

The
Numerical Solution of Hammerstein's Equation

by

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ABSTRACT

This thesis is concerned with the utilization of the variational approach to the Hammerstein equation based on some general results that were obtained by M. Vainberg.

In chapter 1 a summary of Hammerstein results concerning the existence and uniqueness of the non-linear integral equation

$$u(x) = \int_G K(x,t)F(t,u(t))dt, \quad t \in G$$

in Hilbert spaces and particularly in $L_2[0,1]$ are given, and its functional analytic formulation which forms the bases for the variational approach by M. Vainberg. Vainberg's work involves the square root of the linear integral operator as an argument in the functional. This leads to difficulties in numerical work and we resolve it by considering two other functionals, the first of which is based on a sequence of linear operators approximating the square root operator, and appears in this chapter.

The second chapter is devoted to the second functional which is based on a linear perturbation in the linear operator of the Hammerstein equation. This chapter is not written in the usual way of definitions, theorems etc. but all rather in an "integrated form".

In chapter 3 we give two algorithms for the construction of minimizing sequences for these functionals. The first algorithm is based on the theory of discretization for functionals developed by Aubin, Lion and Daniel. The second algorithm is based on the denseness of the piecewise polynomials in $L_2[0,1]$. Some numerical results follow this chapter.

Finally a word about the notation. $(Bu)(x)$ means the value of a linear operator at the point x , i.e. if B is a linear integral operator then

$$(Bu)(x) = \int_0^1 B(x,t)u(t)dt.$$

The numbering of the equations is within the chapters so that if for example we find in chapter 3 a reference to equation 2.8 it means equation 8 of section 2 of this chapter, otherwise the number of the chapter is quoted.

Remark: To make the presentation more complete, a supplement of some of the definitions and theorems of which use is made in the thesis is given at the end. If a reference is made to this supplement then it is denoted by the letter S.

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Chapter 1: The Variational approach to the Hammerstein equation:

Vainberg's theory and the problem of the square root of the linear operator in the equation.

1. Historical Development

The development of the theory of non-linear integral equations began in the early part of this century. After Fredholm completed his algebraic theory for linear system of equations, first attempts were made to apply results from the theory of non-linear systems to non-linear equations.

The first serious studies were carried out by Liapunov in 1906. His studies were partly motivated by problems from theoretical physics such as the equilibrium of a rotating fluid. Generalizations of Liapunov's work were developed by E. Schmidt and L. Lichtenstein. But it was not until 1930 that Hammerstein made an important step forward.

Hammerstein treated a class of non-linear equations of the form

$$u(t) = \int_a^b K(t, \tau) F(\tau, u(\tau)) d\tau. \quad (1.1)$$

These equations are a special case of a more general class of equations, the Urysohn equations:

$$v(t) = \int_a^b K(t, \tau, v(\tau)) d\tau.$$

What may have motivated Hammerstein (and others) was an attempt to solve non-linear boundary value problems for differential equations. Minty and Dolph [13] state that two such problems are thought to have influenced Hammerstein most. The first one was a non-linear elliptic boundary value problem: $\Delta u = e^u$ with $u=f$ on the boundary of a finite domain G in two or three dimensions. This problem arises in the theory of automorphic functions, the theory of surfaces of constant negative curvature and the theory of gas discharges. L. Bieberbach in 1912 and Frank von-Mises in 1916 were able to invert this equation into a non-linear integral equation: using the solution to the problem $\Delta w=0$ and $w=f$ on the boundary they obtained a non-linear integral equation by applying appropriate Green's function:

$$y(t) = \int_G K(t, \tau) \exp w(\tau) [\exp y(\tau) - y(\tau)] d\tau$$

where $\Delta y = (\exp w)(\exp y)$ and $y=0$ on the boundary for the function $y=u-w$.

The second problem thought to have influenced Hammerstein was the problem of a forced oscillation of a finite amplitude of

a pendulum, first treated by G. Duffin in 1918 and G. Hamel in 1922. This problem can be formulated by the following non-linear differential equation: $\frac{d^2 y}{dt^2} + \alpha^2 \sin y = f(t)$, where the driving function $f(t)$ is odd. Again by applying the Green's function for the problem $y''(t)=0$; $y(0)=y(1)=0$ they obtained the following Hammerstein equation:

$$y(t) = - \int_0^1 T(t, \tau) [f(\tau) - \alpha^2 \sin y(\tau)] d\tau.$$

Hammerstein's treatment of the existence and uniqueness problems to these type of equations now known as 'Hammerstein's equations' are important to the understanding of the basis of this thesis and hence they will be outlined in the following section.

2. 'Hammerstein's Contribution

The assumptions made by Hammerstein concerning the linear integral operator in equation (1.1) were the following:

1) The kernel is of class $\mathcal{L}^2$, i.e. the composite kernel is bounded and continuous:

$$\|K\|^2 = \int_a^b \int_a^b K^2(x, y) dx dy = \int_a^b A^2(x) dx < N^2$$

where

$$A(x) = \left[\int_a^b K^2(x, y) dy \right]^{\frac{1}{2}} \quad \text{and } N^2 \text{ is a positive constant.}$$

2) The kernel is symmetric:

$$K(x, y) = K(y, x)$$

3) The kernel is positive definite, that is all its eigenvalues are positive.

A non-linear integral equation possessing a linear integral operator satisfying these assumptions is said to be of Hammerstein type.

Hammerstein's treatment of these equations employed the Hilbert-Schmidt theory of the expansion of the kernel in a Fourier series of its eigenfunctions corresponding to its eigenvalues. Let the kernel K be defined on $L_2[0, 1] \times L_2[0, 1]$. Suppose there exists a function $y \in L_2[0, 1]$ such that

$$y(t) = \int_0^1 K(t, \tau) g(\tau) d\tau \quad (2.1)$$

where $g(\tau) = -F(y(\tau), \tau)$ and $g(\tau) \in L_2[0, 1]$. If such a function $y(t)$ exists then according to the Hilbert-Schmidt theory y can be expressed by a convergent series of the orthonormalized

eigenfunctions of the kernel K corresponding to its eigenvalues:

$$y(t) = \sum_{m=1}^{\infty} c_m \phi_m(t) \quad (2.2)$$

where $\{\phi_i\}_{i=1}^{\infty}$ are the eigenfunctions corresponding to $\{\lambda_i\}_{i=1}^{\infty}$ its eigenvalues. Explicit expressions for the $\{c_i\}_{i=1}^{\infty}$ are obtained by taking the inner product of y with each of the eigenfunctions:

$$c_m = \int_0^1 y(t) \phi_m(t) dt = - \frac{1}{\lambda_m} \int_0^1 F(\tau, y(\tau)) \phi_m(\tau) d\tau. \quad (2.3)$$

Substituting the series (2.2) for $y(\tau)$ in (2.3), we obtain:

$$c_m = - \frac{1}{\lambda_m} \int_0^1 F(\tau, \sum_{i=1}^{\infty} c_i \phi_i(\tau)) \phi_m(\tau) d\tau; \quad m=1, 2, \dots \quad (2.4)$$

In other words the problem of solving a non-linear integral equation has been reduced to a problem of solving an infinite system of non-linear equations with infinite number of unknowns.

It was quite natural therefore to approximate the solution y by a truncated sum;

$$y_n(t) = \sum_{m=1}^n C_{n,m} \phi_m(t) \quad (2.5)$$

where the constants $C_{n,1}, \dots, C_{n,n}$ satisfy the system of n equations with n unknowns

$$C_{n,m} + \frac{1}{\lambda_m} \int_0^1 F(\tau, \sum_{i=1}^n C_{n,i} \phi_i(\tau)) \phi_m(\tau) d\tau = 0. \quad (2.6)$$

Hammerstein showed that for every n , system (2.6) has a solution provided he assumed that $|F(x,u)|$ was majorized by a linear function of $|u|$:

$$|F(x,u)| \leq C_1 |u| + C_2 \quad (2.7)$$

where C_1 and C_2 are positive constants and C_1 is less than the first eigenvalue λ_1 of the kernel K .

This was done by considering a suitable minimum problem for a continuous function of n variables whose partial derivatives at the minimum point of the function, provided that it attained it, coincided with system (2.6). The function used was the following (continuous) function

$$H(x_1, \dots, x_n) = \sum_{m=1}^n \lambda_m x_m^2 + 2 \int_0^1 f(\tau, \sum_{i=1}^n x_i \phi_i(\tau)) d\tau \quad (2.8)$$

where

$$f(\tau, u) = \int_0^u F(\tau, v) dv. \quad (2.9)$$

Indeed, if $H(x_1, \dots, x_n)$ has a minimum which it reaches then its partial derivatives at that point satisfy the relationship:

$$\frac{1}{2\lambda_m} \frac{\partial H}{\partial x_m} = x_m + \frac{1}{\lambda_m} \int_0^1 F(\tau, \sum_{i=1}^n x_i \phi_i(\tau)) d\tau = 0; \quad m=1, \dots, n \quad (2.10)$$

and hence satisfy system (2.6).

From inequality (2.7) and from (2.9) follows: $|f(x, u)| \leq \frac{1}{2} C_1 u^2 + C_2 |u|$. In addition using the fundamental inequality:

$ax - bx^2 \leq \frac{a^2}{4b}$ for $b > 0$, Hammerstein was able to show that $H(x_1, \dots, x_n)$ is bounded below and reaches its lower bound for every n .

Consequently, for every n there is at least one set of values $(x_1^0, \dots, x_n^0)$ for which H becomes an absolute minimum, and at that point

$$\frac{\partial H}{\partial x_1} = \frac{\partial H}{\partial x_2} = \dots = \frac{\partial H}{\partial x_n} = 0$$

so that by letting $C_{n,m} = x_m^0$ for $m=1, \dots, n$, system (2.6) is

satisfied. Having substituted the values $x_m^{(0)}$ for $C_{n,m}$ in (2.6), Hammerstein showed that if $\{y_n\}$, the sequence of the truncated sums, converges to a limit function $\bar{y}(t)$ then $\bar{y}(t)$ is the solution to the non-linear integral equation. This he did by showing that the sum $S_n = \sum_{m=1}^n \lambda_m C_{n,m}^2$ has an upper bound independent of n , from which he deduced that $\int_0^1 y_n^2(t) dt$ is bounded for every n , and hence he was able to show using the Schwartz inequalities for infinite sum and for integrals, that the sum

$$\chi_n(t) = y_n(t) + \int_0^1 K(t, \tau) F(\tau, y_n(\tau)) d\tau$$

approaches zero as n tends to infinity.

Thus Hammerstein proved that if the sequence of the partial sums $\{y_n\}_{n=1}^{\infty}$ converges then the limit function is the solution

to the non-linear integral equation. Under the assumptions made so far it is not possible in general to show that $\{y_n\}_{n=1}^{\infty}$

does converge, however Hammerstein showed that a subsequence

$\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ may be extracted which converges

to a continuous limit function $\bar{y}(t)$. This he did by applying the Arzela-Ascoli theorem to the sequence

$$W_n(t) = \chi_n(t) - y_n(t) = \int_0^1 K(t, \tau) F(\tau, y_n(\tau)) d\tau; \quad n=1, 2, \dots$$

This sequence is uniformly bounded and equicontinuous and therefore conditionally compact. Since $\lim_{n \rightarrow \infty} \chi_n(t) = 0$, the passage to the limit of $y_n(t)$ is obvious.

Concerning the uniqueness of the solution, Hammerstein's results may be summed up in the following theorem: The non-linear integral equation (1.1) has at most one solution provided the function $F(x, u)$ satisfies uniformly a Lipschitz condition of the type:

$$|F(x, u_1) - F(x, u_2)| \leq \alpha |u_1 - u_2|$$

where the positive constant α is less than the first eigenvalue λ_1 of the kernel K .

A fuller account of Hammerstein's work is given in Tricomi's 'Integral Equations' [35].

3. Generalization of the Hammerstein Equation

A more general formulation of the Hammerstein equation could be done using various methods of representation of mapping as is usually done in functional analysis. It may be pointed out that one may wish to solve Hammerstein's equation in different Banach

spaces: finite dimensional spaces, the space of continuous functions with the maximum norm or L_p spaces. In this thesis Hammerstein's equation will be investigated in $L_2[0,1]$, which is a Hilbert space, with the inner product

$$(f,g) = \int_0^1 f(t)g(t)dt$$

and the norm

$$\|u\| = (u,u)^{\frac{1}{2}} = \left\{ \int_0^1 |u(t)|^2 dt \right\}^{\frac{1}{2}}.$$

The generalization is obtained by noting that the operator involved in equation (1.1) could be separated into two parts, the linear part which is the linear integral operator in the equation and the non-linear part represented by the operator $F(t,u(t))$. Let B represent the linear part and F represent the non-linear part, then Hammerstein's equation can be written in the form

$$x = BFx \tag{3.1}$$

where $x \in L_2[0,1]$. That is the operator BF must map the space $L_2[0,1]$ into (or onto) itself. However since BF is a composite operator it will be assumed that F maps the space $L_2[0,1]$ into itself and so does B .

The non-variational approach to the problem of existence of solution for equation (3.1) has been through methods such as fixed point

theorems, resolvents of non-linear operators and so on. Generally it has been found necessary to assume complete continuity of the operator BF in order to establish the existence of a solution. A general investigation of complete continuity in different spaces is given in Krasnoselskii [2]. Krasnoselskii states that in order that Hammerstein operator BF be completely continuous in $L_2[0,1]$ it is sufficient that one of the operators be completely continuous while the other be continuous and bounded. It is generally easier to treat the case of a completely continuous linear operator and a bounded and continuous non-linear operator. It is a sufficient condition for boundedness and continuity of the non-linear operator that it satisfies a Lipschitz condition. The following theorem is due to Nemytskii and Vainberg:

Theorem 3.1: Let G be a domain of finite measure and let

$$\iint_G K^2(s,t) ds dt < \infty$$

Let the function $F(t,u)$, where $t \in G$ and $-\infty < u < \infty$ be continuous in u and measurable in t , while

$$|F(t,u)| \leq g(t) + a|u|$$

where $g(t) \in L_2[0,1]$. Then the Hammerstein operator acts in $L_2[0,1]$ and is completely continuous.

The conditions assumed in this theorem are much the same as the conditions laid down by Hammerstein, and they are present in the variational approach of M. Vainberg which will be presented in the next section.

4. Vainberg's Variational Approach, [36]

The introduction of variational methods provided a very powerful technique for establishing existence of solution to various operator equations. These methods in fact proved valuable in two ways, firstly by providing a theoretical tool, as mentioned above, for proving existence theorems and secondly they suggest numerical algorithms for obtaining an extremal point for a functional which is directly related to the solution of an operator equation such as the Hammerstein one. In this thesis an attempt will be made to construct such algorithms that allow us to approximate the solution via an approximation to an extremal point of a functional whose gradient is closely related to our equation.

Vainberg's theory relies on two main results from the theory of functionals in Hilbert spaces*. The first one is that every weakly lower semicontinuous functional defined on a set $D(\|x\| \leq r)$ that is weakly closed in a weakly compact space has a lower bound which it attains on the set. The second result is that a necessary condition for an inner point in a set

*These results appear in more detail in the supplement.

$D(\|x\| \leq r)$, at which the functional has a linear Gateaux differential, to be an extremal point is that it be a critical point. To prove that an extremal point is in fact an inner point of some set some sort of a growth property is assumed for the functional. This assumption restricts the class of Hammerstein equations treated by Vainberg and also those treated in this thesis.

Vainberg proves the following theorem:

Theorem 4.1 : Let B be a positive self-adjoint operator defined on $L_2[0,1]$ and let $F = \text{grad } f$, where f satisfies the condition

$$2f(x) \leq a_1(x,x) + a_2(x,x)^\gamma + a_3 \quad (4.1)$$

where $a_2, a_3 > 0$; $0 < \gamma < 1$ and $a_1 \|B\| < 1$ if $a_1 > 0$.

Suppose further, that at least one of the following conditions is satisfied:

1) f is a continuous functional, and B is completely continuous.

2) f is weakly continuous (or weakly upper semi-continuous).

Then the operator equation $x = Bfx$ has at least one solution.

Remark: Vainberg actually proves this theorem for any separable Hilbert space H ; in the above theorem H was taken to be $L_2[0,1]$.

Proof: Vainberg considers the following functional;

$$\varphi(x) = (x, x) - 2f(Ax)$$

where A is the positive square root of B (according to [3] p.422 if $B \in L(\bar{X}, \bar{X})$ is positive then there exists a unique positive operator A such that $A^2 = B$). If condition (1) is fulfilled, then the completely continuous operator A carries every weakly convergent sequence into a strongly convergent sequence and hence $f(Ax)$ is weakly continuous. If condition (2) is satisfied, then the bounded self-adjoint operator A carries every weakly convergent sequence into a weakly convergent sequence and since f is weakly upper semi-continuous, $f(Ax)$ is also weakly upper semi-continuous. Consequently, as the sum of the weakly lower semi-continuous functional (x, x) and the weakly lower semi-continuous functional $-2f(Ax)$, the functional φ is weakly lower semi-continuous. From inequality (4.1) satisfied by f we have if $a_1 > 0$:

$$\begin{aligned} \varphi(x) &= (x, x) - 2f(Ax) \geq (x, x) - a_1(Ax, Ax) - a_2(Ax, Ax)^\gamma \\ &\quad - a_3 \geq (x, x) - a_1 \|B\| (x, x) - a_2 \|B\|^\gamma (x, x)^\gamma - a_3 \\ &= (x, x)^\gamma [(1 - a_1 \|B\|) (x, x)^{1-\gamma} - a_2 \|B\|^\gamma] - a_3 \end{aligned}$$

while if $a_1 \leq 0$ then $1 - a_1 \|B\| > 1$ so that

$$\varphi(x) \geq (x,x)^\gamma [(x,x)^{1-\gamma} - a_2 \|B\|^\gamma] - a_3$$

hence φ is unbounded and increases with $\|x\|$. Consequently, there exists a sphere $\|x\| = r$ on which $\varphi(x) > \varphi(\theta)$ and so has a t-property*, φ is a weakly differentiable functional, and hence according to theorem [88] there exists a point $x_0 \in L_2[0,1]$ such that

$$\nabla \varphi(x_0) = \theta$$

and x_0 is an inner point of the sphere of radius r .

But

$$\nabla \varphi(x_0) = 2x_0 - 2AFx_0 = \theta.$$

Applying the operator A to both sides we get

$$Ax_0 - A^2 Fx_0 = \theta.$$

Let $Z_0 = Ax_0$ then recalling that $A^2 = B$ we finally get that

$$Z_0 - BFZ_0 = \theta,$$

*see supplement, p. 115.

hence z_0 solves the Hammerstein equation.

For uniqueness we need only demand the operator F to satisfy some type of Lipschitz condition:

$$\|F(x_2) - F(x_1)\| \leq a_0 \|x_2 - x_1\|$$

where $a_0 \|B\| < 1$. Since if z_0 and y_0 are both solutions to the equations then

$$\|z_0 - y_0\| = \|Bz_0 - By_0\| \leq \|B\| \|Fz_0 - Fy_0\|$$

$$\leq a_0 \|B\| \|z_0 - y_0\| < \|z_0 - y_0\|$$

and this is a contradiction, hence $z_0 = y_0$ or z_0 is the unique solution of $x = BFx$. Throughout this thesis we shall assume uniqueness of the solution through the assumption that the operator F fulfills a Lipschitz condition of the type mentioned above.

Vainberg gives a few equivalent or somewhat different conditions for the existence of a solution, ^{use of} one of which will be made in chapter 3 and will be presented here, without the whole proof:

Theorem 4.2 Let B be a positive self adjoint operator defined on all of $L_2[0,1]$ and let the potential operator F satisfy the following condition:

$$(DF(x,h),h) \leq a_1(h,h) \quad \forall h \in L_2[0,1]$$

where $a_1 \|B\| < 1$ if $a_1 > 0$. Then the Hammerstein equation $x = B F x$ has at least one solution.

For the proof itself Vainberg constructs the same functional as in the proof of theorem 4.1: $\varphi(x) = (x,x) - 2f(Ax)$, then using the definitions for the first and second Gateaux differentials of the functional φ and also inequality (4.2) we obtain that

$$D^2 \varphi(x,h,h) \geq (1 - a_1 \|B\|)(h,h) \quad \text{if } a_1 > 0$$

and

$$D^2 \varphi(x,h,h) \geq (h,h) \quad \text{if } a_1 \leq 0.$$

According to theorem 5.9 there exists an extremal point x_0 for φ and since φ is weakly differentiable functional the rest of the proof follows exactly like the proof for theorem 4.1.

5. The problem of the square root.

Vainberg's variational approach to the problem of existence and uniqueness of a solution to the Hammerstein equation does not furnish us directly with a constructive method for finding the true solution itself. However from the numerical analysis point of view one is not so much concerned with finding the true solutions but approximating them. If methods such as Ritz-Galerkin were used to approximate the minimizing function and thus reducing the problem of approximating the minimizing function to a problem of solving a non-linear set of equations, the first problem one would have to face in using Vainberg's functional would be that the square root of the original linear operator in the Hammerstein equation appears in the argument of the functional.

In order to understand the problem of the square root involved, a summary of the proof by Bachman and Narici [3] of its existence will be given. The theorem, quoted in the last section states that if an operator $B \in L(\bar{X}, \bar{X})$ is positive then there exists a unique operator $A \in L(\bar{X}, \bar{X})$ such that $A^2 = B$.

Definition 5.1: Let P and Q be two positive operators defined on a Hilbert space H . We say that $Q \gg P$ (in the operator sense) if

$$(Pu, u) \leq (Qu, u) \quad \forall u \in H.$$

For the proof itself Bachman and Narici assume that $B \leq I$, however if not then define the following operators: $C = B/\|B\|$ and then $A = \|B\|^{\frac{1}{2}} C^{\frac{1}{2}}$. Consider the sequence of operators defined as follows:

$$A_0 = 0$$

$$A_{n+1} = A_n + \frac{1}{2}(B - A_n^2); n=0,1,\dots$$

The sequence $\{A_n\}_{n=0}^{\infty}$ can be shown to be a monotonically

increasing sequence, i.e. $A_n \leq A_{n+1}$ for all n , which converges to an operator A such that $A^2 = B$.

From the above, a simple analytical representation of the square root of an operator for numerical use is not available (there are different analytical representations for the square root such as in Kato^{*} p.28), where the square root of a coercive positive operator is expressed as an infinite integral again not numerically feasible). One way to overcome this difficulty would be to replace the operator A in the functional by its approximation, and this approach will be tested below.

