



WEAK CONVERGENCE OF UTILITY-RISK PORTFOLIOS

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In Celebration of Alain Bensoussan on the Occasion of his 85th Birthday

ABSTRACT. In this article, we establish the validity of the Donsker invariance principle for weak convergence of the optimal utility-downside-risk portfolio payoff. Thus, the optimal solution is robust to changes in the market model via the pricing kernel. The convergence result is valid for all feasible combinations of utility functions satisfying the Inada conditions, convex downside deviation risk measures, and pricing kernels satisfying integrability conditions. Thus, our results provide a unified framework to understand whether utility-downside-risk solutions are stable under mild misspecifications of asset price processes. As another application, we further obtain a result that the optimal utility-risk values and policies driven by discrete-time binomial models converge weakly to those of the limiting continuous-time Black-Scholes model. We demonstrate the convergence result using numerical examples and find that the optimal utility-risk value converges as the time interval of the binomial model gets smaller.

1. Introduction. Since the celebrated work of Markowitz [30], the portfolio selection problem has become one of the most important research topics in finance and economics. Expected utility and mean-variance are two common criteria for assessing portfolio performance. Merton [32] and Samuelson [36] investigate utility maximisation problems in continuous-time and multi-period settings, respectively, by considering them as stochastic control problems. The advantage of formulating the stochastic control problems allows a direct application of the dynamic programming principle (DPP) to get the Hamilton-Jacobi-Bellman (HJB) equation, leading to a non-linear partial differential equation (PDE) by invoking the inherent tower property; see, for examples, [5, 8, 32, 37] for a detailed introduction. Without assuming a specific form of utility functions, such as hyperbolic or constant absolute risk aversion, the solvability of non-linear PDEs is barely discussed in general. Bensoussan et al. [5] established a special transformation to linearise the primal HJB

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equation into a semi-linear PDE, which can be solved via the weak solution approach in a weighted Sobolev space. This novel method is further applied to resolve more advanced optimisation problems with general utility functions and model settings, like [29]. Alternatively, the martingale method can be applied to solve utility maximisation problems in a complete market, where the existence of the optimal terminal payoff can be shown by convex analysis and that of the optimal trading strategy by the Martingale Representation Theorem; see [12, 23, 27, 34] for details.

As an aftermath of several major financial crises in the last two decades, tighter government regulations, including adequate liquidity requirements, have been enforced to cope with the possibility of extreme losses in the market. As a consequence, there is a trend towards incorporating enterprise risk management practice into portfolio selection problems. Value-at-Risk (V@R) constraints have been implemented into portfolio optimisation in light of the prevailing regulation using V@R as the method of risk assessment; see [3, 13, 40, 41]. Furthermore, more general goal-reaching criteria, such as concerning whether or not the terminal cash flow exceeds a certain benchmark, have also been applied to portfolio selection; see [14, 16, 23, 28]. Apart from the quantile-based risk measures, another more commonly used risk measure is the deviation risk measure, which measures the deviation of the random return from its own mean. Variance is a special example of the deviation risk measure. Markowitz [30] adopted the mean-variance criteria for evaluating portfolio performance. Li and Ng [26] and Richardson [35] extended the mean-variance problem into dynamic Black-Scholes models. From that point on, more sophisticated mean-variance frameworks have also been investigated, such as [1, 9, 10, 11]. In particular, Bensoussan et al. [10] established a time-consistent mean-variance equilibrium portfolio under short-selling prohibition in both discrete and continuous models and showed that discrete-time equilibrium portfolios converge to the continuous-time counterpart in a linear order. Apart from the usual dynamic programming approach or martingale approach, Bensoussan and his collaborators innovated a novel mean-field technique to solve the classical continuous-time mean-variance problem [9] and a mean-variance capital investment optimisation [7], which cannot be solved by the classical stochastic control methods.

A main criticism against variance as a risk measure is that the variance incurs penalty on the upside profitable return. In contrast, the downside-risk measures, which penalise only when the return is below its expectation, can remedy the common criticism against variance, thus Markowitz [31] also claimed that semivariance, an example of downside risk measures, could be a more sensible risk measure than variance. However, the continuous-time mean-downside-risk problem under the Black-Scholes model was shown to have no optimal solution in Jin et al. [20], as the optimal value function is unattainable by any of the admissible portfolios.

Wong et al. [39] first proposed applying the deviation-risk measure to the utility-risk¹ portfolio selection framework, in which the objective function is the difference of the deviation risk from the utility function, under the continuous-time Black-Scholes market. Wong et al. [39] showed the existence of the optimal solutions for various utility-risk problems, including the utility-semivariance problem, which is in sharp contrast to the non-existence for continuous-time mean-downside-risk problem in [20]. The obtained existence result in [39] suggests the revival of using

¹Jin et al. [20] called their problem formulation mean-risk problem, though they only considered deviation risk measure in their work, so to avoid ambiguity, we use a single word “risk” to stand for the deviation risk measure in the rest of this article.

semivariance as a proper risk measure in the portfolio management paradigm. The general utility-deviation-risk framework in Wong et al. [39] is developed into more interesting studies, like time-consistent strategies in Gu et al. [18] and consumption and investment settings in Bensoussan et al. [6]. In particular, Bensoussan et al. [6] converted the utility-variance problem with consideration of intermediate consumption into a mean field-type control problem, which can be tackled using the Hamilton-Jacobi-Bellman and Fokker-Planck equations framework.

To bring the dynamic utility-semivariance framework to use in practice, we could calibrate the model in [39] using past market data, yet the estimated parameters may be different from their actual values during the investment horizon due to the limitation of estimation from past data on future predictions. This deviation could be a subtle issue because most theoretical claims in portfolio selection are mostly model-dependent, and so any tiny deviations in the parameters could lead to substantial inaccuracy in the optimal solution if it is sensitive. However, the lack of explicit forms of the optimal solutions in [39] to study their functional dependence on parameters leads to an extra hurdle to the investigation of the robustness of optimal utility-risk solutions.

To address the foregoing issue, we shall establish the weak convergence of optimal utility-risk payoffs, each corresponding to an approximating market such that the sequence of the corresponding pricing kernels converges to a limiting one. This convergence result implies that the optimal solution is insensitive to changes in the underlying market parameter values. In other words, if we slightly perturb the market model, the projections on the average generated by the model do not change drastically.

In comparison with some existing results of weak convergence in the literature, the convergence of optimal payoff for utility maximisation (without other risk measures) was studied by He [19] for complete market setups and then by Larsen and Zitkovic [25] for incomplete market setups. In particular, Larsen and Zitkovic [25] investigated the stability of the utility maximisation, which is how small perturbations of the market coefficient processes can cause a change in the optimal solution. More research studies regarding the convergence and stability of utility maximisation in incomplete markets have been done; to name a few, see [4, 22, 24, 33]. As the utility-deviation-risk setting is recently established in [39], its convergence and stability have not been studied.

As an application of the weak convergence theorem established in this article, we shall verify that the conditions for the theorem are satisfied by the pricing kernels in connection with the discrete binomial tree model. Hence, the corresponding optimal terminal payoffs converge in distribution to that under the continuous-time Black-Scholes setting. This convergence result will be further illustrated by numerical examples, in which the optimal values converge in a linear order as the time interval of the binomial model gets smaller – the same convergence order found in [10]. Furthermore, under some regularity assumptions, we can further establish the weak convergence of the optimal portfolio and the wealth process by using a similar argument as in [19].

The rest of the paper is organised as follows. In Section 2, we first introduce the problem formulation and some assumptions on the market model. Then, we establish the unique existence of the optimal solution for the utility-downside-risk framework. Section 3 lists certain conditions on a sequence of pricing kernels under which the optimal terminal payoffs converge in distribution. In Section 4, we shall

demonstrate the verification of the aforementioned conditions by the binomial tree models. Hence, with our main claim, we can conclude that the optimal terminal wealth for utility-risk problems under the Black-Scholes model can be approximated by that under binomial tree models. Furthermore, under some mild regularity assumptions, we further establish weak convergence of the optimal portfolio weight and wealth processes. Section 5 demonstrates our established convergence results. The technical proofs will be provided in the Appendix.

2. Problem setting and preliminaries.

2.1. Utility-risk portfolio selection. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space. Suppose that the market has $m + 1$ assets with the joint price process $(S(t))_{t \geq 0}$ with $S(t) := (S_0(t), \dots, S_m(t))^t$; here M^t denotes the transpose of a matrix M . Define the information filtration up to time t as $\mathcal{F}_t := \sigma(S(s) : s \leq t)$, satisfying the usual conditions.

Let $\pi(t) := (\pi_1(t), \dots, \pi_m(t))^t$, where $\pi_k(t)$ is the money amount invested in the k -th risky asset of the portfolio at time t . The dynamics of controlled wealth process $(X^\pi(t))_{t \geq 0}$ is given by

$$dX^\pi(t) = (X^\pi(t) - \pi(t)^t \mathbf{1}_m) \frac{dS_0(t)}{S_0(t)} + \sum_{k=1}^m \pi_k(t) \frac{dS_k(t)}{S(t)}, \quad (1)$$

and $X^\pi(0) = x_0 > 0$, where $\mathbf{1}_m$ is an m -dimensional vector with all components 1.

Let U be a utility function satisfying the following property:

Property 2.1. The utility function $U : (0, \infty) \rightarrow \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable and $U'(0) := \lim_{x \downarrow 0} U'(x) = \infty$ and $U'(\infty) := \lim_{x \uparrow \infty} U'(x) = 0$, i.e. the Inada condition holds.

For the ease of discussion, we extend the domain of U from $\mathbb{R}_+$ to $\mathbb{R}$ so that $U(0) := \lim_{x \downarrow 0} U(x)$ and $U(x) := -\infty$ for $x < 0$.

The deviation risk function D is used to measure the difference of the random portfolio return and its own expected value. In the present article, we focus on downside risk functions defined as follows:

Definition 2.2. $D : \mathbb{R} \rightarrow \mathbb{R}_+$ is said to be a downside risk function if it is positive, strictly increasing, strictly convex, and continuously differentiable on $(0, \infty)$ and $D(x) = 0$ for $x \leq 0$ such that $D'(\infty) = \infty$ and $D'(0) = 0$.

The objective functional is given by

$$J(\pi(\cdot)) := \mathbb{E}[U(X^\pi(T))] - \gamma \mathbb{E}[D(\mathbb{E}[X^\pi(T)] - X^\pi(T))], \quad (2)$$

where the terminal time $T < \infty$ and $\gamma > 0$ denotes the risk aversion coefficient. From that point on, we denote the process $(X(t))_{t \in [0, T]}$ in short as $X(\cdot)$ unless otherwise specified.

The investor aims to simultaneously maximize the expected value of the utility and minimize the deviation risk. For example, if the utility function is $U(x) = \frac{1}{p}x^p$ while the downside risk function is $D(x) = x_+^2$, then our problem becomes the power-utility-semivariance problem. Given $p \geq 1$, let $\mathcal{L}^p := \{Z \mid \|Z\|_p := (\mathbb{E}[|Z|^p])^{\frac{1}{p}} < \infty\}$. Define $\mathcal{H}^2$ to be the class of all $\mathcal{F}_t$ -adapted processes equipped with the norm

$\|\pi(\cdot)\|_{\mathcal{H}^2} := (\mathbb{E}[\int_0^T \pi(t)^t \pi(t) dt])^{\frac{1}{2}} < \infty$. As a mild technical requirement, we require the terminal wealth of every admissible control to satisfy the following conditions:

$$X \geq 0 \text{ a.s.}; \quad U(X) \in \mathcal{L}^1; \quad D(\mathbb{E}[X] - X) \in \mathcal{L}^1. \quad (3)$$

The non-negativity of terminal wealth X ensures that $U(X)$ can be properly evaluated. The integrability of both $U(X)$ and $D(\mathbb{E}[X] - X)$ guarantees both terms in the objective functional (2) take finite values. With these conditions on the terminal wealth, we consider the class of all admissible controls

$$\mathcal{A} := \{\pi(\cdot) \in \mathcal{H}^2 \mid X^\pi(T) \in \mathcal{L}^2 \cap \mathcal{F}_T \text{ and } X^\pi(T) \text{ satisfies (3)}\}.$$

Our utility risk problem can then be formulated as follows:

Problem. Maximize $J(\pi(\cdot))$, subject to $\pi(\cdot) \in \mathcal{A}$ and $(X^\pi(\cdot), \pi(\cdot))$ follows (1) with an initial wealth x_0 .

We impose further assumptions in the market:

Assumption 2.3 (Pricing kernel). *There exists a pricing kernel $\xi(\cdot) \in \mathcal{H}^2$, $\xi(0) = 1$ and $\xi(t) \in \mathbb{R}_+$ a.s. for all $t \in [0, T]$, such that $\xi(T) \in \mathcal{L}^2$ and $(\xi(t)X^\pi(t))_{t \in [0, T]}$ is a martingale for all $\pi(\cdot) \in \mathcal{A}$.*

Assumption 2.4 (Market completeness). *For every $X \in \mathcal{L}^2 \cap \mathcal{F}_T$, there exists a self-financing strategy $\pi(\cdot) \in \mathcal{H}^2$ such that $X^\pi(T) = X$.*

Moreover, we assume the following on the utility function which has also been adopted in [39].

Assumption 2.5. *There exists $k_0 > 0$ so that the inverse of the first-order derivative of U , $(U')^{-1}$, satisfies $(U')^{-1}(k_0 \xi(T)) \in \mathcal{L}^2$.*

2.1.1. Examples of Market Models.

Example 2.6 (Continuous-time Black-Scholes model). Suppose that the market has one bond and one stock, we consider the bond price and stock price satisfying the following:

$$\begin{aligned} dB(t) &= rB(t)dt, & B(0) &= b_0 > 0, \\ dS(t) &= \mu S(t)dt + \sigma S(t)dW(t), & S(0) &= s_0 > 0, \end{aligned}$$

where $W(\cdot)$ is the standard Brownian motion. Let $\pi(t)$ be the money amount of risky investment at time t , so that the dynamics of the wealth process become:

$$dX^\pi(t) = (rX^\pi(t) + \alpha\pi(t))dt + \sigma\pi(t)dW(t), \quad X^\pi(0) = x_0 > 0,$$

where $\alpha := \mu - r$. The continuous-time pricing kernel is a process $\xi^*(\cdot)$ such that:

$$\xi^*(t) := \exp \left[-rt - \frac{\alpha^2}{2\sigma^2}t - \frac{\alpha}{\sigma}W_t \right]. \quad (4)$$

Clearly, $\xi^*(\cdot)$ satisfies Assumption 2.3. Assumption 2.4 can be validated by an application of the Martingale Representation Theorem. Note that all assumptions are satisfied even when the market parameters are $\mathcal{F}_t$ -adapted processes in general.

Example 2.7 (Binomial tree model). We next consider a N -period discrete time framework with time interval $\Delta t := \frac{T}{N}$. The price processes of bond $B^{(N)}(\cdot)$ and stock $S^{(N)}(\cdot)$ are given by $B^{(N)}(t) := B_{\lfloor t/\Delta t \rfloor}^{(N)}$ and $S^{(N)}(t) := S_{\lfloor t/\Delta t \rfloor}^{(N)}$ respectively, where $B_n^{(N)}$ and $S_n^{(N)}$ have the following dynamics:

$$B_{n+1}^{(N)} = (1 + r\Delta t) B_n^{(N)}, \quad B_0^{(N)} = b_0;$$

$$S_{n+1}^{(N)} = \left(1 + \mu\Delta t + \sigma\sqrt{\Delta t}Z_{n+1}^{(N)}\right) S_n^{(N)}, \quad S_0^{(N)} = s_0;$$

where $Z_n^{(N)}$'s are iid Bernoulli random variables for $n = 1, \dots, N$, with probability: $\mathbb{P}[Z_n^{(N)} = 1] = \mathbb{P}[Z_n^{(N)} = -1] = \frac{1}{2}$. Let $\pi_n^{(N)}$ be the money amount invested in the risky asset over the n^{th} period, the corresponding wealth process $X^{(N),\pi}(\cdot)$ with $X^{(N),\pi}(t) := X_{\lfloor t/\Delta t \rfloor}^{(N),\pi}$ has the following dynamics:

$$X_{n+1}^{(N),\pi} = (1 + r\Delta t) X_n^{(N),\pi} + \left(\alpha\Delta t + \sigma\sqrt{\Delta t}Z_{n+1}^{(N)}\right) \pi_n^{(N)}, \quad (5)$$

$X_0^{(N),\pi} = x_0$, where $\alpha = \mu - r$.

In this binomial tree model, the pricing kernel process $\xi^{(N)}(\cdot)$ is given by $\xi^{(N)}(t) := \xi_{\lfloor t/\Delta t \rfloor}^{(N)}$, such that:

$$\xi_n^{(N)} := \prod_{k=1}^n \left(\frac{1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}Z_k^{(N)}}{1 + r\Delta t} \right) \text{ for } n=1, \dots, N. \quad (6)$$

Hence, Assumption 2.3 is obviously satisfied.

