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THE DRELL - YAN PROCESS IN QUANTUM FIELD THEORY

By

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A thesis presented for

the degree of Doctor of Philosophy of the University of London and the Diploma of Membership of Imperial College.

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January, 1981

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ABSTRACT

In this thesis we study two aspects of the Drell-Yan process using perturbative methods in quantum field theory.

The first part of the thesis is an investigation of the analytic structure of the Drell-Yan amplitude in a ϕ^3-model. The motivation for this study is to obtain an understanding of the Drell-Yan process through the method of operator product expansion (OPE) which has been so successfully applied to the process of electroproduction. The Drell-Yan process and the electroproduction process are very intimately related but to date no OPE of the Drell-Yan matrix element exists. Our analysis strongly suggests that such an expansion cannot be carried out because of an infinite time scale hidden in our amplitude.

The infrared nature of non-abelian gauge theories is the subject of study in the second half of the thesis. We investigate the Drell-Yan process and find that at order α^2 where α is the square of the coupling constant in the cross-section, there is no infrared cancellation in the spirit of the Block-Nordsieck Theorem which is true in abelian gauge theories.

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Dedicated to my dear parents, Désiré and Gitane

PREFACE

This work was carried out in the Theoretical Physics Department of Imperial College, London, under the supervision of Dr. I.G. Halliday between October 1977 and October 1980. Except where otherwise stated, this work is original, and has not been submitted for a degree of this or any other university. The material of Chapters 3 and 4 of Part I, except for section 4.3 of Chapter 4, was carried out in collaboration with my supervisor. Some of the work in Part II, namely that directly connected with section 3.4 of Chapter 3 was done in collaboration with C. D'ilieto, a student, and Drs. I.G. Halliday and C.T. Sachrajda.

I would like to thank Janie Coghill for typing this thesis.

I should like to thank Dr. I.G. Haliday for his constant guidance and encouragement during the course of the work and for many valuable discussions and suggestions. I am also grateful to Dr. H.F. Jones, John Butcher, Christine D'ilieto and Jeremy Wyndham for very useful discussions.

I am very much indebted to the Seychelles Government for allowing me to be away from the Seychelles to pursue this research.

The financial support of the University of London Postgraduate Scholarships Board and that of my parents is gratefully acknowledged.

Notation

(1) The Kronecker delta function δ_{il} is defined as

$$\delta_{il} = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1 \end{cases}$$

(2) The θ-function is defined as

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

(3) The Scalar product of two four-vectors $p = (p^0, \underline{p})$ and $q = (q^0, \underline{q})$ is defined by $p.q$ where

$$p.q = p^0 q^0 - \underline{p}.\underline{q}$$

(4) We are using the same notation as "Bjorken and Drell" [10] for spinor normalization.

(5) We are using units in which $h = c = 1$

(6) We are adopting the convention that repeated indices are to be summed over.

(7) $\not{p} \equiv p_\mu \gamma^\mu$

PART I

The Analytic Structure of the Drell-Yan Amplitude

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CHAPTER I

INTRODUCTION

Recently there has been rapid progress in expanding the list of genuine asymptotic freedom predictions in Quantum chromodynamics (QCD) [1]. For several years we only had the pioneering predictions in the total electron-positron annihilation cross-sections and deep inelastic scattering (DIS) [2]. These involve only 2-point and 4-point amplitudes. The generalisations to Drell-Yan [3] and large $p_\perp$-processes, for example, seemed impossible. This impasse was broken by leading log calculations in quantum electrodynamics (QED) and QCD [4]. Subsequently more general arguments have been given [5]. However, in spite of the fact that Drell-Yan and DIS are very intimately related the generalisation of the operator product expansion (OPE) [6] technique to 6-point amplitudes remains obscure to this day.

In the theory of multi-particle production by exchange of Regge poles there was a similar problem. The structure of higher order amplitudes such as $2 \to 4$ and their unitarity equations seemed incompatible with the factorized $2 \to 2$ and $2 \to 3$ amplitudes [7]. This was eventually resolved in an elegant manner by a deeper understanding of the analytic structure of the $2 \to 4$ amplitude.

Since the canonical application of the OPE also requires the use of dispersion relations in x, the Feynman scaling variable, we are led to study the analytic properties of the Drell-Yan amplitude in its asymptotic large s and large Q^2 region. We immediately run into surprises. The Regge limits were characterised by asymptotic formulae

which would be continued round large circles and behaved smoothly as epitomised by $s^{\alpha(t)}$ where $\alpha(t)$ is some function of t, a finite momentum transfer. The Drell-Yan amplitude behaves totally non-uniformly. The characteristic scaling form only appears at very special kinematical points and away from these points the amplitude is down by two powers of s. For example in the Euclidean region of fig 3.1.3 where all invariants are negative, but in the asymptotic scaling region, there is no Drell-Yan scaling term. The amplitude thus has a very different structure from our Regge paradigm. This is because the scaling form comes from a pinch in Feynman parameter space rather than end-point singularities.

As we do not believe that the spin structure affects the analyticity we study the φ^3-model which contains only spinless particles.

In Chapter 2 we review the basic ideas and tools that will be required in this part, and also in the last half, of this thesis. These are the S-matrix and its representation in perturbation theory, the differential cross-sction, the Feynman representation of amplitudes and their analytic continuation and the generalised optical theorem.

The Drell-Yan model is recalled and its kinematics described in Chapter 3. We will be considering only the bare ladder-type diagram contributions to the amplitude. We demonstrate how this amplitude is dominated by a pinch singularity.

Most of the analytic features of the general ladder diagram are present at the one-loop diagram. In Chapter 4 we discuss in detail the analytic properties of the latter. This of course simplifies the the general problem considerably as we do not have to deal with the

multitude of Feynman parameters there. Here we discover an infinite time scale hidden in the problem which turns out to have a simple physical interpretation. This is discussed in the conclusions. This infinite time scale is absent in electroproduction as we will discuss, and we believe it has profound consequences for an OPE treatment of the Drell-Yan amplitude.

We discuss the analytic structure of the general ladder graph in Chapter 5. Apart from a technical difficulty exactly the same features present at the one-loop level emerge.

Chapter 6 contains our conclusions.

CHAPTER II

BASIC PRELIMINARIES

This chapter is essentially a brief review of some of the ideas and tools that will be required in this thesis. It consists of four sections in which we discuss the following topics: the scattering matrix, the differential cross-section, Feynman's representation of S-matrix elements and their analytic continuations from the Euclidean region and the generalised optical theorem. Most of the contents of this chapter are to be found in [8, 9, 10].

2.1 The Scattering Matrix

In using the scattering or S-matrix [8] to describe a scattering experiment we will assume that the forces are sufficiently short range that the initial and final states consist of free particles. These states can then be specified by the momentum of each particle together with certain discrete quantum numbers. Accordingly we assume that in the extreme past ($t \to -\infty$) the particles may be described by states $|in\rangle$ with the following properties.

(i) The Superposition Principle: if $|\psi, in\rangle$ and $|\ell, in\rangle$ are physically occuring states so is $\lambda|\psi, in\rangle + \mu|\ell, in\rangle$ where λ and μ are any complex numbers.

(ii) The orthonormality and completeness properties: the set of physical states is normalizable and complete so that a subset may be chosen satisfying the orthornormality conditions

(2.1.1(a)) $$\langle m, in \mid n, in \rangle = \delta_{mn}$$

and the completeness conditions

(2.1.1(b)) $$\sum_m |m, in\rangle\langle m, in| = \mathbb{1}$$

where m and n are labels describing particular configurations of free particles. δ_{mn} is the usual Kronecker-delta function and $\mathbb{1}$ is the unit operator.

Similarly we assume that in the infinite future ($t \to +\infty$) the particle may be described by states $|$ out$\rangle$ bearing corresponding labels and with exactly similar properties as the states $|$ in$\rangle$.

We define an operator, S, by

(2.1.2(a)) $$S \equiv \sum_m |m, in\rangle\langle m, out|$$

so that

(2.1.2(b)) $$S^\dagger \equiv \sum_m |m, out\rangle\langle m, in|$$

Because of the orthonormality and completeness conditions (2.1.1) on the states $|$ m, in$\rangle$ and the analogous ones on the states $|$ m, out$\rangle$ the matrix elements satisfy

(2.1.3) $$\langle \ell, in | S | \psi, in\rangle = \langle \ell, out | \psi, in\rangle = \langle \ell, out | S | \psi, out\rangle$$

where the labels ℓ, ψ refer to any physical state. Further the

orthonormality and completeness relations imply that S is unitary (if one assumes these may be written):

$$S S^{\dagger} = S^{\dagger} S = \mathbb{1} \tag{2.1.4}$$

If we assume that the matrix element (2.1.3) is a probability amplitude from the state labelled ψ to that labelled φ then the unitary conditions (2.1.4) correspond to the physical requirement of propability conservation i.e. the total probability of the system ending up in some other state must be unity.

We will be considering spin-zero-type particles and use momentum eigenstates as a complete set of physical states.

Using the standard annihilation-creation operator formalism [9] we write the single particle state $|\, p\rangle$ describing a particle with moment $\underset{\sim}{p}$, mass m and energy $p^0 = \left[p^2 + m^2\right]^{1/2}$ as

$$|\, p\rangle = a^{\dagger}(\underset{\sim}{p})\,|\, 0\rangle \tag{2.1.5}$$

where $|\, 0\rangle$ is the vacuum state with no particles i.e.

$$a(\underset{\sim}{p})\,|\, 0\rangle = 0 \tag{2.1.6}$$

and $a^{+}(\underset{\sim}{p})$ and $a(\underset{\sim}{p})$ are creation and annihilation operators respectively.

We shall choose the relativistic normalization

$$\langle p' | p\rangle = \left[a(\underset{\sim}{p}'), a^{\dagger}(\underset{\sim}{p})\right] = \triangle(\underset{\sim}{p}', \underset{\sim}{p}) \tag{2.1.7}$$

where

$$\Delta(p',p) = (2\pi)^3 \cdot 2p_0 \delta^{(3)}(p'-p) \tag{2.1.8}$$

For discrete states with unit normalisation we have the summation property

$$\sum_b f_b \delta_{ab} = f_a \tag{2.1.9}$$

The integration yielding the corresponding property with $\Delta(p',p)$

$$\int_p f(p)\,\Delta(p',p) = f(p') \tag{2.1.10}$$

is given by

$$\int_p \equiv \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p_0} = (2\pi)^{-3}\int d^4p\,\delta^{(+)}(p^2-m^2) \tag{2.1.11}$$

where

$$\delta^{(+)}(p^2-m^2) \equiv \theta(p_0)\,\delta(p^2-m^2) \tag{2.1.12}$$

We define the state of n-particles with momenta p_1, p_2, ...p_n as

$$|p_1,p_2,\cdots,p_n\rangle = a^\dagger(p_1)a^\dagger(p_2)\cdots a^\dagger(p_n)|0\rangle \tag{2.1.13}$$

Because of (2.1.7) the normalisation of this state is

$$\langle p_1',p_2',\cdots,p_n'|p_1,p_2,\cdots,p_n\rangle = \sum_{(\nu)}\{\Delta(p_1',p_{\nu_1})\cdots\Delta(p_n',p_{\nu_n})\} \tag{2.1.14}$$

while states containing different numbers of particles are orthogonal. In (2.1.14) the summation is over the n! ways of pairing off the momenta in the two states. The corresponding n-particle phase space integral is

$$(2.1.15)\qquad \sum_{p_1 \cdots p_n} = \frac{1}{n!} \int_{p_1} \cdots \int_{p_n}$$

so that the completeness relation reads

$$(2.1.16)\qquad \sum_{n=0}^{\infty} \frac{1}{(2\pi)^{3n}} \frac{1}{n!} \int d^4q_1 \cdots d^4q_n \, \delta^{(+)}(q_1^2 - m^2) \cdots \delta^{(+)}(q_n^2 - m^2) \, |q_1 \cdots q_n\rangle\langle q_1 \cdots q_n| = \mathbb{1}$$

For spinless particles if we insist that our S-matrix elements are Lorentz invariant then they depend on the four momenta only through their invariant scalar products. The elements of the S-matrix cannot be analytic due to the occurrence of Dirac δ-functions. These occur due to overall energy-momentum conservation because of which a matrix element $\langle p_f | S | p_i \rangle$ will contain a factor $\delta^{(4)}(P_i - P_f)$. Also note that because of the probability that the particles do not interact at all (due to quantum mechanics) the four-momentum of each particle may remain unchanged. The S-matrix, therefore, can be separated into two parts by writing

$$(2.1.17)\qquad S = \mathbb{1} + iR$$

We then define the transition or T-amplitude by

$$(2.1.18)\qquad \langle p_f | R | p_i \rangle = (2\pi)^4 \delta^{(4)}(p_i - p_f) \langle p_f | T | p_i \rangle$$

It is generally believed that the causality property requires the transition amplitudes to be real-boundary values of analytic functions of complex variables. In perturbation theory Feynman's prescription for obtaining physical amplitudes is giving a small negative imaginary part $(-i\varepsilon)$ to the mass of each particle in any internal line of a Feynman diagram.

2.2 The differential cross-section

Cross-sections, σ, are measured by experimentalists. They give us the rate at which particles are scattered into unit volume per unit incident flux. In terms of transition amplitudes the differential cross-section, $d\sigma$, for spinless particles is

$$(2.2.1)\quad d\sigma = \frac{S}{|V_1 - V_2|(2\omega_{p_1})(2\omega_{p_2})} \sum_{n=2}^{\infty} \frac{d^3k_1}{2\omega_1(2\pi)^3} \cdots \frac{d^3k_n}{2\omega_n(2\pi)^3} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_{i=1}^{n} k_i) \left| \langle k_1 \ldots k_n | T | p_1 p_2 \rangle \right|^2$$

where

$$(2.2.2)\qquad \omega_{p_i} = \left[p_i^2 + m^2 \right]^{1/2}$$

$\underline{V}_1$, $\underline{V}_2$ are the velocities of the incident colinear particles p_1, p_2. This expression is then integrated over all the undetected momenta of the final states $k_1, \ldots k_n$. The statistical factor S is obtained by including a factor m! if there are m identical particles in the final state. The proof of (2.2.1) is to be found in most standard

texts of particle physics [9].

2.3 Feynman representation of S-matrix elements and their analytic continuation from the Euclidean region. [8, 10]

We will be using perturbation theory to evaluate the S-matrix as an infinite number of terms, the expansion parameter being the coupling constant that appears in the interaction part of the Lagrangian. We are of course assuming that the coupling constant is small in some sense. In practice in evaluating the S-matrix elements one represents each term in the perturbation expansion by what is called a Feynman diagram and one then uses the Feynman rules (given in Appendix A for φ^3) to write down the mathematical expression corresponding to a given diagram.

The general Feynman diagram in a $\lambda\varphi^3$ theory, suppressing coupling constants and other numerical factors, is of the form

$$F = \lim_{\varepsilon \to 0^+} \int \frac{\prod_{j=1}^{\ell} d^4 k_j}{\prod_{n=1}^{N} (q_n^2 - m_n^2 + i\varepsilon)} \tag{2.3.1}$$

where $k_1, \ldots k_\ell$ denote ℓ independent loop momenta, $q_1 \ldots q_n$ are the momenta of the internal lines. m_r are the masses of the internal lines. Each q_r depends linearly on one or more of the k_j and possibly on one or more of the E external momenta which we shall denote by p_a taken to enter the graph. These dependences are a consequence of momenta conservation at each vertex of the graph. We now introduce Feynman parameters α_i through the identity

$$(2.3.2)\qquad \frac{1}{f_1 \cdots f_N} = (N-1)! \int_0^1 \frac{\prod_{i=1}^{N} d\alpha_i \; \delta\left(1 - \sum_{n=1}^{N} \alpha_n\right)}{\left[\sum_{n=1}^{N} \alpha_n f_n\right]^N}$$

where $f_r \neq 0$ which allows us to write (2.3.1) as

$$(2.3.3)\qquad F = (n-1)! \lim_{\varepsilon \to 0^+} \int_0^1 \prod_{i=1}^{n} d\alpha_i \; \delta\left(1 - \sum_{n=1}^{n} \alpha_n\right) \int \frac{\prod_{j=1}^{l} d^4 k_j}{\left[\sum_{n=1}^{n} \alpha_n (q_n^2 - m_n^2) + i\varepsilon\right]^n}$$

The linear terms in k in the quadratic form

$$(2.3.4)\qquad \psi(k, p, \alpha) \equiv \sum_{n=1}^{n} \alpha_n \left(q_n^2 - m_n^2\right)$$

are eliminated by a translation in momentum space. The resultant form may be diagonalised through an orthogonal transformation of the k_r. The resultant k-integrals are performed using the identity:

$$(2.3.5)\qquad \int \frac{d^4 k}{\left(a - b k^2\right)^\lambda} = \frac{\pi^2 i}{(\lambda - 1)(\lambda - 2)\, b^2 a^{\lambda - 2}}$$

where

$$(2.3.6)\qquad \begin{cases} \mathrm{Im}\, a & < 0 \\ b & > 0 \\ \mathrm{Re}\, \lambda & > 2 \end{cases}$$

The k_0-integral must first be done by rotating the k_0-contour into the imaginary axis by $\pi/2$ in an anti-clockwise direction. We then obtain

$$(2.3.7)\qquad F = \left(\pi^2 i\right)^l \lim_{\varepsilon \to 0^+} \int_0^1 \frac{\prod_{i=1}^{n} d\alpha_i \; \delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) C^{n-2l-2}}{\left[D + i\varepsilon C\right]^{n-2l}}$$

where

$$(2.3.8) \qquad D = \sum_{i,j=1}^{E} \zeta_{ij}\, p_i \cdot p_j \;-\; \sum_{r=1}^{n} \alpha_r C m_r^2$$

ζ_{ij} and C are positive-definite, linear and homogeneous functions of the α_i's . There exists cutting rules which allow us to write down immediately the C and D functions for any Feynman graph. These are given in appendix B.

In performing the k_0-integral in (2.3.5) we Wick-rotated the k_0-contour:

$$(2.3.9) \qquad k_0 \longrightarrow k_0 e^{i\varphi} \quad , \quad 0 \leq \varphi \leq \pi/2$$

The external momenta are inextricably linked to the k_r via momentum conservation at each vertex of the diagram. Hence, whatever transformation the k_r suffer the external momenta do as well. Thus

$$(2.3.10) \qquad \zeta_{ij}\, p_i \cdot p_j + i\varepsilon C \longrightarrow \zeta_{ij}\left[p_i^0 p_j^0 \cos 2\varphi - \vec{p}_i \cdot \vec{p}_j\right] + i\left[\varepsilon C + \zeta_{ij}\, p_i^0 p_j^0 \sin 2\varphi\right]$$

We see then from (2.3.10) that the imaginary part of D remains positive for all α_i. Then (2.3.7) exists for $0 < \varphi < \pi/2$. In the limit $\varphi = \pi/2$, D becomes negative definite (i.e. we are now in the Euclidean region) and, therefore, we have no singularities plaguing us and we may drop the $i\varepsilon$ term of D. The resultant expression is now valid in the Euclidean region and is real modulo $(i)^{\ell}$. To get back to the kinematical region defined in (2.3.1) we must analytically continue all scalar products, but not internal masses, through $e^{-i\pi}$. This

follows from (2.3.10).

In the region where D is negative definite it may be useful to write (2.3.7) as

$$F \propto \int_0^\infty \frac{\prod_{i=1}^{n} d\alpha_i \; e^{D/C}}{C^2} \tag{2.3.11}$$

This is best seen by scaling all Feynman parameters in (2.3.11) by ρ and then performing the resultant ρ-integral.

2.4 The generalised optical theorem [11]

Let us first recall how the optical theorem relates the total cross-section σ_{TOT} for $p_1 + p_2 \longrightarrow$ anything to the $(p_1 + p_2)^2 = s$ -discontinuity of the $2 \longrightarrow 2$ forward scattering amplitude. It follows from (2.2.1) that

$$\sigma_{TOT} = \frac{1}{F} \sum_{n=2}^{\infty} \frac{1}{n!} \int \prod_{j=1}^{n} \frac{d^4 k_j}{(2\pi)^3} \delta^+(k_j^2 - m_j^2)(2\pi)^4 \delta^{(4)}\left(p_1 + p_2 - \sum_{j=1}^{n} k_j\right) \langle p_1 p_2 | T^\dagger | k_1 \ldots k_n \rangle \langle k_1 \ldots k_n | T | p_1 p_2 \rangle \tag{2.4.1}$$

where F is the flux factor. The unitarity equation (2.1.4) may be written as

$$R - R^\dagger = i R^\dagger R = i R R^\dagger \tag{2.4.2}$$

on using (2.1.17). From (2.1.18) we can write (2.4.2) as

(2.4.3) $\langle P_1P_2|T|P_1P_2\rangle - \langle P_1P_2|T^+|P_1P_2\rangle$

$$= i\sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_{j=1}^{n}\frac{d^4k_j}{(2\pi)^3}\,\delta^{(+)}(k_j^2-m^2)(2\pi)^4\delta^{(4)}(P_1+P_2-\sum_{j=1}^{n}k_j)\,\langle P_1P_2|T|k_1\ldots k_n\rangle\langle k_1\ldots k_n|T^+|P_1P_2\rangle$$

Because of (2.4.2) the order of T and T^+ is irrelevant in (2.4.3). The left hand side (l.h.s.) of (2.4.3) is defined to be the discontinuity in s of the $2\to 2$ forward scattering amplitude. This we denote by

$\Delta_s\{T_{2\to 2}(s,t=0)\}$ where $t = (P_1-P_1')^2=(P_2-P_2')^2 = 0$.

Since

(2.4.4) $$\langle P_1P_2|T|P_1P_2\rangle^* = \langle P_1P_2|T^+|P_1P_2\rangle$$

it follows that

(2.4.5) $$\Delta_s\{T_{2\to 2}(s,t=0)\} = 2i\,Im\{T_{2\to 2}(s,t=0)\}$$

In the summation on the r.h.s. of (2.4.3) the n = 0, 1 terms are each zero as we are assuming that all masses are identical and hence their corresponding T-matrices cannot conserve four-momentum. By comparing (2.4.1) and (2.4.3) we arrive at the optical theorem, namely

(2.4.6) $$\Delta_s\{T_{2\to 2}(s,t=0)\} = i\,F\,\sigma_{TOT}$$

If p_1^2 and $p_2^2 \geqslant 4m^2$ where m are the internal masses, the $2\to 2$ forward amplitude will have branch cuts in these variables. In $\langle k_1\ldots k_n|T|p_1p_2\rangle$ these must be evaluated above their cuts and hence in $\langle p_1p_2|T^+|k_1..k_n\rangle$ they must be evaluated below their cuts. We saw at the very end of section 2.3. that the T-matrix in perturbation theory is real in

the Euclidean region where all invariants are real and negative. It follows from the Schwarz reflection principle in the theory of functions of a complex variably [12] that $T(-se^{-i\theta}) = \overset{*}{T}(-se^{i\theta})$ where $T(-se^{i\theta})$ is the analytic continuation through θ off the negative real axis into the complex s-plane. Hence the l.h.s. of (2.4.3) is obtained by analytic continuation. The r.h.s. of (2.4.3) has the following graphical interpretation: it consists of all $2 \longrightarrow 2$ connected graphs; each graph is cut into two in all possible ways such that s flows into each half and the cut lines are put on mass shell through the delta-functions, $\delta^{(+)}(k_j^2 - m^2)$; everything on the r.h.s. of the cut must be complex conjugated.

The extension of the above formalism to a one-particle inclusive cross-section

(2.4.7) $p_1 + p_2 \longrightarrow A + \text{anything}$

requires greater care. It follows from (2.2.1) that

$$(2.4.8)\quad 2\omega_A (2\pi)^3 \frac{d\sigma}{d^3 k_A} = \frac{1}{F} \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{j=2}^{n} \frac{d^4 k_j}{(2\pi)^3} \delta^{(+)}(k_j^2 - m^2)(2\pi)^4 \delta^{(+)}\Big(p_1 + p_2 - k_A - \sum_{j=2}^{n} k_j\Big) \langle p_1 p_2 | T | k_A k_2 \ldots k_n \rangle \langle k_A k_2 \ldots k_n | T^\dagger | p_1 p_2 \rangle$$

If we now assume that crossing symmetry exists i.e. $\langle k_A k_2 \ldots k_n | T | p_1 p_2 \rangle$ which describes the process (2.4.7) also describes the unphysical process

(2.4.9) $p_1 + p_2 + \bar{A} \longrightarrow \text{anything}$

where $\bar{A}$ denotes the anti-particle of A, by an appropriate analytic continuation then we may replace $\langle k_A k_2 \ldots k_n | T | p_1 p_2 \rangle$ by the unphysical amplitude $\langle p_1 p_2 k_{\bar{A}} | T | k_2 \ldots k_N \rangle$ and obtain

$$(2.4.10)\quad 2\omega_A (2\pi)^3 \frac{d\sigma}{d^3 k_A} = \frac{1}{F} \sum_{n=2}^{\infty} \frac{1}{n!} \int \prod_{j=2}^{n} \frac{d^4 k_j}{(2\pi)^3} \delta^{(+)}(k_j^2 - m^2)(2\pi)^4 \delta^{(4)}(p_1 + p_2 + k_{\bar{A}} - \sum_{j=2}^{n} k_j) \langle p_1 p_2 k_{\bar{A}} | T | k_2 \ldots k_n \rangle \langle k_2 \ldots k_n | T^+ | p_1 p_2 k_{\bar{A}} \rangle$$

where $K_{\bar{A}} = K_A$ but flowing outwards.

As in the case of the total cross-section for $2 \to 2$ scattering amplitude we may, by using the unitarity equation, identify the r.h.s. of (2.4.10) with the discontinuity in $s = (p_1 + p_2 - k_A)^2$ of the connected part of the $3 \to 3$ forward scattering amplitude but with the external invariants s^L in $\langle p_1 p_2 k_{\bar{A}} | T | k_2 \ldots k_n \rangle$ evaluated above their cuts and those s^R in $\langle k_2 \ldots k_n | T^+ | p_1 p_2 k_{\bar{A}} \rangle$ evaluated below their cuts. For invariants attached to both T and T^+ it is irrelevant which boundary value we take. This is Muellers Theorem. We note that two features absent in the $2 \to 2$ forward scattering amplitude have arisen here: firstly we have to state the boundary values of the invariants on which the s-discontinuity depends - this manifested itself in the total cross-section of the $2 \to 2$ amplitude when this had branch cuts in its external masses; secondly, the $3 \to 3$ amplitude has semi-disconnected parts - for example particles 1, 2, 3 come in but only 2 and 3 scatter off each other. 1 then interacts with the products of the $2 \to 3$ collisions. Such disconnected diagrams do not have an s-discontinuity and therefore cannot contribute to the cross-section. With all these considerations in mind we arrive at the following generalization of the optical theorem:

$$(2.4.11)\quad 2\omega_A (2\pi)^3 \frac{d\sigma}{d^3 k_A} = \frac{1}{F} \Delta_s \left\{ T_{3\to 3}(s_+^L, s, s_-^R) \right\}$$

The + and - indicate whether the boundary value of the invariant is to be evaluated above or below its cut respectively.

The above theorem can of course be easily extended to 2, 3, .. n - particle inclusive cross-sections.

CHAPTER 3

A BRIEF REVIEW OF THE DRELL-YAN PROCESS

This chapter has three sections. The Drell-Yan process, its kinematics and its cross-section are reviewed in section one. Section two discusses the Feynman representation of the general ladder diagram. The dominance of the Drell-Yan amplitude by a pinch singularity in Feynman parameter space is shown in section three.

3.1 The Drell-Yan Process, its Cross-section and its Kinematics

The Drell-Yan process [3] is a model for the interaction

(3.1.1) $$P_1 + P_2 \longrightarrow (\ell^+, \ell^-) + \text{anything}$$

where p_1 and p_2 are colliding hadrons and (ℓ^+, ℓ^-) represents a lepton pair. In this model, shown in fig. 3.1.1, one imagines a quark q from hadron p_1 annihilating an anti-quark $\bar{q}$ from the other hadron p_2

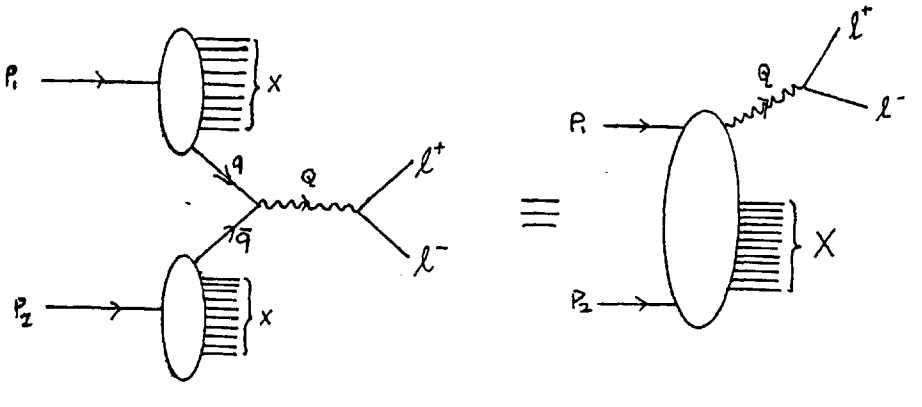


Fig 3.1.1 The Drell-Yan process. X represents anything

into a time-like virtual photon Q such that $Q^2 \gg 4\mu^2$ where μ represents the mass of each lepton. The amplitude describing process (3.1.1) factorises into a hadronic part ($p_1 + p_2 \longrightarrow Q + x$) and a leptonic part ($Q \rightarrow \ell^+\ell^-$). The leptonic part is given completely by Quantum Electrodynamics (QED) and it will not concern us here. We shall be considering the hadronic half of the process namely

(3.1.2) $\quad p_1 + p_2 \longrightarrow Q + \text{anything}$

The differential corss-section $d\sigma/d^3Q$ for this process is given by the generalised optical theorem:

(3.1.3) $$2\omega_q (2\pi)^3 \frac{d\sigma}{d^3q} = \frac{1}{F} \Delta_s \left\{ T_{3\to 3}(s_+^L, s, s_-^R) \right\}$$

$T_{3\to 3}$ represents the amplitude shown in fig 3.1.2

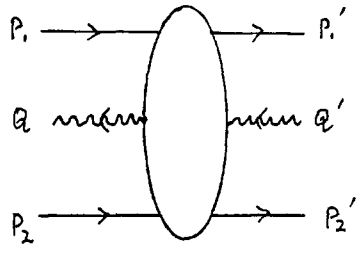


Fig 3.1.2 The 3→3 forward scattering Mueller amplitude for the differential cross-section of 3.1.2

Remember that the primed variables are equal in magnitude to their unprimed counterparts. Our interst lies in the analytic structure of $T_{3\to 3}$. We shall be studying this amplitude in a φ^3 model. All the

particles are identical and spinless. The analyticity properties come mainly from the D-function. D does not change by the introduction of spin which introduces extra factors in the numerator which may enhance the asymptotic behaviour. We shall only be considering only ladder-type contributions to this amplitude in perturbation theory. See fig 3.1.3.

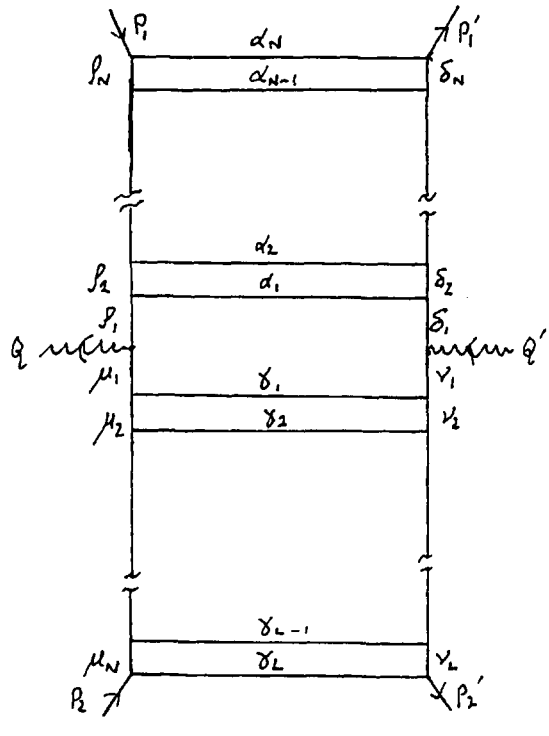


Fig 3.1.3 Ladder-type contributions to $T_{3\to 3}$

The amplitude depicted in fig 3.1.3 is a function of the following Lorentz scalars

(3.1.4)

$$\begin{cases} s = (p_1+p_2-Q)^2 = (p_1'+p_2'-Q')^2, \\ s_1 = (p_1-Q)^2, \quad s_1' = (p_1'-Q')^2, \\ s_2 = (p_2-Q)^2, \quad s_2' = (p_2'-Q')^2, \\ Q^2, Q'^2, \bar{Q}^2 = (p_1-p_1'+Q')^2, \tilde{Q}^2 = (p_1-p_1'-Q)^2, \\ t_1 = (p_1-p_1')^2 = 0 = (p_2-p_2')^2 = t_2, \\ p_1^2 = p_1'^2 = M_1^2, \quad p_2^2 = p_2'^2 = M_2^2. \end{cases}$$

with

$$(3.1.5.)\qquad \begin{cases} |s_1| = |s_1'| \;, \quad |s_2| = |s_2'| \\ |Q^2| = |Q'^2| = |\bar{Q}^2| = |\hat{Q}^2| \end{cases}$$

In terms of these invariants we can translate equation 3.1.3 as

$$(3.1.6)\qquad 2\omega_Q (2\pi)^3 \frac{d\sigma}{d^3Q} = \frac{1}{F}\Delta_s \left\{ T_{3\to 3}\left(s_{1+}, s_{2+}, Q_+^2, s, s'_{1-}, s'_{2-}, Q'^2_-, \bar{Q}^2_\pm, \hat{Q}^2_\pm\right)\right\}$$

where the +, - indicate that s_1, s_2, Q^2 must be analytically continued to their physical values from above their cuts whereas s_1', s_2', and Q'^2 must be continued to below their cuts and the continuation of $\bar{Q}^2$ and $\hat{Q}^2$ is irrelevant so long as we adopt one prescription in evaluating the s-discontinuity.

Let us now consider the kinematics of fig 3.1.2 in order to define the scaling regions which interest us. In the centre of mass of p_1 and p_2 let Q have a momentum Λ transverse to p_1 and p_2. If we have a four vector (a_0, a_1, a_2, a_3) then we define new co-ordinates, called lightcone co-ordinates by (a^+, a_1, a_2, a^-) where $a^\pm = \frac{1}{\sqrt{2}}(a_0 \pm a_3)$. In terms of x_1 and x_2, the lightcone-cordinates of Q in this frame scaled by $\sqrt{u}$ where

$$(3.1.7)\qquad u = (p + p_2)^2 \;,$$

we have

$$(3.1.8) \qquad Q \equiv \left((x_1 + x_2)\frac{\sqrt{u}}{2}\ ,\ \Lambda\ ,\ (x_1 - x_2)\frac{\sqrt{u}}{2} \right)$$

$$(3.1.9) \qquad p_1 = \left(\sqrt{M_1^2 + p^2}\ ,\ p\right)$$

$$(3.1.10) \qquad p_2 = \left(\sqrt{M_2^2 + p^2}\ ,\ -p\right)$$

We will be considering $u \gg M_1^2$ and M_2^2. We can then show that

$$(3.1.11) \qquad \begin{cases} Q^2 = x_1 x_2 u - \Lambda^2 \\ S = (p_1 + p_2 - Q)^2 \simeq (1-x_1)(1-x_2)u - \Lambda^2 \\ S_1 = (p_1 - Q)^2 \simeq -x_2(1-x_1)u\left[1 - \frac{M_1^2}{ux_2} - \frac{M_2^2}{u(1-x_2)} + \frac{\Lambda^2}{x_2(1-x_1)u}\right] \\ S_2 = (p_2 - Q)^2 \simeq -x_1(1-x_2)u\left[1 - \frac{M_2^2}{ux_1} - \frac{M_1^2}{u(1-x_1)} + \frac{\Lambda^2}{x_1(1-x_2)u}\right] \end{cases}$$

We have dropped terms which tend to zero as $u \to \infty$ at fixed x_1, x_2 and Λ^2. We are keeping away from regions where x_1 x_2 tend to 1 or zero. From (3.1.11) we can show that

$$(3.1.12) \qquad \frac{S_1 S_2}{S} - Q^2 \simeq \frac{M_1^2 x_1}{x_2 - 1} + \frac{M_2^2 x_2}{x_1 - 1} + \frac{\Lambda^2}{(1-x_1)(1-x_2)}$$

from which it follows that

$$(3.1.13) \qquad \tau \equiv \frac{S_1 S_2}{Q^2 S} = 1 + O\left(\frac{1}{u}\right)$$

But as we wish to continue s_1, s_2, Q^2 and s independently we will have to consider $\tau = e^{i\mathcal{Q}}$.

Since Q decays into a real lepton pair, each member being of mass μ then $Q^2 \geq 4\mu^2$. Q lies in the forward light cone. Thus both x_1 and x_2 are positive. Since s must be positive (it represents the $(\text{mass})^2$ of anything in (3.1.2)) then it follows that both x_1 and x_2 are positive and less than one in the physical region. It thus follows that s_1 and s_2 are negative in the physical region.

3.2 Feynman representation of the Drell-Yan Ladder Diagram

The general ladder diagram is shown in fig 3.1.3. Using the cutting rules given in appendix B and simplifying to the case of not keeping any distinction between invariants of identical magnitudes, the D and C functions are given by

(3.2.1) $$D = f_u f_L S + f_u g_L s_1 + f_L g_u s_2 + g_u g_L Q^2 + d_0$$

(3.2.2) $$C = (f_u + g_u)\, C(\gamma_1 \ldots \gamma_m) + (f_L + g_L)\, C(\alpha_1 \ldots \alpha_n)$$

(3.2.3) $$d_0 = d_u M_1^2 + d_L M_2^2 - (\textstyle\sum \text{Feynman parameters})\, C m^2$$

(3.2.4) $$P_1^2 = M_1^2, \quad P_2^2 = M_2^2, \quad m^2 = (\text{internal mass})^2$$

f_u, f_L, g_u, g_L, d_u and d_L are positive-definite, linear and homogeneous combinations of the Feynman parameters corresponding to cuts shown in fig 3.2.1. $C(\gamma_1 \ldots \gamma_m)$ means the corresponding

C-function for the graph between the γ_i^{th}, and γ_m^{th} ring inclusive. $C(\alpha_i \ldots \alpha_n)$ is similarly defined. Here we do not give the explicit form of the coefficients of the invariants - they are to be found in section 5.1.

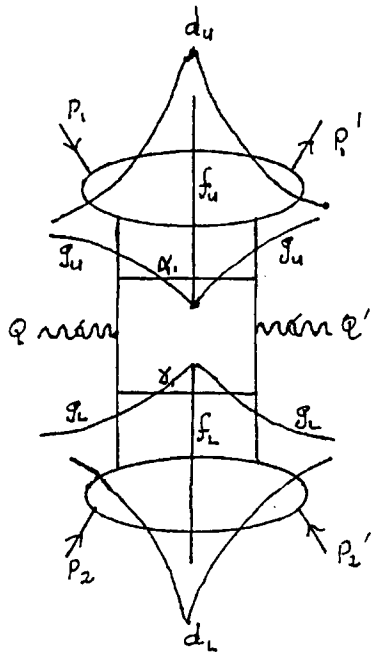


Fig 3.2.1 Diagram showing all the possible cuts corresponding to the general ladder diagram when no distinction between invariants of the same magnitude is kept.

Let us now turn our attention to the simplest of the ladder diagrams, namely, the one-loop diagram shown in fig 3.2.2. The C and D functions are given by :

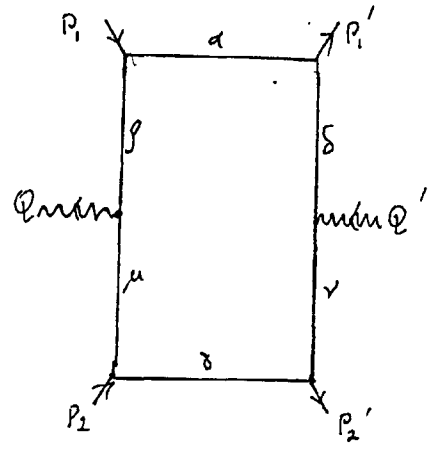


Fig 3.2.2 The one-loop Drell-Yan diagram

(3.2.5) $$D = \alpha\gamma s + \alpha\mu s_1 + \alpha\nu s_1' + \gamma\beta s_2 + \gamma\delta s_2' + \mu\beta p^2 + \nu\delta p'^2 + \nu\beta\bar{p}^2 + \mu\delta\tilde{p}^2 + d_0$$

(3.2.6) $$C = \alpha + \beta + \delta + \gamma + \mu + \nu$$

(3.2.7) $$d_0 = \alpha(\beta+\delta)M_1^2 + \gamma(\mu+\nu)M_2^2 - C^2 m^2$$

It follows from (2.3.7) that the amplitude, A, is essentially

(3.2.8) $$A = \int_0^1 \frac{(\pi d\alpha)\,\delta(1-\Sigma\alpha)}{D^4}$$

where

(3.2.9) $$\begin{cases} (\pi d\alpha) \equiv d\alpha\, d\beta\, d\delta\, d\gamma\, d\mu\, d\nu \\ \Sigma\alpha \equiv \alpha+\beta+\delta+\gamma+\mu+\nu \end{cases}$$

and we have used the constraint imposed by the δ-function to set C = 1. Also factors of 2π, i, g coming from the Feynman rules have been ignored.

