ABSTRACT.

In Part 1 of this work, new definitions are proposed for astigmatism, sagittal image curvature, and tangential image curvature. The definitions give rise to a unique number for the aberration encountered at any refracting or reflecting surface, and remain finite, determinate, and meaningful in all cases. The aberrations described by the new definitions are complete in that they contain all orders.

Several new relations are obtained for refraction in the sagittal section, and an extension to the well-known equation for refraction in this secyion is made. A complete and unique definition is described for distortion, and a simple relation between distortion and sagittal curvature is evaluated,- a relation which corresponds to the first order equation connecting these two aberrations.

The complete astigmatism and sagittal image curvature are evaluated, and it is shown how the first order terms for the aberration are contained in the complete expressions.

It is shown that it is possible to obtain aberration values which approach the exact values much more closely than those obtained from first order theory, by substituting suitable approximate values for the constants and variables in the closed expressions.

Verifying calculations are carried out on a three lens system of the Cooke type.

ABSTRACT (Continued)

Part 2 is concerned with the fine correction of optical systems using differential formulae.
The method proposed eliminates the necessity for tracing a large number of rays and consists essentially of finding the partial differential coefficients of the aberrations for a typical set of rays which have been traced through the system established by first order theory.

It is shown that in general, a change in a construct--ional parameter of the system gives rise to two types of focal shift, longitudinal and transverse, and these are discussed in detail.

The increments in sagittal and tangential image curvature and distortion are evaluated for changes in the three constructional parameters, curvature, axial displacement, and refractive index.

Verifying calculations are carried out on the three lens system already used and it is shown that the differential method greatly facilitates the final correction of a lens system.

THE THEORY OF ASTIGMATISM IN OPTICAL SYSTEMS.

by

Gordon Black, B.Sc., A.Inst.P.,

Imperial College, London.

Thesis submitted for the degree of Doctor of Philosophy in the Faculty of Science in the University of London.

CONTENTS

CHAPTER 1. Page

Historical Notes.....1.

Wave and ray optics.....2.

Measures of astigmatism.....3.

The requirements of a measure of astigmatism.....5.

Paraxial angular variables.....8.

The definition of complete astigmatism.....11.

The definition of complete sagittal curvature.....12.

The definition of complete tangential curvature.....12.

CHAPTER 2.

The evaluation of the complete astigmatism.....14.

Refraction in the sagittal section.....20.

Reference ray identities.....27.

The evaluation of the complete sagittal curvature.....28.

The relation between sagittal curvature and distortion.....37.

An indirect method for tracing sagittal pencils.....42.

CHAPTER 3.

The calculation of the complete astigmatism, sagittal curvature and distortion.....45.

Approximate calculations.....56.

CONTENTS

PART 2 CHAPTER 1. Page

Introduction and general formulae.....63.

The evaluation of $\partial W_o / \partial c$ in the meridian plane.....70.

Focal shift formulae.....73.

CHAPTER 2.

Proof that (hh_s) and (hh_T) at the exit pupil can be regarded as constant.....79.

The increment in $(nuu_T \delta T)$ produced by a small change in the constructional parameter.....81.

The evaluation of $\partial W_o / \partial c$ in the sagittal section.....85.

The evaluation of ∂W_o for small changes in axial displacement and refractive index in the meridian section.....90.

The evaluation of ∂W_o for small changes in axial displacement and refractive index in the sagittal section.....94.

Note on the change in distortion due to a small change in a constructional parameter.....96.

CHAPTER 3.

The calculation of sagittal and tangential image curvature by differential formulae.....98.

The theory of astigmatism in optical systems.

PART 1.

Introduction

Historical notes.

The discovery of the principles of astigmatism must be credited to Newton (1), though it seems likely that the extensive aberration encountered when observing an object by light reflected at large angles of incidence and refraction from a spherical surface must have been noticed earlier.

The history of astigmatism begins with the investigations carried out by Isaac Barrow (2), Newton's tutor, who considered by graphical methods the positions of the images formed by oblique rays incident on the plane and the sphere. The work of Barrow is confined entirely to the tangential section or plane of incidence, and the rule which he gives for the location of the image point is contained in the present day equation for refraction in the tangential section.

Newton was intimately connected with this work and continued it by drawing attention to the fact that the foci formed by image forming rays in the sagittal and tangential sections would not, in general be coincident. This result is in fact the first account of the aberration, through Newton attached little importance to his work.

The work of Newton and Barrow was extended by Robert Smith (3) who again concerned himself mainly with refraction at planes and spheres in the tangential section. Smith apparently did not realise the importance of Newton's work and regarded the subject of astigmatism as little more than an optical curiosity.

It was not until more than a century after Newton's work that the

astigmatism of the human eye was discovered first by Thomas Young in 1801, when an account of it appears (4), and later in 1807. Young drew attention to the importance of Newton's work in the field, and produced the equation for refraction in the sagittal section.

The two equations for refraction in the tangential and sagittal sections will be called the Barrow-Young equations and these will be used in much of the following work.

The astigmatism of the eye was discovered, probably independently by Airy (5), who found that one of his own eyes suffered from both myopia and astigmatism. These defects he corrected using a sphero-cylindrical lens which he had made to his own specification. Airy then turned his attention to eyepieces (6), and investigated the influence of astigmatism in the eye with respect to the performance of various types of eyepieces and objectives used with the camera obscura. The work of Smith, Young, and more especially Airy formed the basis of Coddington's work on optics (7), where the Barrow-Young equations appear in their modern form.

From this time onwards the study of astigmatism ceases to be a geometrical excercise, and enters the field of practical optics. In the first half of the 19th century much was written on the subject by Malus, Sturn, Hamilton, and Kummer, and in the latter half by Helmholtz, Quincke, Cornu, Maxwell, Lippich and others. The investigations of the above mentioned writers were mainly mathematical, although several gave rise to results which led to improvments in photographic lens design.

Wave and Ray optics.

In optical theory we regard a point source of light as the centre of a series of expanding spherical waves and a ray of light as a line drawn

from the source normal to the wave fronts. In other words the connection between wave and ray optics is that the wave-fronts are parallel surfaces orthotomic to the ranges of rays, and rays are wave-normals.

Thus in ray optics we may define the paraxial focus of an axially symmetrical optical system as the limiting intersection point of rays which become increasingly near to the optical axis. This point is thus the centre of curvature of spherical image forming wave-fronts.

Now if we consider a ray from an extra axial object point, we no longer have symmetry about the ray, and foci will be formed along the ray corresponding to the different sections. The wave surface is astigmatic in this case, and the maximum and minimum curvatures of the wave surface lie in the principal sections, the astigmatic line foci passing through the principal centres of curvature.

Measures of astigmatism.

Astigmatism has been defined by previous writers in terms of ray optics. Taylor (8) defines the aberration for a centred optical system as the difference of curvature of circles drawn through the two astigmatic foci, tangent to the paraxial image plane. Fig (1) shows a reference ray PQ, intersecting the paraxial image plane at Q and having astigmatic foci situated at S and T.

In this case $\left[\frac{1}{e_T} - \frac{1}{e_S}\right]$ is the measure of astigmatism. The measure however applies only to systems which are symmetrical about the optical axis, and furthermore, it becomes meaningless when the paraxial image plane for the system is formed at infinity. When this occurs other measures must be employed.

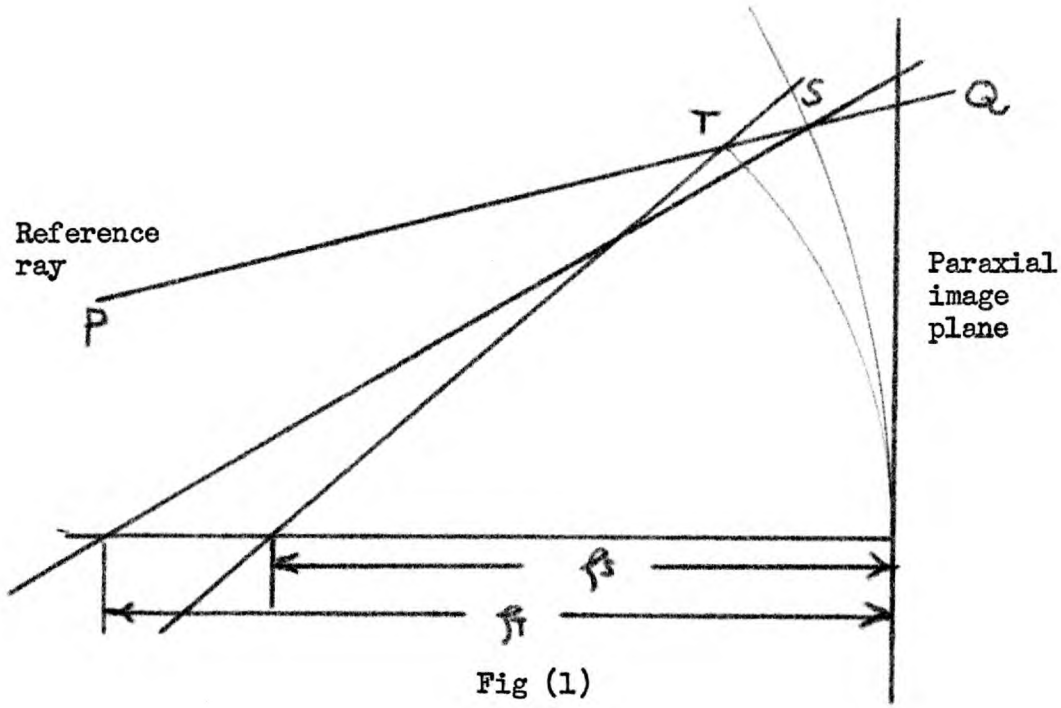


Fig (1)

Conrady (9) related the primary astigmatism to the Petzval curvature of the system and drew attention to the fact that the Petzval curve was the locus of points free from primary astigmatism.

M. von Rohr (10) defined astigmatism in terms of the reciprocal of the distances along the reference ray of the astigmatic foci from the surface concerned. If these distances are s and t, then the surface contribution to astigmatism is given by the change on refraction of :-

$$n\left[\frac{1}{t} - \frac{1}{s}\right]$$

where n is the refractive index of the medium concerned. When this definition of astigmatism is used, a difficulty arises when the sagittal or tangential object point is very near to the surface; then the amount of aberration as defined here becomes infinite, and in fact the measure of the astigmatism encountered is directly related to the position of

the surface concerned.

In the case of a field lens when either the in-coming sagittal or tangential focus may be situated on the refracting surface the above measure is infinite, when the amount of aberration arising at the surface is in fact finite.

The requirements of a measure of astigmatism.

Given optical space extending from $-\infty$ to $+\infty$, we can say that the rays in this space form a normal congruence, and that the associated wave surfaces are orthotomic to the rays. Thus any measure of astigmatism should describe at least implicitly, the disturbance at any point in the space, since the associated diffraction image is a fixed and definite focal distribution of light.

Further it is necessary that the measure should be proportional to the asphericity of the wave-front, i.e., the measure should become greater when the asphericity increases. Finally it is desirable that the definition describing the aberration should remain finite, determinate and meaningful in all cases, and should be independent of the wave-front position, i.e., independent of which wave-front is chosen.

It is clear that these conditions can never be fulfilled by a ray-intersection definition of astigmatism and that we must think of the aberration as an asphericity of the wave-front with respect to some chosen reference sphere.

Now the diffraction image is directly related to the complex amplitude at points on any chosen wave-front and this amplitude may be written $e^{ikr}/r \cdot e^{ik\omega}$ where r is the radius of a reference sphere, and ω measures the asphericity of the wave-front at any point on it, $k = 2\pi/\lambda$ being the propagation constant. Hopkins (11)

has defined the aberration of a wave-front as this asphericity (ω) relative to a suitably chosen reference sphere. This definition can be applied to astigmatism if say the sagittal focus is taken as the centre of a reference sphere and the astigmatism is measured by the intercept along the ray in the tangential section between the reference sphere and the wave-front. Fig (2).

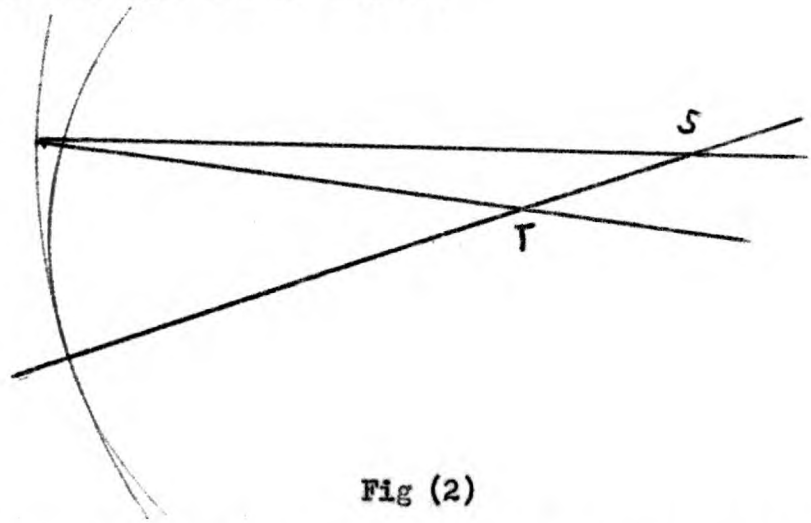


Fig (2)

The wave-front aberration in this section is then exactly similar to that due to a longtitudinal shift in focus. Later it is shown that for small apertures such a wave-front aberration is given by:-

$$\tfrac{1}{2} n u^2 \delta A \qquad \text{----------(1)}$$

where (δ A) is the distance between the astigmatic foci, and (u) the paraxial angle between the line drawn from the centre to a point on the reference sphere, and (n) the refractive index of the medium concerned.

It will be necessary now to investigate the change in the wave-front aberration defined above as the wave-front progresses. In Fig. 3. A_1 S is a reference ray of a pencil having astigmatic foci at S and T. A_1 R_1 and A_2 R_2 are tangential sections of successive wave fronts having their focus at T.

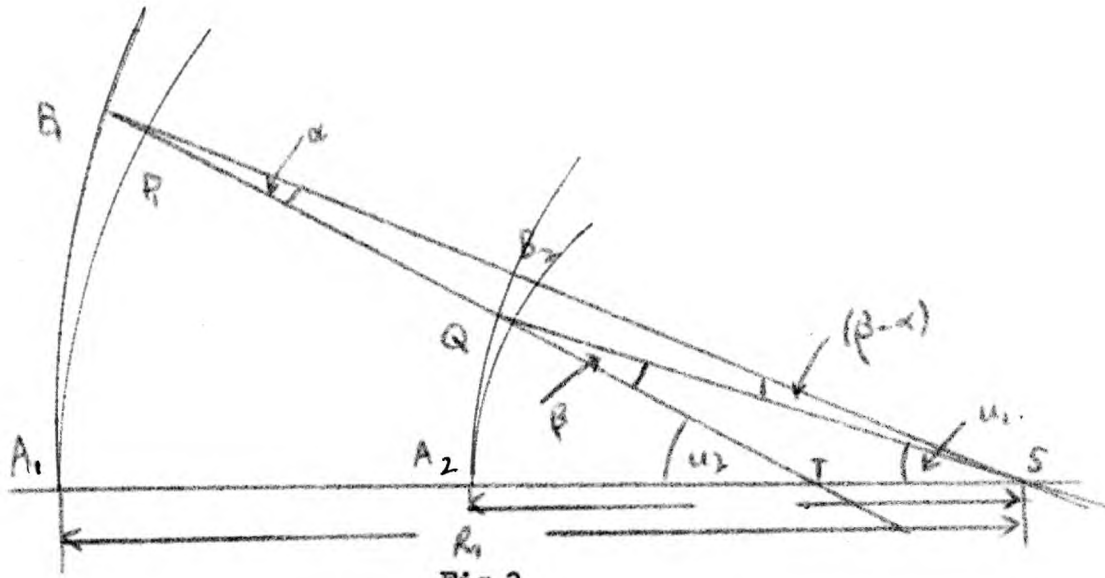


Fig 3.

Let $A_1 S$ and $A_2 S = R_1$ and R_2; $\hat{TB_1 S} = \alpha$ and $\hat{TQS} = \beta$

Let $\hat{B_1 SA_1} = u_1$ and $\hat{QTA_2} = u_2$. Then if $A_1 A_2 = \delta R$ we have that:-

$$\frac{B_1 Q}{\sin(\beta-\alpha)} = \frac{R_1}{\sin\beta} = \frac{R_2}{\sin\alpha} \quad \text{and}$$

we may write:-

$B_1 Q = \delta R + \delta W$ where δW is the change in the wave-front aberration due to the progression of the wave front along $A_1 S$ and $\delta R = R_1 - R_2$, ~~and $R = (R_1 - R_2)$~~.

Thus we have that:-

$$\frac{\delta R + \delta W}{\sin(\beta-\alpha)} = \frac{R_1}{\sin\beta} = \frac{R_2}{\sin\alpha},$$

and subtracting expressions for R_1 and R_2 we have:-

$$\delta R = (\delta R + \delta W)\left[\frac{\sin\beta - \sin\alpha}{\sin(\beta-\alpha)}\right]$$

from which it is clear that:-

$$\delta W = \delta R\left[\frac{\sin(\beta-\alpha)}{\sin\beta - \sin\alpha} - 1\right]$$

When powers of α and β higher than the square are ignored the above

equation reduces to:-

$$\delta W = \tfrac{1}{2} \alpha . \beta . \delta R$$

which shows that the wave-front aberration changes as the wave-front progresses even if only powers of the aperture up to and including the square are retained.

Paraxial angular variables.*

a) The tangential section.

In fig. (4), PT is a reference ray and P_T T another ray close to it in the tangential section. The tangential focus is at T, and P_T T is regarded as a paraxial tangential ray making an angle $(u)_T$ with the reference ray, the linear semi-aperture of the paraxial pencil being P_T Q or y_T .

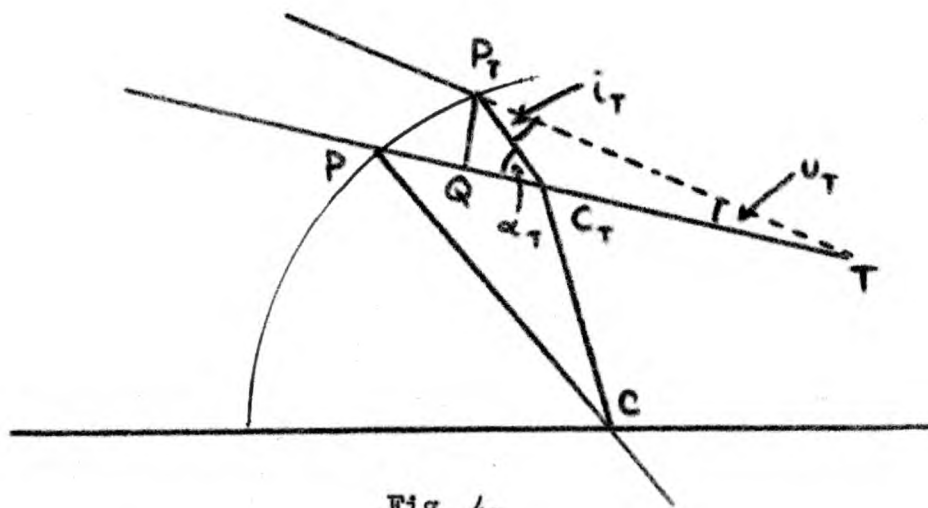


Fig 4.

In the fig. PT = t and P_T Q = y_T .

Thus t is given by (y_T / u_T) and P_T C_T can be regarded as the effective radius of curvature in the tangential section and has the value:-

r. Cos $\bar{I}$ = PC_T, where $\bar{I}$ is the angle of incidence of the reference ray $\hat{CPC}_T$.

* (11)

A series of relations exactly analogous to those for a paraxial ray can be introduced for the paraxial tangential ray these being:-

$$\alpha_T = \frac{y_T}{r.\cos\bar{I}} = \frac{h_T}{r} \; ; \; \alpha_T = u_T + i_T$$

and similarly for refracted quantities. Assuming that PP_T is small compared with PC_T and $P_T C_T$ it will be seen that

$$\frac{y_T}{\cos\bar{I}} = \frac{y_T'}{\cos\bar{I}'}$$

i.e., that h_T is constant on refraction and that

$$\alpha_T = u_T + i_T = u_T' + i_T'$$

The general equation for oblique refraction in the tangential section is:-

$$\Delta\left[\frac{n.\cos^2\bar{I}}{t}\right] = \Delta\left[\frac{n.\cos\bar{I}}{r}\right] \qquad \text{----------(2)}$$

where r is the surface radius of curvature, the equation of ordinary paraxial refraction being a special case of this in that:-

$$\Delta\left[\frac{n}{l}\right] = \Delta\left[\frac{n}{r}\right]$$

The symbol Δ is used, as usual, to denote the difference between quantities after and before refraction or reflection.

These equations can be written

$$\frac{1}{h_T}\Delta(n.\cos\bar{I}.u_T) = \frac{1}{r}\Delta(n.\cos\bar{I}) \; ; \; \frac{1}{h}\Delta(n.u) = \frac{1}{r}\Delta(n).$$

showing that the refraction law in the tangential section is similar to the paraxial law except that the refractive index in the tangential section is modified, and the 'fictitious' value ($n\cos\bar{I}$) is used. If i and i' are the ordinary paraxial angles of incidence and refraction at a surface, then the invariance of the quantity A = ni = n'i' expresses Snell's Law.

In the tangential section this is modified and we have:-

$$A_T = n.\cos\bar{I}.i_T = n'.\cos\bar{I}'.i'_T.$$

b) The sagittal section

Fig (5) shows a reference ray PS and the astigmatic focus in the sagittal section at S, i.e., P_S s is a close ray in the sagittal section.

As in the previous case h_S is the semi-aperture of the paraxial saggittal ray which makes an angle (u_S) with the reference ray PS.

Introducing the paraxial notation as before we have:-

$(u_S) = (h_S/s)$; $\alpha_S = u_S + i_S$

and that:-

$(\alpha_S) = h_S \cos\bar{I} / r$, and similarly for refracted quantities.

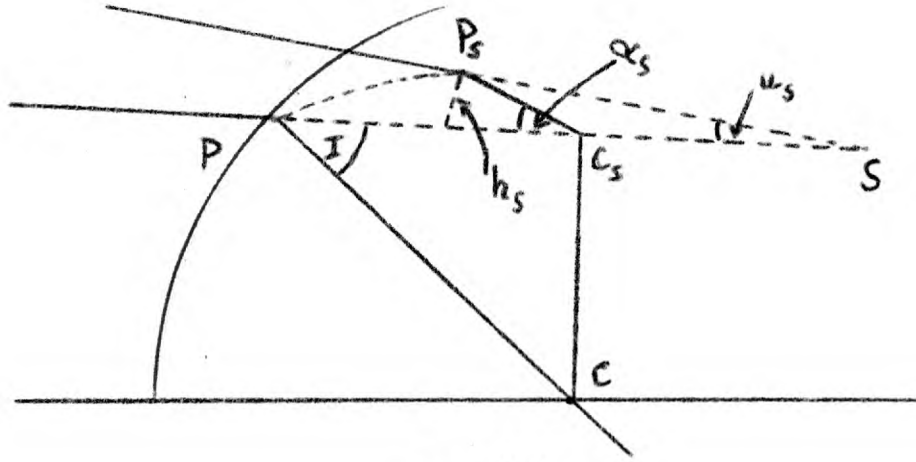


Fig. (5)

The general equation for refraction in the sagittal section is:-

$$\Delta\left[\frac{n}{s}\right] = \Delta\left[\frac{n.\cos\bar{I}}{r}\right]$$

or

$$\frac{1}{h_S}\Delta(n.u_S). = \frac{1}{r}\Delta(n.\cos\bar{I}).$$

Now $u_S = (\alpha_S - i_S)$ or $\Delta(n\alpha_S) - \Delta(ni_S) = h_S \Delta\left[\frac{n.\cos\bar{I}}{r}\right]$

and as $\Delta(n\alpha_S) = h_S \Delta\left(\frac{n.\cos\bar{I}}{r}\right)$ we have that:-

$$\Delta(ni_S) = 0 \qquad \text{and} \qquad A_S = ni_S = n'i'_S$$

the effective radius of curvature in this case being:-

$$PC_S = P_S C_S = (r/\cos\bar{I}).$$

The definition of complete astigmatism.

