Imperial College of Science, Technology and Medicine
Department of Electrical and Electronic Engineering
Control and Power Research Group

Large-Scale Distributed State Estimation: Theory, Methodology and Algorithm

Guitao Yang

Submitted in fulfilment of the requirements for the degree of
Doctor of Philosophy at Imperial College London, January 2022

Declaration of Originality

I hereby declare that this thesis is the work of the author's own work. Any ideas, concepts or results presented in this thesis are appropriately referenced if they are the work from other people.

Copyright Declaration

Abstract

In this thesis, the topics of state estimation for large-scale systems (LSSs) with distributed observer design are studied using notions and tools from classical control theory and graph theory. Over the last few decades, there has been a remarkable growth of LSSs which are essential to people's daily lives, such as smart power grids, water networks, industrial plants, etc. However, these systems are susceptible to faults or attacks, and it is crucial to monitor the status of such systems by estimating the state vector. Due to the vast space that a typical LSS occupies, measurements for such systems are usually distributed over the entire system plant. Our objective is to design a network of observers such that the state vector of the entire system can be estimated, while each observer (node) has access to only local output measurement that may not be sufficient on its own to reconstruct the whole system state. Moreover, we address several categories of faults and disturbances at LSSs and their observers, and the devised distributed estimation schemes are desired to be robust accordingly.

Firstly, we consider a scenario that the links among the distributed observers may fail and rebuild over time, which implies that the communication network does not stay connected constantly. We propose a distributed estimation approach with a guarantee on the feasibility of the design such that the state vector of the system is reconstructed by each observer. Then, we assume that the local observer at each node may not be accessible to all input signals, which could be caused by an unknown input fault or disturbance. We propose a distributed observer capable of estimating the overall system's state in the presence of input, while each node only has some information on input and its local measurements. We provide a design method that guarantees the convergence of the estimation errors under some mild joint detectability conditions, which is also proved to be necessary to obtain asymptotic estimates. After that, we focus on the distributed state estimation with additive fault and noise at the measurements. We assume that local sensor redundancy exists at each node and propose a distributed estimation strategy that exploits this redundancy to provide robustness against faults in the measurement. Under suitable conditions on the redundant sensors, we show that it is possible to mitigate the effects of a class of unbounded sensor faults on the state estimation. Finally, we build a simplified heat exchange model to verify the effectiveness of all of the proposed distributed state estimation methods.

Acknowledgements

Before I started my PhD, I understood this would be a life-changing experience. However, I would never realise how much the world and I changed during my PhD. I am grateful to all the people who have supported me throughout this incredible journey.

In particular, I wish to express my sincerest and deepest gratitude to Prof. Thomas Parisini, for providing me with the opportunity to work with him and to learn from him. He always believes in me and is willing to listen to my opinions patiently. I also want to thank Dr Hamed Rezaee for his generous support and encouragement throughout my PhD. From him, I not only learned to become a better researcher, but also became a man of integrity and kindness.

Special thanks are extended to Prof. Angelo Alessandri, for the fruitful discussions during the lockdown. Also, I would like to thank Prof. Gilberto Pin, for the very prompt support and insightful suggestions.

I want to say a big thank you to my friends and colleagues in the Control and Power Group at Imperial College London, especially Kaiwen, Angelo and Alex. I always enjoy the time when we work together, and I have learnt a lot from you. Also, I want to thank Junhong and Tianwei for the company and support.

During the lockdown and the uncertain time due to the pandemic, I received lots of support and kind help. Among those, I wish to thank Cooper, Junjie, Zaici, Eva, Jerry and Yi for the company and fun time together (including the virtual ones). I also want to thank Wenbo for the deep chat we had, which enlightened me a lot. Luckily because of you, the lockdown was not a horrible memory to me.

I wish to express my appreciation to Yuxin Ji, for supporting me and educating me selflessly, without whom I would never go to this step. I also have to thank my sister Zhen, for all the overnight talks (I am truly sorry for letting you stay up late often because of the time difference). I am deeply grateful to my parents for always caring for me, supporting me, and always being proud of my every achievement.

Lastly but most importantly, I want to thank Zhengming, my dearest, for all the company, support, for all the happy and sad moments we spent together. I cherish you always.

Contents

List of Figures

Chapter 1

Introduction

1.1 Motivations and Objectives

Large-Scale Systems (LSSs) refer to systems in which a large number of states interact over a huge area [1, 2]. Thus, conventional strategies of modelling, analysis and design for such systems usually cannot provide desirable solutions with reasonable efforts. Moreover, a system is considered large-scale if it can be decomposed into subsystems, and LSSs are distributed over a vast space with complicated interconnections among their sub-systems in many cases.

In recent years, there has been a significant growth of LSSs such as smart power grids, water networks, manufacturing plants, etc [3, 4, 5]. These systems are indeed vital to people's daily life, and their importance is even further highlighted in major cities. However, many LSSs are susceptible to faults and cyber attacks, hence the operation status of such systems is required to be monitored with reasonable precision. Thanks to the increasing ubiquity of cyber-physical systems which empowers sensing equipment with strong communication and computation capabilities, complex algorithms are possible to be deployed on sensor equipment, which remarkably benefits the state monitoring of LSSs.

Indeed, the state estimation can be conducted in a centralised way, which indicates that there is a central computation unit collecting all the measurements from sensors at different areas.

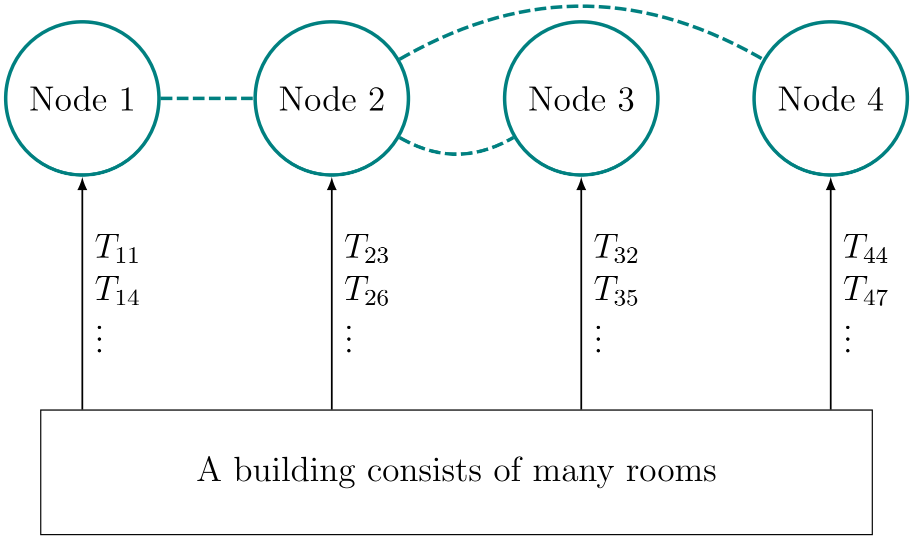


Figure 1.1: An example of a distributed estimation scheme consisting of 4 nodes: each local observer at Node i has access to the measurements of the thermometer at Room i (T_i). Furthermore, neighbouring estimates are exchanged over an undirected communication network (dashed line).

However, centralised state estimation methods may result in additional complexity and information coordination problem, hence running distributed algorithms is a more efficient and effective design choice. Therefore, this thesis addresses the state estimation problem for a classical Linear Time-Invariant (LTI) dynamical system with sensors distributed over the entire system. More specifically, the *distributed state estimation problem* is to design a group of N observers co-located with sensors such that each observer computes an estimate of the state vector of the entire system, while only having access to measurements that are local to each sensor node. Indeed most LSSs have very large state space, the local measurements at each node may not be sufficient to estimate the state of the entire system. Hence, each observer has to exploit the shared information of the state estimate from its neighbouring observers over a communication network to achieve the state estimation of the entire system plant. In this thesis, we focus on the *distributed state estimation problem* where communication of the estimated states among nodes is necessary to achieve state omniscience.

To illustrate the motivation of the distributed estimation problem, we introduce the following example. Suppose a large building consists of hundreds of rooms, and we intend to monitor the temperature of each room. However, only a few rooms are equipped with thermometers, and their real-time measurements are locally available at the nearest node. The nodes can transfer information to each other following a given communication topology. Moreover, parameters of the thermal-dynamical model are time-invariant and known (such as the heat exchange rate of each room). The objective is to build observers co-located at each node where all of the room temperatures can be estimated asymptotically (see Figure 1.1). By investigating the distributed state estimation problem, we can achieve this objective with the developed strategies. Indeed, we will establish a simplified model based on this illustrative example in Chapter 6, and employ our devised strategies to validate their effectiveness. We will also compare our distributed estimation schemes with the centralised ones to demonstrate the pitfalls of the latter design in detail.

Moreover, as mentioned before, LSSs are prone to suffer from faults and disturbances, which may lead to catastrophes if no correcting action is taken promptly. To obtain distributed observer design with robust state estimation performance, we need to take those faults and disturbances into consideration. More specifically, we consider three main different types of faults or disturbances to the target LSSs and their corresponding distributed state estimation schemes:

1. The communication network among the distributed observers is faulty. More specifically, the communication links among distributed observers may fail and rebuild over time and the communication graph does not stay connected constantly.

2. The input of the LSSs has some unknown faults or subsets of the input signals are inaccessible to some of the observers. For example, there may be some disturbances or faults injected into the monitored LSS that are unknown to all of the distributed observers.

3. The measurements of the LSSs are contaminated, i.e., there are some additive faults with unknown bounds or noise at sensor measurements.

With the rapidly increasing computational power and ubiquity of sensing equipment, it is possible to provide constructive solutions to the distributed state estimation problem where the aforementioned faults and disturbances occur in LSS, and in this work, we focus on these problems. The organisation of this thesis is briefly illustrated in the next section.

1.2 Thesis Outline

Before diving into the technical content, we provide a comprehensive literature review in Chapter 2, which teases out the very relevant and up-to-date results to our objectives. Moreover, contributions of this work are presented in detail with comparison to the literature.

The technical part focuses on the distributed state estimation problem with three scenarios of faults and disturbances mentioned previously in Section 1.1. Regarding the first type of fault, we show in Chapter 3 that communication network among nodes is not required to be time-invariant to guarantee the performance of the state estimation if distributed observers exchange their estimated state vectors with their neighbouring observers. Indeed, if the joint of communication links over finite intervals of time make the network communication topology connected, the proposed distributed observer design is capable of achieving the asymptotic state estimate with guaranteed feasibility. Moreover, the results are extended to a scenario when the target LSS is subjected to $\mathscr{L}_2$ external disturbances and measurement noise. Accordingly, gains of the distributed observers are re-designed such that the desired $\mathscr{H}_\infty$ performance is obtained to attenuate the effect of $\mathscr{L}_2$ disturbances and noise.

For the second type of fault, we propose in Chapter 4 a Distributed Unknown Input Observer (DUIO) such that the full state vector of the system is estimated by each local observer in spite of the assumption that each observer do not have access to all input signals (such as faults). The feasible solution of an LMI provides adequate choices of parameters that guarantee the convergence of estimation errors. Moreover, necessary and sufficient existence conditions are drawn for the proposed DUIO design with a rigorous analysis. After that, a modified version of the main result is provided such that a scenario with directed communication graph among distributed observers is included based on the proposed design.

In Chapter 5, we address the distributed state estimation problem with faulty or noisy output measurements for the third type of fault. We define the local sensor redundancy of an LSS with distributed measurements and propose a robust estimation strategy based on this defined redundancy. Under certain conditions, the rejection of a range of fault in the sensors without detecting the unhealthy sensors is proved to be possible, and the transient abnormal behavior of the estimation is completely avoided. Moreover, the feasibility of the proposed robust estimation design is discussed after the main result. Next, we show that the main result can be extended to a class of nonlinear dynamical systems with certain Lipschitz condition.

After unveiling the main technical results, we evaluate all the proposed distributed observer designs in Chapter 6. We consider an LTI system model based on heat exchange in multi-zone building. All of the proposed distributed state estimation schemes are applied on this model to estimate the real-time temperature of all rooms in the building, which provides the evidence of the real-world applicability of the proposed distributed estimation strategies.

In Chapter 7, we state some concluding remarks and discuss some possible directions for future researches.

1.3 Publications

Some of the material presented in this thesis is also featured in the following works:

1. G. Yang, H. Rezaee and T. Parisini, "Sensor Redundancy for Robustness in Nonlinear State Estimation," in *IEEE 58th Conference on Decision and Control (CDC)*, Nice, France, 2019, pp. 3865-3870

2. G. Yang, H. Rezaee and T. Parisini, "Distributed State Estimation for a Class of Jointly Observable Nonlinear Systems," in *21st IFAC World Congress*, IFAC, 2020, Berlin, Germany, pp. 5045-5050

3. G. Yang, H. Rezaee, A. Alessandri and T. Parisini, "State Estimation Using a Network of Distributed Observers With Switching Communication Topology," *Automatica* (provisionally accepted), 2021

4. G. Yang, A. Barboni, H. Rezaee and T. Parisini, "State Estimation Using a Network of Distributed Observers With Unknown Inputs," *Automatica* (provisionally accepted),

2021

5. G. Yang, H. Rezaee, A. Serrani and T. Parisini, "Sensor Fault-Tolerant State Estimation by Networks of Distributed Observers," *IEEE Transaction on Automatic Control* (accepted as a full paper), 2022.

1.4 Preliminaries

Notation and some concepts and definitions are presented in this section.

1.4.1 Notation

Let $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ denote the set of integer numbers and the set of non-negative integer numbers, respectively. Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the sets of positive and non-negative real numbers, respectively. By denoting $\mathbb{C}$ as the set of complex numbers, we define a complex number set with negative real part as $\mathbb{C}^- := \{s : \mathbb{R}(s) < 0\}$ and a complex number set with non-negative real part as $\mathbb{C}^+ := \{s : \mathbb{R}(s) \geq 0\}$.

A closed interval with bounds a and b, $a < b$ on an ordered and closed set is denoted as $[a, b]$, an open interval as (a, b), and an interval closed on left and open on the right is denoted as $[a, b)$.

$\mathscr{L}_2$ is the space of square integrable functions $t \longmapsto \delta(t) \in \mathbb{R}$ such that $\int_0^\infty \delta(t)^2 dt < \infty$. $\|\cdot\|$ is the standard Euclidean norm. $\|\cdot\|_{\mathscr{L}_2}$ is the $\mathscr{L}_2$ norm, i.e., $\|\delta(t)\|_{\mathscr{L}_2} = \sqrt{\int_0^\infty \delta(t)^2 dt}$. $|\cdot|$ is the absolute value of a real scalar.

I_n stands for the $n \times n$ identity matrix. $\mathbf{0}_{n\times m}$ is an $n \times m$ all-zeros matrix. $\mathbf{1}_n$ is an $n \times 1$ all-ones vector. $\otimes$ stands for the Kronecker product. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\dagger$ represents the pseudo inverse of A such that if A is full row rank, $A^\dagger = A^\top(AA^\top)^{-1}$ and if A is full column rank, $A^\dagger = (A^\top A)^{-1}A^\top$. We say $M \succ 0$, if M is a symmetric positive definite matrix. $\lambda_2(\cdot)$ is the second smallest eigenvalue of a real symmetric matrix. For a square matrix M, let $M \succ 0$

or $M \succeq 0$ if it is symmetric positive definite or symmetric positive semi-definite, and let $M \prec 0$ or $M \preceq 0$ if it is symmetric negative definite or symmetric negative semi-definite. For a square matrix M, M^* and M^{-1*} stand for the conjugate transpose and the inverse of the conjugate transpose, respectively. $\text{diag}(M_1, M_2, \ldots, M_n)$ represents a block diagonal matrix composed of the matrices $M_1, M_2, \ldots, M_n$. Similarly, $\text{diag}_{i \in \mathcal{I}}(M_i)$ is a short-hand notation when the matrices are indexed by a set $\mathcal{I}$. The image of a matrix $M \in \mathbb{R}^{n \times m}$ is defined as

$$\text{Im}M := \{y \in \mathbb{R}^n \,|\, y = Mx, x \in \mathbb{R}^m\}.$$

The kernel (null space) of a matrix $M \in \mathbb{R}^{n \times m}$ is defined as

$$\text{Ker}M := \{x \in \mathbb{R}^m \,|\, Mx = 0\}.$$

A 'non-trivial' (sub)space $\mathscr{V}$ is such that $\dim(\mathscr{V}) > 0$. Moreover, if $\mathscr{R}, \mathscr{S} \subseteq \mathscr{X}$, we define the subspace $\mathscr{R} + \mathscr{S} \subseteq \mathscr{X}$ as

$$\mathscr{R} + \mathscr{S} = \{r + s : r \in \mathscr{R} \text{ and } s \in \mathscr{S}\}.$$

We also define the subspace $\mathscr{R} \cap \mathscr{S} \subseteq \mathscr{X}$ as

$$\mathscr{R} \cap \mathscr{S} = \{x : x \in \mathscr{R} \text{ and } x \in \mathscr{S}\}.$$

Accordingly, the symbol $\oplus$ indicates that the subspaces being added are independent. $\mathscr{V}^\perp$ denotes the orthogonal complement of the subspace $\mathscr{V}$. We indicate that two vector spaces $\mathscr{V}$ and $\mathscr{W}$ are isomorphic by $\mathscr{V} \simeq \mathscr{W}$. $\alpha_A(s)$ is the minimal polynomial of A, which is factored as follows:

$$\alpha_A(s) = \alpha_A^+(s)\alpha_A^-(s),$$

where the roots of α_A^+ belong to $\mathbb{C}^+$, and the roots of α_A^- belong to $\mathbb{C}^-$ [6, Chap. 3.6]. $\mathscr{UO}(C, A)$

denotes the unobservable subspace of the pair (C, A) and is defined by

$$\mathscr{U}\mathscr{O}(C, A) = \bigcap_{k=1}^{n} \text{Ker } CA^{k-1}.$$

Moreover, $\mathscr{U}\mathscr{D}(C, A)$ denotes the undetectable subspace of the pair (C, A) and is defined by

$$\mathscr{U}\mathscr{D}(C, A) = \left(\bigcap_{k=1}^{n} \text{Ker } CA^{k-1} \right) \bigcap \text{ Ker } \alpha_A^{+}(A).$$

1.4.2 Elements of Graph Theory

Communication among the observers is described by an unweighted undirected graph $\mathcal{G} = (\mathbf{N}, \mathcal{E}, \mathcal{A})$ where $\mathbf{N} = \{1, 2, \ldots, N\}$ is the set of nodes (denoting N observers with local sensors), $\mathcal{E} \subseteq \mathbf{N} \times \mathbf{N}$ is the set of edges (denoting communication links) where $(i, j) \in \mathcal{E}$ denotes an edge between node i and node j. Moreover, $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ denotes the adjacency matrix where $a_{ij} = a_{ji} = 1$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = a_{ji} = 0$ otherwise. Now, the neighbouring set of Node i is described as $\mathcal{N}_i = \{j | (i, j) \in \mathcal{E}\}$.

We define an undirected graph as *connected* if there exists a path of edges connecting each pair of the nodes of the graph. Accordingly, a set of undirected graphs $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ with the node set $\mathbf{N}$ is defined to be *jointly connected* if their union is connected, while none of them may be connected. A graph is defined to be *strongly connected* if it is directed and has a path from each node to every other node.

The Laplacian matrix associated with the undirected graph $\mathcal{G}$ is a symmetric matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{N \times N}$ described as

$$\ell_{ij} = \begin{cases} \sum_{j=1, i \neq j}^{N} a_{ij} & i = j \\ -a_{ij} & i \neq j, \end{cases}$$

having rows/columns with zero entries summation. Therefore, $\mathcal{L}$ always has a zero eigenvalue, and if $\mathcal{G}$ is connected, all the other eigenvalues are positive. Moreover, if $\mathcal{G}$ is connected, the right and left eigenvectors associated with the zero eigenvalue are $\mathbf{1}_N / \sqrt{N}$ [7]. Furthermore, the Laplacian matrix associated with any undirected graph is positive semi-definite [8].

For any connected graph $\mathcal{G}$, let $\lambda_2(\mathcal{L})$ denote the *algebraic connectivity* of the graph, which is a positive real number [9]. Accordingly, we define $\mathcal{C}(N)$ as a lower bound for the algebraic connectivity of all graphs with N nodes. It should be noted there exist several ways in the literature to obtain a $\mathcal{C}(N)$ based on N (for instance, see [10] and [11]).

Chapter 2

Literature Review and Contributions

In this chapter, we present the general overview of the literature related to our objectives. We begin with a brief introduction of the distributed state estimation problem. Following this, we focus on distributed estimation with different types of faults or disturbances as stated in Section 1.1 by summarising the existing results of the latest literature. Finally, we present the contributions of this thesis in detail with comparison of the reviewed literature.

2.1 Summary of Distributed State Estimation Problem

As mentioned previously, the measurements for an LSS are usually realised by a group of sensors distributed across N nodes that co-located by the corresponding distributed observers. We assume that these nodes are connected via a communication network, and each observer located at the corresponding node only has access to the local measurement and the information of neighbouring nodes that are directly connected to it. Under these assumptions, we intend to estimate the whole state vector of the LSS from the observer at each node. This problem is termed as the *distributed state estimation problem* [12].

Let us suppose the monitored LSS is an LTI system, together with the measurements at different nodes described by

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y_i &= C_i x, \quad i \in \mathbf{N}\end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$ is the state matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix, and $\mathbf{N} := \{1, 2, \ldots, N\}$ is the set of the nodes. Moreover, $y_i \in \mathbb{R}^{p_i}$ is the measurement output at the ith node, and $C_i \in \mathbb{R}^{p_i\times n}$ is the associated output matrix where $\sum_{i=1}^{N} p_i = p$. Accordingly, the collection of all the outputs can be written as

$$y = \begin{bmatrix} y_1^\top & y_2^\top & \ldots & y_N^\top \end{bmatrix}^\top,$$

which corresponds to the following integrated output matrix:

$$C = \begin{bmatrix} C_1^\top & C_2^\top & \cdots & C_N^\top \end{bmatrix}^\top.$$

The goal of distributed state estimation is to build a network of observers such that the estimated state vector of the ith observer denoted by $\hat{x}_i$ converges to x, by only accessing the input u, the local measurement y_i and the estimated state vectors of neighboring nodes. An example of the distributed state estimation scheme is shown in Fig. 2.1.

As the objective is to reconstruct the entire state vector x, we consider a distributed observer $\mathcal{O} = \{\mathcal{O}_i\}_{i\in\mathbf{N}}$ comprising N local nodes (or observers) $\mathcal{O}_i$ located at each sensor node, where each observer has access to just its local outputs y_i and inputs u. Furthermore, the local observers exchange their state estimates through a communication network.

In general, the local measurement at each node is insufficient to estimate the state vector x considering that a typical LSS has huge number of states. Therefore, to achieve the state estimate at each node, each local observer should exploit the estimate from the neighbouring observers (i.e., $\hat{x}_i$) over the communication network when developing the distributed estimation strategies. Before reviewing the existing results for the distributed state estimation problem, we introduce the following definition.

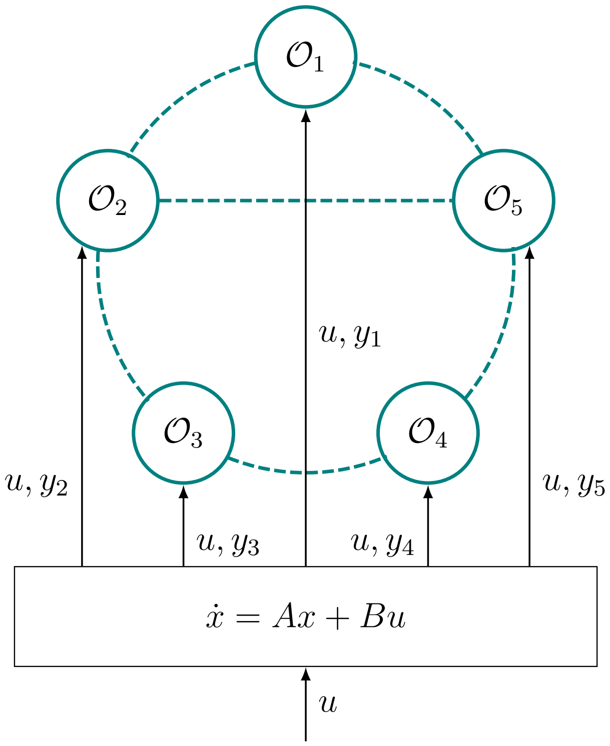


Figure 2.1: An example of distributed observer consisting of 5 nodes: each local observer $\mathcal{O}_i$ has access to the input and local measurement u and y_i. Furthermore, neighbouring estimates are exchanged over a undirected communication network (dashed line).

Definition 2.1. *A system with distributed measurements described in* (2.1) *is* ***jointly detectable (observable)*** *if the pair* (C, A) *is detectable (observable), while the pair* (C_i, A) *is not necessarily detectable (observable) for all* $i \in \mathbf{N}$*. While the pair* (C, A) *is detectable (observable), some nodes even may have no sensors/measurements. In general, being jointly detectable is a weaker condition compared with being jointly observable.*

Indeed, the distributed state estimation problem is studied under the condition that the system is jointly detectable (observable) and it has been an active area of research in the last decade.

