit performance of an obligor; institutions who buy CDS protection do not have to hold any position in the obligations of the reference obligor.

CDS are the most important class of credit derivatives in the market because:

- They are the most actively traded credit derivative instrument, see von Kleist and Mallo (2010).
- They allow one to fully specify the marginal distributions of the portfolio of obligors one seeks to model.
- They serve as a reference product for credit default swaptions, FTDs and CDOs.

A CDS is triggered when there is a default. A CDS contract specifies a set of events (which we call *default events*) that must occur before the protection seller compensates the buyer for losses. The events that are incorporated in contracts varies according to the reference obligor and the bilateral agreement reached by counterparties.

This calls into question assumptions that may be made about the generic structure of CDS contracts. For example O’Kane et al. (2003) consider CDS contracts that incorporate restructuring as a default event. Restructuring occurs when the terms of the obligations of an obligor are re-negotiated between the obligor and its debt holders. O’Kane et al. (2003) show that restructuring in most cases leads to a greater recovery rate value on default than other default events. Yet, not all CDS contracts may have restructuring as an event that triggers a default. Hence care must be taken when we model CDS contracts with the assumption that they all default in the same way. This is not the case.

If a default occurs then depending on whether the contract is specified as *physical delivery* or *cash settled* there will be different mechanisms for settling the transaction:

- If the contract is specified as physical delivery, the protection buyer will deliver an obligation (called *the cheapest to deliver obligation*²) of the obligor’s from an approved list of deliverable obligations for which he will be paid the notional of the CDS contract by the protection seller. The obligations will usually be trading at a significant discount to par, therefore the protection buyer will have a significant gain on default.

front as well as premium payments. As we will assume in this thesis that interest rates and default probabilities are independent, the change in convention does not affect our analysis. However, in a model setting where interest rates and default probabilities are correlated the impact may be significant.

²Usually the protection buyer has a range of obligations issued by the obligor which he can deliver; for no arbitrage he will deliver the cheapest obligation which is allowed in the contract.

- If cash settled is specified then the recovery rate is determined by an auction process run by the International Swaps and Derivatives Association (ISDA), see Cadwalader (2009). The protection buyer will be paid an amount by the protection seller equal to the difference of par minus the recovery rate (the recovery determined in the auction process) multiplied by the notional of the CDS contract.

Practitioners typically ignore these subtle contractual issues and make generic assumptions:

- For example emerging market reference obligors are often assumed to have 25% fixed recoveries on default (Singh and Andritzky (2005)) without specification of the type of default that may occur or whether it is physically or cash settled.
- These assumptions carry over into the pricing and modelling of more complex structures, such as collateralised debt obligations, for which CDS are reference underlying products. In Chapter 4 the impact of these assumptions are partially examined. In that chapter random recovery modelling is considered for the valuation of collateralised debt obligations.
- In the same chapter we consider the work of Bennani and Maetz (2009), who discuss stochastic recovery modelling and consider the significant mis-pricing of CDOs that occurs because of the market standard constant recovery assumptions. To see the significance of making recovery assumptions, note that the recovery assumption for Lehman Brothers was 40% but the actual realised recovery rate was 8.625% (see Singh and Spackman (2009)).

In contrast to *interest rate swaps* the risks assumed by a protection buyer and protection seller in a CDS contract are not symmetrical (see Mingle (2007)). The protection buyer takes a short position in the credit risk of an obligor. The protection seller in contrast has a long position in the credit risk of the obligor. The protection seller exposes himself to potentially huge losses (under events with usually small probability of occurring) relative to the premium he receives, whereas the opposite is true for the protection buyer. The structural characteristics of a CDS are therefore more like that of an out-of-the-money put option than an interest rate swap.

For the protection seller entering into a CDS has the advantage of not requiring any upfront capital because it is in swap format. Without the availability of CDS the protection seller would have to fund the purchase of an obligation of an obligor in order to speculate on the credit performance of the obligor. For the protection buyer having the CDS in swap format

means relying on the protection seller to make good when there is a default and pay what could be large sums. Therefore CDS transactions involve assuming counterparty risk for the protection buyer.

In Chapter 5 we develop our modelling framework to account for counterparty risk. We find intuitive results which show that if a counterparty is highly positively correlated with the reference obligor then the fair CDS swap spread dramatically reduces. This means that on a default of an obligor it is likely the counterparty will not be able to make good on his payment obligations i.e. the counterparty is likely to fail. The protection buyer should pay less for purchasing protection whenever the counterparty's credit worthiness is low and (or) whenever the counterparty is highly positively correlated with the reference obligor. Determining how much the protection buyer should pay in the presence of counterparty risk is addressed in Chapter 5.

The fair market CDS swap spread (when there is no counterparty risk) provides a dynamic assessment of an obligor's credit worthiness. In this thesis we are interested in giving a representation to the dynamics that CDS swap spreads produce. The purpose of Figure 2.1, below, is to demonstrate the dramatic credit spread moves that occurred after the Lehman crisis. We see how quickly an investor's perception of credit risk can change and that credit swap spreads are dynamic.

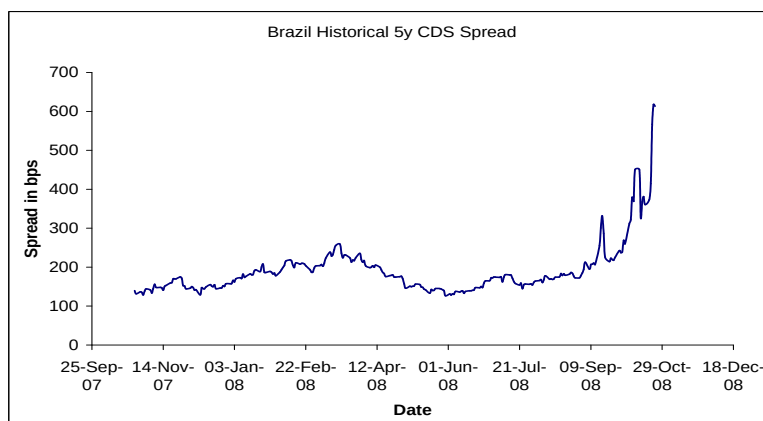


Figure 2.1: The historical 5 year CDS swap spread of Brazil. Data from *MarkIt*

2.2.1 Forward CDS and credit default swaptions

An extension of CDS are forward credit default swaps. FCDS are contracts to enter into a CDS at a pre-defined future time (*exercise time*) for a pre-defined maturity and swap spread (*strike*). A further extension of CDS are credit default swaptions. Credit default swaptions are options which are defined similarly to FCDS. However a credit default swaption contract gives the buyer the right but not the obligation to enter into a CDS at a pre-defined future time for a pre-defined maturity and swap spread. If one has the option to enter as a protection buyer then the swaption is called a payer swaption, if one has the option to enter as a protection seller then the swaption is called a receiver swaption.

The FCDS and credit default swaptions we will consider in this thesis have *knock-out* features; meaning if there has been a default before the exercise time, the contract ends with no exchange of cash flows between the parties to the contract.

2.3 Portfolio Derivatives

In this section we consider portfolio credit derivatives. Portfolio credit derivatives have payoffs that depend on a pool of credit instruments from different obligors. They represent a modelling challenge in part because of:

- The lack of data available on default correlation (see Schönbucher (2003a)).
- The complex default dependence structures that needs to be assessed in order to price the products effectively.

2.3.1 NTDs

Examples of portfolio derivatives are n of m basket default swaps. In these products one has m obligors in a basket and the seller of protection is exposed to default losses only if $n < m$ defaults have occurred.

As an illustration, suppose we sell protection on an NTD basket swap that has 3 reference obligors A , B and C and suppose the transaction is a 2 of 3 (*second to default*) contract on the obligors. If any one of A , B or C defaults the protection seller will still receive the stipulated premium, but if a second default occurs amongst any of the remaining obligors then the protection seller loses $1 - R$ and stops receiving the premium (R is the recovery of the second obligor to default).

The premium the protection seller receives for selling protection is (much) less than for a 1 of 3 (first to default), as in that case the protection seller is exposed to the first obligor to default. On the other hand the premium paid to the 2 of 3 protection seller is (much) higher than that paid to a 3 of 3 protection seller as in the latter there is a cushion of 3 defaults before the protection seller has to make a payment to the protection buyer.

The critical challenge is to assess how default dependence affects the valuation of these products. Let us assume all 3 obligors have the same recovery rate, R . By considering what the value of an FTD (1 of 3) is in the following two cases we establish that accounting for default dependence is important:

- (i) The CDS swap spread of reference obligors A , B and C are the same with a level of 100bps³ (=1%).
- (ii) The CDS swap spread of reference obligors A , B and C are the same with a level of 150bps.

A priori we would expect the premium on an FTD in case (i) to be lower than it would be in case (ii) since in case (ii) the reference obligors have higher credit spreads. However, if in case (i) the default times of A , B and C are independent and in case (ii) the default times are perfectly positively correlated then our initial assumptions do not hold true.

To demonstrate this let us assume that in case (i) default times are independent and in case (ii) default times are perfectly positively correlated:

- In case (i) the effect of independent defaults on an FTD is to compound the instances in which a default can occur. To show why, assume default probabilities are equal amongst the 3 obligors (A , B and C). Let p be the default probability of these obligors over a horizon. For an FTD we need each reference obligor to survive, hence the survival probability of the FTD, because of independence, is $(1 - p)^3 = q$.
- In case (ii) A , B and C effectively become one asset so the default of one reference obligor leads to the default of all other obligors in the basket. Therefore (again assuming default probabilities are equal amongst the 3 obligors and is p) in this case the FTD survival probability is $1 - p$ (the same as any of the reference obligors). With perfect positive correlation there is no compounding of risk when an investor speculates on an FTD.

³bps means basis points. A basis point is $\frac{1}{10,000}$.

Equation (2.1.1) in Section 2.1 established a relationship between continuously compounded credit spreads and survival probabilities. In general CDS premiums are not paid continuously but discretely. For the purpose of the analysis below we will assume CDS premiums are being paid continuously so that we can apply Equation (2.1.1).

For a concrete numerical example assume we are considering an FTD with a maturity of 5 years and the recovery rate of the reference obligors are zero. Equation (2.1.1) says that the fair FTD spread⁴, s , holds the relationship $q = e^{-sT}$, with the survival probability of the FTD being q .

In case (i) where the CDS swap spread is 100bps we recover an implied survival probability for each reference obligor of $1 - p = e^{-100\text{bps} \times 5} \approx 0.607$. By independence we have the survival probability of the FTD is $q = (1 - p)^3 \approx 0.223$. From this we can imply a spread for the FTD of 300bps i.e the sum of the 3 CDS swap spreads. The result for case (i) has an intuitive interpretation: an FTD, when defaults are independent, is equivalent to selling protection on the 3 CDS in the basket and immediately buying protection on the remaining 2 surviving obligors as soon as an obligor defaults. On average doing this should be cost free since the obligors default independently.

In case (ii) where the CDS swap spread for each reference obligor is 150bps and we have perfect positive correlation we know an FTD has the same survival probability as the individual reference obligors. Since there is no compounding of credit risk by Equation (2.1.1) the FTD swap spread is 150bps .

The point of this analysis is to demonstrate that default protection for an FTD is more costly (much more) when we have independence as opposed to having perfect positive correlation. Therefore accounting for the observed correlation in the market is very important.

Below we provide Tables 2.1, 2.2 and Figure 2.2 to show the spectrum of sensitivity of the fair NTD credit spread for changes in default correlation. In this case we have a basket where $m = 4$ and n varies from 1 to 4. The model used to generate these results is the OFGC model and is considered in Chapter 4. At this stage we are not interested in giving an analysis of the model but providing the reader with a sense of the behaviour of such products.

Table 2.1 gives the CDS market data which is used as an input for pricing. Table 2.2 and Figure 2.2 gives the fair NTD credit spread for a particular level of default correlation. There is clear monotonicity of credit spreads as a function of default correlation for the first

⁴There is a close relationship between the spread to Libor on a bond and that on a default swap, differences arise because of liquidity and other market conditions, but in essence a default swap spread can be analysed as a spread to Libor of the default liability.

to default as well as the third and fourth to default swaps. Ambiguity as to the impact of default correlation exists with the second to default since high positive default correlation or independence does not seem to benefit or penalise the protection seller.

Entities	6m	1y	2y	3y	4y	5y	7y	10y
Brazil	202bps	238bps	300bps	338bps	362bps	377bps	389bps	400bps
Russia	963bps	963bps	952bps	890bps	816bps	769bps	730bps	695bps
Hungary	543bps	543bps	543bps	539bps	534bps	528bps	511bps	498bps
Indonesia	634bps	634bps	634bps	634bps	634bps	634bps	630bps	628bps

Table 2.1: Federative Republic of Brazil, Federation of Russia, Republic of Hungary and Indonesia CDS swap spreads as of February 2009. Data source from *MarkIt*.

Correlation	1 of 4	2 of 4	3 of 4	4 of 4
3%	2277bps	662bps	152bps	17bps
20%	1966bps	678bps	214bps	41bps
50%	1511bps	683bps	319bps	112bps
70%	1373bps	673bps	386bps	185bps
97%	826bps	636bps	502bps	358bps

Table 2.2: Sensitivities of n^{th} - to default spreads to changes in the OFGC default correlation. The basket of obligors are from Table 2.1.

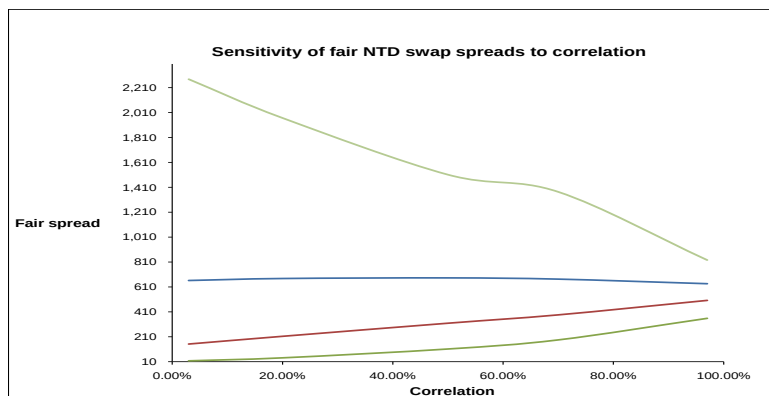


Figure 2.2: Sensitivities of n^{th} -to default spreads to changes in default correlation. The basket of obligors are from Table 2.1.

2.3.2 Collateralised debt obligations

In this section we discuss collateralised debt obligations (CDOs). CDOs are highly dependent on default correlation. Whenever we talk about a CDO in this thesis we will mean a product that restructures the default exposure of a portfolio by exposing the protection seller to losses on a specific number of ordered defaults e.g. the next 5 defaults after there have been 2 defaults or the next 7 defaults after there have been 10 defaults etc.

CDOs offer investors specific and targeted risk reward profiles ranging from the first loss *equity* tranche to the last loss *super senior* tranche. There are a range of motivations for the construction of CDOs, from *arbitrage* CDOs which aim to arbitrage the price difference between the components of the underlying credits and the CDO; to *balance sheet* CDOs which aim to free up regulatory capital by repackaging and selling off some of the risks on the balance sheet. The underlying components of a CDO can be bonds, mortgages, loans or other credit derivatives. In this thesis we will focus on the case of *synthetic* CDOs which have as underlying components a portfolio of CDS. Following Schönbucher (2003a), we give a generic example of a CDO:

Example 1 (CDO).

- Suppose we have 100 reference obligors in a portfolio composed of CDS. Each obligor has an identity i and a notional in the portfolio of N_i where $i \in \{1, 2, 3, \dots, 100\}$. The full notional of the portfolio is $N = \sum_{i=1}^{100} N_i$.
- We will assume three tranches for illustrative purposes: an equity (most risky) tranche with notional E , a mezzanine tranche with notional M and a senior tranche with notional S (least risky) which together constitute the capital structure. Note $E + M + S = N$.
- The key point is that there is a redistribution of the portfolio loss according to the seniority of the tranche. In terms of premiums, the senior tranche investor will be prioritised, then the mezzanine tranche investor and finally the equity tranche investor. For example when there is a default the equity tranche investor (who is a protection seller) will incur a reduction in premium received as the notional amount of the tranche reduces to $(E - \tilde{N}_i)$, where $\tilde{N}_i$ is the loss on the defaulting obligor. In terms of default losses the equity tranche investor is exposed to the first losses of the portfolio. On a default the equity tranche investor will lose par minus recovery on each obligor defaulting multiplied by the notional of the obligor in the portfolio. When enough losses have occurred to deplete the equity tranche notional the mezzanine tranche

investor will begin to incur losses for any defaults of the remaining obligors in the portfolio and then the senior tranche investor will incur losses.

- A tranche can be defined in terms of its upper and lower strike points, k_1 and k_2 , which are expressed as a percentage of the entire portfolio notional. k_1 is called the tranche subordination and k_2 is called the tranche detachment point. Let $(L_t)_{t \geq 0}$ denote the portfolio loss as a proportion of the portfolio notional. Without loss of generality we can assume the portfolio notional is 1, then the tranche loss to the seller of protection as a proportion of the tranche notional is:

$$\frac{(L_t - k_1)^+ - (L_t - k_2)^+}{k_2 - k_1}, \quad (2.3.1)$$

where $(x)^+ = \max(x, 0)$.

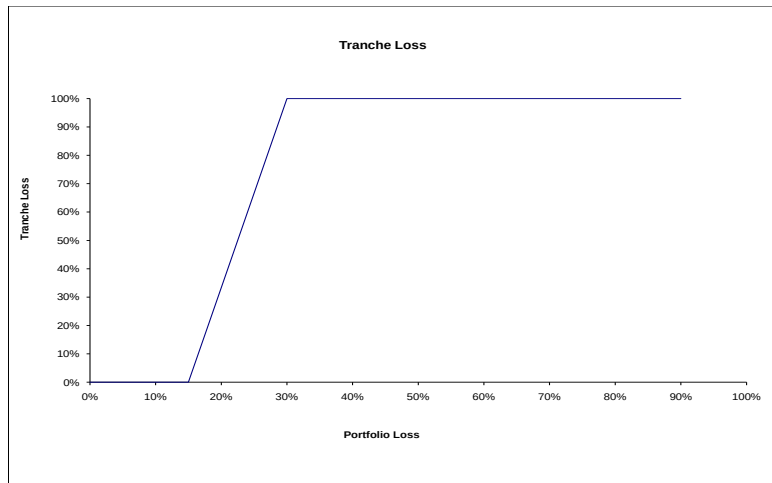


Figure 2.3: The loss of a tranche per unit of tranche notional for each percentage loss of the portfolio. Here $k_1 = 15\%$ and $k_2 = 30\%$.

2.3.3 Portfolio risk characteristics

We have now discussed NTDs and CDOs. The risk characteristics of these products are broadly similar, in this sub-section we discuss these characteristics.

When defaults are independent instances where defaults can occur increases significantly. In contrast if correlation is high and positive instances where defaults occur are more rare with defaults tending to cluster.

We call portfolio products that are affected by the first defaults in a portfolio *first loss products*. Equity investors and FTD investors are first loss investors. First loss investors are relatively invariant to the number of defaults that occur together as their losses are significant even if only one default occurs; their concern is to have reduced instances where defaults can occur. Hence first loss investors are averse to situations where defaults are independent.

On the other hand senior tranches are *last loss products*. For last loss investors it is not the instances of defaults that concern them, but the number of obligors defaulting. The scenario in which last loss investors incur losses is when many defaults have occurred. With high positive correlation multiple defaults are more likely. Therefore last loss investors have the opposite sensitivity to default correlation to first loss investors.

2.3.4 Standardised portfolios

Standardised portfolios for trading single tranche CDOs and credit default index swaps with pre-defined reference obligors exist. In this sub-section we discuss these commonly traded portfolio products and how credit modelling benefits from the liquidity of such products. We also define the structure of credit default index swaps (CDIS).

Standard credit portfolios are called credit indices. *Itraxx* is the name given to a set of indices with European obligors and *CDX* is the name given to a set of indices with North American obligors. *Markit Partners* own the credit indices (see Markit (2008)) and create a new series of an index with updated obligors every six months. The current series at any point in time is called *on-the-run*. Any previous series continues trading but usually with less liquidity as trading activity is concentrated with the on-the-run series.

The Itraxx and CDX each contain investment grade indices. In this thesis we focus on the investment grade indices; they contain 125 obligors that have ratings of BBB- or higher by Standard & Poor's. They are called *Itraxx Main* and *CDX Main* respectively for the Itraxx and CDX. Credit indices have been useful in the following two ways:

- The first way credit indices have been useful is to enable the trading of credit default index swaps. CDIS are similar to CDS, however with CDIS losses are accounted for as a proportion of the portfolio. The protection seller in these contracts has expressed a view on the entire portfolio. We detail the product features in the following:

- A protection seller of an index swap agrees to absorb all losses due to defaults in the portfolio, occurring between the trade date and maturity date. For this he is paid a fixed spread on the notional of the index swap which is agreed at trade inception.
 - The protection seller will experience losses if any of the obligors default. For example if there are 125 obligors equally weighted in the portfolio and the portfolio notional is 1 then on a default the protection seller will make a payment of $\frac{1}{125} \times (1 - R)$ to the protection buyer. R is the recovery of the obligor that has defaulted.
 - The notional on which the protection seller receives his fixed spread, called the *residual notional* is reduced by $\frac{1}{125}$ after each default.
- The second way credit indices have been useful is in the development of index tranche trading. Index tranches are CDOs, therefore they depend on default correlation. Hence market prices of these index tranches can provide information as to the current view of default correlation. By using models such as the OFGC model we can get implied default correlations from traded index tranches.

In Table 2.3 below we provide the traded tranches for the Itraxx main and CDX main:

The tranches available on the Itraxx main

Tranches	Subordination	Detachment
Tranche1	0%	3%
Tranche2	3%	6%
Tranche3	6%	9%
Tranche4	9%	12%
Tranche5	12%	22%

The tranches available on the CDX main

Tranches	Subordination	Detachment
Tranche1	0%	3%
Tranche2	3%	7%
Tranche3	7%	10%
Tranche4	10%	15%
Tranche5	15%	30%

Table 2.3: The tranches available on Itraxx and CDX

2.4 Credit linked notes

In this section we discuss funded credit derivatives. In this thesis we call funded credit derivatives *credit linked notes*. We place particular focus on leveraged credit linked notes. In this thesis LCLNs are used to test the behaviour of models we propose.

Large financial institutions are often reticent to deal in credit linked swaps, e.g. CDS, FTDs or CDOs in swap format. Financial institutions are especially reluctant when the counterparty is linked by geography or sector to the underlying obligors. The counterparty risk that may arise from such transactions could be large, particularly when buying protection (see ECB (2009)).

To mitigate the counterparty risk posed CLNs are sold by financial institutions to investors rather than entering into credit linked swaps with them. In the case of CLNs the protection buyer is the issuer of the note and the protection seller is the buyer of the note. CLNs are economically similar to having funded credit linked swaps, where the protection seller provides funding upfront to cover any future potential losses that may arise from defaults of reference obligors.

CLNs are funded products, hence the interest earned on CLNs should be quoted as a spread above Libor⁵. The spread represents the compensation paid to the investor for assuming the credit risk of a reference obligor(s) and Libor is the investor's compensation for committing capital upfront at trade inception.

CLNs are documented under different agreements to credit linked swaps and are securities rather than bilateral contracts, meaning the notes can be sold on to third parties easily. Credit linked swaps are bilateral swap agreements that require a convoluted process called *novation* in order to transfer a position to a third party. For the purpose of this thesis it is sufficient to think of CLNs as funded versions of credit linked swaps.

The term CLN is sometimes used to refer in the literature to a specific version of a CLN that is linked to only one reference obligor i.e. a funded single obligor CDS. In this thesis we will call CLNs that are linked to only one reference obligor and have no extra characteristics basic credit linked notes (BCLNs). They are economically equivalent to having a funded CDS. We provide a definition for clarity:

Definition 2.1 (The Basic Credit Linked Note). *A BCLN is a note whose redemption and interest payments are linked to an obligor's credit performance*⁶. *On a default depending*

⁵It is feasible to also quote the interest earned as a fixed rate without the floating component of Libor.

⁶The redemption and interest payments of a CLN also depend on the credit performance of the institution issuing the note. For the buyer of the note, he is also assuming the risk of the issuer. In the market this

on whether the note is physically or cash settled there will be different processes for the notes early redemption. If the contract is specified as physical delivery, the protection buyer will deliver the cheapest to deliver debt of the obligor. If cash settled is specified then the recovery rate is determined by an auction process run by the International Swaps and Derivatives Association. The protection buyer will keep the notional invested on trade inception by the protection seller and pay the protection seller the recovery rate multiplied by the notional of the CLN. The gain for the CLN issuer on default is the same as the gain for the protection buyer of a CDS and it is $(1 - \text{recovery}) \times \text{notional}$.

2.4.1 Leveraged credit linked notes

Leveraged credit linked notes are a type of CLN which are similar to BCLNs. Redemption and interest payments are linked to a single obligor's credit performance. But for LCLNs, unlike BCLNs, it is not only a default event that redemption and interest payments are linked to. With LCLN structures redemption and interest payments also depend on where current CDS swap spreads are trading.

To illustrate how an LCLN works let us assume we have an LCLN transaction with maturity T and notional N . LCLNs are characterised by a *leverage factor*, $F \in \mathbb{N}$ (usually F is no more than 10). We define:

$$\text{Reference Notional} = F \times N. \quad (2.4.1)$$

LCLNs incorporate the notion of a Reference CDS. This is best viewed as a shadow CDS position of size Reference Notional. The Reference CDS references the same obligor and has the same maturity as the LCLN. The Reference CDS is defined and tracked for the purposes of being a benchmark. LCLNs have two forms:

1. *Non-recourse version*. In this case no additional investment other than the initial purchase amount (which we will take to be the note notional in this thesis) can be demanded from the investor.
2. *Recourse version*. In this case the note issuer has recourse to the investor for losses that may exceed the initial purchase amount.

There are two ways for an early redemption to occur and for the investor to incur potential losses. We will detail them below as payouts to the issuer.

feature is often not accounted for by investors e.g. some investors bought Lehman issued CLNs without real consideration for the issuer risk.

Payout 1: trigger event

When the fair swap spread of the Reference CDS has reached a certain level we say a trigger event has occurred and the LCLN redeems early with a gain for the note issuer. Given a fair market CDS swap spread, $s(t, T)$, for the Reference CDS at some future time $t > 0$, the payout to the protection buyer on a trigger event is determined by a trigger, k . If $s(t, T)$ breaches the level k , we say that the note has been triggered. When a trigger is breached the LCLN will redeem early, all interest payments to the protection seller stop and the issuer will make a gain based on the current market value of the Reference CDS:

- Non-recourse LCLN = $\min(N, \text{MTM of Reference CDS})$.
- Recourse LCLN = MTM of Reference CDS.

MTM means mark-to-market and is the current market value from the perspective of the protection buyer of the Reference CDS, see Sub-section 3.2.6 (Chapter 3) for a definition.

Payout 2: default event

On a default, again the LCLN will redeem early and all interest payments will stop and the issuer will make a gain of:

- Non-recourse LCLN = $\min(N, \text{Reference Notional} \times (1 - R))$.
- Recourse LCLN = $\text{Reference Notional} \times (1 - R)$.

R is the recovery rate of the obligor.

The non-recourse version of LCLNs is examined in Chapter 3. A non-recourse LCLN can be seen as a combination of an American digital call option on the swap spread of the Reference CDS (**Payout 1**) and a BCLN with a modified recovery (**Payout 2**):

- The modified recovery level depends on the leverage factor, F , and on the recovery rate of the reference obligor, R . As an example, assume the notional of an LCLN is 1, then on a default the gain for the issuer is $\min(1, F \times (1 - R))$. This amount for sufficiently large leverage factor and small assumed market recovery will be 1, which implies the note has a modified recovery of zero i.e. the investor gets nothing back on default. To clarify this point, if we assume a standard market recovery rate of 25% (recovery assumption for emerging market obligors) then we need the leverage factor to be equal to or greater than $\frac{4}{3}$ to get a modified recovery of zero.

- The American digital option depends on the level of the trigger and credit spread dynamics. The setting of the trigger and the levels of market volatility can mean the LCLN has a significantly different value from a BCLN (with the aforementioned modified recovery), see Chapter 3 (Section 3.4).

LCLNs with recourse introduce counterparty risk because on a trigger event or default event the note issuer has recourse to the investor for potentially more than his initial purchase amount. In Chapter 5 we show how to introduce spread dynamics in a model setting that also accounts for default dependence between the note investor and the reference obligor. For LCLNs with recourse default dependence and spread dynamics have an influence on the value of the product. In Chapter 5 we discover, by analysing this product, that having highly volatile credit spreads for either a reference obligor, a note investor or both creates a de-correlating impact reducing instances where we observe joint defaults.

2.5 Chapter conclusion

In this chapter we have extensively studied the products we will consider throughout this thesis and the risk characteristics they exhibit:

- There is a strong link between credit spreads and default probabilities. This enables one to focus on modelling credit spreads rather than default events. This is useful because we have abundant data for credit spreads via CDS.
- CDS are bilateral agreements and therefore are not generic. In addition the events that trigger a default can imply very different outcomes for the recovery rate of a reference obligor.
- CDS provide the forward looking view today of an investor's perception of the likelihood of an obligor defaulting over some horizon. The investor's view is constantly changing giving rise to dynamics in the fair CDS swap spread.
- Credit default swaptions require an assessment of spread dynamics.
- Credit portfolio products are sensitive to default correlation. Investors of first loss products are averse to low default correlation. Investors of last loss products are averse to high positive default correlation.
- Standardised credit portfolios exist and are liquidly traded. This enable the trading of CDIS and CDOs.

- Credit linked swaps introduce counterparty risk for protection buyers. Credit linked notes help mitigate counterparty risk for protection buyers. Protection sellers who purchase CLNs, however, assume the credit risk of the issuer as well as the risk of the reference obligors of the CLN.
- The key risks that need to be captured to effectively price the products discussed in this chapter are single obligor default probabilities, default dependence of reference obligors in a portfolio, default dependence between obligors and counterparties and credit spread dynamics.

Chapter 3

Single Name Valuation, Modelling and Calibration

In this chapter we consider a continuous time economy and the valuation of single obligor credit products. We distinguish between two categories of single obligor credit products:

- **Category 1:** *Products not influenced by stochasticity of credit spreads.* By this we mean that the valuation of these types of products do not require that we put a representation to credit spread evolution. We only need to know the current market for credit swap spreads. Products of this type, considered in this chapter, are bonds, CDS and FCDS.
- **Category 2:** *Products influenced by stochasticity of credit spreads.* In this category of products valuation depends on the dynamics of credit spreads; for example, its volatility. We not only need to know the current credit spread market, we also need to have a sense of the evolution of credit spreads. These products have convex payoffs as a function of credit spreads. Products of this type, considered in this chapter, are credit default swaptions and LCLNs without recourse (both defined in Chapter 2).

3.0.1 Summary of sections in this chapter

This chapter is split into six sections:

- In Section 3.1 we present the modelling framework which is utilised in this chapter:
 - We consider two filtrations: the enlarged filtration, $\mathbb{G}$, and the sub-filtration, $\mathbb{F}$.
 - Under an assumption, Assumption 3.1, we are able to provide results that enable the switching from computation with respect to $\mathbb{G}$ to $\mathbb{F}$.

- The main results in Section 3.1 are Proposition 5 and Corollary 6, which form the basis of pricing results used in this chapter and subsequent chapters of this thesis.
- In Section 3.2 we introduce methods of pricing **Category 1** products via a direct application of the results in Section 3.1. In Section 3.2 we also establish the market standard method for pricing credit default swaptions, which are **Category 2** products. The Section includes the following sub-sections:
 - In Sub-section 3.2.1 we discuss modelling assumptions that may be made on losses on default. To quantify losses one must make an assessment of what the recovery rate on default will be. We introduce the notion of *fractional recovery*, which becomes the recovery rate assumption of this thesis.
 - In Sub-section 3.2.2 we discuss the valuation of basic credit products. These products are shown to be components of more complicated products such as CDS. Hence their valuation is useful in valuing more complicated structures.
 - Sub-sections 3.2.3 and 3.2.4 are concerned with the valuation of CDS. In these sub-sections we describe the different legs of a CDS and how to value them in the context of intensity modelling.
 - Sub-section 3.2.5 provides numerical results for CDS valuation.
 - In Sub-section 3.2.6 we examine the valuation of FCDS and credit default swaptions.
 - In Sub-section 3.2.7 we introduce the notion of *pre-default* quantities. The results on pre-default quantities rely on Corollary 4 in Section 3.1. The important pre-default quantities are: the pre-default premium leg, the pre-default default leg and the pre-default credit swap spread. Using pre-default quantities are important because one removes arbitrage from the definition of the fair swap spread process (see Morini and Brigo (2007)).
- In Section 3.3 a detailed study is done on the extended Vasicek model as a stochastic model for the intensity process. The section includes the following sub-sections:
 - In Sub-section 3.3.1 we describe the general extended Vasicek model.
 - In Sub-section 3.3.2 we provide a new analytical method of calibrating credit default swaptions under an intensity process driven by the extended Vasicek model. This means we can calibrate the volatility parameter of the extended Vasicek model exactly and quickly.

- In Sub-section 3.3.3 under an assumption, Assumption 3.4, we demonstrate the relationship between credit spreads and intensities. This enables the use of historical credit spreads as a proxy for the intensity process and allows one to use historical credit swap spreads to calibrate to the speed of mean reversion parameter of the extended Vasicek model (see Appendix D for a definition of speed of mean reversion).
- In Sub-section 3.3.4 we discuss the calibration of the extended Vasicek model.
- In Sub-section 3.3.5 we provide numerical results on the extended Vasicek model.
- In Section 3.4 we consider the valuation of LCLNs without recourse using the extended Vasicek model. We analyse the behaviour of the product to changes in model parameters and provide various risk metrics to test the sensitivity of the product to market changes as implied by the model.
- In Section 3.5 we examine the CIR and pure jump models as alternatives to the extended Vasicek model.
- In Section 3.6 we provide a conclusion.

3.1 Preliminary results

In this section we introduce results from the literature that enables us to price credit products under a stochastic intensity modelling framework.

3.1.1 No arbitrage pricing

Consider a fixed finite time horizon $T^* > 0$, a complete probability space, $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ ¹. $\mathbb{F}$ and all other filtrations used in this thesis are assumed to satisfy the usual conditions (see Jacod and Shiryaev (2002)). All processes we introduce in the following are assumed to be square-integrable, adapted to $\mathbb{F}$ or its enlarged version, $\mathbb{G}$ (to be defined below), and continuous from the right with left limits. $\mathbb{F}$ provides the evolution of information about asset prices and other economic factors excluding the time of default.

In our specific treatment, in this chapter, we can think of $\mathbb{F}$ as being generated by the prices of assets or products related to an obligor (such as bond prices or CDS swap spreads) and interest rates.

¹All processes are defined up until time T^* , however we use the notation $(\cdot)_{t \geq 0}$ throughout this thesis.

From this we can consider an economy with $M + 1$ non-dividend paying assets, traded continuously from time 0 to time T^* . Their prices are modelled by adapted semi-martingales (see Revuz and Yor (2001) for a definition of a semi-martingale), $\{S^i\}_{i=0,\dots,M}^2$. The assets and their prices, $\{S^i\}_{i=0,\dots,M}$, are termed the *financial model*.

Definition 3.1. *A numeraire is any positive non-dividend paying asset, which other assets are denominated in.*

We can take the asset $(S_t^0)_{t \geq 0}$ (without loss of generality) to be the numeraire asset. $(S_t^0)_{t \geq 0}$ is called the *deflator* in the financial model. In our specific case we take the asset price process, $(S_t^0)_{t \geq 0}$, to evolve according to:

$$dS_t^0 = r_t S_t^0 dt,$$

with $S_0^0 = 1$ and $(r_t)_{t \geq 0}$ the instantaneous spot interest rate (see Appendix C). Hence we can set:

$$\beta(t) \equiv (S_t^0)^{-1} \equiv e^{-\int_0^t r_s ds} \quad \forall t \geq 0.$$

In this instance the deflator is called the discount process. From this we can let $\forall t \geq 0$:

$$\mathbf{S}_t = \{S_t^0, \dots, S_t^M\},$$

and:

$$\tilde{\mathbf{S}}_t = \{\tilde{S}_t^0, \dots, \tilde{S}_t^M\},$$

where $\tilde{S}_t^i = \beta(t) S_t^i$, $0 \leq i \leq M$ and $\tilde{S}_t^0 = 1$. $(\tilde{\mathbf{S}}_t)_{t \geq 0}$ is the *discounted price process* of the financial model.

Definition 3.2. *A trading strategy, $(\boldsymbol{\mu}_t)_{t \geq 0}$, is an $\mathbb{R}^{M+1}$ - valued, locally bounded, $\mathbb{F}$ - predictable process, with:*

$$\boldsymbol{\mu}_t = \{\mu_t^0, \dots, \mu_t^M\},$$

where $(\boldsymbol{\mu}_t)_{t \geq 0}$ is integrable with respect to the semi-martingale $(\mathbf{S}_t)_{t \geq 0}$ (see Yan (1998)).

$(\mu_t^i)_{t \geq 0}$, for all $i \in \{0, \dots, M\}$, represents the number of units of asset i held at time $t \geq 0$.

Definition 3.3. *The corresponding value process of a trading strategy at time $t \geq 0$ is:*

$$V_t(\boldsymbol{\mu}) = \boldsymbol{\mu}_t \cdot \mathbf{S}_t = \sum_{i=0}^M \mu_t^i S_t^i,$$

and the discounted value process is:

$$\tilde{V}_t(\boldsymbol{\mu}) = \beta(t) V_t(\boldsymbol{\mu}).$$

²Here we are suppressing dependence on time for clarity.

Definition 3.4. A trading strategy, $(\mu_t)_{t \geq 0}$, is said to be self-financing if $\forall t \geq 0$:

$$V_t(\mu) = V_0(\mu) + \sum_{i=0}^M \int_0^t \mu_t^i dS_u^i.$$

In economic terms the existence of an *arbitrage* in a financial model means that there is a chance to invest zero today and receive at some later time a non-negative amount that can be positive with positive probability. Mathematically this means there exists a self-financing strategy, $(\mu_t)_{t \geq 0}$, such that $V_0(\mu) = 0$, but $\mathbb{Q}(V_t(\mu) > 0) > 0$ for some future time $t > 0$. A key result of Harrison and Pliska (1981) is in connecting the absence of arbitrage in a financial model with the existence of an *equivalent martingale measure*:

Definition 3.5. A probability measure, $\mathbb{Q}$, is an *equivalent martingale measure* if:

- i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$, that is $\mathbb{Q}(A) = 0$ if and only if $\mathbb{P}(A) = 0 \forall A \in \mathcal{F}$.
- ii) All discounted price processes, $\tilde{S}_t^i = \beta(t)S_t^i$ for $0 \leq i \leq M$, are martingales under the measure $\mathbb{Q}$.

We will assume the existence of an equivalent martingale measure, $\mathbb{Q}$, in this thesis. This assumption implies the absence of arbitrage in the financial model (see Harrison and Pliska (1981)).

3.1.2 Intensity modelling

In the intensity modelling approach we consider additional information arising from the observation of the time representing the default time of an obligor.

Denote by τ a non-negative random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, satisfying $\mathbb{Q}(\tau = 0) = 0$ and $\mathbb{Q}(\tau > t) > 0 \forall t > 0$. τ is taken to mean the default time of an obligor. Introduce the right continuous *default indicator* process, $(D_t)_{t \geq 0}$, by setting:

$$D_t = \mathbb{I}_{\{\tau \leq t\}} \forall t > 0.$$

In this thesis we will require that $(D_t)_{t \geq 0}$ be an adapted process. In other words we require that τ be a stopping time (we give the definition in the following) with respect to some enlarged filtration, $\mathbb{G}$ (to be determined), of $\mathbb{F}$:

Definition 3.6. τ is a *stopping time* if $\{\tau \leq t\} \in \mathcal{G}_t \forall t \geq 0$, where $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is an expanded version of $\mathbb{F}$.

When τ is a stopping time, it's realisation depends exclusively on the past information generated by $\mathbb{G}$.

Various filtration expansion methods are considered in the literature in order to make τ a stopping time with respect to an enlarged filtration (for examples see Guo et al. (2007) and Guo and Zeng (2006)). The method we will use is called the *minimal filtration expansion method*. The minimal filtration expansion is obtained by enlarging $\mathbb{F}$ in the minimal way in order to make τ a stopping time under the expanded filtration. To do this, define $\forall t \geq 0$:

$$\begin{aligned}\mathcal{G}_t &= \mathcal{F}_t \vee \sigma(D_s : s \leq t) \\ &= \mathcal{F}_t \vee \mathcal{D}_t,\end{aligned}\tag{3.1.1}$$

where $\mathcal{D}_t \equiv \sigma(D_s : s \leq t)$ is the smallest sigma-algebra containing information about the default of the obligor up to time t . From this we may define the minimal filtration as:

$$\mathbb{G} = (\mathcal{F}_t \vee \mathcal{D}_t)_{t \geq 0} \equiv \mathbb{F} \vee \mathbb{D}.$$

Under this definition τ is a stopping time with respect to the filtration $\mathbb{G}$.

Definition 3.7. The $\mathbb{F}$ – conditional cumulative distribution process of the default time τ , $\forall t \geq 0$, is given by:

$$F_t = \mathbb{Q}(\tau \leq t | \mathcal{F}_t).\tag{3.1.2}$$

Definition 3.8. The $\mathbb{F}$ – conditional survival process of the default time τ , $\forall t \geq 0$, is given by:

$$Q_t = \mathbb{Q}(\tau > t | \mathcal{F}_t).\tag{3.1.3}$$

Note that $Q_t = 1 - F_t \forall t \geq 0$.

Assumption 3.1. Assume $(F_t)_{t \geq 0}$ is absolutely continuous, $Q_t > 0 \forall t \geq 0$ and all $\mathbb{F}$ – martingales are $\mathbb{G}$ – martingales³.

Under Assumption 3.1 we can define the $\mathbb{F}$ – integrated hazard rate process, $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, via:

$$\Gamma_t = -\ln(Q_t) = -\ln(1 - F_t) \forall t > 0.$$

³These are standard assumptions when modelling under the intensity framework, see Jeanblanc and Cam (2007).

Since the $\mathbb{F}$ – conditional cumulative distribution process is absolutely continuous, we have that $F_t = \int_0^t f(u)du$ for some integrable function f (see Bielecki and Rutkowski (2002)). From this we get:

$$F_t = 1 - e^{-\Gamma_t} = 1 - e^{-\int_0^t \lambda_u du}.$$

Standard manipulations yield, $\forall u \geq 0$ & $F_u \neq 1$:

$$\lambda_u = \frac{f(u)}{(1 - F_u)}.$$

Proposition 1. *When we assume the $\mathbb{F}$ – conditional cumulative distribution process is absolutely continuous we recover a unique $\mathbb{F}$ – intensity, $(\lambda_t)_{t \geq 0}$, for the default time τ .*

Proof. see Bielecki and Rutkowski (2002). □

Proposition 2. *Under Assumption 3.1, $\forall t \geq 0$, we have:*

$$M_t = \mathbb{I}_{\{\tau < t\}} - \Gamma_{t \wedge \tau} \text{ is a } \mathbb{G} - \text{martingale.} \quad (3.1.4)$$

Proof. see Bielecki and Rutkowski (2002). □

$(M_t)_{t \geq 0}$ is called the $(\mathbb{G}, \tau)$ – martingale process.

Note 1 (The case of meaningful stopping times). *We note there are various types of stopping times. For example a stopping time, τ , is:*

- *Predictable if there exists stopping times $\tau_n \uparrow \tau$ with $\tau_n < \tau$ on the set $\{\omega \in \Omega | \tau < \infty\}$. $(\tau_n)_{n \in \mathbb{N}}$ is said to announce τ ,*
- *Accessible if there exists countably many predictable stopping times $(\tilde{\tau}_n)_{n \in \mathbb{N}}$ such that:*

$$\mathbb{P}(\tau \neq \tilde{\tau}_n) = 0 \quad \forall n \in \mathbb{N}.$$

- *Conversely a stopping time, τ , is totally inaccessible if*

$$\mathbb{P}(\tau \neq \tau^*) = 1,$$

for any predictable stopping time, τ^* .

Now:

- *If τ is an $\mathbb{F}$ – stopping time then the $\mathbb{F}$ – conditional cumulative distribution process of the default time τ is $\mathbb{I}_{\{\tau \leq t\}}$ i.e. $F_t = \mathbb{I}_{\{\tau \leq t\}} \quad \forall t \geq 0$. Hence the $\mathbb{F}$ – integrated hazard rate process is not well defined whenever τ is an $\mathbb{F}$ – stopping time.*

- If τ is a $\mathbb{G}$ – predictable stopping time we have that $\Gamma_t = \mathbb{I}_{\{\tau \leq t\}}$ (see Bielecki and Rutkowski (2002)). In this instance Equation 3.1.4 is not meaningful.
- Under Assumption 3.1 τ will necessarily be a $\mathbb{G}$ – stopping time which is totally inaccessible. This is because (under Assumption 3.1) the intensity process exists, and if an intensity exists the stopping time is totally inaccessible (see Guo and Zeng (2006)).
- Hence, under our setting all default models we consider have stopping times that are totally inaccessible stopping times with respect to $\mathbb{G}$. This means that the default models generate surprise in the timing of default.

To underscore the nature of the surprise (of the default time occurrence) under Assumption 3.1, consider the following example:

Assume that $\mathbb{G}$ is a filtration generated by a Brownian motion, $(W_t)_{t \geq 0}$. Given that $(M_t)_{t \geq 0}$ (the $(\mathbb{G}, \tau)$ – martingale process) is a $\mathbb{G}$ – martingale, then from the martingale representation theorem $(M_t)_{t \geq 0}$ can be represented as a stochastic integral with respect to $(W_t)_{t \geq 0}$ (see Oksendal (2005) p.53). Since Brownian motions are diffusive and do not jump, integrals with respect to them also do not jump. Yet, under Assumption 3.1, $(\Gamma_t)_{t \geq 0}$ (the $\mathbb{F}$ – integrated hazard rate process), is absolutely continuous. Duffie (2002) demonstrates that in this case $(M_t)_{t \geq 0}$ must jump at time τ . By contradiction we conclude that the information pattern of an enlarged filtration with a default stopping time, under Assumption 3.1, lacks continuity in the way it evolves.

We have now collected all the necessary background concepts to present the key results used throughout this chapter and developed in the rest of this thesis. We do this in the following sub-section.

3.1.3 Filtration switching

In this sub-section we show how to switch between the sub-filtration, $\mathbb{F}$, and the enlarged filtration, $\mathbb{G}$.

The two filtrations $\mathbb{F}$ and $\mathbb{G}$ are separated by the information revealing the actual default time:

- Lemma 3, below, puts this notion into a mathematical context.
- Lemma 3 leads to Corollary 4. Corollary 4 is important because it shows that $\mathbb{G}$ – adapted processes have versions which are $\mathbb{F}$ – adapted up until default.

- Corollary 4 is employed later in this chapter (Sub-section 3.2.6) to enable a definition of the swap spread that is $\mathbb{F}$ – adapted and defined in all states.
- As most credit derivative products in the single obligor setting are defined only whilst a default has not occurred, switching filtration creates little problems in this setting.
- Proposition 5 and Corollary 6 provide a mechanism of switching from $\mathbb{G}$ to $\mathbb{F}$. Doing this significantly increases tractability in the valuation of credit products (see Jeanblanc and Cam (2007)).

Lemma 3. Assume $\mathbb{G} = \mathbb{F} \vee \mathbb{D}$. $\forall t \geq 0$ given $A \in \mathcal{G}_t \exists B \in \mathcal{F}_t$ such that:

$$A \cap \{\tau > t\} = B \cap \{\tau > t\}.$$

Proof. See Bielecki and Rutkowski (2002). □

Corollary 4. Assume that $(Y_t)_{t \geq 0}$ is a $\mathbb{G}$ – adapted process, then $\exists$ a unique $\mathbb{F}$ – adapted process, $(\tilde{Y}_t)_{t \geq 0}$, such that $\forall t \geq 0$:

$$Y_t \mathbb{I}_{\{\tau > t\}} = \tilde{Y}_t \mathbb{I}_{\{\tau > t\}},$$

Proof. See Rutkowski and Armstrong (2008). □

The $\mathbb{F}$ – adapted process, $(\tilde{Y}_t)_{t \geq 0}$, which is a version of the $\mathbb{G}$ – adapted process, $(Y_t)_{t \geq 0}$, up until the time of default is called the *pre-default process*.

Proposition 5. Let Y be an $\mathcal{F}$ – measurable random variable, then $\forall t \geq 0$ we have:

$$\mathbb{I}_{\{\tau > t\}} \mathbb{E}[Y | \mathcal{G}_t] = \mathbb{I}_{\{\tau > t\}} \frac{\mathbb{E}[Y \mathbb{I}_{\{\tau > t\}} | \mathcal{F}_t]}{\mathbb{E}[\mathbb{I}_{\{\tau > t\}} | \mathcal{F}_t]} = \mathbb{I}_{\{\tau > t\}} Q_t^{-1} \mathbb{E}[Y \mathbb{I}_{\{\tau > t\}} | \mathcal{F}_t] = \mathbb{I}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}[Y \mathbb{I}_{\{\tau > t\}} | \mathcal{F}_t], \quad (3.1.5)$$

where $(Q_t)_{t \geq 0}$ is the $\mathbb{F}$ – conditional survival process of the default time τ and $(\Gamma_t)_{t \geq 0}$ is the $\mathbb{F}$ – integrated hazard rate process.

Proof. See Bielecki and Rutkowski (2002). □

This leads to:

Corollary 6. Let $\tilde{Y}$ be $\mathcal{F}_T$ measurable for some $T > 0$, then $\forall 0 \leq t \leq T$:

$$\mathbb{E}[\tilde{Y} \mathbb{I}_{\{\tau > T\}} | \mathcal{G}_t] = \mathbb{I}_{\{\tau > t\}} Q_t^{-1} \mathbb{E}[\tilde{Y} Q_T | \mathcal{F}_t] = \mathbb{I}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}[\tilde{Y} e^{-\Gamma_T} | \mathcal{F}_t]. \quad (3.1.6)$$

Proof. Invoking Proposition 5, using $Y = \tilde{Y}\mathbb{I}_{\{\tau > T\}}$ and the tower property of conditional expectations gives:

$$\begin{aligned}
\mathbb{I}_{\{\tau > t\}}\mathbb{E}[Y|\mathcal{G}_t] &= \mathbb{I}_{\{\tau > t\}}e^{\Gamma_t}\mathbb{E}\left[\mathbb{E}[\tilde{Y}\mathbb{I}_{\{\tau > T\}}|\mathcal{F}_T]|\mathcal{F}_t\right] \\
&= \mathbb{I}_{\{\tau > t\}}e^{\Gamma_t}\mathbb{E}\left[\mathbb{E}[\tilde{Y}e^{-\Gamma_T}|\mathcal{F}_T]|\mathcal{F}_t\right] \\
&= \mathbb{I}_{\{\tau > t\}}e^{\Gamma_t}\mathbb{E}[\tilde{Y}e^{-\Gamma_T}|\mathcal{F}_t].
\end{aligned} \tag{3.1.7}$$

□

Recall in Chapter 1 we noted that a key part of credit modelling was to quantify default probabilities, Corollary 7 below provides this for stochastic intensity models. Corollary 7 is a first important consequence of Proposition 5 and Corollary 6.

Corollary 7. *Under Assumption 3.1 we have $\forall 0 \leq t \leq s$:*

$$\begin{aligned}
\mathbb{Q}(\tau > s|\mathcal{G}_t) &= \mathbb{I}_{\{\tau > t\}}\mathbb{E}\left[e^{-(\Gamma_s - \Gamma_t)}|\mathcal{F}_t\right] \\
&= \mathbb{I}_{\{\tau > t\}}\mathbb{E}\left[e^{-\int_t^s \lambda_u du}|\mathcal{F}_t\right],
\end{aligned} \tag{3.1.8}$$

$$\begin{aligned}
\mathbb{Q}(t \leq \tau \leq s|\mathcal{G}_t) &= \mathbb{I}_{\{\tau > t\}}\mathbb{E}_{\mathbb{Q}}\left[1 - e^{-(\Gamma_s - \Gamma_t)}|\mathcal{F}_t\right] \\
&= \mathbb{I}_{\{\tau > t\}}\mathbb{E}\left[1 - e^{-\int_t^s \lambda_u du}|\mathcal{F}_t\right].
\end{aligned} \tag{3.1.9}$$

Proof. This is a direct consequence of Equation (3.1.6). □

To conclude, the enlarged filtration does not evolve in a continuous way, it is partly generated by sudden events: defaults. The difference between credit and other asset classes is the possibility of default. At each time, $t \geq 0$, the realisation or not of a default is known only in $\mathcal{G}_t$ and not in $\mathcal{F}_t$. Valuation with respect to the filtration $\mathbb{G}$ is difficult because we need to account for defaults. Default is not a product and is not observable in the market until it occurs.

One of the main challenges in credit is that pricing using the enlarged filtration $\mathbb{G}$ leads to non-tractable computations (see Jeanblanc and Cam (2007)). This challenge is manageable in the single obligor case, as we have only one default to consider and we can apply the filtration switching formula (Proposition 5). As we show later, in Chapters 5 and 6, in the multi-obligor case the interaction and dependence structure of default events of multiple obligors makes the problem significantly more complex.

What we are doing in Proposition 5 is projecting from one filtration, $\mathbb{G}$, to another, $\mathbb{F}$, and by doing this re-weighting events and payoffs (by the conditional survival process). Note

the projection ensures that we recover expectations, yet it does not prescribe a mechanism of capturing sudden market changes due to default events.

This discussion leads to the notion of jump to default (JTD) risk, which is the risk of a sudden change in the value of a product after a default. The idea stipulates that upon a default of an obligor an investor, invested into the assets of the obligor, will incur a sudden loss, this gives rise to jump to default risk. We note that the modelling framework we have adopted (Assumption 3.1) is the common approach adopted in the literature (see Jeanblanc and Cam (2007)). By working under this framework, where we make use of the sub-filtration, $\mathbb{F}$, our modelling framework is not able to account for JTD risk (see Couculescu (2010)) ⁴.