In the case where we do not keep any distinction between left and right invariants which are equal in magnitude the D-function of (3.2.5) collapses to

(3.2.10) $$D = \alpha\gamma s + \alpha(\mu+\nu)s_1 + \gamma(\beta+\delta)s_2 + (\mu+\nu)(\beta+\delta)p^2 + d_0$$

3.3 The dominance of the Drell-Yan amplitude by a pinch singularity in Feynman parameter space

In general if some analytic function has an integral representation

there will be poles in the complex plane of the integration variables. The position of these poles is usually well defined by $i\varepsilon$ – prescriptions when they lie close to the contour of integration. In general one may distort the contours away from such points by using Cauchy's theorem. It may happen, with multidimensional integrals, that the contour gets trapped by poles lying on opposite sides of it - this is called a "pinch". We illustrate this by an example [13] . Consider the integral

$$(3.3.1) \qquad G(s) = \int_0^1 dx_1 \int_0^1 dx_2 \; \frac{1}{\left[(a x_1 x_2 + b_1 x_1 + b_2 x_2 + c)s + d + i\varepsilon\right]^2}$$

On doing the x_2 integral we get

$$(3.3.2) \qquad G(s) = \int_0^1 \frac{dx_1}{\left[(a+b_1)x_1 s + (b_2+c)s + d\right]\left[b_1 x_1 s + cs + d\right]}$$

The integral then has two poles at

$$x_1' = -\frac{(b_2+ac)}{(a_1+b_1)} - \frac{1}{s}\frac{d+i\varepsilon}{a+b_1}$$

$$x_1'' = -\frac{c}{b_1} - \frac{1}{s}\frac{d+i\varepsilon}{b_1}$$

The x_1 contour gets "pinched" if

$$(3.3.3) \qquad \begin{cases} -1 < \dfrac{b_2+c}{a_1+b_1} = \dfrac{c}{b_1} < 0 \\ \\ (a+b_1)\,b_1 < 0 \end{cases}$$

with analogous conditions for $b_1 \longleftrightarrow b_2$. The above conditions are

equivalent to

$$(3.3.4) \qquad ax_1x_2 + b_1x_1 + b_2x_2 + c = a^{-1}(ax_1 + b_1)(ax_2 + b_1)$$

If the pinch conditions are satisfied then G(s) will have an asymptotic behaviour of $1/s$, otherwise it behaves like $1/s^2$.

The asymptotic behaviour of the Drell-Yan amplitude is determined by the D-function. It follows from (3.2.1) that if s, s_1, s_2, Q^2 all have the same sign then in the scaling region defined by (3.1.11) we must set f_u and g_u (or f_L and g_L) equal to zero. This will be achieved by setting (α_1, β_1, δ_1) (or γ_1, μ_1, ν_1) simultaneously to zero. This yields a $1/s^3$ asymptotic behaviour. We may write (3.2.1) as

$$(3.3.5) \qquad D = (f_u + g_u s_2/s)(f_L + g_L s_1/s)s + g_u g_L (Q^2 - s_1 s_2/s) + d_0$$

Then if s_1, s_2 are negative but Q^2, s are positive then $(Q^2 - s_1 s_2/s)$ is finite at fixed x_1 and x_2 (see 3.1.12) and we have a pinch when

$$(3.3.6) \qquad \begin{cases} f_u + g_u s_2/s = 0 \\ f_L + g_L s_1/s = 0 \end{cases}$$

This pinch yields a $1/s$ asymptotic behaviour and will obviously dominate over the $1/s^3$ term.

CHAPTER 4

THE ANALYTIC STRUCTURE OF THE ONE-LOOP DIAGRAM

In this chapter we discuss in detail the analytic structure of the one-loop diagram. It proceeds in three stages. In section 4.1 we set $s_1 = s_1'$, $s_2 = s_2'$, $Q^2 = Q'^2 = \bar{Q}^2 = \tilde{Q}^2$. This reduces the complexity of the problem enormously. It allows us to check the factorized Drell-Yan form of the differential cross-section. In section 4.2 we keep $Q^2 = \tilde{Q}^2$ and $Q'^2 = \bar{Q}^2$ but keep Q^2 and Q'^2 distinct. There we discover an infinite time scale hidden in our amplitude. Finally in section 4.3 we keep the distinction between all the Q^2. We still keep $s_1 = s_1'$ and $s_2 = s_2'$ as we need not continue in these variables.

4.1 Keeping no distinction between invariants of the same magnitude

(a) The Drell-Yan analytic structure and its cross-section

We recall from (3.2.8) that our amplitude A is essentially

$$(4.1.1)\qquad A = \int_0^1 \frac{d\alpha\, d\beta\, d\delta\, d\gamma\, d\mu\, d\nu\ \delta(1-\alpha-\beta-\delta-\gamma-\mu-\nu)}{\left[\alpha\gamma\delta + \alpha(\mu+\nu)s_1 + \gamma(\beta+\delta)s_2 + \mu\nu(\beta+\delta)Q^2 + d_0\right]^4} \equiv \int_0^1 \frac{(\pi d\alpha)\,\delta(1-\Sigma\alpha)}{D^4}$$

where

$$(4.1.2)\qquad d_0 = \alpha(\beta+\delta)M_1^2 + \gamma(\mu+\nu)M_2^2 - (\alpha+\beta+\delta+\gamma+\mu+\nu)^2 m^2$$

Analogously to (3.3.5) we write the D-function which appears in (4.1.1)

as

$$(4.1.3)\quad D = \left(\alpha + (\beta+\delta)s_2/s\right)\left(\gamma + (\mu+\nu)s_1/s\right)s + (\mu+\nu)(\beta+\delta)\left(Q^2 - s_1s_2/s\right) + d_0$$

The second term in the above equation is finite in the scaling region but we keep it displayed as it contains all the Q^2 phase information. Our pinch conditions of (3.3.6) become

$$(4.1.4)\quad \begin{cases} \alpha + (\beta+\delta)\, s_2/s = 0 \\ \gamma + (\mu+\nu)\, s_1/s = 0 \end{cases}$$

These only lie on the original contour of integration defined in (4.1.1) if s_1/s and s_2/s are negative. We now scale α, β, δ by η and γ, μ, ν by $\bar{\eta}$. Then (4.1.1) becomes

$$(4.1.5)\quad A = \int_0^1 d\eta d\bar{\eta}\,(\eta\bar{\eta})^2\,\delta(1-\eta-\bar{\eta})\int_0^1 d\alpha d\beta d\delta\,\delta(1-\alpha-\beta-\delta)\int_0^1 \frac{d\gamma d\mu d\nu\,\delta(1-\gamma-\mu-\nu)}{\bar{D}^4}$$

where

$$(4.1.6)\quad \bar{D} = \eta\bar{\eta}\left\{\left(\alpha+(1-\alpha)s_2/s\right)\left(\gamma+(1-\gamma)s_1/s\right)s + (1-\alpha)(1-\gamma)\left(Q^2 - s_1s_2/s\right)\right\} + \bar{d}_0$$

where

$$(4.1.7)\quad \bar{d}_0 = \eta^2\alpha(1-\alpha)M_1^2 + \bar{\eta}^2\gamma(1-\gamma)M_2^2 - (\eta+\bar{\eta})^2 m^2$$

We perform the β, δ, μ, ν - integrals trivially giving a factor $(1-\alpha)(1-\gamma)$ in the numerator. Then (4.1.5) becomes

$$(4.1.8)\qquad A = \int_0^1 d\eta\, d\bar{\eta}\,(\eta\bar{\eta})^2\,\delta(1-\eta-\bar{\eta}) \int_0^1 \frac{d\alpha\, d\gamma\,(1-\alpha)(1-\gamma)}{\bar{D}^4}$$

Our pinch conditions (4.1.4) have now become

$$(4.1.9)\qquad \begin{cases} \alpha + (1-\alpha)S_2/S = 0 \\ \gamma + (1-\gamma)S_1/S = 0 \end{cases}$$

The solutions to the above equations in the scaling regions are

$$(4.1.10)\qquad \begin{cases} \alpha \simeq \alpha_1 + O(1/u) \\ \gamma \simeq \alpha_2 + O(1/u) \end{cases}$$

However, for reasons which will become apparent later we write (4.1.9) as

$$(4.1.11)\qquad \begin{cases} \alpha = \bar{\alpha} \equiv \dfrac{-S_2/S}{1 - S_2/S} \\ \gamma = \bar{\gamma} \equiv \dfrac{-S_1/S}{1 - S_1/S} \end{cases}$$

Since we are primarily interested in the pinch configuration which gives us our scaling $1/S$ term, we lose nothing by replacing α, γ, everywhere, except in the curly bracket of (4.1.6) by their pinch values given by (4.1.11). The zeroes of the coefficient of u due to $\eta = \bar{\eta} = 0$ do not contribute at the $1/S$ level because of the η^2 and $\bar{\eta}^2$ in the numberator. We do not set α and γ at their pinch values in the coefficient of $(Q^2 - S_1 S_2/S)$ as we will be forced to consider large values of this combination during the analytic continuation of the function.

We now define

$$(4.1.12)\qquad \bar{A}(\lambda) \equiv \int_0^1 d\eta\, d\bar{\eta}\, (\eta\bar{\eta})^2 \delta(1-\eta-\bar{\eta}) \int_0^1 \frac{d\alpha\, d\gamma\, (1-\bar{\alpha})(1-\bar{\gamma})}{[\bar{D}+\lambda]^2}$$

It then follows from (4.1.8) that

$$(4.1.13)\qquad A = \frac{1}{3!}\frac{\partial^2 \bar{A}}{\partial \lambda^2}\Big|_{\lambda=0}$$

With $\bar{d}_0(\eta, \bar{\eta}, \alpha, \gamma)$ now given by $\bar{d}_0(\eta, \bar{\eta}, \bar{\alpha}, \bar{\gamma})$ we may do the α, γ, integrations exactly in (4.1.12) using

$$(4.1.14)\qquad \int_0^1 \frac{d\alpha\, d\gamma}{[A\alpha\gamma + B\alpha + C\gamma + D]^2} = \frac{1}{[AD-BC]} \log\left\{\frac{[A+B+C+D][D]}{[B+D][C+D]}\right\}$$

We have

$$(4.1.15)\qquad \begin{cases} A = \eta\bar{\eta}(s - s_1 - s_2 + Q^2) \\ B = \eta\bar{\eta}(s_1 - Q^2) \\ C = \eta\bar{\eta}(s_2 - Q^2) \\ D = \eta\bar{\eta}Q^2 + \bar{d}_0 + \lambda \end{cases}$$

From (4.1.12) we have therefore

$$(4.1.16)\qquad \bar{A}(\lambda) = \int_0^1 \frac{d\eta\, d\bar{\eta}\, (\eta\bar{\eta})\, \delta(1-\eta-\bar{\eta})\, (1-\bar{\alpha})\, (1-\bar{\gamma})}{[\eta\bar{\eta}(Q^2 s - s_1 s_2) + \bar{d}_0(s - s_1 - s_2 + Q^2) + \lambda(s - s_1 - s_2 + Q^2)]} \log\left\{\frac{[\eta\bar{\eta}s + \bar{d}_0 + \lambda][\eta\bar{\eta}Q^2 + \bar{d}_0 + \lambda]}{[\eta\bar{\eta}s_1 + \bar{d}_0 + \lambda][\eta\bar{\eta}s_2 + \bar{d}_0 + \lambda]}\right\}$$

So using (4.1.13) we see that the dominant contribution to A comes from the differentiation with respect to λ of the denominator in (4.1.16) rather than the differentiation of the logarithmic term. Substituting for $\bar{\alpha}$ and $\bar{\gamma}$ from (4.1.11) we arrive at

$$(4.1.17)\quad A \simeq \frac{1}{3}\frac{(s-s_1-s_2+Q^2)^3}{(1-s_1/s)(1-s_2/s)}\log\left\{\frac{[s][Q^2]}{[s_1][s_2]}\right\}\int_0^1 \frac{d\eta d\bar{\eta}\,\eta\bar{\eta}\,\delta(1-\eta-\bar{\eta})}{D^3}$$

where

$$(4.1.18)\quad D = \eta\bar{\eta}\,(Q^2 s - s_1 s_2) + \bar{d}_0\,(s-s_1-s_2+Q^2)$$

$$(4.1.19)\quad = \eta\bar{\eta}\,(Q^2 - s_1 s_2/s)s + \bar{d}_0\left((1-s_1/s)(1-s_2/s)s' + (Q^2 - s_1 s_2/s)\right)$$

This formula has the expected form. If $(Q^2 - s_1 s_2/s) \longrightarrow \infty$ then we get an answer much smaller than $1/s$ (see (4.1.19). Otherwise we pick up the phase of the logarithm. Remember that $|sQ^2/s_1 s_2| = 1 + O(\frac{1}{u})$ in the regions of interest.

We assume for the moment that the denominator term is innocuous and may be ignored during the continuations. We shall consider this problem in (c) and indeed prove that such is the case.

Thus if we start off in the Euclidean region where all invariants are negative we clearly do not pick up any phase in the logarithm. In other words there is no pinch if all the invariants are negative. As we continue to the Drell-Yan physical region with s_1 and s_2 held fixed and negative but $(-s) \longrightarrow (-s e^{\mp i\pi})$, $(-Q^2) \rightarrow (-Q^2 e^{-i\pi})$ say then the log gives a phase of $(-2i\pi)$ and 0 respectively. This gives the

s-discontinuity for all Q^2 on one side of their cuts. However, for our graph once we have taken the s-discontinuity the Q^2 $i\varepsilon$'s do not matter. Thus this gives the physical cross-section as well. The cross-section is of course positive.

We may also take the Q^2-discontinuity over <u>all</u> the Q^2 cuts. This is finite.

The asymptotic behaviour of A at intermediate points depends on how we continue the invariants. At the start of the continuation $|Q^2 - s_1 s_2/s|$ is finite. Then the pinch term behaves like $1/u^2$ since $\log\left|\frac{sQ^2}{s_1 s_2}\right| = \log(1 + O(\frac{1}{u})) \simeq 1/u$. If we continue $-s \longrightarrow -se^{-i\pi}$ and $-Q^2 \longrightarrow -Q^2 e^{-i\pi}$ simultaneously the $|Q^2 - s_1 s_2/s| \longrightarrow u$ at intermediate points of the continuation and the pinch term behaves like $1/u^4$. At the end of the continuation the logarithm has picked up a phase of $-2i\pi$ and we get back the $1/u$ term. If however, we continue $-s \rightarrow -se^{+i\pi}$ and $-Q^2 \longrightarrow -Qe^{-i\pi}$ simultaneously the $(Q^2 - s_1 s_2/s)$ remains finite but the logarithm is <u>not</u> picking up any phase and the pinch behaves uniformly like $1/u^2$ until the end of the continuation.

(b) <u>Factorization of the cross-section in the physical region</u>

In the Drell-Yan model [3] the cross-section factorises into electroproduction structure functions when the integration of the transverse component, Λ, of the momentum Q is performed.

From (4.1.17) we have

$$(4.1.20) \qquad \Delta_s\{A\} \simeq \frac{(s - s_1 - s_2 + Q^2)^3}{3(1 - s_1/s)(1 - s_2/s)} (-2i\pi) \int_0^1 \frac{d\eta d\bar{\eta}\, \eta\bar{\eta}\, \delta(1 - \eta - \bar{\eta})}{D^3}$$

where D is given by (4.1.18) or (4.1.19). Remember that we are now in the physical region. In D we can neglect the last $(Q^2 - s_1 s_2/s)$ term as it is down by 1/u compared to the other terms. For the same reason the numerator may be written as

$$(4.1.21) \qquad s - s_1 - s_2 + Q^2 = (1 - s_1/s)(1 - s_2/s)s + Q^2 - s_1 s_2/s \simeq (1 - s_1/s)(1 - s_2/s)s$$

From (3.1.11) it follows that

$$(4.1.22) \qquad \begin{cases} (1 - s_1/s) \simeq 1/(1 - x_2) + O(1/u) \\ (1 - s_2/s) \simeq 1/(1 - x_1) + O(1/u) \end{cases}$$

Thus from (4.1.20) we have

$$(4.1.23) \qquad \int \Delta_s\{A\} d^2\Lambda \simeq \frac{(-2i\pi)(1-x_1)^2(1-x_2)^2}{3} \int_0^1 d\eta d\bar{\eta}\, \eta\bar{\eta}\delta(1-\eta-\bar{\eta}) \int \frac{d^2\Lambda}{s\left[\eta\bar{\eta}(1-x_1)(1-x_2)(Q^2 - s_1 s_2/s) + \bar{d}_0\right]^3}$$

where

$$(4.1.24) \qquad \bar{d}_0 \simeq \eta^2 x_1(1-x_1)M^2 + \bar{\eta}^2 x_2(1-x_2)M_2^2 - (\eta+\bar{\eta})^2 m^2$$

on making use of (4.1.7) and (4.1.10). Combining this with (3.1.12) we find

$$(4.1.25) \qquad \eta\bar{\eta}(1-x_1)(1-x_2)(Q^2 - s_1 s_2/s) + \bar{d}_0 \simeq (\eta+\bar{\eta})\left\{\eta\left(M_1^2 x_1(1-x_1) - m^2\right) + \bar{\eta}\left(M_2^2 x_2(1-x_2) - m^2\right)\right\} - \eta\bar{\eta}\Lambda^2$$

We set $\eta+\bar{\eta}=1$ because of the δ- function in (4.1.23). We may neglect the Λ^2-dependence in S. The Λ^2 integral is done using

$$\int \frac{d^2\Lambda}{[a-b\Lambda^2]^3} = \frac{-\pi}{2a^2b} \tag{4.1.26}$$

Note that since the residue of the pole in the Λ^2-plane is zero it does not matter how we go round the pole. From (4.1.23) we then obtain

$$\int_0^\infty d\Lambda^2 \Delta_s\{A\} = \frac{i\pi^2(1-x_1)^2(1-x_2)^2}{3S}\int_0^1 \frac{d\eta d\bar{\eta}\ \delta(1-\eta-\bar{\eta})}{[\eta(M_1^2x_1(1-x_1)-m^2)+\bar{\eta}(M_2^2x_2(1-x_2)-m^2)]^2} \tag{4.1.27}$$

The η, $\bar{\eta}$ integrals are performed trivially using the Feynman identity

$$\frac{1}{ab} = \int_0^1 \frac{d\eta d\bar{\eta}\,\delta(1-\eta-\bar{\eta})}{[\eta a+\bar{\eta}b]^2} \tag{4.1.28}$$

We thus arrive at the factorized form of the Drell-Yan Cross-section.

$$\int d^2\Lambda\,\Delta_s\{A\} \propto \frac{1}{u}\cdot\frac{(1-x_1)}{[M_1^2x_1(1-x_1)-m^2]}\cdot\frac{(1-x_2)}{[M_2^2x_2(1-x_2)-m^2]} \tag{4.1.29}$$

where we have used

$$S \simeq (1-x_1)(1-x_2)u \tag{4.1.30}$$

from (3.1.11) in the final step.

(c) The analytic structure of the $\eta,\bar{\eta}$-integral

There are two problems present when dealing with the integral.

The first stems from the identity (4.1.14) and its subsequent use in arriving at (4.1.17). We note from (4.1.14) that the denominator (AD - BC) which appears on the r.h.s. can vanish. This apparent pole is however killed by the behaviour of the logarithm as AD $\longrightarrow$ BC. This becomes more transparent if we substitute for D = BC/A in the integrand on the l.h.s.. Some elementary algebra reveals that the α and β integrals factorise to give a differnent analytic form which corresponds to the (limit$_{AD\to BC}$) of the log on the r.h.s.. Now in (4.1.17) the analogue of (AD -BC) is the denominator D. But observe that there it is to the cube and therefore the logarithmic term that appears in (4.1.16) cannot kill this singularity - it would have had we kept all the correction terms to (4.1.17) though. So on the face of it we have a badly defined η-$\bar{\eta}$ integral in the Euclidean region as a result of our approximation scheme.

The second problem is the possible existence of an imaginary part, or other analytic structure, from the term generated by the η-$\bar{\eta}$-integral after its analytic continuation to the physical region. This would of course change the value of the cross-section calculated in (a).

Fortunately both of these problems may be avoided by the following arguments:

When we reach (4.1.16) both of the poles of the denominator (note

that after doing the $\bar{\eta}$ -integral through the δ -Function our denominator becomes a quadratic in η) may be made to lie far away from the real η-contour of integration, if they lie close to it, by continuing one or more of the invariants away from the real axis, say, $-s_1 \longrightarrow -s_1 e^{i\theta}$. It is insufficient to continue in the masses because in our asymptotic limits the poles will still lie just off the real axis. We have, however, paid a temporary price for this: our logarithmic term appearing outside (4.1.17) will, in general, have picked up a phase according to the analytic continuation of the invariants made above.

To handle the second problem we must now perform the η - $\bar{\eta}$ integral in (4.1.17) in order to determine its precise analytic structure. This integral from (4.1.17) is of the form

$$(4.1.31) \qquad I = \int_0^1 \frac{d\eta d\bar{\eta}\ \eta\bar{\eta}\ \delta(1-\eta-\bar{\eta})}{\left[\eta\bar{\eta}(Q^2 s - s_1 s_2) + \bar{d}_0 (s - s_1 - s_2 + Q^2)\right]^3}$$

where

$$(4.1.32) \qquad \bar{d}_0 = \eta^2 \bar{\alpha}(1-\bar{\alpha}) M_1^2 + \eta^2 \bar{\beta}(1-\bar{\beta}) M_2^2 - (\eta+\bar{\eta})^2 m^2$$

We now define

$$(4.1.33) \qquad \bar{I}(\lambda) = \int_0^1 \frac{d\eta d\bar{\eta}\ \delta(1-\eta-\bar{\eta})}{\left[\eta\bar{\eta}(Q^2 s - s_1 s_2 + \lambda) + \bar{d}_0 (s - s_1 - s_2 + Q^2)\right]^2}$$

It then follows by comparison with (4.1.31) that

$$(4.1.34) \qquad I = -\frac{1}{2} \left.\frac{\partial \bar{I}}{\partial \lambda}\right|_{\lambda=0}$$

In (4.1.33) we do the $\bar{\eta}$-integral trivially using the δ-function. We then change variables to $\eta = \frac{1}{2} + \eta'$ and some elementary algegra reveals that

$$\text{(4.1.35)} \qquad \bar{I}(\lambda) = \int_{-1/2}^{1/2} \frac{d\eta}{\left[A\eta^2 + B\eta + C\right]^2}$$

where

$$\text{(4.1.36)} \quad \begin{cases} A = (s - s_1 - s_2 + Q^2)\left(\bar{\alpha}(1-\bar{\alpha})M_1^2 + \bar{\gamma}(1-\bar{\gamma})M_2^2\right) - \left(Q^2 s - s_1 s_2 + \lambda\right) \\ B = (s - s_1 - s_2 + Q^2)\left(\bar{\alpha}(1-\bar{\alpha})M_1^2 - \bar{\gamma}(1-\bar{\gamma})M_2^2\right) \\ C = \frac{1}{4}\left\{(s - s_1 - s_2 + Q^2)\left(\bar{\alpha}(1-\bar{\alpha})M_1^2 + \bar{\gamma}(1-\bar{\gamma})M_2^2 - 4m^2\right) + Q^2 s - s_1 s_2 + \lambda\right\} \end{cases}$$

where $\bar{\alpha}$ and $\bar{\gamma}$ are defined by (4.1.11). We write (4.1.35) as

$$\text{(4.1.37)} \qquad \bar{I}(\lambda) = \frac{1}{A^2} \int_{-1/2}^{1/2} \frac{d\eta}{(\eta - \eta_+)^2 (\eta - \eta_-)^2}$$

The cases $A = 0$ and $\eta_+ = \eta_-$ will be discussed later and we have

$$\text{(4.1.38)} \qquad \eta_\pm = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Doing the η-integral in (4.1.37) gives

$$\text{(4.1.39)} \quad \bar{I}(\lambda) = \frac{-1}{A^2(\eta_+ - \eta_-)^2}\left\{\frac{2}{(\eta_+ - \eta_-)} \log\left\{\frac{(\frac{1}{2} - \eta_+)(\frac{1}{2} + \eta_-)}{(\frac{1}{2} - \eta_-)(\frac{1}{2} + \eta_+)}\right\} + \left(\frac{1}{\frac{1}{2} - \eta_-} + \frac{1}{\frac{1}{2} + \eta_-} + \frac{1}{\frac{1}{2} - \eta_+} + \frac{1}{\frac{1}{2} + \eta_+}\right)\right\}$$

It follows from (4.1.35) that when A = 0

$$\bar{I}(\lambda)\Big|_{A=0} = \frac{1}{(C+\frac{1}{2}B)(C-\frac{1}{2}B)} \tag{4.1.40}$$

Also when $\eta_+ = \eta_-$ we see from (4.1.37) that

$$\bar{I}(\lambda)\Big|_{\eta_+=\eta_-} = -\frac{1}{3A^2}\left(\frac{1}{(\frac{1}{2}-\eta_\pm)^3} + \frac{1}{(\frac{1}{2}+\eta_\pm)^3}\right) \tag{4.1.41}$$

To obtain I from $\bar{I}$ we use (4.1.34). The exact form of I is really of no relevance to us. We are only interested in its singularity structure and it is precisely that of $\bar{I}(\lambda)\Big|_{\lambda=0}$ as the differentiation w.r.t. λ will not change that. It follows from (4.1.39), (4.1.40),(4.1.41) that there are three types of singularities we have to worry about:

(i) there is a logarithmic branch cut running between $-\frac{1}{2} \leq \eta_\pm \leq \frac{1}{2}$

(ii) the $(\eta_+ - \eta_-)$ term translates into a square root when we use (4.1.38) and therefore results into another branch cut in the s, Q^2, s_1, s_2 complex planes.

(iii) there are poles at $\eta_\pm = \pm\frac{1}{2}$ and $C = \pm\frac{1}{2}B$

From (4.1.36) and using the constraint that A = 0 for the poles at C = $\frac{1}{2}$B, these correspond to the conditions

$$\begin{cases} \bar{\alpha}(1-\bar{\alpha})M_1^2 - m^2 = 0 \\ \bar{\gamma}(1-\bar{\gamma})M_2^2 - m^2 = 0 \end{cases} \tag{4.1.42}$$

From (4.1.38) it also follows that the poles at $\eta_\pm = \pm\frac{1}{2}$ correspond to conditions (4.1.42). For M_1^2, $M_2^2 < 4m^2$ these conditions can only be satisfied for complex values of $\bar{\alpha}$ and $\bar{\gamma}$, that is, for complex values

s, s_1 and s_2. There we may go round the poles in some arbitrary manner. Of course when M_1^2, M_2^2 are above their $4m^2$ threshold we then have to take more care about their continuations. We have the case $M_1^2 = M_2^2 = m^2$ and so we never hit these poles or branch cuts on the real axis.

We now turn our attention to the cut running between $-\frac{1}{2} \leq \eta_\pm \leq \frac{1}{2}$. Having done the η-$\bar{\eta}$ integral we continue s_1, and any of the other invariants that were continued back to the Euclidean region, thus killing all the phase picked up by the $\log\left\{\frac{sQ^2}{s_1 s_2}\right\}$ term in the first place. We may now continue s, Q^2 to the physical region as may be desired. The term resulting from the η-$\bar{\eta}$ integral <u>will</u> in general pick up a phase becuase of its logarithmic singularity. This phase, if any, is however a constant independent of the invariants and since it has to be multiplied by $\log\left|\frac{sQ^2}{s_1 s_2}\right|$ it will be down by a factor of $1/u$ compared to the pinch term discussed in section (a) and is, therefore, negligible.

Finally we deal with the square root coming from

$$\eta_+ - \eta_- = \frac{\sqrt{B^2 - 4AC}}{A} \tag{4.1.43}$$

This comes from (4.1.38). We define

$$\begin{cases} \Sigma = s - s_1 - s_2 + Q^2 \\ x = \bar{\alpha}(1-\bar{\alpha}) M_1^2 \\ y = \bar{\gamma}(1-\bar{\gamma}) M_2^2 \\ \omega = Q^2 s - s_1 s_2 \end{cases} \tag{4.1.44}$$

Hence from (4.1.36) and setting $\lambda = 0$ we find

$$(4.1.45) \qquad \begin{cases} A = z(x+y) - \omega \\ B = z(x-y) \\ C = \frac{1}{4}\left\{ z(x+y-4m^2) + \omega \right\} \end{cases}$$

We then find that

$$(4.1.46) \qquad B^2 - 4AC = \left[(\omega - 2zm^2)^2 - (m^2-x)(m^2-y) \right] \simeq (\omega - 2zm^2)^2$$

in our asymptotic limit. Thus from (4.1.43) the $(\eta_+ - \eta_-)$ does NOT behave like a square root. Using (4.1.44) we also see that $(w - 2zm^2)$ is negative definite in the physical region. Hence we need not worry about this term.

Thus although the term from the $\eta\cdot\bar{\eta}$ integral is a potential source of trouble, its analytic structure is such that its contribution is irrelevant as long as M_1^2, $M_2^2 < 4m^2$.

So to calculate in any region we first check that $(Q^2 - s_1 s_2/S)$ is finite. If it is then we compute the phase of the log in (4.1.17) and multiply by the $\eta\cdot\bar{\eta}$ integral which we do naively. If it is not then the answer is zero.

4.2 A more general analytic continuation

We return to (3.2.5) and keep Q^2 and Q'^2 distinct but let

$\tilde{Q}^2 = Q^2$ and $\tilde{Q}^2 = Q'^2$. We still neglect the distinctions between s_1 and s_1' and s_2 and s_2'. This is a great simplification compared to the case of keeping all the Q^2 distinct. The D-function may then be written as

$$(4.2.1)\qquad D = \left(\alpha + (\beta+\delta)s_2/s\right)\left(\gamma + (\mu+\nu)s_1/s\right)s + \beta(\mu+\nu)(Q^2 - s_1s_2/s) + \delta(\mu+\nu)(Q'^2 - s_1s_2/s) + d_0$$

where d_0 is given by (4.1.2). The amplitude is still given by (4.1.1) i.e.

$$(4.2.2)\qquad A = \int_0^1 \frac{(d\alpha)\,\delta(1-\Sigma\alpha)}{D^4}$$

Scale α, β, δ by η and γ, μ, ν by $\bar{\eta}$. We get

$$A = \int_0^1 d\eta\, d\bar{\eta}\,(\eta\bar{\eta})^2\,\delta(1-\eta-\bar{\eta})\int_0^1 d\alpha\, d\beta\, d\delta\;\delta(1-\alpha-\beta-\delta)\int_0^1 d\gamma\, d\mu\, d\nu\;\delta(1-\gamma-\mu-\nu)$$

$$(4.2.3)\qquad \times \frac{1.}{\left[\eta\bar{\eta}\left\{\left(\alpha+(1-\alpha)s_2/s\right)\left(\gamma+(1-\gamma)s_1/s\right)s + \beta(1-\gamma)(Q^2 - s_1s_2/s) + \delta(1-\gamma)(Q'^2 - s_1s_2/s)\right\} + \bar{d}_0(\eta,\bar{\eta},\alpha,\gamma)\right]^4}$$

$\bar{d}_0$ is now independent of β, δ, μ, ν. We do the μ, ν and δ-integrals trivially giving

$$A = \int_0^1 d\eta\, d\bar{\eta}\,(\eta\bar{\eta})^2\,\delta(1-\eta-\bar{\eta})\int_0^1 d\alpha\int_0^{1-\alpha} d\beta\int_0^1 d\gamma\;(1-\gamma)$$

$$(4.2.4)\qquad \times \frac{1.}{\left[\eta\bar{\eta}\left\{\left(\alpha+(1-\alpha)s_2/s\right)\left(\gamma+(1-\gamma)s_1/s\right)s + \beta(1-\gamma)(Q^2 - s_1s_2/s) + (1-\alpha-\beta)(1-\gamma)(Q'^2 - s_1s_2/s)\right\} + \bar{d}_0(\eta,\bar{\eta},\alpha,\gamma)\right]^4}$$

The difference between here and the previous section is the presence of the explicit ρ-dependence due to $Q^2 \neq Q'^2$. Performing the ρ-integral we get

$$A = -\frac{1}{3(Q^2-Q'^2)} \int_0^1 d\eta d\bar{\eta}\, \eta\bar{\eta}\, \delta(1-\eta-\bar{\eta}) \int_0^1 d\alpha d\gamma$$

$$\times \left[\frac{1}{\left[\eta\bar{\eta}\left\{(\alpha+(1-\alpha)s_1/s)(\gamma+(1-\gamma)s_2/s)s + (1-\alpha)(1-\gamma)(Q^2-s_1s_2/s)\right\} + \bar{d}_0\right]^3} - \left(\text{Diff}_0\, Q^2 \leftrightarrow Q'^2\right) \right] \tag{4.2.5}$$

We now set the pinch conditions (4.1.11) in $\bar{d}_0$ as we did in section 4.1.(a). These have not changed. Each of the two integrals in (4.2.5) is similar to that encountered in section 4.1.(a) - see (4.1.8) - except that our denominator is to the cube rather than to the fourth power. We proceed in exactly analogous steps to be found from (4.1.12) to (4.1.19) which allow us to arrive at

$$A \simeq \frac{1}{6(Q^2-Q'^2)} \int_0^1 d\eta d\bar{\eta}\, \delta(1-\eta-\bar{\eta}) \left\{ \frac{(s-s_1-s_2+Q^2)}{\left[\eta\bar{\eta}(Q^2s-s_1s_2)+\bar{d}_0(s-s_1-s_2+Q^2)\right]^2} \times \log\left\{\frac{sQ^2}{s_1s_2}\right\} - \text{Diff}_0\left(Q^2 \leftrightarrow Q'^2\right) \right\} \tag{4.2.6}$$

The nature of the η-$\bar{\eta}$ integrals are identical to those discussed in section 4.1.(c) and hence are irrelevant to the continuations which interest us.

If we take a Q^2-discontinuity the first term only contributes the second being independent of Q^2 and the $1/(Q^2 - Q'^2)$ gives us an infinite contribution. This behaviour as we will discuss and show in the concluding chapter has a rather trivial physical explanation and

will lead to an infinite time scale hidden in our amplitude. The s-discontinuity is still finite and gives the same cross-section as before.

4.3 The most general analytic continuation

In this section we maintain the distinction between all the Q^2's. However, we keep $s_1 = s_1'$ and $s_2 = s_2'$ as there is no need to continue in these variables. If the distinction is kept the algebra does not get more complicated. The algebra involved in this section is extremely tedious and consequently a lot of it will be left out.

From (3.2.8) we write our amplitude A as

$$(4.3.1)\quad A = \int_0^1 \frac{d\alpha d\beta d\delta d\gamma d\mu d\nu\, \delta(1-\alpha-\beta-\delta-\gamma-\mu-\nu)}{\left[\alpha\gamma s + \alpha(\mu+\nu)s_1 + \gamma(\beta+\delta)s_2 + \mu\beta Q^2 + \nu\delta Q'^2 + \nu\beta\bar{Q}^2 + \mu\delta\hat{Q}^2 + d_0\right]^4}$$

where

$$(4.3.2)\quad d_0 = \alpha(\beta+\delta)M_1^2 + \gamma(\mu+\nu)M_2^2 - (\alpha+\beta+\delta+\gamma+\mu+\nu)^2 m^2$$

We start off by scaling α, β, δ by η and γ, μ, ν by $\bar{\eta}$. Hence, re-writing the D-function using the constraints imposed by the resulting δ-functions as a result of the scaling trick we find

$$A = \int_0^1 d\eta\, d\bar{\eta}\, (\eta\bar{\eta})^2\, \delta(1-\eta-\bar{\eta}) \int_0^1 d\alpha\, d\beta\, d\delta\; \delta(1-\alpha-\beta-\delta) \int_0^1 d\gamma\, d\mu\, d\nu\; \delta(1-\gamma-\mu-\nu)$$

$$(4.3.3) \qquad \times \frac{1}{\left[\eta\bar{\eta}\left\{\alpha\gamma S + \alpha(1-\gamma)S_1 + \gamma(1-\alpha)S_2 + \mu\beta Q^2 + (1-\gamma-\mu)(1-\alpha-\beta)Q'^2 + (1-\gamma-\mu)\beta\bar{Q}^2 + \mu(1-\alpha-\beta)\bar{Q}'^2\right\} + \bar{d}_0\right]^4}$$

where

$$(4.3.4) \qquad \bar{d}_0 = \eta^2\alpha(1-\alpha)M_1^2 + \bar{\eta}^2\gamma(1-\gamma)M_2^2 - m^2$$

We trivially do the δ and ν integrals using the δ-functions. This causes the upper limits of the β and μ integrals to change to $(1-\alpha)$ and $(1-\gamma)$ respectively. $\bar{d}_0$ is a function of η, $\bar{\eta}$, α and γ. We suppress its dependence on α and γ by substituting them for the pinch conditions (4.1.11). We now have to make a fortunate choice of the order of doing the integrations. We start off by doing the μ-integral followed by the γ-integral and then the β-integral. The integrals involved are elementary but the algebra extremely tedious. The result is

$$A = \frac{1}{6}\int_0^1 d\eta\, d\bar{\eta}\; \eta\bar{\eta}\,\delta(1-\eta-\bar{\eta}) \int_0^1 d\alpha$$

$$(4.3.5) \qquad \times \left\{ \frac{[Q^2 + Q'^2 - \bar{Q}^2 - \bar{Q}'^2]}{(a\alpha+b)(a'\alpha+b')^2} \log\left\{\frac{[a_1''\alpha+b_1''][a_2''\alpha+b_2'']}{[a_3''\alpha+b_3''][a_4''\alpha+b_4'']}\right\} \right.$$

$$+ \frac{1}{(a'\alpha+b')(a\alpha+b)^2} \log\left\{\frac{[a_1''\alpha+b_1''][a_2''\alpha+b_2'']}{[a_3''\alpha+b_3''][a_4''\alpha+b_4'']}\right\}$$

$$\left. - \frac{1}{(a\alpha+b)(a'\alpha+b')}\left(\frac{1}{[a_1''\alpha+b_1'']} + \frac{1}{[a_2''\alpha+b_2'']} - \frac{1}{[a_3''\alpha+b_3'']} - \frac{1}{[a_4''\alpha+b_4'']} \right) \right\}$$

where

$$
\begin{cases}
a = \eta\bar{\eta}(s-s_2), \\
b = \eta\bar{\eta}s_2 + \bar{d}_0, \\
a' = \eta\bar{\eta}\left\{s_1(Q^2+Q'^2-\bar{Q}^2-\tilde{Q}^2) - (Q^2Q'^2-\bar{Q}^2\tilde{Q}^2)\right\}, \\
b' = \eta\bar{\eta}(Q^2Q'^2-\bar{Q}^2\tilde{Q}^2) + (Q^2+Q'^2-\bar{Q}^2-\tilde{Q}^2)\bar{d}_0, \\
a_1'' = \eta\bar{\eta}(s_1-Q^2), \quad b_1'' = \eta\bar{\eta}Q^2 + \bar{d}_0, \\
a_2'' = \eta\bar{\eta}(s_1-Q'^2), \quad b_2'' = \eta\bar{\eta}Q'^2 + \bar{d}_0 \\
a_3'' = \eta\bar{\eta}(s_1-\tilde{Q}^2), \quad b_3'' = \eta\bar{\eta}\tilde{Q}^2 + \bar{d}_0 \\
a_4'' = \eta\bar{\eta}(s_1-\bar{Q}^2), \quad b_4'' = \eta\bar{\eta}\bar{Q}^2 + \bar{d}_0
\end{cases}
\tag{4.3.6}
$$

Some of the α-integrals are of the type

$$
\int \frac{d\alpha}{(a\alpha+b)} \log(a'\alpha+b') \tag{4.3.7}
$$

These cannot be expressed in terms of elementary functions. They are given in terms of Spence functions [14]. Had we done our integrations in a different order these Spence functions would have appeared at a very early stage and the subsequent integrations would have been considerably more difficult.

To do the α-integration we will use a possible standard form of the Spence functions, f, given by

$$(4.3.8)\qquad \int \frac{\log(ax+b)}{(cx+d)}\,dx = \frac{1}{c}\log\left|b-\frac{ad}{c}\right|\log\left|\frac{a(cx+d)}{ad-bc}\right| - \frac{1}{c}f\left\{\frac{a(cx+d)}{ad-bc}\right\}$$

where

$$(4.3.9)\qquad f(x) \equiv -\int_0^x \frac{\log|1-y|}{y}\,dy$$

for real x. As discussed in [14] this may be extended to complex values of x.

The next obvious step is to use (4.3.8) and 'turn the handle' to work out the α-integration in (4.3.5). Having done that we discovered that Spence-type integrals were irrelevant to the continuations of interest. With hindsight we split (4.3.5) into terms that generate these Spence-type integrals and those that do not.

The identity

$$(4.3.10)\qquad \frac{1}{(a\alpha+b)(a'\alpha+b')^2} \equiv \frac{a^2}{(ab'-a'b)^2}\cdot\frac{1}{(a\alpha+b)} - \frac{aa'}{(ab'-a'b)^2}\cdot\frac{1}{(a'\alpha+b')} - \frac{a'}{(ab'-a'b)}\cdot\frac{1}{(a'\alpha+b')^2}$$

allows us to rewrite (4.3.5) as

$$(4.3.11)\qquad A = \frac{1}{6}\int_0^1 d\eta\,d\bar{\eta}\;\eta\bar{\eta}\;\delta(1-\eta-\bar{\eta})\,[A_1+A_2]$$

where

$$(4.3.12)\qquad A_1 \equiv \int_0^1 \frac{d\alpha}{(ab'-a'b)^2}\left\{\left((Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)\left(\frac{a^2}{a\alpha+b}-\frac{aa'}{a'\alpha+b'}\right)+\frac{a'^2}{(a'\alpha+b')}-\frac{aa'}{(a\alpha+b)}\right)\log\left\{\frac{[a_1''\alpha+b_1''][a_2''\alpha+b_1']}{[a_3''\alpha+b_3'][a_4''\alpha+b_4'']}\right\}\right\}$$

$$(4.3.13)\quad A_2 = \int_0^1 d\alpha \left\{ \frac{1}{(ab'-a'b)} \left(\frac{a}{(a\alpha+b)^2} - \frac{a'(Q^2+Q'^2-\tilde{Q}^2-\tilde{\tilde{Q}}^2)}{(a'\alpha+b')^2} \right) \log\left\{ \frac{[a_1''\alpha+b_1''][a_2''\alpha+b_2'']}{[a_3''\alpha+b_3''][a_4''\alpha+b_4'']} \right\} \right.$$

$$\left. - \frac{1}{(a\alpha+b)(a'\alpha+b')} \left(\frac{1}{(a_1''\alpha+b_1'')} + \frac{1}{(a_2''\alpha+b_2'')} - \frac{1}{(a_3''\alpha+b_3'')} - \frac{1}{(a_4''\alpha+b_4'')} \right) \right\}$$

It will turn out that all the phase information, insofar as continuation to the physical region is concerned will be contained in A_2. A_1 will be irrelevant.