At the beginning of this century, some distributed observer designs were proposed via the extension of the classical Kalman filter to a distributed observer network [13, 14, 15]. These designs implement a local Kalman filter based on its own measurements and combine information from other sensors using a consensus term. The conditions for those designs are restrictive. For example, the observers are assumed to have more sensors than the dimension of the system states [14]. After that, distributed estimation algorithms have been developed based on the concept of moving horizon estimation [16, 17]. By assuming the existence of some regional observable nodes, the convergence of the estimates is guaranteed. Moreover, A linear suboptimal consensus-based distributed estimation methodology was addressed in [18] with the assumption of distributed detectability (local detectability).

Recently, more results that focused directly on the distributed state estimation problem have risen. Extending the classical Luenberger observer to a distributed estimation scheme has been a popular direction to solve this problem [12, 19, 20]. These proposed estimation strategies are composed of a conventional Luenberger-based part and a consensus term. Moreover, Kalman observable canonical decomposition played a key role in proving the convergence of those distributed observers. Furthermore, distributed estimation schemes for autonomous LTI plants were derived by adopting the augmented states [21, 22] at each local observer. Under specific design, the necessary and sufficient conditions for state estimation convergence are determined to achieve asymptotic omniscience [21]. In [23], the author presented an interesting distributed observer design such that the estimation error converges to zero at a preassigned but arbitrarily fast exponential convergence rate.

All the studies that mentioned earlier are related to the distributed state estimation problem without considering faults and attacks. Although some results we went through involved $\mathcal{L}_2$ or zero mean white Gaussian noise and disturbance [13, 16, 18], however, the proposed designs did not consider faults with large magnitude in the actuators or measurements of the system plants. Considering that the LSS is prone to suffering from those uncertain elements, it is worthy of viewing the related results for distributed estimation, which will be presented in the next section.

2.2 Distributed Estimation With Faults and Disturbances

In this section, we tease out the literature which focuses on the robust distributed estimation strategies against the faults and disturbances that we state in the main objectives.

We first consider the scenario that the network of the distributed schemes cannot enjoy a steady and fixed communication topology. In this case, the communication links among the nodes may fail and rebuild at some unknown time instants, and the topology of the graph switches over time. This type of faults inevitably affect the effectiveness of the conventional designs, and novel distributed estimation strategies are required to obtain resilient estimation. To address this problem, a class of distributed estimators in the presence of switching topologies was constructed for spacecraft formations [24]. However, based on this design, distributed estimation relied on the observability at each of the nodes (i.e., (C_i, A) is assumed to be observable for all $i \in \mathbf{N}$). Moreover, in this study, all switching topologies are supposed to be connected, which is a bit restrictive to be satisfied in practice. Furthermore, a robust distributed consensus estimators in the presence of Markovian switching topologies was proposed in [25]. The sufficient conditions for the synthesis was provided, and the proposed design can be constructed by solving certain LMIs. However, the strategies required global knowledge of the network topology in advance, which limits the application of the proposed scheme for various scenarios if we consider link failure as a fault. Following the same line, [26] addressed the network topology described by a Markov chain. However, this work focused on the estimation of unknown parameters instead of state vectors of the system, which is not consistent with the problem we consider here.

Very recently, a structured distributed observer was described for estimating the state of a discrete-time, LTI autonomous system in [27]. Assuming the network is switching but always strongly connected, the proposed design guarantees the asymptotic convergence of the estimation error without high gains. Moreover, in [28], distributed state estimation in the presence of jointly connected switching topologies was studied. Indeed, [28] illustrated that asymptotic state estimation is obtained under certain constraints on the dynamical system parameters and the switching topologies. However, the feasibility of the proposed design remains unknown, hence the distributed estimation design is not guaranteed for all scenarios.

Another limitation of the existing works on the distributed estimation problem is to assume that the systems are autonomous, or the entire inputs of the plants are known at all of the nodes (i.e., the input information is available globally). However, when a system is disturbed or driven by some external actuators, it is often the case that each node cannot get access to all of the input signals. Instead, at each node, the local observer may merely have access to a part of the system inputs, and the rest of the input signals remain unknown. In this scenario, the existing distributed estimation schemes in the literature are no longer applicable.

The distributed state estimation problem is still open when unknown inputs at each node are assumed. Without considering the distributed measurements, the Unknown Input Observer (UIO) design existed in the literature have been an essential tool for systems with completely unknown exogenous disturbance (for example, [29, 30]). Among those results, the coupling between subsystems is modelled as the unknown inputs such that the conventional DUIO strategy is applied for state estimation [31]. Few results exist where these observers are instead extended to distributed scenarios, such as [32]. However, in that work, the objective is to estimate the local state vector for each subsystem rather than the entire state vector of the system, which is distinct with our objective in this work.

Nevertheless, few papers have been published yet concerning the robustness of distributed state estimation against faulty measurements. For instance, in [33] and [34], the problem of resilient state estimation of linear systems by increasing the robustness of communication graphs has been studied. It should be noted that many existing works assume that local detectability/observability holds, which is different from the methodology we desire to develop. This work, however, assumes that joint detectability remains over all nodes, and the robust state estimation is realised by exploiting the local sensor redundancy. Other papers dealt with distributed filtering over sensor networks such that a desired stochastic performance is achieved [35, 36]. Recently, [37] has proposed a secure state estimation algorithm in the presence of sensor attacks, however, the observer is not designed in a distributed way.

2.3 Contributions

The main objective of this work is to develop distributed state estimation methodologies for LSSs under certain types of faults and disturbances. By investigating the existing results in the literature, we conclude that it is still an open problem to propose a systematic design framework that guarantees the asymptotic/robust state estimates of jointly detectable LTI systems in the presence of fault and disturbance described in Section 1.1. We briefly state the contributions of this thesis as follows.

Communication links failure

Firstly, in this thesis, we propose design methods that solve the distributed state estimation problem in LTI systems in the presence of communication links failure. We construct a network of observers distributed over N nodes such that each observer has access to its local measurements and the estimated state vectors of neighbouring nodes. We show that under some Linear Matrix Inequality (LMI) conditions, if the ensemble of all the measurements at all the nodes makes the system detectable, and if the network communication graph is jointly connected within bounded time intervals, the proposed distributed estimation scheme guarantees the asymptotic state estimates at each node. Then, by employing canonical decomposition of the dynamical system, we show that the LMI conditions always have solutions just if the pairs $(A, C_i^\top), i \in \mathbf{N}$ are marginally stabilisable. Moreover, we modify the proposed strategy such that distributed state estimation is achieved in the presence of external disturbances. More specifically, we modify the proposed estimation strategy such that if the switching communication topology stays connected during links failure, a desired $\mathscr{H}_\infty$ performance in order to attenuate the influence of external disturbances and measurement noise on the estimation error is obtained.

It should be noted that, although a similar problem is considered in [16] and [25], their design methods depend on conditions for which feasibility is not guaranteed. Moreover, different from the estimation scheme introduced in [24] which is only applicable to connected switching topologies, our proposed strategy allows to perform state estimation in the presence of jointly connected switching topologies (the graph is not necessarily connected at each time instant). Also, differently to [24], the local measurement at each node is not required to ensure the detectability of the entire system in this study, and the joint detectability of the nodes is adequate for estimating the state vector of the system. Furthermore, despite the distributed estimation scheme introduced in [25] which is only applicable for classes of networks with rooted spanning trees, the method we proposed is well suited for networks with jointly connected switching topologies, which is a weaker condition concerning the communication network of the distributed observers. Nevertheless, in [28], the problem of state estimation in the presence of jointly connected switching topologies has been addressed, where the pairs $(A, C_i^\top), i \in \mathbf{N}$, are also marginally stabilisable. In that study, some inequality assumptions on the system parameters and observer gains are required for state estimation. However, here in our design, the marginal stabilisability is the only condition to guarantee state estimation in the presence of jointly connected switching topologies. Hence, the proposed strategy has a wider range of application scenarios.

Unknown inputs and disturbances

After that, we propose a DUIO design for an LTI system, where the information of only a part of the input signals is accessible at each node. Under our devised design, we provide rigorous (necessary and sufficient) conditions for the existence of such DUIO, depending on a rank and a specific detectability criterion. We also show that any feasible solution satisfying a certain LMI condition guarantees asymptotic convergence of the estimates to the real state of the system. Therefore, such LMI condition is constructively applied to compute the gains of the DUIO, provided that the existence conditions are satisfied. Furthermore, when the aforementioned detectability criterion is satisfied, the feasibility of the LMI condition is always guaranteed. Compared to the existing literature, our design does not require access to the entire input signals of the system, but rather only the subset of it is assumed to be available at each node.

Faults and noises at local measurements

Finally, we focus on the distributed state estimation with sensor faults and noise. We propose a robust estimation strategy against a range of additive faults in the sensors under some conditions on the sensor redundancy. We derive sufficient conditions on the proposed observers to guarantee the asymptotic estimates rejecting the effects of sensor faults by using the estimates of its neighbouring observers and a subset of the local measurements when sensor redundancy exists at each node. Moreover, we extend this estimation scheme to the scenario in which a Lipschitz continuity on the nonlinearity exists in the dynamical system model. This extension is a step beyond most of the existing results (for example [19, 23]), which are limited to the linear system when available sensors are fully precise.

Furthermore, from a complexity perspective, all distributed observers proposed in this work do not require augmented or auxiliary states. Moreover, compared to the introduced distributed estimation schemes in [21] and [22], our proposed observers do not require any augmented states.

Chapter 3

Distributed State Estimation With Switching Communication Topology

In this chapter, state estimation of LTI systems by using a network of distributed observers with switching topology is studied. We consider a scenario where the communication links may fail and rebuild over time, and the communication network does not stay constantly connected. Accordingly, the main idea is to propose a distributed approach (with guarantees on the feasibility of the design) such that the state vector of the system is estimated by each observer if the union/joint of communication links in bounded intervals of time makes the network communication graph connected. Moreover, we also consider a scenario where the LTI system is subject to external disturbances and measurement noise. In this case, we derive sufficient conditions on the proposed approach such that, if the communication topology stays connected during links failure, an $\mathscr{H}_\infty$ performance to attenuate the effect of external disturbances and measurement noise on estimation errors is guaranteed.

In Section 3.1, distributed state estimation under jointly connected switching topology is addressed. The problem is formulated with certain assumptions, and the observer design is proposed in Section 3.1.1 with convergence analysis on estimation errors. Moreover, the feasibility of the proposed design is guaranteed in Section 3.1.2 based on the given assumptions. In Section 3.2, we focus on distributed state estimation under switching topology with existence of the disturbances and measurement noise on dynamical systems. The problem is formulated with new assumptions and the revised observer design is presented in Section 3.2.1. Moreover, the feasibility of the proposed design is discussed in Section 3.2.2 based on the stated new assumptions.

3.1 Distributed State Estimation Under Jointly Connected Switching Topology

One of important concerns in communication among the observers is the discontinuity in the communication links such that the links may not stay connected constantly, and they can fail and rebuild over time. In this condition, the network communication graph is switching. This is expressed by $\mathcal{G}(t) = (\mathbf{N}, \mathcal{E}(t), \mathcal{A}(t))$, where $\mathcal{A}(t) = [a_{ij}(t)]$ such that $a_{ij}(t) = 1$ describes a communication link between nodes i and j at time t, and it is zero otherwise. Now, we consider an infinite time sequence t_0, t_1, t_2, ... starting at $t_0 = 0$, where the intervals $[t_0, t_1)$, $[t_1, t_2)$, $[t_2, t_3)$, ... are bounded and, in each interval, the graph is switching and *jointly connected*. In other words, in the interval $[t_k, t_{k+1})$, we consider a finite time sequence t_k^0, t_k^1, ... , $t_k^{m_k}$ where $t_k = t_k^0$ and $t_{k+1} = t_k^{m_k}$ such that $\mathcal{G}(t)$ switches to $\mathcal{G}_k^j$ at t_k^j, $j \in \{0, 1, \ldots, m_k - 1\}$, and stays fixed during the subinterval $[t_k^j, t_k^{j+1})$. Now, the set including the graphs $\mathcal{G}_k^j$, $j \in \{0, 1, \ldots, m_k - 1\}$, is assumed to be jointly connected. In this condition, considering the LTI system (2.1), the objective is to exploit the joint detectability of the nodes along with the joint connectivity of the communication topology to design a network of distributed observers such that the estimated state vector of each observer converges to the state vector of the system as

$$\lim_{t \to \infty} (x(t) - \hat{x}_i(t)) = \mathbf{0}_{n \times 1},\ i \in \mathbf{N}. \tag{3.1}$$

Remark 3.1. Note that the sequence t_k is infinite and does not have accumulation points, i.e., $\lim_{k \to +\infty} t_k = +\infty$. Moreover, the graph topology staying fixed during $[t_k^j, t_k^{j+1})$ implies that the subinterval between any switching is strictly larger than zero.

3.1.1 Observer Design and Analysis

The proposed distributed estimation strategy for the system described in (2.1) is presented in this section such that the objective in (3.1) is achieved. To this end, the following assumptions on the LTI system (2.1) are considered.

Assumption 3.1. *The system described in* (2.1) *is jointly detectable, i.e., the pair* (C, A) *is detectable, but* (C_i, A) *is not necessarily detectable for all* $i \in \mathbf{N}$.

Assumption 3.2. *The pairs* $(A, C_i^\top), i \in \mathbf{N}$, *are marginally stabilizable.*

Assumption 3.2 implies that if the state matrix A is unstable, there should exist an output gain L_i such that $A + L_iC_i$ is stable or marginally stable. This assumption is always satisfied if the state matrix A is stable or marginally stable.

Under the assumptions that previously mentioned, the proposed observer for the system described in (2.1) at Node i can be written as:

$$\dot{\hat{x}}_i = A\hat{x}_i + L_i(C_i\hat{x}_i - y_i) + Bu + P_i^{-1}\sum_{j=1}^{N} a_{ij}(t)(\hat{x}_j - \hat{x}_i),\ i \in \mathbf{N}, \tag{3.2}$$

where $L_i \in \mathbb{R}^{n\times p_i}$ and $P_i \in \mathbb{R}^{n\times n}$ denote the observer gains to be selected.

The mathematical analysis of the proposed distributed observers is given in Theorem 3.1. Before stating this theorem, the following useful lemmas are presented.

Lemma 3.1. *[38, Chap. 5] Let* $v(t_0), v(t_1),\ v(t_2), \ldots$ *be a converging sequence of numbers. For all* $\zeta \in \mathbb{R}_+$, *there exists* $k_\zeta \in \mathbb{N}$ *such that for all* $k \geq k_\zeta$, $|v(t_{k+1}) - v(t_k)| < \zeta$ *(Cauchy's convergence criterion).*

Lemma 3.2. *Consider two LTI autonomous systems described by*

$$\begin{aligned} \dot{\xi}_1 &= A_1\xi_1, \\ \dot{\xi}_2 &= A_2\xi_2, \end{aligned} \tag{3.3}$$

where $\xi_1, \xi_2 \in \mathbb{R}^n$ *are state vectors and* $A_1, A_2 \in \mathbb{R}^{n\times n}$ *are state matrices. Moreover, let* $[t_1, t_2)$ *be a finite time interval with* $0 \leq t_1 < t_2$. *In this condition, if* $\xi_1(t) = \xi_2(t)$ *for* $t \in [t_1, t_2)$*; then,* $\xi_1(t) = \xi_2(t)$ *for all* $t \geq t_2$ *according to the analytic continuation property [39].*

Theorem 3.1. *Consider the jointly detectable LTI system described in* (2.1) *under Assumption 3.2 and the distributed observers given in* (3.2). *Moreover, let* $\mathcal{G}(t)$ *be switching, but jointly connected within infinite sequence of bounded time intervals* $[t_0, t_1)$, $[t_1, t_2)$, $[t_2, t_3)$, $\ldots$ *starting at* t_0. *Under these conditions, the estimation errors* $e_i, i \in \mathbf{N}$, *converge to zero by obtaining* $P_i \succ 0$ *and* $L_i = P_i^{-1} Y_i$ *from the solution of the following LMIs:*

$$A^\top P_i + C_i^\top Y_i^\top + P_i A + Y_i C_i \preceq 0, \;\; i \in \mathbf{N}, \tag{3.4a}$$

$$\sum_{i=1}^{N} (A^\top P_i + C_i^\top Y_i^\top + P_i A + Y_i C_i) \prec 0. \tag{3.4b}$$

Proof. Considering the LTI system with the form of (2.1) and based on the devised observer in (3.2), the differential equation describing $e_i = x - \hat{x}_i, i \in \mathbf{N}$, is given by

$$\dot{e}_i = (A + L_i C_i) e_i + \underbrace{P_i^{-1} \sum_{j=1}^{N} a_{ij}(t)(e_j - e_i)}_{=: \, \varepsilon_i}, \;\; i \in \mathbf{N}. \tag{3.5}$$

To analyze the evolution of the observers' estimation errors, we consider the following Lyapunov function:

$$V = \sum_{i=1}^{N} e_i^\top P_i e_i.$$

The time derivation of V along (3.5) yields

$$\dot{V} = \sum_{i=1}^{N} e_i^\top \Lambda_i e_i + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(t) \left(e_j - e_i\right)^\top e_i, \tag{3.6}$$

where

$$\Lambda_i := A^\top P_i + C_i^\top Y_i^\top + P_i A + Y_i C_i. \tag{3.7}$$

Now, by considering $\mathcal{L}_k^j$ as the fixed Laplacian matrix associated with $\mathcal{G}_k^j$ in the subinterval

$[t_k^j, t_k^{j+1})$, (3.6) can be restated as follows:

$$\dot{V} = e^\top \Lambda e - \underbrace{2e^\top \left(\mathcal{L}_k^j \otimes I_n\right) e}_{=: \, \epsilon}, \tag{3.8}$$

where

$$\Lambda = \operatorname{diag}\left(\Lambda_1, \Lambda_2, \ldots, \Lambda_N\right). \tag{3.9}$$

According to (3.4a), Λ is negative semi-definite, and since $-\mathcal{L}_k^j$ is negative semi-definite, from (3.8) it follows the boundedness of $e_i,\ i \in \mathbf{N}$. We continue the proof in two steps. In the first step, by invoking Lemma 3.1 we show that the term $\sum_{j=1}^N a_{ij}(t)(e_j - e_i)$ in (3.5) converges to zero. In the next step, by invoking Lemma 3.2, we show that $e_j - e_i,\ i, j, \in \mathbf{N}$, converge to zero, which implies that $e_i, i \in \mathbf{N}$, converge to a common variable $\omega \in \mathbb{R}^n$. Then, from (3.4b) and (3.8), we conclude that $\dot{V}$ is negative definite with respect to e.

Step 1: According to (3.4a), Λ is negative semi-definite. Therefore, from (3.8) it follows that

$$\dot{V} \leq -2e^\top \left(\mathcal{L}_k^j \otimes I_n\right) e. \tag{3.10}$$

Since $\dot{V} \leq 0$, based on Lemma 3.1, it follows that, for any $\zeta \in \mathbb{R}_+$, there exists a k_ζ such that, for all $k \geq k_\zeta$, $|V(t_{k+1}) - V(t_k)| < \zeta$, which implies that

$$\left| \int_{t_k}^{t_{k+1}} \dot{V}(t)\, dt \right| < \zeta. \tag{3.11}$$

By considering the time sequence $t_k^0, t_k^1, \ldots, t_k^{m_k}$ in the interval $[t_k, t_{k+1})$ and since $\dot{V} \leq 0$, from (3.11), one gets

$$-\sum_{j=0}^{m_k-1} \int_{t_k^j}^{t_k^{j+1}} \dot{V}(t)\, dt < \zeta,$$

where, by defining $\tau_k := \min_{j \in \{0,1,2,\ldots,m_k-1\}} t_k^{j+1} - t_k^j$, it yields

$$-\sum_{j=0}^{m_k-1} \int_{t_k^j}^{t_k^j+\tau_k} \dot{V}(t)\, dt < \zeta.$$

Since $\dot{V}$ is non-positive, by considering Lemma 3.1 and (3.10), as $\zeta \to 0$, follows that

$$\lim_{t\to\infty} \sum_{j=0}^{m_k-1} \int_t^{t+\tau_k} e(\tau)^\top \left(\mathcal{L}_k^j \otimes I_n\right) e(\tau)\, d\tau = 0. \tag{3.12}$$

Note that during the interval $[t, t+\tau_k)$, the topology of the graph $\mathcal{G}_k^j$ is not switching.

Since the communication graph is undirected, hence the Laplacian is always symmetric and therefore always diagonalisable. By using an orthogonal similarity transformation matrix P_k^j, we diagonalise $\mathcal{L}_k^j$ as $\mathcal{L}_k^j = P_k^j J_k^j P_k^{j\top}$, where $J_k^j = \mathrm{diag}(0, \eta_k^j)$ is the diagonal form of $\mathcal{L}_k^j$ in which η_k^j has a diagonal form with non-negative entries and P_k^j is composed of the right eigenvectors of $\mathcal{L}_k^j$. Thus, if we define the new variable $q = (P_k^{j\top} \otimes I_n)e$, (3.12) can be restated as follows:

$$\lim_{t\to\infty} \sum_{j=0}^{m_k-1} \int_t^{t+\tau_k} q(\tau)^\top \left(J_k^j \otimes I_n\right) q(\tau)\, d\tau = 0. \tag{3.13}$$

If we decompose q as $\begin{bmatrix} \varrho^\top & \rho^\top \end{bmatrix}^\top$ and $\rho \in \mathbb{R}^{(N-1)n}$, since $J_k^j = \mathrm{diag}\left(0, \eta_k^j\right)$, (3.13) yields

$$\lim_{t\to\infty} \sum_{j=0}^{m_k-1} \int_t^{t+\tau_k} \rho(\tau)^\top \left(\eta_k^j \otimes I_n\right) \rho(\tau)\, d\tau = 0. \tag{3.14}$$

Now, by decomposing ρ as $\rho = [\rho_1^\top \;\; \rho_2^\top \;\; \ldots \;\; \rho_{N-1}^\top]^\top$ and by joint connectivity arguments, from (3.14) it follows that

$$\lim_{t\to\infty} \int_t^{t+\tau_k} \rho_i(\tau)^\top \rho_i(\tau)\, d\tau = 0\,,\; i = 1, \ldots, N-1. \tag{3.15}$$

According to the definition of τ_k, there is no graph switching from t to $t+\tau_k$. Moreover, we obtain that e is bounded and hence from (3.5) it follows that also $\dot{e}$ is bounded in $[t, t+\tau_k)$. Now, since $\dot{e}$ is bounded, $\rho_i^\top \rho_i$ is uniformly continuous from t to $t+\tau_k$. Hence, by considering (3.15) and invoking the Barbalat lemma [40], we obtain

$$\lim_{t\to\infty} \rho_i(t)^\top \rho_i(t) = 0,$$

which implies that

$$\lim_{t\to\infty} \rho(t) = \mathbf{0}_{(N-1)n\times 1}. \tag{3.16}$$

Since $e = \left(P_k^j \otimes I_n\right) q$ and $q = \begin{bmatrix} \varrho^\top & \rho^\top \end{bmatrix}^\top$, from (3.16) it follows that

$$\lim_{t\to\infty} (e(t) - p_k^j \otimes \varrho(t)) = \mathbf{0}_{Nn\times 1},$$

where p_k^j is the first column of P_k^j, which is the right eigenvector associated with a zero eigenvalue of $\mathcal{L}_k^j$. If $\mathcal{G}_k^j$ is connected, p_k^j spans $\mathbf{1}_N$. Otherwise, $\mathcal{G}_k^j$ can be considered as the combination of some subgraphs which are connected or have no links. Note that the right eigenvector associated with the zero eigenvalue of the Laplacian matrix of each connected subgraph has identical entries. Hence, if $\mathcal{G}_k^j$ is the combination of some subgraphs which are connected or have no links, p_k^j can be considered as the combination of some coefficients of $\mathbf{1}$ vectors with compatible dimensions. In this condition, for each $i, j \in \mathbf{N}$, if $a_{ij}(t) = 1$, we have $\lim_{t\to\infty}(e_i(t) - e_j(t)) = \mathbf{0}_{n\times 1}$. Moreover, $a_{ij}(t) = 0$ if $a_{ij}(t) \neq 1$. Hence:

$$\lim_{t\to\infty} P_i^{-1} \sum_{j=1}^{N} a_{ij}(t)(e_j(t) - e_i(t)) = \mathbf{0}_{n\times 1}. \tag{3.17}$$

Step 2: According to (3.5) and (3.17), we have

$$\dot{e}_i = (A + L_i C_i) e_i + \varepsilon_i, i \in \mathbf{N},$$

where ε_i (defined in (3.5)) is bounded and converges to zero and $\lim_{t\to\infty}(e_i(t) - e_j(t)) = \mathbf{0}_{n\times 1}$, if $a_{ij}(t) = 1$. Based on Lemma 3.2 and since $\mathcal{G}(t)$ is jointly connected within bounded time intervals, one gets

$$\lim_{t\to\infty} (e_i(t) - e_j(t)) = \mathbf{0}_{n\times 1}, i, j \in \mathbf{N},$$

implying that there exists an $\omega \in \mathbb{R}^n$ such that

$$\lim_{t\to\infty} (e_i(t) - \omega(t)) = \mathbf{0}_{n\times 1}, i \in \mathbf{N}. \tag{3.18}$$

Now, from (3.8) and (3.18) we have

$$\dot{V} = e^\top \Lambda e - \epsilon, \tag{3.19}$$

where e has the form of $\mathbf{1}_N \otimes \omega$, and ϵ (defined in (3.8)) is bounded and converges to zero. By considering (3.9) and since e has the form of $\mathbf{1}_N \otimes \omega$, one gets

$$e^\top \Lambda e = \omega^\top \left(\sum_{i=1}^{N} \Lambda_i \right) \omega,$$

where according to (3.4b), $\sum_{i=1}^{N} \Lambda_i$ is negative definite. Therefore, $e^\top \Lambda e$ is negative definite with respect to e and since ϵ is bounded and converges to zero, the convergence of e to $\mathbf{0}_{Nn\times 1}$ follows from (3.19). $\square$

3.1.2 Feasibility Analysis of the Design

The effectiveness of the proposed distributed state estimation strategy depends on the observer gains that result from the solution of the LMIs (3.4). Next, based on detectability canonical decomposition, we show that the LMI conditions of Theorem 3.1 are always feasible. Let us define a similarity transformation matrix $T_i \in \mathbb{R}^{n\times n}$ as

$$T_i = \begin{bmatrix} T_{id} & T_{iu} \end{bmatrix}, \tag{3.20}$$

in which $T_{iu} \in \mathbb{R}^{n\times v_i}$ is an orthonormal basis of the undetectable subspace of (C_i, A) where v_i is the dimension of the undetectable subspace and $T_{id} \in \mathbb{R}^{n\times(n-v_i)}$ is an orthonormal basis such that Im T_{id} is orthogonal to Im T_{iu}. According to the structure of T_i, we perform the coordinate transformation adapted to $\mathscr{X} = \text{Im } T_{id} \oplus \text{Im } T_{iu}$ as follows [19] ($\mathscr{X}$ is the n-dimensional state space of the system):

$$T_i^\top A T_i = \begin{bmatrix} A_{id} & \mathbf{0}_{(n-v_i)\times v_i} \\ A_{ir} & A_{iu} \end{bmatrix}, \qquad C_i T_i = \begin{bmatrix} C_{id} & \mathbf{0}_{p_i \times v_i} \end{bmatrix}, \tag{3.21}$$

where the pair (C_{id}, A_{id}) is detectable. By using the aforementioned formulation, the feasibility of the LMIs (3.4) is proved in Theorem 3.2. Before presenting the theorem, we introduce the following lemmas.

