ON

STARLIKE UNIVALENT

FUNCTIONS

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Abstract

Let G be a domain in the w-plane starlike with respect to the origin. Let $f(z) = \sum_{1}^{\infty} a_n z^n$, $a_1 \neq 0$, be a regular univalent function in the unit disc whose image domain is G . Then $f(z)$ is called starlike univalent in $|z| < 1$, and satisfies $\operatorname{Re} \frac{zf'(z)}{f(z)} > 0$ in $|z| < 1$. In this thesis we develop the theory of starlike univalent functions from its beginnings by means of the principle of subordination and with the help of a theorem on bounded functions due to G. Julia. By these means we by-pass and indeed essentially reprove the well known representation theorem

$$\frac{zf'(z)}{f(z)} = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 + e^{-it} z}{1 - e^{-it} z} \, d V(t) \qquad (1)$$

where $V(t)$ is an increasing function of t , $(-\infty < t < \infty)$, and $V(t) - t$ has period 2π .

We consider the starlike domain G , and the sectors in G of the form

$$S = \{ w / \ell_1 < \arg w < \ell_2 \}$$

In Chapter 1 we show that if $\pi\alpha$ is the angle of the largest such sector in G then

$$\alpha = \lim_{r\to 1} \frac{\log M(r)}{\log \frac{1}{1-r}} \qquad (2)$$

where $M(r) = \max_{|z|=r} |f(z)|$. In Chapter 3 we show that for the same number α ,

$$\alpha = \lim_{r\to 1} (1 - r) \frac{M'(r)}{M(r)} \qquad (3)$$

These two results are both due to Ch. Pommerenke who proves them by means of the representation theorem, and where in his proof $\pi\alpha$ is the size of the largest jump of $V(t)$. In Chapter 2 we prove the existence of the limit

$$V(t) = \lim_{r\to 1} \arg f(re^{it}) \qquad (4)$$

and show that the angles of the sectors in G are just the jumps of $V(t)$. We also prove the existence of similar limits to (2) and (3) for local maxima in a particular sector of G , the limits in this case being $\frac{1}{\pi}$. angle of the sector in question.

In Chapter 4 we relate the length $L(r)$ of the image of $|z| = r$ to the maximum modulus $M(r)$ and obtain sharp asymptotic estimates. We treat the area $A(r)$ enclosed by the image of $|z| = r$ in a similar fashion, and go on to

consider more general integral means in the same way. In Chapter 5 we consider the class of bounded convex functions $g(z) = z + \sum_{2}^{\infty} b_n z^n$ and show that if $|g(z)| < B$ in $|z| < 1$ then $b_n = o(n^{-1-1/2B})$, thus improving an earlier result due to Pommerenke.

I should like to express my gratitude and acknowledge my considerable indebtedness to Dr. J. Clunie for his assistance and advice throughout the preparation of this thesis. Numerous ideas scattered through the following pages are due to Dr. Clunie; in particular the idea of applying subordination to the study of starlike functions was suggested by him.

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On Starlike Univalent Functions

Chapter 1 The Growth of the Maximum Modulus

Section 1.1 Introduction

Suppose $w = f(z)$ is regular and univalent in $|z| < 1$ and maps the unit disc on to a domain G which is starlike with respect to the origin $w = 0$. Suppose also that $f(0) = 0$. Suppose G contains sectors S of the form

$$S = \{ w \mid \ell_1 < \arg w < \ell_2 \}$$

Then for some α satisfying $0 < \alpha \leq 2$, G contains a sector of angle $\pi\alpha$, and no sector of greater angle. Otherwise G contains no sectors at all, and we then write $\alpha = 0$.

The main theorem of this chapter is that

$$\alpha = \lim_{r \to 1} \log M(r,f)/\log\left(\frac{1}{1-r}\right) \qquad (1.1.1)$$

where $$M(r,f) = \max_{|z|=r} |f(z)| .$$

Christian Pommerenke [9] has proved essentially this result as follows. It is well known (e.g. [8] page 221 or [5] page 15) that a necessary and sufficient condition for

the function

$$f(z) = a_1 z + a_2 z^2 + \dots \qquad (a_1 \neq 0)$$

to be starlike univalent in $|z| < 1$ is that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \qquad (|z| < 1) \tag{1.1.2}$$

Hence by the Herglotz Representation Theorem (e.g. [7] page 68), the function $zf'(z)/f(z)$ can be represented by means of a Poisson-Stieltjes integral

$$\frac{zf'(z)}{f(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + e^{-it} z}{1 - e^{-it} z} \, d V(t) \tag{1.1.3}$$

where $V(t)$ is an increasing function $(-\infty < t < \infty)$ such that $V(t) - t$ has period 2π. Pommerenke proves that with a suitable normalisation,

$$V(t) = \lim_{r \to 1} \arg f(re^{it}) \tag{1.1.4}$$

He then goes on to show that (1.1.1) is true for the number $\pi\alpha$ which is the greatest jump of the function $V(t)$.

Our method of approach is somewhat different to Pommerenke's. We use subordination to prove (1.1.1), and in chapter 2 we will prove that the limit (1.1.4) exists,

and relate the jumps of $V(t)$ to the sectors of G. We will not use anywhere in the thesis the representation (1.1.3).

Section 1.2

The first stage in the proof of (1.1.1) is the following result.

$$M(r) = M(r,f) \geq \frac{c|f'(o)|r}{(1-r)^{\alpha}} \qquad (1.2.1)$$

where c is a positive constant. Pommerenke [9] proves this with

$$c = \frac{2^{\alpha}}{4} \geq \frac{1}{4} \qquad (1.2.2)$$

We will show that for $f(z)$ schlicht (not necessarily starlike) (1.2.1) is true with

$$c = K(\alpha) \geq \frac{9 - \sqrt{17}}{32} > \frac{1}{7} \qquad (1.2.3)$$

where $K(\alpha)$ will be an explicit function of α, $0 \leq \alpha \leq 2$. We will regain Pommerenke's result (1.2.2) in chapter 5, for the starlike case. We do not think (1.2.3) is the best possible value of c for general univalent functions, and conjecture that in general (1.2.1) is true with $c = \frac{1}{4}$. We make no attempt to substantiate this conjecture.

Consider then $f(z)$ univalent in $|z| < 1$ with $f(0) = 0$, and suppose the image domain G of f contains a sector S of angle $\pi\alpha > 0$ where $S = \{ w / |\arg w| < \frac{\pi\alpha}{2} \}$. Let G_0 be the domain

$$G_0 = S \cup D$$

where D is the disc radius $\frac{1}{4}$ centre $w = 0$. Then by the Koebe $\frac{1}{4}$-theorem G_0 is contained in the image domain of the function $f(z)/|f'(0)|$. By the Riemann mapping theorem $\exists$ a unique regular univalent function $F(z)$ which maps $|z| < 1$ on to G_0 with $F(0) = 0$, $F'(0) > 0$. By the subordination principle (e.g. [8] page 227) we then have

$$|f'(0)| M(r,F) \leq M(r,f), \quad (0 \leq r < 1) \qquad (1.2.4)$$

The proof of (1.2.1) thus reduces to explicitly calculating the function $F(z)$. The cases $\alpha = 0$, $\alpha = 2$ are anyway trivial, and we assume until further notice that $0 < \alpha < 2$.

Section 1.3

The essential steps in the calculation of $F(z)$ are given in [12] pages 221-3. We reproduce here the complete argument.

Let us define three regions in the t, s and ζ planes as follows:

$$E_t = \{ t/|t| > 1 \} \cup \{ t/ - \pi\alpha < \arg t < 0 \} \cup \{ t = \infty \}$$

$$E_s = \{ s/(- \tfrac{\pi\alpha}{2} \leq \operatorname{Im} s \leq 2\pi - \tfrac{\pi\alpha}{2}) \text{ and } (\operatorname{Re} s > 0) \}$$

$$\cup \{ s/(- \tfrac{\pi\alpha}{2} \leq \operatorname{Im} s \leq 2\pi - \tfrac{\pi\alpha}{2}) \text{ and } (\operatorname{Re} s \leq 0) \text{ and } s \text{ does not satisfy } (0 \leq \operatorname{Im} s \leq 2\pi - \pi\alpha) \}$$

$$E_\zeta = \{ \zeta / \operatorname{Im} \zeta > 0 \}$$

Since E_ζ and E_t are simply connected domains by the Riemann mapping theorem $\exists$ a unique mapping function

$$t = T(\zeta)$$

which maps E_ζ 1:1 conformally on to E_t, and which maps

$$\begin{aligned} \zeta = 0 \quad &\text{on to} \quad t = 1 \\ \zeta = 1 \quad &\text{on to} \quad t = 0 \\ \zeta = \infty \quad &\text{on to} \quad t = e^{-i\pi\alpha} \\ \zeta = e_\infty \quad &\text{on to} \quad t = \infty \end{aligned} \tag{1.3.1}$$

e_∞ being defined by this last requirement.

Let

$$s = S(\zeta) = \log T(\zeta)$$

where we take that branch of the logarithm such that the argument of $T(\zeta)$ runs from $-\frac{\pi\alpha}{2}$ to $2\pi - \frac{\pi\alpha}{2}$. Then

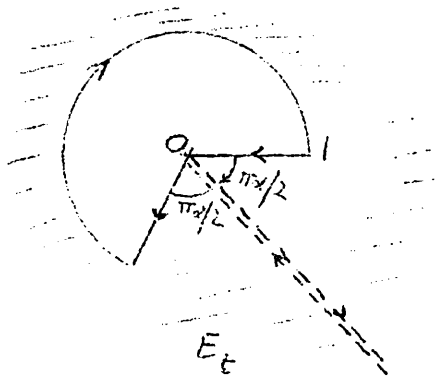


Figure 1

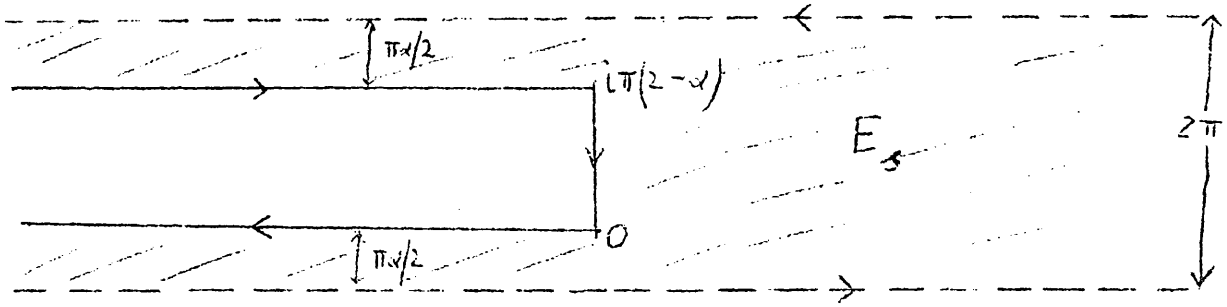


Figure 2

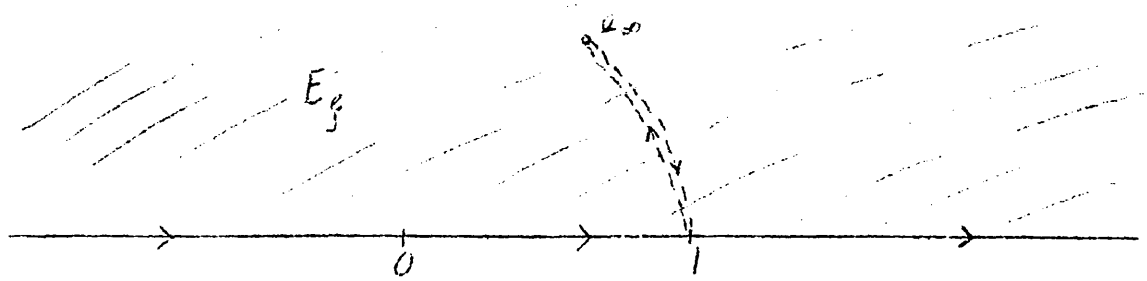


Figure 3

$S(\zeta)$ maps E_ζ on to the region E_s. The line $\arg t = -\frac{\pi\alpha}{2}$ is mapped twice into E_s, on to the lines $\operatorname{Im} s = -\frac{\pi\alpha}{2}$ and $\operatorname{Im} s = 2\pi - \frac{\pi\alpha}{2}$. The line $\arg t = -\frac{\pi\alpha}{2}$ is the image of a section of a curve in E_ζ running from 1 to e_∞. Thus this arc is mapped twice into E_s by $S(\zeta)$. Every other point in E_ζ is mapped once into E_s.

Thus $S(\zeta)$ is regular in E_ζ except on this arc from 1 to e_∞. This arc is actually an arc of the semi-circle

$$\operatorname{Im} \zeta > 0 , \quad |\zeta| = 1 .$$

To see this consider the two regions

$$E_1 = \{ \zeta / \operatorname{Im} \zeta > 0 \text{ and } |\zeta| < 1 \}$$

$$E_2 = \left\{ E_t \cap \left\{ t / -\frac{\pi\alpha}{2} < \arg t < \pi - \frac{\pi\alpha}{2} \right\} \right\} - \{ t = \infty \}$$

E_1 is the upper semi-disc in the ζ-plane. E_2 is that part of E_t to the right of the dotted line extended across the t-plane in both directions. This line divides E_t into two symmetric regions, E_2 and its reflections through this line. Since E_1 and E_2 are simply connected, $\exists$ a unique mapping function $H(\zeta)$ which maps E_1 on to E_2 and sends $\zeta = 0, 1, e^{i\pi(1-\alpha/2)}$ on to $t = 1, 0, -1$ respectively. The dotted line and its extension in the t-plane is then exactly the image under $H(\zeta)$ of the

semi-circle $|\zeta| = 1$, $\operatorname{Im}\zeta > 0$. Hence by the reflection principle we can continue $H(\zeta)$ across this semi-circle into the remainder of E_ζ. The image domain is then extended to the whole of E_t. Further, by symmetry, $H(\infty) = e^{-i\pi\alpha}$, since $H(0) = 1$ and $t = e^{-i\pi\alpha}$ is the reflection point of $t = 1$. It follows that $H(\zeta) \equiv T(\zeta)$ in E'. In particular the dotted line $\arg t = -\frac{\pi\alpha}{2}$ is the image of that part of the semi-circle $|\zeta| = 1$ going from $\zeta = 1$ to $\zeta = e_\infty$.

Thus

$$e_\infty = e^{i\phi} \quad \text{where} \quad 0 < \phi < \pi \tag{1.3.2}$$

We now determine the function $S(\zeta)$.

$S(\zeta)$ is regular on the real axis except at the points $\zeta = 0, 1, \infty$, since it maps linear segments in the ζ-plane on to linear segments in the s-plane. $S(\zeta)$ is also regular everywhere in E_ζ except on the semi-circular arc from $\zeta = 1$ to $\zeta = e_\infty$.

By the reflection principle we can continue $S(\zeta)$ across the linear segments $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$. These reflections correspond to reflections across the linear segments in the s-plane. We obtain in this way a mapping of $\operatorname{Im}\zeta < 0$ on to a region which is a reflection of E_s through a linear segment. We can by the same argument reflect $S(\zeta)$ across another of the three linear segments

on the real axis, thus returning to the upper half ζ-plane. This will give us a region in the s-plane congruent to E_s. We obtain in this way another branch of the mapping function, say $S_1(\zeta)$, where

$$S_1(\zeta) = a\, S(\zeta) + b, \quad a, b \text{ constants.}$$

Hence $$\frac{S_1''(\zeta)}{S_1'(\zeta)} = \frac{S''(\zeta)}{S'(\zeta)}$$

An even number of such reflections always returns a point ζ in E_ζ into itself. An odd number always sends ζ into $\bar{\zeta}$. Meanwhile in the s-plane we always obtain regions congruent to E_s or to a reflection of E_s through one of three lines. It is clear that all these latter reflections, which are maps of $\operatorname{Im} \zeta < 0$, are congruent to one another.

Thus the function $\frac{S''(\zeta)}{S'(\zeta)}$ is single-valued in the whole ζ-plane except possibly at the points of the circular arc going from $\bar{e}_\infty$ to 1 to e_∞. Now $S(\zeta) = \log T(\zeta)$,

hence $$\frac{S''(\zeta)}{S'(\zeta)} = \frac{T''(\zeta)\, T(\zeta) - T'^2(\zeta)}{T(\zeta)\, T'(\zeta)}$$

Since $T(\zeta)$ is regular, univalent and non-zero in E_ζ, but ∞ at e_∞, it follows that $S''(\zeta)/S'(\zeta)$ is regular

at every point of the circular arc from 1 to e_∞ except at 1 and e_∞. Similarly for the lower half of the circular arc from 1 to $\bar{e}_\infty$. Thus $S''(\zeta)/S'(\zeta)$ has its only singularities at the points 0, 1, ∞, $\bar{e}_\infty$, e_∞. Also $S''(\zeta)/S'(\zeta)$ is single-valued on the circular arc.

Now near e_∞

$$\frac{1}{T(\zeta)} = (\zeta - e_\infty)\,\phi(\zeta)$$

where $\phi(\zeta)$ is regular and non-zero at e_∞.

Hence $$\frac{S''(\zeta)}{S'(\zeta)} = -\frac{1}{\zeta - e_\infty} + \text{regular fn at } e_\infty \quad (1.3.3)$$

so that $$\frac{S''(\zeta)}{S'(\zeta)} + \frac{1}{\zeta - e_\infty} \text{ remains regular at } e_\infty.$$

Similarly, by the relection principle, either $e^{S(\zeta)}$ or $e^{-S(\zeta)}$ is zero at $\bar{e}_\infty$, and in either case we easily deduce that $\frac{S''(\zeta)}{S'(\zeta)} + \frac{1}{\zeta - \bar{e}_\infty}$ remains regular at $\bar{e}_\infty$. (1.3.4)

It remains to consider the behaviour of $\frac{S''(\zeta)}{S'(\zeta)}$ near the three points $\zeta = 0, 1, \infty$. $S(\zeta)$ maps linear segments making angle π on each side of a $(= 0, 1, \infty)$ on to linear segments making angle $\pi\lambda$ $(= \frac{3\pi}{2}, 0, \frac{3\pi}{2})$ on each side of b $(= 0, -\infty, \pi i(2 - \alpha))$.

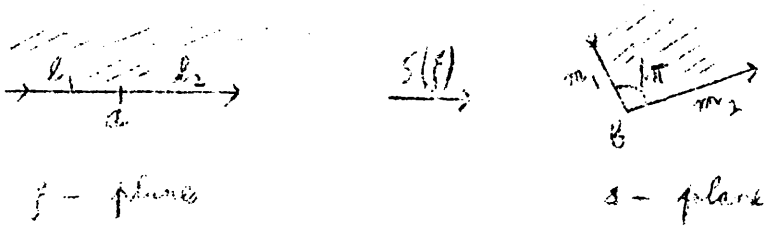


Figure 4

Consider then $V(\zeta) = \{S(\zeta) - b\}^{\frac{1}{\lambda}}$, ζ near to $a(\neq \infty)$, where $b = S(a)$. $S(\zeta)$ maps ℓ_1, on to m_1, ℓ_2 on to m_2. Hence $V(\zeta)$ maps ℓ_1 on to n_1, ℓ_2 on to n_2 where the angle between n_1 and n_2 is π, i.e. $V(\zeta)$ remains regular at a. Therefore we have

$$S(\zeta) = V(\zeta)^{\lambda} + S(a)$$

$$= (\zeta - a)^{\lambda} \chi^{\lambda}(\zeta) + S(a)$$

where $\chi(\zeta)$ is regular and non-zero at a. We deduce

$$\frac{S''(\zeta)}{S'(\zeta)} = \frac{\lambda - 1}{\zeta - a} + \text{regular fn at } a$$

and so

$$\left.\begin{aligned} &\frac{S''(\zeta)}{S'(\zeta)} - \frac{\frac{1}{2}}{\zeta} \quad \text{remains regular at } 0 \\ &\frac{S''(\zeta)}{S'(\zeta)} + \frac{1}{\zeta - 1} \quad \text{remains regular at } 1 \end{aligned}\right\} \qquad (1.3.5)$$

In the case $a = \infty$, $V(\zeta)$ is regular and zero at ∞.

i.e. $$V(\zeta) = \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \dots \quad \text{near } \infty, \quad (a_1 \neq 0).$$

We easily deduce that

$$\frac{S''(\zeta)}{S'(\zeta)} \quad \text{is regular and zero at } \infty. \tag{1.3.6}$$

We obtain from (1.3.3-6)

$$\frac{S''(\zeta)}{S'(\zeta)} + \frac{1}{\zeta - e_\infty} + \frac{1}{\zeta - \bar{e}_\infty} - \frac{1}{2\zeta} + \frac{1}{\zeta - 1}$$

is regular in the whole plane and zero at ∞. Hence by Liouville's theorem this function is identically zero, and we have

$$\frac{S''(\zeta)}{S'(\zeta)} = \frac{1}{2\zeta} - \frac{1}{\zeta - 1} - \frac{1}{\zeta - e_\infty} - \frac{1}{\zeta - \bar{e}_\infty} \tag{1.3.7}$$

Integrating this we deduce

$$S'(\zeta) = \frac{c\sqrt{\zeta}}{(\zeta - 1)(\zeta - e_\infty)(\zeta - \bar{e}_\infty)} \tag{1.3.8}$$

$$= \sqrt{\zeta}\left\{\frac{a_1}{\zeta - 1} + \frac{a_2}{\zeta - e^{i\phi}} + \frac{a_3}{\zeta - e^{-i\phi}}\right\}$$

where c is a constant of integration to be determined, and a_1, a_2, a_3 satisfy

$$\left.\begin{aligned} a_1 &= \frac{c}{(1 - e^{i\phi})(1 - e^{-i\phi})} \\ a_2 &= \frac{c}{(e^{i\phi} - 1)(e^{i\phi} - e^{-i\phi})} \\ a_3 &= \frac{c}{(e^{-i\phi} - 1)(e^{-i\phi} - e^{i\phi})} \end{aligned}\right\} \qquad (1.3.9)$$

Now

$$S'(\zeta) = -\frac{1}{\zeta - e_\infty} + \text{regular fn at } e_\infty \qquad (1.3.10)$$

Let $U(\zeta) = T(\zeta)^{1/\alpha}$ near $\zeta = 1$. Then

$$U(\zeta) = (1 - \zeta)\,\rho(\zeta)$$

where $\rho(\zeta)$ is regular and non-zero at $\zeta = 1$. We easily deduce that

$$S'(\zeta) = \frac{\alpha}{1 - \zeta} + \text{regular fn at } \zeta = 1 \qquad (1.3.11)$$

Hence from (1.3.8) and (1.3.9) we obtain

$$\left.\begin{aligned} a_1 &= \alpha = \frac{c}{(1 - e^{i\phi})(1 - e^{-i\phi})} \\ a_2 &= -\alpha \frac{1 - e^{-i\phi}}{e^{i\phi} - e^{-i\phi}}, \quad a_3 = \bar{a}_2 \end{aligned}\right\} \qquad (1.3.12)$$

And so from (1.3.8) and (1.3.10) we have

$$a_2 = e^{-i\phi/2}$$

and so $$a_3 = e^{i\phi/2}$$

Finally from (1.3.9)

$$a_1 + a_2 + a_3 = 0$$

and therefore

$$\alpha = e^{i\phi/2} + e^{-i\phi/2} = 2 \cos \tfrac{1}{2} \phi \qquad (1.3.13)$$

We have thus proved

$$S'(\zeta) = \sqrt{\zeta} \left\{ \frac{\alpha}{\zeta - 1} - \frac{e^{-i\phi/2}}{\zeta - e^{-i\phi}} - \frac{e^{i\phi/2}}{\zeta - e^{-i\phi}} \right\}$$

hence $$S(\zeta) = \int_0^{\zeta} \sqrt{\zeta} \left\{ \frac{\alpha}{\zeta - 1} - \frac{e^{-i\phi/2}}{\zeta - e^{i\phi}} - \frac{e^{i\phi/2}}{\zeta - e^{-i\phi}} \right\} d\zeta + B$$

(1.3.14)

Let $I_K(\zeta) = \int_0^{\zeta} \frac{\sqrt{\zeta}\, d\zeta}{\zeta - K}$

Then setting $r = \sqrt{\zeta}$ in the integrand,

$$I_K(\zeta) = \int_0^{\sqrt{\zeta}} \frac{2\gamma^2\, d\gamma}{\gamma^2 - K} = 2 \int_0^{\sqrt{\zeta}} \left\{ 1 + \frac{K}{\gamma^2 - K} \right\} d\gamma$$

$$= 2\sqrt{\zeta} + \sqrt{K} \int_0^{\sqrt{\zeta}} \left\{ \frac{1}{\gamma - \sqrt{K}} - \frac{1}{\gamma + \sqrt{K}} \right\} d\gamma$$

$$= 2\sqrt{\zeta} + \sqrt{K} \log \left(\frac{\sqrt{K} - \sqrt{\zeta}}{\sqrt{K} + \sqrt{\zeta}} \right) \qquad (1.3.15)$$

Now $S(0) = 0$, so that $B = 0$, and we have

$$S(\zeta) = \alpha I_1(\zeta) - e^{-i\phi/2} I_{e^{i\phi}}(\zeta) - e^{i\phi/2} I_{e^{-i\phi}}(\zeta)$$

$$= \alpha \log \frac{1 - \sqrt{\zeta}}{1 + \sqrt{\zeta}} - \log \left[\frac{e^{i\phi/2} - \sqrt{\zeta}}{e^{i\phi/2} + \sqrt{\zeta}} \cdot \frac{e^{-i\phi/2} + \sqrt{\zeta}}{e^{-i\phi/2} - \sqrt{\zeta}} \right]$$

$$= \log \left\{ \left(\frac{1 - \sqrt{\zeta}}{1 + \sqrt{\zeta}} \right)^{\alpha} \frac{1 + \alpha\sqrt{\zeta} + \zeta}{1 - \alpha\sqrt{\zeta} + \zeta} \right\}$$

We have therefore proved that

$$t = T(\zeta) = \left(\frac{1 - \sqrt{\zeta}}{1 + \sqrt{\zeta}} \right)^{\alpha} \left(\frac{1 + \alpha\sqrt{\zeta} + \zeta}{1 - \alpha\sqrt{\zeta} + \zeta} \right)$$

maps E_ζ onto E_t so that

$$\left.\begin{aligned} \zeta &= 0 \quad \rightarrow \quad t = 1 \\ \zeta &= 1 \quad \rightarrow \quad t = 0 \\ \zeta &= \infty \quad \rightarrow \quad t = e^{-i\pi\alpha} \\ \zeta &= e^{i\phi} \quad \rightarrow \quad t = \infty \end{aligned}\right\} \qquad (1.3.16)$$

where $\alpha = 2 \cos \frac{1}{2}\phi$ and $0 < \phi < \pi$.

Section 1.4

We now complete the calculation of $F(z)$. It is easy to see that

$$w = g(\zeta) = \frac{1}{4} \frac{e^{i\pi\alpha/2}}{T(\zeta)}$$

$$= \frac{1}{4} \left(\frac{1}{i} \frac{1 + \sqrt{\zeta}}{1 - \sqrt{\zeta}} \right)^{\alpha} \left(\frac{1 - \alpha\sqrt{\zeta} + \zeta}{1 + \alpha\sqrt{\zeta} + \zeta} \right) \qquad (1.4.1)$$

maps E_ζ onto G_o such that

$$\left.\begin{aligned} \zeta = 0 \quad &\rightarrow \quad w = \frac{1}{4} e^{i\pi\alpha/2} , \qquad \zeta = \infty \quad \rightarrow \quad w = \frac{1}{4} e^{-i\pi\alpha/2} \\ \zeta = 1 \quad &\rightarrow \quad w = \infty , \qquad \zeta = e^{i\phi} \quad \rightarrow \quad w = 0 \end{aligned}\right\} \qquad (1.4.2)$$

$$\left.\begin{aligned} \text{Set} \quad z &= e^{-i\phi} \frac{\zeta - e^{i\phi}}{\zeta - e^{-i\phi}} \\ \text{so} \quad \zeta &= \frac{e^{i\phi} + z}{1 + e^{i\phi} z} \end{aligned}\right\}$$

Then $\zeta(z)$ maps $|z| < 1$ onto E_ζ such that

$$\left.\begin{aligned} z = 0 &\rightarrow \zeta = e^{i\phi} \\ z = 1 &\rightarrow \zeta = 1 \\ z = -e^{+i\phi} &\rightarrow \zeta = 0 \\ z = -e^{-i\phi} &\rightarrow \zeta = \infty \end{aligned}\right\} \qquad (1.4.3)$$

Hence $w = g(\zeta(z)) = F(z)$ is the required function.

To calculate $F(z)$ explicitly,

$$\left.\begin{aligned} \text{set} \quad p &= e^{i\phi} + z \\ q &= 1 + e^{i\phi} z \end{aligned}\right\} \qquad (1.4.4)$$

so $\zeta = \frac{p}{q}$.

Then $p + q = (1 + e^{i\phi})(1 + z)$.

$$\left.\begin{aligned} pq &= e^{i\phi}(1 + 2z\cos\phi + z^2) \end{aligned}\right\} \qquad (1.4.5)$$

and $2\cos\phi = \alpha^2 - 2$.

Let $$\lambda(z) = \sqrt{\left[1 + (\alpha^2 - 2)z + z^2\right]} \qquad (1.4.6)$$

So $$pq = e^{i\phi}\lambda^2 \qquad (1.4.7)$$

Let $$\beta = 2\sin\tfrac{1}{2}\phi = \sqrt{(4 - \alpha^2)} \qquad (1.4.8)$$

then $$\frac{1}{i}\,\frac{1 + e^{i\phi}}{1 - e^{i\phi}} = \frac{\alpha}{\beta} \qquad (1.4.9)$$

Also $$(1 + e^{i\phi})^2 = \alpha^2 e^{i\phi} \qquad (1.4.10)$$

$$\text{Consider } \frac{1}{i}\,\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}} = \frac{1}{i}\,\frac{\sqrt{q}+\sqrt{p}}{\sqrt{q}-\sqrt{p}}$$

$$= \frac{1}{i}\,\frac{(\sqrt{q}+\sqrt{p})^2}{q-p} = \frac{1}{i}\,\frac{(1+e^{i\phi})(1+z)+2e^{i\phi/2}\lambda}{(1-e^{i\phi})(1-z)}$$

$$= \frac{\alpha/\beta\,.\,(1+z)+2e^{i\phi/2}\left(\dfrac{1}{i(1-e^{i\phi})}\right)\lambda}{1-z}$$

Now

$$\frac{1}{i(1-e^{i\phi})} = \frac{e^{-i\phi/2}}{\beta}$$

$$\text{Hence } \frac{1}{i}\,\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}} = \frac{\alpha(1+z)+2\lambda(z)}{\beta(1-z)} \qquad (1.4.11)$$

Again

$$\frac{1-\alpha\sqrt{\zeta}+\zeta}{1+\alpha\sqrt{\zeta}+\zeta} = \frac{q-\alpha\sqrt{pq}+p}{q+\alpha\sqrt{pq}+p}$$

$$= \frac{(q+p)^2-\alpha^2 qp}{(q+\alpha\sqrt{pq}+p)^2} = \frac{\beta^2 z}{(1+z+\lambda)^2} \qquad (1.4.12)$$

Hence from (1.4.11) and (1.4.12) we obtain

$$F(z) = \frac{1}{4}\beta^{2-\alpha}\,h(z)\,\frac{z}{(1-z)^{\alpha}} \qquad (1.4.13)$$

$$\left.\begin{aligned} \text{where} \qquad h(\mathbf{z}) &= \frac{\{\alpha(1+\mathbf{z}) + 2\lambda(\mathbf{z})\}^{\alpha}}{\{1+\mathbf{z}+\lambda(\mathbf{z})\}^{2}} \\ \text{and} \qquad \lambda(Z) &= \sqrt{\{1 + (\alpha^2 - 2)\mathbf{z} + \mathbf{z}^2\}} \\ \text{and} \qquad \beta &= \sqrt{(4-\alpha^2)} \ . \end{aligned}\right\} \qquad (1.4.14)$$

Lemma 1.4.A

Let G_o be the domain in the w-plane defined by

$$G_o = \left\{ w/\ |\arg w| < \frac{\pi\alpha}{2} \right\} \ \cup \ \left\{ w/\ |w| < \frac{1}{4} \right\}$$

where $0 < \alpha < 2$.

Then $w = F(\mathbf{z})$, defined by (1.4.13) and (1.4.14), maps $|\mathbf{z}| < 1$ onto G_o 1:1 conformally such that

$$\mathbf{z} = 0 \qquad \rightarrow \quad w = 0$$

$$\mathbf{z} = -e^{i\phi} \qquad \rightarrow \quad w = \frac{1}{4} e^{i\pi\alpha/2}$$

$$\mathbf{z} = -e^{-i\phi} \qquad \rightarrow \quad w = \frac{1}{4} e^{-i\pi\alpha/2}$$

and $F'(0) > 0$. ϕ satisfies, $\alpha = 2\cos\frac{1}{2}\phi$, $0 < \phi < \pi$.

Section 1.5

We continue as in sections 1.3 and 1.4 to take α fixed and satisfying $0 < \alpha < 2$. We now examine the behaviour of

$$h(r;\alpha) = \frac{\{\alpha(1 + r) + 2\lambda(r)\}^{\alpha}}{\{1 + r + \lambda(r)\}^{2}} \qquad (0 \leq r < 1) \qquad (1.5.1)$$

We prove that for each α, $0 < \alpha < 2$, $h(r;\alpha)$ decreases as r increases from 0 to 1. We first show that $h'(r)$ is non-zero for $0 \leq r < 1$. To prove this we write

$$h(1 + r + \lambda)^{2} = \{\alpha(1 + r) + 2\lambda\}^{\alpha}$$

hence
$$h'(1 + r + \lambda)^{2} + 2h(1 + \lambda')(1 + r + \lambda)$$
$$= \alpha(\alpha + 2\lambda')\{\alpha(1 + r) + 2\lambda\}^{\alpha-1}$$

i.e.
$$h'(1 + r + \lambda)^{2}\{\alpha(1 + r) + 2\lambda\}$$
$$+ 2h(1 + \lambda')(1 + r + \lambda)\{\alpha(1 + r) + 2\lambda\}$$
$$= \alpha(\alpha + 2\lambda')\, h(1 + r + \lambda)^{2}$$

Hence if $h' = 0$ then, since $h > 0$ and $1 + r + \lambda > 0$, we obtain

$$2(1 + \lambda')\{\alpha(1 + r) + 2\lambda\} = \alpha(\alpha + 2\lambda')(1 + r + \lambda)$$

i.e. $\alpha(2 - \alpha)(1 + r) + (2 - \alpha)(2 + \alpha)\lambda + 2(2 - \alpha)\lambda\lambda' = 0$

and since $\alpha \neq 2$,

$$\alpha(1 + r) + (2 + \alpha)\lambda + 2\lambda\lambda' = 0 .$$

Now $2\lambda\lambda' = \alpha^2 - 2 + 2r$

hence $(\alpha + 2)(\lambda + r + \alpha - 1) = 0$

hence $\lambda = 1 - r - \alpha$

Thus $1 + (\alpha^2 - 2)r + r^2 = 1 + r^2 + \alpha^2 - 2r + 2\alpha r - 2\alpha$

i.e. $\alpha^2 r = \alpha^2 + 2\alpha r - 2\alpha$

i.e. $\alpha(\alpha - 2)r = \alpha(\alpha - 2)$

Since $\alpha \neq 0$ or 2, we obtain

$$r = 1 .$$

This proves that $h'(r)$ is non-zero for $0 \leq r < 1$. Hence $h(r)$ either decreases or increases as r increases from 0 to 1. Suppose $\exists\ \alpha_o$, $0 < \alpha_o < 2$, such that $h(r;\alpha_o)$ is decreasing. Then $h(r;\alpha)$ is decreasing for every value of α, $0 < \alpha < 2$. To see this suppose $\exists\ \alpha_1$, $0 < \alpha_1 < 2$, such that $h(r;\alpha_1)$ is increasing. Then

$$\left.\begin{array}{l} h'(r;\alpha_o) < 0 \\ h'(r;\alpha_1) > 0 \end{array}\right\} \quad 0 \leq r < 1$$

As a function of α, $h'(r;\alpha)$ is continuous, so $\exists\ \alpha_2$ between α_o and α_1 such that $h'(r;\alpha_2) = 0$, and this is a contradiction. Now consider $h(r;1)$. Then

$$h(0;1) = \frac{3}{4}\,, \qquad h(1,1) = \frac{4}{9} < \frac{3}{4}\,.$$

Hence $h(r;1)$ decreases as r increases. Hence $h(r;\alpha)$ decreases as r increases, for each α, $0 < \alpha < 2$.