For Assumption 2.4, given a $\mathcal{F}_T$ -measurable $X \in \mathcal{L}^2$, the corresponding attainable wealth process can be obtained by

$$X_n^{(N),\pi} = \mathbb{E} \left[\frac{\xi_N^{(N)}}{\xi_n^{(N)}} X \middle| \xi_n^{(N)} \right] = \mathbb{E} \left[\prod_{k=n+1}^N \left(\frac{1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}Z_k^{(N)}}{1 + r\Delta t} \right) X \right],$$

and the underlying trading strategy can be obtained in accordance with (5).

Example 2.8 (Stochastic interest rate model). Adopting the market setting as specified in [2], suppose that the market has one riskless money account, one stock, and one zero coupon bond fund; their prices at t are denoted by $S_0(t)$, $S_1(t)$, and $S_2(t)$, respectively. The asset prices satisfy the following:

$$\begin{aligned} dS_0(t) &= r(t)S_0(t)dt, & S_0(t) &= s_0 > 0, \\ dS_1(t) &= (r(t) + \alpha_1)S_1(t)dt + \sigma_{11}S_1(t)dW_1(t) + \sigma_{12}S_1(t)dW_2(t), & S_1(0) &= s_1 > 0, \\ dS_2(t) &= (r(t) + \alpha_2)S_2(t)dt + \sigma_{21}S_2(t)dW_1(t), & S_2(0) &= s_2 > 0, \end{aligned}$$

where $W_1(\cdot)$ and $W_2(\cdot)$ are two independent standard Brownian motions, and the interest rate $r(\cdot)$ follows an Ornstein-Uhlenbeck process with constant parameters given by

$$dr(t) = a_r(b_r - r(t))dt - \sigma_r dW_1(t), \quad r(0) = r_0.$$

Let $\pi(\cdot) := (\pi_1(t), \pi_2(t))_{t \in [0, T]}$ and $\pi_1(t)$ and $\pi_2(t)$ be the respective money amount invested in the stock and the bond at time t , so that the dynamics of wealth process becomes:

$$\begin{cases} dX^\pi(t) = (X^\pi(t) - \pi_1(t) - \pi_2(t)) \frac{dS_0(t)}{S_0(t)} + \pi_1(t) \frac{dS_1(t)}{S_1(t)} + \pi_2(t) \frac{dS_2(t)}{S_2(t)} \\ \quad = (r(t)X^\pi(t) + \alpha_1\pi_1(t) + \alpha_2\pi_2(t))dt + (\sigma_{11}\pi_1(t) + \sigma_{21}\pi_2(t))dW_1(t) \\ \quad \quad + \sigma_{12}\pi_1(t)dW_2(t), \\ X^\pi(0) = x_0 > 0. \end{cases}$$

The pricing kernel for the market setting becomes the following:

$$\xi(t) := e^{-\int_0^t r(s)ds - \frac{1}{2} \left(\frac{(\alpha_1 - \alpha_2 \frac{\sigma_{12}}{\sigma_{21}})^2}{\sigma_{11}^2} + \frac{\alpha_2^2}{\sigma_{21}^2} \right) t - \frac{\alpha_2}{\sigma_{21}} W_1(t) - \frac{\alpha_1 - \alpha_2 \frac{\sigma_{12}}{\sigma_{21}}}{\sigma_{12}} W_2(t)},$$

$(\xi(t)X^\pi(t))_{t \in [0, T]}$ is a martingale in light of an application of Itô's lemma, and so Assumption 2.3 follows.

Both Brownian noises $W_1(\cdot)$ and $W_2(\cdot)$ can be hedged out perfectly through trading the stock and the zero-coupon bond simultaneously, so the market is complete and Assumption 2.4 is satisfied.

2.2. Unique existence of optimal solution. In this subsection, we state the existence results of present utility-risk problems. Though our present setting is more general than the one in [39], all results in this subsection can be established using the same arguments. Our Problem 2.1 can be converted into the following static one:

Problem.

$$\text{Maximize } \Psi(X), \quad \text{subject to } X \in \mathcal{X} \text{ and } \mathbb{E}[\xi(T)X] = x_0,$$

where $\Psi : \mathcal{X} \rightarrow \mathbb{R}$ is defined by $\Psi(X) := \mathbb{E}[U(X)] - \gamma \mathbb{E}[D(\mathbb{E}[X] - X)]$ and $\mathcal{X} := \{X \in \mathcal{L}^2 | X \in \mathcal{F}_T \text{ and } X \text{ satisfies (3)}\}$.

Under Assumptions 2.3 and 2.4, each feasible solution to Problem 2.2 is attainable by an admissible control with the initial wealth x_0 , so it can be shown that the optimal solution of Problem 2.2 is the optimal terminal wealth of Problem 2.1:

Theorem 2.9. *If $\pi(\cdot) \in \mathcal{A}$ is optimal for Problem 2.1, then $X^\pi(T)$ is optimal for Problem 2.2. Conversely, if $X \in \mathcal{X}$ is optimal for Problem 2.2, there exists $\pi(\cdot) \in \mathcal{A}$ such that $X^\pi(T) = X$ and $\pi(\cdot)$ is optimal for Problem 2.1.*

Note that the maximisation in Problem 2.2 is confined to the set $\mathcal{X}$, so that the solution obtained in Problem 2.2 is an admissible terminal wealth in Problem 2.1. Since Theorem 2.9 provides the equivalence between Problems 2.1 and 2.2, our present paper now aims to establish an admissible terminal wealth $X \in \mathcal{X}$ that maximizes $\Psi(X)$.

Theorem 2.10. *Under Assumption 2.5 so that U satisfies Property 2.1 and D is given by Definition 2.2, $\hat{X} = I(M, \gamma R + Y\xi(T))$ is the unique optimal terminal wealth of the utility-downside-risk problem where I is an implicit function $I : \mathbb{R} \times (0, \infty) \rightarrow (0, \infty)$ satisfying:*

$$U'(I(m, y)) + \gamma D'(m - I(m, y)) - y = 0, \quad \text{for any } (m, y) \in \mathbb{R} \times (0, \infty), \quad (7)$$

and $(Y, M, R) \in \mathbb{R}^3$ is the unique solution to the system of nonlinear equations

$$\mathbb{E}[\xi(T)I(M, \gamma R + Y\xi(T))] = x_0, \quad (8)$$

$$\mathbb{E}[I(M, \gamma R + Y\xi(T))] = M, \quad (9)$$

$$\mathbb{E}[D'(M - I(M, \gamma R + Y\xi(T)))] = R. \quad (10)$$

Proof. The outline of the proof is given in Appendix A. $\square$

3. Main result: Weak convergence of optimal payoffs. In this section, we first establish the convergence of optimal terminal payoffs for a sequence of utility-risk problems, namely, if a sequence of pricing kernels converges weakly to a limit, then the respective optimal terminal payoffs will converge in distribution to that under the limiting pricing kernel.

We consider a sequence of markets, each of which satisfies Assumptions 2.3 and 2.4; denote them by $\{\Pi^{(N)}\}_{N \in \mathbb{N}}$, and each market Π_N has its own unique pricing

kernel $\xi^{(N)}(\cdot)$. Meanwhile, we have a sequence of the random terminal values of pricing kernels (or simply call it the terminal pricing kernel for convenience), namely, $\{\xi^{(N)}\}_{N \in \mathbb{N}}$ with $\xi^{(N)} := \xi^{(N)}(T)$. By Theorem 2.10, we know that each market possesses a unique optimal terminal wealth for the utility risk problem, denoted by $\hat{X}^{(N)} \in \mathcal{L}^2$. We aim to show that if this sequence of terminal pricing kernels converges in distribution to a terminal pricing kernel $\xi^* \in \mathcal{L}^2$ corresponding to a market Π^* , then this limiting optimal terminal wealth solves for the utility-downside-risk problem under the market Π^* .

Theorem 3.1. *Given that U and D respectively satisfy Property 2.1 and Definition 2.2. Under all markets $\Pi^{(N)}$ and Π^* satisfying Assumptions 2.3, 2.4, and 2.5 with their corresponding unique terminal pricing kernels $\xi^{(N)}$ and $\xi^* \in \mathcal{L}^2$, and optimal terminal wealths are given by $\hat{X}^{(N)}$ and $\hat{X}^*$ in Theorem 2.10. Suppose that a sequence of their terminal pricing kernels $\{\xi^{(N)}\}_{N \in \mathbb{N}}$ satisfy the following conditions:*

- (i) $\{\xi^{(N)}\}_{N \in \mathbb{N}}$ converges in distribution to ξ^* as $N \rightarrow \infty$.
- (ii) For any $K > 0$, $\{\xi^{(N)}((U')^{-1}(\xi^{(N)}) + K)\}_{N \in \mathbb{N}}$ is uniformly integrable.

Then the sequence $\{\hat{X}^{(N)}\}_{N \in \mathbb{N}}$ converges in distribution to $\hat{X}^*$. Hence, $\Psi(\hat{X}^{(N)}) \rightarrow \Psi(\hat{X}^*)$, i.e., the sequence of optimal value functions of the utility-risk problem under $\Pi^{(N)}$ converges to that under Π^* .

Proof. By Theorem 2.10, the unique optimal terminal wealth for the utility-downside-risk problem under $\Pi^{(N)}$ is given by $\hat{X}^{(N)} = I(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \xi^{(N)})$, where the function I is given in (7) and the triple $(Y^{(N)}, M^{(N)}, R^{(N)})$ satisfies the system of equations in (8)-(10) under $\xi(T) = \xi^{(N)}$. Similarly, the unique optimal terminal wealth under Π^* is given by $\hat{X}^* = I(M^*, \gamma R^* + Y^* \xi^*)$. Note that the choice of I in (7) is independent of the choice of pricing kernel ξ .

By Skorokhod's representation theorem and condition (i), there exists a sequence of random variables $\{\zeta^{(N)}\}_{N \in \mathbb{N}}$ in which $\zeta^{(N)}$ has the same distribution as $\xi^{(N)}$ such that $\{\zeta^{(N)}\}_{N \in \mathbb{N}}$ almost surely converges to a random variable ζ^* which has the same distribution as ξ^* . To simplify the notation, we denote

$$\tilde{X}^{(N)} := I(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)}) \text{ and } \tilde{X}^* := I(M^*, \gamma R^* + Y^* \zeta^*). \quad (11)$$

The following proposition shows that the sequence of $\{(Y^{(N)}, M^{(N)}, R^{(N)})\}_{N \in \mathbb{N}}$ converges to (Y^*, M^*, R^*) ; its technical proof is postponed to Appendix B.

Proposition 3.2. $\lim_{N \rightarrow \infty} (Y^{(N)}, M^{(N)}, R^{(N)}) = (Y^*, M^*, R^*)$.

By Proposition 3.2, the regularity of I ensures that $\tilde{X}^{(N)} \rightarrow \tilde{X}^*$ a.s.; together with the fact that $\tilde{X}^{(N)}$ and $\hat{X}^{(N)}$ (including $N = *$) have the same distribution, $\hat{X}^{(N)} \rightarrow \hat{X}^*$ in distribution. Since for each m , $I(m, y)$ is decreasing in y and so, for each N ,

$$\tilde{X}^{(N)} = I(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)}) \leq I(M^{(N)}, \gamma R^{(N)}).$$

By Proposition 3.2, $I(M^{(N)}, \gamma R^{(N)}) \rightarrow I(M^*, \gamma R^*) < \infty$, so we have $\tilde{X}^{(N)} \leq I(M^*, \gamma R^*) + 1$ a.s., for all large enough N . A simple application of the dominated convergence theorem, $\Psi(\tilde{X}^{(N)}) \rightarrow \Psi(\tilde{X}^*)$. Together with the fact that $\tilde{X}^{(N)}$ and $\hat{X}^{(N)}$ (including $N = *$) have the same distribution, the convergence of value functions follows. $\square$

Theorem 3.1 implies that the optimal solution for the utility-downside-risk problem is robust with respect to the change in pricing kernel. If the “change” in pricing kernel is small enough, the optimal terminal wealth does not deviate much, and so does the change in value function.

4. Applications: Convergence from discrete to continuous setting. In this section, we shall show that the sequence of terminal pricing kernels in the binomial tree models satisfies the conditions stated in Theorem 3.1, with the limit of the sequence of terminal pricing kernels being that under the Black-Scholes model. We first recall the celebrated martingale central limit theorem, which is vital to our main results in this section.

Theorem 4.1 (Theorem 7.4.1 of [15]). *Assume that $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions such that the following SDE*

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x_0,$$

admits a unique weak solution $X^*(\cdot)$. Suppose that $X^{(N)}(\cdot)$ is a sequence of Markov processes with sample path in $D[0, T]$, which is the space of functions from $[0, T]$ to $\mathbb{R}$ that are right continuous with left limits under the Skorokhod's topology. If there exist processes $L^{(N)}(\cdot)$ and $A^{(N)}(\cdot)$ with sample paths in $D[0, T]$, such that the following conditions hold:

1. $\{X^{(N)}(0)\}_{N \in \mathbb{N}}$ converges to x_0 in distribution;
2. $A^{(N)}(t)$ is non-decreasing in t a.s.;
3. $M^{(N)}(\cdot) := (X^{(N)} - L^{(N)})(\cdot)$ and $((M^{(N)})^2 - A^{(N)})(\cdot)$ are martingales;
4. for all $q > 0$,

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E} \left[\sup_{t \leq \tau_q^{(N)}} |X^{(N)}(t) - X^{(N)}(t-)|^2 \right] &= 0; \\ \lim_{N \rightarrow \infty} \mathbb{E} \left[\sup_{t \leq \tau_q^{(N)}} |L^{(N)}(t) - L^{(N)}(t-)|^2 \right] &= 0; \\ \lim_{N \rightarrow \infty} \mathbb{E} \left[\sup_{t \leq \tau_q^{(N)}} |A^{(N)}(t) - A^{(N)}(t-)| \right] &= 0, \end{aligned}$$

where $\tau_q^{(N)} := \inf \{t \leq T : |X^{(N)}(t)| \geq q \text{ or } |X^{(N)}(t-)| \geq q\} \wedge T$;

5. for all $q > 0$, the following two expressions:

$$\begin{aligned} i) \sup_{t \leq \tau_q^{(N)}} \left| L^{(N)}(t) - \int_0^t b(X^{(N)}(s)) ds \right|, \\ ii) \sup_{t \leq \tau_q^{(N)}} \left| A^{(N)}(t) - \int_0^t (\sigma(X^{(N)}(s)))^2 ds \right| \end{aligned}$$

both converge to 0 in probability as $N \rightarrow \infty$.

Then the process $\{X^{(N)}(\cdot)\}_{N \in \mathbb{N}}$ converges weakly to $X^*(\cdot)$ in the sense that, for any bounded continuous mapping h in $D([0, T])$ to $\mathbb{R}$, we have $\mathbb{E}[h(X^{(N)}(\cdot))] \rightarrow \mathbb{E}[h(X^*(\cdot))]$ as $N \rightarrow \infty$.

To have the main result of this section, the next assumption stronger than Assumption 2.5, which has been adopted in Remark 4.4 in [21], is required:

Assumption 4.2. *There exist $\eta \in (0, 1)$ and $\beta > 1$ such that for any $y > 0$, $U'(\beta y) \leq \eta U'(y)$.*

Theorem 4.3. *Given that Assumption 4.2 holds, U satisfies Property 2.1 and D is given by Definition 2.2, and $\xi_n^{(N)}$ is given in (6). The sequence of the terminal pricing kernels $\{\xi_N^{(N)}\}_{N \in \mathbb{N}}$ satisfies the conditions stated in Theorem 3.1 with $\xi^*(T)$ where $\xi^*(t)$ is defined in (4). Hence, the optimal terminal wealths corresponding to the sequence of binomial tree models converge in distribution to that under the Black-Scholes model.*

Before we proceed to the proof, we first need the following two technical lemmas for verifying the conditions specified in Theorem 4.3.

Lemma 4.4. *Define $\bar{\xi}^*(t) := e^{-\frac{\alpha^2}{2\sigma^2}t - \frac{\alpha}{\sigma}W(t)}$ and $\bar{\xi}^{(N)}(t) = \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)}$ for $0 \leq t \leq T$, where $\bar{\xi}_n^{(N)} := \prod_{k=1}^n (1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} Z_k^{(N)})$. The sequence of processes $\{\bar{\xi}_{\lfloor \cdot / \Delta t \rfloor}^{(N)}\}_{N \in \mathbb{N}}$ converges weakly to $\bar{\xi}^*(\cdot)$.*

Proof. We first note that $\bar{\xi}^*(\cdot)$ satisfies the following SDE:

$$d\bar{\xi}^*(t) = -\frac{\alpha}{\sigma} \bar{\xi}^*(t) dW(t), \quad \bar{\xi}^*(0) = 1;$$

and so, $\bar{\xi}^*(0) = \bar{\xi}^{(N)}(0) = \bar{\xi}_0^{(N)} = 1$, and hence Condition 1 is fulfilled.

