

GEOMETRY OF MINIMIZERS FOR THE INTERACTION ENERGY WITH MILDLY REPULSIVE POTENTIALS

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ABSTRACT. We show that the support of any local minimizer of the interaction energy consists of isolated points whenever the interaction potential is of class C^2 and is mildly repulsive at the origin; if moreover a minimizer is global, then its support is finite. For a particular class of potentials we prove the existence of a uniform upper bound on the cardinal of the support of a global minimizer. In the one-dimensional case we give quantitative bounds.

1. INTRODUCTION

Consider the *interaction energy* $E: \mathcal{P}(\mathbb{R}^n) \rightarrow \mathbb{R} \cup \{+\infty\}$, defined on the set of Borel probability measures $\mathcal{P}(\mathbb{R}^n)$ by

$$(1.1) \quad E(\mu) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W(x-y) d\mu(x) d\mu(y) \quad \text{for all } \mu \in \mathcal{P}(\mathbb{R}^n),$$

where $W: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is an *interaction potential*. The study of local and global minimizers of energies of the form (1.1) has in recent years been in the spotlight of applied mathematics, in particular in the context of the variational approach to partial differential equations. The main reason for this interest is that E is a Lyapunov functional for the continuity equation

$$\partial_t \mu = \nabla \cdot (\mu \nabla W * \mu) \quad \text{on } \mathbb{R}^n, \text{ for } t > 0,$$

called the *aggregation equation*, where $*$ is the convolution operator and $\mu: [0, \infty) \rightarrow \mathcal{P}(\mathbb{R}^n)$ is here a probability curve. These equations describe the continuum behavior of agents interacting via the potential W , and are at the core of many applications ranging from mathematical biology to granular media and economics, see [16, 14, 11, 17, 4] and the references therein. They can also be obtained as dissipative limits of hydrodynamic equations for collective behavior [13].

Typically, interaction potentials are repulsive towards the origin and attractive towards infinity; this reproduces the “social”, or natural, behavior of the agents that are usually considered in applications. In [2] the authors showed that the dimension of the support of a minimizer of E is directly related to the repulsiveness of the potential at the origin, i.e., to the strength of the repulsion of two very close particles. What they showed is that the stronger the repulsion (up to Newtonian), the higher the dimension of the support. In particular, in the case of *mild repulsion*—when the potential behaves like a power of order α , with $\alpha > 2$, near the origin—the Hausdorff dimension of each smooth enough component of the support has to be zero, see [2, Theorem 2]. The smoothness assumption on the connected components of the support was essential in the proof and more than C^2 -smoothness of the parameterizations of the manifolds was needed. Two open problems immediately arise: Is it possible to have minimizers whose supports lie on sets of non-integer Hausdorff dimension? If it is of zero dimension, can we say more about the distribution of the points?

Both questions are answered in this work. Let us first mention that extensive simulations [10, 12, 3, 2, 1, 8] showed that fractal supports were not numerically observed and, moreover, these numerical simulations were consistently giving minimizers supported on finite numbers of points. This is precisely the rigorous result we show in this work under suitable assumptions

on the repulsive-attractive potential W . In the rest of the paper we always assume that W is radially symmetric, of class C^2 , and such that

$$(1.2) \quad W(0) = 0, \text{ and there is } R > 0 \text{ with } W(x) < 0 \text{ if } 0 < |x| < R \text{ and } W(x) \geq 0 \text{ if } |x| \geq R.$$

Let us write $w(|x|) := W(x)$ for all $x \in \mathbb{R}^n$. Remark that, by convention, w being repulsive (resp. attractive) at $r > 0$ means $w'(r) < 0$ (resp. $w'(r) > 0$). We suppose that W is mildly repulsive, that is,

$$(1.3) \quad \text{there exist } \alpha \geq 2 \text{ and } C > 0 \text{ such that } w'(r)r^{1-\alpha} \rightarrow -C \text{ as } r \rightarrow 0.$$

Note that, by L'Hôpital's rule, (1.3) implies that $w(r)r^{-\alpha} \rightarrow -C/\alpha$ as $r \rightarrow 0$.

When we refer to minimizers of the energy (1.1), we either refer to *global* minimizers, in which case no underlying topology is required, or to *local* minimizers, in which case we need to specify the topology. In [2] it was proven that the right topology to obtain suitable Euler-Lagrange conditions was that induced by the ∞ -Wasserstein distance. We define the ∞ -Wasserstein distance between two probability measures μ and ν by

$$(1.4) \quad d_\infty(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \sup_{(x, y) \in \text{supp } \pi} |x - y|,$$

where $\Pi(\mu, \nu)$ is the space of probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with first marginal μ and second marginal ν . We therefore say that $\mu \in \mathcal{P}(\mathbb{R}^n)$ is a d_∞ -local minimizer of E if there exists $\varepsilon > 0$ such that $E(\mu) \leq E(\nu)$ for all $\nu \in \mathcal{P}(\mathbb{R}^n)$ with $d_\infty(\mu, \nu) < \varepsilon$. We refer to [2, 7, 18] and the references therein for a good account on the properties of this distance and its relation to more classical metrics in optimal transport.