Define the following Hammerstein equations:

$$(1) \quad u = BFu = A^2Fu$$

$$(2) \quad v_{n+1} = B_{n+1}Fv_{n+1} = (A_{n+1})^2Fv_{n+1}; \quad n=0,1,\dots$$

Equation (1) is the original Hammerstein equation and equations (2) are Hammerstein equations based on the positive operator obtained by squaring the $(n+1)^{\text{th}}$ approximation to the square root of the operator B . We shall show that these equations are closely related and in fact the solutions of (2) converge to the solution of (1) in the L_2 -norm.

First it will be shown that without changing the assumptions on the functional f and the operator B in theorem 4.1 and under the same conditions of uniqueness of the solution to equation (1) we can prove existence and uniqueness of a solution to equation (2) for every n . For this the following two lemmas are required:

Lemma 5.1: If B is a positive self-adjoint operator with A as its square root then

$$\|A\|^2 = \|B\|.$$

Proof: If B is self-adjoint so is its square root, hence $A=A^*$.

Now

$$\begin{aligned}
 \|A\|^2 &= \left(\sup_{\|x\|=1} \|Ax\| \right)^2 = \sup_{\|x\|=1} \|Ax\|^2 \\
 &= \sup_{\|x\|=1} (Ax, Ax) = \sup_{\|x\|=1} (AAx, x) \\
 &\leq \sup_{\|x\|=1} \|AAx\| \|x\| = \|AA\| = \|B\|.
 \end{aligned}$$

On the other hand

$$\|B\| = \|AA\| \leq \|A\| \|A\| = \|A\|^2$$

from which ^{it} follows that $\|A\|^2 = \|B\|$.

Lemma 5.2: Let Q and P be two bounded and self-adjoint operators defined on a Hilbert space H such that $P \leq Q$. Then $\|P\| \leq \|Q\|$.

Proof: From the theory of linear self-adjoint operators we have:

$$\|Q\| = \inf \left\{ M \geq 0 \mid (Qu, u) \leq M(u, u) \quad \forall u \in H \right\}$$

and similarly

$$\|P\| = \inf \left\{ M' \geq 0 \mid (Pu, u) \leq M'(u, u) \quad \forall u \in H \right\}.$$

From the assumption of the lemma $P \leq Q$, which by definition 5.1

implies that

$$(Pu, u) \leq (Qu, u) \quad \forall u \in H$$

but

$$(Qu, u) \leq \|Q\| (u, u) \quad \forall u \in H$$

and therefore

$$(Pu, u) \leq \|Q\| (u, u) \quad \forall u \in H,$$

so that

$$\|Q\| \in \{M' \geq 0\} \quad \text{such that}$$

$$(Pu, u) \leq M' (u, u) \quad \forall u \in H.$$

The norm of P was the infimum of this set, hence $\|P\| \leq \|Q\|$.

From the last two lemmas and from the monotonicity of the sequence $\{A_n\}$ and its convergence to the square root operator A follows that:

$$\|A_n\| \leq \|A_{n+1}\| \leq \dots \leq \|A\|,$$

Now $\|A\|^2 = \|B\|$ and $\|A_{n+1}\|^2 = \|B_{n+1}\|$ so consequently

$$\|B_{n+1}\| \leq \|B\| \quad \forall n=0, 1, 2, \dots$$

Theorem 5.1: Let the condition of theorem 4.1 hold true for the function f such that $\nabla f = F$ and for the operator B. Then

for every n a solution for equation (2) exists. Moreover, assuming a Lipschitz condition on the operator F sufficient for the solution of equation (1) to be unique, implies the uniqueness of the solution of (2) for every n .

Proof: For the existence it is sufficient to show that for every n theorem 4.1 is applicable to the Hammerstein equation

$$v_{n+1} = B_{n+1} Fv_{n+1}.$$

Now if $a_1 > 0$ is such that $a_1 \|B\| < 1$ then from $\|B_{n+1}\| \leq \|B\|$ for all n follows $a_1 \|B_{n+1}\| < 1$ for all n , so that a_1, a_2, a_3 and γ remain valid for this case. If B is completely continuous then from the theorem by Bachman and Narici we may deduce the complete continuity of B_{n+1} for every n while if B is positive then B_{n+1} is positive for every n . Hence either condition one or condition two of theorem 4.1 for the equation $v_{n+1} = B_{n+1} Fv_{n+1}$ is satisfied for every n . Thus there exists a solution to the equation (2) for every n .

For uniqueness if $\|Fx_1 - Fx_2\| \leq a_0 \|x_1 - x_2\|$ where $a_0 \|B\| < 1$ then by the same argument as above $a_0 \|B_{n+1}\| < 1$ for all n , and hence if the solution for (1) is unique so is the solution of (2) for every n .

With the existence and uniqueness theorems on the solutions of (1) and (2) in mind we may proceed to prove the convergence of the solutions of (2) to the solution of (1).

Theorem 5.2: Let u^0 be the unique solution of (1) and for every n let u_{n+1}^0 be the unique solution of (2) (under the appropriate conditions for existence and uniqueness). Then $\{u_{n+1}^0\}$ strongly converge to u^0 in the norm of $L_2[0,1]$.

Proof: If u^0 is the solution of (1) and u_{n+1}^0 is the solution of (2) then

$$\|u^0 - u_{n+1}^0\| = \|BFu^0 - B_{n+1}Fu_{n+1}^0\| = \|A^2Fu^0 - A_{n+1}^2Fu_{n+1}^0\|$$

$$= \|(A^2Fu^0 - A_{n+1}^2Fu^0) + (A_{n+1}^2Fu^0 - A_{n+1}^2Fu_{n+1}^0)\|$$

$$\leq \|A^2 - A_{n+1}^2\| \|Fu^0\| + \|A_{n+1}^2\| \|Fu^0 - Fu_{n+1}^0\|$$

$$\leq \|A + A_{n+1}\| \|A - A_{n+1}\| \|Fu^0\| + \|A^2\| a_0 \|u^0 - u_{n+1}^0\|$$

Denote by $\delta = \|A^2\| a_0 = \|B\| a_0 < 1$, hence $1 - \delta > 0$ from which follows

$$(1 - \delta) \|u^0 - u_{n+1}^0\| \leq (\|A\| + \|A_{n+1}\|)(\|A - A_{n+1}\|) \|Fu_0\|$$

but $\|A_{n+1}\| \leq \|A\| \quad \forall n \geq 0$ so we further obtain

$$\leq 2 \|A\| \|A - A_{n+1}\| \|Fu^0\|.$$

Finally

$$\|u^0 - u_{n+1}^0\| \leq \frac{2 \|A\| \|Fu^0\|}{1 - \delta} \|A - A_{n+1}\|.$$

Now as $\|Fu^0\|$ is bounded and $A - A_{n+1} \rightarrow 0$ as $n \rightarrow \infty$ it follows that

$\|u^0 - u_{n+1}^0\| \rightarrow 0$ as $n \rightarrow \infty$, hence

$$\lim_{n \rightarrow \infty} u_{n+1}^0 = u^0.$$

Q.E.D.

The last theorem validates the use of the $(n+1)$ -th approximation to the square root to approximate the solution. In practice one substitutes the original operator B in the recursive formula which is actually a polynomial in that operator and then uses the polynomial as an argument in the functional. This method, although rather laborious, yields good results once a numerical algorithm is set up to reduce the problem of finding the minimizing function to a problem of solving a non-linear system of equation or to a 'search minimum' problem. In fact such algorithms are suggested in chapter 3. However it is possible at this stage to estimate the difference in norm between A and A_{n+1} . This difference as we have seen appears in the proof of convergence of the solutions of equations (2) to the solution of (1). For that matter let us observe the recursive relationship of the sequence $\{A_n\}_{n=0}^{\infty}$ again:

$$A_0 = 0$$

$$A_{n+1} = A_n + \frac{1}{2} (B - A_n^2); \quad n=0,1,\dots \quad (5.1)$$

Subtracting both sides of (5.1) from the operator A

$$\begin{aligned} A - A_{n+1} &= A - A_n - \frac{1}{2} (B - A_n^2) = (A - A_n) - \frac{1}{2} (A^2 - A_n^2) \\ &= (A - A_n) \left[I - \frac{1}{2} (A + A_n) \right] \end{aligned} \quad (5.2)$$

As $A+A_n$ is a positive operator, if we replace it in the square brackets by an operator which is smaller in the operator sense, we only get that the right hand side of (5.2) increases in the operator sense. $A \geq A_n$ for all n implies $A+A_n \geq A_n+A_n$, hence

$$I - \frac{1}{2}(A+A_n) \leq I - \frac{1}{2}(A_n+A_n) = I - A_n$$

thus we get

$$A-A_{n+1} \leq (A-A_n)(I-A_n).$$

Applying the same argument to the difference $A-A_n$ as we do to $A-A_{n+1}$ using the recursive relationship for A_n in terms of A_{n-1} we obtain:

$$(A-A_{n+1}) \leq (A-A_n)(I-A_n)$$

$$\leq (A-A_{n-1})(I-A_{n-1})(I-A_n)$$

but $I-A_{n-1} \geq I-A_n$, hence

$$A-A_{n+1} \leq (A-A_{n-1})(I-A_{n-1})^2.$$

Continuing this process for $n-1, n-2, \dots, 1$ we get

$$A - A_{n+1} \leq (A - A_1) (I - A_1)^n$$

but according to the definition of the sequence $\left\{ A_n \right\}_{n=0}^{\infty}$,

follows that $A_1 = \frac{1}{2}B$, from which

$$A - A_{n+1} \leq (A - \frac{1}{2}B) (I - \frac{1}{2}B)^n .$$

By lemma 5.2 this inequality can be written in terms of the norm:

$$\| A - A_{n+1} \| \leq \| (A - \frac{1}{2}B) (I - \frac{1}{2}B)^n \|$$

$$\leq \| A - \frac{1}{2}B \| \| I - \frac{1}{2}B \|^n .$$

We assumed $B \leq I$ so that $I - \frac{1}{2}B \leq I$, hence $\| I - \frac{1}{2}B \| \leq 1$, in most cases it is a fraction, so that the error behaves like a fraction to the $(n)^{\text{th}}$ power or actually to the $(n+1)^{\text{th}}$ since $A - \frac{1}{2}B \leq I - \frac{1}{2}B$ and so

$$\begin{aligned} \|A - A_{n+1}\| &\leq \|A - \frac{1}{2}B\| \left\| I - \frac{1}{2}B \right\|^n \leq \|I - \frac{1}{2}B\| \left\| I - \frac{1}{2}B \right\|^n \\ &= \left\| I - \frac{1}{2}B \right\|^{n+1} . \end{aligned}$$

To recapitulate, in order to approximate the solution of the Hammerstein equation $x = Bx$ we solve a 'neighbouring' equation $V_{n+1} = B_{n+1} F V_{n+1}$. This method, especially the error analysis involved in it reminds us very much of the 'backwards error analysis' used many times (in different contexts) in numerical analysis where we look at the approximation obtained as an exact solution to some neighbouring problem and carry out the error analysis from that point.

Chapter 2: The Variational approach to the Hammerstein equation:

A perturbation technique.

The practical problem that arises from the presence of the square root of the linear operator B in the argument of the functional $\varphi(x) = (x, x) - 2f(Ax)$ leads us to attempt to construct a different functional which make use of the original linear operator B of the Hammerstein equation or some operator close in form to it.

Consider the following functional

$$\tilde{\varphi}(x) = (Bx, x) - 2f(Bx) ; x \in L_2[0,1] \quad (1)$$

where $F = \nabla f$ as in theorem 4.1 of chapter 1. If this functional has a critical point x_0 then its gradient at that point equals to the zero element of the space, θ , i.e. $\exists x_0 \in L_2[0,1]$ such that

$$\nabla \tilde{\varphi}(x_0) = 2Bx_0 - 2BF Bx_0 = \theta.$$

Let $z_0 = Bx_0$, then

$$z_0 - BFz_0 = \theta,$$

in other words z_0 is the solution to the Hammerstein equation.

Obviously the form of the functional $\tilde{\varphi}$ is easier to work with numerically than it is with the functional φ or φ_n suggested in the last chapter.

The proposed functional would therefore be satisfactory if it could be shown to possess a critical point. In chapter 1 we saw that if the functional has a suitable growth property, then there exists a sphere $D(\|x\| \leq r)$ such that the functional has a minimum at some inner point and due to its weak differentiability this point is a critical point. Assuming that f satisfies inequality (4.1) of chapter 1, we have if $a_1 > 0$:

$$\tilde{\varphi}(x) = (Bx, x) - 2f(Bx) \geq (Bx, x) - a_1(Bx, Bx)$$

$$- a_2(Bx, Bx)^\gamma - a_3 \geq (Bx, x) - a_1\|B\|(Bx, x)$$

$$- a_2\|B\|^\gamma (Bx, x)^\gamma - a_3 = (Bx, x)^\delta [(1 - a_1\|B\|)(Bx, x)^{1-\gamma}$$

$$- a_2\|B\|^\delta] - a_3,$$

while if $a_1 \leq 0$

$$\tilde{\varphi}(x) \geq (Bx, x)^\gamma [(Bx, x)^{1-\gamma} - a_2 \|B\|^\gamma] - a_3.$$

Now in infinite dimensional spaces not every positive definite operator is positively bounded below, hence in general we are not able to show that $\tilde{\varphi}(x)$ is unbounded and increase with the norm of x , i.e. (Bx, x) does not generally tend to infinity as $\|x\|$ tend to infinity. We therefore seek a linear operator $\tilde{B}$, close in form to B , that will ensure the desired growth property. It is quite natural to take for $\tilde{B}$ some sort of a perturbation of B .

Let I be the identity operator in $L_2[0,1]$. For every $\varepsilon > 0$ we have

$$((B + \varepsilon I)x, x) = (Bx, x) + \varepsilon (Ix, x)$$

$$= (Bx, x) + \varepsilon (x, x) \geq \varepsilon (x, x)$$

this follows from the positivity of B . Hence if we denote by $B_\varepsilon = B + \varepsilon I$ then for every $\varepsilon > 0$ B_ε is a positive definite operator bounded from below:

$$(B_\varepsilon x, x) \geq (\varepsilon^{\frac{1}{2}})^2 \|x\|^2$$

consequently we have $(B_\epsilon x, x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$. In addition since B and ϵI are both self-adjoint operators their sum B_ϵ is a self-adjoint operator in $L_2[0,1]$.

Let η be a positive number and let $[0, \eta]$ be an interval from which the perturbation parameter will be chosen. In what follows some restriction will be imposed on the magnitude of η . For every $\epsilon \in [0, \eta]$ consider the following Hammerstein equation

$$v = (B + \epsilon I)Fv = B_\epsilon Fv. \quad (2)$$

For $\epsilon = 0$ this equation becomes the original Hammerstein equation.

Our first concern is to establish the existence and uniqueness for the solution of the perturbed equation. Vainberg's theorem, theorem 4.1 of chapter 1, allows two possibilities: either the linear operator is completely continuous and the functional f (such that $F = Vf$) is continuous or the linear operator is positive self-adjoint and the functional f is weakly continuous or weakly upper semi-continuous. Now the operator $B_\epsilon = B + \epsilon I$ is not a completely continuous operator. Even if B is completely continuous, B_ϵ as the sum of a completely continuous operator and a non-completely continuous one, is not a completely continuous operator. Thus if we are to use Vainberg's theorem f will have to satisfy

the requirement that it be a weakly continuous functional or weakly upper semi-continuous (in this chapter it will be assumed that f is weakly continuous). If this condition is satisfied then according to Vainberg's theorem a solution will exist if $a_1 \|B_\epsilon\| < 1$ for $a_1 > 0$ and will be unique if $a_0 \|B_\epsilon\| < 1$. Now $a_1 \|B_\epsilon\| = a_1 \|B + \epsilon I\| \leq a_1 \|B\| + a_1 \epsilon \|I\| = a_1 \|B\| + a_1 \epsilon$. As $a_1 \|B\| < 1$ if $a_1 > 0$ there exists a positive number η' such that $a_1 \|B\| + a_1 \eta' < 1$, hence for all $\epsilon \in [0, \eta']$ there exists a solution to the perturbed Hammerstein equation $v = B_\epsilon Fv$. Likewise for uniqueness, as $a_0 \|B\| < 1$, there exists a positive number η'' such that $a_0 \|B\| + a_0 \eta'' < 1$, hence if we let $\eta = \min[\eta', \eta'']$ then for every $\epsilon \in [0, \eta]$ a solution for (2) exists and is unique.

The use of Vainberg's theorem invokes the following functional:

$$\varphi_\epsilon(x) = (x, x) - 2f(A_\epsilon x) \quad (3)$$

where A_ϵ is the square root of the operator B_ϵ . However since for every $\epsilon > 0$ B_ϵ is positively bounded below, we may construct a different functional for the proof of existence of a solution. A functional which will be closer in nature to the functional suggested in the beginning of this chapter: Let η be defined as above, i.e. $\eta = \min[\eta', \eta'']$, for every $\epsilon \in (0, \eta]$

construct the following functional:

$$\tilde{\varphi}_\varepsilon(x) = (B_\varepsilon x, x) - 2f(B_\varepsilon x) ; \quad x \in L_2[0,1]. \quad (4)$$

By arguments similar to those used in the proof of Vainberg's theorem $\tilde{\varphi}_\varepsilon$ is weakly lower semi-continuous and if as in the conditions stated by Vainberg $F = \inf f$ such that f satisfies

$$2f(x) \leq a_1(x, x) + a_2(x, x)^\gamma + a_3$$

where $a_2, a_3 > 0$; $0 < \gamma < 1$ and $a_1 \|B\| < 1$ if $a_1 > 0$ then for every $x \in L_2[0,1]$ we have:

$$\tilde{\varphi}_\varepsilon(x) = (B_\varepsilon x, x) - 2f(B_\varepsilon x) \geq (B_\varepsilon x, x)$$

$$- a_1(B_\varepsilon x, B_\varepsilon x) - a_2(B_\varepsilon x, B_\varepsilon x)^\gamma - a_3 \geq (B_\varepsilon x, x)$$

$$- a_1 \|B_\varepsilon\| (B_\varepsilon x, x) - a_2 \|B_\varepsilon\|^\gamma (B_\varepsilon x, x)^\gamma - a_3$$

$$= (B_\varepsilon x, x)^\gamma [(1 - a_1 \|B_\varepsilon\|) (B_\varepsilon x, x)^{1-\gamma} - a_2 \|B_\varepsilon\|^\gamma] - a_3,$$

while if $a_1 \leq 0$ then

$$\tilde{\varphi}_\varepsilon(x) \geq (B_\varepsilon x, x)^\gamma [(B_\varepsilon x, x)^{1-\gamma} - a_2 \|B_\varepsilon\|^\gamma] - a_3.$$

Hence $\tilde{\varphi}_\varepsilon$ is unbounded and increases with $\|x\|$ for every $\varepsilon \in (0, n]$, from which follows that for every $\varepsilon \in (0, n]$ there exists a sphere $\|x\| = r_\varepsilon$ such that $\tilde{\varphi}_\varepsilon(x) > \tilde{\varphi}_\varepsilon(0)$, i.e. $\tilde{\varphi}_\varepsilon$ has a t-property. According to theorem 88 since $\tilde{\varphi}_\varepsilon$ is weakly differentiable functional, there exists a point $\tilde{x}_\varepsilon^0$ such that

$$\text{grad } \tilde{\varphi}_\varepsilon(\tilde{x}_\varepsilon^0) = 0$$

and $\tilde{x}_\varepsilon^0$ is an inner point in the sphere $\|x\| \leq r_\varepsilon$. Now

$$\text{grad } \tilde{\varphi}_\varepsilon(\tilde{x}_\varepsilon^0) = 2B_\varepsilon \tilde{x}_\varepsilon^0 - 2B_\varepsilon F B_\varepsilon \tilde{x}_\varepsilon^0 = 0,$$

so if we let $\tilde{z}_\varepsilon^0 = B_\varepsilon \tilde{x}_\varepsilon^0$ we get

$$\tilde{z}_\varepsilon^0 - B_\varepsilon F \tilde{z}_\varepsilon^0 = 0,$$

that is $\tilde{z}_\varepsilon^0$ is a solution of the perturbed equation. Similarly, the conditions for uniqueness are the same as in Vainberg's theorem: a Lipschitz condition of the form $\|Fx_1 - Fx_2\| \leq a_0 \|x_1 - x_2\|$ where $a_0 \|B_\varepsilon\| < 1$ which is fulfilled upon assuming that $\varepsilon \in (0, \eta]$.

The fact that we cannot in general use the original operator as in functional (1) to obtain an extremal point that will, by applying the original operator to the extremal point, yield the solution suggests that some difficulty might arise when we attempt to prove convergence of the solutions of the perturbed equations to the solution to the original one. Let z^0 be the solution to the original Hammerstein equation arrived at using Vainberg's functional, and for every $\varepsilon \in (0, \eta]$ let $\tilde{z}_\varepsilon^0$ be the solution to the corresponding Hammerstein equation $v = B_\varepsilon Fv$. Hence

$$\begin{aligned}
 \|z^0 - \tilde{z}_\varepsilon^0\| &= \|BFz^0 - B_\varepsilon F\tilde{z}_\varepsilon^0\| \\
 &= \|BFz^0 - BF\tilde{z}_\varepsilon^0 - \varepsilon F\tilde{z}_\varepsilon^0\| \\
 &\leq \|BFz^0 - BF\tilde{z}_\varepsilon^0\| + \varepsilon \|F\tilde{z}_\varepsilon^0\| \\
 &\leq \|B\| \|Fz^0 - F\tilde{z}_\varepsilon^0\| + \varepsilon \|F\tilde{z}_\varepsilon^0\| \\
 &\leq a_0 \|B\| \|z^0 - \tilde{z}_\varepsilon^0\| + \varepsilon \|F\tilde{z}_\varepsilon^0\|
 \end{aligned} \tag{5}$$

Since we assumed that F satisfies a Lipschitz condition of the form $\|Fx_1 - Fx_2\| \leq a_0 \|x_1 - x_2\|$ where $a_0 \|B\| + a_0 \varepsilon < 1 \quad \forall \varepsilon \in [0, \eta]$ and hence $a_0 \|B\| < 1$, so if $\delta = a_0 \|B\|$ then inequality (5) becomes:

$$(1 - \delta) \|z^0 - \tilde{z}_\varepsilon^0\| \leq \varepsilon \|F\tilde{z}_\varepsilon^0\|,$$

now $1 - \delta > 0$, thus

$$\|z^0 - \tilde{z}_\varepsilon^0\| \leq \varepsilon \cdot \frac{1}{1 - \delta} \|F\tilde{z}_\varepsilon^0\|. \quad (6)$$

If $\|F\tilde{z}_\varepsilon^0\|$ is bounded for all ε then it is evident that as ε tends to zero the solutions of the perturbed equations tend to the solution of the original one in the L_2 -norm. To show that $\|F\tilde{z}_\varepsilon^0\|$ is bounded in ε it is sufficient to show that $\tilde{z}_\varepsilon^0$ is bounded since F is Lipschitzian and therefore a bounded (non-linear) operator. In the general case, that is when B is not a positive bounded below operator, we cannot establish the existence of $\tilde{z}_0^0 = B\tilde{x}_0^0$, since the functional $\tilde{\phi}_0(x)$ which corresponds to $\varepsilon = 0$ may not possess a growth of property of the form we require, and thus the product $\varepsilon \|F\tilde{z}_\varepsilon^0\|$ may not tend to zero as ε tends to zero.