Thus

$$h(1;\alpha) \leq h(r;\alpha) \leq h(0;\alpha)$$

i.e.
$$\frac{(4\alpha)^{\alpha}}{(\alpha+2)^{2}} \leq h(r;\alpha) \leq \frac{(\alpha+2)^{\alpha}}{4} \tag{1.5.2}$$

for $0 \leq r < 1$, $0 < \alpha < 2$.

It follows from (1.4.13) that

$$M(r,F) \geq \frac{1}{4}(4\alpha)^{\alpha}\sqrt{\frac{(2-\alpha)^{2-\alpha}}{(2+\alpha)^{2+\alpha}}}\;\frac{r}{(1-r)^{\alpha}}$$

$$= K(\alpha)\,\frac{r}{(1-r)^{\alpha}} \tag{1.5.3}$$

It is easy to see that (1.5.3) holds for $\alpha = 0, 2$ if we define $K(\alpha)$ at these points as the respective limits,

$$\left.\begin{aligned} \text{i.e.} \qquad K(0) &= \lim_{\alpha\to 0} K(\alpha) = \frac{1}{4} \\ \text{and} \qquad K(2) &= \lim_{\alpha\to 2} K(\alpha) = 1 \end{aligned}\right\} \tag{1.5.4}$$

Section 1.6

We now determine the minimum value of $K(\alpha)$ for $0 \leq \alpha \leq 2$. We have

$$\log K(\alpha) = \log \frac{1}{4} + \alpha \log 4\alpha + (1 - \frac{\alpha}{2}) \log (2 - \alpha) - (1 + \frac{\alpha}{2}) \log (2 + \alpha)$$

Hence $$\frac{K'(\alpha)}{K(\alpha)} = \log \frac{4\alpha}{\sqrt{(4 - \alpha^2)}}$$

Since $K(\alpha) > 0$, if $K'(\alpha) = 0$ then

$$4\alpha = \sqrt{(4 - \alpha^2)}$$

$$\alpha = \frac{2}{\sqrt{17}} = \alpha_o \text{ say} \qquad (1.6.1)$$

Also

$$K''(\alpha_o) = K(\alpha_o)\left[\frac{1}{\alpha_o} + \frac{\alpha_o}{4 - \alpha_o^2}\right] > 0$$

Hence α_o is a minimum and

$$K(\alpha) \geq K(\alpha_o), \quad \left(0 \leq \alpha \leq 2\right),$$

$$= \frac{1}{4} \frac{2 - \alpha_o}{2 + \alpha_o} = \frac{9 - \sqrt{17}}{32} \qquad (1.6.2)$$

$$\simeq .1524$$

$$> \frac{1}{7} .$$

Theorem 1.6.A

If $w = f(z)$ is regular and univalent in $|z| < 1$, and $f(0) = 0$, and $|z| < 1$ is mapped onto a domain containing a sector of angle $\pi\alpha$ measured from the origin, then

$$M(r,f) \geq K(\alpha) \frac{|f'(0)| \, r}{(1 - r)^{\alpha}} > \frac{|f'(0)| \, r}{7(1 - r)^{\alpha}}$$

for $0 \leq r < 1$. $K(\alpha)$ is given by (1.5.3) and satisfies (1.6.2).

Proof: Result follows from (1.2.4) and (1.5.3).

Section 1.7

Lemma 1.7.A

Suppose $w = G(z)$ gives a conformal univalent map of $|z| < 1$ onto the full w-plane cut along n radial slits, and suppose $G(0) = 0$. Then $G(z)$ has the form

$$G(z) = c_3 \prod_{\nu=1}^{n} (1 - \kappa_\nu z)^{-\lambda_\nu}$$

where the κ_ν are distinct constants satisfying

$$|\kappa_\nu| = 1, \quad \nu = 1, \ldots n$$

and the λ_ν are positive real numbers satisfying

$$\sum_{\nu=1}^{n} \lambda_\nu = 2$$

and c is some non-zero constant.

Proof

Let $\mathcal{Y}'$ be the image domain of $G(z)$, so $\mathcal{Y}'$ is the full plane cut along n radial slits. Let the angles between consecutive slits be $\pi\lambda_1$, $\pi\lambda_2$, ... $\pi\lambda_n$, so each λ_ν is positive and $\sum_{\nu=1}^{n} \pi\lambda_\nu = 2\pi$. By the Riemann mapping theorem we can find a function $T(\zeta)$ which maps $E_\zeta = \{\zeta / \mathrm{Im}\,\zeta > 0\}$ 1:1 onto $\mathcal{Y}'$ conformally. So we set $w = T(\zeta)$, and as we travel from left to right along the real ζ-axis this corresponds to travelling twice along each cut in the w-plane - once towards the origin and once away from it, in that order moving anti-clockwise about the origin.

Let a_1 a_2, ... a_n be the points on the real ζ-axis which are mapped onto the pt at ∞ in $\mathcal{Y}'$. There is one such a_ν corresponding to each slit ℓ_ν, i.e. a_ν corresponds to ∞ at the end of ℓ_ν. Let b_ν be the point on the real ζ-axis corresponding to the finite end of the slit ℓ_ν, this latter point being called c_ν. Thus $T(a_\nu) = \infty$, $T(b_\nu) = c$, $(\nu = 1, \ldots n)$. Each linear segment $a_\nu b_\nu$

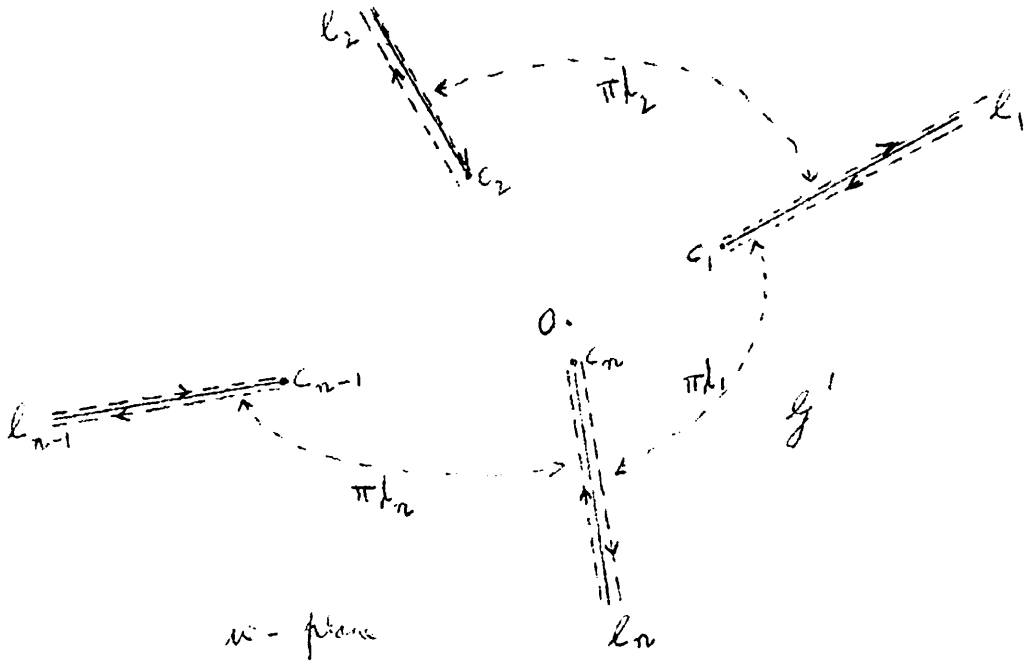


Figure 5

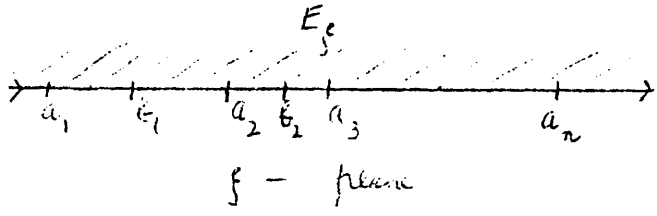


Figure 6

maps onto $\ell_\nu = \infty c_\nu$. Each linear segment $b_\nu a_{\nu+1}$ maps onto $\ell_\nu = c_\nu \infty$, where we adopt the convention $n + 1 = 1$. Thus $w = T(\zeta)$ maps linear segments onto linear segments, and is regular on the real axis except at the points a_ν , b_ν

(supposed finite for the time being). By the reflection principle $T(\zeta)$ can be continued across each such segment giving a map of $\operatorname{Im} \zeta < 0$ onto the reflection of $\mathcal{G}'$ through a radial line ℓ_ν. We can then, by the same argument continue the function thus obtained back into E_ζ, and in the w-plane this will give a map of E_ζ onto a domain congruent to $\mathcal{G}'$, still consisting of radial slits.

The new function $T_1(\zeta)$ thus obtained must have the form

$$T_1(\zeta) = A\, T(\zeta)$$

and so

$$\frac{T_1'(\zeta)}{T_1(\zeta)} = \frac{T'(\zeta)}{T(\zeta)}$$

Hence $\frac{T'(\zeta)}{T(\zeta)}$ is single-valued in the whole ζ-plane and regular except at the points a_ν, b_ν $(\nu = 1, \ldots n)$, and the two points e_0, $\bar{e}_0$ where

$$T(e_0) = 0 .$$

Consider first the points b_ν, $\nu = 1, \ldots n$. Set

$$V(\zeta) = \left\{ T(\zeta) - c_\nu \right\}^{\frac{1}{2}} .$$

Then $V(\zeta)$ maps the segments $a_\nu b_\nu$, $b_\nu a_{\nu+1}$ onto a single linear segment through c_ν. Hence $V(\zeta)$ is

regular at b_ν, and

$$V(\zeta) = (\zeta - b_\nu)\,\phi(\zeta)$$

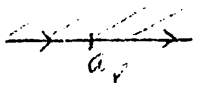
where $\phi(\zeta)$ is regular at b_ν. Thus

$$\frac{T'(\zeta)}{T(\zeta)} = 0 \quad \text{at} \quad b_\nu \text{ and is regular there.} \qquad (1.7.1)$$

(1.7.1) also applies if $b_\nu = \infty$, for then $V(\zeta)$ is regular and zero at ∞.

Next consider the points a_ν $(\nu = 1, \ldots n)$.

Let $U(\zeta) = \left\{ \dfrac{1}{T(\zeta)} \right\}^{1/\lambda_\nu}$



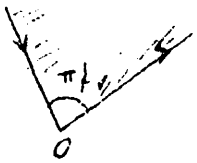


Figure 7

Then it is easy to see that $U(\zeta)$ is regular and zero at a_ν, in fact

$$U(\zeta) = (\zeta - a_\nu)\,\psi(\zeta)$$

where $\psi(\zeta)$ is regular and non-zero at a_ν.

We deduce that

$$\frac{T'(\zeta)}{T(\zeta)} + \frac{\lambda_\nu}{\zeta - a_\nu} \quad \text{remains regular at} \quad a_\nu. \qquad (1.7.2)$$

At both e_o and $\bar{e}_o$ $T(\zeta)$ is regular, zero, and $T'(\zeta) \neq 0$. Hence

$$\frac{T'(\zeta)}{T(\zeta)} - \frac{1}{\zeta - e_o} \quad \text{remains regular at } e_o$$

$$\text{and} \quad \frac{T'(\zeta)}{T(\zeta)} - \frac{1}{\zeta - \bar{e}_o} \quad \text{remains regular at } \bar{e}_o \tag{1.7.3}$$

Finally $\frac{T'(\zeta)}{T(\zeta)}$ is zero at infinity, hence (1.7.1) - (1.7.3) prove that

$$\frac{T'(\zeta)}{T(\zeta)} = \frac{1}{\zeta - e_o} + \frac{1}{\zeta - \bar{e}_o} - \sum_{\nu=1}^{n} \frac{\lambda_\nu}{\zeta - a_\nu} \tag{1.7.4}$$

Thus

$$T(\zeta) = K(\zeta - e_o)(\zeta - \bar{e}_o) \prod_{\nu=1}^{n} (\zeta - a_\nu)^{-\lambda_\nu} \tag{1.7.5}$$

where K is a non-zero constant.

Set

$$z = \frac{\zeta - e_o}{\zeta - \bar{e}_o}, \quad k_\nu = \frac{a_\nu - \bar{e}_o}{a_\nu - e_o} \tag{1.7.6}$$

Since a_ν is real we have

$$|k_\nu| = 1, \quad k_\nu \neq 1. \tag{1.7.7}$$

Also

$$\zeta = \frac{e_o - \bar{e}_o z}{1 - z}, \qquad a_\nu = \frac{\bar{e}_o - \kappa_\nu e_o}{1 - \kappa_\nu} \tag{1.7.8}$$

$\zeta = \zeta(z)$ maps $|z| < 1$ onto E_ζ such that

$$z = 0 \rightarrow \zeta = e_o .$$

Hence $w = G_o(z) = T(\zeta(z))$
maps $|z| < 1$ onto ζ' such that

$$z = 0 \rightarrow w = 0 .$$

Since the mapping of $|z| < 1$ onto ζ' with $z = 0 \rightarrow w = 0$ is unique except for a rotation in the z-plane, which does not affect the form of the result, we may suppose $G_o(z) = G(z)$. We then obtain by means of (1.7.8),

$$G(z) = Kz \prod_{\nu=1}^{n} \left(\frac{1 - \kappa_\nu}{1 - \kappa_\nu z} \right)^{\lambda_\nu} \tag{1.7.9}$$

This proves the lemma in the case when all the a_ν are finite points of the real ζ-axis. To complete the proof we now suppose $a_1 = \infty$. Then it is easily shown that $T'(\zeta)/T(\zeta) = 0$ at ∞.
Hence (1.7.5) becomes

$$T(\zeta) = K(\zeta - e_o)(\zeta - \bar{e}_o) \prod_{\nu=2}^{n} (\zeta - a_\nu)^{-\lambda_\nu} \tag{1.7.10}$$

As before we set

$$z = \frac{\zeta - e_o}{\zeta - \overline{e}_o}, \quad \kappa_\nu = \frac{a_\nu - \overline{e}_o}{a_\nu - e_o}, \quad (\nu = 2, 3, \ldots n)$$

and in addition we set

$$\kappa_1 = 1 .$$

This gives

$$G(z) = K(e_o - \overline{e}_o)^{\lambda_1} \prod_{\nu=2}^{n} (1 - \kappa_\nu)^{\lambda_\nu} \frac{z}{\prod_{\nu=1}^{n} (1 - \kappa_\nu z)^{\lambda_\nu}} \quad (1.7.11)$$

which proves the lemma.

Section 1.8

Theorem 1.8.A

Suppose $w = f(z)$, $f(o) = 0$, is a 1:1 conformal map of $|z| < 1$ onto a starlike domain $\mathcal{G}$, and let $\pi\alpha$ be the angle of the largest sector contained in $\mathcal{G}$. Then for each $\varepsilon > 0$ $\exists$ a positive constant $k = k(\varepsilon; f)$ such that

$$M(r,f) \leq \frac{kr}{(1 - r)^{\alpha+\varepsilon}} \qquad (0 \leq r < 1)$$

Proof

Choose N an integer such that

$$\varepsilon > \frac{4}{N} = 2\eta \quad \text{say}$$

We divide the w-plane into N sectors each of angle $\pi\eta$. In one of the N sectors, or on the boundary of one of these sectors there is a finite boundary point of $\mathcal{G}$. Choose such a point and from it construct a radial cut away from the origin and extending to infinity. Since $\mathcal{G}$ is starlike no point of this cut is in $\mathcal{G}$. We now examine the neighbouring sector in an anti-clockwise direction. Either there is a boundary point of $\mathcal{G}$ in this sector or on its boundary not common to the previous sector, or there is no such point. In the former case we construct another radial cut as before. In the latter case, or after the construction if the former case, we move on to the next sector and examine it in the same way. We repeat this process for each of the N sectors, and obtain in this way n radial slits in the w-plane, where $1 \leq n \leq N$.

Let $\mathcal{G}'$ be the whole w-plane cut along these n radial slits. Then $\mathcal{G} \subset \mathcal{G}'$. Hence if $w = G(z)$ is a 1:1 conformal map of $|z| < 1$ onto $\mathcal{G}'$ such that $G(0) = 0$,

then $f(z)$ is subordinate to $G(z)$, and

$$M(r,f) \leq M(r,G) \qquad (1.8.1)$$

From lemma 1.7.A we have

$$G(z) = \frac{Cz}{\prod_{\nu=1}^{n} (1 - \kappa_\nu z)^{\lambda_\nu}}$$

where $|\kappa_\nu| = 1$, the κ_ν are distinct, $\sum_{\nu=1}^{n} \lambda_\nu = 2$. Let $\lambda_K = \sup_{1 \leq \nu \leq n} \lambda_\nu$. Then for each ν, $1 \leq \nu \leq n$,

$$\lambda_\nu = \lambda_K - \delta_\nu, \quad \delta_\nu \geq 0.$$

Hence setting $|z| = r$ we have

$$|G(z)| = |c| r \prod_{\nu=1}^{n} |1 - \kappa_\nu z|^{\delta_\nu} \left\{ \prod_{\nu=1}^{n} |1 - \kappa_\nu z| \right\}^{-\lambda_K}$$

$$\leq Ar \left\{ \prod_{\nu=1}^{n} |1 - \kappa_\nu z| \right\}^{-\lambda_K}$$

where A is a positive constant.

Consider

$$H(z) = \frac{1}{\prod_{\nu=1}^{n} (1 - \kappa_\nu z)}$$

Since the κ_ν are all distinct we can split $H(z)$ into partial fractions as follows :

$$H(z) = \sum_{\nu=1}^{n} \frac{a_\nu}{1 - \kappa_\nu z}$$

where the a_ν are non-zero constants. Hence

$$|H(z)| \leq \sum_{\nu=1}^{n} \frac{a_\nu}{|1 - \kappa_\nu z|} \leq \frac{B}{1 - r}$$

where B is a constant. We thus have

$$|G(z)| \leq \frac{k\, r}{(1 - r)^{\lambda_K}} \tag{1.8.2}$$

where k is a positive constant. Hence from (1.8.1) we obtain

$$M(r,f) \leq \frac{k\, r}{(1 - r)^{\lambda_K}} \tag{1.8.3}$$

Now the numbers $\pi\lambda_\nu$ are the angles between the cuts of $\mathcal{G}'$. Hence

$$\pi\lambda_K = m\,\pi\eta + \pi\eta_1 + \pi\eta_2$$

where m is an integer, and $0 \leq \eta_1 < \eta$, $0 \leq \eta_2 < \eta$. The largest sector contained in $\mathcal{G}$ has angle $\pi\alpha$, and since there is no boundary pt of $\mathcal{G}$ inside the above $m\,\pi\eta$ sector, it follows that

$$m\,\pi\eta \leq \pi\alpha$$

Thus

$$\pi\lambda_K \leq \pi\alpha + \pi\eta_1 + \pi\eta_2$$

$$< \pi\alpha + 2\pi\eta$$

$$< \pi\alpha + \pi\xi$$

i.e. $$\lambda_K < \alpha + \xi$$

Hence from (1.8.3), since $0 < 1-r \leq 1$,

$$M(r,f) \leq \frac{kr}{(1-r)^{\alpha+\xi}}$$

which proves the theorem.

Section 1.9

Theorem 1

Suppose $w = f(z)$, $f(o) = 0$, is a 1:1 conformal map of $|z| < 1$ onto a starlike domain $\mathcal{G}$, and let $\pi\alpha$ be the angle of the largest sector contained in $\mathcal{G}$. Then

$$\alpha = \lim_{r\to 1} \frac{\log M(r)}{\log \frac{1}{1-r}}$$

Proof

Applying theorems 1.6.A and 1.8.A we have

$$\log c_1 r + \alpha \log \frac{1}{1-r} \leq \log M(r) \leq \log c_2 r + (\alpha+\varepsilon)\log\frac{1}{1-r}$$

where c_1, c_2 are constants. We obtain

$$\varliminf_{r\to 1} \frac{\log M(r)}{\log \frac{1}{1-r}} \geq \alpha$$

and

$$\varlimsup_{r\to 1} \frac{\log M(r)}{\log \frac{1}{1-r}} \leq \alpha + \varepsilon$$

Since ε is arbitrary the result follows.

Section 1.10

We conclude this chapter with the remark that theorem 1 is valid for a wider class of univalent functions than the starlike ones. Indeed since theorem 1.6.A is valid for all univalent functions, it follows that theorem 1 is valid wherever theorem 1.8.A is valid. This latter theorem in turn is valid if we can enclose $\mathcal{G}$ in a domain $\mathcal{G}'$ consisting of radial slits in such a way that for given $\varepsilon > 0$ the largest angle of the sectors of $\mathcal{G}' = \mathcal{G}'(\varepsilon)$ is smaller than $\pi(\alpha+\varepsilon)$. In other words theorem 1 is valid if every sector S_1 of angle greater than $\pi\alpha$ is such that $S_1 \cap \mathcal{G}^c$ contains a radial slit, where $\mathcal{G}^c$ is the complement of $\mathcal{G}$.

Thus for each $\varepsilon > 0$ we can find radial segments making angles with each other smaller than $\pi(\alpha + \varepsilon)$ on which $w = f(z)$ does not take values of arbitrarily large modulus. The sort of functions for which we have not proved theorem 1 are such that in some sector of angle greater than $\pi\alpha$, $|f(z)|$ takes arbitrarily large values on every radial line in the sector.

Chapter 2 General Properties of Starlike Functions with Sectors

Section 2.1 Introduction

As in chapter 1 we suppose

$$f(z) = a_1 z + a_2 z^2 + \ldots \qquad (a_1 \neq 0) \qquad (2.1.1)$$

is a starlike univalent function in $|z| < 1$ whose image domain is $\mathcal{G}$. In this chapter we develop further the properties of $f(z)$ by consideration of the sectors of $\mathcal{G}$. We obtain a representation for $f(z)$ equivalent to the integral representation (1.1.3). We show that the sectors of $\mathcal{G}$ are precisely the jumps of the function

$$V(\theta) = \lim_{r \to 1} \arg f(re^{i\theta})$$

where $V(\theta)$ exists everywhere and is monotonic increasing with θ. We also obtain analytic conditions on the function $\frac{zf'(z)}{f(z)}$ which are necessary and sufficient for $\mathcal{G}$ to contain a sector of given angle $\pi\alpha^{\star}$.

Besides using subordination as in chapter 1 we also make use of a theorem on bounded functions due to G. Julia. This theorem and related results are set out in full in Appendix 1. We remark here that the main interest of our approach to these functions is that only the classical results of

Complex Function theory are involved. In particular the Schwarz Lemma lies at the root of our investigation.

Suppose now that $\mathcal{Y}$ contains a sector S of angle $\pi\alpha^{\star}$ where

$$S = \left\{ w / \left|\arg w\right| < \frac{\pi\alpha^{\star}}{2} \right\} \qquad (2.1.2)$$

and that points of the complement of $\mathcal{Y}$ lie in any larger sector containing S. Then $f(z)$ is subordinate to a function $G(z)$ mapping $|z| < 1$ onto $\mathcal{Y}' > \mathcal{Y}$, where $\mathcal{Y}'$ is the whole plane cut along N radial slits. In particular there is a sector $S' < \mathcal{Y}'$ and $S' > S$. S' has angle $\pi(\alpha^{\star} + \varepsilon)$ where we may suppose $\varepsilon \geq 0$ is small.

Throughout this thesis $S' = S'(\varepsilon)$ will always be assumed to be a sector of the superordinate domain $\mathcal{Y}'$. In Appendix 2 we have set out properties of $G(z)$ further to those of lemma 1.7.A.

Section 2.2

Theorem 2.2.A

Suppose $\mathcal{Y}$ contains a sector S of the form (2.1.2). Then $\exists\ \mathcal{K}$ with $|\mathcal{K}| = 1$ such that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \frac{\alpha^{\star}}{2} \operatorname{Re} \frac{1 + \mathcal{K}z}{1 - \mathcal{K}z}$$

for all z in $|z| < 1$.

Proof

Let $\{ w_n \}$ be a sequence of points on the positive real axis in the w-plane, and suppose the sequence is unbounded. Setting

$$w_n = f(z_n) , \qquad n = 1, 2, \ldots$$

it is clear that

$$| z_n | \to 1 \quad \text{as} \quad n \to \infty .$$

Since $f(z)$ is subordinate to $G(z)$ it follows that

$$f(z) = G(\sigma(z)) , \qquad (|z| < 1)$$

where $\sigma(z)$ is regular in $|z| < 1$ and satisfies

$$|\sigma(z)| < 1 \quad \text{in} \quad |z| < 1$$

and $$\sigma(0) = 0 .$$

Hence

$$\frac{zf'(z)}{f(z)} = \frac{z\,\sigma'(z)\,G'(\sigma(z))}{G(\sigma(z))} \qquad (|z| < 1)$$

It follows from Pick's lemma, (c.f. lemma 3 in Appendix 1), that

$$|\sigma'(z)| \leq \frac{1 - |\sigma(z)|}{1 - |z|} \qquad (|z| < 1)$$

And so we obtain

$$(1-|z|)\left|\frac{zf'(z)}{f(z)}\right| \leq (1-|\sigma(z)|)\left|\frac{G'(\sigma(z))}{G(\sigma(z))}\right| \qquad (2.2.1)$$

in $|z| < 1$.

Now the points $w_n = G(\sigma(z_n))$ lie in S' . It immediately follows from (7) of appendix 2 that

$$\overline{\lim_{n\to\infty}}\ (1-|\sigma(z_n)|)\left|\frac{G'(\sigma(z_n))}{G(\sigma(z_n))}\right| \leq \alpha^{*} + \varepsilon$$

where $\pi(\alpha^{*} + \varepsilon)$ is the angle of S' . Since ε is arbitrarily small we obtain from (2.2.1)

$$\overline{\lim_{n\to\infty}}\ (1-|z_n|)\left|\frac{z_n f'(z_n)}{f(z_n)}\right| \leq \alpha^{*} \qquad (2.2.2)$$

Now $\exists$ a domain $\mathcal{G}_0 \subset \mathcal{G}$ where $\mathcal{G}_0$ is the union of a disc centred at $w = 0$ and the sector S . By lemma 1.4.A the function $F(z)$ defined by (1.4.13) and (1.4.14) (apart from multiplication by a positive real constant) maps $|z| < 1$ onto $\mathcal{G}_0$ and is subordinate to $f(z)$. The points w_n lie in $\mathcal{G}_0$. We may write

$$f(\gamma(z)) = F(z) \qquad (|z| < 1)$$

where $\gamma(0) = 0$, $|\gamma(z)| < 1$ in $|z| < 1$, and $\gamma(z)$ is regular.

Then

$$\frac{\gamma'(z)\; z\; f'(\gamma(z))}{f(\gamma(z))} = \frac{z\, F'(z)}{F(z)} \qquad (|z| < 1) .$$

Applying Pick's lemma to $|\gamma'(z)|$ we obtain

$$(1 - |\gamma(z)|) \left| \frac{f'(\gamma(z))}{f(\gamma(z))} \right| \geq (1 - |z|) \left| \frac{zF'(z)}{F(z)} \right| \qquad (2.2.3)$$

in $|z| < 1$. From (1.4.13) and (1.4.14) we deduce

$$(1 - |z|) \left| \frac{zF'(z)}{F(z)} \right| \geq \alpha^{*} \frac{1 - |z|}{|1 - z|} - o(1) \quad \text{as} \quad z \to 1 .$$

Now since $w_n \in \mathcal{G}_0$ it follows that $\exists$ points ζ_n st

$$\left.\begin{aligned} \gamma(\zeta_n) &= z_n \\ \text{and} \qquad w_n &= F(\zeta_n) \end{aligned}\right\} \quad n = 1, 2, \ldots$$

The points w_n are real and positive, hence we see from (1.4.13) and (1.4.14) that the points ζ_n are real and positive. Thus setting $\gamma(z) = z_n$ in (2.2.3) we obtain

$$(1 - |z_n|) \left| \frac{f'(z_n)}{f(z_n)} \right| \geq \alpha^{*} - o(1) \quad \text{as} \quad n \to \infty$$

since $|F(\zeta_n)| \to \infty$ and so $\zeta_n \to 1$ as $n \to \infty$.

Thus

$$\varliminf_{n\to\infty} (1 - |z_n|) \left| \frac{f'(z_n)}{f(z_n)} \right| \geq \alpha^{*} \qquad (2.2.4)$$

Together with (2.2.2) this yields

$$\lim_{n\to\infty} (1 - |z_n|) \left| \frac{f'(z_n)}{f(z_n)} \right| = \alpha^{*} \qquad (2.2.5)$$

Now since $f(z)$ is starlike in $|z| < 1$ we have

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > 0 \qquad (|z| < 1)$$

Hence we may write

$$\frac{z f'(z)}{f(z)} = \frac{1 + \omega(z)}{1 - \omega(z)} \qquad (|z| < 1)$$

where $\omega(z)$ is regular in $|z| < 1$ and satisfies

$$\omega(0) = 0 , \quad |\omega(z)| < 1 \quad \text{in} \quad |z| < 1 .$$

From (2.2.5) we have

$$\lim_{n\to\infty} (1 - |z_n|) \frac{|1 + \omega(z_n)|}{|1 - \omega(z_n)|} = \alpha^{*}$$

Since $\alpha^{*} > 0$ it follows that

$$\omega(z_n) \to 1 \quad \text{as} \quad n \to \infty$$

Hence

$$\lim_{n\to\infty} \frac{1 - |z_n|}{|1 - \omega(z_n)|} = \frac{\alpha^{*}}{2}$$

And so

$$\overline{\lim_{n\to\infty}} \frac{1 - |\omega(z_n)|}{1 - |z_n|} \leq \overline{\lim_{n\to\infty}} \frac{|1 - \omega(z_n)|}{1 - |z_n|} = \frac{2}{\alpha^*}$$

Hence the limes superior exists and equals γ, where

$$1 \leq \gamma \leq \frac{2}{\alpha^*}$$

since $|\omega(z_n)| \leq |z_n|$ by the Schwarz lemma. We may suppose without loss of generality that $\lambda z_n \to 1$ as $n \to \infty$ where λ is some fixed quantity st $|\lambda| = 1$. We then apply Julia's theorem 1 of Appendix 1 and deduce

$$\frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} \leq \gamma \frac{|1 - \lambda z|^2}{1 - |z|^2} \leq \frac{2}{\alpha^*} \frac{|1 - \lambda z|^2}{1 - |z|^2}$$

for all z in $|z| < 1$. We then obtain

$$\operatorname{Re} \frac{z f'(z)}{f(z)} = \frac{1 - |\omega(z)|^2}{|1 - \omega(z)|^2} \geq \frac{\alpha^*}{2} \frac{1 - |z|^2}{|1 - \lambda z|^2}$$

$$= \frac{\alpha^*}{2} \operatorname{Re} \frac{1 + \lambda z}{1 - \lambda z}, \quad (|z| < 1). \tag{2.2.6}$$

which completes theorem 2.2.A.

If we suppose for convenience that $\lambda = 1$ then when $z = r$ is real and positive we obtain

$$\lim_{r\to 1} (1 - r) \operatorname{Re} \frac{rf'(r)}{f(r)} \geq \alpha^* \tag{2.2.7}$$

Hence given $\varepsilon > 0$ and satisfying $\alpha^{*} - \varepsilon > \delta > 0$, $\exists\ r_o < 1$ st if $r_o \leq r < 1$ then

$$\operatorname{Re} \frac{f'(r)}{f(r)} \geq \frac{\alpha^{*} - \varepsilon}{1 - r}$$

By the Cauchy-Riemann equations it is easy to show that

$$\operatorname{Re} \frac{rf'(r)}{f(r)} = r \frac{\partial}{\partial r} \log |f(r)|$$

Hence integrating from r_o to r,

$$\log |f(r)| - \log |f(r_o)| \geq (\alpha^{*} - \varepsilon) \log \frac{1}{1-r} - (\alpha^{*} - \varepsilon) \log \frac{1}{1-r_o}$$

and so

$$\lim_{r \to 1} \frac{\log |f(r)|}{\log \frac{1}{1-r}} \geq \alpha^{*} \tag{2.2.8}$$

In particular we note that

$$|f(r)| \to \infty \quad \text{as} \quad r \to 1 \tag{2.2.9}$$

Consider now the curve

$$C_o = \{ w/w = f(r), \quad 0 \leq r < 1 \}$$

Then C_o is a continuous curve in the w-plane lying in $\mathcal{G}$, and by (2.2.8) extending to infinity. Furthermore since

$f(z)$ is starlike C_o crosses each circle $\gamma = \gamma(M) = \{w/|w| = M\}$ for $0 < M < \infty$, once and only once. We show now that the points of C_o are in S' for $r_1 < r < 1$, some r_1. To see this we suppose the converse, so that $\exists$ M as large as we please such that the point P where C_o crosses $\gamma(M)$ does not lie in S'. If M is large enough $\gamma(M)$ contains points of the complement of $\mathcal{G}$ such that travelling round γ in either direction from P we meet these points before meeting S'. Since $P \in \mathcal{G}$ a neighbourhood U of P is in $\mathcal{G}$. P is the image of a pt Q on the positive real axis, and a neighbourhood V of Q is mapped by f onto U. V contains points $z_1 = \rho_1 e^{i\theta_1}$, $z_2 = \rho_2 e^{i\theta_2}$ where $\theta_1 > 0$, $\theta_2 < 0$ such that the curves $C_1 = \{w/w = f(re^{i\theta_1}),\ \rho_1 < r < 1\}$ and $C_2 = \{w/w = f(re^{i\theta_2}),\ \rho_2 < r < 1\}$ do not meet S', for like C_o the curves C_1, C_2 are continuous and cannot cross any circle $\gamma(M)$ more than once. Also as r increases the values M of the circles which they do cross increase. It follows that all curves $C = \{w/w = f(re^{i\theta})\}$ for $r > r_2$ say, and $\theta_1 \leq \theta \leq \theta_2$ have the same property and do not meet S'. This ensures that there is no sequence of points $z_n \to 1$ such that $w_n = f(z_n)$ lie in S' for large n, in contradiction to our original sequence on the positive real w-axis. This shows that for $r > r_1$ say the pts $w = f(r)$ are in S'.

It follows from (2.2.1) and the properties of $G(z)$ that

$$\overline{\lim_{r\to 1}} \ (1 - r) \left| \frac{rf'(r)}{f(r)} \right| \leq \alpha^{*} \qquad (2.2.10)$$

(2.2.7) and (2.2.10) together imply that

$$\lim_{r\to 1} \ (1 - r) \ \frac{rf'(r)}{f(r)} = \alpha^{*} \qquad (2.2.11)$$

Consider now a sequence of points $\{z_n\}$ in $|z| < 1$ tending to $z = 1$ as $n \to \infty$ in such a way that all the points lie inside a sector with vertex at $z = 1$, angle less than π so that the arms of the sector lie in $|z| < 1$. Such a sector will be called a Stolz sector, and the sequence $\{z_n\}$ will be said to converge to $z = 1$ (Stolz).