The Euler-Lagrange conditions for d_∞ -local minimizers of E were used to give necessary and sufficient conditions on repulsive-attractive potentials to have existence of global minimizers [5, 15], and to analyze the regularity of the d_∞ -local minimizers for potentials which are as repulsive as, or more singular than, the Newtonian potential [7]. In both cases, d_∞ -local minimizers are solutions of some related obstacle problems for Laplacian or nonlocal Laplacian operators, implying that they are bounded and smooth in their support, or even continuous up to the boundary [7, 9]. Similar Euler-Lagrange equations were also used for nonlinear versions of the Keller-Segel model in order to characterize minimizers of related functionals [6].

Our main theorem in this work is the following.

Theorem 1.1. *Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ be a d_∞ -local minimizer of the interaction energy with $E(\mu) < \infty$. Then every point in the support of μ is isolated. If moreover μ is a global minimizer, then the support of μ is a finite set of points.*

In Section 2 we show some preliminary results concerning minimizers that are needed for the proof of Theorem 1.1, which we give in Section 3. In Section 4 we prove upper estimates on the cardinal of the support of a global minimizer.

2. PRELIMINARY RESULTS

We give in this section the preliminary results needed to prove Theorem 1.1. For every $\mu \in \mathcal{P}(\mathbb{R}^n)$, let us write

$$V_\mu(x) := W * \mu(x) = \int_{\mathbb{R}^n} W(x - y) d\mu(y) \quad \text{for all } x \in \mathbb{R}^n.$$

By the continuity of W it can be checked that, for any $\mu \in \mathcal{P}(\mathbb{R}^n)$, $V_\mu: \mathbb{R}^n \rightarrow [0, +\infty]$ is continuous. From [2, Proposition 2] we have the following lemma.

Lemma 2.1. *Let μ be a d_∞ -local minimizer of the interaction energy with $E(\mu) < \infty$. Then there exists $\varepsilon > 0$ such that any point $x_0 \in \text{supp } \mu$ is a local minimizer of V_μ for a radius ε , that is, $V_\mu(x_0) \leq V_\mu(x)$ for every $x \in B(x_0, \varepsilon)$.*

Remark 2.2. *The ε in the definition of a d_∞ -local minimizer (see below (1.4)) and the one in Lemma 2.1 are the same. This directly comes from the proofs of [2, Propositions 1 and 2].*

By [2, Remark 3, Theorem 4] and the continuity of W we get the result below.

Lemma 2.3. *Let μ be a d_∞ -local minimizer with $E(\mu) < \infty$. Let $K \subset \text{supp } \mu$ be a bounded connected set. There exists $C_K \geq 0$ such that the Euler-Lagrange condition writes*

$$V_\mu(x) = C_K \quad \text{for all } x \in K.$$

If moreover μ is a global minimizer, then we know that $C_K = 2E(\mu)$, which is independent of K , and we have

$$(2.1) \quad \begin{cases} V_\mu(x) = 2E(\mu) & \text{for all } x \in \text{supp } \mu, \\ V_\mu(x) \geq 2E(\mu) & \text{for all } x \in \mathbb{R}^n. \end{cases}$$

We now give the computation of the second variation of the energy functional (1.1), which is a crucial point of the proof of Theorem 1.1.

Lemma 2.4. *Let μ be a d_∞ -local minimizer of the interaction energy with $E(\mu) < \infty$. There exists $\delta > 0$ such that for all $x_0 \in \text{supp } \mu$ we have*

$$(2.2) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W(x-y) d\nu(x) d\nu(y) \geq 0$$

for any signed measure ν verifying $\text{supp } \nu \subset \text{supp } \mu \cap B(x_0, \delta)$ and $\nu(\mathbb{R}^n) = 0$.

Proof. Let $\varepsilon > 0$ be the constant of Lemma 2.1 for the local minimizer μ . Fix $x_0 \in \text{supp } \mu$ and $\delta \leq \varepsilon/2$, and define

$$\mu' := \mu_0 + \mu_1,$$

where μ_0 is an arbitrary nonnegative measure with

$$\mu_0(\mathbb{R}^n) = \mu(B(x_0, \delta)), \quad \text{supp } \mu_0 \subset \text{supp } \mu \cap B(x_0, \delta),$$

and μ_1 is defined as

$$\mu_1(A) := \mu(A \setminus B(x_0, \delta)) \quad \text{for any Borel set } A \subset \mathbb{R}^n.$$

Clearly $\mu' \in \mathcal{P}(\mathbb{R}^n)$ and $d_\infty(\mu, \mu') < \varepsilon$. Let us write $B_0 := B(x_0, \delta)$. Then, since $E(\mu) \leq E(\mu')$,

$$(2.3) \quad \begin{aligned} & \int_{B_0} \int_{B_0} W(x-y) d\mu(y) d\mu(x) + 2 \int_{B_0} \int_{\mathbb{R}^n \setminus B_0} W(x-y) d\mu(y) d\mu(x) \\ & \leq \int_{B_0} \int_{B_0} W(x-y) d\mu_0(y) d\mu_0(x) + 2 \int_{B_0} \int_{\mathbb{R}^n \setminus B_0} W(x-y) d\mu(y) d\mu_0(x). \end{aligned}$$

