

VORTEX-WAVE INTERACTIONS AND EXACT COHERENT STRUCTURES IN SHEAR FLOWS

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Declaration of Originality

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Joseph Lee Maestri
January 4, 2016

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“Confidence is what you have before you understand the problem.”

Woody Allen

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Abstract

Recent studies of transition to turbulence in linearly stable shear flows have been dominated by finding, and characterising, so-called exact coherent structures which exist as either equilibrium or travelling wave solutions of the governing Navier-Stokes equations. These structures have been shown to act as either edge states between laminar and turbulent flow or become associated with attractors for turbulent flows, making the quest to understand these structures equally as mathematically interesting as it is industrially relevant.

In this thesis, vortex-wave interaction theory, an asymptotic approach based on the assumption of large Reynolds number, is used as a means of finding such structures in plane Couette flow. A new, iterative, numerical method will be developed to solve the governing interaction equations that will allow us to circumvent many of the difficulties in calculating such structures (in particular at high Reynolds numbers) and perform analyses into the state-space of solutions, as well as explore their stability properties. A number of previously found exact coherent structures will be shown to be vortex-wave interaction states, including the so-called upper branch sinuous mode and the class of solutions called mirror-symmetric modes.

A number of receptivity type problems will also be considered. Namely, the use of periodic blowing/suction on the channel walls and the introduction of variations in the wall shape. The effect of these two techniques on the skin friction drag and the wave amplitude will be discussed in the context of whether they are beneficial in terms of laminar flow control. Further, it will be shown that the use of periodic blowing/suction leads to a new class of synchronous mirror-symmetric modes and variations in the wall shape lead to a new class of inhomogeneous vortex-wave interaction states.

Contents

List of Tables	9
List of Figures	10
1. Introduction	17
1.1. Transition to Turbulence	17
1.2. Linear Stability of Shear Flows	19
1.3. Nonlinear Stability Theory	22
1.4. Exact Coherent Structures	24
1.5. Vortex-Wave Interaction Theory	29
1.6. Layout of Thesis	30
2. Vortex-Wave Interaction Theory for Shear Flows	32
2.1. The Governing Interaction Equations	32
2.1.1. The High Reynolds Number Asymptotic Theory . . .	33
2.1.2. Classification of VWI States	36
2.1.3. Symmetries of the VWI Equations	37
2.2. Numerical Scheme for Solving the Interaction Equations . . .	38
2.2.1. The Vortex System: Collocation Method	39
2.2.2. The Vortex System: Boundary Integral Method	42
2.2.3. The Wave System: Compact Finite-Difference Method	48
2.2.4. Iteration Procedure on Interaction Equations	55
3. Sinuous Modes	58
3.1. Accuracy of the Numerical Scheme	58
3.2. Exploration of State Space	60
3.2.1. The First Branch	60
3.2.2. The Second Branch	64
3.3. Linear Stability Properties of Sinuous Streaks	67
3.3.1. Stability of the First Branch of VWI States	67

3.3.2. Stability of the Second Branch of VWI States	68
4. Mirror-Symmetric Modes	72
4.1. Accuracy of the Numerical Scheme	72
4.2. Exploration of State Space	75
4.2.1. The First Sub-2 Branch	75
4.2.2. The First Sub-4 Branch	86
4.3. Linear Stability Properties of Mirror-Symmetric VWI Streaks	90
5. Receptivity Theory for VWI States	93
5.1. The Effect of Spanwise Wall Blowing/Suction	94
5.1.1. Flow Modification	94
5.1.2. Sinuous Modes	95
5.1.3. Mirror-Symmetric Modes	95
5.2. The Effect of Wall Imperfections	110
5.2.1. Flow Modification	110
5.2.2. A Class of Inhomogeneous VWI States	114
6. Conclusions and Outlook	121
6.1. Summary of Research	121
6.2. Future Work	122
Bibliography	124
A. Derivation of the VWI Jump Conditions	133
A.1. Local Critical Layer Analysis	133
A.2. Conversion from Local to Cartesian Coordinates	136
B. Derivation of Some Useful Green's Function	138
B.1. The free-space Green's Function	138
B.2. Periodic Array of Point Forces in a Channel	139

List of Tables

3.1.	Convergence of wave amplitude ρ with the number of grid points in y , N_y , and the number of Fourier modes in z , N_z , using the numerical approach described in section 2.2.4 with the collocation approach 2.2.1 when calculating the first branch sinuous VWI state with $(\alpha, \beta; R) = (0.6, 2.0; 16000)$	59
3.2.	Convergence of wave amplitude ρ with regularisation Reynolds number R used in solving the wave system (2.68) with the numerical approach described in section 2.2.4 when calculating the first branch sinuous VWI state with $(\alpha, \beta) = (0.6, 2.0)$. . .	59
4.1.	Convergence of wave amplitude ρ as the number of grid points in y , N_y , and the number of Fourier modes in z , N_z , is increased when calculating the mirror-symmetric state with $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$ by imposing the jump conditions (2.9) directly using the (a) collocation approach outlined in section 2.2.1, and (b) boundary integral approach outlined in section 2.2.2.	73
4.2.	Convergence of the wave amplitude ρ as the number of grid points in y , N_y , and the number of Fourier modes in z , N_z , is increased when calculating the mirror-symmetric state with $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$ by imposing the hybrid body forcing term (2.14) using the collocation approach outlined in section 2.2.1.	74
4.3.	Convergence of wave amplitude ρ with regularisation Reynolds number R used in solving the wave system (2.68), with the numerical approach of section 2.2.4, for the mirror-symmetric VWI state with $(\alpha, \beta) = (0.6, 2.0)$	74

List of Figures

1.1.	Cartoon of the experiment of Reynolds (1883) on the transition of turbulence in a pipe. Water flows down a long pipe and ink is introduced at the centerline. On the right are three drawings showing a) the dye filament in laminar flow, b) the rapid mixing of the dye and the water when the Reynolds number is larger than 2000 and c) details the vortical structures present in the turbulent flow, visualised with a spark. .	18
1.2.	Bifurcation diagrams for the (a) supercritical pitchfork bifurcation, $\dot{x} = rx - x^3$ where for $r < 0$ there is one stable equilibrium at $x = 0$ and for $r > 0$ there is an unstable equilibrium at $x = 0$ and two stable equilibria at $x = \pm\sqrt{r}$, and (b) subcritical pitchfork bifurcation, $\dot{x} = rx + x^3$ where for $r < 0$ there is a stable equilibrium at $x = 0$ and two unstable equilibria at $x = \pm\sqrt{-r}$ and for $r > 0$ the equilibrium at $x = 0$ is unstable.	24
1.3.	Schematic highlighting the tripartite interaction of the self-sustaining process described by Waleffe (1997).	28
2.1.	Schematic of the iterative numerical procedure used to solve the vortex-wave interaction equations.	57
3.1.	Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $ P(y, z) $, and (f) the hybrid body forcing term (2.14) associated with a sinuous VWI state on the first branch at $(\alpha, \beta; R) = (0.7, 2.0; 16000)$. The solid bold curve represents the critical layer location.	61

- 3.2. Variation of the wave amplitude ρ against the streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$ for the sinuous VWI state. The inset figure shows the variation on a log-log scale when $\alpha \rightarrow 0$ 62
- 3.3. The stress jump conditions, (a) $J(z)$ and (b) $K(z)$, given by (2.9) for a sinuous VWI state at $\alpha = 0.2, 0.6, 1.0$ and $\beta = 2.0$. 63
- 3.4. Variation of the wall skin friction $\mathcal{S}$ against the streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$ for the sinuous VWI state. 64
- 3.5. Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $|P(y, z)|$, and (f) the hybrid body forcing term (2.14) associated with a sinuous VWI state on the second branch at $(\alpha, \beta; R) = (0.7, 2.0; 50000)$. The solid bold curve represents the critical layer location. 66
- 3.6. Variation of the linear growth rate $\hat{\sigma} = \hat{\alpha}c_i$ against the streamwise wavenumber $\hat{\alpha}$ of the most unstable eigenmode of the streak $u(y, z)$ associated with the VWI state at $(\alpha, \beta; R) = (0.7, 2.0; 16000)$ 69
- 3.7. Variation of the linear growth rate $\hat{\sigma} = \hat{\alpha}c_i$ against the Reynolds number R of the most unstable eigenmode of the streak $u(y, z)$ associated with the VWI state at $(\alpha, \beta; R) = (0.7, 2.0; 16000)$ at $\hat{\alpha} = 0.35$ 69
- 3.8. The leading (most unstable) eigenvalues for the streak profile associated with a VWI state on the (a) first branch at $(\alpha, \beta; R) = (0.7, 2.0, 16000)$ and (b) the second branch at $(\alpha, \beta; R) = (0.7, 2.0, 50000)$ 70

- 3.9. (a) Variation of the linear growth rate $\hat{\sigma} = \hat{\alpha}c_i$ against the streamwise wavenumber $\hat{\alpha}$ for a number of eigenmodes of the streak $u(y, z)$ associated with the VWI state at $(\alpha, \beta; R) = (0.7, 2.0, 50000)$. $S1, S2, S3$ correspond to stationary eigenmodes and $T1, T2$ correspond to pairs of travelling eigenmodes (with opposite wavespeeds). (b) Close-up of the region of transition from $T1 \rightarrow (S2, S3)$. (c) Close-up of the region of transition from $(S1, S2) \rightarrow T2$ 71
- 4.1. Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $|P(y, z)|$, and (f) the hybrid body forcing term (2.14) associated with a sub-2 mirror-symmetric VWI state at $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$. The solid bold curve represents the critical layer location. 76
- 4.2. Variation of the wave amplitude ρ against the streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$. The inset figure shows the variation on a log-log scale when $\alpha \rightarrow 0$. 77
- 4.3. (a) Variation of the wave amplitude ρ against the spanwise wavenumber β for fixed streamwise wavenumber $\alpha = 0.76$. (b) Variation on a log-log scale when $\beta \rightarrow 0$. (c) Variation on a log-log scale when $\beta \rightarrow \infty$ 78
- 4.4. The stress jump condition $J(z)$ given by (2.9) at different values of the streamwise wavenumber α for a sub-2 mirror-symmetric VWI states with $\beta = 2.0$ 79
- 4.5. Equispaced contours of the roll velocity field for a sub-2 mirror-symmetric vortex-wave interaction states in plane Couette flow. Left, wall-normal roll velocity field; right, spanwise roll velocity field. (a) $(\alpha, \beta) = (0.2, 1.0)$, (b) $(\alpha, \beta) = (0.8, 1.0)$, (c) $(\alpha, \beta) = (1.4, 1.0)$ 82

4.6.	Equispaced contours of the roll velocity field for a sub-2 mirror-symmetric vortex-wave interaction states in plane Couette flow. Left, wall-normal roll velocity field; right, spanwise roll velocity field. (a) $(\alpha, \beta) = (0.76, 2.0)$, (b) $(\alpha, \beta) = (0.76, 4.0)$, (c) $(\alpha, \beta) = (0.76, 8.0)$	83
4.7.	(a) Variation of the wave amplitude ρ against the streamwise wavenumber α for large values of the spanwise wavenumber β . (b) Collapse of the large- β nonlinear eigenrelation data in agreement with the relationship (4.1).	84
4.8.	Variation of the skin friction drag $\mathcal{S}$ against the (a) streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$ and (b) spanwise wavenumber β for fixed streamwise wavenumber $\alpha = 0.76$ for a sub-2 mirror-symmetric VWI state.	85
4.9.	Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $ P(y, z) $, and (f) the hybrid body forcing term (2.14) associated with a sub-4 mirror-symmetric VWI state at $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$. The solid bold curve represents the critical layer location.	88
4.10.	Variation of the wave amplitude ρ against the streamwise wavenumber α for a sub-4 mirror-symmetric VWI state with fixed spanwise wavenumber $\beta = 2.0$	89
4.11.	Variation of the skin friction drag $\mathcal{S}$ against the streamwise wavenumber α for a sub-4 mirror-symmetric VWI state with fixed spanwise wavenumber $\beta = 2.0$	89
4.12.	Variation of the linear growth rate $\hat{\sigma} = \hat{\alpha}c_i$ against the streamwise wavenumber $\hat{\alpha}$ of the most unstable eigenmode of the streak $u(y, z)$ associated with the VWI state on the first sub-2 branch at $(\alpha, \beta; R) = (0.6, 2.0, 10^6)$	91
4.13.	Variation of the linear growth rate $\hat{\sigma} = \hat{\alpha}c_i$ against the Reynolds number R of the most unstable mode of the streak profile $u(y, z)$ associated with the mirror-symmetric VWI state on the first sub-2 branch at $(\alpha, \beta) = (0.6, 2.0)$ at $\hat{\alpha} = 0.35$	91

4.14. The leading (most unstable) eigenvalues for the streak profile associated with a mirror-symmetric VWI state at $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$ on the (a) first sub-2 branch and the (b) first sub-4 branch.	92
5.1. Variation of (a) the wave amplitude ρ and (b) the wall skin friction $\mathcal{S}$ against the amplitude $\mathcal{A}$ of the wall suction (5.1) with $\gamma = \beta$ for a sinuous VWI state at $(\alpha, \beta) = (0.6, 2.0)$. . .	96
5.2. Equispaced contours of the roll velocity fields for sinuous VWI states at $(\alpha, \beta) = (0.6, 2.0)$ in plane Couette flow with wall suction of the form (5.1) with $\gamma = \beta$. Left, wall-normal roll velocity field $v(y, z)$; right, spanwise roll velocity field $w(y, z)$. (a) $\mathcal{A} = -0.5$, (b) $\mathcal{A} = -0.25$, (c) $\mathcal{A} = 0.5$. The solid bold curve represents the critical layer.	97
5.3. Variation of (a) the wave amplitude ρ and (b) the wall skin friction $\mathcal{S}$ against the amplitude $\mathcal{A}$ of the wall suction (5.2) with $\gamma = \beta$ for a mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$	99
5.4. Equispaced contours of the streak and roll velocity fields for principal mirror-symmetric VWI states at $(\alpha, \beta) = (0.8, 2.0)$ in plane Couette flow with wall suction of the form (5.2) with $\gamma = \beta$. Left, $\mathcal{A} = -2.0$; right, $\mathcal{A} = +2.0$. (a) Streak profile $u(y, z)$, (b) wall-normal roll velocity field $v(y, z)$, (c) spanwise roll velocity field $w(y, z)$. The solid bold curve represents the critical layer.	100
5.5. Relationship between the wall suction amplitude $\mathcal{A}$ and the wall skin-friction $\mathcal{S}$ at different values of the streamwise wavenumber α for a mirror-symmetric VWI state at $\beta = 2.0$	101
5.6. Variation of (a) the wave amplitude ρ and (b) the wall drag $\mathcal{S}$ against the amplitude $\mathcal{A}$ of the wall suction (5.2) with $\gamma = 2\beta$ for a mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$	102

- 5.7. Equispaced contours of the streak and roll velocity fields for principal mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$ in plane Couette flow with wall suction of the form (5.2) with $\gamma = 2\beta$. Left, $\mathcal{A} = -2.0$; right, $\mathcal{A} = +2.0$. (a) streak profile $u(y, z)$, (b) wall-normal roll velocity field $v(y, z)$, (c) spanwise roll velocity field $w(y, z)$. The solid bold curve represents the critical layer. 103
- 5.8. Variation of (a) the wave amplitude ρ and (b) the wall drag $\mathcal{S}$ against the amplitude $\mathcal{A}$ of the wall suction (5.2) with $\gamma = \frac{1}{2}\beta$ for a mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$ 105
- 5.9. Equispaced contours of the streak and roll velocity fields for principal mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$ in plane Couette flow with wall suction of the form (5.2) with $\gamma = \frac{1}{2}\beta$. Left, $\mathcal{A} = -2.0$; right, $\mathcal{A} = +2.0$. (a) streak profile $u(y, z)$, (b) wall-normal roll velocity field $v(y, z)$, (c) spanwise roll velocity field $w(y, z)$. The solid bold curve represents the critical layer. 106
- 5.10. Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $|P(y, z)|$, and (f) the hybrid body forcing term (2.14) associated with a synchronous mirror-symmetric VWI state at $(\alpha, \beta; R) = (0.6, 2.0, 10^6)$. The solid bold curve represents the critical layer location. 108
- 5.11. Variation of the wave amplitude ρ against the streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$ for the synchronous mirror-symmetric VWI state. 109
- 5.12. Variation of the skin friction drag $\mathcal{S}$ against the streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$ for the synchronous and subharmonic mirror-symmetric VWI state. . 109

- 5.13. Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $|P(y, z)|$, and (f) the hybrid body forcing term (2.14) associated with a mirror-symmetric VWI state at $(\alpha, \beta; R) = (0.8, 2.0; 10^4)$ with inhomogeneous wall conditions of the form (5.20) with $\mathcal{A} = 25$ and $\gamma = \beta$. The solid bold curve represents the critical layer location. 116
- 5.14. Relationship between the wall shape amplitude, $\mathcal{A}$, and the maximum value in the stress jump condition (2.9), $\mathcal{J}$, for different values of the streamwise wavenumber $\alpha = 0.2, 0.4, 0.8$, fixed $\beta = 2.0$ and $R = 10^4$. The variations in wall shape are given by (5.25) with $\gamma = \beta$ 117
- 5.15. Relationship between the wall shape amplitude, $\mathcal{A}$, and the maximum value in the stress jump condition (2.9), $\mathcal{J}$, for different values of the spanwise wavenumber $\beta = 2.0, 4.0, 8.0$ and fixed $\alpha = 0.8$ and $R = 10^4$. The variations in wall shape are given by (5.25) with $\gamma = \beta$ 118
- 5.16. Equispaced contours of the roll velocity field for inhomogeneous vortex-wave interaction states in plane Couette flow with wall variations of the form (5.25) with $\mathcal{A}$ chosen such that $\mathcal{J} = 20$. Left, wall-normal roll velocity field $v(y, z)$; right, spanwise roll velocity field $w(y, z)$. (a) $(\alpha, \beta) = (0.8, 2.0)$, (b) $(\alpha, \beta) = (0.8, 4.0)$, (c) $(\alpha, \beta) = (0.8, 8.0)$ 119
- 5.17. Relationship between the wall shape amplitude $\mathcal{A}$ and the maximum value in the stress jump condition (2.9), $\mathcal{J}$, for different values of the regularisation Reynolds number $R = 5000, 10000, 20000, 40000$ and fixed $(\alpha, \beta) = (0.8, 2.0)$ 120

1. Introduction

1.1. Transition to Turbulence

The study of fluid mechanics dates back to at least the days of Ancient Greece when Archimedes of Syracuse investigated fluid statics and formulated his famous law now commonly known as Archimedes' principle. Since then, fluid mechanics has been studied by a number of prolific and influential scientists including Leonardo da Vinci, Issac Newton and Leonhard Euler amongst many others.

One of the most active areas of current research in fluid mechanics is that of transition of a laminar flow to a turbulent flow. Turbulence is characterised by irregular, time-dependent, spatially complex looking flow and is generally observed in fast moving fluids. Although the equations that describe the motion of fluids have been known for more than a century, the consensus is that there remains much to be understood about the phenomena of transition to turbulence.

Understanding the nature of turbulence is of great practical importance since turbulent flows cause a massive increase in drag compared to their laminar flow counterparts. In the design of vehicles, engineers want to prevent the onset of turbulence using laminar flow control techniques in order to reduce fuel consumption providing both economic and environmental benefits. Conversely, engineers may also be interested in triggering turbulent flow in order to prevent the occurrence of stall in flight at low speeds, or to promote the rapid mixing of air and fuel in combustion engines.

One of the earliest demonstrations of transition to turbulence was the celebrated series of experiments on the instability of flow in a pipe by Reynolds (1883), see figure 1.1. He demonstrated that the laminar flow broke down when the dimensionless parameter

$$Re = UL/\nu, \tag{1.1}$$

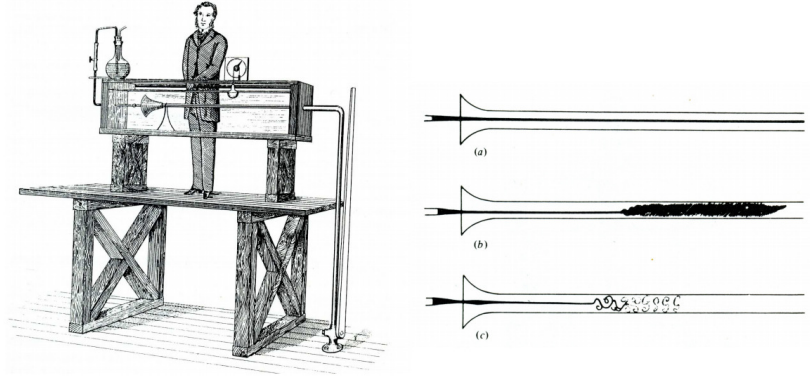


Figure 1.1. Cartoon of the experiment of Reynolds (1883) on the transition of turbulence in a pipe. Water flows down a long pipe and ink is introduced at the centerline. On the right are three drawings showing a) the dye filament in laminar flow, b) the rapid mixing of the dye and the water when the Reynolds number is larger than 2000 and c) details the vortical structures present in the turbulent flow, visualised with a spark.

subsequently called the Reynolds number, exceeded a critical value. Here U and L are characteristic scales of the velocity and length respectively, such as the bulk/mean velocity and the diameter of the pipe, and ν is the kinematic viscosity of the fluid (typically 10^{-5} m²/s for air and 10^{-6} m²/s for water). The Reynolds number is the ratio between inertial forces to viscous forces and consequently quantifies the relative importance of these two types of forces for given flow conditions. Whilst pipe flow will typically be in a turbulent state when the Reynolds number is larger than 2000, Reynolds noted that the exact value at which transition occurred could vary considerably depending on how settled the initial flow was. Now, more than over a century after the work of Reynolds, the nature of transition to turbulence in pipe flow is still a major research interest in the fluid dynamics community, see the recent studies provided by Faisst & Eckhardt (2003) and Wedin & Kerswell (2004) for example.

Although most fluids are compressible, the study of turbulence can be simplified if we consider idealised incompressible flow. This is equivalent to assuming that the speed of sound in our medium, $c = \sqrt{(\partial p / \partial \rho)_S}$ (≈ 340 m/s in air and 1500 m/s in water), is much greater than any velocity in the fluid. The dimensional Navier-Stokes equations for constant density flow in

the absence of body forces may be written as

$$\rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad (1.2)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the fluid velocity field and $p(\mathbf{x}, t)$ is the kinematic pressure field. The kinematic viscosity $\nu = \mu/\rho$ is the same as in the definition of the Reynolds number (1.1) where ρ and μ are the density and the dynamic viscosity of the fluid respectively. For incompressible flow the pressure is purely mechanical, not thermodynamic, and is in fact a non-local, non-linear term for the Navier-Stokes equations which can be seen after re-arranging the above equations to find

$$\nabla^2 p = -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}). \quad (1.3)$$

The Navier-Stokes equations can be non-dimensionalised by scaling every variable by a characteristic scale. If the velocity components are scaled by U , the independent variables by L , the time by L/U and the pressure by ρU^2 , the non-dimensional form of the Navier-Stokes equations is

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0. \quad (1.4)$$

Here, Re is the Reynolds number defined earlier in (1.1). This means that for two flows with differing values of U , L and ν which combine to give the same value of the Reynolds number have equivalent non-dimensional solutions, such flows are said to be dynamically similar.

1.2. Linear Stability of Shear Flows

The earliest attempts at understanding transition in shear flows involved using ideas from linear stability theory. Rayleigh (1880) considered the linear inviscid problem of an incompressible fluid confined between two parallel boundaries and derived an ordinary differential equation describing the disturbance evolution as a two-dimensional travelling wave. Using this equation Rayleigh showed that for an inviscid parallel flow to be unstable it must have an inflection point in the velocity profile.

The mistakenly used assumption that viscosity has a stabilising effect due to the diffusion in the perturbation energy was later resolved by Prandtl

(1921) when he deduced that the instability of non-inflectional shear flows, such as plane Poiseuille flow, was caused by viscosity. The viscous analogue of Rayleigh's equation was derived independently by Orr (1907) and Sommerfeld (1908).

To conduct linear stability analysis the flow is represented in the form of a parallel base flow and a small amplitude perturbation

$$(u, v, w, p) = (u_b(y), 0, 0, p_b) + (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}), \quad (1.5)$$

where u , v , and w are the components of the velocity in the x , y and z directions respectively and p is the pressure. The base flow $(u_b, 0, 0, p_b)$ is obtained as a solution of the linearised Navier-Stokes equations. After substituting expression (1.5) into the non-dimensional Navier-Stokes equations (1.4), neglecting terms which are quadratic and re-arranging, the following equation is found

$$\left(\left(\frac{\partial}{\partial t} + u_b \frac{\partial}{\partial x} \right) \nabla^2 - u_b'' \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^4 \right) \tilde{v} = 0. \quad (1.6)$$

This equation admits a solution in the form of a travelling wave

$$\tilde{v} = \bar{v}(y) e^{i(\alpha x + \beta z - \omega t)} + c.c., \quad (1.7)$$

where α and β are wavenumbers in the streamwise and spanwise directions respectively, ω is the frequency and $c.c.$ denotes the complex conjugate. The above equation for $\tilde{v}$ then reduces to the Orr-Sommerfeld equation

$$(u_b - c) \left(\frac{d^2}{dy^2} - \gamma^2 \right) v(y) - u_b'' v(y) = \frac{1}{i\alpha Re} \left(\frac{d^2}{dy^2} - \gamma^2 \right)^2 \bar{v}(y), \quad (1.8)$$

where $\gamma^2 = \alpha^2 + \beta^2$ and $\omega = \alpha c$. In the limit $Re \rightarrow \infty$ the Orr-Sommerfeld equation reduces to the Rayleigh equation which is singular when $u_b(y_c) = c$ giving what is termed a critical layer at $y = y_c$. This singularity, located at the critical layer, is smoothed out by viscosity in the Orr-Sommerfeld equation.

In equation (1.8), α and γ have been taken to be real and c to be complex resulting in the temporal instability problem. For the spatial instability problem, ω and β are taken to be real whilst α is complex. In either case, the

Orr-Sommerfeld equation coupled with homogeneous boundary conditions is an eigenvalue problem which can be reduced to solving a dispersion relation of the form

$$f(\alpha, \beta, \omega; Re) = 0. \quad (1.9)$$

For the temporal instability problem, the frequency ω is a complex eigenvalue with ω_i representing the temporal growth rate. For given values of α , β and Re there exists a number of eigensolutions often called primary modes. If a primary mode has a positive growth rate then the disturbance will grow in time and the flow is said to be linearly unstable.

Further developments to linear stability theory were obtained by Heisenberg (1924), who discovered instability in plane Poiseuille flow, and by Tollmien (1929) and Schlichting (1933) who solved the Orr-Sommerfeld equation for the Blasius boundary layer flow. In order to solve the equation they used the so-called parallel flow approximation to neglect the streamwise varying properties of the Blasius base flow. Squire (1933) deduced from the Orr-Sommerfeld equation that any three-dimensional instability wave can be associated with a two-dimensional instability wave at a lower Reynolds number, therefore the critical Reynolds number, Re_c , below which no instability can be found must correspond to a two-dimensional mode (Squire's Theorem). The first finite amplitude solutions were found by Meksyn & Stuart (1951) by considering the periodic motion of a viscous fluid between two parallel walls. Objections to the relevance of linear stability theory persisted, see Taylor (1938), until the experiments of Liepmann (1943) and Schubauer & Skramstad (1948) verified the existence of the Tollmien-Schlichting instability. The waves, however, were observed to grow spatially rather than the theoretical calculations which were modelled temporally. The first accurate numerical computations of the critical Reynolds number were obtained by Orszag (1971), who found $Re_c = 5772.2$ for plane Poiseuille flow, and Jordinson (1970), who found $Re_c = 520$ for the Blasius boundary layer.

Since the linear stability of a base flow can be easily calculated it has been used in the past to create transition prediction tools. The e^N method developed independently by Smith & Gamberoni (1956) and Van Ingen (1956) uses the integrated growth rates of linear theory to predict the location of onset of turbulence and is still the de-facto used tool in industrial transition

prediction despite heavy criticism by Reed *et al.* (1996).

There are a number of reasons why developing transition tools based on linear stability theory is problematic. The most alarming is the fact that many dubious simplifications have been used in the past, often in an ad-hoc and heuristic manner, in order to employ linear stability theory. The most drastic of which is the parallel flow approximation where the base flow is assumed to depend on one spatial coordinate.

1.3. Nonlinear Stability Theory

As the two-dimensional instability wave grows the flow becomes more susceptible to the effects of non-linearity and three-dimensional instabilities which evolve on a faster timescale. When the amplitude of an instability wave exceeds some critical value, non-linear effects can vary the evolution of the disturbance in a number of ways. For the case of a single unstable mode the effect of non-linearity can either be stabilising, leading to saturation of the wave, or destabilising, leading to an amplification of the growth rate. If a number of modes exist they may interact causing certain modes with smaller linear growth rates to grow faster than what would be expected from linear theory.

Landau (1944) predicted that the evolution of the disturbance amplitude of a two-dimensional wave was governed by the amplitude itself, leading to a differential equation for the amplitude in time. Extending the work of Meksyn & Stuart (1951), Stuart (1960) and Watson (1960) rederived this equation in the context of weakly non-linear stability theory, valid for disturbances close to the linear neutral stability curve. Stuart noted that since the disturbance is close to the critical value of the control parameter (Reynolds number) the disturbance is slowly evolving. By introducing an amplitude function, $A(\tau)$, dependent on a slow time variable, τ , and expanding the Navier-Stokes equations in powers of the amplitude function a different set of equations need to be solved at successively higher powers of A . These are found to admit valid solutions provided that A satisfies the following equation

$$\frac{dA}{d\tau} = \sigma A - \lambda A|A|^2 + \dots \quad (1.10)$$

Here, σ is the linear growth rate and $\lambda = l_r + il_i$ is the first Landau coefficient which is a complex number determined from the linear eigenfunction and its adjoint. The cubic non-linearity arises through an interaction of the $O(A)$ fundamental mode with the $O(A^2)$ mean-flow distortion and first harmonic components which are themselves driven by the fundamental. Higher order nonlinear terms can be included which represent higher order modal interactions, see Craik (1988) for more details.

If the Stuart-Landau equation is truncated to the first nonlinear term it has the solution

$$|A|^2 = \frac{\sigma K e^{2\sigma\tau}}{1 + l_r K e^{2\sigma\tau}}, \quad (1.11)$$

where K is an arbitrary constant.

For a linearly unstable disturbance, $\sigma > 0$, the corresponding nonlinear disturbance tends to an equilibrium value of

$$|A_e|^2 = \frac{\sigma}{l_r}, \quad (1.12)$$

as $\tau \rightarrow \infty$ provided that $l_r > 0$. Therefore, linearly unstable disturbances can be stabilised by the effects of nonlinearity. This type of stabilising behaviour is known as a supercritical bifurcation. If $l_r < 0$ there are no equilibrium solutions since $|A|^2$ must be positive and so the linearly unstable disturbance has unbounded growth.

For a linearly stable disturbance, $\sigma < 0$, the equilibrium amplitude is again given by (1.12) but provided that $l_r < 0$. This equilibrium is unstable and acts as a threshold amplitude above which disturbances grow unboundedly. Disturbances below this threshold decay to zero as $\tau \rightarrow \infty$, according to the linear growth rate $|A|^2 \sim e^{-2|\sigma|\tau}$. This type of behaviour where a linearly stable disturbance can become unstable is known as a subcritical bifurcation. If $l_r > 0$ there are no equilibrium solutions and so nonlinear disturbances are damped along with linear disturbances.

The behaviour described in terms of the Stuart-Landau equation is exactly the same as that found when considering the canonical pitchfork bifurcation. The behaviour of a supercritical and a subcritical bifurcation can be conveniently visualised in a bifurcation diagram as shown in figure 1.2.

The unbounded growth of the finite amplitude solution is a consequence

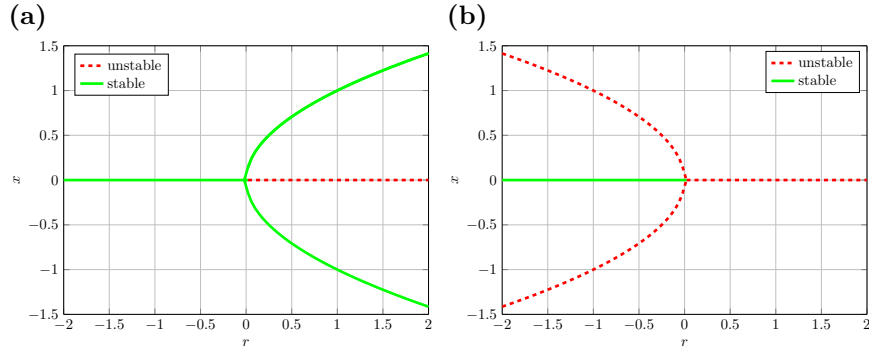


Figure 1.2. Bifurcation diagrams for the (a) supercritical pitchfork bifurcation, $\dot{x} = rx - x^3$ where for $r < 0$ there is one stable equilibrium at $x = 0$ and for $r > 0$ there is an unstable equilibrium at $x = 0$ and two stable equilibria at $x = \pm\sqrt{r}$, and (b) subcritical pitchfork bifurcation, $\dot{x} = rx + x^3$ where for $r < 0$ there is a stable equilibrium at $x = 0$ and two unstable equilibria at $x = \pm\sqrt{-r}$ and for $r > 0$ the equilibrium at $x = 0$ is unstable.

of truncating the Stuart-Landau equation at cubic order. As the amplitude increases, higher order terms become larger and should be included. In principle, including all the terms in the Stuart-Landau equation should result in an exact solution to the Navier-Stokes equations which is two-dimensional and periodic in the streamwise direction. In practice this method is not viable and so Herbert (1976) approached the problem by using spectral methods and arc-length continuation. Using this method both stable and unstable equilibria could be found, demonstrating that there exists a surface in (Re, α, A) -space which separates regions of stability and instability. The neutral stability curve from linear theory can be found on this surface at $A = 0$. For plane Poiseuille flow it was found that the critical Reynolds number reduced from the linear value of 5772.2 to approximately 2900.

1.4. Exact Coherent Structures

A major challenge in the latter half of the 20th century was to bridge the link between the growth of large-wavelength, two-dimensional primary modes from linear stability theory and the three-dimensional, small-scale motion associated with turbulent flow. The majority of theoretical studies into this stage of transition were guided by the hot-wire experiments of Klebanoff *et al.* (1962) who investigated the rapid amplification of a three-

dimensional disturbance by introducing a strong two-dimensional perturbation to the boundary layer of a flat plate. They discovered that the instability arose once the two-dimensional Tollmien-Schlichting wave reached a critical value of the free-stream velocity. The instability took the form of alternating regions of low- and high-speed fluid orientated in the streamwise direction, so-called streaks, due to linear advection through spanwise orientated counter-rotating vortices with the same streamwise periodicity as the fundamental Tollmien-Schlichting wave. The spanwise spacing between the streaks is best expressed in wall units, which are based on the wall shear stress $\tau_w = \mu \partial u / \partial y|_{y=0}$, where $\mu = \nu \rho$ is the dynamic viscosity of the fluid. If we define the length scale $l^+ = \nu / \sqrt{\tau_w / \rho}$ then the spacing is universally observed to be $\sim 100l^+$, see Kline *et al.* (1967). These investigations revealed that at least for wall-bounded turbulent flows, there exist regions in which the flow is highly organised, called coherent structures, where the fluid is more strongly correlated than in fully fledged turbulence.

Early theoretical attempts at understanding the observations made by Klebanoff *et al.* (1962) were conducted by using low-order perturbation analysis of Orr-Sommerfeld modes to various degrees of success. The most notable of which was the resonant triad model due to Craik (1971) which exploited resonance between a planar wave with frequency f_0 together with a pair of oblique waves with frequency $\frac{1}{2}f_0$. It was found that a powerful resonant interaction, in which the oblique waves were strongly amplified whilst the planar wave showed little deviation from linear theory, occurred when the spanwise wavelengths of all three waves were adjusted to give an equal phase speed. Despite heavy criticism, due to the fact that valley-peak splitting is not associated with subharmonic waves, interest remained due to the experiments of Kachanov & Levchenko (1984) who discovered a type of Tollmien-Schlichting wave deformation in which the valley-peak pattern aligned itself in a staggered pattern with a streamwise periodicity twice the Tollmien-Schlichting wavelength, indicating the presence of a subharmonic mode. However, the fact that Nayfeh & Bozatli (1979) found subharmonic resonance at amplitudes of 29%, which far exceeds observed values, strains the validity of low-order perturbation analysis.