If the quantity $(u_S)(u_T)$ is substituted for the quantity $(u)^2$ in equation (1), the difficulty of using the measure for large aberrations is overcome and the value of $(nu_S u_T \delta A)$ becomes the measure of the aberration. The inclusion of u_S and u_T has the advantage that, should either the sagittal or tangential focus be formed at infinity, the quantity remains finite and meaningful. This can be shown, using the paraxial angular variables described above, as follows:-

$$nu_S u_T \delta A = nu_S u_T (s-t) = nh_S u_T - ny_T u_S$$
$$= h_S h_T \Delta\left[n \cos\bar{I} \left(\frac{1}{r} - \frac{1}{s}\right)\right]$$

Though we have considered spherical surfaces only, the definition applies equally well to aspheric surfaces and to optical systems lacking any sort of symmetry. The proposed definition in fact describes the complete aberration of the whole family of wave-fronts and is a unique number for any particular small pencil of light. Given a set of refraction conditions, the change in this quantity is independent of the position of the wave fronts relative to the refracting surface and of

The amount of aberration before or after refraction.

The definition of complete sagittal curvature.

The sagittal curvature of an optical system is usually thought of as the distance along the reference ray from the sagittal focus to the intersection point of the reference ray and the paraxial image plane. This measure suffers from the same faults as the distance (s-t) as a measure of the astigmatism.

In an exactly similar manner to that shown in the case of astigmatism we can both convert the geometrical length (δ S) along the reference ray to the paraxial image plane into a wave front aberration and at the same time obviate the difficulty encountered when the paraxial image plane, or the sagittal focus is formed at infinity. This is achieved by multiplying δ S by (u) (u_s). It is proposed then to define the sagittal curvature* as $nuu_s \delta S$.

This definition is complete and precise, and again the change in it is a unique number for a particular set of refraction conditions. The definition is applicable to aspheric surfaces and, with simple modifications to systems lacking symmetry.

The definition of the complete tangential curvature.

Although it is usually sufficient to obtain the values ($nu_s . u_T \delta A$) and ($nu . u_s \delta S$) to assess the complete astigmatism and field curvature of a system, the tangential curvature is sometimes required independently, and this can be defined in an exactly similar manner to that for the sagittal curvature. If the distance along the reference ray between the tangential focus and the paraxial image plane is δT, then the appropriate definition of the tangential curvature is ($nuu_T \delta T$).

* Apologies are made for following the unfortunate usage of this term. In aberration theory certain measures of the departure of an image surface from the one desired are described as a curvature, although the quantities used are generally lengths.

The proposed definitions enable us to measure the aberration at any point in an optical system, and they can be used to measure the amount of aberration arising at one of the surfaces. This is obtained by subtracting the amount of aberration found before refraction at the surface from that found after refraction. The quantity so obtained will be referred to as the 'surface contribution' to the aberration and

$$\sum_1^K \Delta(n u_S u_T \delta A) \quad ; \quad \sum_1^K \Delta(n u u_S \delta S) \quad ; \quad \sum_1^K \Delta(n u u_T \delta T) \quad .$$

represent the sums of the surface contributions to the three aberrations after refraction through k surfaces. Formulae for these quantities will be investigated in some detail in the work that follows.

Chapter 2.

The evaluation of the Complete Astigmatism.

The quantity ($nu_s\, u_T\, \delta A$) will now be expressed analytically in three different ways, a) as a series involving $\bar{B} = n.\sin \bar{I}$. ($\bar{B}$ being the reference ray counterpart of A = ni for paraxial rays), b) as the complete aberration modified by multiplication by $\cos \bar{U}$ so that the resulting aberration is then axially projected, and c) as the complete aberration measured along the reference ray.

---

a) The general astigmatic equations, (the Barrow-Young equations) can be written:-

$$\Delta\left[\frac{n.\cos^2 \bar{I}}{t}\right] = \Delta\left[\frac{n.\cos \bar{I}}{r}\right] = \Delta\left[\frac{n}{s}\right] \qquad \text{----------(3)}$$

which we may rewrite,

$$\Delta\left[\frac{n.\cos^2 \bar{I}}{t} - \frac{n(\cos \bar{I} - \cos^2 \bar{I})}{t}\right] = \Delta\left[\frac{n.\cos \bar{I}}{s} - \frac{n.(\cos \bar{I} - 1)}{s}\right]$$

or

$$\Delta\left[\frac{n \cos \bar{I}}{t} - \frac{n.\cos \bar{I}}{s}\right] = \Delta\left[\frac{n.\cos \bar{I}(1 - \cos \bar{I})}{t} - \frac{n(\cos \bar{I} - 1)}{s}\right]$$

and multiplying both sides by $h_s\, h_T$ we have:-

$$h_s h_T \Delta\left[n.\cos \bar{I}\left(\frac{1}{t} - \frac{1}{s}\right)\right] = h_s h_T \Delta\left[n(1 - \cos \bar{I})\left(\frac{\cos \bar{I}}{t} + \frac{1}{s}\right)\right]$$

or

$$\Delta\left[n.h_s y_T \left(\frac{s-t}{s.t}\right)\right] = \Delta\left[n(1 - \cos \bar{I})(h_s u_T + h_T u_s)\right]$$

Now $s = h_s / u_s$ and $t = y_T / u_T$, hence we may write:-

$$\Delta\left[n u_s u_T (s-t)\right] = \Delta\left[n(1-\cos\bar{I})(h_s u_T + h_T u_s)\right]$$

Expanding Cos $\bar{I}$ in terms of Sin $\bar{I}$ we have that:-

$$(1-\cos\bar{I}) = \tfrac{1}{2}\sin^2\bar{I} + \tfrac{1}{8}\sin^4\bar{I} + \ldots\ldots \text{ etc.}$$

and thus $\Delta\left[n u_s u_T (s-t)\right]$

$$= \Delta\left[\tfrac{1}{2} n.\sin^2\bar{I}(h_s u_T + h_T u_s)\right]$$

$$+\Delta\left[\tfrac{1}{8} n.\sin^4\bar{I}(h_s u_T + h_T u_s)\right] + \quad \ldots\ldots\text{etc.}$$

and writing $\bar{B} = n.\text{Sin}\,\bar{I}$ we have:-

$$\Delta(n u_s u_T \delta A) = \tfrac{1}{2}\bar{B}^2\left[h_s \Delta\left(\frac{u_T}{n}\right) + h_T \Delta\left(\frac{u_s}{n}\right)\right]$$

$$+\tfrac{1}{8}\bar{B}^4\left[h_s \Delta\left(\frac{u_T}{n^3}\right) + h_T \Delta\left(\frac{u_T}{n^3}\right)\right] + \quad \text{---------(4)}$$

$\ldots\ldots$etc

If now we let

$$h_s = h + \delta h_s \; ; \qquad u_s = u + \delta u_s$$

$$h_T = h + \delta h_T \qquad u_T = u + \delta u_T$$

h being the incident height of a corresponding ordinary paraxial ray.
and $\bar{B}^{-2} = (B + \delta B)^2 = (B^2 + 2\,\delta B)$ where B is the value of (ni) for a paraxial reference ray then:-

$$\Delta(n u_s u_T \delta A) = \tfrac{1}{2}(B^2 + 2\delta B)\left[h \Delta\left(\frac{u+\delta u_T}{n}\right) + \delta h_s \Delta\left(\frac{u+\delta u_T}{n}\right)\right.$$

$$\left. + h \Delta\left(\frac{u+\delta u_s}{n}\right) + \delta h_T \Delta\left(\frac{u+\delta u_s}{n}\right)\right]_{\ldots\text{etc.}}$$

which equation reduces to

$$\Delta(u_3 u_7 \delta A) = B^2 h\, \Delta\left(\frac{u}{n}\right) + 2\delta B . h\, \Delta\left(\frac{u}{n}\right)$$

$$+ \tfrac{1}{4} B^4 h\, \Delta\left(\frac{u}{n}\right) + \tfrac{1}{2} B^2 h\, \Delta\left(\frac{\delta u_T + \delta u_S}{n}\right)$$

$$+ \quad \text{.....etc.}$$

when terms of the type (δ h. δ u.) are ignored.

The first term in this expression for the surface contribution to the astigmatism is the first order term and it is identical with the third sum of von Seidel (s iii), in the form used by Hopkins.

In general, expansions of this sort are not profitable, for they lead to expressions which contain a large number of terms which would be considerably more difficult to calculate than by simply putting in correction terms into a precise expression for the aberration.

In equation (4) one may have a value of $\bar{B}$ at a surface between two media of unity, or even slightly greater, and in a case such as this the value of the fourth power term in $\bar{B}$ is quite appreciable in comparison with the term in the square of $\bar{B}$. For this reason, even when moderate quantities of aberration are present, the use of such series leads to a most unsatisfactory account of the aberration and it is therefore proposed to investigate closed formulas for the complete aberration at a surface in order that a new method of approximation may be employed.

b) The evaluation of the axially projected astigmatism

The definition in this case is modified in an obvious and simple manner. The linear astigmatic difference is projected along the optical axis, so

that we use (δA. Cos $\bar{U}$) instead of (δA), $\bar{U}$ being the angle the reference ray makes with the optical axis. Retaining the same symbol δA for the new quantity, the surface contribution to the astigmatism is:-

$$\Delta(n u_s u_T \delta A) = \Delta\left[n u_s u_T (s-t).\cos\bar{U}\right]$$

Writing Cos $\bar{U}$ = Cos ($\bar{\alpha}$ - $\bar{I}$) we have that:-

$$\Delta(n u_s u_T \delta A) = \cos\bar{\alpha}\,\Delta\left[n u_s u_T \cos\bar{I}\,(s-t)\right] + \sin\bar{\alpha}\,\Delta\left[n u_s u_T \sin\bar{I}\,(s-t)\right]$$

Since $u_S \cdot s = h_S$ and $u_T . t = y_T = h/\text{Cos}\,\bar{I}$ the second term in this expression may be written:-

$$\bar{B}.\sin\bar{\alpha}\,\Delta\left[h_S.(u_T) - h_T(u_S.\cos\bar{I})\right]$$

The first term in the expression is developed as follows:-

The general astigmatic equations can be written:-

$$\Delta\left(\frac{n}{s}\right) = \frac{1}{r}\Delta(n.\cos\bar{I}) = \Delta\left[\frac{n.\cos^2\bar{I}}{s} + \frac{n(1-\cos^2\bar{I})}{s}\right]$$

and

$$\Delta\left(\frac{n.\cos^2\bar{I}}{t}\right) = \frac{1}{r}\Delta(n.\cos\bar{I}) = \Delta\left(\frac{n}{s}\right) = \frac{1}{h_S}\Delta(n u_s).$$

Subtracting these we have that:-

$$\Delta\left[\frac{n.\cos^2\bar{I}}{t}\right] - \Delta\left[\frac{n.\cos^2\bar{I}}{s}\right] = \Delta\left[\frac{n(1-\cos^2\bar{I})}{s}\right]$$

or

$$\Delta\left[n.\cos^2\bar{I}\left(\frac{s-t}{st}\right)\right] = \bar{B}^2\,\Delta\left(\frac{1}{n.s.}\right)$$

and multiplying both sides by $h_s y_T / \cos \bar{I} = h_s h_T$ we have:-

$$\Delta\left[n u_s u_T \cos \bar{I}(s-t)\right] = \Delta\left[\frac{h_s y_T \cdot \bar{B}.}{n.s.\cos \bar{I}}\right]$$

or

$$\Delta\left[n.u_s u_T \cos \bar{I}(s-t)\right] = \bar{B}^2 h_T \Delta\left(\frac{u_s}{n}\right).$$

Consequently the total surface contribution to astigmatism given in equation (4) can be written:-

$$\Delta\left(n u_s u_T \delta A\right) = \bar{B}^2 \cos \bar{\alpha}\, h_T\, \Delta\left(\frac{u_s}{n}\right)$$

$$+ \bar{B}.\sin \bar{\alpha}\left[h_s \Delta(u_T) - h_T \Delta(u_s \cos \bar{I})\right] \quad ---(5)$$

In the paraxial case Cos $\bar{\alpha}$ = 1 and Sin $\bar{\alpha}$ = 0 and the whole expression reduces to $\bar{B}^2 h . \Delta\left(\frac{u}{n}\right)$, the first order astigmatism.

When the reference ray passes through the centre of curvature of the surface concerned then $\bar{B}$ = 0, and there is no contribution to the astigmatism as defined here, no matter what the astigmatism of the incident wave is. Equation (5) reveals other cases when the surface contribution to astigmatism is expressible in very simple form. If the value of h_T in the equation is zero then the first order term becomes zero and the second of the two higher order terms also. We are then left with a value $\bar{B}.\sin \bar{\alpha}\ h_s\ \Delta(u)_T$ which is the contribution to astigmatism under these

conditions.

Similarly when $h_S = 0$ we have that:-

$$\Delta\left(nu_S u_T \delta A\right) = \bar{B}^{\nu} \cos\bar{\alpha}\, h_T\, \Delta\left(\frac{u_S}{\bar{n}}\right) - \bar{B}.\sin\bar{\alpha}\, h_T.\, \Delta\left(u_S \cos\bar{I}\right).$$

c) <u>The evaluation of the complete astigmatism along the reference ray</u>. In the last section we were concerned with the measure of astigmatism along the optical axis. In this section we shall evaluate the surface contribution along the reference ray i.e., the quantity (δ A) is simply (s-t). The analysis is developed so as to include the aberration of aspheric surfaces.

Let the curvatures of the refracting surface in the sagittal and tangential sections be c_S and c_T .

The Barrow-Young equations can now be written:-

$$\Delta\left(\frac{n.\cos^2\bar{I}}{t}\right) = \Delta\left(n.\cos\bar{I}\right) c_T$$

$$\Delta\left(\frac{n}{s}\right) = \Delta\left(n.\cos\bar{I}\right) c_S$$

or

$$\Delta\left[\frac{n.\cos\bar{I}}{t} - \frac{n.\cos\bar{I}\left(1-\cos\bar{I}\right)}{t}\right] = \Delta\left(n.\cos\bar{I}\right) c_T$$

and

$$\Delta\left[\frac{n.\cos\bar{I}}{s} + \frac{n\left(1-\cos\bar{I}\right)}{s}\right] = \Delta\left(n.\cos\bar{I}\right) c_S.$$

Subtracting these equations we have that:-

$$\Delta\left[n.\cos\bar{I}\left(\frac{1}{t}-\frac{1}{s}\right) - n\left(1-\cos\bar{I}\right)\left(\frac{\cos\bar{I}}{t}+\frac{1}{s}\right)\right] = \left(c_T - c_S\right)\Delta\left(n.\cos\bar{I}\right)$$

or $$\Delta(n u_s u_T \delta A) = \Delta\left[n(1-\cos\bar{I})(h_s u_T + h_T u_s)\right]$$

$$+ h_s h_T (c_T - c_s)\,\Delta(n.\cos\bar{I}).$$

giving finally $$\Delta(n u_s u_T \delta A)$$

$$= \bar{B}^2 \Delta\left[\frac{1}{n(1+\cos\bar{I})}(h_s u_T + h_T u_s)\right]$$

$$+ h_T h_s (c_T - c_s)\,\Delta(n.\cos\bar{I}). \qquad \text{----------(5a)}$$

In the case of refraction at a spherical surface $c_T = c_s$ and the equation reduces to:-

$$\Delta(n u_s u_T \delta A) = \bar{B}^2 \Delta\left[\frac{1}{n(1+\cos\bar{I})}(h_s u_T + h_T u_s)\right]$$

Under first order conditions $\dfrac{1}{n(1+\cos\bar{I})} = \dfrac{1}{2n}$ and $(h_s\ u_T + h_T\ u_s) = 2\ hu$, and the whole expression can be written:-

$$\Delta(n u_s u_T \delta A) = B^2 h\,\Delta\left(\frac{u}{n}\right)$$

which is the first order astigmatism contribution in the form used by Hopkins.

<u>Refraction in the Sagittal section</u>.

Before evaluating the sagittal image curvature it will be necessary to note some of the properties of rays refracted in the sagittal section.

It is well known that the angle made between the optical axis and

the line drawn from the centre of curvature of the surface through the sagittal focus is constant on refraction. In fig (6) PS is a reference ray incident at the surface OP at P. The sagittal focus is at S, and the angle mentioned above is $\hat{SCN}$, CS being the 'auxiliary axis'.

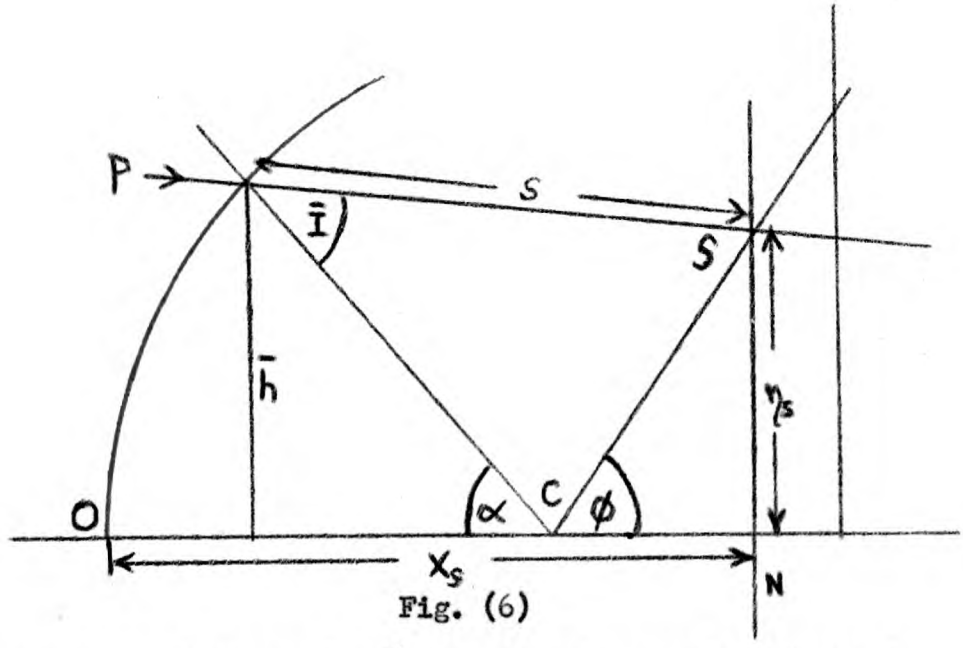


Fig. (6)

The constancy of the angle ϕ on refraction may be expressed:-

$$\Delta(\tan\phi) = 0 = \Delta\left(\frac{\eta_s}{X_s - r}\right) \qquad \text{----------(6)}$$

It will now be shown that the Smith-Helmholtz invariant for paraxial rays applies to rays in the sagittal section.*

For paraxial rays the Smith-Helmholtz invariant is written $H = nu\eta$ where η is the intersection height of the paraxial reference ray in the paraxial image plane. The sagittal optical invariant for finite object heights can be written $H_s = nu\eta_s$ where η_s is the distance of the sagittal focus from the optical axis.

From the geometry of fig. (6) we have that:-

* cf. H. H. Hopkins, Nature Vol. 159. p.573. 1947.

$$\sin \bar{u} = \frac{\bar{h} - \eta_s}{s} \qquad \text{or} \qquad \eta_s = \bar{h} - s.\sin \bar{u}$$

Now if we have $H_s = n u_s \eta_s$

then $H_s = \bar{h}(nu_s) - nu_s.s.\sin \bar{u}.$

or $H_s = \bar{h}(nu_s) - h_s(n.\sin \bar{u}).$ ----------(7)

Now Sin $\bar{U}$ can be written $\sin(\bar{\alpha} - \bar{I})$.

and hence $n.\text{Sin } \bar{U} = (\text{Sin}\bar{\alpha}.n\text{Cos } \bar{I} - \text{Cos}\bar{\alpha}.n.\text{ Sin } \bar{I})$.

and taking the change on refraction we have that:-

$$\Delta(n.\sin \bar{u}.) = \sin \bar{\alpha}\, \Delta(n.\cos \bar{I})$$

or

$$\Delta(n\sin \bar{u}) = \bar{h}\, \Delta\left(\frac{n.\cos \bar{I}}{v}\right)$$

and substituting this value of (Δ n. Sin $\bar{U}$) in equation (7) we have finally that:-

$$\Delta(H_s) = 0$$

Thus it is shown that the Smith-Helmholtz invariant for paraxial rays has an exactly similar counterpart for paraxial sagittal rays. The quantity H_s will be used frequently in the work that follows.

It is frequently found necessary to calculate H_s at the first surface of a system when the object is at infinity. This cannot be done directly for (u_s η_s) is indeterminate in this case. $H_s$is then determined in the following manner:-

From equation (7) we have that:-

$$H_s = \bar{h}(nu_s) - h_s(n.\sin\bar{u})$$

When the object is at infinity $s = \infty$ and $u_s = 0$

hence $H_s = -h_s.u.\sin\bar{u}.$

Now for an element of the object plane situated on the axis

$H = -h.u.\tan\bar{u}$, and for H to be equal to H_s then the values of h and h_s must be in the ratio $1/\cos\bar{U}$. The paraxial sagittal invariant will now be used to prove a useful identity involving an extension to the Young equation for refraction in the sagittal section.

We have from equation (6) that;-

$$\Delta\left[\frac{\eta_s}{x_s - r}\right] = 0 \quad ; \quad \frac{x'_s - r}{x_s - r} = \frac{\eta'_s}{\eta_s}$$

where dashed quantities refer to quantities after refraction. Now $H_s = nu_s\eta_s$ so we may write:-

$$n'.x'_s.s - n'.s.r = n.x_s.s' - n.s'.r.$$

and dividing throughout by $s.s'$ we have:-

$$\frac{n\,x'_s}{s'} - \frac{n'r}{s'} = \frac{n\,x_s}{s} - \frac{n.r.}{s}$$

or

$$\Delta\left[\frac{n\,x_s}{s}\right] = r.\Delta\left[\frac{n}{s}\right]$$

This last equation can be regarded as an extension to the Young equation ~~for refraction in the sagittal section, and/~~ the modified Young equation

for refraction in the sagittal section, and the modified Young equation becomes:-

$$\Delta\left[\frac{n}{s}\right] = \Delta\left[\frac{n.\cos\bar{I}}{r}\right] = \frac{1}{r}\Delta\left[\frac{n.X_s}{s}\right] \qquad \text{----------(8)}$$

It has been found necessary in the following evaluation of the sagittal curvature to obtain an expression for $\bar{B} = n.\text{Sin}\,\bar{I}$, in terms of the sagittal refraction constants and this will be derived now.

<u>The equation for $\bar{B}$ in terms of sagittal refraction constants</u>.

Hopkins (loc.cit) has shown that in first order theory, the value of B (this being the product of n and i for a paraxial reference ray) can be written:-

$$B = \frac{H}{h}(1 - AhE) \qquad \text{----------(11)}$$

where H is the Smith-Helmholtz invariant for the paraxial reference ray, and E is a quantity defined by:-

$$E = 1/H(h^*/h)$$

where h^* is the height at which the paraxial reference ray meets the surface. An expression for $\bar{B}$ exactly similar to that for B will now be determined, the value of $\bar{B}$ being the value of $n.\text{Sin}\,\bar{I}$ for an accurately traced reference ray. Fig. (7) shows a refracting surface and a reference ray refracted at A having a sagittal focus at P.