Theorem 2.2.B

If f contains a sector S of the form (2.1.2) and k is the number of theorem 2.2.A then

$$\text{(i)} \quad (1 - k z) \ \frac{zf'(z)}{f(z)} \to \alpha^{*} \text{ as } kz \to 1 \quad \text{(Stolz)}$$

and (ii) α^{*} is the largest number such that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \frac{\alpha^{*}}{2} \operatorname{Re} \frac{1 + kz}{1 - kz}$$

for all z in $|z| < 1$.

Proof

Assuming for convenience that $\mathcal{K} = 1$ and setting

$$\frac{zf'(z)}{f(z)} = \frac{1 + \omega(z)}{1 - \omega(z)}$$

we have by (2.2.11)

$$\lim_{r\to 1} \frac{1 - \omega(r)}{1 - r} = \frac{2}{\alpha^{\mathcal{K}}}$$

Hence applying theorem 3, Appendix 2 the result follows.

Corollary 2.2.C

Setting $F(z) = \frac{zf'(z)}{f(z)}$ we have

(i) $(1 - \mathcal{K} z) \frac{zF'(z)}{F(z)} \to 1$ as $\mathcal{K} z \to 1$ (Stolz)

(ii) $(1 - \mathcal{K} z) \frac{zf''(z)}{f'(z)} \to 1 + \alpha^{\mathcal{K}}$ as $\mathcal{K} z \to 1$ (Stolz)

(iii) $(1 - \mathcal{K} z)^2 \frac{z^2 f''(z)}{f(z)} \to \alpha^{\mathcal{K}}(1 + \alpha^{\mathcal{K}})$ as $\mathcal{K} z \to 1$ (Stolz)

Proof

Setting $\mathcal{K} = 1$, from theorem 3, Appendix 2,

$$\omega'(z) \to \frac{2}{\alpha^{\varkappa}} \quad \text{as} \quad z \to 1 \quad \text{(Stolz)}.$$

The results are immediate from this and the theorem 2.2.B.

Section 2.3

Taking $\varkappa = 1$ again we now show that

$$\arg f(r) \to 0 (\mathrm{mod}\ 2\pi) \quad \text{as} \quad r \to 1 . \qquad (2.3.1)$$

We actually show that the curve

$$C_o = \left\{ w/w = f(r) , \quad 0 \leq r < 1 \right\}$$

is such that the angle the ray to $w_r = f(r)$ makes with the positive real w-axis as $r \to 1$ converges to zero. (2.3.1) then follows since $|f(r)|$ is increasing with r and hence cannot leave S' for large enough $r < 1$. In other words at least for r near to 1 $C(r)$ cannot wind about the origin so that the limit (2.3.1) exists. By $\to 0 (\mathrm{mod}\ 2\pi)$ we mean $\to 2n\pi$ for some integer n.

If $G(z)$ is the superordinate function then we have

$$\arg f(r) = \arg G(\sigma(r))$$

and from (6) of Appendix 2,

$$\arg G(\sigma(r)) = \arg K + (\alpha^{\varkappa} + \varepsilon) \arg \frac{1}{1-\sigma(r)} + o(1) \quad \text{as} \quad r \to 1 . \qquad (2.3.2)$$

where $|\arg K| < \pi\varepsilon$. Also

$$(1 - r)\left|\frac{f'(r)}{f(r)}\right| \leq (\alpha^{*} + \xi)\,\frac{1 - |\sigma(r)|}{|1 - \sigma(r)|} + o(1)$$

also from (6), Appendix 2. Hence

$$\frac{\alpha^{*}}{\alpha^{*} + \xi} \leq \lim_{r\to 1}\frac{1 - |\sigma(r)|}{|1 - \sigma(r)|} \leq \overline{\lim_{r\to 1}}\,\frac{1 - |\sigma(r)|}{|1 - \sigma(r)|} \leq 1$$

It follows that $\arg \frac{1}{1 - \sigma(r)}$ is small with ξ, and so from (2.3.2) letting $\xi \to 0$ the result follows.

Now from (2.2.11) we have by integration,

$$\lim_{r\to 1}\frac{\log|f(r)|}{\log\frac{1}{1-r}} = \alpha^{*} \qquad (2.3.3)$$

We now show that

$$\frac{\log|f(z)|}{\log\frac{1}{1-|z|}} \to \alpha^{*} \text{ as } z \to 1 \text{ (Stolz)} \qquad (2.3.4)$$

By theorem 2.2.B we have

$$\frac{f'(z)}{f(z)} = \frac{\alpha^{*} + \xi(z)}{1 - z} \qquad (2.3.5)$$

where $\xi(z) \to 0$ uniformly as $z \to 1$ in some fixed Stolz sector. Set $z = re^{i\theta}$, $\zeta = re^{i\phi}$, and assume $\theta > 0$. Then

$$\log f(z) - \log f(r) = \alpha^{\varkappa} \log \frac{1}{1 - z} - \alpha^{\varkappa} \log \frac{1}{1 - r} + \int_0^{\theta} \frac{\xi(\varphi)d\varphi}{1 - \varphi} \tag{2.3.6}$$

Hence

$$\left| \frac{\log |f(z)|}{\log \frac{1}{|1-z|}} - \alpha^{\varkappa} \right| \leq \left| \frac{\log |f(r)|}{\log \frac{1}{|1-z|}} - \alpha^{\varkappa} \frac{\log \frac{1}{1-r}}{\log \frac{1}{|1-z|}} \right| + \frac{\theta \xi}{1 - r}$$

where ξ is small for r near to 1, and $\frac{\theta}{1-r}$ remains bounded as $z = re^{i\theta} \to 1$ since the convergence is Stolz.

Also $\dfrac{\log \frac{1}{1-r}}{\log \frac{1}{|1-z|}} \to 1$ as $z \to 1$ (Stolz). Hence (2.3.4) follows.

Also from (2.3.6) we obtain

$$\left| \arg f(z) - \alpha^{\varkappa} \arg \frac{1}{1 - z} \right| \leq \left| \arg f(r) \right| + \frac{\theta \xi}{1 - r}$$

and so

$$\arg f(z) = \alpha^{\varkappa} \arg \frac{1}{1-z} + o(1) \quad \text{as} \quad z \to 1 \quad \text{(Stolz)} \tag{2.3.7}$$

We also have from (2.3.6)

$$\left| \log \left| \frac{f(z)}{f(r)} \right| - \alpha^{\varkappa} \log \frac{1 - r}{|1 - z|} \right| \leq \frac{\xi \theta}{1 - r} \qquad (2.3.8)$$

and so

$$\log \left| \frac{f(z)}{f(r)} \right| = \alpha^{\varkappa} \log \frac{1 - r}{|1 - z|} + o(1) \quad \text{as} \quad z \to 1 \quad \text{(Stolz)} \qquad (2.3.9)$$

$$\text{i.e.} \quad |f(z)| = \left(\frac{1 - r}{|1 - z|} \right)^{\alpha^{\varkappa}} \cdot |f(r)| \cdot (1 + o(1)) \qquad (2.3.10)$$

as $z \to 1$ (Stolz).

For later reference we collect up these results in

Theorem 2.3.A

If f has a sector S of the form (2.1.2) and κ is the number of theorem 2.2.A then as $\kappa z \to 1$ inside a fixed Stolz sector we have

$$\text{(i)} \quad \frac{\log |f(z)|}{\log \frac{1}{1 - |z|}} \to \alpha^{\varkappa}$$

$$\text{(ii)} \quad \arg f(z) = \alpha^{\varkappa} \arg \frac{1}{1 - \kappa z} + o(1)$$

$$\text{(iii)} \quad |f(z)| = \left(\frac{1 - r}{|1 - \kappa z|} \right)^{\alpha^{\varkappa}} |f(\kappa^{-1} r)| \ (1 + o(1)) .$$

Lemma 2.3.B

Let $\{ z_n \}$ be a sequence of points such that

(i) $\quad |f(z_n)| \to \infty$ as $n \to \infty$

(ii) $\quad |\arg f(z_n)| \to \frac{\pi\alpha^*}{2}$ as $n \to \infty$

(iii) $\quad w_n = f(z_n) \in S'$ for $n > N$, where N depends on $S' = S'(\epsilon)$.

Then $(1 - |z_n|) \frac{z_n f'(z_n)}{f(z_n)} \to 0$ as $n \to \infty$.

Proof

The result follows immediately from (2.2.1) and (8) of Appendix 2.

Section 2.4

Lemma 2.4.A

Let $e^{i\delta_1}$, $e^{i\delta_2}$ be distinct points on $|z| = 1$, and suppose $-\pi \leq \delta_1 < \delta_2 < \pi$. Then $\exists\ \delta$ satisfying $\delta_1 \leq \delta \leq \delta_2$ such that

$$\overline{\lim_{r \to 1}} |f(re^{i\delta})| < \infty .$$

Proof

Suppose the lemma is false. Then $\forall\ \theta$ in $[\delta_1 , \delta_2]$ we have

$$\lim_{r\to 1} \left| f(re^{i\delta}) \right| = \infty .$$

Let T_η be the truncated sector

$$T_\eta = \left\{ z/\delta_1 < \arg z < \delta_2 , \ \eta < |z| < 1 \right\}$$

Then

$$g(z) = \frac{1}{f(z)}$$

is regular and univalent in T_η for each η satisfying $0 \leq \eta < 1$. Let U_η be the image of T_η under g. Let $\{ z_n \}$ be a sequence of points in T_o such that

$$|z_n| \to 1 \quad \text{as} \quad n \to \infty .$$

Then for each η, $0 \leq \eta < 1$, $\exists\ N = N(\eta)$ st $z_n \in T_\eta$ for all $n > N$. It follows that for $n > N(\eta)$ the points $w_n = g(z_n)$ are in U_η. The domains T_η are decreasing as η increases, and since g is univalent the same property holds for the U_η. For each θ, $\delta_1 \leq \theta \leq \delta_2$ we have

$$g(re^{i\theta}) \to 0 \quad \text{as} \quad r \to 1 .$$

For given $\eta > 0$ the circular arc $\{ z/|z| = \eta , \ \delta_1 \leq \arg z \leq \delta_2 \}$

is mapped by $g(z)$ onto a continuous Jordan curve in the w-plane. From each end pt the rays $\arg z = \delta_1$, $\arg z = \delta_2$ drawn to $|z| = 1$ are mapped by $g(z)$ onto two curves in the w-plane with a common end pt at the origin. Neither curve in the w-plane meets elsewhere nor meets again the image of the above circular arc. These three curves in the w-plane thus form a closed Jordan curve, and form the boundary of U_η. It is thus clear that $w = 0$ is the only common boundary pt of all the U_η. Since w_n must tend to the boundary of U_η for each η it follows that

$$g(z_n) = w_n \to 0 \quad \text{as} \quad n \to \infty .$$

Thus for $\eta > 0$, $g(z)$ is regular in T_η, continuous on the boundary of T_η, and zero on $\{ |z| = 1 \} \cap \overline{T}_\eta$. Let

$$h(z) = \begin{cases} g(z) , & z \in T_0 \\ 0 , & z \in T_1 \end{cases}$$

where $T_1 = \{ z/\delta_1 < \arg z < \delta_2 , \ |z| \geq 1 \}$. Then if γ is any closed, rectifiable curve in $T_0 \cup T_1$ our above results yield

$$\int_\gamma h(z)\, dz = 0$$

Hence $h(z)$ is regular in $T_0 \cup T_1$ and hence identically zero. Hence $g(z) \equiv 0$, a contradiction.

Lemma 2.4.B

Suppose $\mathcal{G}$ contains a sector S of the form (2.1.2). Let K be the number of theorem 2.2.A. Let S' be a sector of $\mathcal{G}'$, the superordinate domain, where $S' > S$. Let $\{z_n\}$ be a sequence of points in $|z| < 1$ such that

$$K z_n \to 1 \quad \text{as} \quad n \to \infty$$

Then the sequence $\{w_n\} = \{f(z_n)\}$ satisfies

$$w_n \in S' \quad \text{for} \quad n > n_o(S') \tag{2.4.1}$$

$$|w_n| \to \infty \quad \text{as} \quad n \to \infty \tag{2.4.2}$$

Conversely if (2.4.1) and (2.4.2) are satisfied for every possible $S' > S$, over the set of all superordinate domains, then $K z_n \to 1$ as $n \to \infty$.

Proof

Take $K = 1$. Suppose first that $z_n \to 1$ as $n \to \infty$. Let $z_n = r_n e^{i\theta_n}$ so that $r_n \to 1$, $\theta_n \to 0$ as $n \to \infty$. We first prove (2.4.2). Suppose $\exists$ a subsequence Λ of integers n such that $|w_n| < W$, $n \in \Lambda$. Since $|f(r)| \to \infty$ as $r \to 1$ $\exists$ N st $\forall n \geq N$, $|f(r_n)| > 2W$. Hence by continuity $\exists$ $\delta > 0$ st $|f(re^{i\theta})| > W$ for all $r > r_N$, $-\delta < \theta < \delta$, since $|f(re^{i\theta})|$ increases with r

for each θ. But $\exists$ $n > N$ with $n \in \Lambda$ and θ_n satisfying $-\delta < \theta_n < \delta$, and $r_n > r_N$ such that $|f(r_n e^{i\theta_n})| < W$, a contradiction, which proves (2.4.2). To prove (2.4.1) suppose $\exists$ a subsequence Λ of integers n st for $n \in \Lambda$, $w_n \notin S'$. Consider the circles $\gamma = \gamma(M) = \{w/|w| = M\}$. Choose M so large that $\gamma(M)$ meets the arms of S' at points of the complement of $\mathcal{G}$. Since $|f(r)| \to \infty$ and $\arg f(r) \to 0$ as $r \to 1$ it follows that $\exists$ $N \in \Lambda$ st $|f(r_N)| > 2M$ and $w_{r_N} = f(r_N) \in S < S'$. Then by continuity and the fact that $|f(re^{i\theta})|$ increases with r for each θ, it follows that $\exists$ $\delta > 0$ st the continuous curves

$$C(\theta) = \{w/w = f(re^{i\theta}), \quad r_N \leq r < 1\}$$

lie in S', for $-\delta \leq \theta \leq \delta$. Now choose $n \in \Lambda$ and $n > N$ such that $|\theta_n| < \delta$ and $r_n > r_N$. Then $w_n \in S'$, a contradiction. This proves (2.4.2).

Next suppose that (2.4.1) and (2.4.2) are satisfied for every possible $S' > S$, where $S' < \mathcal{G}'$. (2.4.2) implies that $|z_n| \to 1$ as $n \to \infty$. If z_n does not converge to 1 $\exists$ a subsequence, which we suppose without loss of generality is the actual sequence, such that $z_n \to e^{i\gamma}$ as $n \to \infty$, $\gamma \neq 0$. Choose $\delta > 0$ such that $|\gamma| > 2\delta$. By lemma 2.4.A $\exists$ $\delta_1 > 0$ st $-\delta < -\delta_1 < 0$ and $|f(re^{-i\delta_1})| < K_1$, $0 \leq r < 1$. Similarly $\exists$ $\delta_2 > 0$ st $0 < \delta_2 < \delta$ and

$|f(re^{i\delta_2})| < K_2$, $0 \leq r < 1$. Let

$$C(\theta) = \{ w/w = f(re^{i\theta}) , \quad 0 \leq r < 1 \} \tag{2.4.3}$$

Let $z_n = r_n e^{i\theta_n}$, $n = 1, 2, \ldots$. Then $\exists$ N st $\forall$ n > N

$$|\theta_n| > \delta$$

Let

$$T(\phi_1, \phi_2) = \{ z/ + \phi_1 < \arg z < \phi_2 , \quad 0 < |z| < 1 \}$$

and let $U(\phi_1, \phi_2)$ be the image under $f(z)$ of $T(\phi_1, \phi_2)$. Then for $n > N$ the points z_n are not in $T(-\delta, \delta)$ and so not in $T(-\delta_1, \delta_2)$. Hence the points w_n for $n > N$, $w_n = f(z_n)$, are not in $U(-\delta_1, \delta_2)$. The arms of $U(-\delta_1, \delta_2)$ are $C(-\delta_1)$ and $C(\delta_2)$, and are bounded and cross the circles $\gamma = \gamma(M) = \{ w/|w| = M \}$ not more than once. Hence $C(-\delta_1)$ possesses a limit point W_1 which is a boundary pt of $\mathcal{G}$ and so not in S . Similarly $C(\delta_2)$ possesses a limit pt W_2 which is a boundary pt of $\mathcal{G}$, so not in S . We can construct radial slits to infinity from W_1 and W_2 and obtain a superordinate domain $\mathcal{G}' > \mathcal{G}$ with a sector $S' > S$. Suppose w is a point of $\mathcal{G} \cap S'$ st $|w| > \max(|W_1|, |W_2|)$. We show that $w \notin U(-\delta_1, \delta_2)$.

We consider two cases. (i) W_1 and W_2 do not lie on the same ray through the origin, i.e. arg $W_1 \neq$ arg W_2 .

Then the superordinate domain $\mathcal{G}'$ contains two sectors whose union is the plane cut along the rays through W_1 and W_2 from the origin. One of these sectors is $S' > S$, and $w \in S'$. Also $w \in \mathcal{G}$, hence $w = f(z) = f(re^{i\theta})$ for some pt $z = re^{i\theta}$, $|z| < 1$. Consider the curve $C(\theta)$ given by (2.4.3). Since $|f(re^{i\theta})|$ increases with r, $0 \leq r < 1$, and $|w| > \max(|W_1|, |W_2|)$, clearly $\theta \neq -\delta_1$ or δ_2. Hence either $-\delta_1 < \theta < \delta_2$, when $z \in T(-\delta_1, \delta_2)$ so $w \in U(-\delta_1, \delta_2)$, or either $\theta < -\delta_1$ or $\theta > \delta_2$. In the latter cases $C(\theta)$ can never enter that part of S' beyond the circle $|w| = \max(|W_1|, |W_2|)$, since $C(\theta)$ starts at the origin outside $U(-\delta_1, \delta_2)$ and can only enter the above-named part of S' by crossing $C(-\delta_1)$, $C(\delta_2)$ or the boundary of the sector S', and all three possibilities are clearly impossible. (ii) W_1 and W_2 lie on the same ray, i.e. $\arg W_1 = \arg W_2$. In this case the curves $C(-\delta_1)$, $C(\delta_2)$ are the boundary together with part of the ray joining W_1 to W_2 of the interior of the complement of $U(-\delta_1, \delta_2)$. This latter domain is bounded by $\max(|W_1|, |W_2|)$ so $w \notin U(-\delta_1, \delta_2)$. The "sector" S' in case (ii) is the plane cut along one radial slit only.

Thus for $n > N$, $w_n \notin S'$ unless $|w_n| \leq \max(|W_1|, |W_2|)$. Since $|w_n| \to \infty$ as $n \to \infty$, it follows that $w_n \notin S'$ for $n > N_1$, some N_1. This contradicts our hypothesis, and proves the lemma.

Section 2.5

Theorem 2

For each real θ ,

$$V(\theta) = \lim_{r \to 1} \arg f(re^{i\theta}) \tag{2.5.1}$$

exists and is a monotonic increasing function such that $V(\theta) - \theta$ is periodic with period 2π . $V(\theta)$ is continuous except possibly at a countable number of points. If $V(\theta)$ has a jump $\pi\alpha^{\varkappa}$ at $\theta = \theta_o$ then the image domain $\mathcal{G}$ of $f(z)$ possesses a sector of angle $\pi\alpha^{\varkappa}$, and the number $\mathcal{K}$ of theorem 2.2.A is given by

$$\mathcal{K} = e^{-i\theta_o} \tag{2.5.2}$$

Conversely such a sector in $\mathcal{G}$ implies $V(\theta)$ has a jump corresponding to the angle of the sector and occurring at $\theta = \theta_o$ where $\mathcal{K} = e^{-i\theta_o}$. Finally at each θ , $V(\theta)$ satisfies

$$V(\theta) = \frac{1}{2} (V(\theta + 0) + V(\theta - 0)) . \tag{2.5.3}$$

Proof

We first prove (2.5.1) and consider two cases.

(i) Suppose $|f(re^{i\theta})|$ is bounded as $r \to 1$. Since

$f(z)$ is starlike $|f(re^{i\theta})|$ is increasing with r and so tends to a limit M as $r \to 1$. Suppose $\lim_{r\to 1} \arg f(re^{i\theta})$ does not exist. Then $\exists$ two distinct limit pts w_1, w_2 on $|w| = M$ of the curve $C(\theta)$ given by (2.4.3). Since $C(\theta)$ is continuous and crosses each circle $|w| = m$, $m < M$, once and only once, it follows that $\exists$ an arc γ of $|w| = M$ with end pts w_1, w_2 st every pt on γ is a limit pt of $C(\theta)$. Assume

$$\gamma = \left\{ w/\arg w_1 < \arg w < \arg w_2 , \quad |w| = M \right\}$$

and let

$$\mathcal{G}_o = \left\{ w/\arg w_1 < \arg w < \arg w_2 , \quad 0 < |w| < M \right\}$$

Let $g(w)$ be the inverse function of $f(z)$. Then $g(w)$ is regular and univalent in $\mathcal{G}_o$. Let $\{w_n\}$ be a sequence of pts in $\mathcal{G}_o$ st $|w_n| \to M$ as $n \to \infty$. The image pts $z_n = g(w_n)$ lie in $|z| < 1$, and have no pt of accumulation in $|z| < 1$. Hence $|z_n| \to 1$ as $n \to \infty$. Thus the limit pts of $g(w)$ as w approaches the analytic arc γ all lie on the analytic curve $|z| = 1$. Hence by the reflection principle ([1] th. 10 page 193) $g(w)$ can be continued across γ and is regular in $\mathcal{G}_o \cup \gamma$, so by continuity $g(w) = e^{i\theta}$ on γ, so $g(w) \equiv e^{i\theta}$, $w \in \mathcal{G}_o$ which is absurd. Hence (2.5.1) follows in this case.

(ii) Suppose $|f(re^{i\theta})| \to \infty$ as $r \to 1$. Since f is starlike $\exists$ a boundary pt W of $\mathcal{G}$, $(W \neq 0, \infty)$, st every pt on the radial slit ℓ from W to infinity is in the complement of $\mathcal{G}$. Also $\exists$ $r_0 < 1$ st $\forall r$ satisfying $r_0 \leq r < 1$,

$$|f(re^{i\theta})| > |W|$$

Let

$$C(\theta) = \{ w/w = f(re^{i\theta}), \quad r_0 \leq r < 1 \}$$

Because of the slit ℓ, $C(\theta)$ cannot wind round the origin so that if $w \notin C(\theta)$ then

$$|\arg w - \arg W| \leq \pi$$

Suppose $\lim_{r\to 1} \arg f(re^{i\theta})$ does not exist. Then $\exists$ δ_1, δ_2 with $\delta_1 < \delta_2$, and two sequences $\{w_n\}$, $\{w'_n\}$ of $C(\theta)$ st

$$\left.\begin{array}{ll} \arg w_n \to \delta_1 & \arg w'_n \to \delta_2 \\ |w_n| \to \infty & |w'_n| \to \infty \end{array}\right\} \text{ as } n \to \infty$$

and $|\delta_1 - \arg W| \leq \pi$, $|\delta_2 - \arg W| \leq \pi$. We may suppose that every pt of the interval (δ_1, δ_2) is a limit pt of the argument of some sequence $\to \infty$ on $C(\theta)$, for there is certainly some such interval. Consider the sector

$$S = \{ w/\delta_1 < \arg w < \delta_2 \}$$

Then $S < \mathcal{G}$. For if a pt ζ in S is not in $\mathcal{G}$ then every pt on the radial slit from ζ to infinity is also not in $\mathcal{G}$, which is impossible since $C(\theta)$ crosses this slit. Let S_o be the largest sector contained in $\mathcal{G}$ and containing S. Suppose S_o has angle $\pi\alpha^*$. $\exists$ $r_1 < 1$ st for $r > r_1$, $w = f(re^{i\theta})$ is in S_o. Hence by theorem 2.3.A and lemma 2.4.B

$$\lim_{r\to 1} \arg f(re^{i\theta}) = \arg (\text{bisector of } S_o).$$

i.e. the limit exists, a contradiction. Hence (2.5.1) follows. Thus $V(\theta)$ exists and is monotonic increasing with θ since $\arg f(re^{i\theta})$ is increasing with θ for each fixed r, $0 < r < 1$. Also for each r, $0 < r < 1$, $\arg f(re^{i\theta}) - \theta$ is periodic in θ with period 2π. Hence $V(\theta) - \theta$ is periodic with period 2π.

Before completing the proof of the theorem we prove

Lemma 2.5.A

Let ζ be an accessible boundary point of $\mathcal{G}$, i.e. a finite point from which we can construct a line segment lying in $\mathcal{G}$, except for ζ itself. Then $\exists$ θ_o, $-\pi \leq \theta_o < \pi$, such that

$$\zeta = \lim_{r\to 1} f(re^{i\theta_o})$$

Proof

There is a line segment from ζ into $\mathcal{G}$, and from the end pt in $\mathcal{G}$ of this segment we can construct the line segment to the origin and this also is in $\mathcal{G}$. This curve consisting of two line segments joining 0 to ζ we denote by C_w. Let C_z be the inverse image curve under f of C_w, so C_z lies in $|z| < 1$. As $w \to \zeta$ along C_w, $|z| \to 1$ along C_z. Hence given $\varepsilon > 0$ $\exists$ $\delta > 0$ st $|w - \zeta| < \delta$ and $w \in C_w \Rightarrow 1 > |z| > 1-\varepsilon$ and $z \in C_z$. Since C_z is continuous this property ensures that if there are two distinct limit pts of C_z on $|z| = 1$, then $\exists$ an arc Γ of $|z| = 1$ the whole of which consists of limit pts of C_z. For each $e^{i\theta} \in \Gamma$ $\exists$ a sequence $\{z_n\} = \{r_n e^{i\theta}\}$ of C_z st $r_n \to 1$ as $n \to \infty$. Hence

$$\zeta = \lim_{r \to 1} f(re^{i\theta}), \quad e^{i\theta} \in \Gamma.$$

Consider the domain

$$D_o = \{ z / e^{i \arg z} \in \Gamma, \quad 0 < |z| < 1 \}$$

is mapped by f onto a domain in $\mathcal{G}$, and $f(z) \to \zeta$ as $|z| \to 1$ in D_o. By the reflection principle we easily deduce $f(z) \equiv \zeta$, which is absurd. Hence C_z has a single limit pt on $|z| = 1$, say $z = 1$.

Suppose $\exists\ r_o < 1$ st $|f(r_o)| > |\zeta|$. Then $\exists\ \delta > 0$ st $|f(r_o)| > |\zeta| + 2\delta$. Hence $\exists\ \gamma > 0$ st

$$|f(r_o e^{i\theta})| > |\zeta| + \delta$$

for all θ st $|\theta| < \gamma$. Hence

$$|f(re^{i\theta})| > |\zeta| + \delta$$

for all $1 > r \geq r_o$, $|\theta| < \gamma$. Hence there cannot be a sequence of pts $z_n \to 1$ st $f(z_n) \to \zeta$ as $n \to \infty$, i.e. C_z does not have $z = 1$ as a limit pt, a contradiction. Hence $\lim_{r\to 1} |f(r)|$ is finite and not greater than $|\zeta|$. Hence

$$\zeta_o = \lim_{r\to 1} f(r)$$

exists, and $|\zeta_o| \leq |\zeta|$. To prove the lemma we show that $\zeta_o = \zeta$. Let

$$C_o = \{w/w = f(r),\ 0 \leq r < 1\}$$

and $$C_\theta = \{w/w = f(re^{i\theta}),\ 0 \leq r < 1\}$$

Suppose $\zeta_o \neq \zeta$. Then on the curves C_o, C_w travelling from $w = 0$ there is a last pt w_o of intersection of the two curves. Let $f(z_o) = w_o$. Then C_z, the positive real z-axis and z_o, the last pt of intersection of C_z and the axis, define a simply connected domain D_o in which

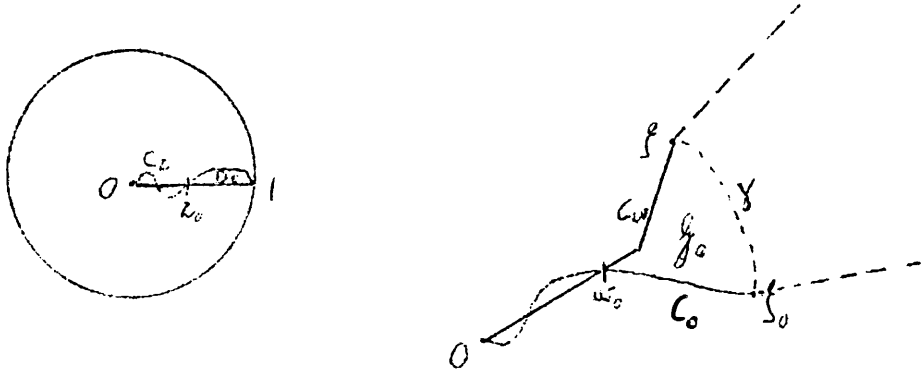


Figure 8

$f(z)$ is univalent and maps D_0 onto $\mathcal{G}_0$. Let $g(w)$ be the inverse function mapping $\mathcal{G}_0$ onto D_0. Let γ be the boundary of $\mathcal{G}_0$ excluding C_w and C_0. Then γ is bounded and a Jordan curve joining ζ to ζ_0. To see this observe that as $w \to \gamma$ in $\mathcal{G}_0$, $z = g(w) \to 1$ in D_0. If $w_1 \in \mathcal{G}_0$ then

$$|w_1| \leq \max_{w \in C_w} |w|$$

for $w_1 = f(r_1 e^{i\theta_1})$, $\theta_1 \neq 0$, the curve $C(\theta_1)$ must cross C_w for some $r > r_1$ and $|f(re^{i\theta_1})|$ increases with r. Thus γ is bounded, and is a Jordan curve (Jordan, since every pt on γ is a bdy pt of the starlike domain $\mathcal{G}$). Let $w_2 \in \mathcal{G}_0$ be a pt st the disc $\Delta : |w - w_2| < \rho$ contains pts of γ. Clearly $\exists\, w_2 \in \mathcal{G}_0$ and $\rho > 0$ st this is possible. Then the circumference of Δ contains points not in $\mathcal{G}_0$ and not on γ, and so a whole arc is so situated. So there is an angle $\alpha > \frac{2\pi}{n}$, for a positive integer n, subtended by this arc at w_2. Let us rotate the domain $\mathcal{G}_0$ about w_2

through successive angles $\frac{2\pi}{n}, \frac{4\pi}{n}, \ldots \frac{2(n-1)\pi}{n}$. We obtain $n-1$ new domains $\mathcal{G}_1, \ldots \mathcal{G}_{n-1}$. The intersection $\mathcal{G}_0 \cap \mathcal{G}_1 \cap \ldots \cap \mathcal{G}_{n-1}$ is an open set containing w_2 and containing no pt of the circle $|w - w_2| = \rho$. Consider the domain $\mathcal{G}'$ defined as the domain containing w_2 which is contained in the intersection $\mathcal{G}_0 \cap \mathcal{G}_1 \cap \ldots \mathcal{G}_{n-1}$, and whose boundary points are obtained by considering each ray from w_2 and taking as the bdy pt of $\mathcal{G}'$ on this ray the first boundary pt of any of the $\mathcal{G}_k$, $k = 0, 1, \ldots n-1$. Then the boundary Γ of $\mathcal{G}'$ lies in the disc $|w - w_2| < \rho$. Consider

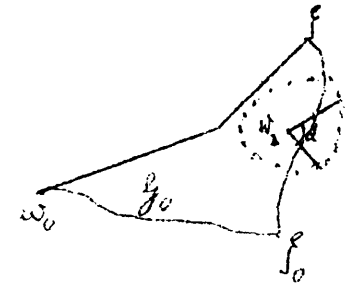


Figure 9

$$h(w) = g(w) - 1$$

Then $$|h(w)| < 2, \quad w \in \mathcal{G}_0.$$

Consider the functions

$$h_k(w) = h(w_2 + e^{\frac{2\pi i k}{n}}(w - w_2)), \quad k = 0, 1, \ldots n-1.$$

Then $h_k(w)$ is regular in $\mathcal{G}_k$, $k = 0, 1, \ldots n-1$.

Hence

$$H(w) = \prod_{k=0}^{n-1} h_k(w)$$

is regular in $\mathcal{G}'$. Clearly

$$|h_k(w)| < 2 , \quad k = 0, 1, \ldots n-1 , \quad w \in \mathcal{G}_k .$$

Let $\{w_n\}$ be a sequence of pts of $\mathcal{G}'$ tending to W , $W \in \Gamma$. Then for one of the functions $h_k(w)$,

$$h_k(w_n) \to 0 \quad \text{as} \quad n \to \infty$$
$$\therefore \quad H(w_n) \to 0 \quad \text{as} \quad n \to \infty$$

Since this is true for every bdy pt $W \in \Gamma$, by the maximum principle

$$H(w_2) = 0$$

But $H(w_2) = h(w_2)^n$, hence $g(w_2) = 1$, a contradiction. Hence $\zeta_0 = \zeta$ and the lemma is proved.

The above argument involving the rotation of $\mathcal{G}_0$ is due to Lindelöf (see [2] page 81).

We now complete the proof of theorem 2. Suppose first $V(\theta)$ has a jump $\pi\alpha^*$ at $\theta = \theta_0$. Since $V(\theta)$ is monotonic the limits $V(\theta_0 + 0)$, $V(\theta_0 - 0)$ exist. We may suppose

$$V(\theta_0 + 0) = \frac{\pi\alpha^*}{2} , \quad V(\theta_0 - 0) = -\frac{\pi\alpha^*}{2} \qquad (2.5.4)$$

Let

$$S = \left\{ w / |\arg w| < \frac{\pi\alpha^*}{2} \right\} \qquad (2.5.5)$$

Suppose S is not wholly contained in $\mathcal{G}$. Then $\exists\ \zeta \in S$ an accessible boundary pt of $\mathcal{G}$. Hence $\exists\ \phi$ st

$$\zeta = \lim_{r\to 1} f(re^{i\phi})$$

and $\arg \zeta = V(\phi)$, hence $\phi = \theta_0$. Now there must be another accessible boundary pt of $\mathcal{G}$ in S. For consider any ray from $w = 0$ in S. Either it is clear of boundary pts or not. If not then the first boundary pt is accessible. If every ray except the ray on which ζ lies is clear of boundary pts, then the ray on which ζ lies consists from ζ onwards of accessible boundary pts. However if ζ_1 is another such bdy pt then we obtain as before

$$\zeta_1 = \lim_{r\to 1} f(re^{i\theta_0}) = \zeta$$

a contradiction. Hence $S < \mathcal{G}$. Now by (2.5.4) and lemma 2.4.A, given $\varepsilon > 0\ \exists\ \delta > 0$ st

$$\frac{\pi\alpha^{*}}{2} < V(\theta_0 + \delta) < \frac{\pi\alpha^{*}}{2} + \varepsilon$$

and

$$\lim_{r\to 1} \left| f(re^{i(\theta_0+\delta)} \right| < \infty$$

since $V(\theta)$ is monotonic increasing. And then

$$\zeta = \lim_{r\to 1} f(re^{i(\theta_0+\delta)}$$

is a finite boundary pt of $\mathcal{G}$, so that $\mathcal{G}$ contains no

sector which properly contains S. By theorem 2.3.A the number $\mathcal{K}$ is clearly $e^{-i\theta_o}$, and $V(\theta_o) = 0$.