Numerical computations by Orszag & Patera (1983) revealed that three-dimensional growth rate to be exponential in time, indicating a linear mechanism, which led Herbert (1981) to attribute peak-valley splitting to the

parametric excitation of a periodic flow, and not from the spanwise differential amplification of a Tollmien-Schlichting wave. This new understanding led to the use of Floquet theory to understand the stability of a base flow that has been modulated by a finite-amplitude Tollmien-Schlichting wave. The well-known mathematical properties of Floquet systems were used to identify classes of solutions whilst numerical methods were used to provide quantitative details to great success, see Herbert (1988).

For all its merit, linear stability theory (and the subsequent analyses that are dependent upon it) completely fails to predict the instability of a number of flows which have been experimentally observed to be unstable and undergo transition. This class of shear flows, which includes plane Couette flow, plane Poiseuille flow and Hagen-Poiseuille flow, undergo transition in a process seemingly unrelated to the local stability properties of the low Reynolds number base flow solution. This discrepancy is all the more alarming given that plane Couette flow and Hagen-Poiseuille flow are both truly parallel and the Orr-Sommerfeld equation is exact, yet both of these flows are believed to be linearly stable against infinitesimal perturbations, see Garg & Rouleau (1972).

Current thinking views transition for these types of flows as an issue of finding and characterising other fully three-dimensional solutions to the Navier-Stokes equations within the same flow configuration. These solutions can take the form of time-independent states (i.e., equilibria) or time-periodic states (e.g., travelling waves) and are called exact coherent structures (ECS) due to their remarkable resemblance to coherent structures found in experiments. This type of transition to turbulence is sometimes labelled ‘subcritical transition’ since these structures appear abruptly (in contrast to ‘supercritical transition’ where the flow undergoes a sequence of supercritical bifurcations) and is analogous to discussion on the Stuart-Landau equation and pitchfork bifurcations in section 1.3.

The existence of such solutions has now been demonstrated for a number of different wall-bounded shear flows. For plane Couette flow, Nagata (1990) found steady solutions at $Re = 100$ compared to the transitional value of $Re \approx 350$ observed experimentally by Tillmark & Alfredsson (1992). For plane Poiseuille flow, Waleffe (2003) found travelling wave solutions at $Re = 977$ compared to the transitional value of $Re \approx 2200$ observed experimentally by Keefe *et al.* (1992).

Nagata (1990) is credited for finding the first equilibrium solution in plane Couette flow by using the homotopy method that utilises bifurcation analysis and numerical continuation techniques. This approach relies on the existence of a non-linear solution in a related problem, such as one which has additional forces or different boundary conditions, in which a linear instability usually exists. A solution to the original problem is then obtained by numerically continuing the solution by using a Newton-Raphson iteration method. Since the existence of a branch in which to continue the solution from a bifurcation point cannot be guaranteed *a priori*, the method is not a robust means of finding equilibrium solutions.

Waleffe (1995, 1998) alleviated these issues by revealing the physical mechanism behind the structures found in turbulent flows. This physical mechanism, the so-called self-sustaining process (SSP), provided the mechanistic underpinning in which to develop the homotopy method into a robust means of finding equilibrium solutions. The central idea behind the SSP was the tripartite interaction of the form:

1. Streamwise rolls sustain streaks by the redistribution of momentum to create spanwise fluctuations.
2. Streaks suffer a wake-like instability due to spanwise inflections that lead to streamwise undulation.
3. Non-linear self-interaction of that streamwise undulation directly regenerates the streamwise rolls.

An instructive pictorial description of the SSP was given by Waleffe (1997) which is reproduced in figure 1.3. The SSP provided an explanation to how structures appearing in turbulent flows could maintain themselves against viscous decay. Waleffe exploited the mechanism to find equilibrium solutions by using an artificial body forcing term to mimic the instability of the streak in order to generate a bifurcation point from which a solution could be smoothly traced back to the zero-forcing problem. Waleffe (2003) also showed that equilibria captured some statistical features of turbulent flow including, at the lowest Reynolds numbers, streak spacing of $100l^+$ as observed by Kline *et al.* (1967) in experiments.

Gibson *et al.* (2008) used a different approach to finding and characterising equilibrium solutions by treating the fully-resolved Navier-Stokes

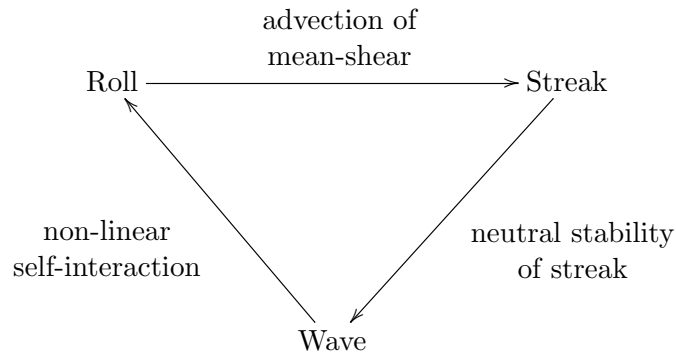


Figure 1.3. Schematic highlighting the tripartite interaction of the self-sustaining process described by Waleffe (1997).

equations as a high dimensional, in principle infinite dimensional, dynamical system. By using methods from dynamical systems theory, invariant solutions and the state space of the system could be found. Here, the state space is the space of all admissible velocity fields, subject to the appropriate boundary conditions, and so includes the laminar profile, turbulent flow profiles and all previously found equilibrium solutions. Using this approach an extensive database of equilibrium solutions has been catalogued, see Gibson (2012) for details.

The terminology ‘lower branch’ and ‘upper branch’ is used to refer to equilibrium solutions that arise from the lower or upper part of the amplitude-Reynolds number curve respectively following the genesis of the solutions from a saddle-node bifurcation. Direct numerical simulations have suggested that these two classes of solutions play two fundamental roles. The lower branch solutions have been shown by Itano & Toh (2001) and Skufca *et al.* (2006) to act as edge-states separating disturbances that evolve into turbulence or return to the laminar state. The upper branch solutions have been shown by Gibson *et al.* (2008) and Kreilos & Eckhardt (2012) to eventually become associated with attractors for turbulent flows.

ECS have also been found in other flows. Faisst & Eckhardt (2003) and Wedin & Kerswell (2004) found families of travelling-wave solutions in pipe flow by applying the continuation strategy developed by Waleffe. The high speed streaks associated with the travelling wave solutions have been observed in experiments using stereoscopic particle image velocimetry by Hof

et al. (2004) to remarkable agreement.

1.5. Vortex-Wave Interaction Theory

Around the same time as Nagata (1990) found the first equilibrium solution for plane Couette flow, Hall & Smith (1988, 1989, 1990, 1991) laid down a formal asymptotic framework for what was described as vortex-wave interaction theory (VWI). The framework was, in some sense, quite general in that it described both inviscid Rayleigh and viscous Tollmien-Schlichting waves interacting with streamwise vortices. The key process identified was that tiny waves could drive the rolls which themselves drive the streaks which are unstable to the wave. Further, the vortex-wave interaction theories describe states where a streak is driven by a wave field that satisfies a linear eigenvalue partial differential equation with coefficients that depend on the streak, which itself is driven by a roll forced by the wave field. Subsequent asymptotic studies of the VWI equations have been conducted by Bennett *et al.* (1991), Walton & Smith (1992), Brown *et al.* (1993), Blackaby (1994) and Timoshin & Smith (1997).

Motivated by the numerical results of Wang *et al.* (2007), Hall & Sherwin (2010) applied the VWI framework to plane Couette flow and demonstrated that the high Reynolds number limit of the lower branch equilibrium solutions were simply vortex-Rayleigh wave interaction states which could be found by numerical solution of the VWI equations. Further, it was found that VWI states agree well to Reynolds numbers all the way down to where the nonlinear solutions begin to exist, suggesting that VWI theory may provide a viable means to finding ECS. Blackburn *et al.* (2013) extended this work further by implementing a hybrid method to solve the equations and examining a range of spanwise wavenumbers showing that as the spanwise wavenumber β increases the states deform into canonical states localised away from any boundaries satisfying the Kolmogorov 5/3 law.

At long streamwise wavelengths of size comparable to the Reynolds number, that is $\alpha \sim R^{-1}$, the VWI theory fails and the states become long-wavelength asymptotic structures as described by Deguchi *et al.* (2013). These structures are found to become spatially localised in the streamwise direction with an $O(1)$ streamwise length scale excluded from the long wavelength theory. This distinguished limit provides a minimum box length, or

critical streamwise wavenumber, for a given high Reynolds number necessary for the emergence of localised structures. Further, the localised solutions were found to look remarkably similar to computationally generated or experimentally observed turbulent spots suggesting that these solutions could play a similar role to previously found equilibrium solutions, which act as edge-states or turbulent attractors, but for turbulent spots.

Deguchi & Hall (2014*c*) performed an in-depth study of the relationship between nonlinear equilibrium solutions of the full Navier-Stokes equations and the high-Reynolds number asymptotic VWI states. It was shown that as the Reynolds number is increased, all nonlinear equilibrium solutions having $O(1)$ wavenumbers converge to VWI states. Further, the question of whether states existed beyond the turning point $\alpha = \alpha_c$ was resolved. In fact, an upper branch of solutions was found that were quite distinct from the previously found lower branch states with the main distinction being the fact that the critical layer, where $u(y, z) = c$, is now multivalued and so the new states are associated with instabilities of jet-like structures.

The VWI description of ECS was further extended by the examination of structures in the asymptotic suction boundary layer flow (ASBL) by Deguchi & Hall (2014*b*). These structures, called freestream coherent structures (FCS), are quite distinct from previously found ECS since it involves a wave, roll and streak interacting in a layer located in the freestream. This structure is convected by almost the freestream speed and is located at distance $O(\ln R)$ away from the wall. Deguchi & Hall (2015) examined these structures in a growing boundary layer and showed that they could only exist over an interval of finite length and led to flow structures reminiscent of turbulent boundary layer numerical simulations.

Deguchi & Hall (2014*a*) demonstrated how previously found ECS and FCS for plane Couette flow and ASBL flows respectively could be locally embedded in arbitrary shear flows. In particular, ECS were shown to exist in unbounded flow domains and FCS in a Burgers vortex sheet, thus demonstrating the universality of the theories and their widespread applicability.

1.6. Layout of Thesis

This thesis is organised as follows: chapter 2 builds the mathematical and numerical framework which is used in chapters 3–5, where a number of

results are presented. Some concluding remarks are presented in chapter 6.

Chapter 2 presents the mathematical framework used throughout the rest of the thesis. In particular, we review the high Reynolds number asymptotic theory of vortex-wave interaction by Hall & Smith (1991) and Hall & Sherwin (2010) before discussing a new numerical technique to solve the vortex-wave interaction equations.

In chapter 3 and 4 we use the numerical technique to calculate VWI states that belong to the sinuous and mirror-symmetric class of solutions respectively. The class of sinuous solutions have been computed in the past by a number of authors and so can be used to verify the numerical technique and provide a benchmark for our computations. The class of mirror-symmetric modes have not been documented in the context of vortex-wave interaction theory in the past and so we present an in-depth study.

In chapter 5 we explore two drag reduction techniques, namely the effect of periodic suction/blowing on the channel walls and the effect of spanwise variations in the channel walls. In either case, results are presented which suggest that there are suitable methods to reduce the wall skin friction of VWI states and alter the basin of attraction for turbulent flow in a channel.

Finally, in chapter 6 we make some concluding remarks and discussions on the work of this thesis.

2. Vortex-Wave Interaction Theory for Shear Flows

In this chapter we focus on building the asymptotic and numerical framework for which studies in subsequent chapters are built upon. We first review the work on vortex-wave interaction theory by Hall & Sherwin (2010) before discussing numerical techniques to solve the vortex-wave interaction equations.

2.1. The Governing Interaction Equations

In the usual notation, the non-dimensional momentum and continuity equations for an unsteady viscous fluid of constant density are

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} = -\nabla \tilde{p} + \frac{1}{R} \nabla^2 \tilde{\mathbf{u}}, \quad \nabla \cdot \tilde{\mathbf{u}} = 0, \quad (2.1)$$

where R is a suitably scaled Reynolds number. In the usual convention we call the x , y and z -coordinates the streamwise, wall-normal and spanwise directions respectively. The corresponding coordinates of the velocity field $\tilde{\mathbf{u}}$ are denoted $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$.

One of the simplest solutions of the above equations is for the case of plane Couette flow where the fluid is driven by two parallel plates moving in opposite directions. In this case, the Reynolds number is based upon the maximum flow velocity and the channel half-height. The flow field is found by solving the above equations subject to the no-slip boundary conditions

$$\tilde{\mathbf{u}} = (\pm 1, 0, 0), \quad y = \pm 1, \quad (2.2)$$

so that plane Couette flow, $\tilde{\mathbf{u}} = (y, 0, 0)$, is a possible solution.

2.1.1. The High Reynolds Number Asymptotic Theory

Since we are interested in other, non-trivial, solutions we follow the high Reynolds number asymptotic theory of Hall & Smith (1991) and note that equations (2.1) admit stationary VWI solutions of the form

$$\begin{aligned}\tilde{\mathbf{u}} &= [u(y, z), R^{-1}v(y, z), R^{-1}w(y, z)] + \dots \\ &\quad + \rho R^{-7/6}[U(y, z), V(y, z), W(y, z)]e^{i\alpha(x-ct)} + c.c.,\end{aligned}\quad (2.3a)$$

$$\tilde{p} = R^{-2}p(y, z) + \rho R^{-7/6}P(y, z)e^{i\alpha(x-ct)} + c.c., \quad (2.3b)$$

where u is the streak flow, $[v, w, p]$ is the two-dimensional roll flow and $[U, V, W, P]$ is the three-dimensional wave with amplitude ρ and complex wavespeed c . The streak, roll and wave are taken to be periodic in $z \in [-\pi/\beta, \pi/\beta]$ with wavelength $L_z = 2\pi/\beta$. The wave is also periodic in $x \in [0, 2\pi/\alpha]$ with wavelength $L_x = 2\pi/\alpha$.

Inserting (2.3) into (2.1) and neglecting higher harmonics of the wave, the momentum and continuity equations can be decomposed into those for the $O(1)$ two-dimensional streak,

$$vu_y + wu_z = u_{yy} + u_{zz}, \quad (2.4)$$

the $O(R^{-1})$ two-dimensional roll,

$$vv_y + ww_z = -p_y + v_{yy} + v_{zz}, \quad (2.5a)$$

$$vw_y + ww_z = -p_z + w_{yy} + w_{zz}, \quad (2.5b)$$

$$v_y + w_z = 0, \quad (2.5c)$$

and the $O(R^{-7/6})$ three-dimensional wave,

$$i\alpha(u - c)U + u_yV + u_zW = -i\alpha P, \quad (2.6a)$$

$$i\alpha(u - c)V = -P_y, \quad (2.6b)$$

$$i\alpha(u - c)W = -P_z, \quad (2.6c)$$

$$i\alpha U + V_y + W_z = 0, \quad (2.6d)$$

which can be re-written conveniently as the pressure form of the generalised

Rayleigh's equation

$$P_{yy} + P_{zz} - \alpha^2 P - \frac{2u_y P_y}{u - c} - \frac{2u_z P_z}{u - c} = 0. \quad (2.7)$$

This equation must be solved subject to the normal derivative of P vanishing at any rigid boundary. Since the wave pressure equation is homogeneous we choose to normalise the pressure eigenfunction by

$$\int_A |P|^2 dA = A \quad (2.8)$$

where A denotes the domain of the solution over one period in the spanwise direction, so $A = 2L_z$. The normalisation of the pressure eigenfunction is the reason behind the introduction of the wave amplitude ρ and whilst the choice in normalisation is in fact arbitrary, (2.8) is the same normalisation used by Hall & Sherwin (2010) and so provides a means to directly compare our results.

The roll and streak equations, often referred to as the vortex system, are characteristic of equations found for centrifugal instabilities in the small gap and thin boundary layer limit. In this case however, due to the absence of any forcing through wall curvature we would anticipate that the solution to the roll equations (2.5) is simply $v = w = p = 0$ since the effective Reynolds number is unity and the unperturbed state is stable.

The wave system provides the forcing to drive the roll flow in a manner analagous to steady streaming processes, see Stuart (1966) or Riley (1967) for example. The wave pressure equation (2.7) becomes singular at $y = f(z)$ where $u(y, z) = c$ and is smoothed out by a thin viscous critical layer of thickness $O(R^{-1/3})$ resulting from the balance between $i\alpha(u - c) = i\alpha(\mathbf{x} - \mathbf{x}_c) \cdot \nabla u$ and $R^{-1}\nabla^2$. In this region the equations for the roll and wave need to be solved in the manner described by Hall & Smith (1991) and Hall & Sherwin (2010), an in-depth discussion of this analysis is provided in appendix A. It can be shown that if the size of the wave is $O(R^{-7/6})$ then within the critical layer the wave's Reynolds stresses drive the roll flow such that equations (2.5) need to be solved subject to the following stress jumps

across the critical layer,

$$[\partial_n v^s(n, s)]_-^+ = n_0 \rho^2 \alpha^{-5/3} \partial_s \left(\mu^{-5/3} |P_s|^2 \right), \quad (2.9a)$$

$$[p]_-^+ = -n_0 \rho^2 (\alpha \mu)^{-5/3} \Lambda_0 |P_s|^2. \quad (2.9b)$$

Here, (n, s) are the normal and tangential coordinates along the critical layer respectively, v^s is the roll velocity in the tangential direction in the $y - z$ plane, Λ_0 is the local curvature of the layer, $\mu = \partial_n u$ and $n_0 = 2\pi(2/3)^{2/3}\Gamma(1/3)$.

It is convenient to state the jump conditions in Cartesian form for computational purposes,

$$\frac{[v_y]_-^+}{f_z} = [w_y]_-^+ = J(z), \quad (2.10a)$$

$$[p]_-^+ = K(z) \quad (2.10b)$$

where

$$J(z) = \frac{n_0 \rho^2}{\Theta^{5/3} \Delta^5} \left[\left(-\frac{7\Delta_z}{2\Delta} - \frac{5\Theta_z}{3\Theta} \right) |P_z|^2 + \partial_z |P_z|^2 \right], \quad (2.11a)$$

$$K(z) = -\frac{n_0 \rho^2}{\Theta^{5/3} \Delta^5} f_{zz} |P_z|^2, \quad (2.11b)$$

and

$$\Delta = 1 + f_z^2, \quad \Theta = \frac{\alpha u_y}{\Delta}, \quad (2.12)$$

here, all z -derivatives are to be evaluated along the critical layer, that is

$$\frac{\partial}{\partial z} \Big|_{y=f(z)} = \frac{\partial}{\partial z} \Big|_y + f_z \frac{\partial}{\partial y} \Big|_z. \quad (2.13)$$

Hall & Sherwin (2010) enforced the jump conditions through a delta-function-type regularisation of forcing terms that involved using a regularisation parameter ξ . Blackburn *et al.* (2013) used a hybrid approach where the forcing on the roll system was computed directly from the wave velocity field by calculating the divergence of the streamwise averaged Reynolds stresses

$$\mathbf{F}_{vwi} = \rho^2 R^{-1/3} [\nabla \cdot (\bar{\mathbf{V}} \mathbf{V} + \mathbf{V} \bar{\mathbf{V}})], \quad (2.14)$$

where $\mathbf{V} = (0, V, W)$ and $\bar{\mathbf{V}}$ denotes complex conjugate. The method employed by Hall & Sherwin (2010) is sometimes called the *regularised* approach and the method employed by Blackburn *et al.* (2013) is sometimes called the *hybrid* approach.

Finally, since we are interested in equilibrium solutions that do not grow or decay we stipulate that the complex wavespeed c should be a purely real quantity. If the real part of the complex wavespeed is non-zero then we have a travelling wave; if the real part of the complex wavespeed is zero then we have a stationary wave. In this thesis, we restrict our focus to stationary waves and so for a stationary equilibrium solution the complex wavespeed will be identically zero.

We are now in a position to summarise the governing equations for vortex-wave interaction states in shear flows. Firstly, the roll given by equations (2.5) must be solved subject to no-slip boundary conditions and periodicity in the spanwise direction. In addition, at the critical layer $y = f(z)$ the velocity and pressure field must satisfy stress jump conditions (2.9) that are a function of the wave pressure, which satisfies equation (2.7), evaluated on the critical layer. The wave pressure arises from an instability of the streak, which satisfies equation (2.4), where the streak arises from the advection of mean-shear due to the roll. The governing equations can also be thought of as a nonlinear eigenvalue problem where the eigenvalue is the wave amplitude ρ , which measures the amplitude of the wave required to drive the interaction.

2.1.2. Classification of VWI States

For a given value of the wave amplitude ρ we may anticipate that a number of stationary eigenmodes will exist. We therefore have a choice of which stationary eigenmode to associate with the VWI state and this leads to a natural taxonomy of VWI states depending on the linear stability properties of the streak field of the VWI state. We will therefore call the class of solutions associated with the most unstable eigenmode of the streak field of a VWI state the first branch of solutions; the class of solutions associated with the second most unstable stationary eigenmode of the streak field of a VWI state the second branch of solutions; and so on. The link between our choice of nomenclature and the nomenclature used in the theory of exact

coherent structures will become clear in chapters 3 and 4.

VWI states from the same class can be characterised by the relationship between the parameters $\rho = \rho(\alpha, \beta)$. The parameter ρ is effectively an eigenvalue that quantifies, for given wavenumbers, the amplitude of a wave required to drive a streak neutrally stable to the wave.

The kinetic energy density, $\mathcal{E}$, the bulk viscous dissipation rate, $\mathcal{D}$, and the mean shear-rate on the walls, $\mathcal{S}$, can also be used to characterise VWI states. These measures may be of potential interest in characterising VWI states in the context of their desirability for laminar-flow control purposes and are given by

$$\mathcal{E} = \frac{1}{A} \int_{-L_z/2}^{L_z/2} \int_{-1}^1 \frac{1}{2} |u|^2 dy dz, \quad (2.15)$$

$$\mathcal{D} = \frac{1}{A} \int_{-L_z/2}^{L_z/2} \int_{-1}^1 |\nabla \times u|^2 dy dz, \quad (2.16)$$

$$\mathcal{S} = \frac{1}{2L_z} \int_{-L_z/2}^{L_z/2} \left(\left. \frac{\partial u}{\partial y} \right|_{y=-1} + \left. \frac{\partial u}{\partial y} \right|_{y=+1} \right) dz, \quad (2.17)$$

where $A = 2L_z$. For an equilibrium state the bulk dissipation rate and the mean shear rate on the walls, which can be thought of as the wall-shear power input, are in balance and so $\mathcal{D} = \mathcal{S}$. The normalisations are chosen so that $\mathcal{D} = \mathcal{S} = 1$ for plane Couette flow.

2.1.3. Symmetries of the VWI Equations

Gibson *et al.* (2008) showed that plane Couette flow is invariant under two discrete symmetries, σ_1 , σ_2 , and a continuous translation $\tau(\ell_z)$, namely

$$\sigma_1[u, v, w](y, z) = [u, v, -w](y, -z), \quad (2.18)$$

$$\sigma_2[u, v, w](y, z) = [-u, -v, w](-y, z), \quad (2.19)$$

$$\tau(\ell_z)[u, v, w](y, z) = [u, v, w](y, z + \ell_z). \quad (2.20)$$

An important subgroup is given by $\mathcal{S} = \{1, s_1, s_2, s_3\}$ where $s_1 = \sigma_1$, $s_2 = \tau(L_z/2)\sigma_2$ and $s_3 = s_1s_2$. The s_1 and s_2 symmetries are often referred to the ‘shift-reflect’ and ‘shift-rotate’ symmetries respectively. The group

actions on the velocity fields are given by

$$s_1[u, v, w](y, z) = [u, v, -w](y, -z), \quad (2.21)$$

$$s_2[u, v, w](y, z) = [-u, -v, w](-y, z + L_z/2), \quad (2.22)$$

$$s_3[u, v, w](y, z) = [-u, -v, -w](-y, -z + L_z/2), \quad (2.23)$$

recalling that L_z is the spanwise wavelength of the state.

The wave pressure equation (2.7) has been studied previously by Hall & Horseman (1991) and Li & Malik (1995) but in the context of the inviscid analysis of streaks. They noted that when the downstream velocity profile $u(y, z)$ corresponds to a Görtler vortex of spanwise wavelength Λ , both synchronous and subharmonic eigenmodes with wavelengths Λ and 2Λ respectively are possible. The synchronous modes are the most unstable, of which the wavy one, sometimes referred to as the sinuous mode, is usually most unstable and causes the onset of wavy vortices in the Taylor-Couette problem. The subgroup given by $\mathcal{S}$ permits both synchronous and subharmonic modes to exist with the symmetry

$$P(y, z) = -\bar{P}(-y, z + L_z/2), \quad (2.24)$$

recalling that L_z is the spanwise wavelength of the state. The synchronous and subharmonic modes are sometimes referred to as the “sinuous mode” and “mirror-symmetric mode” respectively.

2.2. Numerical Scheme for Solving the Interaction Equations

Before discussing the numerical technique used to find VWI states in detail it is first instructive to be clear about the nature of the nonlinear eigenvalue problem we are trying to solve. Suppose we fix the spanwise wavenumber β of the roll-streak flow and the streamwise wavenumber α of the inviscid wave system, then we may anticipate that for some values of the wave amplitude, ρ , and complex wavespeed, c , solutions will exist. Due to the symmetry of the problem we know that purely imaginary wavespeeds are possible which correspond to stationary waves. For each value of α and β there will, in principle, be a corresponding eigenvalue $\rho(\alpha, \beta)$ such that

$c = 0$. The scheme we use is based on guessing a value of $\rho = \rho(\alpha, \beta)$ and finding the corresponding eigenvalue $c = 0 + ic_i$ and then varying ρ until we drive c_i to zero.

In order to perform this iteration procedure we must be able to numerically solve the streak, roll and wave equations subject to their appropriate boundary conditions. Each of these equations can be considered separately since within the VWI framework they are driven by quantities that are known from the previous step in the VWI mechanism. Therefore, before outlining any further details about the full numerical scheme we use to find VWI states we will describe the individual numerical schemes used to find the streak, roll and wave fields.

2.2.1. The Vortex System: Collocation Method

We consider the case when the critical layer is flat, $y = f(z) = 0$, which corresponds to solutions which are mirror-symmetric modes, see chapter 4 for more details, and results in a number of simplifications to the vortex system problem.

Firstly, the form of the stress jumps simplify greatly to

$$[w_y]_{-}^{+} = n_0 \rho^2 \alpha^{-5/3} \partial_z (u_y^{-5/3} |P_z|^2) = J(z). \quad (2.25)$$

Since the right-hand side of (2.25) does not depend on any of the roll field quantities it can be considered as a given function that depends only on the spanwise coordinate z .

If we seek solutions that belong to the symmetry sub-group $\mathcal{S}$ then we can solve the vortex system equations (2.5) in the half-channel $y \in [-1, 0]$ subject to the usual no-slip boundary conditions at the lower wall,

$$v = w = 0, \quad y = -1, \quad (2.26)$$

and the stress jump conditions at the channel mid-height,

$$v = 0, \quad w_y = \frac{1}{2} J(z), \quad y = 0. \quad (2.27)$$

Here, the jump in w_y has been re-written in terms of a Neumann boundary condition by exploiting the symmetry of the solution.

We introduce a stream function $\psi(y, z)$ in the usual way,

$$v = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial y}, \quad (2.28)$$

such that equations (2.5) now read

$$\psi_{yyyy} + 2\psi_{yyzz} + \psi_{zzzz} = \psi_z (\psi_{yyy} + \psi_{zzy}) - \psi_y (\psi_{yyz} + \psi_{zzz}). \quad (2.29)$$

We seek solutions periodic in the spanwise coordinate and write

$$\psi(y, z) = \sum_{k=-N_z}^{k=+N_z} \psi_k(y) e^{ik\beta z}, \quad (2.30)$$

where β is the spanwise wavenumber and the limit $N_z \rightarrow \infty$ is to be taken to yield the full expansion. In practice however, the number of Fourier modes used, N_z , is truncated at some finite value. Substituting (2.30) into equation (2.29) results in a set of differential equations for each mode k of the form

$$\begin{aligned} \psi_k'''' - 2(k\beta)^2 \psi_k'' + (k\beta)^4 \psi_k = \\ \sum_{\substack{m, n \\ m+n=k}} (im\beta) \psi_m (\psi_n''' - (n\beta)^2 \psi_n') - (in\beta) \psi_m' (\psi_n'' - (n\beta)^2 \psi_n), \end{aligned} \quad (2.31)$$

where $(\cdot)'$ denotes differentiation with respect to the wall-normal coordinate y . This fourth-order differential equation can now be formulated in terms of four first-order differential equations by defining

$$\phi_k^1 = \psi_k, \quad \phi_k^2 = \psi_k', \quad \phi_k^3 = \psi_k'', \quad \phi_k^4 = \psi_k''', \quad (2.32)$$

such that

$$\frac{d\phi_k^1}{dy} = \phi_k^2, \quad (2.33a)$$

$$\frac{d\phi_k^2}{dy} = \phi_k^3, \quad (2.33b)$$

$$\frac{d\phi_k^3}{dy} = \phi_k^4, \quad (2.33c)$$

$$\begin{aligned} \frac{d\phi_k^4}{dy} = & 2(k\beta)^2\phi_k^3 - (k\beta)^4\phi_k^1 \\ & + \sum_{\substack{p,q \\ p+q=k}} (ip\beta)\phi_p^1(\phi_q^4 - (q\beta)^2\phi_q^2) - (iq\beta)\phi_p^2(\phi_q^3 - (q\beta)^2\phi_q^1), \end{aligned} \quad (2.33d)$$

and the boundary conditions at the lower wall now read

$$\phi_k^1 = \phi_k^2 = 0, \quad y = -1, \quad (2.34)$$

and the stress jump conditions now read

$$\phi_k^1 = 0, \quad \phi_k^3 = \frac{1}{2}J_k, \quad y = 0, \quad (2.35)$$

where J_k is the k -th mode of the Fourier transform of the stress jump condition $J(z)$.

This approach can also be used if we want to impose the body forcing term (2.9) rather than the jump conditions directly. In this case, the system of first-order differential equations (2.33a) should be solved in the full-channel domain with the no-slip boundary conditions at the upper and lower walls,

$$\phi_k^1 = 0, \quad \phi_k^2 = 0, \quad y = \pm 1. \quad (2.36)$$

Further, an additional term in equation (2.33d) arising from the body forcing term should be included.

In either case, the resulting system of first-order differential equations can be solved using an implicit Runge-Kutta method, namely the four-stage Lobatto IIIa collocation formula that is fifth-order accurate as outlined by Shampine *et al.* (2000).

The streak equation (2.4), which is simply an advection-diffusion equation, can be solved in a similar manner once the roll quantities have been

found. To compute a solution in the half-channel domain the no-slip boundary condition should be used on the lower wall,

$$u = 0, \quad y = -1, \quad (2.37)$$

and at the channel mid-height,

$$u = 0, \quad y = 0, \quad (2.38)$$

which follows from the fact that we are considering the case when the critical layer is located at $y = 0$. The streak equation can be written in the form,

$$\frac{d\varphi_k^1}{dy} = \varphi_k^2, \quad (2.39)$$

$$\frac{d\varphi_k^2}{dy} = (k\beta)^2 \varphi_k^1 + \sum_{\substack{p,q \\ p+q=k}} v_p \varphi_q^2 + (iq\beta) w_p \varphi_q^1, \quad (2.40)$$

where

$$u(y, z) = \sum_{k=-N_z}^{k=+N_z} \varphi_k(y) e^{ik\beta z}, \quad (2.41)$$

in a similar manner to (2.30).

2.2.2. The Vortex System: Boundary Integral Method

We now make no restrictions on the location of the critical layer and propose a numerical scheme based on the use of Green's functions. The governing equations for the roll field are nonlinear whilst the method of Green's functions are based on linear superposition. Thus, solutions of the roll equations cannot be described by using Green's function. However, if we discretise the roll equations such that the nonlinear advection term is treated explicitly and the pressure and viscous diffusion terms are treated implicitly, then we can perform an iterative procedure to find the solution. Using this type of discretisation the roll equations (2.5) are written in the form

$$((\mathbf{v} \cdot \nabla_{2D}) \mathbf{v})^n = -\nabla_{2D} p^{n+1} + \nabla_{2D}^2 \mathbf{v}^{n+1}, \quad \nabla_{2D} \cdot \mathbf{v}^{n+1} = 0, \quad (2.42)$$

where $\mathbf{v} = (0, v, w)$, $\nabla_{2D} = (0, \partial_y, \partial_z)$, $\nabla_{2D}^2 = \partial_{yy}^2 + \partial_{zz}^2$ and the superscripts indicate the iteration. This is a linear boundary value problem subject to no-slip boundary conditions and stress jump conditions (2.9) across $y = f(z)$.

To solve the linear boundary value problem (2.42) using Green's functions we will first consider the singularly forced two-dimensional Stokes flow equations

$$-\nabla_{2D} p + \mu \nabla_{2D}^2 \mathbf{v} + \mathbf{g} \delta(\mathbf{x} - \mathbf{x}_0) = 0, \quad \nabla_{2D} \cdot \mathbf{v} = 0, \quad (2.43)$$

where $\mathbf{g}$ is an arbitrary constant, $\mathbf{x}_0$ is an arbitrary point and δ is the two-dimensional Dirac delta function.

Introducing the Green's function the solution can be written in the form

$$v_i(\mathbf{x}) = \frac{1}{4\pi\mu} \mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0) g_j(\mathbf{x}_0), \quad (2.44)$$

where $\mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0)$ is the Green's function and the subscripts i, j denote the components of the function. The point $\mathbf{x}$ is said to be the observation or field point and the point $\mathbf{x}_0$ is said to be the pole or source point. We represent the vorticity, pressure and stress fields in a similar manner,

$$\omega_i(\mathbf{x}) = \frac{1}{4\pi\mu} \mathcal{V}_{ij}(\mathbf{x}, \mathbf{x}_0) g_j(\mathbf{x}_0), \quad (2.45)$$

$$p(\mathbf{x}) = \frac{1}{4\pi} \mathcal{P}_j(\mathbf{x}, \mathbf{x}_0) g_j(\mathbf{x}_0), \quad (2.46)$$

$$\sigma_{ij}(\mathbf{x}) = \frac{1}{4\pi} \mathcal{S}_{ijk}(\mathbf{x}, \mathbf{x}_0) g_k(\mathbf{x}_0). \quad (2.47)$$

Here $\mathcal{V}$, $\mathcal{P}$ and $\mathcal{S}$ are the vorticity tensor, pressure vector and stress tensor associated with the Green's function respectively.

It can be shown that the Green's function corresponding to a point force in a doubly-infinite domain is given by

$$\mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0) = -\delta_{ij} \ln r + \frac{\hat{x}_i \hat{x}_j}{r^2}, \quad (2.48)$$

and the velocity profile can then found by use of equation (2.44). Here, δ_{ij} is the Kronecker delta, $\hat{\mathbf{x}} = (\mathbf{x} - \mathbf{x}_0)$ and $r = |\hat{\mathbf{x}}|$. The vorticity, pressure

and stress fields associated with the flow may be written as

$$\mathcal{V}_{ij}(\mathbf{x}, \mathbf{x}_0) = 2\epsilon_{ijk} \frac{\hat{x}_k}{r^2}, \quad (2.49)$$

$$\mathcal{P}_i(\mathbf{x}, \mathbf{x}_0) = 2 \frac{\hat{x}_i}{r^2}, \quad (2.50)$$

$$\mathcal{S}_{ijk}(\mathbf{x}, \mathbf{x}_0) = -4 \frac{\hat{x}_i \hat{x}_j \hat{x}_k}{r^4}, \quad (2.51)$$

where ϵ_{ijk} is the Levi-Cevita symbol.