From the fig. we have that:-

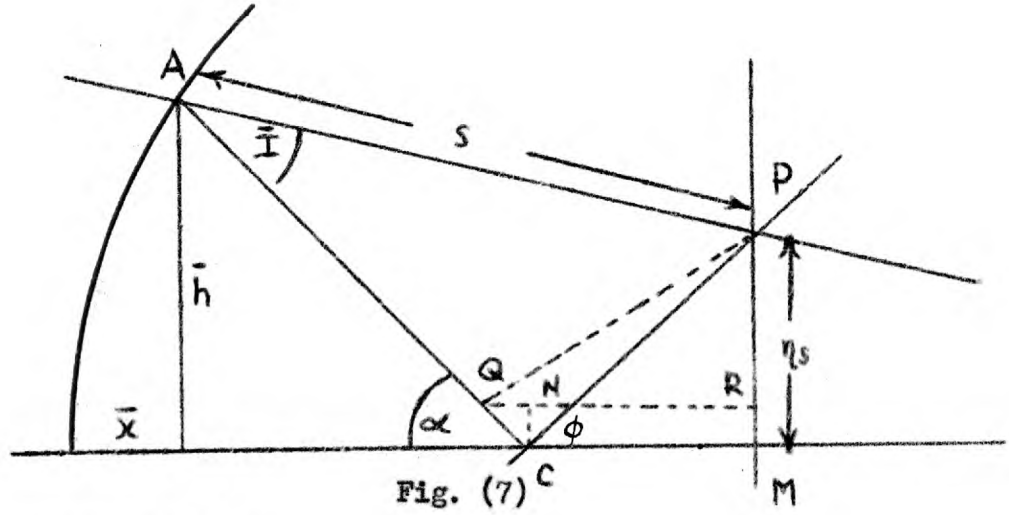


Fig. (7)

$QC = r - s \cos \bar{I}.$

Using the paraxial sagittal formulae we have:-

$$n\alpha_s = nu_s + A_s \quad ; \quad \frac{h_s . n . \cos \bar{I}}{r} = nu_s + A_s$$

and writing $h_s = s . u_s$ we may write:-

$$\frac{s . \cos \bar{I}}{r} . nu_s = nu_s + A_s$$

or $$(s . \cos I - r) = A_s r / nu_s \qquad \text{----------(12)}$$

Now $$PR = PQ . \cos \hat{QPR} = PQ . \cos \bar{\alpha} \; ; \; PQ = s . \sin \bar{I}$$

so we may write

$$PR = s . \bar{B} \cos \bar{\alpha} / n$$

In triangle QNC, $\hat{QCN} = (90 - \bar{\alpha})$ and $NC = QC . \cos (90 - \bar{\alpha})$

or $NC = (r - s \cos \bar{I}) \sin \bar{\alpha}$

Now $PM = \eta_s = PR + NC$

$$\text{or} \quad \eta_s = \frac{s.\bar{B}\cos\bar{\alpha}}{n} + \sin\bar{\alpha}(r - s.\cos\bar{I})$$

Using (12) we have that:-

$$n u_s \eta_s = \bar{B}\cos\bar{\alpha}\, h_s - A_s.r.\sin\bar{\alpha}$$

$$\text{or} \quad \bar{B} = \frac{1}{\cos\bar{\alpha}}\left[\frac{H_s}{\bar{h}_s} + \frac{A_s\bar{h}}{h_s}\right] \qquad \text{----------(13)}$$

It will be seen that this expression for $\bar{B}$ is exactly similar to the expression for B in equation (11) if we write $E_s = \frac{1}{H_s}\left(\frac{\bar{h}}{\bar{h}_s}\right)$. obtaining:-

$$\bar{B} = \frac{1}{\cos\bar{\alpha}}\left[\frac{H}{h_s} + A_s E_s h_s\right]$$

$$\bar{B} = \frac{1}{\cos\bar{\alpha}}\frac{H_s}{h_s}\left(1 + A_s h_s E_s\right)$$

A further identity which can be proved in the sagittal section concerns the relation between the angle the auxiliary axis makes with the optical axis, H_s and $\bar{B}$. This relation is proved as follows:-

In fig. (7) we have that:-

$$\frac{CP}{\sin I} = \frac{s}{\sin(\bar{\alpha} + \phi)}.$$

or

$$\frac{n.\eta_s}{\bar{B}.\sin\phi} = \frac{s.}{\sin(\bar{\alpha} + \phi)}$$

Expanding $\sin(\bar{\alpha} + \phi)$ this gives:-

$$H_s.\sin\bar{\alpha}.\cos\phi + H_s.\cos\bar{\alpha}.\sin\phi = \bar{B}\,h_s\sin\phi.$$

and dividing throughout by Sin ϕ we have:-

$$\frac{H_s \sin\bar{\alpha}}{\tan\phi} + H_s\cos\bar{\alpha} = \bar{B}l_s$$

or
$$\frac{1}{\tan\phi} = \frac{\bar{B}l_s - H_s\cos\bar{\alpha}}{H_s \sin\bar{\alpha}} \qquad \text{----------(14)}$$

Reference ray identities.

There follows now a number of minor identities which are in constant use in the evaluation of accurate expressions for the complete aberration. The proofs of these are very simple and are given below.

1. The Conrady ray tracing formulae give:-

$$(L-r)\sin\bar{u} = r.\sin\bar{I} = \frac{r\bar{B}}{n}$$

and if we write $\bar{\eta} = \bar{L}.\ \mathrm{Sin}\ \bar{U}$ we have:-

$$n\bar{\eta} = \bar{B}r + n.r.\sin\bar{u}$$

Taking the change on refraction this gives:-

$$c\,\Delta(n\bar{\eta}) = \Delta(n.\sin\bar{u}). \qquad \text{----------(18)}$$

where c = 1/r.

---

2. We may write Cos $\bar{U}$ as Cos $(\bar{\alpha} - \bar{I})$ and expanding we have:-

$$\cos\bar{u} = \cos\bar{\alpha}.\cos\bar{I} + \sin\bar{\alpha}.\sin\bar{I}$$

Multiplying both sides by n, and taking the change on refraction we have finally that:-

$$\Delta(n.\cos\bar{u}) = \cos\bar{\alpha}\,\Delta(n.\cos\bar{I}) = r.\cos\bar{\alpha}\,\Delta\left(\frac{n}{s}\right) \qquad \text{---------- (16)}$$

A similar relation for Δ (n.Sin $\bar{U}$) has already been used and we have from equation (7) that:-

$$\Delta(n.\sin\bar{u}) = \bar{h}\,\Delta\left(\frac{n.\cos\bar{I}}{r}\right) = \bar{h}\,\Delta\left(\frac{n}{s}\right)$$

The quantity $\bar{x}$ (fig. 7), being the 'sagitta' or depth of curvature of the surface is one which frequently occurs in the evaluation of complete aberration expressions. The following expression for $\bar{x}$ has been found useful. x is usually given by :- $r(1 - \cos\bar{\alpha})$
which can be written:- $r.\sin^2\bar{\alpha} \,/\, (1+\cos\bar{\alpha})$

or
$$\bar{x} = \frac{1}{(1+\cos\bar{\alpha})}\,\frac{\bar{h}^2}{r} \qquad \text{----------(17)}$$

<u>The evaluation of the complete sagittal image curvature</u>.

It is proposed now to obtain expressions for the surface contribution to sagittal image curvature measured along the ray and projected along the axis and to compare the values so obtained with the first order value for the aberration. It is well known that the first order sagittal, curvature can be regarded as being made up of two portions, the astigmatism and the Petzval curvature. This has been expressed by Hopkins (loc. cit) as follows:-

Sag. Curv = $\sum' \frac{1}{2}\left[B^2 l\, \Delta\left(\frac{u}{n}\right) + H^2 P\right]$

where P is the contribution to the Petzval sum for the surface given by $P = -\frac{1}{r}\Delta\left(\frac{1}{n}\right)$ and H the Smith-Helmholtz invariant. In the following work it will be shown that an expression for the complete aberration can be obtained which is analogous apart from higher order terms to the first order relationship above.

The surface contributions to the sagittal and tangential image curvature are more difficult to evaluate than that for pure astigmatism. The surface contribution to astigmatism is of course independent of the position of the paraxial image plane, and concerns only the reference ray refraction conditions and the positions of the astigmatic foci.

Sagittal and tangential image curvatures however, are dependent on both the positions of the astigmatic foci and on the position of the paraxial image plane.

Both of these aberrations are thus dependent on the distortion of the surface or system concerned, in that directly, or indirectly, a knowledge of the height at which the reference ray strikes the paraxial image plane is essential.

The primary term is of course included in the complete expression for the aberration together with other higher order terms.
It will be shown now that the sagittal curvature contribution at a surface can be expressed in a very simple manner when the aberration is in the 'axially projected' form. In this case the original definition is modified by multiplying by Cos $\bar{U}$. so that the distance between the sagittal focus and the paraxial image plane is measured along the optical axis, i.e., the original length δS is projected on to the optical axis.

If ℓ is the distance of the paraxial image plane from the surface then we may write:-

$$\Delta(nuu_s\delta s) = \Delta\left[nuu_s(\ell - x_s)\right]$$

$$= h.h_s\,\Delta\left[\left(\frac{n}{s}\right) - \left(\frac{nx_s}{h_s}\right)\right] \qquad \text{----------(9)}$$

Paraxial formulae at a single surface give:- $u = \alpha - i$

; and dividing throughout by h we have that:- $\frac{1}{\ell} = \frac{1}{r} - \frac{i}{h}$.

and substituting this value of $\left(\frac{1}{\ell}\right)$ into equation (9) we may write:-

$$\Delta(nuu_s\delta s) = h.h_s\,\Delta\left(\frac{n}{s}\right) - h.h_s\,\Delta\left(\frac{nx_s}{r.s}\right) + h.h_s\,\Delta\left(\frac{nx_s i}{h.s}\right).$$

From equation (8) we have that:-

$$\Delta\left(\frac{n}{s}\right) = \frac{1}{r}\,\Delta\left(\frac{nx_s}{s}\right)$$

and thus

$$\Delta(nuu_s\delta s) = h.h_s\,\Delta\left(\frac{nx_s i}{h.s}\right) = h_s\,\Delta\left(\frac{nx_s i}{s}\right)$$

and writing A = ni we have finally:-

$$\Delta(nuu_s\delta s) = A\,\Delta(x_s u_s). \qquad \text{----------(10)}$$

This is a particularly simple result for the complete surface contribution to sagittal curvature in the axially projected form.

The result is important because the quantities (X_s) and (s) dp not

refer in any way to the paraxial refraction conditions, and further, one can now see that the amount of sagittal curvature introduced by a surface depends entirely on the semi-aperture of the sagittal fan (h_s) and on the constant (A) which contains all the paraxial refraction conditions separately.

The following analysis shows how it is possible to obtain an expression for the complete sagittal curvature along the reference ray which includes terms of the same general form as the primary astigmatism and Petzval terms. It will be shown too that the expression for the sagittal curvature in the axially projected form can be obtained from this analysis. The complete aberration along the ref. ray can be written;-

$$\Delta(nuu_s\delta S) = \frac{nuu_s(\ell - X_s)}{\cos\bar{u}}$$

$$= \frac{nu(\ell - r)u_s}{\cos\bar{u}} - \frac{nu_s(X_s - r)u}{\cos\bar{u}}$$

We have however that:-

$$nu_s(X_s - r) = \frac{H_s(X_s - r)}{\eta_s} = \frac{H_s}{\tan\phi}.$$

and from paraxial formulae

$$nu(\ell - r) = Ar$$

These last two expressions are thus constant on refraction and we may write for the total surface contribution:-

$$\Delta(nuu_s\delta S) = nu(\ell - r)\Delta\left(\frac{u_s}{\cos\bar{u}}\right) - nu_s(X_s - r)\Delta\left(\frac{u}{\cos\bar{u}}\right).$$

Now $$\frac{H_s}{\tan\phi} = \frac{\bar{B}h_s - H_s\cos\bar{\alpha}}{\sin\bar{\alpha}}$$

----------(14)

and $$\bar{B}h_s\cos\bar{\alpha} = H_s + A_s\bar{h}$$

from equation (13)

or $$\bar{B}h_s - H_s\cos\bar{\alpha} = A_s\bar{h}.\cos\bar{\alpha} + \bar{B}h_s\sin^2\bar{\alpha}$$

Thus $nu_s\ (X_s - r)$ can be written:-

$$A_s r.\cos\bar{\alpha} + \bar{B}h_s\sin\bar{\alpha}$$

and thus:-

$$\Delta(nuu_s\delta s) = A_r\Delta\left(\frac{u_s}{\cos\bar{u}}\right) - \left[A_s r.\cos\bar{\alpha} + \bar{B}h_s\sin\bar{\alpha}\right]\Delta\left(\frac{u}{\cos\bar{u}}\right).$$

$$= A_r\Delta\left(\frac{u_s}{\cos\bar{u}}\right) - A_s r\left(\frac{u}{\cos\bar{u}}\right) + \left[A_s\bar{x} - \bar{B}h_s\sin\bar{\alpha}\right]\Delta\left(\frac{u}{\cos\bar{u}}\right)$$

Taking the first two terms separately and omitting the Δ's we may write:-

$$A_r\left(\frac{u_s}{\cos\bar{u}}\right) - A_s r\left(\frac{u}{\cos\bar{u}}\right) = \frac{(n\alpha - nu).r.u_s}{\cos\bar{u}} - \frac{(n\alpha_s - nu_s)r.u.}{\cos\bar{u}}$$

which after expanding gives:-

$$\left[\frac{n.h.h_s}{r.s.} - \frac{nh.h_s}{ls.} - \frac{nhh_s\cos\bar{I}}{r.l.} + \frac{nh.h_s}{l.s.}\right]\frac{r}{\cos\bar{u}}$$

$$= \frac{n h h_s}{r}\left[\frac{1}{s} - \frac{\cos\bar{I}}{l}\right]\frac{r}{\cos\bar{u}}$$

which can be written:-

$$= \left[n.h.h_s\left(\frac{1}{s}-\frac{1}{l}\right) + n h.h_s\left(\frac{1-\cos\bar{I}}{l}\right)\right]\frac{1}{\cos\bar{u}}$$

or

$$\left[n(h u_s - h_s u) + \frac{1}{1+\cos\bar{I}}\cdot\frac{h h_s \bar{B}^2}{n l}\right]\frac{1}{\cos\bar{u}}$$

Thus we have that:-

$$A_r\,\Delta\left(\frac{u_s}{\cos\bar{u}}\right) - A_{s_r}\,\Delta\left(\frac{u}{\cos\bar{u}}\right)$$

$$= \Delta\left[\frac{n(h u_s - h_s u)}{\cos\bar{u}}\right] + \tfrac{1}{2}\bar{B}^2 h_s\,\Delta\left[\frac{2}{\cos\bar{u}(1+\cos\bar{I})}\left(\frac{u}{n}\right)\right]$$

giving for the complete aberration:-

$$\Delta(n u u_s \delta S) = \tfrac{1}{2}\bar{B}^2 h_s\,\Delta\left[\frac{2}{\cos\bar{u}(1+\cos\bar{I})}\left(\frac{u}{n}\right)\right]$$

$$+ \Delta\left[\frac{n(h_s u_s - h_s u)}{\cos\bar{u}}\right] + \left[A_{s\bar{x}} - \bar{B} h_s \sin\bar{\alpha}\right]\Delta\left(\frac{u}{\cos\bar{u}}\right). \qquad \text{-----(18)}$$

The first term in this expression is the 'astigmatism' part of the complete sagittal curvature, and the second and third terms will be combined to form the 'Petzval portion of the aberration. To do this we will treat the second and third terms separately.

Expanding the second term we have that:-

$$\Delta\left[\frac{n(h u_s - h_s u)}{\cos\bar{u}}\right] = \Delta\left[h.nu_s - h_s nu + (hnu_s - h_s nu)\left(\frac{1-\cos\bar{u}}{\cos\bar{u}}\right)\right]$$

or

$$\Delta\left[\frac{n(h u_s - h_s u)}{\cos\bar{u}}\right]$$

$$= \Delta\left[\frac{h h_s n.\cos\bar{I}}{r} - \frac{h.h_s n}{r} + \frac{n(hu_s - h_s u)\sin^2\bar{u}}{\cos\bar{u}(1+\cos\bar{u})}\right]$$

$$= -\frac{h h_s}{r}\Delta\left[n(1-\cos\bar{I})\right] + \Delta\left[\frac{n(hu_s - h_s u).\sin^2\bar{u}}{\cos\bar{u}(1+\cos\bar{u})}\right]$$

$$\Delta\left[\frac{n(hu_s - h_s u)}{\cos\bar{u}}\right] = -\frac{\bar{B}^2 h.h_s}{r}\Delta\left[\frac{1}{n(1+\cos\bar{I})}\right]$$

$$+ \Delta\left[\frac{n(hu_s - h_s u)\sin^2\bar{u}}{\cos\bar{u}(1+\cos\bar{u})}\right]$$

The terms in this expression will be called terms (1) and (2). The second term is a higher order term, and the first term will be used later to form the Petzval term in the final expression.

The third term in equation (18) can be expanded as follows

$$\Delta\left(\frac{u}{\cos\bar{u}}\right) = \Delta\left[(u) + (u)\left(\frac{1-\cos\bar{u}}{\cos\bar{u}}\right)\right]$$

$$= \Delta(i) + \Delta\left[\frac{u.\sin^2\bar{u}}{\cos\bar{u}(1+\cos\bar{u})}\right]$$

which can be written:-

$$\Delta\left(\frac{u}{\cos\bar{u}}\right) = -A\,\Delta\left(\frac{1}{n}\right) + \Delta\left[\frac{u.\sin^2\bar{u}}{\cos\bar{u}\,(1+\cos\bar{u})}\right]$$

So that the complete third term in equation (18) becomes:-

$$-A\left(A_s\bar{x} - \bar{B}h_s\sin\bar{\alpha}\right)\Delta\left(\frac{1}{n}\right)$$

$$+\left(A_s\bar{x} - \bar{B}h_s\sin\bar{\alpha}\right)\Delta\left[\frac{u.\sin^2\bar{u}}{\cos\bar{u}(1+\cos\bar{u})}\right]$$

These terms will be called terms (3) and (4), the second of these being a higher order term.

Terms (1) and (3) can now be combined in the following manner:- Omitting the Δ's we have that:-

$$(1)+(3) = -\frac{\bar{B}^2h.h_s}{r}\left[\frac{1}{n(1+\cos\bar{I})}\right.$$

$$\left. - A\left[A_s\bar{x} - \bar{B}h_s.\sin\bar{\alpha}\right]\Delta\left(\frac{1}{n}\right)\right.$$

$$= -\frac{1}{n.r.}\left(\frac{1}{1+\cos\bar{\alpha}}\right)\left[\frac{\bar{B}^2h.h_s(1+\cos\bar{\alpha})}{(1+\cos\bar{I})} + AA_s\bar{x}(1+\cos\bar{\alpha}).r.\right.$$

$$\left. - A\bar{B}h_s\bar{h}(1+\cos\bar{\alpha})\right]$$

which may be written:-

$$-\frac{1}{n.r.}\left(\frac{1}{1+\cos\bar{\alpha}}\right)\left[\frac{\bar{B}^2 h.h_s(1+\cos\bar{\alpha})}{(1+\cos\bar{I})} + AA_s\bar{h}^2\right.$$

$$\left. -Ah_s\bar{h}(1+\cos\bar{\alpha})\left(\frac{H_s}{h_s\cos\bar{\alpha}} + \frac{A_s\bar{h}}{h_s\cos\bar{\alpha}}\right)\right]$$

i.e., substituting values for $\bar{x}$ and $\bar{B}$ given in equations (17) and (13).

The whole expression thus reduces to:-

$$-\frac{1}{nr}\left(\frac{1}{1+\cos\bar{\alpha}}\right)\left[\frac{\bar{B}^2 h.h_s(1+\cos\bar{\alpha})}{(1+\cos\bar{I})} + AA_s\left(1-\frac{1+\cos\bar{\alpha}}{\cos\bar{\alpha}}\right)\right.$$

$$\left. -AH_s\bar{h}\left(\frac{1+\cos\bar{\alpha}}{\cos\bar{\alpha}}\right)\right]$$

or

$$-\frac{1}{n.r.}\left(\frac{1}{1+\cos\bar{\alpha}}\right)\left[\bar{B}^2 h.h_s\left(\frac{1+\cos\bar{\alpha}}{1+\cos\bar{I}}\right) - \frac{AH_s\bar{h}(1+\cos\bar{\alpha})}{\cos\bar{\alpha}}\right.$$

$$\left. -\frac{AA_s\bar{h}^2}{\cos\bar{\alpha}}\right]$$

Thus the complete expression for the sagittal curvature along the reference ray is:-

$$\Delta(nuu_s\delta S) = \tfrac{1}{2}\bar{B}^2 h_s \Delta\left[\frac{2}{\cos\bar{u}(1+\cos\bar{I})}\left(\frac{u}{n}\right)\right]$$

$$-\Delta\left(\frac{1}{nv}\right)\left(\frac{1}{1+\cos\bar{\alpha}}\right)\left[\bar{B}^2 h h_s\left(\frac{1+\cos\bar{\alpha}}{1+\cos\bar{I}}\right)\right.$$

$$\left. - \frac{AH_s\bar{h}(1+\cos\bar{\alpha})}{\cos\bar{\alpha}} - \frac{AA_s\bar{h}^2}{\cos\bar{\alpha}}\right]$$

$$+\Delta\left[\frac{n(hu_s + h_s u)\sin^2\bar{u}}{\cos\bar{u}(1+\cos\bar{u})}\right]$$

$$+\left(A_s\bar{x} - \bar{B}h_s\sin\bar{\alpha}\right)\Delta\left[\frac{u\cdot\sin^2\bar{u}}{\cos\bar{u}(1+\cos\bar{u})}\right] \qquad \text{---------(18a)}$$

The first term in this expression is the astigmatism part of the sagittal curvature, which developes into $\frac{1}{2}B^2 h\Delta\left(\frac{u}{n}\right)$ in the first order case. The second term in the expression reduces to $\frac{1}{2}H^2 P$ in the first order case, for the expression in the bracket of the second term reduces to

$$\left[B^2h^2 - 2AH\bar{h} - A^2\bar{h}^2\right] \equiv H^2$$

The next two terms are terms which vanish when the aberration is measured along the optical axis as does the (Cos $\bar{U}$) in the first term for the complete aberration.

The formulae for the complete aberration will be used later in exact and approximate calculation of sagittal curvature.

<u>The relation between sagittal curvature and distortion</u>.

If first order theory it has been shown that the primary distortion (S v)

is related to the sagittal curvature by a simple formula:-

$$S\bar{V} = \sum_{1}^{K} \frac{B}{A}\left[H^2 P + B^2 h \Delta\left(\frac{u}{n}\right)\right]$$

or that Distortion = (B/A) (Sagittal curvature). An exact relation of this kind will now be shown to exist. This relation leads to a new and simple tracing for sagittal pencils.

For this we require a new definition of the complete distortion contribution at a surface. Hopkins has shown that a transverse focal shift gives rise to a primary aberration quantity of the form $(nu\,\delta\eta)$ where $(\delta\eta)$ is the difference in the intersection heights of an accurately traced reference ray and that of a corresponding paraxial reference ray.

It is proposed to use the definition $\Delta(n u \delta\eta)$ for the complete distortion contribution of a given surface. The angle $\overset{(u)}{\wedge}$ is that associated with an ordinary paraxial ray. The quantity $nu\delta\eta$ remains finite and meaningful no matter what the position of the object or image plane from the refracting surface. This expression is exactly analogous to the primary aberration form, and will be again a unique number for the complete aberration arising at a surface.

(in fig. (8) AB is a reference ray having a sagittal focus at S.

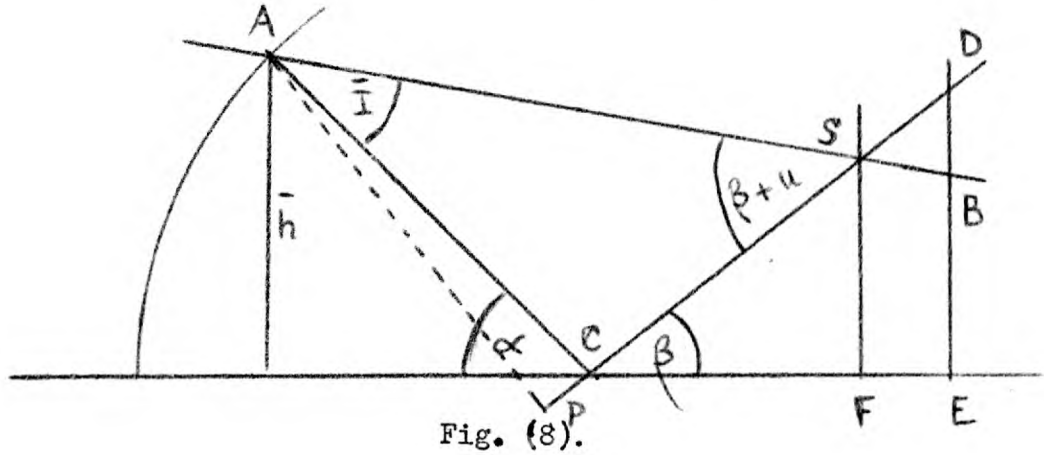


Fig. (8).