3.2 Pricing in the single obligor case

In this section we use the results provided in Section 3.1 to price **Category 1** products and credit default swaptions.

We note that it is feasible to correlate the spot interest rate process, $(r_t)_{t \geq 0}$, and the intensity process, $(\lambda_t)_{t \geq 0}$; however, in this thesis we will assume they are independent.

Assumption 3.2. *The spot interest rate process, $(r_t)_{t \geq 0}$, and the intensity process, $(\lambda_t)_{t \geq 0}$ evolve independently.*

3.2.1 Losses

In Chapter 1 we highlighted that it is generally very difficult to know the extent of losses on a default. Losses in this instance are defined as the difference between contracted payment obligations to an investor from an obligor (we will assume this is par) and the recovery amount of the obligor when it defaults. Although in the literature there are many assumptions on the recovery rate that can be made, we will assume in this thesis fractional recovery on default.

⁴As an example of JTD risk consider an FTD on 3 obligors: A , B and C with notional 1 and some model describing the nature of the co-dependence amongst the obligors. Suppose the model produces deltas, Δ_A , Δ_B and Δ_C , for the equivalent single obligor CDS hedge; all deltas will have a notional less than 1. If name A defaults, we will not be able to anticipate it's default time (as default times are totally inaccessible). Hence, as we have sold protection on the FTD, we will lose 1 (assuming zero recovery) but only make $\Delta_A < 1$ on our hedge. No meaningful model can immunise this lumpy risk; therefore we can never fully re-balance our credit hedges to eliminate JTD risk.

Definition 3.9 (Fractional Recovery). *Let τ be a stopping time that defines the default of an obligor. At default, the face value of a defaultable bond is reduced by a factor $R \in [0, 1)$ (R may be random). Hence a defaultable bond's final value is R .*

R will generally be a constant, however in Chapter 4 (Section 4.6) an example where R is random is considered. The loss given default (LGD) is defined as $1 - R$.

3.2.2 Pricelets

We are now in a position to give examples of products which are best described as *pricelets* (small pricing results that are components to more complicated product structures).

Corollary 8. *The price, $v(t, T)$, at time $t \geq 0$ of a defaultable zero coupon bond with maturity $T \geq t$ and face value of 1 which has no recovery value on default is:*

$$\begin{aligned} v(t, T) &= \mathbb{I}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E} \left[e^{-\int_t^T r_s ds} e^{-\Gamma_T} | \mathcal{F}_t \right] \\ &= \mathbb{I}_{\{\tau > t\}} \mathbb{E} \left[e^{-\int_t^T (r_s + \lambda_s) ds} | \mathcal{F}_t \right], \end{aligned} \quad (3.2.1)$$

where $(\Gamma_t)_{t \geq 0}$ is the $\mathbb{F}$ – integrated hazard rate process of the default time τ (see Equation (3.1.4) in Section 3.1), $(r_t)_{t \geq 0}$ is the spot interest rate and $(\lambda_t)_{t \geq 0}$ is the intensity of the default time τ .

Proof. The risky bond price can be recovered using Corollary 6:

$$\begin{aligned} v(t, T) &= \mathbb{E} \left[\frac{\beta(T)}{\beta(t)} \mathbb{I}_{\{\tau > T\}} | \mathcal{G}_t \right] \\ &= \mathbb{E} \left[e^{-\int_t^T r_s ds} \mathbb{I}_{\{\tau > T\}} | \mathcal{G}_t \right] \\ &= \mathbb{I}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E} \left[e^{-\int_t^T r_s ds} e^{-\Gamma_T} | \mathcal{F}_t \right] \\ &= \mathbb{I}_{\{\tau > t\}} \mathbb{E} \left[e^{-\int_t^T (r_s + \lambda_s) ds} | \mathcal{F}_t \right]. \end{aligned} \quad (3.2.2)$$

Recall $\beta(t) = e^{-\int_0^t r_s ds} \forall t \geq 0$ is the discount process. □

The result shows the key benefit of pricing under Assumption 3.1. We note that effectively we have replaced a complicated payoff of $\mathbb{I}_{\{\tau > T\}}$ by a payoff of 1 at maturity. This has been achieved by adjusting the spot interest rate from r_t to $(r_t + \lambda_t)$ and switching filtration.

The following corollary gives the expected value of a cash pay out which occurs at default. This is useful for valuation of the default leg of a CDS.

Corollary 9. Let Z_τ be a payment made at time τ , $(Z_u)_{u \geq 0}$, for $0 \leq u \leq T$, be $\mathbb{F}$ -adapted with $Z_u = 0$ if $u > T$ and define:

$$Z_\tau = \begin{cases} Z_u & \text{if } u = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the value of this payoff at time $t \geq 0$ is:

$$\bar{\mathbf{D}}_L(t, T) = \mathbb{I}_{\{\tau > t\}} \mathbb{E} \left[\int_t^T Z_s \lambda_s e^{-\int_t^s (r_u + \lambda_u) du} ds | \mathcal{F}_t \right]. \quad (3.2.3)$$

Proof.

$$\begin{aligned} \bar{\mathbf{D}}_L(t, T) &= \mathbb{E} \left[\frac{\beta(\tau)}{\beta(t)} Z_\tau \mathbb{I}_{\{t < \tau \leq T\}} | \mathcal{G}_t \right] \\ &= \mathbb{E} \left[Z_\tau e^{-\int_t^\tau r_u du} \mathbb{I}_{\{t < \tau \leq T\}} | \mathcal{G}_t \right] \\ &= \mathbb{E} \left[\int_t^T Z_s e^{-\int_t^s r_u du} \mathbb{I}_{\{\tau \in \{s, s+ds\}\}} | \mathcal{G}_t \right] \\ &= \mathbb{I}_{\{\tau > t\}} \mathbb{E} \left[e^{\int_0^t \lambda_u du} \int_t^T Z_s e^{-\int_t^s r_u du} \mathbb{I}_{\{\tau \in \{s, s+ds\}\}} | \mathcal{F}_t \right] \\ &= \mathbb{I}_{\{\tau > t\}} \mathbb{E} \left[\mathbb{E} \left[e^{\int_0^t \lambda_u du} \int_t^T Z_s e^{-\int_t^s r_u du} \mathbb{I}_{\{\tau \in \{s, s+ds\}\}} | \mathcal{F}_s \right] | \mathcal{F}_t \right] \\ &= \mathbb{I}_{\{\tau > t\}} \mathbb{E} \left[\mathbb{E} \left[e^{\int_0^t \lambda_u du} \int_t^T Z_s e^{-\int_t^s r_u du} \lambda_s e^{\int_0^s \lambda_u du} ds | \mathcal{F}_s \right] | \mathcal{F}_t \right] \\ &= \mathbb{I}_{\{\tau > t\}} \mathbb{E} \left[\int_t^T Z_s \lambda_s e^{-\int_t^s (r_u + \lambda_u) du} ds | \mathcal{F}_t \right], \end{aligned} \quad (3.2.4)$$

where the fourth equality follows from an application of Proposition 5 and the sixth equality follows from noting that:

$$\begin{aligned} \mathbb{E} [\mathbb{I}_{\{\tau \in \{s, s+ds\}\}} | \mathcal{F}_s] &= \mathbb{E} [\lambda_s e^{-\int_0^s \lambda_u du} ds | \mathcal{F}_s] \\ &= \lambda_s e^{-\int_0^s \lambda_u du} ds. \end{aligned} \quad (3.2.5)$$

□

By taking $Z_u = 1 - R$, $\forall u \in [0, T]$ in Corollary 9 we recover the default leg of a CDS (see Sub-section 3.2.3).

3.2.3 Valuation of a CDS

In this sub-section we detail the valuation methodology for CDS under the intensity framework described in Section 3.1. Let us assume the maturity of a CDS is $T > 0$ and there are N payments at times $T_1, \dots, T_N$ with $T_N = T$. The buyer of protection, PB, will pay a spread s to the protection seller, PS. There are two legs to value:

- (1) A default leg.
- (2) A premium leg.

Assume today is $t_0 < T$ then we have that PB gets a payment of:

$$(1 - R)\mathbb{I}_{\{t_0 < \tau < T\}}, \quad (3.2.6)$$

at time τ . Under the equivalent martingale measure, $\mathbb{Q}$, we can value the payoff in Equation (3.2.6) as being:

$$\begin{aligned} \overline{\mathbf{D}}_L(t_0, T) &= (1 - R)\mathbb{E} \left[\frac{\beta(\tau)}{\beta(t_0)} \mathbb{I}_{\{t_0 < \tau \leq T\}} | \mathcal{G}_{t_0} \right] \\ &= (1 - R)\mathbb{E} \left[e^{-\int_{t_0}^{\tau} r_s ds} \mathbb{I}_{\{t_0 < \tau \leq T\}} | \mathcal{G}_{t_0} \right] \\ &= (1 - R)\mathbb{E} \left[\int_{t_0}^T \lambda_s e^{-\int_{t_0}^s (r_u + \lambda_u) du} ds | \mathcal{F}_{t_0} \right]. \end{aligned} \quad (3.2.7)$$

In addition the premium leg is paid by PB to PS at payment dates $T_1, \dots, T_N$, with payoff given by:

$$s\Delta_i \mathbb{I}_{\{T_i < \tau\}} \quad \text{at} \quad T_i, \quad i \in \{1, \dots, N\}, \quad (3.2.8)$$

where $\Delta_i = T_i - T_{i-1}$ and we take $T_0 = t_0$. Also on a default PB must make a final payment to PS called the accrual with payoff given by:

$$s\Delta^* \mathbb{I}_{\{t_0 < \tau \leq T\}} \quad \text{at} \quad \tau, \quad (3.2.9)$$

where $\Delta^* = \tau - T_r$, $r \in \{0, \dots, N-1\}$, T_r is the last date before default at time τ . The value of this leg is:

$$s \times \overline{\mathbf{P}}_L(t_0, T) = s\mathbb{E} \left[\sum_{i=1}^N \Delta_n \mathbb{I}_{\{\tau > T_n\}} e^{-\int_{t_0}^{T_n} r_s ds} + e^{-\int_{t_0}^{\tau} r_s ds} \Delta^* \mathbb{I}_{\{t_0 < \tau \leq T\}} | \mathcal{G}_{t_0} \right]. \quad (3.2.10)$$

3.2.4 Market standard for CDS valuation

In this sub-section we will consider the valuation of CDS under current market conventions. These conventions are (see Brigo and Alfonsi (2005)) that:

- The intensity of the obligor is deterministic.
- The spot interest rate is deterministic.
- In the valuation of the legs of the CDS certain discretisations are made in order to construct the intensity curve as a function of time (we detail the procedure below).

In order to imply information about the intensity of an obligor, it is standard to *strip* the CDS curve. Stripping is a method of iteratively implying the initial term structure of a process from quoted market data:

- The method enables one to build an implied initial forward hazard rate curve, $\lambda(0, t)$, as in the Heath-Jarrow-Morton (HJM) framework (see Appendix C).
- Generally a *curve* is defined by an interpolation rule and a set of parameters specific to the interpolation rule. The interpolation can be fully parametric or defined by two vectors of associated numbers.
- In the latter case the abscissa and ordinate values have a specifically chosen interpolation method such as piecewise constant, piecewise linear, cubic spline etc (see Hagan and West (2006)).
- In the following, we will assume that the hazard rate curve is defined by a vector of abscissa-ordinate pairs, and an interpolation rule.
- The abscissa values are fixed with times $t_0, t_1, \dots, t_n$, with $t_0 = 0$ and the other times representing the maturity dates of CDS- these times are called *nodes*.
- The ordinate values are the hazard rate values, $(\lambda(0, t_r))_{r \in \{1, \dots, n\}}$, at the node points.
- The CDS market is defined by a set of market tenors e.g. 1y, 3y, 5y, 7y and 10y⁵. These tenors will be taken as the time nodes.
- The market standard for specifying the initial hazard rate is to use a piecewise constant interpolation rule with left continuous time nodes. It is assumed that the hazard rate is piecewise constant between the maturity dates $t_1, t_2, \dots, t_n$ of the standard CDS. So the survival probability from t_0 to time t , where $t_{j-1} < t < t_j$, is given by:

$$Q(t_0, t) = \exp \left\{ - \sum_{i=1}^{j-1} \lambda_i (t_i - t_{i-1}) - \lambda_j (t - t_{j-1}) \right\}. \quad (3.2.11)$$

⁵y=years

- Hence we have that valuation of the premium leg of a CDS is:

$$s \times \bar{\mathbf{P}}_{\mathbf{L}}(t_0, T_N) = s \sum_{i=1}^N \Delta(T_{i-1}, T_i) Q(t_0, T_i) P(t_0, T_i) + s \sum_{i=1}^m \int_{T_{i-1}}^{T_i} \Delta(T_{i-1}, u) P(t_0, u) Q(t_0, u) \lambda(u) du, \quad (3.2.12)$$

where $P(u, t)$ is the discount rate between times $u \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$ with $t > u$. $T_1, \dots, T_N$ are the dates of the payment of the premium leg, the maturity of the swap is $T = T_N$ and s is the coupon amount.

Lastly it is market convention to discretise the integral in Equation (3.2.12) and make the assumption that if a default occurs it will be, on average, half way in between premium dates. This gives:

$$\bar{\mathbf{P}}_{\mathbf{L}}(t_0, T) = \sum_{i=1}^N \Delta(T_{i-1}, T_i) Q(t_0, T_i) P(t_0, T_i) + \sum_{i=1}^N \frac{\Delta(T_{i-1}, T_i)}{2} P(t_0, T_i) (Q(t_0, T_{i-1}) - Q(t_0, T_i)), \quad (3.2.13)$$

and similarly we have that the protection leg is discretised as follows:

$$\bar{\mathbf{D}}_{\mathbf{L}}(t_0, T) = (1 - R) \sum_{i=1}^{\tilde{N}} P(t_0, \tilde{T}_i) (Q(t_0, \tilde{T}_{i-1}) - Q(t_0, \tilde{T}_i)), \quad (3.2.14)$$

where $\tilde{N}$ does not have to be the same as N and times $\tilde{T}_1, \dots, \tilde{T}_{\tilde{N}}$ are a different set of discretisations to the ones used for the premium leg. This gives:

$$s(t_0, T) = \frac{\bar{\mathbf{D}}_{\mathbf{L}}(t_0, T)}{\bar{\mathbf{P}}_{\mathbf{L}}(t_0, T)}, \quad (3.2.15)$$

which is the fair premium for a CDS with maturity $T = T_N$.

3.2.5 Numerical example: CDS valuation

Below we give numerical examples of the process termed stripping:

- Table 3.1 provides the CDS curve of the Federative Republic of Brazil as of February 2009.
- Table 3.2 collects a series of quantities implied by the CDS spread curve provided in Table 3.1. The results are recovered assuming market recovery for the Republic of Brazil of 25%. Provided are implied hazard rates (intensities) for the corresponding time nodes, survival probabilities as of today to maturity of the market tenors, the value of the protection leg per unit notional of CDS and the value of the premium leg. The premium leg is also called the risky duration; it may be looked at as a quantity possessing dimension in time and representing the expected average life of a CDS.
- Figures 3.1, 3.2, 3.3 and 3.4 are graphs of the quantities in Table 3.2.

6m	1y	2y	3y	4y	5y	7y	10y
202bps	238bps	300bps	338bps	362bps	377bps	389bps	400bps

Table 3.1: Federative Republic of Brazil CDS spreads as of February 2009. Data source from *MarkIt*

Tenors	Hazard rate nodes	Survival function	Protection Leg	Premium Leg
6months	2.72%	98.66%	1.00%	0.49
1year	3.69%	96.85%	2.37%	1.00
2years	4.92%	92.20%	5.78%	1.93
3years	5.68%	87.12%	9.42%	2.79
4years	5.99%	77.72%	12.95%	3.58
5years	5.41%	69.75%	15.86%	4.30
7years	5.41%	58.59%	20.98%	5.57
10years	5.81%	54.14%	27.49%	7.07

Table 3.2: Hazard rate node values, survival probability values, protection leg values and premium leg values. Computed using the Federative Republic of Brazil CDS curve.

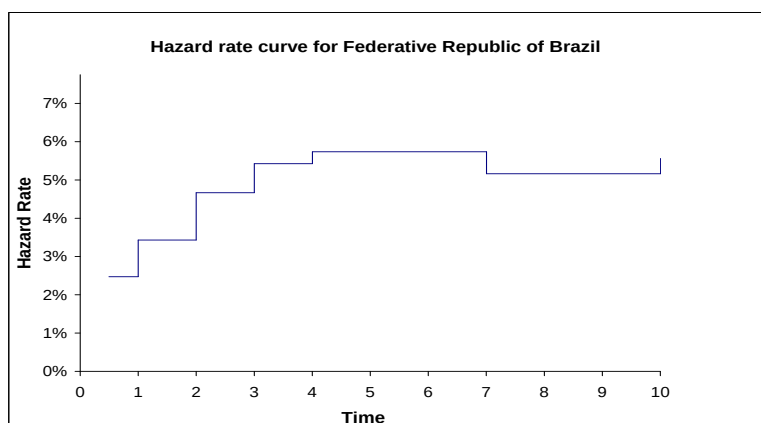


Figure 3.1: Stripped hazard rate curve for the Federative Republic of Brazil with a piece-wise constant intensity curve.

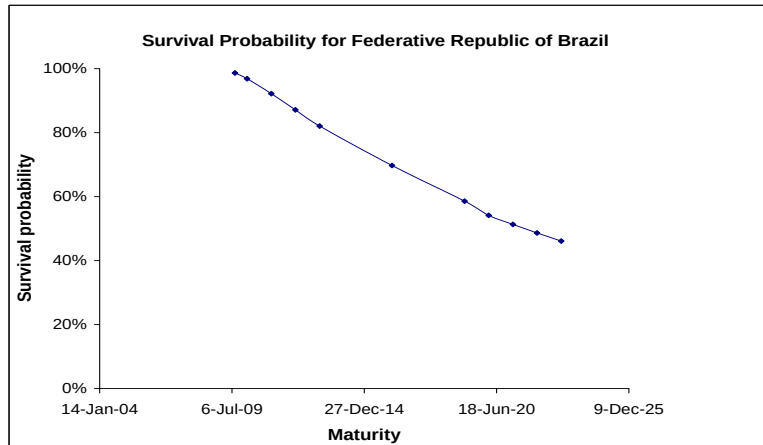


Figure 3.2: Survival probability.

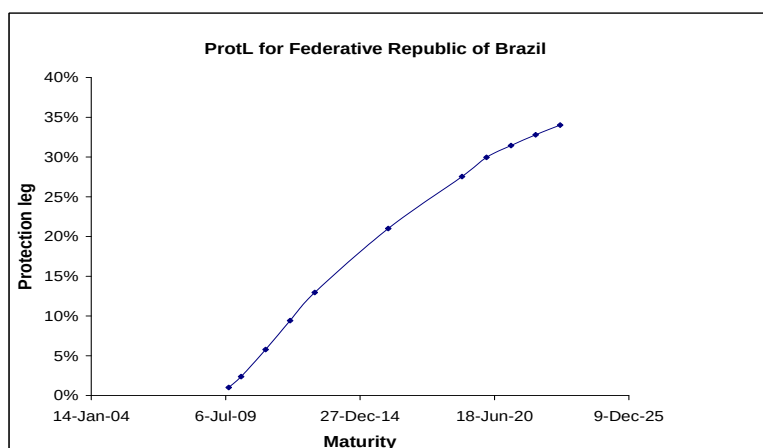


Figure 3.3: Protection leg.

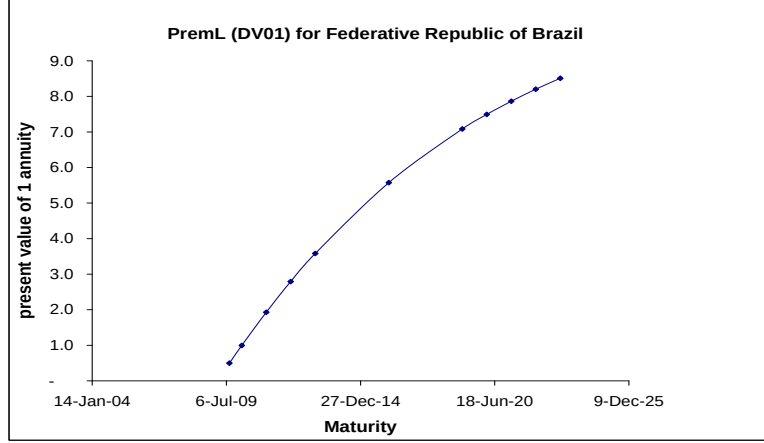


Figure 3.4: Premium leg.

3.2.6 Forwards and Swaptions

In this section we review the pricing of single obligor forward CDS and credit default swaptions, see Schönbucher (2003b). We present the current standard model for swaption valuation. We no longer assume deterministic intensities, but maintain the assumption of deterministic interest rates. This will be an assumption of this thesis going forward.

Assumption 3.3. *The spot interest rate is deterministic.*

In the following we will detail the payoff of an FCDS. Recall that an FCDS is a CDS that starts its life at a time $t > t_0$, t_0 being the trade date and t the forward start date (or expiry). If there is a default at any time in $[t_0, t]$ then an FCDS contract is cancelled and no cash flows are exchanged. We have that a protection seller will receive contingent premiums at dates $T_1 < \dots < T_N$ with $t < T_1$. The payments at $T_i \forall i \in \{1, \dots, N\}$ are:

$$k\Delta_i\mathbb{I}_{\{T_i < \tau\}}, \quad (3.2.16)$$

where k is the spread PB agrees to pay PS at trade expiry, Δ_i is the accrual. In addition, as already discussed, an accrual payment is made on default; it has payoff:

$$k\Delta^*\mathbb{I}_{\{t < \tau < T\}}, \quad (3.2.17)$$

at τ . Hence the value of an FCDS premium leg is:

$$\begin{aligned} k \times \bar{\mathbf{P}}_L(t, T) &= k \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=1}^N \Delta_i \mathbb{I}_{\{\tau > t_i\}} e^{-\int_t^{t_i} r_s ds} + e^{-\int_t^{\tau} r_s ds} \Delta^* \mathbb{I}_{\{t < \tau < T_N\}} | \mathcal{G}_t \right], \\ &= k \mathbb{E}^{\mathbb{Q}} [\mathbf{P}(t, T) | \mathcal{G}_t], \end{aligned} \quad (3.2.18)$$

where T is the maturity date of the contract. Similar arguments give the value of the protection leg:

$$\begin{aligned} \bar{\mathbf{D}}_L(t, T) &= (1 - R) \mathbb{E}^{\mathbb{Q}} [e^{-\int_t^{\tau} r_s ds} \mathbb{I}_{\{t < \tau \leq T\}} | \mathcal{G}_t] \\ &= \mathbb{E}^{\mathbb{Q}} [\mathbf{D}(t, T) | \mathcal{G}_t]. \end{aligned} \quad (3.2.19)$$

$\mathbf{P}(t, T)$ and $\mathbf{D}(t, T)$ are respectively the discounted payoffs of the premium leg and default leg.

In the following we demonstrate the conventional pricing result for credit default swaptions. We first note that at time $t > t_0$ the market value of a CDS with a fair swap spread, $s(t, T)$, is given by:

$$\mathbb{I}_{\{\tau > t\}} (s(t, T) - k) \bar{\mathbf{P}}_L(t, T). \quad (3.2.20)$$

This follows because whenever $\tau > t$ the fair swap spread is:

$$s(t, T) = \frac{\bar{\mathbf{D}}_L(t, T)}{\bar{\mathbf{P}}_L(t, T)}. \quad (3.2.21)$$

If we had the right but not the obligation to enter into an FCDS contract at time $t > t_0$ (trade date being t_0) then we will have a payer credit default swaption (which is a European call option). The option knocks out if default occurs before time t and so has the following payoff:

$$\mathbb{I}_{\{\tau > t\}} (s(t, T) - k)^+ \bar{\mathbf{P}}_L(t, T). \quad (3.2.22)$$

The value of the option at t_0 is:

$$V_{t_0} = \mathbb{I}_{\{\tau > t_0\}} \mathbb{E}^{\mathbb{Q}} [\mathbb{I}_{\{\tau > t\}} \frac{\beta(t)}{\beta(t_0)} (s(t, T) - k)^+ \bar{\mathbf{P}}_L(t, T) | \mathcal{G}_{t_0}]. \quad (3.2.23)$$

In order to value this option it is standard to change measure (see Schönbucher (2003b)). The connection between the equivalent martingal measure, $\mathbb{Q}$, and the new measure can be characterised via the Radon-Nikodým theorem:

Theorem 10 (Radon-Nikodým Theorem). *Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be two probability measures on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\tilde{\mathbb{P}} \ll \mathbb{P}$ (that is, every set in $\mathcal{F}$ with a $\mathbb{P}$ -measure of zero also has a $\tilde{\mathbb{P}}$ -measure of zero). Then there exists a unique (up to $\mathbb{P}$ and $\tilde{\mathbb{P}}$ indistinguishability) $\mathbb{P}$ -martingale, $(E_t)_{t \geq 0}$ (called the Radon-Nikodým density), such that $\forall t \geq 0$ and all $\mathcal{F}_t$ -measurable and $\tilde{\mathbb{P}}$ -integrable random variables, Y , we have:*

$$\mathbb{E}^{\tilde{\mathbb{P}}} [Y] = \mathbb{E}^{\mathbb{P}} [Y E_t].$$

Moreover when $\mathbb{E}^{\mathbb{P}}[E_t] = 1$, $\tilde{\mathbb{P}}$ is a probability measure.

For this reason we write:

$$E_{T^*} = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}},$$

and:

$$E_t = \mathbb{E}\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \middle| \mathcal{F}_t\right].$$

Returning to the valuation of the credit default swaption, we can take:

$$E_t = P(t_0, t) \frac{\mathbb{I}_{\{\tau > t\}} \bar{\mathbf{P}}_L(t, T)}{\mathbb{I}_{\{\tau > t_0\}} \bar{\mathbf{P}}_L(t_0, T)}, \quad (3.2.24)$$

from this we can define a non-negative process, $(E_s)_{s \geq 0}$, by:

$$E_s = \mathbb{E}[E_t | \mathcal{F}_s] \quad \forall s \in [0, t].$$

$(E_s)_{s \geq 0}$ is well defined and non-negative (zero on default) since the value of the annuity is strictly positive when no defaults have occurred. By Theorem 10 we may define a new measure, $\tilde{\mathbb{Q}}$ with the Radon-Nikodým density given by $(E_s)_{s \geq 0}$. Moreover $\tilde{\mathbb{Q}}$ defines a probability measure because:

$$\begin{aligned} \tilde{\mathbb{Q}}(\Omega) &= \mathbb{E}^{\mathbb{Q}} \left[P(t_0, t) \frac{\mathbb{I}_{\{\tau > t\}} \bar{\mathbf{P}}_L(t, T)}{\mathbb{I}_{\{\tau > t_0\}} \bar{\mathbf{P}}_L(t_0, T)} \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{Q}} \left[P(t_0, t) \frac{\mathbb{I}_{\{\tau > t\}} \bar{\mathbf{P}}_L(t, T)}{\mathbb{I}_{\{\tau > t_0\}} \bar{\mathbf{P}}_L(t_0, T)} \middle| \mathcal{G}_t \right] \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{\mathbb{I}_{\{\tau > t_0\}} \bar{\mathbf{P}}_L(t_0, T)} \mathbb{E}^{\mathbb{Q}} [P(t_0, t) \mathbb{I}_{\{\tau > t\}} \bar{\mathbf{P}}_L(t, T) | \mathcal{G}_t] \right] \\ &= 1. \end{aligned} \quad (3.2.25)$$

Hence by Theorem 10 we can simplify Equation (3.2.23) for the price of the swaption by a change of measure to get:

$$\begin{aligned} V_{t_0} &= \mathbb{I}_{\{\tau > t_0\}} \mathbb{E}^{\mathbb{Q}} [\mathbb{I}_{\{\tau > t\}} (s(t, T) - k)^+ \bar{\mathbf{P}}_L(t, T)] \\ &= \mathbb{I}_{\{\tau > t_0\}} \bar{\mathbf{P}}_L(t_0, T) \mathbb{E}^{\tilde{\mathbb{Q}}} [(s(t, T) - k)^+]. \end{aligned} \quad (3.2.26)$$

Under the new measure the only quantity that needs to be modelled is the $\mathbb{G}$ – adapted process $s(t, T)$.

Example 2 (The canonical credit default swaption model). Assume for some fixed maturity T the process $s(t, T) = s_t$ follows a geometric Brownian motion under $\tilde{\mathbb{Q}}$, so that:

$$ds_t = \sigma dW_t,$$

where σ is a positive constant called the log-normal volatility and $(W_t)_{t \geq 0}$ is a $\tilde{\mathbb{Q}}$ -Brownian motion. We get by the usual Black-Scholes formula, (see Black and Scholes (1973)):

$$V_{t_0} = \mathbb{I}_{\{\tau > t_0\}} \bar{\mathbf{P}}_L(t_0, T) [s_t \Phi(d_1) - k \Phi(d_2)], \quad (3.2.27)$$

where,

$$d_1 = \frac{\ln\left(\frac{s_{t_0}}{k} + \frac{\sigma^2(t-t_0)}{2}\right)}{\sigma\sqrt{t-t_0}},$$

$$d_2 = d_1 - \sigma\sqrt{t-t_0},$$

and $\Phi(\cdot)$ is the standard cumulative normal distribution.

To conclude, finding analytical valuation results for credit default swaptions makes the calibration process of a model quick and exact. Currently the standard approach for modelling credit default swaptions has been to model the swap spread process of a CDS as a geometric Brownian motion, see Jackson (2005). Whilst such a model is sufficient for European style options on default swaps, the model is restricted in that it does not consider the term structure of credit spreads and therefore products such as LCLNs cannot be valued under such models.

3.2.7 Pre-default quantities and equilibrium swap spreads

In this sub-section we present the concept of pre-default quantities, which reduces instances of arbitrage and provides a definition of the credit swap spread which is valid in all states.

For an FCDS with expiry $t > t_0$ and swap maturity $T > t$ we have the trivial but important relationship that:

$$\mathbf{P}(t, T) = \mathbb{I}_{\{\tau > t\}} \mathbf{P}(t, T), \quad \mathbf{D}(t, T) = \mathbb{I}_{\{\tau > t\}} \mathbf{D}(t, T). \quad (3.2.28)$$

The relationship in Equation (3.2.28) follows as a direct application of Corollary 4 in Section 3.1. The value of an FCDS from the perspective of a protection buyer, paying a rate of k , is:

$$\overline{\mathbf{val}}_L(t, T, k) = \mathbb{E}^{\mathbb{Q}}[\mathbf{val}(t, T, k) | \mathcal{G}_t] = \mathbb{E}^{\mathbb{Q}}[\mathbf{D}(t, T) | \mathcal{G}_t] - k \mathbb{E}^{\mathbb{Q}}[\mathbf{P}(t, T) | \mathcal{G}_t] = \bar{\mathbf{D}}_L(t, T) - k \bar{\mathbf{P}}_L(t, T), \quad (3.2.29)$$

where $\mathbf{val}(t, T, k)$ is the discounted payoff of an FCDS from the perspective of a protection buyer. We recall that the fair swap spread to enter into an FCDS or CDS is the spread which solves $\overline{\mathbf{val}}_L(t, T, k) = 0$ and can be calculated as:

$$s(t, T) = \frac{\bar{\mathbf{D}}_L(t, T)}{\bar{\mathbf{P}}_L(t, T)}. \quad (3.2.30)$$

$s(t, T)$ is the equilibrium credit swap spread setting the price of a default swap to zero. This equilibrium swap spread is not guaranteed to exist at a time $t > t_0$ past today since $\bar{\mathbf{P}}_L(t, T)$ can be zero. If a default has occurred in between t_0 and t the equilibrium swap spread is not valid (see Morini and Brigo (2007)).

We can avoid the non-existence problem by switching filtration from the enlarged filtration, $\mathbb{G}$, to the sub-filtration, $\mathbb{F}$. Doing this means that we observe only the probabilities of default rather than actual default events so that under the sub-filtration the swap spread is valid with probability one (see Rutkowski and Armstrong (2008)). This is formalised in the following lemma:

Lemma 11. *The price at time $t > t_0$ of an FCDS traded at time $t_0 \geq 0$ with expiry t , satisfies:*

$$\bar{\mathbf{val}}_L(t, T, k) = \mathbb{I}_{\{\tau > t\}} Q_t^{-1} \mathbb{E}[\mathbf{val}(t, T, k) | \mathcal{F}_t] = \mathbb{I}_{\{\tau > t\}} \widetilde{\mathbf{val}}(t, T, k), \quad (3.2.31)$$

where the pre-default value $\widetilde{\mathbf{val}}(t, T, k)$, satisfies $\widetilde{\mathbf{val}}(t, T, k) = \tilde{\mathbf{D}}(t, T) - k\tilde{\mathbf{P}}(t, T)$ with

$$\tilde{\mathbf{D}}(t, T) = Q_t^{-1} \mathbb{E}[\mathbf{D}(t, T) | \mathcal{F}_t], \quad \tilde{\mathbf{P}}(t, T) = Q_t^{-1} \mathbb{E}[\mathbf{P}(t, T) | \mathcal{F}_t],$$

where $(Q_t)_{t \geq 0}$ is the $\mathbb{F}$ – conditional survival process of the default time τ (see Section 3.2).

Proof. See Rutkowski and Armstrong (2008). □

Lemma 11 allows us to give a definition of the fair swap spread which Rutkowski and Armstrong (2008) call the *pre-default swap spread*. The pre-default swap spread is:

$$\tilde{s}(t, T) = \frac{\tilde{\mathbf{D}}(t, T)}{\tilde{\mathbf{P}}(t, T)} = \frac{(1 - R) \mathbb{E}^\mathbb{Q} \left[e^{-\int_t^\tau r_s ds} \mathbb{I}_{\{t < \tau \leq T\}} | \mathcal{F}_t \right]}{\mathbb{E}^\mathbb{Q} \left[\sum_{i=1}^N \Delta_i \mathbb{I}_{\{\tau > t_i\}} e^{-\int_t^{t_i} r_s ds} + e^{-\int_t^\tau r_s ds} \Delta^* \mathbb{I}_{\{t < \tau < T_N\}} | \mathcal{F}_t \right]}, \quad (3.2.32)$$

where $\mathbb{I}_{\{\tau > t\}} s(t, T) = \mathbb{I}_{\{\tau > t\}} \tilde{s}(t, T)$ and $s(t, T)$ (given in Equation (3.2.30)) is undefined under the event $\{\tau \leq t\}$. The pre-default premium leg is always positive i.e. $\tilde{\mathbf{P}}(t, T) > 0$; therefore the pre-default swap spread is valid in all states.

In the below corollary, Corollary 12, we make use of pre-default quantities in order to establish an alternative pricing result for credit default swaptions. In Section 3.3 we use this result to recover analytical valuation for credit default swaptions for the extended Vasicek model.

Corollary 12. *Let $t_0 \geq 0$ be the valuation date and $t > t_0$ the expiry of a payer credit default swaption which gives the owner of the option the right to enter into a CDS with maturity T at a strike of k . The value of the credit default swaption is:*

$$V_{t_0} \mathbb{I}_{\{\tau > t_0\}} \frac{1}{Q_{t_0} \beta(t_0)} \mathbb{E}^\mathbb{Q} [Q_t \beta(t) ((\tilde{s}(t, T) - k)^+ \tilde{\mathbf{P}}(t, T) | \mathcal{F}_{t_0})], \quad (3.2.33)$$

where $(Q_t)_{t \geq 0}$ is the $\mathbb{F}$ – conditional survival process of the default time τ , $\tilde{s}_t = \tilde{s}(t, T)$ is the pre-default fair swap spread as defined in Equation (3.2.32) and $\tilde{\mathbf{P}}(t, T)$ is the pre-default premium leg as defined in Lemma 11.

Proof. By applying Proposition 5, Equation (3.2.28) and Lemma 11 we have:

$$\begin{aligned}
V_{t_0} &= \mathbb{E}^{\mathbb{Q}}[\mathbb{I}_{\{\tau > t\}} \frac{\beta(t)}{\beta(t_0)} (s(t, T) - k)^+ \bar{\mathbf{P}}_{\mathbf{L}}(t, T) | \mathcal{G}_{t_0}] \\
&= \mathbb{E}^{\mathbb{Q}}[\mathbb{I}_{\{\tau > t\}} \frac{\beta(t)}{\beta(t_0)} (\tilde{s}(t, T) - k)^+ \bar{\mathbf{P}}_{\mathbf{L}}(t, T) | \mathcal{G}_{t_0}] \\
&= \mathbb{I}_{\{\tau > t_0\}} \frac{1}{Q_{t_0} \beta(t_0)} \mathbb{E}^{\mathbb{Q}}[Q_t \beta(t) (\tilde{s}(t, T) - k)^+ \tilde{\mathbf{P}}(t, T) | \mathcal{F}_{t_0}] \\
&= \mathbb{I}_{\{\tau > t_0\}} \frac{1}{Q_{t_0} \beta(t_0)} \mathbb{E}^{\mathbb{Q}}[Q_t \beta(t) (\tilde{s}_t - k)^+ \tilde{\mathbf{P}}(t, T) | \mathcal{F}_{t_0}]. \tag{3.2.34}
\end{aligned}$$

□

In the rest of this thesis we will make use of only pre-default quantities for valuation purposes as we have established that they reduce arbitrage by making the fair swap spread valid in all states.

3.3 The extended Vasicek Model

In this section we consider the extended Vasicek model which is used to create a stochastic intensity model. In general, the specification of any stochastic model and its parameters are a trade off between tractability and economic meaningfulness. For example, an important economic feature of this model is mean reversion (a theory that implies that processes eventually move back to some mean level). There are compelling arguments for mean reversion: the pro-cyclical nature of default arrival (see Figure 1.1 in Chapter 1 on Moody's historical default rates) means that a mean reverting model encourages periods of stress where an obligor is more likely to default. Yet, if the obligor survives, the mean reverting property ensures there is a tendency within the model for the arrival intensity to move back to its long term mean level. On the other hand by specifying the models to follow an affine structure, one ensures that we can exploit the tractability afforded in Appendix D (Proposition 32).

3.3.1 Model setup

The extended Vasicek model is described by the SDE:

$$d\lambda_t = (b(t) - a(t)\lambda_t) dt + \sigma(t) dW_t, \tag{3.3.1}$$

where:

1. $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. a is a positive deterministic function that represents the speed of mean reversion of the model.
2. $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. b is a positive deterministic function and $\frac{b}{a}$ represents the long run mean of the model.
3. $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. σ is a positive deterministic function and is the spot volatility of the model.

As in Heath et al. (1992) the term structure volatility specification fully describes a model in a no arbitrage setting (this is detailed in Appendix C (Section C.2)) . The extended Vasicek term structure volatility, for $0 < t < T$, is (see Baxter and Rennie (1996)):

$$\sigma(t, T) = \sigma(t) e^{-\int_t^T a(u) du}. \quad (3.3.2)$$

Equation (3.3.2) shows that the term structure volatility, $\sigma(t, T)$, is influenced by the speed of mean reversion parameter, a . The larger the speed of mean reversion the smaller the term structure volatility. Intuitively this means that the process reverts back to its long run mean quickly since any perturbation by the spot volatility is cancelled by the speed of mean reversion over the period $T - t$, whenever the speed of mean reversion is sufficiently large.

Let us define:

$$Q(t, T) = \mathbb{E} \left[e^{-\int_t^T \lambda_u du} | \mathcal{F}_t \right], \quad (3.3.3)$$

where $(\lambda_t)_{t \geq 0}$ is the intensity process of an obligor. $Q(t, T)$ is the conditional survival probability of the obligor between times t and T . The notation is similar to the $\mathbb{F}$ – conditional survival process of the default time τ defined in Section 3.1, which was denoted $(Q_t)_{t \geq 0}$. However, the conditional survival probability and the $\mathbb{F}$ – conditional survival process of the default time τ have different meanings and uses in this thesis. Appendix E provides solutions for the model and establishes that the conditional survival probability, $Q(t, T)$, has the form:

$$Q(t, T) = e^{\lambda_t B(t, T) + A(t, T)}, \quad (3.3.4)$$

with:

$$A(t, T) = - \int_t^T \left[e^{K(v)} b(v) \left(\int_v^T e^{-K(t)} dt \right) - \frac{1}{2} e^{2K(v)} \sigma^2(v) \left(\int_v^T e^{-K(y)} dy \right)^2 \right] dv,$$

and:

$$B(t, T) = -e^{K(t)} \int_t^T e^{-K(s)} ds,$$

where $K(t) = \int_0^t a(s) ds$.

3.3.2 Analytical valuation for the extended Vasicek model

In this section we will recover new analytical pricing results for credit default swaptions for an intensity process that is driven by the extended Vasicek model.

The extended Vasicek model is Gaussian (see Appendix E); this means the conditional survival function, $Q(t, T)$, is log-normally distributed. Jamshidian (1989) used this property of the model to recover semi-analytic pricing for interest rate swaptions. We extend his method to recover semi-analytical pricing for credit default swaptions.

Proposition 13. *Let us assume today is $t_0 = 0$, the expiry of a credit default swaption is $t > 0$, the maturity of the CDS to which the credit default swaption refers is T , $t < T_1, \dots, T_N = T$ are the coupon payment dates and the strike of the swaption is $k > 0$. Then the price of a payer credit default swaption is given by:*

$$P(0, t)Q(0, t) \left(K\Phi(x^*) - \sum_{i=1}^N \bar{\omega}_i \Phi(x^* - d_i) \right), \quad (3.3.5)$$

where d_i and $\bar{\omega}_i$ for $i \in 1, \dots, N$ are defined in the proof below, $P(0, t)$ is the risk free discount factor to time t , $Q(0, t)$ is the survival probability of the reference obligor to time t and x^* is chosen so that the function $f : \mathbb{R} \rightarrow \mathbb{R}_+$ given by:

$$f(x) = \sum_{i=1}^N \bar{\omega}_i e^{-\frac{1}{2}d_i^2 + d_i x},$$

is such that $f(x^*) = K$, where:

$$K = (1 - R) - \frac{1}{2}k\Delta_1,$$

with $\Delta_1 = T_1 - t$ the first accrual period.

The condition $K > 0$ is required (note $k \neq K$).

Proof. Recalling the pre-default quantities, as defined in Sub-section 3.2.7 of this chapter, we have that the payoff of a credit default swaption is:

$$\mathbb{I}_{\{\tau > t\}}(\tilde{s}_t - k)^+ \tilde{\mathbf{P}}(t, T) = \mathbb{I}_{\{\tau > t\}}(\tilde{s}_t \tilde{\mathbf{P}}(t, T) - k \tilde{\mathbf{P}}(t, T))^+ = \mathbb{I}_{\{\tau > t\}}(\tilde{\mathbf{D}}(t, T) - k \tilde{\mathbf{P}}(t, T))^+, \quad (3.3.6)$$

with $\tilde{\mathbf{P}}(t, T) > 0$ and $\tilde{s}_t = \frac{\tilde{\mathbf{D}}(t, T)}{\tilde{\mathbf{P}}(t, T)}$. These pre-default quantities can be approximated in the following way:

$$\begin{aligned} \tilde{\mathbf{P}}(t, T) &= Q_t^{-1} \mathbb{E}[\mathbf{P}(t, T) | \mathcal{F}_t] \\ &= Q_t^{-1} \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=1}^N \Delta_i \mathbb{I}_{\{\tau > t_i\}} e^{-\int_t^{t_i} r_s ds} + e^{-\int_t^T r_s ds} \Delta^* \mathbb{I}_{\{t < \tau < T_N\}} | \mathcal{F}_t \right] \\ &\approx \sum_{i=1}^N \Delta(i-1, i) Q(t, t_i) P(t, t_i) + \sum_{i=1}^N \frac{\Delta(i-1, i)}{2} P(t, t_{i-1}) [Q(t, t_{i-1}) - Q(t, t_i)], \end{aligned} \quad (3.3.7)$$

and

$$\begin{aligned}
\tilde{\mathbf{D}}(t, T) &= Q_t^{-1} \mathbb{E}[\mathbf{D}(t, T) | \mathcal{F}_t] \\
&= (1 - R) Q_t^{-1} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \mathbb{I}_{\{t < \tau \leq T\}} | \mathcal{F}_t \right] \\
&\approx (1 - R) \sum_{i=1}^M P(t, t_{i-1}) [Q(t, t_{i-1}) - Q(t, t_i)].
\end{aligned} \tag{3.3.8}$$

We have used $\approx$ to indicate that we are using the market approximation to the pre-default premium and protection legs (as detailed in Sub-section 3.2.4).

In the following, for ease of exposition we take $M = N$.

$$\begin{aligned}
(\tilde{\mathbf{D}}(t, T) - k\tilde{\mathbf{P}}(t, T))^+ &= ((1 - R) \sum_{i=1}^N P(t, t_{i-1}) [Q(t, t_{i-1}) - Q(t, t_i)] - k \sum_{i=1}^N \Delta(i-1, i) Q(t, t_i) P(t, t_i) \\
&\quad - k \sum_{i=1}^N \frac{\Delta(i-1, i)}{2} P(t, t_i) [Q(t, t_{i-1}) - Q(t, t_i)])^+.
\end{aligned} \tag{3.3.9}$$

A little algebra allows us to eventually recover that⁶:

$$(\tilde{s}_t - k)^+ \tilde{\mathbf{P}}(t) = \left(\sum_{i=1}^N \omega_i Q(t, t_i) + K \right)^+,$$

here:

$$\omega_i = \begin{cases} (1 - R)(P(t, t_i) - P(t, t_{i-1})) - \frac{1}{2}k(\Delta_{i+1}P(t, t_i) + \Delta_i P(t, t_{i-1})) & \text{if } i < N - 1, \\ \frac{1}{2}k\Delta_N P(t, t_{N-1}) - (1 - R)P(t, t_{N-1}) & \text{if } i = N, \end{cases}$$

and:

$$K = (1 - R) - \frac{1}{2}k\Delta_1 > 0.$$

Recall that K is different from k . We have that $\omega_i < 0 \forall i$, so we can define $\tilde{\omega}_i = -\omega_i > 0$.

This gives:

$$(\tilde{s}_t - k)^+ \tilde{\mathbf{P}}(t, T) = (K - \sum_{i=1}^N \tilde{\omega}_i Q(t, t_i))^+. \tag{3.3.10}$$

Since $t_0 < t$, we have that $Q(t, t_i) \forall i \in \{1, \dots, N\}$ are stochastic and log-normally distributed hence, Equation (3.3.10) is a put option on their sum. Given that $Q(t, t_i) \sim e^{b_i + d_i X}$, where X is normally distributed, $b_i \in \mathbb{R}$ and $d_i \in \mathbb{R}_+$, we can define $\bar{\omega}_i = \tilde{\omega}_i e^{b_i + \frac{1}{2}d_i^2}$ to get:

$$(K - \sum_{i=1}^N \tilde{\omega}_i Q(t, t_i))^+ = (K - \sum_{i=1}^N \bar{\omega}_i e^{-\frac{1}{2}d_i^2 + d_i X})^+_{a.s.} \tag{3.3.11}$$

⁶In the discretised premium leg, in Equation (3.3.9) we have discounted at time i , it may be better to discount at time $\frac{i+i-1}{2}$; however equivalent results can easily be found in this case.

Now assume x^* is chosen such that:

$$f(x^*) = \sum_{i=1}^N \bar{\omega}_i e^{-\frac{1}{2}d_i^2 + d_i x^*} = K.$$

The structure of Equation (3.3.11) is similar to the work of Jamshidian (1989) for interest rate swaptions. Without loss of generality and for ease of exposition assume that $X \sim \Phi(x)$ is standard normal (see Jaeckel (2004)). Applying the Jamshidian (1989) result, Corollary 12, and defining $(\forall t \geq 0) \tilde{\beta}(t) = Q_t \beta(t) = e^{-\Gamma_t} \beta(t) = e^{-\int_0^t \lambda_s ds} \beta(t)$ (which we call the *risky discounting* term), we have the price of the credit default swaption is:

$$\begin{aligned} V_{t_0} &= \mathbb{I}_{\{\tau > t_0\}} \frac{1}{Q_{t_0} \beta(t_0)} \mathbb{E}^{\mathbb{Q}}[Q_t \beta(t) (\tilde{s}_t - k)^+ \tilde{\mathbf{P}}(t, T) | \mathcal{F}_{t_0}] \\ &= \mathbb{I}_{\{\tau > t_0\}} \frac{e^{\Gamma_{t_0}}}{\beta(t_0)} \mathbb{E}^{\mathbb{Q}}[e^{-\Gamma_t} \beta(t) (\tilde{s}_t - k)^+ \tilde{\mathbf{P}}(t, T) | \mathcal{F}_{t_0}] \\ &= \mathbb{I}_{\{\tau > t_0\}} (\tilde{\beta}(t_0))^{-1} \mathbb{E}^{\mathbb{Q}}[\tilde{\beta}(t) (\tilde{s}_t - k)^+ \tilde{\mathbf{P}}(t, T) | \mathcal{F}_{t_0}] \\ &\approx \mathbb{I}_{\{\tau > t_0\}} (\tilde{\beta}(t_0))^{-1} \mathbb{E}^{\mathbb{Q}}[\tilde{\beta}(t) (K - \sum_{i=1}^N \bar{\omega}_i e^{-\frac{1}{2}c_i^2 + c_i x})^+ | \mathcal{F}_{t_0}]. \end{aligned} \quad (3.3.12)$$

Recall $(\Gamma_t)_{t \geq 0}$ is the $\mathbb{F}$ – *integrated hazard rate process* of the default time τ . In the last equality of the above equation, we have used $\approx$ to indicate that the equation holds up to a discretisation error because we are following market convention in the discretisation of the legs of a CDS. Structurally what we have recovered is the form given in Jamshidian (1989), which establishes the result, Equation (3.3.5). $\square$

We make the following remarks on Proposition 13:

- The value of K will be positive whenever we have $\frac{k\Delta_1}{2} < (1 - R)$. In credit markets, this will almost certainly be true since R , the recovery rate of the obligor, is usually no more than 50% and so $1 - R \geq 50\%$; k will be restricted to never be more than 10000bps = 100% and $\Delta_1 \leq 0.5$. Hence $\frac{k\Delta_1}{2} < (1 - R)$ will almost certainly hold, giving $K > 0$ (see Zhang et al. (2005) who analyse credit swap spread levels and Singh and Spackman (2009) who look at typical recovery values post default).
- Up to the discretisation discussed in the proof of Proposition 13 we have established that analytical results are possible for the valuation of credit default swaptions. Since we can take the discretisation of the protection leg as small as we desire in a natural way, it remains to show that we can do so similarly for the premium leg. The premium leg in Equation (3.3.7) can be more accurately approximated by a minor adjustment in its definition. To see this take $M \in \mathbb{N}$ and consider the premium leg defined in the following way:

$$\sum_{i=1}^{MN} \Delta \left(\left\lfloor \frac{i}{M} \right\rfloor, \left\lfloor \frac{i}{M} \right\rfloor + 1 \right) \frac{(i - \lfloor \frac{i}{N} \rfloor M)}{M} P(t, t_{i-1}) (Q(t, t_{i-1}) - Q(t, t_i)), \quad (3.3.13)$$

where $\lfloor \cdot \rfloor$ means the integer part. Moreover note that now $t_i \in \{t_1, \dots, t_{NM}\}$ and we have a finer discretisation than the one we had in Proposition 13. All the other elements of the above equation are as in the proof of Proposition 13. Equation (3.3.13) establishes that we can take the required level of accuracy for the premium leg by making M larger. In numerical results we will consider a version of the analytical credit default swaption formula, Proposition 13, that follows the market assumed discretisation setup (see Sub-section 3.2.4).

3.3.3 Historical credit spreads, convexity adjustments and the real world measure

So far we have assumed the speed of mean reversion parameter is a positive deterministic function, in the following and the rest of this thesis the speed of mean reversion will be taken to be a positive constant.

In this sub-section we seek to establish a relationship between the credit swap spread of a CDS and the spot intensity process. This is important because we do not observe the historical spot intensity process, instead we observe historical credit swap spreads. Establishing a relationship between the credit swap spread and the spot intensity enables us to use historical credit swap spreads to estimate the speed of mean reversion parameter of the extended Vasicek model.

The assumption stated below is needed to recover the relationship between the spot intensity and credit swap spreads:

Assumption 3.4. *The swap spread of a CDS contract that pays coupons continuously, $s^{cts}(t, T)$, is the same as that of a normal market CDS swap spread, $s^{mkt}(t, T)$, up to a constant multiple, so that:*

$$s^{mkt}(t, T) = \kappa s^{cts}(t, T),$$

where κ is some positive constant.

Assumption 3.4 is not controversial under the assumptions of our thesis. Recall in Section 3.2 (Equation (3.2.15)) we derived the standard market method of determining the fair spread of a CDS which trades with the required standard coupon payment frequency. The question arises as to how different the CDS swap spread will be for a CDS that is paid continuously. Using the standard arguments we have (for a CDS with maturity T that pays coupons continuously):

$$\bar{\mathbf{P}}_L^C(0, T) = \int_0^T Q(0, s) P(0, s) ds,$$

and

$$\bar{\mathbf{D}}_L^C(0, T) = (1 - R) \sum_{i=1}^{\tilde{N}} P(0, \tilde{T}_i) (Q(0, \tilde{T}_{i-1}) - Q(0, \tilde{T}_i)).$$

There is no change to the default leg of a CDS as a result of paying coupons continuously. With regard to the premium leg the impact arises whenever the functional form of $Q(0, s)$ and $P(0, s)$ display significant curvature over a short time period (such as over a 3 month or 6 month horizon- the standard market CDS frequency period). As the quantities $Q(0, s)$ and $P(0, s)$ are monotonically decreasing we are very unlikely to experience significant curvature changes in the quantities over a short time horizon. In this context the assumption that $s^{mkt}(t, T) = \kappa s^{cts}(t, T)$ is not a controversial one.