Let us first turn our attention to A_2. The α-integrals are done using the following

$$(4.3.14)\quad \int_0^1 \frac{d\alpha}{(a\alpha+b)^2} \log(a''\alpha+b'') = \frac{1}{a}\left[\frac{1}{b}\log b'' - \frac{1}{(a+b)}\log(a''+b'') + \frac{a''}{(ab''-a''b)} \log\left\{ \frac{[a+b][b'']}{[a''+b''][b]} \right\} \right]$$

and

$$(4.3.15)\quad \int_0^1 \frac{d\alpha}{(a\alpha+b)(a'\alpha+b')(a''\alpha+b'')} = \frac{a}{(ab'-a'b)(ab''-a''b)} \log\left\{ \frac{a+b}{b} \right\}$$

$$+ \frac{a'}{(a'b-ab')(a'b''-a''b')} \log\left\{ \frac{a'+b'}{b'} \right\}$$

$$+ \frac{a''}{(a''b-ab'')(a''b'-a'b'')} \log\left\{ \frac{a''+b''}{b''} \right\}$$

Some lengthy and tedious algebra eventually reveals that

$$A_2 = -\frac{1}{6}\left[\eta\bar{\eta}\left\{s(Q^2Q'^2-\bar{Q}^2\tilde{Q}^2)-s_1s_2(Q^2+Q'^2-\bar{Q}^2-\tilde{Q}^2)\right\}+\bar{d}_0\left\{(s-s_1-s_2)(Q^2+Q'^2-\bar{Q}^2-\tilde{Q}^2)+Q^2Q'^2-\bar{Q}^2\tilde{Q}^2\right\}\right]^{-1}$$

$$\times\Bigg(\frac{(s-s_1)(s-s_2)\left(a^2\{Q'^2\bar{Q}^2(\tilde{Q}^2-Q^2)+Q^2\tilde{Q}^2(\bar{Q}^2-Q'^2)\}+2ab\{\bar{Q}^2\tilde{Q}^2-Q^2Q'^2\}+b^2\{\tilde{Q}^2-Q^2+\bar{Q}^2-Q'^2\}\right)}{\left[aQ^2+b\right]\left[aQ'^2+b\right]\left[a\bar{Q}^2+b\right]\left[a\tilde{Q}^2+b\right]}$$

$$\times\log\left\{\frac{\eta\bar{\eta}s+\bar{d}_0}{\left[\eta\bar{\eta}s_1+\bar{d}_0\right]\left[\eta\bar{\eta}s_2+\bar{d}_0\right]}\right\}$$

(4.3.16)

$$+\Bigg\{\frac{\left(Q^2+Q'^2-\bar{Q}^2-\tilde{Q}^2\right)\left[(\eta\bar{\eta})^2(Q^2s-s_1s_2)s_1s_2+2\eta\bar{\eta}s_1s_2(s-s_1-s_2+Q^2)\bar{d}_0+\left(s_1+s_2\right)(s-s_1-s_2+Q^2)+s_1s_2\right)\bar{d}_0^2\Big]}{\left(\eta\bar{\eta}s_1+\bar{d}_0\right)\left(\eta\bar{\eta}s_2+\bar{d}_0\right)\left[\eta\bar{\eta}(Q^2s-s_1s_2)+\bar{d}_0(s-s_1-s_2+Q^2)\right]\left[\eta\bar{\eta}(Q^2Q'^2-\bar{Q}^2\tilde{Q}^2)+\bar{d}_0(Q^2+Q'^2-\bar{Q}^2-\tilde{Q}^2)\right]}$$

$$+\frac{\left(Q^2Q'^2-\bar{Q}^2\tilde{Q}^2\right)\left[(\eta\bar{\eta})^2\left(s_1s_2(s+Q^2)-Q^2s\left(s_1+s_2\right)\right)+2\eta\bar{\eta}\left(s_1s_2-Q^2s\right)\bar{d}_0+(s-s_1-s_2+Q^2)\bar{d}_0^2\right]}{\left(\eta\bar{\eta}s_1+\bar{d}_0\right)\left(\eta\bar{\eta}s_2+\bar{d}_0\right)\left[\eta\bar{\eta}(Q^2s-s_1s_2)+\bar{d}_0(s-s_1-s_2+Q^2)\right]\left[\eta\bar{\eta}(Q^2Q'^2-\bar{Q}^2\tilde{Q}^2)+\bar{d}_0(Q^2+Q'^2-\bar{Q}^2-\tilde{Q}^2)\right]}\Bigg\}$$

$$\times\log\left[\eta\bar{\eta}Q^2+\bar{d}_0\right]$$

$$+\left(\text{ditto } Q^2\leftrightarrow Q'^2\right)-\left(\text{ditto } Q^2\leftrightarrow\bar{Q}^2\right)-\text{ditto}\left(Q^2\leftrightarrow\tilde{Q}^2\right)\Bigg)$$

where

(4.3.17)

$$\begin{cases} a = \eta\bar{\eta}s+\bar{d}_0 \\ b = \bar{d}_0(s-s_1-s_2)-\eta\bar{\eta}s_1s_2 \end{cases}$$

To analyse the large Q^2, Q'^2, $\bar{Q}^2$, $\tilde{Q}^2$ behaviour it is perhaps useful to make note of one of the identities :

$$
(4.3.18)\quad
\begin{cases}
Q^2Q'^2 - \bar{Q}^2\hat{Q}^2 = (Q^2+Q'^2-\bar{Q}^2-\tilde{Q}^2)Q^2 - (Q^2-\hat{Q}^2)(Q^2-\bar{Q}^2) \\
\qquad = (Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)Q'^2 - (Q'^2-\hat{Q}^2)(Q'^2-\bar{Q}^2) \\
\qquad = (Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)\bar{Q}^2 - (\bar{Q}^2-Q'^2)(Q^2-\bar{Q}^2) \\
\qquad = (Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)\tilde{Q}^2 - (\tilde{Q}^2-Q'^2)(Q^2-\tilde{Q}^2)
\end{cases}
$$

We then observe that the coefficient of $\log\left\{\dfrac{\eta\bar{\eta}s+\bar{d}_0}{[\eta\bar{\eta}s_1+\bar{d}_0][\eta\bar{\eta}s_2+\bar{d}_0]}\right\}$

behaves smoothly as all the Q^2's are made equal whereas the coefficient of $\log Q^2$'s term will contain infinite discontinuities in that limit.

We now turn our attention to A_1 and show that it does not contribute anything in the physical region. If it did then it would lead to a change in the value of the cross-section evaluated in section 4.1. It turns out, with hindsight, to make use of the property of Spence functions, f, that for complex z, [14].

$$
(4.3.19)\qquad f\left(\frac{1}{z}\right) + f(z) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z)
$$

Making use of (4.3.8) and (4.3.19) some more lengthy and tedious algebra allows us to write A_1 from (4.3.12) as

$$\frac{(ab'-a'b)^2 A_1}{\left[a(Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)-a'\right]}$$

$$= \log\left\{\frac{[\eta\bar{\eta}s+\bar{d}_0][\eta\bar{\eta}(Q^2Q'^2-\bar{Q}^2\hat{Q}^2)+(Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)\bar{I}_0]}{[\eta\bar{\eta}s_2+\bar{d}_0][\eta\bar{\eta}s_1+\bar{d}_0][Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2]}\right\} \times \log\left\{\frac{[s_1-Q^2][s_1-Q'^2]}{[s_1-\bar{Q}^2][s_1-\hat{Q}^2]}\right\}$$

$$+\left(f\left\{\frac{-[\eta\bar{\eta}(Q^2s-s_1s_2)+\bar{d}_0(s-s_1-s_2+Q^2)]}{[s_1-Q^2][\eta\bar{\eta}s+\bar{d}_0]}\right\}+\text{Ditto}(Q^2\leftrightarrow Q'^2)-\text{Ditto}(Q^2\leftrightarrow\bar{Q}^2)-\text{Ditto}(Q^2\leftrightarrow\hat{Q}^2)\right)$$

$$-\left(f\left\{\frac{-[\eta\bar{\eta}(Q^2s-s_1s_2)+\bar{d}_0(s-s_1-s_2+Q^2)]}{[s_1-Q^2][\eta\bar{\eta}s_2+\bar{d}_0]}\right\}+\text{Ditto}(Q^2\leftrightarrow Q'^2)-\text{Ditto}(Q^2\leftrightarrow\hat{Q}^2)-\text{Ditto}(Q^2\leftrightarrow\bar{Q}^2)\right)$$

(4.3.20)

$$-\left(f\left\{\frac{-[Q^2-\bar{Q}^2][Q'^2-\hat{Q}^2]}{[s_1-Q^2][Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2]}\right\}+\text{Ditto}(Q^2\leftrightarrow Q'^2)\right)$$

$$+\left(f\left\{-\frac{[\hat{Q}^2-Q'^2][Q^2-\bar{Q}^2]}{[s_1-\hat{Q}^2][Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2]}+\text{Ditto}(\hat{Q}^2\rightarrow\bar{Q}^2)\right)\right.$$

$$+\left(f\left\{\frac{-[\eta\bar{\eta}s_1+\bar{d}_0][Q^2-\bar{Q}^2][Q'^2-\hat{Q}^2]}{[s_1-Q^2][\eta\bar{\eta}(Q^2Q'^2-\bar{Q}^2\hat{Q}^2)+(Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)\bar{I}_0]}\right\}+\text{Ditto}(Q^2\leftrightarrow Q'^2)\right)$$

$$-\left(f\left\{\frac{-[\eta\bar{\eta}s_1+\bar{d}_0][\hat{Q}^2-Q'^2][Q^2-\hat{Q}^2]}{[s_1-Q^2][\eta\bar{\eta}(Q^2Q'^2-\bar{Q}^2\hat{Q}^2)+(Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)\bar{I}_0]}\right\}+\text{Ditto}(\hat{Q}^2\leftrightarrow\bar{Q}^2)\right)$$

In the physical region we can write

(4.3.21)

$$\begin{cases}\eta\bar{\eta}(Q^2Q'^2-\bar{Q}^2\hat{Q}^2)+(Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)\bar{d}_0 \\ =(Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)(\eta\bar{\eta}Q^2+\bar{d}_0)+\eta\bar{\eta}(Q^2-\hat{Q}^2)(Q^2-\bar{Q}^2) \\ \sim (Q^2+Q'^2-\bar{Q}^2-\hat{Q}^2)(\eta\bar{\eta}Q^2+\bar{d}_0)\end{cases}$$

where we have made use of (4.3.18). We see, using (4.3.21), that the arguments of BOTH logarithmic terms in (4.3.20) are of order unity.

Hence, up to corrections of O(1/u) these will not contribute to the cross-section.

The arguments of the Spence functions are either zero or of order $1/S$. It is proved in [14] that

$$(4.3.22) \qquad f(re^{i\theta}) = \frac{1}{2}\int_0^{\pi} -\frac{\log(1-2\rho\cos\theta+\rho^2)}{\rho}d\rho + i\int_0^{\pi}\arctan\left\{\frac{\rho\sin\theta}{1-\cos\theta}\right\}\frac{d\rho}{\rho}$$

from which it follows that the imaginary part tends to zero as $r\to 0$. Thus in the physical region the Spence functions in (4.3.20) have negligible contributions.

Hence insofar as continuations to the physical region are concerned A given by (4.3.11) is effectively

$$(4.3.23) \qquad A = \int_0^1 d\eta d\bar{\eta}\,\eta\bar{\eta}\,\delta(1-\eta-\bar{\eta})\,A_2$$

where A_2 is given by (4.3.16).

CHAPTER 5

The Analytic Structure of the General Ladder Diagram

This chapter consists of two sections. In the first one we show that the analytic structure of the general ladder graph is identical to that of the one-loop diagram. However, a difficulty not present at the one-loop level will emerge. In the second section we show that the pinch condition leads to factorization into electroproduction structure functions.

5.1 The analytic structure of the general ladder graph

Fig 5.1.1 shows the general ladder graph

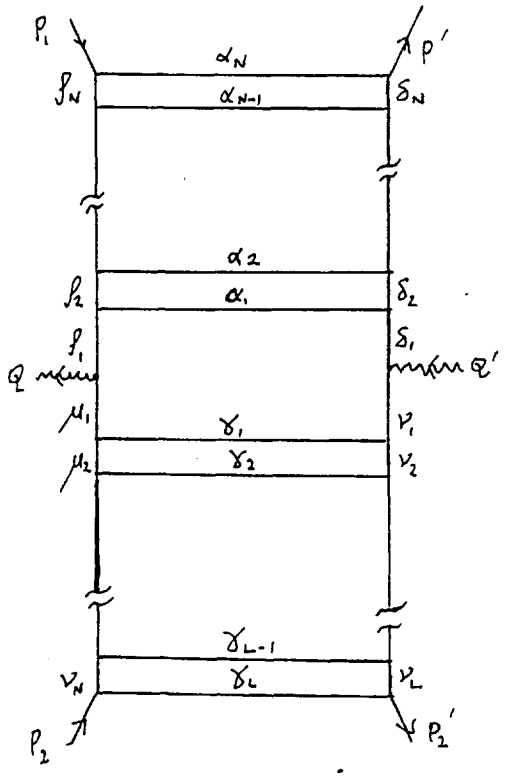


Fig 5.1.1 The general Drell-Yan ladder

The amplitude A is given by (2.3.7),

$$(5.1.1) \qquad A = \int_0^1 \frac{(d\xi)\,\delta(1-\Sigma\xi)\,C^n}{D^{n+2}}$$

where $n = p - 2L - 2$

p = no. of internal lines

L = no. of independent loops

(ξ) represents collectively all the Feynman parameters (indicated by Greek letters in Fig 5.1.1) of the graph

The D and C functions as we have seen from (3.2.1) and (3.2.2) are given by

$$(5.1.2) \qquad D = f_u f_L s + f_u g_L s_1 + f_L g_u s_2 + g_u g_L Q^2 + d_0$$

$$(5.1.3) \qquad C = (f_u + g_u)\,C(\gamma_1 \ldots \gamma_L) + (f_L + g_L)\,C(\alpha_1 \ldots \alpha_N)$$

f_u, f_L, g_u and g_L are represented by the cuts of fig 3.2.1

At this stage we are neglecting all the distinction between left and right hand invariants of equal magnitude. For undressed ladder graphs this distinction is irrelevant insofar as the s-discontinuity is concerned.

We define the amplitude $\bar{A}(\lambda)$ by

$$(5.1.4) \qquad \bar{A}(\lambda) = \int_0^1 \frac{(d\xi)\,\delta(1-\Sigma\xi)\,C^n}{[D+\lambda]^2}$$

Thus from (5.1.1) we have

$$(5.1.5) \qquad A = \frac{(-1)^{n+1}}{(n+1)!} \frac{\partial^n \bar{A}(\lambda)}{\partial \lambda^n}\bigg|_{\lambda=0}$$

We need to know the explicit form of f_u, f_L, g_u, g_L to make progress. These are obtained from the cutting rules. They allow us to write

$$f_u = \prod_{i=1}^{N} \alpha_i = \alpha_N \prod_{i=1}^{N-1} \alpha_i \equiv \alpha_N f_{1\ldots N-1} \tag{5.1.6}$$

$$f_L = \prod_{j=1}^{L} \gamma_j = \gamma_L \prod_{j=1}^{L-1} \gamma_j \equiv \gamma_L f_{1\ldots L-1} \tag{5.1.7}$$

$$g_u = (\rho_1+\delta_1) C(\alpha_1 \ldots \alpha_N) + \alpha_1(\rho_2+\delta_2) C(\alpha_2 \ldots \alpha_N) + \ldots \tag{5.1.8}$$

$$\ldots + \alpha_1\alpha_2\ldots\alpha_{N-2}(\rho_{N-1}+\delta_{N-1}) C(\alpha_{N-1}\alpha_N) + \alpha_1\alpha_2\ldots\alpha_{N-1}(\rho_N+\delta_N)$$

$$= \sum_{i=1}^{N-1} \left(\prod_{j=0}^{i-1} \alpha_j\right)(\rho_i+\delta_i) C(\alpha_i \ldots \alpha_N) + \left(\prod_{j=1}^{N-1} \alpha_j\right)(\rho_N+\delta_N) \tag{5.1.9}$$

with

$$\alpha_0 = 1 \tag{5.1.10}$$

$C(\alpha_i \ldots \alpha_N)$ is the determinant of the graph from the α_i^{th} to α_N^{th} rung inclusive. Using the cutting rules we may write

$$C(\alpha_i \ldots \alpha_N) = (\alpha_N+\rho_N+\delta_N) C(\alpha_i \ldots \alpha_{N-1}) + f_{i\ldots N-1} + \alpha_{N-1} g_{(N-2)\ldots i} \tag{5.1.11}$$

where $f_{i\ldots.(N-1)}$ and $g_{(N-2)\ldots.i}$ are defined by the cuts of fig 5.1.2

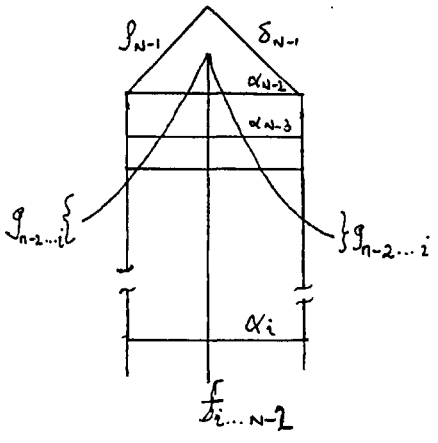


Fig 5.1.2 Diagram defining the $f_{i..(N-2)}$ and $g_{(N-2)..i}$ cuts.

$f_{i\ldots N-1} = \alpha_{N-1} f_{i\ldots N-2}$

The reason for writing $C(\alpha_i \cdots \alpha_N)$ in the form given by (5.1.11) is to show explicitly ALL of its α_N, β_N and δ_N dependences. It will become apparent later why we want to do this.

Using (5.1.11) we can re-write (5.1.9) as

$$(5.1.12)\quad g_u = \sum_{i=1}^{N-1} \left(\prod_{j=0}^{i-1} \alpha_j\right)(\beta_i+\delta_i)\left[(\alpha_N+\beta_N+\delta_N)\,C(\alpha_i \cdots \alpha_{N-1}) + f_{i \cdots N-1} + \alpha_{N-1}\, g_{(N-2)\cdots i}\right] + \left(\prod_{j=1}^{N-1} \alpha_j\right)(\beta_N+\delta_N)$$

Similarly we have

$$(5.1.13)\quad g_L = \sum_{i=1}^{L-1} \left(\prod_{j=0}^{i-1} \gamma_j\right)(\mu_i+\nu_i)\left[(\gamma_L+\mu_L+\nu_L)\,C(\gamma_i \cdots \gamma_{L-1}) + f_{i \cdots L-1} + \gamma_{L-1}\, g_{(L-2)\cdots i}\right] + \left(\prod_{j=1}^{L-1} \gamma_j\right)(\mu_L+\nu_L)$$

where ALL the γ_L, μ_L and ν_L dependences have been explicitly demonstrated

In (5.1.4) we scale α_N, β_N and δ_N by η and γ_L, μ_L and ν_L by $\bar{\eta}$ and we see from (5.1.12)

$$(5.1.14)\quad g_u \longrightarrow g_u(\eta) = g_u'(\eta) + \eta f_{1 \cdots N-1}\,(1-\alpha_N)$$

where

$$(5.1.15)\quad g_u'(\eta) = \sum_{i=1}^{N-1} \left(\prod_{j=0}^{i-1} \alpha_j\right)(\beta_i+\delta_i)\left[\eta\, C(\alpha_i \cdots \alpha_{N-1}) + f_{i \cdots N-1} + \alpha_{N-1}\, g_{(N-2)\cdots i}\right]$$

$$(5.1.16)\quad f_u \longrightarrow \eta\, \alpha_N\, f_{1 \cdots N-1}$$

$$(5.1.17)\quad C(\xi) \longrightarrow C(\eta, \bar{\eta}, \xi)$$

Analogous expressions exist for g_L and f_L. To arrive at the above we

have made use of the fact that after scaling $\alpha_N+\beta_N+\delta_N = 1 = \gamma_L+\mu_L+\nu_L$ The set $(\alpha_N, \beta_N, \delta_N)$ is unique in that it is the only one among $(\alpha_i, \beta_i, \delta_i)$ which leaves us with a linear dependence in α_i after scaling. The rest have a quadratic dependence. The same is true for (γ_L, μ_L, ν_L) when compared with (γ_i, μ_i, ν_i). The resultant pinch integral is then identical to the one-loop case. The difference between the general loop and the one-loop diagram, which will lead to a slight difficulty, is the presence of $g_U'(\eta)$ (and $g_L'(\bar{\eta})$) in (5.1.14). They are zero in the one-loop case. After scaling we can trivially do the β_N, δ_N, μ_L and ν_L integrals as they have disappeared from the D and C functions. We may then write (5.1.4) as

$$\bar{A}(\lambda) = \int_0^1 d\xi' \int d\eta\, d\bar{\eta}\, (\eta\bar{\eta})^2 \delta(1-\eta-\bar{\eta}-\Sigma\xi') \int_0^1 d\alpha_N d\gamma_L (1-\alpha_N)(1-\gamma_L) \times \frac{C^n(\xi', \eta, \bar{\eta}, \alpha_N, \gamma_L)}{\left[\left(\eta\alpha_N f_{1\ldots N-1} + g_U(\eta)S_2/S\right)\left(\bar{\eta}\gamma_L f_{1\ldots L-1} + g_L(\bar{\eta})S_1/S\right)S + g_U(\eta)g_L(\bar{\eta})(Q^2 - S_1S_2/S) + d_0(\eta,\bar{\eta},\alpha_N,\gamma_L,\xi')\right]^{\lambda}} \tag{5.1.18}$$

where (ξ') represents the unscaled Feynman parameters. The pinch conditions (3.3.6) now read

$$\begin{cases} x \equiv \eta\alpha_N f_{1\ldots N-1} + g_U(\eta)S_2/S = 0 \\ y \equiv \bar{\eta}\gamma_L f_{1\ldots L-1} + g_L(\bar{\eta})S_1/S = 0 \end{cases} \tag{5.1.19}$$

where $g_U(\eta)$ and $g_L(\bar{\eta})$ are given by (5.1.14) thus allowing us to solve for the pinch values $\bar{\alpha}_N$ and $\bar{\gamma}_L$ of α_N and γ_L respectively. These values of α_N and γ_L are substituted everywhere except where they appear as coefficients of the large variables or of $(Q^2 - S_1S_2/S)$

We now have our α_N and γ_L integrals of the form given by (4.1.14). Some elementary but tedious algebra eventually yields

$$\bar{A}(\lambda) \simeq \frac{1}{(1-s_1/s)(1-s_2/s)} \int_0^1 d\xi' d\eta d\bar{\eta}\, \delta(1-\eta-\bar{\eta}-\Sigma'\xi')\, \bar{C}^n(\xi',\eta,\bar{\eta},x=y=0)$$

$$\times \frac{[\eta f_{1\cdots N-1} + g_u'(\eta) s_2/s][\bar{\eta} f_{1\cdots L-1} + g_L'(\bar{\eta}) s_1/s]}{f^2_{1\cdots N-1}\, f^2_{1\cdots L-1}\, D} \log\left\{\frac{ab}{cd}\right\} \tag{5.1.20}$$

where

$$\begin{cases} a = [\eta f_{1\cdots N-1} + g_u'(\eta) s_2/s][\bar{\eta} f_{1\cdots L-1} + g_L'(\bar{\eta}) s_1/s] s + g_u'(\eta)\, g_L'(\bar{\eta})\,(Q^2 - s_1 s_2/s) + \bar{d}_0 + \lambda \\ b = Q^2 + \bar{d}_0 + \lambda \\ c = [\eta f_{1\cdots N-1} + g_u'(\eta) s_2/s] s_1 + g_u'(\eta)\,(Q^2 - s_1 s_2/s) + \bar{d}_0 + \lambda \\ d = [\bar{\eta} f_{1\cdots L-1} + g_L'(\bar{\eta}) s_1/s] s_2 + g_L'(\bar{\eta})\,(Q^2 - s_1 s_2/s) + \bar{d}_0 + \lambda \\ D = \{(g_L'(\bar{\eta}) + \bar{\eta} f_{1\cdots L-1})(g_u'(\eta) + \eta f_{1\cdots N-1})(Q^2 - s_1 s_2/s) + (1-s_1/s)(1-s_2/s)(\bar{d}_0 + \lambda)\} s \\ \qquad + (Q^2 - s_1 s_2/s)(\bar{d}_0 + \lambda) \end{cases} \tag{5.1.21}$$

We may neglect the λ-dependence in a, b, c, d in our asymptotic limit so that when A given by (5.1.5) is evaluated only D need be differentiated w.r.t. λ.

For the one-loop diagram we have

$$\begin{cases} g_u'(\eta) = g_L'(\bar{\eta}) = 0 \\ f_{1\cdots N-1} = f_{1\cdots L-1} = 1 \end{cases} \tag{5.1.22}$$

and we are left with only the $\eta, \bar{\eta}$-integrals after having done the

pinch. Then (5.1.20) reduces to (4.1.16).

It will turn out to be useful to write a, b, c, d in (5.1.21) in a different form.

$$(5.1.23)\quad \begin{cases} a = \eta\bar{\eta} f_{1\ldots N-1} f_{1\ldots L-1} s + \eta f_{1\ldots N-1} g_L'(\bar{\eta}) s_1 + \bar{\eta} f_{1\ldots L-1} g_u'(\eta) s_2 + g_u'(\eta) g_L'(\bar{\eta})\, Q^2 \\ b = Q^2 \\ c = \eta f_{1\ldots L-1} s_1 + g_u'(\eta)\, Q^2 \\ d = \bar{\eta} f_{1\ldots L-1} s_2 + g_L'(\bar{\eta})\, Q^2 \end{cases}$$

where we have set $\hat{\lambda} = 0$ and neglected the finite $\bar{d}_0$ term.

We now wish to calculate the phase picked up by the logarithmic term in (5.1.20) as we continue to the physical region. This is not as trivial as the one-loop case where the logarithmic term behaved as $\log\{sQ^2/s_1s_2\}$. Recall that to calculate the cross-section we essentially want to know

$$(5.1.24)\quad \Delta_s\{\bar{A}\} = \bar{A}(\lambda=0, -s_1, -s_2, -se^{-i\pi}, -Q^2e^{2-i\pi}) - \bar{A}(\lambda=0, -s_1, -s_2, -se^{i\pi}, -Q^2e^{2-i\pi})$$

When we continue - $s \to -se^{i\pi}$ and $-Q^2 \longrightarrow -Q^2e^{-i\pi}$ we can ALWAYS keep $Q^2 - s_1s_2/s$ small by continuing s and Q^2 simultaneously. In (5.1.21) we may, therefore, neglect the $Q^2 - s_1s_2/s$ term and we see that our logarithmic term does behave like $\log Q^2s/s_1s_2$. This term is phaseless for the continuation $s \to se^{i\pi}$, $-Q^2 \longrightarrow -Q^2e^{-i\pi}$. This is NOT manifestly obvious when we write a, b, c, d in the form (5.1.23) whose usefulness will become apparent shortly.

When we continue - $s \to - se^{-i\pi}$ and $-Q^2 \longrightarrow - Q^2 e^{-i\pi}$ we CANNOT neglect $Q^2 - s_1 s_2/s$ as it becomes large during the continuation. Fortunately we can determine the phase picked up by the log by observing from (5.1.23) that:

(i) the phase of b is always given by that of Q^2.

(ii) (a) if $\eta f_{1\ldots N-1} |s_1| > g_u'(\eta) |Q^2|$

(b) if $\eta f_{1\ldots N-1} |s_1| < g_u'(\eta) |Q^2|$

then the phase of c is given by that of s_1 if (a) or that of Q^2 if (b) is satisfied.

(iii) (a) if $\bar{\eta} f_{1\ldots L-1} |s_2| > g_L'(\bar{\eta}) |Q^2|$

(b) if $\bar{\eta} f_{1\ldots L-1} |s_2| < g_L'(\bar{\eta}) |Q^2|$

then the phase of d is given by that of s_2 if (a) or that of Q^2 if (b) is satisfied

(iv) Since we know that for the continuation - $s \to - se^{i\pi}$ and - $Q^2 \longrightarrow -Q^2 e^{-i\pi}$ the logarithmic term is phaseless we can determine what phase a picks up in the four regions defined by (ii)(a), (ii)(b), (iii)(a) and (iii)(b)

The four regions, and the phase of a in each region, are:

(1)

$$\begin{cases} \eta f_{1\ldots N-1} |s_1| > g_u'(\eta) |Q^2| \\ \bar{\eta} f_{1\ldots N-1} |s_2| > g_L'(\bar{\eta}) |Q^2| \end{cases} \tag{5.1.25}$$

Then b/cd picks up a phase of $(-i\pi)$. Hence a must pick up a phase of $(i\pi)$, i.e. the phase of a is being determined by that of s.

(2)

$$\begin{cases} \eta f_{1\ldots N-1} |s_1| > g_u'(\eta) |Q^2| \\ \bar{\eta} f_{1\ldots L-1} |s_2| < g_L'(\bar{\eta}) |Q^2| \end{cases} \tag{5.1.26}$$

then b/cd is phaseless and so must be a. Thus the phase of a is being determined by $\{\eta f_{i..N-1}\, g_L'(\bar{\eta})s_1 + \bar{\eta} f_{i..(L-1)}\, g_u'(\eta)\, s_2\}$ which is <u>not</u> being continued.

(3)

$$(5.1.27)\qquad \begin{cases} \eta f_{1\ldots N-1}\,|s_1| < g_u'(\eta)\,|Q^2| \\ \bar{\eta} f_{1\ldots L-1}\,|s_2| > g_L'(\bar{\eta})\,|Q^2| \end{cases}$$

The argument is identical to that in (2).

(4)

$$(5.1.28)\qquad \begin{cases} \eta f_{1\ldots N-1}\,|s_1| < g_u'(\eta)\,|Q^2| \\ \bar{\eta} f_{1\ldots L-1}\,|s_2| < g_L'(\bar{\eta})\,|Q^2| \end{cases}$$

then b/cd picks up a phase of $(-i\pi)$. Hence a must pick up a phase of $i\pi$, i.e. the phase of a is being determined by that of Q^2.

Observe that all the s-dependence is in the a term and we see from the above that the only case the phase of a is being determined by s is when (5.1.25) is satisfied. It, therefore, follows that for the continuation $-s \to -se^{-i\pi}$, $-Q^2 \longrightarrow -Q^2 e^{-i\pi}$ the logarithm only picks up a phase of $(-2i\pi)$ provided (5.1.25) is satisfied.

We may, therefore, write (5.1.24) as

$$(5.1.29)\qquad \Delta_s\{\bar{A}(\lambda)\} = (-2i\pi)\int_0^1 \frac{(d\xi')d\eta d\bar{\eta}\,\delta(1-\eta-\bar{\eta}-\Sigma\xi')\,\bar{C}^{\wedge}(\xi',\eta,\bar{\eta},x=y=0)}{(1-s_1/s)(1-s_2/s)}$$
$$\times \frac{\theta(\eta f_{1\ldots N-1}|s_1| - g_u'(\eta)|Q^2|)\,\theta(\bar{\eta} f_{1\ldots L-1}|s_2| - g_L'(\bar{\eta})|Q^2|)\,[\eta f_{1\ldots N-1} + g_u'(\eta)s_2/s][\bar{\eta} f_{1\ldots L-1} + g_L'(\bar{\eta})s_1/s]}{f_{1\ldots N-1}^2\, f_{1\ldots L-1}^2}\cdot D$$

where the conditions (5.1.25) have been incorporated through the use of the θ -function defined by

$$\theta(x) = \left\{ \begin{array}{l} 1, \; x > 0 \\ 0, \; x < 0 \end{array} \right\} \tag{5.1.30}$$

We see that as in the one-loop case we have picked up an overall phase of $(-2i\pi)$, albeit with certain restrictions, in spite of the more complicated looking logarithmic term. Note that in the one-loop case the θ -functions do not impose any cut offs on the range of the integration variables. We will see in the next section that it is precisely these restrictions that will allow us to write $\Delta_S\{A\}$ in a factorized form. Otherwise the analytic structure of the general ladder graph is precisely that of the one-loop diagram.

We will not be considering the case of keeping any of the Q^2's distinct as we do not believe any new features other than those that have appeared at the one-loop level will emerge.

5.2 Factorization of the cross-section into electroproduction structure functions

Using (5.1.5) we get from (5.1.29)

$$\Delta_S\{A\} \propto \frac{[S-S_1-S_2+Q^2]^\eta}{(1-S_1/S)(1-S_2/S)} \int_0^1 d\xi' \, d\eta \, d\bar{\eta} \, \delta(1-\eta-\bar{\eta}) \, C^\eta(\xi', \eta, \bar{\eta}, x=y=0)$$

$$\times \frac{\theta(\eta f_{\ldots N-1} - g_u'(\eta)|Q_1^2|/|S_{21}|)\,\theta(\bar{\eta} f_{\ldots L-1} - g_L'(\bar{\eta})|Q_1^2|/|S_{21}|)\,[\eta f_{\ldots N-1} + g_u'(\eta)\,S_2/S][\bar{\eta} f_{\ldots L-1} + g_L'(\bar{\eta})\,S_1/S]}{f_{\ldots N-1}^2 \, f_{\ldots L-1}^2 \, D^{\eta+1}}\Big|_{\lambda=0} \tag{5.2.1}$$

D is defined by (5.1.21). The above may be written in the following neat form

$$(5.2.2)\quad \Delta_s\{A\} \propto \frac{1}{s}\int_0^1 \frac{(d\xi)\, C^n(\xi)\, \delta(1-\Sigma\xi)\, \delta(f_u+g_u Q^2/s_1)\,\delta(f_L+g_L Q^2/s_2)}{D^{n+1}}$$

where we have re-introduced all the Feynmam parameters present before scaling and extracted a large s term from D as defined by (5.1.21). The proof of (5.2.2) is best seen by scaling $\alpha_N, \beta_N, \delta_N$ and γ_L, μ_L, ν_L in (5.2.2). We then do the β_N , δ_N , μ_L , ν_L integrals trivially picking up $(1-\alpha_N)(1-\gamma_L)$ in the numerator. We then do the α_L and γ_L integrals through the δ -functions thus recovering the θ -functions which appear in (5.2.1). These cut offs will be obtained from the upper limit of the α_L and γ_L integrals. All the other factors are reproduced faithfully. D in (5.2.2) is given by

$$(5.2.3)\qquad D \simeq g_u g_L (Q^2 - s_1 s_2/s) + d_0$$

where a term $(Q^2 - s_1 s_2/s) d_0/s$ has been neglected. All our factors are now as defined in (5.1.2) and (5.1.3). Summarising we have

$$(5.2.4)\quad \Delta_s\{A\} \propto \frac{1}{s}\int_0^1 \frac{(d\xi)\, C^n\, \delta(1-\Sigma\xi)\, \delta(f_u+g_u Q^2/s_1)\,\delta(f_L+g_L Q^2/s_2)}{\left[g_u g_L (Q^2 - s_1 s_2/s) + d_0\right]^{n+1}}$$

We would like to stress that the representation (5.2.2) is only valid in the physical region where

$$(5.2.5)\qquad \frac{Q^2}{s_1} \simeq \frac{s_2}{s} < 0$$

From the kinematics, (3.1.11), we have

(5.2.6) $$\frac{Q^2}{S_1} \simeq \frac{-x_1}{1-x_1} \quad , \quad \frac{Q^2}{S_2} \simeq \frac{-x_2}{1-x_2} \; .$$

From the δ-function constraints in (5.2.4) we have using (5.2.6)

(5.2.7) $$x_1 \simeq \frac{f_u}{f_u+g_u} \quad , \quad x_2 \simeq \frac{f_L}{f_L+g_L}$$

Thus from (3.1.12) it follows that

(5.2.8) $$Q^2 - \frac{S_1 S_2}{S} \simeq \frac{(f_u+g_u)(f_L+g_L)}{g_u g_L} \left\{ \frac{f_u g_u}{(f_u+g_u)^2} M_1^2 + \frac{f_L g_L}{(f_L+g_L)^2} M_2^2 - \Delta^2 \right\}$$

which we use to prove that (from (5.2.4))

(5.2.9) $$\int d^2\Lambda \, \Delta_S\{A\} \simeq \frac{\pi}{\eta S} \int_0^1 \frac{(d\xi)\, C^\eta \, \delta(f_u + g_u Q^2/S_1)\, \delta(f_L + g_L Q^2/S_2)}{(f_u+g_u)(f_L+g_L)\, D^\eta}$$

Resorting to the exponential representation (see (2.3.11))we may write (5.2.9) as

(5.2.10) $$\int d^2\Lambda \, \Delta_S\{A\} \propto \int_0^\infty \frac{(d\xi)\, \delta(f_u + g_u Q^2/S_1)\, \delta(f_L+g_L)\, e^{-D/C}}{(f_u+g_u)(f_L+g_L)}$$

where

(5.2.11) $$D = (f_u+g_u)(f_L+g_L) \left\{ \frac{f_u g_u}{(f_u+g_u)^2} M_1^2 + \frac{f_L g_L}{(f_L+g_L)^2} M_2^2 \right\} + d_0$$

We now interpret, in a natural way the upper and lower halves of the Drell-Yan diagram, fig 5.1.1, as the electroproduction diagrams shown in fig 5.2.1

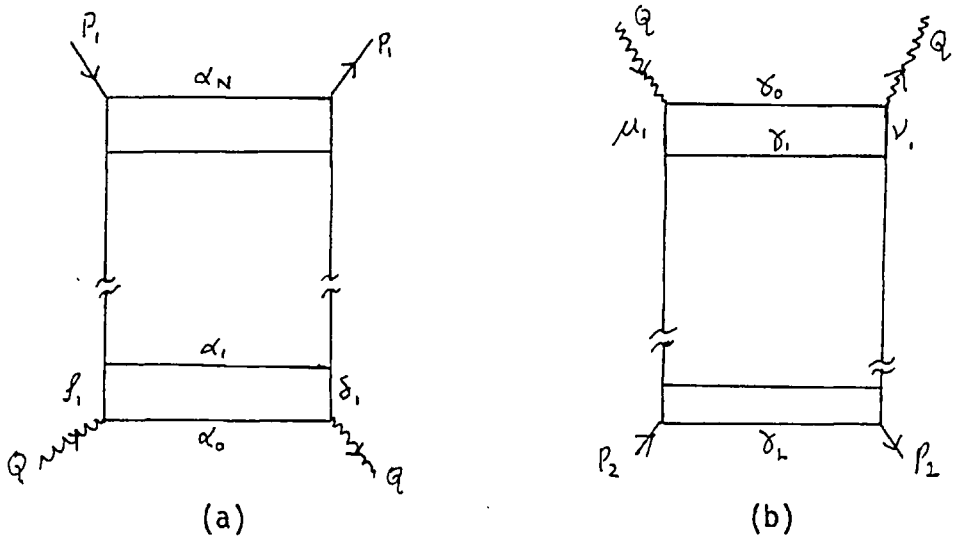


Fig 5.2.1: These two diagrams illustrate how we associate an electroproduction diagram to each half of the Drell-Yan diagram. α_0 and γ_0 are faked lines that have been introduced as we wish to use the results of appendix C.

We refer to the amplitude in fig 5.2.2(a) as $A^u_{e.p}$ and fig 5.2.2(b) as $A^L_{e.p}$ where U and L refer to upper and lower halves and e.p. stands for electroproduction in an obvious notation. Borrowing the results of Appendix C we write

$$(5.2.12)\quad \Delta_{s_1}\{A^u_{e.p}\} = \frac{1}{s_1}\int_0^\infty (d\xi_u)\,\delta(f_u + g_u Q^2/s_1)\,\frac{e}{C_u}\Big|^{-d^u_{e.p.}/C_u}_{\alpha_0=0} \equiv \frac{1}{s_1}F(x_1)$$

$$(5.2.13)\quad \Delta_{s_2}\{A^L_{e.p.}\} = \frac{1}{s_2}\int_0^\infty (d\xi_L)\,\delta(f_L + g_L Q^2/s_2)\,\frac{e}{C_L}\Big|^{-d^L_{e.p.}/C_L}_{\gamma_0=0} \equiv \frac{1}{s_2}F(x_2)$$

$$(5.2.14)\quad \begin{cases} C_u|_{\alpha_0=0} = f_u + g_u \\ C_L|_{\gamma_0=0} = f_L + g_L \end{cases}$$

where f_u, g_u, f_L, g_L, are identical to those that appear in (5.2.10). $F(x_1)$ and $F(x_2)$ are obviously defined.

We prove in Appendix D that

$$\frac{D}{C} = \frac{d^{u}_{e.p}}{C_u}\bigg|_{\alpha_0=0} + \frac{d^{L}_{e.p}}{C_L}\bigg|_{\gamma_0=0} \tag{5.2.15}$$

So using the above results in (5.2.10) we immediately obtain factorization, namely,

$$\int d^2\Lambda\ \Delta_s\{A\} \propto F(x_1)\,F(x_2) \tag{5.2.16}$$

CHAPTER 6

CONCLUSIONS AND DISCUSSIONS

Two features absent in electroproduction have emerged in this analytic study of the Drell-Yan amplitude: the totally non-uniform behaviour of the amplitude as a function of the invariants and the presence of infinite Q^2 - discontinuities.

The first of these should not be totally surprising as we have seen the dominant contributions to the amplitude is due to a pinch in Feynman parameter space. The pinch only occurs when the invariants take on some very special values. Infinitessimally away from these points the pinch term behaves like $((1/s)^4$ or $(1/s)^2$(for the one-loop diagram). The dominant contribution to the amplitude then comes from the end-point singularities of the Feynman parameters. This results in an asymptotic $1/s^3$ behaviour.

