

IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

On spectral instability and kinematic dynamo action in three-dimensional incompressible flows

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April 2026

Submitted in part fulfilment of the requirements for the degree of Doctor of Philosophy in
Mathematics of Imperial College London and the Diploma of Imperial College London.

Declaration of Originality

I, David Villringer, declare that the work in this thesis is my own. The work of others has been appropriately referenced. All material for which I am not the author is explicitly indicated by the appropriate use of citations and references.

Parts of this thesis are based on jointly authored works. Chapter 3 is based on [CSV25], joint with Michele Coti Zelati and Massimo Sorella. Chapter 4 is based on [SV25], joint with Massimo Sorella. Chapter 5 is based on [CSV26], joint with Michele Coti Zelati and Massimo Sorella; and Chapters 6-7 are based on [NV25b; NV25a], joint with Víctor Navarro–Fernández. In each case, my contribution was substantial. The mathematical results were developed collaboratively with the listed co-authors. My contributions included formulating the problems, developing the main proof strategies, proving substantial parts of the results, carrying out technical estimates, and contributing substantially to the writing and revision of the manuscripts.

I acknowledge the use of ChatGPT 5.4 for language-level feedback, checking for typos, and for improving TikZ Figure 2.1. Generative AI tools were not used to generate mathematical results or proofs, and the final text is entirely my own.

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Abstract

This thesis studies the kinematic dynamo problem, and related instabilities in magnetohydrodynamics. Its central question is whether a divergence-free velocity field can sustain exponential growth of a magnetic field, particularly as magnetic resistivity tends to zero. This phenomenon, known as the *dynamo effect*, is a classical explanation for magnetic fields in celestial bodies such as the Earth and the Sun, but comparatively few rigorous results are available.

After introducing the kinematic dynamo equations, as well as multiple foundational results in their theory, we present several new rigorous constructions. We construct an explicit example of a *fast dynamo* on $\mathbb{R}^3$, by considering spatially periodic velocity fields exhibiting the α -effect at fixed resistivity. Via rescaling and spatial localisation, these building blocks are glued together to produce fast dynamo action. Similarly, we also construct a subsequential fast dynamo on $\mathbb{T}^3$, with growth along a subsequence of times.

We then study a pulsed-diffusion model on $\mathbb{T}^3$, proving fast dynamo action under a time-periodic, Lipschitz-continuous velocity field. The argument is inherently perturbative in the magnetic resistivity, using *anisotropic Banach spaces* to employ techniques from the singular perturbation theory.

Finally, we study slow dynamo action in a class of helical velocity fields, known as the *Ponomarenko dynamo*. We establish the existence of growing modes with growth rate of order $\epsilon^{1/3}$ in the resistivity ϵ , for a broad class of velocity fields. The proof relies on resolvent analysis of the associated linear operators, yielding bounds on the associated spectral projectors. Using this, we then prove nonlinear instability of the full three-dimensional MHD equations around the Taylor–Couette flow and the trivial magnetic field.

Overall, this thesis provides new rigorous examples of slow and fast dynamo action, and develops tools for the study of instabilities in the singular limit of vanishing resistivity.

Acknowledgements

I am very grateful to my advisors Michele Coti Zelati and Martin Hairer for their unwavering support during my PhD, and for providing me with guidance and invaluable insights when working on problems. I also thank Angeliki Menegaki, David Gérard-Varet and Greg Pavliotis for agreeing to examine this (rather lengthy) thesis.

Furthermore, I would like to thank Augusto, Haoling, Luca, Lucas, Marc, Marco, Massimo, Niklas, Raj, Víctor, and Yupei for contributing to an environment at Imperial that was both intellectually stimulating and genuinely enjoyable, with many spontaneous conversations and entertaining moments along the way. I particularly thank Augusto and Marc for creating a welcoming environment during my first years as a PhD student, and Massimo and Víctor for collaborating with me in multiple works, and being willing to humour my (sometimes nonsensical) ideas for approaching problems.

I also thank my friends Julius, Marta, Max and Pedro for the many great times we shared together. Finally, I thank my parents Kersten and Arno for their unwavering encouragement and support, and my girlfriend Karen for her love, care and companionship throughout my PhD journey. Her consideration and patience was instrumental in getting me through the tough parts of my PhD, and I am truly grateful for her help.

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Chapter 1

Introduction

1.1 The equations of magnetohydrodynamics

This thesis studies the equations governing the evolution of a magnetic field B , interacting with an electrically conducting fluid u . In full generality, this interaction is described by the equations of *magnetohydrodynamics* (MHD), which govern the evolution of unknown velocity and magnetic fields $u(t, x), B(t, x)$, and are given by

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u + (B \cdot \nabla)B + f, \\ \partial_t B + (u \cdot \nabla)B - (B \cdot \nabla)u = \varepsilon \Delta B, \\ \operatorname{div}(u) = 0, \quad \operatorname{div}(B) = 0, \\ (u(0, x), B(0, x)) = (u_{\text{in}}, B_{\text{in}}), \end{cases} \quad (1.1.1)$$

with $(t, x) \in (0, \infty) \times \Omega$. Here, $\Omega \subset \mathbb{R}^3$ is some physical domain, p is the pressure, f is some body force acting on the fluid, and ν, ε are respectively the kinematic viscosity and the magnetic resistivity. An important property of (1.1.1) is that, if $u(t, x)$ is a solution to the Navier–Stokes equations, then $(u, 0)$ solves (1.1.1). In other words, a vanishing initial magnetic field remains zero under the evolution of the equation. Magnetic fields are, however, ubiquitous in astrophysical and geophysical settings involving electrically conducting fluids. This motivates the search for mechanisms by which a weak magnetic field can be amplified and sustained despite resistive dissipation. In the mathematical theory, this corresponds to the study of instabilities of the zero-field state $(u, 0)$, first at the linear level and, where possible, at the nonlinear level. Indeed, as early as the work of Larmor’s 1919 [Lar19], it was proposed that the instabilities of the MHD system could explain the mechanism behind the maintenance of the Sun’s magnetic field, despite the presence of non-zero magnetic resistivity. However, the study of the full 3-dimensional MHD equations (1.1.1) presents major analytical difficulties; in particular, global existence of solutions for smooth initial data is open. To circumvent this obstruction, the study of MHD instability has historically proceeded via linearisation around a vanishing magnetic field. More concretely, if one considers a solution (1.1.1) of the form $(u(t, x), \delta B(t, x))$, where $\delta \ll 1$ is a small parameter, then the Lorentz force feedback term $B \cdot \nabla B$ in (1.1.1) becomes of order δ^2 . Hence, the equations for the

fluid velocity and the magnetic field formally decouple as $\delta \rightarrow 0$: the fluid evolves independently of the magnetic field B , which in turn evolves as a passive transport equation with velocity field u :

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u + f, \\ \partial_t B + (u \cdot \nabla)B - (B \cdot \nabla)u = \varepsilon \Delta B, \end{cases}$$

At the linear level, the study of magnetic-field amplification is therefore reduced to the system of equations

$$\begin{cases} \partial_t B + (u \cdot \nabla)B - (B \cdot \nabla)u = \varepsilon \Delta B, \\ \operatorname{div} B = 0, \\ B|_{t=0} = B_{\text{in}}, \end{cases} \quad (t, x) \in (0, \infty) \times \Omega, \quad (\text{KDE})$$

which are known as the *kinematic dynamo equations*. The velocity $u(t, x)$ is now a *given* divergence free velocity field. That is, it is supplied as part of the problem statement and does not depend on the magnetic field B , which is consistent with the linearised regime discussed above. In fact, whilst it would certainly be desirable for u to be a solution to the Navier–Stokes equations, the equation (KDE) is perfectly well-defined for any divergence-free velocity field u , and the study of its instabilities is of independent interest.

1.2 The kinematic dynamo equations

At the level of the kinematic dynamo equations (KDE), the central question is whether the magnetic field can exhibit exponential growth in time. This phenomenon is referred to as the dynamo effect. It is precisely this effect that was proposed by Larmor [Lar19] as an explanation for the maintenance of the Earth’s and the Sun’s magnetic fields, despite non-zero magnetic resistivity. However, as pointed out in [CG95], the structure of the magnetic fields of the Earth and the Sun are rather distinct. While the Earth’s magnetic field appears to exist mostly on large scales, in the form of a mean magnetic field of order unity, the situation for the Sun is quite different. Here one instead observes a rather modestly sized mean magnetic field, around which fine scale fluctuations occur that are of intensity $\sim 10^3$ times that of the mean field [CG95]! Kinematic dynamo theory offers an explanation for this discrepancy, in the form of the distinct magnetic Reynolds numbers of the Sun and the Earth respectively. While that of the Earth is estimated to be of order $\sim 10^3$ ([Chr06]), the Sun has a magnetic Reynolds number of order 10^6 ([Hoo12]). In the kinematic dynamo equations, the magnetic Reynolds number enters as the reciprocal of the resistivity ε . That is, the larger the magnetic Reynolds number, the smaller the resistivity in the kinematic dynamo equations. Heuristically, since the action of diffusion is stronger on smaller scales, larger magnetic Reynolds numbers allow for finer scale structures to be maintained over time. However, the implication appears to be more subtle— a classical heuristic, going back to Moffatt and Proctor [MP85], suggests that persistent dynamo action in the large magnetic Reynolds number regime *requires* the magnetic field to develop increasingly fine spatial structure as resistivity vanishes ¹.

¹For the reader familiar with passive scalar turbulence, it may be of interest to learn that the analysis in [MP85] suggests that the typical spatial scale of fast dynamo action is of order $\varepsilon^{1/2}$, which corresponds precisely to the Batchelor scale in passive scalar turbulence [Hai+24].

This (possible) requirement also places a substantial burden on the velocity field u , which must maintain these small scales that would otherwise be preferentially dampened by the diffusivity $\varepsilon\Delta$. As such, it appears that, in order for dynamo action to persist in the infinite magnetic Reynolds number limit (or equivalently in the limit of vanishing resistivity ε), it may be useful to study velocity fields exhibiting a hierarchy of active scales, as is characteristic of turbulent flows and the Richardson–Kolmogorov cascade [Fri95]. This requirement appears to be completely absent for more modestly sized magnetic Reynolds numbers, where the growth may be dominated by larger scale structures. The above heuristic suggests that any mathematical formalisation of this physical intuition should distinguish in the study of the dynamo problem between the case of order unity resistivity ε , and that of the vanishing resistivity limit $\varepsilon \rightarrow 0$. Therefore, for the remainder of this thesis, we explicitly denote the dependence on ε in (KDE):

$$\begin{cases} \partial_t B^\varepsilon + (u \cdot \nabla) B^\varepsilon - (B^\varepsilon \cdot \nabla) u = \varepsilon \Delta B^\varepsilon, \\ \operatorname{div} B^\varepsilon = 0, \\ B^\varepsilon|_{t=0} = B_{\text{in}}^\varepsilon, \end{cases} \quad (t, x) \in (0, \infty) \times \Omega. \quad (\text{KDE-}\varepsilon)$$

We remark that, for divergence-free u, B^ε , there holds

$$-u \cdot \nabla B^\varepsilon + B^\varepsilon \cdot \nabla u = \nabla \times (u \times B^\varepsilon),$$

and so we shall sometimes employ this identity in the formulation of the dynamo equations. Corresponding to the set of equations (KDE- ε), we furthermore define the ε -dependent dynamo growth rate as

$$\gamma(\varepsilon) := \sup_{B_{\text{in}}^\varepsilon \in L^2(\Omega), \nabla \cdot B_{\text{in}}^\varepsilon = 0} \liminf_{t \rightarrow \infty} \frac{1}{t} \log \|B^\varepsilon(t)\|_{L^2(\Omega)}. \quad (1.2.1)$$

With this established, dynamo action can be classified according to a rather clean trichotomy:

Definition 1.2.1. Let $u \in W^{1,\infty}(\Omega)$ be a Lipschitz, divergence free vector field, that has vanishing normal component at the boundary of Ω . Let $\gamma(\varepsilon)$ be defined in (1.2.1). Then, we say that

1. u generates an *ideal dynamo* if $\gamma(0) > 0$
2. u generates a *slow dynamo* if there exists $\varepsilon > 0$ so that $\gamma(\varepsilon) > 0$, but $\liminf_{\varepsilon \rightarrow 0} \gamma(\varepsilon) = 0$
3. u generates a *fast dynamo* if $\liminf_{\varepsilon \rightarrow 0} \gamma(\varepsilon) > 0$.

While all three of these phenomena are of interest in their own right, fast dynamo action in particular has received sustained attention in both mathematics and physics over the last half-century or so [CG95; AK98]. It is the subject of a conjecture, originally posed by Ya. B. Zeldovich and A. D. Sakharov in the 1970s, which appears in Arnold’s influential book of problems (see [AK98, Chapter V] and [Arn04, Pb. 1994–28]), and can be formulated as follows:

Conjecture 1 (Ya. B. Zeldovich, A. D. Sakharov, V. Arnold). *There exists a divergence-free vector field u on $\mathbb{T}^3$ that generates a fast dynamo.*

Despite considerable effort, this conjecture remains open to this day.

In contrast to this, the problems of ideal and slow dynamo action appear to be much more amenable

to classical forms of analysis. The property of ideal dynamo action in particular can be deduced from purely local, geometric properties of the velocity field u , see Lemma 2.1.3. Slow dynamo action occupies an intermediate position between the ideal and fast regimes: unlike ideal dynamo action, it depends essentially on the presence of diffusion, yet unlike fast dynamo action, it remains accessible to comparatively classical analytical methods. In particular, the phenomenon of slow dynamo action exhibits many similarities to that of boundary layer instabilities in fluids, where instabilities are (somewhat paradoxically) *caused* by the presence of diffusion, and vanish in the inviscid limit. This connection was employed for instance in the papers [Gér05; GR07] in order to prove nonlinear instability for the MHD equations via the dynamo effect.

Throughout this manuscript, we will touch on various aspects of the theory of both fast and slow kinematic dynamo action, as well as explicit nonlinear instabilities for the MHD equations. We note that several later chapters rely heavily on perturbation-theoretic arguments. Therefore, for the reader’s convenience, we have collected the operator-theoretic and semigroup-theoretic results used repeatedly throughout the thesis in Appendix A.

1.3 Contributions

We now summarise the main contributions of this thesis. All of the following papers and results were obtained during the author’s stay at Imperial College London, from September 2022 – April 2026. We begin by outlining the contributions toward fast dynamo theory, and then those towards the theory of slow dynamos and MHD instability.

In a joint work with Michele Coti Zelati and Massimo Sorella [CSV25], we establish slow dynamo action for a generic class of spatially periodic velocity fields, and, via a rescaling argument, glue them together to provide a proof of fast dynamo action on $\mathbb{R}^3$. The proof proceeds via an application of perturbation theory to show that generic periodic velocity fields generically admit linear instabilities for the kinematic dynamo operator *at fixed resistivity* $\varepsilon > 0$. Such instabilities are used as building blocks for the subsequent rescaling and gluing arguments. The results of this paper are contained in Chapter 3. Building on similar ideas, in the joint work with Massimo Sorella [SV25], we employ the same rescaled “building blocks”, which are now glued in a temporal fashion to obtain a *subsequentially* fast dynamo on $\mathbb{T}^3$ — that is, a dynamo where growth occurs on a subsequence of times $t_n \rightarrow \infty$. This work will be outlined in Chapter 4.

Following this, in the joint work with Michele Coti Zelati and Massimo Sorella [CSV26], we provide a proof of fast dynamo action on $\mathbb{T}^3$ with pulsed diffusion. This result is perhaps the closest, in spirit, to the classical fast dynamo conjecture 1. Notably, the argument proceeds by establishing a suitable space of distributions where the ideal dynamo operator admits an unstable eigenvalue, and then treating the Laplacian using the singular perturbation theory. This work is the subject of Chapter 5.

A principal contribution of this thesis to slow dynamo theory is the joint work [NV25b] with Víctor Navarro–Fernández. In particular, it rigorously establishes slow dynamo action (with a growth rate $\gamma(\varepsilon) \sim \varepsilon^{1/3}$) for a wide class of helical velocity fields. The proof proceeds via a

carefully constructed asymptotic expansion of the resolvents of the associated linear operators, which are used to estimate spectral projectors. The class of admissible velocity fields includes the *Taylor–Couette flow*, which is an explicit solution of the 3D Navier–Stokes equations. Using this result, in the follow-up work [NV25a], we prove nonlinear instability of the 3D MHD equations around the explicit stationary solution of the Taylor–Couette flow. These results will be covered respectively in Chapters 6 and 7.

We also note that the background Chapter 2 on dynamo action contains a number of new results which had not previously appeared in the literature. For instance, the examples of fast dynamo action in the stretch-fold-shear (SFS) model are, to the best of the author’s knowledge, entirely new.

Finally, we remark on two papers that were produced during the author’s PhD that will not be included in this manuscript, as their topic slightly deviates from that of dynamo theory. The papers are [Vil25] and [CHV25], and deal with the phenomenon of *enhanced dissipation*.

Chapter 2

Background theory and preliminary results

This chapter reviews fundamental results on the kinematic dynamo equations and also records several results and proofs that either do not seem to have appeared previously in the literature, or have appeared only as sketches. A central theme of the chapter that we wish to highlight is the set of obstructions that make dynamo action at non-zero resistivity difficult to establish.

2.1 Basic results on the kinematic dynamo equations

We begin with basic analytical properties of the kinematic dynamo equations, including well-posedness, energy estimates, and some remarks on the ideal dynamo problem.

2.1.1 Notes on well-posedness and basic energy estimates

We recall that the kinematic dynamo equations take the form

$$\begin{cases} \partial_t B^\varepsilon + (u \cdot \nabla) B^\varepsilon - (B^\varepsilon \cdot \nabla) u = \varepsilon \Delta B^\varepsilon, \\ \operatorname{div} B^\varepsilon = 0, \\ B^\varepsilon|_{t=0} = B_{\text{in}}^\varepsilon, \end{cases} \quad (t, x) \in (0, \infty) \times \Omega,$$

for some subdomain $\Omega \subset \mathbb{R}^3$. We shall typically study these equations in $L^2(\Omega)$, together with the physically relevant *perfectly conducting boundary conditions* ([CG95; ST83])

$$u \cdot \hat{n} = 0, \quad B \cdot \hat{n} = 0, \quad (\nabla \times B) \times \hat{n} = 0 \quad (2.1.1)$$

on $\partial\Omega$, whenever $\partial\Omega \neq \emptyset$. In particular, these boundary conditions ensure that the Laplacian $\Delta = -\nabla \times \nabla \times$ acting on divergence-free vector fields is a self-adjoint, negative operator. In fact, there holds the following general well-posedness result:

Lemma 2.1.1. *Let Ω either be a bounded, smooth subdomain of $\mathbb{R}^3$ ¹, or one of $\mathbb{T}^3$ or $\mathbb{R}^3$. Let $u \in L^\infty(0, \infty; W_x^{1, \infty}(\Omega))$ with $u \cdot \hat{n} = 0$ on $\partial\Omega$ be divergence-free. Then, there exists a unique weak solution $B^\varepsilon \in L_{loc}^\infty(0, \infty; L^2(\Omega)) \cap L_{loc}^2(0, \infty; \dot{H}^1(\Omega))$ to (KDE- ε) with initial condition B_{in}^ε , and satisfying the perfectly conducting boundary conditions $B \cdot \hat{n} = 0$, $(\nabla \times B) \times \hat{n} = 0$ on $\partial\Omega$. In fact, there holds the energy estimate*

$$\|B^\varepsilon(t)\|_{L^2} \leq \|B_{in}^\varepsilon\|_{L^2} \exp(\|\nabla u\|_{L^\infty} t), \quad \forall t \geq 0. \quad (2.1.2)$$

Finally, if u is smooth then $B^\varepsilon(t, x)$ is smooth for all $t > 0$.

Proof. Existence follows by a standard approximation argument, so we restrict attention to the formal a priori estimates.

Let $B \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ be a weak solution to (KDE- ε), with initial datum B_{in} . Testing the equation with B in $L^2(\Omega)$, and using that u is divergence-free with vanishing normal component at $\partial\Omega$, gives the identity

$$\frac{1}{2} \frac{d}{dt} \|B^\varepsilon\|_{L^2}^2 + \varepsilon \|\nabla \times B^\varepsilon\|_{L^2}^2 = \int_{\Omega} (B^\varepsilon \cdot \nabla) u \cdot B^\varepsilon.$$

Now (2.1.2) follows immediately from the above identity, by estimating the right-hand side as

$$\int_{\Omega} (B \cdot \nabla) u \cdot B \leq \|\nabla u\|_{L^\infty} \|B\|_{L^2}^2.$$

Then, an application of Grönwall's inequality yields the desired result. Finally, if u is smooth, smoothness of solutions follows from standard parabolic smoothing. $\square$

We draw particular attention to the estimate (2.1.2), which implies a uniform in ε growth bound for the dynamo equation. In fact, while well-posedness can also be established for $u \in L^\infty$, the corresponding energy balance for $\|B^\varepsilon\|_{L^2}$ degenerates as $\varepsilon \rightarrow 0$, making the assumption $u \in W^{1, \infty}(\Omega)$ the most natural for studying the dynamo problem with vanishing diffusivity.

A key feature of the kinematic dynamo equations is that, similar to the case of the transport equations, they admit an explicit solution formula depending solely on the flow generated by the velocity field u . Indeed, there holds the following result:

Lemma 2.1.2. *Let $u \in L_t^\infty W_x^{1, \infty}(\mathbb{T}^3)$ be divergence free. Let $X_t^\varepsilon(x)$ be the stochastic flow generated by the SDE*

$$X_t^\varepsilon(x) = x + \int_0^t u(s, X_s^\varepsilon(x)) ds + \sqrt{2\varepsilon} W_t,$$

where $(W_t)_{t \geq 0}$ is a standard 3-dimensional Wiener process. Then, the solution to (KDE- ε) with initial condition B_{in}^ε is given by

$$B^\varepsilon(t, x) = \mathbb{E} \left((DX_t^\varepsilon B_{in}^\varepsilon) \circ (X_t^\varepsilon)^{-1}(x) \right). \quad (2.1.3)$$

In particular, this formula is also valid in the case when $\varepsilon = 0$, in which case one no longer needs

¹This assumption on Ω guarantees that $\|\nabla \times B\|_{L^2(\Omega)}^2 + \|B\|_{L^2(\Omega)}^2 \sim \|B\|_{H^1(\Omega)}^2$ for divergence-free B satisfying the perfectly conducting boundary conditions, see e.g. [ADN64].

to take expectations.

Proof. The equation (2.1.3) can be justified directly using the backwards Itô formula [Kun02]. However, in the interest of keeping this manuscript self-contained, we shall proceed using the more well-known standard Itô formula. More general results on stochastic flows and their inverses can be found in the monograph [Kun02]. Let $\phi(t, x) \in C_c^\infty([0, \infty) \times \mathbb{T}^3; \mathbb{R}^3)$, and consider

$$\int_{\mathbb{T}^3} B^\varepsilon(t, x) \cdot \phi(t, x) dx = \mathbb{E} \int_{\mathbb{T}^3} DX_t^\varepsilon(x) B_{\text{in}}^\varepsilon(x) \cdot \phi(t, X_t^\varepsilon(x)) dx,$$

where we have changed variables and used that X_t^ε is volume preserving, since u is divergence-free. Applying Itô's formula, we deduce that

$$\begin{aligned} \int_{\mathbb{T}^3} B^\varepsilon(T, x) \cdot \phi(T, x) dx &= \int_{\mathbb{T}^3} B_{\text{in}}^\varepsilon(x) \cdot \phi(0, x) dx + \mathbb{E} \int_0^T \int_{\mathbb{T}^3} Du(t, X_t^\varepsilon(x)) DX_t^\varepsilon(x) B_{\text{in}}^\varepsilon(x) \cdot \phi(t, X_t^\varepsilon(x)) dx \\ &+ \mathbb{E} \int_0^T \int_{\mathbb{T}^3} DX_t^\varepsilon(x) B_{\text{in}}^\varepsilon(x) \cdot (\partial_t \phi(t, X_t^\varepsilon(x)) + \nabla \phi(t, X_t^\varepsilon(x)) \cdot u(t, X_t^\varepsilon(x)) + \Delta \phi(t, X_t^\varepsilon(x))) dx dt, \end{aligned}$$

where we have used that the Martingale term has vanishing expectation. Changing variables back, we find that

$$\begin{aligned} \int_{\mathbb{T}^3} B^\varepsilon(T, x) \cdot \phi(T, x) dx &= \int_{\mathbb{T}^3} B_{\text{in}}^\varepsilon(x) \cdot \phi(0, x) dx + \int_0^T \int_{\mathbb{T}^3} Du(t, x) B^\varepsilon(t, x) \cdot \phi(t, x) dx \\ &+ \int_0^T \int_{\mathbb{T}^3} B^\varepsilon(t, x) \cdot (\partial_t \phi(t, x) + \nabla \phi(t, x) \cdot u(t, x) + \Delta \phi(t, x)) dx dt. \end{aligned}$$

But this is nothing but the definition of a weak solution to (KDE- ε), and so the proof follows by Lemma 2.1.1. We note that in the case $\varepsilon = 0$, the proof is much more direct: applying the chain rule, we observe that

$$\partial_t B(t, X_t(x)) = (B \cdot \nabla u)(t, X_t(x)),$$

which is an ODE for the function $t \mapsto B(t, X_t(x))$. In particular, it admits the unique solution (with initial condition $B_{\text{in}}(x)$)

$$B(t, X_t(x)) = DX_t(x) B_{\text{in}}(x).$$

Hence, inverting the flow, the claim follows. $\square$

2.1.2 Remarks on ideal dynamo action

Lemma 2.1.2 reduces the study of ideal dynamo action to the stretching properties of the underlying flow map, and hence to its Lyapunov exponents. Recall that ideal dynamo action concerns exponential growth of solutions to (KDE- ε) with $\varepsilon = 0$. Applying the solution formula (2.1.3), we deduce that the solution to (KDE- ε) with initial datum B_{in} at time t is given by $B(t, x) = (DX_t B_{\text{in}})(X_t^{-1}(x))$, and so its L^2 norm is nothing but

$$\|B(t)\|_{L^2(\Omega)}^2 = \int_{\Omega} |(DX_t B_{\text{in}})(X_t^{-1}(x))|^2 dx = \int_{\Omega} |DX_t(x) B_{\text{in}}(x)|^2 dx, \quad (2.1.4)$$

where in the last equality, we have changed variables and used the fact that, since u is divergence-free, its flow map is volume preserving. From here, one can deduce the following classical result, which appears for instance in [AK98].

Lemma 2.1.3 ([AK98, Chapter V, Theorem 1.7]). *Let $u \in C^\infty(\Omega)$ be divergence-free, and have vanishing normal component at the boundary. Assume furthermore that u has a stagnation point, where its derivative admits a unique positive eigenvalue. Then, u generates an ideal dynamo.*

Proof: Sketch in a model case. Consider the velocity field $u(x, y)$ on $\mathbb{R}^2$ given by

$$u(x, y) = \begin{pmatrix} -x \\ y \end{pmatrix}. \quad (2.1.5)$$

Then, the flow map generated by u is nothing but

$$X_t(x, y) = \begin{pmatrix} xe^{-t} \\ ye^t \end{pmatrix}.$$

Hence, appealing to equation (2.1.4), we have

$$\|B(t)\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |B_{\text{in}}^{(1)}(x, y)|^2 e^{-2t} + |B_{\text{in}}^{(2)}(x, y)|^2 e^{2t} dx dy.$$

Here $B_{\text{in}}(x, y) = (B_{\text{in}}^{(1)}(x, y), B_{\text{in}}^{(2)}(x, y))$. Hence, as soon as $B_{\text{in}}^{(2)}(x, y) \neq 0$, we observe ideal dynamo action. $\square$

Remark 2.1.4. Although instructive, the above computation does not fully capture the intricacies of ideal dynamo action for “real” velocity fields that admit a hyperbolic stagnation point. Generically, one expects the velocity field to “look like” the simple flow (2.1.5) only for values of x, y very close to the stagnation point (which we take to be the origin here). Thus, any particle with a non-zero initial y value will in fact leave the stagnation point in finite time, and no longer experience exponential stretching. Therefore, it may initially be unclear if ideal dynamo action persists—however, a Taylor approximation argument shows that the mass of particles near the hyperbolic point at time t is of order e^{-t} . Thus, the total growth one observes is

$$\|B(t)\|_{L^2}^2 \sim \int_{A(t)} |e^t B_{\text{in}}|^2 dx \sim e^{2t} \times e^{-t} = e^t,$$

where we have denoted by $A(t)$ the set of particles near the hyperbolic point at time t . As such, while the exponential growth persists in L^2 (and in fact, in any L^p space with $p > 1$), one *does not* expect exponential growth in L^1 . This dependence on the underlying function space suggests that L^2 may not be the most natural setting for perturbative analysis of the ideal dynamo problem. We will elaborate in substantial detail on this notion later on.

There are several natural variants of the ideal dynamo problem that are also of interest. A natural question is whether one can find so-called *universal ideal dynamos*. That is, a velocity field so that

$$\inf_{B_{\text{in}} \in L^2} \liminf_{t \rightarrow \infty} \frac{1}{t} \log \|B(t)\|_{L^2} > 0.$$

Recalling once again formula (2.1.4), this problem is most naturally approached via the study of Lyapunov exponents of the flow generated by u . In fact, it turns out that, under the imposition of suitable non-degeneracy assumptions, *random* velocity fields provide good candidates for studying this problem. Indeed, there holds the following “meta” result due to [BR93; CN24].

Remark 2.1.5 ([BR93; CN24]). Assume that the flow map generated by the random velocity field u admits a strictly positive top Lyapunov exponent. Assume furthermore, that the Markov transition kernel associated to the projective process of this flow admits a unique invariant measure. Then, u generates a universal ideal dynamo, and the growth rate is independent of the initial condition.

We remark that the assumptions of this theorem are verified, respectively, in [BR93] for transport noise-type velocity fields satisfying a variant of Hörmander’s bracket condition (more precisely, the generator of the projective process satisfies the bracket condition), and in [CN24] for ABC flows with randomly selected coefficients.

Perfect dynamo action and the flux conjecture

While ideal dynamo action has the advantage of being fairly simple to analyse and to establish, it turns out to be a rather poor approximation to the behaviour of solutions with non-zero resistivity—one can construct examples of velocity fields that generate an ideal dynamo, but yield *exponential decay* of all (mean free) solutions once any resistivity is present, as illustrated for instance in Example 2.2.10 in the next subsection. To overcome this issue, a useful variant of inviscid dynamo action, which appears to be more closely linked to fast dynamo action, is given by *perfect dynamo action*.

Definition 2.1.6 ([CG95]). A divergence-free velocity field u generates a perfect dynamo, if there exists $B_{\text{in}} \in L^2(\Omega)$, and $\psi \in C^\infty(\Omega)$ so that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \left| \int_{\Omega} B(t, x) \cdot \psi(x) dx \right| > 0,$$

where $B(t, x)$ denotes the solution to (KDE- ε) with $\varepsilon = 0$ and initial datum B_{in} .

Establishing perfect dynamo action is substantially harder than establishing ideal dynamo action—none of the examples of ideal dynamo action outlined in the previous section are known to be perfect dynamos. In fact, perfect dynamo action is the subject of a famous conjecture in dynamo theory, known as the flux conjecture, which can be stated as follows:

Conjecture 2 (Flux conjecture, [CG95]). *Let Ω be a bounded domain. Then, if u is autonomous or time-periodic, and generates a perfect dynamo, it also generates a fast dynamo.*

While open to this day, the flux conjecture is well-supported by numerical evidence [CG95]. Furthermore, throughout this manuscript, we shall provide a couple of examples of fast dynamo action, which turn out to also be perfect dynamos, suggesting a close relationship between the two notions.

2.2 On anti-dynamo theorems

Throughout the previous sections, we have typically implicitly posed the kinematic dynamo equations (KDE) on subdomains of $\mathbb{R}^3$, despite the fact that the formulation is perfectly well-defined in any dimension. In particular, as observed in Lemma 2.1.3, the equations can be analysed, and growth for $\varepsilon = 0$ can be obtained already in dimension 2. However, this is a phenomenon specific to the ideal problem: once $\varepsilon > 0$, dynamo growth cannot occur for (KDE- ε) in dimension two. Indeed, there holds the following result, which we state on the torus for simplicity.

Theorem 2.2.1 (Zeldovich [ZR80]). *Consider the kinematic dynamo equations (KDE- ε) on $\mathbb{T}^2$, with divergence-free, autonomous velocity field $u \in W_x^{1,\infty}(\mathbb{T}^2)$. Then, $\gamma(\varepsilon) \leq 0$ for all $\varepsilon > 0$.*

Proof. Suppose there exists an exponentially growing solution to (KDE- ε). Then, since the Laplacian has a compact resolvent, and the generator of (KDE- ε) is a relatively compact perturbation of the Laplacian, it follows that there exists $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > 0$, and $B \neq 0$ so that

$$-u \cdot \nabla B + B \cdot \nabla u + \varepsilon \Delta B = \lambda B, \quad (2.2.1)$$

with $\text{div} B = 0$ (see Appendix A for more details). Integrating this equation in space, we deduce that $\int_{\mathbb{T}^2} B dx dy = 0$. Therefore, there exists $\psi \in C^\infty(\mathbb{T}^2)$, mean free on $\mathbb{T}^2$ so that $B = \nabla^\perp \psi$, where $\nabla^\perp = (-\partial_y, \partial_x)$. Inserting this into (2.2.1), we observe that the first component of B satisfies

$$u_1 \partial_x \partial_y \psi - u_2 \partial_y^2 \psi - \partial_x u_1 \partial_y \psi + \partial_y u_1 \partial_x \psi - \varepsilon \Delta (\partial_y \psi) = -\lambda \partial_y \psi.$$

Since u is divergence-free, this simplifies to

$$\partial_y (-u \cdot \nabla \psi + \varepsilon \Delta \psi - \lambda \psi) = 0.$$

Similarly, the equation for the second component of B becomes

$$\partial_x (-u \cdot \nabla \psi + \varepsilon \Delta \psi - \lambda \psi) = 0.$$

Therefore,

$$-u \cdot \nabla \psi + \varepsilon \Delta \psi - \lambda \psi = \text{constant}.$$

Integrating in space and using that ψ is mean-free, we finally deduce that

$$-u \cdot \nabla \psi + \varepsilon \Delta \psi - \lambda \psi = 0.$$

But this can only occur if $\psi = 0$, since the operator $-u \cdot \nabla + \varepsilon \Delta$ is dissipative on $L^2(\mathbb{T}^2)$. $\square$

This is an instance of an *anti-dynamo theorem*: a result showing that excessive symmetry in the model prevents dynamo action. These results play an important role in the theory of kinematic dynamos, since they essentially show that one needs to consider “truly 3-dimensional” models if one seeks to obtain dynamo action for non-zero resistivity. Throughout this subsection, we shall present a number of such anti-dynamo results, some of which (such as Theorem 2.2.1) rule out

any dynamo action from taking place with non-zero resistivity $\varepsilon > 0$, whereas others specifically rule out fast dynamo action in flows without sufficient chaotic behaviour.

Remark 2.2.2. The statement of Theorem 2.2.1 can be translated essentially verbatim to planar flows in 3 dimensions. That is, dynamo action on $\mathbb{T}^3$ with $\varepsilon > 0$ *cannot* occur with a velocity field of the form $u(x, y, t) = (u_1(x, y), u_2(x, y), 0)$. In fact, there holds the following vast generalisation of this result, which is known as *Zeldovich's theorem*.

Theorem 2.2.3 (Zeldovich, [ZR80]). *Let $u(t, x, y, z) = (u_1(t, x, y, z), u_2(t, x, y, z), 0)$ be divergence-free and $u \in L_t^\infty W_x^{1,\infty}(\mathbb{T}^3)$. Then it holds $\gamma(\varepsilon) \leq 0$ for all $\varepsilon > 0$.*

Remark 2.2.4. The proof of this result is rather cumbersome at the level of generality at which we have stated it. If either of the dependencies on z or t of u are removed, it becomes significantly easier, owing (respectively) to the facts that the equations effectively become 2-dimensional, and the spectral formulation employed in Theorem 2.2.1. However, since the proof in the original paper [ZR80] is somewhat lacking in mathematical rigor, and no fully rigorous account has (to the best of the author's knowledge) appeared in the literature since, it is worth proving the result in full generality. The following proof is therefore included for completeness; those mainly interested in the conceptual structure of the chapter may safely skip it on a first reading.

Throughout the proof of Theorem 2.2.3, we will frequently rely on the following result on the equation for a passive scalar.

Lemma 2.2.5. *Let f solve*

$$\partial_t f + u \cdot \nabla f = \varepsilon \Delta f + g \quad (2.2.2)$$

on $\mathbb{T}^d$, where $u \in L_t^\infty W_x^{1,\infty}(\mathbb{T}^d)$ is divergence-free, and $\sup_{t \geq 0} \|g(t)\|_{L^2(\mathbb{T}^d)} < \infty$. Suppose furthermore that $\int_{\mathbb{T}^d} g(t, x) dx = 0$ for all t . Then $\sup_{t \geq 0} \|f(t)\|_{H^1(\mathbb{T}^d)} < \infty$.

Proof. Integrating the equation in space shows that the spatial average of f is conserved. Thus, we may write $f(t) = \bar{f} + f^\perp$, where $\bar{f}$ is a constant independent of time, and $f^\perp$ is mean-free. Inserting this into (2.2.2), we observe that $f^\perp$ still solves exactly the same equation (2.2.2). Hence, we may from now on work directly with $f^\perp$, which we denote by f for simplicity. We now multiply the equation by f and integrate by parts, yielding by Cauchy-Schwarz

$$\frac{1}{2} \partial_t \|f(t)\|_{L^2(\mathbb{T}^d)}^2 + \varepsilon \|\nabla f(t)\|_{L^2(\mathbb{T}^d)}^2 \leq \|f(t)\|_{L^2} \|g(t)\|_{L^2(\mathbb{T}^d)}.$$

Since f is mean-free, we apply Young's inequality followed by the Poincaré inequality to deduce that

$$\frac{1}{2} \partial_t \|f(t)\|_{L^2(\mathbb{T}^d)}^2 \leq -c\varepsilon \|f(t)\|_{L^2(\mathbb{T}^d)}^2 + C(\varepsilon) \|g\|_{L_t^\infty L_x^2(\mathbb{T}^d)}^2.$$

Hence, applying the Grönwall inequality, we deduce the desired result for the L^2 norm. Next, testing the equation with $-\Delta f$, and integrating by parts yields

$$\frac{1}{2} \partial_t \|\nabla f\|_{L^2(\mathbb{T}^d)}^2 + \int_{\mathbb{T}^d} \nabla f \cdot \nabla u \cdot \nabla f dx + \varepsilon \|\Delta f\|_{L^2(\mathbb{T}^d)}^2 = - \int_{\mathbb{T}^d} \Delta f g dx.$$

Integrating by parts again and applying Cauchy-Schwarz, we obtain

$$\frac{1}{2} \partial_t \|\nabla f\|_{L^2(\mathbb{T}^d)}^2 - \int_{\mathbb{T}^d} (Du : D^2 f) f dx + \varepsilon \|\Delta f\|_{L^2(\mathbb{T}^d)}^2 \leq \|\Delta f\|_{L^2(\mathbb{T}^d)} \|g\|_{L^2(\mathbb{T}^d)}.$$

Applying Young's inequality following by Poincaré, we bound this by

$$\frac{1}{2} \partial_t \|\nabla f\|_{L^2(\mathbb{T}^d)}^2 \leq C(\varepsilon) \|u\|_{L_t^\infty W_x^{1,\infty}(\mathbb{T}^d)}^2 \|f\|_{L^2(\mathbb{T}^d)}^2 - c\varepsilon \|\nabla f\|_{L^2(\mathbb{T}^d)}^2 + C(\varepsilon) \|g\|_{L_t^\infty L_x^2(\mathbb{T}^d)}^2.$$

Applying the Grönwall Lemma, we complete the proof. $\square$

Hence we may now prove Theorem 2.2.3.

Proof of Theorem 2.2.3. The proof, while lengthy, is conceptually very much in line with that of Theorem 2.2.1. In essence, we shall attempt to reduce the equation to one on $\mathbb{T}^2$ —however, owing to the dependencies of u on t and z , the reduction requires some care. The B_3 component will in general be non-zero, and will enter as a forcing in the (B_1, B_2) equations. Furthermore, the (B_1, B_2) components will in general not be divergence-free, and therefore they do not admit a streamfunction. The strategy will be to first show that the B_3 component and the divergence part of (B_1, B_2) cannot grow, then that the divergence-free part of the (B_1, B_2) equations are forced by B_3 and by the divergence part of (B_1, B_2) , and finally that, due to its boundedness, this forcing cannot overcome the decay that one would expect in view of Theorem 2.2.1.

We begin by noting that the component B_3 of the solution to (KDE- ε) solves

$$\partial_t B_3 + u \cdot \nabla B_3 = \varepsilon \Delta B_3,$$

which is nothing but the equation for a passive scalar. Therefore, we deduce immediately from Lemma 2.2.5 that

$$\|B_3(t)\|_{H^1(\mathbb{T}^3)} \lesssim 1. \quad (2.2.3)$$

Now, for every $z \in \mathbb{T}$ fixed, we may write $(B_1, B_2) = \nabla_h^\perp \psi + \nabla_h \phi + \bar{B}$, where ψ, ϕ are mean-free (in (x, y)) scalar functions, $\bar{B}$ is the (x, y) spatial average of the solution, and we denote by $\nabla_h = (\partial_x, \partial_y)$ the horizontal gradient. Then, note that since $\operatorname{div} B = 0$, there holds

$$\partial_x B_1 + \partial_y B_2 = \Delta_h \phi = -\partial_z B_3.$$

Inverting the Laplacian (and using that ϕ is mean-free in (x, y) for all z), we deduce from (2.2.3) that

$$\|D_{x,y}^2 \phi(t)\|_{L^2(\mathbb{T}^3)}^2 = \int_{\mathbb{T}} \|D_{x,y}^2 \phi(t, z)\|_{L^2(\mathbb{T}^2)}^2 dz \lesssim \int_{\mathbb{T}} \|\partial_z B_3(t, z)\|_{L^2(\mathbb{T}^2)}^2 dz = \|\partial_z B_3(t)\|_{L^2(\mathbb{T}^3)}^2 \lesssim 1. \quad (2.2.4)$$

Next, a computation shows that $\bar{B}$ satisfies the equation

$$\partial_t \bar{B} = \oint_{\mathbb{T}^2} (\nabla_h^\perp \psi \cdot \nabla u + \nabla_h \phi \cdot \nabla u + \bar{B} \cdot \nabla u + B_3 e_z \cdot \nabla u) + \varepsilon \partial_z^2 \bar{B}.$$

Since $\nabla_h^\perp \psi$ is divergence-free in (x, y) , the first term in the integral vanishes. Furthermore, since $\bar{B}$ does not depend on (x, y) , the term $\int_{\mathbb{T}^2} \bar{B} \cdot \nabla u$ also vanishes. In fact, there holds the general identity

$$H \cdot \nabla u = \nabla \cdot (u \otimes H) - (\nabla \cdot H)u,$$

where the divergence is taken row-wise in the tensor product. Applying this with the 2D divergence, we deduce that

$$\nabla_h \phi \cdot \nabla u = \nabla_h \cdot (u \otimes \nabla_h \phi) - \Delta_h \phi u = \nabla_h \cdot (u \otimes \nabla_h \phi) + \partial_z B_3 u.$$

Thus, the evolution equation for the average simplifies to

$$\partial_t \bar{B} = \int_{\mathbb{T}^2} \partial_z (B_3 u) dx dy + \varepsilon \partial_z^2 \bar{B}.$$

In the notation of Lemma 2.2.5, we set $g(t, z) = \partial_z \int_{\mathbb{T}^2} B_3 u dx dy$. Certainly this is mean-free in z , and since B_3 solves a passive scalar equation, and u is uniformly Lipschitz in time, it is clear that $\sup_{t \geq 0} \|g(t)\|_{L^2(\mathbb{T})} < \infty$. Thus, $\|\bar{B}(t)\|_{L^2(\mathbb{T})} \lesssim 1$. Now, there holds $\nabla_h^\perp \cdot (B_1, B_2) = \Delta_h \psi$, and so, differentiating this in time yields

$$\Delta_h (\partial_t \psi + u \cdot \nabla \psi - \varepsilon \Delta \psi) = \nabla_h^\perp \cdot \left(-(u \cdot \nabla) \nabla_h \phi + (\nabla_h \phi + \bar{B} + B_3 e_z) \cdot \nabla u + \varepsilon \Delta \nabla_h \phi + \varepsilon \partial_z^2 \bar{B} \right).$$

Certainly, the right-hand side is mean free in (x, y) , so we may invert the Laplacian to deduce (since ψ is also mean-free in (x, y) by construction)

$$\partial_t \psi + u \cdot \nabla \psi - \varepsilon \Delta \psi = \Delta_h^{-1} \nabla_h^\perp \cdot \left(-(u \cdot \nabla) \nabla_h \phi + (\nabla_h \phi + \bar{B} + B_3 e_z) \cdot \nabla u + \varepsilon \Delta \nabla_h \phi + \varepsilon \partial_z^2 \bar{B} \right).$$

We denote the right-hand side of this equation by $g(t, x, y, z)$. By Lemma 2.2.5, if we can show that $\sup_{t \geq 0} \|g(t)\|_{L^2(\mathbb{T}^3)} < \infty$, the proof is completed. Firstly, note that the term involving $\partial_z^2 \bar{B}$ is annihilated by the $\nabla_h^\perp$. Similarly, it also holds that $\nabla_h^\perp \cdot \nabla_h \phi = 0$. Next, it is clear that, for any fixed z , there holds

$$\|g(z)\|_{L^2(\mathbb{T}^2)}^2 \leq \|-(u \cdot \nabla) \nabla_h \phi + (\nabla_h \phi + \bar{B} + B_3 e_z) \cdot \nabla u\|_{L^2(\mathbb{T}^2)}^2.$$

Note that, since u has vanishing z -component, the term $(u \cdot \nabla) \nabla_h \phi$ is equal to $(u \cdot \nabla_h) \nabla_h \phi$. Thus, integrating over z , it suffices to estimate

$$\|(u \cdot \nabla) \nabla_h \phi\|_{L^2(\mathbb{T}^3)} + \|\nabla_h \phi \cdot \nabla u\|_{L^2(\mathbb{T}^3)} + \|\bar{B} \cdot \nabla u\|_{L^2(\mathbb{T}^3)} + \|B_3 e_z \cdot \nabla u\|_{L^2(\mathbb{T}^3)}.$$

The first two of these terms are bounded by

$$\|u\|_{L_t^\infty W_x^{1,\infty}(\mathbb{T}^3)} \|D_{x,y}^2 \phi\|_{L^2(\mathbb{T}^3)} \lesssim 1,$$

by (2.2.4). The term $\|\bar{B} \cdot \nabla u\|_{L^2(\mathbb{T}^3)}$ is bounded above by $\|\bar{B}\|_{L_z^2(\mathbb{T})} \|u\|_{L_t^\infty W_x^{1,\infty}(\mathbb{T}^3)}$. Hence, by Lemma 2.2.5, the proof is completed. $\square$

While the case of planar velocity fields is in some sense the most canonical for anti-dynamo results, the arguments presented above can be adapted to hold in a variety of cases where symmetries are present. We mention in particular the celebrated result of Cowling [Cow33] which rules out dynamo action in axis-symmetric magnetic fields.²

Anti-dynamo results for fast dynamo action

While the anti-dynamo results mentioned in the previous section already rule out a substantial class of velocity fields from generating kinematic dynamos with non-zero resistivity, the situation is even more complicated if one wants to achieve fast dynamo action in particular. As mentioned in the introduction to this monograph, one expects that fast dynamo action requires some degree of “turbulent” behaviour in the underlying fluid. Building on this intuition, there have been a number of results proven that specifically rule out fast dynamo action for velocity fields that are non-chaotic. The prototypical result of this nature is the following, due to Vishik.

Theorem 2.2.6 (Vishik, [Vis89]). *Let u be a smooth, autonomous velocity field on $\mathbb{T}^3$. Then, there holds*

$$\liminf_{\varepsilon \rightarrow 0} \gamma(\varepsilon) \leq \Lambda_{\max}^{\text{Lyap}},$$

where

$$\Lambda_{\max}^{\text{Lyap}} := \sup_{x \in \mathbb{T}^3, v \in \mathbb{R}^3} \limsup_{t \rightarrow \infty} \frac{1}{t} \log |DX_t(x)v|.$$

In view of the formula (2.1.3), this result is heuristically natural — in order to obtain fast dynamo action, one needs exponential stretching of particle trajectories. The result is, however, rather non-trivial in a couple of aspects. Firstly, despite diffusion generally acting in a dampening manner when averaging over Brownian trajectories, the pathwise behaviour can be rather different. For instance, applying the methods of [BBP22], one can show that the addition of Brownian noise can lead to exponential stretching of particle trajectories, even if the deterministic flow does not yield exponential stretching. This illustrates a mechanism consistent with *slow dynamo action*.³ Secondly, the definition of fast dynamo action involves taking first the limit in time, and then the limit as $\varepsilon \rightarrow 0$. Therefore, the proof of Theorem 2.2.6 requires a careful justification of why these limits may be interchanged (at least when one is taking upper bounds, for lower bounds the limits are strictly non-interchangeable), which is achieved by a carefully constructed asymptotic expansion of the Green’s function associated to (KDE- ε).

While Theorem 2.2.6 rules out a wide class of velocity fields from inducing fast dynamo action, it is still rather easy to construct examples of velocity fields that have a positive top Lyapunov exponent, but are not fast dynamos. For instance, owing to Lemma 2.1.3 and Theorem 2.2.1, any 2D flow with a hyperbolic point falls into this category. Thus, it appears that, in order to sustain dynamo action at vanishingly small diffusivities, one in fact requires even more chaotic particle

²It is worth pointing out that, while conceptually sound, many anti-dynamo results in the literature lack full mathematical rigor. In particular, it often requires a substantial amount of effort to “tease out” the exact regularity assumptions needed on u, B for these results to hold. This was already remarked in [CG95]. As such, it seems (at least on the surface) plausible that one could construct examples of dynamo action with pathological velocity or magnetic fields, which seemingly violate some of these classical anti-dynamo results.

³The Ponomarenko dynamo discussed in Chapter 6 exhibits precisely this phenomenon.

trajectories. Notable progress towards finding a criterion in this vein was made by Klapper and Young.

Theorem 2.2.7 (Klapper, Young, [KY95]). *Let $u \in C^{k+1}(\mathbb{T}^3)$ be an autonomous, divergence free vector field, for some $k \geq 2$. Assume that $B_{\text{in}} \in C^k$ is divergence-free. Then, there holds*

$$\limsup_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{t} \log \|B^\varepsilon(t)\|_{L^1(\mathbb{T}^3)} \leq h(X_1) + \frac{r(X_1)}{k},$$

where B^ε denotes the solution to (KDE- ε) with diffusivity ε , velocity field u , and initial data B_{in} , and X_1 is the time-1 flow map associated to u . Furthermore,

$$r(X_1) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sup_{x \in \mathbb{T}^3} |DX_n(x)|,$$

and $h(X_1)$ is the topological entropy of the flow map X_1 .

We shall define the precise meaning of topological entropy in due course. Before that, it is however pertinent to discuss a few implications of this result. One limitation of Theorem 2.2.6 is that the initial datum B_{in} is fixed as $\varepsilon \rightarrow 0$. Since, as we shall discuss in more detail later on, one expects the “limiting eigenmode” associated to fast dynamo action to live in a space of distributions, there is a-priori no reason to expect that the same initial data will grow for all $\varepsilon \rightarrow 0$. Therefore, it is not clear whether the upper bound of Theorem 2.2.7 truly provides a constraint on fast dynamo action. However, owing to the autonomous nature of the velocity field in Theorem 2.2.7, one can employ techniques from spectral theory, to deduce the following corollary, which reformulates Theorem 2.2.6 in terms of the growth-rate notion $\gamma(\varepsilon)$ used throughout this thesis. To the best of the author’s knowledge, the proof of this corollary (though likely known to experts in the field) has not appeared in the literature before.

Corollary 2.2.8. ⁴ *Let $u \in C^{k+1}(\mathbb{T}^3)$ be as in the setting of Theorem 2.2.7. Then, it holds*

$$\limsup_{\varepsilon \rightarrow 0} \gamma(\varepsilon) \leq h(X_1) + \frac{r(X_1)}{k}.$$

Proof. The proof is a functional-analytic argument using spectral projectors and the Baire category theorem.

Denote by $S^\varepsilon(t)$ the semigroup generated by the kinematic dynamo operator $\mathcal{L}^\varepsilon : B \mapsto -u \cdot \nabla B + B \cdot \nabla u + \varepsilon \Delta B$. Since this operator is a relatively compact perturbation of the Laplacian, it has compact resolvent, and the semigroup S^ε is analytic. As such, invoking Lemma A.4.5 it holds for any $\gamma > 0$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|S^\varepsilon(t)B_{\text{in}}\|_{L^2(\mathbb{T}^3)} \geq \gamma,$$

if and only if $P^\varepsilon B_{\text{in}} \neq 0$, where P^ε is the *Riesz projector* onto all eigenvalues of $\mathcal{L}^\varepsilon$ with real part at least γ . More explicitly, we let $\Gamma^\varepsilon \subset \mathbb{C}$ be some contour in the resolvent set of $\mathcal{L}^\varepsilon$, so that $\Gamma^\varepsilon \subset \{z : \text{Re}(z) \geq \gamma\}$, and all eigenvalues of $\mathcal{L}^\varepsilon$ with real part at least γ are contained in the

⁴The author thanks Paolo Ventura for helpful comments on an initial draft of this result, which helped streamline the presentation.

interior of Γ^ε .⁵ Then we set

$$P^\varepsilon = \frac{1}{2\pi i} \int_{\Gamma^\varepsilon} (\lambda - \mathcal{L}^\varepsilon)^{-1} d\lambda.$$

Properties of these projectors are reviewed in Appendix A. In particular, let $\alpha > 0$ and pick $\gamma = h(X_1) + \frac{r(X_1)}{k} + \alpha$. Suppose the statement of the Corollary was false. Then there exists a sequence $\varepsilon_n \rightarrow 0$, so that $\gamma(\varepsilon_n) > h(X_1) + \frac{r(X_1)}{k} + \alpha$, for all n , for α sufficiently small. Therefore, there exists $\alpha > 0$ small enough so that $P^{\varepsilon_n} \neq 0$ for all n . We now claim there exists $B_{\text{in}} \in C_\sigma^k(\mathbb{T}^3)$ so that $P^{\varepsilon_n} B_{\text{in}} \neq 0$ for all n , where we denote by $C_\sigma^k(\mathbb{T}^3)$ the space of divergence-free C^k vector fields. Suppose this was not the case. Then, it holds

$$C_\sigma^k(\mathbb{T}^3) = \bigcup_{n \in \mathbb{N}} \ker P^{\varepsilon_n} \cap C_\sigma^k(\mathbb{T}^3).$$

Note that all of the sets $\ker P^{\varepsilon_n} \cap C_\sigma^k(\mathbb{T}^3)$ are closed in the $C^k(\mathbb{T}^3)$ topology. If $B_n \in \ker P^{\varepsilon_n} \cap C_\sigma^k(\mathbb{T}^3)$ converges, then the limit will certainly be in $C_\sigma^k(\mathbb{T}^3)$, but also, since the $C^k(\mathbb{T}^3)$ topology is finer than the $L^2(\mathbb{T}^3)$ topology, and $\ker P^{\varepsilon_n}$ is $L^2(\mathbb{T}^3)$ -closed, it is also closed in the $C^k(\mathbb{T}^3)$ topology. Therefore, we have written the complete metric space $C_\sigma^k(\mathbb{T}^3)$ as a countable union of closed sets, so by the Baire category theorem, there exists some $n \in \mathbb{N}$, and a ball $B_\delta^{\|\cdot\|_{C^k}} \subset \ker P^{\varepsilon_n} \cap C_\sigma^k(\mathbb{T}^3)$. But then, $\ker P^{\varepsilon_n} \cap C_\sigma^k(\mathbb{T}^3)$ is a linear subspace of $C_\sigma^k(\mathbb{T}^3)$ that contains an open ball, and so it is all of $C_\sigma^k(\mathbb{T}^3)$. But then, since $C_\sigma^k(\mathbb{T}^3)$ is a dense subspace of $L_\sigma^2(\mathbb{T}^3)$, and P^{ε_n} is a continuous linear operator on $L_\sigma^2(\mathbb{T}^3)$, we deduce that $P^{\varepsilon_n} \equiv 0$, yielding a contradiction. Therefore, there exists $B_{\text{in}} \in C_\sigma^k(\mathbb{T}^3)$ so that $P^{\varepsilon_n} B_{\text{in}} \neq 0$ for all $n \in \mathbb{N}$. But then it holds

$$h(X_1) + \frac{r(X_1)}{k} + \alpha \leq \lim_{t \rightarrow \infty} \frac{1}{t} \log \|S^{\varepsilon_n}(t) B_{\text{in}}\|_{L^2(\mathbb{T}^3)}$$

for all n . Taking the limit as $n \rightarrow \infty$ and employing Theorem 2.2.7, there holds

$$h(X_1) + \frac{r(X_1)}{k} + \alpha \leq \limsup_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{t} \log \|S^\varepsilon(t) B_{\text{in}}\|_{L^2(\mathbb{T}^3)} \leq h(X_1) + \frac{r(X_1)}{k},$$

a contradiction, and so the result is proven.⁶ □

Thus, we can immediately deduce the following generalisation of Vishik's theorem 2.2.6.

Corollary 2.2.9. *Let u be a smooth, autonomous, divergence-free velocity field on $\mathbb{T}^3$. Then, if u generates a fast dynamo, it must have strictly positive topological entropy.*

⁵If there is an eigenvalue of $\mathcal{L}^\varepsilon$ with real part exactly γ , one can introduce a slight kink in the contour in order to also enclose it, in which case the growth rate will be at least $\gamma - \delta$, for all $\delta > 0$.

⁶We have used here without proof that Theorem 2.2.7 still holds if one measures growth in $L^2(\mathbb{T}^3)$ rather than $L^1(\mathbb{T}^3)$. Since the initial datum is C^k , it certainly also lives in $L^2(\mathbb{T}^3)$, and therefore, following Lemma A.4.5, one can decompose the solution at time t into a component living in the finite dimensional span of all generalised eigenfunctions of $\mathcal{L}^\varepsilon$ with eigenvalues of real part at least $h(X_1) + \frac{r(X_1)}{k} + \delta$, which are stretched by a factor at least $e^{t(h(X_1) + \frac{r(X_1)}{k} + \delta)}$, and a component that is bounded in L^2 by $Ce^{t(h(X_1) + \frac{r(X_1)}{k})}$. Since the L^1 norm is strictly weaker than the L^2 norm, this L^2 bound also holds in L^1 , and on the finite dimensional space corresponding to generalised eigenspaces with “large” eigenvalue, the L^1 and L^2 norms are equivalent. Thus, if the component of the solution living on the generalised eigenspaces with “large” eigenvalue is non-zero, one would observe a contradiction to Theorem 2.2.7.

An aside on topological entropy

Though an in-depth treatment of the notion of topological entropy is well beyond the scope of this manuscript, we shall briefly give its definition here and give a couple of small remarks towards its relationship to the dynamo. We follow here the definition of topological entropy given in [KY95].

Suppose one has a continuous, invertible map $f : (X, d) \rightarrow (X, d)$, where (X, d) is some compact metric space. We define a collection of new metrics on this space, given by

$$d_n(x, y) = \max_{0 \leq i \leq n} d(f^i(x), f^i(y)).$$

Given $\delta > 0$, $n \in \mathbb{N}$, we say a set $S \subset X$ (n, δ) -spans X if for all $y \in X$, there exists $x \in S$ so that $d_n(x, y) \leq \delta$. Let $\tau_n(\delta)$ be the smallest possible cardinality of any (n, δ) spanning set. Then, we define the topological entropy of f to be

$$h(f) = \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \tau_n(\delta).$$

The property of having non-zero topological entropy is a hallmark of chaotic behaviour in a map [CG95]. Indeed, for smooth, volume preserving maps, it is a *strictly* stronger property than having a positive top Lyapunov exponent $\Lambda_{\max}^{\text{Lyp}}$ [New80; Yom87]. Thus, Theorem 2.2.7 can be seen as a generalisation of Theorem 2.2.6. The notion of topological entropy is also strong enough to rule out fast dynamo action in 2D flows with a hyperbolic stagnation point, such as those mentioned in Lemma 2.1.3. Indeed, the classical result [You77] rules out smooth, autonomous, two-dimensional velocity fields from generating a flow with non-zero topological entropy. As such, Theorem 2.2.7 provides an alternative proof of the Zeldovich theorem 2.2.1 for fast dynamo action.

On spectral discontinuity in the dynamo equations

As pointed out following the statement of Theorem 2.2.6, a key consequence of the anti-dynamo theorems of the previous subsection, is that they allow one to easily construct examples of velocity fields that generate ideal dynamos, but actually yield no dynamo action as soon as $\varepsilon > 0$. The canonical example in this vein is the following.

Example 2.2.10. Let $u(x, y)$ be a divergence-free velocity field with a hyperbolic stagnation point. Then owing to Lemma 2.1.3, $\gamma(0) > 0$, but due to Theorem 2.2.3 it holds $\gamma(\varepsilon) \leq 0$ for all $\varepsilon > 0$.

This is an example of a *spectral discontinuity* of the family of unbounded operators $\mathcal{L}^\varepsilon B = -u \cdot \nabla B + B \cdot \nabla u + \varepsilon \Delta B$ on $L^2(\mathbb{T}^2)$. Ideal dynamo action implies ⁷ that the $L^2(\mathbb{T}^2)$ spectrum of $\mathcal{L}^0$ protrudes into the right-half plane, but for all $\varepsilon > 0$, the spectrum of $\mathcal{L}^\varepsilon$ is contained in the left-half plane. This rather degenerate spectral behaviour is a large part of the reason why establishing fast dynamo action has been so elusive—the result appears inherently perturbative in nature (if one gets uniform growth as $\varepsilon \rightarrow 0$, there really “ought” to be a “limiting eigenmode” for $\varepsilon = 0$), but the highly degenerate spectral behaviour of the involved operators makes rigorous applications of perturbation theory very challenging. From a more technical point of view, this difficulty in-

⁷The implication is somewhat more subtle than this, since one (a-priori) does not know if the spectral mapping theorem for semigroups holds for the ideal dynamo. However, the result [CLM95] ensures that it does in fact hold.

herently arises due to the fact that the Laplacian is a *singular perturbation* of the ideal dynamo operator $\mathcal{L}^0$. As seen in example 2.2.10, such perturbations need not conserve spectral structures of operators, whereas “regular perturbations” (for instance ones where the perturbation is relatively bounded with respect to the unperturbed operator) do not exhibit this issue [Kat76], see also Appendix A. Of course, the difficulty arising from singular perturbations is by no means unique to dynamo theory. For instance, in the topic of hydrodynamical (in)stability, a common question is whether linear instabilities for the Euler equations persist for the linearised Navier–Stokes operator for sufficiently small $\nu > 0$. To deal with such questions, the *singular perturbation theory* (see for instance [Kat76]) was developed, and it has found great success particularly in hydrodynamic stability. Notably, as shown in [SF08], any linear instability for the Euler equations on $L^2(\mathbb{T}^d)$ that does not lie in the essential spectrum persists for the linearised Navier–Stokes equations with sufficiently small viscosity $\nu > 0$. In particular, this implies that in dimension 2, any linearised Euler instability (in vorticity form) is also linearly unstable for Navier–Stokes with sufficiently small viscosity $\nu > 0$! [SF08; ABC22]. What then makes the perturbative analysis of the dynamo equations so much harder than that of the Euler/Navier–Stokes equations? A hint at the issue can be glimpsed from a reading of the paper [SF08]: the result’s validity depends on the mode one wants to perturb not being embedded into the essential spectrum. In other words, in order for the instability to persist under a singular perturbation, it must (heuristically speaking) be “finite dimensional” in nature.⁸ For the linearised equations of a fluid, this is rather natural: the 2D Euler equations in vorticity form, linearised around some divergence-free u are given by

$$\omega \mapsto u \cdot \nabla \omega + BS(w) \cdot \nabla(\nabla^\perp u),$$

where BS is the Biot-Savart operator. This operator is nothing but a skew-symmetric operator ($\omega \mapsto u \cdot \nabla \omega$) perturbed by a compact operator ($\omega \mapsto BS(w) \cdot \nabla(\nabla^\perp u)$). Therefore, its essential spectrum is confined to the imaginary axis, and so any unstable eigenmode is not embedded into the essential spectrum, see Appendix A for more details. In other words, the instability is *caused* by the compact Biot-Savart operator. In contrast to this, the situation of the ideal dynamo operator is substantially more difficult. It was shown in [CLM95] that, generically, the L^2 spectrum of the ideal dynamo operator $\mathcal{L}^0$ is given by a *single vertical strip*. In other words, the operator admits no discrete spectrum, and so there are no isolated eigenvalues to which one could apply the singular perturbation theory.

A clue to a possible resolution to this issue can once again be seen in the example of ideal dynamo equation of Lemma 2.1.3. We showed that ideal dynamo action occurred in L^p for all $p > 1$, *but not* for $p = 1$. That is, the space on which the equations are considered can have vast consequences for the spectral behaviour of the operators involved. In fact, a result due to Moffatt and Proctor [MP85] shows that the ideal dynamo operator *cannot* admit smooth eigenfunctions. Instead, in view of the highly filamented and chaotic behaviour expected of fast dynamo action, and

⁸This requirement is completely natural if one wants stability under singular perturbations. For instance, the resolvent of $\mathcal{L}^\varepsilon$ is compact for any $\varepsilon > 0$, but not compact for $\varepsilon = 0$. Thus, the resolvents *cannot* converge in operator norm. Instead, one typically expects (at best) pointwise convergence. However, on finite-dimensional spaces, this pointwise convergence is equivalent to operator-norm convergence, and so it is more reasonable to expect a notion of spectral continuity in such spaces.

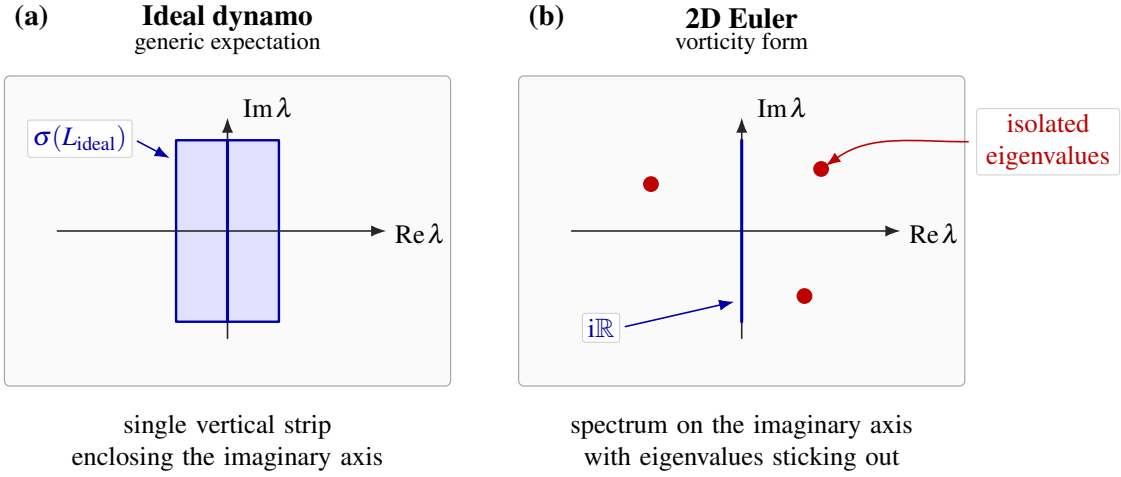


Figure 2.1: Schematic comparison of the spectral geometry for linear instabilities in (a) the ideal dynamo operator and (b) the linearized 2D Euler operator in vorticity form.

reinforced in the result Corollary 2.2.8, one expects the “limiting eigenmode” associated to fast dynamo action to live in a suitable space of distributions. This viewpoint has long been suggested in the physics literature, see e.g. [CG95; Sow93]. The hope is then that, passing to a suitable space of distributions, one observes precisely the “finite dimensional instability” of the ideal dynamo operator, which can be treated using the singular perturbation theory. Chapter 5 constructs such a distribution space for a specific flow and uses singular perturbation methods to treat the pulsed Laplacian.

2.3 Dynamo action in maps

Due to the inherent difficulty in establishing dynamo action for the full kinematic dynamo equations (KDE), a useful model, popularised by Childress and Bayly in [BC88a], is to study dynamo action in *maps*. Starting from the flow formulation of Lemma 2.1.2 $B(t, x) = (DX_t B_{\text{in}}) \circ X_t^{-1}(x)$, one simply replaces $X_t(x)$ by some diffeomorphism $T : \mathbb{T}^3 \rightarrow \mathbb{T}^3$, obtaining the map

$$B \mapsto (DTB) \circ T^{-1}.$$

In the language of dynamical systems, this is nothing but the *vector transfer operator* acting on the vector field B . In order to model diffusive action, one then simply post-composes this operator with the heat semigroup of strength ε , obtaining the one-step discrete dynamo operator

$$S_\varepsilon B = e^{\varepsilon \Delta} ((DTB) \circ T^{-1}).$$

We then have the following definition.

Definition 2.3.1. We say that the volume preserving map T generates a dynamo for $\varepsilon > 0$ if

$$\gamma(\varepsilon) = \sup_{B_{\text{in}} \in L^2, \nabla \cdot B_{\text{in}} = 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \|S_\varepsilon^n B_{\text{in}}\|_{L^2} > 0.$$

If furthermore $\liminf_{\varepsilon \rightarrow 0} \gamma(\varepsilon) > 0$, we say that the map T generates a fast dynamo.

It turns out that, relaxing the requirement that T need be the time-1 map of some flow generated by a velocity field *substantially* simplifies the analysis of dynamo action with $\varepsilon > 0$. Most notably, it is now entirely possible to obtain fast dynamo action in 2-dimensional maps, as seen for instance in the following example based on Arnold's CAT map.

Example 2.3.2. Consider the map

$$Tx = \begin{pmatrix} M^2 + 1 & M \\ M & 1 \end{pmatrix} x \mod 1$$

on $\mathbb{T}^2$, for $M \in \mathbb{N}$. Then T generates a fast dynamo. To begin, note that T is indeed volume preserving. Let now $B_{\text{in}} = (\frac{M + \sqrt{M^2 + 4}}{2}, 1)$. Then

$$(DTB_{\text{in}}) \circ T^{-1} = (1 + M \frac{M + \sqrt{M^2 + 4}}{2}) B_{\text{in}} = S_\varepsilon B_{\text{in}},$$

which certainly grows exponentially in time, independently of $\varepsilon > 0$.

While instructive, the above example clearly illustrates a rather unphysical nature of dynamo action in maps, namely that mean fields can be amplified. By contrast, in the full kinematic dynamo equations (KDE), the mean magnetic field is conserved. In fact, owing to this amplification of the mean field, fast dynamo action in 2-dimensional maps can be entirely classified, as done in the paper [Ose93].

Theorem 2.3.3 (Osedelets, [Ose93]). *Let $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a volume-preserving diffeomorphism, so it may be written in the form*

$$Tx = Ax + \phi(x),$$

where

$$\int_{\mathbb{T}^2} D\phi(x) dx = 0,$$

and A is an integer matrix. Then, T generates a fast dynamo if and only if A has an eigenvalue with modulus strictly greater than 1.

The requirement that the matrix A have an eigenvalue of modulus strictly greater than 1 ensures that T cannot arise as the time-1 map of a flow generated by a velocity field. Indeed, such a time-1 map must be isotopic to the identity map, so the matrix A would only have eigenvalue 1. As such, if one restricts the map T to arise from a flow, the anti-dynamo theorem 2.2.1 continues to hold, even though diffusion now acts in a pulsed fashion, rather than continuously. Theorem 2.2.7 (and its associated corollary 2.2.8) also remain valid in the setting of map dynamos [KY95]. Thus, physically relevant models of fast dynamo action in maps again force one into dimension three. Nevertheless, this formulation provides a much more tractable setting for rigorous analysis. In fact, there has been strong numerical evidence of fast dynamo action in maps arising from flows in the theoretical physics community [Sow93; Ota93], and in Chapter 5 we will provide a rigorous proof of precisely such action. Nevertheless, it is instructive to discuss here a couple of simple models of dynamo action based on the so-called stretch-fold-shear (SFS) mechanism, which dates back to the

papers [BC88a; Chi93; Gil05; Sow93]. This mechanism is constructed by considering a chaotic map $T_{SF} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, followed by a shearing in the z -direction $T_{Sh}(x, y, z) = (0, 0, z - g(x, y))$, for some $g : \mathbb{T}^2 \rightarrow \mathbb{R}$. The map $T := T_{Sh} \circ T_{SF}$ is then a diffeomorphism of the torus, and the associated dynamo operator acts on initial data $B_{in}(x, y, z) = e^{2\pi iz} H(x, y)$ via

$$(DTB_{in}) \circ T^{-1} = e^{2\pi iz} e^{2\pi ig(x, y)} \tilde{S}H(x, y),$$

where $\tilde{S}$ acts on the first two components of H through $\tilde{S}H^{(1,2)} = (DT_{SF}H^{(1,2)}) \circ T_{SF}^{-1}$. In other words, the map acts as the 2D dynamo operator associated to T_{SF} , modulated by a phase $e^{2\pi ig(x, y)}$. The hope is then that this modulation by a phase prevents the destructive interference of heat semigroup smoothing, and allows the map to retain dynamo action in the limit as $\varepsilon \rightarrow 0$.⁹ Although the SFS mechanism essentially reduces the 3D dynamo problem to a 2D operator problem, the resulting analysis remains difficult. Some insight can be gained however by letting T_{SF} be given by perhaps the simplest chaotic map on the torus, namely the 2D *folded Baker's map*.

2.3.1 Fast dynamo action in Baker's map models.

The purpose of this subsection is twofold. First, it gives explicit model problems in which fast dynamo action can be proved rigorously. Second, it illustrates, in a simplified setting, the perturbative mechanisms that will reappear later in the thesis in more regular and physically relevant models.

The folded baker's map $\Phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a prototypical chaotic dynamical system defined piecewise by

$$\Phi(x, y) = \begin{cases} (2x, \frac{y}{2}), & \text{for } 0 \leq x < \frac{1}{2}, \\ (2 - 2x, 1 - \frac{y}{2}), & \text{for } \frac{1}{2} \leq x < 1, \end{cases}$$

where the first coordinate is understood modulo 1. Geometrically, Φ stretches the unit square by a factor of 2 in the x -direction and contracts it by a factor of $1/2$ in the y -direction, followed by a “folding” operation that maps the second half of the rectangle back onto the unit square.

A crucial property of the map Φ is that it is *uniformly hyperbolic*. For any point where the derivative is defined (i.e., $x \neq 1/2$), the tangent space admits a $D\Phi$ -invariant splitting $E^s \oplus E^u$. Here, the splitting is global: the unstable distribution E^u is horizontal with expansion factor $\lambda_u = 2$, and the stable distribution E^s is vertical with contraction factor $\lambda_s = 1/2$.

We consider a generalization to the N -folded baker's map Φ_N , which we lift to the three-dimensional torus $\mathbb{T}^3$ by acting as the identity on the third component. For $N \in \mathbb{N}$, we define

$$\Phi_N(x, y, z) = \begin{cases} (2Nx - j, \frac{y+j}{2N}, z), & \frac{j}{2N} \leq x < \frac{j+1}{2N}, \quad j = 0, \dots, N-1, \\ (j+1 - 2Nx, \frac{j+1-y}{2N}, z), & \frac{j}{2N} \leq x < \frac{j+1}{2N}, \quad j = N, \dots, 2N-1. \end{cases}$$

⁹The idea of modulating a 2-dimensional map with a small shear in the z -component turns out to be rather ubiquitous in dynamo theory. For instance, a similar trick is employed in the study of the *alpha-effect* in chapters 3, 4.

By construction, Φ_N is a volume-preserving map on $\mathbb{T}^2$. Its inverse, Φ_N^{-1} , is given by

$$\Phi_N^{-1}(x, y, z) = \begin{cases} (\frac{j+x}{2N}, 2Ny - j, z), & \frac{j}{2N} \leq y < \frac{j+1}{2N}, \quad j = 0, \dots, N-1, \\ (\frac{j+1-x}{2N}, j+1 - 2Ny, z), & \frac{j}{2N} \leq y < \frac{j+1}{2N}, \quad j = N, \dots, 2N-1. \end{cases}$$

To introduce the necessary coupling between the horizontal dynamics and the vertical component, we define the sheared N -folded baker's map $T_N : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ as

$$T_N = \Phi_g \circ \Phi_N, \quad \Phi_g(x, y, z) = (x, y, z - g(y)),$$

where $g : \mathbb{T} \rightarrow \mathbb{R}$ is a smooth function and the last coordinate is understood modulo 1. We shall then prove the following result:

Proposition 2.3.4. *Suppose that $\int_0^{1/2} e^{-2\pi i g(y)} dy \neq \int_{1/2}^1 e^{-2\pi i g(y)} dy$. Then, there exists $N_0 \geq 0$ so that for all $N \geq N_0$, the map T_N generates a fast dynamo on $\mathbb{T}^3$.*

We analyse the evolution of vector fields of the form $F(x, y, z) = e^{2\pi i z} f(y) \hat{x}$. The one-step dynamo transition operator S_N , defined by the push-forward

$$(DT_N F) \circ T_N^{-1} = e^{2\pi i z} (S_N f)(y) \hat{x}, \quad (2.3.1)$$

acts on the profile f as

$$S_N f(y) = e^{2\pi i g(y)} N \begin{cases} 2f(2Ny - j), & y \in (\frac{j}{2N}, \frac{j+1}{2N}), \quad j = 0, \dots, N-1, \\ -2f(j+1 - 2Ny), & y \in (\frac{j}{2N}, \frac{j+1}{2N}), \quad j = N, \dots, 2N-1. \end{cases}$$

In particular, S_N admits an L^2 -adjoint given by

$$S_N^* f(y) = \sum_{j=0}^{N-1} f\left(\frac{j+y}{2N}\right) e^{-2\pi i g(\frac{j+y}{2N})} - \sum_{j=N}^{2N-1} f\left(\frac{j+1-y}{2N}\right) e^{-2\pi i g(\frac{j+1-y}{2N})}. \quad (2.3.2)$$

It turns out that working with this adjoint formulation (originally due to [BC88a]) significantly simplifies the forthcoming analysis.

Proposition 2.3.5. *Let $e^{\varepsilon \Delta}$ be the toroidal heat semigroup of strength $\varepsilon > 0$. Then, uniformly in $\varepsilon > 0$, there holds the estimate*

$$\left\| \frac{1}{N} S_N^* e^{\varepsilon \partial_y^2} f - P_g e^{\varepsilon \Delta} f \right\|_{H^1} \lesssim \frac{1}{N} \|f\|_{H^1}, \quad (2.3.3)$$

where P_g is the projection operator given by

$$P_g f = 2 \int_0^{1/2} f(y) e^{-2\pi i g(y)} dy - 2 \int_{1/2}^1 f(y) e^{-2\pi i g(y)} dy.$$

Proof. We first consider the estimate when $\varepsilon = 0$. We estimate the H^1 norm by bounding the L^2 norm of the derivative and the function separately. Let us consider the first summand in the

definition of S_N^*f in (2.3.2), denoted by $\Sigma_1(y)$. Differentiating with respect to y , we have

$$\partial_y \Sigma_1(y) = \frac{1}{2N} \sum_{j=0}^{N-1} \left[f' \left(\frac{j+y}{2N} \right) - 2\pi i g' \left(\frac{j+y}{2N} \right) f \left(\frac{j+y}{2N} \right) \right] e^{-2\pi i g \left(\frac{j+y}{2N} \right)}.$$

Applying the Cauchy-Schwarz inequality $|\sum a_j|^2 \leq N \sum |a_j|^2$ and integrating over $y \in [0, 1]$, we obtain

$$\|\partial_y \Sigma_1\|_{L^2}^2 \leq \frac{1}{4N^2} \int_0^1 N \sum_{j=0}^{N-1} \left[\left| f' \left(\frac{j+y}{2N} \right) \right|^2 + 4\pi^2 \|g'\|_{L^\infty}^2 \left| f \left(\frac{j+y}{2N} \right) \right|^2 \right] dy.$$

We perform the change of variables $z = \frac{j+y}{2N}$ for each j . Summing the integrals over j yields an integral over $[0, 1/2]$:

$$\|\partial_y \Sigma_1\|_{L^2}^2 \leq \frac{1}{4N} \cdot 2N \int_0^{1/2} (|f'(z)|^2 + C_g |f(z)|^2) dz \lesssim \|f\|_{H^1}^2.$$

A similar computation holds for the second summand in (2.3.2). Since $P_g f$ is constant, $\partial_y(P_g f) = 0$. Therefore, for the normalized operator, we have

$$\left\| \partial_y \left(\frac{1}{N} S_N^* f - P_g f \right) \right\|_{L^2} = \frac{1}{N} \|\partial_y S_N^* f\|_{L^2} \lesssim \frac{1}{N} \|f\|_{H^1}.$$

Let us now turn to the L^2 norm. Let $h(y) = f(y)e^{-2\pi i g(y)}$. We define the error terms for the two parts of the operator as

$$\left\| \frac{1}{N} S_N^* f - P_g f \right\|_{L^2} \leq \|\mathcal{E}_1\|_{L^2} + \|\mathcal{E}_2\|_{L^2},$$

where

$$\mathcal{E}_1(y) = \frac{1}{N} \sum_{j=0}^{N-1} h \left(\frac{j+y}{2N} \right) - 2 \int_0^{1/2} h(x) dx,$$

and

$$\mathcal{E}_2(y) = \frac{1}{N} \sum_{j=N}^{2N-1} h \left(\frac{j+1-y}{2N} \right) - 2 \int_{1/2}^1 h(x) dx.$$

We analyze $\mathcal{E}_1$; the bound for $\mathcal{E}_2$ is analogous. Using the identity

$$2 \int_0^{1/2} h(x) dx = \frac{1}{N} \sum_{j=0}^{N-1} 2N \int_{j/2N}^{(j+1)/2N} h(x) dx,$$

we can rewrite the error as a sum of local deviations

$$\begin{aligned} |\mathcal{E}_1(y)| &= \left| \frac{1}{N} \sum_{j=0}^{N-1} 2N \int_{\frac{j}{2N}}^{\frac{j+1}{2N}} \left[h \left(\frac{j+y}{2N} \right) - h(x) \right] dx \right| \\ &\leq \frac{1}{N} \sum_{j=0}^{N-1} 2N \int_{\frac{j}{2N}}^{\frac{j+1}{2N}} \left| \int_x^{\frac{j+y}{2N}} h'(t) dt \right| dx \\ &\leq \frac{1}{N} \sum_{j=0}^{N-1} 2N \left(\frac{1}{2N} \int_{\frac{j}{2N}}^{\frac{j+1}{2N}} |h'(t)| dt \right) = \frac{1}{N} \|h'\|_{L^1(0,1/2)}. \end{aligned}$$

Squaring and integrating over y gives $\|\mathcal{E}_1\|_{L^2} \lesssim \frac{1}{N} \|h\|_{H^1} \lesssim \frac{1}{N} \|f\|_{H^1}$. Combining the derivative and

L^2 estimates concludes the proof for $\varepsilon = 0$. But now, to deduce the result for $\varepsilon > 0$, it suffices to apply the $\varepsilon = 0$ estimate (2.3.3) to $e^{\varepsilon \partial_y^2}$, and to note that the heat semigroup is non-increasing on $H^1(\mathbb{T}^2)$. $\square$

The limiting operator $P_g e^{\varepsilon \partial_y^2}$ is of rank one, mapping any function f into the space of constant functions. Therefore, its only potentially non-zero eigenvalue λ must be associated with a constant eigenfunction. By evaluating the operator on the constant function 1, we obtain the eigenvalue equation $P_g e^{\varepsilon \partial_y^2} 1 = \lambda 1$, where

$$\lambda = 2 \int_0^{1/2} e^{-2\pi i g(y)} dy - 2 \int_{1/2}^1 e^{-2\pi i g(y)} dy.$$

For a generic shear profile g (specifically, one where the phase cancellation is not symmetric across $y = 1/2$), we have $\lambda \neq 0$. Thus, the spectrum of $P_g e^{\varepsilon \partial_y^2}$ consists of the isolated eigenvalue λ and zero (with infinite multiplicity). Therefore, we can now establish the following result.

Proposition 2.3.6. *For all N large enough, and all $\varepsilon > 0$, the operator $S_N^* e^{\varepsilon \partial_y^2}$ admits an eigenvalue of modulus strictly larger than 1.*

Proof. This is an application of Lemma A.3.3 with $T_0 = P_g e^{\varepsilon \partial_y^2}$, $T_1 = \frac{1}{N} S_N^* e^{\varepsilon \partial_y^2} - P_g e^{\varepsilon \partial_y^2}$, as soon as we can prove that the resolvents of $P_g e^{\varepsilon \partial_y^2}$ are bounded uniformly in $\varepsilon \rightarrow 0$ away from λ and 0. Suppose that

$$\mu f - P_g e^{\varepsilon \partial_y^2} f = g. \quad (2.3.4)$$

Averaging this equality in y , and subtracting this average from (2.3.4), we deduce that

$$\mu(f - \bar{f}) = g - \bar{g}.$$

Hence, reinserting this into the averaged version of (2.3.4), we deduce that

$$\mu \bar{f} - P_g e^{\varepsilon \partial_y^2} (\bar{f} + \mu^{-1}(g - \bar{g})) = \bar{g}.$$

$P_g e^{\varepsilon \partial_y^2}$ acts via multiplication by λ on constant functions, so we can rearrange this expression to yield

$$(\mu - \lambda) \bar{f} + \mu^{-1} \lambda \bar{g} - P_g e^{\varepsilon \partial_y^2} g = \bar{g}.$$

Hence, we obtain that

$$(\mu - P_g e^{\varepsilon \partial_y^2})^{-1} g = \frac{1}{\mu - \lambda} \left(\bar{g} \left(1 - \frac{\lambda}{\mu} \right) + P_g e^{\varepsilon \partial_y^2} g \right),$$

which is uniformly bounded in H^1 for μ uniformly bounded away from $0, \lambda$. Thus, applying Lemma A.3.3, we deduce that for all N large enough, $\frac{1}{N} S_N^* e^{\varepsilon \partial_y^2}$ has an eigenvalue of modulus at least $\frac{|\lambda|}{2}$, so that, multiplying by N , $S_N^* e^{\varepsilon \partial_y^2}$ has an eigenvalue of modulus strictly larger than 1 for all $\varepsilon > 0$ and N large enough. Here we have denoted the y -average of a function f via $\bar{f}$. $\square$

With this result in the bag, the proof of Proposition 2.3.4 follows immediately.

Proof of Proposition 2.3.4. Since $S_N^* e^{\varepsilon \partial_y^2}$ is a compact operator on L^2 that maps from L^2 into C^∞ (due to the smoothing effect of the heat semigroup), its spectrum in L^2 consists entirely of discrete eigenvalues, to which are associated smooth (generalised) eigenfunctions. As such, the L^2 and H^1 spectra of $S_N^* e^{\varepsilon \partial_y^2}$ coincide. Therefore, for all N large enough and all $\varepsilon > 0$, $S_N^* e^{\varepsilon \partial_y^2}$ admits an L^2 -eigenvalue with modulus strictly larger than 1. But now, for a general bounded operator it holds by classical functional analysis that $\sigma(T^*) = \overline{\sigma(T)}$, where $\overline{\sigma(T)}$ denotes the complex conjugate of the spectrum of T . Hence, $(S_N^* e^{\varepsilon \partial_y^2})^* = e^{\varepsilon \partial_y^2} S_N$ admits an eigenvalue of modulus strictly larger than 1 in L^2 for N large enough. But then, from (2.3.1), we observe that the dynamo operator associated to the N -Baker's map with a shear, followed by the heat semigroup acts on $e^{2\pi i z} f(y) \hat{x}$ via

$$e^{2\pi i z} e^{-4\pi^2 \varepsilon} e^{\varepsilon \partial_y^2} (S_N f)(y) \hat{x}.$$

Therefore, for all $\varepsilon > 0$ small enough, we deduce the existence of an eigenvalue of modulus strictly larger than 1 (and bounded away from 1 as $\varepsilon \rightarrow 0$) for the map

$$B \mapsto e^{\varepsilon \Delta} (DT_N B) \circ T_N^{-1},$$

which shows the desired fast dynamo action. $\square$

Remark 2.3.7. Using a similar “large N ” asymptotic analysis, it was proven in [Chi93] that the N -folded Baker's map is a *perfect dynamo* (see Definition 2.1.6). However, to the best of the author's knowledge, fast dynamo action had not been proven before in this model. Thus, combining Proposition 2.3.4 with [Chi93] yields a rigorous example of a map that generates both a perfect dynamo and a fast dynamo, yielding further evidence in favour of the flux conjecture.

Remark 2.3.8. Due to the very simple nature of the Baker's map, the above argument could be carried out entirely in the Sobolev space of functions H^1 . At first sight, this may appear to contradict the earlier discussion that the limiting dynamo eigenmode should be a distribution. However, we remark that working in H^1 was only possible due to the *adjoint* formulation of (2.3.2), (which in turn only simplified the problem due to the exceedingly simple nature of the Baker's map). Recalling the L^2 inner-product characterisation of the H^{-1} norm, this shows that the limiting fast dynamo eigenmode of $e^{\varepsilon \Delta} S_N$ lives in the *distribution* space H^{-1} , and is therefore not expected to be a function.

The above example had the distinct advantage that *both* the existence of a growing eigenmode for the non-resistive ($\varepsilon = 0$) dynamo operator, and the perturbative argument in ε could be undertaken analytically. However, this degree of explicitness is not always available, even in other natural Baker's map-based models. A good example of this is the following “prototypical” model of the SFS map, which was analysed for instance in [BC88a; Gil05]. The model is based once again on the simple folded Baker's map. However, this time only a single fold is undertaken, and the map acts on $\mathbb{T} \times [-1, 1]$ (with the torus in x identified with $[-1, 1]$) via

$$T_{SF}(x, y) = \begin{cases} (1 + 2x, \frac{1}{2}(y - 1)) & x < 0 \\ (1 - 2x, \frac{1}{2}(1 - y)) & x \geq 0 \end{cases}.$$

This time, the shear is given by $T_{Sh}(x, y, z) = (0, 0, z + \alpha y)$. Undertaking precisely the same adjoint-

formulation approach as in the proof of Proposition 2.3.4, one can show that dynamo action on vectors of the form $e^{iz}h(y)\hat{x}$ yields the L^2 adjoint operator

$$S^*h = e^{i\alpha\frac{(y-1)}{2}}h\left(\frac{y-1}{2}\right) - e^{i\alpha\frac{(1-y)}{2}}h\left(\frac{1-y}{2}\right). \quad (2.3.5)$$

This operator was analysed in the recent work [PMG23a], and by means of a computer-assisted proof, the following was shown.

Theorem 2.3.9 ([PMG23a]). *Let $\alpha \in (\frac{\pi}{2}, 5]$. Then, the operator (2.3.5) admits an analytic eigenfunction, with associated eigenvalue of modulus strictly larger than 1.*

The goal is then to show that this instability persists when diffusion is introduced. Proving this requires substantially more machinery than the rather “standard” application of perturbation theory in Proposition 2.3.4, and as such is perhaps less intuitive. Nevertheless, one can show the following.

Proposition 2.3.10. *Let $\alpha \in (\frac{\pi}{2}, 5]$. Then, the map $T_{Sh} \circ T_{SF}$ generates a fast dynamo on $\mathbb{T} \times [-1, 1] \times \mathbb{T}$, equipped with perfectly conducting boundary conditions (2.1.1).*

Following the ideas of Proposition 2.3.4, it is sufficient to show that, for all $\varepsilon > 0$ small enough, the map $S^*e^{\varepsilon\partial_y^2}$ admits an eigenvalue on $H^1([-1, 1])$ of modulus uniformly bounded away from 1, with associated eigenmode satisfying the perfectly conducting boundary conditions. To construct such eigenmodes, we rely on the perturbative framework of Keller and Liverani, outlined in Proposition A.3.7. In particular, we will first show the following.

Proposition 2.3.11 (Lasota-Yorke inequality). *Let $\alpha > 0$. Then, there exists a constant C depending only on α so that for all $f \in H^1([-1, 1])$ it holds*

$$\|S^*h\|_{H^1[-1,1]} \leq \|h\|_{H^1[-1,1]} + C\|h\|_{L^2[-1,1]}.$$

Proof. Certainly, bounding the L^2 norm of S^*f yields terms that can be absorbed into the constant C . Thus, we focus on bounding the derivative in x . Taking the derivative in (2.3.5) yields

$$\partial_y S^*h = \frac{1}{2}e^{i\alpha\frac{(y-1)}{2}}h'\left(\frac{y-1}{2}\right) - \frac{1}{2}e^{i\alpha\frac{(1-y)}{2}}h'\left(\frac{1-y}{2}\right) + \frac{i\alpha}{2}e^{i\alpha\frac{(y-1)}{2}}h\left(\frac{y-1}{2}\right) - \frac{i\alpha}{2}e^{i\alpha\frac{(1-y)}{2}}h\left(\frac{1-y}{2}\right).$$

The last two terms may be bounded in L^2 , and so we focus on the first two terms. Arguing as in the proof of Proposition 2.3.4, we bound

$$\begin{aligned} & \int_{-1}^1 \left| \frac{1}{2}e^{i\alpha\frac{(y-1)}{2}}h'\left(\frac{y-1}{2}\right) - \frac{1}{2}e^{i\alpha\frac{(1-y)}{2}}h'\left(\frac{1-y}{2}\right) \right|^2 dy \\ & \leq \frac{1}{2} \int_{-1}^1 \left| h'\left(\frac{y-1}{2}\right) \right|^2 dy + \frac{1}{2} \int_{-1}^1 \left| h'\left(\frac{1-y}{2}\right) \right|^2 dy. \end{aligned}$$

Changing variables, this is bounded above by $\int_{-1}^1 |h'(y)|^2 dy$. Therefore, the proof of the proposition follows. $\square$

From here, noting that S^* also defines a bounded operator on $L^2([-1, 1])$, and that $H^1([-1, 1])$ embeds compactly into $L^2([-1, 1])$, we can apply the classical result due to Hennion [Hen93] (see

Proposition A.2.8 for more details) to deduce that the essential spectral radius of S^* on $H^1([-1, 1])$ is contained in the unit ball in the complex plane. In particular, the eigenvalues constructed in Theorem 2.3.9 are isolated from the essential spectrum, and we can hope to apply the singular perturbation theory. To do so, we consider the action of the heat semigroup with perfectly conducting boundary conditions. Note that for a vector of the form $e^{iz}h(y)\hat{x}$ on $\mathbb{T} \times [-1, 1] \times \mathbb{T}$, the perfectly conducting boundary conditions (2.1.1) become

$$h'(y)e^{iz} = 0, \quad y = \pm 1.$$

Hence, the heat semigroup is given by the semigroup generated by the Neumann Laplacian, which is still self-adjoint on L^2 . Therefore, it suffices to prove the following:

Proposition 2.3.12. *For all $\alpha \in (\frac{\pi}{2}, 5]$, there exists $\varepsilon_0 > 0$, so that for all $\varepsilon < \varepsilon_0$, $S^*e^{\varepsilon\partial_{yy}^N}$ admits an eigenvalue of modulus at least $\lambda_\alpha > 1$. Here ∂_{yy}^N denotes the Neumann Laplacian.*

Proof. We aim to apply the perturbative result of Keller-Liverani [KL99] outlined in Proposition A.3.7. To do so, it suffices to show that $e^{\varepsilon\partial_{yy}^N}$ is bounded uniformly in ε on $L^2([-1, 1])$ and on $H^1([-1, 1])$, with operator norm 1, and that

$$\|e^{\varepsilon\partial_{yy}^N}h - h\|_{L^2([-1, 1])} \leq \varepsilon^{1/2}\|f\|_{H^1([-1, 1])}. \quad (2.3.6)$$

The uniform in ε bounds on L^2 and H^1 are straightforward consequences of heat equation estimates with Neumann boundary conditions. Thus it remains to prove (2.3.6). Let h^ε, h^0 respectively solve

$$\partial_t h^\varepsilon = \varepsilon \partial_{yy} h^\varepsilon, \quad \partial_t h^0 = 0,$$

with initial datum $h \in H^1([-1, 1])$, and h^ε satisfying Neumann boundary conditions. Subtracting the equations, multiplying by $h^\varepsilon - h^0$ and integrating yields

$$\frac{1}{2} \partial_t \|h^\varepsilon(t) - h^0\|_{L^2([-1, 1])}^2 = \varepsilon \int_{-1}^1 \partial_y^2 h^\varepsilon (h^\varepsilon - h^0) dy.$$

Integrating by parts yields a boundary term of the form $\int_{\partial[-1, 1]} \partial_y h^\varepsilon (h^\varepsilon - h^0) = 0$, since h^ε satisfies the Neumann boundary conditions. Thus, we obtain (upon adding and subtracting $-\partial_y h^0 \partial_y (h^\varepsilon - h^0)$)

$$\frac{1}{2} \partial_t \|h^\varepsilon(t) - h^0\|_{L^2([-1, 1])}^2 = -\varepsilon \|\partial_y (h^\varepsilon(t) - h^0)\|_{L^2([-1, 1])}^2 + \varepsilon \int_{-1}^1 \partial_y h^0 \partial_y (h^\varepsilon - h^0) dy.$$

In the final term, we employ Young's inequality to deduce that

$$\frac{1}{2} \partial_t \|h^\varepsilon(t) - h^0\|_{L^2([-1, 1])}^2 \leq -\frac{\varepsilon}{2} \|\partial_y (h^\varepsilon(t) - h^0)\|_{L^2([-1, 1])}^2 + \frac{\varepsilon}{2} \|\partial_y h^0\|_{L^2([-1, 1])}^2.$$

Integrating this in time, we deduce that (since the initial conditions of h^ε, h^0 coincide)

$$\|h^\varepsilon(t=1) - h^0\|_{L^2([-1, 1])}^2 \leq \varepsilon \|\partial_y h^0\|_{L^2([-1, 1])}^2,$$

completing the proof of (2.3.6), since $h^\varepsilon(t=1) = e^{\varepsilon \partial_{yy}^N} h$. Therefore, set now $\mathcal{P}_\varepsilon = S^* e^{\varepsilon \partial_{yy}^h}$, and $\mathcal{P}_0 = S^*$. Then, we deduce that uniformly in $\varepsilon > 0$ small enough, there holds

1. $\|\mathcal{P}_\varepsilon h\|_{L^2([-1,1])} \leq C\|h\|_{L^2([-1,1])}$
2. $\|\mathcal{P}_\varepsilon h - h\|_{L^2([-1,1])} \leq C\varepsilon^{1/2}\|h\|_{H^1([-1,1])}$
3. $\|\mathcal{P}_\varepsilon h\|_{H^1([-1,1])} \leq \|h\|_{H^1([-1,1])} + C\|h\|_{L^2([-1,1])}$.

Therefore, we may apply Proposition A.3.7 to deduce the result of Proposition 2.3.12, and hence Proposition 2.3.10. $\square$

Propositions 2.3.4, 2.3.10 provide relatively simple models for which fast dynamo action can be proven analytically. However, they are still rather unphysical, since the Baker's map is discontinuous in space. It is worth noting, however, that one can recover the folded Baker's map as a time-1 flow map of a Sobolev regular velocity field, as shown in [EZ19]. Nevertheless, due to their discontinuous nature, models based on the Baker's map are not fully satisfactory for fast dynamo action. Therefore, in the upcoming chapters, we endeavour to construct examples of more regular velocity fields that generate dynamos with non-zero $\varepsilon > 0$.

Chapter 3

Alpha-unstable flows, and a fast dynamo on $\mathbb{R}^3$.

In this chapter we will present a rather general method of constructing slow dynamos with spatially periodic velocity fields, by exploiting the so-called *alpha-effect*, first introduced in the context of dynamo theory in [Rob70]. We then use these velocity fields as building blocks to construct a velocity field that acts on all spatial scales, which will be shown to yield an example of a fast dynamo on $\mathbb{R}^3$ with a bounded, Lipschitz velocity field. The content of this chapter is contained in the paper [CSV25].

3.1 Introduction and main results

Throughout this chapter, we consider the kinematic dynamo equations (KDE) posed on $\Omega = \mathbb{R}^3$. That is, we consider

$$\begin{cases} \partial_t B^\varepsilon = \nabla \times (u \times B^\varepsilon) + \varepsilon \Delta B^\varepsilon, \\ \nabla \cdot B^\varepsilon = 0, \\ B^\varepsilon|_{t=0} = B_{\text{in}}^\varepsilon, \end{cases} \quad (t, x) \in (0, \infty) \times \mathbb{R}^3.$$

Here, $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a given, divergence-free and time-independent velocity field, and $\varepsilon > 0$ is the magnetic diffusivity, inversely proportional to the magnetic Reynolds number. The main result of this chapter is the following, which resolves the fast dynamo conjecture 1 on the domain $\Omega = \mathbb{R}^3$.

Theorem 1. *There exist an autonomous, divergence-free velocity field $u \in W^{1,\infty}(\mathbb{R}^3)$ and constants $\gamma, c_0 > 0$ with the following property. For any $\varepsilon \in (0, 1)$ there exists a non-zero, divergence-free initial datum $B_{\text{in}}^\varepsilon \in L^2(\mathbb{R}^3)$ such that the corresponding solution of (KDE- ε) satisfies $B^\varepsilon \in L_{\text{loc}}^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L_{\text{loc}}^2(0, \infty; \dot{H}^1(\mathbb{R}^3))$ and*

$$\|B^\varepsilon(t)\|_{L^2(\mathbb{R}^3)} \geq c_0 e^{\gamma t} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)}, \quad (3.1.2)$$

for every $t \geq 0$.

Remark 3.1.1. Due to the gluing procedure and spatial micro-scaling used to define the velocity

field (3.4.4), the resulting velocity field is not $C^{1,\alpha}(\mathbb{R}^3)$ for any $\alpha > 0$. It is however in $C_{loc}^n(\mathbb{R}^3)$, for any $n \in \mathbb{N}$.

This result provides an example of *fast dynamo action* by a Lipschitz, time-independent velocity field on the whole space $\mathbb{R}^3$. The exponential growth, as in (3.1.2), is the fastest possible that a finite-energy initial datum can experience under the dynamics of (KDE- ϵ) with a Lipschitz velocity field.

While we outline the proof of Theorem 1 in Section 3.1.4, we mention that the construction of u is based on the concept of *alpha-unstable flows* on the periodic domain $\mathbb{T}^3 \cong [0, 2\pi]^3 / \sim$, defined as follows.

Definition 3.1.2 (Alpha-unstable flow). Let $U \in C^\infty(\mathbb{T}^3)$ be a mean-free, incompressible velocity field with $\|\nabla U\|_{L^\infty} < 1$. For $v \in \mathbb{C}^3$, consider the solution map $\mathcal{S} : \mathbb{C}^3 \rightarrow L^2(\mathbb{T}^3)$, $v \mapsto \mathcal{S}(v) = S$ of the elliptic problem

$$\nabla \times (U \times S) + \Delta S = \nabla \times (v \times U), \quad \int_{\mathbb{T}^3} S(x) dx = 0, \quad (3.1.3)$$

and define the $\mathbb{C}^{3 \times 3}$ matrix $A = A(U, \mathbf{j})$ via

$$Av = i \frac{\mathbf{j}}{|\mathbf{j}|} \times \int_{\mathbb{T}^3} U \times \mathcal{S}(v), \quad \mathbf{j} \in \mathbb{R}^3 \setminus \{0\}, \quad v \in \mathbb{C}^3. \quad (3.1.4)$$

We say that U is *alpha-unstable* if there exist a nonzero $\mathbf{j} \in \mathbb{R}^3$ and a simple eigenvalue $\mu \in \mathbb{C}$ of A with $\text{Re}(\mu) > 0$.

The assumption $\|\nabla U\|_{L^\infty} < 1$ in Definition 3.1.2 is only needed to ensure existence and uniqueness of solutions to equation (3.1.3), so that $\mathcal{S}$ is well-defined. Although the matrix in (3.1.4) may seem obscure at first, it arises naturally when analyzing the spectrum of the linear operator governing the dynamics of (KDE- ϵ), as well as its perturbations. It also connects with the so-called alpha-effect, as we explain in the next Section 3.1.2. To the best of our knowledge, this is the first rigorous justification of this effect for (KDE- ϵ).

Alpha-unstable flows can be proved to be generic in the space of smooth, mean-free and incompressible velocities (c.f. Proposition 3.2.7). However, their existence does not immediately imply fast dynamo action on $\mathbb{T}^3$. This remains to this date an outstanding open problem, see [Arn04, Pb. 1994-28].

Remark 3.1.3. The main obstruction to constructing our velocity field u , defined in (3.4.4), on $\mathbb{T}^3$ lies in the fact that it requires an infinite family of disjoint, well-separated balls $\{Q_{n,\ell}\}_{n,\ell}$, which cannot be embedded within $\mathbb{T}^3$.

3.1.1 A note on related literature

We remark here specifically on results relating to fast dynamo action with autonomous vector fields, or vector fields on the whole space $\mathbb{R}^3$. In this vein, the only examples of fast dynamo on $\mathbb{R}^3$ that we are aware of are the following:

- In [Zel+84], with a velocity field of the form $u(t, x) = C(t)x$, where $C(t)$ is a traceless

random matrix. This construction relies heavily on the *unboundedness* of the velocity field, since it requires the support of the initial datum to grow exponentially. This cannot happen with a bounded velocity field as in our Theorem 1;

- In [Gil88], with the so-called Ponomarenko model. This is a *discontinuous* helical flow, for which exponential growth happens in a neighborhood of the discontinuity.

The lack of regularity of the above velocity fields further feeds into the paradigm established in Section 2.3.1, which suggests that dynamo action at lower regularity is somewhat easier (though still far from simple) to establish. A further result that is certainly worth mentioning is the work of Soward [Sow87], which, via a very sophisticated formal asymptotic expansion in the resistivity ε , deduces that a spatially periodic velocity field with a logarithmic singularity placed at its hyperbolic point should generate a fast dynamo on $\mathbb{T}^3$. In particular, the results [Gil88; Sow93] rely on the growing modes asymptotically localising near the singularities of the velocity fields as $\varepsilon \rightarrow 0$. In contrast, the construction of the velocity field in Theorem 1 proceeds by iteratively introducing smaller and smaller scales at which the field acts, matching the intuition outlined in the introductory section.

3.1.2 The alpha-effect

The concept of *alpha-instability* in Definition 3.1.2 is named after classical works that sought to exploit the so-called *alpha-effect* - a mechanism introduced to circumvent anti-dynamo theorems while avoiding excessive complexity. The most relevant work for our study is Roberts' 1970 analysis of dynamo action in periodic velocity fields [Rob70], later refined in [Rob72]. Roberts considered a periodic magnetic field $B(t, x_1, x_2)$ that, by itself, cannot generate a dynamo due to the anti-dynamo theorems. However, introducing a slow variation in the x_3 -direction allows the system to escape this constraint, with the alpha-effect determining the extent of dynamo action. While highly effective, Roberts' approach did not yield quantitative bounds on the dynamo growth rate in terms of the magnetic resistivity ε . In particular, it remained unclear whether dynamo action persisted in the limit $\varepsilon \rightarrow 0$, see further studies [Chi79; Sow87].

The importance of the alpha-effect in the dynamo problem can be justified heuristically following [Rob70; Rob72; Mof78; Chi79; CG95]. The crucial mechanism underlying the alpha-effect is the interaction between small and large scales. Specifically, if the velocity field U is concentrated at high frequencies (i.e., small scales) while the initial magnetic field is only at low frequencies (i.e., large scales), the term $\nabla \times (U \times B^\varepsilon)$ in (KDE- ε) induces the creation of high frequencies in the magnetic field.

To clarify this mechanism, we introduce the two-scale approach in the context of the passive vector equation, first developed by Steenbeck, Krause, and Rädler [SK69; SKR66]. We decompose the magnetic field into small and large scales components:

$$B^\varepsilon = B_S + B_L, \quad U = U_S, \quad (3.1.5)$$

where B_L represents the large-scale part, and B_S represents the small-scale part. A common ho-

mogenization technique is to assume that B^ε has the two-scale structure $B^\varepsilon(x) = B_L(x) + B_S(x, \lambda x)$, where $y = \lambda x$ is the small scales variable for $\lambda \gg 1$. Using this approach, the passive vector equation decomposes into separate equations as

$$\begin{cases} \partial_t B_L = \nabla \times \langle U_S \times B_S \rangle_y + \varepsilon \Delta B_L, \\ \partial_t B_S = \nabla \times (U_S \times B_L) + \nabla \times (U_S \times B_S - \langle U_S \times B_S \rangle_y) + \varepsilon \Delta B_S, \end{cases} \quad (3.1.6)$$

where $\langle \cdot \rangle_y$ denotes averaging in the small scales.

The key term here is the mean electromotive force $\mathcal{E} = \langle U_S \times B_S \rangle_y$, which plays a central role in dynamo theory by potentially sustaining the large scales part B_L . A crucial insight is that the growth of B_L and B_S reinforce each other [Mof78]. In particular, it is predicted that the relation

$$\langle U_S \times B_S \rangle_y \approx \alpha(B_L)$$

holds for some matrix $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, known as the *alpha-matrix*. This is the so-called alpha-effect, from which the matrix derives its name.

If this analysis is correct, the first equation in (3.1.6) on the Fourier side $x \mapsto \mathbf{k}$ reads

$$\partial_t \widehat{B}_L = i\mathbf{k} \times \alpha(\widehat{B}_L) - \varepsilon |\mathbf{k}|^2 \widehat{B}_L.$$

If the matrix $i\mathbf{k} \times (\alpha(\cdot))$ has an eigenvalue μ with a positive real part for some $\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}$, then the solution with corresponding eigenvector as initial datum exhibits exponential growth for sufficiently small ε . This heuristic argument, originating from [Rob70; Rob72; Sow87; CG95; Chi79], provides an important qualitative prediction about dynamo behavior. However, to the best of our knowledge, there are no rigorous justifications of this mechanism in a general setting. In Section 3.2, we provide a rigorous justification in a specific rescaled framework, where the notion of alpha-unstable flows naturally arises. In our setting, the decomposition (3.1.5) approximately takes the form

$$B_S = \mathcal{S}(v) \exp(i\mathbf{j} \cdot x + \mu t), \quad B_L = v \exp(i\mathbf{j} \cdot x + \mu t), \quad \mathbf{j} \in \mathbb{R}^3 \setminus \{0\},$$

where $v \in \mathbb{C}^3$ is related to the mean magnetic field. Under suitable assumptions, we obtain a solution to (KDE- ε) up to lower-order terms in $|\mathbf{j}| \ll 1$ using a perturbative approach. The corresponding alpha-matrix, up to the large-scale function $\exp(i\mathbf{j} \cdot x + \mu t)$, is given by

$$\langle U_S \times B_S \rangle_y \approx \int_{\mathbb{T}^3} U \times \mathcal{S}(v)$$

and appears in Definition 3.1.2. Alpha-instability can be explicitly verified for the classical ABC flows, and it is in fact a generic property, see Proposition 3.2.7.

3.1.3 Notation

The symbol $Q_R(x) \subset \mathbb{R}^3$ denotes the closed ball centered at $x \in \mathbb{R}^3$ of radius $R > 0$. When x is the origin, we simply write Q_R . The solution semigroup generated by (KDE- ε) with vector field

$u \in W^{1,\infty}(\mathbb{R}^3)$ is denoted $S_\varepsilon^u(t) : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ and acts as

$$B_{\text{in}}^\varepsilon \mapsto S_\varepsilon^u(t)B_{\text{in}}^\varepsilon = B^\varepsilon(t).$$

Furthermore, we shall denote by $C_c^\infty(\mathbb{R}^3)$ the space of smooth functions with compact support in $\mathbb{R}^3$. Given an operator $\mathcal{L}$, the symbols $\sigma(\mathcal{L})$ and $\rho(\mathcal{L})$ denote the spectrum and resolvent set, respectively. Throughout, whenever we integrate along a contour $\Gamma \subset \mathbb{C}$, we assume that Γ is oriented counterclockwise.

3.1.4 Strategy of the proof

The proof of Theorem 1 combines classical ideas originally introduced by G.O. Roberts in [Rob70; Rob72], a novel rescaling of both the velocity and magnetic fields to derive an ε -independent eigenvalue problem, and the construction of a velocity field on the whole space by appropriately gluing together rescaled building blocks.

Modal form and the rescaled spectral problem. In 1970, G.O. Roberts [Rob70] proposed to approach the problem by looking at solutions with a particular structure that exhibits exponential growth, similar to the classical normal mode form used in countless linear problems. This constitutes the main idea to construct the *building block* eventually used in our velocity field. We take a periodic, bounded and Lipschitz vector field $U : \mathbb{T}^3 \rightarrow \mathbb{R}^3$, rescale it as

$$U_\varepsilon(x) = \sqrt{\varepsilon}U\left(\frac{x}{\sqrt{\varepsilon}}\right) \in W^{1,\infty}(\mathbb{T}^3), \quad \|U_\varepsilon\|_{W^{1,\infty}} \leq \|U\|_{W^{1,\infty}}, \quad (3.1.7)$$

and look initially for solutions to (KDE- ε) with $u = U_\varepsilon$ of the form

$$B^\varepsilon(t, x) = H\left(\frac{x}{\sqrt{\varepsilon}}\right) e^{ie^{-\frac{1}{2}}\mathbf{j} \cdot x + pt}, \quad (3.1.8)$$

for $p \in \mathbb{C}$, $\mathbf{j} \in \mathbb{R}^3$, and $H : \mathbb{T}^3 \rightarrow \mathbb{R}^3$. Notice that at this stage, B^ε may not be periodic, nor finite-energy, hence it is not the solution we find in Theorem 1. However, a computation (c.f. Section 3.2) shows that H solves the ε -independent eigenvalue problem on $\mathbb{T}^3$

$$(\nabla + i\mathbf{j}) \times (U \times H) + (\nabla + i\mathbf{j})^2 H = pH, \quad (3.1.9)$$

$$(\nabla + i\mathbf{j}) \cdot H = 0. \quad (3.1.10)$$

If U is *alpha-unstable* (c.f. Definition 3.1.2), then there exists a triple $(H, \mathbf{j}, p)$ satisfying the above problem with $\text{Re}(p) > 0$.

Proposition 3.1.4. *Suppose that $U \in W^{1,\infty}(\mathbb{T}^3)$ is alpha-unstable. Then there exist $\mathbf{j}_\star \in \mathbb{R}^3$ with $|\mathbf{j}_\star| \leq 1$, $p_\star = p(\mathbf{j}_\star) \in \mathbb{C}$ with $\text{Re}(p_\star) > 0$ and $H(\cdot; \mathbf{j}_\star) \in L^2(\mathbb{T}^3)$ solution to (3.1.9)-(3.1.10).*

The proof of this result is inspired by the works of Roberts [Rob70; Rob72] and relies on classical tools from perturbation theory [Kat76], which are briefly introduced in Appendix A. The perturbation parameter in our analysis is $|\mathbf{j}| \ll 1$, and the key step involves characterizing the kernel of the linear operator in (3.1.9) at $\mathbf{j} = 0$. This kernel is three-dimensional, reducing the problem to

studying the eigenvalues of a 3×3 matrix. Explicit computations in Example 3.2.6 for the so-called ABC flows reveal a broad class of alpha-unstable flows for which Proposition 3.1.4 holds. This insight motivates the construction of a fundamental *building block* that plays a central role in the final vector field u of Theorem 1.

A Bloch-type theorem and extension to $\mathbb{R}^3$. As mentioned above, formula (3.1.8) alone does not directly yield a finite-energy, exponentially growing solution of (KDE- ε). The issue arises from the j -dependent plane wave and the periodicity of H , which prevent square-integrability. To overcome this, we establish a Bloch-type result (see Lemma 3.2.12), which, when combined with Proposition 3.1.4, ensures the existence of a finite-energy, exponentially growing solution of (KDE- ε) on the whole space $\mathbb{R}^3$:

$$\|B^\varepsilon(t)\|_{L^2(\mathbb{R}^3)} \geq e^{2\gamma t} \|B_{\text{in}}\|_{L^2(\mathbb{R}^3)} \quad (3.1.11)$$

where $\gamma > 0$ is independent of ε , and the velocity field is given by the periodic extension of the rescaled vector field U_ε in (3.1.7) - see Theorem 2. This constitutes the main building block in proving Theorem 1. However, the rescaled velocity field U_ε in (3.1.7) still depends on $\varepsilon > 0$. To construct a velocity field u that is independent of ε , we glue together multiple translated and compactly supported copies of U_ε for a carefully chosen sequence of values $\varepsilon \in \{\zeta^n\}_{n \in \mathbb{N}}$ with $\zeta \in (0, 1)$. The details of this gluing procedure are described below.

A key challenge in this construction is ensuring that the exponential growth rate in (3.2.1) remains *uniform* in ε over each interval $(\zeta^{n+1}, \zeta^n]$. Additionally, we must control the concentration of the initial data (3.2.2) uniformly in $\varepsilon > 0$. These considerations are crucial for maintaining robust control over the dynamics of (KDE- ε) under the “glued” velocity field u .

The glued vector field and heuristics of the proof. We now construct the ε -independent glued vector field and outline the proof of Theorem 1. The rescaled flow U_ε in (3.1.7), derived from an alpha-unstable flow U gives rise - thanks to Proposition 3.1.4 and the Bloch-type result described above - to an exponentially growing solution for any $\varepsilon > 0$, with growth rate $2\gamma = \text{Re}(p_\star) > 0$ independent of ε , see (3.1.11).

To construct an ε -independent divergence-free vector field, we assemble rescaled and compactly supported copies of U using the sequence $\{\zeta^n\}_{n \in \mathbb{N}}$, defining

$$u = \sum_{n, \ell=1}^{\infty} u_{n, \ell}, \quad u_{n, \ell}(x) \approx \zeta^{n/2} U(\zeta^{-n/2} x) \mathbb{1}_{Q_{n, \ell}}(x),$$

for appropriately chosen disjoint balls $\{Q_{n, \ell}\}_{n, \ell=1}^{\infty}$ (see (3.4.4) for the precise definition). At this stage, $\ell \in \mathbb{N}$ is a free parameter. For any $\varepsilon \in (0, 1)$ and $\zeta \in (0, 1)$, there exists a unique $n_\varepsilon \in \mathbb{N}$ so that $\varepsilon \in (\zeta^{n_\varepsilon+1}, \zeta^{n_\varepsilon}]$. Accordingly, we define the initial datum as

$$B_{\text{in}}^\varepsilon = \sum_{\ell=1}^{\infty} \ell^{-2} F_\ell^\varepsilon$$

where each F_ℓ^ε is *concentrated* on $Q_{n_\varepsilon, \ell}$ (see (3.4.6) for a precise definition). Now, for any $t > 0$, there exists an index $\ell_t \in \mathbb{N}$ such that $t \in [\ell_t, \ell_t + 1)$. We will show that the solution to (KDE- ε)

with initial data $B_{\text{in}}^\varepsilon$ is *quantitatively* exponentially large around $Q_{n_\varepsilon, \ell_t}$ for any $t \in [\ell_t, \ell_t + 1)$ and $\varepsilon \in (\zeta^{n_\varepsilon+1}, \zeta^{n_\varepsilon}]$, due to the velocity field $u_{n_\varepsilon, \ell_t}$ and the contribution of the initial datum $F_{\ell_t}^{n_\varepsilon}$. More precisely, in Proposition 3.4.1 we establish estimates that imply

$$\|S_\varepsilon^u(t)B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)} \geq \|S_\varepsilon^u(t)B_{\text{in}}^\varepsilon\|_{L^2(Q_{n_\varepsilon, \ell_t})} \gtrsim \ell_t^{-2} \|S_\varepsilon^{u_{n_\varepsilon, \ell_t}}(t)F_{\ell_t}^\varepsilon\|_{L^2(Q_{n_\varepsilon, \ell_t})} \gtrsim e^{\gamma(\ell_t+1)} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)},$$

where the constant $\gamma > 0$ as well as the implicit constant in $\gtrsim$ are independent of $t > 0$ and $\varepsilon \in (0, 1)$. This chain of inequalities confirms that the solution exhibits exponential growth around the ball $Q_{n_\varepsilon, \ell_t}$ for any $t \in [\ell_t, \ell_t + 1)$ and $\varepsilon \in (\zeta^{n_\varepsilon+1}, \zeta^{n_\varepsilon}]$. Since $t > 0$ and $\varepsilon \in (0, 1)$ are arbitrary, we conclude the proof. To rigorously justify these estimates, we rely on suitable local energy bounds (see Section 3.3), together with the careful choice of disjoint balls $\{Q_{n, \ell}\}_{n, \ell=1}^\infty$ and appropriate cutoffs.

3.2 Spectral analysis and modal form

We now state the main theorem we will prove in this section, which is a key tool used in the proof of Theorem 1 in Section 3.4.

Theorem 2. *Let $U = \nabla \times \Psi \in C^\infty(\mathbb{T}^3)$ be an alpha-unstable velocity. Then, there exist $\zeta \in (1/2, 1)$ and $\gamma > 0$ such that*

$$U_n(x) = \zeta^{n/2} U(\zeta^{-n/2} x), \quad U_n = \nabla \times \Psi_n,$$

satisfies the following properties.

- *For every $\varepsilon \in (0, 1]$ there exists $F^\varepsilon \in L^2(\mathbb{R}^3)$, normalized to $\|F^\varepsilon\|_{L^2(\mathbb{R}^3)} = 1$, such that if $\varepsilon \in (\zeta^{n_\varepsilon+1}, \zeta^{n_\varepsilon}]$ for some $n_\varepsilon \in \mathbb{N}$ then*

$$\|S_\varepsilon^{U_{n_\varepsilon}}(t)F^\varepsilon\|_{L^2(\mathbb{R}^3)} \geq e^{2\gamma t}, \quad \forall t \geq 0. \quad (3.2.1)$$

- *For every $\delta \in (0, 1)$, there exists $R > 0$ independent of ε such that*

$$\|F^\varepsilon\|_{L^2(Q_R)} \geq 1 - \delta, \quad \forall \varepsilon > 0. \quad (3.2.2)$$

A consequence of this theorem, considering the case $n = 0$ for the velocity field, i.e. $U_0 = U$ and $\varepsilon = 1$ for the magnetic diffusivity, is that there exists $F^1 \in L^2(\mathbb{R}^3)$ such that

$$\|S_1^U(t)F^1\|_{L^2(\mathbb{R}^3)} \geq e^{2\gamma t} \|F^1\|_{L^2(\mathbb{R}^3)}.$$

In other words, the equation (KDE- ε) admits a solution that grows exponentially in time for $\varepsilon = 1$. The theorem above generalizes this property to a specific rescaling of the velocity field, which preserves a uniform bound on the Lipschitz norm. Furthermore, the initial data corresponding to exponentially growing solutions to (KDE- ε) satisfy a concentration of mass property that holds *uniformly* in ε .

The Section is now divided in several subsections. First, in Subsection 3.2.1, we derive an ansatz

in the modal form for an exponentially growing solution with a rescaled velocity field. Then, in Subsection 3.2.2, we analyze the spectral properties of the operator that naturally arises from the eigenvalue problem associated with the exponentially growing solution. Next, in Subsection 3.2.3, we prove Proposition 3.1.4 for the problem on the three-dimensional torus and show that a typical choice of an ABC flow is alpha-unstable. In Subsection 3.2.4, we study the continuity of the operator arising from the eigenvalue problem with respect to magnetic diffusivity. In Subsection 3.2.5, we prove a Bloch-type result, extending our ansatz from the three-dimensional torus to the whole space. Finally, in Subsection 3.2.6, we use these tools to prove Theorem 2.

3.2.1 Modal form

The goal of this section is to prove Proposition 3.1.4 and construct a solution to (KDE- ε) in $L^2(\mathbb{R}^3)$. To derive the spectral problem (3.1.9)-(3.1.10), we first set $B^\varepsilon(t, x) = H\left(\frac{x}{\sqrt{\varepsilon}}\right) e^{i\varepsilon^{-\frac{1}{2}}j \cdot x + pt}$ as in (3.1.8), use vector calculus identities¹ and the divergence free property of u to get

$$\begin{aligned}\nabla \times (u \times B^\varepsilon) &= \varepsilon^{-\frac{1}{2}} e^{i\varepsilon^{-\frac{1}{2}}j \cdot x + pt} \left(\varepsilon^{\frac{1}{2}} (H \cdot \nabla) u + (\nabla \cdot H) u - (u \cdot \nabla) H + (i\mathbf{j} \cdot H) u - (i\mathbf{j} \cdot u) H \right) \\ \Delta B^\varepsilon &= \varepsilon^{-1} e^{pt + \varepsilon^{-\frac{1}{2}}i\mathbf{j} \cdot x} ((\nabla + i\mathbf{j})^2 H), \\ \nabla \cdot B^\varepsilon &= (\nabla + i\mathbf{j}) \cdot H = 0.\end{aligned}$$

Thus, (3.1.9)-(3.1.10) is derived by defining $u = U_\varepsilon$ as in (3.1.7)-(3.1.8).

The main object of our focus is the linear operator $\mathcal{L}$ in (3.1.9) defined as

$$\mathcal{L}H = (\nabla + i\mathbf{j}) \times (U \times H) + (\nabla + i\mathbf{j})^2 H. \quad (3.2.3)$$

We then define

$$\mathcal{L}_0 H = \nabla \times (U \times H) + \Delta H$$

and its first order perturbation

$$\mathcal{L}_1 H = i \frac{\mathbf{j}}{|\mathbf{j}|} \times (U \times H) + 2i \frac{\mathbf{j}}{|\mathbf{j}|} \cdot \nabla H. \quad (3.2.4)$$

In this way,

$$\mathcal{L} = \mathcal{L}_0 + |\mathbf{j}| \mathcal{L}_1 - |\mathbf{j}|^2, \quad (3.2.5)$$

and we can appeal to tools from perturbation theory [Kat76] using $|\mathbf{j}| \ll 1$ as a small parameter, independently of ε .

3.2.2 Spectral analysis

To analyze the spectral properties of $\mathcal{L}$ in (3.2.5), on L^2 , with domain $D(\mathcal{L}) = H^2$, we first give a fairly accurate description of $\mathcal{L}_0$. Its kernel $\text{Ker}(\mathcal{L}_0)$ was already characterized in the original work of Roberts [Rob70], in a perturbative way.

¹ $\nabla \times (A \times C) = (C \cdot \nabla)A + (\nabla \cdot C)A - (A \cdot \nabla)C - (\nabla \cdot A)C$ and $A \times (C \times D) = (A \cdot D)C - (A \cdot C)D$

Lemma 3.2.1. *There exists $\delta_0 > 0$ so that for any divergence-free $U \in W^{1,\infty}(\mathbb{T}^3)$ with $\|U\|_{W^{1,\infty}} < \delta_0$ and any $v \in \mathbb{C}^3$, there exists a unique $L^2(\mathbb{T}^3)$ -solution to the problem*

$$\mathcal{L}_0 S = \nabla \times (v \times U), \quad \int_{\mathbb{T}^3} S(x) dx = 0. \quad (3.2.6)$$

Calling $\mathcal{S} : \mathbb{C}^3 \rightarrow L^2(\mathbb{T}^3)$ the corresponding solution map $v \mapsto \mathcal{S}(v) = S$, we have that

$$\text{Ker}(\mathcal{L}_0) = \{v + \mathcal{S}(v) : v \in \mathbb{C}^3\}, \quad (3.2.7)$$

and it is therefore three-dimensional.

Proof. We preliminarily notice that S is a mean free solution to (3.2.6) if and only if

$$(1 + \nabla \times (U \times \Delta^{-1})) \Delta S = \nabla \times (v \times U). \quad (3.2.8)$$

In particular, note that the invertibility in the space of mean zero functions of the left-hand side of (3.2.8) is sufficient to guarantee a unique solution to (3.2.6). Furthermore, we have for some $\delta_0 \in (0, 1)$ that

$$\|\nabla \times (U \times \Delta^{-1} S)\|_{L^2} = \|\Delta^{-1} S \cdot \nabla U + \nabla \cdot (\Delta^{-1} S) U - U \cdot \nabla \Delta^{-1} S\|_{L^2} \leq \frac{1}{2\delta_0} \|U\|_{W^{1,\infty}} \|S\|_{L^2},$$

and therefore, as soon as $\|U\|_{W^{1,\infty}} < \delta_0$, the operator $\Delta + \nabla \times (U \times \cdot)$ is invertible and

$$(\Delta + \nabla \times (U \times \cdot))^{-1} = \Delta^{-1} \sum_{n \geq 0} (-1)^n (\nabla \times (U \times \Delta^{-1} \cdot))^n. \quad (3.2.9)$$

To prove (3.2.7), we first observe that any element of the form $v + \mathcal{S}(v)$ belongs to the kernel by a straightforward computation. To prove the $\subset$ inclusion, we consider $H \in L^2(\mathbb{T}^3)$ and write it as

$$H = \langle H \rangle + H_{\neq}, \quad \langle H \rangle := \oint_{\mathbb{T}^3} H(x) dx, \quad H_{\neq} = H - \langle H \rangle.$$

Then $\mathcal{L}_0 H = 0$ implies

$$\mathcal{L}_0 H_{\neq} = -\nabla \times (U \times \langle H \rangle) = \nabla \times (\langle H \rangle \times U),$$

so that $H_{\neq} = \mathcal{S}(\langle H \rangle)$. Therefore $H = \langle H \rangle + \mathcal{S}(\langle H \rangle)$, and the proof is over. $\square$

The Riesz projection of $\mathcal{L}_0$ onto $0 \in \sigma(\mathcal{L}_0)$ is defined as

$$P = \frac{1}{2\pi i} \int_{\Gamma} (\mu - \mathcal{L}_0)^{-1} d\mu,$$

where Γ is a contour enclosing a region entirely contained in the resolvent set $\rho(\mathcal{L}_0)$, and whose interior contains no spectral points of $\mathcal{L}_0$ other than zero. This is particularly useful to characterize the spectrum of $\mathcal{L}$ as follows.

Lemma 3.2.2. *Let μ_1, μ_2, μ_3 be the (possibly repeated) eigenvalues $P\mathcal{L}_1P$ on $PL^2(\mathbb{T}^3)$. Then P*

maps onto $\text{Ker}(\mathcal{L}_0)$ and there exists $\delta_1 \in (0, 1)$ such that for all $|\mathbf{j}| < \delta_1$ there exist three eigenvalues of $\mathcal{L}$, denoted $p_\ell(\mathbf{j})$, $\ell = 1, 2, 3$, which are of the form

$$p_\ell(\mathbf{j}) = \mu_\ell |\mathbf{j}| + o(|\mathbf{j}|).$$

Proof. The proof is an application of Theorem A.3.4. Recalling (3.2.5), it is enough to apply Theorem A.3.4 to the operator $\mathcal{L}_0 + |\mathbf{j}|\mathcal{L}_1$, since $\sigma(\mathcal{L}) = \sigma(\mathcal{L}_0 + |\mathbf{j}|\mathcal{L}_1) - |\mathbf{j}|^2$. Note that $\mathcal{L}_0$ has a compact resolvent, i.e. $(\mu - \mathcal{L}_0)^{-1}$ is a compact operator for any $\mu \in \rho(\mathcal{L}_0)$, so that in particular, all its eigenvalues are isolated, and the corresponding generalized eigenspaces are finite dimensional, see Lemma A.3.1. In particular, since $0 \in \sigma(\mathcal{L}_0)$, we only need to check that the eigenvalue 0 is semisimple (which also implies that P maps onto $\text{Ker}(\mathcal{L}_0)$), and that the operator $\mathcal{L}_1$ satisfies $\|\mathcal{L}_1 H\| \lesssim \|\mathcal{L}_0 H\| + \|H\|$.

Step 1: 0 is semisimple. Suppose that $\mathcal{L}_0^2 v = 0$. In view of Lemma 3.2.1, $\mathcal{L}_0 v = w + \mathcal{S}(w)$, where $w \in \mathbb{C}^3$. Since $\mathcal{L}_0 v$ and $\mathcal{S}(w)$ have zero average, this implies that $w = 0$. In turn, $\mathcal{S}(w) = 0$ and $\mathcal{L}_0 v = 0$ as well.

Step 2: Estimate on $\mathcal{L}_1$. By a straightforward computation,

$$-2\langle \nabla \times (U \times H), H \rangle + 2\|\nabla H\|_{L^2}^2 = -2\langle \mathcal{L}_0 H, H \rangle \leq \|\mathcal{L}_0 H\|_{L^2}^2 + \|H\|_{L^2}^2$$

An integration by parts shows that

$$2|\langle \nabla \times (U \times H), H \rangle| \lesssim \|U\|_{L^\infty} \|H\|_{L^2} \|\nabla H\|_{L^2},$$

so that

$$\|\nabla H\|_{L^2}^2 \lesssim \|\mathcal{L}_0 H\|_{L^2}^2 + (1 + \|U\|_{L^\infty}^2) \|H\|_{L^2}^2 \quad (3.2.10)$$

It is clear from (3.2.4) that

$$\|\mathcal{L}_1 H\|^2 \lesssim \|\nabla H\|^2 + \|U\|_{L^\infty}^2 \|H\|^2,$$

and thus the conclusion of the proof follows from combining the above estimate with (3.2.10). $\square$

Thanks to Lemmas 3.2.1-3.2.2, the eigenvalue problem is reduced to a finite dimensional one. The following property is crucial to understand it in detail.

Lemma 3.2.3. *The Riesz projector P and averaging $\langle \cdot \rangle$ satisfy*

$$\langle PH \rangle = \langle H \rangle, \quad PH = \langle H \rangle + \mathcal{S}(\langle H \rangle),$$

for all $H \in L^2(\mathbb{T}^3)$.

Proof. Fix $\mu \in \mathbb{C} \setminus \{0\}$. By definition of the resolvent $R(\mu; \mathcal{L}_0) = (\mu - \mathcal{L}_0)^{-1}$, we have $(\mu - \mathcal{L}_0)R(\mu; \mathcal{L}_0)H = H$, which we can write explicitly as

$$\mu R(\mu; \mathcal{L}_0)H - \Delta(R(\mu; \mathcal{L}_0)H) - \nabla \times (u \times (R(\mu; \mathcal{L}_0)H)) = H.$$

Applying $\langle \cdot \rangle$ to this equation, we obtain

$$\mu \langle R(\mu; \mathcal{L}_0)H \rangle = \langle H \rangle, \quad \mu \in \mathbb{C} \setminus \{0\},$$

so that $\langle R(\mu; \mathcal{L}_0)H \rangle = \mu^{-1} \langle H \rangle$ for any $H \in L^2(\mathbb{T}^3)$. Hence, it remains to apply $\langle \cdot \rangle$ to the definition of P which yields

$$\langle PH \rangle = \frac{1}{2\pi i} \int_{\Gamma} \langle R(\mu; \mathcal{L}_0)H \rangle d\mu = \frac{1}{2\pi i} \int_{\Gamma} \mu^{-1} \langle H \rangle d\mu = \langle H \rangle,$$

as required. Thanks to Lemma 3.2.2, P maps onto $\text{Ker}(\mathcal{L}_0)$ which is precisely those vector fields of the form $v + \mathcal{S}(v)$, for some $v \in \mathbb{C}^3$, thanks to (3.2.7). Since $PH \in \text{Ker}(\mathcal{L}_0)$ and $\langle PH \rangle = \langle H \rangle$, the expression for P follows immediately. $\square$

It remains to characterize the eigenvalues of $P\mathcal{L}_1P$ in a way that gives us the possibility to control their behavior and, in particular, the positivity of their real part.