By the very definition, $\bar{\xi}^{(N)}(\cdot)$ is a martingale. We then set $L^{(N)}(t) := 0$ and $A^{(N)}(t) := A_{\lfloor t/\Delta t \rfloor}^{(N)}$ where $A_k^{(N)}$ is defined recursively via the relation:

$$A_{k+1}^{(N)} = A_k^{(N)} + \left(\bar{\xi}_k^{(N)}\right)^2 \frac{\alpha^2}{\sigma^2} \Delta t, \quad A_0^{(N)} = 0.$$

Clearly, $A^{(N)}(t)$ is non-decreasing, and Condition 2 is satisfied.

Besides,

$$\begin{aligned} & \mathbb{E} \left[\left(\bar{\xi}_{k+1}^{(N)} \right)^2 - A_{k+1}^{(N)} \middle| \bar{\xi}_k^{(N)}, A_k^{(N)} \right] \\ &= \left(\bar{\xi}_k^{(N)} \right)^2 \mathbb{E} \left[\left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} Z_{k+1}^{(N)} \right)^2 \right] - A_k^{(N)} - \left(\bar{\xi}_k^{(N)} \right)^2 \frac{\alpha^2}{\sigma^2} \Delta t = \left(\bar{\xi}_k^{(N)} \right)^2 - A_k^{(N)}, \end{aligned}$$

which implies that the process $((\bar{\xi}^{(N)})^2 - A^{(N)})(\cdot)$ is a martingale, and thus Condition 3 is validated too.

For any $N \in \mathbb{N}$ and $q > 0$, since $(\bar{\xi}^{(N)}(t-))^2 \leq q^2(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t})^2 \leq 2q^2(1 + \frac{\alpha^2}{\sigma^2} \frac{T}{N})$ by the definitions of $\bar{\xi}^{(N)}(t)$ and $\tau_q^{(N)}$, we have

$$\left| \bar{\xi}^{(N)}(t) - \bar{\xi}^{(N)}(t-) \right|^2 \leq \left| \bar{\xi}^{(N)}(t-) \frac{\alpha}{\sigma} \sqrt{\Delta t} \right|^2 \leq 2q^2 \left(1 + \frac{\alpha^2}{\sigma^2} \frac{T}{N} \right) \frac{\alpha^2}{\sigma^2} \frac{T}{N},$$

for any $t \leq \tau_q^{(N)}$, thus

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sup_{t \leq \tau_q^{(N)}} \left| \bar{\xi}^{(N)}(t) - \bar{\xi}^{(N)}(t-) \right|^2 \right] \leq \lim_{N \rightarrow \infty} 2q^2 \left(1 + \frac{\alpha^2}{\sigma^2} \frac{T}{N} \right) \frac{\alpha^2}{\sigma^2} \frac{T}{N} = 0.$$

Since $|A^{(N)}(t) - A^{(N)}(t-)| \leq (\bar{\xi}^{(N)}(t-))^2 \frac{\alpha^2}{\sigma^2} \Delta t$, similar derivations give

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sup_{t \leq \tau_q^{(N)}} |A^{(N)}(t) - A^{(N)}(t-)| \right] = 0$$

which justify the validity of Condition 4.

Finally, for any $q > 0$,

$$\begin{aligned} & \sup_{t \leq \tau_q^{(N)}} \left| A^{(N)}(t) - \int_0^t \left(\frac{\alpha}{\sigma} \bar{\xi}^{(N)}(s) \right)^2 ds \right| = \sup_{t \leq \tau_q^{(N)}} \frac{\alpha^2}{\sigma^2} \left(\bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \right)^2 \left(t - \lfloor \frac{t}{\Delta t} \rfloor \Delta t \right) \\ & \leq \sup_{t \leq \tau_q^{(N)}} \frac{2\alpha^2}{\sigma^2} q^2 \left(1 + \frac{\alpha^2}{\sigma^2} \frac{T}{N} \right) \Delta t = \frac{2\alpha^2}{\sigma^2} q^2 \left(1 + \frac{\alpha^2}{\sigma^2} \frac{T}{N} \right) \frac{T}{N} \rightarrow 0, \end{aligned}$$

as $N \rightarrow \infty$. Thus, Condition 5 is also fulfilled.

Therefore, we can now apply Theorem 4.1 to conclude that $\{\bar{\xi}^{(N)}(\cdot)\}_{N \in \mathbb{N}}$ converge weakly to $\bar{\xi}^*(\cdot)$ as $N \rightarrow \infty$. $\square$

Lemma 4.5. (i) $\mathbb{E}[(\xi_N^{(N)})^4] < e^{KT}$, where K is a constant independent of N ;
(ii) For every $n \in \mathbb{N}$, there exists a finite constant $M_n > 0$ such that $\mathbb{E}[(\xi_N^{(N)})^{-n}] \leq M_n$ for all $N \in \mathbb{N}$.

Proof. Define $R_k^{(N)} := -\frac{\alpha}{\sigma} \sqrt{\Delta t} Z_k^{(N)}$, we then rewrite $\xi_N^{(N)} = \prod_{k=1}^N \frac{1+R_k^{(N)}}{1+r\Delta t}$.

Thus, provided that N is large enough, for each k ,

$$\begin{aligned} & \mathbb{E} \left[\left(1 + R_k^{(N)} \right)^4 \right] \\ & = \mathbb{E} \left[1 - 4 \frac{\alpha}{\sigma} \sqrt{\Delta t} Z_k^{(N)} + 6 \frac{\alpha^2}{\sigma^2} \Delta t - 4 \frac{\alpha^3}{\sigma^3} (\Delta t)^{3/2} \left(Z_k^{(N)} \right)^3 + \frac{\alpha^4}{\sigma^4} (\Delta t)^2 \right] \\ & \leq 1 + \left(6 \frac{\alpha^2}{\sigma^2} + \frac{\alpha^4}{\sigma^4} \right) \Delta t. \end{aligned}$$

Therefore,

$$\mathbb{E} \left[\left(\xi_N^{(N)} \right)^4 \right] \leq \mathbb{E} \left[\prod_{i=1}^N \left(1 + R_i^{(N)} \right)^4 \right] \leq \left(1 + \left(6 \frac{\alpha^2}{\sigma^2} + \frac{\alpha^4}{\sigma^4} \right) \Delta t \right)^N \leq e^{\left(6 \frac{\alpha^2}{\sigma^2} + \frac{\alpha^4}{\sigma^4} \right) T},$$

for all N . Hence, the claim (i) follows.

Fix $n \in \mathbb{N}$, by direct computations with an application of Taylor series expansions, we have:

$$\begin{aligned} \mathbb{E} \left[\left(1 + R_k^{(N)} \right)^{-n} \right] & = \frac{1}{2} \left(\left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{-n} + \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{-n} \right) \\ & = 1 + \frac{(n+1)n\alpha^2}{2\sigma^2} \Delta t + O(\Delta t^{\frac{3}{2}}). \end{aligned}$$

Then, there exists a constant K_n independent of N such that

$$\begin{aligned} & \mathbb{E} \left[\left(\xi_N^{(N)} \right)^{-n} \right] \\ & = \mathbb{E} \left[\prod_{i=1}^N \left(\frac{1 + R_i^{(N)}}{1 + r\Delta t} \right)^{-n} \right] = \left(1 + \left(rn + \frac{(n+1)n\alpha^2}{2\sigma^2} \right) \Delta t + O(\Delta t^{\frac{3}{2}}) \right)^N \leq e^{K_n T}. \end{aligned}$$

$\square$

Proof of Theorem 4.3. By Lemma 4.4, $\bar{\xi}^{(N)}(T) = \bar{\xi}_N^{(N)} \rightarrow \bar{\xi}^*(T)$ in distribution as $N \rightarrow \infty$. Since $\lim_{N \rightarrow \infty} (1+r\Delta t)^N = e^{rT}$, we also have $\xi_N^{(N)} = (1+r\Delta t)^{-N} (\bar{\xi}_N^{(N)}) \rightarrow$

$e^{-rT}\bar{\xi}^*(T) = \xi^*(T)$ in distribution as $N \rightarrow \infty$. Hence, Condition (i) in Theorem 3.1 follows.

By Assumption 4.2, its inverse function $(U')^{-1}$ satisfies the property: $(U')^{-1}(\eta x) \leq \beta(U')^{-1}(x)$ for all $x > 0$. Further, $(U')^{-1}(\eta^y x) \leq \beta^{y+1}(U')^{-1}(x)$ for all $x, y > 0$ and we also have $(U')^{-1}(\eta^y x) \leq \beta^{y+1}(U')^{-1}(x) + (U')^{-1}(x)$ for all $y \in \mathbb{R}, x > 0$; therefore, we have

$$(U')^{-1}(z) \leq \beta z^{\frac{\ln \beta}{\ln \eta}} (U')^{-1}(1) + (U')^{-1}(1) \text{ for any } z > 0.$$

Then, by Minkowski inequality, we have

$$\left(\mathbb{E} \left[\left((U')^{-1} \left(\xi_N^{(N)} \right) \right)^4 \right] \right)^{\frac{1}{4}} \leq (U')^{-1}(1) \left(\beta^4 \mathbb{E} \left[\left(\xi_N^{(N)} \right)^{4 \frac{\ln \beta}{\ln \eta}} \right] + 1 \right)^{\frac{1}{4}}.$$

According to Lemma 4.5(ii), we now have $\mathbb{E}[(U')^{-1}(\xi_N^{(N)})^4] < M_0$ for some constant $M_0 \in (0, +\infty)$ independent of N . Consider that, for any $K > 0$,

$$\begin{aligned} & \mathbb{E} \left[\left(\xi_N^{(N)} \left((U')^{-1} \left(\xi_N^{(N)} \right) + K \right) \right)^2 \right] \\ & \leq \mathbb{E} \left[\left(\frac{1}{2} \left(\left(\xi_N^{(N)} \right)^2 + \left((U')^{-1} \left(\xi_N^{(N)} \right) + K \right)^2 \right) \right)^2 \right] \\ & \leq \mathbb{E} \left[\left(\frac{1}{2} \left(\left(\xi_N^{(N)} \right)^2 + 2 \left((U')^{-1} \left(\xi_N^{(N)} \right) \right)^2 + 2K^2 \right) \right)^2 \right]. \end{aligned}$$

Then, by a simple application of Minkowski inequality together with Lemma 4.5(i), we can see that $\mathbb{E}[(\xi_N^{(N)}((U')^{-1}(\xi_N^{(N)}) + K))^2] < M$ for some constant $M \in (0, +\infty)$ independent of N . Hence, Condition (ii) in Theorem 3.1 follows by page 127 to 128 in [38]. By Theorem 3.1, the desired convergence result for the optimal terminal wealth follows. $\square$

Furthermore, a similar approximation result for the optimal value function can be obtained:

Corollary 4.6. *Under Assumption 4.2, the optimal value functions for the utility-downside-risk problem corresponding to the sequence of binomial tree models as in Example 2.7 converges to that under the continuous-time Black-Scholes model as in Example 2.6 as the time interval mesh size decreases to 0 (i.e. $N \rightarrow \infty$).*

4.1. Weak convergence of optimal wealth and investment processes. In this subsection, we establish the weak convergence of the optimal wealth processes and the corresponding investment strategies of a sequence of binomial tree models to those under the Black-Scholes model in (2.6), provided that some regularity conditions are satisfied. More precisely, if we consider the optimal wealth at t , $\hat{X}^*(t)$, under the Black-Scholes model to be a function in terms of both the current time t and the time- t pricing kernel $\xi^*(t)$, and such a function is a third-order continuously differentiable one, and all derivatives are of polynomial growth, then the weak convergence of both wealth processes and optimal investment strategies follows.

Indeed, under the Black-Scholes model in Example 2.6, the optimal terminal wealth is given by $\hat{X}^* = I(M^*, \gamma R^* + Y^* \xi^*(T))$, where I is given in (7), and (Y^*, M^*, R^*) is the solution of the system of nonlinear equations of (8)-(10) under

$\xi = \xi^*(T)$. Then, the corresponding wealth process $\hat{X}^*(\cdot)$ that attains $\hat{X}^*$ at T is given by

$$\begin{aligned}\hat{X}^*(t) &= (\xi^*(t))^{-1} \mathbb{E} [\xi^*(T) I(M^*, \gamma R^* + Y^* \xi^*(T)) | \xi^*(t)] \\ &= e^{rt} \left(\bar{\xi}^*(t) \right)^{-1} \mathbb{E} \left[e^{-rT} \bar{\xi}^*(T) I \left(M^*, \gamma R^* + Y^* e^{-rT} \bar{\xi}^*(T) \right) \middle| \bar{\xi}^*(t) \right] \\ &= h^{(Y^*, M^*, R^*)} \left(t, \bar{\xi}^*(t) \right),\end{aligned}\quad (12)$$

where recall $\bar{\xi}^*(t) := e^{-\frac{\alpha^2}{2\sigma^2}t - \frac{\alpha}{\sigma}W(t)}$ and

$$h^{(Y, M, R)}(t, \bar{\xi}) := e^{rt} \bar{\xi}^{-1} \mathbb{E} \left[e^{-rT} \bar{\xi}^*(T) I \left(M, \gamma R + Y e^{-rT} \bar{\xi}^*(T) \right) \middle| \bar{\xi}^*(t) = \bar{\xi} \right]. \quad (13)$$

By an application of Ito's Lemma, we can obtain the investment strategy:

$$\pi^*(t) = -\frac{\alpha}{\sigma^2} \bar{\xi}^*(t) \frac{\partial}{\partial \bar{\xi}} h^{(Y^*, M^*, R^*)} \left(t, \bar{\xi}^*(t) \right). \quad (14)$$

Similarly, under the binomial tree model specified in Example 2.7, the optimal wealth process is given by

$$\begin{aligned}\hat{X}^{(N)}(t) &= \left(\xi^{(N)}(t) \right)^{-1} \mathbb{E} \left[\xi^{(N)}(T) I \left(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \xi^{(N)}(T) \right) | \xi^{(N)}(t) \right] \\ &= (1 + r\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor} \left(\bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} \right)^{-1} \\ &\quad \times \mathbb{E} \left[(1 + r\Delta t)^{-N} \bar{\xi}_N^{(N)} I \left(M^{(N)}, \gamma R^{(N)} + Y^{(N)} e^{-rT} \bar{\xi}_N^{(N)} \right) \middle| \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} \right] \\ &= h^{(N, Y^{(N)}, M^{(N)}, R^{(N)})} \left(t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \right),\end{aligned}\quad (15)$$

where $(Y^{(N)}, M^{(N)}, R^{(N)})$ is the solution of (8)-(10) under $\xi = \xi^{(N)}(T)$, $\bar{\xi}_{n+1}^{(N)} := (1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} Z_{n+1}^{(N)}) \bar{\xi}_n^{(N)}$, $\bar{\xi}_0^{(N)} := 1$, and

$$\begin{aligned}h^{(N, Y, M, R)}(t, \bar{\xi}) \\ := (1 + r\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor} \bar{\xi}^{-1} \mathbb{E} \left[(1 + r\Delta t)^{-N} \bar{\xi}_N^{(N)} I \left(M, \gamma R + Y e^{-rT} \bar{\xi}_N^{(N)} \right) \middle| \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} = \bar{\xi} \right].\end{aligned}$$

For simplicity, we denote $h^{(N, Y^{(N)}, M^{(N)}, R^{(N)})}$ by $h^{(N)}$ unless otherwise specified. By considering the dynamics (5), the investment strategy for the binomial tree model is given by

$$\begin{aligned}\pi^{(N)}(t) &= \frac{1}{2\sigma\sqrt{\Delta t}} \left(h^{(N)} \left(t + \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right) \right. \\ &\quad \left. - h^{(N)} \left(t + \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right) \right).\end{aligned}\quad (16)$$

Theorem 4.7. *Suppose that the optimal wealth process $X^*(\cdot)$ and portfolio $\pi^*(\cdot)$ under the Black-Scholes model are given by (12) and (14) respectively, while the optimal wealth process $X^{(N)}(\cdot)$ and portfolio $\pi^{(N)}(\cdot)$ under the binomial tree model are given by (15) and (16) respectively. If $h^{(Y, M, R)} \in C_p^{3,3}([0, T] \times \mathbb{R}_+)$ as defined in (13), i.e. h is differentiable with respect to both arguments up to the third order, and all of these derivatives up to the third order are continuous and satisfy the following polynomial growth condition:*

There exist $A > 0$ and $q > 1$ such that

$$\left| h^{(Y,M,R)}(t, \bar{\xi}) \right| \leq A(1 + |\bar{\xi}|^q), \quad \forall (t, \bar{\xi}, Y, M, R) \in [0, T] \times \mathbb{R}_+^4.$$

Further suppose that $h_{\bar{\xi}\bar{\xi}\bar{\xi}}$ is Lipschitz continuous with respect to $\bar{\xi}$. Then the processes $\{X^{(N)}(\cdot)\}_{N \in \mathbb{N}}$ and $\{\pi^{(N)}(\cdot)\}_{N \in \mathbb{N}}$ converge weakly to $X^*(\cdot)$ and $\pi^*(\cdot)$ as $N \rightarrow \infty$.