From the diagram we have that:-

$$SF = \eta_s \quad , \quad DE = \eta_c \quad , \text{ and } DB = \delta\eta_c \, , \quad SB = \delta\bar{s} \quad .$$

In triangle SDB we have that:-

$$\delta\eta_c = \delta\bar{s} . \sin \bar{u} + \delta\bar{s} \cos \bar{u} . \tan \beta$$

or

$$n_u \delta\eta_c = n_u \delta\bar{s} (\sin \bar{u} + \cos \bar{u} \tan \beta).$$

$$= n_u u_s \delta\bar{s} \left(\frac{s . \sin \bar{u} + s . \cos \bar{u} \tan \beta}{h_s} \right)$$

$$= n_u u_s \delta\bar{s} . \left[\frac{s}{h_s \cos \beta} (\sin \bar{u} . \cos \beta + \cos \bar{u} . \sin \beta) \right]$$

$$= n_u u_s \delta\bar{s} \left[\frac{s . \sin (\beta + \bar{u})}{h_s \cos \beta} \right]$$

Now s.Sin (β + $\bar{U}$) = AP, and AP is constant on refraction because A is a fixed point and β is constant on refraction.

Thus we have:-

$$\Delta(n_u \delta\eta_c) = \frac{s . \sin (\beta + \bar{u})}{h_s \cos \beta} \Delta (n_u u_s \delta\bar{s}) \qquad \text{---------(19).}$$

Having defined the surface contribution to distortion to be $\Delta(n_u \delta\bar{\eta})$, and having defined ($\delta\bar{\eta}$) as the difference in the intersection heights in the paraxial image plane of the reference ray and the paraxial reference ray we may write:-

$$(\delta\bar{\eta}) = \eta - \bar{\eta} = \eta - \eta_c + (\eta_c - \bar{\eta}).$$

where η is the paraxial reference ray intersection height. Taking the change on refraction we have:-

$$\Delta(nu\delta\bar{\eta}) = \Delta(nu\eta) - \Delta(nu\eta_c) + \Delta(nu\delta\eta_c) \qquad \text{--------(20)}$$

Now $\Delta(nu\eta) = 0$ ($nu\eta$) being the Smith-Helmholtz invariant for the paraxial reference ray, and since the angle β is constant on refraction then the intersection height (η_c) must be directly proportional to the ordinary paraxial magnification, and thus we may write:-

$$\frac{\eta'_c}{\eta_c} = \frac{nu}{n'u'} \quad ; \quad \Delta(nu\eta_c) = 0$$

and equation (20) reduces to :-

$$\Delta(nu\delta\bar{\eta}) = \Delta(nu\delta\eta_c).$$

Thus equation (19) becomes:-

$$\Delta(nu\delta\bar{\eta}) = \frac{s.\sin(\beta+\bar{u})}{h_s\cos\beta}\Delta(nuu_s\bar{\delta s}) \qquad \text{---------(21)}$$

It now remains for us to evaluate the constant in this equation in terms of $(\bar{B}/A_s)$, so that we have an accurate expression for the complete distortion contribution in terms of the complete sagittal curvature along the reference ray, exactly analogous to the first order relation.

If the constant be denoted by γ then we have:-

$$\gamma = \frac{s.\sin(\beta+\bar{u})}{h_s\cos\beta}$$

Now in triangle (CAS) fig. (8) it is clear that:-

$$\frac{r}{\sin(\beta+\bar{u})} = \frac{\eta_s}{\sin\beta.\sin\bar{I}}$$

or

$$\eta_s.\sin(\beta+\bar{u}) = r.\sin\bar{I}.\sin\beta.$$

Thus

$$s.\sin(\beta+\bar{u}) = \frac{r.\bar{B}.\sin\beta.h_s}{H_s}$$

or

$$\gamma = \frac{r.\bar{B}.\sin\beta.h_s}{H_s h_s \cos\beta} = \frac{r.\bar{B}\eta_s}{H_s(x_s-r)}$$

and since $(x_s - r) = (s.\cos\bar{U} - r.\cos\bar{\alpha})$

we have that:-

$$\gamma = \frac{r.\bar{B}}{n.\cos\bar{u}.h_s - nu_s r.\cos\bar{\alpha}}$$

The denominator on expanding becomes:-

$$n h_s.\cos\bar{\alpha}.\cos\bar{I} + \bar{B}h_s\sin\bar{\alpha} - n h_s\cos\bar{I}.\cos\bar{\alpha} + A_s r.\cos\bar{\alpha}$$

$$= A_s r.\cos\bar{\alpha} + \bar{B}h_s\sin\bar{\alpha}$$

Thus we may write:-

$$\gamma = \frac{\bar{B}}{A_s\cos\bar{\alpha} + \frac{\bar{B}h_s\sin\bar{\alpha}}{r}}$$

and substituting this value for $\frac{s.\sin(\beta+\bar{u})}{h_s.\cos\beta}$ in equation (21) we finally have that:-

$$\Delta(nu\delta\eta) = \frac{\bar{B}}{A_s\cos\bar{\alpha} + \frac{\bar{B}h_s\sin\bar{\alpha}}{r}}\Delta(nuu_s\delta\bar{S}) \qquad (22)$$

Thus we have an accuract relation between the complete distortion and complete sagittal curvature contributions similar to the first order relation.

The constant (γ) can be put into a form which is more convenient for some purposes.

Using the relation

$$\bar{B}\bar{h}_s\cos\bar{\alpha} = H_s + A_s\bar{h} \qquad \text{equation (13)}$$

we may write:-

$$\gamma = \frac{\bar{B}}{\frac{\bar{B}\bar{h}_s\cos^2\bar{\alpha}}{\bar{h}} - \frac{H_s\cos\bar{\alpha}}{\bar{h}} + \frac{\bar{B}\bar{h}_s\sin\bar{\alpha}}{v}}$$

$$= \frac{\bar{B}\bar{h}}{\bar{B}\bar{h}_s - H_s\cos\bar{\alpha}}$$

and finally:-

$$\text{Distortion} = \left[\frac{\bar{B}\bar{h}}{\bar{B}\bar{h}_s - H_s\cos\bar{\alpha}}\right]_{\text{(Sagittal curvature)}} \text{---------(23)}$$

<u>An indirect method for tracing sagittal pencils</u>.

Examination of the above equation for distortion in terms of the sagittal curvature will show that the only quantities involving a knowledge of the sagittal refraction conditions are h_s and H_s .

The complete distortion which is a desirable quantity to calculate may be obtained directly from the reference ray, and a knowledge of h_s is all that is required to calculate the complete sagittal curvature; H_s being constant throughout the system.

We will now show how the value $(h_s)_2$ can be obtained from the value $(h_s)_1$ using only reference ray quantities and H_s, the subscripts denoting the number of the surface concerned.

Using the relation that:-

$$H_s = n u_s \eta_s = \bar{h}(nu_s) - h_s(n \cdot \sin \bar{u}) \qquad \text{equation (7)}$$

we have that:-

$$u_s = \frac{H_s}{n\bar{h}} + \frac{h_s \cdot \sin \bar{u}}{\bar{h}}$$

Now u_s does not alter in passing from one surface to another so we may write:-

$$\frac{H_s}{n(\bar{h})_2} + \frac{(h_s)_2 \sin \bar{u}}{(\bar{h})_2} = \frac{H_s}{n(\bar{h})_1} + \frac{(h_s)_1 \sin \bar{u}}{(\bar{h})_1}$$

the only quantities changing between surfaces being $(h_s)_2$ and $(h_s)_1$.

Thus we have a relation between $(h_s)_2$ and $(h_s)_1$ in terms of the known constant H_s and reference ray quantities in that:-

$$(h_s)_2 = (h_s)_1 \left(\frac{\bar{h}_2}{\bar{h}_1}\right) - \frac{H_s}{n \cdot \sin \bar{u}} \left(1 - \frac{\bar{h}_2}{\bar{h}_1}\right) \qquad \text{----------(24)}$$

The distortion and sagittal curvature of a system can thus be calculated with the aid of equations (23) and (24) in the following manner:-

From a traced reference ray the complete distortion is calculated, H_s is found from a knowledge of the object height and distance. The sagittal paraxial invariant need only be calculated once for any system.

The value of h_s is known at the first surface, and now a subsidiary calculation is necessary to calculate the values of h_s at every surface until the system is traversed. This is done using equation (24), the values of $(\bar{h})_1$, $(\bar{h})_2$, and $(\text{Sin } \bar{U})_{12}$ being taken directly from

the reference ray.

If s'_K , the distance from the reference ray intersection point with the last surface to the final sagittal focus is required then this is given by:-

$$s'_K = (n'_K \bar{l}_K) \frac{H_S}{(l_S)_K} + (n \sin \bar{u})'_K .$$

which equation follows directly from equation (7). In the work which follows various calculations are described which have been done to verify that the analytical results obtained are correct, and secondly to determine some method of calculating the surface contributions to astigmatism and sagittal curvature to fourth powers of the field, having first computer the first order values.

Chapter 3.

The calculation of $\Delta(nu_s\ u_I\ \delta A)$, $\Delta(nuu_s\ \delta S)$, $\Delta(nu\bar{\eta})$.

Calculations have been carried out on a three lens system of the Cooke type the specification of which appears below.

Radii of Curvature	Thickness of lens	Glass	√Value
$R_1 = 0.3900$	0.0512	1.615	56.3
$R_2 = -7.6000$			
$R_3 = -0.4700$	0.0100	1.621	36.1
$R_4 = 0.4000$			
$R_5 = 2.3000$	0.0512	1.613	59.3
$R_6 = -0.3788$			

Air separations. Lens 1 to lens 2 = 0.1000
Lens 2 to lens 3 = 0.1080

All dimensions are in inches.

The lens is a 1" focus "Cooke" type F/3.5, full field of 52. (fig. 9).

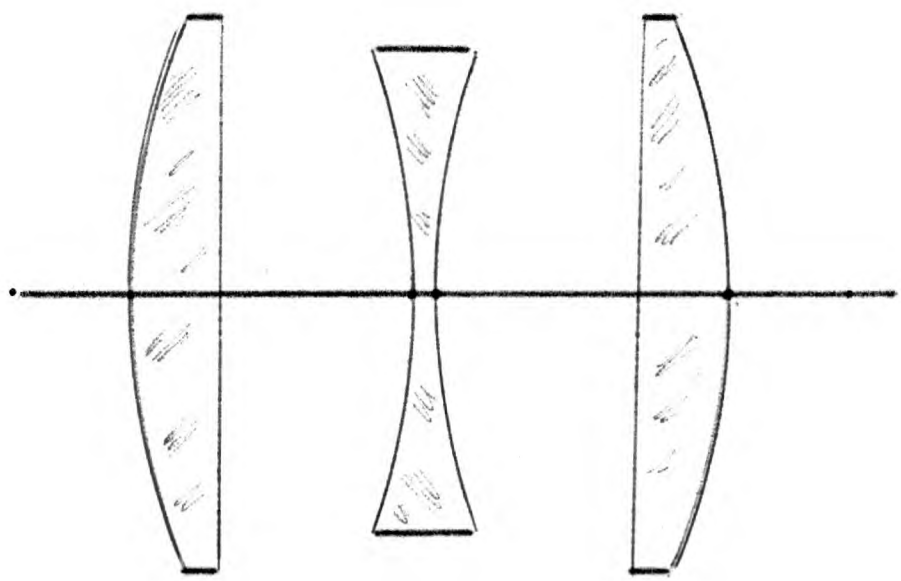


Fig. (9)

A paraxial ray was first traced through the system with (h) = 0.1410, and the first order aberrations calculated the field angle of the paraxial reference ray being 18°.

The method used is due to Hopkins (10), and does not in fact involve the tracing of a paraxial reference ray as such.

The first order sumnation for the system are:-

$$S\ 1 = \sum A^2 h \Delta\left(\frac{u}{n}\right) = 0.0^3\ 649$$

$$S\ 11 = \sum AB h \Delta\left(\frac{u}{n}\right) = \bar{0}.0^3\ 075$$

$$S111 = \sum B^2 h \Delta\left(\frac{u}{n}\right) = \bar{0}.0^3\ 350$$

$$S\ 1v = \sum H^2 P = 0.0^3\ 886$$

$$S\ v = \sum \frac{B}{A}\left[H^2 P + B^2 h \Delta\left(\frac{u}{n}\right)\right] = 0.0^3\ 081$$

Two reference rays were now traced through the system one at 18°, the other at 23°, and the sagittal and tangential foci along these rays determined. The reference ray trace used was a modification of a method due to T. Smith, and a resume of the formulae involved and the schedule for the calculation is given below.

The formulae include the use of $\bar{\eta} = \bar{L}.\mathrm{Sin}\ \bar{U}$, being the length of the line drawn from the pole of the surface concerned perpendicular to the reference ray. Fig. (10) shows a reference ray incident at the point P on the surface AP, intersecting the optical axis at O. The reference sphere AB is drawn with centre O and radius AO, and AN $= \bar{\eta} = \bar{L}.\mathrm{Sin}\ \bar{U}$, BP $= \epsilon$.

From the figure we have that:-

$$(\bar{L}-r)\ \mathrm{Sin}\ \bar{U} = r.\mathrm{Sin}\ \bar{I}$$

or

$$\bar{\eta}.c = \mathrm{Sin}\ \bar{I} + \mathrm{Sin}\ \bar{U} \qquad \text{----------(25)}$$

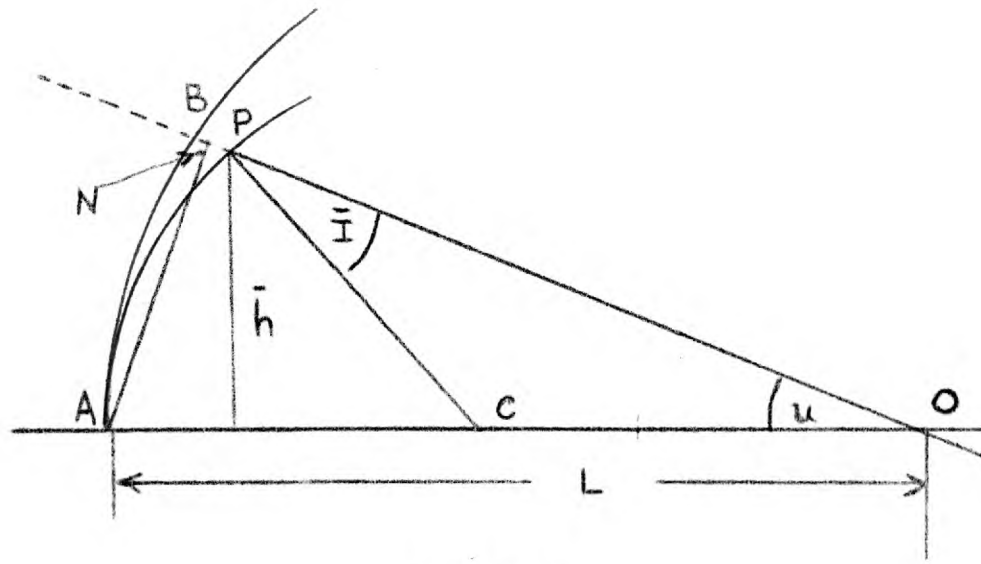


Fig. (10)

Thus one obtains Sin $\bar{I}$ from a knowledge of $\bar{r}_2$, c, and Sin $\bar{U}$ which are all starting quantities. This method of obtaining Sin $\bar{I}$ at the first surface is to be preferred because it eliminates inaccuracies which arise in the usual form $(\bar{L}-r)$ Sin $\bar{U}$ = r.Sin $\bar{I}$, when $\bar{L}$ becomes large and Sin $\bar{U}$ small.

From Sin $\bar{I}$ one obtains Sin $\bar{I}'$, Cos $\bar{I}$, and Cos $\bar{I}'$, and one can then calculate the distance between the reference sphere and the surface given by:-

$$\epsilon = 2\bar{r}_2 \sin \bar{I} / G$$ where G is given by:-

$$G = (1+\cos\bar{u})(1+\cos\bar{I}) + \sin\bar{u}.\sin\bar{I}$$

The height of the ray intersection point with the surface is now calculated by:-

$$(\bar{h}) = \bar{r}_2 - \epsilon.\sin\bar{u} \qquad \text{----------(25a)}$$

and Sin U' calculated using identity (1) equation (15) where we have that:-

$$\Delta(n.\sin\bar{u}) = \bar{h}\,\Delta\left(\frac{n.\cos\bar{I}}{r}\right)$$

The value of Cos $\bar{U}'$ is obtained from Sin $\bar{U}'$, and the refracted value of $\bar{h}_2$ is obtained from:-

$$(\bar{h}') = (\bar{h}_2)\left[\frac{\cos\bar{u}' + \cos\bar{I}'}{\cos\bar{u} + \cos I}\right]$$

This completes all the refraction calculations, and in passing from one surface to the next we have the following:-

$$D - d = \epsilon - \epsilon'_{-1}$$

$$\bar{h}_2 = (\bar{h}_2)_{-1} - d.\sin\bar{u}$$

$$\bar{h} = (\bar{h})_{-1} - D.\sin\bar{u}$$

where D is the distance along the reference ray from one surface to the next, and d is the axial separation between the surfaces. The value of the quantities obtained after refraction can be checked with the following formulae:- the equations being the same as the starting ones, primed quantities being used.

$$\sin\bar{u}' = \bar{h}_2'.c - \sin\bar{I}'$$

$$G' = (1+\cos\bar{u})(1+\cos\bar{I}') + \sin\bar{u}'.\sin\bar{I}'$$

$$\epsilon = 2\bar{h}'.\sin\bar{I}'/G' \quad ; \quad \bar{h}_2' = \bar{h} + \epsilon.\sin\bar{u}'$$

It is to be noted that tracing by this method gives the surface contributions to marginal chromatic aberra tion, and spherical aberration. These can be written:-

$$\Delta(\delta\eta_c) \quad \text{and} \quad \delta(n\epsilon)$$

where δn is the difference in refractive index for the two wavelengths concerned.

Having completed the reference ray traces, the sagittal and tangential foci for infinite object distances were determined using the following schedules:- the traces involving the use of h_S, h_T, u_S, and u_T described already.

Sagittal ray. **Tangential ray.**

On refraction.

$$-A_S = nu_S - h_S K \qquad -A_T = n.\cos\bar{I}\, u_T - h_T K.$$

Sagittal ray **Tangential ray.**

$$n'u'_S = -A_S + h_S K' \qquad n'\cos I'\, u'_T = -A_T + h_T K'$$

Between surfaces.

$$(h_S)_2 = (h_S)_{-1} - D(u_S)_2 \; ; \; (h_T)_2 = \frac{(h_T \cos\bar{I}')_1 - D(u_T)_2}{(\cos\bar{I})_2}$$

Check calculations.

Sagittal ray

$$\frac{\alpha_S}{\cos\bar{I}} = \frac{\alpha_S'}{\cos\bar{I}'} = \frac{u_S}{\cos\bar{I}} + \frac{A_S}{\cos\bar{I}} = \frac{u'_S}{\cos\bar{I}'} + \frac{A_S}{n'\cos I'}$$

and $$(h_S)_K = (h_S)_{K-1} - \sum_1^K D(u_S)_K .$$

Tangential ray.

$$\alpha_T = u_T + \frac{A_T}{n.\cos\bar{I}} = u'_T + \frac{A_T}{n'.\cos\bar{I}'},$$

$$y_T = h_T \cos\bar{I} \quad ; \quad y'_T = h_T \cos\bar{I}'$$

and $$(y_T)_2 = (y_T)_1 - D(u_T)_2$$

The sagittal and tangential foci having been calculated, the s' and t' lengths were projected on to the optical axis, and the distances δ S and δ T determined. These quantities are the distances from the axially projected astigmatic foci to the paraxial image plane, and are given by:-

$$\delta S = l - (s'\cos\bar{u}' + \bar{x})$$

$$\delta T = l - (t'\cos\bar{u}' + \bar{x}).$$

where l is the distance of the paraxial image plane from the last surface of the system, and $\bar{x}$ the sagitta or 'depth of curvature' of the last surface.

The sagittal and tangential field curves for the three lens system are shown in fig. (11)

18°

$\delta X_S = -0.0^2 6736$

$\delta X_T = 0.0^2 4092$

23°

$\delta X_S = -0.0^2 3621$

$\delta X_T = -0.0^2 4620$

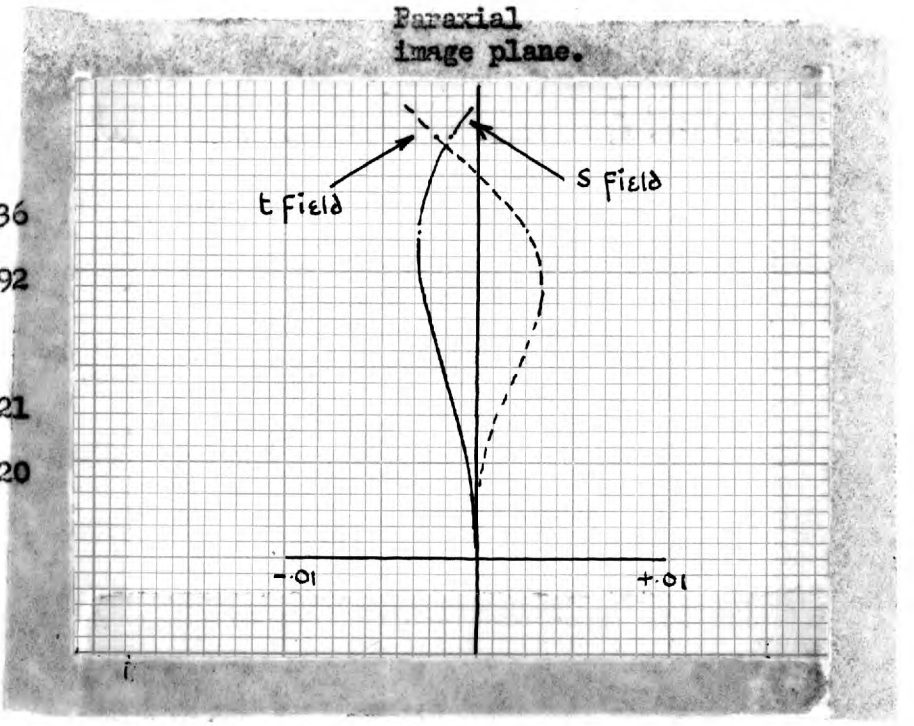


Fig. (11)

We have now sufficient numerical data to obtain values of $\Delta(nu_S u_T \delta A)$, $\Delta(nuu_S \delta S)$.

The following table gives the values of the three terms in equation (5) - that for the surface contribution to astigmatism along the optical axis, at each surface in the three lens system at 18° and 23°.

Surface	1	2	3
Term			
1.	$0.0^3 165$	$-0.0^2 4087$	$-0.0^2 4838$
2.	$-0.0^3 449$	$0.0^3 058$	$-0.0^3 353$
3.	$-0.0^3 447$	$0.0^3 040$	$-0.0^3 273$
Surface Sum	$0.0^3 163$	$0.0^2 4105$	$-0.0^2 4918$

Surface	4	5	6
Term			
1.	$-0.0^{2}1759$	$0.0^{2}1443$	$0.0^{3}739$
2.	$0.0^{3}197$	$0.0^{3}0048$	$-0.0^{3}495$
3.	$0.0^{3}171$	$0.0^{3}036$	$-0.0^{3}487$
Surface Sum	$-0.0^{2}1733$	$0.0^{2}1455$	$0.0^{3}731$

Thus the sum for the system is $\sum_{1}^{6}\Delta(nu_s u_T \delta A) = -0.000197.$ In order that this value be comparable with the first order sum it must be multiplied by a factor $(u')^2/(u'_T u'_s)$ which ensures that the value of $\sum\Delta(nu_s u_T \delta A)$. is concerned with the same emergent pencil as the first order value.