Lemma 3.3. *For any marginally stable real matrix $\mathcal{M}$, the LMI*

$$\mathcal{M}^\top \mathcal{P} + \mathcal{P}\mathcal{M} \preceq 0 \tag{3.22}$$

has a symmetric positive definite solution for $\mathcal{P}$.

Proof. There is a similarity transformation matrix $\mathcal{T}$ such that $\mathcal{M} = \mathcal{T}\mathcal{J}\mathcal{T}^{-1}$ where $\mathcal{J}$ is the Jordan form of $\mathcal{M}$. Since $\mathcal{M}$ is marginally stable, $\mathcal{J}$ can be written as:

$$\mathcal{J} = \mathrm{diag}(\mathcal{W}, \mathcal{H}),$$

where $\mathcal{W}$ is a diagonal matrix containing the eigenvalues of $\mathcal{M}$ on the imaginary axis, and $\mathcal{H}$ is a Jordan matrix containing the eigenvalues of $\mathcal{M}$ on the open left half plane. Since the eigenvalues of $\mathcal{H}$ lie on the open left half plane, according to the Lyapunov stability criterion, for a Hermitian positive definite matrix $\tilde{\mathcal{Q}}$, there exists a Hermitian positive definite matrix $\tilde{\mathcal{P}}$ such that [41, p. 2]

$$\mathcal{H}^* \tilde{\mathcal{P}} + \tilde{\mathcal{P}}\mathcal{H} = -\tilde{\mathcal{Q}}. \tag{3.23}$$

We define a matrix $\acute{\mathcal{Q}} := \mathrm{diag}(\mathbf{0}, \tilde{\mathcal{Q}})$, where $\mathbf{0}$ is a zeros matrix whose dimension is the same as the dimension of $\mathcal{W}$ and define a matrix $\acute{\mathcal{P}} := \mathrm{diag}(I, \tilde{\mathcal{P}})$, where I is an identity matrix whose dimension is the same as the dimension of $\mathcal{W}$. Now, according to (3.23), it follows that:

$$\mathcal{J}^{*}\acute{\mathcal{P}} + \acute{\mathcal{P}}\mathcal{J} = -\acute{\mathcal{Q}}. \tag{3.24}$$

By pre and post multiplication of the both sides of (3.24) by $\mathcal{T}^{-1*}$ and $\mathcal{T}^{-1}$, one gets

$$\mathcal{T}^{-1*}\mathcal{J}^{*}\acute{\mathcal{P}}\mathcal{T}^{-1} + \mathcal{T}^{-1*}\acute{\mathcal{P}}\mathcal{J}\mathcal{T}^{-1} = -\mathcal{T}^{-1*}\acute{\mathcal{Q}}\mathcal{T}^{-1},$$

which, since $\mathcal{M} = \mathcal{T}\mathcal{J}\mathcal{T}^{-1}$, can be rewritten as follows:

$$\mathcal{M}^{\top}\mathcal{T}^{-1*}\acute{\mathcal{P}}\mathcal{T}^{-1} + \mathcal{T}^{-1*}\acute{\mathcal{P}}\mathcal{T}^{-1}\mathcal{M} = -\mathcal{T}^{-1*}\acute{\mathcal{Q}}\mathcal{T}^{-1}. \tag{3.25}$$

By defining $\bar{\mathcal{P}} = \mathcal{T}^{-1*}\acute{\mathcal{P}}\mathcal{T}^{-1}$ and $\bar{\mathcal{Q}} = \mathcal{T}^{-1*}\acute{\mathcal{Q}}\mathcal{T}^{-1}$, we rewrite (3.25) as

$$\mathcal{M}^{\top}\bar{\mathcal{P}} + \bar{\mathcal{P}}\mathcal{M} = -\bar{\mathcal{Q}}. \tag{3.26}$$

Now, from (3.26) we obtain

$$\mathcal{M}^{\top}(\bar{\mathcal{P}} + \bar{\mathcal{P}}^{\top}) + (\bar{\mathcal{P}} + \bar{\mathcal{P}}^{\top})\mathcal{M} = -(\bar{\mathcal{Q}} + \bar{\mathcal{Q}}^{\top}). \tag{3.27}$$

Since $\acute{\mathcal{P}}$ is Hermitian positive definite, one can derive that $\bar{\mathcal{P}} = \mathcal{T}^{-1*}\acute{\mathcal{P}}\mathcal{T}^{-1}$ is Hermitian positive definite implying that $\bar{\mathcal{P}} + \bar{\mathcal{P}}^{\top} \succ 0$. Moreover, since $\acute{\mathcal{Q}}$ is Hermitian positive semi-definite, $\bar{\mathcal{Q}} = \mathcal{T}^{-1*}\acute{\mathcal{Q}}\mathcal{T}^{-1}$ is Hermitian positive semi-definite implying that $\bar{\mathcal{Q}} + \bar{\mathcal{Q}}^{\top} \succeq 0$. Now, by considering (3.27) and by defining $\mathcal{P} = \bar{\mathcal{P}} + \bar{\mathcal{P}}^{\top}$, the LMI (3.22) is satisfied. Based on all the above-mentioned derivations, (3.22) has a solution for $\mathcal{P} \succ 0$. □

Lemma 3.4. *Let the system* (2.1) *satisfy Assumption 3.1. Then, by letting*

$$T_d = \begin{bmatrix} T_{1d} & T_{2d} & \dots & T_{Nd} \end{bmatrix}, \tag{3.28}$$

we have

$$\operatorname{Im} T_d = \mathscr{X}.$$

□

Proof. Since T_{iu} is a orthonormal basis of the undetectable subspace of (C_i, A), we have

$$\text{Im } T_{iu} = \mathscr{U}\mathscr{D}(C_i, A). \tag{3.29}$$

Moreover, since T_{id} is the orthonormal basis such that T_i has full rank, one gets

$$\text{Im } T_{iu} = (\text{Im } T_{id})^{\perp}. \tag{3.30}$$

According to the definition of T_d, it can be said that [6, Chap. 0.12]

$$\text{Im } T_d = \left(\left(\sum_{i=1}^{N} \text{Im } T_{id}\right)^{\perp}\right)^{\perp}$$

implying that

$$\text{Im } T_d = \left(\bigcap_{i=1}^{N} (\text{Im } T_{id})^{\perp}\right)^{\perp}. \tag{3.31}$$

From (3.29), (3.30), and (3.31), one gets

$$\text{Im } T_d = \left(\bigcap_{i=1}^{N} \mathscr{U}\mathscr{D}(C_i, A)\right)^{\perp}. \tag{3.32}$$

Hence, from (3.32), we have

$$\text{Im } T_d = (\mathscr{U}\mathscr{D}(C, A))^{\perp}. \tag{3.33}$$

Finally, by (3.33) and under Assumption 3.1, we have $\text{Im } T_d = 0^{\perp} = \mathscr{X}$ [6, Chap. 0.12], which completes the proof. □

Theorem 3.2. *Consider the jointly detectable pairs* (C_i, A), $i \in \mathbf{N}$, *described in* (2.1) *under Assumptions 3.1 and 3.2. Then, the LMIs* (3.4) *always have solution for* $P_i \succ 0$ *and* $Y_i = P_i L_i$, $i \in \mathbf{N}$.

Proof. Let the observer gains $L_i \in \mathbb{R}^{n\times p_i}$ and $P_i \in \mathbb{R}^{n\times n}$ be as follows:

$$\begin{aligned} L_i &= T_i \begin{bmatrix} L_{id} \\ \mathbf{0}_{v_i\times p_i} \end{bmatrix}, \\ P_i &= T_i \begin{bmatrix} P_{id} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & P_{iu} \end{bmatrix} T_i^\top, \end{aligned} \tag{3.34}$$

where T_i is the similarity transformation matrix defined in (3.20), $L_{id} \in \mathbb{R}^{(n-v_i)\times p_i}$, $P_{id} \in \mathbb{R}^{(n-v_i)\times(n-v_i)} \succ 0$, and $P_{iu} \in \mathbb{R}^{v_i\times v_i} \succ 0$. From the definition of Λ_i in (3.7), the definition of L_i and P_i in (3.34), and the detectable canonical decomposition in (3.21), we have

$$\Lambda_i = T_i \begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} T_i^\top, \tag{3.35}$$

where $\Lambda_{id} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}$, $\Lambda_{ir} \in \mathbb{R}^{v_i\times(n-v_i)}$, and $\Lambda_{iu} \in \mathbb{R}^{v_i\times v_i}$ are as follows:

$$\begin{aligned} \Lambda_{id} &= M_{id}^\top P_{id} + P_{id} M_{id}, \\ \Lambda_{ir} &= P_{iu} A_{ir}, \\ \Lambda_{iu} &= A_{iu}^\top P_{iu} + P_{iu} A_{iu}, \end{aligned}$$

in which $M_{id} = A_{id} + L_{id} C_{id}$. Since T_i is of full rank, according to (3.35) the LMIs (3.4) have solution if and only if the following LMIs have solution:

$$\begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} \preceq 0, i \in \mathbf{N}, \tag{3.36a}$$

$$\sum_{i=1}^{N} \left(T_i \begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} T_i^\top \right) \prec 0. \tag{3.36b}$$

Since $T_i = \begin{bmatrix} T_{id} & T_{iu} \end{bmatrix}$, one gets:

$$T_i \begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} T_i^\top = T_{id}\Lambda_{id}T_{id}^\top + T_{iu}\Lambda_{ir}T_{id}^\top + T_{id}\Lambda_{ir}^\top T_{iu}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top.$$

Then, we have

$$\sum_{i=1}^{N}\left(T_i \begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} T_i^\top\right) = T_d\Lambda_d T_d^\top + \sum_{i=1}^{N}\Big(T_{iu}\Lambda_{ir}T_{id}^\top + T_{id}\Lambda_{ir}^\top T_{iu}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top\Big), \tag{3.37}$$

where T_d is defined as in (3.28), and Λ_d is a block diagonal matrix as follows:

$$\Lambda_d = \mathrm{diag}(\Lambda_{1d}, \Lambda_{2d}, \dots, \Lambda_{Nd}).$$

As stated in Lemma 3.4, $\mathrm{rank}(T_d) = n$. Hence, according to (3.37), the LMI (3.36b) is satisfied if the following LMI is satisfied:

$$T_d\left(\Lambda_d + H\right)T_d^\top \prec 0, \tag{3.38}$$

where

$$H = T_d^{-\dagger}\sum_{i=1}^{N}\Big(T_{iu}\Lambda_{ir}T_{id}^\top + T_{id}\Lambda_{ir}^\top T_{iu}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top\Big)T_d^{-\dagger\top}.$$

Hence, according to (3.36) and (3.38), the LMIs (3.4) have solution if and only if the following LMIs have solution:

$$\begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} \preceq 0, i \in \mathbf{N}, \tag{3.39a}$$

$$T_d\left(\Lambda_d + H\right)T_d^\top \prec 0. \tag{3.39b}$$

Let us consider a vector $\sigma_i = [\sigma_{i1}^\top \;\; \sigma_{i2}^\top]^\top$ where $\sigma_{i1} \in \mathbb{R}^{n-v_i}$ and $\sigma_{i2} \in \mathbb{R}^{v_i}$. The LMIs (3.39a) hold if for all nonzero vectors $\sigma_i, i \in \mathbf{N}$,

$$\begin{bmatrix} \sigma_{i1}^\top & \sigma_{i2}^\top \end{bmatrix} \begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} \begin{bmatrix} \sigma_{i1} \\ \sigma_{i2} \end{bmatrix} \preceq 0, i \in \mathbf{N},$$

which can be restated as follows:

$$\sigma_{i1}^\top \Lambda_{id} \sigma_{i1} + 2\sigma_{i1}^\top \Lambda_{ir}^\top \sigma_{i2} + \sigma_{i2}^\top \Lambda_{iu} \sigma_{i2} \preceq 0, i \in \mathbf{N}. \tag{3.40}$$

It should be noted that, since the pair (C_{id}, A_{id}) is detectable, there exist some L_{id} such that M_{id} is stable. Thus, according to the Lyapunov stability criterion, for any $\Lambda_{id} \prec 0$, the equation $\Lambda_{id} = M_{id}^\top P_{id} + P_{id} M_{id}$ has a solution for $P_{id} \succ 0$. Moreover, since A_{iu} is marginally stable (from Assumption 3.2), by invoking Lemma 3.3, there always exists $P_{iu} \succ 0$ such that $\Lambda_{iu} = A_{iu}^\top P_{iu} + P_{iu} A_{iu} \preceq 0$. Now, to show (3.40), for each $i \in \mathbf{N}$, first we consider a case when σ_{i1} is nonzero. If σ_{i1} is nonzero, for Λ_{id} with large enough negative eigenvalues, (3.40) is satisfied implying the satisfaction of the LMIs (3.39a). Moreover, if σ_{i1} is a zeros vector, since $\Lambda_{iu} \preceq 0$, (3.40) is still satisfied implying the satisfaction of the LMIs (3.39a). For $\Lambda_{id}, i \in \mathbf{N}$, with large enough negative eigenvalues, the LMI (3.39b) also holds. Based on the aforementioned issues, there exist L_{id}, P_{id}, and P_{iu} such that the LMIs (3.4) hold. Then, we construct L_i and P_i as in (3.34). Note that, since $P_{id} \succ 0$ and $P_{iu} \succ 0$ by design, it follows that $P_i \succ 0$ according to (3.34). Therefore, there exist $P_i \succ 0$ and $Y_i = P_i L_i$ such that the LMIs (3.4) are satisfied. $\square$

3.2 Distributed State Estimation With Input and Output Noise Under Switching Topology

In this section, we investigate the distributed state estimation problem in the presence of disturbances and measurement noise where the communication is switching but connected over time. We consider an infinite time sequence t_0, t_1, t_2, $\ldots$ starting at $t_0 = 0$, at which $\mathcal{G}(t)$ switches to $\mathcal{G}_k, k = 0, 1, 2, \ldots$, but stays connected, while the LTI system is subject to external disturbances and measurement noises. In the presence of external disturbance and measurement noises, the LTI system described in (2.1) can be rewritten as follows:

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw_0, \\ y_i &= C_i x + E_i w_i, i \in \mathbf{N}, \end{aligned} \tag{3.41}$$

where $w_0 \in \mathbb{R}^{n_0}$ denotes the system external disturbance, $w_i \in \mathbb{R}^{n_i}$ denotes the measurement noise at the ith node, $D \in \mathbb{R}^{n \times n_0}$, and $E_i \in \mathbb{R}^{p_i \times n_i}$. Under these conditions, we aim at exploiting the joint observability property of the system to design a network of distributed observers such that the estimated state vector of the ith observer converges to the state vector in a noise-free setting. Indeed, by defining $e_i = x - \hat{x}_i, i \in \mathbf{N}$, and by considering $w \in \mathbb{R}^{Nn+p}$ as

$$\begin{aligned} e &= \begin{bmatrix} e_1^\top & e_2^\top & \cdots & e_N^\top \end{bmatrix}^\top \\ w &= \begin{bmatrix} w_0^\top D^\top & w_1^\top E_1^\top & \dots & w_0^\top D^\top & w_N^\top E_N^\top \end{bmatrix}^\top \end{aligned} \tag{3.42}$$

where $p := \sum_{i=1}^{N} p_i$, the objective is to design a set of distributed observers such that the effect of the exogenous signals vector w on e is attenuated. To achieve this goal, we minimise the $\mathscr{H}_\infty$ norm of the mapping from w to e as the induced matrix norm of the $\mathscr{L}_2$ norm of e with respect to the $\mathscr{L}_2$ norm of w as follows:

$$\text{minimise } \mu \text{ s.t. } \left(\|e\|_{\mathscr{L}_2} - \mu\|w\|_{\mathscr{L}_2}\right) < 0, \tag{3.43}$$

where $\mu \in \mathbb{R}_{>0}$.

Remark 3.2. The minimisation for μ in (3.43) is with respect to the parameters of the devised distributed observer, which are illustrated explicitly after we reveal the observer design (see (3.45b)). Moreover, we show that the minimisation is valid for any set of switching connected topologies in later analysis.

3.2.1 Observer Design and Analysis

In this section, the design of the proposed distributed state observers (3.2) is addressed in the presence of external disturbances and measurement noises such that the objective (3.43) can be achieved. Toward this end, we introduce the following assumption.

Assumption 3.3. *The exogenous signals $t \longmapsto w_i(t)$, $i \in \{0, 1, \ldots, N\}$, belong to the $\mathscr{L}_2$ space.*

Remark 3.3. The assumption above allows to analyze how the noises affect the estimation errors. More specifically, the $\mathscr{H}_\infty$ approach is considered, which is in general aimed at attenuating the effect of the $\mathscr{L}_2$ norm of some exogenous disturbances (w in our case) on the $\mathscr{L}_2$ norm of some regulated outputs (i.e., e) as much as possible. Therefore, the main assumption behind the $\mathscr{H}_\infty$ design is that the exogenous signals are $\mathscr{L}_2$ functions of time. In practice, this means that the noises have bounded energy, which does not appear to be particularly restrictive. Generally speaking, any approach in the literature on the analysis of the impact of disturbances demands for suitable assumptions on the noises, in that, for example, they may be regarded as unknown bounded inputs. Such assumptions as well as that of dealing with $\mathscr{L}_2$ signals are instrumental to the analysis of the effect of disturbances on the estimation errors.

Now, the distributed state observers proposed in (3.2) are modified as follows:

$$\dot{\hat{x}}_i = A\hat{x}_i + L_i(C_i\hat{x}_i - y_i) + Bu + \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(t)(\hat{x}_j - \hat{x}_i), i \in \mathbf{N}, \tag{3.44}$$

where $\chi \in \mathbb{R}_{>0}$. The observer gains L_i, P_i and χ can be selected by solving an $\mathscr{H}_\infty$ optimization problem in such a way as to attenuate the effect of the exogenous signals vector w on the observer errors vector e in terms of the induced matrix norm of the $\mathscr{L}_2$ norm of e with respect to the $\mathscr{L}_2$ norm of w as in (3.43).

Theorem 3.3. *Consider the jointly observable LTI system described in* (2.1) *under Assumption 3.3 and the distributed observers given in* (3.44)*. Moreover, let $\mathcal{G}(t)$ be switching, but connected within infinite sequence of bounded time intervals $[t_0, t_1)$, $[t_1, t_2)$, $[t_2, t_3)$, $\ldots$ starting at t_0. Under these conditions, the $\mathscr{H}_\infty$ performance* (3.43) *is satisfied if $\acute{\mu}$, $\chi \in \mathbb{R}_{>0}$, $P_i \succ 0$, and*

$Y_i, i \in \mathbf{N}$, *are obtained from the solution of the following LMI:*

$$\underset{Y_i, P_i, \chi}{\text{minimise}}\ \acute{\mu} \ \ \text{subject to} \tag{3.45a}$$

$$\begin{bmatrix} -\sum_{i=1}^{N} \Lambda_i - NI_n & -\Lambda_P & \Xi_P \\ -\Lambda_P^\top & \begin{matrix} 2\chi\mathcal{C}(N)I_{Nn} \\ -\Lambda - I_{Nn} \end{matrix} & \Xi \\ \Xi_P^\top & \Xi^\top & \acute{\mu}\, I_{p+Nn} \end{bmatrix} \succ 0, \tag{3.45b}$$

in which Λ *is defined in* (3.9) *and* $\Lambda_P = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \dots & \Lambda_N \end{bmatrix}$, *where* Λ_i *is defined in* (3.7) *and* $\Xi = \mathrm{diag}(\Xi_1, \Xi_2, \dots, \Xi_N)$ *and* $\Xi_P = \begin{bmatrix} \Xi_1 & \Xi_2 & \dots & \Xi_N \end{bmatrix}$, *where* $\Xi_i = \begin{bmatrix} P_i & Y_i \end{bmatrix}$. *In this condition,* $\mu = \sqrt{\acute{\mu}}$ *and* $L_i = P_i^{-1} Y_i, i \in \mathbf{N}$.

Proof. Considering the LTI system with the form of (3.41) and based on the observers (3.44), the differential equation describing $e_i, i \in \mathbf{N}$, is given by

$$\dot{e}_i = (A + L_i C_i)e_i - L_i E_i w_i - D w_0 + \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(t)(e_j - e_i), i \in \mathbf{N}. \tag{3.46}$$

To analyze the evolution of the estimation errors of all the nodes, we consider the following Lyapunov function:

$$V = \sum_{i=1}^{N} e_i^\top P_i e_i.$$

The time derivation of V along (3.46) yields

$$\dot{V} = \sum_{i=1}^{N} e_i^\top \Lambda_i e_i + 2\chi \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(t)(e_j - e_i)^\top e_i - 2 \sum_{i=1}^{N} e_i^\top P_i (L_i E_i w_i + D w_0). \tag{3.47}$$

From (3.47) and according to the definition of Λ, Ξ, e and w, it follows that

$$\dot{V} \leq e^\top \Lambda e - 2\chi e^\top (\mathcal{L}_k \otimes I_n) e - 2 e^\top \Xi w, \tag{3.48}$$

where $\mathcal{L}_k$ is the Laplacian matrix associated with $\mathcal{G}_k$. Let us denote by $\mathscr{E}$ the Nn dimensional error space. Since $\mathcal{G}_k$ is connected, according to the properties of the Laplacian matrix of a connected undirected graph, the kernel of $\mathcal{L}_k \otimes I_n$ has the form of $\mathbf{1}_N \otimes \omega$, where $\omega \in \mathbb{R}^n$ is an arbitrary vector. Now, by letting $\mathscr{E}_c \subseteq \mathscr{E}$ ($\dim(\mathscr{E}_c) = n$) as the null space of $\mathcal{L}_k \otimes I_n$, we consider another subspace $\mathscr{E}_r \subseteq \mathscr{E}$ ($\dim(\mathscr{E}_r) = Nn - n$), which is the orthogonal complement of $\mathscr{E}_c$, i.e., $\mathscr{E}_c \oplus \mathscr{E}_r = \mathscr{E}$. Let the vectors e_c and e_r be the elements of the subspaces $\mathscr{E}_c$ and $\mathscr{E}_r$ respectively, i.e., $e_c \in \mathscr{E}_c$ and $e_r \in \mathscr{E}_r$. The intuition behind the aforementioned decomposition is to exploit the properties of the Laplacian matrix associated with a connected graph. Since $e_c = \mathbf{1}_N \otimes \omega$, it will belong to the kernel of $\mathcal{L} \otimes I_n$ and this simplifies the mathematical analysis of (3.48). Based on $e_c = \mathbf{1}_N \otimes \omega$, later we also show that $e_c^\top \mathit{\Lambda} e_c$ is negative definite with respect to e_c. Moreover, since e_r is orthogonal to the kernel of $\mathcal{L} \otimes I_n$ and the second least eigenvalues of the Laplacian matrix is positive, $-2\chi e^\top (\mathcal{L} \otimes I_n) e$ is negative definite with respect to e_r.