Lemma 3.2.4. *Fix $\mathbf{j} \in \mathbb{R}^3 \setminus \{0\}$ with $|\mathbf{j}| < \delta_1$ for some $\delta_1 > 0$ sufficiently small. Assume that $U \in W^{1,\infty}(\mathbb{T}^3)$ is zero-average. Then, μ_ℓ is an eigenvalue of $P\mathcal{L}_1P$ on $PL^2(\mathbb{T}^3)$ if and only if there exists some $v_\ell \in \mathbb{C}^3$ satisfying*

$$i \frac{\mathbf{j}}{|\mathbf{j}|} \times \oint_{\mathbb{T}^3} U \times \mathcal{S}(v_\ell) = \mu_\ell v_\ell.$$

Proof. Fix $\mathbf{j} \in \mathbb{R}^3$ with $|\mathbf{j}| < \delta_1$ as in Lemma 3.2.2, and suppose that $P\mathcal{L}_1Pv = \lambda v$ for some $v \in PL^2(\mathbb{T}^3)$. We observe that P maps onto the kernel of $\mathcal{L}_0$ and by Lemma 3.2.1 we have that $v \in \text{Ker}(\mathcal{L}_0)$ can be decomposed as $v = w + \mathcal{S}(w)$ for some $w \in \mathbb{C}^3$ with $\langle \mathcal{S}(w) \rangle = 0$. Then, by Lemma 3.2.3, we may apply $\langle \cdot \rangle$ to both sides of the equation to see

$$\langle \mathcal{L}_1Pv \rangle = \lambda w. \tag{3.2.11}$$

Substituting this into the expression (3.2.11) and noting that the $\frac{\mathbf{j}}{|\mathbf{j}|} \cdot \nabla$ contribution of $\mathcal{L}_1$ maps onto zero average functions, we see that

$$i \frac{\mathbf{j}}{|\mathbf{j}|} \times \oint_{\mathbb{T}^3} U \times (w + \mathcal{S}(w)) = \lambda w.$$

Finally, since U is zero average, we may further reduce this to

$$i \frac{\mathbf{j}}{|\mathbf{j}|} \times \oint_{\mathbb{T}^3} U \times \mathcal{S}(w) = \lambda w,$$

showing the first direction of the implication. To prove the reverse direction, suppose that

$$i \frac{\mathbf{j}}{|\mathbf{j}|} \times \oint_{\mathbb{T}^3} U \times \mathcal{S}(v_\ell) = \mu_\ell v_\ell, \quad \ell = 1, 2, 3.$$

Set $H = v_\ell + \mathcal{S}(v_\ell)$. Applying $\mathcal{L}_1$ and decomposing the result into a mean-free and constant parts

yields

$$\mathcal{L}_1 H = i \frac{\mathbf{j}}{|\mathbf{j}|} \times \langle U \times \mathcal{S}(v_\ell) \rangle + i \frac{\mathbf{j}}{|\mathbf{j}|} \times (U \times \mathcal{S}(v_\ell))_\# + i \frac{\mathbf{j}}{|\mathbf{j}|} \times (U \times v_\ell) + 2 \frac{\mathbf{j}}{|\mathbf{j}|} \cdot \nabla \mathcal{S}(v_\ell).$$

Thus, applying P and using its explicit form we found in Lemma 3.2.3, we deduce that

$$P \mathcal{L}_1 H = \langle \mathcal{L}_1 H \rangle + \mathcal{S}(\langle \mathcal{L}_1 H \rangle) = i \frac{\mathbf{j}}{|\mathbf{j}|} \times \langle U \times \mathcal{S}(v_\ell) \rangle + \mathcal{S} \left(i \frac{\mathbf{j}}{|\mathbf{j}|} \times \langle U \times \mathcal{S}(v_\ell) \rangle \right).$$

Now, $i \frac{\mathbf{j}}{|\mathbf{j}|} \times \langle U \times \mathcal{S}(v_\ell) \rangle = \mu_\ell v_\ell$ by assumption. Therefore, we get

$$P \mathcal{L}_1 H = \mu_\ell (v_\ell + \mathcal{S}(v_\ell)) = \mu_\ell H,$$

which completes the proof. $\square$

3.2.3 Existence of an unstable eigenvalue and alpha-unstable flows

The characterization of eigenvalues given in Lemma 3.2.4 is precisely the motivation behind the instability condition (3.1.4) in Definition 3.1.2. With this definition at hand, the proof of Proposition 3.1.4 readily follows.

Proof of Proposition 3.1.4. By assumption we know that there exist $\mathbf{j}, v \in \mathbb{C}^3$ and $\mu \in \mathbb{C}$ with $\operatorname{Re}(\mu) > 0$, such that (3.1.4) holds. Using Lemma 3.2.2 and Lemma 3.2.4, we have that μ is an eigenvalue of $P \mathcal{L}_1 P$, and therefore $\mathcal{L}$ has an eigenvalue of the form

$$p(\mathbf{j}) = \mu |\mathbf{j}| + o(|\mathbf{j}|).$$

Up to replacing $\mathbf{j}$ by $\beta \mathbf{j}$ for $\beta \in (0, 1)$ small enough, we get that $p(\mathbf{j})$ is such that $\operatorname{Re}(p(\mathbf{j})) > 0$. Notice that the equation for $\mathcal{S}(v)$ (3.2.6) is independent of $\mathbf{j}$.

To conclude, we need to verify the modal divergence condition (3.1.10). This is in fact equivalent to verifying the usual divergence-free condition for $B(x) = e^{ij \cdot x} H(x)$, which solves

$$\nabla \times (U \times B) + \Delta B = pB$$

in a distributional sense. In particular, we may take divergence of the equation, to see that

$$\Delta \nabla \cdot B = p \nabla \cdot B$$

holds in a distributional sense on $\mathbb{R}^3$. Since $\operatorname{Re}(p) > 0$, and $e^{ij \cdot x} H(x)$ is a tempered distribution, we may take the distributional Fourier transform and conclude $\nabla \cdot B = 0$. This concludes the proof. $\square$

Finally, we provide an observation suggesting that the property of having a positive eigenvalue is in some sense generic for the profile U .

Lemma 3.2.5. *Let μ_ℓ be as in Lemma 3.2.2. Then one and only one of the following is true:*

1. For all $\mathbf{j} \in \mathbb{R}^3$ and all $\ell = 1, 2, 3$, it holds $\operatorname{Re}(\mu_\ell) = 0$.
2. There exists a nonzero $\mathbf{j} \in \mathbb{R}^3$ and $\ell = 1, 2, 3$, so that $\operatorname{Re}(\mu_\ell) > 0$.

Proof. This is a straightforward result. Indeed, suppose that there exists some μ_ℓ with nonzero real part, i.e. there exists a nonzero $\mathbf{j}$ in $\mathbb{R}^3$, and a vector $v \in \mathbb{C}^3$ so that

$$\oint_{\mathbb{T}^3} i \frac{\mathbf{j}}{|\mathbf{j}|} \times (U \times \mathcal{S}(v)) = \mu_\ell v,$$

If $\operatorname{Re}(\mu_\ell) > 0$ we are done. Otherwise, simply replace $\mathbf{j} \mapsto -\mathbf{j}$, which completes the proof. $\square$

As we shall see in Proposition 3.2.7, alpha-instability is a *generic* property of smooth velocity fields. Heuristic evidence supporting this perspective already appears in [Rob70; Rob72; Chi70]. Before turning to this result, we show that the classical ABC flow provides an explicit example of an alpha-unstable velocity field.

Example 3.2.6 (ABC flows). The ABC flows (named after Arnold, Beltrami, and Childress) are a family of steady, three-dimensional velocity fields that solve the 3d Euler equations. Given amplitude parameters $a, b, c \in \mathbb{R}$, the corresponding velocity field is given by

$$U_{\text{abc}}(x_1, x_2, x_3) = \begin{pmatrix} a \sin(x_3) + c \cos(x_2) \\ b \sin(x_1) + a \cos(x_3) \\ c \sin(x_2) + b \cos(x_1) \end{pmatrix}. \quad (3.2.12)$$

Since

$$\|U_{\text{abc}}\|_{W^{1,\infty}(\mathbb{T}^3)} = \max\{|a| + |c|, |a| + |b|, |b| + |c|\},$$

we consider a small positive parameter $\delta_0 \ll 1$ to be able to apply Lemma 3.2.1 for $U = \delta_0 U_{\text{abc}}$. In particular, the solution operator of (3.2.6) can be written as in (3.2.9) as

$$\mathcal{S}(v) = \Delta^{-1} \sum_{n \geq 0} (-1)^n (\nabla \times (\delta_0 U_{\text{abc}} \times \Delta^{-1}))^n G_{\delta_0}(v), \quad (3.2.13)$$

where $G_{\delta_0}(v) = -\nabla \times (\delta_0 U_{\text{abc}} \times v)$. Inserting this into the expression (3.1.4) for the alpha-instability, we see that we need to compute

$$\oint_{\mathbb{T}^3} i \frac{\mathbf{j}}{|\mathbf{j}|} \times (\delta_0 U_{\text{abc}} \times \mathcal{S}(v)) = \sum_{n \geq 0} (-1)^n i \frac{\mathbf{j}}{|\mathbf{j}|} \times \oint_{\mathbb{T}^3} \delta_0 U_{\text{abc}} \times \Delta^{-1} (\nabla \times (\delta_0 U_{\text{abc}} \times \Delta^{-1}))^n G_{\delta_0}(v). \quad (3.2.14)$$

In particular, at lowest order in δ_0 , i.e. the $n = 0$ term above, this is

$$i \delta_0^2 \frac{\mathbf{j}}{|\mathbf{j}|} \times \mathcal{S}(v), \quad \mathcal{S}(v) := - \oint_{\mathbb{T}^3} U_{\text{abc}} \times \Delta^{-1} (\nabla \times (U_{\text{abc}} \times v)). \quad (3.2.15)$$

For sufficiently small δ_0 we can apply the perturbative arguments from Lemma A.3.3 to show that if (3.2.15) has a simple eigenvalue with a positive real part, then the full matrix (3.2.14) will as well.

Using the fact that U_{abc} is divergence-free and v is constant, we have

$$\mathcal{J}(v) = - \oint_{\mathbb{T}^3} U_{\text{abc}} \times \Delta^{-1}((v \cdot i\nabla)U_{\text{abc}}) = \sum_{\mathbf{k} \in \mathbb{Z}^3, |\mathbf{k}|=1} \widehat{U}_{\text{abc}}(\mathbf{k}) \times \left((v \cdot \mathbf{k}) \widehat{U}_{\text{abc}}(-\mathbf{k}) \right),$$

where we used that $\widehat{U}_{\text{abc}}$ is concentrated in the modes with $|\mathbf{k}| = 1$. Indeed, we can compute explicitly

$$\widehat{U}_{\text{abc}}(\mathbf{k}) = \frac{a}{2} \begin{pmatrix} \pm i \\ 1 \\ 0 \end{pmatrix} \mathbb{1}_{(0,0,\pm 1)}(\mathbf{k}) + \frac{b}{2} \begin{pmatrix} 0 \\ \pm i \\ 1 \end{pmatrix} \mathbb{1}_{(\pm 1,0,0)}(\mathbf{k}) + \frac{c}{2} \begin{pmatrix} 1 \\ 0 \\ \pm i \end{pmatrix} \mathbb{1}_{(0,\pm 1,0)}(\mathbf{k}).$$

From this, it follows that

$$\widehat{U}_{\text{abc}}(\mathbf{k}) \times \widehat{U}_{\text{abc}}(-\mathbf{k}) = \pm i \frac{a^2}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathbb{1}_{(0,0,\pm 1)}(\mathbf{k}) \pm i \frac{b^2}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbb{1}_{(\pm 1,0,0)}(\mathbf{k}) \pm i \frac{c^2}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mathbb{1}_{(0,\pm 1,0)}(\mathbf{k}),$$

and thus a computation shows

$$\mathcal{J}(v) = Jv, \quad J := \begin{pmatrix} b^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}, \quad v \in \mathbb{C}^3.$$

In turn,

$$i \frac{\mathbf{j}}{|\mathbf{j}|} \times \mathcal{J}(v) = Lv, \quad L := \frac{i}{|\mathbf{j}|} \begin{pmatrix} 0 & -c^2 j_3 & a^2 j_2 \\ b^2 j_3 & 0 & -a^2 j_1 \\ -b^2 j_2 & c^2 j_1 & 0 \end{pmatrix}, \quad \mathbf{j} = (j_1, j_2, j_3) \in \mathbb{R}^3.$$

In this case, the alpha-instability condition (3.1.4) is equivalent to finding a nonzero $\mathbf{j} \in \mathbb{R}^3$ such that (recall (3.2.15)) the matrix L above has an eigenvalue with positive real part. The three eigenvalues of L can be explicitly computed as

$$\mu_0 = 0, \quad \mu_{\pm} = \pm \frac{1}{|\mathbf{j}|} \sqrt{a^2 b^2 j_2^2 + b^2 c^2 j_3^2 + a^2 c^2 j_1^2}.$$

In particular, as long as any two of a, b, c are nonzero, there exists a nonzero $\mathbf{j} \in \mathbb{R}^3$ so that the δ_0 -multiple of the ABC flow is alpha-unstable.

Let $\dot{H}_{\text{div}}^{\infty}$ the space of divergence-free H^{∞} vector fields with zero average. The topology on $\dot{H}_{\text{div}}^{\infty}$ is induced by the countable family of seminorms $\|\cdot\|_{H^n}$, for $n \in \mathbb{N}$. Consequently, $\dot{H}_{\text{div}}^{\infty}$ is metrizable, with a metric given, for instance, by

$$d_{H^{\infty}}(U, V) = \sum_{n \geq 0} 2^{-n} \frac{\|U - V\|_{H^n}}{1 + \|U - V\|_{H^n}}.$$

With this example in hand, we can establish that this property holds for a dense set of velocity

profiles U . More precisely, for any $U \in \dot{H}_{\text{div}}^\infty$, we can use the matrix A defined in (3.1.4) and define

$$\Omega := \{U \in \dot{H}_{\text{div}}^\infty : \exists \delta_0 > 0, \mathbf{j} \in \mathbb{R}^3 \text{ s.t. } A(\delta_0 U, \mathbf{j}) \text{ has a simple eigenvalue } p \text{ with } \text{Re}(p) > 0\}.$$

Proposition 3.2.7. *The set Ω is open and dense in $\dot{H}_{\text{div}}^\infty$.*

Remark 3.2.8. The small parameter $\delta_0 > 0$ primarily serves to simplify computations by enabling the justification of the series expansion (3.2.13). This, in turn, provides a first-order approximation of the operator $S(v)$ making its analysis more tractable.

Proof of Proposition 3.2.7. This proof proceeds along much the same lines as that of Lemma 3.2. in [Gér05]. To establish the density claim, we begin by noting that, as in (3.2.9), for any $U \in \dot{H}_{\text{div}}^\infty$ and sufficiently small $\delta_0 \ll 1$, we have

$$\oint_{\mathbb{T}^3} (\delta_0 U \times S(v)) = \sum_{n \geq 0} (-1)^n \oint_{\mathbb{T}^3} \delta_0 U \times \Delta^{-1} (\nabla \times (\delta_0 U \times \Delta^{-1}))^n G_{\delta_0}(v)$$

with $G_{\delta_0}(v) = -\nabla \times (\delta_0 U \times v)$. In particular, we may write

$$\oint_{\mathbb{T}^3} (\delta_0 U \times S(v)) = \delta_0^2 \left[-\oint_{\mathbb{T}^3} U \times \Delta^{-1} \nabla \times (U \times v) + B_{\delta_0}(v) \right] = \delta_0^2 [\mathcal{I}_U(v) + B_{\delta_0}(v)]$$

where $\|B_{\delta_0}\| = O(\delta_0)$, and

$$\mathcal{I}_U(v) := -\oint_{\mathbb{T}^3} U \times \Delta^{-1} (\nabla \times (U \times v)). \quad (3.2.16)$$

Taking the cross product with $\mathbf{j} \in \mathbb{S}^2$, Lemma A.3.3 implies that if $i\mathbf{j} \times \mathcal{I}_U(v)$ has a simple eigenvalue with positive real part, then for sufficiently small δ_0 the same holds for $A(\delta_0 U, \mathbf{j})$. Thus, to establish density, it suffices to show that the set

$$\Omega_0 := \{U \in \dot{H}_{\text{div}}^\infty : i\mathbf{j} \times \mathcal{I}_U \text{ has a simple eigenvalue with positive real part for some } \mathbf{j} \in \mathbb{S}^2\}$$

is dense in $\dot{H}_{\text{div}}^\infty$.

However, we can further reduce the complexity of this problem. Specifically, for any real velocity field U , the matrix $\mathcal{I}_U$ is self-adjoint. To see this, note that since $\mathcal{I}_U$ is real-valued, it suffices to show that it is symmetric. Given $v, w \in \mathbb{R}^3$, we use the anti-self-adjoint property of the cross product and the self-adjointness of the curl operator in L^2 to obtain

$$\langle v, \mathcal{I}_U(w) \rangle = \oint_{\mathbb{T}^3} \langle U \times v, \nabla \times (\Delta^{-1} U \times w) \rangle = -\oint_{\mathbb{T}^3} \langle U \times (\Delta^{-1} \nabla \times (U \times v)), w \rangle = \langle \mathcal{I}_U(v), w \rangle.$$

Thus, $\mathcal{I}_U$ is symmetric, allowing us to write it in the form $\mathcal{I}_U = O^\top D O$, where O is an orthogonal matrix and D is a diagonal matrix given by $D = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$, with its diagonal entries representing the real eigenvalues of $\mathcal{I}_U$.

We now analyze the matrix $i\mathbf{j} \times \mathcal{J}_U = i\mathbf{j} \times O^\top DO$. Using the identity

$$Mv \times Mw = \det(M)M^{-T}(v \times w)$$

for any matrix M and viewing $O\mathbf{j} \times D$ as a linear map², we obtain

$$i\mathbf{j} \times (O^\top DOv) = O^\top (iO\mathbf{j} \times DOv) = O^\top (iO\mathbf{j} \times D)Ov,$$

Thus, it suffices to study the eigenvalues of $i(O\mathbf{j}) \times D$. Finally, since O is an isometry of $\mathbb{R}^3$, it maps $\mathbb{S}^2$ onto itself, allowing us to replace $O\mathbf{j}$ with any $\mathbf{j} \in \mathbb{S}^2$. This flexibility means we retain full freedom in choosing $\mathbf{j}$ in the Definition 3.1.2 of alpha-instability. Hence, it remains to study the spectrum of

$$i\mathbf{j} \times D = i \begin{pmatrix} 0 & -\alpha_2 j_3 & \alpha_3 j_2 \\ \alpha_1 j_3 & 0 & -\alpha_3 j_1 \\ -\alpha_1 j_2 & \alpha_2 j_1 & 0 \end{pmatrix}.$$

A simple computation shows that the eigenvalues are given by

$$\mu_0 = 0, \quad \mu_{\pm} = \pm \sqrt{\alpha_1 \alpha_3 j_2^2 + \alpha_1 \alpha_2 j_3^2 + \alpha_2 \alpha_3 j_1^2}.$$

As long as all of $\alpha_1, \alpha_2, \alpha_3$ are non-zero, at least one product - without loss of generality we can take $\alpha_1 \alpha_3$ - must be strictly positive. Setting $\mathbf{j} = (1, 0, 0)$, we obtain a simple eigenvalue with a positive real part.

Thus, it suffices to prove that the set

$$\Omega_1 := \{U \in \dot{H}_{\text{div}}^\infty : \text{Ker}(\mathcal{J}_U) = \{0\}\}$$

is dense in $\dot{H}_{\text{div}}^\infty$. Fix $\varepsilon > 0$, and let $U_N = P_{|\mathbf{k}| \leq N} U$, where $P_{|\mathbf{k}| \leq N}$ is the Fourier projection onto $|\mathbf{k}| \leq N$ modes, and $N = N(\varepsilon)$ is big enough such that

$$\text{d}_{H^\infty}(U_N, U) < \frac{\varepsilon}{2}.$$

If $\mathcal{J}_{U_N}$ has a trivial kernel, the proof is complete. Otherwise, for $\tilde{\varepsilon} \in (0, 1)$, we consider

$$\tilde{U}_N = U_N + \tilde{\varepsilon} U_1^{N+1}$$

where U_1 is the ABC flow in (3.2.12) with $a = b = c = 1$ and

$$U_1^M(\cdot) = U_1(M\cdot), \quad \forall M \in \mathbb{N}.$$

For sufficiently small $\tilde{\varepsilon}$, it follows that

$$\text{d}_{H^\infty}(\tilde{U}_N, U_N) < \frac{\varepsilon}{2},$$

²Denoting by D_n the n^{th} column of the matrix D , the matrix $O\mathbf{j} \times D$ can be explicitly computed to be $(O\mathbf{j} \times D_1, O\mathbf{j} \times D_2, O\mathbf{j} \times D_3)$.

so it remains to show that $\tilde{U}_N$ has a simple eigenvalue with a positive real part. To do this, we claim from (3.2.16) that

$$\mathcal{J}_{\tilde{U}_N} = \mathcal{J}_{U_N} + \tilde{\varepsilon}^2 \mathcal{J}_{U_1^{N+1}}.$$

Indeed, we have the following expression:

$$\begin{aligned} \mathcal{J}_{\tilde{U}_N}(v) &= \mathcal{J}_{U_N}(v) + \tilde{\varepsilon}^2 \mathcal{J}_{U_1^{N+1}}(v) \\ &\quad - \tilde{\varepsilon} \left[\oint_{\mathbb{T}^3} U_N \times \Delta^{-1} \nabla \times (U_1^{N+1} \times v) + \oint_{\mathbb{T}^3} U_1^{N+1} \times \Delta^{-1} \nabla \times (U_N \times v) \right]. \end{aligned} \quad (3.2.17)$$

In Fourier space, we compute

$$\oint_{\mathbb{T}^3} U_N \times \Delta^{-1} \nabla \times (U_1^{N+1} \times v) = - \sum_{\mathbf{k} \in \mathbb{Z}^3} \hat{U}_N(\mathbf{k}) \times i\mathbf{k} \times \left(|\mathbf{k}|^{-2} \hat{U}_1^{N+1}(\mathbf{k}) \times v \right).$$

Now, observe that U_N is supported on Fourier modes with $|\mathbf{k}| \leq N$, whereas U_1^{N+1} is supported solely on Fourier modes with $|\mathbf{k}| = N+1$. Thus, the sum vanishes. The second term in (3.2.17) is handled in an identical manner. Finally, since $\mathcal{J}_{U_1^{N+1}} = \text{Id}$, it follows that we can take $\tilde{\varepsilon}$ sufficiently small so that $\mathcal{J}_{\tilde{U}_N}$ has trivial kernel, thus completing the proof of density. Next, we prove the openness. By a direct computation similar to (3.2.17), we know that if $U \in \Omega$ and $\tilde{U} \in H^\infty$, then

$$\mathcal{J}_{U+\tilde{U}}(v) - \mathcal{J}_U(v) = \mathcal{J}_{\tilde{U}}(v) - \oint_{\mathbb{T}^3} U \times \Delta^{-1} \nabla \times (\tilde{U} \times v) - \oint_{\mathbb{T}^3} \tilde{U} \times \Delta^{-1} \nabla \times (U \times v).$$

By applying Cauchy-Schwarz, we obtain

$$|\mathcal{J}_{U+\tilde{U}}(v) - \mathcal{J}_U(v)| \leq \|U\|_{L^2} \|\tilde{U}\|_{L^2} |v|,$$

and similarly for the other terms. Therefore, as long as $d_{H^\infty}(U, U + \tilde{U}) < \varepsilon$ we have

$$\|\mathcal{J}_{U+\tilde{U}} - \mathcal{J}_U\| \lesssim (1 + \|U\|_{L^2}) \varepsilon.$$

Thus, by Lemma A.3.3, for sufficiently small ε depending only on $\|U\|_{L^2}$ and the spectrum of $\mathcal{J}_U$, the matrix $\mathcal{J}_{U+\tilde{U}}$ still has an empty kernel, so that $i\mathbf{j} \times A(\delta_0 U, \mathbf{j})$ has a simple eigenvalue with positive real part for some $\mathbf{j} \in \mathbb{S}^2$, $\delta_0 > 0$, completing the proof of openness. $\square$

3.2.4 Continuity with respect to diffusivity

We now undertake a more detailed analysis of the spectral properties of the passive vector operator. Thus far, we have shown that alpha-instability guarantees the existence of an unstable eigenvalue for the operator $\mathcal{L}$ when $|\mathbf{j}|$ is sufficiently small and the diffusivity parameter is fixed. However, it is essential to establish that this unstable eigenvalue persists under small perturbations of both the mode $\mathbf{j}$ and the diffusivity parameter ε .

Since these perturbations are relatively bounded with respect to $\mathcal{L}$, we can invoke the classical perturbation theory of Kato [Kat76], as outlined in Appendix A. With this in mind, we first derive an elementary resolvent bound estimate for $\mathcal{L}$. Given that our primary focus is on the continuity

of $\mathcal{L}$ with respect to $\mathbf{j}$ and ε , we will make this dependence explicit in the notation and write

$$\mathcal{L}(\mathbf{j}, \varepsilon)H = (\nabla + i\mathbf{j}) \times (U \times H) + \varepsilon(\nabla + i\mathbf{j})^2 H.$$

We now prove that $\mathcal{L}(\mathbf{j}, \varepsilon)$ is Lipschitz continuous in $\mathbf{j}, \varepsilon$ upon composing on the right with its resolvent.

Lemma 3.2.9. *Let $U \in C^\infty(\mathbb{T}^3)$ be a divergence-free velocity field with zero average, and let $\varepsilon > 0$, $\mathbf{j} \in \mathbb{R}^3$. For any $\mu \in \rho(\mathcal{L}(\mathbf{j}, \varepsilon))$, the following bound holds:*

$$\|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mathcal{L}(\mathbf{j}', \varepsilon'))(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\|_{L^2 \rightarrow L^2} \leq C(|\mathbf{j} - \mathbf{j}'| + |\varepsilon - \varepsilon'|) (\|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\|_{L^2 \rightarrow L^2} + 1) \quad (3.2.18)$$

for any $\mathbf{j}, \mathbf{j}', \varepsilon, \varepsilon'$, where the constant $C > 0$ depends continuously on $\mathbf{j}, \mathbf{j}', \varepsilon, \varepsilon', \mu$. In particular, $(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}$ is jointly continuous in its arguments, so that for any $\mathbf{j}, \varepsilon, \mu$ with $\mu \in \rho(\mathcal{L}(\mathbf{j}, \varepsilon))$, there holds

$$\|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mathcal{L}(\mathbf{j}', \varepsilon'))(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\|_{L^2 \rightarrow L^2} \leq C(|\mathbf{j} - \mathbf{j}'| + |\varepsilon - \varepsilon'|) \quad (3.2.19)$$

where now the constant $C > 0$ depends continuously on $\mathbf{j}, \mathbf{j}', \varepsilon, \varepsilon', \mu$ with $\mu \in \rho(\mathcal{L}(\mathbf{j}, \varepsilon))$.

Remark 3.2.10. The constant $C > 0$ in Lemma 3.2.9 may diverge as $\varepsilon, \varepsilon' \rightarrow 0$. However, in our analysis, we will only apply the lemma for the case $\varepsilon = 1$ and $\varepsilon' \rightarrow 1$.

Proof of Lemma 3.2.9. A computation shows that $(\mathcal{L}(\mathbf{j}, \varepsilon) - \mathcal{L}(\mathbf{j}', \varepsilon'))H$ is equal to

$$i(\mathbf{j} - \mathbf{j}') \times (U \times H) + \varepsilon(\Delta H + 2i\mathbf{j} \cdot \nabla H - |\mathbf{j}|^2 H) - \varepsilon'(\Delta H + 2i\mathbf{j}' \cdot \nabla H - |\mathbf{j}'|^2 H)$$

and so, bounding this expression we get

$$\|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mathcal{L}(\mathbf{j}', \varepsilon'))H\|_{L^2} \lesssim [|\mathbf{j} - \mathbf{j}'| (1 + |\mathbf{j}'| + |\mathbf{j}|) + |\varepsilon - \varepsilon'| (1 + |\mathbf{j}|) + \varepsilon' |\mathbf{j}' - \mathbf{j}|] (\|H\|_{L^2} + \|\Delta H\|_{L^2}).$$

If we set $h = (\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)H$, estimate (3.2.18) is proven once we prove a bound on ΔH . Rearranging the equation for $\mathcal{L}(\mathbf{j}, \varepsilon)$, we see that for any H there holds

$$\varepsilon \Delta H = (\mathcal{L}(\varepsilon, \mathbf{j}) - \mu)H - \varepsilon(2i\mathbf{j} \cdot \nabla - |\mathbf{j}|^2)H + \mu H - (\nabla + i\mathbf{j}) \times (U \times H).$$

Therefore, using the definition of the resolvent and an interpolation inequality, there holds

$$\begin{aligned} \|\Delta(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}h\|_{L^2} &\leq C(\varepsilon, \mathbf{j}) (\|h\|_{L^2} + \|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}h\|_{L^2} + \|\nabla(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}h\|_{L^2}) \\ &\leq C(\varepsilon, \mathbf{j}) (\|h\|_{L^2} + \|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}h\|_{L^2}) + \frac{1}{2}\|\Delta(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}h\|_{L^2}. \end{aligned}$$

Rearranging, this proves the first claim of the theorem. It thus remains to prove (3.2.19). To do so, it suffices to prove that the term $\|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\|_{L^2 \rightarrow L^2}$ is jointly continuous in $\mathbf{j}, \varepsilon, \mu$. To do so,

note that

$$\begin{aligned}\mathcal{L}(\mathbf{j}', \varepsilon') - \mu' &= \mathcal{L}(\mathbf{j}, \varepsilon) - \mu + [\mathcal{L}(\mathbf{j}', \varepsilon') - \mathcal{L}(\mathbf{j}, \varepsilon) + (\mu - \mu')] \\ &= \{1 + [\mathcal{L}(\mathbf{j}', \varepsilon') - \mathcal{L}(\mathbf{j}, \varepsilon) + (\mu - \mu')](\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\} (\mathcal{L}(\mathbf{j}, \varepsilon) - \mu).\end{aligned}$$

We now recall the bound (3.2.18) and defining

$$M = \|[\mathcal{L}(\mathbf{j}', \varepsilon') - \mathcal{L}(\mathbf{j}, \varepsilon) + (\mu - \mu')](\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\|_{L^2 \rightarrow L^2},$$

it holds that

$$M \leq C(|\mathbf{j} - \mathbf{j}'| + |\varepsilon - \varepsilon'| + |\mu - \mu'|)(\|\mathcal{L}(\mathbf{j}, \varepsilon) - \mu\|_{L^2 \rightarrow L^2}^{-1} + 1).$$

In particular, as $(\mathbf{j}', \varepsilon', \mu') \rightarrow (\mathbf{j}, \varepsilon, \mu)$, it follows that $M \rightarrow 0$. Therefore, as long as $M < 1$, we can use a Neumann series expansion to write

$$\begin{aligned}(\mathcal{L}(\mathbf{j}', \varepsilon') - \mu')^{-1} - (\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1} &= \\ (\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1} \sum_{n \geq 1} (-1)^n \{[\mathcal{L}(\mathbf{j}', \varepsilon') - \mathcal{L}(\mathbf{j}, \varepsilon) + (\mu - \mu')](\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\}^n.\end{aligned}$$

Taking norms of this equation, we see that

$$\|(\mathcal{L}(\mathbf{j}', \varepsilon') - \mu')^{-1} - (\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\|_{L^2 \rightarrow L^2} \leq \|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\|_{L^2 \rightarrow L^2} \frac{M}{1 - M}.$$

Therefore, using that $M \rightarrow 0$ as $(\mathbf{j}', \varepsilon', \mu') \rightarrow (\mathbf{j}, \varepsilon, \mu)$, we deduce the joint continuity of $(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}$ and we conclude the proof. $\square$

We now prove Lipschitz continuity of the eigenfunction H with respect to ε , as well as a uniform lower bound on the real part of the eigenvalue with respect to ε and $\mathbf{j}$.

Lemma 3.2.11. *Let $U \in C^\infty(\mathbb{T}^3)$ be a smooth divergence free, zero average velocity field and $\varepsilon \in (0, 1]$. Assume that there exists $\mathbf{j}_*$, a function $H_{\mathbf{j}_*}$ and a simple eigenvalue $p(\mathbf{j}_*)$ with strictly positive real part so that*

$$\mathcal{L}(\mathbf{j}_*, 1)H_{\mathbf{j}_*} = p(\mathbf{j}_*)H_{\mathbf{j}_*}.$$

Then, there exists a curve $\Gamma \subset \{\operatorname{Re}(z) > 0\}$, and $J, \eta > 0$, so that for all $|\mathbf{j} - \mathbf{j}_*| \leq J$, $\varepsilon \in [1 - \eta, 1]$ we have that

$$H(x; \mathbf{j}, \varepsilon) = \frac{1}{2\pi i} \int_{\Gamma} (\mu - \mathcal{L}(\mathbf{j}, \varepsilon))^{-1} H_{\mathbf{j}_*} d\mu$$

is an eigenfunction of $\mathcal{L}(\mathbf{j}, \varepsilon)$ with simple eigenvalue $p(\mathbf{j}, \varepsilon)$ satisfying

$$\operatorname{Re}(p(\mathbf{j}, \varepsilon)) \geq \frac{1}{2} \operatorname{Re}(p(\mathbf{j}_*)).$$

Furthermore, the function

$$\varepsilon \mapsto H(x; \mathbf{j}, \varepsilon) = \frac{1}{2\pi i} \int_{\Gamma} (\mu - \mathcal{L}(\mathbf{j}, \varepsilon))^{-1} H_{\mathbf{j}_*} d\mu$$

satisfies

$$\|H(x; \mathbf{j}, \varepsilon') - H(x; \mathbf{j}, \varepsilon)\|_{L^2(\mathbb{T}^3)} \leq C|\varepsilon' - \varepsilon|$$

for any $\varepsilon, \varepsilon' \in [1 - \eta, 1]$ and $|\mathbf{j} - \mathbf{j}_*| \leq J$, where the constant C is independent of $\mathbf{j}, \varepsilon, \varepsilon'$.

Proof of Lemma 3.2.11. We aim to apply Lemma A.3.3 to the operators $T_0 = \mathcal{L}(\mathbf{j}_*, 1)$, $T = \mathcal{L}(\mathbf{j}, \varepsilon)$, using the bounds from Lemma 3.2.9. Indeed, start by fixing some curve $\Gamma \subset \rho(\mathcal{L}(\mathbf{j}_*, 1))$ that is entirely contained in the half plane $\{z \in \mathbb{C} : \operatorname{Re}(z) > \frac{1}{2}\operatorname{Re}(p(\mathbf{j}_*))\}$, and whose interior contains no spectral points of $\mathcal{L}(\mathbf{j}_*, 1)$ other than $p(\mathbf{j}_*)$. By property (3.2.18) of Lemma 3.2.9 we find that for any $\mu \in \Gamma$, there holds that

$$\|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mathcal{L}(\mathbf{j}_*, 1))(\mathcal{L}(\mathbf{j}_*, 1) - \mu)^{-1}\|_{L^2 \rightarrow L^2} \leq C(|\mathbf{j} - \mathbf{j}_*| + |\varepsilon - 1|)(1 + \|(\mathcal{L}(\mathbf{j}_*, 1) - \mu)^{-1}\|_{L^2 \rightarrow L^2}),$$

where the constant C depends continuously on $\mathbf{j}, \varepsilon, \mu$. Since the resolvent is a continuous function in μ , $\|(\mathcal{L}(\mathbf{j}_*, 1) - \mu)^{-1}\|_{L^2 \rightarrow L^2}$ attains a maximum on the compact set Γ . Hence, there holds uniformly for $\mu \in \Gamma$ that

$$\|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mathcal{L}(\mathbf{j}_*, 1))(\mathcal{L}(\mathbf{j}_*, 1) - \mu)^{-1}\|_{L^2 \rightarrow L^2} \leq C(|\mathbf{j} - \mathbf{j}_*| + |\varepsilon - 1|),$$

where C is continuous in $\mathbf{j}, \varepsilon$. We can thus pick $J > 0, \eta > 0$ so that for any $|\mathbf{j} - \mathbf{j}_*| \leq J, |\varepsilon - 1| \leq \eta$ there holds

$$M := \sup_{\mu \in \Gamma} \|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mathcal{L}(\mathbf{j}_*, 1))(\mathcal{L}(\mathbf{j}_*, 1) - \mu)^{-1}\|_{L^2 \rightarrow L^2} < \frac{1}{1 + |\Gamma| \sup_{\mu \in \Gamma} \|(\mathcal{L}(\mathbf{j}_*, 1) - \mu)^{-1}\|_{L^2 \rightarrow L^2}}, \quad (3.2.20)$$

so that $\Gamma \subset \rho(\mathcal{L}(\mathbf{j}, \varepsilon))$ for any $|\mathbf{j} - \mathbf{j}_*| \leq J, |\varepsilon - 1| \leq \eta$. Furthermore, by Lemma A.3.3, this bound implies that the operator $\mathcal{L}(\mathbf{j}, \varepsilon)$ has at least one eigenfunction with eigenvalue $p(\mathbf{j}, \varepsilon)$ contained in the interior region confined by Γ . In particular, if the eigenvalue $p(\mathbf{j}_*, 1)$ is simple, then the eigenspace corresponding to $p(\mathbf{j}, \varepsilon)$ is also simple.

Next, we shall show the continuity of the map $\varepsilon \mapsto H(x; \mathbf{j}, \varepsilon)$. Indeed, we once again apply Lemma A.3.3 with $T_0 = \mathcal{L}(\mathbf{j}, \varepsilon)$, $T = \mathcal{L}(\mathbf{j}, \varepsilon')$. By Lemma 3.2.9, there exists a constant $C(\mathbf{j}, \varepsilon, \varepsilon', \mu)$ depending continuously on its arguments (as long as $\mu \in \rho(\mathcal{L}(\mathbf{j}, \varepsilon))$) so that it holds

$$\|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mathcal{L}(\mathbf{j}, \varepsilon'))(\mu - \mathcal{L}(\mathbf{j}, \varepsilon))^{-1}\|_{L^2 \rightarrow L^2} \leq C(\mathbf{j}, \varepsilon, \varepsilon', \mu)(|\varepsilon - \varepsilon'|).$$

But note that by (3.2.20), for any $|\mathbf{j} - \mathbf{j}_*| \leq 1, |\varepsilon - 1| \leq \eta$ it holds that $\Gamma \subset \rho(\mathcal{L}(\mathbf{j}, \varepsilon))$. Therefore, $C(\mathbf{j}, \varepsilon, \varepsilon', \mu)$ is a continuous function on a compact set, and thus attains a maximum on it. Therefore, by the estimate (A.3.3), it holds true for any $|\mathbf{j} - \mathbf{j}_*| \leq J, |\varepsilon - 1| \leq \eta, |\varepsilon' - 1| \leq \eta$ that

$$\|H(\cdot; \mathbf{j}, \varepsilon) - H(\cdot; \mathbf{j}, \varepsilon')\|_{L^2(\mathbb{T}^3)} \leq C \sup_{\mu \in \Gamma} \|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\|_{L^2 \rightarrow L^2} |\varepsilon - \varepsilon'| \|H_{\mathbf{j}_*}\|_{L^2(\mathbb{T}^3)}.$$

By Lemma 3.2.9 we know that $\|(\mathcal{L}(\mathbf{j}, \varepsilon) - \mu)^{-1}\|_{L^2 \rightarrow L^2}$ is also jointly continuous in its arguments, and therefore we may bound it uniformly for $(\mathbf{j}, \varepsilon, \mu) \in [\mathbf{j}_* - J, \mathbf{j}_* + J] \times [1 - \eta, 1 + \eta] \times \Gamma$, which completes the proof. $\square$

3.2.5 A Bloch theorem and extension to the whole space

Under the assumptions of Proposition 3.1.4, the function $B(t, x) = H(x)e^{ij \cdot x + pt}$ is an exponentially growing plane wave modulated by a 2π -periodic function H , which solves the passive vector equation (KDE- ε) with $\varepsilon = 1$ and $u = U$. In general, B is not itself 2π -periodic, and therefore the rescaling in (3.1.8) does not provide an example of fast dynamo on $\mathbb{T}^3$. However, we can use Proposition 3.1.4 to construct a proper, finite-energy solution to (KDE- ε) on $\mathbb{R}^3$, in the spirit of the analysis in [Rob70]. This is in the spirit of a Bloch-type result in quantum physics [TZ23, Section 5, Theorem 4]. The result reads as follows.

Lemma 3.2.12. *Let $\mathbb{J} = \mathbb{T}^3$, $G \in L^2(\mathbb{T}^3 \times \mathbb{J})$ and define*

$$F(x) = \int_{\mathbb{J}} G(x; \mathbf{j}) e^{ij \cdot x} d\mathbf{j}.$$

Then $F \in L^2(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} |F(x)|^2 dx = \int_{\mathbb{J}} \int_{\mathbb{T}^3} |G(x; \mathbf{j})|^2 dx d\mathbf{j}.$$

Proof. For any $\mathbf{k} \in 2\pi\mathbb{Z}^3$ and $R \in \{k : \exists n \in \mathbb{N} \text{ such that } k = (2n+1)\pi\}$ we denote by

$$|\mathbf{k}| = \sup_{i=1,2,3} |\mathbf{k}_i|, \quad Q_R = \{\mathbf{k} \in \mathbb{R}^3 : |\mathbf{k}| \leq R\},$$

and $\mathbb{Z}_R = Q_R \cap 2\pi\mathbb{Z}^3$. Then

$$\int_{Q_R} |F(x)|^2 dx = \sum_{\mathbf{k} \in \mathbb{Z}_R} \int_{\mathbf{k} + \mathbb{T}^3} \left| \int_{\mathbb{J}} G(x; \mathbf{j}) e^{ix \cdot \mathbf{j}} d\mathbf{j} \right|^2 dx.$$

We now note that

$$\int_{\mathbf{k} + \mathbb{T}^3} \left| \int_{\mathbb{J}} G(x; \mathbf{j}) e^{ix \cdot \mathbf{j}} d\mathbf{j} \right|^2 dx = \int_{\mathbf{k} + \mathbb{T}^3} \int_{\mathbb{J}} G(x; \mathbf{j}) e^{ix \cdot \mathbf{j}} d\mathbf{j} \cdot \int_{\mathbb{J}} \overline{G(x; \mathbf{j}')} e^{-ix \cdot \mathbf{j}'} d\mathbf{j}' dx.$$

Furthermore, we can change variables, $x = y + \mathbf{k}$, and note that G is 2π -periodic, to get that this expression is equal to

$$\int_{\mathbf{k} + \mathbb{T}^3} \left| \int_{\mathbb{J}} G(x; \mathbf{j}) e^{ix \cdot \mathbf{j}} d\mathbf{j} \right|^2 dx = \int_{\mathbb{T}^3} \int_{\mathbb{J}} G(x; \mathbf{j}) e^{ix \cdot \mathbf{j}} d\mathbf{j} \cdot \int_{\mathbb{J}} \overline{G(x; \mathbf{j}')} e^{-ix \cdot \mathbf{j}'} e^{i\mathbf{k} \cdot (\mathbf{j} - \mathbf{j}')} d\mathbf{j}' d\mathbf{j} dx$$

Hence, reintroducing the summation in $\mathbf{k}$, we get

$$\int_{Q_R} |F(x)|^2 dx = \int_{\mathbb{T}^3} \int_{\mathbb{J}} G(x; \mathbf{j}) e^{ix \cdot \mathbf{j}} d\mathbf{j} \cdot \int_{\mathbb{J}} \overline{G(x; \mathbf{j}')} e^{-ix \cdot \mathbf{j}'} \sum_{\mathbf{k} \in \mathbb{Z}_R^3} e^{i\mathbf{k} \cdot (\mathbf{j} - \mathbf{j}')} d\mathbf{j}' d\mathbf{j} dx.$$

Now, for fixed $x \in \mathbb{T}^3$, set $g(x; \mathbf{j}') = \overline{G(x; \mathbf{j}')} e^{-ix \cdot \mathbf{j}'}$. Then, note that

$$\int_{\mathbb{J}} g(x; \mathbf{j}') \sum_{\mathbf{k} \in \mathbb{Z}_R^3} e^{i\mathbf{k} \cdot (\mathbf{j} - \mathbf{j}')} d\mathbf{j}' = (P_{|\mathbf{k}| \leq R} g(x; \cdot))(\mathbf{j}),$$

where $P_{|\mathbf{k}| \leq R}$ is the projection onto the first R Fourier modes. Hence, we have

$$\int_{B_R} |F(x)|^2 dx = \int_{\mathbb{J}} \int_{\mathbb{T}^3} G(x; \mathbf{j}) e^{ix \cdot \mathbf{j}} \cdot (P_{|\mathbf{k}| \leq R} g(x; \cdot))(\mathbf{j}) d\mathbf{j} dx. \quad (3.2.21)$$

Finally, note that by definition of g we have

$$\int_{\mathbb{J}} |G(x; \mathbf{j}) e^{ix \cdot \mathbf{j}} \cdot (P_{|\mathbf{k}| \leq R} g(x; \cdot))(\mathbf{j})| d\mathbf{j} \leq \int_{\mathbb{J}} |G(x; \mathbf{j})|^2 d\mathbf{j} \in L^2(\mathbb{T}^3)$$

so that taking the limit $R \rightarrow \infty$ in (3.2.21), by dominated convergence theorem and again the definition of g we have

$$\lim_{R \rightarrow \infty} \int_{Q_R} |F(x)|^2 dx = \int_{\mathbb{T}^3} \lim_{R \rightarrow \infty} \int_{\mathbb{J}} G(x; \mathbf{j}) e^{ix \cdot \mathbf{j}} \cdot (P_{|\mathbf{k}| \leq R} g(x; \cdot))(\mathbf{j}) d\mathbf{j} dx.$$

Observing that for every fixed $x \in \mathbb{T}^3$ we have

$$(P_{|\mathbf{k}| \leq R} g(x; \cdot))(\mathbf{j}) \rightharpoonup \overline{G(x; \mathbf{j})} e^{-ix \cdot \mathbf{j}}$$

weakly in $L^2(\mathbb{J})$ as $R \rightarrow \infty$, we conclude the proof. $\square$

3.2.6 Proof of Theorem 2

We now have all the tools needed to provide a proof of the main result of this section. To avoid confusion, we once again specify the dependence on $\mathbf{j}, \varepsilon$ of the operator $\mathcal{L}$ in (3.2.3) and write

$$\mathcal{L}(\mathbf{j}, \varepsilon)H = (\nabla + i\mathbf{j}) \times (U \times H) + \varepsilon(\nabla + i\mathbf{j})^2 H.$$

Proof of Theorem 2. Let us start off the proof with a computation. Indeed, suppose that we have constructed a solution $f \in L_{loc}^\infty(0, \infty; L^2(\mathbb{R}^3))$ to the equation

$$\partial_t f = \nabla \times (U \times f) + \varepsilon \Delta f$$

for some $\varepsilon \in [\zeta, 1]$, and some velocity field U . If we then set $g(t, x) = f(t, \zeta^{-\frac{n}{2}} x)$, and define $V = \zeta^{\frac{n}{2}} U(x \zeta^{-\frac{n}{2}})$, it holds that g solves

$$\partial_t g = \nabla \times (V \times g) + \varepsilon \zeta^n \Delta g. \quad (3.2.22)$$

From this, we deduce that as long as we are able to construct exponentially growing solutions to (KDE- ε) with fixed velocity field U , and diffusivity in the compact set $[\zeta, 1]$, a simple rescaling argument allows us to construct solutions with magnetic diffusivity $\varepsilon \zeta^n$ and velocity field $\zeta^{\frac{n}{2}} U(x \zeta^{-\frac{n}{2}})$. We thus aim to use this observation to reduce to study the case $\varepsilon \in [\zeta, 1]$. *Step 1.*

We firstly aim at proving the exponential growth of solutions to (KDE- ε) with velocity field U_{n_ε} . To do so, note that the assumptions of Lemma 3.2.11 are satisfied for any $U = \nabla \times \Psi \in W^{1, \infty}(\mathbb{T}^3)$ that is alpha-unstable, thanks to Proposition 3.1.4. Hence, by Lemma 3.2.11, there exist $J, \eta > 0$,

so that we may define the function

$$H(x; \mathbf{j}, \varepsilon) = \frac{1}{2\pi i} \int_{\Gamma} (\mu - \mathcal{L}(\mathbf{j}, \varepsilon))^{-1} H_{\mathbf{j}_*} d\mu$$

for $|\mathbf{j} - \mathbf{j}_*| < J$, $|\varepsilon - 1| < \eta$, and

$$\overline{H(x; \mathbf{j}, \varepsilon)} = \frac{1}{2\pi i} \int_{\Gamma} (\mu - \mathcal{L}(\mathbf{j}, \varepsilon))^{-1} \overline{H}_{\mathbf{j}_*} d\mu$$

for $|\mathbf{j} + \mathbf{j}_*| < J$, $|\varepsilon - 1| < \eta$. We claim that these are eigenfunctions of $\mathcal{L}(\mathbf{j}, \varepsilon)$. Indeed, by Lemma 3.2.11, as soon as we know that $\overline{H}_{\mathbf{j}_*}$ is an eigenfunction of $\mathcal{L}(-\mathbf{j}_*, \varepsilon)$, it follows that $H(x; \mathbf{j}, \varepsilon)$ are eigenfunctions of $\mathcal{L}(\mathbf{j}, \varepsilon)$ with simple eigenvalues $p(\mathbf{j}, \varepsilon)$ of real part at least $\frac{1}{2}\text{Re}(p(\mathbf{j}_*))$. Furthermore, it holds that for $|\mathbf{j} - \mathbf{j}_*| < J$, there holds

$$\overline{H(x; \mathbf{j}, \varepsilon)} = H(x, -\mathbf{j}, \varepsilon), \quad \overline{p(\mathbf{j}, \varepsilon)} = p(-\mathbf{j}, \varepsilon).$$

Indeed, to see this, note that if $\mathcal{L}(\mathbf{j}, \varepsilon)H = \lambda H$, then taking complex conjugates, and noting that $\overline{\mathcal{L}(\mathbf{j}, \varepsilon)} = \mathcal{L}(-\mathbf{j}, \varepsilon)$, there holds $\mathcal{L}(-\mathbf{j}, \varepsilon)\overline{H} = \overline{\lambda}\overline{H}$. Hence, $\overline{H}_{\mathbf{j}_*}$ is a simple eigenfunction of $\mathcal{L}(-\mathbf{j}_*, \varepsilon)$, and therefore for $|\mathbf{j} + \mathbf{j}_*| < \eta$, $H(x; \mathbf{j}, \varepsilon)$ is still a simple eigenfunction of $\mathcal{L}(\mathbf{j}, \varepsilon)$ with eigenvalue $p(\mathbf{j}, \varepsilon) = \overline{p(-\mathbf{j}, \varepsilon)}$. Therefore, we may now fix any $\zeta \in [1 - \eta, 1)$, and let $\varepsilon \in (\zeta^{n_\varepsilon+1}, \zeta^{n_\varepsilon}]$ for some $n_\varepsilon \in \mathbb{N}_0$. We define the Ansatz solution B^ε of (KDE- ε) with velocity field $\zeta^{n/2}U(\frac{x}{\zeta^{n/2}})$ as follows. Note first that $\frac{\varepsilon}{\zeta^n} \in (\zeta, 1]$. Thus, it holds

$$e^{\mathcal{L}(\mathbf{j}, \frac{\varepsilon}{\zeta^n})t} H\left(x; \mathbf{j}, \frac{\varepsilon}{\zeta^n}\right) = e^{p(\mathbf{j}, \frac{\varepsilon}{\zeta^n})t} H\left(x; \mathbf{j}, \frac{\varepsilon}{\zeta^n}\right)$$

by our construction. Then we set

$$B^\varepsilon(t, x) = \frac{\zeta^{-3n/2}}{\|H(\cdot; \cdot, \frac{\varepsilon}{\zeta^n})\|_{L^2(\mathbb{T}^3 \times Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*))}} \int_{Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)} e^{p(\mathbf{j}, \frac{\varepsilon}{\zeta^n})t} H\left(\frac{x}{\zeta^{n/2}}; \mathbf{j}, \frac{\varepsilon}{\zeta^n}\right) e^{i\mathbf{j}\zeta^{-n/2} \cdot x} d\mathbf{j},$$

with initial data

$$F^\varepsilon(x) = B^\varepsilon(0, x). \quad (3.2.23)$$

Note that for all $t \geq 0, x \in \mathbb{R}^3$, $B^\varepsilon(t, x)$ is *real valued*, since $\overline{p(\mathbf{j}, \frac{\varepsilon}{\zeta^n})} = p(-\mathbf{j}, \frac{\varepsilon}{\zeta^n})$, $\overline{H(x; \mathbf{j}, \varepsilon \zeta^{-n})} = H(x, -\mathbf{j}, \varepsilon \zeta^{-n})$. Since $e^{p(\mathbf{j}, \frac{\varepsilon}{\zeta^n})t} H(x; \mathbf{j}, \frac{\varepsilon}{\zeta^n}) e^{i\mathbf{j} \cdot x}$ is a solution of (KDE- ε) with magnetic diffusivity $\varepsilon \zeta^{-n}$ and velocity field U for every $\mathbf{j}$, it follows that

$$\int_{Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)} e^{p(\mathbf{j}, \frac{\varepsilon}{\zeta^n})t} H\left(x; \mathbf{j}, \frac{\varepsilon}{\zeta^n}\right) e^{i\mathbf{j} \cdot x} d\mathbf{j}$$

is as well a solution of (KDE- ε) with diffusivity $\varepsilon \zeta^{-n}$ and velocity field U . Hence, the computation (3.2.22) at the beginning of the proof implies that B^ε is a solution to (KDE- ε) with velocity field $\zeta^{n/2}U(\frac{x}{\zeta^{n/2}})$ and diffusivity parameter ε . Furthermore, Lemma 3.2.11 further implies that $\text{Re}(p(\mathbf{j}, \frac{\varepsilon}{\zeta^n})) \geq \frac{1}{2}\text{Re}(p(\mathbf{j}_*))$. Hence, by this inequality, Lemma 3.2.12 and a change of variable, it

holds that

$$\begin{aligned}
\|B^\varepsilon(t)\|_{L^2}^2 &= \frac{\zeta^{-3n}}{\|H(\cdot; \cdot, \frac{\varepsilon}{\zeta^n})\|_{L^2}^2} \int_{\mathbb{R}^3} \left| \int_{Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)} H\left(x\zeta^{-n/2}; \mathbf{j}, \frac{\varepsilon}{\zeta^n}\right) e^{i\mathbf{j}\zeta^{-n/2} \cdot x} d\mathbf{j} \right|^2 dx \\
&= \frac{1}{\|H(\cdot; \cdot, \frac{\varepsilon}{\zeta^n})\|_{L^2}^2} \int_{\mathbb{R}^3} \left| \int_{Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)} H\left(x; \mathbf{j}, \frac{\varepsilon}{\zeta^n}\right) e^{i\mathbf{j} \cdot x} e^{p(\mathbf{j}, \frac{\varepsilon}{\zeta^n})t} d\mathbf{j} \right|^2 dx \\
&\geq e^{\frac{1}{2}\text{Re}(p(\mathbf{j}_*))t} \frac{1}{\|H(\cdot; \cdot, \frac{\varepsilon}{\zeta^n})\|_{L^2}^2} \int_{Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)} \int_{\mathbb{T}^3} \left| H\left(x; \mathbf{j}, \frac{\varepsilon}{\zeta^n}\right) \right|^2 dx d\mathbf{j} \\
&= e^{\frac{1}{2}\text{Re}(p(\mathbf{j}_*))t}.
\end{aligned}$$

Here we identified the 3-dimensional torus in $\mathbf{j}$ with $[-\pi, \pi]^3$. Since $\mathbb{1}_{Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)} H(x; \mathbf{j}, \frac{\varepsilon}{\zeta^n})$ has compact support in this region, it may be identified with a 2π -periodic function upon extending it to all of $\mathbb{R}^3$, and hence we may indeed apply Lemma 3.2.12. Applying Lemma 3.2.12 once again to deduce $\|B^\varepsilon(0)\|_{L^2} = 1$ we conclude the proof of the first property.

Step 2. We now move on to proving the second part. Note that it suffices to prove the claim for $\varepsilon \in [\zeta, 1]$. Indeed, suppose that it is proven in this case. Then, by (3.2.23), for any $\varepsilon \in (\zeta^{n+1}, \zeta^n]$, F^ε is nothing but

$$F^\varepsilon(x) = \zeta^{-\frac{3n}{2}} F^{\frac{\varepsilon}{\zeta^n}}(x\zeta^{-\frac{n}{2}}).$$

Hence, changing variables, for any $R > 0$ so that $\|F^{\frac{\varepsilon}{\zeta^n}}\|_{L^2(Q_R)} \geq 1 - \delta$, the same holds for F^ε . Hence, let $\varepsilon \in [\zeta, 1]$. Define now a map $I : [\zeta, 1] \times \mathbb{R}_+$ to $L^2(\mathbb{R}^3)$ given by

$$I : (\varepsilon, R) \mapsto \int_{Q_R} |F^\varepsilon(x, 0)|^2 dx.$$

Note that for fixed ε , I is increasing in R . We claim that $\inf_{\varepsilon \in [\zeta, 1]} I(\varepsilon, R) \rightarrow 1$ as $R \rightarrow \infty$. Indeed, suppose that there exists some ε_m, R_m where $R_m \rightarrow \infty$, so that $I(\varepsilon_m, R_m) < 1 - \delta$ for all m . Then, there is a non-relabeled converging subsequence $\varepsilon_m \rightarrow \bar{\varepsilon} \in [\zeta, 1]$. Pick now $\bar{R}$ so that $I(\bar{\varepsilon}, \bar{R}) > 1 - \frac{\delta}{2}$. Then, we have

$$I(\varepsilon_m, R_m) \geq I(\bar{\varepsilon}, \bar{R}) - |I(\bar{\varepsilon}, \bar{R}) - I(\varepsilon_m, R_m)|$$

For m large enough, $R_m \geq \bar{R}$, and $I(\varepsilon_m, R_m) < I(\bar{\varepsilon}, \bar{R})$ by assumption. Thus,

$$|I(\bar{\varepsilon}, \bar{R}) - I(\varepsilon_m, R_m)| = I(\bar{\varepsilon}, \bar{R}) - I(\varepsilon_m, R_m) \leq I(\bar{\varepsilon}, R_m) - I(\varepsilon_m, R_m).$$

Finally, we note that

$$\begin{aligned}
|I(\varepsilon_m, R_m) - I(\bar{\varepsilon}, R_m)| &\leq \int_{E_{R_m}} |F^{\varepsilon_m}(x) + F^{\bar{\varepsilon}}(x)| |F^{\varepsilon_m}(x) - F^{\bar{\varepsilon}}(x)| dx \\
&\leq (\|F^{\varepsilon_m}\|_{L^2(\mathbb{R}^3)} + \|F^{\bar{\varepsilon}}\|_{L^2(\mathbb{R}^3)}) \|F^{\varepsilon_m} - F^{\bar{\varepsilon}}\|_{L^2(\mathbb{R}^3)}. \tag{3.2.24}
\end{aligned}$$

Since the F^ε are normalized, it remains to compute

$$\|F^{\varepsilon_m} - F^{\bar{\varepsilon}}\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \left| \int_{Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)} \left[\frac{H(x; \mathbf{j}, \varepsilon_m)}{\|H(\cdot; \cdot, \varepsilon_m)\|_{L^2}} - \frac{H(x; \mathbf{j}, \bar{\varepsilon})}{\|H(\cdot; \cdot, \bar{\varepsilon})\|_{L^2}} \right] e^{i\mathbf{j} \cdot x} d\mathbf{j} \right|^2 dx.$$

Using Lemma 3.2.12, this is nothing but

$$\int_{Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)} \int_{\mathbb{T}^3} \left| \frac{H(x; \mathbf{j}, \varepsilon_m)}{\|H(\cdot; \cdot, \varepsilon_m)\|_{L^2}} - \frac{H(x; \mathbf{j}, \bar{\varepsilon})}{\|H(\cdot; \cdot, \bar{\varepsilon})\|_{L^2}} \right|^2 dx d\mathbf{j}.$$

However, Lemma 3.2.11 implies that there exists some constant $C > 0$ so that

$$\|H(x; \mathbf{j}, \varepsilon_m) - H(x; \mathbf{j}, \bar{\varepsilon})\|_{L^2(\mathbb{T}^3)} \leq C|\varepsilon_m - \bar{\varepsilon}|$$

for any $\mathbf{j} \in Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)$, $\varepsilon_m, \bar{\varepsilon} \in [1 - \eta, 1]$. Therefore, by the reverse triangle inequality, it further holds that

$$|\|H(\cdot; \cdot, \varepsilon_m)\|_{L^2(\mathbb{T}^3 \times Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*))} - \|H(\cdot; \cdot, \bar{\varepsilon})\|_{L^2(\mathbb{T}^3 \times Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*))}| \lesssim |Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)|^{\frac{1}{2}} |\varepsilon_m - \bar{\varepsilon}|.$$

Therefore, we estimate

$$\begin{aligned} & \int_{Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)} \int_{\mathbb{T}^3} \left| \frac{H(x; \mathbf{j}, \varepsilon_m)}{\|H(\cdot; \cdot, \varepsilon_m)\|_{L^2}^2} - \frac{H(x; \mathbf{j}, \bar{\varepsilon})}{\|H(\cdot; \cdot, \bar{\varepsilon})\|_{L^2}^2} \right|^2 dx d\mathbf{j} \\ & \leq \int_{Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)} \int_{\mathbb{T}^3} \left| \frac{H(x; \mathbf{j}, \varepsilon_m) - H(x; \mathbf{j}, \bar{\varepsilon})}{\|H(\cdot; \cdot, \bar{\varepsilon})\|_{L^2}} \right|^2 dx d\mathbf{j} \\ & \quad + \|H(\cdot; \cdot, \varepsilon_m)\|_{L^2}^2 \left(\frac{1}{\|H(\cdot; \cdot, \varepsilon_m)\|_{L^2}^2} - \frac{1}{\|H(\cdot; \cdot, \bar{\varepsilon})\|_{L^2}^2} \right) \\ & \lesssim \frac{|Q_J(\mathbf{j}_*) \cup Q_J(-\mathbf{j}_*)|}{\|H(\cdot; \cdot, \bar{\varepsilon})\|_{L^2}} |\varepsilon_m - \bar{\varepsilon}|^2. \end{aligned}$$

Taking $m \rightarrow \infty$, we thus see from (3.2.24) that eventually

$$I(\varepsilon_m, R_m) \geq 1 - \frac{\delta}{2}$$

which yields a contradiction. □

3.3 Properties and energy estimates for the kinematic dynamo equation

This section is devoted to the proof of weighted energy estimates of (KDE- ε). Although some results hold in greater generality, we restrict ourselves to the setting in which we are given a time-independent, divergence-free streamfunction $\psi \in C^\infty(\mathbb{T}^3)$ and the associated incompressible velocity field $u = \nabla \times \psi$. When considering (KDE- ε) on $\mathbb{R}^3$, such functions are periodically extended to the whole space to functions that are bounded, along with their derivatives. We now establish some more refined estimates than the ones in (2.1.2), which will be useful in the sequel. Roughly speaking, these give quantitative control over the solution and its tails at spatial infinity. The main result reads as follows.

Lemma 3.3.1. *Fix a positive $\varphi \in C_c^\infty(\mathbb{R}^3)$ with $\|\varphi\|_{L^\infty} = 1$ and $\|\nabla \varphi\|_{W^{1,\infty}} \leq 1$, and define $u^\varphi = \nabla \times (\psi \varphi)$. There exists a constant $C_\psi = C(\|\psi\|_{W^{2,\infty}}) > 0$, which depends continuously on its*

argument, such that

$$\|\varphi S_\varepsilon^u(t)B_{\text{in}}\|_{L^2}^2 \leq C_\psi [\|\varphi B_{\text{in}}\|_{L^2}^2 + \|\nabla \varphi\|_{L^\infty} \|B_{\text{in}}\|_{L^2}^2] \exp(C_\psi t), \quad (3.3.1)$$

and

$$\|(1-\varphi)S_\varepsilon^u(t)B_{\text{in}}\|_{L^2}^2 \leq C_\psi [\|(1-\varphi)B_{\text{in}}\|_{L^2}^2 + \|\nabla \varphi\|_{L^\infty} \|B_{\text{in}}\|_{L^2}^2] \exp(C_\psi t), \quad (3.3.2)$$

and

$$\|S_\varepsilon^u(t)B_{\text{in}} - S_\varepsilon^{u_\varphi}(t)B_{\text{in}}\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{C_\psi}{\varepsilon} [\|(1-\varphi)B_{\text{in}}\|_{L^2}^2 + \|\nabla \varphi\|_{W^{1,\infty}} \|B_{\text{in}}\|_{L^2}^2] \exp(C_\psi t), \quad (3.3.3)$$

for every $t \geq 0$.

Proof. The estimates leading to (3.3.1) are the same as (3.3.2), and for (3.3.3) are similar. For the sake of clarity we prove (3.3.1) and (3.3.3).

Proof of (3.3.1). Let $B(t) = S_\varepsilon^u(t)B_{\text{in}}$. Thanks to Lemma 2.1.1, we can multiply the equation (KDE- ε) by $B\varphi^2$ and integrate to get

$$\frac{1}{2} \frac{d}{dt} \|\varphi B\|_{L^2}^2 + \varepsilon \|\varphi \nabla B\|_{L^2}^2 \leq \int_{\mathbb{R}^3} |u| |\nabla \varphi| |B|^2 \varphi + \int_{\mathbb{R}^3} |\nabla u| |\varphi B|^2 + \varepsilon \int_{\mathbb{R}^3} |\varphi \nabla B| |\nabla \varphi| |B|. \quad (3.3.4)$$

Thus, Young's inequality and the bound on φ imply

$$\frac{d}{dt} \|\varphi B\|_{L^2}^2 \leq 2 \|\nabla u\|_{L^\infty} \|\varphi B\|_{L^2}^2 + 2 (\|u\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} + \|\nabla \varphi\|_{L^\infty}^2) \|B\|_{L^2}^2.$$

Using (2.1.2), we can directly integrate the above inequality to get

$$\|\varphi B(t)\|_{L^2}^2 \leq [\|\varphi B_{\text{in}}\|_{L^2}^2 + 2t (\|u\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} + \|\nabla \varphi\|_{L^\infty}^2) \|B_{\text{in}}\|_{L^2}^2] \exp(2 \|\nabla u\|_{L^\infty} t), \quad (3.3.5)$$

and we conclude the proof of (3.3.2) by the inequality $te^{\alpha t} \leq \alpha^{-1} e^{2\alpha t}$.

Proof of (3.3.3). Let $B(t) = S_\varepsilon^u(t)B_{\text{in}}$ and $B_\varphi(t) = S_\varepsilon^{u_\varphi}(t)B_{\text{in}}$. Their difference satisfies

$$\partial_t(B_\varphi - B) = \nabla \times (u_\varphi \times (B_\varphi - B)) + \nabla \times ((u_\varphi - u) \times B) + \varepsilon \Delta(B_\varphi - B).$$

A standard energy estimate similar to (3.3.4) entails

$$\frac{1}{2} \frac{d}{dt} \|B_\varphi - B\|_{L^2}^2 + \varepsilon \|B_\varphi - B\|_{H^1}^2 \leq \int_{\mathbb{R}^3} |\nabla u_\varphi| |B_\varphi - B|^2 + \int_{\mathbb{R}^3} |u_\varphi - u| |B| |\nabla(B_\varphi - B)|.$$

Thus, using the identity $u_\varphi - u = u(\varphi - 1) + \nabla \varphi \times \psi$ we deduce that $\|\nabla u_\varphi\|_{L^\infty} \leq 2\|\psi\|_{W^{2,\infty}}$ and

$$\begin{aligned} \frac{d}{dt} \|B_\varphi - B\|_{L^2}^2 &\leq 2 \int_{\mathbb{R}^3} |\nabla u_\varphi| |B_\varphi - B|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} |u|^2 |(1-\varphi)B|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} |\nabla \varphi \times \psi|^2 |B|^2 \\ &\leq 2 \|\nabla u_\varphi\|_{L^\infty} \|B_\varphi - B\|_{L^2}^2 + \frac{4\|\psi\|_{W^{1,\infty}}^2}{\varepsilon} [\|(1-\varphi)B\|_{L^2}^2 + \|\nabla \varphi\|_{L^\infty} \|B\|_{L^2}^2] \\ &\leq 4\|\psi\|_{W^{2,\infty}} \|B_\varphi - B\|_{L^2}^2 + \frac{4\|\psi\|_{W^{1,\infty}}^2}{\varepsilon} [\|(1-\varphi)B\|_{L^2}^2 + \|\nabla \varphi\|_{L^\infty} \|B\|_{L^2}^2]. \end{aligned}$$

Now, estimate (2.1.2) and (3.3.5) in particular imply that

$$\|(1 - \varphi)B\|_{L^2}^2 + \|\nabla \varphi\|_{L^\infty} \|B\|_{L^2}^2 \leq C_\Psi [\|(1 - \varphi)B_{\text{in}}\|_{L^2}^2 + \|\nabla \varphi\|_{L^\infty} \|B_{\text{in}}\|_{L^2}^2] \exp(4\|\psi\|_{W^{2,\infty}} t).$$

The Grönwall lemma together with the fact that $B_\varphi(0) - B(0) = 0$ then gives

$$\|B_\varphi(t) - B(t)\|_{L^2}^2 \leq \frac{C_\Psi}{\varepsilon} [\|(1 - \varphi)B_{\text{in}}\|_{L^2}^2 + \|\nabla \varphi\|_{L^\infty} \|B_{\text{in}}\|_{L^2}^2] t \exp(4\|\psi\|_{W^{2,\infty}} t).$$

Using that the inequality $te^{\alpha t} \leq \alpha^{-1} e^{2\alpha t}$, we conclude the proof of (3.3.3), and we are done. $\square$

3.4 Construction of the velocity field and proof of Theorem 1

To achieve the proof of Theorem 1, we need to define an ε -independent velocity field $u \in W^{1,\infty}(\mathbb{R}^3)$ such that there are ε -independent constants $c_0, \gamma > 0$ with the following property: for any $\varepsilon > 0$, there exists $B_{\text{in}}^\varepsilon \in L^2(\mathbb{R}^3)$ satisfying

$$\|S_\varepsilon^u(t)B_{\text{in}}^\varepsilon\|_{L^2} \geq c_0 e^{\gamma t} \|B_{\text{in}}^\varepsilon\|_{L^2}, \quad \forall t \geq 0. \quad (3.4.1)$$

This requires a construction involving a (periodic extension of any) alpha-unstable velocity field $U \in C^\infty(\mathbb{T}^3)$ with a simple eigenvalue as given by Theorem 2, suitably rescaled, localized and copied in the whole space. The main ideas are explained in the next section, while the proof of the main technical cornerstone (Proposition 3.4.1 below) is postponed to the later sections.

3.4.1 Main proposition and proof of Theorem 1

Our starting point is Theorem 2, so that we are given an alpha-unstable velocity field $U \in C^\infty(\mathbb{T}^3)$ with a simple eigenvalue, given by a streamfunction $\Psi \in C^\infty(\mathbb{T}^3)$, i.e. $U = \nabla \times \Psi$. Let $\gamma > 0$ and $\zeta \in (0, 1)$ be the constants devised in Theorem 2, and set

$$U_n(x) = \zeta^{n/2} U(\zeta^{-n/2} x), \quad U_n = \nabla \times \Psi_n. \quad (3.4.2)$$

In light of Theorem 2, for any $\varepsilon \in (\zeta^{n_\varepsilon+1}, \zeta^{n_\varepsilon}]$ there exists an initial condition $F^\varepsilon \in L^2(\mathbb{R}^3)$ such that

- $\|F^\varepsilon\|_{L^2(\mathbb{R}^3)} = 1$ and

$$\|S_\varepsilon^{U_n}(t)F^\varepsilon\|_{L^2(\mathbb{R}^3)} \geq e^{2\gamma t}, \quad \forall t \geq 0;$$

- for any $\delta > 0$, there exists $R > 0$ such that

$$\|F^\varepsilon\|_{L^2(Q_R)}^2 \geq 1 - \delta \quad (3.4.3)$$

for any $\varepsilon \in (0, 1]$.

The proof of (3.4.1) not only requires a choice of u , but also of the initial datum $B_{\text{in}}^\varepsilon$. The construction takes Ψ_n and F^ε as the building blocks for the velocity and initial datum, respectively. We give a more precise idea of the construction of both of them below.

Construction of u . The autonomous, incompressible velocity field u is defined through (3.4.2) as a sum

$$u = \sum_{n,\ell=1}^{\infty} \nabla \times (\Psi_n \phi_{n,\ell}) = \sum_{n,\ell=1}^{\infty} u_{n,\ell}, \quad (3.4.4)$$

where $\phi_{n,\ell} \in C_c^\infty(\mathbb{R}^3)$, defined in Subsection 3.4.2, are suitable compactly supported cutoffs with disjoint supports with the property that there exists a ball $Q_{n,\ell} \subset \{\phi_{n,\ell} \equiv 1\}$ of sufficiently large radius.

Construction of $B_{\text{in}}^\varepsilon$. The initial datum is constructed as the sum

$$B_{\text{in}}^\varepsilon = \sum_{\ell=1}^{\infty} \ell^{-2} F_\ell^\varepsilon \quad (3.4.5)$$

with F_ℓ^ε “concentrated” on $Q_{n_\varepsilon,\ell}$ and is made rigorous in Subsection 3.4.2.

The main task now is to properly choose $\{\phi_{n,\ell}\}_{n,\ell}, \{Q_{n,\ell}\}_{n,\ell}$. Roughly speaking, to check (3.4.1) for some $\varepsilon > 0$ and some $t \geq 0$, we restrict our focus on a specific ball Q_{n_ε,ℓ_t} , and observe that the growth happens there. It is crucial that the constants c_0, γ in (3.4.1) are independent of the choices above. The important features of this construction are in the result below.

Proposition 3.4.1. *There exist constants $c_0, \gamma > 0$, collections of balls $\{Q_{n,\ell}\} \subset \mathbb{R}^3$ and cut-offs $\{\phi_{n,\ell}\} \subset C_c^\infty(\mathbb{R}^3)$ such that the velocity $u \in W^{1,\infty}(\mathbb{R}^3)$ given in (3.4.4) satisfies the following properties.*

For every $\varepsilon > 0$ there exists $B_{\text{in}}^\varepsilon \in L^2(\mathbb{R}^3)$ as in (3.4.5) such that if $\varepsilon \in (\zeta^{n_\varepsilon+1}, \zeta^{n_\varepsilon}]$ for some $n_\varepsilon \in \mathbb{N}$ and $t \in [\ell_t, \ell_t + 1)$ for some $\ell_t \in \mathbb{N}$, then

$$(P1) \quad \ell_t^{-2} \|S_\varepsilon^{U_{n_\varepsilon}}(t) F_{\ell_t}^\varepsilon\|_{L^2(Q_{n_\varepsilon,\ell_t})} \geq 2c_0 e^{\gamma t} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)},$$

$$(P2) \quad \ell_t^{-2} \|(S_\varepsilon^{U_{n_\varepsilon}}(t) - S_\varepsilon^{U_{n_\varepsilon,\ell_t}}(t)) F_{\ell_t}^\varepsilon\|_{L^2(Q_{n_\varepsilon,\ell_t})} \leq \frac{c_0}{3} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)},$$

$$(P3) \quad \ell_t^{-2} \|(S_\varepsilon^{U_{n_\varepsilon,\ell_t}}(t) - S_\varepsilon^u(t)) F_{\ell_t}^\varepsilon\|_{L^2(Q_{n_\varepsilon,\ell_t})} \leq \frac{c_0}{3} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)},$$

$$(P4) \quad \left\| \sum_{\ell \neq \ell_t} \ell^{-2} S_\varepsilon^u(t) F_\ell^\varepsilon \right\|_{L^2(Q_{n_\varepsilon,\ell_t})} \leq \frac{c_0}{3} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)}.$$

We now prove Theorem 1 assuming Proposition 3.4.1.

Proof of Theorem 1. Let $u \in W^{1,\infty}(\mathbb{R}^3)$ be the velocity field given by Proposition 3.4.1 through (3.4.4). We need to prove that there exist $c_0, \gamma > 0$ such that for any $\varepsilon > 0$ there exists $B_{\text{in}}^\varepsilon \in L^2(\mathbb{R}^3)$ satisfying the growth estimate (3.4.1).

Fix $\varepsilon > 0$, choose $n_\varepsilon \in \mathbb{N}$ such that $\varepsilon \in (\zeta^{n_\varepsilon+1}, \zeta^{n_\varepsilon}]$, and let $B_{\text{in}}^\varepsilon$ be the initial datum given by Proposition 3.4.1 via (3.4.5). For any $t > 0$, let $\ell_t \in \mathbb{N}$ such that $t \in [\ell_t, \ell_t + 1)$. Then, by the

triangle inequality and linearity of $S_\varepsilon^u(t)$ we have

$$\begin{aligned}
\|S_\varepsilon^u(t)B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)} &\geq \|S_\varepsilon^u(t)B_{\text{in}}^\varepsilon\|_{L^2(Q_{n_\varepsilon, \ell_t})} \\
&\geq \ell_t^{-2} \|S_\varepsilon^u(t)F_{\ell_t}^\varepsilon\|_{L^2(Q_{n_\varepsilon, \ell_t})} - \left\| \sum_{\ell \neq \ell_t} \ell^{-2} S_\varepsilon^u(t)F_\ell^\varepsilon \right\|_{L^2(Q_{n_\varepsilon, \ell_t})} \\
&\geq \ell_t^{-2} \|S_\varepsilon^{U_{n_\varepsilon}}(t)F_{\ell_t}^\varepsilon\|_{L^2(Q_{n_\varepsilon, \ell_t})} - \ell_t^{-2} \| (S_\varepsilon^{U_{n_\varepsilon}}(t) - S_\varepsilon^{U_{n_\varepsilon, \ell_t}}(t))F_{\ell_t}^\varepsilon \|_{L^2(Q_{n_\varepsilon, \ell_t})} \\
&\quad - \ell_t^{-2} \| (S_\varepsilon^{U_{n_\varepsilon, \ell_t}}(t) - S_\varepsilon^u(t))F_{\ell_t}^\varepsilon \|_{L^2(Q_{n_\varepsilon, \ell_t})} - \left\| \sum_{\ell \neq \ell_t} \ell^{-2} S_\varepsilon^u(t)F_\ell^\varepsilon \right\|_{L^2(Q_{n_\varepsilon, \ell_t})}.
\end{aligned}$$

Applying Proposition 3.4.1 we deduce that for any $\varepsilon \in (\zeta^{n_\varepsilon+1}, \zeta^{n_\varepsilon}]$ and $t \in [\ell_t, \ell_t + 1)$ it holds that

$$\|S_\varepsilon^u(t)B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)} \geq c_0 e^\gamma \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)}.$$

Since t, ε are arbitrary and $c_0, \gamma > 0$ are independent of ℓ_t, n_ε we conclude the proof. $\square$

3.4.2 Definition of the velocity field and the initial data

In this section, we define the velocity field and the initial data. In what follows, $\mathfrak{U} = \mathfrak{U}(\|\Psi\|_{W^{2,\infty}}) \geq 10$ is a sufficiently large constant solely depending on the norms of Ψ , whose value will be fixed later.

We first dwell on the construction of the sequence of cut-off functions $\phi_{n,\ell}$. For a fixed $\varepsilon > 0$, consider the initial condition F^ε given in Theorem 2. For each $\ell \in \mathbb{N}$, we appeal to (3.4.3) and find $R_{n,\ell} > 1$ such that

$$R_{n,\ell}^{-1} + \|F^\varepsilon \mathbb{1}_{Q_{R_{n,\ell}}} - F^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{1}{\mathfrak{U}} \exp(-(n+1) - \mathfrak{U}(\ell+1)), \quad (3.4.6)$$

uniformly in $\varepsilon \in (0, 1]$. We then take a countable collection of compactly supported smooth functions satisfying the following properties:

$$0 \leq \phi_{n,\ell} \leq 1, \quad \phi_{n,\ell} \equiv 1 \quad \text{on } Q_{n,\ell} := Q_{2R_{n,\ell}}(x_{n,\ell}), \quad (3.4.7a)$$

$$\text{dist}(\text{supp}(\phi_{n,\ell}), \text{supp}(\phi_{n',\ell'})) \geq 2R_{n,\ell}, \quad \text{for all } (n,\ell) \neq (n',\ell'), \quad (3.4.7b)$$

$$\|\nabla \phi_{n,\ell}\|_{L^\infty} + \|D^2 \phi_{n,\ell}\|_{L^\infty} \leq \frac{1}{\mathfrak{U}} \exp(-(n+1) - \mathfrak{U}(\ell+1)). \quad (3.4.7c)$$

The existence of such collection of functions and balls is fairly easy to prove: once the radii are fixed by (3.4.6), the condition (3.4.7c) is imposing that the balls are well-separated, say by ten times the right-hand side of (3.4.7c). It is then sufficient to fix a suitable well-separated sequence of centers $\{x_{n,\ell}\}$. Properties (3.4.7a)-(3.4.7b) follow immediately. While we now have u as in (3.4.4) at our disposal, it remains to construct the building blocks F_ℓ^ε in (3.4.5). Given $\varepsilon > 0$ and picking $n_\varepsilon \in \mathbb{N}$ such that $\varepsilon \in (\zeta^{n_\varepsilon+1}, \zeta^{n_\varepsilon}]$, we define

$$F_\ell^\varepsilon(\cdot) = F^\varepsilon(\cdot - x_{n_\varepsilon, \ell}).$$

The initial datum $B_{\text{in}}^\varepsilon$ is then defined according to (3.4.5).

3.4.3 Proof of Proposition 3.4.1

The proof of the main Proposition 3.4.1 is split across several steps in the section, each one of them establishing (P1)-(P4). We preliminarily notice that if $\mathfrak{U} \geq 10$, it holds that

$$\frac{1}{C} \|\Psi\|_{W^{2,\infty}} \leq \|\Psi_n \phi_{n,\ell}\|_{W^{2,\infty}}, \left\| \sum_{n,\ell} \Psi_n \phi_{n,\ell} \right\|_{W^{2,\infty}}, \|\Psi_n\|_{W^{2,\infty}} \leq C \|\Psi\|_{W^{2,\infty}}, \quad (3.4.8)$$

for some constant C independent of $n, \ell, \mathfrak{U}$. This follows from the periodicity of Ψ , the fact that the rescaling (3.4.2) do not increase the Lipschitz norm of $\nabla \Psi$, and the properties (3.4.7) of the cut-offs. Furthermore, using that $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$, for any $\ell \geq 1$ and $\varepsilon \in (0, 1)$, we deduce from (3.4.6) that

$$\frac{1}{2} \leq \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)} \leq 2. \quad (3.4.9)$$

Indeed, by the summability of $\sum_{j \geq 1} j^{-2}$ the upper bound is elementary. To show the lower bound, note that

$$\|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)} \geq \|F_1^\varepsilon\|_{L^2(Q_{n,1})} - \left\| \sum_{\ell \geq 2} \frac{1}{\ell^2} F_\ell^\varepsilon \right\|_{L^2(Q_{n,1})}.$$

By the condition (3.4.6), as long as $\mathfrak{U}$ is sufficiently large we have that $\|F_1^\varepsilon\|_{L^2(Q_{n,1})} \geq \frac{9}{10}$. Furthermore, since $\text{dist}(Q_{n,\ell}, Q_{n,1}) \geq 2R_{n,1}$ for all $\ell \neq 1$, it holds that $\|F_\ell^\varepsilon\|_{L^2(Q_{n,1})} \leq \|F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3 \setminus Q_{R_{n,1}})} \leq \frac{1}{10}$. Hence, $\left\| \sum_{\ell \geq 2} \ell^{-2} F_\ell^\varepsilon \right\|_{L^2(Q_{n,1})} \leq \frac{1}{10} \sum_{\ell \geq 2} \ell^{-2} < \frac{2}{5}$. Therefore, it holds that

$$\|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)} \geq \frac{1}{2}.$$

In other words, the Lipschitz (resp. L^2) norms of the building blocks of the velocity field (resp. initial data) are comparable. These facts will frequently be used in the following proofs below.

Proof of Proposition 3.4.1. To ease notation, we will omit some of the subscripts and write n for n_ε and ℓ for ℓ_t in the proof. Also, the symbol C_Ψ denotes a *generic* constant depending continuously on $\|\Psi\|_{W^{2,\infty}}$, that may change within the proof.

Proof of (P1). Let $\tilde{\phi}$ be a standard smooth cut-off such that $\tilde{\phi} \equiv 1$ outside $Q_{n,\ell}$, and $\tilde{\phi} \equiv 0$ on $Q_{\frac{3}{2}R_{n,\ell}}(x_{n,\ell})$, and $\|\nabla \tilde{\phi}\|_{L^\infty} \leq 4R_{n,\ell}^{-1}$. Then, we have that

$$\|S_\varepsilon^{U_n}(t) F_\ell^\varepsilon\|_{L^2(Q_{n,\ell})} \geq \|S_\varepsilon^{U_n}(t) F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)} - \|\tilde{\phi} S_\varepsilon^{U_n}(t) F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)}. \quad (3.4.10)$$

In view of (3.3.2) in Lemma 3.3.1, there holds

$$\|\tilde{\phi} S_\varepsilon^{U_n}(t) F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \leq C_{\Psi_n} [\|\tilde{\phi} F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\phi}\|_{L^\infty} \exp(C_{\Psi_n} t)].$$

for some universal constants C_{Ψ_n} depending only on $\|\Psi_n\|_{W^{2,\infty}}$. However, in view of (3.4.8) and the continuous dependence of the constants on the norms of Ψ as in Lemma 3.3.1, we may replace all the constants C_{Ψ_n} by new universal constants C_Ψ depending only on $\|\Psi\|_{W^{2,\infty}}$. Furthermore, by

the choice of $R_{n,\ell}$ in (3.4.6), we have that

$$R_{n,\ell}^{-1} + \|F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3 \setminus Q_{R_{n,\ell}}(x_{n,\ell}))} \leq \frac{1}{\mathfrak{U}} \exp(-(n+1) - \mathfrak{U}(\ell+1)),$$

and since the support of $\tilde{\varphi}$ is contained in $\mathbb{R}^3 \setminus Q_{R_{n,\ell}}(x_{n,\ell})$, it holds for any $t \in [\ell, \ell+1)$

$$\begin{aligned} \|\tilde{\varphi} S_\varepsilon^{U_n}(t) F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 &\leq C_\Psi \left[\|F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3 \setminus Q_{R_{n,\ell}}(x_{n,\ell}))}^2 + \frac{4}{R_{n,\ell}} \right] \exp(C_\Psi(\ell+1)) \\ &\leq C_\Psi \mathfrak{U}^{-1} \exp((C_\Psi - \mathfrak{U})(\ell+1)). \end{aligned}$$

Hence, picking $\mathfrak{U}$ large depending only on $\|\Psi\|_{W^{2,\infty}}$, it holds by (3.4.9) that

$$\|\tilde{\varphi} S_\varepsilon^{U_n}(t) F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{4} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)}, \quad \forall t \in [\ell, \ell+1). \quad (3.4.11)$$

Furthermore, thanks to the growth estimate (3.2.1) in Theorem 2 and (3.2.1) it holds that

$$\|S_\varepsilon^{U_n}(t) F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)} \geq e^{2\gamma t} \geq \frac{1}{2} e^{2\gamma t} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)}. \quad (3.4.12)$$

Therefore, by combining (3.4.11) and (3.4.12) with (3.4.10) we deduce that

$$\|S_\varepsilon^{U_n}(t) F_\ell^\varepsilon\|_{L^2(Q_{n,\ell})} \geq \frac{1}{2} \left(e^{2\gamma t} - \frac{1}{2} \right) \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)} \geq \frac{1}{4} e^{2\gamma t} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)} \geq \frac{\gamma^2}{4} \ell^2 e^{\gamma t} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)},$$

for all $t \in [\ell, \ell+1)$, as we wanted.

Proof of (P2). The proof of (P2) is an application of Lemma 3.3.1. Indeed, note that up to translating $x_{n,\ell}$ to the origin, we can just apply (3.3.3) in Lemma 3.3.1 with $u = U_n$ to deduce

$$\|(S_\varepsilon^{U_n}(t) - S_\varepsilon^{u_{n,\ell}}(t)) F_\ell^\varepsilon\|_{L^2(Q_{n,\ell})}^2 \leq \frac{C_{\Psi_n}}{\varepsilon} [\|(1 - \phi_{n,\ell}) F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \phi_{n,\ell}\|_{W^{1,\infty}}] \exp(C_{\Psi_n} t).$$

Once again, we may appeal to (3.4.8) to replace the constants C_{Ψ_n} by constants C_Ψ depending only on $\|\Psi\|_{W^{2,\infty}}$. Next, recall that $Q_{n,\ell} \subset \{\phi_{n,\ell} \equiv 1\}$. In particular, (3.4.6) implies

$$\|(1 - \phi_{n,\ell}) F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \leq \|F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3 \setminus Q_{n,\ell})}^2 \leq \mathfrak{U}^{-1} \exp(-(n+1) - \mathfrak{U}(\ell+1)). \quad (3.4.13)$$

Furthermore, by the assumption (3.4.7c), we may bound

$$\|\nabla \phi_{n,\ell}\|_{W^{1,\infty}} \leq \mathfrak{U}^{-1} \exp(-(n+1) - \mathfrak{U}(\ell+1)).$$

Hence, since we assumed that $\varepsilon \in (\zeta^{n+1}, \zeta^n]$, there holds for $t \in [\ell, \ell+1]$

$$\|(S_\varepsilon^{U_n}(t) - S_\varepsilon^{u_{n,\ell}}(t)) F_\ell^\varepsilon\|_{L^2(Q_{n,\ell})}^2 \leq C_\Psi \mathfrak{U}^{-1} (\zeta e)^{-(n+1)} \exp(-(\mathfrak{U} - C_\Psi)(\ell+1)).$$

Since $\zeta > \frac{1}{e}$ by Theorem 2, we obtain

$$\|(S_\varepsilon^{U_n}(t) - S_\varepsilon^{u_{n,\ell}}(t)) F_\ell^\varepsilon\|_{L^2(Q_{n,\ell})}^2 \leq C_\Psi \mathfrak{U}^{-1} \exp((C_\Psi - \mathfrak{U})(\ell+1)).$$

Hence, upon taking $\mathfrak{U}$ large enough depending only on $\|\Psi\|_{W^{2,\infty}}$, we can bound the above quantity with the help of (3.4.9) by

$$\|(S_\varepsilon^u(t) - S_\varepsilon^{u_{n,\ell}}(t))F_\ell^\varepsilon\|_{L^2(Q_{n,\ell})} \leq \frac{c_0}{16} \leq \frac{c_0}{8} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)},$$

for $t \in [\ell, \ell + 1]$.

Proof of (P3). Recall that u is given in (3.4.4), so that we can write $\psi = \sum_{n', \ell'} \Psi_{n'} \phi_{n', \ell'}$ and $u = \nabla \times \psi$. Moreover, $u_{n,\ell} = \nabla \times (\phi_{n,\ell} \Psi_n) = \nabla \times (\tilde{\phi} \psi)$ where $\tilde{\phi} \in C_c^\infty(\mathbb{R})$ is any function so that $\text{supp}(\tilde{\phi}) \cap \text{supp}(\phi_{n,\ell'}) = \emptyset$ for all $\ell' \neq \ell$, and $\tilde{\phi} \equiv 1$ on $\text{supp}(\phi_{n,\ell})$. In particular, by the assumption (3.4.7b), we can find such a $\tilde{\phi}$ that furthermore satisfies $\|\nabla \tilde{\phi}\|_{W^{1,\infty}} \leq 10R_{n,\ell}^{-1}$. Thus, we may once again apply Lemma 3.3.1 to obtain (replacing all constants C_ψ by constants C_Ψ depending only on $\|\Psi\|_{W^{2,\infty}}$ via (3.4.8)):

$$\|(S_\varepsilon^u(t) - S_\varepsilon^{u_{n,\ell}}(t))F_\ell^\varepsilon\|_{L^2(Q_{n,\ell})}^2 \leq \frac{C_\Psi}{\varepsilon} \left[\|(1 - \tilde{\phi})F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\phi}\|_{W^{1,\infty}} \right] \exp(C_\Psi t).$$

As we did in (3.4.13), since $\tilde{\phi} \equiv 1$ on $\text{supp}(\phi_{n,\ell}) \supset Q_{n,\ell}$ as in (3.4.7a), in view of (3.4.6) it holds that

$$\|(1 - \tilde{\phi})F_\ell^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\phi}\|_{W^{1,\infty}} \leq \frac{1}{\mathfrak{U}} \exp(-(n+1) - \mathfrak{U}(\ell+1)).$$

Therefore, we see that

$$\|(S_\varepsilon^u(t) - S_\varepsilon^{u_{n,\ell}}(t))F_\ell^\varepsilon\|_{L^2(Q_{n,\ell})}^2 \leq C_\Psi \mathfrak{U}^{-1} (\zeta e)^{-(n+1)} \exp((C_\Psi - \mathfrak{U})(\ell+1))$$

for any $t \in [\ell, \ell + 1]$. Hence, arguing exactly as for the previous term, we see that for all $\mathfrak{U}$ large enough depending only on $\|\Psi\|_{W^{2,\infty}}$, and for any time $t \in [\ell, \ell + 1)$ this is bounded by

$$\|(S_\varepsilon^u(t) - S_\varepsilon^{u_{n,\ell}}(t))F_\ell^\varepsilon\|_{L^2(Q_{n,\ell})} \leq \frac{c_0}{8} \|B_{\text{in}}^\varepsilon\|_{L^2(\mathbb{R}^3)}.$$

which completes the proof.

Proof of (P4). Once again, this is an application of Lemma 3.3.1, equation (3.3.1). Indeed, note that

$$\left\| \sum_{j \neq \ell} j^{-2} S_\varepsilon^u(t) F_j^\varepsilon \right\|_{L^2(Q_{n,\ell})} \leq \sum_{j \neq \ell} j^{-2} \|\phi_{n,\ell} S_\varepsilon^u(t) F_j^\varepsilon\|_{L^2(\mathbb{R}^3)}.$$

From (3.3.1) deduce that

$$\|\phi_{n,\ell} S_\varepsilon^u(t) F_j^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \leq C_\Psi \left[\|\phi_{n,\ell} F_j^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \phi_{n,\ell}\|_{L^\infty} \right] \exp(C_\Psi t).$$

By definition of F_j^ε we now observe that $(F_j^\varepsilon \phi_{n,\ell})(\cdot + x_{n,j}) = F^\varepsilon(\cdot) \phi_{n,\ell}(\cdot + x_{n,j})$. We also observe that $\phi_{n,\ell}(\cdot + x_{n,j}) \equiv 0$ on $Q_{R_{n,\ell}}$ for any $j \neq \ell$ by condition (3.4.7b). Therefore, using (3.4.6) we obtain that for $j \neq \ell$ it holds

$$\|\phi_{n,\ell} F_j^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \leq \|F^\varepsilon\|_{L^2(\mathbb{R}^3 \setminus Q_{R_{n,\ell}})}^2 \leq \frac{1}{\mathfrak{U}} \exp(-\mathfrak{U}(\ell+1)).$$

Hence, for any time $t \in [\ell, \ell + 1)$, by the properties of the $\phi_{n,\ell}$ in (3.4.7) we deduce

$$\left\| \sum_{j \neq \ell} j^{-2} S_{\varepsilon}^u(t) F_j^{\varepsilon} \right\|_{L^2(Q_{n,\ell})} \leq (C_{\Psi} \mathfrak{U}^{-1})^{1/2} \sum_j j^{-2} \exp((C_{\Psi} - \mathfrak{U})(\ell + 1)/2)$$

Thanks to (3.4.9), we have that $\|B_{\text{in}}^{\varepsilon}\|_{L^2(\mathbb{R}^3)} \geq \frac{1}{2}$. By possibly increasing that value of $\mathfrak{U}$, we therefore conclude that

$$\left\| \sum_{j \neq \ell} j^{-2} S_{\varepsilon}^u(t) F_j^{\varepsilon} \right\|_{L^2(Q_{n,\ell})} \leq \frac{c_0}{8} \|B_{\text{in}}^{\varepsilon}\|_{L^2(\mathbb{R}^3)}$$

for $t \in [\ell, \ell + 1)$, completing the proof. □

Chapter 4

A limsup fast dynamo on $\mathbb{T}^3$.

In this chapter we construct what we shall call a limsup fast dynamo on $\mathbb{T}^3$: a time-dependent velocity field for which exponential magnetic growth occurs in the $\limsup_{t \rightarrow \infty}$ sense, uniformly as $\varepsilon \rightarrow 0$. That is, we will allow for the $\liminf_{t \rightarrow \infty}$ in (1.2.1) to be replaced by a $\limsup_{t \rightarrow \infty}$, matching the definition of fast dynamo action given for instance in [CG95]. The chapter thus has a twofold purpose. First, it constructs a dynamo satisfying the classical limsup formulation used in parts of the physics literature such as [CG95]. Second, it shows that for time-dependent flows this notion is strictly weaker than the liminf-based fast dynamo notion adopted in [AK98] and in this thesis. Therefore, to distinguish our result from the classical fast dynamo, we give it the name of *limsup fast dynamo*. The proof of this Chapter's main result is once again based on rescaled alpha-unstable flows from Chapter 3. However, this time we carefully classify the set of initial data which experience exponential growth under an alpha-unstable velocity field, and construct a fully time-dependent u which visits each rescaled alpha-unstable flow infinitely many times. The content of this Chapter is included in the paper [SV25].

4.1 Introduction

We study the kinematic dynamo equation on $\mathbb{T}^3$, with a given, fully time-dependent velocity field $u : (0, \infty) \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$. That is, we study

$$\begin{cases} \partial_t B^\varepsilon = \nabla \times (u \times B^\varepsilon) + \varepsilon \Delta B^\varepsilon \\ \nabla \cdot B = 0. \end{cases} \quad (t, x) \in (0, \infty) \times \mathbb{T}^3,$$

where $\varepsilon > 0$ is the magnetic diffusivity. As in the previous chapters, we will be interested in the asymptotically largest growth rate of solutions to this equation in the singular limit as $\varepsilon \rightarrow 0$. Interestingly enough, several mathematical definitions have been proposed in the literature (see, for instance, [CG95; AK98; Arn04; CSV25; Row25; NV25b]) for capturing the precise meaning of this asymptotic growth rate, which, as will be shown in this chapter, are *not* equivalent when the velocity field u is time-dependent. In particular, these definitions differ in how the limits $\varepsilon \rightarrow 0$ and $t \rightarrow \infty$ are taken. Let u be a bounded, Lipschitz continuous, divergence-free velocity field,

$\varepsilon \in (0, 1)$ and $B_{\text{in}}^\varepsilon \in L^2(\mathbb{T}^3)$ the initial datum. Then, as in Definition 1.2.1, we define the *dynamo growth rate* $\gamma(\varepsilon)$ of (KDE- ε) as

$$\gamma(\varepsilon) := \sup_{B_{\text{in}}^\varepsilon \in L^2(\mathbb{T}^3)} \liminf_{t \rightarrow \infty} \frac{1}{t} \log \|B^\varepsilon(t)\|_{L^2(\mathbb{T}^3)}. \quad (4.1.2)$$

Conversely, we define the *limsup dynamo growth rate* $\overline{\gamma}(\varepsilon)$ by

$$\overline{\gamma}(\varepsilon) := \sup_{B_{\text{in}}^\varepsilon \in L^2(\mathbb{T}^3)} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|B^\varepsilon(t)\|_{L^2(\mathbb{T}^3)}, \quad (4.1.3)$$

We say that u is a fast dynamo (correspondingly limsup fast dynamo) if the following holds true.

$$\textbf{Fast dynamo:} \quad \liminf_{\varepsilon \rightarrow 0} \gamma_\varepsilon > 0$$

$$\textbf{limsup Fast dynamo:} \quad \liminf_{\varepsilon \rightarrow 0} \overline{\gamma}_\varepsilon > 0$$

The definition of a fast dynamo adopted here coincides with that given by Arnold in [Arn04; AK98], whilst the definition of a limsup fast dynamo agrees with the fast dynamo definition introduced by Childress and Gilbert [CG95, see (4.1.2) and (4.1.4)]. We observe that every fast dynamo is necessarily a limsup fast dynamo, since by definition $\gamma_\varepsilon \leq \overline{\gamma}_\varepsilon$; however, the converse does not always hold, unless the velocity field is autonomous or time-periodic, see Lemma 4.1.6. In this work, we resolve the problem of finding a limsup fast dynamo, which corresponds to the formulation of the fast dynamo problem given in [CG95]; however, we believe that establishing the existence of a genuine fast dynamo on $\mathbb{T}^3$ is substantially more difficult.

Theorem 4.1.1. *There exists a divergence free velocity field $u \in L^\infty((0, \infty); W^{1,\infty}(\mathbb{T}^3))$ that is a limsup fast dynamo on $\mathbb{T}^3$.*

As a corollary, since our velocity field u satisfies $\gamma_\varepsilon = 0$ for all $\varepsilon > 0$, we also show that $\gamma_\varepsilon < \overline{\gamma}_\varepsilon$ is possible for time-dependent velocity fields.

We mention that this work was inspired by the recent result of Rowan [Row25], who constructs a smooth velocity field on $\mathbb{T}^3$ that produces exponential magnetic growth along a diverging sequence of times and a vanishing subsequence of diffusivities. That is, it is a limsup fast dynamo, with the liminf in ε being replaced by a limsup $_{\varepsilon \rightarrow 0}$. The improvement in Theorem 4.1.1, compared to the result in [Row25], is that the growth we obtain is uniform as $\varepsilon \rightarrow 0$ and does not occur only along a subsequence. However, it is worth noting that the velocity field constructed in [Row25] is more regular than ours, being $L^\infty([0, \infty); H^m(\mathbb{T}^3))$, for any $m \in \mathbb{N}$, and also possessing temporal smoothness.

4.1.1 Strategy of the proof

The proof starts from a concrete alpha-unstable building block furnished by the ABC family. We consider, as a building block, an ABC flow with $a = 0$ and $b = c = 1$ and with small amplitude

$\delta > 0$, i.e.

$$U(\mathbf{x}) = \delta \begin{pmatrix} \cos(y) \\ \sin(x) \\ \sin(y) + \cos(x) \end{pmatrix}.$$

Recalling the results of Chapter 3, and in particular Example 3.2.6, we know that the alpha-effect gives rise to solutions of (KDE- ε) with velocity field $U(\mathbf{x})$ in the modal form

$$B(t, \mathbf{x}) = \exp(i\mathbf{j} \cdot \mathbf{x} + pt)H(\mathbf{x}), \quad \mathbf{j} \in \mathbb{R}^3, |\mathbf{j}| \ll 1$$

that are exponentially growing in time with $\varepsilon = 1$. Moreover, as noted for instance in the proof of Corollary 2.2.8, any initial datum such that $P_\lambda B \neq 0$, where P is the Riesz projector onto an eigenvalue λ with $\text{Re}(\lambda) > 0$, also gives rise to an exponentially growing solution. In particular, via a perturbative expansion for such a Riesz projector, we prove that the specific mode $B_{\text{in}} = e^{ijz}(-i, 1, 0)$ has a non-zero projection onto the unstable eigenspace, and hence produces exponential growth.

Notably, we will deduce that solutions B^ε to (KDE- ε) with $|\varepsilon - 1| \ll 1$ and initial datum $B(0) \in L^2$ such that $\|B_{\text{in}} - B(0)\|_{L^2} \ll 1$ satisfy

$$\|B^\varepsilon(t)\|_{L^2} \gtrsim e^{\frac{\text{Re}(p)}{2}t} \|B_{\text{in}}\|_{L^2}.$$

Furthermore, we deduce this property also for the rescaled vectors

$$B(t, \mathbf{x}) \rightarrow B^{(n)} := B(t, n\mathbf{x}), \quad U(\mathbf{x}) \rightarrow U^{(n)} := \frac{1}{n}U(n\mathbf{x}), \quad \varepsilon \rightarrow \varepsilon_n := \frac{\varepsilon}{n^2} \quad \forall n \in \mathbb{N},$$

see Proposition 4.3.2 for a precise statement¹. Note that the Lipschitz norm of U with this rescaling is non-increasing. We then construct the time-dependent velocity field u for which growth of solutions will be guaranteed for $\varepsilon \in (\varepsilon_n)_n$, which is inspired by [Row25]. We first observe that the vector $(0, 0, 1)$ is a stationary solution to (KDE- ε) with such velocity fields depending only on (x, y) , and thus take this constant vector as our initial condition in Theorem 4.1.1. In fact, no matter the choice of velocity field, the spatial average $(0, 0, 1)$ of solutions is preserved along the evolution of (KDE- ε). We now fix a sequence that visits each $n \in \mathbb{N}$ infinitely many times – more precisely, we fix a sequence $(a_k)_{k \geq 1}$ of natural numbers such that

$$\forall n \in \mathbb{N}, \quad \#\{k \in \mathbb{N} : a_k = n\} = \infty,$$

and denote by $(t_k)_k$ a suitable sequence of times to be chosen later.

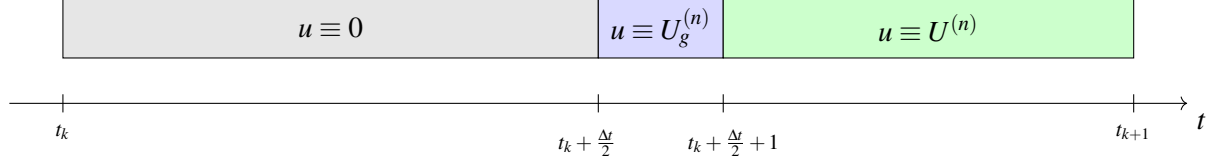
We then use the velocity field $U_g^{(n)}$, defined in (4.4.2), which generates

$$B_{\text{in}}^{(n)}(\mathbf{x}) = B_{\text{in}}(n\mathbf{x})$$

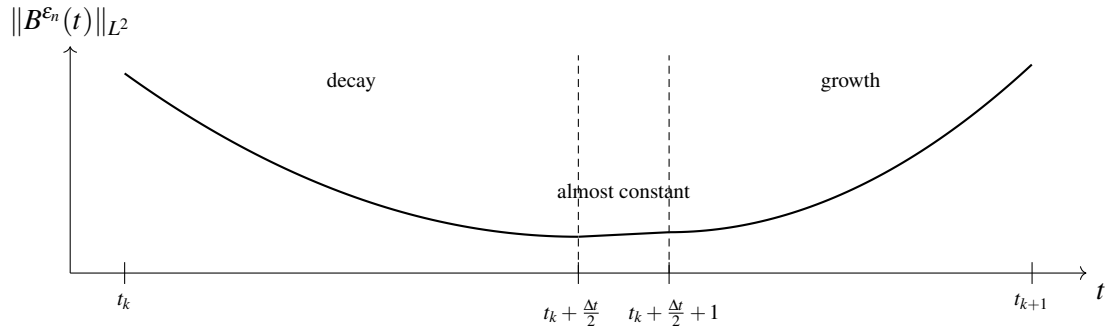
within time 1, starting from the spatially averaged initial state $(0, 0, 1)$. With this newly generated

¹Technically, we also rescale U and B with a fixed number $N_0 \in \mathbb{N}$ sufficiently large so that they become periodic functions. Indeed, B_{in} is not periodic because $|\mathbf{j}| \ll 1$.

initial condition, the solution to (KDE- ε) exhibits exponential growth under the action of $U^{(n)}$. More precisely, if $k \in \mathbb{N}$ is such that $a_{k+1} = n$, then we define the velocity field u on $[t_k, t_{k+1}]$ as follows



Assuming again that $a_{k+1} = n$, then if we choose the sequence $(t_k)_k$ as in (4.4.1), we can prove that the evolution of the magnetic energy of B^{ε_n} , solution to (KDE- ε), satisfies the following



In particular, we can prove that

$$\|B^{\varepsilon_n}(t_{k+1})\|_{L^2} \gtrsim e^{\frac{\text{Re}(p)}{3}t_{k+1}} \|B_{\text{in}}\|_{L^2}.$$

Finally, with Proposition 4.3.2 in hand we can deduce a similar evolution for the solution B^ε with $\varepsilon \in [\varepsilon_n, \varepsilon_{n-1})$. Indeed, Proposition 4.3.2 encodes the continuity of the growth with respect to small perturbation of initial datum and of Δ in the equation (KDE- ε), and thus we deduce that

$$\|B^\varepsilon(t_{k+1})\|_{L^2} \gtrsim e^{\frac{\text{Re}(p)}{4}t_{k+1}} \|B_{\text{in}}\|_{L^2}, \quad \forall \varepsilon \in [\varepsilon_n, \varepsilon_{n-1}).$$

4.1.2 Preliminaries on spectral theory and sectorial operators

Throughout this section, we shall recall some background material from Appendix A that will be needed throughout the chapter. As per usual, we have the following definition of a sectorial operator.

Definition 4.1.2 (Definition A.4.1). Let $(X, \|\cdot\|)$ be a separable Banach space and let $T : \mathcal{D}(T) \subset X \rightarrow X$ be a closed operator with dense domain. We say that T is Sectorial of angle $\theta_0 \in (0, \frac{\pi}{2})$ if there exists a constant $b_0 > 0$ so that the spectrum of T is contained in the sector $S_{\theta_0, b_0} = \{z \in \mathbb{C} : \text{Arg}(b_0 - z) \in [-\frac{\pi}{2} + \theta_0, \frac{\pi}{2} - \theta_0]\}$, and there exists a constant $C > 0$ so that

$$\|(\lambda - T)^{-1}\| \leq C \text{dist}(\lambda, S_{\theta_0, b_0})^{-1}$$

for all $\lambda \notin S_{\theta_0, b_0}$.

We shall need two key results associated to such operators, which are the following.

Lemma 4.1.3 (Lemma A.4.5). *Let $(T, \mathcal{D}(T))$ be a sectorial operator with compact resolvent on some Hilbert space H , with $\sigma(T) \subset S_{\theta_0, b_0}$ as in Definition A.4.1. Then, for all $p < b_0$, and all constants $\zeta > 0$ small enough, there exist contours γ_1, γ_2 so that $\gamma_1 \subset \{\operatorname{Re}(z) \leq p - \zeta\}$, $\gamma_2 \subset \{\operatorname{Re}(z) \geq p - \frac{\zeta}{2}\}$ and the Semigroup generated by T is equal to*

$$e^{Tt} = \frac{1}{2\pi i} \int_{\gamma_1} (z - T)^{-1} e^{zt} dz + \frac{1}{2\pi i} \int_{\gamma_2} (z - T)^{-1} e^{zt} dz =: S_1(t) + S_2(t).$$

Furthermore, there holds $\liminf_{t \rightarrow \infty} \frac{1}{t} \|e^{Tt} x\| \geq p - \frac{\zeta}{2}$ if and only if $Px \neq 0$, where P is the Riesz projector associated to γ_2 as in (A.3.1).

As a consequence of the previous lemma, we obtain a continuity property for sectorial operators with compact resolvent of a particular structure. Specifically, we will apply the lemma below with

$$T_1 = \nabla \times (U \times \cdot), \quad T_0 = \varepsilon \Delta, \quad \varepsilon > 0,$$

to deduce that the operator

$$\kappa T_0 + T_1$$

exhibits exponential growth for $\kappa \approx 1$, assuming the exponential growth for $\kappa = 1$. Moreover, this growth is uniform with respect to small perturbations of the initial data. This lemma is crucial in order to obtain the result $\liminf_{\varepsilon \rightarrow 0}$ in Theorem 4.1.1.

Lemma 4.1.4 (Lemma A.4.6). *Let $T(\kappa) = \kappa T_0 + T_1$, where $(T_0, \mathcal{D}(T_0))$ is a negative, self-adjoint operator with compact resolvent, and for any $\delta > 0$ there exists a constant $C(\delta)$, continuous in δ so that for all $x \in \mathcal{D}(T_0)$ it holds*

$$\|T_1 x\| \leq \delta \|T_0 x\| + C(\delta) \|x\|.$$

Suppose further that there exists $x_0 \in H$ so that $\liminf_{t \rightarrow \infty} \frac{1}{t} \|e^{T(1)t} x_0\| = p > 0$. Then, there exist constants $\kappa_0 > 0$, $\eta_0 > 0$, $C > 0$, $T > 0$ depending only on $\frac{x_0}{\|x_0\|}, T(1)$, so that for all $|1 - \kappa| \leq \kappa_0$, $\|x - x_0\| \leq \eta_0 \|x_0\|$ it holds

$$\|e^{T(\kappa)t} x\| \geq C e^{\frac{p}{2}t} \|x\|, \quad \forall t \geq T.$$

Furthermore, we develop a Taylor expansion of the eigenvalues with respect to j of the operator $T_0 + jT_1$ as $j \rightarrow 0$.

Lemma 4.1.5 (Lemma A.3.6). *Let $T(j) = T_0 + jT_1$, where T_1 is relatively bounded with respect to the closed operator $(T_0, \mathcal{D}(T_0))$ on some Banach space X . Assume that p is an isolated, semisimple eigenvalue of T_0 , and assume further that the eigenvalues of the operator $PT_1P : PX \rightarrow PX$ are all distinct, where P is the Riesz projector corresponding to the eigenvalue p of T_0 . Then, $T(j)$ admits eigenvalues of the form*

$$p_i(j) = p + j\mu_i + o(j),$$

where μ_i are the eigenvalues of the operator $PT_1P : PX \rightarrow PX$. Let $P_i^{(1)}$ be the Riesz projectors associated to the eigenvalues of PT_1P , and denote by $P_i(j)$ the projectors associated to the eigen-

values $p_i(j)$, which act on X by $P_i^{(1)}x = P_i^{(1)}Px$. Then, in the strong operator topology

$$P_i(j) \rightarrow P_i^{(1)}, \quad \text{as } j \rightarrow 0.$$

Finally, we show that for autonomous or time-periodic vector fields on $\mathbb{T}^3$ (or more generally on any compact domain), the dynamo growth rate γ_ε in (4.1.2) coincides with the limsup dynamo growth rate $\overline{\gamma}_\varepsilon$ in (4.1.3).

Lemma 4.1.6. *Let $u \in L^\infty([0, \infty), W^{1,\infty}(\mathbb{T}^3))$ be a time-periodic divergence-free velocity field. Then, the following holds true*

$$\gamma_\varepsilon = \overline{\gamma}_\varepsilon, \quad \forall \varepsilon > 0.$$

Proof. For the proof we set $\varepsilon = 1$ and u be a 1 periodic velocity field to ease readability. Let $S(t, s)$ be the propagator associated to the

$$\partial_t B = \nabla \times (u \times B) + \Delta B,$$

in other words, $S(t, s)B_0$ is the solution to (KDE- ε) with velocity field u and initial condition B_0 starting from time s and posing the equation for $t > s$. Furthermore, let $T = S(1, 0)$ be the monodromy operator associated to the problem. We claim that

$$\log r(T) \leq \gamma_1 \leq \overline{\gamma}_1 \leq \log r(T),$$

where $r(T)$ denotes the spectral radius of T . We firstly prove the lower bound and without loss of generality we assume that $r(T) > 0$. Since T is compact, we can pick an eigenfunction B with eigenvalue $|\lambda| = r(T)$, and then set $B(t) = S(t, 0)B$. Clearly $\|B(n)\|_{L^2} = |\lambda|^n \|B\|_{L^2}$, so it remains to obtain a lower bound in the intermediate region $t \in (n, n+1)$. By standard well-posedness theory, the map $t \mapsto S(t, 0)B$ is continuous from $[0, 1] \mapsto L^2(\mathbb{T}^3)$ and never vanishing. Thus, $\|B(t)\|_{L^2}$ attains a strictly positive minimum on $[0, 1]$, which we call $m > 0$. Thus, for $t \in [n, n+1]$ we bound

$$\|B(t)\|_{L^2} = \|S(t, n)\lambda^n B\|_{L^2} = |\lambda|^n \|S(t-n, 0)B\|_{L^2} \geq m|\lambda|^n.$$

Therefore, it follows that $\gamma_1 \geq \log(|\lambda|) = \log(r(T))$. The upper bound is a straightforward application of the Gelfand spectral radius formula (see e.g. [Kat76])

$$r(T) = \limsup_{n \rightarrow \infty} \|T^n\|^{1/n}$$

and the upper bound $\|B(t)\|_{L^2} \leq \|B(n)\| \exp(\|u\|_{L^\infty W_x^{1,\infty}})$ for any $t \in [n, n+1]$ by energy estimates. Noting that this upper bound also holds in the case $r(T) = 0$, we conclude the proof. $\square$

4.2 Preliminaries on rescaled ABC flows

In this section, we recall some spectral properties of ABC flows that have been studied in Chapter 3 (see also [CSV25]), and prove a few more properties we need for proving Theorem 4.1.1 about

ABC flows with $a = 0$ and $b = c = 1$. We look for solutions in the modal form

$$B(t, \mathbf{x}) = \exp(i\mathbf{j} \cdot \mathbf{x} + pt)H(\mathbf{x})$$

of the kinematic dynamo equation (KDE- ε) with $\varepsilon = 1$. The goal is to prove existence of a modal form solution with $\operatorname{Re}(p) > 0$. We observe that B is a solution to (KDE- ε) with $\varepsilon = 1$ if and only if H solves the eigenvalue problem

$$\mathcal{L}(\mathbf{j})H = pH \quad \text{and} \quad (\nabla + i\mathbf{j}) \cdot H = 0,$$

where

$$\mathcal{L}(\mathbf{j}) = (\nabla + i\mathbf{j}) \times (u \times H) + (\nabla + i\mathbf{j})^2 H.$$

For convenience, we expand $\mathcal{L}(\mathbf{j})$ in $\mathbf{j}$ as

$$\mathcal{L}(\mathbf{j}) = \mathcal{L}_0 + |\mathbf{j}|\mathcal{L}_1 - |\mathbf{j}|^2 \operatorname{Id}$$

where

$$\begin{aligned} \mathcal{L}_0 &= \nabla \times (u \times H) + \Delta H \\ \mathcal{L}_1 &= i \frac{\mathbf{j}}{|\mathbf{j}|} \times (u \times H) + 2i \frac{\mathbf{j}}{|\mathbf{j}|} \cdot \nabla H. \end{aligned}$$

We recall that for any u such that $\|u\|_{W^{1,\infty}} < 1$ we have that $\operatorname{Ker}(\mathcal{L}_0)$ is three dimensional, see Chapter 3, Lemma 3.2.1. We also recall the following result from Chapter 3, Lemma 3.2.2, which we restate here for convenience.

Lemma 4.2.1. *There exist $\delta, \delta_1 > 0$ small enough such that the following holds true. For any $|\mathbf{j}| < \delta_1$ and $u \in W^{1,\infty}(\mathbb{T}^3)$ with zero average and $\|u\|_{W^{1,\infty}} < \delta$, $\mathcal{L}(\mathbf{j})$ admits eigenvalues of the form*

$$p_\ell(\mathbf{j}) = |\mathbf{j}|\mu_\ell + o(|\mathbf{j}|) \in \mathbb{C}, \quad \ell = 1, 2, 3,$$

where μ_ℓ are the (possibly repeated) eigenvalues of the operator

$$P\mathcal{L}_1P : PL^2(\mathbb{T}^3) \rightarrow PL^2(\mathbb{T}^3),$$

where P is the Riesz projector, associated to the eigenvalue 0, meaning that it projects onto $\operatorname{Ker}(\mathcal{L}_0)$ which is a three dimensional space and $PL^2(\mathbb{T}^3) = \operatorname{Ker}(\mathcal{L}_0) \cong \mathbb{C}^3$.

From now on, for notational convenience, we assume that $\delta \in (0, 1)$ and

$$u = \delta U, \quad \text{with} \quad \|U\|_{W^\infty} \leq 1.$$

Under this assumption on U we have that the operator $\Delta + \nabla \times (\delta U \times \cdot) = \Delta(\operatorname{Id} + \Delta^{-1} \nabla \times (\delta U \times \cdot))$ is invertible. For any $H \in L^2(\mathbb{T}^3)$ we denote the average of the vector field as

$$\langle H \rangle = \oint_{\mathbb{T}^3} H$$

and we denote by $L_0^2(\mathbb{T}^3)$ the space of L^2 vector field with zero average. Then, we define

$$\mathcal{S} : \mathbb{C}^3 \rightarrow L_0^2(\mathbb{T}^3)$$

so that for any v , $\mathcal{S}(v)$ is the unique zero average solution to

$$\nabla \times (\delta U \times \mathcal{S}(v)) + \Delta \mathcal{S}(v) = \nabla \times (v \times \delta U).$$

By Neumann series expansion using that $\Delta + \nabla \times (\delta U \times \cdot) = \Delta(\text{Id} + \delta \Delta^{-1} \nabla \times (U \times \cdot))$ we have that $\mathcal{S}(v)$ satisfies

$$\|\mathcal{S}(v) - \delta \Delta^{-1}(\nabla \times (v \times U))\|_{L^2} \leq C \delta^2 |v|, \quad \|\mathcal{S}(v)\|_{L^2} \leq C \delta |v|. \quad (4.2.1)$$

with a constant $C > 0$ depending only on $\|U\|_{W^\infty} \leq 1$. Therefore, from this and Lemma 3.2.3, we deduce the following result.

Lemma 4.2.2. *Let $U \in W^{1,\infty}(\mathbb{T}^3)$ with $\|U\|_{W^{1,\infty}} \leq 1$ and $\delta \in (0, 1)$. Then, the Riesz projector P acts on $L^2(\mathbb{T}^3)$ via*

$$PH = \langle H \rangle + \mathcal{S}(\langle H \rangle),$$

In particular, the Riesz projector P admits the asymptotic expansion for any $\delta \in (0, 1)$, meaning that there exists a universal constant $C > 0$ such that

$$\|PH - \langle H \rangle - \delta \Delta^{-1} \nabla \times (\langle H \rangle \times U)\|_{L^2} \leq C \delta^2 |\langle H \rangle|.$$

We now fix the ABC flow with $a = 0$ and $b = c = 1$ that is independent of z . More precisely, we define

$$U(x, y) = \begin{pmatrix} \cos(y) \\ \sin(x) \\ \sin(y) + \cos(x) \end{pmatrix}. \quad (4.2.2)$$

For such velocity field, the matrix $M_0 \in \mathbb{C}^{3 \times 3}$, defined by

$$M_0(v) = i \frac{\mathbf{j}}{|\mathbf{j}|} \times \oint (\delta U \times (\delta \Delta^{-1} \nabla \times (v \times U))), \quad \forall v \in \mathbb{C}^3,$$

can be computed explicitly. Indeed, fixing $\mathbf{j} = (0, 0, j_3)$ and using Fourier series (thanks to the fact that U is supported on modes $|k| = 1$), we can compute

$$M_0 = i \delta^2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

whose eigenvalues are $\pm \delta^2$ and 0. We now compute the eigenvalues μ_ℓ given in Lemma 4.2.1 of $P \mathcal{L}_1 P$, where P is the Riesz projector onto $\text{Ker}(\mathcal{L}_0)$. By Lemma 4.2.2 we have $P \mathcal{L}_1 PH = \langle \mathcal{L}_1 PH \rangle + \mathcal{S}(\langle \mathcal{L}_1 PH \rangle)$. Using also that U is zero average we deduce that $\langle \mathcal{L}_1 PH \rangle = \frac{i \mathbf{j}}{|\mathbf{j}|} \times \langle \delta U \times$

$\mathcal{S}(\langle H \rangle)$ and by the expansion of $\mathcal{S}(v)$ given by (4.2.1) we conclude

$$\|M_0(\langle H \rangle) - P\mathcal{L}_1PH\|_{L^2} \leq \|M_0(\langle H \rangle) - \langle P\mathcal{L}_1PH \rangle\|_{L^2} + \|\mathcal{S}(\langle P\mathcal{L}_1PH \rangle)\|_{L^2} \leq C\delta^3.$$

Therefore, since the operator M_0 has a positive eigenvalue δ^2 we also deduce that $P\mathcal{L}_1P$ has an eigenvalue μ_1 such that $\operatorname{Re}(\mu_1) > \delta^2/2$ for $\delta > 0$ sufficiently small.

Lemma 4.2.3. *Let $P^{(1)}$ be the Riesz projector of $P\mathcal{L}_1P$ onto the eigenvalue μ_1 such that $\operatorname{Re}(\mu_1) > \delta^2/2$. Then, for any $H \in L^2(\mathbb{T}^3)$, there exists $\mathcal{O}(\delta)H \in L^2(\mathbb{T}^3)$ with $\|\mathcal{O}(\delta)H\|_{L^2(\mathbb{T}^3)} \leq C\delta\|\langle H \rangle\|$ such that the following holds true*

$$P^{(1)}H = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \langle H \rangle + \mathcal{O}(\delta)H.$$

Proof. We consider $\gamma : [0, 1] \rightarrow \mathbb{C}$ to be a contour enclosing the eigenvalue μ_1 , which contains only the eigenvalue μ_1 , and such that $\Gamma = \{\lambda \in \mathbb{C} : \exists t \in [0, 1] : \gamma(t) = \lambda\} \subset \rho(P\mathcal{L}_1P)$. To do so one can just consider a circle of radius $\frac{\delta^2}{2}$ around the point δ^2 , since μ_1 is in the discrete spectrum. We now use the short-hand notation for the matrix $M \in \mathbb{C}^{3 \times 3}$ defined as

$$M(v) = \frac{ij}{|j|} \times \langle \delta U \times \mathcal{S}(v) \rangle, \quad \forall v \in \mathbb{C}^3.$$

A direct computation shows that for any $H \in PL^2(\mathbb{T}^3)$, i.e. by Lemma 4.2.2, $H = \langle H \rangle + \mathcal{S}(\langle H \rangle)$, and any $\lambda \in \Gamma$ we have $(P\mathcal{L}_1P - \lambda)^{-1}H = (M - \lambda)^{-1}\langle H \rangle + \mathcal{S}((M - \lambda)^{-1}\langle H \rangle)$. Indeed, by observing that

$$P\mathcal{L}_1PF = \langle \mathcal{L}_1PF \rangle + \mathcal{S}(\langle \mathcal{L}_1PF \rangle) = M(\langle F \rangle) + \mathcal{S}(M(\langle F \rangle)), \quad \forall F \in L^2$$

if we set $F = (M - \lambda)^{-1}\langle H \rangle + \mathcal{S}((M - \lambda)^{-1}\langle H \rangle)$, we deduce

$$\begin{aligned} (P\mathcal{L}_1P - \lambda)F &= M\langle F \rangle + \mathcal{S}(M\langle F \rangle) - \lambda\langle F \rangle - \lambda\mathcal{S}(\langle F \rangle) \\ &= (M - \lambda)\langle F \rangle + \mathcal{S}((M - \lambda)\langle F \rangle) = \langle H \rangle + \mathcal{S}(\langle H \rangle). \end{aligned}$$

Then, we have

$$\frac{1}{2\pi i} \int_{\Gamma} (P\mathcal{L}_1P - \lambda)^{-1}H d\gamma = \frac{1}{2\pi i} \int_{\Gamma} (M - \lambda)^{-1}\langle H \rangle d\gamma + \mathcal{S} \left(\frac{1}{2\pi i} \int_{\Gamma} (M - \lambda)^{-1}\langle H \rangle d\gamma \right)$$

and using (4.2.1) we just have to consider

$$\frac{1}{2\pi i} \int_{\Gamma} (M - \lambda)^{-1}\langle H \rangle d\lambda$$

up to an error of size $\mathcal{O}(\delta)H$. Thus we compute

$$(M - \lambda)^{-1} = (M_0 - \lambda + (M - M_0))^{-1} = (\operatorname{Id} + (M_0 - \lambda)^{-1}(M - M_0))^{-1}(M_0 - \lambda)^{-1}$$

and since $\|(M_0 - \lambda)^{-1}(M - M_0)\|_{L^2 \rightarrow L^2} \leq C\delta^{-2}\delta^3 = C\delta$ we reduce the computation to

$$\frac{1}{2\pi i} \int_{\Gamma} (M_0 - \lambda)^{-1} \langle H \rangle d\lambda,$$

up to errors of size $\mathcal{O}(\delta)H$. Parametrizing the contour as $\gamma = \delta^2 + \frac{1}{2}\delta^2\eta$, where η traces out a circle of radius 1 in the complex plane denoted by Γ_1 , by changing variables we deduce

$$\frac{1}{2\pi i} \int_{\Gamma} (M_0 - \lambda)^{-1} d\gamma = \frac{1}{2\pi i} \int_{\Gamma_1} (M_0 - (1 + \frac{1}{2}\eta))^{-1} d\eta.$$

By an explicit computation $(M_0 - (1 + \frac{1}{2}\eta))^{-1}$ is given by

$$(M_0 - (1 + \frac{1}{2}\eta))^{-1} = \begin{pmatrix} \frac{1+\frac{1}{2}\eta}{1-(1+\frac{1}{2}\eta)^2} & \frac{i}{(1+\frac{1}{2}\eta)^2-1} & 0 \\ \frac{-i}{(1+\frac{1}{2}\eta)^2-1} & \frac{1+\frac{1}{2}\eta}{1-(1+\frac{1}{2}\eta)^2} & 0 \\ 0 & 0 & -\frac{1}{(1+\frac{1}{2}\eta)} \end{pmatrix},$$

so integrating this around a circle Γ_1 and using the Cauchy integral formula yields

$$\frac{1}{2\pi i} \int_{\Gamma} (M_0 - (1 + \frac{1}{2}\eta))^{-1} d\eta = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

concluding the proof. $\square$

Lemma 4.2.4. *Let $B_{in}(x, y, z) = e^{ijz}(-i, 1, 0)$, then there exist $\delta, j \in (0, 1)$ sufficiently small such that the following holds true. There exists $\lambda, C > 0$ such that the solution B to (KDE- ε), with $\varepsilon = 1$ and velocity field $u = \delta U$ with U the ABC flow with $a = 0$ and $b = c = 1$ defined in (4.2.2), satisfies*

$$\|B(t)\|_{L^2} \geq Ce^{\lambda t} \|B_{in}\|_{L^2} \quad \forall t \geq 0.$$

Proof. Let P be the Riesz projector of $\mathcal{L}_0$ onto $\text{Ker}(\mathcal{L}_0)$ and $P^{(1)}$ be the Riesz projector of $P\mathcal{L}_1P$ onto the eigenvalue μ_1 with $\text{Re}(\mu_1) > \delta^2/2$, which, thanks to Lemma 4.2.3 is of the form

$$P^{(1)}H = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \langle H \rangle + \mathcal{O}(\delta)H, \quad \forall H \in L^2.$$

We define $H_{in} = (-i, 1, 0)$ and we notice that $PH_{in} \neq 0$, since $H_{in} = \langle H_{in} \rangle \neq 0$. Then, $P^{(1)}H_{in} = P^{(1)}PH_{in}$ and

$$P^{(1)}PH_{in} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} + \mathcal{O}(\delta) = -\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} + \mathcal{O}(\delta).$$

Therefore, for $\delta > 0$ small enough, $P^{(1)}H_{in} \neq 0$. Hence, letting P_j be the Riesz projection onto

the eigenvalue $|\mathbf{j}|(\delta^2 + \mathcal{O}(\delta^3))$ of $\mathcal{L}(\mathbf{j})^2$, we deduce that for $|\mathbf{j}|$ small enough it holds $P_{\mathbf{j}}H_{\text{in}} \neq 0$ thanks to Lemma A.3.6. Finally, we note that $\mathcal{L}(\mathbf{j})$ generates an analytic semigroup, and has compact resolvent. Therefore, by Lemma A.4.5 we immediately deduce the claimed growth for $|\mathbf{j}|$, δ small enough. $\square$

4.3 Main proposition

In this section, we state and prove the main proposition concerning our building block velocity field, which asserts that the exponential growth of solutions persists under small perturbations of ε and of the initial data. More precisely, we fix $N_0 \in \mathbb{N}$ and $\delta > 0$ given in Proposition 4.3.2 and for any $n \in \mathbb{N}$ let $U^{(n)} \in C^\infty(\mathbb{T}^3)$ be the divergence-free velocity field

$$U^{(n)}(\mathbf{x}) = \frac{\delta}{N_0 n} \begin{pmatrix} \cos(nN_0 y) \\ \sin(nN_0 x) \\ \sin(nN_0 y) + \cos(nN_0 x) \end{pmatrix} \quad (4.3.1)$$

which is a rescaling of a particular choice of an ABC flow. We also denote by $\mathcal{S}_n^\varepsilon(t)$ the semigroup generated by equation (KDE- ε) with velocity field $U^{(n)}$. In particular, we denote by

$$\mathcal{S}_n^\varepsilon(t)B_{\text{in}} := B^\varepsilon(t)$$

the solution to (KDE- ε) under the action of the velocity field $\bar{u}_n$, initial datum B_{in} and magnetic diffusivity $\varepsilon > 0$. Let $j_0 \in \mathbb{N}$, then we slightly abuse the notation and we denote by

$$B_{\text{in}}^{(n)}(\mathbf{x}) = e^{inz}(-i, 1, 0) + (0, 0, 1). \quad (4.3.2)$$

Remark 4.3.1. We add the constant vector $(0, 0, 1)$ to the initial datum, unlike in Section 4.2. Although it is a stationary solution to (KDE- ε) for the velocity field $U^{(n)}$ and thus irrelevant here, it will become essential in Section 4.4. We include it now for convenience.

Proposition 4.3.2. *There exist $\lambda > 0$, $\delta > 0$, $N_0, \bar{n} \in \mathbb{N}$ such that the following holds true. For any $n \geq \bar{n}$ sufficiently large there exist $\eta_n, c_n \in (0, 1/2)$ such that the solution B to (KDE- ε) with velocity field $U^{(n)}$ defined in (4.3.1), initial datum B_{in} satisfying $\|B_{\text{in}} - B_{\text{in}}^{(n)}\|_{L^2} < \eta_n \|B_{\text{in}}^{(n)}\|_{L^2}$ and magnetic diffusivity $\varepsilon \in [\frac{1}{N_0^2 n^2}, \frac{1}{N_0^2 (n-1)^2})$ satisfies*

$$\|B(t)\|_{L^2} \geq c_n e^{\lambda t} \|B_{\text{in}}\|_{L^2} \quad \forall t \geq 0.$$

The proof of this proposition follows from the following lemma that is a consequence of the analysis carried out in Section 4.2.

Lemma 4.3.3. *There exist $N_0 \in \mathbb{N}$, $\delta > 0$ and $\lambda > 0^3$ such that the solution B^{ε_1} to (KDE- ε) with*

²Here $\mathbf{j} = (0, 0, j)$.

³This $\lambda > 0$ is not the same the one in Proposition 4.3.2.