By Lemma 2.1,

$$V_\mu(x) \leq V_\mu(y) \quad \text{for all } x \in \text{supp } \mu \cap B_0 \text{ and } y \in B_0.$$

Then, since $\mu_0(\mathbb{R}^n) = \mu(B_0)$, it follows from integration against $\int_{B_0} \int_{B_0} d\mu_0(x) d\mu(y)$ that

$$\begin{aligned} & \int_{B_0} \int_{\mathbb{R}^n \setminus B_0} W(x-y) d\mu(y) d\mu_0(x) - \int_{B_0} \int_{\mathbb{R}^n \setminus B_0} W(x-y) d\mu(y) d\mu(x) \\ & \leq \int_{B_0} \int_{B_0} W(x-y) d\mu(y) d\mu(x) - \int_{B_0} \int_{B_0} W(x-y) d\mu(y) d\mu_0(x). \end{aligned}$$

Hence (2.3) implies

$$\begin{aligned} & 2 \int_{B_0} \int_{B_0} W(x-y) d\mu(y) d\mu_0(x) \\ & \leq \int_{B_0} \int_{B_0} W(x-y) d\mu(y) d\mu(x) + \int_{B_0} \int_{B_0} W(x-y) d\mu_0(y) d\mu_0(x). \end{aligned}$$

Then the above inequality becomes

$$\int_{B_0} \int_{B_0} W(x-y) d[\mu - \mu_0](y) d[\mu - \mu_0](x) \geq 0,$$

which, by the arbitrariness of μ_0 , gives the result. $\square$

Remark 2.5. In Lemma 2.4, if μ is a global minimizer of the interaction energy, then the result is slightly stronger. Indeed, in this case, for all $x_0 \in \text{supp } \mu$ we have

$$(2.4) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W(x-y) d\nu(x) d\nu(y) \geq 0$$

for any signed measure ν verifying $\text{supp } \nu \subset \text{supp } \mu$ and $\nu(\mathbb{R}^n) = 0$. This can be seen simply by adapting the proof of Lemma 2.4 to global minimizers.

Remark 2.5 yields the following lemma.

Lemma 2.6. Let μ be a global minimizer of the interaction energy. Then μ is compactly supported with $\text{diam}(\text{supp } \mu) \leq R$.

Proof. Let $x, y \in \text{supp } \mu$. Take $\nu = \delta_x - \delta_y$ in (2.4) and get that $W(x-y) \leq 0$. By the assumption given in (1.2), we are done. $\square$

3. PROOF OF THEOREM 1.1

Suppose that μ is a d_∞ -local minimizer of the interaction energy, and let $y_0 \in \text{supp } \mu$. Then, without loss of generality, for any $\delta > 0$ smaller than the one in Lemma 2.4, there are two different points $y_1, y_2 \in \text{supp } \mu \cap B(y_0, \delta)$, or there is nothing to prove.

Step 1.- Geometric constraint. We show that there is a geometric constraint for the points in the support of μ stemming from the minimality condition of Lemma 2.4. For convenience, let us write $y_0 = 0$, $y_1 = x_1$ and $y_2 = -x_2$, and consider the measure

$$\nu_\lambda = -\delta_0 + \lambda\delta_{x_1} + (1-\lambda)\delta_{-x_2} \quad \text{for any } \lambda \in [0, 1].$$

By plugging ν_λ in (2.2) in place of ν we get

$$\lambda(1-\lambda)W(x_1+x_2) \geq \lambda W(x_1) + (1-\lambda)W(x_2).$$

By the assumption in (1.2), we can choose δ small enough such that $W(x_1), W(x_2)$ and $W(x_1+x_2)$ are negative. Then, set $a := W(x_1)/W(x_1+x_2)$ and $b := W(x_2)/W(x_1+x_2)$, so that the above inequality is equivalent to

$$(3.1) \quad \lambda(1-\lambda) \leq \lambda a + (1-\lambda)b,$$

and implies

$$(3.2) \quad \sqrt{a} + \sqrt{b} \geq 1.$$

To see this, assume $|b-a| \leq 1$ (otherwise the inequality is trivial, since $a, b \geq 0$). Then, choosing $\lambda = (1+b-a)/2$, the inequality (3.1) becomes, after some rearrangement,

$$(1-a)^2 - 2(1+a)b + b^2 \leq 0.$$

This implies in particular that

$$b \geq 1+a-2\sqrt{a} = (1-\sqrt{a})^2,$$

which is the desired inequality (3.2). Now, recalling the definition of a and b , we get, from (3.2),

$$\sqrt{-W(x_1)} + \sqrt{-W(x_2)} \geq \sqrt{-W(x_1+x_2)}.$$

Assume, by homogeneity, that $x_1+x_2 = pe_1$, where e_1 is the first unit vector of the orthonormal basis of $\mathbb{R}^n$, and $p > 0$ is a small rescaling parameter. Then the above inequality becomes

$$\sqrt{-W(x_1)} + \sqrt{-W(pe_1-x_1)} \geq \sqrt{-W(pe_1)}.$$

Write $x_1 = p(te_1 + y)$, where $y \in \mathbb{R}^n$ has zero first coordinate, and, by homogeneity, $t \in [0, 1]$. Then, using that $|x_1| \leq pt + p|y|$ and $|pe_1 - x_1| \leq p(1-t) + p|y|$, and that, for p small enough, $x \mapsto \sqrt{-W(px)}$ is radially non-decreasing in $B(0, 1+|y|)$, we get