Therefore, from (3.48) one gets

$$\dot{V} \leq e_c^\top \mathit{\Lambda} e_c + 2e_r^\top \mathit{\Lambda} e_c + e_r^\top \mathit{\Lambda} e_r - 2\chi e_r^\top (\mathcal{L}_k \otimes I_n) e_r - 2e_c^\top \mathit{\Xi} w - 2e_r^\top \mathit{\Xi} w. \tag{3.49}$$

A necessary condition to guarantee the negative definiteness of $\dot{V}$ is to ensure that $e_c^\top \mathit{\Lambda} e_c$ is negative definite with respect to $e_c \in \mathscr{E}_c$. Since e_c has the form of $\mathbf{1}_N \otimes \omega$, from the definition of $\mathit{\Lambda}$ we have

$$e_c^\top \mathit{\Lambda} e_c = \omega^\top \left(\sum_{i=1}^{N} \mathit{\Lambda}_i \right) \omega. \tag{3.50}$$

Considering the LMI condition (3.45b), $\sum_{i=1}^{N} \mathit{\Lambda}_i$ should be negative definite. Thus, according to (3.50), $e_c^\top \mathit{\Lambda} e_c$ is negative definite with respect to $e_c \in \mathscr{E}_c$. Based on the definition of $\mathit{\Lambda}_P$ and since $e_c = \mathbf{1}_N \otimes \omega$, we have

$$e_r^\top \mathit{\Lambda} e_c = e_r^\top \mathit{\Lambda}_P^\top \omega = \omega^\top \mathit{\Lambda}_P e_r. \tag{3.51}$$

Moreover, since $\mathscr{E}_r$ is the orthogonal complement of $\mathscr{E}_c$, it follows that $\forall e_r \in \mathscr{E}_r$ and $\forall \omega \in \mathbb{R}^n$,

$$e_r^\top (\mathbf{1}_N \otimes \omega) = 0.$$

Thus, as $\mathbf{1}_N$ is the eigenvector associated with the zero eigenvalue of $\mathcal{L}_k$, one can get that

$\forall e_r \in \mathscr{E}_r$ [9],

$$e_r^\top(\mathcal{L}_k \otimes I_n)e_r \geq \lambda_2(\mathcal{L}_k)e_r^\top e_r. \tag{3.52}$$

Now, according to (3.50), (3.51) and (3.52) and since $e_c = \mathbf{1}_N \otimes \omega$, from (3.49) it follows that

$$\begin{aligned}\dot{V} \leq &-\begin{bmatrix}\omega^\top & e_r^\top\end{bmatrix}\begin{bmatrix}-\sum_{i=1}^N \varLambda_i & -\varLambda_P \\ -\varLambda_P^\top & 2\chi\,\lambda_2(\mathcal{L}_k)I_{Nn} - \varLambda\end{bmatrix}\begin{bmatrix}\omega \\ e_r\end{bmatrix} \\ &- 2(\mathbf{1}_N \otimes \omega)^\top \varXi w - 2e_r^\top \varXi w,\end{aligned}$$

which using the definition of $\varXi_P$ and since $\lambda_2(\mathcal{L}_k) \geq \mathcal{C}(N)$, can be restated as follows:

$$\begin{aligned}\dot{V} \leq &-\begin{bmatrix}\omega^\top & e_r^\top\end{bmatrix}\begin{bmatrix}-\sum_{i=1}^N \varLambda_i & -\varLambda_P \\ -\varLambda_P^\top & 2\chi\,\mathcal{C}(N)I_{Nn} - \varLambda\end{bmatrix}\begin{bmatrix}\omega \\ e_r\end{bmatrix} \\ &- 2\omega^\top \varXi_P w - 2e_r^\top \varXi w.\end{aligned} \tag{3.53}$$

Now, by invoking [42], we need to impose

$$J = \dot{V} + e^\top e - \mu^2 w^\top w < 0. \tag{3.54}$$

According to the definition of e_c and e_r and by defining $\acute{\mu} = \mu^2$, from (3.54) it follows that

$$J = \dot{V} + e_c^\top e_c + e_r^\top e_r + 2e_c^\top e_r - \acute{\mu} w^\top w. \tag{3.55}$$

Since $e_c = \mathbf{1}_N \otimes \omega$ and $e_r^\top e_c = 0$, (3.55) can be rewritten as

$$J = \dot{V} + N\omega^\top \omega + e_r^\top e_r - \acute{\mu} w^\top w. \tag{3.56}$$

Thus, by considering (3.53) and (3.56), we have

$$J \leq -\begin{bmatrix}\omega^\top & e_r^\top & w^\top\end{bmatrix} M \begin{bmatrix}\omega^\top & e_r^\top & w^\top\end{bmatrix}^\top, \tag{3.57}$$

where

$$M = \begin{bmatrix} -\sum_{i=1}^{N} \Lambda_i - NI_n & -\Lambda_P & \Xi_P \\ -\Lambda_P^\top & 2\chi\mathcal{C}(N)I_{Nn} - \Lambda - I_{Nn} & \Xi \\ \Xi_P^\top & \Xi^\top & \acute{\mu} I_{p+Nn} \end{bmatrix}.$$

According to the LMI condition (3.45b), we have $M \succ 0$. Thus, from (3.57) it follows that $J < 0$, which implies that the $\mathscr{H}_\infty$ performance (3.43) is satisfied. Thus, the proof is completed.

□

Remark 3.4. From (3.53), it follows that for each $\varepsilon \in \mathbb{R}_{>0}$,

$$\begin{aligned} \dot{V} \leq & -\begin{bmatrix} \omega^\top & e_r^\top \end{bmatrix} \begin{bmatrix} -\sum_{i=1}^{N} \Lambda_i & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\mathcal{C}(N)I_{Nn} - \Lambda \end{bmatrix} \begin{bmatrix} \omega \\ e_r \end{bmatrix} \\ & + \frac{1}{\varepsilon}\omega^\top\omega + \varepsilon w^\top \Xi_P^\top \Xi_P w + \frac{1}{\varepsilon} e_r^\top e_r + \varepsilon w^\top \Xi^\top \Xi w, \end{aligned}$$

which can be restated as follows:

$$\dot{V} \leq -\begin{bmatrix} \omega^\top & e_r^\top \end{bmatrix} Q \begin{bmatrix} \omega \\ e_r \end{bmatrix} + d, \tag{3.58}$$

where

$$\begin{aligned} Q &= \begin{bmatrix} -\sum_{i=1}^{N} \Lambda_i - \frac{1}{\varepsilon} I_n & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\mathcal{C}(N)I_{Nn} - \Lambda - \frac{1}{\varepsilon} I_{Nn} \end{bmatrix}, \\ d &= \varepsilon w^\top \Xi_P^\top \Xi_P w + \varepsilon w^\top \Xi^\top \Xi w. \end{aligned}$$

Note that if the LMI (3.45b) is satisfied, then there exists ε such that $Q \succ 0$. Now, if $w_i, i \in \{0, 1, \ldots, N\}$, are bounded, d is bounded as well. Therefore, by defining $\bar{d}$ as a supremum of d, according to (3.58), the estimation errors are uniformly ultimately bounded such that

$$\limsup_{t \to \infty} \left\| \begin{bmatrix} \omega(t)^\top & e_r(t)^\top \end{bmatrix} \right\| \leq \left(\frac{\bar{d}}{\lambda_{\min}(Q)} \right)^{\frac{1}{2}},$$

which (since $e_c = \mathbf{1}_N \otimes \omega$) implies that $\|e\|$ is uniformly ultimately bounded as follows:

$$\limsup_{t\to\infty}\|e(t)\|\leq \left(\frac{N\bar{d}}{\lambda_{\min}(Q)}\right)^{\frac{1}{2}}.$$

Remark 3.5. In contrast to the connectivity condition (jointly connected) in Theorem 3.1, we assume that the communication graph is always connected for each switched topology in this section. However, we relax the assumption of the marginal detectability of the pairs (C_i, A), i.e., Assumption 3.2 is no longer necessary to be satisfied in Theorem 3.3.

3.2.2 Feasibility Analysis of the Design

The effectiveness of the proposed distributed state estimation strategy depends on the successful selection of the observer gains according to (3.45b). In the following theorem, we show that the LMI condition of Theorem 3.3 is always feasible.

Theorem 3.4. *Consider the jointly observable pairs* (C_i, A), $i \in \mathbf{N}$, *described in* (3.41)*. Then, for each* $\acute{\mu} \in \mathbb{R}_{>0}$*, there exist* $\chi \in \mathbb{R}_{>0}$ *and matrices* $P_i \succ 0$ *and* $Y_i = P_i L_i, i \in \mathbf{N}$*, satisfying the LMI* (3.45b)*.*

Proof. Let us consider the matrix M as follows:

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12}^\top & M_{22} & M_{23} \\ M_{13}^\top & M_{23}^\top & M_{33} \end{bmatrix},$$

where

$$\begin{aligned} &M_{11} = -\sum_{i=1}^{N} \Lambda_i - N I_n, \quad M_{12} = -\Lambda_P, \\ &M_{13} = \Xi_P, \quad M_{22} = 2\,\chi\,\mathcal{C}(N) I_{Nn} - \Lambda - I_{Nn}, \\ &M_{23} = \Xi, \quad M_{33} = \acute{\mu} I_{p+Nn}. \end{aligned} \tag{3.59}$$

Based on the Schur complement [43], M is positive definite if and only if

$$M_{33} \succ 0, \tag{3.60a}$$

$$M_{22} - M_{23}M_{33}^{-1}M_{23}^{\top} \succ 0, \tag{3.60b}$$

$$M_{11} - \begin{bmatrix} M_{12} & M_{13} \end{bmatrix} \begin{bmatrix} M_{22} & M_{23} \\ M_{23}^{\top} & M_{33} \end{bmatrix}^{-1} \begin{bmatrix} M_{12}^{\top} \\ M_{13}^{\top} \end{bmatrix} \succ 0. \tag{3.60c}$$

For any $\acute{\mu} \in \mathbb{R}_{>0}$, the inequality (3.60a) is satisfied. Since the term $M_{23}M_{33}^{-1}M_{23}^{\top}$ does not depend on χ and $M_{22} = 2\,\chi\,\mathcal{C}(N)I_{Nn} - \Lambda - I_{Nn}$, there exists a large enough $\chi \in \mathbb{R}_{>0}$ such that (3.60b) is satisfied as well. Moreover, for some $\alpha \in \mathbb{R}_{>0}$,

$$\begin{bmatrix} M_{12} & M_{13} \end{bmatrix} \begin{bmatrix} M_{22} & M_{23} \\ M_{23}^{\top} & M_{33} \end{bmatrix}^{-1} \begin{bmatrix} M_{12}^{\top} \\ M_{13}^{\top} \end{bmatrix} \preceq \alpha I_n.$$

Now, to show that the inequality (3.60c) has a solution, we show that $\forall \alpha \in \mathbb{R}_{>0}$, there exist $P_i \succ 0$ and $L_i = P_i^{-1}Y_i, i \in \mathbf{N}$, such that

$$M_{11} \succ \alpha I_n,$$

which, according to (3.59), can be rewritten as follows:

$$-\sum_{i=1}^{N} \Lambda_i \succ \alpha I_n + N I_n. \tag{3.61}$$

By considering $L_i \in \mathbb{R}^{n \times p_i}$ and $P_i \in \mathbb{R}^{n \times n}$ as (3.34), following a procedure similar to the proof of Theorem 3.2, one gets that (3.61) has a solution if the following LMI has a solution:

$$-T_d\,(\Lambda_d + H)\,T_d^{\top} \succ \alpha I_n + N I_n. \tag{3.62}$$

Since the pair (C_{id}, A_{id}) is detectable, there exists an L_{id} such that M_{id} is stable. Thus, according to the Lyapunov stability criterion, for any $\Lambda_{id} \prec 0$, the equation $\Lambda_{id} = M_{id}^{\top}P_{id} + P_{id}M_{id}$, has a solution for $P_{id} \succ 0$. Therefore, for $\Lambda_{id}, i \in \mathbf{N}$, with large enough negative eigenvalues, (3.62) will hold implying that (3.61) holds. Therefore, by arbitrarily choosing $P_{iu} \succ 0$, there exist L_{id}, P_{id} and P_{iu} such that (3.61) holds. Then, we construct L_i and P_i as given in (3.34). Therefore, for each $\acute{\mu} \in \mathbb{R}_{>0}$, there exist $\chi \in \mathbb{R}_{>0}$, $P_i \succ 0$, and $Y_i = P_i L_i, i \in \mathbf{N}$, satisfying the LMIs (3.60), which implies that the LMI (3.45b) is satisfied. Therefore, the proof is completed. □

Remark 3.6. It should be noted that, according to Theorem 3.4, for any performance index $\acute{\mu} > 0$, there always exists a set of gains for the proposed distributed observers satisfying the LMI (3.45b) with a dimension that scales quadratically with the number of the nodes N. Therefore, for any size of the network, by employing the proposed distributed estimation scheme, a desired performance can be guaranteed. However, if we push $\acute{\mu}$ very close to zero, the gains of the proposed distributed observer in (3.44) would be very large. In this case, even though the effects of $\mathscr{L}_2$ noise from the given channels (as stated in (3.41)) are attenuated, the large gains obtained from the LMI (3.45b) may weaken the robustness of the observer design against the potential noise, disturbances or systematical uncertainties. Usually, it is preferable to put certain constraints on the gain norms to prevent a set of large gains.

3.3 Summary

In this chapter, we have addressed the distributed estimation problem for LTI systems with switching communication communication topology. We have developed a method to design a network of distributed observers where the output measurement available for each observer may not be sufficient for the observability of the system. We have shown that if the observers can exchange their estimated state vectors with other neighboring observers, the observability of the system under the joint of all the measurements in the network is sufficient for each observer to estimate the full state vector of the system. We have also shown that, to guarantee the performance of the state estimation under the proposed strategy, communication among the agents is not required to be continuous, and distributed state estimation can be performed if the joint of communication links over finite intervals of time makes the network communication topology connected. Despite similar existing studies in the literature limited to LTI systems under certain assumptions and conditions in terms of systems parameters and communication topology (whose feasibility is not guaranteed for many scenarios), distributed state estimation for a wider class of LTI systems in the presence of jointly connected switching topology is guaranteed. Moreover, the results are extended to a scenario when the LTI system is subject to $\mathscr{L}_2$ external disturbances and measurement noises. Accordingly, observer gains are designed such that a desired $\mathscr{H}_\infty$ performance to attenuate the effect of external disturbances and measurement noises is guaranteed. Despite similar existing results in the literature on $\mathscr{H}_\infty$-based distributed state estimation, limited to LTI systems under certain conditions not guaranteed by design, in our work the distributed state estimation of all LTI systems in the presence of connected switching topology is ensured.

Chapter 4

Distributed State Estimation With Unknown Inputs

In this chapter, state estimation for a class of LTI systems with distributed output measurements (distributed sensors) and unknown inputs is addressed. Existing results in the literature on distributed state estimation assume either that the system does not have inputs, or that all the system's inputs are precisely known to all the observers. Accordingly, we address this gap by proposing a distributed observer capable of estimating the overall system's state in the presence of inputs, while each node may have limited access to the reliable input signals due to faults and/or disturbances.

We provide a design method that guarantees convergence of the estimation error under some mild joint detectability conditions. This design suits undirected communication graphs and also applies to strongly connected directed graphs with some proper modification on the observer gains. We also give existence conditions that harmonise with existing results on unknown input observers.

This chapter is organised as follows. Section 4.1 illustrates the design of a distributed state estimation scheme under connected networks. Moreover, in Section 4.2 the feasibility of the design is guaranteed and the necessity and sufficiency of the corresponding condition are discussed. Section 4.3 deals with the directed networks by modifying the proposed distributed estimation scheme.

4.1 DUIO Design

Based on the system described in (2.1), we consider input disturbance into the system, which is given by

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw, \\ y_i &= C_i x \end{aligned} \tag{4.1}$$

where $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is an unknown external disturbance, and $D \in \mathbb{R}^{n \times q}$ is the disturbance matrix gain. The outputs of the system are measured via a distributed measurement system comprising of a group of sensors distributed over N nodes, which is identical to the case in (2.1).

In order to further distinguish the locally available signals, we partition the system's inputs into a component u_i that is the reliable and accessible input signal at Node i, and is assumed to be known by the observer – and a component $\acute{u}_i$ – that is unknown and can instead be assimilated to an exogenous disturbance or fault. In symbols, we then have

$$Bu = B_i u_i + \acute{B}_i \acute{u}_i,$$

where $u_i \in \mathbb{R}^{r_i}$, $B_i \in \mathbb{R}^{n \times r_i}$, $\acute{u}_i \in \mathbb{R}^{d_i}$, and $\acute{B}_i \in \mathbb{R}^{n \times d_i}$, with $r_i + d_i = m$. Then, as w is also unknown, we define

$$\acute{w}_i = \begin{bmatrix} \acute{u}_i^\top & w^\top \end{bmatrix}^\top$$

as the locally unknown inputs and $\bar{B}_i = \begin{bmatrix} \acute{B}_i & D \end{bmatrix}$ as the known gain of the unknown input terms.

Assumption 4.1. *Matrix $\bar{B}_i$ is full column rank for all $i \in \mathbf{N}$.*

Remark 4.1. Note that Assumption 4.1 does not cause any loss of generality and is typically

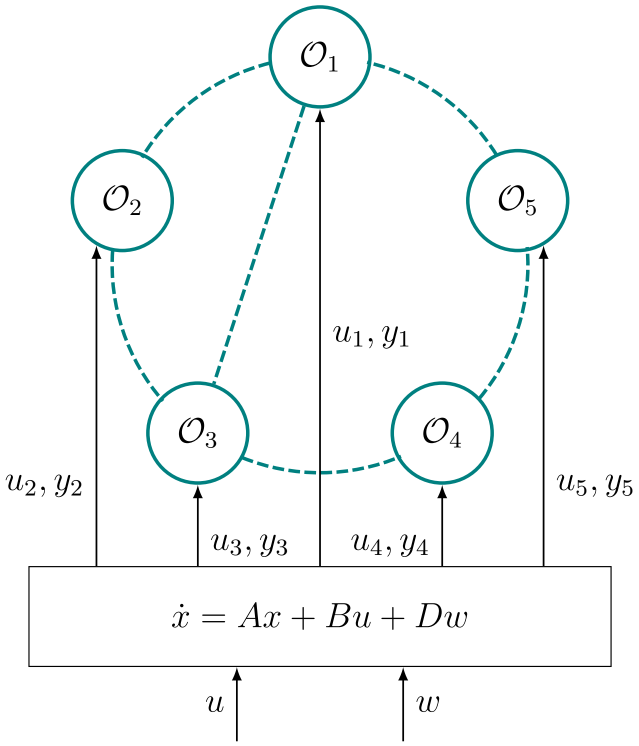


Figure 4.1: A distributed observer consisting of 5 nodes: each local observer $\mathcal{O}_i$ has access to local inputs and measurements u_i and y_i. Furthermore, neighbouring estimates are exchanged over a undirected communication network (dashed line).

made in the literature of estimation with unknown disturbances [30]. In fact, it is always possible (by means of singular value decomposition, for instance) to decompose $\bar{B}_i$ in a product $\bar{B}_i = \bar{B}_i' \bar{B}_i''$, where $\bar{B}_i'$ is full column rank and $\bar{w}_i' = \bar{B}_i'' \acute{w}_i$ constitutes the new unknown input.

Assumption 4.2. *The network communication graph is connected.*

To provide a visual example of the proposed architecture, in Fig. 4.1, an undirected network of distributed observers with 5 nodes is shown, where the local information of each node includes the local output measurement vector y_i and the local known control input vector u_i.

We can finally characterize the distributed estimation problem. Let $\hat{x}_i$ denote the estimate of x produced by the local observer $\mathcal{O}_i$, then we define the estimation error as

$$e_i = x - \hat{x}_i. \tag{4.2}$$

A DUIO is hence defined as follows.

Definition 4.1. *The set of observers* $\{\mathcal{O}_i\}_{i\in\mathbf{N}}$ *is a DUIO for system* (4.1) *if for all* $i \in \mathbf{N}$,

$$\lim_{t\to+\infty} \|e_i(t)\| = 0,$$

for all locally unknown inputs $\acute{w}_i$.

That is to say a distributed observer is a DUIO if the local estimation error terms are decoupled from the disturbances and the input components that are not locally available.

The basic principle to design a UIO is to derive some algebraic conditions that decouple the observer's error from the unknown disturbances and inputs (see e.g., [30]). Along this pattern, we propose the following full-order local observer $\mathcal{O}_i, i \in \mathbf{N}$:

$$\begin{aligned}
\dot{z}_i &= N_i z_i + M_i B_i u_i + L_i y_i + \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i), \\
\hat{x}_i &= z_i + H_i y_i,
\end{aligned} \tag{4.3}$$

where $z_i \in \mathbb{R}^n$ is the state vector of the observer $\mathcal{O}_i$, the matrices N_i, M_i, L_i, P_i, and H_i are to be designed, and χ is a real-valued design parameter.

We note that

$$e_i = x - z_i - H_i y_i = (I_n - H_i C_i)x - z_i. \tag{4.4}$$

Taking the time derivative of (4.4) yields

$$\begin{aligned}
\dot{e}_i =& (I_n - H_i C_i)(Ax + B_i u_i + \bar{B}_i \acute{w}_i) \\
& - N_i z_i - M_i B_i u_i - L_i y_i - \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i),
\end{aligned}$$

which by adding and subtracting the term $(I_n - H_i C_i)A\hat{x}_i$ to the right-hand side and since

$\hat{x}_i = z_i + H_i y_i$, can be restated as follows:

$$\begin{aligned}\dot{e}_i =&(I_n - H_iC_i)Ae_i + (I_n - H_iC_i - M_i)B_iu_i \\ &+ (I_n - H_iC_i)\bar{B}_i\acute{w}_i + (I_n - H_iC_i)A(z_i + H_iy_i) \\ &- N_iz_i - L_iy_i - \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i).\end{aligned} \tag{4.5}$$

According to the definition of e_i, one gets

$$-K_iC_ie_i + K_iC_i(x - \hat{x}_i) = \mathbf{0}_{n\times 1}. \tag{4.6}$$

Since $\hat{x}_i = z_i + H_iy_i$, from (4.6), it follows that

$$-K_iC_ie_i + K_iy_i - K_iC_iz_i - K_iC_iH_iy_i = \mathbf{0}_{n\times 1}. \tag{4.7}$$

We note that

$$\hat{x}_j - \hat{x}_i = x - \hat{x}_i - (x - \hat{x}_j) = e_i - e_j. \tag{4.8}$$

Now, by adding the zero term (4.7) to the right-hand side of (4.5) and by considering (4.8), after grouping similar terms, we finally obtain the following differential equation (detailed derivation is provided in Appendix A):

$$\begin{aligned}\dot{e}_i =& [(I_n - H_iC_i)A - K_iC_i]e_i \\ &+ (I_n - H_iC_i - M_i)B_iu_i + (I_n - H_iC_i)\bar{B}_i\acute{w}_i \\ &+ [(I_n - H_iC_i)A - K_iC_i - N_i]z_i \\ &+ [K_i + ((I_n - H_iC_i)A - K_iC_i)H_i - L_i]y_i \\ &+ \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(e_j - e_i).\end{aligned} \tag{4.9}$$

Now, we impose the following conditions:

$$(I_n - H_iC_i)\bar{B}_i = \mathbf{0}_{n\times 1}, \tag{4.10a}$$

$$M_i = I_n - H_iC_i, \tag{4.10b}$$

$$N_i = M_iA - K_iC_i, \tag{4.10c}$$

$$L_i = K_i + N_iH_i. \tag{4.10d}$$

For convenience, and as a starting point to tackle the solution of (4.10), we recall the following lemma.

Lemma 4.1. *[30] Equation* (4.10a) *is solvable with respect to* H_i *if and only if*

$$\mathrm{rank}(C_i\bar{B}_i) = \mathrm{rank}(\bar{B}_i), \tag{4.11}$$

and the general solution is given by

$$H_i = U_i + Y_iV_i, \tag{4.12}$$

where $Y_i \in \mathbb{R}^{n\times p_i}$ *is an arbitrary matrix, and* U_i *and* V_i *are defined as follows:*

$$U_i = \bar{B}_i(C_i\bar{B}_i)^\dagger,$$

$$V_i = I_{p_i} - C_i\bar{B}_i(C_i\bar{B}_i)^\dagger.$$

Remark 4.2. The condition in (4.11) is a geometric condition to filter out the unknown input in channel $\bar{B}_i$, which implies that the unknown input in channel $\bar{B}_i$ should be "reflected" at the local output at Node i. Moreover, U_i is a special solution to the decoupling equation (4.10a), it may be beneficial to also consider Y_i as a design parameter that provides additional degrees of freedom without affecting the decoupling property. This enlarged solution space is useful as it may provide better (e.g., with lower gains) solutions to optimization problems such as the one of Theorem 4.1, or achieving secondary objectives like noise attenuation [44].

Lemma 4.1 provides a geometric condition that allows (4.10a) in particular to be satisfied. If one satisfies also the other equations in the group (4.10), then the estimation error in (4.9) takes the following form:

$$\dot{e}_i = N_i e_i + \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(e_j - e_i). \tag{4.13}$$

It should be noted that under the condition (4.10a), $\bar{B}_i$ will be in the null space of $I_n - H_i C_i$, such that the term $(I_n - H_i C_i)\bar{B}_i \acute{w}_i$ appearing in (4.9) does not enter (4.13). Therefore, any arbitrary w will not have any effect on the estimation errors.

Before introducing the main results on the design and existence of the DUIO, we investigate the detectability properties of the system. For convenience, we first introduce the following definition.

Definition 4.2 (Extended joint detectability)**.** *Let*

$$A_i = (I_n - U_i C_i)A. \tag{4.14}$$

System (4.1) *is extended jointly detectable from Node i if*

$$\bigcap_{i=1}^{N} \mathscr{UD}(C_i, A_i) = 0. \tag{4.15}$$

◁

Remark 4.3. If there is no unknown input channel at all the nodes, i.e., if $\bar{B}_i$ does not exist for all $i \in \mathbf{N}$, then we have $A_i = A$, $\forall i \in \mathbf{N}$. In this case, the definition of the extended joint detectability is the same as the joint detectability as we stated in Definition 2.1.