Proof. See Appendix C. $\square$

The polynomial growth condition can be guaranteed if I in (7) is differentiable with respect to the second argument up to the sixth order, and all of these derivatives are continuous and satisfy the polynomial growth condition; indeed, by this time, h is continuously differentiable with respect to the argument $\bar{\xi}$ up to the sixth order by Theorem 5.5 in [17]. Then, as h satisfies the following boundary value problem by Kolmogorov's forward equation:

$$\begin{cases} rh = \frac{\partial h}{\partial t} + \frac{\alpha^2}{\sigma^2} \bar{\xi} \frac{\partial h}{\partial \bar{\xi}} + \frac{1}{2} \frac{\alpha^2}{\sigma^2} \bar{\xi}^2 \frac{\partial^2 h}{\partial \bar{\xi}^2} \\ h^{(Y,M,R)}(T, \bar{\xi}) = I(M, \gamma R + Y \bar{\xi}), \end{cases} \quad (17)$$

we can then deduce the continuous differentiability with respect to the argument t up to the third order by expressing $\frac{\partial h}{\partial t}$, $\frac{\partial^2 h}{\partial t^2}$, and $\frac{\partial^3 h}{\partial t^3}$ in terms of the special partial derivatives of h up to the sixth order. The idea of proof of the weak convergence in Theorem 4.7 is based on the arguments proposed in [19]. We first consider:

$$\begin{aligned} X^{(N)}(t) &= h^{(Y^{(N)}, M^{(N)}, R^{(N)})} \left(\left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \right) \\ &\quad - \left[h^{(Y^{(N)}, M^{(N)}, R^{(N)})} \left(\left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \right) - h^{(N)} \left(\left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \right) \right]. \end{aligned} \quad (18)$$

The first term in (18) can be shown to converge weakly to $X^*(\cdot)$ by establishing that the sequence of process $\{\bar{\xi}^{(N)}(\cdot)\}_{N \in \mathbb{N}}$ converges weakly to $\bar{\xi}^*(\cdot)$ and then applying the continuous mapping theorem. The second term in (18) converges weakly to the zero process via an application of the martingale central limit theorem (Theorem 4.1).

5. Numerical example. In this section, we shall numerically illustrate Corollary 4.6 that the sequence of optimal utility risk terminal wealths and value functions under binomial tree models converge as the number of period N increases to infinity, or equivalently, $\Delta = \frac{T}{N} \rightarrow 0$. To be realistic in any modest economic condition, we set the parameters to be: $r = 0.03, \alpha = 0.07, \sigma = 0.2, T = 8, x_0 = 1$ and $\gamma = 0.1$. Besides, we set the utility function to be a power function: $U(x) = 2x^{\frac{1}{2}}$, and so $U'(x) = x^{-\frac{1}{2}}$, and the risk function adopted is semivariance ($D(x) = \frac{x_+^2}{2}$). We compute each solution of the nonlinear system in (8)-(10) ($Y^{(N)}, M^{(N)}, R^{(N)}$) for every N , optimal control at $t = 0$ ($\pi^{(N)}(0)$), and then determine the optimal expected utility ($U^{(N)}$), the semivariance ($SV^{(N)}$), the optimal objective value ($J^{(N)}$), and the sequential change in optimal objective value for different values of N .

According to the numerical results as quoted in Table 1, whenever we double the number of periods N , the change in value function is approximately halved. This result suggests that the convergence of the value function is in the linear order of the time interval of the binomial model. Following this trend pattern, the result

N	$Y^{(N)}$	$M^{(N)}$	$R^{(N)}$	$\pi^{(N)}(0)$	$U^{(N)}$	$SV^{(N)}$	$J^{(N)}$	Change
25	1.2547	3.9077	1.5571	2.3042	3.4072	2.2446	3.182717	
50	1.2561	3.8816	1.5456	2.2517	3.3959	2.2093	3.174965	-0.007752
100	1.2570	3.8666	1.5428	2.2262	3.3902	2.1905	3.171126	-0.003839
200	1.2573	3.8607	1.5394	2.2135	3.3874	2.1823	3.169219	-0.001908
400	1.2575	3.8573	1.5383	2.2073	3.3861	2.1779	3.168274	-0.000945
800	1.2575	3.8557	1.5375	2.2041	3.3854	2.1758	3.167795	-0.000479
1600	1.2576	3.8550	1.5371	2.2025	3.3850	2.1748	3.167557	-0.000238
3200	1.2576	3.8546	1.5369	2.2018	3.3849	2.1742	3.167439	-0.000118
6400	1.2576	3.8544	1.5368	2.2014	3.3848	2.1740	3.167379	-0.000060

TABLE 1. Convergence of optimal solutions under the case of utility-semivariance.

$N = 50$ could serve as a good approximation to the optimal solution for the utility risk problem under the continuous-time Black-Scholes model since the possible error is less than 0.25% in terms of the optimal objective value.

Appendix A. Outline of proof of Theorem 2.10. We outline the main idea of the proof, all the claims in this section can be established using the same idea in [39].

First, we recall Theorem 4.2 in [39], which derive a static problem, called *Nonlinear Moment Problem*, and the corresponding verification theorem to characterize the optimal solution for Problem 2.2:

Theorem A.1. *Suppose that there exists $\hat{X} \in \mathcal{X}$ satisfying $U'(\hat{X}) \in \mathcal{L}^2$ and $D'(\mathbb{E}[\hat{X}] - \hat{X}) \in \mathcal{L}^2$ and there exist constants $Y, M, R \in \mathbb{R}$ so that the quadruple $(\hat{X}, Y, M, R)$ solves for the Nonlinear Moment Problem, $NMP(\xi(T))$, involving*

$$Y\xi(T) = U'(\hat{X}) - \gamma R + \gamma D'(M - \hat{X}) \quad a.s., \quad (19)$$

subject to the nonlinear moment constraints

$$\mathbb{E}[\xi(T)\hat{X}] = x_0, \quad (20)$$

$$\mathbb{E}[\hat{X}] = M, \quad (21)$$

$$\mathbb{E}[D'(M - \hat{X})] = R. \quad (22)$$

Then, $\hat{X}$ is the unique optimal solution for Problem 2.2, and it is also the unique optimal terminal wealth of Problem 2.1.

Note that we denote (19)-(22) as $NMP(\xi(T))$ since its solution depends on the choice of the terminal random variable of the pricing kernel, $\xi(T)$. With Theorem A.1, it is sufficient for us to solve NMP. To this end, we eliminate (19) with the following result:

Proposition A.2. *There exists an implicit function $I : \mathbb{R} \times (0, \infty) \rightarrow (0, \infty)$ satisfying (7). Moreover, this function I possesses the following regularities:*

- (a) For each $y \in (0, \infty)$,
 - (i) $I(m, y) < m$ for all $m \in \{m \in \mathbb{R} : y > U'(m)\}$
 - (ii) $I(m, y) = (U')^{-1}(y) \in (0, \infty)$ for all $m \in \{m \in \mathbb{R} : y \leq U'(m)\}$
- (b) (i) For each $m \in \mathbb{R}$, $I(m, y)$ is strictly decreasing in y on $(0, \infty)$;

- (ii) For each $y \in (0, \infty)$, $I(m, y)$ is strictly increasing in m on $\{m \in \mathbb{R} \mid y \geq U'(m)\}$.
- (c) $I(m, y)$ is jointly continuous in (m, y) on $\mathbb{R} \times (0, \infty)$.

Proof. We only show the assertion (a)(i) as other assertions have been established in Proposition 4.5 in [39]. Assume a contrary that the assertion (a)(i) does not hold, i.e. $I(m, y) \geq m$. We have

$$U'(I(m, y)) \leq U'(I(m, y)) + \gamma D'(m - I(m, y)) = y < U'(m) \leq U'(I(m, y)),$$

which is absurd. $\square$

With the implicit function I , Nonlinear Moment Problem (19)-(22) is further reduced into the nonlinear programming problem (8)-(10).

Define $\kappa := \text{esssup } \xi(T)$. Now, we shall solve for roots Y, M and R for (8)-(10) one by one via applying the intermediate value theorem successively as in Proposition 4.6 in [39]:

Lemma A.3. *Given $Y \in (0, \infty)$, we consider two exclusive cases: (i) $M \in ((U')^{-1}(Y\kappa), \infty)$ and (ii) $M \in (0, (U')^{-1}(Y\kappa))$.*

- (i) *there exists a unique $R = R_{Y,M} \in (0, D'(M))$ satisfying*

$$\mathbb{E}[D'(M - I(M, \gamma R + Y\xi(T)))] = R; \quad (23)$$

or equivalently by (7):

$$\mathbb{E}[U'(I(M, \gamma R + Y\xi(T)))] = Y\mathbb{E}[\xi(T)]. \quad (24)$$

Furthermore, for every fixed Y , $R_{Y,M}$ is strictly increasing in M , and it is also strictly increasing in Y for each fixed M .

- (ii) $R = R_{Y,M} = 0$ uniquely solves (23).

Lemma A.4. *Given $Y \in (0, \infty)$ and $R_{Y,M}$ as specified for each $M \in (0, \infty)$ in Lemma A.3, there exists a unique $M = M_Y \in ((U')^{-1}(Y\kappa), \infty)$ such that*

$$\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi(T))] = M; \quad (25)$$

furthermore, M_Y is strictly decreasing in Y .

Lemma A.5. *Given $R_{Y,M}$ and M_Y as specified in Lemmas A.3 and A.4 respectively for each $Y, M \in (0, \infty)$, there exists a $Y^* \in (0, \infty)$ such that*

$$\mathbb{E}[\xi(T)I(M_Y, \gamma R_{Y^*, M_Y} + Y^*\xi(T))] = x_0.$$

According to lemmas A.3, A.4 and A.5, the triple $(Y^*, M_{Y^*}, R_{Y^*, M_{Y^*}})$ solves the system of nonlinear equations in (8)-(10). Thus, the quadruple, $(\hat{X}, Y^*, M_{Y^*}, R_{Y^*, M_{Y^*}})$ with

$$\hat{X} := I(M_{Y^*}, \gamma R_{Y^*, M_{Y^*}} + Y^*\xi(T))$$

solves the Nonlinear Moment Problem (19)-(22). Next, it is clear that

$$0 \leq \hat{X} = I(M_{Y^*}, \gamma R_{Y^*, M_{Y^*}} + Y^*\xi(T)) \leq I(M_{Y^*}, \gamma R_{Y^*, M_{Y^*}}) < \infty. \quad (26)$$

With the uniform boundedness of $\hat{X}$ in (26), we can verify $U'(\hat{X}), D'(\mathbb{E}(\hat{X}) - \hat{X}) \in \mathcal{L}^2$ and $\hat{X} \in \mathcal{X}$. Then, by Theorem A.1, $\hat{X}$ is the unique optimal solution for Problem 2.2.

Since $\hat{X}$ is the unique solution and $Y = \mathbb{E}[U'(\hat{X})]/\mathbb{E}[\xi(T)]$ by (24), Y obtained in Lemma A.5 is unique. Hence, only a unique set of triple (Y, M, R) solves (8)-(10). Finally, by Theorem 2.9, $\hat{X}$ is an optimal terminal wealth of utility-downside-risk problem.

Appendix B. Proof of Proposition 3.2. Clearly, for each N , $\zeta^{(N)}$ has the same distribution as $\xi^{(N)}$, the respective systems of nonlinear equations in (8)-(10) with $\xi(T) = \zeta^{(N)}$ and $\xi(T) = \xi^{(N)}$ share the same set of solution $(Y^{(N)}, M^{(N)}, R^{(N)})$, i.e. $(Y^{(N)}, M^{(N)}, R^{(N)})$ satisfy:

$$\mathbb{E} \left[D' \left(M^{(N)} - I \left(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)} \right) \right) \right] = R^{(N)}, \quad (27)$$

$$\mathbb{E} \left[I \left(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)} \right) \right] = M^{(N)}, \quad (28)$$

$$\mathbb{E} \left[\zeta^{(N)} I \left(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)} \right) \right] = x_0. \quad (29)$$

Given the terminal pricing kernel $\xi(T) = \xi^*$, we can define $R_{Y,M}^*$ and M_Y^* from Lemmas A.3 and A.4. With the fact that ξ^* and ζ^* have the same distribution, We know that $(Y^*, M_{Y^*}^*, R_{Y^*, M_{Y^*}^*}^*)$ solves the system of nonlinear equations of (8)-(10) with $\xi(T) = \xi^*$ and $\xi(T) = \zeta^*$. Since the solution of the system is unique, we have $(Y^*, M^*, R^*) = (Y^*, M_{Y^*}^*, R_{Y^*, M_{Y^*}^*}^*)$.

Note that the following simplified notation in (11) applies in this section:

$$\tilde{X}^{(N)} := I \left(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)} \right). \quad (30)$$

We first consider the next three lemmas which establish the convergences of $R^{(N)}$, $M^{(N)}$ and $Y^{(N)}$.

Lemma B.1. *Suppose there is a subsequence $\{(M^{(N_k)}, Y^{(N_k)})\}_{k \in \mathbb{N}}$ and $M_0, Y_0 \in (0, \infty)$ such that $M_0 = \lim_{k \rightarrow \infty} M^{(N_k)}$ and $Y_0 = \lim_{k \rightarrow \infty} Y^{(N_k)}$, then $\lim_{k \rightarrow \infty} R^{(N_k)} = R_{Y_0, M_0}^*$ where $R_{Y, M}^*$ is defined in Lemma A.3 corresponding to $\xi(T) = \zeta^*$ and $(Y^{(N)}, M^{(N)}, R^{(N)})$ satisfy system of equations (27)-(29).*

Proof. By Bolzano-Weierstrass theorem, there exists a subsequence $\{N_{k_i}\}_{i \in \mathbb{N}} \subset \{N_k\}_{k \in \mathbb{N}}$ such that $R_0 := \lim_{i \rightarrow \infty} R^{(N_{k_i})} \in [0, D'(M_0)]$ exists. To facilitate the proof, without loss of generality, we just assume $\{N_i\}_{i \in \mathbb{N}}$ to be the notation of a particular chosen subsequence of $\{N_k\}_{k \in \mathbb{N}}$. Since I and D' are continuous, thus

$$\lim_{i \rightarrow \infty} D' \left(M^{(N_i)} - \tilde{X}^{(N_i)} \right) = D' (M_0 - I (M_0, \gamma R_0 + Y_0 \zeta^*)) \text{ a.s.}$$

Then, by the dominated convergence theorem and (27),

$$\begin{aligned} \mathbb{E} [D' (M_0 - I (M_0, \gamma R_0 + Y_0 \zeta^*))] &= \lim_{i \rightarrow \infty} \mathbb{E} \left[D' \left(M^{(N_i)} - \tilde{X}^{(N_i)} \right) \right] = \lim_{i \rightarrow \infty} R^{(N_i)} \\ &= R_0. \end{aligned}$$

Since Lemma A.3 with $\xi(T) = \zeta^*$ is uniquely solved by R_{Y_0, M_0}^* , we have $R_0 = R_{Y_0, M_0}^*$. Since every subsequential limit of each subsequence $\{R^{(N_k)}\}_{k \in \mathbb{N}}$ agrees with R_{Y_0, M_0}^* , the result follows. $\square$

Lemma B.2. *Suppose that there is a subsequence $\{Y^{(N_k)}\}_{k \in \mathbb{N}}$ and $Y_0 \in (0, \infty)$ such that $Y_0 = \lim_{k \rightarrow \infty} Y^{(N_k)}$, then $\lim_{k \rightarrow \infty} M^{(N_k)} = M_{Y_0}^*$ where M_Y^* is defined in Lemma A.4 corresponding to $\xi(T) = \zeta^*$ and $(Y^{(N)}, M^{(N)}, R^{(N)})$ satisfy system of equations (27)-(29).*

Proof. By Bolzano-Weierstrass theorem, there is a subsequence $\{N_{k_i}\} \subset \{N_k\}$ such that $M_0 := \lim_{i \rightarrow \infty} M^{(N_{k_i})} \in [0, \infty]$ exists. To facilitate the proof, without loss of generality, we just assume $\{N_i\}_{i \in \mathbb{N}}$ to be the notation of a particular chosen subsequence of $\{N_k\}_{k \in \mathbb{N}}$.

We first show that $M_0 \in (0, \infty)$ by following the next 4 steps:

- Step 1: Prove if $M_0 = 0$, then $\lim_{i \rightarrow \infty} \tilde{X}^{(N_i)} > 0$ a.s.; note that $\tilde{X}^{(N)}$ is given by (30);
- Step 2: Check that $M_0 > 0$;
- Step 3: Show that if $M_0 = \infty$, then $\lim_{i \rightarrow \infty} R^{(N_i)} = \infty$;
- Step 4: Establish that $M_0 < \infty$.