$$\sqrt{-w(p(t+|y|))} + \sqrt{-w(p(1-t+|y|))} \geq \sqrt{-w(p)}.$$

We write $w_p(r) := w(pr)p^{-\alpha}$ for any $r \geq 0$. Then, dividing the inequality above by $p^{\alpha/2}$ yields

$$\sqrt{-w_p(t + |y|)} + \sqrt{-w_p(1 - t + |y|)} \geq \sqrt{-w_p(1)}.$$

Observe that since the inequality above is invariant under the transformation $t \leftrightarrow 1 - t$, we could have assumed $t \in [0, 1/2]$ without loss of generality when writing $x_1 = p(te_1 + y)$. Define, for all $s \in [0, 1]$ and $z \in \mathbb{R}^n$, and for $p > 0$ small enough,

$$\eta_p(s, z) = \sqrt{-w_p(s + |z|)} + \sqrt{-w_p(1 - s + |z|)} - \sqrt{-w_p(1)}.$$

For any two $v, v' \in \mathbb{R}^n$ distinct, define the open set

$$\mathcal{S}_p(v, v') := \left\{ u \in \mathbb{R}^n \mid \pi_{(v, v')}u \in [v, v'] \text{ and } \eta_p\left(\frac{|\pi_{(v, v')}u - v|}{|v - v'|}, \pi_{(v, v')}u - u\right) < 0 \right\},$$

where $\pi_{(v, v')}$ denotes the orthogonal projection on the line (v, v') .

Since the point y_0 , taken above to be the origin, can be an arbitrary point of the support of μ , the geometric constraint result shown above is: for any $y_0, y_1 \in \text{supp } \mu$ such that $|y_0 - y_1| = p$, where p is small enough for the above calculations to make sense, we have

$$\text{supp } \mu \cap \mathcal{S}_p(y_0, y_1) = \emptyset.$$

Step 2.- Asymptotic geometric constraint. Define, for all $s \in [0, 1]$ and $z \in \mathbb{R}^n$,

$$(3.3) \quad \eta_{0, \alpha}(s, z) = (s + |z|)^{\alpha/2} + (1 - s + |z|)^{\alpha/2} - 1.$$

For any two $v, v' \in \mathbb{R}^n$ distinct, define the open set

$$\mathcal{S}_{0, \alpha}(v, v') := \left\{ u \in \mathbb{R}^n \mid \pi_{(v, v')}u \in [v, v'] \text{ and } \eta_{0, \alpha}\left(\frac{|\pi_{(v, v')}u - v|}{|v - v'|}, \pi_{(v, v')}u - u\right) < 0 \right\}.$$

By the assumption given in (1.3), and assuming with no loss of generality that $C = \alpha$, we have $w_p(r) \rightarrow -r^\alpha$ and $w'_p(r) \rightarrow -\alpha r^{\alpha-1}$ as $p \rightarrow 0$, and

$$(3.4) \quad \begin{cases} \eta_p \rightarrow \eta_{0, \alpha}, \\ \nabla \eta_p \rightarrow \nabla \eta_{0, \alpha}, \end{cases} \quad \text{pointwise as } p \rightarrow 0.$$

This shows that, for any $y_0, y_1 \in \text{supp } \mu$ asymptotically close, we have

$$(3.5) \quad \text{supp } \mu \cap \mathcal{S}_{0, \alpha}(y_0, y_1) = \emptyset$$

The boundary $\eta_{0, \alpha}$ is represented in Figure 1 for $n = 2$ and different values of α . It is important to keep in mind that this shape only depends on α and not on the choice of the minimizer μ .

Step 3.- Understanding the geometric constraint. Given $v, v' \in \mathbb{R}^n$, define the open “double cone” with opening $\tau > 0$ by

$$C_\tau(v, v') := \left\{ u \in \mathbb{R}^n \mid \pi_{(v, v')}u \in [v, v'] \text{ and } |\pi_{(v, v')}u - u| < \tau \min(|\pi_{(v, v')}u - v|, |\pi_{(v, v')}u - v'|) \right\}.$$

Since $\alpha > 2$, then $\mathcal{S}_{0, \alpha}(v, v')$ is non-empty for any $v, v' \in \mathbb{R}^n$ distinct and $r \mapsto r^{\alpha/2}$ is a strictly convex function on $[0, \infty)$, which implies that $\mathcal{S}_{0, \alpha}(v, v')$ is a convex set. Thus $\gamma_\alpha > 0$, the opening of the cone with maximum volume that we can fit inside $\mathcal{S}_{0, \alpha}(v, v')$, is well-defined. By symmetry of $\mathcal{S}_{0, \alpha}(v, v')$, γ_α can be obtained by using that $|z| = \gamma_\alpha s$ when $s = 1/2$ and $\eta_{0, \alpha}(s, z) = 0$ in (3.3):

$$\gamma_\alpha = 2^{1-2/\alpha} - 1.$$

Note that γ_α only depends on α (and not on the choice of v and v'). Therefore, for any $y_0, y_1 \in \text{supp } \mu$ asymptotically close, (3.5) gives

$$\text{supp } \mu \cap C_{\gamma_\alpha}(y_0, y_1) = \emptyset.$$

Furthermore, $\partial_s \eta_{0, \alpha}(0, 0) = -\alpha/2$ and $\|\nabla \eta_{0, \alpha}(0, 0)\| = \alpha/\sqrt{2}$, so that no tangent plan to $\mathcal{S}_{0, \alpha}(v, v')$ at $s = 0$ and $z = 0$ contains $v' - v$. By symmetry, this also holds at $s = 1$ and $z = 0$.