We now turn our attention to the existence of the infinite Q^2-discontinuities and consider their physical interpretation. One of the Q^2-discontinuities corresponds to cutting the graph as shown in fig 6.1 Because we are studying the forward scattering amplitude putting β and μ on man-shell automatically forces δ, ν to be on-shell. The

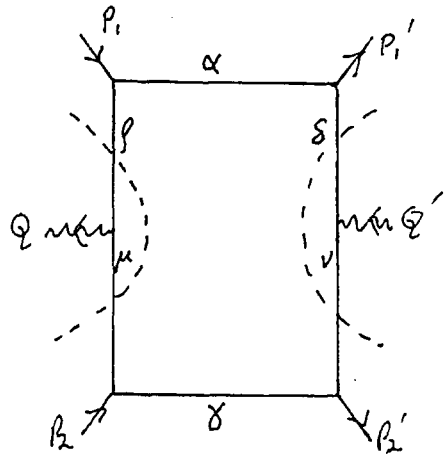


Fig 6.1 A Q^2-discontinuity of the one-loop diagram

infinity coming from the $1/(Q^2 - Q'^2)$ term is related to the fact that in this kinematical configuration β, μ, δ, ν, are on shell and can propagate for large times. Thus the distinct collisions A, B, C and D of fig 6.2

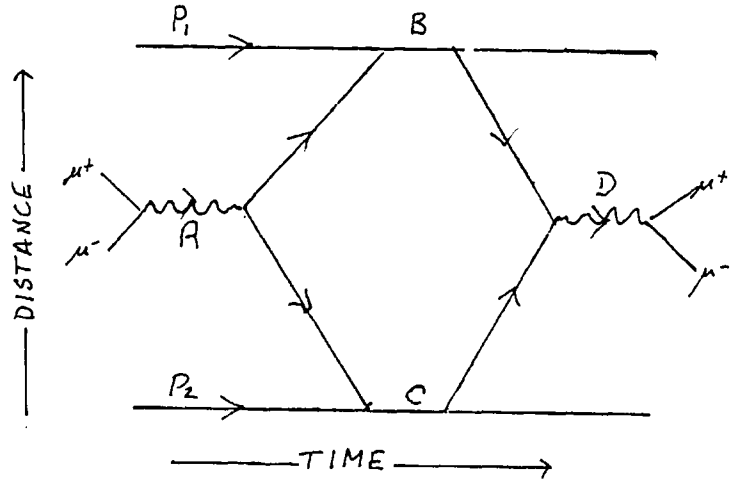


Fig 6.2 Space-Time Structure of the Drell-Yan one-loop amplitude

may occur at large space and time separations. The existence of these infinite time scales hidden inside the analytic structure of our amplitude hardly increases our confidence in the use of the operator product expansion for this amplitude. We will come back to this point later. In electroproduction these infinite time and distance scales do not exist for consider the one-loop electroproduction diagram in fig 6.3. In electroproduction Q^2 is negative and thus we are always below the normal threshold of this

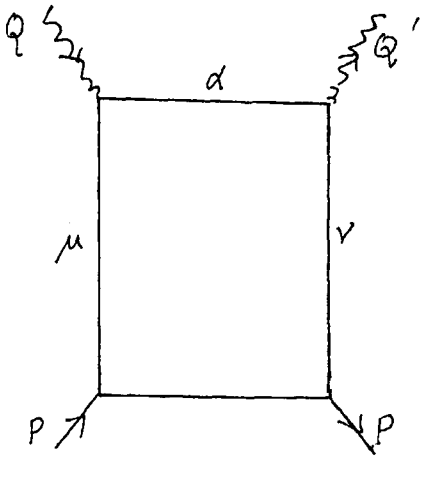


Fig 6.3 The one-loop 2→2 forward scattering amplitude.

invariant. We must look for other processes where Q^2 is large and positive. This is to be found in e^+e^- annihilation. The total cross-section for this process has been analysed by an OPE treatment but there the amplitude of interest is the $1 \longrightarrow 1$ forward scattering of fig 6.4

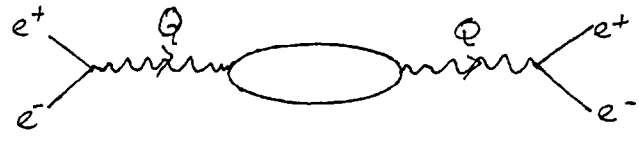


Fig 6.4 $1 \longrightarrow 1$ forward scattering in total e^+e^- annihilation cross-section

and obviously the distinction between Q^2-left and Q^2-right is irrelevant as we are interested in the discontinuity of this variable. However, the problem of infinite Q^2-discontinuities arises in the one-particle inclusive cross-section for

$$e^+e^- \longrightarrow h + \text{anything}$$

The relevant amplitude for this process is fig 6.3. Now we must keep the distinction between Q^2 and Q'^2 and we see that in the forward direction when μ goes on-shell so does ν . This will give rise to a $1/(Q^2 - Q'^2)$ term. Unfortunately no OPE treatment of the process exists.

The hadronic tensor, $W^{D.Y.}_{\mu\nu}$, for the Drell-Yan process [3] is essentially

$$(6.1)\quad W^{D.Y.}_{\mu\nu} \equiv \sum_n \int d^4q\, \delta(q^2-Q^2) \int d^4x\, e^{-iq\cdot x}\, \delta^{(4)}(P_1+P_2-q-P_n)\, \langle P_1P_2^{(in)}|J_\mu(0)|n\rangle\langle n|J_\nu(0)|P_1P_2^{(in)}\rangle$$

$$= \int d^4q\, \delta(q^2-Q^2) \int d^4x\, e^{-iq\cdot x}\, \langle P_1P_2^{(in)}|J_\mu(x)J_\nu(0)|P_1P_2^{(in)}\rangle$$

The second step follows from the completeness relation for the intermediate states $|n\rangle$. The diagram corresponding to (6.1) is shown in fig 6.5.

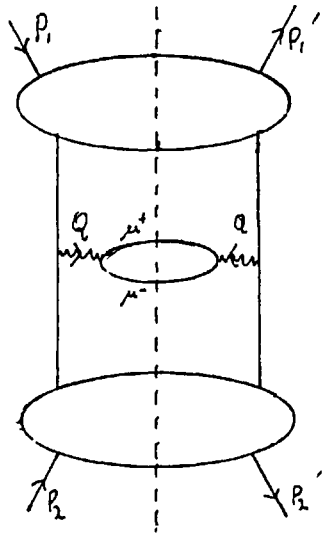


Fig 6.5 Diagramatic representation of $W_{\mu\nu}^{D.Y.}$. The dashed line represents the taking of the $s = (p_1 + p_2 - Q)^2$-discontinuity.

We are interested in an OPE for the current product $J_\mu(x)\, J_\nu(o)$ either as $x_\mu \longrightarrow 0$ or $x_\mu x^\mu \longrightarrow 0$. The analogous tensor for electroproduction [15] is

$$W_{\mu\nu}^{e.p.} = \sum_n \delta^{(4)}(p+q-p_n)\, \langle p|J_\mu(o)|n\rangle\langle n|J_\nu(o)|p\rangle = \int d^4x\, e^{iq\cdot x} \langle p|J_\mu(x) J_\nu(o)|p\rangle \tag{6.2}$$

Its diagramatic representation is fig 6.6, What allows progress to be made in electroproduction is the ability to replace the product $J_\mu(x)J_\nu(o)$ by the commutator $[J_\mu(x), J_\nu(o)]$. This is because the tensor $W_{\nu\mu}^{e.p.}$ corresponding to sandwiching $J_\nu(o)\, J_\mu(x)$ between the proton physical states gives zero as a result of the spectral properties of the states [15]. As we now have a commutator, we may by causality only consider values of x such that $x^2 \geqslant 0$. It is this constraint that allows us to prove that electroproduction is a light-done dominated

process.

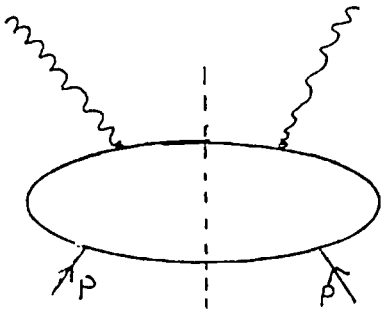


Fig 6.6 Diagramatic representation of $W_{\mu\nu}^{e.p.}$

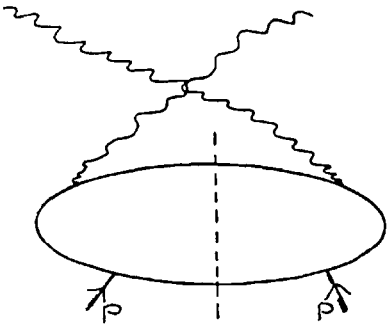


Fig 6.7 Diagramatic representation of $W_{\nu\mu}^{e.p.}$

For the Drell-Yan process this is not the case. If we wish to replace the product $J_\mu(x)\, J_\nu(0)$ by a commutator we must subtract from $W_{\mu\nu}^{D.Y.}$ the tensor $W_{\nu\mu}^{D.Y.}$ where

$$(6.3)\quad \left\{ \begin{aligned} W_{\nu\mu}^{D.Y.} &= \int d^4q\, \delta(q^2-Q^2) \int d^4x\, e^{-iq\cdot x} \langle P_1 P_2^{in} | J_\nu(0) J_\mu(x) | P_1 P_2^{in} \rangle \\ &= \sum_n \int d^4q\, \delta(q^2-Q^2)\, \delta^{(4)}(P_n - P_1 - P_2 - Q) \langle P_1 P_2^{in} | J_\nu(0) | n \rangle \langle n | J_\mu(0) | P_1 P_2^{in} \rangle \end{aligned} \right.$$

This term corresponds diagramatically to fig 6.8 and there is no reason why this should be zero.

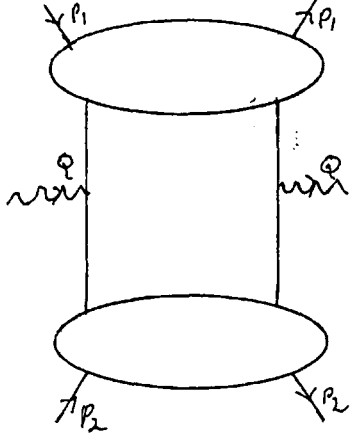


Fig 6.8 Diagramatic representation of $W_{\nu\mu}^{D.Y.}$

However, note the difference between fig 6.5 and fig 6.8. The former corresponds to taking the discontinuity in $s = (p_1 + p_2 - Q)^2$ of the $3 \longrightarrow 3$ forward amplitude where $s_1 = (p_1 - q)^2$, $s_2 = (p_2 - q)^2$ are negative and s, Q^2 are positive. The latter corresponds to taking the $s = (p_1 + p_2 + Q)^2$ - discontinuity of the $3 \longrightarrow 3$ forward amplitude where $s_1 = (p_1 + q)^2$, $s_2 = (p_2 + q)^2$, s, Q^2 are all positive. We know from our analysis of the $3 \longrightarrow 3$ amplitude that fig 6.8 cannot be dominated by a pinch whereas fig 6.5 is. Consequently fig 6.8 is down by a factor $1/u^3$ compared to 6.5 <u>at least in our φ^3-model and considering only bare ladder diagrams</u>. Hence to a good approximation we may write (6.1) as

$$(6.4) \qquad W_{\mu\nu}^{DY.} \simeq \int d^4q\, \delta(q^2 - Q^2) \int d^4x\, e^{-iq\cdot x} \langle P_1 P_2^{(in)} | [J_\mu(x), J_\nu(0)] | P_1 P_2^{(in)} \rangle$$

Since $q^2 > 0$ we may go to the frame $q = (q^0, \underset{\sim}{0})$ to do the x-integral. There we may expect that the integral would be dominated by

$$x_0 \sim \frac{1}{q_0} \longrightarrow 0$$

in the asymptotic large Q^2 limit. Moreover since we must have

$$x^2 = x_0^2 - \underset{\sim}{x}^2 > 0$$

by the causality property of the commutator, then

$$x^2 \leq x_0^2 \longrightarrow \frac{1}{q_0^2} \longrightarrow 0$$

Hence $\underset{\sim}{x} \to 0$. Thus we expect the contribution to (6.4) to come mainly from the tip of the light cone $x^\mu \longrightarrow 0$, i.e. a short-distance dominated process, as in total e^+e^- annihilation cross-section [15].

But this is as far as we can go. We are now faced with two major

difficulties. The first was pointed out by Drell and Yan [3]. The limite $q^\mu \to \infty$ with fixed p_1 and p_2 in (6.1) corresponds to $Q^2/u \to \infty$ and this is very far removed from the experimental regime of study and interest. But it is just this region of asymptotically large Q^2/u that is probed by taking $q^\mu \to \infty$ and expanding the matrix element about the origin in a power series of $x^\mu x_\mu$ and $x\cdot(p_1 + p_2)$. Somehow one has to relate the $Q^2/u \to \infty$ limit to the $Q^2/u \to$ finite limit. In the former region the pinch term is absent and it is very difficult to see how these two regions can be bridged by some form of analytic continuation like dispersion techniques. This is the second problem.

Even if this were successful somehow it seems impossible to reconcile the existence of large time and distance scales with a short distance expansion of the matrix element. But an observation is in line here. If we go back to the cases where we kept the Q^2's distinct we note that the infinite Q^2-discontinuties separate out in an additive manner from the s-dependant terms yielding the cross-section. Although a full short-distance expansion of the matrix elements seems impossible a partial one may be successful to the $e^{i(p_1 + p_2)x}$ term of $\langle p_1 p_2^{(in)} | [J_\mu(x), J_\nu(0)] | p_1 p_2^{(in)} \rangle$ which leads to the s-dependance of the tensor.

APPENDIX A: Feynman Rules in φ^3 [8]

Corresponding to a given Feynman diagram there is a contribution to the S-matrix given by the following rules: It is an integral taken over the four momentum of each internal line in the diagram, these four-momenta being labelled independantly in the first instance.

The integrand contains the following factors

(1) $\frac{i}{(2\pi)^4} \frac{1}{k^2 - \mu^2 + i\varepsilon}$ for each internal line of four-momentum k

(2) 1 for each external line

(3) $(-i)^n$ corresponding to the diagram being n^{th} order in the perturbation expansion

(4) $(2\pi)^4 \delta^{(4)}(p - p' + k)$ for each vertex so as to give energy-momentum conservation for each line p entering and p' leaving and k entering (or -k leaving) the vertex.

The S-matrix element for a given process is the sum of all the Feynman integrals allowed for that process

APPENDIX B: Cutting rules for graphs in ϕ^3 [8]

We first give the rules for writing down the function $C(\alpha)$. A set of cuts is made in the internal lines such that: (i) every vertex is still connected to every other vertex by a sequence of uncut lines; (ii) no further cuts can be made without violating (i). The set of cuts corresponds to a contribution to C that is the product of the α_i corresponding to the cut lines. The complete expression for C arises from adding in the products of α for every possible set of cuts that obey (i) and (ii).

We now give the rules for writing down the D-function. This expression is a linear function of the various Lorentz scalars (but not all of them) that can be manufactured from the external momenta p. The rules are: By a set of cuts on the internal lines divide the graph into two disjoint parts such that within each part the previous rules (i) and (ii) are obeyed and such that at least one external momentum line is connected to each part. Any such set of cuts yield a product of α_i for the two disjoint parts exactly as before. These products are multiplied together and are also multiplied by the Lorentz scalar $(\Sigma' p)^2$ where $(\Sigma' p)$ denotes the sum of the external momenta coming into either of the disjoint parts (because of energy-momentum conservation, $(\Sigma' p)^2$ is the same for both parts). Do not forget the possibility that $(\Sigma' p)$ may include only the four momenta in one external line for an appropriate set of cuts. We are not including the contribution from the internal masses in the D-function. This can be obtained from (2.3.8)

APPENDIX C: The electroproduction structure function

There is nothing new in the following but we include it here for the sake of completeness, [16].

Consider the general ladder electroproduction graph in φ^3 of fig. C.1.

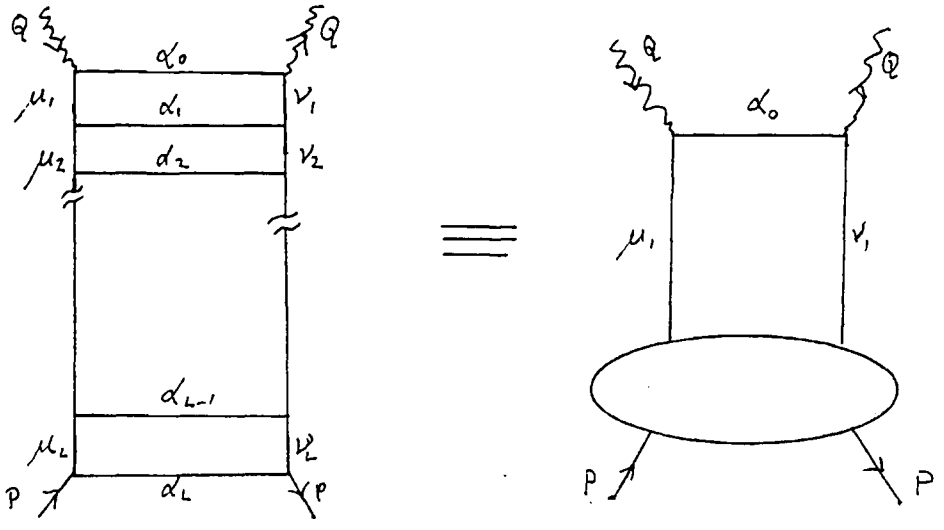


Fig C.1 The general ladder graph for electroproduction.
The right hand diagram is an obvious short hand rotation for the l.h. one.

Our Feynman amplitude will depend on the invariants Q^2 and $s = (p + Q)^2$. In the electroproduction physical region $Q^2 < 0$ and Q^2/s is fixed. We define the Feynman scaling variable x by

C.1 $$x = -Q^2/2p.Q$$

We will use the exponential representation for the amplitude which is

C.2 $$A(Q^2,s) \equiv \int_0^\infty \frac{(\pi d\xi)\, d\alpha_0\, e^{-D/C}}{C^2}$$

where

$$D = \alpha_0 f s + \alpha_0 G Q^2 + d_0$$

and C are given by the usual cutting rules. The cuts representing f and G are shown in fig C.2

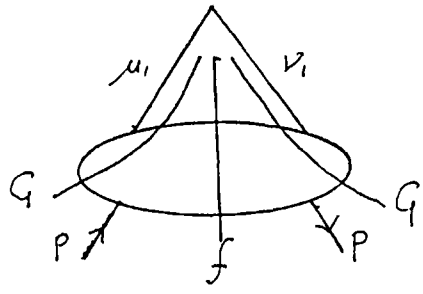


Fig C.2 Diagram illustrating the cuts f and G

To obtain the asymptotic behaviour of A we shall use the Mellin transform [17] technique. This is defined through the following pair of equations

C.3

$$F(\beta) = \int_0^\infty f(t)\, t^{-\beta-1}\, dt$$

$$f(t) = \frac{1}{2\pi i}\int_C F(\beta)\, t^{\beta}\, d\beta$$

where $F(\beta)$ is the Mellin transform of the function, f(t). The contour C which appears in the inversion formula is parallel to the imaginary axis and $F(\beta)$ is analytic along C.

We take Mellin transforms w.r.t. s and Q^2. Let their transform variables be j and k respectively. Then

C.4

$$\begin{cases} \tilde{\tilde{A}}(j,k) = \int_0^\infty ds\, dQ^2\, s^{-j-1}\,(Q^2)^{-k-1}\, A(s,Q^2) \\ A(s,Q^2) = \frac{1}{(2\pi i)^2}\int_{C_2} dk \int_{C_1} dj\; s^j\,(Q^2)^k\, \hat{\tilde{A}}(j,k) \end{cases}$$

where C_1 and C_2 are inversion contours defined in C3. Substituting for C.3 and C.2 in the top equation of C.4 we find

$$\text{C.5} \qquad \tilde{\tilde{A}}(j,k) = \Gamma(-j)\Gamma(-k)\int_0^\infty (\pi d\xi)\, d\alpha_0\, \alpha_0^{j+k} f^j g^k \frac{e^{-d_0/c}}{c^{j+k+2}}$$

Integrating by parts w.r.t. α_0 we get

$$\text{C.6} \qquad \tilde{\tilde{A}}(j,k) = -\frac{\Gamma(-j)\Gamma(-k)}{(j+k+1)}\int_0^\infty (\pi d\xi) d\alpha_0 .. \alpha_0^{j+k+1} f^j g^k \frac{\partial}{\partial \alpha_0}\left(\frac{e^{-d_0/c}}{c^{j+k+2}}\right)$$

The regions of analyticity in the j and k complex planes are

$$\text{C.7} \qquad \begin{cases} -1 < \mathrm{Re}\, j < 0 \\ -1 < \mathrm{Re}\, k < c \end{cases}$$

C.7 defines the regions where our inversion contours will lie.

We first of all take the inversion w.r.t. s and for large s the leading term comes from picking up the residue of the rightmost pole on the left of the inversion contour which is at

$$j = -k - 1$$

Hence

$$\text{C.8} \qquad \tilde{A}(s,k) \equiv \frac{1}{2\pi i}\int_{C_1} dj\, s^j \tilde{\tilde{A}}(j,k) \simeq -\Gamma(k+1)\Gamma(-k)\int_0^\infty (\pi d\xi) d\alpha_0 (fs)^{-k-1} g^k \frac{\partial}{\partial \alpha_0}\left[\frac{e^{-d_0/c}}{c}\right]$$

where we have set $j = -k-1$ everywhere except where it appears as the pole and as the exponent of s and where we have used the result

$$\text{C.9} \qquad \frac{1}{2\pi i}\int_C \frac{d\beta\, t^\beta}{(\beta-\beta_0)^n} = \frac{t^{\beta_0}(\log t)^{n-1}}{(n-1)!}$$

The inversion formula w.r.t. Q^2 gives us

$$A(s,Q^2) = \frac{1}{2\pi i}\int_{C_2} dk\,(Q^2)^k\,\hat{A}(s,k)$$

Hence, from C.8 we have

$$A(s,Q^2) \simeq -\frac{1}{2\pi i}\int_0^\infty \frac{(\pi d\xi)\,d\alpha_0}{f_s}\,\frac{\partial}{\partial\alpha_0}\left[\frac{e}{C}^{-d_0/C}\right]\int_{C_2} dk\,\left(\frac{sQ^2}{fs}\right)^k \Gamma(-k)\,\Gamma(k+1)$$

Since Q^2/s is fixed we are not allowed to pick up the pole at $k = -1$ to get the leading large behaviour. The k-integral may be done exactly through the formula:

C.10
$$\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(-s)\,\Gamma(\beta+s)\,t^s\,ds = \frac{2\pi i\,\Gamma(\beta)}{(1+t)^\beta}$$

$$0 > \gamma < \mathrm{Re}(1-\beta), \quad |\arg t| < \pi$$

Using C.10 and doing the α_0 integration we get

C.11
$$A(s,Q^2) \simeq \frac{1}{s}\int_0^\infty (\pi d\xi)\,\frac{1}{(f+gQ^2/s)}\,\frac{e}{C}^{-d_0/C}\Big|_{\alpha_0=0}$$

from which it follows that

C.12
$$\Delta_s\{A(s,Q^2)\} \simeq -\frac{2\pi i}{s}\int_0^\infty (\pi d\xi)\,\delta(f+gQ^2/s)\,\frac{e}{C}^{-d_0/C}\Big|_{\alpha_0=0}$$

$\Delta_s\{A(s,Q^2)\}$ is related by the optical theorem to the total forward compton scattering of $p + Q \to$ anything. $\Delta_s\{A(s,Q^2)\}$ is called the electroproduction structure function.

APPENDIX D: Essentially the proof of the factorization of ladder graphs.

The notation here is identical to that of Chapter 5.

We wish to prove that

D.1 $$\frac{D}{C} = \frac{d^{u}_{e.p}}{C_u}\Big|_{\alpha_0=0} + \frac{d^{L}_{e.p.}}{C_L}\Big|_{\gamma_0=0}$$

We have seen from Chapter 5 (see equation (5.2.11)) that

D.2 $$D = (f_u+g_u)(f_L+g_L)\left\{\frac{f_u g_u}{(f_u+g_u)^2}M_1^2 + \frac{f_L g_L}{(f_L+g_L)^2}M_2^2\right\} + d_0$$

d_0 may be written in the form

D.3 $$d_0 = d_u M_1^2 + d_L M_2^2 - C\left(\sum_u \xi_u + \sum_L \xi_L\right)m^2$$

M_1^2 and M_2^2 are the external masses and m is the mass of each internal line. $\sum_u \xi_u$, $\sum_L \xi_L$ represent sums over the all the Feynman parameter of the upper (U) and lower half (L) of the diagram where the photon vertex represents the dividing line between upper and lower halves of the diagram (fig.3.2.1) we have

D.4 $$C = (f_u+g_u)\,C(\gamma_1\ldots\gamma_L) + (f_L+g_L)\,C(\alpha_1\ldots\alpha_N)$$

D.5 (or 5.2.14)
$$\begin{cases} C_u\big|_{\alpha_0=0} = f_u + g_u \\ C_L\big|_{\gamma_0=0} = f_L + g_L \end{cases}$$

The cuts corresponding to f_u, f_L, g_u and g_L have been defined in fig. 3.2.1. $C_u\big|_{\alpha_0=0}$ and $C_L\big|_{\gamma_0=0}$, we recall (fig 5.2.1) are the determinants of the electroproduction graphs we associated with the general Drell-Yan

ladder graph. We can write

D.6
$$\begin{cases} d^{u}_{e.p.} = \tilde{d}^{u}_{e.p.} M_1^2 - C_u \left(\sum_u \xi_u\right) m^2 \\ d^{L}_{e.p} = \tilde{d}^{L}_{e.p.} M_2^2 - C_L \left(\sum_L \xi_L\right) m^2 \end{cases}$$

Insofar as the internal masses are concerned we see from D.2, D.3 and D.6 that they satisfy identity D.1. Moreover, we need only show that this identity is true for the term proportional to M_1^2. The proof for the term proportional to M_2^2 then follows.

Hence proving D.1 is now equivalent to proving

D.7
$$\tilde{d}^{u}_{e.p} \left.\frac{C}{C_u}\right|_{\alpha_0=0} = \frac{(f_L+g_L) f_u g_u}{f_u+g_u} + d_u$$

where we have also made use of D.1, D.2, and D.3. Substituting for C from D.4 and C_u from D.5 we can re-write D.7 as

D.8
$$\tilde{d}^{u}_{e.p}\Big|_{\alpha_0=0}\left[(f_u+g_u) C(\gamma_1 \ldots \gamma_L) + (f_L+g_L) C(\alpha_1 \ldots \alpha_N)\right] = (f_L+g_L) f_u g_u + d_u (f_u+g_u)$$

The cutting rules with reference to fig 5.1.1 allow us to write

D.9
$$d_u = (\rho_1+\delta_1) f_{1\ldots N} C(\gamma \ldots \gamma_L) + (\rho_2+\delta_2) f_{2\ldots N} C(\alpha_1 \ldots \gamma_L) + (\rho_3+\delta_3) f_{3\ldots N} C(\alpha_2 \ldots \gamma_L) + \ldots + (\rho_N+\delta_N) f_N C(\alpha_{N-1} \ldots \gamma_L)$$

where

D.10
$$f_{r\ldots s} \equiv \prod_{i=r}^{s} \alpha_r \quad , \quad r \geq 1$$

$C(\alpha_i \ldots \gamma_L)$ is the determinant of the graph from the α_i^{th} to the γ_L^{th}

rung inclusive. For clarity we depict the cuts corresponding to d_u in fig D.1

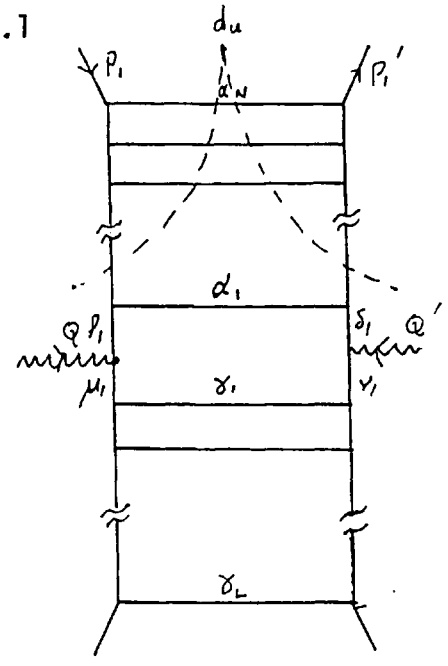


Fig D.1 The cuts corresponding to d_u

Using D.4 we have

D.11 $$C(\alpha_i \ldots \gamma_L) = (f_{1\ldots i} + g_{1\ldots i})\, C(\gamma_1 \ldots \gamma_L) + (f_L + g_L)\, C(\alpha_1 \ldots \alpha_i)$$

The cuts corresponding to $g_{1\ldots i}$ are depicted in fig D.2

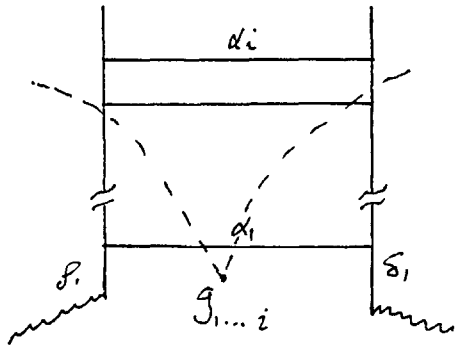


Fig D.2 The ucts corresponding to $g_{1\ldots i}$

We also have

D.12 $$\tilde{d}^{u}_{e.p}\Big|_{\alpha_0=0} = (p_1+\delta_1) f_{1\ldots N} + (p_2+\delta_2) f_{2\ldots N}\, C(\alpha_0 \alpha_1)\Big|_{\alpha_0=0} + (p_3+\delta_3) f_{3\ldots N}\, C(\alpha_0 \ldots \alpha_2)\Big|_{\alpha_0=0} + \ldots + (p_N+\delta_N) f_N\, C(\alpha_0 \ldots \alpha_{N-1})\Big|_{\alpha_0=0}$$

The cuts corresponding to $\tilde{d}^{u}_{e\text{-}p}$ are shown in Fig D.3

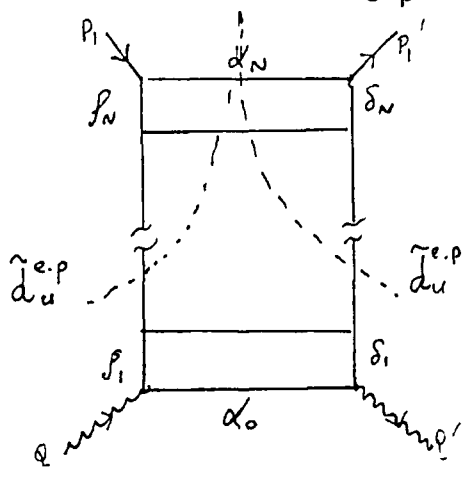


Fig D.3 Cuts corresponding to $\tilde{d}_u^{e.p}$

Using D.5 we have

D.13 $$C(\alpha_0 \ldots \alpha_i)\Big|_{\alpha_0=0} = f_{1\ldots i} + g_{1\ldots i}$$

where $f_{1\ldots i}$ and $g_{1\ldots i}$ have been defined by D.10 and fig D.2

Substituting for D.13 in D.12 we find

D.14 $$\tilde{d}^{u}_{e.p.}\Big|_{\alpha_0=0} = (\beta_1+\delta_1) f_{1\ldots N} + (\beta_2+\delta_2) f_{2\ldots N}(f_1+g_1) + (\beta_3+\delta_3) f_{3\ldots N}(f_{1.2.}+g_{1.2.}) + \ldots$$
$$\ldots + (\beta_N+\delta_N)\left(f_{1\ldots(N-1)} + g_{1\ldots(N-1)}\right)$$

Note that we have now reduced the proof of D.1 to proving D.8. Substitute for d_u from D.9 into the r.h.s. of D.8 and we find

r.h.s. of D.8

$$= (f_L+g_L) f_u g_u + (f_u+g_u)\Big\{(\beta_1+\delta_1) f_{1\ldots N} C(\gamma_1 \ldots \gamma_L) + (\beta_2+\delta_2) f_{2\ldots N} C(\alpha_1 \ldots \gamma_L) + (\beta_3+\delta_3) f_{3\ldots N} C(\alpha_2 \ldots \gamma_L) + \ldots$$
$$\ldots + (\beta_N+\delta_N) f_N C(\alpha_{N-1} \ldots \gamma_L)\Big\}$$

Using D.11 in the above we find

r.h.s. of D.8

$$= (f_L + g_L) f_u g_u$$

$$+ (f_u + g_u) \Big\{ (\rho_1 + \delta_1) f_{1\ldots N} C(\gamma_1 \ldots \gamma_L)$$

$$+ (\rho_2 + \delta_2) f_{2\ldots N} \left[(f_1 + g_1) C(\gamma_1 \ldots \gamma_L) + (f_L + g_L) C(\alpha_1) \right]$$

$$+ (\rho_3 + \delta_3) f_{3\ldots N} \left[(f_{1.2} + g_{1.2}) C(\gamma_1 \ldots \gamma_L) + (f_L + g_L) C(\alpha_1 \alpha_2) \right]$$

$$+ \ldots$$

$$+ (\rho_N + \delta_N) f_N \left[(f_{1\ldots N-1} + g_{1\ldots N-1}) C(\gamma_1 \ldots \gamma_L) + (f_L + g_L) C(\alpha_1 \ldots \alpha_{N-1}) \right] \Big\}$$

where $C(\alpha_1) \equiv 1$

We re-write the above as

r.h.s. of D.8

$$= (f_L + g_L) f_u g_u$$

$$+ (f_u + g_u) C(\gamma_1 \ldots \gamma_L) \Big\{ (\rho_1 + \delta_1) f_{1\ldots N} + (\rho_2 + \delta_2) f_{2\ldots N} (f_1 + g_1) + (\rho_3 + \delta_3) f_{3\ldots N} (f_{1.2} + g_{1.2}) + \ldots$$

$$+ (\rho_N + \delta_N) f_N (f_{1\ldots N-1} + g_{1\ldots N-1}) \Big\}$$

$$+ (f_u + g_u)(f_L + g_L) \Big\{ (\rho_2 + \delta_2) f_{2\ldots N} C(\alpha_1) + (\rho_3 + \delta_3) f_{3\ldots N} C(\alpha_1 \alpha_2) + \ldots + (\rho_N + \delta_N) f_N C(\alpha_1 \ldots \alpha_{N-1}) \Big\}$$

Since the first term in curly brackets is $\hat{a}^u_{e.p}\Big|_{d_0=0}$, using D.14 we find

r.h.s. of D.8

$$= (f_L+g_L)\Big[f_u g_u + (f_u+g_u)\Big\{(\rho_2+\delta_2)f_{2\ldots N}\,C(\alpha_1)+(\rho_3+\delta_3)f_{3\ldots N}\,C(\alpha_1\alpha_2)+\ldots+(\rho_N+\delta_N)f_N\,C(\alpha_1\ldots\alpha_{N-1})\Big\}\Big]$$

$$+(f_u+g_u)\,C(\gamma_1\ldots\gamma_L)\,\tilde{d}^{\,u}_{e.p.}\Big|_{\alpha_0=0}$$

Comparing this with the l.h.s. of D.8 we see that D.8 is proved if

D.15 $$\hat{d}^{\,u}_{e.p.}\Big|_{\alpha_0=0} C(\alpha_1\ldots\alpha_N) = f_u g_u + (f_u+g_u)\Big\{(\rho_2+\delta_2)f_{2\ldots N}\,C(\alpha_1) + (\rho_3+\delta_3)f_{3\ldots N}\,C(\alpha_1\alpha_2)+\ldots$$
$$\ldots+(\rho_N+\delta_N)f_N\,C(\alpha_1\ldots\alpha_{N-1})\Big\}$$

We re-write D.14 as follows:

D.16 $$\tilde{d}^{\,u}_{e.p.}\Big|_{\alpha_0=0} = (\rho_1+\delta_1)f_{1\ldots N} + (\rho_2+\delta_2)\Big\{f_{1\ldots N}+f_{2\ldots N}g_1\Big\}$$
$$+(\rho_3+\delta_3)\Big\{f_{1\ldots N}+f_{3\ldots N}g_{1.2.}\Big\}+\cdots+(\rho_N+\delta_N)\Big\{f_{1\ldots N}+f_N g_{1\ldots N-1}\Big\}$$

We have

D.17 $$C(\alpha_1\ldots\alpha_N) = (f_{2\ldots N}+g_{2\ldots N}) + \alpha_1\,C(\alpha_2\ldots\alpha_N)$$

We iterate this to give

D.18

$$
\begin{cases}
C(\alpha_1 \ldots \alpha_N) = (f_{2\ldots N} + g_{2\ldots N}) + \alpha_1 C(\alpha_2 \ldots \alpha_N) \\
= (f_{2\ldots N} + g_{2\ldots N}) + \alpha_1 (f_{3\ldots N} + g_{3\ldots N}) + \alpha_1\alpha_2 C(\alpha_3 \ldots \alpha_N) \\
\vdots \\
= (f_{2\ldots N} + g_{2\ldots N}) + \alpha_1 (f_{3\ldots N} + g_{3\ldots N}) + \alpha_1\alpha_2 (f_{4\ldots N} + g_{4\ldots N}) \\
\quad + \ldots + \alpha_1\alpha_2 \ldots \alpha_{N-1}(f_N + g_N) + \alpha_1\alpha_2 \ldots \alpha_{N-1}
\end{cases}
$$

Using D.16 we re-write the l.h.s.of D.15 as

l.h.s. of D.15

$$
\begin{aligned}
&= (\beta_1 + \delta_1) f_{1\ldots N} C(\alpha_1 \ldots \alpha_N) \\
&+ (\beta_2 + \delta_2) \{ f_{1\ldots N} + f_{2\ldots N} g_1 \} C(\alpha_1 \ldots \alpha_N) \\
&+ (\beta_3 + \delta_3) \{ f_{1\ldots N} + f_{3\ldots N} g_{1,2} \} C(\alpha_1 \ldots \alpha_N) \\
&+ \ldots \\
&+ (\beta_N + \delta_N) \{ f_{1\ldots N} + f_N g_{1\ldots N-1} \} C(\alpha_1 \ldots \alpha_N)
\end{aligned}
$$

Using D.18 we re-write the above as follows: we leave the first $C(\alpha_1 \ldots \alpha_N)$ unchanged, we re-write the second $C(\alpha_1 \ldots \alpha_N)$ using the first of equation D.18 the third $C(\alpha_1 \ldots \alpha_N)$ using the second of equation D.18 and so on. We substitute for $C(\alpha_1 \ldots \alpha_N)$ only when they appear as coefficients of $f_{1\ldots N}$.