$\varepsilon = \frac{1}{N_0^2} > 0$, $u = U_1$ defined in (4.3.1) with $n = 1$, and $B_{\text{in}}^{(1)}(x, y, z) = e^{iz}(-i, 1, 0) + (0, 0, 1)$ satisfies

$$\|B^{\varepsilon_1}(t)\|_{L^2(\mathbb{T}^3)} \geq \frac{1}{2}e^{\lambda t}\|B_{\text{in}}\|_{L^2(\mathbb{T}^3)}, \quad \forall t \geq 0.$$

Proof. We notice that $B^{\varepsilon_1}(t, \mathbf{x})$ is a solution to (KDE- ε) with $\varepsilon = \frac{1}{N_0^2}$, velocity field $u(\mathbf{x}) = U^{(n)}(\mathbf{x})$ and initial datum $B_{\text{in}}(\mathbf{x})$ if and only if $B(t, \mathbf{x}) = B^{\varepsilon_1}(t, \frac{\mathbf{x}}{N_0})$ solves (KDE- ε) with $\varepsilon = 1$, and velocity field

$$N_0 U^{(1)}(\frac{\mathbf{x}}{N_0}) = \delta \begin{pmatrix} \cos(y) \\ \sin(x) \\ \sin(y) + \cos(x) \end{pmatrix},$$

and initial datum $B_{\text{in}}(\frac{\mathbf{x}}{N_0})$. In particular, choosing N_0 sufficiently large so that $j = \frac{1}{N_0}$ is sufficiently small and $\delta > 0$ is sufficiently small we can apply Lemma 4.2.4 and deduce

$$\|B(t)\|_{L^2((0, 2\pi)^3)} \geq \frac{1}{2}e^{\lambda t}\|B_{\text{in}}(\frac{\cdot}{N_0})\|_{L^2((0, 2\pi)^3)}, \quad \forall t \geq 0.$$

Noticing that B takes the form $B(t, x, y, z) = e^{ijz}e^{\mathcal{L}(j)t}(-i, 1, 0) + (0, 0, 1)$ and in particular is 2π periodic in x, y , we deduce that

$$\|B^{\varepsilon_1}(t)\|_{L^2(\mathbb{T}^3)} = \|B(t)\|_{L^2((0, 2\pi)^3)} \geq \frac{1}{2}e^{\lambda t}\|B_{\text{in}}(\frac{\cdot}{N_0})\|_{L^2((0, 2\pi)^3)} = \frac{1}{2}e^{\lambda t}\|B_{\text{in}}\|_{L^2(\mathbb{T}^3)},$$

where the last hold true thanks to the explicit formula $|B_{\text{in}}(\mathbf{x})| = \sqrt{3}, \forall \mathbf{x}$. □

Proof of Proposition 4.3.2. By Lemma A.4.6 applied to the result from Lemma 4.3.3, there exists some constants $\omega, c_1 > 0$ such that the following holds true. The solution $B^{(1)}$ to (KDE- ε) with $\varepsilon \in [\frac{1}{N_0^2} - \omega, \frac{1}{N_0^2} + \omega]$, velocity field U_1 defined in (4.3.1) with $n = 1$ and initial datum $B_{\text{in}}^{(1)} = e^{iz}(-i, 1, 0) + (0, 0, 1)$ satisfies

$$\|B^{(1)}(t)\|_{L^2(\mathbb{T}^3)} \geq c_1 e^{\frac{\lambda}{2}t}, \quad \forall t \geq 0.$$

Let $B^{(n)}(t, \mathbf{x}) = B(t, n\mathbf{x})$, it follows that $B^{(n)}$ solves

$$\partial_t B^{(n)} = \nabla \times (U^{(n)} \times B^{(n)}) + \frac{\varepsilon}{n^2} \Delta B^{(n)}$$

with initial condition $B_{\text{in}}^{(n)} = e^{izn}(-i, 1, 0) + (0, 0, 1)$. By periodicity on $\mathbb{T}^3$ we also have $\|B^{(n)}(t)\|_{L^2(\mathbb{T}^3)} = \|B^{(1)}(t)\|_{L^2(\mathbb{T}^3)}$ for all $t \geq 0$. Hence, we have shown that for any $n \in \mathbb{N}$, $\varepsilon \in [\frac{1}{n^2}(\frac{1}{N_0^2} - \omega), \frac{1}{n^2}(\frac{1}{N_0^2} + \omega)]$, there exists a solution to

$$\partial_t B = \nabla \times (U^{(n)} \times B) + \varepsilon \Delta B,$$

that grows exponentially at rate at least $\frac{\lambda}{2}$. Fix now $n \in \mathbb{N}$. For all $\varepsilon \in [\frac{1}{n^2}(\frac{1}{N_0^2} - \omega), \frac{1}{n^2}(\frac{1}{N_0^2} + \omega)]$ we now apply Lemma A.4.6 to deduce the existence of positive constants $\eta_\varepsilon, \omega_\varepsilon, c_\varepsilon > 0$ so that

the following holds true. The solution to

$$\partial_t B = \nabla \times (U^{(n)} \times B) + \varepsilon' \Delta B$$

with ε' satisfying $|\varepsilon' - \varepsilon| < \omega_\varepsilon$, initial datum B_{in} satisfying $\|B_{\text{in}} - e^{inz}(-i, 1, 0) - (0, 0, 1)\|_{L^2(\mathbb{T}^3)} \leq \eta_\varepsilon$ grows exponentially as

$$\|B(t)\|_{L^2(\mathbb{T}^3)} \geq c_\varepsilon e^{\frac{\lambda}{4}t} \|B_{\text{in}}\|_{L^2}, \quad \forall t \geq 0.$$

Since we can cover the set $I = [\frac{1}{n^2}(\frac{1}{N_0^2} - \omega), \frac{1}{n^2}(\frac{1}{N_0^2} + \omega)]$ by

$$I \subset \bigcup_{\varepsilon \in I} (\varepsilon - \omega_\varepsilon, \varepsilon + \omega_\varepsilon),$$

then, by compactness of I , we can extract a finite subcover so that

$$I \subset \bigcup_{i=1}^N (\varepsilon_i - \omega_{\varepsilon_i}, \varepsilon_i + \omega_{\varepsilon_i})$$

We define $\eta_n = \frac{1}{\sqrt{2}} \inf_{i=1, \dots, N} \eta_{\varepsilon_i}$, $c_n = \inf_{i=1, \dots, N} c_{\varepsilon_i}$. Thus, we have proven that for any $\varepsilon \in [\frac{1}{n^2}(\frac{1}{N_0^2} - \omega), \frac{1}{n^2}(\frac{1}{N_0^2} + \omega)]$ there exist $\eta_n, c_n > 0$ such that the following holds true. The solution B of

$$\partial_t B = \nabla \times (U^{(n)} \times B) + \varepsilon \Delta B$$

with $\varepsilon \in [\frac{1}{n^2}(\frac{1}{N_0^2} - \omega), \frac{1}{n^2}(\frac{1}{N_0^2} + \omega)]$, initial datum B_{in} satisfying $\|B_{\text{in}} - B_{\text{in}}^{(n)}\|_{L^2(\mathbb{T}^3)} < \sqrt{3}\eta_n = \eta_n \|B_{\text{in}}^{(n)}\|_{L^2}$ grows exponentially as

$$\|B(t)\|_{L^2(\mathbb{T}^3)} \geq c_n e^{\frac{\lambda}{4}t} \|B_{\text{in}}\|_{L^2}.$$

Noticing that $[\frac{1}{N_0^2 n^2}, \frac{1}{N_0^2 (n-1)^2}] \subset [\frac{1}{n^2}(\frac{1}{N_0^2} - \omega), \frac{1}{n^2}(\frac{1}{N_0^2} + \omega)]$ for any $n \geq \bar{n}$ where $\bar{n}$ depends only on ω , we conclude the proof. $\square$

4.4 Construction of the velocity field

4.4.1 Parameters

Firstly, we define the sequence of diffusivity parameters

$$\varepsilon_n = \frac{1}{N_0^2 n^2} \quad \forall n \in \mathbb{N}.$$

Then, we fix a sequence that visits each $n \in \mathbb{N}$ infinitely many times. More precisely, we fix a sequence $(a_k)_{k \geq 1}$ of natural numbers such that

$$\forall n \in \mathbb{N}, \quad \#\{k \in \mathbb{N} : a_k = n\} = \infty,$$

where $\#$ denotes the cardinality of the set. Furthermore, for every $n \in \mathbb{N}$, we denote by $(k_j)_{j \geq 1}$ an infinite strictly increasing sequence of indices such that

$$a_{k_j} = n \quad \text{for all } j \geq 1,$$

where we drop the dependence on n of the sequence $(k_j)_j$ to simplify the notation.

One explicit example is the “diagonal” sequence

$$a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 1, a_5 = 2, a_6 = 3, a_7 = 1, \dots$$

Finally, we set $t_0 = 0$ and define recursively the sequence of times $(t_k)_{k \geq 1}$ as follows. Suppose we have defined $t_0, t_1, \dots, t_{k-1}$. Let $n \in \mathbb{N}$ so that $a_k = n$, we choose t_k large enough so that

$$t_k \geq t_{k-1} + 2n^2 t_{k-1} + 2n^2 \log\left(\frac{10}{\eta_n}\right) + \frac{10}{\lambda} \log(C_n), \quad (4.4.1)$$

where $\lambda, C_n, \eta_n > 0$ are the constants given in Proposition 4.3.2.

4.4.2 Definition of the velocity field

Firstly, we define another building block $U_g^{(n)} : \mathbb{T}^3 \rightarrow \mathbb{R}^3$ which is needed to create an initial datum close to (4.3.2) so that we can apply Proposition 4.3.2. For any $n \in \mathbb{N}$ we simply define the 1-Lipschitz velocity field

$$U_g^{(n)}(x, y, z) = \frac{e^{inz}}{in}(-i, 1, 0). \quad (4.4.2)$$

We define the velocity field on each time interval $t \in [t_{k-1}, t_k]$ for any $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ so that $a_k = n$. Then, denoting by

$$\Delta t = 4n^2 N_0^2 t_{k-1} + 4n^2 N_0^2 \log\left(\frac{10}{\eta_n}\right) + \frac{4}{\lambda} \log(C_n),$$

and we define

$$u(t, \cdot) = \begin{cases} 0 & \text{if } t \in [t_{k-1}, t_{k-1} + \frac{\Delta t}{2}) \\ U_g^{(n)}(\cdot) & \text{if } t \in [t_{k-1} + \frac{\Delta t}{2}, t_{k-1} + \frac{\Delta t}{2} + 1) \\ U^{(n)}(\cdot) & \text{if } t \in [t_{k-1} + \frac{\Delta t}{2} + 1, t_k) \end{cases} \quad (4.4.3)$$

Lemma 4.4.1. *The velocity field defined in (4.4.3) satisfies*

$$\|u(t)\|_{W^{1,\infty}(\mathbb{T}^3)} \leq 2 \quad \forall t > 0.$$

Lemma 4.4.2. *Let $u = U_n^{(c)} \in C^\infty(\mathbb{T}^3)$ and $B_{in} = (0, 0, 1) + B_{in, err}$ with $\|B_{in, err}\|_{L^2} < \eta_n/10$ and B be the solution to (KDE- ε) with $\varepsilon \in [\frac{1}{N_0^2 n^2}, \frac{1}{N_0^2 (n-1)^2})$ and $n \geq 2$ where $\varepsilon_\star$ is given by Proposition*

4.3.2. Then there exists $\alpha \in [1, 2]$ such that

$$\|\alpha B(1) - B_{in}^{(n)}\|_{L^2} < \eta_n \|B_{in}^{(n)}\|_{L^2}$$

Proof. Let $\tilde{B}$ be the solution to (KDE- ε) with initial condition $\tilde{B} = (0, 0, 1)$. Then, by explicit computation we have

$$\tilde{B}(t, z) = \varphi(t, z)(-i, 1, 0)$$

where

$$\varphi(t, z) = \int_0^t e^{-\varepsilon n^2(t-s)} e^{inz} ds = \frac{1}{\varepsilon n^2} (1 - e^{-\varepsilon n^2}) e^{inz}.$$

Since $\varepsilon \in [\frac{1}{N_0^2 n^2}, \frac{1}{N_0^2 (n-1)^2})$ we deduce that $\alpha^{-1} = \frac{1}{\varepsilon n^2} (1 - e^{-\varepsilon n^2}) \in [1/2, 1]$. Hence, $\alpha \tilde{B}(1) = B_{in}^{(n)}$ with $\alpha \in [1, 2]$.

Finally, we observe that $\|u\|_{W^{1,\infty}} \leq 2$ and by energy estimate the solution B_{err} with initial condition $B_{in, err}$ satisfies

$$\|B_{err}(t)\|_{L^2} \leq e^t \|B_{in, err}\|_{L^2}.$$

Therefore, we conclude by the linearity of the equation. $\square$

4.5 Proof of Theorem 4.1.1

We fix $\varepsilon_* \in (0, \frac{1}{N_0^2 \bar{n}^2})$ where $N_0 \in \mathbb{N}$ and $\bar{n} \geq 2$ are natural numbers so that Proposition 4.3.2 applies. We define the initial condition

$$B_{in} = (0, 0, 1)$$

and we denote by B the solution of the passive vector equation (KDE- ε) with velocity field $u \in L^\infty$ and diffusivity parameter $\varepsilon \in (0, \varepsilon_*)$. Since $\varepsilon \in (0, \varepsilon_*)$, there exists $n \in \mathbb{N}$ so that $\varepsilon \in [\frac{1}{N_0^2 n^2}, \frac{1}{N_0^2 (n-1)^2})$, where $n \geq 2$ is sufficiently large so that Proposition 4.3.2 applies. We now fix an increasing subsequence $(k_j)_j \subset \mathbb{N}$ such that the sequence $(a_k)_k$ defined in Section 4.4.1 satisfies

$$a_{k_j} = n \quad \forall j.$$

Then, for the sequence of times $(t_k)_{k \in \mathbb{N}}$ defined in Section 4.4.1 we claim that

$$\|B(t_{k_j})\|_{L^2} \geq e^{\frac{\lambda t_{k_j}}{4}} \|B_{in}\|_{L^2} \quad \forall j \in \mathbb{N}$$

from which we deduce the theorem, since $\lambda > 0$ given by Proposition 4.3.2. Let $j \in \mathbb{N}$ be arbitrary. By standard energy estimates using that $\|\nabla u\|_{L^\infty} \leq 2$ we have that

$$\|B(t_{k_{j-1}})\|_{L^2} \leq e^t \|B_{in}\|_{L^2}.$$

Thanks to $u \equiv 0$ on $t \in [t_{k_{j-1}}, t_{k_{j-1}} + \Delta t/2)$ by definition (4.4.3) and the fact that the average of B is conserved in time we conclude by the heat semigroup property that

$$\|B(t) - (0, 0, 1)\|_{L^2} \leq e^{-\varepsilon(t-t_{k_{j-1}})/2} \|B(t_{k_{j-1}}) - (0, 0, 1)\|_{L^2} \quad \forall t \in [t_{k_{j-1}}, t_{k_{j-1}} + \frac{\Delta t}{2}).$$

Thanks to $\varepsilon \geq \frac{1}{N_0^2 n^2}$ and $\Delta t \geq 4N_0^2 n^2 t_{k-1} + 4N_0^2 n^2 \log(\frac{10}{\eta_n})$ we deduce

$$\|B(t_{k_j-1} + \frac{\Delta t}{2}) - (0, 0, 1)\|_{L^2} \leq \frac{\eta_n}{10}.$$

By definition of u on $t \in [t_{k_j-1} + \frac{\Delta t}{2}, t_{k_j-1} + \frac{\Delta t}{2} + 1)$ and Lemma 4.4.2, recalling the notation for $B_{\text{in},n}^{(p)}$ in (4.3.2) we deduce that there exists $\alpha \in [1, 2]$ such that

$$\|\alpha B(t_{k_j-1} + \frac{\Delta t}{2} + 1) - B_{\text{in}}^{(n)}\|_{L^2} < \eta_n \|B_{\text{in}}^{(n)}\|_{L^2}.$$

Finally, using the linearity of the equation we apply Proposition 4.3.2 and using the bound on t_{k_j} given by (4.4.1) we deduce that

$$\begin{aligned} \|B(t_{k_j})\|_{L^2} &\geq \frac{C_n}{\alpha} e^{\lambda(t_{k_j} - (t_{k_j-1} + \frac{\Delta t}{2} + 1))} \|B(t_{k_j-1} + \frac{\Delta t}{2} + 1)\|_{L^2} \\ &\geq \frac{C_n}{4} e^{\lambda t_{k_j}/3} \|B_{\text{in}}^{(n)}\|_{L^2} \\ &\geq e^{\lambda t_{k_j}/4} \|B_{\text{in}}\|_{L^2}. \end{aligned}$$

Chapter 5

A pulsed fast dynamo on $\mathbb{T}^3$.

In this chapter, we construct a map-based fast dynamo on $\mathbb{T}^3$ (as in Definition 2.3.1), where the map we employ arises as the time-1 flow map of a Lipschitz continuous velocity field on $\mathbb{T}^3$. As such, besides being the first (to the best of our knowledge) rigorous fast dynamo result with a map arising from a flow on $\mathbb{T}^3$, we also bridge the gap between the map-based dynamos of Definition 2.3.1 and the continuous time dynamo action of 1.2.1. The contents of this chapter are contained in the paper [CSV26].

5.1 Introduction

In this Chapter, we resolve the fast dynamo conjecture for a model involving *pulsed* diffusion. Specifically, we consider an approximation where advection and diffusion act alternately over unit time intervals. The pulsed kinematic dynamo equation is defined by

$$\begin{cases} \partial_t B = \nabla \times (u \times B), & t \in [2k, 2k+1) \\ \partial_t B = \varepsilon \Delta B, & t \in [2k+1, 2k+2), \end{cases} \quad (\text{P-KDE})$$

for any $k \in \mathbb{N}$. Our main result is as follows.

Theorem 3. *The fast dynamo conjecture holds for the pulsed model (P-KDE) with a time-periodic, Lipschitz, and divergence-free velocity field.*

The significance of this result is threefold. First, the growth parameters we obtain are independent of $\varepsilon \in (0, 1]$, demonstrating the sharpness of the general upper bound (2.1.2), which is valid for any solution to (P-KDE). Second, to the best of our knowledge, this is the first result—even in the context of pulsed diffusion—that constructs a genuine fast dynamo on $\mathbb{T}^3$ driven by a Lipschitz velocity field, rather than a discrete-time map. Third, our proof rigorously establishes the existence of an isolated eigenvalue for the ideal dynamo operator (for $\varepsilon = 0$, see Theorem 5.2.7) in a suitable space of distributions and in the limit of asymptotically large stretching—a result long conjectured for flows [CG95; Sow93], but never previously proven.

5.1.1 The Lagrangian picture and the transfer operator approach

Our approach is intrinsically Lagrangian, hinging on the representation formula for the ideal dynamo equation ($\varepsilon = 0$), given by Lemma 2.1.2, which we recall here for convenience.

$$B(t, x) = (D\Phi_t B_{\text{in}}) \circ \Phi_t^{-1}(x), \quad \frac{d}{dt}\Phi_t(x) = u(t, \Phi_t(x)), \quad \Phi_0(x) = x, \quad (5.1.1)$$

where $\nabla \cdot B_{\text{in}} = 0$. Consequently, the ideal dynamo evolution is determined by the flow map Φ_t and its derivative. By restricting attention to 1-periodic velocity fields, this evolution defines a linear operator $\mathcal{L}h := (D\Phi_1 h) \circ \Phi_1^{-1}$, which is the pushforward operator induced by Φ_1 acting on vector fields. In the language of dynamical systems, $\mathcal{L}$ may be viewed as a *vector-valued transfer operator*—the natural extension to vector fields of the scalar transfer operator $h \circ \Phi_1^{-1}$ acting on functions.

To identify a regime where $\mathcal{L}$ admits unstable eigenvalues, we construct a velocity field u that implements the *stretch-fold-shear* (SFS) mechanism [CG95] (see Section 5.2.1). This model utilizes a pair of alternating shear flows to generate chaotic advection in the (x, y) -plane, a strategy that has been used extensively to establish optimal exponential mixing and enhanced dissipation rates [ELM25; Pie94; BCG23; CIS25; Che+23]. However, such two-dimensional dynamics are insufficient for dynamo action; as mandated by the Zeldovich anti-dynamo theorem 2.2.3, any purely planar flow eventually leads to the decay of the mean-free magnetic field. To circumvent this topological restriction, we introduce a third, out-of-plane shear component. This SFS paradigm has long been a cornerstone of numerical and theoretical investigations into the fast dynamo problem [BC88b; Gil93; DO93; FO90; Gil05; PMG23b; Sow94], as it provides a minimal setting that reconciles the simplicity of alternating shears with the three-dimensional requirements for magnetic growth. Notably, in Section 2.3.1 we have already shown how this paradigm can be implemented in simple maps to rigorously establish fast dynamo action.

5.1.2 Uniform hyperbolicity and anisotropic Banach spaces

As remarked in great detail in Section 2.2, the primary obstacle in establishing the fast dynamo property of Definition 1.2.1, is the singular nature of the vanishing resistivity limit $\varepsilon \rightarrow 0$. To overcome this spectral instability, we move beyond the L^2 framework and utilize the *uniform hyperbolicity* of the flow. The guiding heuristic is that fast dynamo action should be encoded in a limiting eigenmode as $\varepsilon \rightarrow 0$; however, it has long been known that such a limiting eigenfunction cannot be smooth [MP85], see also [CG95, Section 4.3]. Rather, as already noted in [CG95; Ose93; Chi93], one expects the limiting eigenfunction to belong to a suitable space of distributions, adapted to the chaotic features of the underlying velocity field [Sow93]. This suggests a clear strategy: first identify the correct distribution space on which the limiting eigenfunction lives, then establish favorable spectral properties of the ideal operator on that space, and finally treat $\varepsilon\Delta$ via singular perturbation theory.

The strategy of extending operators to distributional spaces in order to recover improved spectral properties has been extremely successful in hyperbolic dynamics. Without attempting an exhaustive overview, we mention for instance [Rug92; Kit99; BKL02; GL06; DL08; BT07], where

so-called *anisotropic Banach spaces* were employed. The key insight behind these works is that, for uniformly hyperbolic maps (with invariant expanding and contracting directions), the associated *scalar transfer operator* acts anisotropically: it improves the regularity of functions along the expanding direction, and the regularity of distributions along the contracting direction. This anisotropy can be exploited by working on Banach spaces that impose different regularity requirements in different tangent directions. In our vector-valued context, this ensures that the essential spectrum is restricted to a disk of radius strictly less than the spectral radius, leaving the leading spectral behavior to be governed by isolated eigenvalues—the so-called *Ruelle resonances*—which remain stable under the diffusive perturbation [KL99]. This perspective is reinforced by [DZ15], which shows that for smooth contact Anosov flows, resonances are stable under the addition of a sufficiently small elliptic perturbation. The comparison with the fast dynamo problem is therefore quite suggestive.

There are, however, two substantial obstacles in our setting. First, since smooth Anosov flows do not exist on $\mathbb{T}^3$, we must develop anisotropic Banach spaces adapted to *piecewise smooth* dynamics. Second, although existing anisotropic space theory provides sharp control over the essential spectral radius, much less is known regarding the identification of the *leading* spectral radius. For our purposes, it is crucial not only to produce an isolated resonance, but to rigorously establish that its modulus is *strictly larger than 1*. To achieve this, we analyze the vector-valued transfer operator in the strong-chaos limit, namely, when both stretching and mixing tend to infinity (see Section 5.2.1). In this regime, the leading resonance can be identified explicitly (Theorem 5.2.7). Notably, as a by-product of our analysis, the velocity field constructed in Theorem 3 can also be shown to be a *perfect dynamo*; see Corollary 5.2.10. In particular, it provides an explicit example of a velocity field for which the flux conjecture 2 is valid.

5.2 Main ideas and outline of the proof

The proof of our main Theorem 3 relies on a quantitative analysis of a family of transfer operators in the so-called strong-chaos regime. This section provides the geometric intuition behind the construction of our velocity field (Section 5.2.1), the general perturbative setting developed by Keller and Liverani [KL99] (Section 5.2.2), and the key points of the proof of our main Theorem 3 (Section 5.2.3).

5.2.1 The stretch-fold-shear velocity field

To generate the hyperbolic dynamics required for our analysis, we draw inspiration from the recent study of optimal mixing in [ELM25] and the classical *stretch-fold-shear* (SFS) model of dynamo theory [CG95].

Fix $\alpha \in 2\mathbb{N}$. For $(x, y, z) \in \mathbb{T}^3$, we define the constituent shear flows as

$$V_\alpha(x, y) = \begin{pmatrix} 0 \\ 4\alpha|x - 1/2| \\ 0 \end{pmatrix}, \quad H_\alpha(x, y) = \begin{pmatrix} 4\alpha|y - 1/2| \\ 0 \\ 0 \end{pmatrix}, \quad Z(x, y) = \begin{pmatrix} 0 \\ 0 \\ -2g(x, y) \end{pmatrix}, \quad (5.2.1)$$

where $g : \mathbb{T}^2 \rightarrow \mathbb{R}$ is a C^1 function. The time-periodic velocity field u_α is defined piecewise over the unit interval $[0, 1)$ by:

$$u_\alpha(t, x, y, z) = \begin{cases} V_\alpha(x, y), & t \in [0, 1/4), \\ H_\alpha(x, y), & t \in [1/4, 1/2), \\ Z(x, y), & t \in [1/2, 1), \end{cases} \quad (5.2.2)$$

and is extended periodically for all $t \geq 0$. By construction, u_α is divergence-free, as each constituent flow is a shear depending only on the coordinates transverse to its direction. The sequence of V_α and H_α constitutes a piecewise-linear approximation of a stretch-and-fold mechanism in the (x, y) -plane, while the third component Z provides the necessary shearing displacement.

Remark 5.2.1. The velocity field can be made smooth in time, without affecting the proofs, but not in space, where the arguments break down.

The first two components of u_α generate, through their flow map over $t \in [0, 1/2]$, a class of *uniformly hyperbolic maps* $T_\alpha : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ (see Section 5.3 for details). This specific map was used in [ELM25] to prove optimal exponential mixing. While one might expect such highly chaotic maps to naturally admit growing resonances, classical anti-dynamo theorems [Zel92; Ose93] preclude dynamo action for purely two-dimensional velocity fields. Crucially, as shown in [Ose93], this restriction extends to discrete-time maps. To circumvent this, we employ the SFS model [CG95] by introducing the large-scale shear Z in the z -direction. Physically, this out-of-plane shear periodically modulates the phase of the magnetic field, preventing the destructive interference that would otherwise lead to field decay.

To formalize the Lagrangian intuition, we compute the α -dependent time-1 map associated to u_α as

$$\mathcal{T}_\alpha(x, y, z) = \begin{pmatrix} T_\alpha(x, y) \\ z - g(T_\alpha(x, y)) \end{pmatrix}.$$

Next, we formulate an Ansatz for the initial magnetic field B_{in} . Let $H : \mathbb{T}^2 \rightarrow \mathbb{R}^3$ have planar components $h : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ and vertical component $H^{(3)} : \mathbb{T}^2 \rightarrow \mathbb{R}$. We consider an initial datum of the form

$$B_{\text{in}}(x, y, z) = e^{2\pi iz} \begin{pmatrix} h(x, y) \\ H^{(3)}(x, y) \end{pmatrix}, \quad (5.2.3)$$

which we ensure is divergence-free. By pushing forward this initial condition under the ideal flow, the evolution of the field components is governed by the transfer operator. We define the normalized $2d$ vector-valued transfer operator $\mathcal{L}_\alpha$ on a suitable Banach space X (see Section 5.4) as

$$\mathcal{L}_\alpha h = \frac{1}{\alpha^2} (DT_\alpha h) \circ T_\alpha^{-1}. \quad (5.2.4)$$

Via the representation formula (5.1.1), there exists a bounded linear operator $K_\alpha : X \rightarrow X$ so that

$$\begin{aligned} (D\mathcal{T}_\alpha B_{\text{in}}) \circ \mathcal{T}_\alpha^{-1}(x, y, z) &= e^{2\pi i(z+g(x,y))} (D\mathcal{T}_\alpha H) \circ T_\alpha^{-1}(x, y) \\ &= e^{2\pi i z} e^{2\pi i g(x,y)} \begin{pmatrix} \alpha^2 \mathcal{L}_\alpha h \\ K_\alpha h + H^{(3)} \circ T_\alpha^{-1} \end{pmatrix}. \end{aligned} \quad (5.2.5)$$

While the operator $\alpha^2 \mathcal{L}_\alpha$ (associated with the purely $2d$ flow) cannot admit growing modes due to anti-dynamo constraints, the shear-modulated operator $e^{2\pi i g(x,y)} \alpha^2 \mathcal{L}_\alpha$ will be shown to admit unstable eigenvalues in the strong-chaos limit $\alpha \rightarrow \infty$, see Theorem 5.2.7 below.

Remark 5.2.2 (On the strong-chaos limit). The strong-chaos limit $\alpha \rightarrow \infty$ is non-trivial precisely because of the phase-shearing effect. For our construction, introducing the shear g at each iteration with $\alpha \gg 1$ yields fundamentally different spectral behavior compared to applying a shear only after $n \gg 1$ iterations of a fixed map. Although both scenarios formally correspond to a regime of “infinite mixing”, we show in Remark 5.6.2 that even two iterations of the advection step performed before a single shear would result in a limiting operator with no non-trivial eigenvalues. This highlights the extreme sensitivity of the dynamo mechanism to the fine-scale temporal structure of the flow.

5.2.2 The spectral perturbation framework

As motivated above, we endeavor to construct a suitable anisotropic Banach space in which our ideal dynamo operator admits Ruelle resonances—discrete eigenvalues strictly separated from the essential spectrum. To achieve this, we prove a *Lasota–Yorke inequality* which, via the Nussbaum formula [Nus70], bounds the essential spectral radius of the ideal dynamo operator. These results are introduced in more detail in Appendix A, but we recall them here for convenience. Further details can be also found in, e.g., [Bal00].

Proposition 5.2.3. *Let $(X, \|\cdot\|)$ and $(X_w, |\cdot|_w)$ be Banach spaces such that $X \subset X_w$, with continuous inclusion $|\cdot|_w \leq \|\cdot\|$. Assume that the unit ball $\{f \in X : \|f\| \leq 1\}$ is relatively compact in $(X_w, |\cdot|_w)$.*

Let $\mathcal{P} : X \rightarrow X$ be a bounded linear operator that also extends to a bounded operator $\mathcal{P} : X_w \rightarrow X_w$. Furthermore, assume $\mathcal{P}$ satisfies a Lasota–Yorke inequality: there exist constants $0 \leq a < M$ such that

$$\|\mathcal{P}f\| \leq a\|f\| + M|f|_w \quad \forall f \in X.$$

Then, the essential spectrum $\sigma_{\text{ess}}(\mathcal{P})$ of $\mathcal{P}$ on X is contained in a disk of radius a , i.e., $\sigma_{\text{ess}}(\mathcal{P}) \subset \overline{B_a(0)}$. In particular, any spectral point of $\mathcal{P}$ of modulus strictly larger than a corresponds to an isolated eigenvalue of finite multiplicity.

The benefits of establishing a Lasota–Yorke inequality are twofold. First, if we can construct an anisotropic Banach space X with norms $\|\cdot\|$ and $|\cdot|_w$ such that (A.2.1) holds true with some $a \geq 0$, we immediately deduce that any eigenvalue of the ideal dynamo operator with modulus strictly larger than a is isolated. Second, these isolated eigenmodes are stable under a wide class of singular perturbations, thanks to the following seminal result by Keller and Liverani [KL99].

Proposition 5.2.4 (Keller–Liverani). *Let $(\mathcal{P}_\varepsilon)_{\varepsilon \geq 0}$ be a family of bounded linear operators acting*

on Banach spaces $(X, \|\cdot\|)$ and $(X_w, |\cdot|_w)$ satisfying the assumptions of Proposition A.2.8. Suppose there exist constants $0 \leq a < M$ such that the following hold for all $f \in X$:

- (i) $|\mathcal{P}_\varepsilon f|_w \leq M|f|_w$;
- (ii) $|\mathcal{P}_\varepsilon f - \mathcal{P}_0 f|_w \leq \tau_\varepsilon \|f\|$, for some modulus of continuity $\tau_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$;
- (iii) $\|\mathcal{P}_\varepsilon f\| \leq a\|f\| + M|f|_w$;
- (iv) There exists an eigenvalue $\lambda \in \mathbb{C}$ of $\mathcal{P}_0$ with $|\lambda| > a$.

Then, there exists $\delta > 0$ such that $\sigma(\mathcal{P}_\varepsilon) \cap \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \delta\}$ consists of finitely many eigenvalues (counted with algebraic multiplicity) that converge to λ as $\varepsilon \rightarrow 0$.

The application of this framework to Theorem 3 proceeds by a double-limit argument. By first working in the strong-chaos regime ($\alpha \rightarrow \infty$), we establish the existence of a spectral gap and identify a leading resonance λ_α such that $|\lambda_\alpha| > \alpha^2/4$ for the ideal operator $\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha$, see (5.2.4)-(5.2.5) and Theorem 5.2.7. We then prove a uniform-in- ε Lasota–Yorke inequality, ensuring that this leading eigenvalue is spectrally isolated and stable under the singular perturbation of diffusion.

Remark 5.2.5. Our analysis is further complicated by the fact that the functional spaces we work on depend intrinsically on the stretching parameter α . Extracting useful spectral behavior from the limiting object as $\alpha \rightarrow \infty$ requires careful, quantitative (in α) estimates on the resolvent and an adaptation of the ideas in [KL99]. The resonances obtained in this limit provide the necessary spectral anchor for the perturbative treatment of the diffusive term.

5.2.3 Outline of the proof of Theorem 3

As hinted throughout the introduction, our proof relies on three key propositions. The goal of this section is to demonstrate how, once these building blocks are established, one can deduce the existence of a fast dynamo via the abstract results of Propositions A.2.8 and A.3.7.

In Section 5.4, we construct suitable anisotropic Banach spaces $(X, \|\cdot\|)$ and $(X_w, |\cdot|_w)$ that satisfy the assumptions of Proposition A.2.8 and depend intrinsically on the large parameter $\alpha \in 2\mathbb{N}$. We suppress this dependence for brevity, though we note that extracting uniform spectral behavior as $\alpha \rightarrow \infty$ requires careful, quantitative resolvent estimates.

The first crucial ingredient is a Lasota–Yorke inequality, which provides the spectral gap necessary for isolated eigenvalues.

Proposition 5.2.6 (Lasota–Yorke inequality). *There exist constants $C, \eta > 0$, independent of α , such that for all α sufficiently large, the operator $e^{2\pi i g} \mathcal{L}_\alpha$ satisfies*

$$\|e^{2\pi i g} \mathcal{L}_\alpha h\| \leq C\alpha^{-\eta} \|h\| + C|h|_w, \quad |e^{2\pi i g} \mathcal{L}_\alpha h|_w \leq C|h|_w, \quad \forall h \in X. \quad (5.2.6)$$

Consequently, by Proposition A.2.8, any eigenvalue of $e^{2\pi i g} \mathcal{L}_\alpha$ with modulus strictly greater than $C\alpha^{-\eta}$ is isolated and of finite multiplicity.

In Section 5.6, we characterize the limiting operator $e^{2\pi i g} \mathcal{L}_\infty$ of $e^{2\pi i g} \mathcal{L}_\alpha$ as $\alpha \rightarrow \infty$. With convergence understood in a suitable sense (see Lemma 5.6.3), the limiting operator takes the form

$$e^{2\pi i g} \mathcal{L}_\infty h = e^{2\pi i g} \begin{pmatrix} \left(\int_{x \geq 1/2} h^{(1)} - \int_{x < 1/2} h^{(1)} \right) (\mathbb{1}_{y \geq 1/2} - \mathbb{1}_{y < 1/2}) \\ 0 \end{pmatrix}, \quad h = (h^{(1)}, h^{(2)})$$

Using this characterization alongside Proposition 5.2.6, we prove in Section 5.6 the existence of a discrete eigenvalue with modulus larger than 1 (in fact, of order $\approx \alpha^2$ for $\alpha \gg 1$).

Theorem 5.2.7. *There exist $g \in C^1(\mathbb{T}^2)$ and $\alpha_0 \geq 1$ such that, for every $\alpha \geq \alpha_0$ with $\alpha \in 2\mathbb{N}$, the operator $\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha : X \rightarrow X$ admits a discrete eigenvalue $\lambda_\alpha \in \mathbb{C}$ satisfying $|\lambda_\alpha| \geq \alpha^2/4$.*

Finally, we require a bound on the heat semigroup to ensure the diffusive perturbation is small in the weak norm. This is proved in Section 5.7.

Proposition 5.2.8. *Let $e^{\varepsilon \Delta}$ be the time-1 solution operator of the heat equation with diffusion $\varepsilon > 0$ on $\mathbb{T}^2$. There exists $\beta > 0$ such that*

$$\|e^{\varepsilon \Delta} f\| \leq \|f\|, \quad |e^{\varepsilon \Delta} f|_w \leq |f|_w, \quad |e^{\varepsilon \Delta} f - f|_w \leq 2\varepsilon^{\frac{\beta}{4}} \|f\|, \quad (5.2.7)$$

for all $f \in X$.

With these three propositions in hand, we can now establish our main result.

Proof of Theorem 3. We consider the time-1 periodic velocity field u_α and the magnetic field initial datum $B_{\text{in}}(x, y, z) = e^{2\pi i z} H(x, y)$, where $H = (h, H^{(3)})$. The time-1 ideal dynamo solution operator $\mathcal{P}_{3d,0}$ acts as¹

$$(D\mathcal{T}_\alpha B_{\text{in}}) \circ \mathcal{T}_\alpha^{-1} = e^{2\pi i z} e^{2\pi i g(x,y)} \begin{pmatrix} \alpha^2 \mathcal{L}_\alpha h \\ K_\alpha h + H^{(3)} \circ T_\alpha^{-1} \end{pmatrix},$$

where K_α is a bounded linear operator. The time-2 solution operator $\mathcal{P}_{3d,\varepsilon}$ for the pulsed dynamo equations (P-KDE) is then

$$\mathcal{P}_{3d,\varepsilon}(e^{2\pi i z} H) = e^{-4\pi^2 \varepsilon} e^{2\pi i z} \begin{pmatrix} e^{\varepsilon \Delta} e^{2\pi i g} \alpha^2 \mathcal{L}_\alpha h \\ e^{\varepsilon \Delta} e^{2\pi i g} K_\alpha h + e^{\varepsilon \Delta} (e^{2\pi i g} H^{(3)} \circ T_\alpha^{-1}) \end{pmatrix}.$$

To prove fast dynamo action, we first show that the 2×2 operator

$$\mathcal{P}_\varepsilon := e^{\varepsilon \Delta} e^{2\pi i g} \alpha^2 \mathcal{L}_\alpha$$

admits an eigenvalue with modulus strictly larger than 1. This will rely on the Keller–Liverani framework (Proposition A.3.7).

Firstly, by Theorem 5.2.7, for $\alpha \geq \alpha_0$, the ideal operator $\mathcal{P}_0$ has an eigenvalue $|\lambda| \geq \alpha^2/4$, thus verifying (iv). We now combine Propositions 5.2.6 and 5.2.8. The uniform Lasota–Yorke

¹This is the solution formula for the PDE $\partial_t B + u \cdot \nabla B - B \cdot \nabla u = 0$ which is equivalent to $\partial_t B = \nabla \times (u \times B)$ if both u and B_{in} are divergence-free.

inequality (iii) follows directly from the strong norm inequalities in (5.2.6) and (5.2.7), since

$$\|\mathcal{P}_\varepsilon h\| \leq \|\mathcal{P}_0 h\| \leq C\alpha^{2-\eta}\|h\| + C\alpha^2|h|_w.$$

Similarly, the weak norm counterparts give (i). The weak-strong continuity (ii) is a consequence of

$$\|\mathcal{P}_\varepsilon h - \mathcal{P}_0 h\|_w \leq \|e^{\varepsilon\Delta} - \text{Id}\|_{X \rightarrow w} \|\mathcal{P}_0 h\| \leq C\alpha^2 \varepsilon^{\frac{\beta}{4}} \|h\|,$$

where we used the standard operator norm notation, see Section 5.2.5. Hence, an application of Proposition A.3.7 gives that for all $\varepsilon \leq \varepsilon_0$, the operator $e^{-4\pi^2\varepsilon} \mathcal{P}_\varepsilon$ admits a discrete eigenvalue λ_ε with $|\lambda_\varepsilon| \geq \alpha^2/8 > 1$.

To lift this to the full system, let h be the unstable eigenfunction of $\mathcal{P}_\varepsilon$. We define

$$J_\varepsilon(H^{(3)}) = e^{-4\pi^2\varepsilon} e^{\varepsilon\Delta} (e^{2\pi i g} H^{(3)} \circ T_\alpha^{-1}).$$

Since $\|J_\varepsilon\| < 1$ and $|\lambda_\varepsilon| > 1$, the operator $(\lambda_\varepsilon - J_\varepsilon)$ is invertible. Setting

$$H^{(3)} = (\lambda_\varepsilon - J_\varepsilon)^{-1} e^{-4\pi^2\varepsilon} e^{\varepsilon\Delta} e^{2\pi i g} K_\alpha h$$

yields a genuine growing eigenmode for the full operator. Smoothness follows from the parabolic regularity of $e^{\varepsilon\Delta}$.

Finally, we confirm the divergence-free condition. For the ideal kinematic dynamo equation, we have that

$$\partial_t B + u_\alpha \cdot \nabla B - B \cdot \nabla u_\alpha = 0 \quad \Rightarrow \quad (\partial_t + u_\alpha \cdot \nabla)(\nabla \cdot B) = 0,$$

implying $\|\nabla \cdot \mathcal{P}_{3d,0} B_{\text{in}}\|_{L^2} \leq \|\nabla \cdot B_{\text{in}}\|_{L^2}$. For our eigenfunction, the commutation of the divergence and heat operators gives

$$|\lambda_\varepsilon| \|\nabla \cdot (e^{2\pi i z} H)\|_{L^2} = \|e^{\varepsilon\Delta_{\mathbb{T}^3}} \nabla \cdot (\mathcal{P}_{3d,0}(e^{2\pi i z} H))\|_{L^2} \leq \|\nabla \cdot (e^{2\pi i z} H)\|_{L^2}.$$

Since $|\lambda_\varepsilon| > 1$, this inequality holds if and only if $\|\nabla \cdot (e^{2\pi i z} H)\|_{L^2} = 0$, completing the proof. $\square$

5.2.4 Connection with the flux conjecture

As a consequence of our spectral approach, we are able to show that the velocity field u_α , for α sufficiently large but fixed, is a *perfect dynamo*. This property, which concerns the growth of magnetic flux in the ideal limit, is generally more difficult to establish than energy growth due to the potential for fine-scale cancellations. We first recall the definition.

Definition 5.2.9 (Perfect dynamo, [CG95]). Let $\psi : \mathbb{T}^3 \rightarrow \mathbb{R}^3$ be a smooth vector field. The *perfect dynamo growth rate relative to ψ* , denoted by $\Gamma(\psi)$, is defined as

$$\Gamma(\psi) = \sup_{B_{\text{in}} \in L^2(\mathbb{T}^3)} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left| \int_{\mathbb{T}^3} B(t, x) \cdot \psi(x) dx \right|,$$

where $B(t, x)$ is the solution to the ideal equation (KDE- ε) ($\varepsilon = 0$) with divergence-free initial

datum B_{in} . A time-periodic, divergence-free velocity field $u \in L_t^\infty W_x^{1,\infty}$ is called a *perfect dynamo* if there exists some $\psi \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ such that $\Gamma(\psi) > 0$.

The relationship between ideal flux growth and diffusive energy growth is encapsulated in the following conjecture 2, which we recall here for convenience.

Flux conjecture. *Every perfect dynamo is a fast dynamo.*

The flux conjecture suggests that the growth required for a perfect dynamo is sufficient to overcome the destructive effects of small diffusion [CG95, Chapter 4]. In Section 5.6.3, we prove that our velocity field u_α satisfies the perfect dynamo condition. Combined with Theorem 3, this result provides a concrete example in which the flux conjecture holds for the pulsed-diffusion model (P-KDE) on the torus.

Corollary 5.2.10. *The velocity field u_α defined in (5.2.2) is a perfect dynamo for any $\alpha \in 2\mathbb{N}$ sufficiently large.*

5.2.5 Notation

We denote by $(X, \|\cdot\|)$ the anisotropic Banach space endowed with the strong norm, and by $|\cdot|_w$ a weak norm on X defined in Section 5.4. For a bounded linear operator $\mathcal{L} : X \rightarrow X$, we define the operator norms

$$\|\mathcal{L}\|_{X \rightarrow X} = \sup_{\|x\| \leq 1} \|\mathcal{L}x\|, \quad \|\mathcal{L}\|_{w \rightarrow w} = \sup_{|x|_w \leq 1} |\mathcal{L}x|_w, \quad \|\mathcal{L}\|_{X \rightarrow w} = \sup_{\|x\| \leq 1} |\mathcal{L}x|_w,$$

For any vector $\mathbf{v} \in \mathbb{R}^d$ with $d \geq 2$, we write $\mathbf{v}^{(i)}$ for its i -th component. We denote by $d_{\mathbb{T}^d}$ the standard toroidal distance on the d -dimensional torus $\mathbb{T}^d \cong [0, 1]^d / \sim$ with $d \geq 2$. Throughout the manuscript, the constant $C > 0$ appearing in the estimates may change from line to line but is always independent of all relevant parameters, in particular independent of α . We abuse the notation and we denote by $C^q(W)$ the closure of the space of Lipschitz continuous functions with respect to the $\|\cdot\|_{C^q(W)}$ norm. In particular, we denote by $C^1(\mathbb{T}^2, \mathbb{C})$ the space of Lipschitz functions.

5.3 A family of hyperbolic maps on $\mathbb{T}^2$.

In this section, we analyze a family of hyperbolic maps $(T_\alpha)_{\alpha \geq \alpha_0} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, given by the first two components of the time- $\frac{1}{2}$ flow map of the velocity field u_α defined in (5.2.2), i.e. without the shearing action induced by $Z(x, y)$.

We define the indicator functions on the circle $\mathbb{T} = [0, 1)$ as

$$\mathbb{1}_{\geq a}(x) = \begin{cases} 1, & \text{if } x \in [a, 1), \\ 0, & \text{otherwise,} \end{cases}$$

and let $\mathbb{1}_{< a}(x) = 1 - \mathbb{1}_{\geq a}(x)$. Given the velocity field defined in (5.2.1)–(5.2.2) with shear strength

$\alpha \in 2\mathbb{N}$, we define the horizontal and vertical shearing flow maps, T_{H_α} and T_{V_α} respectively, as:

$$T_{H_\alpha}(x, y) = \begin{pmatrix} (x + \alpha y) \mathbb{1}_{\geq 1/2}(y) - (x - \alpha y) \mathbb{1}_{< 1/2}(y) \\ y \end{pmatrix} \pmod{1}$$

and

$$T_{V_\alpha}(x, y) = \begin{pmatrix} x \\ (y + \alpha x) \mathbb{1}_{\geq 1/2}(x) + (y - \alpha x) \mathbb{1}_{< 1/2}(x) \end{pmatrix} \pmod{1}.$$

Let $T_\alpha : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be the (x, y) -components of the time-1 flow map associated with the velocity field u . This map is given by the composition

$$T_\alpha = T_{H_\alpha} \circ T_{V_\alpha}, \quad T_\alpha^{-1} = T_{V_\alpha}^{-1} \circ T_{H_\alpha}^{-1}.$$

By construction, T_α is a piecewise linear, volume-preserving map. There exists a partition of the torus into four regions $\mathcal{M}_\ell \subset \mathbb{T}^2$ for $\ell = 1, \dots, 4$, such that the restriction $T_\alpha|_{\mathcal{M}_\ell}$ coincides with a linear transformation $A_\ell \in SL(2, \mathbb{Z})$. We refer to these $\mathcal{M}_\ell$ as the *forward smoothness regions* of the map.

Correspondingly, by the inverse function theorem, the inverse map $T_\alpha^{-1} : T_\alpha(\mathcal{M}_\ell) \rightarrow \mathcal{M}_\ell$ is given by the inverse matrices A_ℓ^{-1} . We refer to the sets $T_\alpha(\mathcal{M}_\ell)$ as the *backward smoothness regions*. We now proceed to characterize the geometry of these regions and the explicit form of the matrices A_ℓ .

The forward map

By computing the composition $T_\alpha = T_{H_\alpha} \circ T_{V_\alpha}$ explicitly, we find that the second component is determined solely by the initial vertical shear, namely

$$T_\alpha^{(2)}(x, y) = (y + \alpha x) \mathbb{1}_{\geq 1/2}(x) + (y - \alpha x) \mathbb{1}_{< 1/2}(x) \pmod{1}.$$

The first component, which incorporates the horizontal shear acting on the already-sheared vertical coordinate, is given by:

$$\begin{aligned} T_\alpha^{(1)}(x, y) = & [(1 + \alpha^2)x + \alpha y] \mathbb{1}_{\mathcal{M}_1}(x, y) + [(1 - \alpha^2)x + \alpha y] \mathbb{1}_{\mathcal{M}_2}(x, y) \\ & + [(1 - \alpha^2)x - \alpha y] \mathbb{1}_{\mathcal{M}_3}(x, y) + [(1 + \alpha^2)x - \alpha y] \mathbb{1}_{\mathcal{M}_4}(x, y) \pmod{1}, \end{aligned}$$

where the four forward smoothness regions $\mathcal{M}_\ell \subset \mathbb{T}^2$ are defined as

$$\begin{aligned} \mathcal{M}_1 &= \{x \geq 1/2, y + \alpha x \geq 1/2 \pmod{1}\}, & \mathcal{M}_2 &= \{x < 1/2, y - \alpha x \geq 1/2 \pmod{1}\}, \\ \mathcal{M}_3 &= \{x \geq 1/2, y + \alpha x < 1/2 \pmod{1}\}, & \mathcal{M}_4 &= \{x < 1/2, y - \alpha x < 1/2 \pmod{1}\}. \end{aligned}$$

Consequently, the map T_α is a piecewise linear automorphism of the torus. On each region $\mathcal{M}_\ell$, the map acts as $T_\alpha(x, y) = A_\ell \begin{pmatrix} x \\ y \end{pmatrix} \pmod{1}$, where the matrices $A_\ell \in SL(2, \mathbb{Z})$ are given by:

$$A_1 = \begin{pmatrix} 1 + \alpha^2 & \alpha \\ \alpha & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 - \alpha^2 & \alpha \\ -\alpha & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 - \alpha^2 & -\alpha \\ \alpha & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 + \alpha^2 & -\alpha \\ -\alpha & 1 \end{pmatrix}. \quad (5.3.1)$$

This characterization confirms that the smoothness region of T_α is the union of these four components, each corresponding to a specific combination of the directional shears.

The backward map

Following a computation analogous to the forward map, we determine the inverse map $T_\alpha^{-1} = T_{V_\alpha}^{-1} \circ T_{H_\alpha}^{-1}$. The first component of the inverse is determined by the horizontal inverse shear acting on the y -coordinate:

$$(T_\alpha^{-1})^{(1)}(x, y) = (x - \alpha y) \mathbb{1}_{\geq 1/2}(y) + (x + \alpha y) \mathbb{1}_{< 1/2}(y) \pmod{1},$$

and the second component, which accounts for the vertical inverse shear acting on the result of the horizontal shear, is given by:

$$\begin{aligned} (T_\alpha^{-1})^{(2)}(x, y) = & [(1 + \alpha^2)y - \alpha x] \mathbb{1}_{T_\alpha(\mathcal{M}_1)}(x, y) + [(1 - \alpha^2)y + \alpha x] \mathbb{1}_{T_\alpha(\mathcal{M}_2)}(x, y) \\ & + [(1 - \alpha^2)y - \alpha x] \mathbb{1}_{T_\alpha(\mathcal{M}_3)}(x, y) + [(1 + \alpha^2)y + \alpha x] \mathbb{1}_{T_\alpha(\mathcal{M}_4)}(x, y) \pmod{1}. \end{aligned}$$

The *backward smoothness regions* $T_\alpha(\mathcal{M}_\ell)$ partition the torus according to the continuity of the inverse map:

$$\begin{aligned} T_\alpha(\mathcal{M}_1) &= \{y \geq 1/2, x - \alpha y \geq 1/2 \pmod{1}\}, & T_\alpha(\mathcal{M}_2) &= \{y \geq 1/2, x - \alpha y < 1/2 \pmod{1}\}, \\ T_\alpha(\mathcal{M}_3) &= \{y < 1/2, x + \alpha y \geq 1/2 \pmod{1}\}, & T_\alpha(\mathcal{M}_4) &= \{y < 1/2, x + \alpha y < 1/2 \pmod{1}\}. \end{aligned}$$

The structure of these backward smoothness sets is critical, as they define the regions where the vector-valued transfer operator remains regular. Just as the forward regions $\mathcal{M}_\ell$ are skewed vertically, these backward regions are skewed horizontally, reflecting the time-reversal symmetry of the shear operations. We refer to Figure 5.1 to visualize these sets.

5.3.1 The hyperbolic structure.

The crucial property of the map T_α is that it is *uniformly hyperbolic*. Notably, the hyperbolic structure is significantly more rigid than the general case outlined in [DL08]; here, the unstable and stable cone fields are independent of the spatial coordinates (x, y) . The following lemma summarizes the spectral and geometric properties required for our analysis.

Lemma 5.3.1. *There exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$, there exist two constant cones*

$$C_s = \{\mathbf{v} = (v_1, v_2) : |v_1| \leq 2\alpha^{-1}|v_2|\}, \quad C_u = \{\mathbf{v} = (v_1, v_2) : |v_2| \leq 2\alpha^{-1}|v_1|\} \quad (5.3.2)$$

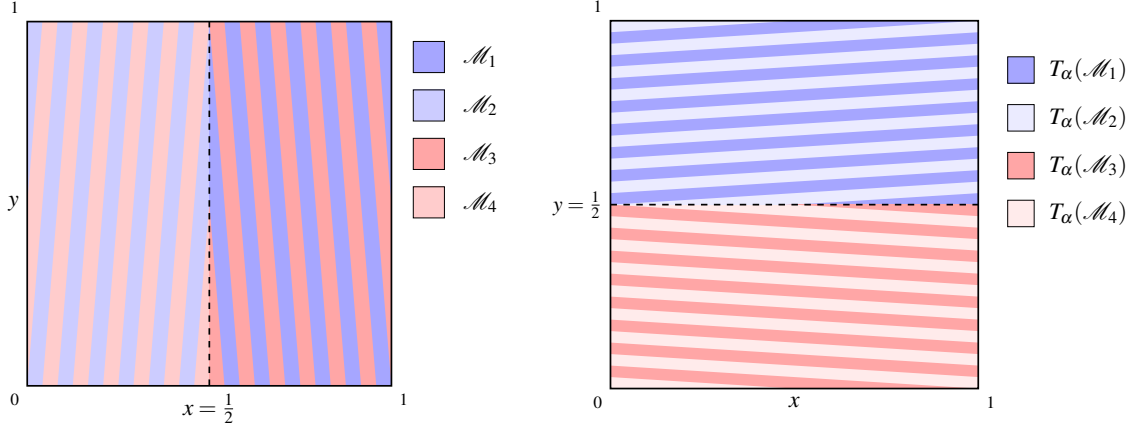


Figure 5.1: The partition of $[0,1]^2$ into the regions $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ and the sets $T_\alpha(\mathcal{M}_1), T_\alpha(\mathcal{M}_2), T_\alpha(\mathcal{M}_3), T_\alpha(\mathcal{M}_4)$, with $\alpha = 16$.

satisfying the invariance properties

$$DT_\alpha C_u \subset C_u, \quad DT_\alpha^{-1} C_s \subset C_s.$$

Furthermore, we have the expansion estimates

$$|DT_\alpha \mathbf{v}| \geq \frac{1}{2} \alpha^2 |\mathbf{v}|, \quad \forall \mathbf{v} \in C_u, \quad |DT_\alpha^{-1} \mathbf{v}| \geq \frac{1}{2} \alpha^2 |\mathbf{v}|, \quad \forall \mathbf{v} \in C_s.$$

Proof. We provide the details for the forward map, as the argument for the inverse is analogous. Recall that on each forward smoothness region $\mathcal{M}_\ell$, the derivative DT_α is given by one of the matrices $A_1, \dots, A_4$ derived in (5.3.1).

We demonstrate the argument for $A_1 = \begin{pmatrix} 1 + \alpha^2 & \alpha \\ \alpha & 1 \end{pmatrix}$; the remaining cases follow by identical estimates. Let $\mathbf{v} = (v_1, v_2) \in C_u$. By definition, $|v_2| \leq 2\alpha^{-1}|v_1|$. The image vector $\mathbf{v}' = A_1 \mathbf{v}$ has components

$$\begin{aligned} |v'_1| &= |(1 + \alpha^2)v_1 + \alpha v_2| \geq (1 + \alpha^2)|v_1| - \alpha|v_2| \geq (1 + \alpha^2)|v_1| - 2|v_1| = (\alpha^2 - 1)|v_1|, \\ |v'_2| &= |\alpha v_1 + v_2| \leq \alpha|v_1| + |v_2| \leq (\alpha + 2\alpha^{-1})|v_1|. \end{aligned}$$

To satisfy $DT_\alpha C_u \subset C_u$, we require $|v'_2| \leq 2\alpha^{-1}|v'_1|$. Substituting the bounds

$$(\alpha + 2\alpha^{-1})|v_1| \leq 2\alpha^{-1}(\alpha^2 - 1)|v_1| = (2\alpha - 2\alpha^{-1})|v_1|.$$

For large α , this simplifies to $\alpha \leq 2\alpha$, which is clearly satisfied. For the expansion estimate, since $|v| \approx |v_1|$ for vectors in C_u , we have:

$$|DT_\alpha \mathbf{v}| \geq |v'_1| \geq (\alpha^2 - 1)|v_1| \geq \frac{1}{2} \alpha^2 |\mathbf{v}|, \quad \forall \alpha \geq \alpha_0,$$

concluding the proof. $\square$

The existence of these constant cones facilitates the construction of *anisotropic Banach spaces*. Because the expansion factor $\approx \alpha^2$ grows quadratically with the shear strength, the essential spectral radius of the vector-valued transfer operator can be made arbitrarily small relative to the dominant eigenvalue as $\alpha \rightarrow \infty$.

5.4 Functional setting and anisotropic Banach spaces

Having outlined the hyperbolic structure of the map, we now define suitable anisotropic Banach spaces. These are essentially those of [DL08], with adaptations made to account for our strictly linear cone structures.²

We begin by defining a class of admissible leaves against which we shall integrate. Recall from Lemma 5.3.1 that our dynamics admit constant stable/unstable cones given in (5.3.2), which are invariant under the backwards and forwards dynamics of T_α , respectively. For any basepoint $\mathbf{x} \in \mathbb{T}^2$, unit direction vector $\mathbf{v} \in \hat{C}_s = \{\mathbf{v} \in C_s : |\mathbf{v}| = 1, \mathbf{v}^{(2)} > 0\}$, and length $S \in (0, 1]$, we define a parameterized curve $\gamma_{\mathbf{x}, \mathbf{v}, S}(t) = \mathbf{x} + t\mathbf{v}$ for $t \in [0, S]$. This yields a line segment $W \subset \mathbb{T}^2$ given by $W = \gamma_{\mathbf{x}, \mathbf{v}, S}([0, S])$.

We first verify that such lines admit a canonical parameterization.

Lemma 5.4.1. *Let $W \subset \mathbb{T}^2 \cong [0, 1]^2 / \sim$ be a line segment of length at most 1, with tangent direction in C_s . If W is not a vertical segment of length 1, then there exist unique $\mathbf{x} \in \mathbb{T}^2$, $\mathbf{v} \in \hat{C}_s$, and $S \in [0, 1]$ such that $W = \gamma_{\mathbf{x}, \mathbf{v}, S}([0, S])$. If W is a vertical line segment of length 1, we fix the convention that its base point is $\mathbf{x} = (x_0, 0)$ for some $x_0 \in \mathbb{T}$; under this convention, the above parametrization is unique as well.*

Proof. Assume that W is not a vertical line segment of length 1, then it cannot wrap around the torus and self-intersect; therefore, the set W is a simple line segment with exactly two distinct endpoints on $\mathbb{T}^2$. Since any tangent vector $\mathbf{v} \in \hat{C}_s$ must satisfy $|\mathbf{v}^{(1)}| \leq 2\alpha^{-1}|\mathbf{v}^{(2)}|$, its vertical component cannot be zero. Consequently, between the two possible unit vectors parallel to W , exactly one satisfies $\mathbf{v}^{(2)} > 0$. Choosing this orientation uniquely determines the starting endpoint $\mathbf{x} \in \mathbb{T}^2$, the direction $\mathbf{v}$, and the arc length $S \in (0, 1)$, completing the proof. If W is a vertical line segment of length 1 it is straightforward to see that the convention restores a unique parametrization. $\square$

The class of admissible leaves

Having established a canonical parameterization for segments with tangents in $\hat{C}_s$, we may now define a metric space of admissible leaves. Indeed, define first the space Λ to be

$$\Lambda = \mathbb{T}^2 \times \hat{C}_s \times [0, 1],$$

with metric

$$d_\Lambda((\mathbf{x}_1, \mathbf{v}_1, S_1), (\mathbf{x}_2, \mathbf{v}_2, S_2)) = d_{\mathbb{T}^2}(\mathbf{x}_1, \mathbf{x}_2) + |\mathbf{v}_1 - \mathbf{v}_2| + |S_1 - S_2|.$$

²These adaptations are, in fact, essential in order to treat the heat semigroup as a perturbation. The class of admissible perturbations in [DL08] does not natively include composition with the heat semigroup.

Clearly the space (Λ, d_Λ) is a compact metric space. We then define Γ to be a subset of Λ given by the union of non-vertical line segments of length at most one, perfectly vertical line segments of length strictly less than one, and finally perfectly vertical line segments of length exactly 1

$$\Gamma = (\mathbb{T}^2 \times \hat{C}_s \setminus \{(0, 1)\} \times (0, 1]) \cup (\mathbb{T}^2 \times \{(0, 1)\} \times (0, 1)) \cup ((\mathbb{T} \times \{0\}) \times \{(0, 1)\} \times \{1\}).$$

In particular, (Γ, d_Λ) is a pre-compact metric space. We now define the class of admissible stable leaves as:

$$\Sigma = \Sigma_s = \{W \subset \mathbb{T}^2 : \exists \mathbf{v} \in \hat{C}_s, \exists \mathbf{x} \in \mathbb{T}^2, \exists S \in (0, 1] \text{ such that } W = \gamma_{\mathbf{x}, \mathbf{v}, S}([0, S])\}. \quad (5.4.1)$$

Now, by Lemma 5.4.1, we note that the map $\Psi : \Gamma \rightarrow \Sigma$ given by $\Psi(\mathbf{x}, \mathbf{v}, S) = \gamma_{\mathbf{x}, \mathbf{v}, S}([0, S])$ is a bijection. Therefore, we may induce a metric on Σ via the map Ψ , given by

$$d_\Sigma(W_1, W_2) = d_\Lambda(\Psi^{-1}(W_1), \Psi^{-1}(W_2)).$$

Under this metric, Ψ becomes a homeomorphism, and thus the metric space (Σ, d_Σ) inherits the pre-compactness property of (Γ, d_Λ) , which will be of use in Lemma 5.4.3. From now on, we abuse notation and identify elements in Σ with their parameterisation. Thus, for $W_1 = \gamma_{\mathbf{x}_1, \mathbf{v}_1, S_1}([0, S_1])$ and $W_2 = \gamma_{\mathbf{x}_2, \mathbf{v}_2, S_2}([0, S_2])$, we have

$$d_\Sigma(W_1, W_2) = d_{\mathbb{T}^2}(\mathbf{x}_1, \mathbf{x}_2) + |\mathbf{v}_1 - \mathbf{v}_2| + |S_1 - S_2|,$$

where $d_{\mathbb{T}^2}(\mathbf{x}_1, \mathbf{x}_2)$ is the standard toroidal distance. We now define a pseudo-metric on test func-

tions. From now on, we abuse notation and use $C^1(\mathbb{T}^2; \mathbb{C})$ as the space of Lipschitz functions. We define a distance between test functions defined on these leaves. For any $q \in (0, 1]$, $W \in \Sigma$, and $\varphi \in C^1(\mathbb{T}^2, \mathbb{C})$, we define the Hölder seminorm and norm along W as:

$$[\varphi]_{C^q(W)} = \sup_{\substack{x, y \in W \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^q}, \quad \|\varphi\|_{C^q(W)} = \sup_{y \in W} |\varphi(y)| + [\varphi]_{C^q(W)},$$

and define, for $\sigma \in (0, 1)$, a geometrically weighted norm

$$\|\varphi\|_{\sigma, W} = \|\varphi\|_{C^q(W)} |W|^\sigma.$$

We denote by $C^q(W)$ the closure of the space of Lipschitz continuous functions, $C^1(\mathbb{T}^2, \mathbb{C})$, with respect to the $\|\cdot\|_{C^q(W)}$ norm.³ To compare functions on different leaves, let $W_i = \gamma_{\mathbf{x}_i, \mathbf{v}_i, S_i}([0, S_i]) \in \Sigma$ for $i = 1, 2$. For test functions $\varphi_i \in C^q(W_i)$, we define the pseudo metric

$$d_q(\varphi_1, \varphi_2) = \|\varphi_1(\gamma_{\mathbf{x}_1, \mathbf{v}_1, S_1}(\cdot)) - \varphi_2(\gamma_{\mathbf{x}_2, \mathbf{v}_2, S_2}(\cdot))\|_{C^q([0, S_1] \wedge S_2]}.$$

³We remark that $C^q(W)$ defined via this closure is smaller than the full standard Hölder space when $q \in (0, 1)$.

Definitions of the norms

We fix positive parameters $\sigma, \beta, q \in (0, 1)$ satisfying the strict inequalities:

$$\sigma > \beta, \quad 1 - q > \beta. \quad (5.4.2)$$

For any vector valued function $f \in C^1(\mathbb{T}^2; \mathbb{C}^2)$, we define the *weak stable norm* by

$$|f|_w := \sup_{W \in \Sigma} \sup_{\|\varphi\|_{C^1(W)} \leq 1} \left| \int_W f \varphi \right|,$$

where we denote by $|\cdot|$ the euclidean norm on $\mathbb{C}^2$. Next, we define the *strong stable norm* as

$$\|f\|_s := \sup_{W \in \Sigma} \sup_{\|\varphi\|_{\sigma, W} \leq 1} \left| \int_W f \varphi \right|,$$

and the *strong unstable norm*, which measures the regularity of the distributions across stable leaves

$$\|f\|_u := \sup_{0 < \delta \leq \frac{2}{\alpha}} \sup_{d_\Sigma(W^1, W^2) < \delta} \sup_{\substack{\|\varphi_1\|_{C^1}, \|\varphi_2\|_{C^1} \leq 1 \\ d_q(\varphi_1, \varphi_2) < \delta}} \frac{1}{\delta^\beta} \left| \int_{W^1} f \varphi_1 - \int_{W^2} f \varphi_2 \right|.$$

Finally, for any $h \in C^1(\mathbb{T}^2, \mathbb{C}^2)$, the full *strong norm* is given by

$$\|h\| = \|h\|_s + \|h\|_u. \quad (5.4.3)$$

We define the Banach space X as the completion of $C^1(\mathbb{T}^2; \mathbb{C}^2)$ with respect to the strong norm $\|\cdot\|$, and the weak space X_w as the completion with respect to $|\cdot|_w$.

Remark 5.4.2. We emphasize that these norms inherently depend on the shear parameter $\alpha \geq 1$.

On the choice of anisotropic Banach spaces

While all constructions of anisotropic Banach spaces in the literature follow the philosophy of enforcing “negative regularity in the contracting direction, positive regularity in the expanding direction of the map”, they are by no means a monolith. Over the last few decades, there has been a wide proliferation of different anisotropic spaces, each adapted to particular dynamical systems and coming with distinct advantages and drawbacks. For a comprehensive overview of these spaces, we refer the reader to the excellent survey by Baladi [Bal17]. The vast majority of spaces in the literature were constructed to handle relatively smooth transfer operators, such as those arising from smooth Anosov flows. For instance, the microlocal anisotropic spaces utilized in [BT07; BT08; DZ15; DZ16] rely heavily on the high regularity of the underlying map. Even spaces designed to accommodate piecewise smooth maps, such as [BG09], often assume some regularity on the stable foliation, which is absent in our setting. Therefore, the functional spaces adapted to our problem must be flexible enough to handle piecewise smooth transfer operators with merely measurable stable foliations. Following the taxonomy in [Bal17], candidates with these properties include the “Triebel” space approach of [BG10], the geometric spaces of Demers and Liverani [DL08], and the spaces introduced in [Bal17]. While it is likely that any of these

frameworks could be adapted to our arguments, the present article employs spaces closely modeled on the geometric approach of [DL08]. The simple piecewise-linear geometry of our map makes the analysis of the strong-chaos limit ($\alpha \rightarrow \infty$) in these spaces significantly more tractable.

5.4.1 Compact embeddings

Having defined the weak and strong norms on our Banach space, we now establish the fundamental functional-analytic properties required for spectral analysis. Specifically, the spectral gap of the dynamo operator is contingent upon the Lasota-Yorke inequalities. We seek to show that the pair $(X_w, |\cdot|_w)$ and $(X, \|\cdot\|)$ satisfy the following.

1. *Norm ordering:* $|f|_w \leq \|f\|$ for all $f \in X$.
2. *Compact embedding:* The unit ball $\{f \in X : \|f\| \leq 1\}$ is relatively compact in $(X_w, |\cdot|_w)$.

The first property is immediate from the definition of the norms, as the test function space for the weak norm is a subset of that for the strong norm (with a stronger topology). We now prove the second property.

Lemma 5.4.3. *The unit ball of X embeds compactly into X_w .*

Proof. Let $\mathcal{B}_1 = \{h \in X : \|h\| \leq 1\}$ and fix $\varepsilon > 0$. Since (Σ, d_Σ) is pre-compact, it is totally bounded. Hence, there exists a finite ε -net of leaves $\{(\mathbf{x}_i, \mathbf{v}_i, S_i)\}_{i=1}^n = \{W_i\}_{i=1}^n$. For each leaf W_i in our net, the space of test functions $\{\varphi \in C^1(W_i) : \|\varphi\|_{C^1} \leq 1\}$ is compactly embedded in $C^q(W_i)$ by the Arzela-Ascoli Theorem since $q \in (0, 1)$. Thus, we can select a finite ε -net of test functions $\{\varphi_{i,j}\}_{j=1}^{n_i}$ for each W_i . In other words, for any $\varphi \in C^1(W_i)$ with $\|\varphi\|_{C^1(W_i)} \leq 1$, there exists some $\varphi_{i,j}$ so that $\|\varphi - \varphi_{i,j}\|_{C^q(W_i)} < \varepsilon$. Let $h \in \mathcal{B}_1$. For any $W \in \Sigma$ and $\varphi \in C^1(W)$ with $\|\varphi\|_{C^1} \leq 1$, we estimate $\int_W h\varphi$ by comparing it to an integral over some W_i from our net $\{W_i\}_{i=1}^n$. We parametrize W, W_i respectively by

$$x + t\mathbf{v}, \quad t \leq S; \quad x_i + t\mathbf{v}_i, \quad t \leq S_i.$$

hence we can define $\tilde{\varphi} \in C^1(W_i)$

$$\tilde{\varphi}(x_i + t\mathbf{v}_i) = \begin{cases} \varphi(x + t\mathbf{v}) & \forall t \in (0, S \wedge S_1) \\ \varphi(x + S\mathbf{v}) & \forall t \in (S \wedge S_1, S_1), \end{cases}$$

and we have that the Lipschitz norm of $\tilde{\varphi}$ satisfies $\|\tilde{\varphi}\|_{C^1(W_i)} \leq \|\varphi\|_{C^1(W)} \leq 1$ and $d_q(\varphi, \tilde{\varphi}) = 0$. Picking $\varphi_{i,j}$ from our ε -net of test functions such that $\|\varphi_{i,j} - \tilde{\varphi}\|_{C^q(W_i)} < \varepsilon$, we decompose the error as

$$\int_W h\varphi - \int_{W_i} h\varphi_{i,j} = \left(\int_W h\varphi - \int_{W_i} h\tilde{\varphi} \right) + \int_{W_i} h(\tilde{\varphi} - \varphi_{i,j}).$$

The first term is bounded by $\varepsilon^\beta \|h\|_u$ via the strong unstable norm. The second term is bounded by $\varepsilon |W_i|^\sigma \|h\|_s$. Combining these, we find that for any $\varepsilon > 0$, there exists a finite collection of linear functionals $\ell_{i,j}(h) = \int_{W_i} h\varphi_{i,j} : (X, \|\cdot\|) \rightarrow \mathbb{C}^2$ such that

$$|h|_w \leq \max_{i,j} |\ell_{i,j}(h)| + C(\varepsilon^\beta + \varepsilon) \|h\|.$$

Since $C(\varepsilon^\beta + \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, a standard diagonal argument shows that any sequence in $\mathcal{B}_1$ has a subsequence that is Cauchy in X_w , completing the proof. $\square$

5.4.2 Some technical lemmas

In this section we collect a few results on our anisotropic Banach spaces and admissible curves, which we shall need throughout the manuscript. We begin with a lemma, which shows that indicator functions of half-planes live in our spaces, and have uniformly bounded norms in α . Whilst somewhat involved, the proof is in effect a simpler version of the matching of curves we shall do in more generality later.

Lemma 5.4.4. *Let $\mathbf{w} \in \mathbb{C}^2$ be a normalised vector $|\mathbf{w}| = 1$ and $\mathbb{1}_{y \geq 1/2}$ be the periodised indicator function of $\{y \geq 1/2\}$. Then the following hold true*

$$\|\mathbf{w}\mathbb{1}_{y \geq 1/2}\|_s \leq C, \quad \|\mathbf{w}\mathbb{1}_{y \geq 1/2}\|_u \leq C\alpha^{-1+\beta}.$$

Furthermore, for any $h \in X$ it holds that

$$\left| \int_{x \geq 1/2} h \right| \leq \frac{1}{2} |h|_w.$$

Proof. We shall prove the result for $\mathbf{w} = (1, 0)$ for the sake of convenience. We estimate the strong stable norm. Let $W \in \Sigma$ be an admissible curve, then there exist at most two connected components W_1, W_2 so that $W \cap \{y \geq 1/2\} = W_1 \cup W_2$. Then, for $\varphi \in C^q(W)$ with $\|\varphi\|_{C^q(W)} \leq |W|^{-\gamma}$, we have

$$\int_W \mathbb{1}_{y \geq 1/2} \varphi = \int_{W_1} \varphi + \int_{W_2} \varphi \leq (|W_1| + |W_2|) \|\varphi\|_{C^q} \leq 2|W|^{1-\gamma} \leq 2.$$

We estimate the strong unstable norm. We consider two admissible curves $W_1, W_2 \in \Sigma$, with $d_\Sigma(W_1, W_2) < \delta < 1/\alpha$ and $\varphi_1 \in C^1(W_1)$, $\varphi_2 \in C^1(W_2)$ with $d_q(\varphi_1, \varphi_2) < \delta$ and $\|\varphi_1\|_{C^1(W_1)} \leq 1$, $\|\varphi_2\|_{C^1(W_2)} \leq 1$. Since we are on $\mathbb{T}^2$ there exist at most two connected curves $(W_{1,j})_{j=1,2}$ and $(W_{2,j})_{j=1,2}$ so that $\cup_{j=1,2} W_{1,j} = W_1 \cap \{y \geq 1/2\}$ and $\cup_{j=1,2} W_{2,j} = W_2 \cap \{y \geq 1/2\}$. A computation similar to that in Lemma 5.4.9 implies that

$$d_\Sigma(W_{1,j}, W_{2,j}) \leq C d_\Sigma(W_1, W_2),$$

uniformly in α . Thus, we estimate

$$\int_{W_1 \cap \{y \geq 1/2\}} \varphi_1 - \int_{W_2 \cap \{y \geq 1/2\}} \varphi_2 = \sum_{j=1}^2 \int_{W_{1,j}} \varphi_1 - \int_{W_{2,j}} \varphi_2.$$

Choosing the parametrizations $W_{i,j}$ as $\gamma_{i,j}(t) = \mathbf{x}_{i,j} + t\mathbf{v}_{i,j}$ with $t \in (0, S_{i,j})$ for $i, j = 1, 2$ we deduce,

for e.g. comparing $W_{1,1}$ and $W_{2,1}$ and assuming that $S_{1,1} \leq S_{2,1}$ we have

$$\begin{aligned} \int_{W_{1,1}} \varphi_1 - \int_{W_{2,1}} \varphi_2 &= \int_0^{S_{1,1}} \varphi_1(\mathbf{x}_{1,1} + t\mathbf{v}_{1,1}) dt - \int_0^{S_{2,1}} \varphi_2(\mathbf{x}_{2,1} + t\mathbf{v}_{2,1}) dt \\ &= \int_0^{S_{1,1}} \varphi_1(\mathbf{x}_{1,1} + t\mathbf{v}_{1,1}) dt - \int_0^{S_{1,1}} \varphi_2(\mathbf{x}_{2,1} + t\mathbf{v}_{2,1}) dt + \int_{S_{1,1}}^{S_{2,1}} \varphi_2(\mathbf{x}_{2,1} + t\mathbf{v}_{2,1}) dt \\ &\leq d_q(\varphi_1, \varphi_2) S_{1,1} + |S_{2,1} - S_{1,1}| \|\varphi_2\|_{C^1(W_{2,1})} \leq d_q(\varphi_1, \varphi_2) + d_\Sigma(W_{1,1}, W_{2,1}) \leq 2\delta. \end{aligned}$$

Since the argument works the same for $W_{1,2}$ and $W_{2,2}$ we deduce that $\left| \int_{W_1} \mathbb{1}_{y \geq 1/2} \varphi_1 - \int_{W_2} \mathbb{1}_{y \geq 1/2} \varphi_2 \right| \leq 4\delta$. Hence, dividing through by δ^β we obtain an upper bound of order $4\delta^{1-\beta} \leq 4\alpha^{\beta-1}$, showing the claimed estimate for the strong unstable norm. The proof of the second assertion follows from

$$\left| \int_{x \geq 1/2} h \right| \leq \int_{x \geq 1/2} \left| \int_{\mathbb{T}} h(x, y) dy \right| dx \leq \int_{x \geq 1/2} |h|_w dx = \frac{1}{2} |h|_w.$$

□

Next, we show that multiplication by Lipschitz continuous functions is a bounded operator on our space.

Lemma 5.4.5. *Let $g \in C^1(\mathbb{T}^2, \mathbb{C})$. Then, the following hold true*

$$|gh|_w \leq C \|g\|_{C^1} |h|_w, \quad \|gh\| \leq C \|g\|_{C^1} \|h\|, \quad \forall h \in X.$$

Proof. Let $W \in \Sigma$ be an admissible curve, and let $\varphi \in C^1(W)$ with C^1 norm at most 1. Then, we have

$$\left| \int_W gh\varphi \right| = \left| \int_W h(g\varphi) \right| \leq |h|_w \|g\varphi\|_{C^1}.$$

Since $\|g\varphi\|_{C^1} \leq 2\|g\|_{C^1} \|\varphi\|_{C^1}$, it follows that $|gh|_w \leq 2\|g\|_{C^1} |h|_w$. Next, we prove the strong norm bound. As above, we consider W admissible and $\varphi \in C^1(W)$ with $\|\varphi\|_{\sigma, W} \leq 1$. Then we estimate

$$\left| \int_W gh\varphi \right| \leq \|h\|_s \|g\varphi\|_{\sigma, W}.$$

But now, $\|g\varphi\|_{\sigma, W} = |W|^\sigma \|g\varphi\|_{C^q(W)} \leq 2|W|^\sigma \|g\|_{C^1} \|\varphi\|_{C^q(W)} \leq 2\|g\|_{C^1}$. Hence, it holds $\|gh\|_s \leq 2\|g\|_{C^1} \|h\|_s$. Finally, we bound the strong unstable norm. We consider $W_1, W_2 \in \Sigma$ with $d_\Sigma(W_1, W_2) < \delta$ and $\varphi_1 \in C^1(W_1)$, $\varphi_2 \in C^1(W_2)$ with $d_q(\varphi_1, \varphi_2) < \delta$ and C^1 norm bounded by 1. Suppose that the W_i are parameterised respectively by $\mathbf{x}_i + t\mathbf{v}_i$, $t \leq S_i$. Say $S_1 \geq S_2$. Then, define $\widetilde{g\varphi_1} \in C^1(W_2)$ by

$$\widetilde{g\varphi_1}(\mathbf{x}_2 + t\mathbf{v}_2) = g\varphi_1(\mathbf{x}_1 + t\mathbf{v}_1).$$

This is a well-defined function, and by construction it holds $d_q(g\varphi_1, \widetilde{g\varphi_1}) = 0$. Thus, we have

$$\int_{W_1} gh\varphi_1 - \int_{W_2} gh\varphi_2 = \int_{W_1} h(g\varphi_1) - \int_{W_2} h(\widetilde{g\varphi_1}) + \int_{W_2} h(\widetilde{g\varphi_1} - g\varphi_2).$$

Firstly, it follows from the definition of the strong unstable norm that

$$\left| \int_{W_1} h(g\varphi_1) - \int_{W_2} h(\widetilde{g\varphi_1}) \right| \leq \delta^\beta \|h\|_u \max\{\|g\varphi_1\|_{C^1(W_1)}, \|\widetilde{g\varphi_1}\|_{C^1(W_2)}\} \leq 2\delta^\beta \|h\|_u \|g\|_{C^1}.$$

Next, we claim that $\|g\varphi_2 - \widetilde{g\varphi_1}\|_{C^q(W_2)} \leq 12\|g\|_{C^1} \delta^{1-q}$. To do so, note first that $\|g\varphi_2 - \widetilde{g\varphi_1}\|_{C^1(W_2)} \leq 4\|g\|_{C^1}$. Next, note that

$$\begin{aligned} \sup_{t \leq S_2} |g\varphi_2(\mathbf{x}_2 + t\mathbf{v}_2) - \widetilde{g\varphi_1}(\mathbf{x}_2 + t\mathbf{v}_2)| &\leq \sup_{t \leq S_2} |g(\mathbf{x}_2 + t\mathbf{v}_2)(\varphi_2(\mathbf{x}_2 + t\mathbf{v}_2) - \varphi_1(\mathbf{x}_1 + t\mathbf{v}_1))| \\ &\quad + \sup_{t \leq S_2} |\varphi_1(\mathbf{x}_1 + t\mathbf{v}_1)(g(\mathbf{x}_2 + t\mathbf{v}_2) - g(\mathbf{x}_1 + t\mathbf{v}_1))| \\ &\leq \|g\|_{C^1} \delta + \|\varphi_1\|_{C^1(W_1)} \|g\|_{C^1} (|\mathbf{x}_1 - \mathbf{x}_2| + S_2|\mathbf{v}_1 - \mathbf{v}_2|) \\ &\leq 3\delta \|g\|_{C^1}, \end{aligned}$$

where we used that $S_2 \leq 1$ by the definition of our set of admissible leaves Σ . Therefore, interpolating yields

$$\|g\varphi_2 - \widetilde{g\varphi_1}\|_{C^q(W_2)} \leq \|g\varphi_2 - \widetilde{g\varphi_1}\|_{C^1(W_2)}^q \|g\varphi_2 - \widetilde{g\varphi_1}\|_{C^0(W_2)}^{1-q} \leq 12\|g\|_{C^1} \delta^{1-q},$$

as claimed. Therefore, all in all

$$\frac{1}{\delta^\beta} \left| \int_{W_1} gh\varphi_1 - \int_{W_2} gh\varphi_2 \right| \leq C\|h\|_u \|g\|_{C^1} (1 + \delta^{1-q-\beta}) \leq C\|g\|_{C^1} \|h\|,$$

since $\beta < 1 - q$ by assumption. Hence, the claim is proven. $\square$

The next result we consider governs the way integration against stable curves interacts with changes of variables through the map T_α .

Lemma 5.4.6. *Let $W \in \Sigma$ be an admissible curve parametrized by $\gamma: (0, S) \rightarrow \mathbb{T}^2$ $\gamma(t) = \{(x, y) + t\mathbf{v}, t \in (0, S)\}$, Let $\tilde{W} = T_\alpha^{-1}(W)$ be parametrized by $\tilde{\gamma} = T_\alpha^{-1}(\gamma)$. We define $J_{\tilde{W}}T_\alpha = |DT_\alpha(\tilde{\gamma}(t))\tilde{\tau}(t)|$ with $\tilde{\tau}(t) := \frac{\tilde{\gamma}'(t)}{|\tilde{\gamma}'(t)|}$. Then for any C^1 vector field h , the following holds true:*

$$\int_W h = \int_{\tilde{W}} h \circ T_\alpha |J_{\tilde{W}}T_\alpha|.$$

Furthermore, if $W \in \Sigma$ is such that $W \subset \overline{T_\alpha(\mathcal{M}_\ell)}$ then $J_{\tilde{W}}T_\alpha$ is a constant value satisfying $|J_{\tilde{W}}T_\alpha| \leq 2\alpha^{-2}$.

Remark 5.4.7. Note that $\tilde{W}$ need not be admissible, since $|\tilde{W}| \approx \alpha^2|W|$ may be too long. Crucially however, it may always be written as a disjoint union of admissible curves.

Proof. Since T_α is a bijection we have that $\gamma'(t) = DT_\alpha(\tilde{\gamma}(t))\tilde{\gamma}'(t)$. Let $h: \mathbb{T}^2 \rightarrow \mathbb{C}^2$ be a C^1 function. Then using the definition of integral along curves with parametrizations γ and $\tilde{\gamma}$ we

deduce

$$\begin{aligned} \int_W h &= \int_0^S h(\gamma(t)) |\gamma'(t)| dt = \int_0^S h(T_\alpha(\tilde{\gamma}(t))) |DT_\alpha(\tilde{\gamma}(t)) \tilde{\gamma}'(t)| dt \\ &= \int_0^S h(T_\alpha(\tilde{\gamma}(t))) |DT_\alpha(\tilde{\gamma}(t)) \tilde{\tau}(t)| |\tilde{\gamma}'(t)| dt = \int_{\tilde{W}} h \circ T_\alpha |J_{\tilde{W}} T_\alpha|. \end{aligned}$$

For the estimate of the Jacobian along the curve we use $\gamma(t) = T_\alpha(\tilde{\gamma}(t))$ and assume $|\gamma'(t)| = 1$. Then, it holds that

$$T_\alpha \circ T_\alpha^{-1}(\gamma(t)) = \gamma(t),$$

and taking the derivative in time we have

$$|DT_\alpha(\tilde{\gamma}(t)) \tilde{\gamma}'(t)| = |\gamma'(t)| = 1. \quad (5.4.4)$$

By definition and by the fact that $\gamma'(t)$ is in the stable cone, hence expanding through T_α^{-1} , we obtain

$$|\tilde{\gamma}'(t)| = |DT_\alpha^{-1}(\gamma(t)) \gamma'(t)| \geq \frac{1}{2} \alpha^2.$$

Hence, by this bound and (5.4.4) we conclude $|J_{\tilde{W}} T_\alpha| \leq 2\alpha^{-2}$ for any t . $\square$

In the next lemma we show how compositions with T_α affect the Hölder norms of test functions.

Lemma 5.4.8. *Let $W \in \Sigma$ such that $W \subset T_\alpha(\mathcal{M}_\ell)$ for some ℓ and let $\tilde{W} \subset T_\alpha^{-1}(W)$ be an admissible curve in Σ . Then, for any $x, y \in \tilde{W}$ we have*

$$|T_\alpha(x) - T_\alpha(y)| \leq 2\alpha^{-2} |x - y|.$$

In particular, $[T_\alpha]_{C^1(\tilde{W})} \leq 2\alpha^{-2}$ and $[\varphi \circ T_\alpha]_{C^r(\tilde{W})} \leq 2\alpha^{-2r} [\varphi]_{C^r(W)}$, for $r \in [0, 1]$.

Proof. Since T_α is a bijection for any $x, y \in \tilde{W}$ we can define $x = T_\alpha^{-1}(z)$ and $y = T_\alpha^{-1}(w)$ with $z, w \in W$. Hence,

$$|T_\alpha(x) - T_\alpha(y)| = |z - w|$$

and using that $z - w$ is a vector in the stable cone and T_α^{-1} expand such vector by α^2 we have

$$|T_\alpha^{-1}(z) - T_\alpha^{-1}(w)| \geq \frac{\alpha^2}{2} |z - w|$$

hence

$$|T_\alpha(x) - T_\alpha(y)| = |z - w| \leq 2\alpha^{-2} |T_\alpha^{-1}(z) - T_\alpha^{-1}(w)| = 2\alpha^{-2} |x - y|.$$

The estimate for $\varphi \circ T_\alpha$ is an immediate consequence. $\square$

Finally, we state a crucial technical lemma which concerns matching of curves according to smoothness strips in the spirit of the proof of Lemma 5.4.4, as well as the behaviour of matched curves under preimages of T_α . The proof of this result is quite lengthy and technical, and will thus be postponed to Appendix 5.7.

Lemma 5.4.9. *There exist $\alpha_0 \geq 1$ such that for all $\alpha \geq \alpha_0$, and for any two admissible curves $W_1, W_2 \in \Sigma$ with $d_\Sigma(W_1, W_2) \leq \frac{2}{\alpha}$, the following hold true:*

1. *For any $W \in \Sigma$ there exist $(W_j)_{j=1}^{M_\alpha}$ with $M_\alpha \leq 3 + \lceil \alpha |W| \rceil$ such that $W_j \in \Sigma$, $W = \bigcup_j^{M_\alpha} W_j$ and $|W_j| \leq \frac{2}{\alpha}$, and for any j there exists $\ell = 1, 2, 3, 4$ such that $W_j \subset \overline{T_\alpha(\mathcal{M}_\ell)}$. Furthermore, for each W_j we can write $T_\alpha^{-1}(W_j) = \bigcup_{k=1}^{N_\alpha} \tilde{W}_{j,k}$ with $N_\alpha \leq 3 + \lceil \alpha^2 |W_j| \rceil$ and $\tilde{W}_{j,k} \in \Sigma$ for any j, k .*
2. *For any $i = 1, 2$ we have $W_i = \bigcup_{j=1}^{M_\alpha} W_{i,j} \cup \bigcup_{j=1}^8 U_{i,j}$ with $W_{i,j}, U_{i,j} \in \Sigma$ for any i, j and $W_{i,j} \subset \overline{T_\alpha(\mathcal{M}_\ell)}$ for some $\ell = 1, 2, 3, 4$ and $M_\alpha \leq 2\alpha$. Furthermore, $|W_{i,j}| \leq \frac{2}{\alpha}$, $|U_{i,j}| \leq C d_\Sigma(W_1, W_2)$ and $d_\Sigma(W_{1,j}, W_{2,j}) \leq C d_\Sigma(W_1, W_2)$.*
3. *For any $i = 1, 2$ and j we have $T_\alpha^{-1}(W_{i,j}) = \bigcup_{k=1}^{N_\alpha} \tilde{W}_{i,j,k} \cup \bigcup_{k=1}^8 \tilde{U}_{i,j,k}$ with $N_\alpha \leq 2\alpha$ and*

$$d_\Sigma(\tilde{W}_{1,j,k}, \tilde{W}_{2,j,k}) \leq C \alpha^{-2} d_\Sigma(W_{1,j}, W_{2,j}), \quad \forall j, k,$$

$\tilde{W}_{i,j,k}, \tilde{U}_{i,j,k} \in \Sigma$ for all i, j, k and

$$|\tilde{U}_{i,j,k}| \leq C \alpha^{-1} d_\Sigma(W_{1,j}, W_{2,j}), \quad \forall i, j, k.$$

Furthermore, if $W_{i,j}$ are parameterised by $\gamma_{i,j}(t)$, then $\tilde{W}_{i,j,k} = T_\alpha^{-1} \circ \gamma_{i,j}([t_{1,j,k}, t_{1,j,k+1}])$, with $|t_{1,j,k} - t_{2,j,k}| \leq C \alpha^{-1} d_\Sigma(W_{1,j}, W_{2,j})$ and $|t_{1,j,k}| + |t_{2,j,k}| \leq C \alpha$.

5.5 Lasota-Yorke inequality

In this section, we establish the spectral properties of the vector-valued transfer operator in (5.2.4), namely

$$\mathcal{L}_\alpha : X \rightarrow X, \quad \mathcal{L}_\alpha h = \frac{1}{\alpha^2} (DT_\alpha h) \circ T_\alpha^{-1},$$

where $T_\alpha : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is the uniformly hyperbolic map defined in Section 5.3. In particular, we shall prove the Lasota-Yorke inequality stated in Proposition 5.2.6 with $(X, \|\cdot\|)$, $(X_w, |\cdot|_w)$ defined in Section 5.4. Notation-wise, when we write $\mathcal{L}_\alpha h$, we highlight that h is a vector-valued distribution, and its integration against scalar test functions is performed component-wise. To facilitate the estimates in the following subsections, we recall several key properties of the map T_α and the geometry of the leaves in Σ : $\diamond$ *Jacobian Scaling*: For any leaf $\tilde{W}$, the Jacobian $J_{\tilde{W}} T_\alpha$ is constant on each partition element and satisfies (see Lemma 5.4.6)

$$|J_{\tilde{W}} T_\alpha| \leq 2\alpha^{-2}. \quad (5.5.1)$$

$\diamond$ *Matrix norms*: On each partition element $\mathcal{M}_\ell$, the derivative DT_α is a constant matrix A_ℓ with magnitude (see (5.3.1))

$$|A_\ell| \leq 2\alpha^2. \quad (5.5.2)$$

$\diamond$ *Test function pullback*: For any $\varphi \in C^1(W)$, in light of Lemma 5.4.8, its pullback satisfies the non-expansion property

$$\|\varphi \circ T_\alpha\|_{C^1(T_\alpha^{-1}(W))} \leq \|\varphi\|_{C^1(W)}. \quad (5.5.3)$$

◇ *Leaf averages:* We denote the average of a function f over a leaf W by

$$\bar{f}^W = \frac{1}{|W|} \int_W f. \quad (5.5.4)$$

The main result of this section is summarized in the following proposition, which provides the quantitative bounds necessary to prove the existence of a spectral gap. Note that the following proposition implies Proposition 5.2.6 by recalling (5.4.2) and defining $\eta = \min\{2q, 2 - 2\sigma, 2 - \sigma - \beta, 1 - \beta\} > 0$.

Proposition 5.5.1. *There exists a constant $\alpha_0 > 0$, so that for all $\alpha \geq \alpha_0$, and any $h \in X$, the following hold. The operator $\mathcal{L}_\alpha$ is continuous in the weak norm, namely*

$$|\mathcal{L}_\alpha h|_w \leq C|h|_w. \quad (5.5.5)$$

Moreover, it satisfies the strong stable norm estimate

$$\|\mathcal{L}_\alpha h\|_s \leq C(\alpha^{-2q} + \alpha^{2\sigma-2})\|h\|_s + C|h|_w, \quad (5.5.6)$$

and strong unstable norm estimate

$$\|\mathcal{L}_\alpha h\|_u \leq C \left[(\alpha^{-2+\sigma+\beta} + \alpha^{-1+\beta} + \alpha^{-2q+\beta-\sigma})\|h\|_s + \alpha^{-2\beta}\|h\|_u + \alpha^{\beta-\sigma}|h|_w \right]. \quad (5.5.7)$$

The proof of Proposition 5.5.1 is organized as follows: Section 5.5.1 establishes the weak norm continuity (5.5.5), while Sections 5.5.2 and 5.5.3 provide the bounds (5.5.6)-(5.5.7) for the strong stable and unstable norms, respectively.

5.5.1 Weak norm continuity

We begin by establishing the bound (5.5.5). Let $W \in \Sigma$ be an admissible leaf and $\varphi \in C^1(W)$ be a test function such that $\|\varphi\|_{C^1(W)} \leq 1$. By Lemma 5.4.9-(1), we partition W into segments $\{W_j\}_{j=1}^{M_\alpha}$, each contained in a single image partition $\overline{T_\alpha(\mathcal{M}_{\ell_j})}$. Each segment pulls back to N_α pre-image leaves $\{\tilde{W}_{j,k}\}_{k=1}^{N_\alpha}$. Applying the definition of $\mathcal{L}_\alpha$ and the change of variables formula, we have

$$\begin{aligned} \left| \int_W \mathcal{L}_\alpha h \varphi \right| &= \left| \frac{1}{\alpha^2} \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} \int_{\tilde{W}_{j,k}} DT_\alpha h \cdot (\varphi \circ T_\alpha) J_{\tilde{W}_{j,k}} T_\alpha \right| \\ &\leq \frac{1}{\alpha^2} \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} |A_{\ell_j}| \cdot |J_{\tilde{W}_{j,k}} T_\alpha| \cdot \left| \int_{\tilde{W}_{j,k}} h(\varphi \circ T_\alpha) \right|, \end{aligned}$$

where we used that on each pre-image leaf $\tilde{W}_{j,k}$, the derivative DT_α is a constant matrix A_{ℓ_j} and the Jacobian is a constant. Using the Jacobian bound (5.5.1) and the matrix bound (5.5.2), and noting that $\|\varphi \circ T_\alpha\|_{C^1} \leq 1$ by (5.5.3), we obtain

$$\left| \int_W \mathcal{L}_\alpha h \varphi \right| \leq \frac{C}{\alpha^2} \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} |h|_w. \quad (5.5.8)$$

From the counting bounds in Lemma 5.4.9, the total number of pre-image components satisfies

$$\sum_{j=1}^{M_\alpha} N_\alpha \leq \sum_{j=1}^{M_\alpha} (\alpha^2 |W_j| + 1) \leq \alpha^2 |W| + M_\alpha \leq C\alpha^2,$$

since $|W| \leq 1$ and $M_\alpha \sim \alpha$. Substituting this into (5.5.8) completes the proof of (5.5.5).

5.5.2 Strong stable norm

We now establish the estimate (5.5.6). Let $W \in \Sigma$ and $\varphi \in C^q(W)$ such that $\|\varphi\|_{\sigma, W} = \|\varphi\|_{C^q(W)} |W|^\sigma \leq$

1. Using the leaf fragmentation and pullback from Lemma 5.4.9-(1), we write

$$\begin{aligned} \left| \int_W \mathcal{L}_\alpha h \varphi \right| &= \left| \frac{1}{\alpha^2} \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} \int_{\tilde{W}_{j,k}} DT_\alpha h \cdot (\varphi \circ T_\alpha) J_{\tilde{W}_{j,k}} T_\alpha \right| \\ &\leq \frac{4}{\alpha^2} \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} \left| \int_{\tilde{W}_{j,k}} h(\varphi \circ T_\alpha) \right| \\ &\leq \underbrace{\frac{4}{\alpha^2} \sum_{j,k} \left| \int_{\tilde{W}_{j,k}} h(\varphi \circ T_\alpha - \overline{\varphi \circ T_\alpha}^{\tilde{W}_{j,k}}) \right|}_I + \underbrace{\frac{4}{\alpha^2} \sum_{j,k} \left| \int_{\tilde{W}_{j,k}} h \overline{\varphi \circ T_\alpha}^{\tilde{W}_{j,k}} \right|}_{II}, \end{aligned}$$

where the leaf-average notation is that of (5.5.4). We firstly estimate term II . Note that $|\overline{\varphi \circ T_\alpha}^{\tilde{W}_{j,k}}| \leq \|\varphi\|_{C^0(W)} \leq |W|^{-\sigma}$ given our normalization of φ . Recall that $j \in \{1, \dots, M_\alpha\}$ and $k \in \{1, \dots, N_\alpha\}$. We consider three regimes for the length of the original leaf $|W|$ using the bound for M_α, N_α in Lemma 5.4.9: $\diamond$ *Large leaves* ($|W| \geq \alpha^{-1}$): In this case, $M_\alpha \leq 4\alpha|W|$ and $N_\alpha \leq 4\alpha$. Since $\overline{\varphi \circ T_\alpha}^{\tilde{W}_{j,k}}$ is constant, the integral is bounded by $|h|_w$. Summing over the fragments yields

$$II \leq \frac{4}{\alpha^2} (4\alpha|W|)(4\alpha) |h|_w |W|^{-\sigma} \leq 64|W|^{1-\sigma} |h|_w \leq C|h|_w.$$

$\diamond$ *Medium leaves* ($\alpha^{-2} \leq |W| < \alpha^{-1}$): Here, $M_\alpha \leq 4$ and $N_\alpha \leq 3 + \lceil \alpha^2 |W_j| \rceil \leq 4\alpha^2 |W|$, where the last holds true for any j thanks to $|W| \geq \alpha^{-2}$. We have

$$II \leq \frac{4}{\alpha^2} (3)(4\alpha^2 |W|) |h|_w |W|^{-\sigma} \leq 48|W|^{1-\sigma} |h|_w \leq C\alpha^{\sigma-1} |h|_w.$$

$\diamond$ *Small leaves* ($|W| < \alpha^{-2}$): At this scale, $M_\alpha, N_\alpha \leq 4$. We utilize the definition of the strong stable norm directly. Since $|\tilde{W}_{j,k}| \leq 2\alpha^2 |W|$ and $\|1\|_{\sigma, \tilde{W}_{j,k}} = |\tilde{W}_{j,k}|^\sigma$:

$$II \leq \frac{C}{\alpha^2} \|h\|_s \max_{j,k} |\tilde{W}_{j,k}|^\sigma |W|^{-\sigma} \leq C\alpha^{-2} (\alpha^2 |W|)^\sigma |W|^{-\sigma} \|h\|_s \leq C\alpha^{2\sigma-2} \|h\|_s.$$

We now estimate I , which measures the oscillation of the test function. By Lemma 5.4.8 we have $[\varphi \circ T_\alpha]_{C^q(\tilde{W}_{j,k})} \leq C\alpha^{-2q} \|\varphi\|_{C^q}$. Furthermore, using that $|T_\alpha(\tilde{W}_{j,k})| \leq C\alpha^{-2}$, hence the fluctuation satisfies $\|\varphi \circ T_\alpha - \overline{\varphi \circ T_\alpha}^{\tilde{W}_{j,k}}\|_{C^0(\tilde{W}_{j,k})} \leq C\alpha^{-2q} \|\varphi\|_{C^q}$. Therefore, we deduce

$$\|\varphi \circ T_\alpha - \overline{\varphi \circ T_\alpha}^{\tilde{W}_{j,k}}\|_{\sigma, \tilde{W}_{j,k}} \leq C\alpha^{-2q} \|\varphi\|_{\sigma, W} |W|^{-\sigma} |\tilde{W}_{j,k}|^\sigma$$

Repeating the same analysis as above, the dominant factor is the contraction of the C^q norm. From the definition of $\|h\|_s$ and the properties of the test function on the pre-image leaves we have

$$I \leq \frac{4}{\alpha^2} \sum_{j,k} \|h\|_s \|\varphi \circ T_\alpha - \overline{\varphi \circ T_\alpha}\|_{\sigma, \tilde{W}_{j,k}} \leq \frac{C}{\alpha^2} \sum_{j,k} \|h\|_s (\alpha^{-2q} |W|^{-\sigma} |\tilde{W}_{j,k}|^\sigma).$$

As before, we split the cases of large leaves, medium leaves and small leaves. If $|W| \geq \alpha^{-1}$, we use $M_\alpha \leq 4\alpha|W|$ and $N_\alpha \leq 4\alpha$ to deduce that $I \leq C\alpha^{-2q}|W|^{1-\sigma}\|h\|_s$. If $\alpha^{-2} \leq |W| < \alpha^{-1}$, we use $M_\alpha \leq 3$, and $N_\alpha \leq 4\alpha^2|W|$ to deduce that $I \leq \alpha^{-2q}|W|^{1-\sigma}\|h\|_s$. If $|W| < \alpha^{-2}$, we use $M_\alpha, N_\alpha \leq 3$ and $|\tilde{W}_{j,k}| \leq 2\alpha^2|W|$ to deduce that $I \leq \alpha^{-2-2q+2\sigma}\|h\|_s$. Combining the bounds for I and II results in we conclude the proof of (5.5.6).

5.5.3 Strong unstable norm

To establish the unstable norm estimate (5.5.7), we consider $W_1, W_2 \in \Sigma$ two admissible curves with $d_\Sigma(W_1, W_2) < \delta \leq \alpha^{-1}$. We use property (2) in Lemma 5.4.9 to decompose each W_i as

$$W_i = \bigcup_{j=1}^{M_\alpha} W_{i,j} \cup \bigcup_{j=1}^8 U_{i,j},$$

where $M_\alpha \leq \alpha$ and $|U_{i,j}| \leq C\delta$. Let φ_1, φ_2 be test functions satisfying $d_q(\varphi_1, \varphi_2) \leq \delta$ and $\|\varphi_i\|_{C^1(W_i)} \leq 1$. The integral difference is split into matched and unmatched components

$$\begin{aligned} & \frac{1}{\alpha^2} \left| \int_{W_1} (DT_\alpha h) \circ T_\alpha^{-1} \varphi_1 - \int_{W_2} (DT_\alpha h) \circ T_\alpha^{-1} \varphi_2 \right| \\ & \leq \sum_{j=1}^{M_\alpha} \frac{1}{\alpha^2} \left| A_j \int_{W_{1,j}} h \circ T_\alpha^{-1} \varphi_1 - A_j \int_{W_{2,j}} h \circ T_\alpha^{-1} \varphi_2 \right| \quad (\text{Matched}) \\ & \quad + \sum_{i=1,2} \sum_{j=1}^8 \frac{1}{\alpha^2} \left| A_j \int_{U_{i,j}} h \circ T_\alpha^{-1} \varphi_i \right| \quad (\text{Unmatched}). \end{aligned}$$

We treat the two components separately.

The unmatched curves

We firstly estimate the unmatched curves, tackling the integral over $U_{i,j}$ for $i = 1$ (the case $i = 2$ is symmetric). Using that $|A_j| \leq C\alpha^2$ and the strong stable bound (5.5.6) we deduce

$$\begin{aligned} \frac{1}{\alpha^2} \left| \int_{U_{i,j}} A_j h \circ T_\alpha^{-1} \varphi_i \right| & \leq \|\varphi_i\|_{\sigma, U_{i,j}} \|\mathcal{L}_\alpha h\|_s \leq |U_{i,j}|^\sigma ((\alpha^{-2q} + \alpha^{2\sigma-2}) \|h\|_s + C|h|_w) \\ & \leq \delta^\beta \alpha^{\beta-\sigma} ((\alpha^{-2q} + \alpha^{2\sigma-2}) \|h\|_s + C|h|_w). \end{aligned}$$

The matched curves

We now turn to the matched curves $W_{i,j}$. We will repeatedly use that $M_\alpha, N_\alpha \leq 2\alpha$, $|A_j| \leq 2\alpha^2$ for any j , and that the Jacobian satisfies $|J_{\tilde{W}} T_\alpha| \leq \frac{2}{\alpha^2}$. By property (3) of Lemma 5.4.9, for any

$j = 1, \dots, M_\alpha$, the pre-image decomposes as

$$T_\alpha^{-1}(W_{i,j}) = \bigcup_{k=1}^{N_\alpha} \tilde{W}_{i,j,k} \cup \bigcup_{k=1}^8 \tilde{U}_{i,j,k},$$

where $|\tilde{U}_{i,j,k}| \leq C\alpha^{-1}\delta$ for any i, j, k and $N_\alpha \leq 2\alpha$. The difference over the matched segments expands to

$$\begin{aligned} & \sum_{j=1}^{M_\alpha} \frac{1}{\alpha^2} \left| A_j \int_{W_{1,j}} h \circ T_\alpha^{-1} \varphi_1 - A_j \int_{W_{2,j}} h \circ T_\alpha^{-1} \varphi_2 \right| \\ & \leq \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} \frac{2}{\alpha^4} \left| A_j \int_{\tilde{W}_{1,j,k}} h(\varphi_1 \circ T_\alpha) - A_j \int_{\tilde{W}_{2,j,k}} h(\varphi_2 \circ T_\alpha) \right| \end{aligned} \quad (5.5.9)$$

$$+ \sum_{i=1,2} \sum_{j=1}^{M_\alpha} \sum_{k=1}^8 \frac{2}{\alpha^4} \left| A_j \int_{\tilde{U}_{i,j,k}} h(\varphi_i \circ T_\alpha) \right|. \quad (5.5.10)$$

We may bound the second term by

$$\begin{aligned} \sum_{j=1}^{M_\alpha} \sum_{k=1}^8 \frac{1}{\alpha^4} \left| A_j \int_{\tilde{U}_{i,j,k}} h(\varphi_i \circ T_\alpha) \right| & \leq C\alpha^{-1} \|h\|_s \max_{j,k} \|\varphi_i \circ T_\alpha\|_{\sigma, \tilde{U}_{i,j,k}} \leq C\alpha^{-1} \|h\|_s \|\varphi_i\|_{C^q(W_i)} |\tilde{U}_{i,j,k}|^\sigma \\ & \leq C\alpha^{-1} (\alpha^{-1}\delta)^\sigma \|h\|_s \leq C\alpha^{-1-\sigma} \delta^{\sigma-\beta} \delta^\beta \|h\|_s \leq C\alpha^{-1-2\sigma+\beta} \delta^\beta \|h\|_s. \end{aligned}$$

This takes care of (5.5.10) in a way that is consistent with (5.5.7). We now turn to (5.5.9). We parametrize $W_{i,j}$ as

$$\gamma_{i,j}(t) = \mathbf{x}_{i,j} + t\mathbf{v}_i, \quad t \in [0, t_{fin,i,j}].$$

Hence $\tilde{W}_{i,j,k}$ is parametrized by

$$\tilde{\gamma}_{i,j,k}(t) = T_\alpha^{-1}(\mathbf{x}_{i,j}) + t_{i,j,k} \frac{T_\alpha^{-1}(\mathbf{v}_i)}{|T_\alpha^{-1}(\mathbf{v}_i)|} + t \frac{T_\alpha^{-1}(\mathbf{v}_1)}{|T_\alpha^{-1}(\mathbf{v}_1)|}, \quad t \in [0, \tilde{t}_{fin,i,j,k}].^4$$

For any i, j, k we extend the curves $\tilde{W}_{i,j,k}$ as affine curves on $\mathbb{R}^2$. From now on we assume without loss of generality that $|\tilde{W}_{2,j,k}| \leq |\tilde{W}_{1,j,k}|$. We define an affine map $C_{j,k} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ so that $C_{j,k}(\tilde{\gamma}_{2,j,k}(t)) = \tilde{\gamma}_{1,j,k}(t)$ for all $t \leq t_{fin,2,j,k}$. We define the map $\varphi_{1,C_{j,k}} = \varphi_1(T_\alpha \circ C_{j,k} \circ T_\alpha^{-1})$ which satisfies

$$\begin{aligned} \varphi_{1,C_{j,k}}(T_\alpha \circ \tilde{\gamma}_{2,j,k}(t)) & = \varphi_1(T_\alpha \circ C_{j,k}(\tilde{\gamma}_{2,j,k}(t))) = \varphi_1(T_\alpha(\tilde{\gamma}_{1,j,k}(t))) \\ & = \varphi_1 \left(T_\alpha \circ \left(T_\alpha^{-1}(\mathbf{x}_{1,j}) + t_{1,j,k} \frac{T_\alpha^{-1}(\mathbf{v}_1)}{|T_\alpha^{-1}(\mathbf{v}_1)|} + t \frac{T_\alpha^{-1}(\mathbf{v}_1)}{|T_\alpha^{-1}(\mathbf{v}_1)|} \right) \right) \\ & = \varphi_1 \left(\mathbf{x}_{1,j} + t_{1,j,k} \frac{\mathbf{v}_1}{|T_\alpha^{-1}(\mathbf{v}_1)|} + t \frac{\mathbf{v}_1}{|T_\alpha^{-1}(\mathbf{v}_1)|} \right). \end{aligned}$$

Hence we deduce

$$d_q(\varphi_1 \circ T_\alpha, \varphi_{1,C_{j,k}} \circ T_\alpha) = 0.$$

⁴In the case where T_α flips orientations, the starting and ending times, as well as the beginning and end points should be reversed.

Summing and subtracting the function $\varphi_{1,C_{j,k}}$ we have

$$\begin{aligned} & \frac{2}{\alpha^4} \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} \left| A_j \int_{\tilde{W}_{1,j,k}} h \varphi_1 \circ T_\alpha - A_j \int_{\tilde{W}_{2,j,k}} h \varphi_2 \circ T_\alpha \right| \\ & \leq \frac{2}{\alpha^4} \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} \left| A_j \int_{\tilde{W}_{1,j,k}} h \varphi_1 \circ T_\alpha - A_j \int_{\tilde{W}_{2,j,k}} h \varphi_{1,C_{j,k}} \circ T_\alpha \right| \\ & \quad + \frac{2}{\alpha^4} \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} \left| A_j \int_{\tilde{W}_{2,j,k}} h (\varphi_{1,C_{j,k}} \circ T_\alpha - \varphi_2 \circ T_\alpha) \right|. \end{aligned}$$

The first summand can be estimated thanks to $d_q(\varphi_1 \circ T_\alpha, \varphi_{1,C_{j,k}} \circ T_\alpha) = 0$, and properties (2), (3) in Lemma 5.4.9

$$\begin{aligned} \frac{2}{\alpha^4} \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} \left| A_j \int_{\tilde{W}_{1,j,k}} h \varphi_1 \circ T_\alpha - A_j \int_{\tilde{W}_{2,j,k}} h \varphi_{1,C_{j,k}} \circ T_\alpha \right| & \leq C \|h\|_u \max_{j,k} d_\Sigma^\beta(\tilde{W}_{1,j,k}, \tilde{W}_{2,j,k}) \\ & \leq C \|h\|_u \alpha^{-2\beta} d_\Sigma^\beta(W_1, W_2) \\ & \leq C \alpha^{-2\beta} \delta^\beta \|h\|_u. \end{aligned}$$

For the second summand we claim that

$$\|\varphi_{1,C_{j,k}} \circ T_\alpha - \varphi_2 \circ T_\alpha\|_{C^q(\tilde{W}_{2,j,k})} \leq C \delta^{1-q} \alpha^{-2q} + \delta. \quad (5.5.11)$$

If the claim holds true, recalling that $N_\alpha, M_\alpha \leq 2\alpha$ and $|A_j| \leq 2\alpha^2$ we bound the second summand

$$\begin{aligned} \frac{2}{\alpha^4} \sum_{j=1}^{M_\alpha} \sum_{k=1}^{N_\alpha} \left| A_j \int_{\tilde{W}_{2,j,k}} h (\varphi_{1,C_{j,k}} \circ T_\alpha - \varphi_2 \circ T_\alpha) \right| & \leq C \|h\|_s \|\varphi_{1,C_{j,k}} \circ T_\alpha - \varphi_2 \circ T_\alpha\|_{C^q(\tilde{W}_{2,j,k})} \\ & \leq C \delta^\beta \alpha^{-1+\beta} \|h\|_s \end{aligned}$$

and we conclude the proof. We now prove the claim. We firstly estimate the $C^0(\tilde{W}_{2,j,k})$ norm, assuming without loss of generality that $|W_{2,j}| \leq |W_{1,j}|$.

$$\begin{aligned} & \sup_t |\varphi_{1,C_{j,k}}(T_\alpha \circ \tilde{\gamma}_{2,j,k}(t)) - \varphi_2(T_\alpha(\tilde{\gamma}_{2,j,k}(t)))| \\ & = \sup_t \left| \varphi_1 \left(\mathbf{x}_{1,j} + t_{1,j,k} \frac{\mathbf{v}_1}{|T_\alpha^{-1}(\mathbf{v}_1)|} + t \frac{\mathbf{v}_1}{|T_\alpha^{-1}(\mathbf{v}_1)|} \right) - \varphi_2 \left(\mathbf{x}_{2,j} + t_{2,j,k} \frac{\mathbf{v}_2}{|T_\alpha^{-1}(\mathbf{v}_2)|} + t \frac{\mathbf{v}_2}{|T_\alpha^{-1}(\mathbf{v}_2)|} \right) \right| \\ & \leq \sup_t \left| \varphi_1 \left(\mathbf{x}_{1,j} + t_{1,j,k} \frac{\mathbf{v}_1}{|T_\alpha^{-1}(\mathbf{v}_1)|} + t \frac{\mathbf{v}_1}{|T_\alpha^{-1}(\mathbf{v}_1)|} \right) - \varphi_1 \left(\mathbf{x}_{1,j} + t_{2,j,k} \frac{\mathbf{v}_1}{|T_\alpha^{-1}(\mathbf{v}_2)|} + t \frac{\mathbf{v}_1}{|T_\alpha^{-1}(\mathbf{v}_2)|} \right) \right| \\ & \quad + \sup_t \left| \varphi_1 \left(\mathbf{x}_{1,j} + t_{2,j,k} \frac{\mathbf{v}_1}{|T_\alpha^{-1}(\mathbf{v}_2)|} + t \frac{\mathbf{v}_1}{|T_\alpha^{-1}(\mathbf{v}_2)|} \right) - \varphi_2 \left(\mathbf{x}_{2,j} + t_{2,j,k} \frac{\mathbf{v}_2}{|T_\alpha^{-1}(\mathbf{v}_2)|} + t \frac{\mathbf{v}_2}{|T_\alpha^{-1}(\mathbf{v}_2)|} \right) \right| \end{aligned}$$

The final term is bounded by δ by definition of the strong unstable norm. Now, using (3) in Lemma 5.4.9, we deduce

$$|t_{1,j,k} - t_{2,j,k}| \leq C \alpha^{-1} \delta, \quad |t_{1,j,k}| + |t_{2,j,k}| \leq C \alpha,$$

and

$$\left| \frac{1}{|T_\alpha^{-1} \mathbf{v}_2|} - \frac{1}{|T_\alpha^{-1} \mathbf{v}_1|} \right| \leq \frac{|T_\alpha^{-1}(\mathbf{v}_1 - \mathbf{v}_2)|}{|T_\alpha^{-1} \mathbf{v}_1| |T_\alpha^{-1} \mathbf{v}_2|} \leq C \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{|T_\alpha^{-1} \mathbf{v}_1|} \leq C \alpha^{-2} \delta,$$

and using also $\|\varphi_1\|_{C^1(W_1)} \leq 1$ we deduce

$$\|\varphi_{1,C_{j,k}} \circ T_\alpha - \varphi_2 \circ T_\alpha\|_{C^0(\tilde{W}_{2,j,k})} \leq C\delta.$$

Now, from Lemma 5.4.8 we deduce that the Lipschitz seminorms satisfy the bound

$$[\varphi_{1,C_{j,k}} \circ T_\alpha]_{C^1} + [\varphi_2 \circ T_\alpha]_{C^1} \leq C\alpha^{-2}.$$

Hence, using the interpolation inequality $[\cdot]_{\dot{C}^q} \leq [\cdot]_{C^1}^q [\cdot]_{C^0}^{1-q}$ we deduce that the claim (5.5.11) holds true.

5.6 Spectral theory of the ideal dynamo operator

We recall that $(X, \|\cdot\|)$ is the anisotropic Banach space defined in Section 5.4. Our focus is the operator:

$$\mathcal{L}_\alpha : X \rightarrow X, \quad \mathcal{L}_\alpha h = \frac{1}{\alpha^2} (DT_\alpha h) \circ T_\alpha^{-1},$$

where $T_\alpha : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is the uniformly hyperbolic map defined in Section 5.3 and $h : \mathbb{T}^2 \rightarrow \mathbb{C}^2$. To perform the analysis, we partition the torus into four quadrants:

$$\begin{aligned} Q_1 &:= \{(x, y) : x \geq \tfrac{1}{2}, y \geq \tfrac{1}{2}\}, & Q_2 &:= \{(x, y) : x < \tfrac{1}{2}, y \geq \tfrac{1}{2}\}, \\ Q_3 &:= \{(x, y) : x < \tfrac{1}{2}, y < \tfrac{1}{2}\}, & Q_4 &:= \{(x, y) : x \geq \tfrac{1}{2}, y < \tfrac{1}{2}\}. \end{aligned}$$

We denote the integral of a function f over a set Q by

$$(f)_Q = \int_Q f, \quad \forall f \in L^1, \forall Q \subset \mathbb{T}^2.$$

In this section, we prove Theorem 5.2.7 by establishing the existence of a leading eigenvalue through a limiting rank-1 operator.

Proposition 5.6.1. *Let $g \in C^1(\mathbb{T}^2)$ be such that $(e^{2\pi i g})_{Q_1} + (e^{2\pi i g})_{Q_3} - (e^{2\pi i g})_{Q_4} - (e^{2\pi i g})_{Q_2} \neq 0$. There exists $\alpha_0 \geq 1$ such that for any $\alpha \geq \alpha_0$, the operator $e^{2\pi i g} \mathcal{L}_\alpha$ admits a simple, discrete eigenvalue $\mu_\alpha \in \mathbb{C}$ satisfying*

$$|\mu_\alpha| \geq \frac{1}{2} |(e^{2\pi i g})_{Q_1} + (e^{2\pi i g})_{Q_3} - (e^{2\pi i g})_{Q_4} - (e^{2\pi i g})_{Q_2}|.$$

In particular, there exists $g \in C^1(\mathbb{T}^2)$ such that the operator $\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha$ admits a leading eigenvalue $\lambda_\alpha = \alpha^2 \mu_\alpha$ with $|\mu_\alpha| \geq 1/4$.

A suitable $g \in C^1(\mathbb{T}^2)$ can be constructed via mollification

$$g = f \star \varphi_\ell, \quad f(x, y) = \frac{1}{2} \mathbb{1}_{Q_2 \cup Q_4}(x, y),$$

where φ_ℓ is a standard periodic Friedrichs mollifier. Then,

$$(e^{2\pi i f})_{Q_1} + (e^{2\pi i f})_{Q_3} - (e^{2\pi i f})_{Q_4} - (e^{2\pi i f})_{Q_2} = 1/4 + 1/4 - (-1/4) - (-1/4) = 1,$$

which implies that $|(e^{2\pi ig})_{Q_1} + (e^{2\pi ig})_{Q_3} - (e^{2\pi ig})_{Q_4} - (e^{2\pi ig})_{Q_2}| \geq 1/2$ for ℓ sufficiently small.

The core of the proof is the existence of a rank-1 “limiting operator” $\mathcal{L}_\infty : X \rightarrow X$ that approximates $\mathcal{L}_\alpha$ as $\alpha \rightarrow \infty$:

$$\mathcal{L}_\infty h = \begin{pmatrix} \left(\int_{x \geq 1/2} h^{(1)} - \int_{x < 1/2} h^{(1)} \right) (\mathbb{1}_{y \geq 1/2} - \mathbb{1}_{y < 1/2}) \\ 0 \end{pmatrix}, \quad h = (h^{(1)}, h^{(2)}).$$

While it is tempting to assume that the spectral properties of $\mathcal{L}_\alpha$ must mirror those of $\mathcal{L}_\infty$ for large α , this convergence is non-trivial in our setting. Specifically, because the anisotropic Banach space $(X, \|\cdot\|)$ is itself α -dependent, the very notion of operator convergence requires a careful, quantitative treatment. To bridge this gap, we proceed via uniform resolvent estimates, rigorously tracking the dependencies on α to ensure the stability of the leading resonance.

Remark 5.6.2 (The role of g). In the absence of the phase factor g (which represents the out-of-plane shear), the operator $\mathcal{L}_\alpha$ has no non-trivial eigenvalues. This is consistent with anti-dynamo theorems, as the purely $2d$ map cannot sustain growth. Our proof relies on introducing the shear $e^{2\pi ig}$ after only *one* iteration of the $2d$ operator. If the shear were applied after two iterations, we would be considering $e^{2\pi ig} \mathcal{L}_\infty^2$. However, $\mathcal{L}_\infty^2 \equiv 0$, as the range of $\mathcal{L}_\infty$ is simply the span of $\begin{pmatrix} \mathbb{1}_{y \geq 1/2} - \mathbb{1}_{y < 1/2} \\ 0 \end{pmatrix}$, and

$$\left(\mathcal{L}_\infty \begin{pmatrix} \mathbb{1}_{y \geq 1/2} \\ 0 \end{pmatrix} \right)^{(1)} = \left(\int_{x \geq 1/2, y \geq 1/2} 1 - \int_{x < 1/2, y \geq 1/2} 1 \right) (\mathbb{1}_{y \geq 1/2} - \mathbb{1}_{y < 1/2}) = 0,$$

and the same holds true with $\mathbb{1}_{y < 1/2}$. Hence, the behavior of $e^{2\pi ig} \mathcal{L}_\alpha^n$ for fixed α as $n \rightarrow \infty$ is indeed entirely distinct from that of $e^{2\pi ig} \mathcal{L}_\alpha$ as $\alpha \rightarrow \infty$, mechanism.

5.6.1 Technical lemmas

The following two lemmas provide the rigorous bridge from $\alpha = \infty$ to finite α . The convergence of $e^{2\pi ig} \mathcal{L}_\alpha$ as $\alpha \rightarrow \infty$ is established in the following lemma and it represents the main technical result of this section.

Lemma 5.6.3. *Let $e^{2\pi ig} \in C^1(\mathbb{T}^2)$. Then, there exist a constant $\tilde{\eta}$ depending only on $\|e^{2\pi ig}\|_{C^1}$, and $\alpha_0 \geq 1$ so that for all $\alpha \geq \alpha_0$, it holds*

$$\|e^{2\pi ig} \mathcal{L}_\alpha h - e^{2\pi ig} \mathcal{L}_\infty h\| \leq C \alpha^{-\tilde{\eta}} \|h\|, \quad \forall h \in X.$$

The proof of this result is delayed to Section 5.6.2. In order to deduce the convergence of spectra from the above, we need to ensure that the resolvents of our limiting operator are uniformly bounded in the strong norm, despite the fact that $\|\cdot\|$ intrinsically depends on α . The following Lemma ensures this is the case.

Lemma 5.6.4 (Resolvent Bound). *Let $g \in C^1(\mathbb{T}^2)$ be a non zero function. There exists a matrix $M \in \mathbb{C}^{2 \times 2}$ with eigenvalues 0 and $\mu = (e^{2\pi ig})_{Q_1} + (e^{2\pi ig})_{Q_3} - (e^{2\pi ig})_{Q_2} - (e^{2\pi ig})_{Q_4}$, so that the following holds true. There exists $\alpha_0 > 1$ so that for any $\alpha \geq \alpha_0$ and for any $\lambda \in \mathbb{C} \setminus \{0, \mu\}$ it*

holds

$$\|(\lambda - e^{2\pi ig} \mathcal{L}_\infty)^{-1}\|_{X \rightarrow X} \leq C|\lambda|^{-1}(1 + \|(\lambda - M)^{-1}\|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2}).$$

Proof. Since the operator $e^{2\pi ig} \mathcal{L}_\infty$ is diagonal in its components, and its second component is the zero map, it suffices to compute the inverse of its first component. Thus, for the remainder of the proof, we abuse notation and identify $e^{2\pi ig} \mathcal{L}_\infty$ with its first component.

We express $e^{2\pi ig} \mathcal{L}_\infty$ as the product

$$e^{2\pi ig} \mathcal{L}_\infty = A_g B,$$

where $A_g : \mathbb{R}^2 \rightarrow X$ and $B : X \rightarrow \mathbb{C}^2$ are defined respectively by

$$A_g(v_1, v_2) = e^{2\pi ig}(v_1 - v_2)(\mathbb{1}_{y \geq 1/2} - \mathbb{1}_{y < 1/2}), \quad (5.6.1)$$

and

$$Bf = \begin{pmatrix} \int_{x \geq 1/2} f \\ \int_{x < 1/2} f \end{pmatrix}. \quad (5.6.2)$$

Applying the Sherman-Morrison-Woodbury finite dimensional reduction formula to compute

$$f = (\lambda - e^{2\pi ig} \mathcal{L}_\infty)^{-1} h.$$

For some $c \in \mathbb{C}^2$ we use the ansatz $f = \lambda^{-1}(h + A_g c)$ then $f = (\lambda - e^{2\pi ig} \mathcal{L}_\infty)^{-1} h$ if and only if

$$A_g c - \lambda^{-1} A_g B h - \lambda^{-1} A_g B A_g c = 0$$

hence we impose that $c \in \mathbb{C}^2$ is such that $(\text{Id} - \lambda^{-1} B A_g) c = \lambda^{-1} B h$. Hence assuming that the 2×2 matrix $(\text{Id} - \lambda^{-1} B A_g)$ is invertible for any $\lambda \in \Gamma$ we choose

$$c = \lambda^{-1} (\text{Id} - \lambda^{-1} B A_g)^{-1} B h.$$

We deduce the Sherman-Morrison-Woodbury finite dimensional reduction formula

$$f = (\lambda - \mathcal{L}_\infty)^{-1} h = \lambda^{-1} (h + \lambda^{-1} A_g (\text{Id} - \lambda^{-1} B A_g)^{-1} B h).$$

We define $M = B A_g \in \mathbb{C}^{2 \times 2}$. Hence, we have

$$\|(\lambda - \mathcal{L}_\infty)^{-1} h\| \leq |\lambda|^{-1} (\|h\| + \|A_g\|_{\mathbb{C}^2 \rightarrow X} \|(\lambda - M)^{-1}\|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2} \|B\|_{X \rightarrow \mathbb{C}^2} \|h\|).$$

Using Lemma 5.4.4, we deduce $\|B\|_{X \rightarrow \mathbb{C}^2} \leq 2$ and using Lemma 5.4.4 and Lemma 5.4.5 we deduce $\|A_g\|_{\mathbb{C}^2 \rightarrow X} \leq C$ uniformly in α . Finally, we note that the matrix $M = B A_g$ can be computed to be

$$M = B A_g = \begin{pmatrix} (e^{2\pi ig})_{Q_1} - (e^{2\pi ig})_{Q_4} & (e^{2\pi ig})_{Q_4} - (e^{2\pi ig})_{Q_1} \\ (e^{2\pi ig})_{Q_2} - (e^{2\pi ig})_{Q_3} & (e^{2\pi ig})_{Q_3} - (e^{2\pi ig})_{Q_2} \end{pmatrix} \in \mathbb{C}^{2 \times 2},$$

which indeed possesses the claimed eigenvalues of 0 and $\mu = (e^{2\pi ig})_{Q_1} + (e^{2\pi ig})_{Q_3} - (e^{2\pi ig})_{Q_2} -$

$(e^{2\pi ig})_{Q_4}$. Thus, we conclude the proof. $\square$

With these results under our belt, the proof of Proposition 5.6.1 may now be deduced in a way analogous to the standard result on isomorphisms of “close” Riesz projectors.

Proof of Proposition 5.6.1. Let Γ be some circle in the complex plane enclosing $(e^{2\pi ig})_{Q_1} - (e^{2\pi ig})_{Q_2} - ((e^{2\pi ig})_{Q_3} - (e^{2\pi ig})_{Q_4})$, with radius $r > 0$ small enough so that the circle does not enclose or pass through the origin. For any $\alpha \geq 1$ (including $\alpha = \infty$), we define the Riesz projector

$$P_\alpha h = \frac{1}{2\pi i} \int_\Gamma (\lambda - e^{2\pi ig} \mathcal{L}_\alpha)^{-1} h d\lambda$$

We claim that P_∞ is a non-zero operator. We recall that $(e^{2\pi ig} \mathcal{L}_\infty)^{(1)}$ may be written as the composition of two operators, namely

$$(e^{2\pi ig} \mathcal{L}_\infty)^{(1)} = A_g B,$$

where $A_g : \mathbb{R}^2 \rightarrow X$, $B : X \rightarrow \mathbb{R}^2$ are defined above in (5.6.1), (5.6.2). In particular, note that the non-zero eigenvalues of $e^{2\pi ig} \mathcal{L}_\infty$ are actually equal to those of the 2×2 matrix BA_g . Indeed, if $BA_g v = \lambda v$ and $\lambda \neq 0$, $v \neq 0$ (which implies $A_g v \neq 0$), then $A_g BA_g v = \lambda A_g v$, and since $A_g v \neq 0$, it follows that λ is also an eigenvalue of $A_g B$. Similarly, if $A_g B f = \lambda f$ and $\lambda \neq 0$, $f \neq 0$ (which implies $B f \neq 0$) it holds that $BA_g B f = \lambda B f$ and so λ is an eigenvalue of BA_g . As in the proof of Lemma 5.6.4, we see that

$$BA_g = \begin{pmatrix} (e^{2\pi ig})_{Q_1} - (e^{2\pi ig})_{Q_4} & (e^{2\pi ig})_{Q_4} - (e^{2\pi ig})_{Q_1} \\ (e^{2\pi ig})_{Q_2} - (e^{2\pi ig})_{Q_3} & (e^{2\pi ig})_{Q_3} - (e^{2\pi ig})_{Q_2} \end{pmatrix} \in \mathbb{C}^{2 \times 2},$$

which has eigenvalues 0 and $(e^{2\pi ig})_{Q_1} + (e^{2\pi ig})_{Q_3} - (e^{2\pi ig})_{Q_2} - (e^{2\pi ig})_{Q_4}$, and so as soon as the latter quantity is non-zero, we deduce the existence of a non-zero eigenvalue for $A_g B$. Now, we develop the resolvent of $e^{2\pi ig} \mathcal{L}_\alpha$ in a Neumann series

$$\begin{aligned} (\lambda - e^{2\pi ig} \mathcal{L}_\alpha)^{-1} &= (\lambda - e^{2\pi ig} \mathcal{L}_\infty)^{-1} (\text{Id} + (e^{2\pi ig} \mathcal{L}_\infty - e^{2\pi ig} \mathcal{L}_\alpha)(\lambda - e^{2\pi ig} \mathcal{L}_\infty)^{-1})^{-1} \\ &= (\lambda - e^{2\pi ig} \mathcal{L}_\infty)^{-1} \sum_{n \geq 0} (-1)^n ((e^{2\pi ig} \mathcal{L}_\infty - e^{2\pi ig} \mathcal{L}_\alpha)(\lambda - e^{2\pi ig} \mathcal{L}_\infty)^{-1})^n. \end{aligned}$$

Indeed, by Lemma 5.6.3 combined with Lemma 5.6.4, it holds that

$$\|(e^{2\pi ig} \mathcal{L}_\infty - e^{2\pi ig} \mathcal{L}_\alpha)(\lambda - e^{2\pi ig} \mathcal{L}_\infty)^{-1}\|_{X \rightarrow X} \leq C \alpha^{-\tilde{\eta}} (1 + \|(\lambda - M)^{-1}\|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2}),$$

which for all α large enough is strictly less than 1, uniformly for $\lambda \in \Gamma$. As such, the resolvent of $e^{2\pi ig} \mathcal{L}_\alpha$ exists on Γ , and we deduce that $\|P_\alpha - P_\infty\|_{X \rightarrow X} \leq C |\Gamma| \alpha^{-\tilde{\eta}}$, where C depends only on the choice of Γ . We claim that this implies that the dimension of the range of P_α is bigger or equal than the one of P_∞ , i.e. $\dim(R(P_\alpha)) \geq \dim(R(P_\infty))$, for $\alpha \geq \alpha_0$. We consider the operator $P_\alpha : R(P_\infty) \rightarrow R(P_\alpha)$. Suppose $P_\alpha x = 0$. Then, it holds

$$\|x\| = \|P_\infty x - P_\alpha x\| \leq C \alpha^{-\tilde{\eta}} \|x\|,$$

Thus, taking α large enough, we deduce that $x = 0$, and hence that $\dim(R(P_\alpha)) \geq \dim(R(P_\infty)) = 1$. Exchanging the roles of P_α and P_∞ we obtain the opposite inequality, and thus deduce that for all α large enough, $e^{2\pi i g} \mathcal{L}_\alpha$ has a simply eigenvalue with the claimed property. $\square$

It thus remains to prove Lemma 5.6.3, which will be the focus of the following section.