For the proposition considered below we can assume without loss of generality that $\kappa = 1$ (where κ is the constant in Assumption 3.4), the recovery rate of an obligor is zero and interest rates are zero.

Proposition 14. *Under Assumption 3.4 we get the following results:*

1.

$$s^{mkt}(t, T) = \frac{\int_t^T \mathbb{E}^{\mathbb{Q}^s}[\lambda_s | \mathcal{F}_t] Q(t, s) ds}{\int_t^T Q(t, s) ds}. \quad (3.3.14)$$

2.

$$s^{mkt}(t, T) = \int_t^T \mathbb{E}^{\mathbb{Q}^s}[\lambda_s | \mathcal{F}_t] ds + \int_t^T \mathbb{E}^{\mathbb{Q}^s}[\lambda_s | \mathcal{F}_t] \left(\frac{Q(t, s)}{\int_t^T Q(t, s) ds} - 1 \right) ds, \quad (3.3.15)$$

where:

$$\begin{aligned} Q(t, s) &= \mathbb{E}[e^{-\int_t^s \lambda_u du} | \mathcal{F}_t], \\ E_s &= \frac{\tilde{\beta}(t) e^{-\int_t^s \lambda_u du}}{Q(0, s)}, \\ E_t &= \frac{\tilde{\beta}(t) Q(t, s)}{Q(0, s)} = \frac{\mathbb{E}[e^{-\int_0^s \lambda_u du} | \mathcal{F}_t]}{\mathbb{E}[e^{-\int_0^s \lambda_u du}]} = \exp \left\{ -\frac{1}{2} \int_0^t \sigma^*(u, s)^2 du - \int_0^t \sigma^*(u, s) dW_u \right\}, \end{aligned}$$

$\tilde{\beta}(t) = e^{-\int_0^t \lambda_u du}$ and $\sigma^*(t, s)$ is defined in Appendix C (Theorem 31).

Proof. We have that,

$$\begin{aligned}
s^{mkt}(t, T) &= \kappa s^{cts}(t, T) \\
&= \kappa \frac{\tilde{\mathbf{D}}^{cts}(t, T)}{\tilde{\mathbf{P}}^{cts}(t, T)} \\
&= \frac{\tilde{\mathbf{D}}^{cts}(t, T)}{\tilde{\mathbf{P}}^{cts}(t, T)} \\
&= \frac{\int_t^T \mathbb{E}[\lambda_s \mathbf{e}^{-\int_t^s \lambda_u du} | \mathcal{F}_t] ds}{\int_t^T \mathbb{E}[\mathbf{e}^{-\int_t^s \lambda_u du} | \mathcal{F}_t] ds} \\
&= \frac{\int_t^T \mathbb{E}^{\mathbb{Q}^s}[\lambda_s | \mathcal{F}_t] \mathbb{E}[\mathbf{e}^{-\int_t^s \lambda_u du} | \mathcal{F}_t] ds}{\int_t^T Q(t, s) ds} \\
&= \frac{\int_t^T \mathbb{E}^{\mathbb{Q}^s}[\lambda_s | \mathcal{F}_t] Q(t, s) ds}{\int_t^T Q(t, s) ds} \\
&= \int_t^T \mathbb{E}^{\mathbb{Q}^*}[\lambda_s | \mathcal{F}_t] ds + \int_t^T \mathbb{E}^{\mathbb{Q}^s}[\lambda_s | \mathcal{F}_t] \left(\frac{Q(t, s)}{\int_t^T Q(t, s) ds} - 1 \right) ds. \quad (3.3.16)
\end{aligned}$$

The measure $\mathbb{Q}^s$ has the Radon-Nikodým density process at t given by:

$$\begin{aligned}
E_s &= \frac{\tilde{\beta}(t) \mathbf{e}^{-\int_t^s \lambda_u du}}{Q(0, s)} \\
&= \frac{\tilde{\beta}(t) \mathbf{e}^{-\int_t^s \lambda_u du}}{\mathbb{E}[\mathbf{e}^{-\int_0^s \lambda_u du}]},
\end{aligned}$$

and for $0 < t < s$ we have:

$$\begin{aligned}
E_t &= \mathbb{E} \left[\frac{d\tilde{\mathbb{Q}}^s}{d\mathbb{Q}} | \mathcal{F}_t \right] \\
&= \frac{\tilde{\beta}(t) Q(t, s)}{Q(0, s)} \\
&= \frac{\mathbb{E}[\mathbf{e}^{-\int_0^s \lambda_u du} | \mathcal{F}_t]}{\mathbb{E}[\mathbf{e}^{-\int_0^s \lambda_u du}]} \\
&= \exp \left\{ -\frac{1}{2} \int_0^t \sigma^*(u, s)^2 du - \int_0^t \sigma^*(u, s) dW_u \right\}.
\end{aligned}$$

The process $(E_t)_{t \geq 0}$ is a strictly positive $\mathbb{F}$ – *martingale* under $\mathbb{Q}$ and moreover $\mathbb{E}[E_t] = 1$ ensuring that $\mathbb{Q}^s$ is a probability measure by Theorem 10 (in Section 3.2). $\square$

Equation (3.3.14) indicates that the swap spread to maturity can be seen as an average of the conditional expectation of the intensity under measure $\mathbb{Q}^s$, weighted by the conditional survival probabilities under measure $\mathbb{Q}$. Equation (3.3.15) shows that, up to a correction, the swap spread is the integral of the conditional expected value of the intensity in the interval $[t, T]$ under measure $\mathbb{Q}^s$. Letting $T \rightarrow t$ we see that the correction term reduces and the swap spread scales like $\mathbb{E}^{\mathbb{Q}^s}[\lambda_{t+} | \mathcal{F}_t]$.

3.3.4 Model calibration

In previous sub-sections of this section we have described the general structure of the extended Vasicek model. In this sub-section we collect the elements of the work done so far in order to construct a calibration routine for an intensity process driven by the extended Vasicek model. We make the following refining assumptions on the model parameters:

Assumption 3.5. *We assume the following:*

- a is a positive constant.
- $(b(t))_{t \geq 0}$ is a positive deterministic function.
- $(\sigma(t))_{t \geq 0}$ is piecewise constant with left continuous time nodes⁷.

Such a parameterisation setup enables one to exactly calibrate to all traded credit default swaptions across different expiry points.

In the following, for ease of exposition assume that $(\sigma(t))_{t \geq 0}$ is constant. Since we are considering a parameterisation that is piecewise constant on arbitrary time intervals, e.g. $t_1, \dots, t_n$, this assumption does not lose the generality of our parameterisation assumption.

We now discuss the calibration process. It can be shown (see Carr (2005)) that:

$$b(t) = \frac{\partial \lambda(0, t)}{\partial t} + a\lambda(0, t) + \frac{\sigma^2}{2a}(1 - e^{-2at}), \quad (3.3.17)$$

with $\lambda(0, t) = -\frac{\partial \log Q(0, t)}{\partial t}$. The main problem with Equation (3.3.17) is that we have to differentiate the initial forward hazard rate term structure function, $\lambda(0, t)$. Since $\lambda(0, t)$ is an interpolated curve any differentiation of it amplifies interpolation error (see Hagan and West (2006)). We can restructure Equation (3.3.1) so that:

$$\lambda(X_t, t) = f(t, \lambda(0, t)) + X_t, \quad (3.3.18)$$

with $f(t, \lambda(0, t))$ deterministic and X_t solving:

$$dX_t = -aX_t dt + \sigma dW_t,$$

and $X_0 = 0$. We have decomposed $(\lambda_t)_{t \geq 0}$ to be described in one part by $f(t, \lambda(0, t))$ which is a function of the initial term structure, $\lambda(0, t)$, and a stochastic part which initialises at 0 and perturbs the initial term structure. It is relatively straightforward to show, by using Equations E.2.2 (in Appendix E) and Equation (3.3.17), that:

$$f(t, \lambda(0, t)) = \lambda_0 e^{-at} + \int_0^t \frac{\partial [e^{-a(t-u)} \lambda(0, u)]}{\partial u} + \frac{\sigma^2}{2a} e^{-a(t-u)} (1 - e^{-2au}) du,$$

⁷We choose this parameterisation because it allows calibration to the market where multiple credit default swaptions with different expiries exist.

which simplifies to give:

$$f(t, \lambda(0, t)) = \lambda(0, t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2.$$

Note 2 (The state variable construction). Equation (3.3.18) is a practical way, in line with Appendix D of representing a process which is driven by a state variable. We can therefore represent the intensity in the form of Equation (3.3.18), with $f(t, \lambda(0, t))$ functionally dependent on the initial forward hazard rate term structure of the intensity. In addition as we have specified $\lambda(X_t, t)$ to be an affine function of X_t and have that $(X_t)_{t \geq 0}$ is an affine process (see Appendix D), we can follow the notation in Appendix D and define $R(X_t) = \lambda(X_t, t)$ and from this we can introduce the characteristic function:

$$\psi(u, X_t, t, T) = \mathbb{E}[e^{-\int_t^T R(X_s) ds} e^{uX_t} | \mathcal{G}_t],$$

which by Proposition 32 has solution of the form:

$$\psi^X(u, x, t, T) = e^{A(t, T) + B(t, T)x},$$

with $A(T, T) = 0$ and $B(T, T) = u$ and A and B satisfying the PDE given in Equation (D.2.3) of Appendix D.

Under the conditions of Note 2 we have that for the extended Vasicek model, taking $u = 0$ (so that $B(T, T) = 0$), the conditional survival probability is recovered via the solution of the following PDE:

$$\begin{aligned} \frac{\partial A(t, T)}{\partial t} &= f(t, \lambda(0, t)) - \frac{1}{2}B^2\sigma^2, \\ \frac{\partial B(t, T)}{\partial t} &= 1 + aB, \end{aligned}$$

with:

$$\lambda(0, T) = -\frac{\partial \ln Q(0, T)}{\partial T},$$

so that:

$$\ln \left[\frac{Q(0, T)}{Q(0, t)} \right] = - \int_t^T \lambda(0, u) du. \quad (3.3.19)$$

Hence integrating the differential equation gives:

$$\begin{aligned} B(t, T) &= \frac{1}{a}(e^{-a(T-t)} - 1), \\ A(t, T) &= - \int_t^T (f(s, \lambda(0, s)) - \frac{1}{2}B^2\sigma^2) ds \\ &= \ln \left[\frac{Q(0, T)}{Q(0, t)} \right] + \frac{\sigma^2}{2a^2} \int_t^T ((1 - e^{-as})^2 - (1 - e^{-a(T-s)})^2) ds. \end{aligned} \quad (3.3.20)$$

To conclude, in order to calibrate the model one only needs to determine $\lambda(0, t)$, a and $(\sigma(t))_{t \geq 0}$. We have $\lambda(0, t)$ by bootstrapping (a method of calibrating or constructing a curve

iteratively, see Hagan and West (2006)), see Equation (3.2.11) in Sub-section 3.2.4. Hence the aim of the calibration is to recover $\sigma(t)$ and a . We do this in the following way:

1. Use statistical inference and historical spreads to recover an estimate of the speed of mean reversion parameter, a . The spot intensity version of the extended Vasicek model under these assumptions are:

$$d\lambda_t = a(\tilde{b}(t) - \lambda_t) dt + \sigma(t) dW_t,$$

where $\tilde{b}(t) = \frac{b(t)}{a}$. The discrete version of this spot model is:

$$\lambda_{t_i} - \lambda_{t_{i-1}} = \Delta\lambda_{t_i} = (\nu_0 + \nu_1\lambda_{t_i}) + \epsilon_t, \quad (3.3.21)$$

where $\nu_0 = a\tilde{b}\Delta_t$, $\nu_1 = -a\Delta_t$ and ϵ_t is a Gaussian white noise with zero mean. Hence if we regress $\Delta\lambda_{t_i}$ against λ_{t_i} we can get ordinary least squares estimates for the speed of mean reversion parameter.

2. It remains to find the piecewise constant values of the spot volatility $(\sigma(t))_{t \geq 0}$ in Equation (3.3.2). This is done by iteratively solving for the time nodes of the spot volatility $(\sigma(t))_{t \geq 0}$, with the expiry of the credit default swaptions taken as the time nodes. We can do this efficiently because we have developed an analytic valuation method.

3.3.5 Numerical results

In this sub-section we provide results that demonstrate the calibration accuracy of the calibration procedure proposed in Sub-section 3.3.4. We also illustrate the relationship between the speed of mean reversion parameter and the spot volatility. Further numerical results are provided in Section 3.4 that explains the behaviour of the model when pricing credit products.

- Table 3.3 provides calibration results for the spot volatilities of the extended Vasicek model, where the speed of mean reversion parameter has been estimated to be 5.04. The model is calibrated to the Federative Republic of Brazil CDS given in Table 3.1. We have calibrated to credit default swaptions which we assume trade with an implied log-normal volatility of 60% (this is used to imply quoted prices for credit default swaptions, see Sub-section 3.2.6). These prices are then used to calibrate the spot volatility of the extended Vasicek model. The table is provided to demonstrate that we can calibrate to multiple credit default swaption points.

- Table 3.4 provides a simulation of a 3m expiry and 5y maturity (3m5y) payer swaption with *at-the-money* (ATM) Forward strike (391bps). The analytic value as prescribed by Theorem 13 is 2.28%. We observe that as we increase the iterations there is convergence to the analytic value of 2.28% to within a small error. The table is provided to show, even with the discretisation assumption of Proposition 13, the accuracy of the analytic calibration.
- Figure 3.5 provides the forward hazard rate of the Brazil sovereign CDS curve and a simulated hazard rate path with model parameterisation as calibrated in Table 3.3. Figure 3.6 provides the integrated forward hazard rate and a simulated integrated hazard rate path again with parameterisation as calibrated in Table 3.3.
- Table 3.5 and Figure 3.7 are the calibrated ATM Brazil spot volatilities matrix for the extended Vasicek model. The result is constructed by assuming the market trades the following credit default swaptions: 6m5y, 1y5y, 18m5y and 2y5y with a 60% implied log-normal volatility. We then vary the speed of mean reversion parameter to see how the calibrated spot volatility nodes are affected. The table and figure show that as speed of mean reversion increases the spot volatility must increase in order to ensure that we still calibrate to credit default swaptions. The pattern is consistent across time.

6m	1y	18m	2Y	30m
95.7%	125.6%	138.0%	141.5%	144.3%

Table 3.3: Calibrated ATM Brazil spot volatility curve with an inferred speed of mean reversion of 5.04.

Iterations	Price	Error
127	2.32%	1.20%
511	2.33%	1.90%
2,047	2.26%	-0.70%
8,191	2.28%	-0.11%
32,767	2.28%	-0.05%
131,071	2.28%	0.01%
524,287	2.28%	0.01%

Table 3.4: Simulation of a 3m5y payer swaption valuation with ATM forward strike.

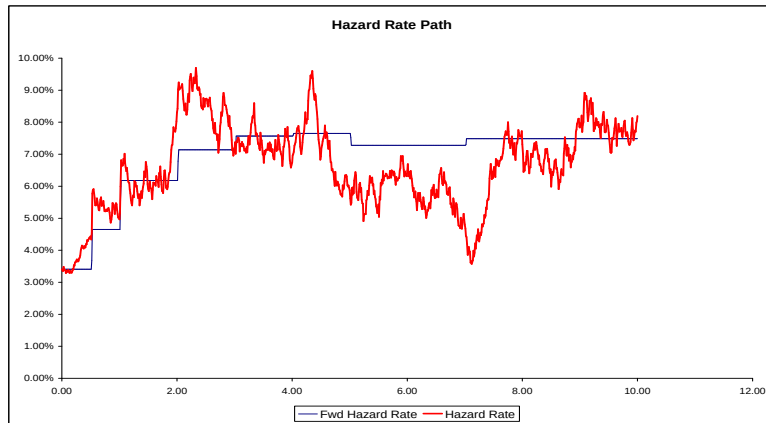


Figure 3.5: Forward hazard rate of Federative Republic of Brazil CDS curve and a simulated hazard rate path.

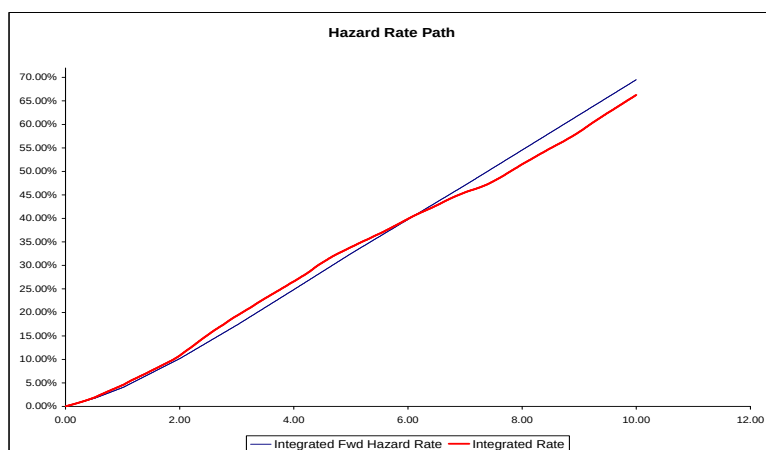


Figure 3.6: Integrated forward hazard rate and a simulated integrated hazard rate.

Speed of mean reversion	6m	1y	18m	2y
0	2.2%	2.3%	2.4%	2.5%
0.25	3.8%	4.1%	4.1%	4.3%
0.5	5.9%	6.5%	6.7%	6.9%
0.75	8.5%	9.7%	10.1%	10.3%
1	11.4%	13.5%	14.2%	14.4%
1.25	14.8%	17.8%	18.9%	19.2%
1.5	18.4%	22.6%	24.2%	24.5%

Table 3.5: ATM spot volatilities matrix for different speed of mean reversion levels

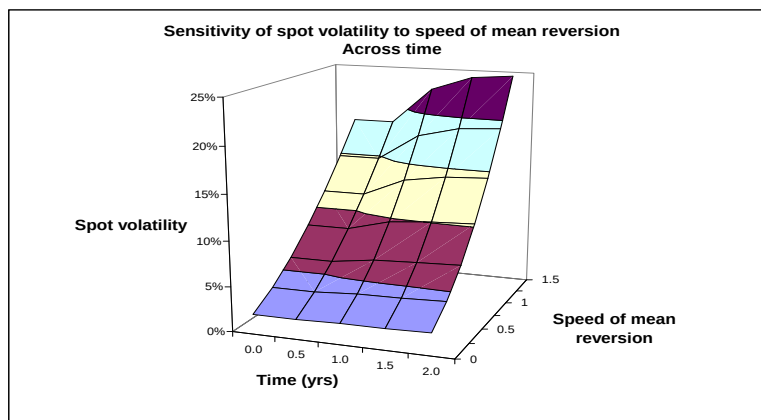


Figure 3.7: Surface of spot volatility against option expiry and speed of mean reversion.

3.3.6 Analysis of the extended Vasicek model

The extended Vasicek model is the model we will use as the stochastic driver for intensities in later chapters of this thesis. However the model is a Gaussian process (see Appendix E) and so can assume negative states. This has led, according to Rebonato (1998), the model meeting with “less interest from academics and practitioners” than it should. Rebonato (1998) clarifies that (for interest rates) in the use of a model what is really important for the purpose of option evaluation is not so much the possibility of the spot interest rate assuming negative values, but the integrated interest rate process assuming negative or zero values.

The same is true in credit modelling: we are concerned with the compensated process (integrated process), $(\Gamma_t)_{t>0}$. $(\Gamma_t)_{t>0}$ is used to evaluate default probabilities and generate default times. So on the one hand we can confirm the analysis of Rebonato (1998) also holds true in credit modelling. However since the intensity can be negative a constructed default time (we demonstrate how this is done in Sub-section 3.4.1 below) may not be unique because $(\Gamma_t)_{t>0}$ is no longer guaranteed to be strictly increasing.

Another issue with the extended Vasicek model is that it is non-stationary. Non-stationarity arises when either $(a(t))_{t\geq 0}$, the speed of mean reversion or (and) $(\sigma(t))_{t\geq 0}$, the spot volatility are time dependent. When this occurs, it means that the future structure of the volatility of the model, see Equation (3.3.2), can vary from that observed today. Given that $(\sigma(t))_{t\geq 0}$ is time dependent in our model setup we have a non-stationary model. Hull and White (1996) argue that this is likely to lead to mis-pricing of products whose value depend on the future volatility structure such as, for example, American style options. The argument is that having a volatility structure in the future which is functionally the same to today’s up to a time increment is desirable for a model to possess. The cost of this is that if we set the parameters to be constant we are unable to fit to the prices of all the swaptions in the market. Philosophically we believe that calibrating to all the traded products holds more value than re-tracing in the future the same volatility structure as today.

A final issue surrounds the use of historical spreads to estimate the speed of mean reversion. It can be argued that this implies a cross pollination between measures: the real world measure, $\mathbb{P}$, and the equivalent martingale measure, $\mathbb{Q}$. By using historical credit spreads to estimate the speed of mean reversion we are operating under the real world measure, yet by using credit default swaptions to calibrate to the spot volatility curve we are switching to the equivalent martingale measure. However this is not a significant problem since the speed of mean reversion is a variance parameter (see Equation (3.3.2)), we can assume

it stays unaffected through changes of equivalent measure (see Duffee and Stanton (2004) who show that the drift of a process under the Vasicek model shares in the physical measure the same speed of mean reversion as in the equivalent martingale measure).

3.4 Leveraged credit linked note without recourse

In this section we will price an LCLN without recourse using the extended Vasicek model. The definition of LCLNs is provided in Chapter 2 (Sub-section 2.4.1).

The analysis of the product in this section is new. Recall that in Chapter 2 (Sub-section 2.4.1) we said that an LCLN without recourse was similar to having a basic credit linked note (BCLN) with an embedded American digital call option on the credit spread. The option is held by the issuer. However with this type of product, once the trigger is breached, the issuer must unwind the structure and realise the gain on a trigger. Strictly speaking, this is not so much an option as he must always unwind on a trigger breach, although it always leads to a positive gain for the issuer. For this reason we will call the protection buyer a *pseudo option holder*.

One of the main findings in this section is that there is a non-monotonic relationship between the level of volatility and the credit spread gamma of the product (we define this precisely below). Moreover we find that the credit spread gamma is not strictly positive in all states of volatility. With other asset classes there is always a clear positive value for the holder of an option.

Our analysis concludes that the main reason for this is that the pseudo option holder does not always want there to be a trigger event as he will forego any possible gain on default.

Typically an issuer will gain significantly more on a default event than he would on a trigger event. Hence at times when volatility is high the probability of a default may increase sufficiently so that the issuers preference is to wait to see if a default will occur rather than taking the immediate gain ensured from a trigger being hit. Hence we see non-monotonic (and non-positive) relationships between credit spread gamma and volatility.

3.4.1 Default time construction

In this sub-section we show how to construct default times given under the intensity modelling set-up. Doing this is important because it provides a clear method of simulating default times when valuing credit products.

We assume that we have an $\mathbb{F}$ – adapted absolutely continuous process, $(\Gamma_t)_{t \geq 0}$, with $\Gamma_0 = 0$, $\Gamma_\infty = +\infty$ and:

$$\Gamma_t = \int_0^t \lambda_s ds,$$

where $(\lambda_t)_{t \geq 0}$ is the intensity process. Further let ζ be a uniform random variable taking values in $[0, 1]$ and assume that $(\Omega, \mathcal{F}, \mathbb{Q})$ is sufficiently rich to support the random variable ζ , and that ζ is independent of $\mathbb{F}$ under $\mathbb{Q}$. Now define $\tau : \Omega \rightarrow \mathbb{R}_+$, under $\mathbb{Q}$, to be such that:

$$\tau = \inf\{t \in \mathbb{R}_+ : e^{-\Gamma_t} \leq \zeta\} = \inf\{t \in \mathbb{R}_+ : \Gamma_t \geq -\ln(\zeta)\}, \quad (3.4.1)$$

with $-\ln(\zeta) = \eta$ having unit exponential distribution under $\mathbb{Q}$.

As in Section 3.1 we set $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t \forall t \geq 0$. Moreover we have that $\{\tau > t\} = \{\zeta < e^{-\Gamma_t}\}$ and Γ_t is $\mathcal{F}_\infty$ – measurable $\forall t \geq 0$. Recall $(F_t)_{t \geq 0}$ is given in Definition 3.1.2 as the $\mathbb{F}$ – conditional cumulative distribution process of the default time of an obligor. Below we establish that the default time construction above leads to a setting similar to that of Section 3.1. To show this note that:

$$\mathbb{Q}(\tau > t | \mathcal{F}_\infty) = \mathbb{Q}(\zeta \leq e^{-\Gamma_t} | \mathcal{F}_\infty) = e^{-\Gamma_t},$$

therefore:

$$1 - F_t = \mathbb{Q}(\tau > t | \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}[\mathbb{Q}(\tau > t | \mathcal{F}_\infty) | \mathcal{F}_t] = e^{-\Gamma_t}. \quad (3.4.2)$$

Hence τ , the time constructed, is a stopping time with respect to $\mathbb{G}$ and satisfies Assumption 3.1 (see Schönbucher (2003a)).

3.4.2 The pricing method

We consider as an obligor the Federative Republic of Brazil and take the credit spreads as given in Table 3.1. Let $t_1, \dots, t_M$ be the discretisation dates, let $D_{t_j} \ j \in 1, \dots, M$ be an indicator that is 1 if default occurs and 0 otherwise at time t_j .

In Sub-section 3.4.1 we showed how to simulate default times. In order to simulate default times in this case we simulate ζ , a uniform random variable, and at each discretisation point we test the condition $\Gamma_{t_j} \geq -\ln(\zeta)$. $(\Gamma_t)_{t \geq 0}$ is the $\mathbb{F}$ – integrated hazard rate process with an intensity process generated by the extended Vasicek model. In this setting (when the intensity process is generated by an extended Vasicek model) $(\Gamma_t)_{t \geq 0}$ is given analytically (see Appendix E).

Note, if the first time $\Gamma_{t_j} \geq -\ln(\zeta)$ holds is at time $t_j > 0$ we know default has occurred some time in between t_{j-1} and $t_j \forall j \in 1, \dots, M$; then $D_{t_j} = 1$ holds.

If no defaults have occurred, we do however test the further condition of $\tilde{s}(t_j, T) > k$ (recalling $\tilde{s}(t_j, T) = \frac{\tilde{\mathbf{D}}(t_j, T)}{\mathbf{P}(t_j, T)}$). This test confirms whether the trigger has been breached or not. Under each of the scenarios, default or trigger breach, the trade ends with no more coupons paid to the investor. Below we summarise the cash flow of the protection leg under different events, assume a nominal of \$1:

Event A : No defaults or trigger breaches on dates $t_1, \dots, t_M$. No gain for the note issuer (recall the note issuer is the protection buyer).

Event B : A default occurs before the trigger is breached. This has a gain for the note issuer of:

$$\min(1, F \times (1 - R)),$$

where F is the leverage factor and R is the recovery rate.

Event C : A trigger breach occurs before any default at time $t_j : j \in 1, \dots, M$. This has a gain for the note issuer of:

$$\min(1, F \times (\tilde{s}(t_j, T) - \tilde{s}(0, T))\bar{\mathbf{P}}_L(t_j, T)).$$

Under events A to C different payments are made for the default leg of an LCLN. By simulation we can value the default leg by consideration of the cash flows that occur under each of these events. In order to arrive at a fair spread for the LCLN we solve for a spread, s^* , which equates the value of the premium leg to the value of the default leg. s^* is the fair spread above Libor assuming the issuing bank's borrowing costs are at Libor (this is the rate the issuer receives for depositing the capital the note investor has paid upfront for the LCLN).

3.4.3 Fair spread and risk analysis

The product specifics for this numerical example are:

Example 3 (LCLN without recourse). *The trade has the following features (based on the Federative Republic of Brazil CDS spreads in Table 3.1):*

1. Maturity (T) = 2y
2. Notional (N) = USD 10,000,000
3. $k = 1000\text{bps}$
4. $\tilde{s}(0, T) = 300\text{bps}$ (from 2y quote in Table 3.1)

5. $F = 5$

6. *Reference Notional* $= 5 \times N = \text{USD } 50,000,000$

3.4.4 Numerical results

In this sub-section we present numerical results for the example LCLN product, Example 3. We present the results in separate sub-headings, each sub-heading introduces a different risk-metric.

Fair spread

The calculation method of the fair spread of an LCLN has been discussed in Sub-section 3.4.2. The table and figure below, Table 3.6 and Figure 3.8, demonstrate how the fair swap spread of an LCLN is affected by changes in both the speed of mean reversion and the spot volatility of the extended Vasicek model. We observe the following:

- The results show that at a speed of mean reversion level of 4.5 the fair swap spread of an LCLN becomes invariant to spot volatility. This seems to be because the high speed of mean reversion level is suppressing the term structure volatility (see Equation (3.3.2)) and therefore not giving much value to the embedded pseudo American digital option in the product. The trigger (which is 1000bps) is not being reached when the speed of mean reversion is at 4.5 relative to the spot volatility (which does not exceed 8% in our results).
- When the speed of mean reversion is 4.5 the fair swap spread is either 4.01% or 4.00%. Since the market 2yr spread is 300bps, the spread level of around 400bps for the LCLN is what one expects. This is because the pseudo optionality (value of hitting the trigger) is close to zero and we recall from discussions in Chapter 2 (Sub-section 2.4.1) a leverage without recourse where there is no optionality value is a BCLN with modified recovery (in our example the modified recovery is zero). This leads to a fair spread of $\frac{300bps}{1-R} = \frac{300bps}{0.75} = 400bps$ (where R is the recovery assumption for the obligor and is 25%), which conforms to our numerical results.

Spot volatility (σ)	Speed of mean reversion			
	0	1.5	3	4.5
1%	4.02%	4.01%	4.01%	4.01%
2%	4.50%	4.01%	4.01%	4.01%
3%	5.26%	4.01%	4.01%	4.01%
4%	6.21%	4.11%	4.01%	4.01%
5%	7.22%	4.28%	4.00%	4.01%
6%	8.15%	4.53%	4.03%	4.01%
7%	9.32%	4.80%	4.08%	4.00%
8%	10.72%	5.08%	4.17%	4.01%

Table 3.6: LCLN spread premium above the bench mark interest rate, Libor, as a function of varying speed of mean reversion and extended Vasicek spot volatilities.

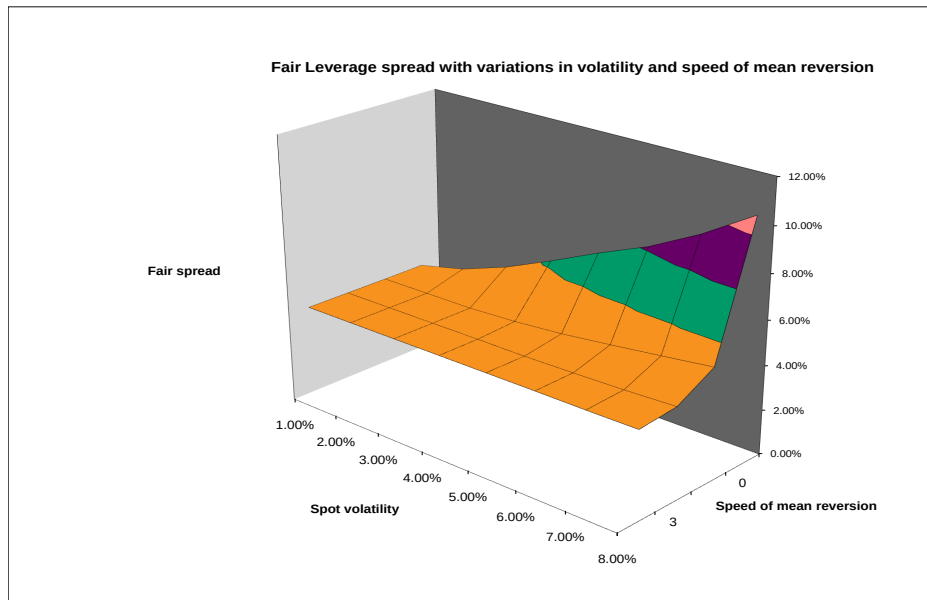


Figure 3.8: LCLN spread premium above the bench mark interest rate, Libor, as a function of varying speed of mean reversion and extended Vasicek spot volatilities

Credit spread delta (CSD)

The risk CSD is used to refer to the change in value of a product due to a parallel change in the credit swap spread curve. Let V_t denote the MTM of the product at time $t \geq 0$. Then the CSD of the product is the change in the MTM for an infinitesimal parallel (uniform) change of the points of the credit curve. Below we examine a specific form of the CSD , the credit spread of 1bps ($CS01$), which is the change in the value of a product for a 1bps parallel change in the credit swap curve.

We begin the analysis by highlighting that the $CS01$ of an LCLN is influenced by the coupon paid to the LCLN investor. Hence, if we assume different coupons are paid to the investor we get different values for the $CS01$. The results are split into two sets to demonstrate this:

- In the first set of numerical results- Tables 3.7, 3.8 and Figures 3.9, 3.10- we hold fixed a spread (not the fair LCLN spread) and take this plus Libor as the coupon paid to the investor. We then consider how the $CS01$ of the product (with this coupon) varies when we change the spot volatility and the speed of mean reversion parameter of the model. The first table and the first figure (Table 3.7 and Figure 3.9) provide the $CS01$ numbers; the second table and second figure (Table 3.8 and Figure 3.10) provide the equivalent amount of CDS hedges one would need to do in order to immunise the $CS01$ produced by the product.
- In the second set of numerical results- Tables 3.9, 3.10 and Figures 3.11, 3.12- we first solve for the fair spread of the LCLN product. Then using this spread plus Libor as the coupon paid to the investor we consider how $CS01$ varies when we vary the spot volatility and the speed of mean reversion parameter of the model. The first table and the first figure (Table 3.9 and Figure 3.11) provide the $CS01$ numbers; the second table and the second figure (Table 3.10 and Figure 3.12) provide the equivalent amount of CDS hedges one would need to do in order to immunise the $CS01$ produced by the product.

We observe the following from the results:

- There is an increasing monotonic relationship between the $CS01$ of the product and the term structure volatility i.e. as the speed of mean reversion reduces or the spot volatility increases the $CS01$ increases.
- The effects of the parameters on the level of the $CS01$ are significant:

1. *Case of high volatility.* In Table 3.8 we see that with a speed of mean reversion of 0 and a spot volatility of 8% (scenario in which the term structure volatility is high) the amount of CDS protection (referencing the same obligor and having the same maturity as the LCLN) one has to purchase as an investor of an LCLN in order to immunise the LCLN $CS01$ is \$32.27mil.
2. *Case of low volatility.* On the other hand, with a speed of mean reversion of 4.5 and a volatility of 1% (scenario in which the term structure volatility is low), the equivalent maturity hedge is only \$13.91mil.

The degree of difference in $CS01$ (similarly equivalent hedges) to changes in the spot volatility or speed of mean reversion shows the strong dependence of the product on spread dynamics.

- Whenever the spot volatility is zero and there is no probability of the trigger being hit, an LCLN can be considered a modified recovery BCLN. The modified recovery is zero if we have an assumed market recovery of 25% for the Federative Republic of Brazil and a leverage factor of at least $\frac{4}{3}$ -see discussions in Chapter 2 (Sub-section 2.4.1). This is the case in our example, Example 3. Hence, to hedge an LCLN in this scenario with an equivalent maturity CDS in order to immunise the $CS01$ of the LCLN we need only do $\frac{1}{1-R}$ amount of equivalent maturity CDS in the case of zero volatility. In the current example this amount is \$13.33mil which conforms with the numerical results in the low volatility case (when volatility is 1% and the speed of mean reversion is 4.5) where the equivalent hedge amounts were found to be either \$13.91mil or \$13.83mil in respectively Tables 3.8 and 3.10.

Spot volatility (σ)	Speed of mean reversion			
	0	1.5	3	4.5
1%	2,862	2,597	2,590	2,592
2%	3,933	2,598	2,597	2,594
3%	4,032	2,764	2,596	2,592
4%	4,512	3,422	2,597	2,597
5%	4,794	3,495	2,599	2,582
6%	5,892	3,914	2,896	2,599
7%	4,971	4,056	2,896	2,588
8%	6,013	4,410	3,060	2,600

Table 3.7: LCLN $CS01$ with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities.

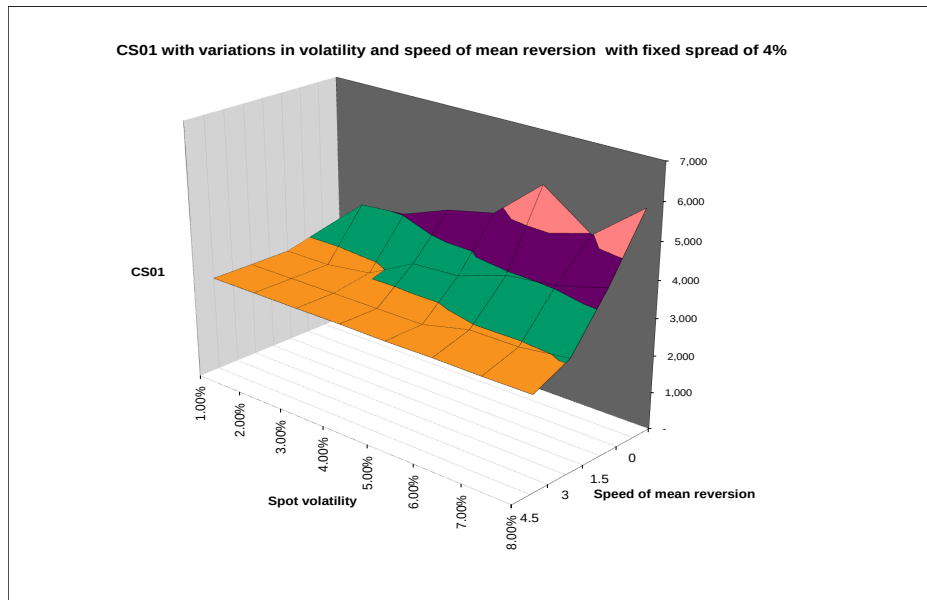


Figure 3.9: LCLN $CS01$ with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities

Spot volatility (σ)	Speed of mean reversion			
	0	1.5	3	4.5
1%	15.36m	13.94m	13.90m	13.91
2%	21.11m	13.94m	13.94m	13.92m
3%	21.64m	14.83m	13.93m	13.91m
4%	24.21m	19.36m	13.94m	13.94m
5%	25.73m	18.76m	13.95m	13.86m
6%	31.62m	21.01m	15.54m	13.95m
7%	26.68m	21.77m	15.54m	13.89m
8%	32.27m	23.67m	16.43m	13.95m

Table 3.8: LCLN hedge position of equivalent maturity CDS with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities.

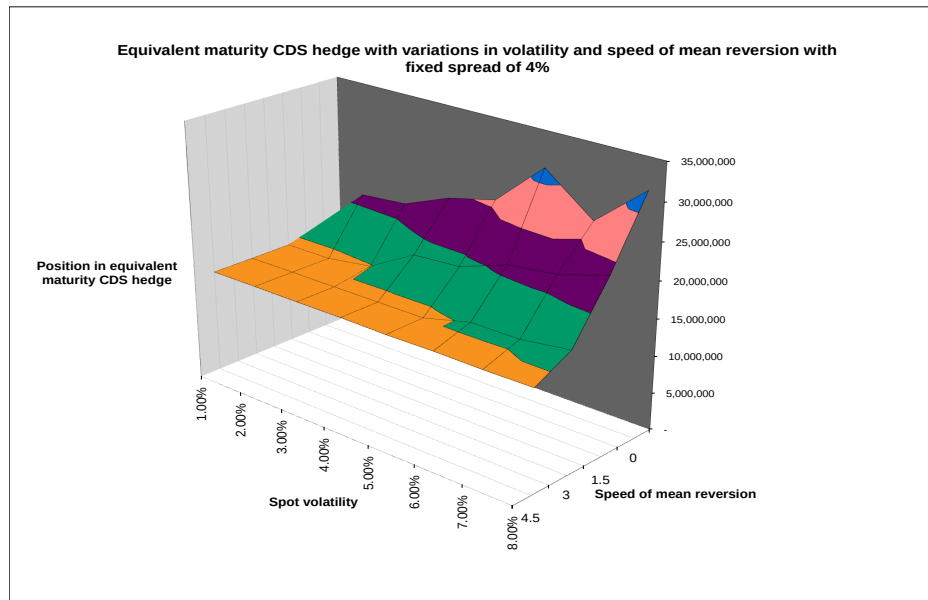


Figure 3.10: LCLN hedge position of equivalent maturity CDS with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities

Spot volatility (σ)	Speed of mean reversion			
	0	1.5	3	4.5
1%	2,840	2,581	2,572	2,577
2%	4,473	2,858	2,945	2,932
3%	3,948	2,722	2,582	2,574
4%	5,192	4,088	2,913	2,883
5%	4,883	3,351	2,581	2,562
6%	6,959	4,596	3,362	2,835
7%	5,360	3,870	2,835	2,575
8%	6,926	5,106	3,490	2,853

Table 3.9: LCLN $CS01$ with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.

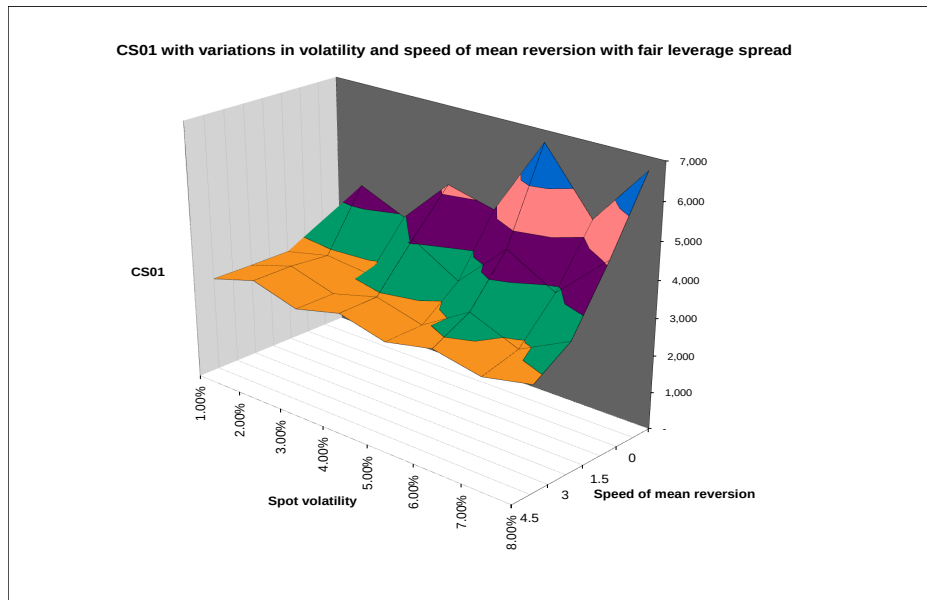


Figure 3.11: LCLN $CS01$ with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.

Spot volatility (σ)	Speed of mean reversion			
	0	1.5	3	4.5
1%	15.24m	13.85m	13.80m	13.83m
2%	24.01m	15.34m	15.81m	15.74m
3%	21.19m	14.61m	13.86m	13.81m
4%	27.87m	21.94m	15.63m	15.47m
5%	26.21m	17.98m	13.85m	13.75m
6%	37.35m	24.67m	18.04m	15.22m
7%	28.77m	20.77m	15.22m	13.82m
8%	37.17m	27.40m	18.73m	15.31m

Table 3.10: LCLN hedge position of equivalent maturity CDS with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.

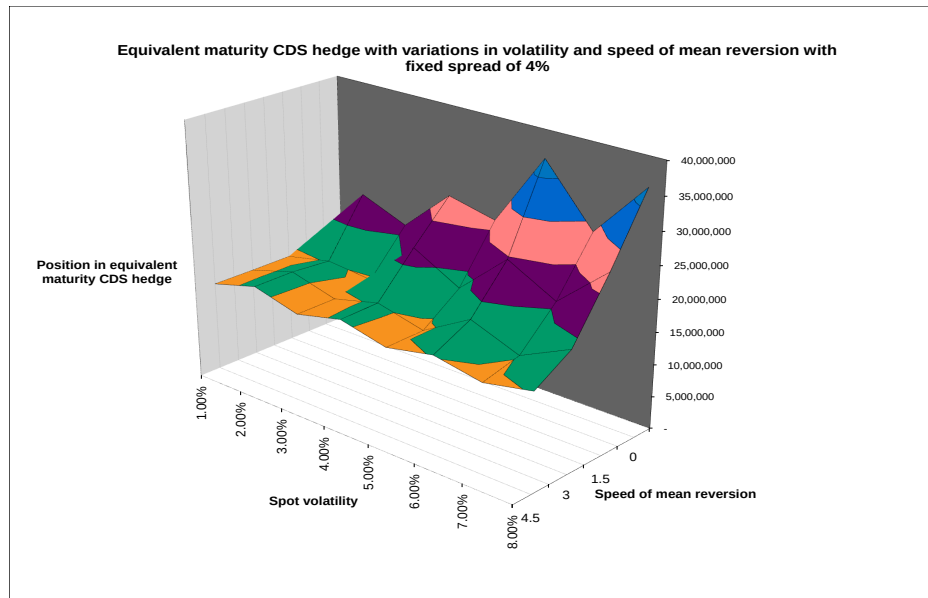


Figure 3.12: LCLN hedge position of equivalent maturity CDS with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.

Credit spread gamma

We now consider the convexity of the LCLN value as a function of the credit curve. Let $CS01(t)$ be the $CS01$ of the product at time $t \geq 0$. Define the *Gamma* of the product at time t to be the change in value of $CS01(t)$ for an infinitesimal positive parallel shift of the credit curve. In our results we observe the change in $CS01$ for a 1bps parallel change in the credit swap curve.

We split the analysis in two:

- In the first set of numerical results- Table 3.11 and Figure 3.13- we hold fixed a spread and take this plus Libor as the coupon paid to the investor. We then consider how the *Gamma* of the product (with this coupon) varies when we vary the spot volatility and the speed of mean reversion parameter of the model.
- In the next set of numerical results- Table 3.12 and Figure 3.14- we first solve for the fair spread of the LCLN product. Then using this spread plus Libor as the coupon paid to the investor we consider how *Gamma* varies when we vary the spot volatility and the speed of mean reversion parameter of the model.

Broadly, we observe from the numerical results that there is no monotonicity of *Gamma* as a function of the term structure volatility and that *Gamma* can be negative in certain states of volatility. This observation is true for the results in Tables 3.11 and 3.12.

We believe the reason for this can be found in an interpretation of what *Gamma* means in this case and with a clarification of the LCLN product:

- *Gamma* is a measure of how fast the value of an LCLN product changes for a parallel change in the credit swap spread curve. Typically in the financial literature it is understood that *Gamma* is always positive for the holder of an option. Moreover deep out of the money options (as we have in this case) have *Gamma* increasing as a function of volatility.
- An LCLN protection buyer has two ways of experiencing a gain:
 1. A default leg payment.
 2. A trigger leg payment.

If we were only concerned with the trigger leg, then we would expect to see a clear positive monotonic relationship between *Gamma* and the term structure volatility, as given in Equation (3.3.2) above. This is because the term structure volatility would increase the chance

of the trigger being breached and the note issuer (who is the protection buyer) making a gain of some value. However the value of an LCLN is also influenced at the same time by how likely it is that one will have a default and make a gain (which will have a different value to the gain on a trigger event). At any one point in time it is difficult to judge which event is more likely to occur (trigger or default) and which one when it occurs holds more value to the protection buyer.

So we believe that the somewhat paradoxical non-monotonic results are reasonable:

Although with increasing volatility it is more likely that we will breach a trigger and make a gain on the trigger leg, as protection buyer, there may be a point when we do not want the trigger to occur. We may be at a point where the default probability has increased significantly relative to the increase in the likelihood of a trigger event occurring. Hence breaching the trigger to get a gain on the trigger leg, when default probabilities are high, means foregoing any gain that one may get if an actual default was to occur. In our view this explains why *Gamma* seems to switch from negative to positive with changes in speed of mean reversion or spot volatility.

This only partially explains the results because we observe that *Gamma*, in the case of zero speed of mean reversion and when the spot volatility is between 6% and 8%, in both Tables 3.11 and 3.12 assumes high positive values. Again our view is that the rationale for this behaviour can be found from consideration of:

- Which leg provides a larger gain: a default or a trigger.
- And at any given time, which of the two events is more likely to occur.

Therefore as an explanation of the high positive values when the spot volatility is between 6% and 8% we suggest the following argument:

- Once the term structure volatility is sufficiently high, the chance of breaching a trigger becomes much more likely than a default occurring.
- The amount gained on a trigger event may be less than on a default yet the overwhelming opportunity to realise a gain changes the pseudo option buyers view.
- Hence with sufficiently high term structure volatility the value of an LCLN has a clear positive reaction to increasing likelihoods of hitting the trigger and therefore positive *Gamma*.

Spot volatility (σ)	Speed of mean reversion			
	0	1.5	3	4.5
1%	7	0	-1	2
2%	-28	9	0	1
3%	-107	-51	4	-7
4%	-202	16	-19	8
5%	-624	-31	80	-9
6%	571	50	37	-39
7%	334	-124	-75	-24
8%	877	-126	-84	-54

Table 3.11: LCLN *Gamma* with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities.

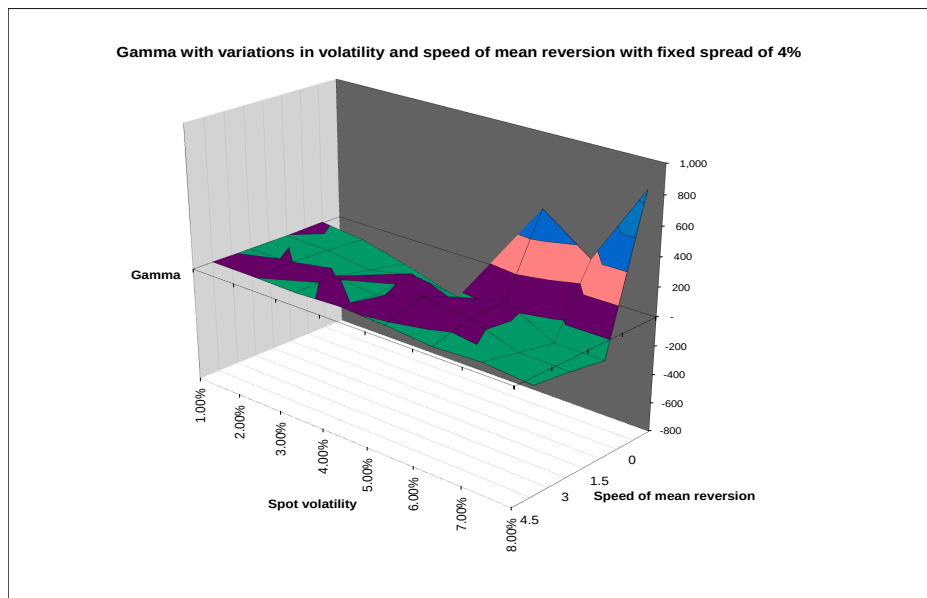


Figure 3.13: LCLN *Gamma* with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities

Spot volatility (σ)	Speed of mean reversion			
	0	1.5	3	4.5
1%	48	3	-2	0
2%	-79	-37	49	-26
3%	-336	-71	2	-6
4%	-235	132	-18	-72
5%	-739	6	82	-9
6%	676	-66	-11	-33
7%	273	-128	-80	-25
8%	1,016	-199	-134	20

Table 3.12: LCLN *Gamma* with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.

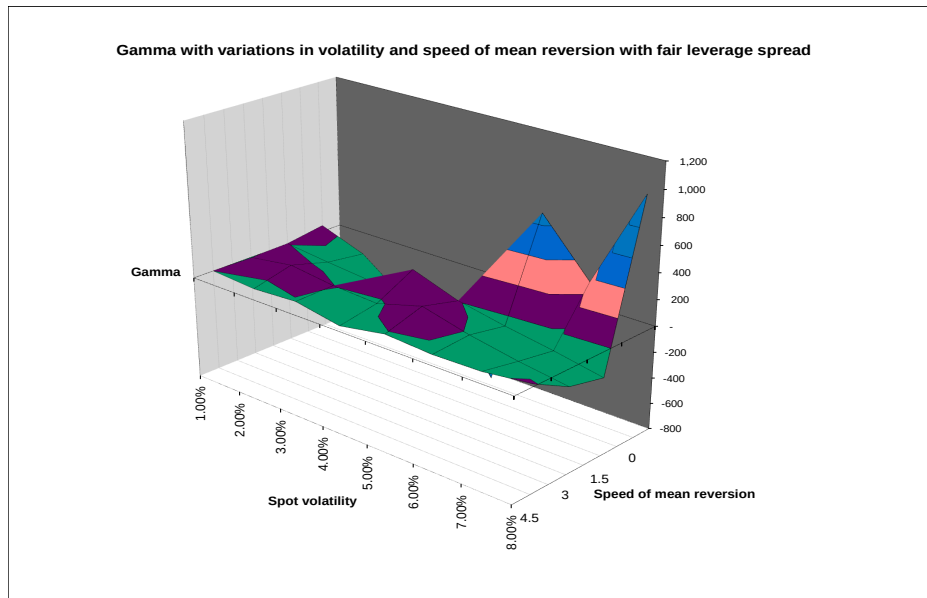


Figure 3.14: LCLN *Gamma* with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.

Credit spread vega

The *Vega* of the product is the sensitivity of the LCLN product for a small parallel shift in the spot volatility. In our case we consider a 1% parallel change in the spot volatility curve of the model.

Again we split the analysis in two:

- In the first set of numerical results- Table 3.13 and Figure 3.15- we hold fixed a spread and take this plus Libor as the coupon paid to the investor. We then consider how the *Vega* of the product (with this coupon) varies when we vary the spot volatility and the speed of mean reversion parameter of the model.
- In the next set of numerical results- Table 3.14 and Figure 3.16- we first solve for the fair spread of the LCLN product. Then using this spread plus Libor as the coupon paid to the investor we consider how *Vega* varies when we vary the spot volatility and the speed of mean reversion parameter of the model.