Also these values must be multiplied by 1/ Cos $\bar{U}'_6$ so that they refer to the aberration measured along the referance ray.

We have then finally that $\sum_{1}^{6}\Delta(nu_s u_T \delta A) = -0.0^{3}230$, the higher order contribution being $0.0^{3}120$. It will be noticed that the contribution of the second two terms is quite small at this angular field, the sum of the last two terms being -0,0000 34.

The corresponding values for the 23° ray are as follows:-

Surface	1	2	3
Term			
1.	$0.0^{3}196$	$0.0^{2}7612$	$-0.0^{2}8089$
2.	$-0.0^{3}681$	$0.0^{3}125$	$-0.0^{3}733$
3.	$-0.0^{3}677$	$0.0^{3}064$	$-0.0^{3}454$
Surface Sum	$0.0^{3}192$	$0.0^{2}7223$	$-0.0^{2}8368$

Surface	4	5	6
Term			
1.	$-0.0^{2}2826$	$0.0^{2}2553$	$0.0^{2}1159$
2.	$0.0^{3}381$	$0.0^{3}093$	$-0.0^{3}811$
3.	$0.0^{3}302$	$0.0^{3}059$	$-0.0^{3}783$
Surface Sum	$-0.0^{2}2747$	$0.0^{2}2587$	$0.0^{2}1131$

The sum for this ray is 0.0000023 (after modification), the higher order contribution being 0.000638.

Astigmatism along the reference ray.

From the ray tracing data the astigmatism along the reference ray has been calculated using equation (5a), and the surface sums at 18° and 23° are shown in the table below:-

18°

Surface	1	2	3
	$0.0^{3}168$	$0.0^{2}4570$	$-0.0^{2}5412$
	4	5	6
	$-0.0^{2}1919$	$0.0^{2}1641$	$0.0^{3}747$

23°

Surface	1	2	3
	$0.0^{3}204$	$0.0^{2}8778$	$-0.0^{2}9992$
	4	5	6
	$-0.0^{2}3261$	$0.0^{2}3102$	$0.0^{2}1187$

At 18° the sum for the system is 0.000205 and if this value is multiplied by $(\text{Cos } U')_{6}$ we obtain 0.000195 which agrees with the value obtained using the formula (5) from which we obtained the value 0.000197.

Similarly at 23° the sum for the system is 0.000018 which when

converted into an axial value by multiplication by Cos $\overline{U}_2'$ gives 0.000017, which again agrees with the former value 0.000018. The values given on pages 53-55 for the surface contributions have been checked using the information directly from the trigonometrical ray traces and they are found to agree with the results obtained by these formulae to a similar order of accuracy, i.e., to one or two in the sixth place.

Sagittal curvature.

The formulae giving the surface contributions to sagittal curvature have been evaluated numerically in a similar manner to those for astigmatism, and the results for the aberration as expressed in equation (18a) are given as follows:-

18°

Surface	1	2	3
	$0.0^2 1035$	$0.0^2 2155$	$-0.0^2 3415$
Surface	4	5	6
	$-0.0^2 1836$	$0.0^2 0834$	$0.0^2 1360$

23°

Surface	1	2	3
	$0.0^2 1644$	$0.0^2 3870$	$-0.0^2 6044$
Surface	4	5	6
	$-0.0^2 3020$	$0.0^2 1409$	$0.0^2 2209$

At 18° the sum for the system is 0.000133, and measured along the reference ray this becomes 0.000140. The first order value given by:-

$$\tfrac{1}{2}\left[B^2 h\,\Delta\left(\tfrac{u}{n}\right) + H^2 P\right]$$ is 0.000262.

In order that these values be comparable the exact sum must be multiplied by $(u'/u_5')_2 = 1.043789$ and becomes 0.000146, so that the values to be

compared are 0.000146 and 0.000262, i.e., the higher order contribution is -0.000116.

Similarly for the 23° ray the sum for the system is 0.000069 measured along the axis and 0.000073 measured along the reference ray.

The first order sum for this angular field is 0.000449 and the final values to be compared are: 0.000449 and 0.000079, the higher order contribution being -0.000370.

The results obtained using these formulae for the surface contributions to the sagittal curvature have been checked against values calculated directly from the trigonometrical ray traces - these being inexact agreement.

Numerical calculation of distortion and sagittal curvature

It remains now to check the equation given on page (42) equation (23). For this it is proposed to use the indirect sagittal ray trace, and to check the calculated value of the sagittal curvature against the value obtained above.

For surface 1. the value of $(h_s)_1$ is 0.1410, the value of $(h_s)_2$ is obtained from equation (24), and so on through the system. The values of h_s at each surface are as follows:-

Surface	1	2	3
	0.0141000	0.134766	0.108671
Surface	4	5	6
	0.108133	0.119233	0.121113

The values of $\Delta(nu\bar{\eta})$ are now calculated at each surface; for the 18° ray they are as follows:-

Surface	1	2	3
	$0.0^{3}369$	$-0.0^{2}4355$	$0.0^{2}3960$

Surface	4	5	6
	$-0.0^{2}2118$	$0.0^{2}2784$	$0.0^{3}617$

Now the values of the constant $\left[\frac{\bar{B}h_s - H_s \cos\bar{\imath}}{\bar{B}\bar{h}}\right]$ can be calculated at each surface giving the following results:-

Surface	1	2	3
	2.914098	-0.566956	-0.953149
Surface	4	5	6
	0.948028	0.351117	2.268261

and the sagittal curvature contributions are as follows:-

Surface	1	2	3
	$0.0^{2}1075$	$0.0^{2}2469$	$-0.0^{2}3774$
Surface	4	5	6
	$-0.0^{2}2008$	$0.0^{3}978$	$0.0^{2}1400$

The sum for the system is 0.000140 for this angular field. If this sum is multiplied by Cos $\bar{U}'_6$ we can transfer the aberration to the optical axis giving the result $\sum\Delta(nuu,\delta s)$ = 0.000133 which is the value obtained using the algebraic formulae (page 37). This completes the numerical checks on the formulae and we proceed now to approximate calculations.

Approximate calculations.

It is well known that first order aberration theory gives results which are reasonably accurate for systems having not more than a few degrees of angular field.

At even moderate field sizes first order aberration theory becomes inadequate and other measures must be taken so that a good account of the aberrations can be obtained.

Generally, the algebraic evaluation of aberrations above the first order is complicated and the terms giving the second and third order aberrations are both numerous and complex, so that they do not lend themselves to easy calculation.

We have defined the complete aberration contributions for astigmatism, sagittal and tangential image curvature, and distortion, and have obtained simple expressions for these total aberrations. It is proposed now to use these expressions to obtain numerical values for the aberration contributions which approach the correct values much more closely than the first order ones. The calculation is that of the insertion of approximate values for the constants and variables of the final expressions for these aberrations so that the final results contain those terms which depend on the square and the fourth power of the field. The error in this is at most $\underline{o}(\sigma^6)$. where (σ) is the ratio between the image height, (or field size), and the equivalent focal length of the system.

The approximations are based on the "$\bar{t}_2$" trace already mentioned. It is not necessary to calculate the value of ϵ however, and the calculations are very simple and depend simply on a series of approximations for Cos $\bar{I}$, $\bar{x}$, and $\bar{h}$.

a) The relation between $(\bar{h})$ and $(\bar{t}_2)$.

We have from equation (25a) (page 47) that:-

$$\bar{t}_2 = \bar{h} + \epsilon \cdot \sin \bar{u}$$

where ϵ is given by:-

$$\frac{2 \cdot \bar{t}_2 \cdot \sin \bar{I}}{(1+\cos \bar{u})(1+\cos \bar{I}) + \sin \bar{u} \cdot \sin \bar{I}}$$

and to sufficient accuracy we have that:-

$$\bar{h}_2 = \bar{h} + \tfrac{1}{2}\,\bar{h}_2 . \sin \bar{I} . \sin \bar{u} \qquad \text{----------(26)}$$

$$\bar{h} = \bar{h}_2 \left(1 - \tfrac{1}{2} \sin \bar{I} . \sin \bar{u}\right).$$

---

b) An expression for Cos $\bar{\alpha}$.

From the previous equation we have that:-

$$\bar{h} = r . \sin \bar{\alpha} = \bar{h}_2 \left(1 - \tfrac{1}{2} \sin \bar{I} . \sin \bar{u}\right)$$

and

$$\left(1 - \tfrac{1}{2} \sin^2 \bar{\alpha}\right) = \cos \bar{\alpha}$$

or

$$\text{Cos } \bar{\alpha} = 1 - \tfrac{1}{2} \left[\bar{h}_2 c \left(1 - \tfrac{1}{2} \sin \bar{I} . \sin \bar{u}\right)\right]^2 \qquad \text{----------(27)}$$

which reduces to:-

$$\text{Cos } \bar{\alpha} = 1 - \tfrac{1}{2} (\bar{h}_2 c)^2$$

Now Cos $\bar{\alpha}$ can be written $(\mathbf{r}-\bar{\mathbf{x}}/\mathbf{r}) = 1 - \bar{\mathbf{x}}.\mathbf{c}.$

or $1 - \bar{x}.c. = 1 - \frac{1}{2} (\bar{h}_2.c.)^2$

or $\bar{x} = \frac{1}{2} (\bar{h}_2)^2 . c.$ ----------(28)

The approximate calculation continues in the same way as the reference ray trace, and is in effect a "second order" paraxial reference ray trace. Thus:-

From Sin $\bar{I}$ one can determine Sin $\bar{I}'$ and then Cos $\bar{I}$ given by $(1-\frac{1}{2}\ \mathrm{Sin}^2\bar{I})$. The value of $\bar{h}$ is obtained from equation (26) and the value of Sin $\bar{U}'$ calculated from the relation:-

$$\Delta(n.\sin\bar{u}) = \bar{h}\,\Delta\left(\frac{n.\cos\bar{I}}{r}\right)$$

The value of $\bar{h}$ for the next surface is given by:-

$$(\bar{h})_2 = (\bar{h}')_1 - d.(\sin\bar{u})_{12}$$

where d is the axial separation between the surfaces as before.

Calculations of this nature have been carried out on the three lens system used before, the sagittal curvature being determined by the accurate formulae. The astigmatism has been calculated using the accurate formulae given in equation (5).

There follows now tables of results in tabular form obtained by these approximate methods and a comparison with the first order values is made.

<u>Astigmatism 18°</u>

Surface	1	2	3
	0.0^3163	$0.0^2 4094$	$-0.0^2 4928$
Surface	4	5	6
	$-0.0^2 1717$	$0.0^2 1431$	$0.0^3 748$

The sum being -0.0^3209, which is to be compared with the trigonometrical value of -0.0^3197.

Astigmatism 23°

Surface	1	2	3
	$0.0^{3}191$	$0.0^{2}7168$	$-0.0^{2}8478$
Surface	4	5	6
	$-0.0^{2}2715$	$0.0^{2}2480$	$0.0^{2}1229$

The sum being $-0.0^{3}124$, which is to be compared with the trigonometrical value of $0.0^{3}018$

Sagittal curvature 18°

Surface	1	2	3
	$0.0^{2}1037$	$0.0^{2}2135$	$-0.0^{2}3387$
Surface	4	5	6
	$-0.0^{2}1824$	$0.0^{3}826$	$0.0^{2}1366$

The sum being $0.0^{3}153$, which is to be compared with the trigonometrical value of $0.0^{3}139$

Sagittal curvature 23°

Surface	1	2	3
	$0.0^{2}1640$	$0.0^{2}3766$	$-0.0^{2}5908$
Surface	4	5	6
	$-0.0^{2}2960$	$0.0^{2}1371$	$0.0^{2}2244$

The sum being $0.0^{3}154$, which is to be compared with the trigonometrical value of $0.0^{3}069$

Fig. (12) shows these results in graphical form and a comparison is made between a) accurate values b) approximate values obtained by the method outlined above and c) the first order value.

The first order aberrations were calculated for the 18° field, and the 23° values have been obtained by multiplying the 18° values by $(\text{Tan } 23^{\circ} / \text{Tan } 18^{\circ})^{2}$ In the figure the numbers plotted are:-

Accurate values.	18°	23°
Astigmatism	$-0.0^{3}197$	$0.0^{3}018$
Sagittal curvature	$0.0^{3}139$	$0.0^{3}069$
Approximate values.		
Astigmatism	$-0.0^{3}209$	$-0.0^{3}124$
Sagittal curvature	$0.0^{3}153$	$0.0^{3}154$
First Order values.		
Astigmatism	$-0.0^{3}350$	$-0.0^{3}600$
Sagittal curvature	$0.0^{3}262$	$0.0^{3}419$

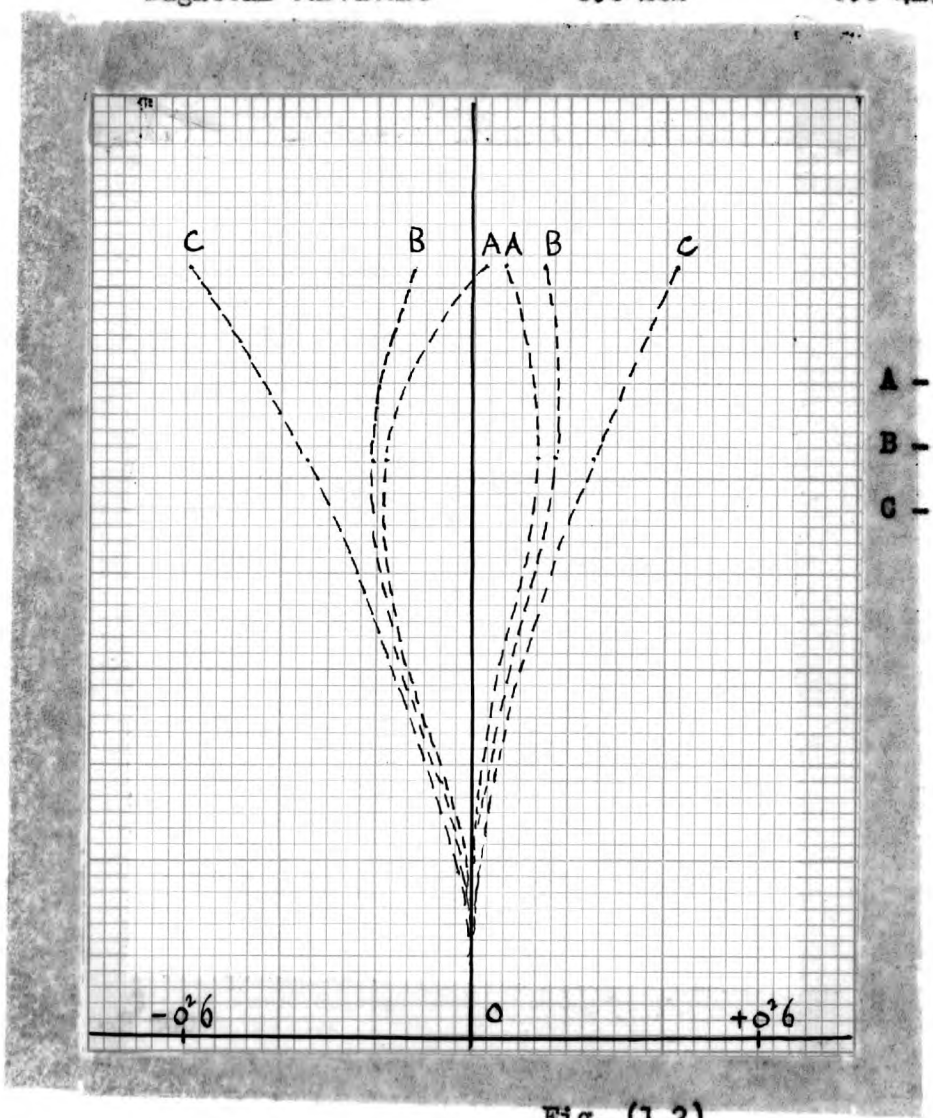


Fig. (1 2)

From the curves it will be seen that the approximate method gives adequate results up to an angular field of 18^{o} on this particular system, but at larger angular fields the higher order aberrations become more important, and one must use accurate methods to determine the complete aberration. The approximate values are however a great improvement on the first order values, and in cases when only the approximate values are known the above method of substituting these into the complete aberration expression is clearly a most valuable one.

PART II

CHAPTER 1.

The fine correction of astigmatism of lens systems using differential formulae for wave aberrations.

The design of a lens system consists essentially of three main parts. Firstly a preliminary survey is made of the lens type which is to be designed and the various glass types are chosen so that the chromatic aberrations are minimised. This survey usually fixes the focal lengths of the elements to be used and determines their positions in the system.

Secondly the first order aberrations are calculated and the system is then adjusted so that these aberrations are minimised. The adjustment usually consists of 'bending' the components so that their equivalent focal lengths and their relative positions remain fixed while the first order aberrations change. This ensures that the initial chromatic corrections are kept. This does not mean however that the first order aberrations should be all zero, for in many cases a residual of first order aberration is left in the system so that a known amount of higher order aberration can be corrected. This type of correction can of course only be executed if the designer has previous knowledge of the amount of higher order aberration likely to be encountered.

The result of this sort of adjustment is a 'first order solution' to the design, and this being done the third part of the design can be executed. This consists essentially of determining the complete aberrations at finite apertures and field angles, and finally minor adjustments are made to the system so that the complete aberrations are minimised while keeping the overall state of correction across the field.

Until comparitively recently the only method available for this

last stage of the design was that of tracing a large number of finite rays so that the state of correction could be determined. This work is necessarily laborious for each minor adjustment of the system requires in effect a further complete survey of the system by ray tracing.

The next part of this work is concerned with the third part of the design programme i.e., that of the final adjustment of the system.

The method proposed eliminates the necessity for tracing a large number of rays and consists essentially of finding the partial differential coefficients of the aberrations for a typical set of rays which have been traced through the design established when part two of the procedure is completed. The differential coefficients are obtained with respect to the design parameters, i.e., curvature of surfaces, lens thicknesses, axial separations etc.

Having selected the changes in the parameters which minimise the aberrations of the system, - (the selection depending on the residual of undesirable aberration, and the differential coefficients), the cycle of operations can be repeated, thereby obtaining an iterative method for the final adjustment of the system.

Variational formulae of this kind were first proposed by Cruikshank and M'Aulay (Proc.Phys.Soc. 1945) and they have been reformulated by o ther writers. These calculations are concerned with the displacement of a typical ray due to a change in the design parameter, the ray displacement being defined by two quantities δl and δu Fig. (1). These displacement variables have then to be transferred surface by surface to the image s pace. For this, a series of 'transfer coefficients' are required, and these calculations constitute the major part of the labour.

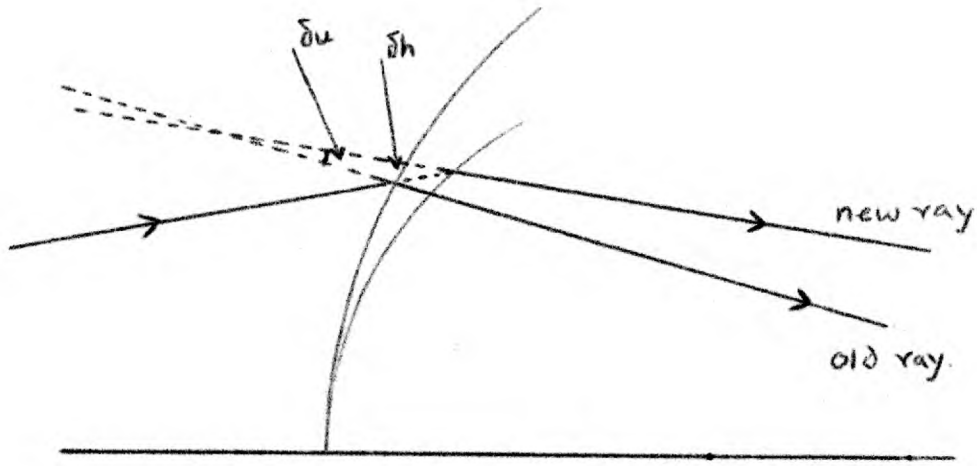


Fig. (1)

A method which eliminates the use of 'transfer coefficients' would be a great improvement on this method and such is the case when the differential coefficients concern the wave aberration change due to a change in one of the system parameters. A method of this nature follows from the theorem of neighbouring rays given by Hopkins (11), of which a brief account now follows:-

If we use the wave aberration in place of the ray aberration along a given ray of an oblique pencil then the difference in optical path which determines the aberration is given by:-

$$W = \left[\bar{A}\,\bar{A}_1 \ldots\ldots \bar{A}_K \bar{A}'\right] - \left[A A_1 \ldots\ldots A_K A'\right] \qquad \text{——————(1)}$$

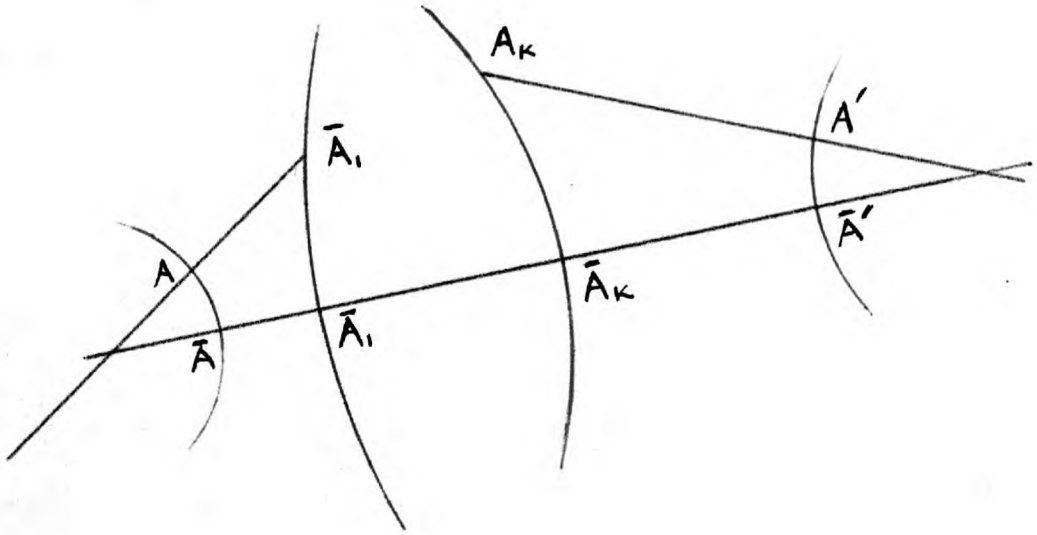


Fig. (2)

where the bar denotes a reference ray and the subscripts 1 and K refer to the initial and final surfaces of the system, and $\bar{A}A$, $\bar{A}'A'$ are the reference spheres in the object and image space respectively.

Any variation of equation (1) can now be expressed as

$$\delta W = \delta\left[\bar{A}\bar{A}_1 \ldots\ldots \bar{A}_K\bar{A}'\right] - \delta\left[AA_1 \ldots\ldots A_KA'\right] \qquad \text{——————(2)}$$

and we now consider the change in optical path length along the ray rather than the change in the path of the ray from the changed surface to the final image space.

If the change in the system parameter is small then the displacement of the path of the ray is a first order quantity, and the new and old ray paths are in fact neighbouring rays. Thus, by the theorem mentioned above the optical paths along them between perpendicular intercepts are equal. The variations in the optical path length in equation (2) can be found from calculations which are confined to the changed surface, and the variation

in the subsequent paths of the ray may be ignored. The need for transfer coefficients is thereby obviated, and the calculations are greatly simplified.

General Formulae.

We shall now develop variational formulae of this kind for astigmatism, i.e., for sagittal and tangential field curvatures.

It is required first to find a formula for the influence of the displacement of a refracting interface on the optical path length along a ray. In fig (3) A_0 A_1 .....P_s.....A_K A'_0 is a traced ray through an optical system of k surfaces and A_0 A'_0 are on the reference spheres.