Note that we do not assume that each pair (C_i, A) is observable, or even detectable, i.e., a single output measurement y_i may not be sufficient in general to observe the system's state. In the following, we derive some further results on joint detectability.

By virtue of the definition of A_i in (4.14) and by recalling (4.12), we define

$$\bar{A}_i = (I_n - H_iC_i)A = A_i - Y_iV_iC_iA, \tag{4.16}$$

so that we can express $N_i = \bar{A}_i - K_iC_i$. With this, the convergence of the estimation errors in terms of the detectability properties of the pair $(C_i, \bar{A}_i)$ will be investigated. Accordingly, we introduce a similarity transformation matrix $T_i \in \mathbb{R}^{n\times n}, i \in \mathbf{N}$, as $T_i = \begin{bmatrix} T_{id} & T_{iu} \end{bmatrix}$ in which $T_{iu} \in \mathbb{R}^{n\times v_i}$ is an orthonormal basis of the undetectable subspace of $\left(C_i, \bar{A}_i\right)$, where v_i is the dimension of the undetectable subspace of the pair $(C_i, \bar{A}_i)$, and $T_{id} \in \mathbb{R}^{n\times(n-v_i)}$ is an orthonormal basis such that Im T_{id} is orthogonal to Im T_{iu} [19]. Note that by defining $\mathscr{X} \simeq \mathbb{R}^n$ as the n-dimensional state space of the system, we have $\mathscr{X} = \text{Im } T_{id} \oplus \text{Im } T_{iu}$. In the following lemmas, we investigate such detectability features using a geometric approach. We first prove that the detectability properties of the pairs $(C_i, \bar{A}_i)$ and (C_i, A_i) are equivalent, and then we provide a condition for which all the estimation errors can be steered to zero.

Lemma 4.2. *The undetectable subspaces of the pairs $(C_i, \bar{A}_i)$ and (C_i, A_i) are identical for all $Y_i \in \mathbb{R}^{n\times p_i}$.*

Proof. By considering (4.16), for some $F_i \in \mathbb{R}^{n\times p_i}$, one gets

$$\begin{aligned} \bar{A}_i + F_iC_i &= A_i - Y_iV_iC_iA_i + F_iC_i \\ &= A_i + \begin{bmatrix} F_i & -Y_iV_i \end{bmatrix} \begin{bmatrix} C_i \\ C_iA_i \end{bmatrix}. \end{aligned} \tag{4.17}$$

From (4.17), it follows that

$$\mathscr{UD}\left(C_i, \bar{A}_i\right) = \mathscr{UO}\left(\begin{bmatrix} C_i \\ C_iA_i \end{bmatrix}, A_i\right) \bigcap \text{ Ker } \alpha^+_{A_i}(A_i). \tag{4.18}$$

Meanwhile,

$$\mathscr{UD}\left(C_i, A_i\right) = \mathscr{UO}(C_i, A_i) \bigcap \text{ Ker } \alpha^+_{A_i}(A_i). \tag{4.19}$$

By comparing (4.18) and (4.19), to show that $\mathscr{UD}\left(C_i, \bar{A}_i\right) = \mathscr{UD}\left(C_i, A_i\right)$, we can show that the unobservable subspaces of $\left(\begin{bmatrix} C_i \\ C_iA_i \end{bmatrix}, A_i\right)$ and (C_i, A_i) are identical. It can be said that

$$\mathscr{UO}\left(\begin{bmatrix} C_i \\ C_iA_i \end{bmatrix}, A_i\right) = \bigcap_{k=1}^{n} \text{Ker} \begin{bmatrix} C_i \\ C_iA_i \end{bmatrix} A_i^{k-1}. \tag{4.20}$$

One can observe that

$$\text{Ker} \begin{bmatrix} C_i \\ C_iA_i \end{bmatrix} A_i^{k-1} = \text{Ker } C_iA_i^{k-1} \bigcap \text{ Ker } C_iA_i^{k},$$

which implies that

$$\bigcap_{k=1}^{n} \text{Ker} \begin{bmatrix} C_i \\ C_iA_i \end{bmatrix} A_i^{k-1} = \bigcap_{k=1}^{n+1} \text{Ker } C_iA_i^{k-1}. \tag{4.21}$$

Moreover, the unobservable subspace of the pair (C_i, A_i) is

$$\mathscr{UO}\left(C_i, A_i\right) = \bigcap_{k=1}^{n} \text{Ker } C_iA_i^{k-1}. \tag{4.22}$$

Now, from (4.20), (4.21), and (4.22), it follows that

$$\mathscr{UO}\left(C_i, A_i\right) = \mathscr{UO}\left(\begin{bmatrix} C_i \\ C_iA_i \end{bmatrix}, A_i\right),$$

and thus from (4.18) and (4.19), we have

$$\mathscr{UD}\left(C_i, A_i\right) = \mathscr{UD}\left(C_i, \bar{A}_i\right),$$

which completes the proof. □

Lemma 4.3. *Let system* (4.1) *be extended jointly detectable. Then, by letting*

$$T_d = \begin{bmatrix} T_{1d} & T_{2d} & \dots & T_{Nd} \end{bmatrix},$$

we have

$$\mathrm{Im}\ T_d = \mathscr{X}.$$

Proof. The proof is similar to the proof of Lemma 3.4. □

Let us investigate the stability of the estimation errors, with the hypotheses that a solution to (4.10) exists. In this case, we leverage standard Lyapunov arguments to obtain an LMI condition that guarantees stability of the (collective) error e, defined as the stacked vector of local observers' errors as follows:

$$e = \begin{bmatrix} e_1^\top & e_2^\top & \dots & e_N^\top \end{bmatrix}^\top. \tag{4.23}$$

The stability of the proposed distributed estimation scheme is analysed in the following theorem.

Theorem 4.1 (Stability)**.** *Consider the DUIO described in* (4.3) *under Assumption 4.2 and the conditions* (4.10)*. By letting*

$$\Lambda_i = A^\top(I_n - C_i^\top U_i^\top)P_i + P_i(I_n - U_iC_i)A - A^\top C_i^\top V_i^\top \bar{Y}_i^\top - \bar{Y}_i V_i C_i A - C_i^\top \bar{K}_i^\top - \bar{K}_i C_i, \tag{4.24}$$

the estimation error e converges to zero if the matrices $P_i \succ 0$, $Y_i = P_i^{-1}\bar{Y}_i$, and $K_i = P_i^{-1}\bar{K}_i$ are a feasible solution of the following LMI

$$\sum_{i=1}^{N} \Lambda_i \prec 0, \tag{4.25}$$

and the gain χ satisfies

$$\chi > \frac{\left\| \Lambda + \Lambda_P^\top \left(\sum_{i=1}^N \Lambda_i \right)^{-1} \Lambda_P \right\|}{2\lambda_2(\mathcal{L})}, \tag{4.26}$$

where

$$\begin{aligned} \Lambda &= \mathrm{diag}(\Lambda_1, \Lambda_2, \dots, \Lambda_N), \\ \Lambda_P &= \begin{bmatrix} \Lambda_1 & \Lambda_2 & \dots & \Lambda_N \end{bmatrix}. \end{aligned} \tag{4.27}$$

Proof. We show that along (4.13), the estimation errors converge to zero. Accordingly, we consider the following Lyapunov candidate of the estimation errors:

$$V = \sum_{i=1}^{N} e_i^\top P_i e_i, \tag{4.28}$$

which is a positive definite function of the estimation errors. The time derivative of V along (4.13) can be stated as follows:

$$\dot{V} = e^\top \operatorname*{diag}_{i\in\mathbf{N}}(N_i^\top P_i + P_i N_i)e - 2\chi e^\top(\mathcal{L}\otimes I_n)e. \tag{4.29}$$

Based on the conditions on M_i and N_i in (4.10b) and (4.10c), it follows that

$$\begin{aligned} N_i^\top P_i + P_i N_i =& \Big((I_n - H_iC_i)A - K_iC_i\Big)^\top P_i \\ &+ P_i\Big((I_n - H_iC_i)A - K_iC_i\Big). \end{aligned} \tag{4.30}$$

According to (4.30) and the definition of H_i in (4.12), one gets

$$\begin{aligned} N_i^\top P_i + P_i N_i = & A^\top P_i + P_iA - A^\top C_i^\top U_i^\top P_i - P_iU_iC_iA - A^\top C_i^\top V_i^\top Y_i^\top P_i - P_iY_iV_iC_iA \\ & - C_i^\top K_i^\top P_i - P_iK_iC_i, \end{aligned}$$

which by considering (4.24) with $\bar{Y}_i = P_iY_i$ and $\bar{K}_i = P_iK_i$, can be rewritten as

$$N_i^\top P_i + P_i N_i = \varLambda_i. \tag{4.31}$$

Now, from (4.31), (4.29) can be restated as follows:

$$\dot{V} = e^\top \varLambda e - 2\chi e^\top(\mathcal{L}\otimes I_n)e, \tag{4.32}$$

where $\varLambda$ is defined in (4.27). To analyze (4.32), we decompose the error space to two subspaces. By defining the error space as $\mathscr{E} \simeq \mathbb{R}^{Nn}$, one of these subspaces is denoted by $\mathscr{E}_c \subseteq \mathscr{E}$ ($\dim(\mathscr{E}_c) = n$) which is the kernel of $\mathcal{L}\otimes I_n$ and has the form of $\mathbf{1}_N \otimes \omega, \omega \in \mathbb{R}^n$. Accordingly, the other subspace is the orthogonal complement subspace of $\mathscr{E}_c$ which is denoted by $\mathscr{E}_r \subseteq \mathscr{E}$

($\dim(\mathscr{E}_c) = Nn - n$) such that $\mathscr{E}_c \oplus \mathscr{E}_r = \mathscr{E}$. Thus, by considering $e_c \in \mathscr{E}_c$ and $e_r \in \mathscr{E}_r$, from (4.32) we get:

$$\dot{V} = e_c^\top \Lambda e_c + 2e_r^\top \Lambda e_c + e_r^\top(\Lambda - 2\chi(\mathcal{L} \otimes I_n))e_r,$$

which, since $e_c = \mathbf{1}_N \otimes \omega$, can be restated as follows:

$$\dot{V} = \omega^\top \Big(\sum_{i=1}^{N} \Lambda_i\Big)\omega + 2e_r^\top \Lambda_P^\top \omega + e_r^\top(\Lambda - 2\chi(\mathcal{L} \otimes I_n))e_r. \tag{4.33}$$

Moreover, as e_r is orthogonal to the kernel of $\mathcal{L} \otimes I_n$, one gets [9]

$$-e_r^\top(\mathcal{L} \otimes I_n)e_r \le -\lambda_2(\mathcal{L})e_r^\top e_r. \tag{4.34}$$

Note that, since the graph is connected from Assumption 4.2, $\lambda_2(\mathcal{L}) \in \mathbb{R}_{>0}$. By considering (4.33) and (4.34), we have

$$\dot{V} \le -\begin{bmatrix} \omega \\ e_r \end{bmatrix}^\top \begin{bmatrix} -\sum_{i=1}^{N} \Lambda_i & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\lambda_2(\mathcal{L})I_{Nn} - \Lambda \end{bmatrix} \begin{bmatrix} \omega \\ e_r \end{bmatrix}. \tag{4.35}$$

From the inequality (4.26), one gets:

$$2\chi\lambda_2(\mathcal{L})I_{Nn} - \Lambda - \Lambda_P^\top \left(\sum_{i=1}^{N} \Lambda_i\right)^{-1} \Lambda_P \succ 0. \tag{4.36}$$

Finally, according to (4.36) and by invoking the Schur complement [43], the negative definiteness of $\dot{V}$ in (4.35) can be concluded. Thus, V asymptotically converges to zero, which implies that the estimation error e (and therefore all its components $e_i, \forall i \in \mathbf{N}$) converges to zero. □

Remark 4.4. *From* (4.32)*, it follows that*

$$\dot{V} = -e^\top \left(2\chi(\mathcal{L} \otimes I_n) - \Lambda\right) e,$$

where, according to the proof of Theorem 4.1, $2\chi(\mathcal{L} \otimes I_n) - \Lambda \succ 0$*. Now, by considering* (4.28)*,* *one gets*

$$\dot{V} \leq -\mu V,$$

where

$$\mu = \frac{\lambda_{\min}\big(2\chi(\mathcal{L} \otimes I_n) - \Lambda\big)}{\max\limits_{i \in \mathbf{N}} \big(\lambda_{\max}(P_i)\big)}. \tag{4.37}$$

From the comparison theorem for scalar ordinary differential equations [45, Chap. 3], one obtains $0 \leq V \leq v$ *for all* $t \geq 0$*, where* v *is given by*

$$v(t) = e^{-\mu t} v(0).$$

Namely v *converges to zero with time constant* $1/\mu$*, where, since* $0 \leq V \leq v$*,* $1/\mu$ *implies an upper bound for the time constant of the convergence of* V *to zero as well.*

4.2 Feasibility and Necessity Analysis of the Design

In this section, the feasibility of the LMI in (4.25) is discussed. Moreover, the necessary conditions for the guarantee of the solutions of the LMI is provided with proof.

Theorem 4.2 (Feasibility)**.** *If system* (4.1) *is extended jointly detectable, then the LMI* (4.25) *is always feasible for some* P_i*,* $\bar{Y}_i = P_i Y_i$*, and* $\bar{K}_i = P_i K_i$*.*

Proof. From (4.14), (4.16), and (4.24), Λ_i can be written as

$$\Lambda_i = (\bar{A}_i - K_i C_i)^\top P_i + P_i(\bar{A}_i - K_i C_i). \tag{4.38}$$

By considering the similarity transformation matrix $T_i \in \mathbb{R}^{n \times n}$, one can observe that [19]

$$\begin{aligned} T_i^\top \bar{A}_i T_i &= \begin{bmatrix} \bar{A}_{id} & \mathbf{0}_{(n-v_i)\times v_i} \\ \bar{A}_{ir} & \bar{A}_{iu} \end{bmatrix}, \\ C_i T_i &= \begin{bmatrix} C_{id} & \mathbf{0}_{p_i \times v_i} \end{bmatrix}, \end{aligned} \tag{4.39}$$

where the pair $(C_{id}, \bar{A}_{id})$ is detectable. Based on the aforementioned formulation, without loss of generality, let the observer gains $K_i \in \mathbb{R}^{n\times p_i}$ and $P_i \in \mathbb{R}^{n\times n}$ be written as follows:

$$\begin{aligned} K_i &= T_i \begin{bmatrix} K_{id} \\ \mathbf{0}_{v_i\times p_i} \end{bmatrix}, \\ P_i &= T_i \begin{bmatrix} P_{id} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & P_{iu} \end{bmatrix} T_i^\top, \end{aligned} \tag{4.40}$$

where $K_{id} \in \mathbb{R}^{(n-v_i)\times p_i}$, $P_{id} \in \mathbb{R}^{(n-v_i)\times(n-v_i)} \succ 0$, and $P_{iu} \in \mathbb{R}^{v_i\times v_i} \succ 0$. From the definition of Λ_i in (4.38), the definition of K_i and P_i in (4.40), and the decomposition performed in (4.39), we have

$$\Lambda_i = T_i \begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} T_i^\top, \tag{4.41}$$

where $\Lambda_{id} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}$, $\Lambda_{ir} \in \mathbb{R}^{v_i\times(n-v_i)}$, and $\Lambda_{iu} \in \mathbb{R}^{v_i\times v_i}$ are as follows:

$$\begin{aligned} \Lambda_{id} &= \Gamma_{id}^\top P_{id} + P_{id}\Gamma_{id}, \\ \Lambda_{ir} &= P_{iu}\bar{A}_{ir}, \\ \Lambda_{iu} &= \bar{A}_{iu}^\top P_{iu} + P_{iu}\bar{A}_{iu}, \end{aligned}$$

with $\Gamma_{id} = \bar{A}_{id} - K_{id}C_{id}$. Since $T_i = \begin{bmatrix} T_{id} & T_{iu} \end{bmatrix}$, one gets:

$$T_i \begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} T_i^\top = T_{id}\Lambda_{id}T_{id}^\top + T_{iu}\Lambda_{ir}T_{id}^\top + T_{id}\Lambda_{ir}^\top T_{iu}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top. \tag{4.42}$$

Now, from (4.42), and by defining

$$\begin{aligned} T_d &= \begin{bmatrix} T_{1d} & T_{2d} & \dots & T_{Nd} \end{bmatrix}, \\ \Lambda_d &= \mathrm{diag}(\Lambda_{1d}, \Lambda_{2d}, \dots, \Lambda_{Nd}), \end{aligned}$$

it follows that

$$\sum_{i=1}^{N}\left(T_i\begin{bmatrix}\Lambda_{id} & \Lambda_{ir}^{\top}\\ \Lambda_{ir} & \Lambda_{iu}\end{bmatrix}T_i^{\top}\right) = T_d\Lambda_d T_d^{\top} + \sum_{i=1}^{N}\left(T_{iu}\Lambda_{ir}T_{id}^{\top} + T_{id}\Lambda_{ir}^{\top}T_{iu}^{\top} + T_{iu}\Lambda_{iu}T_{iu}^{\top}\right). \tag{4.43}$$

If the system is extended jointly detectable it follows by Lemma 4.3 that $\mathrm{rank}(T_d) = n$. Hence, from (4.41) and (4.43), we have

$$\sum_{i=1}^{N}\Lambda_i = T_d\left(\Lambda_d + S\right)T_d^{\top}, \tag{4.44}$$

where

$$S = T_d^{\dagger}\sum_{i=1}^{N}\left(T_{iu}\Lambda_{ir}T_{id}^{\top} + T_{id}\Lambda_{ir}^{\top}T_{iu}^{\top} + T_{iu}\Lambda_{iu}T_{iu}^{\top}\right)T_d^{\dagger\top}.$$

Considering (4.44), since T_d is row independent, the LMI (4.25) is feasible if the following inequality has solution:

$$\Lambda_d + S \prec 0. \tag{4.45}$$

Let us recall that $\Lambda_d = \mathrm{diag}(\Lambda_{1d}, \Lambda_{2d}, \ldots, \Lambda_{Nd})$, where

$$\Lambda_{id} = \Gamma_{id}^{\top}P_{id} + P_{id}\Gamma_{id}$$

$$\Gamma_{id} = \bar{A}_{id} - K_{id}C_{id}.$$

Because of the detectability of the pair $(C_{id}, \bar{A}_{id})$, there exists K_{id} such that Γ_{id} is Hurwitz. In this condition, according to the Lyapunov stability criterion [46, Chap. 6], for each $\beta \in \mathbb{R}_{>0}$ there exists $P_{id} \succ 0$ such that $\Lambda_{id} = \Gamma_{id}^{\top}P_{id} + P_{id}\Gamma_{id} = -\beta I_{n-v_i}$. On the other hand, there exists a large enough β such that (4.45) has solutions, which guarantees the feasibility of the LMI (4.25). Hence, by selecting $P_{iu} \succ 0$ and Y_i arbitrarily, and according to the definition of K_i and P_i in (4.40), the LMI (4.25) always has solutions for P_i, $\bar{Y}_i$, and $\bar{K}_i$. □

Theorems 4.1 and 4.2 give *constructive* sufficient conditions which can be effectively used to compute the design parameters that achieve error convergence to zero. In the next theorem, we provide necessary and sufficient existence conditions for the proposed observer to be a DUIO in the sense of Definition 4.1.

Theorem 4.3 (Existence)**.** *Under Assumption 4.2, the observer $\mathcal{O} = \{\mathcal{O}_i\}_{i\in\mathbf{N}}$ comprising local observers in the form* (4.3) *is a DUIO for the LTI system* (4.1) *if and only if the following conditions hold:*

(i) $\mathrm{rank}(C_i\bar{B}_i) = \mathrm{rank}(\bar{B}_i), \forall i \in \mathbf{N}$,

(ii) $\bigcap_{i=1}^{N} \mathscr{U}\mathscr{D}(C_i, A_i) = 0$.

Proof. (Sufficiency) – If (i) holds, (4.10a) is solvable as stated in Lemma 4.1. If (ii) is true, then by Theorem 4.2 we conclude that the LMI (4.25) admits a solution. Therefore, we can also apply Theorem 4.1 and conclude that such solution renders e asymptotically stable (as in (3.1)). Therefore, $\mathcal{O}$ is a DUIO for (4.1), according to Definition 4.1.

(Necessity) – Assume now that $\mathcal{O} = \{O_i\}_{i\in\mathbf{N}}$ is a DUIO for (4.1), i.e., $\forall i \in \mathbf{N}, \lim_{t\to+\infty}\|e_i(t)\|= 0$. This immediately implies that (4.10a) is solvable, since it is a necessary condition (see [30]) for any arbitrary disturbance term $\acute{w}$ in (4.9) to be exactly cancelled. Hence, according to Lemma 4.1, (i) holds. To prove the necessity of (ii), we proceed by contradiction and assume that there exists a non-trivial subspace $\mathscr{S} \subset \mathscr{X}$ such that

$$\bigcap_{i=1}^{N} \mathscr{U}\mathscr{D}(C_i, A_i) = \mathscr{S} \neq 0,$$

which, according to (4.19), is equivalent to

$$\mathscr{S} = \left(\bigcap_{i=1}^{N} \mathscr{U}\mathscr{O}(C_i, A_i)\right) \bigcap \left(\bigcap_{i=1}^{N} \mathrm{Ker}\ \alpha^+_{A_i}(A_i)\right). \tag{4.46}$$

Without loss of generality and for convenience of notation, we consider eigenvalues with unit multiplicity, so that the factorization of $\alpha^+_{A_i}(s)$ is

$$\alpha^+_{A_i}(s) = \underbrace{(s - \mu^+_{i,1})}_{\alpha^+_{A_i,1}(s)} \cdots \underbrace{(s - \mu^+_{i,q_i})}_{\alpha^+_{A_i,q_i}(s)}, \tag{4.47}$$

for some positive $q_i \leq n$, where $\mu_{i,k} \in \mathbb{C}^+$ for $k = 1, \ldots, q_i$. An analogous factorization exists for $\alpha^-_{A_i}(s)$. By the primary decomposition theorem for linear transformations [46, Chap. 2.2.M], $\mathscr{X}$ is decomposed into linearly independent subspaces as

$$\mathscr{X} \simeq \mathscr{X}_i^- \oplus \mathscr{X}_i^+ = \mathscr{X}_i^- \oplus \mathscr{X}_{i,1}^+ \oplus \cdots \oplus \mathscr{X}_{i,q_i}^+, \tag{4.48}$$

where $\mathscr{X}_i^- = \text{Ker } \alpha^-_{A_i}(A_i)$ and $\mathscr{X}_{i,k}^+ = \text{Ker } \alpha^+_{A_i,k}(A_i)$. Therefore, thanks to the linear independence of the modes, we have

$$\text{Ker } \alpha^+_{A_i}(A_i) = \text{Ker } \alpha^+_{A_i,1}(A_i) \oplus \cdots \oplus \text{Ker } \alpha^+_{A_i,q_i}(A_i). \tag{4.49}$$

Thus, the second term in the right hand side of (4.46) expands as follows:

$$\bigcap_{i=1}^{N} \text{Ker } \alpha^+_{A_i}(A_i) = \bigcap_{i=1}^{N} \mathscr{X}_{i,1}^+ \oplus \cdots \oplus \mathscr{X}_{i,q_i}^+.$$

Since $\mathscr{S} \neq 0$, there exists $\mathscr{X}_\cap^+ \subseteq \mathscr{X}_i^+, \forall i \in \mathbf{N}$, whose intersection with the unobservable subspaces is non-trivial. Namely, $\mathscr{X}_\cap^+ \subseteq \mathscr{S}$ is by construction an A_i-invariant subspace of an undetectable mode that is common to all nodes, i.e., $\forall i \in \mathbf{N}$, it satisfies

$$\mathscr{X}_\cap^+ = \{x \in \mathscr{X} : (A_i - \mu_i I_n)x = \mathbf{0}_{n\times 1}, x \neq \mathbf{0}_{n\times 1}\}, \tag{4.50}$$

for some $\mu_i \in \mathbb{C}^+$. By Lemma 4.2, (4.46)–(4.50) hold for $\bar{A}_i$ as well, thus we let $v \in \mathscr{X}_\cap^+$ be one of such common undetectable modes, and since $\mathscr{S} \subseteq \text{Ker } C_i$ for all $i \in \mathbf{N}$, it holds that

$$(\bar{A}_i - K_i C_i)v = \bar{A}_i v. \tag{4.51}$$

By stacking the error components and from (4.13), we obtain

$$\begin{aligned} \dot{e} &= \left(\operatorname*{diag}_{i\in\mathbf{N}}(N_i) - \chi \operatorname*{diag}_{i\in\mathbf{N}}\left(P_i^{-1}\right)(\mathcal{L} \otimes I_n)\right) e \\ &= (\Phi - \Pi)e, \end{aligned} \tag{4.52}$$

where the definitions of Φ and Π follow trivially from the equality. Let $\bar{e} = \mathbf{1}_N \otimes v$. Thanks to Assumption 4.2, $\bar{e} \in \operatorname{Ker} \Pi$, that is $\Pi\bar{e} = \mathbf{0}_{Nn\times 1}$. Moreover, for each block of Φ, $\bar{e}$ satisfies (4.51) and the eigenvalue relation (4.50). Therefore, by considering (4.10), we have

$$\begin{aligned}(\Phi - \Pi)\bar{e} &= \operatorname*{diag}_{i\in\mathbf{N}}(\bar{A}_i - K_i C_i)\bar{e} = \operatorname*{diag}_{i\in\mathbf{N}}(\bar{A}_i)\bar{e} \\ &= \left(\operatorname*{diag}_{i\in\mathbf{N}}(\mu_i) \otimes I_n\right)\bar{e}.\end{aligned}$$

Now, choosing $\bar{e}_0 = \mathbf{1}_N \otimes v$ as the initial condition for (4.52) produces an error along the direction of the unstable mode v. This contradicts the asymptotic stability hypothesis, and therefore (ii) must be true. □

It should be noted that we have formulated Theorem 4.3 in a way to express the similarities of the conditions derived in our approach to the classical existence conditions [30, Theorem 1] for the centralised case. We remark as well that (ii) is a necessary and sufficient condition also appearing in [25].