The above Green's functions are often called the free-space Green's functions since they are for infinite unbounded flow. When the domain of the flow is partly bounded by a solid surface we often call the Green's function semi-infinite and when the fluid is completely confined by a solid surface we often call the Green's function internal. For semi-infinite and internal Green's functions we require that the Green's function vanish over all the boundaries of the flow where the no-slip boundary condition needs to be satisfied. Green's functions of this type are often constructed by the superposition of the free-space Green's function. Finally, as the observation point $\mathbf{x}$ approaches the pole $\mathbf{x}_0$ all Green's functions exhibit singular behaviour and, to leading order, behave like the free-space Green's function.

The boundary integral method involves expressing the solution to a linear elliptic partial differential equation as an integral over the flow boundary. To develop the boundary integral formulation we can follow the well-established theory for steady Stokes flows, see Pozrikidis (1992), by first writing the reciprocal identity,

$$\frac{\partial}{\partial x_j} (v'_i \sigma_{ik} - v_i \sigma'_{ik}) = 0. \quad (2.52)$$

Here, v_i and v'_i are arbitrary non-singular solutions to (2.43) with corresponding stress fields σ_{ik} and σ'_{ik} .

If we now select a point $\mathbf{x}_0$ in the domain of the flow and integrate (2.52) over a closed volume D with boundary S and then apply the divergence theorem we obtain an integral expression at a point $\mathbf{x}_0$ in terms of single-layer and double-layer distributions. For points that lie on the flow boundary this

expression takes the form

$$v_j(\mathbf{x}_0) = -\frac{1}{2\pi\mu} \int_S \mathcal{F}_i(\mathbf{x}) \mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0) dS(\mathbf{x}) + \frac{1}{2\pi} \int_S^{\mathcal{PV}} v_i(\mathbf{x}) \mathcal{S}_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}), \quad (2.53)$$

where $\mathcal{F}_i = \sigma_{ik} n_k$, which is often called the surface traction, and n_k is the unit normal vector orientated into the area of the flow. The superscript $\mathcal{PV}$ indicates that the principal value of the double-layer potential should be used, defined as the improper double-layer integral when the point $\mathbf{x}_0$ is right on S . If the point lies inside the flow domain then the boundary integral expression takes the form

$$v_j(\mathbf{x}_0) = -\frac{1}{4\pi\mu} \int_S \mathcal{F}_i(\mathbf{x}) \mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0) dS(\mathbf{x}) + \frac{1}{4\pi} \int_S v_i(\mathbf{x}) \mathcal{S}_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}). \quad (2.54)$$

The boundary integral method involves first solving the integral equation (2.53) for the surface traction, $\mathcal{F}_i$, on the boundary of the flow domain S . The fluid velocity inside the domain can then be found by evaluating integral (2.54).

The boundary integral method can also be formulated for multi-phase fluid problems where there may be two or more fluids of different viscosities. In this case the boundary will be the interface between the two adjacent regions of fluid.

Suppose we have two fluids of different viscosities, μ_1 and μ_2 , separated by an interface given by L . The boundary integral equation for the velocity field of fluid 1 at a point $\mathbf{x}_0$ in the domain of fluid 1 is

$$v_j^{(1)}(\mathbf{x}_0) = -\frac{1}{4\pi\mu_1} \int_L \Delta \mathcal{F}_i(\mathbf{x}) \mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0) dL(\mathbf{x}) + \frac{1-\nu}{4\pi} \int_L v_i(\mathbf{x}) \mathcal{S}_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}_0) dL(\mathbf{x}), \quad (2.55)$$

where $\nu = \mu_2/\mu_1$ and $\Delta \mathcal{F}$ is the discontinuity in the interfacial surface force, defined as

$$\Delta \mathcal{F}_i = \mathcal{F}_i^{(1)} - \mathcal{F}_i^{(2)} = \left(\sigma_{ik}^{(1)} - \sigma_{ik}^{(2)} \right) n_k. \quad (2.56)$$

Similarly, the boundary integral equation for the velocity field of fluid 2 at a point $\mathbf{x}_0$ in the domain of fluid 2 is

$$\begin{aligned} v_j^{(2)}(\mathbf{x}_0) = & -\frac{1}{4\pi\mu_2} \int_L \Delta\mathcal{F}_i(\mathbf{x}) \mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0) dL(\mathbf{x}) \\ & + \frac{1-\nu}{4\pi} \int_L v_i(\mathbf{x}) \mathcal{S}_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}_0) dL(\mathbf{x}). \end{aligned} \quad (2.57)$$

If we now let the point $\mathbf{x}_0$ approach the interface from either fluid 1 or fluid 2 we find that equations (2.55) and (2.57) reduce to

$$\begin{aligned} v_j(\mathbf{x}_0) = & -\frac{1}{2\pi(\mu_1 + \mu_2)} \int_L \Delta\mathcal{F}_i(\mathbf{x}) \mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0) dL(\mathbf{x}) \\ & + \frac{(1-\nu)}{2\pi(1+\nu)} \int_L^{\mathcal{PV}} v_i(\mathbf{x}) \mathcal{S}_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dL(\mathbf{x}), \end{aligned} \quad (2.58)$$

which is the boundary integral equation for the fluid velocity on the interface.

Prescribing the discontinuity in surface force $\Delta\mathcal{F}$ renders equations (2.59) a Fredholm integral equation of the second kind for the interfacial velocity profile $\mathbf{v}$. Once this integral equation is solved, the velocities in fluids 1 and 2 can be computed using the representations (2.55) and (2.57).

If fluid 1 and 2 have the same viscosity, $\mu_1 = \mu_2 = \mu$, then $\nu = 1$ and equations (2.55), (2.57) and (2.59) all reduce to

$$v_j(\mathbf{x}_0) = -\frac{1}{2\pi\mu} \int_L \Delta\mathcal{F}_i(\mathbf{x}) \mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0) dL(\mathbf{x}), \quad (2.59)$$

which is a line integral that can be calculated directly after prescribing the discontinuity in surface force $\Delta\mathcal{F}$ across the given interface location L .

We now return to our main problem of interest, that is, to solve the roll equations (2.5) subject to no-slip boundary conditions on the walls and a given stress jump over a given layer $y = f(z)$. To achieve this we will use integral equation (2.59) and prescribe the surface force discontinuity in terms of the stress jumps from VWI theory with an appropriate Green's function. Recall that the stress jumps given by VWI theory in Cartesian

coordinates are of the form

$$\frac{[v_y]_-^+}{f_z} = [w_y]_-^+ = J(z), \quad (2.60a)$$

$$[p]_-^+ = K(z), \quad (2.60b)$$

where $J(z)$ and $K(z)$ are two given functions of the spanwise coordinate z . From relation (2.13) we also have that $[\cdot]_-^+ = -f_z[\cdot]_-^+$ for jumps across the critical layer. If we now define the normal vector to the critical layer as

$$\mathbf{n} = (1 + f_z^2)^{-\frac{1}{2}} \begin{pmatrix} 1 \\ -f_z \end{pmatrix} \quad (2.61)$$

then the VWI stress jumps can be expressed in terms of the surface force discontinuity as

$$\begin{aligned} \Delta \mathcal{F}^{vwi} &= \Delta \sigma_{ij} \cdot n_j, \\ &= (1 + f_z^2)^{-\frac{1}{2}} \begin{pmatrix} -K + 2f_z J & (1 - f_z^2)J \\ (1 - f_z^2)J & -K - 2f_z J \end{pmatrix} \begin{pmatrix} 1 \\ -f_z \end{pmatrix} \\ &= (1 + f_z^2)^{-\frac{1}{2}} \begin{pmatrix} -K + f_z(1 + f_z^2)J \\ f_z K + (1 + f_z^2)J \end{pmatrix}. \end{aligned} \quad (2.62)$$

All that remains to evaluate (2.59) is to declare the appropriate Green's function. By the superposition of the free-space Green's function (2.48) we can construct a Green's function, say $\mathcal{G}_{ij}^{2W1P}$, corresponding to the flow due to a periodic array of point forces confined by two parallel walls. The use of this Green's function automatically takes care of the no-slip boundary conditions at $y = \pm 1$ and by using a periodic array we only have to perform the integration over one spanwise period of the critical layer $y = f(z)$. A detailed derivation and discussion of the Green's function used is provided in appendix B.

The line integral (2.59) can now be evaluated using the surface force discontinuity given by (2.62) and the Green's function $\mathcal{G}_{ij}^{2W1P}$ to find the Stokes flow due to VWI stress jumps across the critical layer in a channel with walls located at $y = \pm 1$. The line integral reads

$$v_j(\mathbf{x}_0) = -\frac{1}{2\pi} \int_L \Delta \mathcal{F}_i^{vwi}(\mathbf{x}) \mathcal{G}_{ij}^{2W1P}(\mathbf{x}, \mathbf{x}_0) dL(\mathbf{x}), \quad (2.63)$$

which can be re-written as

$$v_j(y_0, z_0) = -\frac{1}{2\pi} \int_{-\frac{\pi}{\beta}}^{\frac{\pi}{\beta}} \Delta \mathcal{F}_i^{vwi}(z) \mathcal{G}_{ij}^{2W1P}(f(z), z, y_0, z_0) (1 + f_z^2)^{\frac{1}{2}} dz, \quad (2.64)$$

after using the parameterisation of the critical layer. All quantities in the integrand are known and so this expression can be integrated using the composite Simpson's rule to find the associated velocity field.

Now, revisiting expression (2.42) the solution to the roll equations can be found by performing an iterative procedure by writing

$$\mathbf{v}^{(n)} = \mathbf{v}_{jump} + \mathbf{v}^{(n-1)}, \quad (2.65)$$

where $\mathbf{v}^{(n)}$ is found by evaluating

$$v_j^{(n)}(\mathbf{x}_0) = \int_D F_i(\mathbf{v}^{(n-1)}(\mathbf{x})) \mathcal{G}_{ij}^{2W1P}(\mathbf{x}, \mathbf{x}_0) dS(\mathbf{x}), \quad (2.66)$$

and $F(\mathbf{v}^{(n-1)})$ is the nonlinear advection term which is treated explicitly, that is

$$F(\mathbf{v}^{(n-1)}) = \mathbf{v}^{(n-1)} \cdot \nabla_{2D} \mathbf{v}^{(n-1)}. \quad (2.67)$$

The iteration procedure can be stopped once a norm on the relative error between velocity profiles at successive iterations reaches a tolerance criteria. The converged solution (2.65) is then a solution of the roll equations (2.5) subject to the given stress jumps (2.62) across the critical layer at $y = f(z)$ and no-slip boundary conditions on the walls at $y = \pm 1$.

2.2.3. The Wave System: Compact Finite-Difference Method

The wave system given by equation (2.7) is singular at the critical layer given by $y = f(z)$ where $u(y, z) = 0$. In order to take care of the singularity at the critical layer we will follow Hall & Sherwin (2010) and solve the viscous form of the wave equations (i.e., a more generalised Orr-Sommerfeld equation). However, in this treatment the Reynolds number is treated as a regularisation parameter and is set sufficiently large such that with any further increase the numerical results are independent of the Reynolds num-

ber.

The viscous wave equations are

$$i\alpha(u-c)U + u_y V + u_z W = -i\alpha P + \frac{1}{R}(U_{yy} + U_{zz} - \alpha^2 U), \quad (2.68a)$$

$$i\alpha(u-c)V = -P_y + \frac{1}{R}(V_{yy} + V_{zz} - \alpha^2 V), \quad (2.68b)$$

$$i\alpha(u-c)W = -P_z + \frac{1}{R}(W_{yy} + W_{zz} - \alpha^2 W), \quad (2.68c)$$

$$i\alpha U + V_y + W_z = 0, \quad (2.68d)$$

subject to the no-slip boundary conditions

$$U = V = W = 0, \quad y = \pm 1. \quad (2.69)$$

For disturbances that are periodic in the spanwise coordinate z we can write

$$[U, V, W, P, u](y, z) = \sum_{k=-N_z}^{k=+N_z} [U_k, V_k, W_k, P_k, u_k](y) e^{ik\beta z}, \quad (2.70)$$

where β is the spanwise wavenumber and the limit $N_z \rightarrow \infty$ is to be taken to yield the full expansion. Since the streak profile $u(y, z)$ is given, the values of u_k can be computed using a fast Fourier transform. The value of β is also given since the periodicity of the streak profile is known, however it is worth noting that subharmonic modes may exist and so the spanwise wavenumber should be changed accordingly.

Substituting (2.70) into equations (2.68) results in a set of differential

equations for each Fourier mode k of the form

$$i\alpha c U_k = \sum_{\substack{m,n \\ m+n=k}} (i\alpha u_m U_n + u'_m V_n + im\beta u_m W_n) + i\alpha P_k - \dots \\ \frac{1}{R} (U_k'' - (\alpha^2 + (k\beta)^2) U_k), \quad (2.71a)$$

$$i\alpha c V_k = \sum_{\substack{m,n \\ m+n=k}} (i\alpha u_m V_n) + P'_k - \frac{1}{R} (V_k'' - (\alpha^2 + (k\lambda)^2) V_k), \quad (2.71b)$$

$$i\alpha c W_k = \sum_{\substack{m,n \\ m+n=k}} (i\alpha u_m W_n) + ik\lambda P_k - \frac{1}{R} (W_k'' - (\alpha^2 + (k\lambda)^2) W_k), \quad (2.71c)$$

$$0 = i\alpha U_k + V'_k + ik\lambda W_k, \quad (2.71d)$$

where $(\cdot)'$ denotes differentiation with respect to the wall-normal coordinate y .

To discretise these equations we will use a fourth order accurate two-point scheme of the form developed by Malik *et al.* (1982) that is based on the Euler-Maclaurin formula

$$\psi_j - \psi_{j-1} = \frac{h}{2} \left(\frac{d\psi_j}{dy} + \frac{d\psi_{j-1}}{dy} \right) - \frac{h^2}{12} \left(\frac{d^2\psi_j}{dy^2} - \frac{d^2\psi_{j-1}}{dy^2} \right) + O(h^5), \quad (2.72)$$

where $\psi_j = \psi(y_j)$ and h is the uniform step-size between discretisation points in y . If we use N_y discretisation points then $h = 2/(N_y - 1)$ and the discretisation points are given by

$$y_j = -1 + (j - 1)h, \quad j = 1 : N_y. \quad (2.73)$$

In order to apply this scheme to equations (2.71) we must reformulate them as a set of first order differential equations of the form

$$\frac{d\chi_j}{dy} = D_j \chi_j, \quad (2.74)$$

where D_j is a matrix representing the linear operator given by equations (2.71) at discretisation point y_j and χ_j is a column vector of the unknown disturbance quantities at discretisation point y_j . We define χ_j in the fol-

lowing manner

$$\chi_j = (\chi_j^1, \chi_j^2, \chi_j^3, \chi_j^4, \chi_j^5, \chi_j^6)^T, \quad (2.75)$$

where

$$\begin{aligned} \chi_j^1 &= (U_{-N}(y_j), \dots, U_0(y_j), \dots, U_N(y_j))^T, \\ \chi_j^2 &= (U'_{-N}(y_j), \dots, U'_0(y_j), \dots, U'_N(y_j))^T, \\ \chi_j^3 &= (V_{-N}(y_j), \dots, V_0(y_j), \dots, V_N(y_j))^T, \\ \chi_j^4 &= (W_{-N}(y_j), \dots, W_0(y_j), \dots, W_N(y_j))^T, \\ \chi_j^5 &= (W'_{-N}(y_j), \dots, W'_0(y_j), \dots, W'_N(y_j))^T, \\ \chi_j^6 &= (P_{-N}(y_j), \dots, P_0(y_j), \dots, P_N(y_j))^T, \end{aligned} \quad (2.76)$$

recalling that the subscript of the disturbance quantities denotes the Fourier mode number. The entries of the matrix D_j in (2.74) can now be found by re-writing equations (2.71) using the above notation. For convenience, we just consider the equations at an arbitrary Fourier mode k , and use the notation

$$\begin{aligned} \chi_{j,k}^1 &= U_k(y_j), \\ \chi_{j,k}^2 &= U'_k(y_j), \\ \chi_{j,k}^3 &= V_k(y_j), \\ \chi_{j,k}^4 &= W_k(y_j), \\ \chi_{j,k}^5 &= W'_k(y_j), \\ \chi_{j,k}^6 &= P_k(y_j), \end{aligned} \quad (2.77)$$

and so equations (2.71) now read

$$\frac{d\chi_{j,k}^1}{dy} = \chi_{j,k}^2, \quad (2.78a)$$

$$\begin{aligned} \frac{d\chi_{j,k}^2}{dy} = R \sum_{\substack{m,n \\ m+n=k}} (i\alpha u_{j,m} \chi_{j,n}^1 + u'_{j,m} \chi_{j,n}^3 + im\beta u_{j,m} \chi_{j,n}^4) + i\alpha R \chi_{j,k}^6 \\ + (\alpha^2 + (k\beta)^2) \chi_{j,k}^1 - i\alpha R c \chi_{j,k}^1, \end{aligned} \quad (2.78b)$$

$$\frac{d\chi_{j,k}^3}{dy} = -i\alpha \chi_{j,k}^1 - ik\beta \chi_{j,k}^4, \quad (2.78c)$$

$$\frac{d\chi_{j,k}^4}{dy} = \chi_{j,k}^5, \quad (2.78d)$$

$$\begin{aligned} \frac{d\chi_{j,k}^5}{dy} = R \sum_{\substack{m,n \\ m+n=k}} (i\alpha u_{j,m} \chi_{j,n}^4) + ik\beta R \chi_{j,k}^6 + (\alpha^2 + (k\beta)^2) \chi_{j,k}^4 \\ - i\alpha R c \chi_{j,k}^4, \end{aligned} \quad (2.78e)$$

$$\begin{aligned} \frac{d\chi_{j,k}^6}{dy} = - \sum_{\substack{m,n \\ m+n=k}} (i\alpha u_{j,m} \chi_{j,n}^3) - \frac{1}{R} (i\alpha \chi_{j,k}^2 + ik\beta \chi_{j,k}^5 + (\alpha^2 + (k\beta)^2) \chi_{j,k}^3) \\ + i\alpha c \chi_{j,k}^3. \end{aligned} \quad (2.78f)$$

The boundary conditions (2.69) in this notation are

$$\chi_1^1 = \chi_1^3 = \chi_1^4 = \chi_{N_y}^1 = \chi_{N_y}^3 = \chi_{N_y}^4 = \mathbf{0} = (0, \dots, 0, \dots, 0)^T. \quad (2.79)$$

If we now write

$$\psi_j = \chi_j, \quad \frac{d\psi_j}{dy} = D_j \chi_j, \quad \frac{d^2\psi_j}{dy^2} = H_j \chi_j, \quad (2.80)$$

then (2.72) can be written as

$$\left[I - \frac{h}{2} D_j + \frac{h^2}{12} H_j \right] \chi_j - \left[I + \frac{h}{2} D_{j-1} + \frac{h^2}{12} H_{j-1} \right] \chi_{j-1} = 0, \quad (2.81)$$

where I denotes the identity matrix. The entries of the matrix H_j , which represent the second order system of differential equations at discretisation point y_j , can be found by differentiating (2.74).

This two-point difference can be written in block-tridiagonal form,

$$A_j \chi_{j-1} + B_j \chi_j + C_j \chi_{j+1} = \mathbf{0}, \quad (2.82)$$

that is,

$$\begin{pmatrix} B_0 & C_0 & & & \\ \ddots & \ddots & \ddots & & \\ & A_j & B_j & C_j & \\ & & \ddots & \ddots & \ddots \\ & & & A_{N_y} & B_{N_y} \end{pmatrix} \chi = \mathbf{0}. \quad (2.83)$$

where A_j , B_j , C_j are square matrices of order $6N_z$ and so the order of the full system is $6N_y N_z$. The sub-matrices B_0 and B_{N_y} are altered to represent the boundary conditions at $y = \pm 1$. For a detailed account of how the matrix is formulated see Malik *et al.* (1982).

The resulting matrix system can be written as a generalised eigenvalue problem of the form

$$A\chi = cB\chi \quad (2.84)$$

for the complex wavespeed c . This eigenvalue problem is solved using the implicitly restarted Arnoldi method as implemented by Lehoucq *et al.* (1998).

The Arnoldi method is a technique for approximating a few eigenvalues and eigenvectors of interest and is part of a larger class of iterative linear algebra solvers that are particularly efficient for large-scale problems, see Arnoldi (1951) and Saad (2003) for more details. The method requires the action of a matrix on a vector to build a Krylov subspace defined as

$$\mathcal{K}_n = \text{span} \{ \mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, \dots, A^n\mathbf{x} \}, \quad (2.85)$$

whilst these vectors are not necessarily orthogonal, by performing the stabilised Gram-Schmidt process an orthogonal set of vectors that span the Krylov subspace can be found. During this process an upper Hessenberg matrix is formed which describes the orthogonal projection of the matrix A onto the Krylov subspace. The eigenvalues can then be found from the upper Hessenberg matrix efficiently with the QR algorithm. The number of

eigenvalues delivered is determined by the (freely chosen) dimension of the Krylov subspace. Since the Arnoldi method does not require the input of the full matrix to find eigenvalues and is therefore very attractive to problems where the matrices are sparse and the matrix-vector product $\mathbf{y} \leftarrow A\mathbf{x}$ requires $O(n)$, rather than the usual $O(n^2)$ floating point operations.

We can introduce a known shift σ to equation (2.84)

$$(A - \sigma B) \boldsymbol{\chi} = (c - \sigma) B \boldsymbol{\chi} \quad (2.86)$$

which can be rearranged to give

$$(A - \sigma B)^{-1} B \boldsymbol{\chi} = \lambda \boldsymbol{\chi}, \quad (2.87)$$

where

$$\lambda = \frac{1}{c - \sigma}. \quad (2.88)$$

This is now a standard eigenvalue problem for the eigenvalue λ . To solve this standard eigenvalue problem using the Arnoldi method the operation

$$\mathbf{y} \leftarrow (A - \sigma B)^{-1} B \mathbf{x}, \quad (2.89)$$

needs to be supplied, which is equivalent to solving the linear system of equations of the form

$$(A - \sigma B) \mathbf{y} = B \mathbf{x}. \quad (2.90)$$

The introduction of the known shift, σ , to the system has mapped the eigenvalues, c_j , of the generalised problem that are nearest to σ in absolute value to the eigenvalues of largest magnitude, λ_j , in the standard problem. It is precisely the eigenvalues of largest magnitude which are the most efficient to compute using the Arnoldi method. The original eigenvalues are then given by

$$c_j = \sigma + 1/\lambda_j, \quad (2.91)$$

and the corresponding eigenvector of the eigenvalue λ_j of the transformed problem is also the eigenvector of the original problem corresponding to c_j .

This method of introducing a known shift is sometimes called the shift and invert strategy and can be used to obtain specific eigenvalues of interest. For example, if we are interested in the most unstable eigenmodes, which correspond to eigenvalues with the largest imaginary part, then a shift of $\sigma = i$ would be suitable.

2.2.4. Iteration Procedure on Interaction Equations

There are a number of different approaches to solving the VWI equations. Rather than performing a full Newton method on the governing interaction equations we perform an iterative procedure on the wave amplitude ρ , that is, whilst keeping the streamwise and spanwise wavenumbers (α, β) fixed we find the value of the wave amplitude ρ such that the linear growth rate of a stationary eigenmode is zero.

The scheme is initialised by providing an initial streak profile $u^0(y, z)$ from either a previous study or by constructing a streak profile with the correct form of symmetry. For a sinuous state a suitable initial profile is

$$u_S^0(y, z) = y + \epsilon(1 - y^2) \cos(\beta z) + O(\epsilon^2), \quad (2.92)$$

and for a mirror-symmetric state a suitable initial profile is

$$u_M^0(y, z) = y + \epsilon y(1 - y^2) \cos(2\beta z) + O(\epsilon^2). \quad (2.93)$$

Here ϵ is a parameter that measures the strength of the perturbation away from plane Couette flow and should be chosen such that a neutral eigenmode exists suitably close to the value of the wavenumbers (α, β) to ensure convergence of the scheme.

Using the initial streak profile $u^0(y, z)$ as a base flow, the inviscid wave equation (2.7) can be solved using the method outlined in section 2.2.3 to obtain the wave pressure field $P^0(y, z)$ corresponding to a stationary eigenmode at (α, β) whose values are fixed.

Given a value of the wave amplitude ρ , the initial streak profile $u^0(y, z)$ and the wave pressure field $P^0(y, z)$ can be used to compute the jump conditions given by (2.9) or alternatively, given the wall-normal and spanwise components of the wave field, $V^0(y, z)$ and $W^0(y, z)$ respectively, the body forcing term (2.14) can be computed. In either case, given a value of ρ

we can generate the appropriate VWI forcing and subsequently calculate the resulting roll and streak fields using methods outlined in sections 2.2.1 and 2.2.2. The wave equation (2.7) is solved for the new streak field and the value of ρ is updated in order to drive the growth rate of the selected stationary eigenmode to zero.

After the value of ρ has converged and the growth rate of the selected stationary eigenmode is zero, the iteration procedure is repeated for the newly converged streak profile $u^1(y, z)$ (which is not necessarily the same as $u^0(y, z)$). This process is repeated until the error on a quantity relating to the VWI state is sufficiently small. For example, the difference in velocity fields, $\|u^{k+1}(y, z) - u^k(y, z)\|$, the wave amplitude, $\rho^{k+1} - \rho^k$, or the wall skin-friction, $\mathcal{S}^{k+1} - \mathcal{S}^k$, are all valid choices.

Figure 2.1 provides a schematic overview of the numerical scheme in the format of a flowchart.

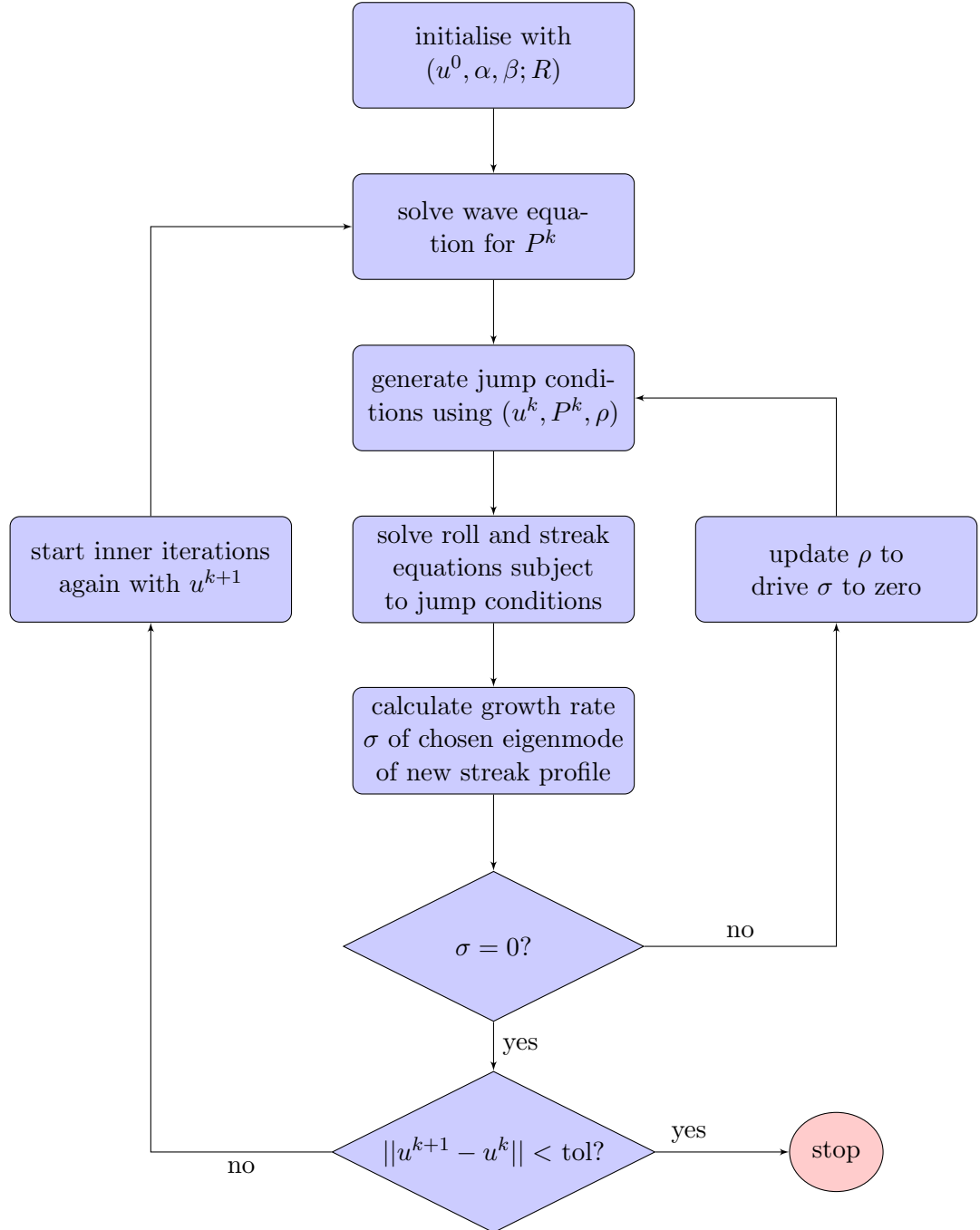


Figure 2.1. Schematic of the iterative numerical procedure used to solve the vortex-wave interaction equations.

3. Sinuous Modes

In this chapter we begin our exploration of VWI states by calculating solutions that belong to the sinuous class of solutions. This class of solutions includes the lower and upper branch equilibria originally found by Nagata (1990) which correspond to EQ1/EQ2 in the terminology of Gibson *et al.* (2008). The lower branch sinuous VWI state was also originally shown to be a VWI state by Hall & Sherwin (2010).

3.1. Accuracy of the Numerical Scheme

To validate the numerical technique we use to calculate VWI states we will first make some remarks about the accuracy of the numerical scheme when calculating a typical sinuous VWI state. In section 2.2 a number of different methods to solve the vortex system equations were discussed; for a sinuous VWI state we use either the collocation approach outlined in section 2.2.2 and impose hybrid body forcing term (2.14), or the boundary integral approach outlined in section 2.2.2, imposing the jump conditions (2.9) directly.

Table 3.1 contains tabulated values of the converged wave amplitude ρ after computing a sinuous VWI state by imposing the jump conditions directly by using the boundary integral approach. The data demonstrates that, for fixed streamwise and spanwise wavenumbers (α, β) , the value of the wave amplitude ρ becomes independent of the grid resolution as the number of grid points in y or Fourier modes in z is increased. For two significant figures of accuracy in ρ a resolution of $(N_y, N_z) = (129, 33)$ was found to be suitable.

The effect of the regularisation Reynolds number used in solving the wave equation (2.7) is shown in table 3.2. Whilst VWI states can be found at quite low Reynolds numbers, we observe that at increasingly high values of the Reynolds number, corresponding to the inviscid limit, the value of

		N_z			
		17	33	49	65
N_y	129	0.36521	0.35565	0.35433	0.35387
	257	0.36542	0.35433	0.35483	0.35438
	513	—	0.35387	0.35493	0.35469

Table 3.1. Convergence of wave amplitude ρ with the number of grid points in y , N_y , and the number of Fourier modes in z , N_z , using the numerical approach described in section 2.2.4 with the collocation approach 2.2.1 when calculating the first branch sinuous VWI state with $(\alpha, \beta; R) = (0.6, 2.0; 16000)$.

R	1000	5000	10000	20000	30000	40000	50000
ρ	0.39670	0.36716	0.35926	0.35426	0.35182	0.35031	0.34965

Table 3.2. Convergence of wave amplitude ρ with regularisation Reynolds number R used in solving the wave system (2.68) with the numerical approach described in section 2.2.4 when calculating the first branch sinuous VWI state with $(\alpha, \beta) = (0.6, 2.0)$.

the wave amplitude ρ starts to become independent of the regularisation Reynolds number. The wave amplitude ρ is accurate to two significant figures when $R \sim O(20000)$. In each calculation the number of grid points in y is chosen such that the critical layer structure, of size $O(R^{-1/3})$, is fully resolved.

3.2. Exploration of State Space

3.2.1. The First Branch

The first branch of sinuous VWI states are associated with the most unstable eigenmode of the streak. To find VWI states belonging to this class of solutions we follow the numerical procedure outlined in section 2.2.4 and drive the linear growth rate of the most unstable eigenmode to zero.

Using an initial streak profile of the form (2.92) and running exploratory cycles for different values of the wave amplitude ρ , a VWI state at $(\alpha, \beta; R) = (0.7, 2.0; 16000)$ belonging to the first branch was found. Figure 3.1 illustrates the structure associated with this VWI state, similar features are observed at other values of (α, β) . The streaky flow $u(y, z)$ and the roll flow $[0, v, w](y, z)$ are very similar to the lower branch ECS discussed by Wang *et al.* (2007) and the synchronous VWI states found by Hall & Sherwin (2010). In terms of the library of solutions provided by Gibson (2012), this state corresponds to EQ1. Notable features of this state include the updraft at $z = 0$ and downdraft at $z = \pm\pi/\beta$ of the roll field that sustains the modulation of the streak and the concentration of the wave field about the critical layer location, which varies in z .

The variation of the wave amplitude ρ against the streamwise wavenumber α is shown in figure 3.2. The data was obtained by using the converged VWI state at $(\alpha, \beta) = (0.7, 2.0)$ as an initial guess and running exploratory cycles for different values of (α, β, ρ) .

The $\alpha - \rho$ relationship follows the already established behaviour for sinuous VWI states as noted by Hall & Sherwin (2010). At small values of the streamwise wavenumber α we observe the balance $\rho \sim \alpha^{5/6}$ that ensures the stress jump conditions (2.9) remain $O(1)$, this is shown in the inset figure where the variation is shown on a log-log scale at small values of α . At large values of the streamwise wavenumber α we find that obtaining converged values of ρ becomes increasingly more difficult. In fact, as noted by Hall & Sherwin (2010) there is a critical value of the streamwise wavenumber, $\alpha_c \approx 1.4$, where the $\alpha - \rho$ curve seemingly approaches a vertical asymptote. Deguchi & Hall (2014c) demonstrated that, by using a Newton iteration method, this critical value of α was in fact a turning point and that the curve turns around into the so-called upper branch curve which describes a new set of solutions.

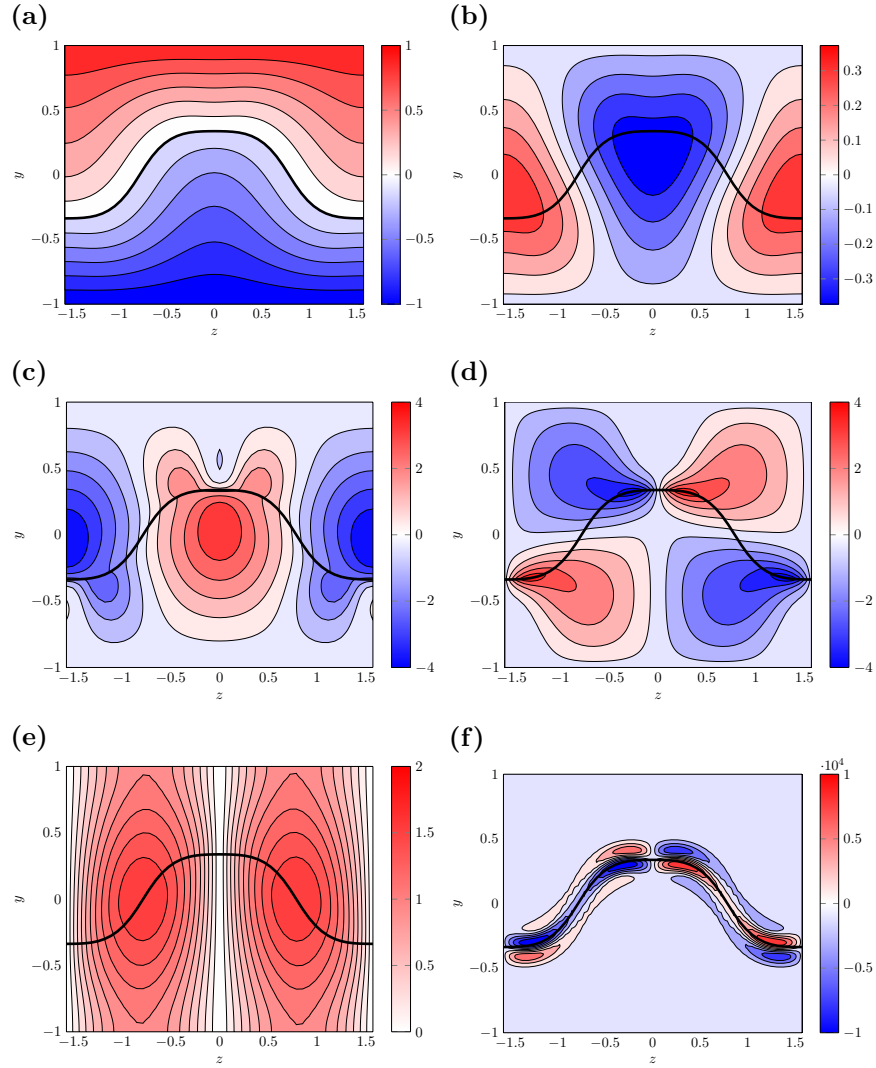


Figure 3.1. Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $|P(y, z)|$, and (f) the hybrid body forcing term (2.14) associated with a sinuous VWI state on the first branch at $(\alpha, \beta; R) = (0.7, 2.0; 16000)$. The solid bold curve represents the critical layer location.