Thus L.H.S. of D.15

$$= (\beta_1+\delta_1) f_{1\ldots N}\, C(\alpha_1 \ldots \alpha_N)$$

$$+ (\beta_2+\delta_2)\left\{ f_{1\ldots N}\left[f_{2\ldots N} + g_{2\ldots N} + \alpha_1 C(\alpha_2 \ldots \alpha_N)\right] + f_{2\ldots N} + g_1 C(\alpha_1 \ldots \alpha_N)\right\}$$

$$+ (\beta_3+\delta_3)\left\{ f_{1\ldots N}\left[f_{2\ldots N} + g_{2\ldots N} + \alpha_1 (f_{3\ldots N} + g_{3\ldots N}) + \alpha_1\alpha_2 C(\alpha_2 \ldots \alpha_N)\right] + f_{3\ldots N} g_{1.2.} C(\alpha_1 \ldots \alpha_N)\right\}$$

$$+ \ldots.$$

$$+ (\beta_i+\delta_i)\left\{ f_{1\ldots N}\left[f_{2\ldots N} + g_{2\ldots N} + \alpha_1 (f_{3\ldots N} + g_{3\ldots N}) + \ldots + \alpha_1 \alpha_2 \ldots \alpha_{i-2} (f_{i\ldots N} + g_{i\ldots N}) \right.\right.$$

$$\left.\left. + \alpha_1 \alpha_2 \ldots \alpha_{i-1} C(\alpha_i \ldots \alpha_N)\right] + g_{1\ldots i-1} f_{i\ldots N} C(\alpha_1 \ldots \alpha_N)\right\}$$

$$+ \ldots.$$

$$+ (\beta_N+\delta_N)\left\{ f_{1\ldots N}\left[f_{2\ldots N} + g_{2\ldots N} + \alpha_1 (f_{3\ldots N} + g_{3\ldots N}) + \ldots + \alpha_1\alpha_2 \ldots \alpha_{N-2} (f_N + g_N)\right]\right.$$

$$\left. + f_N g_{1\ldots N-1} C(\alpha_1 \ldots \alpha_N)\right\}$$

We now re-arrange the above as

L.H.S. of D.15

$$= f_{1\ldots N}\left[(\beta_1+\delta_1) C(\alpha_1 \ldots \alpha_N) + \alpha_1 (\beta_2+\delta_2) C(\alpha_2 \ldots \alpha_N) + \alpha_1\alpha_2 (\beta_3+\delta_3) C(\alpha_3 \ldots \alpha_N) + \ldots \right.$$

$$\left. + \alpha_1\alpha_2 \ldots \alpha_{i-1} (\beta_i + \delta_i) C(\alpha_i \ldots \alpha_N) + \ldots + \alpha_1\alpha_2 \ldots \alpha_{N-1} (\beta_N+\delta_N) C(\alpha_N)\right]$$

$$+ (\beta_2+\delta_2)\left[f_{1\ldots N} (f_{2\ldots N} + g_{2\ldots N}) + g_1 f_{2\ldots N} C(\alpha_1 \ldots \alpha_N)\right]$$

$$+ (\beta_3+\delta_3)\left[f_{1\ldots N}\left\{ (f_{2\ldots N} + g_{2\ldots N}) + \alpha_1 (f_{3\ldots N} + g_{3\ldots N})\right\} + g_{1.2} f_{3\ldots N} C(\alpha_1 \ldots \alpha_N)\right]$$

(D.19)

$$+ \ldots.$$

$$+ (\beta_i+\delta_i)\left[f_{1\ldots N}\left\{ f_{2\ldots N} + g_{2\ldots N} + \alpha_1 (f_{3\ldots N} + g_{3\ldots N}) + \ldots + \alpha_1\alpha_2 \ldots \alpha_{i-2} (f_{i\ldots N} + g_{i\ldots N})\right.\right.$$

$$\left.\left. + \ldots + g_{1\ldots i-1} f_{i\ldots N} C(\alpha_1 \ldots \alpha_N)\right]\right.$$

$$+ (\beta_N+\delta_N)\left[f_{1\ldots N}\left\{ f_{2\ldots N} + g_{2\ldots N} + \alpha_1 (f_{3\ldots N} + g_{3\ldots N}) + \ldots + \alpha_1\alpha_2 \ldots \alpha_{N-2} (f_N + g_N)\right\}\right.$$

$$\left. + g_{1\ldots N-1} f_N C(\alpha_1 \ldots \alpha_N)\right]$$

We now recall equation (5.1.8) and we see that the first term in square brackets in D.19 is g_u. Moreover we use D.18 in the following form

D.20
$$\begin{cases} f_{2\ldots N} + g_{2\ldots N} = C(\alpha_1 \ldots \alpha_N) - \alpha_1 C(\alpha_2 \ldots \alpha_N) \\ f_{2\ldots N} + g_{2\ldots N} + \alpha_1 (f_{3\ldots N} + g_{3\ldots N}) = C(\alpha_1 \ldots \alpha_N) - \alpha_1\alpha_2 C(\alpha_3 \ldots \alpha_N) \\ \vdots \\ f_{2\ldots N} + g_{2\ldots N} + \alpha_1 (f_{3\ldots N} + g_{3\ldots N}) + \alpha_1\alpha_2 (f_{4\ldots N} + g_{4\ldots N}) + \cdots + \alpha_1\alpha_2\ldots\alpha_{N-1}(f_N + g_N) \\ \qquad = C(\alpha_1 \ldots \alpha_N) - \alpha_1\alpha_2 \ldots \alpha_{N-1} \end{cases}$$

Hence, we rewrite D.19 as

l.h.s. of D.15

$$= f_{1\ldots N} g_u$$
$$+ (g_2 + \delta_2) f_{2\ldots N} \left[f_1 \{ C(\alpha_1 \ldots \alpha_N) - \alpha_1 C(\alpha_2 \ldots \alpha_N) \} + g_1 C(\alpha_1 \ldots \alpha_N) \right]$$
$$+ (g_3 + \delta_3) f_{3\ldots N} \left[f_{1,2} \{ C(\alpha_1 \ldots \alpha_N) - \alpha_1\alpha_2 C(\alpha_3 \ldots \alpha_N) \} + g_{1,2} C(\alpha_1 \ldots \alpha_N) \right]$$
$$+ \ldots .$$

D.21
$$+ (g_i + \delta_i) f_{i\ldots N} \left[f_{1\ldots i-1} \{ C(\alpha_1 \ldots \alpha_N) - \alpha_1\alpha_2 \ldots \alpha_{i-1} C(\alpha_i \ldots \alpha_N) \} + g_{1\ldots i-1} C(\alpha_1 \ldots \alpha_N) \right]$$
$$+ \ldots .$$
$$+ (g_N + \delta_N) f_N \left[f_{1\ldots N-1} \{ C(\alpha_1 \ldots \alpha_N) - \alpha_1\alpha_2 \ldots \alpha_{N-1} C(\alpha_N) \} + g_{1\ldots N-1} C(\alpha_1 \ldots \alpha_N) \right]$$

where $C(\alpha_N) = 1$

Comparing D.21 with the r.h.s. of D.15 we see that D.15 is proved if the following can be proved:

$$(f_u + g_u) = (f_1 + g_1)\, C(\alpha_1 \ldots \alpha_N) - \alpha_1^2\, C(\alpha_2 \ldots \alpha_N)$$

$$(f_u + g_u)\, C(\alpha_1 \alpha_2) = (f_{12} + g_{12})\, C(\alpha_1 \ldots \alpha_N) - \alpha_1^2 \alpha_2^2\, C(\alpha_3 \ldots \alpha_N)$$

$$\vdots$$

or generally

D.22 $$(f_u + g_u)\, C(\alpha_1 \ldots \alpha_i) = (f_{1\ldots i} + g_{1\ldots i})\, C(\alpha_1 \ldots \alpha_N) - \alpha_1^2 \alpha_2^2 \ldots \alpha_i^2\, C(\alpha_{i+1} \ldots \alpha_N)$$

We now proceed to prove D.22

From D.18 we have

$$C(\alpha_1 \ldots \alpha_N) = f_{2\ldots N} + g_{2\ldots N} + \alpha_1\, C(\alpha_2 \ldots \alpha_N)$$

Hence multiplying both sides by α_1 we get

D.23 $$\alpha_1\, C(\alpha_1 \ldots \alpha_N) = f_{1\ldots N} + \alpha_1\, g_{2\ldots N} + \alpha_1^2\, C(\alpha_2 \ldots \alpha_N)$$

We can write (see equation 5.1.8)

D.24 $$g_{1\ldots N} = (\beta_1 + \delta_1)\, C(\alpha_1 \ldots \alpha_N) + \alpha_1\, g_{2\ldots N}$$

Substituting for $\alpha_1\, g_{2\ldots N}$ from D.24 in D.23 we get

$$\alpha_1\, C(\alpha_1 \ldots \alpha_N) = f_{1\ldots N} + g_{1\ldots N} - (\beta_1 + \delta_1)\, C(\alpha_1 \ldots \alpha_N) + \alpha_1^2\, C(\alpha_2 \ldots \alpha_N)$$

$$\therefore \quad f_{1\ldots N} + g_{1\ldots N} \equiv f_u + g_u = (\alpha_1 + \beta_1 + \delta_1)\, C(\alpha_1 \ldots \alpha_N) - \alpha_1^2\, C(\alpha_2 \ldots \alpha_N)$$

But $$\alpha_1 + \beta_1 + \delta_1 = f_1 + g_1$$

D.25 $$\therefore \quad f_u + g_u = (f_1 + g_1)\, C(\alpha_1 \ldots \alpha_N) - \alpha_1^2\, C(\alpha_2 \ldots \alpha_N)$$

Note that D.25 is the case of D.22 with i = 1

It follows from D.25 that

D.26 $$(f_u + g_u)\, C(\alpha_1 \ldots \alpha_i) = (f_1 + g_1)\, C(\alpha_1 \ldots \alpha_N)\, C(\alpha_1 \ldots \alpha_i) - \alpha_1^2 C(\alpha_1 \ldots \alpha_i) C(\alpha_2 \ldots \alpha_N)$$

Using

D.27 $$C(\alpha_1 \ldots \alpha_i) = f_{2 \ldots i} + g_{2 \ldots i} + \alpha_1\, C(\alpha_2 \ldots \alpha_i)$$

D.28 $$g_{1 \ldots i} = g_1\, C(\alpha_1 \ldots \alpha_i) + \alpha_1 g_{2 \ldots i}$$

we find

r.h.s. of D.26

$$= \left\{ f_1 \left(f_{2 \ldots i} + g_{2 \ldots i} + \alpha_1 C(\alpha_2 \ldots \alpha_i) \right) + g_{1 \ldots i} - \alpha_1 g_{2 \ldots i} \right\} C(\alpha_1 \ldots \alpha_N)$$
$$- \alpha_1^2\, C(\alpha_1 \ldots \alpha_i)\, C(\alpha_2 \ldots \alpha_N)$$

$$= (f_{1 \ldots i} + g_{1 \ldots i})\, C(\alpha_1 \ldots \alpha_N) + \alpha_1^2\, C(\alpha_2 \ldots \alpha_i)\, C(\alpha_1 \ldots \alpha_N)$$
$$- \alpha_1^2 C(\alpha_1 \ldots \alpha_i) C(\alpha_2 \ldots \alpha_N)$$

Using D.27 for $C(\alpha_1 \ldots \alpha_i)$ and $C(\alpha_1 \ldots \alpha_N)$ we get

r.h.s. of D.26

$$= [f_{1 \ldots i} + g_{1 \ldots i}]\, C(\alpha_1 \ldots \alpha_N) - \alpha_1^2 \left[(f_{2 \ldots i} + g_{2 \ldots i})\, C(\alpha_2 \ldots \alpha_N) - (f_{2 \ldots N} + g_{2 \ldots N}) C(\alpha_2 \ldots \alpha_i) \right]$$

Adding and subtracting $\alpha_1^2 \alpha_2^2 \ldots \alpha_i^2\, C(\alpha_{i+1} \ldots \alpha_N)$ to the above we get

r.h.s. of D.26

D.29
$$= [f_{1 \ldots i} + g_{1 \ldots i}]\, C(\alpha_1 \ldots \alpha_N) - \alpha_1^2 \alpha_2^2 \ldots \alpha_i^2\, C(\alpha_{i+1} \ldots \alpha_N)$$
$$- \alpha_1^2 \left[f_{2 \ldots i} + g_{2 \ldots i} \right] C(\alpha_2 \ldots \alpha_N) - (f_{2 \ldots N} + g_{2 \ldots N}) C(\alpha_2 \ldots \alpha_i)$$
$$- f_{2 \ldots i}^2\, C(\alpha_{i+1} \ldots \alpha_N) \Big] = \text{L.H.S. of (D.26)} = (f_u + g_u)\, C(\alpha_1 \ldots \alpha_i)$$

Comparing the above to D.22 we find that D.22 , is proved if the last term in square brackets in D.29 is zero i.e. if

$$(f_{2\ldots N} + g_{2\ldots N})\, C(\alpha_2 \ldots \alpha_i)$$

D.30

$$= (f_{2\ldots i} + g_{2\ldots i})\, C(\alpha_2 \ldots \alpha_N) - f_{2\ldots i}^{\,2}\, C(\alpha_{i+1} \ldots \alpha_N)$$

Recalling that $f_u \equiv f_{1\ldots N}$ and $g_u \equiv g_{1\ldots N}$ and comparing D.30 and D.22 we see that their structure is very similar. So if we repeat all the steps that led to D.30 from D.22 we see that proving D.30 is equivalent to proving

$$(f_{3\ldots N} + g_{3\ldots N})\, C(\alpha_3 \ldots \alpha_i)$$

$$= (f_{3\ldots i} + g_{3\ldots i})\, C(\alpha_2 \ldots \alpha_N) - f_{3\ldots i}^{\,2}\, C(\alpha_{i+1} \ldots \alpha_N)$$

Iterating we will eventually find that proving the above is equivalent to proving

$$(f_{i\ldots N} + g_{i\ldots N})\, C(\alpha_i) = (f_i + g_i)\, C(\alpha_2 \ldots \alpha_N) - f_i^{\,2}\, C(\alpha_{i+1} \ldots \alpha_l)$$

and since $C(\alpha_i) = 1$ we will have to prove that

D.31 $$(f_{1\ldots N} + g_{1\ldots N}) = (f_i + g_i)\, C(\alpha_2 \ldots \alpha_N) - f_i^{\,2}\, C(\alpha_{i+1} \ldots \alpha_l)$$

But D.31 is precisely equation D.25 with

$$f_u \equiv f_{1\ldots N} \rightarrow f_{i\ldots N}$$

$$g_u \equiv g_{1\ldots N} \rightarrow g_{i\ldots N}$$

$$C(\alpha_1 \ldots \alpha_N) \rightarrow C(\alpha_i \ldots \alpha_N)$$

$$\alpha_1^{\,2} \rightarrow \alpha_i^{\,2}$$

Hence, D.31 is proved and consequently all the previous equations we .

wished to prove, in particular D.1.

This completes the proof of factorization of Drell-Yan .

PART II

The Infrared Nature of the Drell-Yan Process

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3.4 The Non-Cancelling sector at $O(\alpha^2)$

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Chpater 4 Conclusions and discussions

Appendices: A: QCD Feynman rules

B: The mass counterterm at $O(\alpha)$

D: The integrals of Section 3.4

CHAPTER 1

INTRODUCTION

Much of the phenomonology of Quantumchromodynamics (QCD) [1] is based on the theorem that mass singularities factorize [18] and are universal in all hard scattering processes. A necessary condition for this theorem to be valid is that the Bloch-Nordsieck mechanism [19] for the cancellation of infrared (i.r.) divergences holds at least up to the accuracy required. This has not been proved in general and indeed recently Doria et al [20] have shown that the i.r. structure of inclusive QCD processes is very different from the structure in quantumelectrodynamics (QED). In QED the Bloch-Nordsieck mechanism cancellation between real and virtual photon processes is known to hold for all orders and for all powers of the logarithmic divergences. In QCD it is apparently still true for processes with no hadronic interactions in the initial state, for example, $e^+e^- \longrightarrow$ anything [21] or in which there is only one incoming coloured particle e.g. $\gamma^* q \longrightarrow$ anything [21]. — there are certain errors in this paper because the $i\varepsilon$ terms in propagators have not been taken care of properly. However, the results are still true according to a check by J.C. Taylor (private communication). For processes where there are two incoming particles which can radiate soft gluons it is false at order α^2 where $\alpha = g^2$, g being the QCD coupling constant.

We will be investigating two such processes

P.1 quark + quark $\longrightarrow (\mu^+,\mu^-)$ + anything

P.2 quark + antiquark $\longrightarrow (\mu^+,\mu^-)$ + anything

Process P.1 is shown in fig 1.a. It will be referred to as Drell-Yan bremsstrahlung. A lepton pair is being produced from a radiated photon, γ^*, from a quark. In order to conserve four-moment a "hard" gluon, g must be exchanged between the two quarks. By "hard" we mean that the gulon is a long way off mass-shell. Process P.2 is shown in fig 1.b and this is the usual Drell-Yan annihilation diagram. A

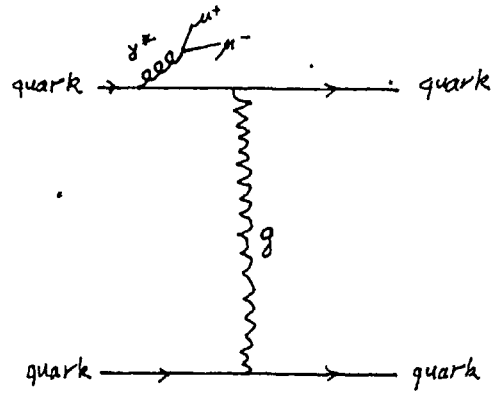


Fig. 1.a: Lepton pair production via photon bremsstrahlung

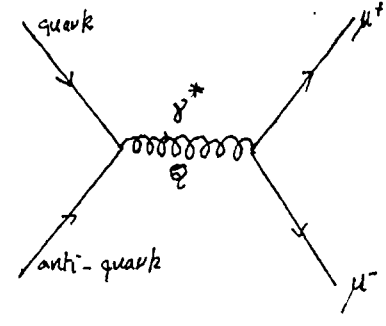


Fig. 1.b: Lepton-pair production via annihilation

quark-anti-quark pair annihilate to produce a virtual photon which then decays into a lepton pair. Again Q is "hard" in the sense defined above.

Some of this second half of the thesis is a repetition of the Doria et al calculations [20] which deals with P.1. We agree with these authors that at order α^2 the i.r. divergences do not cancel in the colour averaged cross-section. However, this agreement is fortuitous since several of the intermediate steps are marred by several major errors. These errors are removed and by some minor miracle our answer is identical to that of Doria et al.

In Chapter 2 we discuss the assumptions that are made to obtain the answer. The problem of renormalization is discussed there. The

necessary tools that will be required to orgainise the graphs as to leave us with as little calculations as possible are developed in this chapter.

The details of the calculations are to be found in Chapter 3 together with Appendix B.

Chapter 4 contains our conclusions and discussions.

CHAPTER 2

BASIC REQUIREMENTS

This chapter is a review of the basic assumptions that go into the calculation and the development of certain tools that will be required eventually.

2.1 Basic assumptions and the necessary preliminaries

(i) The Cross-section

Our basic method is to assume that we are given the cross-section in terms of the discontinuities of some hard diagram H as shown in fig 2.1.1 Recall from our discussion of the optical theorem [11]

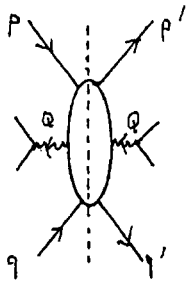


Fig 2.1.1: The cross-section of a basic hard diagram H. The dashed line indicates that the intermediate states must be put on mass-shell

in the first part of this thesis that fig 2.1.1 represents a forward scattering amplitude. Fig 2.1.2 explicitly shows the structure of our hard diagram for P.1 and P.2. p and q represent the momentum of the

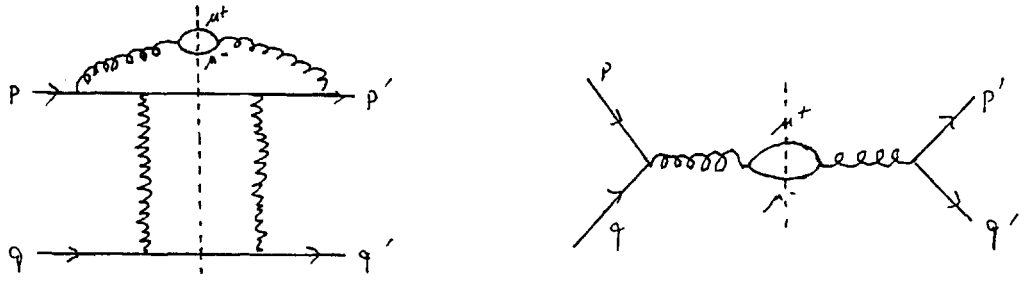


Fig 2.1.2 The corresponding hard diagrams for P.1 and P.2

incoming particles. p' and q' have the same magnitude as p and q respectively but the distinction between them has to be kept for reasons dictated by the optical theorem.

By hard we mean that the cross-section of fig 2.1.1 is infrared finite and is assumed not to have any singular behaviour as the momenta entering the hard diagram vary slightly off mass-shell.

It is known that if we dress the hard diagrams to order α then the i.r. divergences cancel. We show this explicitly in Chapter 3 using the techniques we are about to develop.

So we dress our hard diagram to order α^2 by adding soft (i.e.not hard) gluons in all possible ways. These gluons need only interact with the external legs. This is manifestly obvious in the case of P.2 as the gluons have no quarks or other gluons to couple with in our specif hard diagram. The Drell-Yan annihilation process is in this sense a much cleaner process to study than Drell-Yan bremsstrahlung. The argument for P.2 is basically unitarity - the i.r. divergences cancel when a given amplitude is cut in all possible ways to give all its contribution to the cross-section [20, and references therein].

An example of dressing up our hard diagram with soft gluons is given in fig 2.1.3. Throughout this calculation we insist that our hard

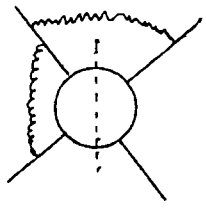


Fig 2.1.3. Dressing up a hard diagram H with 2 soft gluons.

diagram is ALWAYS cut. From now on this will be taken as given and

will not be indicated in the diagrams. Fig2.1.3 is NOT what we wish to calculate. Not only must the hard blob be cut but also any gluon CROSSING it. This is illustrated in Fig 2.1.4. The asterisk means that everything on the r.h.s of the cut must be complex conjugated.

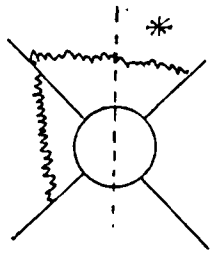


Fig 2.1.4 Contribution of dressed hard blob to the cross-section

(ii) Colour factors

The fundamental difference between calculating Feynman diagrams in abelian and in non-abelian gauge theories is the introduction of a colour matrix factor at each interaction vertex because our fermions and gauge fields have additional quantum numbers known as 'colour'. Our hard blob will, therefore, have colour indices $H_{ab}^{\alpha\beta}$ with α, β, a, b, representing the colour of the quark lines entering the hard blob H.

In the physical world "coloured" particles are not observed. It is therefore natural to calculate a colour averaged cross-section. This means we perform an average over colour indices in the final state. We are also averaging over spins. The colour sums will be incoherent ones meaning that the quarks (p, q) will have the SAME colour as the "final" quarks (p', q'). Be careful about the meaning of "final' as fig2.1.3 is a back-to-back amplitude and thus all our external quarks are really initial states particles.

When we add 4-gluon vertices and perform the sum over colour we

obtain the following group theoretic weights as a result of the colour matrix multiplication (the Feynman rules are given in Appendix A):

$$(2.1.1)\qquad \begin{cases} C_2(G)\,(T_i)^{\beta\alpha}(T_i)_{b\alpha}\,H^{\alpha\beta}_{ab} \\ \{T_i,T_j\}^{\beta\alpha}\,\{T_i,T_j\}_{b\alpha}\,H^{\alpha\beta}_{ab} \\ C_2(F)\,(T_i)^{\beta\alpha}(T_i)_{b\alpha}\,H^{\alpha\beta}_{ab} \\ [C_2(F)]^2\,H^{\alpha\alpha}_{aa} \\ C_2(F)C_2(G)\,H^{\alpha\alpha}_{aa} \end{cases}$$

$$(2.1.2)\qquad \begin{cases} C_2(F)C_2(G)H \\ [C_2(F)]^2 H \end{cases}$$

T_i are the generators of the colour group SU(N). $C_2(F)$ and $C_2(G)$ are the Casimir factors defined by

$$(2.1.3)\qquad f_{ijk}f_{ijl} = C_2(G)\delta_{kl}$$

$$(2.1.4)\qquad (T_i \cdot T_i)_{ab} = C_2(F)\delta_{ab}$$

f_{ijk} are the SU(N) structure constants and δ_{ij} is the usual Kronecker-delta function. Repeated indices are to be summed over.

(2.1.1) are the colour factors associated with P.1 whereas (2.1.2) are those associated with P.2. The reason there are fewer types of group theoretic weights for P.2 is the quark-anti-quark pair annihilates into a photon, a colourless object. Hence when these enter the hard blob they must have the same colour.

As an illustration of the calculations leading to (2.1.1) and (2.1.2) consider, firstly, the contribution of fig 2.1.5 to P.1

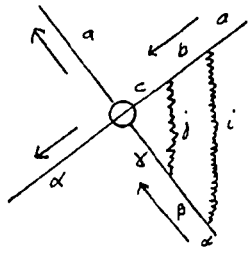


Fig2.1.5 Colour labels for a dressed hard blob of P.1. The arrows indicate the order in which the colour matrix multiplication has to be done.

The colour factor $C|_{P.1}$ for fig 2.1.5 is

$$C|_{P.1} = (T_i)_{ab}(T_j)_{bc}(T_i)_{\alpha\beta}(T_j)_{\beta\gamma} H^{ca}_{\gamma\alpha} \tag{2.1.5}$$

Using the identity

$$T_i \cdot T_j = \tfrac{1}{2}\{T_i, T_j\} + \tfrac{1}{2}[T_i, T_j] \tag{2.1.6}$$

where

$$\begin{cases} \{A,B\} = AB + BA \\ [A,B] = AB - BA \end{cases} \tag{2.1.7}$$

we find

$$(2.1.8)\qquad C|_{P.1.} = \frac{1}{4}\left(\{T_i,T_j\}_{ac} + i f_{ijk}(T_k)_{ac}\right)\left(\{T_i,T_j\}_{\beta\alpha} + i f_{ijl}(T_l)_{\beta\alpha}\right) H^{ca}_{\alpha\beta}$$

on using

$$(2.1.9)\qquad [T_i, T_j] = i f_{ijk} T_k$$

Making use of the antisymmetry properties of the structure constants we eventually arrive at

$$(2.1.10)\qquad C|_{P.1.} = \frac{1}{4}\{T_i,T_j\}_{ac}\{T_i,T_j\}_{\beta\alpha} H^{ca}_{\alpha\beta} - \frac{1}{4} C_2(G)(T_i)_{ac}(T_i)_{\beta\alpha} H^{ca}_{\alpha\beta}$$

Let us now repeat the calculations of P.2. The relevant diagram is shown in fig 2.1.6. Note how the colour labels have changed and also how the order of matrix multiplication has changed.

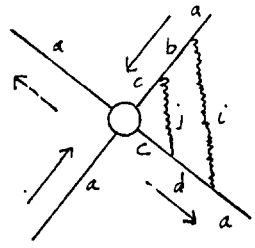


Fig 2.1.6 Colour labels for P.2 for a given dressed hard diagram

We now have

$$C|_{P.2.} = (T_i)_{ab}(T_j)_{bc}(T_j)_{cd}(T_i)_{da} H$$

On using (2.1.4) we find

$$(2.1.11)\qquad C|_{P.2.} = N\left[C_2(F)\right]^2 H$$

In Table I we list which graphs contribute to which weights using calculations illustrated immediately above. Note that since the

initial and final quarks are constrained to have the same colour label then the graph shown in fig2.1.7,which is obtained from fig 2.1.5 by swinging the outermost gluon from the q' to the q line, has an identical colourfactor to that of fig 2.1.5. By continued cyclic permutation of the gluons we can generate all the graphs of a given colour factor. It

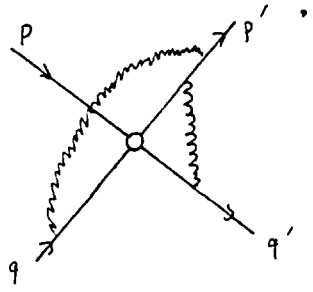


Fig 2.1.7 An example of a graph that has the same colour structure as fig 2.1.5.

is on this basis that we classify the graphs into the different groups of Table 1. Care must be taken not to double count certain graphs as a result of this classification scheme.

(iii) The eikonal approximation

Throughout we will be using the eikonal approximation. This means that since the infrared divergences of our integrals will be coming from $k^\mu \to 0$ where k^μ is the gluons 4-momentum then we may write a fermion propagator like

$$(2.1.12) \qquad \left[(p-k)^2 - m^2 + i\varepsilon \right] \simeq -2p\cdot k + i\varepsilon$$

on neglecting quadratic terms and taking $p^2 = m^2$. Also we will be neglecting all k-dependences in the numerator so that a fermion numerator becomes $(\not{p} + m)$. This allows us to replace all the spinology by the rule "$2p^\mu$ for each fermion-gluon-fermion vertex if we are on a p-line". A further simplification resulting from the eikonal approximation is

to write, for a fermion on mass-shell,

$$(2.1.13)\qquad \delta^{(+)}((p-k)^2-m^2) \equiv \theta(p^0-k^0)\,\delta((p-k)^2-m^2) \simeq \delta(2p\cdot k)$$

We will return to the problem of the eikonal approximation again.

(iv) The frame of reference and the gauge

We will be working in the Feynman gauge. This means that a gluon propagator is $-ig_{\mu\nu}/(k^2+i\varepsilon)$ where k is the gluon's 4-momentum.

We will be working in the Lorentz frame

$$(2.1.14)\qquad p = (M, \underline{0}),\quad q = (q_0, 0, 0, q_z)$$

We define the following quantities

$$(2.1.15)\qquad K = \frac{q\cdot k}{q_0} = \frac{|\underline{q}||\underline{k}|\cos\theta}{q_0} = \beta\,|\underline{k}|\,x$$

(2.1.16) where $\beta = \dfrac{|\underline{q}|}{q_0}$ = velocity of q relative to p

$$(2.1.17)\qquad x = \cos\theta$$

Hence

$$(2.1.18)\qquad K < |\underline{k}|$$

2.2 The Libby-Sterman Identity

This identity, due to Libby and Sterman [21] allows us to cancel sets of graphs at the level of the integrands as long as we are careful

about $i\epsilon$ -prescriptions in the quark propagators. Consider the diagram shown in Fig 2.2.1. The following analysis holds as long as there is

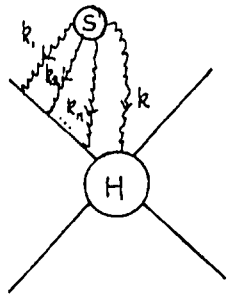


Fig 2.2.1: An arbitrarily dressed hard blob with at least one gluon k entering it

at least one gluon k going into the hard blob H from the gluonic blob S so that no self-energy diagrams ever arise. The latter are dealt with in the next section.

Define

$$a_i \equiv 2p\cdot k_i \tag{2.2.1}$$

From Cauchy's Theorem we find, provided there are at least two denominators,

$$\int_{-\infty}^{\infty} \frac{dx}{(x+i\epsilon)(x+a_1+2i\epsilon)(x+a_1+a_2+3i\epsilon)\cdots(x+a_1+a_2+\cdots+a_n+(n+1)i\epsilon)} = 0 \tag{2.2.2}$$

Hence if we pull the contour past the poles we find

$$0 = \frac{1}{[a_1+i\epsilon][a_1+a_2+2i\epsilon][a_1+a_2+a_3+3i\epsilon]\cdots[a_1+a_2+\cdots+a_n+ni\epsilon]}$$

$$+ \frac{1}{[-a_1-i\epsilon][a_2+i\epsilon][a_2+a_3+2i\epsilon]\cdots[a_2+a_3+\cdots+a_n+(n-1)i\epsilon]}$$

$$+ \frac{1}{[-a_1-a_2-2i\epsilon][-a_2-i\epsilon][a_3+i\epsilon]\cdots[a_3+a_4+\cdots+a_n+(n-2)i\epsilon]}$$

$$+ \cdots$$

$$+ \frac{}{[-a_1-a_2-\cdots-a_n-ni\epsilon]\cdots[-a_1-i\epsilon]}$$

These correspond precisely to the sum of diagrams shown in fig 2.2.2 in the eikonal approximation, first with no gluons on the right, then only k_1 then k_1 and k_2 and so on. Notice the gluon's momenta is always outgoing from the hard blob. We have (+ $i\varepsilon$) prescriptions on the left and (- $i\varepsilon$) to the right. By changing the $2i\varepsilon$, $3i\varepsilon$, etc. in (2.2.2) we may obtain many $i\varepsilon$ prescriptions. All the graphs of fig (2.2.2) have the same colour factor.

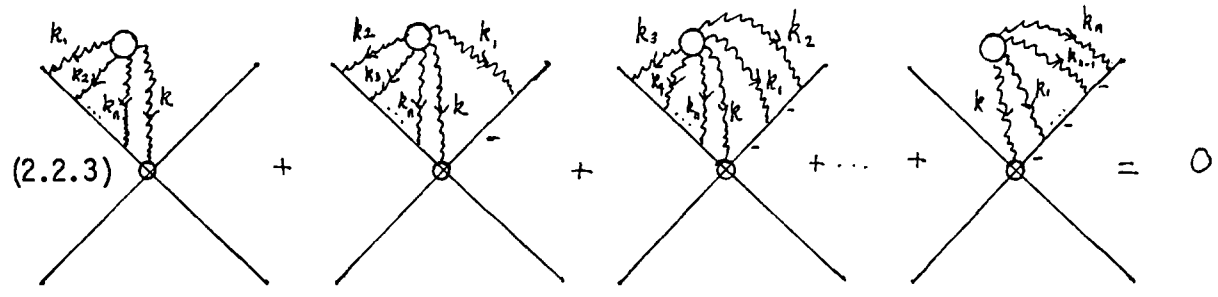


Fig 2.2.2 Graphical interpretaion of the Libby-Sterman identity

The - indicates anti-fermion $i\varepsilon$ prescription

2.3 The Self-energy cancellation identity

Because some of our diagrams have self-energy insertions on extrenal legs we are forced to consider the question of renormalization. In the Libby-Sterman paper [21] this was somehow dealt with by a faked momentum prescription. It turns out in our approach that the sign of the $i\varepsilon$ in the fermion propagators is crucial. It is not at all obvious how the faked momentum prescription deals with the $i\varepsilon$ terms.

We have to go back to the Lehman-Symmanzik-Zimmerman (LSZ) formalism [10] and introduce a mass counterterm renormalization prescription.

Implicit in the eikonal approximation is on mass-shell renormalization. This is best seen if we return to (2.1.12). The mass term m^2 which appears in the propagator is the mass in the free part of the Lagrangian. To write down (2.1.12) we had to assume that $p^2 = m^2$. But p^2 is the "physical" mass of the quark.

Let us recall the formula for the S-matrix, S, in the LSZ-formalism:

$$S \equiv \langle p_1 \cdots p_n, \text{out} | q_1 \cdots q_m, \text{in} \rangle$$

$$= \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \prod_{i=1}^{m} \int d^4x_i \prod_{j=1}^{n} \int d^4y_j f_{q_i}(x_i) \overrightarrow{(\Box_{x_i} + m^2)} \langle 0 | T(\varphi(y_1) \cdots \varphi(y_n) \varphi(x_1) \cdots \varphi(x_m)) | 0 \rangle \overleftarrow{(\Box_{y_j} + m^2)} f_{p_j}(y_j) \tag{2.3.1}$$

for all $p_i \neq q$, in the case of BARE scalar fields φ. We are using the same notation as in [10]. Z is the wavefunction renormalization constant. (2.3.1) can easily be extended to our case of fermionic fields. In (2.3.1) we have re-written the bare mass m_0 that appears in the free part of the Lagrangian as

$$m_0 = m + (m_0 - m) \tag{2.3.2}$$

where m is the physical mass. We now have a mass counterterm which is absorbed into the interaction part of the Lagrangian. Otherwise all other quantities such as fields and coupling constants are bare objects.

The full fermion propagator $S_F'(p)$ defined by

(2.3.3) $$S_F'(\not p) \equiv \langle 0|T(\phi(x_1)\phi(x_2))|0\rangle$$

may be written as [10]

(2.3.4) $$S_F'(\not p) \equiv \frac{i}{\not p - m - \Sigma(\not p)}$$

where $-i\Sigma'(\not p)$ represents all the one-particle imeducible graphs with external legs amputated. Near the mass shell we can write $\Sigma(\not p)$ as

(2.3.5) $$\Sigma(\not p) = \delta m(m, \Delta m) - (Z_2^{-1} - 1)(\not p - m) + O(\not p - m)^2$$

where Δm represents the mass counterterm introduced previously and Z_2 is the fermionic field wavefunction renormalization constant analogous to Z in (2.3.1). We now choose our renormalization condition to be

(2.3.6) $$\delta m(m, \Delta m)\Big|_{\not p = m} = 0$$

in order to keep the pole of $S_F'(\not p)$ at the physical mass. Thus substituting for $\Sigma'(\not p)$ from (2.3.5) and using the renormalization condition (2.3.6) in (2.3.4) we find

(2.3.7) $$\lim_{\not p \to m} S_F'(\not p) = \frac{i}{\not p - m} Z_2 .$$

(2.3.6) defines our mass counterterm represented graphically by a $(-i\Delta m)$ insertion into a fermion line as shown in fig 2.3.1 to all orders in perturbation theory. (2.3.7) tells us that the effect

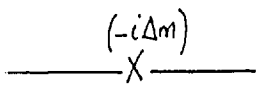


Fig 2.3.1: Feynman graphical representation of mass counterterm

of self-energy insertions in a propagator near mass shell is to multiply the bare propagator by Z_2. Thus the $1/\sqrt{Z_2}$ that appears outside (2.3.1) (if we have fermion fields) for each external field is cancelled leaving us with an overall multiplicative factor of $\sqrt{Z_2}$ for each external leg.

We can write

$$(2.3.8) \qquad Z_2^{1/2} = \left(1 + (Z_2^{-1} - 1)\right)^{-1/2} = 1 - \frac{1}{2}(Z_2^{-1} - 1) + \frac{1 \cdot 3}{2 \cdot 4}(Z_2^{-1} - 1)^2 + \cdots$$

This is a valid expansion for small α as the first term in $(Z_2^{-1} - 1)$ is of order α.

From (2.3.5) we have

$$(2.3.9) \qquad \delta m = \bar{u}(p)\,\Sigma(\not{p})\,u(p)\Big|_{\not{p}=m}$$

where we have made use of

$$(2.3.10) \qquad \begin{cases} \bar{u}(p)(\not{p} - m) = 0 \\ \bar{u}(p)\,u(p) = 1 \end{cases}$$

Also from (2.3.5) we have

$$(2.3.11) \qquad Z_2^{-1} - 1 = -\left.\frac{\bar{u}(p)\left(\frac{\partial}{\partial p^\nu}\Sigma(\not{p})\right)u(p)}{\bar{u}(p)\,\gamma^\nu\,u(p)}\right|_{\not{p}=m}$$

Hence to calculate the contribution of a self-energy insertion on an external leg we must work out all the one-particle irreducible graphs that contribute to the full fermion propagator to the required order in α and then use (2.3.8). We will be drawing graphs with self-energy insertions on external legs for the sake of book-keeping.

In the eikonal approximation $(Z_2^{-1} - 1)$ turns out to have a simple graphical interpretation. This will allow us to write an identity analogous to (2.2.3).

Consider the diagram of fig 2.3.2 having a one-particle irreducible self-energy insertions on one of its external legs to an arbitrary

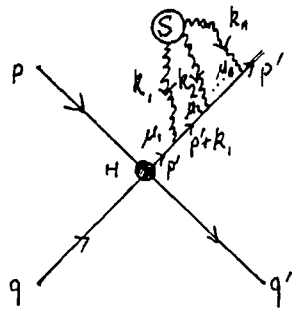


Fig 2.3.2 Our hard diagram dressed by an arbitrary self-energy one-particle irreducible insertion on one of its external leg.

order in α. $\mu_1, \mu_2, \ldots \mu_n$ are Lorentz indices of the gluons coming from the gluonic blob S. The fermion propagators take the form

$$(2.3.12) \quad \not{p}_{\mu_1 \ldots \mu_n} \equiv \gamma_{\mu_1} \frac{1}{\not{p}_1} \gamma_{\mu_2} \frac{1}{\not{p}_2} \gamma_{\mu_3} \frac{1}{\not{p}_3} \gamma_{\mu_4} \ldots \gamma_{\mu_{n-1}} \frac{1}{\not{p}_{n-1}} \gamma_{\mu_n}$$

where

$$(2.3.13) \quad \not{p}_i = \not{p} + \sum_{r=1}^{i} \not{k}_r - m + i\varepsilon$$

Using

(2.3.14) $$\frac{\partial}{\partial p^\nu}\left\{\frac{1}{\not p+\not k-m+i\varepsilon}\right\} = -\frac{1}{\not p+\not k-m+i\varepsilon}\gamma^\nu\frac{1}{\not p+\not k-m+i\varepsilon}$$

we find

$$\frac{\partial}{\partial p^\nu}\{p_{\mu_1},\dots,v_n\}$$

$$= -\left\{\gamma_{\mu_1}\frac{1}{\not p_1}\gamma^\nu\frac{1}{\not p_1}\gamma_{\mu_2}\frac{1}{\not p_2}\gamma_{\mu_3}\cdots\gamma_{\mu_{n-1}}\frac{1}{\not p_{n-1}}\gamma_{\mu_n}\right.$$

(2.3.15)

$$+\gamma_{\mu_1}\frac{1}{\not p_1}\gamma_{\mu_2}\frac{1}{\not p_2}\gamma^\nu\frac{1}{\not p_2}\gamma_{\mu_3}\frac{1}{\not p_3}\gamma_{\mu_4}\cdots\gamma_{\mu_{n-1}}\frac{1}{\not p_{n-1}}\gamma_{\mu_n}$$

$$+\cdots$$

$$\left.+\gamma_{\mu_1}\frac{1}{\not p_1}\gamma_{\mu_2}\cdots\gamma_{\mu_{n-1}}\frac{1}{\not p_{n-1}}\gamma^\nu\frac{1}{\not p_{n-1}}\gamma_{\mu_n}\right\}$$

So from (2.3.11) and using the eikonal approximation it follows from the above that

$$(Z_2-1) = \int\prod_{i=1}^{n}\frac{d^4k_i}{(k_i^2+i\varepsilon)}\delta^{(4)}\left(\sum_{i=1}^{n}k_i\right)\left\{\frac{2p_{\mu_1}\cdot 2p_{\mu_2}}{[2p\cdot k_1+i\varepsilon]^2}\cdot\frac{2p_{\mu_3}}{[2p\cdot(k_1+k_2)+i\varepsilon]}\cdots\frac{2p_{\mu_n}}{[2p\cdot(k_1+k_2+\cdots+k_{n-1})+i\varepsilon]}\right.$$

(2.3.16)

$$+\frac{2p_{\mu_1}}{[2p\cdot k_1+i\varepsilon]}\cdot\frac{2p_{\mu_2}\,2p_{\mu_3}}{[2p\cdot(k_1+k_2)+i\varepsilon]^2}\frac{2p_{\mu_4}}{[2p\cdot(k_1+k_2+k_3)+i\varepsilon]}\cdots\frac{2p_{\mu_n}}{[2p\cdot(k_1+k_2+\cdots+k_{n-1})+i\varepsilon]}$$

$$+\cdots$$

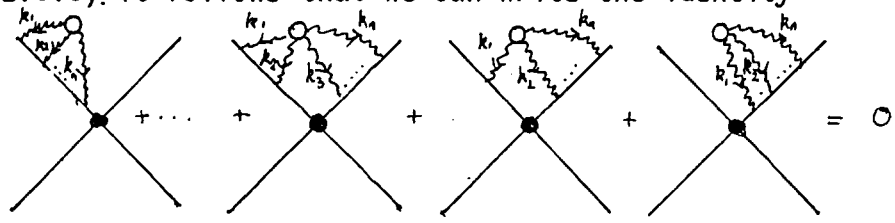
$$\left.+\frac{2p_{\mu_1}}{[2p\cdot k_1+i\varepsilon]}\cdot\frac{2p_{\mu_2}}{[2p\cdot(k_1+k_2)+i\varepsilon]}\cdots\frac{2p_{\mu_{n-1}}\cdot 2p_{\mu_{n-1}}}{[2p\cdot(k_1+k_2+\cdots+k_{n-1})+i\varepsilon]^2}\right\}S(k_i=0)$$

The terms in curly brackets above correspond to respectively the graphs obtained from fig 2.3.2 by swinging the gluons across to the p-leg, first with only k_1 on the p'-leg, then k_1 and k_2 and so on until the last term which correspond with only k_n on the p-leg. Note that all these graphs have the same colour structure even though we are swinging the gulons from the "inside" rather than from the "outside" as we did for

the Libby-Sterman identity. The reason for this is self energy graphs have an additional constraint on their colour labels, namely, the propagator nearest to the hard blob on the p'-leg (see fig 2.3.2) has the same colour as the external p'-leg. Note that UNLIKE the Libby-Sterman identity we have $(+ i\varepsilon)$ in the propagators to the right AND to the left.

From (2.3.8) and (2.3.16.) it follows that we can write the identity

(2.3.17)



(2.3.17) shall be referred to as the self-energy cancellation identity.