If $\mathcal{G}$ contains a sector S, say given by (2.5.5), of exactly angle $\pi\alpha^{\mathcal{K}}$, and we set $\mathcal{K} = e^{-i\theta_o}$, then by theorem 2.3.A

$$V(\theta_o) = 0$$

If $V(\delta_o + \delta)$, for some $\delta > 0$, is less than $\frac{\pi\alpha^{\mathcal{K}}}{2}$ then by lemma 2.4.A we may suppose

$$\zeta = \lim_{r\to 1} f(re^{i(\theta_o+\delta)} \text{ is finite}$$

and then $\zeta \in S$, a contradiction. Hence

$$V(\theta_o + 0) \geq \frac{\pi\alpha^{\mathcal{K}}}{2}$$

Similarly $$V(\theta_o - 0) \leq -\frac{\pi\alpha^{\mathcal{K}}}{2}$$

We thus obtain (2.5.4) since otherwise, by our previous argument, $\mathcal{G}$ would contain a sector larger than $\pi\alpha^{\mathcal{K}}$ containing S. Thus $V(\theta)$ has a jump $\pi\alpha^{\mathcal{K}}$ at θ_o.

This completes theorem 2.

We remark here that we can deduce the well-known representation (1.1.3) for $\frac{zf'(z)}{f(z)}$ by means of theorem 2. We here briefly indicate one possible proof. With $V(\theta)$ defined by (2.5.1) consider

$$F_o(z) = \frac{1}{2\pi} \int_o^{2\pi} \frac{1 + e^{-it} z}{1 - e^{-it} z} \, dV(t)$$

Then clearly $\operatorname{Re} F_o(z) > 0$ in $|z| < 1$ and $F_o(z)$ is regular in $|z| < 1$. Also we easily see that $F_o(0) = 1$. Define $f_o(z)$ by the equation

$$\frac{zf_o'(z)}{f_o(z)} = F_o(z), \quad f_o'(0) = 1$$

and we also assume $f'(o) = 1$. Our aim is to show that $f_o(z) \equiv f(z)$. We refer to [9], page 210 where Pommerenke shows that

$$\lim_{r\to 1} \arg f_o(re^{i\theta}) = V(\theta)$$

Assuming this result we have

$$\lim_{r\to 1} \left\{ \arg \frac{f_o(re^{i\theta})}{re^{i\theta}} - \arg \frac{f(re^{i\theta})}{re^{i\theta}} \right\} = 0$$

for each θ. Since the function inside the curly brackets is harmonic in the unit disc it follows that it is identically zero.

Thus

$$\log \frac{f_o(z)}{z} \equiv \log \frac{f(z)}{z} + \text{const}$$

and since $f_o'(o) = 1 = f'(o)$ we deduce

$$f_o(z) \equiv f(z) \quad \text{as required.}$$

Section 2.6

In this section we continue with the methods we have developed, and we obtain a representation for $f(z)$ equivalent to that of (1.1.3), and also prove another necessary and sufficient condition for $\mathcal{F}$ to contain a sector of angle $\pi\alpha^{*}$.

Theorem 2.6.A

Suppose $\mathcal{F}$ contains sectors S_1, S_2, ... of angles $\pi\lambda_1$, $\pi\lambda_2$, ... respectively. Then

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \frac{1}{2} \sum_{\nu} \lambda_\nu \operatorname{Re} \frac{1 + \kappa_\nu z}{1 - \kappa_\nu z} \qquad (|z| < 1)$$

where the κ_ν are distinct quantities satisfying $|\kappa_\nu| = 1$. In fact for each ν , κ_ν is the number of theorem 2.2.A for the sector of angle $\pi\lambda_\nu$.

Proof

The numbers λ_ν are all positive and $\sum_{\nu} \lambda_\nu \leq 2$. By theorem 2.2.A for each sector S_ν we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \frac{\lambda_\nu}{2} \operatorname{Re} \frac{1 + \kappa_\nu z}{1 - \kappa_\nu z} \qquad (|z| < 1).$$

where $|\kappa_\nu| = 1$. The numbers κ_ν are distinct, for by theorem 2.3.A

$$\arg f(\kappa_\nu^{-1} r) \to \text{bisecting angle of } S_\nu \text{ as } r \to 1$$

and the sectors S_ν being disjoint these limits are all different.

Consider a finite number $N > 1$ of sectors S_ν, $\nu = 1, \ldots N$. Suppose we have proved

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \sum_{\nu=1}^{N-1} \frac{\lambda_\nu}{2} \operatorname{Re} \frac{1 + \kappa_\nu z}{1 - \kappa_\nu z} \qquad (|z| < 1).$$

Let

$$G(z) = \frac{zf'(z)}{f(z)} - \sum_{\nu=1}^{N-1} \frac{\lambda_\nu}{2} \frac{1 + \kappa_\nu z}{1 - \kappa_\nu z} \qquad (|z| < 1).$$

so that $\operatorname{Re} G(z) \geq 0$ in $|z| < 1$. If for some z in $|z| < 1$, $\operatorname{Re} G(z) = 0$, then by the maximum principle

$$\operatorname{Re} G(o) = 0$$

$$\text{i.e.} \quad \sum_{\nu=1}^{N-1} \lambda_\nu = 2$$

and so $\lambda_N = 0$, a contradiction. Hence $\operatorname{Re} G(z) > 0$ in $|z| < 1$. So we may write

$$G(z) = \frac{1 + \omega(z)}{1 - \omega(z)} \qquad (|z| < 1)$$

where $\omega(z)$ is regular, $|\omega(z)| < 1$ in $|z| < 1$. Consider the sector S_N. By theorem 2.3.A we have

$$\lim_{r\to 1} (1 - r) \frac{\kappa_N^{-1} r f'(\kappa_N^{-1} r)}{f(\kappa_N^{-1} r)} = \lambda_N$$

hence

$$\lim_{r\to 1} (1 - r) G(\kappa_N^{-1} r) = \lambda_N$$

since $\kappa_N \neq \kappa_\nu$, $\nu = 1, \ldots N-1$. Thus

$$\lim_{r\to 1} (1 - r) \frac{1 + \omega(\kappa_N^{-1} r)}{1 - \omega(\kappa_N^{-1} r)} = \lambda_N$$

and so $\omega(\kappa_N^{-1} r) \to 1$ as $r \to 1$. Hence

$$\lim_{r\to 1} \frac{1 - \omega(\kappa_N^{-1} r)}{1 - r} = \frac{2}{\lambda_N}$$

and since λ_N is real

$$\lim_{r\to 1} \frac{1 - |\omega(\kappa_N^{-1} r)|}{1 - r} = \frac{2}{\lambda_N}$$

Hence by Julia's theorem

$$\frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} \leq \frac{2}{\lambda_N} \frac{|1 - \kappa_N z|^2}{1 - |z|^2} \qquad (|z| < 1)$$

and λ_N is the largest number st this holds for every z in $|z| < 1$. Thus we obtain

$$\operatorname{Re} G(z) \geq \frac{\lambda_N}{2} \operatorname{Re} \frac{1 + \kappa_N z}{1 - \kappa_N z}$$

$$\text{i.e.} \quad \operatorname{Re} \frac{zf'(z)}{f(z)} \geq \sum_{\nu=1}^{N} \frac{\lambda_N}{2} \operatorname{Re} \frac{1 + \kappa_\nu z}{1 - \kappa_\nu z} \qquad (|z| < 1).$$

It follows by induction that for any finite set Λ of positive integers ν,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \sum_{\nu \in \Lambda} \frac{\lambda_\nu}{2} \operatorname{Re} \frac{1 + \kappa_\nu z}{1 - \kappa_\nu z} \qquad (|z| < 1). \quad (2.6.1)$$

If Λ is an infinite such set then it is countable and for every finite subset (2.6.1) holds. Each term on the right is positive so the infinite sum exists and is bounded above by $\operatorname{Re} \frac{zf'(z)}{f(z)}$. This proves the theorem.

Corollary 2.6.B

Let Λ be any set of positive integers ν corresponding to sectors S_ν of $\mathcal{F}$. Then the function

$$h(z) = f(z) \prod_{\nu \in \Lambda} (1 - \kappa_\nu z)^{\lambda_\nu} \qquad (2.6.2)$$

is starlike in $|z| < 1$.

Proof

We easily obtain

$$\operatorname{Re}\frac{zh'(z)}{h(z)} = \operatorname{Re}\frac{zf'(z)}{f(z)} - \sum_{\nu\in\Lambda}\frac{\lambda_\nu}{2}\operatorname{Re}\frac{1+\kappa_\nu z}{1-\kappa_\nu z}$$

$$+ \sum_{\nu\in\Lambda}\frac{\lambda_\nu}{2} \qquad (|z|<1)$$

$$\geq \sum_{\nu\in\Lambda}\frac{\lambda_\nu}{2} > 0 \qquad (2.6.3)$$

Also $h(o) = 0$, $h'(o) = f'(o) \neq 0$.

Theorem 2.6.C

The following two conditions are necessary and sufficient for f to contain a sector S of exactly angle $\pi\alpha^*$.

(i) $\exists\,\kappa$, $|\kappa| = 1$, such that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{\alpha^*}{2}\operatorname{Re}\frac{1+\kappa z}{1-\kappa z} \qquad (|z|<1) \qquad (2.6.4)$$

and α^* is the largest number such that (2.6.4) holds for all z in $|z|<1$.

(ii) with the same number $\mathcal{K}$ as in (i),

$$\lim_{r\to 1} (1 - r) \frac{\mathcal{K}^{-1}r\, f'(\mathcal{K}^{-1}r)}{f(\mathcal{K}^{-1}r)} = \alpha^{\mathcal{K}} \qquad (2.6.5)$$

Proof

We have already proved the necessity of both conditions (theorems 2.2.A and 2.2.B). Suppose (i) is satisfied, and assume for convenience $\mathcal{K} = 1$. Set

$$\frac{zf'(z)}{f(z)} = \frac{1 + \omega(z)}{1 - \omega(z)}$$

Then (2.6.4) implies

$$\frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} \leq \frac{2}{\alpha^{\mathcal{K}}} \frac{|1 - z|^2}{1 - |z|^2}$$

where $\frac{2}{\alpha^{\mathcal{K}}}$ is the smallest number st this holds for all z in $|z| < 1$. As shown in the proof of theorem 2, Appendix 1, this implies

$$\frac{1 - \omega(r)}{1 - r} \to \frac{2}{\alpha^{\mathcal{K}}} \quad \text{as} \quad r \to 1 .$$

Thus $\omega(r) \to 1$ as $r \to 1$ and so

$$(1 - r) \frac{1 + \omega(r)}{1 - \omega(r)} \to \alpha^{\mathcal{K}} \quad \text{as} \quad r \to 1$$

$$\text{i.e.} \qquad \lim_{r\to 1} \ (1 - r) \ \frac{rf'(r)}{f(r)} \ = \ \alpha^{*} \qquad (2.6.6)$$

hence (ii) is satisfied. To complete the proof it is enough to show that (2.6.6) implies $\mathcal{G}$ contains a sector of exactly angle $\pi\alpha^{*}$. From (2.6.6) we obtain

$$|f(r)| \ \to \ \infty \quad \text{as} \quad r \to 1 \ . \qquad (2.6.7)$$

Consider the pts $w_r = f(r)$, $0 \leq r < 1$. Let $\mathcal{G}'$ be a superordinate domain containing $\mathcal{G}$. Then $\exists \ r_o < 1$ st $\forall r$ st $r_o \leq r < 1$, $w_r \in S'$ where S' is one of the sectors of $\mathcal{G}'$. If $G(z)$ is the superordinate fn corresponding to $\mathcal{G}'$, then

$$f(z) = G(\sigma(z))$$

where $\sigma(z)$ is regular, $|\sigma(z)| < 1$ in $|z| < 1$. As in obtaining (2.2.1) we have

$$(1 - |z|) \ \left| \frac{zf'(z)}{f(z)} \right| \ \leq \ (1 - |\sigma(z)|) \ \left| \frac{G'(\sigma(z))}{G\ (\sigma(z))} \right|$$

and hence

$$\alpha^{*} \ \leq \ \alpha_o \qquad (2.6.8)$$

where S' has angle $\pi\alpha_o$. Consider the set of all sectors

$S' = S'(\xi)$ contained in the superordinate domain $\mathcal{G}' = \mathcal{G}'(\xi)$, which are such that they eventually contain the pts w_r. By virtue of (2.6.8) the intersection of all such sectors is a sector S of angle $\pi\beta$, where

$$\beta \geq \alpha^{\varkappa} .$$

If γ is the angle which the bisector of S makes with the + ve real w-axis then

$$\overline{\lim_{r\to 1}} \; |\arg w_r - \gamma| \leq \tfrac{1}{2}\pi\beta . \qquad (2.6.9)$$

Also $S < \mathcal{G}$. For if $\zeta \in S$ and $\zeta \notin \mathcal{G}$ then a radial slit from ζ is not in $\mathcal{G}$, hence the curve w_r eventually lies on one side only of this slit, and we could construct an S' containing the pts w_r, for $r > r_1$ say, and of smaller angle than $\pi\beta$. This contradicts the definition of S.

Now $\exists\ \delta$ st $|f(re^{i\delta})| \to \infty$ and $\arg f(re^{i\delta}) \to \gamma$ as $r \to 1$. By lemma 2.4.B, $\delta = 0$.

We thus obtain by theorem 2.2.B

$$\lim_{r\to 1} (1 - r) \frac{rf'(r)}{f(r)} = \beta$$

and so $\beta = \alpha^{\varkappa}$

i.e. $\mathcal{G}$ contains a sector of angle $\pi\alpha^{\varkappa}$.

Lemma 2.6.D

Consider the starlike function $h(z)$ of (2.6.2). Let H be the image domain of h. Then H contains a sector S of angle $\pi\alpha^{*}$ if, and only if, $\mathcal{G}$ contains a sector S_μ of angle $\pi\alpha^{*}$ and $\mu \notin \Lambda$.

Proof

(i) Suppose $\mathcal{G} > S_\mu$ of angle $\pi\alpha^{*}$, $\mu \notin \Lambda$. Then

$$\frac{zh'(z)}{h(z)} = \frac{zf'(z)}{f(z)} - \sum_{\nu\in\Lambda} \frac{\lambda_\nu}{2} \frac{1 + \kappa_\nu z}{1 - \kappa_\nu z} + \sum_{\nu\in\Lambda} \frac{1}{2}\lambda_\nu$$

Now $$(1 - r) \sum_{\nu\in\Lambda} \frac{\lambda_\nu}{2} \frac{1 + \kappa_\nu \kappa_\mu^{-1} r}{1 - \kappa_\nu \kappa_\mu^{-1} r} \to 0 \quad \text{as} \quad r \to 1 .$$

To see this choose $\varepsilon > 0$. Assume Λ is infinite, since otherwise result trivial, as $\kappa_\mu \neq \kappa_\nu$, $\nu \in \Lambda$. For convenience let ν range over 1, 2, ... $\exists$ N st

$$\sum_{\nu=N+1}^{\infty} \lambda_\nu < \varepsilon$$

$$\therefore \quad \left| (1 - r) \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{2} \frac{1 + \kappa_\nu \kappa_\mu^{-1} r}{1 - \kappa_\nu \kappa_\mu^{-1} r} \right| \leq (1-r) \sum_{\nu=1}^{N} \frac{\lambda_\nu}{2} \frac{|1+\kappa_\nu \kappa_\mu^{-1} r|}{|1-\kappa_\nu \kappa_\mu^{-1} r|} + \varepsilon$$

$\to \varepsilon$ as $r \to 1$. We thus have

$$\lim_{r\to 1}(1-r)\frac{\kappa_\mu^{-1}r\,h'(\kappa_\mu^{-1}r)}{h(\kappa_\mu^{-1}r)} = \lim_{r\to 1}(1-r)\frac{\kappa_\mu^{-1}r\,f'(\kappa_\mu^{-1}r)}{f(\kappa_\mu^{-1}r)}$$

$$= \alpha^*$$

Hence by theorem 2.6.C, H contains a sector of angle $\pi\alpha^*$.

(ii) Suppose $H > S$, a sector of angle $\pi\alpha^*$. Then $\exists\ \kappa$, $|\kappa| = 1$, st

$$(1 - r)\frac{\kappa^{-1}r\,h'(\kappa^{-1}r)}{h(\kappa^{-1}r)} \to \alpha^* \quad \text{as} \quad r \to 1 .$$

hence $(1 - r)\dfrac{\kappa^{-1}r\,f'(\kappa^{-1}r)}{f(\kappa^{-1}r)} \to \alpha^*$ or $\alpha^* + \lambda_\nu$ as $r \to 1$,

$\alpha^* + \lambda_\nu$ in the case when $\kappa = \kappa_\nu$ for some $\nu \in \Lambda$. But if $\kappa = \kappa_\nu$ then

$$(1 - r)\frac{\kappa^{-1}r\,f'(\kappa^{-1}r)}{f(\kappa^{-1}r)} \to \lambda_\nu \quad \text{as} \quad r \to 1$$

so $\alpha^* = 0$, a contradiction. Hence

$$(1 - r)\frac{\kappa^{-1}r\,f'(\kappa^{-1}r)}{f(\kappa^{-1}r)} \to \alpha^* \quad \text{as} \quad r \to 1 .$$

Hence by theorem 2.6.C $\mathcal{G}$ contains a sector S of angle $\pi\alpha^*$, so $S = S_\mu$ and $\mu \notin \Lambda$. This proves lemma 2.6.D.

Finally we have the following theorem giving a representation for $f(z)$.

Theorem 3

Suppose $f(z)$ is starlike in $|z| < 1$, and suppose the image domain $\mathcal{G}$ of f contains sectors S_1, S_2, ... of exact angles $\pi\lambda_1$, $\pi\lambda_2$, ... associated with the numbers $\mathcal{K}_1$, $\mathcal{K}_2$, ... of theorem 2.2.A, and no other sectors. Then we may write

$$f(z) = h(z) \prod_{\nu} (1 - \mathcal{K}_\nu z)^{-\lambda_\nu} \qquad (2.6.10)$$

where $h(z)$ is a starlike function whose image domain contains no sectors. $h(z)$ satisfies

$$\lim_{|z| \to 1} \frac{\log |h(z)|}{\log \frac{1}{1 - |z|}} = 0 \qquad (2.6.11)$$

$$\lim_{|z| \to 1} (1 - |z|) \frac{zh'(z)}{h(z)} = 0 \qquad (2.6.12)$$

Proof

(2.6.11) follows from theorem 1. We have only (2.6.12) to prove. If (2.6.12) is false then $\exists$ sequence $\{z_n\} \to 1$, say, st

$$(1 - |z_n|) \left| \frac{z_n h'(z_n)}{h(z_n)} \right| \to \beta \quad \text{as} \quad n \to \infty$$

where $\beta > 0$. As in obtaining (2.2.6) from (2.2.5) we deduce

$$\operatorname{Re} \frac{zh'(z)}{h(z)} \geq \frac{\beta}{2} \operatorname{Re} \frac{1 + z}{1 - z} \qquad (|z| < 1).$$

Hence by theorem 2.6.C the image domain of $h(z)$ contains a sector of angle at least $\pi\beta$, a contradiction. Hence (2.6.12) follows.

Appendix 1 to Chapter 2

Bounded Functions and Julia's Theorem

In this appendix we assume throughout that $\omega(z)$ is a regular function in $|z| < 1$ and satisfies

$$|\omega(z)| < 1 \qquad (|z| < 1).$$

We aim to prove a theorem due to G. Julia, and related consequences. Our work is adapted from that of Caratheodory ([2] pages 53 and 96).

Let M be the class of regular univalent functions which map the unit disc onto itself. The following three lemmas are well-known.

Lemma 1

$\omega(z) \in M$ if, and only if, $\exists$ constants a, λ satisfying

$$0 \leq \lambda < 2\pi$$
$$|a| < 1$$

such that

$$\omega(z) = e^{i\lambda} \frac{z - a}{1 - \bar{a} z} \qquad (1)$$

Lemma 2

If z_1 , z_2 are two distinct points in $|z| < 1$, then

$$\left| \frac{\omega(z_1) - \omega(z_2)}{1 - \overline{\omega}(z_1)\omega(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z}_1 z_2} \right|$$

where equality occurs if, and only if, $\omega(z) \in M$.

Lemma 3

$$|\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2}$$

and if $\omega(o) = 0$

$$|\omega'(z)| \leq \frac{1 - |\omega(z)|}{1 - |z|}$$

Proofs

For the sake of completeness we give brief indications of the proofs of lemmas 1, 2, 3. First note that

(i) if $f \in M$ then the inverse function $f^{-1} \in M$,

(ii) if $f, g \in M$ then $h = f(g) \in M$.

To prove functions of the form (1) are in M it is a trivial matter to verify that $|\omega(z)| < 1$ in $|z| < 1$ if $\omega(z)$ is given by (1). Also solving (1) for z in terms of ω we

can then show $|z| < 1$ for all ω st $|\omega| < 1$. Next set

$$\omega = h(z) = \frac{\omega(z) - \omega(z_o)}{1 - \overline{\omega}(z_o)\omega(z)}$$

$$\zeta = k(z) = \frac{z - z_o}{1 - \overline{z}_o z}$$

where z_o is some pt in $|z| < 1$. Then $k(z) \in M$ and $|h(z)| < 1$ in $|z| < 1$. Thus

$$h(z) = h\left[k^{-1}(\zeta)\right] = g(\zeta) \quad \text{say}$$

and $g(\zeta)$ is regular, $|g(\zeta)| < 1$ in $|\zeta| < 1$. Also $g(o) = h(z_o) = 0$. Hence applying the Schwarz lemma we easily obtain

$$|h(z)| \leq |k(z)|$$

and further it is easy to show that equality occurs if, and only if $h(z) \in M$, i.e. if, and only if, $\omega(z) \in M$. This proves lemma 2. Setting $z_o = 0$, and supposing $\omega(z) \in M$ so $h(z) = \frac{\omega(z) - \omega(o)}{1 - \overline{\omega}(o)\omega(z)} \in M$ we easily deduce the form (1) for $\omega(z)$. This completes lemma 1. Finally lemma 3 is a trivial consequence of lemma 2 and the Schwarz lemma for the second part of the lemma.

In preparation for Julia's theorem we prove two simple lemmas.

Lemma 4

Let

$$K_o = \left\{ z / \left| \frac{z - z_o}{1 - \overline{z}_o z} \right| < \gamma_o \right\}$$

where $|z_o| < 1$, $0 < \gamma_o < 1$. Then

$$K_o = \left\{ z / |z - Z_o| < \rho_o \right\}$$

where $$Z_o = \frac{z_o(1 - \gamma_o^2)}{1 - \gamma_o^2 |z_o|^2}$$

and $$\rho_o = \frac{\gamma_o(1 - |z_o|^2)}{1 - \gamma_o^2 |z_o|^2}$$

Proof

$$z \in K_o \iff (z - z_o)(\overline{z} - \overline{z}_o) < \gamma_o^2 (1 - \overline{z}_o z)(1 - z_o \overline{z})$$

$$\iff |z|^2 - (z \overline{Z}_o + \overline{z} Z_o) + \frac{|z_o|^2 - \gamma_o^2}{1 - \gamma_o^2 |z_o|^2} < 0$$

$$\iff |z|^2 - (z \overline{Z}_o + \overline{z} Z_o) + |Z_o|^2$$

$$< \frac{|z_o|^2 (1 - \gamma_o^2)^2}{(1 - \gamma_o^2 |z_o|^2)^2} - \frac{|z_o|^2 - \gamma_o^2}{1 - \gamma_o^2 |z_o|^2} = \rho_o^2$$

$$\iff |z - Z_o| < \rho_o .$$

Lemma 5

Suppose

$$K_n = \left\{ z / \left| \frac{z - z_n}{1 - \overline{z}_n z} \right| = \gamma_n \right\} , \quad n = 1, 2, \ldots$$

is a sequence of sets where

$$|z_n| < 1 , \quad 0 < \gamma_n < 1 , \quad n = 1, 2, \ldots$$

And suppose $\exists \zeta$ such that $\zeta \in K_n$ for every n. Then the sets K_n are circles with centres Z_n, radii ρ_n where

$$Z_n = \frac{z_n(1 - |\zeta|^2)}{1 - z_n\overline{\zeta} - \overline{z}_n\zeta + |z_n|^2}$$

and

$$\rho_n = \left| \frac{(\zeta - z_n)(1 - \overline{z}_n\zeta)}{1 - z_n\overline{\zeta} - \overline{z}_n\zeta + |z_n|^2} \right|$$

Proof

Clearly $$\gamma_n = \left| \frac{\zeta - z_n}{1 - \overline{z}_n\zeta} \right|$$

and, as in lemma 4, K_n is a circle with centre Z_n, radius ρ_n where

$$Z_n = \frac{z_n(1 - \gamma_n^2)}{1 - \gamma_n^2 |z_n|^2} , \qquad \rho_n = \frac{\gamma_n(1 - |z_n|^2)}{1 - \gamma_n^2 |z_n|^2}$$

To obtain the result we simply substitute for γ_n in these two relations.

We also note

Lemma 6

Suppose $g(z)$ is a function defined by the equation

$$\frac{1 - g(z)}{1 + g(z)} = \alpha \frac{1 - z}{1 + z} \qquad (|z| < 1 , \ \alpha \text{ real}).$$

Then $g(z) \in M$ if, and only if, $\alpha > 0$.

Proof

Solving for $g(z)$ we obtain

$$g(z) = \frac{z - \frac{\alpha-1}{\alpha+1}}{1 - \frac{\alpha-1}{\alpha+1} z}$$

Hence by lemma 1, $g(z) \in M$ if, and only if

$$\left| \frac{\alpha-1}{\alpha+1} \right| < 1$$

$$\iff \alpha^2 - 2\alpha + 1 < \alpha^2 + 2\alpha + 1$$

$$\iff \alpha > 0 .$$

Theorem 1 (Julia)

Suppose $\omega(z)$ is a function regular in $|z| < 1$ and satisfying

$$|\omega(z)| < 1 \qquad (|z| < 1)$$

Suppose further $\exists$ a sequence $\{z_n\}$ and a number α such that

(i) $|z_n| < 1$, $n = 1, 2, \ldots$

(ii) $z_n \to 1$ as $n \to \infty$

(iii) $\omega(z_n) \to 1$ as $n \to \infty$

(iv) $\dfrac{1 - |\omega(z_n)|}{1 - |z_n|} \to \alpha$ as $n \to \infty$

Then

$$\frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} \leq \alpha \frac{|1 - z|^2}{1 - |z^2|}$$

for all z in $|z| < 1$.

Before proving theorem 1 we prove

Lemma 7

$$|\omega(z)| \leq \frac{|z| + |\omega(o)|}{1 + |z||\omega(o)|}$$

Proof

Set $h(z) = \dfrac{z + \omega(o)}{1 + \overline{\omega}(o)z}$

Then $h(z) \in M$ and $h(o) = \omega(o)$.

Hence $\omega(z)$ is subordinate to $h(z)$, and so

$$|\omega(z)| \leq \max_{|z|=r} |h(z)|$$

Set $z = re^{i\theta}$, $\omega(o) = ae^{i\gamma}$, $\zeta = re^{i(\theta-\gamma)}$

Then $$\left| h(re^{i\theta}) \right| = \left| \frac{\zeta + a}{1 + a\zeta} \right|$$

If the lemma is false then $\exists\ \zeta$ st

$$\left| \frac{\zeta + a}{1 + a\zeta} \right| > \frac{|\zeta| + a}{1 + a|\zeta|}$$

Squaring both sides we then obtain

$$(|\zeta| - \mathrm{Re}\,\zeta)(|\zeta|^2 - 1)(1 - a^2) > 0$$

Since $|\zeta|^2 < 1$, $a^2 < 1$, $|\zeta| \geq \mathrm{Re}\,\zeta$ this gives a contradiction. Hence lemma 7 follows.

Proof of Julia's Theorem

Set $$\left.\begin{aligned} z_n &= r_n e^{i\theta_n} \\ \omega(z_n) &= |\omega(z_n)|\, e^{i\psi_n} \end{aligned}\right\} \quad n = 1, 2, \ldots$$

Define $g_n(z)$ in $|z| < 1$ by the equation

$$\frac{1 - e^{-i\psi_n} g_n(z)}{1 + e^{-i\psi_n} g_n(z)} = \frac{1 - |\omega(z_n)|}{1 - |z_n|} \cdot \frac{1 + |z_n|}{1 + |\omega(z_n)|} \cdot \frac{1 - e^{-i\theta_n} z}{1 + e^{-i\theta_n} z} \qquad (2)$$

Then it is clear from lemma 6 that $g_n(z) \in M$. By lemma 7 we have

$$\frac{1 - |\omega(z_n)|}{1 - |z_n|} \geq \frac{1}{1 - |z_n|} \left(1 - \frac{|z_n| + |\omega(o)|}{1 + |z_n||\omega(o)|} \right)$$

$$= \frac{1 - |\omega(o)|}{1 + |z_n||\omega(o)|} > \frac{1 - |\omega(o)|}{1 + |\omega(o)|}$$

Hence letting $n \to \infty$ we deduce

$$\alpha \geq \frac{1 - |\omega(o)|}{1 + |\omega(o)|} > 0$$

As $n \to \infty$, $g_n(z) \to g(z)$ where $g(z)$ is defined by

$$\frac{1 - g(z)}{1 + g(z)} = \alpha \frac{1 - z}{1 + z} \qquad (|z| < 1) \tag{3}$$

and by lemma 6, $g(z) \in M$.

Note also that

$$g_n(z_n) = \omega(z_n), \qquad n = 1, 2, \ldots \tag{4}$$

Let x be a point on the real axis in $|z| < 1$. Set

$$K_n = \left\{ z / \left| \frac{z - \overline{z}_n}{1 - \overline{z}_n z} \right| \leq \gamma_n = \left| \frac{x - z_n}{1 - \overline{z}_n x} \right| \right\} \tag{5}$$

Hence by lemmas 4 and 5, K_n is the closed disc in $|z| < 1$ whose boundary includes the point x for every n, and whose centre is Z_n, radius ρ_n where

$$Z_n = \frac{z_n(1 - x^2)}{1 - z_n x - \overline{z}_n x + |z_n|^2}, \quad \rho_n = \left| \frac{(x - z_n)(1 - \overline{z}_n x)}{1 - z_n x - \overline{z}_n x + |z_n|^2} \right| \tag{6}$$

Consider the mapping

$$w = g_n(z)$$

of K_n into the w-plane. Let the domain so defined be C_n. Since $g_n(z) \in M$ we have by lemma 2 that

$$\left| \frac{g_n(z) - g_n(z_n)}{1 - \overline{g}_n(z_n)\, g_n(z)} \right| = \left| \frac{z - z_n}{1 - \overline{z}_n\, z} \right|$$

Hence if z is a boundary point of K_n we obtain

$$\left| \frac{g_n(z) - \omega(z_n)}{1 - \overline{\omega}(z_n)\, g_n(z)} \right| = \left| \frac{x - z_n}{1 - \overline{z}_n\, x} \right|$$

$$= \left| \frac{g_n(x) - g_n(z_n)}{1 - \overline{g}_n(z_n)\, g_n(x)} \right| = \left| \frac{y_n - \omega(z_n)}{1 - \overline{\omega}(z_n)\, y_n} \right| = \gamma_n$$

where $y_n = g_n(x)$. Similarly if z is in the interior of K_n, then

$$\left| \frac{g_n(z) - \omega(z_n)}{1 - \overline{\omega}(z_n)\, g_n(z)} \right| < \gamma_n$$

Thus if we think of $\omega(z_n)$ as a point in the w-plane, it is clear that $g_n(z)$ maps K_n onto C_n where C_n is a disc of centre W_n, radius R_n given by

$$\left.\begin{aligned} W_n &= \frac{\omega(z_n)(1 - |y_n|^2)}{1 - \omega(z_n)\overline{y}_n - \overline{\omega}(z_n)y_n + |\omega(z_n)|^2} \\ R_n &= \left| \frac{(y_n - \omega(z_n))(1 - \overline{\omega}(z_n)y_n)}{1 - \omega(z_n)\overline{y}_n - \overline{\omega}(z_n)y_n + |\omega(z_n)|^2} \right| \end{aligned}\right\} \quad (7)$$

Next let $z \in K_n$ and consider the image in the w-plane under the mapping

$$w = \omega(z)$$

By lemma 2,

$$\left| \frac{w - \omega(z_n)}{1 - \overline{\omega}(z_n)w_n} \right| \leq \left| \frac{z - z_n}{1 - \overline{z}_n z} \right| = \gamma_n$$

Hence $w \in C_n$, so $w = \omega(z)$ maps K_n into C_n.

Define

$$K = \left\{ z / \left| z - \tfrac{1}{2}(1 + x) \right| \leq \tfrac{1}{2}(1 - x) \right\} \quad (8)$$

Let z be an interior point of K. Then $\exists\ \gamma > 0$ st

$$\left| z - \tfrac{1}{2}(1 + x) \right| < \tfrac{1}{2}(1 - x) - \gamma$$

We note that

$$Z_n \to \tfrac{1}{2}(1 + x) \qquad \text{as} \quad n \to \infty$$
$$\rho_n \to \tfrac{1}{2}(1 - x) \qquad \text{as} \quad n \to \infty$$

Choose ε st $0 < \varepsilon < \frac{1}{2}\gamma$. Then $\exists\ N_1$ st $\forall\ n > N_1$

$$\left| Z_n - \tfrac{1}{2}(1 + x) \right| < \varepsilon$$

and $\exists\ N_2$ st $\forall\ n > N_2$

$$\left| \rho_n - \tfrac{1}{2}(1 - x) \right| < \varepsilon$$

Hence for $n > \max(N_1, N_2)$

$$\left| z - Z_n \right| = \left| z - \tfrac{1}{2}(1 + x) + \tfrac{1}{2}(1 + x) - Z_n \right|$$

$$< \tfrac{1}{2}(1 - x) - \gamma + \varepsilon$$

$$< \rho_n - \gamma + 2\varepsilon < \rho_n$$

Thus if $z \in$ interior of K then $\exists\ N$ st $\forall\ n > N$, $z \in K_n$.
Conversely, if $z \in K_n$ for infinitely many n, then

$$\left| z - Z_n \right| \leq \rho_n \quad \text{for each such } n$$

and so

$$\left|z - \tfrac{1}{2}(1 + x)\right| = \left|z - Z_n + Z_n - \tfrac{1}{2}(1 + x)\right|$$

$$\leq \rho_n + \left|Z_n - \tfrac{1}{2}(1 + x)\right|$$

$$\to \tfrac{1}{2}(1 - x) \quad \text{as} \quad n \to \infty$$

hence $z \in K$.

In a similar manner the discs C_n have a limiting circular region C whose centre is given by $\frac{1}{2}(1 + y)$ and radius $\frac{1}{2}(1 - y)$, where

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} g_n(x) = g(x)$$

and so y satisfies

$$\frac{1 - y}{1 + y} = \alpha \frac{1 - x}{1 + x} \tag{9}$$

Suppose now z is an interior point of K . Then for large n , $z \in K_n$. Hence if $w = \omega(z)$ then $w \in C_n$ for large n , and so $w \in C$. Thus $w = \omega(z)$ maps the interior of K into C . Since $\omega(z)$ is regular at each point of K , except at $z = 1$, it follows that if z is a boundary pt of K , not $z = 1$, then $w = \omega(z)$ is in C . Thus if

$$\left|z - \tfrac{1}{2}(1 + x)\right| = \tfrac{1}{2}(1 - x)$$

then $$\left|\omega(z) - \tfrac{1}{2}(1+y)\right| \leq \tfrac{1}{2}(1 - y)$$

where y is given by (9). Now consider z an arbitrary point in $|z| < 1$. Define x by the equation

$$\frac{1 - x}{1 + x} = \frac{|1 - z|^2}{1 - |z|^2}$$

Then we easily obtain

$$\left| z - \tfrac{1}{2}(1 + x) \right| = \tfrac{1}{2}(1 - x)$$

Hence $$\left| \omega(z) - \tfrac{1}{2}(1 + y) \right| \leq \tfrac{1}{2}(1 - y)$$

and again this implies

$$\frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} \leq \frac{1 - y}{1 + y}$$

But $$\frac{1 - y}{1 + y} = \alpha \frac{1 - x}{1 + x} = \alpha \frac{|1 - z|^2}{1 - |z|^2}$$

Hence we deduce theorem 1.