- **Step 1.** Assume the contrary that there exists a sample $\omega \in \{\omega \in \Omega : \zeta^*(\omega) < \text{esssup } \zeta^*\}$ and a subsequence $\{N_{i_j}\}_{j \in \mathbb{N}} \subset \{N_i\}_{i \in \mathbb{N}}$ such that $\lim_{j \rightarrow \infty} \tilde{X}^{(N_{i_j})}(\omega) = 0$. To facilitate the proof, without loss of generality, we just assume $\{N_j\}_{j \in \mathbb{N}}$ to be the notation of a particular chosen subsequence of $\{N_i\}_{i \in \mathbb{N}}$. Then $\lim_{j \rightarrow \infty} D'(M^{(N_j)} - \tilde{X}^{(N_j)}(\omega)) = D'(M_0) = D'(0)$ and $\lim_{j \rightarrow \infty} U'(\tilde{X}^{(N_j)}(\omega)) = \infty$. But by the fact that I satisfies (7) with $m = M^{(N_j)}$ and $y = \gamma R^{(N_j)} + Y^{(N_j)}\zeta^{(N_j)}(\omega)$ and the fact that $R^{(N_j)} \leq D'(M^{(N_j)})$ from (27), we have:

$$\begin{aligned}
\infty &= \lim_{j \rightarrow \infty} U'(\tilde{X}^{(N_j)}(\omega)) \\
&= \lim_{j \rightarrow \infty} Y^{(N_j)}\zeta^{(N_j)}(\omega) + \gamma \lim_{j \rightarrow \infty} \left(R^{(N_j)} - D'(M^{(N_j)} - \tilde{X}^{(N_j)}(\omega)) \right) \\
&\leq \zeta^*(\omega)Y_0 + \gamma \lim_{j \rightarrow \infty} \left(D'(M^{(N_j)}) - D'(M^{(N_j)} - \tilde{X}^{(N_j)}(\omega)) \right) \\
&\leq \zeta^*(\omega)Y_0 + \gamma (D'(0) - D'(0)) < \infty,
\end{aligned}$$

which leads to a contradiction. Hence, $\lim_{i \rightarrow \infty} \tilde{X}^{(N_i)} > 0$ a.s.

- **Step 2.** Assume a contrary that $M_0 = 0$. By Step 1, we have $\lim_{i \rightarrow \infty} \tilde{X}^{(N_i)} > 0$ a.s. By the Fatou's lemma and (28), we have

$$0 = M_0 = \liminf_{i \rightarrow \infty} M^{(N_i)} = \liminf_{i \rightarrow \infty} \mathbb{E} \left[\tilde{X}^{(N_i)} \right] \geq \mathbb{E} \left[\liminf_{i \rightarrow \infty} \tilde{X}^{(N_i)} \right] > 0,$$

which leads to a contradiction. Hence $M_0 > 0$.

- **Step 3.** By (7) with $m = M^{(N_i)}$ and $y = \gamma R^{(N_i)} + Y^{(N_i)}\zeta^{(N_i)}$, we first have

$$U'(\tilde{X}^{(N_i)}) - Y^{(N_i)}\zeta^{(N_i)} = \gamma R^{(N_i)} - \gamma D'(M^{(N_i)} - \tilde{X}^{(N_i)}). \quad (31)$$

When the left hand side of (31) is positive infinite as taking limit infimum with $i \rightarrow \infty$, we have $\lim_{i \rightarrow \infty} R^{(N_i)} = \infty$ since the second term of the right hand side is always negative, therefore, the claim follows immediately. On the other hand, when the left hand side of (31) is almost surely finite as taking limit infimum with $i \rightarrow \infty$, we further assume a contrary that there exists a subsequence $\{N_j\}_{j \in \mathbb{N}} \subset \{N_i\}_{i \in \mathbb{N}}$ such that $\lim_{j \rightarrow \infty} R^{(N_j)} < \infty$, then, due to $M_0 = \lim_{j \rightarrow \infty} M^{(N_j)} = \infty$, we have $\lim_{j \rightarrow \infty} \tilde{X}^{(N_j)} = \infty$ a.s. Thus, $\lim_{j \rightarrow \infty} (U'(\tilde{X}^{(N_j)}) - Y^{(N_j)}\zeta^{(N_j)}) = -Y_0\zeta^*$ a.s.

Then, by (27), the reverse Fatou's lemma, and (31), we further obtain:

$$\begin{aligned} 0 &> -Y_0 \mathbb{E}[\zeta^*] = \mathbb{E} \left[\limsup_{j \rightarrow \infty} \left(U' \left(\tilde{X}^{(N_j)} \right) - Y^{(N_j)} \zeta^{(N_j)} \right) \right] \\ &\geq \limsup_{j \rightarrow \infty} \mathbb{E} \left[U' \left(\tilde{X}^{(N_j)} \right) - Y^{(N_j)} \zeta^{(N_j)} \right] \\ &= \limsup_{j \rightarrow \infty} \mathbb{E} \left[\gamma R^{(N_j)} - \gamma D' \left(M^{(N_j)} - \tilde{X}^{(N_j)} \right) \right] = 0, \end{aligned}$$

which is a contradiction, therefore, $\lim_{i \rightarrow \infty} R^{(N_i)} = \infty$.

• **Step 4.** Assume that $M_0 = \infty$, we aim to show that

$$\lim_{i \rightarrow \infty} \left(M^{(N_i)} - \tilde{X}^{(N_i)} \right) = \infty, \text{ a.s.} \quad (32)$$

Indeed, given a sample $\omega \in \Omega$, assume the contrary that there exists a subsequence $\{N_{i_j}\}_{j \in \mathbb{N}} \subset \{N_i\}_{i \in \mathbb{N}}$ such that $\lim_{j \rightarrow \infty} (M^{(N_{i_j})} - \tilde{X}^{(N_{i_j})}(\omega)) < \infty$. To facilitate the proof, without loss of generality, we just assume $\{N_j\}_{j \in \mathbb{N}}$ to be the notation of a particular chosen subsequence of $\{N_i\}_{i \in \mathbb{N}}$. Then clearly $\lim_{j \rightarrow \infty} D'(M^{(N_j)} - \tilde{X}^{(N_j)}(\omega)) < \infty$. Since $\lim_{j \rightarrow \infty} R^{(N_j)} = \infty$ by Step 3 and I satisfies (7), we have $\lim_{j \rightarrow \infty} U'(\tilde{X}^{(N_j)}(\omega)) = \infty$, thus $\lim_{j \rightarrow \infty} \tilde{X}^{(N_j)}(\omega) = 0$, and hence $\lim_{j \rightarrow \infty} (M^{(N_j)} - \tilde{X}^{(N_j)}(\omega)) = \infty$, which contradicts to the assumption on its finiteness. Hence, (32) holds.

By (28), the Fatou's lemma, and then (32), we now have

$$0 = \liminf_{i \rightarrow \infty} \mathbb{E} \left[M^{(N_i)} - \tilde{X}^{(N_i)} \right] \geq \mathbb{E} \left[\liminf_{i \rightarrow \infty} \left(M^{(N_i)} - \tilde{X}^{(N_i)} \right) \right] = \infty, \quad (33)$$

which leads to a contradiction.

In conclusion, by Steps 2 and 4, we have $M_0 \in (0, \infty)$. By Lemma B.1, $\lim_{i \rightarrow \infty} R^{(N_i)} = R_{Y_0, M_0}^*$, thus

$$\lim_{i \rightarrow \infty} I \left(M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} \right) = I \left(M_0, \gamma R_{Y_0, M_0}^* + Y_0 \zeta^* \right).$$

By the dominated convergence theorem and (28),

$$\begin{aligned} \mathbb{E} \left[I \left(M_0, \gamma R_{Y_0, M_0}^* + Y_0 \zeta^* \right) \right] &= \lim_{i \rightarrow \infty} \mathbb{E} \left[I \left(M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} \right) \right] \\ &= \lim_{i \rightarrow \infty} M^{(N_i)} = M_0. \end{aligned}$$

By the uniqueness of Lemma A.4 with $\xi(T) = \zeta^*$, we have $M_0 = M_{Y_0}^*$. Since every subsequential limit of $\{M_{N_k}\}_{k \in \mathbb{N}}$ agrees with $M_0 = \lim_{i \rightarrow \infty} M^{(N_i)} = M_{Y_0}^*$, the claim follows. $\square$

Lemma B.3. $\lim_{N \rightarrow \infty} Y^{(N)} = Y^*$.

Proof. By Bolzano-Weierstrass theorem, there exists a subsequence $\{N_k\}_{k \in \mathbb{N}}$ such that $Y_0 := \lim_{k \rightarrow \infty} Y^{(N_k)} \in [0, \infty]$ exists.

We will prove $Y_0 \in (0, \infty)$ by using following 4 steps:

- Step 1: Prove if $Y_0 = 0$, $\mathbb{P}[\lim_{k \rightarrow \infty} \tilde{X}^{(N_k)} = \infty] > 0$, note that $\tilde{X}^{(N)}$ is given by (30);
- Step 2: Check that $Y_0 > 0$.
- Step 3: Show that if $Y_0 = \infty$, $\lim_{k \rightarrow \infty} M^{(N_k)} < \infty$.

- Step 4: Establish that $Y_0 < \infty$.

- **Step 1.** Assume the contrary that there exists an arbitrary a sample $\omega_0 \in \{\omega \in \Omega : \zeta^*(\omega) < \text{esssup } \zeta^*\}$ and a sequence $\{N_{k_i}\}_{i \in \mathbb{N}} \subset \{N_k\}_{k \in \mathbb{N}}$ such that $\lim_{i \rightarrow \infty} \tilde{X}^{(N_{k_i})}(\omega_0) < \infty$. To facilitate the proof, without loss of generality, we just assume $\{N_i\}_{i \in \mathbb{N}}$ to be the notation of a particular chosen subsequence of $\{N_k\}_{k \in \mathbb{N}}$. For any sample $\omega \in \{\omega \in \Omega : \zeta^*(\omega) > \zeta^*(\omega_0)\}$, we have $\zeta^{(N_i)}(\omega) > \zeta^{(N_i)}(\omega_0)$ for large enough i , hence, by the fact that $\tilde{X}^{(N_i)} = I(M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)})$ is strictly decreasing in $\zeta^{(N_i)}$ according to Prop A.2 (b)(i), we have $U'(\tilde{X}^{(N_i)}(\omega)) > U'(\tilde{X}^{(N_i)}(\omega_0))$.

Therefore, for any $\omega \in \{\omega \in \Omega : \zeta^*(\omega) > \zeta^*(\omega_0)\}$,

$$\liminf_{i \rightarrow \infty} U'(\tilde{X}^{(N_i)}(\omega)) \geq \lim_{i \rightarrow \infty} U'(\tilde{X}^{(N_i)}(\omega_0)) > 0.$$

Fulfilling Condition (ii) in Theorem 3.1 implies that $\mathbb{E}[\zeta^{(N_i)}]$ is uniformly bounded from above, then, by (7) with $m = M^{(N_i)}$ and $y = \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)}$, (27), and the Fatou's lemma,

$$\begin{aligned} 0 &= \liminf_{i \rightarrow \infty} Y^{(N_i)} \mathbb{E}[\zeta^{(N_i)}] = \liminf_{i \rightarrow \infty} \mathbb{E} \left[U' \left(\tilde{X}^{(N_i)} \right) \right] \\ &\geq \mathbb{E} \left[\liminf_{i \rightarrow \infty} U' \left(\tilde{X}^{(N_i)} \right) \right] \geq \lim_{i \rightarrow \infty} U' \left(\tilde{X}^{(N_i)}(\omega_0) \right) \mathbb{P}[\zeta^* > \zeta^*(\omega_0)] > 0, \end{aligned}$$

which results in a contradiction, hence $\lim_{k \rightarrow \infty} \tilde{X}^{(N_k)}(\omega_0) = \infty$ for all $\omega_0 \in \{\omega \in \Omega : \zeta^*(\omega) < \text{esssup } \zeta^*\}$.

- **Step 2.** Assume the contrary that $Y_0 = 0$, by Step 1, we further have $\mathbb{P}[\lim_{k \rightarrow \infty} (\zeta^{(N_k)} \tilde{X}^{(N_k)}) = \infty] > 0$. By (29) and the Fatou's lemma, we have

$$x_0 = \liminf_{k \rightarrow \infty} \mathbb{E} \left[\zeta^{(N_k)} \tilde{X}^{(N_k)} \right] \geq \mathbb{E} \left[\liminf_{k \rightarrow \infty} \zeta^{(N_k)} \tilde{X}^{(N_k)} \right] = \infty,$$

which is impossible and thus $Y_0 > 0$.

- **Step 3.** Assume that $\lim_{k \rightarrow \infty} M^{(N_k)} = \infty$. Following the similar argument as Step 4 of the proof of Lemma B.2, we can derive the same absurd inequality as (33).
- **Step 4.** Assume $Y_0 = \infty$. We claim that $\lim_{k \rightarrow \infty} \tilde{X}^{(N_k)} = 0$ a.s. Assume the contrary that there exists a sample $\omega_0 \in \Omega$ and a sequence $\{N_{k_i}\}_{i \in \mathbb{N}} \subset \{N_k\}_{k \in \mathbb{N}}$ with $Y^{(N_{k_i})} \rightarrow \infty$ such that $\lim_{i \rightarrow \infty} \tilde{X}^{(N_{k_i})}(\omega_0) > 0$. To facilitate the proof, without loss of generality, we just assume $\{N_i\}_{i \in \mathbb{N}}$ to be the notation of a particular chosen subsequence of $\{N_k\}_{k \in \mathbb{N}}$. By (7) with $m = M^{(N_i)}$ and $y = \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)}(\omega_0)$, we first have

$$\begin{aligned} &U' \left(\tilde{X}^{(N_i)}(\omega_0) \right) - Y^{(N_i)} \zeta^{(N_i)}(\omega_0) \\ &= \gamma R^{(N_i)} - \gamma D' \left(M^{(N_i)} - \tilde{X}^{(N_i)}(\omega_0) \right) \geq -\gamma D' \left(M^{(N_i)} \right). \end{aligned}$$

Taking limit to the both sides of above with the fact that $\lim_{k \rightarrow \infty} M^{(N_k)} < \infty$ from Step 3, we have

$$\begin{aligned} -\infty &= \lim_{i \rightarrow +\infty} \left(U' \left(\tilde{X}^{(N_i)}(\omega_0) \right) - Y^{(N_i)} \zeta^{(N_i)}(\omega_0) \right) \\ &\geq -\gamma \lim_{i \rightarrow +\infty} D' \left(M^{(N_i)} \right) > -\infty, \end{aligned}$$

which is absurd. Therefore, we have

$$\lim_{k \rightarrow \infty} \tilde{X}^{(N_k)} = 0 \text{ a.s.} \quad (34)$$

By Prop A.2 (a), we have

$$\begin{aligned} \tilde{X}^{(N_k)} &= I \left(M^{(N_k)}, Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)} \right) \\ &\leq M^{(N_k)} + (U')^{-1} \left(Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)} \right) \\ &\leq 2 \lim_{k \rightarrow \infty} M^{(N_k)} + (U')^{-1} (\zeta^{(N_k)}), \end{aligned}$$

for large enough k . Due to Condition (ii) in Theorem 3.1, $\{\zeta^{(N_k)} \tilde{X}^{(N_k)}\}_{k \in \mathbb{N}}$ is uniformly integrable. By the dominated convergence theorem under the uniform integrability assumption, (29), and (34),

$$x_0 = \lim_{k \rightarrow \infty} \mathbb{E} \left[\zeta^{(N_k)} \tilde{X}^{(N_k)} \right] = \mathbb{E} \left[\lim_{k \rightarrow \infty} \zeta^{(N_k)} \tilde{X}^{(N_k)} \right] = 0,$$

which is absurd, and hence $Y_0 < \infty$.

In conclusion, by Steps 2 and 4, we have $Y_0 \in (0, \infty)$. By Lemmas B.1 and B.2, $\lim_{k \rightarrow \infty} (Y^{(N_k)}, M^{(N_k)}, R^{(N_k)}) = (Y_0, M_{Y_0}^*, R_{Y_0, M_{Y_0}^*}^*)$, therefore, we have

$$\lim_{k \rightarrow \infty} \zeta^{(N_k)} \tilde{X}^{(N_k)} = \zeta^* I \left(M_{Y_0}^*, \gamma R_{Y_0, M_{Y_0}^*}^* + Y_0 \zeta^* \right).$$

Due to Condition (ii) in Theorem 3.1, we have $\{\zeta^{(N_k)} \tilde{X}^{(N_k)}\}_{k \in \mathbb{N}}$ uniformly integrable, and by dominated convergence theorem and (29),

$$x_0 = \lim_{k \rightarrow \infty} \mathbb{E} \left[\zeta^{(N_k)} \tilde{X}^{(N_k)} \right] = \mathbb{E} \left[\zeta^* I \left(M_{Y_0}^*, \gamma R_{Y_0, M_{Y_0}^*}^* + Y_0 \zeta^* \right) \right].$$

Together with the fact that $R_{Y, M}^*$ and M_Y^* solve (23) and (25) with $\xi = \zeta^*$, $(Y_0, M_{Y_0}^*, R_{Y_0, M_{Y_0}^*}^*)$ solving for the system of equations (8)-(10) with $\xi = \zeta^*$. Since the solution of the system is unique, so we have $(Y_0, M_{Y_0}^*, R_{Y_0, M_{Y_0}^*}^*) = (Y^*, M^*, R^*)$. Since every subsequential limit of $\{Y^{(N)}\}_{N \in \mathbb{N}}$ agrees with $Y_0 = \lim_{k \rightarrow \infty} Y^{(N_k)} = Y^*$, therefore, $Y^* = \lim_{N \rightarrow \infty} Y^{(N)}$. $\square$

By Lemmas B.1 and B.2, we have

$$\lim_{N \rightarrow \infty} (Y^{(N)}, M^{(N)}, R^{(N)}) = (Y^*, M_{Y^*}^*, R_{Y^*, M_{Y^*}^*}^*) = (Y^*, M^*, R^*),$$

consequently, the proposition follows.