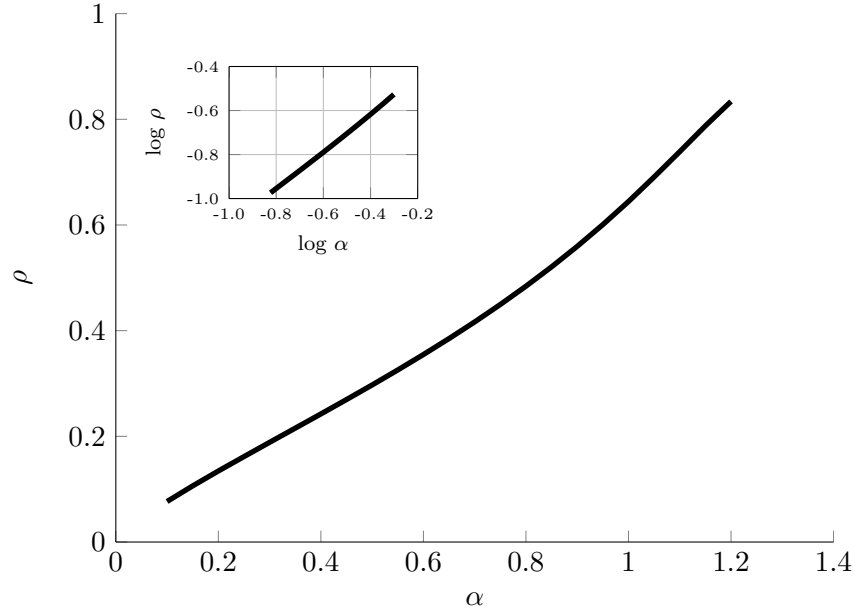


Figure 3.2. Variation of the wave amplitude ρ against the streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$ for the sinuous VWI state. The inset figure shows the variation on a log-log scale when $\alpha \rightarrow 0$.

Further, the $\alpha - \rho$ relationship indicates the magnitude of forcing required in order to drive a neutrally stable wave given the streamwise wavenumber of the VWI state. Indeed, as the streamwise extent of a VWI state decreases the forcing required to drive the VWI state is much larger. Figure 3.3 illustrates the variation of the stress jump conditions $J(z)$ and $K(z)$ for different values of the streamwise wavenumber and corroborates these observations.

Figure 3.4 shows the variation of the skin friction drag $\mathcal{S}$, given by (2.17), against the streamwise wavenumber α . We observe that the normalised skin friction drag has a minimum which is approximately 17.5% higher than the drag for plane Couette flow as $\alpha \rightarrow 0$. Over the range of values of the streamwise wavenumber α considered, we find that the skin friction drag can vary by 20% but is always much lower than the corresponding value for turbulent drag in channel flow as noted by Hall & Sherwin (2010).

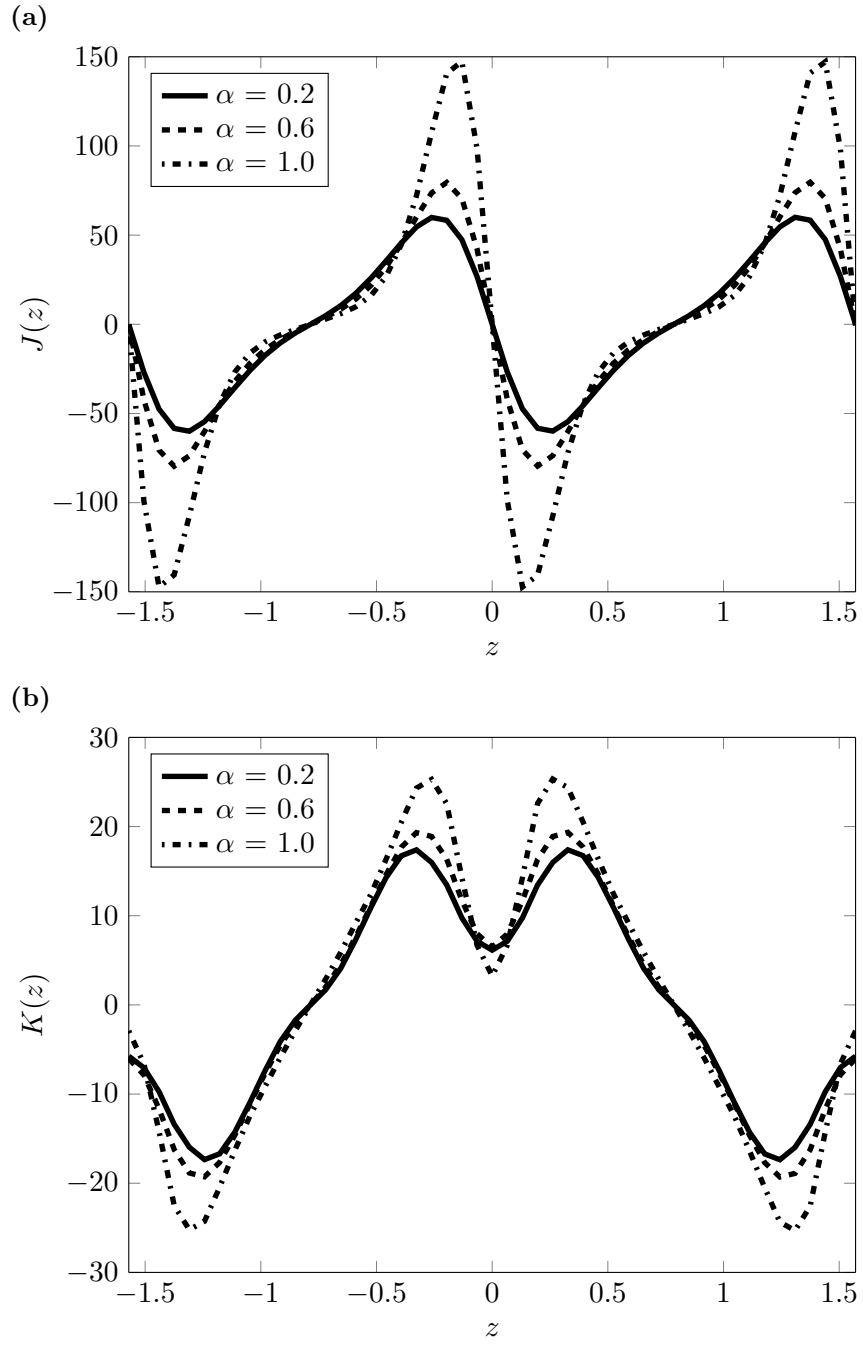


Figure 3.3. The stress jump conditions, (a) $J(z)$ and (b) $K(z)$, given by (2.9) for a sinuous VWI state at $\alpha = 0.2, 0.6, 1.0$ and $\beta = 2.0$.

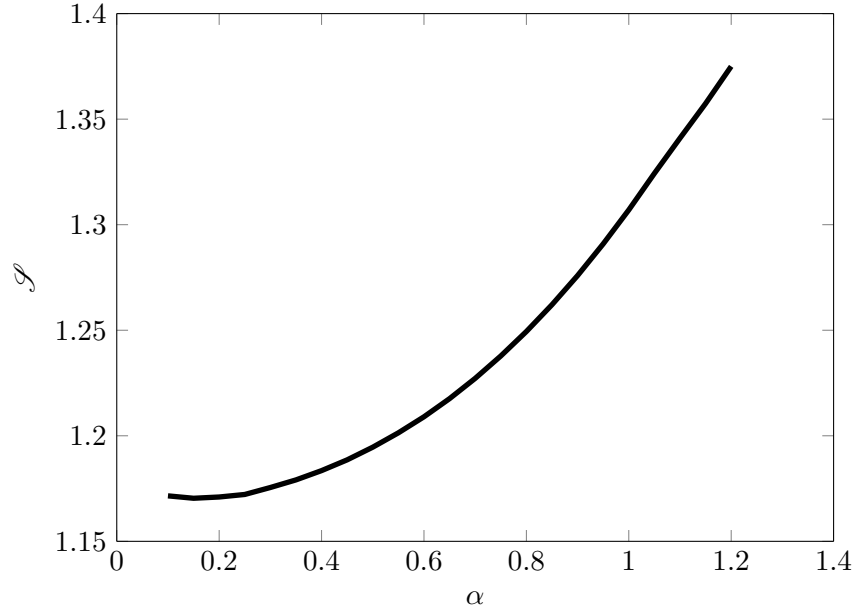


Figure 3.4. Variation of the wall skin friction $\mathcal{S}$ against the streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$ for the sinuous VWI state.

3.2.2. The Second Branch

Now let us turn our attention to the second branch of sinuous VWI states that are associated with the second most unstable stationary eigenmode of the streak field. The numerical procedure in section 2.2.4 is followed but now the linear growth rate of the second most unstable stationary eigenmode is driven to zero.

In order to initialise the numerical scheme we use the forcing from a VWI state from the first branch and artificially increase the wave amplitude to drive growth rate of the second most unstable stationary eigenmode to zero. The numerical procedure is then initialised with this streak profile and the second most unstable stationary eigenmode is used in all subsequent computations (i.e., when calculating the forcing terms).

It is worth noting that from a computational perspective, it is much easier to impose the hybrid body forcing term (2.14) since the location of the critical layer is handled automatically. Conversely, if the jump conditions (2.9) were imposed directly then not only does the critical layer location have to be parameterised, but a line integral over this parameterisation

needs to be calculated. Whilst not unattainable, difficulties arise when the critical layer parameterisation, $y = f(z)$, is a multivalued function in z . In this scenario great care has to be taken to ensure the parameterisation is sufficiently refined to ensure subsequent calculations are accurate. Further difficulties that arise in computing VWI states on the second branch are discussed in section 3.3.2.

A number of exploratory cycles were conducted with different initial conditions and a VWI state on the second branch was found at $(\alpha, \beta; R) = (0.7, 2.0; 50000)$. The resulting converged VWI state has skin-friction drag $\mathcal{S} = 4.633$ which compares well against the DNS solutions found by Deguchi & Hall (2014*c*) for these parameter values. The solution found corresponds to the so-called upper branch solution found by Nagata (1990) and corresponds to the solution EQ2 in the library of solutions provided by Gibson (2012).

Figure 3.5 illustrates the structure associated with the VWI state on the second branch at $(\alpha, \beta; R) = (0.7, 2.0; 50000)$. Examining the roll field $[0, v, w](y, z)$ we observe that it is much stronger in comparison to solutions from the first branch and that the bulk of the vorticity is now concentrated in a region near the channel walls. The appearance of the streak field $u(y, z)$ is also distinctively different from solutions on the first branch, notably the strong updraft at $z = 0$ and downdraft at $z = \pm\pi/\beta$ results in a streak profile with a plume-like appearance and the location of the critical layer is now multivalued in the spanwise direction z .

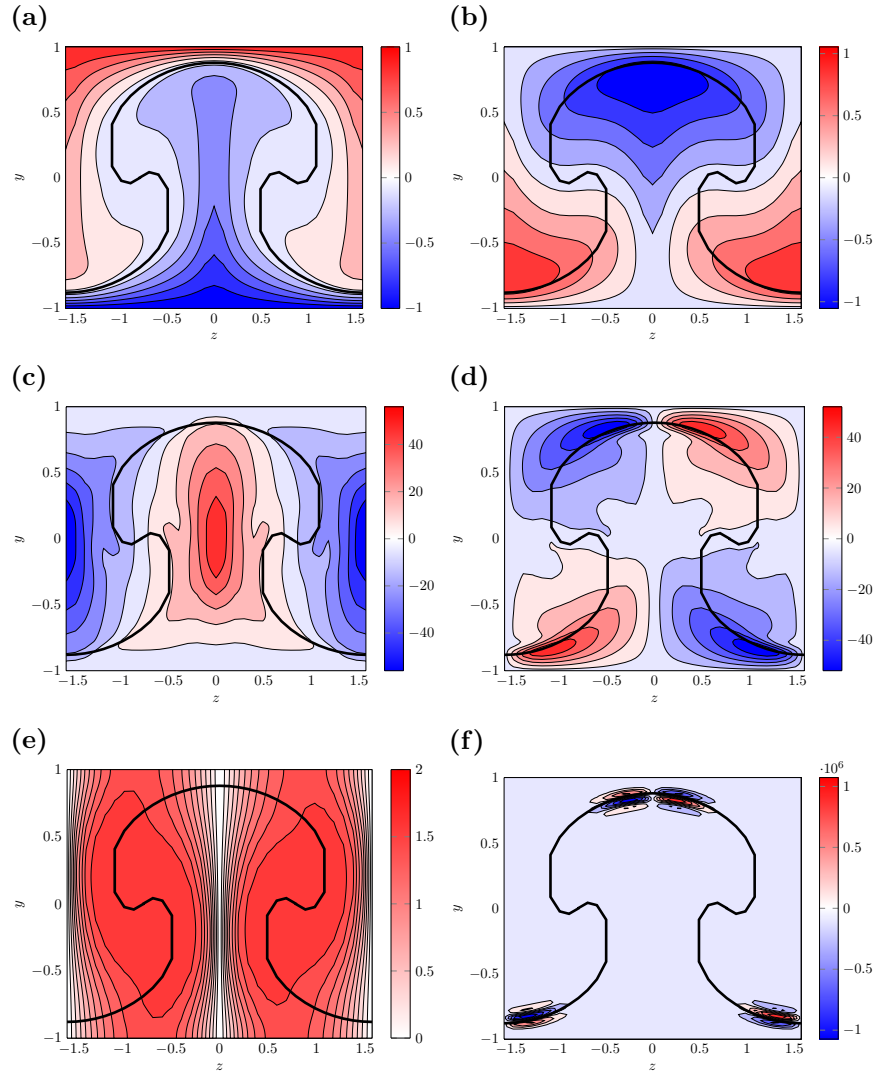


Figure 3.5. Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $|P(y, z)|$, and (f) the hybrid body forcing term (2.14) associated with a sinuous VWI state on the second branch at $(\alpha, \beta; R) = (0.7, 2.0; 50000)$. The solid bold curve represents the critical layer location.

3.3. Linear Stability Properties of Sinuous Streaks

VWI states exist due to the nonlinear self-interaction of a neutrally stable inviscid Rayleigh wave generating a stress jump across the critical layer. Whilst the wave associated with the VWI state is neutrally stable, the state may be unstable to waves at different wavenumbers or to viscous Tollmien-Schlichting waves.

To examine the linear stability of VWI states we can perturb the velocity and pressure fields of a VWI state, given by (2.3), by a perturbation of size ϵ of the form

$$\begin{aligned}\tilde{\mathbf{u}} = & [u(y, z), R^{-1}v(y, z), R^{-1}w(y, z)] + \dots \\ & + \rho R^{-7/6} [U(y, z), V(y, z), W(y, z)] e^{i\alpha(x-ct)} + c.c.\end{aligned}\quad (3.1)$$

$$+ \epsilon [\hat{U}(y, z), \hat{V}(y, z), \hat{W}(y, z)] e^{i\hat{\alpha}(x-\hat{c}t)} + c.c., \quad (3.2)$$

$$\begin{aligned}\tilde{p} = & R^{-2}p(y, z) + \rho R^{-7/6} P(y, z) e^{i\alpha(x-ct)} + c.c. \\ & + \epsilon \hat{P}(y, z) e^{i\hat{\alpha}(x-\hat{c}t)} + c.c.,\end{aligned}\quad (3.3)$$

where we have assumed the perturbation grows temporally and has the same spanwise periodicity of the base streak flow $u(y, z)$ but not necessarily the same streamwise wavenumber. We have denoted the wave-like disturbance quantities by a hat to distinguish these quantities against the wave associated with the VWI state.

If we assume that the size of the disturbance is smaller than the size of the wave associated with the VWI state, that is $\epsilon < \rho R^{-7/6}$, then the equations that describe the disturbance at leading order are exactly (2.68) with the only difference being notation (the disturbance quantities here have a hat). The numerical method outlined in section 2.2.3 can therefore be used to calculate the stability properties of the VWI state.

3.3.1. Stability of the First Branch of VWI States

Figure 3.6 shows how the linear growth rate varies with the streamwise wavenumber $\hat{\alpha}$ of the most unstable eigenmode for the streak profile $u(y, z)$ associated with a sinuous VWI state at $(\alpha, \beta; R) = (0.7, 2.0, 16000)$. It is precisely the most unstable eigenmode which is associated with the first branch of solutions and so, unsurprisingly, the linear growth rate of the

eigenmode is zero at $\hat{\alpha} = 0.7$. The streak is unstable to long waves and the linear growth rate has a maximum of $\sigma = 0.0066$ at $\hat{\alpha} \approx 0.35$.

Figure 3.7 shows the linear growth rate of the leading eigenvalue of the streak profile at $\hat{\alpha} = 0.35$ for different values of the Reynolds number R . As noted previously in section 3.1 we observe that there is a well-defined limit for large values of the Reynolds number where the growth rate remains unchanged, corresponding to the inviscid limit.

3.3.2. Stability of the Second Branch of VWI States

Now let us examine the linear stability properties of the second branch of sinuous VWI states by performing analyses on the streak profile associated with the second branch state shown in figure 3.5.

The leading (most unstable) eigenvalues of the streak profile associated with the first and second branch sinuous VWI states at $(\alpha, \beta) = (0.7, 2.0)$ is shown in figure 3.8. The figure highlights the major difference between first and second branch solutions, namely that the eigenspectrum of the second branch contains a number of linearly unstable modes, including travelling modes, whereas the eigenspectrum of the first branch just contains linearly stable modes.

To further appreciate the behaviour of the second branch of VWI states, figure 3.9 shows how the linear growth rate varies for a number of eigenmodes as the streamwise wavenumber $\hat{\alpha}$ varies. In the figure, the growth rate associated with a stationary eigenmode is denoted by a solid curve and labelled with the prefix S and the growth rate associated with a pair of travelling waves (with opposite wavespeeds) is denoted by a dashed curve and labelled with the prefix T .

The $S1$ curve, which exists in the range $\hat{\alpha} \in [0, 1.01]$, corresponds to the most unstable stationary eigenmode and is precisely the eigenmode driven to zero to obtain the first branch of VWI states as discussed in section 3.2.1. The $S2$ curve corresponds to the second most unstable stationary eigenmode and is therefore responsible for the second branch of VWI states as discussed in section 3.2.2 and the VWI state shown in figure 3.5. It is worth noting that the $S2$ curve exists in the smaller range $\hat{\alpha} \in [0.59, 1.01]$ and that $\hat{\sigma} = 0.0$ at $\hat{\alpha} = 0.7$ as expected. The genesis of the $S2$ curve around $\hat{\alpha} = 0.59$ is due to a pair of travelling eigenmodes $T1$ (with opposite

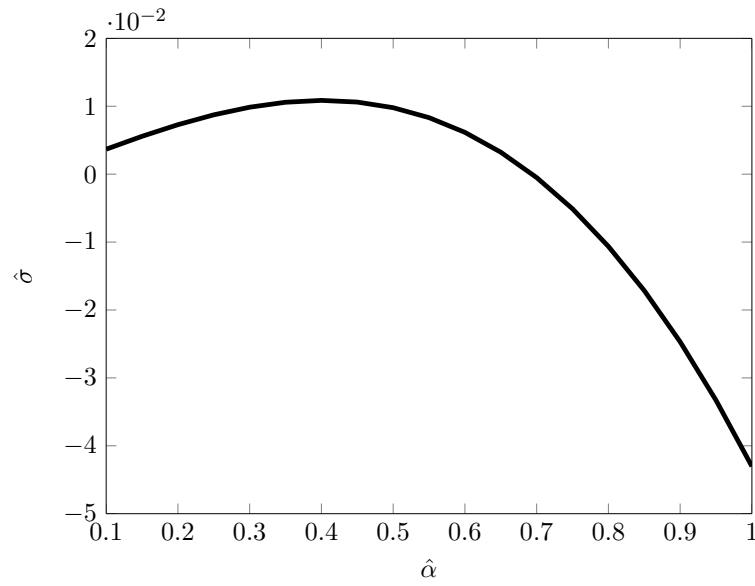


Figure 3.6. Variation of the linear growth rate $\hat{\sigma} = \hat{\alpha}c_i$ against the stream-wise wavenumber $\hat{\alpha}$ of the most unstable eigenmode of the streak $u(y, z)$ associated with the VWI state at $(\alpha, \beta; R) = (0.7, 2.0; 16000)$.

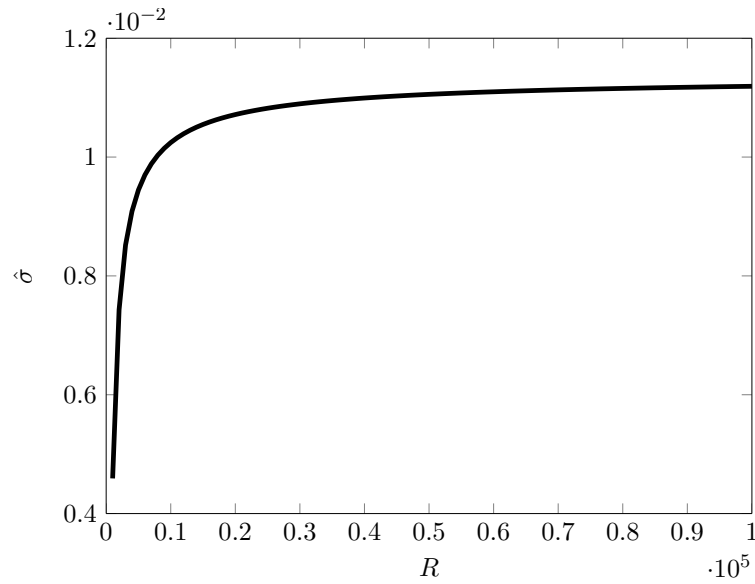


Figure 3.7. Variation of the linear growth rate $\hat{\sigma} = \hat{\alpha}c_i$ against the Reynolds number R of the most unstable eigenmode of the streak $u(y, z)$ associated with the VWI state at $(\alpha, \beta; R) = (0.7, 2.0; 16000)$ at $\hat{\alpha} = 0.35$.

wavespeeds) undergoing a transition to two stationary eigenmodes ($S2, S3$). A detailed view of the transition $T1 \rightarrow (S2, S3)$ is presented in figure 3.8a. In a similar manner, at $\hat{\alpha} = 1.01$ the $S1$ and $S2$ eigenmodes undergo a transition to a pair of travelling waves denoted $T2$, a detailed view of the transition can be seen in figure 3.8b.

A consequence of the eigenmode behaviour of streak profiles associated with second branch VWI states is the difficulty in constructing an efficient autonomous iteration procedure to drive the linear growth rate to zero. For the case of finding VWI states on the first branch, the most unstable stationary eigenmode is well-defined over all values of the streamwise wavenumber, see figure 3.6, and so for each value of the wave amplitude ρ the resulting streak profile has a well-defined growth rate which can be driven to zero. In contrast, when conducting the same iteration procedure for VWI states on the second branch, for each value of the wave amplitude ρ , the growth rate of the resulting streak profile exists in a small range of values of the streamwise wavenumber. Therefore, depending on the value of ρ the growth rate associated with the eigenmode on $S2$ at the desirable value of the streamwise wavenumber may not exist. Whilst difficulties arising from this can be alleviated by appropriate exception handling, a more systematic, although computationally expensive, approach would be to perform a sweep over a set of values of the wave amplitude ρ to find the range over which the $S2$ curve exists before performing the iteration procedure.

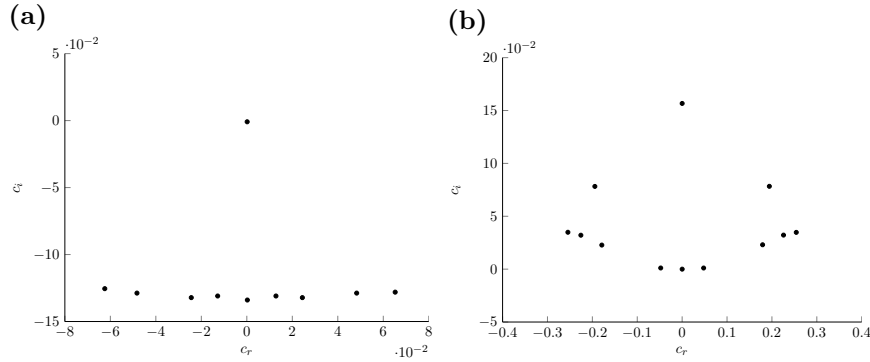


Figure 3.8. The leading (most unstable) eigenvalues for the streak profile associated with a VWI state on the (a) first branch at $(\alpha, \beta; R) = (0.7, 2.0, 16000)$ and (b) the second branch at $(\alpha, \beta; R) = (0.7, 2.0, 50000)$.

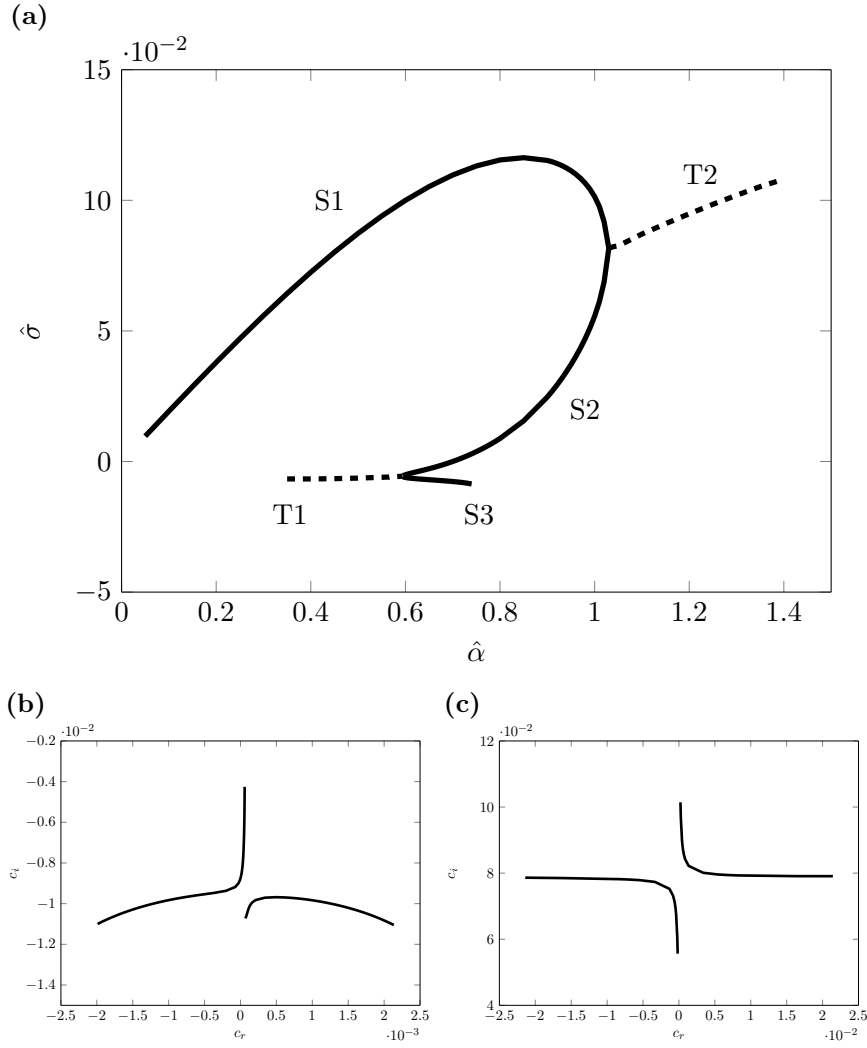


Figure 3.9. (a) Variation of the linear growth rate $\hat{\sigma} = \hat{\alpha}c_i$ against the streamwise wavenumber $\hat{\alpha}$ for a number of eigenmodes of the streak $u(y, z)$ associated with the VWI state at $(\alpha, \beta; R) = (0.7, 2.0, 50000)$. $S1, S2, S3$ correspond to stationary eigenmodes and $T1, T2$ correspond to pairs of travelling eigenmodes (with opposite wavespeeds). (b) Close-up of the region of transition from $T1 \rightarrow (S2, S3)$. (c) Close-up of the region of transition from $(S1, S2) \rightarrow T2$.

4. Mirror-Symmetric Modes

In this chapter we continue our exploration of VWI states by calculating solutions that are often called mirror-symmetric states. This class of solutions corresponds to EQ7/EQ8 in Gibson *et al.* (2008), HVS in Itano & Generalis (2009) and has been briefly documented in the context of VWI theory by Deguchi & Hall (2014*a*).

4.1. Accuracy of the Numerical Scheme

Before discussing the nature of mirror-symmetric VWI states we will make a few remarks about the accuracy of our numerical scheme. In section 2.2 a number of different methods to solve the vortex equations were discussed; to solve the vortex equations we can either impose the jump conditions (2.9) directly or impose the hybrid body forcing term (2.14) and we have the choice between using either the collocation approach outlined in section (2.2.1) or the boundary integral approach outlined in section (2.2.2).

Table 4.1 contains tabulated values of the converged wave amplitude ρ after computing a mirror-symmetric VWI state by imposing the jump conditions directly using the collocation and boundary integral approaches. The data demonstrates that, at fixed values of the streamwise and spanwise wavenumbers (α, β) , the value of the wave amplitude ρ becomes independent of the grid resolution as the number of grid points in y or Fourier modes in z is increased. Further, the different methods utilised to solve the vortex equations give comparable values of ρ despite imposing the jump conditions directly using different numerical methods. For two significant figures of accuracy in ρ a resolution of $(N_y, N_z) = (257, 33)$ was found to be suitable.

Table 4.2 contains tabulated values of the converged wave amplitude ρ after computing a mirror-symmetric VWI state by imposing the hybrid body forcing term (2.14) using the collocation approach at a number of different grid resolutions. It was found that at least 257 grid points in y

were required to capture the body forcing term, which is concentrated in a region of size $O(R^{-1/3})$ at the critical layer location, to sufficient accuracy. For large grid resolutions it was found that the numerical procedure to solve the roll equations subject to the body forcing term became computationally prohibitive, at least with the computing resources available at the time of conducting these studies, and so we will make no further remarks about the convergence of ρ when imposing the hybrid forcing. Nevertheless, the converged values of ρ obtained are comparable to the values obtained by imposing the jump conditions directly as shown in table 4.1.

The effect of the regularisation Reynolds number used in solving the wave equation (2.6) is tabulated in table 4.3. Whilst mirror-symmetric states can be found at moderate values of the regularisation Reynolds number, we observe that the value of ρ is independent of the regularisation Reynolds number when $R \sim O(10^6)$.

(a)

		N_z			
		17	33	49	65
N_y	257	0.32345	0.30942	0.30916	0.30902
	513	0.32333	0.30931	0.30902	0.30888
	1025	0.32330	0.30928	0.30888	—

(b)

		N_z			
		17	33	49	65
N_y	257	0.35886	0.31755	0.30900	0.30517
	513	0.35875	0.31752	0.30901	0.30507
	1025	0.35873	0.31750	0.30899	0.30506

Table 4.1. Convergence of wave amplitude ρ as the number of grid points in y , N_y , and the number of Fourier modes in z , N_z , is increased when calculating the mirror-symmetric state with $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$ by imposing the jump conditions (2.9) directly using the (a) collocation approach outlined in section 2.2.1, and (b) boundary integral approach outlined in section 2.2.2.

		N_z	
		17	33
N_y	257	0.34195	0.33586
	513	0.34118	0.33461

Table 4.2. Convergence of the wave amplitude ρ as the number of grid points in y , N_y , and the number of Fourier modes in z , N_z , is increased when calculating the mirror-symmetric state with $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$ by imposing the hybrid body forcing term (2.14) using the collocation approach outlined in section 2.2.1.

R	25000	50000	100000	250000	500000	750000	1000000
ρ	0.34819	0.33439	0.32492	0.31633	0.31216	0.31042	0.30942

Table 4.3. Convergence of wave amplitude ρ with regularisation Reynolds number R used in solving the wave system (2.68), with the numerical approach of section 2.2.4, for the mirror-symmetric VWI state with $(\alpha, \beta) = (0.6, 2.0)$.

4.2. Exploration of State Space

4.2.1. The First Sub-2 Branch

Recall that the first branch of VWI states is the class of VWI states associated with the most unstable eigenmode of the streak. To find VWI states that belong to this class of solutions we follow the numerical procedure outlined in section 2.2.4 and drive the growth rate of the most unstable stationary eigenmode of the streak to zero.

Using an initial streak profile of the form (2.93) and running exploratory cycles for different values of the wave amplitude ρ , a VWI state at $(\alpha, \beta) = (0.6, 2.0)$ belonging to the first branch was found. Figure 4.1 illustrates the structure associated with this VWI state, similar features are observed at other values of (α, β) . Note that the solution corresponds to EQ7/EQ8 in the library of solutions provided by Gibson (2012). The streaky flow $u(y, z)$ and the roll flow $[0, v, w](y, z)$ are quite distinct to that of sinuous VWI states (as seen in chapter 3) in a number of ways, see figure 3.1 for a direct comparison. Notably, the spanwise wavenumber of the streak and roll field is double that of the spanwise wavenumber of the wave field (note that figure 4.1e shows the absolute value of the wave pressure field not the real or imaginary part) and so the wave exists as a subharmonic of the streak field in the spanwise direction. The critical layer is also now located at the channel mid-height, $y = 0$, and the flow field quantities are strongest in this region which can be seen by examining the perturbation of the streak field away from plane Couette flow, $u(y, z) - y$, which illustrates the low- and high-speed regions of the streak field. Note that we have called this branch of solutions the “first sub-2 branch” as the wave is a subharmonic with double the periodicity of the underlying vortex field in the spanwise direction.

The relationship between the wave amplitude ρ and the wavenumbers (α, β) is shown in figures 4.2 and 4.3. The data was obtained by using the converged VWI state at $(\alpha, \beta) = (0.6, 2.0)$ as an initial guess and running exploratory cycles for different values of (α, β, ρ) .