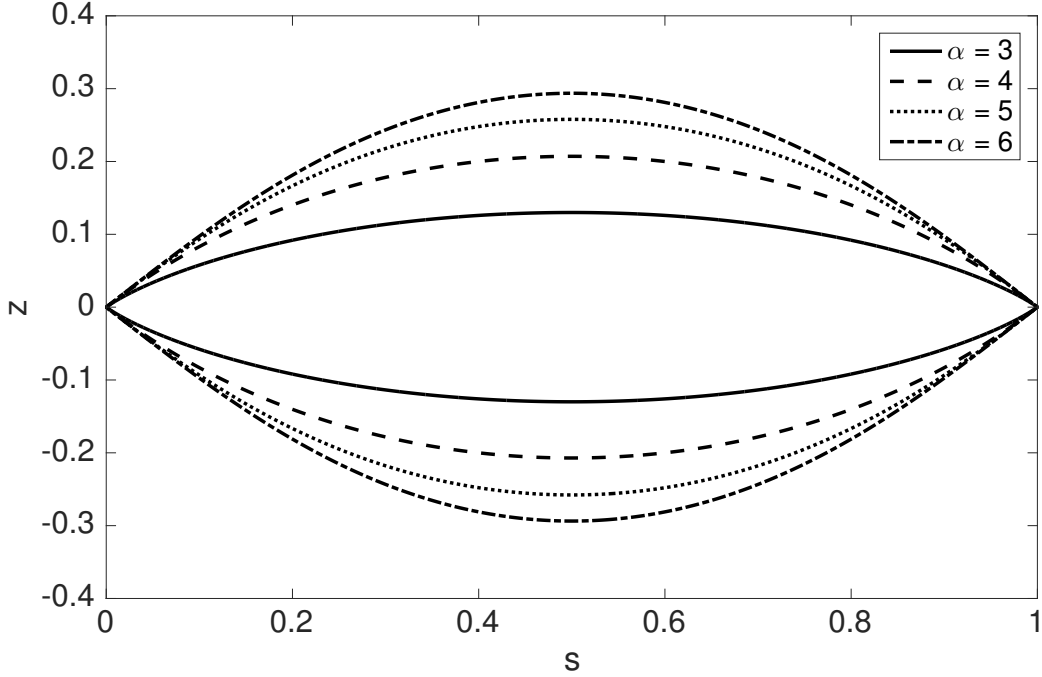


FIGURE 1. The boundary $\eta_{0,\alpha}$ for $n = 2$ and different values of α .

This, along with the convexity of $\mathcal{S}_{0,\alpha}(v, v')$ for any distinct $v, v' \in \mathbb{R}^n$ and (3.4), we get the final form of the geometric constraint that need to conclude: there exist $p' > 0$ and $\tau_{p'} < \gamma_\alpha$ such that, for all $p < p'$ and $y_0, y_1 \in \text{supp } \mu$ with $|y_0 - y_1| = p$, we have $C_{\tau_{p'}}(y_0, y_1) \subset \mathcal{S}_p(y_0, y_1)$.

Step 4.- Conclusion. We now proceed by contradiction. Let us assume that y_0 is an accumulation point in $\text{supp } \mu$. Then there exist a sequence $(x_k)_{k \in \mathbb{N}} \subset \text{supp } \mu$ converging to y_0 and a unit direction e such that

$$\pi_e x_k - x_k \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

where π_e is the orthogonal projection on the line spanned by e . Let $\tau < \tau_{p'}$. We know that, for any $k \in \mathbb{N}$, $C_\tau(y_0, \pi_e x_k)$ contains a point of the support of μ . In addition, there exists $k' > 0$ large enough such that $C_\tau(y_0, \pi_e x_{2k}) \subset C_{\tau_{p'}}(y_0, x_k)$ for all $k > k'$. Therefore, for all $k > k'$, $C_{\tau_{p'}}(y_0, x_k)$ contains a point of the support of μ .

This is the wanted contradiction since we know that, for any k large enough such that $p_k := |y_0 - x_k| < p'$, we have $C_{\tau_{p'}}(y_0, x_k) \subset \mathcal{S}_{p_k}(y_0, x_k)$ by Step 3 and $\text{supp } \mu \cap \mathcal{S}_{p_k}(y_0, x_k) = \emptyset$ by Step 2. Therefore, any $y_0 \in \text{supp } \mu$ is isolated.

Finally, the finiteness of $\text{supp } \mu$ when μ is a global minimizer of the interaction energy directly stems from Lemma 2.6, which tells us that $\text{supp } \mu$ is compact, and the fact that every point in $\text{supp } \mu$ is isolated, as just shown.