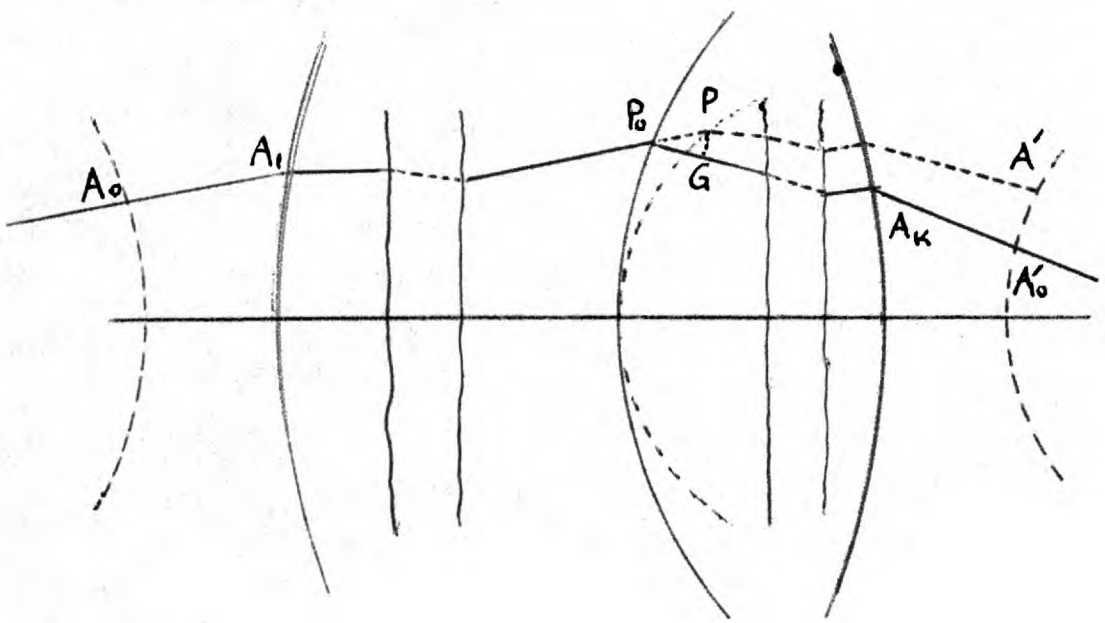


Fig.(3)

Using square brackets to denote optical path lengths we require the change:-

$$\delta\left[A_0 \ldots\ldots A'_0\right]$$

when any surface of the system is displaced due to a change in curvature or a lateral displacement as a whole. If this leads to the ray being incident at P instead of P_0 then we have that:-

$$\delta\left[A_0 \ldots\ldots A_0'\right] = \left[A_0 \ldots P_0 P \ldots\ldots A'\right] - \left[A_0 \ldots P_0 \ldots A_0'\right] \qquad (3)$$

and if G is the foot of the perpendicular from P on the original ray we have:-

$$\left[P \ldots A'\right] = \left[G \ldots\ldots A_0'\right]$$

a statement which follows from the theorem of neighbouring rays.
We may write then that:-

$$\delta\left[A_0 \ldots A_0'\right] = \left[A_0 \ldots P_0\right] + \left[P_0 P\right] + \left[P \ldots\ldots A'\right] - \left[A_0 \ldots P_0\right] - \left[P_0 G\right] - \left[G \ldots A'\right]$$

or

$$\delta\left[A_0 \ldots\ldots A_0'\right] = \left[P_0 P\right] - \left[P_0 G\right] \qquad (4)$$

which demonstrates that the change in optical path along a ray may be found from simple geometrical considerations at the surface displaced.

The surface displacement will be specified by the distance measured from P_0, along the normal to the original surface to H, the point where this normal cuts the displaced surface. fig. (4).

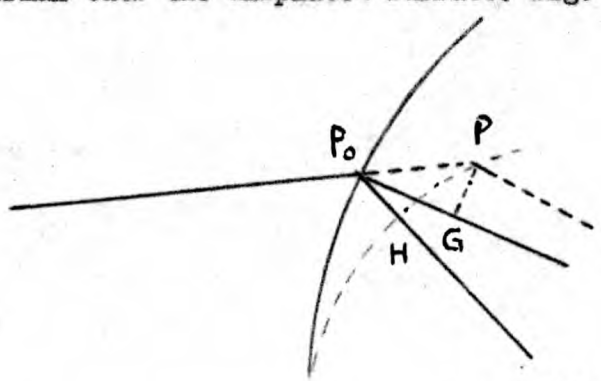


Fig. (4)

This displacement can occur either because of a change of curvature of the surface, or because of a change in eccentricity (if the surface is non-spherical), or because of a lateral displacement of the surface due to a change in axial separation or a decentering of the surface.

In any of these cases an increment in the constructional parameter will be used to specify the magnitude of the change. If the increment in the constructional parameter is δp, we may write:-

$$P_oH = f.\delta p$$

This quantity can now be used in equation (4) for we have:-

$$P_oP = P_oH/\cos\bar{I} \quad ; \quad P_oG = P_oP_1\cos(\bar{I}'-\bar{I})$$

$$P_oG = P_oH/\cos\bar{I} \cdot [\cos(\bar{I}'-\bar{I})]$$

where $\bar{I}$ and $\bar{I}'$ are angles of incidence and refraction at P_o. Hence the optical path difference (4) may be written:-

$$\delta[A_o - A_o'] = \delta p.f.\left[\frac{n}{\cos\bar{I}} - \frac{n'\cos(\bar{I}'-\bar{I})}{\cos\bar{I}}\right]$$

or

$$\frac{\partial}{\partial p}[A_o \ldots A_o'] = f\left[\frac{n}{\cos\bar{I}} - n'\cos\bar{I}' - \frac{n'\sin\bar{I}'.\sin\bar{I}}{\cos\bar{I}}\right]$$

and replacing $n'.\sin\bar{I}'$ by $n.\sin\bar{I}$ this reduces to:-

$$\frac{\partial[A_o \ldots A_o']}{\partial p} = -f\Delta(n.\cos I) \qquad \text{——————(6)}$$

In deriving this last equation we have placed no restrictions whatever on the form of the refracting surface, or on the type of ray, or on the displacement. In other words the last equation applies equally well to skew rays, and to non-spherical surfaces, or both, as to rays confined to one section, through spherical surfaces.

One of the advantages of this method over earlier ones is that

systems lacking all symmetry can be included in the calculations.

Combining equations (6) and (2), and extending the formulae to two rays we have that:-

$$\frac{\partial W_o}{\partial \rho} = f\,\Delta(n.\cos I) - \bar{f}\,\Delta(n.\cos \bar{I}) \qquad \text{----------(7)}$$

where quantities having the bar above them refer to a reference ray. The subscript zero has been given to ∂W to denote the fact that equation (7) gives the rate of change of the aberration with respect to the original reference sphere.

We will now investigate the value of $\frac{\partial W_o}{\partial \rho}$ in equation (7) for two close rays in the meridian section. In general, $\frac{\partial W_o}{\partial \rho}$ will contain terms which comprise a) a longtitudinal focal shift, and b) a transverse focal shift, the former being associated with a displacement of the astigmatic foci, and the latter with a change in the image height. It is necessary then to separate these two types of term in the expression for $\frac{\partial W_o}{\partial \rho}$.

The evaluation of $\partial W_o/\partial c$ in the meridian plane.

Fig (5) shows a reference ray incident at the surface OB, at A. EB is another ray close to it in the meridian section. If the radius of the surface is changed by a small amount δc, then $\bar{f}$ the distance along the normal from the point of intersection of the reference ray with the surface to the displaced surface is given by:-

$$\bar{f} = \tfrac{1}{2}(\bar{x}^2 + \bar{y}^2)\,\delta c$$

where $\bar{x}$, and $\bar{y}$ are the coordinates of the point of intersection.

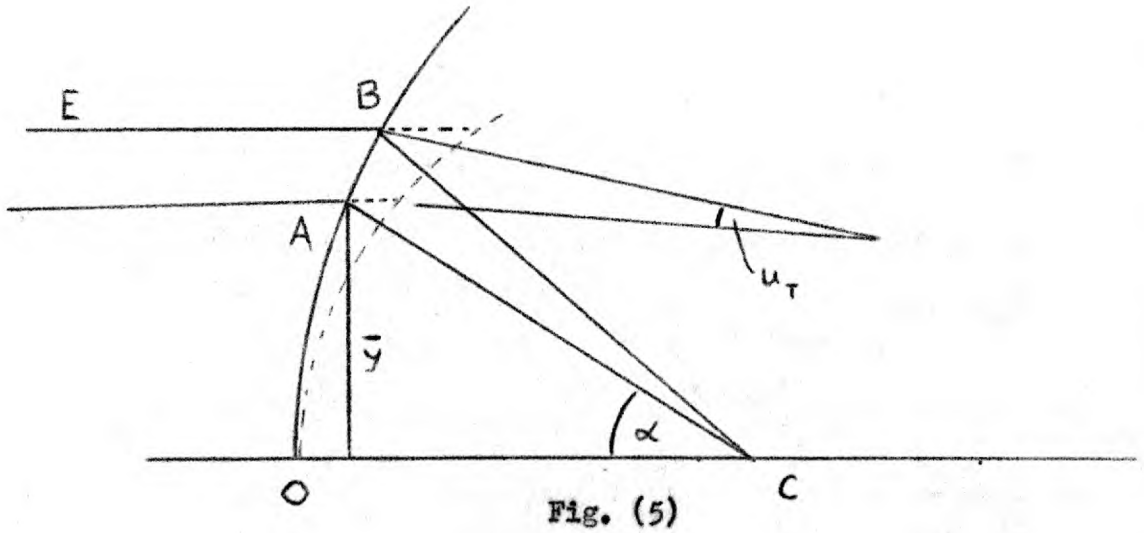


Fig. (5)

The coordinates of the point of intersection of the close ray being

$$x = \bar{x} + h_T \sin\bar{\alpha} \quad ; \quad y = \bar{y} + h_T \cos\bar{\alpha}$$

we have that:-

$$f = \tfrac{1}{2}\left[(\bar{x} + h_T \sin\bar{\alpha})^2 + (\bar{y} + h_T \cos\bar{\alpha})^2\right]$$

where f is the distance already described for the close ray, and h_T the distance AB, the aperture of the 'tangential fan'.

Thus f can be written:-

$$\tfrac{1}{2}(\bar{x}^2 + \bar{y}^2) + \tfrac{1}{2}(h_T)^2 + (h_T \bar{x}.\sin\bar{\alpha} + h_T \bar{y}.\cos\bar{\alpha}).$$

or

$$f = \bar{f} + h_T(\bar{x}\sin\bar{\alpha} + \bar{y}\cos\bar{\alpha}) + \tfrac{1}{2}(h_T)^2 \qquad \text{---------(8)}$$

This expression for f is accurate enough for reasons already stated, in that terms of the order of $(h_T)^3$ can be ignored.

In order that we may evaluate δW_o completely it is necessary to obtain an expression for Cos I in terms of Cos $\bar{I}$ so that we can express

δW_o in terms of reference ray quantities and the aperture (h_T).

Now $I = \bar{I} + \delta I$, and it has been shown (12) that $\delta I = (i_T)$ where (i_T) is the fictitious angle of incidence of the tangential paraxial ray.

Thus we may write equation (7)

$$\frac{\partial W_o}{\partial c} = \left[\bar{f} + h_T(\bar{x}\sin\bar{\alpha} + \bar{y}\cos\bar{\alpha}) + \tfrac{1}{2}(h_T)^2\right]\Delta n\left[\cos(\bar{I}+i_T) - \bar{f}\,\Delta(n.\cos\bar{I}).\right]$$

$$= \bar{f}\,\Delta\left[n.\cos(\bar{I}+i_T)\right] - \bar{f}\,\Delta(n.\cos\bar{I}) + \tfrac{1}{2}(h_T)^2\Delta n(\cos\bar{I}+i_T)$$

$$+ (\bar{y}\cos\bar{\alpha} + \bar{x}.\sin\bar{\alpha})\,h_T\,\Delta\left[n.\cos(\bar{I}+i_T)\right]$$

Expanding Cos $(\bar{I} + i_T)$ and writing Cos $i_T = (1-\frac{1}{2}\,\mathrm{Sin}^2\, i_T) = (1 - \frac{1}{2} i_T^2)$ we have that:-

$$\frac{\partial W_o}{\partial c} = \bar{f}\left[n.\cos\bar{I}\left(1-\tfrac{1}{2}i_T^2\right)\right] - \bar{f}\,\bar{B}\,\Delta(i_T) - \bar{f}\,\Delta(n\cos\bar{I})$$

$$+ \bar{x}\bar{y}\left(\frac{1}{\bar{x}} - \frac{1}{r} + \frac{1}{r}\right)(h_T)\,\Delta\left(n.\cos\bar{I}.\cos i_T - \bar{B}\,i_T\right)$$

$$+ \tfrac{1}{2}(h_T)^2\,\Delta\left(n.\cos\bar{I}.i_T - \bar{B}\,i_T\right)$$

Only terms depending on powers of (h_T) up to and including the square are required so we may write:-

$$\frac{\partial W_o}{\partial c} = \tfrac{1}{2}(h_T)^2\,\Delta(n.\cos\bar{I}) - \bar{y}(h_T)\,\bar{B}\,\Delta(i_T)$$

$$-\tfrac{1}{2}\bar{f}\,\Delta(n.\cos\bar{I}.i_T^2) - \bar{f}\,\bar{B}\,\Delta(i_T) + \bar{y}\,h_T\,\Delta(n\cos\bar{I})$$

and writing

$$A_T = (n i_T \cos\bar{I}) \quad ; \quad \Delta(u_T) = -\Delta(i_T).$$

we have finally:-

$$\frac{\partial W_o}{\partial c} = \tfrac{1}{2}(h_T)^2 \Delta(n.\cos\bar{I}) + \bar{\gamma}\bar{B}(h_T)\Delta(u_T) \cdot$$

$$+ \tfrac{1}{2}\bar{f} A_T \Delta(u_T) + \bar{\gamma}(h_T)\Delta(n.\cos\bar{I})$$

$$+ \bar{f}\bar{B}\Delta(u_T). \qquad \text{---------(9)}$$

This is an important equation for it shows that $\frac{\partial W_o}{\partial c}$ can be expanded into a series of terms some of which (depending on the square of h_T) relate to a longtitudinal shift in focus, and others depending on the first power of (h_T) which relate to a transverse shift in focus. These types of focal shift terms will now be discussed in detail.

Transverse focal shift formulae.

In fig (6) P_o is the point of intersection of a ray having direction cosines (L, M, and N), with a reference sphere of centre O_o and radius R_o . The ray cuts a new sphere of centre O and radius AO in P.

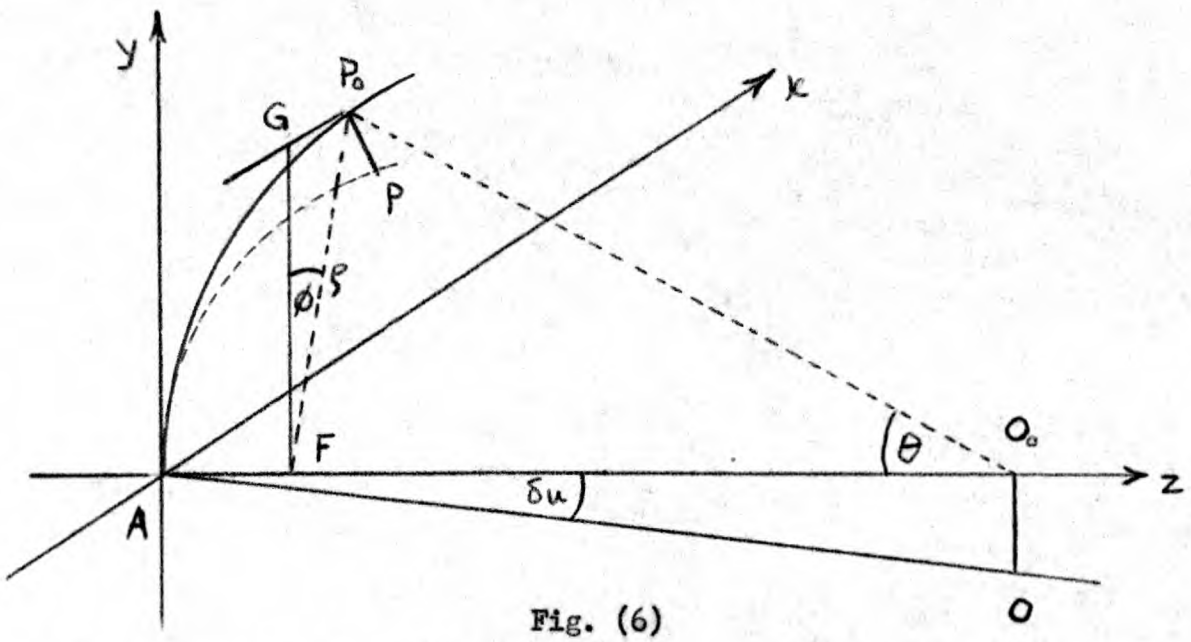


Fig. (6)

The distance O_o O = δY is regarded as a transverse shift in focus.

If P_o P = f, then the coordinates of P are given by:-

(x + Lf), (y + Mf), and (z + Nf), the coordinates of the point P_o being x, y, and z.

For the reference sphere centre Q we have that:-

$$(x + Lf)^2 + (y + Mf + \delta y)^2 + (z + Nf - R_o)^2 = (R_o)^2 + (\delta y)^2$$

which can be written:-

$$x^2 + y^2 + z^2 + f^2 + 2f(Lx + My + N(z - R_o) + M\delta y + 2y\delta Y - 2zR_o = 0$$

or

$$f^2 + 2f(Lx + My + N(z - R_o) + M\delta Y) + 2y\delta Y = 0$$

----------(10)

Now Lx + My + N (z - R) = projection of

$$\overrightarrow{O_o F} + \overrightarrow{FG} + \overrightarrow{G P_o}$$

= projection of O_o P_o on the ray.

Thus Lx + My + N (z - R) can be written:-

= $-R_o \cos \delta U_o$, where δU_o is the angular aberration.

Thus from (10) we have that:-

$$f^2 - 2f\beta R_o + 2y\delta Y = 0$$

where $\beta = \cos \delta u_o - M \tan \delta \bar{u}$; $\delta \bar{u} = \tan^{-1} \frac{\delta Y}{R_o}$.

The smaller root of the above quadratic equation in f can be written:-

$$f = y.\delta Y / \beta R_o + \frac{1}{2} y^2 \delta Y^2 / \beta^3 R_o^3 \quad \ldots\ldots \text{etc.}$$

-------etc.

or to the required degree of approximation:-

$$f = y.\delta Y / R_o$$

Thus the transverse shift in focus from O_o to O gives rise to a change in aberration:-

$$\delta W_T = -nf = -n.\sin\theta.\cos\phi\,\delta Y.$$

or

$$\delta W_T = -n\varsigma.\delta\bar{u}.\cos\phi. \qquad \text{----------(11)}$$

where (n) is the refractive index.

Thus it is shown that a transverse shift in focus gives rise to a change in aberration which depends on the first power of the aperture. This result will be used to distinguish the transverse focal shift components in the general equation for $(\partial W_o/\partial c)$.

Longitudinal shift formulae.

In a similar manner, the change in the wave-front aberration due to a longitudinal shift in focus will now be investigated.

In fig. (7), the new focus is at O, and the radius of the new reference sphere is AO = R.

If the coordinates of P_0 are (x, y, and z) as before, we have that:-

$$x^2 + y^2 + z^2 = 2R_o z.$$

and

$$f^2 + 2f\left[Lx + My + N(z-R)\right] + 2z(R_o - R) = 0 \qquad \text{----------(12)}$$

If the angular aberration of the ray with respect to the new focus is u

then we may write:-

Lx + My + N (z - R) = the projection of O P_o on to the ray,

or $-(R \cos \delta U + f)$

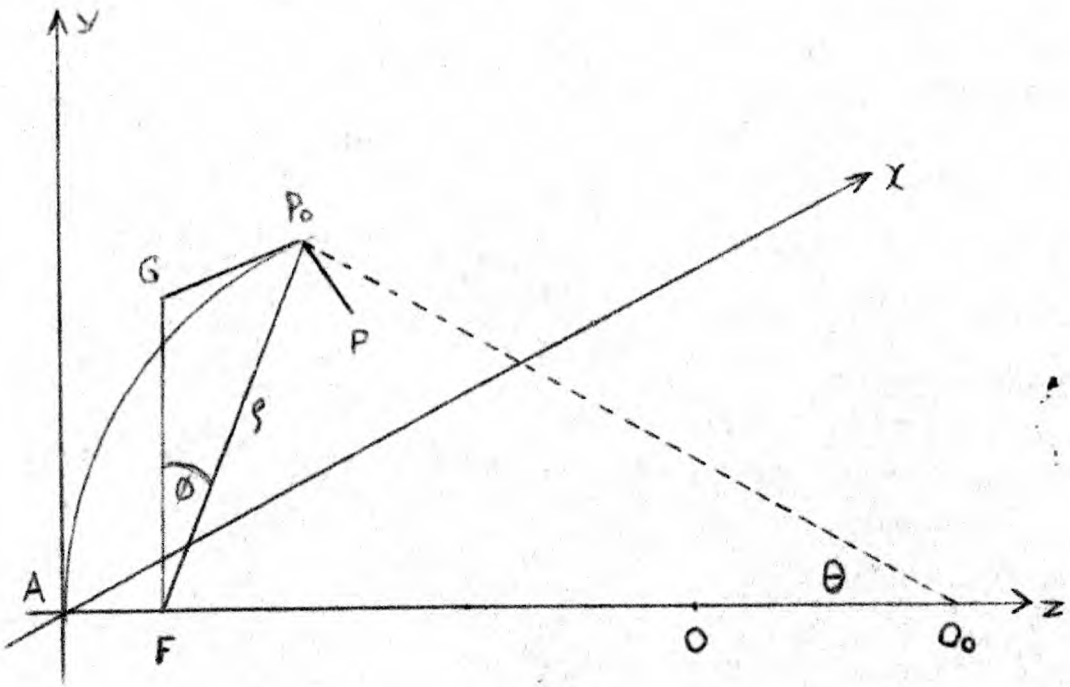


Fig. (7)

Now we have that:-

$$R. \cos \delta u = -f - (Lx + My + Nz) + NR.$$

and

$$R_o \cos \delta \bar{u}_o = (Lx + My + Nz) + NR_o.$$

from which we obtain that:-

$$\cos \delta u - \cos \delta u_o = -\frac{f}{R} - (Lx + My + Nz)\left(\frac{1}{R} - \frac{1}{R_o}\right)$$

or

$$R \cos \delta u + f = R\left[\cos \delta u_o - (Lx + My + Nz)\left(\frac{1}{R} - \frac{1}{R_o}\right)\right] = \gamma R.$$

Thus the quadratic equation in f (equation 12) can be written:-

$$f^2 - 2f\gamma R + 2z(R_o - R) = 0$$

the required solution being:-

$$f = \frac{z(R_o - R)}{\gamma R} + \frac{1}{2}\frac{z^2(R_o - R)^2}{\gamma^3 R^3} + \quad \text{.....etc.}$$

or

$$f = \frac{z(R_o - R)}{\gamma R.}$$ to the required degree of approximation.

Using the identity given on page 28, equation 17, we have that:-

$$z = \frac{R_o \sin^2 \theta}{(1 + \cos\theta)}$$, so that:-

$$f = \frac{\varsigma^2}{(1+\cos\theta)}\left(\frac{1}{R} - \frac{1}{R_o}\right)\gamma$$

If we now assume that δU_o is small we may write that:-

$$(Lx + My + Nz)\left(\frac{1}{R} - \frac{1}{R_o}\right) \equiv z\left(\frac{1}{R} - \frac{1}{R_o}\right)$$

and

$$\gamma \equiv 1$$

giving finally:-

$$\delta W_L = -n.f. = -\frac{n\varsigma^2}{1+\cos\theta}\left(\frac{1}{R} - \frac{1}{R_o}\right)$$

Thus it is shown that a longtitudinal shift in focus gives rise to a change in aberration which depends on the square of the aperture. This

will be used in the further evaluation of $\frac{\partial W_0}{\partial c}$.

Transverse and longtitudinal shift terms in the meridian section.