We show next that theorem 1 has a converse. Suppose $\exists$ α, a real number, such that

$$\frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} \leq \alpha \frac{|1 - z|^2}{1 - |z|^2}, \qquad (|z| < 1)$$

where $\omega(z)$ is regular, $|\omega(z)| < 1$ in $|z| < 1$. Then clearly $\exists$ $\alpha_0 > 0$ such that the above inequality is valid

for $\alpha = \alpha_o$ throughout $|z| < 1$, and for no smaller number α_o. Choose x satisfying $-1 < x < 1$. Define y by the equation

$$\frac{1 - y}{1 + y} = \alpha_o \frac{1 - x}{1 + x}$$

Let $$r = \frac{1-x}{2}, \quad \rho = \frac{1-y}{2}$$

then $$\rho = \frac{\alpha_o r}{1 - r(1 - \alpha_o)} .$$

Let K be the disc of centre $\frac{1}{2}(1 + x) = 1 - r$, radius r.
Let C be the disc of centre $\frac{1}{2}(1 + y) = 1 - \rho$, radius ρ.
Let z be a boundary pt of K, $(z \neq 1)$. Then

$$\left| z - \tfrac{1}{2}(1 + x) \right| = \tfrac{1}{2}(1 - x)$$

hence $$\frac{1 - x}{1 + x} = \frac{|1 - z|^2}{1 - |z|^2}$$

Thus $$\frac{1 - y}{1 + y} = \alpha_o \frac{|1 - z|^2}{1 - |z|^2} \geq \frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2}$$

In particular we may write $z = x$, so that

$$\frac{|1 - \omega(x)|^2}{1 - |\omega(x)|^2} \leq \frac{1 - y}{1 + y}$$

and hence

$$\left|\omega(x) - \tfrac{1}{2}(1 + y)\right| \leq \tfrac{1}{2}(1 - y)$$

Therefore

$$\left|1 - \omega(x)\right| = \left|\omega(x) - 1 + \rho - \rho\right| \leq \left|\omega(x) - 1 + \rho\right| + \rho \leq 2\rho$$

Hence

$$\frac{1 - \left|\omega(x)\right|}{1 - x} \leq \frac{\left|1 - \omega(x)\right|}{1 - x} \leq \frac{2\rho}{1 - x} = \frac{\rho}{r} = \frac{\alpha_o}{1 - r(1 - \alpha_o)}$$

As $x \to 1$, $r \to 0$, and so

$$\overline{\lim_{x\to 1}} \frac{1 - \left|\omega(x)\right|}{1 - x} \leq \overline{\lim_{x\to 1}} \frac{\left|1 - \omega(x)\right|}{1 - x} \leq \alpha_o$$

Next let

$$\beta = \overline{\lim_{x\to 1}} \frac{1 - \left|\omega(x)\right|}{1 - x}$$

Then $\beta \leq \alpha_o$. Also by theorem 1,

$$\frac{\left|1 - \omega(z)\right|^2}{1 - \left|\omega(z)\right|^2} \leq \beta \frac{\left|1 - z\right|^2}{1 - \left|z\right|^2}, \qquad (\left|z\right| < 1).$$

Hence by definition of α_o ,

$$\alpha_o \leq \beta .$$

Thus $\beta = \alpha_o$, and we deduce

$$\overline{\lim_{x\to 1}} \frac{1 - |\omega(x)|}{1 - x} \leq \overline{\lim_{x\to 1}} \frac{|1 - \omega(x)|}{1 - x} \leq \alpha_o$$

$$= \lim_{x\to 1} \frac{1 - |\omega(x)|}{1 - x} \leq \lim_{x\to 1} \frac{|1 - \omega(x)|}{1 - x}$$

It follows that

$$\lim_{x\to 1} \frac{1 - |\omega(x)|}{1 - x} = \lim_{x\to 1} \frac{|1 - \omega(x)|}{1 - x} = \alpha_o$$

and since $\alpha_o > 0$ we also have

$$\lim_{x\to 1} \frac{|1 - \omega(x)|}{1 - |\omega(x)|} = 1$$

and therefore deduce

$$\lim_{x\to 1} \frac{1 - \omega(x)}{1 - x} = \alpha_o .$$

We have thus proved

Theorem 2

Suppose $\omega(z)$ is regular in $|z| < 1$, and suppose $|\omega(z)| < 1$ in $|z| < 1$. Then either

(i) $\dfrac{1 - |\omega(z)|}{1 - |z|} \to \infty$ as $z \to 1$ (in any manner in $|z| < 1$)

or (ii) $\exists\ \alpha_o > 0$ such that

$$\frac{1 - \omega(x)}{1 - x} \to \alpha_o \quad \text{as} \quad x \to 1 , \quad (x \text{ real}, \ |x| < 1),$$

and α_o is the smallest number such that

$$\frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} \leq \alpha_o \frac{|1 - z|^2}{1 - |z|^2}$$

for all z in $|z| < 1$.

If (i) does not hold we can in fact prove rather more than the conclusion of (ii). Let $\{z_n\}$ be a sequence of points such that

(a) $|z_n| < 1 , \quad n = 1, 2, \ldots$

(b) $z_n \to 1$ as $n \to \infty$

(c) $\overline{\lim_{n\to\infty}} \ |\arg(1 - z_n)| < \gamma < \frac{\pi}{2}$

Then for sufficiently large values of n the points z_n lie inside a triangle Δ in the unit disc with vertex at $z = 1$. Further we may suppose Δ has vertices A, B in the unit disc such that AB is perpendicular to the real axis and

A, B lie on opposite sides of the real axis. We call Δ a Stolz triangle. Let h be the distance from $z = 1$ to AB.

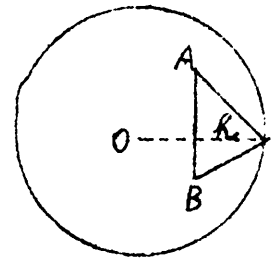


Figure 10

Set

$$\left.\begin{aligned} r_n &= \frac{\mathrm{Re}(1 - z_n)}{h} \\ \text{and} \quad \rho_n &= \frac{\alpha_o r_n}{1 - r_n(1 - \alpha_o)} \end{aligned}\right\} \qquad (10)$$

Define t by the equation

$$(1 - z) = r_n(1 - t) \qquad (11)$$

and $\Phi_n(t)$ by the equation

$$1 - \omega(z) = \rho_n(1 - \Phi_n(t)) \qquad (12)$$

Then $\Phi_n(t)$ is a regular function of t for all t such that in (11) $|z| < 1$. Let

$$K_n = \left\{ z / |z - (1 - r_n)| < r_n \right\}$$

and

$$C_n = \left\{ w / |w - (1 - \rho_n)| < \rho_n \right\}$$

We note that

$$|z - (1 - r_n)| = r_n|t| \quad \text{from (11).}$$

Hence $z \in K_n$ if, and only if, $|t| < 1$.

Consider then $z \in K_n$. Then

$$\frac{|1 - z|^2}{1 - |z|^2} < \frac{1 - \xi_n}{1 + \xi_n}$$

where ξ_n is given by

$$1 - r_n = \frac{1}{2}(1 + \xi_n) .$$

Hence

$$\frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} < \alpha_o \frac{1 - \xi_n}{1 + \xi_n} = \frac{1 - \eta_n}{1 + \eta_n} \quad \text{say}$$

Setting $w = \omega(z)$ we obtain

$$|w - \tfrac{1}{2}(1 + \eta_n)| < \tfrac{1}{2}(1 - \eta_n)$$

Set $\rho_n' = \frac{1}{2}(1 - \eta_n)$. So that

$$|w - (1 - \rho_n')| < \rho_n' .$$

Now

$$\frac{\rho_n'}{1 - \rho_n'} = \frac{1 - \eta_n}{1 + \eta_n} = \alpha_o \frac{1 - \xi_n}{1 + \xi_n} = \alpha_o \frac{r_n}{1 - r_n}$$

We also have from (10) that

$$\frac{\rho_n}{1 - \rho_n} = \alpha_o \frac{r_n}{1 - r_n} .$$

Hence $\rho_n' = \rho_n$, and so $w \in C_n$. Now applying (12) we have, if $w = \omega(z)$,

$$\left| w - (1 - \rho_n) \right| = \rho_n \left| \Phi_n(t) \right|$$

Hence $w \in C_n$ implies $\left| \Phi_n(t) \right| < 1$. It follows that $\Phi_n(t)$ is regular in $|t| < 1$, and satisfies

$$\left| \Phi_n(t) \right| < 1 \qquad (|t| < 1). \tag{13}$$

Suppose t is real in $|t| < 1$. Then z is real in K_n, $z = x_n$ say. Since from (10) and (b) $r_n \to 0$ as $n \to \infty$, it follows from the definition of K_n that $x_n \to 1$ as $n \to \infty$. Hence by theorem 2,

$$\frac{1 - \omega(x_n)}{1 - x_n} \to \alpha_o \quad \text{as} \quad n \to \infty$$

But
$$1 - \Phi_n(t) = \frac{1 - \omega(x_n)}{\rho_n}$$

$$= (1 - t) \frac{1 - \omega(x_n)}{\alpha_o(1 - x_n)} (1 - r_n(1 - \alpha_o))$$

$$\to (1 - t) \quad \text{as} \quad n \to \infty .$$

Thus for each fixed real t in $|t| < 1$,

$$\Phi_n(t) \to t \quad \text{as} \quad n \to \infty. \tag{14}$$

Also by (13) $\Phi_n(t)$ is bounded in $|t| < 1$. Hence by Vitali's theorem, in every compact subset of $|t| < 1$, $\Phi_n(t)$ converges uniformly to a regular function, which is t on the real axis by (14), and hence by analytic continuation

$$\Phi_n(t) \to t \quad \text{as} \quad n \to \infty \tag{15}$$

uniformly in every compact subset of $|t| < 1$.

Next let t_n be defined by

$$r_n(1 - t_n) = 1 - z_n, \quad n = 1, 2, \ldots$$

so that $$(1 - t_n) = h \frac{1 - z_n}{\mathrm{Re}(1 - z_n)}$$

Thus

$$\arg (1 - t_n) = \arg (1 - z_n)$$

and $$\frac{|1 - t_n|}{h} = \frac{|1 - z_n|}{\mathrm{Re}(1 - z_n)}$$

Since the points z_n lie inside Δ it is clear that the

points t_n lie on AB (if we consider the same triangle in $|t| < 1$). Thus the points t_n all lie on a compact subset of $|t| < 1$, namely AB. And from (15)

$$\gamma_n(t) = \Phi_n(t) - t \to 0 \quad \text{as} \quad n \to \infty$$

uniformly on every compact subset of $|t| < 1$. Hence

$$\gamma_n(t_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Now

$$\frac{1 - \omega(z_n)}{1 - z_n} = \frac{\alpha_o}{1 - r_n(1 - \alpha_o)} \frac{1 - \Phi_n(t_n)}{1 - t_n}$$

$$= \frac{\alpha_o}{1 - r_n(1 - \alpha_o)} \left(1 - \frac{\gamma_n(t_n)}{1 - t_n} \right)$$

$$\to \alpha_o \quad \text{as} \quad n \to \infty. \tag{16}$$

Finally we note that since $\Phi_n(t) \to t$ uniformly on every compact subset of $|t| < 1$, so

$$\Phi_n'(t) \to 1 \quad \text{as} \quad n \to \infty$$

uniformly on every compact subset of $|t| < 1$. In particular

$$\Phi_n'(t_n) \to 1 \quad \text{as} \quad n \to \infty.$$

From (11) and (12) we have

$$\frac{dz}{dt} = r_n \ , \quad \omega'(z)\,\frac{dz}{dt} = \rho_n \Phi_n'(t)$$

Hence

$$\omega'(z_n) = \frac{\alpha_o}{1 - r_n(1 - \alpha_o)}\,\Phi_n'(t)$$

Thus

$$\omega'(z_n) \to \alpha_o \quad \text{as} \quad n \to \infty . \tag{17}$$

We have thus proved

Theorem 3

Suppose $\omega(z)$ is regular in $|z| < 1$, and suppose

$$|\omega(z)| < 1 , \qquad (|z| < 1).$$

Then either

(i) $$\frac{1 - |\omega(z)|}{1 - |z|} \to \infty \quad \text{as} \quad z \to 1 \quad \text{inside} \quad |z| < 1$$

or (ii) $\exists\ \alpha_o > 0$ such that

$$\frac{1 - \omega(z)}{1 - z} \to \alpha_o \ , \quad \omega'(z) \to \alpha_o$$

as $z \to 1$ inside any fixed Stolz triangle.

Further α_o is the smallest number such that

$$\frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} \leq \alpha_o \frac{|1 - z|^2}{1 - |z|^2}$$

for all z in $|z| < 1$.

Appendix 2 to Chapter 2

The Superordinate Function

Consider in the w-plane the domain $\mathcal{G}'$ consisting of the whole plane cut along n radial slits, where $w = 0$ is in $\mathcal{G}'$. So $\mathcal{G}'$ contains n sectors measured from $w = 0$, S_1', S_2', ... S_n' of angles $\pi\lambda_1$, $\pi\lambda_2$, ... $\pi\lambda_n$ respectively, where

$$\sum_{\nu=1}^{n} \lambda_\nu = 2$$

We suppose that travelling anti-clockwise about the origin we pass through S_1', S_2', ... S_n', S_1' etc. in that order. Suppose

$$S_1' = \left\{ w/|\arg w| < \frac{\pi\lambda_1}{2} \right\}$$

i.e. S_1' lies astride the positive real axis. Let $G(z)$ be a regular univalent function in $|z| < 1$ which maps $|z| < 1$ onto $\mathcal{G}'$ and $G(o) = 0$. The lower arm of S_1' is the image of two arcs of $|z| = 1$ meeting at a point on $|z| = 1$. The left-hand arc is mapped by G onto this lower arm in such a way that as z moves anti-clockwise, $w = G(z)$ moves up the lower arm to infinity. This "point at infinity" is the image of $z = 1$ say, and to the left of

$z = 1$, $w = G(z)$ moves down the upper arm of S_1' . We can always choose $G(z)$ such that the point is in fact $z = 1$. We may write

$$G(z) = Kz \left(\frac{1}{1 - z}\right)^{\lambda_k} \sum_{\substack{\nu = 1 \\ \nu \neq k}}^{n} \left(\frac{1 - \kappa_\nu}{1 - \kappa_\nu z} \right)^{\lambda_\nu} \qquad (1)$$

where K is a constant, $|\kappa_\nu| = 1$ each $\nu \neq k$, the κ_ν are distinct. This follows from lemma 1.7.A. We prove that

$$\lambda_k = \lambda_1 , \qquad \arg K = 0 \qquad (2)$$

Consider $G(e^{i\theta})$ defined by (1). It is clear that in the interval $0 < \theta < \delta$, for small enough δ ,

$$\arg G(e^{i\theta}) \equiv \frac{\pi\lambda_1}{2} \qquad (\mathrm{mod}\ 2\pi)$$

Similarly in the interval $-\eta < \theta < 0$ for small η ,

$$\arg G(e^{i\theta}) \equiv -\frac{\pi\lambda_1}{2} \qquad (\mathrm{mod}\ 2\pi)$$

Now it is easy to calculate

$$\arg \frac{1 - \kappa_\nu}{1 - \kappa_\nu e^{i\theta}} \equiv -\frac{1}{2}\theta \qquad (\mathrm{mod}\ 2\pi)$$

in both intervals. Also

$$\arg \frac{1}{1 - e^{i\theta}} \equiv \begin{cases} \frac{\pi}{2} - \frac{1}{2}\theta , & 0 < \theta < \delta \\ -\frac{\pi}{2} - \frac{1}{2}\theta , & -\eta < \theta < 0 \end{cases} \pmod{2\pi}$$

Hence if $0 < \theta < \delta$,

$$\arg G(e^{i\theta}) \equiv \theta + \arg K + \frac{\pi\lambda_k}{2} - \frac{\lambda_k}{2}\theta + \sum_{\nu \neq k} \lambda_\nu (-\frac{1}{2}\theta)$$

$$\equiv \arg K + \frac{\pi\lambda_k}{2}$$

and so

$$\frac{\pi\lambda_1}{2} \equiv \arg K + \frac{\pi\lambda_k}{2} \tag{3}$$

Again if $-\eta < \theta < 0$ we obtain

$$\arg G(e^{i\theta}) \equiv \arg K - \frac{\pi\lambda_k}{2}$$

and so

$$-\frac{\pi\lambda_1}{2} \equiv \arg K - \frac{\pi\lambda_k}{2} \tag{4}$$

from (3) and (4) we immediately deduce (2).

Thus we can write

$$G(z) = (\tfrac{1}{1-z})^{\lambda_1} \prod_{\nu=2}^{n} (\frac{1 - k_\nu}{1 - k_\nu z})^{\lambda_\nu} \cdot Kz \tag{5}$$

where $K > 0$.

From (5) we deduce

$$\left.\begin{aligned} \log |G(z)| &= \lambda_1 \log \frac{1}{|1-z|} + O(1) \quad \text{as} \quad z \to 1 \\ \arg\ G(z) &= \lambda_1 \arg \frac{1}{1-z} + o(1) \quad \text{as} \quad z \to 1 \end{aligned}\right\} \qquad (6)$$

Now from (6) and the method of lemma 2.4.B as $z \to 1$ the points $w = G(z)$ are eventually in S_1'. Conversely if $\{w_n\}$ is a sequence in S_1' which diverges to infinity then

$$|G(z_n)| \to \infty \quad \text{as} \quad n \to \infty$$

where $w_n = G(z_n)$, and so from (5) either $z_n \to 1$ as $n \to \infty$ or $\exists$ a subsequence $\{z_{n_p}\}$ such that $z_{n_p} \to \frac{1}{\kappa_\mu}$ for some μ as $p \to \infty$. However in this latter case we could easily show by an argument similar to the above that the points w_{n_p} would be eventually in sector S_μ' of angle $\pi\lambda_\mu$, a contradiction. Hence $z_n \to 1$ as $n \to \infty$.

Now

$$\frac{G'(z)}{G(z)} = \frac{1}{z} + \sum_{\nu=2}^{n} \frac{\lambda_\nu \kappa_\nu}{1 - \kappa_\nu z} + \frac{\lambda_1}{1-z}$$

hence

$$(1 - |z|) \left| \frac{G'(z)}{G(z)} \right| \leq \lambda_1 \frac{1 - |z|}{|1 - z|} + o(1) \quad \text{as} \quad z \to 1 \qquad (7)$$

Suppose $\{w_n\}$ is an unbounded sequence in S_1' such that

$$\overline{\lim_{n\to\infty}} \; |\arg w_n| \; > \; \frac{\pi\lambda_1}{2} - \varepsilon \quad \text{for some small } \varepsilon > 0 .$$

Then from (6)

$$\overline{\lim_{n\to\infty}} \; \left| \arg \frac{1}{1 - z_n} \right| \; > \; \frac{\pi}{2} - \frac{\varepsilon}{\lambda_1}$$

where $w_n = G(z_n)$. Hence

$$\overline{\lim_{n\to\infty}} \; \frac{1 - |z_n|}{|1 - z_n|} \; \leq \; \delta(\varepsilon)$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Hence from (7)

$$\overline{\lim_{n\to\infty}} \; (1 - |z_n|) \left| \frac{G'(z_n)}{G(z_n)} \right| \; \leq \; \delta_o(\varepsilon) \qquad (8)$$

where $\delta_o(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Chapter 3 The Derivative of the Maximum Modulus

Section 3.1

Let

$$M(r) = M(r,f) = \max_{|z|=r} |f(z)|$$

and $$M_{\mathfrak{S}}(r) = M_{\mathfrak{S}}(r,f) = \max_{\substack{|z|=r \\ w=f(z)\in \mathfrak{S}}} |f(z)|$$

where S is a sector in the w-plane. Blumenthal proved that $M(r)$ is a piecewise analytic function of r, $0 \leq r < 1$. Pommerenke [10] has shown that if the largest sector in $\mathfrak{F}$ has angle $\pi\alpha$ then

$$\alpha = \lim_{r\to 1} (1 - r) \frac{M'(r)}{M(r)} .$$

We will give another proof of this result, together with some related results, in this chapter.

If $S \subset \mathfrak{F}$ is a sector from the origin then $M_S(r)$ is piecewise differentiable for $r > r_o$. We use a method in [11], pages 22-27, to prove this.

Lemma 3.1.A

Let S' containing S be a sector of $\mathfrak{F}'$, the domain superordinate to $\mathfrak{F}$. Then $\exists$ $r_o < 1$ such that for $r_o < r < 1$, $\log M_{S'}(r)$ is a convex function of $\log r$.

From this it follows that $M_{S'}(r)$ is continuous and possesses left and right derivatives at each r, $r_o < r < 1$. Furthermore

$$\frac{r\, M_{S'}^{-\prime}(r)}{M_{S'}(r)} \leq \frac{r\, M_{S'}^{\prime -}(r)}{M_{S'}(r)} \qquad (r_o < r < 1) \qquad (3.1.1)$$

where $M_{S'}^{-\prime}$ is the left derivative, $M_{S'}^{\prime -}$ the right derivative.

Proof

On each arm of S' $\exists$ pts ζ_1, ζ_2 the first boundary pts of $\mathcal{G}$ travelling from the origin along each arm. Thus the two lines from $w = 0$ to ζ_1 and from $w = 0$ to ζ_2 are the images of two curves in $|z| < 1$ travelling from $z = 0$ to $e^{i\delta_1}$ and from $z = 0$ to $e^{-i\delta_2}$, where $-\delta_2 < 0 < \delta_1$, under the assumption that the sector S is associated with the pt $z = 1$ in the obvious sense (i.e. $\mathcal{K} = 1$ of Theorem 2.2.A). Calling the curves γ_1 and γ_2 then each curve, being the inverse image of rays from $w = 0$, crosses each circle $|z| = r$, $0 < r < 1$, once and only once. For if we write $g(w) = f^{-1}(w)$ as the inverse function of f then since f is starlike

$$\operatorname{Re} \frac{w\, g'(w)}{g(w)} > 0, \qquad w \in \mathcal{G}.$$

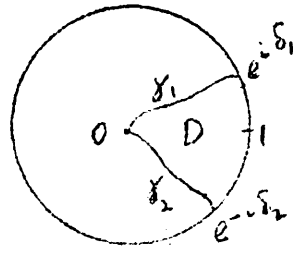


Figure 11

Let D be the domain with boundary γ_1, γ_2 and $|z| = 1$ between $e^{i\delta_1}$ and $e^{-i\delta_2}$, including $z = 1$. The circle $|z| = r$ intersects D and crosses γ_1 and γ_2 once and only once each. $M_{S'}(r)$ is the maximum of $|f(z)|$ on $\{|z| = r\} \cap D$. We note that $\exists\ r_o < 1$ st $\forall\ r_o < r < 1$ the maximum $M_{S'}(r)$ is attained in the interior of D, since $|f(z)|$ is bounded on γ_1 and γ_2 and $|f(r)| \to \infty$ as $r \to 1$. We now apply Hadamard's Three Circle method. Consider three circles $|z| = r_1, r_2, r_3$ where

$$r_o < r_1 < r_2 < r_3 < 1 .$$

Write

$$\phi(z) = z^{\lambda} f(z)$$

where λ is some real constant. Let D_o be that part of D cut off by the outer circles $|z| = r_1$, $|z| = r_3$.

$\phi(z)$ is regular in D_o and its modulus is single-valued. Hence we can apply the maximum principle to $\phi(z)$, and obtain that $|\phi(z)|$ attains its maximum in D_o on one of the two circles $|z| = r_1$ or $|z| = r_3$. Thus

$$r_2^{\lambda} M_{S'}(r_2) \leq \max\left[r_1^{\lambda} M_{S'}(r_1) , r_3^{\lambda} M_{S'}(r_3) \right]$$

Now choose

$$\lambda = - \left(\log \frac{M_{S'}(r_3)}{M_{S'}(r_1)} \right) / \log \frac{r_3}{r_1}$$

and we obtain

$$\log M_{S'}(r_2) \leq \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M_{S'}(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M_{S'}(r_3)$$

the required result.

Lemma 3.1.B

Under the same hypotheses as lemma 3.1.A $M_{S'}(r)$ is piecewise differentiable in any interval (r_o, R) where $R < 1$.

Proof

Let $f(z) = \sum_1^{\infty} a_n z^n$, and write

$$a_n = a_n' + ia_n'', \quad n = 1, 2, \ldots$$

where a_n', a_n'' are real. Then

$$\left.\begin{aligned} u(re^{i\theta}) = \operatorname{Re} f(re^{i\theta}) &= \sum_1^\infty (a_n' \cos n\theta - a_n'' \sin n\theta) r^n \\ \text{and} \quad v(re^{i\theta}) = \operatorname{Im} f(re^{i\theta}) &= \sum_1^\infty (a_n' \sin n\theta + a_n'' \cos n\theta) r^n \end{aligned}\right\} \quad (3.12)$$

Consider the two right hand sums, and allow r, θ to take complex values satisfying

$$\left.\begin{aligned} &0 \le |r| < R < 1 \\ &0 \le |\theta| \le 2\pi , \qquad |\operatorname{Im} \theta| \le |\theta_o| < 2\pi . \end{aligned}\right\} \quad (3.1.3)$$

We obtain

$$\begin{aligned} |\cos n\theta| &\le e^{n|\theta_o|} \\ |\sin n\theta| &\le e^{n|\theta_o|} \end{aligned}$$

And so

$$\left| \sum_1^\infty (a_n' \cos n\theta - a_n'' \sin n\theta) r^n \right| \le 2 \sum_1^\infty |a_n| e^{n|\theta_o|} R^n$$

and the right sum of this expression converges provided $\operatorname{Re}^{|\theta_o|} < 1$. Similarly for the other above sum of (3.1.2). For given $R < 1$ we can choose θ_o st $|\theta_o| > 0$ and

$\mathrm{Re}^{|\theta_o|} < 1$. Then for r, θ satisfying (3.1.3) the complex functions $u(r, \theta)$, $v(r, \theta)$ given by (3.1.2) converge uniformly and are regular functions of r and θ . Let

$$m(r, \theta) = u(r, \theta)^2 + v(r, \theta)^2$$

Then $m(r, \theta)$ is similarly regular and for real r, θ

$$m(r, \theta) = |f(re^{i\theta})|^2 .$$

Also $n(r, \theta) = \frac{\partial}{\partial \theta} m(r, \theta)$ is similarly regular. For fixed $r < R$, $n(r, \theta)$ is a regular fn of θ , and so either $n(r, \theta) \equiv 0$, $(0 \leq |\theta| \leq 2\pi$, $|\mathrm{Im}\ \theta| \leq \theta_o)$ or $n(r, \theta)$ has a finite number of zeros, in particular a finite number of real zeros. Suppose

$$n(r, \theta) \equiv 0 \text{ for some fixed } r < R .$$

i.e. $|f(re^{i\theta})| = K$ on $|z| = r$, K constant. This means that $\frac{\partial}{\partial \theta} |f(re^{i\theta})| = 0$ on $|z| = r$. If $K = 0$ then $f(z)$ is identically zero. If $K > 0$ then since $\frac{zf'(z)}{f(z)}$ is regular in $|z| \leq r$, so $\mathrm{Im} \frac{zf'(z)}{f(z)} = 0$ in $|z| \leq r$, and therefore $\frac{zf'(z)}{f(z)} \equiv \mathrm{Re} \left\{ \frac{zf'(z)}{f(z)} \right\}_{z=0} = 1$, i.e. $f(z) \equiv a_1 z$. Thus we may ignore this case and assume $n(r, \theta)$ has a finite number of zeros on $|z| = r$.

We note next that each zero of $n(r, \theta)$, r, θ real, is an accumulation pt of real zeros of $n(r, \theta)$, i.e. if $z_1 = r_1 e^{i\theta_1}$ is a zero of $n(r, \theta)$ then each disc centred

on z_1 contains a zero of n. other than z_1 . To see this we observe that if r, θ are real then $\frac{\partial}{\partial \theta}$ in the complex sense is the same as $\frac{\partial}{\partial \theta}$ in the real sense, and

$$\frac{\partial m}{\partial \theta} = |f(re^{i\theta})| \frac{\partial}{\partial \theta} |f(re^{i\theta})|$$

hence if $n(r_1, \theta_1) = 0$ then $\operatorname{Im} \frac{z_1 f'(z_1)}{f(z_1)} = 0$. By the maximum principle the harmonic function $\operatorname{Im} \frac{zf'(z)}{f(z)}$ must have a zero on small circle centred at z_1 . In fact $\operatorname{Im} \frac{zf'(z)}{f(z)}$ must take positive and negative values on such a circle, so in fact there are at least two zeros on every such circle.

We consider now $R > r > r_o$ and let $z_1 = r_1 e^{i\theta_1}$, $R > r_1 > r_o$, be a pt on $|z| = r_1$ and in D where the maximum value $M_{S'}(r_1)$ is attained. Then $n(r_1, \theta_1) = 0$ at such a pt. In the nbhd of (r_1, θ_1) we can write

$$n(r, \theta) = \sum_{k,\ell} \alpha_{k,\ell} (r-r_1)^k (\theta-\theta_1)^\ell \qquad (3.1.4)$$

We consider the equation

$$n(r, \theta) = 0 \qquad (3.1.5)$$

as an equation to be solved for θ in terms of r . By a theorem on implicit functions due to Weierstrass (e.g. [4] pages 233-239) equation (3.1.5) has in a domain $|r-r_1| < \delta$, $|\theta-\theta_1| < \delta$, a finite number of solutions which are regular

with respect to the variable $(r - r_1)^{\frac{1}{\rho}}$, ρ being a certain positive integer. We write these solutions

$$\theta = \theta_1 + \chi\left[(r - r_1)^{\frac{1}{\rho}}\right] \qquad (3.1.6)$$

where $\chi(u)$ is regular for $|u| < \delta$. These are the only solutions of (3.1.5) near (r_1, θ_1). Since we know there are an infinity of solutions of (3.1.5) near z_1 which are real, i.e. r, θ are real, then any of the finite set of functions χ which covers an infinite number of all these solutions for $r > r_1$, $|r - r_1| < \delta$, is real for $r > r_1$ and yields solutions for all $r > r_1$, $|r - r_1| < \delta$. (See e.g. [1] page 190). Thus all real solutions, except possibly a finite number, are covered by χ's real for $r > r_1$, or by a similar argument real for $r < r_1$. Assuming δ chosen small enough we can exclude the possible exceptions and assert that all solutions lie on a finite number of analytic arcs radiating from z_1, corresponding to χ's. Substituting into $m(r, \theta)$ the value of θ from (3.1.6) we obtain

$$m(r, \theta) = m(r_1, \theta_1) + \chi_1\left[(r - r_1)^{\frac{1}{\rho}}\right] \qquad (3.1.7)$$

where $\chi_1(u)$ is regular for $|u| < \delta$. We repeat this process for every value on $z = r_1$, $z \in D$, where $m(r_1, \theta)$ has a maximum equal to $M^2_{S'}(r_1)$. We obtain a

finite number of expressions of the form (3.1.7) valid in an interval $(r_1 - \delta_1, r_1 + \delta_1)$. Restricting m to being real now we have two sets of the form (3.1.7), one valid for $r > r_1$, the other for $r < r_1$. These two finite sets yield all the values of the maximum $M_S^2,(r)$ in an interval $(r_1 - \delta_2, r_1 + \delta_3)$, since on $|z| = r_1$ outside the arcs $|\theta - \theta_1| < \delta$, and in D, $m(r_1, \theta) < M_S^2,(r_1) - \gamma$ say. Hence by continuity $\exists$ a small annulus in D, $r_1 - \delta_3$, $r_1 + \delta_3$ st in the portions exterior to $|\theta - \theta_1| < \delta$, $m(r, \theta_1) < M_S^2,(r) - \frac{1}{2}\gamma$ and on the portions interior to $|\theta - \theta_1| < \delta$ there are curves on which $m(r, \theta) > M_S^2,(r_1) - \frac{1}{2}\gamma$.

Consider now the differences taken in pairs of the set (3.1.7) corresponding to $r > r_1$. We obtain a finite number of functions zero at $r = r_1$ and regular in the variable $(r - r_1)^{1/s}$ for an integer s. Each such function is therefore identically zero, greater than zero near $r = r_1$ or less than zero near $r = r_1$. Thus we can find an interval $r_1 \leq r \leq r_1 + \delta_4$ in which one of the functions (3.1.7) is greater than or equal to all the others. Similarly for an interval $r_1 - \delta_5 \leq r \leq r_1$. The functions so obtained are thus the maximum modulus $M_S^2,(r)$ functions, and the derivative exists in $[r_1 - \delta_5, r_1 + \delta_4]$ except possibly at r_1, where by the convexity property the left

and right derivatives exist. Any interval $[r_o, R]$ we can cover by a finite number of such intervals as we have obtained using the Heine-Borel theorem. This completes the proof that $M_{S'}(r)$ is piecewise differentiable for $r > r_o$.

Lemma 3.1.C

If z_1 is a point on $|z| = r_1$ and in D and z_1 lies on one of the analytic arcs (3.1.6), and the maximum $M_{S'}(r_1)$ is attained at z_1 then

$$\frac{r_1 M'_{S'}(r_1)}{M_{S'}(r_1)} = \frac{z_1 f'(z_1)}{f(z_1)}$$

where for definiteness we take $M'_{S'}$ as the left derivative.

Proof

Set $f(z) = f(re^{i\theta}) = Re^{i\phi}$.

From (3.1.6) we see that $\theta = \theta(r)$ along the curve γ on which z_1 lies. Thus $\theta(r)$ is piecewise differentiable with left and right derivatives at every pt.

$$M_{S'}(r) = R(r, \theta(r))$$

$$M'_{S'}(r) = \left[\frac{\partial R}{\partial r} + \theta'(r)\frac{\partial R}{\partial \theta}\right]_{\theta=\theta(r)}$$

$$= \left[\frac{\partial R}{\partial r} \right]_{\theta=\theta(r)} \quad \text{since} \quad \frac{\partial R}{\partial \theta} = 0 \quad \text{at} \quad \theta = \theta(r) .$$

Now $$\frac{zf'(z)}{f(z)} = \frac{r}{R}\frac{\partial R}{\partial r} - \frac{i}{R}\frac{\partial R}{\partial \theta}$$

Hence $$\frac{z_1 f'(z_1)}{f(z_1)} = \frac{r_1 M_S'(r_1)}{M_S(r_1)}$$ as required.