5.6.2 Convergence as $\alpha \rightarrow \infty$.

In order to study the convergence of $\mathcal{L}_\alpha$ as $\alpha \rightarrow \infty$, we define an “intermediate” operator

$$\bar{\mathcal{L}}_\alpha h = 2 \begin{pmatrix} (\mathbb{1}_{T_\alpha(\mathcal{M}_1)} - \mathbb{1}_{T_\alpha(\mathcal{M}_3)}) \int_{x \geq 1/2} h^{(1)} + (\mathbb{1}_{T_\alpha(\mathcal{M}_4)} - \mathbb{1}_{T_\alpha(\mathcal{M}_2)}) \int_{x < 1/2} h^{(1)} \\ 0 \end{pmatrix},$$

where we recall

$$\begin{aligned} \mathcal{M}_1 &= \{x \geq 1/2, y + \alpha x \geq 1/2 \pmod{1}\}, & \mathcal{M}_2 &= \{x < 1/2, y - \alpha x \geq 1/2 \pmod{1}\}, \\ \mathcal{M}_3 &= \{x \geq 1/2, y + \alpha x < 1/2 \pmod{1}\}, & \mathcal{M}_4 &= \{x < 1/2, y - \alpha x < 1/2 \pmod{1}\}. \end{aligned}$$

By the triangle inequality and Lemma 5.4.5 we have

$$\|e^{2\pi i g} \mathcal{L}_\alpha h - e^{2\pi i g} \mathcal{L}_\infty h\| \leq C (\|\mathcal{L}_\alpha h - \bar{\mathcal{L}}_\alpha h\|_s + \|\bar{\mathcal{L}}_\alpha h - \mathcal{L}_\infty h\|_s + \|\mathcal{L}_\alpha h - \mathcal{L}_\infty h\|_u),$$

and so if we prove the convergence to zero of all the summands on the right-hand side, the result follows. By Proposition 5.5.1 and Lemma 5.4.4 we deduce

$$\|\mathcal{L}_\alpha h - \mathcal{L}_\infty h\|_u \leq \|\mathcal{L}_\alpha h\|_u + \|\mathcal{L}_\infty h\|_u \leq C\alpha^{\beta-\sigma}\|h\| + C\alpha^{-1+\beta}\|h\|.$$

To prove Lemma 5.6.3 it is thus sufficient to prove the following two lemmas and to choose $\bar{\eta} = \min\{1 - \sigma, \sigma - \beta, q, \beta\}$.

Lemma 5.6.5. *There exists $\alpha_0 \geq 1$ so that for all $\alpha \geq \alpha_0$, it holds*

$$\|\mathcal{L}_\alpha h - \bar{\mathcal{L}}_\alpha h\|_s \leq C(\alpha^{-q} + \alpha^{-1+\sigma} + \alpha^{-\beta})\|h\|.$$

Lemma 5.6.6. *There exists $\alpha_0 \geq 1$ so that for all $\alpha \geq \alpha_0$, it holds*

$$\|\bar{\mathcal{L}}_\alpha h - \mathcal{L}_\infty h\|_s \leq C(\alpha^{-1+\sigma} + \alpha^{-q})\|h\|.$$

We now commence with the proof of Lemma 5.6.5.

Proof of Lemma 5.6.5. We begin by noting that only the first component of $\mathcal{L}_\alpha h$ matters. Indeed, from the definition of $\mathcal{L}_\alpha$ in (5.2.4) and the matrix decomposition (5.3.1), all components of $\alpha^{-2}DT_\alpha$, other than the top-left entry, are of order at most α^{-1} . Therefore, we first claim that in

this convergence result, we may replace the map $\mathcal{L}_\alpha h$ by the principal part of the map

$$h \mapsto \mathcal{L}_\alpha^{(p)} h := \begin{pmatrix} \text{sign}([DT_\alpha]^{(1,1)} \circ T_\alpha^{-1}) h^{(1)} \circ T_\alpha^{-1} \\ 0 \end{pmatrix}.$$

Indeed, the difference between $\mathcal{L}_\alpha h$ and the above map $\mathcal{L}_\alpha^{(p)} h$ is equal to $(R_\alpha h) \circ T_\alpha^{-1}$, where R_α is a matrix that is constant on each $\mathcal{M}_\ell$ for $\ell = 1, 2, 3, 4$, and is bounded in absolute value by $|R_\alpha| \leq \frac{C}{\alpha}$. In particular, since our estimates in Proposition 5.2.6 contain a multiplicative term of a matrix with entries of order unity, it follows that

$$\|(R_\alpha h) \circ T_\alpha^{-1}\| \leq C\alpha^{-1} \|h \circ T_\alpha^{-1}\| \leq C\alpha^{-1} \|h\|.$$

Hence, in the following estimates, we can replace $\mathcal{L}_\alpha$ by $\mathcal{L}_\alpha^{(p)}$ and denote by $c_\ell = \text{sign}([DT_\alpha]^{(1,1)}(x)) \in \{-1, 1\}$, for $x \in \mathcal{M}_\ell$ for all $\ell = 1, 2, 3, 4$. As such, we abuse notation for the rest of the proof and identify $h = (h^{(1)}, h^{(2)})$ with its first component. Throughout the proof, we further denote by $\mathcal{H}_\ell$ the half-plane $\{x \geq 1/2\}$ or $\{x \leq 1/2\}$ so that $\mathcal{M}_\ell \subset \mathcal{H}_\ell$ for any ℓ . Pick some curve $W \in \Sigma$ and $\varphi \in C^q(W)$ with $\|\varphi\|_{\sigma, W} \leq 1$. If $|W| \leq \alpha^{-2}$, using property (1) in Lemma 5.4.9 we may write each $W = \bigcup_{j=1}^{M_\alpha} W_j$ with $M_\alpha \leq 4$, where each W_j is entirely contained in one of the smoothness strips $T_\alpha(\mathcal{M}_\ell)$ and also $T_\alpha^{-1}(W_j) = \bigcup_{k=1}^{N_\alpha} \tilde{W}_{j,k}$ with $N_\alpha \leq 4$, $\tilde{W}_{j,k} \in \Sigma$ and $|\tilde{W}_{j,k}| \leq 2\alpha^2 |W|$. Then,

$$\left| \int_{W_j} (\mathcal{L}_\alpha^{(p)} h - \bar{\mathcal{L}}_\alpha h) \varphi \right| \leq \left| \int_{W_j} h^{(1)} \circ T_\alpha^{-1} \varphi \right| + \left| \int_{W_j} \left(\int_{\mathcal{H}_\ell} h \right) \varphi \right| \quad (5.6.3)$$

and for the first term

$$\begin{aligned} \left| \int_{W_j} h^{(1)} \circ T_\alpha^{-1} \varphi \right| &\leq \frac{2}{\alpha^2} \sum_k \left| \int_{\tilde{W}_{j,k}} h^{(1)} \varphi \circ T_\alpha \right| \\ &\leq 2\alpha^{-2} \max_k \|\varphi \circ T_\alpha\|_{\sigma, \tilde{W}_{j,k}} \|h\|_s \\ &\leq C\alpha^{-2+2\sigma} \|\varphi\|_{\sigma, W} \|h\|_s, \end{aligned}$$

where in the last we have used Lemma 5.4.8 and $|\tilde{W}_{j,k}|^\sigma \leq 2^\sigma \alpha^{2\sigma} |W|^\sigma$ for any k . From Lemma 5.4.4 the estimate of the second terms follows

$$\left| \int_{W_j} \left(\int_{\mathcal{H}_\ell} h \right) \varphi \right| \leq C \|\varphi\|_{\sigma, W} |W|^{-\sigma} |W_j| |h|_w \leq C\alpha^{-2+2\sigma} |h|_w. \quad (5.6.4)$$

If $\alpha^{-2} \leq |W| \leq \alpha^{-1}$ we may write $W = \bigcup_{j=1}^{M_\alpha} W_j$ with $M_\alpha \leq 5$, where each W_j is entirely contained in one of the smoothness strips $T_\alpha(\mathcal{M}_\ell)$ and also $T_\alpha^{-1}(W_j) = \bigcup_{k=1}^{N_\alpha} \tilde{W}_{j,k}$ with $N_\alpha \leq 4\lceil \alpha^2 |W_j| \rceil \leq 4\lceil \alpha^2 |W| \rceil$ for any j . Hence, we use the bound (5.6.3) and for any $j \leq 5$ we estimate the first term

$$\begin{aligned} \left| \int_{W_j} h^{(1)} \circ T_\alpha^{-1} \varphi \right| &\leq \frac{C}{\alpha^2} \sum_{k=1}^{N_\alpha} \left| \int_{\tilde{W}_{j,k}} h^{(1)} \varphi \circ T_\alpha \right| \\ &\leq C|W| \max_k \|\varphi \circ T_\alpha\|_{\sigma, \tilde{W}_{j,k}} \|h\|_s \\ &\leq C\alpha^{-1+\sigma} \|\varphi\|_{\sigma, W} \|h\|_s. \end{aligned}$$

The estimate of the second term in (5.6.3) is identical to that in (5.6.4), obtaining in this case a bound of order $\alpha^{-1+\sigma}\|h\|_s$. Thus it remains to deal with the case when $|W| \geq \frac{1}{\alpha}$. We apply again Lemma 5.4.9 (1) and we write $W = \cup_{j=1}^{M_\alpha} W_j$ with $M_\alpha \leq 4\lceil \alpha|W| \rceil$. We aim to estimate

$$\sum_{j=1}^{M_\alpha} \left| \int_{W_j} (\mathcal{L}_\alpha h - \bar{\mathcal{L}}_\alpha h) \varphi \right|.$$

We begin by noting that we can further select among the curves W_j those that traverse the smoothness strips $T_\alpha(\mathcal{M}_\ell)$ for some $\ell = 1, 2, 3, 4$ regularly. More precisely, we denote the set of indices $j \in \mathcal{J}$ for the lines W_j parametrized by curve γ so that $\gamma([t_j, t_{j+1}]) = W_j$ so that $\gamma(t_j) \in \{x - \alpha y = \frac{k+1}{2}\}$ and $\gamma(t_{j+1}) \in \{x - \alpha y = \frac{k}{2}\}$ for some $k \in \mathbb{Z}$ or $\gamma(t_j) \in \{x + \alpha y = \frac{k}{2}\}$ and $\gamma(t_{j+1}) \in \{x + \alpha y = \frac{k+1}{2}\}$ for some $k \in \mathbb{Z}$. We note that the cardinality $\#\mathcal{J} \leq 5\alpha|W|$ and $\#\mathcal{J}^c \leq 6$. Hence, we write

$$W = \bigcup_{j \in \mathcal{J}} W_j \cup \bigcup_{j \notin \mathcal{J}} W_j.$$

Since all W_j corresponding to $j \in \mathcal{J}^c$ have length at most $\frac{1}{\alpha}$, we estimate by the previous step

$$\sum_{j \notin \mathcal{J}} \left| \int_{W_j} (\mathcal{L}_\alpha^{(p)} h - \bar{\mathcal{L}}_\alpha h) \varphi \right| \leq 2 \sum_{j \notin \mathcal{J}} \alpha^{-1+\sigma} \|\varphi\|_{\sigma, W} \|h\|_s \leq C \alpha^{-1+\sigma} \|h\|_s.$$

Hence, we reduce to estimating $\int_{W_j} (\mathcal{L}_\alpha h - \bar{\mathcal{L}}_\alpha h) \varphi$, for any $j \in \mathcal{J}$. Writing out the integral, we need to estimate

$$c_j \int_{W_j} \left(h \circ T_\alpha^{-1} - 2 \int_{\mathcal{H}_j} h \right) \varphi.$$

Without loss of generality we suppose for instance that $W_j \in T_\alpha(\mathcal{M}_1)$, the other three cases are similar. Hence $c_1 = 1$ and $\mathcal{H}_1 = \{x \geq 1/2\}$. From the definition of the map T_α it follows that $T_\alpha^{-1}(\gamma(t_j)) \in \{x = 1\}$ and $T_\alpha^{-1}(\gamma(t_{j+1})) \in \{x = 1/2\}$. In this case, if the tangent vector of W is $\mathbf{v} = (a, b)$ with $|a| \leq 2\alpha^{-1}b$, we have that

$$t_{j+1} - t_j = \frac{1}{2(\alpha b - a)}.$$

Furthermore, on $T_\alpha(\mathcal{M}_1)$, the inverse of T_α is the linear map (see (5.3.1))

$$DT_\alpha^{-1}|_{T_\alpha(\mathcal{M}_1)} = \begin{pmatrix} 1 & -\alpha \\ -\alpha & 1 + \alpha^2 \end{pmatrix}$$

then

$$T_\alpha^{-1}(\mathbf{v}) = (a - \alpha b, -\alpha a + (1 + \alpha^2)b).$$

Hence viewing the curve of $\mathbb{R}^2$ and denoting $(x(t), y(t)) = T_\alpha^{-1}(\gamma(t))$ we deduce that

$$|x(t_{j+1}) - x(t_j)| = \frac{1}{2}, \quad \left| y(t_{j+1}) - y(t_j) - \frac{\alpha}{2} \right| \leq 1.$$

We consider $x_0 = \max\{x \in [1/2, 1] : (x, 0) \in T_\alpha^{-1}(W_i)\}$ and define the partition

$$T_\alpha^{-1}(W_j) = \bigcup_{j=1}^{\frac{\alpha}{2}-2} \tilde{W}_{j,k} \cup \tilde{U}_{in} \cup \tilde{U}_{fin}$$

where $\tilde{U}_{in}$ is the curve of length $|U_{in}| \leq 1$ connecting $T_\alpha^{-1}(\gamma(0))$ to $(x_0, 0)$ on $\mathbb{T}^2$ and $\tilde{W}_{j,k}$ are subsequent curves of length 1, whereas $\tilde{U}_{fin}$ is the last curve of length $|\tilde{U}_{fin}| \leq 1$. We define the time S_j to be how long it takes $T_\alpha^{-1}(\gamma(\cdot))$ to move the y -component by 1, namely $S_j = \frac{|T_\alpha^{-1}\mathbf{v}|}{(1+\alpha^2)b-\alpha a}$. Now, set x_k to be the x component of $T_\alpha^{-1}(\gamma(t))$ after the y -component has increased by k , which is given by

$$x_k = x_0 + k S_j \frac{a - \alpha b}{|T_\alpha^{-1}\mathbf{v}|} = x_0 + k \frac{a - \alpha b}{(1 + \alpha^2)b - \alpha a} = x_0 - \frac{k}{\alpha} \left(1 + \frac{b}{(1 + \alpha^2)b - \alpha a} \right),$$

for all $k \in \{1, \dots, \frac{\alpha}{2} - 2\}$. Indeed,

$$x_{\alpha/2} = x_0 - \frac{1}{2} \left(1 + \frac{b}{(1 + \alpha^2)b - \alpha a} \right) < 1/2,$$

since $b > 0$, but

$$x_{\alpha/2-2} = x_0 - \left(\frac{1}{2} - \frac{2}{\alpha} \right) \left(1 + \frac{b}{(1 + \alpha^2)b - \alpha a} \right) > 1/2,$$

upon noting that $1 - x_0 \leq \frac{b}{(1+\alpha^2)b-\alpha a}$. Thus, there are at least $\alpha/2 - 2$ pieces, but no more than $\alpha/2 - 1$. Upon possibly discarding a single piece, we thus assume there are $\alpha/2 - 2$ pieces.

We define the vertical lines $\tilde{V}_{j,k} = \{(x, y) : x = x_k, y \in [0, 1]\}$. Using that the second component of the base point of the curve $\tilde{V}_{j,k+1}$ is 0 (which equals $k \bmod 1$), and the second component of the base point of $\tilde{W}_{j,k+1}$ is $k \frac{-\alpha a + (1+\alpha^2)b}{|T_\alpha^{-1}(\mathbf{v})|}$, we have that

$$\left| k - k \frac{-\alpha a + (1+\alpha^2)b}{|T_\alpha^{-1}(\mathbf{v})|} \right| \leq \frac{4}{\alpha^2} |k| \leq 2\alpha^{-1}.$$

Similarly, the difference in x -components is given by

$$\left| k \frac{a - \alpha b}{|T_\alpha^{-1}\mathbf{v}|} (1 - S_j) \right| \leq C\alpha^{-2}.$$

Hence, it is straightforward to check that

$$d_\Sigma(\tilde{W}_{j,k}, \tilde{V}_{j,k}) \leq 2\alpha^{-1}.$$

From these definitions, for the particular case of W_j considered, using the change of variables T_α^{-1}

we have

$$\begin{aligned} \left| \int_{W_j} \left(h \circ T_\alpha^{-1} - 2 \int_{x \geq 1/2} h \right) \varphi \right| &\leq \frac{2}{\alpha^2} \left| \sum_k \int_{\tilde{W}_{j,k}} \left(h - 2 \int_{x \geq 1/2} h \right) \varphi \circ T_\alpha \right| \\ &\quad + \frac{2}{\alpha^2} \left| \int_{\tilde{U}_{in}} \left(h - 2 \int_{x \geq 1/2} h \right) \varphi \circ T_\alpha \right| \\ &\quad + \frac{2}{\alpha^2} \left| \int_{\tilde{U}_{fin}} \left(h - 2 \int_{x \geq 1/2} h \right) \varphi \circ T_\alpha \right| = I + II + III \end{aligned}$$

and the last two terms are bounded by $C\alpha^{-2+\sigma}\|h\|_s$, so upon reintroducing the summation in $j \in \mathcal{J}$ these terms create an error of order $\mathcal{O}(\alpha^{-1+\sigma})$. Let us thus discuss the first term I . Fix $\bar{x} \in T_\alpha^{-1}(W_j)$ and sum and subtract $\varphi \circ T_\alpha(\bar{x})$ so that

$$\begin{aligned} I &\leq \left| \sum_k \frac{1}{\alpha^2} \int_{\tilde{W}_{j,k}} \left(h - 2 \int_{x \geq 1/2} h \right) \varphi \circ T_\alpha(\bar{x}) \right| \\ &\quad + \left| \sum_k \frac{1}{\alpha^2} \int_{\tilde{W}_{j,k}} \left(h - 2 \int_{x \geq 1/2} h \right) (\varphi \circ T_\alpha - \varphi \circ T_\alpha(\bar{x})) \right| = I_1 + I_2. \end{aligned}$$

We observe that the length of W_j satisfies $|W_j| \leq 2\alpha^{-1}$. Hence

$$\sup_k \sup_{x \in \tilde{W}_{j,k}} |\varphi \circ T_\alpha(x) - \varphi \circ T_\alpha(\bar{x})| \leq \sup_{x,y \in W_j} |\varphi(x) - \varphi(y)| \leq C\alpha^{-q} \|\varphi\|_{C^q}.$$

Using this and Lemma 5.4.8 to estimate $[\varphi \circ T_\alpha]_{C^q}$ we deduce that $\|\varphi \circ T_\alpha - \varphi \circ T_\alpha(\bar{x})\|_{C^q} \leq \frac{C}{\alpha^q} \|\varphi\|_{C^q}$. Thus, denoting by $\tilde{h} = h - 2 \int_{x \geq 1/2} h$, observing that $\|\tilde{h}\|_s \leq C\|h\|_s$ by Lemma 5.4.4 and using that the sum over k contains at most $\alpha/2$ terms, we deduce that the second term is bounded by

$$\begin{aligned} I_2 &\leq \alpha^{-1} \sup_{j,k} \left| \int_{\tilde{W}_{j,k}} \left(h - 2 \int_{x \geq 1/2} h \right) (\varphi \circ T_\alpha - \varphi \circ T_\alpha(\bar{x})) \right| \\ &\leq C\alpha^{-1} \|h\|_s \sup_{j,k} \|\varphi \circ T_\alpha - \varphi \circ T_\alpha(\bar{x})\|_{\sigma, \tilde{W}_{j,k}} \leq C\alpha^{-q-1} |W|^{-\sigma} \|\varphi\|_{\sigma, W}. \end{aligned}$$

Hence, reintroducing the sum over $j \in \mathcal{J}$ and noting that $\#J \leq 5\alpha|W|$, this term becomes an error of order $\mathcal{O}(\alpha^{-q})$. For the term I_1 we note that $|\tilde{W}_{j,k}| = 1$, and thus $\int_{\tilde{W}_{j,k}} \int_{x \geq 1/2} h = \int_{x \geq 1/2} h$. Furthermore, it holds that

$$\left| \int_{\tilde{W}_{j,k}} h - \int_{\tilde{V}_{j,k}} h \right| \leq C\alpha^{-\beta} \|h\|_u,$$

by definition of the strong unstable norm. As such, using that $k \leq N_\alpha \leq C\alpha$ we deduce

$$I_1 \leq \alpha^{-\beta-1} \|h\|_u |W|^{-\sigma} \|\varphi\|_{\sigma, W} + \frac{1}{\alpha^2} \left| \sum_k \left(\int_{\tilde{V}_{j,k}} h - 2 \int_{x \geq 1/2} h \right) \right| |W|^{-\sigma} \|\varphi\|_{\sigma, W},$$

and reintroducing the summation in j , the first of these terms is bounded by $\alpha^{-\beta} \|h\|_u$. Thus, if we

prove that

$$\frac{1}{\alpha^2} \left| \sum_k \left(\int_{\tilde{V}_{j,k}} h - 2 \int_{x \geq 1/2} h \right) \right| \leq C \alpha^{-1-\beta} \|h\|.$$

reintroducing the summation over $j \in \mathcal{J}$ and using that $\#\mathcal{J} \leq 4\alpha|W|$ we bound the second summand of I_1 and we conclude the proof. To do so, we recall that the sum runs over $k = 1, \dots, \frac{\alpha}{2} - 2$ and the base points of $\tilde{V}_{j,k}$ are $x_k = x_{k-1} - \frac{1}{\alpha} \left(1 + \frac{b}{(1+\alpha^2)b - \alpha a} \right)$. From now on, we will repeatedly use the fact that

$$\left| \frac{1}{\alpha} \left(1 + \frac{b}{(1+\alpha^2)b - \alpha a} \right) - \frac{1}{\alpha} \right| \leq C \alpha^{-3}.$$

Using the notation $f_A = \frac{1}{|A|} \int_A$ for any set A , we notice that

$$\begin{aligned} \sum_j 2 \int_{x \geq 1/2} h &= 2 \left(\frac{\alpha}{2} - 2 \right) \int_{x \geq 1/2} h \\ &= (\alpha - 4) \left(\sum_{k=1}^{\alpha/2-2} \int_{x_k}^{x_{k-1}} \int_0^1 h(x, y) dy dx + \int_{x_0}^1 \int_0^1 h(x, y) dy dx + \int_{1/2}^{x_{\frac{\alpha}{2}-2}} \int_0^1 h(x, y) dy dx \right) \\ &= \left(\sum_{k=1}^{\alpha/2-2} \int_{x_k}^{x_{k-1}} \int_0^1 h(x, y) dy dx \right) + \mathcal{O}_{\alpha \rightarrow \infty}(1) |h|_w. \end{aligned}$$

Hence, we deduce that

$$\begin{aligned} \frac{1}{\alpha^2} \left| \sum_{k=1}^{\alpha/2-2} \left(\int_{\tilde{V}_{j,k}} h - 2 \int_{x \geq 1/2} h \right) \right| &\leq \frac{1}{\alpha^2} \sum_{k=1}^{\alpha/2-2} \left| \int_{\tilde{V}_{j,k}} h - \int_{x_k}^{x_{k-1}} \int_0^1 h(x, y) dy dx \right| + \frac{C}{\alpha^2} |h|_w \\ &\leq \frac{1}{\alpha} \sum_{k=1}^{\alpha/2-2} \int_{x_k}^{x_{k-1}} \left| \int_0^1 h(x_k, y) dy - \int_0^1 h(x, y) dy \right| dx \leq C \alpha^{-1-\beta} \|h\|_u + C \alpha^{-2} |h|_w, \end{aligned}$$

where in the last inequality we used the strong unstable norm together with the fact that

$$d_\Sigma(\tilde{V}_{j,k}, \{(x, y) : y \in [0, 1]\}) \leq 2\alpha^{-1}, \quad \forall x \in [x_k, x_{k-1}],$$

concluding the proof. $\square$

Proof of Lemma 5.6.6. This lemma is a direct consequence of the following property: there exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$ and all vectors $\mathbf{w} \in \mathbb{C}^2$ with $|\mathbf{w}| = 1$ it holds true

$$\begin{aligned} \|\mathbf{w}(\mathbb{1}_{T_\alpha(\mathcal{M}_1)} - 1/2 \mathbb{1}_{y \geq 1/2})\|_s &\leq C(\alpha^{-1+\sigma} + \alpha^{-q}), \quad \|\mathbf{w}(\mathbb{1}_{T_\alpha(\mathcal{M}_2)} - 1/2 \mathbb{1}_{y \geq 1/2})\|_s \leq C(\alpha^{-1+\sigma} + \alpha^{-q}), \\ \|\mathbf{w}(\mathbb{1}_{T_\alpha(\mathcal{M}_3)} - 1/2 \mathbb{1}_{y < 1/2})\|_s &\leq C(\alpha^{-1+\sigma} + \alpha^{-q}), \quad \|\mathbf{w}(\mathbb{1}_{T_\alpha(\mathcal{M}_4)} - 1/2 \mathbb{1}_{y < 1/2})\|_s \leq C(\alpha^{-1+\sigma} + \alpha^{-q}). \end{aligned}$$

We shall provide the proof for $T_\alpha(\mathcal{M}_1)$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The general proof is entirely analogous.

Let $W \in \Sigma$. If $|W| \leq \frac{1}{\alpha}$, we split $W = \bigcup_{j=1}^M W_j$, with $M \leq 4$ and using $\|\varphi\|_{\sigma,W} \leq 1$ we obtain

$$\begin{aligned} \left| \int_W (\mathbb{1}_{T_\alpha(\mathcal{M}_1)} - 1/2 \mathbb{1}_{y \geq 1/2}) \varphi \right| &\leq \sum_{i=j}^M \int_{W_j} |(\mathbb{1}_{T_\alpha(\mathcal{M}_1)} - 1/2 \mathbb{1}_{y \geq 1/2}) \varphi| \\ &\leq 2M |W_j| |W|^{-\sigma} \leq 2M |W|^{1-\sigma} \leq 2M \alpha^{-1+\sigma}. \end{aligned}$$

Therefore, we may suppose that $|W| \geq 1/\alpha$, and furthermore we may assume that $W \subset \{y \geq 1/2\}$, since $T_\alpha(\mathcal{M}_1) \subset \{y \geq 1/2\}$. Then, W may be parameterised as $\{(x_1, y_1) + t\mathbf{v} : t \in [0, T]\}$ with $T \leq 1$ and $|\mathbf{v}| = 1$. By Lemma 5.4.9 we can split this curve into its connected components as $W = \bigcup_{j=0}^{M-1} W_j$, where $M \leq 4\alpha|W|$ and $W_j = \{(x_1, y_1) + t\mathbf{v} : t \in [t_j, t_{j+1}]\}$, with $t_0 = 0$ and $|t_{j+1} - t_j| = \frac{1}{2(a_1 + \alpha b_1)} \leq \alpha^{-1}$ for any j , where we have used $\mathbf{v} = (a_1, b_1)$, and $W_j \subset \overline{T_\alpha(\mathcal{M}_\ell)}$ for some ℓ . The integrals over W_0 and W_{M-1} are treated as an error since $|W_0|, |W_{M-1}| \leq \frac{2}{\alpha}$, namely for any φ with $\|\varphi\|_{\sigma,W} \leq 1$, we have

$$\left| \int_{W_0} (\mathbb{1}_{T_\alpha(\mathcal{M}_1)} - 1/2 \mathbb{1}_{y \geq 1/2}) \varphi \right| \leq C |W_0|^{1-\sigma} \|\varphi\|_{\sigma,W} \leq C \alpha^{-1+\sigma},$$

and similarly for W_{M-1} .

Without loss of generality we now suppose that M is even and

$$\mathbb{1}_{T_\alpha(\mathcal{M}_1)} \cap W_j = \emptyset \quad \text{for } j \text{ even}, \quad \mathbb{1}_{T_\alpha(\mathcal{M}_1)} \cap W_j = W_j \quad \text{for } j \text{ odd}.$$

Thus, from the previous estimate, for any $\varphi \in C^q(W)$ with $\|\varphi\|_{\sigma,W} \leq 1$ we have

$$\int_W (\mathbb{1}_{T_\alpha(\mathcal{M}_1)} - 1/2 \mathbb{1}_{y \geq 1/2}) \varphi = \frac{1}{2} \sum_{j=1}^{\frac{M}{2}-2} \int_{W_{2j+1}} \varphi - \int_{W_{2j}} \varphi + \mathcal{O}(\alpha^{\sigma-1}).$$

Using that $|W_{2j+1}| = |W_{2j}| = \frac{1}{2(a_1 + \alpha b_1)} \leq \alpha^{-1}$ for any j , we deduce that

$$\left| \int_{W_{2j+1}} \varphi - \int_{W_{2j}} \varphi \right| \leq C \alpha^{-q-1} \|\varphi\|_{\sigma,W} |W|^{-\sigma},$$

and so reintroducing the summation in j , and noting that $M \leq 4\alpha|W|$, we obtain

$$\left| \int_W (\mathbb{1}_{T_\alpha(\mathcal{M}_1)} - 1/2 \mathbb{1}_{y \geq 1/2}) \varphi \right| \leq C(\alpha^{\sigma-1} + \alpha^{-q}) \|\varphi\|_{\sigma,W},$$

which concludes the proof. $\square$

5.6.3 Relationship to the flux conjecture

In this subsection, we show that the velocity field u_α defined in (5.2.2), for $\alpha \in 2\mathbb{N}$ sufficiently large, is a *perfect dynamo*. This establishes that the magnetic growth in our model is not merely a result of fine-scale energy accumulation, but involves the growth of a macroscopic flux.

Proposition 5.6.7. *There exists $\alpha_0 \geq 1$ so that for every $\alpha \geq \alpha_0$ the following holds true. There*

exists a smooth vector field $\psi \in C^\infty(\mathbb{T}^2; \mathbb{R}^2)$ such that

$$\sup_{B_{\text{in}} \in L^2(\mathbb{T}^2)} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \left| \int_{\mathbb{T}^2} (\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha)^n B_{\text{in}} \cdot \psi \right| > 0.$$

Note that Corollary 5.2.10 follows directly from Proposition 5.6.7. Indeed, in the context of Definition 5.2.9, we may consider the smooth 3d test function $\Psi(x, y, z) = e^{-2\pi i z}(\psi(x, y), 0)$. The factor $e^{-2\pi i z}$ is chosen precisely to compensate for the phase factor in our magnetic field ansatz (5.2.3) and the push-forward formula (5.2.5), effectively reducing the 3d flux integral to the 2d pairing analyzed below.

Proof. Any element of the anisotropic space X naturally defines a distribution in $(C^q(\mathbb{T}^2))^*$. Here, we denote by $C^q(\mathbb{T}^2)$ the closure of Lipschitz continuous functions in the C^q norm. Indeed, for any $h \in C^1(\mathbb{T}^2)$ and $\psi \in C^q(\mathbb{T}^2)$, we have the estimate

$$\left| \int_{\mathbb{T}^2} h \cdot \psi \right| \leq C \|\psi\|_{C^q(\mathbb{T}^2)} \|h\|,$$

where $\|h\|$ is the strong norm (5.4.3) of our anisotropic space. It is straightforward to prove that the inclusion $(X, \|\cdot\|_X) \hookrightarrow (C^q(\mathbb{T}^2))^*$ is injective, see for instance [DL08]. Consequently, any non-zero element $h \in X$ acts as a non-zero functional on $C^q(\mathbb{T}^2)$. By Theorem 5.2.7, for sufficiently large α , the operator $\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha$ admits a unique, simple leading eigenvalue λ of maximum modulus, with $|\lambda| \geq \frac{\alpha^2}{4} > 1$. Let P_α be the Riesz projector onto the corresponding eigenspace. Since P_α is a non-zero projector and $C^1(\mathbb{T}^2)$ is dense in X , there must exist an initial datum $B_{\text{in}} \in C^1(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$ such that $P_\alpha B_{\text{in}} \neq 0$. By the injectivity of the inclusion into the dual space and the density of $C^\infty(\mathbb{T}^2)$ in $C^q(\mathbb{T}^2)$,⁵ we can select a test function $\psi \in C^\infty(\mathbb{T}^2)$ such that the pairing is non-vanishing; without loss of generality, we assume

$$\left| \int_{\mathbb{T}^2} P_\alpha B_{\text{in}} \cdot \psi \right| \geq 1.$$

We now employ the spectral decomposition

$$(\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha)^n B_{\text{in}} = \lambda^n P_\alpha B_{\text{in}} + (\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha)^n (\text{Id} - P_\alpha) B_{\text{in}}.$$

Since the spectral radius of $\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha (\text{Id} - P_\alpha)$ is bounded by $\alpha^{2-\eta}$ for some $\eta > 0$, see Proposition 5.2.6 and its proof, the spectral radius formula implies that

$$\|(\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha)^n (\text{Id} - P_\alpha) B_{\text{in}}\| \leq C_\alpha \alpha^{(2-\eta)n} \|(\text{Id} - P_\alpha) B_{\text{in}}\|$$

⁵this is only true because we defined $C^q(\mathbb{T}^2)$ as the completion of Lipschitz functions in the C^q norm.

for all n . It follows that for sufficiently large n , the dominant eigenmode dictates the growth:

$$\begin{aligned} \left| \int_{\mathbb{T}^2} (\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha)^n B_{\text{in}} \cdot \psi \right| &\geq |\lambda|^n \left| \int_{\mathbb{T}^2} P_\alpha B_{\text{in}} \cdot \psi \right| - \left| \int_{\mathbb{T}^2} (\alpha^2 e^{2\pi i g} \mathcal{L}_\alpha)^n (\text{Id} - P_\alpha) B_{\text{in}} \cdot \psi \right| \\ &\geq |\lambda|^n - C_\alpha \alpha^{(2-\eta)n} \geq \frac{1}{2} |\lambda|^n. \end{aligned}$$

Since $|\lambda| \geq \frac{\alpha^2}{4} > 1$, the growth rate is strictly positive, completing the proof. $\square$

5.7 Heat semigroup bounds

In this section we prove Proposition 5.2.8. More precisely, we denote by $e^{\varepsilon \Delta}$ the time-1 solution operator of the heat equation with diffusion $\varepsilon > 0$ and we aim to prove that for all $f \in X$ it holds true

$$\|e^{\varepsilon \Delta} f\| \leq \|f\|, \quad |e^{\varepsilon \Delta} f|_w \leq |f|_w, \quad |e^{\varepsilon \Delta} f - f|_w \leq 2\varepsilon^{\frac{\beta}{4}} \|f\|, \quad (5.7.1)$$

where we recall the norms $\|\cdot\|$ and $|\cdot|_w$ defined in Section 5.4. We first prove the weak norm

bound. The key observation we will use is that the heat kernel is nothing but $e^{\varepsilon \Delta} f(x) = \mathbb{E} f(x - \sqrt{\varepsilon} B)$, where B is a standard Gaussian and $\mathbb{E}$ denotes the expectation. As such, we consider $\varphi \in C^1(W)$ with $\|\varphi\|_{C^1(W)} \leq 1$ and we have that

$$\int_W e^{\varepsilon \Delta} f \varphi = \mathbb{E} \int_W f(x - \sqrt{\varepsilon} B) \varphi(x) = \mathbb{E} \int_{W - \sqrt{\varepsilon} B} f \varphi(x + \sqrt{\varepsilon} B).$$

Since $W \in \Sigma$ is an admissible curve it follows that $W - \sqrt{\varepsilon} B \in \Sigma$ is still an admissible curve and $\|\varphi(\cdot + \sqrt{\varepsilon} B)\|_{C^1} = \|\varphi(\cdot)\|_{C^1} \leq 1$. Thus, it is straightforward to prove that

$$\sup_{W \in \Sigma} \sup_{\|\varphi\|_{C^1(W)} \leq 1} \left| \int_W e^{\varepsilon \Delta} f \varphi \right| \leq |f|_w.$$

The strong stable norm $\|\cdot\|_s$ is entirely analogous, namely $\|e^{\varepsilon \Delta} f\|_s \leq \|f\|_s$. For the strong unstable norm, let W_1, W_2 such that $d_\Sigma(W_1, W_2) \leq \delta$, and let $\varphi_1 \in C^1(W_1)$, $\varphi_2 \in C^1(W_2)$ such that have $d_q(\varphi_1, \varphi_2) \leq \delta$ and $\|\varphi_1\|_{C^1(W_1)}, \|\varphi_2\|_{C^1(W_2)} \leq 1$. Then, we have

$$\int_{W_1} e^{\varepsilon \Delta} f \varphi_1 - \int_{W_2} e^{\varepsilon \Delta} f \varphi_2 = \mathbb{E} \left(\int_{W_1 - \sqrt{\varepsilon} B} f(\cdot) \varphi_1(\cdot + \sqrt{\varepsilon} B) - \int_{W_2 - \sqrt{\varepsilon} B} f(\cdot) \varphi_2(\cdot + \sqrt{\varepsilon} B) \right).$$

Since we are only doing rigid translations, we have that $d_\Sigma(W_1 - \sqrt{\varepsilon} B, W_2 - \sqrt{\varepsilon} B) = d_\Sigma(W_1, W_2) \leq \delta$, $d_q(\varphi_1(\cdot + \sqrt{\varepsilon} B), \varphi_2(\cdot + \sqrt{\varepsilon} B)) = d_q(\varphi_1, \varphi_2) \leq \delta$ and $\|\varphi_1(\cdot + \sqrt{\varepsilon} B)\|_{C^1} = \|\varphi_1(\cdot)\|_{C^1} \leq 1$, $\|\varphi_2(\cdot + \sqrt{\varepsilon} B)\|_{C^1} = \|\varphi_2(\cdot)\|_{C^1} \leq 1$. Thus, we bound

$$\sup_{0 < \delta < 1/\alpha} \sup_{d_\Sigma(W^1, W^2) < \delta} \sup_{\substack{|\varphi_1|_{C^1} \leq 1, |\varphi_2|_{C^1} \leq 1 \\ d_q(\varphi_1, \varphi_2) < \delta}} \frac{1}{\delta^\beta} \left| \int_{W_1} e^{\varepsilon \Delta} f \varphi_1 - \int_{W_2} e^{\varepsilon \Delta} f \varphi_2 \right| \leq \delta^\beta.$$

To prove the convergence result, we consider

$$|e^{\varepsilon\Delta}f - f|_w = \sup_{W \in \Sigma} \sup_{|\varphi|_{C^1(W)} \leq 1} \left| \mathbb{E} \int_{W - \sqrt{\varepsilon}B} f\varphi(\cdot + \sqrt{\varepsilon}B) - \int_W f\varphi \right|.$$

We fix $W \in \Sigma$ and $\varphi \in C^1(W)$ with $|\varphi|_{C^1(W)} \leq 1$ and we bound this by

$$\mathbb{E} \mathbb{1}_{\sqrt{\varepsilon}|B| \leq \varepsilon^{1/4}} \left| \int_{W - \sqrt{\varepsilon}B} f\varphi(\cdot + \sqrt{\varepsilon}B) - \int_W f\varphi \right| + 2|f|_w \mathbb{P}(\sqrt{\varepsilon}|B| \geq \varepsilon^{1/4}).$$

We observe that $d_q(\varphi(\cdot + \sqrt{\varepsilon}B), \varphi) = 0$ by definition and on the set $\{\sqrt{\varepsilon}|B| \leq \varepsilon^{1/4}\}$, it holds that $d_\Sigma(W - \sqrt{\varepsilon}B, W) \leq \varepsilon^{1/4} < \frac{1}{\alpha}$, where the last holds true if ε is sufficiently small. Then, the first term is bounded by

$$\mathbb{E} \mathbb{1}_{\sqrt{\varepsilon}|B| \leq \varepsilon^{1/4}} \left| \int_{W - \sqrt{\varepsilon}B} f\varphi(\cdot + \sqrt{\varepsilon}B) - \int_W f\varphi \right| \leq \varepsilon^{\beta/4} \|f\|,$$

and using $\mathbb{P}(\sqrt{\varepsilon}|B| \geq \varepsilon^{1/4}) \leq \varepsilon^{\beta/4}$, we conclude the proof of (5.7.1).

Remark 5.7.1. We remark that the pathwise behaviour of the pulsed model is quite distinct to that of the model involving continuous-time diffusion. Indeed, one of the main obstacles to verifying the assumptions of Proposition A.3.7 for the continuous-diffusion model (KDE- ε) is that, with high probability, the pathwise realizations of the stochastic flow map

$$dX_t^\varepsilon = u(t, X_t^\varepsilon) dt + \sqrt{2\varepsilon} dW_t$$

violate our uniform hyperbolicity assumption.

Appendix to Chapter 5

Here we outline the proof of the main technical Lemma 5.4.9 of the work. We recall that the set of admissible curves is denoted by Σ (5.4.1), the stable cone is denoted by C_s (5.3.2), the map T_α and the sets $\mathcal{M}_\ell$ with $\ell = 1, 2, 3, 4$ are defined in Section 5.3.

Proof. The proof of (1) follows from the definition of the $T_\alpha(\mathcal{M}_\ell)$ for $\ell = 1, 2, 3, 4$. Indeed, we have that $\partial T_\alpha(\mathcal{M}_\ell)$ are curves $y = \pm \frac{1}{\alpha}x$ periodized on the torus, which are almost horizontal lines, see Figure 5.2. Also by definition of admissible curves $W \in \Sigma$ we have that such affine lines are almost vertical with tangent vector $\mathbf{v} = (a, b)$ with $|a| \leq \frac{2}{\alpha}b$. Thanks to Lemma 5.3.1 we deduce that the tangent vector $\frac{T_\alpha^{-1}(\mathbf{v})}{|\mathbf{v}|}$ is still in the stable cone C_s if $\alpha \geq \alpha_0$ is large enough and hence each subcurve of $T_\alpha^{-1}(W_j)$ of length less than 1 is an admissible curve. Since $|DT_\alpha^{-1}| \leq 2\alpha^2$ there are at most $N_\alpha \leq 3 + \lceil \alpha^2 |W_j| \rceil$ curves $(\tilde{W}_{j,k})_{k=1}^{N_\alpha}$ such that $\tilde{W}_{j,k} \in \Sigma$ and $T_\alpha^{-1}(W_j) = \cup_{k=1}^{N_\alpha} \tilde{W}_{j,k}$. Hence the properties follow up to choosing $\alpha \geq \alpha_0$ sufficiently large.

We now prove (2). We begin by picking a fundamental domain in x so that both curves are entirely contained in this domain. This is always possible due to the length restrictions on the curves, as well as the fact that they are “almost vertical”. Without loss of generality, we will identify this

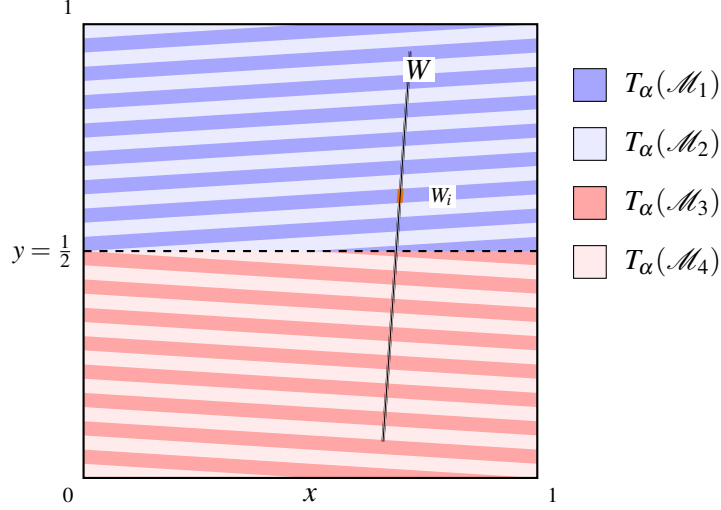


Figure 5.2: The sets $T_\alpha(\mathcal{M}_\ell)$ in $[0,1]^2$ with $\alpha = 16$, with an admissible curve $W \in \Sigma$ and one highlighted subcurve $W_i \subset W \cap T_\alpha(\mathcal{M}_1)$.

fundamental domain with $[0, 1]$. Next, we parameterise W_1, W_2 as two curves $\gamma_1 : [0, S_1] \rightarrow \mathbb{T}^2$ and $\gamma_2 : [0, S_2] \rightarrow \mathbb{T}^2$ with $S_1, S_2 \leq 1$ denoted by

$$\gamma_1(t) = (x_1, y_1) + t(a_1, b_1), \quad \gamma_2(t) = (x_2, y_2) + t(a_2, b_2). \quad (5.7.2)$$

We first observe that, once we choose the unmatched subcurves U so that their total length is bounded by $C d_\Sigma(W_1, W_2)$, the base points of the remaining pieces $W_1 \setminus U$ and W_2 (and vice versa) stay close. More precisely, the distance between the corresponding base points is controlled by the triangle inequality by the distance between the original base points plus the length of the unmatched parts, which is in turn bounded by $(C+1)d_\Sigma(W_1, W_2)$. Therefore, it suffices to estimate the length of the unmatched pieces.

Step 1: Initial pre-processing–Unmatched curves for base points and end points. As a first step, we shall “cut” pieces from the beginnings and ends of W_1, W_2 in order to obtain two curves that are more amenable for matching. In particular, we shall endeavour to ensure that both curves begin in the same half plane ($\{y \geq 1/2\}$ or $\{y < 1/2\}$) and terminate in the same half plane ($\{y \geq 1/2\}$ or $\{y < 1/2\}$), and furthermore begin and terminate on the same strip $\{x \pm \alpha y = q\}$.

More formally, we claim that for any $W_1, W_2 \in \Sigma$ we can define unmatched curves $U_{1,1}, U_{1,2}, U_{2,1}, U_{2,2}$,

$$W_1 \setminus (U_{1,1} \cup U_{1,2}), W_2 \setminus (U_{2,1} \cup U_{2,2}) \in \Sigma$$

and the base points of $W_1 \setminus (U_{1,1} \cup U_{1,2}), W_2 \setminus (U_{2,1} \cup U_{2,2})$ that we call $(\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)$ satisfy

$$\bar{y}_1, \bar{y}_2 \geq 1/2 \quad \text{or} \quad \bar{y}_1, \bar{y}_2 < 1/2$$

and if $\bar{y}_1 \geq 1/2$ then there exists $q \in \mathbb{R}$ such that

$$(\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2) \in \{x - \alpha y = q\}$$

and if $\bar{y}_1 < 1/2$ then there exists $q \in \mathbb{R}$ such that

$$(\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2) \in \{x + \alpha y = q\}.$$

Furthermore, the same holds for the endpoints of $W_i \setminus \{U_{i,1} \cup U_{i,2}\}$. We begin by outlining the argument for the base points.

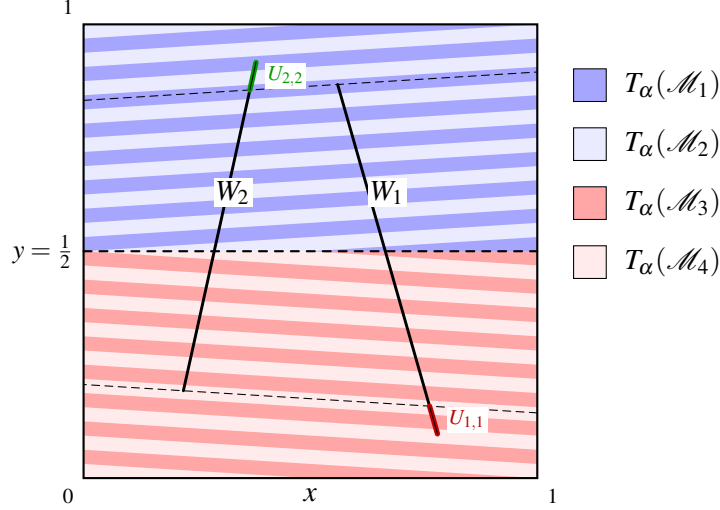


Figure 5.3: Pre-processing of two admissible curves W_1, W_2 : a small initial curve $U_{1,1}$ is cut from W_1 and a small final curve $U_{2,2}$ is cut from W_2 . The base points of $W_1 \setminus U_{1,1}$ and $W_2 \setminus U_{2,2}$ lie in the same half-plane and on the same $\{x + \alpha y = q_{in}\}$ for some $q_{in} \in \mathbb{R}$ (indicated by the dashed strip lines); likewise for the endpoints of $W_1 \setminus U_{1,1}$ and $W_2 \setminus U_{2,2}$ lie in the same half-plane and on the same $\{x - \alpha y = q_{fin}\}$ for some $q_{fin} \in \mathbb{R}$.

Base points. Recall the parametrizations γ_i for $i = 1, 2$ (5.7.2). There are now two options: either $d_{\mathbb{T}}(y_1, 1/2) \leq d_{\mathbb{T}}(y_1, 1)$, or $d_{\mathbb{T}}(y_1, 1) \leq d_{\mathbb{T}}(y_1, 1/2)$. Assume that $d_{\mathbb{T}}(y_1, 1/2) \leq d_{\mathbb{T}}(y_1, 1)$. The other case is handled in exactly the same way by symmetry. In particular, the conditions of $d_{\mathbb{T}}(y_1, 1/2) \leq d_{\mathbb{T}}(y_1, 1)$, together with $d_{\Sigma}(W_1, W_2) \leq 2/\alpha$ imply that the euclidean distance between the base points viewed as elements of $[0, 1]^2$ is the same as the toroidal distance. We now have two cases for the base points of the curves γ_1, γ_2 , which are viewed as elements of $[0, 1]^2$:

- 1) $y_1, y_2 \geq 1/2$ or $y_1, y_2 < 1/2$.
- 2) $(y_1 < 1/2 \text{ and } y_2 \geq 1/2) \text{ or } (y_1 \geq 1/2 \text{ and } y_2 < 1/2)$.

Assume that we are in the second case, for instance $y_1 < 1/2, y_2 \geq 1/2$. Then we define the unmatched curve

$$U_{1,1} = \{\gamma_1(t) : t \in [0, t_1 \wedge S_1]\}$$

where t_1 is chosen so that the second component of $\gamma_1(t_1)$ is equal to $1/2$ and we get $t_1 = \frac{1/2 - y_1}{b_1} \leq \frac{y_2 - y_1}{b_1} \leq 2d_{\Sigma}(W_1, W_2)$.

Therefore, we deal with case 1), assuming for now that $y_1, y_2 < 1/2$. The other case is identical. We can assume without loss of generality that $x_1 + \alpha y_1 \leq x_2 + \alpha y_2$. We define $q_2 = x_2 + \alpha y_2 \in \mathbb{R}$

and we define the unmatched curve

$$U_{1,1} = \{\gamma_1(t) : t \in [0, t_1 \wedge S_1]\}$$

where t_1 is now defined so that $\gamma_1(t_1) \in \{x + \alpha y = q_2, x, y \in [0, 1]\}$. Hence

$$0 \leq t_1 = \frac{x_2 + \alpha y_2 - (x_1 + \alpha y_1)}{a_1 + \alpha b_1} \leq 2d_\Sigma(W_1, W_2).$$

If $y_1 + t_1 b_1 < 1/2$, we are done. If $y_1 + t_1 b_1 > 1/2$, we define another unmatched piece

$$U_{2,2} = \{\gamma_2(t) : t \leq t_2 \wedge S_2\},$$

where $t_2 = \frac{1/2 - y_2}{b_2}$. Now,

$$t_2 \leq \frac{y_1 + t_1 b_1 - y_2}{b_2} \leq Cd_\Sigma(W_1, W_2).$$

Hence, no matter what case we are in, we may either conclude our cutting of the initial points, or reduce to the case where $y_1, y_2 \geq 1/2$, and y_1, y_2 are close to $1/2$. Hence, we assume that both curves start with $y_1, y_2 > 1/2$, with y_1, y_2 a distance less than $\frac{C}{\alpha}$ to $y = 1/2$. Without loss of generality, we assume that $x_1 - \alpha y_1 \geq x_2 - \alpha y_2 =: \tilde{q}_2$, and define the unmatched piece

$$U_{1,1} = \{\gamma_1(t) : t \in [0, t_1 \wedge S_1]\},$$

where $t_1 = \frac{\tilde{q}_2 - (x_1 - \alpha y_1)}{-b_1 \alpha + a_1}$. As before, we estimate $t_1 \leq Cd_\Sigma(W_1, W_2)$, and since both y_1, y_2 are close to $1/2$ it still holds that $y_1 + t_1 b_1 < 1$, so the cutting of base points is complete.

Finally we remark that, although in the previous computations we had always implicitly assumed that $t_1 \wedge S_1 = t_1$ and $t_2 \wedge S_2 = t_2$, in the case where for instance $S_1 \leq t_1$, we may actually view both W_1, W_2 entirely as unmatched. Indeed, we saw that $t_1 \leq Cd_\Sigma(W_1, W_2)$. Thus, $S_1 \leq Cd_\Sigma(W_1, W_2)$. Furthermore, $S_2 \leq |S_1 - S_2| + S_2 \leq (C+1)d_\Sigma(W_1, W_2)$, and thus they are indeed unmatched.

End points.

We now undertake precisely the same cutting procedure that we did for the base points in reverse: we consider the curves $\tilde{\gamma}_i(t) = \{(x_i + S_i a_i, y_i + S_i b_i) + t(-a_i, -b_i) : t \leq S_i - t_i\}$, where the t_i are the new starting points of $W_i \setminus U_{i,1}$. Note that the distance between these “flipped” curves is bounded by the distance between the non-flipped ones. Indeed, the only thing to check is the distance between the base points, which is bounded by

$$|x_1 - x_2| + |y_1 - y_2| + |S_1 a_1 - S_2 a_2| + |S_1 b_1 - S_2 b_2| \leq Cd_\Sigma(W_1, W_2).$$

As such, we now obtain curves $U_{1,2}, U_{2,2}$ with $|U_{i,2}| \leq Cd_\Sigma(W_1, W_2)$ so that the end points $\tilde{x}_i, \tilde{y}_i$ of $W_i \setminus U_{i,2} \cup U_{i,1}$ satisfy either

$$\tilde{y}_1, \tilde{y}_2 \geq 1/2 \quad \text{or} \quad \tilde{y}_1, \tilde{y}_2 < 1/2$$

and if $\tilde{y}_1 \geq 1/2$ then there exists $q \in \mathbb{R}$ such that

$$(\tilde{x}_1, \tilde{y}_1), (\tilde{x}_2, \tilde{y}_2) \in \{x - \alpha y = q\}$$

and if $\tilde{y}_1 < 1/2$ then there exists $q \in \mathbb{R}$ such that

$$(\tilde{x}_1, \tilde{y}_1), (\tilde{x}_2, \tilde{y}_2) \in \{x + \alpha y = q\}.$$

Step 2: Matched curves. We firstly define the matched curves. Without loss of generality and up to discarding some unmatched curves defined above, we suppose that the base points $(x_1, y_1), (x_2, y_2)$ are such that $y_1, y_2 \in [0, 1/2)$ and $(x_1, y_1), (x_2, y_2) \in \{x + \alpha y = q_{in}\}$ for some $q_{in} \in [\frac{j_{in}}{2}, \frac{j_{in}+1}{2})$, where $j_{in} \in \mathbb{Z}$. Without loss of generality we can suppose that $\gamma_1(S_1), \gamma_2(S_2)$ are such that the second component is in $[1/2, 1)$ and $\gamma_1(S_1), \gamma_2(S_2) \in \{x - \alpha y = q_{fin}\}$ for some $q_{fin} \in (\frac{j_{fin}}{2}, \frac{j_{fin}+1}{2}]$, where $j_{fin} \in \mathbb{Z}$. We define accordingly $t_{1,j_{in}} = t_{2,j_{in}} = 0$ and $t_{1,j_{fin}} = S_1$ and $t_{2,j_{fin}} = S_2$. For $j_{in} \leq j$ we also define $t_{1,j} \geq t_{1,j_{in}} = 0$ as the minimal time so that $\gamma_1(t_{1,j}) \in \{x + \alpha y = \frac{j}{2}\}$ and similarly $t_{2,j} > 0$ as the minimal time so that $\gamma_2(t_{2,j}) \in \{x + \alpha y = \frac{j}{2}\}$. Hence

$$t_{1,j} = \frac{\frac{j}{2} - (x_1 + \alpha y_1)}{a_1 + \alpha b_1}, \quad t_{2,j} = \frac{\frac{j}{2} - (x_2 + \alpha y_2)}{a_2 + \alpha b_2}$$

$$|t_{1,j} - t_{2,j}| = \left| \frac{(\frac{j}{2} - (x_1 + \alpha y_1))(a_2 + \alpha b_2) - (\frac{j}{2} - (x_2 + \alpha y_2))(a_1 + \alpha b_1)}{(a_2 + \alpha b_2)(a_1 + \alpha b_1)} \right| \leq 2d_\Sigma(W_1, W_2),$$

where in the last we have used that $x_2 + \alpha y_2 = x_1 + \alpha y_1$ and $|j| \leq C\alpha$. Hence, for all $j \in \mathbb{Z}$ so that $\gamma_1(t_{1,j})$ and $\gamma_2(t_{2,j})$ has the second component less than $1/2$, we define the matched curves as

$$W_{1,j} = \{\gamma_1(t) : t \in [t_{1,j+1}, t_{1,j}]\}, \quad W_{2,j} = \{\gamma_2(t) : t \in [t_{2,j+1}, t_{2,j}]\}. \quad (5.7.3)$$

Defining $(x_{1,j}, y_{1,j})$ the intersection of W_1 with $\{x + \alpha y = \frac{j}{2}\}$ and defining similarly $(x_{2,j}, y_{2,j})$ we deduce that

$$|y_{2,j} - y_{1,j}| = |y_2 + t_{2,j}b_2 - (y_1 + t_{1,j}b_1)| \leq |y_2 - y_1| + |t_{1,j}b_1 - t_{2,j}b_2| \leq 3d_\Sigma(W_1, W_2).$$

The same argument holds true for the x -variable. Hence, these matched curves satisfy all the properties. The above argument may be continued until the second component of either curve reaches $1/2$. We outline here the case of crossing $y = 1/2$. We denote by $\bar{j} \in \mathbb{Z}$ the largest number so that the second component of $\gamma_1(t_{1,\bar{j}})$ and $\gamma_2(t_{2,\bar{j}})$ are less than $1/2$. Hence, at least one of the two curves cross the set $\{y = 1/2\}$ before crossing $\{x + \alpha y = \frac{\bar{j}+1}{2}\}$. Since both curves terminate in $\{y \geq 1/2\}$ by Step 1, there exists $\bar{t}_2 \in (t_{2,\bar{j}}, t_{2,\bar{j}+1}]$ such that the second component of $\gamma_2(\bar{t}_2)$ is $1/2$ and $\bar{t}_1 \in (t_{1,\bar{j}}, t_{1,\bar{j}+1}]$ such that the second component of $\gamma_1(\bar{t}_1)$ is $1/2$. Suppose that $x_1 + \bar{t}_1 a_1 + \alpha/2 \geq x_2 + \bar{t}_2 a_2 + \alpha/2$, see Figure 5.4. We now define $q_{2,up}, q_{2,down} \in \mathbb{R}$ such that

$$\gamma_2(\bar{t}_2) \in \{x + \alpha y = q_{2,down}\} \cap \{x - \alpha y = q_{2,up}\}.$$

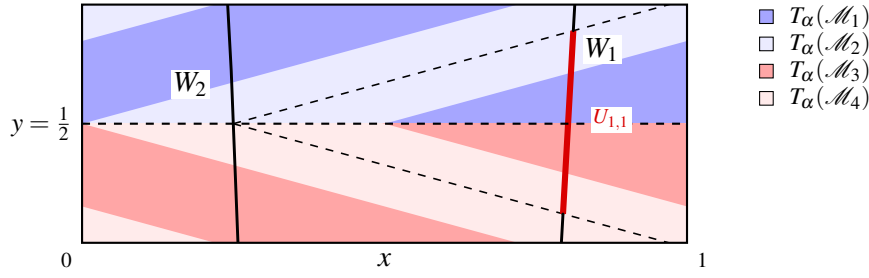


Figure 5.4: We represent W_1, W_2 as two black lines. We zoom near $y = \frac{1}{2}$. The two dashed lines are $\{x + \alpha y = q_{2,down}\}$ and $\{x - \alpha y = q_{2,up}\}$ passing through $W_2 \cap \{y = 1/2\}$. The red segment is the unmatched curve $U_{1,1} \subset W_1$.

Hence we define the unmatched curve

$$U_{1,1} = \{\gamma_1(t) : t \in [t_{1,down}, t_{1,up}]\},$$

so that

$$\gamma_1(t_{1,down}) \in \{x + \alpha y = q_{2,down}\}, \quad \gamma_1(t_{1,up}) \in \{x - \alpha y = q_{2,up}\}.$$

By the assumption that $x_1 + \bar{t}_1 a_1 + \alpha/2 \geq x_2 + \bar{t}_2 a_2 + \alpha/2$, it follows that $y_1 + t_{1,up} b_1 \leq 1/2$, and so the curves $\{\gamma_2(t) : t \in [t_{2,\bar{j}}, \bar{t}_2]\}$, and $\{\gamma_1(t) : t \in [t_{1,\bar{j}}, t_{1,down}]\}$ are indeed matched, with distance bounded as before by $cd_\Sigma(W_1, W_2)$. Next, we have

$$\begin{aligned} x_1 + t_{1,down} a_1 + \alpha(y_1 + t_{1,down} b_1) &= q_{2,down} = x_2 + \bar{t}_2 a_2 + \alpha(y_2 + \bar{t}_2 b_2), \\ x_1 + t_{1,up} a_1 - \alpha(y_1 + t_{1,up} b_1) &= q_{2,up} = x_2 + \bar{t}_2 a_2 - \alpha(y_2 + \bar{t}_2 b_2), \end{aligned}$$

and summing the two we deduce

$$\begin{aligned} |t_{1,up} - t_{1,down}| &= \left| \frac{2(x_2 - x_1) + (\bar{t}_2 a_2 - t_{1,down} a_1) + (\bar{t}_2 a_2 - t_{1,up} a_1)}{\alpha b_1} \right| \\ &\leq C \alpha^{-1} (d_\Sigma(W_1, W_2) + |\bar{t}_2 a_2 - t_{1,up} a_1|) \end{aligned}$$

where in the last we used $|\bar{t}_2 a_2 - t_{1,down} a_1| = |\bar{t}_2(a_2 - a_1) + a_1(\bar{t}_2 - t_{1,down})| \leq C d_\Sigma(W_1, W_2)$ since we have already an upper bound on $|\bar{t}_2 - t_{1,down}|$ by similar computations given above. Hence, using also that $|\bar{t}_2 a_2 - t_{1,up} a_1| \leq |(\bar{t}_2 - t_{1,down}) a_2| + |(a_2 - a_1) t_{1,down}| + |(t_{1,down} - t_{1,up}) a_1| \leq C d_\Sigma(W_1, W_2) + C \alpha^{-1} |(t_{1,down} - t_{1,up})|$ we deduce that up to relabelling the constant $C > 0$

$$|t_{1,up} - t_{1,down}| \leq C \alpha^{-1} d_\Sigma(W_1, W_2) + C \alpha^{-2} |(t_{1,down} - t_{1,up})|$$

and since $\alpha \geq \alpha_0$ is sufficiently large we deduce (up to again relabelling the constant) $|t_{1,up} - t_{1,down}| \leq C \alpha^{-1} d_\Sigma(W_1, W_2)$.

From the new base points given by $\gamma_1(t_{1,up})$ and $\gamma_2(\bar{t}_2)$ we create matched pieces iteratively similarly as in (5.7.3) with iterative times so that the curves cross $\{x - \alpha y = \frac{j}{2}\}$ for $j \in \mathbb{Z}$ for $y \in [1/2, 1)$. We then may define also unmatched curves close to $y = 1$ similarly to the case $y = 1/2$ discussed above.

We now prove (3). We suppose that we have two matched curves $W_{1,1}, W_{2,1}$ parametrized by γ_1, γ_2 , more precisely

$$\gamma_1(t) = (x_1, y_1) + t(a_1, b_1), \quad \gamma_2(t) = (x_2, y_2) + t(a_2, b_2).$$

By (2) we have that $\gamma_1([t_{1,in}, t_{1,fin}]), \gamma_2([t_{2,in}, t_{2,fin}]) \subset T_\alpha(\mathcal{M}_\ell)$ for some $\ell = 1, 2, 3, 4$. It is straightforward to see that $|t_{i,in} - t_{i,fin}| \leq \frac{2}{\alpha}$ for $i = 1, 2$. To simplify the notation and by the matching of the curves given in (2), we can always translate the parametrization and suppose that $\gamma_1(0), \gamma_2(0) \in \{x + \alpha y = q\}$ and $\gamma_1(t_1), \gamma_2(t_2) \in \{x + \alpha y = \bar{q}\}$ with $\bar{q} \geq q$ and $|q - \bar{q}| \leq 1/2$ and $|t_1 - t_2| \leq C\alpha^{-1}d_\Sigma(W_{1,1}, W_{2,1})$. Indeed, $t_i = \frac{\bar{q}-q}{a_i+\alpha b_i}$, so $t_1 - t_2 = (\bar{q} - q) \frac{a_2 - a_1 + \alpha(b_2 - b_1)}{(a_1 + \alpha b_1)(a_2 + \alpha b_2)} \leq C\alpha^{-1}d_\Sigma(W_{1,1}, W_{2,1})$. We suppose that we are working on $T_\alpha(\mathcal{M}_4)$, more precisely

$$DT_\alpha^{-1} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 + \alpha^2 \end{pmatrix}.$$

We compute the first component of $T_\alpha^{-1}(\gamma_1(t))$

$$(T_\alpha^{-1}(\gamma_1(t)))^{(1)} = x_1 + \alpha y_1 + t(a_1 + \alpha b_1)$$

and the second component

$$(T_\alpha^{-1}(\gamma_1(t)))^{(2)} = \alpha x_2 + (1 + \alpha^2)y_2 + t(\alpha a_2 + (\alpha^2 + 1)b_2).$$

Hence, we have that the base point of $T_\alpha^{-1}(\gamma_1(t))$ is $T_\alpha^{-1}(\gamma(0)) = (x_1 + \alpha y_1, \alpha x_1 + (1 + \alpha^2)y_1)$. From which we deduce

$$X_1 = T_\alpha^{-1}(\gamma_1(0)) = (q, y_1 + \alpha q).$$

Similarly we find the parametrization of $T_\alpha^{-1}(\gamma_2(t))$ and its base point is

$$X_2 = (q, y_2 + \alpha q).$$

We deduce that the second component of these base points satisfy

$$|X_1^{(2)} - X_2^{(2)}| \leq |y_1 - y_2| \leq \alpha^{-1}|x_1 - x_2| \leq C\alpha^{-1}d_\Sigma(W_{1,1}, W_{2,1}),$$

where we have used that $x_i + \alpha y_i = q$ for $i = 1, 2$. We now suppose that $y_1 \leq y_2$ and we define the time t^* such that the second component of $T_\alpha^{-1}(\gamma_1(t))$ is the same as the second component of $T_\alpha^{-1}(\gamma_2(0))$. Hence, using again that $|y_2 - y_1| \leq \alpha^{-1}|x_2 - x_1| \leq C\alpha^{-1}d_\Sigma(W_{1,1}, W_{2,1})$ we deduce

$$t^* = \frac{y_2 - y_1}{(1 + \alpha^2)b_1 + \alpha a_1} \leq C\alpha^{-3}d(W_{1,1}, W_{2,1}).$$

We define the unmatched curve

$$\tilde{U}_1 = \{T_\alpha^{-1}(\gamma_1(t)) : t \in [0, t^*]\}.$$

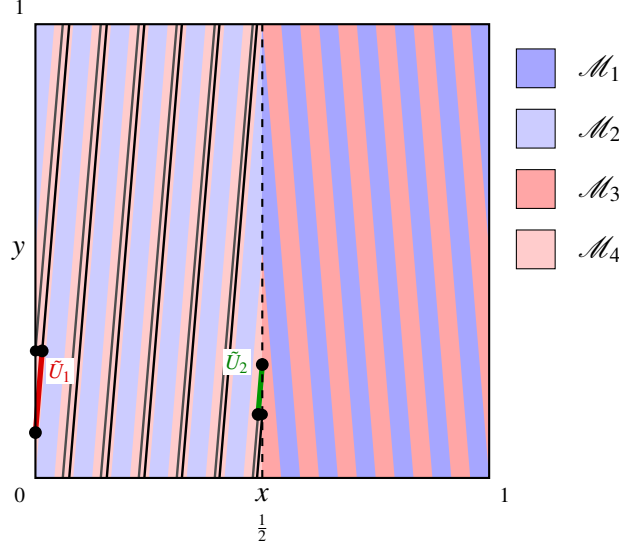


Figure 5.5: We represent $T_\alpha^{-1}(W_1)$ and $T_\alpha^{-1}(W_2)$ as the two black lines in the region $\mathcal{M}_4$. In the proof these lines are cut in $\mathcal{O}(\alpha)$ curves of length 1. The red segment $\tilde{U}_1$ is the initial unmatched curve of $T_\alpha^{-1}(W_1)$, starting from $x = 0$ until it reaches the same y -level as the curve $T_\alpha^{-1}(W_2)$. The green segment $\tilde{U}_2$ is the final unmatched curve of $T_\alpha^{-1}(W_2)$.

By reparametrizing the time $\mathcal{O}(\alpha^2)t$ so that this curve has tangent vector of norm 1 we get that $\tilde{U}_1$ is an admissible curve with length $|\tilde{U}_1| \leq C\alpha^{-1}d_\Sigma(W_{1,1}, W_{2,1})$. We now define the new base points $\tilde{X}_1 = T_\alpha^{-1}(\gamma_1(t^*))$. Since the second component of $\tilde{X}_1$ is the same of the second component of X_2 , their distance is just given by the distance of the first component distance, i.e.

$$|\tilde{X}_1 - X_2| = |t^*(a_1 + \alpha b_1)| \leq C\alpha^{-2}d_\Sigma(W_{1,1}, W_{2,1}).$$

We denote by $\mathbf{v}_1 = (a_1, b_1)$ and $\mathbf{v}_2 = (a_2, b_2)$. The tangent vectors of $T_\alpha^{-1}(\gamma_1(t))$ and $T_\alpha^{-1}(\gamma_2(t))$ will be denoted by

$$\mathbf{w}_1 = T_\alpha^{-1}(\mathbf{v}_1), \quad \mathbf{w}_2 = T_\alpha^{-1}(\mathbf{v}_2),$$

and the matched curves are defined as

$$\tilde{W}_{1,1,k} = \left\{ \tilde{X}_1 + t \frac{\mathbf{w}_1}{|\mathbf{w}_1|} : t \in [k, k+1] \right\}, \quad \tilde{W}_{2,1,k} = \left\{ X_2 + t \frac{\mathbf{w}_2}{|\mathbf{w}_2|} : t \in [k, k+1] \right\},$$

for any k until they reach the end of the curve $T_\alpha^{-1}(\gamma_1(t))$ and $T_\alpha^{-1}(\gamma_2(t))$. Note that, despite these curves being of length 1, they are well-defined objects in Σ . Indeed, a computation shows that if $W \in \Sigma$, then $T_\alpha^{-1}W$ is never a perfectly vertical curve. Note also that there is an unmatched curve $\tilde{U}_2$ for the final part of the curve that satisfies the same properties as $\tilde{U}_1$. We address this curve later.

We now compute the distance of the tangent vectors. We claim that

$$\left| \frac{\mathbf{w}_1}{|\mathbf{w}_1|} - \frac{\mathbf{w}_2}{|\mathbf{w}_2|} \right| = \left| \frac{T_\alpha^{-1}(\mathbf{v}_1)}{|T_\alpha^{-1}(\mathbf{v}_1)|} - \frac{T_\alpha^{-1}(\mathbf{v}_2)}{|T_\alpha^{-1}(\mathbf{v}_2)|} \right| \leq C\alpha^{-4}|\mathbf{v}_1 - \mathbf{v}_2|. \quad (5.7.4)$$

We simply rewrite

$$A_\alpha := \begin{pmatrix} 1 & \alpha \\ \alpha & 1 + \alpha^2 \end{pmatrix}, \quad T_\alpha^{-1}(z) = A_\alpha z, \quad \hat{z} := \frac{z}{|z|} \quad (z \neq 0).$$

Recall that $\alpha \geq 1$ and that $\mathbf{v}_1 = (a_1, b_1)$, $\mathbf{v}_2 = (a_2, b_2)$ satisfy

$$|\mathbf{v}_1| = |\mathbf{v}_2| = 1, \quad |a_1| \leq 2\alpha^{-1}|b_1|, \quad |a_2| \leq 2\alpha^{-1}|b_2|.$$

We observe that since the angle θ between $\mathbf{w}_1, \mathbf{w}_2$ satisfies $\theta \in [-\pi/2, \pi/2]$ we deduce

$$|\hat{\mathbf{w}}_1 - \hat{\mathbf{w}}_2|^2 = 2(1 - \cos \theta) \leq 2\sin^2 \theta,$$

hence

$$|\hat{\mathbf{w}}_1 - \hat{\mathbf{w}}_2| \leq \sqrt{2}|\sin \theta| = \sqrt{2} \frac{|\det(\mathbf{w}_1, \mathbf{w}_2)|}{|\mathbf{w}_1||\mathbf{w}_2|}.$$

Here $(\mathbf{w}_1, \mathbf{w}_2)$ denotes the matrix with columns given by $\mathbf{w}_1, \mathbf{w}_2$ respectively. Since $\det(A_\alpha) = 1$,

$$\det(\mathbf{w}_1, \mathbf{w}_2) = \det(A_\alpha \mathbf{v}_1, A_\alpha \mathbf{v}_2) = \det(\mathbf{v}_1, \mathbf{v}_2).$$

Moreover,

$$|\det(\mathbf{v}_1, \mathbf{v}_2)| = |\det(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2)| \leq |\mathbf{v}_1 - \mathbf{v}_2| |\mathbf{v}_2| = |\mathbf{v}_1 - \mathbf{v}_2|.$$

Therefore

$$|\hat{\mathbf{w}}_1 - \hat{\mathbf{w}}_2| \leq \sqrt{2} \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{|A_\alpha \mathbf{v}_1| |A_\alpha \mathbf{v}_2|}.$$

It is straightforward to check that for α sufficiently large

$$|A_\alpha \mathbf{v}_1| |A_\alpha \mathbf{v}_2| \geq \frac{\alpha^4}{2},$$

hence the claim follows.

We consider the distances of base points. We have already shown that the distance of the base points is bounded by $|\tilde{X}_1 - X_2| \leq C\alpha^{-2}d_\Sigma(W_{1,1}, W_{2,1})$. Furthermore, since $|DT_\alpha^{-1}| \leq C\alpha^2$ and the original curves are of length bounded by $\frac{2}{\alpha}$ we deduce that $T_\alpha^{-1}(W_{1,1}), T_\alpha^{-1}(W_{2,1})$ are made of at most $\mathcal{O}(\alpha)$ admissible curves. By (5.7.4) we deduce that base points on each matched admissible curves can separate at most by $C\alpha^{-4}d_\Sigma(W_{1,1}, W_{2,1})$. Therefore, all matched base points are always distant at most $\leq C\alpha^{-2}d_\Sigma(W_{1,1}, W_{2,1})$.

We now consider the last unmatched curve. Recalling the notation for the curves $\tilde{W}_{1,1,k}$ and $\tilde{W}_{2,1,k}$, their lengths are always 1, until we arrive at the final matched curve of length 1, which terminates at $t = N \in \mathbb{N}$ with $N \leq C\alpha$. Following this, we construct another set of matched curves by $\tilde{W}_{1,1,fin} = \{\tilde{X}_1 + N\hat{\mathbf{w}}_1 + t\hat{\mathbf{w}}_1 : t \leq |\mathbf{w}_1|t_1 \wedge |\mathbf{w}_2|t_2 - N\}$, $\tilde{W}_{2,1,fin} = \{\tilde{X}_2 + N\hat{\mathbf{w}}_2 + t\hat{\mathbf{w}}_2 : t \leq |\mathbf{w}_1|t_1 \wedge |\mathbf{w}_2|t_2 - N\}$.

Recalling that $t_i = \frac{\bar{q}-q}{a_i+\alpha b_i}$ for $i = 1, 2$ we deduce that for $i = 1, 2$ we have

$$\begin{aligned} t_i^2 |DT_\alpha^{-1} \mathbf{v}_i|^2 &= t_i^2 \left((a_i + \alpha b_i)^2 (1 + \alpha^2) + 2\alpha(a_i + \alpha b_i)b_i + b_i^2 \right) \\ &= (\bar{q} - q)^2 \left((1 + \alpha^2) + \frac{2\alpha b_i}{(a_i + \alpha b_i)} + \frac{b_i^2}{(a_i + \alpha b_i)^2} \right) \end{aligned}$$

thus, defining $R_i = \frac{2\alpha b_i}{(1+\alpha^2)(a_i+\alpha b_i)} + \frac{b_i^2}{(1+\alpha^2)(a_i+\alpha b_i)^2}$ for $i = 1, 2$ we deduce

$$|t_1 |DT_\alpha^{-1} \mathbf{v}_1| - t_2 |DT_\alpha^{-1} \mathbf{v}_2|| \leq |\bar{q} - q| \sqrt{1 + \alpha^2} \left| \sqrt{1 + R_1} - \sqrt{1 + R_2} \right| \leq C\alpha |R_1 - R_2|.$$

Using that

$$|R_1 - R_2| \leq C\alpha^{-2} d_\Sigma(W_1, W_2)$$

we conclude that the final unmatched curve defined by (assuming for instance that $|\mathbf{w}_2|t_2 \geq |\mathbf{w}_1|t_1$)

$$\tilde{U}_2 = \{T_\alpha^{-1}(\gamma_2(t)) : t \in [|\mathbf{w}_1|t_1, |\mathbf{w}_2|t_2]\}$$

satisfies $|\tilde{U}_2| \leq C\alpha^{-1} d_\Sigma(W_1, W_2)$. Hence, we conclude the proof. $\square$

Chapter 6

Slow Dynamo action in the Ponomarenko model

In this chapter, we begin our rigorous study of slow dynamos, which will culminate in a proof of non-linear instability for the MHD equations in the following Chapter 7. In particular, we will study dynamo action in a wide class of helical velocity fields, which notably include an explicit stationary solution to the 3d Navier–Stokes equations. Via detailed analysis of the ODE arising from a modal Ansatz to (KDE- ε) with helical velocity field, we prove that slow dynamo action occurs subject to a purely local geometric condition, which can be seen as governing the local relative strengths of stretching and enhanced dissipation. The contents of this chapter are contained in the paper [NV25b].

6.1 Introduction

Throughout this chapter, we consider the kinematic dynamo equations (KDE- ε)

$$\begin{cases} \partial_t B^\varepsilon + (u \cdot \nabla) B^\varepsilon - (B^\varepsilon \cdot \nabla) u &= \varepsilon \Delta B^\varepsilon, \\ \operatorname{div}(B^\varepsilon) &= 0, \\ B^\varepsilon(0, \cdot) &= B_{\text{in}}^\varepsilon, \end{cases} \quad (6.1.1)$$

with $t \in (0, \infty)$ and $x \in \mathcal{M} \times \mathbb{T}$. Here $\mathcal{M} \subseteq \mathbb{R}^2$, $\varepsilon > 0$ denotes the magnetic diffusivity, and $B_{\text{in}}^\varepsilon$ represents an initial configuration for the magnetic field. When $\partial \mathcal{M} \neq \emptyset$, we further impose the *perfectly conducting* boundary conditions (2.1.1), which we recall here for convenience

$$B^\varepsilon \cdot \hat{n}|_{\partial(\mathcal{M} \times \mathbb{T})} = 0, \quad (\nabla \times B^\varepsilon) \times \hat{n}|_{\partial(\mathcal{M} \times \mathbb{T})} = 0, \quad (6.1.2)$$

where $\hat{n}$ denotes the outwards pointing unit normal vector to the boundary of $\mathcal{M} \times \mathbb{T}$. This choice of perfectly conducting walls as boundary conditions is standard for the magnetohydrodynamics (MHD) equations, see e.g. [ST83; Rob67].

In this chapter, we shall focus on the case of slow dynamos, where the growth rate decays to zero

as $\varepsilon \rightarrow 0$. This phenomenon is intricately linked to concepts in boundary layer theory, where, somewhat counter-intuitively, the introduction of the dissipative term $\varepsilon \Delta$ into the equation may actually have a *destabilizing* effect. Analogous phenomena are well documented for the Navier–Stokes equations, where profiles that may be stable for the Euler equations can become linearly unstable once viscosity is introduced, see for instance [GGN16b; GGN16a; Col+25].

Despite the fact that slow dynamos should in principle be less complicated to study than fast dynamos, there are remarkably few rigorous results in the literature. In large part, this lack of results can be attributed to the *anti-dynamo theorems* outlined in Chapter 2, Section 2.2, which rule out dynamo action in velocity fields with “too much symmetry”. Despite these theoretical restrictions, in [Pon73] Ponomarenko gives an example of a relatively simple velocity field under which the analysis of the kinematic dynamo equations reduces to a system of coupled ordinary differential equations, but which still may yield dynamo action. Indeed, in [Pon73], velocities of the form

$$u = r\Omega(r)\hat{\theta} + U(r)\hat{z}, \quad (6.1.3)$$

are considered, where $(r, \theta, z) \in \mathbb{R} \times \mathbb{T} \times \mathbb{T}$ denotes the standard cylindrical coordinate system, and

$$\Omega(r) = \begin{cases} \Omega & r \leq r_0, \\ 0 & r > r_0, \end{cases} \quad U(r) = \begin{cases} U & r \leq r_0, \\ 0 & r > r_0, \end{cases} \quad (6.1.4)$$

for some constant $\Omega, U \neq 0$. In fact, due to the piecewise constant nature of Ω, U , in [Gil88] Gilbert shows that with this choice of axisymmetric and discontinuous velocity field, (KDE- ε) actually becomes explicitly solvable using modified Bessel functions, and fast dynamo action can be shown. However, the growth is asymptotically localised near the discontinuity at $r = r_0$ as $\varepsilon \rightarrow 0$, suggesting that dynamo action occurs *as a result* of the discontinuity.

To rectify the somewhat unphysical nature of a discontinuous velocity field, in [Gil88] the author further considers divergence-free and radial velocity fields of the form (6.1.3), where Ω, U are arbitrary, smooth functions of r . Via a formal asymptotic WKB expansion, Gilbert claims that the maximum growth rate of dynamos driven by velocity fields of the form (6.1.3) is given by $\gamma_\varepsilon \sim \varepsilon^{1/3}$. Henceforth, we shall refer to dynamo action by such velocity fields as “smooth Ponomarenko dynamos”. In particular, $\gamma_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, and so the dynamo action is that of a *slow dynamo*. This statement has since been supported with strong numerical [Wyn17; PGP08; Nor03; RSS88] and experimental evidence [Gai+00; Ver+10]. The only rigorous result on the smooth Ponomarenko dynamo we are aware of is from [GR07], where the authors prove finite time instability for the forced MHD equations with a Ponomarenko-type velocity profile. It is worth remarking that both the lower and upper bounds from [GR07] are in good agreement with the growth rate $\gamma_\varepsilon \sim \varepsilon^{1/3}$, further supporting the idea that the results obtained in [Gil88] are sharp.

In this chapter, we shall provide a rigorous proof of dynamo action for a large class of functions Ω, U in the Ponomarenko dynamo (6.1.3), for $\mathcal{M} \times \mathbb{T}$ being either the full space $\mathbb{R}^2 \times \mathbb{T}$, a cylinder, an annular cylinder, or the exterior of a cylinder, equipped with perfectly conducting boundary conditions. Furthermore, we shall give a precise description of the growing magnetic field profile in the limit $\varepsilon \rightarrow 0$.

6.1.1 Preliminaries and main results

At this point, it is worth outlining the ways in which the choice of Ponomarenko velocity field (6.1.3) simplifies the analysis of the kinematic dynamo equations. Writing the magnetic vector in cylindrical components $B = B_r \hat{r} + B_\theta \hat{\theta} + B_z \hat{z}$, we find that the kinematic dynamo equations (6.1.1) become

$$\begin{aligned}\partial_t B_r + \Omega(r) \partial_\theta B_r + U(r) \partial_z B_r &= \varepsilon \left(\Delta B_r - \frac{B_r}{r^2} - \frac{2}{r^2} \partial_\theta B_\theta \right), \\ \partial_t B_\theta + \Omega(r) \partial_\theta B_\theta + U(r) \partial_z B_\theta &= \varepsilon \left(\Delta B_\theta - \frac{B_\theta}{r^2} + \frac{2}{r^2} \partial_\theta B_r \right) + r \Omega'(r) B_r, \\ \partial_t B_z + \Omega(r) \partial_\theta B_z + U(r) \partial_z B_z &= \varepsilon \Delta B_z + U'(r) B_r,\end{aligned}$$

where $\Delta = \partial_r^2 + r^{-1} \partial_r + r^{-2} \partial_\theta^2 + \partial_z^2$ represents the scalar Laplacian operator in cylindrical coordinates. Observe that the first two equations—for B_r and B_θ —are coupled through the stretching term $r \Omega'(r) B_r$ appearing in the second equation, as well as through the Laplacian coupling, which is a by-product of the cylindrical geometry. Crucially however, the equations corresponding to B_r, B_θ do not contain B_z , and so the PDE admits a Jordan block-like structure. Indeed, our strategy will be to solve the coupled system for the radial and azimuthal components (B_r, B_θ) , and then, only once a solution is found, do we solve for the z component, which satisfies a standard advection-diffusion equation with a forcing of the form $U'(r) B_r$. In this coordinate system, we consider the components $(r, \theta) \in \mathcal{M} = \mathcal{I} \times \mathbb{T}$, where $\mathcal{I} \subseteq [0, \infty)$ is an interval in the positive real line, namely $\mathcal{M} \subseteq \mathbb{R}^2$ consists of a circle, an annulus, or the exterior of a circle centred at the origin. We can further make the modal form Ansatz

$$B(t, r, \theta, z) = b(r) e^{\lambda t + i(m\theta + kz)},$$

for $b : \mathcal{I} \rightarrow \mathbb{C}^3$, $\lambda \in \mathbb{C}$ and $m, k \in \mathbb{Z}$. The components of the vector $b = b_r \hat{r} + b_\theta \hat{\theta} + b_z \hat{z}$ satisfy the following system of ordinary differential equations,

$$\lambda b_r + i(\Omega(r)m + U(r)k) b_r = \varepsilon \left(\partial_r^2 b_r + \frac{1}{r} \partial_r b_r - \left(\frac{m^2 + 1}{r^2} + k^2 \right) b_r - \frac{2im}{r^2} b_\theta \right), \quad (6.1.5)$$

$$\lambda b_\theta + i(\Omega(r)m + U(r)k) b_\theta = r \Omega'(r) b_r + \varepsilon \left(\partial_r^2 b_\theta + \frac{1}{r} \partial_r b_\theta - \left(\frac{m^2 + 1}{r^2} + k^2 \right) b_\theta + \frac{2im}{r^2} b_r \right), \quad (6.1.6)$$

$$\lambda b_z + i(\Omega(r)m + U(r)k) b_z = U'(r) b_r + \varepsilon \left(\partial_r^2 b_z + \frac{1}{r} \partial_r b_z - \frac{m^2}{r^2} b_z - k^2 b_z \right). \quad (6.1.7)$$

The perfectly conducting boundary conditions (6.1.2) translate into

$$b_r|_{\partial \mathcal{I} \setminus \{0\}} = (r b_\theta)'|_{\partial \mathcal{I} \setminus \{0\}} = b_z'|_{\partial \mathcal{I} \setminus \{0\}} = 0 \quad (6.1.8)$$

under this modal Ansatz. If $\mathcal{I} \subseteq [0, \infty)$ is of the form $\mathcal{I} = [0, q]$ for some $0 < q \leq \infty$, it is worth noting that $\{0\} \in \partial \mathcal{I}$ is not an actual element of $\partial(\mathcal{M} \times \mathbb{T})$ and just a fabrication of the coordinate system. Therefore, finding an exponentially growing solution to the kinematic dynamo equations may be achieved by finding an *unstable eigenvalue* of the kinematic dynamo operator, that is, finding a finite energy magnetic field $b(r)$, and $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > 0$ so that equations

(6.1.5)–(6.1.7) are represented by

$$\mathcal{L}^{\text{dyn}} b = \lambda b, \quad (6.1.9)$$

where the differential operator $\mathcal{L}^{\text{dyn}}$ will be referred to as the *Ponomarenko dynamo operator*. Observe that an eigenvalue λ in (6.1.9) with positive real part gives the growth condition $\gamma(\varepsilon) > 0$ in (1.2.1). Indeed, the bulk of this chapter shall be devoted to studying the spectrum of the operator $\mathcal{L}^{\text{dyn}}$, and we shall say that any (smooth) velocity field of the form (6.1.3) for which there is an unstable eigenfunction of $\mathcal{L}^{\text{dyn}}$ is a (smooth) Ponomarenko dynamo.

With this spectral point of view established, we now move on to providing a summary of the main results that we prove in this chapter.

Main results

The main aim of this work is to find conditions on the functions Ω, U which guarantee the existence of an exponentially growing mode for the Ponomarenko dynamo operator defined by (6.1.5)–(6.1.7). Of course the exact conditions depend on the domain $\mathcal{M}$ in question, but there are a few commonalities. In fact, we will show that the existence of a growing mode can be guaranteed by the local properties of Ω, U around some critical radius r_0 determined by the Fourier modes m, k , as long as the behaviours of $\Omega(r), U(r)$ as $r \rightarrow \infty$ are not too degenerate (if the domain we are interested in includes this endpoint). Since the precise form of the degeneracies we need to rule out is somewhat technical, it is instructive to give an informal sketch of the assumptions we shall place on the functions Ω, U . Indeed, we will assume conditions on Ω, U that formally amount to: $\Omega(r), U(r)$ are smooth functions on $[0, \infty)$, which furthermore satisfy

1. Ω, U grow at most polynomially as $r \rightarrow \infty$,
2. Ω, U do not oscillate infinitely often around a fixed value as $r \rightarrow \infty$,
3. As $r \rightarrow \infty$, $|\Omega(r) + U(r)| \gg |r\Omega'(r)|$.

We refer to Assumptions H0–H3 in Section 6.3.4 for a more rigorous statement of these properties. We draw special attention to the last of these assumptions, since (at least heuristically) it gives some insight as to why we can always find a growing *eigenfunction* for (6.1.1), even when we pose the equation on a non-compact domain. Indeed, on an intuitive level, this assumption amounts to saying that for large r , the stretching term $r\Omega'(r)$ is “weaker” than the transport term $u \cdot \nabla$, and therefore it cannot overcome the decay caused by the interplay of the Laplacian and mixing as a result of transport. Therefore, all growth in the equation must happen in the vicinity of the origin, where the Laplacian admits a compact resolvent, and so growing solutions are indeed attained by eigenfunctions. More formally, we can say that the equation (6.1.1) can be treated as an advection diffusion equation, perturbed by the *relatively compact* term $r\Omega'(r)B_r$. Since such a perturbation does not disturb the essential spectrum, which for the advection diffusion equation will be contained in the left half plane, any spectral value in the right half plane will immediately be an eigenvalue for the dynamo equation. Indeed, in Section 6.4 we shall make this intuition rigorous.

Our main results, stated rigorously below, then show that under the aforementioned conditions,

slow dynamo action with rate $\gamma_\varepsilon \sim \varepsilon^{1/3}$, can be deduced from a purely local, geometric condition at some critical radius $r_0 > 0$. Moreover, we give a precise asymptotic expansion of the associated eigenfunction B^ε as $\varepsilon \rightarrow 0$.

Our construction of the dynamo profiles proceeds via the analysis of the ODE system defined by the operator $\mathcal{L}^{\text{dyn}}$, which is a meaningful problem even when its connection to the PDE (6.1.1) is less relevant (for instance when $m, k \notin \mathbb{Z}$, and so do not represent “true” Fourier variables). We therefore split our main result into two theorems, one which focuses exclusively on the ODE, whereas the other synthesises the ODE result into an instability result for our PDE.

Theorem 6.1.1. *Let $\mathcal{J} = [p, q]$ or $\mathcal{J} = [p, \infty)$, where $p \in [0, \infty)$, $q \in (p, \infty)$, and let the velocity field $u = r\Omega(r)\hat{\theta} + U(r)\hat{z}$ satisfy Assumptions H0–H2. Pick any $r_0 \in \text{int}(\mathcal{J})$ so that*

$$r_0 \left| \frac{d}{dr} \log \left| \frac{\Omega'(r_0)}{U'(r_0)} \right| \right| < 4,$$

and any $|M|$ small. Then, for all $\varepsilon > 0$ sufficiently small, there exist a constant $\mu \in \mathbb{C}$ with $\text{Re}(\mu) > 0$, and a C^2 vector field $b^\varepsilon : \mathcal{J} \rightarrow \mathbb{C}^3$ with $b^\varepsilon \in (L^1 \cap L^\infty)(\mathcal{J}, r dr)$, so that $\mathcal{L}^{\text{dyn}} b^\varepsilon = \varepsilon^{1/3} \mu b^\varepsilon$. Furthermore, if $\mathcal{J} \neq [0, \infty)$, b^ε satisfies the perfectly conducting boundary conditions

$$b_r^\varepsilon|_{\partial \mathcal{J} \setminus \{0\}} = 0, \quad (rb_\theta^\varepsilon)'|_{\partial \mathcal{J} \setminus \{0\}} = 0, \quad (b_z^\varepsilon)'|_{\partial \mathcal{J} \setminus \{0\}} = 0.$$

In addition, the growth rate satisfies the asymptotic expansion $\mu = \mu_\star + o_{\varepsilon \rightarrow 0}(1)$, where

$$\text{Re}(\mu_\star) = \left[|M|^{1/2} \left(\frac{|\Omega'(r_0)|^{1/2}}{r_0^{1/2}} - \frac{1}{2} \left| \Omega''(r_0) - \frac{\Omega'(r_0)}{U'(r_0)} U''(r_0) \right|^{1/2} \right) - |M|^2 \left(\frac{1}{r_0^2} + \frac{|\Omega'(r_0)|^2}{|U'(r_0)|^2} \right) \right] > 0.$$

If either $\mathcal{J}$ is compact, or Assumption H3 is satisfied, there further exists $a > 0$ such that for any $N \in \mathbb{N}$ and all $\varepsilon > 0$ small enough depending on N , the (r, θ) components of b^ε admit the asymptotic expansion,

$$\begin{pmatrix} b_r^\varepsilon \\ b_\theta^\varepsilon \end{pmatrix} = \begin{pmatrix} \varepsilon^{1/3} \sqrt{\frac{-2iM}{r_0^3 \Omega'(r_0)}} \\ 1 \end{pmatrix} \left(e^{-\frac{1}{2} \varepsilon^{-2/3} c_2^{1/2} (r-r_0)^2} + \psi_{\text{err}}^\varepsilon \right)$$

for some explicitly computable constant $c_2 \in \mathbb{C}$ with $\text{Re}(c_2^{1/2}) > 0$, and

$$\sup_{r \in \mathcal{J}} \left| \left(1 + (|r - r_0| \varepsilon^{-1/3})^N \right) \psi_{\text{err}}^\varepsilon \right| \lesssim \varepsilon^a.$$

Once the result for the ODE is established, under an additional technical assumption, we can immediately deduce the existence of a slow dynamo, as stated in the next theorem.

Theorem 6.1.2. *If in addition to the assumptions of Theorem 6.1.1, the compact set $\mathcal{R}_0$ from H2 contains an interval around r_0 , then, for all $\varepsilon > 0$ sufficiently small, there exists $m, k \in \mathbb{Z}$ so that*

$$B^\varepsilon(t, r, \theta, z) = e^{\varepsilon^{1/3} \mu t} e^{i(m\theta + kz)} b^\varepsilon(r)$$

is a divergence-free, finite energy solution to the kinematic dynamo equations (6.1.1), where b^ε

and μ are given by Theorem 6.1.1.

Remark 6.1.3. Whilst the existence of a growing mode for the dynamo equation may be deduced already under Assumptions H0–H2, Assumption H3 is needed to ensure that the growing mode is an *isolated eigenvalue* for the kinematic dynamo operator on a suitable space. This is the subject of Section 6.4. Once this is achieved, our asymptotic expansion is deduced using tools from classical spectral theory.

Remark 6.1.4. Although throughout the chapter we make the standing assumption that $z \in \mathbb{T}$, our results readily imply a corresponding dynamo result for the case when $z \in \mathbb{R}$. Indeed, under the assumptions of Theorem 6.1.2, we immediately deduce the existence of an open set $\mathcal{K} \subset \mathbb{R}$, with eigenmode $b^\varepsilon(r, k)$, so that as k varies throughout $\mathcal{K}$, $\text{Re}(\mu(k)) > 0$. Then, setting

$$B(t, r, \theta, z) = \int_{\mathcal{K}} e^{\varepsilon^{1/3} \mu(k)t} e^{i(m\theta + kz)} b^\varepsilon(r, k) dk,$$

we obtain a slow dynamo in $L^2(\mathcal{M} \times \mathbb{R})$.

Our results show that, for a large class of velocity fields, the Ponomarenko dynamo (6.1.3) exhibits L^p norm exponential growth of the magnetic field for any $1 \leq p \leq \infty$. In particular, it displays slow dynamo action with growth rate (1.2.1) of the form

$$\gamma(\varepsilon) = \varepsilon^{1/3}(\text{Re}(\mu_\star) + o_{\varepsilon \rightarrow 0}(1)) > 0.$$

At this point, it is worth providing some relatively simple, explicit examples of velocity fields which satisfy the aforementioned Assumptions H0–H3. Indeed, we have the following result.

Example 6.1.5. In each of the following cases, the conclusions of Theorems 6.1.1, 6.1.2 hold true on $\mathcal{J} = [p, q]$ or $\mathcal{J} = [p, \infty)$ for any $p \in [0, \infty)$, $q \in (p, \infty)$.

1. Simplified Ponomarenko model: $\Omega(r) = 1 - r$, $U(r) = 1 - r^2$
2. Gaussian velocity fields: $\Omega(r) = e^{-ar^2}$, $U(r) = e^{-br^2}$, with $a, b > 0$ and $a \neq b$.
3. Compactly supported velocity fields: $\Omega(r) = (1 - r)\chi(r)$, $U(r) = (1 - r^2)\chi(r)$, where

$$\chi(r) = \begin{cases} 1 & \text{if } r \leq 2, \\ 0 & \text{if } r \geq 3, \end{cases}$$

and $\chi(r)$ is smooth.

Furthermore, the conclusions of Theorems 6.1.1, 6.1.2 hold true on the interval $\mathcal{J} = [p, q]$, for $p > 0$, $q \in (p, \infty)$ for the Taylor–Couette flow

$$u(r) = \left(\frac{a_1}{r} + a_3 r \right) \hat{\theta} + (a_2 \log(r) + a_4) \hat{z},$$

which is an explicit stationary solution to the three-dimensional Navier–Stokes equations.

In particular, examples (2) and (3) are, to the best of our knowledge, the first rigorous proofs of slow dynamo action driven respectively by finite energy and by compactly supported velocity

fields, whereas the Taylor–Couette flow is the first rigorous proof of dynamo action driven by an explicit solution to the Navier–Stokes equations.

Notation

Due to the length of the chapter, we collect here certain notational conventions that will be frequently used.

- We denote by $B(t, r, \theta, z)$ magnetic fields that solve the full kinematic dynamo PDE (6.1.1), whereas $b(r)$ denotes a solution to the modal form ODE (6.1.5)–(6.1.7).
- Throughout the chapter, $\mathcal{I} \subseteq [0, \infty)$ denotes an interval of the form $\mathcal{I} = [p, q]$, with $p \in [0, \infty)$, $q \in (p, \infty]$ (where in the case $q = +\infty$, we take $\mathcal{I} = [p, \infty)$). Since the interval arises from a change of variables into cylindrical coordinates, the point 0 (should it be an endpoint of $\mathcal{I}$) is not in fact a physical boundary. Thus, we make the abuse of notation and denote the boundary $\partial\mathcal{I} = \{q\}$ in the case where $p = 0$. Next, the letters m, k denote Fourier variables for θ, z respectively.
- We will employ the notation $a \lesssim b$ to denote that there exists a constant $C > 0$, independent of all the relevant quantities (in particular always independent of ε), such that $a \leq Cb$. Moreover, in many different estimates we will use a generic constant $C > 0$ that might change from line to line.
- We use the notation “big O ” and “little o ” for the limit $\varepsilon \rightarrow 0$ as it is standard. Namely we say that an object is $O_{\varepsilon \rightarrow 0}(1)$ if it is bounded independently of ε for all $\varepsilon > 0$ small enough, and we say that a function f is $o_{\varepsilon \rightarrow 0}(1)$ if $\lim_{\varepsilon \rightarrow 0} f = 0$.
- In keeping with the notation of [PGP08], we set $m = M\varepsilon^{-1/3}$, $k = K\varepsilon^{-1/3}$, for some real coefficients $M, K = O_{\varepsilon \rightarrow 0}(1)$.
- Given a function space F , if all the components of a vector valued function

$$f = (f_1, \dots, f_n) : \mathcal{M} \times \mathbb{T} \rightarrow \mathbb{R}^n$$

satisfy $f_i \in F(\mathcal{M} \times \mathbb{T})$, then we will abuse the notation and write as well $f \in F(\mathcal{M} \times \mathbb{T})$.

6.2 Strategy

In this section we give an overview on the method employed to establish the main results, i.e. Theorems 6.1.1 and 6.1.2. Let us recall the main ideas presented thus far:

1. We consider helical vector fields—after Ponomarenko—of the form

$$u(r) = r\Omega(r)\hat{\theta} + U(r)\hat{z}.$$

In Section 6.3 we define the class of admissible vector fields.

2. We look for growing modes: solutions to equations (6.1.1) in modal form

$$B(t, r, \theta, z) = b(r)e^{\lambda t}e^{i(m\theta + kz)},$$

for some $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 0$, and $m, k \in \mathbb{Z}$. This Ansatz transforms a PDE problem (6.1.1) into an ODE problem (6.1.5)–(6.1.7).

3. Since the equations for (b_r, b_θ) in (6.1.5)–(6.1.6) are decoupled from b_z , we put our efforts in solving them first. After a suitable change of variables introduced in Section 6.3, we denote the operator associated to such ODE by $\mathcal{L} - \lambda$.
4. In Section 6.4 we present analytical and spectral properties of the operator $\mathcal{L} - \lambda$, and we make sure that all admissible solutions satisfy the divergence-free condition.
5. The main goal and bulk of this chapter is to find V in some suitable functional space, and $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 0$, such that

$$(\mathcal{L} - \lambda)V = 0.$$

This is the main purpose of Sections 6.5, 6.6 and 6.7.

6. Once solutions (b_r, b_θ) are obtained, we construct the vertical component b_z in Section 6.8.

Without further ado, let us present some details on the strategy to find a growing mode of the kinematic dynamo operator.

6.2.1 Resolvent formalism and Green's functions

We begin by discussing the strategy for constructing a growing mode in the case where $\mathcal{M} = \mathbb{R}^2$, in other words, $r \in [0, \infty)$. Hence, we shall study the operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset X \rightarrow X$, where X denotes the Banach space

$$X = L^\infty((0, \infty), \max\{1, r^2\}w_\varepsilon(r - r_0)dr),$$

with the weight function

$$w_\varepsilon(s) = 1 + (\varepsilon^{-1/3}|s|)^N, \tag{6.2.1}$$

for some fixed $N > 0$. In particular, notice that $X \subset (L^1 \cap L^\infty)(0, \infty)$.

The main challenge of the spectral analysis will be to bridge the gap between the formal, local analysis of Gilbert [Gil88]—outlined in Section 6.3.2—and the existence of a genuine, global eigenfunction of the dynamo operator. On one hand, at the level of generality we pose the problem, there is no real hope of finding an explicit solution to the system of ODEs (6.1.5)–(6.1.6). On the other hand, there is no immediate property, such as for instance normality, of the operator $\mathcal{L}$, that would facilitate spectral analysis. Nevertheless, we shall use the fact that our equation involves a small parameter ε to study its spectral properties using tools from perturbation theory. In particular, we will be interested in studying the *resolvent* $(\mathcal{L} - \lambda)^{-1}$ of our operator, in the formal limit when $\varepsilon \rightarrow 0$. This of course carries with it its own difficulties, since the “straightfor-

ward” approach of constructing the inverse of a complicated operator by viewing it as perturbation of a known operator will simply not work for our case. Indeed, whilst for certain subregions of $[0, \infty)$, the operator may be well-approximated by a simpler ODE (as seen for instance in Gilbert’s analysis in Section 6.3.2), the approximations are not global. This issue is well known in the community, and in the case of asymptotic expansions, is typically dealt with using the technique of “matching inner and outer expansions”, i.e. constructing asymptotic expansions which are valid in overlapping spatial regions. In fact, this technique has been rigorously employed to prove nonlinear instability for instance in the seminal works [Gre00; DG03], and in particular in the context of the MHD equations, in the works [Gér05; GR07]. Inspired by this intuition, we employ a technique of “matched local inverses”, namely constructing approximate inverses for our equation in disjoint parts of the domain, which can be glued together to produce a global inverse. Due to the importance of this technique in our work, we now describe it in more detail. We split the interval $[0, \infty)$ into distinct regions, in each of which our ODE is well-approximated (in the strong sense) by a simpler ODE, which we know how to solve. In particular, given an interval, say $[a, b] \subset [0, \infty)$, we seek an approximate operator $\mathcal{L}_{\text{app}}$, so that $\mathcal{L}_{\text{app}}$ is invertible on $L^\infty(-\infty, \infty)$, and $(\mathcal{L}_{\text{app}})^{-1}$ restricted to $L^\infty([a, b])$ satisfies $\|(\mathcal{L} - \mathcal{L}_{\text{app}})(\mathcal{L}_{\text{app}})^{-1}\| = o_{\varepsilon \rightarrow 0}(1)$. Namely, uniformly for all f such that $\text{supp}(f) \subset [a, b]$, there holds

$$\|(\mathcal{L} - \mathcal{L}_{\text{app}})(\mathcal{L}_{\text{app}})^{-1}f\|_{L^\infty([a, b])} \leq o_{\varepsilon \rightarrow 0}(1)\|f\|_{L^\infty([a, b])}.$$

Additionally, we shall require that, for any compactly supported f , $(\mathcal{L}_{\text{app}})^{-1}f$ decays sufficiently quickly (typically this means exponentially with an ε dependent exponent) outside of the support of f . Indeed, this property turns out to be rather generic for elliptic equations on $(-\infty, \infty)$, so long as one is sufficiently far away from the spectrum of the ODE. Supposing further that $\mathcal{L} - \mathcal{L}_{\text{app}}$ is solely a multiplication operator by a function with at most polynomial growth at infinity—call it $R(r)$, we can now hope to deduce uniformly for any f with $\text{supp}(f) \subset [a, b]$

$$\begin{aligned} \|(\mathcal{L} - \mathcal{L}_{\text{app}})(\mathcal{L}_{\text{app}})^{-1}f\|_{L^\infty([0, \infty))} &\leq \|(\mathcal{L} - \mathcal{L}_{\text{app}})(\mathcal{L}_{\text{app}})^{-1}f\|_{L^\infty([a, b])} \\ &\quad + \sup_{r \leq a} e^{-\varepsilon^{-\theta}|r-a|} |R(r)| \|f\|_{L^\infty([a, b])} + \sup_{r \geq b} e^{-\varepsilon^{-\theta}|r-b|} |R(r)| \|f\|_{L^\infty([a, b])} \\ &\lesssim o_{\varepsilon \rightarrow 0}(1)\|f\|_{L^\infty([a, b])}, \end{aligned}$$

for some particular $\theta > 0$, provided $R(r)$ is small near a and b . This means that $\mathcal{L}_{\text{app}}$ is a *global* approximation to the ODE, as long as f is supported in $[a, b]$.

From here, we construct a partition

$$[0, \infty) = \bigcup_{i \in I} [p_i, p_{i+1}),$$

where I denotes a finite set of indices, and to each interval $[p_i, p_{i+1})$ we associate an operator $\mathcal{L}_{\text{app}}^i$ that satisfies

$$\|(\mathcal{L} - \mathcal{L}_{\text{app}}^i)(\mathcal{L}_{\text{app}}^i)^{-1}f\|_{L^\infty([0, \infty))} \lesssim o_{\varepsilon \rightarrow 0}(1)\|f\|_{L^\infty([p_i, p_{i+1}])},$$

uniformly for $\text{supp}(f) \subset [p_i, p_{i+1})$. Then we can in fact compute the true inverse of $\mathcal{L}$ on $[0, \infty)$

via a Neumann series type construction. Set

$$\mathcal{G}_{\text{app}} f = \sum_{i \in I} (\mathcal{L}_{\text{app}}^i)^{-1} (\mathbb{1}_{[p_i, p_{i+1})} f),$$

and then define

$$\mathcal{L}^{-1} = \mathcal{G}_{\text{app}} \sum_{n \geq 0} (-1)^n (\mathcal{L} \mathcal{G}_{\text{app}} - \text{Id})^n.$$

Applying $\mathcal{L}$ to this operator truncated at degree N yields

$$\sum_{n=0}^N (-1)^n (\mathcal{L} \mathcal{G}_{\text{app}} - \text{Id})^{n+1} + \sum_{n=0}^N (-1)^n (\mathcal{L} \mathcal{G}_{\text{app}} - \text{Id})^n = (-1)^N (\mathcal{L} \mathcal{G}_{\text{app}} - \text{Id})^{N+1} + \text{Id}.$$

Hence, as long as $\|\mathcal{L} \mathcal{G}_{\text{app}} - \text{Id}\| < 1$, we indeed have an inverse. But note that

$$\begin{aligned} \|(\mathcal{L} \mathcal{G}_{\text{app}} - \text{Id})f\|_{L^\infty([0, \infty))} &= \left\| \mathcal{L} \sum_{i \in I} (\mathcal{L}_{\text{app}}^i)^{-1} (\mathbb{1}_{[p_i, p_{i+1})} f) - \sum_{i \in I} \mathbb{1}_{[p_i, p_{i+1})} f \right\|_{L^\infty([0, \infty))} \\ &= \left\| \sum_{i \in I} (\mathcal{L} - \mathcal{L}_{\text{app}}) (\mathcal{L}_{\text{app}}^i)^{-1} (\mathbb{1}_{[p_i, p_{i+1})} f) \right\|_{L^\infty([0, \infty))} \\ &\lesssim o_{\varepsilon \rightarrow 0}(1) \|f\|_{L^\infty([0, \infty))}, \end{aligned}$$

and so for all ε small enough, our series converges and defines an inverse. When applying the above heuristic to the operator $\mathcal{L}$, we encounter the issue that $\mathcal{L}$ contains terms of order r^{-2} , which ensure that $|(\mathcal{L} - \mathcal{L}_{\text{app}})(\mathcal{L}_{\text{app}})^{-1}f|$ is always large near the origin, even for compactly supported f . To rectify this, we introduce the auxiliary space

$$Y = L^\infty((0, \infty), r^2 w_\varepsilon(r - r_0) dr),$$

which allows for quadratic divergences as $r \rightarrow 0$, and will be used to measure the size of the “error”. In particular, we will show that $\|\mathcal{L} \mathcal{G}_{\text{app}} - \text{Id}\|_{Y \rightarrow Y} = o_{\varepsilon \rightarrow 0}(1)$. In order for this inverse to be well defined, it must map X to itself, so we further show that $\mathcal{G}_{\text{app}}$ actually defines a bounded map $\mathcal{G}_{\text{app}} : Y \rightarrow X$.

With this in mind, we can now sketch a high level overview of the proof of Theorem 6.1.1.

1. We begin by fixing a curve

$$\Gamma \subset \{z \in \mathbb{C} \mid \text{Re}(z) > 0\}.$$

For $\lambda \in \Gamma$, we then construct a “glued approximate operator” $\mathcal{G}^\lambda$, satisfying

$$\|(\mathcal{L} - \lambda) \mathcal{G}^\lambda - \text{Id}\|_{Y \rightarrow Y} \lesssim o_{\varepsilon \rightarrow 0}(1),$$

uniformly for $\lambda \in \Gamma$. Crucially, $\mathcal{G}^\lambda$ is of the form

$$\mathcal{G}^\lambda f = \mathcal{G}_2^\lambda (f \mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]}) + \mathcal{G}_{\text{rem}} (f \mathbb{1}_{\mathbb{R} \setminus \mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]}),$$

where $\mathcal{G}_2^\lambda$ is the Green’s function of Gilbert’s approximate operator (6.3.8), and $\mathcal{G}_{\text{rem}}$ does

not depend on $\lambda \in \Gamma$.

2. Using $\mathcal{G}^\lambda$, we show that inverse of the operator $\mathcal{L} - \lambda$ can be written as the infinite series

$$(\mathcal{L} - \lambda)^{-1}f = \mathcal{G}^\lambda \sum_{n=0}^{\infty} (-1)^n \left((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id} \right)^n f,$$

uniformly for all $\lambda \in \Gamma$.

3. We define the Riesz projector (see Appendix A)

$$Pf = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \mathcal{L})^{-1} f d\lambda,$$

and compute $P\phi$, where ϕ is the approximate growing mode from Section 6.3.2. We show that

$$P\phi = -\frac{1}{2\pi i} \int_{\Gamma} \mathcal{G}^\lambda \phi d\lambda + \text{error}$$

where the error term is of order $o_{\varepsilon \rightarrow 0}(1)$.

4. Recalling the definition of $\mathcal{G}^\lambda$, it holds by Cauchy's theorem

$$-\frac{1}{2\pi i} \int_{\Gamma} \mathcal{G}^\lambda \phi d\lambda = -\frac{1}{2\pi i} \int_{\Gamma} \mathcal{G}_2^\lambda (\phi \mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]}) d\lambda,$$

and since ϕ is *by its very definition* an eigenfunction of Gilbert's operator (6.3.8), we conclude that $P\phi \neq 0$. Therefore, there exists $\lambda_0 \in \text{Int}(\Gamma)$, and a function $\psi \in X$ such that $\psi \in \ker(\mathcal{L} - \lambda_0)$, and so we have constructed a growing mode.

The majority of the chapter will be devoted to the careful construction of the operator $\mathcal{G}^\lambda$, since we shall need precise, quantitative bounds on it, in order to rigorously carry out the above programme.

Extension to bounded domains

The strategy for proving the existence of a growing mode we have described thus far inherently relies on the problem being posed on an unbounded domain. In particular, the property that $(\mathcal{L}_{\text{app}})^{-1}f$ decays exponentially outside of the support of f is evidently false in general if $(\mathcal{L}_{\text{app}})^{-1}$ is constructed to furthermore satisfy boundary conditions on some $\mathcal{J} \subset [0, \infty)$. Thus, our previous strategy needs to be amended in order to cover domains with boundary.

In particular, we begin by defining a new Banach space, denoted by $X_{\mathcal{J}}$, which consists of

$$X_{\mathcal{J}} = L^\infty(\mathcal{J}, \max\{1, r^2\} w_\varepsilon(r - r_0) dr),$$

and new domain

$$\mathcal{D}(\mathcal{L}_{\mathcal{J}}^{\text{dyn}}) = \{b \in X_{\mathcal{J}} \cap H_{\text{loc}}^2(\mathcal{J}) : \mathcal{L}^{\text{dyn}} b \in X, \quad b_r|_{\partial\mathcal{J}} = (rb_\theta)'|_{\partial\mathcal{J}} = 0\},$$

where we introduce the perfectly conducting walls boundary conditions (6.1.8), and we abuse notation setting $\partial\mathcal{J} = \{q\}$ in the case when $\mathcal{J} = [0, q]$.

The intuition we follow is that the growing mode we constructed is highly concentrated around the critical layer r_0 , i.e. it decays rapidly as the boundaries of the domain are approached. Thus, on an intuitive level, the boundary conditions are “almost” satisfied, and adding a small corrective term may in fact be enough to obtain a growing mode that satisfies the perfectly conducting boundary conditions. Guided by this heuristic, we prove that the inverse operator $(\mathcal{L} - \lambda)_{\text{pc}}^{-1}$ with perfectly conducting boundary conditions can be written as the full inverse $(\mathcal{L} - \lambda)^{-1}|_{\mathcal{J}}$ on $[0, \infty)$, restricted to the interval $\mathcal{J}$, plus corrective terms, which consist of appropriately scaled homogeneous solutions of the equation $(\mathcal{L} - \lambda)V = 0$ on $\mathcal{J}$, ensuring that the boundary conditions are satisfied. In this way, we write

$$(\mathcal{L} - \lambda)_{\text{pc}}^{-1} f = \mathcal{G}^\lambda \sum_{n \geq 0} (-1)^n \left((\mathcal{L} - \lambda) \mathcal{G}^\lambda - \text{Id} \right)^n f + \text{boundary terms},$$

where $\mathcal{G}^\lambda$ is precisely the approximate inverse we constructed for the case $[0, \infty)$, and the boundary terms depend *solely* on the values of $(\mathcal{L} - \lambda)^{-1} f$ at the boundary of the domain. Recalling that the growing mode we constructed for $[0, \infty)$ decays rapidly away from r_0 , when $f = \phi$ is the approximate growing mode from Section 6.3.2 we deduce that the boundary terms may indeed be treated as an error. Thus, we once again write

$$P\phi = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \mathcal{L})^{-1} \phi d\lambda = -\frac{1}{2\pi i} \int_{\Gamma} \mathcal{G}^\lambda \phi d\lambda + \text{error} \neq 0,$$

and deduce the existence of a growing mode, satisfying the perfectly conducting boundary conditions (6.1.8).

6.2.2 Design of the approximate operators

A key point of our proof is the design of the approximated operator $\mathcal{L}_{\text{app}}$. Whilst of course there are some case-by-case subtleties, as a general rule of thumb, we construct the approximate operator as follows.

We determine the “dominant” terms of the operator $\mathcal{L}$ in any spatial region by keeping track of two parameters: the magnetic diffusivity ε , and the radius r . Combined, one may associate to every term in the equation defining $\mathcal{L}$ a fairly crude “relative size” as $\varepsilon \rightarrow 0$, depending on the spatial region. For instance, in the expression $\varepsilon^{1/3} r^{-2} - \varepsilon^{-1/3} r^2$, the dominant term for $r \ll \varepsilon^{1/6}$ is $\varepsilon^{1/3} r^{-2}$, whereas for $r \gg \varepsilon^{1/6}$ it is $-\varepsilon^{-1/3} r^2$. Whilst this approach is crude, due to the perturbative nature of the problem it actually yields remarkable accurate approximate operators throughout overlapping spatial domains, which have relatively simple inverse operators. In fact, our hypotheses H0-H2 will ensure that there is only one subset of $[0, \infty)$ (in the example $\varepsilon^{1/3} r^{-2} - \varepsilon^{-1/3} r^2$ one might think of the “transition point” $r = \varepsilon^{1/6}$), where this naive “order of magnitude analysis” does not give a simple approximate operator, and this will be precisely where we expect the growing mode to localise. Fortunately, in this critical regime, we can make use of Gilbert’s approximation scheme (see Section 6.3.2) in order to derive careful local inverses, for values of the spectral parameter very close to a spectral point of the approximate operator. In our previous notation, this yields the operator $\mathcal{G}_2^\lambda$, and it will allow us to close our argument as explained at in Section 6.2.1.

We start our analysis in Section 6.6 by considering the case $\mathcal{I} = [0, \infty)$ and a simplified scenario for the transport function $T(r)$, that is first assumed to have one single zero at $r = r_0$. In this setting, we show that there exists $\gamma > 0$ such that the domain $[0, \infty)$ can be split as follows.

- *Towards infinity:* $\text{supp}(f) \subset [r_0 + \varepsilon^\gamma, \infty)$. In this region, the operator $\mathcal{L}$ is well approximated by a semiclassical Schrödinger operator with complex, non-vanishing potential. In particular, it will allow us to be very rough in our construction of the approximate inverse, denoted by $\mathcal{G}_3$, which is obtained by freezing the value of the potential $T(r)$ on the boundaries of a very fine partition of $[r_0 + \varepsilon^\gamma, \infty)$, and gluing together the resulting approximate inverses. The resulting map $\mathcal{G}_3$ is independent of λ . Further details about the precise definition and the estimates can be found in Section 6.6.2.
- *Towards zero:* $\text{supp}(f) \subset [\varepsilon^\gamma, r_0 - \varepsilon^\gamma)$. Once again, in this region there is a clear “leading order operator”, which again is a semiclassical Schrödinger operator with complex, non-vanishing potential. Thus, we proceed entirely as in the case towards infinity, taking care this time of the asymptotics as we approach $r = 0$. We denote the Green’s function associated to this regime by $\mathcal{G}_1$, and it is again defined independently of λ . Details about this regime can be found in Section 6.6.3.
- *Near zero:* $\text{supp}(f) \subset [0, \varepsilon^\gamma)$. This regime presents the first part of the domain where additional care must be taken, since our approximate operator will involve terms that diverge quadratically as $r \rightarrow 0$. It is however of crucial importance, since the resulting Green’s function will turn out to have a *regularising effect*, mapping functions that diverge quadratically at $r = 0$ to bounded functions at $r = 0$. We denote the Green’s function associated to this regime by $\mathcal{G}_0$, and we make it such that it does not depend on λ . The approximation procedure yields an ODE that will be solved using modified Bessel functions. The analysis of this part is covered in Section 6.6.4.
- *Around the critical radius:* $\text{supp}(f) \subset [r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma)$. This construction is the crux of the argument, and requires detailed analysis of parabolic cylinder functions near spectral values of the Hermite equation. We denote the Green’s function in this interval by $\mathcal{G}_2^\lambda$, making this the only regime where the approximated operator depends on λ . The approximation procedure will follow Gilbert’s ideas presented in Section 6.3.2. Further details on the definition and precise estimates on $\mathcal{G}_2^\lambda$ can be found in Section 6.6.1.

The form of the approximate Green’s function $\mathcal{G}^\lambda$ in the situation of Section 6.6 will be given by

$$\mathcal{G}^\lambda f = \mathcal{G}_0(f \mathbb{1}_{[0, \varepsilon^\gamma)}) + \mathcal{G}_1(f \mathbb{1}_{[\varepsilon^\gamma, r_0 - \varepsilon^\gamma)}) + \mathcal{G}_2^\lambda(f \mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma)}) + \mathcal{G}_3(f \mathbb{1}_{[r_0 + \varepsilon^\gamma, \infty)}),$$

where $\mathbb{1}_A$ denotes the indicator function of the set A : $\mathbb{1}_A(r) = 1$ if $r \in A$, and $\mathbb{1}_A(r) = 0$ otherwise.

In Section 6.6.5, we extend the arguments from the previous Sections 6.6.1, 6.6.2, 6.6.3, 6.6.4 to the case where the transport function vanishes linearly at a finite set of points, denoted by $s_j > 0$. We study the problem locally around each s_j , specifically in the interval $[s_j - \varepsilon^\omega, s_j + \varepsilon^\omega)$ for some parameter $\omega > 0$. We denote each corresponding approximated Green’s function by $\mathcal{G}_j^{\text{LV}}$, and we solve the problem using Airy functions. All in all, we obtain an approximate inverse operator that

can be iterated to obtain a well-defined inverse for $\mathcal{L} - \lambda$ on $[0, \infty)$.

In particular, this inverse will also be key in allowing us to construct homogeneous solutions to $(\mathcal{L} - \lambda)V = 0$ on intervals $\mathcal{I} \subset [0, \infty)$, by setting (for instance) $V = (\mathcal{L} - \lambda)^{-1}(\mathbb{1}_{[0, \infty) \setminus \mathcal{I}})$. By definition, it holds $(\mathcal{L} - \lambda)V = 0$ on $\mathcal{I}$, and via a careful analysis of the iterative procedure defining V , it can be shown that V is not identically zero on $\mathcal{I}$.

When constructing resolvents for our problem on bounded domains with perfectly conducting boundary conditions, this property will be crucial. Indeed, from the point of view of the ODE, it is clear that if one may find n linearly independent homogeneous solutions to an ODE with n boundary conditions, that are furthermore “linearly independent at the boundary” (in the sense that the matrix consisting of their values at the boundary is invertible), the ODE must be invertible, simply by adding suitable scaled homogeneous solutions to any particular solution to the ODE in order to “fix” the boundary conditions. In the case of the Ponomarenko dynamo, when $\mathcal{I} = [p, q]$ there will generically be four boundary conditions that need to be satisfied. Thus, we endeavour to construct four linearly independent solutions to $(\mathcal{L} - \lambda)v_i = 0$ on $[p, q]$, $i = 1, 2, 3, 4$. The idea will be to set for instance

$$v_1 = (\mathcal{L} - \lambda)^{-1} \begin{pmatrix} \mathbb{1}_{[p-\varepsilon^\delta, p]} \\ 0 \end{pmatrix}$$

for δ small. Certainly $(\mathcal{L} - \lambda)v_1 = 0$ in (p, q) , and by continuity, we can hope to show that for $r \in [p, q]$, r close to p , $v_1(r) \neq 0$, so it is not the trivial solution. Furthermore, we can hope that the inverse operator $(\mathcal{L} - \lambda)^{-1}$ we construct inherits the exponential decay property of its building blocks, namely that $(\mathcal{L} - \lambda)^{-1}v(r)$ decays exponentially away from the support of v .— In particular, this would imply that $|v_1(q)| \ll 1$. Setting thus

$$v_2(r) = (\mathcal{L} - \lambda)^{-1} \begin{pmatrix} \mathbb{1}_{[q, q+\varepsilon^\delta]} \\ 0 \end{pmatrix}$$

and repeating the argument, we deduce $v_2(q) = O(1)$, whereas $v_2(p) \sim 0$. Hence, we deduce that v_1, v_2 are linearly independent. Similarly, we can define v_3, v_4 by localising the initial condition in the second component, i.e.

$$v_3 = (\mathcal{L} - \lambda)^{-1} \begin{pmatrix} 0 \\ \mathbb{1}_{[p-\varepsilon^\delta, p]} \end{pmatrix}$$

and once again showing that the corresponding solutions are small (respectively large) at the end-points. In fact, in Section 6.7 we undertake a detailed dynamical analysis of the iterative procedure defining $(\mathcal{L} - \lambda)^{-1}$, studying the propagation of mass by each term of the series defining $(\mathcal{L} - \lambda)^{-1}$, and conclude that not only do v_i $i = 1, 2, 3, 4$ define linearly independent solutions to $(\mathcal{L} - \lambda)v_i = 0$, but they also satisfy suitable estimates as $\varepsilon \rightarrow 0$, allowing them to be treated perturbatively as $\varepsilon \rightarrow 0$ in the construction of the inverse problem $(\mathcal{L} - \lambda)b = v$ on $[p, q]$, subject to perfectly conducting boundary conditions.

6.2.3 Comparison with existing literature

The (discontinuous) Ponomarenko dynamo was first introduced in [Pon73], and it was subsequently shown to be a fast dynamo in [Gil88]. In that paper, Gilbert goes through the argument sketched in Section 6.3.2 in order to deduce the existence of a growing mode for the smooth Ponomarenko dynamo. This is further expanded upon in the more recent paper [PGP08]. As mentioned in the introduction, the only rigorous result on the Ponomarenko dynamo we are aware of is [GR07], which proves nonlinear instability for the full MHD equations perturbed around Ponomarenko-like velocity fields. The authors consider a sequence of velocity fields u^v that formally converge to the discontinuous Ponomarenko dynamo (6.1.4), and employ the method of matched asymptotic expansions to show that initial data of order v^p for $p > 0$ and $v \ll 1$ can grow to size $O(1)$ in time $O(|\log(v)|)$. This result demonstrates the sort of nonlinear instability that is conjectured to occur as a by-product of the dynamo effect. However, since nonlinear instability results are inherently concerned with short time growth, to the best of the authors' knowledge, it is not possible to directly deduce linear spectral instability from either the results or the methods of [GR07]. Indeed, the method of matched asymptotic expansions yields an exponentially growing first order term, which is dominant for times of order $|\log(v)|$, but for times larger than that, higher order terms may dominate, whose behaviours are quite difficult to discern.

Deducing linear spectral instability for ODEs is itself a rich field, but we shall draw special attention to the paper [GGN16b], which served as an inspiration for some of our techniques. In this chapter, the authors prove spectral instability for the linearised Navier–Stokes equations around a wide class of shear flows, via an analysis of the Orr–Sommerfeld equations. Such analysis carries with it the inherent challenge that the Orr–Sommerfeld instability problem is not a spectral problem in the traditional sense—the spectral parameter enters multiplicatively in the equation. Thus, there are considerably fewer abstract tools available to study such a problem. In [GGN16b] the authors circumvent this issue via an ingenious approach, involving the construction of two homogeneous solutions to the equation using carefully constructed approximate inverses for the operator. By means of studying the dispersion relation of their homogeneous solutions, they are able to construct a growing mode, which satisfies the desired boundary conditions, whilst giving detailed estimates on the spectral parameter in the process. Such an approach is particularly well suited for a boundary layer problem, in which one expects the instability to arise *as a result* of the boundary. Constructing an approximate inverse to the equation which is finite both at the origin and at infinity requires that the ODE is invertible on $[0, \infty)$ provided one does not impose any boundary conditions at 0.

It is precisely this property that fails for our construction—homogeneous solutions to the Ponomarenko dynamo equations are either finite or blow up near zero, and finding a solution that is well behaved both at the origin and at infinity is the entire point of our endeavour. Therefore, the strategy of constructing a homogeneous solution which is well-behaved at infinity, and then using continuity arguments to show that this solution must also be well-behaved at zero for some value of the spectral parameter does not seem to work in our situation. Crucially however, since our problem is indeed a traditional spectral problem, we can fall back to powerful results from classical spectral theory, and deduce the existence of a growing mode simply by finding approximate

inverse operators on a circle enclosing the predicted location of the eigenvalue, and then estimating the resulting construction with care. In particular, we believe our methodology may be applicable in some generality for deducing rigorous spectral instability results from formal asymptotic expansions, provided the problem is posed as an eigenvalue problem. It may be possible to adapt our methods to study further dynamo problems, such as for instance the “almost-fast” dynamo on $\mathbb{T}^3$ constructed in [Sow87].

6.3 Analysis of the equations and admissible vector fields

In this section we explain how the equations under consideration are derived. We begin with a simplified model to motivate the scaling in ε from Theorem 6.1.1 for λ , m and k . We then give an overview on Gilbert’s asymptotic analysis, which leads to a change that is crucial the rest of the analysis presented in this chapter.

6.3.1 A simplified model

Before delving into the Ponomarenko dynamo equations in more detail, it is useful to get some feel for how the construction of the dynamo occurs from a heuristic point of view. So far it is unclear how the critical radius $r_0 > 0$ enters into the equations, as well as what scaling of the Fourier modes $m, k \in \mathbb{Z}$ yields the largest growth rate. We will try to answer these questions in this section, by means of a formal order of magnitude analysis on a simplified model of the dynamo equations, which discards the terms of lesser relevance to the dynamics.

Indeed, consider the system

$$\begin{aligned}\partial_t b_r &= \varepsilon \left(\partial_r^2 b_r - \frac{m^2}{r^2} b_r - \frac{2im}{r^2} b_\theta \right) - i(m\Omega(r) + kU(r))b_r \\ \partial_t b_\theta &= \varepsilon \left(\partial_r^2 b_\theta - \frac{m^2}{r^2} b_\theta + \frac{2im}{r^2} b_r \right) - i(m\Omega(r) + kU(r))b_\theta + r\Omega'(r)b_r.\end{aligned}$$

We split the equation into an “unstable” stretching term, given by

$$\partial_t \begin{pmatrix} b_r \\ b_\theta \end{pmatrix} = -\varepsilon m r^{-2} \begin{pmatrix} b_r \\ b_\theta \end{pmatrix} + \varepsilon \begin{pmatrix} -2imr^{-2}b_\theta \\ 2imr^{-2}b_r \end{pmatrix} + \begin{pmatrix} 0 \\ r\Omega'(r)b_r \end{pmatrix},$$

and a stable term, which is simply the diagonal advection-diffusion equation with velocity field $m\Omega(r) + kU(r)$. Since the advection-diffusion component is formally negative, any growth that occurs must be a result of the unstable term. In addition, since the unstable term does not involve any derivatives in r , we can diagonalize the equation. A computation yields that for $r \gtrsim 1$, the associated diagonal system is well described by the matrix

$$\partial_t \begin{pmatrix} b_r \\ b_\theta \end{pmatrix} = \begin{pmatrix} -\varepsilon m^2 + i\sqrt{-\varepsilon m(\varepsilon m - i)} & 0 \\ 0 & -\varepsilon m^2 - i\sqrt{-\varepsilon m(\varepsilon m - i)} \end{pmatrix} \begin{pmatrix} b_r \\ b_\theta \end{pmatrix}.$$

In other words, the growth rate resulting from the stretching term is of order

$$-\varepsilon m^2 + \sqrt{\varepsilon m} \sim \sqrt{m\varepsilon} \quad \text{as } \varepsilon \rightarrow 0. \quad (6.3.1)$$

It thus remains to show that this growth rate formally dominates the decay from the advection diffusion equation. From classical results in enhanced dissipation, see e.g. [BC17; ABN22; Wei21; Vil25; GLM24], we recall that so long as $m\Omega'(r_0) + kU'(r_0) \neq 0$, $m, k = O_{\varepsilon \rightarrow 0}(1)$, the equation

$$\partial_r b_r = \varepsilon \partial_r^2 b_r + i(m\Omega(r) + kU(r))b_r$$

experiences *enhanced dissipation* of order $\varepsilon^{1/3}$ for functions localised near $r = r_0$. Hence, near such a point, we can expect a competition between two mechanisms in the dynamo equations: a local decay of order $\varepsilon^{1/3}$ from the advection-diffusion equations, and growth of order $\varepsilon^{1/2}$ from the unstable terms. Since $\varepsilon^{1/3} \gg \varepsilon^{1/2}$ for ε small, the decay will be the dominating force near r_0 so long as $m\Omega'(r_0) + kU'(r_0) \neq 0$, and thus no dynamo action can take place. Hence, it will be crucial to designate a so called “critical radius” $r_0 > 0$, determined by the Fourier modes m, k , at which the azimuthal and vertical shears are aligned in such a way that $m\Omega'(r_0) + kU'(r_0) = 0$. In this case, the heuristic enhanced dissipation argument yields local decay of order $\varepsilon^{1/2}$, which is precisely the same order as the stretching from the unstable term. In fact, recall from (6.3.1) that the heuristic growth rate from the stretching term is $-\varepsilon m^2 + \sqrt{\varepsilon m}$, which is maximized at order $\varepsilon^{1/3}$ when $m = O(\varepsilon^{-1/3})$. Therefore, the same argument yields that, so long as $m, k = O(\varepsilon^{-1/3})$, the stretching and diffusion terms are balanced at order $\varepsilon^{1/3}$ as $\varepsilon \rightarrow 0$, and therefore there is potential for exponential growth at rate $\varepsilon^{1/3}$ to occur near such a critical radius r_0 . The precise relative signs of the stretching and diffusion terms will then be determined by the local geometry of the flow, for which a more subtle analysis is needed.

6.3.2 Gilbert’s asymptotic approach

Having provided a heuristic towards the existence of a critical radius r_0 where growth takes place, as well as towards the fact that the optimal growth rate of order $\varepsilon^{1/3}$ is achieved when $m, k \sim \varepsilon^{-1/3}$, we now recall the main ideas from Gilbert’s asymptotic analysis [Gil88], that give conditions for the stretching term to be locally stronger than the diffusion term. These ideas, suitably adapted, will guide the construction of the desired growing mode for the full problem. In particular, they yield a convenient change of variables, which we examine in detail in Section 6.3.3.