Similar analysis can be done for *Vega* as we have done for *Gamma*. In the case of zero speed of mean reversion we observe a relatively strong positive relationship in both Tables 3.13 and 3.14. When the term structure volatility is high and strong, the chance of a trigger being breached and a gain occurring on the trigger leg seems to dominate the value of the LCLN and creates in both tables of results positive *Vega* as a function of volatility, however when the term structure volatility is low, we do not observe a clear monotonic relationship because the value of the default leg also has a strong influence.

Spot volatility (σ)	Speed of mean reversion			
	0	1.5	3	4.5
1%	432	4	-1	-7
2%	1,421	6	-13	8
3%	1,419	104	8	1
4%	1,561	454	14	-7
5%	1,644	394	10	9
6%	2,217	494	72	-6
7%	2,642	583	130	-13
8%	2,788	532	85	3

Table 3.13: LCLN *Vega* with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities.

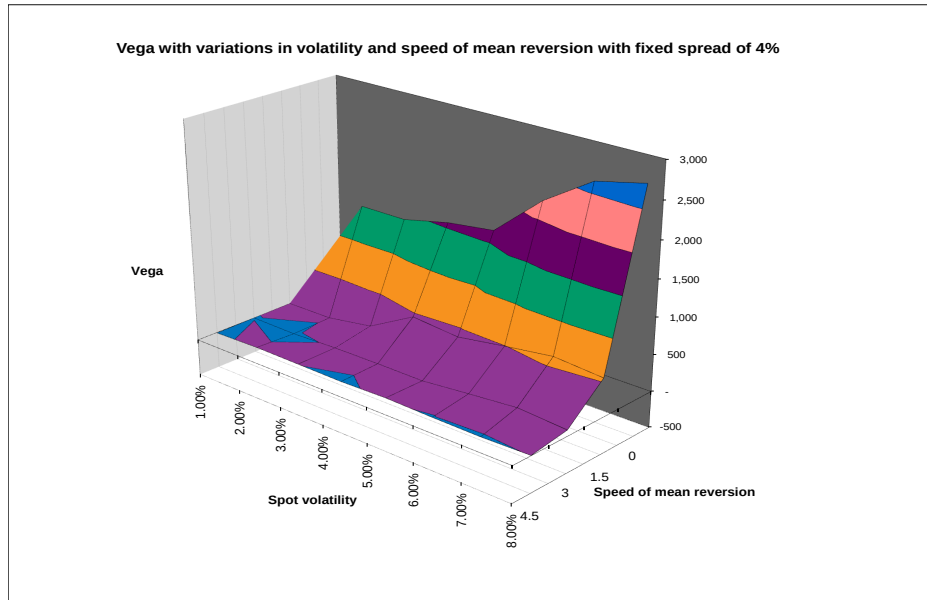


Figure 3.15: LCLN *Vega* with fixed coupon of 4% above Libor and varying speed of mean reversion and extended Vasicek spot volatilities

Spot volatility (σ)	Speed of mean reversion			
	0	1.5	3	4.5
1%	535	4	-1	-2
2%	2,239	6	35	-16
3%	1,757	165	3	1
4%	2,673	848	35	17
5%	2,089	463	5	4
6%	3,486	859	72	19
7%	2,409	672	115	-4
8%	2,542	819	256	3

Table 3.14: LCLN *Vega* with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.

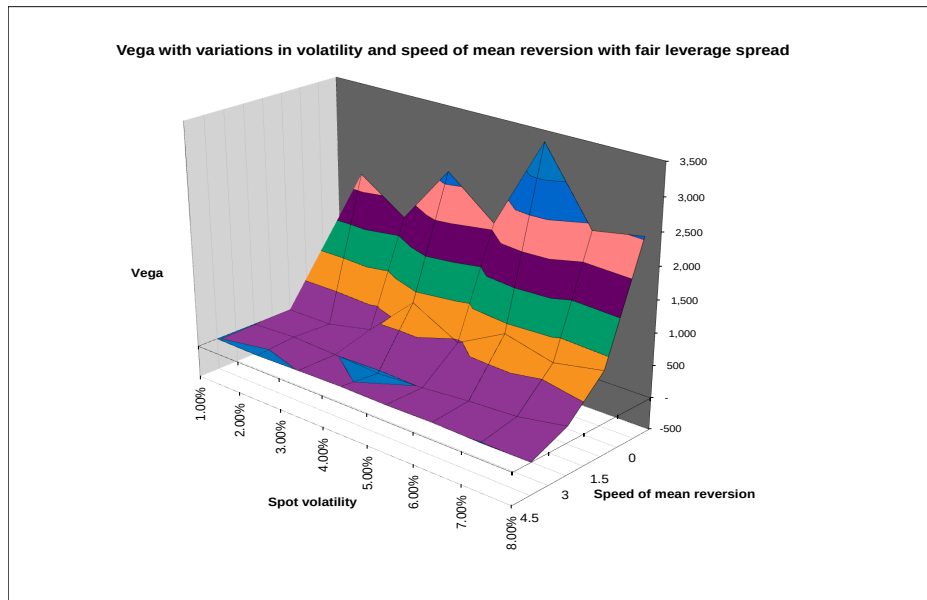


Figure 3.16: LCLN *Vega* with fair coupon and varying speed of mean reversion and extended Vasicek spot volatilities.

3.5 Alternative models for the intensity process

In this section we examine the CIR model and pure jump model as alternatives to the extended Vasicek model.

3.5.1 Cox Ingersoll Ross model

The general CIR model is of the form:

$$d\lambda_t = (b(t) - a(t)\lambda_t) dt + \sigma(t)\sqrt{\lambda_t} dW_t. \quad (3.5.1)$$

The SDE produces a solution that is non-negative almost surely (see Shreve (1998)). The Feller condition, $2ab > \sigma^2$, ensures the process, $(\lambda_t)_{t \geq 0}$, is positive almost surely and has a stationary distribution.

Using the form in Note 2 in Sub-section 3.3.4 introduce the state variable:

$$\lambda(X_t, t) = f(t, \lambda(0, t)) + X_t,$$

with $(X_t)_{t \geq 0}$ solving the SDE:

$$dX_t = (b - aX_t) dt + \sigma\sqrt{X_t}dW_t,$$

and $f(t, \lambda(0, t))$ to be determined but being a function of the initial hazard rate and the parameters of the state variable, $(X_t)_{t \geq 0}$. We again introduce the function, $\psi(u, x, t, T)$, and look for a solution of the form:

$$\psi^X(u, x, t, T) = e^{A(t, T) + B(t, T)x},$$

taking $u = 0$ we recover the conditional survival function, $Q(t, T)$, by solving the following PDE:

$$\begin{aligned} \frac{\partial B(t, T)}{\partial t} &= 1 + aB - \frac{1}{2}B^2\sigma^2, \\ \frac{\partial A(t, T)}{\partial t} &= f(t, \lambda(0, t)) - bB. \end{aligned}$$

$f(t, \lambda(0, t))$ is chosen so that the initial hazard rate, $\lambda(0, T)$, is matched exactly for all $T > 0$; this gives (following the form of the solution in Shreve (1998)):

$$\begin{aligned} B(t, T) &= \frac{-\sinh(\gamma(T-t))}{\gamma \cosh(\gamma(T-t)) + \frac{1}{2}a \sinh(\gamma(T-t))}, \\ A(t, T) &= -\int_t^T f(s, \lambda(0, s)) ds + b \int_t^T B(s, T) ds \\ &= -\int_t^T f(s, \lambda(0, s)) ds + \frac{2ab}{\sigma^2} \ln \left(\frac{\gamma e^{\frac{a(T-t)}{2}}}{\gamma \cosh(\gamma(T-t)) + \frac{1}{2}a \sinh(\gamma(T-t))} \right), \end{aligned} \quad (3.5.2)$$

where $\gamma = \frac{1}{2}\sqrt{a^2 + 2\sigma^2}$.

Many authors regard the Feller condition as necessary for modelling intensity processes (and the spot interest rate process). However, in the advent of the era of *quantitative easing*⁸ and base rates at close to zero it is the author's view that as well as being restrictive these constraints are not necessary for interest rate modelling and credit modelling. In particular, the mean reverting nature of the model pulls the process to its long run mean after it has been perturbed to zero in either periods of deflation, for interest rates, or strong credit out performance, for credits. To clarify this point, we note that it was common in the literature to assume that major economies such as the US or the UK were near risk free entities so that their default arrival intensities were close to zero. When Northern Rock, the UK bank, was close to bankruptcy and then taken over by the UK government it is reasonable to assume that Northern Rock's default arrival intensity became small at least in the short term. This is because the government ownership implied it would not default within any short horizon⁹. Hence zero arrival intensities over short periods is not highly controversial.

3.5.2 Pure jump model

Under this model the intensity follows a jump process with drift. The form of the drift process is again mean reverting and the state variable is a pure jump process with SDE given by:

$$dX_t = -aX_t dt + Sd\tilde{J}_t, \quad (3.5.3)$$

$(\tilde{J}_t)_{t \geq 0}$ is a Poisson process with constant arrival intensity Γ ; S is the jump size distribution which is independent of $(\tilde{J}_t)_{t \geq 0}$. We will consider normally distributed jumps (which means the intensity can take negative values) so that:

$$S \sim N(\mu_S, \sigma_S^2). \quad (3.5.4)$$

The form of the PDE (using Proposition 32 in Appendix D) now becomes:

$$\begin{aligned} \frac{\partial B(t, T)}{\partial t} &= 1 + aB \\ \frac{\partial A(t, T)}{\partial t} &= f(t, \lambda(0, t)) - \Gamma(\theta(\beta) - 1). \end{aligned}$$

The moment generating function, $\theta(\beta) = \mathbb{E}[e^{\beta S}]$, is the transform of the jump density function, S . In our case where S is normal we get that:

$$\theta(\beta) = \int_{-\infty}^{\infty} e^{\beta z} f(z) dz = e^{\beta \mu_S + \frac{1}{2} \beta^2 \sigma_S^2}, \quad (3.5.5)$$

⁸A method of flooding money into the economy to stave off deflation.

⁹The credit crisis has seen a re-assessment by the market of sovereign risk, see ECB (2009).

where $f(z)$ is the density of S . From this we have:

$$\begin{aligned} B(t, T) &= \frac{1}{a}(e^{-a(T-t)}) \\ A(t, T) &= -\int_t^T f(s, \lambda(0, s)) ds + \Gamma \int_t^T (\theta(B) - 1) ds. \end{aligned} \quad (3.5.6)$$

In order to get analytic results for the pure jump model we assume $a = 0$. In this case:

$$\begin{aligned} B(t, T) &= -(T - t) \\ A(t, T) &= -\int_t^T f(s, \lambda(0, s)) ds + \Gamma \int_t^T (\theta(B) - 1) ds. \end{aligned} \quad (3.5.7)$$

Again in order to match exactly the initial hazard rate term structure we let:

$$f(t, \lambda(0, t)) = \lambda(0, t) + \Gamma(\theta(-t) - 1),$$

we then have:

$$A(t, T) = \ln \left[\frac{Q(0, T)}{Q(0, t)} \right] + \Gamma \int_t^T (\theta(s - T) - \theta(-s)) ds. \quad (3.5.8)$$

The integral over jump transforms are (see Duffie et al. (2000), Jaeckel and Mainwaring (2002) and Feng (2008)):

$$\begin{aligned} \int_t^T (\theta(u - T) - \theta(-u)) du &= \frac{\lambda_h}{\sigma_S} e^{-\frac{\mu_S^2}{2\sigma_S^2}} \sqrt{\frac{\pi}{2}} \left(E_i \left(\frac{\mu_S}{\sqrt{2}\sigma_S} \right) - E_i \left(\frac{\mu_S - \sigma_S^2(T-t)}{\sqrt{2}\sigma_S} \right) \right. \\ &\quad \left. + E_i \left(\frac{\mu_S - \sigma_S^2 T}{\sqrt{2}\sigma_S} \right) - E_i \left(\frac{\mu_S - \sigma_S^2 t}{\sqrt{2}\sigma_S} \right) \right), \end{aligned} \quad (3.5.9)$$

where the standard error function is defined to be:

$$E(y) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx, \quad (3.5.10)$$

and $E_i(\cdot)$ for imaginary argument, y , is related to the usual error function by the equation:

$$E_i(y) = E(iy)/i. \quad (3.5.11)$$

3.5.3 Numerical results

In this section we provide some numerical results relating to the CIR model and the pure jump model:

- Figures 3.17 and 3.18 relate to the CIR model. The first figure shows the forward hazard rate of the Federative Republic of Brazil in comparison with a simulated hazard rate path using the CIR model. The second figure gives the integrated versions. The simulations are done with a spot volatility of 35% and mean reversion speed of 1.0 for the CIR model.

- Figures 3.19 and 3.20 relate to the pure jump model. The first figure shows the forward hazard rate of the Federative Republic of Brazil in comparison with a simulated hazard rate path using the pure jump model. The second figure gives the integrated versions. The simulations are done with $\mu_S = 0$, $\sigma_S = 4\%$, constant jump intensity for Γ equal to 4% and mean reversion speed of 0.
- Comparing the two models we see that the simulated hazard rate path for the CIR model is not negative. Moreover the integrated hazard rate path is monotonically increasing with time. In between years 3 and 4 the hazard path produced by the pure jump model (which can assume negative states) is negative. This is a weakness in the model.
- Tables 3.15 and 3.16 are provided to compare the performance of the three models: CIR, pure jump and extended Vasicek in the valuation of LCLNS:
 - Table 3.15 summarises calibrated model parameters for the CIR, pure jump and extended Vasicek model. The models are calibrated to an arbitrary flat CDS curve set at 100bps and a 2y3y payer swaption. The payer swaption has a log-normal volatility of 60%.
 - Table 3.16 provides the valuation results and risk metrics (fair spread, $CS01$ and Γ) of the following LCLN product for the three models:
 1. Maturity (T) = 2y
 2. Notional (N) = USD 10,000,000
 3. $k = 200\text{bps}$
 4. $\tilde{s}(0, T) = 100\text{bps}$ (arbitrary flat CDS curve)
 5. $F = 5$
 6. Reference Notional = $2 \times N = \text{USD } 20,000,000$
 - We notice that the fair spreads are somewhat similar ranging from 187bps to 200bps. The higher the fair spread the more value is attributed to the product by the model. Hence the CIR model attributes the most value to the LCLN product.
 - Although the fair spreads are similar, there is a large divergence in the $CS01$ and the Γ of the three models.
 - * The CIR model attributes a significant risk value ($CS01$ of 6535) to the product, whereas the pure jump model seems to attribute a low risk value ($CS01$ of 2793) to the product.

- * A probable explanation for this is to do with the fact that the CIR model assumes states that are non-negative, whereas the pure jump model in its current form (and the extended Vasicek model) can assume negative states.
 - * Recall that an LCLN is a BCLN with an embedded option. As the CIR has non-negative default probabilities over all periods there is more value (than the other models) attributed to the default leg of the Reference CDS making it more likely the trigger will be breached.
- We conclude that all three models produce consistent fair spreads; yet they also produce significant differences in the key risk metrics (*CS01* and *Gamma*). Hence if we are concerned about risk, the choice of model is very important.

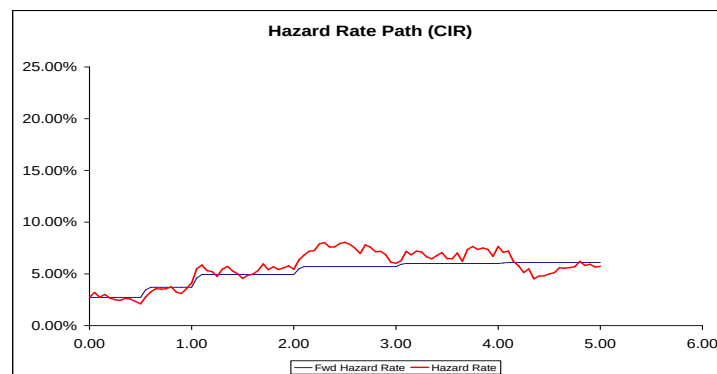


Figure 3.17: The stripped forward hazard rate of the Federative Republic of Brazil alongside a CIR simulated hazard rate path.

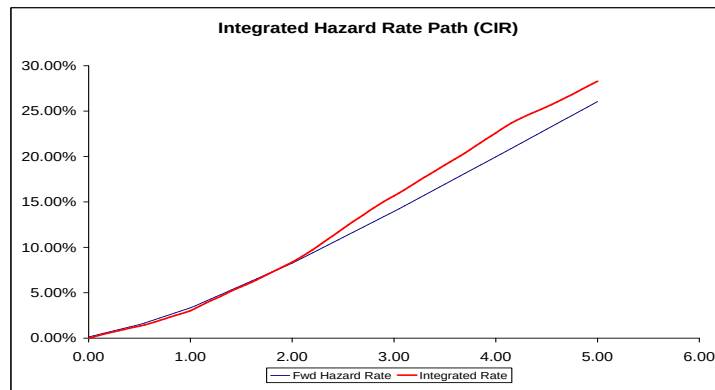


Figure 3.18: The integrated forward hazard rate and a simulated integrated hazard rate path for the CIR model.

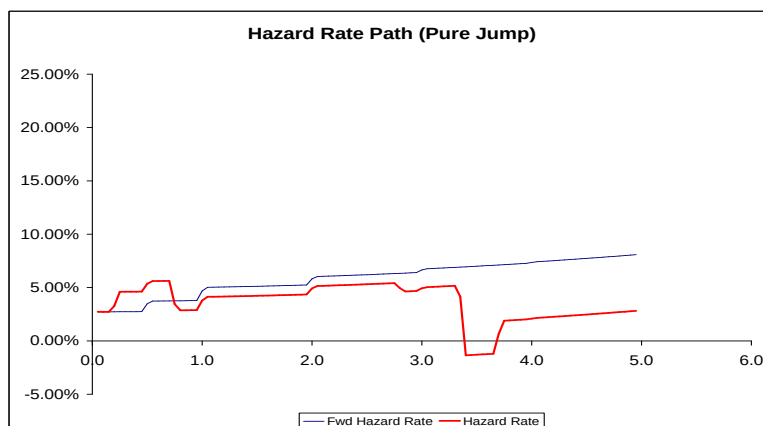


Figure 3.19: The stripped forward hazard rate of the Federative Republic of Brazil along side a pure jump simulated hazard rate path.

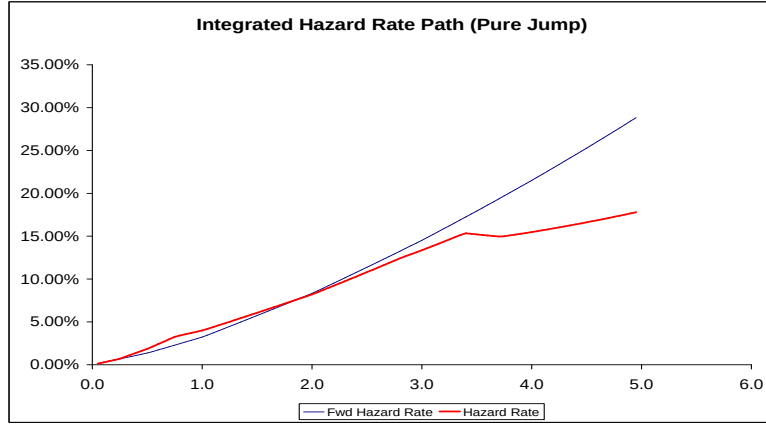


Figure 3.20: The integrated forward hazard rate and a simulated integrated hazard rate path for the pure jump model. The simulation is done with $\mu_S = 0$, $\sigma_S = 4\%$, constant jump intensity for Γ equal to 4% and mean reversion speed of 0.

Model Calibration		
Models	Parameters	Calibrated values
CIR	a	10%
	σ	77.48%
Pure jump	μ_S	0%
	σ_S	10%
	a	0
	Γ	5.84%
Extended Vasicek	a	10%
	σ	0.98%

Table 3.15: Calibrated parameters for the CIR, pure jump and extended Vasicek models respectively. The models have been calibrated to a 2y3y ATM payer swaption with 60% log-normal volatility.

Model results		
Models	Risk metrics	values
CIR	Fair spread	187bps
	<i>CS01</i>	6535
	<i>Gamma</i>	118
Pure jump	Fair spread	200bps
	<i>CS01</i>	2793
	<i>Gamma</i>	-3
Extended Vasicek	Fair spread	193bps
	<i>CS01</i>	4957
	<i>Gamma</i>	39

Table 3.16: Risk metrics for calibrated CIR, pure jump and extended Vasicek models respectively. The models have been calibrated to a 2y3y ATM payer swaption with 60% log-normal volatility. The risks are produced for an LCLN trade.

3.6 Chapter conclusion

In this chapter we build a framework to enable the valuation of single obligor products:

- We considered **Category 1** and **Category 2** products and their valuation. In the former we do not need to give a representation to credit spread dynamics in order to price the products, with the latter we have established the significant impact credit spread dynamics has on valuation.
- We provided market standard methods for valuing CDS, FCDS and credit default swaptions.
- We introduced the notion of pre-default quantities. These quantities are defined with respect to the sub-filtration, $\mathbb{F}$, and so are not impacted by defaults. By defining the premium leg and default leg in terms of pre-default quantities we showed that instances of arbitrage are reduced. In this setting the fair CDS swap spread is well defined in all events including default.
- Analytical results for the valuation of credit default swaptions are developed in Proposition 13. This allows one to build up a calibration routine for an intensity process driven by the extended Vasicek model.
- The speed of mean reversion parameter influences the drift of the spot volatility, it also acts as a dampener to the term structure volatility, see Equation (3.3.2). Hence

the speed of mean reversion affects the variance of the process also.

- We calibrated the speed of mean reversion parameter by showing a relationship between the swap spread and the spot intensity after introducing Assumption 3.4. The relationship is that the swap spread can be seen as the weighted average of the conditional expectation of the intensity under a suitable measure change. We further showed that the swap spread is equal to the integrated conditional expected value of the intensity over the time period for which the swap spread is defined up to a measure change and a correction term which tends to zero as we let the swap spread maturity, T , tend to the expiry date, $t \geq 0$.
- We tested the impact of varying the parameters of the extended Vasicek model on leverage credit linked notes and demonstrated the significant effect of spread dynamics. In our example we show that depending on the levels of the spot volatility and speed of mean reversion the implied fair spread varied from 402bps to 1072bps .
- We considered two alternative models in order to generate spread dynamics: CIR and pure jump model.

The developments in this chapter are important in themselves. The development of an efficient calibration routine for single obligor intensities driven by an extended Vasicek model, in this chapter, is also useful for the work done in Chapters 5 and 6. In those chapters we consider products that are sensitive to both default dependence and credit spread dynamics.

Chapter 4

Multiple Defaults And Dependence

In this chapter we are concerned with multi-obligor pricing and capturing default dependence.

Recalling the discussion in Chapter 1 we highlighted several risk factors that needed to be assessed in order to fully evaluate credit products. Qualitative arguments were made that highlighted the importance of these risk factors in credit modelling. Amongst these risk factors were:

1. Credit spread dynamics.
2. Default correlation amongst reference obligors.

In Chapter 3 we demonstrated how to apply stochastic intensity models in the valuation of single obligor exotic structures such as LCLNs. By examining LCLNs we found that spread dynamics has a significant impact on pricing. In this chapter we review models that capture default dependence and develop new techniques for implying the distribution of losses in a credit portfolio. Once this chapter is concluded we will have collected efficient means of capturing spread dynamics (Chapter 3) and accounting for default dependence (this chapter). This is useful for the work done in Chapters 5 and 6 where we will develop dynamic multi-obligor models.

In general this chapter is similar to Chapter 3 in that it develops a framework for credit modelling; however, the work is done with respect to multi-obligor products.

4.0.1 Summary of sections in this chapter

This chapter is split into nine sections:

- Section 4.1 establishes how some of the work done in the single obligor case can be developed so we can price multi-obligor products. In particular we demonstrate that the work done in Chapter 3 (Sub-sections 3.2.3 and 3.2.4) can be applied to NTD pricing and CDO pricing by interchanging the single obligor default time distribution for the NTD default time distribution or the CDO tranche loss distribution respectively.
- Section 4.2 demonstrates that when default times are mutually independent tractable valuation results are available for multi-obligor products. We then discuss conditionally independent frameworks. Under these frameworks, conditional on a factor or set of factors default times are mutually independent. Hence conditionally independent frameworks often lead to semi-analytic structures for multi-obligor pricing. We also show that attempting to generate default dependence by correlating the intensities of obligors is typically inadequate.
- Section 4.3 introduces copula methods and describes the general one period factor framework. We specifically demonstrate how to construct a portfolio's loss distribution under this framework. The one factor Gaussian copula (OFGC) is the standard model under the factor framework.
- Section 4.4 details the valuation methodology for CDIS. In that section we also provide the method of eliminating the basis that exists between quoted CDIS swap spreads and the individual obligor swap spreads.
- Section 4.5 details how correlation curves are constructed. In that section we analyse the compound correlation and base correlation methods.
- In Section 4.6 we consider the random recovery model which is an extension of the OFGC model. We show that the model accounts for the empirically observed fact that average recovery rates tend to be lower in economically depressed times. In the section we also demonstrate that the base correlation framework in the OFGC model produces instances of arbitrage, which the base correlation framework under the random recovery model does not. Nevertheless we conclude the section by showing that the particular version of the random recovery model we consider also contains arbitrage.
- In Section 4.7 we consider how to generate loss distributions. In Chapter 2 we explained that the OFGC model does not re-price all the traded tranches simultaneously, as it is a one parameter model. The inability of the OFGC model to re-price tranches leads to the need to construct correlation curves. The curve shape of a correlation

curve contains information about the real loss distribution of a portfolio. Hence the literature (Turc et al. (2004)) have developed methods of extracting the loss distribution of a portfolio by manipulating the base correlation curve. In the section:

- We state the current method of recovering the loss distribution of a credit portfolio. By exploiting the fact that the expected loss of an equity tranche must be conserved between models, Turc et al. (2004) are able to establish a method of generating a portfolio's loss distribution.
 - The weakness of this approach is that it requires a very specific form of the base correlation curve in order to work. In addition, the method requires differentiation of an interpolated curve.
 - We provide a new method, *the loss algorithm*. It is an algorithm which iteratively builds up a portfolio's loss density. The only requirement on the method is that we are able to extract the expected loss of an equity tranche (which the Turc et al. (2004) method requires) and for the loss density to have finite support (which will always be the case).
 - We demonstrate, by applying our new loss algorithm, there exists further arbitrages in the OFGC base correlation model. Moreover we show that the random recovery model reduces instances of these potential arbitrages.
- In Section 4.8 we review the random factor loading model. The model is an extension of the OFGC model. In this model default correlation changes as a function of the current economic condition. The model is shown to produce consistent loss distributions with that implied by traded tranches. The RFL model is developed in Chapter 6 where we show how to introduce spread dynamics into the factor framework.
 - Section 4.9 provides a conclusion.

4.1 Portfolio pricing

In this section we seek to connect some of the theoretical and practical results provided in Chapter 3 to the multi-obligor setting.

In Chapter 3 we discussed methods of expanding the sub-filtration, $\mathbb{F}$, so that the default time of an obligor was a stopping time with respect to an enlarged filtration. Now consider the default times $\tau_1, \dots, \tau_n$ of multiple obligors. We can expand $\mathbb{F}$ so that all the default times are stopping times with respect to an enlarged filtration, $\mathbb{G}$. This is done by defining:

$$\mathbb{G} = (\mathcal{G}_t)_{(t \geq 0)} = (\mathcal{D}_t \vee \mathcal{F}_t)_{(t \geq 0)},$$

where $\mathcal{D}_t = \bigvee_{i=1}^n \mathcal{D}_t^i$ and $\mathcal{D}_t^i$ is the smallest sigma-algebra containing information about the default of an obligor $i \in \{1, \dots, n\}$ at time $t \geq 0$. Then $\tau_1, \dots, \tau_n$ are stopping times with respect to $\mathbb{G}$.

We may now associate with the collection of times $\{\tau_1, \dots, \tau_n\}$ the set of times $\{\tau_{(1)}, \dots, \tau_{(n)}\}$, defined recursively as:

$$\tau_{(i+1)} = \min\{\tau_k : k \in \{1, \dots, n\} \text{ \& } \tau_k > \tau_{(i)}\}, \quad (4.1.1)$$

with $\tau_{(1)}$ the first obligor to default and $\tau_{(r)}$, for $r \in \{2, \dots, n\}$, is the r^{th} obligor to default. Equation (4.1.1) is important for two reasons:

- Firstly suppose we have a basket of $n > 1$ obligors and we want to price an m of n basket, with $1 \leq m \leq n$. $\tau_{(m)}$ is the default time of the m of n basket. For example if we wanted to price an FTD then $\tau_{(1)}$ would be the default time of the FTD. Recall that in Chapter 3 (Sub-sections 3.2.3 and 3.2.4) we showed that knowledge of the distribution of the default time of an obligor was required to value CDS i.e. if τ is the default time of an obligor its distribution $\forall t \geq 0$ is $\mathbb{Q}(\tau < t)$. Therefore if we assume recovery rates amongst the n obligors are equal, the valuation of an NTD follows in the same way as for a CDS once we know the distribution of $\tau_{(m)}$.
- Secondly, consider the case $m = n$ and the time $\tau_{(n)}$ i.e. the time of the last obligor to default. $\tau_{(n)}$ is called the *armageddon* time by Morini and Brigo (2007) and the *collapse* time by Rutkowski and Armstrong (2008). Under a suitable redefinition of the sub-filtration (see Chapter 6 (Section 6.2)) Rutkowski and Armstrong (2008) show that we can create quantities, similar to pre-default quantities discussed in Chapter 3 (Sub-section 3.2.7). In Chapter 3 the pre-default quantities we considered were for single obligor valuation models. In Chapter 6 (Section 6.2) these quantities, now called *pre-collapse* quantities, are for multi-obligors. We use pre-collapse quantities in Chapter 6 (Section 6.2) to show that analytical pricing for credit default index swaptions is feasible.

We now consider the valuation setup for CDOs. In the case of m of n to default baskets we required knowledge of only the distribution of $\tau_{(m)}$, since this is the time when a payment is made to the protection buyer and the protection seller stops receiving the swap spread. However with CDOs we require knowledge of the distribution of losses of the tranche. Hence for CDO valuation we need to know the distribution of portfolio losses rather than a default time distribution (as for CDS and NTDs). Nevertheless the valuation methodology

stays similar to that of CDS and NTDs once we can recover the distribution of losses of the CDO tranche.

We can define a portfolio's loss distribution, with n obligors as follows:

$$L_t = \sum_{i=1}^n \omega_i (1 - R_i) \mathbb{I}_{\{\tau_i < t\}}, \quad (4.1.2)$$

where $\omega_i > 0$ is the weight of obligor $i \in \{1, \dots, n\}$ in the portfolio, $\sum_{i=1}^n \omega_i = 1$ and $R_i \in [0, 1)$ is the recovery rate of obligor $i \in \{1, \dots, n\}$.

Recall the definition of the loss of a CDO tranche in Chapter 2 (Equation (2.3.1)), from this we get that the expected loss at time $t \geq 0$ of a CDO tranche is:

$$\frac{\mathbb{E}[(L_t - k_1)^+] - \mathbb{E}[(L_t - k_2)^+]}{k_2 - k_1}, \quad (4.1.3)$$

where k_1 is the tranche subordination and k_2 is the tranche detachment point. This allows us to define:

$$\mathbb{Q}^{tranche}(0, t) = 1 - \frac{\mathbb{E}[(L_t - k_2)^+] - \mathbb{E}[(L_t - k_1)^+]}{k_2 - k_1}. \quad (4.1.4)$$

$\mathbb{Q}^{tranche}(0, t) \in [0, 1]$ is a measure of the expected proportion of a tranche's notional remaining at time $t \geq 0$. When $\mathbb{Q}^{tranche}(0, t)$ goes to zero the tranche is effectively expected, almost surely, not to exist at $t \geq 0$. In this sense $\mathbb{Q}^{tranche}(0, t)$ measures the survival rate of a tranche over the horizon $[0, t]$.

By interchanging the survival probability function of a single obligor CDS for $\mathbb{Q}^{tranche}(0, t) \forall t \geq 0$ we can value CDOs with the methods developed in Chapter 3 (Sub-sections 3.2.3 and 3.2.4). A significant portion of the rest of the work of this chapter is to demonstrate ways of recovering the quantity $\mathbb{Q}^{tranche}(0, t) \forall t \geq 0$. Our main contribution to this field of work is in constructing an effective algorithm to generate portfolio loss distributions.

4.2 Mutually independent defaults and the conditionally independent framework

In this section we provide valuation results for multi-obligor products when default times are mutually independent. When default times are assumed to be mutually independent we generally recover tractable valuation results. We then proceed to consider how to generate default dependence.

4.2.1 The case of mutually independent default times

We consider the case of mutually independent default times because in the classic multi-obligor setup default dependence is typically generated under a conditionally independent framework. Conditionally independent frameworks postulate that obligors have mutually independent default times once we condition on a factor. In this thesis we will consider conditionally independent frameworks as a basis of generating default dependence.

Assume we have the default probabilities $\mathbb{Q}(\tau_i < t) \forall t \geq 0$ and all obligors $i \in \{1, \dots, n\}$. In order to simplify future calculations let:

$$p_i(t) = \mathbb{Q}(\tau_i < t) \forall t \geq 0.$$

The distribution of times $\tau_{(m)} \forall m \in \{1, \dots, n\}$ when default times are mutually independent are:

1. When $m = 1$ (the distribution of an FTD):

$$\mathbb{Q}(\tau_{(1)} \leq t) = 1 - \mathbb{Q}(\tau_{(1)} > t) = 1 - \prod_{i=1}^n (1 - p_i(t)),$$

2. When $m = n$ (the distribution of the collapse time):

$$\mathbb{Q}(\tau_{(n)} \leq t) = \mathbb{Q}(\tau_1 \leq t, \dots, \tau_n \leq t) = \prod_{i=1}^n p_i(t).$$

3. More generally, see Bielecki and Rutkowski (2002):

$$\mathbb{Q}(\tau_{(m)} \leq t) = \sum_{k=m}^n \sum_{\Pi \in \Pi^k} \prod_{j \in \Pi} p_j(t) \prod_{l \notin \Pi} (1 - p_l(t)),$$

where Π^k denotes the family of subsets $\{1, \dots, n\}$ consisting of k elements.

If we now associate with obligors $1, \dots, n$ intensities $(\lambda_t^1)_{t \geq 0}, \dots, (\lambda_t^n)_{t \geq 0}$ we recover (using tools from Chapter 3) the equality:

$$\mathbb{Q}(\tau_{(1)} \geq t) = \mathbb{E} \left[\exp^{-\int_0^t \lambda_{(1)}(s) ds} \right],$$

with $\lambda_t^{(1)} = \lambda_t^1 + \dots + \lambda_t^n$. Similar results can be found for the general m^{th} to default cases, see Bielecki and Rutkowski (2002).

To conclude, from above we can see that when default times are mutually independent, tractable results exist. This is the key motivating reason for multi-obligor credit models that are conditionally independent.

4.2.2 Conditional independence

With conditionally independent frameworks dependence amongst obligors derives from a common factor. Conditional on this factor(s) default times become independent:

Definition 4.1. *The random times $\tau_1, \dots, \tau_n$ are said to be conditionally independent if conditional on a set of random factors the times are mutually independent.*

For example consider the case where the factor is the sub-filtration, $\mathbb{F}$. In this case the random times $\tau_1, \dots, \tau_n$ are said to be conditionally independent with respect to the sub-filtration, $\mathbb{F}$, under $\mathbb{Q}$ if for any $t > 0$ and arbitrary times $t_1, \dots, t_n \in [0, t]$ we have:

$$\mathbb{Q}(\tau_1 > t_1, \dots, \tau_n > t_n | \mathcal{F}_t) = \prod_{i=1}^n \mathbb{Q}(\tau_i > t_i | \mathcal{F}_t).$$

We now illustrate how to construct a conditionally independent framework with respect to the sub-filtration, $\mathbb{F}$:

Note 3 (Canonical Construction Of Conditional Independence). *Let Γ^i , $i \in \{1, \dots, n\}$, be a collection of $\mathbb{F}$ – adapted absolutely continuous stochastic processes defined on a common filtered probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \mathbb{P}^*)$. Further assume $\Gamma_0^i = 0$, $\Gamma_\infty = +\infty$ and:*

$$\Gamma_t = \int_0^t \lambda_s^i ds,$$

where $(\lambda_t^i)_{t \geq 0}$ are also $\mathbb{F}$ – adapted processes.

Let $(\widetilde{\Omega}, \widetilde{\mathcal{D}}, \widetilde{\mathbb{P}})$ be an auxiliary probability space endowed with a sequence ζ_i , for $i \in \{1, \dots, n\}$ of mutually independent random variables uniformly distributed on $[0, 1]$. We consider $(\Omega, \mathcal{F}, \mathbb{Q}) = (\widetilde{\Omega} \times \widehat{\Omega}, \widetilde{\mathcal{D}} \otimes \widehat{\mathcal{F}}, \widetilde{\mathbb{P}} \otimes \mathbb{P}^*)$ and set for $i \in \{1, \dots, n\}$:

$$\tau_i = \inf\{t \in \mathbb{R}_+ : \int_0^t \lambda_s^i ds \geq -\ln(\zeta_i)\}, \quad (4.2.1)$$

with $\eta_i = -\ln(\zeta_i)$, exponentially distributed with unit parameter. Equation (4.2.1) may be re-stated as:

$$\tau_i = \inf\{t \in \mathbb{R}_+ : \Gamma_t^i \geq \eta_i\}. \quad (4.2.2)$$

Note 3 is the standard setup for multi-obligor intensity modelling, see Duffie (2002), and is used in the simulation of default times of conditionally independent defaults generated by correlated intensities. In this setup conditional on $\mathbb{F}$ the integrated hazard rate functions are deterministic and hence independent, no other source of dependence is available.

This specific construction of conditional independence suggests that default dependence can be created by correlating the integrated hazard rate function, $\Gamma^i \forall i \in \{1, \dots, n\}$ or equivalently the intensities of obligors.

In Chapter 1 we discussed the problems of correlating intensities which either do not generate the appropriate level of default correlation or lead to intractable modelling frameworks. We will now demonstrate that when we try to generate default correlation with intensity processes that follow diffusion processes we alter the spread dynamics of the reference obligors.

4.2.3 Generating default dependence by correlating intensities

The following example is an adjusted version of a similar example found in Schönbucher (2003a). Assume that τ_1 , and τ_2 are default times determined by intensity processes $(\lambda_t^1)_{t \geq 0}$ and $(\lambda_t^2)_{t \geq 0}$ we therefore have that their joint distribution is:

$$\begin{aligned}
\mathbb{Q}(\tau_1 < T, \tau_2 < T) &= p_{12}(\tau_1, \tau_2) = p_{12} \\
&= \mathbb{E}[\mathbb{I}_{\{\tau_1 < T\}} \mathbb{I}_{\{\tau_2 < T\}}] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{I}_{\{\tau_1 < T\}} \mathbb{I}_{\{\tau_2 < T\}} | \mathcal{F}_T]] \\
&= \mathbb{E}[\mathbb{E}[(1 - e^{-\int_0^T \lambda_s^1 ds})(1 - e^{-\int_0^T \lambda_s^2 ds}) | \mathcal{F}_T]] \\
&= 1 - (1 - \mathbb{Q}(\tau_1 < T)) - (1 - \mathbb{Q}(\tau_2 < T)) + \mathbb{E}[e^{-\int_0^T \lambda_s^1 + \lambda_s^2 ds}] \\
&= \mathbb{Q}(\tau_1 < T) + \mathbb{Q}(\tau_2 < T) + \mathbb{E}[e^{-\int_0^T \lambda_s^1 + \lambda_s^2 ds}] - 1 \\
&= p_1 + p_2 + \mathbb{E}[e^{-\int_0^T \lambda_s^1 + \lambda_s^2 ds}] - 1, \tag{4.2.3}
\end{aligned}$$

where $\mathbb{Q}(\tau_i < T) = p_i$, $i = \{1, 2\}$. Evidently the strongest level of dependence in Equation (4.2.3) is reached when both intensities are identical, $\lambda_t^1 = \lambda_t^2 = \lambda_t \forall t \geq 0$, then:

$$p_{12} = 2p_1 + \mathbb{E}[e^{-2 \int_0^T \lambda_s ds}] - 1, \tag{4.2.4}$$

since $p_1 = p_2$. We are now in a position to compute the default correlation between indicators $\mathbb{I}_{\{\tau_1 < T\}}$ and $\mathbb{I}_{\{\tau_2 < T\}}$. For $i \in \{1, 2\}$ we have:

$$\mathbb{V}[\mathbb{I}_{\{\tau_i < T\}}] = \mathbb{E}[\mathbb{I}_{\{\tau_i < T\}}^2] - \mathbb{E}[\mathbb{I}_{\{\tau_i < T\}}]^2 = p_1(1 - p_1),$$

where $\mathbb{V}$ is the variance operator. Hence the correlation between $\mathbb{I}_{\{\tau_1 < T\}}$ and $\mathbb{I}_{\{\tau_2 < T\}}$ is:

$$\mathbb{C}(\mathbb{I}_{\{\tau_1 < T\}}, \mathbb{I}_{\{\tau_2 < T\}}) = \frac{p_{12} - p_1^2}{p_1(1 - p_1)} = \frac{\mathbb{E}[e^{-2 \int_0^T \lambda_s ds}] - (1 - p_1)^2}{p_1(1 - p_1)} = \frac{\mathbb{V}[e^{-\int_0^T \lambda_s ds}]}{p_1(1 - p_1)},$$

where $\mathbb{C}(X_1, X_2)$ represents the correlation between random variables X_1 and X_2 . If we compare the above equation with Equations (E.1.2) and (E.2.4) in Appendix E we see that:

$$\mathbb{E}[e^{-2 \int_0^T \lambda(s) ds}] = e^{-2m(T) + 2v(T, T)}, \tag{4.2.5}$$

where m and v are the mean and variance of the integrated intensity process (as set out in Appendix E (Section E.2)). This leads to:

$$\begin{aligned}\mathbb{C}(\mathbb{I}_{\{\tau_1 < T\}}, \mathbb{I}_{\{\tau_2 < T\}}) &= \frac{e^{-2m(T)+2v(T,T)} - e^{-2m(T)+v(T,T)}}{e^{-m(T)+\frac{v(T,T)}{2}} \left(1 - e^{-m(T)+\frac{v(T,T)}{2}}\right)} \\ &= \frac{1-p_1}{p_1}(e^{v(T,T)} - 1).\end{aligned}\tag{4.2.6}$$

We readily see that in cases where $v(T, T)$ is small the level of default correlation could be very small. Moreover Equation (4.2.6) suggests that we must induce dependence elsewhere if we do not want to alter spread dynamics in order to control default dependence.

In Chapter 5 we apply the construction method of Note 3 by considering cases where $\zeta_i \forall i \in \{1, \dots, n\}$ are not mutually independent. This is done in order to have a dynamic intensity framework, where default dependence derives from exogenous factors (correlating $\zeta_i \forall i \in \{1, \dots, n\}$) other than the intensity dynamics. Such a framework allows us to manipulate default dependence without altering spread dynamics.

To conclude, by considering Equation (4.2.1) in Note 3, we see there are three main methods to create default dependence¹:

1. Explicitly correlate the driving intensities $(\lambda_t^i)_{t \geq 0}$ for all obligors $i \in \{1, \dots, n\}$.
2. Induce dependence on the quantities $\zeta_i \forall i \in \{1, \dots, n\}$. These quantities are exogenous to the intensities and are as defined in Note 3.
3. Do both 1 and 2.

Recall in Chapter 1 (Section 1.2) we considered that there had been a departure from attempting to capture default correlation via correlating intensities and a move was made to the factor framework. Under the factor framework a default time copula arises naturally. Mathematically and practically this move can be defined by a departure from method 1 to method 2 above. In practical simulations the factor framework can be structured in terms of method 2. We now consider copula methods and the factor framework.

¹A fourth alternative exists, which involves having a common shock factor, e.g. the Marshall Olkin copula, which considers a common shock factor which affects the reference obligors (see Andersen and Sidenius (2005)).

4.3 Copula and the factor framework

In this section we define copulas, detail the factor framework and relate the concepts to the valuation of multi-obligor credit products.

4.3.1 Copula

Copula functions are used in conjunction with intensity models in order to generate the appropriate levels of default dependence. Brigo and Mercurio (2001) note that in standard modelling of other asset classes we are usually satisfied to:

1. Capture the dynamics of processes under Brownian driven stochastic systems and,
2. Capture the dependence between instantaneous Brownian shocks as being jointly Gaussian.

With other asset classes the process we are modelling is usually the economic variable we are looking to assess e.g. stock prices, commodity prices etc. Hence, linear correlation usually works well. However in credit we are modelling the stochastic intensity of obligors, which is not the final economic variable we are seeking to model. In fact what we are seeking to understand in credit modelling is when a default will occur, this time is non-linearly linked to the intensity process. Therefore in credit modelling, linear correlation between intensities usually does not work well.

As an alternative, in order to introduce default dependence, we can correlate the quantities $\zeta_i \forall i \in \{1, \dots, n\}$ in Note 3 directly. This is essentially the copula approach when it comes to credit modelling.

Before we define copulas, recall that the dependence structure of real valued random variables, $X_1, \dots, X_n$, is described by their joint distribution function $F : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that:

$$F(\mathbf{x}) = \mathbb{Q}(X_1 < x_1, \dots, X_n < x_n),$$

with vector $\mathbf{x} = \{x_1, \dots, x_n\}$. In addition, for some cumulative distribution function $F_X : \mathbb{R} \rightarrow \mathbb{R}_+$ of X , which we assume to be continuous (hence invertible) we can define $\zeta = F_X(x)$, with $\zeta \sim U[0, 1]$.

The utility of the copula derives from two properties:

- Firstly the joint distribution function, F , can be separated in two parts: its marginal distribution functions and the dependence structure between the random variables, which is given by a copula function.

- Secondly the copula approach allows for the full processing of a dependence relationship. Suppose we have a set of continuous random variables $\{X_1, \dots, X_n\}$ where the dependence relationship is defined by a copula. In Embrechts et al. (2001) it is shown that if we have a corresponding set of strictly increasing functions $\{f_1, \dots, f_n\}$ (defined on the respective ranges of the random variables) then the sets $\{f_1(X_1), \dots, f_n(X_n)\}$ and $\{X_1, \dots, X_n\}$ share the same dependence structure (copula).

Definition 4.2. An n – dimensional copula, C , is a multivariate distribution function which:

- Is defined on the hypercube $[0, 1]^n$ with uniform marginal distributions for random variables $\mathbf{U} = \{\zeta_1, \dots, \zeta_n\}$.
- Has marginals C_k , $k \in \{1, \dots, n\}$ which satisfy $C_k(u) = u \ \forall u \in [0, 1]$.

Theorem 15 (Sklar’s Theorem). Let F be an n – dimensional distribution function which has domain $\mathbb{R}^n$ such that:

$$F(\infty, \dots, \infty) = 1,$$

with marginal distributions $F_1, \dots, F_n$, then $\exists$ an n – dimensional copula, C , such that $\forall \mathbf{x}$ in $\mathbb{R}^n$ we have:

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)). \quad (4.3.1)$$

Conversely if C is an n dimensional copula and $F_1, \dots, F_n$ are distribution functions, then the function F defined above is an n – dimensional distribution function with marginals $F_1, \dots, F_n$. Moreover if the marginals are continuous then this copula is unique.

Proof. See Embrechts et al. (2001). □

Example 4 (The canonical copula). The canonical copula is obtained by using a multivariate n – dimensional normal distribution, $\Phi_{\mathbb{R}^n}$, with standard Gaussian marginal correlation matrix $\mathbf{M}$, as the multivariate distribution function:

$$C = \Phi_{\mathbb{R}^n}(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)), \quad (4.3.2)$$

where Φ^{-1} is the inverse of the standard cumulative normal distribution. If we assume that all credits are driven by N common factors, the correlation matrix can be specified with a factor structure. This means that we can represent an n – dimensional standard Gaussian vector $\mathbf{X}$ as:

$$\mathbf{X} = \mathbf{c}\mathbf{Z} + (\mathbf{1} - \mathbf{M})^{\frac{1}{2}} \mathbf{e}, \quad (4.3.3)$$

where $\mathbf{Z}$ is an N – dimensional vector of independent standard Gaussian variables, $\mathbf{c}$ is an $n \times N$ matrix, with the correlation matrix being $\mathbf{M} = \mathbf{c} \mathbf{c}^T$ and $\mathbf{e}$ is an n – dimensional

vector of idiosyncratic Gaussian variables. The case where N is equal to 1 and the pairwise correlations are assumed the same is the one factor Gaussian copula (OFGC), which is the market standard copula.

4.3.2 Factor framework

In order to review the factor framework and its general use in the literature, we will consider the case where $n > 1$ reference obligors have deterministic intensities $\lambda_1, \dots, \lambda_n$. In Chapters 5 and 6 we will ease this restriction to introduce stochastic intensities. In the present chapter we are following the general literature where the concern, in the multi-obligor case, is not to give a representation to spread dynamics but to capture default dependence.

Fix an arbitrary time $t \geq 0$. The factor framework is a one period setup where the occurrence or non occurrence of default is assessed from only one time, t . The unconditional marginal distribution of default times, $\tau_i \forall i \in \{1, \dots, n\}$, is given by:

$$q_i(t) = \mathbb{Q}(\tau_i \geq t) = e^{-\int_0^t \lambda^i(0,s) ds}, \quad (4.3.4)$$

where $\lambda^i(0, s)$ is the initial hazard rate for obligor i and we are using the notation $q_i(t)$ in place of $\mathbb{Q}(\tau_i \geq t)$ because it aides in the clarity of the exposition below.

In the typical setup of the factor framework one introduces a (low dimensional) vector of market drivers, $\mathbf{Z} = \{Z_1, \dots, Z_d\}$. The factor framework is a version of a conditionally independent framework, where the factors are $\mathbf{Z}$. The conditional survival probabilities of obligors are exogenously specified functions of $\mathbf{Z}$. Hence for $i \in \{1, \dots, n\}$ we have:

$$\mathbb{Q}(\tau_i \geq t | \mathbf{Z} = \mathbf{z}) = q_i(t, \mathbf{z}). \quad (4.3.5)$$

We then decide on a choice of density, $\psi(t, \mathbf{z})$, for the driver $\mathbf{Z}$, which in full generality may evolve with time t . The following then holds:

$$q_i(t) = \mathbb{Q}(\tau_i \geq t) = \int_{\mathbb{R}^d} q_i(t, \mathbf{z}) \psi(t, \mathbf{z}) d\mathbf{z}. \quad (4.3.6)$$

Further we introduce random variables $X_1, \dots, X_n$ and values $c_1, \dots, c_n$. $X_i \forall i \in \{1, \dots, n\}$ are proxy asset values, related to obligor $i \in \{1, \dots, n\}$. Conditional on the market factors, $\mathbf{Z}$, the X_i 's are mutually independent. The values $c_i \forall i \in \{1, \dots, n\}$ are thresholds which represent the liquidity tolerance that triggers default if $X_i \forall i \in \{1, \dots, n\}$ falls below it. From this we have that the following holds:

$$\mathbb{I}_{\{\tau_i < t\}} = \mathbb{I}_{\{X_i < c_i\}} \forall i \in \{1, \dots, n\}. \quad (4.3.7)$$

Recall that we have $\mathbb{Q}(\tau_i \geq t)$ from standard stripping of the CDS curve of obligors (see Chapter 3 (Sub-section 3.2.4)). Hence, we have default probabilities of individual obligors; this enables us to use Equation (4.3.6) to calibrate to the parameters of the quantities $q_i(t, z)$ and $\psi(t, z)$.

At this stage it is important to emphasise that the factor framework in this setup is one period. We are considering a structure only relevant to the time point $t \geq 0$. No qualifications are made about how the market factors $Z_1, \dots, Z_d$, the random variables $X_1, \dots, X_n$ and the values $c_1, \dots, c_n$ develop with time.

The aim of the factor framework is to generate a portfolio's loss distribution. The setup, Equation (4.3.6), can only generate a portfolio's loss distribution at a single fixed time $t \geq 0$. Loss distributions at earlier maturities can be implied from the resulting model structure by assuming the factor structure, $\mathbf{Z}$, is valid for earlier maturities and re-calibrating the values $c_1, \dots, c_n$. However, as noted by Andersen (2006) even though loss distributions at earlier maturities can be implied from the resulting model there exists little control over those distributions since we rely on the assumption $Z_t = \mathbf{Z}$. In Chapter 6 we will consider how one can extend the factor framework beyond the one period structure.

We now describe how the factor framework is used to value CDO tranches. If we want to price a tranche, we have established that we need to recover the quantity $\mathbb{Q}^{tranche}(0, t)$ and do so for multiple times $t \geq 0$. In order to do this it is enough to show we can get expectations of functions with the following form (see Equation (4.1.4) (Section 4.1)):

$$f(L_t) = (L_t - k)^+,$$

where L_t is the portfolio loss distribution. We have the following:

$$\mathbb{E}[f(L_t)] = \int_{\mathbb{R}^d} \mathbb{E}[f(L_t) | \mathbf{Z} = \mathbf{z}] \psi(t, \mathbf{z}) d\mathbf{z} = \int_{\mathbb{R}^d} d(t, \mathbf{z}) \psi(t, \mathbf{z}) d\mathbf{z}. \quad (4.3.8)$$

The valuation of this expectation has two stages:

- Andersen et al. (2003) produces an algorithm, the *recursion algorithm*, which can be applied in this case to build up the conditional expectation $\mathbb{E}[f(L_t) | \mathbf{Z} = \mathbf{z}]$.
- Once we have the conditional expectation, $\mathbb{E}[f(L_t) | \mathbf{Z} = \mathbf{z}]$, we can integrate over the density of $\mathbf{Z}$ to get the unconditional expectation $\mathbb{E}[f(L_t)]$.