As a concrete example consider the self-energy insertion to order α of fig 2.3.3. This will be required later. Using (2.3.5) we

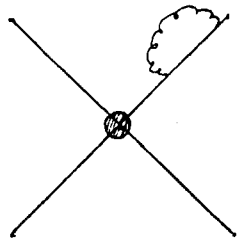
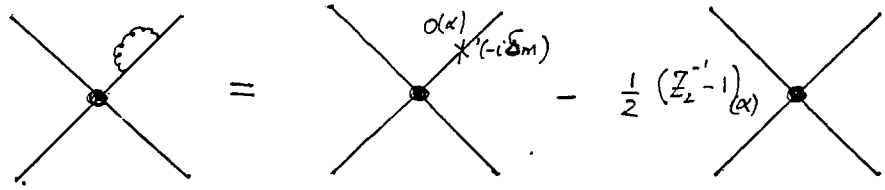


Fig 2.3.3 Self-energy insertion to $O(\alpha)$

may write

(2.3.18)



where $(Z_2^{-1} - 1)_{(\alpha)}$ means evaluating $(Z_2^{-1} - 1)$ to order α. Note that the graphs with mass insertions are not infrared divergent. Anyway they are usually cancelled by a mass counterterm $(-i\Delta m)$. $(Z_2^{-1} - 1)_{(\alpha)}$ has the graphical interpretation of fig 2.3.4. In the eikonal

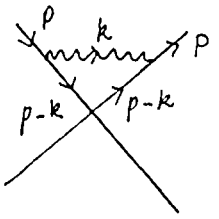


Fig 2.3.4 Graphical interpretation of $(Z_2^{-1} - 1)_{(\alpha)}$

approximation the integral representation for it is given by

$$(2.3.19) \qquad (Z_2^{-1} - 1)_{(\alpha)} = i \int \frac{d^4k \; 4M^2}{[k^2+i\varepsilon][-2p\cdot k+i\varepsilon]^2}$$

which is real as can be seen by working in the frame $p = (M, \underline{0})$. The gluon goes on shell picking up an extra factor of i. The $i\varepsilon$ in the fermion propagator then becomes irrelevant.

2.4 Converting the cut-graphs into Feynman amplitudes

Since we are summing over all possible intermediate states, the inclusive cross-section is given by Mueller's Theorem [11].

$$\text{Flux} \frac{d\sigma}{d^3q} = \frac{1}{2i} \Delta_s (S_L^+, s, s_R^-)$$

where A is the forward scattering amplitude for P.1 and P.2 dressed with two soft gluons. The kinematics are as shown in fig 2.4.1

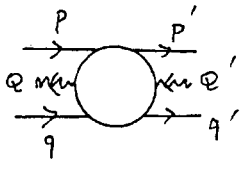


Fig 2.4.1 Kinematics of P.1 and P.2

s is the total missing mass squared, $(p + q - Q)^2$. Δ_s is the discontinuity in that variable. S_L and S_R refer to all the invariants lying to the left and right of the cut C respectively. The $\pm$ means

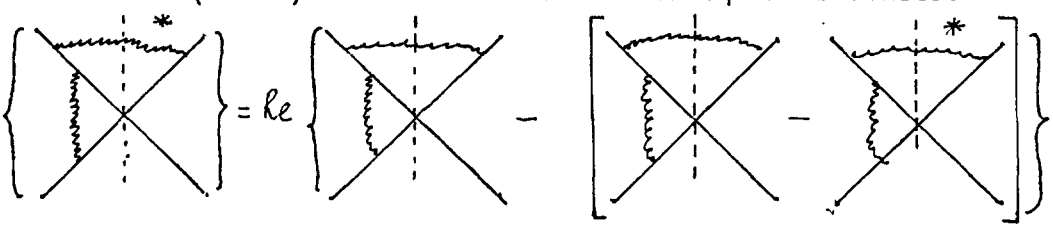
they are evaluated above and below their cuts. Thus S_L includes $(p+q)^2$, p^2, q^2, Q^2 and S_R indicates $(p'+q')^2$, p'^2, q'^2, Q'^2. In [20] it is assumed the one is below the Q^2-thresholds. However, note that since our hard diagram is always cut this assumption is unnecessary for the next step which is writing

$$(2.4.1)\qquad \frac{\Delta_s}{2i}\left\{A(s_L^+, s, s_R^-)\right\} = \frac{\Delta_s}{2i}\left\{A(s_L^+, s, s_R^+) - \left[A(s_L^+, s, s_R^+) - A(s_L^-, s, s_R^-)\right]\right\}$$

The term in square brackets is interpreted as the discontinuity in all the right hand variables. One may very well ask what the discontinuities in p'^2 and q'^2 are? Under normal circumstances the answer is known but here we are sitting right where their threshold values start, namely at $p'^2 = m^2 = q'^2$. It is then not obvious what the answer is. Here we avoid this approach altogether and stick to the graphical counterpart of (2.4.1)

Recall that in graphical language fig 2.1.4 is the object one wants to calculate. Moreover a cross-section is real so we only want the real part of fig 2.1.4. Note also that the Libby-Sterman and the self-energy cancellation identities only apply to Feynman AMPLITUDES and not to cut diagrams. Our aim, therefore, is to re-write all our cut diagrams in terms of Feynman amplitudes.

We start with (2.4.1) by writing its graphical equivalent. As an example consider (2.4.2). Consider the term in square brackets.

(2.4.2) 

Remember that the cut hard diagram is a real number. We will then be applying S-matrix Feynman rules to the rest of the diagram. It can be

proved, except for the graph of fig 2.4.2, that the term in square brackets is either zero or pure imaginary. As illustrations of the

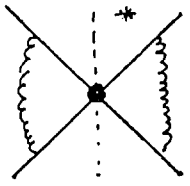


Fig 2.4.2: The graph for which (2.4.5) does not hold

arguments that have to be used consider the graphs of fig 2.4.3

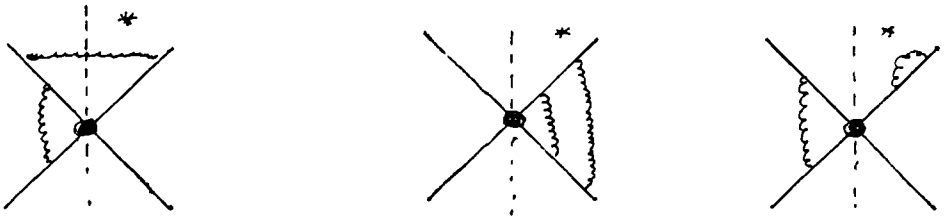
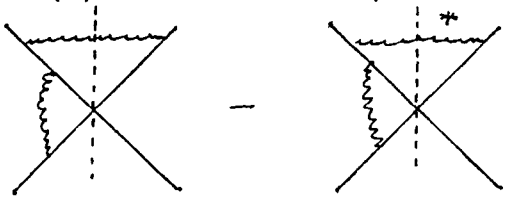


Fig 2.4.3: Graphs serving as illustrations of (2.4.5)

First take fig 2.4.3(a). The term in square bracket form (2.4.2) is

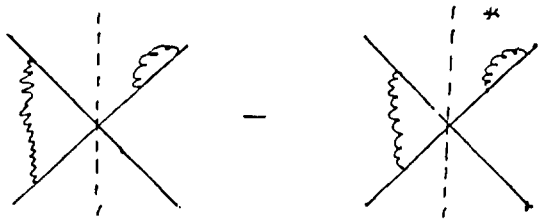
(2.4.3) 

The above is zero for the following reason: the term on the r.h.s. of the cut is a tree graph. As we are using S-matrix rules this is real. The term on the l.h.s. of the cut does not change in both diagrams - Thus (2.4.3) is zero.

The contribution of fig 2.4.3(b) to the square bracket of (2.4.2) is obviously pure imaginary.

With fig 2.4.3(c) greater care is required. We have to use the LSZ-formalism. Its contribution to the square bracket of (2.4.2) is

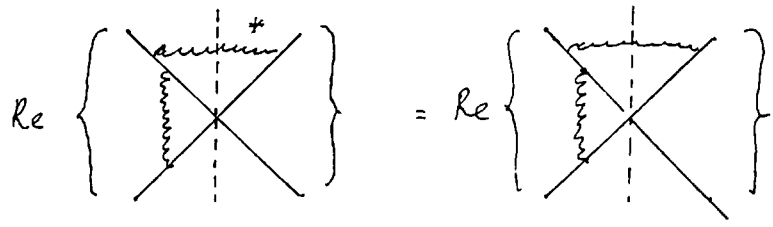
(2.4.4)



The contribution of the graph to the r.h.s. of the cut is $(Z_2^{-1} - 1)_{(\propto)}$ which as we have seen from (2.3.19) is real in the eikonal approximation. Thus (2.4.4) is zero.

So with the exception of fig 2.4.2 we can write for all graphs an identity analogous to (2.4.5)

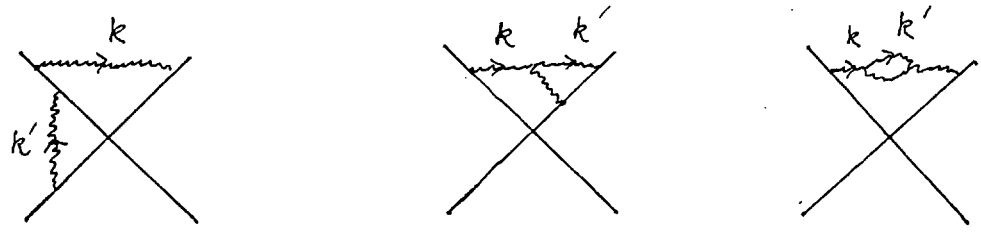
(2.4.5)



Can we now drop the cut that appears on the r.h.s. of (2.4.5)?

For graphs which do not have any gluons crossing the hard vertex - see fig 2.4.3(b) and (c), for example-this is trivially done.

Otherwise we have three general cases each with its own characteristic difficulty. These are illustrated in fig 2.4.4



(a) (b) (c)

Fig 2.4.4: Characteristic graphs used in the illustration of removal of the cuts

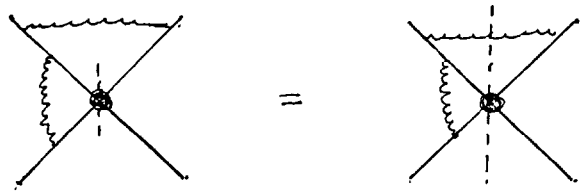
The k_o-integrals for fig. 2.4.4(a),(b), and (c) are given by A_a, A_b and A_c respectively where

$$(2.4.6)\quad \begin{cases} A_a = \displaystyle\int_{-\infty}^{\infty} \frac{dk_0}{[k_0^2 - \underset{\sim}{k}^2 + i\varepsilon][-2p\cdot k + i\varepsilon][-2p\cdot(k+k') + i\varepsilon]} \left[\frac{\Delta_s H}{2i}\right] \\ A_b = \displaystyle\int_{-\infty}^{\infty} \frac{dk_0\ V(k^2)}{[k_0^2 - \underset{\sim}{k}^2 + i\varepsilon][-2p\cdot k + i\varepsilon]^2} \left[\frac{\Delta_s H}{2i}\right] \\ A_c = \displaystyle\lim_{\substack{\lambda_1\to 0 \\ \lambda_2 \to 0}} \int_{-\infty}^{\infty} \frac{dk_0\ U(k^2)}{[k^2 - \lambda_1^2 + i\varepsilon][k^2 - \lambda_2^2 + i\varepsilon][-2p\cdot k + i\varepsilon]^2} \left[\frac{\Delta H}{2i}\right] \end{cases}$$

In the above all the k_0-dependeces have been explicitly shown. $\Delta_s H/2i$ represents cutting the hard diagram and is real.

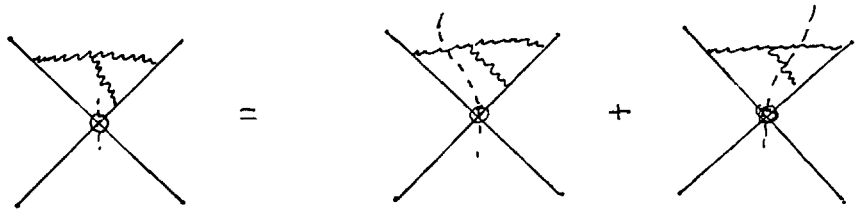
First consider A_a. The fermion poles all lie on one side of the k_0-contour thanks to the eikonal approximation. Pulling the k_0-contour the other way puts k^2 on mass-shell with positive energy. This of course corresponds to cutting the diagram giving us the identity (2.4.7)

(2.4.7)



In A_b again all the fermion poles lie on one side of the k_0-contour. $V(k^2)$ is the term obtained by doing the k'-integral. This has a cut running along the real axis from $k^2 = 0$ to ∞ . So when we pull the contour in the sense opposite to the fermion poles we pick up the k^2-pole with positive energy and the mass-cut in k^2 of the triangle, V. In graphical language we have (2.4.8)

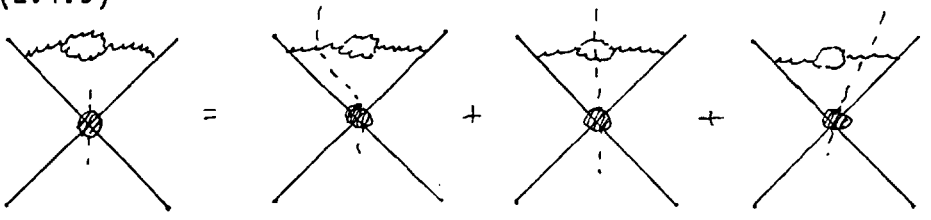
(2.4.8)



In A_c we have had to introduce gluonic masses λ_1 and λ_2 in order to treat the gluons with identical momenta as distinct. In fig 2.4.4(c)

when we put one of the gluons on mass-shell the other one automatically goes on shell. On the face of it we have an infinite contribution to the cross-section. But note that from dimensional arguments $U(k^2)$, which is obtained by doing the loop integral behaves like k^2. Thus there is no infinite discontinuity. As before all our poles lie on one side of the k_0-contour. Pulling the contour the other way we pick up the 2 terms coming from putting the masses λ_1 and λ_2 on mass-shell and another term from the mass-cut in k^2 of $U(k^2)$. Hence we have the identity (2.4.9)

(2.4.9)

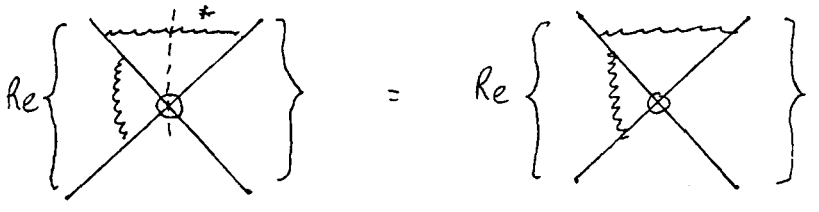


We may now let $\lambda_1^2, \lambda_2^2 \to 0$.

Similar reasoning applies to all other graphs. The above analysis indicates that we must be very careful about removing the cuts from the graphs. (2.4.8) and (2.4.9) clearly show it is a non-trivial matter.

We can now write an identity analogous to (2.4.10) for all graphs with the exception of fig 2.4.2 thanks to the eikonal approximation

(2.4.10)



CHAPTER 3

A COUNTER-EXAMPLE TO THE BLOCH-NORDSIECK THEOREM

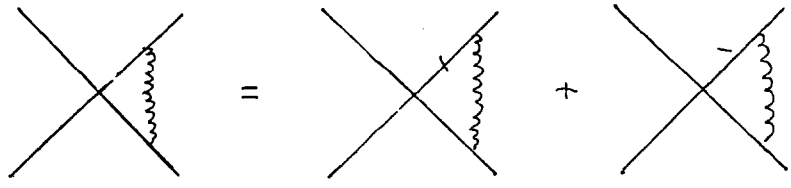
This chapter consists of four sections. In the first section we show how, as an illustration of the techniques developed in Chapter 2, infrared cancellation at $O(\alpha)$ is a trivial matter. Sections two, three and four deal with the $O(\alpha^2)$ calculation. In the first of these we reduce some of our graphs to simpler forms. The cancelling sector of the graphs is dealt with in section three. There are reasons like QED ones why some of the graphs should cancel and thus this section provides a useful check on our method. In section four we derive the analytic form of the non-cancelling sector of the graphs.

Throughout this chapter we will be making liberal use of the identity

3.1(a) $$\frac{1}{k^2-m^2+i\varepsilon} - \frac{1}{k^2-m^2-i\varepsilon} = (-2\pi i)\,\delta(k^2-m^2)$$

whose graphical equivalent is

3.1(b)



The cut means the fermion line is on mass-shell and the (-) indicates we have anti-Fenman propagators.

3.1 Infra-red Cancellation at $O(\alpha)$

The graphs which contribute at $O(\alpha)$ are shown in fig 3.1.1. Each

of the graphs from (v) to (ix) has an inverse. Under colour averaging they all have the same group theoretic weight.

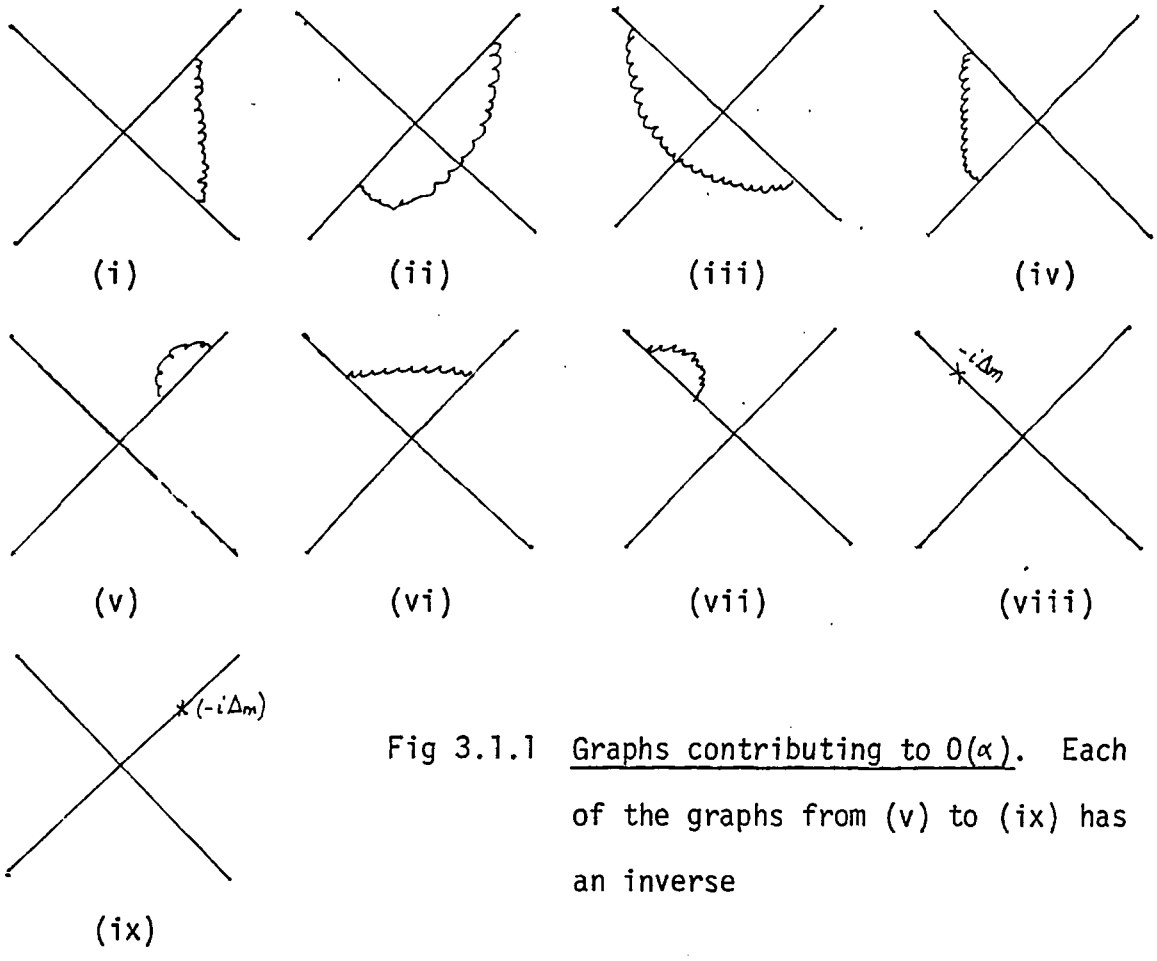
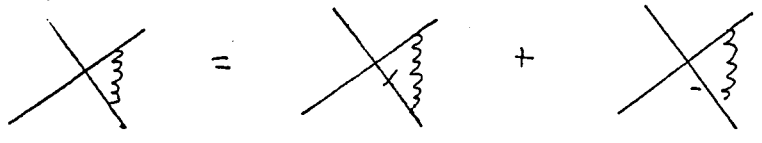
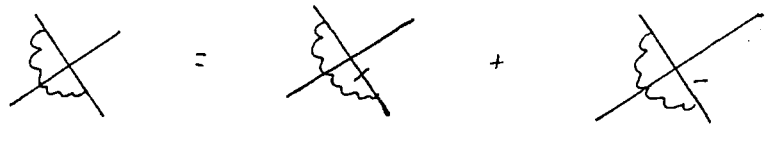


Fig 3.1.1 Graphs contributing to $O(\alpha)$. Each of the graphs from (v) to (ix) has an inverse

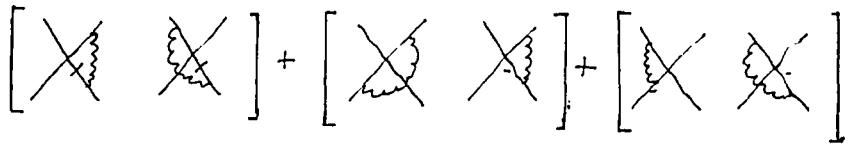
By the self-energy cancellation identity (2.3.17) graphs (v) to (ix) give zero. Similarly their inverses cancel.

Applying the identity 3.1(b) to (i) and (iii) gives

(3.1.1) 

(3.1.2) 

Using the above we arrive at

(3.1.3) $\sum$ graphs at $O(\alpha)$ = 

The last two square brackets of (3.1.3) each gives zero as a result of the Libby-Sterman identity. We re-write the first square bracket using 3.1(b) and get

(3.1.4) $\sum$ graphs at $O(\alpha)$ = 

The last two graphs on the r.h.s. of (3.1.4) sum to zero by the Libby-Sterman identity. The first graph is pure imaginary. Hence,

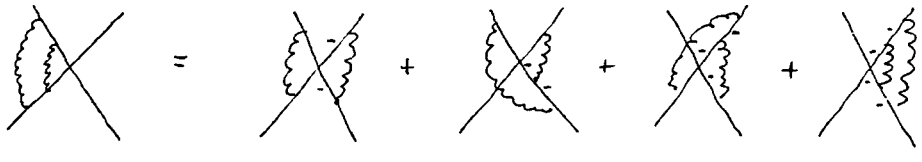
(3.1.5) $$Re\left\{\sum \text{graphs at } O(\alpha)\right\} = 0$$

(3.1.5) is the statement that i.r. divergences cancel at $O(\alpha)$.

3.2 Reduction of some of the graphs at $O(\alpha^2)$

We refer to Table I and specifically to groups 1 , 2 , 3 and 13 which are responsible for the non-cancellation of the i.r. divergences.

We apply the Libby-Sterman identity and also the identity 3.1(b) to the graphs in these groups. As an illustration of this consider

(3.2.1) 

The remaining diagrams are those of fig 3.2.1, 3.2.2, 3.2.3, 3.2.4, where these are obtained from groups 1 , 2 , 3 and 13 respectively.

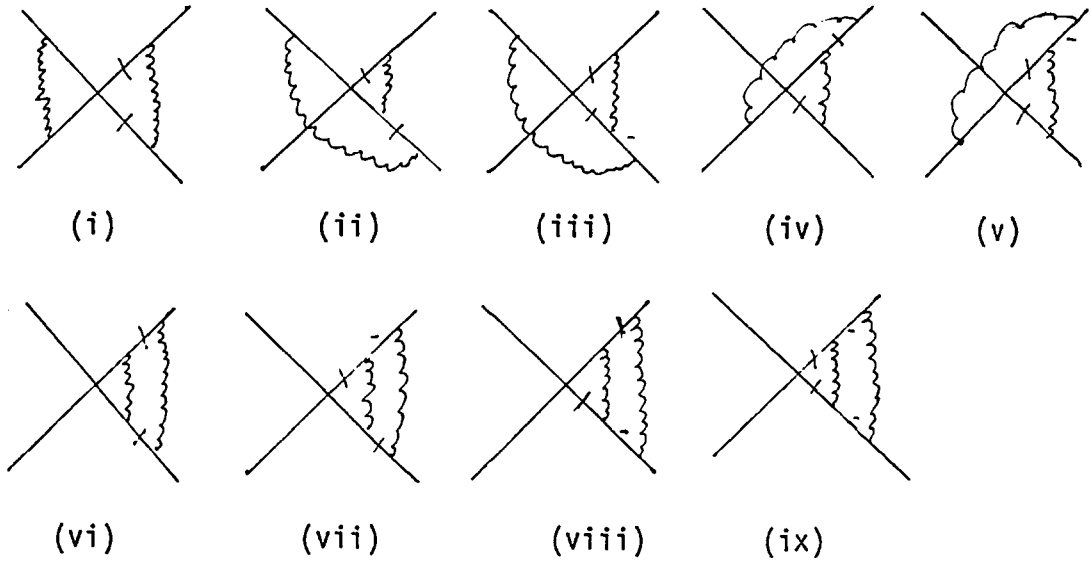


Fig 3.2.1: Graphs remaining after applying the Libby-Sterman identity and the identity 3.1(b) to group 1

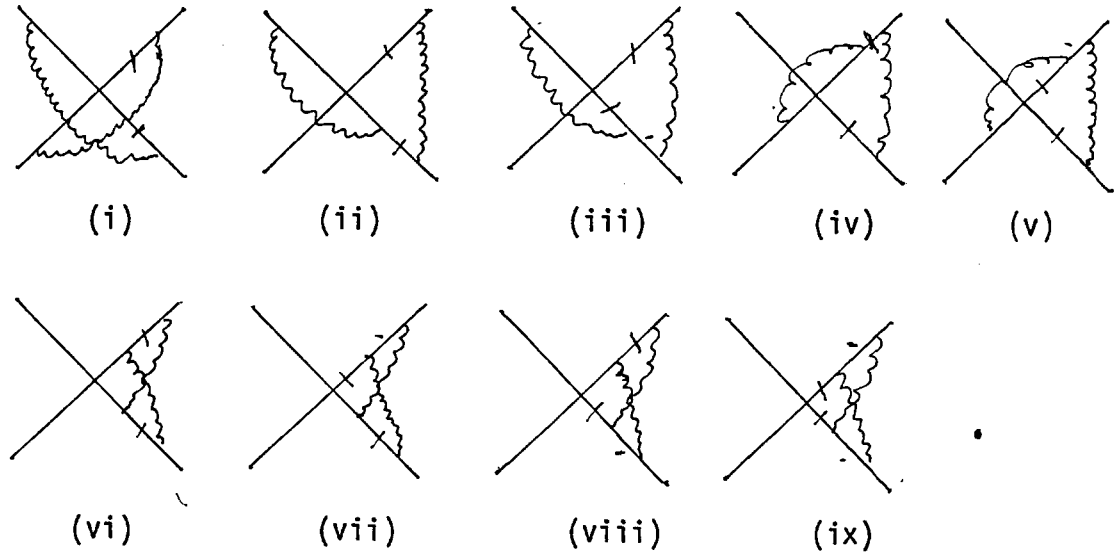


Fig 3.2.2: Graphs remaining after applying the Libby-Sterman identity and the identity 3.1(b) to group 2

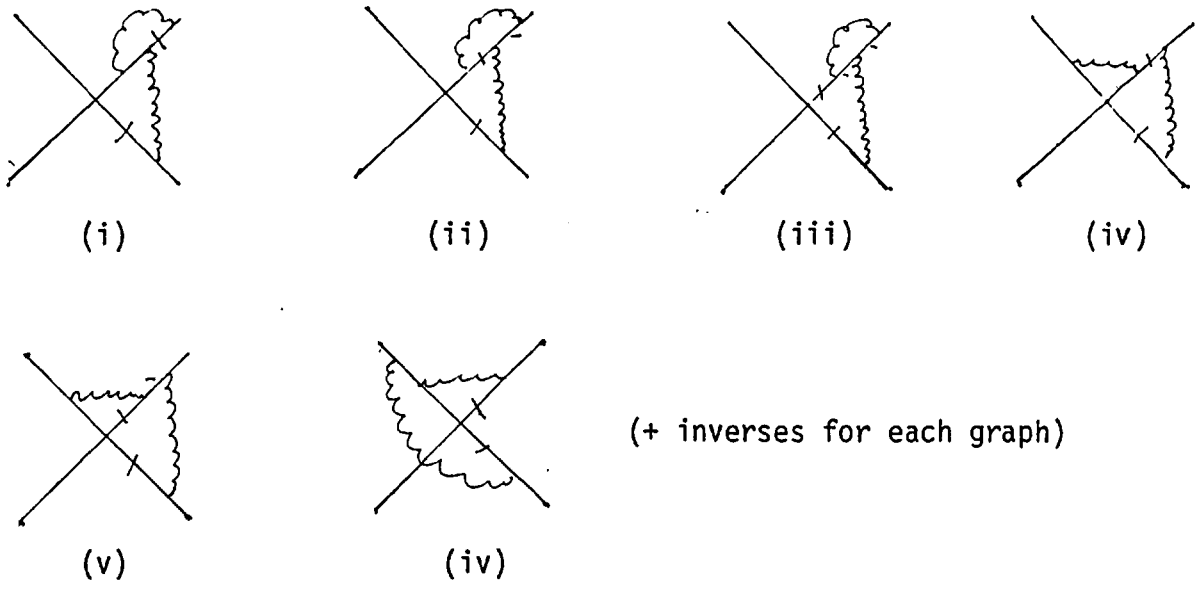


Fig 3.2.3: Graphs remaining after applying the Libby-Sterman identity and the identity 3.1(b) to group 3

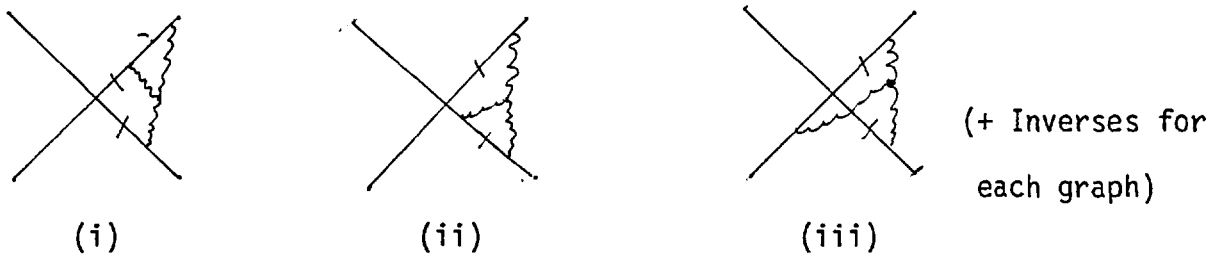


Fig 3.2.4: Graphs remaining after applying the Libby-Sterman identity and the identity 3.1(b) to group 13

In our calculation we will introduce a multiplicative factor of 2 for graphs having inverses. A priori this is NOT justified for reasons which will become apparent soon. Usually one argues that each graph is a Lorentz scalar. Hence in our case it can only be a function of $p^2 = m^2 = q^2$ and p.q. Thus under interchange of p and q the answer should remain invariant. However there is a snag. Recall that we are working in the frame $p = (M, \underset{\sim}{0})$, $q = (q_0, \underset{\sim}{q})$. After the

eikonal approximation we still let our integration variables run between $-\infty$ and $+\infty$. Consequently our graphs diverge at large k, the ultra-violet region. We are not interested in these divergences. We therefore impose an ultra-violet cut-off on our integrals by introducing an exponential factor of e^{-k^2/Λ^2} for each integration variable k where Λ is some arbitrary parameter. In principle therefore each graph is a function of Λ, a manifestly NON-COVARIANT object. Hence it is no longer true to say that each graph is equal to its inverse. However, it is not difficult to observe that our graphs split naturally into two groups, one being the inverse of the other. Of course there are graphs like fig 3.2.2(vi) which is the inverse of itself. In that case one uses the trivial identity $1 = \frac{1}{2} + \frac{1}{2}$ to put half of such graphs in one group and half in the other. It turns out at the end that the I.R. DIVERGENT part is Λ-INDEPENDENT and hence is a Lorentz scalar. So the argument of invariance under interchange of p and q is only true for SETS of graphs rather than individual graphs. Moreover, it only applies to the i.r. divergent part of the answer - the finite bit of the answer is Λ-dependent.

Some of the graphs obtained above do not give any contributions to the cross-section as they are pure imaginary. In fig 3.2.2 (ii), (iv) and (ix) are pure imaginary. In fig 3.2.3 such is the case for (iii), (iv) and their inverses. Also (i) in Fig 3.2.4 is pure imaginary They all can be expressed in the form iT_1T_2 where T_1T_2 are real.

Recall that the graph of fig. 2.4.2, unlike the other graphs, gives a contribution to the term in square bracket of (2.4.2). A simple calculation reveals that this is cancelled by (i) in fig 3.2.1.

In fig 3.2.1 the following pairs are inverses of each other

(ii, iv), (iii, v), (vii, viii). We prove now that (v) = (vii). Consider these as labelled in Fig 3.2.5

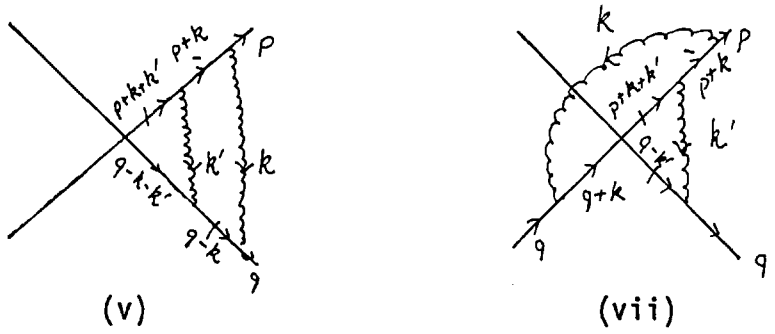


Fig 3.2.5. Two graphs from fig 3.2.1 which are equal

From the Feynman rules and neglecting terms common to both graphs we have

$$(3.2.2)\qquad (v) = \int \frac{d^4k\, d^4k'\, \delta(2p\cdot(k+k'))\, \delta(2q\cdot k)}{[k^2+i\varepsilon][k'^2+i\varepsilon][2p\cdot k - i\varepsilon][-2q\cdot(k+k')+i\varepsilon]}$$

$$(3.2.3)\qquad (vii) = \int \frac{d^4k\, d^4k'\, \delta(2p\cdot(k+k'))\, \delta(2q\cdot k)}{[k^2+i\varepsilon][k'^2+i\varepsilon][2p\cdot k - i\varepsilon][2q\cdot k + i\varepsilon]}$$

Using the fact that 2q.k = 0 from the δ-function we may re-write (3.2.2) as

$$(3.2.4)\qquad (v) = \int \frac{d^4k\, d^4k'\, \delta(2p\cdot(k+k'))\, \delta(2q\cdot k)}{[k^2+i\varepsilon][k'^2+i\varepsilon][2p\cdot k - i\varepsilon][-2q\cdot k' + i\varepsilon]}$$

In (3.2.3) let $k \leftrightarrow k'$. Use the δ -functions then to replace 2p.k' by -2p.k. Then let $k \longrightarrow -k$ and $k' \longrightarrow -k'$ and we then find that (3.2.3) = (3.2.4)

In fig 3.2.2 the following pairs are inverses of each other

(ii, iv), (iii, v), (vii, viii). As immediately above we can prove that (v) = (viii). Similarly it may be proved in fig 3.2.3 that (i) = (vi) and (ii) = (v). Also in fig 3.2.4 (ii) = (iii).

Bearing all the above considerations in mind we see that the graphs of fig 3.2.6, 3.2.7, 3.2.8, and 3.2.9 are the contributions of Groups 1 , 2 , 3 and 13 respectively where labels (a), (b), etc refer to the graph and not to the numerical factors.

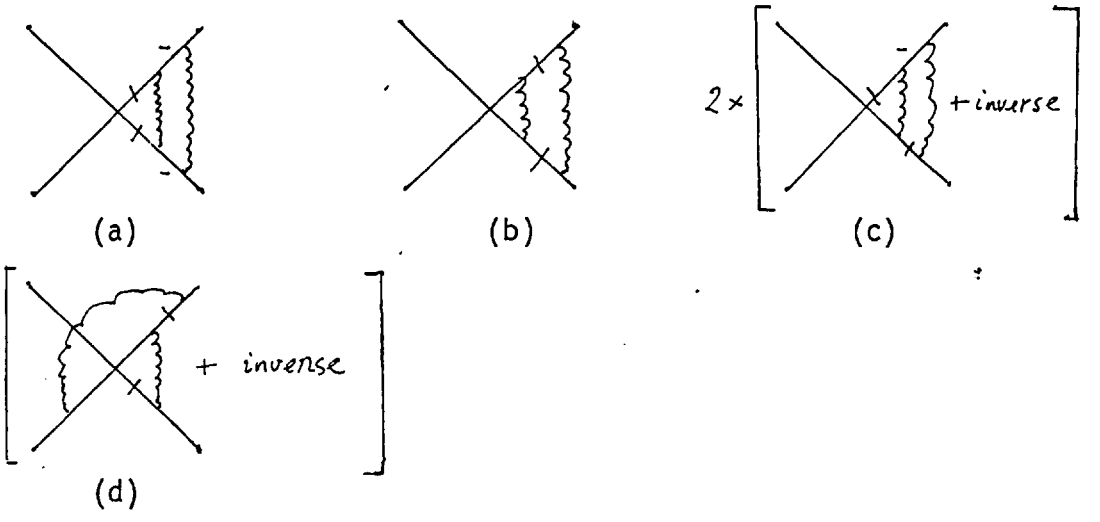


Fig 3.2.6: Final reduction of group 1

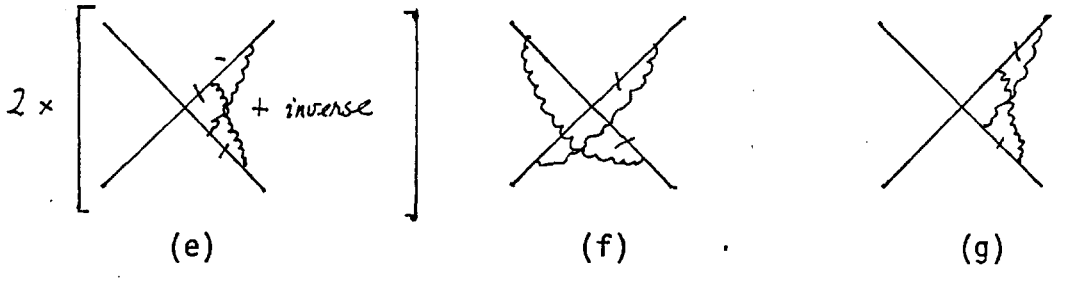


Fig 3.2.7: Final reduction of group 2

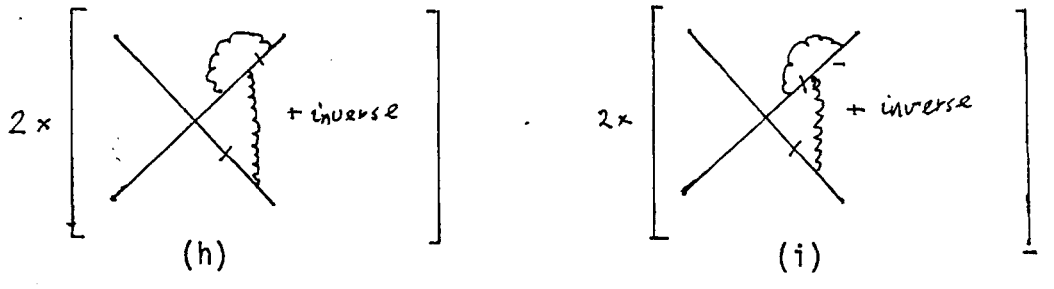
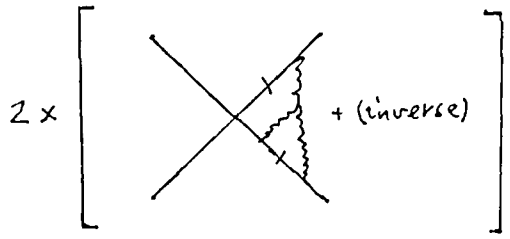


Fig 3.2.8 Final reduction of group 3



(j)

Fig 3.2.9: Final reduction of group 13

3.3 The cancelling Sector at $O(\alpha^2)$

(i) Cancellation of groups 6 , 9 , 14 , 15 , 16 and 17

Each of these groups gives zero trivially through the self-energy cancellation identity (2.3.17).

(ii) Cancellation of groups 7 and 8

These two groups cancel AMONG themselves. They are not really independent of each other - note they have identical colour structure. Also on cyclically permuting the gluons, graphs found in one group are also found in the other. We must therefore be careful not to double count the graphs. As can be seen from Table I some graphs in group 8 have two-particle irreducible self-energy insertions on their external legs. In such cases we must exercise care in the use of the self-energy cancellation identity. We must then take into account the term $\frac{3}{8}(Z_2^{-1} - 1)^2$ in (2.3.8). Otherwise the cancellation is trivial.

(iii) Cancellation of groups 10, 11 and 12

These groups reduce to the graphs shown in fig 3.3.1 by a

calculation identical to that of Section 3.1. From the kinematics of the diagram all the Q^2's are negative. Hence each of these graphs

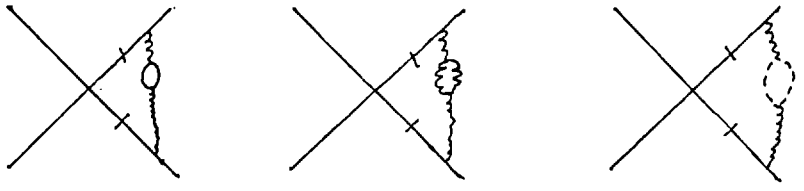


Fig 3.3.1:Reduction of graphs in groups 10 , 11 and 12

is essentially an S-matrix element whose only invariant Q^2 lies below normal threshold. Thus each is pure imaginary.

(iV) Cancellation of $C_2(F)^2$ and $C_2(F)(T_i)_{\alpha\beta}(T_i)_{ab}H^{\alpha\beta}_{ab}$ terms from Groups 3 , 4 and 5

The cancellation here is non-trivial compared to how it has been happening in the previous three cases.