From the exposition in the previous section, we assume that the growing mode will asymptotically concentrate as $\varepsilon \rightarrow 0$ around a critical radius $r_0 > 0$, that will be formally treated as a *boundary layer*. We take the Fourier transform $\theta \mapsto m = \varepsilon^{-1/3}M$, $z \mapsto k = \varepsilon^{-1/3}K$, where $M, K = O_{\varepsilon \rightarrow 0}(1)$, and we further make the modal Ansatz $B(t, r, \theta, z) = e^{\varepsilon^{1/3}\mu t} e^{i\varepsilon^{-1/3}(M\theta + Kz)}$. Thus, the (r, θ) compo-

nents of the kinematic dynamo equations become

$$\begin{aligned} & \varepsilon^{1/3} \mu b_r + i \varepsilon^{-1/3} (M \Omega(r) + K U(r)) b_r \\ &= \varepsilon \left(\partial_r^2 b_r + \frac{1}{r} \partial_r b_r - \varepsilon^{2/3} \frac{M^2}{r^2} b_r - \varepsilon^{2/3} K^2 b_r - \frac{1}{r^2} b_r - \varepsilon^{1/3} \frac{2iM}{r^2} b_\theta \right), \end{aligned} \quad (6.3.2)$$

$$\begin{aligned} & \varepsilon^{1/3} \mu b_\theta + i \varepsilon^{-1/3} (M \Omega(r) + K U(r)) b_\theta \\ &= \varepsilon \left(\partial_r^2 b_\theta + \frac{1}{r} \partial_r b_\theta - \varepsilon^{2/3} \frac{M^2}{r^2} b_\theta - \varepsilon^{2/3} K^2 b_\theta - \frac{1}{r^2} b_\theta + \varepsilon^{1/3} \frac{2iM}{r^2} b_r \right) + r \Omega'(r) b_r. \end{aligned} \quad (6.3.3)$$

Since the dynamo growth rate is entirely determined by $\text{Re}(\lambda) = \text{Re}(\varepsilon^{1/3} \mu)$, we may replace

$$\lambda \mapsto \lambda - \varepsilon^{-1/3} i (M \Omega(r_0) + K U(r_0))$$

without altering the growth rate. Now, in order for growth to occur near r_0 , by Section 6.3.1 we need to impose that the transport term has vanishing derivative at r_0 . Therefore, we assume that

$$M \Omega'(r_0) + K U'(r_0) = 0. \quad (6.3.4)$$

We now Taylor expand the equation around r_0 , making the change of variables $s = \varepsilon^{-1/3}(r - r_0)$. Making the perturbation series Ansatz

$$b_r = \sum_{n \geq 0} \varepsilon^{\frac{n}{3}} b_r^{(n)}, \quad \sum_{n \geq 0} \varepsilon^{\frac{n}{3}} b_\theta^{(n)},$$

it follows that the only term of order 1 in (6.3.2), (6.3.3) is the term from the θ equation given by

$$r_0 \Omega'(r_0) b_r^{(0)} = 0,$$

from which we deduce $b_r^{(0)} = 0$. Next, the $O(\varepsilon^{1/3})$ terms are

$$(\partial_s^2 - M^2 r_0^{-2} - K^2) b_\theta^{(0)} + r_0 \Omega'(r_0) b_r^{(1)} = \lambda b_\theta^{(0)} + i c_2 s^2 b_\theta^{(0)},$$

where

$$c_2 = \frac{i}{2} (M \Omega''(r_0) + K U''(r_0)). \quad (6.3.5)$$

We also find that the $O(\varepsilon^{2/3})$ terms from (6.3.2) are given by

$$\partial_s^2 b_r^{(1)} - M^2 r_0^{-2} b_r^{(1)} - K^2 b_r^{(1)} - 2iM r_0^{-2} b_\theta^{(0)} = \lambda b_r^{(1)} + i c_2 s^2 b_r^{(1)}.$$

We therefore arrive at the closed system of ODEs

$$\begin{aligned} \partial_s^2 b_r^{(1)} - M^2 r_0^{-2} b_r^{(1)} - K^2 b_r^{(1)} - 2iM r_0^{-2} b_\theta^{(0)} &= \lambda b_r^{(1)} + i c_2 s^2 b_r^{(1)} \\ (\partial_s^2 - M^2 r_0^{-2} - K^2) b_\theta^{(0)} + r_0 \Omega'(r_0) b_r^{(1)} &= \lambda b_\theta^{(0)} + i c_2 s^2 b_\theta^{(0)}. \end{aligned}$$

Now, we make the change of variables

$$b_r = \alpha \varepsilon^{1/3} (V_2 - V_1), \quad b_\theta = V_1 + V_2, \quad (6.3.6)$$

with

$$\alpha^2 = -\frac{2iM}{r_0^3 \Omega'(r_0)}. \quad (6.3.7)$$

Furthermore, we set $\lambda = \varepsilon^{1/3} \mu$, with $\mu = O(1)$ as $\varepsilon \rightarrow 0$. We thus see that V_1, V_2 satisfy the ordinary differential equations

$$\begin{aligned} \varepsilon \partial_r^2 V_1 - c_2 \varepsilon^{-1/3} (r - r_0)^2 V_1 - \varepsilon^{1/3} \left(M^2 \left[\frac{1}{r_0^2} + \left(\frac{\Omega'(r_0)}{U'(r_0)} \right)^2 \right] + \mu - \sqrt{\frac{-2iM\Omega'(r_0)}{r_0}} \right) V_1 &= 0, \\ \varepsilon \partial_r^2 V_2 - c_2 \varepsilon^{-1/3} (r - r_0)^2 V_2 - \varepsilon^{1/3} \left(M^2 \left[\frac{1}{r_0^2} + \left(\frac{\Omega'(r_0)}{U'(r_0)} \right)^2 \right] + \mu + \sqrt{\frac{-2iM\Omega'(r_0)}{r_0}} \right) V_2 &= 0. \end{aligned} \quad (6.3.8)$$

We make the change of variables $z = \zeta s$, where recall $s = \varepsilon^{-1/3} (r - r_0)$, with $\zeta = \sqrt{2} c_2^{1/4}$ so that

$$V_1''(z) + \left(-\frac{1}{2} c_2^{-1/2} q(\mu) - \frac{1}{4} z^2 \right) V_1(z) = 0, \quad (6.3.9)$$

and where we set

$$q(\mu) = M^2 \left[\frac{1}{r_0^2} + \left(\frac{\Omega'(r_0)}{U'(r_0)} \right)^2 \right] + \mu - \sqrt{\frac{-2iM\Omega'(r_0)}{r_0}}.$$

Analogously, we define $q(\mu)$ with a $+$ sign in front of the last addend in the equation for V_2 , and we choose the root that satisfies $\text{Re}(\sqrt{-iM\Omega'(r_0)}) > 0$. In this way we see that the corresponding eigenvalue problem (6.3.9) for the second component produces an eigenvalue with a real part strictly larger than for the first component, and therefore we can focus on the analysis of the first component. Solutions to equations of the form (6.3.9) are called *parabolic cylinder functions*—see Appendix 6.8.1 for more details, and they have finite energy on $(-\infty, \infty)$ if and only if

$$\frac{1}{2} c_2^{-1/2} q(\mu) = -j - \frac{1}{2}, \quad j = 0, 1, 2, \dots \quad (6.3.10)$$

Therefore, solving for μ so that (6.3.10) holds, we see indeed that

$$\text{Re}(\mu) = \sqrt{\frac{|M||\Omega'(r_0)|}{r_0}} - M^2 \left(\frac{1}{r_0^2} + \frac{|\Omega'(r_0)|^2}{|U'(r_0)|^2} \right) - \sqrt{2|c_2|} \left(j + \frac{1}{2} \right).$$

To maximize the growth rate, we pick $j = 0$. Moreover, using (6.3.5) and making the appropriate choices of complex roots we obtain,

$$\text{Re}(\mu) = |M|^{1/2} \left(\frac{|\Omega'(r_0)|^{1/2}}{r_0^{1/2}} - \frac{1}{2} \left| \Omega''(r_0) - \frac{\Omega'(r_0)}{U'(r_0)} U''(r_0) \right|^{1/2} \right) - |M|^2 \left(\frac{1}{r_0^2} + \frac{|\Omega'(r_0)|^2}{|U'(r_0)|^2} \right).$$

Hence via (6.3.4) and rewriting $\lambda = \mu \varepsilon^{1/3}$, $m = M \varepsilon^{-1/3}$, $k = K \varepsilon^{-1/3}$, which is precisely the growth rate of our Theorem 6.1.1. Moreover, this shows that upon letting the absolute value of

$M \in \mathbb{R}$ be sufficiently small, we find that $\text{Re}(\mu) > 0$ provided that

$$r_0 \left| \frac{d}{dr} \log \left| \frac{\Omega'(r_0)}{U'(r_0)} \right| \right| < 4. \quad (6.3.11)$$

Hence, we find that locally near r_0 , the stretching term is stronger than the enhanced dissipation, precisely if (6.3.11) is satisfied.

Remark 6.3.1. From our presentation, it may seem that the condition (6.3.4) determines the Fourier modes m, k given the critical radius r_0 , making the value r_0 appear as a free parameter. However, from the point of view of the full dynamo PDEs, it is more apt to view the condition (6.3.4) as the Fourier modes *determining* the critical radius r_0 —In other words, upon picking Fourier modes m, k , the condition (6.3.4) determines the localisation of the growing mode for the Ponomarenko dynamo equations.

6.3.3 Reformulation of the kinematic dynamo equations

The main goal of this chapter is to bridge the gap from this formal analysis to a fully rigorous description of the growing modes of the Ponomarenko dynamo. Whilst the analysis of [Gil88] is very precise in a small neighbourhood of r_0 , for the existence of a genuine eigenfunction of the kinematic dynamo equations, we need to extend the analysis to a global one. To do so, we shall begin by bringing the equations (6.1.5), (6.1.6) into a form more amenable for analysis.

We fix a positive radius $r_0 > 0$, and write $\Omega_0 = \Omega(r_0)$ and $U_0 = U(r_0)$. Furthermore, we write Ω'_0 , Ω''_0 to denote $\Omega'(r_0)$, $\Omega''(r_0)$, and analogously with U . Following Gilbert's analysis, we make the following Ansatz:

- $\lambda = \mu \varepsilon^{1/3}$, where $\mu \in \mathbb{C}$, and $\varepsilon > 0$ denotes the magnetic diffusivity from (6.1.1).
- $m = M \varepsilon^{-1/3}$, and $k = K \varepsilon^{-1/3}$, with $M, K \in \mathbb{R}$. Moreover, we choose the frequencies such that Gilbert's formula (6.3.4) is satisfied,

$$M \Omega'_0 + K U'_0 = 0.$$

Under this choice of parameters λ , m and k , we can rewrite the equations corresponding to the radial and azimuthal components of the magnetic field (6.1.5)–(6.1.6), which take the form,

$$\begin{aligned} \mu \varepsilon^{1/3} b_r + i \varepsilon^{-1/3} M \left(\Omega - \frac{\Omega'_0}{U'_0} U \right) b_r \\ = \varepsilon \left(\partial_r^2 b_r + \frac{1}{r} \partial_r b_r \right) - \varepsilon^{1/3} M^2 \left(\frac{1}{r^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right) b_r - \frac{\varepsilon}{r^2} b_r - \frac{2i \varepsilon^{2/3} M}{r^2} b_\theta, \\ \mu \varepsilon^{1/3} b_\theta + i \varepsilon^{-1/3} M \left(\Omega - \frac{\Omega'_0}{U'_0} U \right) b_\theta \\ = \varepsilon \left(\partial_r^2 b_\theta + \frac{1}{r} \partial_r b_\theta \right) - \varepsilon^{1/3} M^2 \left(\frac{1}{r^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right) b_\theta - \frac{\varepsilon}{r^2} b_\theta + \frac{2i \varepsilon^{2/3} M}{r^2} b_r + r \Omega' b_r. \end{aligned}$$

As mentioned in the previous sections, the z component is decoupled from the r and θ compo-

nents. The bulk of our efforts will go towards finding a solution to this coupled system of ODEs, where $\mu \in \mathbb{C}$ satisfies $\text{Re}(\mu) > 0$. After we solve this problem, in Section 6.8 we will address the behaviour of the vertical component.

We want to write the equation in a more convenient form, following the ideas exposed in Section 6.3.2. In order to do so, we subtract

$$i\varepsilon^{-1/3}M\left(\Omega_0 - \frac{\Omega'_0}{U'_0}U_0\right)b_r \quad \text{and} \quad i\varepsilon^{-1/3}M\left(\Omega_0 - \frac{\Omega'_0}{U'_0}U_0\right)b_\theta$$

from the left hand side in the first and second equations respectively. In this manner, we only modify the imaginary part of the eigenvalue $\mu\varepsilon^{1/3}$, leaving the growth rate invariant. Taking this into account, we find the following system of equations,

$$\begin{aligned} \mu\varepsilon^{1/3}b_r + i\varepsilon^{-1/3}M\left((\Omega - \Omega_0) - \frac{\Omega'_0}{U'_0}(U - U_0)\right)b_r \\ = \varepsilon\left(\partial_r^2 b_r + \frac{1}{r}\partial_r b_r\right) - \varepsilon^{1/3}M^2\left(\frac{1}{r^2} + \left(\frac{\Omega'_0}{U'_0}\right)^2\right)b_r - \frac{\varepsilon}{r^2}b_r - \frac{2i\varepsilon^{2/3}M}{r^2}b_\theta, \end{aligned} \quad (6.3.12)$$

$$\begin{aligned} \mu\varepsilon^{1/3}b_\theta + i\varepsilon^{-1/3}M\left((\Omega - \Omega_0) - \frac{\Omega'_0}{U'_0}(U - U_0)\right)b_\theta \\ = \varepsilon\left(\partial_r^2 b_\theta + \frac{1}{r}\partial_r b_\theta\right) - \varepsilon^{1/3}M^2\left(\frac{1}{r^2} + \left(\frac{\Omega'_0}{U'_0}\right)^2\right)b_\theta - \frac{\varepsilon}{r^2}b_\theta + \frac{2i\varepsilon^{2/3}M}{r^2}b_r + r\Omega'b_r. \end{aligned} \quad (6.3.13)$$

The addition of these purely imaginary terms yields a “transport term” of the form

$$T(r) = \left((\Omega(r) - \Omega_0) - \frac{\Omega'_0}{U'_0}(U(r) - U_0)\right),$$

that crucially vanishes up to second order as $r \rightarrow r_0$. In order to make the notation cleaner, we point out that equations (6.3.12)–(6.3.13) can be rewritten in contracted form as

$$\begin{aligned} \varepsilon\left(\partial_r^2 b_r + \frac{1}{r}\partial_r b_r\right) &= \lambda b_r + A_\varepsilon b_r + B_\varepsilon b_\theta, \\ \varepsilon\left(\partial_r^2 b_\theta + \frac{1}{r}\partial_r b_\theta\right) &= \lambda b_\theta + A_\varepsilon b_\theta + C_\varepsilon b_r, \end{aligned}$$

where $\lambda = \mu\varepsilon^{1/3}$, and with

$$\begin{aligned} A_\varepsilon &= \frac{\varepsilon}{r^2} + \varepsilon^{1/3}M^2\left(\frac{1}{r^2} + \left(\frac{\Omega'_0}{U'_0}\right)^2\right) + i\varepsilon^{-1/3}MT(r), \\ B_\varepsilon &= \frac{2iM\varepsilon^{2/3}}{r^2}, \quad C_\varepsilon = -r\Omega' - \frac{2iM\varepsilon^{2/3}}{r^2}. \end{aligned}$$

Before analysing the structure of solutions to this system of ODEs, we perform one final change of variables, following the ideas outlined in Section 6.3.2. This transformation—originally introduced implicitly in Gilbert’s paper [Gil88]—has the advantage of yielding a symmetric scaling in

ε across both components of the equations. As in (6.3.6) we set $(b_r, b_\theta) \mapsto (V_1, V_2)$, defined by

$$b_r = \alpha \varepsilon^{1/3} (V_2 - V_1), \quad b_\theta = V_1 + V_2,$$

or equivalently

$$V_1 = \frac{1}{2} \left(b_\theta - \frac{1}{\alpha} \varepsilon^{-1/3} b_r \right), \quad V_2 = \frac{1}{2} \left(b_\theta + \frac{1}{\alpha} \varepsilon^{-1/3} b_r \right),$$

where $\alpha \in \mathbb{C}$ is given by (6.3.7). After the change of variables, the equations that V_1 and V_2 satisfy have the form

$$\begin{aligned} \varepsilon \left(\partial_r^2 V_1 + \frac{1}{r} \partial_r V_1 \right) &= \lambda V_1 + A_\varepsilon V_1 - \frac{1}{2} \left(\frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon + \alpha \varepsilon^{1/3} C_\varepsilon \right) V_1 - \frac{1}{2} \left(\frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon - \alpha \varepsilon^{1/3} C_\varepsilon \right) V_2, \\ \varepsilon \left(\partial_r^2 V_2 + \frac{1}{r} \partial_r V_2 \right) &= \lambda V_2 + A_\varepsilon V_2 + \frac{1}{2} \left(\frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon + \alpha \varepsilon^{1/3} C_\varepsilon \right) V_2 + \frac{1}{2} \left(\frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon - \alpha \varepsilon^{1/3} C_\varepsilon \right) V_1. \end{aligned}$$

One can see from here that the cross terms have the same scaling in ε in both components. The above ODE defines a linear operator, which we denote (up to the inclusion of the term $\mu \varepsilon^{1/3}$) by $\mathcal{L}$. In other words, the above ODEs may be written as

$$(\mathcal{L} - \mu \varepsilon^{1/3}) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0.$$

Analysing the precise form of the factors in $\mathcal{L}$ more carefully, we find a useful structure in the cross factors. Notice that we can compute,

$$\begin{aligned} \frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon - \alpha \varepsilon^{1/3} C_\varepsilon &= \frac{2iM\varepsilon^{1/3}}{r^2} \left(\frac{1}{\alpha} + \alpha \varepsilon^{2/3} \right) + \alpha \varepsilon^{1/3} r \Omega' \\ &= \frac{2iM\alpha\varepsilon}{r^2} + \frac{2iM\varepsilon^{1/3}}{\alpha} \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) + \frac{2iM\varepsilon^{1/3}}{\alpha r_0^2} \\ &\quad + \alpha \varepsilon^{1/3} (r \Omega' - r_0 \Omega'_0) + \alpha \varepsilon^{1/3} r_0 \Omega'_0. \end{aligned}$$

Hence, making the choice of α as in (6.3.7), we find that

$$\frac{1}{2} \left(\frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon - \alpha \varepsilon^{1/3} C_\varepsilon \right) = \frac{iM\alpha\varepsilon}{r^2} + \frac{iM\varepsilon^{1/3}}{\alpha} \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) + \frac{\alpha \varepsilon^{1/3}}{2} (r \Omega' - r_0 \Omega'_0),$$

and therefore, we observe that the cross terms will vanish for $r \rightarrow r_0$ and $\varepsilon \rightarrow 0$. This will be precisely the key idea at the core of our approach to the problem: analysing the behaviour of $\mathcal{L}$ for values very close to r_0 , where, according to Gilbert's idea [Gil88], the exponential growth is expected to occur. Following a similar argument we find that this choice of $\alpha \in \mathbb{C}$ yields for the other factor

$$\frac{1}{2} \left(\frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon + \alpha \varepsilon^{1/3} C_\varepsilon \right) = -\frac{iM\alpha\varepsilon}{r^2} + \frac{2iM\varepsilon^{1/3}}{\alpha r_0^2} + \frac{iM\varepsilon^{1/3}}{\alpha} \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) - \frac{\alpha \varepsilon^{1/3}}{2} (r \Omega' - r_0 \Omega'_0).$$

We can now combine all this information and rewrite (6.3.12)–(6.3.13) in the desired form for the analysis we will carry out. Before doing so, however, let us introduce some more shorthand

notation for a more condensed presentation of the equations. We define

$$D_\varepsilon^+ = \frac{iM\varepsilon^{1/3}}{\alpha} \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) + \frac{\alpha\varepsilon^{1/3}}{2} (r\Omega' - r_0\Omega'_0),$$

$$D_\varepsilon^- = \frac{iM\varepsilon^{1/3}}{\alpha} \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) - \frac{\alpha\varepsilon^{1/3}}{2} (r\Omega' - r_0\Omega'_0),$$

and we take without loss of generality the positive sign in the square root when defining α in (6.3.7). Thus, $\mathcal{L} - \mu\varepsilon^{1/3}$ will be the operator associated to the equations

$$\varepsilon \left(\partial_r^2 V_1 + \frac{1}{r} \partial_r V_1 \right) = \left(\lambda + A_\varepsilon + \frac{iM\alpha\varepsilon}{r^2} - \sqrt{\frac{-2iM\Omega'_0}{r_0}} \varepsilon^{1/3} - D_\varepsilon^- \right) V_1 - \left(\frac{iM\alpha\varepsilon}{r^2} + D_\varepsilon^+ \right) V_2, \quad (6.3.14)$$

$$\varepsilon \left(\partial_r^2 V_2 + \frac{1}{r} \partial_r V_2 \right) = \left(\lambda + A_\varepsilon - \frac{iM\alpha\varepsilon}{r^2} + \sqrt{\frac{-2iM\Omega'_0}{r_0}} \varepsilon^{1/3} + D_\varepsilon^- \right) V_2 + \left(\frac{iM\alpha\varepsilon}{r^2} + D_\varepsilon^+ \right) V_1. \quad (6.3.15)$$

From here on, unless otherwise specified, any vector will be assumed to be written in the basis induced by this change of variables,

$$V_r \hat{r} + V_\theta \hat{\theta} + V_z \hat{z} \mapsto \frac{1}{2} \left(V_\theta - \frac{1}{\alpha} \varepsilon^{-1/3} V_r \right) \hat{v}_1 + \frac{1}{2} \left(V_\theta + \frac{1}{\alpha} \varepsilon^{-1/3} V_r \right) \hat{v}_2 + V_z \hat{v}_3.$$

which is well-defined for any $\varepsilon > 0$.

Remark 6.3.2. From the change of variables, we obtain further evidence of the heuristic from Section 6.3.1 that the growth rate of order $\varepsilon^{1/3}$ in our main result Theorem 6.1.1 is sharp. Indeed, the only “unstable” terms (i.e. those that are non-negative when undertaking energy estimates) are of order $O(\varepsilon^{1/3})$, aligning precisely with our result.

6.3.4 Admissible vector fields and examples

Thus far we have avoided giving an explicit description of the necessary conditions needed for Theorem 6.1.1 to hold. In this section we shall outline them, and in particular we will eventually endeavour to provide simple, checkable conditions which imply our hypotheses, as well as some explicit examples of particular interest. For any fixed $r_0 > 0$, we begin by defining the function

$$T(r) = \Omega(r) - \Omega(r_0) + \frac{\Omega'(r_0)}{U'(r_0)} (U(r) - U(r_0)), \quad (6.3.16)$$

which we will henceforth refer to as the *transport function*. Without further ado, we now list the general hypotheses under which we can deduce the existence of a growing mode for the Ponomarenko dynamo on the domain $\mathcal{J} \subseteq [0, \infty)$, with perfectly conducting boundary conditions.

General Hypotheses: Fix a set $\mathcal{J} \subseteq [0, \infty)$ that is open in the topology on $[0, \infty)$, and so that furthermore $\overline{\mathcal{J}} \subseteq \mathcal{J}$. We assume the following.

H0 The velocity field is at least $\Omega, U \in C^3(\overline{\mathcal{J}})$.

H1 The transport function vanishes up to second order only at $r_0 > 0$, namely

$$T(r_0) = T'(r_0) = 0, \quad T''(r_0) \neq 0.$$

There exists a (possibly empty) finite set of points

$$\mathcal{F}_0 = \{s_1, \dots, s_q\} \subset \mathcal{J} \setminus \{0\},$$

where the transport function satisfies $T(s) = 0, T'(s) \neq 0$, for all $s \in \mathcal{F}_0$. Moreover, $T(r) \neq 0$ for all $r \in \mathcal{J} \setminus (r_0 \cup \mathcal{F}_0)$.

H2 There exist numbers $N_3, N_4 \in \mathbb{N}$, a constant $\Xi > 0$, and a compact set $\mathcal{R}_0 \subset \mathcal{J}$ so that uniformly for $r_0 \in \mathcal{R}_0$ it holds for all $r, r' \geq \Xi$

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq C \sum_{n=1}^{N_3} |r - r'|^n, \quad \frac{|r\Omega'(r)|}{|T(r')|} \leq C \sum_{n=1}^{N_4} |r - r'|^n.$$

Furthermore, we assume that $\inf_{r_0 \in \mathcal{R}_0} \inf_{r \geq \Xi} |T(r)| > 0$.

Notice that, in case that $0 \in \overline{\mathcal{J}}$, Assumptions H0 and H1 together imply that there exists $\tau \in \mathbb{R} \setminus \{0\}$ such that

$$\lim_{r \rightarrow 0} T(r) = \tau.$$

These *General Hypotheses* H0–H2 are required for our construction of the growing mode to hold true. From the proof of Theorem 6.5.1 one can appreciate that any vector field satisfying H0–H2 can be proved to be a slow dynamo with growth rate $\text{Re}(\lambda) = \varepsilon^{1/3} \text{Re}(\mu) > 0$. If we desire to obtain further information on the eigenfunction, i.e. an asymptotic expansion as stated in Theorem 6.1.1, we need to impose the following additional assumption on the velocity field. Indeed, provided that $\mathcal{J}$ is an unbounded domain, we have the following further assumption.

H3 The velocity field satisfies either

$$- |r\Omega'(r)| \text{ is bounded and } |r\Omega'(r)| \rightarrow 0 \text{ as } r \rightarrow \infty;$$

or

$$- T(r) \text{ is bounded from above or below, } |T(r)| \rightarrow \infty \text{ as } r \rightarrow \infty \text{ and there exists } C > 0, \Lambda \in \mathbb{R} \text{ such that}$$

$$\sup_{r \in \mathcal{J}} \frac{|r\Omega'(r)|^2}{|T(r) + \Lambda|} \leq C.$$

As mentioned earlier, Assumption H3 should be thought of as a collection of checkable conditions that imply that the kinematic dynamo equations may be treated as a relatively compact perturbation of the advection diffusion equation, thus implying that all growing modes must be isolated eigenvalues. This intuition is made precise in Section 6.4.

Remark 6.3.3. Upon introducing the modal form Ansatz $B(t, r, \theta, z) = b(r)e^{\lambda t + i(m\theta + kz)}$, the kinematic dynamo equations are transformed into a system of ODEs, where the physical meaning of the coefficients m, k is no longer of direct importance. Indeed, it is entirely possible to find a

growing mode for the system of ODEs (6.1.5)–(6.1.7) for $m, k \notin \mathbb{Z}$, which, whilst of independent interest from the point of view of the ODE, is rather irrelevant in the context of the kinematic dynamo PDE (6.1.1). Recalling further that according to Gilbert’s scaling, we pick Fourier modes

$$m = M\varepsilon^{-1/3}, \quad k = -m \frac{\Omega'(r_0)}{U'(r_0)}, \quad (6.3.17)$$

as $\varepsilon \rightarrow 0$, these will generically not be integers. Thus, simply fixing a single $r_0 > 0$, $M \in \mathbb{Z}$ and showing that the associated system of ODEs has a growing mode for all $\varepsilon > 0$ small enough is not enough to deduce the slow dynamo claim: our estimates need to be robust enough to allow for some “wiggle room” in both M and r_0 . Indeed, for $\varepsilon > 0$ small enough, one can always perturb M by a term of order $\varepsilon^{1/3}$ to ensure that $M\varepsilon^{-1/3}$ is an integer. The issue then becomes the coupling between m and k via (6.3.17). Here, we once again make use of the fact that $M\varepsilon^{-1/3}$ is very large for ε small enough, and thus it is generically not too difficult to find *some* $r_0 > 0$ so that

$$-\varepsilon^{-1/3} M \frac{\Omega'(r_0)}{U'(r_0)} \in \mathbb{Z}.$$

Thus, for any $\varepsilon > 0$, to make sure that the ODE result is meaningful in the context of the PDE (6.1.1), we first make a perturbation of order $\varepsilon^{1/3}$ to ensure that $M\varepsilon^{-1/3}$ is an integer, and then make a slight perturbation to r_0 so that k as defined in (6.3.17) is also an integer. The point of the compact set $\mathcal{R}_0$ in H2 is then to guarantee that estimates used in the construction of growing modes for (6.1.5)–(6.1.7) holds *uniformly* for the perturbations needed on M , r_0 as $\varepsilon \rightarrow 0$, so that we indeed find a growing mode for the PDE (6.1.1) *for all* $\varepsilon \rightarrow 0$. That being said, the uniformity in r_0 is not entirely needed for all applications—for instance, if we instead let the variable z live in $\mathbb{R}$, then the assumption $k \in \mathbb{Z}$ can be dispensed with. In keeping with this, assumption H2 could in principle be satisfied for the compact set $\mathcal{R}_0 = \{r_0\}$. In Theorems 6.1.1, 6.1.2 we thus differentiate between this case and the case where $\mathcal{R}_0$ contains a non-empty open interval.

Having outlined our assumptions, we now endeavour to “translate” them into the main properties of the transport function $T(r)$ needed for our proofs. This is the subject of the following lemma.

Lemma 6.3.4 (Consequences of H0–H2). *Assume H0–H2 hold true, and pick any $\gamma, \delta > 0$ so that*

$$\frac{2}{9} < \gamma < \frac{1}{3}, \quad \gamma < \delta, \quad \gamma + \delta < \frac{2}{3}.$$

Then, there exist a compact interval $\mathcal{R} \subset \mathcal{I}$ with $r_0 \in \mathcal{R}$, a finite set $\mathcal{F}_0(r_0)$ depending on $r_0 \in \mathcal{R}$ of constant cardinality, and a constant $C > 0$ independent of $r_0 \in \mathcal{R}$, so that for any $\varepsilon > 0$ small enough the following holds:

P1 $T(s_j) = 0$ for all $s_j \in \mathcal{F}_0(r_0)$, and $|T'(s_j)|$ is bounded below uniformly in $r_0 \in \mathcal{R}$, $s_j \in \mathcal{F}_0(r_0)$. Also, it holds

$$\sup_{r_0 \in \mathcal{R}} \sup_{s_j \in \mathcal{F}_0(r_0)} \sup_{|r - s_j| \leq 1} \frac{|T(r) - T'(s_j)(r - s_j)|}{|r - s_j|^2} < \infty.$$

P2 $|T''(r_0)|$ is bounded below uniformly in $r_0 \in \mathcal{R}$, and there holds

$$\sup_{r_0 \in \mathcal{R}} \sup_{|r-r_0| < 1} \frac{|T(r) - \frac{1}{2}T''(r_0)(r-r_0)^2|}{|r-r_0|^3} < \infty.$$

P3 There exist a positive number $N_1 \in \mathbb{N}$ and a collection of possibly ε -dependent coefficients $B_n(\varepsilon) \geq 0$, $1 \leq n \leq N_1$, such that for any $\varepsilon > 0$ small, and any two points $r, r' \in \mathcal{J}$, $r' \geq \varepsilon^\gamma$ with $|r' - R| \geq \varepsilon^\gamma$ for all $R \in \{r_0\} \cup \mathcal{F}_0$, there holds that

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq \sum_{n=1}^{N_1} B_n(\varepsilon) |r - r'|^n.$$

Additionally, there exists $\beta > 0$ such that the coefficients $B_n(\varepsilon) \geq 0$ satisfy

$$\sum_{n=1}^{N_1} B_n(\varepsilon) \varepsilon^{n\delta} \leq C\varepsilon^\beta,$$

for some β independent of $r_0 \in \mathcal{R}$.

P4 There exist a number $N_2 \in \mathbb{N}$ and a collection of coefficients $D_0, D_n \geq 0$, with $1 \leq n \leq N_2$, such that for any $\varepsilon > 0$ small, and any two points $r, r' \in \mathcal{J}$, $r' \geq \varepsilon^\gamma$ with $|r' - R| \geq \varepsilon^\gamma$ for all $R \in \{r_0\} \cup \mathcal{F}_0(r_0)$, there holds that

$$\frac{|r\Omega'(r)|}{|T(r')|} \leq \varepsilon^{-2\gamma} \left(D_0 + \sum_{n=1}^{N_2} D_n |r - r'|^n \right).$$

P5 If $0 \in \overline{\mathcal{J}}$, then $|\lim_{r \rightarrow 0} T(r)|$ is bounded below uniformly in $r_0 \in \mathcal{R}$, and

$$\sup_{r_0 \in \mathcal{R}} \sup_{r \leq 1} \frac{|T(r) - T(0)|}{r} < \infty.$$

P6 Let $s_1, s_2 \in \mathcal{F}_0(r_0) \cup \{r_0\}$ be neighbouring points, i.e. $(s_1, s_2) \cap (\mathcal{F}_0(r_0) \cup \{r_0\}) = \emptyset$. Then,

– if $s_1, s_2 \neq r_0$,

$$\inf_{r \in [s_1 + \varepsilon^\gamma, s_2 - \varepsilon^\gamma]} |T(r)| \geq C\varepsilon^\gamma,$$

for some constant $C > 0$ independent of $r_0 \in \mathcal{R}$;

– if either of s_1, s_2 are equal to r_0 ,

$$\inf_{r \in [s_1 + \varepsilon^\gamma, s_2 - \varepsilon^\gamma]} |T(r)| \geq C\varepsilon^{2\gamma}$$

for some constant $C > 0$ independent of $r_0 \in \mathcal{R}$.

Proof. Let $p \geq 0$ and $q > 0$ that can be taken to be $q = \infty$, define $\mathcal{I} = [p, q]$, and $\mathcal{J} = [p_1, q_1]$ with $p_1 \in [0, p]$, $q_1 \in [q, \infty]$. Assume that $\mathcal{F}_0$ is nonempty, let $p_2 > \Xi$ and define $\mathcal{K} = [p_1, p_2]$ be a compact interval so that $\mathcal{F}_0 \subset \text{int}(\mathcal{K})$. By the implicit function theorem, for any $s_j \in \mathcal{F}_0$,

there exist an open neighbourhood of (s_j, r_0) , and a differentiable function ϕ_j so that the unique zero of $T(s, r_0)$ is given by $(\phi_j(r_0), r_0)$. Hence, since $\mathcal{F}_0$ is finite, we may find a compact interval $\mathcal{R}_0 \supset \mathcal{R} \ni r_0$, and a number $\omega_0 > 0$ so that for $r_0 \in \mathcal{R}$, $\text{dist}(r, \mathcal{F}_0 \cup \{r_0\}) < \omega_0$, $T(r, r_0) = 0$ if and only if $r = \phi_j(r_0)$, and $|r - s_j| < \omega_0$, for some j . Next, by Taylor's remainder theorem,

$$\left| T(r) - \frac{1}{2} T''(r_0)(r - r_0)^2 \right| \leq \sup_{s \in [r_0, r]} |T^{(3)}(s)| |r - r_0|^3,$$

$$|T(r) - T'(s_j)(r - s_j)| \leq \sup_{\xi \in [s_j, r]} |T''(\xi)| |r - s_j|^2.$$

and $\sup_{\xi \in [s_j, r_0]} |T''(\xi)|$, $\sup_{s \in [r_0, r]} |T^{(3)}(s)|$ are continuous in r_0 , and thus bounded on the compact interval $\mathcal{R}$. Similarly, since $s_j = \phi_j(r_0)$, and $|T''(r_0)|$, $|T'(s_j)|$, are positive functions that are continuous in r_0 , upon possibly shrinking the size of $\mathcal{R}$, they may be bounded below uniformly for $r_0 \in \mathcal{R}$. Hence, points P1, P2 have been proven. Additionally, P5 is deduced in entirely the same way as P1, P2.

Next, upon possibly further shrinking $\mathcal{R}$, there exists some ω_1 so that

$$|T(r)| \geq \frac{1}{4} |T''(r_0)| |r - r_0|^2$$

$$|T(r)| \geq \frac{1}{2} |T'(\phi_j(r_0))| |r - \phi_j(r_0)|,$$

respectively for $|r - r_0| < \omega_1$, $|r - \phi_j(r_0)| < \omega_1$, and $|r - \phi_j(r_0)| < \omega_1$, which implies $|r - s_j| < \omega_0$. Consider now the set $\{(r, r_0) \in \mathcal{R} \times \mathcal{K} : |r - \phi_j(r_0)| \geq \omega_1 \forall j, |r - r_0| \geq \omega_1\}$. This is a compact set, and $T(r, r_0)$ is a non-vanishing continuous function from it to the reals. Thus, $|T(r, r_0)|$ is uniformly bounded below on this set, which combined with P1, P2 immediately yields P6.

There is only P3 and P4 left to be studied, so let us start with the former. We may write,

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq \frac{|T'(r')|}{|T(r')|} |r - r'| + \frac{\sup_{s \in \mathcal{J}} |T''(s)|}{|T(r')|} |r - r'|^2.$$

But now, note that $|T'(r')| \leq \sup_{\xi \in \mathcal{K}} |T''(\xi)| |r' - r_0|$, and as soon as $|r' - r_0| < \omega_1$, we can write

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq C_1 |r' - r_0|^{-1} |r - r'| + C_2 |r' - r_0|^{-2} |r - r'|^2.$$

Thus, for $\omega_1 \geq |r' - r_0| \geq \varepsilon^\gamma$, this is bounded uniformly for $r_0 \in \mathcal{R}$ by

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq C (\varepsilon^{-\gamma} |r - r'| + \varepsilon^{-2\gamma} |r - r'|^2).$$

Similarly, we can show that as soon as $\varepsilon^\gamma \leq |r' - \phi_j(r_0)| < \omega_1$, we have

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq C \varepsilon^{-\gamma} |r - r'|.$$

Furthermore, since away from these two sets, we have shown that $|T(r')|$ is bounded below uni-

formly for $r_0 \in \mathcal{R}$, we have

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq C|r - r'|.$$

An identical argument shows that

$$\frac{|r\Omega(r)|}{|T(r')|} \leq C\varepsilon^{-2\gamma}|r - r'|$$

uniformly for $r, r' \in \mathcal{K}$, $|r - \phi_j(r_0)| \geq \varepsilon^\gamma$, $|r - r_0| \geq \varepsilon^\gamma$. Finally, suppose that $r \geq \Xi$, $r' \leq \Xi$. If both $r, r' \in \mathcal{K}$, then certainly **P3** is satisfied, so assume that $r \notin \mathcal{K}$, so that $|r - r'| \gtrsim 1$. Hence, observing further that by **H2** we have the bound

$$|T(r)| \leq C \sum_{n=0}^{N_3+1} |r - \Xi|^n,$$

we write

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq C\varepsilon^{-2\gamma} \sum_{n=0}^{N_3+1} |r - \Xi|^n \leq C\varepsilon^{-2\gamma} \sum_{n=0}^{N_3+1} |r - r'|^n,$$

where we have used that $|r - \Xi|^n \leq C(n)(|r - r'|^n + |r' - \Xi|^n)$. But now, note that $|r' - \Xi|^n \leq \Xi^n$, and that $1 \lesssim |r - r'|$. Thus, we may easily bound

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq C\varepsilon^{-2\gamma} \sum_{n=2}^{N_3+1} |r - r'|^n.$$

Similarly, if $r' \notin \mathcal{K}$, $r \leq \Xi$, we simply bound

$$\frac{|T(r) - T(r')|}{|T(r')|} \lesssim 1 + \frac{|T(\Xi) - T(r')|}{|T(r')|} \lesssim C \sum_{n=0}^{N_3} |r' - \Xi|^n.$$

Proceeding entirely as before, using that $|r - r'| \gtrsim 1$, this is bounded by

$$C \sum_{n=1}^{N_3} |r' - r|^n,$$

and so in any case we may write

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq C \left(\varepsilon^{-\gamma}|r - r'| + \varepsilon^{-2\gamma}|r - r'|^2 + \varepsilon^{-2\gamma} \sum_{n=3}^{N_3+1} |r - r'|^n \right).$$

Hence,

$$\sum_{n=1}^{N_3+1} B_n(\varepsilon) \varepsilon^{n\delta} \leq C(\varepsilon^{-\gamma+\delta} + \varepsilon^{-2\gamma+2\delta} + 2\varepsilon^{-2\gamma+3\delta}).$$

The argument for $|r\Omega'(r)||T(r')|^{-1}$ is entirely analogous. Hence, this shows that **P3** and **P4** hold true, and so the proof of the lemma is completed. $\square$

Sufficient conditions for Assumption H2.

While conditions H0 and H1 are rather easily checkable, the condition H2 is slightly less intuitive. We show that a simple computation on the derivatives of $T(r)$ is sufficient to verify H2 in many cases.

Lemma 6.3.5. *Let $\Omega, U \in C^\infty([0, \infty))$, and set $T(r)$ as defined in (6.3.16). Suppose that there exists $\Xi > 0$, and a compact set $\mathcal{R}_0 \ni r_0$ so that the following hold:*

1. *There exist $N_1 \geq 2$, $C_i < \infty$ so that for all $0 \leq i \leq N_1 - 1$, $r_0 \in \mathcal{R}_0$ it holds*

$$\sup_{r \geq \Xi} \frac{|T^{(i)}(r)|}{|T(r)|} + \sup_{r \geq \Xi} \frac{|(r\Omega'(r))^{(i)}|}{|T(r)|} \leq C_i.$$

2. *There holds*

$$\sup_{r_0 \in \mathcal{R}_0} \left(\sup_{r \geq \Xi} |T^{(N_1)}(r)| + \sup_{r \geq \Xi} |(r\Omega'(r))^{(N_2)}| \right) < \infty.$$

If further $\inf_{r_0 \in \mathcal{R}_0} \inf_{r \geq \Xi} |T(r)| > 0$, then, condition H2 is satisfied for the interval $\mathcal{R}_0$.

Proof. By a Taylor expansion there holds

$$|T(r) - T(r')| \leq \sup_{\xi_1 \in [r, r']} |T'(\xi_1)| |r - r'| \leq |r - r'| \left(|T'(r')| + |r - r'| \sup_{\xi_2 \in [r, r']} |T''(\xi_2)| \right).$$

Inductively, we deduce that

$$|T(r) - T(r')| \leq |r - r'| \sum_{1 \leq i \leq N_1 - 1} |T^{(i)}(r')| |r - r'|^{i-1} + |r - r'|^{N_1} \sup_{\xi \geq 0} |T^{(N_1)}(\xi)|.$$

Therefore, dividing through by $|T(r')|$, we deduce that for any $r_0 \in \mathcal{R}_0$ there holds

$$\frac{|T(r) - T(r')|}{|T(r')|} \leq |r - r'| \sum_{i \leq N_1} B_i |r - r'|^{i-1},$$

where we have set

$$B_i = \sup_{r_0 \in \mathcal{R}_0} \sup_{r' \geq \Xi} \frac{|T^{(i)}(r')|}{|T(r')|}$$

for $i \leq N_1 - 1$, and

$$B_{N_1} = \sup_{r_0 \in \mathcal{R}_0} \sup_{r \geq 0} |T^{(N_1)}(r)| \sup_{r' \geq \Xi} |T(r')|^{-1}.$$

An identical argument yields the claim for $|r\Omega'(r)| |T(r')|^{-1}$. □

Relevant examples

In this section, we present illustrative examples that fall within the scope of Assumptions H0–H3 and are of particular interest due to their historical significance or physical relevance. Additionally,

we will make sure that these examples also satisfy the condition

$$r_0 \left| \frac{d}{dr} \log \left| \frac{\Omega'(r_0)}{U'(r_0)} \right| \right| < 4 \quad (6.3.18)$$

from Theorem 6.1.1.

Lemma 6.3.6 (Simplified model). *The smooth vector field*

$$\Omega(r) = 1 - r, \quad U(r) = 1 - r^2,$$

satisfies H0-H3 as well as (6.3.18) with any choice of compact set $\mathcal{R}_0 \subset (0, \infty)$.

This example of a polynomial vector field exhibiting slow dynamo action is relevant due to the simplicity of its transport function $T(r) \sim (r - r_0)^2$. It has been widely used in applied and numerical studies of the smooth Ponomarenko dynamo, see e.g. [PGP08; RSS88].

Proof. With Lemma 6.3.5 under our belt, this proof becomes elementary. Observe that

$$T(r) = \frac{1}{2r_0} (r - r_0)^2$$

for any fixed $r_0 > 0$. It remains to note that $T(r)$ only vanishes at $r = r_0$, moreover

$$\frac{|T'(r)|}{|T(r)|} = \frac{2}{(r - r_0)^2},$$

and $T^{(i)} = 0$ for any $i \geq 2$, hence

$$\sup_{r \geq 2r_0+1} \frac{|T^{(i)}(r)|}{|T(r)|} < \infty.$$

Similarly, since $r\Omega'(r) = -r$, conditions (1) and (2) from Lemma 6.3.5 follow accordingly. Moreover, to see that it also satisfies (6.3.18), we compute

$$r \left| \frac{d}{dr} \log \left| \frac{\Omega'(r)}{U'(r)} \right| \right| = 1$$

for all $r \in (0, \infty)$. □

The following relevant examples concern the existence of finite energy—in the sense of finite L^2 norm—velocity fields that exhibit slow dynamo action. The vector fields here presented are in particular in $L^1 \cap L^\infty$.

Lemma 6.3.7 (Finite energy dynamos). *The following velocity fields satisfy Assumptions H0–H3 and condition (6.3.18) from Theorem 6.1.1.*

1. *Gaussian velocity fields*

$$\Omega(r) = e^{-ar^2}, \quad U(r) = e^{-br^2},$$

with $a, b > 0$ and $a \neq b$.

2. Let Ω, U satisfy the assumptions of Lemma 6.3.5, and moreover assume that

$$\Omega(r_0) - \frac{\Omega'(r_0)}{U'(r_0)}U(r_0) \neq 0$$

and (6.3.18) hold true for all $r_0 \in \mathcal{R}_0$, where $\mathcal{R}_0 \subset (0, \infty)$ is given by Lemma 6.3.5. Then, there exists a smooth cut-off $\chi \in C^\infty([0, \infty))$, i.e. $0 \leq \chi \leq 1$ and

$$\chi(r) = \begin{cases} 1 & \text{if } r \leq R_0, \\ 0 & \text{if } r \geq R_1, \end{cases}$$

for some $R_1 > R_0 > 0$, so that $\chi(r)\Omega(r)$, $\chi(r)U(r)$ also satisfy H0–H3 and (6.3.18).

Proof. We begin by checking the case of Gaussian velocity fields. We have

$$T(r) = e^{-ar^2} - \frac{a}{b}e^{-(a-b)r_0^2}e^{-br^2} + \left(\frac{a}{b} - 1\right)e^{-ar_0^2}.$$

A direct computation yields that $T(r) = 0$ if and only if

$$\frac{a}{b} - 1 = \frac{a}{b}e^{-b(r^2-r_0^2)} - e^{-a(r^2-r_0^2)},$$

which shows that $r = r_0$ is the unique point where $T(r)$ vanishes—and it does so up to second order—. Finally, since $T^{(i)}(r) \rightarrow 0$ as $r \rightarrow \infty$ for any i , and the same holds for $r\Omega'(r)$, the assumptions of Lemma 6.3.5 are satisfied. To check that (6.3.18) holds true, we compute

$$r \left| \frac{d}{dr} \log \left| \frac{\Omega'(r)}{U'(r)} \right| \right| = 2r^2|b-a|.$$

Thus, a smart choice of $a, b > 0$ yields that there exists $\mathcal{R}_0 \subset (0, \infty)$ compact and with nonempty interior such that (6.3.18) is satisfied for all $r_0 \in \mathcal{R}_0$.

For the case of compactly supported velocity fields, we just need to pick a smooth cut-off so that H1 is satisfied and $r_0 < R_0$, since the relative derivative bounds from 6.3.5 follow immediately from the condition

$$\Omega(r_0) - \frac{\Omega'(r_0)}{U'(r_0)}U(r_0) \neq 0$$

and from Ω and U being smooth. To obtain the corresponding condition (6.3.18), simply recall that Ω and U satisfy it by assumption, hence upon choosing R_0 and R_1 conveniently, this property readily follows for $\chi\Omega$ and χU . $\square$

Finally, we show that the Taylor–Couette flow indeed satisfies our assumptions.

Lemma 6.3.8 (Taylor–Couette Ponomarenko dynamo). *Let*

$$u(r) = \left(\frac{a_1}{r} + a_3r\right)\hat{\theta} + (a_2 \log(r) + a_4)\hat{z},$$

i.e. $\Omega(r) = a_1r^{-2} + a_3$, $U(r) = a_2 \log(r) + a_4$, with $a_1, a_2 \neq 0$. Then, the Taylor–Couette flow satisfies Assumptions H0–H3 and (6.3.18) on either $\mathcal{J} = [p, q]$ or $\mathcal{J} = [p, \infty)$ with $0 < p < q$, for

any $r_0 \in \text{int}(\mathcal{I})$.

Proof. The transport function is given by

$$T(r) = a_1 \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) + a_1 \frac{2}{r_0^2} \left(\log\left(\frac{r}{r_0}\right) \right).$$

Differentiating this equation in r , we obtain

$$-\frac{2a_1}{r^3} + \frac{2a_1}{rr_0^2},$$

which is equal to zero only if $r^2 = r_0^2$, i.e. when $r = r_0$, since we only consider non-negative values of r , and thus **H0**, **H1** are satisfied. For **H2**, we apply Lemma 6.3.5 and deduce easily from our computation of $T(r)$ that $|T'(r)||T(r)|^{-1}$ is bounded uniformly for r large and r_0 in any compact interval. Certainly also $|T''(r)|$ is bounded uniformly, and $|T(r)|$ is bounded below uniformly for r large, r_0 in a compact interval. Similarly, $r\Omega'(r) = -2a_1r^{-2}$, and so the same considerations can be used in order to conclude (1), (2), and hence **H2**. To check **H3** for the case $\mathcal{I} = [p, \infty)$, we note that $|r\Omega'(r)| \rightarrow 0$ as $r \rightarrow \infty$. Additionally, to obtain condition (6.3.18) we compute

$$r \left| \frac{d}{dr} \log \left| \frac{\Omega'(r)}{U'(r)} \right| \right| = 2,$$

uniformly for an $r > 0$. □

6.4 A-priori estimates on the kinematic dynamo operator

Before commencing with the proof of the existence of a growing mode, we collect here a couple of facts about the kinematic dynamo operator, which will be important for our analysis. In particular, we shall derive an extremely useful a-priori estimates on the behaviour of eigenfunctions of the kinematic dynamo operator, and discuss some of its consequences. These include being able to show that any growing mode of the dynamo equation must be divergence free, as well as begin able to deduce qualitative spectral properties about the kinematic dynamo operator. In particular, we will be able to give sufficient conditions for all unstable eigenvalues of the kinematic dynamo operator to be isolated, and hence for the asymptotic expansion of the growing mode in Theorem 6.1.1 to be valid.

6.4.1 The a-priori estimate

To start with, we provide an a-priori estimate on all eigenfunctions of the operator $\mathcal{L}$ in X , which is a crucial ingredient of the proofs of Lemmas 6.4.2, 6.4.3, as well as of the proof of Lemma 6.4.4 concerning the divergence free property of the growing mode that we construct.

Lemma 6.4.1. *Let $\mathcal{I} = [0, q]$, for $q \in (0, \infty]$, suppose $\Omega, U \in C^3(\mathcal{I})$, and consider the operator*

$$\mathcal{L}^{\text{dyn}} b = \varepsilon \left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{m^2}{r^2} - k^2 - \frac{1}{r^2} - i\varepsilon^{-1}(m\Omega(r) + kU(r)) \right) b + \begin{pmatrix} -2i\varepsilon m r^{-2} b_\theta \\ (2i\varepsilon m r^{-2} + r\Omega'(r)) b_r \end{pmatrix}.$$

If $b = (b_r, b_\theta)$ is an eigenfunction of $\mathcal{L}^{\text{dyn}}$ with eigenvalue $\lambda \in \mathbb{C}$, so that further

$$b \in \{b \in H_{\text{loc}}^2(\mathcal{J}) \cap X : \mathcal{L}^{\text{dyn}} b \in X_{\mathcal{J}}\},$$

then the following holds for all m, N large enough depending only on Ω, U

1. $r^{-2}b \in L^\infty(\mathcal{J}, \max\{1, r^2\}dr)$,
2. $r^{-1}\partial_r b \in L^\infty(\mathcal{J}, \max\{1, r^2\}dr)$,
3. $\partial_r^2 b \in L^\infty(\mathcal{J}, \max\{1, r^2\}dr)$.

In fact, if $b_z \in X_{\mathcal{J}}$ satisfies

$$\varepsilon(\partial_r^2 + \frac{1}{r}\partial_r - \frac{m^2}{r^2} - \frac{1}{r^2} - i\varepsilon^{-1}(m\Omega(r) + kU(r))b_z + U'(r)b_r = \lambda b_z,$$

the same bounds also hold for b_z .

Proof. Set $W_1 = ib_r + b_\theta$ and $W_2 = -ib_r + b_\theta$. Then, we may rewrite the equations as

$$\begin{aligned} \widehat{\mathcal{L}^{\text{dyn}}}W &= \varepsilon \left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{m^2}{r^2} - k^2 - \frac{1}{r^2} - i(m\Omega(r) + kU(r)) \right) W \\ &\quad + \frac{2m}{r^2} \begin{pmatrix} -W_1 \\ W_2 \end{pmatrix} + \frac{r\Omega'(r)}{2i} \begin{pmatrix} W_1 - W_2 \\ W_1 - W_2 \end{pmatrix} \end{aligned} \quad (6.4.1)$$

In particular, if W solves the equation $(\widehat{\mathcal{L}^{\text{dyn}}} - \lambda)W = 0$, we find that the first component must satisfy

$$\varepsilon \left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{m^2 + 2m + 1}{r^2} \right) W_1 = i(m\Omega(r) + kU(r))W_1 - r\Omega'(r)\frac{W_1 - W_2}{2i} + (\lambda + \varepsilon k^2)W_1. \quad (6.4.2)$$

Since by assumption $W \in X = L^\infty(\mathcal{J}, \max\{1, r^2\}w_\varepsilon(r - r_0)dr)$, with $w_\varepsilon(s) = 1 + (\varepsilon^{-1/3}|s|)^N$, we see that as long as the exponent $N > 0$ in the weight is sufficiently large, the right hand side of (6.4.2) is in $L^\infty(\mathcal{J}, \max\{1, r^2\}dr)$. Indeed, from Assumption H2 it is clear that Ω, U and Ω' cannot grow faster than polynomially as $r \rightarrow \infty$. In fact, given Ω and U , we may choose N sufficiently large so that the right hand side in (6.4.2) vanishes at least cubically as $r \rightarrow \infty$. With this in mind, we construct a Green's function for this equation. A simple computation shows that the equation

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\alpha^2}{r^2} \right) v = 0$$

has two linearly independent solutions, given by r^α and $r^{-\alpha}$. Therefore, upon computing the Wronskian, we can construct a Green's function given by

$$\mathcal{G}_\alpha f = -\frac{1}{2\alpha} \left[r^{-\alpha} \int_0^r s^{\alpha+1} f(s) ds + r^\alpha \int_r^\infty s^{-\alpha+1} f(s) ds \right], \quad (6.4.3)$$

where without loss of generality we assume $\alpha > 0$. In particular, we can rewrite the equation for

the first component as

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{(m+1)^2}{r^2}\right)W_1 = f, \quad (6.4.4)$$

with

$$f = \varepsilon^{-1} \left(i(m\Omega(r) + kU(r))W_1 - r\Omega'(r)\frac{W_1 - W_2}{2i} + (\lambda + \varepsilon k^2)W_1 \right),$$

that is clearly in $L^\infty(\mathcal{I})$, and as mentioned before, it decays at least cubically as $r \rightarrow \infty$. A direct computation from (6.4.3) yields the estimate

$$|\mathcal{G}_\alpha f(r)| \lesssim r^2 \|f\|_{L^\infty} \quad (6.4.5)$$

for any $r > 0$, provided that $\alpha > 2$. Since $\alpha = m + 1$ for W_1 , and $\alpha = m - 1$ for W_2 , we need to choose $m > 3$. In addition, using that f decays at least cubically at infinity, we find that as $r \rightarrow \infty$ there holds

$$|\mathcal{G}_\alpha f(r)| \lesssim \frac{1}{r}. \quad (6.4.6)$$

Going back to (6.4.4) and setting $\alpha = m + 1$, we find that

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{(m+1)^2}{r^2}\right)(\mathcal{G}_{m+1}f - W_1) = 0,$$

and so $\mathcal{G}_{m+1}f - W_1$ must be in the kernel of the operator $\partial_r^2 + r^{-1}\partial_r - (m+1)^2r^{-2}$, which we have just shown to be precisely $r^{\pm(m+1)}$. $\mathcal{G}_{m+1}f$ is bounded both at zero and, since it lives in X it cannot grow as the right endpoint of $\mathcal{I}$ is approached. Thus, it follows that $\mathcal{G}_{m+1}f = W_1$, and so we find that W_1 vanishes quadratically at zero, so that $r^{-2}W_1 \in L^\infty(\mathcal{I}, \max\{1, r^2\}dr)$. (In fact, we may iterate this argument further to show it vanishes at zero at order proportional to m). Next, we shall show that b must in fact be differentiable in r . To do so, recall that we have proved that we can write

$$W_1 = -\frac{1}{2\alpha} \left[r^{-\alpha} \int_0^r s^{\alpha+1} f(s) ds + r^\alpha \int_r^\infty s^{-\alpha+1} f(s) ds \right].$$

Note that $f \in H_{\text{loc}}^2$ by assumption, and since we are in one dimension, this in fact implies that $f \in C^1$. Therefore, it is clear that W_1 is continuously differentiable away from zero, and its derivative is given by

$$\partial_r W_1 = -\frac{1}{2} \left[-r^{-\alpha-1} \int_0^r s^{\alpha+1} f(s) ds + r^{\alpha-1} \int_r^\infty s^{-\alpha+1} f(s) ds \right].$$

Hence, arguing as for (6.4.5) and (6.4.6), we deduce that

$$\begin{aligned} |\partial_r W_1| &\lesssim r \|f\|_{L^\infty} \quad \text{for any } r > 0, \\ |\partial_r W_1| &\lesssim \frac{1}{r^2} \quad \text{as } r \rightarrow \infty. \end{aligned}$$

In particular this implies that (at least) $r^{-1}\partial_r W_1 \in L^\infty(\mathcal{I}, \max\{1, r^2\}dr)$. The same argument, (with the sole exception that now $\alpha = m - 1$), carries over for the W_2 using the second component in (6.4.1). All in all we can finally combine the facts that

- $\mathcal{L}b \in L^\infty(\mathcal{I}, \max\{1, r^2\})$,

- $r^{-2}b, r^{-1}\partial_r b \in L^\infty(\mathcal{J}, \max\{1, r^2\}dr)$, and
- $(m\Omega(r) + kU(r))b, r\Omega'(r)b \in L^\infty(\mathcal{J}, \max\{1, r^2\}dr)$ provided that N is large enough,

to deduce from

$$\varepsilon \partial_r^2 b_r = -\varepsilon \left(\frac{1}{r} \partial_r - \frac{m^2}{r^2} - k^2 - \frac{1}{r^2} - i(m\Omega(r) + kU(r)) \right) b_r - \frac{2im}{r^2} b_\theta.$$

that $\partial_r^2 b \in L^\infty(\mathcal{J}, \max\{1, r^2\}dr)$. In fact, from our arguments it actually follows that $\partial_r^2 b$ is continuous. Finally, the same arguments may be repeated for the z -component, which then completes the proof. $\square$

6.4.2 Spectral properties of the dynamo operator

An important application of the a-priori bound from Lemma 6.4.1 is to the question of whether eigenfunctions of the kinematic dynamo operator are isolated. Indeed, whilst this question is an elementary consequence of the compact embedding of H^1 into L^2 on bounded domains, on unbounded domains there is a-priori not much of a reason for this property to hold. In particular, the cases where $r \in [0, \infty)$ or $r \in [p, \infty)$ with perfectly conducting boundary conditions require additional care.

Since the dynamo equations can be viewed as a perturbation of the advection diffusion equation by the stretching term $r\Omega'(r)B_r$, it is natural to expect that spectral considerations are most easily dealt with in L^2 . From this point of view, the role of Lemma 6.4.1 is then quite simply to show that the set of L^2 eigenfunctions of the kinematic dynamo operator contains the set of its eigenfunctions in X . Therefore, if the set of L^2 -eigenvalues is isolated, we conclude a-fortiori that the same holds for the set of X -eigenvalues.

The aim this sub-section will therefore precisely be to find conditions so that the set of (unstable) L^2 -eigenvalues of the kinematic dynamo operator is isolated when the operator acts on functions defined on unbounded domains. Continuing with the intuition that one can treat the stretching term perturbatively, we will mainly be interested in the behaviour of $r\Omega'(r)$, and in fact we deduce two checkable assumptions on this stretching term that ensure that all unstable eigenvalues are isolated. The first of these addresses the case where the stretching decays at infinity, namely

$$\lim_{r \rightarrow \infty} |r\Omega'(r)| = 0.$$

This is certainly satisfied in many cases of interest, such as for compactly supported, or finite energy velocity fields. If on the other hand, the velocity field grows at infinity, the second Lemma of this section still yields relatively mild conditions for all eigenfunctions to be isolated. Indeed, we simply require that the transport function $T(r)$ must diverge as $r \rightarrow \infty$ at a rate exceeding the square of the stretching term, namely $|T(r)| \rightarrow \infty$ and

$$\lim_{r \rightarrow \infty} \frac{|T(r)|}{|r\Omega'(r)|^2} = \infty.$$

We stress once again that these conditions for the velocity fields are required in order to obtain

an asymptotic expression for the growing mode eigenfunction. Any examples of vector fields that satisfy Assumptions [H0–H2](#) but that do not satisfy the conditions here introduced, will still have a growing mode of the form stated in Theorem [6.1.1](#), but we cannot say much about how the eigenfunction looks like.

Without further ado, let us address the main results of this section.

Lemma 6.4.2. *Let $\mathcal{I} = [0, \infty)$ or $\mathcal{I} = [p, \infty)$, for some $p > 0$. Suppose that*

$$\lim_{r \rightarrow \infty} |r\Omega'(r)| = 0.$$

Then, provided k^2 is large enough, there are at most countably many $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 0$ for which $\mathcal{L} - \lambda$ is not injective on $X|_{\mathcal{I}}$, and each such λ is isolated.

Proof. We shall provide explicit details for this proof in the case where $\mathcal{I} = [0, \infty)$, and mention how to modify the argument for $\mathcal{I} = [p, \infty)$ at the end of the proof.

We consider the operator S defined on $L^2((0, \infty), r dr)$ by

$$Sb = \varepsilon \left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{m^2}{r^2} - k^2 - \frac{1}{r^2} - i\varepsilon^{-1}(m\Omega(r) + kU(r)) \right) b + \begin{pmatrix} -2\varepsilon i m r^{-2} b_\theta \\ 2i\varepsilon m r^{-2} b_r \end{pmatrix}, \quad (6.4.7)$$

where we denote $b = (b_r, b_\theta)$. Furthermore, we endow it with the domain

$$\mathcal{D}(S) = H^2((0, \infty), r dr) \cap X,$$

where we denote by $H^2((0, \infty), r dr)$ the space of functions which have weak derivatives up to second order in $L^2((0, \infty), r dr)$. Note that for any $B \in \mathcal{D}(S)$ we may integrate by parts, and obtain that

$$\begin{aligned} \langle Sb, b \rangle_{L^2(r dr)} &= -\varepsilon \int_0^\infty \left[r |\partial_r b|^2 + \frac{m^2}{r^2} |b|^2 + k^2 |b|^2 + \frac{1}{r} |b|^2 + 4 \operatorname{Re} \left(i \frac{m b_\theta \bar{b}_r}{r} \right) \right] dr \\ &\quad - i \int_0^\infty r (m\Omega(r) + kU(r)) |b|^2 dr. \end{aligned}$$

From this it is clear that $(S, \mathcal{D}(S))$ is a dissipative operator. Furthermore, note that it is in fact densely defined, since $\mathcal{D}(S)$ certainly contains $C_c^\infty(0, \infty)$. Given any $H \in L^2((0, \infty), r dr)$, one finds a cut-off χ so that $H\chi$ has compact support in $(0, \infty)$, and such that $\|H - H\chi\|_{L^2(r dr)} \leq \delta/2$. In turn, up to a mollification, one can find a smooth, compactly supported function ϕ so that $\|H\chi - \phi\|_{L^2(r dr)} < \delta/2$, and hence density follows. Therefore, using [[ABN13](#), Lemma 3.44], it follows that $(S, \mathcal{D}(S))$ is closable, and its closure is a dissipative operator. In fact, denoting the closure of S by $(\bar{S}, \mathcal{D}(\bar{S}))$, it holds that for any $b \in \mathcal{D}(\bar{S})$ one has

$$\operatorname{Re} \langle -\bar{S}b, b \rangle_{L^2(r dr)} \geq \varepsilon \int_0^\infty (r |\partial_r b|^2 + r^{-1} |b|^2) dr.$$

Indeed, let $b_n \in \mathcal{D}(S)$ be so that $b_n \rightarrow b$, $Sb_n \rightarrow Sb$. Then

$$\begin{aligned} \operatorname{Re} \langle -\bar{S}b, b \rangle_{L^2(rdr)} &= \lim_{n \rightarrow \infty} \operatorname{Re} \langle -\bar{S}b_n, b_n \rangle_{L^2(rdr)} \\ &\geq \liminf_{n \rightarrow \infty} \varepsilon \int_0^\infty (r|\partial_r b_n|^2 + r^{-1}|b_n|^2) dr \geq \varepsilon \int_0^\infty (r|\partial_r b|^2 + r^{-1}|b|^2) dr. \end{aligned}$$

From now on we abuse notation and identify S with its closure. With this in mind, we will aim to establish the essential spectrum of the operator S , and then treat the operator $\mathcal{L}$ as a relatively compact perturbation of S . Note that we can write

$$\mathcal{L}b = Sb + Kb$$

where

$$K \begin{pmatrix} b_r \\ b_\theta \end{pmatrix} = \begin{pmatrix} 0 \\ r\Omega'(r)b_r \end{pmatrix}$$

in cylindrical coordinates. Furthermore, note that for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 0$, there exists $C > 0$ such that

$$\|(S - \lambda)^{-1}f\|_{H^1(rdr)} \leq C\|f\|_{L^2(rdr)},$$

where we have abused notation and set $\|b\|_{H^1(rdr)} = \int_0^\infty r|\partial_r b|^2 + r^{-1}|b|^2 dr$. Therefore, by a standard argument, it holds that $K(S - \lambda)^{-1}$ is a compact operator. We reproduce this here for the convenience of the reader. We begin by extending functions in $L^2(rdr)$ to radial functions $L^2(\mathbb{R}^2)$. Then, recalling that the vector gradient in of a radial field b on $\mathbb{R}^2$ has modulus given by

$$|\nabla b| = |\partial_r b|^2 + r^{-2}|b|^2,$$

the $H^1(rdr)$ norm of such vector fields corresponds precisely to their $H^1(\mathbb{R}^2)$ norm. Therefore, by Rellich's theorem, if we have a sequence f_n with bounded $L^2(rdr)$ norm, there will exist a (non-relabelled) subsequence, and a radial $f \in H^1(\mathbb{R}^2)$ so that $(S - \lambda)^{-1}f_n \rightarrow f$ in $L^2_{\text{loc}}(\mathbb{R}^2)$. But then, since by assumption $|r\Omega'(r)| \rightarrow 0$ as $r \rightarrow \infty$, we can pick $R > 0$ so that for all n it holds

$$\int_{|x| \geq R} |K(S - \lambda)^{-1}f_n|^2 dx \leq \frac{\delta}{8}.$$

By the local convergence, we may further pick N large enough so that for all $m, n \geq N$,

$$\int_{|x| < R} |K(S - \lambda)^{-1}f_n - K(S - \lambda)^{-1}f_m|^2 dx < \frac{\delta}{2}.$$

Then, for all $m, n \geq N$,

$$\begin{aligned} \int_{\mathbb{R}^2} |K(S - \lambda)^{-1}f_n - K(S - \lambda)^{-1}f_m|^2 dx &\leq \int_{|x| < R} |K(S - \lambda)^{-1}f_n - K(S - \lambda)^{-1}f_m|^2 dx \\ &\quad + 2 \int_{|x| \geq R} |K(S - \lambda)^{-1}f_n|^2 + |K(S - \lambda)^{-1}f_m|^2 dx < \delta. \end{aligned}$$

But note that, since all functions in the above argument are radial, we in fact have shown that there

exists a function $f \in L^2((0, \infty), r dr)$, so that $K(S - \lambda)^{-1} f_n \rightarrow f$ as $n \rightarrow \infty$. Hence, $K(S - \lambda)^{-1}$ is indeed a compact operator. Therefore, we deduce first by [Kat76, IV-1, Theorem 1.11] that $(S + K, \mathcal{D}(S))$ is closed, and by Theorem A.2.1 it follows that the essential spectrum of $\mathcal{L}$ satisfies

$$\sigma_{\text{ess}}(\mathcal{L}) = \sigma_{\text{ess}}(S) \subset \{z \in \mathbb{C} : \text{Re}(z) \leq 0\},$$

where the latter inclusion follows from the fact that S is a dissipative operator. Since furthermore it is clear that $\mathcal{L} - \lambda$ is invertible for $\text{Re}(\lambda)$ sufficiently large, we conclude by Theorem A.1.5 that the component of the spectrum of $\mathcal{L}$ contained in the right half-plane consists of at most countably many points, each being isolated eigenvalues with finite multiplicity. But now, note that by Lemma 6.4.1, any eigenfunction of the operator $\mathcal{L}$ in X is in $\mathcal{D}(S)$, and so the result follows.

Finally, we shall briefly mention how this result may be extended to the scenario when $r \in [p, \infty)$. In this case, whilst integration by parts for the b_r component proceeds without issue, the b_θ component requires some additional care. Indeed, we have

$$\int_p^\infty \frac{d}{dr} (r b'_\theta) \overline{b_\theta} dr = \int_p^\infty \frac{d}{dr} ((r b_\theta)' - b_\theta) \overline{b_\theta} dr = - \int_p^\infty r |b'_\theta|^2 dr - \int_p^\infty 2 \text{Re}(b'_\theta \overline{b_\theta}) dr.$$

Therefore, in totality the energy balance equation reads

$$\text{Re} \langle -Sb, b \rangle_{L^2(r dr)} \geq \varepsilon \int_p^\infty r |\partial_r b|^2 + r^{-1} |b|^2 + k^2 |b|^2 + 2 \text{Re}(b'_\theta \overline{b_\theta}) dr.$$

But we may bound $|2 \text{Re}(b'_\theta \overline{b_\theta})| \leq \frac{1}{2} r |b'_\theta|^2 + 8 p^{-1} |b_\theta|^2$, and so, as soon as k^2 is large enough, it holds

$$\text{Re} \langle -Sb, b \rangle_{L^2(r dr)} \geq \varepsilon \int_p^\infty \frac{1}{2} r |\partial_r b|^2 + r^{-1} |b|^2 dr,$$

from where the proof proceeds entirely as before. $\square$

Lemma 6.4.3. *Let $\mathcal{I} = [0, \infty)$ or $\mathcal{I} = [p, \infty)$, for some $p > 0$. Suppose that*

$$\left| \Omega(r) - \frac{\Omega'(r_0)}{U'(r_0)} U(r) \right| \rightarrow \infty \quad (6.4.8)$$

as $r \rightarrow \infty$, and moreover suppose that there exists $C > 0, \Lambda > 0$ such that

$$\sup_{r \in (0, \infty)} \frac{|r \Omega'(r)|^2}{\left| \Omega(r) - \frac{\Omega'(r_0)}{U'(r_0)} U(r) + \Lambda \right|} \leq C.$$

Assume further that $\Omega(r) - \frac{\Omega'(r_0)}{U'(r_0)} U(r)$ is bounded from below (or from above). Then, for all $|m|, |k|$ sufficiently large, there are at most countably many $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > 0$, where $\mathcal{L} - \lambda$ is not injective on $X_{\mathcal{I}}$, and each such λ is isolated.

Proof. We provide a proof in the case where $\mathcal{I} = [0, \infty)$, $\Omega(r) - \frac{\Omega'(r_0)}{U'(r_0)} U(r)$ is bounded from below, and $m \rightarrow +\infty$. The general case follows similarly. We aim to show that $\mathcal{L}$ has a compact resolvent on $L^2(r dr)$. Once this is established, by Lemma A.2.3 $\mathcal{L}$ has empty essential spectrum on $L^2(r dr)$, and then the result follows exactly as in the previous Lemma. We once again take

$(S, \mathcal{D}(S))$ defined in (6.4.7) as the “base” operator around which we perturb. Recall that by the proof of Lemma 6.4.2 we know that $(S, \mathcal{D}(S))$ is closable, and its closure is dissipative. In fact, we claim that for all $\operatorname{Re}(\lambda)$ large enough, and for all $b \in \mathcal{D}(\bar{S})$ there holds that

$$\Im \langle (i\lambda - \bar{S})b, b \rangle_{L^2(rdr)} \geq \int_0^\infty rm(\Omega(r) - \frac{\Omega'(r_0)}{U'(r_0)}U(r) + \frac{\lambda}{m})|b|^2 dr. \quad (6.4.9)$$

Indeed, let $\operatorname{Re}(\lambda) \geq -\inf_{r \in (0, \infty)} m(\Omega(r) - \frac{\Omega'(r_0)}{U'(r_0)}U(r))$, then the inequality (6.4.9) certainly holds for any $b \in \mathcal{D}(S)$. Proceeding as in Lemma 6.4.2, we take a sequence $b_n \rightarrow b$, $Sb_n \rightarrow Sb$, $b \in \mathcal{D}(\bar{S})$, and observe that there must hold

$$\Im \langle (i\lambda - \bar{S})b, b \rangle_{L^2(rdr)} \geq \liminf_{n \rightarrow \infty} \int_0^\infty rm(\Omega(r) - \frac{\Omega'(r_0)}{U'(r_0)}U(r) + \frac{\lambda}{m})|b_n|^2 dr. \quad (6.4.10)$$

Since the right-hand side of (6.4.10) is non-negative, by Fatou’s Lemma we in fact see that we obtain (6.4.9) as claimed. Furthermore, from Lemma 6.4.2 we also know that

$$\operatorname{Re} \langle -\bar{S}b, b \rangle_{L^2(rdr)} \geq \varepsilon \int_0^\infty r|\partial_r b|^2 dr$$

for all $b \in \mathcal{D}(\bar{S})$. Once again we abuse notation and identify S with its closure. From (6.4.8), (6.4.9) we can conclude by an identical argument as in the proof of Lemma 6.4.2 that S has a compact resolvent on $L^2((0, \infty), rdr)$.

Next, we show that in fact $S + K$ is also a closed operator with compact resolvent. Indeed, let us begin by showing that $(S + K, \mathcal{D}(S))$ is closed. To do so, we let $\lambda = m\lambda_0$, where $\lambda_0 \geq -\inf_{r \in (0, \infty)} (\Omega(r) - \frac{\Omega'(r_0)}{U'(r_0)}U(r)) + 1$. Then, the inequality (6.4.9) yields

$$\|B\|_{L^2(rdr)} \|(i\lambda - S)^{-1}b\|_{L^2(rdr)} \geq m \|(i\lambda - S)^{-1}b\|_{L^2(rdr)}^2,$$

and so

$$\|B\|_{L^2(rdr)} \geq m \|(i\lambda - S)^{-1}b\|_{L^2(rdr)}.$$

In particular, we observe that

$$\|(i\lambda - S)^{-1}\|_{L^2(rdr) \rightarrow L^2(rdr)} \leq m^{-1}.$$

Furthermore, we see from (6.4.9) that

$$\begin{aligned} \|K(i\lambda - S)^{-1}b\|_{L^2(rdr)}^2 &\leq \int_0^\infty r|r\Omega'(r)|^2 |(i\lambda - S)^{-1}b|^2 dr \\ &\leq m^{-1} \sup_{r \geq 0} \frac{|r\Omega'(r)|^2}{|\Omega(r) - \frac{\Omega'(r_0)}{U'(r_0)}U(r) + \lambda|} \int_0^\infty rm|\Omega(r) - \frac{\Omega'(r_0)}{U'(r_0)}U(r) + \lambda| |(i\lambda - S)^{-1}b|^2 dr \\ &\leq Cm^{-1} \|b\|_{L^2(rdr)} \|(i\lambda - S)^{-1}b\|_{L^2(rdr)} \leq Cm^{-2} \|b\|_{L^2(rdr)}^2. \end{aligned}$$

Therefore, we conclude that for $b \in \mathcal{D}(S)$,

$$\|Kb\| \leq Cm^{-1} \|(i\lambda - S)b\|_{L^2(rdr)} \leq Cm^{-1} \|Sb\|_{L^2(rdr)} + Cm^{-1} |\lambda| \|b\|_{L^2(rdr)}.$$

In the language of Appendix A, K is S -bounded with S -bound Cm^{-1} . Thus, picking m large enough, by Theorem A.3.2, it holds that $(S + K, D(S))$ is closed for all m large enough. Next, we show that it still admits a compact resolvent. To do so, we once again use the observation that $\|K(i\lambda - S)\|_{L^2(rdr) \rightarrow L^2(rdr)} \leq Cm^{-1}$. We write

$$(K + S - i\lambda) = (K(S - i\lambda)^{-1} + \text{Id})(S - i\lambda).$$

Since $\|K(i\lambda - S)^{-1}\|_{L^2(rdr) \rightarrow L^2(rdr)} < 1$ for m large, we deduce that $(K(S - i\lambda)^{-1} + \text{Id})$ is invertible, and hence so is $(K + S - i\lambda)$, as a product of invertible operators. In fact, we have

$$(K + S - i\lambda)^{-1} = (S - i\lambda)^{-1}(K(S - i\lambda)^{-1} + \text{Id})^{-1}$$

is a compact operator, as a composition of a compact operator with a bounded operator. Hence, $S + K$ has a compact resolvent, and so the proof is complete. $\square$

6.4.3 The divergence-free condition

The next crucial application of the a-priori bound from Lemma 6.4.1 is to the question of whether a given growing mode is in fact divergence free. It is worth noting that, at least on a formal level, this property should certainly be satisfied. Taking divergence of the equation

$$\varepsilon \Delta B + B \cdot \nabla u - u \cdot \nabla B = \lambda B,$$

we see that $\text{div}(B)$ satisfies the stationary advection diffusion equation

$$\varepsilon \Delta \text{div}(B) - u \cdot \nabla \text{div}(B) = \lambda \text{div}(B), \tag{6.4.11}$$

see also e.g. [CG95]. If it were possible to integrate by parts in (6.4.11), the divergence free property would immediately follow as soon as $\text{Re}(\lambda) > 0$. Therefore, we need at least that $\text{div}(B) \in H^1$, which crucially follows from Lemma 6.4.1. Hence, we have the following result.

Lemma 6.4.4. *Let $b \in X$ define a growing mode for the kinematic dynamo equation with Ponomarenko velocity field via $B(r, \theta, z) = b(r)e^{i(m\theta + kz)}$ on $\mathbb{R}^2 \times \mathbb{T}$. Then, $\text{div}(B) = 0$.*

Proof. Recall that we have found λ with $\text{Re}(\lambda) > 0$, and a vector field B so that

$$\varepsilon \Delta B + B \cdot \nabla u - u \cdot \nabla B = \lambda B,$$

and so from (6.4.11) we have that $\text{div}(B)$ satisfies

$$\varepsilon \Delta \text{div}(B) - u \cdot \nabla \text{div}(B) = \lambda \text{div}(B). \tag{6.4.12}$$

We claim this is enough to deduce that $\text{div}(B) = 0$. Indeed, rewriting this in cylindrical coordinates, we see that

$$\varepsilon \left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{m^2}{r^2} - k^2 \right) \text{div}(B) - i(m\Omega(r) + kU(r)) \text{div}(B) = \lambda \text{div}(B).$$

But now, recall that in cylindrical coordinates, the divergence is given by

$$\operatorname{div}(B) = \frac{1}{r}(B_r + mB_\theta) + \partial_r B_r + kB_z.$$

In particular, following Lemma 6.4.1, we find that if $B \in X = L^\infty((0, \infty), (1 + (\varepsilon^{-1/3}|r - r_0|)^N)dr)$, then there holds that

1. $\operatorname{div}(B) \in L^\infty((0, \infty), (1 + (\varepsilon^{-1/3}|r - r_0|)^{N-1})dr)$,
2. $\partial_r \operatorname{div}(B) \in L^\infty((0, \infty), (1 + (\varepsilon^{-1/3}|r - r_0|)^{N-2})dr)$.

Furthermore, iterating the argument from Lemma 6.4.1, we in fact deduce that $\operatorname{div}(B) \in C^2((0, \infty))$. Hence, we now multiply (6.4.12) by $\overline{\operatorname{div}(B)}$ and integrate by parts in the left-hand side to get, for any $R > 0$,

$$\begin{aligned} & \int_0^R r \left[\varepsilon \left(\frac{1}{r} \partial_r(r \partial_r) - \frac{m^2}{r^2} - k^2 \right) \operatorname{div}(B) - i(m\Omega(r) + kU(r)) \right] \operatorname{div}(B) \overline{\operatorname{div}(B)} dr \\ &= - \int_0^R \left[\varepsilon \left(r |\partial_r \operatorname{div}(B)|^2 + \frac{m^2}{r} |\operatorname{div}(B)|^2 + k^2 r |\operatorname{div}(B)|^2 \right) + ir(m\Omega(r) + kU(r)) |\operatorname{div}(B)|^2 \right] dr \\ & \quad + \varepsilon |R \operatorname{div}(B(R)) \partial_r \operatorname{div}(B(R))|. \end{aligned}$$

Taking $R \rightarrow \infty$, we observe from (6.4.12)

$$\begin{aligned} & - \int_0^\infty \varepsilon \left(\left[r |\partial_r \operatorname{div}(B)|^2 + \left(\frac{m^2}{r} + rk^2 \right) |\operatorname{div}(B)|^2 \right] + ir(m\Omega(r) + kU(r)) |\operatorname{div}(B)|^2 \right) dr \\ &= \lambda \int_0^\infty r |\operatorname{div}(B)|^2 dr. \end{aligned}$$

If $\operatorname{div}(B)$ is non-zero, the real part of the left-hand side is negative, whilst the real part of the right-hand side is positive. Therefore, we deduce that $\operatorname{div}(B) = 0$, and the claim of the lemma follows. $\square$

When proving a similar result on domains with boundaries, we need to take additional care, since the perfectly conducting boundary conditions need to first be transformed into appropriate boundary conditions for the divergence. Nevertheless, we have the following result.

Lemma 6.4.5. *Let b define a growing mode for the kinematic dynamo equations with Ponomarenko velocity field via $B(r, \theta, z) = b(r)e^{i(m\theta + kz)}$ on a domain of the form $[p, q] \times \mathbb{T}^2$, where p may be either zero or a finite number, and q may be either a finite number larger than p or ∞ . Suppose further that b satisfies perfectly conducting boundary conditions on the boundary of the domain. Then, $\operatorname{div}(B) = 0$.*

Proof. Denote by $\hat{n}$ the outward normal to the domain, and take the dot product of the eigenvalue equation with $\hat{n}$, to deduce (using that $B \cdot \hat{n} = 0$ on the boundary)

$$(u \cdot \nabla B - B \cdot \nabla u) \cdot \hat{n} = \varepsilon \Delta B \cdot \hat{n}.$$

Now since $\hat{n}$ is simply a vector pointing in the radial direction, $(B \cdot \nabla u) \cdot \hat{n} = 0$, and similarly, using

the modal form we see $(u \cdot \nabla B) \cdot \hat{n} = (im\Omega(r) + ikU(r))B_r = 0$ on the boundary of the domain. Therefore, on $\partial\mathcal{M}$, it holds $\Delta B \cdot \hat{n} = 0$. Furthermore, note that $\Delta = \nabla \times \nabla - \nabla(\operatorname{div})$. An explicit computation shows that $(\nabla \times \nabla \times B) \cdot \hat{n} = 0$ on $\partial\mathcal{M}$, and so it holds $\nabla(\operatorname{div} B) \cdot \hat{n} = 0$. But this is precisely what we need, since now it remains to multiply (6.4.12) by $\overline{\operatorname{div}(B)}$ and observe that the boundary terms vanish due to the condition $\nabla(\operatorname{div} B) \cdot \hat{n} = 0$ on $\partial\mathcal{M}$. Thus we deduce that $\operatorname{div}(B) = 0$. $\square$

6.5 Proof of Theorems 6.1.1 and 6.1.2 in the full space

This section is devoted to the proof of the main theorems, that provide the existence of a growing mode, and gives an asymptotic profile of such mode as $\varepsilon \rightarrow 0$. For the convenience of the reader, let us state a more concise version of Theorem 6.1.1 specific for the case $\mathcal{M} = \mathbb{R}^2$, whose proof will be the central point of this section.

Theorem 6.5.1. *Under Assumptions H0–H2 for the velocity field, for all ε small enough, the operator $\mathcal{L}$ has an eigenvalue in X of the form*

$$\lambda = \varepsilon^{1/3} \left[-M^2 \left(\frac{1}{r_0^2} + \frac{\Omega'_0}{U'_0} \right) + \sqrt{\frac{-2iM\Omega'_0}{r_0} - c_2^{1/2}} + o_{\varepsilon \rightarrow 0}(1) \right].$$

If moreover the velocity field satisfies H3, then the associated eigenfunction $\psi \in X$ is given by

$$\psi(r) = C \begin{pmatrix} e^{-\frac{1}{2}\varepsilon^{-2/3}c_2^{1/2}(r-r_0)^2} \\ 0 \end{pmatrix} + \psi_{\text{err}}(r),$$

with $C \in \mathbb{C} \setminus \{0\}$ and such that ψ_{err} satisfies $\|\psi_{\text{err}}\|_X = o_{\varepsilon \rightarrow 0}(1)$.

To prove Theorem 6.5.1 we need to introduce an actual inverse of $\mathcal{L} - \lambda$, given in terms of $\mathcal{G}^\lambda$ as the following composition of operators

$$(\mathcal{L} - \lambda)^{-1} = \mathcal{G}^\lambda \sum_{n=0}^{\infty} (-1)^n \left((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \operatorname{Id} \right)^n.$$

We thus crucially need to make use of the results presented in Section 6.4, as well as the following proposition that gives quantitative estimates on the operator norm of $\mathcal{G}^\lambda$, and measures the error produced by this Green's function, as and approximation the actual inverse near Gilbert's eigenvalue

$$\lambda^* = \varepsilon^{1/3} \mu^* = \varepsilon^{1/3} \left[-M^2 \left(\frac{1}{r_0^2} + \frac{\Omega'_0}{U'_0} \right) + \sqrt{\frac{-2iM\Omega'_0}{r_0} - c_2^{1/2}} \right]. \quad (6.5.1)$$

The proposition reads as follows.

Proposition 6.5.2. *Let $\eta \in \mathbb{C} \setminus \{0\}$ be such that $|\operatorname{Re}(-c_2^{1/2}\eta)| \leq 2$, pick $\gamma \in (\frac{2}{9}, \frac{1}{3})$ and $\omega \in (\frac{2}{9}, \gamma)$. Assume H0–H2, and let $\mathcal{R}$ be the compact interval in r_0 guaranteed by Lemma 6.3.4. Fix any*

compact set $\mathcal{N} \subset \mathbb{R} \setminus \{0\}$, and let

$$\lambda = \varepsilon^{1/3} \left(\eta - M^2 \left[\frac{1}{r_0^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right] + \sqrt{\frac{-2iM\Omega'_0}{r_0} - c_2^{1/2}} \right).$$

Then, there exists a Green's function $\mathcal{G}^\lambda$ such that the following estimates are satisfied uniformly for $f \in Y$ with $\text{supp}(f) \subset [0, \infty)$, $\varepsilon > 0$ sufficiently small, $r_0 \in \mathcal{R}$, and $M \in \mathcal{N}$,

$$\|\mathcal{G}^\lambda f\|_X \lesssim \varepsilon^{-1/3} \|f\|_Y, \quad (6.5.2)$$

$$\|((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id})f\|_Y \lesssim \left(\varepsilon^\gamma + \varepsilon^{3\gamma-2/3} + \varepsilon^{2/3-2\gamma} + \varepsilon^\beta + \varepsilon^{2\omega-4/9} \right) \|f\|_Y,$$

where the coefficient $\beta > 0$ is defined in [P3](#) and all bounds are uniform in compact subsets of η . In particular, $\mathcal{G}^\lambda$ may be written as

$$\mathcal{G}^\lambda f = \mathcal{G}_2^\lambda(f \mathbb{1}_{[r_0-\varepsilon^\gamma, r_0+\varepsilon^\gamma]}) + \mathcal{G}_{\text{extra}}(f \mathbb{1}_{\mathbb{R}_+ \setminus [r_0-\varepsilon^\gamma, r_0+\varepsilon^\gamma]}).$$

Here $\mathcal{G}_{\text{extra}}$ does not depend on λ , and $\mathcal{G}_2^\lambda$ is the exact inverse of $\mathcal{L}_2 - \lambda$ on $(-\infty, \infty)$, where $\mathcal{L}_2$ denotes a differential operator for which $\lambda_\star \in \mathbb{C}$ as defined in [\(6.5.1\)](#) is an isolated eigenvalue, with associated eigenfunction

$$f_\star = \begin{pmatrix} e^{-\frac{1}{2}\varepsilon^{-2/3}c_2^{1/2}(r-r_0)^2} \\ 0 \end{pmatrix} \quad (6.5.3)$$

on $L^\infty((-\infty, \infty) \max\{1, r^2\} w_\varepsilon(r - r_0))$.

The proof of this statement shall be postponed to the subsequent [Section 6.6](#), since the construction of the Green's function $\mathcal{G}^\lambda$ entails some of the most technical and convoluted arguments presented in this chapter.

As a last extra step before proceeding with the proof of [Theorem 6.5.1](#), we need a small interlude to discuss some preliminary results about the properties of the operator $\mathcal{L}$.

Lemma 6.5.3. *The unbounded operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ with domain $\mathcal{D}(\mathcal{L})$ given by*

$$\mathcal{D}(\mathcal{L}) = \{b \in H_{\text{loc}}^2(0, \infty) \cap X : \mathcal{L}b \in X\}$$

is closed.

Proof. We begin by noting that, since any $b \in \mathcal{D}(\mathcal{L})$ is in H_{loc}^2 , the action of $\mathcal{L}$ on $\mathcal{D}(\mathcal{L})$ is well defined, and hence the definition of $\mathcal{D}(\mathcal{L})$ makes sense. Suppose now that we have a sequence $(B_n)_n$ and two elements $b, h \in X$ such that

$$b_n \rightarrow b, \quad \mathcal{L}b_n \rightarrow h$$

as $n \rightarrow \infty$ in X . Fix a smooth compactly supported vector valued $\phi \in C_c^\infty(0, \infty)$. Then, it holds

$$\int_0^\infty \mathcal{L}B_n(x) \cdot \phi(x) dx = \int_0^\infty b_n(x) \cdot \mathcal{L}^* \phi(x) dx \rightarrow \int_0^\infty b(x) \cdot \mathcal{L}^* \phi(x) dx$$

as $n \rightarrow \infty$, where $\mathcal{L}^*$ denotes the formal adjoint of $\mathcal{L}$, and convergence follows since $\mathcal{L}^*\phi$ is a compactly supported, bounded function. However, by assumption it also holds that

$$\int_0^\infty \mathcal{L}b_n(x) \cdot \phi(x) dx \rightarrow \int_0^\infty h(x) \cdot \phi(x) dx$$

and so we deduce that

$$\int_0^\infty b(x) \cdot \mathcal{L}^*\phi(x) dx = \int_0^\infty h(x) \cdot \phi(x) dx \quad (6.5.4)$$

for any $\phi \in C_c^\infty(0, \infty)$. We thus conclude that $\mathcal{L}b = h$ on $(0, \infty)$ in a distributional sense. We note that since $H \in X$, it holds that $H \in L_{\text{loc}}^2(0, \infty)$. Therefore, by elliptic regularity, see for instance [Fol96, Theorem 6.33], there holds that $b \in H_{\text{loc}}^2(0, \infty)$, so that in fact $b \in \mathcal{D}(\mathcal{L})$. Finally, since $b \in H_{\text{loc}}^2(0, \infty)$ and ϕ has compact support we may integrate by parts in (6.5.4) to obtain

$$\int_0^\infty \mathcal{L}b \cdot \phi(x) dx = \int_0^\infty h(x) \cdot \phi(x) dx$$

which by the Fundamental Lemma of Calculus of Variations implies that $\mathcal{L}b = h$ almost everywhere on $(0, \infty)$, and this completes the proof. $\square$

With this tool in our bag, we can directly move on to the proof of Theorem 6.5.1.

Proof of Theorem 6.5.1. The proof of this theorem will be split in three different pieces. First of all we will use the approximated Green's functions to construct a resolvent operator, secondly we will define a Riesz projector to prove that $\mathcal{L}$ has a spectral point nearby the candidate eigenvalue, and finally we will prove the existence of the corresponding eigenfunction.

Step 1. We begin by using the approximate Green's function $\mathcal{G}^\lambda$ to construct a resolvent operator. As further elaborated in Section 6.6, $\mathcal{G}^\lambda$ is given in terms of a linear combination of different Green's functions associated to approximations of the problem in different regions,

$$\mathcal{G}^\lambda f = \mathcal{G}_2^\lambda(f \mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]}) + \mathcal{G}_{\text{extra}}(f \mathbb{1}_{\mathbb{R}_+ \setminus [r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]}),$$

where $\mathcal{G}_{\text{extra}}$ is defined accordingly depending on the number of zeroes of the transport function $T(r)$, e.g. if $\mathcal{F}_0$ from Assumption H1 is empty, then

$$\mathcal{G}_{\text{extra}}(f \mathbb{1}_{\mathbb{R}_+ \setminus [r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]}) = \mathcal{G}_0(f \mathbb{1}_{[0, \varepsilon^\gamma]}) + \mathcal{G}_1(f \mathbb{1}_{[\varepsilon^\gamma, r_0 - \varepsilon^\gamma]}) + \mathcal{G}_3(f \mathbb{1}_{[r_0 + \varepsilon^\gamma, \infty)}).$$

Crucially, the superscript λ appears exclusively in $\mathcal{G}_2^\lambda$ since it is the only piece of the approximate Green's function that depends on the choice of $\lambda = \mu \varepsilon^{1/3}$, regardless of the number of zeroes of $T(r)$. Hence, we define μ to be given by an expression of the form

$$\mu = -M^2 \left(\frac{1}{r_0^2} + \frac{\Omega'_0}{U'_0} \right) + \sqrt{\frac{-2iM\Omega'_0}{r_0}} - c_2^{1/2} - \eta,$$

where $\eta \in \mathbb{C}$ is chosen as in Proposition 6.5.2, namely $\eta \neq 0$ and sufficiently small in modulus. For any fixed η , we find that $\lambda = \mu \varepsilon^{1/3}$ traces out a circumference of radius $\varepsilon^{1/3}|\eta| > 0$ around

Gilbert's eigenvalue

$$\varepsilon^{1/3} \mu_* = \varepsilon^{1/3} \left[-M^2 \left(\frac{1}{r_0^2} + \frac{\Omega'_0}{U'_0} \right) + \sqrt{\frac{-2iM\Omega'_0}{r_0}} - c_2^{1/2} \right]$$

in the complex plane. We shall denote this curve by $\Gamma \subset \mathbb{C}$. Next in order, notice from Proposition 6.5.2 that there exists a positive number $a > 0$ such that for any $f \in Y$ and any $\varepsilon > 0$ sufficiently small,

$$\|(\mathcal{L} - \lambda)\mathcal{G}^\lambda f - f\|_Y \lesssim \varepsilon^a \|f\|_Y. \quad (6.5.5)$$

In addition, we claim that the inverse of the exact operator $\mathcal{L} - \lambda$ is given by the map

$$\mathcal{G}^\lambda_{\mathcal{L}} = \mathcal{G}^\lambda \sum_{n=0}^{\infty} (-1)^n \left((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id} \right)^n. \quad (6.5.6)$$

First of all we will show that (6.5.6) is a right inverse. Indeed, notice from (6.5.5) that

$$\left\| \left((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id} \right)^n \right\| \lesssim \varepsilon^{na},$$

which is summable in n for ε small. Here we use $\|\cdot\|$ to denote the operator norm $Y \rightarrow Y$. Moreover, since $\mathcal{G}^\lambda$ maps Y to X , we deduce that the operator $(\mathcal{L} - \lambda)^{-1}$ is well-defined. In addition, for any fixed $N \in \mathbb{N}$ we can estimate

$$\begin{aligned} & \left\| (\mathcal{L} - \lambda)\mathcal{G}^\lambda \sum_{n=0}^N (-1)^n \left((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id} \right)^n f - f \right\|_Y \\ &= \left\| \sum_{n=0}^N (-1)^n \left((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id} \right)^{n+1} f + \sum_{n=0}^N (-1)^n \left((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id} \right)^n f - f \right\|_Y \\ &= \left\| \sum_{n=1}^{N+1} (-1)^{n-1} \left((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id} \right)^n f - \sum_{n=0}^N (-1)^{n-1} \left((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id} \right)^n f - f \right\|_Y \\ &= \left\| (-1)^N \left((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id} \right)^{N+1} f \right\|_Y \lesssim \varepsilon^{(N+1)a} \|f\|_Y, \end{aligned}$$

which converges to zero as $N \rightarrow \infty$. Therefore, mimicking the proof of closedness of the operator from Lemma 6.5.3, there holds that $\mathcal{G}^\lambda_{\mathcal{L}}$ maps X into $\mathcal{D}(\mathcal{L})$, and $(\mathcal{L} - \lambda)\mathcal{G}^\lambda_{\mathcal{L}} = \text{Id}$. We now mention a subtle point that needs to be dealt with. Throughout this section, we will frequently argue by saying that $\mathcal{G}^\lambda_{\mathcal{L}} = (\mathcal{L} - \lambda)^{-1}$, where $(\mathcal{L} - \lambda)^{-1}$ denotes the functional analytic resolvent, and thus derive properties about said resolvent using our explicit form for $\mathcal{G}^\lambda_{\mathcal{L}}$. However, thus far we have only showed that $\mathcal{G}^\lambda_{\mathcal{L}}$ is a *right inverse* to $(\mathcal{L} - \lambda)$, and so $(\mathcal{L} - \lambda)^{-1}$ may not even exist. From the existence of a right inverse, we know that $(\mathcal{L} - \lambda)$ is surjective, but injectivity is still an open question. However, in Lemmas 6.4.2 and 6.4.3 from Section 6.4, we show that under H3, $\mathcal{L} : \mathcal{D}(\mathcal{L}) \rightarrow X$ admits at most countably many, isolated eigenvalues with positive real part, which do not have an accumulation point. Thus, given our Jordan curve $\Gamma \subset \{\text{Re}(z) > 0\}$ for which $\mathcal{G}^\lambda_{\mathcal{L}}$ defines a right inverse of $\mathcal{L} - \lambda$ for all λ in a neighbourhood of Γ , we may simply deform Γ slightly and conclude that there exists a curve $\tilde{\Gamma}$ on which both $\mathcal{L} - \lambda$ is injective, and $\mathcal{G}^\lambda_{\mathcal{L}}$ is a right inverse for $\mathcal{L} - \lambda$. Hence, for any $\lambda \in \tilde{\Gamma}$, by surjectivity we know that there exists a

(possibly unbounded) inverse map $(\mathcal{L} - \lambda)^{-1}$, and for any $f \in X$ it holds

$$(\mathcal{L} - \lambda)((\mathcal{L} - \lambda)^{-1}f - \mathcal{G}_{\mathcal{L}}^{\lambda}f) = 0,$$

so that by the injectivity of $\mathcal{L} - \lambda$, it must hold $(\mathcal{L} - \lambda)^{-1}f = \mathcal{G}_{\mathcal{L}}^{\lambda}f$ for all $f \in X$. Hence $(\mathcal{L} - \lambda)^{-1} = \mathcal{G}_{\mathcal{L}}^{\lambda}$, and since $\mathcal{G}_{\mathcal{L}}^{\lambda}$ is bounded, we conclude that $\lambda \in \rho(\mathcal{L})$, the resolvent set of $\mathcal{L}$. Hence, (6.5.6) gives a well-defined right and left inverse of the operator $\mathcal{L} - \lambda$ for all $\lambda \in \tilde{\Gamma}$ and $\varepsilon > 0$ sufficiently small. From now on, we shall abuse notation and identify Γ with $\tilde{\Gamma}$. We claim that from this, we can deduce the existence of a growing mode for $\mathcal{L}$, and thus complete the proof of Theorem 6.5.1.

Step 2. At this stage we will show that the operator $\mathcal{L}$ has a spectral point inside the curve Γ . In order to do so we define the Riesz projector

$$Pf = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \mathcal{L})^{-1} f d\lambda.$$

We introduce this object due to the fact that standard properties (see Lemma A.3.1) of Riesz projectors ensure that if there exists $\phi \in X$ such that $P\phi \neq 0$, then $\mathcal{L}$ has a spectral point in the interior of Γ , see e.g. [Kat76]. The goal now is to find such ϕ . We use Proposition 6.5.2 to find the estimate

$$\begin{aligned} \left\| \mathcal{G}^{\lambda} \sum_{n=1}^{\infty} (-1)^n \left((\mathcal{L} - \lambda) \mathcal{G}^{\lambda} - \text{Id} \right)^n f \right\|_X &\lesssim \varepsilon^{-1/3} \left\| \sum_{n=1}^{\infty} (-1)^n \left((\mathcal{L} - \lambda) \mathcal{G}^{\lambda} - \text{Id} \right)^n f \right\|_Y \\ &\lesssim \varepsilon^{-1/3} \sum_{n=1}^{\infty} \varepsilon^{na} \|f\|_Y \leq \varepsilon^{a-1/3} \|f\|_Y. \end{aligned}$$

In particular, note that

$$\begin{aligned} Pf - \frac{1}{2\pi i} \int_{\Gamma} (-\mathcal{G}^{\lambda}) f d\lambda &= \frac{1}{2\pi i} \int_{\Gamma} \left[(\lambda - \mathcal{L})^{-1} - (\lambda - \mathcal{L})^{-1} (\lambda - \mathcal{L}) (-\mathcal{G}^{\lambda}) \right] f d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma} (\mathcal{L} - \lambda)^{-1} \left((\mathcal{L} - \lambda) \mathcal{G}^{\lambda} - \text{Id} \right) f d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma} \mathcal{G}^{\lambda} \sum_{n=0}^{\infty} (-1)^n \left((\mathcal{L} - \lambda) \mathcal{G}^{\lambda} - \text{Id} \right)^{n+1} f d\lambda, \end{aligned}$$

hence, combining both expressions we obtain

$$\begin{aligned} \left\| Pf - \frac{1}{2\pi i} \int_{\Gamma} (-\mathcal{G}^{\lambda}) f d\lambda \right\|_X &\lesssim \varepsilon^{1/3} |\eta| \sup_{\lambda \in \Gamma} \left\| \mathcal{G}^{\lambda} \sum_{n=1}^{\infty} (-1)^{n-1} \left((\mathcal{L} - \lambda) \mathcal{G}^{\lambda} - \text{Id} \right)^n f \right\|_X \\ &\lesssim \varepsilon^a |\eta| \|f\|_Y, \end{aligned}$$

where recall that $\varepsilon^{1/3} |\eta|$ is the radius of the circle $\Gamma \subset \mathbb{C}$. Furthermore, note that when integrating $\mathcal{G}^{\lambda}$ in Γ , all components of the Green's function $\mathcal{G}^{\lambda}$ that do not depend on λ —namely G_{extra} —will vanish due to Cauchy's Theorem. Hence, using the decomposition of Proposition 6.5.2, we can write

$$\frac{1}{2\pi i} \int_{\Gamma} (-\mathcal{G}^{\lambda}) f d\lambda = \frac{1}{2\pi i} \int_{\Gamma} (-\mathcal{G}_2^{\lambda}) (f \mathbb{1}_{[r_0 - \varepsilon^{\gamma}, r_0 + \varepsilon^{\gamma}]}) d\lambda.$$

Therefore, as a candidate we want to choose a function that has the property of being an eigenfunction for the approximated operator $\mathcal{L}_2$, namely the operator corresponding to Green's function $\mathcal{G}_2^\lambda$. This operator is defined by

$$\mathcal{L}_2 V = \varepsilon \partial_r^2 V - \left[c_2 \varepsilon^{-1/3} (r - r_0)^2 - \varepsilon^{1/3} M^2 \left(\frac{1}{r_0^2} + \frac{\Omega'_0}{U'_0} \right) \right] V + \sqrt{\frac{-2iM\Omega'_0}{r_0}} \begin{pmatrix} V_1 \\ -V_2 \end{pmatrix}$$

see Section 6.6.1 for further explanation. We pick

$$f_\star = \begin{pmatrix} e^{-\frac{1}{2}\varepsilon^{-2/3}c_2^{1/2}(r-r_0)^2} \\ 0 \end{pmatrix},$$

because it is an eigenfunction of the operator $\mathcal{L}_2$ in $L^\infty(\mathbb{R}, w_\varepsilon(r - r_0) \max\{1, r^2\})$ with an isolated eigenvalue

$$\lambda_\star = \mu_\star \varepsilon^{1/3} = \varepsilon^{1/3} \left[-M^2 \left(\frac{1}{r_0^2} + \frac{\Omega'_0}{U'_0} \right) + \sqrt{\frac{-2iM\Omega'_0}{r_0}} - c_2^{1/2} \right].$$

The obtention of this eigensystem is a straightforward computation that we include in the Appendix 6.8.2 for the convenience of the reader. Since $\mathcal{G}_2^\lambda$ is the Green's function associated to the operator $\mathcal{L}_2 - \lambda$, there holds that

$$\frac{1}{2\pi i} \int_\Gamma (-\mathcal{G}_2^\lambda) f_\star d\lambda = f_\star, \quad (6.5.7)$$

due to standard properties of the Riesz projector, since $\lambda_\star \in \text{int}(\Gamma)$. Moreover, for any $\varepsilon > 0$ sufficiently small we have that

$$\|f_\star(r) - f_\star \mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]}(r)\|_X = \sup_{|r - r_0| \geq \varepsilon^\gamma} |w_\varepsilon(r - r_0) e^{-\varepsilon^{-2/3}(r - r_0)^2 \text{Re}(c_2^{1/2})}| \leq \varepsilon \|f_\star\|_X,$$

where we crucially need $\gamma < 1/3$. Hence, using (6.5.7) and that the Riesz projector is a bounded operator with norm of order $\mathcal{O}(1)$ as $\varepsilon \rightarrow 0$, we observe that

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_\Gamma (-\mathcal{G}^\lambda) f_\star d\lambda \right\|_X &\geq \|f_\star\|_X - \left\| \frac{1}{2\pi i} \int_\Gamma (-\mathcal{G}^\lambda) (f_\star - f_\star \mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]}) d\lambda \right\|_X \\ &\geq (1 - \varepsilon) \|f_\star\|_X. \end{aligned}$$

Finally, we combine all the estimates to conclude with

$$\begin{aligned} \|Pf_\star\|_X &\geq \left\| \frac{1}{2\pi i} \int_\Gamma (-\mathcal{G}^\lambda) f_\star d\lambda \right\|_X - \left\| Pf_\star - \frac{1}{2\pi i} \int_\Gamma (-\mathcal{G}^\lambda) f_\star d\lambda \right\|_X \\ &\gtrsim (1 - \varepsilon - \varepsilon^a) \|f_\star\|_X. \end{aligned}$$

Therefore we see that if $\varepsilon > 0$ is small enough, then $\|Pf_\star\|_X > 0$, and thus the range of the Riesz projector is non-empty. *Step 3.* Last, we move on to showing that we can actually infer the existence of an eigenfunction, since so far we have only shown that there exists a spectral value of $\mathcal{L}$ in the interior of Γ . To show that an eigenfunction exists it suffices to prove that the range of P is finite dimensional, and indeed, we will show that it is one-dimensional. We claim that for

any $\psi \in X$ in the range of P with $\|\psi\|_X = 1$, there exist a constant $C \in \mathbb{C}$, $C \neq 0$, and a function $\psi_{\text{err}} \in X$ so that

$$\psi = Cf_\star + \psi_{\text{err}}.$$

Moreover, ψ_{err} satisfies $\|\psi_{\text{err}}\|_X \leq c_0 \varepsilon^a$ for some $a > 0$, $c_0 > 0$ independent of ψ . Once this is established, an application of Riesz' Lemma immediately implies that the range of P is one dimensional. To see this, on the one hand we notice that the range of the operator

$$\frac{1}{2\pi i} \int_{\Gamma} (-\mathcal{G}_2^\lambda) d\lambda,$$

is precisely $\text{span}(f_\star)$. On the other hand, using these properties we can write

$$\begin{aligned} \|\psi - Cf_\star\|_X &\leq \left\| \psi - \frac{1}{2\pi i} \int_{\Gamma} (-\mathcal{G}_2^\lambda) (\mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]} \psi) d\lambda \right\|_X \\ &\quad + \left\| \frac{1}{2\pi i} \int_{\Gamma} (-\mathcal{G}_2^\lambda) (\mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]} \psi) d\lambda - Cf_\star \right\|_X. \end{aligned}$$

The first addend in the right hand side is bounded by $\varepsilon^a |\eta|$, following the same techniques that we addressed in Step 2. The second addend directly vanishes, upon an appropriate choice of the constant C . All in all, the claim of the theorem follows from here, assuming [H3](#).

Finally, we remark that, if [H3](#) is not satisfied and the injectivity of the operator $\mathcal{L} - \lambda$ cannot be established, we can still guarantee the existence of a growing mode with an eigenvalue $\lambda \in \Gamma$ of the form stated in Theorem [6.5.1](#). Indeed, if injectivity fails for some λ close to $\varepsilon^{1/3} \mu_\star$, then by definition there exists some $\phi \in \ker(\mathcal{L} - \lambda)$, and so we have a growing mode. Hence, the proof of the theorem is complete. $\square$

With this result under our belt, we now prove the existence of growing modes for the kinematic dynamo equations under the additional assumption that the compact set $\mathcal{R}_0$ from [H2](#) contains an interval.

Proof of Theorem [6.1.2](#). Under [H2](#), the bounds in Proposition [6.5.2](#) are uniform for $r_0 \in \mathcal{R}$, and M in any compact subset $\mathcal{N} \subset \mathbb{R} \setminus \{0\}$. Therefore, for any fixed compact $\mathcal{N} \subset \mathbb{R} \setminus \{0\}$, we may pick $\varepsilon > 0$ small enough so that the conclusion of Theorem [6.5.1](#) holds uniformly in $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$. Now, in order for a function of the form $b(r)$ to define a growing mode of the kinematic dynamo equations via the modal form $B(r, \theta, z) = b(r) e^{i\varepsilon^{-1/3}(M\theta + Kz)}$, it must be that $\varepsilon^{-1/3}M, \varepsilon^{-1/3}K \in \mathbb{Z}$. However, note that K is related to M via

$$K = -\frac{\Omega'(r_0)}{U'(r_0)}M.$$

Thus, fix now some $M \in \mathcal{N}$, and note that for all $\varepsilon > 0$ small enough, there exists $\tilde{M}(\varepsilon) \in [-1, 1]$ so that $(M + \varepsilon^{1/3}\tilde{M}(\varepsilon))\varepsilon^{-1/3} \in \mathbb{Z}$. Next, note that the quotient $\Omega'(r_0)/U'(r_0)$ is not constant for $r_0 \in \mathcal{R}$. Indeed, suppose that it was constant for $r_0 \in \mathcal{R}$. Then, we compute

$$T'(r) = U'(r) \left(\frac{\Omega'(r)}{U'(r)} - \frac{\Omega'(r_0)}{U'(r_0)} \right) = 0$$

for $r \in \mathcal{R}$. But this implies that $T(r) = T(r_0) = 0$ for all $r \in \mathcal{R}$, which contradicts [H1](#). Furthermore, $\Omega'(r_0)/U'(r_0)$ defines a continuous function in r_0 , since by assumption Ω, U are C^3 . In particular, there exists $r_0 \in \mathcal{R}$ so that

$$r_0 \left| \frac{d}{dr} \log \left| \frac{\Omega'(r_0)}{U'(r_0)} \right| \right| < 4,$$

and this must further in fact hold true in some subinterval of $\mathcal{R}$, and so we replace $\mathcal{R}$ by this subinterval. By the intermediate value theorem, the image of $\mathcal{R}$ under the map

$$r \mapsto \frac{\Omega'(r)}{U'(r)}$$

contains a non-empty interval $[p_1, q_1] \subset \mathbb{R}$. Thus, having fixed M so that $M\varepsilon^{-1/3} \in \mathbb{Z}$, as soon as $|q_1 - p_1| > M^{-1}\varepsilon^{1/3}$, there exists $r_0 \in \mathcal{R}$ so that

$$\frac{\Omega'(r_0)}{U'(r_0)} M\varepsilon^{-1/3} \in \mathbb{Z}.$$

Hence, for all $\varepsilon > 0$ sufficiently small, we may pick $M \in \mathcal{N}$, $r_0 \in \mathcal{R}$ according to the above recipe, deduce the existence of a growing mode for $\mathcal{L}$ with these choices of M, r_0 , and thus define a growing mode for the kinematic dynamo equations of the form $B(r, \theta, z, t) = b(r)e^{i\varepsilon^{-1/3}(M\theta + Kz) + \lambda t}$, with $\text{Re}(\lambda) = \varepsilon^{1/3}(\mu + o_{\varepsilon \rightarrow 0}(1))$, as desired. $\square$

6.6 Construction of the Green's functions: proof of Proposition 6.5.2

In this section, we conduct a detailed analysis of the operator $\mathcal{G}^\lambda$, which serves as an approximation to the inverse of $\mathcal{L} - \lambda$ in the space Y . As outlined in Section 6.5, this operator plays a crucial role in defining the true inverse (6.5.6), hence this section is mostly devoted to proving the quantitative estimates from Proposition 6.5.2. Following the framework introduced in Section 6.5, we assume throughout this section that the problem is posed on the full space $\mathcal{M} = \mathbb{R}^2$, thereby avoiding complications related to boundary conditions. The arguments here presented will be extended to the case of domains with boundaries in Section 6.7.

The complexity of the Green's function $\mathcal{G}^\lambda$ is directly influenced by the number of zeros of the transport function $T(r)$, which is determined by the (given) velocity field u . Under Assumption [H1](#), $T(r)$ will always vanish up to second order only at $r = r_0$, and it might vanish linearly in a finite collection of points $s_j \neq r_0$. Since the construction of $\mathcal{G}^\lambda$ is intricate, we will first consider the following extra assumption:

SH The transport function $T(r)$ only vanishes at $r = r_0$, i.e. the set $\mathcal{F}_0$ from [H1](#) is empty.

This simplification allows us to present the core ideas in a more streamlined manner. We maintain Assumption [SH](#) throughout Sections 6.6.1, 6.6.2, 6.6.3 and 6.6.4, removing it only for Section 6.6.5, where Proposition 6.5.2 is proved in full generality.

Assumptions [H0–H3](#) together with [SH](#) already gives a very wide class of admissible vector fields, including polynomials, or examples with finite kinetic energy, e.g. $\Omega(r) = e^{-ar^2}$, $U(r) = e^{-br^2}$ with $a \neq b$. In this scenario, the approximate Green's function $\mathcal{G}^\lambda$ is defined as the sum of the four

contributions,

$$\mathcal{G}^\lambda f = \mathcal{G}_0(f\mathbb{1}_{[0,\varepsilon^\gamma)}) + \mathcal{G}_1(f\mathbb{1}_{[\varepsilon^\gamma, r_0 - \varepsilon^\gamma)}) + \mathcal{G}_2^\lambda(f\mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma)}) + \mathcal{G}_3(f\mathbb{1}_{[r_0 + \varepsilon^\gamma, \infty)}), \quad (6.6.1)$$

associated respectively to the contributions near the origin, towards the origin, around the critical radius, and towards infinity. The operators $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2^\lambda$ and $\mathcal{G}_3$ are themselves Green's functions associated to differential equations that approximate the exact problem (6.3.14)–(6.3.15) in a tailor-made manner for each of their domains of definition.

The claim of Proposition 6.5.2 under the additional assumption SH is a straightforward consequence of Propositions 6.6.1, 6.6.7, 6.6.10 and 6.6.11, that will be discussed in the subsequent sections. As it can be observed from the definition of $\mathcal{G}^\lambda$ in (6.6.1), we divide the support of $f \in Y$ in four distinctive parts: $[0, \varepsilon^\gamma)$, $[\varepsilon^\gamma, r_0 - \varepsilon^\gamma)$, $[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma)$, and $[r_0 + \varepsilon^\gamma, \infty)$. Each piece of this puzzle presents its own intrinsic difficulties. In order to prove Proposition 6.5.2 in full generality we need to combine the arguments from Sections 6.6.1–6.6.4 with Lemma 6.6.13. In particular, the complete proof is discussed by the end of Section 6.6.5.

6.6.1 Green's function around the critical radius

We start by studying the problem in the region where we expect the growth we are seeking to occur, that is, near the critical radius $r_0 > 0$. We look for an operator for which we can compute exact inverses, and that approximates the exact operator $\mathcal{L} - \lambda$ for values of r very close to r_0 . We will do so by looking at $\mathcal{L} - \lambda$ in (6.3.14)–(6.3.15) and removing terms that are sufficiently small in this regime. Thus, we drop D_ε^+ and D_ε^- since they will be very small as $r \rightarrow r_0$. Moreover, we eliminate all terms in $\mathcal{L}$ of order *strictly* larger than $1/3$ in ε , which effectively produces an approximation of equations (6.3.14) and (6.3.15) that is fully decoupled. Notice that crucially we do not drop the term $\lambda = \mu\varepsilon^{1/3}$.

We also need to find suitable estimates for the transport term $T(r)$, which corresponds to a term of order $-1/3$ in ε . To do so, we fall back on the intuition provided by Gilbert in [Gil88] (see also Section 6.3.2). Observe that a Taylor expansion for Ω and U around r_0 yields

$$\Omega - \Omega_0 = \Omega'_0(r - r_0) + \frac{1}{2}\Omega''_0(r - r_0)^2 + \Omega_{\text{hot}},$$

$$U - U_0 = U'_0(r - r_0) + \frac{1}{2}U''_0(r - r_0)^2 + U_{\text{hot}},$$

where both Ω_{hot} and U_{hot} are $O(|r - r_0|^3)$. Therefore, we can write

$$i\varepsilon^{-1/3}T(r) = \frac{i\varepsilon^{-1/3}}{2} \left(\Omega''_0 - \frac{\Omega'_0}{U'_0} U''_0 \right) (r - r_0)^2 + T_{\text{hot}},$$

since the linear terms in r vanish thanks to Gilbert's condition (6.3.4), i.e. $M\Omega'_0 + KU'_0 = 0$. It is important to notice that the remainder T_{hot} is of order

$$T_{\text{hot}} = O(\varepsilon^{-1/3}|r - r_0|^3), \quad (6.6.2)$$

and by Lemma 6.3.4, P2, the $O(\varepsilon^{-1/3}|r-r_0|^3)$ is in fact *uniform* in $r_0 \in \mathcal{R}$. Finally, we will also omit the first order derivatives $\varepsilon r^{-1}\partial_r$ for the approximated operator. All in all, we obtain the following equations,

$$\varepsilon \partial_r^2 V_1 - c_2 \varepsilon^{-1/3} (r-r_0)^2 V_1 - \varepsilon^{1/3} \left(M^2 \left[\frac{1}{r_0^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right] + \mu - \sqrt{\frac{-2iM\Omega'_0}{r_0}} \right) V_1 = f_1, \quad (6.6.3)$$

$$\varepsilon \partial_r^2 V_2 - c_2 \varepsilon^{-1/3} (r-r_0)^2 V_2 - \varepsilon^{1/3} \left(M^2 \left[\frac{1}{r_0^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right] + \mu + \sqrt{\frac{-2iM\Omega'_0}{r_0}} \right) V_2 = f_2, \quad (6.6.4)$$

where recall that following Gilbert's notation (6.3.5), we introduce the constant

$$c_2 = \frac{iM}{2} \left(\Omega''_0 - \frac{\Omega'_0}{U'_0} U''_0 \right).$$

We denote by $\mathcal{L}_2$ the operator associated to this system of ODEs, excluding the factor $\lambda = \mu \varepsilon^{1/3}$ in its definition, namely the system (6.6.3)–(6.6.4) can be written as

$$(\mathcal{L}_2 - \lambda)V = f.$$

Throughout this chapter, unless otherwise specified, V_1 and V_2 will denote the components of the vector $V = (V_1, V_2)$, and similarly we write $f = (f_1, f_2)$.

As is standard, one way of inverting the operator $\mathcal{L}_2 - \lambda$ is by computing a Green's operator $\mathcal{G}_2^\lambda$, which is related to its corresponding Green's kernel G_2^λ by

$$(\mathcal{L}_2 - \lambda)V(r) = f(r), \quad V(r) = \mathcal{G}_2^\lambda f(r) = \int_0^\infty G_2^\lambda(r, s) f(s) ds.$$

We now endeavour to find estimates on this Green's operator $\mathcal{G}_2^\lambda$. Indeed, for any f compactly supported in a small neighbourhood of r_0 we find the following result.

Proposition 6.6.1. *Let $\eta \in \mathbb{C}$, $\eta \neq 0$, be such that $|\operatorname{Re}(-c_2^{1/2}\eta)| \leq 2$, assume H0–H2, and choose a coefficient $\gamma > \frac{2}{9}$. Let $\mathcal{R}$ be the compact set from Lemma 6.3.4, and let $\mathcal{N} \subset \mathbb{R} \setminus \{0\}$ be any compact set. Let $\mathcal{G}_2^\lambda$ be the Green's function associated to the system of equations (6.6.3)–(6.6.4), with*

$$\mu = \eta - M^2 \left[\frac{1}{r_0^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right] + \sqrt{\frac{-2iM\Omega'_0}{r_0}} - c_2^{1/2}.$$

Then, uniformly for $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$, and for $f \in Y$ such that $\operatorname{supp}(f) \subset [r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]$, the following estimates hold true for all $\varepsilon > 0$ sufficiently small.

$$\|\mathcal{G}_2^\lambda f\|_X \lesssim \varepsilon^{-1/3} \|f\|_Y, \quad (6.6.5)$$

$$\|\partial_r \mathcal{G}_2^\lambda f\|_X \lesssim \varepsilon^{-1+\gamma} \|f\|_Y, \quad (6.6.6)$$

$$\|((\mathcal{L} - \mu \varepsilon^{1/3})\mathcal{G}_2^\lambda - \operatorname{Id})f\|_Y \lesssim (\varepsilon^\gamma + \varepsilon^{3\gamma-2/3}) \|f\|_Y. \quad (6.6.7)$$

Furthermore, all estimates are uniform in compact subsets of η .

Remark 6.6.2. As it will be clear throughout the proof of the proposition, estimate (6.5.2) corresponds to the worst case scenario, that occurs when $r \in \text{supp}(f)$. If $r \notin \text{supp}(f)$ we crucially find an exponentially decaying estimate of the form

$$|w_\varepsilon(r - r_0)\mathcal{G}_2^\lambda f(r)| \lesssim \varepsilon^{-1/3} e^{-\frac{1}{2}\chi\varepsilon^{-2/3}(|r-r_0|^2 - \varepsilon^{2\gamma})} \|f\|_Y,$$

for some constant $\chi > 0$ defined in Lemma 6.6.3. This property will be essential in order to ensure that the *global* error estimate (6.6.7) holds.

Before proceeding with the proof of the proposition, let us make a comment on how to derive the Green's function associated to the approximated problem (6.6.3)–(6.6.4). Notice that now (6.6.3)–(6.6.4) consists of two decoupled equations that can be studied individually, hence we can derive a separate Green's function for each component,

$$\mathcal{G}_2^\lambda f = \begin{pmatrix} \mathcal{G}_{2,1}^\lambda f_1 \\ \mathcal{G}_{2,2}^\lambda f_2 \end{pmatrix}.$$

We start by analysing $\mathcal{G}_{2,1}^\lambda$, then it will become clear that the analysis of $\mathcal{G}_{2,2}^\lambda$ is completely analogous. In order to derive a Green's function, we first look for two independent solutions of (6.6.3). This equation falls into the umbrella term of *Weber differential equations*, which are ordinary differential equations of the form

$$y''(z) + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2 \right) y(z) = 0, \quad (6.6.8)$$

with $\nu \in \mathbb{C}$. Two independent solutions to this Weber equations can be found by means of the *parabolic cylinder functions*: $y_1(z) = D_\nu(z)$, $y_2(z) = D_\nu(-z)$, and their Wronskian is given by

$$W(y_1(z), y_2(z)) = \frac{\sqrt{2\pi}}{\Gamma(-\nu)},$$

see [Leb72] and Appendix 6.8.1 for extended information about this type of special functions. We can rewrite equation (6.6.3) in the form

$$\nu''(z) - \zeta^2 \varepsilon^{2\beta} \left(q(\mu) \varepsilon^{-2/3} + c_2 \zeta^2 \varepsilon^{2\beta-4/3} z^2 \right) \nu(z) = 0,$$

where we made the change of variables $\zeta \varepsilon^\beta z = r - r_0$, and set

$$q(\mu) = M^2 \left(\frac{1}{r_0^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right) + \mu - \sqrt{\frac{-2iM\Omega'_0}{r_0}}.$$

In order to obtain an equation as (6.6.8), we make the choice $\beta = 1/3$ and $c_2 \zeta^4 = 1/4$, and without loss of generality we pick

$$\zeta = \frac{c_2^{-1/4}}{\sqrt{2}},$$

so that

$$v''(z) + \left(-\frac{1}{2}c_2^{-1/2}q(\mu) - \frac{1}{4}z^2 \right) v(z) = 0.$$

Therefore, we find solutions to this Weber differential equation in terms of parabolic cylinder functions with index

$$v = -\frac{1}{2} \left(c_2^{-1/2}q(\mu) + 1 \right) = -\frac{1}{2}c_2^{-1/2}\eta,$$

for some number $\eta \in \mathbb{C}$ such that

$$\eta = q(\mu) + c_2^{1/2} = \mu + M^2 \left(\frac{1}{r_0^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right) - \sqrt{\frac{-2iM\Omega'_0}{r_0}} + c_2^{1/2}.$$

Notice that to study the second component of the differential equation (6.6.4), we follow the same analysis with the sole exception that in the definition of $q(\mu)$ we write

$$q(\mu) = \mu + M^2 \left(\frac{1}{r_0^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right) + \sqrt{\frac{-2iM\Omega'_0}{r_0}}.$$

We crucially choose the branch $\text{Re}(\sqrt{-i}) > 0$, in this way we observe that the (real part of the) index v corresponding the second component (6.6.4) will be strictly smaller than the one associated to the first component and it suffices to study the first component, see Remark 6.6.5. Putting everything together, we find two independent solutions to the (first component of the) Weber differential equation (6.6.3),

$$v_1(r) = D_{-\frac{1}{2}c_2^{-1/2}\eta} \left(\sqrt{2}c_2^{1/4}\varepsilon^{-1/3}(r-r_0) \right), \quad (6.6.9)$$

$$v_2(r) = D_{-\frac{1}{2}c_2^{-1/2}\eta} \left(-\sqrt{2}c_2^{1/4}\varepsilon^{-1/3}(r-r_0) \right), \quad (6.6.10)$$

with Wronskian given by

$$W(v_1(r), v_2(r)) = W(v_1(z), v_2(z)) \frac{dz}{dr} = \frac{2\sqrt{\pi}c_2^{1/4}}{\Gamma\left(\frac{1}{2}c_2^{-1/2}\eta\right)} \varepsilon^{-1/3} =: \frac{1}{\mathfrak{w}(\eta)} \varepsilon^{-1/3}.$$

The parabolic cylinder functions admit a representation in terms of the *Hermite functions*, denoted by $H_v(z)$. The relation between these special functions is of the form

$$D_v(z) = 2^{-\frac{1}{2}v} e^{-\frac{1}{4}z^2} H_v \left(\frac{z}{\sqrt{2}} \right).$$

Since for some specific calculations it will be more convenient to use this different representation of the two independent solutions. We see thus that (6.6.9)–(6.6.10) can be written as

$$v_1(r) = 2^{\frac{1}{4}c_2^{-1/2}} \eta e^{-\frac{1}{2}c_2^{1/2}\varepsilon^{-2/3}(r-r_0)^2} H_{-\frac{1}{2}c_2^{-1/2}\eta} \left(c_2^{1/4}\varepsilon^{-1/3}(r-r_0) \right),$$

$$v_2(r) = 2^{\frac{1}{4}c_2^{-1/2}} \eta e^{-\frac{1}{2}c_2^{1/2}\varepsilon^{-2/3}(r-r_0)^2} H_{-\frac{1}{2}c_2^{-1/2}\eta} \left(-c_2^{1/4}\varepsilon^{-1/3}(r-r_0) \right),$$

where $H_v(z)$ denotes the Hermite function of order $v \in \mathbb{C}$. For further information about these special functions and some relevant related results we refer to Appendix 6.8.1.

The next step is to define a Green's function associated to equation (6.6.3), as well as to obtain suitable bounds on this Green's function. In order to get there, we begin by noting the following convenient bounds for the solutions $v_1(r)$ and $v_2(r)$,

Lemma 6.6.3. *Let us define*

$$\chi = \frac{1}{2}|c_2|^{-1/2} > 0, \quad \text{and} \quad \nu = -\frac{1}{2}c_2^{-1/2}\eta.$$

Then, $v_1(r)$ and $v_2(r)$ defined in (6.6.9)–(6.6.10) satisfy the following bounds uniformly for c_2, ν in compact sets of $\mathbb{C}$:

1. *If $r < r_0$,*

$$\begin{aligned} |v_1(r)| &\lesssim e^{\chi\epsilon^{-2/3}(r-r_0)^2}, \\ |v_2(r)| &\lesssim e^{-\chi\epsilon^{-2/3}(r-r_0)^2} \left(1 + \epsilon^{-1/3}|r-r_0|\right)^{\operatorname{Re}(\nu)}. \end{aligned}$$

2. *If $r \geq r_0$,*

$$\begin{aligned} |v_1(r)| &\lesssim e^{-\chi\epsilon^{-2/3}(r-r_0)^2} \left(1 + \epsilon^{-1/3}|r-r_0|\right)^{\operatorname{Re}(\nu)}, \\ |v_2(r)| &\lesssim e^{\chi\epsilon^{-2/3}(r-r_0)^2}. \end{aligned}$$

The proof of this lemma follows from standard properties of the parabolic cylinder functions, exploiting the fact that they can be written in terms of Hermite functions. In particular, it is a direct consequence of Lemma 6.8.5 in Appendix 6.8.1.

Remark 6.6.4. As is the case throughout the document, many of our estimates are written using the $\lesssim$ symbol for a more convenient presentation of results. In particular, we typically absorb all constants that do not depend on ϵ into the symbol $\lesssim$. However, it is worth noting that these constants may depend on $\nu \in \mathbb{C}$, and hence on $\eta \in \mathbb{C}$, which is a parameter that must be treated carefully later on. However, it is clear from our estimates that they are upper bounded by the case when $\operatorname{Re}(\nu) = 1$. Thus, proving the result in this case automatically yields the entire range $|\operatorname{Re}(\nu)| \leq 1$.

The last preparation required before proceeding with the proof of Proposition 6.6.1 is the construction of the Green's function, that we do in the usual fashion. Taking the two independent solutions $v_1(r)$ and $v_2(r)$ and its Wronskian, we write a Green's kernel

$$G_{2,1}^\lambda(r,s) = \mathfrak{w}(\eta)\epsilon^{-2/3} \begin{cases} v_1(r)v_2(s) & \text{if } s < r, \\ v_1(s)v_2(r) & \text{if } r \leq s. \end{cases}$$

We find the solution to the first component of our problem (6.6.3) via

$$\begin{aligned} v(r) &= (\mathcal{G}_{2,1}^\lambda f_1)(r) = \int_0^\infty G_{2,1}^\lambda(r,s) f_1(s) ds \\ &= \mathfrak{w}(\eta) \varepsilon^{-2/3} v_1(r) \int_0^r v_2(s) f_1(s) ds + \mathfrak{w}(\eta) \varepsilon^{-2/3} v_2(r) \int_r^\infty v_1(s) f_1(s) ds. \end{aligned} \quad (6.6.11)$$

Using the bounds from Lemma 6.6.3, we can write a suitable pointwise estimate for the Green's function. This estimate is tailor made for the case near r_0 , therefore we crucially assume that

$$\text{supp}(f_1) \subset [r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma],$$

for some $\gamma > 0$ to be specified later. Recall that we use the notation from Lemma 6.6.3,

$$\chi = \frac{1}{2} |c_2|^{-1/2} > 0, \quad \mathbf{v} = -\frac{1}{2} c_2^{-1/2} \eta.$$

Finding upper bounds for the Green's function will turn out to be rather cumbersome, since it requires a careful splitting of the integrals in (6.6.11) depending on the value of r . Indeed, using Lemma 6.6.3 we find the following estimate for the Green's function,

$$|\mathcal{G}_{2,1}^\lambda f(r)| \lesssim \varepsilon^{-2/3} (\mathbf{I}_\varepsilon + \mathbf{I}'_\varepsilon + \mathbf{II}_\varepsilon + \mathbf{III}_\varepsilon + \mathbf{IV}_\varepsilon + \mathbf{IV}'_\varepsilon), \quad (6.6.12)$$

where

$$\begin{aligned} \mathbf{I}_\varepsilon &= \mathbb{1}_{\{r_0 - \varepsilon^\gamma \leq r < r_0\}} e^{\chi \varepsilon^{-2/3} (r-r_0)^2} \int_{r_0 - \varepsilon^\gamma}^r \left(1 + \varepsilon^{-1/3} |s - r_0|\right)^{\text{Re}(\mathbf{v})} e^{-\chi \varepsilon^{-2/3} (s-r_0)^2} |f_1(s)| ds, \\ \mathbf{I}'_\varepsilon &= \mathbb{1}_{\{r_0 \leq r\}} \left(1 + \varepsilon^{-1/3} |r - r_0|\right)^{\text{Re}(\mathbf{v})} e^{-\chi \varepsilon^{-2/3} (r-r_0)^2} \\ &\quad \times \int_{r_0 - \varepsilon^\gamma}^{r_0} \left(1 + \varepsilon^{-1/3} |s - r_0|\right)^{\text{Re}(\mathbf{v})} e^{-\chi \varepsilon^{-2/3} (s-r_0)^2} |f_1(s)| ds, \\ \mathbf{II}_\varepsilon &= \left(1 + \varepsilon^{-1/3} |r - r_0|\right)^{\text{Re}(\mathbf{v})} e^{-\chi \varepsilon^{-2/3} (r-r_0)^2} \int_{\min\{r, r_0\}}^r e^{\chi \varepsilon^{-2/3} (s-r_0)^2} |f_1(s)| ds, \\ \mathbf{III}_\varepsilon &= \left(1 + \varepsilon^{-1/3} |r - r_0|\right)^{\text{Re}(\mathbf{v})} e^{-\chi \varepsilon^{-2/3} (r-r_0)^2} \int_r^{\max\{r, r_0\}} e^{\chi \varepsilon^{-2/3} (s-r_0)^2} |f_1(s)| ds, \\ \mathbf{IV}_\varepsilon &= \mathbb{1}_{\{r_0 \leq r < r_0 + \varepsilon^\gamma\}} e^{\chi \varepsilon^{-2/3} (r-r_0)^2} \int_r^{r_0 + \varepsilon^\gamma} \left(1 + \varepsilon^{-1/3} |s - r_0|\right)^{\text{Re}(\mathbf{v})} e^{-\chi \varepsilon^{-2/3} (s-r_0)^2} |f_1(s)| ds, \\ \mathbf{IV}'_\varepsilon &= \mathbb{1}_{\{r < r_0\}} \left(1 + \varepsilon^{-1/3} |r - r_0|\right)^{\text{Re}(\mathbf{v})} e^{-\chi \varepsilon^{-2/3} (r-r_0)^2} \\ &\quad \times \int_{r_0}^{r_0 + \varepsilon^\gamma} \left(1 + \varepsilon^{-1/3} |s - r_0|\right)^{\text{Re}(\mathbf{v})} e^{-\chi \varepsilon^{-2/3} (s-r_0)^2} |f_1(s)| ds. \end{aligned}$$

We leave it to the reader to verify that these bounds are precisely what one obtains by applying Lemma 6.6.3 carefully to the expression (6.6.11). Note that by our assumptions, $|\text{Re}(\mathbf{v})| \leq 1$, and we shall thus simply prove all our upper bounds for $\text{Re}(\mathbf{v}) = 1$, showing that the estimates are

indeed uniform for $|\operatorname{Re}(\nu)| \leq 1$.