The idea is to look for a different method which will enable us to prove that when ε tends to zero the solutions $\tilde{z}_\varepsilon^0$ will tend to z^0 . The uniqueness of the solution of the perturbed equations for every $\varepsilon > 0$ ensures that the solution resulting from the new functional and from $\tilde{\varphi}_\varepsilon$ will be identical. It must be noted that we want to solve numerically only the perturbed equation and not the original one.

Let $A_\varepsilon = (B + \varepsilon I)^{1/2}$, A_ε is the square root operator of B_ε , and $\underbrace{A_\varepsilon}_{\text{possesses}}$ all the properties of B_ε such as positive bounded below, self adjointness etc. Consider the following functionals

$$\varphi_\varepsilon(x) = (x, x) - 2f(A_\varepsilon x) ; \quad x \in L_2[0, 1]$$

where f satisfies inequality (4.1) of chapter 1 and is weakly continuous. These functionals were already introduced in equation (3). Since these functionals possess a t -property for every ε and since they are weakly differentiable there exists a critical point for every $\tilde{\varphi}_\varepsilon$: x_ε^0 . But $\nabla \varphi_\varepsilon(x_\varepsilon^0) = 2x_\varepsilon^0 - 2A_\varepsilon F A_\varepsilon x_\varepsilon^0 = 0$ so if $z_\varepsilon^0 = A x_\varepsilon^0$ one obtains $z_\varepsilon^0 - B_\varepsilon F z_\varepsilon^0 = 0$. If z^0 is the solution of the original Hammerstein equation then similar to (5) and (6) we get:

$$\|z^0 - z_\varepsilon^0\| = \|BFz^0 - B_\varepsilon Fz_\varepsilon^0\| = \|BFz^0 - BFz_\varepsilon^0 - \varepsilon Fz_\varepsilon^0\|$$

$$\leq \|BFz^0 - BFz_\varepsilon^0\| + \varepsilon \|Fz_\varepsilon^0\|$$

$$\leq \|B\| \|Fz^0 - Fz_\varepsilon^0\| + \varepsilon \|Fz_\varepsilon^0\|$$

Again from F being Lipschitzian

$$\leq a_0 \|B\| \|z^0 - z_\varepsilon^0\| + \varepsilon \|Fz_\varepsilon^0\|$$

Denote by $\delta = a_0 \|B\| < 1$, thus

$$(1 - \delta) \|z^0 - z_\varepsilon^0\| \leq \varepsilon \|Fz_\varepsilon^0\|$$

or

$$\|z^0 - z_\varepsilon^0\| \leq \varepsilon \cdot \frac{1}{1 - \delta} \|Fz_\varepsilon^0\|$$

If $\|Fz_\varepsilon^0\|$ is bounded in ε on the interval $[0, \eta]$ then z^0 is the limit z_ε^0 in the norm of $L_2[0,1]$ when ε tends to zero.

To show the boundedness of $\|Fz_\varepsilon^0\|$ it is sufficient to show that $\|z_\varepsilon^0\|$ is bounded since F is a bounded (non-linear) operator, i.e. it is sufficient that there exists a constant $M > 0$ such that

$$\|z_\varepsilon^0\| \leq M < \infty \quad \forall \varepsilon \in [0, \eta].$$

Now z_ε^0 was obtained as a result of applying the operator A_ε to the critical point of the functional $\varphi_\varepsilon: z_\varepsilon^0 = A_\varepsilon x_\varepsilon^0$, hence

$$\|z_\varepsilon^0\| = \|A_\varepsilon x_\varepsilon^0\| \leq \|A_\varepsilon\| \|x_\varepsilon^0\|$$

from which follows that in order to show the boundedness of $\|z_\varepsilon^0\|$ in ε it is enough to show the boundedness of $\|x_\varepsilon^0\|$ or equivalently show that all critical points of φ_ε , $\varepsilon \in [0, \eta]$, lie in some sphere of finite radius.

For the proof we shall show that for all $\varepsilon \in [0, \eta]$ there exists an increasing function of the norm that is majorized by all φ_ε and that does not depend on ε , this will enable through its growth property to enclose all critical points for φ_ε in one sphere of a finite radius. Suppose that a_1 of theorem 5.1

of chapter 1 is positive. For every $\varepsilon \in [0, \eta]$ we had

$$\begin{aligned} \varphi_\varepsilon(x) &= (x, x) - 2f(A_\varepsilon x) \\ &\geq (x, x)^\gamma [(1 - a_1 \|B_\varepsilon\|)(x, x)^{1-\gamma} - a_2 \|B_\varepsilon\|^\gamma] - a_3. \end{aligned} \quad (7)$$

Thus $\varphi_\varepsilon(x)$ is bounded from below by the right hand side of inequality (7). Let

$$\begin{aligned} \text{lb} \varphi_\varepsilon(x) &\stackrel{\text{def}}{=} (x, x)^\gamma \\ &\cdot [(1 - a_1 \|B_\varepsilon\|)(x, x)^{1-\gamma} - a_2 \|B_\varepsilon\|^\gamma] - a_3 \\ &= (x, x) - a_1 \|B_\varepsilon\| (x, x) - a_2 \|B_\varepsilon\|^\gamma (x, x)^\gamma - a_3; \end{aligned}$$

$$\varepsilon \in [0, \eta] \quad (8)$$

a_1, a_2 are positive and η was such that $a_1 \|B_\varepsilon\| \leq a_1 (\|B\| + \varepsilon \|I\|) \leq a_1 (\|B\| + \eta \|I\|) = a_1 \|B\| + a_1 \eta < 1$ hence we have that

$$\text{lb} \varphi_\varepsilon(x) \geq (x, x) - a_1(\|B\| + \eta)(x, x)$$

$$- a_2(\|B\| + \eta)^\gamma \cdot (x, x)^\gamma - a_3$$

$$= (x, x)^\gamma [(1 - a_1\|B\| - a_1\eta)(x, x)^{1-\gamma}$$

$$- a_2(\|B\| + \eta)^\gamma] - a_3 \stackrel{\text{def}}{=} \text{lb} \hat{\varphi}_\eta(x) \quad \forall \varepsilon \in [0, \eta].$$

since $1 - a_1\|B\| - a_1\eta > 0$, $\text{lb} \hat{\varphi}_\eta(x)$ is an increasing function of $\|x\|$ and is majorized by all functionals φ_ε , $\varepsilon \in [0, \eta]$.

Consider the functionals $\varphi_\varepsilon(x)$, $\forall \varepsilon \in [0, \eta]$, φ_ε is a continuous function in ε for every $x \in L_2[0, 1]$:

$$|\varphi_{\varepsilon_1}(x) - \varphi_{\varepsilon_2}(x)|$$

$$= |((x, x) - 2f(A_{\varepsilon_1} x)) - ((x, x) - 2f(A_{\varepsilon_2} x))|$$

$$= |2f(A_{\varepsilon_2} x) - 2f(A_{\varepsilon_1} x)| \quad (9)$$

The continuity as a function of ε follows from the assumption

that f is weakly continuous functional or from lemma 5.1 in [36] p. 60. The lemma says that if the gradient of f is bounded on some convex set in H , then f satisfies a Lipschitz condition of the form $|f(x_2) - f(x_1)| \leq M \|x_1 - x_2\|$. The set $\{A_\epsilon x\}$ where x is some function in H and ϵ varies between zero and η can always be included in a closed sphere of a finite radius since $\|A_\epsilon x\| \leq (\|B\| + \eta)^{1/2} \|x\|$ and this sphere is convex. On $F = \text{grad } f$ we assumed a Lipschitz condition and therefore in particular it is bounded on any closed and convex set. Thus the conditions of the lemma are satisfied and there exists a constant M such that

$$|f(x_1) - f(x_2)| \leq M \|x_1 - x_2\|$$

Applying it to (9) we get

$$|f(A_{\epsilon_1} x) - f(A_{\epsilon_2} x)|$$

$$\leq M \|A_{\epsilon_1} x - A_{\epsilon_2} x\|$$

$$\leq M \|x\| \|A_{\epsilon_1} - A_{\epsilon_2}\|$$

From the definition of A_ε for all ε it follows that

$\|A_{\varepsilon_1} - A_{\varepsilon_2}\| \rightarrow 0$ when $\varepsilon_1 \rightarrow \varepsilon_2$, so that φ_ε is a continuous function of ε . In particular φ_ε is a continuous function in ε for $u=0$, hence on a closed interval $\varphi_\varepsilon(0)$ is bounded and reaches its bounds: $\exists \varepsilon_m \in [0, \eta]$ such that

$$\varphi_{\varepsilon_m}(0) = \max_{\varepsilon \in [0, \eta]} \varphi_\varepsilon(0). \quad (10)$$

Since $\text{lb } \hat{\varphi}_\eta(x)$ is increasing with respect to the norm of x , there exists a sphere of radius R ; $0 < R < \infty$ such that

$$\text{lb } \hat{\varphi}_\eta(x) > \varphi_{\varepsilon_m}(0), \quad \forall \|x\| = R.$$

But from (7) and the above inequality

$$\varphi_\varepsilon(x) \geq \text{lb } \hat{\varphi}_\eta(x) > \varphi_{\varepsilon_m}(0) \quad \forall \varepsilon \in [0, \eta] \quad (11)$$

and from (10):

$$\varphi_{\varepsilon_m}(0) \geq \varphi_\varepsilon(0) \quad \forall \varepsilon \in [0, \eta]. \quad (12)$$

Thus from inequality (11) we further have for each $\varepsilon \in [0, n]$:

$$\varphi_\varepsilon(x) > \varphi_\varepsilon(\theta), \quad \forall \|x\| = R$$

which implies, as we have already seen, that the critical point for φ_ε lies inside the sphere of radius R . In this way we have proved that all critical points x_ε^0 lie in a sphere of finite radius R if we assume that $a_1 > 0$. If $a_1 \leq 0$ then we can always find $a_1' > 0$ such that $a_1' \|B\| < 1$ and

$$\Delta f(x) \leq a_1(x, x) + a_2(x, x)^\gamma + a_3$$

$$\leq a_1'(x, x) + a_2(x, x)^\gamma + a_3$$

hence

$$\varphi_\varepsilon(x) = (x, x) - 2f(A_\varepsilon x)$$

$$\geq (x, x)^\gamma [(1 - a_1' \|B_\varepsilon\|)(x, x)^{1-\gamma}$$

$$- a_2 \|B_\varepsilon\|^\gamma] - a_3$$

the rest of the proof is then exactly as for the case $a_1 > 0$.

Hence $\exists M' \geq 0$ such that

$$\|Fz_\varepsilon^0\| \leq M' < \infty \quad \forall \varepsilon \in [0, \eta].$$

Finally we have that

$$\|z^0 - z_\varepsilon^0\| \leq \frac{1}{1-\delta} M \cdot \varepsilon \rightarrow 0 \quad (\varepsilon \rightarrow 0)$$

i.e. the solutions of the perturbed equations $v = B_\varepsilon Fv$ converge to the solution of the original Hammerstein equation $x = BFx$ as the perturbation tends to zero.

Consider now the two kinds of functionals:

$$(1) \quad \tilde{\varphi}_\varepsilon(x) = (B_\varepsilon x, x) - 2f(B_\varepsilon x)$$

$$(2) \quad \varphi_\varepsilon(x) = (x, x) - 2f(A_\varepsilon x)$$

where in both (1) and (2) the conditions imposed on the functional

f and the operator B are the same. If $\tilde{x}_\varepsilon^0$ is the critical point for (1) and x_ε^0 is the critical point for (2), then $B_\varepsilon \tilde{x}_\varepsilon^0$ is the solution to the perturbed Hammerstein equation through functional (1) and $A_\varepsilon x_\varepsilon^0$ is the solution to the perturbed Hammerstein equation through functional (2) for the same value of ε . However the solution is unique hence

$$B_\varepsilon \tilde{x}_\varepsilon^0 = A_\varepsilon x_\varepsilon^0$$

so that for a prescribed perturbation or for a prescribed rate of accuracy i.e. for $\varepsilon > 0$ we may use functional (1) which is numerically easier to deal with than functional (2).

CHAPTER 3: Construction of Two Algorithms for the Solution of the Variational Problem and a Discussion on their Applicability.

§1. Introduction

The results of chapters 1 and 2 provide us with two different functionals, defined on the Hilbert space $L_2[0,1]$, that we wish to minimize or rather approximate their minimizing function by methods based on a finite number of operations, usually by posing the minimum problem in some finite dimensional space.

A large amount of work has been done concerning minimization problems for functionals that arise from non-linear differential boundary value problems. However much less has been done to develop algorithms for the minimization of functionals that arise from non-linear integral equations. There may be cases when we have a variational problem for a boundary value problem that we wish to pose as a variational problem for integral equation. For instance, the use of a set of trial functions for the Ritz-Galerkin method in a variational setting for a boundary value problem requires that they satisfy the boundary conditions. If the boundary value problem is turned into an integral equation, then the Green's function involved in the transformation "incorporates" the boundary conditions. Thus if we were to solve the integral equation through its variational formulation using the Ritz-Galerkin method, the set of trial functions do not necessarily have to satisfy the boundary conditions.

The aim of this chapter will be to construct algorithms in order to approximate the minimizing function for the functionals introduced in earlier chapters. In other words our objective is

to construct algorithms for determining minimizing sequences that will give good approximations to the minimizing function in not too large amount of computations.

An important result of which use is made in this chapter is that of Wang Shang-Wang [37], which is based on results by Vainberg [36] and Kacuvovskii [47]. This result concerns Vainberg's functional (see theorem 4.1 of chapter 1) and hence applies to both functionals suggested: Due to the weak semi-continuity of ψ and its convexity, every minimizing sequence for ψ converges strongly in $L_2[0,1]$ to the minimizing function.

§2. An algorithm based on some results from the "Discretization Theory"

1. Daniel's Results

The main idea in this method is to estimate the minimizing function for a functional ψ by replacing the integral in the functional with a quadrature approximation and obtain an approximation at discrete points by minimizing the quadrature formula with respect to the values of the minimizing function at those points. Once we have the values at discrete points, an interpolation scheme may be applied to obtain an approximation over the whole region.

The idea to approximate the value of the functional by replacing the integral with a quadrature formula dates back to Euler [7]. Aubin and Lions [2] and Daniel [9, 10] have developed a more general theory which relates the variational problem to a discrete approximation to the variational problem. Daniel's main results are summarized below and will be applied to the

functional $\tilde{\psi}_E$ of chapter 2.

Let E be a normed linear space and ψ be a real valued functional on E . Let E be also a reflexive space, $E = (E^*)^*$.

Definition 2.1: A discretization for a functional ψ on E consists of a family of normed linear spaces E_n , a family of real valued functionals ψ_n on E_n , a family of mappings $p_n: E_n \rightarrow E$ and a family of mappings $r_n: E \rightarrow E_n$. We shall denote the discretization by

$$\text{disc}(\psi, E) = \{E, E_n, \psi, \psi_n, p_n, r_n\}.$$

An example for the mappings p_n and r_n could be when p_n is a piecewise linear joining of values in E_n and r_n is such that if $u \in E$ then $(r_n u)_i = u_i$, i.e. r_n gives the values of a function at discrete points.

To ensure that the discretization will in fact give some kind of convergence, Daniel defines a few properties such that if satisfied by the discretization scheme, then we have a convergence of the values at discrete points to the values of the minimizing function at these points.

Definition 2.2: A discretization for ψ on E is said to be consistent if

$$\limsup_{n \rightarrow \infty} \psi_n(r_n \hat{u}) \leq \psi(\hat{u})$$

where $\hat{u}$ minimizes ψ over a subset $B \subset E$ (if we wish to minimize over this subset) or over the whole space, and

$$\limsup_{n \rightarrow \infty} [\psi(p_n \hat{u}_n) - \psi_n(\hat{u}_n)] \leq 0$$

where $\hat{u}_n$ minimizes ψ_n over a set $B_n \subset E_n$ and $p_n \hat{u}_n \in B$.

Now we wish to solve the minimization problem over B (which we denote by MPB) by solving a similar problem for ψ_n over B_n . Let ψ_n be bounded below on a set B_n such that $p_n B_n \subset B$ and $r_n B \subset B_n$. Thus we seek $\hat{u}_n \in B_n$ such that $\psi_n(\hat{u}_n) \leq \psi_n(u_n) + \varepsilon_n$ for all $u_n \in B_n$, $0 < \varepsilon_n \downarrow 0$. In this case the sequence $\{p_n \hat{u}_n\}$ forms a minimizing sequence for ψ . In fact we have the following theorem:

Theorem 2.1: Let ψ be W.L.S. (weakly lower semi-continuous) and bounded below on a weakly compact subset B of E . Let

$$\text{disc}(\psi, E) = \{E, E_n, \psi, \psi_n, r_n, p_n\}$$

be a consistent discretization for ψ on E and also let $B_n \subset E_n$, satisfying $p_n B_n \subset B$ and $r_n B \subset B_n$, then there exist $\hat{u}, \hat{u}_n$ solving MPB and MPB_n respectively. Further

$$\lim_{n \rightarrow \infty} \psi(p_n \hat{u}_n) = \lim_{n \rightarrow \infty} \psi_n(\hat{u}_n) = \psi(\hat{u})$$

and all weak limit points of $p_n \hat{u}_n$, at least one of which exists solve MPB. If the solution of MPB is unique then the solution of MPB_n weakly converges to it.

Details of the proof are to be found in [10]. Theorem 2.1 is a "local" theorem since it deals with the minimization over a subset B of the space E . In order to achieve a global theorem Daniel requires that $\text{disc}(\psi, E)$ satisfy two more conditions:

Definition 2.3: A discretization for ψ on E satisfies a uniform growth condition if:

$$\lim_{n \rightarrow \infty} \sup \psi_n(u_n) = +\infty \text{ as } \lim_{n \rightarrow \infty} \sup \|u_n\|_n = +\infty$$

where $\|u_n\|_n$ is the norm defined on E_n .

Definition 2.4: A discretization for ψ on E is said to be stable if there exists a constant A such that

$$\|p_n u_n\| \leq A \|u_n\|_n$$

for every $u_n \in E_n$ and for every n , and A is independent of u_n and n .

With these additional assumptions, Daniel proves the following theorem:

Theorem 2.2: Let ψ be W.L.S. functional on a reflexive space E , let ψ satisfy a t -property at θ with $t = T_0$. Let $\text{disc}(\psi, E)$ be stable, consistent (for uniformly bounded $\|\hat{u}_n\|_n$ and $\|p_n \hat{u}_n\|$) and satisfy a uniform growth condition. Then

$$\lim_{n \rightarrow \infty} \psi(p_n \hat{u}_n) = \lim_{n \rightarrow \infty} \psi_n(\hat{u}_n) = \psi(\hat{u})$$

where $\hat{u}$ solves MPE, and all weak limit points of $\{p_n \hat{u}_n\}$, at least one of which exists, solve MPE; in particular, if $\hat{u}$ is unique, then $p_n \hat{u}_n$ weakly converges to it.

2. An algorithm based on a discretization scheme

In chapter 2, we introduced a perturbed Hammerstein equation such that its solution could be used to approximate the solution of the original equation. The solution to the perturbed equation is obtained by minimizing the functional $\tilde{\varphi}_\varepsilon$, where ε is the perturbation parameter and belongs to some restricted interval $[0, \eta]$, and then applying the perturbed operator B_ε to the minimizing function of $\tilde{\varphi}_\varepsilon$, which is also the critical point for $\tilde{\varphi}_\varepsilon$ in

$L_2[0,1]$. In this section a discretization scheme for the functional $\mathcal{J}_\varepsilon$ will be suggested, and if $\varepsilon \in [0, \eta/2]$, it will be shown to satisfy the consistency, stability and uniform growth condition and by so doing provide us with a practical algorithm based on this scheme. To make the presentation clear, we shall reconstruct the functional $\mathcal{J}_\varepsilon$ from the original equation.

Suppose we have the following Hammerstein equation:

$$u(x) = \int_0^1 B(x,t)F(t,u(t))dt.$$

If the kernel $B(x,t)$ satisfies the conditions of chapter 1 section 2, i.e. $B(x,t)$ is symmetric, positive and continuous in both arguments (and hence belongs to $\mathcal{L}^2$) and if $F(t,u)$ satisfies:

$$|F(t,u)| \leq c_1|u| + c_2 \quad (2.1)$$

where $0 < c_1 < \lambda_1$, λ_1 being the smallest eigenvalue of B and $c_2 > 0$, then there exists at least one solution to the equation.

Further, if F satisfies a Lipschitz condition of the form

$|F(x,u_1) - F(x,u_2)| < a|u_1 - u_2|$ where a is smaller than λ_1 , then the solution is unique. The assumptions on B and on F imply the satisfaction of the Vainberg conditions for existence and uniqueness of the solution to the Hammerstein equation (see theorem 4.1 of chapter 1) and in particular the satisfaction of the inequality

$$2f(u) \leq c_1(u,u) + c_2'(u,u)^\gamma + c_3'$$

where f is a functional such that $F = \nabla f$; $c_1, c_2', c_3' > 0$;

$c_1 \|B\| < 1$ (since $c_1 < \lambda_1$) and $\gamma = \frac{1}{2}$. Indeed this follows from the inequality $|F(x,u)| \leq c_1|u| + c_2$ since

$$\begin{aligned}
f(u) &= \int_0^1 \int_0^1 \frac{u(x)}{F(x,v)} dv dx \leq \int_0^1 \int_0^1 |F(x,v)| dv dx \\
&\leq \int_0^1 \left(\frac{1}{2} c_1 |u(x)|^2 + c_2 |u(x)| \right) dx \\
&\leq \frac{1}{2} c_1 \int_0^1 |u(x)|^2 dx + c_2 \left\{ \int_0^1 |u(x)|^2 dx \right\}^{1/2} + c_3
\end{aligned}$$

where c_3 is an arbitrary positive constant

$$= \frac{1}{2} c_1 (u, u) + c_2 (u, u)^{1/2} + c_3$$

and hence

$$2f(u) \leq c_1 (u, u) + c_2' (u, u)^{1/2} + c_3' \quad (c_2' = 2c_2, \quad c_3' = 2c_3).$$

Likewise for the conditions of uniqueness.