Hence we can construct $\mathbb{Q}^{tranche}(0, t)$ for the particular time point under consideration. The process can be repeated for multiple time points. From this, as discussed, we can value a CDO tranche.

4.3.3 One factor Gaussian copula

The one factor Gaussian copula (OFGC) is the standard default time copula for credit modelling and is constructed under the factor framework. In this case the market factor Z is one dimensional. Since we are working under a one period framework we can consider an arbitrary fixed time $t \geq 0$ and suppress explicit dependence on time. We begin by defining the factor setup of the model:

$$X_i = \rho_i Z + \sqrt{1 - \rho_i^2} \epsilon_i, \quad (4.3.9)$$

where ϵ_i are standard *iid* normal random variables independent of Z . Z is a standard normal random variable and $\rho_i \in [0, 1]$.

The OFGC model is recovered by setting $\rho_i = \sqrt{\rho} \forall i \in \{1, \dots, n\}$. ρ is the linear correlation between the random variables $X_i \forall i \in \{1, \dots, n\}$. Finally $X_i \forall i \in \{1, \dots, n\}$ are standard normals. Under the model we have:

$$1 - q_i = \mathbb{Q}(X_i < c_i) = \Phi(c_i). \quad (4.3.10)$$

Finding c_i requires an inversion of the standard normal cumulative distribution function:

$$c_i = \Phi^{-1}(1 - q_i). \quad (4.3.11)$$

Conditional on the common factor, Z , we have:

$$\begin{aligned} 1 - q_i(z) &= \mathbb{Q}(X_i < c_i | Z = z) \\ &= \mathbb{Q}(\sqrt{\rho}z + \sqrt{1 - \rho} \epsilon_i < c_i | Z = z) \\ &= \mathbb{Q}\left(\epsilon_i < \frac{c_i - \sqrt{\rho}z}{\sqrt{1 - \rho}} | Z = z\right) \\ &= \Phi\left(\frac{\Phi^{-1}(1 - q_i) - \sqrt{\rho}z}{\sqrt{1 - \rho}}\right). \end{aligned} \quad (4.3.12)$$

To conclude, in general, under factor frameworks, pricing of a CDO will require that we recover, over a time grid $t_1, \dots, t_M$, the quantity $\mathbb{Q}^{tranche}(0, t_i) \forall t_i \in \{1, \dots, M\}$. This is the same requirement we had for single obligor CDS where we needed to recover the survival probability of an obligor over a time grid in order to value CDS (see Chapter 3 (Sub-section 3.2.4)). Hence, if we have M discretisation points we will then have to apply the recursion method M times and do M numerical integrations in order to recover the unconditional quantity $\mathbb{Q}^{tranche}(0, t_i) \forall t_i \in \{1, \dots, M\}$.

For example if we were pricing a 7 year trade with quarterly coupons and we discretised the protection leg, $(\tilde{\mathbf{D}}^{tranche}(t, T))_{t \geq 0, T \geq t}$, according to the coupon dates then we have $M = 7 \times 4 = 28$. In the case of the OFGC the only significant computations being performed are:

1. Inversions of a normal distribution (which are done for each obligor in the portfolio).
2. Quadrature integration of a Gaussian density.

Therefore there are no speed issues with the OFGC model in this process other than the fact that the recursion algorithm, see Appendix F (Section F.1), can become slow if the assumed recovery rates of obligors are very different. In this thesis, unless otherwise stated, we assume that reference obligors have the same recovery rates.

For other models (below we consider the random recovery model and random factor loading) the process of constructing the quantity $\mathbb{Q}^{tranche}(0, t)$ can give rise to additional computational issues.

Numerical Example

Figure 4.1 shows the surface of a loss density for varying correlation levels applied to a portfolio. The loss density is constructed with an assumed homogeneous portfolio of 40 obligors each with a flat CDS curve of 50bps. Using the OFGC model we construct the portfolio loss density. The figure shows that correlation significantly affects the shape of the loss density:

- In the case of 90% correlation the probability of no losses is 82% and in the case of 10% correlation, the probability of no losses is 29%.
- This numerical result conforms with the qualitative assessment we made in Chapter 2 (Section 2.3), where we discussed a CDO tranche's risk characteristics.
- The result explains, quantitatively, why senior tranche investors are averse to high correlations. A high OFGC correlation parameter makes the distribution fat tailed, meaning it is more likely that there are larger levels of losses if defaults occur. Note this also means it is more likely that no defaults will occur. This is the case senior investors are averse to.
- With low correlations the probability of large losses reduces, but the instances of defaults occurring increases, which means equity investors are more likely to experience losses.

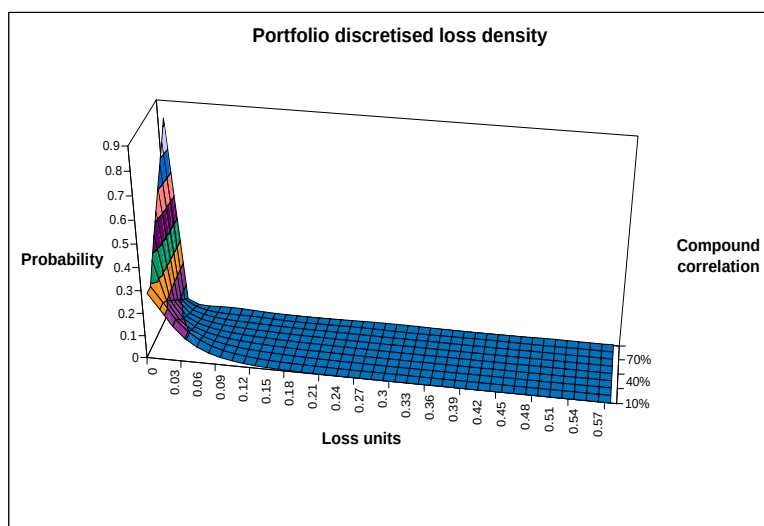


Figure 4.1: Portfolio loss density

4.4 Index swap valuation

In this section we consider the valuation of credit default index swaps (CDIS) and market conventions for index swap pricing:

- In the context of correlation curves (considered in Section 4.5) we want to value CDIS because we need to compare the difference of the fair CDIS swap spread and the (duration adjusted) weighted average CDS swap spreads of individual obligors. This difference is called the *index swap spread basis*, accounting for it ensures proper valuation of CDOs (see Stamicar (2008) and Couderc (2005)).
- In Chapter 6 (Section 6.2) we will apply the results of this section. In that section we show how to construct semi-analytical valuation results for credit default index swaptions.

4.4.1 The standard Index and market conventions

Let us assume a notional of 1 for the CDIS at its creation date and that there are $n > 1$ obligors that make up an index. Recall that buying protection on a CDIS is equivalent to buying $\frac{1}{n}$ of protection on each individual obligor. In addition, for a maturity $T > 0$, a CDIS has its swap spread, s^{fixed} , fixed when the index is created. After the creation of the index s^{fixed} will be the CDIS swap spread paid by the protection buyer. Therefore at a time after the index has been created, unless market conditions are unchanged, there will be a cash payment either by the buyer of protection or the seller of protection to reflect worsening or improving market conditions respectively.

Although market convention is as described above, in this thesis we will consider a theoretical fully running CDIS swap spread, $s^I(t, T)$. $s^I(t, T)$ is the current market fair swap spread of a CDIS for a time $t \geq 0$ and maturity $T > 0$, which is implied by s^{fixed} and the upfront cash payment. In a later chapter (Chapter 6) we will define the notion of a pre-collapse CDIS fair swap spread which reduces instances of arbitrage in a similar way to pre-default CDS fair swap spreads defined in Chapter 3 (Sub-section 3.2.7).

Recall Equation (4.1.2) in Section (4.1) gives the portfolio loss process and is of the form:

$$L_t = \sum_{i=1}^n \omega_i (1 - R_i) \mathbb{I}_{\{\tau_i < t\}}.$$

In the rest of this thesis, unless otherwise stated, we assume that for all portfolios considered $\omega_i = \frac{1}{n}$ and $R_i = R \forall i \in \{1, \dots, n\}$.

We now detail valuation methods for forward CDIS in a similar way as for single obligor CDS in Chapter 3 (Sub-section 3.2.6):

To begin, recall the definition of a CDIS in Chapter 2 (Section 2.3). After each default the notional of the portfolio reduces by the notional of the defaulting obligor. We may define the outstanding notional of a portfolio to be:

$$O_t = 1 - \frac{L_t}{1 - R}. \quad (4.4.1)$$

Let $t_0 \geq 0$ be the trade date, $t \geq t_0$ be the expiry date of the forward CDIS and define the coupon payment dates to be $\{T_1, \dots, T_N\}$ with $t < T_1$. The premium leg pays at time $T_j \in \{T_1, \dots, T_N\}$ a rate, k , on the average $\bar{O}_{(T_{j-1}, T_j]}$ of the outstanding notional O_t for $t \in (T_{j-1}, T_j]$ (see Morini and Brigo (2007)). The discounted payoff of the premium leg is:

$$\begin{aligned} \mathbf{P}^I(t, T) &= \sum_{j=1}^N \Delta(T_{j-1}, T_j) P(t, T_j) \frac{1}{T_j - T_{j-1}} \int_{T_{j-1}}^{T_j} O_s ds \\ &\approx \sum_{j=1}^N \Delta(T_{j-1}, T_j) P(t, T_j) \left(1 - \frac{L_{T_j}}{1 - R} \right), \end{aligned} \quad (4.4.2)$$

where in the last equality the approximation symbol is used as we assume that coupons are paid with the outstanding notional at the end of the coupon period. We will make this assumption in the rest of this thesis. We can define the value of the premium leg as:

$$k \times \bar{\mathbf{P}}_L^I(t, T) = k \times \mathbb{E} [\mathbf{P}^I(t, T) | \mathcal{G}_t] .$$

The discounted payoff of the protection leg is:

$$\begin{aligned} \mathbf{D}^I(t, T) &= \int_t^T P(t, u) dL_u \\ &\approx \sum_{j=1}^N P(t, T_j) (L_{T_j} - L_{T_{j-1}}) , \end{aligned} \quad (4.4.3)$$

where in the last equality the approximation discretises the loss payments to be made on the premium payment dates. We will make this assumption in the rest of this thesis. We can define the value of the protection leg as:

$$\bar{\mathbf{D}}_L^I(t, T) = \mathbb{E} [\mathbf{D}^I(t, T) | \mathcal{G}_t] .$$

Whenever $\tau_{(n)} > t$ (recall $\tau_{(n)}$ is the time of default of the last surviving obligor) we can define:

$$s^I(t, T) = \frac{\bar{\mathbf{D}}_L^I(t, T)}{\bar{\mathbf{P}}_L^I(t, T)} . \quad (4.4.4)$$

In the market CDIS swap spreads are quoted with a few additional assumptions (see Morini and Brigo (2007)):

1. The intensities of reference obligors are deterministic.
2. The credit risk of reference obligors are the same. In this case the portfolio is said to be *homogeneous* and each reference obligor shares the same deterministic intensity.

Under these assumptions we have:

$$\bar{\mathbf{D}}_L^I(t, T) = O_t(1 - R) \sum_{j=1}^N P(t, T_j) \left(e^{-\int_t^{T_{j-1}} \lambda_s ds} - e^{-\int_t^{T_j} \lambda_s ds} \right) ,$$

and:

$$\bar{\mathbf{P}}_L^I(t, T) = O_t \sum_{j=1}^N \Delta(T_{j-1}, T_j) P(t, T_j) e^{-\int_t^{T_j} \lambda_s ds} .$$

From this the deterministic intensities can be calibrated to the spot CDIS swap spreads trading in the market in a similar way to single obligor CDS.

4.4.2 The credit index swap basis

To extract the correlation curve from market prices of index tranches, it is standard practice to scale the underlying CDS spreads to eliminate the difference, called the *index basis*, that exists between the index swap and the duration weighted average spread of the obligors contained within the index.

Definition 4.3. The basis, $Bas(t_0, T)$, at maturity T is given by:

$$Bas(t_0, T) = s^I(t_0, T)\bar{\mathbf{P}}_L^I(t_0, T) - \sum_{i=1}^N \omega_i s^i(t_0, T)\bar{\mathbf{P}}_L^i(t_0, T),$$

where

- $s^I(t_0, T)$ is the index spread for maturity T .
- $\bar{\mathbf{P}}_L^I(t_0, T)$ is the premium leg of the index using the market assumption of a homogeneous portfolio.
- ω_i is the weight of obligor $i \in \{1, \dots, n\}$ in the index.
- $\bar{\mathbf{P}}_L^i(t_0, T)$ is the risky duration of obligor i .

To eliminate the basis we have to modify each CDS curve of the underlying obligor in the basket so that we have:

$$0 = s^I(t_0, T)\bar{\mathbf{P}}_L^I(t_0, T) - \sum_{i=1}^N \omega_i \tilde{s}^i(t_0, T)\tilde{\mathbf{P}}_L^i(t_0, T),$$

where $\tilde{s}^i(t_0, T)$ is now a modified CDS swap spread. Various basis elimination methods exist.

We consider the following:

1. Assuming the typical 1y, 3y, 5y, 7y and 10y CDS quotes are available and that the index has the usual 3y, 5y, 7y and 10y quotes:
2. Scale the 1y and 3y underlying CDS swap spreads by a scalar value such that the basis in Definition 4.3 is eliminated.
3. Given the scaled 1y and 3y CDS swap spreads, now scale the 5y CDS swap spread to match the 5y index to further eliminate the basis there, noting that the 1y and 3y CDS swap spreads will influence the 5y scaling.
4. Continue on for the remaining maturities².

²Scaling by spreads may be unstable and can introduce arbitrage especially after having bootstrapped a few maturities to then recover the later maturities may push the later maturity spreads down in a way that introduces arbitrage. A possible fix is to scale hazard rates instead directly.

4.5 Correlation Curves

In this section we consider two curve construction types, base correlation and compound correlation. The base correlation approach is an evolution of the compound correlation approach. It overcomes some significant shortcomings of the compound correlation approach.

This section is an overview of the literature (Andersen and Sidenius (2004), McGinty and Beinstein (2004), Turc et al. (2004), Turc et al. (2006), O’Kane and Livesey (2004) and others). We provide the overview on the one hand for the convenience of the reader, but also because:

- Later in Section 4.7 we show how to imply the loss distribution of a portfolio as implied by a base correlation curve. This is achieved because, in Section 4.7, we develop a new algorithm, the loss algorithm, which generates in an efficient way a portfolio’s loss distribution.
- Then in Chapter 6 (Section 6.2) we show how one can find semi-analytical results for credit default index swaptions. Finding tractable results for index swaptions requires we are able to construct a portfolio’s loss distribution. This section and Section 4.7 allows us to construct the loss distribution of a portfolio.

4.5.1 Calibrating the correlation curve

Recall in Chapter 2 (Section 2.3) we detailed that in the market a number of traded CDO tranches exist (both Itraxx and CDX have 5). Further we also mentioned that the OFGC model as a one parameter model could not re-price the CDO tranches simultaneously. The method of correlation curve construction is to create a curve with the correlation values (from the OFGC model) that re-prices the traded tranches as ordinates. The abscissas of the correlation curves are related to the tranche subordination and detachment points. In this way one can pass in a particular non-standard tranche into the correlation curve function and recover a correlation value which can be used to value the non-standard tranche. In this sense the correlation curve method for CDOs is similar to volatility curves that use the Black and Scholes model in equities.

We now detail the curve construction method. Define the value of a tranche from the perspective of a protection buyer as $\overline{\text{val}}_{\text{L}}^{\text{tranche}}(t, T, k_1, k_2, \rho, s^{[k_1, k_2]})$, where:

- $t \geq 0$ is the trade start date, $T > t$ is the trade maturity.
- k_1 and k_2 are the tranche’s subordination and detachment points respectively.

- ρ is the OFGC correlation parameter.
- $s^{[k_1, k_2]}$ is the swap spread paid by the protection buyer to the protection seller for the CDO tranche $[k_1, k_2]$.

The following additional inputs will be required to value a CDO tranche using the OFGC model (although we do not explicitly show the dependence of these on CDO valuation):

- **S**: a matrix of credit spreads. We require a matrix of credit spreads as each obligor has a credit curve (typically with maturities 1y, 3y, 5y, 7y, and 10y, see Chapter 3 (Sub-section 3.2.4)). Hence the matrix will have a row of length equal to the number of obligors, n , and columns equal to the number of points on the CDS curve, which is usually 5.
- **R**: a vector of length n representing the recovery rates of each obligor. The recovery rates are assumed to be homogeneous and equal to 40%.
- $(P(t, T))_{t \geq 0, T \geq t}$: the deterministic discount factor defined in Appendix C (Definition C.1).

As the fair swap spread of a traded tranche is provided for in the market, the only unknown is the correlation parameter, ρ . With correlation curve constructions the real work we do is solve for a correlation value that makes $\overline{\text{val}}_L^{\text{tranche}}(t, T, k_1, k_2, \rho, s^{[k_1, k_2]})$ zero.

4.5.2 Compound correlation

The compound correlation method consists in calculating the correlation value, ρ , in the OFGC model that re-prices each tranche in line with market quotes. The difference in implied correlations across different tranches gives rise to a curve. In the case of compound correlation the abscissas of the curve are the detachment points of the tranches.

In Table 4.1 and Figure 4.2 we provide a numerical example of the compound correlation method:

- The portfolio we consider is the Itraxx, the maturity of the index is 5 years.
- We have provided the example using data as of 09 June 2009 when the index level was at 120bps .
- The first three tranches are quoted as an upfront payment + 500bps running spread; hence, we have only provided the upfront amount. Note the upfront is negative for

the [6%-9%] tranche. The protection seller of this tranche makes an initial payment (represented as a negative value), since he receives more for the risk he takes by getting coupon payments of 500bps .

- The market data that provides the matrix of credit spreads is provided in Appendix G (Figures G.1 and G.2).

Compound correlation has attracted significant criticisms:

- Figure 4.2 shows that it is difficult to make an assessment of which correlation value one should use in order to price non-standard tranches. Figure 4.2 also shows the curve does not have monotonicity.
- Further, O’Kane and Livesey (2004) demonstrate that the method can lead to non-unique implied correlation values. This means more than one correlation value can solve for the market quoted prices.

Tranche	5y Market Price	Compound Correlation
0%-3%	31.50%	44.60%
3%-6%	0.375%	71.16%
6%-9%	-4.25%	5.11%
9%-12%	1.125%	11.48%
12%-22%	0.49%	23.26%

Table 4.1: Itraxx series 10 compound correlation values for 09 June 2009.

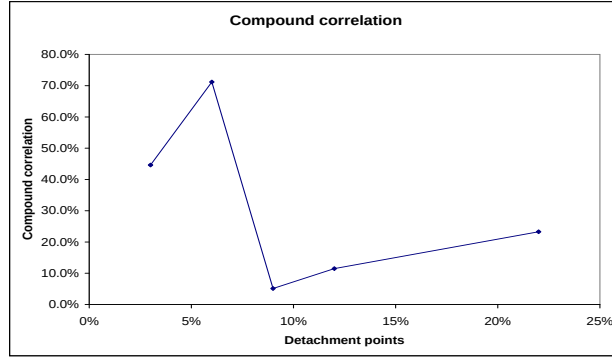


Figure 4.2: The compound correlation curve for data in Table 4.1.

4.5.3 Base correlation

The base correlation framework (BCF) provides a natural way of pricing non-standard tranches using a base correlation curve. Consider a tranche $[k_1, k_2]$, the BCF works by:

- Decomposing the tranche as the difference of two equity (base) tranches $[0, k_2]$ and $[0, k_1]$.
- Employing a bootstrapping procedure that recovers a set of correlation values corresponding to the set of base tranches, see O’Kane and Livesey (2004).
- The base correlation values recovered from the bootstrapping procedure form the ordinate values of the correlation curve. The abscissas of the curve are related to the base tranches, see Turc et al. (2006).
- Finally one must impose an interpolation and extrapolation regime to construct the curve. Turc et al. (2006) consider various interpolation techniques for the BCF and demonstrates that using piecewise linear or any other discontinuous interpolation method generates increased instances of arbitrage. Hence it is common to assume that some form of continuous interpolation regime is followed.

Assume that the standard tranches have the following set of detachment points, $\{k_1, k_2, \dots, k_s\}$, with $s \in \mathbb{N}$. The first traded tranche will be the equity tranche $[0, k_1]$, then $[k_1, k_2]$ is the next tranche and so on. The bootstrapping procedure is:

1. Find the value of ρ_1 so that the base tranche $[0, k_1]$ has zero MTM:

$$\overline{\mathbf{val}}_L^{tranche}(t, T, 0, k_1, \rho_1, s^{[0, k_1]}) = 0.$$

2. Find the value of ρ_2 so that the tranche $[k_1, k_2]$, which is decomposed as the difference of two base tranches ($[0, k_2]$ and $[0, k_1]$) has zero value:

$$\begin{aligned} \overline{\mathbf{val}}_L^{tranche}(t, T, k_1, k_2) &= \\ &= \frac{k_2}{k_2 - k_1} \overline{\mathbf{val}}_L^{tranche}(t, T, 0, k_2, \rho_2, s^{[k_1, k_2]}) \\ &\quad - \frac{k_1}{k_2 - k_1} \overline{\mathbf{val}}_L^{tranche}(t, T, 0, k_1, \rho_1, s^{[k_1, k_2]}) \\ &= 0, \end{aligned} \tag{4.5.1}$$

where the notation $\overline{\mathbf{val}}_L^{tranche}(t, T, k_1, k_2)$ is used to refer to the value of a tranche decomposed as the difference of two base tranches.

3. Continue until we find all the correlations $\rho_1, \dots, \rho_s$.

Equation (4.5.1) is an iterative bootstrapping method that enables us to recover the base correlation values that forms the ordinate values of the base correlation curve. For example, in the case of the Itraxx one can start with the equity tranche, $[0, 3\%]$, then solve for the correlation value for this tranche, we can call this ρ_1 . From this we move on to the junior mezzanine tranche, $[3\%, 6\%]$, using ρ_1 we can solve for ρ_2 . We can continue on in this way until we have recovered all the base correlation values.

There are various forms of base correlation curves used by market practitioners. The difference arise in the way the abscissas of the base correlation curve is defined and in the interpolation and extrapolation methods chosen. The specific curve we will use is referred to as the base correlation *loss ratio curve*, see Turc et al. (2006). We now detail this curve construction method:

- Let $\mathbf{k} = \{k_1, \dots, k_s\}$ be a set of base tranche detachment points which correspond to those observed in the market, where $s \in \mathbb{N}$ is the number of base tranches available. For example in the case of the Itraxx and CDX these base tranches will be $\{3\%, 6\%, 9\%, 12\%, 22\%\}$ and $\{3\%, 7\%, 10\%, 15\%, 30\%\}$ respectively.
- $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_s\}$ is the corresponding set of calibrated correlation values for each base tranche.

The correlation curve construction process is, for a fixed maturity $T > 0$:

1. Calculate the current value of the protection leg for the full portfolio. Call this $\overline{\mathbf{val}}_{\mathbf{L}}^{Prot}(Index)$.
2. Calculate the current value of the protection leg for a base tranche, $[0, k]$, using its respective base correlation. Call this $\overline{\mathbf{val}}_{\mathbf{L}}^{Prot}(k, \rho)$.
3. The loss ratio corresponding to the base detachment point k is defined as:

$$r = \frac{\overline{\mathbf{val}}_{\mathbf{L}}^{Prot}(k, \rho)}{\overline{\mathbf{val}}_{\mathbf{L}}^{Prot}(Index)}. \quad (4.5.2)$$

For each portfolio we can recover a set $\mathbf{r} = \{r_1, \dots, r_s\}$ which is the set of loss ratios corresponding to the base tranches $\mathbf{k} = \{k_1, \dots, k_s\}$.

We choose this form of abscissas as it accounts for spread dispersion (meaning that the curve constructed will vary according to the degree that credit spreads in a portfolio are dispersed³). In addition the loss ratio method accounts for the expected loss level of the index, a feature that not all curve constructions possess, see Turc et al. (2006).

The base correlation curve can finally be constructed by exogenously imposing an interpolation regime. The interpolation regime we follow is monotone cubic (see Hagan and West (2006)) and the extrapolation method is linear to the points (0%,0%) and (100%,100%). We have found that this is the most popular regime followed by practitioners, see Turc et al. (2006). Moreover as discussed the least arbitrage is created when a continuous interpolation regime is followed.

In Table 4.2 and Figure 4.3 we provide a numerical example. We again use data from 09 June 2009 and the 5y Itraxx index at the time. We can see from the results that there is monotonicity in the curve. This enables one to price non-standard tranches by using, as inputs, a tranches subordination and detachment points in order to extract correlation values from the constructed curve. The correlation values are then used to price the tranche (as the difference of two base tranches); this method is detailed in Appendix F (Section F.2). Later in Section 4.6 and 4.7 we will show that the BCF is prone to instances of arbitrage.

³Although Turc et al. (2006) highlight that it may account for spread dispersion in a counter-intuitive way.

Tranche	Loss Ratio	Base Correlation
0%-3%	36%	44.60%
3%-6%	52%	52.51%
6%-9%	62%	57.29%
9%-12%	67%	64.65%
12%-22%	74%	84.21%

Table 4.2: Itraxx series 10 loss ratios and base correlations for 09 June 2009. 5y index level at 120bps .

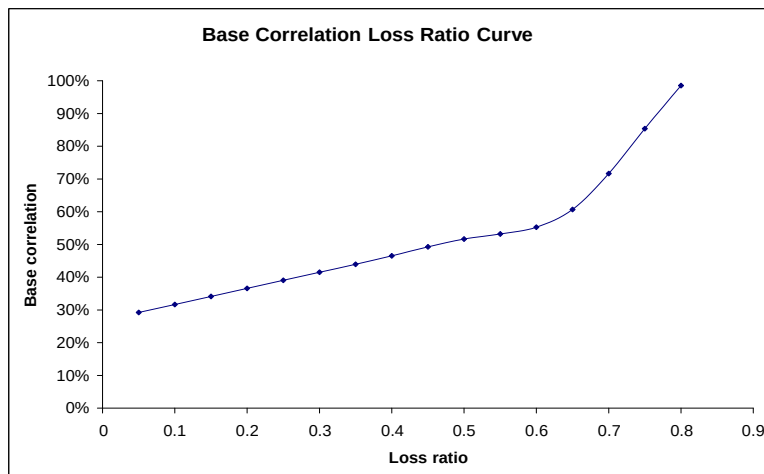


Figure 4.3: The base correlation loss ratio curve for data in Table 4.2 with monotonic cubic interpolation and linear extrapolation.

4.6 Random recovery

So far we have discussed the OFGC model in order to price CDOs. In this section we will detail an extension of the model. The model is the random recovery model; it extends the OFGC model to incorporate random recovery rates. Works by Andersen and Sidenius (2004), Hui (2009), Bennani and Maetz (2009) and others have discussed various versions

of the random recovery model.

We will demonstrate that when the model is used to construct a base correlation curve instead of the OFGC model then instances of arbitrage are reduced. The model is detailed in this thesis for the following reasons:

- Historically average recovery rates tend to be inversely related to default rates so that in bad economic environments not only are there more defaults but the actual recovery rates tend to be lower. In the literature this has been shown empirically by Acharya and Bharath (2003), Altman et al. (2003), and Hu and Perraudin (2002). These authors demonstrate that there is strong empirical support for there to be an inverse relationship between the number of defaults and the average recovery rate in a given period. The random recovery model captures this feature.
- Recent realised recovery rates have been highly dispersed e.g. Tribune (a large multi-media company) defaulted with a recovery rate of 1.5%, Cemex (a cement manufacturer in Mexico) defaulted with a recovery rate of 98% and the weighted average recovery rate over the one year period beginning July 2008 was around 10% compared with market assumptions of recovery rates being between 25% and 40% (see Bennani and Maetz (2009)). This empirical relationship affects the tail behaviour of portfolio loss distributions aggregating more loss to the senior tranches.
- The random recovery model can be calibrated to the CDX market. In most of 2009 with a constant market recovery assumption of 40% for the OFGC model it was not possible to get valid base correlation curves. This was because one was recovering implied correlation values that exceeded 100% for senior tranches, see Hui (2009). This was during the credit crisis when more value was being assigned to senior tranches (as the possibility of a huge systemic failure became more likely). The OFGC model could not re-price certain senior tranches even with a correlation value of 100%. The random recovery model overcomes this problem and preserves average market assumed recovery values; thereby ensuring the valuation of single obligor credit default swaps are unaffected.
- The random recovery model also allows us to obtain non zero prices for the super senior tranche ([60%, 100%]). This is not possible with the OFGC model because the model assumes the maximum loss of a CDO tranche is 60%, see Hui (2009).
- Finally by considering the OFGC and random recovery models we demonstrate in Section 4.7 that the new loss algorithm we devise can generate loss distributions for both models in an efficient way. This allows us to test for arbitrage in the models.

The root of the idea of the random recovery model is attributed to Andersen and Sidenius (2004) who model the recovery rate as being functionally dependent on the systemic factor, Z , in the factor model setting.

Model set up

We consider a two state version of the random recovery model. Assume we have n obligors and let τ_i be the default time of an obligor $i \in \{1, \dots, n\}$. Recall that in the OFGC model obligor i defaults by time $t > 0$ if an abstract random variable, X_i , representing the asset value of obligor i falls below some liquidity threshold, c_i :

$$\mathbb{I}_{\{\tau_i < t\}} = \mathbb{I}_{\{X_i < c_i\}} \quad \forall i \in \{1, \dots, n\}.$$

The asset value is expressed as:

$$X_i = \sqrt{\rho}Z + \sqrt{1 - \rho} \epsilon_i \quad \forall i \in \{1, \dots, n\}.$$

As in Equation (4.3.9) in Section 4.3, Z and ϵ_i , for $i \in \{1, \dots, n\}$, are both independent standard Gaussian random variables and ρ is the correlation parameter. The recovery rate given default of an obligor (recall all obligors are assumed to have the same random recovery rate) is given as a function of the common factor, Z :

$$\tilde{R}(Z) = \begin{cases} \alpha & \text{if } Z < \theta, \\ \beta & \text{if } Z \geq \theta, \end{cases} \quad (4.6.1)$$

where $\alpha, \beta \in [0, 1)$. Equation (4.6.1) tells us that when the market factor, Z , crosses some boundary, θ , the recovery rate will switch from α to β . In order for the random recovery model to be consistent with fixed recovery CDS pricing, we require the expected recovery value given default to match the fixed market recovery assumption denoted by $R^{mkt} \in [0, 1)$:

$$R^{mkt} = \mathbb{E}[\tilde{R}|\tau \leq t] = \mathbb{E}[\tilde{R}|X_i \leq c_i] = \alpha \mathbb{Q}(Z \leq \theta | X_i \leq c_i) + \beta(1 - \mathbb{Q}(Z \leq \theta | X_i \leq c_i)), \quad (4.6.2)$$

where $t \geq 0$ is the fixed time being considered and:

$$\mathbb{Q}(Z \leq \theta | X_i \leq c_i) = \frac{\mathbb{Q}(\{Z \leq \theta\} \cap \{X_i \leq c_i\})}{\mathbb{Q}(X_i \leq c_i)} = \frac{\Phi_2(\theta, c_i, \sqrt{\rho})}{\Phi(c_i)}, \quad (4.6.3)$$

with Φ_2 the bivariate normal distribution (see Hui (2009)). We impose that $\alpha < \beta$. By consideration of Equation (4.6.2), the parameter θ will be computed using a Newton-Raphson search algorithm.

We are not explicitly highlighting the dependence on the time horizon $t > 0$. However in any calibration scheme, ensuring that the model remains consistent with the CDS market for each time will mean that c_i (for $i \in \{1, \dots, n\}$) and therefore θ change with time (if α and β are fixed). Hence within this specification we have that the random recovery, $\tilde{R}$, depends on the event $\{\tau \leq t\}$ and the factor Z . So we may write $\tilde{R} = \tilde{R}(t, z)$ and say it is a deterministic function of t conditional on Z .

4.6.1 Numerical example

In this section we detail numerical results for the random recovery model. We again use data from 09 June 2009, the Itraxx as a portfolio and consider a maturity of 5 years:

- Table 4.3 provides for each tranche the loss ratio (which corresponds to the detachment point of the tranche), the compound correlation and base correlation using the random recovery model:
- Figures 4.4 and 4.5 are graphs of the compound correlation and base correlation curves under the random recovery model.
- Figure 4.6 is a comparison of the OFGC model and the random recovery model. In the figure we provide the fair swap spreads of tranchlets. Tranchlets are CDO tranches where the difference between the subordination and detachment point is 1%. The tranchlets are priced using both the OFGC and random recovery model. The correlation value we use are those implied from the base correlation curves of respectively the OFGC model and the random recovery model.
 - For the OFGC model at a 60% subordination point one gets zeros for the fair swap spread of a tranchlet, since the maximum loss of a CDO is 60% with the OFGC model and a 40% recovery assumption. This implies arbitrage exists in the OFGC model.
 - For the random recovery model we get non zero prices for tranchlets beyond 60% detachment point.
 - We notice that the fair swap spreads for tranchlets in the random recovery model are monotonically decreasing, this is not true for the OFGC model. What this means is that for the OFGC model one can enter into a tranche higher up the capital structure and have to pay more in spread than a tranche lower in the capital structure. This is an arbitrage opportunity.

Hence the base correlation framework under the OFGC model contains multiple instances of arbitrage. These instances of arbitrage are reduced with the base correlation framework using the random recovery model.

- The reason for the pricing difference between using the base correlation framework with the OFGC model as opposed to the random recovery model seems to be that the random recovery model produces a flatter base correlation curve (compare Figures 4.2 and 4.5). This is because the random recovery model introduces tail behaviour into a portfolio's loss distribution, aggregating more loss to senior tranches for any given correlation value. This then reduces instances where one has a negative or zero probability of experiencing losses in thin tranches (tranches where the difference between the detachment and subordination, called the *tranche width*, is small) higher up the capital structure, see Section 4.7.

Tranche	Loss Ratio	Compound Correl. Random Rec	Base Correl Random Rec.
0%-3%	36.40%	39.33%	39.33%
3%-6%	52.53%	59.73%	44.41%
6%-9%	62.55%	21.11%	46.23%
9%-12%	67.25%	0.84%	51.49%
12%-22%	74.66%	5.37%	67.91%

Table 4.3: Itraxx series 10 base correlations and compound correlations for 09 June 2009. With random recovery parameters $\alpha = 0$ and $\beta = 50\%$. Also displayed are the loss ratios.

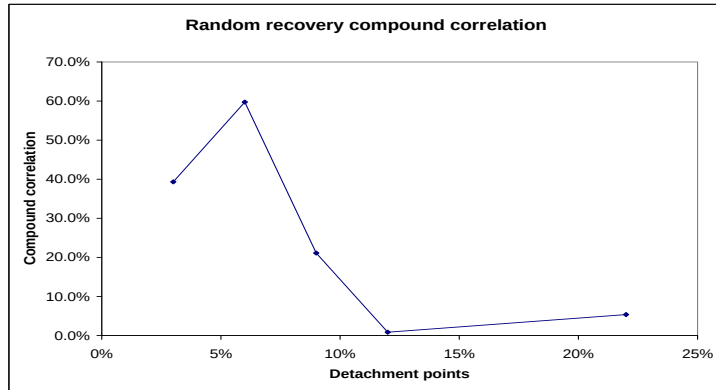


Figure 4.4: The compound correlation random recovery curve for data in Table 4.3.

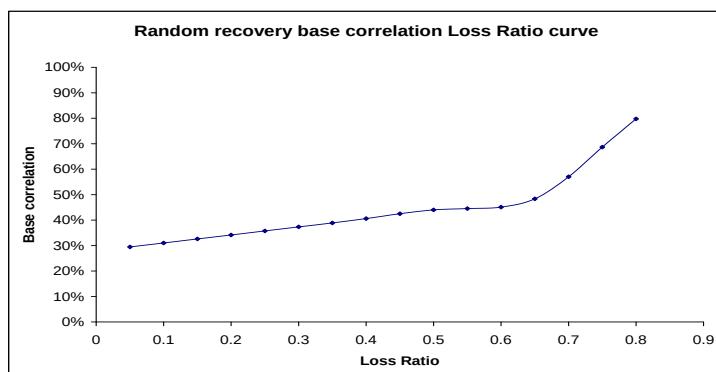


Figure 4.5: The base correlation random recovery loss ratio curve for data in Table 4.3 with monotonic cubic interpolation.

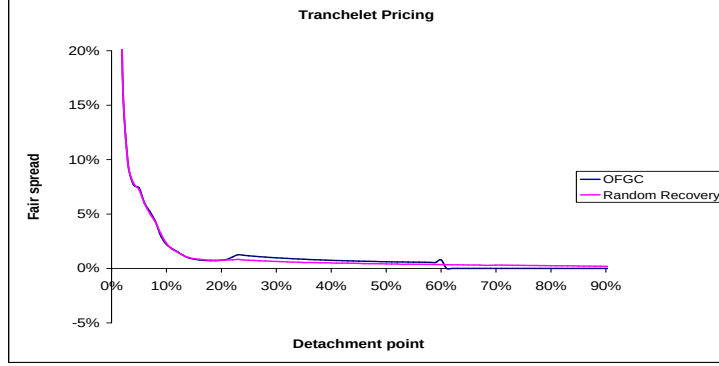


Figure 4.6: Tranchelet prices for Itraxx series 10.

4.6.2 Arbitrage in the random recovery

So far we have demonstrated that the specific form of the random recovery model detailed reduces instances of arbitrage, when compared to the OFGC model. In this section, for completeness, we show that random recovery models of this nature still generate arbitrage. In order to show this we provide a few definitions due to Bennani and Maetz (2009):

Definition 4.4. The spot recovery rate, $\tilde{r}(t)$, is the recovery rate paid if default happens at time $t > 0$:

$$\tilde{r}(t) = \tilde{r}(\tau)|_{\tau = t}.$$

The notation $\tilde{r}(\tau)|_{\tau = t}$ is used to mean the recovery rate conditional on default occurring at time t . In a factor model setting, this translates into the notion of the conditional spot recovery, $r(t, Z)$, which is:

$$\tilde{r}(t, Z) = \mathbb{E}[\tilde{r}(\tau)|_{\tau = t}, Z],$$

where Z is the common factor in the factor framework.

Definition 4.5. The recovery to maturity, $\tilde{R}(t)$, is the recovery rate paid if default happens before time $t > 0$:

$$\tilde{R}(t) = \tilde{R}(\tau)|_{\tau \leq t}.$$

The notation $\tilde{R}(\tau)|_{\tau \leq t}$ is used to mean the recovery rate conditional on default occurring before or at time t . In a factor model setting, this translates into the notion of the conditional

recovery to maturity, $R(t, Z)$, which is:

$$\tilde{R}(t, Z) = \mathbb{E}[\tilde{R}(\tau)|\tau \leq t, Z].$$

Note 4 (The random recovery to maturity model is arbitrageable). We have shown that the random recovery model presented in Section 4.6 (Equation (4.6.1)) depends only on the factor Z and $\{\tau \leq t\}$ (which means it is a random recovery to maturity model). Bennani and Maetz (2009) uses the following relationship between spot recovery and recovery to maturity to show that recovery to maturity models lead to arbitrage:

$$\int_0^t \tilde{r}(s, Z) dp(s, Z) = \tilde{R}(s, Z)p(s, Z),$$

with $p(s, Z)$ being the conditional probability of a default at time $s > 0$. This can be used to show that:

$$\tilde{r}(s, Z) = \tilde{R}(s, Z) + \tilde{R}'(s, Z) \frac{p(s, Z)}{p'(s, Z)}, \quad (4.6.4)$$

where the prime denotes differentiation with respect to the time variable. Requiring that $\tilde{r} \in [0, 1)$ produces a constraint on the recovery to maturity which is that:

$$0 \leq \tilde{R}(s, Z) + \tilde{R}'(s, Z) \frac{p(s, Z)}{p'(s, Z)} < 1. \quad (4.6.5)$$

Using this constraint Bennani and Maetz (2009) show that if, in a factor model, a recovery to maturity model depends only on $\{\tau \leq t\}$ and Z (as the random recovery model we have considered does) then the only valid recovery to maturity model is the one that is constant $R(t, Z) = R^{constant}$. This establishes that such models do produce and introduce arbitrage.

4.7 Generating a portfolio's loss distribution: a new loss algorithm

In this section we show how to get the market implied loss distribution. We will:

- Detail the current approach devised by Turc et al. (2004). Their approach uses a specific base correlation curve construction. From this base correlation curve the method extracts what the implied loss distribution is.
- Detail our new approach. Our approach is an algorithm. The method we derive does not depend on a specific form of the base correlation curve in order to work. In fact it can work with any model.

4.7.1 The current method of extracting the loss distribution

The constraint on the BCF, or indeed any model, is that to be coherent the model should conserve the equity tranche expected loss for base points $k \in (0, 1]$. By recalling Chapter 2 (Equation (2.3.1)), in the BCF this translates to:

$$\mathbb{E}[(L_t - k)^+] = \mathbb{E}[(L_t(\rho_t^{BCF}(k)) - k)^+].$$

This implies:

$$\mathbb{E}[L_t \mathbb{I}_{\{L_t < k\}}] + k[1 - \mathbb{Q}(L_t < k)] = \mathbb{E}[L_t(\rho_t^{BCF}(k)) \mathbb{I}_{\{L_t(\rho_t^{BCF}(k)) < k\}}] + k[1 - \mathbb{Q}(L_t(\rho_t^{BCF}(k)) < k)], \quad (4.7.1)$$

where $\rho_t^{BCF}(k)$ is the correlation under the base correlation framework at maturity $t > 0$ and detachment point k . By $L_t(\rho_t^{BCF}(k))$ we mean the loss process under the OFGC model using the correlation value ρ_t^{BCF} . The constraint in Equation (4.7.1) is what leads to Theorem 16, which is a statement of the method of Turc et al. (2004).

In order to construct the loss distribution, the method devised by Turc et al. (2004) relies on a particular form of the base correlation curve. The base correlation curve needed uses as abscissas the tranche detachment points. This is in contrast to the base correlation curve constructed in Section 4.5 where the abscissas were loss ratios. We now state the method in the following theorem:

Theorem 16. *The following holds:*

$$\mathbb{Q}(L_t < k) = \mathbb{Q}(L_t(\rho_t^{BCF}(k)) < k) - S_t(k, \rho_k) \times E_t(k, \rho_k),$$

where $S_t(k, \rho_k) = \frac{\partial \rho_t^{BCF}(k)}{\partial k}$ and $E_t(k, \rho_k) = \frac{\partial \mathbb{E}[(L_t(\rho_t^{BCF}(k)) - k)^+]}{\partial \rho_t^{BCF}(k)}$. Here $S_t(k, \rho_k)$ is the sensitivity of correlation to a change in k and $E_t(k, \rho_k)$ is the change in the tranche expected loss to a change in correlation.

Proof. Firstly one can note that the loss of an equity tranche with detachment point $k \in [0, 1]$ as a proportion of the portfolio is:

$$L_t - (L_t - k)^+ = \min(L_t, k).$$

Simplifying notation we can let the expected equity loss as a proportion of the portfolio be:

$$\bar{L}_k = \mathbb{E}[\min(L_t, k)].$$

Recall that the BCF must conserve the expected loss of an equity tranche. Using the integral representation, we can represent the expected loss of an equity tranche as:

$$\begin{aligned}
\bar{L}_k &= \int_0^1 \min(x, k) d\mathbb{Q}(L_t < x) \\
&= \int_0^k x d\mathbb{Q}(L_t < x) + k(1 - \mathbb{Q}(L_t < k)) \\
&= k - \int_0^k \mathbb{Q}(L_t < x) dx.
\end{aligned} \tag{4.7.2}$$

Equation (4.7.2) denotes calculation of the expected loss of a tranche using the real loss distribution (model independent). As we have conservation of the expected loss of an equity tranche we have similarly using the BCF that:

$$\bar{L}_k = k - \int_0^k \mathbb{Q}(L_t(\rho_t^{BCF}(x)) < x) dx. \tag{4.7.3}$$

On the one hand we have by differentiating Equation (4.7.2):

$$\frac{\partial \bar{L}_k}{\partial k} = 1 - \mathbb{Q}(L < k),$$

and on the other hand from differentiating Equation (4.7.3):

$$\frac{\partial \bar{L}_k}{\partial k} = 1 - \mathbb{Q}(L_t(\rho_t^{BCF}(k)) < k) - \frac{\partial \rho_t^{BCF}(k)}{\partial k} \int_0^k \frac{\partial \mathbb{Q}(L_t(\rho_t^{BCF}(x)) < x)}{\partial \rho_t^{BCF}(x)} dx.$$

We note that under the BCF the correlation value, $\rho_t^{BCF}(k)$, is a function of strike, k . By definition $\frac{\partial \rho_k}{\partial k} = S_t(k, \rho_k)$ and:

$$E_t(k, \rho_k) = \frac{\partial \bar{L}_k}{\partial \rho_k} = - \int_0^k \frac{\partial \mathbb{Q}(L_t(\rho_t^{BCF}(x)) < x)}{\partial \rho_t^{BCF}(x)} dx,$$

respectively. □

Theorem 16 describes the relationship between the loss distribution that comes from the OFGC model, using different correlation values (i.e. the base correlation framework), and the actual loss distribution of a portfolio. The theorem postulates that the loss distribution implied by the entire base correlation curve represents the real loss distribution of the portfolio. This is because the base correlation curve is constructed to re-price the traded tranches exactly (and therefore the expected losses of equity tranches). We make the following observations:

- The introduction of the quantities $S_t(k, \rho_k)$ and $E_t(k, \rho_k)$ into the theorem is simply a statement that the OFGC model cannot sufficiently account for the complex co-dependence structure that arises from traded tranches.

- Theorem 16 is useful if one wants to understand the constraints imposed on the base correlation curve in order to be arbitrage free as discussed in Turc et al. (2006). The theorem shows that if the base correlation curve is discontinuous around a point k then between k and $k + \epsilon$ (where $\epsilon > 0$) the slope ($S_t(k, \rho_k)$) jumps downwards. In this case the probability for the loss to be below $k + \epsilon$ may be less than it is to be below k . This shows that along with some of the analysis in Section 4.6 the BCF contains a number of arbitrageable situations.
- Theorem 16 relies on approximating varying differential quantities ($S_t(k, \rho_k)$ and $E_t(k, \rho_k)$) in order to work. With any interpolated curve differentiating the curve gives rise to significant errors (see Hagan and West (2006)).

4.7.2 The loss algorithm

To the best of our knowledge the following (approach/idea) is original. Let us consider a portfolio of n obligors. If the portfolio experiences defaults, denote the losses arising by $(k_j)_{j \in \{1, \dots, N\}}$. In a constant recovery framework it is trivial to set $k_j = j(1 - R)$ for $j \in \{1, \dots, N\}$ and $N = n$. For the random recovery case all possible combinations of losses depend upon the choice of the lower and higher recoveries. In our case we choose a state where the recovery could either be 50% or 0%. Hence the possible losses are 1 or 0.5 and therefore $k_j = \frac{j}{2}$ for $j \in \{1, \dots, N\}$ and $N = 2n$. We want to calculate the probability density for the loss of a portfolio to be $k_j \forall j \in \{1, \dots, N\}$, which will give the loss distribution:

Proposition 17 (The loss algorithm). *Assume all the possible losses of a portfolio are given by $(k_j)_{j \in \mathbb{N}}$. Let the density of losses be denoted by $\mathbb{Q}(L_t = k_j) = p_j$. The density of losses is given by:*

$$\begin{aligned}
 p_0 &= 1 - \frac{\bar{L}_{k_1}}{k_1} \\
 p_{i-1} &= 1 - \mathbb{I}_{\{i \geq 2\}} \sum_{j=0}^{i-2} p_j - \frac{\bar{L}_{k_i} - \bar{L}_{k_{i-1}}}{k_i - k_{i-1}}.
 \end{aligned}
 \tag{4.7.4}$$

where $\bar{L}_k = \mathbb{E}[\min(L_t, k)]$.

Proof. The proposition is established in steps:

Step(1) Using the same notation as Theorem 16, we have

$$\begin{aligned}
\bar{L}_{k_1} &= \mathbb{E}[\min(L_t, k_1)] \\
&= 0 \times \mathbb{Q}(L_t = 0) + k_1 \mathbb{Q}(L_t \geq k_1) \\
&= k_1(1 - p_0),
\end{aligned} \tag{4.7.5}$$

where $p_0 = \mathbb{Q}(L_t = 0)$ and by Equation (4.7.5) $p_0 = 1 - \frac{\bar{L}_{k_1}}{k_1}$.

Step(i) (Where $i > 1$) we can show:

$$\begin{aligned}
\bar{L}_{k_i} &= \mathbb{E}[\min(L_t, k_i)] \\
&= \mathbb{E}[L_t \mathbb{I}_{\{L_t < k_i\}}] + k_i \mathbb{Q}(L_t \geq k_i) \\
&= \mathbb{E}[L_t \mathbb{I}_{\{L_t < k_{i-1}\}}] + k_{i-1} p_{i-1} + k_i \mathbb{Q}(L_t \geq k_i),
\end{aligned}$$

where the last equality comes from recognising that:

$$\mathbb{E}[L_t \mathbb{I}_{\{L_t < k_i\}}] = \mathbb{E}[L_t \mathbb{I}_{\{L_t < k_{i-1}\}}] + \mathbb{E}[k_{i-1} \mathbb{I}_{\{L_t = k_{i-1}\}}].$$

Now this implies that:

$$\bar{L}_{k_i} = \bar{L}_{k_{i-1}} - k_{i-1} \mathbb{Q}(L_t \geq k_{i-1}) + k_{i-1} p_{i-1} + k_i \mathbb{Q}(L_t \geq k_i),$$

where $p_i = \mathbb{Q}(L_t = k_i)$ and $\mathbb{Q}(L_t \geq k_{i-1}) = 1 - \mathbb{I}_{\{i \geq 2\}} \sum_{j=0}^{i-2} p_j$ and $\mathbb{Q}(L_t \geq k_i) = \mathbb{Q}(L_t \geq k_{i-1}) - p_{i-1}$. Taken in total this gives:

$$\bar{L}_{k_i} = \bar{L}_{k_{i-1}} - (k_{i-1} - k_i) p_{i-1} + (k_i - k_{i-1}) (1 - \mathbb{I}_{\{i \geq 2\}} \sum_{j=0}^{i-2} p_j),$$

which implies the result:

$$p_{i-1} = 1 - \mathbb{I}_{\{i \geq 2\}} \sum_{j=0}^{i-2} p_j - \frac{\bar{L}_{k_i} - \bar{L}_{k_{i-1}}}{k_i - k_{i-1}}.$$

□

Proposition 17 is more general than Theorem 16 because it does not assume the existence of a base correlation curve, the only requirement to recovering the density of the loss distribution is to have the expected loss of equity tranches at the finite support points $(k_j)_{j \in \{1, \dots, N\}}$. Moreover there is no differentiation involved in the proposition, as opposed to Theorem 16. For these reasons we believe our method of constructing the loss distribution is more applicable.

4.7.3 Numerical example

In this sub-section we will apply Proposition 17 in order to generate the loss distribution implied by a base correlation curve arising from the OFGC model and the random recovery model respectively:

- Figure 4.7 shows the implied loss distribution (both density and cumulative) for the 5y Itraxx on 09 June 2009 using the base correlation curve given in Table 4.2. Note that the density becomes negative, highlighting the embedded arbitrage produced by our particular choice of base correlation curve and interpolation and extrapolation method.
- Figure 4.8 shows the implied loss distribution (both density and cumulative) for the 5y Itraxx on 09 June 2009 using the random recovery base correlation curve as prescribed in Table 4.3. We see the curve shape is much smoother than the loss distribution implied by the base correlation curve under the OFGC model, in Figure 4.7, and there are fewer negative density points.

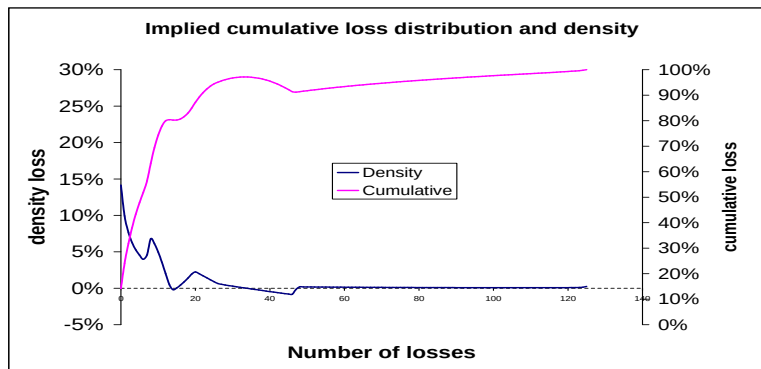


Figure 4.7: Implied loss distribution using the OFGC BCF.

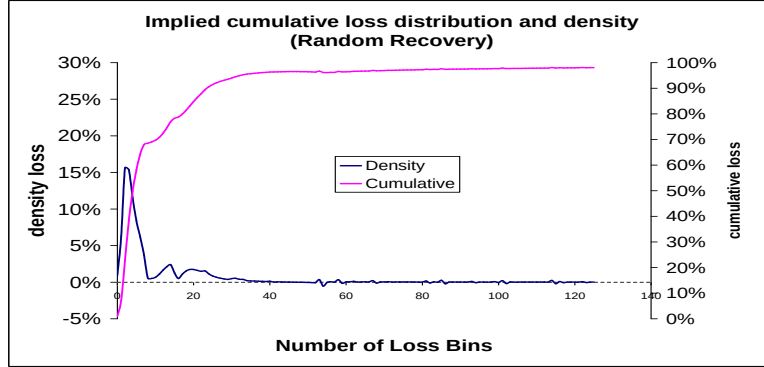


Figure 4.8: Implied loss distribution using the random recovery BCF.

4.8 Random factor loading

Consider Equation (4.3.8) in Section 4.3. Using the arguments of Andersen and Sidenius (2005), calibration to the CDO market requires manipulation of the parameters of quantities involved in Equation (4.3.8) so that we are close to the value of $\mathbb{E}[f(L_t)]$ implied by the market. Recall $f(L_t) = (L_t - k)^+$, for some $k \in (0, 1]$ and $(L_t)_{t \geq 0}$ is the credit portfolio loss process.