In Appendix B we calculate the mass counterterm to $O(\alpha)$ that will be required in what follows. The analytic form of this is frame-dependent and for mass insertions on the p or p' leg these must be evaluated in the p-rest frame. As we have observed earlier, graphs with mass insertions are not infrared divergent. We keep them because they make the cancellation of integrands easier. Should we neglect such diagrams then we have to isolate terms like $\int dk' \int \frac{dk}{(p\cdot k)^2}$ which tend to zero in the i.r. limit, in diagrams with self-energy insertions on internal legs.

From fig 3.2.8 we know what the graphs are from group 3 . Using the Libby-Sterman identity,(3.1.b)and neglecting sums of graphs which are pure imaginary in groups 4 and 5 we find that we need to

calculate the SUM of the graphs of fig 3.3.2. Some of the graphs of groups 4 and 5 are equal like those of fig 3.2.5

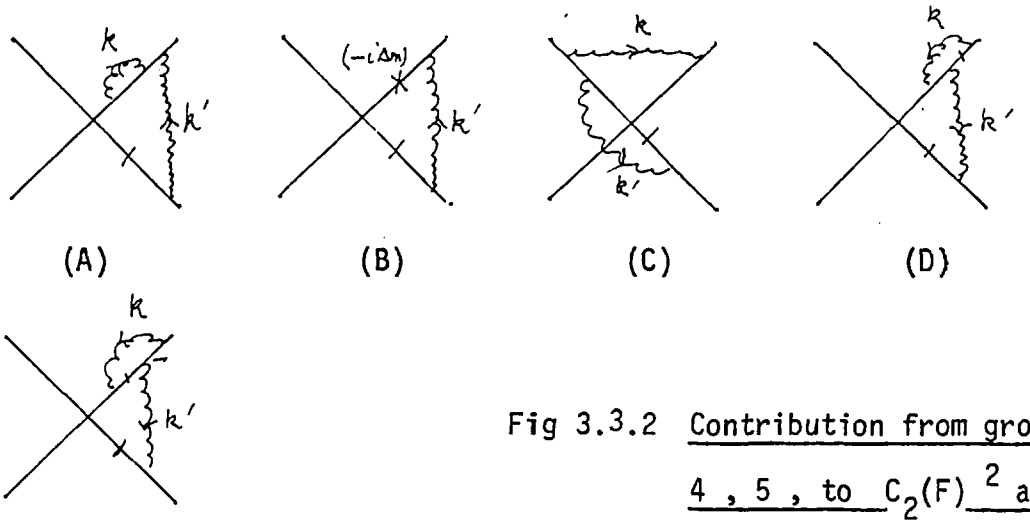


Fig 3.3.2 Contribution from groups 3 , 4 , 5 , to $C_2(F)^2$ and $C_2(F)(T_i)_{\alpha\beta}(T_i)_{ab}H^{\beta\alpha}_{ba}$

The first three graphs come from groups 4 and 5 and the last two from group 3 . Inverses are not being taken into account yet.

We now evaluate the graphs of fig 3.3.2. Doing the k_o and k_o' integrals we find (see (3.4.10) and (3.4.11) also)

$$(3.3.1)\quad \begin{cases} A = \frac{1}{2}\times\frac{1}{(2\pi)^6}\int \frac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'}{|\underset{\sim}{k}|\,[\underset{\sim}{k}'^2-K'^2][K'-i\varepsilon]^2\,[|\underset{\sim}{k}|+K'-i\varepsilon]} \\ B = -\frac{1}{2}\times\frac{1}{(2\pi)^6}\int \frac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'}{[\underset{\sim}{k}'^2-K'^2][K'-i\varepsilon]^2\,\underset{\sim}{k}^2} \\ C = \frac{1}{2}\times\frac{1}{(2\pi)^6}\int \frac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'}{\underset{\sim}{k}^3[\underset{\sim}{k}'^2-K'^2][|\underset{\sim}{k}|+K'-i\varepsilon]} \\ D = \frac{1}{(2\pi)^6}\int \frac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'}{\underset{\sim}{k}^2[\underset{\sim}{k}'^2-K'^2][K'+i\varepsilon]^2} \\ E = \frac{-1}{(2\pi)^6}\int \frac{d^3k\;d^3\underset{\sim}{k}'}{[\underset{\sim}{k}'^2-K'^2][K'+i\varepsilon]^2[\underset{\sim}{k}^2-K'^2+i\varepsilon]} \end{cases}$$

In A let $k' \longrightarrow -k'$, complex conjugate the result and add to A as given in (3.3.1) to yield

(3.3.2) $$\mathrm{Re}\,A = \frac{1}{(2\pi)^6}\,\frac{1}{2}\int \frac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'}{[\underset{\sim}{k}'^2-K'^2][K'-i\varepsilon]^2[\underset{\sim}{k}^2-(K'-i\varepsilon)^2]}$$

Similarly for C we find that

(3.3.3) $$\mathrm{Re}\,C = \frac{1}{(2\pi)^6}\cdot\frac{1}{2}\int \frac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'}{[\underset{\sim}{k}'^2-K'^2][\underset{\sim}{k}^2][\underset{\sim}{k}^2-(K'-i\varepsilon)^2]}$$

We also note that B = B* and D = D*. Hence

(3.3.4) $$\begin{cases} \mathrm{Re}\,B = B \\ \mathrm{Re}\,D = D \end{cases}$$

In D replace (K' + iε) by (K' - iε) as D = D*. Then

(3.3.5) $$\mathrm{Re}\,(B+D) = -B$$

In E first let k'→ -k'. Then we find

(3.3.6) $$E = \frac{-1}{(2\pi)^6}\int \frac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'}{[\underset{\sim}{k}'^2-K'^2][K'-i\varepsilon]^2[\underset{\sim}{k}^2-K'^2-i\varepsilon]}$$

Complex conjugate E using (3.3.1). We find

(3.3.7) $$E^* = \frac{-1}{(2\pi)^6}\int \frac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'}{[\underset{\sim}{k}'^2-K'^2][K'-i\varepsilon]^2[\underset{\sim}{k}^2-K'^2+i\varepsilon]}$$

Now

(3.3.8) $$\mathrm{Re}\,E = \mathrm{Re}\,E^*$$

Comparing (3.3.6) and (3.3.7) we see that insofar as the real part of E is concerned the sign of the iε in $[k^2 - K'^2 \pm i\varepsilon]$ is irrelevant. We,

therefore, choose to write this propagator as $k^2 - (K' - i\varepsilon)^2$ and hence obtain

$$(3.3.9) \qquad Re\, E = -\frac{1}{(2\pi)^6} Re \left\{ \int \frac{d^3\underset{\sim}{k}\, d^3\underset{\sim}{k}'}{[\underset{\sim}{k}'^2 - K'^2][K' - i\varepsilon]^2[\underset{\sim}{k}^2 - (K' - i\varepsilon)^2]} \right\}$$

Comparing (3.3.9) and (3.3.2) we find

$$(3.3.10) \qquad Re\,(E + A) = -Re\, A$$

Using (3.3.5) and (3.3.3) we get

$$(3.3.11) \qquad Re\{B + C + D\} = Re\, A$$

So from (3.3.10) and (3.3.11) we have

$$(3.3.12) \qquad Re\{A + B + C + D + E\} = 0$$

It therefore follows that the set of inverses also gives zero.

(3.3.12) means that the contribution of the $[C_2(F)]^2$ and $C_2(F)(T_i)_{\alpha\beta}(T_i)_{ab}H^{\alpha\beta}_{\alpha b}$ from groups 3 , 4 and 5 is zero.

This is a result to be expected from QED. The graphs contributing to these three groups have a unique property compared to the others in that they pick up a minus sign when we compare the processes e^-e^- scattering or e^+e^- annihilation .

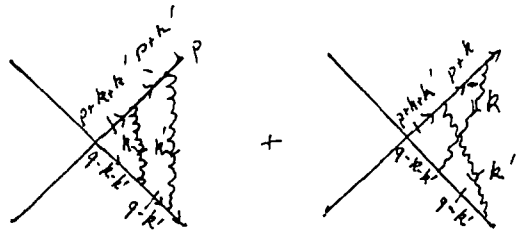
(v) Cancellation of $[C_2(F)]^2$ and $\frac{1}{4}\{T_i,T_j\}_{\alpha\beta}\{T_i,T_j\}_{ab}$ H terms from Groups 1 and 2

Since all the other QED-type graphs cancel (i.e. graphs with no gluon self-interation) these contributions must cancel because of the Bloch-Nordsieck mechanism in QED

We must ADD the graphs of figs 3.2.6 and 3.2.7. In the non-abelian case these get subtracted from each other.

We have

(3.3.13) $c + e =$



$$= \int \frac{d^4k\,d^4k'\,\delta(2p\cdot(k+k'))\,\delta(2q\cdot k')(4p\cdot q)^2}{[k^2+i\varepsilon][k'^2+i\varepsilon][-2q\cdot(k+k')+i\varepsilon]} \left\{ \frac{1}{2p\cdot k'-i\varepsilon} + \frac{1}{2p\cdot k-i\varepsilon} \right\}$$

Using the δ-function constraint we replace 2p.k by -2p.k' in the last term in curly bracket. Using the distribution identity

(3.3.14) $$\frac{1}{x+i\varepsilon} - \frac{1}{x-i\varepsilon} = -2\pi i\,\delta(x)$$

we may replace the term in curly bracket by $2\pi i\,\delta(2p.k')$. Hence we have

(3.3.15) $$c + e = (2\pi i)\int \frac{d^4k\,d^4k'\,\delta(2p.k)\,\delta(2p.k')\,\delta(2q.k')(4p\cdot q)^2}{[k^2+i\varepsilon][k'^2+i\varepsilon][-2q\cdot k+i\varepsilon]}$$

In (3.3.15) we next let $k \to -k$ and adding the resultant expression to it we find using (3.3.14) that

$$c+e = \frac{(4p\cdot q)^2}{2}(2\pi i)(-2\pi i)\int \frac{d^4k\,d^4k'\,\delta(2p\cdot k)\delta(2p\cdot k')\,\delta(2q\cdot k)\,\delta(2q\cdot k')}{[k^2+i\varepsilon][k'^2+i\varepsilon]}$$

We can now do the k_o and k_o' integral to give

$$(3.3.16) \qquad c + e = 2\pi^2 \left[\int \frac{d^3\underset{\sim}{k}\, \delta(K)}{\underset{\sim}{k}^2} \right]^2$$

Now consider

$$(3.3.17) \qquad d = \int \frac{d^4k\, d^4k'\, (4p\cdot q)^2 \delta(2p\cdot k)\, \delta(2q\cdot k')}{[k^2+i\varepsilon][k'^2+i\varepsilon][2q\cdot k+i\varepsilon][2p\cdot(k+k')+i\varepsilon]}$$

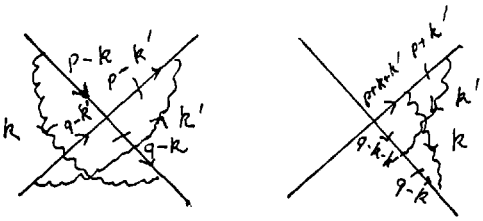
Using steps similar to above we can prove that

$$(3.3.18) \qquad d = -\pi^2 \int \frac{d^4k\, d^4k'\, (4p\cdot q)^2\, \delta(2p\cdot k)\, \delta(2p\cdot k')\, \delta(2q\cdot k)\, \delta(2q\cdot k')}{[k^2+i\varepsilon][k'^2+i\varepsilon]}$$

$$(3.3.19) \qquad \therefore \quad d = -\pi^2 \left[\int \frac{d^3\underset{\sim}{k}\, \delta(K)}{\underset{\sim}{k}^2} \right]^2$$

Next consider

$$(3.3.20) \qquad f+g =$$

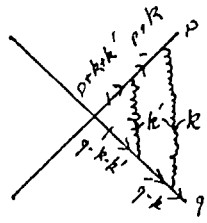


$$= \int \frac{d^4k\, d^4k'\, (4p\cdot q)^2 \delta(2p\cdot k')\, \delta(2q\cdot k)}{[k^2+i\varepsilon][k'^2+i\varepsilon]} \left\{ \frac{1}{[-2p\cdot k+i\varepsilon][-2q\cdot k'+i\varepsilon]} + \frac{1}{[2p\cdot(k+k')+i\varepsilon][-2q\cdot(k+k')+i\varepsilon]} \right.$$

Using steps analogous to those we used to obtain (3.3.16) we find

$$(3.3.21) \qquad f+g = -2\pi^2 \left[\int \frac{d^3\underset{\sim}{k}\, \delta(K)}{\underset{\sim}{k}^2} \right]^2$$

Now consider

(3.3.22) $a =$  $= \int \frac{d^4k\, d^4k'\, \delta(2p\cdot(k+k'))\,\delta(2q\cdot(k+k'))\,(4p\cdot q)^2}{[k^2+i\varepsilon][k'^2+i\varepsilon'][2p\cdot k-i\varepsilon][-2q\cdot k-i\varepsilon]}$

where

(3.3.23) $$\varepsilon' > \varepsilon$$

Thus we have

$$a = \int \frac{d^4k\, d^4k'\, \delta(k_0+k_0')\,\delta(K+K')}{[k^2+i\varepsilon][k'^2+i\varepsilon][k_0-i\varepsilon][-k_0+K-i\varepsilon]}$$

We first do the k_0 integral using the δ-function and we find

$$a = \int \frac{d^3\underset{\sim}{k}\, d^4k'\, \delta(K+K')}{[k_0'^2-\underset{\sim}{k}^2+i\varepsilon][k_0'^2-\underset{\sim}{k}'^2+i\varepsilon'][-k_0'-i\varepsilon][k_0'+K-i\varepsilon]}$$

Do k_0'-integral by closing contour in the lower half plane. By making use of the fact that K = -K' from the δ-function and $\varepsilon' > \varepsilon$ we find

$$a = (-2\pi i)\int d^3\underset{\sim}{k}\, d^3\underset{\sim}{k}'\, \delta(K+K') \times \left\{ \frac{1}{2|\underset{\sim}{k}|[\underset{\sim}{k}^2-\underset{\sim}{k}'^2+i\varepsilon'][-|\underset{\sim}{k}|][|\underset{\sim}{k}|+K]} + \frac{1}{2|\underset{\sim}{k}'|[\underset{\sim}{k}'^2-\underset{\sim}{k}^2-i\varepsilon'][-|\underset{\sim}{k}'|][|\underset{\sim}{k}'|+K']} + \frac{1}{[-\underset{\sim}{k}^2][-\underset{\sim}{k}'^2][K-i\varepsilon]} \right\}$$

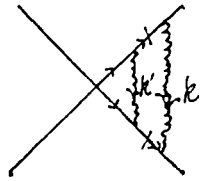
In the first term let $\underset{\sim}{k} \longrightarrow -\underset{\sim}{k}'$ and $-\underset{\sim}{k}' \longrightarrow \underset{\sim}{k}$ It then follows that the sum of the first two terms is imaginary. Moreover since we are only interested in real part of a , in the third term we may let

$$\frac{1}{K-i\varepsilon} \longrightarrow i\pi\,\delta(K)$$

and find

(3.3.24) $$Re\, a = -2\pi^2 \left[\int \frac{d^3\underset{\sim}{k}\, \delta(K)}{\underset{\sim}{k}^2} \right]^2$$

Finally consider

(3.3.25) $b =$  $= \int \frac{d^4k\, d^4k' (4p\cdot q)^2 \delta(2p\cdot k)\delta(2q\cdot k)}{[k^2+i\varepsilon][k'^2+i\varepsilon][2p-(k+k')+i\varepsilon][-2q\cdot(k+k')+i\varepsilon]}$

Doing the k_o integral we find

$$b = \int \frac{d^3\underset{\sim}{k}\, d^4k'\, \delta(K)}{[-\underset{\sim}{k}^2][k'^2+i\varepsilon][k_o'+i\varepsilon][-k_o'+K'+i\varepsilon]}$$

Next closing contour in the lower half plane we get

(3.3.26) $$b = (-2\pi i)\int d^3\underset{\sim}{k}\, d^3\underset{\sim}{k}'\, \delta(K)\left\{ \frac{1}{[-\underset{\sim}{k}^2][2\underset{\sim}{k}'][-\underset{\sim}{k}'][-\underset{\sim}{k}'+K']} + \frac{1}{[-\underset{\sim}{k}^2][-\underset{\sim}{k}'^2][K'+i\varepsilon]} \right\}$$

The first term in (3.3.26) is pure imaginary. Since we only want the real part of b we can replace $1/(K' + i\varepsilon)$ by $-i\pi\delta(K')$ in the last term. We then find

(3.3.27) $$Re\, b = -2\pi^2 \left[\int \frac{d^3\underset{\sim}{k}\, \delta(K)}{\underset{\sim}{k}^2} \right]^2$$

Collecting all these results we find

(3.3.28)
$$\begin{cases} Re\, a = Re\, b = -2\pi^2 \left[\int \frac{d^3\underset{\sim}{k}\, \delta(K)}{\underset{\sim}{k}^2} \right]^2 \\ c + e = 2\pi^2 \left[\int \frac{d^3\underset{\sim}{k}\, \delta(K)}{\underset{\sim}{k}^2} \right]^2 \\ d = -\pi^2 \left[\int \frac{d^3\underset{\sim}{k}\, \delta(K)}{\underset{\sim}{k}^2} \right]^2 \\ f + g = -2\pi^2 \left[\int \frac{d^3\underset{\sim}{k}\, \delta(K)}{\underset{\sim}{k}^2} \right]^2 \end{cases}$$

Note that all the above have been expressed as totally symmetric functions of p and q. Hence we are justified to introduce a factor of 2 for inverses. Going back to fig 3.2.6 and fig 3.2.7 it is a trivial matter to check using (3.3.28) that the sum of the graphs in these two groups gives zero.

3.4 The non-cancelling sector at $O(\alpha^2)$

. We are now left with only the contributions of $C_2(F)C_2(G)$ in the case of Drell-Yan annihilation coming from groups 2 , 3 and 13 and that of $C_2(G)(T_i)^{\alpha\beta}(T_j)_{ab}H^{\beta\alpha}_{ba}$ in the case of Drell-Yan bremsstrahlung coming from groups 1 , 2 , 3 , 13 .

(i) Contribution from groups 1 and 2

We have seen in section 3.3(v) that, neglecting colour factors,

(3.4.1) $\sum'$ graph in group ① + $\sum'$ graphs in group ② = 0

It, therefore, follows that

(3.4.2) $\sum'$ graphs in group ① = $-\sum'$ graphs in group ②

Let us consider the contribution D where, neglecting colour factors,

(3.4.3) $D = \sum'$ graphs in group ① $-\sum'$ graphs in group ②

We go back to (3.3.28) and fig 3.2.6 and 3.2.7 We see that

(3.4.4) $$d + \text{inverse} - f - g = 0$$

(3.4.5) $$Re(a) + Re(b) = -4\pi^2\left[\int \frac{d^3\underset{\sim}{k}\,\delta(K)}{\underset{\sim}{k}^2}\right]^2$$

It is trivial matter to check from (3.3.13) that

(3.4.6) $$c - e = \int \frac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'\,2K'}{[K'^2-\underset{\sim}{k}^2+i\varepsilon][K'^2-\underset{\sim}{k}'^2][K+K'+i\varepsilon][K'^2+\varepsilon^2]}$$

It is shown in Appendix D (see D.6 and D.22)

(3.4.7) $$\lim_{n\to 3}\int \frac{d^n\underset{\sim}{k}\,\delta(K)\,e^{-\underset{\sim}{k}^2/\Lambda^2}}{\underset{\sim}{k}^2} = \frac{2\pi^{(n-1)/2}}{\beta}\frac{(\Lambda^2)^{(n-3)/2}}{(n-3)}$$

(3.4.8) $$\lim_{n\to 3}\int \frac{d^n\underset{\sim}{k}\,d^n\underset{\sim}{k}'\,e^{-(\underset{\sim}{k}^2+\underset{\sim}{k}'^2)/\Lambda^2}\,2K'}{[K'^2-\underset{\sim}{k}^2+i\varepsilon][K'^2-\underset{\sim}{k}'^2][K+K'+i\varepsilon][K'^2+\varepsilon^2]} = \frac{4\pi^{n+1}(\Lambda^2)^{n-3}}{\beta^2(n-3)^2}\left\{1+\frac{(n-3)}{2}\log\left(\frac{1+\beta}{1-\beta}\right)\right\}$$

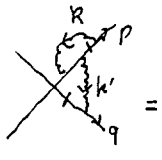
Using these results it immediately follows that

(3.4.9) $$\sum \text{graphs of group ①} = -\sum' \text{graphs of group ②} = \frac{\pi^4}{\beta^2}\cdot\frac{1}{(n-3)}\log\left(\frac{1+\beta}{1-\beta}\right)$$

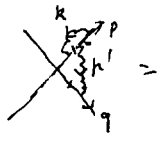
(ii) <u>Contributions from group 3</u>

The graphs contributing are those of fig 3.2.8.

We have

(3.4.10) $h =$ 

$$= \int \frac{d^4k\,d^4k'\,4p\cdot q\,4M^2\,\delta(2p\cdot k)\,\delta(2q\cdot k')}{[k^2+i\varepsilon][k'^2+i\varepsilon][2p\cdot(k+k')+i\varepsilon][2p\cdot k'+i\varepsilon]}$$

(3.4.11) $i =$ 

$$= \int \frac{d^4k\,d^4k'\,4p\cdot q\,4M^2\,\delta(2p\cdot(k+k'))\,\delta(2q\cdot k')}{[k^2+i\varepsilon][k'^2+i\varepsilon][2p\cdot k'+i\varepsilon][2p\cdot k-i\varepsilon]}$$

Doing the k_o and k_o' integrals we find

$$(3.4.12) \qquad h = \int \frac{d^3\underset{\sim}{k}\, d^3\underset{\sim}{k}'}{\underset{\sim}{k}^2 [\underset{\sim}{k}'^2 - K'^2][K' + i\varepsilon]^2}$$

$$(3.4.13) \qquad i = -\int \frac{d^3\underset{\sim}{k}\, d^3\underset{\sim}{k}'}{[\underset{\sim}{k}^2 - K'^2 - i\varepsilon][\underset{\sim}{k}'^2 - K'^2][K' + i\varepsilon]^2}$$

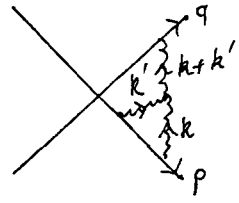
From (3.4.12) h is not i.r. divergent since the factorization of the k and k' integrals implies that the k-integral gives a term proportional to Λ where Δ is some i.r. cut off. In the dimensional regularization scheme this term is regular at n = 3 although it has a pole at n = 4. We shall evaluate it and show that this peculiar behaviour gets killed by an identical term hiding in i. In Appendix D (see D.32) we find that

$$(3.4.14) \qquad \lim_{n \to 3} (h + i) = -\frac{2\pi^{n+1} (\Lambda^2)^{n-3}}{\beta(n-3)}$$

(iii) Contribution from group 13

There is only one graph to be calculated and it is shown in fig 3.2.9. We have

$$(3.4.15) \qquad j =$$



Taking into account the factor of i to be found in Table I we find

$$(3.4.16) \qquad j = -8 \int \frac{d^4k\, d^4k' \, [M^2 q\cdot(k-k') - p\cdot q\; p\cdot(k-k')]\, \delta(2p\cdot k)\, \delta(2q\cdot(k+k'))}{[k^2 + i\varepsilon][k'^2 + i\varepsilon][(k+k')^2 + i\varepsilon][2p\cdot(k+k') + i\varepsilon]}$$

Changing variables to $\bar{k}$ where

$$k' = \bar{k} - k$$

we find

$$j = -8\int \frac{d^4k\,d^4\bar{k}\,[M^2 q\cdot(2k-\bar{k}) - p\cdot q\; p\cdot(2k-\bar{k})]\,\delta(2p\cdot k)\,\delta(2q\cdot\bar{k})}{[k^2+i\varepsilon][(k-\bar{k})^2+i\varepsilon][\bar{k}^2+i\varepsilon][2p\cdot\bar{k}+i\varepsilon]}$$

We now do the k_0 and $\bar{k}_0$ integrals using the δ-functions. This gives

$$j = -\int \frac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'\,(K'-2K)}{\underset{\sim}{k}^2(\underset{\sim}{k}'^2-K'^2)(K'^2-(\underset{\sim}{k}-\underset{\sim}{k}')^2+i\varepsilon)(K'+i\varepsilon)} \tag{3.4.17}$$

wehre we have set $\bar{k} = k'$. In appendix D we evaluate the n-dimensional ultraviolet cut-off analogue of (3.4.17) (see D.50) to get

$$\lim_{n\to 3}\int \frac{d^n\underset{\sim}{k}\,d^n\underset{\sim}{k}'\,(K'-2K)\,e^{-(\underset{\sim}{k}^2+\underset{\sim}{k}'^2)/\Lambda^2}}{\underset{\sim}{k}^2(\underset{\sim}{k}'^2-K'^2)(K'^2-(\underset{\sim}{k}'-\underset{\sim}{k})^2+i\varepsilon)(K'+i\varepsilon)} = \frac{\pi^{n+1}(\Lambda^2)^{n-3}}{(n-3)}\left[\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right)-2\right] \tag{3.4.18}$$

(iv) the total contribution

Collecting together (3.4.9), (3.4.14) and (3.4.18) we find that the infrared divergent part of the answer is of the form

$$\text{flux}\,\frac{d\sigma}{d^3Q} = \mp\frac{g^4 H}{32\pi^2}\cdot\frac{1}{(n-3)}\left[\frac{1}{\beta}-1\right]\left[\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right)-2\right] \tag{3.4.19}$$

where $\mp$ refers to Drell-Yan bremsstrahlung and annihilation respectively. We have also taken into account a factor of $g^4/(2\pi)^6$ coming from the

Feynman rules. Here H represents the discontinuity of our hard diagram. We have neglected the colour factors $C_2(F)C_2(G)$ in the case of bremsstrahlung and $(T_i)_{ab}(T_i)_{\beta\alpha} H^{ba}_{\alpha\beta}$ in the case of annihilation, and factors of 1/N coming from colour averaging.

3.4 The eikonal approximation re-visited

Throughout the above calculation we have adopted the eikonal approximation of writing Fermion numerators as $(\not{p} - m)$ and fermion denominators as $(2p.k + i\varepsilon)$ dropping higher terms in the soft gluon momenta k.

It is rather straightforward to check that the double pole terms coming from (3.4.5) (see 3.4.7also) and (3.4.8) will disappear on keeping corrections to the eikonal approximation. However we see that the coefficients of the double pole terms involve $(\Lambda^2)^{n-3}$ factors where Λ^2 was our k and k' ultra-violet cut-off. In our final answer we assumed that these cut-offs should be the same for all our diagrams. We then find no Λ^2-dependence (and hence obtain Lorentz invariance) of the final answer even at the single pole level.

We believe this to be the current procedure. The Λ^2-dependence is artificial. We should be calculating the (n-3) infrared poles in the full amplitude without the eikonal approximations. However, we know the eikonal approximation is only valid in a given region of the momentum integrals. Thus $\Lambda^2$is really fixed by where we believe the eikonal to be a good approximation to the integrand. In all the double log terms this is fixed by when we may drop k, k' in the same numberators and drop k^2, $(k + k')^2$, k'^2 in the same denominators. Thus Λ^2 will be equal for our various terms.

CHAPTER 4

CONCLUSIONS AND DISCUSSIONS

Infrared divergences must be consistant with the Kinoshita-Lee-Naunberg (KLN) theorem [22] which says that they must cancel when degenerate initial states are taken into account - these involve disconnected graphs as well. The Block-Nordsieck Theorem involves only degenerate final states. This KLN cancellation has been checked [23].

Cross-sections are gauge invariant quantities and consequently our divergent answer should be. In [23] the same calculations have been done in the Coulomb gauge and the results check.

There is one immediate observation to be made. The form of our answer implies that the existence of the i.r. divergences is independent of the number of quark flavours. It is therefore a property of the non-abelian nature of the theory rather than that of asymptotic freedom.

It is perhaps interesting to make the following observation: we have seen that the i.r. divergences at $O(\alpha)$ cancel (see section 3.1). The cancellation there happens as in QED i.e. between real and virtual gluon processes. The graphs after colour averaging are then really no different from QED ones. If however one calculates only the virtual graphs (or only the real ones for that matter) then one finds that the answer is of the form $\frac{\alpha}{(n-3)}\left[\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right)-2\right]$.

This is to be compared with (3.4.19) and we immediately see that the i.r. divergences at $O(\alpha^2)$ is precisely that, up to colour factors, but modified by a factor $\alpha\left(\frac{1}{\beta}-1\right)$. The $1/\beta$ comes from graphs with NO gluon

self-interaction whereas the other term comes from graphs with gluon self-interactions. One immediate consequence of our answer is that one cannot believe the results of perturbation theory in QCD at some finite order in α. One thing that could possibly rectify this unfortunate state of affairs is that the i.r. divergences calculated to all orders in α exponentiate to give a factor $e^{-\infty}$ x S-matrix element (providing confinement?). The form of our answer would perhaps suggest that such an exponentiation CANNOT take place because one way this could happen is that at each order in α^{N+1} the i.r. divergences be of the form: the i.r. divergences at $O(\alpha^N)$ of only the virtual part of the ABELIAN sector of the theory modified by a factor of $(\frac{1}{\beta} - 1)^n$. Note that the i.r. divergences at $O(\alpha^2)$ obeys this ansatz up to colour factors. It would require a miracle for this ansatz to generalise to all orders especially if one expects the colour factors not to mess things up. A study of the i.r. divergences at $O(\alpha^3)$ may throw some light on the matter but with the techniques available it would be a somewhat daunting task. An attempt has been made to investigate the form of the answer at higher orders in [24] and the preliminary results, which deals with the leading non-cancelling i.r. divergences, suggests that such an exponentiation does not take place.

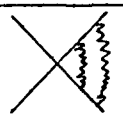
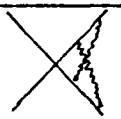
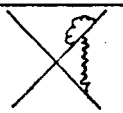
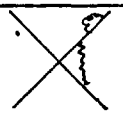
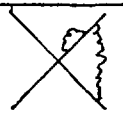
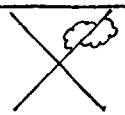
The existence of i.r. divergences is almost certainly related to the problem of confinement. In using perturbation theory in QCD we are not using the real asymptotic states to calculate the S-matrix elements. Presumably if one took into account the fact that there are only colour singlet states in the real world then the i.r. divergences would vanish and manifest themselves as a logR term where R is a measure of the size of the hadronic system. The existence of such terms might have serious implications for leading log calculations [4]

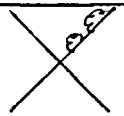
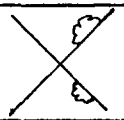
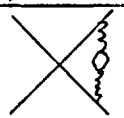
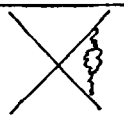
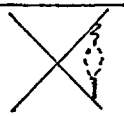
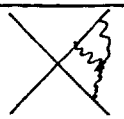
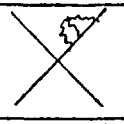
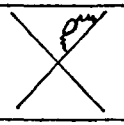
There one neglects $\alpha^k \log Q^2$ relative to $\log^n Q^2$ where α in the running coupling constants and behaves like $1/\log Q^2$ and Q^2 is some high energy scale which characterizes the hadronic process. If $\log RQ^2$ are present one would expect regulated terms like $\alpha \log^n Q^2$ to be modified to $\log RQ^2 \log^n Q^2 \sim \log^n Q^2$.

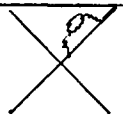
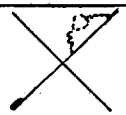
A final observation we would like to make is that the i.r. divergences behaves like $1/(Q^2)^4$. This in OPE language corresponds to "higher twist" effects which in general have not been calculated. It would be interesting to do these calculations there, also a daunting task, and compare the results.

TABLE 1: The Group Theoretic Weight contribution of the graphs. All the graphs that are obtained by cyclic permutations of the gluons are understood to be included. In certain groups one must add inversion graphs and mass insertion ones as well.

Legend : GLUONS FERMIONS GHOSTS BASIC HARD DIAGRAM

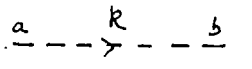
GROUP	GRAPHS	GROUP THEORETIC WEIGHT FOR Q-$\bar{Q}$	GROUP THEORETIC WEIGHT FOR Q-Q
1		$\left[C_2(F)\right]^2$	$\frac{1}{4}\{T_iT_j\}_{\alpha\beta}\{T_iT_j\}_{ab}\,H^{\beta\alpha}_{ba} - \frac{1}{4}C_2(G)(T_i)_{\alpha\beta}(T_j)_{ab}\,H^{\beta\alpha}_{ba}$
2		$\left[C_2\ (F)\right]^2 - \frac{1}{2}C_2(F)C_2(G)$	$\frac{1}{4}\{T_iT_j\}_{\alpha\beta}\{T_iT_j\}_{ab}\,H^{\beta\alpha}_{ba} + \frac{1}{4}C_2(G)(T_i)_{\alpha\beta}(T_j)_{ab}\,H^{\beta\alpha}_{ba}$
3		$\left[C_2\ (F)\right]^2 - \frac{1}{2}C_2(F)C_2(G)$	$C_2(F)(T_i)_{\alpha\beta}(T_i)_{ab}\,H^{\beta\alpha}_{ba} - \frac{1}{2}C_2(G)(T_i)_{\alpha\beta}(T_i)_{ab}\,H^{\beta\alpha}_{ba}$
4		$\left[C_2(F)\right]^2$	$C_2(F)(T_i)_{\alpha\beta}(T_i)_{ab}\,H^{\beta\alpha}_{ba}$
5		$\left[C_2(F)\right]^2$	$C_2(F)(T_i)_{\alpha\beta}(T_i)_{ab}\,H^{\beta\alpha}_{ba}$
6		$\left[C_2\ (F)\right]^2 - \frac{1}{2}C_2(F)C_2(G)$	$\left[C_2\ (F)\right]^2 H^{\alpha\alpha}_{aa} - \frac{1}{2}C_2(F)C_2(G)H^{\alpha\alpha}_{aa}$
7		$\left[C_2\ (F)\right]^2$	$\left[C_2\ (F)\right]^2 H^{\alpha\alpha}_{aa}$

GROUP	GRAPHS	GROUP THEORETIC WEIGHT FOR Q-$\bar{Q}$	GROUP THEORETIC WEIGHT FOR Q-Q
8		$\left[C_2\ (F)\right]^2$	$\left[C_2\ (F)\right]^2 H^{\alpha\alpha}_{aa}$
9		$\left[C_2\ (F)\right]^2$	$\left[C_2\ (F)\right]^2 H^{\alpha\alpha}_{aa}$
10		$\left[C_2\ (F)\right]^2$	$C_2(F)(T_i)_{\alpha\beta}(T_i)_{ab}\ H^{\beta\alpha}_{ba}$
11		$C_2(F)C_2(G)$	$C_2(G)(T_i)_{\alpha\beta}(T_i)_{ab}\ H^{\beta\alpha}_{ab}$
12		$C_2(F)C_2(G)$	$C_2(G)(T_i)_{\alpha\beta}(T_i)_{ab}\ H^{\beta\alpha}_{ab}$
13		$\frac{1}{2}C_2(F)C_2(G)$	$\frac{1}{2}C_2(G)(T_i)_{\alpha\beta}(T_i)_{ab}\ H^{\beta\alpha}_{ba}$
14		$\frac{1}{2}C_2(F)C_2(G)$	$\frac{1}{2}C_2(G)C_2(F)\ H^{\alpha\alpha}_{aa}$
15		$\left[C_2\ (F)\right]^2$	$\left[C_2\ (F)\right]^2 H^{\alpha\alpha}_{aa}$

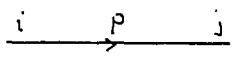
GROUP	GRAPHS	GROUP THEORETIC WEIGHT FOR Q-$\bar{Q}$	GROUP THEORETIC WEIGHT FOR Q-Q
16		$C_2(F)C_2(G)$	$C_2(F)C_2(G)\ H_{aa}^{\alpha\alpha}$
17		$C_2(F)C_2(G)$	$C_2(F)C_2(G)\ H_{aa}^{\alpha\alpha}$

APPENDIX A: QCD Feynman rules 1

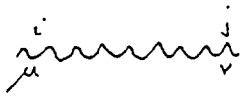
The quantisation of QCD in a renormalisable gauge leads to the Feynman rules summarised below. The solid and wavy lines are quarks and vector gluons respectively. The dashed line represents the Faddeev-Popov ghost particles which are introduced in the theory to preserve unitarity and gauge invariance. Roman letters (a, b, c, i, j) refer to colour indices. Greek letters are Lorentz indices



Ghost propagator $= G^0_{ab}(k) = \dfrac{\delta_{ab}}{k^2 + i\varepsilon}$

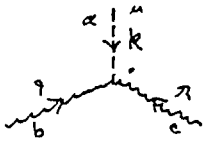


Fermion propagator $= S^0(p) = \dfrac{i\delta_{ij}}{\not{p} - m + i\varepsilon}$

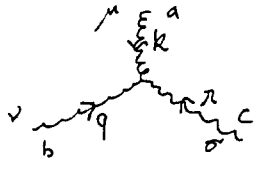


Gauge Field Propagator $= D^{0\,ij}_{\mu\nu}(k) = -i\left((g_{\mu\nu} - \dfrac{k_\mu k_\nu}{k^2}) + \alpha \dfrac{k_\mu k_\nu}{k^2}\right)\dfrac{\delta_{ij}}{k^2 + i\varepsilon}$

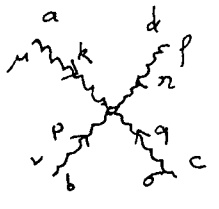
In the Feynman gauge $\alpha = 1$



ghost vertex $g\, C_{abc}\, r_\mu$

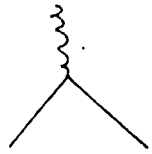


gauge field triple vertex $g\, C_{abc}\left[g_{\mu\nu}(k-q)_\sigma + g_{\nu\sigma}(q-r)_\mu + g_{\sigma\mu}(r-k)_\nu\right]$



gauge field quadruple vertex

$$-ig^2\left[C_{abe}C_{cde}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) + C_{ace}C_{bde}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\nu\sigma}) + C_{ade}C_{cbe}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\sigma\rho})\right]$$



Fermion vertex $ig\gamma_\mu t^a$

In addition

(i) We introduce $\bar{u}(p)$, $u(p)$, $\bar{v}(p)$, $v(p)$ for spinors corresponding to external fermion lines

(ii) for each internal momentum not fixed by momentum conservation constraints at vertices a factor

$$\int \frac{d^4\ell}{(2\pi)^4}$$

(iii) a factor $(2\pi)\,\delta^{(+)}(p^2 - m^2)$ for each on shell fermion of momentum p.

APPENDIX B: The mass counterterm to $O(\alpha)$

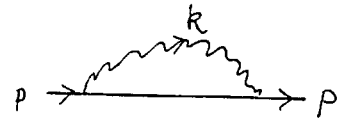
The one-particle irreducible graphs, with external legs amputated to $O(\alpha)$ are given by (see fig B.1)

$$\text{B.1} \qquad -i\sum_{(1)}^{(\alpha)}(\not p) = \frac{\alpha}{(2\pi)^4}\int \frac{d^4k\,(-i)}{k^2+i\varepsilon}\cdot i\gamma_\mu \cdot \frac{i}{\not p-\not k-m+i\varepsilon}\, i\gamma^\mu$$

and

$$\text{B.2} \qquad -i\sum_{(2)}^{(\alpha)}(\not p) = -i\,\Delta m$$

where the colour factors have been ommitted

$-i\sum_{(1)}^{(\alpha)}(\not p) \equiv$ 

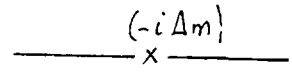
$-i\sum_{(2)}^{(\alpha)}(\not p) \equiv$ 

Fig B.1 One-particle irreducible graphs to $O(\alpha)$

From B.1 we find

$$\sum_{(1)}^{(\alpha)}(\not p) = \frac{-i}{(2\pi)^4}\int \frac{d^4k}{k^2+i\varepsilon}\,\frac{\gamma_\mu(\not p-\not k+m)\gamma^\mu}{[(p-k)^2-m^2+i\varepsilon]}$$

Using the eikonal approximation and working in the rest from of p we find

$$\text{B.3} \qquad \bar u(p)\sum_{(1)}^{(\alpha)}(\not p)\,u(p)\Big|_{\not p=m} = \frac{-i\alpha}{(2\pi)^4}\int \frac{d^4k}{[k^2+i\varepsilon][-k_0+i\varepsilon]}$$

From B.2 we find

$$\bar u(p)\sum_{(2)}^{(\alpha)}(\not p)\,u(p)\Big|_{\not p=m} = \Delta m$$

So from our renormalization condition (2.3.6) and B3 we find

B.4 $$-i\Delta m = \frac{\alpha}{(2\pi)^4}\int \frac{d^4k}{[k^2+i\varepsilon][-k_0+i\varepsilon]}$$

APPENDIX D: The integrals of Section 3.4

In this appendix we evaluate the integrals that appear in section 3.4 using dimensional regularization. The integrals are evaluated in $[n]$ for $n > 3$ and large. We then go back to $n = 3$ in a controlled manner.