Thus if $\varepsilon \in [0, \eta]$, where η is such that $c_1 \|B\| + c_1 \eta < 1$ then according to chapter 2, the functional $\tilde{\varphi}_\varepsilon(u) = (B_\varepsilon u, u) - 2f(B_\varepsilon u)$ is W.L.S. and bounded below in $L_2[0, 1]$ and possess a unique extremal and critical point in $L_2[0, 1]$ such that the operation of B_ε on this point is the solution to the perturbed equation $u_\varepsilon = B_\varepsilon F u_\varepsilon$.

For Hammerstein's equation, the explicit expression for $\tilde{\varphi}_\varepsilon(u)$ is the following:

$$\begin{aligned}
\tilde{\varphi}_\varepsilon(u) &= (B_\varepsilon u, u) - 2f(B_\varepsilon u) \\
&= \int_0^1 (B_\varepsilon u)(x) u(x) dx - 2 \int_0^1 \int_0^1 (B_\varepsilon u)(x) F(x, v) dv dx \\
&= \int_0^1 [(B_\varepsilon u)(x) u(x) - 2 \int_0^1 (B_\varepsilon u)(x) F(x, v) dv] dx \\
&= \int_0^1 \Phi(x; u(x); (B_\varepsilon u)(x)) dx
\end{aligned}$$

where

$$\phi(x; u; z) \stackrel{\text{def}}{=} uz - 2 \int_0^z F(x, v) dv.$$

A trapezoidal approximation to $\tilde{\varphi}_\varepsilon(u)$ can be constructed as follows: Let $0 \equiv x_0^n < x_1^n < \dots < x_n^n \equiv 1$. Denote by $u_{n,i} = u(x_i^n)$ $i = 0, 1, \dots, n$ and by $B_{i,j}^n = B(x_i^n, t_j^n)$ $i, j = 0, 1, \dots, n$ where $t_i^n = x_i^n$ $i = 0, \dots, n$ and let $h_i = x_i^n - x_{i-1}^n = \text{const} = \frac{1}{n}$ (in practice refinement of the mesh will be carried out by halving the distance between the mesh points from the previous partition, thus mesh points from the previous partition remain mesh points in the new one). The approximation to $\tilde{\varphi}_\varepsilon(u)$ based on the trapezoidal quadrature formula using the values of the function u at the nodes x_i^n , $0 \leq i \leq n$, is $\tilde{\varphi}_{n,\varepsilon}(\bar{u}_n)$:

$$\begin{aligned} \tilde{\varphi}_{n,\varepsilon}(\bar{u}_n) &= \frac{1}{2} h \phi(x_0^n; u_{n,0}; \bar{T}_0^n) \\ &+ h \sum_{i=1}^{n-1} \phi(x_i^n; u_{n,i}; \bar{T}_i^n) + \frac{1}{2} h \phi(x_n^n; u_{n,n}; \bar{T}_n^n) \end{aligned}$$

where $\bar{u}_n = (u_{n,0}, u_{n,1}, \dots, u_{n,n})$ is a vector formed from the discrete values of u at the points x_i^n , $0 \leq i \leq n$ and $\bar{T}_i^n$ is the trapezoidal approximation to $(B_\varepsilon u)(x_i^n) = (Bu)(x_i^n) + \varepsilon u(x_i^n) = \int_0^1 B(x_i^n, t) u(t) dt + \varepsilon u_{n,i}$ based on the discrete values of u at the points x_i^n , $0 \leq i \leq n$:

$$\begin{aligned} \bar{T}_i^n &\stackrel{\text{def}}{=} \frac{1}{2} h B_{i,0}^n u_{n,0} + h \sum_{j=1}^{n-1} B_{i,j}^n u_{n,j} + \frac{1}{2} h B_{i,n}^n u_{n,n} \\ &+ \varepsilon u_{n,i}. \end{aligned}$$

We shall proceed now to construct a discretization scheme for $\tilde{\varphi}_\varepsilon$ on $L_2[0,1]$, and for simplicity we shall use Σ' to denote the following

$$\sum_{i=0}^n g_i = \frac{1}{2} g_0 + \sum_{i=1}^{n-1} g_i + \frac{1}{2} g_n.$$

Let $E = L_2[0,1] = \{u \mid \int_0^1 |u(x)|^2 dx < \infty\}$ where the norm is defined in the usual way: $\|u\| = \left\{ \int_0^1 |u(x)|^2 dx \right\}^{1/2}$ and for each n let E_n be the $(n+1)$ -dimensional Euclidean space. Let $\|\underline{u}_n\|_n$ be a norm in E_n defined by:

$$\|\underline{u}_n\|_n = \left\{ h \sum_{i=0}^n |u_i^n|^2 \right\}^{1/2}$$

where $\underline{u}_n$ is an element in E_n : $\underline{u}_n = (u_0^n, u_1^n, \dots, u_n^n)$, (this notation is used to distinguish between a general element in E_n to a discretized function from $E = L_2[0,1]$ that is also an element in E_n). $\|\underline{u}_n\|_n$ is indeed a norm since as can be easily verified it satisfies all three conditions required of a norm.

Let $p_{n-1} \underline{u}_n$ denote the piecewise linear function passing through the points (x_i^n, u_i^n) , $0 \leq i \leq n$. Clearly $p_{n-1} \underline{u}_n \in L_2[0,1]$, and denote by r_n the mapping from $E = L_2[0,1]$ to E_n such that $(r_n u)_i = u_{n,i}$ for $0 \leq i \leq n$. Let the functional $\tilde{\varphi}_{n,\varepsilon}$ be defined on E_n in the following way:

$$\tilde{\varphi}_{n,\varepsilon}(\underline{u}_n) \stackrel{\text{def}}{=} h \sum_{i=0}^n \phi(x_i^n; u_i^n; \underline{T}_i^n)$$

where $\underline{T}_i^n \stackrel{\text{def}}{=} h \sum_{j=0}^n B_{i,j}^n u_j^n + \varepsilon u_i^n$.

Thus we have constructed a discretization scheme for $\tilde{\varphi}_\varepsilon$ on $E = L_2[0,1]$:

$$\text{disc}(\tilde{\mathcal{Q}}_\varepsilon, L_2[0,1]) = \{L_2[0,1], E_n, \tilde{\mathcal{Q}}_\varepsilon, \tilde{\mathcal{Q}}_{n,\varepsilon}, p_n, r_n\}$$

which we shall prove to satisfy the stability, consistency and uniform growth conditions.

1. The discretization scheme for $\tilde{\mathcal{Q}}_\varepsilon$ on $L_2[0,1]$ is stable, i.e. there exists a constant A independent of n and $\underline{u}_n$ such that

$$\|p_n \underline{u}_n\| \leq A \|\underline{u}_n\|_n.$$

Proof:

$$\|p_n \underline{u}_n\|^2 = \int_0^1 |(p_n \underline{u}_n)(x)|^2 dx = \sum_{j=1}^n \int_{x_{j-1}^n}^{x_j^n} |(p_n \underline{u}_n)(x)|^2 dx \quad (2.2)$$

On each subinterval $[x_{j-1}^n, x_j^n]$, $j = 1, \dots, n$ the function $(p_n \underline{u}_n)(x)$ is linear and has the form $(1 - \alpha)u_{j-1}^n + \alpha u_j^n$; $0 \leq \alpha \leq 1$. Thus if we perform the following change of integration variables in the (j) -th subinterval: $x \rightarrow x_{j-1}^n + \alpha h$ then $dx = h d\alpha$ and (2.2) becomes

$$\begin{aligned} &= h \sum_{j=1}^n \int_0^1 |(1 - \alpha)u_{j-1}^n + \alpha u_j^n|^2 d\alpha \\ &\leq h \sum_{j=1}^n \int_0^1 [(1 - \alpha)^2 |u_{j-1}^n|^2 + 2\alpha(1 - \alpha) |u_{j-1}^n| |u_j^n| + \alpha |u_j^n|^2] d\alpha \end{aligned}$$

but $\alpha, 1 - \alpha \leq 1$ and hence

$$\leq h \sum_{j=1}^n [|u_{j-1}^n|^2 + 2|u_{j-1}^n| |u_j^n| + |u_j^n|^2].$$

For all j we have:

$$|u_{j-1}^n| |u_j^n| = \sqrt{|u_{j-1}^n|^2 |u_j^n|^2} \leq \frac{|u_{j-1}^n|^2 + |u_j^n|^2}{2}$$

or
$$2|u_{j-1}^n||u_j^n| \leq |u_{j-1}^n|^2 + |u_j^n|^2.$$

Thus we get that

$$\begin{aligned} \|p_n \underline{u}_n\|^2 &\leq 2h \sum_{j=1}^n [|u_{j-1}^n|^2 + |u_j^n|^2] \\ &= 4(\frac{h}{2}|u_0^n|^2 + h \sum_{j=1}^n |u_j^n|^2 + \frac{h}{2}|u_n^n|^2) \\ &= 4\|\underline{u}_n\|_n^2 \end{aligned}$$

and hence

$$\|p_n \underline{u}_n\| \leq 2\|\underline{u}_n\|_n.$$

2. The discretization for $\tilde{\varphi}_\varepsilon$ on $L_2[0,1]$ satisfies a uniform growth condition, i.e.:

$$\limsup_{n \rightarrow \infty} \tilde{\varphi}_{n,\varepsilon}(\underline{u}_n) = +\infty \quad \text{as} \quad \limsup_{n \rightarrow \infty} \|\underline{u}_n\|_n = +\infty.$$

Proof:

$$\tilde{\varphi}_{n,\varepsilon}(\underline{u}_n) = h \sum_{i=0}^n \Phi(x_i^n; u_i^n; T_i^n)$$

writing the explicit expression for Φ

$$\begin{aligned} &= h \sum_{i=0}^n [u_{i-1}^{n,T_i^n} - 2 \int_0^{T_i^n} F(x_1^n, v) dv] \\ &\geq h \sum_{i=0}^n [u_{i-1}^{n,T_i^n} - 2 \int_0^{T_i^n} |F(x_1^n, v)| dv]. \end{aligned}$$

Now $|F(x, u)| \leq c_1|u| + c_2$ (where $0 < c_1 < \lambda_1$ and $c_2 > 0$), hence

$$\int_0^u |F(x,v)| dv \leq \int_0^u [c_1|v| + c_2] dv$$

$$\leq \frac{1}{2} c_1 |u|^2 + c_2 |u|$$

thus we further have that

$$\begin{aligned} \varphi_{n,\varepsilon}(\underline{u}_n) &\geq h \sum_{i=0}^n [u_{i-1}^{n,Tn} - c_1 |T_{i-1}^n|^2 - 2c_2 |T_{i-1}^n|] \\ &= h \sum_{i=0}^n u_{i-1}^{n,Tn} - c_1 h \sum_{i=0}^n |T_{i-1}^n|^2 - 2c_2 h \sum_{i=0}^n |T_{i-1}^n|. \end{aligned} \quad (2.3)$$

Let us write explicitly each of the sums above (it is possible to carry the proof in a complete matrix form, however for the sake of clarity it will not be done here). For the first sum we have:

$$\begin{aligned} h \sum_{i=0}^n u_{i-1}^{n,Tn} &= h \sum_{i=0}^n u_i^n [h \sum_{j=0}^n B_{i,j}^n u_j^n + \varepsilon u_i^n] \\ &= h \sum_{i=0}^n u_i^n [h \sum_{j=0}^n B_{i,j}^n u_j^n] + \varepsilon h \sum_{i=0}^n |u_i^n|^2. \end{aligned}$$

Note that $h \sum_{i=0}^n u_i^n [h \sum_{j=0}^n B_{i,j}^n u_j^n] \geq 0$ since this expression is a quadratic form: $\underline{u}_n^T B^n \underline{u}_n$, where $B^n \stackrel{\text{def}}{=} ((hB_{i,j}^n))$ is a positive symmetric matrix (the notation B^n is to denote that some of its elements have terms equalling $\frac{h}{2} B_{\dots}^n$ and $\frac{h}{4} B_{\dots}^n$ in an ordered fashion, verified upon careful inspection of the quadratic form). For the second sum we have:

$$\begin{aligned} c_1 h \sum_{i=0}^n |T_{i-1}^n|^2 &= c_1 h \sum_{i=0}^n |h \sum_{j=0}^n B_{i,j}^n u_j^n + \varepsilon u_i^n|^2 \\ &\leq c_1 h \sum_{i=0}^n |h \sum_{j=0}^n B_{i,j}^n u_j^n|^2 + 2c_1 \varepsilon h \sum_{i=0}^n u_i^n [h \sum_{j=0}^n B_{i,j}^n u_j^n] \\ &\quad + c_1 \varepsilon^2 h \sum_{i=0}^n |u_i^n|^2. \end{aligned}$$

Now

$$c_1 h \sum_{i=0}^n |h \sum_{j=0}^n B_{i,j}^n u_j^n|^2 = c_1 h \sum_{i=0}^n (h \sum_{j=0}^n B_{i,j}^n u_j^n) \cdot (h \sum_{k=0}^n B_{i,k}^n u_k^n).$$

This expression, using various methods of proof can be shown to be majorized by the following expression

$$\leq c_1 h \sum_{i=0}^n (h \sum_{j=0}^n B_{i,j}^n u_j^n) (\|B'^n\| \sum_{k=0}^n u_k^n).$$

where $\|B'^n\|$ is the normal "Euclidean" norm of the matrix B'^n ,

$$= c_1 \|B'^n\| h \sum_{i=0}^n [h \sum_{j=0}^n B_{i,j}^n u_j^n].$$

As for the third part of (2.3) we have

$$\begin{aligned} & 2c_2 h \sum_{i=0}^n |h \sum_{j=0}^n B_{i,j}^n u_j^n + \epsilon u_i^n| \\ & \leq 2c_2 [\sum_{j=0}^n h |B_{i,j}^n| |h u_j^n| + 2c_2 \epsilon h \sum_{i=0}^n |u_i^n|] \\ & \leq 2c_2 \{ \sum_{i=0}^n |h B_{i,j}^n|^2 \}^{1/2} \{ h \sum_{j=0}^n |u_j^n|^2 \}^{1/2} + 2c_2 \epsilon \{ h \sum_{i=0}^n |u_i^n|^2 \}^{1/2} \\ & \leq 2c_2 \|B'^n\| \|u_n\|_n + 2c_2 \epsilon \|u_n\|_n \end{aligned}$$

the last inequality follows since the norm of any vector in a matrix is always majorized by the norm of the whole matrix.

Hence, combining all the estimates yields that:

$$\begin{aligned} \varphi_{n,\epsilon}(u_n) & \geq [1 - c_1 \|B'^n\| - 2c_1 \epsilon] u_n^T B'^n u_n \\ & + \|u_n\|_n [(1 - c_1 \epsilon) \epsilon \|u_n\|_n - 2c_2 \|B'^n\| - 2c_2 \epsilon]. \quad (2.4) \end{aligned}$$

Now $\|B^n\|^2 = \sum_{i=0}^n \sum_{j=0}^n h^2 B_{i,j}^n$ which is the trapezoidal approximation to $\int_0^1 \int_0^1 B^2(x,t) dx dt = \|B\|^2$. Therefore as n tends to ∞ , $1 - c_1 \|B^n\| - 2c_1$ tends to $1 - c_1 \|B\| - 2c_1 \varepsilon$. For $\varepsilon \in [0, n/2]$, $1 - c_1 \|B\| - 2c_1 \varepsilon \leq 1 - c_1 \|B\| - c_1 \eta < 1$ (see chapter 2), therefore the first term on the right hand side of (2.4) is non negative. For the second term $1 - c_1 \|B\| - c_1 \varepsilon < 1$ for $\varepsilon \in [0, \eta]$ and hence $1 - c_1 \varepsilon < 1$ thus the second term is bounded below and tends to ∞ as $\|u_n\|_n$ tends to ∞ .

It therefore follows that:

$$\limsup_{n \rightarrow \infty} \tilde{\varphi}_{n,\varepsilon}(u_n) = +\infty \quad \text{as} \quad \limsup_{n \rightarrow \infty} \|u_n\|_n = +\infty,$$

which completes the proof for the uniform growth condition of the discretization scheme.

3. $\text{disc}(\tilde{\varphi}_\varepsilon, L_2[0,1]) = \{L_2[0,1], E_n, \tilde{\varphi}_\varepsilon, \tilde{\varphi}_{n,\varepsilon}, p_n, r_n\}$ is a consistent discretization for $\tilde{\varphi}_\varepsilon$ on $L_2[0,1]$.

Proof: For the first part of the consistency condition we have to show that:

$$\limsup_{n \rightarrow \infty} \tilde{\varphi}_{n,\varepsilon}(r_n u_\varepsilon) \leq \tilde{\varphi}_\varepsilon(\hat{u}_\varepsilon)$$

where $\hat{u}_\varepsilon$ minimizes $\tilde{\varphi}_\varepsilon$ over $L_2[0,1]$, (as was pointed before, $\exists u_\varepsilon \in L_2[0,1]$ that minimizes $\tilde{\varphi}_\varepsilon$ over $L_2[0,1]$, and $\hat{u}_\varepsilon$ is also a unique critical point for $\tilde{\varphi}_\varepsilon$). Let us write the expression for $\tilde{\varphi}_{n,\varepsilon}$:

$$\tilde{\varphi}_{n,\varepsilon}(r_n \hat{u}_\varepsilon) = h \sum_{i=0}^n \phi(x_i^n; (\hat{u}_\varepsilon)_{n,i}; \bar{T}_i^n)$$

and define by $\bar{\varphi}_{n,\varepsilon}$ the true (exact) trapezoidal approximation to $\tilde{\varphi}_{\varepsilon}(\hat{u}_{\varepsilon})$:

$$\bar{\varphi}_{n,\varepsilon}(r_n \hat{u}_{\varepsilon}) = h \sum_{i=0}^n \phi(x_i^n; (\hat{u}_{\varepsilon})_{n,i}; (B_{\varepsilon} \hat{u}_{\varepsilon})(x_i^n)).$$

Now

$$\begin{aligned} |\tilde{\varphi}_{n,\varepsilon}(r_u \hat{u}_{\varepsilon}) - \tilde{\varphi}_{\varepsilon}(\hat{u}_{\varepsilon})| &\leq |\tilde{\varphi}_{n,\varepsilon}(r_n \hat{u}_{\varepsilon}) - \bar{\varphi}_{n,\varepsilon}(r_n \hat{u}_{\varepsilon})| \\ &+ |\bar{\varphi}_{n,\varepsilon}(r_n \hat{u}_{\varepsilon}) - \tilde{\varphi}_{\varepsilon}(\hat{u}_{\varepsilon})|. \end{aligned} \quad (2.5)$$

From the definition of the integral and from the existence of the integral $\tilde{\varphi}_{\varepsilon}(\hat{u}_{\varepsilon})$ follows that

$$\lim_{n \rightarrow \infty} |\bar{\varphi}_{n,\varepsilon}(r_n \hat{u}_{\varepsilon}) - \tilde{\varphi}_{\varepsilon}(\hat{u}_{\varepsilon})| = 0.$$

Let us consider the first modulus on the right hand side of inequality (2.5):

$$\begin{aligned} |\tilde{\varphi}_{n,\varepsilon}(r_n \hat{u}_{\varepsilon}) - \bar{\varphi}_{n,\varepsilon}(r_n \hat{u}_{\varepsilon})| &= |h \sum_{i=0}^n \phi(x_i^n; (\hat{u}_{\varepsilon})_{n,i}; \bar{T}_i^n) \\ &- h \sum_{i=0}^n \phi(x_i^n; (\hat{u}_{\varepsilon})_{n,i}; (B_{\varepsilon} \hat{u}_{\varepsilon})(x_i^n))| \leq h \sum_{i=0}^n |\phi(x_i^n; (\hat{u}_{\varepsilon})_{n,i}; \bar{T}_i^n \\ &- \phi(x_i^n; (\hat{u}_{\varepsilon})_{n,i}; (B_{\varepsilon} \hat{u}_{\varepsilon})(x_i^n))|. \end{aligned} \quad (2.6)$$

We shall prove that the expression on the right hand side of this inequality tends to zero as n tends to ∞ . From the definition of $\phi(x; u; z)$ follows that

$$\begin{aligned} \frac{d}{dz} \phi(x; u; z) &= \frac{d}{dz} [uz - 2 \int_0^z F(x, v) dv] \\ &= u - 2F(x, z) \end{aligned}$$

hence if we apply the mean value theorem to the right hand side of inequality (2.6) we get

$$|\tilde{\varphi}_{n,\varepsilon}(x_n \hat{u}_\varepsilon) - \bar{\varphi}_{n,\varepsilon}(x_n \hat{u}_\varepsilon)| \leq h \sum_{i=0}^n |(\hat{u}_\varepsilon)_{n,i} - 2F(x_i^n, (B_\varepsilon \hat{u}_\varepsilon)(x_i^n)) + \tau_i^n ((B_\varepsilon \hat{u}_\varepsilon)(x_i^n) - \bar{T}_i^n)| | (B_\varepsilon \hat{u}_\varepsilon)(x_i^n) - \bar{T}_i^n |$$

where $0 < \tau_i^n < 1$ for $0 \leq i \leq n$. Furthermore, from inequality (2.1), i.e. that $|F(x, u)| \leq c_1 |u| + c_2$ we obtain

$$\begin{aligned} &\leq h \sum_{i=0}^n [|(\hat{u}_\varepsilon)_{n,i}| + 2c_1 | (B_\varepsilon \hat{u}_\varepsilon)(x_i^n) + \tau_i^n ((B_\varepsilon \hat{u}_\varepsilon)(x_i^n) - \bar{T}_i^n) | \\ &+ 2c_2] | (B_\varepsilon \hat{u}_\varepsilon)(x_i^n) - \bar{T}_i^n | \leq h \sum_{i=0}^n [|(\hat{u}_\varepsilon)_{n,i}| + 2c_1 | (B_\varepsilon \hat{u}_\varepsilon)(x_i^n) | \\ &+ 2c_1 \tau_i^n | (B_\varepsilon \hat{u}_\varepsilon)(x_i^n) - \bar{T}_i^n | + 2c_2] | (B_\varepsilon \hat{u}_\varepsilon)(x_i^n) - \bar{T}_i^n |. \end{aligned}$$

Define the following expressions:

$$W^n(x) \stackrel{\text{def}}{=} (B\hat{u}_\varepsilon)(x) - \bar{T}^n(x)$$

where

$$\bar{T}^n(x) \stackrel{\text{def}}{=} h \sum_{j=0}^n B(x, t_j^n) (\hat{u}_\varepsilon)_{n,j}.$$