Appendix C. Proof of Theorem 4.7. To facilitate the proof, different constants C which appear throughout the following proof may be different line by line. First recall that the following version of Continuous Mapping Theorem will be used to prove this theorem:

Proposition C.1 (Lemma 1 of [19]). *Assume that the sequence of processes in $\mathbb{R}^n$, $\{X^{(N)}(\cdot)\}_{N \in \mathbb{N}}$, converges weakly to $X^*(\cdot)$. Suppose that $h(x)$ is a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}$. Then $\{h(X^{(N)}(\cdot))\}_{N \in \mathbb{N}}$ converges weakly to $h(X^*(\cdot))$.*

First, we telescope the following expression:

$$X^{(N)}(t) = h^{(Y^{(N)}, M^{(N)}, R^{(N)})} \left(\left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \right) - \varepsilon^{(N)}(t), \quad (35)$$

where we define such truncation errors: $\varepsilon^{(N)}(t) := \varepsilon_{\lfloor t/\Delta t \rfloor}^{(N)}$ and

$$\begin{aligned} \varepsilon_k^{(N)} &:= h^{(Y^{(N)}, M^{(N)}, R^{(N)})} \left(k\Delta t, \bar{\xi}_k^{(N)} \right) - h^{(N, Y^{(N)}, M^{(N)}, R^{(N)})} \left(k\Delta t, \bar{\xi}_k^{(N)} \right) \\ &=: h \left(k\Delta t, \bar{\xi}_k^{(N)} \right) - h^{(N)} \left(k\Delta t, \bar{\xi}_k^{(N)} \right), \end{aligned}$$

where for simplicity, we denote $h^{(Y^{(N)}, M^{(N)}, R^{(N)})}$ and $h^{(N, Y^{(N)}, M^{(N)}, R^{(N)})}$ respectively by h and $h^{(N)}$; same type of simplification will be adopted for their derivatives.

By Lemma 4.4, the sequence of processes $\{\bar{\xi}_{\lfloor \cdot / \Delta t \rfloor}^{(N)}\}_{N \in \mathbb{N}}$ converges weakly to $\bar{\xi}^*(\cdot)$, thus the sequence of processes $\{(\lfloor \cdot / \Delta t \rfloor \Delta t, \bar{\xi}_{\lfloor \cdot / \Delta t \rfloor}^{(N)}, Y^{(N)}, M^{(N)}, R^{(N)})\}_{N \in \mathbb{N}}$ in $\mathbb{R}^5$ converges weakly to $(\cdot, \bar{\xi}^*(\cdot), Y^*, M^*, R^*)$. Then, by Proposition C.1, $h^{(Y^{(N)}, M^{(N)}, R^{(N)})} \left(\lfloor \cdot / \Delta t \rfloor \Delta t, \bar{\xi}_{\lfloor \cdot / \Delta t \rfloor}^{(N)} \right)$ converges weakly to $h^{(Y^*, M^*, R^*)}(\cdot, \bar{\xi}^*(\cdot))$. Thus, we only need to prove that the process $\{\varepsilon^{(N)}(\cdot)\}_{N \in \mathbb{N}}$ converges weakly to a zero process as N goes to ∞ . To this end, we need the following lemmas regarding the boundedness of $\varepsilon_k^{(N)}$ (their proofs will be postponed to Subsections C.1 and C.2):

Lemma C.2. *There exist $C > 0$ and a positive integer p such that for any k and N , there are δ and M , which are independent of k and N , such that $|O(\Delta t)| \leq M\Delta t$ when $0 < \Delta t < \delta$, and we also have*

$$|\varepsilon_k^{(N)}| \leq C \left(1 + |\bar{\xi}_k^{(N)}|^p \right) \left(\sqrt{\Delta t} + O(\Delta t) \right).$$

Lemma C.3. *For any $p \in \mathbb{N}$, there exists a $M_p > 0$ such that $\mathbb{E}[\sup_{1 \leq n \leq N} |\bar{\xi}_k^{(N)}|^p] \leq M_p$ for all $N \in \mathbb{N}$.*

Now, we are going to show that $\{\varepsilon^{(N)}(\cdot)\}_{N \in \mathbb{N}}$ converges weakly to 0 by applying martingale central limit theorem as stated in Theorem 4.1.

By Lemma C.2, there exist $C > 0$ and a positive integer p such that

$$|\varepsilon^{(N)}(0)| = |\varepsilon_0^{(N)}| \leq C \left(1 + |\bar{\xi}_0^{(N)}|^p \right) \left(\sqrt{\Delta t} + O(\Delta t) \right) = 2C \left(\sqrt{\Delta t} + O(\Delta t) \right) \rightarrow 0,$$

as $N \rightarrow \infty$, and so Condition 1 in Theorem 4.1 is fulfilled.

By setting $L^{(N)}(t) := \varepsilon^{(N)}(t)$, $A^{(N)}(t) := 0$, and $M^{(N)}(t) := \varepsilon^{(N)}(t) - L^{(N)}(t) = 0$, and obviously, Conditions 2 and 3 in Theorem 4.1 are both satisfied.

By Lemma C.2, for any k , we have

$$\begin{aligned} |\varepsilon_k^{(N)}|^2 &\leq \frac{C}{2} \left(1 + 2|\bar{\xi}_k^{(N)}|^p + |\bar{\xi}_k^{(N)}|^{2p} \right) \left(\Delta t + O\left(\Delta t^{\frac{3}{2}}\right) + O(\Delta t^2) \right) \\ &\leq C \left(1 + |\bar{\xi}_k^{(N)}|^{2p} \right) \left(\Delta t + O\left(\Delta t^{\frac{3}{2}}\right) \right), \end{aligned}$$

for some $C > 0$. By Lemma C.3, we have

$$\begin{aligned} \mathbb{E} \left[\sup_{1 \leq k \leq N} |\varepsilon_k^{(N)}|^2 \right] &\leq C \left(1 + \mathbb{E} \left[\sup_{1 \leq k \leq N} |\bar{\xi}_k^{(N)}|^{2p} \right] \right) \left(\Delta t + O\left(\Delta t^{\frac{3}{2}}\right) \right) \\ &\leq M \left(\Delta t + O\left(\Delta t^{\frac{3}{2}}\right) \right). \end{aligned}$$

Therefore, for any t , we have

$$\begin{aligned} \mathbb{E} \left[\sup_t \left| L^{(N)}(t) - L^{(N)}(t-) \right|^2 \right] &\leq 2\mathbb{E} \left[\sup_t \left(\left| \varepsilon^{(N)}(t) \right|^2 + \left| \varepsilon^{(N)}(t-) \right|^2 \right) \right] \\ &\leq 4\mathbb{E} \left[\sup_{1 \leq k \leq N} \left| \varepsilon_k^{(N)} \right|^2 \right] \leq 4M \left(\Delta t + O \left(\Delta t^{\frac{3}{2}} \right) \right) \rightarrow 0, \end{aligned}$$

as $N \rightarrow \infty$. Thus, Condition 4 in Theorem 4.1 is satisfied.

By Lemmas C.2 and C.3, we have

$$\begin{aligned} \mathbb{E} \left[\sup_t \left| L^{(N)}(t) \right| \right] &\leq \mathbb{E} \left[\sup_{1 \leq k \leq N} \left| \varepsilon_k^{(N)} \right| \right] \\ &\leq C \left(1 + \mathbb{E} \left[\sup_{1 \leq k \leq N} \left| \tilde{\xi}_k^{(N)} \right|^p \right] \right) \left(\sqrt{\Delta t} + O(\Delta t) \right) \leq M \left(\sqrt{\Delta t} + O(\Delta t) \right) \rightarrow 0, \end{aligned}$$

as $N \rightarrow \infty$. Thus, $\sup_t |L^{(N)}(t)| \rightarrow 0$ in probability as $N \rightarrow \infty$. Hence, Condition 5 in Theorem 4.1 is again fulfilled. Hence, by Theorem 4.1, we have $\{\varepsilon^{(N)}(\cdot)\}_{N \in \mathbb{N}}$ converges weakly to zero. Now, we have already shown that the first term in the right hand side of (35) converges weakly to the process $h^{(Y^*, M^*, R^*)}(\cdot, \tilde{\xi}^*(\cdot))$, which is exactly the $X^*(\cdot)$, and the second term on the right hand side of (35) converges weakly to zero, and hence $X^{(N)}(\cdot)$ converges weakly to $X^*(\cdot)$.

For the convergence of investment policy, by (16), we have

$$\begin{aligned} \pi^{(N)}(t) &= \frac{1}{2\sigma\sqrt{\Delta t}} \left(h^{(N)} \left(t + \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right) \right. \\ &\quad \left. - h^{(N)} \left(t + \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right) \right) \\ &=: \frac{1}{2\sigma\sqrt{\Delta t}} \left(f_{\lfloor t/\Delta t \rfloor}^{(N)+}(1) - f_{\lfloor t/\Delta t \rfloor}^{(N)-}(1) - \varepsilon_{\lfloor t/\Delta t \rfloor+1}^{(N)+} + \varepsilon_{\lfloor t/\Delta t \rfloor+1}^{(N)-} \right), \end{aligned} \tag{36}$$

where we set

$$\begin{aligned} \bar{\xi}_{k+1}^{(N)+} &:= \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \bar{\xi}_k^{(N)}, \\ \bar{\xi}_{k+1}^{(N)-} &:= \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \bar{\xi}_k^{(N)}, \\ f_k^{(N)+}(\tau) &:= h \left((k + \tau)\Delta t, \bar{\xi}_k^{(N)} + \tau \left(\bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \right), \\ f_k^{(N)-}(\tau) &:= h \left((k + \tau)\Delta t, \bar{\xi}_k^{(N)} + \tau \left(\bar{\xi}_{k+1}^{(N)-} - \bar{\xi}_k^{(N)} \right) \right), \\ \varepsilon_k^{(N)+} &:= h \left(k\Delta t, \bar{\xi}_k^{(N)+} \right) - h^{(N)} \left(k\Delta t, \bar{\xi}_k^{(N)+} \right), \\ \varepsilon_k^{(N)-} &:= h \left(k\Delta t, \bar{\xi}_k^{(N)-} \right) - h^{(N)} \left(k\Delta t, \bar{\xi}_k^{(N)-} \right). \end{aligned}$$

Since h is continuously differentiable in both t and $\bar{\xi}$ up to the third order, $f_k^{(N)+}$ and $f_k^{(N)-}$ admit the following Taylor expansions:

$$f_k^{(N)+}(1) = f_k^{(N)+}(0) + \frac{\partial}{\partial \tau} f_k^{(N)+}(0) + \frac{1}{2} \frac{\partial^2}{\partial \tau^2} f_k^{(N)+}(0) + P_k^{(N)}, \tag{37}$$

$$f_k^{(N)-}(1) = f_k^{(N)-}(0) + \frac{\partial}{\partial \tau} f_k^{(N)-}(0) + \frac{1}{2} \frac{\partial^2}{\partial \tau^2} f_k^{(N)-}(0) + Q_k^{(N)}, \tag{38}$$

where

$$P_k^{(N)} := \frac{1}{6} \int_0^1 (1-s)^3 \frac{\partial^3}{\partial \tau^3} f_k^{(N)+}(s) ds, \quad (39)$$

$$Q_k^{(N)} := \frac{1}{6} \int_0^1 (1-s)^3 \frac{\partial^3}{\partial \tau^3} f_k^{(N)-}(s) ds. \quad (40)$$

A direct calculation through an application of the law of total derivatives gives the following:

$$\frac{\partial}{\partial \tau} f_k^{(N)+}(0) = h_t \Delta t + \left(\bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) h_{\bar{\xi}} = h_t \Delta t - \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{\bar{\xi}} \sqrt{\Delta t}, \quad (41)$$

$$\frac{\partial}{\partial \tau} f_k^{(N)-}(0) = h_t \Delta t + \left(\bar{\xi}_{k+1}^{(N)-} - \bar{\xi}_k^{(N)} \right) h_{\bar{\xi}} = h_t \Delta t + \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{\bar{\xi}} \sqrt{\Delta t}, \quad (42)$$

$$\frac{\partial^2}{\partial \tau^2} f_k^{(N)+}(0) = h_{tt}(\Delta t)^2 - 2 \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{t\bar{\xi}} \Delta t^{\frac{3}{2}} + \frac{\alpha^2}{\sigma^2} \left(\bar{\xi}_k^{(N)} \right)^2 h_{\bar{\xi}\bar{\xi}} \Delta t, \quad (43)$$

$$\frac{\partial^2}{\partial \tau^2} f_k^{(N)-}(0) = h_{tt}(\Delta t)^2 + 2 \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{t\bar{\xi}} \Delta t^{\frac{3}{2}} + \frac{\alpha^2}{\sigma^2} \left(\bar{\xi}_k^{(N)} \right)^2 h_{\bar{\xi}\bar{\xi}} \Delta t, \quad (44)$$

where all partial derivatives of h are evaluated at the point $(k\Delta t, \bar{\xi}_k^{(N)})$ while all of these functions and derivatives are indexed with superscripts $(Y^{(N)}, M^{(N)}, R^{(N)})$ and we suppress these notations for the sake of convenience. Using these expressions from (37) to (44), we obtain:

$$\begin{aligned} & \frac{1}{2\sigma\sqrt{\Delta t}} \left(f_{\lfloor t/\Delta t \rfloor}^{(N)+}(1) - f_{\lfloor t/\Delta t \rfloor}^{(N)-}(1) \right) \\ &= -\frac{\alpha}{\sigma^2} \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} h_{\bar{\xi}} - \frac{\alpha}{\sigma^2} \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} h_{t\bar{\xi}} \Delta t + \frac{1}{6} \int_0^1 (1-s)^3 \frac{\partial^3}{\partial \tau^3} \left(f_{\lfloor t/\Delta t \rfloor}^{(N)+}(s) - f_{\lfloor t/\Delta t \rfloor}^{(N)-}(s) \right) ds. \end{aligned}$$

Thus, (36) can be rewritten as

$$\pi^{(N)}(t) = -\frac{\alpha}{\sigma^2} \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} h_{\bar{\xi}} \left(\lfloor t/\Delta t \rfloor \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \right) + e_{\lfloor t/\Delta t \rfloor}^{(N)}, \quad (45)$$

where

$$\begin{aligned} e_k^{(N)} &:= -\frac{\alpha}{\sigma^2} \bar{\xi}_k^{(N)} h_{t\bar{\xi}} \Delta t + \frac{1}{6} \int_0^1 (1-s)^3 \frac{\partial^3}{\partial \tau^3} \left(f_k^{(N)+}(s) - f_k^{(N)-}(s) \right) ds \\ &\quad - \frac{1}{2\sigma\sqrt{\Delta t}} \left(\varepsilon_{k+1}^{(N)+} - \varepsilon_{k+1}^{(N)-} \right). \end{aligned} \quad (46)$$

By Theorem C.1, the first term in (45) converges weakly to the process $\left(-\frac{\alpha}{\sigma^2} \bar{\xi}^*(\cdot) \times h_{\bar{\xi}}^{(Y^*, M^*, R^*)}(\cdot, \bar{\xi}^*(\cdot)) \right)$, which is exactly the optimal investment process $\pi^*(\cdot)$.

Hence, we only need to show that the process $\{e_{\lfloor \cdot/\Delta t \rfloor}^{(N)}\}_{N \in \mathbb{N}}$ converges weakly to zero. To this end, we need the following lemma regarding the boundedness of $e_k^{(N)}$ (its proof will be postponed to the next subsection C.3):

Lemma C.4. *There exist $C > 0$ and a positive integer p such that for any k and N , there are δ and M , which are independent of k and N , such that $|O(\Delta t)| \leq M\Delta t$ when $0 < \Delta t < \delta$, and we also have*

$$\left| e_k^{(N)} \right| \leq C \left(1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left(\sqrt{\Delta t} + O(\Delta t) \right).$$

Then, with an application of the martingale central limit theorem (Theorem 4.1) with the same procedure as in the case of $\varepsilon^{(N)}(\cdot)$, which solely depends on the boundedness property but not the expression, one can deduce the process $\{e_{\lfloor \cdot / \Delta t \rfloor}^{(N)}\}_{N \in \mathbb{N}}$ converges weakly to the process of identically zero; and so the weak convergence of optimal investment process also follows.