Throughout we will be working in the frame

$$p = (M, \underset{\sim}{0}) \quad , \quad q = (q_0, \underbrace{0, \ldots 0}_{(n-1)\text{ components}}, q_z)$$

where q_z is chosen to lie along the n^{th} component of q. We have defined the following quantities in the text:

$$(D.1) \quad \begin{cases} \kappa = \dfrac{q \cdot \underset{\sim}{k}}{q_0} \equiv \dfrac{q_z k_z}{q_0} = \beta k_z \\ \kappa' = \dfrac{q \cdot \underset{\sim}{k}'}{q_0} \equiv \dfrac{q_z k_z'}{q_0} \equiv \beta k_z' \\ 0 < \beta < 1 \end{cases}$$

We will be using the following identities

$$(D.2) \quad \begin{cases} \dfrac{1}{x} = \displaystyle\int_0^\infty d\alpha\, e^{-\alpha x} \quad , \quad x > 0 \\ \dfrac{1}{x + i\varepsilon} = -i \displaystyle\int_0^\infty d\alpha\, e^{i\alpha(x+i\varepsilon)} \quad , \quad x \text{ is real} \\ \dfrac{1}{(x+i\varepsilon)^2} = -\displaystyle\int_0^\infty d\alpha\, \alpha\, e^{i\alpha(x+i\varepsilon)} \end{cases}$$

$$(D.3) \quad \int_{-\infty}^{\infty} e^{-p^2 x^2 \pm qx}\, dx = \frac{\sqrt{\pi}}{p}\, e^{q^2/4p^2}$$

(a) The integral $I_1 \equiv \int \dfrac{d^3\underset{\sim}{k}\, \delta(K)}{\underset{\sim}{k}^2}$

Using D.2 we find

D.4 $$I_1 = \int d^3\underset{\sim}{k}\,\delta(K)\int_0^\infty d\alpha\, e^{-\alpha \underset{\sim}{k}^2}$$

We now define the integral

D.5 $$I_1^{[n]} = \int d^n\underset{\sim}{k}\, e^{-\underset{\sim}{k}^2/\Lambda^2}\,\delta(K)\int_0^\infty d\alpha\, e^{-\alpha\underset{\sim}{k}^2}$$

wehre Λ^2 is some arbitrary ultraviolet cut-off. Reversing the order of integration in D.5 we find

$$I_1^{[n]} = \int_0^\infty d\alpha \int d^{(n-1)}\underset{\sim}{k}\; e^{-(\alpha+1/\Lambda^2)\underset{\sim}{k}^2}\int_{-\infty}^{\infty}\frac{dk_z}{\beta}\,\delta(k_z)\,e^{-(\alpha+1/\Lambda^2)k_z^2}$$

So using D.3 we find

$$I_1^{[n]} = \frac{1}{\beta}\int_0^\infty d\alpha\left(\frac{\pi}{\alpha+1/\Lambda^2}\right)^{(n-1)/2}$$

Doing the α-integration we find

$$I_1^{[n]} = \frac{1}{\beta}\cdot\frac{2}{(3-n)}\,\pi^{(n-1)/2}\left[\alpha+1/\Lambda^2\right]^{\frac{3-n}{2}}\Big|_0^\infty$$

thus

D.6 $$\lim_{n=3+\varepsilon} I_1^{[n]} = \frac{1}{\beta}\cdot\frac{2\pi^{(n-1)/2}}{(n-3)}\,(\Lambda^2)^{(n-3)/2}$$

D.7 $$\lim_{n\to 3+\varepsilon}\int\frac{d^n\underset{\sim}{k}\,\delta(K)}{\underset{\sim}{k}^2} = \frac{2\pi}{\beta\varepsilon}$$

(b) the integral $I_2 = \int\dfrac{d^3\underset{\sim}{k}\,d^3\underset{\sim}{k}'\,2K'}{[K'^2-\underset{\sim}{k}^2+i\varepsilon][K'^2-\underset{\sim}{k}'^2][K+K'+i\varepsilon][K'^2+\varepsilon^2]}$

We first write I_2 as

D.8 $$I_2 = \frac{1}{\beta^2}\int \frac{d^3\underset{\sim}{k}\, d^3\underset{\sim}{k}'\, 2k_z'}{[\beta^2 k_z'^2 - \underset{\sim}{k}^2 + i\varepsilon][\beta^2 k_z'^2 - \underset{\sim}{k}'^2][k_z + k_z' + i\varepsilon][k_z'^2 + \varepsilon^2]}$$

We define the equivalent of I_2 in n-dimensions with an u.v. cutoff :

D.9 $$I_2^{[n]} = \frac{1}{\beta^2}\int \frac{d^n\underset{\sim}{k}\, d^n\underset{\sim}{k}'\, e^{-(\underset{\sim}{k}^2 + \underset{\sim}{k}'^2)/\Lambda^2}\, 2k_z'}{[\underset{\sim}{k}^2 - \beta^2 k_z'^2 - i\varepsilon][\underset{\sim}{k}'^2 - \beta^2 k_z'^2][k_z + k_z' + i\varepsilon][k_z'^2 + \varepsilon^2]}$$

Using D.2 we find

$$I_2^{[n]} = \frac{1}{\beta^2}\int_0^\infty d\alpha\, d\mu\, d\nu\, d\gamma \int d^n\underset{\sim}{k}\, d^n\underset{\sim}{k}'\, e^{-\underset{\sim}{k}^2/\Lambda^2 - \underset{\sim}{k}'^2/\Lambda^2}\, e^{-\alpha(\underset{\sim}{k}'^2 - \beta^2 k_z'^2)}\, e^{-i\mu(\underset{\sim}{k}^2 - \beta^2 k_z'^2 - i\varepsilon)}\, e^{-i\nu(k_z + k_z' + i\varepsilon)}\, e^{-\gamma(k_z'^2 + \varepsilon^2)} \times (2k_z').$$

We can replace the $2k_z'$ term in the numerator by writing

$$I_2^{[n]} = \frac{1}{\beta^2}\int_0^\infty d\alpha\, d\mu\, d\nu\, d\gamma\, e^{-(\mu+\nu+\varepsilon\gamma)\varepsilon} \int d^n\underset{\sim}{k}\, e^{-(i\mu + 1/\Lambda^2)\underset{\sim}{k}^2 + i\nu k_z} \times \left(-2i\frac{\partial}{\partial\nu}\right)\int d^n\underset{\sim}{k}'\, e^{-(\alpha+1/\Lambda^2)\underset{\sim}{k}'^2 + (\alpha\beta^2 + i\mu\beta^2 - \gamma)k_z'^2 + i\nu k_z'}$$

We do the $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ integrals using D.3 and obtain

$$I_2^{[n]} = \frac{\pi^n}{\beta^2}\int_0^\infty d\alpha\, d\mu\, d\nu\, d\gamma\, e^{-(\mu+\nu+\varepsilon\gamma)\varepsilon}\, \frac{1}{(i\mu + 1/\Lambda^2)^{n/2}}\, e^{-\frac{\nu^2}{4(i\mu + 1/\Lambda^2)}} \cdot \frac{1}{(\alpha + 1/\Lambda^2)^{(n-1)/2}} \times \left(-2i\frac{\partial}{\partial\nu}\right)\left\{ \frac{e^{-\frac{\nu^2}{4[\alpha(1-\beta^2) + \gamma + \frac{1}{\Lambda^2} - i\mu\beta^2]}}}{[\alpha(1-\beta^2) + \gamma + \frac{1}{\Lambda^2} - i\mu\beta^2]^{1/2}} \right\}$$

The ν-integral is absolutely convergent and we can drop the $e^{-\nu\varepsilon}$ term. We can then do the ν-integral exactly. Differentiating the term in curly bracket in the above brings down a factor of $-\frac{\nu}{2[\alpha(1-\beta^2) + \gamma + \frac{1}{\Lambda^2} - i\mu\beta^2]}$

Since $\nu d\nu = \frac{1}{2}d\nu^2$ the resultant ν-integral is then trivially done to give

$$I_2^{[n]} = \frac{2i\pi^n}{\beta^2}\int_0^\infty \frac{d\alpha\, d\mu\, d\gamma\, e^{-(\mu+\varepsilon\gamma)\varepsilon}\, (i\mu + 1/\Lambda^2)}{(\alpha + 1/\Lambda^2)^{(n-1)/2}\,(i\mu + 1/\Lambda^2)^{n/2}\,(\alpha(1-\beta^2) + \gamma + \frac{1}{\Lambda^2} - i\mu\beta^2)^{1/2}\,(\alpha(1-\beta^2) + i\mu(1-\beta^2) + \gamma + 2/\Lambda^2)}$$

We scale α, γ, μ by $1/\Lambda^2$ and rotate the μ-contour so that it runs just to the right of the negative imaginary axis. We then find

D.10
$$I_2^{[n]} = \frac{2\pi^n (\Lambda^2)^{n-3}}{\beta^2} \int_0^\infty \frac{d\alpha\, d\mu\, d\gamma\; e^{-\varepsilon^2\gamma}}{[\alpha+1]^{(n-1)/2}[\mu+1]^{(n-2)/2}[\alpha(1-\beta^2)+1+\gamma-\mu\beta^2]^{1/2}[(\alpha+\mu)(1-\beta^2)+\gamma+2]}$$

wehre we have dropped the $e^{-\mu\varepsilon}$ term for large n. We may also drop the $e^{-\varepsilon^2\gamma}$ term as the γ- integral is convergent at large γ. Note that the term $[(\alpha+\mu)(1-\beta^2)+\gamma+2]$ is positive definite. Thus the $i\varepsilon$ term that in principle is picked up when rotating the μ-contour may be neglected. We only want the real part of $I_2^{[n]}$. This is fixed by the argument of the square root being positive. This gives two integration regions since we want

D.11
$$\gamma > \mu\beta^2 - 1 - \alpha(1-\beta^2)$$

This is automatically satisfied if

$$\mu < \frac{1+\alpha(1-\beta^2)}{\beta^2}$$

However if $\mu > \frac{1+\alpha(1-\beta^2)}{\beta^2}$ then the γ-integration must be cut off as dictated by D.11. Hence from D.10 we find

$$\mathrm{Re}\{I_2^{[n]}\} = \frac{2\pi^n(\Lambda^2)^{n-3}}{\beta^2}\left\{ \int_0^\infty \frac{d\alpha}{(\alpha+1)^{(n-1)/2}} \int_0^{\frac{1+\alpha(1-\beta^2)}{\beta^2}} \frac{d\mu}{(\mu+1)^{(n-2)/2}} \int_0^\infty \frac{d\gamma}{[\alpha(1-\beta^2)+\gamma-\mu\beta^2+1]^{1/2}[(\alpha+\mu)(1-\beta^2)+2+\gamma]} \right.$$

D.12
$$\left. + \int_0^\infty \frac{d\alpha}{(\alpha+1)^{(n-1)/2}} \int_{\frac{1+\alpha(1-\beta^2)}{\beta^2}}^\infty d\mu\,(1+\mu)^{(2-n)/2} \int_{\mu\beta^2-1-\alpha(1-\beta^2)}^\infty d\gamma \times \frac{}{[\alpha(1-\beta^2)+\gamma-\mu\beta^2+1]^{1/2}[(\alpha+\mu)(1-\beta^2)+2+\gamma]} \right\}$$

Using the identity

D.13 $$\int \frac{d\gamma}{[\alpha+b\gamma]\sqrt{A+B\gamma}} = \frac{2\,\mathrm{Tan}^{-1}\left(\sqrt{\frac{b}{(aB-Ab)}}\sqrt{A+B\gamma}\right)}{\sqrt{b(aB-Ab)}} \quad \text{if } \frac{(aB-Ab)}{b}>0,\ A+B\gamma>0.$$

D.12 becomes

$$\mathrm{Re}\{I_2^{[n]}\} = \frac{4\pi^n(\Lambda^2)^{n-3}}{\beta^2}\left\{\int_0^\infty \frac{d\alpha}{(\alpha+1)^{(n-1)/2}}\int_0^{\frac{1+\alpha(1-\beta^2)}{\beta^2}} \frac{d\mu}{(\mu+1)^{(n-1)/2}}\left[\frac{\pi}{2}-\mathrm{Tan}^{-1}\left(\sqrt{\frac{1+\alpha(1-\beta^2)-\mu\beta^2}{1+\mu}}\right)\right]\right.$$

$$\left. + \frac{\pi}{2}\int_0^\infty \frac{d\alpha}{(\alpha+1)^{(n-1)/2}}\int_{\frac{1+\alpha(1-\beta^2)}{\beta^2}}^\infty \frac{d\mu}{(1+\mu)^{(n-1)/2}}\right\}$$

D.14 $$\equiv \frac{4\pi^n(\Lambda^2)^{n-3}}{\beta^2}\left[\frac{\pi}{2}\bar{I}_1+\bar{I}_2\right]$$

where

D.15 $$\bar{I}_1 \equiv \int_0^\infty \frac{d\alpha}{(\alpha+1)^{(n-1)/2}}\int_{\frac{1+\alpha(1-\beta^2)}{\beta^2}}^\infty \frac{d\mu}{(\mu+1)^{(n-1)/2}}$$

D.16 $$\bar{I}_2 \equiv \int_0^\infty \frac{d\alpha}{(\alpha+1)^{(n-1)/2}}\int_0^{(1+\alpha(1-\beta^2))/\beta^2} \frac{d\mu}{(\mu+1)^{(n-1)/2}}\left(\frac{\pi}{2}-\mathrm{Tan}^{-1}\sqrt{\frac{1+\alpha(1-\beta^2)-\mu\beta^2}{1+\mu}}\right)$$

Consider first $\bar{I}_1$. The μ-integral is easily done to give

D.17 $$\bar{I}_1 = \frac{2}{(n-3)}\left(\frac{\beta^2}{1-\beta^2}\right)^{\frac{(n-3)}{2}}\int_0^\infty \frac{d\alpha}{(1+\alpha)^{(n-1)/2}(\alpha+Z)^{(n-3)/2}}$$

where

D.18 $$Z \equiv \frac{1+\beta^2}{1-\beta^2} > 1$$

Define

$$v = \alpha + \tfrac{1}{2}(1+z)$$

$$\therefore J = \int_0^\infty \frac{d\alpha}{(1+\alpha)^{(n-1)/2}(\alpha+z)^{(n-3)/2}} = \int_{\frac{1}{2}(1+z)}^\infty \frac{dv\,(v+\frac{1}{2}(z-1))}{(v^2-\frac{1}{4}(z-1)^2)^{(n-1)/2}} = \frac{1}{2}\int_{\frac{1}{4}(1+z)^2}^\infty \frac{dv^2\,(1+\frac{1}{2}(z-1)/v)}{[\,v^2-\frac{1}{4}(z-1)^2\,]^{(n-1)/2}}$$

Only the first term in the numerator has a pole at n = 3. In the second term we may set n = 3. Thus

$$\lim_{n\to 3} J = \frac{1}{2}\int_{\frac{1}{4}(1+z)^2}^\infty \frac{dv^2}{[v^2-\frac{1}{4}(z-1)^2]^{(n-1)/2}} + \frac{(z-1)}{2}\int_{\frac{1}{2}(1+z)}^\infty \frac{dv}{[v^2-\frac{1}{4}(z-1)^2]}$$

$$= \frac{z^{(3-n)/2}}{n-3} + \frac{1}{2}\log z$$

D.19 $$\equiv \frac{1}{(n-3)}\,, \quad \text{as the constant terms cancel in the limit } n\to 3.$$

Hence substituting in D.17 we find

D.20 $$\lim_{n\to 3+\varepsilon} \bar{I}_1 = \frac{2}{(n-3)^2}\left[1 + \frac{(n-3)}{2}\log\left\{\frac{\beta^2}{1-\beta^2}\right\}\right]$$

Now consider $\bar{I}_2$. Using the identity

$$\mathrm{Tan}^{-1}x + \mathrm{Cot}^{-1}x = \pi/2\,, \quad \text{for } x > 0$$

we write D.16 as

$$\bar{I}_2 = \int_0^\infty \frac{d\alpha}{(\alpha+1)^{(n-1)/2}} \int_0^{(1+\alpha(1-\beta^2))/\beta^2} \frac{d\mu}{(\mu+1)^{(n-1)/2}}\, \mathrm{Cot}^{-1}\sqrt{\frac{1+\alpha(1-\beta^2)-\mu\beta^2}{1+\mu}}$$

Defining

$$t^2 \equiv \frac{1+\alpha(1-\beta^2)-\mu\beta^2}{1+\mu}$$

and eliminating μ we get

$$\bar{I}_2 = \int_0^\infty \frac{d\alpha}{(\alpha+1)^{(n-1)/2}} \int_0^{\sqrt{1+\alpha(1-\beta^2)}} \frac{2t\,dt\ \cot^{-1}t\ .\ (\beta^2+t^2)^{(n-5)/2}}{[1+\beta^2+\alpha(1-\beta^2)]^{(n-3)/2}}$$

The region $\alpha \leq \Delta$ gives no pole at n = 3. We can, therefore, treat the region $\alpha \geq \Delta$ for many large Δ only. Changing the order of α and t integrations we get

$$\bar{I}_2 \simeq 2\int_0^{\sqrt{1+\Delta(1-\beta^2)}} t\,dt\,(\beta^2+t^2)^{(n-5)/2}\cot^{-1}t \int_\Delta^\infty \frac{d\alpha}{\alpha^{n-2}} + 2\int_{\sqrt{1+\Delta(1-\beta^2)}}^\infty t\,dt\,(\beta^2+t^2)^{(n-5)/2}\cot^{-1}t \int_{\frac{t^2-1}{1-\beta^2}}^\infty \frac{d\alpha}{\alpha^{n-2}}$$

$$= \frac{2}{n-3}\Delta^{3-n}\int_0^{\sqrt{1+\Delta(1-\beta^2)}} t\,dt\,(\beta^2+t^2)^{(n-5)/2}\cot^{-1}t + \frac{2}{n-3}\int_{\sqrt{(1+\Delta(1-\beta^2))}}^\infty t\,dt\,(\beta^2+t^2)^{(n-5)/2}\cot^{-1}t\left(\frac{1-\beta^2}{t^2-1}\right)^{n-3}$$

These are convergent for $n \geq 2$. Thus setting n = 3 we find

$$\bar{I}_2 = \frac{2}{n-3}\int_0^\infty \frac{t\cot^{-1}t\,dt}{t^2+\beta^2}$$

This integral may be done by first doing a by parts integration followed by a contour integration. This gives

D.21 $$\lim_{n\to 3} \bar{I}_2 = \frac{\pi}{n-3}\log\left(\frac{1+\beta}{\beta}\right)$$

Substituting for D.20 and D.21 in D.14 gives

$$\lim_{n\to 3+\varepsilon} Re\{I_2^{[n]}\} = \frac{4\pi^{n+1}(\Lambda^2)^{n-3}}{\beta^2(n-3)}\left[1+\frac{(n-3)}{2}\log\left(\frac{\beta^2}{(1-\beta^2)}\right)+(n-3)\log\left(\frac{1+\beta}{\beta}\right)\right]$$

Hence

D.22 $$\lim_{n\to 3+\varepsilon} \mathrm{Re}\left\{ I_2^{[n]} \right\} = \frac{4\pi^{n+1}(\Lambda^2)^{n-3}}{\beta^2 (n-3)^2} \left\{ 1 + \frac{1}{2}(n-3) \log\left(\frac{1+\beta}{1-\beta}\right) \right\}$$

c) The integrals $I_3 = \int \frac{d^3\underset{\sim}{k}\, d^3\underset{\sim}{k}'}{\underset{\sim}{k}^2 (\underset{\sim}{k}'^2 - K'^2)(K'+i\varepsilon)^2}$

and $I_4 = -\int \frac{d^3\underset{\sim}{k}\, d^3\underset{\sim}{k}'}{[\underset{\sim}{k}^2 - K'^2 + i\varepsilon][\underset{\sim}{k}'^2 - K'^2][K'+i\varepsilon]^2}$

We first of all define the n-dimensional, ultraviolet cut-off integrals

D.23 $$I_3^{[n]} = \int \frac{d^n\underset{\sim}{k} d^n\underset{\sim}{k}'\, e^{-(\underset{\sim}{k}^2 + \underset{\sim}{k}'^2)/\Lambda^2}}{\underset{\sim}{k}^2 (\underset{\sim}{k}'^2 - K'^2)(K'+i\varepsilon)^2}$$

D.24 $$I_4^{[n]} = -\int \frac{d^n\underset{\sim}{k} d^n\underset{\sim}{k}'\, e^{-(\underset{\sim}{k}^2 + \underset{\sim}{k}'^2)/\Lambda^2}}{[\underset{\sim}{k}^2 - K'^2 - i\varepsilon][\underset{\sim}{k}'^2 - K'^2][K'+i\varepsilon]^2}$$

First consider $I_3^{[n]}$. This factors into separate k and k' integrals

$$\int \frac{d^n\underset{\sim}{k}\, e^{-\underset{\sim}{k}^2/\Lambda^2}}{\underset{\sim}{k}^2} = \int_0^\infty d\alpha \int d^n\underset{\sim}{k}\, e^{-(\alpha + 1/\Lambda^2)\underset{\sim}{k}^2} = -\frac{\pi^{n/2}(\Lambda^2)^{\frac{n}{2}-1}}{(1-n/2)}$$

Hence

D.25 $$\int \frac{d^n\underset{\sim}{k}\, e^{-\underset{\sim}{k}^2/\Lambda^2}}{\underset{\sim}{k}^2} = -\frac{\pi^{n/2}(\Lambda^2)^{\frac{n}{2}-1}}{(1-n/2)}$$

This is analytic for $n > 2$

Now

$$\int \frac{d^n\underset{\sim}{k}'\, e^{-\underset{\sim}{k}'^2/\Lambda^2}}{(\underset{\sim}{k}'^2 - K'^2)(K'+i\varepsilon)^2} = -\frac{1}{\beta^2}\int_0^\infty d\alpha\, \gamma d\gamma \int d^n\underset{\sim}{k}'\, e^{-\underset{\sim}{k}'^2/\Lambda^2}\, e^{i\gamma(k_z'+i\varepsilon) - \alpha(\underset{\sim}{k}'^2 - \beta^2 k_z'^2)}$$

$$= -\frac{1}{\beta^2}\int_0^\infty d\alpha\, \gamma d\gamma \left(\frac{\pi}{\alpha+1/\Lambda^2}\right)^{(n-1)/2} \frac{\pi^{1/2}}{[\alpha(1-\beta^2)+1/\Lambda^2]^{1/2}}\, e^{-\frac{\gamma^2}{4[\alpha(1-\beta^2)+1/\Lambda^2]}} = -\frac{\pi^{n/2}}{2\beta^2}\int_0^\infty \frac{d\alpha\, .\, 4[\alpha(1-\beta^2)+1/\Lambda^2]^{1/2}}{(\alpha+1/\Lambda^2)^{(n-1)/2}}$$

Hence

D.26 $$\int \frac{d^n\underset{\sim}{k}'\, e^{-\underset{\sim}{k}'^2/\Lambda^2}}{(\underset{\sim}{k}'^2-K'^2)(K'+i\varepsilon)^2} = \frac{-\pi^{n/2}}{2\beta^2}\int_0^\infty \frac{d\alpha\; A}{(\alpha+1/\Lambda^2)^{(n-1)/2}}\frac{[\alpha(1-\beta^2)+\frac{1}{\Lambda^2}]^{1/2}}{}$$

The above has a pole at n = 4. Combining D.25, D.26, and D.23 we find

D.27 $$I_3^{[n]} = \frac{2\pi^n(\Lambda^2)^{(\frac{n}{2}-1)}}{\beta^2(1-n/2)}\int_0^\infty \frac{d\alpha\;(\alpha(1-\beta^2)+1/\Lambda^2)^{1/2}}{(\alpha+1/\Lambda^2)^{(n-1)/2}}$$

Let us now consider D.24

$$I_4^{[n]} = -\int \frac{d^n\underset{\sim}{k}\, d^n\underset{\sim}{k}'\, e^{-(\underset{\sim}{k}^2+\underset{\sim}{k}'^2)/\Lambda^2}}{[\underset{\sim}{k}^2-\beta^2k_z'^2-i\varepsilon][\underset{\sim}{k}'^2-\beta^2k_z'^2][\beta k_z'+i\varepsilon]^2}$$

$$= i\int_0^\infty d\alpha\, d\mu\, \nu\, d\nu \int d^n\underset{\sim}{k}\, d^n\underset{\sim}{k}'\, e^{-\alpha(\underset{\sim}{k}'^2-\beta^2k_z'^2)-i\mu(\underset{\sim}{k}^2-\beta^2k_z'^2+i\varepsilon)+i\nu(\beta k_z'+i\varepsilon)-(\underset{\sim}{k}^2+\underset{\sim}{k}'^2)/\Lambda^2}$$

on making use of D.2 Using D.3 we do the $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ integrations giving

$$I_4^{[n]} = i\int_0^\infty d\alpha\, d\mu\, \nu\, d\nu \left(\frac{\pi}{i\mu+1/\Lambda^2}\right)^{n/2}\left(\frac{\pi}{\alpha+1/\Lambda^2}\right)^{(n-1)/2}\frac{\pi^{1/2}\; e^{-\frac{\nu^2\beta^2}{4[1/\Lambda^2+\alpha(1-\beta^2)-i\mu\beta^2]}}}{[1/\Lambda^2+\alpha(1-\beta^2)-i\mu\beta^2]^{1/2}}$$

Scaling α, μ by $1/\Lambda^2$ and doing the ν integrals gives

$$I_4^{[n]} = \frac{2i(\Lambda^2)^{n-3}\pi^n}{\beta^2}\int_0^\infty \frac{d\alpha\, d\mu\;[1+\alpha(1-\beta^2)-i\mu\beta^2]^{1/2}}{(1+i\mu)^{n/2}\;(1+\alpha)^{(n-1)/2}}$$

We can rotate the μ-contour until it lies just to the right of the negative imaginary axis. This gives

D.28 $$I_4^{[n]} = \frac{2\pi^n(\Lambda^2)^{n-3}}{\beta^2}\int_0^\infty \frac{d\alpha\, d\mu\;(1+\alpha(1-\beta^2)-\mu\beta^2)^{1/2}}{(1+\mu)^{n/2}\;(1+\alpha)^{(n-1)/2}}$$

We only want the real part of D.28. Thus the square root in the numerator must have a positive argument. Thus we have

$$Re\{I_4^{[n]}\} = \frac{2\pi^n(\Lambda^2)^{n-3}}{\beta^2}\int_0^\infty \frac{d\alpha}{(1+\alpha)^{(n-1)/2}}\int_0^{\frac{1+\alpha(1-\beta^2)}{\beta^2}} \frac{d\mu}{(1+\mu)^{n/2}} \times [1+\alpha(1-\beta^2)-\mu\beta^2]^{1/2}$$

Now do the μ-integral by parts This gives

$$Re\{I_4^{[n]}\} = -\frac{2\pi^n(\Lambda^2)^{n-3}}{\beta^2}\cdot\frac{1}{(1-n/2)}\int_0^\infty \frac{d\alpha}{(\alpha+1)^{(n-1)/2}}[1+\alpha(1-\beta^2)]^{1/2}$$

D.29

$$+\frac{\pi^n(\Lambda^2)^{n-3}}{(1-n/2)}\int_0^\infty \frac{d\alpha}{(1+\alpha)^{(n-1)/2}}\int_0^{[1+\alpha(1-\beta^2)]/\beta^2} \frac{d\mu}{(1+\mu)^{\frac{n}{2}-1}(1+\alpha(1-\beta^2)-\mu\beta^2)^{1/2}}$$

Adding D.29 and D.27 we see that the first term in D.29 is cancelled exactly by D.27. Thus we have

D.30

$$Re\{I_3^{[n]}+I_4^{[n]}\} = \frac{\pi^n(\Lambda^2)^{n-3}}{(1-n/2)}\int_0^\infty \frac{d\alpha}{(1+\alpha)^{(n-1)/2}}\int_0^{[1+\alpha(1-\beta^2)]/\beta^2} \frac{d\mu}{(1+\mu)^{(n-2)/2}(1+\alpha(1-\beta^2)-\mu\beta^2)^{1/2}}$$

Let

D.31

$$t^2 = \frac{1+\alpha(1-\beta^2)-\mu\beta^2}{1+\mu}$$

Eliminating μ in D.30 we find

$$Re\{I_3^{[n]}+I_4^{[n]}\} = \frac{2\pi^n(\Lambda^2)^{n-3}}{(1-n/2)}\int_0^\infty \frac{d\alpha}{(1+\alpha)^{(n-1)/2}(1+\beta^2+\alpha(1-\beta^2))^{(n-3)/2}}\int_0^{\sqrt{1+\alpha(1-\beta^2)}} dt\,(t^2+\beta^2)^{(n-5)/2}$$

The region $\alpha \leq \Delta$ where Δ is large clearly gives no pole at n = 3. We may therefore, write

$$\lim_{n\to 3} Re\{I_3^{[n]}+I_4^{[n]}\} = \frac{2\pi^n(\Lambda^2)^{n-3}}{(1-n/2)}\int_\Delta^\infty \frac{d\alpha}{(1-\beta^2)^{(n-3)/2}\alpha^{n-2}}\int_0^{\sqrt{\alpha(1-\beta^2)}} dt\,(t^2+\beta^2)^{(n-5)/2}$$

Reversing the order of α and t integration we find

$$\lim_{n\to 3} \mathrm{Re}\left\{ I_3^{[n]} + I_4^{[n]} \right\}$$

$$= \frac{2\pi^n (\Lambda^2)^{n-3}}{(1-n/2)(1-\beta^2)^{(n-3)/2}} \left\{ \int_0^{\sqrt{\Delta(1-\beta^2)}} dt\, (t^2+\beta^2)^{(n-5)/2} \int_\Delta^\infty \frac{d\alpha}{\alpha^{n-2}} \right.$$

$$\left. + \int_{\sqrt{\Delta(1-\beta^2)}}^\infty dt\, (t^2+\beta^2)^{(n-5)/2} \int_{t^2/(1-\beta^2)}^\infty \frac{d\alpha}{\alpha^{n-2}} \right.$$

Doing the α-integration gives

$$\lim_{n\to 3} \mathrm{Re}\left\{ I_3^{[n]} + I_4^{[n]} \right\}$$

$$= \frac{2\pi^n (\Lambda^2)^{n-3}}{(1-n/2)(1-\beta^2)^{(n-3)/2}} \left\{ \int_0^{\sqrt{\Delta(1-\beta^2)}} dt\, (t^2+\beta^2)^{(n-5)/2} \frac{\Delta^{3-n}}{n-3} \right.$$

$$\left. + \int_{\sqrt{\Delta(1-\beta^2)}}^\infty dt\, (t^2+\beta^2)^{(n-5)/2} \left(\frac{t^2}{1-\beta^2}\right)^{3-n} \times \frac{1}{(n-3)} \right\}$$

The second integral is well defined for $n \geqslant 3$. Setting $n = 3$ in the t inegral gives

$$\lim_{n\to 3} \mathrm{Re}\left\{ I_3^{[n]} + I_4^{[n]} \right\} = \frac{2\pi^n (\Lambda^2)^{n-3}}{(1-n/2)(n-3)} \int_0^\infty \frac{dt}{(t^2+\beta^2)}$$

Now

$$\int_0^\infty \frac{dt}{t^2+\beta^2} = \frac{\pi}{2\beta}$$

Hence

D.32 $$\lim_{n\to 3} \mathrm{Re}\left\{ I_3^{[n]} + I_4^{[n]} \right\} = -\frac{2\pi^{n+1}(\Lambda^2)^{n-3}}{\beta\,(n-3)}$$

d) The integral $I_5^{[n]} = \int \frac{d^n\underset{\sim}{k}\, d^n\underset{\sim}{k}'\,(\kappa'-2\kappa)\, e^{-(\underset{\sim}{k}^2+\underset{\sim}{k}'^2)/\Lambda^2}}{\underset{\sim}{k}^2\,(\underset{\sim}{k}'^2-\kappa'^2)(\kappa'^2-(\underset{\sim}{k}'-\underset{\sim}{k})^2+i\varepsilon)(\kappa'+i\varepsilon)}$

Now

D.33 $$I_5^{[n]} = \int \frac{d^n\underset{\sim}{k}\, d^n\underset{\sim}{k}'\,(k_z'-2k_z)\, e^{-(\underset{\sim}{k}^2+\underset{\sim}{k}'^2)/\Lambda^2}}{\underset{\sim}{k}^2\,(\underset{\sim}{k}'^2-\beta^2k_z'^2)(\beta^2k_z'^2-(\underset{\sim}{k}'-\underset{\sim}{k})^2+i\varepsilon)(k_z'+i\varepsilon)}$$

We write this as

$$I_5^{[n]} = -\int_0^\infty d\alpha\, d\gamma\, d\mu\, d\delta \int d^n\underset{\sim}{k}\, d^n\underset{\sim}{k}'\, e^{-(\underset{\sim}{k}^2+\underset{\sim}{k}'^2)/\Lambda^2 - \alpha\underset{\sim}{k}^2 - \gamma(\underset{\sim}{k}'^2-\beta^2k_z'^2) + i\delta(k_z'+i\varepsilon)} e^{i\mu(\beta^2k_z'^2-(\underset{\sim}{k}'-\underset{\sim}{k})^2+i\varepsilon)} e^{i\sigma k_z} (k_z'-2k_z)\Big|_{\sigma=0}$$

where we have introduced σ so that we obtain

$$I_5^{[n]} = -\int_0^\infty d\alpha\, d\gamma\, d\mu\, d\delta\, e^{-(\delta+\mu)\varepsilon} \frac{1}{i}\left(\frac{\partial}{\partial\delta} - 2\frac{\partial}{\partial\sigma}\right) \int d^n\underset{\sim}{k}\, d^n\underset{\sim}{k}'\, e^{-(\underset{\sim}{k}^2+\underset{\sim}{k}'^2)/\Lambda^2} \times e^{-\alpha\underset{\sim}{k}^2 - \gamma(\underset{\sim}{k}'^2-\beta^2k_z'^2) + i\delta(k_z'+i\varepsilon) + i\mu(\beta^2k_z'^2-(\underset{\sim}{k}'-\underset{\sim}{k})^2+i\sigma k_z}\Big|_{\sigma=0}$$

We do the $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ integral to give

D.34 $$I_5^{[n]} = -\frac{1}{i}\int_0^\infty \frac{d\alpha\, d\gamma\, d\mu\, d\delta\, e^{-(\mu+\delta)\varepsilon}}{C^{(n-1)/2}}\,\pi^n\left(\frac{\partial}{\partial\delta} - 2\frac{\partial}{\partial\sigma}\right)\frac{e^{-D/C'}}{\sqrt{C'}}\Big|_{\sigma=0}$$

where

D.35
$$\begin{cases} C = (\alpha+1/\Lambda^2)(\gamma+1/\Lambda^2) + i\mu\,(\alpha+\gamma+2/\Lambda^2) \\ C' = (\alpha+i\mu+1/\Lambda^2)(\gamma(1-\beta^2)+i\mu(1-\beta^2)+1/\Lambda^2) + \mu^2 \\ D = \frac{\delta^2}{4}(\alpha+i\mu+1/\Lambda^2) - i\mu\frac{\sigma\delta}{2} \end{cases}$$

where we have kept only linear terms in σ. We can drop the ε's in the exponent since for large n the convergence of the μ, δ, integrals is

good.

Now

D.36 $$\frac{1}{i}\int_0^\infty d\delta \frac{\partial}{\partial\delta} e^{-D/C'}\Big|_{\sigma=0} = -\frac{1}{i} e^{-D/C'}\Big|_{\sigma=\delta=0} = -\frac{1}{i}$$

The $\frac{\partial}{\partial\sigma}$ term gives

$$-\frac{2}{i}\frac{\partial}{\partial\sigma}\int_0^\infty d\delta \frac{e^{-D/C'}}{\sqrt{C'}}\Big|_{\sigma=0} = \frac{\mu}{2C'^{3/2}}\int_0^\infty d\delta^2 e^{-\frac{\delta^2}{4C'}\left(\frac{1}{\Lambda^2}+\alpha+i\mu\right)}$$

D.37 $$= \frac{2\mu}{\sqrt{C'}\,(\alpha+i\mu+1/\Lambda^2)}$$

Using D.34 to D.37 we get

$$I_5^{[n]} = -\pi^n \int_0^\infty \frac{d\alpha\, d\gamma\, d\mu \left[-\frac{1}{i} + \frac{2\mu}{\alpha+i\mu+1/\Lambda^2}\right]}{C^{(n-1)/2}\sqrt{C'}}$$

Scale α, γ, μ, by $1/\Lambda^2$ and we obtain

$$I_5^{[n]} = -i\pi^n(\Lambda^2)^{n-3}\int_0^\infty d\alpha\, d\gamma\, d\mu \left(\frac{\alpha - i\mu+1}{\alpha+i\mu+1}\right)\times\frac{1}{C^{(n-1)/2}\sqrt{C'}}$$

We can now rotate the μ-contour until it runs down just to the right of the negative imaginary axis. Hence

D.38 $$I_5^{[n]} = -\pi^n(\Lambda^2)^{n-3}\int_0^\infty \frac{d\alpha\, d\gamma\, d\mu \left(\frac{\alpha-\mu+1}{\alpha+\mu+1}\right)}{\left[(1+\alpha)(1+\gamma)+\mu(\alpha+\gamma+2)\right]^{(n-1)}\left[(1+\alpha+\mu)(\gamma(1-\beta^2)+1+\mu(1-\beta^2)\right]-\mu^2]^{1/2}}$$

Since we only want the real part of D.38 the integration region is limited to make the last square root have positive argument. Since the n-dependent factor has no zero in the integration region the only

n-singularity can come from large α, γ, μ values. It is easily checked (using a scaling trick) that the only regions giving a pole at n = 3 have α, γ, μ all large. Thus we scale α, γ, μ by ρ and obtain

$$\lim_{n\to 3} I_5^{[n]} = -\pi^n (\Lambda^2)^{n-3} \int_N^\infty \frac{\rho^2 d\rho}{\rho^{n-1}} \int_0^1 \frac{d\alpha d\gamma d\mu\, \delta(\alpha+\gamma+\mu-1)}{(\alpha\gamma+\mu(\alpha+\gamma))^{(n-1)/2}}$$

$$\times \left(\frac{\alpha-\mu}{\alpha+\mu}\right) \times \frac{1}{\rho\sqrt{(\alpha+\mu)[\gamma(1-\beta^2)+\mu(1-\beta^2)]-\mu^2}}$$

where we have kept only the dominant large ρ term and N is a lower cut off of ρ.

D.39 $$\therefore \lim_{n\to 3} I_5^{[n]} = \frac{\pi^n(\Lambda^2)^{n-3}}{(n-3)} \int_0^1 \frac{d\alpha d\gamma d\mu\, \delta(\alpha+\gamma+\mu-1)}{(\alpha\gamma+\mu(\alpha+\gamma))\sqrt{(\alpha+\mu)(\gamma+\mu)(1-\beta^2)-\mu^2}} \left(\frac{\alpha-\mu}{\alpha+\mu}\right)$$

For the residue of the pole we have set n = 3 inside the integral. D.39 may be written in the form

D.40 $$\lim_{n\to 3} I_5^{[n]} = \frac{\pi^n 2(\Lambda^2)^{n-3}}{(n-3)} \int_0^\infty \frac{d\alpha d\gamma d\mu\, e^{-(\alpha\gamma+\mu(\alpha+\gamma))}}{\sqrt{(\alpha+\mu)(\gamma+\mu)(1-\beta^2)-\mu^2}} \left(\frac{\alpha-\mu}{\alpha+\mu}\right)$$

That D.40 is equal to D.39 may be proved by scaling α, γ, μ by ρ and then doing the ρ-integral.

Now scale α, γ by μ and from D.40 we can then do the μ-integral giving

D.41 $$\lim_{n\to 3} I_5^{[n]} = \frac{\pi^n(\Lambda^2)^{n-3}}{(n-3)} \int_0^\infty \frac{d\alpha d\gamma \left(\frac{\alpha-1}{\alpha+1}\right)}{(\alpha\gamma+\alpha+\gamma)\sqrt{(1+\alpha)(1+\gamma)(1-\beta^2)-1}}$$

Define

D.42 $x = \alpha + 1 \,, \quad y = \gamma + 1$

$$\therefore \lim_{n\to 3} I_5^{[n]} = \frac{\pi^n (\Lambda^2)^{n-3}}{(n-3)} \int_1^\infty \frac{dx\,dy\,(x-2)}{x(xy-1)\sqrt{xy(1-\beta^2)-1}}$$

Define u = xy and eliminate y from the above.

This gives

$$\lim_{n\to 3} I_5^{[n]} = \frac{\pi^n (\Lambda^2)^{n-3}}{(n-3)} \int_1^\infty \frac{dx\,(x-2)}{x^2} \int_x^\infty \frac{du}{(u-1)\sqrt{u(1-\beta^2)-1}}$$

Interchanging the order of the u and x integrals gives

$$\lim_{n\to 3} I_5^{[n]} = \frac{\pi^n (\Lambda^2)^{n-3}}{(n-3)} \int_1^\infty \frac{du}{(u-1)\sqrt{u(1-\beta^2)-1}} \int_1^u dx\, \frac{x-2}{x^2}$$

D.43 $$= \frac{\pi^n (\Lambda^2)^{n-3}}{(n-3)} \int_1^\infty \frac{du}{(u-1)\sqrt{u(1-\beta^2)-1}} \left\{ \log u + \frac{2}{u} - 2 \right\}$$

D.44 $$\equiv \frac{\pi^n (\Lambda^2)^{n-3}}{(n-3)} \left[\bar{I}_1 + \bar{I}_2 \right]$$

where

D.45 $$\bar{I}_1 \equiv \int_1^\infty \frac{du \log u}{(u-1)\sqrt{u(1-\beta^2)-1}}$$

D.46 $$\bar{I}_2 \equiv -2 \int_1^\infty \frac{du}{u\sqrt{u(1-\beta^2)-1}}$$

Define $$t^2 = u(1-\beta^2) - 1$$

Hence

D.47 $$Re[\bar{I}_1] = \int_0^\infty \frac{2\,dt \log\left(\frac{t^2+1}{1-\beta^2}\right)}{t^2+\beta^2}$$

D.48 $$Re[\bar{I}_2] = -2 \int_0^\infty \frac{2dt}{t^2+1} = -2\pi$$

Now from D.47 we have

$$Re\,[\bar{I}_1] = \int_{-\infty}^{\infty} \frac{dt\ \log\,[(t^2+1)/(1-\beta^2)]}{t^2+\beta^2}$$

$$= \int_{-\infty}^{\infty} \frac{dt\ \log\,\{(1+it)(1-it)/(1-\beta^2)\}}{\beta^2+t^2}$$

$$= 2\,Rl\left\{\int_{-\infty}^{\infty} dt\ \log\left\{\frac{1-it}{\sqrt{1-\beta^2}}\right\}\right/(\beta^2+t^2)$$

Closing contours in the upper half-plane we find

D.49 $$Re\,\bar{I}_1 = 2\,Rl\left[2\pi i\,\log\left(\frac{1+\beta}{\sqrt{1-\beta^2}}\right)\Big/2i\beta\right] = \frac{\pi}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right)$$

So from D.49, D.48 and D.44 we find

D.50 $$\lim_{n\to 3} I_S^{[n]} \equiv \frac{\pi}{(n-3)}^{n+1}(\Lambda^2)^{(n-3)}\left[\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right) - 2\right]$$

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