4.3 Extension to Directed Networks

The results presented in Theorem 4.1 are based on the assumption that the communication graph is undirected. However, by suitably modifying the observer gains, we show that in this section, the obtained results can be extended to a more general scenario, which is the distributed state estimation under directed networks.

Let the network communication graph be fixed and directed. When the graph is undirected, the Laplacian matrix associated with the communication graph is semi-definite, and this property has been used in the proof of Theorem 4.1. However, if the graph is directed and strongly connected (note that the connected graph is a special case of the directed strongly connected graph), it is possible to modify the proposed DUIO in Theorem 4.1 such that the obtained results in Theorem 4.1 are extendable to directed networks as well. In this regard, we first introduce the following lemma.

Lemma 4.4. *[47] Assume $\mathcal{G}$ is a strongly connected directed graph. Then, there exists a unique positive row vector $r = \begin{bmatrix} r_1 & r_2 & \dots & r_N \end{bmatrix}$ such that $r\mathcal{L} = \mathbf{0}_{1\times N}$ and $r\mathbf{1}_N = N$, and by defining $R := \mathrm{diag}(r_1, \dots, r_N)$, the symmetric matrix $\hat{\mathcal{L}} := R\mathcal{L} + \mathcal{L}^\top R$ is positive semi-definite. Furthermore, $\mathbf{1}_N^\top \hat{\mathcal{L}} = \mathbf{0}_{1\times N}$, $\hat{\mathcal{L}}\mathbf{1}_N = \mathbf{0}_{N\times 1}$, and $\lambda_1 = 0$ is a simple eigenvalue of $\hat{\mathcal{L}}$ while the other eigenvalues of $\hat{\mathcal{L}}$ are positive real.*

Based on Lemma 4.4, to extend the result of Theorem 4.1 to strongly connected directed networks, the DUIO (4.3) can be modified as follows:

$$\begin{aligned} \dot{z}_i &= N_i z_i + M_i B_i u_i + L_i y_i + \chi r_i P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i), \\ \hat{x}_i &= z_i + H_i y_i. \end{aligned} \tag{4.53}$$

Corollary 4.1. *Consider the DUIO described in (4.53) under the conditions (4.10) and a strongly connected graph. By letting $\mathit{\Lambda}_i$ as (4.24), the estimation error e converges to zero if the matrices $P_i \succ 0$, $Y_i = P_i^{-1}\bar{Y}_i$, and $K_i = P_i^{-1}\bar{K}_i$ are a feasible solution of the LMI (4.25), and the gain χ satisfies*

$$\chi > \frac{\left\| \mathit{\Lambda} + \mathit{\Lambda}_P^\top \left(\sum_{i=1}^{N} \mathit{\Lambda}_i \right)^{-1} \mathit{\Lambda}_P \right\|}{2\lambda_2(\hat{\mathcal{L}})}, \tag{4.54}$$

where $\mathit{\Lambda}$ and $\mathit{\Lambda}_P$ are defined in (4.27).

Proof. Based on the analytical procedure given in Section 4.1, along the DUIO (4.53), the estimation error takes the following form:

$$\dot{e}_i = N_i e_i + \chi P_i^{-1} \sum_{j=1}^{N} r_i a_{ij}(e_j - e_i).$$

By considering the same Lyapunov function as in (4.28) and following the same procedure as in the proof of Theorem 4.1, one gets

$$\begin{aligned} \dot{V} &= e^\top \mathit{\Lambda} e - 2\chi e^\top \left(R\mathcal{L} \otimes I_n \right) e \\ &= e^\top \mathit{\Lambda} e - \chi e^\top \left((R\mathcal{L} + \mathcal{L}^\top R) \otimes I_n \right) e, \end{aligned}$$

which, after defining $\hat{\mathcal{L}} := R\mathcal{L} + \mathcal{L}^\top R$, can be restated as follows:

$$\dot{V} = e^\top \Lambda e - \chi e^\top \left(\hat{\mathcal{L}} \otimes I_n\right) e. \tag{4.55}$$

Since $\mathcal{G}$ is strongly connected, by considering Lemma 4.4, $\hat{\mathcal{L}}$ is symmetric positive semidefinite, $\mathbf{1}_N^\top \hat{\mathcal{L}} = \mathbf{0}_{1\times N}$, $\hat{\mathcal{L}}\mathbf{1}_N = \mathbf{0}_{N\times 1}$, and $\hat{\mathcal{L}}$ has one zero eigenvalue and $N-1$ positive real eigenvalues. By following the same procedure as in the proof of Theorem 4.1, from (4.55) one gets

$$\dot{V} = \omega^\top \Big(\sum_{i=1}^{N} \Lambda_i\Big)\omega + 2e_r^\top \Lambda_P^\top \omega + e_r^\top(\Lambda - \chi(\hat{\mathcal{L}} \otimes I_n))e_r.$$

Since $\mathcal{G}$ is strongly connected, according to Lemma 4.4, it follows that

$$-e_r^\top(\hat{\mathcal{L}} \otimes I_n)e_r \leq -\lambda_2(\hat{\mathcal{L}})e_r^\top e_r < 0.$$

Then, $\dot{V}$ yields

$$\dot{V} \leq - \begin{bmatrix} \omega \\ e_r \end{bmatrix}^\top \begin{bmatrix} -\sum_{i=1}^{N} \Lambda_i & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\lambda_2(\hat{\mathcal{L}})I_{Nn} - \Lambda \end{bmatrix} \begin{bmatrix} \omega \\ e_r \end{bmatrix}. \tag{4.56}$$

Based on (4.56) and the Schur complement, $\dot{V}$ is negative definite if

$$2\chi\lambda_2(\hat{\mathcal{L}})I_{Nn} - \Lambda - \Lambda_P^\top \left(\sum_{i=1}^{N} \Lambda_i\right)^{-1} \Lambda_P \succ 0,$$

which is guaranteed by (4.54). Hence, as $\dot{V}$ is negative definite, e converges to zero. □

Remark 4.5. Note that in this design, the gains N_i, M_i, L_i, P_i and H_i are selected based on the same LMI as provided in (4.24). Therefore, the feasibility analysis presented in Section 4.2 still applies, which indicates that the solution for the gains of this design is guaranteed.

4.4 Summary

In this chapter, distributed state estimation of a class of LTI systems has been addressed. We have proposed a DUIO consisting of N local observers co-located with the N nodes and connected via a communication network such that the full state vector of the system was estimated by each local observer. The proposed architecture allows to account for partial measurements, but more notably for inputs that may not be available locally at a node, together with other unknown disturbances. The feasible solution of an LMI provided adequate choices of parameters that guarantee convergence of the estimation errors, under some joint detectability conditions. Furthermore, we have provided necessary and sufficient existence conditions that were in line with existing theorems for the centralised case. Finally, we have extended our main result to include a more general scenario where the communication network is directed.

Chapter 5

Distributed State Estimation With Sensor Fault-Tolerance

In this chapter, we address the distributed state estimation for LTI systems with faults and noise at the measurement. We consider the scenario where the measurements are jointly detectable altogether, and locally redundant at each node. The proposed estimation scheme exploits sensor redundancy to provide robustness against fault in the sensors. Under suitable conditions on the redundant sensors, we show that it is possible to mitigate the effects of a class of sensor faults on the state estimation.

We begin this chapter by formulating the problem and defining precisely the sensor redundancy. Then, we propose a distributed state estimation scheme assuming no sensor noise nor faults occurence. Based on this scheme, we present a distributed sensor fault-tolerant observer design with guaranteed solutions. Finally, we show that the proposed designs can be extended to nonlinear LSS plant under certain Lipschitz condition.

5.1 Problem Statement

We consider a class of LTI system with distributed measurements as described in (2.1), given by

$$\dot{x} = Ax + Bu \tag{5.1a}$$

$$y_i = C_i x. \tag{5.1b}$$

Moreover, we assume that observers along with their local sensors are distributed in N nodes, and the sensor redundancy at Node i is r_i. In other words, we assume that r_i clusters of identical sensors are available at Node i. By defining $C_i \in \mathbb{R}^{p_i \times n}$ as the output matrix associated with the ith node, and by defining $\mathbf{r}_i = \{1, 2, \ldots, r_i\}$, the output measurement of the kth cluster of sensors is as follows:

$$y_{i,k} = C_i x + \varpi_{i,k}(t) + \delta_{i,k}(t), \quad i \in \mathbf{N},\ k \in \mathbf{r}_i \tag{5.2}$$

where the vectors $\varpi_{i,k}(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{p_i}$ and $\delta_{i,k}(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{p_i}$ represent unknown bounded signals denoting additive noise and fault respectively in the kth cluster of the sensors of Node i.

Notice that the output redundancy is from the r_i identical local measurements defined in (5.2) for each C_i, whereas joint observability is related to the collective of all $C_i, i \in \mathbf{N}$, together. Fig. 5.1 shows an example of a network of distributed observers with measurement redundancy.

Assumption 5.1. *The communication graph associated with the observers network is connected.*

Assumption 5.2. *We decompose the measurement noise vector $\varpi_{i,k}$ as*

$$\varpi_{i,k} = \begin{bmatrix} \varpi_{i,k,1} & \varpi_{i,k,2} & \ldots & \varpi_{i,k,p_i} \end{bmatrix}^\top,$$

and we let $\mathbf{p}_i = \{1, 2, \ldots, p_i\}$. The measurement noises $\varpi_{i,k,q}(t), i \in \mathbf{N}, k \in \mathbf{r}_i, q \in \mathbf{p}_i$, are bounded for all $t \geq 0$.

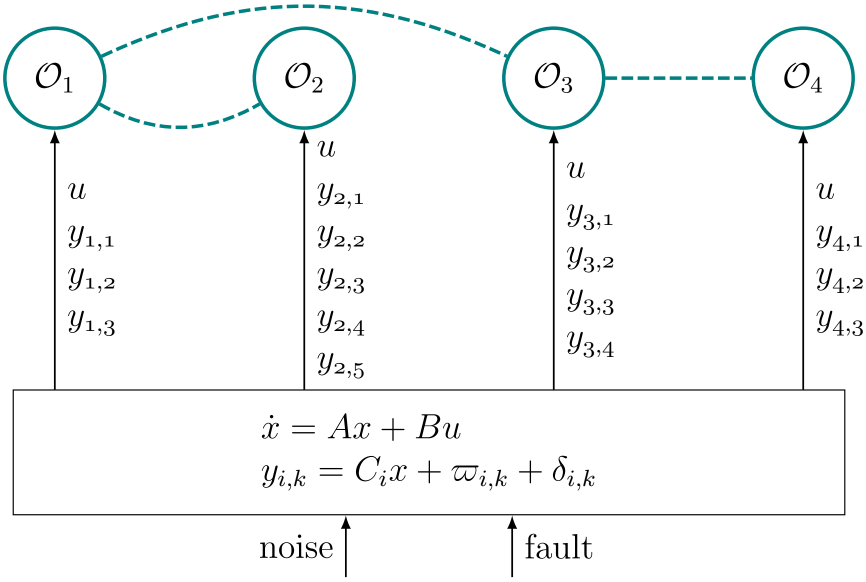


Figure 5.1: A distributed observer consisting of 4 nodes: each local observer $\mathcal{O}_i$ has access to inputs u and measurements with redundancy $y_{i,k}$. Furthermore, neighbouring estimates are exchanged over a undirected communication network (dashed line).

Definition 5.1. *Let us decompose $y_{i,k}$ and $\delta_{i,k}$ as*

$$\begin{aligned} y_{i,k} &= \begin{bmatrix} y_{i,k,1} & y_{i,k,2} & \dots & y_{i,k,p_i} \end{bmatrix}^\top \\ \delta_{i,k} &= \begin{bmatrix} \delta_{i,k,1} & \delta_{i,k,2} & \dots & \delta_{i,k,p_i} \end{bmatrix}^\top. \end{aligned}$$

For an arbitrary scalar constant $\beta_{i,q} \in \mathbb{R}, i \in \mathbf{N}, q \in \mathbf{p}_i$, we define an output measurement $y_{i,k,q}, i \in \mathbf{N}, k \in \mathbf{r}_i, q \in \mathbf{p}_i$, to be healthy if for all $t \geq 0$,

$$|\varpi_{i,k,q}(t) + \delta_{i,k,q}(t)| \leq \beta_{i,q}. \tag{5.3}$$

Otherwise, the measurement $y_{i,k,q}$ is called unhealthy. Accordingly, the associated sensors are called healthy and unhealthy, respectively.

Indeed, according to the required accuracy for state estimation, the value of $\beta_{i,q}$ will be determined, and this value will be considered in designing the proposed distributed observers later. According to Definition 5.1, we provide the following assumption.

Assumption 5.3. *In the set $\{y_{i,1,q}, y_{i,2,q}, \dots, y_{i,r_i,q}\}$, at least* $\mathrm{ceil}\left(\frac{r_i+1}{2}\right)$ *output measurements are healthy (no matter how large the magnitudes of the faults of unhealthy sensors are).*

Assumption 5.3 implies that the number of healthy measurements at each node, i.e., those satisfying (5.3), exceeds that of the unhealthy ones. Thus, for instance, out of 5 measurements of output, at least 3 of those measurements should be healthy at all times.

Note 5.1. *In the rest of the chapter, for any variable $\vartheta_{i,k}$, $\vartheta_{i,q}$, or $\vartheta_{i,k,q}$, we let $i \in \mathbf{N}$ denote the index of the nodes, $k \in \mathbf{r}_i$ denote the index of the clusters of the sensors at Node i, and $q \in \mathbf{p}_i$ denote the index of the measurements at each cluster of the sensors at Node i.*

For a system given by (5.1) under the assumptions above, the objective is to design a distributed sensor fault-tolerant observer such that the estimation error is uniformly ultimately bounded.

To achieve the objectives above, we first devise a linear distributed observer scheme in the next section *without* considering measurement noises and faults.

5.2 Distributed Observer Design Without Faults and Noise

The basic idea in distributed state estimation is that the ith observer updates its estimated state vector $\hat{x}_i$ not just based on local measurement, but also based on estimated state vectors of neighboring observers which is $\sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i)$ (for instance, see [12, 21, 27], and [34]). Based on this general approach, the proposed distributed estimation strategy for the system described in (5.1) and (5.2) when $\varpi_{i,k}(t) = \delta_{i,k}(t) = \mathbf{0}_{p_i \times 1}, \forall i \in \mathbf{N}, \forall k \in \mathbf{r}_i, \forall t \geq 0$, is presented in this section. Since sensor redundancy is unnecessary under this initial assumption, the output measurements of the system can be considered to be modeled as in (2.1).

Now, by using the similarity transformation matrix T_i defined in (3.20), the local observer at Node i is designed as

$$\dot{\hat{x}}_i = A\hat{x}_i + L_i(C_i\hat{x}_i - y_i) + Bu + \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i), \quad i \in \mathbf{N}, \tag{5.4}$$

where $L_i \in \mathbb{R}^{n\times p_i}$ and $P_i \in \mathbb{R}^{n\times n}$ are the observer gains computed as

$$\begin{aligned} L_i &= T_i \begin{bmatrix} P_{id}^{-1} C_{id}^\top H_i^\top \\ \mathbf{0}_{v_i\times p_i} \end{bmatrix} \\ P_i &= T_i \begin{bmatrix} P_{id} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & I_{v_i} \end{bmatrix} T_i^\top \end{aligned} \tag{5.5}$$

where $P_{id} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}$, $P_{id} \succ 0$, and $H_i \in \mathbb{R}^{p_i\times p_i}$ are matrices to be determined.

The analysis of the proposed distributed observer (5.4) is given in the following theorem.

Theorem 5.1. *Consider the noise-free and fault-free dynamical system described in* (5.1) *with the distributed observer given in* (5.4) *and* (5.5), *under Assumption 5.1. The estimation errors* $e_i = x - \hat{x}_i, \forall i \in \mathbf{N}$, *converge to zero if the matrices* P_{id} *and* $H_i, i \in \mathbf{N}$, *can be obtained from the solution of the following LMI:*

$$\begin{aligned} T_d\,(K - M)\,T_d^\top &\succ 0 \\ P_{id} &\succ 0 \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} K &= \operatorname{diag}(K_{1d}, K_{2d}, \ldots, K_{Nd}) \\ K_{id} &= -(A_{id}^\top P_{id} + P_{id}A_{id} + C_{id}^\top (H_i + H_i^\top) C_{id}) \\ M &= T_d^{-r} \sum_{i=1}^{N} \left(T_{iu}A_{ir}T_{id}^\top + T_{id}A_{ir}^\top T_{iu}^\top + T_{iu}(A_{iu} + A_{iu}^\top)T_{iu}^\top \right) \left(T_d^{-r}\right)^\top . \end{aligned} \tag{5.7}$$

Moreover, the gain χ *is chosen as follows:*

$$\chi > \frac{\left\| \varLambda + \varLambda_P^\top Q^{-1} \varLambda_P \right\|}{2\lambda_2(\mathcal{L})} \tag{5.8}$$

where

$$\begin{aligned}
\Lambda &= \operatorname{diag}\left(\Lambda_1, \Lambda_2, \ldots, \Lambda_N\right) \\
\Lambda_i &= (A + L_i C_i)^\top P_i + P_i (A + L_i C_i) \\
Q &= -\sum_{i=1}^{N} \Lambda_i \\
\Lambda_P &= \begin{bmatrix} \Lambda_1 & \Lambda_2 & \ldots & \Lambda_N \end{bmatrix}.
\end{aligned} \tag{5.9}$$

Proof. To begin, from (5.1), (5.2), and (5.4), it follows that the dynamics of $e_i = x - \hat{x}_i$, $i \in \mathbf{N}$, is given by

$$\dot{e}_i = (A + L_i C_i) e_i + \chi P_i^{-1} \sum_{j=1}^{N} a_{ij} (\hat{x}_j - \hat{x}_i), \quad i \in \mathbf{N},$$

which, since $-(\hat{x}_j - \hat{x}_i) = e_j - e_i$, can be restated as follows:

$$\dot{e}_i = (A + L_i C_i) e_i + \chi P_i^{-1} \sum_{j=1}^{N} a_{ij} (e_j - e_i), \quad i \in \mathbf{N}. \tag{5.10}$$

We should show that along (5.10), the estimation errors converge to zero. Consider the following Lyapunov candidate of the estimation errors:

$$V = \sum_{i=1}^{N} e_i^\top P_i e_i \tag{5.11}$$

which since $P_{io} \succ 0, i \in \mathbf{N}$, is a positive definite function of the estimation errors. The derivative of V along (5.10) reads as

$$\dot{V} = \sum_{i=1}^{N} e_i^\top \left((A + L_i C_i)^\top P_i + P_i (A + C_i L_i) \right) e_i + 2\chi \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \left(e_j - e_i\right)^\top e_i \tag{5.12}$$

Moreover, we get

$$2\chi \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \left(e_j - e_i\right)^\top e_i = -2\chi e^\top \left(\mathcal{L} \otimes I_n\right) e \tag{5.13}$$

where $e = \begin{bmatrix} e_1^\top & e_2^\top & \ldots & e_N^\top \end{bmatrix}^\top$. From (5.12) and (5.13), and according to the definition of Λ

in (5.9), it follows that

$$\dot{V} \leq e^\top \Lambda e - 2\chi e^\top (\mathcal{L} \otimes I_n)\, e. \tag{5.14}$$

Note that, since the pair (C_i, A) may not be detectable, Λ may not be negative definite by choosing L_i and $P_i \succ 0$. Thus, to show that the right-hand side of (5.14) is negative definite, in the following, we decompose the error space into two complementary subspaces such that $e^\top \Lambda e$ is negative definite when projected onto one of the subspace, and $-2\chi e^\top (\mathcal{L} \otimes I_n)\, e$ is negative definite when projected onto the other one. According to Assumption 5.1 and the properties of the Laplacian matrix of a connected undirected graph, $\mathcal{L}$ has a zero eigenvalue and $N-1$ positive eigenvalues, and the right eigenvector associated with the zero eigenvalue is $\mathbf{1}_N$. Thus, we decompose the error e as

$$e = e_c + e_r \tag{5.15}$$

where $e_c \in \mathbb{R}^{Nn}$ is the *consensus vector* in the form $e_c = \mathbf{1}_N \otimes \omega,\ \omega \in \mathbb{R}^n$, and $e_r \in \mathbb{R}^{Nn}$ is the *disagreement vector* satisfying $e_r^\top e_c = 0$ [48]. By using the same decomposition procedure as in Section 3.2.1 and considering (5.15), (5.14) yields

$$\dot{V} = e_c^\top \Lambda e_c + 2e_r^\top \Lambda e_c + e_r^\top \Lambda e_r - 2\chi e_r^\top (\mathcal{L} \otimes I_n) e_r. \tag{5.16}$$

To guarantee negative definiteness of $\dot{V}$, one should show that $e_c^\top \Lambda e_c$ is negative definite. Since $e_c = \mathbf{1}_N \otimes \omega$, according to the definition of Λ and $\bar{P}$ in (5.9), one obtains

$$e_c^\top \Lambda e_c = \omega^\top \left(\sum_{i=1}^{N} \Lambda_i \right) \omega. \tag{5.17}$$

In what follows, we show that the right-hand side of (5.17) is negative definite. From (3.21), we have

$$\begin{aligned} A &= T_i \begin{bmatrix} A_{id} & \mathbf{0}_{(n-v_i)\times v_i} \\ A_{ir} & A_{iu} \end{bmatrix} T_i^\top \\ C_i &= \begin{bmatrix} C_{id} & \mathbf{0}_{p_i \times v_i} \end{bmatrix} T_i^\top. \end{aligned} \tag{5.18}$$

Therefore, by considering (5.18) and the design of L_i and P_i in (5.5),

$$\Lambda_i := (A + L_i C_i)^\top P_i + P_i (A + L_i C_i)$$

can be rewritten as follows:

$$\Lambda_i = T_i \begin{bmatrix} -K_{id} & A_{ir}^\top \\ A_{ir} & A_{iu} + A_{iu}^\top \end{bmatrix} T_i^\top, \tag{5.19}$$

and $T_i = \begin{bmatrix} T_{id} & T_{iu} \end{bmatrix}$, (5.19) can be rewritten as

$$\Lambda_i = -T_{id} K_{id} T_{id}^\top + T_{iu} A_{ir} T_{id}^\top + T_{id} A_{ir}^\top T_{iu}^\top + T_{iu}(A_{iu} + A_{iu}^\top) T_{iu}^\top.$$

Therefore,

$$\sum_{i=1}^N \Lambda_i = -T_d K T_d^\top + \sum_{i=1}^N \left(T_{iu} A_{ir} T_{id}^\top + T_{id} A_{ir}^\top T_{iu}^\top \right) + \sum_{i=1}^N T_{iu}(A_{iu} + A_{iu}^\top) T_{iu}^\top \tag{5.20}$$

where $K \in \mathbb{R}^{\left(Nn - \sum_{i=1}^N v_i\right) \times \left(Nn - \sum_{i=1}^N v_i\right)}$ is the block diagonal matrix given in (5.7) and T_d is defined in (3.28). By invoking Lemma 3.1, T_d is full rank and hence its rows are linearly independent. Therefore, there exists $T_d^{-\dagger}$ such that $T_d T_d^{-\dagger} = I_n$. Accordingly, (5.20) can be restated in the following form:

$$\sum_{i=1}^N \Lambda_i = -T_d \left(K - M \right) T_d^\top \tag{5.21}$$

where M is defined in (5.7). Hence,

$$-\sum_{i=1}^N \Lambda_i = T_d \left(K - M \right) T_d^\top. \tag{5.22}$$

From (5.6) and (5.22), it follows that the right-hand side of (5.17) is negative definite, and, as a result, we have

$$e_c^\top \Lambda e_c = \omega^\top \left(\sum_{i=1}^N \Lambda_i \right) \omega < 0, \quad \forall \omega \neq 0. \tag{5.23}$$

Now, we investigate the other terms in the right-hand side of the inequality (5.16). As $e_c = \mathbf{1}_N \otimes \omega$, one has

$$e_r^\top \Lambda e_c = e_r^\top \Lambda_P^\top \omega = \omega^\top \Lambda_P e_r \tag{5.24}$$

where Λ_P is defined in (5.9). Moreover, since $e_r^\top e_c = 0$, it follows that

$$e_r^\top (\mathbf{1}_N \otimes \omega) = 0, \quad \forall \omega \in \mathbb{R}^n.$$

Since $\mathbf{1}_N$ is the eigenvector associated with the zero eigenvalue of $\mathcal{L}$, one obtains [9]

$$-e_r^\top (\mathcal{L} \otimes I_n) e_r \leq -\lambda_2(\mathcal{L}) e_r^\top e_r. \tag{5.25}$$

According to (5.23), (5.24), and (5.25), (5.16) is satisfied if

$$\dot{V} \leq - \begin{bmatrix} \omega \\ e_r \end{bmatrix}^\top \begin{bmatrix} Q & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\lambda_2(\mathcal{L}) I_{Nn} - \Lambda \end{bmatrix} \begin{bmatrix} \omega \\ e_r \end{bmatrix}, \tag{5.26}$$

where $Q \in \mathbb{R}^{n \times n} \succ 0$ is defined in (5.9). From (5.8), it follows that

$$2\chi\lambda_2(\mathcal{L}) I_{Nn} - \Lambda - \Lambda_P^\top Q^{-1} \Lambda_P \succ 0,$$

and thus, by using the Schur Complement, we get

$$\begin{bmatrix} Q & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\lambda_2(\mathcal{L}) I_{Nn} - \Lambda \end{bmatrix} \succ 0. \tag{5.27}$$

According to (5.26) and (5.27), there exists a $k \in \mathbb{R}_{>0}$ such that

$$\dot{V} \leq -k \left\| \begin{bmatrix} \omega \\ e_r \end{bmatrix} \right\|^2 \tag{5.28}$$

where

$$k = \lambda_{\min}\left(\begin{bmatrix} Q & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\lambda_2(\mathcal{L})I_{Nn} - \Lambda \end{bmatrix}\right). \tag{5.29}$$

Since $e = e_r + e_c$ and $e_c = \mathbf{1}_N \otimes \omega$, one gets

$$\|e_i\| \leq \left\| \begin{bmatrix} \omega \\ e_r \end{bmatrix} \right\|,$$

and hence

$$\|e\| \leq \sqrt{N} \left\| \begin{bmatrix} \omega \\ e_r \end{bmatrix} \right\|. \tag{5.30}$$

Now, (5.28) and (5.30) yield

$$\dot{V} \leq -\frac{k}{N}\|e\|^2. \tag{5.31}$$

Moreover, from (5.11), we have

$$V \leq \max_{i\in\mathbf{N}}(\lambda_{\max}(P_i))\|e\|^2. \tag{5.32}$$

According to (5.31) and (5.32), one finally obtains

$$\dot{V} \leq -\frac{k}{N\max_{i\in\mathbf{N}}(\lambda_{\max}(P_i))}V.$$

Thus, V asymptotically converges to zero implying that the estimation errors $e_i, \forall i \in \mathbf{N}$, converge to zero. □

Corollary 5.1. *If system* (5.1) *is jointly detectable, then the LMI* (5.6) *is always feasible for some H_i and P_{id}.*

Proof. Since (C_{id}, A_{id}) is detectable for all $i \in \mathbf{N}$, for any symmetric positive definite matrix K_{id}, there exists an H_i such that the Riccati-like equation

$$K_{id} = -(A_{id}^\top P_{id} + P_{id}A_{id} + C_{id}^\top(H_i + H_i^\top)C_{id})$$

has a symmetric positive definite solution for P_{id}. In this condition, (5.6) is solvable if

$$\lambda_{\min}(K) = \min_{i \in \mathbf{N}} \lambda_{\min}(K_{id}) > \|M\| . \tag{5.33}$$

Thus, by designing K based on (5.33), the distributed estimation problem is always feasible. □

It is important to observe that the basis of Theorem 5.1 is the availability of precise measurements at all nodes. In the next section, the proposed strategy will be generalised to the more realistic case of distributed state estimation in the presence of measurement noise and sensor faults.