In particular $W^n(x_i^n) = (B\hat{u}_\varepsilon)(x_i^n) - \bar{T}^n(x_i^n) = ((B\hat{u}_\varepsilon)(x_i^n) + \varepsilon(\hat{u}_\varepsilon)_{n,i}) - (\bar{T}^n(x_i^n) + \varepsilon(\hat{u}_\varepsilon)_{n,i}) = (B_\varepsilon \hat{u}_\varepsilon)(x_i^n) - \bar{T}_i^n$. Now $\bar{T}^n(x)$ is the trapezoidal approximation to $(B\hat{u}_\varepsilon)(x) = \int_0^1 B(x, t) \hat{u}_\varepsilon(t) dt$ at each point x in the interval $[0, 1]$ and since the integral for each $x \in [0, 1]$ exists and is finite we have that

$$\lim_{n \rightarrow \infty} \bar{T}^n(x) = (B\hat{u}_\varepsilon)(x) \quad x \in [0, 1]$$

and therefore

$$\lim_{n \rightarrow \infty} |W^n(x)| = 0 \quad x \in [0, 1]$$

which implies that

$$\lim_{n \rightarrow \infty} [\sup_{x \in [0, 1]} |W^n(x)|] = 0.$$

Thus, taking into account that $0 < \tau_i^n < 1$ for $0 \leq i \leq n$, we have that

$$|\tilde{\varphi}_{n,\varepsilon}(r_n \hat{u}_\varepsilon) - \bar{\varphi}_{n,\varepsilon}(r_n \hat{u}_\varepsilon)| \leq [h \sum_{i=0}^n |(\hat{u}_\varepsilon)_{n,i}| + 2c_1 h \sum_{i=0}^n |(B_\varepsilon \hat{u}_\varepsilon)(x_i^n)| + 2c_1 \sup_{x \in [0,1]} |w^n(x)|] \sup_{x \in [0,1]} |w^n(x)|.$$

$\underset{=1}{\sum_{i=0}^n h}$

If we turn to the expressions in the square brackets of the above inequality we have the following: By the Schwartz inequality $h \sum_{i=0}^n |(\hat{u}_\varepsilon)_{n,i}| \leq \{h \sum_{i=0}^n |(\hat{u}_\varepsilon)_{n,i}|^2\}^{1/2}$, since $h \sum_{i=0}^n |(\hat{u}_\varepsilon)_{n,i}|^2$ tends to $\int_0^1 |\hat{u}_\varepsilon(x)|^2 dx$ (an integral we know to exist, which is finite) as n tends to ∞ there exists a constant M_1 such that $h \sum_{i=0}^n |(\hat{u}_\varepsilon)_{n,i}| \leq M_1$ for all n . For the second term in the square brackets we have $2c_1 h \sum_{i=0}^n |(B_\varepsilon \hat{u}_\varepsilon)(x_i^n)| = 2c_1 h \sum_{i=0}^n |(B\hat{u}_\varepsilon)(x_i^n) + \varepsilon(\hat{u}_\varepsilon)_{n,i}| \leq 2c_1 h \sum_{i=0}^n |(B\hat{u}_\varepsilon)(x_i^n)| + 2\varepsilon c_1 h \sum_{i=0}^n |(\hat{u}_\varepsilon)_i^n|$. Since $(B\hat{u}_\varepsilon)(x)$ is a continuous function on $[0,1]$, $\exists M_2 \geq 0$ such that $|(B\hat{u}_\varepsilon)(x)| \leq M_2 \forall x \in [0,1]$ and hence $2c_1 h \sum_{i=0}^n |(B\hat{u}_\varepsilon)(x_i^n)| \leq 2c_1 M_2$, and similar to the first term, $2\varepsilon c_1 h \sum_{i=0}^n |(\hat{u}_\varepsilon)_i^n| \leq 2\varepsilon c_1 M_1$. Hence if we denote the sum of these constants by M we get that

$$|\tilde{\varphi}_{n,\varepsilon}(r_n \hat{u}_\varepsilon) - \bar{\varphi}_{n,\varepsilon}(r_n \hat{u}_\varepsilon)| \leq [M + 2c_1 \sup_{x \in [0,1]} |w^n(x)|] \sup_{x \in [0,1]} |w^n(x)|$$

which tends to zero as n tends to ∞ since $\sup_{x \in [0,1]} |w^n(x)| \rightarrow 0$ as n tends to ∞ . Hence from equality (2.5) we get that

$$\lim_{n \rightarrow \infty} |\tilde{\varphi}_{n,\varepsilon}(r_n \hat{u}_\varepsilon) - \tilde{\varphi}_\varepsilon(\hat{u}_\varepsilon)| = 0.$$

Finally

$$0 = \lim_{n \rightarrow \infty} |\tilde{\varphi}_{n,\varepsilon}(r_n \hat{u}_\varepsilon) - \tilde{\varphi}_\varepsilon(\hat{u}_\varepsilon)| = \lim_{n \rightarrow \infty} \sup |\tilde{\varphi}_{n,\varepsilon}(r_n \hat{u}_\varepsilon) - \tilde{\varphi}_\varepsilon(\hat{u}_\varepsilon)| \geq \lim_{n \rightarrow \infty} \sup (\tilde{\varphi}_{n,\varepsilon}(r_n \hat{u}_\varepsilon) - \tilde{\varphi}_\varepsilon(\hat{u}_\varepsilon))$$

from which follows that

$$\lim_{n \rightarrow \infty} \sup \tilde{\varphi}_{n,\varepsilon}(r_n \hat{u}_\varepsilon) \leq \tilde{\varphi}_\varepsilon(\hat{u}_\varepsilon),$$

so the first condition for the consistency is satisfied.

For the second condition for the consistency we have to show that

$$\lim_{n \rightarrow \infty} \sup (\tilde{\varphi}_\varepsilon(p_n \hat{u}_n) - \tilde{\varphi}_{n,\varepsilon}(\hat{u}_n)) \leq 0$$

where $\hat{u}_n$ minimizes the "discrete" functional $\tilde{\varphi}_{n,\varepsilon}$ over E_n .

According to Daniel [10, p.31], the satisfaction of the

stability and uniform growth conditions by the discretization

scheme implies the existence of a uniform bound for $\|\hat{u}_n\|_n$

i.e. there exists a constant M' , $0 \leq M' < \infty$ such that

$\|\hat{u}_n\|_n \leq M'$ for all n (this can be easily verified). We shall

use this result to prove that the second consistency conditions

is satisfied. Though the method of proof will be similar to

that used to prove the first condition for consistency, in the

present case we find it convenient to establish some (in between)

auxiliary results.

(i) Let us define the following expressions:

$$\tilde{\varphi}_{n,\varepsilon}(\hat{u}_n) = h \sum_{i=0}^n \phi(x_i^n; \hat{u}_i^n; (B_\varepsilon p_n \hat{u}_n)(x_i^n))$$

and show that $|\bar{\varphi}_{n,\varepsilon}(\hat{u}_n) - \tilde{\varphi}_{n,\varepsilon}(\hat{u}_n)| \rightarrow 0$ as $n \rightarrow \infty$.

$$\begin{aligned} |\bar{\varphi}_{n,\varepsilon}(\hat{u}_n) - \tilde{\varphi}_{n,\varepsilon}(\hat{u}_n)| &= |h \sum_{i=0}^n \phi(x_i^n; \hat{u}_i^n; (B_\varepsilon p_{n-n})(x_i^n)) \\ &\quad - h \sum_{i=0}^n \phi(x_i^n; \hat{u}_i^n; \underline{T}_i^n)| \leq h \sum_{i=0}^n |\phi(x_i^n; \hat{u}_i^n; (B_\varepsilon p_{n-n})(x_i^n)) \\ &\quad - \phi(x_i^n; \hat{u}_i^n; \underline{T}_i^n)| \end{aligned}$$

which (again) by the mean value theorem is further majorized by

$$\begin{aligned} &\leq h \sum_{i=0}^n |\hat{u}_i^n - 2F(x_i^n, (B_\varepsilon p_{n-n})(x_i^n) + \tau_i^n((B_\varepsilon p_{n-n})(x_i^n) \\ &\quad - \underline{T}_i^n))| |(B_\varepsilon p_{n-n})(x_i^n) - \underline{T}_i^n|. \end{aligned}$$

Furthermore from inequality (2.1) we have

$$\begin{aligned} &\leq h \sum_{i=0}^n [|\hat{u}_i^n| + 2c_1 |(B_\varepsilon p_{n-n})(x_i^n)| + 2c_1 \tau_i^n |(B_\varepsilon p_{n-n})(x_i^n) \\ &\quad - \underline{T}_i^n| + 2c_2] |(B_\varepsilon p_{n-n})(x_i^n) - \underline{T}_i^n|. \end{aligned} \quad (2.7)$$

We shall also define

$$\begin{aligned} v^n(x) &\stackrel{\text{def}}{=} (B p_{n-n})(x) - \underline{T}^n(x); \quad x \in [0,1], \\ \underline{T}^n(x) &\stackrel{\text{def}}{=} h \sum_{i=0}^n B(x, t_j^n) \hat{u}_j^n. \end{aligned}$$

In particular

$$v^n(x_i^n) = ((B p_{n-n})(x_i^n) + \varepsilon \hat{u}_i^n) - (\underline{T}^n(x_i^n) + \varepsilon \hat{u}_i^n) = (B_\varepsilon p_{n-n})(x_i^n) - \underline{T}_i^n$$

for all n and for all i . We shall show that $|v^n(x)|$ tends to zero as n tends to ∞ for all $x \in [0,1]$:

$$|v^n(x)| = |(Bp_{n-\bar{n}}\hat{u})(x) - \underline{T}^n(x)| = \left| \int_0^1 B(x,t) (p_{n-\bar{n}}\hat{u})(t) dt - h \sum_{i=0}^n B(x,t_i^n) \hat{u}_i^n \right|,$$

since $p_{n-\bar{n}}\hat{u}$ is a piecewise linear function

$$\begin{aligned} &= \left| h \sum_{i=1}^n \int_0^1 B(x, t_{i-1}^n + \alpha h) [(1-\alpha)\hat{u}_{i-1}^n + \alpha\hat{u}_i^n] d\alpha \right. \\ &\quad \left. - h \sum_{i=0}^n B(x, t_i^n) \hat{u}_i^n \right| = \left| \left[h \sum_{i=1}^n \int_0^1 B(x, t_{i-1}^n + \alpha h) (1-\alpha) \hat{u}_{i-1}^n d\alpha \right. \right. \\ &\quad \left. + h \sum_{i=1}^n \int_0^1 B(x, t_{i-1}^n + \alpha h) \alpha \hat{u}_i^n d\alpha \right] - \left[\frac{h}{2} \sum_{i=1}^n B(x, t_{i-1}^n) \hat{u}_{i-1}^n \right. \\ &\quad \left. + \frac{h}{2} \sum_{i=1}^n B(x, t_i^n) \hat{u}_{i-1}^n \right] \right|. \end{aligned} \quad (2.8)$$

It is readily seen that $\frac{h}{2} \sum_{i=1}^n B(x, t_{i-1}^n) \hat{u}_{i-1}^n = h \sum_{i=1}^n \int_0^1 B(x, t_{i-1}^n) (1-\alpha) \hat{u}_{i-1}^n d\alpha$

and $\frac{h}{2} \sum_{i=1}^n B(x, t_i^n) \hat{u}_i^n = h \sum_{i=1}^n \int_0^1 B(x, t_i^n) \alpha \hat{u}_i^n d\alpha$. Therefore we have the following inequality from (2.8)

$$\begin{aligned} &\leq h \sum_{i=1}^n \int_0^1 |B(x, t_{i-1}^n + \alpha h) - B(x, t_{i-1}^n)| (1-\alpha) \hat{u}_{i-1}^n d\alpha \\ &\quad + h \sum_{i=1}^n \int_0^1 |B(x, t_{i-1}^n + \alpha h) - B(x, t_i^n)| \alpha \hat{u}_i^n d\alpha. \end{aligned}$$

The kernel function $B(x,t)$ was assumed to be continuous both in x and in t on $[0,1] \times [0,1]$, hence for every fixed $x \in [0,1]$,

$B(x,t)$ is uniformly continuous in t , i.e. for every $\zeta_x > 0$

$\delta(\zeta_x) > 0$ such that $|t' - t''| < \delta(\zeta_x)$ and $t', t'' \in [0,1]$ implies

$$|B(x, t') - B(x, t'')| < \zeta_x.$$

Thus for a sufficiently small refinement of the mesh, that is for

$h < \delta(\zeta_x)$, and owing to $|\alpha|, |1-\alpha| \leq 1$ we obtain that

$$|v^n(x)| \leq 2\zeta_x h \sum_{i=0}^n |\hat{u}_i^n|$$

and by applying Schwartz's inequality

$$\leq 2\zeta_x \{h \sum_{i=0}^n |\hat{u}_i^n|^2\}^{1/2} = 2\zeta_x \|\hat{u}_n\|_n.$$

But for every n , $\|\hat{u}_n\|_n \leq M' < \infty$ and hence

$$|v^n(x)| \leq 2M\zeta_x \rightarrow 0 \text{ as } n \rightarrow \infty,$$

from which follows that

$$\lim_{n \rightarrow \infty} [\sup_{x \in [0,1]} |v^n(x)|] = 0,$$

Turning back to the terms in the square brackets of (2.8)

we have the following (recalling that $|(B_\varepsilon p_{n-\underline{n}})(x)| \leq |(Bp_{n-\underline{n}})(x)| + \varepsilon |(p_{n-\underline{n}})(x)|$; $(Bp_{n-\underline{n}})(x)$ is a continuous function in x for every n and hence bounded, denote by $M_n = \max_{x \in [0,1]} |(Bp_{n-\underline{n}})(x)|$, thus

$$\begin{aligned} M_n &= \max_{x \in [0,1]} \left| \int_0^1 B(x,t) (p_{n-\underline{n}})(t) dt \right| \leq \max_{x \in [0,1]} \int_0^1 |B(x,t)| |p_{n-\underline{n}}(t)| dt \\ &\leq \max_{x,t \in [0,1]} |B(x,t)| \int_0^1 |p_{n-\underline{n}}(t)| dt \leq \max_{x,t \in [0,1]} |B(x,t)| \left\{ \int_0^1 |p_{n-\underline{n}}(t)|^2 dt \right\}^{1/2} \\ &= \max_{x,t \in [0,1]} |B(x,t)| \|p_{n-\underline{n}}\| \end{aligned}$$

since $\|p_{n-\underline{n}}\| \leq 2\|\hat{u}_n\| \leq 2M'$ and since $M''' \geq \max_{x,t \in [0,1]} |B(x,t)|$

and $M''' < \infty$ we get that $|(Bp_{n-\underline{n}})(x)|$ is bounded for all x and for all n by some constant say N' , $0 \leq N' < \infty$. In addition,

$h \sum_{i=0}^n 1 = 1$ and $0 < \tau_1^n < 1$ from which we obtain the following estimate:

$$\begin{aligned} |\bar{\psi}_{n,\varepsilon}(\hat{u}_n) - \tilde{\psi}_{n,\varepsilon}(\hat{u}_n)| &\leq [h \sum_{i=0}^n |\hat{u}_i^n| + 2c_1 N' + 2\varepsilon c_1 h \sum_{i=0}^n |\hat{u}_i^n| \\ &+ 2c_1 \sup_{x \in [0,1]} |v^n(x)| + 2c_2] \cdot \sup_{x \in [0,1]} |v^n(x)|. \end{aligned} \quad (2.9)$$

Applying Schwartz's inequality to $h \sum_{i=0}^n |\hat{u}_i^n|$ in (2.9) we get

$$\begin{aligned} &\leq [\{h \sum_{i=0}^n |\hat{u}_i^n|^2\}^{1/2} + 2c_1 N' + 2\varepsilon c_1 \{h \sum_{i=0}^n |\hat{u}_i^n|^2\}^{1/2} \\ &+ 2c_1 \sup_{x \in [0,1]} |v^n(x)| + 2c_2] \cdot \sup_{x \in [0,1]} |v^n(x)| \\ &\leq [(1 + 2\varepsilon c_1)M' + 2c_1 N' + 2c_1 \sup_{x \in [0,1]} |v^n(x)|] \cdot \sup_{x \in [0,1]} |v^n(x)| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence we obtain the first auxiliary result that

$$|\bar{\psi}_{n,\varepsilon}(\hat{u}_n) - \tilde{\psi}_{n,\varepsilon}(\hat{u}_n)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(ii) Let us define the following expressions:

$$\begin{aligned} \bar{\psi}_{n,\varepsilon}(\hat{u}_n) &= h \sum_{i=1}^n \int_0^1 \bar{\phi}(x_{i-1}^n; (1-\alpha)\hat{u}_{i-1}^n; (B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h)) d\alpha \\ &+ h \sum_{i=1}^n \int_0^1 \bar{\phi}(x_i^n; \alpha \hat{u}_i^n; (B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h)) d\alpha \end{aligned}$$

where $\bar{\phi}(x; u; z) \stackrel{\text{def}}{=} uz - \int_0^z F(x, v) dv$. These expressions, as we shall see later, resemble very much the expression for $\tilde{\psi}_{\varepsilon}(p_n \hat{u}_n)$. In this section we shall prove that $|\bar{\psi}_{n,\varepsilon}(\hat{u}_n) - \tilde{\psi}_{n,\varepsilon}(\hat{u}_n)| \rightarrow 0$ as $n \rightarrow \infty$. However the proof itself may not be simple as in (i) since

it will be observed that for the proof of the second condition for consistency the uniform boundedness of the norm of the minimizing vectors for the approximating functionals $\tilde{\varphi}_{n,\epsilon}$ is not sufficient and one has to show that the components are actually uniformly bounded, i.e. $|\hat{u}_i^n| \leq \text{constant}$ for all n and for all i .

Now

$$\begin{aligned} \tilde{\varphi}_{n,\epsilon}(\hat{u}_n) &= h \sum_{i=0}^n \bar{\varphi}(x_i^n; \hat{u}_i^n; (B_\epsilon p_{n-n})(x_i^n)) \\ &= \frac{h}{2} \sum_{i=1}^n [\hat{u}_{i-1}^n (B_\epsilon p_{n-n})(x_{i-1}^n) - 2 \int_0^{(B_\epsilon p_{n-n})(x_{i-1}^n)} F(x_{i-1}^n, v) dv] \\ &\quad + \frac{h}{2} \sum_{i=1}^n [\hat{u}_i^n (B_\epsilon p_{n-n})(x_i^n) - 2 \int_0^{(B_\epsilon p_{n-n})(x_i^n)} F(x_i^n, v) dv] \\ &= h \sum_{i=1}^n \int_0^1 [(1-\alpha) \hat{u}_{i-1}^n (B_\epsilon p_{n-n})(x_{i-1}^n) - \int_0^{(B_\epsilon p_{n-n})(x_{i-1}^n)} F(x_{i-1}^n, v) dv] d\alpha \\ &\quad + h \sum_{i=1}^n \int_0^1 [\alpha \hat{u}_i^n (B_\epsilon p_{n-n})(x_i^n) - \int_0^{(B_\epsilon p_{n-n})(x_i^n)} F(x_i^n, v) dv] d\alpha \end{aligned}$$

and by the definition of $\bar{\varphi}$

$$\begin{aligned} &= h \sum_{i=1}^n \int_0^1 \bar{\varphi}(x_{i-1}^n; (1-\alpha) \hat{u}_{i-1}^n; (B_\epsilon p_{n-n})(x_{i-1}^n)) d\alpha \\ &\quad + h \sum_{i=1}^n \int_0^1 \bar{\varphi}(x_i^n; \alpha \hat{u}_i^n; (B_\epsilon p_{n-n})(x_i^n)) d\alpha. \end{aligned}$$

Thus we obtain the following estimate:

$$\begin{aligned} |\tilde{\varphi}_{n,\epsilon}(\hat{u}_n) - \bar{\varphi}_{n,\epsilon}(\hat{u}_n)| &\leq h \sum_{i=1}^n \int_0^1 |\bar{\varphi}(x_{i-1}^n; (1-\alpha) \hat{u}_{i-1}^n; (B_\epsilon p_{n-n})(x_{i-1}^n) \\ &\quad + \alpha h) - \bar{\varphi}(x_{i-1}^n; (1-\alpha) \hat{u}_{i-1}^n; (B_\epsilon p_{n-n})(x_{i-1}^n))| d\alpha + \end{aligned}$$

$$+ h \sum_{i=1}^n \int_0^1 \left| \bar{\Phi}(x_i^n; \alpha \hat{u}_i^n; (B_{\varepsilon} p_{n-\underline{n}})(x_{i-1}^n + \alpha h)) - \bar{\Phi}(x_i^n; \alpha \hat{u}_i^n; (B_{\varepsilon} p_{n-\underline{n}})(x_i^n)) \right| d\alpha.$$

By the use of the mean value theorem

$$\begin{aligned} &\leq h \sum_{i=1}^n \int_0^1 [(1 - \alpha) |\hat{u}_{i-1}^n| + |F(x_{i-1}^n, (B_{\varepsilon} p_{n-\underline{n}})(x_{i-1}^n + \alpha h) \\ &+ \tau_i^n ((B_{\varepsilon} p_{n-\underline{n}})(x_{i-1}^n + \alpha h) - (B_{\varepsilon} p_{n-\underline{n}})(x_{i-1}^n))| \cdot |(B_{\varepsilon} p_{n-\underline{n}})(x_{i-1}^n + \alpha h) \\ &- (B_{\varepsilon} p_{n-\underline{n}})(x_{i-1}^n)| + S_1 \end{aligned}$$

where S_1 is a similar expression to the one before it and

$0 < \tau_i^n < 1$ for all $0 \leq i \leq n$.

$$\begin{aligned} \text{Now: } &(B_{\varepsilon} p_{n-\underline{n}})(x_{i-1}^n + \alpha h) - (B_{\varepsilon} p_{n-\underline{n}})(x_{i-1}^n) = [(B p_{n-\underline{n}})(x_{i-1}^n + \alpha h) \\ &+ \varepsilon((1 - \alpha)\hat{u}_{i-1}^n + \alpha \hat{u}_i^n)] - [(B p_{n-\underline{n}})(x_{i-1}^n) + \varepsilon \hat{u}_{i-1}^n] \\ &= [(B p_{n-\underline{n}})(x_{i-1}^n + \alpha h) - (B p_{n-\underline{n}})(x_{i-1}^n)] + \varepsilon \alpha [\hat{u}_i^n - \hat{u}_{i-1}^n] \end{aligned}$$

and likewise in S_1 , from which together with inequality (2.1) yields

$$\begin{aligned} &\leq h \sum_{i=1}^n \int_0^1 \{ (1 - \alpha) |\hat{u}_{i-1}^n| + c_1 |(B_{\varepsilon} p_{n-\underline{n}})(x_{i-1}^n + \alpha h)| \\ &+ c_1 \tau_i^n [|(B p_{n-\underline{n}})(x_{i-1}^n + \alpha h) - (B p_{n-\underline{n}})(x_{i-1}^n)| + \varepsilon \alpha |\hat{u}_i^n - \hat{u}_{i-1}^n|] \\ &+ c_2 [|(B p_{n-\underline{n}})(x_{i-1}^n + \alpha h) - (B p_{n-\underline{n}})(x_{i-1}^n)| + \varepsilon \alpha |\hat{u}_i^n - \hat{u}_{i-1}^n|] \\ &+ S_2 \end{aligned} \quad (2.10)$$

where S_2 is similar to the first term. Noting that $\alpha, 1 - \alpha \leq 1$, $\tau_i^n \leq 1$ and also rearranging all triangular inequalities and all

terms like

$$|(\hat{B}_{\varepsilon} p_{n-\bar{n}})(x_{i-1}^n + \alpha h)| \leq |(B p_{n-\bar{n}})(x_{i-1}^n + \alpha h)| + \varepsilon |(1 - \alpha) \hat{u}_{i-1}^n|$$