C.1. Proof of Lemma C.2. We proceed the proof by splitting into 3 steps as follows:

- **Step 1.** To establish that

$$\varepsilon_k^{(N)} = \mathbb{E}^{(N)} \left[\sum_{n=k}^{N-1} \left(\beta(n\Delta t, \bar{\xi}_n^{(N)}) \frac{1}{(1+r\Delta t)^{n-k}} (\Delta t)^2 + \gamma_n^{(N)} \frac{1}{(1+r\Delta t)^{n-k}} \right) \middle| \bar{\xi}_k^{(N)} \right], \quad (47)$$

where $\mathbb{E}^{(N)}$ is the expectation with respect to the probability measure so that $\mathbb{P}[Z_n^{(N)} = 1] = \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2}$ and $\mathbb{P}[Z_n^{(N)} = -1] = \frac{1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2}$; $\beta(t, \xi) := -\frac{(h_{tt}(t, \xi) + 2\frac{\alpha^2}{\sigma^2} \xi h_{t\xi}(t, \xi))}{2(1+r\Delta t)}$ and $\gamma_k^{(N)} := -\frac{(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t})P_k^{(N)} + (1 + \frac{\alpha}{\sigma} \sqrt{\Delta t})Q_k^{(N)}}{2(1+r\Delta t)}$.

- **Step 2.** Then we aim to show that

$$\sum_{n=k}^{N-1} \mathbb{E}^{(N)} \left[\left| \beta(n\Delta t, \bar{\xi}_n^{(N)}) \right| \left| \bar{\xi}_k^{(N)} \right| (\Delta t)^2 \right] \leq A \left(1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \Delta t, \quad (48)$$

for some $A > 0$, which is independent of k and N , and a positive integer p .

- **Step 3.** Finally, as stated in the lemma, there are δ and M , which are independent of k and N , such that $|O(\Delta t)| \leq M\Delta t$ when $0 < \Delta t < \delta$, we need to have

$$\sum_{n=k}^{N-1} \mathbb{E}^{(N)} \left[\left| \gamma_n^{(N)} \right| \left| \bar{\xi}_k^{(N)} \right| \right] \leq A \left(1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left(\sqrt{\Delta t} + O(\Delta t) \right)$$

for some $A > 0$ and a positive integer p .

- **Step 1.** In order to derive the expression of $\varepsilon_k^{(N)}$, we should first calculate two items, namely, $\mathbb{E}^{(N)}[h((k+1)\Delta t, \bar{\xi}_{k+1}^{(N)}) | \bar{\xi}_k^{(N)}]$ and $h^{(N)}(k\Delta t, \bar{\xi}_k^{(N)})$. Using the expressions as obtained in (37) – (44) with $f_k^{(N)\pm}(0) = h(k\Delta t, \bar{\xi}_k^{(N)})$, which is clear from the definitions, we first have

$$\begin{aligned} & \mathbb{E}^{(N)} \left[\frac{1}{1+r\Delta t} h((k+1)\Delta t, \bar{\xi}_{k+1}^{(N)}) \middle| \bar{\xi}_k^{(N)} \right] \\ &= \frac{1}{2} \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{1+r\Delta t} f_k^{(N)+}(1) + \frac{1}{2} \frac{1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}}{1+r\Delta t} f_k^{(N)-}(1) \\ &= \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1+r\Delta t)} \left(h + h_t \Delta t - \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{\bar{\xi}} \sqrt{\Delta t} + \frac{1}{2} h_{tt} (\Delta t)^2 \right. \\ & \quad \left. - \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{t\bar{\xi}} (\Delta t)^{\frac{3}{2}} + \frac{1}{2} \frac{\alpha^2}{\sigma^2} \left(\bar{\xi}_k^{(N)} \right)^2 h_{\bar{\xi}\bar{\xi}} \Delta t + P_k^{(N)} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}}{2(1 + r\Delta t)} \left(h + h_t \Delta t + \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{\bar{\xi}} \sqrt{\Delta t} + \frac{1}{2} h_{tt} (\Delta t)^2 \right. \\
& \quad \left. + \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{t\bar{\xi}} (\Delta t)^{\frac{3}{2}} + \frac{1}{2} \frac{\alpha^2}{\sigma^2} \left(\bar{\xi}_k^{(N)} \right)^2 h_{\bar{\xi}\bar{\xi}} \Delta t + Q_k^{(N)} \right) \\
& = \frac{1}{1 + r\Delta t} \left(h + h_t \Delta t + \frac{\alpha^2}{\sigma^2} \bar{\xi}_k^{(N)} h_{\bar{\xi}} \Delta t + \frac{1}{2} h_{tt} (\Delta t)^2 \right. \\
& \quad \left. + \frac{\alpha^2}{\sigma^2} \bar{\xi}_k^{(N)} h_{t\bar{\xi}} (\Delta t)^2 + \frac{1}{2} \frac{\alpha^2}{\sigma^2} \left(\bar{\xi}_k^{(N)} \right)^2 h_{\bar{\xi}\bar{\xi}} \Delta t \right) \\
& \quad + \frac{1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}}{2(1 + r\Delta t)} P_k^{(N)} + \frac{1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}}{2(1 + r\Delta t)} Q_k^{(N)} \\
& = h \left(k\Delta t, \bar{\xi}_k^{(N)} \right) + \frac{h_{tt} + 2 \frac{\alpha^2}{\sigma^2} \bar{\xi}_k^{(N)} h_{t\bar{\xi}}}{2(1 + r\Delta t)} (\Delta t)^2 \\
& \quad + \frac{\left(1 - \frac{\alpha}{\sigma}\sqrt{\Delta t} \right) P_k^{(N)} + \left(1 + \frac{\alpha}{\sigma}\sqrt{\Delta t} \right) Q_k^{(N)}}{2(1 + r\Delta t)}, \tag{49}
\end{aligned}$$

where we use the PDE (17) satisfied by h .

Next, consider the recursive relation for $X_k^{(N)}$, by Tower property, we have:

$$\begin{aligned}
& h^{(N)} \left(k\Delta t, \bar{\xi}_k^{(N)} \right) \\
& = (1 + r\Delta t)^k \left(\bar{\xi}_k^{(N)} \right)^{-1} \\
& \quad \times \mathbb{E} \left[(1 + r\Delta t)^{-N} \bar{\xi}_N^{(N)} I \left(M, \gamma R + Y e^{-rT} \bar{\xi}_N^{(N)} \right) \mid \bar{\xi}_k^{(N)} \right] \\
& = \left(\frac{1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}}{2(1 + r\Delta t)} (1 + r\Delta t)^{k+1} \left(\bar{\xi}_{k+1}^{(N)+} \right)^{-1} \right. \\
& \quad \times \mathbb{E} \left[(1 + r\Delta t)^{-N} \bar{\xi}_N^{(N)} I \left(M, \gamma R + Y e^{-rT} \bar{\xi}_N^{(N)} \right) \mid \bar{\xi}_{k+1}^{(N)+} \right] \\
& \quad + \frac{1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}}{2(1 + r\Delta t)} (1 + r\Delta t)^{k+1} \left(\bar{\xi}_{k+1}^{(N)-} \right)^{-1} \\
& \quad \times \mathbb{E} \left[(1 + r\Delta t)^{-N} \bar{\xi}_N^{(N)} I \left(M, \gamma R + Y e^{-rT} \bar{\xi}_N^{(N)} \right) \mid \bar{\xi}_{k+1}^{(N)-} \right] \tag{50} \\
& = \frac{1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}}{2(1 + r\Delta t)} h^{(N)} \left((k+1)\Delta t, \bar{\xi}_{k+1}^{(N)+} \right) \\
& \quad + \frac{1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}}{2(1 + r\Delta t)} h^{(N)} \left((k+1)\Delta t, \bar{\xi}_{k+1}^{(N)-} \right) \\
& = \mathbb{E}^{(N)} \left[\frac{1}{1 + r\Delta t} h^{(N)} \left((k+1)\Delta t, \bar{\xi}_{k+1}^{(N)} \right) \mid \bar{\xi}_k^{(N)} \right].
\end{aligned}$$

Note that $\varepsilon_N^{(N)} = 0$, combining (49) and (50), we have

$$\begin{aligned}
\varepsilon_k^{(N)} & := h \left(k\Delta t, \bar{\xi}_k^{(N)} \right) - h^{(N)} \left(k\Delta t, \bar{\xi}_k^{(N)} \right) \\
& = \mathbb{E}^{(N)} \left[\frac{1}{1 + r\Delta t} h \left((k+1)\Delta t, \bar{\xi}_{k+1}^{(N)} \right) \mid \bar{\xi}_k^{(N)} \right] + \beta \left(k\Delta t, \bar{\xi}_k^{(N)} \right) (\Delta t)^2 \\
& \quad + \gamma_k^{(N)} - \mathbb{E}^{(N)} \left[\frac{1}{1 + r\Delta t} h^{(N)} \left((k+1)\Delta t, \bar{\xi}_{k+1}^{(N)} \right) \mid \bar{\xi}_k^{(N)} \right] \\
& = \mathbb{E}^{(N)} \left[\frac{1}{1 + r\Delta t} \varepsilon_{k+1}^{(N)} \mid \bar{\xi}_k^{(N)} \right] + \beta \left(k\Delta t, \bar{\xi}_k^{(N)} \right) (\Delta t)^2 + \gamma_k^{(N)}.
\end{aligned}$$

Then we can prove (47) by backward induction.

- **Step 2.** Since, for any $q > 1$,

$$\begin{aligned} & \mathbb{E}^{(N)} \left[\left| \bar{\xi}_{k+1}^{(N)} \right|^q \middle| \bar{\xi}_k^{(N)} \right] \\ &= \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^q \left| \bar{\xi}_k^{(N)} \right|^q + \frac{1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^q \left| \bar{\xi}_k^{(N)} \right|^q \\ &= \left(1 + \frac{\alpha^2 (q+1)q}{\sigma^2} \Delta t + O(\Delta t^{\frac{3}{2}}) \right) \left| \bar{\xi}_k^{(N)} \right|^q, \end{aligned}$$

by Tower property, for any $n > k$, we have, by a simple application of Markovian property,

$$\begin{aligned} \mathbb{E}^{(N)} \left[\left| \bar{\xi}_n^{(N)} \right|^q \middle| \bar{\xi}_k^{(N)} \right] &= \left(1 + \frac{\alpha^2 (q+1)q}{\sigma^2} \Delta t + O(\Delta t^{\frac{3}{2}}) \right)^{n-k} \left| \bar{\xi}_k^{(N)} \right|^q \\ &\leq e^{KT} \left| \bar{\xi}_k^{(N)} \right|^q, \end{aligned} \quad (51)$$

for some $K > 0$. Since h_{tt} and $h_{t\bar{\xi}}$ satisfy the polynomial growth condition, β also satisfies the polynomial growth condition of the same power as well, and thus we have

$$\begin{aligned} \mathbb{E}^{(N)} \left[\left| \beta \left(n\Delta t, \bar{\xi}_n^{(N)} \right) \right| \middle| \bar{\xi}_k^{(N)} \right] &\leq A \left(1 + \mathbb{E}^{(N)} \left[\left| \bar{\xi}_n^{(N)} \right|^p \middle| \bar{\xi}_k^{(N)} \right] \right) \\ &\leq A \left(1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \end{aligned}$$

for some $p \in \mathbb{Z}^+$ and $A > 0$, which confirms the claim as in Step 2.

- **Step 3.** First, differentiate $f_k^{(N)+}(\tau)$ up to the third order, we have, for $s \in (0, 1)$,

$$\begin{aligned} & \frac{\partial^3}{\partial \tau^3} f_k^{(N)+}(s) \\ &= -\frac{\alpha^3}{\sigma^3} \left(\bar{\xi}_k^{(N)} \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left((k+s)\Delta t, \bar{\xi}_k^{(N)} + s \left(\bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \right) \Delta t^{\frac{3}{2}} \\ &\quad + 3\frac{\alpha^2}{\sigma^2} \left(\bar{\xi}_k^{(N)} \right)^2 h_{t\bar{\xi}\bar{\xi}} \left((k+s)\Delta t, \bar{\xi}_k^{(N)} + s \left(\bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \right) \Delta t^2 \\ &\quad - 3\frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{tt\bar{\xi}} \left((k+s)\Delta t, \bar{\xi}_k^{(N)} + s \left(\bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \right) \Delta t^{\frac{5}{2}} \\ &\quad + h_{ttt} \left((k+s)\Delta t, \bar{\xi}_k^{(N)} + s \left(\bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \right) \Delta t^3. \end{aligned}$$

Under the polynomial growth condition on the partial derivatives of h up to the third order, we have, for $s \in (0, 1)$,

$$\begin{aligned} & \left| \frac{\partial^3}{\partial \tau^3} f_k^{(N)+}(s) \right| \\ &\leq A \left(1 + \left| \bar{\xi}_k^{(N)} \right|^3 \right) \left(1 + \left| \bar{\xi}_k^{(N)} + s \left(\bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \right|^q \right) \\ &\quad \times \left(\Delta t^{\frac{3}{2}} + O(\Delta t^2) \right) \\ &= A \left(1 + \left| \bar{\xi}_k^{(N)} \right|^3 \right) \left(1 + \left| \bar{\xi}_k^{(N)} \right|^q \left(1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right)^q \right) \left(\Delta t^{\frac{3}{2}} + O(\Delta t^2) \right) \\ &\leq A \left(1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left(\Delta t^{\frac{3}{2}} + O(\Delta t^2) \right), \end{aligned} \quad (52)$$

for some positive integer p . As a result, for any N and k , we have

$$\begin{aligned} |P_k^{(N)}| &\leq \frac{1}{6} \int_0^1 (1-s)^3 \left| \frac{\partial^3}{\partial \tau^3} f_k^{(N)+}(s) \right| ds \\ &\leq A \left(1 + |\bar{\xi}_k^{(N)}|^p \right) \left(\Delta t^{\frac{3}{2}} + O(\Delta t^2) \right). \end{aligned}$$

Similar estimate for $|Q_k^{(N)}|$ can be obtained via the same approach, and thus we have, $|\gamma_k^{(N)}| \leq A(1 + |\bar{\xi}_k^{(N)}|^p)(\Delta t^{\frac{3}{2}} + O(\Delta t^2))$, for some $p \in \mathbb{Z}^+$ and $A > 0$. Then, by using the estimate (51), we attain

$$\mathbb{E}^{(N)} \left[|\gamma_n^{(N)}| |\bar{\xi}_k^{(N)}| \right] \leq A \left(1 + |\bar{\xi}_k^{(N)}|^p \right) \left(\Delta t^{\frac{3}{2}} + O(\Delta t^2) \right),$$

which confirms the validity of Step 3.

And hence we conclude with the claim of the lemma.

C.2. Proof of Lemma C.3. We first define $S_n^{(N)} := -\sum_{k=1}^n Z_k^{(N)}$ and $S_*^{(N)} := \max_{1 \leq n \leq N} S_n^{(N)}$, and aim to establish the following lemma beforehand:

Lemma C.5. *For any $N \in \mathbb{N}$, either one of the followings is true:*

1. *if N is odd, $\mathbb{P}(2k - \frac{1}{2} < S_*^{(N)} < 2k + \frac{3}{2}) = 2\mathbb{P}(S_N^{(N)} = 2k + 1)$ for any non-negative integer k ;*
2. *if N is even, $\mathbb{P}(2k - \frac{3}{2} < S_*^{(N)} < 2k + \frac{1}{2}) = 2\mathbb{P}(S_N^{(N)} = 2k)$ for any non-negative integer k .*

Proof. Define a stopping time $\tau_a = \inf\{n \in \mathbb{N} \mid S_n^{(N)} \geq a\}$. We here only show that the case of odd value of N ; the case for even value of N is similar, and we therefore omit the details. For any non-negative integer k ,

$$\begin{aligned} \mathbb{P}\left(S_N^{(N)} > 2k - \frac{1}{2}\right) &= \mathbb{P}\left(S_N^{(N)} \geq 2k + 1\right) = \mathbb{P}\left(S_N^{(N)} \geq 2k + 1, \tau_{2k} < N\right) \\ &= \mathbb{P}\left(S_N^{(N)} - S_{\tau_{2k}}^{(N)} \geq 1, \tau_{2k} < N\right) = \mathbb{P}\left(S_N^{(N)} - S_{\tau_{2k}}^{(N)} \geq 1 \mid \tau_{2k} < N\right) \mathbb{P}(\tau_{2k} < N) \\ &= \frac{1}{2} \mathbb{P}\left(S_*^{(N)} > 2k - \frac{1}{2}\right) \end{aligned}$$

where the first equality follows because $S_N^{(N)}$ can only be equal to an odd number, the third equality holds because $S_{\tau_{2k}}^{(N)} = 2k$ by the definition of τ_{2k} . For the last equality, note that $\mathbb{P}(S_N^{(N)} - S_{\tau_{2k}}^{(N)} \geq 1 \mid \tau_{2k} < N)$ and $\mathbb{P}(S_N^{(N)} - S_{\tau_{2k}}^{(N)} \geq 0 \mid \tau_{2k} < N)$ are equivalent since $S_N^{(N)}$ is odd and $S_{\tau_{2k}}^{(N)}$ is even, while $\mathbb{P}(S_N^{(N)} \geq S_{\tau_{2k}}^{(N)} \mid \tau_{2k} < N) = \frac{1}{2}$ because of the future independent increments after τ_{2k} and the symmetry of binomial tree model.