Remark 6.6.5. In the proof of Proposition 6.6.1 we will always argue for (6.6.3), namely only the first component of equations. We claim that our arguments readily carry over to the second component (6.6.4) because of the choice of the branch

$$\operatorname{Re} \left(\sqrt{\frac{-2iM\Omega'_0}{r_0}} \right) > 0,$$

in equations (6.6.3)–(6.6.4). This option ensures that the real part of the index ν corresponding to the second component (6.6.4) will be strictly smaller, and therefore the condition

$$\operatorname{Re}(\nu) = \operatorname{Re} \left(-\frac{1}{2} c_2^{1/2} \eta \right) \leq 1,$$

required for (6.6.3), will suffice as well for (6.6.4). Everything else follows analogously.

We now proceed with the proof of Proposition 6.6.1. Since many of the arguments throughout Section 6.6 will be quite similar in flavour to those appearing in this proof, we endeavour to provide a substantial amount of detail in the following.

Proof of Proposition 6.6.1. We will argue for equation (6.6.3), since, as mentioned in Remark 6.6.5, the same arguments carry over to the analysis of equation (6.6.4). Let $f \in Y$ be such that $\operatorname{supp}(f) \subset [r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]$. First of all we address (6.6.5), and in order to do so we will find suitable bounds for each of the six addends in (6.6.12). To do so, we must carefully split our analysis into sections, depending on the value of r . Furthermore, we shall take care to ensure that our estimates hold true *uniformly* for $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$. In particular, we this dependency primarily shows up in the guise of the constant χ .

Step 1. Assume $0 \leq r < r_0 - \varepsilon^\gamma$, then all the addends in (6.6.12) vanish except for III_ε and IV'_ε . For the third addend we can write

$$\begin{aligned} \varepsilon^{-2/3} \text{III}_\varepsilon &\leq \varepsilon^{-2/3} \left(1 + \varepsilon^{-1/3} |r - r_0| \right) e^{-\chi \varepsilon^{-2/3} (r - r_0)^2} \int_{r_0 - \varepsilon^\gamma}^{r_0} e^{\chi \varepsilon^{-2/3} (s - r_0)^2} |f(s)| ds \\ &\leq \|f\|_Y \frac{\varepsilon^{-2/3} (1 + \varepsilon^{-1/3} |r - r_0|)}{(r_0 - \varepsilon^\gamma)^2} e^{-\chi \varepsilon^{-2/3} (r - r_0)^2} \int_{r_0 - \varepsilon^\gamma}^{r_0} \frac{e^{\chi \varepsilon^{-2/3} (s - r_0)^2}}{1 + (\varepsilon^{-1/3} |s - r_0|)^N} ds. \end{aligned}$$

The main goal is to bound each of these addends in X , hence we need to make sure that when multiplied by $\max\{1, r^2\} w_\varepsilon(r - r_0)$, everything stays uniformly bounded in r and behaves well for ε small. First of all notice that if $r < r_0 - \varepsilon^\gamma$ we have the trivial bound

$$\max\{1, r^2\} \leq 1 + r_0^2 \lesssim 1.$$

After performing a change of variables $z = \varepsilon^{-1/3}(r_0 - r)$, respectively $x = \varepsilon^{-1/3}(r_0 - s)$ for the integral variable, we can write

$$\varepsilon^{-2/3} \max\{1, r^2\} w_\varepsilon(r - r_0) \text{III}_\varepsilon \lesssim \varepsilon^{-1/3} \|f\|_Y (1 + z) (1 + z^N) e^{-\chi z^2} \int_0^{\varepsilon^{-1/3 + \gamma}} \frac{e^{\chi x^2}}{1 + x^N} dx.$$

Notice that since $r \in [0, r_0 - \varepsilon^\gamma]$, this corresponds to $z \in [\varepsilon^{-1/3+\gamma}, r_0 \varepsilon^{-1/3}]$. Therefore, for all z in this region, we can write the estimate

$$\varepsilon^{-2/3} \max\{1, r^2\} w_\varepsilon(r - r_0) \text{III}_\varepsilon \lesssim \varepsilon^{-1/3} \|f\|_Y P(z),$$

where

$$P(z) = (1+z) (1+z^N) e^{-\chi z^2} \int_0^z \frac{e^{\chi x^2}}{1+x^N} dx. \quad (6.6.13)$$

We claim that $P : [0, \infty) \rightarrow \mathbb{R}_+$ is a bounded function for any $\chi > 0$. Since it is a smooth, non-negative function with $P(0) = 0$, it suffices to show that the limit as $z \rightarrow \infty$ remains bounded. Indeed, let us use L'Hôpital's rule,

$$\begin{aligned} \frac{d}{dz} \int_0^z \frac{e^{\chi x^2}}{1+x^N} dx &= \frac{e^{\chi z^2}}{1+z^N}, \\ \frac{d}{dz} \left[(1+z)^{-1} (1+z^N)^{-1} e^{\chi z^2} \right] &= -(1+z)^{-2} (1+z^N)^{-1} e^{\chi z^2} \\ &\quad - N z^{N-1} (1+z)^{-1} (1+z^N)^{-2} e^{\chi z^2} \\ &\quad + 2\chi z (1+z)^{-1} (1+z^N)^{-1} e^{\chi z^2}, \end{aligned}$$

and hence,

$$\lim_{z \rightarrow \infty} \frac{\frac{d}{dz} \int_0^z (1+x^N)^{-1} e^{\chi x^2} dx}{\frac{d}{dz} \left[(1+z)^{-1} (1+z^N)^{-1} e^{\chi z^2} \right]} = \lim_{z \rightarrow \infty} \frac{(1+z)}{2\chi z} = \frac{1}{2\chi}.$$

This argument shows that, under the assumption that for any fixed $\chi > 0$, we have $P(z) \lesssim 1$. Next, to show uniformity in χ , simply differentiate $P(z)$ in χ , yielding

$$\frac{d}{d\chi} P(z) = -z^2 P(z) + (1+z)(1+z^N) e^{-\chi z^2} \int_0^z \frac{e^{\chi x^2} x^2}{1+x^N} dx.$$

But we can simply bound

$$(1+z)(1+z^N) \int_0^z \frac{e^{\chi x^2} x^2}{1+x^N} dx \leq z^2 (1+z)(1+z^N) \int_0^z \frac{e^{\chi x^2} x^2}{1+x^N} dx = z^2 P(z),$$

and so we deduce $\frac{d}{d\chi} P(z) \leq 0$, for all $z > 0$. Thus, for χ in any compact interval of $(0, \infty)$, we have that $P(z) \lesssim 1$ *uniformly* in χ . Therefore we can write

$$\varepsilon^{-2/3} \max\{1, r^2\} w_\varepsilon(r - r_0) \text{III}_\varepsilon \lesssim \varepsilon^{-1/3} \|f\|_Y.$$

In fact, note that

$$\begin{aligned} &\varepsilon^{-2/3} w_\varepsilon(r - r_0) \text{III}_\varepsilon(r) \\ &\leq \|f\|_Y \varepsilon^{-2/3} w_\varepsilon(\varepsilon^\gamma) (1 + \varepsilon^{-1/3+\gamma}) e^{-\chi \varepsilon^{-2/3+2\gamma}} \int_{r_0-\varepsilon^\gamma}^{r_0} \frac{e^{\chi \varepsilon^{-2/3}(s-r_0)^2}}{1 + (\varepsilon^{-1/3}|s-r_0|)^N} ds \times L(r, \varepsilon), \end{aligned}$$

where

$$L(r, \varepsilon) = \frac{w_\varepsilon(r - r_0)(1 + \varepsilon^{-1/3}|r - r_0|)}{w_\varepsilon(\varepsilon^\gamma)(1 + \varepsilon^{-1/3+\gamma})} e^{-\chi \varepsilon^{-2/3}((r - r_0)^2 - \varepsilon^{2\gamma})}.$$

But by Lemma 6.8.8, we know that there exists a constant $C > 0$ independent of ε, r , depending continuously on χ , so that

$$\frac{w_\varepsilon(r - r_0)(1 + \varepsilon^{-1/3}|r - r_0|)}{w_\varepsilon(\varepsilon^\gamma)(1 + \varepsilon^{-1/3+\gamma})} \leq C e^{\frac{1}{2}\chi \varepsilon^{-2/3}((r - r_0)^2 - \varepsilon^{2\gamma})}.$$

Hence, we conclude in fact that

$$\varepsilon^{-2/3} w_\varepsilon(r - r_0) \text{III}_\varepsilon(r) \lesssim \|f\|_Y \varepsilon^{-2/3} w_\varepsilon(\varepsilon^\gamma) \text{III}_\varepsilon(r_0 - \varepsilon^\gamma) L(r, \varepsilon) \lesssim \|f\|_Y \varepsilon^{-1/3} e^{-\frac{1}{2}\chi \varepsilon^{-2/3}((r - r_0)^2 - \varepsilon^{2\gamma})},$$

uniformly for $r \leq r_0 - \varepsilon^\gamma$, $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$.

For the term IV'_ε we use that $s^{-2} w_\varepsilon(s - r_0)^{-1}$ is a decreasing function for all $s > r_0$, together with the bound

$$\left(1 + \varepsilon^{-1/3}|s - r_0|\right) e^{-\chi \varepsilon^{-2/3}(s - r_0)^2} \lesssim e^{-\frac{2}{3}\chi \varepsilon^{-2/3}(s - r_0)^2}$$

to get the estimate

$$\begin{aligned} \varepsilon^{-2/3} \text{IV}'_\varepsilon &= \varepsilon^{-2/3} \frac{1}{r_0^2 w_\varepsilon(0)} e^{-\frac{2}{3}\chi \varepsilon^{-2/3}(r - r_0)^2} \int_{r_0}^{r_0 + \varepsilon^\gamma} e^{-\frac{2}{3}\chi \varepsilon^{-2/3}(s - r_0)^2} |f(s)| ds \\ &\lesssim \|f\|_Y \varepsilon^{-1/3} e^{-\frac{2}{3}\chi \varepsilon^{-2/3}(r - r_0)^2}. \end{aligned}$$

Hence

$$\varepsilon^{-2/3} |\max\{1, r^2\} w_\varepsilon(r - r_0) \text{IV}'_\varepsilon| \lesssim \varepsilon^{-1/3} \|f\|_Y w_\varepsilon(r - r_0) e^{-\frac{2}{3}\chi \varepsilon^{-2/3}(r - r_0)^2}.$$

Finally, note that $w_\varepsilon(r - r_0) e^{-\frac{2}{3}\chi \varepsilon^{-2/3}(r - r_0)^2} \lesssim e^{-\frac{1}{2}\chi \varepsilon^{-2/3}(r - r_0)^2}$, which decays very fast as $\varepsilon \rightarrow 0$ due to the effect of the exponential. Putting all pieces together, we find that uniformly for $0 \leq r < r_0 - \varepsilon^\gamma$, $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$, there holds

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_{2,1}^\lambda f(r)| \lesssim \varepsilon^{-1/3} e^{-\frac{1}{2}\chi \varepsilon^{-2/3}((r - r_0)^2 - (\varepsilon^\gamma)^2)} \|f\|_Y. \quad (6.6.14)$$

Step 2. Assume now that $r_0 - \varepsilon^\gamma \leq r < r_0$. Note that in this regime, we once again can neglect the $\max\{1, r^2\}$ weight up to a multiplicative constant independent of ε . Furthermore, in this scenario we have $\text{I}'_\varepsilon = 0$, $\text{II}_\varepsilon = 0$, and $\text{IV}_\varepsilon = 0$, but the other three terms contribute to the error. For the first addend we write

$$\begin{aligned} \varepsilon^{-2/3} \text{I}_\varepsilon &\leq \varepsilon^{-2/3} e^{\chi \varepsilon^{-2/3}(r - r_0)^2} \int_{r_0 - \varepsilon^\gamma}^r \left(1 + \varepsilon^{-1/3}|s - r_0|\right) e^{-\chi \varepsilon^{-2/3}(s - r_0)^2} |f(s)| ds \\ &\leq \|f\|_Y \frac{\varepsilon^{-2/3}}{(r_0 - \varepsilon^\gamma)^2} e^{\chi \varepsilon^{-2/3}(r - r_0)^2} \int_{r_0 - \varepsilon^\gamma}^r \frac{(1 + \varepsilon^{-1/3}|s - r_0|)}{1 + (\varepsilon^{-1/3}|s - r_0|)^N} e^{-\chi \varepsilon^{-2/3}(s - r_0)^2} ds. \end{aligned}$$

We want to argue using L'Hôpital's rule as for Step 1. In this regard, notice that after performing the change of variables $z = \varepsilon^{-1/3}(r_0 - r)$, and $x = \varepsilon^{-1/3}(r_0 - s)$ respectively for the integral

variables, we obtain the bound

$$\varepsilon^{-2/3} w_\varepsilon(r-r_0) \mathbf{I}_\varepsilon \lesssim \varepsilon^{-1/3} \|f\|_Y Q(z),$$

with

$$Q(z) = (1+z^N) e^{\chi z^2} \int_z^\infty \frac{(1+x)}{1+x^N} e^{-\chi x^2} dx. \quad (6.6.15)$$

The upper bound of the integral after the change of variables corresponds to $\varepsilon^{-1/3+\gamma}$, however we can brutally upper-bound it by infinity. As before, we want to show that $Q(z) \lesssim 1$, indeed, this is a non-negative smooth function with the property

$$0 \leq Q(0) = \int_0^\infty \frac{(1+x)}{1+x^N} e^{-\chi x^2} dx \lesssim \int_0^\infty e^{-\chi x^2} dx \lesssim 1.$$

Therefore, as before, it suffices to show that $Q(z)$ remains bounded in the limit $z \rightarrow \infty$ in order to obtain the desired result. To see this we make use again of L'Hôpital's rule,

$$\frac{d}{dz} \int_z^\infty \frac{(1+x)}{1+x^N} e^{-\chi x^2} dx = -\frac{(1+z)}{1+z^N} e^{-\chi z^2},$$

$$\frac{d}{dz} \left[(1+z^N)^{-1} e^{-\chi z^2} \right] = -N z^{N-1} (1+z^N)^{-2} e^{-\chi z^2} - 2\chi z (1+z^N)^{-1} e^{-\chi z^2},$$

and therefore

$$\lim_{z \rightarrow \infty} \frac{\frac{d}{dz} \int_z^\infty (1+x) (1+x^N)^{-1} e^{-\chi x^2} dx}{\frac{d}{dz} \left[(1+z^N)^{-1} e^{-\chi z^2} \right]} = \lim_{z \rightarrow \infty} \frac{(1+z)}{2\chi z},$$

which, as in the previous case, is bounded by a constant, provided that $\operatorname{Re}(v) \leq 1$. All in all, we derived the estimate

$$\varepsilon^{-2/3} |\max\{1, r^2\} w_\varepsilon(r-r_0) \mathbf{I}_\varepsilon| \lesssim \varepsilon^{-1/3} \|f\|_Y,$$

for all $r \in [r_0 - \varepsilon^\gamma, r_0]$. Finally, we deduce that $\frac{d}{d\chi} Q(z) \leq 0$, and so we can extend the bound to be uniform in compact intervals in χ . Next, for the third addend we write

$$\begin{aligned} \varepsilon^{-2/3} \mathbf{III}_\varepsilon &= \varepsilon^{-2/3} \left(1 + \varepsilon^{-1/3} |r-r_0| \right) e^{-\chi \varepsilon^{-2/3} (r-r_0)^2} \int_r^{r_0} e^{\chi \varepsilon^{-2/3} (s-r_0)^2} |f(s)| ds \\ &\leq \|f\|_Y \frac{\varepsilon^{-2/3} \left(1 + \varepsilon^{-1/3} |r-r_0| \right)}{(r_0 - \varepsilon^\gamma)^2} e^{-\chi \varepsilon^{-2/3} (r-r_0)^2} \int_r^{r_0} \frac{e^{\chi \varepsilon^{-2/3} (s-r_0)^2}}{1 + (\varepsilon^{-1/3} |s-r_0|)^N} ds \end{aligned}$$

which, after multiplication by the weights from the X norm, and the usual change of variables $z = \varepsilon^{-1/3} (r_0 - r)$, $x = \varepsilon^{-1/3} (r_0 - s)$, can be bounded by

$$\varepsilon^{-2/3} |\max\{1, r^2\} w_\varepsilon(r-r_0) \mathbf{III}_\varepsilon| \lesssim \varepsilon^{-1/3} \|f\|_Y P(z),$$

where $P(z)$ is defined in (6.6.13). In particular, since we have already proved in Step 1 that $P(z) \lesssim 1$, we directly obtain the estimate

$$\varepsilon^{-2/3} |\max\{1, r^2\} w_\varepsilon(r-r_0) \mathbf{III}_\varepsilon| \lesssim \varepsilon^{-1/3} \|f\|_Y.$$

Lastly, for the fourth term we write just as in Step 1,

$$\begin{aligned} \varepsilon^{-2/3} \text{IV}'_\varepsilon &\leq \|f\|_Y \frac{\varepsilon^{-2/3} (1 + \varepsilon^{-1/3} |r - r_0|)}{r_0^2} e^{-\chi \varepsilon^{-2/3} (r - r_0)^2} \\ &\quad \times \int_{r_0}^{r_0 + \varepsilon^\gamma} (1 + \varepsilon^{-1/3} |r - r_0|) \frac{e^{-\chi \varepsilon^{-2/3} (s - r_0)^2}}{1 + (\varepsilon^{-1/3} |s - r_0|)^N} ds. \end{aligned}$$

As before, we make the change of variables $z = \varepsilon^{-1/3} (r_0 - r)$ and $x = \varepsilon^{-1/3} (r_0 - s)$ so that

$$\varepsilon^{-2/3} w_\varepsilon(r - r_0) \text{IV}'_\varepsilon \lesssim \varepsilon^{-1/3} \|f\|_Y (1 + z) (1 + z^N) e^{-\chi z^2} \int_0^\infty (1 + x)^{\text{Re}(v)} \frac{e^{-\chi x^2}}{1 + x^N} dx.$$

Due to the finiteness of the integral and the boundedness on the prefactor we can write

$$(1 + z) (1 + z^N) e^{-\chi z^2} \int_0^\infty (1 + x) \frac{e^{-\chi x^2}}{1 + x^N} dx \lesssim 1,$$

which readily yields

$$\varepsilon^{-2/3} |\max\{1, r^2\} w_\varepsilon(r - r_0) \text{IV}'_\varepsilon| \lesssim \varepsilon^{-1/3} \|f\|_Y.$$

All in all, we find that for any $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$, $r \in [r_0 - \varepsilon^\gamma, r_0)$, the approximate Green's function satisfies the estimate

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_{2,1}^\lambda f(r)| \lesssim \varepsilon^{-1/3} \|f\|_Y. \quad (6.6.16)$$

Step 3. We assume now that $r_0 \leq r < r_0 + \varepsilon^\gamma$, which directly implies $\text{I}_\varepsilon = 0$, $\text{III}_\varepsilon = 0$, and $\text{IV}'_\varepsilon = 0$. The analysis in this regime follows very closely the ideas introduced in the previous two steps. First of all we have,

$$\begin{aligned} \varepsilon^{-2/3} \text{I}'_\varepsilon &\leq \|f\|_Y \frac{\varepsilon^{-2/3} (1 + \varepsilon^{-1/3} |r - r_0|)}{(r_0 - \varepsilon^\gamma)^2} e^{-\chi \varepsilon^{-2/3} (r - r_0)^2} \\ &\quad \times \int_{r_0 - \varepsilon^\gamma}^{r_0} (1 + \varepsilon^{-1/3} |r - r_0|) \frac{e^{-\chi \varepsilon^{-2/3} (s - r_0)^2}}{1 + (\varepsilon^{-1/3} |r - r_0|)^N} ds. \end{aligned}$$

Once again, the usual change of variables $z = \varepsilon^{-1/3} (r_0 - r)$, and $x = \varepsilon^{-1/3} (r_0 - s)$ gives the estimate

$$\varepsilon^{-2/3} w_\varepsilon(r - r_0) \text{I}'_\varepsilon \lesssim \varepsilon^{-1/3} \|f\|_Y (1 + z) (1 + z^N) e^{-\chi z^2} \int_0^\infty \frac{e^{-\chi x^2}}{1 + x^N} (1 + x) dx,$$

hence using that this is a product of a finite integral and a bounded function, we readily obtain the bound

$$\varepsilon^{-2/3} |\max\{1, r^2\} w_\varepsilon(r - r_0) \text{I}'_\varepsilon| \lesssim \varepsilon^{-1/3} \|f\|_Y.$$

Next in order, for the second term we observe that

$$\begin{aligned}\varepsilon^{-2/3} \Pi_\varepsilon &= \varepsilon^{-2/3} \left(1 + \varepsilon^{-1/3} |r - r_0|\right) e^{-\chi \varepsilon^{-2/3} (r - r_0)^2} \int_{r_0}^r e^{\chi \varepsilon^{-2/3} (s - r_0)^2} |f(s)| ds \\ &\leq \|f\|_Y \frac{\varepsilon^{-2/3} (1 + \varepsilon^{-1/3} |r - r_0|)}{r_0^2} e^{-\chi \varepsilon^{-2/3} (r - r_0)^2} \int_{r_0}^r \frac{e^{\chi \varepsilon^{-2/3} (s - r_0)^2}}{1 + (\varepsilon^{-1/3} |r - r_0|)^N} ds,\end{aligned}$$

so that, invoking once again the boundedness of $P(z)$ defined in (6.6.13), we directly obtain the estimate

$$\varepsilon^{-2/3} |\max\{1, r^2\} w_\varepsilon(r - r_0) \Pi_\varepsilon| \lesssim \varepsilon^{-1/3} \|f\|_Y.$$

Finally, the forth term in the definition of the approximate Green's function can be controlled instead by using the boundedness of $Q(z)$, defined in (6.6.15), that has been proved in Step 2. We thus write,

$$\begin{aligned}\varepsilon^{-2/3} \text{IV}_\varepsilon &= \varepsilon^{-2/3} e^{\chi \varepsilon^{-2/3} (r - r_0)^2} \int_r^{r_0 + \varepsilon^\gamma} \left(1 + \varepsilon^{-1/3} |s - r_0|\right) e^{-\chi \varepsilon^{-2/3} (s - r_0)^2} |f(s)| ds \\ &\leq \|f\|_Y \frac{\varepsilon^{-2/3}}{r^2} e^{\chi \varepsilon^{-2/3} (r - r_0)^2} \int_r^{r_0 + \varepsilon^\gamma} \frac{(1 + \varepsilon^{-1/3} |s - r_0|)}{1 + (\varepsilon^{-1/3} |r - r_0|)^N} e^{-\chi \varepsilon^{-2/3} (s - r_0)^2} ds\end{aligned}$$

Therefore, since $r \in [r_0, r_0 + \varepsilon^\gamma]$ we get

$$\varepsilon^{-2/3} |\max\{1, r^2\} w_\varepsilon(r - r_0) \text{IV}_\varepsilon| \lesssim \varepsilon^{-1/3} \|f\|_Y Q(z) \lesssim \varepsilon^{-1/3} \|f\|_Y.$$

Putting everything together, we find that for the values $r_0 \leq r < r_0 + \varepsilon^\gamma$, the approximate Green's function, weighted with $\max\{1, r^2\} w_\varepsilon(r - r_0)$, is pointwise bounded by

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_{2,1}^\lambda f(r)| \lesssim \varepsilon^{-1/3} \|f\|_Y. \quad (6.6.17)$$

Step 4. The last piece missing concerns the regime $r \geq r_0 + \varepsilon^\gamma$, where we need to be slightly more careful due to the effect of the weight $\max\{1, r^2\}$. We have $\text{I}_\varepsilon = 0$, $\text{III}_\varepsilon = 0$, and $\text{IV}_\varepsilon = \text{IV}'_\varepsilon = 0$, for the other two terms we argue as before. On the one hand, for I'_ε we can write

$$\begin{aligned}\varepsilon^{-2/3} \text{I}'_\varepsilon &\leq \|f\|_Y \frac{\varepsilon^{-2/3} (1 + \varepsilon^{-1/3} |r - r_0|)}{(r_0 - \varepsilon^\gamma)^2} e^{-\chi \varepsilon^{-2/3} (r - r_0)^2} \\ &\quad \times \int_{r_0 - \varepsilon^\gamma}^{r_0} (1 + \varepsilon^{-1/3} |s - r_0|) \frac{e^{-\chi \varepsilon^{-2/3} (s - r_0)^2}}{1 + (\varepsilon^{-1/3} |r - r_0|)^N} ds.\end{aligned}$$

Since both exponentials have the correct sign, as in Step 1 we find that

$$\max\{1, r^2\} \varepsilon^{-2/3} w_\varepsilon(r - r_0) \text{I}'_\varepsilon \lesssim \varepsilon^{-1/3} \|f\|_Y,$$

and in fact, we note that

$$\begin{aligned} \max\{1, r^2\} \varepsilon^{-2/3} I'_\varepsilon(r) &\lesssim \|f\|_Y \varepsilon^{-2/3} (1 + \varepsilon^{-1/3+\gamma}) w_\varepsilon(\varepsilon^\gamma) \max\{1, r^2\} L(r, \varepsilon) \\ &\quad \times e^{-\chi \varepsilon^{-2/3}(r-r_0)^2} \int_{r_0-\varepsilon^\gamma}^{r_0} (1 + \varepsilon^{-1/3} |s-r_0|) \frac{e^{-\chi \varepsilon^{-2/3}(s-r_0)^2}}{1 + (\varepsilon^{-1/3} |r-r_0|)^N} ds, \end{aligned}$$

where as before we define

$$L(r, \varepsilon) = \frac{w_\varepsilon(r-r_0)(1 + \varepsilon^{-1/3} |r-r_0|)}{w_\varepsilon(\varepsilon^\gamma)(1 + \varepsilon^{-1/3+\gamma})} e^{-\chi \varepsilon^{-2/3}((r-r_0)^2 - \varepsilon^{2\gamma})}.$$

Once again, we observe by Lemma 6.8.8 that

$$L(r, \varepsilon) \lesssim e^{-\frac{1}{2} \chi \varepsilon^{-2/3}((r-r_0)^2 - \varepsilon^{2\gamma})},$$

uniformly in r, ε , as well as χ in compact intervals of $(0, \infty)$, and so we deduce that

$$\varepsilon^{-2/3} w_\varepsilon(r-r_0) I'_\varepsilon \lesssim \varepsilon^{-1/3} e^{-\frac{1}{2} \chi \varepsilon^{-2/3}((r-r_0)^2 - (\varepsilon^\gamma)^2)} \|f\|_Y,$$

if $r \geq r_0 + \varepsilon^\gamma$. Finally, to include the $\max\{1, r^2\}$ weight, we note that

$$\max\{1, r^2\} e^{-\frac{1}{2} \chi \varepsilon^{-2/3}((r-r_0)^2 - \varepsilon^{2\gamma})} \lesssim e^{-\frac{1}{4} \chi \varepsilon^{-2/3}((r-r_0)^2 - \varepsilon^{2\gamma})},$$

and so we deduce

$$\max\{1, r^2\} \varepsilon^{-2/3} w_\varepsilon(r-r_0) I'_\varepsilon \lesssim \varepsilon^{-1/3} e^{-\frac{1}{4} \chi \varepsilon^{-2/3}((r-r_0)^2 - (\varepsilon^\gamma)^2)} \|f\|_Y,$$

Next, we analyse Π_ε . We write,

$$\begin{aligned} \varepsilon^{-2/3} w_\varepsilon(r-r_0) \Pi_\varepsilon &\lesssim \varepsilon^{-2/3} r^2 w_\varepsilon(r-r_0) \left(1 + \varepsilon^{-1/3} |r-r_0|\right) \\ &\quad \times e^{-\chi \varepsilon^{-2/3}(r-r_0)^2} \int_{r_0}^r e^{\chi \varepsilon^{-2/3}(s-r_0)^2} |f(s)| ds \\ &\leq \|f\|_Y \varepsilon^{-2/3} w_\varepsilon(r-r_0) \left(1 + \varepsilon^{-1/3} |r-r_0|\right) e^{-\chi \varepsilon^{-2/3}(r-r_0)^2} \int_{r_0}^{r_0+\varepsilon^\gamma} \frac{e^{\chi \varepsilon^{-2/3}(s-r_0)^2}}{w_\varepsilon(s-r_0)} ds. \end{aligned}$$

On the one hand we can see that via the usual change of variables,

$$\varepsilon^{-2/3} w_\varepsilon(r-r_0) \Pi_\varepsilon \lesssim \|f\|_Y \varepsilon^{-1/3} P(z)$$

where $P(z)$ is the uniformly bounded function defined in (6.6.13). On the other hand, we once again notice that by Lemma 6.8.8, the following estimate holds true,

$$L(r, \varepsilon) e^{-\chi \varepsilon^{-2/3}(r^2 - (r_0 + \varepsilon^\gamma)^2)} \lesssim e^{-\frac{1}{2} \chi \varepsilon^{-2/3}((r-r_0)^2 - (\varepsilon^\gamma)^2)}.$$

Therefore we can write

$$\varepsilon^{-2/3} w_\varepsilon(r-r_0) \Pi_\varepsilon \lesssim \|f\|_Y \varepsilon^{-1/3} e^{-\frac{1}{2} \chi \varepsilon^{-2/3}((r-r_0)^2 - (\varepsilon^\gamma)^2)},$$

for any $r \geq r_0 + \varepsilon^\gamma$, $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$, and where the constants absorbed in $\lesssim$ are independent of all the relevant quantities. Finally, as in the previous step, we can absorb the $\max\{1, r^2\}$ weight up to reducing the coefficient in the decaying exponential, and so we obtain the estimate

$$\varepsilon^{-2/3} \max\{1, r^2\} w_\varepsilon(r - r_0) \Pi_\varepsilon \lesssim \|f\|_Y \varepsilon^{-1/3} e^{-\frac{1}{4}\chi \varepsilon^{-2/3}((r-r_0)^2 - (\varepsilon^\gamma)^2)},$$

uniformly for $r \geq r_0 + \varepsilon^\gamma$. Hence, putting all these estimates together, we find that

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_{2,1}^\lambda f(r)| \lesssim \varepsilon^{-1/3} e^{-\frac{1}{4}\chi \varepsilon^{-1/3}((r-r_0)^2 - \varepsilon^{2\gamma})} \|f\|_Y, \quad (6.6.18)$$

for $r \geq r_0 + \varepsilon^\gamma$. Therefore, by combining (6.6.14), (6.6.16), (6.6.17) and (6.6.18), we arrive at the first claim in the proposition (6.6.5). As mentioned earlier, the argument for the second component of the approximate Green's function is completely analogous.

Estimate for $\partial_r \mathcal{G}_2^\lambda$. In order to derive the sought estimate for the derivatives of $\mathcal{G}_2^\lambda$, it is convenient to regard the independent solutions $v_1(r)$ and $v_2(r)$, (6.6.9)–(6.6.10), in terms of Hermite functions instead of parabolic cylinder functions. In particular, we know that we can represent these solutions as

$$\begin{aligned} v_1(r) &= D_{-\frac{1}{2}c_2^{-1/2}\eta} \left(\sqrt{2}c_2^{1/4} \varepsilon^{-1/3}(r - r_0) \right) \\ &= 2^{\frac{1}{4}} c_2^{-1/2} \eta e^{-\frac{1}{2}c_2^{1/2} \varepsilon^{-2/3}(r-r_0)^2} H_{-\frac{1}{2}c_2^{-1/2}\eta} \left(c_2^{1/4} \varepsilon^{-1/3}(r - r_0) \right), \\ v_2(r) &= D_{-\frac{1}{2}c_2^{-1/2}\eta} \left(-\sqrt{2}c_2^{1/4} \varepsilon^{-1/3}(r - r_0) \right) \\ &= 2^{\frac{1}{4}} c_2^{-1/2} \eta e^{-\frac{1}{2}c_2^{1/2} \varepsilon^{-2/3}(r-r_0)^2} H_{-\frac{1}{2}c_2^{-1/2}\eta} \left(-c_2^{1/4} \varepsilon^{-1/3}(r - r_0) \right), \end{aligned}$$

where $H_\nu(z)$ denote the Hermite functions. Once we have the bounds for the Green's operator $\mathcal{G}_2^\lambda$, we can now easily bound its derivatives using a special relation

$$H'_\nu(z) = 2\nu H_{\nu-1}(z). \quad (6.6.19)$$

We refer to [Leb72] and Appendix 6.8.1 for more detailed information. Again, we argue for the first component of the approximate Green's function, and the second component follows readily. Differentiating the Green's function $\mathcal{G}_{2,1}^\lambda$ from (6.6.11) we find that

$$\partial_r(\mathcal{G}_{2,1}^\lambda f)(r) = \mathfrak{w}(\eta) \varepsilon^{-2/3} \left(v'_1(r) \int_0^r v_2(s) f(s) ds + v'_2(r) \int_r^\infty v_1(s) f(s) ds \right).$$

In addition, using (6.6.19) we get

$$\begin{aligned} v'_1(r) &= -c_2^{1/2} \varepsilon^{-2/3}(r - r_0) v_1(r) \\ &\quad - 2^{\frac{1}{4}} c_2^{-1/2} \eta^{-1} c_2^{-1/4} \eta \varepsilon^{-1/3} e^{-\frac{1}{2}c_2^{1/2} \varepsilon^{-2/3}(r-r_0)^2} H_{-\frac{1}{2}c_2^{-1/2}\eta-1} \left(c_2^{1/4} \varepsilon^{-1/3}(r - r_0) \right), \\ v'_2(r) &= -c_2^{1/2} \varepsilon^{-2/3}(r - r_0) v_2(r) \\ &\quad + 2^{\frac{1}{4}} c_2^{-1/2} \eta^{-1} c_2^{-1/4} \eta \varepsilon^{-1/3} e^{-\frac{1}{2}c_2^{1/2} \varepsilon^{-2/3}(r-r_0)^2} H_{-\frac{1}{2}c_2^{-1/2}\eta-1} \left(-c_2^{1/4} \varepsilon^{-1/3}(r - r_0) \right). \end{aligned}$$

If we want to compare this with the result we just proved for the Green's operator before taking the derivative, we see that $\partial_r(\mathcal{G}_{2,1}^\lambda f)(r)$ will have two main contributions: $\partial_r(\mathcal{G}_{2,1}^\lambda f)(r) = A'_\varepsilon + A''_\varepsilon$. Here we define A'_ε to be the collection of terms that arise when the derivative hits the exponential functions, whereas A''_ε consists of the collection of terms arising when the derivative hits the Hermite functions. For the first of the addends we can write

$$\begin{aligned} A'_\varepsilon &= -\mathfrak{w}(\eta)\varepsilon^{-2/3}c_2^{1/2}\varepsilon^{-2/3}(r-r_0)\left(v_1(r)\int_0^r v_2(s)f(s)ds + v_2(r)\int_r^\infty v_1(s)f(s)ds\right) \\ &= -c_2^{1/2}\varepsilon^{-2/3}(r-r_0)(\mathcal{G}_{2,1}^\lambda f)(r). \end{aligned}$$

Adding the weights corresponding to X , we arrive at the estimate

$$|\max\{1, r^2\}w_\varepsilon(r-r_0)A'_\varepsilon| \lesssim \varepsilon^{-2/3}|r-r_0||\max\{1, r^2\}w_\varepsilon(r-r_0)\mathcal{G}_{2,1}^\lambda f(r)|.$$

In particular, we see that if r is sufficiently close to r_0 , e.g. $|r-r_0| \leq 2\varepsilon^\gamma$, then we can directly use (6.6.5) and write the estimate

$$|\max\{1, r^2\}w_\varepsilon(r-r_0)A'_\varepsilon| \lesssim \varepsilon^{-1+\gamma}\|f\|_Y.$$

For the remaining uncovered regions for r we need to be make use of the exponential decay. If $r < r_0 - 2\varepsilon^\gamma$, then we are strictly inside the regime covered in Step 1 of this proof. Therefore, a direct reproduction of the proof in such case yields that $I_\varepsilon = I'_\varepsilon = 0$, $\Pi_\varepsilon = 0$ and $IV_\varepsilon = 0$. Hence, recall from Step 1 that we have the following estimate for III_ε ,

$$\varepsilon^{-2/3}|\max\{1, r^2\}w_\varepsilon(r-r_0)III_\varepsilon| \lesssim \varepsilon^{-1/3}e^{-\frac{1}{2}\chi\varepsilon^{-2/3}((r-r_0)^2-(\varepsilon^\gamma)^2)}\|f\|_Y,$$

and thus we write

$$|\max\{1, r^2\}w_\varepsilon(r-r_0)A'_\varepsilon| \lesssim \varepsilon^{-1/3}e^{-\frac{1}{2}\chi\varepsilon^{-2/3}((r-r_0)^2-(\varepsilon^\gamma)^2)}\|f\|_Y.$$

This argument carries over identically for the term IV'_ε . Moreover, following a completely analogous argument for the regime $r > r_0 + 2\varepsilon^\gamma$, we observe that, for any r it holds

$$|\max\{1, r^2\}w_\varepsilon(r-r_0)A_\varepsilon| \lesssim \varepsilon^{-1+\gamma}\|f\|_Y \left(1 + e^{-\frac{1}{4}\chi\varepsilon^{-2/3}((r-r_0)^2-(\varepsilon^\gamma)^2)}\mathbb{1}_{\{|r-r_0| \geq 2\varepsilon^\gamma\}}(r)\right).$$

The second contribution to the derivative of the Green's function concerns the case in which the derivatives fall into the Hermite functions. We find thus an expression of the form

$$\begin{aligned} A''_\varepsilon &= C\mathfrak{w}(\eta)\varepsilon^{-1}2^{\nu/2}e^{-\frac{1}{2}\zeta(r)^2}H_{\nu-1}(\zeta(r))\int_{\zeta(0)}^{\zeta(r)} e^{-\frac{1}{2}\zeta(s)^2}H_\nu(-\zeta(s))f(s)ds \\ &\quad - C\mathfrak{w}(\eta)\varepsilon^{-1}2^{\nu/2}e^{-\frac{1}{2}\zeta(r)^2}H_{\nu-1}(-\zeta(r))\int_{\zeta(0)}^{\zeta(r)} e^{-\frac{1}{2}\zeta(s)^2}H_\nu(\zeta(s))f(s)ds, \end{aligned}$$

where in order to shorten the notation, we define $\zeta(r) = c_2^{1/4}\varepsilon^{-1/3}(r-r_0)$, and C is a new constant absorbing only ε and r -independent terms. Now, in order to find appropriate estimates for A''_ε , we simply notice that the bounds from Lemma 6.6.3 are still applicable to this new Green's function,

with the exception that, when dealing with the derivatives, the exponent $\text{Re}(v)$ turns into $\text{Re}(v) - 1$. This yields no additional constraints since the only condition that we needed to impose on $\text{Re}(v)$ is $\text{Re}(v) \leq 1$. All in all, we can yet again use the bounds we derived for $\mathcal{G}_2^\lambda$ in (6.6.5), keeping in mind that now there is an extra $\varepsilon^{-1/3}$ multiplying everything. Therefore, we find as before that the largest contribution comes from the values $|r - r_0| \leq \varepsilon^\gamma$ and, it is bounded by

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) A_\varepsilon''| \lesssim \varepsilon^{-2/3} \|f\|_Y.$$

Since we assume $\gamma < 1/3$, the estimate from the A_ε' is indeed worse, and we obtain

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \partial_r \mathcal{G}_{2,1}^\lambda f(r)| \lesssim \varepsilon^{-1+\gamma} \|f\|_Y.$$

Once again, all these estimates hold uniformly in $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$. As pointed before, the estimate for the second component of the approximate Green's function follows by a completely analogous argument. *The error estimate.* The last bit remaining to be proved from Proposition 6.6.1 is the error estimate (6.6.7), that quantifies how good the approximated operator $\mathcal{L}_2 - \lambda$ is. For this we compare equations (6.3.14) associated to the operator $\mathcal{L} - \lambda$, and (6.6.3) associated to the Green's function $\mathcal{G}_2^\lambda$. On the one hand notice that by definition of the Green's function $\mathcal{G}_2^\lambda$, we have the identities

$$((\mathcal{L} - \lambda)\mathcal{G}_2^\lambda - \text{Id})f = (\mathcal{L} - \lambda)V - (\mathcal{L}_2 - \lambda)V = (\mathcal{L} - \mathcal{L}_2)\mathcal{G}_2^\lambda f.$$

Notice that, using the representation of the operator developed at the beginning of the section, for the first component we may write

$$((\mathcal{L} - \mathcal{L}_2)V)_1 = \frac{\varepsilon}{r} \partial_r V_1 - T_{\text{hot}} V_1 - \frac{iM\alpha\varepsilon}{r^2} V_1 + D_\varepsilon^- V_1 + D_\varepsilon^+ V_2 + \frac{iM\alpha\varepsilon}{r^2} V_2,$$

and analogously for the second component. Taking this into account, we write for $|r - r_0| \leq 2\varepsilon^\gamma$

$$|(\mathcal{L} - \mathcal{L}_2)\mathcal{G}_2^\lambda f| \lesssim \frac{\varepsilon}{r} |\partial_r \mathcal{G}_2^\lambda f| + \frac{\varepsilon}{r^2} |\mathcal{G}_2^\lambda f| + \varepsilon^{-1/3} |r - r_0|^3 |\mathcal{G}_2^\lambda f| + (|D_\varepsilon^-| + |D_\varepsilon^+|) |\mathcal{G}_2^\lambda f|,$$

where in particular, following (6.6.2) we use that $T_{\text{hot}} \lesssim \varepsilon^{-1/3} |r - r_0|^3$, uniformly for $r_0 \in \mathcal{R}$. Now, by definition of D_ε^+ , D_ε^- we derive the bounds

$$|D_\varepsilon^+|, |D_\varepsilon^-| \lesssim \varepsilon^{1/3} \left(\left| \frac{1}{r^2} - \frac{1}{r_0^2} \right| + |r\Omega' - r_0\Omega'_0| \right) \lesssim \varepsilon^{1/3} |r - r_0|,$$

therefore, using (6.6.5)–(6.6.6) we obtain the estimate for $|r - r_0| \leq 2\varepsilon^\gamma$,

$$\begin{aligned} |r^2(\mathcal{L} - \mathcal{L}_2)\mathcal{G}_2^\lambda f| &\lesssim \frac{\varepsilon}{r} |\partial_r \mathcal{G}_2^\lambda f| + \left(\frac{\varepsilon}{r^2} + \varepsilon^{-1/3} |r - r_0|^3 + \varepsilon^{1/3} |r - r_0| \right) |\mathcal{G}_2^\lambda f| \\ &\lesssim \varepsilon |\partial_r \mathcal{G}_2^\lambda f| + \left(\varepsilon + \varepsilon^{-1/3+3\gamma} + \varepsilon^{1/3+\gamma} \right) |\mathcal{G}_2^\lambda f| \\ &\lesssim \left(\varepsilon^{2/3} + \varepsilon^{-2/3+3\gamma} + \varepsilon^\gamma \right) \|f\|_Y. \end{aligned}$$

Observe that the right hand side converges to zero as $\varepsilon \rightarrow 0$ provided that $3\gamma - 2/3 > 0$. Next, for

$|r - r_0| \geq 2\varepsilon^\gamma$, noting that by Assumption H2, T and Ω may be bounded by some polynomial of the form $C(1+r)^{N_1}$, $C(1+r)^{N_2}$ respectively for some $N_1, N_2 \in \mathbb{N}$ and some constant $C > 0$, we find the error estimate

$$\begin{aligned} |r^2(\mathcal{L} - \mathcal{L}_2)\mathcal{G}_2^\lambda f| &\lesssim \varepsilon |r\partial_r \mathcal{G}_2^\lambda f| + \left(\varepsilon + r^2 \varepsilon^{-1/3} (1+r)^{N_1} + \varepsilon^{1/3} r^2 (1+r)^{N_2} \right) |\mathcal{G}_2^\lambda f| \\ &\lesssim \varepsilon^\gamma \|f\|_Y + \left(\varepsilon^{2/3} + \varepsilon^{-2/3} r^2 (1+r)^{N_1} + r^2 (1+r)^{N_2} \right) e^{-\frac{1}{4}\chi \varepsilon^{-\frac{2}{3}}((r-r_0)^2 - \varepsilon^{2\gamma})} \|f\|_Y \\ &\lesssim \varepsilon^\gamma \|f\|_Y, \end{aligned}$$

uniformly for $\varepsilon > 0$ small enough, $r_0 \in \mathcal{R}$, and $M \in \mathcal{N}$, and thus, the proof of Proposition 6.6.1 is complete. $\square$

6.6.2 Green's function towards infinity

In this section we will take f to be supported in the interval $[r_0 + \varepsilon^\gamma, \infty)$. The key idea to investigate the problem in this region is similar to how argued in the previous section, but with suitable changes to account for the unbounded domain and possible growth at infinity. We need to consider an appropriate approximation of (6.3.14)–(6.3.15) in this regime, and therefore, the behaviour of the general transport functions Ω and U will play a crucial role.

The strategy to find such approximated version of (6.3.14)–(6.3.15) will be to neglect all terms that converge to zero as $r \rightarrow \infty$ and $\varepsilon \rightarrow 0$, up to order $\varepsilon^{1/3}$ (included). We find the system

$$\begin{aligned} \varepsilon \partial_r^2 V_1 - iM \varepsilon^{-1/3} \left((\Omega - \Omega_0) - \frac{\Omega'_0}{U'_0} (U - U_0) \right) V_1 + \frac{\alpha \varepsilon^{1/3}}{2} r \Omega' V_2 &= f_1, \\ \varepsilon \partial_r^2 V_2 - iM \varepsilon^{-1/3} \left((\Omega - \Omega_0) - \frac{\Omega'_0}{U'_0} (U - U_0) \right) V_2 - \frac{\alpha \varepsilon^{1/3}}{2} r \Omega' V_1 &= f_2. \end{aligned}$$

Additionally, we will neglect in our approximated equation the coupled terms involving Ω' . The possible growth of Ω' at infinity must be addressed later in the analysis, but will turn that we can treat it as an error, as long as the transport term $T(r)$ grows at least as quickly as $r\Omega'(r)$. In this way we have decoupled the ODE, and moreover, we observe that both components V_1 and V_2 satisfy the same equation

$$\varepsilon \partial_r^2 V - iM \varepsilon^{-1/3} T(r) V = f, \quad (6.6.20)$$

where, as usual,

$$T(r) = \left((\Omega(r) - \Omega_0) - \frac{\Omega'_0}{U'_0} (U(r) - U_0) \right).$$

The transport term $T(r)$ at this stage could be very complex, so our strategy must be general enough to allow for a variety of behaviours of T . The crucial insight is that, as long as T is non-vanishing, the equation (6.6.20) is “very far away” from its spectrum. Therefore, we most certainly expect the operator associated to (6.6.20) to be invertible, and in fact we may hope that its inverse is very well behaved. As such, we can get away with a rather simple approach for constructing an approximate inverse, which consists of pointwise linear approximations to the function $T(r)$, which, using the exponential decay of the associated Green's functions, may be glued together to produce a global inverse. Indeed, let $\delta > 0$ to be chosen later. Select a sequence of points

$\{r_j\}_{j \geq 1}$ such that the domain $[r_0 + \varepsilon^\gamma, \infty)$ is partitioned in intervals of the form $[r_j, r_j + \varepsilon^\delta)$ where we identify

$$r_{j+1} = r_j + \varepsilon^\delta,$$

namely $r_1 = r_0 + \varepsilon^\gamma$ and

$$[r_0 + \varepsilon^\gamma, \infty) = \bigcup_{j=1}^{\infty} [r_j, r_j + \varepsilon^\delta).$$

We will denote this partition by $\mathcal{P}_3(\varepsilon^\delta)$, i.e. $r_j = r_1 + (j-1)\varepsilon^\delta \in \mathcal{P}_3(\varepsilon^\delta)$ for all $j \geq 1$. Now we evaluate the transport term precisely at r_j on each interval to get rid of the r -dependent coefficients. Indeed, we can approximate (6.6.20) by

$$\varepsilon \partial_r^2 V(r) - i\varepsilon^{-1/3} MT(r_j) V(r) = f, \quad r \in [r_j, r_j + \varepsilon^\delta). \quad (6.6.21)$$

Setting $f = 0$, this ODE has two nontrivial independent solutions

$$v_1(r) = e^{\sqrt{iMT(r_j)}\varepsilon^{-2/3}r}, \quad v_2(r) = e^{-\sqrt{iMT(r_j)}\varepsilon^{-2/3}r},$$

with Wronskian given by $W(v_1(r), v_2(r)) = 2\sqrt{iMT(r_j)}\varepsilon^{-2/3}$. Therefore, following the usual method, we can construct a Green's function $\mathcal{G}_{3,j}$ associated to the linearly approximated ODE (6.6.21) in the domain $[r_j, r_j + \varepsilon^\delta)$ with the correct decay at infinity, that is

$$\mathcal{G}_{3,j}f(r) = \frac{\varepsilon^{-1/3}}{2\sqrt{iMT(r_j)}} \left(\int_0^r e^{-\sqrt{iMT(r_j)}\varepsilon^{-2/3}(r-s)} f(s) ds + \int_r^\infty e^{-\sqrt{iMT(r_j)}\varepsilon^{-2/3}(s-r)} f(s) ds \right),$$

where we always choose the branch $\text{Re}(\sqrt{iMT(r_j)}) > 0$, regardless of the sign of $MT(r_j)$. Then we define the Green's function associated to the complete interval $[r_0 + \varepsilon^\gamma, \infty)$ by a linear combination

$$\mathcal{G}_3 f(r) = \sum_{j=1}^{\infty} \mathcal{G}_{3,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^\delta)})(r). \quad (6.6.22)$$

Observe that to have a well-defined Green's function, we need to make sure, among other things, that $T(r_j) \neq 0$ for all $j \geq 1$, which is directly ensured by Assumptions H0–H3 together with the extra assumption SH.

Without further ado, let us introduce the main results corresponding to the solution towards infinity. We start with a preliminary pointwise estimate for each term $\mathcal{G}_{3,j}$.

Lemma 6.6.6. *Let $\gamma, \delta > 0$ and consider the partition $\mathcal{P}_3(\varepsilon^\delta)$ of the interval $[r_0 + \varepsilon^\gamma, \infty)$ in subintervals of size ε^δ . Let $\mathcal{G}_{3,j}$ be the Green's function associated to equation (6.6.21), and let $f \in Y$. Then, under Assumptions H0–H2, and SH, if*

$$\frac{2}{3} - \gamma - \delta > 0, \quad \gamma < \delta, \quad \gamma < \frac{1}{3},$$

there exists $c_0 > 0$ independent of $\varepsilon, r > 0, r_0 \in \mathcal{R}, M \in \mathcal{N}$, where $\mathcal{R}$ is from Lemma 6.3.4, and $\mathcal{N} \subset \mathbb{R} \setminus \{0\}$ is a compact set, so that for any $\varepsilon > 0$ sufficiently small independently of $r, r_j \geq$

$r_0 + \varepsilon^\gamma$, and any $r \geq 0$ the following estimates hold true,

$$\begin{aligned}
|w_\varepsilon(r-r_0)\mathcal{G}_{3,j}(f\mathbb{1}_{[r_j, r_j+\varepsilon^\delta]})(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \mathbb{1}_{\{r < r_j - \varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} \mathbb{1}_{\{r_j - \varepsilon^\delta \leq r \leq r_j + 2\varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \mathbb{1}_{\{r > r_j + 2\varepsilon^\delta\}}(r),
\end{aligned} \tag{6.6.23}$$

$$\begin{aligned}
|w_\varepsilon(r-r_0)\partial_r \mathcal{G}_{3,j}(f\mathbb{1}_{[r_j, r_j+\varepsilon^\delta]})(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{-1/3}}{r_j^2 \sqrt{|MT(r_j)|}} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \mathbb{1}_{\{r < r_j - \varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{-1/3}}{r_j^2 \sqrt{|MT(r_j)|}} \mathbb{1}_{\{r_j - \varepsilon^\delta \leq r \leq r_j + 2\varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{-1/3}}{r_j^2 \sqrt{|MT(r_j)|}} e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \mathbb{1}_{\{r > r_j + 2\varepsilon^\delta\}}(r),
\end{aligned} \tag{6.6.24}$$

where the implicit constants from $\lesssim$ are independent of $r, r_j \geq r_0 + \varepsilon^\gamma, \varepsilon, r_0 \in \mathcal{R}, M \in \mathcal{N}$.

Proof. Let $\mathcal{G}_{3,j}$ be the Green's function associated to the linearised equation (6.6.21). Fix $j \in \mathbb{N}$, and take $f \in L^\infty$ with $\text{supp}(f) \subset [r_j, r_j + \varepsilon^\delta]$. Using the branch $\text{Re}(\sqrt{iMT(r_j)}) > 0$, and since $\text{Re}(\sqrt{i}) = 1/\sqrt{2}$ regardless of the sign of $T(r_j)$, we find the pointwise estimate

$$\begin{aligned}
|\mathcal{G}_{3,j}f(r)| &\leq \frac{\varepsilon^{-1/3}}{2\sqrt{|MT(r_j)|}} \int_0^r e^{-\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3}(r-s)} |f(s)| ds \\
&+ \frac{\varepsilon^{-1/3}}{2\sqrt{|MT(r_j)|}} \int_r^\infty e^{-\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3}(s-r)} |f(s)| ds.
\end{aligned}$$

Since $\text{supp}(f) \subset [r_j, r_j + \varepsilon^\delta]$, if $r < r_j - \varepsilon^\delta$ we can write

$$\begin{aligned}
|w_\varepsilon(r-r_0)\mathcal{G}_{3,j}f(r)| &\leq \frac{\varepsilon^{-1/3}w_\varepsilon(r-r_0)}{2\sqrt{|MT(r_j)|}} \int_{r_j}^{r_j+\varepsilon^\delta} e^{-\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3}(s-r)} |f(s)| ds \\
&\leq \|f\|_Y \frac{\varepsilon^{-1/3}w_\varepsilon(r-r_0)}{2r_j^2 \sqrt{|MT(r_j)|}} \int_{r_j}^\infty w_\varepsilon(s-r_0)^{-1} e^{-\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3}(s-r)} ds
\end{aligned}$$

and therefore, using that $w_\varepsilon(s-r_0)^{-1}$ is a decreasing function for all $s > r_0$, we obtain

$$|w_\varepsilon(r-r_0)\mathcal{G}_{3,j}f(r)| \lesssim \|f\|_Y \frac{\varepsilon^{1/3}w_\varepsilon(r-r_0)}{w_\varepsilon(r_j-r_0)r_j^2 |MT(r_j)|} e^{-\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3}(r_j-r)}. \tag{6.6.25}$$

Now, by Lemma 6.8.8, there exists a constant $C > 0$ so that uniformly for $r_j \geq r_0 + \varepsilon^\gamma, r \leq r_0$ it holds

$$\frac{w_\varepsilon(r-r_0)}{w_\varepsilon(r_j-r_0)} \leq C e^{\varepsilon^{-\gamma}(r_j-r_0)}.$$

In particular, notice that $|T(r_j)| \gtrsim \varepsilon^\gamma$ uniformly for $r_j \geq r_0 + \varepsilon^\gamma, r_0 \in \mathcal{R}$ by Lemma 6.3.4. Therefore, $\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3+\gamma} \gtrsim \varepsilon^{-2/3+2\gamma} \rightarrow \infty$ as $\varepsilon \rightarrow 0$, provided that $\gamma < \frac{1}{3}$. In particular, for all

$\varepsilon > 0$ sufficiently small, it holds $\varepsilon^{-\gamma} \leq \frac{1}{2\sqrt{2}} \sqrt{|MT(r_j)|} \varepsilon^{-2/3}$, uniformly for $r_j \geq r_0 + \varepsilon^\gamma$, $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$. Hence, we may further upper bound the expression from (6.6.25) by

$$\|f\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-(\frac{1}{\sqrt{2}} \sqrt{|MT(r_j)|} \varepsilon^{-\frac{2}{3}} - \varepsilon^{-\gamma})(r_j - r)} \leq C \|f\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-\frac{1}{2\sqrt{2}} \sqrt{|MT(r_j)|} \varepsilon^{-\frac{2}{3}}(r_j - r)}. \quad (6.6.26)$$

Certainly this argument still works uniformly for $r_j - \varepsilon^\delta \leq r < r_j$, but in that particular case the contribution of the exponential may be negligible, since in the limit $r \rightarrow r_j$ we do not gain any decay. Thus, for $r_j - \varepsilon^\delta \leq r < r_j$ we write

$$|w_\varepsilon(r - r_0) \mathcal{G}_{3,j} f(r)| \lesssim \|f\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|}.$$

If $r \in (r_j, r_j + 2\varepsilon^\delta)$ we write,

$$\begin{aligned} |w_\varepsilon(r - r_0) \mathcal{G}_{3,j} f(r)| &\leq \frac{\varepsilon^{-1/3} w_\varepsilon(r - r_0)}{2\sqrt{|MT(r_j)|}} \int_{r_j}^r e^{-\frac{1}{\sqrt{2}} \sqrt{|MT(r_j)|} \varepsilon^{-2/3}(r-s)} |f(s)| ds \\ &\quad + \frac{\varepsilon^{-1/3} w_\varepsilon(r - r_0)}{2\sqrt{|MT(r_j)|}} \int_r^{r_j + \varepsilon^\delta} e^{-\frac{1}{\sqrt{2}} \sqrt{|MT(r_j)|} \varepsilon^{-2/3}(s-r)} |f(s)| ds. \end{aligned}$$

We deal with the terms separately. Note first that

$$\begin{aligned} &\frac{\varepsilon^{-1/3} w_\varepsilon(r - r_0)}{2\sqrt{|MT(r_j)|}} \int_r^{r_j + \varepsilon^\delta} e^{-\frac{1}{\sqrt{2}} \sqrt{|MT(r_j)|} \varepsilon^{-2/3}(s-r)} f(s) ds \\ &\leq \|f(r)\|_Y \frac{\varepsilon^{-1/3} w_\varepsilon(r - r_0)}{2\sqrt{|MT(r_j)|}} \int_r^{r_j + \varepsilon^\delta} e^{-\frac{1}{\sqrt{2}} \sqrt{|MT(r_j)|} \varepsilon^{-2/3}(s-r)} w_\varepsilon(s - r_0)^{-1} ds \\ &\leq \|f(r)\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|}. \end{aligned}$$

Here we used again that $w_\varepsilon(s - r_0)^{-1}$ is decreasing for $s > r_0$. Next, we estimate

$$\begin{aligned} |w_\varepsilon(r - r_0) \mathcal{G}_{3,j} f(r)| &\leq \frac{\varepsilon^{-1/3} w_\varepsilon(r - r_0)}{2\sqrt{|MT(r_j)|}} \int_{r_j}^r e^{-\frac{1}{\sqrt{2}} \sqrt{|MT(r_j)|} \varepsilon^{-2/3}(r-s)} f(s) ds \\ &\lesssim \|f\|_Y \frac{\varepsilon^{-1/3} w_\varepsilon(r - r_0)}{2r_j^2 \sqrt{|MT(r_j)|}} \int_{r_j}^r w_\varepsilon(s - r_0)^{-1} e^{-\frac{1}{\sqrt{2}} \sqrt{|MT(r_j)|} \varepsilon^{-2/3}(r-s)} ds \\ &\lesssim \|f\|_Y \frac{\varepsilon^{1/3} w_\varepsilon(r - r_0)}{w_\varepsilon(r_j - r_0) r_j^2 |MT(r_j)|}. \end{aligned}$$

By Lemma 6.8.8, we find that the quotient

$$\frac{w_\varepsilon(r - r_0)}{w_\varepsilon(r_j - r_0)}$$

is bounded by a constant, uniformly for $r_j \geq r_0 + \varepsilon^\gamma$, $|r - r_j| < 2\varepsilon^\delta$ (provided $\gamma < \delta$), and thus we can write

$$|w_\varepsilon(r - r_0) \mathcal{G}_{3,j} f(r)| \lesssim \|f\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|}.$$

Finally, if $r \geq r_j + 2\varepsilon^\delta$, then we estimate

$$\begin{aligned} |w_\varepsilon(r-r_0)\mathcal{G}_{3,j}f(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{1/3}w_\varepsilon(r-r_0)}{w_\varepsilon(r_j-r_0)r_j^2|MT(r_j)|} e^{-\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3}(r-r_j)} e^{\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3+\delta}} \\ &= \|f\|_Y \frac{\varepsilon^{1/3}w_\varepsilon(r-r_0)}{w_\varepsilon(r_j-r_0)r_j^2|MT(r_j)|} e^{-\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3}(r-r_j-\varepsilon^\delta)}. \end{aligned}$$

It remains to note the bound

$$\frac{w_\varepsilon(r-r_0)}{w_\varepsilon(r_j-r_0)} e^{-\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3}(r-r_j-\varepsilon^\delta)} \leq C e^{-\frac{1}{2\sqrt{2}}\sqrt{|MT_\varepsilon|}\varepsilon^{-2/3}(r-r_j-\varepsilon^\delta)},$$

for some $C > 0$ independent of $r > 0$, $r_j \geq r_0 + \varepsilon^\gamma$, and $\varepsilon > 0$ small enough, which can be deduced from Lemma 6.8.8 in precisely the same way as (6.6.26). Noting finally that by Lemma 6.3.4, $|MT(r_j)| \gtrsim \varepsilon^{2\gamma}$ uniformly for $r_j \geq r_0 + \varepsilon^\gamma$, $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$, we arrive at the first claim of the lemma, estimate (6.6.23).

In order to deal with the r derivatives of the Green's functions, we directly compute them using the definition of $\mathcal{G}_{3,j}$,

$$\partial_r \mathcal{G}_{3,j}f(r) = -\frac{\varepsilon^{-1}}{2} \left(\int_0^r e^{-\sqrt{iMT(r_j)}\varepsilon^{-2/3}(r-s)} f(s) ds + \int_r^\infty e^{-\sqrt{iMT(r_j)}\varepsilon^{-2/3}(s-r)} f(s) ds \right),$$

Therefore,

$$|\partial_r \mathcal{G}_{3,j}f(r)| \leq \frac{\varepsilon^{-1}}{2} \left(\int_0^r e^{-\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3}(r-s)} |f(s)| ds + \int_r^\infty e^{-\frac{1}{\sqrt{2}}\sqrt{|MT(r_j)|}\varepsilon^{-2/3}(s-r)} |f(s)| ds \right),$$

and similarly to how we argued for the previous case but noticing the different exponents on ε and $|T(r_j)|$ now, we obtain the analogous pointwise estimate (6.6.24). $\square$

Lemma 6.6.6 yields a pointwise estimate in $L^\infty((0, \infty), w_\varepsilon(r-r_0)dr)$ for each addend $\mathcal{G}_{3,j}$ that constitute the Green's function $\mathcal{G}_3$. In order to obtain an analogous estimate in the full norm X we must include the extra weight $\max\{1, r^2\}$. In this aspect, it is convenient to obtain estimates—akin to (6.6.23) and (6.6.24)—with an r^2 weight. From Lemma 6.6.6, it is rather direct to see that we have the following pointwise bounds

$$\begin{aligned} |r^2 w_\varepsilon(r-r_0)\mathcal{G}_{3,j}(f\mathbb{1}_{[r_j, r_j+\varepsilon^\delta)})(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{1/3}}{|MT(r_j)|} e^{-c_0\varepsilon^{-2/3+\gamma}(r_j-r)} \mathbb{1}_{\{r_j-r>\varepsilon^\delta\}}(r) \\ &\quad + \|f\|_Y \frac{\varepsilon^{1/3}}{|MT(r_j)|} \mathbb{1}_{\{r_j-\varepsilon^\delta \leq r \leq r_j+2\varepsilon^\delta\}}(r) \\ &\quad + \|f\|_Y \frac{\varepsilon^{1/3}}{|MT(r_j)|} \left(1 + \frac{|r-r_j|^2}{r_j^2} \right) e^{-c_0\varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \mathbb{1}_{\{r-r_j>2\varepsilon^\delta\}}(r), \end{aligned} \tag{6.6.27}$$

as well as,

$$\begin{aligned}
|r^2 w_\varepsilon(r - r_0) \partial_r \mathcal{G}_{3,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^\delta)})(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{-1/3}}{\sqrt{|MT(r_j)|}} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j - r)} \mathbb{1}_{\{r_j - r > \varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{-1/3}}{\sqrt{|MT(r_j)|}} \mathbb{1}_{\{r_j - \varepsilon^\delta \leq r \leq r_j + 2\varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{-1/3}}{\sqrt{|MT(r_j)|}} \left(1 + \frac{|r - r_j|^2}{r_j^2}\right) e^{-c_0 \varepsilon^{-2/3 + \gamma}(r - r_j - \varepsilon^\delta)} \mathbb{1}_{\{r - r_j > 2\varepsilon^\delta\}}(r).
\end{aligned} \tag{6.6.28}$$

Notice that, after introducing the r^2 factor, when it is not comparable to r_j^2 we use the relation

$$\frac{r^2}{r_j^2} \leq 2 \left(1 + \frac{|r - r_j|^2}{r_j^2}\right).$$

These estimates, together with Lemma 6.6.6, provide the key to prove the main proposition of the section, that we present next.

Proposition 6.6.7. *Let $\gamma, \delta > 0$ be such that*

$$\gamma < \frac{1}{3}, \quad \gamma + \delta < \frac{2}{3}, \quad \gamma < \delta,$$

consider the partition $\mathcal{P}_3(\varepsilon^\delta)$ of the interval $[r_0 + \varepsilon^\gamma, \infty)$ in subintervals of length ε^δ , and let $\mathcal{G}_3$ be the Green's function introduced in (6.6.22). Let $f \in Y$ with $\text{supp}(f) \subset [r_0 + \varepsilon^\gamma, \infty)$. Under Assumptions H0–H2 and SH, the following estimates hold true uniformly for any $\varepsilon > 0$ sufficiently small, $r_0 \in \mathcal{R}$, and M in any compact set $\mathcal{N} \subset \mathbb{R} \setminus \{0\}$.

$$\|\mathcal{G}_3 f\|_X \lesssim \varepsilon^{1/3 - 2\gamma} \|f\|_Y, \tag{6.6.29}$$

$$\|\partial_r \mathcal{G}_3 f\|_X \lesssim \varepsilon^{-1/3 - \gamma} \|f\|_Y. \tag{6.6.30}$$

Moreover, the error caused by the approximated operator can be estimated by

$$\|((\mathcal{L} - \lambda)\mathcal{G}_3 - \text{Id})f\|_Y \lesssim \left(\varepsilon^{2/3 - 2\gamma} + \varepsilon^{2/3 - 2\gamma} + \varepsilon^\beta\right) \|f\|_Y,$$

where the coefficient $\beta > 0$ is from P3.

Proof. Part 1: Estimates on $\mathcal{G}_3$ and $\partial_r \mathcal{G}_3$. First of all we will derive the X and Y bounds on the Green's function and its first order derivatives. These will be a direct consequence of the estimates derived in Lemma 6.6.6. In particular, for $\mathcal{G}_3$ we see that we can combine (6.6.23) and (6.6.27) as

follows

$$\begin{aligned}
|\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_{3,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^\delta]})(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{1/3}}{|MT(r_j)|} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j - r)} \mathbb{1}_{\{r_j - r > \varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{1/3}}{|MT(r_j)|} \mathbb{1}_{\{r_j - \varepsilon^\delta \leq r \leq r_j + 2\varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{1/3}}{|MT(r_j)|} (1 + |r - r_j|^2) e^{-c_0 \varepsilon^{-2/3 + \gamma}(r - r_j - \varepsilon^\delta)} \mathbb{1}_{\{r - r_j > 2\varepsilon^\delta\}}(r).
\end{aligned}$$

Using linearity from the definition (6.6.22) we can write

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_3 f(r)| \leq \sum_{j=1}^{\infty} |\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_{3,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^\delta]})(r)|.$$

Now, say that $r \in [r_k, r_k + \varepsilon^\delta)$, where $r_k = r_1 + (k-1)\varepsilon^\delta$ is an element of the partition $\mathcal{P}_3(\varepsilon^\delta)$. Then the pointwise estimate after summing over all j can be bounded by

$$\begin{aligned}
|\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_3 f(r)| &\lesssim \|f\|_Y \sum_{j=1}^{k-2} \varepsilon^{1/3-2\gamma} (1 + |r - r_j|^2) e^{-c_0 \varepsilon^{-2/3 + \gamma}(r - r_j - \varepsilon^\delta)} \\
&+ \|f\|_Y \sum_{j=k+2}^{\infty} \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j - r)} \\
&+ \|f\|_Y \varepsilon^{1/3} \left(\frac{1}{|MT(r_{k-1})|} + \frac{1}{|MT(r_k)|} + \frac{1}{|MT(r_{k+1})|} \right).
\end{aligned}$$

Naturally, if $r < r_0 + \varepsilon^\gamma$ we cannot write $r \in [r_k, r_k + \varepsilon^\delta)$, but in such case one can simply obtain the corresponding contributions using the same bounds, namely if $r < r_1 - 2\varepsilon^\delta$

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_3 f(r)| \lesssim \|f\|_Y \sum_{j=1}^{\infty} \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j - r)},$$

and the claim of the proposition follows readily. Let us thus analyse the case $r \in [r_k, r_{k+1})$, by starting with the contributions of $j = k-1$, $j = k$ and $j = k+1$. By Lemma 6.3.4, these may be estimated by

$$\|f\|_Y \varepsilon^{1/3} \left(\frac{1}{|MT(r_{k-1})|} + \frac{1}{|MT(r_k)|} + \frac{1}{|MT(r_{k+1})|} \right) \lesssim \|f\|_Y \varepsilon^{1/3-2\gamma}.$$

For the other two terms we observe that we have a converging geometric series that goes to zero quickly as $\varepsilon \rightarrow 0$, namely using that $r_j = r_1 + (j-1)\varepsilon^\delta$,

$$\begin{aligned}
\sum_{j=k+2}^{\infty} \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j - r)} &\leq \sum_{j=k+2}^{\infty} \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_1 + (j-1)\varepsilon^\delta - r)} \\
&= \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_1 - \varepsilon^\delta - r)} \sum_{j=k+2}^{\infty} e^{-c_0 \varepsilon^{-2/3 + \gamma + \delta} j} \\
&\lesssim \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_1 + (k+1)\varepsilon^\delta - r)},
\end{aligned}$$

where the last inequality holds uniformly in $\varepsilon > 0$ sufficiently small, provided that $\gamma + \delta < 2/3$.

Taking into account that $r_1 + (k+1)\varepsilon^\delta = r_{k+2}$, and $r < r_{k+1}$ we write

$$\sum_{j=k+2}^{\infty} \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \lesssim \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_{k+2}-r)} \leq \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3+\gamma+\delta}}.$$

Hence, crucially using the condition $\gamma + \delta < 2/3$, we obtain the quantitative converging estimate as $\varepsilon \rightarrow 0$. In particular, this bound holds *uniformly* for $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$. The estimate for the remaining term follows from the estimate,

$$\sum_{j=1}^{k-2} \varepsilon^{1/3-2\gamma} (1 + |r - r_j|^2) e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \lesssim \sum_{j=1}^{k-2} \varepsilon^{1/3-2\gamma} e^{-\frac{1}{2}c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)},$$

thus, since $r \geq r_k$ we have that $r - r_{k-2} - \varepsilon^\delta \geq \varepsilon^\delta$, and

$$\begin{aligned} \sum_{j=1}^{k-2} \varepsilon^{1/3-2\gamma} e^{-\frac{1}{2}c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} &\lesssim \varepsilon^{1/3-2\gamma} e^{-\frac{1}{2}c_0 \varepsilon^{-2/3+\gamma}(r-r_{k-2}-\varepsilon^\delta)} \lesssim \varepsilon^{1/3-2\gamma} e^{-\frac{1}{2}c_0 \varepsilon^{-2/3+\gamma+\delta}}. \\ \varepsilon^{1/3-2\gamma} \sum_{j=1}^{k-2} e^{-\frac{1}{2}c_0 \varepsilon^{-2/3+\gamma}(r-r_1-(j-1)\varepsilon^\delta-\varepsilon^\delta)} &\lesssim \varepsilon^{1/3-2\gamma} e^{-\frac{1}{2}c_0 \varepsilon^{-2/3+\gamma}(r-r_1)} e^{\frac{1}{2}c_0 \varepsilon^{-2/3+\gamma+\delta}(k-2)} \\ &\leq \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3+\gamma+\delta}}. \end{aligned}$$

All in all, putting everything together we find for $r \in [r_k, r_{k+1})$,

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_3 f(r)| \lesssim \|f\|_Y \varepsilon^{1/3-2\gamma} \left(\mathbb{1}_{[r_k, r_k+\varepsilon^\delta)}(r) + e^{-\frac{1}{2}c_0 \varepsilon^{-2/3+\gamma+\delta}} \mathbb{1}_{[0, r_k) \cap [r_{k+1}, \infty)}(r) \right),$$

with the implicit constants in the $\lesssim$ symbol independent of $\varepsilon > 0$, $r_k \geq r_0 + \varepsilon^\gamma$, which in particular implies (6.6.29).

The estimate on $\partial_r \mathcal{G}_2$ follows similarly, we can put together the pointwise estimates for $\partial_r \mathcal{G}_{r_j}$ from (6.6.28), i.e.

$$\begin{aligned} |\max\{1, r^2\} w_\varepsilon(r - r_0) \partial_r \mathcal{G}_{r_j}(f \mathbb{1}_{[r_j, r_j+\varepsilon^\delta)})(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{-1/3}}{\sqrt{|MT(r_j)|}} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \mathbb{1}_{\{r_j-r > \varepsilon^\delta\}}(r) \\ &+ \|f\|_Y \frac{\varepsilon^{-1/3}}{\sqrt{|MT(r_j)|}} \mathbb{1}_{\{r_j-\varepsilon^\delta \leq r \leq r_j+2\varepsilon^\delta\}}(r) \\ &+ \|f\|_Y \frac{\varepsilon^{-1/3}}{\sqrt{|MT(r_j)|}} \left(1 + \frac{|r - r_j|^2}{r_j^2} \right) e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \mathbb{1}_{\{r-r_j > 2\varepsilon^\delta\}}(r). \end{aligned}$$

Therefore, a completely analogous argument to that used for $\mathcal{G}_3$ yields

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \partial_r \mathcal{G}_2 f(r)| \lesssim \|f\|_Y \varepsilon^{-1/3-\gamma} \left(\mathbb{1}_{[r_k, r_k+\varepsilon^\delta)}(r) + e^{-\frac{1}{2}c_0 \varepsilon^{-2/3+\gamma+\delta}} \mathbb{1}_{[0, r_k) \cap [r_{k+1}, \infty)}(r) \right),$$

and thus, as a by-product, we obtain (6.6.30).

Part 2: The error estimate. We now move on to computing the error committed by the linear

approximation in the regime $r \geq r_0 + \varepsilon^\gamma$. We define,

$$\mathcal{L}_{3,j}V(r) = \varepsilon \partial_r^2 V(r) - i\varepsilon^{-1/3}MT(r_j)V(r),$$

so that

$$\mathcal{L}_{3,j}\mathcal{G}_{3,j}(f\mathbb{1}_{[r_j, r_j+\varepsilon^\delta]})(r) = (f\mathbb{1}_{[r_j, r_j+\varepsilon^\delta]})(r),$$

for any $j \in \mathbb{N}$. The exact operator $\mathcal{L} - \lambda$ has two components, and we want to approximate both with the same operator $\mathcal{L}_3$. We can write the exact operator as

$$(\mathcal{L} - \lambda) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \varepsilon \left(\partial_r^2 + \frac{1}{r} \partial_r \right) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} - \begin{pmatrix} A_\varepsilon + \lambda + P_\varepsilon & -Q_\varepsilon \\ Q_\varepsilon & A_\varepsilon + \lambda - P_\varepsilon \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix},$$

where we define

$$\begin{aligned} P_\varepsilon(r) &= \frac{iM\alpha\varepsilon}{r^2} - \sqrt{\frac{-2iM\Omega'_0}{r_0}}\varepsilon^{1/3} - D_\varepsilon^-(r), \quad Q_\varepsilon(r) = \frac{iM\alpha\varepsilon}{r^2} + D_\varepsilon^+(r), \\ A_\varepsilon(r) &= iM\varepsilon^{-1/3}T(r) + \frac{\varepsilon}{r^2} + \varepsilon^{1/3}M^2 \left(\frac{1}{r^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right), \end{aligned} \tag{6.6.31}$$

with D_ε defined as in Section 6.3.3. With this information we can now evaluate the size of the error. For the first component, let $j \in \mathbb{N}$, so that we can write

$$\begin{aligned} (\mathcal{L} - \lambda - \mathcal{L}_{3,j})V_1(r) &= \frac{\varepsilon}{r} \partial_r V_1(r) - iM\varepsilon^{-1/3}(T(r) - T(r_j))V_1(r) \\ &\quad - \left[\frac{\varepsilon}{r^2} + \varepsilon^{1/3}M^2 \left(\frac{1}{r^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right) + \lambda + P_\varepsilon(r) \right] V_1 + Q_\varepsilon(r)V_2(r). \end{aligned}$$

Now, we get the following pointwise estimates

$$\begin{aligned} |D_\varepsilon^+(r)|, |D_\varepsilon^-(r)| &\lesssim \varepsilon^{1/3} \left(\frac{r^2 - r_0^2}{r_0^2 r^2} + |r\Omega'(r) - r_0\Omega'_0| \right) \\ &\lesssim \varepsilon^{1/3} \left(1 + \frac{1}{r^2} + |r\Omega'(r)| \right), \\ |P_\varepsilon(r)| &\lesssim \frac{\varepsilon}{r^2} + \varepsilon^{1/3} + |D_\varepsilon^-(r)| \lesssim \frac{\varepsilon}{r^2} + \varepsilon^{1/3} \left(1 + \frac{1}{r^2} + |r\Omega'(r)| \right), \\ |Q_\varepsilon(r)| &\lesssim \frac{\varepsilon}{r^2} + |D_\varepsilon^+(r)| \lesssim \frac{\varepsilon}{r^2} + \varepsilon^{1/3} \left(1 + \frac{1}{r^2} + |r\Omega'(r)| \right). \end{aligned}$$

Therefore, noting that

$$(\mathcal{L} - \lambda)\mathcal{G}_3 f - f = \sum_{j \geq 1} (\mathcal{L} - \lambda)\mathcal{G}_{3,j}(f\mathbb{1}_{[r_j, r_j+\varepsilon^\delta]}) - \sum_{j \geq 1} \mathcal{L}_{3,j}\mathcal{G}_{3,j}(f\mathbb{1}_{[r_j, r_j+\varepsilon^\delta]}),$$

and using $\lambda = \mu \varepsilon^{1/3}$ we obtain

$$\begin{aligned}
|r^2 w_\varepsilon(r-r_0)((\mathcal{L} - \lambda)\mathcal{G}_3 - \text{Id})f| &\lesssim \varepsilon |r w_\varepsilon(r-r_0) \partial_r \mathcal{G}_3 f| + \varepsilon^{1/3} |\max\{1, r^2\} w_\varepsilon(r-r_0) \mathcal{G}_3 f| \\
&\quad + \varepsilon^{-1/3} \sum_{j=1}^{\infty} \left| r^2 w_\varepsilon(r-r_0) M(T(r) - T(r_j)) \mathcal{G}_{3,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon \delta]}) \right| \\
&\quad + \varepsilon^{1/3} |r \Omega'(r)| |r^2 w_\varepsilon(r-r_0) \mathcal{G}_3 f|.
\end{aligned} \tag{6.6.32}$$

For the term involving the derivative of the Green's function we find that

$$\varepsilon |r w_\varepsilon(r-r_0) \partial_r \mathcal{G}_3 f| \leq \varepsilon \|\partial_r \mathcal{G}_3 f\|_X \lesssim \varepsilon^{2/3-2\gamma} \|f\|_Y,$$

by a direct application of (6.6.30). The second addend in the right hand side in (6.6.32) can be bounded using (6.6.29) by

$$\varepsilon^{1/3} |\max\{1, r^2\} w_\varepsilon(r-r_0) \mathcal{G}_3 f| \leq \varepsilon^{2/3-2\gamma} \|f\|_Y.$$

We further see that all of the above bounds are uniform for $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$. In order to estimate the terms involving $T(r)$ it will not suffice to use (6.6.29), and so we shall employ the more precise bound (6.6.27). We write for the first component

$$\begin{aligned}
&\varepsilon^{-1/3} \sum_{j=1}^{\infty} \left| r^2 w_\varepsilon(r-r_0) M(T(r) - T(r_j)) \mathcal{G}_{3,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon \delta]})(r) \right| \\
&\leq \|f_1\|_Y \sum_{j=1}^{\infty} \frac{|T(r) - T(r_j)|}{|T(r_j)|} \left(1 + \frac{|r - r_j|^2}{r_j^2} \right) e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \mathbb{1}_{\{r > r_{j+1}\}} \\
&\quad + \|f_1\|_Y \sum_{j=1}^{\infty} \frac{|T(r) - T(r_j)|}{|T(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \mathbb{1}_{\{r < r_{j-1}\}} \\
&\quad + \|f_1\|_Y \sum_{j=1}^{\infty} \frac{|T(r) - T(r_j)|}{|T(r_j)|} \mathbb{1}_{\{r_{j-1} \leq r \leq r_{j+1}\}} \\
&= \text{I}_\varepsilon(r) + \text{II}_\varepsilon(r) + \text{III}_\varepsilon(r).
\end{aligned}$$

To deal with the first two terms in the right hand side, we will use the fast decay of the exponentials to compensate for the (possible) growth of $T(r)$ as $r \rightarrow \infty$. Throughout this analysis we shall assume that $r \geq r_0 + \varepsilon^\gamma$, since the general case is analogous (and in fact easier). Let us start analysing the first addend. On the one hand, notice that there must exist some $k \geq 0$ so that $r \in [r_k, r_k + \varepsilon^\delta]$ with $k > 2$, and hence by Property P3 we have

$$\begin{aligned}
\text{I}_\varepsilon(r) &= \|f_1\|_Y \sum_{j=1}^{k-2} \frac{|T(r) - T(r_j)|}{|T(r_j)|} \left(1 + \frac{|r - r_j|^2}{r_j^2} \right) e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \\
&\leq \|f_1\|_Y \sum_{n=1}^{N_1} B_n(\varepsilon) \sum_{j=1}^{k-2} |r - r_j|^n (1 + |r - r_j|^2) e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)}.
\end{aligned}$$

In order to compute the partial sums of a polynomial multiplying a decaying exponential we argue

as follows. Let $n \in \mathbb{N}$ and $r_j = r_1 + (j-1)\varepsilon^\delta$, then we write

$$\begin{aligned} \sum_{j=1}^{k-2} |r - r_j|^n e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} &\leq n! \sum_{j=1}^{k-2} e^{n(r-r_j)-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \\ &= n! e^{-(c_0 \varepsilon^{-2/3+\gamma}-n)(r-r_1)} \sum_{j=1}^{k-2} e^{(c_0 \varepsilon^{-2/3+\gamma}-n)j\varepsilon^\delta} \\ &\lesssim e^{-(c_0 \varepsilon^{-2/3+\gamma}-n)(r-r_1)} e^{(c_0 \varepsilon^{-2/3+\gamma}-n)(k-2)\varepsilon^\delta} \\ &= e^{-(c_0 \varepsilon^{-2/3+\gamma}-n)(r-r_{k-1})} \lesssim e^{-(c_0 \varepsilon^{-2/3+\gamma}-n)\varepsilon^\delta} \end{aligned}$$

uniformly in $k \geq 0$, provided that we choose $\varepsilon > 0$ small enough so that $\varepsilon^{-1/3+\gamma} > c_0^{-1}n$. Therefore we arrive at the estimate

$$\mathbf{I}_\varepsilon(r) \lesssim \|f_1\|_Y e^{-(c_0 \varepsilon^{-2/3+\gamma}-2-N_1)\varepsilon^\delta} \sum_{n=1}^{N_1} B_n(\varepsilon) \lesssim \|f_1\|_Y \varepsilon^{-N_1\delta+\beta} e^{-(c_0 \varepsilon^{-2/3+\gamma}-2-N_1)\varepsilon^\delta},$$

uniformly in k , and where $\beta > 0$ comes from Property P3. Notice that it might possible that $-N_1\delta + \beta < 0$, however this estimate is still rapidly converging to zero as $\varepsilon \rightarrow 0$. We apply a similar argument for the second term,

$$\begin{aligned} \mathbf{II}_\varepsilon(r) &= \|f_1\|_Y \sum_{j=k+2}^{\infty} \frac{|T(r) - T(r_j)|}{|T(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \\ &\leq \|f_1\|_Y \sum_{n=1}^{N_1} B_n(\varepsilon) \sum_{j=k+2}^{\infty} |r_j - r|^n e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \\ &\lesssim \|f_1\|_Y e^{-(c_0 \varepsilon^{-2/3+\gamma}-N)\varepsilon^\delta} \sum_{n=1}^{N_1} B_n(\varepsilon) \\ &\lesssim \|f_1\|_Y \varepsilon^{-N_1\delta+\beta} e^{-(c_0 \varepsilon^{-2/3+\gamma}-N_1)\varepsilon^\delta} \end{aligned}$$

Last, for the third addend in the error estimate we use Property P3 once again to achieve

$$\begin{aligned} \mathbf{III}_\varepsilon(r) &= \|f_1\|_Y \left(\frac{|T(r) - T(r_{k-1})|}{|T(r_{k-1})|} + \frac{|T(r) - T(r_k)|}{|T(r_k)|} + \frac{|T(r) - T(r_{k+1})|}{|T(r_{k+1})|} \right) \\ &\leq \|f_1\|_Y \sum_{n=1}^{N_1} B_n(\varepsilon) (|r - r_{k-1}|^n + |r - r_k|^n + |r - r_{k+1}|^n). \end{aligned}$$

Moreover, since $r \in [r_k, r_k + \varepsilon^\delta)$, we can use that $|r - r_{k-1}| \leq 2\varepsilon^\delta$, $|r - r_k| \leq \varepsilon^\delta$, and $|r - r_{k+1}| \leq 2\varepsilon^\delta$, to write

$$\mathbf{III}_\varepsilon(r) \lesssim \|f_1\|_Y \sum_{n=1}^{N_1} B_n(\varepsilon) \varepsilon^{n\delta} \lesssim \varepsilon^\beta \|f_1\|_Y.$$

Since $r \geq r_0 + \varepsilon^\gamma$ was arbitrary, all in all we may write

$$\begin{aligned} & \varepsilon^{-1/3} \sum_{j=1}^{\infty} \left| w_\varepsilon(r-r_0) r^2 (T(r) - T(r_j)) \mathcal{G}_{r_j}(f_1 \mathbb{1}_{[r_j, r_j + \varepsilon^\delta]})(r) \right| \\ & \lesssim \varepsilon^\beta \|f_1\|_Y \mathbb{1}_{[r_{k-1}, r_{k+1}]}(r) \\ & \quad + \|f_1\|_Y \varepsilon^{-N\delta + \beta} e^{-(c_0 \varepsilon^{-2/3 + \gamma - 2 - N}) \varepsilon^\delta} \mathbb{1}_{[r_0 + \varepsilon^\gamma, r_{k-1}] \cap [r_{k+1}, \infty)}(r) \\ & \lesssim \varepsilon^\beta \|f_1\|_Y, \end{aligned}$$

for all $r \geq r_0 + \varepsilon^\gamma$, and it is straightforward to see that it holds in fact for all $r \geq 0$. There is only one more term to be addressed in (6.6.32), namely the term concerning $\Omega'(r)$. For this term, we once again need to make use of the more precise estimate (6.6.27). As before, we argue for the first component. Let $r \in [r_k, r_k + \varepsilon^\delta]$, then we write

$$\begin{aligned} \varepsilon^{1/3} |r \Omega'(r)| |r^2 w_\varepsilon(r-r_0) \mathcal{G}_\infty f_1(r)| & \leq \varepsilon^{2/3} \|f_1\|_Y \sum_{j=1}^{k-2} \left(1 + \frac{|r-r_j|^2}{r_j^2} \right) \frac{|r \Omega'(r)|}{|MT(r_j)|} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r-r_j - \varepsilon^\delta)} \\ & \quad + \varepsilon^{2/3} \|f_1\|_Y \sum_{j=k+2}^{\infty} \frac{|r \Omega'(r)|}{|MT(r_j)|} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j-r)} \\ & \quad + \varepsilon^{2/3} \|f_1\|_Y \left(\frac{|r \Omega'(r)|}{|T(r_{k-1})|} + \frac{|r \Omega'(r)|}{|MT(r_k)|} + \frac{|r \Omega'(r)|}{|T(r_{k+1})|} \right) \\ & = \text{I}_{k,\varepsilon}(r) + \text{II}_{k,\varepsilon}(r) + \text{III}_{k,\varepsilon}(r). \end{aligned}$$

In order to deal with these terms we need to impose a new set of conditions on the function Ω and its relation with the function T , and this is where Property P4 comes into play. Using this hypothesis we can directly estimate the third term by noticing that

$$\frac{|r \Omega'(r)|}{|T(r_{k-1})|} \leq \varepsilon^{-2\gamma} \left(D_0 + \sum_{n=1}^{N_2} D_n |r-r'|^n \right) \lesssim \varepsilon^{-2\gamma} \left(1 + \sum_{n=1}^{N_2} \varepsilon^{n\delta} \right) \lesssim \varepsilon^{-2\gamma},$$

and analogously with r_k and r_{k+1} . Thus, we arrive at the estimate,

$$\text{III}_{k,\varepsilon}(r) \lesssim \varepsilon^{2/3} \|f_1\|_Y \left(\frac{|r \Omega'(r)|}{|T(r_{k-1})|} + \frac{|r \Omega'(r)|}{|T(r_k)|} + \frac{|r \Omega'(r)|}{|T(r_{k+1})|} \right) \lesssim \varepsilon^{2/3-2\gamma} \|f_1\|_Y.$$

For the second addend we write,

$$\text{II}_{k,\varepsilon}(r) \leq \varepsilon^{2/3-2\gamma} \|f_1\|_Y \sum_{j=k+2}^{\infty} \left(D_0 + \sum_{n=1}^{N_2} D_n |r-r'|^n \right) e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j-r)}.$$

The interactions between the growing polynomial and decreasing exponential inside the infinite sum can be controlled for any $n \in \mathbb{N}$ by

$$\begin{aligned} \sum_{j=k+2}^{\infty} (r_j-r)^n e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j-r)} & \leq n! \sum_{j=k+2}^{\infty} e^{n(r_j-r) - c_0 \varepsilon^{-2/3 + \gamma}(r_j-r)} \\ & \lesssim e^{-(c_0 \varepsilon^{-2/3 + \gamma} - n)(r_{k+2}-r)} \lesssim e^{-(c_0 \varepsilon^{-2/3 + \gamma} - n) \varepsilon^\delta} \end{aligned}$$

provided that $\varepsilon > 0$ is sufficiently small in comparison to N_2 . Notice that in order to obtain this

upper bound, we are only using that $x^n \leq n!e^x$ for any $x \geq 0$, which is not very sharp, but sufficient for our purposes. Putting things together we find that

$$\begin{aligned}\Pi_{\varepsilon,k}(r) &\lesssim \varepsilon^{2/3-2\gamma} \|f_1\|_Y e^{-(c_0 \varepsilon^{-2/3+\gamma-N_2}) \varepsilon^\delta} \sum_{n=1}^{N_2} D_n \\ &\lesssim \varepsilon^{2/3-2\gamma-N_2\delta} e^{-(c_0 \varepsilon^{-2/3+\gamma-N_2}) \varepsilon^\delta} \|f_1\|_Y.\end{aligned}$$

Finally, for the remaining term we argue similarly,

$$\begin{aligned}\text{III}_{\varepsilon,k}(r) &\leq \varepsilon^{2/3-2\gamma} \|f_1\|_Y \sum_{j=1}^{k-2} \left(1 + \frac{|r-r_j|^2}{r_j^2}\right) \left(D_0 + \sum_{n=1}^{N_2} D_n |r-r'|^n\right) e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \\ &\lesssim \varepsilon^{2/3-2\gamma} \|f_1\|_Y e^{-(c_0 \varepsilon^{-2/3+\gamma-(N_2+2)}) \varepsilon^\delta} \sum_{n=1}^{N_2} D_n \\ &\lesssim \varepsilon^{2/3-2\gamma-N_2\delta} \|f_1\|_Y e^{-(c_0 \varepsilon^{-2/3+\gamma-(N_2+2)}) \varepsilon^\delta}.\end{aligned}$$

Putting all these estimates together, we find the bound

$$\begin{aligned}\varepsilon^{1/3} |r\Omega'(r)| |r^2 w_\varepsilon(r-r_0) \mathcal{G}_3 f_1(r)| &\lesssim \|f_1\|_Y \varepsilon^{2/3-2\gamma} \mathbb{1}_{[r_{k-1}, r_{k+1})}(r) \\ &\quad + \|f_1\|_Y \varepsilon^{2/3-2\gamma-N_2\delta} e^{-(c_0 \varepsilon^{-2/3+\gamma-(N_2+2)}) \varepsilon^\delta} \mathbb{1}_{[r_0+\varepsilon^\gamma, r_{k-1}) \cap [r_{k+2}, \infty)}(r),\end{aligned}$$

namely,

$$\varepsilon^{1/3} |r\Omega'(r)| |r^2 w_\varepsilon(r-r_0) \mathcal{G}_3 f_1(r)| \lesssim \varepsilon^{2/3-2\gamma} \|f_1\|_Y.$$

Coming back to (6.6.32) and introducing all the estimates we were able to derive, we can conclude that for any $r \geq r_0 + \varepsilon^\varepsilon$ the following estimate holds true uniformly for $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$,

$$|r^2 w_\varepsilon(r-r_0) (\mathcal{L} - \lambda - \mathcal{L}_2) \mathcal{G}_3 f| \lesssim \left(\varepsilon^{2/3-2\gamma} + \varepsilon^\beta + \varepsilon^{2/3-2\gamma} \right) \|f_1\|_Y.$$

We remark here that if $r < r_0 + \varepsilon^\gamma$ we cannot say that $r \in [r_k, r_{k+1})$ for some $k \in \mathbb{N}$. However, the estimates (6.6.23), (6.6.24), (6.6.27) and (6.6.24) still hold true: with only an exponentially decaying term surviving in $r < r_0 + \varepsilon^\gamma - \varepsilon^\delta$, or at most one of the constant contributions if the distance between r and $r_1 = r_0 + \varepsilon^\gamma$ is sufficiently small. Therefore, all these arguments carry over for the scenario $r < r_0 + \varepsilon^\gamma$, and we arrive at the claim of the proposition. $\square$

6.6.3 Green's function towards zero

Next in order we focus on studying the solution towards zero, that is, the solution when the support of f is contained in an interval of the form $[\varepsilon^\gamma, r_0 - \varepsilon^\gamma)$, for some $\gamma > 0$ to be chosen accordingly. The method we will use is analogous to the method introduced for the solution towards infinity. Again, our approximate operator will be “strongly invertible”, and so it suffices to approximate the exact operator by means of linear interpolation on a partition in the same vein as for the previous section.

Fix $r_0 > 0$, $\gamma, \delta > 0$, and let $\varepsilon > 0$ be arbitrarily small, then we define $\ell = \ell(\varepsilon)$ to be

$$\ell = \left\lceil \frac{r_0 - 2\varepsilon^\gamma}{\varepsilon^\delta} \right\rceil.$$

Here $\lceil a \rceil$ denotes the smallest $n \in \mathbb{N}$ for which $n \geq a$. Then, we define the partition

$$[\varepsilon^\gamma, r_0 - \varepsilon^\gamma) = \bigcup_{j=1}^{\ell-1} [r_j, r_j + \varepsilon^\delta) \cup [r_\ell, r_0 - \varepsilon^\gamma),$$

where we identify $r_1 = \varepsilon^\gamma$, and $r_{j+1} = r_j + \varepsilon^\delta$. We denote this partition by $\mathcal{P}_1(\varepsilon^\delta)$, which is finite for any fixed value of $\varepsilon > 0$.

Following the ideas and the notation of Section 6.6.2, we approximate the exact equations (6.6.9)–(6.6.10) for the case of f supported in $[\varepsilon^\gamma, r_0 - \varepsilon^\gamma)$, with the simpler ordinary differential equation,

$$\varepsilon \partial_r^2 V - i\varepsilon^{-1/3} MT(r)V = f,$$

where the transport function is defined as usual by

$$T(r) = \left(\Omega(r) - \Omega_0 - \frac{\Omega'_0}{U'_0} (U(r) - U_0) \right).$$

As before, we further approximate solutions to this simpler ODE by a piecewise linear evaluation of the transport function $T(r)$ in the points r_j of the partition $\mathcal{P}_1(\varepsilon^\delta)$, that is

$$\varepsilon \partial_r^2 V - i\varepsilon^{-1/3} MT(r_j)V = f, \quad r \in [r_j, r_j + \varepsilon^\delta), \quad (6.6.33)$$

which, as shown in the previous section, has two nontrivial independent solutions of the form

$$v_1(r) = e^{\sqrt{iMT(r_j)}\varepsilon^{-2/3}r}, \quad v_2(r) = e^{-\sqrt{iMT(r_j)}\varepsilon^{-2/3}r}.$$

In the same manner as for the section towards infinity, and taking into account that their Wronskian is given by $W(v_1(r), v_2(r)) = 2\sqrt{iMT(r_j)}\varepsilon^{-2/3}$, we construct a Green's function associated to (6.6.33) with the correct decay at infinity as

$$(\mathcal{G}_{1,j}f)(r) = \frac{\varepsilon^{-1/3}}{2\sqrt{iMT(r_j)}} \left(\int_0^r e^{-\sqrt{iMT(r_j)}\varepsilon^{-2/3}(r-s)} f(s) ds + \int_r^\infty e^{-\sqrt{iMT(r_j)}\varepsilon^{-2/3}(s-r)} f(s) ds \right),$$

where we make the choice of the branch $\text{Re}(\sqrt{iMT(r_j)}) > 0$ regardless of the sign of $MT(r_j)$, which again we know it is never zero because of Assumptions H0–H2 and SH.

Since the approximated operator is exactly the same as the one defined for the solution towards infinity, we quickly review the essential definitions before analysing the error produced by the approximation in this regime. We define the Green's function associated to the whole interval

$[\varepsilon^\gamma, r_0 + \varepsilon^\gamma)$ by the linear combination

$$\mathcal{G}_1 f(r) = \sum_{j=1}^{\ell-1} \mathcal{G}_{1,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^\delta)})(r) + \mathcal{G}_{1,\ell}(f \mathbb{1}_{[r_\ell, r_0 - \varepsilon^\gamma)})(r). \quad (6.6.34)$$

The first result we present here is a pointwise estimate for the terms $\mathcal{G}_{1,j}$ constituting the Green's function $\mathcal{G}_1$. Since the definition of the operator is exactly the same, and the only change comes from the domain in which this operator is applied, these estimates presented now follow trivially from the proof of Lemma 6.6.6, simply being extra careful with the fact that both extrema of the interval now are ε dependent.

Lemma 6.6.8. *Let $\gamma, \delta > 0$ and consider the partition $\mathcal{P}_1$ of the interval $[\varepsilon^\gamma, r_0 - \varepsilon^\gamma)$ in subintervals of length ε^δ . Let $\mathcal{G}_{1,j}$ be the Green's function associated to equation (6.6.33), and let $f \in Y$. Then, under Assumptions H0–H2 and SH, if*

$$\frac{2}{3} - \gamma - \delta > 0, \quad \gamma < \delta,$$

there exists $c_0 > 0$ independent of $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$, so that for any $\varepsilon > 0$ sufficiently small, independent of j , and any $r \geq 0$ the following estimates hold true,

$$\begin{aligned} |w_\varepsilon(r - r_0) \mathcal{G}_{1,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^\delta)})(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j - r)} \mathbb{1}_{\{r_j - r > \varepsilon^\delta\}}(r) \\ &\quad + \|f\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} \mathbb{1}_{\{r_j - \varepsilon^\delta \leq r \leq r_j + 2\varepsilon^\delta\}}(r) \\ &\quad + \|f\|_Y \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r - r_j - \varepsilon^\delta)} \mathbb{1}_{\{r - r_j > 2\varepsilon^\delta\}}(r), \end{aligned} \quad (6.6.35)$$

$$\begin{aligned} |w_\varepsilon(r - r_0) \partial_r \mathcal{G}_{1,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^\delta)})(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{-1/3}}{r_j^2 \sqrt{|MT(r_j)|}} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j - r)} \mathbb{1}_{\{r_j - r > \varepsilon^\delta\}}(r) \\ &\quad + \|f\|_Y \frac{\varepsilon^{-1/3}}{r_j^2 \sqrt{|MT(r_j)|}} \mathbb{1}_{\{r_j - \varepsilon^\delta \leq r \leq r_j + 2\varepsilon^\delta\}}(r) \\ &\quad + \|f\|_Y \frac{\varepsilon^{-1/3}}{r_j^2 \sqrt{|MT(r_j)|}} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r - r_j - \varepsilon^\delta)} \mathbb{1}_{\{r - r_j > 2\varepsilon^\delta\}}(r). \end{aligned} \quad (6.6.36)$$

The proof of this result follows straightforwardly from the proof of Lemma 6.6.6, so we omit details here for brevity. Nonetheless, we make the following remark regarding the contribution of the smaller extra element in the partition of this domain.

Remark 6.6.9. As before, one of the conditions for this lemma to hold true is the relation $\gamma < \delta$, which comes from the fact that the length of our partition should not exceed ε^γ . Now, our partition has an extra interval $[r_\ell, r_0 - \varepsilon^\gamma)$, but this interval by definition is *at most* of size $\varepsilon^\delta < \varepsilon^\gamma$, and hence this extra piece of the partition does not cause any issues.

As in the previous section, we also want to estimate the Green's function with an r^2 weight to fully

recover the X norm. As a direct consequence of (6.6.35) and (6.6.36) we find that

$$\begin{aligned}
|r^2 w_\varepsilon(r - r_0) \mathcal{G}_{1,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^\delta]})(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{1/3}}{|T(r_j)|} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j - r)} \mathbb{1}_{\{r_j - r > \varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{1/3}}{|T(r_j)|} \mathbb{1}_{\{r_j - \varepsilon^\delta \leq r \leq r_j + 2\varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{1/3}}{|T(r_j)|} \left(1 + \frac{|r - r_j|^2}{r_j^2}\right) e^{-c_0 \varepsilon^{-2/3 + \gamma}(r - r_j - \varepsilon^\delta)} \mathbb{1}_{\{r - r_j > 2\varepsilon^\delta\}}(r),
\end{aligned} \tag{6.6.37}$$

whereas for the derivative,

$$\begin{aligned}
|r^2 w_\varepsilon(r - r_0) \partial_r \mathcal{G}_{1,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^\delta]})(r)| &\lesssim \|f\|_Y \frac{\varepsilon^{-1/3}}{\sqrt{|MT(r_j)|}} e^{-c_0 \varepsilon^{-2/3 + \gamma}(r_j - r)} \mathbb{1}_{\{r_j - r > \varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{-1/3}}{\sqrt{|MT(r_j)|}} \mathbb{1}_{\{r_j - \varepsilon^\delta \leq r \leq r_j + 2\varepsilon^\delta\}}(r) \\
&+ \|f\|_Y \frac{\varepsilon^{-1/3}}{\sqrt{|MT(r_j)|}} \left(1 + \frac{|r - r_j|^2}{r_j^2}\right) e^{-c_0 \varepsilon^{-2/3 + \gamma}(r - r_j - \varepsilon^\delta)} \mathbb{1}_{\{r - r_j > 2\varepsilon^\delta\}}(r).
\end{aligned} \tag{6.6.38}$$

With these estimates in hand we proceed with the main result of the section, that provides quantitative estimates for the X norm of $\mathcal{G}_1$ and $\partial_r \mathcal{G}_1$, and gives an estimate for the error produced by the approximated operator.

Proposition 6.6.10. *Let $\gamma, \delta > 0$ be such that*

$$\gamma < \frac{1}{3}, \quad \gamma < \delta, \quad \gamma + \delta < \frac{2}{3},$$

consider the partition $\mathcal{P}_1(\varepsilon^\delta)$ of the interval $[\varepsilon^\gamma, r_0 - \varepsilon^\gamma]$ in subintervals of length (at most) ε^δ , and let $\mathcal{G}_1$ be the Green's function introduced in (6.6.34). Let $f \in Y$ with $\text{supp}(f) \subset [\varepsilon^\gamma, r_0 - \varepsilon^\gamma]$. Under Assumptions H0–H2 and SH, the following estimates hold true uniformly for any $\varepsilon > 0$ sufficiently small, $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$,

$$\|\mathcal{G}_1 f\|_X \lesssim \varepsilon^{1/3 - 2\gamma} \|f\|_Y, \tag{6.6.39}$$

$$\|\partial_r \mathcal{G}_1 f\|_X \lesssim \varepsilon^{-1/3 - 2\gamma} \|f\|_Y, \tag{6.6.40}$$

Moreover, the error carried out by the approximated operator can be estimated by

$$\|((\mathcal{L} - \lambda)\mathcal{G}_1 - \text{Id})f\|_Y \lesssim \left(\varepsilon^{2/3 - 2\gamma} + \varepsilon^{2/3 - 2\gamma} + \varepsilon^\beta\right) \|f\|_Y,$$

where the coefficient $\beta > 0$ is from P3.

Proof. Part 1: Estimates on $\mathcal{G}_1$. We will obtain the bounds on the corresponding weighted L^∞ spaces using linearity

$$(\mathcal{G}_1 f)(r) = \sum_{j=1}^{\ell_\varepsilon} \mathcal{G}_{1,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^\delta]})(r),$$

and Lemma 6.6.8. We begin by assuming that $r \in [\varepsilon^\gamma, r_0 - \varepsilon^\gamma]$, and hence $r \in [r_k, r_k + \varepsilon^\delta]$ for some k . By means of (6.6.35), we find the estimate

$$\begin{aligned} |w_\varepsilon(r - r_0)\mathcal{G}_1 f(r)| &\lesssim \|f\|_Y \sum_{j=1}^{k-2} \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \\ &\quad + \|f\|_Y \sum_{j=k+2}^{\ell} \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \\ &\quad + \|f\|_Y \varepsilon^{1/3} \left(\frac{1}{r_{k-1}^2 |MT(r_{k-1})|} + \frac{1}{r_k^2 |MT(r_k)|} + \frac{1}{r_{k+1}^2 |MT(r_{k+1})|} \right). \end{aligned}$$

Let us start analysing the third term in the estimate. By Assumptions H1 and SH there holds that $|MT(r_k)| \gtrsim \varepsilon^{2\gamma}$, with $|T(r_k)| \sim \varepsilon^{2\gamma}$ only near r_0 . Since $r_k \geq \varepsilon^\gamma$, but $r_k \sim \varepsilon^\gamma$ only near the origin, i.e. far away from r_0 , we can write

$$\frac{\varepsilon^{1/3}}{r_k^2 |MT(r_k)|} \lesssim \varepsilon^{1/3-2\gamma},$$

uniformly in k . The other two addends follow readily as in the proof of Proposition 6.6.7. That is, on the one hand we have

$$\begin{aligned} \|f\|_Y \sum_{j=k+2}^{\ell} \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} &\lesssim \|f\|_Y \varepsilon^{1/3-2\gamma} \sum_{j=k+2}^{\infty} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \\ &\lesssim \|f\|_Y \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_{k+2}-r)} \\ &\leq \|f\|_Y \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3+\gamma+\delta}}. \end{aligned}$$

On the other hand we also have the bound,

$$\begin{aligned} \|f\|_Y \sum_{j=1}^{k-2} \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} &\lesssim \|f\|_Y \varepsilon^{1/3-2\gamma} \sum_{j=1}^{k-2} e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \\ &\lesssim \|f\|_Y \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_{k-1})} \\ &\leq \|f\|_Y \varepsilon^{1/3-2\gamma} e^{-c_0 \varepsilon^{-2/3+\gamma+\delta}}. \end{aligned}$$

Note that for the above bounds we once again require the condition $\gamma + \delta < 2/3$. Since all of these bounds are uniform in k , we arrive at the estimate

$$|w_\varepsilon(r - r_0)\mathcal{G}_1 f(r)| \lesssim \varepsilon^{1/3-2\gamma} \|f\|_Y,$$

which in tandem with the observation that $\max\{1, r^2\} \lesssim 1$ for $r \leq r_0 - \varepsilon^\gamma$ implies that

$$|\max\{1, r^2\} w_\varepsilon(r - r_0)\mathcal{G}_1 f(r)| \lesssim \varepsilon^{1/3-2\gamma} \|f\|_Y$$

for $\varepsilon^\gamma \leq r \leq r_0 - \varepsilon^\gamma$.

If now $r \in [0, \varepsilon^\gamma]$, using (6.6.35) we find the corresponding estimate

$$\begin{aligned} |\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_1 f(r)| &\lesssim \|f\|_Y \sum_{j=3}^{\ell} \frac{\varepsilon^{1/3}}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \\ &\quad + \|f\|_Y \varepsilon^{1/3} \left(\frac{1}{r_1^2 |MT(r_1)|} + \frac{1}{r_2^2 |MT(r_2)|} \right). \end{aligned}$$

Proceeding exactly as before, this is bounded in magnitude by $\varepsilon^{-1/3+2\gamma} \|f\|_Y$. On the contrary, if $r > r_0 + \varepsilon^\gamma$ we find an analogous expression via a combination of (6.6.35) and (6.6.37). We write

$$\begin{aligned} |\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_1 f(r)| &\lesssim \|f\|_Y \sum_{j=1}^{\ell-1} \frac{\varepsilon^{1/3}}{|MT(r_j)|} \left(1 + \frac{|r - r_j|^2}{r_j^2} \right) e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \\ &\quad + \|f\|_Y \frac{\varepsilon^{1/3}}{|MT(r_\ell)|} \mathbb{1}_{\{|r-r_\ell| \leq 2\varepsilon^\delta\}}. \end{aligned}$$

Observe that the last addend vanishes if r and the last point of the partition r_ℓ are not sufficiently close. Arguing as in the previous section, we thus see that

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_1 f(r)| \lesssim \|f\|_Y \varepsilon^{1/3-2\gamma},$$

uniformly for $r \geq r_0 - \varepsilon^\gamma$. Combining the estimates for the intervals $[0, \varepsilon^\gamma]$, $[\varepsilon^\gamma, r_0 - \varepsilon^\gamma]$, $[r_0 - \varepsilon^\gamma, \infty)$, we find the estimate (6.6.39) in the claim of the proposition.

Part 2: Estimates on $\partial_r \mathcal{G}_1$. We want to derive bounds for $\partial_r \mathcal{G}_1$ in X . We start from the pointwise bound (6.6.35) from Lemma 6.6.8. Arguing as in the previous part of the proof, for any $r \in [r_k, r_{k+1})$ we obtain an estimate of the form

$$\begin{aligned} |\max\{1, r^2\} w_\varepsilon(r - r_0) \partial_r \mathcal{G}_1 f(r)| &\lesssim \|f\|_Y \sum_{j=1}^{k-2} \frac{\varepsilon^{-1/3}}{r_j^2 \sqrt{|MT(r_j)|}} e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)} \\ &\quad + \|f\|_Y \sum_{j=k+2}^{\ell} \frac{\varepsilon^{-1/3}}{r_j^2 \sqrt{|MT(r_j)|}} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)} \\ &\quad + \|f\|_Y \varepsilon^{-1/3} \left(\frac{1}{r_{k-1}^2 \sqrt{|MT(r_{k-1})|}} + \frac{1}{r_k^2 \sqrt{|MT(r_k)|}} + \frac{1}{r_{k+1}^2 \sqrt{|MT(r_{k+1})|}} \right). \end{aligned}$$

Since $r_j \in [\varepsilon^\gamma, r_0 - \varepsilon^\gamma]$ we can write

$$\frac{\varepsilon^{-1/3}}{r_j^2 \sqrt{|MT(r_j)|}} \lesssim \max \left\{ \varepsilon^{-1/3-2\gamma}, \varepsilon^{-1/3-\gamma} \right\} = \varepsilon^{1/3-2\gamma},$$

where again, we are using Assumptions H1 and SH to argue that r_j cannot be close to ε^γ at the same time as $|T(r_j)|$ is close to $\varepsilon^{2\gamma}$. Therefore, following the same arguments that we used for the bound of $\mathcal{G}_1$ in X , we find

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \partial_r \mathcal{G}_1 f(r)| \lesssim \varepsilon^{-1/3-2\gamma} \|f\|_Y,$$

uniformly for $r \in [\varepsilon^\gamma, r_0 - \varepsilon^\gamma]$. Similarly, for the case $r \notin [\varepsilon^\gamma, r_0 + \varepsilon^\gamma]$ we obtain again

$$|\max\{1, r^2\} w_\varepsilon(r - r_0) \partial_r \mathcal{G}_1 f(r)| \lesssim \|f\|_Y \varepsilon^{-1/3-2\gamma},$$

and thus we arrive at the claim of the proposition (6.6.40).

Part 3: The error estimate. This part of the claim entails the biggest differences with respect to the case of the solution towards infinity. Even though the setting is initially the same, now we need to address the terms involving $T(r)$ and $\Omega'(r)$, whilst bearing in mind the degeneracies that may occur as $r \rightarrow 0$. As before, we define the operator $\mathcal{L}_{1,j}$ by (6.6.33). We present here the proof for the first component of the operator, since the argument for the second component is identical. The difference between the exact and the approximate operators is given by

$$\begin{aligned} (\mathcal{L} - \lambda - \mathcal{L}_{1,j})V_1(r) &= \frac{\varepsilon}{r} \partial_r V_1(r) - iM\varepsilon^{-1/3}(T(r) - T(r_j))V_1(r) \\ &\quad - \left[\frac{\varepsilon}{r^2} + \varepsilon^{1/3}M^2 \left(\frac{1}{r^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right) + \lambda + P_\varepsilon(r) \right] V_1 + Q_\varepsilon(r)V_2(r), \end{aligned}$$

where $P_\varepsilon(r)$ and $Q_\varepsilon(r)$ are defined by (6.6.31). Let $f \in Y$ with $\text{supp}(f) \subset [\varepsilon^\gamma, r_0 - \varepsilon^\delta]$. Arguing as in the proof of Proposition 6.6.7, we get that

$$\begin{aligned} |r^2 w_\varepsilon(r - r_0)((\mathcal{L} - \mu\varepsilon^{1/3})\mathcal{G}_1 - \text{Id})f| &\lesssim \varepsilon |w_\varepsilon(r - r_0)r \partial_r \mathcal{G}_1 f| + \varepsilon^{1/3} |\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_1 f| \\ &\quad + \varepsilon^{-1/3} \sum_{j=1}^{\ell} \left| r^2 w_\varepsilon(r - r_0) M(T(r) - T(r_j)) \mathcal{G}_{1,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon\delta]}) \right| \\ &\quad + \varepsilon^{1/3} |r \Omega'(r)| |w_\varepsilon(r - r_0) r^2 \mathcal{G}_1 f|. \end{aligned}$$

To deal with the first two addends we can directly apply the bounds previously derived in this proposition. On the one hand,

$$\varepsilon |w_\varepsilon(r - r_0)r \partial_r \mathcal{G}_1 f| \lesssim \varepsilon^{2/3-2\gamma} \|f\|_Y.$$

On the other hand, by (6.6.39) we obtain the estimate

$$\varepsilon^{1/3} |\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_1 f| \lesssim \varepsilon^{2/3-2\gamma} \|f\|_Y.$$

Only the terms involving $T(r)$ and $\Omega'(r)$ are left to be analysed. Let us start with the addend concerning the transport function $T(r)$. Assume for now that $r \in [r_k, r_k + \varepsilon^\delta]$ for some $k \in \mathbb{N}$. Hence, by means of Lemma 6.6.8 we write

$$\varepsilon^{-1/3} \sum_{j=1}^{\ell} r^2 M |T(r) - T(r_j)| |w_\varepsilon(r - r_0) \mathcal{G}_{1,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon\delta]})| \leq \text{I}_{\varepsilon,k}(r) + \text{II}_{\varepsilon,k}(r) + \text{III}_{\varepsilon,k}(r),$$

where

$$\text{I}_{\varepsilon,k}(r) = \|f\|_Y \sum_{j=k+2}^{\ell} \frac{r^2 |T(r) - T(r_j)|}{r_j^2 |T(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)},$$

$$\begin{aligned}\Pi_{\varepsilon,k}(r) &= \|f\|_Y \left(\frac{r^2|T(r) - T(r_{k-1})|}{r_{k-1}^2|T(r_{k-1})|} + \frac{r^2|T(r) - T(r_k)|}{r_k^2|T(r_k)|} + \frac{r^2|T(r) - T(r_{k+1})|}{r_{k+1}^2|T(r_{k+1})|} \right), \\ \text{III}_{\varepsilon,k}(r) &= \|f\|_Y \sum_{j=1}^{k-2} \frac{r^2|T(r) - T(r_j)|}{r_j^2|T(r_j)|} e^{-c_0\varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^\delta)}.\end{aligned}$$

We start by analysing the second addend. Observe that since $r \in [r_k, r_{k+1})$, then

$$\frac{r^2}{r_k^2}, \frac{r^2}{r_{k-1}^2}, \frac{r^2}{r_{k+1}^2} \lesssim 1.$$

This together with Property P3 yields

$$\Pi_{\varepsilon,k}(r) \lesssim \|f\|_Y \frac{r^2|T(r) - T(r_k)|}{r_k^2|T(r_k)|} \lesssim \|f\|_Y \sum_{n=1}^{N_1} B_n(\varepsilon) |r - r_k|^n \leq \|f\|_Y \sum_{n=1}^{N_1} B_n(\varepsilon) \varepsilon^{n\delta} \lesssim \|f\|_Y \varepsilon^\beta.$$

Next in order we analyse the first term, corresponding to the indices $k+2 \leq j \leq \ell$. We write

$$\begin{aligned}\text{I}_{\varepsilon,k}(r) &\leq \|f\|_Y \sum_{j=k+2}^{\ell} \sum_{n=1}^{N_1} B_n(\varepsilon) \frac{r^2|r - r_j|^n}{r_j^2} e^{-c_0\varepsilon^{-2/3+\gamma}(r_j-r)} \\ &\leq \|f\|_Y \sum_{n=1}^{N_1} B_n(\varepsilon) \sum_{j=k+2}^{\infty} |r - r_j|^n (1 + \varepsilon^{-2\gamma}|r - r_j|^2) e^{-c_0\varepsilon^{-2/3+\gamma}(r_j-r)},\end{aligned}$$

and therefore, up to a constant depending exclusively on N_1 that we absorb with the symbol $\lesssim$,

$$\text{I}_{\varepsilon,k}(r) \leq \|f\|_Y \varepsilon^{-2\gamma} \sum_{n=1}^{N_1} B_n(\varepsilon) \sum_{j=k+2}^{\infty} e^{-\frac{1}{2}c_0\varepsilon^{-2/3+\gamma}(r_j-r)} \lesssim \|f\|_Y \varepsilon^{-2\gamma} e^{-\frac{1}{2}c_0\varepsilon^{-2/3+\gamma}(r_{k+2}-r)} \sum_{n=1}^{N_1} B_n(\varepsilon).$$

Additionally, notice that Property P3 implies that

$$\sum_{n=1}^{N_1} B_n(\varepsilon) \leq \varepsilon^{-N_1\delta} \sum_{n=1}^{N_1} B_n(\varepsilon) \varepsilon^{n\delta} \lesssim \varepsilon^{\beta-N_1\delta},$$

and thus

$$\text{I}_{\varepsilon,k}(r) \lesssim \|f\|_Y \varepsilon^{\beta-2\gamma-N_1\delta} e^{-\frac{1}{2}c_0\varepsilon^{-2/3+\gamma+\delta}},$$

which quickly converges to zero as $\varepsilon \rightarrow 0$ provided that $\delta + \gamma < 2/3$ due to the effect of the exponential. Last, we address the third addend using a completely analogous argument,

$$\begin{aligned}\text{III}_{\varepsilon,k}(r) &\lesssim \|f\|_Y \sum_{j=1}^{k-2} \sum_{n=1}^{N_1} B_n(\varepsilon) (1 + \varepsilon^{-2\gamma}|r - r_j|^2) |r - r_j|^n e^{-c_0\varepsilon^{-2/3+\gamma}(r-r_{j+1})} \\ &\lesssim \|f\|_Y \varepsilon^{\beta-2\gamma-N_1\delta} e^{-\frac{1}{2}c_0\varepsilon^{-2/3+\gamma+\delta}}.\end{aligned}$$

Observe that if $r \notin [\varepsilon^\gamma, r_0 - \varepsilon^\gamma)$ the estimates obtained are analogous, since (6.6.35), (6.6.36), (6.6.37) and (6.6.38) hold true still in this range and the bounds the produce are analogous. The principle one should bear in mind is that if there exists $r_k \in \mathcal{P}_1(\varepsilon^\delta)$ such that $|r - r_k| \leq 2\varepsilon^\delta$, then the error is bounded by ε^β as shown before. Otherwise the error will be bounded by a more rapidly decaying exponential function, and can be neglected. In particular, we find that for the term with

the transport function, the error is bounded by

$$\varepsilon^{-1/3} \sum_{j=1}^{\ell} M|T(r) - T(r_j)| |w_{\varepsilon}(r - r_0) \mathcal{G}_{1,j}(f \mathbb{1}_{[r_j, r_j + \varepsilon^{\delta}])}| \lesssim \varepsilon^{\beta},$$

where β is defined in Property P3. The last contribution to be addressed is the one corresponding to the terms involving Ω' . Again, for any $r \in [r_k, r_k + \varepsilon^{\delta})$, we can write

$$\varepsilon^{1/3} |r \Omega'(r)| |r^2 w_{\varepsilon}(r - r_0) \mathcal{G}_1 f| \lesssim \text{I}_{\varepsilon,k}(r) + \text{II}_{\varepsilon,k}(r) + \text{III}_{\varepsilon,k}(r),$$

where

$$\begin{aligned} \text{I}_{\varepsilon,k}(r) &= \varepsilon^{2/3} \|f\|_Y \sum_{j=k+2}^{\ell} \frac{|r \Omega'(r)|}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r_j-r)}, \\ \text{II}_{\varepsilon,k}(r) &= \varepsilon^{2/3} \|f\|_Y \sum_{j \in \{k-1, k, k+1\}} \frac{|r \Omega'(r)|}{|MT(r_j)|}, \\ \text{III}_{\varepsilon,k}(r) &= \varepsilon^{2/3} \|f\|_Y \sum_{j=1}^{k-2} \frac{r^2 |r \Omega'(r)|}{r_j^2 |MT(r_j)|} e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^{\delta})}. \end{aligned}$$

As before, we start by the second addend. On the one hand, observe that by means of Property P4 we can write

$$\begin{aligned} \text{II}_{\varepsilon,k}(r) &\lesssim \varepsilon^{2/3-2\gamma} \|f\|_Y \left(D_0 + \sum_{n=1}^{N_2} D_n |r - r_k|^n \right) \\ &\lesssim \varepsilon^{2/3-2\gamma} \|f\|_Y \left(D_0 + \sum_{n=1}^{N_2} D_n \varepsilon^{n\delta} \right) \lesssim \varepsilon^{2/3-2\gamma} \|f\|_Y. \end{aligned}$$

Next in order, we look at the term $\text{I}_{\varepsilon,k}(r)$, for which Lemma 6.3.4 yields the estimate

$$\begin{aligned} \text{I}_{\varepsilon,k}(r) &\lesssim \varepsilon^{2/3-4\gamma} \|f\|_Y \sum_{j=k+2}^{\infty} \left(D_0 + \sum_{n=1}^{N_2} D_n |r_j - r|^n \right) e^{-c_0 \varepsilon^{-2/3+\gamma+\delta}} \\ &\lesssim \|f\|_Y \varepsilon^{2/3-4\gamma} e^{-c_0 \varepsilon^{-2/3+\gamma+\delta}}, \end{aligned}$$

which quickly converges to zero as $\varepsilon \rightarrow 0$ due to the effect of the exponential. The last term to be addressed can be analogously bounded by

$$\begin{aligned} \text{III}_{\varepsilon,k}(r) &\lesssim \varepsilon^{2/3-2\gamma} \|f\|_Y \sum_{j=1}^{k-2} (1 + \varepsilon^{-2\gamma} |r - r_j|^2) \left(D_0 + \sum_{n=1}^{N_2} D_n |r - r_j|^n \right) e^{-c_0 \varepsilon^{-2/3+\gamma}(r-r_j-\varepsilon^{\delta})} \\ &\lesssim \|f\|_Y \varepsilon^{2/3-4\gamma} e^{-\frac{1}{2} c_0 \varepsilon^{-2/3+\gamma+\delta}}. \end{aligned}$$

As argued before, we find the “worst” estimates for values of r such that $|r - r_k| \leq 2\varepsilon^{\delta}$ for some $r_k \in \mathcal{P}_1(\varepsilon^{\delta})$, whereas if $|r - r_k| > 2\varepsilon^{\delta}$. This argument yields analogous bounds for the case $r \notin [\varepsilon^{\gamma}, r_0 - \varepsilon^{\gamma})$, and thus we can write in general

$$\varepsilon^{1/3} |r \Omega'(r)| |w_{\varepsilon}(r - r_0) r^2 \mathcal{G}_1 f| \lesssim \varepsilon^{2/3-2\gamma} \|f\|_Y.$$

Combining all the estimates together we find the claim of the proposition,

$$|r^2 w_\varepsilon(r-r_0)((\mathcal{L}-\lambda)\mathcal{G}_1 f - f)| \lesssim \max \left\{ \varepsilon^{2/3-2\gamma}, \varepsilon^\beta, \varepsilon^{2/3-2\gamma} \right\} \|f\|_Y,$$

which, under the assumption $\gamma < \delta$, converges to zero as $\varepsilon \rightarrow 0$. $\square$

6.6.4 Green's function near zero

In this section we consider the last bit of the (positive) real line that is left to study, namely $[0, \varepsilon^\gamma)$. This last bit of the Green's function will turn out to be of vital importance, since it can be shown to have a *regularizing* effect on the solution. Indeed, when computing the error for the Green's functions we have defined so far in Section 6.6, the error bounds we have derived are all in the space Y , which allows for quadratic blow up as $r \rightarrow 0$. Since we would like to invert $\mathcal{L}$ on the space X , it is however essential that our inverse does not blow up near $r = 0$. In this regard, we show that the approximate Green's function near zero not only has the properties we expect of "generic" approximate Green's functions (such as rapid decay outside of the support of the input), but also defines a bounded map from $Y \rightarrow X$. This allows us to close the construction of the operator $\mathcal{G}^\lambda$ in such a way that it defines a bounded linear map $\mathcal{G}^\lambda : Y \rightarrow X$, and hence terms of the form $\mathcal{G}^\lambda(((\mathcal{L}-\lambda)\mathcal{G}^\lambda - \text{Id})^n)$ are well defined maps.

That being said, in order to derive such a regularising map, substantial care has to be taken in defining the approximate Green's function, since the terms of order r^{-2} in the definition of $\mathcal{L}$ may no longer be treated as errors. In fact, as $r \rightarrow 0$, these will be the dominant terms in $\mathcal{L}$. We thus begin with some algebraic manipulations of the equation defining $\mathcal{L}$ in order to obtain an expression more amenable to analysis.

Recall from Section 6.3.3, that the kinematic dynamo equations can be written in the following form,

$$\begin{aligned} \varepsilon \left(\partial_r^2 V_1 + \frac{1}{r} \partial_r V_1 \right) &= (A_\varepsilon + \lambda) V_1 - \frac{1}{2} \left(\frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon + \alpha \varepsilon^{1/3} C_\varepsilon \right) V_1 - \frac{1}{2} \left(\frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon - \alpha \varepsilon^{1/3} C_\varepsilon \right) V_2, \\ \varepsilon \left(\partial_r^2 V_2 + \frac{1}{r} \partial_r V_2 \right) &= (A_\varepsilon + \lambda) V_2 + \frac{1}{2} \left(\frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon + \alpha \varepsilon^{1/3} C_\varepsilon \right) V_2 + \frac{1}{2} \left(\frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon - \alpha \varepsilon^{1/3} C_\varepsilon \right) V_1, \end{aligned}$$

where

$$A_\varepsilon = \frac{\varepsilon}{r^2} + \varepsilon^{1/3} M^2 \left(\frac{1}{r^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right) + i \varepsilon^{-1/3} M T(r),$$

$$B_\varepsilon = \frac{2iM\varepsilon^{2/3}}{r^2}, \quad C_\varepsilon = -r\Omega' - \frac{2iM\varepsilon^{2/3}}{r^2},$$

and

$$\alpha^2 = -\frac{2iM}{r_0^3 \Omega'_0}.$$

We now conjugate this equation by the invertible matrix

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{6.6.41}$$

and define the operator $\tilde{\mathcal{L}}$ via

$$(\tilde{\mathcal{L}} - \lambda) \begin{pmatrix} V_+ \\ V_- \end{pmatrix} = P(\mathcal{L} - \lambda)P^{-1} \begin{pmatrix} V_+ \\ V_- \end{pmatrix},$$

with

$$(\tilde{\mathcal{L}} - \lambda) \begin{pmatrix} V_+ \\ V_- \end{pmatrix} = \left(\varepsilon \partial_r^2 + \varepsilon \frac{1}{r} \partial_r - (A_\varepsilon + \lambda) \right) \begin{pmatrix} V_+ \\ V_- \end{pmatrix} + \begin{pmatrix} \alpha \varepsilon^{1/3} C_\varepsilon V_- \\ \frac{1}{\alpha} \varepsilon^{-1/3} B_\varepsilon V_+ \end{pmatrix}, \quad (6.6.42)$$

where we denote $V_+ = V_1 + V_2$ and $V_- = V_1 - V_2$. Recall now that—in the coordinate system $V = (V_1, V_2)$ and $f = (f_1, f_2)$ —we want to solve the equation $(\mathcal{L} - \lambda)V = f$. Suppose for a second that we manage to find an operator $\tilde{\mathcal{G}}_0$, so that for all g supported in $[0, \varepsilon^\gamma)$, it holds

$$\|((\tilde{\mathcal{L}} - \lambda)\tilde{\mathcal{G}}_0 - \text{Id})g\|_Y \lesssim \varepsilon^a \|g\|_Y, \quad (6.6.43)$$

for some $a > 0$. Then, setting $f = P^{-1}g$, $\mathcal{G}_0 = P^{-1}\tilde{\mathcal{G}}_0P$, we have

$$\begin{aligned} ((\mathcal{L} - \lambda)\mathcal{G}_0 - \text{Id})f &= (\mathcal{L} - \lambda)P^{-1}\tilde{\mathcal{G}}_0g - P^{-1}g \\ &= P^{-1}(P(\mathcal{L} - \lambda)P^{-1}\tilde{\mathcal{G}}_0g - g) = P^{-1}((\tilde{\mathcal{L}} - \lambda)\tilde{\mathcal{G}}_0g - g). \end{aligned}$$

Therefore, since as a linear mapping the matrix P satisfies $\|P\|, \|P^{-1}\| \lesssim 1$, there holds

$$\|((\mathcal{L} - \lambda)\mathcal{G}_0 - \text{Id})f\|_Y \leq \|P^{-1}\| \|((\tilde{\mathcal{L}} - \lambda)\tilde{\mathcal{G}}_0 - \text{Id})g\|_Y \lesssim \varepsilon^a \|Pf\|_Y \lesssim \varepsilon^a \|f\|_Y. \quad (6.6.44)$$

Thus, it suffices to find an operator $\tilde{\mathcal{G}}_0$ so that for any g supported in $[0, \varepsilon^\gamma)$, estimate (6.6.43) holds true for some $a > 0$. As we have been doing so far, we want to find $\tilde{\mathcal{G}}_0$ as an exact inverse of an approximate ODE for the system $\tilde{\mathcal{L}} - \lambda$. In order to write the appropriate approximated operator we use Assumptions H0 and H1, which in particular ensure that $T \in C^3([0, \infty))$ is well defined up to $r = 0$, and moreover

$$\lim_{r \rightarrow 0} T(r) = \tau \neq 0.$$

Taking this into account, we may now define the operator

$$\tilde{\mathcal{L}}_0 \begin{pmatrix} V_+ \\ V_- \end{pmatrix} = \left(\varepsilon \partial_r^2 + \varepsilon \frac{1}{r} \partial_r - \frac{\varepsilon^{1/3} M^2}{r^2} + iM\varepsilon^{-1/3} \tau \right) \begin{pmatrix} V_+ \\ V_- \end{pmatrix} - \frac{\varepsilon^{1/3}}{r^2} \sqrt{-2iMr_0^3 \Omega'_0} \begin{pmatrix} 0 \\ V_+ \end{pmatrix} \quad (6.6.45)$$

which approximates (6.6.42). To derive it, we follow the same criteria as for the previous sections, neglecting the contributions of the terms in (6.6.42) that are either multiplied by a sufficiently large power of ε , or in this case, by a non-negative power of r , since this region corresponds to the limit $r \rightarrow 0$. We denote this approximated operator by $\tilde{\mathcal{L}}_0$, and its corresponding Green's function by $\tilde{\mathcal{G}}_0$, namely

$$\tilde{\mathcal{G}}_0 \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

solves

$$\tilde{\mathcal{L}}_0 \begin{pmatrix} V_+ \\ V_- \end{pmatrix} = \begin{pmatrix} f_+ \\ f_- \end{pmatrix},$$

with the further boundary condition that $(V_+, V_-) \rightarrow 0$ as $r \rightarrow \infty$.