Section 3.2

Theorem 3.2.A

Suppose $\mathcal{G}$ contains a sector S given by (2.1.2). Then

$$\text{(i)} \quad \lim_{r\to 1} \frac{\log M_S(r)}{\log \frac{1}{1-r}} = \alpha^* \qquad (3.2.1)$$

$$\text{(ii)} \quad \lim_{r\to 1} (1 - r) \frac{M_S'(r)}{M_S(r)} = \alpha^* \qquad (3.2.2)$$

(iii) the maxima $M_S(r)$ are attained at points z, $|z| = r$, such that

$$\left.\begin{aligned} \arg f(z) &\to 0 \\ \text{and} \quad \frac{|1 - \kappa z|}{1 - |z|} &\to 1 \end{aligned}\right\} \quad \text{as} \quad |z| \to 1 . \qquad (3.2.3)$$

Proof

Let the number κ corresponding to S be 1. Let $\mathcal{G}'$

be a superordinate sector containing S, and suppose S' has angle $\pi(\alpha^* + \varepsilon)$ where ε is small. We first show that the points z where $M_{S'}(r)$ is attained, $(|z| = r)$, are such that $z \to 1$ as $r \to 1$. Clearly

$$M_S'(r) \to \infty \text{ as } r \to 1$$

so that $|f(z)| \to \infty$ as $r \to 1$ for such pts z. Hence if $z \not\to 1$ then by lemma 2.4.B, $w = f(z)$ satisfies

$$\frac{\pi(\alpha^* + \varepsilon)}{2} \geq \arg w > \frac{\pi(\alpha^* + \eta)}{2}$$

for some $\eta > 0$, $r > r_1$. Since $|f(z)| \to \infty$ any sequence $\{w_n\} = \{f(z_n)\}$ of max. mod. pts, $|z_n| \to 1$, possesses a subsequence st we can construct a superordinate function of f whose domain contains a sector of angle less than $\pi\varepsilon$ which contains the subsequence. We then obtain

$$\overline{\lim_{n\to\infty}} \frac{\log |f(z_n)|}{\log \frac{1}{1 - |z_n|}} \leq \varepsilon$$

However we know that

$$\lim_{r\to 1} \frac{\log |f(r)|}{\log \frac{1}{1-r}} = \alpha^* \qquad (3.2.4)$$

so $|f(r_n)| > |f(z_n)|$ for large n, which is in

contradiction to the assumption that $|f(z_n)| = M_{S'}(r_n)$. This proves that $z \to 1$ for pts where $M_{S'}(r)$ is attained, as asserted.

Clearly we have

$$\lim_{r\to 1} \frac{\log M_{S'}(r)}{\log \frac{1}{1-r}} = \alpha^{*} \qquad (3.2.5)$$

For the limes sup follows from subordination, and the lim inf from (3.2.4). We also have by subordination and lemma 3.1.C

$$\overline{\lim_{r\to 1}} \; (1-r) \frac{r M_{S'}'(r)}{M_{S'}(r)} \leq \alpha^{*}$$

We show that

$$\overline{\lim_{r\to 1}} \; (1-r) \frac{r M_{S'}'(r)}{M_{S'}(r)} = \alpha^{*} \qquad (3.2.6)$$

For if the limes sup is $\beta < \alpha^{*}$, so $\alpha^{*} - \beta = \delta > 0$, then $\exists$ $R < 1$ st for r satisfying $R \leq r < 1$,

$$(1-r) \frac{M_{S'}'(r)}{M_{S'}(r)} < \beta + \tfrac{1}{2}\delta$$

and on integrating

$$\frac{\log M_{S'}(r)}{\log \frac{1}{1-r}} \leq \frac{\log M_{S'}(R)}{\log \frac{1}{1-r}} + \beta + \tfrac{1}{2}\delta - (\beta + \tfrac{1}{2}\delta) \frac{\log \frac{1}{1-R}}{\log \frac{1}{1-r}}$$

for $R \leq r < 1$. Letting $r \to 1$ we deduce from (3.2.5)

$$\alpha^{*} \leq \beta + \tfrac{1}{2}\delta = \alpha^{*} - \tfrac{1}{2}\delta$$

i.e. $\delta \leq 0$, a contradiction.

So (3.2.6) follows, and $\exists$ sequence $\{r_n\} \to 1$ st

$$\left.\begin{aligned} &\lim_{n\to\infty} (1 - r_n) \frac{r_n M_S'(r_n)}{M_S(r_n)} = \alpha^{*} \\ &\text{and so} \\ &\lim_{n\to\infty} (1 - |z_n|) \frac{z_n f'(z_n)}{f(z_n)} = \alpha^{*} \end{aligned}\right\} \qquad (3.2.7)$$

where $|f(z_n)| = M_S(r_n)$, $n = 1, 2, \ldots$.

Now suppose

$$\overline{\lim_{n\to\infty}} \frac{|1 - z_n|}{1 - |z_n|} = \infty$$

Since $z_n \to 1$ as $n \to \infty$ it follows by lemma 2.4.B that either (a) the hypothesis of lemma 2.3.B is satisfied, whose conclusion contradicts (3.2.7), or (b) the pts $w_n = f(z_n)$ are eventually inside a smaller sector contained in S (or a subsequence has this property). But by theorem 2.3.A (ii) and the univalence of f it is clear that this latter condition implies $z_n \to 1$ (Stolz), again a contradiction. Hence

$$\overline{\lim_{n\to\infty}} \quad \frac{|1 - z_n|}{1 - |z_n|} \quad < \quad \infty \qquad (3.2.8)$$

so the points s_n converge to 1 in a Stolz sector. Hence by theorem 2.2.B (i)

$$\lim_{n\to\infty} \quad \frac{|1 - z_n|}{1 - |z_n|} \quad = \quad 1 \qquad (3.2.9)$$

We also observe that any sequence $\{ z_n \} \to 1$ at which the maximum is attained, and we have 3.1.C, and (3.2.8) is satisfied, implies (3.2.9) from theorem 2.2.B(i), and hence (3.2.7). To see this it is enough to note that $\frac{z_n f'(z_n)}{f(z_n)}$ and α^* are real.

We thus have two possibilities:

(i) $$\lim_{r\to 1} \quad (1 - r) \quad \frac{r\, M_S'(r)}{M_S(r)} \quad = \quad \alpha^*$$

or (ii) $\exists$ sequence $\{ r'_n \} \to 1$ st

(a) $$\frac{|1 - z'_n|}{1 - r'_n} \to \infty \text{ as } n \to \infty, \quad (|z'_n| = r'_n)$$

and (b) $$(1 - r'_n) \frac{r_n M_S'(r_n)}{M_S(r_n)} \to 0 \text{ as } n \to \infty$$

(b) following from lemma 2.3.B. We proceed to show that

(ii) is impossible. Consider a Stolz sector S_o at $z = 1$ of positive angle including the positive real axis. Assume (ii) is true. For $r > R_1$ the analytic curves containing the sequence $\{z_n\}$ of (3.2.7) lie in S_o, and the analytic curves containing $\{z'_n\}$ are outside S_o. Let us call the two sets of curves Γ_o, Γ_o' respectively, both sets being non-empty. Let Γ be the set Γ_o plus all the analytic curves eventually in S_o on which the maximum is attained. Let Γ' be the set Γ_o' plus all the analytic curves eventually not in S_o on which the maximum is attained. Γ covers an infinite set of intervals (R_3, R_4), $R_4 \to 1$ but since (ii) is satisfied Γ does not cover all values of r, $1 > r > R_2$ for any $R_2 < 1$. Similarly Γ' does not cover all values of r, $1 > r > R_2$ for any $R_2 < 1$, but Γ' does cover an infinite set of intervals (R_5, R_6), $R_6 \to 1$. Consider any R_4, a right end pt of an interval covered by a $\gamma \in \Gamma$ st (R_4, R_7) for any R_7 is not covered by a $\gamma \in \Gamma$. Then (R_4, R_8) is covered by a $\gamma' \in \Gamma'$ for some $R_8 < 1$. The left derivative of $M_{S'}(r)$ at R_4 is given by a γ, and the right derivative by a γ'. We have

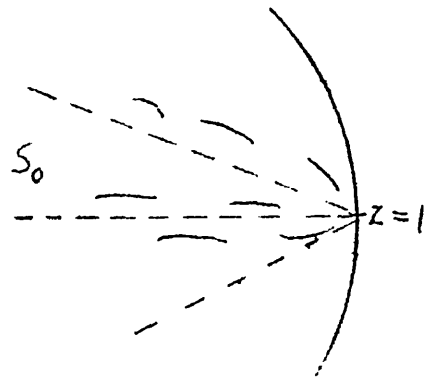


figure 12

$$\frac{R_4\, M_S^{-\prime}(R_4)}{M_S(R_4)} \leq \frac{R_4\, M_S^{\prime -}(R_4)}{M_S(R_4)}$$

by convexity. However there is an infinite sequence of such R_4 tending to 1. Since the left side corresponds to a γ we get

$$(1 - R_4)\frac{R_4\, M_S^{-\prime}(R_4)}{M_S(R_4)} \to \alpha^* \quad \text{as} \quad R_4 \to 1$$

Since the right side corresponds to a γ' we get

$$(1 - R_4)\frac{R_4\, M_S^{\prime -}(R_4)}{M_S(R_4)} \to 0 \quad \text{as} \quad R_4 \to 1$$

We deduce $\alpha^* \leq 0$, a contradiction. Hence (ii) is impossible, and (i) holds. We also have that the maxima are attained at pts z st

$$\frac{|1 - z|}{1 - r} \to 1 \quad \text{as} \quad r = |z| \to 1 .$$

And so $\exists$ $R_o < 1$ st $\forall$ $r > R_o$

$$M_S(r) = M_S(r) .$$

This completes the proof of theorem 3.2.A.

Section 3.3

Theorem 4

Suppose $w = f(z)$, $f(o) = 0$, is a 1:1 conformal map of $|z| < 1$ onto a starlike domain $\mathcal{G}$, and let $\pi\alpha$ be the angle of the largest sector, measured from $w = 0$, contained in $\mathcal{G}$. Then

$$\alpha = \lim_{r\to 1} (1 - r) \frac{M'(r)}{M(r)}$$

Proof

If $\alpha = 0$, $\mathcal{G}$ contains no sectors and the result follows from (2.6.12). Assume $\alpha > 0$ and let $S_1, S_2, \ldots S_n$ be the sectors in $\mathcal{G}$ of angle $\pi\alpha$. Choose a superordinate domain $\mathcal{G}' > \mathcal{G}$ containing sectors $S_1', S_2', \ldots S_n'$ where $S_i' > S_i$, $i = 1, \ldots n$. Since the number of sectors of angle $\pi\alpha$ is finite, $\exists\ R_o < 1$ st $\forall$ r satisfying $R_o < r < 1$

$$M_{S_i}(r) = M_{S_i'}(r), \qquad i = 1, \ldots n. \tag{3.3.1}$$

Suppose $\{z_k\}$ is a sequence of points in $|z| < 1$ st

$$|z_k| \to 1 \quad \text{as} \quad k \to \infty$$

$$|f(z_k)| \to \infty \quad \text{as} \quad k \to \infty$$

$$w_k = f(z_k) \notin S_i', \quad i = 1, \ldots n.$$

Then $\exists$ a subsequence of $\{z_k\}$ st $w_k = f(z_k) \in T$ for $k > k_o$, k in the subsequence, where T is a sector of $\mathcal{G}'$ not one of the S_i'. We may suppose T has angle $\pi\beta$ where $\beta < \alpha$. We then deduce

$$\overline{\lim_{k\to\infty}} \frac{\log |f(z_k)|}{\log \frac{1}{1-|z_k|}} < \alpha$$

Hence by theorem 1 the sequence $\{z_k\}$ cannot be a sequence of maximum modulus points, and we deduce that the maximum modulus points z are such that the image points $w = f(z)$ eventually lie in one or some of the sectors S_i'. And then from (3.3.1) we deduce

$$M(r) = M_{S_i}(r), \qquad r > R_1 > R_o \qquad (3.3.2)$$

and some i, $1 \leq i \leq n$, not necessarily the same i for each r. In any case since the S_i all have the same angle $\pi\alpha$ the result follows from theorem 3.2.A.

Section 3.4

In this section we return to theorem 2.3.A (ii) and (iii) and show that the hypothesis of the theorem that $\not\propto z \to 1$ (Stolz) can be dropped for (ii) and partially dropped for (iii), i.e. we will allow $\not\propto z$ to tend to 1 in any manner, from inside $|z| < 1$. More precisely we prove

Theorem 3.4.A

If g has a sector S of the form (2.1.2) and $\kappa = 1$ is the number of theorem 2.2.A then

(i) $\arg f(z) = \alpha^{\kappa} \arg \frac{1}{1-z} + o(1)$ as $z \to 1$

(ii) for each ε, $0 < \varepsilon < \alpha^{\kappa}$,

$$\log \left| \frac{f(r)}{f(z)} \right| \geq (\alpha^{\kappa} - \varepsilon) \log \frac{|1 - z|}{1 - r} - o(1)$$

as $z \to 1$. In particular if $z \to 1$ but not inside any Stölz sector, (and of course $|z| < 1$), then

$$|f(z)| = o(M_S(r))$$

for such z as $|z| = r \to 1$.

Proof

(i) is clear from theorem 2.3.A and lemma 2.4.B.

(ii) Set

$$g(z) = (1 - z)^{\alpha^{\kappa} - \varepsilon} f(z)$$

Then

$$\frac{zg'(z)}{g(z)} = \frac{zf'(z)}{f(z)} + \frac{\alpha^{\kappa} - \varepsilon}{2} - \frac{\alpha^{\kappa} - \varepsilon}{2} \frac{1 + z}{1 - z}$$

Hence

$$\operatorname{Re}\frac{zg'(z)}{g(z)} = \operatorname{Re}\frac{zf'(z)}{f(z)} - \frac{\alpha^{*}}{2}\operatorname{Re}\frac{1+z}{1-z} + \frac{\xi}{2}\operatorname{Re}\frac{1+z}{1-z} + \frac{\alpha^{*}-\xi}{2}$$

$$\geq \frac{\xi}{2}\operatorname{Re}\frac{1+z}{1-z} + \frac{\alpha^{*}-\xi}{2} > 0$$

applying theorem 2.2.A. Also $g'(o) = f'(o) \neq 0$, $g(o) = 0$. So $g(z)$ is starlike and applying theorem 2.6.C maps $|z| < 1$ onto a domain D containing a sector T of angle $\pi\xi$ associated with the pt $z = 1$. We have

$$\log\left|\frac{f(r)}{f(z)}\right| = \log\left|\frac{g(r)}{g(z)}\right| + (\alpha^{*}-\xi)\log\frac{|1-z|}{1-r}$$

Now by theorem 2.3.A (iii)

$$\left|\frac{g(r)}{M_T(r,g)}\right| \to 1 \quad \text{as} \quad r \to 1$$

since the maximum points are attained near the centre of T.

Thus

$$\log\left|\frac{g(r)}{g(z)}\right| = \log\left|\frac{g(r)}{M_T(r,g)} \cdot \frac{M_T(r,g)}{g(z)}\right|$$

$$\geq \log\left|\frac{g(r)}{M_T(r,g)}\right| \to 0 \quad \text{as} \quad r \to 1,$$

for pts $z \to 1$. Hence

$$\log \left| \frac{f(r)}{f(z)} \right| \geq (\alpha^{*} - \xi) \log \frac{|1 - z|}{1 - r} - o(1)$$

as $z \to 1$.

Section 3.5

In this section we write

$$\left.\begin{aligned} F(z) &= \frac{zf'(z)}{f(z)} \\ \text{and } h(z) &= zf'(z) \end{aligned}\right\} \quad (|z| < 1) \qquad (3.5.1)$$

Lemma 3.5.A

For each θ , $0 \leq \theta < 2\pi$,

$$\lim_{r \to 1} (1 - r) F(re^{i\theta}) \quad \text{exists.} \qquad (3.5.2)$$

Proof

Suppose the limit is not zero for some θ . Then $\exists$ a sequence $\{r_n\} \to 1$ as $n \to \infty$ st

$$\lim_{n \to \infty} (1 - r_n) \left| F(r_n e^{i\theta}) \right| = \beta > 0$$

As in obtaining (2.2.6) from (2.2.5) we deduce

$$\operatorname{Re} F(z) \geq \frac{\beta}{2} \operatorname{Re} \frac{1 + e^{-i\theta} z}{1 - e^{-i\theta} z} \qquad (|z| < 1)$$

Hence by Theorem 2.6.C $\mathcal{G}$ contains a sector of angle at least $\pi\beta$ associated with the pt $z = e^{+i\theta}$, and the limit exists and is therefore β.

Theorem 3.5.B

Let $\pi\alpha$, $0 \leq \alpha \leq 2$, be the angle of the largest sector contained in $\mathcal{G}$. Then

$$\lim_{r \to 1} (1 - r) M(r, F) = \alpha \qquad (3.5.3)$$

If $\alpha > 0$ then the set of points z at which the maximum is attained has a finite number of limit points ζ on $|z| = 1$ and

$$\frac{|1 - \zeta^{-1} z_n|}{1 - |z_n|} \to 1 \quad \text{as} \quad n \to \infty \qquad (3.5.4)$$

for any sequence $\{z_n\}$ of such pts tending to ζ. Furthermore

$$\arg F(z_n) \to 0 \quad \text{as} \quad n \to \infty \qquad (3.5.5)$$

for such points.

Proof

If $\alpha = 0$ (3.5.3) follows from the proof of lemma 3.5.A. Assume $\alpha > 0$. Then by theorem 2.6.C the limit (3.5.2) for some value of θ is α, and so

$$\lim_{r\to 1} (1 - r)\, M(r, F) \geq \alpha . \qquad (3.5.6)$$

If $\{z_n\}$ is a sequence of pts at which the maximum is attained, and $z_n \to \zeta$, $(|\zeta| = 1)$, as $n \to \infty$, and

$$\lim_{n\to\infty} (1 - |z_n|)\, M(|z_n|, F) = \beta$$

then as before

$$\operatorname{Re} F(z) \geq \frac{\beta}{2} \operatorname{Re} \frac{1 + \zeta^{-1} z}{1 - \zeta^{-1} z} \qquad (|z| < 1).$$

Hence $\mathcal{G}$ contains a sector of angle at least $\pi\beta$ associated with $z = \zeta$, and since by (3.5.6) $\beta \geq \alpha$ we deduce $\beta = \alpha$. This proves (3.5.3). By lemma 2.4.B the points $w_n = f(z_n)$ are eventually in $S' > S$, where $S < \mathcal{G}$ is the sector angle $\pi\alpha$ associated with ζ, and S' a superordinate sector, constructed in the usual manner. By lemma 2.3.B the points z_n tend to ζ from inside a Stolz sector at ζ, and so by theorem 2.2.B (i)

$$(1 - \zeta^{-1} z_n)\, F(z_n) \to \alpha \quad \text{as} \quad n \to \infty$$

Also $(1 - |z_n|)\ |F(z_n)| \to \alpha$ as $n \to \infty$.
Hence (3.5.4) and (3.5.5) follow.

Theorem 3.5.C

Let $\pi\alpha$, $0 \leq \alpha \leq 2$, be the angle of the largest sector contained in $\mathcal{G}$. Then

$$\lim_{r\to 1} (1 - r) \frac{M(r, h)}{M(r, f)} = \alpha \qquad (3.5.7)$$

If $\alpha > 0$ then the set of points z at which the maximum $M(r,h)$ is attained has a finite number of limit points ζ on $|z| = 1$, and the properties (3.5.4) and (3.5.5) are also satisfied.

Proof

Since $h(z) = f(z)\ F(z)$ we have

$$M(r,h) \leq M(r,f)\ M(r,F)$$

and so

$$\overline{\lim_{r\to 1}} (1 - r) \frac{M(r,h)}{M(r,f)} \leq \alpha \qquad (3.5.8)$$

by theorem 3.5.B. Hence (3.5.7) follows for $\alpha = 0$.
Assume $\alpha > 0$. Let $\{z_n\}$ be a sequence of points at which the maximum $M(r,f)$ is attained, and where $z_n \to \zeta$, $(|\zeta| = 1)$, as $n \to \infty$. Then

$$(1 - |z_n|)\ \frac{M(|z_n|, h)}{M(|z_n|, f)} \geq (1 - |z_n|)\ |F(z_n)|$$

$$\to \alpha \quad \text{as} \quad n \to \infty .$$

Hence

$$\lim_{r\to 1}\ (1 - r)\ \frac{M(r, h)}{M(r, f)} \geq \alpha$$

and with (3.5.8) we obtain (3.5.7).

Next let $\{ z_n \}$, $|z_n| \to 1$, be a sequence of points at which the maximum $M(r,h)$ is attained. Then

$$\lim_{n\to\infty}\ (1 - |z_n|)\ |F(z_n)| = \lim_{n\to\infty} (1 - |z_n|)\ \frac{M(|z_n|, h)}{|f(z_n)|}$$

$$\geq \lim_{n\to\infty}\ (1 - |z_n|)\ \frac{M(|z_n|, h)}{M(|z_n|, f)} \to \alpha \quad \text{as} \quad n \to \infty$$

Hence if $z_n \to \zeta$ say then $\mathcal{G}$ has a sector of angle at least $\pi\alpha$, and hence of $\pi\alpha$, associated with ζ .

Thus it easily follows that

$$(1 - |z_n|)\ |F(z_n)| \to \alpha \quad \text{as} \quad n \to \infty$$

and $$(1 - \zeta^{-1} z_n)\ F(z_n) \to \alpha \quad \text{as} \quad n \to \infty .$$

and so the theorem follows.

Theorem 3.5.D

Let $\pi\alpha$, $0 \leq \alpha \leq 2$, be the angle of the largest sector contained in $\mathcal{G}$. Then

$$\overline{\lim_{r\to 1}} \; (1 - r) \; \frac{M'(r, F)}{M(r, F)} \leq 1 \qquad (3.5.9)$$

and

$$\lim_{r\to 1} \; (1 - r)^2 \; M'(r, F) = \alpha \qquad (3.5.10)$$

If $\alpha > 0$ then

$$\lim_{r\to 1} \; (1 - r) \; \frac{M'(r, F)}{M(r, F)} = 1 \qquad (3.5.11)$$

If $\alpha = 0$ then (3.5.11) does not hold in general. However there are functions F in this case such that (3.5.9) is sharp.

Theorem 3.5.E

Let $\pi\alpha$, $0 \leq \alpha \leq 2$, be the angle of the largest sector contained in $\mathcal{G}$. Then

$$\overline{\lim_{r\to 1}} \; (1 - r) \; \frac{M'(r, h)}{M(r, h)} \leq 1 + \alpha \qquad (3.5.12)$$

and

$$\lim_{r\to 1} \; (1 - r)^2 \; \frac{M'(r, h)}{M(r, f)} = \alpha(\alpha + 1) \qquad (3.5.13)$$

If $\alpha > 0$ then

$$\lim_{r \to 1} (1 - r) \frac{M'(r, h)}{M(r, h)} = \alpha + 1 \qquad (3.5.14)$$

If $\alpha = 0$ then (3.5.14) does not hold in general. However there are functions h in this case such that (3.5.12) is sharp.

Proofs

For definiteness we suppose the above derivatives are left derivatives, which exist for each r, $0 < r < 1$. Assume first that $\alpha > 0$. On each circle $|z| = r$ we can find z st

$$\frac{r\, M'(r, F)}{M(r, F)} = \frac{z\, F'(z)}{F(z)}$$

and $$M(r, F) = |F(z)| .$$

Any sequence of such pt z as $|z| \to 1$ possesses a subsequence converging to a pt ζ on $|z| = 1$, and g has a sector of angle $\pi\alpha$ associated with ζ in the usual sense. Furthermore the subsequence converges to ζ along the radial direction at ζ. We deduce from 2.2.C(i)

$$(1 - r) \frac{zF'(z)}{F(z)} \to 1 \quad \text{as} \quad r \to 1$$

and this proves (3.5.11). Similarly we have

$$\frac{r\,M'(r,\,h)}{M(r,\,h)} = \frac{z\,h'(z)}{h(z)}$$

and $$M(r,\,h) = |h(z)|$$

for some z on $|z| = r$, each r, $0 < r < 1$. Now

$$\frac{z\,h'(z)}{h(z)} = F(z) + \frac{z\,F'(z)}{F(z)} \qquad (3.5.15)$$

The points z have the same property as before, so that

$$(1 - r)\,F(z) \to \alpha \quad \text{as} \quad r \to 1$$

and $$(1 - r)\,\frac{zF'(z)}{F(z)} \to 1 \quad \text{as} \quad r \to 1$$

and so (3.5.14) follows.

In the case $\alpha = 0$ these two results do not hold in general as can be seen from the function

$$f(z) = z$$

when $h(z) = z$, $F(z) = 1$ and both limits (3.5.11) and (3.5.14) are zero. To prove (3.5.9) we note the inequality

$$|F'(z)| \leq \frac{2}{1 - |z|^2}\,\mathrm{Re}\,F(z) \qquad (3.5.16)$$

whose proof we leave to lemma 4.1.C. We deduce

$$(1 - r)\left|\frac{F'(z)}{F(z)}\right| \leq \frac{2}{1 + r} \to 1 \quad \text{as} \quad r \to 1 .$$

This proves (3.5.9). Also if $\alpha = 0$ then

$$(1 - r)\, F(z) \to 0 \quad \text{as} \quad r \to 1$$

so (3.5.12) follows from (3.5.15) and the above inequality. If we consider the function

$$f(z) = \log \frac{1}{1-z}$$

then $$h(z) = \frac{z}{1-z}$$

is starlike, so $f(z)$ is convex, and in particular starlike, and clearly $\alpha = 0$. In this case the limits (3.5.11) and (3.5.14) are valid and so (3.5.9) and (3.5.12) are sharp.

If $\alpha > 0$ then (3.5.10) follows from (3.5.3) and (3.5.11). Also (3.5.13) follows from (3.5.7) and (3.5.14). If $\alpha = 0$ then (3.5.10) follows from (3.5.3) and (3.5.9). Also (3.5.13) follows from (3.5.7) and (3.5.12).

This completes the proofs of theorems 3.5.D and E, apart from showing (3.5.16).

Section 3.6

We complete chapter 3 with a consideration of the

higher derivatives of $M(r) = M(r, f)$. According to theorem 3 we may write

$$f(z) = h(z) \prod_{\nu} (1 - \zeta_\nu z)^{-\lambda_\nu}$$

where $|\zeta_\nu| = 1$, the ζ_ν are distinct, $\sum_{\nu} \lambda_\nu \leq 2$, and

$$(1 - |z|) \frac{z\, h'(z)}{h(z)} \to 0 \quad \text{as} \quad |z| \to 1 .$$

We set

$$F(z) = \frac{z\, f'(z)}{f(z)}, \quad H(z) = \frac{z\, h'(z)}{h(z)} .$$

Then

$$F(z) = H(z) + \sum_{\nu} \frac{\lambda_\nu \zeta_\nu z}{1 - \zeta_\nu z}$$

and $(1 - r)\, M(r, H) \to 0$ as $r \to 1$.

Lemma 3.6.A

Let z, $|z| \to 1$, be maximum modulus points of f. Then

$$(1 - |z|)^{n+1} z^n F^{(n)}(z) \to \alpha n! \quad \text{as} \quad |z| \to 1 \qquad (n = 0, 1, 2, \ldots)$$

where $\pi\alpha$, $0 \leq \alpha \leq 2$, is the angle of the largest sector in $\mathcal{G}$.

Proof

$|z| \to 1$ such that convergent sequences $\{z_k\}$ tend to a κ_μ^{-1} where $\lambda_\mu = \alpha$ for each such μ, and z_k approaches κ_μ^{-1} along the radial direction at κ_μ^{-1}. We have

$$F^{(n)}(z) = H^{(n)}(z) + n! \sum_\nu \frac{\lambda_\nu \kappa_\nu^n}{(1 - \kappa_\nu z)^{n+1}}$$

Hence the lemma follows provided

$$(1 - |z|)^{n+1} H^{(n)}(z) \to 0 \quad \text{as} \quad |z| \to 1$$

By Cauchy's integral formula

$$H^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{H(\zeta)\, d\zeta}{(\zeta - z)^{n+1}}$$

where C is a circle centre z whose circumference is contained in $|z| < 1$. Let $z = re^{i\theta}$ and choose C to have radius $\delta = \frac{1}{2}(1-r)$. Then

$$H^{(n)}(re^{i\theta}) = \frac{n!}{2\pi} \int_0^{2\pi} \frac{H(re^{i\theta} + \delta e^{i\phi})\, d\phi}{\delta^n e^{in\phi}}$$

$$= \frac{2^n n!}{2\pi(1-r)^n} \int_0^{2\pi} \frac{H(re^{i\theta} + \delta e^{i\phi})\, d\phi}{e^{in\phi}}$$

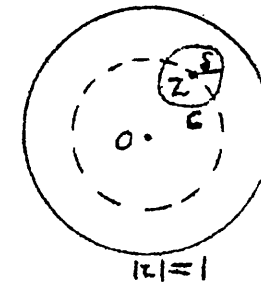


Figure 13

Hence $(1 - r)^{n+1} \left| H^{(n)}(re^{i\theta}) \right| \leq 2^n n! \, (1 - r) \, M(r+\delta, H)$

$= 2^{n+1} \, n! \, (1 - R) \, M(R, H)$ where $R = r+\delta$

$\to 0$ as $R \to 1$, and $R \to 1$ as $r \to 1$.

This proves the lemma.

[Note our convention $u^{(o)}(z) \equiv u(z)$].

Corollary 3.6.B

Under the same hypothesis as lemma 3.6.A

$$\lim_{|z| \to 1} (1 - |z|)^{n+1} \frac{z^{n+1} f^{(n+1)}(z)}{f(z)} = \alpha(\alpha+1) \ldots (\alpha+n)$$

$(n = 0, 1, \ldots)$.

Proof

The result is already proved for $n = 0$. So we assume $n \geq 1$ and that the result is true for values up to $n - 1$. According to Leibniz Theorem

$$\frac{d^n}{dz^n} (u(z) \, v(z)) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} u^{(n-k)} v^{(k)}$$

We apply this result as follows. Set

$$z f'(z) = f(z) F(z)$$

Then

$$z\, f^{(n+1)} + n\, f^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, f^{(n-k)}\, F^{(k)}$$

Let $$P_k = \frac{z^k\, f^{(k)}}{f}, \quad Q_k = z^k\, F^{(k)}$$

Then

$$P_{n+1} + n\, P_n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, P_{n-k}\, Q_k .$$

Hence applying lemma 3.6.A and our induction hypothesis

$$\lim_{|z|\to 1} (1-|z|)^{n+1}\, P_{n+1} = \sum_{k=0}^{n-1} \frac{\alpha\, n!}{(n-k)!}\, \alpha(\alpha+1) \ldots (\alpha+n-k-1) + \alpha n!$$

$$= \sum_{m=0}^{n} \frac{\alpha\, n!}{m!}\, \alpha_m ,$$

where $\alpha_o = 1$, $\alpha_m = \alpha(\alpha+1) \ldots (\alpha+m-1)$, $(m \geq 1)$.

We now proceed by a second induction argument to show that

$$\alpha_{n+1} = \sum_{m=0}^{n} \frac{\alpha\, n!}{m!}\, \alpha_m .$$

Suppose for some $n \geq 1$ we know that

$$\alpha_n = \sum_{m=0}^{n-1} \frac{\alpha(n-1)!}{m!}\, \alpha_m$$

Now

$$\alpha_{n+1} = (\alpha + n)\,\alpha_n$$

$$= -\alpha\,\alpha_n + \sum_{m=0}^{n-1} \frac{\alpha\, n!}{m!}\,\alpha_m$$

$$= \sum_{m=0}^{n} \frac{\alpha\, n!}{m!}\,\alpha_m \quad \text{as required}$$

But $\alpha_1 = \alpha$, $\alpha_2 = \alpha(\alpha + 1) = \alpha^2 + \alpha$ satisfy our induction hypothesis, hence

$$\lim_{|z|\to 1} (1 - |z|)^{n+1} P_{n+1} = \alpha_{n+1}$$

and the corollary follows by induction.

Theorem 3.6.C

Let $\pi\alpha$, $0 \leq \alpha \leq 2$, be the angle of the largest sector in $\mathcal{G}$. Then

$$\lim_{r\to 1} (1 - r)^2 \frac{M''(r)}{M(r)} = \alpha(\alpha + 1)$$

Remark

$M''(r)$ is well-defined except possibly at a countable number of isolated points. It will turn out from the proof that $M''(r)$ defined by continuity from the left (or from the

right) at these points must remain finite, so that we can attach a significant meaning to $M''(r)$ for each r, $r_o < r < 1$.

Proof

Let γ be a piecewise differentiable curve on which $|f(z)|$ attains its maximum. We can write

$$\gamma = \left\{ z/z = z(r) = re^{i\theta(r)} , \quad 0 < r < 1 \right\}$$

where $\theta(r)$ is piecewise differentiable. Then on γ we have

$$r \frac{d}{dr} = (1 + ir\theta'(r)) z \frac{d}{dz} \qquad (3.6.1)$$

As equivalent operators. We also have on γ

$$P(r) = \frac{r M'(r)}{M(r)} = \frac{z f'(z)}{f(z)} = F(z)$$

Hence $rP'(r) = (1 + ir\theta'(r) z F'(z)$

Therefore

$$(1 - r)^2 r P'(r) = (1 + ir\theta'(r))(\alpha + o(1)) \quad \text{as} \quad r \to 1 ,$$

by lemma 3.6.A. However the left-hand side is real, and so

$$(1 - r)^2 r P'(r) = \alpha + o(1) \quad \text{as} \quad r \to 1$$

and the result follows by the argument of corollary 3.6.B.

We also have proved

$$(1 - r)^2 r\Theta'(r)\, z\, F'(z) = o(1) \quad \text{as} \quad r \to 1 . \qquad (3.6.2)$$

If $\alpha > 0$ this implies

$$\Theta'(r) = o(1) \quad \text{as} \quad r \to 1 . \qquad (3.6.3)$$

Theorem 3.6.C leads us to conjecture that

$$\lim_{r\to 1} (1 - r)^{n+1} \frac{M^{(n+1)}(r)}{M(r)} = \alpha(\alpha + 1) \ldots (\alpha + n)$$

for $n = 0, 1, 2, \ldots$. A proof, if the result is true, by our method would need a step by step application of (3.6.1) and we cannot see how to obtain a general result. However by routine calculation we can carry out the next step, and prove

Theorem 3.6.D

Let $\pi\alpha$, $0 < \alpha \leq 2$, be the angle of the largest sector in $\mathcal{G}$. Then

$$\lim_{r\to 1} (1 - r)^3 \frac{M'''(r)}{M(r)} = \alpha(\alpha + 1)(\alpha + 2)$$

Remark

$M'''(r)$ as before is well defined except at the exceptional isolated pts where again it can be seen from

the proof that $M'''(r)$ does not become infinite in $r_o < r < 1$.

Proof

We again apply (3.6.1) and obtain

$$r \frac{d}{d r} (r P'(r)) = \left\{ (1+ir\theta'(r)) z \frac{d}{d z} \right\} ((1+ir\theta'(r)) z F'(z))$$

$$= (1+ir\theta'(r))^2 (z F'(z)+z^2F''(z)) + r \frac{d}{d r} (1+ir\theta'(r)).z F'(z)$$

Hence

$$(1-r)^3 r\frac{d}{d r} (rP'(r)) = 2\alpha+o(1) + 2ir\theta'(r)(1-r)^3(zF'(z)+z^2F''(z))$$

$$- r^2\theta'^2(r)(1-r)^3(zF'(z) + z^2 F''(z))$$

$$+ i(1-r)^3r\theta'(r)zF'(z)+i(1-r)^3r^2\theta''(r)zF'(z)$$

Since by hypothesis $\alpha > 0$, we have $\theta'(r) = o(1)$, and we deduce

$$\left.\begin{aligned} (1 - r)^3 r \frac{d}{d r} (r P'(r)) &= 2\alpha + o(1) \\ \text{and} \qquad (1 - r) \theta''(r) &= o(1) \end{aligned}\right\} \text{ as } r\to 1. \quad (3.6.4)$$

The theorem now follows easily. We would expect the theorem to hold in the case $\alpha = 0$, but the difficulty in proving this arises from the term

$$-r^2 \theta'^2(r) (1 - r)^3 \left\{ z F'(z) + z^2 F''(z) \right\}$$

which would need to be $o(1)$ for the result to hold.