Remark 3.1. Let $n = 2$. One has $\nabla \eta_{0,\alpha}(0,0) = \alpha/2(-1, \pm 1)$, meaning that the direction of any tangent plan to the shape $\mathcal{S}_{0,\alpha}(x, y)$ at $s = 0$ and $z = 0$ does not depend on α , for any two points $x, y \in \mathbb{R}^n$. By symmetry, this is also true at $s = 1$ and $z = 0$. Moreover, the angle that these tangent plans make with the base of the shape is $\pi/4$.

4. ESTIMATE ON THE CARDINAL OF THE SUPPORT OF A GLOBAL MINIMIZER

We give here upper estimates on the cardinal of the support of a global minimizer of the interaction energy. We obtain a non-quantitative bound in any dimension, and a quantitative one in the one-dimensional case.

For any $m \in \mathbb{N}$ and unit direction e , we write

$$D_e^m W := \sum_{\{\gamma \in \mathbb{N}_0^n \mid |\gamma| = m\}} e^\gamma D^\gamma W,$$

where, for each multi-index $\gamma \in \mathbb{N}_0^n$, $e^\gamma := e_1^{\gamma_1} \dots e_n^{\gamma_n}$ and $D^\gamma := \partial^{\gamma_1} / \partial x_1^{\gamma_1} \dots \partial^{\gamma_n} / \partial x_n^{\gamma_n}$. First note that, for every unit direction e , Lemma 2.1 implies that for any d_∞ -local minimizer μ ,

$$(4.1) \quad \begin{cases} D_e^1 V_\mu(x) = D_e^1 W * \mu(x) = 0, \\ D_e^2 V_\mu(x) = D_e^2 W * \mu(x) \geq 0, \end{cases} \quad \text{for all } x \in \text{supp } \mu.$$

4.1. Non-quantitative estimate. In addition to the hypotheses on the potential W given in Section 1, we consider in this section the following.

$$(4.2) \quad \begin{aligned} & W \in C^4(\mathbb{R}^n \setminus \{0\}), W \text{ satisfies (1.3) with } \alpha < 4, \\ & \text{and there exists } M > 0 \text{ such that, for any } \gamma \in \mathbb{N}_0^n \text{ with } |\gamma| = 4, D^\gamma W \leq M. \end{aligned}$$

We give below an example of potential satisfying (4.2).

Example 4.1. Assume that there exists $U \in C^4(\mathbb{R}^n)$ such that

$$(4.3) \quad W(x) = -|x|^\alpha / \alpha + U(x) \quad \text{for all } x \in \mathbb{R}^n, \text{ with } 3 \leq \alpha < 4,$$

and that all the assumptions discussed in Section 1 are verified. We claim that W satisfies (4.2) as well. To show this, compute, for all $x \in \mathbb{R}^n$ and unit direction e ,

$$D_e^4 W(x) = -(\alpha - 2)(3|x|^{\alpha-4} + 6(\alpha - 4)(x \cdot e)^2|x|^{\alpha-6} + (\alpha - 4)(\alpha - 6)(x \cdot e)^4|x|^{\alpha-8}) + D_e^4 U(x).$$

Therefore $W \in C^4(\mathbb{R}^n \setminus \{0\})$ and $D_e^4 W$ blows up at the origin (since $\alpha < 4$). We thus want the term within parentheses to be non-negative. Up to multiplying the above expression by a non-negative constant, we can assume $|x| = 1$. Then, since $(x \cdot e)^2 \in [0, 1]$, it is necessary and sufficient to show that

$$(4.4) \quad f(\xi) := 3 - 6\beta\xi + \beta(\beta + 2)\xi^2 \geq 0 \quad \text{for all } \xi \in [0, 1],$$

where $\beta := 4 - \alpha$. Since $3 \leq \alpha < 4$, $\beta \in (0, 1]$. Now, one can check that the minimum of f is reached at $\bar{\xi} = 1$ and $f(\bar{\xi}) = (\beta - 3)(\beta - 1) \geq 0$, which proves the claim. We can also show that the condition $3 \leq \alpha$ in (4.3) is sharp. Indeed, let $2 < \alpha < 3$; then $\beta \in (1, 2)$, and the minimum of f is reached at $\bar{\xi} = 3/(\beta + 2) \in (3/4, 1)$ and $f(\bar{\xi}) = 6(1 - \beta)/(\beta + 2) < 0$, which violates (4.4).

Lemma 4.2. Suppose that W satisfies (4.2) and let $\mu \in \mathcal{P}(\mathbb{R}^n)$ be a global minimizer of the interaction energy. Then $D_e^2 V_\mu(x) > 0$ for all $x \in \text{supp } \mu$ and unit direction e .