We observe that in the small streamwise wavenumber limit, $\alpha \rightarrow 0$, the balance $\rho \sim \alpha^{5/6}$ such that the jump conditions (2.9) remain $O(1)$ can be identified and is shown in the inset figure where the variation in ρ is shown on a log-log scale. In this distinguished limit the vortex and wave

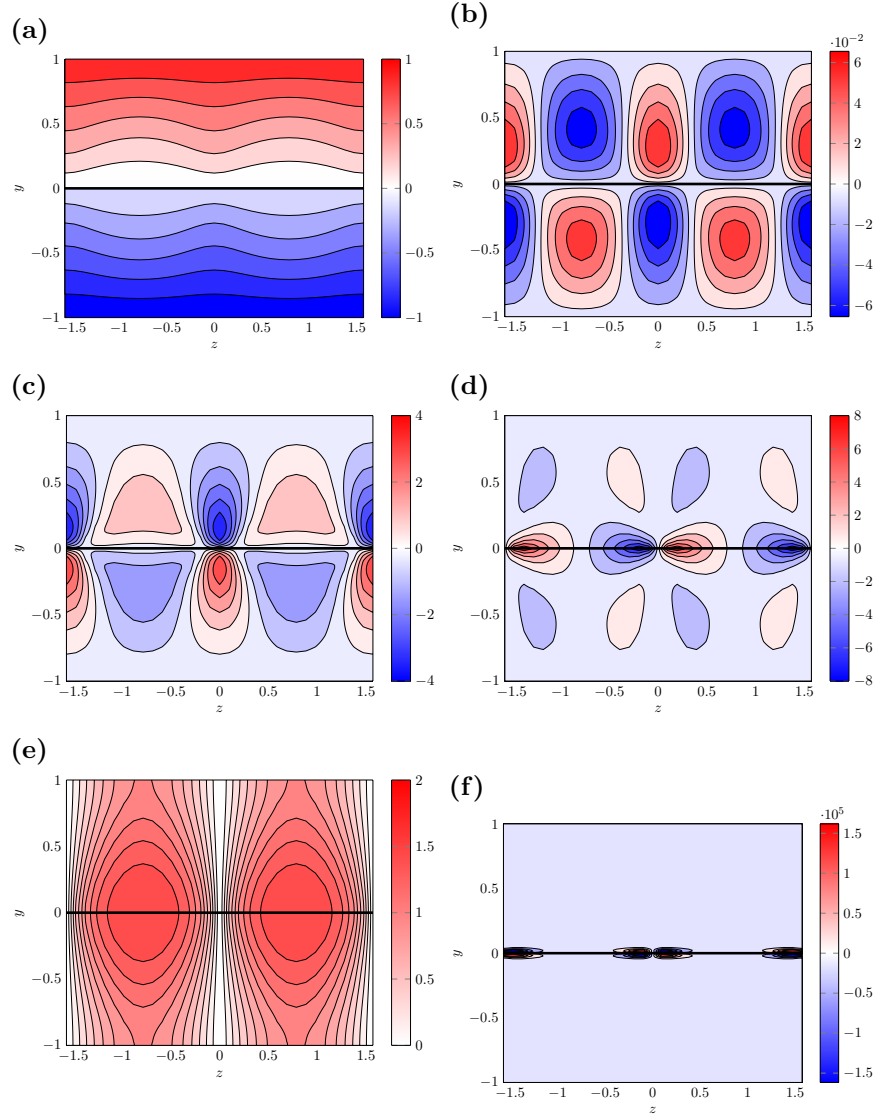


Figure 4.1. Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $|P(y, z)|$, and (f) the hybrid body forcing term (2.14) associated with a sub-2 mirror-symmetric VWI state at $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$. The solid bold curve represents the critical layer location.

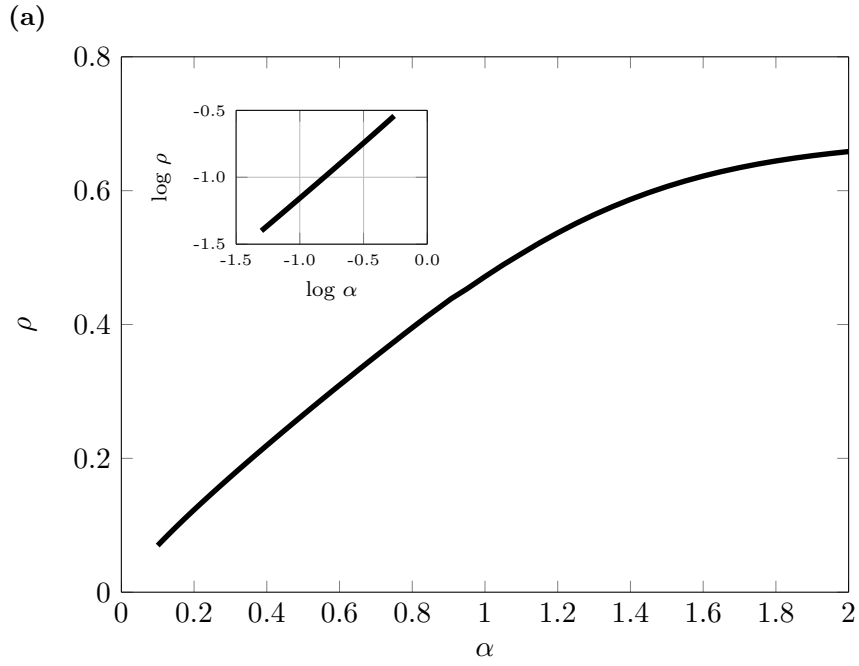


Figure 4.2. Variation of the wave amplitude ρ against the streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$. The inset figure shows the variation on a log-log scale when $\alpha \rightarrow 0$.

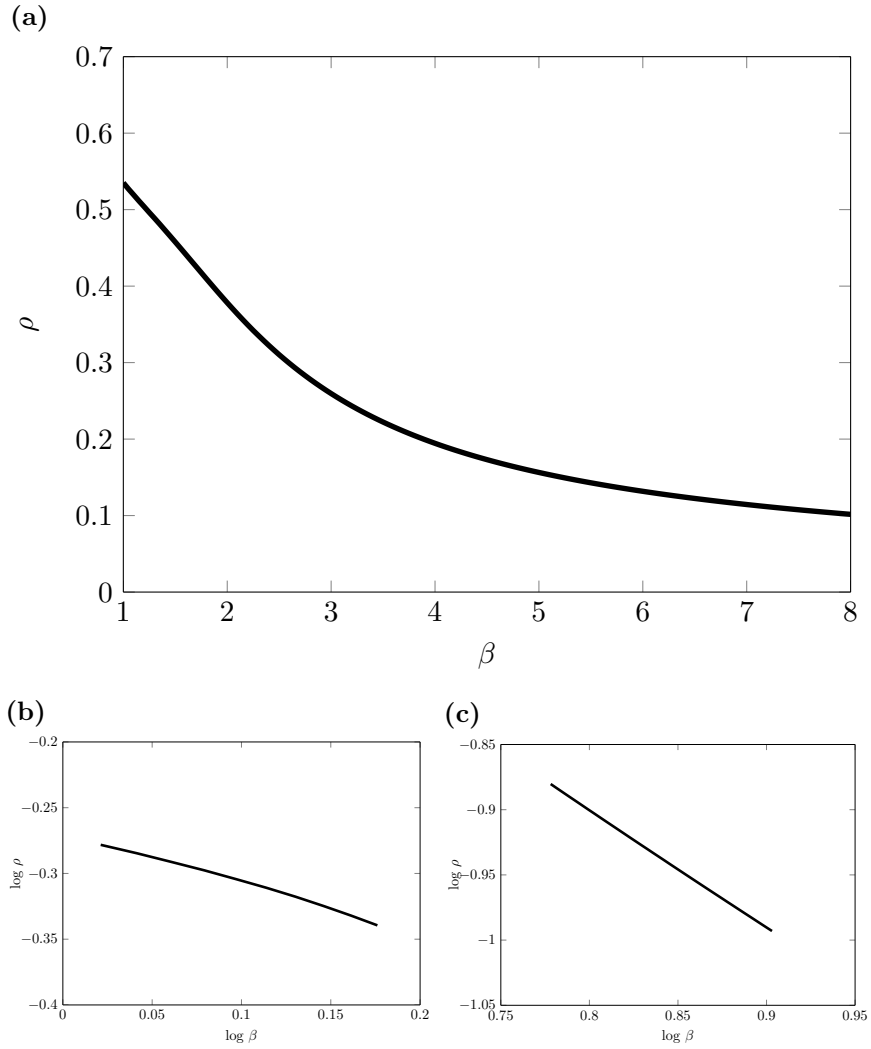


Figure 4.3. (a) Variation of the wave amplitude ρ against the spanwise wavenumber β for fixed streamwise wavenumber $\alpha = 0.76$. (b) Variation on a log-log scale when $\beta \rightarrow 0$. (c) Variation on a log-log scale when $\beta \rightarrow \infty$.

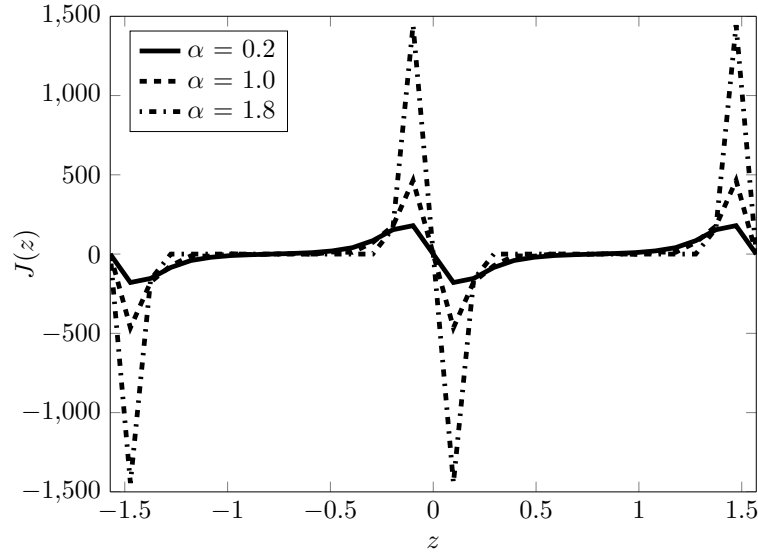


Figure 4.4. The stress jump condition $J(z)$ given by (2.9) at different values of the streamwise wavenumber α for a sub-2 mirror-symmetric VWI states with $\beta = 2.0$.

decouple and become comparable in size. Comparing the pressure from the roll and wave we find $R^{-2} \sim \rho R^{-7/6}$, hence $\alpha \sim R^{-1}$ provides the critical value, proportional to the Reynolds number, at which streamwise localisation occurs and VWI theory fails to describe the states, see Deguchi *et al.* (2013) for more details.

At large values of the streamwise wavenumber α we observe that the forcing in the critical layer region becomes much stronger. This can be seen in figure 4.4 where the stress jump condition $J(z)$ given by (2.9) is plotted at different values of the streamwise wavenumber α for fixed $\beta = 2.0$. The convergence of the Fourier series representation of $J(z)$ for large values of α is quite poor and so it is advantageous to use the boundary integral approach outlined in section 2.2.2 (where the Fourier transform of $J(z)$ is not required) when computing VWI states with large values of α . Unlike the sinuous VWI states, where there is a definitive turning point at $\alpha_c \approx 1.4$, there does not appear to be any indication that there is a critical value of α for mirror-symmetric states. In fact, for large values of α we observe from the computations conducted that the value of ρ asymptotes to $\rho \approx 0.675$ for fixed $\beta = 2.0$.

In the small spanwise wavenumber limit, $\beta \rightarrow 0$, we observe that the

VWI states begin to localise in the spanwise direction. It can be seen that in this limit that $\rho \sim \beta$, which can be seen in the subfigure where the variation on a log-log scale is shown. In fact, as the box-size gets larger the states seemingly cannot maintain themselves over the full spanwise extent of their domain and for large values of α we find that the states occupy a region of size $z \sim O(\alpha^{-1})$ compared to the $O(\beta^{-1})$ domain size. This is illustrated in figure 4.5 where the roll velocity components have been plotted for $\alpha = 0.2, 0.8, 1.4$ and $\beta = 1.0$.

In the large spanwise wavenumber limit, $\beta \rightarrow \infty$, the asymptotic structure identified by Blackburn *et al.* (2013) can be identified, namely the balance $\rho \sim \beta^{-1}$ which can be seen in the subfigure where the variation on a log-log scale is shown. For large values of the spanwise wavenumber β the vertical extent of the states become proportional to the spanwise wavelength L_z and as a result the solutions become independent of the constraint at the walls. This is illustrated in figure 4.6 where the roll velocity components have been plotted for $\beta = 2, 4, 8$ and $\alpha = 0.76$. In fact, as shown by Blackburn *et al.* (2013) for large values of β the nonlinear eigenrelation between the wave amplitude ρ and the wavenumbers (α, β) takes on the functional form

$$\rho\beta^{1/6} = \left(\frac{\alpha}{\beta}\right)^{5/6} S\left(\frac{\alpha}{\beta}\right) \quad (4.1)$$

where $(\alpha/\beta)^{5/6}$ is the behaviour required for $(\alpha/\beta) \rightarrow 0$ and the function S represents the deviation from this scaling when (α/β) is large. The collapse of the large β eigenrelation is shown in figure 4.7 for $\beta = 5, 6, 7, 8$.

Figure 4.8 shows the variation of the skin friction drag $\mathcal{S}$, given by (2.17), with respect to the streamwise and spanwise wavenumbers (α, β) . The skin friction drag $\mathcal{S}$ increases monotonically with the streamwise wavenumber α . This is to be expected as the wave amplitude ρ also increases monotonically with the streamwise wavenumber α and the wave amplitude can be thought of as a measure of the strength of forcing required to maintain an equilibrium state, or indirectly the strength of the perturbation to the streak field away from plane Couette flow. In terms of the spanwise wavenumber β , we observe that for fixed streamwise wavenumber α there is a maxima of the skin friction drag for some value of β . For large values of β we have already noted that the states begin to localise in a region at

the channel mid-height, refer to figure 4.6, and so we would not anticipate any variations in the streak profile near the channel walls, hence the skin friction drag decreases with increasing β . For small values of β we have already noted that the states begin to undergo localisation in the spanwise direction, so whilst both the wave amplitude and spanwise wavelength increase, the net effect is a decrease in the skin friction drag over one spanwise period due to any variations in the streak profile being concentrated where the states have localised, rather than over the full spanwise extent of the channel. Finally, note that the skin friction drag associated with this branch of mirror-symmetric VWI states is much lower than that of the sinuous VWI states due to the main activity of the interaction taking place in the middle of the channel rather than the full extent of the channel.

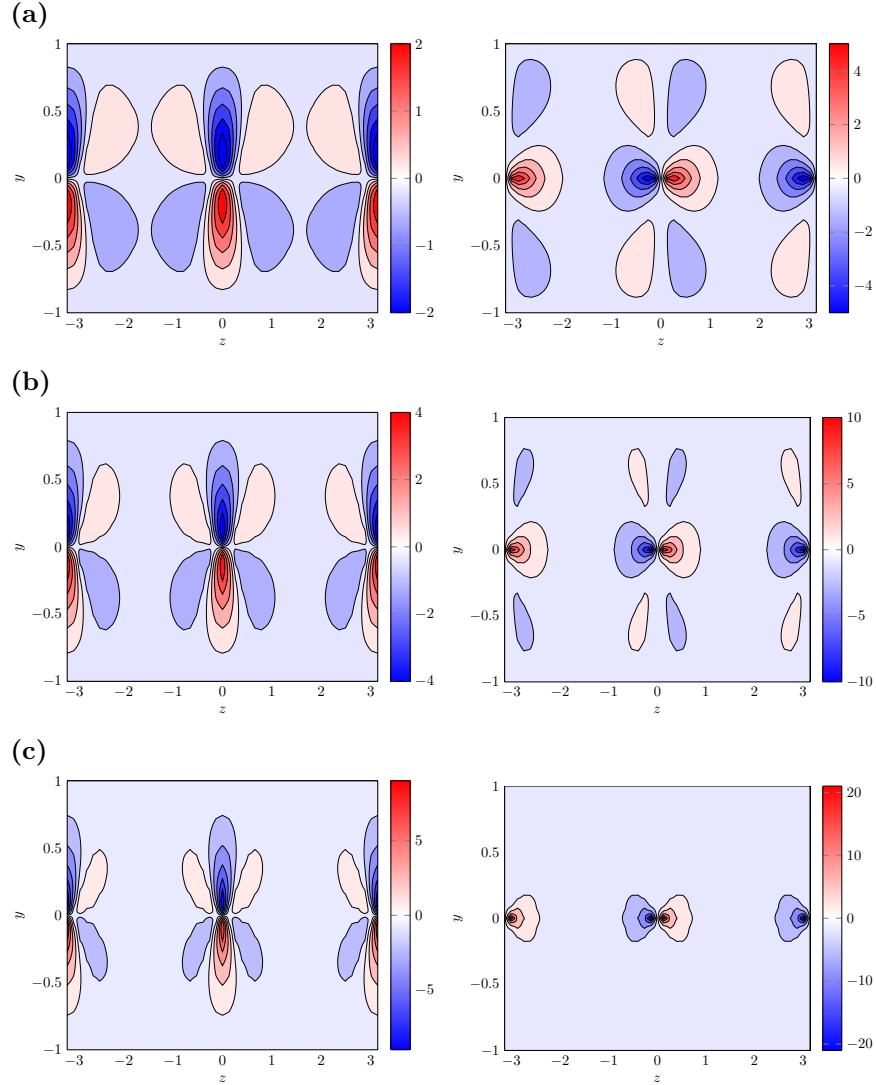


Figure 4.5. Equispaced contours of the roll velocity field for a sub-2 mirror-symmetric vortex-wave interaction states in plane Couette flow. Left, wall-normal roll velocity field; right, spanwise roll velocity field. (a) $(\alpha, \beta) = (0.2, 1.0)$, (b) $(\alpha, \beta) = (0.8, 1.0)$, (c) $(\alpha, \beta) = (1.4, 1.0)$.

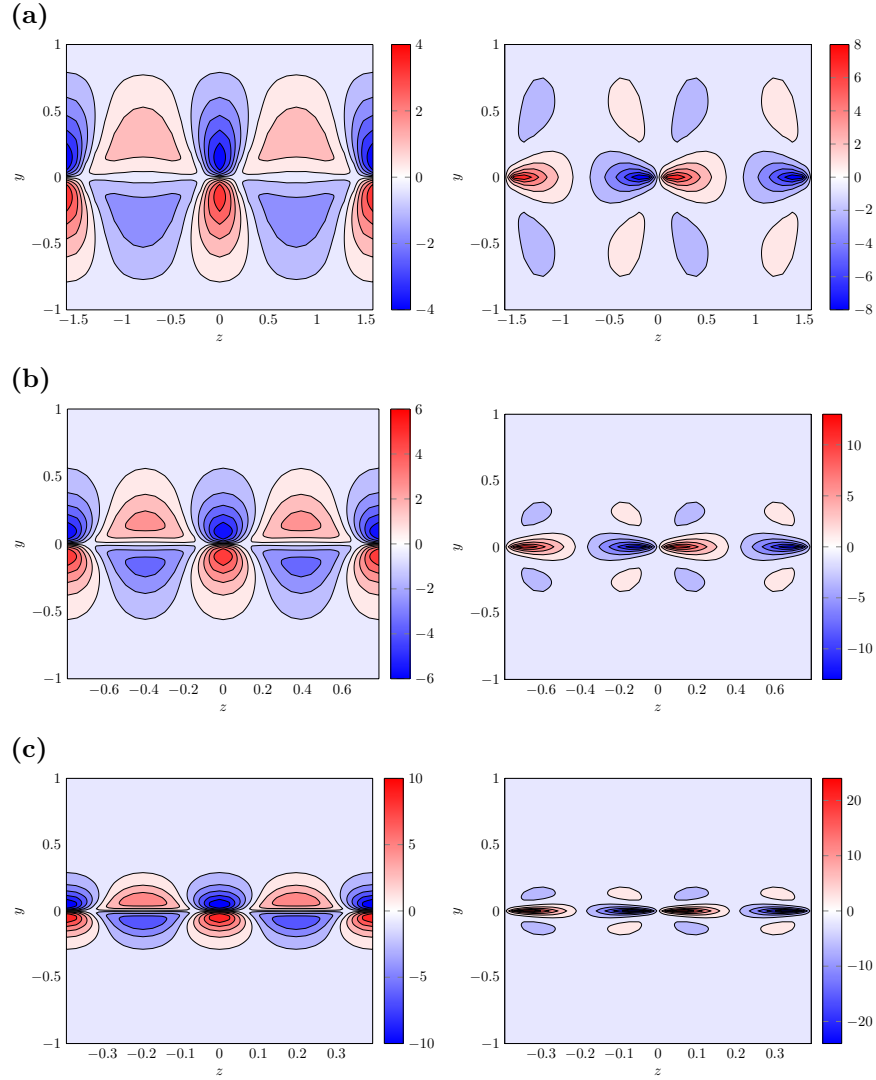


Figure 4.6. Equispaced contours of the roll velocity field for a sub-2 mirror-symmetric vortex-wave interaction states in plane Couette flow. Left, wall-normal roll velocity field; right, spanwise roll velocity field. (a) $(\alpha, \beta) = (0.76, 2.0)$, (b) $(\alpha, \beta) = (0.76, 4.0)$, (c) $(\alpha, \beta) = (0.76, 8.0)$.

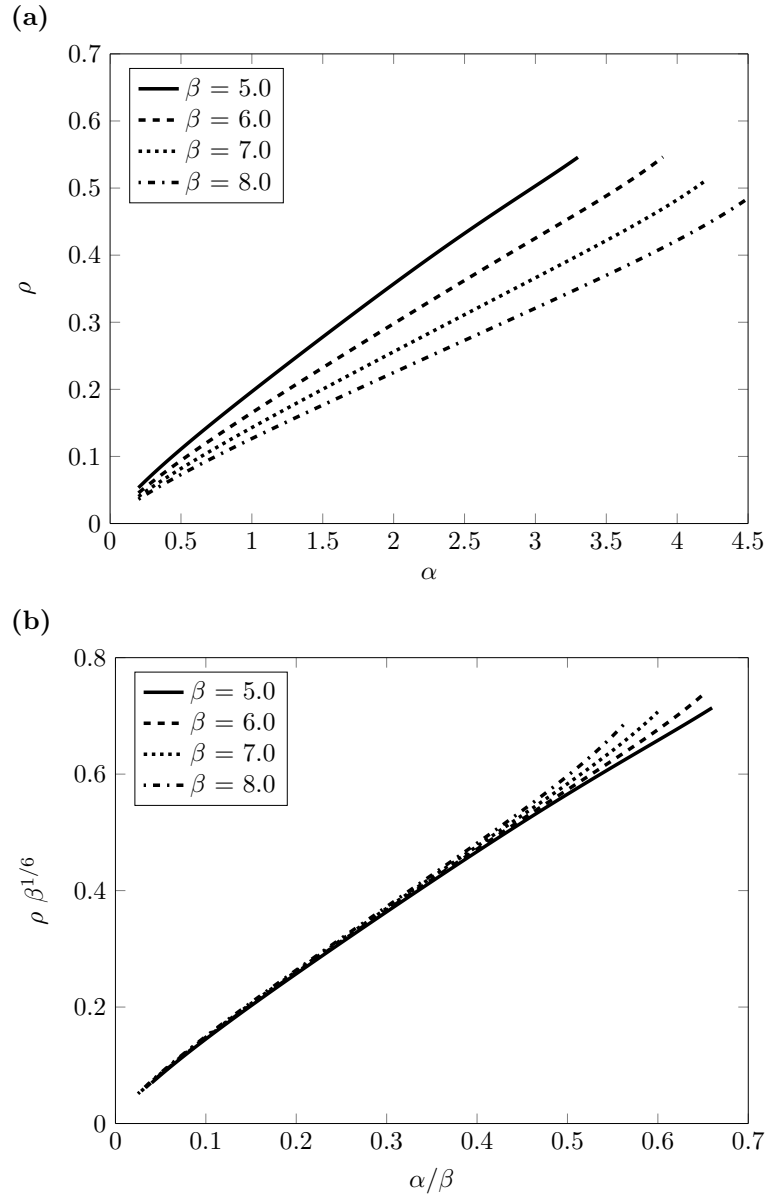


Figure 4.7. (a) Variation of the wave amplitude ρ against the stream-wise wavenumber α for large values of the spanwise wavenumber β . (b) Collapse of the large- β nonlinear eigenrelation data in agreement with the relationship (4.1).

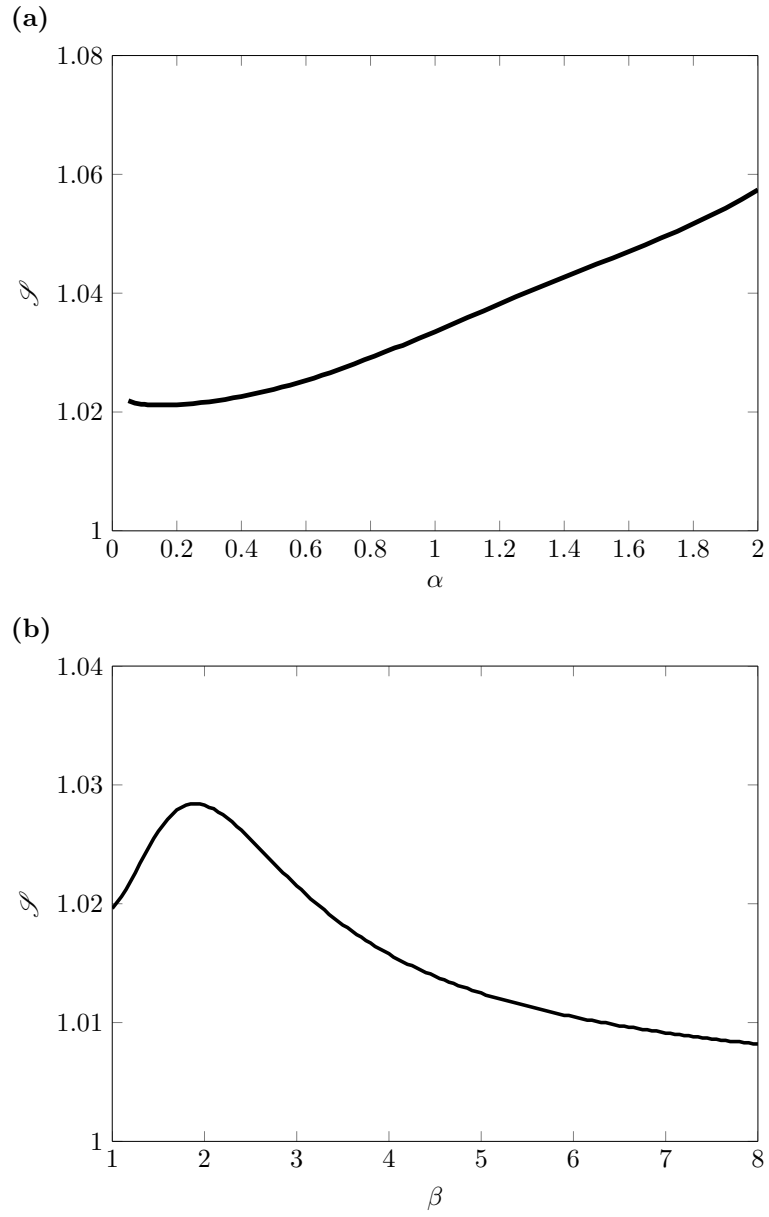


Figure 4.8. Variation of the skin friction drag $\mathcal{S}$ against the (a) streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$ and (b) spanwise wavenumber β for fixed streamwise wavenumber $\alpha = 0.76$ for a sub-2 mirror-symmetric VWI state.

4.2.2. The First Sub-4 Branch

The second branch of mirror-symmetric VWI states is the class of VWI states associated with the second most unstable stationary eigenmode of the streak. To find VWI states that belong to this class of solutions we follow the numerical procedure outlined in section 2.2.4 and drive the linear growth rate of the second most unstable stationary mode to zero. A number of exploratory cycles were conducted with different initial conditions to ensure the numerical scheme converged to the same solution. Further, the two approaches of imposing the jump conditions (2.9) directly and imposing the hybrid body forcing term (2.14) resulted in the same solution giving further confidence in the converged solution.

Despite the number of studies conducted, solutions that exist on the second branch in an analogous manner to those found previously in section 3.2.2 could not be found. Instead, a class of solutions were found that exist with two stationary neutral eigenmodes¹ where the wave exists as a subharmonic with quadruple periodicity of the underlying vortex field, rather than double in the case of the sub-2 branch. Therefore, we have refrained from calling this branch of solutions the second branch and instead have called this branch of solutions the “first sub-4 branch”.

Figure 4.9 illustrates the structure associated with a mirror-symmetric VWI states at $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$ on the first sub-4 branch, similar features hold for other values of (α, β) . Notably, we observe that the wave is a subharmonic with a periodicity of a quarter of the underlying vortex field as suggested by the name of this branch of solutions. Further, the streaky flow $u(y, z)$ and the roll flow $[0, v, w](y, z)$ are strongly concentrated in a region centered around the channel mid-height.

In a similar manner to the sub-2 branch of mirror-symmetric VWI states, the relationship between the wave amplitude ρ and the wavenumbers (α, β) can be found by running exploratory cycles for different values of (α, β, ρ) . Figure 4.10 shows the variation of the wave amplitude ρ as a function of the streamwise wavenumber α for fixed $\beta = 2.0$ for this branch of solutions.

We observe that in the small streamwise wavenumber limit, $\alpha \rightarrow 0$, the balance $\rho \sim \alpha^{5/6}$ such that the jump conditions (2.9) remain $O(1)$ can be identified and is shown in the inset figure where the variation in ρ is shown

¹See section 4.3 for further discussion on the eigenmodes.

on a log-log scale. For large values of α there is no indication of a horizontal asymptote as in the case of the first sub-2 branch, at least for the values of α considered.

Examination of the $\alpha - \rho$ relationships for both the sub-2 and sub-4 branches suggests that the curves may connect for some small value of α . Whether this point is within the context of VWI theory or is beyond the $\alpha \sim R^{-1}$ distinguished limit where the theory of Deguchi *et al.* (2013) should be followed is beyond the scope of the current study, but we can postulate that the sub-2 branch of solutions undergoes a period doubling bifurcation.

Figure 4.11 shows the variation of the skin friction drag $\mathcal{S}$ of the sub-4 branch of mirror-symmetric VWI states as the streamwise wavenumber α is varied for fixed spanwise wavenumber $\beta = 2.0$. The skin friction drag $\mathcal{S}$ increases monotonically as the streamwise wavenumber α is increased and has a minimum which is approximately 1.5% higher than the drag for plane Couette flow as $\alpha \rightarrow 0$. Finally, note that the skin friction drag associated with this branch of VWI states is much lower than that for the sub-2 mirror-symmetric branch due to the increase in periodicity leading to solutions which are more strongly concentrated around the channel mid-height.

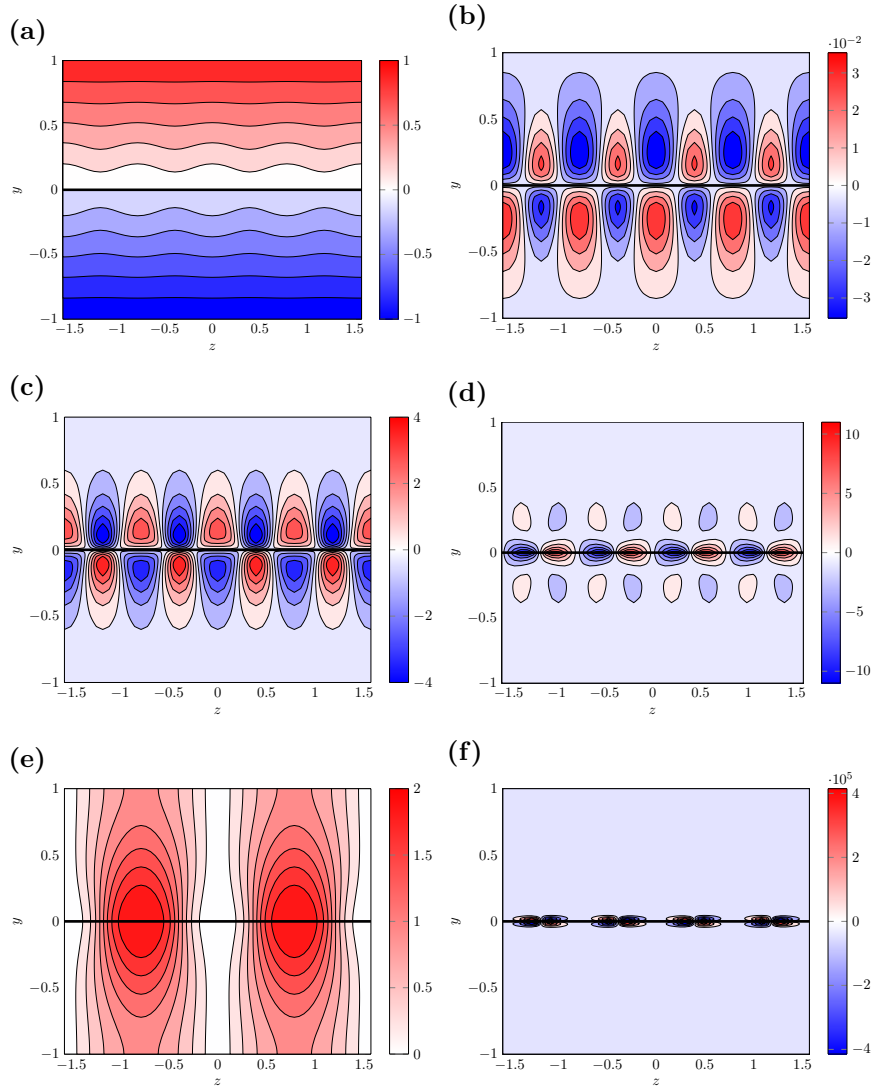


Figure 4.9. Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $|P(y, z)|$, and (f) the hybrid body forcing term (2.14) associated with a sub-4 mirror-symmetric VWI state at $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$. The solid bold curve represents the critical layer location.

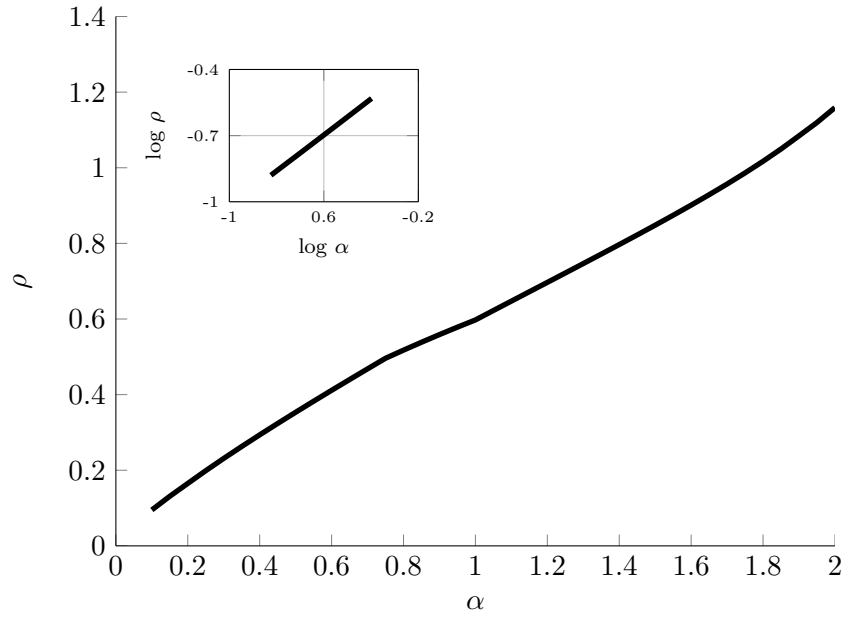


Figure 4.10. Variation of the wave amplitude ρ against the streamwise wavenumber α for a sub-4 mirror-symmetric VWI state with fixed spanwise wavenumber $\beta = 2.0$.

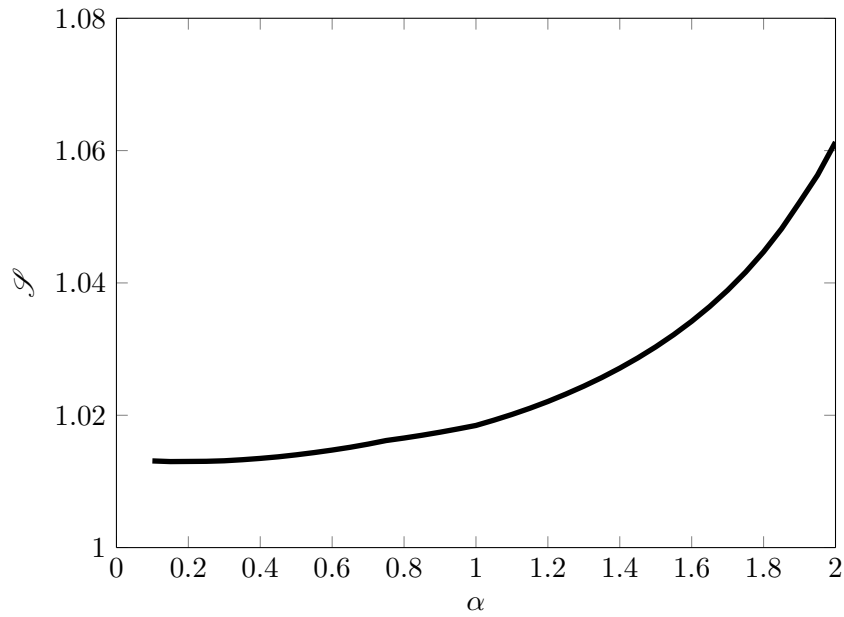


Figure 4.11. Variation of the skin friction drag $\mathcal{S}$ against the streamwise wavenumber α for a sub-4 mirror-symmetric VWI state with fixed spanwise wavenumber $\beta = 2.0$.