+ $\alpha \hat{u}_i^n| \leq |(B p_{n-\bar{n}})(x_{i-1}^n + \alpha h)| + \varepsilon |\hat{u}_{i-1}^n| + \varepsilon |\hat{u}_i^n|$, we obtain a further inequality for (2.10)

$$\begin{aligned} &\leq h \sum_{i=1}^n \int_0^1 [|\hat{u}_{i-1}^n| + c_1 |(B p_{n-\bar{n}})(x_{i-1}^n + \alpha h)| + 2\varepsilon c_1 |\hat{u}_{i-1}^n| \\ &+ 2\varepsilon c_1 |\hat{u}_i^n| + c_1 |(B p_{n-\bar{n}})(x_{i-1}^n + \alpha h) - (B p_{n-\bar{n}})(x_{i-1}^n)| + c_2] \cdot \\ &\cdot [|(B p_{n-\bar{n}})(x_{i-1}^n + \alpha h) - (B p_{n-\bar{n}})(x_{i-1}^n)| + \varepsilon |\hat{u}_i^n - \hat{u}_{i-1}^n|] d\alpha \\ &+ S_3 \end{aligned}$$

where S_3 is similar to the first term. Since $|(B p_{n-\bar{n}})(x_{i-1}^n + \alpha h)|$ is easily proved to be bounded and the same applies to the difference $|(B p_{n-\bar{n}})(x_{i-1}^n + \alpha h) - (B p_{n-\bar{n}})(x_{i-1}^n)|$, after cross multiplication we get the following basic types of terms:

$$\begin{aligned} &\text{const} \cdot h \sum_{i=1}^n \int_0^1 |(B p_{n-\bar{n}})(x_{i-1}^n + \alpha h) - (B p_{n-\bar{n}})(x_{i-1}^n)| d\alpha, \\ &h \sum_{i=1}^n \int_0^1 |\hat{u}_{i-1}^n| |(B p_{n-\bar{n}})(x_{i-1}^n + \alpha h) - (B p_{n-\bar{n}})(x_{i-1}^n)| d\alpha \text{ (or } |\hat{u}_i^n| \end{aligned}$$

times the difference as the case may be), $\text{const} \cdot h \sum_{i=1}^n \int_0^1 |\hat{u}_{i-1}^n| |\hat{u}_i^n - \hat{u}_{i-1}^n| d\alpha$ (or $|\hat{u}_i^n| |\hat{u}_i^n - \hat{u}_{i-1}^n|$), and $\text{const} \cdot h \sum_{i=1}^n |\hat{u}_i^n - \hat{u}_{i-1}^n|$, (the same for S_3). We shall prove that all these expressions tend to zero as n tends to ∞ . For the first term we have:

$$\begin{aligned} &\text{const} \cdot h \sum_{i=1}^n \int_0^1 |(B p_{n-\bar{n}})(x_{i-1}^n + \alpha h) - (B p_{n-\bar{n}})(x_{i-1}^n)| d\alpha \\ &\leq \text{const} \cdot h \sum_{i=1}^n \int_0^1 \left[\int_0^1 |B(x_{i-1}^n + \alpha h, t) - B(x_{i-1}^n, t)| |(p_{n-\bar{n}})(t)| dt \right] d\alpha. \end{aligned}$$

If we apply now Schwartz' inequality for integrals to the integral inside the square brackets we get

$$\leq \text{const} \cdot h \sum_{i=1}^n \int_0^1 \left[\int_0^1 |B(x_{i-1}^n + \alpha h, t) - B(x_{i-1}^n, t)|^2 dt \right]^{1/2} \\ \left[\int_0^1 |(p_n \hat{u}_n)(t)|^2 dt \right]^{1/2} d\alpha.$$

Now $\left\{ \int_0^1 |(p_n \hat{u}_n)(t)|^2 dt \right\}^{1/2} = \|p_n \hat{u}_n\|$ which by the stability condition is majorized by $2 \|\hat{u}_n\|_n$ and therefore also uniformly bounded, hence if we denote the new constant also by const we get

$$\leq \text{const} \cdot h \sum_{i=1}^n \int_0^1 \left[\int_0^1 |B(x_{i-1}^n + \alpha h, t) - B(x_{i-1}^n, t)|^2 dt \right]^{1/2} d\alpha,$$

but since $B(x, t)$ is continuous in both variables on $[0, 1]$

$$\leq \text{const} \cdot h \sum_{i=1}^n \int_0^1 \sup_{t \in [0, 1]} |B(x_{i-1}^n + \alpha h, t) - B(x_{i-1}^n, t)| d\alpha. \quad (2.11)$$

Again from the properties of the kernel function

$$\sup_{t \in [0, 1]} |B(x_{i-1}^n + \alpha h, t) - B(x_{i-1}^n, t)| \leq \sup_{x \in [0, 1]} \left[\sup_{t \in [0, 1]} |B(x + \alpha h, t) - B(x, t)| \right]$$

$$= |B(x, t)|$$

which clearly tends to zero as $n \rightarrow \infty$ since $\alpha h \leq h \rightarrow 0$ as $n \rightarrow \infty$.

Thus (2.11) becomes

$$\leq \text{const} \cdot \sup_{x \in [0, 1]} \sup_{t \in [0, 1]} |B(x + \alpha h, t) - B(x, t)| h \sum_{i=1}^n 1.$$

Since $h \sum_{i=1}^n 1 = 1$ for all n , we get that the first of the basic terms tends to zero as n tends to ∞ .

The second basic term can be reduced to a similar one to the first case since:

$$\begin{aligned}
& h \sum_{i=1}^n \int_0^1 | (B_{p_{n-n}} \hat{u}_n)(x_{i-1}^n + \alpha h) - (B_{p_{n-n}} \hat{u}_n)(x_{i-1}^n) | |\hat{u}_{i-1}^n| d\alpha \\
& \leq h \sum_{i=1}^n \int_0^1 \left\{ \int_0^1 |B(x_{i-1}^n + \alpha h, t) - B(x_{i-1}^n, t)|^2 dt \right\}^{1/2} \left\{ \int_0^1 |(p_{n-n} \hat{u}_n)(t)|^2 dt \right\}^{1/2} \\
& \quad |\hat{u}_{i-1}^n| d\alpha
\end{aligned}$$

since $\left\{ \int_0^1 |(p_{n-n} \hat{u}_n)(t)|^2 dt \right\}^{1/2} \leq \text{const}$, we have that the above

$$\leq \text{const} \sup_{x \in [0,1]} \left[\sup_{t \in [0,1]} |B(x + \alpha h, t) - B(x, t)| \right] \cdot h \sum_{i=1}^n |\hat{u}_{i-1}^n|. \quad (2.12)$$

Using the Schwartz's inequality for sums one can show that

$$h \sum_{i=1}^n |\hat{u}_{i-1}^n| \leq \left\{ h \sum_{i=1}^n |\hat{u}_{i-1}^n|^2 \right\}^{1/2}$$

and this is majorized by $\{2h \sum_{i=0}^n |\hat{u}_i^n|^2\}^{1/2} = \text{const} \cdot \|\hat{u}_n\|_n$

which is uniformly bounded by the uniform bound for $\|\hat{u}_n\|_n$

times a constant (the factor 2 enters because of the end term $|\hat{u}_n^n|$).

Thus we obtain that the right hand side of inequality (2.12) tends to zero as n tends to ∞ , i.e. the second basic term tends to zero as n tends to ∞ .

The third basic term is:

$$\text{const} \cdot h \sum_{i=1}^n \int_0^1 |\hat{u}_{i-1}^n| |\hat{u}_i^n - u_{i-1}^n| d\alpha = \text{const} \cdot h \sum_{i=1}^n |\hat{u}_{i-1}^n| |\hat{u}_i^n - \hat{u}_{i-1}^n|$$

We shall prove that $|\hat{u}_i^n - \hat{u}_{i-1}^n|$ is majorized by an expression that tends to zero as n tends to ∞ , and thus using the uniform boundedness of $h \sum_{i=1}^n |\hat{u}_{i-1}^n|$, their product will tend to zero as n tends to ∞ . To show this we shall first prove that $|\hat{u}_i^n|$, $i = 0, \dots, n$; $n = 2, 3, \dots$ are uniformly bounded. Since $\tilde{\varphi}_{n,\varepsilon}(\underline{u}_n) = \psi(u_0^n, \dots, u_n^n)$ (where ψ is just used for clarity) are continuous in u_i^n , $i = 0, \dots, n$; bounded below and attain their absolute minimum at

$\hat{u}_n = (\hat{u}_1^n, \dots, \hat{u}_n^n)$, it follows that

$$\frac{\partial \tilde{\varphi}_{n, \varepsilon}(\hat{u}_n)}{\partial \hat{u}_k^n} = \frac{\partial \psi(\hat{u}_0^n, \dots, \hat{u}_n^n)}{\partial \hat{u}_k^n} = 0; \quad k = 0, \dots, n.$$

Now

$$\begin{aligned} \frac{\partial \tilde{\varphi}_{n, \varepsilon}(\hat{u}_n)}{\partial \hat{u}_k^n} &= \frac{\partial}{\partial \hat{u}_k^n} \left[h \sum_{i=0}^n \Phi(x_i^n, \hat{u}_i^n, \underline{T}_i^n) \right] \\ &= h \sum_{i=0}^n \frac{\partial}{\partial \hat{u}_k^n} \left[\hat{u}_i^n \underline{T}_i^n - 2 \int_0^{\underline{T}_i^n} F(x_i^n, v) dv \right]. \end{aligned}$$

As we recall $\underline{T}_i^n = h \sum_{j=0}^n B_{ij}^n \hat{u}_j^n + \varepsilon \hat{u}_i^n$, so after differentiating and collecting the terms we get that the above

$$\begin{aligned} &= h \left[2\varepsilon \hat{u}_k^n + 2h \sum_{j=0}^n B_{kj}^n \hat{u}_j^n - 2h \sum_{i=0}^n B_{ik}^n F(x_i^n, \underline{T}_i^n) \right. \\ &\quad \left. - 2\varepsilon F(x_k^n, \underline{T}_k^n) \right]. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \cancel{h} \left[2\varepsilon \hat{u}_k^n + 2h \sum_{j=0}^n B_{kj}^n \hat{u}_j^n - 2h \sum_{i=0}^n B_{ik}^n F(x_i^n, \underline{T}_i^n) \right. \\ &\quad \left. - 2\varepsilon F(x_k^n, \underline{T}_k^n) \right] \geq 2\varepsilon |\hat{u}_k^n| - 2h \sum_{j=0}^n |B_{kj}^n| |\hat{u}_j^n| \\ &\quad - 2h \sum_{i=0}^n |B_{ik}^n| |F(x_i^n, \underline{T}_i^n)| - 2\varepsilon |F(x_k^n, \underline{T}_k^n)|. \end{aligned}$$

From inequality (2.1) and by the boundedness of the kernel function $B(x, t)$ on $[0, 1] \times [0, 1]$, a bound which will be denoted by D , follows

$$\begin{aligned}
&\geq 2\varepsilon |\hat{u}_k^n| - 2Dh \sum_{j=0}^n |\hat{u}_j^n| - 2Dh \sum_{i=0}^n (c_1 |\hat{u}_i^n| + c_2) \\
&- 2\varepsilon (c_1 |\hat{u}_k^n| + c_2) \geq 2\varepsilon |\hat{u}_k^n| - 2Dh \sum_{j=0}^n |\hat{u}_j^n| \\
&- 2Dh \sum_{i=0}^n [c_1 h \sum_{j=0}^n |B_{ij}^n| |\hat{u}_j^n| + c_1 \varepsilon |\hat{u}_i^n| + c_2] \\
&- 2\varepsilon [c_1 h \sum_{j=0}^n |B_{kj}^n| |\hat{u}_j^n| + c_1 \varepsilon |\hat{u}_k^n| + c_2] \\
&\geq 2\varepsilon (1 - c_1 \varepsilon) |\hat{u}_k^n| - (2D + 4Dc_1 \varepsilon) h \sum_{i=0}^n |\hat{u}_i^n| \\
&- 2D^2 h \sum_{i=0}^n [h \sum_{j=0}^n |\hat{u}_j^n|] - 2Dc_2 h \sum_{i=0}^n 1 - 2\varepsilon c_2
\end{aligned}$$

hence

$$\begin{aligned}
&(2D + 4Dc_1 \varepsilon) h \sum_{i=0}^n |\hat{u}_i^n| + 2D^2 h \sum_{i=0}^n [h \sum_{j=0}^n |\hat{u}_j^n|] \\
&+ 2Dc_2 + 2\varepsilon c_2 \geq 2\varepsilon (1 - c_1 \varepsilon) |\hat{u}_k^n|.
\end{aligned}$$

Due to the uniform boundedness of $h \sum_{i=0}^n |\hat{u}_i^n|$ and of $h \sum_{i=0}^n [h \sum_{j=0}^n |\hat{u}_j^n|]$ which follows from the uniform boundedness of $\|\hat{u}_n\|_n = \{h \sum_{i=0}^n |\hat{u}_i^n|^2\}^{1/2}$ (as we have seen before), the right hand side of the last inequality is uniformly bounded for all n and the bound is independent of k . On the other hand from the construction of the perturbation $0 < 1 - c_1 \varepsilon < 1$ and hence $|\hat{u}_k^n|$ are uniformly bounded for all n and for all k . Turning back to the third basic term we have the following:

$$\begin{aligned}
&\text{const} \cdot h \sum_{i=1}^n |\hat{u}_{i-1}^n| |\hat{u}_i^n - \hat{u}_{i-1}^n| \\
&= \text{const} h \sum_{i=1}^n |\hat{u}_{i-1}^n| |(p_{n-\hat{u}})(x_{i-1}^n + h) - (p_{n-\hat{u}})(x_{i-1}^n)|.
\end{aligned}$$

Consider the family of functions $\{p_{n-\underline{n}}^{\hat{u}}\}_{n=1}^{\infty}$ which were obtained by the piecewise linear joining of the minimizing values for $\varphi_{n,\varepsilon}$. Since (as we have proved) $|\hat{u}_i^n| \leq \text{constant}$ for $0 \leq i \leq n$ and for $n = 1, 2, \dots$, the constant being independent of i and of n , the functions $p_{n-\underline{n}}^{\hat{u}}$ are uniformly bounded in the maximum norm, hence we can write

$$\begin{aligned}
 & \text{const. } h \sum_{i=1}^n |\hat{u}_{i-1}^n| \left| (p_{n-\underline{n}}^{\hat{u}})(x_{i-1}^n + \alpha h) - (p_{n-\underline{n}}^{\hat{u}})(x_{i-1}^n) \right| \\
 & \leq \text{const. } \max_{x \in [0,1]} \left| (p_{n-\underline{n}}^{\hat{u}})(x+h) - (p_{n-\underline{n}}^{\hat{u}})(x) \right| \left(2h \sum_{i=0}^n |\hat{u}_i^n| \right) \\
 & \leq \text{const. } \sup_{k \geq 1} \left[\max_{x \in [0,1]} \left| (p_{k-\underline{k}}^{\hat{u}})(x+h) - (p_{k-\underline{k}}^{\hat{u}})(x) \right| \right] \\
 & \quad \cdot (2 \|\hat{u}_n\|_n). \tag{2.13}
 \end{aligned}$$

Since $\|\hat{u}_n\|_n$ are uniformly bounded, the right hand side of inequality (2.13) tends to zero as n tends to ∞ , i.e.

const. $h \cdot \sum_{i=1}^n |\hat{u}_{i-1}^n| |\hat{u}_i^n - \hat{u}_{i-1}^n| \rightarrow 0$ as $n \rightarrow \infty$. The fourth basic term is clearly a special case of the third basic term and hence tends to zero as n tends to ∞ . Thus we have proved the second auxiliary result that

$$\lim_{n \rightarrow \infty} \left| \varphi_{n,\varepsilon}^{\hat{u}}(\hat{u}_n) - \varphi_{n,\varepsilon}^{\hat{u}}(\hat{u}_n) \right| = 0.$$

(iii) In this part we shall prove that

$$\lim_{n \rightarrow \infty} \left| \varphi_{n,\varepsilon}^{\hat{u}}(\hat{u}_n) - \tilde{\varphi}_{\varepsilon}(p_{n-\underline{n}}^{\hat{u}}) \right| = 0.$$

Now

$$\tilde{\varphi}_\varepsilon(p_n \hat{u}_n) = \int_0^1 \bar{\Phi}(x, (p_n \hat{u}_n)(x), (B_\varepsilon p_n \hat{u}_n)(x)) dx.$$

Performing the change of variable in each of the subintervals:

$x \rightarrow x_{i-1}^n + \alpha h$ yields

$$= h \sum_{i=1}^n \int_0^1 \bar{\Phi}(x_{i-1}^n + \alpha h; (1 - \alpha) \hat{u}_{i-1}^n + \alpha \hat{u}_i^n; (B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h))$$

$$\cdot d\alpha = h \sum_{i=1}^n \int_0^1 \{[(1 - \alpha) \hat{u}_{i-1}^n + \alpha \hat{u}_i^n] (B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h)$$

$$- 2 \int_0^{(B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h)} F(x_{i-1}^n + \alpha h, v) dv\} d\alpha$$

which by the definition of $\bar{\Phi}$ becomes

$$= h \sum_{i=1}^n \int_0^1 \bar{\Phi}(x_{i-1}^n + \alpha h; (1 - \alpha) \hat{u}_{i-1}^n; (B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h)) d\alpha$$

$$+ h \sum_{i=1}^n \int_0^1 \bar{\Phi}(x_{i-1}^n + \alpha h; \alpha \hat{u}_i^n; (B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h)) d\alpha.$$

Hence:

$$|\tilde{\varphi}_\varepsilon(p_n \hat{u}_n) - \varphi_{n, \varepsilon}(\hat{u}_n)|$$

$$\leq h \sum_{i=1}^n \int_0^1 |\bar{\Phi}(x_{i-1}^n + \alpha h, (1 - \alpha) \hat{u}_{i-1}^n; (B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h))$$

$$- \bar{\Phi}(x_{i-1}^n; (1 - \alpha) \hat{u}_{i-1}^n; (B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h))| d\alpha + h \sum_{i=1}^n \int_0^1 |$$

$$\bar{\Phi}(x_{i-1}^n + \alpha h; \alpha \hat{u}_i^n; (B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h))$$

$$- \bar{\Phi}(x_i^n; \alpha \hat{u}_i^n; (B_\varepsilon p_n \hat{u}_n)(x_{i-1}^n + \alpha h))| d\alpha.$$

If we inspect the differences carefully we find that they differ only in the first argument which in turn implies differences of the form

$$|F(x_{i-1}^n + \alpha h, v) - F(x_{i-1}^n, v)| \quad \text{or} \quad |F(x_{i-1}^n + \alpha h, v) - F(x_i^n, v)|$$

integrated from zero to the same limit $(B_\varepsilon p_{n-n})(x_{i-1}^n + \alpha h)$.

The assumption that $\Phi(x; u; v)$ is continuous in all three arguments implies in particular continuity of F in the first argument x (which is the usual case in most Hammerstein equations) and hence

$$|F(x_{i-1}^n + \alpha h, v) - F(x_{i-1}^n, v)| \leq \max_{x \in [0, 1]} |F(x + h, v) - F(x, v)|$$

which due to the continuity of F in both arguments implies that this difference tends to zero as n tends to ∞ and hence

$$|\tilde{\varphi}_\varepsilon(p_n \hat{u}_n) - \bar{\bar{\varphi}}_{n, \varepsilon}(\hat{u}_n)| \rightarrow 0 \text{ as } n \rightarrow \infty. \text{ Finally}$$

$$\begin{aligned} |\tilde{\varphi}_\varepsilon(p_n \hat{u}_n) - \tilde{\varphi}_{n, \varepsilon}(\hat{u}_n)| &\leq |\tilde{\varphi}_\varepsilon(p_n \hat{u}_n) - \bar{\bar{\varphi}}_{n, \varepsilon}(\hat{u}_n)| \\ &+ |\bar{\bar{\varphi}}_{n, \varepsilon}(\hat{u}_n) - \bar{\varphi}_{n, \varepsilon}(\hat{u}_n)| + |\bar{\varphi}_{n, \varepsilon}(\hat{u}_n) - \tilde{\varphi}_{n, \varepsilon}(\hat{u}_n)|, \end{aligned}$$

hence by (i), (ii) and (iii) $|\tilde{\varphi}_\varepsilon(p_n \hat{u}_n) - \tilde{\varphi}_{n, \varepsilon}(\hat{u}_n)| \rightarrow 0$ as $n \rightarrow \infty$. But

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} |\tilde{\varphi}_\varepsilon(p_n \hat{u}_n) - \tilde{\varphi}_{n, \varepsilon}(\hat{u}_n)| = \lim_{n \rightarrow \infty} \sup |\tilde{\varphi}_\varepsilon(p_n \hat{u}_n) \\ &- \tilde{\varphi}_{n, \varepsilon}(\hat{u}_n)| \geq \lim_{n \rightarrow \infty} \sup (\tilde{\varphi}_\varepsilon(p_n \hat{u}_n) - \tilde{\varphi}_{n, \varepsilon}(\hat{u}_n)), \end{aligned}$$

i.e. the second condition for consistency is satisfied. Thus the discretization scheme fulfils all three requirements of theorem 2.2 of this chapter.

In practice, we construct the functionals $\tilde{\varphi}_{n, \varepsilon}$ and differentiate with respect to the values $\hat{u}_i^n$ to obtain a system of $n + 1$ equations with $n + 1$ unknowns, a system of which a solution

is known to exist and then piecewise linearly join these values to obtain an approximation over the whole region. Alternatively, having constructed the functionals $\tilde{\varphi}_{n,\varepsilon}$ we may use search-minimum methods to obtain the minimizing vector (or an approximation to it) and then piecewise linearly join these values. In the appendix of numerical results to follow this chapter, we shall give results obtained using this algorithm.