Proof. By Theorem 1.1 we know that the support of μ is discrete. Also, by the minimality conditions (4.1) we know that the Hessian of V_μ is positive semi-definite on the support of μ . Let $\{x_k\}_{k \geq 1} := \text{supp } \mu$ and denote by m_k the mass of particle x_k for any $k \geq 1$. Without loss of generality, take x_1 to be so that $D_e^2 V_\mu(x_1) = 0$. We want to show that this yields a contradiction by constructing a probability measure with smaller energy than μ . Define, for all $\varepsilon > 0$,

$$\mu_\varepsilon := \sum_{i \geq 2} m_i \delta_{x_i} + \frac{m_1}{2} \delta_{y_\varepsilon} + \frac{m_1}{2} \delta_{z_\varepsilon},$$

where y_ε and z_ε are distinct points belonging to the boundary of $B(x_1, \varepsilon/2)$ such that $x_\varepsilon - x_1$ and $y_\varepsilon - x_1$ are collinear to e , and which we can suppose to be different from x_k for all $k \geq 2$. Clearly μ_ε is the probability measure resulting from splitting the mass present at x_1 under μ and moving it evenly to two opposite points in the direction e at a distance $\varepsilon/2$ from x_1 . Then,

$$\begin{aligned} E(\mu_\varepsilon) &= \frac{1}{2} \sum_{i \geq 2} m_i \left(\sum_{\substack{j \geq 2 \\ j \neq i}} m_j W(x_i - x_j) + m_1 W(x_i - y_\varepsilon) + m_1 W(x_i - z_\varepsilon) \right) + \frac{m_1^2}{4} W(y_\varepsilon - z_\varepsilon) \\ &= E(\mu) + \frac{m_1}{2} \sum_{i \geq 2} m_i \left(W(x_i - y_\varepsilon) + W(x_i - z_\varepsilon) - 2W(x_i - x_1) \right) + \frac{m_1^2}{4} w(\varepsilon). \end{aligned}$$

Using the upper boundedness of $D_e^4 W$, and the fact that, by (1.2), (1.3) and (4.2), $D_e^2 W(0) = 0$,

$$\begin{aligned} E(\mu_\varepsilon) &\leq E(\mu) + \frac{m_1}{8} \sum_{i \geq 2} m_i D_e^2 W(x_i - x_1) \varepsilon^2 + \frac{m_1^2}{4} w(\varepsilon) + C' \varepsilon^4 \\ &= E(\mu) + \frac{m_1}{8} D_e^2 V_\mu(x_1) \varepsilon^2 - \frac{m_1}{8} D_e^2 W(0) \varepsilon^2 + \frac{m_1^2}{4} w(\varepsilon) + C' \varepsilon^4 = E(\mu) + \frac{m_1^2}{4} w(\varepsilon) + C' \varepsilon^4, \end{aligned}$$

for some constant $C' \in \mathbb{R}$ depending on M . Then

$$E(\mu_\varepsilon) \leq E(\mu) - \frac{m_1^2}{4} (C/\alpha) \varepsilon^\alpha + C' \varepsilon^4,$$

where C is as in (1.3). Since $\alpha < 4$ we therefore know that, for ε small enough, $E(\mu_\varepsilon) < E(\mu)$, which contradicts the fact that μ is a global minimizer of E . $\square$

Theorem 4.3. *Let $R^*, M^* > 0$. There exists $N \in \mathbb{N}$ such that, for any potential W satisfying the assumptions (1.2) with $R = R^*$ and (4.2) with $M = M^*$, any global minimizer $\mu \in \mathcal{P}(\mathbb{R}^n)$ of the interaction energy satisfies $\text{card}(\text{supp } \mu) \leq N$.*

Proof. We proceed by contradiction. Let $(W_k)_{k \in \mathbb{N}}$ be a sequence of potentials satisfying the assumptions in (1.2) and (4.2) with $R = R^*$ and $M = M^*$. Then, $\{W_k\}_{k \in \mathbb{N}}$ is compact in $C^2(\mathbb{R}^n)$ with limit some W satisfying (1.2) and (4.2) again with $R = R^*$ and $M = M^*$. Let $(\mu_k)_{k \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n)$ be such that μ_k is a global minimizer of the interaction energy associated to W_k for every $k \in \mathbb{N}$ and $\text{card}(\text{supp } \mu_k) \rightarrow \infty$ as $k \rightarrow \infty$. By Lemma 2.6, $\text{supp } \mu_k \subset B(0, R^*)$ for all $k \in \mathbb{N}$, so that, up to a subsequence, $(\mu_k)_{k \in \mathbb{N}}$ converges narrowly to some $\mu \in \mathcal{P}(\mathbb{R}^n)$ as $k \rightarrow \infty$, which, by narrow lower semi-continuity of the energy (since W is bounded from below and lower semi-continuous), is a global minimizer associated to W . We can pick $x_k, y_k \in \text{supp } \mu_k$ for all $k \in \mathbb{N}$ such that the sequences $(x_k)_{k \in \mathbb{N}}$ and $(y_k)_{k \in \mathbb{N}}$ converge to some $x \in \text{supp } \mu$ as $k \rightarrow \infty$. Let $e_k := (x_k - y_k)/|x_k - y_k|$ for all $k \in \mathbb{N}$, and, up to a subsequence, define $e := \lim_{k \rightarrow \infty} e_k$. Thanks to the first line of (4.1) and C^2 -continuity of W ,

$$0 = \frac{D_{e_k}^1 V_{\mu_k}(x_k) - D_{e_k}^1 V_{\mu_k}(y_k)}{|x_k - y_k|} = \int_0^1 D_{e_k}^2 V_{\mu_k}(x_k + t(y_k - x_k)) dt \xrightarrow[k \rightarrow \infty]{} D_e^2 V_\mu(x).$$

However, Lemma 4.2 gives $D_e^2 V_\mu(x) > 0$, which leads to the wanted contradiction. $\square$

4.2. Quantitative estimate in the one-dimensional case. We fix $n = 1$. In addition to the assumptions discussed in 1 we consider in this section the following.