5.3 Sensor Fault-Tolerant Distributed State Estimation Scheme

In this section, we present a distributed sensor fault-tolerant state observer for jointly observable nonlinear systems. To tolerate the effect of sensor faults in the state estimation, we use redundant sensors and resilient selection of the measurements. Indeed, the main idea of using sensor redundancy for state estimation has been the detection of the faulty sensors and the reconfiguration of the observer after the occurrence of the faults (for instance, see [49, 50, 51, 52, 53, 54]). There exists an inevitable transient time for fault detection and observer reconfiguration in sequence, which may not be efficient in some practical applications, especially for fast varying state systems. However, by considering all the sensors' measurements, we show that it is possible to design an observer that can reject the effect of a set of sensors' faults in state estimation.

According to Definition 5.1, an output measurement $y_{i,k,q}(t)$ $i \in \mathbf{N}, k \in \mathbf{r}_i, q \in \mathbf{p}_i$ is healthy if the norm of the corresponding measurement error is smaller than $\beta_{i,q}$. Now, if we decompose $C_i x(t)$ as

$$C_i x(t) = \begin{bmatrix} \zeta_{i,1}(t) & \zeta_{i,2}(t) & \dots & \zeta_{i,p_i}(t) \end{bmatrix}^\top ,$$

according to Definition 5.1, an output measurement $y_{i,k,q}(t)$ is healthy if

$$|y_{i,k,q}(t) - \zeta_{i,q}(t)| \leq \beta_{i,q}, \ \forall t \geq 0. \tag{5.34}$$

Based on Assumption 5.3, for each $q \in \mathbf{p}_i$, thanks to Algorithm 1, it is possible for Node i to ignore some output measurements and just employ a healthy output measurement for which (5.34) is satisfied. In practice, the computations of the algorithm cause a delay which is accounted for by assuming that the observers use the information of the output measurement and the control input with an unknown constant time delay $\tau \in \mathbb{R}_{>0}$. It is worth mentioning that since increasing r_i may lead to more computation, increasing r_i may lead to increment in such delays.

Algorithm 1 Selection of measurements

1: For each $q \in \mathbf{p}_i$ and Node i, obtain and hold $u(t)$ and $y_{i,k,q}(t), k \in \mathbf{r}_i$.
2: Sort $y_{i,k,q}(t), k \in \mathbf{r}_i$, from the largest to the smallest one, and select $y_{i,m,q}(t)$ where $m = \text{ceil}\left(\frac{r_i}{2}\right)$.
3: Return $y_{i,m,q}(t), q \in \mathbf{p}_i$, and $u(t)$.

To explain the algorithm with an example, consider a case where $r_i = 5$. Based on Assumption 5.3, at each time instant, at most two output measurements can be outside the bound described in (5.34). Hence, in terms of the values of output measurements of unhealthy sensors, at each time instant, four cases need to be considered as shown in Fig. 5.2. In the first case shown in Fig. 5.2-(a), the output measurements of unhealthy sensors are in the domain described by the bound in (5.34). In the second case shown in Fig. 5.2-(b), one of them is outside the domain. In the third one shown in Fig. 5.2-(c), two of them are outside the domain with different directions from the actual output. Finally, in the fourth one shown in Fig. 5.2-(d), two of them are outside the domain with the same direction from the actual output. In all the cases, according to Assumption 5.3 and based on the algorithm, only one output measurement in the domain (5.34) will be selected. According to the first two cases, in some time instants, the chosen output measurement can be the output measurement of an unhealthy sensor whose measurement is in the domain (5.34). Note that according to the above-mentioned issues, Assumption 5.3 is needed to guarantee that Algorithm 1 can always choose the measurements

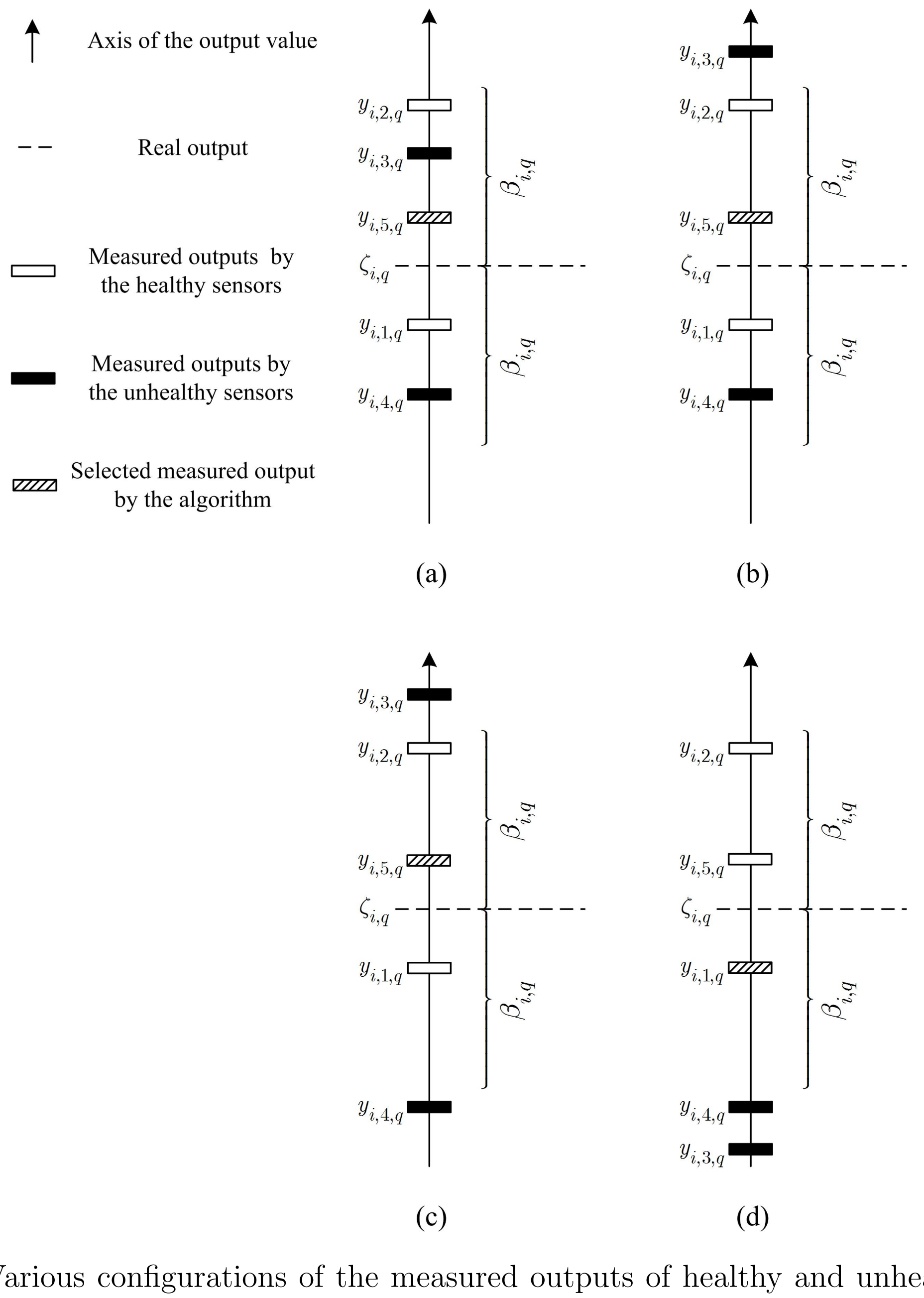


Figure 5.2: Various configurations of the measured outputs of healthy and unhealthy sensors when $r_i = 5$.

satisfying (5.34). Indeed, Assumption 5.3 is a sufficient condition under which distributed estimation in the presence of sensor faults will be possible. Nevertheless, this assumption is reasonable and practical in real-world applications, when fusion of redundant sensors is needed (for instance, see [34, 33]).

Now, by employing the output measurement chosen by Algorithm 1 and by considering the delay due to the computation of the algorithm, we formulate distributed observers such that

$$\lim_{t\to\infty}\sup \sum_{i\in\mathbf{N}} \|x(t-\tau)-\hat{x}_i(t)\| \leq \rho\left(\sum_{i\in\mathbf{N},\, q\in\mathbf{p}_i} \beta_{i,q}^2\right),$$

where $\rho(\cdot)\in[0,\infty)$ is a class-$\mathcal{K}$ function of the bounds $\beta_{i,q}\in\mathbb{R}$, $i\in\mathbf{N}, q\in\mathbf{p}_i$. To employ the output measurement chosen, we consider a nonlinear robustifying function depicted in Fig. 5.3 which is denoted by msgn$(\cdot)$. This function has a dead zone, which by tuning the dead zone domain, it is possible to omit the sensitivity of the function to a range of output measurement errors if an output measurement error lies inside the dead zone. Moreover, tuning the function's gain can provide robustness against a range of measurement noise and faults if an output measurement error lies outside the dead zone. We will study these features in more details later.

Remark 5.1. According to Assumption 5.3, for the measurement set $\{y_{i,1,q}, y_{i,2,q}, \dots, y_{i,r_i,q}\}$, at least ceil $\left(\frac{r_i+1}{2}\right)$ output measurements are expected to be healthy, that is the number of healthy output measurements should be more than the number of unhealthy ones. Therefore, in the presence of faults, r_i should be at least 3 such that Assumption 5.3 can be satisfied. In this condition, any increasing in r_i, according to Algorithm 1, increases the chance of choosing healthy output measurements which leads to more sensor fault tolerance in state estimation. However, increasing r_i needs more sensors and leads to more cost in the implementation.

For the system described in (5.1) and (5.2), we propose a fault-tolerant distributed observer as

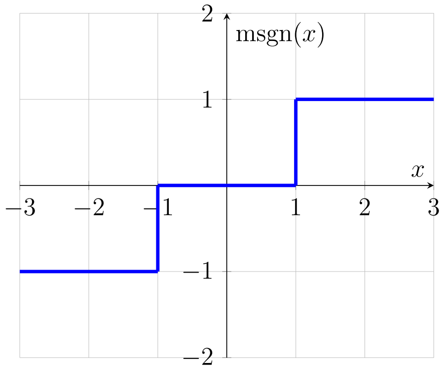


Figure 5.3: msgn(·) function.

follows:

$$
\begin{aligned}
\dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + L_i\left(C_i\hat{x}_i(t) - y_{i,m}(t-\tau) - \nu_i(t)\right) \\
&\quad + \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(\hat{x}_j(t) - \hat{x}_i(t)) + Bu(t-\tau), \quad i \in \mathbf{N}_i
\end{aligned}
\tag{5.35}
$$

where by considering $\tilde{P}$, K, M, Λ, $\bar{P}$, Q, and Λ_P the same as (5.7) and (5.9) and L_i and P_i the same as (5.5), χ is the same as (5.8), in which $P_{io} \succ 0$ and H_i is a diagonal matrix obtained from the solution of the LMI (5.6) (note that in Theorem 5.1, H_i is not needed to be diagonal). Moreover, $\nu_i(t)$ is a robustifying term designed as follows:

$$
\nu_i(t) = \boldsymbol{\eta_i}\mathrm{msgn}\left(\boldsymbol{\beta_i}^{-1}\mathrm{sgn}(H_i)(C_i\hat{x}_i(t) - y_{i,m}(t-\tau))\right) \tag{5.36}
$$

where $\boldsymbol{\beta_i} \in \mathbb{R}^{p_i \times p_i}$ and $\boldsymbol{\eta_i} \in \mathbb{R}^{p_i \times p_i}$ are diagonal matrices defined as

$$
\begin{aligned}
\boldsymbol{\beta_i} &= \mathrm{diag}(\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,p_i}) \\
\boldsymbol{\eta_i} &= \mathrm{diag}(\eta_{i,1}, \eta_{i,2}, \ldots, \eta_{i,p_i}).
\end{aligned}
$$

The entries of $\boldsymbol{\eta_i}$ are determined later in the analysis. Note that since Algorithm 1 is used for obtaining $y_{i,m}$ and it is with computational delay, in (5.35) and (5.36), the time delay τ is considered in $y_{i,m}$. Hence instead of pursuing the estimates of the real-time system state vector $x(t)$, we intend to estimate the system state vector with that time delay, i.e., $x(t-\tau)$, and thus in the observer design, the input u in (5.35) is delayed with time τ via the algorithm.

Remark 5.2. It should be noted that (5.35) has a discontinuous right-hand side due to the msgn(·) function and Algorithm 1. However, since msgn(·) is locally bounded and measurable, a solution in the sense of Fillipov exists for (5.35) [55, 56, 57].

Theorem 5.2. *Consider the dynamical system described in* (5.1a) *and* (5.2) *under the distributed observer given in* (5.35) *and* (5.36)*, when Assumptions 5.1, 5.2, and 5.3 are satisfied, K, M, $\mathit{\Lambda}$, Q, and $\mathit{\Lambda}_P$ are the same as* (5.7) *and* (5.9)*, and L_i and P_i are the same as* (5.5)*. Moreover, let the symmetric positive definite matrices P_{io} and the diagonal matrices $H_i, i \in \mathbf{N}$, be obtained from the solution of the LMI* (5.6)*, χ is chosen as* (5.8)*, and $\eta_{i,q} \geq \beta_{i,q}$, $i \in \mathbf{N}, q \in \mathbf{p}_i$. Under these conditions, the estimation errors $e_i(t) = x(t-\tau) - \hat{x}_i(t), i \in \mathbf{N}, t \geq \tau$, are uniformly ultimately bounded such that*

$$\lim_{t\to\infty} \sup \|e(t)\| \leq 2\sqrt{\frac{\sum_{i=1}^{N}\sum_{q=1}^{p_i} |H_{i,q}|\beta_{i,q}^2}{\mu \min\limits_{i\in\mathbf{N}}(\lambda_{\min}(P_i))}}, \tag{5.37}$$

where

$$\begin{aligned} e(t) &= \begin{bmatrix} e_1(t)^\top & e_2(t)^\top & \cdots & e_N(t)^\top \end{bmatrix}^\top, \\ \mu &= \frac{k}{N \max\limits_{i\in\mathbf{N}}(\lambda_{\max}(P_i))} \end{aligned}$$

in which k is defined in (5.29).

Proof. From (5.1), it follows that

$$\dot{x}(t-\tau) = Ax(t-\tau) + Bu(t-\tau). \tag{5.38}$$

Since

$$-(\hat{x}_j(t) - \hat{x}_i(t)) = e_j(t) - e_i(t),$$

from (5.2), (5.35), (5.36), and (5.38), and the basis of Algorithm 1, the differential equation describing $e_i(t) = x(t-\tau) - \hat{x}_i(t), i \in \mathbf{N}$, is given by

$$\begin{aligned}\dot{e}_i(t) = (A + L_iC_i)e_i(t) + \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(e_j(t) - e_i(t)) - L_i\left(\varpi_{i,m}(t-\tau) + \delta_{i,m}(t-\tau)\right) \\ - L_i\boldsymbol{\eta_i}\mathrm{msgn}\Big(\boldsymbol{\beta_i}^{-1}\mathrm{sgn}(H_i)\big(C_ie_i(t) - \varpi_{i,m}(t-\tau) - \delta_{i,m}(t-\tau)\big)\Big).\end{aligned} \tag{5.39}$$

To analyze the evolution of $e_i(t)$ along (5.39), we consider the same Lyapunov candidate as (5.11). Thus, following a procedure similar to that in the proof of Theorem 5.1, the time derivation of V along (5.39) yields

$$\begin{aligned}\dot{V}(t) \leq e^\top(t)\Lambda e(t) - 2\chi e^\top(t)(\mathcal{L}\otimes I_n)e(t) - \sum_{i=1}^{N} 2e_i^\top P_iL_i\Big(\varpi_{i,m}(t-\tau) + \delta_{i,m}(t-\tau) \\ + \boldsymbol{\eta_i}\mathrm{msgn}\big(\boldsymbol{\beta_i}^{-1}\mathrm{sgn}(H_i)(C_ie_i(t) - \varpi_{i,m}(t-\tau) - \delta_{i,m}(t-\tau))\big)\Big).\end{aligned} \tag{5.40}$$

Due to the structure of P_i and L_i given in (5.5) and since $T_i^\top T_i = I_n$, one obtains

$$P_iL_i = T_i\begin{bmatrix} P_{id} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & I_{v_i}\end{bmatrix}\begin{bmatrix} P_{id}^{-1}C_{id}^\top H_i \\ \mathbf{0}_{v_i\times p_i}\end{bmatrix}$$

which can be simplified as follows:

$$P_iL_i = T_i\begin{bmatrix} C_{id}^\top \\ \mathbf{0}_{v_i\times p_i}\end{bmatrix}H_i. \tag{5.41}$$

Since $T_i\begin{bmatrix} C_{id} & \mathbf{0}_{p_i\times v_i}\end{bmatrix}^\top = C_i^\top$ and H_i is diagonal, from (5.41), one obtains

$$P_iL_i = C_i^\top H_i^\top. \tag{5.42}$$

From (5.40) and (5.42), and following a procedure similar to that in the proof of Theorem 5.1

for (5.14), one gets

$$\begin{aligned}\dot{V}(t) \leq -\mu V(t) - 2\sum_{i=1}^{N} (H_i C_i e_i(t))^\top \Big(\varpi_{i,m}(t-\tau) + \delta_{i,m}(t-\tau) \\ + \boldsymbol{\eta_i}\text{msgn}\big(\boldsymbol{\beta_i}^{-1}\text{sgn}(H_i)(C_i e_i(t) - \varpi_{i,m}(t-\tau) - \delta_{i,m}(t-\tau))\big)\Big).\end{aligned} \tag{5.43}$$

We analyze the other term in the right-hand side of (5.43). To analyze

$$\text{msgn}\big(\boldsymbol{\beta_i}^{-1}\text{sgn}(H_i)(C_i e_i(t) - \varpi_{i,m}(t-\tau) - \delta_{i,m}(t-\tau))\big),$$

let us decompose H_i as follows:

$$H_i = \text{diag}(H_{i,1}, H_{i,2}, \cdots, H_{i,p_i}).$$

According to Assumption 5.3 and Algorithm 1, for each $i \in \mathbf{N}$ and $q \in \mathbf{p}_i$, (5.3) is satisfied for the selected output measurements. Now, by decomposing $C_i e_i(t)$ as

$$C_i e_i(t) = \begin{bmatrix} \xi_{i,1}(t) & \xi_{i,2}(t) & \ldots & \xi_{i,p_i}(t) \end{bmatrix}^\top,$$

for each $i \in \mathbf{N}$ and $q \in \mathbf{p}_i$, two cases arise. One case is when $|\xi_{i,q}(t)| > 2\beta_{i,q}$ and the other one is when $|\xi_{i,q}(t)| \leq 2\beta_{i,q}$. If $|\xi_{i,q}(t)| > 2\beta_{i,q}$, according to the definition of the msgn$(\cdot)$ function, it can be concluded that

$$\text{msgn}\big(\beta_{i,q}^{-1}\text{sgn}(H_{i,q})(\xi_{i,q}(t) - \varpi_{i,m,q}(t-\tau) - \delta_{i,m,q}(t-\tau))\big) = \text{sgn}(H_{i,q}\xi_{i,q}(t)).$$

In this case, according to (5.3) and since $\eta_{i,q} \geq \beta_{i,q}, i \in \mathbf{N}, q \in \mathbf{p}_i$, there is a positive scalar $\alpha_{i,q}$ such that

$$\begin{aligned}&- H_{i,q}\xi_{i,q}(t)\Big(\varpi_{i,m,q}(t-\tau) + \delta_{i,m,q}(t-\tau) + \eta_{i,q}(t)\text{msgn}\big(\beta_{i,q}^{-1}\text{sgn}(H_{i,q})(\xi_{i,q}(t) \\ &- \varpi_{i,m,q}(t-\tau) - \delta_{i,m,q}(t-\tau))\big)\Big) = -H_{i,q}\xi_{i,q}(t)\alpha_{i,q}\text{sgn}(H_{i,q}\xi_{i,q}(t)).\end{aligned} \tag{5.44}$$

Conversely, if $|\xi_{i,q}(t)| \leq 2\beta_{i,q}$, according to the definition of the msgn$(\cdot)$ function, the measurement errors satisfying (5.3) cannot invert the sign of

$$\mathrm{msgn}\big(\beta_{i,q}^{-1}\mathrm{sgn}(H_{i,q})(\xi_{i,q}(t)-\varpi_{i,m}(t-\tau)-\delta_{i,m}(t-\tau))\big).$$

Hence, (5.44) still is satisfied or

$$\mathrm{msgn}\Big(\beta_{i,q}^{-1}\mathrm{sgn}(H_{i,q})(\xi_{i,q}(t)-\varpi_{i,m,q}(t-\tau)-\delta_{i,m,q}(t-\tau))\Big)=0$$

which happens when

$$|\xi_{i,q}(t)-\varpi_{i,m,q}(t-\tau)-\delta_{i,m,q}(t-\tau)|<\beta_{i,q}.$$

If this is the case, according to (5.3) and since $|\xi_{i,q}(t)|\leq 2\beta_{i,q}$, one obtains, in place of (5.44),

$$\begin{aligned}
&-H_{i,q}\xi_{i,q}(t)\Big(\varpi_{i,m,q}(t-\tau)+\delta_{i,m,q}(t-\tau)+\eta_{i,q}\mathrm{msgn}\big(\beta_{i,q}^{-1}\mathrm{sgn}(H_{i,q})(\xi_{i,q}(t)\\
&-\varpi_{i,m,q}(t-\tau)-\delta_{i,m,q}(t-\tau))\big)\Big)\\
\leq\ &-H_{i,q}\xi_{i,q}(t)(\varpi_{i,m,q}(t-\tau)+\delta_{i,m,q}(t-\tau))\leq 2|H_{i,q}|\beta_{i,q}^2.
\end{aligned}\tag{5.45}$$

Now, from (5.43), (5.44), and (5.45), it follows that

$$\dot{V}(t)\leq-\mu V(t)+4\sum_{i=1}^{N}\sum_{q=1}^{p_i}|H_{i,q}|\beta_{i,q}^2\tag{5.46}$$

and since

$$V(t)\geq\min_{i\in\mathbf{N}}(\lambda_{\min}(P_i))\|e(t)\|^2,$$

one obtains

$$\dot{V}(t)\leq-\mu\min_{i\in\mathbf{N}}(\lambda_{\min}(P_i))\|e(t)\|^2+4\sum_{i=1}^{N}\sum_{q=1}^{p_i}|H_{i,q}|\beta_{i,q}^2.$$

Therefore, $e(t)$ is uniformly ultimately bounded such that

$$\limsup_{t\to\infty}\|e(t)\|\leq 2\sqrt{\frac{\sum_{i=1}^{N}\sum_{q=1}^{p_i}|H_{i,q}|\beta_{i,q}^2}{\mu\min\limits_{i\in\mathbf{N}}(\lambda_{\min}(P_i))}}.$$

Therefore, the proof is completed. □

Remark 5.3. It should be noted that according to (5.35) and (5.36), the time delay is not in the loop of the observer, because the delay just affects the output measurement available for the observer. In this condition, as the delay is fixed, the delay value does not have any effect on the convergence of the observer. However, since the delay leads to error in output measurement, it can lead to the convergence of the estimated state vector to an erroneous state vector. This implies that larger τ may lead to increments in such errors.