We now begin the analysis of the operator $\tilde{\mathcal{L}}_0$. Observe that the first component of $\tilde{\mathcal{L}}_0$ in (6.6.45) can be studied independently of the second component. We introduce a new Green's function associated just to this first component. We consider the operator

$$\varepsilon \left(\partial_r^2 V + \frac{1}{r} \partial_r V \right) - \left(\frac{\varepsilon^{1/3} M^2}{r^2} + i \varepsilon^{-1/3} M \tau \right) V, \quad (6.6.46)$$

and we write $\mathcal{G}_0^1$ to denote its Green's function, i.e.

$$(\tilde{\mathcal{L}}_0)_1 \mathcal{G}_0^1 f = f.$$

We start by studying this first Green's function. Just like in the previous sections, we look for two independent solutions to the approximated ODE (6.6.46). In general, ODEs of the form

$$v''(r) + \frac{1}{r} v'(r) = \left(\frac{a}{r^2} + b \right) v(r)$$

have two independent solutions given by Bessel functions of the first and second kind, see [Olv74] for detailed information. Since our arguments are necessarily complex, we make the choice of defining the two independent solutions as *modified Bessel functions*,

$$v_1(r) = I_{a^{1/2}} \left(b^{1/2} r \right), \quad v_2(r) = K_{a^{1/2}} \left(b^{1/2} r \right),$$

where $I_\nu(x)$ represents a modified Bessel function of the first kind, whereas $K_\nu(x)$ represents a modified Bessel function of the second kind. For more details about these special functions, we refer the reader to Appendix 6.8.1. For our specific case (6.6.46), notice that

$$a_\varepsilon = M^2 \varepsilon^{-2/3}, \quad b_\varepsilon = i \tau M \varepsilon^{-4/3},$$

where we include the subindex to stress their ε dependency now. The Wronskian between the modified Bessel functions I_ν and K_ν is given by

$$W(I_\nu(z), K_\nu(z)) = I_\nu(z) K'_\nu(z) - I'_\nu(z) K_\nu(z) = -\frac{1}{z},$$

so therefore, applied to the solutions found for (6.6.46) we get,

$$W(v_1(r), v_2(r)) = -\frac{1}{r}.$$

With this in mind, we can construct the following Green's function associated to the problem (6.6.46). In this section we take $\text{supp}(f) \subset [0, \varepsilon^\gamma]$, for some $\gamma > 0$ to be chosen accordingly. Then

we write,

$$\mathcal{G}_0^1 f(r) = -\varepsilon^{-1} v_2(r) \int_0^r s v_1(s) f(s) ds - \varepsilon^{-1} v_1(r) \int_r^\infty s v_2(s) f(s) ds, \quad (6.6.47)$$

namely

$$\begin{aligned} \mathcal{G}_0^1 f(r) &= -\varepsilon^{-1} K_{a_\varepsilon^{1/2}} \left(b_\varepsilon^{1/2} r \right) \int_0^r s I_{a_\varepsilon^{1/2}} \left(b_\varepsilon^{1/2} s \right) f(s) ds \\ &\quad - \varepsilon^{-1} I_{a_\varepsilon^{1/2}} \left(b_\varepsilon^{1/2} r \right) \int_r^\infty s K_{a_\varepsilon^{1/2}} \left(b_\varepsilon^{1/2} s \right) f(s) ds. \end{aligned}$$

Since all we want to compute now are weighted L^∞ bounds for this Green's function, first of all what we need are suitable asymptotic bounds for the modified Bessel functions. From Lemma 6.8.2 we have the bounds which hold uniformly as $v \rightarrow \infty$ for all $z \in \mathbb{C}$ with $|\arg(z)| = \frac{\pi}{4}$.

$$|I_v(vz)| \leq C \frac{e^{v\xi}}{(2\pi v)^{1/2} (1+z^2)^{1/4}}, \quad (6.6.48)$$

$$|K_v(vz)| \leq C \left(\frac{\pi}{2v} \right)^{1/2} \frac{e^{-v\xi}}{(1+z^2)^{1/4}}, \quad (6.6.49)$$

where

$$\xi = (1+z^2)^{1/2} + \log \left(\frac{z}{1+(1+z^2)^{1/2}} \right).$$

Since v_1 and v_2 are represented in terms of modified Bessel function with indices of the form $v = a_\varepsilon^{1/2} = M\varepsilon^{-1/3}$, and $|\arg(b_\varepsilon^{1/2} r)| = \frac{\pi}{4}$, these are indeed the bounds we are interested in. For simplicity's sake, and for the sake of a clearer presentation of the results, we will select the value $M = 1$ in the following. The analysis remains largely unchanged for other values of M , and it will be clear from the analysis that all our bounds depend continuously on M . With this in mind we obtain from (6.6.48), (6.6.49) the estimates

$$\begin{aligned} |v_1(r)| &\lesssim \varepsilon^{1/6} \frac{|\tau|^{\frac{1}{2}\varepsilon^{-1/3}} \varepsilon^{-\frac{1}{3}\varepsilon^{-1/3}} r^{\varepsilon^{-1/3}}}{|1+i\tau\varepsilon^{-2/3}r^2|^{1/4} |1+(1+i\tau\varepsilon^{-2/3}r^2)^{1/2}|^{\varepsilon^{-1/3}}} e^{\operatorname{Re}[\varepsilon^{-1/3}(1+i\tau\varepsilon^{-2/3}r^2)^{1/2}]}, \\ |v_2(r)| &\lesssim \varepsilon^{1/6} \frac{|1+(1+i\tau\varepsilon^{-2/3}r^2)^{1/2}|^{\varepsilon^{-1/3}}}{|1+i\tau\varepsilon^{-2/3}r^2|^{1/4} |\tau|^{\frac{1}{2}\varepsilon^{-1/3}} \varepsilon^{-\frac{1}{3}\varepsilon^{-1/3}} r^{\varepsilon^{-1/3}}} e^{-\operatorname{Re}[\varepsilon^{-1/3}(1+i\tau\varepsilon^{-2/3}r^2)^{1/2}]}. \end{aligned}$$

In order to obtain precise bounds from here, we shall break down the required computations into the following pieces

$$\begin{aligned} \operatorname{Re} \left[\varepsilon^{-1/3} (1+i\tau\varepsilon^{-2/3}r^2)^{1/2} \right] &= \frac{\varepsilon^{-1/3}}{\sqrt{2}} \left(\left(1+\varepsilon^{-4/3}r^4\tau^2 \right)^{1/2} + 1 \right)^{1/2}; \\ \left| 1+(1+i\tau\varepsilon^{-2/3}r^2)^{1/2} \right|^{\varepsilon^{-1/3}} &= \left(1 + \left[2+2 \left(1+\varepsilon^{-4/3}r^4\tau^2 \right)^{1/2} \right]^{1/2} + \left(1+\varepsilon^{-4/3}r^4\tau^2 \right)^{1/2} \right)^{\frac{1}{2}\varepsilon^{-1/3}}; \\ \left| 1+i\tau\varepsilon^{-2/3}r^2 \right|^{1/4} &= \left(1+\varepsilon^{-4/3}r^4\tau^2 \right)^{1/8}. \end{aligned}$$

Putting everything together, we find the estimates

$$|v_1(r)| \lesssim \frac{|\tau|^{\frac{1}{2}\varepsilon^{-1/3}} \varepsilon^{\frac{1}{6}-\frac{1}{3}} \varepsilon^{-1/3} r^{\varepsilon^{-1/3}} e^{\frac{1}{\sqrt{2}}\varepsilon^{-1/3}} \left((1+\varepsilon^{-4/3} r^4 \tau^2)^{1/2} + 1 \right)^{1/2}}{(1+\varepsilon^{-4/3} r^4 \tau^2)^{1/8} \left(1 + \left[2 + 2(1+\varepsilon^{-4/3} r^4 \tau^2)^{1/2} \right]^{1/2} + (1+\varepsilon^{-4/3} r^4 \tau^2)^{1/2} \right)^{\frac{1}{2}\varepsilon^{-1/3}}};$$

$$|v_2(r)| \lesssim \frac{|\tau|^{-\frac{1}{2}\varepsilon^{-1/3}} \varepsilon^{\frac{1}{6}+\frac{1}{3}} \varepsilon^{-1/3} r^{-\varepsilon^{-1/3}} e^{-\frac{1}{\sqrt{2}}\varepsilon^{-1/3}} \left((1+\varepsilon^{-4/3} r^4 \tau^2)^{1/2} + 1 \right)^{1/2}}{(1+\varepsilon^{-4/3} r^4 \tau^2)^{1/8} \left(1 + \left[2 + 2(1+\varepsilon^{-4/3} r^4 \tau^2)^{1/2} \right]^{1/2} + (1+\varepsilon^{-4/3} r^4 \tau^2)^{1/2} \right)^{-\frac{1}{2}\varepsilon^{-1/3}}}.$$

In order to plug in these bounds into an estimate for the Green's function (6.6.47), we first realise that given any positive n , the function

$$x \mapsto \frac{1}{x^{1/8} \left(1 + \sqrt{2}(1+x)^{1/2} + x \right)^n} e^{\sqrt{2}n(1+x)^{1/2}}$$

is monotone increasing for all $x > 1$. In particular, setting $x = (1 + \varepsilon^{-4/3} r^4 \tau^2)^{1/2}$, $n = \frac{1}{2}\varepsilon^{-\frac{1}{3}}$, for the first addend in Green's function (6.6.47) we can write

$$\begin{aligned} \left| \varepsilon^{-1} v_1(r) \int_0^r s v_1(s) f(s) ds \right| &\lesssim \frac{|\tau|^{-\frac{1}{2}\varepsilon^{-1/3}} \varepsilon^{-\frac{5}{6}+\frac{1}{3}} \varepsilon^{-1/3} r^{-\varepsilon^{-1/3}}}{(1+\varepsilon^{-4/3} r^4 \tau^2)^{1/4}} \int_0^r |\tau|^{\frac{1}{2}\varepsilon^{-1/3}} \varepsilon^{\frac{1}{6}-\frac{1}{3}} \varepsilon^{-1/3} s^{1+\varepsilon^{-1/3}} |f(s)| ds \\ &\leq \|f\|_Y \frac{\varepsilon^{-2/3} r^{-\varepsilon^{-1/3}}}{(1+\varepsilon^{-4/3} r^4 \tau^2)^{1/4}} \int_0^r w_\varepsilon(s-r_0)^{-1} s^{-1+\varepsilon^{-1/3}} \mathbb{1}_{\{s < \varepsilon^r\}}(s) ds. \end{aligned}$$

An analogous game allows us to obtain a similar bound for the second addend in (6.6.47). Here we simply use that the mapping

$$x \mapsto \frac{1}{\left(1 + \sqrt{2}(1+x)^{1/2} + x \right)^{-n}} e^{-\sqrt{2}n(1+x)^{1/2}}$$

is monotone decreasing for all $x \geq 1$ and n positive, and that $x(r) = (1 + \varepsilon^{-4/3} r^4 \tau^2)^{1/2}$ is monotone increasing for all $r > 0$. Therefore, the second addend may be bounded above by

$$\begin{aligned} \left| \varepsilon^{-1} v_2(r) \int_r^\infty s v_2(s) f(s) ds \right| &\lesssim \frac{|\tau|^{\frac{1}{2}\varepsilon^{-1/3}} \varepsilon^{-\frac{5}{6}-\frac{1}{3}} \varepsilon^{-1/3} r^{\varepsilon^{-1/3}}}{(1+\varepsilon^{-4/3} r^4 \tau^2)^{1/8}} \int_r^\infty \frac{|\tau|^{-\frac{1}{2}\varepsilon^{-1/3}} \varepsilon^{\frac{1}{6}+\frac{1}{3}} \varepsilon^{-1/3} s^{1-\varepsilon^{-1/3}}}{(1+\varepsilon^{-4/3} s^4 \tau^2)^{1/8}} |f(s)| ds \\ &\leq \|f\|_Y \frac{\varepsilon^{-2/3} r^{\varepsilon^{-1/3}}}{(1+\varepsilon^{-4/3} r^4 \tau^2)^{1/4}} \int_r^\infty w_\varepsilon(s-r_0)^{-1} s^{-1-\varepsilon^{-1/3}} \mathbb{1}_{\{s < \varepsilon^r\}}(s) ds. \end{aligned}$$

Hence, going back to the expression for the Green's function (6.6.47) we arrive at the estimate

$$\begin{aligned} |\mathcal{G}_0^1 f(r)| &\lesssim \frac{\varepsilon^{-2/3} \|f\|_Y}{(1+\varepsilon^{-4/3} r^4 \tau^2)^{1/4}} r^{-\varepsilon^{-1/3}} \int_0^r w_\varepsilon(s-r_0)^{-1} s^{-1+\varepsilon^{-1/3}} ds \\ &\quad + \frac{\varepsilon^{-2/3} \|f\|_Y}{(1+\varepsilon^{-4/3} r^4 \tau^2)^{1/4}} r^{\varepsilon^{-1/3}} \int_r^\infty w_\varepsilon(s-r_0)^{-1} s^{-1-\varepsilon^{-1/3}} ds, \end{aligned} \tag{6.6.50}$$

for any function f . Next in order, we want to recover an expression for the Green's function $\tilde{\mathcal{G}}_0$ associated to the full approximated variables-changed operator $\tilde{\mathcal{L}}_0$ defined in (6.6.45),

$$\tilde{\mathcal{L}}_0 \begin{pmatrix} V_+ \\ V_- \end{pmatrix} = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}.$$

The first component in (6.6.45) is already sorted out, i.e. $V_+ = \mathcal{G}_0^1 f_+$. For the second component we can use the structure of the operator we want to invert,

$$f_- = (\tilde{\mathcal{L}}_0)_2 V_- = (\tilde{\mathcal{L}}_0)_1 V_- - \frac{\varepsilon^{1/3}}{r^2} \sqrt{-2iMr_0^3 \Omega'_0} V_+ = (\tilde{\mathcal{L}}_0)_1 V_- - \frac{\varepsilon^{1/3}}{r^2} \sqrt{-2iMr_0^3 \Omega'_0} \mathcal{G}_0^1 f_+,$$

namely,

$$V_- = \mathcal{G}_0^1 f_- + \mathcal{G}_0^1 \left(\frac{\varepsilon^{1/3}}{r^2} \sqrt{-2iMr_0^3 \Omega'_0} \mathcal{G}_0^1 f_+ \right).$$

Therefore, collecting everything in a main expression we obtain that the total Green's function is given by

$$\begin{pmatrix} V_+ \\ V_- \end{pmatrix} = \tilde{\mathcal{G}}_0 \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} \mathcal{G}_0^1 f_+ \\ \mathcal{G}_0^1 f_- + \varepsilon^{1/3} \sqrt{-2iMr_0^3 \Omega'_0} \mathcal{G}_0^1 (r^{-2} \mathcal{G}_0^1 f_+) \end{pmatrix}. \quad (6.6.51)$$

An important observation here is that the Green's function $\mathcal{G}_0^1$ now acts additionally on the function

$$h(r) = \frac{1}{r^2} \mathcal{G}_0^1(f_-)(r),$$

which is no longer supported in the interval $[0, \varepsilon^\gamma]$. This should be carefully taken into account when addressing the corresponding estimates.

Without further ado, let us introduce the main result about the weighted L^∞ estimates on the Green's function $\tilde{\mathcal{G}}_0$, and the error produced by the approximated operator $\tilde{\mathcal{L}}_0$. Notice, that in comparison with the previous sections, we keep the first order derivative in the approximated operator. This implies that we will not need to estimate $\partial_r \tilde{\mathcal{G}}_0$ as before in order to compute the error of the approximation.

Proposition 6.6.11. *Let $\gamma > 2/9$. Define $\tilde{\mathcal{G}}_0$ as in (6.6.51) and set $\mathcal{G}_0 = P^{-1} \tilde{\mathcal{G}}_0 P$, where P is the invertible matrix defined in (6.6.41). Let $f \in Y$ with $\text{supp}(f) \subset [0, \varepsilon^\gamma]$. Under Assumptions H0–H2, the following estimates hold true uniformly for all $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$, and for any $\varepsilon > 0$ sufficiently small,*

$$\begin{aligned} |\max\{1, r^2\} w_\varepsilon(r - r_0) \mathcal{G}_0 f(r)| &\lesssim \varepsilon^{-1/3} \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) \|f\|_Y \\ &\quad + \varepsilon^{-1/3} \left(\frac{\varepsilon^\gamma}{r} \right)^{\frac{1}{4}\varepsilon^{-1/3}} \mathbb{1}_{\{r \geq 2\varepsilon^\gamma\}}(r) \|f\|_Y, \end{aligned} \quad (6.6.52)$$

$$\begin{aligned} |r^2 w_\varepsilon(r - r_0) \mathcal{G}_0 f(r)| &\lesssim \varepsilon^{-1/3+2\gamma} \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) \|f\|_Y \\ &\quad + \varepsilon^{-1/3+2\gamma} \left(\frac{\varepsilon^\gamma}{r} \right)^{\frac{1}{4}\varepsilon^{-1/3}} \mathbb{1}_{\{r \geq 2\varepsilon^\gamma\}}(r) \|f\|_Y, \end{aligned} \quad (6.6.53)$$

and the same bounds also hold true with $\mathcal{G}_0$ replaced by $\tilde{\mathcal{G}}_0$. Moreover, the error produced by the approximated operator is bounded by

$$\|((\mathcal{L} - \lambda)\mathcal{G}_0 - \text{Id})f\|_Y \lesssim \left(\varepsilon^{2/3} + \varepsilon^{2\gamma} + \varepsilon^{3\gamma-2/3}\right) \|f\|_Y. \quad (6.6.54)$$

Remark 6.6.12. Notice that from (6.6.52) we straightforwardly obtain the following estimate for the approximate Green's function in X ,

$$\|\mathcal{G}_0 f\|_X \lesssim \varepsilon^{-1/3} \|f\|_Y,$$

provided that $\text{supp}(f) \subset [0, \varepsilon^\gamma]$.

Proof. Part 1: Estimate on $\tilde{\mathcal{G}}_0$. We want to derive estimates for a Green's function that can be written as in equation (6.6.52). This implies that we need to find appropriate bounds for expressions of the form $|\mathcal{G}_0^1 f(r)|$, and $|\mathcal{G}_0^1(r^{-2}\mathcal{G}_0^1 f)(r)|$, where in both cases f is a bounded function supported in the interval $[0, \varepsilon^\gamma]$. Let us begin with the first case, since we have already presented suitable estimates at the beginning of Section 6.6.4. In particular, since $\text{supp}(f) \subset [0, \varepsilon^\gamma]$, we can use the estimate (6.6.50) to write

$$\begin{aligned} |w_\varepsilon(r-r_0)\mathcal{G}_0^1 f(r)| &\lesssim \frac{\varepsilon^{-2/3}\|f\|_Y w_\varepsilon(r-r_0)}{(1+\varepsilon^{-4/3}r^4\tau^2)^{1/4}} r^{-\varepsilon^{-1/3}} \int_0^r w_\varepsilon(s-r_0)^{-1} s^{-1+\varepsilon^{-1/3}} ds \\ &\quad + \frac{\varepsilon^{-2/3}\|f\|_Y w_\varepsilon(r-r_0)}{(1+\varepsilon^{-4/3}r^4\tau^2)^{1/4}} r^{\varepsilon^{-1/3}} \int_r^{\varepsilon^\gamma} w_\varepsilon(s-r_0)^{-1} s^{-1-\varepsilon^{-1/3}} ds. \end{aligned} \quad (6.6.55)$$

The second addend is nonzero only when $r < \varepsilon^\gamma$, therefore let us start by taking $r \in [0, \varepsilon^\gamma]$ and estimate both terms separately. Firstly, we note that $w_\varepsilon(s-r_0)^{-1}$ is increasing for $s \in [0, r_0]$, and so the first addend from (6.6.55) may be bounded by

$$w_\varepsilon(r-r_0)\varepsilon^{-2/3}\|f\|_Y r^{-\varepsilon^{-1/3}} \int_0^r s^{-1+\varepsilon^{-1/3}} w_\varepsilon(s-r_0)^{-1} ds \leq \varepsilon^{-1/3}\|f\|_Y,$$

for $r \leq \varepsilon^\gamma$. The second term in (6.6.55) will require slightly more care. Indeed, we want to bound

$$w_\varepsilon(r-r_0)r^{\varepsilon^{-1/3}} \int_r^{\varepsilon^\gamma} w_\varepsilon(s-r_0)^{-1} s^{-1-\varepsilon^{-1/3}} ds.$$

Since $w_\varepsilon(s-r_0)^{-1}$ is increasing, we estimate it by its value at the right endpoint, ending up with an upper bound of the form

$$\frac{w_\varepsilon(r-r_0)}{w(\varepsilon^\gamma-r_0)} r^{\varepsilon^{-1/3}} \int_r^{\varepsilon^\gamma} s^{-1-\varepsilon^{-1/3}} ds \leq \frac{w_\varepsilon(r-r_0)}{w(\varepsilon^\gamma-r_0)} \varepsilon^{1/3}.$$

For $r \in [0, \varepsilon^\gamma]$, we further show in Lemma 6.8.8 that the quotient

$$\frac{w_\varepsilon(r-r_0)}{w(\varepsilon^\gamma-r_0)}$$

is bounded for any ε sufficiently small by a constant that depends only on N . Therefore, we deduce

that there exists a constant $C > 0$ independent of $\varepsilon \rightarrow 0$, so that for all $r \in [0, \varepsilon^\gamma]$,

$$|w_\varepsilon(r - r_0)\mathcal{G}_0^1 f(r)| \leq C\varepsilon^{-1/3}\|f\|_Y,$$

for any f with support contained in $[0, \varepsilon^\gamma]$. Furthermore, if $r > \varepsilon^\gamma$, the second addend in (6.6.55) vanishes and we can estimate

$$\begin{aligned} |w_\varepsilon(r - r_0)\mathcal{G}_0^1 f(r)| &\leq w_\varepsilon(r - r_0)\varepsilon^{-2/3}\|f\|_Y r^{-\varepsilon^{-1/3}} \int_0^{\varepsilon^\gamma} w_\varepsilon(s - r_0)^{-1} s^{-1+\varepsilon^{-1/3}} ds \\ &\lesssim \frac{w_\varepsilon(r - r_0)}{w_\varepsilon(\varepsilon^\gamma - r_0)} \varepsilon^{-1/3}\|f\|_Y \left(\frac{\varepsilon^\gamma}{r}\right)^{\varepsilon^{-1/3}}. \end{aligned}$$

Now, thanks to Lemma 6.8.8 about the properties of the weight function, we argue that there exists a (uniform) constant C such that for all ε small and all $r \geq \varepsilon^\gamma$ there holds

$$\frac{w_\varepsilon(r - r_0)}{w_\varepsilon(\varepsilon^\gamma - r_0)} \leq \left(\frac{\varepsilon^\gamma}{r}\right)^{-\frac{1}{2}\varepsilon^{-1/3}},$$

and thus we can write

$$\frac{w_\varepsilon(r - r_0)}{w_\varepsilon(\varepsilon^\gamma - r_0)} \left(\frac{\varepsilon^\gamma}{r}\right)^{\varepsilon^{-1/3}} \leq C \left(\frac{\varepsilon^\gamma}{r}\right)^{\frac{1}{2}\varepsilon^{-1/3}}.$$

All in all, we obtain the estimate

$$|w_\varepsilon(r - r_0)\mathcal{G}_0^1 f(r)| \lesssim \varepsilon^{-1/3}\|f\|_Y \left(\mathbb{1}_{\{0 \leq r < \varepsilon^\gamma\}}(r) + \left(\frac{\varepsilon^\gamma}{r}\right)^{\frac{1}{2}\varepsilon^{-1/3}} \mathbb{1}_{\{r \geq \varepsilon^\gamma\}}(r) \right).$$

From this, we also easily obtain bounds with an extra r^2 weight in the L^∞ norm. Indeed, if $r \in [0, \varepsilon^\gamma]$, then trivially we obtain

$$|r^2 w_\varepsilon(r - r_0)\mathcal{G}_0^1 f(r)| \lesssim r^2 \varepsilon^{-1/3}\|f\|_Y \leq \varepsilon^{2\gamma-1/3}\|f\|_Y.$$

On the other hand, for $r \geq \varepsilon^\gamma$ we write $r^2 = \varepsilon^{2\gamma}(r\varepsilon^{-\gamma})^2$ so that

$$|r^2 w_\varepsilon(r - r_0)\mathcal{G}_0^1 f(r)| \lesssim \varepsilon^{-1/3}\|f\|_Y r^2 \left(\frac{\varepsilon^\gamma}{r}\right)^{\frac{1}{2}\varepsilon^{-1/3}} \lesssim \varepsilon^{-1/3+2\gamma}\|f\|_Y \left(\frac{\varepsilon^\gamma}{r}\right)^{\frac{1}{4}\varepsilon^{-1/3}}. \quad (6.6.56)$$

With this, we obtained the desired estimate for the X norm of $\mathcal{G}_0^1 f$. To find the sought estimate for $\mathcal{G}_0 f$ we also need to control terms of the form

$$\varepsilon^{1/3}\mathcal{G}_0^1 \left(\frac{1}{r^2} \mathcal{G}_0^1 f \right),$$

where $\text{supp}(f) \subset [0, \varepsilon^\gamma]$, but $\text{supp}(r^{-2}\mathcal{G}_0^1 f)$ is not contained in the interval $[0, \varepsilon^\gamma]$. In this case we split the function inside the Green's function in two contributions,

$$\frac{1}{r^2} \mathcal{G}_0^1 f = h_1(r) + h_2(r),$$

with

$$h_1(r) = \left(\frac{1}{r^2} \mathcal{G}_0^1 f \right) \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r), \quad h_2(r) = \left(\frac{1}{r^2} \mathcal{G}_0^1 f \right) \mathbb{1}_{\{r \geq 2\varepsilon^\gamma\}}(r).$$

The term corresponding to h_1 is handled identically to before, and we obtain the pointwise upper bound

$$|w_\varepsilon(r-r_0) \mathcal{G}_0^1 h_1(r)| \lesssim \varepsilon^{-1/3} \|h_1(r)\|_Y \left(\mathbb{1}_{\{r \leq 2\varepsilon^\gamma\}} + \left(\frac{2\varepsilon^\gamma}{r} \right)^{\frac{1}{2}\varepsilon^{-1/3}} \mathbb{1}_{\{r > 2\varepsilon^\gamma\}} \right).$$

Recall further that we already showed that $\|h_1(r)\|_Y \lesssim \varepsilon^{-1/3} \|f(r)\|_Y$. Therefore, we observe

$$|w_\varepsilon(r-r_0) \mathcal{G}_0^1 h_1(r)| \lesssim \varepsilon^{-2/3} \|f(r)\|_Y \left(\mathbb{1}_{\{r \leq 2\varepsilon^\gamma\}} + \left(\frac{2\varepsilon^\gamma}{r} \right)^{\frac{1}{2}\varepsilon^{-1/3}} \mathbb{1}_{\{r > 2\varepsilon^\gamma\}} \right).$$

The remaining bit $h_2(r)$ is no longer supported in a small interval around zero. On the contrary, it is supported strictly away from zero, and it is precisely this property that we seek to exploit in order to derive suitable bounds. From (6.6.50), the Green's function $\mathcal{G}_0^1$ applied to $h_2(r)$ is bounded by an expression of the form

$$\begin{aligned} & |w_\varepsilon(r-r_0) \mathcal{G}_0^1 h_2(r)| \\ & \lesssim \frac{\varepsilon^{-2/3} w_\varepsilon(r-r_0)}{(1 + \varepsilon^{-4/3} r^4 \tau^2)^{1/4}} r^{-\varepsilon^{-1/3}} \int_{2\varepsilon^\gamma}^{\max\{r, 2\varepsilon^\gamma\}} w_\varepsilon(s-r_0)^{-1} s^{-1+\varepsilon^{-1/3}} |s^2 w_\varepsilon(s-r_0) h_2(s)| ds \\ & \quad + \frac{\varepsilon^{-2/3} w_\varepsilon(r-r_0)}{(1 + \varepsilon^{-4/3} r^4 \tau^2)^{1/4}} r^{\varepsilon^{-1/3}} \int_{\max\{r, 2\varepsilon^\gamma\}}^{\infty} w_\varepsilon(s-r_0)^{-1} s^{-1-\varepsilon^{-1/3}} |s^2 w_\varepsilon(s-r_0) h_2(s)| ds. \end{aligned}$$

Recalling the bounds (6.6.56) for $|\mathcal{G}_0^1 f(s)|$ when $s \geq \varepsilon^\gamma$, we have that for $r \in [0, 2\varepsilon^\gamma]$,

$$\begin{aligned} |w_\varepsilon(r-r_0) \mathcal{G}_0^1 h_2(r)| & \lesssim \varepsilon^{-2/3} w_\varepsilon(r-r_0) r^{\varepsilon^{-1/3}} \int_{2\varepsilon^\gamma}^{\infty} w_\varepsilon(s-r_0)^{-1} s^{-1-\varepsilon^{-1/3}} |w_\varepsilon(s-r_0) \mathcal{G}_0^1 f(s)| ds \\ & \lesssim \varepsilon^{-1} \|f\|_Y w_\varepsilon(r_0) \int_{2\varepsilon^\gamma}^{\infty} s^{-1-\varepsilon^{-1/3}} \left(\frac{\varepsilon^\gamma}{s} \right)^{\frac{1}{2}\varepsilon^{-1/3}} ds \\ & \lesssim \varepsilon^{-\frac{2}{3}} \|f\|_Y w_\varepsilon(r_0) 2^{-\frac{1}{2}\varepsilon^{-1/3}}. \end{aligned}$$

In the second inequality we used both the facts that $w_\varepsilon(r-r_0) \leq w_\varepsilon(r_0)$ for $r \leq r_0$, and that $w_\varepsilon(s-r_0)^{-1} \leq 1$ for all $s \geq 0$. From the definition of the weight function (6.2.1) it automatically follows that $w_\varepsilon(r_0) \lesssim \varepsilon^{-N/3}$. We conclude that, since the prefactor $2^{-\frac{1}{2}\varepsilon^{-1/3}}$ decays geometrically fast to zero as $\varepsilon \rightarrow 0$, we can bound this term uniformly as $\varepsilon \rightarrow 0$ for any $r \leq 2\varepsilon^\gamma$ by

$$|w_\varepsilon(r-r_0) \mathcal{G}_0^1 h_2(r)| \leq \varepsilon^{-2/3} 2^{-\frac{1}{4}\varepsilon^{-1/3}} \|f\|_Y$$

If $r \geq 2\varepsilon^\gamma$ instead, we can compute the estimate

$$\begin{aligned} |w_\varepsilon(r-r_0) \mathcal{G}_0^1 h_2(r)| & \lesssim \varepsilon^{-2/3} w_\varepsilon(r-r_0) r^{-\varepsilon^{-1/3}} \int_{2\varepsilon^\gamma}^r s^{-1+\varepsilon^{-1/3}} w_\varepsilon(s-r_0)^{-1} |w_\varepsilon(s-r_0) \mathcal{G}_0^1 f(s)| ds \\ & \quad + \varepsilon^{-2/3} w_\varepsilon(r-r_0) r^{\varepsilon^{-1/3}} \int_r^{\infty} s^{-1-\varepsilon^{-1/3}} w_\varepsilon(s-r_0)^{-1} |w_\varepsilon(s-r_0) \mathcal{G}_0^1 f(s)| ds, \end{aligned}$$

thus, using the previous estimates for $\mathcal{G}_0^1 f$ we obtain

$$\begin{aligned} |w_\varepsilon(r-r_0)\mathcal{G}_0^1 h_2(r)| &\lesssim \varepsilon^{-1} \|f\|_Y w_\varepsilon(r-r_0) r^{-\varepsilon^{-1/3}} \int_{2\varepsilon^\gamma}^r s^{-1+\varepsilon^{-1/3}} \left(\frac{\varepsilon^\gamma}{s}\right)^{\frac{1}{2}\varepsilon^{-1/3}} ds \\ &\quad + \varepsilon^{-1} \|f\|_Y w_\varepsilon(r-r_0) r^{\varepsilon^{-1/3}} \int_r^\infty s^{-1-\varepsilon^{-1/3}} \left(\frac{\varepsilon^\gamma}{s}\right)^{\frac{1}{2}\varepsilon^{-1/3}} ds \\ &\lesssim \varepsilon^{-2/3} \|f\|_Y w_\varepsilon(r-r_0) \left(\frac{\varepsilon^\gamma}{r}\right)^{\frac{1}{2}\varepsilon^{-1/3}}. \end{aligned}$$

But observe from Lemma 6.8.8 about the properties of the weight function, since $w_\varepsilon(\varepsilon^\gamma - r_0) \geq 1$, we can readily deduce that for any $r \geq 2\varepsilon^\gamma$ there holds

$$w_\varepsilon(r-r_0) \leq \left(\frac{\varepsilon^\gamma}{r}\right)^{-\frac{1}{4}\varepsilon^{-1/3}}.$$

Therefore, we deduce that

$$|w_\varepsilon(r-r_0)\mathcal{G}_0^1 h_2(r)| \lesssim \varepsilon^{-2/3} \left(\frac{\varepsilon^\gamma}{r}\right)^{\frac{1}{4}\varepsilon^{-1/3}} \|f\|_Y$$

for any $r \geq 2\varepsilon^\gamma$. All in all, we obtain a final estimate for the Green's function $\mathcal{G}_0^1$ acting on $r^{-2}\mathcal{G}_0^1 f$ of the form

$$\begin{aligned} \left| w_\varepsilon(r-r_0)\mathcal{G}_0^1 \left(\frac{1}{r^2} \mathcal{G}_0^1 f \right) \right| &\lesssim \varepsilon^{-2/3} \|f\|_Y \left(1 + 2^{-\frac{1}{4}\varepsilon^{-1/3}} \right) \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) \\ &\quad + \varepsilon^{-2/3} \|f\|_Y \left(\frac{\varepsilon^\gamma}{r} \right)^{\frac{1}{4}\varepsilon^{-1/3}} \mathbb{1}_{\{r \geq 2\varepsilon^\gamma\}}(r), \end{aligned}$$

which implies in particular

$$\begin{aligned} \left| w_\varepsilon(r-r_0)\mathcal{G}_0^1 \left(\frac{1}{r^2} \mathcal{G}_0^1 f \right) \right| &\lesssim \varepsilon^{-2/3} \|f\|_Y \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) \\ &\quad + \varepsilon^{-2/3} \|f\|_Y \left(\frac{\varepsilon^\gamma}{r} \right)^{\frac{1}{4}\varepsilon^{-1/3}} \mathbb{1}_{\{r \geq 2\varepsilon^\gamma\}}(r). \end{aligned}$$

Finally, in order to obtain the claim for the X norm, we need to obtain an analogous estimate with an additional r^2 weight. This is thus straightforwardly deduced from the previous estimate,

$$\begin{aligned} \left| r^2 w_\varepsilon(r-r_0)\mathcal{G}_0^1 \left(\frac{1}{r^2} \mathcal{G}_0^1 f \right) \right| &\lesssim \varepsilon^{2\gamma-2/3} \|f\|_Y \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) \\ &\quad + \varepsilon^{2\gamma-2/3} \|f\|_Y \left(\frac{\varepsilon^\gamma}{r} \right)^{\frac{1}{4}\varepsilon^{-1/3}} \mathbb{1}_{\{r \geq 2\varepsilon^\gamma\}}(r). \end{aligned}$$

Now we can go back to the explicit expression for the Green's function $\tilde{\mathcal{G}}_0$ in (6.6.51) and bring together all the estimates we derived thus far. Something to bear in mind is that the first estimate derived for $\mathcal{G}_0^1 f$ is split into the intervals $[0, \varepsilon^\gamma)$ and $[\varepsilon^\gamma, \infty)$. However a quick inspection reveals that this estimate is also true if split into the intervals $[0, 2\varepsilon^\gamma)$ and $[2\varepsilon^\gamma, \infty)$. We analyse the second

component of (6.6.51), since it is ever so slightly more complicated and the estimate for the first component follows analogously.

$$\begin{aligned} |w_\varepsilon(r-r_0)\tilde{\mathcal{G}}_0 f_\pm| &\lesssim |w_\varepsilon(r-r_0)\mathcal{G}_0^1 f_-| + \varepsilon^{1/3} \left| w_\varepsilon(r-r_0) \left(\mathcal{G}_0^1 \left(\frac{1}{r^2} \mathcal{G}_0^1(f_+) \right) \right) \right| \\ &\lesssim \left(\varepsilon^{-1/3} \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) + \varepsilon^{-1/3} \left(\frac{\varepsilon^\gamma}{r} \right)^{\frac{1}{4}\varepsilon^{-1/3}} \mathbb{1}_{\{r \geq 2\varepsilon^\gamma\}}(r) \right) \|f_\pm\|_Y. \end{aligned}$$

Here we have used the notation $f_\pm = (f_+, f_-)$. Moreover, if we put together the pointwise estimates with the r^2 weight, we can write

$$\begin{aligned} |r^2 w_\varepsilon(r-r_0)\tilde{\mathcal{G}}_0 f_\pm| &\lesssim |w_\varepsilon(r-r_0)\mathcal{G}_0^1 f_-| + \varepsilon^{1/3} \left| w_\varepsilon(r-r_0) \left(\mathcal{G}_0^1 \left(\frac{1}{r^2} \mathcal{G}_0^1(f_+) \right) \right) \right| \\ &\lesssim \left(\varepsilon^{2\gamma-1/3} \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) + \varepsilon^{2\gamma-1/3} \left(\frac{\varepsilon^\gamma}{r} \right)^{\frac{1}{4}\varepsilon^{-1/3}} \mathbb{1}_{\{r \geq 2\varepsilon^\gamma\}}(r) \right) \|f_\pm\|_Y. \end{aligned}$$

We obtained in fact (6.6.53) as claimed in the lemma, and combining the last two estimates we arrive as well at (6.6.52).

Part 2: The error estimate. The next claim to be addressed corresponds to the error estimate (6.6.54). We need to compare the exact operator with the approximated operator that we constructed for functions f supported in a neighbourhood of $r = 0$. The goal is to find an estimate for

$$((\tilde{\mathcal{L}} - \lambda) - \tilde{\mathcal{L}}_0) \tilde{\mathcal{G}}_0 f_\pm.$$

Indeed, if we find a suitable estimate on this difference, by (6.6.44) we conclude (6.6.54). Combining (6.6.42) with (6.6.45) and writing $\lambda = \mu\varepsilon^{1/3}$, we find that

$$\begin{aligned} (\tilde{\mathcal{L}} - \lambda) \begin{pmatrix} V_+ \\ V_- \end{pmatrix} - \tilde{\mathcal{L}}_0 \begin{pmatrix} V_+ \\ V_- \end{pmatrix} &= - \left(\frac{\varepsilon}{r^2} + \varepsilon^{1/3} M^2 \left(\frac{\Omega'_0}{U'_0} \right)^2 + \mu\varepsilon^{1/3} \right) \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \\ &\quad - i\varepsilon^{-1/3} (T(r) - \tau) \begin{pmatrix} V_+ \\ V_- \end{pmatrix} - \alpha\varepsilon^{1/3} \left(r\Omega' + \frac{2iM\varepsilon^{2/3}}{r^2} \right) \begin{pmatrix} V_- \\ 0 \end{pmatrix}. \end{aligned}$$

Introducing weights corresponding to the Y norm, we observe that it suffices to estimate

$$\begin{aligned} |r^2 w_\varepsilon(r-r_0) ((\tilde{\mathcal{L}} - \lambda) - \tilde{\mathcal{L}}_0) \tilde{\mathcal{G}}_0 f_\pm| &\lesssim \left| r^2 w_\varepsilon(r-r_0) \varepsilon^{-1/3} (T(r) - \tau) \tilde{\mathcal{G}}_0 f_\pm \right| \\ &\lesssim \left| r^2 w_\varepsilon(r-r_0) \left(\frac{\varepsilon}{r^2} + \varepsilon^{1/3} M^2 \left(\frac{\Omega'_0}{U'_0} \right)^2 + \mu\varepsilon^{1/3} \right) \tilde{\mathcal{G}}_0 f_\pm \right| \\ &\quad + \left| w_\varepsilon(r-r_0) \left(\alpha\varepsilon^{1/3} (r^3 \Omega'(r) + 2iM\varepsilon^{2/3}) \right) \tilde{\mathcal{G}}_0 f_\pm \right|. \end{aligned} \tag{6.6.57}$$

Using (6.6.52), (6.6.53) the first second from (6.6.57) may be controlled by

$$\varepsilon |w_\varepsilon(r-r_0) \tilde{\mathcal{G}}_0 f_\pm| + \varepsilon^{1/3} \left| M^2 \left(\frac{\Omega'_0}{U'_0} \right)^2 + \mu \right| |r^2 w_\varepsilon(r-r_0) \tilde{\mathcal{G}}_0 f_\pm| \lesssim (\varepsilon^{2/3} + \varepsilon^{2\gamma}) \|f_\pm\|_Y.$$

To find the corresponding bound for the first addend in (6.6.57), we first note that by a Taylor

expansion near zero, there holds $|T(r) - \tau| \lesssim r \mathbb{1}_{\{0 \leq r \leq 2\varepsilon^\gamma\}} + \mathcal{O}_{\varepsilon \rightarrow 0}(\varepsilon^{2\gamma})$. On the other hand, far away from the origin we use Hypothesis H2, which in turn produces the estimate

$$|T(r) - \tau| \lesssim r \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) + \left(1 + \sum_{n=1}^{N_1} B_n(\varepsilon) r^n\right) \mathbb{1}_{\{r \geq 2\varepsilon^\gamma\}}(r)$$

Putting this together with (6.6.53) we find

$$\begin{aligned} \varepsilon^{-1/3} |T(r) - \tau| \left| r^2 w_\varepsilon(r - r_0) \tilde{\mathcal{G}}_0 f_\pm \right| &\lesssim \varepsilon^{2\gamma-2/3} r \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) \|f_\pm\|_Y \\ &\quad + \varepsilon^{2\gamma-2/3} \left(1 + \sum_{n=1}^{N_1} B_n(\varepsilon) r^n\right) \left(\frac{\varepsilon^\gamma}{r}\right)^{\frac{1}{4}\varepsilon^{-1/3}} \mathbb{1}_{\{r \geq 2\varepsilon^\gamma\}}(r) \|f_\pm\|_Y \\ &\lesssim \varepsilon^{3\gamma-2/3} \|f_\pm\|_Y, \end{aligned}$$

where we used the estimate from Property P3

$$\sum_{n=1}^{N_1} B_n(\varepsilon) \lesssim \varepsilon^{\beta - N_1 \delta}.$$

Hence, we obtain

$$\varepsilon^{-1/3} |T(r) - \tau| \left| r^2 w_\varepsilon(r - r_0) \tilde{\mathcal{G}}_0 f_\pm \right| \lesssim \varepsilon^{3\gamma-2/3} \|f_\pm\|_Y.$$

The last term to be addressed can be split in two. On the one hand, the term without the Ω' can be directly bounded by

$$\varepsilon^{1/3} \left| 2iM\varepsilon^{2/3} \right| \left| w_\varepsilon(r - r_0) \tilde{\mathcal{G}}_0 f_\pm \right| \lesssim \varepsilon \left| w_\varepsilon(r - r_0) \tilde{\mathcal{G}}_0 f_\pm \right|,$$

which together with (6.6.52) yields

$$\varepsilon \left| w_\varepsilon(r - r_0) \tilde{\mathcal{G}}_0 f_\pm \right| \lesssim \varepsilon^{2/3} \|f_\pm\|_Y.$$

On the other hand, the term with the azimuthal component of the vector field can be estimated by using Property P4 far away from the origin, and the mean value theorem near the origin. Namely, there exist $N_2 \in \mathbb{N}$ such that

$$|r\Omega'(r)| \lesssim r \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) + C \left(1 + \sum_{n=1}^{N_2} D_n r^n\right) \mathbb{1}_{\{2\varepsilon^\gamma \leq r\}}(r).$$

Hence, putting this together with (6.6.53) we can write the estimate

$$\begin{aligned} \varepsilon^{1/3} |r\Omega'| \left| r^2 w_\varepsilon(r - r_0) \tilde{\mathcal{G}}_0 f_\pm \right| &\lesssim r \varepsilon^{2\gamma} \mathbb{1}_{\{0 \leq r < 2\varepsilon^\gamma\}}(r) \|f_\pm\|_Y \\ &\quad + \varepsilon^{-1/3} \left(1 + \sum_{n=1}^{N_2} D_n r^n\right) \left(\frac{\varepsilon^\gamma}{r}\right)^{\frac{1}{4}\varepsilon^{-1/3}} \mathbb{1}_{\{2\varepsilon^\gamma \leq r\}}(r) \|f_\pm\|_Y \\ &\lesssim \left(\varepsilon^{3\gamma} + \varepsilon^{-1/3 - N_2 \delta} \left(\frac{1}{2}\right)^{\frac{1}{8}\varepsilon^{-1/3}} \right) \|f_\pm\|_Y. \end{aligned}$$

As in the previous scenario, the term that is exponentially decaying becomes negligible in comparison with the other term. All in all we find that

$$r^2 w_\varepsilon(r - r_0) |(\tilde{\mathcal{L}} - \lambda) \tilde{\mathcal{G}}_0 f_\pm - f_\pm| \lesssim \left(\varepsilon^{2/3} + \varepsilon^{2\gamma} + \varepsilon^{3\gamma-2/3} \right) \|f_\pm\|_Y,$$

and so we conclude the proof of the Proposition by (6.6.44). $\square$

6.6.5 Transport functions with linear zeroes.

Thus far we have assumed that $T(r) \neq 0$ for all $r \neq r_0$, however Assumption SH is not strictly necessary. In fact, we shall show that if there exists a (finite) collection of points $s_j > 0$ with $T(s_j) = 0$, and $T'(s_j) \neq 0$, we can still construct a local inverse to our operator in a neighbourhood of s_j , which can be glued together with the remaining inverses as before. This yields the only bit remaining in order to obtain the claim of Proposition 6.5.2

As mentioned at the beginning of Section 6.6, the form of the Green's operator $\mathcal{G}^\lambda$ depends on the specific form of the transport function $T(r)$. We want to adapt the proof of the previous sections to the general case allowing for linear zeros, and we will do so by dividing the support of f in different regions as earlier, but taking into account the new zeros. In particular, around the origin $\text{supp}(f) \subset [0, \varepsilon^\gamma]$, and around the critical radius $\text{supp}(f) \subset [r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]$, we define $\mathcal{G}_0$ and $\mathcal{G}_2^\lambda$ just like in Sections 6.6.4 and 6.6.1 respectively, whereas around every linear zero s_j we introduce new operators $\mathcal{G}_j^{\text{LV}}$ defined for $\text{supp}(f) \subset [s_j - \varepsilon^\omega, s_j + \varepsilon^\omega]$. The regions between the origin and the first zero of T , as well as between any two zeros, will be covered by $\mathcal{G}_1$. The region from the largest zero $\max\{r_0, s_j\}$ to infinity will be covered by $\mathcal{G}_3$ as in the previous section. All in all, we can write the Green's function as in Section 6.5

$$\mathcal{G}^\lambda f = \mathcal{G}_2^\lambda (f \mathbb{1}_{[r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]}) + \mathcal{G}_{\text{extra}} (f \mathbb{1}_{\mathbb{R}_+ \setminus [r_0 - \varepsilon^\gamma, r_0 + \varepsilon^\gamma]}),$$

where crucially all Green's operators included in $\mathcal{G}_{\text{extra}}$ are independent of the choice of λ .

Notice that we can glue together all these Green's function due to the exponential decay exhibited for all $r \notin \text{supp}(f)$ on each corresponding case—see Remark 6.6.2 for further details. Therefore, to prove Proposition 6.5.2 it only remains to focus on the local problem around one of the linear zeros s_j , since every other region follows from the analysis in the previous sections.

Suppose that $s_j > 0$ is a linear zero of $T(r)$, and set $MT'(s_j) = \tau_j$, which without loss of generality we take $\tau_j > 0$. In this section we will assume that $\text{supp}(f) \subset [s_j - \varepsilon^\omega, s_j + \varepsilon^\omega]$, for some $\omega > 0$ to be specified later, and we define the following operator

$$\mathcal{L}_j^{\text{LV}} = \varepsilon \partial_r^2 - i \varepsilon^{-1/3} \tau_j (r - s_j). \quad (6.6.58)$$

that we claim it approximates the true $\mathcal{L}$ sufficiently well if f is supported around s_j . We now set $v(r)$ as a solution of the differential equation

$$\partial_r^2 v - r v = 0,$$

and let $w(r) = v(\gamma(r - s_j))$. Then,

$$\mathcal{L}_j^{\text{LV}} w(r) = \varepsilon \gamma^2 v''(\gamma(r - s_j)) - i \varepsilon^{-1/3} \gamma^{-1} \tau_j (\gamma(r - s_j) v(\gamma(r - s_j))).$$

In particular, as soon as $i \varepsilon^{-4/3} \gamma^{-3} \tau_j = 1$, then $\mathcal{L}_j^{\text{LV}} w = 0$. Therefore, we pick $\gamma = i^{1/3} \tau_j^{1/3} \varepsilon^{-4/9}$, where we also make the choice of the branch in the cube root so that $\arg(i^{1/3}) = \pi/6$. Now, the solutions of

$$\partial_r^2 v - r v = 0$$

are the so called *Airy functions*. In particular, as further elaborated in Appendix 6.8.1, we have two linearly independent solutions given by

$$v_1(r) = \text{Ai}(r), \quad v_2(r) = \text{Ai}(e^{2\pi i/3} r).$$

Furthermore, subtracting the equations

$$v_2 v_1'' - r v_1 v_2 = 0$$

$$v_1 v_2'' - r v_2 v_1 = 0$$

we see that $\frac{d}{dr} W(v_1, v_2)(r) = 0$, where $W(v_1, v_2) = v_1' v_2 - v_2' v_1$ is the Wronskian. Hence it is clear that the Wronskian is given by $W(\text{Ai}(r), \text{Ai}(e^{2\pi i/3} r)) = \text{constant}$. Furthermore, from Lemma 6.8.6 in Appendix 6.8.1, we have the following asymptotic bounds valid uniformly in z with $|\arg(z)| = \frac{\pi}{6}$,

$$\begin{aligned} |\text{Ai}(z)|, |\text{Ai}(-e^{2\pi i/3} z)| &\leq C e^{-\frac{\sqrt{2}}{3}|z|^{3/2}}, \\ |\text{Ai}(-z)|, |\text{Ai}(e^{2\pi i/3} z)| &\leq C e^{\frac{\sqrt{2}}{3}|z|^{3/2}}, \\ |\text{Ai}'(z)|, |\text{Ai}'(-e^{2\pi i/3} z)| &\leq C(1 + |z|^{\frac{1}{4}}) e^{-\frac{\sqrt{2}}{3}|z|^{3/2}}, \\ |\text{Ai}'(-z)|, |\text{Ai}'(e^{2\pi i/3} z)| &\leq C(1 + |z|^{\frac{1}{4}}) e^{\frac{\sqrt{2}}{3}|z|^{3/2}}, \end{aligned}$$

where we write $\text{Ai}'(z)$ to denote the (complex) derivative of $\text{Ai}(z)$ with respect to z . Therefore, under the boundary condition that our solutions should decay as $r \rightarrow \pm\infty$, we obtain the Green's function

$$\mathcal{G}_j^{\text{LV}} f(r) = C \varepsilon^{-5/9} \left[a_1(r) \int_{s_j - \varepsilon^\omega}^r a_2(s) f(s) ds + a_2(r) \int_r^{s_j + \varepsilon^\omega} a_1(s) f(s) ds \right],$$

where

$$a_1(r) = \text{Ai}\left(i^{1/3} \tau_j^{1/3} \varepsilon^{-4/9} (r - s_j)\right), \quad a_2(r) = \text{Ai}\left(e^{2\pi i/3} i^{1/3} \tau_j^{1/3} \varepsilon^{-4/9} (r - s_j)\right).$$

Recall that here we assume that $\text{supp}(f) \subset [s_j - \varepsilon^\omega, s_j + \varepsilon^\omega]$, for some $\omega > 0$ to be specified later.

We now shall state the main result of the section.

Lemma 6.6.13. *Suppose there exists a point $s_j > 0$ so that $T(s_j) = 0$, but $T'(r_j) \neq 0$, and let $\omega > \frac{2}{9}$. Then, there exists an operator $\mathcal{G}_j^{\text{LV}}$ so that for all f with $\text{supp}(f) \subset [r_j - \varepsilon^\omega, r_j + \varepsilon^\omega]$ there*

holds

$$\|\mathcal{G}_j^{\text{LV}} f\|_X \lesssim \varepsilon^{-1/9} \|f\|_Y, \quad \|\partial_r \mathcal{G}_j^{\text{LV}} f\|_X \lesssim \varepsilon^{-5/9} \|f\|_Y. \quad (6.6.59)$$

Furthermore, we have the following error estimate

$$\|((\mathcal{L} - \lambda)\mathcal{G}_j^{\text{LV}} - \text{Id})f\|_Y \lesssim \left(\varepsilon^{2/9} + \varepsilon^{2\omega-4/9}\right) \|f\|_Y. \quad (6.6.60)$$

Remark 6.6.14. An interesting consequence of our analysis of the Airy functions is its connection to Gilbert's condition $M\Omega'(r_0) + KU'(r_0) = 0$ from Section 6.3.2, see (6.3.4). Indeed, if we assume $M\Omega'(r_0) + KU'(r_0) \neq 0$, then by an order of magnitude analysis, the dominant contribution of the equation near r_0 would be given by $\varepsilon \partial_r^2 - i(M\Omega'(r_0) + KU'(r_0))\varepsilon^{-1/3}(r - r_0)$, just as in (6.6.58). Hence, to first order, any growing mode must look like a

$$v(i^{1/3}\tau_j^{1/3}\varepsilon^{-4/9}(r - r_0))$$

near r_0 , where v is a solution to Airy's equation. But from Lemma 6.8.6, we see that any such solution will grow exponentially either for $r > r_0$ or $r < r_0$. Therefore, we do not expect such an approximate to be a good candidate for a "boundary layer" solution, i.e. a solution for which the growth occurs in a neighbouring area of a critical radius r_0 .

Proof. We proceed following the ideas of Section 6.6.1. Set

$$\kappa = \frac{\sqrt{2}}{3}\tau_j^{1/3},$$

then we find the estimate for the Green's function

$$|\mathcal{G}_j^{\text{LV}} f(r)| \lesssim C \|f\|_Y \varepsilon^{-5/9} (\text{I}_\varepsilon + \text{I}'_\varepsilon + \text{II}_\varepsilon + \text{III}_\varepsilon + \text{IV}_\varepsilon + \text{IV}'_\varepsilon),$$

where

$$\begin{aligned} \text{I}_\varepsilon &= \mathbb{1}_{\{s_j - \varepsilon^\omega \leq r < s_j\}} e^{\kappa \varepsilon^{-2/3}(s_j - r)^{3/2}} \int_{s_j - \varepsilon^\omega}^r \frac{e^{-\kappa \varepsilon^{-2/3}|s_j - s|^{3/2}}}{s^2 w_\varepsilon(s - r_0)} ds, \\ \text{I}'_\varepsilon &= \mathbb{1}_{\{s_j \leq r\}} e^{-\kappa \varepsilon^{-2/3}(r - s_j)^{3/2}} \int_{s_j - \varepsilon^\omega}^{s_j} \frac{e^{-\kappa \varepsilon^{-2/3}(s_j - s)^{3/2}}}{s^2 w_\varepsilon(s - r_0)} ds, \\ \text{II}_\varepsilon &= e^{-\kappa \varepsilon^{-2/3}(r - s_j)^{3/2}} \int_{\min\{r, s_j\}}^{\min\{r, s_j + \varepsilon^\omega\}} \frac{e^{\kappa \varepsilon^{-2/3}(s - s_j)^{3/2}}}{s^2 w_\varepsilon(s - r_0)} ds, \\ \text{III}_\varepsilon &= e^{-\kappa \varepsilon^{-2/3}(s_j - r)^{3/2}} \int_{\max\{r, s_j - \varepsilon^\omega\}}^{\max\{r, s_j\}} \frac{e^{\kappa \varepsilon^{-2/3}(s_j - s)^{3/2}}}{s^2 w_\varepsilon(s - r_0)} ds, \\ \text{IV}_\varepsilon &= \mathbb{1}_{\{s_j \leq r < s_j + \varepsilon^\omega\}} e^{\kappa \varepsilon^{-2/3}(r - s_j)^{3/2}} \int_r^{s_j + \varepsilon^\omega} \frac{e^{-\kappa \varepsilon^{-2/3}|s_j - s|^{3/2}}}{s^2 w_\varepsilon(s - r_0)} ds, \\ \text{IV}'_\varepsilon &= \mathbb{1}_{\{r < s_j\}} e^{-\kappa \varepsilon^{-2/3}(s_j - r)^{3/2}} \int_{s_j}^{s_j + \varepsilon^\omega} \frac{e^{-\kappa \varepsilon^{-2/3}(s - s_j)^{3/2}}}{s^2 w_\varepsilon(s - r_0)} ds. \end{aligned}$$

It thus only remains to estimate these terms. Since the arguments are very similar to those we

have seen before in Section 6.6.1, we shall restrict ourselves to showing the required bounds for $r \in \text{supp}(f)$. For $r \notin \text{supp}(f)$, the Green's function will decay exponentially as in Section 6.6.1, and thus the errors are negligible. Without loss of generality we also assume that $s_j > r_0$, since the case $s_j < r_0$ follows in a similar manner.

We start by considering $r \in [s_j - \varepsilon^\omega, s_j]$. In this case, $I'_\varepsilon = 0$, $\Pi_\varepsilon = 0$ and $\text{IV}_\varepsilon = 0$. For the first term we find, after introducing the weight corresponding to the X norm,

$$w_\varepsilon(r - r_0) \text{I}_\varepsilon = w_\varepsilon(r - r_0) e^{\kappa \varepsilon^{-2/3} (s_j - r)^{3/2}} \int_{s_j - \varepsilon^\omega}^r \frac{e^{-\kappa \varepsilon^{-2/3} |s_j - s|^{3/2}}}{s^2 w_\varepsilon(s - r_0)} ds.$$

However s^{-2} is a decreasing function bounded by its value in the left extreme of the integral, and moreover, as we show in Lemma 6.8.8, there exists a constant C only depending on N such that

$$\frac{w_\varepsilon(r - r_0)}{w_\varepsilon(s - r_0)} \leq C$$

for any $s_j - \varepsilon^\omega \leq s \leq r < s_j$. Notice that C is independent of ε provided that ε is small enough. Up to the change of variables $v = \varepsilon^{-4/9} (s_j - s)$, $u = \varepsilon^{-4/9} (s_j - r)$, we find the estimate

$$w_\varepsilon(r - r_0) \text{I}_\varepsilon \lesssim \varepsilon^{4/9} e^{\kappa u^{3/2}} \int_u^{\varepsilon^{\omega-4/9}} e^{-\kappa v^{3/2}} dv \lesssim \varepsilon^{4/9} e^{\kappa u^{3/2}} \int_u^\infty e^{-\kappa v^{3/2}} dv.$$

Furthermore, arguing as in Section 6.6.1, an application of L'Hôpital's rule shows that the function

$$u \mapsto e^{\kappa u^{3/2}} \int_u^\infty e^{-\kappa z^{3/2}} dz \quad (6.6.61)$$

is uniformly bounded in u , and decays like $u^{-1/2}$ as $u \rightarrow \infty$. Therefore, we observe that the first term is bounded by

$$\|f\|_Y \varepsilon^{-5/9} w_\varepsilon(r - r_0) \text{I}_\varepsilon \lesssim \varepsilon^{-1/9} \|f\|_Y,$$

and differentiating the expression in κ , we observe uniformity of the estimate in compact intervals of κ as in Section 6.6.1. For the third term we argue similarly,

$$\begin{aligned} w_\varepsilon(r - r_0) \text{III}_\varepsilon &\leq w_\varepsilon(r - r_0) e^{-\kappa \varepsilon^{-2/3} (s_j - r)^{3/2}} \int_r^{s_j} \frac{e^{\kappa \varepsilon^{-2/3} (s_j - s)^{3/2}}}{s^2 w_\varepsilon(s - r_0)} ds \\ &= \varepsilon^{4/9} e^{-\kappa u^{3/2}} \int_0^u e^{\kappa v^{3/2}} dv, \end{aligned}$$

under the change of variables $v = \varepsilon^{-4/9} (s_j - s)$, $u = \varepsilon^{-4/9} (s_j - r)$. This time, via L'Hôpital we find as well that

$$u \mapsto e^{-\kappa u^{3/2}} \int_0^u e^{\kappa z^{3/2}} dz \quad (6.6.62)$$

is also uniformly bounded in u , and moreover, it decays like $u^{-1/2}$ as $u \rightarrow \infty$. Hence

$$\|f\|_Y \varepsilon^{-5/9} w_\varepsilon(r - r_0) \text{III}_\varepsilon \lesssim \varepsilon^{-1/9} \|f\|_Y.$$

Again, uniformity in compact intervals of κ follows as in Section 6.6.1. Lastly, the fourth term is

easily boundable since both exponentials are decreasing towards infinity,

$$\begin{aligned} w_\varepsilon(r-r_0) \text{IV}'_\varepsilon &\leq w_\varepsilon(r-r_0) e^{-\kappa \varepsilon^{-2/3}(s_j-r)^{3/2}} \int_{s_j}^{s_j+\varepsilon^\omega} \frac{e^{-\kappa \varepsilon^{-2/3}(s-s_j)^{3/2}}}{s^2 w_\varepsilon(s-r_0)} ds \\ &\lesssim \varepsilon^{4/9} e^{-\kappa u^{3/2}} \int_0^{\varepsilon^{\omega-4/9}} e^{-\kappa v^{3/2}} dv \lesssim \varepsilon^{4/9}. \end{aligned}$$

All in all, we obtain the estimate

$$\|f\|_Y \varepsilon^{-5/9} w_\varepsilon(r-r_0) \text{IV}'_\varepsilon \lesssim \varepsilon^{-1/9} \|f\|_Y e^{-\varepsilon^{-2/3}(s_j-r)^{3/2}},$$

which quickly converges to zero as $\varepsilon \rightarrow 0$ if $|r-s_j| \gtrsim O_{\varepsilon \rightarrow 0}(1)$ due to the effect of the exponential, but is only bounded by $\varepsilon^{-1/9}$ for the limit case $r = s_j$. The argument exposed for this region carries over to the domain $r \in [s_j, s_j + \varepsilon^\omega]$ by symmetry. Moreover, as mentioned before, if $r \notin \text{supp}(f)$, following the ideas from Section 6.6.1 we find that the Green's function decays exponentially, hence we obtain the first estimate in (6.6.59).

Next we investigate the derivative of the Green's function. We recall the asymptotic bounds for the derivative of the Airy function from Appendix 6.8.1, i.e.

$$\text{Ai}'(z) \sim z^{1/4} e^{-cz^{3/2}}.$$

Observing that the decay of the functions in (6.6.61) and (6.6.62) is like $u^{-1/2}$ as $u \rightarrow \infty$, it follows that upon multiplying the function by $u^{1/4}$, it will still remain bounded uniformly in u , and therefore exactly the same arguments as before carry over and yield pointwise bounds for the derivative of the Green's function as stated in (6.6.59).

It remains to comment on the error estimates (6.6.60). A direct computation reveals that the error may be bounded by

$$\begin{aligned} \|((\mathcal{L} - \lambda) \mathcal{G}_j^{\text{LV}} - \text{Id})f\|_Y &\lesssim \|\varepsilon \partial_r \mathcal{G}_j^{\text{LV}} f\|_Y + \|\varepsilon^{1/3} \mathcal{G}_j^{\text{LV}} f\|_Y \\ &\quad + \|r^{-2} \varepsilon^{1/3} \mathcal{G}_j^{\text{LV}} f\|_Y + \||r-s_j|^2 \varepsilon^{-1/3} \mathcal{G}_j^{\text{LV}} f\|_Y, \end{aligned}$$

where the $|r-s_j|^2$ contribution is *uniform* in $r_0 \in \mathcal{R}$ due to Lemma 6.3.4, P1. Due to the bounds on $\mathcal{G}_j^{\text{LV}} f$, the first three error terms are certainly controlled by

$$\|\varepsilon \partial_r \mathcal{G}_j^{\text{LV}} f\|_Y \lesssim \varepsilon^{4/9} \|f\|_Y,$$

$$\|\varepsilon^{1/3} \mathcal{G}_j^{\text{LV}} f\|_Y \leq \varepsilon^{1/3} \|\mathcal{G}_j^{\text{LV}} f\|_X \lesssim \varepsilon^{2/9} \|f\|_Y,$$

$$\|r^{-2} \varepsilon^{1/3} \mathcal{G}_j^{\text{LV}} f\|_Y \leq \varepsilon^{1/3} \|\mathcal{G}_j^{\text{LV}} f\|_X \lesssim \varepsilon^{2/9} \|f\|_Y.$$

Finally, by the L^∞ bounds on $\mathcal{G}_j^{\text{LV}} f$, we observe for the final term that we can write

$$\left| |r-s_j|^2 \varepsilon^{-1/3} r^2 w_\varepsilon(r-r_0) \mathcal{G}_j^{\text{LV}} f \right| \lesssim \varepsilon^{2\omega} \varepsilon^{-4/9} \|f\|_Y$$

if $|r-s_j| < \varepsilon^\omega$, and due to the exponential decay of the Green's function away from $|r-s_j| < \varepsilon^\omega$,

this bound can be extended to all of $[0, \infty)$, which yields the claimed result (6.6.60). $\square$

Proof of Proposition 6.5.2. With this result in hand, we can now readily conclude the proof of Proposition 6.5.2. Indeed, by Assumptions H0–H2, we can cover any interval without zeroes of T that is at least a distance ε^γ away from $\mathcal{F}_0 \cup \{r_0\}$ by linear interpolation steps as in Lemma 6.6.6. Thus, it remains to note that if $|r - s_j| < \varepsilon^\gamma$, $s_j \in \mathcal{F}_0$, then as long as $\varepsilon^\gamma < \varepsilon^\omega$, this regime is covered by Lemma 6.6.13. But recall that $\gamma > 2/9$, and in Lemma 6.6.13, ω can be chosen arbitrarily in $(2/9, \infty)$. Therefore, simply picking $2/9 < \omega < \gamma$, the claim of Proposition 6.5.2 follows. $\square$

6.7 Extension to domains with boundaries

In this section, we extent our analysis to cover the case where the dynamo equations are posed on cylindrical domains with boundary. Given some interval $\mathcal{J} \subset [0, \infty)$ of the form $[0, q]$, $[p, q]$, or $[p, \infty)$, with $0 < p < q < \infty$, we consider domains of the form

$$\mathcal{M} \times \mathbb{T} = \mathcal{J} \times \mathbb{T} \times \mathbb{T} \subseteq \mathbb{R}^3,$$

i.e. (periodic) cylinders, annular cylinders, or the exterior of cylinders respectively. We hence attempt to solve $(\mathcal{L}^{\text{dyn}} - \lambda)B = 0$ with suitable and physically relevant boundary conditions.

Note that by construction the velocity field u will always be tangential to the boundary, namely $u \cdot \hat{n} = 0$ on $\partial(\mathcal{M} \times \mathbb{T})$, where we recall that $\hat{n}$ denotes the outward normal to the boundary unit vector. Thus, for the magnetic field we can naturally consider the boundary condition corresponding to *perfectly conducting walls*, namely

$$B \cdot \hat{n} = 0, \quad (\nabla \times B) \times \hat{n} = 0,$$

we refer to [GLL07; Rob67] for further information. Componentwise, for cylindrical boundaries of the type here considered where the θ and z variables are periodic, and solutions in modal form

$$B(t, r, \theta, z) = b(r)e^{\lambda t + i(m\theta + kz)},$$

this reads,

$$b_r|_{\partial\mathcal{J}} = 0, \quad \partial_r(rb_\theta)|_{\partial\mathcal{J}} = 0, \quad \partial_rb_z|_{\partial\mathcal{J}} = 0.$$

From our analysis it will become clear that we could cover some other mathematically relevant, albeit not so physically interesting scenarios, such as homogeneous Dirichlet boundary conditions $B = 0$ on $\partial\mathcal{M}$. Since they are not standard for the MHD equations, we shall not provide any details however, and focus exclusively on the perfectly conducting walls scenario.

The boundary condition over b_z will be omitted for the moment, and we will only address it later on when we discuss the z component of the equation. We thus consider the r and θ components of the operator $\mathcal{L}^{\text{dyn}}$, i.e. corresponding to (6.1.5) and (6.1.6), which, abusing notation, we will still

denote by $\mathcal{L}^{\text{dyn}}$ for readability. The operator is thus defined on $\mathcal{D}(\mathcal{L}^{\text{dyn}}) \subset L^\infty(\mathcal{J})$, with

$$\mathcal{D}(\mathcal{L}^{\text{dyn}}) = \{B \in L^\infty(\mathcal{J}) \cap H_{\text{loc}}^2 : \mathcal{L}^{\text{dyn}} B \in L^\infty(\mathcal{J}), B_r|_{\partial\mathcal{J}} = (rB_\theta)'|_{\partial\mathcal{J}} = 0\}.$$

As before, it is standard to show that $\mathcal{L}^{\text{dyn}}$ is closed on $\mathcal{D}(\mathcal{L}^{\text{dyn}})$. The main result of this section will be the following.

Theorem 6.7.1 (Growing modes with boundary conditions). *Let $\mathcal{J} \subset [0, \infty)$ be a domain either of the form $[0, q]$, $[p, q]$, or $[p, \infty)$ with $0 < p < q$, and assume H0–H2. Let $\mathcal{R}$ be the compact set provided by Lemma 6.3.4, and let $\mathcal{N} \subset \mathbb{R} \setminus \{0\}$ be compact. Then, for all $\varepsilon > 0$ small, uniformly for $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$, the operator $\mathcal{L}^{\text{dyn}}$ defined in (6.1.5)–(6.1.6) has an eigenvalue of the form $\lambda = \varepsilon^{1/3}(\mu_\star + o_{\varepsilon \rightarrow 0}(1))$ as in Theorem 6.1.1, with an associated finite energy eigenfunction $b \in X_{\mathcal{J}}$, satisfying the boundary conditions*

$$b_r|_{\partial\mathcal{J}} = 0, \quad (rb_\theta)'|_{\partial\mathcal{J}} = 0.$$

Furthermore, assuming H3 only if $\mathcal{J}$ is unbounded, the asymptotic expansion of Theorem 6.1.1 for the eigenfunction b is valid if $\text{Re}(\mu_\star) > 0$.

The strategy we follow will be similar to the case of the full space: we construct a resolvent for $\mathcal{L}^{\text{dyn}} - \lambda$ in a contour enclosing Gilbert’s approximate eigenvalue $\lambda_\star = \mu_\star \varepsilon^{1/3}$, and show that the associated Riesz projector has non-empty range. In order to construct the inverse operator, we shall simply employ the inverse we defined for the global equation, and add homogeneous solutions to correct the boundary conditions. Since the previous sections have emphasized uniformity in $r_0 \in \mathcal{R}$, $M \in \mathcal{N}$ in great deal already, we shall omit any mention of it here, since our analysis builds entirely on that in Section 6.6.

We give now full details of the proof of Theorem 6.7.1 for the case of the annular cylinder $\mathcal{J} = [p, q]$, and we will comment on the other two scenarios—corresponding to a cylinder $\mathcal{J} = [0, q]$ and the exterior of a cylinder $\mathcal{J} = [p, \infty)$ —later on. Furthermore, throughout this section we shall make the standing assumption that Hypotheses H0–H2 are satisfied, and in particular that the closed set $\mathcal{J}$ from the Hypotheses is given by the interval $[\frac{p}{2}, 2q]$.

Since there are a total of four different boundary conditions to satisfy in the case where $\mathcal{J} = [p, q]$, we shall construct four homogeneous solutions to $(\mathcal{L}^{\text{dyn}} - \lambda)B = 0$ with $\lambda \in \mathbb{C}$ being some contour enclosing Gilbert’s approximate eigenvalue $\lambda_\star$. From there, it is a matter of inverting a 4×4 matrix in order to fix the boundary conditions for our previously constructed inverse. It thus remains to construct precisely these homogeneous solutions. In order to make sure these are linearly independent, we shall attempt to construct solutions for which either the first or the second component is “large” at p , and “small” at q , or vice versa. Indeed, the main technical result of this section reads as follows.

Lemma 6.7.2. *Let $[p, q] \subset (0, \infty)$, $r_0 \in (p, q)$, and let $\lambda = \lambda_\star + \varepsilon^{1/3}\eta \in \mathbb{C}$ as in Proposition 6.6.1. Then, for ε sufficiently small, there exist four homogeneous solutions $v_i = (v_{i,1}, v_{i,2})$ with $i = 1, 2, 3, 4$, to $(\mathcal{L}^{\text{dyn}} - \lambda)v_i = 0$ on (p, q) such that $\|v_i\|_{L^\infty} \lesssim \varepsilon^{1/3}$ for all i , and so that further*

$$1. \quad |v_{1,1}(p)| \gtrsim \varepsilon^{1/3}, \quad |v_{1,1}(q)| \lesssim \varepsilon^{2/3}, \quad |\partial_r v_{1,2}(p)| \lesssim 1, \quad |\partial_r v_{1,2}(q)| \lesssim \varepsilon^{2/3}.$$

$$2. |v_{2,1}(q)| \gtrsim \varepsilon^{1/3}, |v_{2,1}(p)| \lesssim \varepsilon^{2/3}, |\partial_r v_{2,2}(q)| \lesssim 1, |\partial_r v_{2,2}(p)| \lesssim \varepsilon^{2/3}.$$

$$3. |v_3(p)| \lesssim \varepsilon^{2/3}, |v_3(q)| \lesssim \varepsilon^{2/3}, |\partial_r v_{3,2}(p)| \gtrsim 1, |\partial_r v_{3,2}(q)| \lesssim \varepsilon^{2/3},$$

$$4. |v_4(q)| \lesssim \varepsilon^{2/3}, |v_4(p)| \lesssim \varepsilon^{2/3}, |\partial_r v_{4,2}(q)| \gtrsim 1, |\partial_r v_{4,2}(p)| \lesssim \varepsilon^{2/3}.$$

We remark that the first two estimates in (1) and (2) refer to the first component of v_1 and v_2 respectively, whereas the first two estimates in (3) and (4) refer to both components of the vectors v_3 and v_4 . Since the arguments for both endpoints will be essentially identical, we shall only prove the result for the left endpoint $p > 0$. Let us outline the main ideas of this proof, taking v_1 as an example. For this case, we shall set

$$v_1 = (\mathcal{L}^{\text{dyn}} - \lambda)^{-1} \begin{pmatrix} \mathbb{1}_{[p-\varepsilon^\delta, p]} \\ 0 \end{pmatrix},$$

where $(\mathcal{L}^{\text{dyn}} - \lambda)^{-1}$ is the inverse operator constructed on $L^\infty(\mathcal{J}, \max\{1, r^2\} w_\varepsilon(r - r_0))$.—In other words, it is the inverse constructed throughout section 6.6, but restricted to functions supported on $\mathcal{J}$. In particular, this inverse still exists for functions Ω, U that are highly degenerate at infinity, since we only care about the local behaviour in the compact interval $\mathcal{J}$. By the construction of v_1 , it must hold that $(\mathcal{L}^{\text{dyn}} - \lambda)v_1 = 0$ on (p, q) . However, simply using the brute force iterative bounds from the construction of $(\mathcal{L}^{\text{dyn}} - \lambda)^{-1}$ in the previous sections will not be enough to obtain the more subtle quantitative bounds required for Lemma 6.7.2—indeed, the most we could say would be that globally, $\|v_1\|_X \lesssim \varepsilon^{-1/3}$, which is nowhere near sufficient for our purposes. Instead, we shall undertake a more subtle analysis, which carefully studies the dynamics of the iterative procedure defining $(\mathcal{L}^{\text{dyn}} - \lambda)^{-1}$. The key insight here is that each “linear interpolation” step as in Sections 6.6.2 and 6.6.3 propagates the mass of the input forwards by at most ε^δ , subject to an exponentially small correction. Therefore, after N iterations, the majority of the mass of

$$\sum_{n=0}^N \mathcal{G}^\lambda ((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}^\lambda - \text{Id})^n \begin{pmatrix} \mathbb{1}_{[p-\varepsilon^\delta, p]} \\ 0 \end{pmatrix}$$

will be contained in $[p - (N+2)\varepsilon^\delta, p + (N+1)\varepsilon^\delta]$ —the additional ε^δ term comes from the fact that the action of $\mathcal{G}^\lambda$ can itself be shown to propagate mass by ε^δ . Since simultaneously each iterative term yields progressive geometric decay, we can conclude that an initial datum concentrated near p will have an inverse with very little mass at q (and vice versa), provided ε is small enough. This intuition is the essential driving force behind our proof of Lemma 6.7.2.

Before proceeding further with the proof, we must however acknowledge that we have only constructed an inverse for the operator $\mathcal{L} - \lambda$ so far, which arises from $\mathcal{L}^{\text{dyn}} - \lambda$ via a change of variables. More precisely, from Section 6.3.3 we see that

$$\mathcal{L}V = Q^{-1} \mathcal{L}^{\text{dyn}} QV,$$

where Q denotes the invertible matrix

$$Q = \begin{pmatrix} -\alpha\varepsilon^{1/3} & \alpha\varepsilon^{1/3} \\ 1 & 1 \end{pmatrix}.$$

Therefore, we see that

$$(\mathcal{L}^{\text{dyn}} - \lambda)^{-1} = Q(\mathcal{L} - \lambda)^{-1}Q^{-1}.$$

Furthermore, since our homogeneous solutions need only be local—and we focus only in the cylindrical shell defined by the radial interval $\mathcal{I} = [p, q]$ —we may define $(\mathcal{L} - \lambda)^{-1}$ using the operator

$$\mathcal{G}^\lambda f = \mathcal{G}_1 \left(f \mathbb{1}_{[\frac{1}{2}p, r_0 - \varepsilon\gamma]} \right) + \mathcal{G}_2^\lambda \left(f \mathbb{1}_{[r_0 - \varepsilon\gamma, r_0 + \varepsilon\gamma]} \right) + \mathcal{G}_3 \left(f \mathbb{1}_{[r_0 + \varepsilon\gamma, 2q]} \right),$$

i.e. we discard the components near zero and at infinity. Notice that now the operator $\mathcal{G}^\lambda$ depends on the choice of p and q , nonetheless we retain the same notation as before to avoid introducing additional cumbersome symbols. Furthermore, we can write

$$(\mathcal{L} - \lambda)^{-1} = \mathcal{G}^\lambda \sum_{n \geq 0} (-1)^n Q^{-1} (Q((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id})Q^{-1})^n Q,$$

and so

$$Q(\mathcal{L} - \lambda)^{-1}Q^{-1} = \mathcal{G}_{\text{dyn}}^\lambda \sum_{n \geq 0} (-1)^n ((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n,$$

where

$$\mathcal{G}_{\text{dyn}}^\lambda f = Q\mathcal{G}^\lambda Q^{-1}f = \mathcal{G}_1 \left(f \mathbb{1}_{[\frac{1}{2}p, r_0 - \varepsilon\gamma]} \right) + Q\mathcal{G}_2^\lambda (Q^{-1}f \mathbb{1}_{[r_0 - \varepsilon\gamma, r_0 + \varepsilon\gamma]}) + \mathcal{G}_3 \left(f \mathbb{1}_{[r_0 + \varepsilon\gamma, 2q]} \right).$$

For this last expression we have used that $\mathcal{G}_1, \mathcal{G}_3$ are diagonal operators with identical entries on the diagonal. The error bounds transfer immediately from Section 6.6 into the weighted space

$$L^\infty \left(\left[\frac{p}{2}, 2q \right], Q^{-1}dr \right).$$

Namely, using Proposition 6.5.2, we can write for some suitable $a > 0$,

$$\begin{aligned} \|(\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id}\|_{L^\infty([\frac{1}{2}p, 2q], Q^{-1}dr)} &= \|Q^{-1}((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})f\|_{L^\infty([\frac{1}{2}p, 2q])} \\ &= \|((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id})Q^{-1}f\|_{L^\infty([\frac{1}{2}p, 2q])} \\ &\lesssim \varepsilon^a \|f\|_{L^\infty([\frac{1}{2}p, 2q], Q^{-1}dr)}. \end{aligned} \tag{6.7.1}$$

However, the unweighted L^∞ bounds for the error are slightly different. Indeed, we collect them first for the Green's operator $\mathcal{G}_1$ —that we recall it is defined in Section 6.6.3—in the following lemma.

Lemma 6.7.3 (Unweighted bounds). *Fix $k_1, k_2 > 0$ such that $r_0 - k_1 \geq k_2$. Let $c_0 > 0$ be as defined in Section 6.6.3. If f is such that $\text{supp}(f) \subset [\frac{1}{2}p, k_1]$, then the following bounds hold true.*

1. Let $f = (f_1, 0)$, then the pointwise error bound by component is given by

$$\begin{aligned} |((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_1 f - f)_1|(r) &\lesssim \varepsilon^\beta \|f\|_{L^\infty} \left(\mathbb{1}_{[\frac{1}{2}p, k_1 + 2\varepsilon^\delta]} + e^{-c_0 \varepsilon^{-2/3} |r - (k_1 + \varepsilon^\delta)|} \right), \\ |((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_1 f - f)_2|(r) &\lesssim \varepsilon^{1/3} \|f\|_{L^\infty} \left(\mathbb{1}_{[\frac{1}{2}p, k_1 + 2\varepsilon^\delta]} + e^{-c_0 \varepsilon^{-2/3} |r - (k_1 + \varepsilon^\delta)|} \right), \end{aligned}$$

where β is defined in Property P3. The second component of the Green's function satisfies $|(\mathcal{G}_1 f)_2|(r) = 0$, whereas for the first component we obtain,

$$\begin{aligned} |(\mathcal{G}_1 f)_1|(r) &\lesssim \varepsilon^{1/3} \|f\|_{L^\infty} \left(\mathbb{1}_{[\frac{1}{2}p, k_1 + 2\varepsilon^\delta]} + e^{-c_0 \varepsilon^{-2/3} |r - (k_1 + \varepsilon^\delta)|} \right), \\ |\partial_r (\mathcal{G}_1 f)_1|(r) &\lesssim \varepsilon^{-1/3} \|f\|_{L^\infty} \left(\mathbb{1}_{[\frac{1}{2}p, k_1 + 2\varepsilon^\delta]} + e^{-c_0 \varepsilon^{-2/3} |r - (k_1 + \varepsilon^\delta)|} \right). \end{aligned}$$

2. If instead $f = (0, f_2)$, the pointwise error bound reads

$$\begin{aligned} |((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_1 f - f)_1|(r) &\lesssim \varepsilon \|f\|_{L^\infty} \left(\mathbb{1}_{[\frac{1}{2}p, k_1 + 2\varepsilon^\delta]} + e^{-c_0 \varepsilon^{-2/3} |r - (k_1 + \varepsilon^\delta)|} \right), \\ |((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_1 f - f)_2|(r) &\lesssim \varepsilon^\beta \|f\|_{L^\infty} \left(\mathbb{1}_{[\frac{1}{2}p, k_1 + 2\varepsilon^\delta]} + e^{-c_0 \varepsilon^{-2/3} |r - (k_1 + \varepsilon^\delta)|} \right), \end{aligned}$$

where again β comes from Property P3. The first component of the Green's function satisfies $|(\mathcal{G}_1 f)_1|(r) = 0$, whereas for the second component we obtain

$$\begin{aligned} |(\mathcal{G}_1 f)_2|(r) &\lesssim \varepsilon^{1/3} \|f\|_{L^\infty} \left(\mathbb{1}_{[\frac{1}{2}p, k_1 + 2\varepsilon^\delta]} + e^{-c_0 \varepsilon^{-2/3} |r - (k_1 + \varepsilon^\delta)|} \right), \\ |\partial_r (\mathcal{G}_1 f)_2|(r) &\lesssim \varepsilon^{-1/3} \|f\|_{L^\infty} \left(\mathbb{1}_{[\frac{1}{2}p, k_1 + 2\varepsilon^\delta]} + e^{-c_0 \varepsilon^{-2/3} |r - (k_1 + \varepsilon^\delta)|} \right). \end{aligned}$$

These bounds are straightforward consequences of our analysis in Section 6.6, taking into account that $\text{supp}(f) \subset [\frac{1}{2}p, k_1]$, $k_1 < r_0$, where the distance between k_1 and r_0 is independent of ε .

At this point we want to introduce a partition of the interval $[\frac{1}{2}p, r_0 - \varepsilon^\gamma]$, but notice that we can always do it so that p is a boundary point of the partition. Indeed, we set

$$r_j = \frac{p}{2} + (j-1)\varepsilon^\delta, \quad j \geq 1,$$

until $j_0 \in \mathbb{N}$ such that $p - 2\varepsilon^\delta \leq \frac{1}{2}p + j_0\varepsilon^\delta < p - \varepsilon^\delta$. Then simply set $r_{j_0+1} = p - \varepsilon^\delta$, and inductively $r_{j+1} = r_j + \varepsilon^\delta$ from there. This way we create a partition $\bigcup_j [r_j, r_{j+1})$, which always contains the interval $[p - \varepsilon^\delta, p)$. With this partition defined, we now define

$$\mathcal{P}_{\leq J} := \bigcup_{j=1}^J [r_j, r_{j+1}),$$

and $\mathcal{P}_{> J} := [\frac{1}{2}p, 2q] \setminus \mathcal{P}_{\leq J}$. With these preliminaries out of the way, we now state the lemma which will be the main workhorse of this section.

Lemma 6.7.4. *Let $f \in L^\infty([\frac{1}{2}p, 2q])$ be of the form $f = (f_1, f_2)$, and fix $k_2 > 0$ and J such that*

$r_0 - r_{J+1} \geq k_2$. Then, there hold the estimates

$$\begin{aligned} \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})f\|_{L^\infty(\mathcal{P}_{\leq J+2})} &\leq \varepsilon^\beta \|f_1\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon^{1/3} \|f_2\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon^a \|Q^{-1}f\|_{L^\infty(\mathcal{P}_{>J})}, \\ \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})f\|_{L^\infty(\mathcal{P}_{\leq J+2})} &\leq \varepsilon^\beta \|f_2\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon \|f_1\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon^a \|Q^{-1}f\|_{L^\infty(\mathcal{P}_{>J})}, \\ \|Q^{-1}((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})f\|_{L^\infty(\mathcal{P}_{>J+2})} &\leq e^{-\frac{1}{2}c_0\varepsilon^{-2/3+\delta}} \|f\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon^a \|Q^{-1}f\|_{L^\infty(\mathcal{P}_{>J})}. \end{aligned}$$

Proof. Writing $f = f\mathbb{1}_{\mathcal{P}_{\leq J}} + f\mathbb{1}_{\mathcal{P}_{>J}}$, we have

$$|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})f(r)| \leq |((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_1^\lambda - \text{Id})\mathbb{1}_{\mathcal{P}_{\leq J}}f(r)| + |((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})\mathbb{1}_{\mathcal{P}_{>J}}f(r)|.$$

For the second addend in the right hand side we can directly deduce from (6.7.1),

$$|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})\mathbb{1}_{\mathcal{P}_{>J}}f(r)| \leq \|Q\| \|Q^{-1}((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})\mathbb{1}_{\mathcal{P}_{>J}}f(r)\| \lesssim \varepsilon^a \|Q^{-1}f\|_{L^\infty(\mathcal{P}_{>J})}.$$

For the first addend we use Lemma 6.7.3, so that all in all we can write for any $r \leq r_J + 3\varepsilon^\delta$,

$$\begin{aligned} |(((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})f)_1(r)| &\lesssim \varepsilon^\beta \|f_1\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon^{1/3} \|f_2\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon^a \|Q^{-1}f\|_{L^\infty(\mathcal{P}_{>J})}, \\ |(((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})f)_2(r)| &\lesssim \varepsilon^\beta \|f_2\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon \|f_1\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon^a \|Q^{-1}f\|_{L^\infty(\mathcal{P}_{>J})}. \end{aligned}$$

Similarly, for $r > r_J + 3\varepsilon^\delta$, again using Lemma 6.7.3 we can write

$$\begin{aligned} |Q^{-1}((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})f(r)| &\lesssim \|Q^{-1}\| \left(\varepsilon^\beta + \varepsilon^{1/3} \right) e^{-c_0\varepsilon^{-2/3+\delta}} \|f\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon^a \|Q^{-1}f\|_{L^\infty(\mathcal{P}_{>J})} \\ &\lesssim \left(\varepsilon^{\beta-1/3} + 1 \right) e^{-c_0\varepsilon^{-2/3+\delta}} \|f\|_{L^\infty(\mathcal{P}_{\leq J})} + \varepsilon^a \|Q^{-1}f\|_{L^\infty(\mathcal{P}_{>J})}, \end{aligned}$$

where we used that $\|Q^{-1}\| \lesssim \varepsilon^{-1/3}$. From here the proof follows provided that $\delta < 2/3$, which is ensured by Lemma 6.3.4. $\square$

Before proceeding with the proof of Lemma 6.7.2 we first need to prove some technical results that will be required in the coming pages.

Lemma 6.7.5. *Let $\sigma > 0$, $a \in (0, 1/3)$, and consider the iteratively defined sequence*

$$\begin{aligned} a_{n+1} &\leq \varepsilon^a (a_n + c_n) + \varepsilon b_n, \\ b_{n+1} &\leq \varepsilon^a (b_n + c_n) + \varepsilon^{1/3} a_n, \\ c_{n+1} &\leq \varepsilon^a (c_n + \sigma(a_n + b_n)), \end{aligned}$$

with initial configuration $a_0, b_0 > 0$, $c_0 = 0$. Then there holds for all $n \geq 0$

$$\begin{aligned} a_n &\leq 2(\varepsilon^{an/2} a_0 + \varepsilon^{1/3+an(n-1)/2} b_0 + \sqrt{\sigma} \varepsilon^{an/2} (a_0 + b_0)), \\ b_n &\leq 2(\varepsilon^{an/2} b_0 + \varepsilon^{1/3+an(n-1)/2} a_0 + \sqrt{\sigma} \varepsilon^{an/2} (a_0 + b_0)), \\ c_n &\leq 2\sqrt{\sigma} \varepsilon^{an/2} (a_0 + b_0). \end{aligned}$$

Proof. Since all constants are non-negative, the sequence is pointwise bounded above by the same sequence, but with the inequalities replaced by equalities, and further replacing the εb_n contribu-

tion by $\varepsilon^{1/3}b_n$ i.e.

$$\begin{aligned}a_{n+1} &= \varepsilon^a(a_n + c_n) + \varepsilon^{1/3}b_n \\b_{n+1} &= \varepsilon^a(b_n + c_n) + \varepsilon^{1/3}a_n \\c_{n+1} &= \varepsilon^a(c_n + \sigma(a_n + b_n)).\end{aligned}$$

Whilst this system may be solved via a matrix diagonalisation, the computation becomes extremely messy. Thus, we instead employ the following approach, which yields slightly less sharp bounds, but is sufficient for our purposes. We first set $d_n = a_n + b_n$, and so obtain a reduced system

$$\begin{aligned}d_{n+1} &\leq 2\varepsilon^a(d_n + c_n) \\c_{n+1} &\leq 2\varepsilon^a(c_n + \sigma d_n),\end{aligned}$$

which we may once again upper bound by replacing the inequalities by equalities. Via a matrix diagonalization, we see that

$$\begin{pmatrix} d_n \\ c_n \end{pmatrix} = \frac{1}{2}(2\varepsilon^a)^n \begin{pmatrix} ((1 - \sqrt{\sigma})^n + (1 + \sqrt{\sigma})^n)d_0 + \frac{1}{\sqrt{\sigma}}(-(1 - \sqrt{\sigma})^n + (1 + \sqrt{\sigma})^n)c_0 \\ \sqrt{\sigma}(-(1 - \sqrt{\sigma})^n + (1 + \sqrt{\sigma})^n)d_0 + ((1 - \sqrt{\sigma})^n + (1 + \sqrt{\sigma})^n)c_0 \end{pmatrix},$$

and so, since by assumption $c_0 = 0$,

$$d_n \leq (2\varepsilon^a(1 + \sqrt{\sigma}))^n d_0, \quad c_n \leq \sqrt{\sigma}(2\varepsilon^a(1 + \sqrt{\sigma}))^n d_0.$$

In particular, we can now use these bounds to study the equations for a_n, b_n , which now, via the trivial estimate $(2\varepsilon^a(1 + \sqrt{\sigma}))^n \leq \varepsilon^{an/2}$ for ε sufficiently small, may be bounded by

$$a_{n+1} = \varepsilon^a a_n + \varepsilon^{1/3} b_n + \sqrt{\sigma} \varepsilon^{an/2+a} d_0, \quad b_{n+1} = \varepsilon^a b_n + \varepsilon^{1/3} a_n + \sqrt{\sigma} \varepsilon^{an/2+a} d_0.$$

Solving this difference equation is a rather arduous task, but still considerably easier than solving the full 3×3 system directly. Defining v_n such that

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = P v_n = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_n^1 \\ v_n^2 \end{pmatrix}$$

we find that v_n satisfies the difference equation

$$v_{n+1} = \varepsilon^a \begin{pmatrix} 1 - \varepsilon^{1/3-a} & 0 \\ 0 & 1 + \varepsilon^{1/3-a} \end{pmatrix} v_n + \varepsilon^{an/2+a} \sqrt{\sigma} d_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We thus directly observe that $v_n^1 = \varepsilon^{na}(1 - \varepsilon^{1/3-a})^n v_0^1$. The computation for v_n^2 is more challenging, and can be done using the z -transform, a discrete time analogue of the Laplace transform, see [Fle22, Chapter 5]. Indeed, multiplying the n^{th} term in the difference equation by z^{-n} and summing over n , we obtain

$$zY(z) - z v_0^2 = \varepsilon^a(1 + \varepsilon^{1/3-a})Y(z) + \sqrt{\sigma} \varepsilon^a \sum_{n \geq 0} \varepsilon^{an/2} z^{-n} = \varepsilon^a(1 + \varepsilon^{1/3-a})Y(z) + \sqrt{\sigma} d_0 \varepsilon^a \frac{z}{z - \varepsilon^{a/2}},$$

where we have set

$$Y(z) = \sum_{n \geq 0} v_n^2 z^{-n}.$$

We thus solve this algebraic equation, and via a partial fractions computation, we thus have

$$Y(z) = \frac{z}{z - \varepsilon^a(1 + \varepsilon^{1/3-a})} v_0^2 + \frac{\sqrt{\sigma} d_0 \varepsilon^a}{\varepsilon^{a/2}(1 - \varepsilon^{a/2}(1 + \varepsilon^{1/3-a}))} \left(\frac{z}{z - \varepsilon^{a/2}} - \frac{z}{z - \varepsilon^a(1 + \varepsilon^{1/3-a})} \right).$$

We recognise the right hand side as the z -transform of

$$v_0^2 (\varepsilon^a(1 + \varepsilon^{1/3-a}))^n + \frac{\sqrt{\sigma} d_0 \varepsilon^a}{\varepsilon^{a/2}(1 - \varepsilon^{a/2}(1 + \varepsilon^{1/3-a}))} \left(\varepsilon^{an/2} - (\varepsilon^a(1 + \varepsilon^{1/3-a}))^n \right),$$

and so undoing the change of variables yields

$$\begin{aligned} \begin{pmatrix} a_n \\ b_n \end{pmatrix} &= \frac{1}{2} \varepsilon^{na} \begin{pmatrix} (a_0 + b_0)(1 + \varepsilon^{1/3-a})^n - (b_0 - a_0)(1 - \varepsilon^{1/3-a})^n \\ (a_0 + b_0)(1 + \varepsilon^{1/3-a})^n + (b_0 - a_0)(1 - \varepsilon^{1/3-a})^n \end{pmatrix} \\ &\quad + \frac{\sqrt{\sigma} d_0 \varepsilon^a}{\varepsilon^{a/2}(1 - \varepsilon^{a/2}(1 + \varepsilon^{1/3-a}))} \left(\varepsilon^{an/2} - (\varepsilon^a(1 + \varepsilon^{1/3-a}))^n \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned} \quad (6.7.2)$$

We see that we readily obtain from here the claim of the lemma using on the one hand the straightforward estimates

$$\begin{aligned} \varepsilon^{na} \left((1 + \varepsilon^{1/3-a})^n + (1 - \varepsilon^{1/3-a})^n \right) &\leq \varepsilon^{an/2}, \\ \frac{\varepsilon^a}{\varepsilon^{a/2}(1 - \varepsilon^{a/2}(1 + \varepsilon^{1/3-a}))} \left(\varepsilon^{an/2} - (\varepsilon^a(1 + \varepsilon^{1/3-a}))^n \right) &\leq \varepsilon^{an/2}, \end{aligned}$$

and on the other hand we claim

$$\varepsilon^{na} \left((1 + \varepsilon^{1/3-a})^n - (1 - \varepsilon^{1/3-a})^n \right) \leq 4\varepsilon^{1/3+a(n-1)/2},$$

for all n and ε small enough. Indeed, from the binomial theorem, it follows that

$$\left((1 + \varepsilon^{1/3-a})^n - (1 - \varepsilon^{1/3-a})^n \right) \leq 2((1 + \varepsilon^{1/3-a})^n - 1).$$

In turn, we note the expression valid for any $x \neq 1$,

$$\frac{x^n - 1}{x - 1} = \sum_{k=0}^{n-1} x^k,$$

and so

$$((1 + \varepsilon^{1/3-a})^n - 1) \leq \varepsilon^{1/3-a} \sum_{k=0}^{n-1} (1 + \varepsilon^{1/3-a})^k.$$

Therefore,

$$\begin{aligned} \varepsilon^{na} ((1 + \varepsilon^{1/3-a})^n - 1) &\leq \varepsilon^{1/3-a+na} + \varepsilon^{1/3-a} \varepsilon^{na/2} \sum_{k=1}^{\infty} (\varepsilon^{a/2} + \varepsilon^{1/3-a/2})^k \\ &\leq \varepsilon^{1/3-a+na} + \varepsilon^{1/3-a+na/2} \frac{\varepsilon^{a/2} + \varepsilon^{1/3-a/2}}{1 - (\varepsilon^{a/2} + \varepsilon^{1/3-a/2})} \leq 2\varepsilon^{1/3+a(n-1)/2}, \end{aligned}$$

as soon as $1/3 - a/2 > a/2$, i.e. as soon as $a < 1/3$, and $\varepsilon > 0$ is small enough. Thus, plugging this into (6.7.2), the proof is complete. $\square$

Combining the previous lemmas, we arrive at the following result.

Lemma 6.7.6. Fix $k_2, N > 0$ and suppose that $f = (f_1, 0)$ is supported in $\mathcal{P}_{\leq J}$, where

$$r_J + (2N + 1)\varepsilon^\delta \leq r_0 - k_2.$$

Then, for all $n \leq N$ there holds

$$\begin{aligned} \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f)_1\|_{L^\infty(\mathcal{P}_{\leq J+2n})} &\lesssim \varepsilon^{an/2} \|f\|_{L^\infty(\mathcal{P}_{\leq J})}, \\ \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f)_2\|_{L^\infty(\mathcal{P}_{\leq J+2n})} &\lesssim \varepsilon^{1/3+a(n-1)/2} \|f\|_{L^\infty(\mathcal{P}_{\leq J})}, \\ \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f\|_{L^\infty(\mathcal{P}_{>J+2n}, Q^{-1}dr)} &\lesssim \varepsilon^{\frac{1}{2}(\delta-1/3+na)} e^{-\frac{1}{2}c_0\varepsilon^{-2/3+\delta}} \|f\|_{L^\infty(\mathcal{P}_{\leq J})}. \end{aligned}$$

If instead $f = (0, f_2)$, then

$$\begin{aligned} \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f)_1\|_{L^\infty(\mathcal{P}_{\leq J+2n})} &\lesssim \varepsilon^{1/3+a(n-1)/2} \|f\|_{L^\infty(\mathcal{P}_{\leq J})}, \\ \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f)_2\|_{L^\infty(\mathcal{P}_{\leq J+2n})} &\lesssim \varepsilon^{an/2} \|f\|_{L^\infty(\mathcal{P}_{\leq J})}, \\ \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f\|_{L^\infty(\mathcal{P}_{>J+2n}, Q^{-1}dr)} &\lesssim \varepsilon^{\frac{1}{2}(\delta-1/3+na)} e^{-\frac{1}{2}c_0\varepsilon^{-2/3+\delta}} \|f\|_{L^\infty(\mathcal{P}_{\leq J})}. \end{aligned}$$

Proof. We set

$$\begin{aligned} a_n &= \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f)_1\|_{L^\infty(\mathcal{P}_{\leq J+2n})}, \\ b_n &= \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f)_2\|_{L^\infty(\mathcal{P}_{\leq J+2n})}, \\ c_n &= \|((\mathcal{L}^{\text{dyn}} - \lambda)\mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f\|_{L^\infty(\mathcal{P}_{>J+2n}, Q^{-1}dr)}. \end{aligned}$$

Then, using the bounds from Lemma 6.7.4 and setting $\sigma = \varepsilon^{\delta-1/3} e^{-c_0\varepsilon^{-2/3+\delta}}$, we may directly apply Lemma 6.7.5 and deduce the result. $\square$

With this result in hand, we may now commence the proof of Lemma 6.7.2.

Proof of Lemma 6.7.2. The proof will be divided into multiple steps.

Step 1. We begin by constructing v_1 . Take $f = (f_1, f_2)$ with $f_1(r) = \mathbb{1}_{[p-\varepsilon^\delta, p]}(r)$, $f_2 = 0$, and set $v_1(r) = (\mathcal{L}^{\text{dyn}} - \lambda)^{-1}f$. In particular, it holds that $(\mathcal{L}^{\text{dyn}} - \lambda)v_1 = 0$ in (p, q) , and so we have a homogeneous solution. It remains to prove the claimed bounds at the endpoints. We begin by constructing the first iterate by hand. Indeed, we have via a direct computation that

$$f^0(r) := \mathcal{G}_{\text{dyn}}^\lambda f = \begin{pmatrix} \frac{\varepsilon^{1/3}}{2iT(p-\varepsilon^\delta)} e^{-\sqrt{iT(p-\varepsilon^\delta)}\varepsilon^{-2/3}r} \left(e^{-\sqrt{iT(p-\varepsilon^\delta)}\varepsilon^{-2/3}p} - e^{-\sqrt{iT(p-\varepsilon^\delta)}\varepsilon^{-2/3}(p-\varepsilon^\delta)} \right) \\ 0 \end{pmatrix},$$

for $r \geq p$. In particular, at $r = p$ we have

$$|f_0^1(r)| = \left| \frac{\varepsilon^{1/3}}{2|T(p-\varepsilon^\delta)|} (1 - e^{-\sqrt{iT(p-\varepsilon^\delta)}\varepsilon^{-2/3+\delta}}) \right| \gtrsim \varepsilon^{1/3}.$$

Fix now N large enough so that $Na > 4/3$, and $a + 3N\varepsilon^\delta < r_0 - |r_0 - p|/2$. Then, we write

$$v_1(r) = f^0(r) + \sum_{n=1}^N (-1)^n \mathcal{G}_{\text{dyn}}^\lambda ((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f + \sum_{n \geq N+1}^{\infty} (-1)^n \mathcal{G}_{\text{dyn}}^\lambda ((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f.$$

By our error bounds, there holds

$$\begin{aligned} & \left\| \sum_{n \geq N+1}^{\infty} (-1)^n \mathcal{G}_{\text{dyn}}^\lambda ((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f \right\|_{L^\infty([\frac{p}{2}, 2q])} \\ & \leq \|Q\| \left\| \sum_{n \geq N+1}^{\infty} (-1)^n \mathcal{G}_{\text{dyn}}^\lambda ((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f \right\|_{L^\infty([\frac{p}{2}, 2q], Q^{-1}dr)} \\ & \lesssim \varepsilon^{-1/3} \sum_{n \geq N+1} \varepsilon^{na} \|f\|_{L^\infty([\frac{p}{2}, 2q], Q^{-1}dr)} \lesssim \varepsilon^{-1/3+(N+1)a} \|Q^{-1}f\|_{L^\infty([\frac{p}{2}, 2q])} \lesssim \varepsilon^{-2/3+(N+1)a}. \end{aligned}$$

Next, we note that we can split the error produced at the n^{th} iteration into two parts as follows:

$$\begin{aligned} ((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f &= \left(((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f \right) \mathbf{1}_{r \leq p+2n\varepsilon^\delta} \\ &+ \left(((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f \right) \mathbf{1}_{r > p+2n\varepsilon^\delta} =: f_\leq^n + f_\>^n. \end{aligned}$$

By Lemma 6.7.6, we observe that

$$\|f_\leq^n\|_{L^\infty([\frac{p}{2}, 2q])} \lesssim \varepsilon^{\frac{na}{2}}, \quad \|f_\>^n\|_{L^\infty([\frac{p}{2}, 2q], Q^{-1}dr)} \lesssim \varepsilon^{\frac{an}{2}} e^{-\frac{c_0}{4}\varepsilon^{-2/3+\delta}},$$

so long as ε is small enough. Furthermore, $f_\leq^n$ is supported in $r < r_0 - |r_0 - p|/2$, meaning $\mathcal{G}_{\text{dyn}}^\lambda f_\leq^n = \mathcal{G}_1 f_\leq^n$, and the bounds from Lemma 6.7.3 are applicable. We begin by estimating

$$|\mathcal{G}_{\text{dyn}}^\lambda ((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^\lambda - \text{Id})^n f|(r) \leq |\mathcal{G}_1 f_\leq^n|(r) + \|Q\| \|\mathcal{G}_{\text{dyn}}^\lambda f_\>^n\|_{L^\infty([\frac{p}{2}, 2q], Q^{-1}dr)}.$$

In particular, for $r \leq p + 3n\varepsilon^\delta$ we deduce from Lemma 6.7.6 the bound

$$|\mathcal{G}_{\text{dyn}}^\lambda f_\leq^n|(r) \leq \varepsilon^{1/3+\frac{na}{2}} + \varepsilon^{-1/3+\frac{na}{2}} e^{-\frac{c_0}{4}\varepsilon^{-2/3+\delta}} \lesssim \varepsilon^{1/3+\frac{na}{2}},$$

whereas for $r > p + 3n\varepsilon^\delta$ it holds

$$|\mathcal{G}_{\text{dyn}}^\lambda f_\>^n| \leq \varepsilon^{1/3+\frac{na}{2}} e^{-c_0\varepsilon^{-2/3+\delta}} + \varepsilon^{-1/3+\frac{na}{2}} e^{-\frac{c_0}{4}\varepsilon^{-2/3+\delta}} \lesssim \varepsilon^{\frac{na}{2}} e^{-\frac{c_0}{8}\varepsilon^{-2/3+\delta}}.$$

Therefore, we deduce

$$|v_{1,1}(p)| \gtrsim \varepsilon^{1/3} - \sum_{n=1}^{\infty} \varepsilon^{1/3+\frac{na}{2}} - \varepsilon^{-2/3+(N+1)a} \gtrsim \varepsilon^{1/3},$$

and similarly

$$|v_{1,1}(q)| \lesssim e^{-\frac{c_0}{4}\varepsilon^{-2/3+\delta}} \sum_{n=0}^{\infty} \varepsilon^{\frac{an}{2}} + \varepsilon^{-2/3+(N+1)a} \lesssim \varepsilon^{2/3}.$$

It thus remains to verify the derivative bounds. Recalling that we split

$$f_{\leq}^n = \left(((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^{\lambda} - \text{Id})^n f \right) \mathbf{1}_{r \leq p+2n\epsilon^\delta}, \quad f_{>}^n = \left(((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^{\lambda} - \text{Id})^n f \right) \mathbf{1}_{r > p+2n\epsilon^\delta},$$

we now denote the components of $f_{\leq}^n$ by $f_{\leq,1}^n, f_{\leq,2}^n$, and analogously for $f_{<}^n$. By Lemma 6.7.6, recalling that $f_{\leq,2}^0 = 0$, we get the more precise bounds for $f_{\leq,1}^n, f_{\leq,2}^n$

$$\|f_{\leq,1}^n\|_{L^\infty([\frac{p}{2}, 2q])} \lesssim \epsilon^{\frac{na}{2}}, \quad \|f_{\leq,2}^n\|_{L^\infty([\frac{p}{2}, 2q])} \lesssim \epsilon^{1/3+(n-1)\frac{a}{2}}.$$

Therefore, we may bound the derivative of the second component for $r < a + 3n\epsilon^\delta$ by

$$\begin{aligned} |\partial_r(\mathcal{G}_{\text{dyn}}^{\lambda}((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^{\lambda} - \text{Id})^n f)^2|(r) &\leq |\partial_r \mathcal{G}_1 f_{\leq,2}^n|(r) + \|Q\| \|\partial_r \mathcal{G}_{\text{dyn}}^{\lambda} f_{>}^n\|_{L^\infty([\frac{p}{2}, 2q], Q^{-1} dr)} \\ &\lesssim \epsilon^{(n-1)\frac{a}{2}} + \epsilon^{-2/3+\frac{na}{2}} e^{-\frac{c_0}{4}\epsilon^{-2/3+\delta}}. \end{aligned}$$

Thus,

$$|(v_{1,2})'(p)| \lesssim \sum_{n=1}^{\infty} \epsilon^{\frac{(n-1)a}{2}} + \epsilon^{-4/3+(N+1)a} \lesssim 1.$$

Similarly, we can show that $|(v_{1,2})'(q)| \lesssim \epsilon^{2/3}$, and so the construction of v_1 is complete.

Step 2. We now construct v_3 . To do so, we pick $g = (g_1, g_2)$ with $g_1 = 0$, $g_2 = \epsilon^{1/3} \mathbf{1}_{[p-\epsilon^\delta, p]}(r)$, and set $v_3(r) = (\mathcal{L}^{\text{dyn}} - \lambda)^{-1} g$. Following an entirely analogous strategy as for the construction of v_1 , we compute the first iterate by hand, yielding for $r \geq p$

$$g^0 := \mathcal{G}_{\text{dyn}}^{\lambda} g = \left(\begin{array}{c} 0 \\ \frac{\epsilon^{2/3}}{2iT(p-\epsilon^\delta)} e^{-\sqrt{iT(p-\epsilon^\delta)}\epsilon^{-2/3}r} \left(e^{-\sqrt{iT(p-\epsilon^\delta)}\epsilon^{-2/3}p} - e^{-\sqrt{iT(p-\epsilon^\delta)}\epsilon^{-2/3}(p-\epsilon^\delta)} \right) \end{array} \right).$$

In particular, its derivative at $r = p$ is of size $O(1)$ in ϵ . Writing as before

$$g_{\leq}^n = \left(((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^{\lambda} - \text{Id})^n g \right) \mathbf{1}_{r \leq p+2n\epsilon^\delta}, \quad g_{>}^n = \left(((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^{\lambda} - \text{Id})^n g \right) \mathbf{1}_{r > p+2n\epsilon^\delta},$$

with respective components $g_{\leq}^n = (g_{\leq,1}^n, g_{\leq,2}^n)$, an analogously for $g_{>}^n$, we can bound

$$\begin{aligned} |\partial_r(\mathcal{G}_{\text{dyn}}^{\lambda}((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^{\lambda} - \text{Id})^n g)_2|(r) &\leq |\partial_r \mathcal{G}_1 g_{\leq,2}^n|(r) + \|Q\| \|\partial_r \mathcal{G}_{\text{dyn}}^{\lambda} g_{>}^n\|_{L^\infty([\frac{p}{2}, 2q], Q^{-1} dr)} \\ &\lesssim \epsilon^{(n-1)\frac{a}{2}} + \epsilon^{-1/3+\frac{na}{2}} e^{-\frac{c_0}{4}\epsilon^{-2/3+\delta}}. \end{aligned}$$

Following the arguments of Step 1, we can show that $|(v_{3,2})'(p)| \gtrsim 1$. For the bound of $|v_3(p)|$, we note that at the first iterate, its value is $\epsilon^{2/3}$. Furthermore, we have the bounds

$$\begin{aligned} |\mathcal{G}_{\text{dyn}}^{\lambda}((\mathcal{L}^{\text{dyn}} - \lambda) \mathcal{G}_{\text{dyn}}^{\lambda} - \text{Id})^n g|(r) &\leq |\mathcal{G}_1 g_{\leq}^n|(r) + \|Q\| \|\mathcal{G}_{\text{dyn}}^{\lambda} g_{>}^n\|_{L^\infty([\frac{p}{2}, 2q], Q^{-1} dr)} \\ &\lesssim \epsilon^{2/3+\frac{na}{2}} + \epsilon^{-2/3+\frac{na}{2}} e^{-\frac{c_0}{4}\epsilon^{-2/3+\delta}} \lesssim \epsilon^{2/3+(n-1)\frac{a}{2}}. \end{aligned}$$

Hence, summing over n once again yields the desired upper bound $|v_3(p)| \lesssim \epsilon^{2/3}$, and following the arguments from Step 1 verbatim, we also see that $|v_3(q)| \lesssim \epsilon^{2/3}$, as desired. The bounds for v_2 and v_4 thus follow analogously. $\square$

Finally, we can prove the analogue of Theorem 6.1.1. Indeed, we begin by setting

$$(\mathcal{L}^{\text{dyn}} - \lambda)_{\text{bd}}^{-1} f = (\mathcal{L}^{\text{dyn}} - \lambda)^{-1} f - \mathcal{J}^{-1} \begin{pmatrix} ((\mathcal{L}^{\text{dyn}} - \lambda)^{-1} f)_1(p) \\ ((\mathcal{L}^{\text{dyn}} - \lambda)^{-1} f)_1(q) \\ \partial_r(r((\mathcal{L}^{\text{dyn}} - \lambda)^{-1} f)_2(p)) \\ \partial_r(r((\mathcal{L}^{\text{dyn}} - \lambda)^{-1} f)_2(q)) \end{pmatrix} \cdot \begin{pmatrix} v_1(r) \\ v_2(r) \\ v_3(r) \\ v_4(r) \end{pmatrix},$$

where $\mathcal{J}$ is the matrix given by

$$\mathcal{J} = \begin{pmatrix} v_{1,1}(p) & v_{2,1}(p) & v_{3,1}(p) & v_{4,1}(p) \\ v_{1,1}(q) & v_{2,1}(q) & v_{3,1}(q) & v_{4,1}(q) \\ (rv_{1,2})'(p) & (rv_{2,2})'(p) & (rv_{3,2})'(p) & (rv_{4,2})'(p) \\ (rv_{1,2})'(q) & (rv_{2,2})'(q) & (rv_{3,2})'(q) & (rv_{4,2})'(q) \end{pmatrix}.$$

A computation reveals that with our definition, $(\mathcal{L}^{\text{dyn}} - \lambda)_{\text{bd}}^{-1}$ indeed satisfies the desired boundary conditions on $[p, q]$. We note the following: $\mathcal{J}$ is invertible, and its inverse has operator norm of order $\varepsilon^{-1/3}$. Indeed, write $\mathcal{J} = \mathcal{J}_{\text{main}} + \mathcal{J}_{\text{error}}$, with

$$\mathcal{J}_{\text{main}} = \begin{pmatrix} v_{1,1}(p) & 0 & 0 & 0 \\ 0 & v_{2,1}(q) & 0 & 0 \\ (rv_{1,2})'(p) & 0 & (rv_{3,2})'(p) & 0 \\ 0 & (rv_{2,2})'(q) & 0 & (rv_{4,2})'(q) \end{pmatrix}.$$

Using Lemma 6.7.2, we see that $\|\mathcal{J}_{\text{error}}\| \lesssim \varepsilon^{2/3}$. Furthermore, a computation yields that

$$\mathcal{J}_{\text{main}}^{-1} = \begin{pmatrix} \frac{1}{v_{1,1}(p)} & 0 & 0 & 0 \\ 0 & \frac{1}{v_{2,1}(q)} & 0 & 0 \\ -\frac{(rv_{1,2})'(p)}{v_{1,1}(p)(rv_{3,2})'(p)} & 0 & \frac{1}{(rv_{3,2})'(p)} & 0 \\ 0 & -\frac{(rv_{2,2})'(q)}{v_{2,1}(q)(rv_{4,2})'(q)} & 0 & \frac{1}{(rv_{4,2})'(q)} \end{pmatrix},$$

which has components of order at most $\varepsilon^{-1/3}$, and so its operator norm is bounded by $\varepsilon^{-1/3}$. Thus, we write

$$\mathcal{J}^{-1} = \mathcal{J}_{\text{main}}^{-1} (\text{Id} + \mathcal{J}_{\text{error}} \mathcal{J}_{\text{main}}^{-1})^{-1},$$

where $(\text{Id} + \mathcal{J}_{\text{error}} \mathcal{J}_{\text{main}}^{-1})^{-1}$ is well-defined and has operator norm of order $O(1)$ in ε because of $\|\mathcal{J}_{\text{error}} \mathcal{J}_{\text{main}}^{-1}\| \lesssim \varepsilon^{1/3}$. Hence, we obtain $\|\mathcal{J}^{-1}\| \lesssim \varepsilon^{-1/3}$ as claimed. We can now finally proceed with the proof of the existence of growing modes in the case where $\mathcal{J} = [p, q]$, that we state in more detail in the following proposition.

Proposition 6.7.7 (Growing mode with boundaries). *Consider the operator $\mathcal{L}^{\text{dyn}}$ from (6.1.5)–(6.1.6) on the interval $\mathcal{J} = [p, q]$ with boundary conditions $b_r|_{\partial\mathcal{J}} = 0$, $(rb_\theta)'|_{\partial\mathcal{J}} = 0$. Assume the conditions of 6.7.1. Then, there exists an eigenvalue λ of $\mathcal{L}^{\text{dyn}}$ that satisfies $\lambda = \varepsilon^{1/3}(\lambda_\star +$*

$o_{\varepsilon \rightarrow 0}(1)$). Furthermore, there holds the asymptotic expansion for the growing mode

$$\begin{pmatrix} (b_{\text{in}}^\varepsilon)_r \\ (b_{\text{in}}^\varepsilon)_\theta \end{pmatrix} = \begin{pmatrix} \varepsilon^{1/3} \sqrt{\frac{-2iM}{r_0^3 \Omega'(r_0)}} \\ 1 \end{pmatrix} \left(e^{-\frac{1}{2}\varepsilon^{-2/3}c_2^{1/2}(r-r_0)^2} + \psi_{\text{err}}^\varepsilon \right)$$

for some explicitly computable constant $c_2 \in \mathbb{C}$ with $\text{Re}(c_2^{1/2}) > 0$, and

$$\sup_{r \in [0, \infty)} \left| \left(1 + (|r - r_0| \varepsilon^{-1/3})^N \right) \psi_{\text{err}}^\varepsilon \right| \lesssim \varepsilon^a,$$

for some $a > 0$ and some $N > 0$ large enough.

Proof. In this proof, we will make explicit the power of the weight w_ε in the definition of $X_{\mathcal{J}}$ by writing

$$X_{\mathcal{J}}^N = L^\infty \left([p, q], \max\{1, r^2\} (1 + (\varepsilon^{-1/3}|r - r_0|)^N dr) \right).$$

Analogously, we do the same with the Banach space Y and denote

$$Y_{\mathcal{J}}^N = L^\infty \left([p, q], r^2 (1 + (\varepsilon^{-1/3}|r - r_0|)^N dr) \right).$$

Given $N \in \mathbb{N}$, consider ε small enough so that $(\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id}$ is contractive on Y_{2N} . We then define the Riesz projector

$$P_{\text{bd}} = \frac{1}{2\pi i} \int_{\Gamma} -(\mathcal{L}^{\text{dyn}} - \lambda)_{\text{bd}}^{-1} d\lambda = QPQ^{-1} + \text{boundary terms},$$

where, following the conventions of this section we abuse notation slightly and set

$$P = \frac{1}{2\pi i} \int_{\gamma} -(\mathcal{L} - \lambda)^{-1} d\lambda,$$

with

$$(\mathcal{L} - \lambda)^{-1} = \mathcal{G}^\lambda \sum_{n \geq 0} (-1)^n ((\mathcal{L} - \lambda)\mathcal{G}^\lambda - \text{Id})^n.$$

We aim to show that there exists ϕ so that $P\phi \neq 0$. From the proof of Theorem 6.1.1 in Section 6.5, it is direct that there exists $a > 0$ such that $\|Q^{-1}(QPQ^{-1})Qf_\star\|_{X_{\mathcal{J}}^N} \gtrsim 1 - \varepsilon^a$, where $f_\star$ is defined in (6.5.3). Therefore, it only remains to deal with the contributions from the boundary. Note that $\|v_i\|_{L^\infty} \lesssim \varepsilon^{1/3}$, so that

$$\|v_i\|_{X_{\mathcal{J}}^N} \lesssim \varepsilon^{1/3} \varepsilon^{-N/3} \max\{|p - r_0|^N, |q - r_0|^N\}.$$

Thus, we estimate

$$\left\| \mathcal{J}^{-1} \begin{pmatrix} ((\mathcal{L}^{\text{dyn}} - \lambda)^{-1}f)^1(p) \\ ((\mathcal{L}^{\text{dyn}} - \lambda)^{-1}f)^1(q) \\ \partial_r(r((\mathcal{L}^{\text{dyn}} - \lambda)^{-1}f)^2(p)) \\ \partial_r(r((\mathcal{L}^{\text{dyn}} - \lambda)^{-1}f)^2(q)) \end{pmatrix} \cdot \begin{pmatrix} v_1(r) \\ v_2(r) \\ v_3(r) \\ v_4(r) \end{pmatrix} \right\|_{X_{\mathcal{J}}^N} \lesssim \varepsilon^{-N/3} \left\| \begin{pmatrix} ((\mathcal{L}^{\text{dyn}} - \lambda)^{-1}f)^1(p) \\ ((\mathcal{L}^{\text{dyn}} - \lambda)^{-1}f)^1(q) \\ \partial_r(r((\mathcal{L}^{\text{dyn}} - \lambda)^{-1}f)^2(p)) \\ \partial_r(r((\mathcal{L}^{\text{dyn}} - \lambda)^{-1}f)^2(q)) \end{pmatrix} \right\|.$$

However, note from our discussion in Section 6.6, and the assumption at the beginning of the proof that $\|(\mathcal{L} - \lambda)^{-1}f\|_{X_{\mathcal{J}}^{2N}} \lesssim \varepsilon^{-1/3}\|f\|_{X_{\mathcal{J}}^{2N}}$, $\|\partial_r(\mathcal{L} - \lambda)^{-1}f\|_{X_{\mathcal{J}}^{2N}} \lesssim \varepsilon^{-1}\|f\|_{X_{\mathcal{J}}^{2N}}$. Recalling that we can recover $(\mathcal{L}^{\text{dyn}} - \lambda)^{-1}$ from $(\mathcal{L} - \lambda)^{-1}$ via conjugation by Q , we thus have

$$\|(\mathcal{L}^{\text{dyn}} - \lambda)^{-1}Qf\|_{X_{\mathcal{J}}^{2N}} \lesssim \varepsilon^{-1/3}\|f\|_{X_{\mathcal{J}}^{2N}}, \quad \|\partial_r(\mathcal{L}^{\text{dyn}} - \lambda)^{-1}Qf\|_{X_{\mathcal{J}}^{2N}} \lesssim \varepsilon^{-1}\|f\|_{X_{\mathcal{J}}^{2N}}.$$

But now, note that $\|f_{\star}\|_{X_{\mathcal{J}}^{2N}} \lesssim 1$. Thus, it holds that

$$\varepsilon^{-N/3}|(\mathcal{L}^{\text{dyn}} - \lambda)^{-1}Qf_{\star}|(p) \leq \frac{\varepsilon^{-N/3}}{w_{\varepsilon}(p - r_0)}\|(\mathcal{L}^{\text{dyn}} - \lambda)^{-1}Qf_{\star}\|_{X_{\mathcal{J}}^{2N}} \lesssim \frac{\varepsilon^{-(1+N)/3}}{1 + \varepsilon^{-2N/3}|r_0 - p|^{2N}} \lesssim \varepsilon^{\frac{N-1}{3}},$$

and the same game can be played both for other endpoint and for the derivatives. Thus, for N large enough, we see that the boundary terms are of order $\varepsilon^{1/3}$ in X_N , and thus it holds

$$\|Q^{-1}PQf_{\star}\|_{X_{\mathcal{J}}^N} \gtrsim 1 - \varepsilon^a - \varepsilon^{2/3} \gtrsim 1,$$

and the result is proven, with the asymptotic expansion being deduced precisely as in the proof of Theorem 6.1.1 for the full space in Section 6.5. $\square$

Remark 6.7.8. To extend the result to other domains with boundaries, the construction used above is entirely sufficient. Indeed, for domains of the form $[0, q]$ or $[p, \infty)$, we only need to “correct” the boundary conditions at $r = q$ and $r = p$ respectively, which can be done using v_2 and v_4 for $r = q$, and v_1 and v_3 for $r = p$. From there, the proof proceeds exactly as for the case $\mathcal{J} = [p, q]$.

6.8 The vertical component

This section is devoted to the completion of the missing bit that is needed for the slow dynamo effect enunciated in Theorem 6.1.1. So far we have shown that the two-dimensional problem (6.3.14)–(6.3.15) has a growing mode with an eigenvalue of the form $\lambda = \mu\varepsilon^{1/3}$. This yields the result for the radial and azimuthal components of the magnetic field B_r and B_{θ} . However, the dynamo problem is three dimensional, and the behaviour of the z component must thus also be addressed. Recall from (6.1.7) that the z component solves a forced advection-diffusion equation, with forcing given by the B_r component of our previously constructed growing mode.

$$\partial_t B_z + (u \cdot \nabla) B_z = \varepsilon \Delta B_z + U'(r) B_r. \quad (6.8.1)$$

Thus, to construct such z component, we need not worry about finding an eigenvalue, but instead it suffices to simply find a finite energy solution to (6.8.1). In other words, we want to invert the operator $\mathcal{L}_z - \lambda$, where, up to a change of the imaginary part of $\lambda \mapsto \lambda + i\varepsilon^{-1/3}(M\Omega(r_0) + KU(r_0))$, we define

$$\mathcal{L}_z = \varepsilon \left(\partial_r^2 + \frac{1}{r} \partial_r - \varepsilon^{-2/3} M^2 - \varepsilon^{-2/3} K^2 \right) - i\varepsilon^{-1/3} (M(\Omega(r) - \Omega(r_0)) + K(U(r) - U(r_0))).$$

Recall that we have showed that

$$\lambda = \varepsilon^{1/3} \left(\eta - M^2 \left[\frac{1}{r_0^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right] + \sqrt{\frac{-2iM\Omega'_0}{r_0}} - c_2^{1/2} \right), \quad (6.8.2)$$

for some η so that $|\operatorname{Re}(\eta c_2^{1/2})| \leq 2$. From here, note that we can proceed in essentially the same way as for the (B_r, B_z) components. For the sake of brevity, we therefore shall only comment on the slight differences we encounter here. We once again subdivide the interval $[0, \infty)$ into a number of intervals depending on the number of zeros of $T(r)$ —four intervals, if $T(r)$ only vanishes at r_0 —, and construct approximate Green's functions for each of them. It will turn out that we may in fact simply use the same approximate inverses as for the B_r, B_θ components. The only real difference occurs in the range $r \sim r_0$, since there the cross terms from Section 6.6.1 disappear when studying the vertical component. Namely, in this regime the approximate operator is given by

$$\mathcal{L}_{z,2} - \lambda = \varepsilon \partial_r^2 - i\varepsilon^{-1/3} c_2 (r - r_0)^2 - \varepsilon^{1/3} \left(M^2 \left[\frac{1}{r_0^2} + \left(\frac{\Omega'_0}{U'_0} \right)^2 \right] + \mu \right),$$

or equivalently

$$\mathcal{L}_{z,2} - \lambda = \varepsilon \partial_r^2 - i\varepsilon^{-1/3} c_2 (r - r_0)^2 - \varepsilon^{1/3} \left(\eta + \sqrt{\frac{-2iM\Omega'(r_0)}{r_0}} - c_2^{1/2} \right).$$

Proceeding as in Section 6.6.1, we see that the homogeneous solutions to this operator are given in terms of the parabolic cylinder functions by

$$\begin{aligned} v_1(r) &= D_{-\frac{1}{2}c_2^{-1/2} \left(\eta + \sqrt{\frac{-2iM\Omega'(r_0)}{r_0}} \right)} \left(\sqrt{2} c_2^{1/4} \varepsilon^{-1/3} (r - r_0) \right), \\ v_2(r) &= D_{-\frac{1}{2}c_2^{-1/2} \left(\eta + \sqrt{\frac{-2iM\Omega'(r_0)}{r_0}} \right)} \left(-\sqrt{2} c_2^{1/4} \varepsilon^{-1/3} (r - r_0) \right). \end{aligned}$$

From here, the estimates proceed exactly as in the case of the B_r and B_θ components, see Section 6.6.1 for further details. All in all, we obtain the following result, where only for this statement, we make explicit the dependence of the Banach space X on the parameter $N \in \mathbb{N}$ by writing

$$X_N = L^\infty \left((0, \infty), \max\{1, r^2\} (1 + (\varepsilon^{-1/3} |r - r_0|)^N) dr \right).$$

Lemma 6.8.1. *Assume H0–H2, and let (B_r, B_θ) be the solution we constructed in the previous sections to the two-dimensional problem (6.6.3)–(6.6.4). Assume further that there exists $N' \in \mathbb{N}$ such that $|U'(r)| \lesssim r^{N'}$ as $r \rightarrow \infty$. Then, letting $\mathcal{R}$ be the compact interval from Lemma 6.3.4, and taking $\mathcal{N} \subset \mathbb{R} \setminus \{0\}$ to be any compact set, for any $N > N'$, and any $\varepsilon > 0$ sufficiently small (depending on N), there exists an operator $\mathcal{G}_z : X_{\mathcal{G}}^N \rightarrow \mathcal{D}(\mathcal{L}_{\mathcal{G}}^{\text{dyn}}) \subset X_{\mathcal{G}}^N$ so that $\mathcal{L}_z \mathcal{G}_z = \text{Id}$. In particular, setting $B_z = \mathcal{G}_z(-U'(r)B_r)$, we find that*

$$B = (B_r, B_\theta, B_z) \in X_{\mathcal{G}}^{N-N'}$$

defines an eigenfunction to the equations (6.1.1) with eigenvalue λ as defined in (6.8.2), and satisfies perfectly conducting boundary conditions in the case where $\partial \mathcal{J} \neq \emptyset$.

The proof of this lemma in the case where $\mathcal{J} = [0, \infty)$ follows from the arguments indicated prior to the statement, and from the observation that if $B_r \in X_N$ and $U' \in X_{\mathcal{J}}^{N'}$, then $f = U'(r)B_r \in X_{\mathcal{J}}^{N-N'}$. When $\partial \mathcal{J} \neq \emptyset$, the only adjustment that needs to be made is to add homogeneous solutions in order to correct the boundary condition for B_z , which reads $B'_z|_{\partial \mathcal{J}} = 0$. But this can be done in precisely the same way as the construction of v_3, v_4 in Section 6.7.

Appendix to Chapter 6

6.8.1 Special functions

In this section we shall lay out the basic theory of the modified Bessel functions, the parabolic cylinder and Hermite functions, as well as the Airy functions, which play a crucial role in the analysis of the approximated operators defined along this chapter. In particular we shall obtain results regarding their asymptotic expansions. Throughout this section, we shall work frequently with complex numbers and their fractional powers. To clarify notation, we thus fix the following convention:

- $\arg(z) \in (-\pi, \pi]$ denotes the *principal argument* of a complex number.
- When writing z^α , for $\alpha \in \mathbb{R} \setminus \mathbb{N}$, we are referring to the *principal value* of z^α , i.e. for $z = |z|e^{i\theta}$, $\theta \in (-\pi, \pi]$, $z^\alpha = |z|^\alpha e^{i\alpha\theta}$.

Modified Bessel functions

The modified Bessel functions arise as solutions to the differential equation

$$z^2 \frac{d^2 v_v}{dz^2} + z \frac{dv_v}{dz} - z^2 v_v - v^2 v_v = 0. \quad (6.8.3)$$

In particular, they can be recovered from solutions to Bessel's equation

$$z^2 \frac{d^2 w_v}{dz^2} + z \frac{dw_v}{dz} - z^2 w_v - v^2 w_v = 0$$

via the change of variables $v_v(z) = w_v(iz)$.

As a second order ODE, the solution space of (6.8.3) is a two-dimensional vector space. In particular, it is customary to pick the basis vectors $I_v(z)$ and $K_v(z)$, which are known as the *modified Bessel functions of the first (resp. second) kind*. For real values of the parameter v and all z in a sector of the form $\{z \in \mathbb{C} : \arg(z) < \pi/2\}$, $I_v(z)$ is bounded at zero, and grows exponentially as $|z| \rightarrow \infty$, whereas $K_v(z)$ diverges near zero, and decays exponentially as $|z| \rightarrow \infty$. In, particular, from [Olv74, Chapter 7.8] there holds

$$K_v(z) \sim \frac{1}{2} \Gamma(v) \left(\frac{z}{2}\right)^{-v}, \quad \text{for } v > 0 \text{ and } z \rightarrow 0,$$

and from the series expansion

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n \geq 0} \frac{(\frac{1}{4}z^2)^n}{n! \Gamma(\nu + n + 1)}$$

it is clear that $I_\nu(z) \sim (\frac{1}{2}z)^\nu$ near zero.

In this work, our particular interests lie in the asymptotic behaviour of $I_\nu(z)$, $K_\nu(z)$ as $\nu \rightarrow \infty$. We therefore require bounds on these functions in the limit $\nu \rightarrow \infty$ which hold *uniformly* for all values of z in some complex sector as specified above. Fortunately, such bounds have been constructed in [Olv74, Chapter 10.7], and we aim to give a brief summary here of the results therein. In particular, since the bounds we require for the modified Bessel functions are the most subtle of all the special function bounds we employ, we shall sketch the proof of the following result.

Lemma 6.8.2. *There exists a constant $C > 0$ so that uniformly in $z \in \mathbb{C}$ with $\arg(z) = \pi/4$, and for all $\nu \geq 1$ there holds*

$$\begin{aligned} |I_\nu(\nu z)| &\leq C \frac{e^{\nu \xi}}{(2\pi\nu)^{1/2} (1+z^2)^{1/4}}, \\ |K_\nu(\nu z)| &\leq C \left(\frac{\pi}{2\nu}\right)^{1/2} \frac{e^{-\nu \xi}}{(1+z^2)^{1/4}}, \end{aligned}$$

where we have set

$$\xi(z) = (1+z^2)^{1/2} + \log \left(\frac{z}{1+(1+z^2)^{1/2}} \right).$$

Remark 6.8.3. The above result is actually valid in a much larger context, namely such a constant can be found uniformly for any sector of the form $|\arg(z)| \leq \pi/2 - \delta$ for arbitrarily small $\delta > 0$, see [Olv74, Chapter 10.7]. However, since for our applications we only require $|\arg(z)| = \pi/4$, we shall only prove the result in this simplified case.

We start by noting that in [Olv74, Chapter 10.7.4] it is shown that for any $\nu > 0$, $|\arg(z)| < \pi/2$, there holds for any $n \in \mathbb{N}$

$$I_\nu(\nu z) = \frac{1}{1 + \eta_{n,1}(\nu, \infty)} \frac{e^{\nu \xi}}{(2\pi\nu)^{1/2} (1+z^2)^{1/4}} \left(\sum_{s=0}^{n-1} \frac{U_s(p)}{\nu^s} + \eta_{n,1}(\nu, z) \right), \quad (6.8.4)$$

$$K_\nu(\nu z) = \left(\frac{\pi}{2\nu}\right)^{1/2} \frac{e^{-\nu \xi}}{(1+z^2)^{1/4}} \left(\sum_{s=0}^{n-1} (-1)^s \frac{U_s(p)}{\nu^s} + \eta_{n,2}(\nu, z) \right), \quad (6.8.5)$$

where $p = (1+z^2)^{-1/2}$, and the $U_s(p)$ are polynomials defined recursively by

$$U_{s+1}(p) = \frac{1}{2} p^2 (1-p^2) U'_s(p) + \frac{1}{8} \int_0^p (1-5q^2) U_s(q) dq$$

and $U_0(p) = 1$. Finally, the error terms $\eta_{n,j}(\mathbf{v}, z)$ satisfy

$$|\eta_{n,1}(\mathbf{v}, z)| \leq 2 \exp\left(\frac{2\mathcal{V}_{1,p}(U_1)}{\mathbf{v}}\right) \frac{\mathcal{V}_{1,p}(U_n)}{\mathbf{v}^n}$$

$$|\eta_{n,2}(\mathbf{v}, z)| \leq 2 \exp\left(\frac{2\mathcal{V}_{0,p}(U_1)}{\mathbf{v}}\right) \frac{\mathcal{V}_{0,p}(U_n)}{\mathbf{v}^n}$$

where $\mathcal{V}_{1,p}$ (resp. $\mathcal{V}_{0,p}$) denotes the 1-variation from 1 (resp. from 0) to p along so called ξ -progressive paths. It thus remains to derive uniform bounds on the variational terms. To do so, it is first imperative to define the notion of a ξ -progressive path. Indeed, we have the following definition from [Olv74, Chapter 6.11.4].

Definition 6.8.4. Let $s : [0, 1] \rightarrow \mathbb{C}$ be a twice differentiable function so that

1. $s''(\tau)$ is continuous
2. $s'(\tau) \neq 0$ for all $\tau \in [0, 1]$.

Then, s is called an R_2 -arc. Let now $\mathcal{P}_j$, $j = 1, 2$ be paths in the complex plane connecting two points $a_j, b_j \in \mathbb{C}$, and let $\xi : \mathbb{C} \rightarrow \mathbb{C}$ be a conformal map. $\mathcal{P}_j$ is called a ξ -progressive path if

1. $\mathcal{P}_j$ consists of a finite number of R_2 -arcs
2. As t passes along $\mathcal{P}_j$ from b_j to a_j , $\operatorname{Re}(\xi(t))$ is non-decreasing for $j = 1$, and non-increasing for $j = 2$.

With this in mind, we now provide a proof of Lemma 6.8.2.