Chapter 4. Estimates of Length, Area and Integral Means

Section 4.1

In this chapter we concern ourselves with the estimation of (i) the length of the image curve of $|z| = r$ under the starlike mapping $w = f(z)$. This is given by

$$L(r) = r \int_0^{2\pi} |f'(re^{i\theta})| \, d\theta \qquad (4.1.1)$$

(ii) the area enclosed by the same image curve. This is given by

$$A(r) = \int_0^r \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 \, \rho \, d\rho \, d\theta \qquad (4.1.2)$$

(iii) the integral means defined by

$$I_p(r,f) = I_p(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \qquad (4.1.3)$$

for $p > 0$.

We use the same notation as earlier. The following lemma will be fundamental to our investigation.

Lemma 4.1.A

Suppose $w = f(z)$ is a starlike map of $|z| < 1$ onto

a domain $\mathcal{G}$ and suppose the largest sector with vertex at the origin contained in $\mathcal{G}$ has angle $\pi\alpha$, where $\alpha > 0$. Suppose there are n sectors in $\mathcal{G}$ of angle $\pi\alpha$, and let $\zeta_1, \zeta_2, \ldots \zeta_n$ be the n distinct points on $|z| = 1$ corresponding to the n sectors $S_1, S_2, \ldots S_n$ in $\mathcal{G}$ respectively. Choose ε satisfying $0 < \varepsilon < \alpha$. Then at ζ_i, $(1 \leq i \leq n)$, $\exists$ a Stolz sector T_i, which includes the radial direction from ζ_i to $z = 0$, and $\exists$ $R < 1$ such that for all r, $R < r < 1$, we have

$$|f(z)| < \varepsilon M(r) \qquad (4.1.4)$$

for all z on $|z| = r$ outside the sectors T_i, $i = 1, .. n$. Furthermore the angles of the Stolz sectors may be chosen to depend on ε and tend to π as $\varepsilon \to 0$.

Proof

Let $\{z_k\}$ be a sequence of points in $|z| < 1$ such that

(i) $|z_k| \to 1$ as $k \to \infty$

(ii) $|f(z_k)| \geq \varepsilon M(r_k)$, $k = 1, 2, \ldots$

where $r_k = |z_k|$.

We then have

$$\varepsilon M(r_k) \leq |f(z_k)| \leq M(r_k)$$

and so

$$\frac{\log \varepsilon}{\log \frac{1}{1-r_k}} + \frac{\log M(r_k)}{\log \frac{1}{1-r_k}} \leq \frac{\log |f(z_k)|}{\log \frac{1}{1-|z_k|}} \leq \frac{\log M(r_k)}{\log \frac{1}{1-r_k}}$$

The outer terms each tend to α as $k \to \infty$. Hence

$$\lim_{k\to\infty} \frac{\log |f(z_k)|}{\log \frac{1}{1-|z_k|}} = \alpha \qquad (4.1.5)$$

For a given superordinate domain $\mathcal{G}' > \mathcal{G}$ with sectors $S_i' > S_i$, $(i = 1, \ldots n)$, as we showed in the proof of theorem 4, (4.1.5) implies that for $k > k_o$ the points $w_k = f(z_k)$ lie in the union $\bigcup_{i=1}^{n} S_i'$. From lemma 2.4.B it follows that any convergent subsequence of $\{z_k\}$ converges to one of the points ζ_i, $(i = 1, \ldots n)$. We may assume for convenience that

$$z_k \to \zeta_1 = 1 \text{ say as } k \to \infty$$

Writing $S_1 = S$ we have by (ii)

$$|f(z_k)| \geq \varepsilon M(r_k) \geq \varepsilon M_S(r_k)$$

hence

$$\left| \frac{f(z_k)}{f(r_k)} \right| \geq \varepsilon \frac{M_S(r_k)}{|f(r_k)|} \geq \varepsilon, \quad \text{for } k > k_1 .$$

Hence from theorem 3.4.A we see that

$z_k \to 1$ as $k \to \infty$ inside a Stolz sector at $z = 1$.

Hence we can apply Theorem 2.3.A (iii) and obtain

$$\left|\frac{f(z_k)}{f(r_k)}\right| = \left(\frac{(1 - r_k)}{|1 - z_k|}\right)^{\alpha} (1 - o(1)) \geq \xi \quad \text{as} \quad k \to \infty$$

i.e. for $k > k_2$ say

$$\frac{1 - r_k}{|1 - z_k|} \geq \frac{\xi^{\frac{1}{\alpha}}}{(1 - o(1))^{\frac{1}{\alpha}}} > \frac{1}{2}\xi^{\frac{1}{\alpha}} \tag{4.1.6}$$

Thus for $k > k_2$, z_k lies inside a Stolz sector which is independent of the particular sequence $\{z_k\}$, i.e. depends only on ξ. We define our Stolz sector T_1 by (4.1.6) and similarly for the other points ζ_i. We choose $R_o < 1$ so that if $|z| > R_o$ then z is either in one and only one of the sectors T_i, or in none of them. It remains to show that $\exists$ R, $R_o < R < 1$, such that for the Stolz sectors we have constructed (4.1.4) is satisfied. Let us call the set of points on $|z| = r$, $R_o < r < 1$, which are in one of the Stolz sectors, $E = E(r)$. And let $E'(r)$ be the complement of $E(r)$ wrt $|z| = r$. Consider $\max_{z \in E'(r)} |f(z)|$. If the lemma were false then this

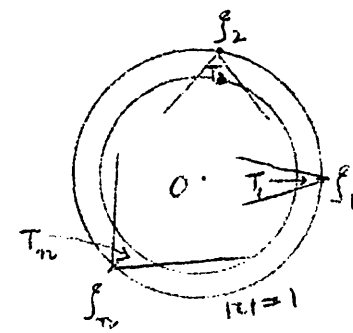


Figure 14

maximum, by our previous argument, would be attained at points inside one of the Stolz sectors and at no points outside nor on the boundary of any of the sectors, for r near to 1. This is clearly impossible, and the lemma follows.

Our later estimates require

Lemma 4.1.B

Suppose δ satisfies $0 < \delta < \pi$ and $\beta > \frac{1}{2}$. Then

$$\left\{ \int_0^{\delta} \frac{d\theta}{|1-re^{i\theta}|^{2\beta}} \leq \frac{1}{\sqrt{r}\cos\frac{1}{2}\delta.(1-r)^{2\beta-1}} J(\beta) \right\} \quad (4.1.7)$$

where $$J(\beta) = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\beta - \frac{1}{2})}{\Gamma(\beta)}$$

Proof

$$|1 - re^{i\theta}|^2 = (1 - r)^2 + 4r \sin^2\tfrac{1}{2}\theta$$

$$= (1 - r)^2 \left\{ 1 + \frac{4r \sin^2 \frac{1}{2}\theta}{(1 - r)^2} \right\}$$

$$\therefore \int_0^{\delta} \frac{d\theta}{|1 - re^{i\theta}|^{2\beta}} = \frac{1}{(1-r)^{2\beta}} \int_0^{\delta} \frac{d\theta}{\left\{ 1 + \frac{4r \sin^2 \frac{1}{2}\theta}{(1-r)^2} \right\}^{\beta}}$$

Set $$\phi(\theta) = \frac{2\sqrt{r} \sin \frac{1}{2}\theta}{1 - r}$$

so $$\frac{d\phi}{d\theta} = \frac{\sqrt{r} \cos \frac{1}{2}\theta}{1 - r}$$

$$\text{hence} \qquad \int_0^{\delta} \frac{d\theta}{|1-re^{i\theta}|^{2\beta}} = \frac{1}{(1-r)^{2\beta-1}} \int_0^{\frac{2\sqrt{r}\sin\frac{1}{2}\delta}{1-r}} \frac{1}{\sqrt{r}\cos\frac{1}{2}\theta} \frac{d\phi}{(1+\phi^2)^{\beta}}$$

$$\leq \frac{1}{\sqrt{r}\cos\frac{1}{2}\delta} \frac{1}{(1-r)^{2\beta-1}} \int_0^{\infty} \frac{d\phi}{(1+\phi^2)^{\beta}}$$

Set $\phi = \tan x$, then

$$\int_0^{\infty} \frac{d\phi}{(1+\phi^2)^{\beta}} = \int_0^{\pi/2} \cos^{2(\beta-1)}x \, dx$$

This is a standard integral, see e.g. [13], page 256.

We obtain

$$\int_0^{\pi/2} \cos^{2(\beta-1)}x \, dx = \frac{1}{2} \frac{\Gamma(\beta-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\beta)}$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, this proves the lemma.

Lemma 4.1.C

Suppose $F(z)$ is regular in $|z| < 1$, and satisfies

$$\text{Re } F(z) > 0 , \quad |F(o)| \leq 1 \quad \text{in} \quad |z| < 1 .$$

Then we have

$$\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2\, d\theta \le \frac{1+3r^2}{1-r^2} \qquad (4.1.8)$$

and $$\frac{1}{2\pi}\int_0^{2\pi} |F'(re^{i\theta})|\, d\theta \le \frac{2}{1-r^2} \qquad (4.1.9)$$

Proof

Let $F(z) = c_0 + c_1 z + c_2 z^2 + \ldots \qquad (|z| < 1).$

It is well known that

$$|c_n| \le 2 \operatorname{Re} F(0), \qquad n = 1, 2, \ldots$$

(e.g. [5] page 15). Hence $|c_0| \le 1$, $|c_n| \le 2$, $n = 1, 2, \ldots$

Thus $$\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}$$

$$\le 1 + 4\sum_{n=1}^{\infty} r^{2n} = \frac{1+3r^2}{1-r^2} \qquad \text{as required.}$$

Next write

$$F(z) = \frac{1+\omega(z)}{1-\omega(z)}$$

Then $\omega(z)$ is regular, $|\omega(z)| < 1$ in $|z| < 1$.

$$F'(z) = \frac{2\,\omega'(z)}{(1 - \omega(z))^2}$$

Hence applying Pick's lemma

$$|F'(z)| \leq \frac{2}{|1 - \omega(z)|^2} \frac{1 - |\omega(z)|^2}{1 - |z|^2}$$

$$= \frac{2}{1 - |z|^2} \operatorname{Re} \frac{1 + \omega(z)}{1 - \omega(z)} = \frac{2}{1 - |z|^2} \operatorname{Re}\ F(z)$$

Thus

$$\frac{1}{2\pi} \int_0^{2\pi} |F'(re^{i\theta})|\, d\theta \leq \frac{2}{1-r^2} \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\ F(re^{i\theta})\, d\theta$$

$$= \frac{2}{1-r^2} \operatorname{Re}\ F(o)$$

$$\leq \frac{2}{1-r^2}, \quad \text{as required.}$$

Section 4.2 A Length Estimate

Theorem 5

If the largest sector in $\mathcal{G}$ has angle $\pi\alpha > 0$, then

$$\overline{\lim_{r\to 1}} \frac{L(r)}{M(r)} \leq \left[\frac{2}{\alpha}\right] \alpha \sqrt{\pi}\, \frac{\Gamma(\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}\alpha + \frac{1}{2})} \qquad (4.2.1)$$

and $$\lim_{r\to 1} \frac{L(r)}{M(r)} \geq \alpha\sqrt{\pi}\, \frac{\Gamma(\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}\alpha + \frac{1}{2})} \qquad (4.2.2)$$

For each $\alpha > 0$ these estimates are sharp.

Proof

We define

$$L(r; G) = r\int_G |f'(re^{i\theta})|\, d\theta$$

where G is a measurable subset of $(0, 2\pi)$. The proof will be divided into estimating $L(r; E)$ and $L(r; E')$ where E is the set defined in section 4.1. We use a method due to Pommerenke [10] to estimate $L(r; E')$. Clearly

$$L(r) = L(r; E) + L(r; E') .$$

We write

$$F(z) = \frac{zf'(z)}{f(z)}, \quad h(z) = z\, f'(z) .$$

Then $$h'(z) = f(z)\left\{ \frac{F^2(z)}{z} + F'(z) \right\}$$

By lemma 4.1.A for each $\varepsilon > 0$ $\exists$ $R < 1$ st

$$|f(z)| < \varepsilon M(r), \quad R < r < 1, \quad z \in E'(r) .$$

Hence

$$\frac{1}{2\pi}\int_{E'} |h'(re^{i\theta})|\,d\theta < \frac{\varepsilon M(r)}{r}\,\frac{1}{2\pi}\int_{E'} |F(re^{i\theta})|^2\,d\theta \qquad (R < r < 1)$$
$$+ \varepsilon M(r)\,\frac{1}{2\pi}\int_{E'} |F'(re^{i\theta})|\,d\theta$$

Hence applying lemma 4.1.C

$$\frac{1}{2\pi}\int_{E'} |h'(re^{i\theta})|\,d\theta \le \varepsilon M(r)\,\frac{1+2r+3r^2}{r(1-r^2)}, \qquad (R < r < 1)$$

Now

$$L(r;E') = \int_{E'} |h(re^{i\theta})|\,d\theta = \int_{E'} \left|\int_0^r h'(\rho e^{i\theta})d\rho\right| d\theta$$
$$\le \int_0^r \int_{E'} |h'(\rho e^{i\theta})|\,d\rho\,d\theta$$
$$\le \int_0^R \int_{E'} |h'(\rho e^{i\theta})|\,d\rho\,d\theta + 2\pi\varepsilon \int_R^r M(\rho)\,\frac{1+2\rho+3\rho^2}{\rho(1-\rho^2)}\,d\rho$$
$$\le B(R) + 2\pi\varepsilon \int_R^r \frac{3\,M(\rho)\,d\rho}{\rho(1-\rho)}$$

We may suppose R chosen large enough so that

$$(1-\rho)\,\frac{\rho\,M'(\rho)}{M(\rho)} > \alpha - \varepsilon, \qquad (R < \rho < 1).$$

Then

$$L(r;E') \le B(R) + \frac{6\pi\varepsilon}{\alpha-\varepsilon}\int_R^r M'(\rho)\,d\rho$$

$$\leq B(R) + \frac{6\pi\xi}{\alpha-\xi} M(r) , \quad (R < r < 1). \qquad (4.2.3)$$

We now estimate $L(r; E)$. We suppose ξ fixed, $0 < \xi < \alpha$, and $R < 1$ chosen so that lemma 4.1.A is satisfied. We have n sectors $S_1, \ldots S_n$ in $\mathcal{G}$ of angle $\pi\alpha > 0$. Let $S = S_1$ and suppose for convenience that S is associated with $z = 1$ in the usual sense. Let T be the Stolz sector at $z = 1$ determined as in lemma 4.1.A. Then by theorem 2.3.A (iii), for $r > r_o$ we have

$$|f(z)| = e^{\eta(z)} \left(\frac{1-r}{|1-z|} \right)^{\alpha} |f(r)|$$

for $|z| = r$, $z \in T$ and where $|\eta(z)| < \xi$.
Also for $r > r_1$,

$$1 \geq \frac{|f(r)|}{M_S(r)} \geq 1 - \xi ,$$

hence

$$|f(z)| = e^{\eta_o(z)} \left(\frac{1-r}{|1-z|} \right)^{\alpha} M_S(r) \qquad (4.2.4)$$

for $r > \max(r_o, r_1)$, and where $|\eta_o(z)| < \xi$, $z \in T$.

Also we have

$$(1 - z) \frac{zf'(z)}{f(z)} \to \alpha \quad \text{as} \quad z \to 1 \quad \text{in} \quad T .$$

Hence for $r > r_2$, since the convergence is clearly uniform in $\overline{T}$,

$$\left| \frac{z\, f'(z)}{f(z)} \right| = \frac{\alpha + \eta_1(z)}{|1 - z|}$$

where $|\eta_1(z)| < \xi$, for $r > r_2$, $z \in T$. We may suppose R has been chosen greater than $\max (r_o, r_1, r_2)$, and then we have for $r > R$,

$$|z\, f'(z)| = \frac{(\alpha + \eta_1(z))\, e^{\eta_o(z)}\, (1 - r)^{\alpha}\, M_S(r)}{|1 - z|^{\alpha+1}} , \quad (4.2.5)$$

where $|\eta_o(z)| < \xi$, $|\eta_1(z)| < \xi$, $z \in T$.

Let $F_1, F_2, \ldots F_n$ be the respective contributions to E on $|z| = r$ of the Stolz sectors $T_1 = T, T_2, \ldots T_n$. So that

$$E = \bigcup_{i=1}^{n} F_i \text{ is a union of disjoint arcs.}$$

Write $F = F_1$. We estimate $L(r; F)$. It is clear that we may choose each arc F_i symmetric wrt the point ξ_i , so that

$$L(r; F) = 2r \int_0^{\delta} |f'(re^{i\theta})|\, d\theta$$

where $\delta = \delta(r) \to 0$ as $r \to 1$, $\delta(r) > 0$, and $\frac{\delta(r)}{1-r}$ remains bounded as $r \to 1$. Applying lemma 4.1.B, we have for $r > R$,

$$L(r;\,F) \;\le\; 2(\alpha+\varepsilon)\,e^{\varepsilon}\,M_S(r)(1-r)^{\alpha}\int_0^{\delta}\frac{d\theta}{|1-re^{i\theta}|^{\alpha+1}}$$

$$\le\; \frac{2(\alpha+\varepsilon)e^{\varepsilon}\,M_S(r)}{\sqrt{r}\,\cos\frac{1}{2}\delta(r)}\;J\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\right) \qquad (4.2.6)$$

Hence

$$L(r;\,E) \;\le\; \frac{2n(\alpha+\varepsilon)e^{\varepsilon}\,M(r)}{\sqrt{r}\,\cos\frac{1}{2}\delta(r)}\;J\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\right) \qquad (4.2.7)$$

for $r > R$. From (4.2.3) and (4.2.7) we obtain

$$\overline{\lim_{r\to 1}}\;\frac{L(r)}{M(r)} \;\le\; 2n(\alpha+\varepsilon)e^{\varepsilon}\;J\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\right)+\frac{6\pi\varepsilon}{\alpha-\varepsilon}$$

Since $\varepsilon > 0$ can be made as small as we please, we deduce

$$\left.\begin{aligned}\overline{\lim_{r\to 1}}\;\frac{L(r)}{M(r)} &\le 2n\alpha\;J\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\right)\\ &\le 4\;J\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\right)\end{aligned}\right\} \qquad (4.2.8)$$

since clearly $n\alpha\pi \le 2\pi$. This proves (4.2.1).

To complete the theorem we prove the following lemma.

Lemma 4.2.A

$$\lim_{r\to 1}\;\frac{L_S(r)}{M_S(r)} \;=\; 2\alpha\;J\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\right) \qquad (4.2.9)$$

where $L_S(r)$ is the length of that part of the image curve $C(r)$ of $|z| = r$ which lies in the sector S.

Proof

From (4.2.6) we have

$$\overline{\lim_{r\to 1}} \frac{L(r; F)}{M_S(r)} \leq 2\alpha J(\tfrac{1}{2}\alpha + \tfrac{1}{2})$$

That part of $C(r)$ in S not covered by the image of F is clearly such that $|f(z)| < \xi M_S(r)$, for $R < r < 1$, over this part of $C(r)$. This follows from the argument of lemma 4.1.A. We then argue as in obtaining (4.2.3) and see that the length over this part has the order $\xi M_S(r)$, and so

$$\overline{\lim_{r\to 1}} \frac{L_S(r)}{M_S(r)} \leq 2\alpha J(\tfrac{1}{2}\alpha + \tfrac{1}{2}) .$$

Now from (4.2.5) we have

$$|z f'(z)| \geq \frac{(\alpha - \xi) e^{-\xi} (1 - r)^{\alpha} M_S(r)}{|1 - z|^{\alpha+1}} \qquad (4.2.10)$$

for $z \in T$, $r > R$. Hence for $r > R$,

$$L(r; F) \geq 2(\alpha - \xi) e^{-\xi} M_S(r)(1 - r)^{\alpha} \int_0^{\delta} \frac{d\theta}{|1 - re^{i\theta}|^{\alpha+1}}$$

$$= 2(\alpha-\xi)e^{-\xi} M_S(r) \int_0^{\frac{2\sqrt{r}\sin\frac{1}{2}\delta}{1-r}} \frac{1}{\sqrt{r}\cos\frac{1}{2}\theta} \frac{d\phi}{(1+\phi)^{\frac{1}{2}(\alpha+1)}}$$

$$\geq 2(\alpha-\xi)e^{-\xi} M_S(r) \int_0^{\frac{2\sqrt{r}\sin\frac{1}{2}\delta}{1-r}} \frac{d\phi}{(1+\phi^2)^{\frac{1}{2}(\alpha+1)}} \quad , \quad \text{as in lemma 4.1.B.}$$

Now by lemma 4.1.A the Stolz sector T depends on ξ and can be made with angle tending to π as $\xi \to 0$. And so

$$\lim_{r\to 1} \frac{2\sqrt{r}\sin\frac{1}{2}\delta(r)}{1-r} \geq K = K(\xi) ,$$

where $K(\xi) \to \infty$ as $\xi \to 0$.

It follows that

$$\lim_{r\to 1} \frac{L_S(r)}{M_S(r)} \geq \lim_{r\to 1} \frac{L(r;F)}{M_S(r)}$$

$$\geq 2(\alpha-\xi)\, e^{-\xi} \int_0^K \frac{d\phi}{(1+\phi^2)^{\frac{1}{2}\alpha+\frac{1}{2}}}$$

$$\to 2\alpha\, J(\tfrac{1}{2}\alpha+\tfrac{1}{2}) \quad \text{as} \quad \xi \to 0 .$$

This proves the lemma.

Now for $r > r_o$ we have

$$M(r) = M_{S_i}(r) ,$$

where $i = i(r)$ and $1 \leq i \leq n$.

Hence $$\frac{L(r)}{M(r)} \geq \frac{L_{S_i}(r)}{M_{S_i}(r)} \to 2\alpha\, J(\tfrac{1}{2}\alpha + \tfrac{1}{2}) \quad \text{as} \quad r \to 1$$

and so (4.2.2) follows.

If $n = 1$ then (4.2.2) is sharp. For $n > 1$ we can easily construct functions with n sectors of angle $\pi\alpha$ in their image domains st

$$M_S(r) \sim M(r) \quad \text{as} \quad r \to 1 ,$$

for each such sector S . So (4.2.1) is best possible.

Section 4.3 An Area Estimate

Theorem 6

If the largest sector in $\mathcal{Y}$ has angle $\pi\alpha \geq 0$, then

$$\overline{\lim_{r\to 1}} \frac{A(r)}{M^2(r)} \leq [\tfrac{2}{\alpha}] \frac{\alpha\sqrt{\pi}}{2} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 1)} \tag{4.3.1}$$

and $$\underline{\lim_{r\to 1}} \frac{A(r)}{M^2(r)} \geq \frac{\alpha\sqrt{\pi}}{2} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 1)} \tag{4.3.2}$$

For each α , $0 \leq \alpha \leq 2$, these estimates are sharp.

Proof

In the case $\alpha = 0$ both estimates are trivial. (4.3.1) is sharp for the function $f(z) = z$, and (4.3.2) is sharp for the function $f(z) = \log \frac{1}{1-z}$.

Assume $\alpha > 0$. With the same notation as section 4.2 we have

$$A(r) = A(R) + \int_R^r \frac{d\rho}{\rho} \int_0^{2\pi} \left|\rho\, f'(\rho e^{i\theta})\right|^2 d\theta$$

$$\leq A(R) + \pi\ell^2\, M^2(r) + \int_R^r \frac{d\rho}{\rho} \int_E \left|\rho\, f'(\rho e^{i\theta})\right|^2 d\theta \qquad (4.3.3)$$

Using equation (4.2.5) we have

$$\int_0^{\delta} \left|\rho f'(\rho e^{i\theta})\right|^2 d\theta \leq (\alpha+\ell)^2\, e^{2\ell}\, M_S^2(\rho)(1-\rho)^{2\alpha} \int_0^{\delta} \frac{d\theta}{\left|1-\rho e^{i\theta}\right|^{2(\alpha+1)}}$$

$$\leq (\alpha+\ell)^2\, e^{2\ell}\, J(\alpha+1)\, \frac{1}{\sqrt{\rho}\cos\frac{1}{2}\delta}\, \frac{M_S^2(\rho)}{1-\rho}$$

Now $(1-\rho)\,\rho\sqrt{\rho}\cos\frac{1}{2}\delta(\rho)\,\dfrac{M_S'(\rho)}{M_S(\rho)} \to \alpha$ as $\rho \to 1$.

Hence we may suppose R has been chosen large enough so that

$$(1-\rho)\ \rho\sqrt{\rho}\ \cos\tfrac{1}{2}\delta\ \frac{M_S'(\rho)}{M_S(\rho)} \geq \alpha-\xi\ , \quad (R < \rho < 1).$$

We then have

$$2\int_R^r \frac{d\rho}{\rho}\int_0^\delta |\rho\ f'(\rho e^{i\theta})|^2\ d\theta$$

$$\leq \frac{(\alpha+\xi)^2\ e^{2\xi}\ J(\alpha+1)}{\alpha-\xi}\int_R^r 2\ M_S(\rho)\ M_S'(\rho)\ d\rho$$

$$\leq \frac{(\alpha+\xi)^2\ e^{2\xi}\ J(\alpha+1)}{\alpha-\xi}\ M_S^2(r) \qquad (4.3.4)$$

From (4.3.3) and (4.3.4) we easily deduce

$$\left.\begin{aligned} \overline{\lim_{r\to 1}}\ \frac{A(r)}{M^2(r)} &\leq n\ \frac{(\alpha+\xi)^2\ e^{2\xi}}{\alpha-\xi}\ J(\alpha+1) + \pi\xi^2 \\ &\to n\ \alpha\ \ J(\alpha+1) \quad \text{as} \quad \xi \to 0 \\ &\leq 2\ J(\alpha+1) \end{aligned}\right\} \qquad (4.3.5)$$

which proves (4.3.1).

Lemma 4.3.A

$$\lim_{r\to 1}\ \frac{A_S(r)}{M_S^2(r)} = \pi\ J(\alpha+1) \qquad (4.3.6)$$

where $A_S(r)$ is the area of that part of $\mathcal{S}$ cut off by $C(r)$ and S .

Proof

From (4.3.4) and the fact that $|f(z)| < \xi M_S(r)$ over that part of the inverse image of $C(r) \cap S$ not covered by F , it is clear that

$$\overline{\lim_{r\to 1}} \frac{A_S(r)}{M_S^2(r)} \leq \alpha J(\alpha+1) \tag{4.3.7}$$

Let

$$A(r; F) = 2 \int_R^r \int_0^\delta |f'(\rho e^{i\theta})|^2 \rho \, d\rho \, d\theta$$

$$\geq 2 \int_R^r \frac{d\rho}{\rho} \int_0^\delta \frac{(\alpha-\xi)^2 e^{-2\xi} (1-\rho)^{2\alpha} M_S^2(\rho) \, d\phi}{|1-\rho e^{i\theta}|^{2(\alpha+1)}},$$

applying (4.2.10). As in lemma 4.1.B,

$$A(r; F) \geq 2(\alpha-\xi)^2 e^{-2\xi} \int_R^r \frac{M_S^2(\rho)}{1-\rho} d\rho \int_0^{\frac{2\sqrt{\rho} \sin \frac{1}{2}\delta}{1-\rho}} \frac{d\phi}{(1+\phi^2)^{\alpha+1}}$$

$$\geq 2(\alpha-\xi)^2 e^{-2\xi} \int_0^K \frac{d\phi}{(1+\phi^2)^{\alpha+1}} \cdot \int_R^r \frac{M_S^2(\rho)}{1-\rho} d\rho$$

where $K = K(\xi) \to \infty$ as $\xi \to 0$. Assuming R chosen large enough

$$2 \int_R^r \frac{M_S^2(\rho)}{1-\rho}\, d\rho \geq \frac{1}{\alpha+\xi} \int_R^r 2\, M_S'(\rho)\, M_S(\rho)\, d\rho$$

$$= \frac{1}{\alpha+\xi} \left(M_S^2(r) - M_S^2(R)\right) .$$

Thus

$$\lim_{r\to 1} \frac{A(r;\, F)}{M_S^2(r)} \geq \frac{(\alpha-\xi)^2\, e^{-2\xi}}{\alpha+\xi} \int_0^K \frac{d\phi}{(1+\phi^2)^{\alpha+1}}$$

Also clearly $\lim_{r\to 1} \frac{A_S(r)}{M_S^2(r)} \geq \lim_{r\to 1} \frac{A(r;\, F)}{M_S^2(r)}$. Hence letting $\xi \to 0$ we deduce,

$$\lim_{r\to 1} \frac{A_S(r)}{M_S^2(r)} \geq \alpha\, J(\alpha+1) \qquad (4.3.8)$$

and so with (4.3.7) the lemma follows.

Theorem 6 now is completed as we completed theorem 5.

Section 4.4

We consider now more general integral means. The idea remains the same, namely to connect the integral means with the maximum modulus using (4.2.4) and (4.2.5), and to show

that contributions over E' do not affect our estimates. We confine ourselves to proving two results.

Theorem 7

If the largest sector in $\mathcal{G}$ has angle $\pi\alpha > 0$, and if $p\alpha > 1$, then

$$\overline{\lim_{r\to 1}} \frac{I_p(r,f)}{(1-r)M^p(r,f)} \leq [\tfrac{2}{\alpha}] \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}p\alpha - \frac{1}{2})}{\Gamma(\frac{1}{2}p\alpha)} \qquad (4.4.1)$$

and $$\underline{\lim_{r\to 1}} \frac{I_p(r,f)}{(1-r)M^p(r,f)} \geq \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}p\alpha - \frac{1}{2})}{\Gamma(\frac{1}{2}p\alpha)} \qquad (4. .2)$$

For each α, $0 < \alpha \leq 2$, these estimates are sharp.

Proof

With our usual notation and R near enough to 1, we have for $R < r < 1$,

$$I_p(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta = \frac{1}{2\pi}\int_E + \frac{1}{2\pi}\int_{E'} = I_1 + I_2 \text{ say.}$$

To estimate I_1 we use (4.2.4) and consider

$$\int_0^{\delta} |f(re^{i\theta})|^p \, d\theta \leq (1+\xi)^p \int_0^{\delta} \frac{(1-r)^{\alpha p} M_S^p(r)}{|1-re^{i\theta}|^{\alpha p}} \, d\theta$$

$$\leq (1+\epsilon)^p \frac{(1-r)^{\alpha p} M_S^p(r)}{\sqrt{r}\cos\frac{1}{2}\delta\,(1-r)^{\alpha p-1}} J(\tfrac{1}{2}\alpha p) \quad \text{by lemma 4.1.B}$$

$$= \frac{(1+\epsilon)^p}{\sqrt{r}\cos\frac{1}{2}\delta}(1-r)\,M_S^p(r)\;J(\tfrac{1}{2}\alpha p) \qquad (4.4.3)$$

Thus (4.4.1) follows provided we can prove

$$I_2 \leq K\epsilon(1-r)\,M^p(r) + O(1)$$

for some constant K. We remark that a $O(1)$ does not affect matters since if A is any constant

$$\frac{A}{(1-r)\,M^p(r)} \to 0 \quad \text{as} \quad r \to 1 .$$

To see this we note that

$$(1-r)\,M^p(r) = \left\{(1-r)^\alpha M^{p\alpha}(r)\right\}^{\frac{1}{\alpha}}$$

$$= \left\{(1-r)^\alpha M(r)\,M^\delta(r)\right\}^{\frac{1}{\alpha}} \to \infty \quad \text{as} \quad r \to 1$$

since $p\alpha > 1$, so $\delta > 0$, and $M^\delta(r) \to \infty$, since $\alpha > 0$, and $(1-r)^\alpha M(r) > C > 0$ as $r \to 1$.

Let

$$h(z) = (f(z))^p$$

so $$h'(z) = p\,f'(z)\,f^{p-1}(z)$$

Thus

$$2\pi I_2 = \int_{E'} \left| h(re^{i\theta}) \right| d\theta = \int_{E'} \left| \int_0^r h'(\rho e^{i\theta}) d\rho \right| d\theta$$

$$\leq \int_0^r d\rho \int_{E'} \left| h'(\rho e^{i\theta}) \right| d\theta$$

$$= p \int_R^r d\rho \int_{E'} \left| f'(\rho e^{i\theta}) \right| \left| f(\rho e^{i\theta}) \right|^{p-1} d\theta + O(1)$$

$$= p \int_R^r \frac{d\rho}{\rho} \int_{E'} \left| F(\rho e^{i\theta}) \right| \left| f(\rho e^{i\theta}) \right|^{p} d\theta + O(1)$$

where $F(z) = \dfrac{z\, f'(z)}{f(z)}$.

Let $g(z) = F(z)\, f^p(z)$

so $g'(z) = F'\, f^p + p\, F\, f^{p-1}\, f'$

$$= f^p \left\{ F' + \frac{p}{z} F^2 \right\}$$

Thus

$$\int_{E'} \left| g(\rho e^{i\theta}) \right| d\theta = \int_{E'} \left| \int_0^\rho g'(\sigma e^{i\theta})\, d\sigma \right| d\theta$$

$$\leq \int_R^\rho \int_{E'} \left| g'(\sigma e^{i\theta}) \right| d\sigma\, d\theta + O(1)$$

$$\leq \quad \xi \int_R^\rho M^p(\sigma)\, d\sigma \int_{E'} \left\{ |F'| + \frac{p}{\sigma} |F|^2 \right\} d\theta + O(1)$$

$$\leq \quad A\xi \int_R^\rho \frac{M^p(\sigma)\, d\sigma}{1 - \sigma} + O(1) \,, \quad \text{using lemma 4.1.C.}$$

Now for $R < \sigma < 1$,

$$(1 - \sigma) \frac{M'(\sigma)}{M(\sigma)} > \alpha - \xi$$

so $$\int_R^\rho \frac{M^p(\sigma) d\sigma}{1 - \sigma} \leq \frac{1}{\alpha - \xi} \int_R^\rho M^{p-1}(\sigma)\, M'(\sigma)\, d\sigma$$

$$\leq \frac{1}{p(\alpha - \xi)} M^p(\rho)$$

Thus

$$\int_{E'} \left| g(\rho e^{i\theta}) \right| d\theta \leq A_1 \xi\, M^p(\rho) + O(1)$$

Hence

$$2\pi I_2 \leq A_3 \xi \int_R^r M^p(\rho)\, d\rho + O(1)$$

Consider

$$u(\rho) = (1 - \rho)\, M^p(\rho)$$

then $$u'(\rho) = - M^p(\rho) + p(1-\rho)\, M^{p-1}(\rho)\, M'(\rho)$$

Hence

$$\int_R^r M^p(\rho)\, d\rho = -(1-r)\, M^p(r) + (1-R)\, M^p(R) + p \int_R^r (1-\rho)\, M^{p-1}(\rho)\, M'(\rho)\, d\rho$$

$$\geq (1-R)\, M^p(R) - (1-r)\, M^p(r) + p \int_R^r (\alpha-\xi)\, M^p(\rho)\, d\rho$$

Hence $\{p(\alpha-\xi) - 1\} \cdot \int_R^r M^p(\rho)\, d\rho \leq (1-r)\, M^p(r)$.

Since $p\alpha - 1 > 0$ we may suppose ξ has been chosen sufficiently small so that $p\alpha - 1 > p\xi$. We then have

$$I_2 \leq A_4 \xi\, (1 - r)\, M^p(r) + O(1)$$

as required. Thus (4.4.1) follows.