In this calibration exercise we typically consider, simultaneously, the fit of multiple CDOs, perhaps constrained to the same maturity, but attempting to re-price as much traded tranches as possible (see Andersen and Sidenius (2005)). To obtain a good fit under the factor framework one must look to develop the density function, $\psi(t, z)$, or the conditional survival function, $q_i(t, z)$, as defined in Equation (4.3.6). The OFGC model can change the function $q_i(t, z)$ only through a constant parameter, ρ . The degree to which this is a highly limited model comes through Theorem 16 or Proposition 17 and the fact that the model leads to a correlation curve. In this section we introduce a model with more free parameters in $q_i(t, z)$ in order to obtain better market fits.

Model setup

The Random Factor Loading (RFL) model attempts to extend the OFGC model yet retaining broadly the structure of the OFGC model. The idea is to take the parameter of the OFGC,

ρ , and make it a function of the systemic variable, Z . This notion is similar to that of local volatility in equities, where the volatility parameter is a function of the underlying. Empirically both ideas are well founded; in our case firms are more correlated in bad economic times than in good times, i.e. when Z is low ρ should be high and when Z is high ρ should be lower.

Hence the RFL model makes correlation functionally dependent on the systemic factor and it is suggested that correlation should be decreasing as a function of Z .

Following Andersen and Sidenius (2004), we start generally by defining for all obligors, $i \in \{1, \dots, n\}$:

$$X_i = \rho_i(Z) + v_i \epsilon_i + m_i, \quad (4.8.1)$$

where $v_i := \sqrt{1 - \mathbb{V}[\rho_i(Z)]}$ and $m_i := -\mathbb{E}[\rho_i(Z)]$ are chosen so that X_i has zero mean and unit variance. We note that:

$$\begin{aligned} v_i &= \sqrt{1 - \int_{\mathbb{R}^d} \rho_i(z)^2 \psi(z) dz + m_i^2}; \\ m_i &= - \int_{\mathbb{R}^d} \rho_i(z) \psi(z) dz; \\ p_i(z) &= G_i^\epsilon \left(\frac{c_i - \rho_i(z) - m_i}{v_i} \right), \end{aligned} \quad (4.8.2)$$

also we have:

$$\begin{aligned} p_i = \mathbb{Q}(\tau_i < t) = \mathbb{Q}(X_i < c_i) &= \mathbb{E} \left[\mathbb{Q}(\epsilon_i \leq \frac{c_i - \rho_i(z) - m_i}{v_i}) | Z \right] \\ &= \int_{\mathbb{R}^d} G_i^\epsilon \left(\frac{c_i - \rho_i(z) - m_i}{v_i} \right) d\psi(z)(z), \end{aligned} \quad (4.8.3)$$

where $\psi(z)$ is the distribution function for Z and we have suppressed dependence on time. G_i^ϵ is the cumulative distribution of ϵ_i . In this specific form of the RFL model we:

- Let Z and ϵ_i have the standard normal distribution.
- Make $\rho_i(Z)$ homogeneous and equal to $\rho(Z)$ for all $i \in \{1, \dots, n\}$.
- Take $\rho(Z)$ to be a two point distribution so that:

$$\rho(Z) = \begin{cases} \alpha Z & \text{if } Z < \theta, \\ \beta Z & \text{if } Z \geq \theta. \end{cases} \quad (4.8.4)$$

From our discussion we want $\beta < \alpha$ to reflect empirical observations. With these restrictions we recover the following:⁴

$$\begin{aligned} v &= \sqrt{1 - [\alpha^2(\Phi(\theta) - \theta\phi(\theta)) + \beta^2(\theta\phi(\theta) + 1 - \Phi(\theta)) - (\beta\phi(\theta) - \alpha\phi(\theta))^2]}, \\ m &= (\alpha\phi(\theta) - \beta\phi(\theta)), \\ p_i(Z) &= \Phi\left(\frac{c_i - \rho(Z) - m}{v}\right), \end{aligned}$$

where ϕ is the standard normal density. We also have that the default probabilities (suppressing dependence on obligors $i \in \{1, \dots, n\}$) are:

$$\begin{aligned} p &= \mathbb{Q}(\tau < t) \\ &= \mathbb{Q}(X < c) \\ &= \Phi_2\left(\frac{c - m}{\sqrt{(v^2 + \alpha^2)}}, \theta, \frac{\alpha}{\sqrt{(v^2 + \alpha^2)}}\right) + \Phi\left(\frac{c - m}{\sqrt{(v^2 + \beta^2)}}\right) - \Phi_2\left(\frac{c - m}{\sqrt{(v^2 + \beta^2)}}, \theta, \frac{\beta}{\sqrt{(v^2 + \beta^2)}}\right). \end{aligned} \tag{4.8.5}$$

Numerical example

The last equality in Equation (4.8.5) is a function which is not easily invertible, compared to the OFGC case where we needed to invert a standard Gaussian distribution. Here the route to recovering the thresholds, c , is to perform a root search. This can become time consuming as we do the inversion for each obligor and at multiple time points. For example suppose we have a portfolio of 125 obligors, a CDO with a maturity of 10 years and we wanted to discretise the protection leg according to the coupon payment dates. Then we would need to do $125 \times 10 \times 4 = 5000$ (where 4 is the coupon frequency for payments).

We can reduce this computational problem by exploiting the monotonicity of the default threshold c as a function of time, i.e. the default thresholds increase monotonically with time as default probabilities also increase. We can then generate default thresholds, c , for fewer points than required up until the trade maturity and apply some interpolation. From this we can find the other thresholds, c , for different times.

Figure 4.9 shows the implied loss distribution using the RFL model applied to index market data, Appendix G, for varying α and β with θ taken to be 0:

- Curve C1 corresponds to empirical observation, here $\alpha = 80\%$ and $\beta = 20\%$. By comparing the shape of the curve with that of Figure 4.7 in Section 4.7 curve C1 shows that the RFL model with such a parameterisation has a loss distribution which is qualitatively consistent with the existence of a base correlation curve.

⁴The case $\alpha = \beta$ recovers the OFGC.

- In curve C2 we take $\alpha = \beta = \sqrt{44.6\%}$. This reduces down to the OFGC model. Here 44.6% is the compound (and base) correlation of the equity tranche which we recovered in Section 4.5 (Table 4.1).
- Curve C3 is for completeness ($\alpha = 20\%$ and $\beta = 80\%$) and is counter-empirical. The curve is counter-empirical as it is constructed by assuming that in good economic times the default of obligors are more correlated than in bad economic times.

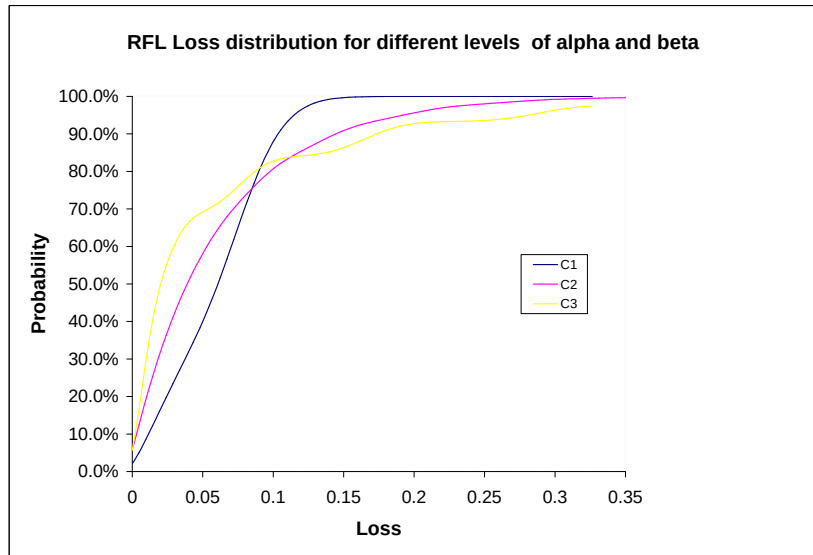


Figure 4.9: RFL loss distributions.

4.9 Chapter conclusion

In this chapter we have detailed methods of pricing multi-obligor products. Moreover we provide a new algorithm to generate a portfolio's loss distribution:

- We considered methods of generating default dependence. We provided a quantitative argument to show that correlating stochastic intensities that are driven by diffusion processes leads to a loss of control over spread dynamics.

- We demonstrated that under the conditionally independent framework one can have tractable results.
- We highlighted that the aim of multi-obligor models was to effectively account for the loss distribution of a credit portfolio and capture the dependence relationship between obligors.
- We reviewed the standard model under the factor framework, the one factor Gaussian copula (OFGC). Under this model it is necessary to construct correlation curves.
- This is because the OFGC model does not re-price all traded tranches. Correlation curves are used in conjunction with the OFGC model (but are exogenous to the model) in order to re-price all traded tranches.
- We considered the compound and base correlation curve construction approaches. The compound correlation method was shown to be weak. Base correlation was shown to be more useful. Nevertheless it contained significant instances of arbitrage.
- We considered the random recovery model, which is an extension to the OFGC model. The model contains economically relevant and empirically observed features and reduces instances of arbitrage in comparison to the OFGC model.
- We demonstrated how to generate the loss distribution of a portfolio using the base correlation curve. Two methods were provided:
 - The standard method developed by Turc et al. (2004).
 - The loss algorithm, which we develop.

We established that our method is more efficient and that it is more broadly applicable than the method devised by Turc et al. (2004).

- We reviewed the random factor loading model. An extension of the model is considered in Chapter 6.
- The loss algorithm enables us to generate a portfolio's loss distribution as implied by the base correlation curve. We argued that the loss distribution generated should be representative of the real loss distribution of the credit portfolio, since the base correlation curve allows for exact re-pricing of traded tranches.
- By constructing a portfolio's loss distribution we were able to show that the OFGC model produces arbitrage and that these instances of arbitrage are reduced with the random recovery model. Moreover from this work we were able to establish that the

RFL model qualitatively produces a loss distribution which is similar to the market implied loss distribution.

- The work done in this chapter is subsequently built upon in Chapters 5 and 6

Chapter 5

Spread Dynamics With Default Correlation

In this chapter we will examine methods of generating both spread dynamics and default dependence in order to value multi-obligor products that are influenced by spread dynamics.

Recall in Chapter 1 we provided an overview of the literature. In that chapter we discussed the development of single obligor models and multi-obligor models. We argued that:

- The literature for single obligor models has looked to capture spread dynamics.
- The literature for multi-obligor models has placed a focus on accounting for default dependence.

In this thesis, so far, we have examined methods of capturing spread dynamics and default dependence separately:

- In Chapter 3 we discussed the valuation of single obligor credit products. We demonstrated by valuing LCLNs without recourse the significant influence of spread dynamics. In that chapter the emphasis was on putting in place appropriate dynamics on the intensity process.
- In Chapter 4 we detailed methods of pricing multi-obligor products. We demonstrated the importance of being able to capture the distribution of losses in a portfolio and the large effects of default correlation. Arguments were provided to show the difficulty in extending the single obligor framework by correlating stochastic intensities.

- Moreover in Chapter 4 we demonstrated how dependence could be generated using the factor framework setting, under which a default time copula arises naturally. Under the factor framework, a factor (exogenous to the intensity process) is introduced in order to generate default dependence. In its classic setup the framework is a one period model and hence one cannot introduce spread dynamics easily into the framework.

5.0.1 Summary of sections in this chapter

This chapter is split into four sections:

- In Section 5.1 we detail how a dynamic multi-obligor modelling framework can be constructed when we have $n > 1$ obligors:
 - The model considers $\mathbb{F}$ – *adapted* continuous stochastic processes, $(\tilde{\lambda}_t^i)_{t \geq 0}$ with $i \in \{1, \dots, n\}$, which Schubert and Schönbucher (2001) call pseudo intensities of the obligors. Default dependence is generated by comparing the value of the integrated $\mathbb{F}$ – *adapted* pseudo intensities with exogenously generated correlated uniform variables, $\zeta_1, \dots, \zeta_n$.
 - We define two filtrations: the *partial filtration* and the *full filtration*. The former contains information about the default (or not) of only one obligor; the latter contains information about the default (or not) of all obligors.
 - Under the partial filtration $(\tilde{\lambda}_t^i)_{t \geq 0}$ is the intensity of obligor $i \in \{1, \dots, n\}$. Under the full filtration the intensity of obligor i is influenced by quantities related to the other obligors.
 - We define two copulas, the *survival copula* and the *threshold copula*. The survival copula is the copula function of default times and the threshold copula is the copula function of the exogenous uniform random variables, $\zeta_1, \dots, \zeta_n$, related to obligors $i \in \{1, \dots, n\}$. The survival copula is the conditional expectation of the threshold copula.
- In Section 5.2 we consider securitised loans. These are loans given to a borrower who is asked to provide collateral in order to mitigate losses that may occur because the borrower fails to pay:
 - The collateral provided by the borrower are debt instruments issued by a third party reference obligor. Therefore there is a need to account for default correlation between the borrower and the third party reference obligor.

- These types of agreements are often linked to a dynamic form of risk mitigation called *margin calling*. Margin calling requires the borrower to provide more collateral whenever the value of the collateral she had originally posted falls in value by a pre-specified amount. The impact of this is to ensure in a worsening market the lender has collateral of sufficient value to reduce any potential losses from a borrower default. This means we need to consider the dynamics of the price of the collateral.
 - Hence such products require that we account for default correlation as well as spread dynamics.
 - By an application of the dynamic modelling framework detailed in Section 5.1 we show how we can value secured loan products.
- In Section 5.3 we consider the valuation of leverage credit linked notes with recourse:
 - Recall in Chapter 3 (Section 3.4) we valued leverage credit linked notes without recourse. By considering recourse, where the issuer of an LCLN has recourse to the note investor (the counterparty), we need to account for default correlation.
 - Default correlation between the counterparty and the reference obligor (the obligor which the LCLN refers to) is important as on a default of the reference obligor the counterparty may have to make additional payments to the issuer.
 - Since we are valuing an LCLN which has an embedded American digital option, accounting for spread dynamics is also important.
 - Some significant conclusions are reached in the section. In particular we find that when volatility is high it acts like a dampener on default correlation reducing instances of joint defaults.
 - In Section 5.4 we provide a conclusion.

5.1 Framework

In this section we detail the dynamic copula modelling framework first developed by Schubert and Schönbucher (2001).

Assume we have n obligors and recall in Chapter 4 (Note 3) we demonstrated how a conditionally independent framework could be constructed. For obligors $i \in \{1, \dots, n\}$ we determined that default times could be constructed as:

$$\tau_i = \inf\{t \in \mathbb{R}_+ : \int_0^t \tilde{\lambda}_s^i ds \geq -\ln(\zeta_i)\}, \quad (5.1.1)$$

where ζ_i are random variables with uniform distribution on $[0, 1]$. In Chapter 4 (Note 3) we assumed these random variables were mutually independent, here we relax this assumption. Moreover in the models subsequently considered in Chapter 4 we assumed that $(\tilde{\lambda}_t^i)_{t \geq 0}$ were deterministic, in this chapter we take $(\tilde{\lambda}_t^i)_{t \geq 0}$ to be $\mathbb{F}$ – *adapted* continuous stochastic processes.

We will use the notation λ and $\tilde{\lambda}$ to denote intensities in the setting where we have full information on the default times of all obligors and the setting where we only have information on a specific obligor respectively. The process $(\tilde{\lambda}_t^i)_{t \geq 0}$ will be shown to be the intensity of obligor i in the setting where we only have information about the occurrence (or not) of default of obligor i . For this reason Schubert and Schönbucher (2001) call $(\tilde{\lambda}_t^i)_{t \geq 0}$ the *pseudo* intensity of obligor i . We will subsequently denote the intensity of obligors defined on the filtration that contains the default information of all obligors by $(\lambda_t^i)_{t \geq 0}$. We call this intensity the *full intensity*.

In this chapter the setup of Note 3 (Chapter 4) will enable us to generate spread dynamics using stochastic intensities. Although it is feasible to correlate the processes $(\tilde{\lambda}_t^i)_{t \geq 0}$ in order to induce some level of default correlation, we will not do this. Default dependence is generated by the joint distribution of $\zeta_i \forall i \in \{1, \dots, n\}$ using a copula function.

This setup works well because a dependence structure is completely characterised by a copula function. By Sklar's theorem (Theorem 15 in Chapter 4), the use of a copula function does not affect the marginal distribution of default times. Hence in this setup the calibration of single obligor intensities are not affected by the default dependence relationship between obligors as it is given by a copula. This is important as it allows for efficient multi-obligor modelling. In contrast, in Chapter 4 (Section 4.2), we showed that when one tries to describe the default dependence relationship between obligors by correlating intensities the calibration of the volatility parameter is affected.

5.1.1 The full intensity of an obligor

Recall the definition of the enlarged filtration:

$$\mathbb{G} = \mathbb{F} \vee \mathbb{D},$$

where $\mathbb{D} = (\mathcal{D}_t)_{t \geq 0}$, $\mathcal{D}_t = \vee_{i=1}^n \mathcal{D}_t^i$ and $\mathcal{D}_t^i$ is the smallest sigma-algebra containing information about the default of an obligor $i \in \{1, \dots, n\}$ at time $t \geq 0$. $\mathbb{G}$ contains information about the default of all obligors. We can also define a smaller filtration:

$$\tilde{\mathbb{G}} = \mathbb{F} \vee \mathbb{D}^i,$$

where $\mathbb{D}^i = (\mathcal{D}_t^i)_{t \geq 0}$. $\tilde{\mathbb{G}}$ contains information about the default of only obligor i .

- The filtration $\tilde{\mathbb{G}}$ is the one we used in Chapter 3, where only one obligor was considered.
- The filtration $\mathbb{G}$ is the one used in Chapter 4 and includes information on the default times of all obligors. We call this filtration the *full filtration*.

Under $\tilde{\mathbb{G}}$ we have established in Chapter 3 (Section 3.4.1) that the intensity of obligor i is the $\mathbb{F}$ – adapted process $(\tilde{\lambda}_t^i)_{t \geq 0}$ in Note 3 (Chapter 4).

We now consider the default intensity of obligor i under the full filtration, $\mathbb{G}$, where we have default information about all obligors.

Assumption 5.1. Let $\mathbf{U} = \{\zeta_1, \dots, \zeta_n\}$. Under $\mathbb{G}$, $\mathbf{U}$ is distributed with an n – dimensional copula which is twice differentiable. We call the copula function for $\mathbf{U}$ the threshold copula, denoted by C^T .

Recall $\mathbf{U}$ is independent of $\mathbb{F}$.

Theorem 18. Let $(\lambda_t^i)_{t \geq 0}$ be the intensity of obligor i under $\mathbb{G}$, then $\forall t \geq 0$, If no defaults have occurred, we have:

- The intensity of obligor i is:

$$\lambda_t^i = \tilde{\lambda}_t^i e^{-\int_0^t \tilde{\lambda}_s^i ds} \frac{\frac{\partial}{\partial x_i} C^T(x_1, \dots, x_n)}{C^T(x_1, \dots, x_n)}, \quad (5.1.2)$$

where $x_i = e^{-\int_0^t \tilde{\lambda}_s^i ds} \forall i \in \{1, \dots, n\}$.

- The dynamics of the intensity of obligor i is:

$$\frac{d\lambda_t^i}{\lambda_t^i} = \frac{d\tilde{\lambda}_t^i}{\tilde{\lambda}_t^i} + \left(\lambda_t^i \left(1 - \frac{C_{x_i x_i} C}{C_{x_i}^2} \right) - \tilde{\lambda}_t^i \right) dt - dN^i + \sum_{j=1, j \neq i}^n \left(\frac{C_{x_i x_j} C}{C_{x_i} C_{x_j}} - 1 \right) (dN^j - \lambda_t^j dt), \quad (5.1.3)$$

where $C = C^T(x_1, \dots, x_n)$, C_{x_i} denotes the partial derivative of C with respect to the i^{th} argument, $C_{x_i x_j}$ is the second order partial derivative with respect to the i^{th} and j^{th} argument, N^i is the point process of obligor i .

Proof. See propositions 4.3 and 4.7 in Schubert and Schönbucher (2001). □

Similar results can be found for the case where defaults have occurred, see Schubert and Schönbucher (2001). A few important points arise from Theorem 18:

- Under $\mathbb{G}$ the intensity of obligor i still depends on $(\tilde{\lambda}_t^i)_{t \geq 0}$. However, now, the additional information from $\zeta_j \forall j \in \{1, \dots, n\} \& j \neq i$ is incorporated.
- Consider the case of two obligors. Let us assume ζ_1 and ζ_2 (as given in Note 3 (Chapter 4)) have perfect positive dependence. If obligor 1 has not defaulted, then by Note 3 this implies that the realisation of ζ_1 was not high, as no default has been triggered e.g. $e^{-\int_0^t \tilde{\lambda}_s^1 ds} > \zeta_1$ holds. As a result of perfect positive correlation, we have that $\zeta_1 = \zeta_2$. Hence ζ_2 is also not high, meaning it is less likely obligor 2 has defaulted. This is the effect that comes through when the intensity of obligor i is considered in the full filtration: the conditional value of ζ_1 influences directly the full intensity of obligor 2.
- Let:

$$\pi_{ij} = \left(\frac{C_{x_i x_j} C}{C_{x_i} C_{x_j}} - 1 \right).$$

Schönbucher (2003a) shows that π_{ij} is positive if and only if there is locally positive dependence between obligors j and i . Hence in the case of positive dependence between obligors we have $\pi_{ij} > 0 \forall i, j \in \{1, \dots, n\}$. From Theorem 18 we can consider the influence of obligor j on obligor i , with $j \neq i$:

- If obligor j has not defaulted then $dN_j = 0$ and the only influence from obligor j is $-\pi_{ij} \lambda_t^j < 0$. Hence, in the case of no default of obligor j the intensity of obligor i reduces. This is what we expect.
- If obligor j has defaulted then $dN_j = 1$ and the influence from obligor j is to add $\pi_{ij} > 0$ to the intensity of obligor j . Again a desired feature.
- Therefore default contagion feeds naturally into this model setup. When there is positive correlation in the quantities $\zeta_i \forall i \in \{1, \dots, n\}$, then upon a default of an obligor, all other obligors' full intensities increase leading to greater chances of further defaults.

5.1.2 The survival and threshold copula

In this sub-section we make a connection between the threshold copula and a copula called *the survival copula*. Recall that the threshold copula was the copula that described the relationship between quantities $\zeta_i \forall i \in \{1, \dots, n\}$ and was denoted by C^T .

Now consider the conditional survival functions, $Q_i(t, t_i) \forall i \in \{1, \dots, n\}$, defined in Chapter 3 (Equation (3.3.3)) with $t_i \geq t$. The conditional joint survival function, S , of default times

$\tau_1, \dots, \tau_n$ can be denoted by:

$$S(t_1, \dots, t_n) = \mathbb{Q}(\tau_1 > t_1, \dots, \tau_n > t_n | \mathcal{F}_t). \quad (5.1.4)$$

By Sklar's theorem (Theorem 15) we have:

$$S(t_1, \dots, t_n) = C^S(Q_1(0, t_1), \dots, Q_n(0, t_n)). \quad (5.1.5)$$

C^S is called the survival copula. Jouanin et al. (2001) provide the following important result:

Proposition 19. *The relationship between C^T and C^S is given, $\forall t \geq 0$, by:*

$$C^S(Q_1(t, t_1), \dots, Q_n(t, t_n)) = \mathbb{E}[C^T(e^{-\int_t^{t_1} \tilde{\lambda}_s^1 ds}, \dots, e^{-\int_t^{t_n} \tilde{\lambda}_s^n ds}) | \mathcal{F}_t], \quad (5.1.6)$$

where the notation $(\tilde{\lambda}_t^i)_{t \geq 0}$ refers to the intensity of obligor i with respect to the filtration $\tilde{\mathbb{G}}$ that contains default information only about obligor i .

Proof. See Jouanin et al. (2001). □

We note that C^T and C^S are equivalent when intensities are deterministic, which is the case we considered in Chapter 4. The theorem tells us that the survival copula, which is the natural default time copula (see Andersen (2006)), can be generated as the conditional expectation of the threshold copula.

5.2 Securitised loans

In this section we present a method to price loans which are secured with obligations from a third party. We are concerned here with how effective certain types of collateral are in reducing the *counterparty risk* posed by a *borrower* in a loan agreement.

We will analyse and quantify how far default losses are reduced by collateralising the loan given to the borrower. We consider the way in which default correlation and the credit worthiness of the collateral affects the *post-default value of the collateralisation*. By post-default value of the collateralisation we mean the value of the collateral when the borrower has defaulted.

When an institution gives a loan, as *lender*, to another financial institution, as borrower, the lender is exposed to the potential default of the borrower. Without any collateral, the lender has outright counterparty risk. If the borrower defaults large losses are experienced by the lender. In order to mitigate this default risk, the lender requests collateral. In the event of a default of the borrower the lender can take possession of the collateral in order to offset against losses due to the failure of the borrower to fulfill its payment obligations.

The value of the collateral after the borrower has defaulted depends on a range of factors. The main factors are the quality (in terms of credit worthiness of the collateral) and the tendency of the collateral security and the counterparty to default together i.e. their default correlation.

5.2.1 Collateralisation

When a borrower defaults the lender recovers R times the loan amount, R is the recovery rate on default of the borrower.

As we have discussed, where there is concern about counterparty default risk, collateral will be requested by the lender from the borrower. When collateral in the form of a debt security has been provided, the financing transaction is called a repurchase order agreement (or *repo*). This is the transaction type we will consider. Recall that we call the obligor who has issued the debt security that forms the collateral the reference obligor.

The lender typically requests that the value of the collateral posted is more than the value of the loan given, see Krishnamurthy et al. (2010). In this case the borrower is said to take a haircut:

Definition 5.1 (Haircut). *Let the loan amount be X and let the value of the collateral be V ; the haircut is defined as the percentage given by $\frac{V-X}{V}$.*

For example if the loan amount is \$80m and the collateral value is \$100m then the haircut is 20%.

Often the collateral agreement is dynamic, so that the amount of collateral posted changes with the changes in value of the collateral, this is achieved through a process called *margin calling*. Margin calls require borrowers to post more collateral as the value of the original collateral posted falls¹. This process reduces risk for the lender as in a worsening market scenario he is being given additional collateral to offset against potential losses due to the default of the borrower.

Margin calls ensure that the ratio of the outstanding loan value to the value of collateral stays roughly constant throughout the life of the financing. Margin calls generally occur weekly or monthly and in some cases daily (see Choudry (2006)). Margin will be called if the value of the collateral posted falls below a pre-agreed *threshold*.

¹The borrower may receive collateral if the collateral value appreciates.

5.2.2 Documentation

Repo agreements are transacted under the *Global Master Repo Agreement* (GMRA). The document allows, in the case where a borrower has defaulted:

- For the lender to take possession of any posted collateral and liquidate the collateral. The lender is entitled to keep from the liquidation proceeds an amount equal to that which is due and payable by the borrower.
- If the proceeds from the collateral does not cover the due amount the lender can also claim the difference from the defaulted borrower. The lender's claim will rank as *senior unsecured debt* in the capital structure.

5.2.3 Risk characteristics

In a repo transaction, a lender has two main risks:

1. The first occurs when there is a slow deterioration of both the borrower and the underlying collateral which eventually leads to a default of the borrower and or the reference obligor. In this scenario margin calling should have helped recoup significant portions of any losses. The post-default value of the collateral should be similar to its pre-default value since the deterioration was slow and the subsequent default eventually expected.
2. The second, and by far the greater risk, is when there is a sudden joint default of both the borrower and the reference obligor. In this scenario the lender has not had time to call any margin and is exposed to potentially significant losses.

Other scenarios such as a default of the borrower and no default or significant impairment of the reference obligor may produce losses but, in general, not significant ones relative to the above two.

A few factors should be considered when evaluating repos. In the following we list these factors and the way in which we capture them:

- Credit quality of the borrower. We use the CDS curve in the market for the borrower.
- Credit quality of the reference obligor. We assume the reference obligor has a tradeable CDS curve and proxy MTM changes in the collateral by corresponding MTM changes in the CDS. Doing this induces a basis risk, since debt securities do not move

perfectly with CDS movements. The MTM changes form the test to assess whether a change in value of the collateral has crossed the threshold to call margin.

- Recovery rate of the borrower. We take those assumed in the market (usually 25% or 40%).
- Recovery rate of the reference obligor. We take those assumed in the market (usually 25% or 40%).
- Default correlation between the borrower and the reference obligor. This can be proxied by the implied correlation from NTD data.
- Order of defaults. We can simulate default times via Monte Carlo, Note 3 (Chapter 4) and Section 5.1 establishes how this is achieved.
- The volatility structure of the reference obligor (the ability to call margin can be viewed like an option for the lender and will be affected by the price volatility of the collateral). Data from swaptions can be used here.

To conclude, in order to analyse counterparty risk we can simulate the intensity of the borrower and the intensity of the reference obligor. As marginal distributions do not influence the joint distribution of default times, under the copula approach, we can calibrate the individual intensities separately. The framework detailed in Section 5.1 prescribes how this can be done in a consistent way using the threshold copula, C^T .

5.2.4 The pricing method and model

In this section we consider a specific model within the framework in Section 5.1:

- There are 2 obligors: the counterparty denoted with the subscript C and a reference obligor, who issues the collateral security, denoted with the subscript R .
- The intensities $(\tilde{\lambda}_t^i)_{t \geq 0}$, for $i \in \{C, R\}$, as defined in Section 5.1 will be driven by an extended Vasicek model as in Chapter 3 (Section 3.3).
- The threshold copula, C^T , as defined in Section 5.1 will have a joint Gaussian distribution. In the case of a Gaussian copula the joint distribution becomes:

$$C^T(\zeta_C, \zeta_R) = \Phi_2(Z_C, Z_R; \rho), \quad (5.2.1)$$

where Z_C and Z_R are standard normal random variables, ρ is the correlation parameter and $\zeta_i = \Phi^{-1}(Z_i)$ for $i \in \{C, R\}$.

We will simulate the intensity process of both the counterparty and the reference obligor on arbitrary time intervals $t_1, \dots, t_M$, with $t_M = T$ the maturity of the repo transaction.

We will assume that:

- Without loss of generality the price at inception of the collateral is 100% (its face value) and the haircut taken is $H \in [0, 1)$.
- All margin calls are done with the posting (or drawing) of cash by the counterparty.
- The counterparty has borrowed \$1 in the repo transaction.
- Let $(\mathbf{Call}(t))_{t \geq 0}$ stand for the total amount of cash posted (a positive value) or drawn (a negative value) by the counterparty in a repo transaction. The value of $(\mathbf{Call}(t))_{t \geq 0}$ is determined by the value of the collateral, $(V_t)_{t \geq 0}$, and the level of the threshold, $\mathbf{K}$. Let $t > 0$ be a time when the value of the collateral changes by an amount equal to $\mathbf{K}$. At this time the counterparty will post cash in an amount equal to $\mathbf{K}$ increasing $\mathbf{Call}(t)$ if the the value of the collateral has fallen (respectively receive cash in an amount equal to $\mathbf{K}$ reducing $\mathbf{Call}(t)$ if the value of the collateral has risen). This process will happen for each additional fall (rise) in value of the collateral equal to $\mathbf{K}$.
- $(\mathbf{Call}(t))_{t \geq 0}$ can be positive or negative. We will assume that $(\mathbf{Call}(t))_{t \geq 0}$ is updated only at the times $t_1, \dots, t_M$.
- Let R_R and R_C be the recovery rates of the reference obligor and the counterparty respectively.

Now, applying Note 3 (Chapter 4) we simulate correlated ζ_R and ζ_C which are the default thresholds for the reference obligor and the counterparty respectively.

We are interested in accounting for the default leg of a repo transaction. In this instance the default leg quantifies the losses incurred by the lender. We can then solve for a spread (above Libor) that makes the premium leg² (paid according to a frequency) equal to the default leg. The following events specify the payoff profile of the default leg of a repo transaction with margin calling:

Event A : No defaults on dates $t_1, \dots, t_M$; the borrower pays back \$1 at maturity of the repo.

Event B : If at any of the times $t_1, \dots, t_M$ there is deemed to be a default of the reference obligor and no default of the counterparty, we assume no losses. The lender is repaid

²Note that the premium leg pays coupons until the counterparty defaults or the repo transaction ends.

\$1 at the default time, $\tau_R \in \{t_1, \dots, t_M\}$, of the reference obligor and the transaction ends.

Event C : At time $t_i \in \{t_1, \dots, t_M\}$ the counterparty experiences his first default but the reference obligor does not. In this case the lender takes possession of the collateral and attempts to recover as much value as possible from the collateral. This will have a value to the lender of:

$$\min(1, V_{t_i} + \mathbf{Call}(t_i)).$$

If the lender has not recovered \$1 he can seek from the borrower the residual amount owed. This is:

$$\max(0, 1 - V_{t_i} - \mathbf{Call}(t_i)) \times R_C.$$

The total cash flow under this event is:

$$\min(1, V_{t_i} + \mathbf{Call}(t_i)) + \max(0, 1 - V_{t_i} - \mathbf{Call}(t_i)) \times R_C. \quad (5.2.2)$$

Event D : At time $t_i \in \{t_1, \dots, t_M\}$ the counterparty and the reference obligor experience their first defaults. In this case the lender takes possession of the collateral and recovers:

$$\min(1, \frac{R_R}{1 - H} + \mathbf{Call}(t_i)),$$

if after liquidating the collateral and accounting for the margin called the lender has not recovered his due and payable amount, which is \$1, he can go after the counterparty for the difference, which will have payoff:

$$\max(0, 1 - \frac{R_R}{1 - H} - \mathbf{Call}(t_i)) \times R_C.$$

The total cash flow under this event is:

$$\min(1, \frac{R_R}{1 - H} + \mathbf{Call}(t_i)) + \max(0, 1 - \frac{R_R}{1 - H} - \mathbf{Call}(t_i)) \times R_C.$$

As discussed events A to D constitute the cash flow for the default leg of a repo transaction. From this we can solve for a spread, r , which equates the premium leg to the default leg. r can be viewed as the risk premium above Libor that a lender would take in order to lend to a borrower for the specific collateral the borrower provides. Since the borrower is providing collateral, the lender is taking less risk than if he were to give an *unsecured* loan to the borrower. Therefore, for any given tenor, r will be less than (or equal to) the fair CDS swap spread of the borrower in the market. We call r the *repo spread*.

5.2.5 Numerical example

In this section we present numerical results for the valuation of repo transactions:

- Table 5.1 provides the CDS swap spread curves for Goldman Sachs (the counterparty) and Gazprom (the reference obligor). We assume that Goldman Sachs is a counterparty who posts a bond issued by Gazprom, a Russian Oil and Gas company as collateral.
- Figure 5.1 shows that as the default correlation between the asset and the counterparty increases, so does the required repo spread. The repo spread increases to compensate the lender for assuming greater joint default risk. The figure is produced assuming a 25% recovery rate for both the reference obligor and the counterparty, a repo maturity of 1yr, a collateral maturity of 5yrs, a threshold of 2% and a log-normal volatility of 10% for both the reference obligor and the counterparty. Market spread data is from Table 5.1.
- Figure 5.2 shows the change in the repo spread as the margin calling threshold changes. We see that the higher the threshold level the more the lender requires in compensation, with increased repo spreads. This relationship exists because the value of the collateral must change by an amount greater than the threshold in order for the lender to make a margin call. When the threshold is large the value of the collateral must change significantly before a margin call can be made. This means the scenario of a sudden joint default without any posted margin by the borrower is more likely. The figure is produced assuming a 25% recovery for both the reference obligor and the counterparty, a repo maturity of 1yr, a collateral maturity of 5yrs, a default correlation of 50% and a log-normal volatility of 10% for both the counterparty and the reference obligor. Market swap spread data is from Table 5.1.
- Figure 5.3 shows the sensitivity of the repo spread against changes in volatility. Here we have marked the changes against the log-normal Black and Scholes volatility. In order to do this for each log-normal volatility we assume that all credit default swaptions trade with a fixed log-normal volatility. From this we imply the value of the credit default swaption; using these credit default swaption values we calibrate the extended Vasicek model. Here we have calibrated the extended Vasicek model using ATM 6m5y and ATM 1y5y credit default swaptions. The figure is produced assuming a 25% recovery for both the reference obligor and the counterparty, a repo maturity of 1yr and a collateral maturity of 5yrs. We plot three graphs for three different

correlation values (5%, 50% and 95%). Market spread data is from Table 5.1.

Entities	6m	1y	2y	3y	5y	7y	10y
Goldman Sachs	341bps	344bps	312bps	301bps	285bps	257bps	237bps
Gazprom	1361bps	1376bps	1260bps	1139bps	1024bps	943bps	874bps

Table 5.1: Goldman Sachs and Gazprom CDS swap spreads. Data source from *MarkIt*, February 2009

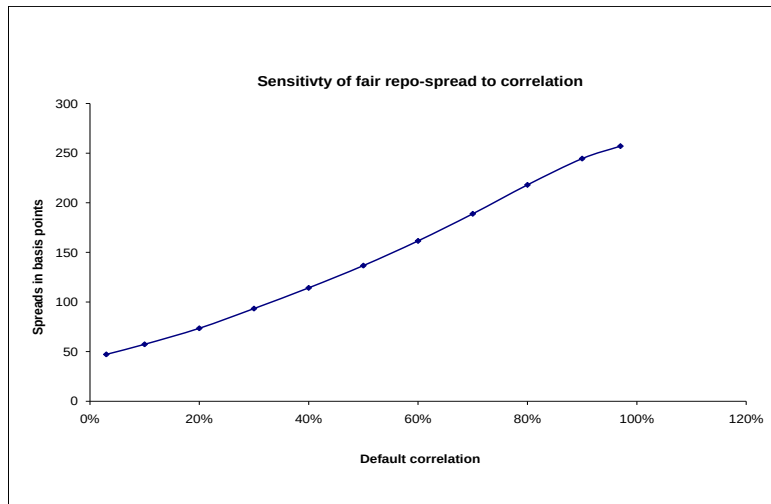


Figure 5.1: Sensitivity of the repo spread to changes in volatility

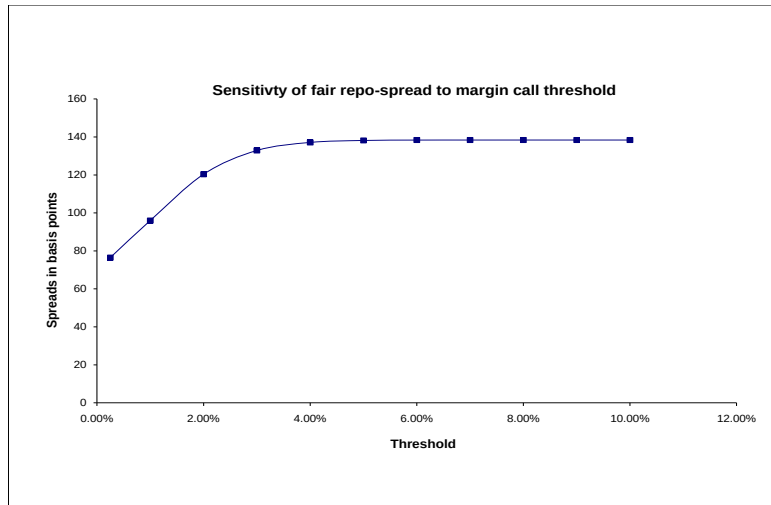


Figure 5.2: Sensitivity of the repo spread to changes in the margin calling threshold.

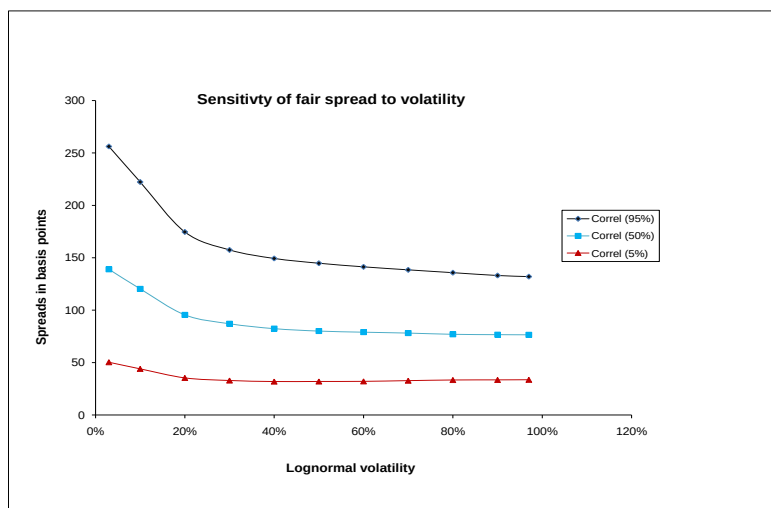


Figure 5.3: Sensitivity of the repo spread to changes in volatility for various default correlation values.

5.2.6 Remarks

To conclude, the modelling setup and numerical implementation we have considered is flexible and accounts for the risk factors highlighted in Sub-section 5.2.3:

- We have shown how to account for margin calling and the frequency in which it is called.
- We have accounted for haircuts and the volatility of the collateral.
- We have taken into account the credit quality of both the reference obligor and the counterparty.
- We have incorporated default correlation and accounted for the payoffs that occur on a counterparty default or on a joint default of both the counterparty and the reference obligor.
- This development is in contrast to other works such as Ghosh et al. (2008). Ghosh et al. (2008) do make use of a Gaussian copula setup in order to correlate default times, yet they do not incorporate spread dynamics and hence do not capture the value of margin calling.

5.3 Leverage with recourse credit linked notes

Before the credit crisis, that occurred towards the end of 2008, interest had developed in methods to extract leverage against capital invested in the credit markets (see Scott-Quinn and Walmsley (1998)).

A good example of such products are leveraged with recourse credit linked notes defined in Chapter 2 (Sub-section 2.4.1). The following analysis of LCLNs with recourse is new and sheds light on the influence counterparty default correlation (with a reference obligor) on the level of leverage an issuer of an LCLN is willing to provide.

The product requires:

- A description of the default dependence between the counterparty and the reference obligor.
- An assessment of the embedded optionality in the product.

We apply the threshold approach, again using the extended Vasicek model as the model for the driving intensity, and the threshold copula, C^T , having the one factor Gaussian copula structure.

5.3.1 The pricing method and model

In this section we consider a similar modelling setup to that in Section 5.2:

- There are 2 obligors: the counterparty denoted with the subscript C and a reference obligor, which the LCLN contract references, denoted with the subscript R .
- The intensities $(\tilde{\lambda}_t^i)_{t \geq 0}$, for $i \in \{C, R\}$, as defined in Section 5.1, will be driven by an extended Vasicek model as in Chapter 3 (Section 3.3).
- The threshold copula, C^T , as defined in Section 5.1 will again have a bivariate Gaussian distribution.

Following similar steps to the pricing methods in Chapter 3 (Section 3.4) and Section 5.2 we will simulate the intensity process of both the counterparty and the reference obligor on arbitrary time intervals $t_1, \dots, t_M$, with $t_M = T$ (the maturity of the LCLN transaction). We can let:

- $D_{t_j}^R$ and $D_{t_j}^C$ $j \in 1, \dots, M$ be the default indicators at time t_j for respectively the reference obligor and the counterparty.
- We also simulate correlated (under the Gaussian copula) ζ_R and ζ_C , which are the default thresholds for the underlying and the counterparty respectively.
- At each discretisation point we test whether the conditions:

$$\Gamma_{t_j}^R \geq -\ln(\zeta_R) \text{ or } \Gamma_{t_j}^C \geq -\ln(\zeta_C) \quad \forall j \in \{1, \dots, M\},$$

hold. $(\Gamma_t^i)_{t \geq 0}$ for $i \in \{R, C\}$ is the integrated hazard rate process. If the first time any one of the conditions hold is at time t_j we know a default has occurred some time in between t_{j-1} and $t_j \quad \forall j \in \{1, \dots, M\}$. In this case $D_{t_j}^i = 1$, where $i \in \{R, C\}$.

- If no defaults have occurred, we test the further condition of $\tilde{s}^R(t_j, T) > k$ (recalling $\tilde{s}^R(t_j, T) = \frac{\tilde{\mathbf{D}}^R(t_j, T)}{\tilde{\mathbf{P}}^R(t_j, T)}$ is the fair CDS swap spread of the reference obligor³).

Under each of the scenarios, default or trigger breach, the trade ends with no more coupons to be paid with the following payoff profile:

Event A : No defaults or trigger breaches on dates $t_1, \dots, t_M$.

³We could additionally place a trigger on the counterparty's spread level; but do not here.

Event B : A default occurs before the trigger is breached. This has a payoff of:

$$F \times (1 - R),$$

where F is the leverage factor and R is the recovery rate of the reference obligor. Note, however, that the issuer of the LCLN (who is the protection buyer) is guaranteed only the principal and is exposed to the risk that the counterparty will not pay the net difference of the principal, \$1, and $F \times (1 - R)$.

Event C : A trigger breach occurs before any default at time $t_j : j \in 1, \dots, M$. This has payoff:

$$F \times (\tilde{s}(t_j, T) - \tilde{s}(0, T))\tilde{\mathbf{P}}(t_j, T). \quad (5.3.1)$$

Again the issuer is guaranteed only the principal and is exposed to the risk that the counterparty will not pay the net difference of the principal, \$1, and $F \times (\tilde{s}(t_j, T) - \tilde{s}(0, T))\tilde{\mathbf{P}}(t_j, T)$.

Events A to C constitute the only events that can occur on the default leg. From this we can solve for a spread, s^* , which equates the premium leg to the default leg. s^* is the fair spread above Libor of the LCLN.

5.3.2 Numerical Example

In this numerical example we take a reference obligor and a counterparty that both have constant CDS spreads of 400bps across maturity. In addition we assume they have market recoveries of 25% each.

Example 5 (LCLN with recourse). *The specific trade we consider has the following features:*

1. Maturity (T) = 3y.
2. Notional (N) = USD 10,000,000
3. $k = 1000\text{bps}$
4. $\tilde{s}^R(0, T) = 400\text{bps}$ (The initial swap spread of the reference obligor).
5. $s^C(0, T) = 400\text{bps}$ (The initial swap spread of the counterparty).
6. $F = 5$.
7. Reference Notional = $5 \times N = \text{USD } 50,000,000$

Table 5.2 and Figure 5.4 shows an important relationship between correlation and volatility:

- The table and figure provide values for the fair LCLN spread for various spot volatilities, speed of mean reversion and correlations.
- Note that LCLN fair spread levels above 20% are produced. At first this seems wrong since one may expect the maximum the fair spread can be is 5 times the fair CDS swap spread which is $5 \times 4\% = 20\%$. However this can be explained in the following manner:
 - Recall that the fair swap spread of a CDS is $\tilde{s}(t, T) = \frac{\tilde{\mathbf{D}}(t, T)}{\tilde{\mathbf{P}}(t, T)}$.
 - With an LCLN the value of the default leg, call it $\tilde{\mathbf{D}}^{lev}(t, T)$, is unlikely to be more than 5 times (the leverage factor) of the un-leveraged CDS default leg.
 - Yet the premium leg of a CDS (which is its risky duration), $\tilde{\mathbf{P}}(t, T)$, is larger than the premium leg of an equivalent maturity LCLN, call it $\tilde{\mathbf{P}}^{lev}(t, T)$.
 - The reason for this is that the premium leg of a CDS is only extinguished upon default, whereas the premium leg of an LCLN is extinguished upon a default event or a trigger event. Hence the trigger event is effectively creating an additional dampening impact to the risky duration of an LCLN making it less than that of a comparable CDS.
 - This gives rise to the possibility of the fair margin being more than 20%.
- Consider Figure 5.4 and the graph with 90% default correlation. We note that as volatility increases the fair LCLN spread increases up to 5 times of the fair CDS swap spread (which is 400bps).
- Hence the effect of volatility seems to be like a de-correlator. This can be understood from the following perspective: when two obligors are highly correlated if we add volatility it creates spread dispersion. This makes the event of joint defaults less likely, which subsequently means the counterparty can receive a higher spread premium.
- There also seems to be little or no variation in fair LCLN spreads for correlations below 60% (we see a variation in premium from 17% to 23%). Again this seems to be because of the de-correlating effects of volatility.

Spot volatility (σ)	Correlation			
	20%	40%	60%	90%
0%	19.30%	18.37%	16.99%	8.04%
1%	19.57%	18.76%	17.51%	10.71%
2%	19.62%	19.01%	17.90%	12.94%
3%	19.74%	19.71%	18.85%	15.19%
4%	20.33%	20.73%	20.16%	16.79%
5%	20.97%	20.87%	20.97%	18.72%
6%	21.59%	21.96%	21.25%	20.76%
7%	22.51%	22.29%	22.10%	22.02%
8%	22.58%	22.57%	22.62%	21.77%

Table 5.2: Leverage with recourse spread premium above the bench mark interest rate, Libor, as a function of varying correlation and extended Vasicek spot volatilities with speed of mean reversion fixed at 0. We use a hypothetical flat CDS spread of 400bps for both counterparty and underlying.

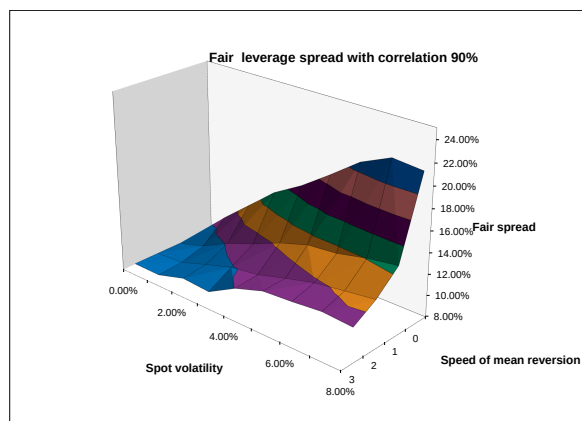
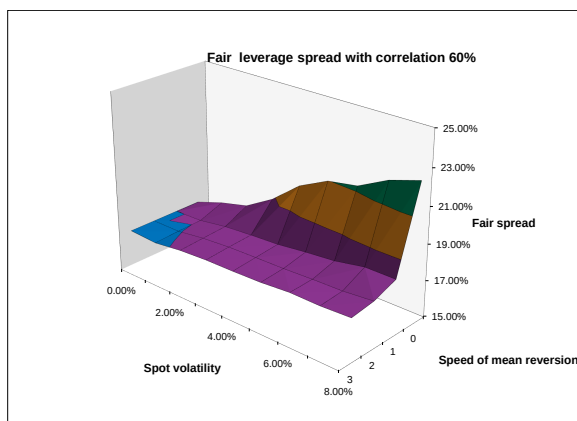
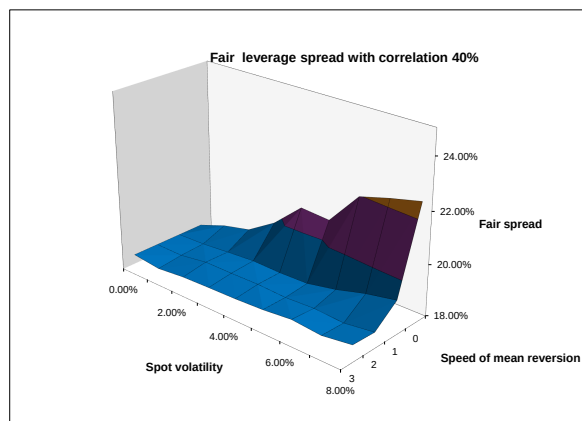
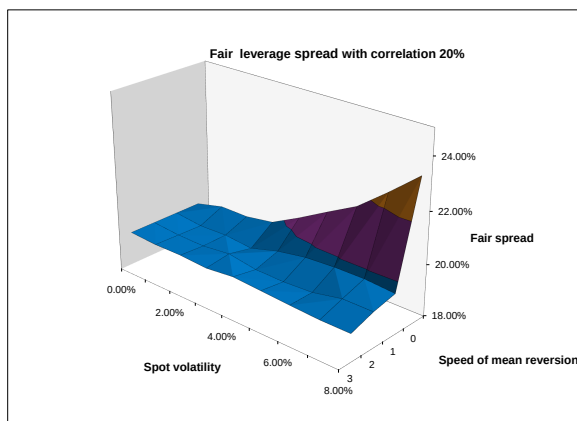


Figure 5.4: Leverage with recourse spread premium above the bench mark interest rate, Libor, as a function of varying speed of mean reversion and extended Vasicek spot volatilities with correlation fixed at 20%, 40%, 60% and 90% respectively.

5.4 Chapter conclusion

In this chapter we have considered a framework that contains both default dependence and spread dynamics:

- The modelling framework is efficient because it allows one to calibrate separately the dynamics of individual obligors and the dependence between obligors.
- This means the work done in Chapter 3 can be directly applied. In that chapter we constructed a calibration routine for an intensity process driven by the extended Vasicek model.
- We considered two products:
 - Securitised loans.
 - Leverage credit linked notes.

Both products require we account for default correlation and credit spread dynamics.

- We have found:
 - Higher default correlation increases counterparty risk.
 - Embedding thresholds in products that exhibit counterparty risk is important. Thresholds enable margin to be called whenever collateral posted by a counterparty falls in value. This significantly reduces the credit risk a lender takes in transactions that exhibit counterparty risk.
 - If all things are kept equal increasing the volatility parameter of the model dampens the effects of default correlation. This is because higher volatility reduces instances where both the reference obligor and counterparty have similar spreads. This reduces the likelihood that (even with high default correlation) the condition for a joint default to occur (detailed in Note 3 (Chapter 4)) will hold.

Part of the working assumptions of this thesis has been that trading in credit markets does not occur in liquid and transparent markets (recall in Chapter 1 we explained that Duffie and Singleton (2003) had made this evaluation about the credit markets). The implications of this is that models have to be clear and interpretable, since a black box calibration routine will mean little without the products and liquidity to calibrate them to.

In this chapter we have considered two products modelled with an intensity model driven by the extended Vasicek model. The model is parsimonious and allows for the calibration to the term structure of credit spreads and credit default swaptions.