Besides, for any odd number N ,

$$\begin{aligned} \mathbb{P}\left(S_N^{(N)} = 2k + 1\right) &= \mathbb{P}\left(S_N^{(N)} > 2k - \frac{1}{2}\right) - \mathbb{P}\left(S_N^{(N)} > 2k + \frac{3}{2}\right) \\ &= \frac{1}{2} \mathbb{P}\left(S_*^{(N)} > 2k - \frac{1}{2}\right) - \frac{1}{2} \mathbb{P}\left(S_*^{(N)} > 2k + \frac{3}{2}\right) \\ &= \frac{1}{2} \mathbb{P}\left(2k - \frac{1}{2} < S_*^{(N)} < 2k + \frac{3}{2}\right), \end{aligned}$$

for every non-negative integer k , therefore the claim follows. $\square$

Upon the event $\{S_*^{(N)} = k\}$, for any integer $1 \leq n \leq N$

$$\begin{aligned}\bar{\xi}_n^{(N)} &= \prod_{k=1}^n \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} Z_k^{(N)}\right) \\ &= \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^k \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^a \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^b \quad \text{for some integers } a \leq b \\ &\leq \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^k.\end{aligned}$$

Thus, $\sup_{1 \leq n \leq N} (\bar{\xi}_n^{(N)})^p \leq (1 + \frac{\alpha}{\sigma} \sqrt{\Delta t})^{pk}$ upon the event $\{S_*^{(N)} = k\}$. Then, we have

$$\mathbb{E} \left[\sup_{1 \leq n \leq N} \left(\bar{\xi}_n^{(N)} \right)^p \right] \leq \sum_{k=0}^N \mathbb{P} \left(S_*^{(N)} = k \right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{pk}.$$

i) For if N is odd, there is an integer m such that $N = 2m + 1$, then, by Lemma C.5, we have

$$\begin{aligned}& \sum_{k=0}^N \mathbb{P} \left(S_*^{(N)} = k \right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{pk} \\ &= \sum_{k=0}^m \mathbb{P} \left(S_*^{(2m+1)} = 2k \right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{p(2k)} \\ & \quad + \sum_{k=0}^m \mathbb{P} \left(S_*^{(2m+1)} = 2k + 1 \right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{p(2k+1)} \\ &\leq \sum_{k=0}^m 2\mathbb{P} \left(S_{2m+1}^{(2m+1)} = 2k + 1 \right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{p(2k)} \\ & \quad + \sum_{k=0}^m 2\mathbb{P} \left(S_{2m+1}^{(2m+1)} = 2k + 1 \right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{p(2k+1)} \\ &\leq 4 \sum_{k=0}^m \frac{C_{m+k+1}^{2m+1}}{2^{2m+1}} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{p(2k+1)} \\ &= \frac{4}{2^{2m+1}} \sum_{k=0}^m C_k^{2m+1} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{p(2m+1-2k)} \\ &\leq \frac{4}{2^N} \sum_{k=0}^N C_k^N \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{p(N-k)} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^{-pk} \\ &= \frac{4}{2^N} \left(\left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^p + \frac{1}{\left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^p} \right)^N \\ &= 4 \left(1 + p^2 \frac{\alpha^2}{\sigma^2} \Delta t + O \left(\Delta t^{\frac{3}{2}} \right) \right)^N \leq 4e^{KT},\end{aligned}$$

for some $K > 0$.

ii) The case for even value of N can be resolved similarly.

C.3. Proof of Lemma C.4. On the one hand, the polynomial growth condition on $h_{t\bar{\xi}}$ implies that

$$\left| \bar{\xi}_k^{(N)} h_{\bar{\xi}\bar{\xi}} \cdot \Delta t \right| \leq C \left(1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \Delta t, \quad (53)$$

for some $p \in \mathbb{Z}^+$ and $C > 0$. On the other hand, by (52), we have

$$\int_0^1 (1-s)^3 \frac{\partial^3}{\partial \tau^3} \left(f_k^{(N)+}(s) - f_k^{(N)-}(s) \right) ds \leq C \left(1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left(\Delta t^{\frac{3}{2}} + O(\Delta t^2) \right), \quad (54)$$

for some $C > 0$ and positive integer p . According to the definition of $e_k^{(N)}$, we have showed the boundedness of the first two terms in (46). Therefore, the proof of this lemma can be accomplished if we can show that

$$\left| \frac{1}{2\sigma\sqrt{\Delta t}} \left(\varepsilon_{k+1}^{(N)+} - \varepsilon_{k+1}^{(N)-} \right) \right| \leq C \left(1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left(\sqrt{\Delta t} + O(\Delta t) \right),$$

for some $C > 0$ and positive integer p . To this end, we use the expression (47), and then we have

$$\begin{aligned} & \left| \frac{1}{2\sigma\sqrt{\Delta t}} \left(\varepsilon_{k+1}^{(N)+} - \varepsilon_{k+1}^{(N)-} \right) \right| \\ &= \sum_{n=k+1}^{N-1} \frac{1}{2\sigma\sqrt{\Delta t}} \left(\left| \mathbb{E}^{(N)} \left[\beta \left(n\Delta t, \bar{\xi}_n^{(N)} \right) \left| \bar{\xi}_{k+1}^{(N)+} \right| \right] - \mathbb{E}^{(N)} \left[\beta \left(n\Delta t, \bar{\xi}_n^{(N)} \right) \left| \bar{\xi}_{k+1}^{(N)-} \right| \right] \right| \\ & \quad \times \frac{1}{(1+r\Delta t)^{m-k-1}} (\Delta t)^2 \\ & \quad + \left| \mathbb{E}^{(N)} \left[\gamma_n^{(N)} \left| \bar{\xi}_{k+1}^{(N)+} \right| \right] - \mathbb{E}^{(N)} \left[\gamma_n^{(N)} \left| \bar{\xi}_{k+1}^{(N)-} \right| \right] \right| \frac{1}{(1+r\Delta t)^{m-k-1}} \right), \end{aligned} \quad (55)$$

where $\beta(t, \xi) := -\frac{(h_{tt}(t, \xi) + 2\frac{\alpha^2}{\sigma^2} \xi h_{t\xi}(t, \xi))}{2(1+r\Delta t)}$ and $\gamma_k^{(N)} := -\frac{(1-\frac{\alpha}{\sigma}\sqrt{\Delta t})P_k^{(N)} + (1+\frac{\alpha}{\sigma}\sqrt{\Delta t})Q_k^{(N)}}{2(1+r\Delta t)}$.

To bound the first term on the right hand side of (55), by (48),

$$\begin{aligned} & \sum_{n=k+1}^{N-1} \frac{1}{2\sigma\sqrt{\Delta t}} \left(\left| \mathbb{E}^{(N)} \left[\beta \left(n\Delta t, \bar{\xi}_n^{(N)} \right) \left| \bar{\xi}_{k+1}^{(N)+} \right| \right] - \mathbb{E}^{(N)} \left[\beta \left(n\Delta t, \bar{\xi}_n^{(N)} \right) \left| \bar{\xi}_{k+1}^{(N)-} \right| \right] \right| \\ & \quad \times \frac{1}{(1+r\Delta t)^{m-k-1}} (\Delta t)^2 \right) \\ & \leq \sum_{n=k+1}^{N-1} \frac{1}{2\sigma\sqrt{\Delta t}} \left(\left| \mathbb{E}^{(N)} \left[\beta \left(n\Delta t, \bar{\xi}_n^{(N)} \right) \left| \bar{\xi}_{k+1}^{(N)+} \right| \right] \right| \right. \\ & \quad \left. + \left| \mathbb{E}^{(N)} \left[\beta \left(n\Delta t, \bar{\xi}_n^{(N)} \right) \left| \bar{\xi}_{k+1}^{(N)-} \right| \right] \right| \right) (\Delta t)^2 \\ & \leq \frac{1}{2\sigma\sqrt{\Delta t}} \left(A \left(1 + \left| \bar{\xi}_{k+1}^{(N)+} \right|^p \right) \Delta t + A \left(1 + \left| \bar{\xi}_{k+1}^{(N)-} \right|^p \right) \Delta t \right) \\ & \leq A \left(1 + \left| \bar{\xi}_k^{(N)} \right| \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^p + \left| \bar{\xi}_k^{(N)} \right| \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^p \sqrt{\Delta t} \\ & \leq A \left(1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left(\sqrt{\Delta t} + O(\Delta t) \right) \end{aligned} \quad (56)$$

for some $p \in \mathbb{Z}^+$ and $A > 0$.

To bound the second term on the right hand side of (55), we first fix any n and k with $n > k$, at the same, for the sake of notation, we suppress the indices by taking $\bar{\xi}_k^{(N)} = \bar{\xi} > 0$. For any $s \in (0, 1)$, by the Lipschitz continuity and the polynomial growth of $h_{\bar{\xi}\bar{\xi}\bar{\xi}}$,

$$\begin{aligned}
& \left| \mathbb{E}^{(N)} \left[\left(\bar{\xi}_n^{(N)} \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left((s+n)\Delta t, \bar{\xi}_n^{(N)} + s \left(\bar{\xi}_{n+1}^{(N)+} - \bar{\xi}_n^{(N)} \right) \right) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right. \\
& \quad \left. - \mathbb{E}^{(N)} \left[\left(\bar{\xi}_n^{(N)} \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left((s+n)\Delta t, \bar{\xi}_n^{(N)} + s \left(\bar{\xi}_{n+1}^{(N)+} - \bar{\xi}_n^{(N)} \right) \right) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
& \leq \left| \mathbb{E}^{(N)} \left[\left(\frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left(\bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left((s+n)\Delta t, \bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left(1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right) \right] \right| \\
& \quad \left| \mathbb{E}^{(N)} \left[\left(\frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left(\bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left((s+n)\Delta t, \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left(1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right) \right] \right| \\
& \quad + \left| \mathbb{E}^{(N)} \left[\left(\frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left(\bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left((s+n)\Delta t, \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left(1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right) \right] \right| \\
& \quad \left| \mathbb{E}^{(N)} \left[\left(\frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left(\bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left((s+n)\Delta t, \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left(1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right) \right] \right| \\
& \leq \mathbb{E}^{(N)} \left[L \left(\frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left(\bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^3 2 \bar{\xi} \frac{\alpha}{\sigma} \sqrt{\Delta t} \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left(1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right] \\
& \quad + \mathbb{E}^{(N)} \left[\left(\frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \bar{\xi}^3 \left(6 \frac{\alpha}{\sigma} \sqrt{\Delta t} + O(\Delta t^{\frac{3}{2}}) \right) \right. \\
& \quad \left. \times A \left(1 + \left| \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left(1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right|^q \right) \right] \\
& \leq C \bar{\xi}^4 \left(\sqrt{\Delta t} + O(\Delta t) \right) \mathbb{E}^{(N)} \left[\left(\frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^4 \right] \\
& \quad + C \bar{\xi}^3 \left(\sqrt{\Delta t} + O(\Delta t) \right) \mathbb{E}^{(N)} \left[\left(\frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left(1 + \bar{\xi}^q \left| \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right|^q \right) \right] \\
& \leq C \left(1 + \bar{\xi}^p \right) \left(\sqrt{\Delta t} + O(\Delta t) \right),
\end{aligned}$$

for some $p \in \mathbb{Z}^+$ and $C > 0$, where the last inequality follows since $\mathbb{E}^{(N)} \left[\left(\frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^q \right] \leq M_q$ for some $M_q > 0$ by (51). Note that similar approaches can be applied to other terms in $\frac{\partial^3}{\partial \tau^3} f_n^{(N)+}$, therefore we have

$$\begin{aligned}
& \left| \mathbb{E}^{(N)} \left[\frac{\partial^3}{\partial \tau^3} f_n^{(N)+}(s) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] - \mathbb{E}^{(N)} \left[\frac{\partial^3}{\partial \tau^3} f_n^{(N)+}(s) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
& \leq \frac{\alpha^3}{\sigma^3} (\Delta t)^{\frac{3}{2}} \left| \mathbb{E}^{(N)} \left[\left(\bar{\xi}_n^{(N)} \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left((s+n)\Delta t, \bar{\xi}_n^{(N)} + s \left(\bar{\xi}_{n+1}^{(N)+} - \bar{\xi}_n^{(N)} \right) \right) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right. \\
& \quad \left. - \mathbb{E}^{(N)} \left[\left(\bar{\xi}_n^{(N)} \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left((s+n)\Delta t, \bar{\xi}_n^{(N)} + s \left(\bar{\xi}_{n+1}^{(N)+} - \bar{\xi}_n^{(N)} \right) \right) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
& \quad + (\Delta t)^2 \left| \mathbb{E}^{(N)} \left[3 \frac{\alpha^2}{\sigma^2} \left(\bar{\xi}_n^{(N)} \right)^2 h_{t\bar{\xi}\bar{\xi}} - 3 \frac{\alpha}{\sigma} \bar{\xi}_n^{(N)} h_{t\bar{\xi}} \sqrt{\Delta t} + h_{ttt} \Delta t \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right. \\
& \quad \left. - \mathbb{E}^{(N)} \left[3 \frac{\alpha^2}{\sigma^2} \left(\bar{\xi}_n^{(N)} \right)^2 h_{t\bar{\xi}\bar{\xi}} - 3 \frac{\alpha}{\sigma} \bar{\xi}_n^{(N)} h_{t\bar{\xi}} \sqrt{\Delta t} + h_{ttt} \Delta t \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
& \leq A \left(1 + \bar{\xi}^p \right) \left(\Delta t^2 + O(\Delta t^{\frac{5}{2}}) \right),
\end{aligned}$$

for some $A > 0$. Then,

$$\begin{aligned}
& \left| \mathbb{E}^{(N)} \left[P_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] - \mathbb{E}^{(N)} \left[P_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
& \leq \frac{1}{6} \int_0^1 (1-s)^3 \left| \mathbb{E}^{(N)} \left[\frac{\partial^3}{\partial \tau^3} f_n^{(N)+}(s) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right. \\
& \quad \left. - \mathbb{E}^{(N)} \left[\frac{\partial^3}{\partial \tau^3} f_n^{(N)+}(s) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| ds \\
& \leq \frac{1}{6} \int_0^1 (1-s)^3 A \left(1 + \bar{\xi}^p \right) \left(\Delta t^2 + O \left(\Delta t^{\frac{5}{2}} \right) \right) ds \leq A \left(1 + \bar{\xi}^p \right) \left(\Delta t^2 + O \left(\Delta t^{\frac{5}{2}} \right) \right).
\end{aligned}$$

The same approach can be applied to $Q_n^{(N)}$, and thus we finally have

$$\begin{aligned}
& \left| \mathbb{E}^{(N)} \left[\gamma_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] - \mathbb{E}^{(N)} \left[\gamma_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
& \leq \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1+r\Delta t)} \left| \mathbb{E}^{(N)} \left[P_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right. \\
& \quad \left. - \mathbb{E}^{(N)} \left[P_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
& \quad + \frac{1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1+r\Delta t)} \left| \mathbb{E}^{(N)} \left[Q_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right. \\
& \quad \left. - \mathbb{E}^{(N)} \left[Q_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
& \leq A \left(1 + \bar{\xi}^p \right) \left((\Delta t)^2 + O \left((\Delta t)^{\frac{5}{2}} \right) \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sum_{n=k+1}^{N-1} \frac{1}{2\sigma\sqrt{\Delta t}} \left(\left| \mathbb{E}^{(N)} \left[\gamma_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)+} \right] - \mathbb{E}^{(N)} \left[\gamma_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)-} \right] \right| \frac{1}{(1+r\Delta t)^{m-k-1}} \right) \\
& \leq A \left(1 + \left(\bar{\xi}_k^{(N)} \right)^p \right) \left(\sqrt{\Delta t} + O(\Delta t) \right),
\end{aligned} \tag{57}$$

and so substituting (56) and (57) into (55), we obtain that

$$\left| \frac{1}{2\sigma\sqrt{\Delta t}} \left(\varepsilon_{k+1}^{(N)+} - \varepsilon_{k+1}^{(N)-} \right) \right| \leq C \left(1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left(\sqrt{\Delta t} + O(\Delta t) \right),$$

for some $p \in \mathbb{Z}^+$ and $C > 0$. Together with (53) and (54), the claim of this lemma is concluded.

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