4.3. Linear Stability Properties of Mirror-Symmetric VWI Streaks

As previously discussed in section 3.3.2 we can use the numerical method outlined in section 2.2.3 to perform analyses into the linear stability of the streak profiles associated with VWI states. In this section we examine the linear stability of streak profiles associated with mirror-symmetric VWI states.

Figure 4.12 shows how the linear growth rate of the most unstable eigenmode varies with the streamwise wavenumber $\hat{\alpha}$ for the streak profile $u(y, z)$ associated with a mirror-symmetric VWI state on the sub-2 branch at $(\alpha, \beta; R) = (0.6, 2.0, 10^6)$. It is precisely the most unstable eigenmode which is associated with this branch of VWI states and so, unsurprisingly, the linear growth rate of the eigenmode is zero at $\hat{\alpha} = 0.6$. We observe that the streak is unstable to long waves and the linear growth rate has a maxima of $\hat{\sigma} = 0.0037$ at $\hat{\alpha} \approx 0.35$.

Figure 4.13 shows the linear growth rate of the leading (most unstable) eigenvalue of the streak profile at $\hat{\alpha} = 0.35$ for different values of the Reynolds number R . We observe that the streak profile is neutrally stable at $R = 20000$ and as the Reynolds number R is increased, the linear growth rate becomes independent of the value of R corresponding to the inviscid limit.

Now let us turn our attention to the linear stability properties of the streak profiles associated with mirror-symmetric VWI states on the sub-4 branch. Figure 4.14 illustrates the spectrum of the leading (most unstable) eigenvalues associated with the streak profile of both the sub-2 and sub-4 branch mirror-symmetric VWI states at $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$. In both cases we observe that all eigenmodes are linearly stable with the exception of the stationary neutral eigenmode which is associated with the VWI state. For the sub-4 case however, there exist two stationary neutral eigenmodes and whilst the associated eigenfunctions are different they both lead to exactly the same stress jumps and VWI states.

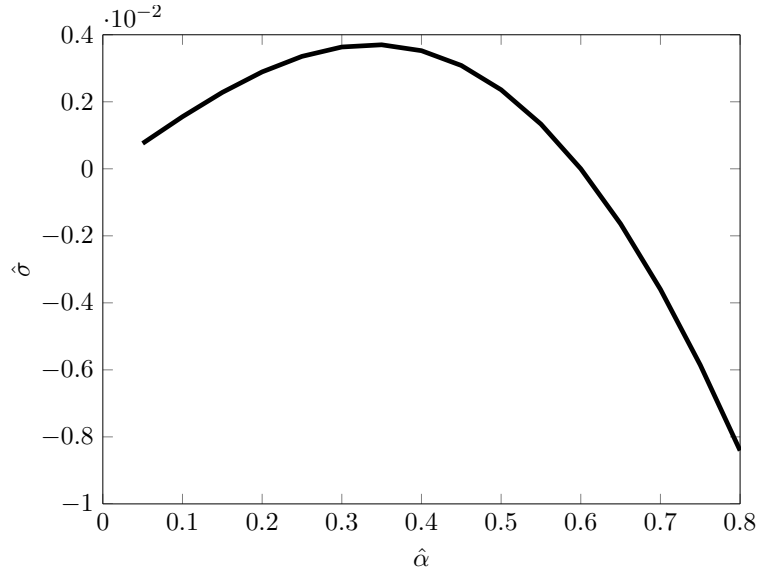


Figure 4.12. Variation of the linear growth rate $\hat{\sigma} = \hat{\alpha}c_i$ against the streamwise wavenumber $\hat{\alpha}$ of the most unstable eigenmode of the streak $u(y, z)$ associated with the VWI state on the first sub-2 branch at $(\alpha, \beta; R) = (0.6, 2.0, 10^6)$.

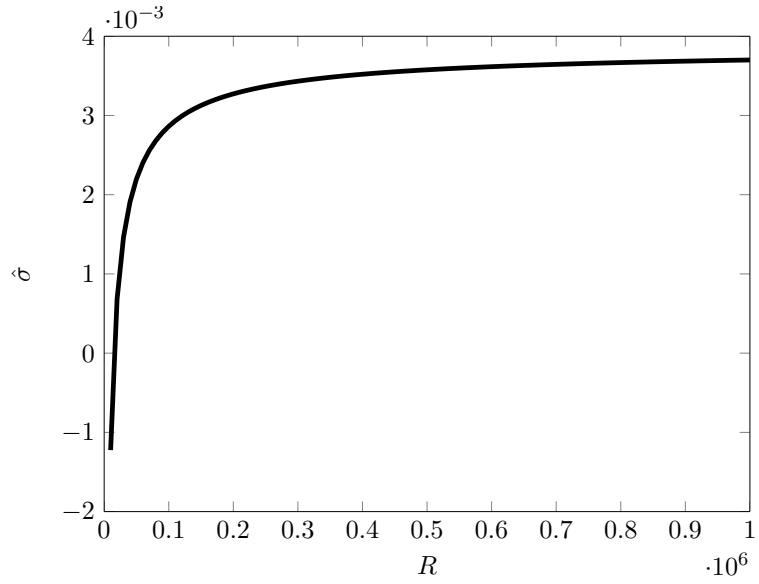


Figure 4.13. Variation of the linear growth rate $\hat{\sigma} = \hat{\alpha}c_i$ against the Reynolds number R of the most unstable mode of the streak profile $u(y, z)$ associated with the mirror-symmetric VWI state on the first sub-2 branch at $(\alpha, \beta) = (0.6, 2.0)$ at $\hat{\alpha} = 0.35$.

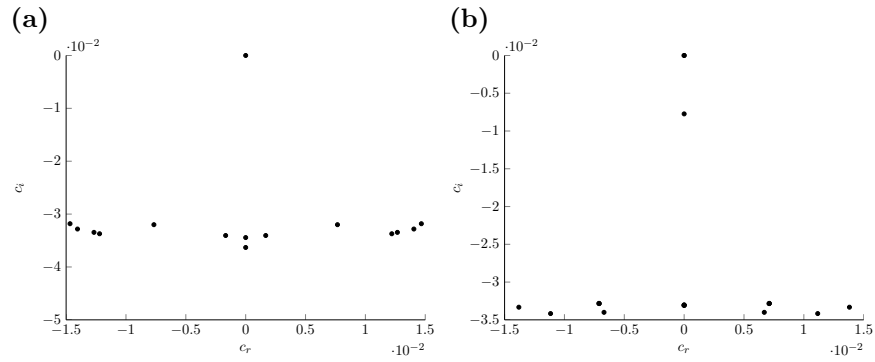


Figure 4.14. The leading (most unstable) eigenvalues for the streak profile associated with a mirror-symmetric VWI state at $(\alpha, \beta; R) = (0.6, 2.0; 10^6)$ on the (a) first sub-2 branch and the (b) first sub-4 branch.

5. Receptivity Theory for VWI States

In this chapter we turn our attention to understanding how external disturbances can alter the flow structure associated with VWI states. The motivation behind studying such problems is often due to aeronautical applications where the design of laminar aerofoils has reduced fuel consumption providing both economical and environmental benefits. There exist several different methods to control a laminar boundary layer including blowing and suction, cooling the wall and introducing bumps to the wall surface. The main aim of these controls has been to either prevent laminar separation or delay the transition from laminar to turbulent flow.

In the context of VWI states we will consider two different methods, the use of periodic blowing/suction and the introduction of wall imperfections. In each case we will analyse the effect on the skin friction drag associated with the $O(1)$ streak profile and also the value of the wave amplitude ρ . The skin friction drag is a natural metric to examine since it directly contributes to the parasitic drag. The reason why the wave amplitude ρ is of interest is less intuitive but recent work by Pringle *et al.* (2012) on understanding the minimal seed required for turbulent flow provides our motivation. Since the first branch VWI states we have found act as edge states between laminar and turbulent flow, the wave amplitude quantifies the size of the wave required to drive a VWI state. Therefore, increasing the wave amplitude is equivalent to increasing the magnitude of the minimal seed required to reach turbulence, or alternatively the basins of attraction for laminar and turbulent flow. The analyses that follow could be conducted for other branches of VWI states, which have been shown to be associated with turbulent attractors, but that is beyond the scope of this work.

5.1. The Effect of Spanwise Wall Blowing/Suction

The effect of applying suction, spatially periodic blowing/suction or variants thereof at a wall to stabilise or destabilise a fluid flow has been extensively studied in the past usually in the context of the attachment-line instability, see Hall *et al.* (1984) and Balakumar & Hall (1999) for details. Whereas these studies have focused on delaying transition by reducing the thickness of the boundary layer, our approach is to apply spanwise periodic blowing/suction in order to excite/suppress the vortical flow of the roll field.

5.1.1. Flow Modification

The $O(R^{-1})$ roll field can be directly forced by applying spanwise periodic blowing/suction of size $O(R^{-1})$. The new boundary conditions at the upper and lower walls for the roll equations (2.5) are of the form

$$v = \mathcal{A} \cos(\gamma z), \quad y = \pm 1, \quad (5.1)$$

for sinuous VWI states and are of the form

$$v = \mp \mathcal{A} \cos(\gamma z), \quad y = \pm 1, \quad (5.2)$$

for mirror-symmetric VWI states. Here $\mathcal{A}$ represents the magnitude and direction of the suction and γ is the spanwise wavenumber of the suction. In each case, the form of the boundary conditions is chosen such that the net mass flux through each channel wall over one spanwise period is zero and the symmetry conditions (2.21) are obeyed.

In order to compute VWI states with spanwise blowing/suction the numerical procedure in section 2.2.4 is followed but the boundary conditions on the roll equations are now inhomogeneous and are given by either (5.1) or (5.2). This forcing will directly alter the roll field, which in turn will alter the streak and wave fields, and so the wave amplitude will adjust in order to generate the necessary forcing such that the streak remains neutrally stable to the wave.

5.1.2. Sinuous Modes

We first explore the effect of applying spanwise periodic blowing/suction on sinuous VWI states which typically exhibit larger skin friction drag compared to their mirror-symmetric counterparts, compare figures 3.4 and 4.8 for instance. The numerical procedure outlined in section 2.2.4 was used with the boundary conditions (5.1) being implemented using the collocation approach outlined in section 2.2.1 with a hybrid body forcing term (2.14).

Figure 5.1 shows the variation of the wave amplitude ρ and the skin-friction drag $\mathcal{S}$ with respect to the suction amplitude $\mathcal{A}$ for a sinuous VWI state at $(\alpha, \beta) = (0.6, 2.0)$. Here, the spanwise periodic blowing/suction is of the form (5.1) with spanwise wavenumber $\gamma = \beta = 2.0$. Note the existence of two curves corresponding to the two different solutions that exist due to the underlying symmetry of the problem. The curve denoted principal solution corresponds to solutions of the form presented in chapter 3 whilst the complementary solution can be found by reflecting the principal solution in the channel mid-height $y = 0$.

The figure shows that the wave amplitude has a quadratic dependence on the suction amplitude and that the minimum in wave amplitude can be found at $\mathcal{A} = \pm 0.5$. The skin friction drag follows similar behaviour but the relationship is not quadratic. In fact, increasing or decreasing the suction amplitude corresponds to either exciting or suppressing the vortex structure which results in either a stronger or weaker lift-up mechanism, thus resulting in an increase or decrease in the value of the skin friction drag. When conducting computations, it was observed that finding solutions beyond the local minimum proved difficult since the solutions would preferentially converge to solutions on the other curve. Figure 5.2 shows the roll velocity field at different values of the suction amplitude $\mathcal{A}$, namely at $\mathcal{A} = -0.25, 0.5$ on the principal solution branch and $\mathcal{A} = -0.5$ on the complementary solution branch. The critical layer is represented by a solid bold curve and highlights the variation in the critical layer location as the suction amplitude is varied.

5.1.3. Mirror-Symmetric Modes

Firstly, recall that the wave associated with a mirror-symmetric VWI state is a subharmonic of the underlying streak field and therefore if the streak is periodic with spanwise wavenumber β then the

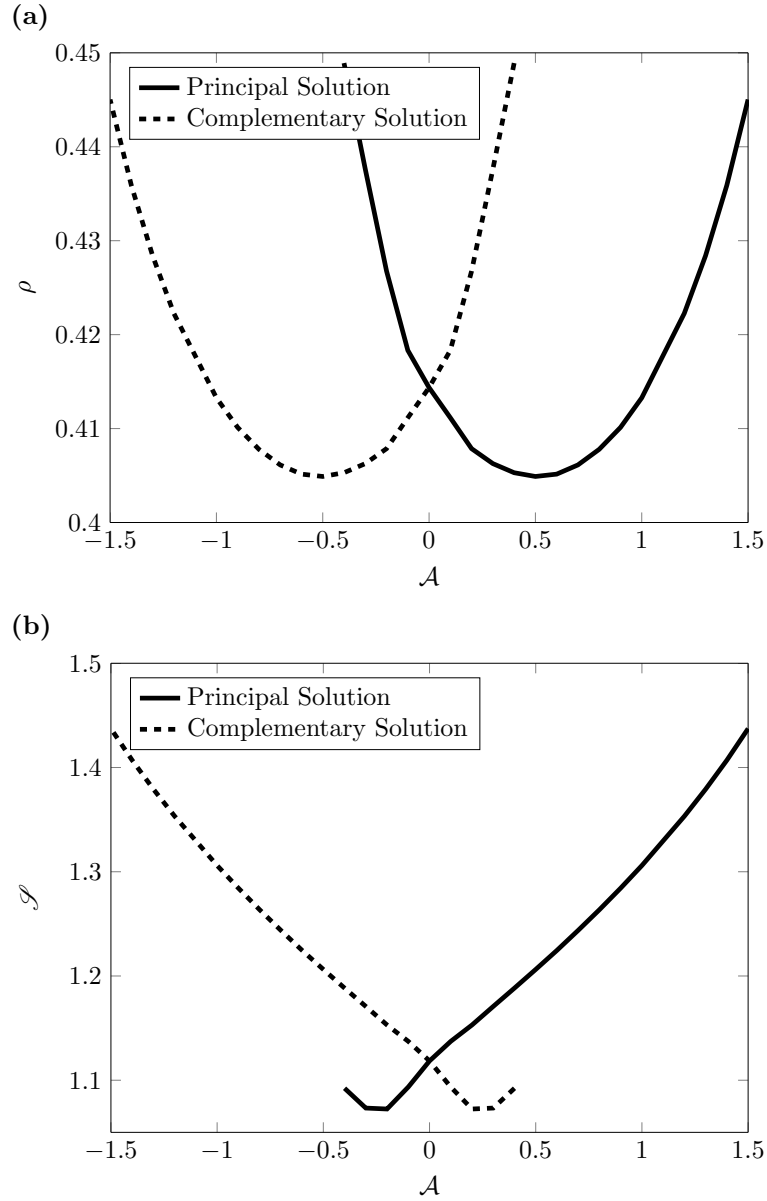


Figure 5.1. Variation of (a) the wave amplitude ρ and (b) the wall skin friction $\mathcal{S}$ against the amplitude $\mathcal{A}$ of the wall suction (5.1) with $\gamma = \beta$ for a sinuous VWI state at $(\alpha, \beta) = (0.6, 2.0)$.

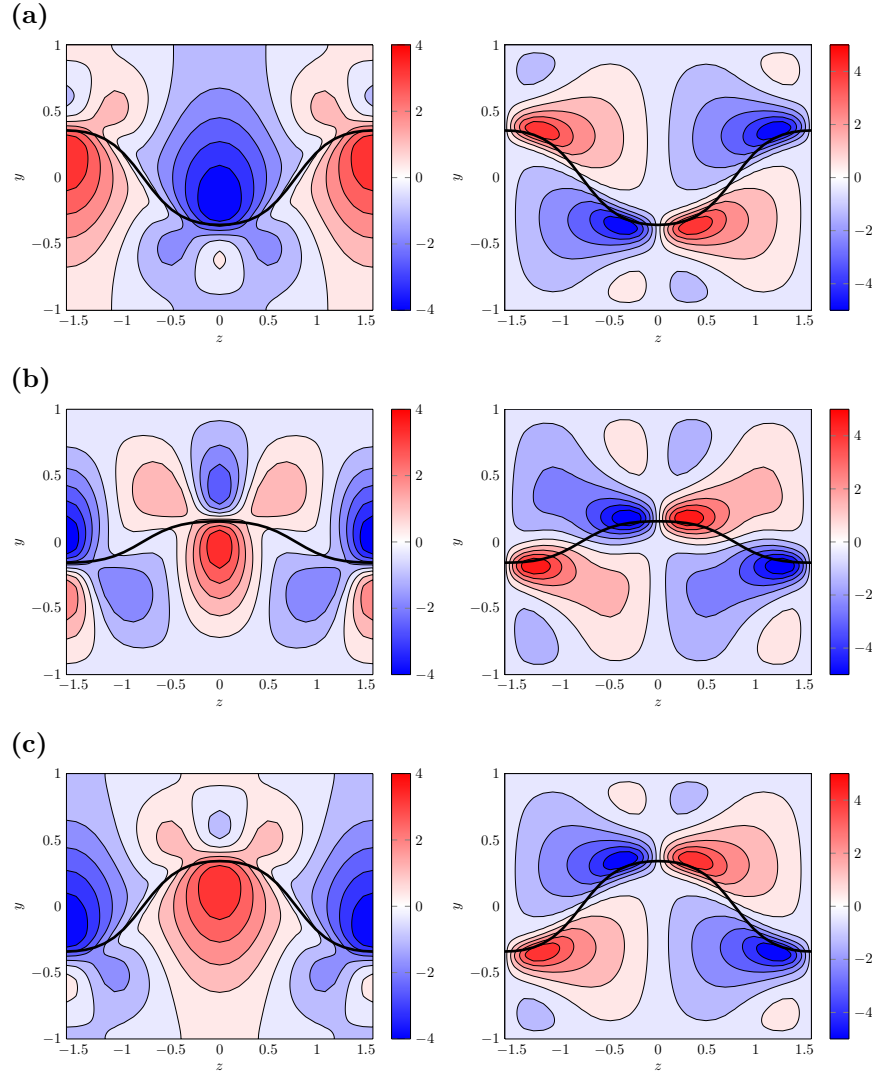


Figure 5.2. Equispaced contours of the roll velocity fields for sinuous VWI states at $(\alpha, \beta) = (0.6, 2.0)$ in plane Couette flow with wall suction of the form (5.1) with $\gamma = \beta$. Left, wall-normal roll velocity field $v(y, z)$; right, spanwise roll velocity field $w(y, z)$. (a) $\mathcal{A} = -0.5$, (b) $\mathcal{A} = -0.25$, (c) $\mathcal{A} = 0.5$. The solid bold curve represents the critical layer.

spanwise wavenumber of the wave field is $\frac{1}{2}\beta$. We will consider applying spanwise periodic blowing/suction with three different spanwise wavenumbers: at the subharmonic wavenumber $\frac{1}{2}\beta$, at the fundamental wavenumber β , and at a higher harmonic wavenumber 2β .

Fundamental Wavenumber $\gamma = \beta$

Figure 5.3 shows the variation of the wave amplitude ρ and skin friction drag $\mathcal{S}$ with respect to the suction amplitude $\mathcal{A}$ for a mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$ with spanwise periodic blowing/suction of the form (5.2) with spanwise wavenumber $\gamma = \beta$.

Once again, the solution branch denoted the principal solution corresponds to the mirror-symmetric VWI states shown previously in chapter 4 whilst the branch denoted complementary solution can be found by reflecting the principal solution in the channel mid-height $y = 0$. If we consider the principal solution curve then we observe that as the suction amplitude is increased both the wave amplitude and skin friction drag also increase. This is due to the suction exciting the vortex structure in the roll field resulting in a stronger lift-up mechanism. Conversely, as the suction amplitude is decreased and the vortex structure in the roll field is suppressed, the opposite is observed and the wave amplitude and skin friction drag decrease. A minimum in the skin friction drag is reached at $\mathcal{A} \approx -1.9$, thereafter the effect of the suction increases the shear stress in the fluid near the wall and so the skin friction drag increases. These observations can also be seen in figure 5.4 where the streak and roll velocity fields are shown for when $\mathcal{A} = -2.0$ and $\mathcal{A} = +2.0$.

Figure 5.5 shows the relationship between the wall suction amplitude and the skin friction drag for mirror-symmetric VWI states at different values of the streamwise wavenumber. For clarity only the principal solution curve is shown. We observe that regardless of the streamwise wavenumber, applying spanwise periodic suction on the channel walls can reduce the wall skin friction associated with a VWI state by 1-2.5% and can also be used to increase the wave amplitude associated with the VWI state.

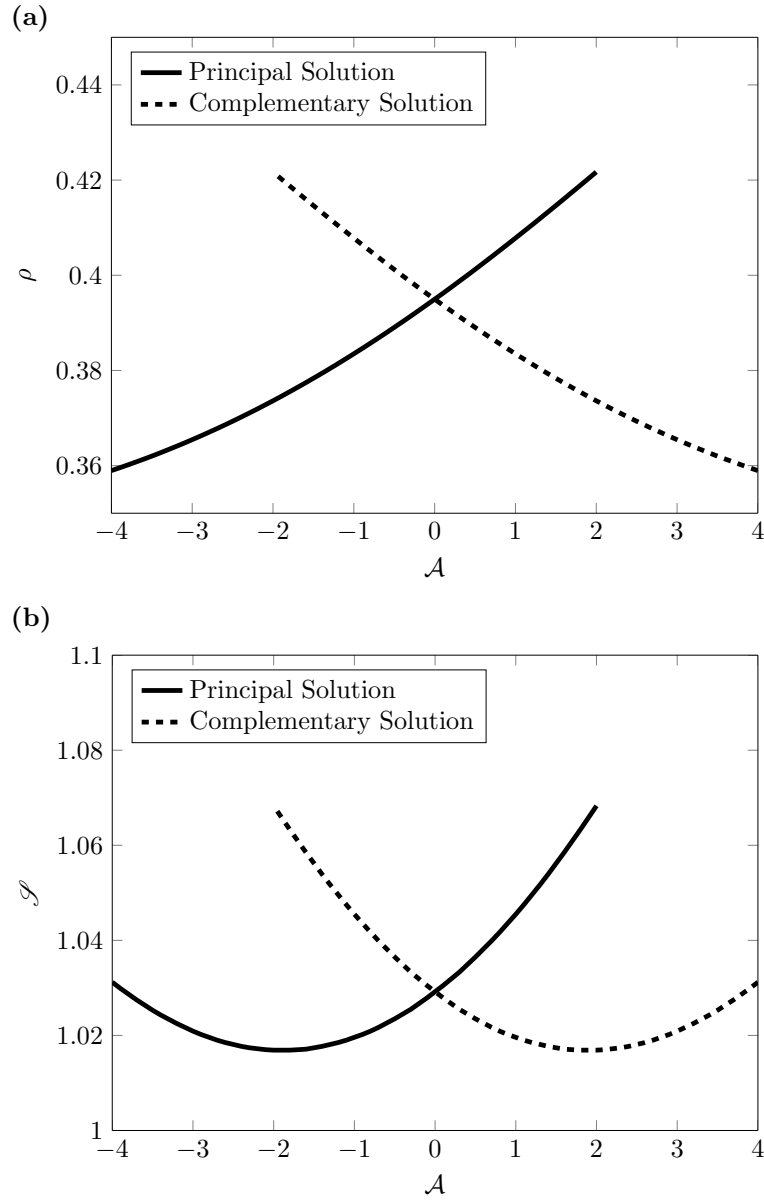


Figure 5.3. Variation of (a) the wave amplitude ρ and (b) the wall skin friction $\mathcal{S}$ against the amplitude $\mathcal{A}$ of the wall suction (5.2) with $\gamma = \beta$ for a mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$.

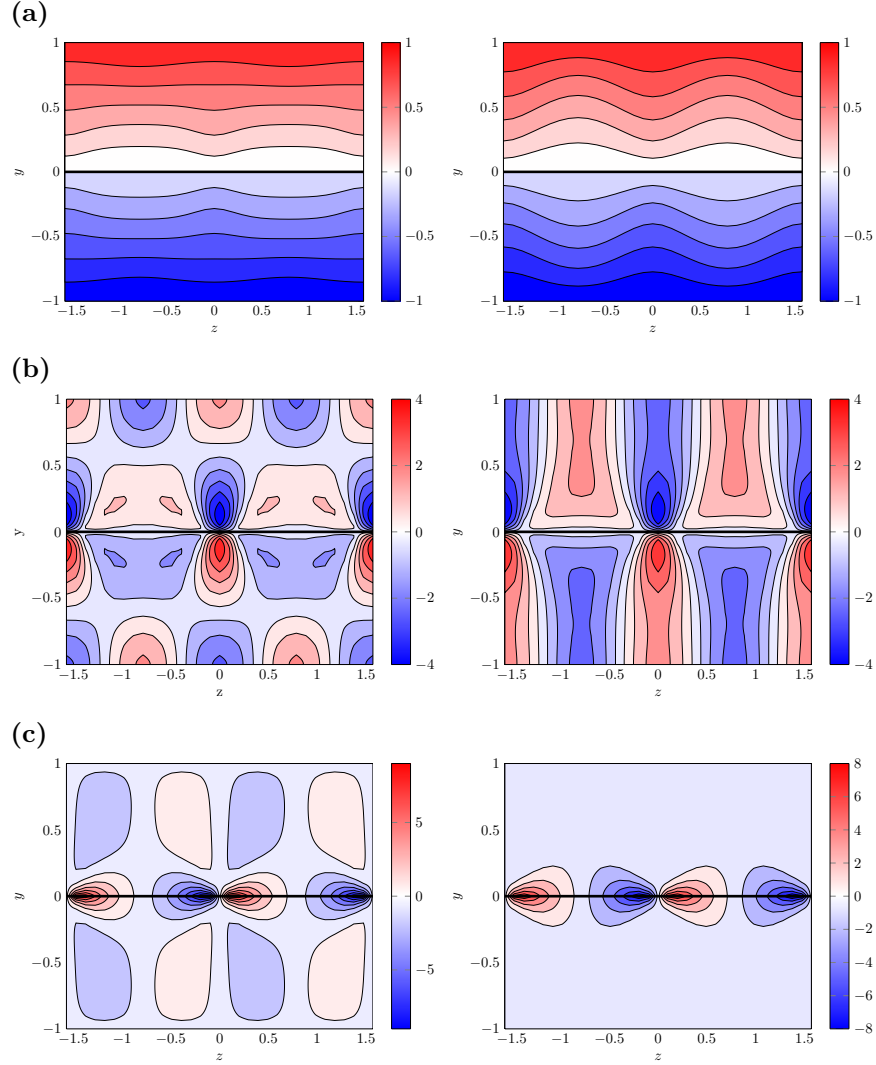


Figure 5.4. Equispaced contours of the streak and roll velocity fields for principal mirror-symmetric VWI states at $(\alpha, \beta) = (0.8, 2.0)$ in plane Couette flow with wall suction of the form (5.2) with $\gamma = \beta$. Left, $\mathcal{A} = -2.0$; right, $\mathcal{A} = +2.0$. (a) Streak profile $u(y, z)$, (b) wall-normal roll velocity field $v(y, z)$, (c) spanwise roll velocity field $w(y, z)$. The solid bold curve represents the critical layer.

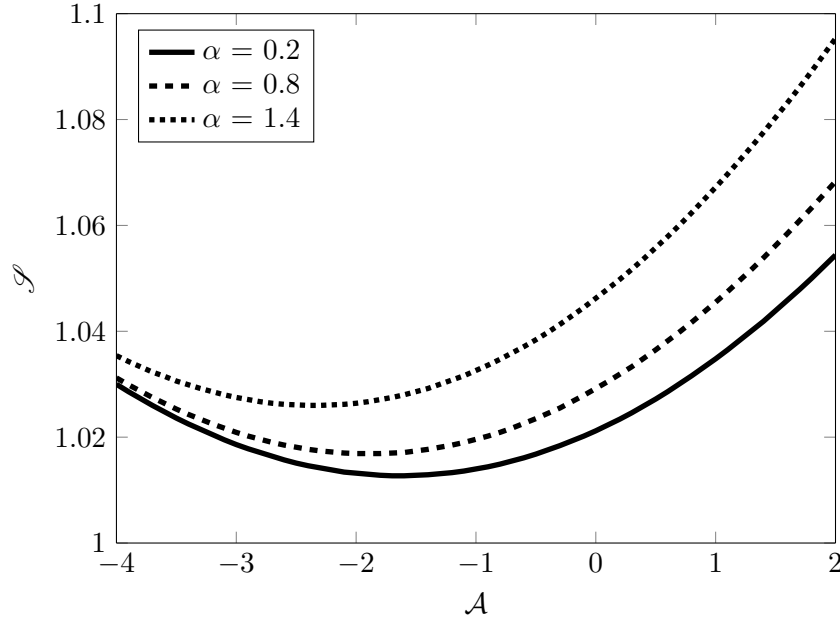


Figure 5.5. Relationship between the wall suction amplitude $\mathcal{A}$ and the wall skin-friction $\mathcal{S}$ at different values of the streamwise wavenumber α for a mirror-symmetric VWI state at $\beta = 2.0$.

Higher Wavenumber $\gamma = 2\beta$

Figure 5.6 shows the variation of the wave amplitude ρ and the skin friction drag $\mathcal{S}$ of a mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$ as a function of the suction amplitude $\mathcal{A}$ for spanwise periodic blowing/suction of the form (5.2) with wavenumber $\gamma = 2\beta$.

We observe that the effect of applying suction is much less pronounced than in the case of applying suction with the fundamental wavenumber $\gamma = \beta$. In fact, regardless of whether we increase or decrease the suction amplitude the skin friction drag increases and so the introduction of spanwise blowing/suction at a higher wavenumber than the fundamental is ineffective in reducing the drag associated with mirror-symmetric VWI states. Further, there is only a small difference in the principal and complementary solution curves. These observations can be elucidated by examining the streak and roll velocity fields at $\mathcal{A} = -2.0$ and $\mathcal{A} = +2.0$ corresponding to the principal solution, as shown in figure 5.7. In this figure it can be seen that the action of the suction creates a secondary flow structure near the channel walls rather than directly exciting or suppressing the vortical struc-

ture. Therefore, since the near-wall structure is responsible for the value of the skin-friction drag there is little difference between the two solutions at the same value of the suction amplitude.

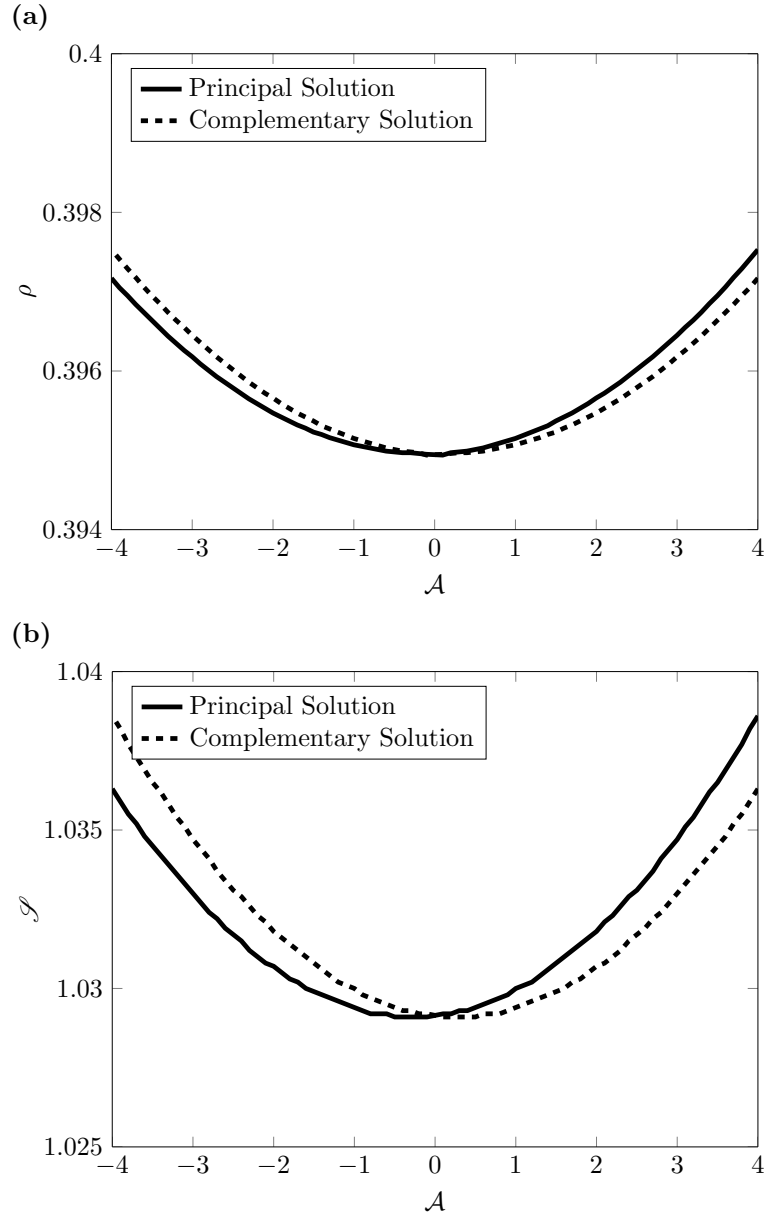


Figure 5.6. Variation of (a) the wave amplitude ρ and (b) the wall drag $\mathcal{S}$ against the amplitude $\mathcal{A}$ of the wall suction (5.2) with $\gamma = 2\beta$ for a mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$.

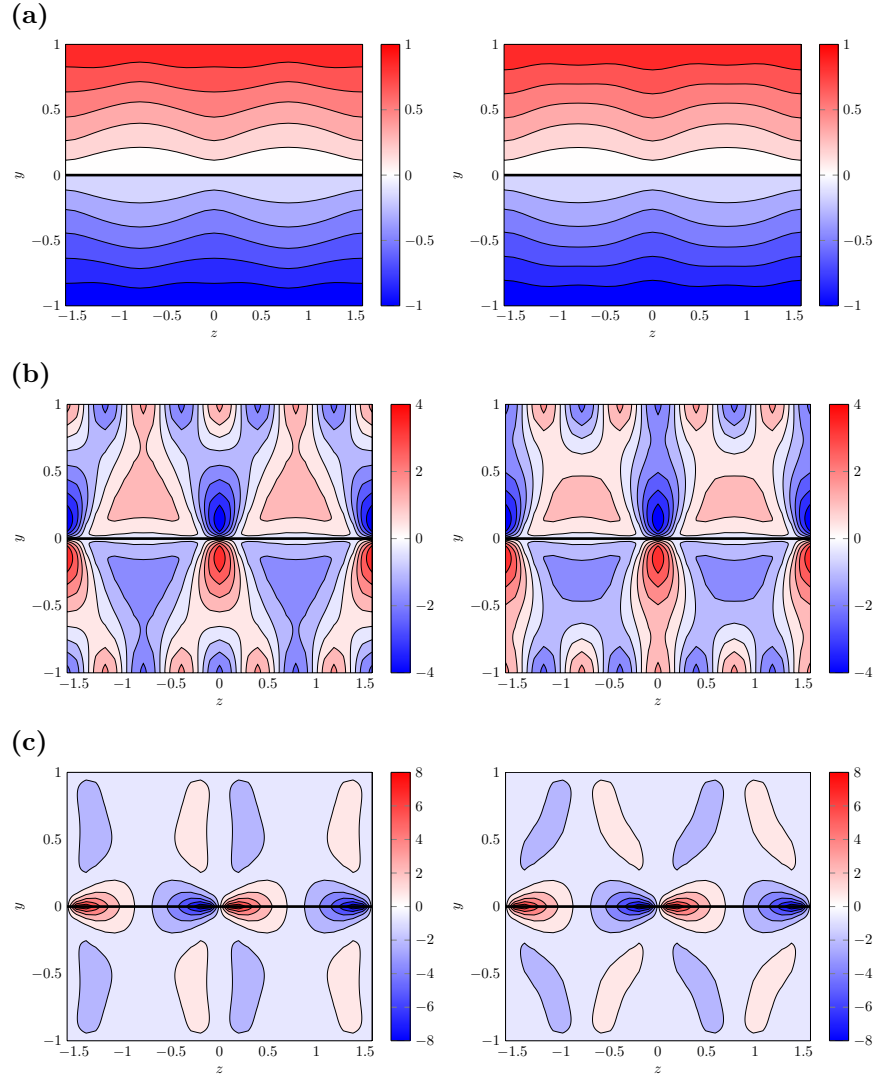


Figure 5.7. Equispaced contours of the streak and roll velocity fields for principal mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$ in plane Couette flow with wall suction of the form (5.2) with $\gamma = 2\beta$. Left, $\mathcal{A} = -2.0$; right, $\mathcal{A} = +2.0$. (a) streak profile $u(y, z)$, (b) wall-normal roll velocity field $v(y, z)$, (c) spanwise roll velocity field $w(y, z)$. The solid bold curve represents the critical layer.