In order to create default dependence we are using a Gaussian copula for the threshold copula. Again the Gaussian copula is parsimonious in that it accounts for dependence via a single parameter, ρ .

Arguments may still be put forward that the intensity model should be more developed, for instance perhaps by introducing a stochastic volatility model or adding a jump component, etc. Moreover it may be asserted that having a more complex dependence structure (such as the ones considered in Chapter 4) would be better. However what we give up in this process is interpretability. What we gain are more parameters to fit without the products to fit them to⁴.

In Chapter 6, however, we will consider products which require more developed dependence structures and spread dynamics to be accounted for. We will develop explicit methods which advance the Schubert and Schönbucher (2001) framework.

⁴Even though the CDO market displays heterogeneity in default correlation across time and capital structure (see Schönbucher (2006)), those CDOs are for a specific portfolio. Here we are trying to describe the dynamics and default correlation between a counterparty and a reference obligor.

Chapter 6

Future Research

In this chapter we consider two models which will require future research. These models attempt to develop the multi-obligor framework by accounting for spread dynamics via intensity modelling.

6.0.1 Summary of sections in this chapter

This chapter is split into three sections:

- In Section 6.1 we detail the framework developed by Andersen (2006), which introduces the notion of *inter-temporal* dynamics. By this we mean correlating the systemic common factor in the factor framework over multiple time horizons. This enables the level of common factors at earlier maturities to impact common factors at later maturities. From this we develop the notion of the *stochastic liquidity threshold* approach. This method makes the liquidity threshold in the factor framework setting (Chapter 4 (Sub-section 4.3.2)) stochastic.
- In Section 6.2 we consider the valuation of credit default index swaptions. We detail three distinct evolutions in the pricing of credit default index swaptions. Further, by:
 1. A redefinition of the enlarged filtration, $\mathbb{G}$.
 2. A consideration of pre-collapse quantities (which reduce instances of arbitrage).
 3. And by assuming portfolio losses occur on a discretised set independently of the CDIS swap spread process,

we develop, using intensity dynamics, a semi-analytical valuation of credit default index swaptions. By introducing intensity dynamics we can account for the term struc-

ture of credit spreads and develop a model which will be able to calibrate to multiple credit default index swaptions.

- In Section 6.3 we provide a conclusion.

6.1 Factor framework: a new way forward

In this section we develop an extension of the factor framework to account for spread dynamics.

The shortcomings of the OFGC model were highlighted in Chapter 4 (both in the case of compound correlation and base correlation):

- In order to account for some of the problems of the model, several modifications have been sought in the literature (the RFL and random recovery models have been considered in this thesis).
- In essence, base correlation (and compound correlation as an earlier version) attempted to overcome the failings of the OFGC model by constructing exogenous correlation curves that describe how the correlation value of the OFGC model changes with the respective tranche.
- A critical assessment made by Schönbucher (2006) is that historical price dynamics show that implied correlation parameters of tranches can vary strongly over time as well as over the capital structure.
- Schönbucher (2006) suggests there is a need for some form of stochastic correlation modelling, which allows for more general dynamics of relative prices seen in the market. We agree with Schönbucher (2006) but differ on his approach in how to account for the dynamics of dependence over time.
- The argument Schönbucher (2006) makes is that we should view a credit index, e.g. the Itaxx, in a similar light to an equity index, e.g. the Euro Stoxx 50. In the latter it is common to account for the dynamics of the index and not the constituent equities. Schönbucher (2006) suggests to model credit products from this perspective. He produces a form of a class of models called *Loss Market Models*. These models directly model the loss process, rather than the spread dynamics of individual obligors. The model proposed by Schönbucher (2006) does calibrate the stated index tranches completely, yet he concedes: "some information on the individuals is lost in the transition to aggregate".

- The question here is are the two markets, equity and credit, comparable in this respect? In light of the work done in Chapter 3 (Section 3.1) we suggest that the two markets are not comparable because the information sets are different. Conditional on $\mathbb{F}$, the equity markets are determined, which roughly suggests that aggregating stochasticity to the index carries some regularity, if we assume some process $(X_t)_{t \geq 0}$ generates $\mathbb{F}$. However to assess multi-obligor credit products we also need the conditioning information $\mathcal{D}_t = \bigvee_{i=1}^n \mathcal{D}_t^i \forall i \in \{1, \dots, n\}$ discussed in the introduction to Chapter 4.

What Loss Market Models do, which in the equity world has no comparative equivalent, is collapse the full information source of default times by direct modelling of the loss process of a portfolio. In the following we want to extend the factor framework to account for spread dynamics, whilst still accounting for the full information evolution. We suggest a new way to extend the factor framework, detailed in Chapter 4 (Sub-section 4.3.2).

Before proceeding we give an exposition of the work of Andersen (2006) where the current idea stems from:

The argument made by Andersen (2006) is that the factor models as they stand are essentially one period models since the conditioning (systemic) factor, $\mathbf{Z}$, has no dimension in time. In fact to price even plain vanilla CDOs, knowledge of the portfolio loss distribution, $(L_t)_{t \geq 0}$, on either a continuum of dates or at least on a discretised time line, $\{T_1, \dots, T_M\}$, is what is needed (where M is the discretisation size). In order to address the issue, Andersen (2006) suggests introducing M different $N - dimensional$ conditioning variables $\mathbf{Z}_{T_1}, \dots, \mathbf{Z}_{T_M}$ with separate densities, $\psi_{T_1}, \dots, \psi_{T_M}$. For each reference obligor we then need M conditional survival probabilities:

$$q_{T_j}^i(\mathbf{z}), \forall j \in \{1, \dots, M\},$$

and we still impose the consistency condition (see Chapter 4 (Sub-section 4.3.2)):

$$\mathbb{Q}(\tau_i \geq T_j) = \int_{\mathbb{R}^d} q_{T_j}^i(\mathbf{z}) \psi_{T_j}(\mathbf{z}) d\mathbf{z}. \quad (6.1.1)$$

Andersen (2006) also imposes the arbitrage reducing condition:

$$T_k > T_j \Rightarrow \mathbb{Q}(L_{T_k} > u) \geq \mathbb{Q}(L_{T_j} > u),$$

where $u \geq 0$. The argument further asserted is that the multi-period setting, Equation (6.1.1), suffers severe shortcomings when it comes to trying to price a non-static product such as a forward start CDO or options on a CDO. To show this, consider the case where

$M = 2$ and $N = 1$. We want to compute $\mathbb{Q}(\tau_r \geq T_j, \tau_s \geq T_k)$ such that $r, s \in \{1, \dots, n\}$. The case $j = k = 1$ is easily accomplished by taking:

$$\begin{aligned}\mathbb{Q}(\tau_r \geq T_1, \tau_s \geq T_1) &= \int_{\mathbb{R}} \mathbb{Q}(\tau_r \geq T_1, \tau_s \geq T_1 | Z = z) \psi_{T_1}(z) dz \\ &= \int_{\mathbb{R}} q_{T_1}^r(z) q_{T_1}^s(z) \psi_{T_1}(z) dz,\end{aligned}\tag{6.1.2}$$

where the last expression of Equation (6.1.2) holds because of assumed conditional independence. However extending to two horizons i.e. $\mathbb{Q}(\tau_r \geq T_1, \tau_s \geq T_2)$ produces the question: “how do we describe the way the event $\mathbb{I}_{\{\tau_s > T_2\}}$ may be affected by Z_{T_1} ?” Andersen (2006) suggests a natural way forward is to assume the events $\mathbb{I}_{\{\tau_r > T_1\}}$ and $\mathbb{I}_{\{\tau_s > T_2\}}$ are independent conditional on Z_{T_1} and Z_{T_2} , so that we have something of the form:

$$\begin{aligned}\mathbb{Q}(\tau_r \geq T_1, \tau_s \geq T_2) &= \\ &\int_{\mathbb{R}^2} \mathbb{Q}(\tau_r \geq T_1 | Z_{T_1} = z_1, Z_{T_2} = z_2) \mathbb{Q}(\tau_s \geq T_2 | Z_{T_1} = z_1, Z_{T_2} = z_2) \psi_{T_1, T_2}(z_1, z_2) dz,\end{aligned}$$

where, $\psi_{T_1, T_2}(z_1, z_2)$, is the joint density of Z_{T_1} and Z_{T_2} . Now the standard factor setup has not prescribed the joint densities of $Z_{T_1}, \dots, Z_{T_M}$ and nor does it specify how to compute probabilities of the form:

$$\mathbb{Q}(\tau_r \geq T_1 | Z_{T_1} = z_1, Z_{T_2} = z_2).$$

Andersen (2006) considers that “the lack of a default time copula is a rather severe practical drawback, as Monte Carlo simulation of joint default times of portfolio obligors becomes impossible”. To address the issue, Andersen (2006) considers aggregating all Z_{T_j} s into a vector, $\mathbf{S} = \{Z_{T_1}, \dots, Z_{T_M}\}$, and constructing a joint density $\psi_{\mathbf{S}}$ for $\mathbf{S}$. Assume $\mathbf{S}$ has domain $\mathcal{D}^* \subset \mathbb{R}^M$, then the obligors are conditionally independent given $\mathbf{S}$ and we have that the conditional survival probabilities are characterised by functions $q^i : \mathbb{R}_+ \times \mathcal{D}^* \rightarrow [0, 1]$:

$$\mathbb{Q}(\tau_i > t | \mathbf{S} = s) = q^i(s),\tag{6.1.3}$$

which is subject to the usual constraint:

$$\mathbb{Q}(\tau_i > t) = \int_{\mathcal{D}^*} q^i(s) \psi_{\mathbf{S}}(s) ds.\tag{6.1.4}$$

Andersen (2006) calls this model a *complete factor setup* which he demonstrates a true default time copula naturally arises from.

6.1.1 Toward a model with spread evolution

We can start by considering the canonical factor framework setup:

$$X_i = \rho_i(Z)Z + \epsilon_i,\tag{6.1.5}$$

where again Z is the common factor, $\rho_i(Z)$ is some firm specific function of Z and ϵ_i is the idiosyncratic factor. Note that both the OFGC and RFL models fall under this setup. Under this setup $\forall i \in \{1, \dots, n\}$ and suppressing the dependence of Z in ρ_i we have:

$$\begin{aligned}\mathbb{Q}(\tau_i < t) &= \mathbb{Q}(X_i < c_i(t)), \\ \mathbb{Q}(\tau_i < t) &= H_i(c_i(t)), \\ \mathbb{Q}(\tau_i < t | Z = z) &= 1 - G_i(c_i(t) - \rho_i z),\end{aligned}$$

where G_i is the distribution function for ϵ_i , H_i is the distribution function for X_i and $c_i(t)$ is the liquidity threshold. Conditional independence gives:

$$\mathbb{Q}(\tau_1 > t_1, \dots, \tau_n > t_n) = \int_{\mathbb{R}} \prod_{i=1}^n (1 - G_i(c_i(t) - \rho_i z)) \psi_Z(z) dz, \quad (6.1.6)$$

where ψ_Z is the density of Z . This setting creates a "degenerate" complete factor setting, in the sense that we may take:

$$\mathcal{S} = \{Z_{T_1}, \dots, Z_{T_M}\} = \{Z, \dots, Z\} = Z.$$

To evolve the framework, Andersen (2006), suggests a two period construction of the type:

$$\begin{aligned}X_i^{(1)} &= \rho_i(T_1)Z^{(1)} + \epsilon_i^{(1)}, \\ X_i^{(2)} &= \rho_i(T_2)Z^{(2)} + \epsilon_i^{(2)},\end{aligned} \quad (6.1.7)$$

where the density of $Z^{(j)}$ is $\psi_{Z^{(j)}}$, for $j \in \{1, 2\}$, and their joint density is $\psi_Z^{(1,2)}$. Similarly introduce $G_i^{(1,2)}$ to be the joint density of $\epsilon_i^{(1)}$ and $\epsilon_i^{(2)}$. From this Andersen (2006) sets $\mathcal{S} = \{Z^{(1)}, Z^{(2)}\}$ and considers the model with the usual constraint:

$$\mathbb{Q}(\tau_i > t) = \int_{\mathcal{D}^*} q^i(s) \psi_Z^{(1,2)}(s_1, s_2) ds.$$

Control of inter-temporal co-dependence of losses on $[0, T_1]$ and $[T_1, T_2]$ is accomplished through specification of the joint distributions $G_i^{(1,2)}$ and $\psi_Z^{(1,2)}$. However we note the framework does not address the issue of giving a representation to the credit spread dynamics.

Consider again the setup of Equation (6.1.5), we are interested in constructing a mechanism to induce spread dynamics into this setting. We notice the only quantity which has dimension in time is the liquidity threshold, $c_i(t)$. Now assume we have stochastic intensities, $(\lambda_t^i)_{t \geq 0} \forall i \in \{1, \dots, n\}$, and the corresponding integrated hazard rate process, $(\Gamma_t^i)_{t \geq 0}$. We still have the constraint:

$$\mathbb{Q}(X_i < c_i(t)) = \mathbb{Q}(\tau_i < t) = e^{-\int_0^t \lambda^i(0,s) ds}, \quad (6.1.8)$$

where $\lambda^i(0, s)$ is the forward hazard rate as of today and this holds regardless of whether we add stochasticity or not. However by Equation (C.2.1), in Appendix C, we have:

$$\mathbb{Q}(X_i < c_i(t)) = e^{-\int_0^t \lambda^i(0,s)ds} = \mathbb{E}[e^{-\int_0^t \lambda_s^i ds}], \quad (6.1.9)$$

where $(\lambda_t^i)_{t \geq 0}$ is the spot intensity. We consider Equation (6.1.9) to be our alternative constraint.

In the interest of exposition we take two horizons: today, $t_0 = 0$, and $t_1 > t_0$. Now suppose we were to simulate the integrated intensity to the time t_1 so that we have $\Gamma_{t_1}^{simulation}$ (we have suppressed the i notation for clarity). Our constraint means that:

$$H_i(c_i(t_1)) = \mathbb{Q}(\tau_i < t_1) = e^{-\int_0^{t_1} \lambda^i(0,s)ds}. \quad (6.1.10)$$

If we assume H_i is invertible we have a further restatement of our constraint:

$$c_i(t_1) = H_i^{-1}(\mathbb{Q}(\tau_i < t)) = H_i^{-1}(e^{-\int_0^t \lambda(0,s)ds}). \quad (6.1.11)$$

Now we take inspiration from Equation (6.1.11) and develop a random variable $\tilde{c}_i(t_1)$ defined by:

$$\tilde{c}_i(t_1) = H_i^{-1}(e^{-\Gamma_{t_1}^{simulation}}). \quad (6.1.12)$$

By the law of large numbers we know that we recover an average of $\mathbb{E}[e^{-\int_0^{t_1} \lambda_s ds}]$ as the number of simulations of the intensity tends to infinity. Let us assume independence between $\tilde{c}_i(t_1)$ and X_i , by Equation (6.1.12) we have constructed a random variable, $\tilde{c}_i(t_1)$, which is influenced by spread dynamics. In this case the second moments of $\tilde{c}_i(t_1)$ now depend on the variance structure of $(\Gamma_t^i)_{t \geq 0}$ and hence the spread dynamics are directly feeding into the setup of Equation (6.1.5). We call this approach the *stochastic liquidity threshold* approach. If we assume that $(\Gamma_t^i)_{t \geq 0}$ is strictly monotonic (which is not true for the extended Vasicek model, since it can take negative states for the intensity) we recover one quick advantage of doing this: one automatically satisfies the condition that default probabilities are non-decreasing with time under a stochastic threshold model. This may not have been the case if the stochastic thresholds had been developed exogenously to the calibrated intensities. We can now consider a structure of the form $S_i = \tilde{c}_i(t_1)$ where S_i has density $\psi_{S_i} \forall i \in \{1, \dots, n\}$ and the joint density of S_i with Z is given by $\psi_{(S_i, Z)}$:

$$\mathbb{Q}(\tau_i > t) = \int_{\mathcal{D}^*} q^i(t, s_1, s_2) \psi_{(S_i, Z)}(s_1, s_2) ds_1 ds_2,$$

where $\mathcal{D}^*$ is now the domain of the joint density of $\psi_{(S_i, Z)}$. This structure can potentially be used to price exotic CDO structures and can be considered for future research.

6.2 Index swaptions with intensity models

In this section we propose a method of valuing credit default index swaptions under the intensity setting.

Assume we have n obligors in an index, let $t_0 \geq 0$ be the trade date, $t > t_0$ be the expiry date of the credit default index swaption and $T > t$ be the maturity of the underlying CDIS. As in the case of the single obligor credit default swaption, the holder of a credit default index swaption has the right to enter into a CDIS at expiry T at time t for a strike (swap spread) of k . Hence credit default index swaptions are similar to single obligor credit default swaptions, yet they have some significant differences, which are:

- The underlying in this case is a CDIS rather than a CDS. CDIS are more complex than CDS.
- Most credit default index swaptions are not knock out on defaults. Even in the case where all n obligors have defaulted the credit default index swaption will still exist.
- Included in the payoff of the credit default index swaption is the notion of *front end protection* (FEP). By this we mean that the buyer of a payer credit default index swaption is compensated for any losses occurring between trade date, t_0 , and expiry t (only on exercising the option).

The valuation of a credit default index swaption has had three distinct evolutions:

- Initially the standard modelling approach was to treat credit default index swaptions as single obligor default swaptions. In this approach only the swap spread process was considered in the option valuation. Losses accumulated in the index from trade date to expiry date (the FEP) are added onto the value of the option ex-post.
- The initial approach had significant flaws, particularly because it ignored the fact that the FEP was part of the option and could not be separated and subsequently added ex-post:
 - If the buyer of the payer swaption does not exercise the option, she does not get the FEP.
 - The issue of the FEP was first highlighted by Pedersen (2003) who improved the initial approach by redefining the CDIS swap spread to take into account losses from defaults.
 - The redefined CDIS swap spread is called the *loss adjusted index spread* (LAIS).

- Recent works by Jackson (2005), Morini and Brigo (2007) and Rutkowski and Armstrong (2008) have shown that the approach by Pedersen (2003) still has some problems:
 - Firstly the definition of the CDIS swap spread is not valid globally. The CDIS swap spread is valid only in the case where the expected premium leg of the underlying CDIS is not zero.
 - When the expected premium leg value is zero the option formula given by Pedersen (2003) is undefined. Yet Morini and Brigo (2007) demonstrate that in this scenario (when the premium leg is zero) the option formula is known exactly.
- The third evolution of credit default index swaption pricing postulated (in part or in full) by the three aforementioned authors deals with this issue.
 - The works establish that from a redefinition of the enlarged filtration, $\mathbb{G}$, the value of the option can be defined in all states.
 - The works also highlight that if one wants to reach a consistent definition of the index swap spread and an arbitrage free valuation formula, one will need a specific treatment of the information sets involved (i.e. redefining the enlarged filtration). In the presence of multiple defaults and non-knock out options, doing so here, is much more subtle than in the single obligor case.
 - The works are connected to the notion of pre-default quantities, which was useful in Chapter 3 to enable the valuation of single obligor default swaptions. In this case we will consider *pre-collapse* quantities.

We will specifically detail the third evolution in credit default index swaption pricing:

- Our exposition is centered on the introduction of pre-collapse quantities which was developed by Rutkowski and Armstrong (2008).
- However the development by Rutkowski and Armstrong (2008), Morini and Brigo (2007) and Jackson (2005), who establish the third evolution of credit default index swaption pricing, is still based on the explicit modelling of the CDIS swap spread.
- Doing so does not enable a term structure of the CDIS to be accounted for. We will show how this framework can be extended into an intensity model setting.
- By modelling the intensity we can give a specific representation to the term structure of CDIS swap spreads.

- Further by considering an intensity process driven by the extended Vasicek model we recover semi-analytical pricing.

6.2.1 Framework

Recall the definition of the enlarged filtration:

$$\mathbb{G} = \mathbb{F} \vee \mathbb{D},$$

where $\mathbb{D} = (\mathcal{D}_t)_{t \geq 0}$, $\mathcal{D}_t = \bigvee_{i=1}^n \mathcal{D}_t^i$ and $\mathcal{D}_t^i$ is the smallest sigma-algebra containing information about the default of an obligor $i \in \{1, \dots, n\}$ at time $t \geq 0$. $\mathbb{G}$ contains information about the default of all obligors. Moreover recall the definition of the i^{th} obligor to default:

$$\tau_{(i+1)} = \min\{\tau_k : k \in \{1, \dots, n\} \text{ \& } \tau_k > \tau_{(i)}\},$$

with $\tau_{(1)}$ the first obligor to default and $\tau_{(r)}$, for $r \in \{2, \dots, n\}$, the r^{th} obligor to default. We have that $\tau_{(1)} < \dots < \tau_{(n)}$. We can use this definition of the i^{th} obligor to default to redefine the enlarged filtration in the following way:

$$\begin{aligned} \mathbb{G} &= \mathbb{F} \vee \mathbb{D}^{(1)} \vee \dots \vee \mathbb{D}^{(n)} \\ &= \hat{\mathbb{F}} \vee \mathbb{D}^{(n)}, \end{aligned} \tag{6.2.1}$$

where $\mathbb{D}^{(k)}$, for $k \in \{1, \dots, n\}$, is the filtration generated by the k^{th} obligor to default and $\hat{\mathbb{F}} = \mathbb{F} \vee \mathbb{D}^{(1)} \vee \dots \vee \mathbb{D}^{(n-1)}$.

This redefinition pivots the enlarged filtration around the last obligor to default, $\tau_{(n)} = \hat{\tau}$. $\hat{\tau}$ is called the *armageddon* time by Morini and Brigo (2007) and the *collapse* time by Rutkowski and Armstrong (2008). We will adopt the term collapse time.

Examining Equation (6.2.1) we see that the intention is to establish a functional form similar to the single obligor modelling case considered in Chapter 3. In place of the sub-filtration, used in earlier chapters, is now the *adjusted* sub-filtration, $\hat{\mathbb{F}}$. $\hat{\mathbb{F}}$ accounts for default times other than the collapse time. Nevertheless, just as in the single obligor case, we have a sub-filtration, $\hat{\mathbb{F}}$, which is enlarged using only one default time: the collapse time.

From this a significant amount of the work done in Chapter 3 can be applied, with some adjustments. Firstly we can define the $\hat{\mathbb{F}}$ – *conditional survival process* of the collapse time, $\hat{\tau}$, as:

$$\hat{Q}_t = \mathbb{Q}(\hat{\tau} > t | \hat{\mathcal{F}}_t). \tag{6.2.2}$$

Correspondingly we can take $\hat{F}_t = 1 - \hat{Q}_t$ to be the $\hat{\mathbb{F}}$ – *conditional cumulative distribution process* of the collapse time $\hat{\tau}$.

The following results hold analogously to Corollary 4 and Proposition 5 in Chapter 3:

Lemma 20. For a $\mathbb{Q}$ – integrable and $\hat{\mathcal{F}}_t$ – measurable random variable, Y , we have that:

$$\mathbb{E}[\mathbb{I}_{\{\hat{\tau} > t\}} Y | \mathcal{G}_{t_0}] = \mathbb{I}_{\{\hat{\tau} > t_0\}} Q_{t_0}^{-1} \mathbb{E}[Y Q_t | \hat{\mathcal{F}}_{t_0}]. \quad (6.2.3)$$

Proof. See Rutkowski and Armstrong (2008). □

Lemma 21. Assume that $(Y_t)_{t \geq 0}$ is a $\mathbb{G}$ – adapted stochastic process, then there exists a unique $\hat{\mathbb{F}}$ – adapted process, $(\hat{Y}_t)_{t \geq 0}$, such that $\forall t > 0$:

$$Y_t \mathbb{I}_{\{\hat{\tau} > t\}} = \hat{Y}_t \mathbb{I}_{\{\hat{\tau} > t\}},$$

Proof. See Rutkowski and Armstrong (2008). □

The process $\hat{Y}_t$ is the pre-collapse value of Y .

6.2.2 Pricing

Recall the definition of the discounted payoff of the CDIS premium leg, $\mathbf{P}^I(t, T)$, and default leg, $\mathbf{D}^I(t, T)$ in Chapter 4 (Sub-section 4.4.1). By lemma 21 the following holds:

$$\mathbf{P}^I(t, T) = \mathbb{I}_{\{\hat{\tau} > t\}} \mathbf{P}^I(t, T), \quad \mathbf{D}^I(t, T) = \mathbb{I}_{\{\hat{\tau} > t\}} \mathbf{D}^I(t, T).$$

The above leads to the subsequent lemma which is almost exactly the same as in Lemma 11 in Chapter 3:

Lemma 22. The price at time t of a CDIS traded at time t_0 with expiry t satisfies:

$$\overline{\mathbf{val}}_L^I(t, T, k) = \mathbb{I}_{\{\hat{\tau} > t\}} \hat{Q}_t^{-1} \mathbb{E}[\mathbf{val}^I(t, T, k) | \mathcal{F}_t] = \mathbb{I}_{\{\hat{\tau} > t\}} \widetilde{\mathbf{val}}^I(t, T, k), \quad (6.2.4)$$

where the pre-collapse value, $\widetilde{\mathbf{val}}^I(t, T, k)$, satisfies $\widetilde{\mathbf{val}}^I(t, T, k) = \tilde{\mathbf{D}}^I(t, T) - k \tilde{\mathbf{P}}^I(t, T)$ with:

$$\tilde{\mathbf{D}}^I(t, T) = \hat{Q}_t^{-1} \mathbb{E}[\mathbf{D}^I(t, T) | \hat{\mathcal{F}}_t], \quad \tilde{\mathbf{P}}^I(t, T) = \hat{Q}_t^{-1} \mathbb{E}[\mathbf{P}^I(t, T) | \hat{\mathcal{F}}_t],$$

where $\tilde{\mathbf{D}}^I(t, T)$ is the pre-collapse present value of the default leg of a CDIS and $\tilde{\mathbf{P}}^I(t, T)$ is the pre-collapse present value of the premium leg of a CDIS.

Proof. See Rutkowski and Armstrong (2008). □

The pre-collapse fair CDIS swap spread can thus be computed as:

$$\tilde{s}^I(t, T) = \frac{\tilde{\mathbf{D}}^I(t, T)}{\tilde{\mathbf{P}}^I(t, T)} = \frac{\hat{Q}_t^{-1} \mathbb{E}[\int_t^T P(t, u) dL_u | \hat{\mathcal{F}}_t]}{\hat{Q}_t^{-1} \mathbb{E}[\sum_{j=1}^N \Delta(T_{j-1}, T_j) P(t, T_j) \left(1 - \frac{L_{T_j}}{1-R}\right) | \hat{\mathcal{F}}_t]}. \quad (6.2.5)$$

As in the single obligor case the pre-collapse premium leg is always positive i.e. $\tilde{\mathbf{P}}^I(t, T) > 0$, therefore the pre-collapse swap spread, $\tilde{s}^I(t, T)$, is valid in all states. Note that in the denominator of the last equality, above, we have assumed that coupons are paid with the outstanding notional at the end of the coupon period in line with Chapter 4 (Section 4.3).

Hence we have that the value of a CDIS at time $t \in [t_0, T]$ is:

$$\overline{\mathbf{val}}_L^I(t, T, k) = \mathbb{I}_{\{\hat{\tau} > t\}} \tilde{\mathbf{P}}^I(t, T) (\tilde{s}^I(t, T) - k).$$

From this we can define the payoff of a credit index default swaption, remembering that it includes the notion of the FEP:

Definition 6.1. *The payoff of a payer credit default index swaption traded at t_0 with maturity t , strike k and CDIS swap maturity T is:*

$$\begin{aligned} V_t &= (\overline{\mathbf{val}}_L^I(t, T, k) + L_t)^+ \\ &= (\tilde{\mathbf{P}}^I(t, T) (\tilde{s}^I(t, T) - k) + L_t)^+ \\ &= (\tilde{\mathbf{D}}^I(t, T) - k \tilde{\mathbf{P}}^I(t, T) + L_t)^+, \end{aligned} \quad (6.2.6)$$

where L_t is the portfolio loss at time t .

Morini and Brigo (2007) demonstrate that the value of a payer credit default index swaption can be decomposed into three components:

Corollary 23. *Let $t_0 \geq 0$ be the trade date and $t > t_0$ the expiry of a credit default index swaption, which gives the owner of the option the right to buy CDIS protection with maturity $T > t$ at a strike of k . The value of the swaption is:*

$$\begin{aligned} V_{t_0} &= \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0} \beta(t_0)} \mathbb{E}^{\mathbb{Q}}[\hat{Q}_t \beta(t) ((\tilde{s}_t^I - k) \tilde{\mathbf{P}}^I(t, T) + L_t)^+ | \hat{\mathcal{F}}_{t_0}] \\ &\quad + \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0} \beta(t_0)} \mathbb{E}^{\mathbb{Q}}[\beta(t) \mathbb{I}_{\{\hat{\tau} < t\}} (1 - R) | \hat{\mathcal{F}}_{t_0}] \\ &\quad + \mathbb{I}_{\{\hat{\tau} < t_0\}} (1 - R) P(t, T) \\ &= O_1 + O_2 + O_3, \end{aligned} \quad (6.2.7)$$

where R is the recovery rate of all obligors in the portfolio and $P(t, T)$ is the risk free discount factor between t and T .

Proof. See Morini and Brigo (2007). □

We will assume that at trade date all the obligors in the portfolio have not defaulted, so that $\mathbb{I}_{\{\hat{\tau} < t_0\}} = 0$ holds; hence we can assume the quantity O_3 is zero.

With regards to O_2 , Jackson (2005) finds it has analytic value if:

- Interest rates are assumed deterministic (an assumption of this thesis).
- Losses in the portfolio occur on a discretised set $\{l_1, \dots, l_N\}$ with corresponding probabilities $\{p_1, \dots, p_N\}$, where $N \in \mathbb{N}$. Recall that in Chapter 4 (Section 4.7) we constructed an explicit algorithm to find a portfolio's loss distribution.

With these assumptions Jackson (2005) shows O_2 has value:

$$\begin{aligned} O_2 &= P(t, T) \mathbb{E}^{\mathbb{Q}}[L_t = l_N] \\ &= p_N P(t, T). \end{aligned} \tag{6.2.8}$$

It remains to determine a value for O_1 . In order to do this Jackson (2005) uses:

- Iterated expectations and the tower property (conditional on the loss process) and approximates the annuity leg, $\bar{\mathbf{P}}_L^I(t, T)$, without consideration for pre-collapse quantities.
- His method is very intuitive and attractive as it connects directly with the base correlation approach.
- The connection is in his assumption about the portfolio loss process and the fact that he assumes the CDIS swap spread process evolves independently from the portfolio loss process. This enables him to use the loss distributions generated under the BCF.
- Hence he uses the probabilities $\{p_1, \dots, p_N\}$ corresponding to loss states at $\{l_1, \dots, l_N\}$.

Morini and Brigo (2007) and Rutkowski and Armstrong (2008) apply:

- Pre-collapse quantities directly and adjust the CDIS swap spread process (similarly to Pedersen (2003)) to account for portfolio losses.
- By establishing that under an appropriate change of measure the loss adjusted CDIS swap spread is a martingale they are able to model the loss adjusted swap spread directly.

Rather than directly modelling the CDIS swap spread, we are interested in seeing if we could model the dynamics of spread evolution via intensity modelling. Doing so would enable one to account for the term structure of credit spreads in addition to having a fuller set of parameters to calibrate to credit default index swaptions. We begin with the following corollary:

Corollary 24. Let $t_0 \geq 0$ be the trade date and $t > t_0$ the expiry of a payer credit default index swaption which gives the owner of the option the right to enter into a CDIS with maturity $T > t$ at a strike of k . Let us assume:

- Portfolio losses occur on a discretised set $\{l_1, \dots, l_N\}$ with corresponding probabilities $\{p_1, \dots, p_N\}$.
- The loss process is independent of the CDIS swap spread process and $(\hat{Q}_t)_{t \geq 0}$ (the $\hat{\mathbb{F}}$ – conditional survival process of the collapse time, $\hat{\tau}$).

Then we have:

$$\begin{aligned} V_{t_0} &= \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0}\beta(t_0)} \sum_{i=1}^{N-1} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{I}_{\{L_t=l_i\}} | \hat{\mathcal{F}}_{t_0} \right] \mathbb{E}^{\mathbb{Q}} \left[\hat{Q}_t\beta(t)(\tilde{\mathbf{D}}^I(t, T) - k\tilde{\mathbf{P}}^I(t, T) + l_i)^+ | \hat{\mathcal{F}}_{t_0} \right] + p_N P(t, T) \\ &= \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0}\beta(t_0)} \hat{Q}_t\beta(t) \sum_{i=1}^{N-1} p_i \mathbb{E}^{\mathbb{Q}} \left[\hat{Q}_t\beta(t)(\tilde{\mathbf{D}}^I(t, T) - k\tilde{\mathbf{P}}^I(t, T) + l_i)^+ | \hat{\mathcal{F}}_{t_0} \right] + p_N P(t, T). \end{aligned} \quad (6.2.9)$$

Proof. Recall $O_3 = 0$ and $O_2 = p_N P(t, T)$, hence we have that:

$$\begin{aligned} V_{t_0} &= \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0}\beta(t_0)} \mathbb{E}^{\mathbb{Q}} \left[\hat{Q}_t\beta(t)((\tilde{s}_t^I - k)\tilde{\mathbf{P}}^I(t, T) + L_t)^+ | \hat{\mathcal{F}}_{t_0} \right] + p_N P(t, T) \\ &= \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0}\beta(t_0)} \mathbb{E}^{\mathbb{Q}} \left[\hat{Q}_t\beta(t) \sum_{i=1}^{N-1} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{I}_{\{L_t=l_i\}}((\tilde{s}_t^I - k)\tilde{\mathbf{P}}^I(t, T) + l_i)^+ | \hat{\mathcal{F}}_{t_0} \right] | \hat{\mathcal{F}}_{t_0} \right] + p_N P(t, T) \\ &= \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0}\beta(t_0)} \mathbb{E}^{\mathbb{Q}} \left[\hat{Q}_t\beta(t) \sum_{i=1}^{N-1} \mathbb{I}_{\{L_t=l_i\}} \mathbb{E}^{\mathbb{Q}} \left[((\tilde{s}_t^I - k)\tilde{\mathbf{P}}^I(t, T) + l_i)^+ | \hat{\mathcal{F}}_t \right] | \hat{\mathcal{F}}_{t_0} \right] + p_N P(t, T) \\ &= \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0}\beta(t_0)} \sum_{i=1}^{N-1} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{I}_{\{L_t=l_i\}} \mathbb{E}^{\mathbb{Q}} \left[\hat{Q}_t\beta(t)((\tilde{s}_t^I - k)\tilde{\mathbf{P}}^I(t, T) + l_i)^+ | \hat{\mathcal{F}}_t \right] | \hat{\mathcal{F}}_{t_0} \right] + p_N P(t, T) \\ &= \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0}\beta(t_0)} \sum_{i=1}^{N-1} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{I}_{\{L_t=l_i\}} | \hat{\mathcal{F}}_{t_0} \right] \mathbb{E}^{\mathbb{Q}} \left[\hat{Q}_t\beta(t)((\tilde{s}_t^I - k)\tilde{\mathbf{P}}^I(t, T) + l_i)^+ | \hat{\mathcal{F}}_{t_0} \right] + p_N P(t, T) \\ &= \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0}\beta(t_0)} \sum_{i=1}^{N-1} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{I}_{\{L_t=l_i\}} | \hat{\mathcal{F}}_{t_0} \right] \mathbb{E}^{\mathbb{Q}} \left[\hat{Q}_t\beta(t)(\tilde{\mathbf{D}}^I(t, T) - k\tilde{\mathbf{P}}^I(t, T) + l_i)^+ | \hat{\mathcal{F}}_{t_0} \right] + p_N P(t, T) \\ &= \mathbb{I}_{\{\hat{\tau} > t_0\}} \frac{1}{\hat{Q}_{t_0}\beta(t_0)} \sum_{i=1}^{N-1} p_i \mathbb{E}^{\mathbb{Q}} \left[\hat{Q}_t\beta(t)(\tilde{\mathbf{D}}^I(t, T) - k\tilde{\mathbf{P}}^I(t, T) + l_i)^+ | \hat{\mathcal{F}}_{t_0} \right] + p_N P(t, T). \end{aligned} \quad (6.2.10)$$

□

In the third equality we recognise that conditional on $\hat{\mathcal{F}}_t \mathbb{I}_{\{L_t=l_i\}}$, for $i \in \{1, \dots, N-1\}$, is determined by virtue of the definition of $\hat{\mathbb{F}}$. The fourth equality is an application of the tower property. Notice that unlike in Jackson (2005), by using pre-collapse quantities, we do not run into the problem of having to account for how the loss density changes with a change of probability measure.

Corollary 24 is the weighted sum (weighted by loss densities) of expressions very similar to the swaption valuation structure established in Corollary 12 (Chapter 3). The main difference is that in Corollary 24 there is the addition of the quantities l_i , for $i \in \{1, \dots, N-1\}$. It is important to note again that we have developed in Chapter 4 (Section 4.7) an efficient method of generating the loss density, p_i , for $i \in \{1, \dots, N\}$.

Let us assume that in each loss state, $l_i \forall i \in \{1, \dots, N-1\}$, spread dynamics can be implied from intensity dynamics. This is achieved by noting again that:

$$\tilde{s}^I(t, T) = \frac{\tilde{\mathbf{D}}^I(t, T)}{\tilde{\mathbf{P}}^I(t, T)} = \frac{\hat{Q}_t^{-1} \mathbb{E}[\int_t^T P(t, u) dL_u | \hat{\mathcal{F}}_t]}{\hat{Q}_t^{-1} \mathbb{E}[\sum_{j=1}^N \Delta(T_{j-1}, T_j) P(t, T_j) \left(1 - \frac{L_{T_j}}{1-R}\right) | \hat{\mathcal{F}}_t]}.$$

For brevity we assume that the pre-collapse default leg and premium leg, in each loss state, $l_i \forall i \in \{1, \dots, N-1\}$, can be approximated in the same way as we did in Proposition 13 (Chapter 3).

In Proposition 13 we arrived at a basket option of log-normally distributed quantities from which the results of Jamshidian (1989) could be applied to recover semi-analytic valuation results. In this case we have the inclusion of the loss states, $l_i \forall i \in \{1, \dots, N-1\}$, as an additive, i.e.:

$$(\tilde{\mathbf{D}}^I(t, T) - k\tilde{\mathbf{P}}^I(t, T) + l_i)^+.$$

If we follow the same steps as in Proposition 13, in this case (Corollary 24) we will arrive at the sum of a series of basket options on *displaced* log-normally distributed quantities. Let these log-normal quantities be denoted g_j , for $j \in \{1, \dots, W\}$. In each state, $l_i \forall i \in \{1, \dots, N-1\}$, we can define the displaced values to be:

$$s_i = \frac{l_i}{W},$$

that is, we allocate to each log-normal quantity the same displacement, s_i , which changes in each state, $l_i \forall i \in \{1, \dots, N-1\}$. Hence the displaced diffusion is $g_j + s_i$, for $j \in \{1, \dots, W\}$. Jaeckel (2004) notes that in practical terms it is enough to assume the displaced quantities are almost log-normal:

$$(g_j + s_i) \approx (g_j(0) + s_i) e^{-\frac{1}{2}c^2 + cX} \text{ with } X \sim \Phi(x),$$

where $g_j(0)$ is the initial value of g_j and c^2 is the variance of g_j . Jaeckel (2004) demonstrates that for the purpose of valuing caplets in the interest rate setting, making such an approximation typically works well.

We are interested, for future research, to consider whether approximating the displaced log-normal quantities as log-normal quantities also works well in the valuation of credit default index swaptions.

To conclude, we note that it may be argued that our approach hinges on a few assumptions and approximations. Yet, in the alternative, we believe there is a structural problem with the construction of pre-collapse quantities and the manipulation of the sub-filtration $\hat{\mathbb{F}}$. The alternative in this case is the Rutkowski and Armstrong (2008) and Morini and Brigo (2007) approach. This problem has also been noted by Rutkowski and Armstrong (2008):

By postulating that the pre-collapse loss adjusted CDIS swap spread process can be modelled as a log-normal process, driven by an $\hat{\mathbb{F}}$ -Brownian motion, $(\hat{W}_t)_{t \geq 0}$, under a respective change of an equivalent measure, either:

- $(\hat{W}_t)_{t \geq 0}$ generates the filtration $\hat{\mathbb{F}}$: in which case, as noted in Chapter 3 (Note 1), the times $\tau_{(1)}, \dots, \tau_{(n-1)}$ are not meaningful stopping times.
- Or $\hat{\mathbb{F}}$ will need support from other discontinuous processes to generate it, implying that the loss-adjusted swap spread as defined (accounting for losses up until the $(n - 1)^{th}$ to default) does not carry the full information of the evolution of $\hat{\mathbb{F}}$.

In each case we believe this fact (in addition to the series of assumptions we have needed to make to arrive at our results) highlights a problem with the use of collapse times. Recall that the effort was to arrive at a setting similar to the single obligor case; however, in doing this we have constructed the sub-filtration, $\hat{\mathbb{F}}$, which *does* contain information about defaults. The good thing in the single obligor case was that we were able to operate in a sub-filtration, $\mathbb{F}$, where defaults were not considered.

6.3 Chapter conclusion

In this chapter we have considered two models for pricing more exotic multi-obligor products:

- In Section 6.1 we provided an explicit method of inducing spread dynamics into the factor framework. To the best of our knowledge this has not been done before in the literature:
 - This property enables one to account for small idiosyncratic spread movements; which tend to get cancelled out in large portfolios, unless the spread dynamics of obligors are linked.
 - What we also do is link this to the work of Andersen (2006) via the introduction of inter-temporal dynamics.

- By combining the stochastic liquidity threshold approach with inter-temporal portfolio dynamics, we account for idiosyncratic dynamics and systemic dynamics.
- In Section 6.2 we considered the valuation of credit default index swaptions:
 - We devised a mechanism of introducing stochastic intensity modelling to the pricing of credit default index swaptions.
 - We established that it is feasible to recover semi-analytic valuation of credit default index swaptions with the extended Vasicek model.

Chapter 7

Conclusion Of Thesis

In this chapter we provide an evaluation of this thesis, which has addressed the issue of credit derivatives modelling.

In Chapter 1 we discussed the fact that credit modelling is unique in that it attempts to assess the likelihood and timing of default of an obligor. The challenges of modelling credit derivatives were set out. Moreover we highlighted the different evolutions of single obligor modelling and multi-obligor modelling.

In Chapter 2 we detailed the characteristics of various credit products. We found that the key risks that needed to be captured to effectively value credit products were:

- Single obligor default probabilities.
- Default dependence of reference obligors in a portfolio.
- Default dependence between reference obligors and a counterparty in the presence of counterparty risk.
- Credit spread dynamics.

In Chapter 3 we considered the valuation of credit products:

- Our main interest was to give a representation to credit spread dynamics and the term structure of credit spreads that exists in the market. We did this by stochastic intensity modelling:
 - The main model developed to account for the intensity process was the extended Vasicek model.
 - We provided a new analytical valuation result for the model and developed a full calibration routine.

- The extended Vasicek model is weakened by the possibility of assuming negative states. Alternative models were considered: the CIR model, for example, never assumes negative states and can be parameterised to ensure it is always positive.
- However, with the extended Vasicek model it is possible to calibrate to multiple swaption expiries analytically. To the best of our knowledge this has not been shown to be feasible in the literature (analytically) for the CIR model with non-constant parameters.

In Chapter 4 we considered the multi-obligor valuation of credit products:

- An exposition of the factor framework was provided. We focused on the OFGC model and detailed the construction of correlation curves. Such curves are needed since the OFGC model is a one parameter model and therefore does not re-price the traded tranches.
- The correlation curves were shown to exhibit multiple instances of arbitrage. We demonstrated that the random recovery model reduced such instances of arbitrage. Nevertheless we established the random recovery model still contains arbitrage.
- The main contribution made in the chapter was to develop an efficient method of constructing a portfolio's loss distribution.

In Chapter 5 we introduced spread dynamics into the pricing of multi-obligor products:

- The dynamic multi-obligor modelling setup devised by Schubert and Schönbucher (2001) was detailed and shown to exhibit the correct features and behaviour.
- The key utility of the framework is that one is able to separate the modelling of the intensity process of each obligor from that of the dependence structure that defines the relationship between the obligors.
- We are therefore able to apply the work done in Chapter 3 to efficiently calibrate the intensity process (using the extended Vasicek model) of each obligor in a portfolio. From this we can control the dependence structure of the obligors independently of the intensity calibration. Overall this leads to efficient multi-obligor valuation.
- The model is tested by valuing two products (securitised loans and LCLNs with recourse). The results are qualitatively intuitive:
 - There is increasing monotonicity between default correlation and counterparty risk.

- Adding triggers to unwind trading positions significantly reduces counterparty risk.
- We find a new evaluation of the nature of counterparty risk and the effects of spread dynamics: all things being equal increasing volatility reduces instances of joint default, thereby reducing counterparty risk.

In Chapter 6 we detail some specific ideas for future research:

- Section 6.1 adds to the factor framework in a way that introduces inter-temporal dynamics that is now reliant on the spread evolution of the underlying intensities. Hence the model is directly capturing the covariance between factors over a time period, but also accounts for the spread dynamics emanating from the reference obligors.
- Section 6.2 shows how to value CDIS swaptions using the notion of pre-collapse quantities and intensity modelling. By assuming that losses occur independently of the CDIS swap spread level and that the term structure of the CDIS can be described by an intensity process, we are able to provide a method to recover semi-analytic valuation results for the CDIS swaption using the extended Vasicek model.

To conclude, we have examined various credit models and discussed the complexity that surrounds credit derivatives as an asset class. The uniqueness of credit derivatives can be understood from the connection of the enlarged filtration, $\mathbb{G}$ (which provides the full set of information flow), and the sub-filtration, $\mathbb{F}$ (that in some sense represents the economy and its information flow). Conditional on $\mathbb{F}$ all risk factors in other asset classes (equities, interest rates etc) are determined; but not credit. Additional risk factors remain in credit (the exact timing of defaults) and this represents the continued complexity in the asset class. A complexity which we view as lacking any meaningful solution: precisely because defaults occur as a surprise.

Appendix A

The Structural Model

In this appendix we detail the classic version of the structural approach which was first proposed by Merton (1974). The key to the structural approach is to consider the asset value of a firm or the GDP of a sovereign as being an indicator of default whenever its value falls significantly. The first use of the structural models found its root in the Black and Scholes (1973) modelling framework. The following relation is assumed to hold in the model proposed by Merton (1974):

$$V_t = D_t + S_t \quad \forall t \geq 0, \quad (\text{A.0.1})$$

where $(V_t)_{t \geq 0}$ is the asset value of an obligor, $(D_t)_{t \geq 0}$ is the total debts of the obligor and $(S_t)_{t \geq 0}$ is the equity value of the obligor. Using the standard assumption of Black and Scholes (1973), Merton (1974) assesses the default likelihood of an obligor. The following assumptions hold:

- The process $(V_t)_{t \geq 0}$ is continuously observable and continuously tradeable.
- There exists no arbitrage and the market is complete (meaning the equivalent martingale measure is unique, see Yan (1998)).
- For a given obligor there exists only one debt instrument, a zero coupon bond, which matures at time $T > 0$ with a redemption value of $K > 0$.

T is the specified time we want to assess credit risk from. When the total value, V_T , of an obligor at time T is less than the redemption value of the zero coupon bond, K , the obligor is said to have defaulted. We define the default probability to be:

$$\mathbb{Q}(\tau < T) = \mathbb{Q}(V_T < K). \quad (\text{A.0.2})$$

K is called *the default boundary* (or liquidity threshold). This setup is used in the one factor Gaussian copula model (OFGC), where normal random variables (proxy asset values) are considered (see Chapter 4 (Section 4.3)). With the OFGC model obligors are said to be in a state of default if they fall below a pre-specified level, which is exogenously determined. Under the equivalent martingale measure, $\mathbb{Q}$, $(V_t)_{t \geq 0}$, has SDE of the form:

$$dV_t = rV_t dt + \sigma V_t dW_t, \quad (\text{A.0.3})$$

where $\sigma \in \mathbb{R}_+$ is a constant and $r \in \mathbb{R}_+$ is the spot interest rate which we take to be constant.

By Equation (A.0.1), $S_t = V_t - D_t$, but we also have the economic constraint that the equity value is always non-negative, $S_t \geq 0$. Therefore we may write:

$$S_t = \max(V_t - D_t, 0) \quad \forall t \geq 0.$$

Hence the equity value is an option on the underlying asset value process. From the above and the Black and Scholes (1973) modelling framework we have that:

$$S_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} \max(V_T - K, 0) | \mathcal{G}_t].$$

The Merton (1974) model suffers significant problems, some of which are stated in the following:

- All the traditional Black and Scholes (1973) assumptions and the additional assumption that there is only one debt type (a zero coupon bond): in reality there are no obligors like this.
- The asset value is modelled as a diffusion process, see Equation (A.0.3), which means that the default probability, Equation (A.0.2), goes to zero as time goes to zero. This is not perceived in the market where short dated CDS imply default probabilities within a few months which are significant (see Gemmill (2002)).

Various extensions have been suggested to overcome these issues:

- Firstly Black and Cox (1976) suggests making the liquidity threshold time dependent, $(K(t))_{t \geq 0}$. $(K(t))_{t \geq 0}$ can then be calibrated exogenously e.g. one may use as a proxy for the volatility process in Equation (A.0.3) the realised equity returns variance. Once the volatility is recovered, CDS quotes can be used to calibrate a piecewise constant deterministic function for $(K(t))_{t \geq 0}$.
- Other attempts were made to overcome the short term zero default probability implied by a diffusion process by adding jumps to the process in Equation (A.0.3). In general such models gain significantly in complexity (see Bielecki and Rutkowski (2002)).

Appendix B

Mathematical Importance Of Conditional Poisson Process

In this appendix we consider some important results relating to point processes, particularly the conditional Poisson process (CPP):

- In Section B.1 we detail the characterisation of CPPs formulated by Bremaud (1981). The characterisation demonstrates that CPPs arise naturally, but also that one can focus on stochastic intensities when modelling CPPs.
- In Section B.2 we detail Girsanov's theorem for point processes. The theorem defines how a stochastic intensity changes under a measure change.
- In Section B.3 we detail the *correlation neutral measure*. By appropriate changes of measure Giesecke (2009) establishes that the only point processes that do not produce feedback between the state of the intensity and the state of the point process are CPPs. Moreover for each point process there exists a family of measures under which the feedback of any point process to its intensity is neutralised.

B.1 Bremaud's characterisation

In this section we discuss the classical characterisation of a conditionally Poisson process (CPP) in the work of Bremaud (1981).

The characterisation shows that the expected value of the integral representation of a predictable process with respect to a point process is equal to the expected value of the integral representation of the same predictable process with respect to the intensity of the

point process whenever the point process is a CPP (with the relatively strong restriction that the intensity be $\mathcal{G}_0$ measurable).

The characterisation adds justification to the use of CPPs as the canonical point process (see Lando (1998)) as it shows that the intensity preserves the expected rate of decay of the point process. This enables the modelling of credit products using the intensity process rather than directly working with the point process.

Definition B.1. Let $(N_t)_{t \geq 0}$ be a point process adapted to $\mathbb{G}$ and let $(\lambda_t)_{t \geq 0}$ be a non-negative measurable process defined on $\{\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}\}$ and suppose that $(\lambda_t)_{t \geq 0}$ is $\mathcal{G}_0$ -measurable $\forall t \geq 0$ (recall that $\mathbb{G}$ is an enlarged version of $\mathbb{F}$) and:

$$\int_0^t \lambda_s ds < \infty \quad \mathbb{Q} - a.s.,$$

if $\forall 0 \leq s \leq t$ and $\forall u \in \mathbb{R}$ we have:

$$\mathbb{E} \left[e^{iu(N_t - N_s)} | \mathcal{G}_s \right] = \exp \left\{ (e^{iu} - 1) \int_s^t \lambda_v dv \right\},$$

then $(N_t)_{t \geq 0}$ is called a $(\mathbb{Q}, \mathbb{G})$ - conditional Poisson process.

Definition B.1 is a more restrictive condition than we would like in so far as it requires $(\lambda_t)_{t \geq 0}$ to be $\mathcal{G}_0$ measurable. Schönbucher (2003a) suggests that such a restriction is useful in the context of actuarial analysis where “it does not matter when the information is revealed as long as the initial distribution is realistic”.

However with the need to model credit derivative products which exhibit convexity, the formation of information patterns in the intensity must carry some element of stochasticity. To this end the analysis below is more of an instructive analysis highlighting the significance of CPPs.

We state two lemmas leading up to Theorem 28 below, the reader is referred to Bremaud (1981) pg 24 for the proof of the lemmas.

Lemma 25. Let $(N_t)_{t \geq 0}$ be a point process adapted to the history $\mathbb{G}$ and let $(\lambda_t)_{t \geq 0}$ be a non-negative process which is $\mathbb{F}$ - progressive (see Revuz and Yor (2001) for a definition of a progressive process) such that:

$$\int_0^t \lambda_s ds < \infty \quad \mathbb{Q} - a.s.,$$

and $\forall \mathbb{G}$ - predictable processes $(P_t)_{t \geq 0}$ if:

$$\mathbb{E} \left[\int_0^\infty P_s dN_s \right] = \mathbb{E} \left[\int_0^\infty P_s d\lambda_s \right],$$

then $(N_t)_{t \geq 0}$ is $\mathbb{Q}$ – non – explosive and $(M_t)_{t \geq 0}$ is a $\mathbb{G}$ – local martingale (again see Revuz and Yor (2001) for a definition of a local martingale) such that:

$$M_t = N_t - \int_0^t \lambda_s ds,$$

where a *non-explosive* point process is such that the intensity $(\lambda_t)_{t \geq 0}$ is predictable and non-negative with $(\forall t \geq 0)$:

$$\int_0^t \lambda_s ds < \infty \quad \mathbb{Q} - a.s.$$

Lemma 26. Under the conditions of Lemma 25 if we let $(X_t)_{t \geq 0}$ be a $\mathbb{G}$ – predictable process such that $\forall t \geq 0$:

$$\mathbb{E} \left[\int_0^t |X_s| \lambda_s ds \right] < \infty;$$

then:

$$Y_t = \int_0^t X_s dM_s ,$$

is a $\mathbb{G}$ – local martingale where:

$$M_t = N_t - \int_0^t \lambda_s ds.$$

Before proceeding to the main theorem we state one other theorem (see Bremaud (1981) pg 34) called the exponential formula.