(4.5) There exists $r \in (0, R]$ such that $\sqrt{-w}$ is strictly convex on $(0, r)$.

Lemma 4.4. *Suppose that W satisfies (4.5). Let μ be a global minimizer of the interaction energy. Then, any interval of size strictly less than r contains at most one point of $\text{supp } \mu$.*

Proof. Without loss of generality, we can take three distinct points $0, x_1, -x_2 \in \text{supp } \mu$ with $-x_2 < 0 < x_1$. By applying the reasoning led in the proof of Theorem 1.1 this time to global minimizers, we get

$$(4.6) \quad \sqrt{-w(x_1)} + \sqrt{-w(x_2)} \geq \sqrt{-w(x_1 + x_2)}.$$

By contradiction, suppose that $x_1 + x_2 \in (0, r)$. Then, since $\sqrt{-w}$ is strictly convex on $(0, r)$,

$$\sqrt{-w(x_1)} < \frac{x_1}{x_1 + x_2} \sqrt{-w(x_1 + x_2)} + \frac{x_2}{x_1 + x_2} \sqrt{-w(0)},$$

which, since $w(0) = 0$, yields

$$\sqrt{-w(x_1)} < \frac{x_1}{x_1 + x_2} \sqrt{-w(x_1 + x_2)}.$$

Similarly, for $-x_2$,

$$\sqrt{-w(x_2)} < \frac{x_2}{x_1 + x_2} \sqrt{-w(x_1 + x_2)}.$$

Thus, by adding up the last two inequalities,

$$\sqrt{-w(x_1)} + \sqrt{-w(x_2)} < \sqrt{-w(x_1 + x_2)},$$

which contradicts (4.6). Therefore $x_1 + x_2 \geq r$, which is the desired result. $\square$

As a direct consequence of Lemmas 2.6 and 4.4, we get the theorem below.

Theorem 4.5. *Suppose that W satisfies (4.5). Let $\mu \in \mathcal{P}(\mathbb{R})$ be a global minimizer of the interaction energy. Then $\text{card}(\text{supp } \mu) \leq 2\lceil 2R/r \rceil$, where $\lceil \cdot \rceil$ is the ceiling function.*

We give now an example of potential for which we can apply the results above to estimate the number of points in the supports of global minimizers.

Example 4.6. Consider the power-law potential

$$(4.7) \quad W(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b} \quad \text{for all } x \in \mathbb{R} \text{ with } 2 < b < a.$$

We want to use Theorem 4.5 to estimate the number of points in the support of a global minimizer μ for the potential (4.7) as a function of a and b . Clearly, W satisfies the assumptions in (1.2), with $R = R(a, b) = (a/b)^{1/(a-b)}$, and in (1.3). After some computation, we get that the condition in (4.5) is fulfilled for $r = r(a, b)$, where

$$r(a, b) = \left(\frac{a(a-1) + b(b-1) - ab}{b(a-2)} - \sqrt{\left(\frac{a(a-1) + b(b-1) - ab}{b(a-2)} \right)^2 - \frac{a(b-2)}{b(a-2)}} \right)^{1/(a-b)}$$

Then, Theorem 4.5 gives us an upper estimate on $\text{card}(\text{supp } \mu)$. To make it simpler, let us assume that $a = 2b$. By injecting this into the above equation,

$$r(a, b) = r(b) = \left(\frac{3}{2} - \frac{1}{2} \sqrt{\frac{5b-1}{b-1}} \right)^{1/b}.$$

Since now $R(a, b) = R(b) = 2^{1/b}$, Theorem 4.5 gives

$$\text{card}(\text{supp } \mu) \leq 2 \left\lceil 2 \left(\frac{4\sqrt{b-1}}{3\sqrt{b-1} - \sqrt{5b-1}} \right)^{1/b} \right\rceil.$$

Note that the right-hand side of the formula is infinity for the limit case $b = 2$. This suggests, as also do numerical simulations (see [2]), that if the potential behaves like $-|\cdot|^\alpha$ with $\alpha \leq 2$, then the support of a global minimizer is no more atomic. When $b \rightarrow \infty$, however, the right-hand side of the above formula gives an upper bound of 4 points in the support.

Remark 4.7. *When W is a power-law potential, as in (4.7), with even powers, we can estimate the number of points in the support of a global minimizer by another consideration than the one of Theorem 4.5. Indeed, if a and b are even integers and μ is a global minimizer, then V_μ is a polynomial of degree a , and thus we cannot expect the first line of (2.1) to be true more than $a/2$ times. Therefore, $\text{card}(\text{supp } \mu) \leq a/2$. Note that this estimate, unlike the one of Theorem 4.5, is independent of b , and is thus valid for $b = 2$ (and $a \geq 4$).*

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