Subharmonic Wavenumber $\gamma = \frac{1}{2}\beta$

Let us now consider applying periodic blowing/suction at the subharmonic wavenumber $\gamma = \frac{1}{2}\beta$. This form of blowing/suction does not satisfy the symmetry conditions given by (2.21), therefore in the results that follow we have relaxed all symmetry constraints in the computations. However, note that as in the previous cases two solution curves exist.

Figure 5.8 shows the variation of the wave amplitude and the skin friction drag of a mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$ as a function of the suction amplitude. Examining the principal solution curve we observe that there exists a suction amplitude, $\mathcal{A} \approx \pm 0.45$, in which the skin friction drag reaches a minimum for a given streamwise wavenumber α . Whilst the reduction in the skin friction drag is not particularly noteworthy, since it is less than 1%, the structure of the VWI state has changed significantly and the wave is now a synchronous mode, rather than a subharmonic mode, of the streak. Figure 5.9 shows the streak and roll velocity fields corresponding to a principal solution when $\mathcal{A} = -2.0$ and $\mathcal{A} = +2.0$.

This new synchronous mirror-symmetric VWI state can be numerically continued to the case when the suction amplitude is zero, $\mathcal{A} = 0$, suggesting that such states exist in plane Couette flow in the absence of any forcing. Figure 5.10 illustrates the structure associated with this synchronous mirror-symmetric VWI state at $(\alpha, \beta) = (0.6, 2.0)$ in the absence of any wall suction/blowing. Note that we have performed the translation $z \rightarrow z + \pi/\beta$ to center the interaction region at $z = 0$ in the plots. Most noteworthy is perhaps the fact that the VWI state is now seemingly confined to a region centered at $z = 0$ and as $z \rightarrow \pm\pi/2$ the flow field quantities tend to that of plane Couette flow. This suggests the possibility that non-periodic VWI states may exist in plane Couette flow, particularly in the $\beta \rightarrow 0$ limit, and could be embedded into other shear flows in a similar manner to that shown by Deguchi & Hall (2014a).

The variation in the wave amplitude ρ against the streamwise wavenumber α is shown in figure 5.11. The $\alpha - \rho$ relationship follows the already established behaviour expected for mirror-symmetric VWI states in the small and large streamwise wavenumber limits. In particular, at small values of the streamwise wavenumber we observe that $\rho \sim \alpha^{5/6}$ such that the jump conditions (2.9) remain $O(1)$, which is shown in the inset figure where the

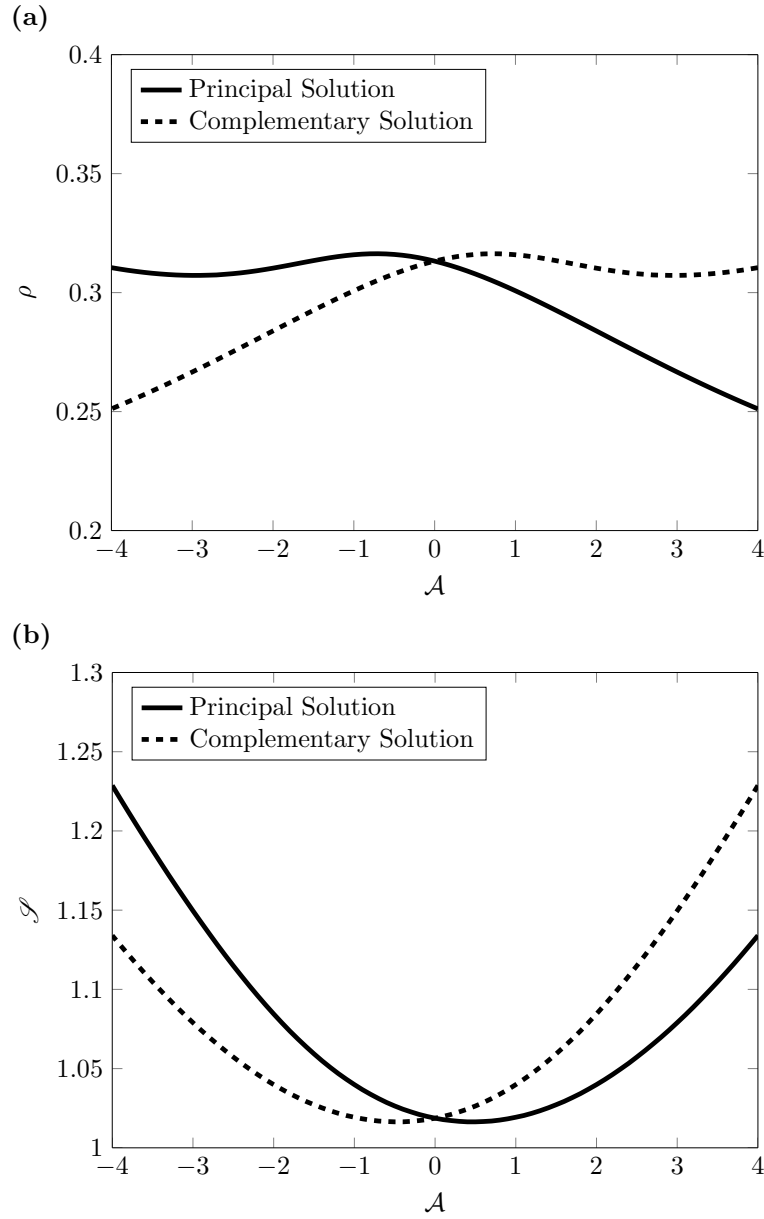


Figure 5.8. Variation of (a) the wave amplitude ρ and (b) the wall drag $\mathcal{S}$ against the amplitude $\mathcal{A}$ of the wall suction (5.2) with $\gamma = \frac{1}{2}\beta$ for a mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$.

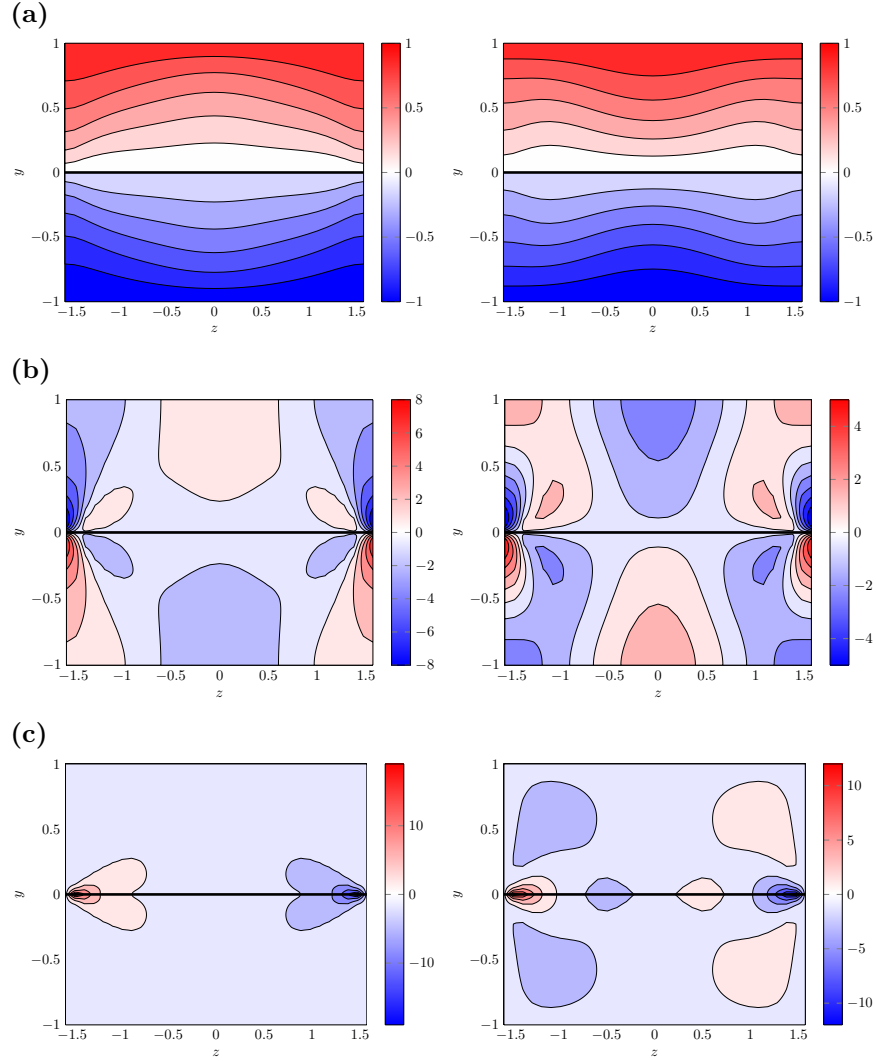


Figure 5.9. Equispaced contours of the streak and roll velocity fields for principal mirror-symmetric VWI state at $(\alpha, \beta) = (0.8, 2.0)$ in plane Couette flow with wall suction of the form (5.2) with $\gamma = \frac{1}{2}\beta$. Left, $\mathcal{A} = -2.0$; right, $\mathcal{A} = +2.0$. (a) streak profile $u(y, z)$, (b) wall-normal roll velocity field $v(y, z)$, (c) spanwise roll velocity field $w(y, z)$. The solid bold curve represents the critical layer.

variation on a log-log scale is shown at small values of α , and at large values of the streamwise wavenumber we observe that the value of ρ asymptotes to a value of $\rho \approx 0.45$, which is shown by the dotted line in the plot.

The variation of the skin friction drag against the streamwise wavenumber for fixed spanwise wavenumber $\beta = 2.0$ is shown in figure 5.12 for both the synchronous and subharmonic modes. This is of particular interest since the skin friction drag associated with the synchronous mirror-symmetric state is lower than that of its subharmonic counterpart. Therefore, the use of periodic blowing/suction to drive a subharmonic mirror-symmetric VWI state to its synchronous counterpart could be used as a drag reduction technique.

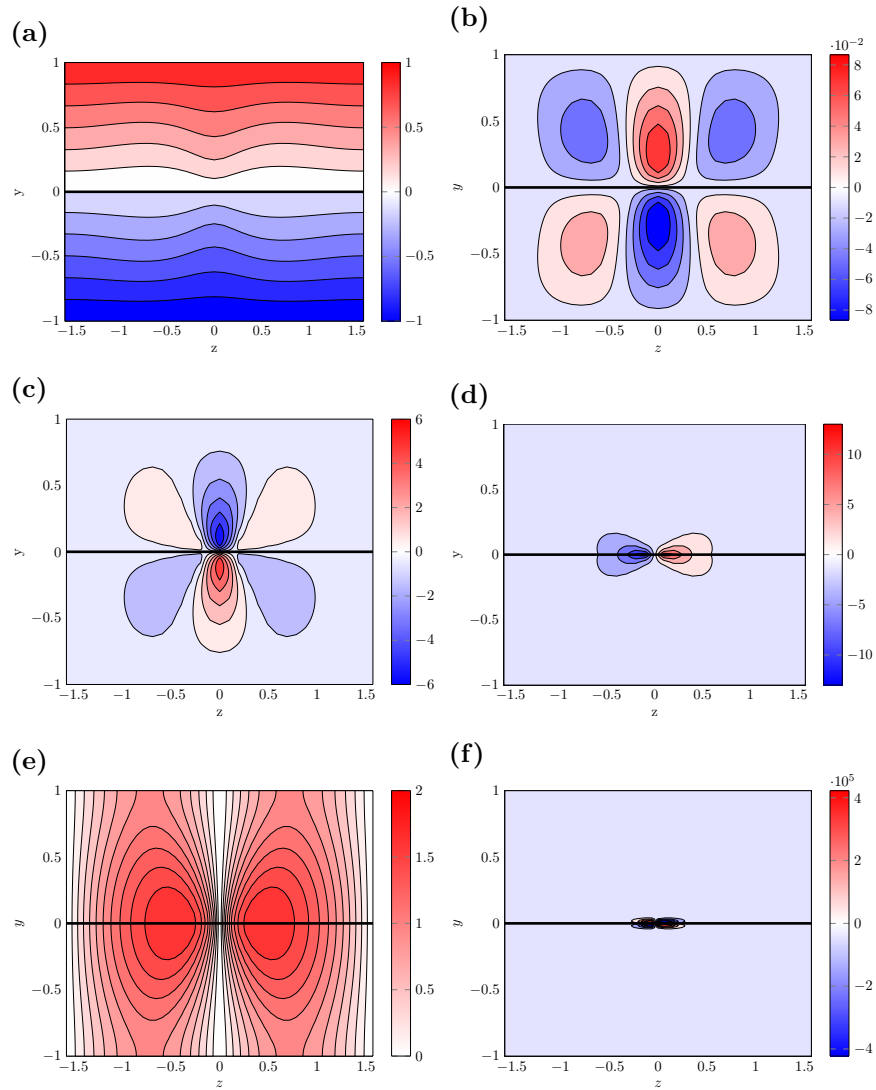


Figure 5.10. Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $|P(y, z)|$, and (f) the hybrid body forcing term (2.14) associated with a synchronous mirror-symmetric VWI state at $(\alpha, \beta; R) = (0.6, 2.0, 10^6)$. The solid bold curve represents the critical layer location.

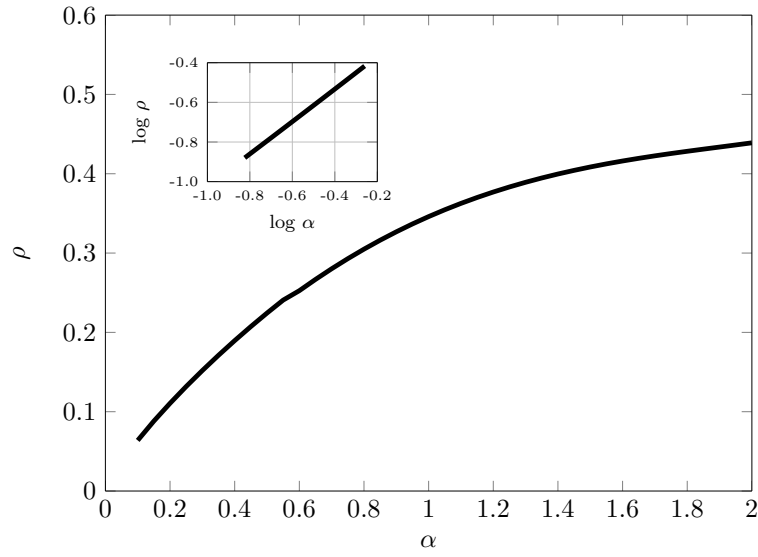


Figure 5.11. Variation of the wave amplitude ρ against the streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$ for the synchronous mirror-symmetric VWI state.

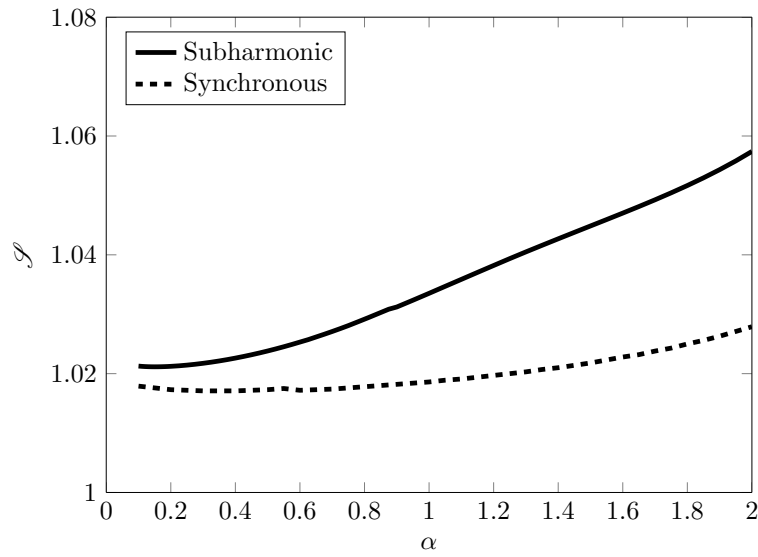


Figure 5.12. Variation of the skin friction drag $\mathcal{S}$ against the streamwise wavenumber α for fixed spanwise wavenumber $\beta = 2.0$ for the synchronous and subharmonic mirror-symmetric VWI state.

5.2. The Effect of Wall Imperfections

The effect of small-scale variations in the shape of a wall on the stability of a fluid flow has been studied extensively in the past. One of the chief motivations for this problem is due to aeronautical applications where surface roughness can be present due to a manufacturing process, for example rivets in the construction of an aerofoil.

A high Reynolds number asymptotic theory developed simultaneously by Neiland (1969), Stewartson (1969) and Messiter (1970), now commonly known as triple-deck theory, has been used in the past to successfully model the behaviour of a fluid in these types of scenarios. Smith *et al.* (1981) and Sychev *et al.* (1987) provide many examples in which to understand the effect of discrete roughness elements, slopes and humps have on the stability of the flow and provide a framework to understand when laminar separation or transition to turbulence may occur. It is worthwhile to note that in some cases the introduction of wall variations or surface roughness can help delay the onset of transition, for example see the work of Saric *et al.* (2011) on discrete roughness elements, or provide other benefits, such as the work of Gaster (1967) on prevention of attachment-line contamination.

Our approach is to consider a parameterisation of the wall shape that is quite general but choose the magnitude of the variations such that it will directly force the wave system.

5.2.1. Flow Modification

Suppose that the walls are now given by

$$y_{\text{wall}} = \pm 1 + \delta g_{\pm}(x, z), \quad (5.3)$$

where $g(x, z)$ is a given function that captures the variations in the wall shape and δ is a small constant that represents the amplitude of the wall variations. Our main aim in the analysis that follows is to find the distinguished limit of $\delta = \delta(R)$ that results in the wall variations forcing the wave system governed by equations (2.7), which is of size $O(R^{-7/6})$ within the VWI framework.

The no-slip condition (2.2) evaluated at the wall location given by (5.3)

can be Taylor expanded to find

$$\tilde{\mathbf{u}}(y_{\text{wall}}, z) = \tilde{\mathbf{u}}|_{y=\pm 1} + \delta g_{\pm}(x, z) \frac{\partial \tilde{\mathbf{u}}}{\partial y} \Big|_{y=\pm 1} + O(\delta^2) = \pm 1, \quad (5.4)$$

where $\tilde{\mathbf{u}}$ is the velocity field. As the new boundary condition (5.4) is an asymptotic expansion in powers of δ we can take a similar asymptotic expansion for the velocity field,

$$\tilde{\mathbf{u}} = \mathbf{u}_0 + \delta \mathbf{u}_1 + O(\delta^2). \quad (5.5)$$

Substituting (5.5) into expression (5.4) and collecting like powers of δ , we find the following conditions for the streamwise velocity component

$$u_0|_{y=\pm 1} = \pm 1, \quad (5.6)$$

$$u_1|_{y=\pm 1} = -g_{\pm}(x, z) \frac{\partial u_0}{\partial y}. \quad (5.7)$$

The boundary conditions for the wall-normal and spanwise velocity components are found in a similar manner with the leading order condition being homogeneous

In the large Reynolds number limit, $R \rightarrow \infty$, a boundary layer will develop near the walls that results from the balance between $i\alpha u U$ and $R^{-1}U_{yy}$ which, since $\alpha \sim O(1)$, leads to a boundary layer of thickness $O(R^{-1/2})$. At the lower wall we have $(y + 1) \sim R^{-1/2}$ and so a suitable rescaling inside the boundary layer is

$$\eta = R^{1/2}(y + 1). \quad (5.8)$$

In this region the streak field $u(y, z)$ can be expressed in terms of η as

$$u(\eta, z) = -1 + R^{-1/2}\eta + O(R^{-1}), \quad (5.9)$$

and so

$$\frac{\partial u}{\partial y} = R^{1/2} \frac{\partial u}{\partial \eta} = 1 + O(R^{-1/2}). \quad (5.10)$$

Now, the streamwise velocity component should be $O(\delta)$ in order to match the inhomogeneous boundary condition given by (5.6). It follows from ex-

amining the continuity equation that the wall-normal velocity component should be $O(\delta R^{-1/2})$ and the spanwise velocity component should be $O(\delta)$. Examination of the wall-normal momentum equation suggests that the pressure should be $O(\delta R^{-1/2})$. Therefore, a suitable rescaling of the velocity and pressure fields is

$$U = \delta U_0 + \dots, \quad (5.11a)$$

$$V = \delta R^{-1/2} V_0 + \dots, \quad (5.11b)$$

$$W = \delta W_0 + \dots, \quad (5.11c)$$

$$P = \delta R^{-1/2} W_0 + \dots, \quad (5.11d)$$

and so the leading order equations from equations (2.68) are now

$$U_{0,\eta\eta} + i\alpha U_0 = 0, \quad (5.12a)$$

$$P_{0,\eta} = 0, \quad (5.12b)$$

$$W_{0,\eta\eta} + i\alpha W_0 = 0, \quad (5.12c)$$

$$i\alpha U_0 + V_{0,\eta} + W_{0,z} = 0, \quad (5.12d)$$

which are to be solved subject to the following boundary conditions

$$(U_0, V_0, W_0) = (g_-(x, z), 0, 0), \quad \eta = 0, \quad (5.13)$$

$$(U_0, V_0, W_0) \text{ bounded as } \eta \rightarrow \infty. \quad (5.14)$$

This system of equations are similar to those that govern a Stokes boundary layer and can be solved to find

$$U_0 = g_-(x, z) e^{-(\frac{\alpha}{2})^{1/2}(1+i)\eta}, \quad (5.15)$$

$$V_0 = g_-(x, z) \left(\frac{\alpha}{2}\right)^{1/2} (1+i) \left[1 - e^{-(\frac{\alpha}{2})^{1/2}(1+i)\eta}\right], \quad (5.16)$$

$$W_0 = 0. \quad (5.17)$$

Then in the limit $\eta \rightarrow \infty$ we find that

$$V_0 \rightarrow g_-(x, z) \left(\frac{\alpha}{2}\right)^{1/2} (1+i), \quad (5.18)$$

which is the displacement caused by the variations in the wall shape. If we

wish the wall variations to provide a forcing to the wave system then (5.18) should match onto the $O(R^{-7/6})$ inviscid wave velocity field. Expressing (5.18) in terms of y rather than wall layer coordinate η we obtain

$$V \rightarrow \delta R^{-1/2}(1+i) \left(\frac{\alpha}{2}\right)^{1/2} g_-(x, z), \quad (5.19)$$

and so to match onto the inviscid wave field we must have that $\delta R^{-1/2} \sim R^{-7/6}$ and so $\delta \sim R^{-2/3}$. This is the distinguished limit of δ that ensures that the variations in the wall shape force the wave system. For values of δ smaller than $R^{-2/3}$ there is no effect at leading order and for values of δ larger than $R^{-2/3}$ the order of magnitude analysis needs to be reconsidered.

The numerical method outlined in section 2.2.3 solves the inviscid wave pressure equation (2.7) by solving the viscous form of the wave equations at increasingly higher values of the Reynolds number such that the solution is no longer dependent on its value. As the viscous form of the equations are solved, the boundary condition given by (5.7) should be used to impose the effect of the wall variations. Taking into the account the size of δ we find that the inhomogeneous boundary condition on the wave system at the lower wall is

$$U|_{y=-1} = R^{1/2} \mathcal{A} G_-(z) \frac{\partial u}{\partial y} \Big|_{y=-1} \quad (5.20)$$

where we have taken

$$g_-(x, z) = -\rho \mathcal{A} G_-(z) e^{i\alpha x} + c.c., \quad (5.21)$$

and the x -dependence is chosen such that it matches onto the wave system and $\mathcal{A}$ is an $O(1)$ parameter. A similar condition can be found for the upper wall.

The boundary condition (5.20) can be incorporated into the numerical scheme outlined in section 2.2.3 resulting in an inhomogeneous generalised eigenvalue problem of the form

$$A\chi = cB\chi + \mathbf{f} \quad (5.22)$$

where $\mathbf{f}$ is a vector that represents the boundary conditions (5.20). For stationary neutral eigenmodes we have that $c = 0$ and so we have a linear

system of the form

$$A\chi = \mathbf{f}, \quad (5.23)$$

which can be solved using the GMRES approach to solving sparse non-symmetric linear systems of equations.

The governing equations for the inviscid wave pressure field are no longer homogeneous and so the resulting pressure field $P(y, z)$ does not need to be normalised using condition (2.8). As a result, the wave amplitude in this inhomogeneous VWI problem is a redundant parameter and is set equal to unity. The magnitude of the wave field comes directly from the amplitude in the wall variations.

To compute VWI states with this form of wall variations the numerical procedure outlined in section 2.2.4 is used but the inhomogeneous wave system governed by (5.23) is solved in place of the generalised eigenvalue problem (2.84).

To characterise VWI states for this new inhomogeneous VWI problem we define a metric $\mathcal{J}$ based on the maximum value of the stress jump condition $J(z)$ over one spanwise period, that is

$$\mathcal{J} = \max_{z \in [-\pi/\beta, \pi/\beta]} J(z). \quad (5.24)$$

Whilst this metric is not physically meaningful, it does provide a measure of the magnitude of the jump conditions in a similar manner to how the wave amplitude did previously.

5.2.2. A Class of Inhomogeneous VWI States

We will consider wall variations of the form (5.21) with

$$G_{\pm}(z) = \pm \sin(\gamma z) \quad (5.25)$$

where γ is the spanwise wavenumber of the wall variations. This form of $G_{\pm}(z)$ ensures the symmetry of the wave pressure field is the same as that in previously found VWI states.

A number of exploratory computations were conducted using a large set of different initial conditions to gain insight into this inhomogeneous VWI

problem. From these initial computations it was found that regardless of the initial streak profile used, the numerical procedure preferentially found inhomogeneous VWI states in a neighbourhood of plane Couette flow and that they took the form of “weak” mirror-symmetric VWI states. That is, for all computations we found that if we followed the solution to the zero amplitude scenario then plane Couette flow was recovered, rather than a VWI state.

Figure 5.13 illustrates the structure associated with the converged inhomogeneous VWI state at $(\alpha, \beta; R) = (0.8, 2.0; 10^4)$ with wall variations of the form (5.25) with $\mathcal{A} = 25$. Examining the flow field quantities of the inhomogeneous state, a definitive structure that is very similar to the first branch of mirror-symmetric VWI states seen previously in section 4.2.1 is observed, see figure 4.1 for a comparison.

Figure 5.14 shows the variation of $\mathcal{J}$, the maximum value in the stress jump condition $J(z)$, as the amplitude $\mathcal{A}$ of wall variations of the form (5.25), with $\gamma = \beta = 2.0$, is increased for different values of the streamwise wavenumber α . Following the solution to when the amplitude is zero, plane Couette flow is recovered. Over the range of streamwise wavenumbers considered the same behaviour is observed, namely that a branch of inhomogeneous VWI states slowly bifurcates away from the zero amplitude base state, plane Couette flow, before reaching a vertical asymptote at some value of the amplitude. As α is increased it appears that the curve approaches a limiting configuration. For $\beta = 2.0$, when $\mathcal{J} > 25$ the numerical procedure fails to converge suggesting that our procedure either fails to capture such structures at large amplitudes or that a reconsideration of the boundary conditions is required. The variation in the wall skin friction $\mathcal{S}$ over the same parameter values is less than 1%, hence a plot of the variation in $\mathcal{S}$ is omitted.

The variation of $\mathcal{J}$ against the amplitude of wall variations $\mathcal{A}$ for different values of the spanwise wavenumber β is shown in figure 5.15. The wall variations are of the form (5.25) with $\gamma = \beta$ for $\beta = 2, 4, 8$. Note that the natural logarithm of the amplitude has been taken to rescale the horizontal axis to make the plot much clearer. Examining the $\log(\mathcal{A}) - \mathcal{J}$ curves we observe similar behaviour as in figure 5.14 but both $\mathcal{A}$ and $\mathcal{J}$ vary over a much larger scale as β is increased. For large values of β , the inhomogeneous VWI states localise in a region around the channel mid-height similar

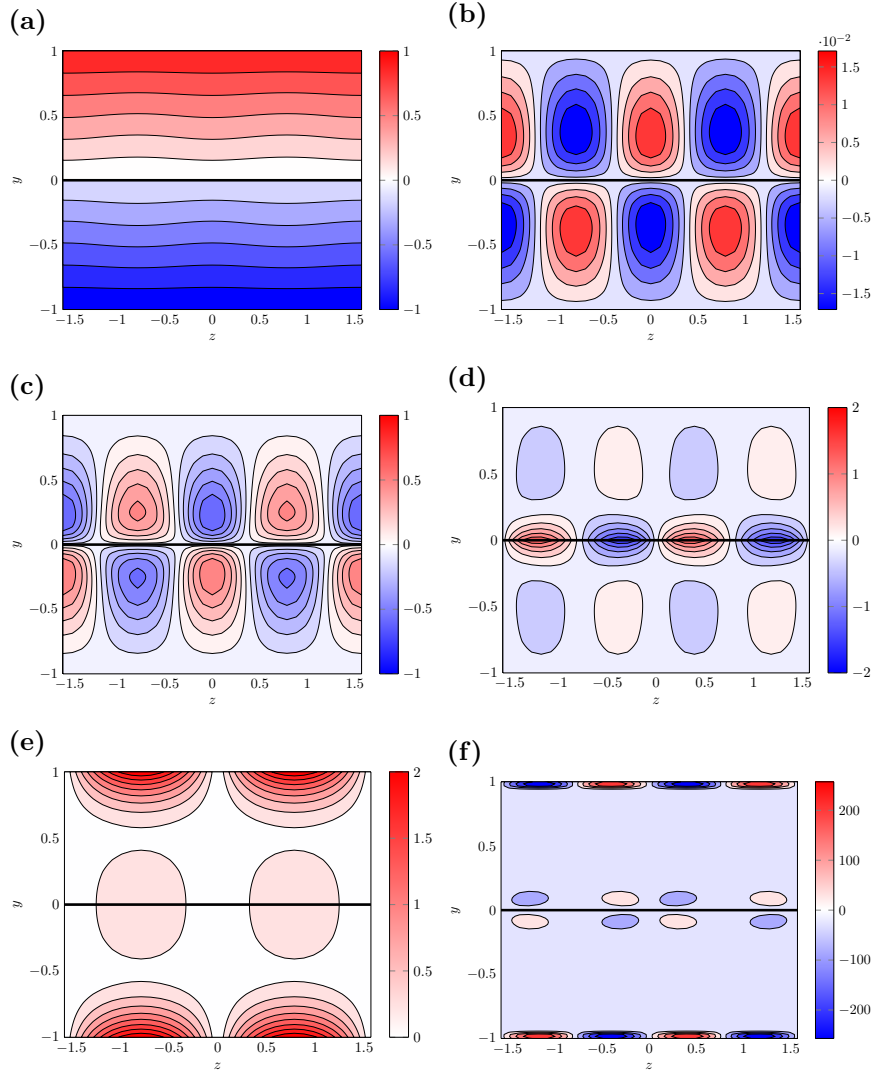


Figure 5.13. Equispaced contours of the (a) streak field $u(y, z)$, (b) perturbation of the streak away from plane Couette flow $u(y, z) - y$, (c) wall-normal roll velocity field $v(y, z)$, (d) spanwise velocity field $w(y, z)$, (e) absolute value of the wave pressure field $|P(y, z)|$, and (f) the hybrid body forcing term (2.14) associated with a mirror-symmetric VWI state at $(\alpha, \beta; R) = (0.8, 2.0; 10^4)$ with inhomogeneous wall conditions of the form (5.20) with $\mathcal{A} = 25$ and $\gamma = \beta$. The solid bold curve represents the critical layer location.

to homogeneous VWI states and so a much larger amplitude in wall variations is required in order to sufficiently force the pressure field to drive the state. Figure 5.16 shows the roll velocity components for $\beta = 2, 4, 8$ with $\alpha = 0.8$ and $\mathcal{J} = 20$, illustrating the localisation.

Figure 5.17 highlights the effect of the regularisation Reynolds number R on $\mathcal{J}$, the maximum value of the stress jump conditions (2.9), for the inhomogeneous VWI state at $(\alpha, \beta) = (0.8, 2.0)$. Unsurprisingly, we observe that at large values of the regularisation Reynolds number the relationship between $\mathcal{J}$ and $\mathcal{A}$ becomes independent of the value of R and approaches a limiting configuration corresponding to the inviscid limit.

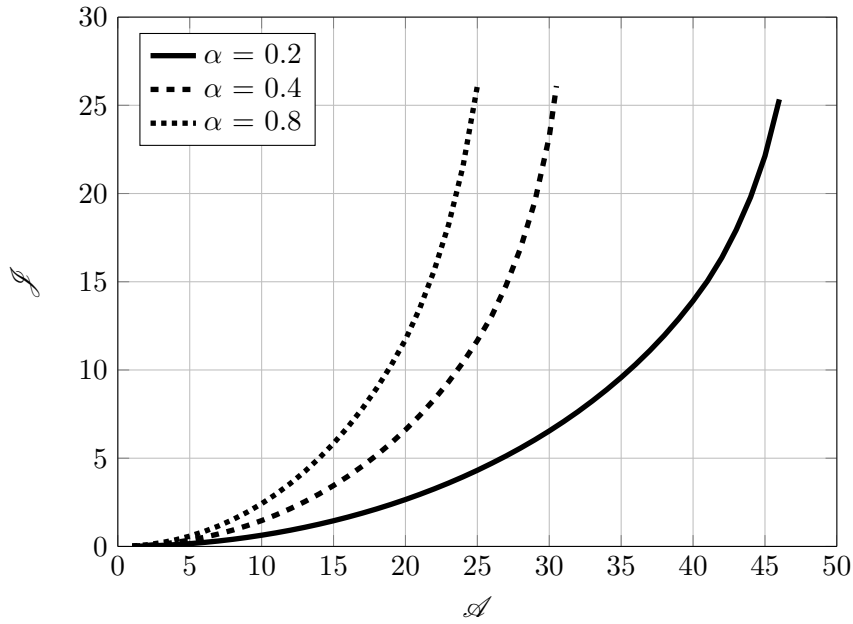


Figure 5.14. Relationship between the wall shape amplitude, $\mathcal{A}$, and the maximum value in the stress jump condition (2.9), $\mathcal{J}$, for different values of the streamwise wavenumber $\alpha = 0.2, 0.4, 0.8$, fixed $\beta = 2.0$ and $R = 10^4$. The variations in wall shape are given by (5.25) with $\gamma = \beta$.