Remark: It is quite possible that such discretization scheme could be applied to the functionals φ_{n+1} of chapter 1 which were based on the approximation to the square root of the operator B. However due to the nature of these approximations, i.e. being polynomials in the operator B the discretization scheme may be of a more complicated form to work with.

On $[t_{j-1}, t_j]$ we construct a quadratic interpolation polynomial for ϕ_{s_0} based on the values of ϕ_{s_0} at the points t_{j-1} , $t_{j-1/2}$, t_j :

$$L_h^j(t) = \sum_{i=1}^3 \phi_{s_0}(t_{i,j}) l_{i,j}(t)$$

where $t_{1,j} \equiv t_{j-1}$, $t_{2,j} \equiv t_{j-1/2}$, $t_{3,j} \equiv t_j$ and $l_{i,j}(t)$ are the co-ordinate functions: $l_{i,j}(t_{k,j}) = \delta_{i,k}$.

The error due to the interpolation to ϕ_{s_0} on the (j) -th interval is then:

$$E_j^{s_0}(t) = \frac{p_3^j(t) \phi_{s_0}^{(3)}(\xi_t^j)}{3!}$$

where $p_3^j(t) = (t - t_{1,j})(t - t_{2,j})(t - t_{3,j})$ and ξ_t^j is some inner point in $[t_{j-1}, t_j]$ that depends on t . Since $\phi_{s_0}^{(3)}$ is continuous on $[0,1]$ there exists a constant $M_3^{s_0}$ such that $|\phi_{s_0}^{(3)}(t)| \leq M_3^{s_0}$ for all $t \in [0,1]$, so we have the following error bound on the (j) -th interval

$$|E_j^{s_0}(t)| \leq \frac{|p_3^j(t)| |\phi_{s_0}^{(3)}(\xi_t^j)|}{3!} \leq \frac{h^3 M_3^{s_0}}{3!}.$$

Hence we can choose $h_0 = \frac{1}{n_0}$ small enough such that

$$|E_j^{s_0}(t)| \leq \frac{h_0^3 M_3^{s_0}}{3!} < \frac{\varepsilon}{2}$$

in each sub-interval $[t_{j-1}, t_j]$; $j=1,2,\dots,n$. Let $\chi_j(t)$ be the characteristic function for the (j) -th interval:

$$\chi_j(t) = \begin{cases} 1 & \text{if } t \in [t_{j-1}, t_j] \\ 0 & \text{if } t \notin [t_{j-1}, t_j] \end{cases}$$

and form the following combination

$$L_{n_0}(t) = \sum_{j=1}^{n_0} L_{h_0}^j(t) \chi_j(t).$$

L_{n_0} is then a piecewise quadratic interpolation polynomial for $\phi_{s_0}(t)$ on $[0,1]$ such that

$$|\phi_{s_0}(t) - L_{n_0}(t)| < \frac{\varepsilon}{2} \quad \forall t \in [0,1],$$

and therefore

$$\int_0^1 |\phi_{s_0}(t) - L_{n_0}(t)|^2 dt < \frac{\varepsilon^2}{4}$$

or

$$\|\phi_{s_0} - L_{n_0}\| < \frac{\varepsilon}{2}.$$

Finally we obtain

$$\|x - L_{n_0}\| \leq \|x - \phi_{s_0}\| + \|\phi_{s_0} - L_{n_0}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e. the class of piecewise quadratic polynomials on $[0,1]$ is dense in $L_2[0,1]$.

An immediate consequence of this lemma is that for every $x \in L_2[0,1]$ there exists at least one sequence of piecewise quadratic polynomials that converges to x in the norm of $L_2[0,1]$. Now suppose that the functional f of theorem 4.1, chapter 1 is either continuous or weakly continuous. Then if a sequence $\{x_n\}_{n=1}^{\infty} \in L_2[0,1]$ strongly converges to x in $L_2[0,1]$ then

$$\lim_{n \rightarrow \infty} |\varphi(x_n) - \varphi(x)| = 0,$$

(and this statement is true for either functionals suggested in chapters 1 and 2). Let x_0 be the unique critical point for φ in

$L_2[0,1]$, which implies that

$$\varphi(x_0) = \min_{x \in L_2[0,1]} \varphi(x),$$

(see supplement). Then by the consequence of the last lemma, there exists a sequence of piecewise quadratic polynomials

$\{\alpha_n\}_{n=1}^{\infty}$ such that $\alpha_n \rightarrow x_0$ in the L_2 -norm. Hence

$$\varphi(\alpha_n) \rightarrow \varphi(x_0) \text{ as } n \rightarrow \infty,$$

i.e. $\{\alpha_n\}_{n=1}^{\infty}$ is a minimizing sequence for φ in $L_2[0,1]$.

Consider the class of piecewise quadratic polynomials defined for a given partitioning of the interval and denote this class by $Cl(L_n)$, where L_n has the form:

$$L_n(t) = \sum_{j=1}^n (a_1^j t^2 + a_2^j t + a_3^j) \chi_j(t)$$

where a_1^j , a_2^j and a_3^j are real. It is not very difficult to show that $Cl(L_n)$ form a linear subspace, provided we define the sum of two elements, the multiplication of an element by a real etc. in the proper way. In $Cl(L_n)$ we shall define the following norm:

$L_n \in Cl(L_n)$ then

$$\|L_n\|_{Cl(L_n)} = \left\{ \sum_{j=1}^n \left(\sum_{i=1}^3 |a_i^j|^2 \right) \right\}^{1/2}$$

(which is indeed a Euclidean norm with $3n$ elements). We shall show that φ is bounded below and tends to ∞ as $\|L_n\|_{Cl(L_n)}$ tends to ∞ . Moreover, since φ is easily seen to be continuous in a_i^j $i = 1, 2, 3$; $j = 1, \dots, n$, it attains its absolute minimum. Since $\varphi \rightarrow \infty$ if $\|L_n\|_{L_2[0,1]}$ tends to ∞ , it is sufficient to show that

$\|L_n\|_{Cl(L_n)} \rightarrow \infty$ implies $\|L_n\|_{L_2[0,1]} \rightarrow \infty$. Now

$$\begin{aligned} \|L_n\|_{L_2[0,1]}^2 &= \int_0^1 \left(\sum_{j=1}^n L_h^j(t) \chi_j(t) \right)^2 dt \\ &= \sum_{j=1}^n \int_0^1 (L_h^j(t) \chi_j(t))^2 dt + 2 \sum_{i \neq j} \int_0^1 (L_h^i(t) L_h^j(t) \chi_i(t) \cdot \\ &\quad \cdot \chi_j(t)) dt. \end{aligned} \quad (3.1)$$

The second sum on the right hand side is equal to zero since $\chi_i(t) \chi_j(t) = 0$ for all $t \in [0,1]$, hence (3.1) becomes

$$= \sum_{j=1}^n \int_0^1 (L_h^j(t) \chi_j(t))^2 dt = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (a_1^j t^2 + a_2^j t + a_3^j)^2 dt$$

which by inequality (2.12.1) of [16] p.32 majorizes

$$\begin{aligned} &\geq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} [a_1^{j2} t^4 + a_2^{j2} t^2 + a_3^{j2}] dt \\ &= \sum_{j=1}^n [a_1^{j2} \int_{t_{j-1}}^{t_j} t^4 dt + a_2^{j2} \int_{t_{j-1}}^{t_j} t^2 dt + a_3^{j2} \int_{t_{j-1}}^{t_j} dt]. \end{aligned} \quad (3.2)$$

The set of numbers $\left\{ \int_{t_{j-1}}^{t_j} t^4 dt, \int_{t_{j-1}}^{t_j} t^2 dt, \int_{t_{j-1}}^{t_j} dt \right\}_{j=1}^n$ is a set of positive numbers that does not change for each $Cl(L_n)$ (that is for a fixed n), hence there exists at least one which is the smallest, denote it by α , thus we have further for inequality (3.2)

$$\begin{aligned} &\geq \alpha \sum_{j=1}^n (a_1^{j2} + a_2^{j2} + a_3^{j2}) = \alpha \sum_{j=1}^n \left(\sum_{i=1}^3 a_i^{j2} \right) \\ &= \alpha \|L_n\|_{Cl(L_n)}^2 \end{aligned}$$

i.e. $\lim \|L_n\|_{Cl(L_n)} \rightarrow \infty$ implies $\lim \|L_n\|_{L_2[0,1]} \rightarrow \infty$ and hence $\lim \|L_n\|_{Cl(L_n)} \rightarrow \infty$ implies that $\varphi(L_n) \rightarrow \infty$. (Note! the above could be proved using other methods usually used in such cases, however this one was chosen in order not to introduce additional terminology). On the other hand equality between the norms exists when both are equal to zero (which occurs when all the a_1^j , $i = 1, 2, 3$; $j = 1, 2, \dots, n$) and hence φ is bounded below on $Cl(L_n)$. Thus φ attains its absolute minimum on $Cl(L_n)$. Denote by w_n the minimizing element for φ in $Cl(L_n)$:

$$\varphi(w_n) = \min_{L_n \in Cl(L_n)} \varphi(L_n),$$

and recall the sequence of piecewise quadratic polynomials $\{\alpha_n\}_{n=1}^{\infty}$ that forms a minimizing sequence for φ in $L_2[0,1]$. For all n we have that

$$\varphi(w_n) \leq \varphi(\alpha_n) \quad (3.3)$$

but w_n is also an element of $L_2[0,1]$, it follows that

$$\varphi(x_0) \leq \varphi(w_n) \leq \varphi(\alpha_n) \rightarrow \varphi(x_0).$$

Hence $\{w_n\}_{n=1}^{\infty}$ forms a minimizing sequence for φ in $L_2[0,1]$.

Using Wang Shang-Wang result we conclude that w_n strongly converge to x_0 in the norm of $L_2[0,1]$.

If we write

$$w_n(t) = \sum_{j=1}^n (\hat{a}_1^j t^2 + \hat{a}_2^j t + \hat{a}_3^j) \chi_j(t)$$

then $\hat{a}_i^j$; $i = 1, 2, 3$; $j = 1, \dots, n$ are obtained by differentiating $\varphi(L_n)$ with respect to a_i^j : $i = 1, 2, 3$; $j = 1, \dots, n$ and solving the following system of $3n$ equations with $3n$ unknowns:

$$\frac{\partial \varphi(L_n)}{\partial a_i^j} = 0: \quad i = 1, 2, 3; \quad j = 1, \dots, n.$$

This system has a solution since the minimum point is also a critical one. Furthermore, the uniqueness of the solution follows from the operator F satisfying a Lipschitz condition with a proper constant.

To obtain an approximation to the solution of the Hammerstein equation we have yet to apply the appropriate linear operator (see chapters 1 and 2), which due to either continuity and the strong convergence of w_n to x_0 yields strong convergence to the solution.

In the following section we give an estimate of the rate of convergence of the above algorithm in certain cases and discuss some of the difficulties from the computational point of view. This algorithm yields in the case of our limited numerical experiments good results which will be presented in an appendix to follow this chapter. We should also remark that unlike finite element method for differential equations with boundary conditions, the construction of the piecewise polynomials did not require that they be continuous or smooth across the boundaries of the subregions.

§4. A rough estimate for the rate of convergence for the algorithms and some discussion on the numerical problems in utilizing these algorithms

The algorithms suggested in this chapter to approximate the critical point for the functional of chapters 1 and 2, and hence to obtain an approximate solution to the Hammerstein equation, need further investigation as to their practicality: how accurate are they, what difficulties occur in the numerical computation, etc? This work does not give a complete answer to all these questions. However, an attempt will be made to clarify some relevant points and by so doing partially answer some of these questions.

In chapters 1 and 2 we obtained an estimate of the error due to the use of the functionals suggested. For instance (see chapter 1 section 5) if we use the functional based on the (n) -th approximation to the square root of the linear operator in the Hammerstein then we can give an approximation to the error incurred due to actually solving a "neighbouring" equation to the original Hammerstein equation we wish to solve. Likewise we can do for the perturbed Hammerstein equation of chapter 2.

Our next step is to try to estimate the error or at least the rate of convergence of the algorithms suggested for the construction of minimizing sequences for these functionals. For this let us make use of the following theorem in Daniel [10] p.21, based on complementary variational principle:

Let ψ be a real-valued non-linear functional on a real Hilbert space H , and let u minimize ψ on the norm-closed convex set S . For each w in some given $S^* \subset H$, let a self-adjoint operator

P_w be defined, satisfying $\langle P_w h, h \rangle \geq a_w \langle h, h \rangle$ for all h in H , with $a_w > 0$. Suppose that ψ is twice continuously Frechet-differentiable on the set of points of the form $\lambda v + (1 - \lambda)w$ for $v \in S$ and $w \in S^*$, and λ in $[0, 1]$; and that

$$\langle [\psi''_{\lambda v + (1-\lambda)w} - P_w](v - w), (v - w) \rangle \geq 0$$

for all v in S , w in S^* and λ in $[0, 1]$. For each w in S^* , let v_w in S be defined as the (unique) point in S minimizing

$$\frac{1}{2} \langle P_w v_w, v_w \rangle + \langle \nabla \psi(w), v_w - w \rangle,$$

and let ψ^* be defined on S^* as

$$\psi^*(w) = \psi(w) + \frac{1}{2} \langle v_w - w, P_w(v_w - w) \rangle + \frac{1}{2} \langle v_w - w, \nabla \psi(w) \rangle.$$

If $\hat{u}$ is in S^* , then $\psi^*(w)$ is maximized by $w = \hat{u}$ and

$\psi^*(\hat{u}) = \psi(\hat{u})$. For all u in S and w in S^* we have the estimates:

$$\psi(u) - \psi^*(w) \geq \frac{1}{2} \langle P_w(\hat{u} - v_w), \hat{u} - v_w \rangle$$

and

$$\psi(u) - \psi^*(w) \geq \frac{1}{2} \langle P_{\hat{u}}(\hat{u} - u), \hat{u} - u \rangle.$$

We shall want to make use of these inequalities in particular the second one, since

$$\begin{aligned} \psi(u) - \psi(\hat{u}) &= \psi(u) - \psi^*(\hat{u}) \geq \frac{1}{2} \langle P_{\hat{u}}(\hat{u} - u), \hat{u} - u \rangle \\ &\geq \frac{1}{2} a_{\hat{u}} \langle \hat{u} - u, \hat{u} - u \rangle = \frac{1}{2} a_{\hat{u}} \|\hat{u} - u\|^2, \end{aligned} \quad (4.1)$$

so we shall see that if we can estimate the difference $\psi(u) - \psi(\hat{u})$ then in some cases we may be able to estimate the rate of convergence of an algorithms such as suggested in the preceding section.

In chapter 1, §4, an additional condition was given for the existence of a solution to the Hammerstein equation $x = B F x$: B is a linear positive definite and self-adjoint operator defined

on a real Hilbert space H and F is a potential operator satisfying $(DF(x,h),h) \leq a_1(h,h)$ where $a_1 \|B\| < 1$ if $a_1 > 0$. This condition implies that $D^2\varphi(x,h,h) \geq (h,h)(1 - a_1 \|B\|)$ if $a_1 > 0$ and $D^2\varphi(x,h,h) \geq (h,h)$ if $a_1 \leq 0$, so that by theorem S.10 in the supplement there exists a critical point for φ in H which is unique if F is Lipschitzian with the appropriate constant. We shall assume the following: a) that the above conditions hold true for the cases being treated, which in fact holds true in most of the common cases; b) that $D^2\varphi$ which is the second Gateaux derivative is also the second Frechet derivative which again holds true in most cases. We also remark that the above conditions imply the "usual" Vainberg conditions used in this thesis (see theorem 4.1 of chapter 1), and though they may be a bit stronger they are usually satisfied whenever the "usual" Vainberg conditions are satisfied.

To apply Daniel's theorem: Let φ be defined on $H = L_2[0,1]$, and let $\hat{u}$ minimize φ over the norm-closed convex set $S \equiv H$, i.e. we choose the convex set to be the whole space. Choose $S^* \equiv H$ and for all $w \in S^* \equiv H$ we take P_w to be the self adjoint operator μI , where I is the identity operator in $L_2[0,1]$ and $\mu > 0$ yet to be determined. Hence

$$\langle P_w h, h \rangle = \langle \mu I h, h \rangle = \mu \langle h, h \rangle \quad \text{for all } h \in H.$$

Since φ is twice Frechet differentiable on the whole space, it is particular twice differentiable on the set of points

$$\lambda v + (1 - \lambda)w; \quad v \in S = H, w \in S^* = H, \lambda \in [0,1].$$

Now

$$\begin{aligned} & \langle [\varphi''_{\lambda v + (1-\lambda)w} - \mu I](v - w), v - w \rangle \\ &= \langle [\varphi''_{\lambda v + (1-\lambda)w}](v - w), v - w \rangle - \mu \langle v - w, v - w \rangle \end{aligned}$$

which by the above assumption

$$\begin{aligned} & \geq (1 - a_1 \|B\|) \langle v - w, v - w \rangle - \mu \langle v - w, v - w \rangle \\ &= (1 - a_1 \|B\| - \mu) \langle v - w, v - w \rangle. \end{aligned}$$

Since $1 - a_1 \|B\| > 0$ if $a_1 > 0$, we can choose $\mu > 0$ such that $1 - a_1 \|B\| - \mu \geq 0$ and hence

$$\langle [\varphi''_{\lambda v + (1-\lambda)w} - \mu I](v - w), v - w \rangle \geq 0$$

for all $v \in S = H$, $w \in S^* = H$. Hence all the conditions of Daniel theorem are satisfied and we can construct the corresponding functional φ^* and show that it possesses the desired properties. Hence according to (4.1) we obtain the following estimate

$$\varphi(u) - \varphi(\bar{u}) \geq \frac{1}{2}\mu \|u - \bar{u}\|^2.$$

In order to demonstrate the way in which we may apply this result to the two kinds of functionals, let us take the general result ((in the sense that Vainberg's functional was used) in the last section, where it was proved that by minimizing φ over a class of piecewise quadratic polynomials we obtained a minimizing sequence that strongly converges to the critical point of φ . Indeed from inequality (3.3) we have that

$$\varphi(\mathcal{L}_n) - \varphi(\bar{u}) \geq \varphi(w_n) - \varphi(\bar{u})$$

where $\{\mathcal{L}_n\}_{n=1}^\infty$ is some sequence (or any sequence for that matter) of piecewise quadratic polynomials that forms a minimizing sequence

for φ and that strongly converges to the critical point and $\{w_n\}_{n=1}^{\infty}$ is the minimizing sequence obtained by the algorithm. By inequality (4.1) we have in particular for w_n that

$$\varphi(w_n) - \varphi(\hat{u}) \geq \frac{1}{2}\mu \|w_n - \hat{u}\|^2$$

but from (3.3) follows that

$$\varphi(\mathcal{L}_n) - \varphi(\hat{u}) \geq \varphi(w_n) - \varphi(\hat{u}) \geq \frac{1}{2}\mu \|w_n - \hat{u}\|^2$$

and hence

$$\varphi(\mathcal{L}_n) - \varphi(\hat{u}) \geq \frac{1}{2}\mu \|w_n - \hat{u}\|^2.$$

By Vainberg [36] p. 60 in every closed ball in $L_2[0,1]$, (due to the differentiability of φ) φ satisfies a Lagrange condition of the sort

$$|\varphi(x_1) - \varphi(x_2)| \leq M \|x_1 - x_2\|$$

where M is a constant that depends on the magnitude of the bound for $D\varphi$ on the closed ball which can be determined in certain cases.

Hence (since $\varphi(\mathcal{L}_n) - \varphi(\hat{u}) \geq 0$ and therefore $|\varphi(\mathcal{L}_n) - \varphi(\hat{u})| = \varphi(\mathcal{L}_n) - \varphi(\hat{u})$)

$$M \|\mathcal{L}_n - \hat{u}\| \geq \varphi(\mathcal{L}_n) - \varphi(\hat{u}) \geq \frac{1}{2}\mu \|w_n - \hat{u}\|^2.$$

Thus if we have any further knowledge about $\hat{u}$ such as $\hat{u}$ being an element of $C^k[0,1]$; $k = 0, 1, \dots$ then it is possible to construct a sequence of piecewise interpolation polynomials

$\{\mathcal{L}_n\}_{n=1}^{\infty}$ that will converge to $\hat{u}$ in some fashion such as $\mathcal{O}(h^1)$;

$l = 1, 2, \dots$ depending on k . But in general, all we can say, since we only know $\hat{u}$ to be continuous is that for every $\varepsilon > 0$ we can construct a sequence of piecewise quadratic polynomials

$\{\mathcal{L}_n\}_{n=1}^{\infty}$ such that

$$\|\hat{u} - \mathcal{L}_n\| \leq \varepsilon + M_{(\varepsilon)} \mathcal{O}(h^3) \quad (h = \frac{1}{n})$$

where $M_{(\varepsilon)} = M_{\frac{\varepsilon}{2}}$ of lemma 3.1 of this chapter. This does not give us the precise rate of convergence of the algorithm. To obtain a more precise result we should have a better idea of the dependency of $M_{(\varepsilon)}$ on ε . What we know is that on the one hand the dependency is determined by the particular sequence $\{\phi_k\} \subset C^\infty[0,1]$ such that $\phi_k \xrightarrow{s} \hat{u}$ in the L_2 -norm and on the other hand by the continuity properties of $\hat{u}$. It is hoped that in the future such better result will be obtained.

(Note that the actual numerical results will appear in an appendix to follow this chapter).

The algorithm based on the discretization method and the algorithm based on the denseness of the class of piecewise polynomials in $L_2[0,1]$ both are fairly simple particularly because one does not require that the element "shape functions" possess inter element smoothness or even continuity. That is, as was previously mentioned in section 3 of this chapter, in contrast to the usual applications of the finite element method. One does not have to impose further conditions on the elements. Now if we observe the functionals suggested in chapter 1 and 2:

$$(i) \quad \varphi_{n+1}(u) = (u, u) - 2f(A_{n+1}u); \quad u \in L_2[0,1]$$

$$(ii) \quad \tilde{\varphi}_\varepsilon(u) = (B_\varepsilon u, u) - 2f(B_\varepsilon u); \quad u \in L_2[0,1]$$

where A_{n+1} is the $(n+1)$ -th approximation to the square root of the linear operator B in the Hammerstein equation $x = Bfx$ and B_ε is a perturbation for $B: B_\varepsilon = B + \varepsilon I$, then we see that both functionals have as an argument an expression which involves an integration over the whole region, i.e. the interval $[0,1]$. For functional (ii) this is obvious and for functional (i) this follows since

A_{n+1} is a polynomial in the operator B . Hence, owing to the functional being an integral over the whole region of an integrand that includes as an argument an expression itself involving integration over the whole region, any reduction of the problem of minimizing the functional to that, for instance, of solving a system of non-linear equations yields full matrices. So that applying such methods as SOR in order to obtain an approximation to the solution (or the solution itself) does not give any real advantage over such methods as Gaussian elimination.