Proof of Lemma 6.8.2. It is clear that it suffices to bound $\mathcal{V}_{1,p}(U_1)$ uniformly for $|\arg(z)| = \pi/4$. For simplicities sake, we shall assume that $\arg(z) = \pi/4$, since the general case follows similarly. First, note that we may compute

$$U_1(p) = \frac{1}{8} \left(p - \frac{5}{3} p^3 \right).$$

Next, we consider the image of $\arg(z) = \pi/4$ under the map $p(z) = (1 + z^2)^{-1/2}$. Letting $z = ae^{i\pi/4}$, $a > 0$, we have

$$p(ae^{i\pi/4}) = \left(\frac{1 - ia^2}{1 + a^4} \right)^{1/2}.$$

Next, consider

$$\xi(a) = \left(\frac{2 + a^4 - ia^2}{1 + a^4} \right)^{1/2} + \log \left(\frac{p(ae^{i\pi/4})}{1 + \left(\frac{2 + a^4 - ia^2}{1 + a^4} \right)^{1/2}} \right).$$

A computation shows that $\operatorname{Re}(\xi(a))$ is in fact a non-increasing function in a for positive values of a . Therefore, given any point $z = ae^{i\pi/4}$, ξ -progressive paths from 1 (resp. 0) to z can be constructed simply by taking $\gamma(t) = p(te^{i\pi/4})$. Given any smooth function $f : \mathbb{C} \rightarrow \mathbb{C}$, it therefore

holds

$$\mathcal{V}_{1,p}(f) \leq \int_0^\infty \left| \frac{it}{(1+it^2)^{3/2}} f' \left(\frac{1}{1+it^2} \right) \right| dt \leq \|f'\|_{L^\infty(|z|\leq 1)} \int_0^\infty \frac{t}{(1+t^4)^{3/4}} dt \leq C \|f'\|_{L^\infty(|z|\leq 1)}.$$

Similarly, it also holds $\mathcal{V}_{0,p}(f) \leq C \|f'\|_{L^\infty(|z|\leq 1)}$, uniformly for all $\arg(z) = \pi/4$. Therefore, we observe that the error bounds satisfy

$$\begin{aligned} |\eta_{1,1}(v, z)| &\leq 2v^{-1} C \exp(2v^{-1} C \|U_1'\|_{L^\infty(|z|\leq 1)}) \|U_1'\|_{L^\infty(|z|\leq 1)}, \\ |\eta_{1,2}(v, z)| &\leq 2v^{-1} C \exp(2v^{-1} C \|U_1'\|_{L^\infty(|z|\leq 1)}) \|U_1'\|_{L^\infty(|z|\leq 1)}, \end{aligned}$$

and so the result follows immediately from (6.8.4)–(6.8.5). $\square$

Parabolic cylinder and Hermite functions

In this section, we summarise basic facts about parabolic cylinder and Hermite functions. Most of this material may be found in [Leb72, Chapter 10.6]. We consider the solutions to the second order differential equation

$$\frac{d^2 w_v}{dz^2} + \left(v + \frac{1}{2} - \frac{z^2}{4} \right) w_v = 0.$$

The solution space of this equation is spanned by the functions $D_v(z)$, $D_v(-z)$, where we call $D_v(z)$ a parabolic cylinder function. Making the change of variables

$$w_v = e^{-\frac{1}{4}z^2} u_v \left(\frac{z}{\sqrt{2}} \right),$$

yields that u_v satisfies the ODE

$$\frac{d^2 u_v}{dz^2} - 2z \frac{du_v}{dz} + 2v u_v = 0, \quad (6.8.6)$$

which is called the *Hermite's equation*. Solutions to (6.8.6) are called Hermite functions, and in particular, the solution space is spanned by $H_v(z)$, $H_v(-z)$ so long as $v \neq 0, 1, 2, \dots$. In the case where $v = n = 0, 1, 2, \dots$, H_n corresponds to the n^{th} Hermite polynomial. The Hermite functions enjoy the following pointwise asymptotics, which may be found in [Leb72, Chapter 10.6].

$$H_v(z) = (2z)^v (1 + \mathcal{O}(z^{-2})), \quad \text{if } \arg(z) \in \left(-\frac{3\pi}{4}, \frac{3\pi}{4} \right), \quad (6.8.7)$$

$$H_v(z) = ((2z)^v - C(v)e^{z^2})(1 + \mathcal{O}(z^{-2})), \quad \text{if } \arg(z) \in \left(\frac{\pi}{4}, \frac{5\pi}{4} \right). \quad (6.8.8)$$

(in (6.8.8) we have made the slight abuse of notation by writing $\arg(z) \in (\frac{\pi}{4}, \frac{5\pi}{4})$ to mean $\arg(z) \in (\frac{\pi}{4}, \pi] \cup (-\pi, -\frac{3\pi}{4})$). With these asymptotic representations under our belt, we may now prove the following pointwise bounds for the parabolic cylinder functions.

Lemma 6.8.5. *For any fixed $v \neq 0, 1, 2, \dots$, there exists a constant $C > 0$ so that*

$$|D_v(z)| \leq C \left| e^{-\frac{1}{4}z^2} \right| (1 + |z|)^{\text{Re}(v)}, \quad \text{if } |\arg(z)| < \frac{3\pi}{4}, \quad (6.8.9)$$

$$|D_\nu(z)| \leq C \left| e^{\frac{1}{4}z^2} \right|, \quad \text{if } \arg(z) \in \left(\frac{3\pi}{4}, \frac{5\pi}{4} \right). \quad (6.8.10)$$

Proof. We begin by noting that we have the following relation

$$D_\nu(z) = 2^{-\nu/2} e^{-\frac{1}{4}z^2} H_\nu \left(\frac{z}{\sqrt{2}} \right).$$

On the one hand, we know that $H_\nu(z)$ is a continuous function in $z \in \mathbb{C}$, and moreover $|H_\nu(0)| \lesssim 1$ for any $\nu \in \mathbb{C}$. On the other hand, we may now use (6.8.7), to deduce that for $|\arg(z)| \leq 3\pi/4$, it holds

$$|H_\nu(z)| \lesssim (1+|z|)^\gamma (1+(1+|z|)^{-2}) \lesssim (1+|z|)^\gamma,$$

and hence we obtain (6.8.9). Finally, in order to prove (6.8.10) we use (6.8.8) in a similar way. This time we write

$$\left| e^{-\frac{1}{4}z^2} H_\nu \left(\frac{z}{\sqrt{2}} \right) \right| \lesssim \max \left\{ \left| e^{-\frac{1}{4}z^2} \right| (1+|z|)^\gamma, \left| e^{\frac{1}{4}z^2} \right| \right\} (1+(1+|z|)^{-2}) \lesssim \left| e^{\frac{1}{4}z^2} \right|.$$

At this point we observe that if $|z| \sim 1$ then both quantities in the maximum operator are comparable. However, for values of z with $|z| \rightarrow \infty$ we have

$$\left| e^{-\frac{1}{4}z^2} \right| (1+|z|)^\gamma \lesssim \left| e^{\frac{1}{4}z^2} \right|$$

provided that $\operatorname{Re}(z)^2 \geq \Im(z)^2$, or equivalently

$$\arg(z) \in \left[-\frac{\pi}{4}, \frac{\pi}{4} \right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4} \right].$$

Since the first interval is not compatible with equation (6.8.8), we discard it, and hence we find the sought estimate (6.8.10). $\square$

Airy functions

We finally give a brief introduction to the theory of Airy functions. These are solutions to the second order differential equation

$$\frac{d^2 v}{dz^2} - zv = 0. \quad (6.8.11)$$

The typical convention (see e.g. [Olv74]) is to make the choice of basis vectors $\operatorname{Ai}(z)$, $\operatorname{Bi}(z)$ for the solution space of (6.8.11). With this choice, for real values of x , $\operatorname{Ai}(x)$ decays exponentially for $x > 0$ and oscillates for $x < 0$, whereas $\operatorname{Bi}(x)$ grows exponentially for $x > 0$ and oscillates for $x < 0$. However, for the applications we have in mind, we are primarily interested in *complex* values of z . For such values, it turns out (see [Olv74, Chapter 11.8]) that $\operatorname{Bi}(z)$ is exponentially large, as long as $\arg(z) \neq \pi$. In order to derive a suitable Green's function for Section 6.6.5, we require that at all times, at least one of the solutions to (6.8.11) remains small. Therefore, we shall pick the function $\operatorname{Ai}(e^{2\pi i/3}z)$ as a second solution of (6.8.11) instead of $\operatorname{Bi}(z)$. To verify that indeed $\operatorname{Ai}(z)$, $\operatorname{Ai}(e^{2\pi i/3}z)$ are linearly independent, we examine the asymptotic bounds for the Airy functions

$\text{Ai}(z)$. Indeed, from [Olv74, Chapter 11.8] as well as [Olv+20] we set

$$\zeta = \frac{2}{3}z^{3/2},$$

where $z^{3/2}$ denotes the *principal branch* of $z^{3/2}$. Then, we have the asymptotic expansion

$$\begin{aligned} \text{Ai}(z) &\sim \frac{e^{-\zeta}}{2\sqrt{\pi}z^{1/4}} \sum_{n \geq 0} (-1)^n \frac{u_n}{\zeta^n}, \quad |\arg(z)| \leq \pi - \delta, \\ \text{Ai}'(z) &\sim -\frac{z^{1/4}e^{-\zeta}}{2\sqrt{\pi}} \sum_{n \geq 0} (-1)^n \frac{v_n}{\zeta^n}, \quad |\arg(z)| \leq \pi - \delta, \end{aligned} \quad (6.8.12)$$

where $u_0 = v_0 = 1$, and recursively

$$u_k = \frac{(6k-5)(6k-3)(6k-1)}{(2k-1)216k} u_{k-1}, \quad v_k = \frac{6k+1}{1-6k} u_k.$$

From this, we deduce the following lemma.

Lemma 6.8.6. *Let $z = re^{i\frac{\pi}{6}}$. Then, there exists a constant $C > 0$ uniform in $r \in [0, \infty)$ such that the following bounds hold true,*

$$\begin{aligned} |\text{Ai}(z)|, |\text{Ai}(-e^{2\pi i/3}z)| &\leq Ce^{-\frac{\sqrt{2}}{3}r^{3/2}}, \\ |\text{Ai}(-z)|, |\text{Ai}(e^{2\pi i/3}z)| &\leq Ce^{\frac{\sqrt{2}}{3}r^{3/2}}, \\ |\text{Ai}'(z)|, |\text{Ai}'(-e^{2\pi i/3}z)| &\leq C(1+r^{1/4})e^{-\frac{\sqrt{2}}{3}r^{3/2}}, \\ |\text{Ai}'(-z)|, |\text{Ai}'(e^{2\pi i/3}z)| &\leq C(1+r^{1/4})e^{\frac{\sqrt{2}}{3}r^{3/2}}. \end{aligned} \quad (6.8.13)$$

Proof. We begin by understanding how to truncate the asymptotic series (6.8.12). From [Olv+20], we may truncate them at degree N , up to introducing an error of order $|u_N||\zeta|^{-N}$ multiplied by

1. 1, if $|\arg(z)| < \pi/3$,
2. $\min\{\csc(\arg(\zeta)), \chi(N+\sigma)+1\}$ if $\pi/3 < |\arg(z)| < 2\pi/3$,
3. $\sqrt{2\pi(N+\sigma)}|\sec(\arg(\zeta))|^{N+\sigma} + \chi(N+\sigma)+1$ if $2\pi/3 < |\arg(z)| < \pi$.

Here we set,

$$\chi(x) = \pi^{1/2} \frac{\Gamma(\frac{1}{2}x+1)}{\Gamma(\frac{1}{2}x+\frac{1}{2})},$$

$$N \geq 0, \quad \sigma = \frac{1}{6} \quad \text{for the bounds on } \text{Ai},$$

$$N \geq 1, \quad \sigma = 0 \quad \text{for the bounds on } \text{Ai}'.$$

It is easy to see that, as long as $\arg(z)$ is not near $\pm\pi$, $\sec(\arg(\zeta))$ is bounded. Hence, we see that

$$\begin{aligned} |\text{Ai}(z)| &\leq C \left| \frac{e^{-\zeta}}{2\sqrt{\pi}z^{1/4}} \right|, \quad |\arg(z)| \leq \pi - \delta, \\ |\text{Ai}'(z)| &\leq C \left| \frac{z^{1/4}e^{-\zeta}}{2\sqrt{\pi}} \right|, \quad |\arg(z)| \leq \pi - \delta. \end{aligned} \tag{6.8.14}$$

Furthermore, $\text{Ai}(z), \text{Ai}'(z)$ are always bounded near zero (see [Olv74]), so to deduce (6.8.13) from (6.8.14), it suffices to study the behaviour at infinity. In particular, we will check if the choice of $\sqrt{2}/3$ in the exponential indeed works for all cases. For $r > 0$, the principal value of $(re^{i\pi/6})^{3/2}$ is given by $r^{3/2}e^{i\pi/4}$. Thus, in this case the real part of $e^{i\pi/4}$ is indeed $\sqrt{2}/2$, so

$$\text{Re}(\zeta) = \frac{2}{3}\text{Re}(z^{2/3}) = \frac{\sqrt{2}}{3}r.$$

Similarly, for $-e^{2\pi i/3}e^{i\pi/6}r = e^{-i\pi/6}r$, we compute $(e^{-i\pi/6}r)^{3/2} = r^{3/2}e^{-i\pi/4}$, so once again $\sqrt{2}/2$ is the relevant coefficient of the real part. Next, we consider $\text{Ai}(-re^{i\pi/6})$. Here, we have

$$(-re^{i\pi/6})^{3/2} = (re^{-5\pi i/6})^{3/2} = e^{3\pi i/4}r^{3/2}.$$

In particular, we deduce that the exponential has real part $-\sqrt{2}/2$. Finally, we have that

$$(e^{2\pi i/3}e^{i\pi/6}r)^{3/2} = e^{-3\pi i/4}r^{3/2},$$

and so again the choice of $-\sqrt{2}/2$ works. Crucially, as stated in [Olv+20], we may truncate the asymptotic expansions (6.8.12) at degree N , introducing an error of order $u_N\zeta^{-N}$, as long as $|\arg(z)| \leq \pi/3$. Thus, for all $|z|$ sufficiently large, we deduce the bounds required for our purposes in Section 6.6.5. $\square$

6.8.2 The operator around the critical radius

In this appendix we will show that the approximated operator $\mathcal{L}_2$ around the critical radius r_0 has an isolated eigenvalue and an eigenfunction of the form stated in the proof of Theorem 6.5.1. Since this is a key step in the derivation of the profiles of the growing mode for the operator $\mathcal{L}$, we give more details about the derivation of this result in the following proposition.

Proposition 6.8.7. *The operator*

$$\mathcal{L}_2 = \varepsilon \partial_r^2 - c_2 \varepsilon^{-1/3} (r - r_0)^2 + \varepsilon^{1/3} \left(-M^2 \left[\frac{1}{r_0^2} + \frac{\Omega'_0}{U'_0} \right] + \sqrt{\frac{-2iM\Omega'_0}{r_0}} \right)$$

has an eigenfunction in $L^\infty(\mathbb{R}, \max\{1, r^2\}w_\varepsilon(r - r_0)dr)$ of the form

$$f_\star(r) = e^{-\frac{1}{2}\varepsilon^{-2/3}c_2^{1/2}(r-r_0)^2},$$

with eigenvalue

$$\lambda_\star = \varepsilon^{1/3} \left[-M^2 \left(\frac{1}{r_0^2} + \frac{\Omega'_0}{U'_0} \right) + \sqrt{\frac{-2iM\Omega'_0}{r_0}} - c_2^{1/2} \right],$$

where $c_2^{1/2}$ represents a complex number with positive real part. In fact, this is the spectral value of $\mathcal{L}_2$ with largest real part. Moreover, the eigenvalue $\lambda_\star$ is isolated.

Proof. First of all, a direct computation shows that

$$f_\star''(r) = \varepsilon^{-2/3} c_2^{1/2} \left(\varepsilon^{-2/3} c_2^{1/2} (r - r_0)^2 - 1 \right) f_\star(r),$$

so that plugging it into the definition of the operator, we find

$$\varepsilon f_\star''(r) - c_2 \varepsilon^{-1/3} (r - r_0)^2 f_\star(r) = -\varepsilon^{1/3} c_2^{1/2} f_\star(r),$$

and therefore,

$$\mathcal{L}_2 f_\star(r) = \varepsilon^{1/3} \left[-M^2 \left(\frac{1}{r_0^2} + \frac{\Omega'_0}{U'_0} \right) + \sqrt{\frac{-2iM\Omega'_0}{r_0}} - c_2^{1/2} \right] f_\star(r).$$

Next, we prove that the eigenvalue is indeed isolated and the largest one of $\mathcal{L}_2$ in real part. Indeed, most of the work for this has already been done in Section 6.6.1. Recall that for $\mu = \lambda_\star + \varepsilon^{1/3} \eta$, $\eta \neq 0$ and $|\eta|$ small enough, there exist two linearly independent solutions to $(\mathcal{L}_2 - \mu)f = 0$ given by

$$\begin{aligned} v_1(r) &= D_{\frac{1}{2}c_2^{-1/2}\eta} \left(\sqrt{2} c_2^{1/4} \varepsilon^{-1/3} (r - r_0) \right), \\ v_2(r) &= D_{\frac{1}{2}c_2^{-1/2}\eta} \left(-\sqrt{2} c_2^{1/4} \varepsilon^{-1/3} (r - r_0) \right), \end{aligned}$$

where

$$D_\nu(z) = 2^{-\nu/2} e^{-z^2/4} H_\nu \left(\frac{z}{\sqrt{2}} \right).$$

Therefore, one can construct a Green's function for this equation given by

$$(\mathcal{G}_{r_0} f)(r) = \mathfrak{w}(\eta) \varepsilon^{-2/3} v_1(r) \int_0^r v_2(s) f(s) ds + \mathfrak{w}(\eta) \varepsilon^{-2/3} v_2(r) \int_r^\infty v_1(s) f(s) ds.$$

where $\mathfrak{w}(\eta)$ is the Wronskian. Recall the bounds on the Hermite functions

$$|D_\nu(z)| \lesssim \left| e^{-z^2/4} \right| (1 + |z|)^{\operatorname{Re}(\nu)}, \quad \text{if } |\arg(z)| < \frac{3\pi}{4},$$

$$|D_\nu(z)| \lesssim \left| e^{z^2/4} \right|, \quad \text{if } \arg(z) \in \left(\frac{3\pi}{4}, \frac{5\pi}{4} \right),$$

In particular, as long as $\operatorname{Re}(\nu) < 1$, we can redo the arguments in Section 6.6.1 to show that $\mathcal{G}_{r_0}$ is a well defined operator on $L^\infty(\mathbb{R}, \max\{1, r^2\} w_\varepsilon(r - r_0) dr)$, which acts as a right inverse to $\mathcal{L}_2$. Hence, surjectivity of $\mathcal{L}_2 - \lambda$ both for λ near $\lambda_\star$, and also for any λ with $\operatorname{Re}(\lambda) > \operatorname{Re}(\lambda_\star)$ is proven. Finally, injectivity is an immediate consequence of the asymptotic properties of the

functions D_v , which are bounded on $\mathbb{R}$ if and only if $v = 0, 1, 2, \dots$. This completes the proof. $\square$

6.8.3 Properties of the weight function

In this final section, we state a lemma that compiles a set of useful properties of the weight function w_ε , which appears in the definitions of the Banach spaces X and Y . These properties play an important role in the proofs of Sections 6.6 and 6.6.5. For the convenience of the reader—and to avoid repetitive arguments in the course of those longer proofs—we have chosen to collect all relevant facts about w_ε in one place here.

Recall that the weight function is defined for any $\varepsilon > 0$ and any fixed $N \in \mathbb{N}$ as the following polynomial of degree N ,

$$w_\varepsilon(s) = 1 + (\varepsilon^{-1/3}|s|)^N.$$

We now present the following set of properties of the weight function.

Lemma 6.8.8. *The weight $w_\varepsilon(\cdot)$ satisfies the following properties for any $\varepsilon > 0$ sufficiently small.*

1. *Let $\gamma, \chi > 0$ and $\vartheta \in \mathbb{R}$ be fixed. For any $|r - r_0| \geq \varepsilon^\gamma$ there holds the estimate*

$$\frac{w_\varepsilon(r - r_0)}{w_\varepsilon(\varepsilon^\gamma)} \frac{(1 + \varepsilon^{-1/3}|r - r_0|)^\vartheta}{(1 + \varepsilon^{-1/3+\gamma})^\vartheta} \leq C e^{\frac{1}{2}\chi \varepsilon^{-2/3}((r-r_0)^2 - (\varepsilon^\gamma)^2)},$$

where $C > 0$ is a uniform constant that depends continuously on N , ϑ , and χ .

2. *Fix $a > b \geq 0$, let $r_0 < s \leq r$ (analog. $r \leq s < r_0$) be such that $|r - s| \leq \varepsilon^a$ and assume that $|s - r_0| \geq \varepsilon^b$. Then,*

$$\frac{w_\varepsilon(r - r_0)}{w_\varepsilon(s - r_0)} \leq C$$

where $C > 0$ is a uniform constant, independent of all parameters.

3. *Let $a > 0$, for any $r \geq \varepsilon^a$ there holds the estimate*

$$\frac{w_\varepsilon(r - r_0)}{w_\varepsilon(\varepsilon^a - r_0)} \leq \left(\frac{r}{\varepsilon^a} \right)^N.$$

4. *For any $\gamma > 0$, and $s, r \geq 0$ such that $|s - r_0| \geq \varepsilon^\gamma$ there holds*

$$\frac{w_\varepsilon(r - r_0)}{w_\varepsilon(s - r_0)} \leq C e^{\varepsilon^{-\gamma}|s-r|}.$$

where $C > 0$ is a uniform constant that only depends on N .

Proof. (1) Let us define $x = \varepsilon^{-1/3}|r - r_0|$ and $y = \varepsilon^{-1/3+\gamma}$, which satisfies $x \geq y > 1$ since $\gamma < 1/3$ by assumption. With this notation we have

$$\frac{w_\varepsilon(r - r_0)}{w_\varepsilon(\varepsilon^\gamma)} \frac{(1 + \varepsilon^{-1/3}|r - r_0|)^\vartheta}{(1 + \varepsilon^{-1/3+\gamma})^\vartheta} = \frac{(1 + x^N)(1 + x)^\vartheta}{(1 + y^N)(1 + y)^\vartheta}.$$

Now observe that if $\vartheta \geq 0$ we have that $(1 + x)^\vartheta \leq (2x)^\vartheta$ and $(1 + y)^\vartheta \geq (y)^\vartheta$, whereas if $\vartheta < 0$

we have that $(1+x)^\vartheta \leq (x)^\vartheta$ and $(1+y)^\vartheta \geq (2y)^\vartheta$. Therefore, for any $\vartheta \in \mathbb{R}$ we can write

$$\frac{(1+x)^\vartheta}{(1+y)^\vartheta} \leq 2^{|\vartheta|} \left(\frac{x}{y}\right)^\vartheta,$$

and moreover

$$\frac{(1+x^N)(1+x)^\vartheta}{(1+y^N)(1+y)^\vartheta} \leq 2^{1+|\vartheta|} \left(\frac{x}{y}\right)^{N+\vartheta}.$$

This in turn implies that there exists a constant $C > 0$ depending exclusively on $N \in \mathbb{N}$ and $\vartheta \in \mathbb{R}$ such that the following estimate

$$\frac{(1+x^N)(1+x)^\vartheta}{(1+y^N)(1+y)^\vartheta} \leq C e^{\frac{1}{2}(x^2 y^{-2} - 1)} = C e^{\frac{1}{2} y^{-2}(x^2 - y^2)} \leq C e^{\frac{1}{2}(x^2 - y^2)}$$

holds true uniformly in $x, y \geq 1$, and thus the claim of the lemma follows.

(2) Take $r_0 < s \leq r$, since $|r-s| \leq \varepsilon^a$ and $|s-r_0| \geq \varepsilon^b$, we can write via triangle inequality

$$\frac{w_\varepsilon(r-r_0)}{w_\varepsilon(s-r_0)} \leq 2 \frac{(\varepsilon^{-1/3}|r-r_0|)^N}{(\varepsilon^{-1/3}|s-r_0|)^N} \leq 2 \left(1 + \frac{|r-s|^N}{|s-r_0|^N}\right) \leq 2(1 + \varepsilon^{N(a-b)}).$$

Thus, the right hand side can be uniformly bounded by 4 provided that $a > b$. The case $r \leq s < r_0$ follows by symmetry.

(3) Take $r \geq \varepsilon^a$ and write the trivial bound

$$\frac{w_\varepsilon(r-r_0)}{w_\varepsilon(\varepsilon^a-r_0)} \leq \frac{1 + (\varepsilon^{-1/3}|r-r_0|)^N}{(\varepsilon^{-1/3}|\varepsilon^a-r_0|)^N}.$$

On the one hand, the numerator can be estimated by

$$1 + (\varepsilon^{-1/3}|r-r_0|)^N \lesssim (\varepsilon^{-1/3}r)^N + (\varepsilon^{-1/3}r_0)^N \lesssim \varepsilon^{-(\frac{1}{3}-a)N} \left(\frac{r}{\varepsilon^a}\right)^N + \varepsilon^{-\frac{1}{3}N}.$$

On the other hand, for the denominator we simply use that $|\varepsilon^a - r_0| \gtrsim r_0$ to write

$$(\varepsilon^{-1/3}|\varepsilon^a - r_0|)^N \gtrsim \varepsilon^{-\frac{1}{3}N},$$

so that combining both estimates we obtain

$$\frac{w_\varepsilon(r-r_0)}{w_\varepsilon(\varepsilon^a-r_0)} \lesssim \varepsilon^{aN} \left(\frac{r}{\varepsilon^a}\right)^N + 1 \lesssim \left(\frac{r}{\varepsilon^a}\right)^N$$

where the last inequality comes from the observation that $r \geq \varepsilon^a$.

(4) We start by estimating

$$\frac{w_\varepsilon(r-r_0)}{w_\varepsilon(s-r_0)} \leq \frac{1 + (\varepsilon^{-1/3}|s-r_0|)^N + (\varepsilon^{-1/3}|s-r|)^N}{1 + (\varepsilon^{-1/3}|s-r_0|)^N} \leq 1 + \frac{(\varepsilon^{-1/3}|s-r|)^N}{1 + (\varepsilon^{-1/3}|s-r_0|)^N}.$$

Now, since $|s - r_0| \geq \varepsilon^\gamma$ we get

$$1 + \frac{(\varepsilon^{-1/3}|s - r|)^N}{1 + (\varepsilon^{-1/3}|s - r_0|)^N} \leq 1 + \frac{|s - r|^N}{|s - r_0|^N} \leq 1 + \varepsilon^{-\gamma N}|s - r|^N \leq C e^{\varepsilon^{-\gamma}|s - r|}.$$

where for the last inequality we used the fact that there exists a constant $C > 0$ only depending on N so that $1 + x^N \leq C e^x$. □

Chapter 7

Nonlinear instability for the 3D MHD equations around the Taylor–Couette flow.

In this chapter, we finally connect the linear instability analysis of the kinematic dynamo equations back to the non-linear theory of the MHD equations (1.1.1). In particular, we outline a linear-to-nonlinear instability argument, to deduce nonlinear instability for the 3d MHD equations around the Taylor–Couette flow, which was proven to be a dynamo in Chapter 6. The contents of this chapter are contained in the paper [NV25a].

7.1 Introduction

Consider the three-dimensional incompressible magnetohydrodynamics (MHD) equations (1.1.1) describing the evolution of a magnetic field $B(t, x) \in \mathbb{R}^3$ generated by an electrically charged fluid moving with velocity $u(t, x) \in \mathbb{R}^3$, which we recall here

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u + (B \cdot \nabla)B, \\ \partial_t B + (u \cdot \nabla)B - (B \cdot \nabla)u = \varepsilon \Delta B, \\ \operatorname{div}(u) = 0, \\ \operatorname{div}(B) = 0, \end{cases} \quad (7.1.1)$$

with $(t, x) \in (0, \infty) \times \mathcal{M}$. Here $p(t, x)$ denotes the pressure of the fluid, $\nu > 0$ its kinematic viscosity, and $\varepsilon > 0$ the magnetic resistivity. The fluid is contained in a periodic annular cylinder, that we denote by $\mathcal{M} \subset \mathbb{R}^3$, which in cylindrical coordinates is defined in terms of the inner and outer radii $0 < R_1 < R_2$ by

$$\mathcal{M} = \{(r, \theta, z) \in (0, \infty) \times \mathbb{T} \times \mathbb{T} : R_1 \leq r \leq R_2\}. \quad (7.1.2)$$

The boundaries of this annular cylinder are physical—thus, no fluid can go across—rotate with a constant speed, and move vertically at a constant relative velocity. Upon picking a suitable reference frame, we can always assume that the cylinder corresponding to the boundary $r = R_1$ is stationary, whereas the cylinder corresponding to $r = R_2$ is moving. This, on the one hand, translates into the following boundary conditions for the velocity field

$$u \cdot \hat{n}|_{\partial \mathcal{M}} = 0, \quad u_\theta|_{r=R_1} = u_z|_{r=R_1} = 0, \quad u_\theta|_{r=R_2} = \beta_1, \quad u_z|_{r=R_2} = \beta_2, \quad (7.1.3)$$

where $\beta_1, \beta_2 \in \mathbb{R} \setminus \{0\}$, and $\hat{n}$ denotes the unit normal outwards vector to the boundary $\partial \mathcal{M}$. On the other hand, for the magnetic field we consider the boundary conditions of a perfectly conducting wall (2.1.1), which are modeled by

$$B \cdot \hat{n}|_{\partial \mathcal{M}} = 0, \quad \hat{n} \times (\nabla \times B)|_{\partial \mathcal{M}} = 0. \quad (7.1.4)$$

Provided that we take some initial configuration (u_0, B_0) in a suitable Banach space, e.g. $L_\omega^p(\mathcal{M})$ —the space of divergence-free vector valued functions in $L^p(\mathcal{M})$ —we find that (7.1.1)–(7.1.4) defines a locally well-posed Cauchy problem, see [DL72; ST83]. For further information about the MHD equations and physically relevant boundary conditions, we refer to the monographs [Rob67; Dav01].

In this setting, we introduce a special stationary solution to equations (7.1.1) with the boundary conditions (7.1.3)–(7.1.4). We denote this solution by $(u_{\text{TC}}, 0)$, it consists of the trivial magnetic field $B = 0$, together with the three-dimensional *Taylor–Couette* radial vector field, which in cylindrical coordinates has the form

$$u_{\text{TC}}(r) = \left(\frac{a_1}{r} + a_3 r \right) \hat{\theta} + (a_2 \log(r) + a_4) \hat{z}. \quad (7.1.5)$$

The coefficients $a_1, a_2, a_3, a_4 \in \mathbb{R}$, with $a_1, a_2 \neq 0$, are chosen to satisfy the boundary conditions (7.1.3). Given any viscosity parameter $\nu > 0$, the Taylor–Couette velocity field (7.1.5) is an explicit solution to the Navier–Stokes equations in $\mathcal{M}$ with boundary conditions (7.1.3), without forcing, and with a radial pressure term p_{TC} given by

$$\partial_r p_{\text{TC}}(r) = \frac{1}{r} (u_{\text{TC}})_\theta^2 = \frac{a_1^2}{r^3} + a_3^2 r.$$

The main goal of this chapter is to show that the MHD equations (7.1.1) are *nonlinearly unstable* around the steady state $(u_{\text{TC}}, 0)$ in L^p for every $p \in (1, \infty)$. Nonlinear instability in this context refers to the existence of an initial configuration (u_0, B_0) arbitrarily close to the steady state in a suitable Banach space, and a time $t_\star > 0$ at which the corresponding solution $(u(t_\star), B(t_\star))$ has drifted a distance of order 1 from the steady state. In our particular case we can allow for perturbations around $(u_{\text{TC}}, 0)$ that are small in any Sobolev space $W^{s,p}$ with $s \geq 0$ and $p \in (1, \infty)$, and we find a solution (u, B) that departs from the steady state in L^p . Our line of reasoning exploits the recent linear instability result for the magnetic field [NV25b]—see details in Section 7.2—and adapts the arguments from Friedlander, Pavlović and Shvydkoy [FPS06] to the full MHD equations (7.1.1) with the physically relevant boundary conditions (7.1.3)–(7.1.4).

Nonlinear instability results for (7.1.1) around any linearly unstable flow for the Navier–Stokes equations (combined with the trivial background magnetic field $B = 0$) may immediately be deduced from corresponding linear instability via [FPS06]. More popular in the literature seems to be the opposite question: are there stability thresholds for the MHD equations when perturbed around some steady state? This has been typically answered for the *Couette flow*. In two-dimensions and Cartesian coordinates, the Couette flow in $\mathbb{T} \times \mathbb{R}$ has the form $u = (y, 0)$. When combined with the constant magnetic field $B = (\beta, 0)$, $\beta \neq 0$, it has been shown to be a steady state of the 2D MHD equations that is stable in L^2 under sufficiently small perturbations in H^s , with $s > 0$ big enough [Dol24; Kno25b; WZ24; JRW25]. Similar results but in Gevrey spaces exist if one of the diffusion parameters vanishes, i.e. $\nu = 0$ [KZ23; ZZ24], $\varepsilon = 0$ [DKZ24], or $\nu = \varepsilon = 0$ [Kno25a]. For the 3D MHD equations in $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$, one can find analogous thresholds for the Couette flow $u = (y, 0, 0)$ together with a constant background magnetic field of the form $B = \alpha(\sigma, 0, 1)$, for $\alpha \in \mathbb{R}$ sufficiently large and $\sigma \in \mathbb{R} \setminus \mathbb{Q}$ satisfying some Diophantine condition—which has been shown to be stable in L^2 for certain regimes of the diffusion parameters $\nu, \varepsilon > 0$ [Lis20; RZZ25; WXZ25]. The stability of static steady states $u = 0$ has also been studied together with a constant nontrivial magnetic fields in $\mathbb{R}^2$ or $\mathbb{R}^3$ [BSS88; HXY18], or with a monotone shear magnetic field $B = (b(y), 0)$ in $\mathbb{T} \times [0, 1]$ [RZ17]. Closer to the purposes of this note, we mention the works of Gérard-Varet and Rousset [Gér05; GR07], who proved nonlinear instability in L^2 for the 3D MHD equations that includes a forcing in the Navier–Stokes equations. They made use of the dynamo mechanism through which the action of a velocity field u makes the magnetic field B grow exponentially. We draw special attention to the paper [GR07], where velocity fields of a similar form to those in [NV25b] are considered. In particular, the authors define a sequence of vector fields u^κ that converge formally to the discontinuous Ponomarenko velocity field as $\kappa \rightarrow 0$, see [Gil88; NV25b]. Via the method of matched asymptotic expansions, they show that the MHD equations near this (forced) steady state are nonlinearly unstable, in the sense that a perturbation of size $O(\kappa^p)$, with $p > 0$ and $\kappa \ll 1$, grows to $O(1)$ in times $O(|\log(\kappa)|)$.

Notation

We use the symbol $a \lesssim b$ to denote that there exists a constant $C > 0$, independent of all the relevant quantities, such that $a \leq Cb$. Moreover, we use the symbol $a \sim b$ to denote that $a \lesssim b$ and $b \lesssim a$. In many different estimates we will use a generic constant $C > 0$ that might change from line to line. Given a Banach space X of vector valued functions $w : \mathcal{M} \rightarrow \mathbb{R}^3$, we define

$$X_\omega = \{w \in X : \nabla \cdot (w) = 0\}.$$

Given a function space X and a vector valued function $w = (w_1, \dots, w_n)$ such that $w_i \in X$ for all $1 \leq i \leq n$, we abuse the notation and write $w \in X$.

7.1.1 Main result

As in [Gér05; GR07], we derive a nonlinear instability result via the dynamo mechanism, which provides exponential growth for the linearised magnetic field. In this regard, we write an equation

for the perturbation $(v, B) = (u - u_{\text{TC}}, B)$, which has the form

$$\partial_t \begin{pmatrix} v \\ B \end{pmatrix} = A \begin{pmatrix} v \\ B \end{pmatrix} + \text{nonlinear terms}.$$

Here, $A = A_1 \oplus A_2$ defines a linear operator that can be written as the direct sum of the linearised Navier–Stokes operator A_1 ,

$$A_1 v = \mathbb{P}(-(u_{\text{TC}} \cdot \nabla)v - (v \cdot \nabla)u_{\text{TC}} + \nu \Delta v),$$

where $\mathbb{P}$ denotes the Leray projector onto the space of divergence-free vector fields, and the kinematic dynamo operator A_2 ,

$$A_2 B = -(u_{\text{TC}} \cdot \nabla)B + (B \cdot \nabla)u_{\text{TC}} + \varepsilon \Delta B.$$

If we can show that either A_1 or A_2 admits a growing mode—as [NV25b] ensures for kinematic dynamo operator A_2 —we say that the corresponding equations are *linearly unstable*. Then, via the arguments in [FPS06], we can upgrade the instability for the full nonlinear equations.

Theorem 7.1.1. *Let u_{TC} denote the Taylor–Couette velocity field (7.1.5). For any $s \geq 0$, $\nu > 0$, and any $\varepsilon > 0$ small enough, there exist $\chi, c > 0$ so that the following holds true: For all $\delta > 0$, there exists an initial configuration $(u_0, B_0) \in W^{s,p}$ with*

$$\left\| \begin{pmatrix} u_0 \\ B_0 \end{pmatrix} - \begin{pmatrix} u_{\text{TC}} \\ 0 \end{pmatrix} \right\|_{W^{s,p}} \leq \delta,$$

and a time $t_\star = c|\log(\delta)|$, so that the solution to the MHD equations (7.1.1) satisfies

$$\left\| \begin{pmatrix} u(t_\star) \\ B(t_\star) \end{pmatrix} - \begin{pmatrix} u_{\text{TC}} \\ 0 \end{pmatrix} \right\|_{L^p} \geq \chi.$$

Furthermore, if $\nu \gg \|u_{\text{TC}}\|_{W^{1,\infty}}$, there holds $\|B(t_\star)\|_{L^p} \geq \chi$.

Theorem 7.1.1 shows that the MHD equations (7.1.1) with sufficiently small magnetic resistivity $0 < \varepsilon \ll 1$ and boundary conditions given by (7.1.3) and (7.1.4) are nonlinearly unstable in L^p , for any $1 < p < \infty$, around the Taylor–Couette velocity field (7.1.5) and the trivial magnetic field $B = 0$.

Remark 7.1.2. From the proof of Theorem 7.1.1 in Section 7.4, it will become clear that the actual behaviour of $(u(t_\star), B(t_\star))$ depends primarily on the spectral picture of the linearised operator $A = A_1 \oplus A_2$. In fact, $(u(t_\star), B(t_\star))$ will look like the eigenmode of A associated to the eigenvalue with largest real part, perturbed by the nonlinear terms from the mild formulation. In particular, if the operator A_1 is linearly stable, which from energy estimates can be seen to occur for instance when $\nu \gg \|u_{\text{TC}}\|_{W^{1,\infty}}$, then $\|B(t_\star)\|_{L^p} \sim \chi$. In other words, provided ν is not too small, we see that the nonlinear instability develops due to a transfer of energy from the velocity field to the magnetic field, as conjectured in the physical literature, see e.g. [BS05; Mof78; RH06; CG95]. We stress that, after a suitable rewriting of the equations in terms of standard dimensionless quantities,

the parameter ν , that in our nomenclature denotes the kinematic viscosity of the fluid, can be interpreted as the inverse of the fluid's Reynolds number $\nu \sim \text{Re}^{-1}$. Hence the condition $\nu \gg \|u_{\text{TC}}\|_{W^{1,\infty}}$ implies that the Reynolds number of the fluid must be sufficiently small when compared to $\|u_{\text{TC}}\|_{W^{1,\infty}}$, i.e. $\text{Re} \ll \|u_{\text{TC}}\|_{W^{1,\infty}}$.

An interesting follow-up question would be to study the dependence of the coefficients χ and t_* from Theorem 7.1.1 on the resistivity ε , in the case where $\nu \gg \|u_{\text{TC}}\|_{W^{1,\infty}}$. Our methods show that

$$t_* \sim \frac{1}{\lambda} \log \left(\frac{\chi}{\delta} \right),$$

where $\lambda \gtrsim \varepsilon^{1/3}$ denotes the eigenvalue corresponding to the fastest growing mode of the linear operator A_2 . However, the value of χ depends on the constants appearing in Lemma 7.3.1. Quantifying these in ε would necessitate precise resolvent estimates for the linearised operator A_2 in L^p , which do not appear to follow from our analysis in [NV25b].

7.2 Linear instability and the kinematic dynamo

The main cornerstone of our proof is the linear instability result that was obtained in Chapter 6. We shall dedicate this section to a brief overview of the constructions and theorems from that chapter in order to yield a better understanding of the result. This work is concerned with spectral instability of the so-called *kinematic dynamo equations*, namely the following linear problem,

$$\begin{aligned} \partial_t B + (\bar{u} \cdot \nabla) B - (B \cdot \nabla) \bar{u} &= \varepsilon \Delta B, \\ \nabla \cdot B &= 0, \end{aligned} \tag{7.2.1}$$

with $(t, x) \in (0, \infty) \times \mathcal{M}$, endowed with the perfectly conducting boundary conditions $B \cdot \hat{n}|_{\partial \mathcal{M}} = 0$, $\hat{n} \times (\nabla \times B)|_{\partial \mathcal{M}} = 0$, and an initial datum $B(0, \cdot) = B_0$. As before, $\hat{n}$ denotes the unit outwards pointing normal vector to $\partial \mathcal{M}$. In this problem, the magnetic field B is *passively* transported by a given autonomous and divergence-free velocity field $\bar{u}$ of the form

$$\bar{u}(r) = r\Omega(r)\hat{\theta} + U(r)\hat{z},$$

with U, Ω smooth radial functions. This type of vector fields were introduced by Ponomarenko in the context of the dynamo problem [Pon73; Gil88], which asks whether it is possible to find examples of velocity fields $\bar{u}$ such that the magnetic energy associated to the linear problem (7.2.1) grows exponentially in time,

$$\|B(t)\|_{L^2} \geq C e^{\lambda t} \|B_0\|_{L^2},$$

for some constants $C, \lambda > 0$.

The results of Chapter 6 can be applied to the case of $\mathcal{M} = [R_1, R_2] \times \mathbb{T} \times \mathbb{T}$ as defined in (7.1.2) with the Taylor–Couette velocity field $\bar{u} = u_{\text{TC}}$ (7.1.5). Therefore we see that the equations (7.2.1) can be written in terms of the kinematic dynamo operator A_2 introduced in the previous section: $\partial_t B = A_2 B$. In this setting we find the following proposition.

Proposition 7.2.1 (Chapter 6, Theorem 6.1.1). *For every $\varepsilon > 0$ sufficiently small, there exist a C^2*

divergence-free vector field $B_0 : \mathcal{M} \rightarrow \mathbb{R}^3$ and a constant $\lambda \sim \varepsilon^{1/3} > 0$ so that

$$B(t, r, \theta, z) = e^{\lambda t} B_0(r, \theta, z)$$

defines a solution to the linear problem (7.2.1) with Taylor–Couette velocity field $\bar{u} = u_{\text{TC}}$ (7.1.5) and perfectly conducting boundary conditions (7.1.4) in the annular cylinder $\mathcal{M}$.

Proposition 7.2.1 proceeds via establishing the existence of an unstable eigenmode $B_0 \in L^2_\omega(\mathcal{M})$ for the kinematic dynamo operator A_2 . The proof of this result revolves around detailed resolvent estimates that can be carried out via the introduction of tailor-made Green’s functions, used to invert A_2 near the unstable eigenvalue λ . Proposition 7.2.1 ensures the existence of a growing mode, however it is currently unknown whether this one is the eigenvalue with the largest real part.

The aim of this note will be to “upgrade” the linear instability result of Proposition 7.2.1 to a nonlinear instability result for the full nonlinear 3D MHD equations.

7.3 Setup and technical lemmas

We consider a divergence-free perturbation (v, B) around the steady state $(u_{\text{TC}}, 0)$ with Dirichlet boundary conditions for the velocity field $v|_{\partial\mathcal{M}} = 0$, and perfectly conducting boundary conditions for the magnetic field $B \cdot \hat{n}|_{\partial\mathcal{M}} = \hat{n} \times (\nabla \times B)|_{\partial\mathcal{M}} = 0$. Let $\mathbb{P}$ denote the Leray projector operator onto the space of divergence-free vector fields. Then, the perturbation satisfies the equation

$$\partial_t \begin{pmatrix} v \\ B \end{pmatrix} = A \begin{pmatrix} v \\ B \end{pmatrix} + \begin{pmatrix} -N(v, v) + N(B, B) \\ M(v, B) \end{pmatrix}, \quad (7.3.1)$$

where A denotes the following linear operator

$$A \begin{pmatrix} v \\ B \end{pmatrix} = \begin{pmatrix} \mathbb{P}(-(u_{\text{TC}} \cdot \nabla)v - (v \cdot \nabla)u_{\text{TC}} + v\Delta v) \\ -(u_{\text{TC}} \cdot \nabla)B + (B \cdot \nabla)u_{\text{TC}} + \varepsilon\Delta B \end{pmatrix}, \quad (7.3.2)$$

The nonlinear part is given by $N(v, v) = \mathbb{P}(\nabla \cdot (v \otimes v))$ and $M(v, B) = \nabla \times (v \times B)$. In particular, the operator A admits a rather simple, diagonal structure, in the sense that it may be written as $A = A_1 \oplus A_2$, where A_1 is the linearised Navier–Stokes operator with homogeneous Dirichlet boundary conditions, as studied already in [FPS06], and A_2 is the kinematic dynamo operator with perfectly conducting boundary conditions around the Taylor–Couette flow, for which Proposition 7.2.1 yields an unstable eigenmode. This diagonal structure simplifies certain computations substantially—in particular, for any holomorphic $f : \mathbb{C} \rightarrow \mathbb{C}$ for which the functional calculus for sectorial operators is well defined, $f(A)$ is nothing but the operator $f(A_1) \oplus f(A_2)$. That being said, the following result shall be the main technical ingredient of this section.

Lemma 7.3.1. *The operator $(A, \mathcal{D}(A))$ defined in (7.3.2) has compact resolvent, and admits at most finitely many eigenvalues with positive real part. Let λ be the largest real part of all such eigenvalues, let $\eta > 0$ and set $A_\eta = A - \lambda - \eta$. Then,*

- for any $\alpha \in (0, 1)$, $p > 1$ there exists a constant $C > 0$ so that

$$\|A_\eta^\alpha e^{A_\eta^\alpha t}\|_{L^p \rightarrow L^p} \leq \frac{C}{t^\alpha};$$

- for any $\alpha \in (1/2, 1)$, $p > 1$ and any $q \leq 3p/(3 - (2\alpha - 1)p)$, there exists a constant $C > 0$ so that

$$\|A_\eta^{-\alpha} \nabla H\|_{L^q} \leq C \|H\|_{L^p}.$$

Here we abuse notation and write $\nabla H = \begin{pmatrix} \nabla H_1 \\ \nabla H_2 \end{pmatrix}$ for some vector fields H_1, H_2 .

The usefulness of this technical lemma will become apparent in the proof of Theorem 7.1.1 in Section 7.4. In particular, within the mild formulation, when the semigroup $e^{A(t-\tau)}$ for $0 \leq \tau \leq t$ acts on the nonlinear terms, we employ the identity

$$e^{A(t-\tau)} = e^{(\lambda+\eta)(t-\tau)} A_\eta^\alpha e^{A_\eta(t-\tau)} A_\eta^{-\alpha}$$

to control the corresponding L^p norms.

Via the diagonal decomposition $A = A_1 \oplus A_2$, it suffices to prove the result separately for A_1, A_2 . Furthermore, the result for A_1 has already been shown in [FPS06, Lemmas 3.1, 3.2], so it remains to cover the B component, i.e. to prove the corresponding result for the operator A_2 .

Analysing this operator will turn out to be somewhat non-trivial, since it is inherently a perturbation of the *Hodge Laplacian*, rather than the Dirichlet Laplacian. In particular, this operator is no longer self-adjoint on L^2 , unless it is considered on the space of divergence free vector fields, where it acts as $\Delta = -\nabla \times \nabla \times$. As such, we take A_2 to have domain

$$\mathcal{D}(A_2) = \{B \in W_\omega^{2,p}(\mathcal{M}) : B \cdot \hat{n}|_{\partial \mathcal{M}} = \hat{n} \times (\nabla \times B)|_{\partial \mathcal{M}} = 0\}.$$

In order to prove Lemma 7.3.1, the goal will then be to show that A_2 behaves “as though it was a Dirichlet Laplacian”. In particular, its fractional powers should generate suitable fractional Sobolev spaces.

It is worth mentioning that from a differential geometry perspective, the Hodge Laplacian is defined in terms of the exterior derivative d and its formal adjoint Δ as $\Delta B = (d\delta + \delta d)B$, where now B denotes a k -form in the Riemannian manifold $\mathcal{M}$. In this formulation, one finds the following natural set of boundary conditions, so-called *absolute* or *Neumann-type*,

$$\iota_{\hat{n}} B|_{\partial \mathcal{M}} = 0, \quad \iota_{\hat{n}} dB|_{\partial \mathcal{M}} = 0,$$

where the symbol $\iota_{\hat{n}}$ denotes the inner product with the unit vector $\hat{n}$ that is normal to $\partial \mathcal{M}$. In particular, if B is a 1-form like a three dimensional vector field, these boundary conditions coincide with the perfectly conducting setting introduced in (7.1.4). For further information about the Hodge Laplacian in Riemannian manifolds we refer to [GS03].

To start, let us recall classical results on the L^p theory of systems of elliptic equations. In particular,

we shall briefly mention the classical work of [ADN64], which is concerned with elliptic systems of differential equations satisfying so called *complementing boundary conditions*. The verification of these conditions for the Laplacian is a rather simple, although somewhat arduous endeavour. As a consequence, we shall state here a list of properties of the Hodge Laplacian with perfectly conducting boundary conditions.

Lemma 7.3.2 (Properties of the Hodge Laplacian ([ADN64; See71; See72])). *Let $\mathcal{M} \subset \mathbb{R}^3$ be a smooth domain, and consider the operator Δ on $L^p(\mathcal{M})$, with domain*

$$\mathcal{D}(\Delta) = \{B \in W^{2,p}(\mathcal{M}) : B \cdot \hat{n}|_{\partial\mathcal{M}} = \hat{n} \times (\nabla \times B)|_{\partial\mathcal{M}} = 0\}.$$

Then, the following statements hold true.

- *There exists a constant $K_1 > 0$ so that for any $F \in L^p(\mathcal{M})$ such that $\Delta B = F$, $B \in \mathcal{D}(\Delta)$ there holds*

$$\|B\|_{W^{2,p}(\mathcal{M})} \leq K_1 (\|F\|_{L^p(\mathcal{M})} + \|B\|_{L^p(\mathcal{M})}). \quad (7.3.3)$$

- *For all $c > 0$ large enough, and for any $\theta \in [0, 1]$, the interpolation space $\mathcal{D}((-c + \Delta)^\theta)$ is well-defined, equal to a subspace of $W^{2\theta,p}(\mathcal{M})$, and there exists a constant $K_2 > 0$ so that for all $B \in \mathcal{D}((-c + \Delta)^\theta)$ it holds*

$$\|B\|_{W^{2\theta,p}(\mathcal{M})} \leq K_2 \|(-c + \Delta)^\theta B\|_{L^p(\mathcal{M})}. \quad (7.3.4)$$

Proof. The estimate (7.3.3) follows immediately from [ADN64, Theorem 10.5]. To prove (7.3.4), we first note that by [See71], we know that $\mathcal{D}((-c + \Delta)^\theta) = [L^p(\mathcal{M}), \mathcal{D}(\Delta)]_\theta$, and from a reading of the proof of [See71, Theorem 3], there exists a constant $C > 0$ so that $\|B\|_\theta \leq C \|(-c + \Delta)^\theta B\|_{L^p(\mathcal{M})}$, where $\|\cdot\|_\theta$ denotes the interpolation norm of $[L^p(\mathcal{M}), \mathcal{D}(\Delta)]_\theta$, see e.g. [Cal64]. Furthermore, from [See72], we further deduce the existence of a constant $C_2 > 0$ so that for all $B \in [L^p(\mathcal{M}), \mathcal{D}(\Delta)]_\theta$, there holds $\|B\|_{W^{2\theta,p}(\mathcal{M})} \leq C_2 \|B\|_\theta$. Hence, (7.3.4) follows. $\square$

We remark that in [ADN64; See71; See72] very general sets of boundary conditions are considered—so-called *complementary* [ADN64, Section 2] or *normal* [See72, Definition 3.1] boundary conditions—which in particular include the perfectly conducting case (7.1.4). These are given locally via an algebraic relation, and ensure that the boundary value problem for the elliptic operator is well-posed. In addition, we want to highlight that the perfectly conducting boundary conditions for divergence-free vector fields lead to a dissipative realization of the (vector) Laplacian, since we have the identity

$$\int_{\mathcal{M}} \Delta B \cdot B dx = - \int_{\mathcal{M}} |\nabla \times B|^2 dx.$$

In particular, from this expression it is clear that the Laplacian is a negative operator, and as noted already in [ST83], $\|\nabla \times B\| \sim \|B\|_{\dot{H}^1}$ for divergence-free vector fields. It is clear that the kinematic dynamo operator admits a compact resolvent on the space of divergence-free vector fields satisfying the perfectly conducting boundary conditions. Since the same property holds true for A_1 , it follows that all eigenvalues of A are isolated and discrete. With these preliminaries out of the way, we now turn to the proof of Lemma 7.3.1.

7.3.1 Proof of Lemma 7.3.1

The proof will be split into multiple shorter results, the first and most essential of which is the following.

Lemma 7.3.3. *Let $1 < p < \infty$, and consider $A_2 B = \nabla \times (u_{\text{TC}} \times B) + \varepsilon \Delta B$ on $L_\omega^p(\mathcal{M})$, with domain*

$$\mathcal{D}(A_2) = \{B \in W_\omega^{2,p}(\mathcal{M}) : B \cdot \hat{n}|_{\partial \mathcal{M}} = \hat{n} \times (\nabla \times B)|_{\partial \mathcal{M}} = 0\}.$$

Then, A_2 generates an analytic semigroup on $L_\omega^p(\mathcal{M})$. Furthermore, for any $\mu \in \rho(A_2)$, where $\rho(A_2)$ denotes the resolvent set, there exists a constant $C > 0$ depending only on μ , so that for all $B \in W^{1,p}(\mathcal{M})$ there holds

$$\|(A_2 - \mu)^{-1} \nabla B\|_{W^{1,p}(\mathcal{M})} \leq C \|B\|_{L^p(\mathcal{M})}.$$

Proof. We begin by noting that by [Miy80, Theorem 3.10], the operator Δ on L_ω^p , with domain $\mathcal{D}(A_2)$ generates an analytic semigroup and has compact resolvent. Hence, it remains to show that the addition of the terms $\nabla \times (u_{\text{TC}} \times \cdot)$ does not disturb this property. Once this is established, the property of having at most finitely many eigenvalues with positive real part follows immediately. To do so, note that certainly for any $B \in \mathcal{D}(A_2)$ there holds that

$$\|\nabla \times (u_{\text{TC}} \times B)\|_{L^p(\mathcal{M})} \leq C(\|u_{\text{TC}}\|_{W^{1,\infty}}) \|B\|_{W^{1,p}}.$$

But now, by the Gagliardo–Nirenberg inequality, we can write

$$\|B\|_{W^{1,p}(\mathcal{M})} \leq C \|B\|_{W^{2,p}(\mathcal{M})}^{1/2} \|B\|_{L^p(\mathcal{M})}^{1/2}.$$

Hence, for any $\eta > 0$, there exists $C(\eta) > 0$ so that for all $B \in W^{2,p}(\mathcal{M})$ we can bound

$$\|\nabla \times (u_{\text{TC}} \times B)\|_{L^p(\mathcal{M})} \leq \eta \|B\|_{W^{2,p}(\mathcal{M})} + C(\eta) \|B\|_{L^p(\mathcal{M})}. \quad (7.3.5)$$

Next, from (7.3.3) we deduce the existence of a *uniform* constant $K > 0$ so that if $\Delta B = F$, complemented with perfectly conducting boundary conditions, there holds

$$\|B\|_{W^{2,p}(\mathcal{M})} \leq K (\|F\|_{L^p} + \|B\|_{L^p}).$$

In other words, we have that for any $B \in \mathcal{D}(A_2)$

$$\|B\|_{W^{2,p}(\mathcal{M})} \leq K (\|\Delta B\|_{L^p} + \|B\|_{L^p}).$$

Hence, combining this with (7.3.5), for any $\eta > 0$ there exists $C(\eta) > 0$ so that for all $B \in \mathcal{D}(A_2)$,

$$\|\nabla \times (u_{\text{TC}} \times B)\|_{L^p(\mathcal{M})} \leq \eta \|\Delta B\|_{L^p(\mathcal{M})} + C(\eta) \|B\|_{L^p(\mathcal{M})}. \quad (7.3.6)$$

Consequently, by classical functional analysis results, see for instance [Kat76], the operator

$$A_2 = \varepsilon \Delta + \nabla \times (u_{\text{TC}} \times \cdot)$$

is closed and has compact resolvent when given the domain $\mathcal{D}(A_2)$, and generates an analytic semigroup (see also Lemma A.4.4), completing the proof of the first claim in view of Lemma A.4.3.

To show the second claim of the lemma, we once again use the inequality (7.3.6) to note that for any $\eta > 0$, there exists $C(\eta) > 0$ so that for all $B \in \mathcal{D}(A_2)$ there holds

$$\|\nabla \times (u \times B)\|_{L^p(\mathcal{M})} \leq \eta \|\Delta B\|_{L^p(\mathcal{M})} + C(\eta) \|B\|_{L^p(\mathcal{M})}.$$

In particular, taking $\Delta \mapsto \varepsilon \Delta$, for $\mu > 0$ very large we find that

$$\begin{aligned} \|\nabla \times (u \times (\varepsilon \Delta - \varepsilon \mu)^{-1} B)\|_{L^p} &\leq \varepsilon \eta \|\Delta (\varepsilon \Delta - \varepsilon \mu)^{-1} B\|_{L^p} + C(\varepsilon \eta) \|(\varepsilon \Delta - \varepsilon \mu)^{-1} B\|_{L^p} \\ &\leq \eta \|B\|_{L^p} + \eta \mu \|(\Delta - \mu)^{-1} B\|_{L^p} + \varepsilon^{-1} C(\varepsilon \eta) \|(\Delta - \mu)^{-1} B\|_{L^p} \\ &\leq \eta(1 + C) \|B\|_{L^p(\mathcal{M})} + \varepsilon^{-1} C(\varepsilon \eta) C \mu^{-1} \|B\|_{L^p}, \end{aligned}$$

where we used the fact that, since Δ generates an analytic semigroup, there exists $C > 0$ so that for all $\mu > 0$ large enough it holds $\|(\Delta - \mu)^{-1}\| \leq C \mu^{-1}$. Therefore, taking first $\eta > 0$ sufficiently small, and then μ sufficiently large, we get that

$$\|\nabla \times (u \times (\varepsilon \Delta - \varepsilon \mu)^{-1} B)\|_{L^p} \leq \frac{1}{2} \|B\|_{L^p}$$

for all $B \in L_\omega^p(\mathcal{M})$. Thus, for this value of μ , $A_2 - \varepsilon \mu$ exists and is given by

$$(A_2 - \varepsilon \mu)^{-1} = (\varepsilon \Delta - \varepsilon \mu)^{-1} \sum_{n \geq 0} (-1)^n (\nabla \times (u \times (\varepsilon \Delta - \varepsilon \mu)^{-1}))^n.$$

We apply this operator now to ∇B ,

$$(A_2 - \varepsilon \mu)^{-1} \nabla B = (\varepsilon \Delta - \varepsilon \mu)^{-1} \nabla B + (\varepsilon \Delta - \varepsilon \mu)^{-1} \sum_{n \geq 0} (-1)^{n+1} (\nabla \times (u \times (\varepsilon \Delta - \varepsilon \mu)^{-1}))^n z, \quad (7.3.7)$$

where $z = \nabla \times (u \times (\varepsilon \Delta - \varepsilon \mu)^{-1} \nabla B)$. In particular, it follows that $\|z\|_{L^p} \leq C \|B\|_{L^p}$, for some constant C independent of $B \in L_\omega^p(\mathcal{M})$. Hence, taking $W^{1,p}(\mathcal{M})$ norms in (7.3.7), there holds

$$\begin{aligned} \|(A_2 - \varepsilon \mu)^{-1} \nabla B\|_{W^{1,p}} &\leq \|(\varepsilon \Delta - \varepsilon \mu)^{-1} \nabla B\|_{W^{1,p}} \\ &\quad + \|(\varepsilon \Delta - \varepsilon \mu)^{-1}\|_{L^p \rightarrow W^{1,p}} \sum_{n \geq 0} \|(\nabla \times (u \times (\varepsilon \Delta - \varepsilon \mu)^{-1}))\|_{L^p \rightarrow L^p}^n \|z\|_{L^p} \\ &\leq C \|B\|_{L^p}, \end{aligned}$$

with C independent of $B \in L_\omega^p(\mathcal{M})$. In other words, the claim holds true for this particular value of μ . But now, recalling the resolvent identity $R_\mu = R_s + (s - \mu)R_\mu R_s$, for $s, \mu \in \rho(A_2)$, we find

$$\|(A_2 - s)^{-1} \nabla B\|_{W^{1,p}} \leq \|(A_2 - \varepsilon \mu)^{-1} \nabla B\|_{W^{1,p}(\mathcal{M})} + |\varepsilon \mu - s| \|(A_2 - \varepsilon \mu)^{-1} (A_2 - s)^{-1} \nabla B\|_{W^{1,p}}.$$

But we showed that $\|(A_2 - \varepsilon \mu)^{-1} B\|_{W^{1,p}} \leq C \|B\|_{L^p}$, so we conclude, using that by definition

$(A_2 - s)^{-1} : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$ is bounded, that

$$\|(A_2 - s)^{-1} \nabla B\|_{W^{1,p}} \leq C(\mu, s, \varepsilon) \|B\|_{L^p},$$

for all $B \in L^p_\omega(\mathcal{M})$, and the proof is complete. $\square$

Finally, we need the following classical result on interpolation spaces, which is contained in Lemma A.4.4 in the Appendix A. For the convenience of the reader, we recall it here.

Lemma 7.3.4 ([Hai], Proposition 4.45). *Let A generate an analytic semigroup on some separable Banach space X , and suppose $S : \mathcal{D}((-A)^\alpha) \rightarrow X$ is a bounded operator, for some $\alpha \in [0, 1]$. Then, $A + S$ generates an analytic semigroup, and $\mathcal{D}((-A - S)^\alpha) \sim \mathcal{D}((-A)^\alpha)$, for all $\alpha \in (0, 1)$, i.e. the corresponding domains are equal, and the norms $\|(-A - S)^\alpha B\|$, $\|(-A)^\alpha B\|$ are equivalent for any $B \in \mathcal{D}((-A)^\alpha)$.*

With this tool in our hand, we can now commence with the proof of the technical Lemma 7.3.1.

Proof of Lemma 7.3.1. In view of the results in [FPS06] for the linearised Navier–Stokes equations, it only remains to prove the analogous statement for A_2 , i.e. the linearised magnetic operator. The first claim of Lemma 7.3.1 is true for any analytic semigroup, and so follows immediately from Lemma 7.3.3. Next, note that $\nabla \times (u_{\text{TC}} \times \cdot)$ is a bounded operator from $W^{1,p}(\mathcal{M})$ to $L^p(\mathcal{M})$. Hence, for any $B \in \mathcal{D}((\varepsilon \Delta - c_0)^{1/2})$, for any $c_0 > 0$ large enough there holds

$$\|\nabla \times (u_{\text{TC}} \times B)\|_{L^p(\mathcal{M})} \leq C \|B\|_{W^{1,p}(\mathcal{M})} \leq C \|(\varepsilon \Delta - c_0)^{1/2} B\|_{L^p},$$

where the last inequality is from (7.3.4). Hence, $S := \nabla \times (u_{\text{TC}} \times \cdot)$ is indeed bounded from $\mathcal{D}((\varepsilon \Delta - c_0)^{1/2})$ to $L^p_\omega(\mathcal{M})$. Thus, by Lemma A.4.4, for all $\alpha \in [0, 1]$ there exists $C(\alpha)$ so that

$$\|\Delta^\alpha B\|_{L^p} \leq C(\alpha) \|(A_2 - \mu)^\alpha B\|_{L^p}.$$

Thus, let now $B \in L^p(\mathcal{M})$, and define $q \leq 3p/(3 - (2\alpha - 1)p)$ and $1/2 < \alpha < 1$. Using a density argument, we can assume that B is smooth. Then, for $\mu \in \rho(A_2)$ it holds

$$\|(A_2 - \mu)^{-\alpha} \nabla B\|_{L^q} \leq C \|(A_2 - \mu)^{-\alpha} \nabla B\|_{W^{2\alpha-1,p}}, \quad (7.3.8)$$

from the Sobolev inequality. But then, we estimate

$$\begin{aligned} \|(A_2 - \mu)^{-\alpha} \nabla B\|_{W^{2\alpha-1,p}} &= \|(A_2 - \mu)^{1-\alpha} (A_2 - \mu)^{-1} \nabla B\|_{W^{2\alpha-1,p}} \\ &\leq C \|(A_2 - \mu)^{\alpha-1/2} (A_2 - \mu)^{1-\alpha} (A_2 - \mu)^{-1} \nabla B\|_{L^p} \\ &\leq C \|(A_2 - \mu)^{1/2} (A_2 - \mu)^{-1} \nabla B\|_{L^p}, \end{aligned} \quad (7.3.9)$$

where we used (7.3.4) in the second to last inequality. But as noted already in [Miy80], $\mathcal{D}(\Delta^{1/2}) \subset W^{1,p}(\mathcal{M})$, with continuous injection, and by Lemma A.4.4, the same holds for A_2 . Thus, there exists a constant $C > 0$ so that $\|(A_2 - \mu)^{1/2} B\|_{L^p} \leq C \|B\|_{W^{1,p}}$. Hence, we deduce from (7.3.8) and (7.3.9) that

$$\|(A_2 - \mu)^{-\alpha} \nabla B\|_{L^q} \leq C \|(A_2 - \mu)^{-1} \nabla B\|_{W^{1,p}} \leq C \|B\|_{L^p},$$

where the last inequality follows by Lemma 7.3.3. $\square$

7.4 Proof of nonlinear instability

We now have all the ingredients required to prove the claimed nonlinear instability result around the Taylor–Couette flow. We follow very closely the method of [FPS06], making the necessary adaptations to handle the addition of the magnetic field component.

Proof of Theorem 7.1.1. Pick the growing mode ϕ_0 of A corresponding to the eigenvalue with largest real part, which we call λ . Via (7.3.1) and the mild formulation, we write

$$w(t) = \begin{pmatrix} v(t) \\ B(t) \end{pmatrix} = e^{At} \delta \phi_0 + \int_0^t e^{A(t-\tau)} \begin{pmatrix} -N(v, v) + N(B, B) \\ M(v, B) \end{pmatrix} (\tau) d\tau. \quad (7.4.1)$$

We want to show that the nonlinear instability mechanism is directly produced by the linear part of the formulation—that is the first term in the right hand side. In this regard, we start by looking at the nonlinear addend, that for some $q > 1$ to be chosen later, can be estimated via Lemma 7.3.1 by

$$\begin{aligned} & \left\| \int_0^t e^{A(t-\tau)} \begin{pmatrix} -N(v, v) + N(B, B) \\ M(v, B) \end{pmatrix} (\tau) d\tau \right\|_{L^q} \\ & \leq \int_0^t e^{(\lambda+\eta)(t-\tau)} \left\| A_\eta^\alpha e^{A_\eta(t-\tau)} A_\eta^{-\alpha} \begin{pmatrix} -N(v, v) + N(B, B) \\ M(v, B) \end{pmatrix} \right\|_{L^q} d\tau \\ & \lesssim \int_0^t e^{(\lambda+\eta)(t-\tau)} \frac{1}{(t-\tau)^\alpha} \left\| A_\eta^{-\alpha} \begin{pmatrix} -N(v, v) + N(B, B) \\ M(v, B) \end{pmatrix} \right\|_{L^q} d\tau. \end{aligned}$$

Moreover, by letting $\alpha \in (1/2, 1)$ and $q > 3/(2\alpha - 1)$, we can use the second claim from Lemma 7.3.1 to write

$$\|A_\eta^{-\alpha} N(v, v)\|_{L^q} \leq \|A_\eta^{-\alpha} \nabla(v \otimes v)\|_{L^q} \lesssim \|v \otimes v\|_{L^{q/2}} \lesssim \|v\|_{L^q}^2.$$

Analogously, we bound $\|A_\eta^{-\alpha} N(B, B)\|_{L^q} \lesssim \|B\|_{L^q}^2$, and $\|A_\eta^{-\alpha} M(v, B)\|_{L^q} \lesssim \|B\|_{L^q} \|v\|_{L^q}$, which in particular yields the estimate

$$\left\| \int_0^t e^{A(t-\tau)} \begin{pmatrix} -N(v, v) + N(B, B) \\ M(v, B) \end{pmatrix} (\tau) d\tau \right\|_{L^q} \lesssim \int_0^t e^{(\lambda+\eta)(t-\tau)} \frac{1}{(t-\tau)^\alpha} \|w(\tau)\|_{L^q}^2 d\tau.$$

For any $Q > \|\phi_0\|_{L^q}$, we now define a time $T \in (0, \infty]$ given by

$$T = \inf\{t \geq 0 : \|w(t)\|_{L^q} \geq e^{\lambda t} \delta Q\}. \quad (7.4.2)$$

Hence, for any $t \leq T$, there holds

$$\int_0^t e^{(\lambda+\eta)(t-\tau)} \frac{1}{(t-\tau)^\alpha} \|w(\tau)\|_{L^q}^2 d\tau \leq Q^2 \delta^2 \int_0^t e^{(\lambda+\eta)(t-\tau)} \frac{1}{(t-\tau)^\alpha} e^{2\lambda\tau} d\tau \lesssim Q^2 \delta^2 e^{2\lambda t},$$

for all $t \leq T$, provided that we choose η small enough. All in all, we found that there exists a

constant $C > 0$, such that the L^q norm of the nonlinear part is bounded by

$$\left\| \int_0^t e^{A(t-\tau)} \begin{pmatrix} -N(v, v) + N(B, B) \\ M(v, B) \end{pmatrix} (\tau) d\tau \right\|_{L^q} \leq CQ^2 \delta^2 e^{2\lambda t}$$

for any $t \leq T$. This result holds true for $q > 3/(2\alpha - 1)$, $\alpha \in (1/2, 1)$, which in turn implies that $q > 3$. However, in order to obtain a similar estimate for all $p > 1$, we can simply make use of the fact that the domain is bounded to write $\|\cdot\|_{L^p} \leq C'\|\cdot\|_{L^q}$. Going back to (7.4.1) and using the definition of T in (7.4.2), we find that at time $t = T$ there holds

$$\delta Q e^{\lambda T} = \|w(T)\|_{L^q} \leq \|\phi_0\|_{L^q} \delta e^{\lambda T} + CQ^2 \delta^2 e^{2\lambda T}.$$

Here we are formally assuming that $T < \infty$, but the inequality holds true for any $t \leq T$ by (7.4.2). Thus, we find that there exists a constant $\chi_\star > 0$ with the property

$$\delta e^{\lambda T} \geq \frac{Q - \|\phi_0\|_{L^q}}{CQ^2} =: \chi_\star.$$

Therefore, there must exist some $t_\star < T$ so that $\delta e^{\lambda t_\star} = \chi_\star$. Evaluating (7.4.1) at time $t = t_\star$, taking the L^p norm and using the triangle inequality we can write

$$\|w(t_\star)\|_{L^p} \geq \delta e^{\lambda t_\star} \|\phi_0\|_{L^p} - C' CQ^2 \delta^2 e^{2\lambda t_\star} \geq \chi_\star \|\phi_0\|_{L^p} - CC' Q^2 \chi_\star^2 \geq \chi_\star (\|\phi_0\|_{L^p} - CC' Q^2 \chi_\star).$$

All in all, picking Q sufficiently close to $\|\phi_0\|_{L^q}$, we deduce that

$$\|w(t_\star)\|_{L^p} \geq \frac{1}{2} \chi_\star,$$

completing the proof. To deduce that the initial perturbation may be in $W^{s,p}$, note that by elliptic regularity, in particular by Gårding's inequality applied to eigenfunctions ϕ_0 of elliptic operators, see e.g. [Tay96], we can write $\|\phi_0\|_{W^{s,p}} \lesssim (|\lambda| + 1)^{s/2} \|\phi_0\|_{L^p}$. Thus the $W^{s,p}$ norm of any eigenfunction of A is comparable to its L^p norm, and so the result follows immediately.

Furthermore, we readily see that if the growing mode ϕ_0 of A corresponding to the eigenvalue with largest real is of the form $\phi_0 = (0, B_0)$, then the growth occurs for the magnetic component and we can write

$$\|B(t_\star)\|_{L^p} \gtrsim \|w(t_\star)\|_{L^p} \geq \frac{1}{2} \chi_\star$$

as claimed. In particular, via Proposition 7.2.1 we see that this will be the case if the operator A_1 is spectrally stable. One simple way to ensure that A_1 is stable is to let $\nu > 0$ (i.e. the inverse of the fluid's Reynolds number) be large when compared to $\|u_{TC}\|_{W^{1,\infty}}$, since via Poincaré we can write

$$\|A_1 \nu\|_{L^2} \leq -(C\nu - \|u_{TC}\|_{W^{1,\infty}}) \|\nu\|_{L^2}$$

for some constant $C > 0$. Since any eigenfunction of A_1 in L^p must as well be in L^2 , we conclude that no growing mode can exist for A_1 in L^p if $\nu \gg \|u_{TC}\|_{W^{1,p}}$. $\square$

Appendix A

A primer on perturbation theory

Appendix A collects the elements of spectral, perturbation, and semigroup theory used repeatedly throughout the thesis. Most of this material is standard and may be found in Kato [Kat76] and related references. We record here the specific formulations needed later, and include proofs in the cases that are used centrally in the main text.

A.1 Basic definitions and results

Throughout this section, we will be working with an unbounded operator $(T, D(T))$ on a complex Banach space $(X, \|\cdot\|)$. We recall the following elementary definitions.

Definition A.1.1. The operator $(T, D(T))$ is said to be *closed*, if the domain $D(T)$ of T equipped with the graph norm $\|x\|_T = \|x\| + \|Tx\|$ is complete. Given a closed operator $(T, D(T))$, the resolvent set $\rho(T)$ of T is defined as

$$\rho(T) := \{\lambda \in \mathbb{C} \mid (\lambda - T) : D(T) \rightarrow X \text{ has a bounded inverse}\}$$

and the spectrum $\sigma(T)$ of T is the set

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

It is a general fact that $\sigma(T)$ is a closed subset of $\mathbb{C}$. In finite dimensions, the spectrum of a linear operator (a matrix) is simply the set of its eigenvalues. However, upon working in infinite dimensions, the situation becomes substantially more complicated, and one can for instance have elements of the spectrum which *do not* have associated eigenvalues and eigenvectors. In such cases, the spectrum can behave in rather wild ways under perturbations, especially if they are “singular” in nature, as seen for instance in Example 2.2.10. Since such phenomena cannot occur in finite dimensions, this effect is inherently “infinite-dimensional” in nature, and so it is rather useful to distinguish elements of the spectrum by whether the operator acts “truly infinite-dimensional” there. A convenient way of formalising this intuition is via the notion of a *Fredholm operator*.

Definition A.1.2. Let $(T, D(T))$ be a closed operator on a Banach space. We say that T is

Fredholm of index $n \in \mathbb{Z}$ if $\dim \ker(T) < \infty$, $\operatorname{codim} \operatorname{Ran}(T) < \infty$, and $\operatorname{ind}(T) = \dim \ker(T) - \operatorname{codim} \operatorname{Ran}(T) = n$. The quantities $\dim \ker(T)$ and $\operatorname{codim} \operatorname{Ran}(T)$ are known (respectively) as the *nullity* and *deficiency* of T , denoted by $\operatorname{nul}(T)$ and $\operatorname{def}(T)$.

With this definition established, we can now define the so-called *essential spectrum* of an operator, which, heuristically, is the part of the spectrum where the operator does not behave like it would in a finite-dimensional spectral problem.

Definition A.1.3. ¹Let $(T, D(T))$ be a closed operator on a complex Banach space. Then, the *essential spectrum* $\sigma_{\text{ess}}(T)$ of T is defined as

$$\sigma_{\text{ess}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not Fredholm}\}.$$

Since for any $\lambda \in \rho(T)$ it must hold that $\operatorname{nul}(\lambda - T) = \operatorname{def}(\lambda - T) = 0$, certainly $\sigma_{\text{ess}}(T) \subset \sigma(T)$.

The utility of this definition can be seen for instance in the following result, which concerns stability of the nullity and deficiency under small perturbations.

Lemma A.1.4 ([Kat76, IV, Theorem 5.31]). *Let T be Fredholm, and let $|\kappa| > 0$. Then, for all $|\kappa| > 0$ small enough, $T + \kappa$ is Fredholm, and $\operatorname{nul}(T + \kappa)$ and $\operatorname{def}(T + \kappa)$ are constant.*

As a direct consequence, we obtain the following “structure” theorem for the spectrum of a general closed operator.

Theorem A.1.5 ([Kat76, IV, Section 6]). *Let $(T, D(T))$ be a closed operator on a complex Banach space. Then, the essential spectrum of T is a closed subset of $\sigma(T)$, and any connected component of $\mathbb{C} \setminus \sigma_{\text{ess}}(T)$ that contains an element of $\rho(T)$ has at most countably many, isolated points of $\sigma(T)$.*

Proof. By Lemma A.1.4 it follows that $\sigma_{\text{ess}}(T)$ is closed. Next, consider the set $\mathbb{C} \setminus \sigma_{\text{ess}}$. This set is open, and therefore it may be written as a countable union of connected open sets $\{\Delta_n\}_{n \in \mathbb{N}}$, with the Δ_n being pairwise disjoint. If there exists $\lambda \in \rho(T) \cap \Delta_n$, by Lemma A.1.4 there are at most countably many spectral points of T in Δ_n . Indeed, consider Δ_n with the subspace topology, let $V = \{\lambda \in \Delta_n : \operatorname{nul}(\lambda - T) \neq 0\}$, and set

$$U = \{\lambda \in \Delta_n : \lambda \text{ is an accumulation point of } V\}.$$

Note first that this is open, since if there exists a sequence $\lambda_m \subset V$ so that $\lambda_m \rightarrow \lambda \in \Delta_n$, then by Lemma A.1.4 there exists $\kappa_0 > 0$ so that

$$\operatorname{nul}(\lambda + \kappa - T) = \text{constant}$$

for all $|\kappa| \leq \kappa_0$. In particular, $\operatorname{nul}(\lambda + \kappa - T) = \operatorname{nul}(\lambda_m - T) \neq 0$ for all m large enough, so that there exists a punctured open disc around λ on which $\operatorname{nul}(\lambda + \kappa - T) = \text{constant} \neq 0$. Therefore, any point in this disc is an element of U . Furthermore, U is also closed, since if $U \ni \lambda_i \rightarrow \tilde{\lambda} \in \Delta_n$,

¹There are many definitions of the essential spectrum, which are not all equivalent. For instance, in [Kat76] the essential spectrum is defined as those values of λ where $\lambda - T$ is not *semi-fredholm*, which is a strictly weaker condition than the one we impose. However, it will turn out that for “most” parts of the spectrum, the different definitions coincide.

then by the proof of openness, for all the λ_i , there exists a $\tilde{\lambda}_i$ with $|\tilde{\lambda}_i - \lambda_i| < 2^{-i}$ so that $\tilde{\lambda}_i \in V$, and so $\tilde{\lambda} \in U$. Hence, U is an open and closed subset of the connected set Δ_n , and so it is either the empty set or the full set Δ_n . But since Δ_n contains a point of the resolvent, where notably the nullity is zero, it must be that $U = \emptyset$, and so there are no accumulation points of V in Δ_n . Hence, writing $\Delta_n = \bigcup_{m \in \mathbb{N}} \{\lambda \in \Delta_n : \text{dist}(\lambda, \partial \Delta_n) \geq 1/m, |\lambda| \leq m\}$, each element in the union can contain at most finitely many spectral points of T , from where the proof follows. $\square$

This result motivates the following definition, which is in some sense the antithesis of the essential spectrum, since it represents spectral points where the eigenvalue problem is “almost finite-dimensional”.

Definition A.1.6. Let $\lambda \in \sigma(T)$. Then, λ is said to be in the *discrete spectrum*, $\sigma_{\text{disc}}(T)$ if $\lambda \notin \sigma_{\text{ess}}(T)$, and furthermore λ is an isolated point of $\sigma(T)$.

Lemma A.1.5 can thus be read as saying that, any spectral point $\lambda \in \sigma(T)$ that belongs to a connected component of $\mathbb{C} \setminus \sigma_{\text{ess}}(T)$ containing an element of $\rho(T)$ is automatically in the discrete spectrum. This property turns out to be exceedingly useful, since elements of the discrete spectrum behave particularly well under perturbations. However, before delving further into the perturbation theory aspect of spectra, we shall first prove a few more characterisations and stability theorems for the essential spectrum.

A.2 Characterisations and stability of the essential spectrum

Perhaps the most important stability property of the essential spectrum is its relationship with compact operators. Indeed, since one may (morally) view compact operators as acting mostly in a finite-dimensional fashion, it appears reasonable to posit that they should not impact the essential spectrum of an operator. Indeed, it turns out that one may prove the following result.

Theorem A.2.1 ([Kat76, IV, Theorem 5.26]). *Let $(T, D(T))$ be a closed operator, and suppose that A is relatively compact with respect to T . In other words, suppose that for any $\lambda \in \rho(T)$, $A(\lambda - T)^{-1} : X \rightarrow X$ is a compact operator. Then, $\sigma_{\text{ess}}(T + A) = \sigma_{\text{ess}}(T)$.*

An immediate consequence of this result is for instance the following.

Lemma A.2.2. *Let $K : X \rightarrow X$ be a compact operator. Then, its spectrum is entirely discrete, with the exception being $0 \in \sigma_{\text{ess}}(K)$.*

Proof. We can write $K = K + 0$. Then, $\sigma_{\text{ess}}(0) = \{0\}$, and so $\sigma_{\text{ess}}(K) = \sigma_{\text{ess}}(0) = \{0\}$. Furthermore, applying the structure theorem A.1.5, it is clear that any point of $\sigma(K) \setminus \{0\}$ is connected in $\mathbb{C}$ to a point of $\rho(K)$, and so the result follows. $\square$

In fact, this property extends not only to compact operators, but also to operators that have a compact resolvent.

Lemma A.2.3. *Let $(T, D(T))$ be a closed operator on a complex Banach space, so that furthermore T admits a compact resolvent. That is, there exists $\lambda \in \rho(T)$ so that $(\lambda - T)^{-1} : X \rightarrow X$ is a compact operator. In fact, owing to the resolvent identity $(\mu - T)^{-1} = (\lambda - T)^{-1} + (\lambda - \mu)(\mu - T)^{-1}(\lambda - T)^{-1}$ and the fact that the composition of a bounded and a compact operator is*

compact, it holds that $(\mu - T)^{-1}$ is a compact operator for any $\mu \in \rho(T)$. Then, $\sigma_{\text{ess}}(T) = \emptyset$, and so the spectrum of T is entirely discrete.

Proof. Consider $\lambda \in \mathbb{C}$. Then, we note that $T = \lambda + (T - \lambda)$. Then, since $\lambda(T - \mu)^{-1}$ is compact for any $\mu \in \rho(T)$, we deduce that $\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(T - \lambda) = \sigma_{\text{ess}}(T) - \lambda$. Since this is true for any $\lambda \in \mathbb{C}$, it must imply that $\sigma_{\text{ess}}(T) = \mathbb{C}$ or $\sigma_{\text{ess}}(T) = \emptyset$. But we know that $\rho(T) \neq \emptyset$, and so it follows that $\sigma_{\text{ess}}(T) = \emptyset$. $\square$

This property turns out to be especially useful in the study of parabolic PDEs, which notably include the kinematic dynamo equations.

Example A.2.4. Consider the kinematic dynamo operator $\mathcal{L}B = \nabla \times (u \times B) + \varepsilon \Delta$, for some $\varepsilon > 0$ on $L^2(\mathbb{T}^3)$, and $u \in W^{1,\infty}(\mathbb{T}^3)$ divergence-free. Then, $\mathcal{L}$ has a compact resolvent, and its spectrum is entirely discrete. Indeed, energy estimates show that if $\text{Re}(\lambda)$ is sufficiently large, then $\|(\lambda - \mathcal{L})^{-1}B\|_{H^1(\mathbb{T}^3)} \lesssim \|B\|_{L^2(\mathbb{T}^3)}$, and so the resolvent is indeed compact.

Some characterisations of the essential spectrum

Another useful corollary of Theorem A.2.1 is that it allows for a rather clean characterisation of the essential spectrum for bounded operators on a Banach space. Indeed, given a complex Banach space, denote by $\mathcal{K}(X)$ the ideal of compact operators on X . Then, we can define the *Calkin algebra* as $\mathcal{C}(X) = \mathcal{B}(X)/\mathcal{K}(X)$, i.e. the quotient space of the space of bounded linear operators on X with respect to the ideal of compact operators. We then have the following result.

Lemma A.2.5 ([Heu86, Theorem 25.1]). *Let $T \in \mathcal{B}(X)$ be a bounded linear operator. Then, $\lambda \in \sigma_{\text{ess}}(T)$ if and only if $\lambda \in \sigma^{\mathcal{C}(X)}(T)$, where $\sigma^{\mathcal{C}(X)}(T)$ denotes the spectrum of T when viewed as an element of $\mathcal{C}(X)$.*

An immediate corollary of this result is the following characterisation of the essential spectral radius of a bounded operator.

Lemma A.2.6. *Let $T \in \mathcal{B}(X)$. Then, the essential spectral radius of T , defined via $r_{\text{ess}}(T) = \sup\{|\lambda| : \lambda \in \sigma_{\text{ess}}(T)\}$ is equal to*

$$r_{\text{ess}}(T) = \lim_{n \rightarrow \infty} \left(\inf_{K \in \mathcal{K}(X)} \|T^n + K\|_{\mathcal{B}(X)} \right)^{1/n}.$$

Proof. This is an immediate consequence of Lemma A.2.5 combined with the spectral radius formula, valid for any element of a Banach algebra, given by (see e.g. [Heu86, Chapter VII])

$$r(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n}.$$

$\square$

Despite the above formula being of significant utility already, a further characterisation of the essential spectral radius was provided by Nussbaum in [Nus70], using the *Kuratowski ball-measure*

of non-compactness. Indeed, given a bounded subset A of a Banach space X , we define

$$\gamma(A) = \inf\{d > 0 : \text{there exists a finite number of balls } \{B_1, \dots, B_M\} \text{ of radius } d \text{ so that } A \subset \bigcup_{i=1}^M B_i\}.$$

Furthermore, we require that the centers of the balls B_i lie in A . We then define the Kuratowski ball-measure of non-compactness of a bounded operator $T \in \mathcal{B}(X)$ as $\gamma(T) = \gamma(T(B_1(0)))$.² With this definition, we can now state the main result of the paper [Nus70].

Theorem A.2.7 ([Nus70]). *Let $T \in \mathcal{B}(X)$. Then, there holds $r_{\text{ess}}(T) = \lim_{n \rightarrow \infty} \gamma(T^n)^{1/n}$.*

While the above formulas are rather elegant, it may not yet be obvious how they can be of use. However, it turns out that, the formula of Theorem A.2.7 is of crucial importance to the arguments we employ in Chapter 5, since it allows one to establish bounds on the essential spectral radius using a so-called *Lasota-Yorke inequality*. Indeed, the following is essentially due to Hennion [Hen93].

Proposition A.2.8. *Let $(X, \|\cdot\|)$ and $(X_w, |\cdot|_w)$ be Banach spaces such that $X \subset X_w$, with continuous inclusion $|\cdot|_w \leq \|\cdot\|$. Assume that the unit ball $\{f \in X : \|f\| \leq 1\}$ is relatively compact in $(X_w, |\cdot|_w)$.*

Let $T : X \rightarrow X$ be a bounded linear operator that also extends to a bounded operator $T : X_w \rightarrow X_w$. Furthermore, assume T satisfies a Lasota–Yorke inequality: there exist constants $0 \leq a < M$ such that

$$\|Tf\| \leq a\|f\| + M|f|_w \quad \forall f \in X. \quad (\text{A.2.1})$$

Then, the essential spectrum $\sigma_{\text{ess}}(T)$ of T on X is contained in a disk of radius a , i.e., $\sigma_{\text{ess}}(T) \subset \overline{B_a(0)}$. In particular, any spectral point of T of modulus strictly larger than a corresponds to an isolated eigenvalue of finite multiplicity.

Proof. Iterating the inequality (A.2.1), we deduce that for any n , there exists a constant $C_n > 0$ so that

$$\|T^n f\| \leq a^n \|f\| + C_n |f|_w.$$

Since the inclusion $X \hookrightarrow X_w$ is compact, for any $\delta > 0$ we may find $N_\delta \in \mathbb{N}$ and points $f_j \in B_1^{\|\cdot\|}$ so that

$$B_1^{\|\cdot\|} = \bigcup_{j=1}^{N_\delta} B_\delta^w(f_j) \cap B_1^{\|\cdot\|}$$

Applying T^n we deduce that

$$T^n(B_1^{\|\cdot\|}) = \bigcup_{j=1}^{N_\delta} T^n(B_\delta^w(f_j) \cap B_1^{\|\cdot\|}).$$

But now, we claim that as long as δ is small enough, it holds

$$T^n(B_\delta^w(f_j) \cap B_1^{\|\cdot\|}) \subset B_{4a^n}^{\|\cdot\|}(T^n(f_j)).$$

²We remark that in [Nus70] a more general definition of the measure of non-compactness is used. However, for the case of bounded operators, the one we are using will suffice.

Indeed, if $|g - f_j|_w \leq \delta$ and $g \in B_1^{\|\cdot\|}$, then applying the Lasota-Yorke inequality yields

$$\|T^n g - T^n f_j\| \leq a^n \|g - f_j\| + C_n |g - f_j|_w \leq 2a^n + C_n \delta.$$

Hence, choosing δ small enough, we deduce that

$$T^n(B_1^{\|\cdot\|}) \subset \bigcup_{j=1}^N B_{4a^n}^{\|\cdot\|}(T^n(f_j)).$$

Thus, we deduce that $\gamma(T^n) \leq 4a^n$, for all n . Hence, applying the theorem of Nussbaum A.2.7 we deduce that

$$r_{\text{ess}}(T) \leq \limsup_{n \rightarrow \infty} \gamma(T^n)^{1/n} \leq a.$$

□

Therefore, in many settings, estimating the essential spectral radius boils down to establishing a Lasota-Yorke inequality, with sharp bounds on the constants involved. We make heavy use of this in Chapter 5 and also in Section 2.3.1 of Chapter 2.

A.3 On perturbation theory

Having established various ways of ruling out the essential spectrum, it is now worth pointing out what exactly is gained by working for discrete points of the spectrum. Most immediately of course, such points are in fact true eigenvalues, which are defined exactly as in finite dimensions: If $\lambda \in \sigma(T)$, and there exists $x \in X \setminus \{0\}$ so that $\lambda x - Tx = 0$, we say that λ is an eigenvalue of T , and write $\lambda \in \sigma_p(T)$, the point spectrum of T . Due to the structure theorem A.1.5, we note that any point in the discrete spectrum not only corresponds to an eigenvalue, but if it is furthermore connected to the operator's resolvent set, it is in fact an *isolated point of the spectrum* of T , i.e. the case where $\lambda \in \sigma(T)$, and there exists an open set $U \ni \lambda$ so that $U \cap \sigma(T) = \{\lambda\}$. In this case, we may associate to λ an eigenprojection, known as the *Riesz projector*. Let Γ be a simple, closed curve in the complex plane surrounding λ , and so that $\Gamma \subset U$, then, we set

$$P_\lambda = \frac{1}{2\pi i} \int_{\Gamma} (\mu - T)^{-1} d\mu. \quad (\text{A.3.1})$$

We outline a number of the properties of this operator in the following lemma.

Lemma A.3.1. *The bounded operator P_λ satisfies the following:*

1. $P_\lambda^2 = P_\lambda$, i.e. it is a projection operator.
2. T commutes with P_λ , and $T|_{\text{Ran}(P_\lambda)}$ is a bounded operator that maps the range $\text{Ran}(P_\lambda)$ into itself.
3. $\sigma(T|_{\text{Ran}(P_\lambda)}) = \{\lambda\}$.
4. If $\lambda \in \sigma_{\text{disc}}(T)$, then $\text{Ran}(P_\lambda)$ is finite dimensional, λ is an eigenvalue of T , and $\text{Ran}(P_\lambda)$ coincides precisely with the generalized eigenspace of T at λ , i.e. $\text{Ran}(P_\lambda) = \bigcup_{n \in \mathbb{N}} \text{Ker}(T -$

$\lambda)^n$.

5. If there exists $\mu \in \rho(T)$ so that $(T - \mu)^{-1}$ is a compact operator, then $\text{Ran}(P_\lambda)$ is finite dimensional.
6. Generally, if $\Gamma \subset \mathbb{C}$ is any contour contained in $\rho(T)$ and we define P as in (A.3.1), then $\text{Ran}(P) = \{0\}$ if and only if $\sigma(T) \cap \text{int}(\Gamma) = \emptyset$.

The proof of the above Lemma A.3.1 may be found throughout [Kat76, Chapter III, Section 5, 6, 7]. Using the Riesz projector, we can now say that $\lambda \in \sigma(T)$ is a *simple eigenvalue* if $\dim \text{Ran}(P_\lambda) = 1$, and a *semisimple eigenvalue*, if the range of P_λ consists entirely of eigenfunctions of T , or equivalently $(T - \lambda)P_\lambda = 0$.

Riesz projectors and perturbation theory

We now discuss how the Riesz projector can be used in our perturbative arguments. In particular, we will primarily focus on the relatively non-degenerate case, where the perturbation is, in some sense, “at least as regular” as the unperturbed operator.

Given two unbounded operators $(T_0, D(T_0))$, $(T_1, D(T_1))$, we say that T_1 is *relatively bounded* with respect to T_0 if $D(T_0) \subset D(T_1)$, and there exist constants $a, b > 0$ so that for all $x \in D(T_0)$, it holds

$$\|T_1 x\| \leq a\|x\| + b\|T_0 x\|.$$

The greatest lower bound b_0 of all possible values of b in this definition is called the T_0 -*bound* of T_1 . This relative boundedness assumption significantly simplifies many technical difficulties that typically arise when dealing with perturbations of unbounded operators. In particular, it allows us to avoid complications related to the domains of the operators involved.

The following result, found in [Kat76, Chapter III, Section 4], is a fundamental consequence of this framework.

Theorem A.3.2. *Let $(T_0, D(T_0))$, $(T_1, D(T_1))$ be as above. Assume that*

1. $(T_0, D(T_0))$ is a closed operator.
2. T_1 is relatively bounded with respect to T_0 , with T_0 bound strictly less than 1.

Then, the operator $(T_0 + T_1, D(T_0))$ is closed.

Furthermore, under the relative boundedness assumption, it can be shown that for any $\mu \in \rho(T_0)$, the operator $T_1(\mu - T_0)^{-1} : X \rightarrow X$ is bounded. With this result in place, we are now ready to state and prove a key lemma that will serve as the main tool for our perturbative arguments.

Lemma A.3.3. *Let $(T_0, D(T_0))$ be a closed operator, with an isolated, discrete eigenvalue λ , and let Γ be a simple closed contour in the complex plane such that λ is the only spectral point of T_0 in its interior. Assume that $(T_1, D(T_1))$ is relatively bounded with respect to T_0 , with T_0 -bound strictly less than 1, and that*

$$\sup_{\mu \in \Gamma} \|T_1(\mu - T_0)^{-1}\| = M < 1. \quad (\text{A.3.2})$$

Define the Riesz projections

$$P_0 = \frac{1}{2\pi i} \int_{\Gamma} (\mu - T_0)^{-1} d\mu, \quad P_1 = \frac{1}{2\pi i} \int_{\Gamma} (\mu - T)^{-1} d\mu$$

where $T = T_0 + T_1$. Then

$$\|P_0 - P_1\| \leq \frac{1}{2\pi} |\Gamma| \frac{M}{1-M} \sup_{\mu \in \Gamma} \|(T_0 - \mu)^{-1}\| \quad (\text{A.3.3})$$

In particular, if

$$M < \frac{2\pi}{2\pi + |\Gamma| \sup_{\mu \in \Gamma} \|(T_0 - \mu)^{-1}\|},$$

then T has an isolated eigenvalue within Γ , and $\dim(\text{Ran}(P_0)) = \dim(\text{Ran}(P_1))$. Moreover, if λ is a simple eigenvalue of T_0 , then T also only has a simple eigenvalue in the interior of Γ .

Proof. We begin by noting that

$$T - \mu = (\text{Id} + T_1(T_0 - \mu)^{-1})(T_0 - \mu).$$

In view of (A.3.2), $\text{Id} + T_1(T_0 - \mu)^{-1}$ is invertible and therefore

$$(T - \mu)^{-1} = (T_0 - \mu)^{-1} (\text{Id} + T_1(T_0 - \mu)^{-1})^{-1}.$$

Writing the corresponding Neumann series, we find

$$(T - \mu)^{-1} = (T_0 - \mu)^{-1} \sum_{n \geq 0} (-1)^n (T_1(T_0 - \mu)^{-1})^n.$$

From here, we note that for $\mu \in \Gamma$,

$$\begin{aligned} \|(T - \mu)^{-1} - (T_0 - \mu)^{-1}\| &\leq \|(T_0 - \mu)^{-1}\| \left\| \sum_{n \geq 1} (T_1(T_0 - \mu)^{-1})^n \right\| \\ &\leq \frac{M}{1-M} \sup_{\mu \in \Gamma} \|(T_0 - \mu)^{-1}\|. \end{aligned}$$

In particular, (A.3.3) follows. Next, assume that M is small enough so that $\|P_1 - P_0\| < 1$. Since λ is a discrete spectral point of T_0 , it follows that $\dim(\text{Ran}(P_0)) < \infty$. We claim this implies that $\dim(\text{Ran}(P_1)) < \infty$ as well. Consider the map

$$P_0 : \text{Ran}(P_1) \rightarrow \text{Ran}(P_0).$$

Since P_0 is a projection, its range $\text{Ran}(P_0)$ is a closed, finite-dimensional subspace. If P_0 is injective on $\text{Ran}(P_1)$, then it is invertible on its range by the open mapping theorem, ensuring that $\dim(\text{Ran}(P_1)) \leq \dim(\text{Ran}(P_0))$.

To prove injectivity, assume there exists $x \in \text{Ran}(P_1)$ such that $P_0 x = 0$. Since P_1 is a projection,

we have $x = P_1 x$, and thus

$$\|x\| = \|P_0 x - P_1 x\| \leq \|P_0 - P_1\| \|x\| < \|x\|$$

which is a contradiction unless $x = 0$. Therefore, P_0 is injective on $\text{Ran}(P_1)$, establishing that $\dim(\text{Ran}(P_1)) \leq \dim(\text{Ran}(P_0))$. Repeating the argument with P_0 and P_1 interchanged shows the reverse inequality $\dim(\text{Ran}(P_0)) \leq \dim(\text{Ran}(P_1))$, and thus $\dim(\text{Ran}(P_1)) = \dim(\text{Ran}(P_0))$.

Since P_1 is a projection onto a finite-dimensional space, T restricted to $\text{Ran}(P_1)$ is a finite-dimensional operator (i.e. a matrix), meaning all its spectral points are eigenvalues. This completes the proof. $\square$

We now state and prove a simplified version of a result in perturbation theory, tailored to obtaining unstable eigenvalues for the dynamo problem. In [Kat76, Chapter VIII, Theorem 2.6], the following result is stated under more general assumptions, requiring several additional definitions that we prefer to avoid for clarity. Instead, we present a streamlined version sufficient for our purposes.

Theorem A.3.4. *Let $(T_0, D(T_0))$ be a closed operator on a Banach space X , and let $(T_1, D(T_1))$ be another closed operator that is bounded relative to T_0 . For sufficiently small κ , define the operator $T(\kappa) = (T_0 + \kappa T_1, D(T_0))$, which is a closed operator on X . Let λ be an isolated, semisimple eigenvalue of T_0 with finite multiplicity m , and let P_λ be the Riesz projector onto λ . Then, the eigenvalues $\mu_j(\kappa)$ of $T(\kappa)$ near λ satisfy the asymptotic bound*

$$\mu_j(\kappa) = \lambda + \kappa \mu_j + o(\kappa), \quad j \in \{1, \dots, m\}$$

where the μ_j 's are the (repeated) eigenvalues of the operator $P_\lambda T_1 P_\lambda$ restricted to $\text{Ran}(P_\lambda)$.

The proof is based on the following result from finite-dimensional perturbation theory, which can be found in [Kat76, Chapter II, Theorem 5.4]. For further details on the λ -group eigenvalues, which correspond to the eigenvalues of $T(\kappa)$ generated by the splitting of the eigenvalue λ at $\kappa = 0$, we refer to [Kat76, Chapter II, Section 2].

Theorem A.3.5. *Let $T(\kappa)$ be a family of linear operators on a finite dimensional space M , differentiable at $\kappa = 0$. If λ is a semisimple eigenvalue of $T(0)$, then the λ -group eigenvalues μ_j of $T(\kappa)$ are differentiable at $\kappa = 0$ and satisfy the asymptotic expansion*

$$\mu_j(\kappa) = \lambda + \kappa \mu_j^1 + o(\kappa)$$

where the μ_j^1 are the repeated eigenvalues of the operator $P_\lambda T'(0) P_\lambda$ restricted to $\text{Ran}(P_\lambda)$, and P_λ is the Riesz projector associated with λ .

We may now provide a proof of Theorem A.3.4, relying on Theorem A.3.5.

Proof of Theorem A.3.4. Let Γ be a curve around the eigenvalue λ , and let $P_\lambda, P_\lambda(\kappa)$ denote the Riesz projectors associated with the operators $T_0, T(\kappa) = T_0 + \kappa T_1$, respectively, around Γ . Denote the resolvents of T_0 and $T(\kappa)$ by $R(\mu; T_0) = (T_0 - \mu)^{-1}$ and $R(\mu; T(\kappa)) = (T(\kappa) - \mu)^{-1}$,

respectively. We begin by considering the operator

$$V(\kappa) = I - P_\lambda + P_\lambda(\kappa)P_\lambda.$$

Note that $V(\kappa)P_\lambda = P_\lambda(\kappa)P_\lambda$ and $V(\kappa)(I - P_\lambda) = I - P_\lambda$. By Lemma A.3.3, there exists some constant $c > 0$ such that $\|P_\lambda(\kappa) - P_\lambda\| \leq c\kappa$, for κ sufficiently. Consequently, for small κ , $V(\kappa)$ is invertible, and its inverse can be written as

$$V(\kappa)^{-1} = I + (P_\lambda - P_\lambda(\kappa))P_\lambda + o(\kappa)$$

where $o(\kappa)$ denotes any operator with norm of order $o(\kappa)$ as $\kappa \rightarrow 0$. In particular, $V(\kappa)$ is an isomorphism from $\text{Ran}(P_\lambda) \rightarrow \text{Ran}(P_\lambda(\kappa))$. Next, consider the operator

$$R_1(\mu; \kappa) := V(\kappa)^{-1}R(\mu; T(\kappa))V(\kappa)P_\lambda, \quad \text{for } \mu \in \Gamma.$$

This operator annihilates $\text{Ran}(I - P_\lambda)$. Since $V(\kappa)$ maps $\text{Ran}(P_\lambda)$ into $\text{Ran}(P_\lambda(\kappa))$ and $R(\mu; T(\kappa))$ commutes with $P_\lambda(\kappa)$ by functional calculus, we conclude that $R_1(\mu; \kappa)$ maps $\text{Ran}(P_\lambda(\kappa))$ into itself. Furthermore, by the invertibility of $V(\kappa)$, we have $R_1(\mu; \kappa) = P_\lambda R_1(\mu; \kappa)$. We now expand $R_1(\mu; \kappa)$ as

$$R_1(\mu; \kappa) = (I + (P_\lambda - P_\lambda(\kappa))P_\lambda + o(\kappa))(R(\mu; T_0) - \kappa R(\mu; T_0)T_1 R(\mu; T_0) + o(\kappa))(I - (P_\lambda - P_\lambda(\kappa))P_\lambda).$$

This simplifies to

$$R_1(\mu; \kappa) = R(\mu; T_0) - \kappa R(\mu; T_0)T_1 R(\mu; T_0) - R(\mu; T_0)(P_\lambda - P_\lambda(\kappa))P_\lambda + (P_\lambda - P_\lambda(\kappa))P_\lambda R(\mu; T_0) + o(\kappa). \quad (\text{A.3.4})$$

Using the fact that $R(\mu; T_0)P_\lambda = (\lambda - \mu)^{-1}P_\lambda$ due to the semisimplicity of λ for T_0 , we get

$$P_\lambda R(\mu; T_0)(P_\lambda - P_\lambda(\kappa))P_\lambda = P_\lambda(P_\lambda - P_\lambda(\kappa))P_\lambda R(\mu; T_0)P_\lambda.$$

Thus, using that $R_1(\mu; \kappa) = P_\lambda R_1(\mu; \kappa)P_\lambda$ and again $R(\mu; T_0)P_\lambda = (\lambda - \mu)^{-1}P_\lambda$, we deduce from (A.3.4) that

$$R_1(\mu; \kappa) = (\lambda - \mu)^{-1}P_\lambda - \kappa(\lambda - \mu)^{-2}P_\lambda T_1 P_\lambda + o(\kappa).$$

Multiplying by $\frac{-\mu}{2\pi i}$ and integrating around the contour Γ , we obtain the expression

$$V(\kappa)^{-1}T(\kappa)P_\lambda(\kappa)V(\kappa)P_\lambda = \lambda P_\lambda + \kappa P_\lambda T_1 P_\lambda + o(\kappa) \quad (\text{A.3.5})$$

where we have used the fact that

$$T(\kappa)P_\lambda(\kappa) = -\frac{1}{2\pi i} \int_{\Gamma} R(\mu; T(\kappa))\mu d\mu.$$

Observe that the $\mu_j(\kappa)$'s are the eigenvalues of $T(\kappa)P_\lambda(\kappa)$ in the m -dimensional space $\text{Ran}(P_\lambda(\kappa))$. By similarity, they are also the eigenvalues of $V(\kappa)^{-1}T(\kappa)P_\lambda(\kappa)V(\kappa)$ in the m -dimensional space

$\text{Ran}(P_\lambda)$. Since P_λ acts as the identity on $\text{Ran}(P_\lambda)$, they are also equal to the eigenvalues of $V(\kappa)^{-1}T(\kappa)P_\lambda(\kappa)V(\kappa)P_\lambda$ on $\text{Ran}(P_\lambda)$. Finally, applying Theorem A.3.5 and using the asymptotic expression (A.3.5), the proof follows immediately. $\square$

We note that Theorem A.3.4 may in fact be refined in the following way.

Lemma A.3.6. *Assume the setting of Theorem A.3.4, and further that PT_1P has m distinct eigenvalues on PTP . Let $P_j^{(1)}$ be the Riesz projectors associated to the eigenvalues of PT_1P , and denote by $P_j(\kappa)$ the projectors associated to the eigenvalues $p_j(\kappa)$, which act on X by $P_j^{(1)}x = P_j^{(1)}Px$. Then, in the strong operator topology*

$$P_j(\kappa) \rightarrow P_j^{(1)}, \quad \text{as } \kappa \rightarrow 0.$$

Proof of Lemma A.3.6. Continuing from the end of the proof of Theorem A.3.4, we recall the expression (A.3.5)

$$V(\kappa)^{-1}T(\kappa)P(\kappa)V(\kappa)P = pP + \kappa PT_1P + o(\kappa).$$

Now, note that $P_j(\kappa)$ is the one-dimensional eigenprojection for the eigenvalue $\mu_j + o_{\kappa \rightarrow 0}(1)$ of the operator

$$\tilde{T}^{(1)}(\kappa) := \kappa^{-1}(T(\kappa) - p)P(\kappa) : P(\kappa)X \rightarrow P(\kappa)X.$$

By (A.3.5) we have that

$$V(\kappa)^{-1}\tilde{T}^{(1)}(\kappa)P(\kappa)V(\kappa)P = PT_1P + o_{\kappa \rightarrow 0}(1), \quad (\text{A.3.6})$$

since $V(\kappa)^{-1}P(\kappa)V(\kappa)P = P$, as $V(\kappa)$ maps $Px \mapsto P(\kappa)x$. Note also that $V(\kappa)^{-1}\tilde{T}^{(1)}(\kappa)P(\kappa)V(\kappa)P$ is an operator from $PX \mapsto PX$. Thus, since $PT_1P : PX \rightarrow PX$ has m distinct eigenvalues, for all κ small enough it follows from (A.3.6) that the same holds for $V(\kappa)^{-1}\tilde{T}^{(1)}(\kappa)P(\kappa)V(\kappa)P$. Choose its m eigenvectors to be $\psi_i(\kappa)$, $i = 1, \dots, m$. From (A.3.6) it follows that $\psi_i(\kappa) \rightarrow \psi_j$, where ψ_j is the eigenvector corresponding to an eigenvalue μ_j of PT_1P . Next, note that in fact $\phi_j(\kappa) = V(\kappa)\psi_j(\kappa)$ is an eigenvector of $\tilde{T}^{(1)}(\kappa)$ with eigenvalue $\mu_j(\kappa)$. Indeed, from (A.3.6) we see that, since $\psi_j(\kappa) \in PX$, it holds

$$\tilde{T}^{(1)}(\kappa)V(\kappa)\psi_j(\kappa) = \tilde{T}^{(1)}(\kappa)V(\kappa)P\psi_j(\kappa) = \mu_j V(\kappa)\psi_j(\kappa).$$

Furthermore, since $V(\kappa) \rightarrow \text{Id}$ as $\kappa \rightarrow 0$, it follows that $\phi_j(\kappa) \rightarrow \psi_j$ as $\kappa \rightarrow 0$. Since the $\phi_j(\kappa)$ are linearly independent, they form a basis for the m -dimensional space $P(\kappa)X$. Thus, given $x \in X$ it holds

$$P(\kappa)x = \sum_{i=1}^m \alpha_i(\kappa)\phi_i(\kappa),$$

for some uniquely determined coefficients $\alpha_j(j = \kappa)$. From this it follows that $P_j^{(1)}(\kappa)x = \alpha_j(\kappa)\psi_j(\kappa)$, $j = 1, \dots, m$. Furthermore, we claim that the $\alpha_j(\kappa)$ converge as $\kappa \rightarrow 0$. Indeed, consider the following map $M(\kappa) : PX \rightarrow PX$, defined by $M(\kappa)\phi_j = \psi_j(\kappa)$. Since $\psi_j(\kappa) \rightarrow \phi_j$, it follows that

$M(\kappa) \rightarrow \text{Id}$ as $\kappa \rightarrow 0$. Therefore, $M^{-1}(\kappa) \rightarrow \text{Id}$ as well, and it holds

$$M^{-1}(\kappa)V(\kappa)^{-1}P(\kappa)x = M^{-1}(\kappa) \sum_{j=1}^m \alpha_j(\kappa)\psi_j(\kappa) = \sum_{j=1}^m \alpha_j(\kappa)\phi_j.$$

The left-hand side converges to Px as $\kappa \rightarrow 0$, and so, since the ϕ_j form a basis of PX , it follows that the $\alpha_j(\kappa)$ must converge to some limit α_j . Therefore, the projectors $P_j^{(1)}(\kappa)x = \alpha_j(\kappa)\psi_j(\kappa)$ converge pointwise to $\alpha_j\phi_j$. But note that, since

$$Px = \sum_{j=1}^m \alpha_j\phi_j,$$

it follows that $P_j^{(1)}Px = \alpha_j\phi_j$, and so the proof is completed. $\square$

Finally, we shall present the seminal result in perturbation theory of Keller and Liverani, which combines the Lasota-Yorke framework of Proposition A.2.8 with techniques in the *singular perturbation theory*, to treat spectral perturbations that do not fall into the clean framework of Lemma A.3.3.

Proposition A.3.7 (Keller–Liverani). *Let $(T_\varepsilon)_{\varepsilon \geq 0}$ be a family of bounded linear operators acting on Banach spaces $(X, \|\cdot\|)$ and $(X_w, |\cdot|_w)$ satisfying the assumptions of Proposition A.2.8. Suppose there exist constants $0 \leq a < M$ such that the following hold for all $f \in X$:*

- (i) $|T_\varepsilon f|_w \leq M|f|_w$;
- (ii) $|T_\varepsilon f - T_0 f|_w \leq \tau_\varepsilon \|f\|$, for some modulus of continuity $\tau_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$;
- (iii) $\|T_\varepsilon f\| \leq a\|f\| + M|f|_w$;
- (iv) *There exists an eigenvalue $\lambda \in \mathbb{C}$ of T_0 with $|\lambda| > a$.*

Then, there exists $\delta > 0$ such that $\sigma(T_\varepsilon) \cap \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \delta\}$ consists of finitely many eigenvalues (counted with algebraic multiplicity) that converge to λ as $\varepsilon \rightarrow 0$.

A.4 Semigroup theory and sectorial operators

Thus far, we have mainly discussed the spectral theory associated to operators which appear for instance as the generator of (KDE- ε). However, we are fundamentally interested in the behaviour of solutions to these equations over time, so it is pertinent to translate this spectral information into information about the time-evolution of the equations. The canonical tool for doing so is *semigroup theory*, which we shall now discuss. Of particular interest is the theory of *analytic semigroups*, which are (in some sense) a generalisation of semigroups generated by self-adjoint operators. In particular, one expects fairly strong regularising properties under the evolution of such semigroups. The generators of such semigroups are known as *sectorial operators*, the definition of which we shall now recall, see Figure A.1.

Definition A.4.1. Let $(X, \|\cdot\|)$ be a separable Banach space and let $T : \mathcal{D}(T) \subset X \rightarrow X$ be a closed operator with dense domain. We say that T is sectorial of angle $\theta_0 \in (0, \frac{\pi}{2})$ if there exists a

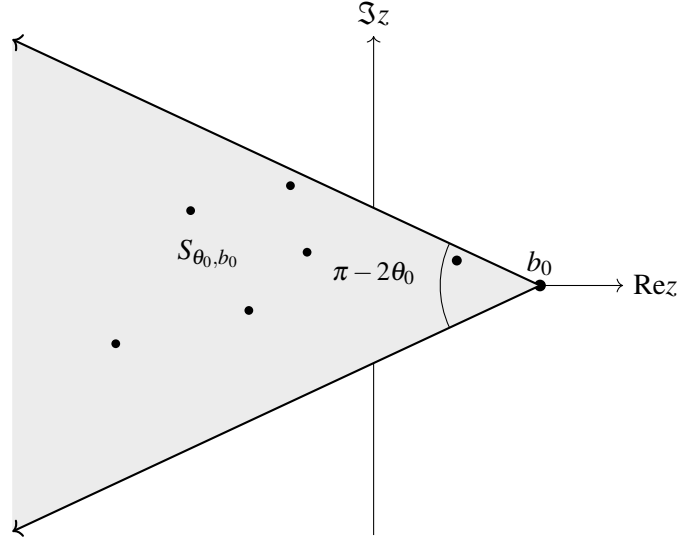


Figure A.1: The spectrum $\sigma(T)$ of a sectorial operator contained in the sector S_{θ_0, b_0} .

constant $b_0 > 0$ so that the spectrum of T is contained in the sector $S_{\theta_0, b_0} = \{z \in \mathbb{C} : \text{Arg}(b_0 - z) \in [-\frac{\pi}{2} + \theta_0, \frac{\pi}{2} - \theta_0]\}$, and there exists a constant $C > 0$ so that

$$\|(\lambda - T)^{-1}\| \leq C \text{dist}(\lambda, S_{\theta_0, b_0})^{-1}$$

for all $\lambda \notin S_{\theta_0, b_0}$.

A key property of sectorial operators is that their generated semigroups admit an explicit representation formula via the holomorphic functional calculus for sectorial operators.

Theorem A.4.2. *Let T be sectorial of angle θ_0 with constant b_0 . For any $\theta < \theta_0$, $b > b_0$, denote by $\gamma_{\theta, b}$ the contour obtained by traversing the boundary of the sector $S_{\theta, b}$ counterclockwise. Then, the semigroup generated by T is given by*

$$S(t) = \frac{1}{2\pi i} \int_{\gamma_{\theta, b}} (z - T)^{-1} e^{zt} dz.$$

As a result of this representation formula, it is possible to show that such analytic semigroups enjoy rather useful regularising properties, which we employ throughout Chapter 7. The following results on interpolation spaces follow the presentation in [Hai]. As a first remark, we note that, up to translation, we may assume that $\{\text{Re}(z) \geq 0\} \subset \rho(T)$. Then, for any $\alpha > 0$ we define the operator $(-T)^{-\alpha}$ via

$$(-T)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t) dt.$$

Of course, if T is a normal operator, fractional powers can just be defined using the functional calculus, but this definition is general enough to encompass a wider class of operators. Furthermore, one can define the unbounded operator $(-T)^\alpha$ as the map from $\text{Ran}(-T)^{-\alpha} \rightarrow X$ defined via $((-T)^{-\alpha})^{-1}$. (this inverse exists as an unbounded operator, since $(-T)^{-\alpha}$ is injective). Given an analytic semigroup $S(t)$ on a Banach space X , we then define the interpolation space X_α as the range $\text{Ran}(-T)^{-\alpha}$, equipped with norm $\|x\|_\alpha = \|(-T)^\alpha x\|$. There then holds the following useful

result.

Lemma A.4.3 ([Hai, Proposition 4.40]). *Let T be sectorial, and let $S(t)$ be its generated analytic semigroup. Then, $S(t)$ maps X into X_α for any $\alpha > 0$, and there holds*

$$\|S(t)x\|_\alpha \leq \frac{C_\alpha}{t^\alpha} \|x\|,$$

for any $t \in (0, 1]$.

In fact, interpolation spaces, and more generally the property of being a sectorial operator are rather stable under perturbations, as seen in the following result.

Lemma A.4.4 ([Hai], Proposition 4.45). *Let T generate an analytic semigroup on some separable Banach space X , and suppose $S: \mathcal{D}((-T)^\alpha) \rightarrow X$ is a bounded operator, for some $\alpha \in [0, 1)$. Then, $T + S$ generates an analytic semigroup, and $\mathcal{D}((-T - S)^\alpha) \sim \mathcal{D}((-T)^\alpha)$, for all $\alpha \in (0, 1)$, i.e. the corresponding domains are equal, and the norms $\|(-T - S)^\alpha B\|$, $\|(-T)^\alpha B\|$ are equivalent for any $B \in \mathcal{D}((-T)^\alpha)$.*

A further useful property of analytic semigroups is that they enjoy the *spectral mapping property*—that is, their growth rates are entirely determined by their spectral behaviour, which is *not* the case for general C^0 semigroups, see for instance [EN00]. We prove a special case of interest in this vein, which further assumes that the generator of the semigroup has a compact resolvent.

Lemma A.4.5. *Let $(T, \mathcal{D}(T))$ be a sectorial operator with compact resolvent on some Hilbert space H , with $\sigma(T) \subset S_{\theta_0, b_0}$ as in Definition A.4.1. Then, for all $p < b_0$, and all constants $\zeta > 0$ small enough, there exist contours γ_1, γ_2 so that $\gamma_1 \subset \{\operatorname{Re}(z) \leq p - \zeta\}$, $\gamma_2 \subset \{\operatorname{Re}(z) \geq p - \frac{\zeta}{2}\}$ and the Semigroup generated by T is equal to*

$$e^{Tt} = \frac{1}{2\pi i} \int_{\gamma_1} (z - T)^{-1} e^{zt} dz + \frac{1}{2\pi i} \int_{\gamma_2} (z - T)^{-1} e^{zt} dz =: S_1(t) + S_2(t).$$

Furthermore, there holds $\liminf_{t \rightarrow \infty} \frac{1}{t} \log \|e^{Tt}x\| \geq p - \frac{\zeta}{2}$ if and only if $Px \neq 0$, where P is the Riesz projector associated to γ_2 as in (A.3.1).

Lemma A.4.6. *Let $T(\kappa) = \kappa T_0 + T_1$, where $(T_0, \mathcal{D}(T_0))$ is a negative, self-adjoint operator with compact resolvent, and for any $\delta > 0$ there exists a constant $C(\delta)$, continuous in δ so that for all $x \in \mathcal{D}(T_0)$ it holds*

$$\|T_1 x\| \leq \delta \|T_0 x\| + C(\delta) \|x\|.$$

Suppose further that there exists $x_0 \in H$ so that $\liminf_{t \rightarrow \infty} \frac{1}{t} \log \|e^{T(1)t} x_0\| = p > 0$. Then, there exist constants $\kappa_0 > 0$, $\eta_0 > 0$, $C > 0$, $T > 0$ depending only on $\frac{x_0}{\|x_0\|}$, $T(1)$, so that for all $|1 - \kappa| \leq \kappa_0$, $\|x - x_0\| \leq \eta_0 \|x_0\|$ it holds

$$\|e^{T(\kappa)t} x\| \geq C e^{\frac{p}{2}t} \|x\|, \quad \forall t \geq T.$$

Proof of Lemma A.4.5. To aid in readability, we shall denote the resolvent $R_z = (z - T)^{-1}$. By standard analytic Semigroup theory, we for any $b > b_0$, $\theta \in (0, \theta_0)$, the contour $\gamma_{\theta, b}$ given by the

boundary of the sector $\{\text{Arg}(b-z) \in [-\frac{\pi}{2} + \theta, \frac{\pi}{2} - \theta]\}$ satisfies

$$e^{Tt} = \frac{1}{2\pi i} \int_{\gamma_{\theta,b}} e^{zt} (z-T)^{-1} dz.$$

Now, take $p < b$. Then, for all $\zeta > 0$ small enough, the strip $\{\text{Re}(z) \in [p - \zeta, p - \frac{\zeta}{2}]\}$ does not intersect the spectrum of T . Indeed, suppose this was not true. Then, there would exist a sequence of eigenvalues λ_n of T , with $\text{Re}(\lambda_n) \in [p - \frac{1}{3^n}, p - \frac{1}{2 \times 3^n}]$. In particular, note that all the λ_n must be distinct, since the intervals $[p - \frac{1}{3^n}, p - \frac{1}{2 \times 3^n}]$ are non-overlapping. Furthermore, due to the sectorial property, the λ_n have uniformly bounded imaginary part. Thus, there exists a (non-relabelled) convergent subsequence, converging to some $\lambda \in \mathbb{C}$, $\text{Re}(\lambda) = p$. By closedness of the spectrum, $\lambda \in \sigma(T)$, and clearly λ is an accumulation point of the spectrum, which cannot occur due to the compactness of the resolvent of T . Therefore, for all ζ small enough, we can construct new contours $\gamma_{\theta,b}^1, \gamma_{\theta,b}^2$, which are constructed as follows. $\gamma_{\theta,b}^1$ proceeds by first traversing $\gamma_{\theta,b}$ along its lower branch, until the line $\{\text{Re}(z) = p - \frac{\zeta}{2}\}$ is reached. Then, traverse the line $\{\text{Re}(z) = p - \frac{\zeta}{2}\}$ upwards until it intersects the upper branch of $\gamma_{\theta,b}$, at which point one traverses this upper branch towards the left. Similarly, $\gamma_{\theta,b}^2$ is obtained by first traversing the contour $\gamma_{\theta,b} \cap \{\text{Re}(z) \geq p - \frac{\zeta}{2}\}$, and then traversing the line $\{\text{Re}(z) = p - \frac{\zeta}{2}\}$ downwards in order to close the contour. Overall, since the traversals along $\{\text{Re}(z) = p - \frac{\zeta}{2}\}$ of the contour cancel, we immediately obtain

$$e^{Tt} = \frac{1}{2\pi i} \int_{\gamma_{\theta,b}^1} R_z e^{zt} dz + \frac{1}{2\pi i} \int_{\gamma_{\theta,b}^2} R_z e^{zt} dz.$$

But note now that since $\{\text{Re}(z) \in [p - \zeta, p - \frac{\zeta}{2}]\}$ contains no spectral points of T , by Cauchy's theorem we may deform $\gamma_{\theta,b}^1$ to traverse the line $\{\text{Re}(z) = p - \zeta\}$ upwards instead of the line $\{\text{Re}(z) = p - \frac{\zeta}{2}\}$. Hence, setting $\gamma_1 = \gamma_{\theta,b}^1$ (the deformed version), and $\gamma_2 = \gamma_{\theta,b}^2$, these satisfy the claimed properties.

Next, we prove the claimed growth bounds. To begin, note that P from (A.3.1) is well defined, and is nothing but the Riesz projector onto the space spanned by eigenvalues of T with real part at least $p - \frac{\zeta}{2}$. By the functional calculus, it holds that $e^{Tt}P = S_2(t)$. In fact, we furthermore deduce that $S_2(t)$ in fact generates a group. In particular, the integral expression defining $S_2(t)$ is well defined if $t < 0$, and by the functional calculus it follows that $S(-t)S(t) = P$. Furthermore, observe the estimate

$$\|S_2(-t)\| \leq \frac{1}{2\pi} \text{len}(\gamma_2) \sup_{z \in \gamma_2} \|R_z\| e^{-t(p - \frac{\zeta}{2})}.$$

Therefore, we deduce that

$$\|Px\| = \|S(-t)S(t)x\| \leq \frac{1}{2\pi} \text{len}(\gamma_2) \sup_{z \in \gamma_2} \|R_z\| e^{-t(p - \frac{\zeta}{2})} \|S(t)x\|,$$

and thus

$$\|S_2(t)x\| \geq C_1(\gamma_2, \sup_{z \in \gamma_2} \|R_z\|) e^{t(p - \frac{\zeta}{2})} \|Px\|.$$

Next, we proceed to bounding $S_1(t)$. To do so, we identify the contour $\gamma_{\theta,b}$ with the lines $\ell_1 = \{x(1 + i\alpha) - i\alpha b, x \leq b\}$, $\ell_2 = \{x(1 - i\alpha) + i\alpha b, x \leq b\}$, for some $\alpha > 0$. Then, $S_1(t)$ is nothing

but

$$\begin{aligned} S_1(t) &= \frac{1}{2\pi i} \int_{-\infty}^{p-\zeta} R_{x(1+i\alpha)-i\alpha b} e^{t(x(1+i\alpha)-i\alpha b)} (1+i\alpha) dx + \frac{1}{2\pi i} \int_{\alpha(p-\zeta-b)}^{\alpha(b-p+\zeta)} i R_{p-\zeta+ix} e^{t(p-\zeta)} dx \\ &+ \frac{1}{2\pi i} \int_{p-\zeta}^{-\infty} R_{x(1-i\alpha)+i\alpha b} e^{t(x(1-i\alpha)+i\alpha b)} (1-i\alpha) dx. \end{aligned}$$

Hence, bounding these terms we obtain

$$\|S_1(t)\| \leq C_2(\gamma_1, \sup_{z \in \gamma_1} \|R_z\|)(1+t^{-1})e^{t(p-\zeta)}.$$

Hence, overall we observe that if $Px = 0$, it holds

$$\|S(t)x\| \leq C_2(\gamma_1, \sup_{z \in \gamma_1} \|R_z\|)(1+t^{-1})e^{t(p-\zeta)}\|x\|,$$

whereas in general we obtain the lower bound

$$\|S(t)x\| \geq C_1(\gamma_2, \sup_{z \in \gamma_2} \|R_z\|)e^{t(p-\frac{\zeta}{2})}\|Px\| - C_2(\gamma_1, \sup_{z \in \gamma_1} \|R_z\|)(1+t^{-1})e^{t(p-\zeta)}\|x\|,$$

and so the claim of the Lemma follows. $\square$

Proof of Lemma A.4.6. Upon replacing $x_0 \mapsto \frac{x_0}{\|x_0\|}$ we can assume that $\|x_0\| = 1$ and deduce the general result by linearity. Note that there holds for $\lambda \notin (-\infty, 0]$

$$\begin{aligned} \|T_1(\kappa T_0 - \lambda)^{-1}\| &\leq \delta \kappa \|T_0(\kappa T_0 - \lambda)^{-1}\| + C(\delta \kappa) \|(\kappa T_0 - \lambda)^{-1}\| \\ &\leq \delta + \delta |\lambda| \|(\kappa T_0 - \lambda)^{-1}\| + C(\delta \kappa) \|(\kappa T_0 - \lambda)^{-1}\| \\ &\leq \delta \left(1 + \frac{|\lambda|}{\text{dist}(\lambda, (-\infty, 0])}\right) + \frac{C(\delta \kappa)}{\text{dist}(\lambda, (-\infty, 0])}. \end{aligned}$$

Thus, there exists some $b_0 > 0$, $0 < \theta_0 < \frac{\pi}{2}$ so that for all $\lambda \notin S_{\theta_0, b_0}$, $\kappa \in (\frac{1}{2}, \frac{3}{2})$, it holds $|\lambda| \leq 2\text{dist}(\lambda, (-\infty, 0])$, and $\frac{C(\frac{\kappa}{6})}{\text{dist}(\lambda, (-\infty, 0])} \leq \frac{1}{3}$. Thus, there holds

$$\|T_1(\kappa T_0 - \lambda)^{-1}\| \leq \frac{5}{6}$$

uniformly for $\lambda \notin S_{\theta_0, b_0}$, $\kappa \in (\frac{1}{2}, \frac{3}{2})$. Hence, we have

$$\|(T(\kappa) - \lambda)^{-1}\| = \|(\kappa T_0 - \lambda)^{-1}(\text{Id} + T_1(\kappa T_0 - \lambda)^{-1})^{-1}\| \leq \frac{6}{\text{dist}(\lambda, (-\infty, 0])},$$

which is the analytic semigroup bound. Therefore, for all $\kappa \in (\frac{1}{2}, \frac{3}{2})$, the semigroup generated by $T(\kappa)$ is given by

$$e^{T(\kappa)t} = \frac{1}{2\pi i} \int_{\gamma} (z - T(\kappa))^{-1} e^{zt} dz,$$

where the contour γ is independent of κ , and is the boundary of some sector $S_{\theta, b}$ with $\theta < \theta_0$,

$b > b_0$. Then, by assumption and Lemma A.4.5, we may write

$$e^{T(1)t} = \frac{1}{2\pi i} \int_{\gamma_1} (z - T(1))^{-1} e^{zt} dz + \frac{1}{2\pi i} \int_{\gamma_2} (z - T(1))^{-1} e^{zt} dz,$$

where γ_1, γ_2 are constructed from γ as in the proof of Lemma A.4.5. In particular, $\gamma_1 \subset \{\operatorname{Re}(z) \leq p - \zeta\}$ for some $\zeta > 0$ and $\gamma_2 \subset \{\operatorname{Re}(z) \geq p - \frac{\zeta}{2}\}$. Now, we claim this decomposition is valid for all κ sufficiently close to 1. Indeed, from the proof of Lemma A.4.5, it is clear that the decomposition follows as soon as $\sigma(T(\kappa)) \cap \{\operatorname{Re}(z) \in [p - \zeta, p - \frac{\zeta}{2}]\} = \emptyset$. But now, we compute

$$(\kappa T_0 + T_1 - \lambda) = (\operatorname{Id} + (\kappa - 1)T_0(T_0 + T_1 - \lambda)^{-1})(T_0 + T_1 - \lambda).$$

As soon as $\|(\kappa - 1)T_0(T_0 + T_1 - \lambda)^{-1}\| < 1$ this is invertible. But now,

$$\|T_0(T_0 + T_1 - \lambda)^{-1}\| \leq 1 + \|(T_1 - \lambda)(T_0 + T_1 - \lambda)^{-1}\| \leq 1 + |\lambda| \|(T_0 + T_1 - \lambda)^{-1}\| + \|T_1(T_0 + T_1 - \lambda)^{-1}\|.$$

But by assumption this last term may be bounded by

$$\frac{1}{2} \|T_0(T_0 + T_1 - \lambda)^{-1}\| + C\left(\frac{1}{2}\right) \|(T_0 + T_1 - \lambda)^{-1}\|.$$

Hence, we observe that

$$\|T_0(T_0 + T_1 - \lambda)^{-1}\| \leq 2 \left(1 + (|\lambda| + C\left(\frac{1}{2}\right)) \|(T_0 + T_1 - \lambda)^{-1}\| \right).$$

Taking suprema over $\lambda \in S_{\theta,b} \cap \{\operatorname{Re}(z) \in [p - \zeta, p - \frac{\zeta}{2}]\}$, we observe from the continuity of the resolvent with respect to λ that

$$\sup_{\lambda \in S_{\theta,b} \cap \{\operatorname{Re}(z) \in [p - \zeta, p - \frac{\zeta}{2}]\}} \|(T_0(T_0 + T_1 - \lambda)^{-1}\| \leq K,$$

for some constant K . Thus, $\{\operatorname{Re}(z) \in [p - \zeta, p - \frac{\zeta}{2}]\} \subset \rho(T(\kappa))$ if $|\kappa - 1| \leq \frac{1}{2}K^{-1}$, and we have the uniform resolvent bound for all λ in this strip

$$\|(T(\kappa) - \lambda)^{-1}\| \leq C(K).$$

Furthermore, combined with the estimates at the beginning of the proof, we deduce that $\sup_{z \in \gamma_1} \|(z - T(\kappa))^{-1}\| + \sup_{z \in \gamma_2} \|(z - T(\kappa))^{-1}\| \leq C(K)$ for all $|\kappa - 1| \leq \frac{1}{2}K^{-1}$. Thus, uniformly for κ in this interval, it holds

$$\|e^{T(\kappa)t}x\| \geq C_1 e^{t(p - \frac{\zeta}{2})} \|P(\kappa)x\| - C_2(1 + t^{-1}) e^{t(p - \zeta)} \|x\|,$$

where

$$P(\kappa) = \frac{1}{2\pi i} \int_{\gamma_2} (z - T(\kappa))^{-1} dz.$$

Now, by the same estimates as above, it follows that $\|P(\kappa) - P(1)\| \rightarrow 0$ as $\kappa \rightarrow 1$. Furthermore,

by Lemma A.4.5 it holds that $\|Px_0\| > 0$. Therefore, we have

$$\|P(\kappa)x\| = \|(P(\kappa) - P)x + Px\| \geq \|Px_0\| - \|P(x - x_0)\| - \|(P - P(\kappa))x\|.$$

Thus, so long as $\|P - P(\kappa)\| \leq \frac{\|Px_0\|}{3\|x\|}$, $\|x - x_0\| \leq \frac{\|Px_0\|}{3\|P\|}$, there holds

$$\|P(\kappa)x\| \geq \frac{\|Px_0\|}{3}.$$

Hence, this is true for all $\|x - x_0\| \leq \eta_0$, for some $\eta_0 > 0$, and for $|\kappa - 1| \leq \kappa_0$ for some $\kappa_0 > 0$. Thus, for all such x, κ it holds

$$\|e^{T(\kappa)t}x\| \geq \frac{1}{3}C_1e^{t(p-\frac{\zeta}{2})}\|Px_0\| - 2C_2(1+t^{-1})e^{t(p-\zeta)}\|x_0\|.$$

Now, since $\|Px_0\| \neq 0$, there exists a constant so that $\|Px_0\| \geq c\|x_0\|$, and so for all t large enough, it holds

$$\|e^{T(\kappa)t}x\| \geq Ce^{t(p-\frac{3\zeta}{4})}\|x_0\|.$$

We can further note (upon possibly reducing the size of η), that $\|x_0\| \geq \frac{1}{2}\|x\|$ for all $\|x - x_0\| \leq \eta_0$, and so the claim of the Lemma follows. $\square$

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