Using (4.4.3) the same argument as in lemma 4.2.A easily yields an analogous result, and the rest of the theorem follows easily.

Theorem 8

If the largest sector in $\mathcal{G}$ has angle $\pi\alpha > 0$, and if $p(\alpha + 1) > 1$, then

$$\overline{\lim_{r\to 1}} \frac{(1-r)^{p-1} I_p(r,f')}{M^p(r,f)} \leq [\tfrac{2}{\alpha}] \frac{\alpha^p}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}p(\alpha+1) - \frac{1}{2})}{\Gamma(\frac{1}{2}p(\alpha+1))} \quad (4.4.4)$$

and

$$\underline{\lim_{r\to 1}} \frac{(1-r)^{p-1} I_p(r,f')}{M^p(r,f)} \geq \frac{\alpha^p}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}p(\alpha+1) - \frac{1}{2})}{\Gamma(\frac{1}{2}p(\alpha+1))} \quad (4.4.5)$$

For each α, $0 < \alpha \leq 2$ these estimates are sharp.

Proof

For $R < r < 1$ we have

$$I_p(r,f') = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^p \, d\theta$$

$$= \frac{1}{2\pi} \int_E + \frac{1}{2\pi} \int_{E'} = I_1 + I_2 \text{ say .}$$

To estimate I_1 we use (4.2.5), so that

$$\int_0^{\delta} |f'(re^{i\theta})|^p \, d\theta \leq (\alpha+\ell)^p e^{p\ell} (1-r)^{\alpha p} M_S^p(r) \int_0^{\delta} \frac{d\theta}{|1-re^{i\theta}|^{p(\alpha+1)}}$$

$$\leq \frac{(\alpha+\ell)^p M_S^p(r)}{(1-r)^{p-1}\sqrt{r}\cos\frac{1}{2}\delta} J(\tfrac{1}{2}p(\alpha+1)) \quad (4.4.6)$$

provided $p(\alpha+1) > 1$. Thus (4.4.4) easily follows if we can show that

$$I_2 \leq K\xi \frac{M^p(r)}{(1-r)^{p-1}} + O(1)$$

As in theorem 7 a $O(1)$ term does not affect matters since

$$\frac{M^p(r)}{(1-r)^{p-1}} \geq \frac{C(1-r)^{-\alpha p}}{(1-r)^{p-1}} = \frac{C}{(1-r)^{p(1+\alpha)-1}}$$

$\to \infty$ as $r \to 1$, since $p(1+\alpha) > 1$.

Consider then

$$2\pi I_2 = \frac{1}{r^p} \int_{E'} \left| rf'(re^{i\theta}) \right|^p d\theta$$

$$= \frac{1}{r^p} \int_{E'} \left| F(re^{i\theta}) \right|^p \left| f(re^{i\theta}) \right|^p d\theta \qquad (4.4.7)$$

where $F(z) = \frac{z\, f'(z)}{f(z)}$. We choose k, k' to satisfy

(i) $\frac{1}{k} + \frac{1}{k'} = 1$

(ii) $pk > 1$

(iii) $pk' > \frac{1}{\alpha}$.

If $\alpha p \geq 1$ we choose k to satisfy

$$k > \max\left(1, \frac{1}{p}\right)$$

and define k' by (i). And then

$$pk' = \frac{p\,k}{k-1} > p \geq \frac{1}{\alpha}$$

so (i), (ii) and (iii) are satisfied.

If $\alpha p < 1$ then since $p(\alpha + 1) > 1$ we have

$$\frac{1}{p} < \frac{1}{1 - \alpha p}$$

and $$1 < \frac{1}{1 - \alpha p}$$

So we choose k to satisfy

$$\max\left(1, \frac{1}{p}\right) < k < \frac{1}{1 - \alpha p}$$

and define k' by (i). We have

$$k(1 - \alpha p) < 1$$

and so $$\alpha p k > k - 1 .$$

Hence $$pk' = \frac{p\,k}{k-1} > \frac{1}{\alpha}$$

so again (i), (ii), and (iii) are satisfied.

We now apply Hölder's Inequality to (4.4.7). We obtain

$$2\pi I_2 \leq \frac{1}{r^p}\left\{\int_{E'} |F|^{pk}\, d\theta\right\}^{\frac{1}{k}} \left\{\int_{E'} |f|^{pk'}\, d\theta\right\}^{\frac{1}{k'}} . \qquad (4.4.8)$$

Let $g(z) = (F(z))^{pk}$

so $g'(z) = pk\, F^{pk-1}\, F'$

Hence

$$\int_{E'} |F^{pk}|\, d\theta \leq \int_0^r d\rho \int_0^{2\pi} pk\,|F(\rho e^{i\theta})|^{pk-1}\,|F'(\rho e^{i\theta})|\, d\theta$$

$$\leq \int_0^r \frac{A_1}{(1-\rho)^{pk-1}} \int_0^{2\pi} |F'(\rho e^{i\theta})|\, d\theta + O(1)$$

$$\leq \frac{A_2}{(1-r)^{pk-1}} \qquad \text{by lemma 4.1.C.}$$

Again, consider

$$\int_{E'} |f(re^{i\theta})|^{pk'} d\theta \leq A_3 \varepsilon\, (1-r)\, M^{pk'}(r) + O(1)$$

for $R < r < 1$, as in the proof of theorem 7.

Hence from (4.4.8) we deduce

$$I_2 \leq \frac{A_4}{(1-r)^{p-\frac{1}{k}}} \cdot \frac{A_5\, \varepsilon^{\frac{1}{k'}}\, M^p(r)}{(1-r)^{-\frac{1}{k'}}}$$

$$= \frac{A_6\, \varepsilon^{\frac{1}{k'}}\, M^p(r)}{(1-r)^{p-1}}, \qquad (R < r < 1).$$

(4.4.4) now follows immediately. The rest of the theorem is again a repeat of the argument of lemma 4.2.A.

For values of $p \leq \frac{1}{\alpha}$ for the means $I_p(r,f)$ or $p \leq \frac{1}{\alpha+1}$ for the means $I_p(r,f')$ we remark that the contribution to the integrals over E' then becomes significant, and our method therefore does not work.

We state finally the following theorem without proof.

Theorem

Let $f(z)$ be starlike in $|z| < 1$, and let $\{\pi\lambda_\nu\}$, $\nu = 1, 2, \ldots$ be the angles of all the sectors in $\mathcal{G}$, the image domain of f. Let $F(z) = z f'(z)/f(z)$. Then for $p > 1$,

$$\lim_{r\to 1} (1 - r)^{p-1} I_p(r,F) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}(p-1))}{\Gamma(\frac{1}{2} p)} \sum_\nu \lambda_\nu^p \quad (4.4.9)$$

This result is due to W. K. Hayman [6]. Professor Hayman employs the integral representation (1.1.3) in his proof. If we use theorem 3 the result follows quite easily by Professor Hayman's method.

Chapter 5. Bounded Convex Functions - a Coefficient Problem

Section 5.1

Let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (|z| < 1) \qquad (5.1.1)$$

be a convex univalent function in $|z| < 1$. We suppose $w = g(z)$ maps $|z| < 1$ onto the convex domain D. This means that if w_1, w_2 are distinct points in D then the line segment joining w_1 and w_2 is in D. It is well known (e.g. [8] page 223) that a necessary and sufficient condition for a function $g(z)$ of the form (5.1.1) to be convex is that the function

$$f(z) = z\, g'(z) \qquad (|z| < 1)$$

be starlike univalent in $|z| < 1$. Regarding the coefficients of $g(z)$ it is well known (e.g. [5] page 12) that

$$|b_n| \leq 1, \qquad (n = 1, 2, \ldots)$$

and this inequality is sharp, since

$$w = g(z) = \frac{z}{1 - z} = \sum_{1}^{\infty} z^n$$

maps $|z| < 1$ onto the convex domain $\operatorname{Re} w > -\frac{1}{2}$.

If $g(z)$ is bounded in $|z| < 1$, J. Clunie and F. R. Keough [3] have shown that

$$b_n = o(n^{-1}) \quad \text{as} \quad n \to \infty .$$

Ch. Pommerenke [9] has further proved that if

$$|g(z)| < B \qquad (|z| < 1)$$

then for each $\varepsilon > 0$

$$|b_n| < A n^{-1-(1-\varepsilon)/\pi B} , \qquad (n = 2,3, \ldots)$$

where A depends only on B and ε.

In this chapter we will prove

Theorem 9

Suppose

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{5.1.2}$$

is a convex univalent function in $|z| < 1$, and suppose

$$|g(z)| < B , \qquad (|z| < 1) . \tag{5.1.3}$$

Then $\exists$ a constant $\delta > 0$, depending on g, such that

$$b_n = o(n^{-1-\frac{1}{2B}-\delta}) \quad \text{as} \quad n \to \infty . \tag{5.1.4}$$

Furthermore for each $\varepsilon > 0$ $\exists$ a convex function $g(z)$ and

a constant B, where $g(z)$ has the form (5.1.2) and $|g(z)| < B$ in $|z| < 1$, and $\exists$ a sequence of increasing natural numbers $\{n_k\}$, $k = 1, 2, \ldots$, such that

$$n_k^{1+\frac{1}{(2-\varepsilon)B}} |b_{n_k}| \to \infty \quad \text{as} \quad k \to \infty. \qquad (5.1.5)$$

Remarks

(5.1.4) is a sharpening of Pommerenke's result and essentially reduces his constant π in the index of n to the constant 2. (5.1.5) shows that this constant 2 cannot be improved. However the existence of δ in (5.1.4) indicates the possibility of further improvement to the index of n, possibly by the introduction of higher powers $-\frac{1}{B^2}$ etc. What is needed to completely solve the problem is the calculation of an extremal function for each fixed B.

Section 5.2

We shall need the following theorem proved by Pommerenke [9], and based on a result of Clunie and Keough [3]. We give the complete proof.

Theorem 5.2.A

Let $$f(z) = \sum_1^\infty a_n z^n$$

be starlike univalent in $|z| < 1$, and suppose

$$M(r) \leq \frac{1}{(1 - r)^{\alpha}}, \qquad 0 \leq r < 1,$$

where $0 \leq \alpha \leq 2$. Then

$$|a_n| < 2(e/\alpha)^{\alpha} (n + 1)^{\alpha-1}$$

for $n > 1$.

Proof

Since $f(z)$ is starlike we may write

$$\frac{z f'(z)}{f(z)} = \frac{1 + \omega(z)}{1 - \omega(z)}$$

where $\omega(z) = \sum_{1}^{\infty} \omega_n z^n$ and $|\omega(z)| < 1$ in $|z| < 1$.
We then have

$$\{z f'(z) + f(z)\} \omega(z) = z f'(z) - f(z)$$

i.e.
$$\left\{ \sum_{k=1}^{\infty} (k+1)a_k z^k \right\} \left\{ \sum_{1}^{\infty} \omega_k z^k \right\} = \sum_{k=2}^{\infty} (k-1)a_k z^k \qquad (5.2.1)$$

Equating coefficients we obtain

$$(n - 1)a_n = 2 a_1 \omega_{n-1} + 3a_2 \omega_{n-2} + \ldots + n a_{n-1} \omega_1$$

Thus in equation (5.2.1) the coefficient a_n on the right

depends only on $a_1, a_2, \ldots a_{n-1}$ on the left. Hence we can write (5.2.1) in the form

$$\left\{ \sum_{k=1}^{n-1} (k+1)a_k z^k \right\} \omega(z) = \sum_{k=2}^{n} (k-1)a_k z^k + \sum_{n+1}^{\infty} c_k z^k$$

for $n > 1$. Squaring the moduli of both sides and integrating round the circle $|z| = r < 1$, while using the fact that $|\omega(z)| < 1$, we easily deduce

$$\sum_{k=2}^{n} (k-1)^2 |a_k|^2 r^{2k} + \sum_{n+1}^{\infty} |c_k|^2 r^{2k} \leq \sum_{k=1}^{n-1} (k+1)^2 |a_k|^2 r^{2k}$$

Letting $r \to 1$ we deduce

$$\sum_{k=2}^{n} (k-1)^2 |a_k|^2 \leq \sum_{k=1}^{n-1} (k+1)^2 |a_k|^2$$

$$\text{i.e.} \qquad (n-1)^2 |a_n|^2 \leq 4 \sum_{k=1}^{n-1} k |a_k|^2$$

Adding $4n|a_n|^2$ to each side we obtain

$$(n+1)^2 |a_n|^2 \leq 4 \sum_{k=1}^{n} k |a_k|^2 \qquad (n > 1) \qquad (5.2.2)$$

Suppose now $\alpha = 0$, so that $|f(z)| < 1$ in $|z| < 1$. Then the image of $|z| < 1$ under f has area $\leq \pi$, so by the area theorem,

$$\sum_{k=1}^{\infty} k\,|a_k|^2 \leq 1$$

and so

$$|a_n| \leq \frac{2}{n+1} \qquad \text{from (5.2.2).}$$

Equality occurs only for $f(z) \equiv a_1 z$, so this proves the theorem in the case $\alpha = 0$.

Assume now $\alpha > 0$. By the area theorem

$$\sum_{k=1}^{\infty} k\,|a_k|^2\, r^{2k} \leq M^2(r) \qquad (5.2.3)$$

and so (5.2.2) yields

$$(n+1)|a_n| \leq 2r^{-n}\left(\sum_{k=1}^{n} k|a_k|^2\, r^{2k}\right)^{\frac{1}{2}}$$

$$\leq 2r^{-n} M(r) \leq 2r^{-n}(1-r)^{-\alpha}$$

Setting $r = \frac{n}{n+\alpha}$ we obtain

$$(n+1)\,|a_n| \leq 2\,\frac{(n+\alpha)^{n+\alpha}}{n^n\,\alpha^{\alpha}}$$

$$< 2\,e^{\alpha}\,\alpha^{-\alpha}\,(n+1)^{\alpha}$$

which proves the theorem.

We observe that for large values of n we can improve theorem 5.2.A by using our estimate (4.3.1) for the area in (5.2.3) instead of the estimate

$$A(r) \leq \pi M^2(r) .$$

In fact we have

$$A(r) \leq \left\{ \frac{\Gamma(\frac{1}{2})\,\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)} + o(1) \right\} M^2(r) \quad \text{as } r \to 1 .$$

Hence $(n+1)\,|a_n| \leq 2r^{-n}\,\frac{1}{\sqrt{\pi}}\,\sqrt{A(r)}$

$$\leq 2r^{-n} \left\{ \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\alpha+1)} + o(1) \right\}^{\frac{1}{2}} M(r)$$

$$\leq 2r^{-n} \left\{ \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\alpha+1)} + o(1) \right\}^{\frac{1}{2}} \frac{1}{(1-r)^{\alpha}} \quad \text{as } r \to 1 .$$

Setting $r = \frac{n}{n+\alpha}$ we then obtain

$$(n+1)|a_n| < 2e^{\alpha}\alpha^{-\alpha} \left\{ \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\alpha+1)} + o(1) \right\} (n+1)^{\alpha} \quad \text{as } n \to \infty .$$

Thus for $n > n_0(\epsilon)$,

$$|a_n| \leq 2e^{\alpha}\,\alpha^{-\alpha} \left\{ \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\alpha+1)} + \epsilon \right\}^{\frac{1}{2}} (n+1)^{\alpha-1} .$$

i.e. $$\overline{\lim_{n\to\infty}}\ (n+1)^{1-\alpha}\,|a_n| \leq 2e^{\alpha}\alpha^{-\alpha} \left\{ \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\alpha+1)} \right\}^{\frac{1}{2}} . \qquad (5.2.4)$$

Lemma 5.2.B

Suppose $\exists\ K > 0$ such that

$$M(r) \geq \frac{Kr}{(1 - r)^{\alpha}} , \quad 0 \leq r < 1 ,$$

where $\alpha \geq 0$. Suppose further that $\exists\ \beta > 0$ such that

$$a_n = o(n^{\beta-1}) \quad \text{as} \quad n \to \infty .$$

Then

$$\alpha < \beta .$$

Proof

Given $\xi > 0$ $\exists$ N st $\forall$ $n > N$

$$|a_n| < \xi n^{\beta-1}$$

Hence

$$M(r) \leq \sum_{1}^{\infty} |a_n| r^n$$

$$< \sum_{1}^{N} |a_n| r^n + \sum_{n=N+1}^{\infty} n^{\beta-1} r^n$$

$$< K_1 + \xi K_2 \frac{1}{(1 - r)^{\beta}}$$

Hence $\overline{\lim_{r\to 1}} (1-r)^{\beta} M(r) = 0$. Thus $\alpha < \beta$ as required. From this lemma it immediately follows that $\exists$ a sequence

of natural numbers $\{n_k\}$, $k = 1, 2, \ldots$ such that

$$|a_{n_k}| \geq c\, n_k^{\alpha-1} , \quad k = 1, 2, \ldots$$

for some $c > 0$.

Section 5.3

In this section we outline Pommerenke's approach to our coefficient problem. Let $g(z)$ satisfy the hypotheses of theorem 9 and write

$$f(z) = z\, g'(z) = z + \sum_{2}^{\infty} a_n z^n \qquad (|z| < 1)$$

so that $f(z)$ is starlike and

$$a_n = n\, b_n , \qquad n = 1, 2, \ldots$$

Let $\pi\alpha$ be the angle of the largest sector contained in the image domain of $f(z)$. Then by theorem 1.8.A, for each $\varepsilon > 0$ $\exists$ a constant $k = k(\varepsilon)$ such that

$$M(r,f) \leq \frac{k}{(1 - r)^{\alpha+\varepsilon}} , \quad (0 \leq r < 1).$$

From theorem 5.2.A it follows that

$$|a_n| < A\, n^{\alpha+\varepsilon-1} , \qquad (n > 1)$$

where A depends on ε , and also on the function f .

Our problem therefore is to determine an upper bound for α in terms of B. We observe incidentally that $B \geq 1$, since if $|g(z)| < B < 1$ in $|z| < 1$ then by the Schwarz lemma we would have $|g'(o)| < 1$ whereas in fact $g'(o) = 1$.

Let D be the image domain of $g(z)$, so that D is a bounded, convex domain. It is well known (e.g. [5] page 13) that D contains a disc of radius $\frac{1}{2}$ centred at the origin. Let ζ be a point on the boundary of D which we will suppose is distant B from $w = 0$. From ζ we construct the two lines tangent to the circle of radius $\frac{1}{2}$ centre the origin. Since D is convex these two lines are in the closure $\overline{D}$. If 2γ is the angle these two lines make at ζ then

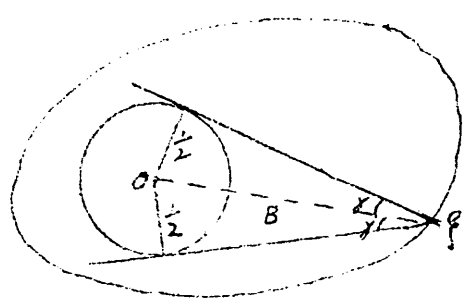


Figure 15

$$2\gamma \geq 2 \sin \gamma = \frac{1}{B}$$

Assuming the existence of a tangent at each point near ζ on the boundary of D, the angle λ which the directed tangents make near ζ and on either side of ζ must therefore satisfy

$$\pi - \lambda > \frac{1 - \varepsilon}{B}$$

for some small $\xi > 0$. On the other hand

$$\int^{\xi} = g(e^{it})$$

and
$$\lambda = \arg g'(e^{i(t+\delta_1)}) - \arg g'(e^{i(t-\delta_2)})$$

$$= \arg f(e^{i(t+\delta_1)}) - \arg f(e^{i(t-\delta_2)})$$

Hence we see that

$$\pi\alpha < \pi - \frac{1-\xi}{B}$$

$$\text{i.e.} \quad \alpha < 1 - \frac{1-\xi}{\pi B} ,$$

since $\pi\alpha$ is the largest jump of the function $V(\theta) = \lim_{r\to 1} \arg f(re^{i\theta})$. This is the essential outline of Pommerenke's argument, and we refer to [9] for a complete and rigorous exposition.

Section 5.4

Our approach to the problem is based on

Lemma 5.4.A

Suppose

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n$$

is starlike in $|z| < 1$ and satisfies

$$\operatorname{Re} \frac{z\, h'(z)}{h\,(z)} \geq \frac{\alpha}{2} \qquad (|z| < 1)$$

where $0 \leq \alpha \leq 2$. Then

$$|h(z)| \geq \frac{|z|}{(1 + |z|)^{2-\alpha}} \qquad (|z| < 1).$$

Proof

Choose κ satisfying $|\kappa| = 1$, and consider

$$f_{\kappa}(z) = \frac{h\,(z)}{(1 - \kappa z)^{\alpha}} , \qquad (|z| < 1).$$

Then differentiating we obtain

$$\frac{z\, f'_{\kappa}(z)}{f_{\kappa}(z)} = \frac{z\, h'(z)}{h\,(z)} + \frac{\alpha \kappa z}{1 - \kappa z}$$

$$= \frac{z\, h'(z)}{h\,(z)} - \frac{\alpha}{2} + \frac{\alpha}{2} \frac{1 + \kappa z}{1 - \kappa z}$$

Hence

$$\operatorname{Re} \frac{z\, f'_{\kappa}(z)}{f_{\kappa}(z)} \geq \frac{\alpha}{2} \operatorname{Re} \frac{1 + \kappa z}{1 - \kappa z} > 0$$

Also $f_{\kappa}(o) = 0$, $f'_{\kappa}(o) = 1$. It follows that $f_{\kappa}(z)$ is

starlike univalent and normalized in $|z| < 1$, and therefore by the well known distortion theorem, ([5] page 4),

$$|f_{\kappa}(z)| \geq \frac{|z|}{(1+|z|)^2}, \quad (|z| < 1).$$

From this we obtain

$$|h(z)| \geq |1-\kappa z|^{\alpha} \frac{|z|}{(1+|z|)^2}, \quad (|z| < 1).$$

Since this is true for every κ on $|\kappa| = 1$, for a particular z it is true when $-\kappa z = |z|$. Hence

$$|h(z)| \geq |1+|z||^{\alpha} \frac{|z|}{(1+|z|)^2}$$

which is the lemma.

Consider now our starlike function $f(z)$, and suppose the image domain of f has a sector of angle $\pi\alpha$ associated with the point $z = 1$ in the usual sense. Then we may write

$$f(z) = \frac{h(z)}{(1-z)^{\alpha}} \tag{5.4.1}$$

where by theorem 2.2.A we obtain

$$\operatorname{Re} \frac{z\,h'(z)}{h(z)} \geq \frac{\alpha}{2} \qquad (|z| < 1).$$

Hence applying lemma 5.4.A

$$|f(z)| \geq \frac{|z|}{(1+|z|)^{2-\alpha}\,|1-z|^{\alpha}}, \quad (|z| < 1). \quad (5.4.2)$$

We note in passing that (5.4.2) implies

$$M(r,f) \geq \frac{2^{\alpha}}{4} \frac{r}{(1-r)^{\alpha}} \quad (5.4.3)$$

thus proving (1.2.2).

Now since $z\,g'(z) = f(z)$ we obtain from (5.4.2)

$$|g'(z)| \geq \frac{1}{(1+|z|)^{2-\alpha}|1-z|^{\alpha}}, \quad (|z| < 1). \quad (5.4.4)$$

Consider now the image domain D of $g(z)$. By the argument of lemma 2.5.A we see that for each point on $|z| = 1$ $\exists$ a unique image point on the boundary of D. Let ζ be the image point of $z = 1$, and let Γ be the line joining ζ to the origin. Then Γ lies in D except for the point ζ. Let γ be the inverse image of Γ under g, so that γ is an analytic arc going from $z = 0$ to $z = 1$. Since the image curves of the circles $|z| = r$ are starlike, Γ crosses each such image curve once and only once. Hence γ crosses the circles $|z| = r$, $0 < r < 1$, once and only once. Therefore we may write

$$\gamma = \{ z/z = z(r), \quad 0 \leq r \leq 1 \}$$

where $z(o) = 0$, $z(1) = 1$, and $z(r)$ is differentiable at each r, $0 \leq r < 1$. The length of Γ is given by

$$\ell(\Gamma) = \int_{\Gamma} |dw| = \int_{\gamma} |g'(z)| \, |dz| \qquad (5.4.5)$$

Hence we obtain applying (5.4.4),

$$B \geq \ell(\Gamma) \geq \int_{\gamma} \frac{|dz|}{(1+|z|)^{2-\alpha}|1-z|^{\alpha}}$$

$$= \int_{o}^{1} \frac{|z'(r)| \, dr}{(1+r)^{2-\alpha}|1-z(r)|^{\alpha}} = I, \text{ say.} \qquad (5.4.6)$$

We now show that I is minimised if γ is the positive real axis from $z = 0$ to $z = 1$. To do this we integrate I by parts as follows. Let

$$u_1(r) = \frac{1}{(1+r)^{2-\alpha}}$$

$$v_1(r) = \int_{o}^{r} \frac{|z'(\rho)| \, d\rho}{|1-z(\rho)|^{\alpha}}$$

We then have

$$I = \frac{2^{\alpha}}{4} \int_{o}^{1} \frac{|z'(\rho)| \, d\rho}{|1-z(\rho)|^{\alpha}} + (2-\alpha) \int_{o}^{1} \left\{ \int_{o}^{r} \frac{|z'(\rho)| \, d\rho}{|1-z(\rho)|^{\alpha}} \right\} \frac{dr}{(1+r)^{3-\alpha}}$$

$$= I_1 + J_1 \quad \text{say} .$$

We also write

$$I' = \int_0^1 \frac{dr}{(1+r)^{2-\alpha}(1-r)^{\alpha}}$$

$$= \frac{2^{\alpha}}{4} \int_0^1 \frac{d\rho}{(1-\rho)^{\alpha}} + (2-\alpha) \int_0^1 \left\{ \int_0^r \frac{d\rho}{(1-\rho)^{\alpha}} \right\} \frac{dr}{(1+r)^{3-\alpha}}$$

$$= I_1' + J_1' \quad \text{say} .$$

We note that

$$I_1 \geq \left| \frac{2^{\alpha}}{4} \int_{\gamma} \frac{dz}{(1-z)^{\alpha}} \right| = I_1'$$

We now integrate J_1 and J_1' by parts similarly, as follows. Let

$$u_2(r) = \frac{1}{(1+r)^{3-\alpha}}$$

$$\text{and } v_2(r) = \int_0^r \left\{ \int_0^R \frac{|z'(\rho)|\, d\rho}{|1-z(\rho)|^{\alpha}} \right\} dR$$

$$= \int_0^r v_1(R)\, dR .$$

Then

$$J_1 = (2-\alpha)\,\frac{2^\alpha}{8}\int_0^1 \left\{ \int_0^R \frac{|z'(\rho)|\,d\rho}{|1-z(\rho)|^\alpha} \right\} dR$$

$$+ (2-\alpha)(3-\alpha)\int_0^1 \left\{ \int_0^r \left\{ \int_0^R \frac{|z'(\rho)|\,d\rho}{|1-z(\rho)|^\alpha} \right\} dR \right\} \frac{dr}{(1+r)^{4-\alpha}}$$

$$= I_2 + J_2 \quad \text{say}$$

Similarly $J_1' = I_2' + J_2'$ where the expression

$$\frac{|z'(\rho)|\,d\rho}{|1-z(\rho)|^\alpha} \quad \text{is replaced by} \quad \frac{d\rho}{(1-\rho)^\alpha}\,.$$

We again note that

$$I_2 \geq I_2'\,.$$

We again repeat this process on J_2 and J_2', and in general we set

$$u_n(r) = \frac{1}{(1+r)^{n+1-\alpha}}$$

$$v_n(r) = \int_0^r v_{n-1}(\rho)\,d\rho$$

obtaining

$$I = \sum_{k=1}^{n} I_k + J_n$$

and $$I' = \sum_{k=1}^{n} I_k' + J_n'$$

and $$I_k \geq I_k' , \qquad k = 1, \ldots n .$$

We now show that

$$J_n' \to 0 \quad \text{as} \quad n \to \infty .$$

We have

$$J_n' = (2-\alpha)(3-\alpha) \ldots (n+1-\alpha) \int_0^1 \left\{ \int_0^{r_1} \left\{ \int_0^{r_2} \ldots \left\{ \int_0^{r_n} \frac{d\rho}{(1-\rho)^\alpha} \right\} dr_n \right\} \ldots \right.$$

$$\ldots \frac{dr_1}{(1+r_1)^{n+2-\alpha}}$$

Now $$\int_0^{r_n} \frac{d\rho}{(1-\rho)^\alpha} \leq \int_0^1 \frac{d\rho}{(1-\rho)^\alpha} = \frac{1}{1-\alpha} .$$

Hence

$$J'_n \leq \frac{(2-\alpha)(3-\alpha) \ldots (n+1-\alpha)}{1-\alpha} \int_0^1 \left\{ \int_0^{r_1} \ldots \left\{ \int_0^{r_{n-1}} dr_n \right\} dr_{n-1} \right\} \ldots$$

$$\ldots \frac{dr_1}{(1+r_1)^{n+2-\alpha}}$$

$$\leq \frac{(n+1)!}{(1-\alpha)(n-1)!} \int_0^1 \frac{r_1^{n-1}\, dr_1}{(1+r_1)^{n+2-\alpha}} \leq \frac{n(n+1)}{1-\alpha} \int_0^1 \frac{r_1^{n-1}\, dr_1}{(1+r_1)^{n+1}}$$

$$= \frac{n(n+1)}{1-\alpha} \int_0^{\frac{1}{2}} u^{n-1}\, du \quad , \quad \text{where} \quad u = \frac{r_1}{1+r_1} \, ,$$

$$= \frac{n(n+1)}{1-\alpha} \cdot \frac{1}{n2^n} = \frac{n+1}{2^n} \cdot \frac{1}{1-\alpha} \to 0 \quad \text{as} \quad n \to \infty \, .$$

Hence

$$J'_n \to 0 \quad \text{as} \quad n \to \infty \, .$$

It follows that

$$I' = \sum_{k=0}^{\infty} I'_k \, .$$

Hence since $J_n \geq 0$ we have

$$I \geq \sum_{k=1}^{\infty} I_k \geq \sum_{k=1}^{\infty} I'_k = I' \, .$$

And thus

$$B \geq \int_0^1 \frac{dr}{(1+r)^{2-\alpha}(1-r)^{\alpha}}$$

$$= \frac{1}{2} \int_0^1 (1-u)^{-\alpha}\, du \, , \text{ where } 1-u = \frac{1-r}{1+r}$$

$$= \frac{1}{2} \cdot \frac{1}{1-\alpha} \, .$$

And we therefore obtain

$$\alpha \leq 1 - \frac{1}{2B} \, . \qquad (5.4.7)$$

We now show that equality in (5.4.7) cannot occur. For suppose that $B = \frac{1}{2}\ \frac{1}{1-\alpha}$. Then

$$B = \ell(F) = I = I_1 + J_1 = I_1' + J_1' .$$

Clearly $J_1 \geq J_1'$ by the same proof that $I \geq I'$.

Hence $I_1' \geq I_1 \geq I_1'$, i.e. $I_1 = I_1'$.

$$\text{i.e.} \qquad \int_\gamma \frac{|dz|}{|1-z|^\alpha} = \int_0^1 \frac{d\rho}{(1-\rho)^\alpha} \tag{5.4.6}$$

Consider the mapping

$$W = q(z) = \frac{1}{1-\alpha}\left\{ 1 - (1-z)^{1-\alpha} \right\}$$

so

$$z\, q'(z) = \frac{z}{(1-z)^\alpha}$$

Hence $q(z)$ is a convex univalent function in $|z| < 1$, and the image of the positive real axis from $z = 0$ to $z = 1$ is mapped onto the positive real W-axis from $W = 0$ to $W = \frac{1}{1-\alpha}$. γ is mapped onto a curve going from $W = 0$ to $W = \frac{1}{1-\alpha}$ which has length given by (5.4.8), i.e. has the length $\frac{1}{1-\alpha}$ of the positive real W-axis. Hence γ is the positive real z-axis. We therefore have that

$$\int_0^1 |g'(x)|\, dx = \int_0^1 \frac{dx}{(1+x)^{2-\alpha}(1-x)^\alpha} .$$

Also from (5.4.4),

$$|g'(x)| \geq \frac{1}{(1+x)^{2-\alpha}(1-x)^{\alpha}}, \quad (0 \leq x < 1).$$

Hence

$$|g'(x)| = \frac{1}{(1+x)^{2-\alpha}(1-x)^{\alpha}}, \quad (0 \leq x < 1).$$

and so

$$g'(x) = \frac{e^{i\phi(x)}}{(1+x)^{2-\alpha}(1-x)^{\alpha}}$$

where $\phi(x)$ is analytic and real, $0 \leq x < 1$.

Hence by analytic continuation

$$g'(-x) = \frac{e^{i\phi(-x)}}{(1-x)^{2-\alpha}(1+x)^{\alpha}}, \quad 0 \leq x < 1$$

where $\phi(-x)$ is analytic and real, $0 \leq x < 1$.

And so

$$|g'(-x)| = \frac{1}{(1-x)^{2-\alpha}(1+x)^{\alpha}}, \quad (0 \leq x < 1).$$

Hence the length of the image of the negative real axis from $z = 0$ to $z = -1$ under the mapping $g(z)$ is given by

$$\int_0^1 \frac{dx}{(1-x)^{2-\alpha}(1+x)^{\alpha}} = \infty$$

since $\alpha < 1$. This is a contradiction to the boundedness of $|g(z)|$.

We therefore deduce that

$$\alpha < 1 - \frac{1}{2B} \qquad (5.4.9)$$

It follows that $\exists\ \delta > 0$, δ depending on the function $g(z)$, such that

$$\alpha < 1 - \frac{1}{2B} - 3\delta$$

Now as we showed in section 5.3, $\exists$ a constant A depending on δ such that

$$|b_n| < A\, n^{\alpha+\delta-2}, \qquad (n > 1)$$

and so

$$|b_n| < A\, n^{-1-\frac{1}{2B}-2\delta}$$
$$= o(n^{-1-\frac{1}{2B}-\delta}) \quad \text{as} \quad n \to \infty .$$

This proves (5.1.4).

To complete the proof of theorem 9, we consider the function

$$g(z) = \int_0^z \frac{dt}{(1-t^2)^\alpha}, \qquad (|z| < 1) \qquad (5.4.10)$$

where $0 < \alpha < 1$. Then

$$f(z) = z\, g'(z) = \frac{z}{(1-z^2)^\alpha}$$

is clearly starlike in $|z| < 1$, for $0 < \alpha < 1$. So $g(z)$ is convex univalent in $|z| < 1$, and its upper bound B is given b

$$B = \int_0^1 \frac{dt}{(1-t^2)^{\alpha}} = \frac{1}{2} \int_0^1 u^{-\frac{1}{2}} (1-u)^{-\alpha} \, du$$

where $t^2 = u$. This last integral (see e.g. [13] page 253) is given by

$$B = \frac{1}{2} \frac{\Gamma(\frac{1}{2}) \Gamma(1-\alpha)}{\Gamma(\frac{1}{2} + 1 - \alpha)}$$

As we have shown

$$(1 - \alpha) B \geq \frac{1}{2} .$$

On the other hand

$$(1 - \alpha) B = \frac{1}{2} \frac{\Gamma(\frac{1}{2}) \Gamma(2-\alpha)}{\Gamma(\frac{1}{2} + 1 - \alpha)}$$

$$\to \frac{1}{2} \quad \text{as} \quad \alpha \to 1 .$$

Thus we see that given $\varepsilon > 0$ $\exists$ a $g(z)$ and a B such that $|g(z)| < B$, and

$$\alpha > 1 - \frac{1}{(2-\varepsilon)B} .$$

Thus applying lemma 5.2.B, (5.1.5) follows, and the proof of theorem 9 is complete.

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