Another problem that requires a further clarification is the dependence of the stability of the numerical process to construct a minimizing sequence for the perturbed functional $\tilde{\varphi}_\varepsilon$ on the magnitude of ε . Clearly this greatly depends on the properties and behaviour of the linear operator B . The closer the operator B is to a positive bounded below operator the lesser we may expect the reduction of the minimization problem from an infinite dimensional problem to a finite dimensional one to be unstable for a small perturbation as the dimensionality is increased.

Appendix of some numerical results

We present here the numerical results of two problems. Both are nonlinear boundary value problems that were posed as nonlinear integral equations of Hammerstein type.

Problem 1:

$$\begin{cases} u'' = e^u \\ u(0) = u(1) = 0 \end{cases}$$

Applying Green's function we get the following nonlinear integral equation:

$$u(x) = \int_0^1 G(x,t) e^{u(t)} dt$$

where

$$G(x,t) = \begin{cases} t(x-1) & 0 \leq t \leq x \leq 1 \\ x(t-1) & 0 \leq x \leq t \leq 1. \end{cases}$$

This equation can be further written as

$$u(x) = \int_0^1 G^*(x,t) [-e^{u(t)}] dt$$

where

$$G^*(x,t) = \begin{cases} t(1-x) & 0 \leq t \leq x \leq 1 \\ x(1-t) & 0 \leq x \leq t \leq 1. \end{cases}$$

Thus we have a positive kernel defined on $[0,1] \times [0,1]$. Now $F(t, u(t)) = -e^{u(t)}$ and hence

$$\begin{aligned} 2f(u) &= 2 \int_0^1 \left[\int_0^{u(x)} -e^v dv \right] dx \\ &= -2 \int_0^1 e^{u(x)} dx, \end{aligned}$$

Since (upon calculating) $\|G^*\| = 0.106$, Vainberg's conditions are satisfied with any $a, \gamma > 0$ and the perturbation parameter, ξ , must only satisfy $(a_1 + \xi) 0.106 < 1$.

The functional we wish to minimize is then

$$\tilde{\varphi}_{\xi}(u) = (G^*u, u) + \xi(u, u) + 2 \int_0^1 \exp[(G^*u)(x) + \xi u(x)] dx.$$

The algorithm used here is the one based on the discretization method for which we have to halve the maximal perturbation parameter, and the discrete functionals have the form

$$\begin{aligned} \tilde{\varphi}_{h, \xi}(u_h) = & h \sum_{i=0}^n \left[h \sum_{j=0}^n G_{ij}^{*n} u_j^n \right] u_i^n + \xi h \sum_{i=0}^n u_i^{n2} \\ & + 2h \sum_{i=0}^n \exp \left[h \sum_{j=0}^n G_{ij}^{*n} u_j^n + \xi u_i^n \right]. \end{aligned}$$

This problem leads to a full matrix of $(n+1) \times (n+1)$.

In the following table the perturbation was reduced each time to a tenth of the one preceeding it, starting with $\xi = 0.01$. The exact results were taken from [5].

x	$\xi = 0.01$	$\xi = 0.001$	$\xi = 0.0001$	u(x)
0.0	-0.010842	-0.001948	-0.000986	0.000000
0.1	-0.050477	-0.043617	-0.041319	-0.041436
0.2	-0.088874	-0.698305	-0.080338	-0.073269
0.3	-0.122214	-0.093624	-0.092163	-0.095800
0.4	-0.147247	-0.113024	-0.101811	-0.109238
0.5	-0.159753	-0.125492	-0.116393	-0.113704
0.6	-0.154642	-0.119819	-0.113121	-0.109238
0.7	-0.125693	-0.105506	-0.103205	-0.095800
0.8	-0.061107	-0.087745	-0.067525	-0.073269
0.9	-0.043256	-0.035169	-0.417219	-0.041436
1.0	-0.000473	-0.000451	-0.000346	0.000000

Note: One has to bear in mind that in addition to the error due to the perturbation there is an additional error due to the "true" minimizing function approximated by trapezoidal quadrature rule and also the functional itself being approximated by such a quadrature rule.

Problem 2:

$$\begin{cases} u'' = -\sin u \\ u(0) = 0, u(3) = 2.7 \end{cases}$$

First, by defining

$$u(t) = v(t) + 0.9t$$

we get the following boundary value problem

$$\begin{cases} v''(t) = -\sin(v(t) + 0.9t) \\ v(0) = v(3) = 0 \end{cases}$$

and then applying Green's function to both sides of the equation we obtain

$$v(t) = -\int_0^3 G_1(x, t) \sin(v(t) + 0.9t) dt$$

$$G_1(x, t) = \begin{cases} t(x-3)/3 & 0 \leq t \leq x \leq 3 \\ x(t-3)/3 & 0 \leq x \leq t \leq 3 \end{cases}$$

which can be further written as

$$v(t) = \int_0^3 G_1^*(x, t) \sin(v(t) + 0.9t) dt$$

where

$$G_1^*(x, t) = \begin{cases} t(3-x)/3 & 0 \leq t \leq x \leq 3 \\ x(t-3)/3 & 0 \leq x \leq t \leq 3 \end{cases}$$

and thus we have a positive kernel defined on $[0, 3] \times [0, 3]$. The nonlinear part is then $F(t, v(t)) = \sin(v(t) + 0.9t)$, hence

$$\begin{aligned} 2f(v) &= 2 \int_0^3 \left[\int_0^{v(t)} \sin(w + 0.9t) dw \right] dt \\ &= -2 \int_0^3 \cos(v(t) + 0.9t) dt \end{aligned}$$

and since the cosine is a bounded function we may choose a_1 to be any positive number such that $a_1 \|G_1^*\| < 1$. Now $\|G_1^*\| = 0.95$ and depending on the choice of a_1 we choose the perturbation parameter such that the maximal value η satisfies $(a_1 + \eta) \times 0.95 < 1$. The functional we wish to minimize then is

$$\tilde{\varphi}_\varepsilon(v) = (G_1^* v, v) + \xi(v, v) + 2 \int_0^3 \cos((G_1^* v)(t) + v(t) + 0.9t) dt$$

For this problem we used the algorithm based on the densness of the piecewise polynomials in $L_2[0, 3]$, where the interval $[0, 3]$ was partitioned to twelve subintervals (in order to compare results with [4]) of equal length and a quadratic polynomial was defined on each one of them using the mid-point of the subintervals. The approximating functionals were as follows:

$$\begin{aligned} \tilde{\varphi}_\varepsilon(L_{12}) &= (G_1^* L_{12}, L_{12}) + \xi(L_{12}, L_{12}) \\ &\quad + 2 \int_0^3 \cos((G_1^* L_{12})(t) + L_{12}(t) + 0.9t) dt \end{aligned}$$

where

$$L_{12}(t) = \sum_{j=1}^{12} (a_1^j t^2 + a_2^j t + a_3^j) \chi_j(t)$$

and

$$\chi_j(t) = \begin{cases} 1 & t \in [0.25(j-1), 0.25j] \\ 0 & t \notin [0.25(j-1), 0.25j] \end{cases}.$$

This problem leads to a full matrix of 36×36 .

In the following table the results appear only in the nodes that appear in [4] and the end nodes - which gave exact answers at the boundaries to the boundary value problem with zero at both boundaries and hence upon addition of $0.9t$ gave the exact values at the boundary.

x	$\xi = 0.01$	$\xi = 0.001$	$\xi = 0.0001$	Results from [4]
0.00000	...	...	...	...
0.75	1.47372	1.37871	1.36011	1.34255
1.50	2.36888	2.26115	2.22978	2.19699
2.25	2.730919	2.63711	2.62921	2.59880
3.00	...	...	...	...

SUPPLEMENT

1. Continuity of operators

Much of the material from functional analysis used in the thesis can be found in the book by Vainberg. For the readers convenience a summary of the definitions and theorems quoted in the thesis is given in this section. Most of the theorems are quoted without proof; these proofs can be found in many of the text books on functional analysis.

Let E_1 and E_2 be two Banach spaces and let $d(E_1)$ be the domain of definition of the operator F on E_1 and $r(E_2)$ be its range in E_2 . We shall also use f as a functional defined on E_1 into R' .

Definition S1: The operator $F:d(E_1) \rightarrow r(E_2)$ is said to be continuous at a point $x_0 \in d(E_1)$ if for any sequence $\{x_n\} \in d(E_1)$ that converges strongly to x_0 (in the norm of E_1) the sequence $\{F(x_n)\}$ strongly converges to $F(x_0)$ (in the norm of E_2), i.e.

$$\lim_{n \rightarrow \infty} \|F(x_n) - F(x_0)\| = 0.$$

Remark S1: Most of the definitions given here could be equivalently defined in terms of neighbourhoods.

Remark S2: The operator $F:d(E_1) \rightarrow r(E_2)$ is said to be continuous on a set $\omega \in d(E_1)$ if it is continuous at every point $x \in \omega$.

Definition S2: A functional $f:d(E_1) \rightarrow R'$ is said to be weakly continuous at a point $x_0 \in d(E_1)$ if for every sequence $\{x_n\}$ that converges weakly to x_0

$$\lim_{n \rightarrow \infty} f(x_n) = f(x_0).$$

Definition S3: $F:d(E_1) \rightarrow r(E_2)$ is said to be compact on a set $\omega \in d(E_1)$ if it transforms every bounded set in ω into a compact set in $r(E_2)$.

Definition S4: The operator $F:d(E_1) \rightarrow r(E_2)$ is said to be completely continuous on a set $\omega \in d(E_1)$ if it is continuous and compact on ω .

Remark S3: In spaces having weakly compact spheres, weak continuity of an operator is a stronger condition than strong continuity. In finite dimensional spaces both forms of continuity are equivalent.

Remark S4: It is important to note the following theorem:
In order that the unit sphere of a separable Banach space be compact, it is necessary and sufficient that the space be reflexive.

Due to the linearity of the space weak compactness of the unit sphere implies the weak compactness of the whole space. The kind of Hilbert spaces we deal with are separable and reflexive, hence they are weakly compact.

2. The Gateaux and Frechet differentials and derivatives

Definition S4: If for some $x \in d(E_1)$ and every $h \in E_1$,

$$\lim_{t \rightarrow 0} \frac{F(x+th) - F(x)}{t} = VF(x, h)$$

the limit exists, then the operator $VF(x, h)$ is called the Gateaux differential or the weak differential of the operator F .

Theorem S1: If the Gateaux differential of a functional f exists at each point of some convex set $\omega \in d(E_1)$ then for any point $x, x+h \in \omega$ the Lagrange formula:

$$f(x+h) - f(x) = Vf(x+\tau h, h), \quad (0 < \tau < 1)$$

is true.

Theorem S2: If the operator $F: d(E_1) \rightarrow r(E_2)$ has a Gateaux differential at each point of some convex set $\omega \in d(E_1)$ then for

every $x, x+h \in \omega$ the corresponding Lagrange formula is:

$$\begin{aligned} & (F(x+h) - F(x), e) \\ &= (VF(x+\tau h, h), e), \quad (0 < \tau < 1) \end{aligned}$$

where e is an arbitrary element of norm one in E_2^* and $\tau \in (0, 1)$ depends on the element e .

Since $e \in E_2^*$ is an arbitrary unit vector in E_2^* , from the theory of conjugate spaces, for fixed x and h , it can be chosen so that

$$\begin{aligned} & |(F(x+h) - F(x), e)| \\ &= \|F(x+h) - F(x)\|. \end{aligned}$$

From the relationship $(F(x+h) - F(x), e) = (VF(x+\tau h, h), e)$ follows that

$$\begin{aligned} & \|F(x+h) - F(x)\| \\ &= |(F(x+h) - F(x), e)| \end{aligned}$$

$$= |(VF(x+\tau h, h), e)|$$

$$\leq \|VF(x+\tau h, h)\| \|e\|$$

$$= \|VF(x+\tau h, h)\| ,$$

$$\text{i.e. } \|F(x+h) - F(x)\| \leq \|VF(x+\tau h, h)\| .$$

The following theorem will deal with the question of the linearity of the Gateaux differential:

Theorem S3: Suppose that the following conditions are satisfied:

1) The Gateaux differential $VF(x, h)$ of the operator F exists in some neighbourhood $U(x_0)$ of x_0 and is continuous in h at point x_0 .

2) $VF(x_0, h)$ is continuous in h at the point $h=0$.

Then the Gateaux differential $VF(x_0, h)$ is a linear operator in h , and is denoted by

$$VF(x_0, h) = DF(x_0, h).$$

There is a stronger type of differential, the Frechet differential:

Definition S5: If at a point $x \in d(E_1)$

$$F(x+h) - F(x) = u(x,h) + \omega(x,h)$$

where $u(x,h)$ is a linear operator for every $h \in d(E_1)$ and

$$\lim_{\|h\| \rightarrow 0} \frac{\|\omega(x,h)\|}{\|h\|} = 0$$

then $u(x,h)$ is called the Frechet differential at a point $x \in d(E_1)$; $\omega(x,h)$ is called the remainder of the Frechet differential.

The existence of a Frechet differential at a point $x_0 \in d(E_1)$ implies the existence of a Gateaux differential at x_0 and that they are equal. The reverse is not always true.

If the Gateaux differential exists and is linear then it is a member of the space of all bounded linear operators from E_1 into E_2 , $B(E_1, E_2)$, which is in itself a Banach space. We call this operator the Gateaux derivative and denote it by $F'(x)$, thus we can write $DF(x,h) = F'(x)h$. In the same way we can define the Frechet derivative.

Theorem S4: If the Gateaux derivative F' exists in some neighbourhood $U(x_0)$ of x_0 and is continuous at x_0 then the Frechet

differential exists and is equal to the Gateaux differential. In other words a continuous Gateaux derivative is a Frechet derivative.

3. Gradients and potential operators

Definition S6: Suppose that a functional f has a linear Gateaux differential $Df(x,h)$ on some set $\omega \subset d(E_1)$. As we have seen $Df(x,h)$ is a linear functional on the domain of definition

$$Df(x,h) = f'(x)h \quad \forall h \in d(E_1)$$

denote $f'(x)$ by $F(x)$, then $F(x)$ is a mapping from ω into E_1^* .

The operator $F(x)$ defined by

$$\lim_{t \rightarrow 0} \left[\frac{f(x+th) - f(x)}{t} \right] = \langle F(x), h \rangle$$

for any $h \in d(E_1)$ is called the gradient of the functional f :

$$F = \text{grad } f = \nabla f .$$

Remark S5: If f has a Frechet differential then the gradient is called a strong gradient, in exact analogy to previous terminology.

Example S1: Let $f(x) = \|x\|$ in a real Hilbert space have a gradient at every point $x \neq 0$. To find the gradient:

$$f(x+th) - f(x) = \|x+th\| - \|x\|$$

$$= \frac{\|x+th\|^2 - \|x\|^2}{\|x+th\| + \|x\|}$$

$$= \frac{2(x, th) + \|th\|^2}{\|x+th\| + \|x\|}$$

hence

$$\lim_{t \rightarrow 0} \frac{f(x+th) - f(x)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{1}{t} \frac{2(x, th) + \|th\|^2}{(\|x+th\| + \|x\|)}$$

$$= \frac{(x, h)}{\|x\|} = \left(\frac{x}{\|x\|}, h \right)$$

From which follows that $\text{grad } \|x\| = \frac{x}{\|x\|}$.

Example S2: Another example of which use is made in the thesis:

$$f(x) = (x, x) = \|x\|^2 :$$

$$f(x+th) - f(x)$$

$$= \|x+th\|^2 - \|x\|^2$$

$$= 2(x, th) + \|th\|^2$$

and hence

$$\lim_{t \rightarrow 0} \frac{f(x+th) - f(x)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{2(x, th) + \|th\|^2}{t}$$

$$= 2(x, h) = (2x, h),$$

hence $\text{grad } f(x) = 2x$.

Definition S7: An operator $F: E \rightarrow E^*$ is said to be a potential operator on some set $\omega \in H$, if there exists a functional f such that

$$\text{grad } f(x) = F(x) \quad \forall x \in \omega.$$

Remark S6: If an operator F acts on a pair of elements such that on each one it acts linearly then it is said to be a bilinear operator. If its values are real it is said to be a bilinear functional.

Definition S8: A bilinear operator F (or a functional) is called symmetric if

$$F(h_1, h_2) = F(h_2, h_1).$$

Definition S9: If $D^2F(x_0, h_1; h_2) = D^2F(x_0, h_2; h_1)$ then F is called a symmetric operator.

The conditions for potentialness of an operator turn out to be closely related to the symmetry of the second Gateaux differential, as we should expect from the corresponding situation in finite dimensional spaces where the Hessian is a symmetric matrix.

Theorem S5: Suppose that the following conditions are fulfilled:

- 1) F is an operator from E into its conjugate space E^* .
- 2) F has a linear Gateaux differential $DF(x, h)$ at every point of the ball $\rho(x, x_0) = \|x - x_0\| < r$.
- 3) The functional $(DF(x, h_1), h_2)$ is continuous for every $x \in \rho(x, x_0)$.

Then in order that the operator F be a potential operator in the ball $\rho(x_1, x_0)$ it is necessary and sufficient that the bilinear functional be symmetric for every $x \in \rho(x, x_0)$, i.e.

$$(DF(x, h_1), h_2)$$

$$= (DF(x, h_2), h_1) \text{ for all } h_1, h_2.$$

One can show that if the functional f whose gradient F is required to assume a value f_0 at some point x_0 then it is unique. In other words there is just one functional f such that $F = \nabla f$ and $f(x_0) = f_0$, and the functional is given by

$$f(x) = f_0 + \int_0^1 (F(x_0 + t(x-x_0)), x-x_0) dt.$$

As we know from elementary calculus if a function $g(x)$ has a derivative $g'(x)$ and a is a constant then $[g(ax)]' = ag'(ax)$. Much the same occurs for a product of operators. Let A be a linear operator and F be an operator possessing either types of differentials, then $D[AF(x, h)] = ADF(x, h)$ and hence the

potentialness of the product AF depends on fulfillment of a condition of the form:

$$\begin{aligned} & (ADF(x, h_1), h_2) \\ &= (ADF(x, h_2), h_1). \end{aligned}$$

If now F is a potential operator then by theorem S5

$$\begin{aligned} & (ADF(x, h_2), h_1) \\ &= (DF(x, h_2), A^*h_1) \\ &= (DF(x, A^*h_1), h_2) \end{aligned}$$

from which one concludes that a necessary condition for the product AF to be potential is that

$$\begin{aligned} & (ADF(x, h_2), h_1) \\ &= (DF(x, A^*h_1), h_2); \quad \forall h_1, h_2 \in H \end{aligned}$$

or more concisely that $ADF(x, h) = DF(x, A^*h)$, $\forall h \in H$. Using this result it follows at once that a necessary and sufficient condition for potentialness of a linear operator A in a Hilbert space is that it be a self-adjoint operator, i.e. $A=A^*$.

Having introduced the notion of gradients of potential operators, we proceed one step further and establish a method for solving functional and operator equations. The weak compactness of Hilbert spaces allows us to obtain quite general results with regards to functionals having an extremal point in some weakly closed and bounded spheres.

4. Weak continuity, extremal and critical points.

Definition S10: A functional f is said to be weakly lower semi-continuous at a point x_0 if for any sequence $\{x_n\}$ which converges weakly to x_0

$$f(x_0) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

It is possible at this stage to give necessary and sufficient conditions for a functional to be weakly semi-continuous. However, first we shall indicate the use of this notion and then come back to these conditions.

Definition S11: Let f be a real functional defined on a Banach space E . A point $x_0 \in E$ is called an extremal point of f if there exists a neighbourhood $U(x_0)$ of x_0 on which: $f(x) \leq f(x_0)$ or $f(x) \geq f(x_0) \quad \forall x \in U(x_0)$.

Theorem S6: Let the functional f be defined on a region ω of a Banach space E , and let x_0 be an interior point of ω , at which f has a linear Gateaux differential. Then in order that the point x_0 be extremal it is necessary that it be critical, i.e. $\nabla f(x_0) = 0$.

This theorem suggests that in some cases when we wish to solve an operator equation we may seek to construct a functional whose gradient operator is that which appears in the operator equation we wish to solve and then try and establish that at some point in the domain or space on which our operator is defined, the functional has an extremal point.

Theorem S7: If the functional f is weakly lower semi-continuous on a bounded weakly closed region ω in a Banach space having weakly compact spheres, then f has a lower bound, which it achieves on ω .

Obviously any Hilbert space is such a space possessing weakly compact spheres and hence any weakly lower semi-continuous functional attains a minimum on every bounded weakly closed set in the space. We saw in theorem S6 that an extremal point is a critical point if it is an inner point. Therefore if we can show that the functional

satisfies some sort of a growth condition that depends on the norm of the elements in the space and also that it has an extremal point in some neighbourhood such that as we move away from this point the functional is not going to decrease and by so doing that point is going to be an inner extremal point and hence a critical point. For this reason we introduce the t-property.

Definition S12: If among those sets ω which are weakly closed, there exists at least one, $\bar{\omega}_0$ such that $f(x) > f(x_0)$ for every $x \in \partial\omega_0$, where x_0 is some point in ω_0 then we say that f has a t-property.

Theorem S8: If a weakly lower semi-continuous functional f , defined on a space E having weakly compact spheres, has a t-property, then f has at least one critical point in the space, provided f is weakly differentiable.

With the aid of the t-property we may also prove the following results:

Theorem S9: Suppose that the functional f , defined on a Banach space E which has weakly compact spheres, has first and second linear Gateaux differentials, and the latter satisfies:

$$D^2 f(x, h, h) \geq \|h\| \gamma(\|h\|)$$

where $\gamma(t)$ is a non-negative continuous function defined for

$t > 0$ such that $\lim_{t \rightarrow \infty} \gamma(t) = \infty$. Then there exists a point at which f has a relative minimum.

To complete this short supplement we shall give conditions for weak lower semi-continuity, which we use in our analysis of the Hammerstein equation.

Theorem S10: Suppose the following conditions are satisfied:

- 1) The functional f has first and second linear Gateaux differentials $Df(x, h)$ and $D^2f(x, h, k)$ for all x in some ball $D(\|x\| \leq r)$; h and k being arbitrary in E .
- 2) For every $x \in D(\|x\| \leq r)$ and $h \in E$

$$D^2f(x, h, h) > 0.$$

Then f is a weakly lower semi-continuous functional in D .

Another theorem which gives sufficient conditions for weak lower semi-continuity through weak continuity and is generally easier to verify is the following:

Theorem S11: From the compactness of $\text{grad } f$ in the ball

$D(\|x\| \leq r)$ follows the weak continuity of f on D .

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