Returning now to the general expression for $\frac{\partial W_0}{\partial c}$ given in equation (9)

$$\frac{\partial W_0}{\partial c} = \tfrac{1}{2}(h_T)^2 \Delta(n\cos\bar{I}) + \bar{y}\,\bar{B}(h_T)\,\Delta(u_T) + \tfrac{1}{2}\bar{f}A_T\Delta(u_T)$$

$$+ \bar{y}(h_T)\,\Delta(n\cos\bar{I}) + \bar{f}\,\bar{B}\,\Delta(u_T)$$

we see that terms 1, 2, and 3, depend on $(h_T)^2$, $(h_T u_T)$ and $(A_T u_T)$, while the last two terms depend on (h_T) and (u_T). Thus the first three terms may be called longtitudinal shift terms being dependent on the square of the aperture of the tangential fan. In this case we may write:-

$$\frac{\partial W_{LT}}{\partial c} = \tfrac{1}{2}(h_T)^2 \Delta(n\cos\bar{I}) + \bar{y}\,\bar{B}(h_T)\,\Delta(u_T) + \tfrac{1}{2}\bar{f}A_T\Delta(u_T) \qquad \text{---------(13)}$$

Similarly the transverse focal shift terms can be written:-

$$\frac{\partial W_T}{\partial c} = \bar{y}(h_T)\,\Delta(n\cos\bar{I}) + \bar{f}\,\bar{B}\,\Delta(u_T) \qquad \text{---------(14)}$$

In the case of a paraxial pencil, it will be seen that the general expression $\frac{\partial W_0}{\partial c}$ reduces to

$$\tfrac{1}{2}(h)^2\,\Delta(n) \qquad \text{---------(15)}$$

for in this case $\bar{y} = 0$ and $\bar{f} = 0$, and further $\Delta(n\cos\bar{I})$ reduces to $\Delta(n)$ while (h_T) becomes the ordinary paraxial semi-aperture.

Equations (13), (14), and (15) describe completely the change in aberration due to a small change in the surface radius.

Chapter 2.

Having calculated the quantity $\frac{\partial W_o}{\partial \rho}$ we desire to know the effect of this aberration change in the final image space. In the case of astigmatism it is thus desirable to know the increments in $(nuu_T \delta T)$ and $(nuu_S \delta S)$ for a change in the constructional parameter.

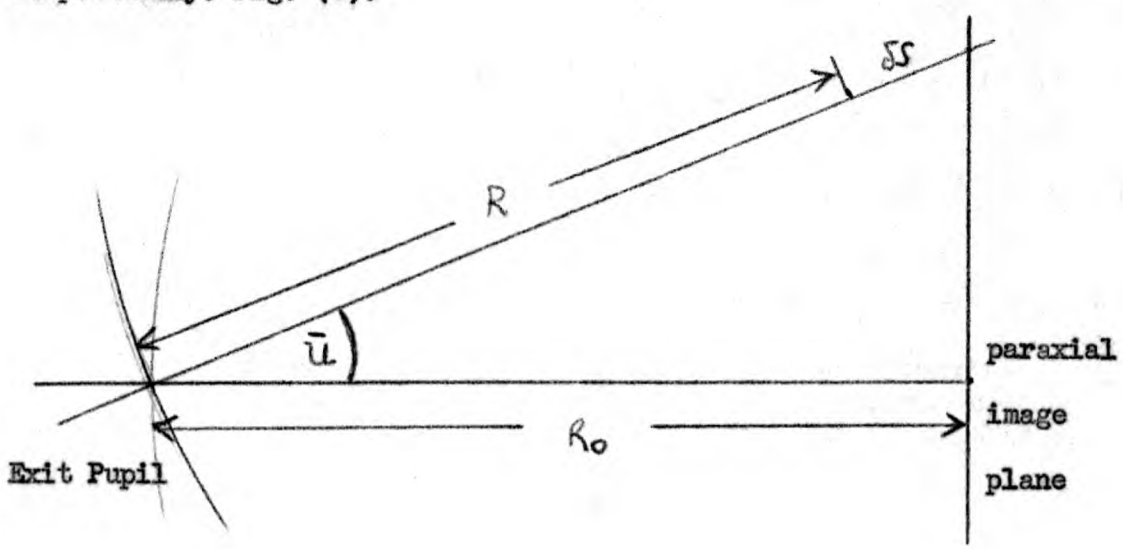
It is necessary in the evaluation of $(nuu_S \delta S)$ and $(nuu_T \delta T)$ to assume that the semi-apertures at the exit pupil in the sagittal and tangential sections are constant throughout the change in the constructional parameter. The justification for this assumption is given below.

Proof that hh_S and hh_T at the exit pupil can be regarded as constant.

Considering the sagittal section we may write:-

$$\delta(nuu_S \delta S) = \delta\left(n h_t h_s \frac{\delta S}{R_o R}\right)$$

where h and h are the semi-apertures at the exit pupil, and R , R$_o$ are the distances from the exit pupil point to the sagittal focus along the reference ray, and to the paraxial image plane along the optical axis respectively. Fig. (8).



Now (δ S) can be written $\left[\frac{R_o}{\cos\bar{u}} - R\right]$

and hence

$$\delta(nuu_s\delta s) = \delta\left[n.h.h_s\left(\frac{1}{R\cos\bar{u}} - \frac{1}{R_o}\right)\right]$$

If we denote values after a change in the contructional parameter by the superscript * we have that:-

$$\delta(nuu_s\delta S) = n.h h_s\left(\frac{1}{R\cos\bar{u}} - \frac{1}{R_o}\right)^*$$

$$+ n\left(h^* h_s^* - h h_s\right)\left(\frac{1}{R\cos\bar{u}} - \frac{1}{R_o}\right)^*$$

$$- n.h.h_s\left(\frac{1}{R\cos\bar{u}} - \frac{1}{R_o}\right)$$

$$= n.h h_s\delta\left(\frac{1}{R\cos u} - \frac{1}{R_o}\right) + n\left(h^* h_s^* - h.h_s\right)\left(\frac{\delta S^*}{R^* R_o^*}\right)$$

In the final correction of the system it is usual to arrange that the distance δS be small, if not zero. In other words we can say that δS^* approaches zero, and the last term in the above equation for $\delta(nuu_s\delta S)$ becomes negligable, the first term being dependent on the values of h and h_s before the change in the constructional parameter is made - i.e., that (hh_s) is constant.

When δS^* is small then the error incurred by assuming that (hh_s) remains constant can be shown as follows:-

We may write the last term in the above expression:-

$$\left(1 - \frac{h h_s}{h^* h_s^*}\right)(nuu_s\delta S)^*$$

The departure of the first bracket from zero depends on the value of (δp), this being not more than one or two per. cent. Similarly the value of $(nuu_s \delta S)^*$ in a lens which is nearly corrected is equally small, the product of the two brackets being smaller still. This is considered enough justification that (hh_s) can be considered constant while a small change is made in the constructional parameter.

Similar reasoning holds in the tangential section and (hh_T) can be considered constant for the same reasons.

The increment in $(nnu_T \delta T)$ produced by a small change in the constructional parameter.

We will consider now the effect of the small change in the constructional parameter in the final image space, and determine the change in $(nuu_T \delta T)$, assuming that the values of (h) and (h_T) at the exit pupil remain constant while the change is made.

In fig. (9) OQ is a reference ray striking the paraxial image plane at Q. The exit pupil is at O. If the tangential focus is at T, then PT = t, PN being the last surface of the system whose sagitta is $\bar{x}$. The 'Black focal length' of the system is the distance NB = ℓ .

Thus we may write:- $u = h/R_o$; $u_T = h_T/R$

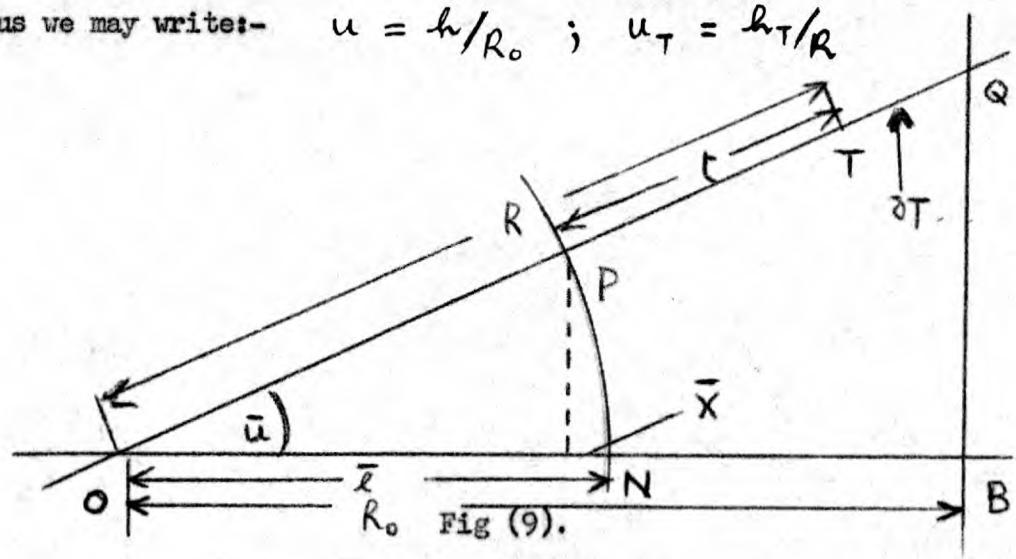


Fig (9).

We have that:-

$$(nuu_T\delta T) = n.h.h_T\left(\delta T/_{RR_o}\right)$$

and as $(\delta T) = (R_o/\cos\bar{u} - R)$, we may write that:-

$$\delta(nuu_T\delta T) = n.h.h_T\,\delta\left(\frac{1}{R\cos\bar{u}}\right) - nh.h_T\,\delta\left(\frac{1}{R_o}\right).$$

The second term in this expression is a longtitudinal focal shift term described by equation (1) Part 1, and equation (15) Part II. If the longtitudinal focal shift associated with the change in parameter is $\delta W_L / \delta p$ then we may write:-

$$\frac{1}{2}\frac{\delta(nuu_T\delta T)}{\delta p} = \frac{1}{2}\frac{n.h.h_T}{\delta p}\,\delta\left(\frac{1}{R\cos\bar{u}}\right) - \frac{h_T}{h}\frac{\delta W_L}{\delta p} \qquad \text{----------(16)}$$

We can expand the first term in this expression as follows:-

$$\frac{1}{2}\frac{n.h.h_T}{\delta p}\,\delta\left(\frac{1}{R.\cos\bar{u}}\right) = \frac{1}{2}n.h.h_T\left[\frac{1}{\cos(\bar{u}+\delta\bar{u})}\,\delta\left(\frac{1}{R}\right)\right]$$

$$-\frac{1}{R}\left[\frac{1}{\cos\bar{u}} - \frac{1}{\cos(\bar{u}+\delta\bar{u})}\right]$$

and expanding the last bracket we have that:-

$$\frac{1}{\cos\bar{u}} - \frac{1}{\cos\bar{u}+\delta\bar{u}} = \frac{1}{\cos\bar{u}}\left(1 - \frac{1}{1-\tan\bar{u}.\delta\bar{u}}\right)$$

$$= -\frac{1}{\cos\bar{u}}\left(\tan\bar{u}.\delta\bar{u}\right)$$

$$= -\frac{\sin\bar{u}}{\cos^2\bar{u}}.\delta\bar{u}$$

to the

required degree of accuracy.

Thus equation (16) can be written:-

$$\tfrac{1}{2}\delta(nuu_T\delta T) = \tfrac{1}{2}nh.h_T\left[\frac{1}{\cos(\bar{u}+\delta\bar{u})}\delta\left(\frac{1}{R}\right) + \frac{\sin\bar{u}\,\delta\bar{u}}{R.\cos^2\bar{u}}\right]$$

$$-\frac{h_T}{h}\delta W_L \qquad \text{----------(17)}$$

The change in the constructional parameter will, in general give rise to a movement in the position of the tangential focus along the reference ray. This can be described as a longtitudinal focal shift, and if we let δW_{LT} be the change in the wave front aberration due to the shift we may write

$$\delta W_{LT} = \tfrac{1}{2}n(h_T)^2\delta\left(\frac{1}{R}\right)$$

a statement which follows from the results obtained in the section on longtitudinal focal shifts.

Thus equation (14) becomes;-

$$\tfrac{1}{2}\delta(nuu_T\delta T) = \left[\frac{h}{\cos(\bar{u}+\delta\bar{u})}\frac{\delta\bar{w}_{LT}}{h_T} + \tfrac{1}{2}\frac{n.h.h_T\sin\bar{u}.\delta\bar{u}}{R.\cos^2\bar{u}}\right.$$

$$\left.-\frac{h_T}{h}\delta W_L\right] \qquad \text{----------(18)}$$

In the section on transverse focal shifts we showed that $\delta\bar{w}_T = nu_T\delta\eta_T$ (the equation immediately preceeding equation 11, Part II) Hence we have that:-

$$\delta\bar{u} = \delta\bar{w}_T/nu_TR. \qquad \text{as} \qquad \delta\bar{u} = \delta\eta_T/R.$$

Thus equation (18) can be written:-

$$\delta \Omega_T = \frac{h}{h_T}\left(\frac{\delta \bar{W}_{LT}}{\cos \bar{u}}\right) + \frac{1}{2}\frac{h.\sin u\,(u_T)\,\delta \bar{W}_{LT}}{h_T.\cos^2 u} \qquad \frac{h_T \delta W_L}{h} \qquad \text{------(19)}$$

to the required degree of accuracy.

Equation (19) shows the change in the tangential image curvature in the final image space due to a small change in the constructional parameter.

An exactly similar analysis can be made for the sagittal section. The only difference is that the longtitudinal shift in the sagittal focus must be inserted in place of $\delta \bar{W}_{LT}$. The transverse focal shift δW_T and the paraxial focal shift δW_L remain as they are, for they concern the transverse movement of the reference ray, and the longtitudinal shift in the paraxial image plane which apply equally to both sagittal and tangential image curvatures.

Thus the increments in $(nuu_S\,\delta S)$ and $(nuu_T\,\delta T)$ in final image space due to a small change in the constructional parameter may be written:-

$$\frac{\partial \Omega_S}{\partial \rho} = \frac{h\ \delta \bar{W}_{LS}}{h_S \cos \bar{u}\ \delta \rho} + \frac{1}{2}\frac{h.\sin \bar{u}\,(u_T)}{h_T \cos^2 \bar{u}}\frac{\partial \bar{W}_T}{\partial \rho} - \frac{h_S}{h}\frac{\partial W_L}{\partial \rho}$$

and

$$\frac{\partial \Omega_T}{\partial \rho} = \frac{h}{h_T \cos \bar{u}}\frac{\partial \bar{W}_{LT}}{\partial \rho} + \frac{1}{2}\frac{h.\sin \bar{u}\,(u_T)}{h_T \cos^2 \bar{u}}\frac{\partial \bar{W}_T}{\partial \rho} - \frac{h_T}{h}\frac{\partial W_L}{\partial \rho}$$

It remains now to evaluate $\partial W_0 / \partial \rho$ with reference to the sagittal section in order that we may calculate the changes in $(nuu_S\,\delta S)$ for small changes in the surface radius.

The quantity (δW_S) which corresponds to (δW_T) is zero, for the

effect of a transverse focal shift is zero in the sagittal section.

The evaluation of $\delta W_o / \delta c$ for the sagittal section.

For a small change in the constructional parameter we may write equation (7) as follows:-

$$\frac{\delta W_o}{\delta \rho} = (\bar{f} + \delta f)\,\Delta\left[n.\cos\bar{I} + \delta(n\cos I)\right] - \bar{f}\,\Delta(n.\cos\bar{I}).$$

and as δf depends on the square of the aperture we can ignore

$\delta f\,\Delta\left[\delta(n\cos I)\right]$ so that we have:-

$$\frac{\delta W_o}{\delta \rho} = \delta f\,\Delta(n\cos\bar{I}) + \bar{f}\,\Delta\left[\delta(n\cos I)\right] \qquad \text{----------(20)}$$

We must now evaluate δf and $\delta(n.\cos\bar{I})$. These quantities will be determined separately.

Fig. (10) shows a reference ray refracted at P and a ray in the sagittal section which strickes the surface at P_s.

The coordinates of the point P are (0, y, and z.), and those of P_s are (h_s), ($y + \delta y$), and ($z + \delta z$). If the radius of the surface is altered by a small amount δc, then we may write:-

$$(\bar{f} + \delta f) = \tfrac{1}{2}\left[(h_s)^2 + (y+\delta y)^2 + (z+\delta z)^2\right]$$

and

$$\bar{f} = \tfrac{1}{2}(y^2 + z^2).$$

Now terms involving $(\delta y)^2$ and $(\delta z)^2$ can be ignored for δy and δz already depend on the square of the aperture and thus we may write:-

$$\delta f = \left[\tfrac{1}{2}(h_s)^2 + y.\delta y + z.\delta z\right] \qquad \text{----------(21)}$$

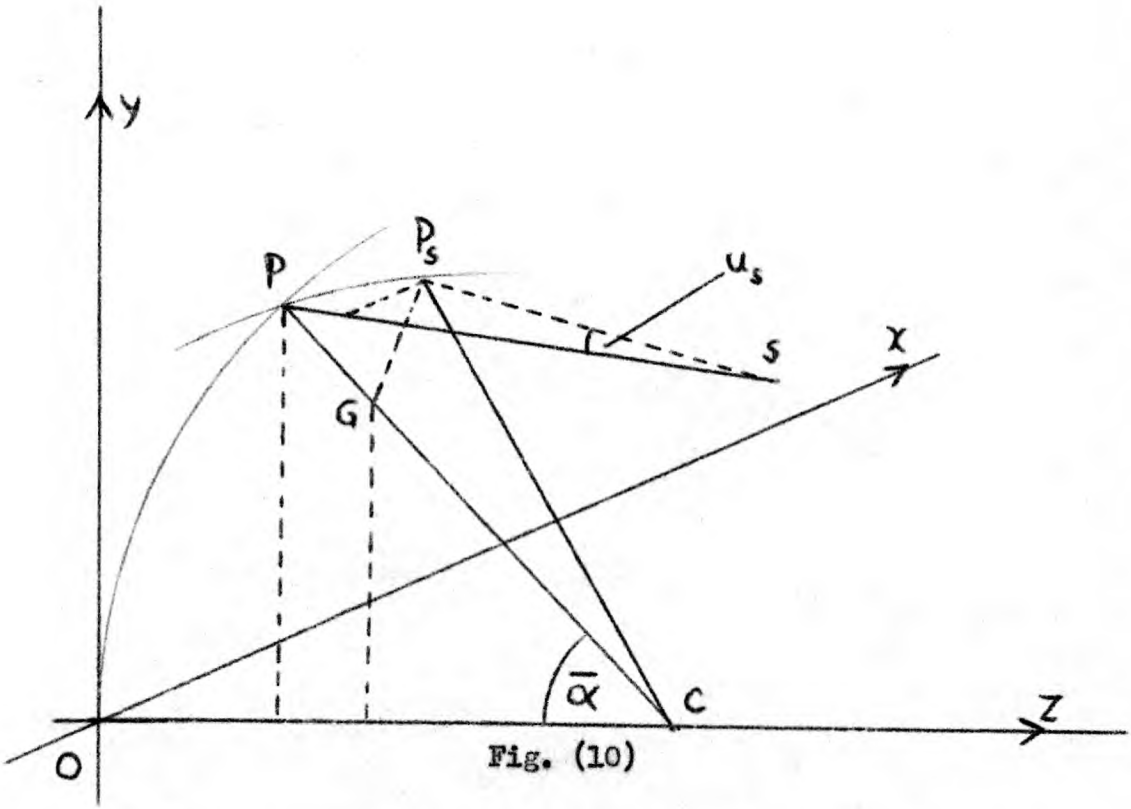


Fig. (10)

Now for the paraxial sagittal ray P $_s$ S we have that:-

$$PG = \tfrac{1}{2}(h_s)^2 . c .$$ and projecting the length PG on to the z and y axes we may write:-

$$\delta z = \frac{1}{2}\frac{(h_s)^2 \cos\bar{\alpha}}{r} \quad ; \quad \delta y = -\frac{1}{2}\frac{(h_s)^2 \sin\bar{\alpha}}{r}$$

and substituting these values for δz and δy in equation (21) we have that:-

$$\delta f = \frac{1}{2}\frac{(h_s)^2}{\bar{r}}\left(r\cos\bar{\alpha} - r\cos^2\bar{\alpha} - r\sin^2\bar{\alpha} + r\right)$$

or

$$\delta f = \tfrac{1}{2}(h_s)^2 \cos\bar{\alpha} \qquad \text{----------(22)}$$

The expression for δ (n Cos I) is obtained as follows:-

In fig. (10) let

$\bar{\lambda}+\delta\lambda$, $\bar{\mu}+\delta\mu$, $\bar{\nu}+\delta\nu$ be the direction cosines of the sagittal ray.

$\bar{\lambda}$, $\bar{\mu}$, $\bar{\nu}$ be the direction cosines of the principal ray (reference ray),

$\bar{l}+\delta l$, $\bar{m}+\delta m$, $\bar{n}+\delta n$ be the direction cosines of the normal at P ,

$\bar{l}$, $\bar{m}$, $\bar{n}$ be the direction cosines of the normal at P.

Thus cos $\bar{I}$ can be written:- $\sum \bar{l}\bar{\lambda}.$

and cos I

$$= \sum (\bar{l}+\delta l)(\bar{\lambda}+\delta\lambda)$$

$$= \cos\bar{I} + \sum \delta l.\bar{\lambda}. + \sum \delta\lambda\,\bar{l} + \sum \delta\lambda.\delta l.$$

and to the required degree of accuracy we have that:-

$$\delta(\cos I) = \sum \delta l.\bar{\lambda}. + \sum \delta\lambda\,\bar{l}.$$

The first term in this expression can be expanded into ($\delta l.\bar{\lambda} + \delta m.\bar{\mu}.$), $\bar{\nu}$ being zero for the reference ray in the y, z, plane.

Now δl the difference between the cosines of the angles $\hat{PCO}$, and $P_S\hat{C}O$, may be written:- $\left[-\delta y.\frac{\bar{\mu}}{\bar{\nu}}\right]$

and similarly δ m being the difference of the sines may be written:-

$$\left[-\delta_2 . \frac{\bar{\lambda}}{\nu} \right]$$

Thus we have that:-

$$\delta(\cos I) = \sum \delta\lambda\, \bar{\ell} - \delta y \frac{\bar{\mu}}{\nu} - \delta z \frac{\bar{\lambda}}{\nu} \qquad \text{----------(23)}$$

If we now project the length P $_s$ S on to the reference ray and then on to the z axis, we have that:-

$$(\bar{\lambda} + \delta\lambda)\beta = \beta \cos u_s \bar{\lambda}$$, where β is the length P_s S. Similarly for the y axis:-

$$(\bar{\mu} + \delta\mu)\beta = \beta \cos u_s \bar{\mu}$$

and for the x axis:-

$$(\bar{\nu} + \delta\nu)\beta = -\beta . \sin u_s$$

Thus we have that:- $\delta\lambda = (\cos u_s - 1)\bar{\lambda}$

$\delta\mu = (\cos u_s - 1)\bar{\mu}$ and $\delta\nu = -\sin u_s$, $\bar{\nu}$ being zero because the reference ray is in the y, z, plane.