Theorem 27. Now let $a(t)$ be a right continuous and increasing function with $a(0) = 0$ and let $v(t)$ be such that $\int_0^t |v(s)| da(s) < \infty$ $t \geq 0$, then the equation:

$$b(t) = b(0) + \int_0^t b(s-) v(s) da(s),$$

admits a unique locally bounded solution, in so far as:

$$\sup_{s \in [0, t]} |b(s)| < \infty,$$

given by:

$$b(t) = b(0) \exp \left\{ \int_0^t v(s) da^c(s) \right\} \prod_{0 \leq s \leq t} (1 + v(s) \Delta a(s)),$$

where $\Delta a(t) = a(t) - a(t-)$ and $a^c(t)$ is the continuous part of $a(t)$:

$$a^c(t) = a(t) - \sum_{s \leq t} \Delta a(s).$$

The result below serves as a powerful argument for centering point processes around CPPs.

Theorem 28. Let $(N_t)_{t \geq 0}$ be a point process adapted to $\mathbb{G}$ and let $(\lambda_t)_{t \geq 0}$ be a non-negative measurable process such that $\forall t \geq 0$ we have:

(i) λ_t is $\mathcal{G}_0$ - measurable.

(ii) $\int_0^t \lambda_s ds < \infty \quad \mathbb{Q} - a.s.$

Then, if the equality,

$$\mathbb{E} \left[\int_0^\infty P_s dN_s \right] = \mathbb{E} \left[\int_0^\infty P_s d\lambda_s \right],$$

is verified $\forall$ non-negative $\mathbb{G}$ - predictable processes $(P_t)_{t \geq 0}$ then $(N_t)_{t \geq 0}$ is a CPP with $\mathbb{F}$ - intensity $(\lambda_t)_{t \geq 0}$.

Proof. For $u \in \mathbb{R}$, define M_t^u by:

$$M_t^u = \frac{e^{iuN_t}}{\exp \left\{ (e^{iu} - 1) \int_0^t \lambda_s ds \right\}}. \quad (\text{B.1.1})$$

In order to keep the form of Theorem 27 let:

$$\begin{aligned} v(t) &= e^{iu} - 1, \text{ constant.} \\ a(t) &= N_t - \int_0^t \lambda_s ds, \text{ right continuous and increasing process.} \\ a^c(t) &= - \int_0^t \lambda_s ds, \text{ as } N_t \text{ is a jump process.} \\ \Delta a(t) &= \Delta(N_t - \int_0^t \lambda_s ds) = \Delta N_t. \\ b(t) &= M_t^u. \end{aligned} \quad (\text{B.1.2})$$

We can use Theorem 27 to show that:

$$\begin{aligned} 1 + \int_0^t (e^{iu} - 1) M_{s-}^u (dN_s - \lambda_s ds) &= 1 + \int_0^t v(s) b(s-) da(s) \\ &= \exp \left\{ \int_0^t v(s) da^c(s) \right\} \prod_{0 \leq s \leq t} (1 + v(s) \Delta a(s)) \\ &= \exp \left\{ - \int_0^t (e^{iu} - 1) d\lambda_s \right\} \prod_{0 \leq s \leq t} (1 + (e^{iu} - 1) \Delta N_s) \\ &= \frac{\prod_{0 \leq s \leq t} (1 + (e^{iu} - 1) \Delta N_s)}{\exp \left\{ (e^{iu} - 1) \int_0^t \lambda_s ds \right\}} = M_t^u. \end{aligned} \quad (\text{B.1.3})$$

Note the numerator in the last equation can be shown to be equal to e^{iuN_t} ; since we have $\Delta N_s \in \{0, 1\}$:

- If $\Delta N_s = 0$ then $1 + (e^{iu} - 1) \Delta N_s = 1$.
- And if $\Delta N_s = 1$ then $1 + (e^{iu} - 1) \Delta N_s = e^{iu}$.

Therefore we have 1 whenever there is no jumps and e^{iu} precisely when we have jumps in the interval $0 \leq s \leq t$ and the number of jumps in this interval is by definition N_t which therefore means that:

$$\prod_{0 \leq s \leq t} (1 + (e^{iu} - 1)\Delta N_s) = e^{iuN_t}.$$

Now, let:

$$A_n = \inf\{t \mid \int_0^t \lambda_s ds \geq n\}, \quad (\text{B.1.4})$$

and define the increasing sequence of stopping times by:

$$S_n = \begin{cases} A_n & \text{if } \{t : \int_0^t \lambda_s ds \geq n\} \neq \emptyset \\ +\infty & \text{otherwise} \end{cases}$$

By Lemmas 25 and 26 we have that $M_{t \wedge S_n}^u$ is a martingale, and moreover:

$$\mathbb{E} \left[\frac{\exp \{iu(N_{t \wedge S_n} - N_{s \wedge S_n})\}}{\exp \left\{ (e^{iu} - 1) \int_{s \wedge S_n}^{t \wedge S_n} \lambda_r dr \right\}} \middle| \mathcal{G}_s \right] = 1, \quad (\text{B.1.5})$$

λ_t and S_n are $\mathcal{G}_0$ -measurable so that:

$$\mathbb{E}[e^{iu(N_{t \wedge S_n} - N_{s \wedge S_n})}] = \exp \left\{ (e^{iu} - 1) \int_{s \wedge S_n}^{t \wedge S_n} \lambda_r dr \right\}, \quad (\text{B.1.6})$$

letting $n \rightarrow \infty$ we recover Definition B.1. □

B.2 Girsanov's theorem for point processes

We assume that the point process is *non-explosive, totally inaccessible*, the exponential of the integrated intensity, $(e^{\int_0^t \lambda_s ds})_{t \geq 0}$, is integrable and relax the need for $(\lambda_t)_{t \geq 0}$ to be $\mathcal{G}_0$ measurable.

Theorem 29 (Girsanov for point process). *Let $(N_t)_{t \geq 0}$ be a point process with intensity $(\lambda_t)_{t \geq 0}$ and let $(\mu_t)_{t \geq 0}$ be a strictly positive predictable process such that for some arbitrary fixed time, $T > 0$, we have:*

$$\int_0^T \mu_s \lambda_s ds < \infty, \quad (\text{B.2.1})$$

from this we may define:

$$E_t = \begin{cases} \exp \left\{ \int_0^t (1 - \mu_s) \lambda_s ds \right\} & \text{if } t < T_1, \\ \left(\prod_{i \geq 1} \mu_{T_i} \mathbb{I}_{\{T_i \leq t\}} \right) \exp \left\{ \int_0^t (1 - \mu_s) \lambda_s ds \right\} & \text{if } t \geq T_1, \end{cases} \quad (\text{B.2.2})$$

$(E_t)_{t \geq 0}$ can be shown to be a local martingale and T_i denotes the i^{th} jump time of the point process. Moreover $(E_t)_{t \geq 0}$ can also be represented in the form:

$$E_t = 1 - \int_0^t E_{s-} (1 - \mu_s) dM_s, \quad (\text{B.2.3})$$

where $M_t = N_t - \int_0^t \lambda_s ds$. Suppose moreover that:

$$\mathbb{E}[E_t] = 1,$$

and define a new probability measure by setting:

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = E_T,$$

Then we have:

$$E_t = \mathbb{E}[E_T | \mathcal{F}_t] \quad \forall t \in [0, T],$$

and $(N_t)_{t \geq 0}$ has $\tilde{\mathbb{Q}}$ – intensity:

$$\tilde{\lambda}_t = \mu_t \lambda_t \quad \forall t \in [0, T].$$

Proof. See Duffie (2002) for an intuitive proof, Bremaud (1981) for a generalised proof for a multi-dimensional point process or Jacod and Shiryaev (2002) for the most general exposition. $\square$

Proposition 30. Suppose $(N_t)_{t \geq 0}$ is a CPP adapted to the filtration $\mathbb{G}$ with $\mathbb{F}$ – intensity $(\lambda_t)_{t \geq 0}$. Let the conditions of Theorem 29 hold, then restricted to the time interval $[0, T]$, $(N_t)_{t \geq 0}$ is a CPP under $\tilde{\mathbb{Q}}$.

Proof. See Duffie (2002). $\square$

B.3 Correlation neutral measure

The idea of the correlation neutral measure is that a CPP is the only point process that does not produce feedback between the level of the point process and the intensity. However for each point process there exists a family of measures under which the feedback of the point process to its intensity is neutralised. We define:

Definition B.2 (Laplace Transform). Let $(N_t)_{t \geq 0}$ be a non-negative random variable with distribution function G and, if it exists with respect to the Lebesgue measure, density function g , then:

(i) The Laplace transform of $N_t \quad \forall t \geq 0$ is defined as:

$$\ell_N(u) \equiv \mathbb{E}[e^{-uN}] = \int_0^\infty e^{-ux} dG(x) = \int_0^\infty e^{-ux} g(x) dx. \quad (\text{B.3.1})$$

(ii) Let $\vartheta : \mathbb{R}_+ \rightarrow [0, 1]$. If a solution exists, the inverse Laplace transform ℓ_ϑ^{-1} of ϑ is defined as the function $\chi : \mathbb{R}_+ \rightarrow [0, 1]$ which solves:

$$\ell_\chi(u) = \int_0^\infty e^{-ux} \chi(x) dx = \vartheta(u), \quad \forall u > 0. \quad (\text{B.3.2})$$

(iii) The distribution of $(N_t)_{t \geq 0}$ is uniquely characterised by its Laplace transform (see Schönbucher (2003a)).

We consider the conditional Laplace transform of $N_t - N_s$, such that $0 \leq s < t$, defined by:

$$\ell(u, s, t) \equiv \mathbb{E}[e^{-u(N_t - N_s)} | \mathcal{G}_s] \quad u \geq 0. \quad (\text{B.3.3})$$

Further recalling Equation (B.2.2) let $\mu_s = e^{-u}$ and let $\varphi(u) = 1 - e^{-u}$. Then we have from Equation (B.2.2) that:

$$E_t^u = \begin{cases} \exp \left\{ \int_0^t (1 - e^{-u}) \lambda_s ds \right\} & \text{if } t < T_1, \\ \left(\prod_{i \geq 1} e^{-u} \mathbb{I}_{\{T_i \leq t\}} \right) \exp \left\{ \int_0^t (1 - e^{-u}) \lambda_s ds \right\} & \text{if } t \geq T_1, \end{cases} \quad (\text{B.3.4})$$

with $\prod_{i \geq 1} e^{-u} \mathbb{I}_{\{T_i \leq t\}} = e^{-u N_t}$ easily shown. From this we recover the familiar form in Giesecke (2009):

$$E_t^u = e^{\varphi(u) \int_0^t \lambda_s ds - u N_t}, \quad (\text{B.3.5})$$

which has the alternative form:

$$E_t^u = 1 - \varphi(u) \int_0^t E_{s-} dM_s. \quad (\text{B.3.6})$$

The set of E^u $u > 0$ creates a family of Radon-Nikodým derivatives, Π , that admit a family of equivalent measures $\mathbb{Q}^u$ (see Giesecke (2009)). We now set:

$$L^u(v, s, t) = \mathbb{E}^u[e^{-v \int_s^t \lambda_r dr} | \mathcal{G}_s] \quad u, v \geq 0,$$

to be the conditional Laplace transform of the integrated intensity relative to $\mathbb{Q}^u$. From this the thrust of the idea in the correlation neutral measure is recovered. The $\mathbb{Q} -$ Laplace transform of $(N_t)_{t \geq 0}$, Equation (B.3.3), can be expressed in terms of the $\mathbb{Q}^u -$ Laplace transform of the integrated intensity, $(\int_0^t \lambda_s ds)_{t \geq 0}$:

$$\begin{aligned} \ell(u, s, t) &\equiv \mathbb{E}^u[e^{-\varphi(u) \int_s^t \lambda_r dr} | \mathcal{G}_s] \\ &= L^u(\varphi(u), s, t). \end{aligned} \quad (\text{B.3.7})$$

Equation (B.3.7) shows that the Laplace transform of a counting process is given by the Laplace transform of the integrated intensity, $(\int_0^t \lambda_s ds)_{t \geq 0}$, under a change of measure. Moreover for a point process that is CPP we know the distribution, $N_t - N_s$, follows a $\mathcal{G}_s \vee \mathcal{F}_t$ conditional Poisson process with parameter $\int_s^t \lambda_r dr$; so that:

$$\mathbb{E}[e^{-u(N_t - N_s)} | \mathcal{G}_s \vee \mathcal{F}_t] = e^{-\varphi(u) \int_s^t \lambda_r dr} = e^{-\varphi(u) \int_s^t \lambda_r dr}. \quad (\text{B.3.8})$$

Taking conditional expectations with respect to $\mathcal{G}_s$ on both sides yields:

$$\ell(u, s, t) = L^0(\varphi(u), s, t) = \mathbb{E}[e^{-\varphi(u) \int_s^t \lambda_r dr} | \mathcal{G}_s],$$

where the Laplace transform is taken under $\mathbb{Q}^0 = \mathbb{Q}$.

The result says that if we have a CPP there is no need to change measure. Uniqueness of the distribution of $(N_t)_{t \geq 0}$ and its Laplace transform confirms that the CPP is the only point process that admits $\mathbb{Q} = \mathbb{Q}^0 \in \Pi$ which satisfies Equation (B.3.7). Giesecke (2009) analyses this result in the following sense: within the CPP there is no feedback between the intensity, $(\lambda_t)_{t \geq 0}$, and the point process, $(N_t)_{t \geq 0}$. Moreover Equation (B.3.7) tells us there is a measure $\mathbb{Q}^u \in \Pi$ for which the Laplace transform, $\ell(u, s, t)$, of any point process is structurally similar to that of a CPP.

Appendix C

Interest Rates: Theory And Application To Credit

In Chapter 3 we demonstrated that a number of results can be derived, such as the survival probability of an obligor, by consideration of stochastic intensities. Lando (1998) demonstrates that a credit risky obligation can be valued in the same way as a risk free obligation by adjusting the spot interest rate (under Assumption 3.1 in Chapter 3). The spot interest rate is adjusted by adding the (scaled) stochastic intensity. Hence we can use in the valuation of credit products a great deal of the interest rate infrastructure that has been developed.

The works of Vasicek (1977), Cox et al. (1985), Hull and White (1990) and Heath et al. (1992) are some of the crucial works in the development of interest rate modelling. This appendix briefly explains the basic interest rate modelling framework.

C.1 Zero coupon bonds and the forward rate

Following Heath et al. (1992) we have:

Definition C.1. A zero coupon bond with maturity $T \geq 0$ guarantees to pay a nominal of 1 at maturity T . The price at time $0 \leq t \leq T$ of a zero coupon bond is denoted by $P(t, T)$, with $P(T, T) = 1$.

Definition C.2. The instantaneous forward rate is defined to be:

$$f(t, T) = -\frac{\partial \ln P(t, T)}{\partial T}. \quad (\text{C.1.1})$$

The spot interest rate is defined to be:

$$r_t = f(t, t). \quad (\text{C.1.2})$$

The forward rate, $f(t, T)$, is always constrained to be positive because the zero price, $P(t, T)$, is a decreasing function of maturity. Practitioners in interest rates initially modelled the spot interest rate directly, Heath et al. (1992) then generalized these models into the HJM framework.

The spot interest rate in the classic setup is modelled as a diffusion process, this implies that its stochastic differential equation is of the following form under $\mathbb{Q}$:

$$dr_t = \mu(t, r_t) dt + \sigma(t, r_t) dW_t, \quad (\text{C.1.3})$$

where $\mu(t, r_t)$ and $\sigma(t, r_t)$ are deterministic drift and volatility functions of t and r_t . We have considered other cases where the formulation of the spot interest rate (conversely the intensity in our setting) does not have to follow a purely diffusion process. An alternative will be for Equation (C.1.3) to follow a pure jump process (replace the Brownian motion with a Poisson process) or a jump-diffusion process (see Appendix D below). Under the $\mathbb{Q}$ measure we have that the zero coupon bond price is given by:

$$P(t, T) = \mathbb{E} \left[e^{-\int_t^T r_s ds} | \mathcal{F}_t \right]. \quad (\text{C.1.4})$$

C.2 The Heath-Jarrow-Morton framework and the T-Forward measure

Heath et al. (1992) study the modelling of forward rates and create a class of models by directly analysing the instantaneous forward rates. Heath et al. (1992) show every spot interest rate model (see Equation (C.1.3)) has an equivalent HJM representation. Lando (1998) discusses the equivalence of models in the form of Equation (C.1.3) with intensity based modelling. The form of the survival probability (a key quantity in credit risk pricing) is then similar to that of the zero bond price equation (a key quantity in rates theory). Therefore our intensity framework can carry over all the infrastructure of the interest rates world, especially the HJM framework. In particular use is made of the adapted $T - forward measure$ approach described below: in Chapter 3 (Sub-section 3.3.3) we applied an adjusted form of the $T - forward measure$ called the $T - forward survival measure$ to recover a measure from which we formulated a relationship between the credit swap spread and the intensity. Given Definition C.2 we can show a relationship between the forward rate and Equation (C.1.4), namely:

$$P(t, T) = e^{-\int_t^T f(t, u) du} = \mathbb{E} \left[e^{-\int_t^T r_u du} | \mathcal{F}_t \right], \quad (\text{C.2.1})$$

$t < s < T$ and f has $\mathbb{Q}$ dynamics given by:

$$f(t, s) = f(0, s) + \int_0^t \alpha(u, s) du + \int_0^t \sigma(u, s) dW_u, \quad (\text{C.2.2})$$

or equivalently:

$$df(t, s) = \alpha(t, s) dt + \sigma(t, s) dW_t. \quad (\text{C.2.3})$$

The key result of the HJM analysis was to recognise that the assumption of no arbitrage implies a constraint on the relationship between the drift $\alpha(t, \cdot)$ and volatility $\sigma(t, \cdot)$. This relationship is defined in the following theorem.

Theorem 31. *If there is no arbitrage and the $\mathbb{Q}$ dynamics of the forwards are given by Equation (C.2.2), then:*

$$\alpha(t, s) = \sigma(t, s) \int_0^t \sigma(u, s) du. \quad (\text{C.2.4})$$

Proof. Let:

$$X_t = \ln(P(t, T)) = - \int_t^T f(u, T) du;$$

then an application of Itô's lemma (see Shreve (1998)) gives:

$$dX_t = f(t, t) dt - \int_t^T df(t, s) ds = r_t dt - \int_t^T (\alpha(t, s) dt + \sigma(t, s) dW_s) ds.$$

Take:

$$\begin{aligned} \alpha^*(t, T) &= \int_t^T \alpha(t, s) ds; \\ \sigma^*(t, T) &= \int_t^T \sigma(t, s) ds. \end{aligned} \quad (\text{C.2.5})$$

This implies after an application of Fubini that:

$$dX_t = (r_t - \alpha^*(t, T)) dt - \sigma^*(t, T) dW_t. \quad (\text{C.2.6})$$

We are operating under $\mathbb{Q}$ hence, tradeable assets discounted to present value must be martingales. From above we have that $P(t, T) = e^{X_t}$, again we can apply Itô's lemma directly to give:

$$\begin{aligned} dP(t, T) &= e^{X_t} (r_t - \alpha^*(t, T) + \frac{1}{2} \sigma^*(t, T)^2) - e^{X_t} \sigma^*(t, T) dW_t \\ &= P(t, T) (r_t - \alpha^*(t, T) + \frac{1}{2} \sigma^*(t, T)^2) dt - P(t, T) \sigma^*(t, T) dW_t. \end{aligned}$$

This implies that:

$$d[e^{\int_0^t -r_s ds} P(t, T)] = e^{\int_0^t -r_s ds} P(t, T) (-\alpha^*(t, T) + \frac{1}{2} \sigma^*(t, T)^2) dt - e^{\int_0^t -r_s ds} P(t, T) \sigma^*(t, T) dW_t.$$

As the discounted bond price is a martingale we have that the drift is zero:

$$\alpha^*(t, T) = \frac{1}{2} \sigma^*(t, T)^2. \quad (\text{C.2.7})$$

Finally by differentiating Equation (C.2.7) we get the result. $\square$

Note also that:

$$\frac{P(t, T)}{P(0, T)} = \exp \left\{ \int_0^t (r_s - \frac{1}{2} \sigma^*(s, T)^2) ds - \int_0^t \sigma^*(s, T) dW_s \right\}.$$

Definition C.3 (The T-Forward measure). *Let $\beta(t) = e^{-\int_0^t r_s ds}$. Under the measure $\mathbb{Q}$ we have that the time t price of a random payoff $X \in \mathcal{F}_T$ is:*

$$V_t = \mathbb{E}^{\mathbb{Q}} \left[\frac{\beta(T)}{\beta(t)} X | \mathcal{G}_t \right].$$

We may define a Radon-Nikodým density process, using notation in Theorem 31, by:

$$E_t = \frac{\beta(t)P(t, T)}{P(0, T)} = \exp \left\{ -\frac{1}{2} \int_0^t \sigma^*(s, T)^2 ds - \int_0^t \sigma^*(s, T) dW_s \right\}.$$

By Girsanov's theorem (see Shreve (1998)) we have the $\mathbb{Q}$ -Brownian motion is transformed to the $\mathbb{Q}^T$ Brownian motion in the following way:

$$dW_t^{\mathbb{Q}^T} = \sigma^*(t, T) dt + dW_t^{\mathbb{Q}}. \quad (\text{C.2.8})$$

And importantly we have that:

$$\tilde{V}_t = \frac{V_t}{P(t, T)} = \mathbb{E}^{\mathbb{Q}} \left[\frac{\beta(T)}{\beta(t)P(t, T)} X | \mathcal{G}_t \right] = \mathbb{E}^{\mathbb{Q}^T} [X | \mathcal{G}_t]. \quad (\text{C.2.9})$$

So we get that the T – forward Brownian motion is defined by a shift of minus the T – maturity bond volatility.

Appendix D

Affine Models

In Chapter 3 a few different models for the intensity process were considered. The intensity models considered in Chapter 3 are affine models. In this appendix we detail the affine modelling framework.

D.1 Diffusion models

Let $(X_t)_{t \geq 0}$ be the driving stochastic process and let $(\lambda_t)_{t \geq 0}$ be the underlying process we seek to model, then assume the following definitions:

Definition D.1. *A model for the intensity is an affine diffusion if:*

- (i) $(\lambda_t)_{t \geq 0}$, the intensity, is an affine function of $(X_t)_{t \geq 0}$.
- (ii) $(X_t)_{t \geq 0}$ is an affine diffusion under $\mathbb{Q}$ with $(\forall t \geq 0)$:

$$dX_t = \mu(X_t) dt + \sigma(X_t) dW_t, \tag{D.1.1}$$

and with the following holding:

- (a) $\mu(X_t) = K_0 + K_1 X_t$;
- (b) $\sigma^2(X_t) = H_0 + H_1 X_t$;
- (c) $\lambda(X_t) = \kappa_0 + \kappa_1 X_t$.

If we take $\kappa_0 = 0$ and $\kappa_1 = 1$ and the drift process μ to be of the form $a(b - X_t)$; where $K_0 = ab$ and $K_1 = -a$ with $a, b > 0$ the structure then ensures that if the current state of $(X_t)_{t \geq 0}$ is below its long run mean, b , then the drift changes sign and pushes the process (on average) back to its long run mean. $(W_t)_{t \geq 0}$ disturbs the process $(X_t)_{t \geq 0}$ from moving

back to b ; the shocks are normally distributed with the magnitude and nature of the shocks being determined by $\sigma(X_t)$. If σ is constant or time dependent then $(X_t)_{t \geq 0}$ is a Gaussian process (defined in Appendix E (Section E.1)). If σ is state dependent then these normally distributed shocks translate into shocks which introduce heteroscedasticity into $(X_t)_{t \geq 0}$. The following figure, Figure D.1, shows the effects of mean reversion:

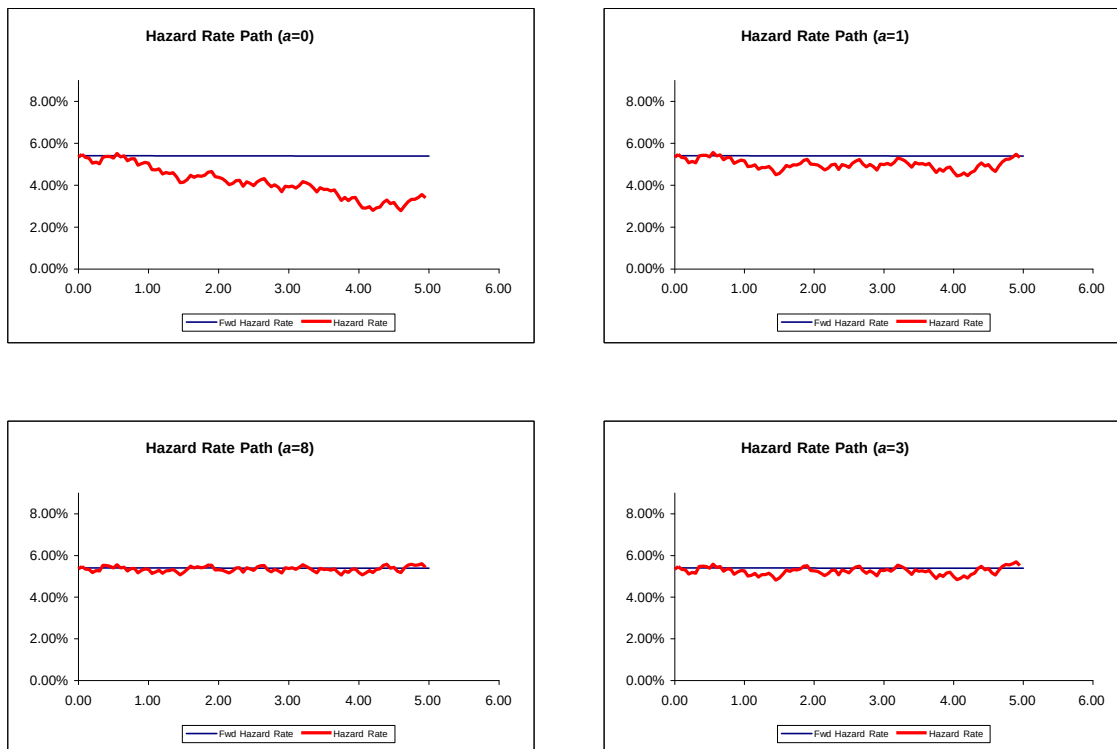


Figure D.1: The effects of varying the parameter a (the speed of mean reversion), defined in Definition D.1. The model used is the extended Vasicek model (see Chapter 3 (Section 3.3) for more details) with a flat credit swap spread curve of 400bps. We use a constant spot volatility, $\sigma = 15\%$.

D.2 Jump diffusion models

Definition D.1 makes the assumption of the model being a diffusion process. It is noteworthy that in CDS¹ data we often have that there are overnight jumps in credit spreads which mean diffusion-based intensities may be inadequate. The reader is reminded that:

- We are not commenting here on the (diffusion-based) structural approach, discussed in Appendix A, where a diffusion variable hits a barrier to generate default times. In that case default times do not occur as a surprise (default times under the setting are predictable stopping times, see Bielecki and Rutkowski (2002)).
- In this case we are considering an intensity model which is generated by a diffusion process.
- Thus the default time, τ , produced in this case is totally inaccessible (see Guo and Zeng (2006)). Hence, defaults occur as a surprise.
- However by specifying the intensity as being generated by a diffusion process we do not generate surprise in the spread dynamics.

If, on the other hand, we were to construct an intensity with jumps we will have a double jump process in so far as the point process jump times arrive with an intensity that is itself a jump process, so that:

$$\mathbb{Q}(\Delta X_t = X_t - X_{t-} > 0) > 0 \quad \forall t > 0.$$

As detailed by Gouriéroux et al. (2005) in affine models if we specify a jump process we must separate out the timing of jumps from the size of the jumps. That is, to retain tractability, either the jump distribution will be state dependent or the jump size distribution will be state dependent but we should not seek both. Formally, then, let:

$$dX_t = \mu(X_{t-}) dt + \sigma(X_{t-}) dW_t + dJ_t, \tag{D.2.1}$$

where all of Definition D.1 is satisfied and $(J_t)_{t \geq 0}$ is a pure jump process with arrival intensity, Λ , and jump distribution, ν , which are independent of each other. We consider the

¹The CDS swap spread represents the riskiness of an obligor and also has a strong relationship with the intensity. In fact, under stylised assumptions we can show that the CDS swap spread, s , and the intensity, λ , are related by the following equation:

$$s = \lambda \times (1 - \text{recovery}).$$

This enables us to interchange our language between spread and intensity when thinking conceptually.

structure presented by Duffie et al. (2000) and state the important *Proposition 1* in their paper. Let:

$$\begin{aligned}
\mu(X_t) = a(b - X_t) &= K_0 + K_1 X_t \text{ with } K = (K_0, K_1) \\
\sigma^2(X_t) &= H_0 + H_1 X_t \text{ with } H = (H_0, H_1) \\
\Lambda(X_t) &= L_0 + L_1 X_t \text{ with } L = (L_0, L_1) \\
R(X_t) &= P_0 + P_1 X_t \text{ with } P = (P_0, P_1),
\end{aligned} \tag{D.2.2}$$

further take, in the most general sense $c \in \mathbb{C}$, a complex number and define:

$$\theta(c) = \int_{\mathbb{R}} e^{(c-z)} d\nu(z),$$

with the process $(J_t)_{t \geq 0}$ a Poisson process with jump intensity that is affine, given in Equalities D.2.2. Duffie et al. (2000) develop the concept of a “discounted conditional transform function” which fully describes the process $(X_t)_{t \geq 0}$ (described in Equation (D.2.1)); let $\chi = (K, H, L, \theta, P)$ be the set of “coefficients”, then χ fully describes the process $(X_t)_{t \geq 0}$.

Proposition 32 (Proposition 1 of Duffie et al. (2000)). *Let:*

$$\psi^\chi(u, X_t, t, T) = \mathbb{E}[e^{-\int_t^T R(X_s) ds} e^{uX_t} | \mathcal{G}_t],$$

we say χ is well behaved at $(u, T) \in \mathbb{C} \times [0, \infty)$ in the sense that:

- (i) $\mathbb{E}[\int_0^T |\gamma_t| dt] < \infty$ where $\gamma_t = \Psi_t(\theta(B(t)) - 1)\Lambda(X_t)$.
- (ii) $\mathbb{E}[\int_0^T |\eta_t| dt] < \infty$ where $\eta_t = \Psi_t B(t)\sigma(X_t)$.
- (iii) $\mathbb{E}[|\Psi_t|] < \infty$ where $\Psi_t = \exp^{-\int_0^t R(X_s) ds} e^{A(t)+B(t)X_t}$.
- (iv) B and A are the unique solutions of:

$$\begin{aligned}
\frac{\partial B(t)}{\partial t} &= P_1 - K_1 B(t) - \frac{1}{2} H_1 B^2(t) - L_1(\theta(B(t)) - 1), \\
\frac{\partial A(t)}{\partial t} &= P_0 - K_0 B(t) - \frac{1}{2} H_0 B^2(t) - L_0(\theta(B(t)) - 1).
\end{aligned} \tag{D.2.3}$$

If χ is well behaved at (u, T) then the transform, ψ^χ , of X has form:

$$\psi^\chi(u, x, t, T) = e^{A(t,T)+B(t,T)x}. \tag{D.2.4}$$

For our purpose, in this thesis, the jump times will be determined by a Poisson process with a deterministic intensity and a jump size with some distribution which produces an analytic transform θ . Equation (D.2.4) and more generally the work of Duffie et al. (2000) is a key achievement that allows us to show that if $(\lambda_t)_{t \geq 0}$ is an affine function of $(X_t)_{t \geq 0}$ then (see Mortensen (2006) who discusses more fully the scope of the affine framework for credit and portfolio pricing):

$$\mathbb{E} \left[e^{-\int_t^T \lambda_s ds} (d + e\lambda_s) e^{uX_T} | \mathcal{G}_t \right], \quad (\text{D.2.5})$$

has tractable solutions (where d , e and u are constants). We can choose the constants in such a way to help solve many pricing problems. In particular the survival function has form:

$$\mathbb{E} \left[e^{-\int_t^T \lambda_s ds} | \mathcal{G}_t \right] = e^{A(t,T) + B(t,T)\lambda_t}. \quad (\text{D.2.6})$$

where A and B solve Equation (D.2.3).

Appendix E

Extended Vasicek Model

In this appendix we consider solutions to the extended Vasicek model.

E.1 Gaussian processes

The extended Vasicek model is a Gaussian process:

Definition E.1. A Gaussian process, $(X_t)_{t \geq 0}$, is a stochastic process with the property that for every set of times $0 \leq t_1 \leq \dots \leq t_n$, $X_{t_1}, \dots, X_{t_n}$ are jointly normal.

Gaussian processes are completely defined by their means and covariances:

$$m(t) = \mathbb{E}[X_t] \quad v(s, t) = \mathbb{E}[(X_s - m(s))(X_t - m(t))].$$

The joint density of $X_{t_1}, \dots, X_{t_n}$ is:

$$\mathbb{Q}[X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n] = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \mathbf{m}(\mathbf{t})) \Sigma^{-1} (\mathbf{x} - \mathbf{m}(\mathbf{t}))^\top \right\} dx_1, \dots, dx_n,$$

where

$$\Sigma = \begin{pmatrix} v(t_1, t_1) & v(t_1, t_2) & \cdots & v(t_1, t_n) \\ v(t_2, t_1) & v(t_2, t_2) & \cdots & v(t_2, t_n) \\ \vdots & \vdots & \ddots & \vdots \\ v(t_m, t_1) & v(t_m, t_2) & \cdots & v(t_m, t_n) \end{pmatrix}, \quad (\text{E.1.1})$$

and $\mathbf{x}$ and $\mathbf{t}$ are row vectors and $\mathbf{m}(\mathbf{t})$ is the row vector of expectations; the moment generating function is:

$$\mathbb{E} \left[\exp \left\{ \sum_{i=1}^n u_i X_{t_i} \right\} \right] = \exp \left\{ \mathbf{u} \cdot \mathbf{m}(\mathbf{t})^\top + \frac{1}{2} \mathbf{u} \Sigma \mathbf{u}^\top \right\}, \quad (\text{E.1.2})$$

with $\mathbf{u} = \{u_1, \dots, u_n\}$ a row vector.

Brownian motion is a Gaussian process with $m(t) = 0$ and $v(s, t) = s \wedge t$.

Theorem 33. *Let W_t be a Brownian motion and let δ and h be non-random functions.*

Define:

$$Z_t = \int_0^t \delta(u) dW_u, \quad Y_t = \int_0^t h(u) Z_u dW_u,$$

then Z_t is a Gaussian process with mean function 0 and covariance:

$$v(t, s) = \int_0^{t \wedge s} \delta(u)^2 du. \quad (\text{E.1.3})$$

Similarly Y_t is a Gaussian process with mean function 0 and covariance:

$$v_Y(s, t) = \int_0^{t \wedge s} \delta(u)^2 \left(\int_u^s h(y) dy \right) \left(\int_u^t h(y) dy \right) du. \quad (\text{E.1.4})$$

Proof. See Shreve (1998). □

E.2 Model solution

The extended Vasicek model is described by the SDE:

$$d\lambda_t = (b(t) - a(t)\lambda_t) dt + \sigma(t) dW_t, \quad (\text{E.2.1})$$

where a, b and σ are non-random positive functions of t . Following Shreve (1998) set:

$$K(t) = \int_0^t a(u) du.$$

Below we provide the standard solution to the SDE:

$$d(e^{K(t)} \lambda_t) = e^{K(t)} (a(t)\lambda_t dt + d\lambda_t) = e^{K(t)} (b(t) dt + \sigma(t) dW_t),$$

integrating we get:

$$e^{K(t)} \lambda_t = \lambda_0 + \int_0^t e^{K(u)} b(u) du + \int_0^t e^{K(u)} \sigma(u) dW_u,$$

which implies:

$$\lambda_t = e^{-K(t)} \left[\lambda_0 + \int_0^t e^{K(u)} b(u) du + \int_0^t e^{K(u)} \sigma(u) dW_u \right], \quad (\text{E.2.2})$$

we have that $e^{K(u)} \sigma(u) \in \mathcal{L}^2$ and by Theorem 33 this implies that Equation (E.2.2) is Gaussian with mean function given by:

$$m_{\lambda_t}(t) = e^{-K(t)} \left[\lambda_0 + \int_0^t e^{K(u)} b(u) du \right],$$

and covariance given by:

$$v_{\lambda_t}(t, s) = e^{-K(t)-K(s)} \int_0^{s \wedge t} e^{2K(u)} \sigma^2(u) du.$$

We are interested in:

$$Q(0, T) = \mathbb{E} \left[e^{-\int_0^T \lambda_t dt} \right],$$

the survival probability of the obligor. We want to find the distribution of $\int_0^T \lambda_t dt$ and we will have $Q(0, T)$ by the moment generating function of a normal distribution, Equation (E.1.2).

Let:

$$Z_t = \int_0^t e^{K(u)} \sigma(u) dW_u \quad Y_T = \int_0^T e^{-K(t)} Z_t dW_t,$$

then we have:

$$\lambda_t = e^{-K(t)} \left[\lambda_0 + \int_0^t e^{K(u)} b(u) du \right] + e^{-K(t)} Z_t,$$

and:

$$\int_0^T \lambda_t dt = \int_0^T e^{-K(t)} \left[\lambda_0 + \int_0^t e^{K(u)} b(u) du \right] dt + Y_T.$$

By Theorem 33 $\int_0^T \lambda_t dt$ is also Gaussian and its mean function is given by:

$$m_{(\int \lambda_t)}(T) = m(T) = \int_0^T e^{-K(t)} \left[\lambda_0 + \int_0^t e^{K(u)} b(u) du \right] dt$$

and covariance given by:

$$v_{(\int \lambda_t)}(T, T) = v(T, T) = \mathbb{E} [Y^2(T)] = \int_0^T e^{2K(v)} \sigma^2(v) \left(\int_v^T e^{K(y)} dy \right)^2 dv$$

The survival probability at time T is thus given by:

$$\begin{aligned} Q(0, T) &= \mathbb{E} \left[e^{-\int_0^T \lambda_t dt} \right] = \exp \left\{ (-1)m(T) + \frac{1}{2}(-1)^2 v(T, T) \right\} \\ &= e^{\lambda_0 B(0, T) - A(0, T)}, \end{aligned} \quad (\text{E.2.3})$$

with:

$$B(0, T) = - \int_0^T e^{-K(t)} dt,$$

and:

$$A(0, T) = - \int_0^T \int_0^t e^{-K(t)+K(u)} b(u) du dt + \frac{1}{2} \int_0^T e^{2K(v)} \sigma^2(v) \left(\int_v^T e^{-K(y)} dy \right)^2 dv.$$

Exchanging the order of integration:

$$A(0, T) = - \int_0^T \left[e^{K(v)} b(v) \left(\int_v^T e^{-K(t)} dt \right) - \frac{1}{2} \int_0^T e^{2K(v)} \sigma^2(v) \left(\int_v^T e^{-K(y)} dy \right)^2 \right] dv.$$

Further Shreve (1998) shows that:

$$Q(t, T) = \mathbb{E} \left[e^{-\int_t^T \lambda_u du} | \mathcal{F}_t \right] = e^{\lambda_t B(t, T) + A(t, T)}, \quad (\text{E.2.4})$$

with:

$$A(t, T) = - \int_t^T \left[e^{K(v)} b(v) \left(\int_v^T e^{-K(t)} dt \right) - \frac{1}{2} e^{2K(v)} \sigma^2(v) \left(\int_v^T e^{-K(y)} dy \right)^2 \right] dv,$$

and:

$$B(t, T) = -e^{K(t)} \int_t^T e^{-K(s)} ds.$$

Appendix F

The Factor Framework And Copula Algorithms

The aim of this appendix is to explain how calculations for CDOs are performed under the factor framework.

F.1 Recursion Method

The basic insight of Andersen et al. (2003) is that the CDO price in the factor framework is essentially a linear combination of the expected loss of the tranche at each time in the calculation time line (assume a discretised set of times $\{t_1, \dots, t_M\}$) similar in spirit to the work done in Chapter 3 (Sub-section 3.3.4), where we calibrated the model by reducing the pricing to a weighted sum of survival probabilities:

$$\text{Expected loss value of a tranche} = \sum_{i=0}^n \alpha_i \mathbb{E}(L_{t_i}^{tranche}),$$

where $t_i \in \{t_1, \dots, t_M\}$. Furthermore, the quadrature methods (in our case we use Gauss Hermite) enables the evaluation of $\mathbb{E}(L_{t_i}^{tranche})$ by calculation of a weighted sum of conditional expected losses $\mathbb{E}(L_{t_i}^{tranche}|z)$ for various values of z .

F.1.1 Generic method of extracting the expected loss

Assume we have probabilities $p_1(z), \dots, p_n(z)$ of independent random variables, $a_1, \dots, a_n$ taking values $\beta_1, \dots, \beta_n$ and zeros otherwise. Define the portfolio loss to be:

$$L = \sum_{i=1}^n a_i.$$

The recursion method allows us to build up the loss distribution, $(L_t)_{t \geq 0}$, of a sum of random variables whose values are β_i given a default. To do this we require an arbitrary loss unit, u , so that the loss amount $\beta_i \forall i \in \{1, \dots, n\}$ can be approximated well by integer multiples of u , so that $\beta_i = k_i u$ for some positive integer k_i , for ease of exposition and without loss of generality assume that $u = 1$.

Let L_m , $m \in \{1, \dots, n\}$ be the portfolio loss, measured in loss units in the sub-set consisting of the first m reference obligors $a_1, \dots, a_m$ (the ordering does not matter):

1. Beginning with $m = 0$ we have:

$$\mathbb{Q}(L_0 = k|z) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

2. If we add an asset, there are three possible outcomes on the loss:

$$\begin{aligned} \mathbb{Q}(L_1 = 0 | z) &= 1 - p_1(z) \\ \mathbb{Q}(L_1 = k_1 | z) &= p_1(z) \\ \mathbb{Q}(L_1 = k_i | z) &= 0 \quad \forall k_i \notin \{k_1, 0\}. \end{aligned} \tag{F.1.1}$$

3. We can build up the conditional loss distribution by adding each asset one by one until we get to the last asset and thus we would have specified the full conditional loss distribution using the following recursion algorithm:

$$\mathbb{Q}(L_{m+1} = k|z) = p_{m+1}(z) \times \mathbb{Q}(L_m = k - k_{m+1}|z) + (1 - p_{m+1}(z)) \times \mathbb{Q}(L_m = k|z) \tag{F.1.2}$$

Once we have the conditional portfolio loss distribution, after using Equation (F.1.2), the conditional loss distribution of a tranche, $L^{tranche}(z)$, can be recovered by transforming the conditional portfolio loss, $L(z)$. This is done by using Equation (2.3.1) in Chapter 2. We can then recover the conditional expected value of the tranche and from this the unconditional expected loss is found via quadrature.

F.2 Bespoke CDO Pricing

Recall that the present value of a bespoke CDO can be decomposed into the MTM of two base tranches via Equation (4.5.1) in Chapter 4.

In this section we consider how one can recover the correlation values of a bespoke tranche. These correlation values will correspond to the subordination and detachment points of a

tranche. Call these values ρ^{k_1} and ρ^{k_2} ; where k_1 is the tranche subordination and k_2 is the tranche detachment.

Given a base correlation curve corresponding to the same portfolio and having the same maturity as the bespoke tranche we seek to value it is relatively easy to recover the correlations ρ^{k_1} and ρ^{k_2} . In order to recover ρ^{k_1} and ρ^{k_2} it is sufficient to recover the loss ratios of the respective bespoke subordination and detachment points, see Chapter 4. This can be done by iteratively looking for a loss-ratio, r^* , which from the base correlation curve is mapped to a correlation value, ρ^* . r^* is constrained to satisfy:

$$r^* = \frac{\overline{\mathbf{val}}_L^{Prot}(k, \rho^*)}{\overline{\mathbf{val}}_L^{Prot}(Index)}.$$

F.2.1 Mixing base correlation

The method above is sufficient whenever we are pricing a tranche with the same portfolio and the same maturity as the constructed base correlation curve. When we want to value tranches of non standard maturities or portfolios (for example a 4y trade, which is between the standard 3y and 5y maturities) the market standard, see Turc et al. (2006) is to:

- Take the interpolated weighted sum of the correlation and loss ratios of the standard maturities to form the new off market maturity base correlation curve. For example, assume the correlation (ordinate) and corresponding loss ratio (abscissa) for a 3y maturity is ρ_3 and r_3 and the similar ordinate correlation and abscissa loss-ratio for the 5y (similar in that it corresponds to the same base tranche as the 3y we are looking at) are ρ_5 and r_5 . From this we can construct the new correlation and loss-ratio for that base tranche with a maturity between 3y and 5y by taking the weighted sum of the 3y and 5y ordinates and abscissas respectively.
- When one has a portfolio of mixed obligors from Europe and North America the practice is to mix the correlation curves, again by taking a weighted average sum of the standard Itraxx and CDX tranches where the weight corresponds to the proportion of European and North American obligors respectively (see Turc et al. (2006)).

These methods are driven more by a need for market practicality than any justifiable arbitrage free construction.

F.3 The basic computational algorithm for the Threshold and Survival copulas

Jouanin et al. (2001) describes the computational algorithm for both the Threshold and Survival copulas. In the case of the threshold copula we have:

1. Simulate $(\zeta_1, \dots, \zeta_n)$ from the copula C^T .
2. Compute $\eta_i = -\ln \zeta_i \forall i \in \{1, \dots, n\}$.
3. For each reference obligor simulate the intensity process to compute Γ_t^i . Stop when $\Gamma_t^i \geq \eta_i$ and take $\tilde{\tau}_i = t$, the default time.

In the case of the survival copula we have:

1. Simulate $(\zeta_1, \dots, \zeta_n)$ from the copula C^S .
2. Compute the default time, $\tilde{\tau}_i = Q_i^{-1}(\zeta_i) \forall i \in \{1, \dots, n\}$.

Appendix G

Market Data

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
ACCOR	105	105	103	101	99	99	99	99	99	99
AEGON	100	105	110	115	120	117	115	115	115	115
AERO	245	245	245	245	245	240	235	230	225	220
AKZO	126	130	132	134	136	132	128	126	124	122
ALZ	74	76	78	80	82	82	82	82	82	82
ASSGEN	100	90	85	82	80	79	78	77	76	75
AUCHAN	95	85	80	82	78	77	76	75	74	74
AVEFP	53	56	60	61	63	63	63	63	63	63
AVLN	54	56	58	59	60	59	58	57	56	55
AXASA	215	205	200	200	200	200	200	200	200	200
AVL	143	143	138	133	128	127	126	125	124	123
BAGR	43	44	45	47	48	48	48	48	48	48
BASF	144	144	144	144	144	144	144	144	144	144
BATSLN	74	76	78	79	80	81	82	83	84	85
BAVB	70	72	74	76	78	79	80	82	84	86
BBVSM	73	68	68	68	68	68	68	68	68	68
BERTEL	102	102	97	92	90	89	88	87	86	85
BMW	235	245	245	245	245	245	245	245	245	245
BNP	123	151	160	157	155	149	142	140	138	136
BPLN	74	73	72	71	70	70	70	70	70	70
BRITEL	38	39	39	40	40	40	41	42	42	43
BSNSA	155	160	167	171	175	176	177	178	179	180
BYIE	50	52	54	58	62	62	62	62	62	62
CARR	52	54	56	58	60	61	62	63	64	65
CBRY	38	42	44	47	50	51	52	53	54	55
CENTRI	36	38	40	42	44	44	44	44	44	44
CLAR	58	63	64	65	65	64	63	61	60	59
CMZB	275	285	295	285	275	270	265	255	245	235
CONTI	95	95	95	95	95	95	95	95	95	95
CPGLN	1,140	1,000	910	835	755	730	695	675	650	625
CRDIT	36	40	44	47	50	50	50	50	50	50
CRDSUI	125	120	115	115	115	115	115	115	115	115
DB	93	93	93	93	93	93	93	93	93	93
DCX	99	99	99	94	89	89	89	89	89	89
DIAG	118	140	145	145	150	145	140	138	135	130
DPW	52	56	59	62	65	65	65	65	65	65
DSM	44	47	50	53	56	57	58	59	60	61
DT	62	64	66	68	70	70	70	71	72	73
EAUG	92	102	110	110	110	111	112	113	114	115
EDF	106	109	112	114	116	116	116	116	116	116
EDNIM	63	63	64	64	65	64	63	62	61	60
ELEPOR	82	82	82	82	80	79	78	77	76	75
ELTLX	72	84	86	86	84	82	80	80	80	80
ENBW	60	70	78	82	85	83	81	81	81	81
ENEL	46	48	49	49	50	49	49	49	48	48
EQAGR	133	144	150	142	130	123	116	113	109	106
ESPAN	70	71	72	73	71	71	71	71	71	71
EXHO	125	125	125	122	120	120	120	120	120	120
FINMEC	35	42	49	52	55	55	55	55	55	55
FRTEL	80	85	90	85	85	80	80	75	70	70
FRUM	36	40	44	47	50	52	54	55	56	57
GASNAT	58	62	62	62	62	62	62	61	61	60
GAZDF	98	128	157	185	210	229	245	261	269	279
GKNLN	55	53	51	50	50	49	48	47	46	45
GLENC	345	350	355	345	335	325	315	310	305	300
GROUPE	675	760	735	705	675	665	655	645	635	625
GUSLN	85	95	110	115	120	117	115	113	113	113
HANRUE	33	36	38	42	46	46	46	46	46	46
HBOS	66	63	63	61	58	56	50	50	50	50
HENKEL	76	100	122	144	164	178	191	203	210	217
HOLZSW	70	72	74	74	74	72	70	68	66	64
IBERDU	246	252	258	264	270	260	250	245	240	235

Figure G.1: Part 1: obligor swap spread data for the Itraxx index on the 09 June 2009. Data source from *MarkIt*

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
JTI	105	113	113	113	113	111	109	109	108	108
KPN	27	28	29	30	31	32	33	34	35	36
LAFCP	51	54	57	60	63	64	65	66	67	68
LINDE	238	248	258	268	278	268	258	253	248	243
LORFP	73	74	75	76	77	78	79	80	81	82
LUFTHA	420	510	600	578	545	522	498	493	488	483
METFNL	155	160	165	165	165	160	155	155	155	155
MICH	220	225	230	233	235	230	225	225	225	225
MKS	165	175	185	180	175	170	165	160	155	150
MOET	175	180	185	190	195	187	185	185	185	185
MONTE	50	55	59	62	65	65	65	65	65	65
MUNRE	87	82	82	82	80	80	80	80	80	80
NESTLE	54	53	52	51	50	49	48	47	46	45
NGGLN	40	43	45	50	55	55	55	55	55	55
NXT	123	128	133	127	120	118	117	115	114	113
OTE	120	125	130	135	140	135	130	130	130	130
PEUGOT	72	81	85	90	95	96	97	98	99	100
PHG	295	305	305	295	285	270	255	250	245	240
PORTEL	90	100	105	108	110	107	104	104	104	104
PRTP	95	97	99	102	105	106	107	108	109	110
PSON	275	285	290	290	290	280	270	270	270	270
RBS	58	61	63	64	65	61	57	57	57	57
REEDLN	170	170	170	170	170	169	168	167	166	165
RENAUL	49	54	59	61	62	63	64	65	66	67
REP	385	405	395	390	380	365	350	340	335	330
RNTKIL	132	138	149	147	145	141	139	135	132	129
RTRGRP	94	124	151	177	202	220	236	251	259	268
RWE	18	23	28	28	28	28	28	28	28	28
SANTAN	48	52	52	52	52	52	51	51	50	50
SBRY	91	91	91	91	89	89	89	89	89	89
SCACAP	68	73	78	80	83	83	83	83	83	83
SCHREI	104	106	108	110	112	108	104	102	100	98
SIEM	245	255	255	245	235	227	225	223	220	215
SOCGEN	86	96	101	103	105	105	105	105	105	105
STGOBN	100	100	100	100	95	95	95	95	95	95
STM	150	160	170	175	180	177	174	170	166	162
SUEDZU	85	95	100	100	100	95	93	93	93	93
SWEMAT	85	90	95	98	100	98	96	96	96	96
TATELY	65	70	75	75	80	80	85	85	85	85
TELEFO	130	135	140	140	140	138	136	134	132	130
TELNOR	94	99	104	107	110	111	112	113	114	115
TTIM	72	94	115	136	154	168	180	192	198	205
TKA	185	185	190	200	210	212	214	216	218	220
TKAGR	78	82	86	87	88	88	88	88	88	88
TLIASS	415	455	480	465	455	440	430	420	415	410
TOTAL	57	62	65	68	70	70	70	70	70	70
TPG	52	57	61	64	64	64	64	64	64	64
TSCO	70	75	79	82	85	85	85	85	85	85
UBS	73	78	83	88	93	93	93	93	93	93
ULVR	135	135	135	130	125	125	125	125	125	125
UNFSM	35	37	38	40	42	43	44	45	46	47
UU	190	204	218	209	200	197	194	190	186	184
VATEAL	55	56	58	59	60	60	60	60	60	60
VIEFP	52	57	60	60	60	60	60	60	60	60
VINCI	120	127	129	125	122	122	122	121	120	119
VLOF	106	110	114	118	122	118	114	112	110	108
VLVY	242	252	242	237	232	222	212	205	198	191
VOD	400	405	400	385	370	360	350	340	335	330
VW	85	88	90	93	95	96	97	98	99	100
WOLKLU	168	178	188	187	190	181	176	173	171	169
WPPLN	48	51	54	57	60	62	63	63	63	63
ZURNVX	230	230	230	235	240	237	235	235	235	235

Figure G.2: Part 2: obligor swap spread data for the Itraxx index on the 09 June 2009. Data source from *MarkIt*

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