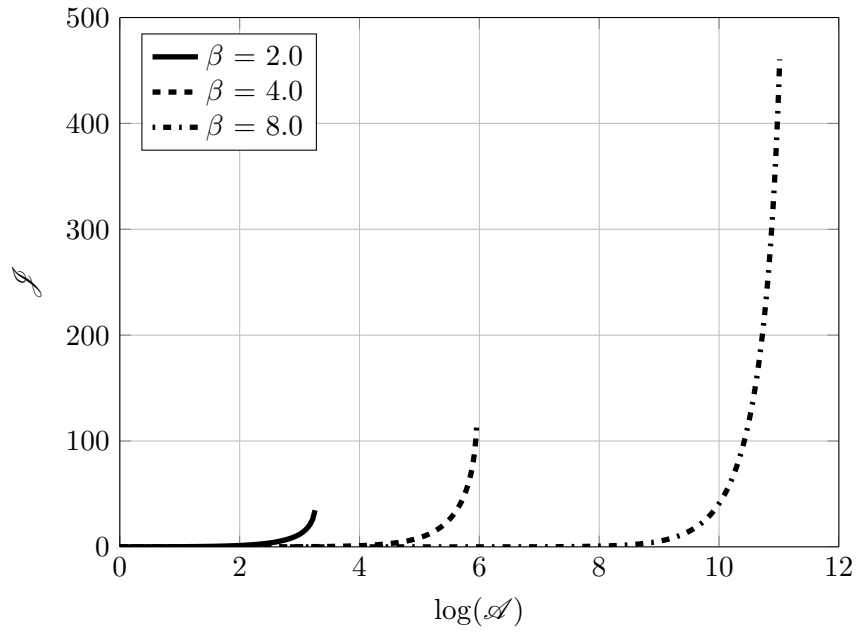


Figure 5.15. Relationship between the wall shape amplitude, $\mathcal{A}$, and the maximum value in the stress jump condition (2.9), $\mathcal{J}$, for different values of the spanwise wavenumber $\beta = 2.0, 4.0, 8.0$ and fixed $\alpha = 0.8$ and $R = 10^4$. The variations in wall shape are given by (5.25) with $\gamma = \beta$.

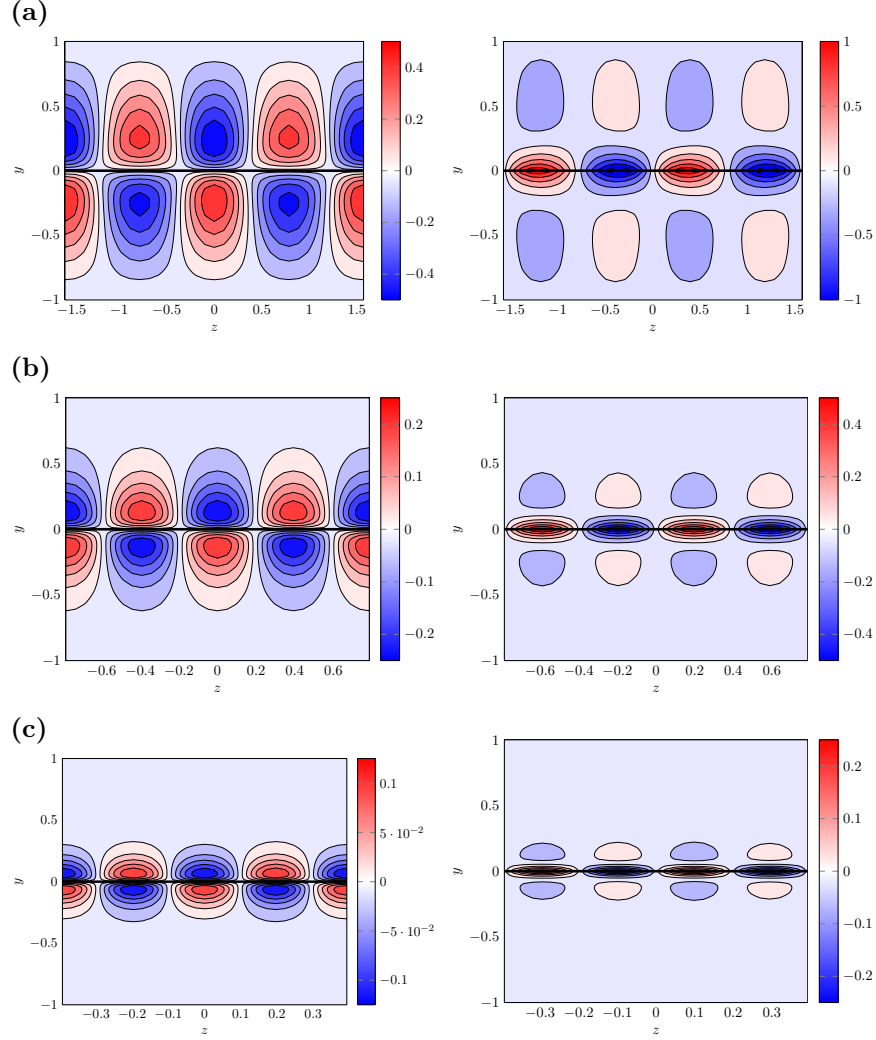


Figure 5.16. Equispaced contours of the roll velocity field for inhomogeneous vortex-wave interaction states in plane Couette flow with wall variations of the form (5.25) with $\mathcal{A}$ chosen such that $\mathcal{J} = 20$. Left, wall-normal roll velocity field $v(y, z)$; right, spanwise roll velocity field $w(y, z)$. (a) $(\alpha, \beta) = (0.8, 2.0)$, (b) $(\alpha, \beta) = (0.8, 4.0)$, (c) $(\alpha, \beta) = (0.8, 8.0)$.

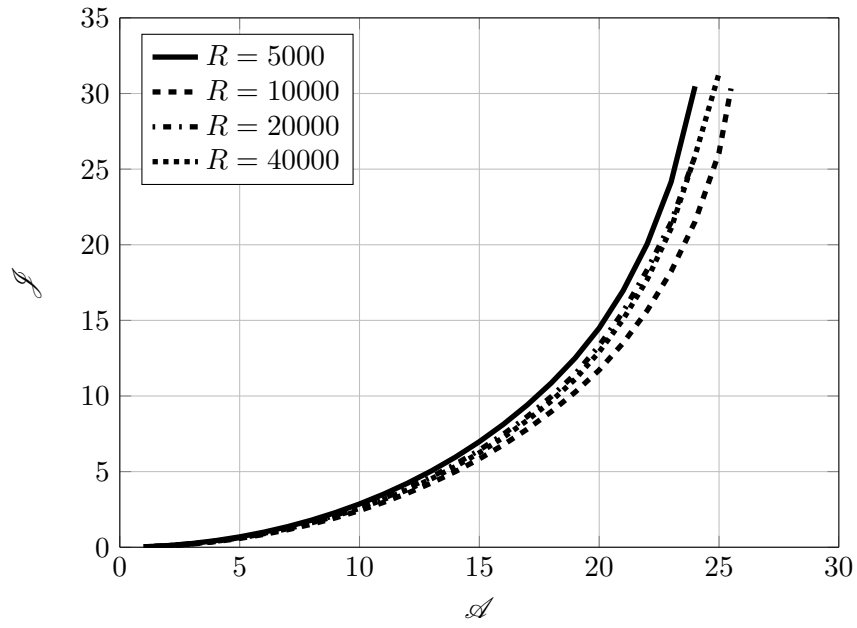


Figure 5.17. Relationship between the wall shape amplitude $\mathcal{A}$ and the maximum value in the stress jump condition (2.9), $\mathcal{J}$, for different values of the regularisation Reynolds number $R = 5000, 10000, 20000, 40000$ and fixed $(\alpha, \beta) = (0.8, 2.0)$.

6. Conclusions and Outlook

This thesis has presented a comprehensive study of the link between vortex-wave interaction theory and the theory of exact coherent structures. In this chapter we provide a summary of the research conducted in section 6.1 before discussing possible extensions and potential research directions to the current work in section 6.2.

6.1. Summary of Research

Building on the work of Hall & Sherwin (2010) we have successfully developed a new, robust, numerical scheme to solve the VWI equations for channel flows. Whereas previous attempts at solving the VWI equations have involved using a body forcing term of some form we have implemented a method that imposes the stress jump conditions exactly through the use of methods based on Green's functions. Arguably, the method developed here is computationally advantageous, especially at high Reynolds numbers, since it is based on numerically solving the formally reduced form of the governing equations rather than conducting full direct numerical simulations of the governing Navier-Stokes equations which is typically computationally prohibitive.

The nature of VWI states and how they are linked to ECS has also been elucidated in a number of ways from the number of analyses presented chapters 3 and 4. Firstly, we believe that we have convincingly demonstrated that a large number of ECS found by other authors can in fact be described by VWI theory in the high Reynolds number limit. Further, and maybe more pertinent, is that VWI theory gives an alternative interpretation of the classification of the so-called lower and upper branch equilibrium solutions. In the ECS community it is well understood that the terminology lower and upper branch originates from the origin of the solutions from a saddle-node bifurcation. In the context of VWI theory, we have shown that these

solutions are associated with different eigenmodes of the streak field. In particular, the most unstable eigenmode corresponds to the lower branch solutions and the second most unstable eigenmode corresponds to the upper branch solutions.

The first drag reduction technique examined in chapter 5 was the inclusion of $O(R^{-1})$ spanwise periodic blowing/suction was used as a means of directly exciting/suppressing the vortex structure in the roll field. It was shown that considerable reductions of up to 5% in the skin friction drag for sinuous-symmetric VWI states could be achieved. Whilst reductions in the skin friction drag for mirror-symmetric VWI states were smaller in comparison, a maximum difference of 2.5% from the no forcing case, a new class of synchronous mirror-symmetric VWI states were found when the forcing was applied at the subharmonic wavenumber. These VWI states are quite interesting in the sense that they typically have smaller drag associated with them and so are interesting from a laminar flow control perspective but they could also indicate the existence of localised structures in the spanwise direction, particularly in the $\beta \rightarrow 0$ limit.

The second drag reduction technique examined in chapter 5 was the effect of wall variations of size $O(R^{-2/3})$ which resulted in an inhomogeneous wave system. The resulting VWI states were shown to bifurcate away from plane Couette flow as the amplitude of the wall variations increased until a vertical asymptote was reached. Interestingly, regardless of the initial conditions used, the numerical procedure converged to the same inhomogeneous VWI states that appeared as weaker forms of the subharmonic mirror-symmetric VWI states discussed in section 4.2.1.

6.2. Future Work

The work presented in this thesis can be extended naturally in a number of different ways.

Within the context of VWI theory, the state space of plane Couette flow has been mapped out considerably for the first branch of solutions with the characteristics of many of the limiting cases now being fully documented. The next natural step would be to conduct the same parameter study for the second branch of solutions, which have been shown to correspond to the upper branch equilibrium solutions in the ECS community, and explore

whether any more branches exist by performing subsequent analyses.

The work of this thesis has also been restricted to stationary VWI states in plane Couette flow. Relative equilibria, or travelling wave solutions, have been documented by the ECS community, see Gibson *et al.* (2009) for example, and we have shown that travelling eigenmodes exist in plane Couette flow, recall the discussion on eigenspectra in section 3.3.2. A natural question is then to ask whether our numerical procedure based on VWI theory can capture these travelling wave equilibria.

In chapter 5, a number of analyses into possible drag reduction techniques were performed. The work presented in this thesis is by no means an exhaustive study into all the possible techniques that could be used to reduce the skin friction drag associated with VWI states. Further studies could focus on conducting a more extensive investigation into different variations in wall shape, in terms of both shape and magnitude, as well as completely different drag reduction techniques. An example of which is the use of compliant walls, rather than rigid walls, which have been shown to have a stabilising effect on the flow, see Davies & Carpenter (1997), could also be conducted. Hall (2012*a,b*) described the governing equations for VWI states with the additional effect of temperature under a Boussinesq approximation but results were limited to certain asymptotic limits, providing scope for a more in-depth study.

Finally, it is worthwhile to note that whilst the results presented in this thesis have mainly been centered on VWI states in plane Couette flow the theory can be applied to many different flow configurations. For instance, see the recent work by Dempsey *et al.* (2015) on VWI theory in plane Poiseuille flow and the work by Ozcakir *et al.* (2015) on travelling wave states in pipe flow.

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A. Derivation of the VWI Jump Conditions

In this appendix we provide a more in-depth description of the derivation of the stress jump conditions (2.9) for completion. The analysis here is similar to the derivation in local coordinates provided by Hall & Sherwin (2010) but note that alternative derivations can be found in appendix B of Hall & Smith (1991) and appendix A of Brown *et al.* (1993).

A.1. Local Critical Layer Analysis

As noted previously, the wave pressure equation (2.6) becomes singular at $y = f(z)$ where $u(y, z) = c$, that is, where speed of the wave coincides with the speed of the streak profile. This singularity is smoothed out by a thin viscous layer resulting from the balance between

$$i\alpha(u - c) = i\alpha(\mathbf{x} - \mathbf{x}_c) \cdot \nabla u \sim R^{-1}\nabla^2, \quad (\text{A.1})$$

from which we find that the suitable scaling is $y \sim R^{-1/3}$.

Our aim now is find the leading order equations within the critical layer region, first we write the asymptotic form of the velocity and pressure fields as

$$\tilde{\mathbf{u}} = [u(y, z), R^{-1}v(y, z), R^{-1}w(y, z)] + \dots$$

$$\rho\delta R^{-1/3}[U(y, z), V(y, z), W(y, z)]e^{i\alpha(x-ct)} + c.c., \quad (\text{A.2a})$$

$$\tilde{p} = R^{-2}p(y, z) + \rho\delta R^{-1/3}P(y, z)e^{i\alpha(x-ct)} + c.c., \quad (\text{A.2b})$$

where we have introduced a parameter δ , the value of which will be fixed in the subsequent analysis. Note that outside of the critical layer the wave is of size $O(\delta R^{-1/3})$ and within the critical layer it will grow to $O(\delta)$.

We will now define a local coordinate system (x, n, s) with n, s measuring

the distance normal and tangential to the critical layer respectively. The governing equations can be re-written in these coordinates, see Rosenhead (1963) for the general form of the Navier-Stokes equations in which to work from. Alternatively, the equations can be thought of as the body fitted equations that arise in the study of flow over a surface with curvature in a similar manner to the Görtler vortex problem. We also introduce the critical layer coordinate $N = R^{1/3}n$ and write the leading order wave velocity and pressure quantities as

$$\rho\delta \left(\mathcal{U}(N, s), R^{-1/3}\mathcal{V}(N, s), \mathcal{W}(N, s), R^{-1/3}\mathcal{P}(N, z) \right) E + c.c. \quad (\text{A.3})$$

where $E = e^{i\alpha(x-ct)}$.

After some cumbersome algebra, we find that the leading order terms in the spanwise momentum equation are

$$\frac{1}{R^2} R^{2/3} \frac{\partial^2 v^s}{\partial N^2} = \delta^2 \rho^2 \left(-i\alpha \mathcal{U} \overline{\mathcal{W}} + \mathcal{V} \overline{\mathcal{W}}_N + \mathcal{W} \overline{\mathcal{W}}_s + c.c. \right) + \dots, \quad (\text{A.4})$$

and likewise, the leading order terms in the wall-normal momentum equation are

$$\frac{1}{R^2} R^{1/3} \frac{\partial p^s}{\partial N} = -\delta^2 \rho^2 \Lambda_0(s) \left(\mathcal{W} \overline{\mathcal{W}} + c.c. \right) + \dots, \quad (\text{A.5})$$

where Λ_0 is the local curvature of the critical layer, the overline denotes the complex conjugate, (v^n, v^s) denotes the roll velocity field (v, w) resolved in the N, s directions respectively. The roll pressure field parameterised by N, s is denoted by p^s .

Equation A.4 can be integrated to find

$$\left[\frac{\partial v^s}{\partial N} \right]_-^+ = R^{4/3} \delta^2 \rho^2 \int_{-\infty}^{+\infty} \left(-i\alpha \mathcal{U} \overline{\mathcal{W}} + \mathcal{V} \overline{\mathcal{W}}_N + \mathcal{W} \overline{\mathcal{W}}_s + c.c. \right) dN, \quad (\text{A.6})$$

using the equation of continuity for the wave we obtain

$$\left[\frac{\partial v^s}{\partial N} \right]_-^+ = R^{4/3} \delta^2 \rho^2 \int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial N} (\mathcal{V} \overline{\mathcal{W}} + \overline{\mathcal{V}} \mathcal{W}) + 2 \frac{\partial}{\partial s} (\mathcal{W} \overline{\mathcal{W}}) \right) dN, \quad (\text{A.7})$$

which simplifies to

$$\left[\frac{\partial v^s}{\partial n} \right]_-^+ = 2R^{5/3} \delta^2 \rho^2 \int_{-\infty}^{+\infty} \frac{\partial}{\partial s} |\mathcal{W}(N, s)|^2 dN, \quad (\text{A.8})$$

since $\lim_{N \rightarrow \pm\infty} \mathcal{W} = 0$. Similarly, integrating (A.5) we obtain

$$[p]_-^+ = R^{5/3} \delta^2 \rho^2 \Lambda_0(s) \int_{-\infty}^{+\infty} |\mathcal{W}(N, s)|^2 dN. \quad (\text{A.9})$$

The choice of δ now becomes apparent since by choosing $\delta = R^{-5/6}$ we see that the outer roll equations must be solved subject to a set of jump conditions in the shear and pressure across the critical layer that depend on the wave quantities within the critical layer region.

The jump conditions (A.8) and (A.9) can be simplified even further by writing down the form of $\mathcal{W}$. From the s momentum equation for the wave in the critical layer we find that at leading order $\mathcal{W}$ satisfies

$$\frac{\partial^2 \mathcal{W}(N, s)}{\partial N^2} - i\alpha\mu(s)N\mathcal{W}(N, s) = \frac{\partial \mathcal{P}(N, s)}{\partial s}, \quad (\text{A.10})$$

and from the N momentum equation we find that

$$\frac{\partial \mathcal{P}(N, s)}{\partial N} = 0. \quad (\text{A.11})$$

Here $\mu(s) = (\partial u / \partial n)(f(s), s)$ is the normal derivative of the streak $u(y, z)$ evaluated on the critical layer and is therefore a function of s only. Equation (A.11) implies that the wave pressure within the critical layer does not vary in the normal direction and therefore matching to the wave outside the critical layer we find

$$\mathcal{P}(N, s) = P(y = f(z(s)), z(s)). \quad (\text{A.12})$$

Equation (A.10) for $\mathcal{W}(N, s)$ should be solved such that $\mathcal{W}$ decays to zero as $N \rightarrow \pm\infty$. The solution can be found in terms of Scorer's functions and after some manipulation, we find

$$\mathcal{W}(N, s) = -(\alpha\mu)^{-2/3} \frac{\partial \mathcal{P}}{\partial s} \int_0^\infty \exp\left(-i(\alpha\mu)^{1/3} Nk - k^3/3\right) dk. \quad (\text{A.13})$$

and so the jump conditions reduce to

$$\left[\frac{\partial v^s}{\partial n} \right]_-^+ = n_0 \rho^2 \alpha^{-5/3} \frac{\partial}{\partial s} \left(\mu^{-5/3} \left| \frac{\partial \mathcal{P}}{\partial s} \right|^2 \right), \quad (\text{A.14})$$

and

$$[p]_-^+ = -n_0 \rho^2 (\alpha \mu)^{-5/3} \Lambda_0 \left| \frac{\partial \mathcal{P}}{\partial s} \right|^2, \quad (\text{A.15})$$

where $n_0 = 2\pi(2/3)^{2/3}\Gamma(1/3)$.

A.2. Conversion from Local to Cartesian Coordinates

It is convenient to convert the jump conditions in local coordinates (n, s) given by (A.14) and (A.15) to Cartesian coordinates (y, z) for computational purposes.

If we define

$$\Delta = (1 + f_z^2) \quad (\text{A.16})$$

then we define the vectors normal and tangential to the critical layer as

$$\mathbf{n} = \Delta^{-1/2} \begin{pmatrix} 1 \\ -f_z \end{pmatrix}, \quad \mathbf{t} = \Delta^{-1/2} \begin{pmatrix} f_z \\ 1 \end{pmatrix}, \quad (\text{A.17})$$

respectively and the curvature of the critical layer is given by

$$\Lambda_0 = \frac{f_{zz}}{(1 + f_z^2)^{3/2}} = \Delta^{-3/2} f_{zz}. \quad (\text{A.18})$$

Noting that

$$\mu = \frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u = \Delta^{-1/2} (u_y - f_z u_z) = \Delta^{1/2} u_y, \quad (\text{A.19})$$

where we have used the fact that $f_z = -u_z/u_y$ and

$$\frac{\partial P}{\partial s} = \mathbf{t} \cdot \nabla P = \Delta^{-1/2} (f_z P_y + P_z) \quad (\text{A.20})$$

then (A.15) can be written as

$$[p]_-^+ = -n_0 \rho^2 \Theta^{-5/3} \Delta^{-5} f_{zz} \left| \frac{\partial P}{\partial z} \right|^2, \quad (\text{A.21})$$

where $\Theta = \alpha u_y / \Delta$ and the derivative of the pressure is to be evaluated on the critical layer, that is

$$\frac{\partial P}{\partial z} = \frac{\partial P}{\partial z} \Big|_{y=f(z)} = f_z \frac{\partial P}{\partial y} \Big|_z + \frac{\partial P}{\partial z} \Big|_y. \quad (\text{A.22})$$

Now let us examine (A.14), after some cumbersome algebra we can find

$$\left| \frac{\partial P}{\partial s} \right|^2 \frac{\partial}{\partial s} (\mu^{-5/3}) = \Delta^{-3/2} \mu^{-5/3} \left[-\frac{5a_z}{3a} - \frac{5\Delta_z}{2\Delta} \right] \left| \frac{\partial P}{\partial z} \right|^2, \quad (\text{A.23})$$

and

$$\mu^{-5/3} \frac{\partial}{\partial s} \left| \frac{\partial P}{\partial s} \right|^2 = \mu^{-5/3} \Delta^{-3/2} \left[-\frac{\Delta_z}{\Delta} \left| \frac{\partial P}{\partial z} \right|^2 + \frac{\partial}{\partial z} \left| \frac{\partial P}{\partial z} \right|^2 \right], \quad (\text{A.24})$$

and so (A.14) now reads

$$\left[\frac{\partial v^s}{\partial n} \right]_-^+ = n_0 \rho^2 \alpha^{-5/3} \mu^{-5/3} \Delta^{-3/2} \left[\left(-\frac{5a_z}{3a} - \frac{7\Delta_z}{2\Delta} \right) \left| \frac{\partial P}{\partial z} \right|^2 + \frac{\partial}{\partial z} \left| \frac{\partial P}{\partial z} \right|^2 \right]. \quad (\text{A.25})$$

and after some manipulation we can obtain

$$\left[\frac{\partial w}{\partial y} \right]_-^+ = n_0 \rho^2 \Theta^{-5/3} \Delta^{-5} \left[\left(-\frac{5a_z}{3a} - \frac{7\Delta_z}{2\Delta} \right) \left| \frac{\partial P}{\partial z} \right|^2 + \frac{\partial}{\partial z} \left| \frac{\partial P}{\partial z} \right|^2 \right], \quad (\text{A.26})$$

which is equivalent to (2.10)

B. Derivation of Some Useful Green's Function

In this appendix we derive and discuss some derivations of Green's functions used in the boundary integral method for solving the roll equations for completeness. Pozrikidis (1992) provides a complete description of the techniques used to find the representations of Green's functions we have used.

B.1. The free-space Green's Function

The two-dimensional Stokeslet given by (2.48) can be found by first noting that the two-dimensional Dirac delta function can be written in the form

$$\delta(\mathbf{x} - \mathbf{x}_0) = \frac{1}{2\pi} \nabla^2 \ln r, \quad (\text{B.1})$$

and that the pressure is a harmonic function and so can be written in the form

$$p = \frac{1}{2\pi} (\nabla \ln r) \cdot \mathbf{g}. \quad (\text{B.2})$$

Substituting (B.1) and (B.2) into the two-dimensional Stokes flow equations (2.43) yields

$$\nabla^2 \mathbf{v} = -\frac{1}{2\pi} \mathbf{g} \cdot (\nabla \nabla - I \nabla^2) \ln r, \quad (\text{B.3})$$

if we express the velocity in terms of a scalar function H as

$$\mathbf{v} = \mathbf{g} \cdot (\nabla \nabla - I \nabla^2) H, \quad (\text{B.4})$$

and substitute (B.4) into equation (B.3), then after discarding the arbitrary constant $\mathbf{g}$ we obtain

$$(\nabla\nabla - I\nabla^2) \left(\nabla^2 H - \frac{1}{2\pi} \ln r \right) = 0, \quad (\text{B.5})$$

which is a Poisson equation for H , namely

$$\nabla^2 H = \frac{1}{2\pi} \ln r. \quad (\text{B.6})$$

Setting

$$H = \frac{1}{8\pi} r^2 (\ln r - 1) \quad (\text{B.7})$$

and substituting into (B.4) we obtain the velocity field in the standard form given by (2.48) where $\mathcal{G}_{ij}$ is the two-dimensional free-space Green's function, namely

$$\mathcal{G}_{ij} = -\delta_{ij} \ln r + \frac{\hat{x}_i \hat{x}_j}{r^2}. \quad (\text{B.8})$$

B.2. Periodic Array of Point Forces in a Channel

If we now consider the Green's function for the point force in a domain that is confined by two parallel walls located at $y = \pm H$ then we can construct one representation of this Green's function by considering an image system with respect to the two walls. The Green's function can then be given by the limit, as N tends to infinity, of

$$\begin{aligned} \mathcal{G}_{ij}^{2W}(\mathbf{x}, \mathbf{x}_0) = & \sum_{n=-N}^N \left(-\delta_{ij} \ln r_n + \frac{\hat{x}_{n,i} \hat{x}_{n,j}}{r_n^2} \right) - \dots \\ & \sum_{n=-N}^N \left(-\delta_{ij} \ln R_n + \frac{\hat{X}_{n,i} \hat{X}_{n,j}}{R_n^2} \right), \end{aligned} \quad (\text{B.9})$$

where

$$\begin{aligned} \hat{\mathbf{x}}_n = \mathbf{x} - \mathbf{x}_n, \quad r_n = |\hat{\mathbf{x}}_n|, \quad \mathbf{x}_n = (x_0, y_0 + 4Hn), \\ \hat{\mathbf{X}}_n = \mathbf{x} - \mathbf{x}_n^{IM}, \quad R_n = |\hat{\mathbf{X}}_n|, \quad \mathbf{x}_n^{IM} = (x_0, -y_0 - 2H + 4Hn). \end{aligned} \quad (\text{B.10})$$

As N tends to infinity, both sums on the right hand side of (B.9) diverge. To obtain a regular expression, we split the Green's function into a fundamental component $\mathbf{G}^F$ and a complimentary componenet $\mathbf{G}^C$. Before writing down expressions for these components we first consider the summation formula

$$A(\hat{\mathbf{x}}_0) = \sum_{n=-\infty}^{\infty} \ln(r_n) = \frac{1}{2} \ln [\cosh(\hat{y}_0) - \cos(\hat{x}_0)] + \frac{1}{2} \ln(2), \quad (\text{B.11})$$

where the $\frac{1}{2} \ln(2)$ term is added to ensure that close to the pole the Green's function will reduce to the Stokeslet. Differentiating (B.11), we obtain

$$\frac{\partial A}{\partial \hat{x}_0} = A_x(\hat{\mathbf{x}}_0) = \sum_{n=-\infty}^{\infty} \frac{\hat{x}_n}{r_n^2} = \frac{1}{2} \frac{\sin(\hat{x}_0)}{\cosh(\hat{y}_0) - \cos(\hat{x}_0)}, \quad (\text{B.12})$$

$$\frac{\partial A}{\partial \hat{y}_0} = A_y(\hat{\mathbf{x}}_0) = \sum_{n=-\infty}^{\infty} \frac{\hat{y}_n}{r_n^2} = \frac{1}{2} \frac{\sinh(\hat{y}_0)}{\cosh(\hat{y}_0) - \cos(\hat{x}_0)}. \quad (\text{B.13})$$

Therefore, for a periodic array of point forces we can write

$$\begin{aligned} \mathcal{G}_{ij}^P(\mathbf{x}, \mathbf{x}_0) &= \sum_{n=-\infty}^{\infty} \left(-\delta_{ij} \ln(r_n) + \frac{\hat{x}_i \hat{x}_j}{r_n^2} \right) \\ &= \begin{bmatrix} -A(\hat{\mathbf{x}}_0) + \hat{x} A_x(\hat{\mathbf{x}}_0) & \hat{y} A_x(\hat{\mathbf{x}}_0) \\ \hat{y} A_x(\hat{\mathbf{x}}_0) & -A(\hat{\mathbf{x}}_0) + \hat{y} A_y(\hat{\mathbf{x}}_0) \end{bmatrix}. \end{aligned} \quad (\text{B.14})$$

Returning to (B.9), we define the fundamental component as

$$\mathbf{G}^F(\mathbf{x}, \mathbf{x}_0) = \mathbf{G}^P(\mathbf{x}, \mathbf{x}_0) - \mathbf{G}^P(\mathbf{x}, \mathbf{x}_0^{IM}). \quad (\text{B.15})$$

Evaluating the fundamental component on the two walls we find that

$$\begin{aligned} G_{xx}^F(y = \pm H) &= G_{yy}^F(y = \pm H) = 0, \\ G_{xy}^F(y = H) &= G_{yx}^F(y = H) = f(\hat{x}_0, h_2), \\ G_{xy}^F(y = -H) &= G_{yx}^F(y = -H) = f(\hat{x}_0, h_1), \end{aligned} \quad (\text{B.16})$$

where $h_1 = H + y_0$ and $h_2 = H - y_0$, and

$$f(x, h) = \frac{\pi}{2} \frac{x}{H} \frac{\sin(\pi h/2H)}{\cosh(\pi x/2H) - \cos(\pi h/2H)}. \quad (\text{B.17})$$

The diagonal componenets of the fundamental Green's function vanish over both walls. To ensure all components of the Green's function $\mathbf{G}^{2W}$ vanish we require that the complimentary Green's function $\mathbf{G}^C$ satisfies the boundary conditions

$$\begin{aligned} G_{xx}^C(y = \pm H) &= G_{yy}^C(y = \pm H) = 0, \\ G_{xy}^C(y = \pm H) &= G_{yx}^C(y = \pm H) = -G_{xy}^F(y = \pm H). \end{aligned} \quad (\text{B.18})$$

The complimentary Green's function can then be found using the method of Fourier transforms. Taking the Fourier transform of (B.17) we find

$$\begin{aligned} \mathcal{F}(f(x, h)) &= \int_{-\infty}^{\infty} f(x, h) \exp(i\omega x) dx \\ &= 2\pi i \frac{\omega}{|\omega|} g(|\omega|, h) \end{aligned} \quad (\text{B.19})$$

where

$$g(z, h) = \frac{d}{dz} \left(\frac{\exp(-(4H - h)z) - \exp(-hz)}{1 - \exp(-4Hz)} \right), \quad (\text{B.20})$$

by the calculus of residues. Now taking the Fourier transform of the Stokes flow equations, imposing the above boundary conditions in Fourier space, and then inverting back to real space we find that

$$G_{xx}^C = \int_0^\infty [ye^{\omega y}, -ye^{-\omega y}, 2e^{\omega y}, 2e^{-\omega y}] \cdot \mathbf{a} \cos(\omega \hat{x}) d\omega, \quad (\text{B.21a})$$

$$G_{xy}^C = \int_0^\infty [ye^{\omega y}, -ye^{-\omega y}, 2e^{\omega y}, 2e^{-\omega y}] \cdot \mathbf{b} \sin(\omega \hat{x}) d\omega, \quad (\text{B.21b})$$

$$G_{yx}^C = \int_0^\infty \left[\frac{(\omega y - 1)}{\omega} e^{\omega y}, \frac{(\omega y + 1)}{\omega} e^{-\omega y}, 2e^{\omega y}, -2e^{-\omega y} \right] \cdot \mathbf{a} \sin(\omega \hat{x}) d\omega, \quad (\text{B.21c})$$

$$G_{yy}^C = \int_0^\infty \left[\frac{(\omega y - 1)}{\omega} e^{\omega y}, \frac{(\omega y + 1)}{\omega} e^{-\omega y}, 2e^{\omega y}, -2e^{-\omega y} \right] \cdot \mathbf{b} \cos(\omega \hat{x}) d\omega. \quad (\text{B.21d})$$

The vectors $\mathbf{a}$ and $\mathbf{b}$ arise from the solution of the system of linear equations

$$B\mathbf{a} = \exp(-\omega H)\mathbf{c}, \quad B\mathbf{b} = \exp(-\omega H)\mathbf{d}, \quad (\text{B.22})$$

where

$$B = \begin{pmatrix} \frac{1}{2}H & -\frac{1}{2}H\theta & 1 & \theta \\ -\frac{1}{2}H\theta & \frac{1}{2}H & \theta & 1 \\ \frac{1}{2}(H - \omega^{-1}) & \frac{1}{2}\theta(H + \omega^{-1}) & 1 & -\theta \\ -\frac{1}{2}\theta(H + \omega^{-1}) & -\frac{1}{2}(H - \omega^{-1}) & \theta & -1 \end{pmatrix}, \quad (\text{B.23})$$

and

$$\begin{aligned} \theta &= \exp(-2\omega H), \\ \mathbf{c} &= [0, 0, -g(\omega, h_2), g(\omega, h_1)], \\ \mathbf{d} &= [g(\omega, h_2), -g(\omega, h_1), 0, 0]. \end{aligned} \quad (\text{B.24})$$

For a periodic array of point forces in a channel with walls located at $y = \pm H$ we simply sum the Green's function corresponding to a single point force in a channel. By analogy with the Green's function for a single point force we decompose the Green's function into a fundamental periodic component $\mathbf{G}^{FP}$ and a complimentary periodic component, $\mathbf{G}^{CP}$. The fundamental periodic component is given by

$$\mathbf{G}^{FP}(\mathbf{x}, \mathbf{x}_0) = \sum_{n=-\infty}^{\infty} \mathbf{G}^F(\mathbf{x}, \mathbf{x}_n), \quad (\text{B.25})$$

where $\mathbf{x}_n = (x_0 + nL, y_0)$ and L is the distance between two successive point forces. The Green's function $\mathbf{G}^F$ is the fundamental component given by (B.15). Furthermore, using (B.21) and the identity

$$H(N) = \sum_{n=-N}^N e^{i(\hat{x}+nl)t} = e^{i\hat{x}t} \frac{\cos(Nlt) - \cos((N+1)lt)}{1 - \cos(lt)} \quad (\text{B.26})$$

such that

$$H(\infty) = \frac{2\pi}{l} e^{i\hat{x}t} \sum_{m=-\infty}^{\infty} \delta\left(t - \frac{2\pi m}{l}\right), \quad (\text{B.27})$$

we find that

$$G_{xx}^{CP} = \frac{2n}{L} \sum_{m=1}^{\infty} [ye^{w_my}, -ye^{-w_my}, 2e^{w_my}, 2e^{-w_my}] \cdot \mathbf{a} \cos(w_m \hat{x}_0), \quad (\text{B.28a})$$

$$G_{xy}^{CP} = \frac{2n}{L} \sum_{m=1}^{\infty} [ye^{w_my}, -ye^{-w_my}, 2e^{w_my}, 2e^{-w_my}] \cdot \mathbf{b} \sin(w_m \hat{x}_0), \quad (\text{B.28b})$$

$$G_{yx}^{CP} = \frac{2n}{L} \sum_{m=1}^{\infty} \left[\frac{w_n y - 1}{w_n} e^{w_n y}, \frac{w_n y + 1}{w_n} e^{-w_n y}, 2e^{w_n y}, -2e^{-w_n y} \right] \cdot \mathbf{a} \cdot \sin(w_n \hat{x}_0), \quad (\text{B.28c})$$

$$G_{yy}^{CP} = \frac{2n}{L} \sum_{m=1}^{\infty} \left[\frac{w_n y - 1}{w_n} e^{w_n y}, \frac{w_n y + 1}{w_n} e^{-w_n y}, 2e^{w_n y}, -2e^{-w_n y} \right] \cdot \mathbf{b} \cos(w_m \hat{x}_0), \quad (\text{B.28d})$$

where $w_m = 2\pi m/L$. To compute $\mathbf{a}$ and $\mathbf{b}$ we solve the linear systems given by (B.22) but with w_m in place of ω .