Equation (23) then becomes:-

$$\delta(\cos I) = (\cos u_s - 1)(\bar{\lambda}\bar{\ell} + \bar{\mu}\bar{m}) - \frac{\delta y \bar{\mu}}{\nu} - \frac{\delta z \bar{\lambda}}{\nu}$$

The angle of incidence of the reference ray is given by:- $\sum \bar{\lambda}\bar{\ell}$, and using the values given on page (86) for δz and δy equation (23)

becomes:-

$$\delta(\cos I) = \cos \bar{I}(\cos u_s - 1) + \tfrac{1}{2}\frac{(h_s)^2 \sin \bar{\alpha}\, \bar{\mu}}{r^2}$$

$$-\tfrac{1}{2}\frac{(h_s)^2 \cos \bar{\alpha}\, \bar{\lambda}}{r^2}$$

Now Cos $\bar{I}$, can be written:-

$$\text{Cos } \bar{\alpha} \text{ Cos } \bar{U} + \text{Sin } \bar{\alpha} \text{ Sin } \bar{U}$$

and as $\bar{\lambda}$ = Cos $\bar{U}$ and $\bar{\mu}$ = - Sin $\bar{U}$, we have that:-

$$\delta(\cos I) = -\tfrac{1}{2}\frac{(h_s)^2}{r^2}(\cos \bar{I}) - \tfrac{1}{2}(u_s)^2 \cos \bar{I}$$

$$\delta(n.\cos I) = -\tfrac{1}{2}\frac{(h_s)^2}{r^2}(n.\cos \bar{I}) - \tfrac{1}{2} n u_s^2 \cos \bar{I}. \qquad \text{----------(24)}$$

Combining equations (20), (22), and (24) we have that:-

$$\frac{\partial W_o}{\partial c} = \tfrac{1}{2}(h_s)^2 \cos \bar{\alpha}\, \Delta(n.\cos \bar{I}) - \tfrac{1}{2}\frac{\bar{f}(h_s)^2}{r}\Delta(n\cos \bar{I})$$

$$-\tfrac{1}{2}\bar{f}\,\Delta(n u_s^2 \cos \bar{I}).$$

$$= \Delta(n\cos \bar{I})\frac{(h_s)^2}{2}\left[\frac{\cos \bar{\alpha}(1+\cos \bar{\alpha}) - (1-\cos^2 \bar{\alpha})}{(1+\cos \bar{\alpha})}\right]$$

$$-\tfrac{1}{2}\bar{f}\,\Delta(n u_s^2 \cos \bar{I}).$$

or $$\frac{\partial \bar{W}_{LS}}{\partial c} = \tfrac{1}{2}(h_s)^2 (2\cos\bar{z} - 1)\,\Delta(n.\cos\bar{I})$$

$$-\tfrac{1}{2}\bar{f}\,\Delta(nu_s^2\cos\bar{I}).$$

----------(25)

We have evaluated aberration derivatives in the sagittal and tangential section for changes in curvature, and it remains for us to evaluate derivatives for the remaining parameters.

<u>The evaluation of δW_o for small changes in axial displacement and refractive index in the meridian section</u>.

Besides surface curvature there are two other parameters which may be adjusted in the final stages of a lens design. Distances between surfaces may be increased or reduced, i.e. the thickness of a component lens may be altered or the air separation between two components adjusted.

These changes of course affect the aberrations in the image plane, and it is desirable to determine what these changes are. This section of the work and the following one will deal with these changes in the tangential and sagittal sections.

In fig. (11) a refracting surface OP is shown displaced to AQ by an amount δz. The tangential focus is at T, PT being the reference ray and RT a close ray in the meridian section.

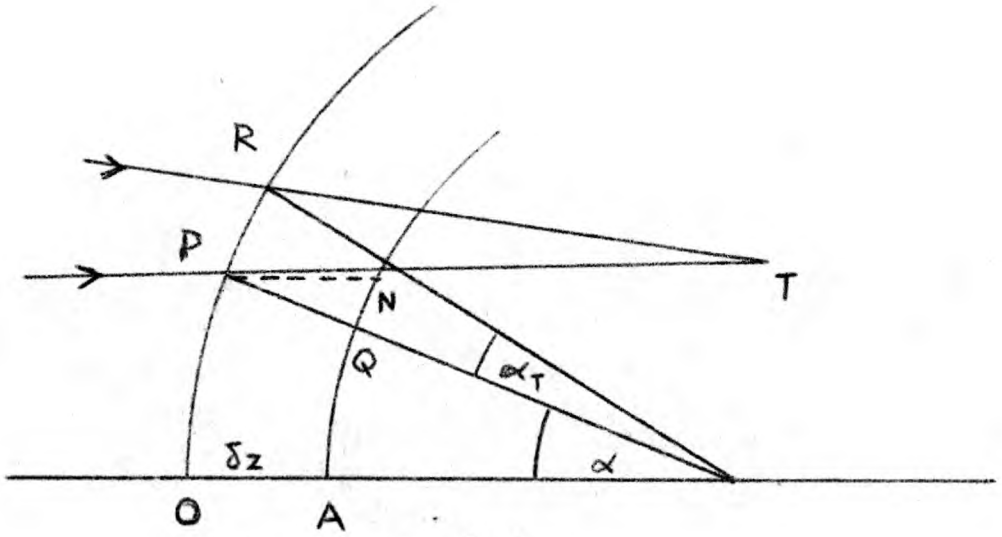


Fig. (11)

We have that PQ by definition is $\bar{f}$. δz, and in triangle PNQ $\cos \bar{\alpha} = PQ/\delta z$, or that:-

$$\bar{f} = \cos \bar{\alpha}$$

For the close ray in the tangential section

$$f = \cos(\bar{\alpha} + \delta\alpha) = \cos(\bar{\alpha} + \alpha_T)$$

and thus we may write equation (7)

$$\frac{\partial W_o}{\partial z} = \cos(\bar{\alpha} + \alpha_T)\,\Delta(n\cos I) - \cos\bar{\alpha}\,\Delta(u.\cos\bar{I}).$$

which on expanding into terms depending on the first and second powers of (u_T), (i_T) and (α_T) developes into:-

$$\frac{\partial W_{LT}}{\partial z} = A_T \cos\bar{\alpha}\,\Delta(u_T) - \tfrac{1}{2}\cos\bar{\alpha}\,(h_T.c)^2\Delta(u.\cos\bar{I})$$

$$-c.\sin\bar{\alpha}.\bar{B}.h_T.\Delta(u_T). \qquad \text{---------(26)}$$

$$\frac{\partial W_T}{\partial z} = \bar{B}\cos\bar{\alpha}\,\Delta(u_T) - c.\sin\bar{\alpha}\,h_T\,\Delta(n.\cos\bar{I}) - \Delta(n.\sin\bar{u}.u_T). \qquad \text{---------(27)}$$

Small change in refractive index.

Using the theorem of neigbouring rays we have that optical path lengths along such rays measured between planes perpendicular to one of them are equal. This leads to a well-known formula for chromatic aberration which can be expressed by writing $\delta W/\delta n = \bar{D} - D$ where D is the distance measured along the ray from one surface to the next, the bar denoting the value for the reference ray.

In Fig (12) PT is a reference ray and RT a close ray in the meridian section. PR is drawn perpendicular to the close ray and QS perpendicular to the reference ray.

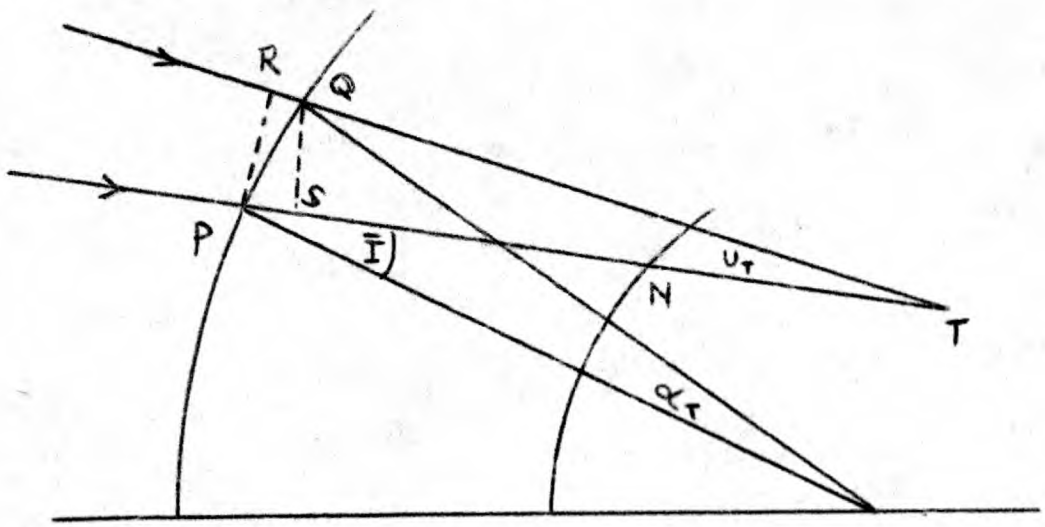


Fig. (12)

It will be seen from the figure that D, the distance along the close ray between two surfaces is given by

$$D = \bar{D}/\cos(u_T) - a_{-1} + a$$

where the subscript (-1) denotes the previous surface

where (u_T) is the angle $\hat{RTP}$ and (a) = RQ, and $\bar{D}$ = PN.

Expanding cos $(\bar{u}_T)$ to the second power we have that:-

$$D = \bar{D}\left[1 + \tfrac{1}{2}(u_T)^2\right] - a_{-1} + a.$$

or

$$\frac{\partial W_o}{\partial n} = (\bar{D} - D) = a_{-1} - a - \tfrac{1}{2}\bar{D}(u_T)^2 \qquad \text{----------(28)}$$

It is thus necessary to evaluate (a) in terms of reference and paraxial tangential ray quantities.

From fig. (12) we have that:- $\hat{PQS} = (\bar{I} + \tfrac{1}{2}\alpha_T)$

and PQ = $2r.\sin\tfrac{1}{2}\alpha_T$ so that:-

$$a.\cos u_T = 2r\sin\tfrac{1}{2}\alpha_T.\sin(\bar{I} + \tfrac{1}{2}\alpha_T).$$

or

$$a = 2r\sin\tfrac{1}{2}\alpha_T(\sin\bar{I} + \cos\bar{I}.\sin\tfrac{1}{2}\alpha_T)$$

ignoring powers of α_T higher than the second, and thus

$$a = r.\sin\bar{I}\ \frac{2.\sin\tfrac{1}{2}\alpha_T.\cos\tfrac{1}{2}\alpha_T}{\cos\tfrac{1}{2}\alpha_T} + 2r.\sin^2\tfrac{1}{2}\alpha_T.\cos\bar{I}.$$

so to the required degree of accuracy we have that:-

$$a = r.\sin\bar{I}.\alpha_T. + 2r.\sin^2\tfrac{1}{2}\alpha_T.\cos\bar{I}.$$

or

$$a = r.\sin\bar{I}.\alpha_T + \tfrac{1}{2}r\alpha_T^2\cos\bar{I}$$

Thus we may write equation (28)

$$\frac{\partial W_o}{\partial n} = \left[h_T \sin \bar{I} + \tfrac{1}{2} (h_T)^2 . c . \cos \bar{I} \right]_{-1}$$

$$- (h_T) \sin \bar{I} + \tfrac{1}{2} (h_T)^2 . c . \cos \bar{I} - \tfrac{1}{2} \bar{D} (u_T)^2$$

which reduces to:-

$$\frac{\partial W_o}{\partial n} = (h_T . \sin \bar{I}')_{-1} - (h_T . \sin \bar{I}) + \frac{1}{2n} \left[(A_T z)_{-1} - (A_T z) \right] \qquad (29)$$

where n is the refractive index of the medium between the surfaces.

This completes the analysis for the tangential section.

The evaluation of δW_o for small changes in axial displacement and refractive index in the sagittal section.

In order to determine completely the changes in astigmatism due to small changes in the above parameters we must determine the values of $\partial W_o / \partial z$, and $\partial W_o / \partial n$ for the sagittal section.

As in the tangential axial displacement we have that:-

$$f = \cos \bar{\alpha} \qquad \delta f = \delta (\cos \alpha) = - c . \delta z$$

We have however shown that δz is given by:-

$$\tfrac{1}{2} (h_s)^2 . c . \cos \bar{\alpha}$$

and thus $f = - \tfrac{1}{2} (h_s)^2 . c^2 . \cos \bar{\alpha}$

This value can now be inserted in the general equation (17) for δW_o, and we have that:-

$$\frac{\partial W_o}{\partial z} = -\tfrac{1}{2}(h_s)^2.c^2.\cos\bar{\alpha}\;\Delta(n.\cos\bar{I}) + \cos\bar{\alpha}\;\Delta\left[-\tfrac{1}{2}(h_s)^2.c^2.n.\cos\bar{I} - \tfrac{1}{2}nu_s^2\cos\bar{I}\right]$$

or finally

$$\frac{\partial W_o}{\partial z} = -(h_s)^2.c^2.\cos\bar{\alpha}\;\Delta(n.\cos\bar{I}) - \tfrac{1}{2}\cos\bar{\alpha}\;\Delta(nu_s^2\cos\bar{I}). \qquad \text{---------(30)}$$

For a glass change the formulae given for the tangential section hold and we have again that:-

$$\frac{\delta W_o}{\delta n} = \bar{D} - D \quad ; \quad \frac{\delta W_o}{\delta n} = a_{-1} - a - \tfrac{1}{2}\bar{D}(u_s)^2$$

but now (u_T) is replaced by (u_s) and the value of (a) at each surface is of course different for the sagittal section. The value of (a) is in fact the (δz) shown in fig. (10), given by:-

$$\tfrac{1}{2}(h_s)^2.c.\cos\bar{\alpha}$$

and we may write:-

$$\frac{\partial W_o}{\partial n} = \left[\tfrac{1}{2}(h_s)^2.c.\cos\bar{\alpha}\right]_{-1} - \left[\tfrac{1}{2}(h_s)^2.c.\cos\bar{\alpha}\right] - \tfrac{1}{2}\bar{D}(u_s)^2.$$

which developes into:-

$$\frac{\partial W_o}{\partial n} = \tfrac{1}{2}(h_s)^2.c.\left[\cos\bar{\alpha} - \cos\bar{I}\right]_{-1} - \tfrac{1}{2}(h_s)^2.c.\left[\cos\bar{\alpha} - \cos\bar{I}\right] + \frac{1}{2n}\left[(A_s h_s)_{-1} - (A_s h_s)\right] \qquad \text{---------(31)}$$

It will be noticed that there are no terms in this expression depending on the first power of the sagittal aperture, all the terms being square terms, or longtitudinal shift terms, -- There being no transverse refraction in the sagittal section.

This completes all the analysis necessary for the calculation of changes in sagittal and tangential image curvature for small changes in curvature, axial displacement and refractive index.

Note on the change in distortion due to a change in the constructional parameter,

The distortion is measured by the difference in two image heights, and involves therefore the transverse position of the reference ray. Hence any variation in distortion arising from a change in a constructional parameter must take account both of the longitudinal displacement of the Gaussian image plane, and also of any transverse displacement of the reference ray, since this latter is itself $\underline{0}(\delta p)$, where δp denotes a change in the constructional parameter.

That the difficulty does not arise in considering the effect of displacement of the Gaussian image plane on other aberrations may be seen from the following consideration of the case of the axial image.

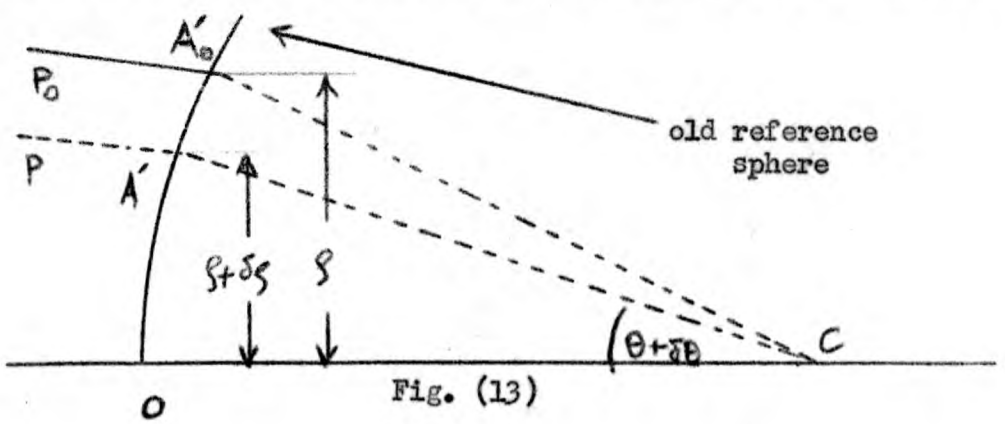


Fig. (13)

In fig. (13) PA' is the displaced new ray and P_0 A_0' the old ray. OA_0' is the old reference sphere.

The changed aberration $W_0 + \delta W_0$ refers to the point A' not to A_0', where $\delta\varsigma = \underline{0}(\delta p)$. If we find δW_L, then it refers to a new paraxial image height $h + \delta h$, where $\delta h = \underline{0}(\delta p)$.

Now

$$\delta W_L = \tfrac{1}{2} n (h + \delta h)^2 \delta \left(\frac{1}{R_0}\right)$$

and

$$\delta W_L = \frac{n(\varsigma + \delta\varsigma)^2}{1 + \cos(\theta + \delta\theta)} \delta \left(\frac{1}{R_0}\right)$$

where

$$\delta\theta = \underline{0}(\delta p)$$

or

$$\delta W_L = \frac{(\varsigma + \delta\varsigma)^2}{1 + \cos(\theta + \delta\theta)} \quad \frac{2.}{(h + \delta h)^2} \delta W_L$$

$$\delta W_L = -\frac{2}{1 + \cos(\theta + \delta\theta)} \left[\frac{(\varsigma + \delta\varsigma)^2}{(h + \delta h)^2}\right] \delta W_L$$

We require δW_L to an accuracy $\underline{0}(\delta p)$. But $\delta W_L = \underline{0}(\delta p)$. Hence we make an error of at least $\underline{0}(\delta p)$ by writing

$$\delta W_L = -\frac{2}{1 + \cos\theta} \left(\frac{\varsigma}{h}\right)^2 \delta W_L.$$

An exactly similar consideration applies, mutatis mutandis, to oblique pencils. In the case of distortion however this does not apply, for the following reason.

In the diagram PA and P'A' are the old and new reference rays, QB and Q'B' the old and new paraxial rays.

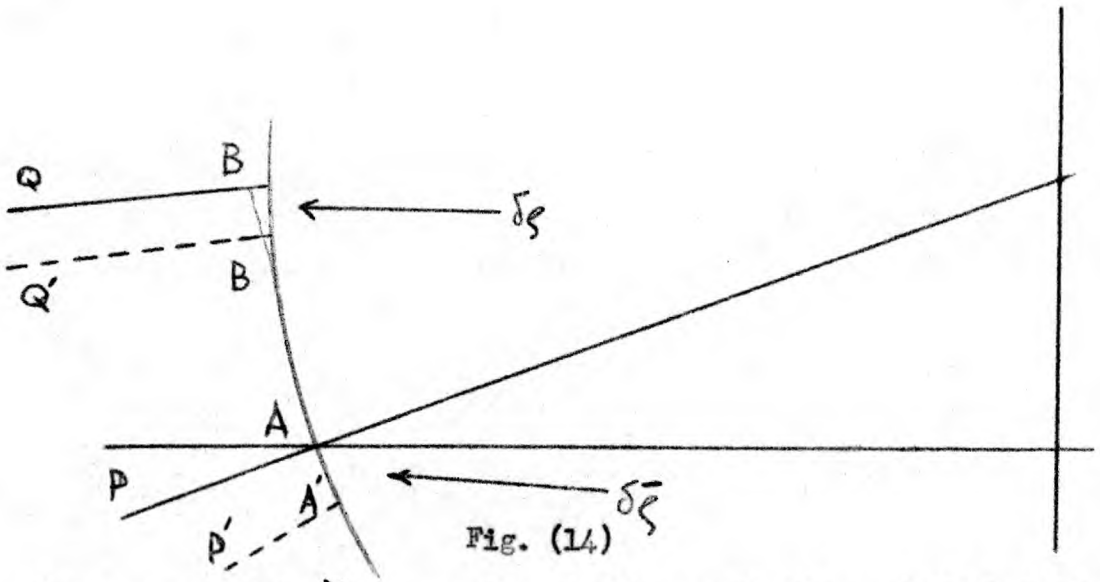


Fig. (14)

Here we see that δW_o, refers to the new incidence height which is $(\delta\varsigma - \delta\bar{\varsigma})$ different from ς.

The old values of ς can be used to get an error of only $\underline{0}$ $(\delta\rho^2)$, but the transverse focal shift gives the tilt of the reference ray, and not its transverse displacement (which is $0(\delta\rho)$). This is necessary for the calculation of the change in the distortion. For these reasons recourse must be made to the formulae of the type developed by Cruickshank and M'Auley (13)

<u>Calculation of sagittal and tangential image curvature by differential formulae</u>.

Curvature changes. Calculations have been carried out on the 23° ray through the system used in the first part of this work. They were done in the following manner:-

The quantities $\delta\bar{W}_T/\partial c$, $\partial W_{LT}/\partial c$ and $\partial W_L/\partial c$ were calculated for each surface of the system using the equations (14), (18) and (15) which led to the determination of $\frac{1}{2}\delta(nuu_T\delta T)$ by means of equation (19).

In this last equation, all the quantities except the ones mentioned above are 'last surface' values and need only be calculated once for any reference ray.

The values of $\frac{1}{2}\delta(nuu_T\,\delta T)$ are shown for each surface in the table given below.

Surface	1	2	3
	0.0^3931	-0.0^24050	0.0^21210
Surface	4	5	6
	-0.0^3108	0.0^21742	-0.0^3426

From these values the variation of the 't' focus with curvature is calculated, again using the last surface value for n, u, and u . These values are as follows:-

Surface	1	2	3
	0.099768	-0.434030	0.129672
Surface	4	5	6
	-0.011568	0.186686	-0.045649

The surface coefficients being highest at surfaces 2 and 5, these surfaces were chosen for experiment.

The 't' focus of the original system at 23° is 0.005019" short of the paraxial image plane along the reference ray. This requires a change in the curvature of surface 2 of 0.011563, in order that the resulting 't' focus will be formed in the paraxial image plane.

When surface 2 was altered to this extent and the calculations repeated to determine the position of the 't' focus it was found that the new focus was formed at a distance of 0.000437" from the paraxial image plane i.e. the formulae give an accuracy in the final result of

92 per. cent. in this case.

Calculations for the variation of the sagittal focus were carried out in a similar manner; the surface coefficients for each surface were determined using equation (25).

The results for the 23° ray are as follows:-

Surface	1	2	3
	0.027924	-0.180927	0.069125
	4	5	6
	-0.039409	0.076606	-0.030031

The curvature of surface 5 was adjusted so that the 's' focus should fall on the final paraxial image plane and it was found that the required value of δc was -0.026884.

The final position of the 's' focus was in fact at a distance of 0.000264 from the final paraxial image plane, a result which shows 95 per. cent. accuracy.

The effect of this change on the sagittal focus was such that the final calculated position agreed with the fore-cast position to the extent of 95 per. cent.

BIBLIOGRAPHY AND REFERENCES.

(1) Newton. "Lectiones opticae annis 1669, 1670, 1671, in scholis publicus Cantabrigiensium ex cathedra Lucasiana habitae".
(English edition 1728. London).

(2) Barrow. "Lectiones opticae et geometricae"
(Godbid, London, 1674)

(3) Smith. "A Compleat System of Opticks" in four books.
(Cambridge, 1738)

(4) Young. "On the mechanism of the eye"
(Phil.Trans. 102, 1801.)

(5) Airy. "On a peculiar defect in the eye and mode of correcting it"
(Camb.Phil.Trans. 2. 1927)

(6) Airy. "On the spherical aberration of the eyepieces of telescopes"
(Camb.Phil.Trans. 3. 1830)

(7) Coddington. "A treatise on the reflexion and refraction of light, being Part 1. of 'A system of Optics'"
(London 1829)

(8) Taylor. "A system of applied optics"
(London, 1906)

(9) Conrady. "Applied optics and optical design"
(Oxford University Press, 1929)

(10) von.Rohr. "The formation of images in optical instruments"
(H.M.Stationary Office, 1920)

BIBLIOGRAPHY AND REFERENCES (Continued)

(11) Hopkins. "Wave theory of aberrations" (Clarendon Press, 1950)

(12) Hopkins. Proc. Phys.Soc. 1946. -p695.

(13) Cruikshank and M'Aulay Proc.Phys.Soc. 1945 et.seq.

ACKNOWLEDGEMENTS.

I am indebted to Dr. H. H. Hopkins, the supervisor of this work, whose guidance and criticism have been invaluable.

I wish to thank the Director of the British Scientific Instrument Research Association, and the Council for financial assistance, and for their allowing my leave of absence to undertake this work.

My thanks are due also to Miss J. M. Drewitt of the Technical Optics section for checking some of the numerical results, and to Miss S. J. Wagstaff of the Computing Machine Laboratory, Manchester University, for preparing the manuscript.