Remark 5.4. The ultimate bound of the estimation errors depends on the accuracy of the available sensors. In this study, in (5.3) we have considered the bounds $\beta_{i,q}, q \in \mathbf{p}_i, i \in \mathbf{N}$, for faults and noise which are acceptable for practical healthy sensors. Indeed, we consider a sensor to be unhealthy if its output measurement does not satisfy (5.3). In this condition, if Assumption 5.3 is satisfied, the ultimate bound of the estimation errors is (5.37) which is a function of $\beta_{i,q}, q \in \mathbf{p}_i, i \in \mathbf{N}$. This directly implies that if healthy measurements have no noises and faults, the estimation error vector e asymptotically converges to zero (in the presence of any fault in possible unhealthy sensors under Assumption 5.3).

Remark 5.5. Without using the robustifying term, the boundedness of the second term in the right-hand side of (5.43) should be guaranteed by the term $-\mu V(t)$, and hence in such case, the ultimate bound (5.37) may not be obtained.

Now, suppose we do not consider ν_i for the observer design in (5.35). Then, $\dot{V}$ yields

$$\dot{V}(t) \leq e^\top(t)\Lambda e(t) - 2\chi e^\top(t)(\mathcal{L}\otimes I_n)e(t) - \sum_{i=1}^{N} 2e_i^\top P_i L_i \left(\varpi_{i,m}(t-\tau) + \delta_{i,m}(t-\tau)\right).$$

By defining $\mathcal{P} := \operatorname{diag}(P_1, P_2, \cdots, P_N)$ and $\mathcal{L} := \operatorname{diag}(L_1, L_2, \cdots, L_N)$, we derive

$$\dot{V}(t) \leq e^\top(t)\Lambda e(t) - 2\chi e^\top(t)(\mathcal{L}\otimes I_n)e(t) + \frac{1}{\epsilon}e^\top\mathcal{P}e + \epsilon\bar{\beta}^\top\mathcal{L}\bar{\beta},$$

where $\bar{\beta} = \left[\beta_{1,1}, \cdots, \beta_{1,p_1}, \beta_{2,1}, \cdots, \beta_{2,p_2}, \cdots, \beta_{N,1}, \cdots, \beta_{N,q_N}\right]^\top$, and ϵ is a scalar by our choice. We can see that a small ϵ may no longer guarantee that

$$\Lambda - 2\chi(\mathcal{L}\otimes I_n) + \frac{\mathcal{P}}{\epsilon} \prec 0,$$

while a large ϵ may lead to a very large ultimate bound of $\limsup_{t\to+\infty}\|e(t)\|$.

Remark 5.6. From (5.46), we have

$$\dot{V}(t) \leq -\mu V(t) + d$$

where

$$d(t) = 4\sum_{i=1}^{N}\sum_{q=1}^{p_i}|H_{i,q}|\beta_{i,q}^2.$$

From the comparison theorem for scalar ordinary differential equations [58], one obtains $V(t) \leq v(t)$ for all $t \geq 0$, where $v(t)$ is given by

$$v(t) = e^{-\mu t}V(0) + \frac{d}{\mu}\left(1 - e^{-\mu t}\right)$$

implying the convergence of $v(t)$ to d/μ with time constant $1/\mu$.

Remark 5.7. It should be noted that the results presented in Theorem 5.1 and Theorem 5.2 can be extended to the scenario with strongly connected (directed) communication network by using Lemma 4.4, and with the same procedure as in Section 4.3.

5.4 Extension to Lipschitz Nonlinear Systems

In this section, we extend our main results to a class of nonlinear system with a global Lipschitz condition. Consider a class of nonlinear system described as

$$\dot{x} = Ax + f(x) + Bu \tag{5.47}$$

where $f(x) \in \mathbb{R}^n$ is a nonlinear vector field. We consider $f(x)$ to satisfy the following global Lipschitz condition.

Assumption 5.4. *The nonlinear vector field $f(x)$ is assumed to be globally Lipschitz, that is, there exists $\gamma \in \mathbb{R}_{>0}$ such that*

$$\|f(x_1) - f(x_2)\| \leq \gamma \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n.$$

Similar to Section 5.2, we firstly proposed the distributed estimation scheme for the system assuming that there are no noise or faults, i.e., $\varpi_{i,k}(t) = \delta_{i,k}(t) = \mathbf{0}_{p_i \times 1}, \forall i \in \mathbf{N}, \forall k \in \mathbf{r}_i, \forall t \geq 0$. Hence in this case, the output measurement of the system is modeled as in (5.1b). Based on (5.35), the local nonlinear observer at Node i is proposed as

$$\dot{\hat{x}}_i = A\hat{x}_i + L_i(C_i\hat{x}_i - y_i) + f(\hat{x}_i) + Bu + \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i), \quad i \in \mathbf{N} \tag{5.48}$$

where L_i and P_i are the same as in (5.5). The analysis of the proposed distributed nonlinear observer (5.48) is given in the next theorem.

Theorem 5.3. *Consider the noise-free and fault-free dynamical system described in* (5.47) *and* (5.1b) *with the distributed nonlinear observer given in* (5.48) *and* (5.5)*, under Assumptions 5.1 and 5.4. The estimation errors $e_i = x - \hat{x}_i, \forall i \in \mathbf{N}$, converge to zero if the matrices P_{io} and $H_i, i \in \mathbf{N}$, can be obtained from the solution of the following LMI:*

$$\begin{bmatrix} I_{Nn} & \sqrt{\gamma}\tilde{P}^\top \\ \sqrt{\gamma}\tilde{P} & T_o\left(K - M\right)T_o^\top - \gamma N I_n \end{bmatrix} \succ 0 \tag{5.49}$$

$$P_{io} \succ 0$$

where K, K_{id} and M are identical to (5.7)*, and $\tilde{P}$ is given by*

$$\tilde{P} = \begin{bmatrix} P_1 & P_2 & \dots & P_N \end{bmatrix}. \tag{5.50}$$

Moreover, the gain χ *must be chosen such that*

$$\chi > \frac{\left\|\varLambda + \gamma \bar{P} + \varLambda_P^\top Q^{-1} \varLambda_P\right\|}{2\lambda_2(\mathcal{L})}, \tag{5.51}$$

where $\varLambda$ *and* $\varLambda_i$ *are the same as in* (5.9)*. Moreover,* Q *and* $\varLambda_P$ *are modified (or newly defined) as*

$$\begin{aligned}
\bar{P} &= \operatorname{diag}\left(P_1^2 + I_n, P_2^2 + I_n, \dots, P_N^2 + I_n\right) \\
Q &= -\sum_{i=1}^{N} \Big(\varLambda_i + \gamma(P_i^2 + I_n)\Big) \\
\varLambda_P &= \Big[\varLambda_1 + \gamma(P_1^2 + I_n) \quad \varLambda_2 + \gamma(P_2^2 + I_n) \quad \dots \quad \varLambda_N + \gamma(P_N^2 + I_n)\Big]
\end{aligned} \tag{5.52}$$

The proof of Theorem 5.3 is presented in Appendix B.

Remark 5.8. It should be noted that the solvability of the LMI (5.49) is a sufficient condition on Theorem 5.3 to guarantee state estimation in the network of observers. From Corollary 5.1, we show that the distributed estimation problem of Theorem 5.3 is always feasible when $\gamma = 0$. Hence, there exists a set of γ such that the LMI (5.49) has solutions for P_{id} and $H_i, i \in \mathbf{N}$. Indeed, since (C_{id}, A_{id}) is detectable for all $i \in \mathbf{N}$, there exist P_{id} and $H_i, i \in \mathbf{N}$, such that the following matrix has positive eigenvalues:

$$M_1 = \begin{bmatrix} I_{Nn} & 0 \\ 0 & T_d\left(K - M\right) T_d^\top \end{bmatrix}.$$

In this condition, if γ is sufficiently small compared with the parameters of M_1, there exists a set of γ such that the eigenvalues of the following matrix remain positive:

$$M_2 = \begin{bmatrix} I_{Nn} & \sqrt{\gamma}\tilde{P}^\top \\ \sqrt{\gamma}\tilde{P} & T_d\left(K - M\right) T_d^\top - \gamma N I_n \end{bmatrix},$$

implying that the LMI (5.49) is satisfied.

For the system described in (5.47) and (5.2), the proposed nonlinear fault-tolerant distributed observer in (5.48) is modified as

$$\begin{aligned}\dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + L_i\left(C_i\hat{x}_i(t) - y_{i,m}(t-\tau) - \nu_i(t)\right) + f(\hat{x}_i(t)) \\ &\quad + \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(\hat{x}_j(t) - \hat{x}_i(t)) + Bu(t-\tau),\ i \in \mathbf{N}_i,\end{aligned} \tag{5.53}$$

where K, M and Λ are the same as in (5.7) and (5.9). $\tilde{P}$, $\bar{P}$, Q and Λ_P are the same as (5.50) and (5.52), L_i and P_i are the same as (5.5), χ is the same as (5.51), and $P_{id} \succ 0$. Moreover, the matrix H_i is a diagonal matrix obtained from the solution of the LMI (5.49), and $\nu_i(t)$ is the same as (5.36).

Theorem 5.4. *Consider the dynamical system described in* (5.47) *and* (5.2) *under the distributed observer given in* (5.53)*, when Assumptions 5.1, 5.2, 5.3 and 5.4 are satisfied, Moreover, let the symmetric positive definite matrices P_{io} and the diagonal matrices $H_i, i \in \mathbf{N}$, be obtained from the solution of the LMI* (5.49)*, and χ be chosen as* (5.51)*, and $\eta_{i,q} \geq \beta_{i,q}, i \in \mathbf{N}, q \in \mathbf{p}_i$. Under these conditions, the estimation errors $e_i(t) = x(t-\tau) - \hat{x}_i(t), i \in \mathbf{N}, t \geq \tau$, are uniformly ultimately bounded and satisfy* (5.37)*.*

The proof of Theorem 5.4 is provided in Appendix C.

5.5 Summary

In this chapter, we have considered a case where the distributed sensors of an LSS are faulty and noisy. We assume the sensor redundancy exists at each node locally where the number of the healthy sensors dominates the number of the unhealthy ones. Firstly, we have provided a distributed estimation strategy assuming that there is no noise or fault with guaranteed observer design. Then, based on this design, we have proposed a sensor fault-tolerant distributed observer design exploiting the local sensor redundancy stated previously. Finally, we have shown that the proposed distributed estimation schemes can be extended to mildly nonlinear system plants satisfying globally Lipschitz condition.

Chapter 6

Simulation Validation: Heat Exchange

In the previous chapters, we have presented several distributed state estimation strategies when certain types of faults or disturbances occur at the system plants. In this chapter we provide some simulation results demonstrating the effectiveness of our design, and we also compare these results with the performance from some conventional designs to address the advantages of the proposed schemes.

The simulation of the proposed design is carried out on an LTI system based on the heat exchange model in multi-zone buildings [59]. For the sake of simplicity, we restrict our formulation to the single floor scenario, which can be extended to multiple floors by adding the heat exchanges dynamics between top and bottom dividers. In this model, an area in a single floor is divided into 9 rooms by walls with different heat exchange rates as shown in Fig. 6.1. We assume that the temperature changing rate in Room k is proportional to the temperature difference between Room k and its adjacent rooms, and the coefficient depends on the corresponding heat exchange rates. Moreover, we only consider the heat exchange inside these nine rooms without concerning the effects of ambient temperature.

The initial temperatures of these nine rooms are different, but they are supposed to be regulated to some desired value. Therefore, heating, ventilation, and air conditioning (HVAC) equipments are available in Room 2, 6 and 9 to adjust the temperatures of all the rooms, which are regarded as the inputs of the system plant. Moreover, four sets of sensors are distributed over this area,

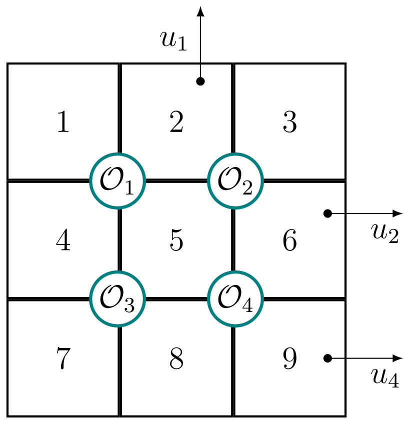


Figure 6.1: Schematic representation of the simulated system. Each square represents a different building zone, and the adjacent rooms exchange heats with different rates. The arrows indicate which rooms are equipped with HVAC.

whose locations are indicated by $\mathcal{O}_i$ in Fig. 6.1. Note that the available measurements at Node i are assumed to be different in each scenario, which will be explicitly stated later. Our objective is to build the distributed observers co-located with these measurements (nodes) such that the convergent state estimates are achieved at each node. Note that the initial conditions of all the observers are set as zero vectors for all simulation runs.

The system model is in the form of

$$\dot{x} = Ax + Bu$$

where A and B are given in Appendix D.1. The lumped inputs u is given by $u = \begin{bmatrix} u_1 & u_2 & u_4 \end{bmatrix}$. The measurements at each node are the temperature of different rooms, which varies in different scenarios, and will be indicated later.

Without loss of generality, the control input is selected as $u = -F(x - x_d)$, where $F \in \mathbb{R}^{3\times 9}$ is provided in Appendix D.1. x_d is the desired state that indicates the desired temperature in each room, which is 18 °C throughout this chapter. The initial temperature of each room is chosen arbitrarily between 7 °C and 32 °C, which is given by

$$x_0 = \begin{bmatrix} 15.8 & 27.5 & 7.4 & 8.1 & 11.2 & 23.2 & 25.3 & 23.2 & 18.3 \end{bmatrix}^\top .$$

Under those conditions and settings, the real state vectors of the system are shown in Fig. 6.2. We observe that without any input disturbances, the temperature of all rooms converges to the

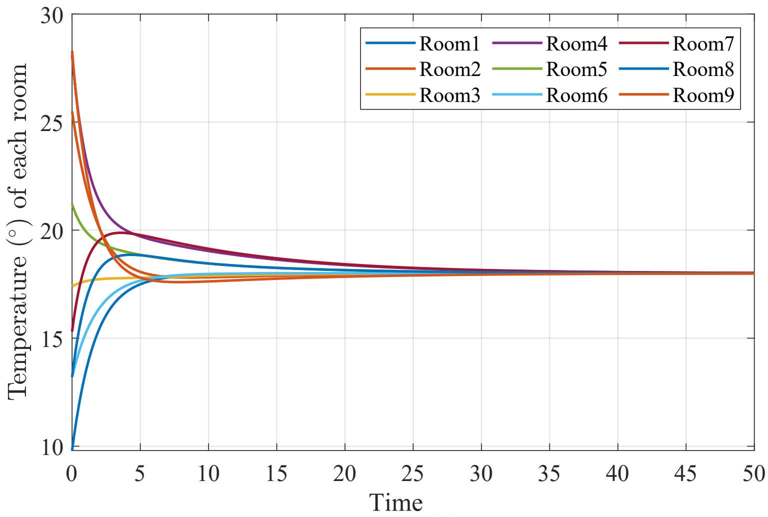


Figure 6.2: True temperature of each room with no input disturbances.

desired value asymptotically as we expected.

In the next few sections, the distributed state estimation strategies proposed in the technical part (Chapter 3-5) are evaluated based on this heat exchange model in sequence. Although the model is simplified by ignoring the less significant factors, it is a good example of all linear LSSs (or the ones which is applicable to be linearised) with distributed inputs and measurements. Certainly, in most of the real applications, the monitored systems are supposed to have significantly larger number of states and nodes. These features, however, highlight the advantages of our distributed based designs, which will be discussed in detail when we compare the proposed designs with centralised based design in Scenario 6.6.

6.1 Simulation Results of Distributed Observers Under Switching Topology

In this section, we illustrate the effectiveness of the proposed distributed estimation strategies in Chapter 3. The measurements at each node are given by

$$y_1 = \begin{bmatrix} y_{r1} \\ y_{r5} \end{bmatrix} \qquad y_2 = \begin{bmatrix} y_{r3} \\ y_{r6} \end{bmatrix} \qquad y_3 = \begin{bmatrix} y_{r4} \\ y_{r7} \end{bmatrix} \qquad y_4 = \begin{bmatrix} y_{r8} \\ y_{r9} \end{bmatrix}$$

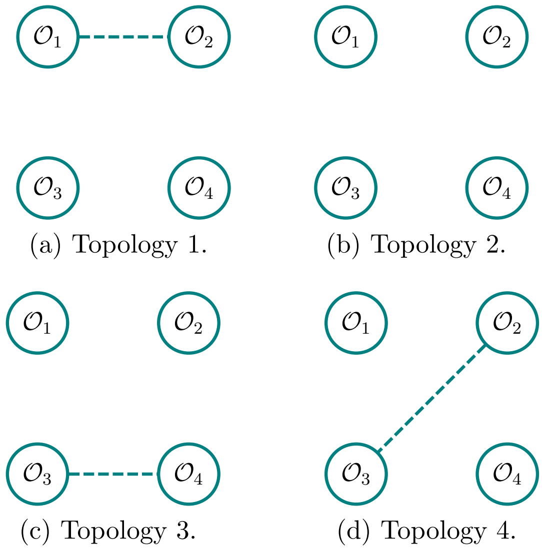


Figure 6.3: Switching set of the network topologies in Scenario 6.1.

where the measurements y_i available to node i are denoted based on the measurements y_{rk} of Room k.

We consider two scenarios in simulation corresponding to both noise-free and noisy settings.

Scenario 6.1 (No noise at input and measurements)**.** In this scenario, we assume that the observers exchange their estimated state vectors over the jointly connected switching communication topologies depicted in Fig. 6.3, such that the information switches to the next topology every 0.5 second, (after Topology 4 the graph switches back to Topology 1). It is worth noticing that the graph in Topology 2 is set to be fully disconnected deliberately.

Distribution estimation scheme is constructed based on the distributed observers given in (3.2) where the parameters L_i and P_i are chosen by solving the LMIs (3.4) as provided in Appendix D.2. Note that the solution of the LMIs is obtained by using the CVX toolbox [60].

Under these conditions, the results of a simulation run are shown in Fig. 6.4. Although the communication topology is not connected at each time instant, we observe that the estimation errors at each node converge to zero, verifying that all local observers provide asymptotic state estimates following the proposed observer design.

Scenario 6.2 ($\mathscr{L}_2$ noise at input and measurements)**.** In this scenario, the target system (heat exchange model) is subject to external disturbances, and the measurements are also

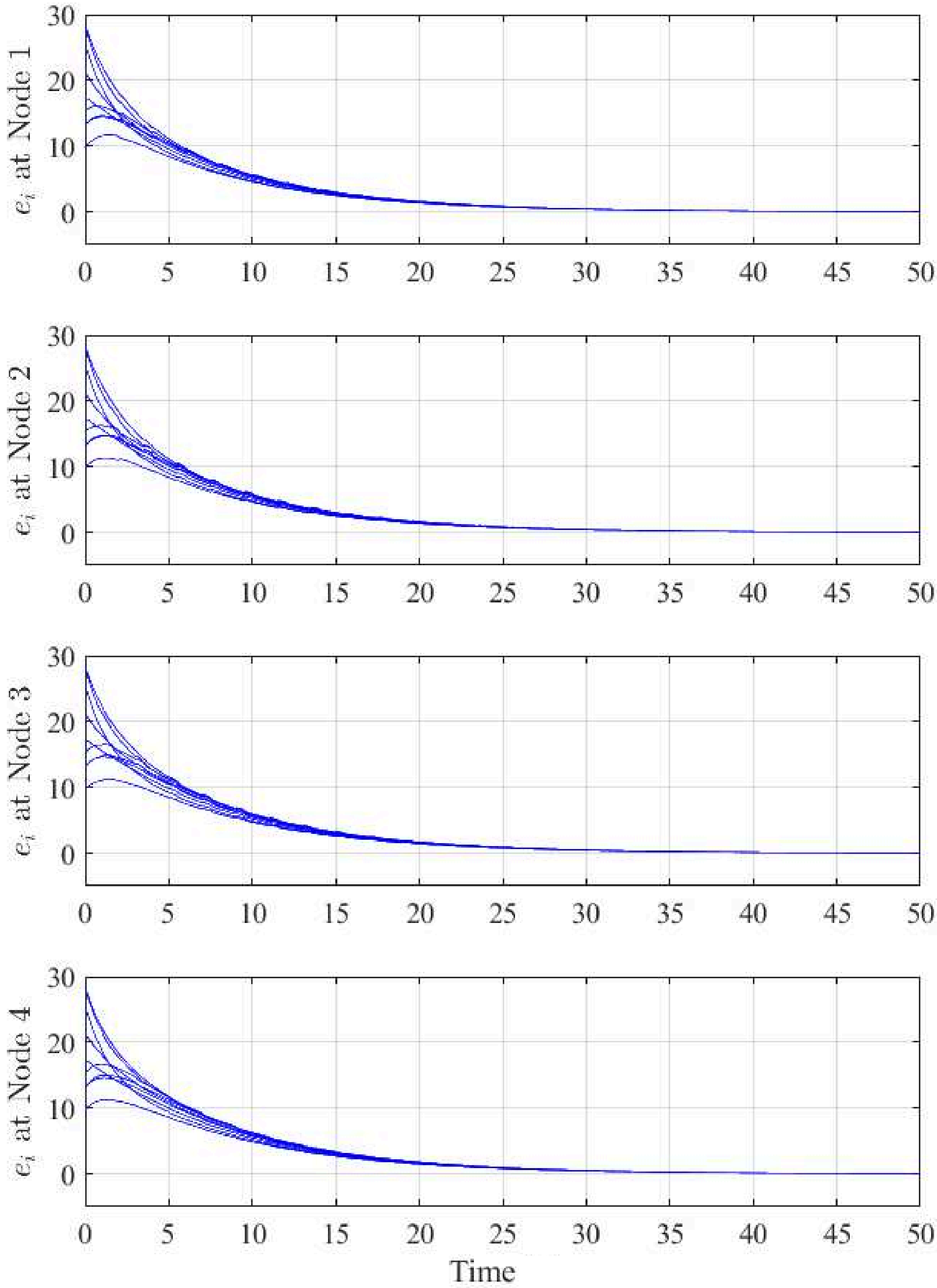
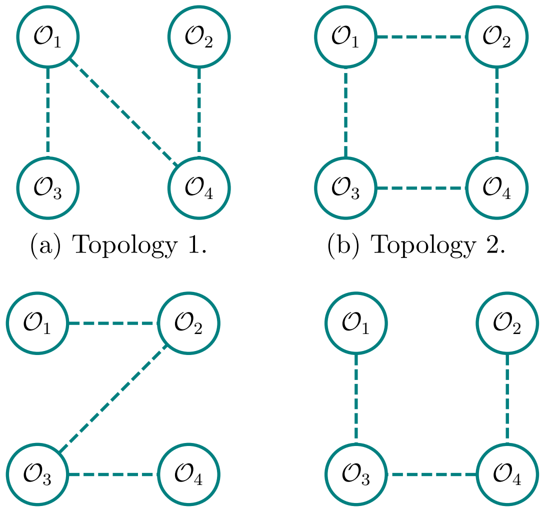


Figure 6.4: Estimation errors generated by the proposed distributed observer with switching communication network in Scenario 6.1



(a) Topology 1. (b) Topology 2.

(c) Topology 3. (d) Topology 4.

Figure 6.5: Switching set of the network topologies in Scenario 6.2.

assumed noisy as described in (3.41) with $D = B$ and $E_i = C_i, \forall i \in \{1, 2, 3, 4\}$. Moreover, to model the external disturbances and measurement noise, the white noise with power of 0.03 is applied. We assume that local observers exchange their estimates over the connected switching communication topologies depicted in Fig. 6.5. The information exchange starts from following Topology 1 and switches to the next topology every 2 seconds (after Topology 4 is Topology 1).

The distributed observer scheme is built following (3.44) where L_i and P_i (given in Appendix D.3) are obtained by solving LMIs (3.45) with $\chi = 58.97$ and $\acute{\mu} = 29.18$ ($\mu = 5.40$). After applying the proposed distributed observers with the aforementioned parameters, the estimation errors at all nodes are shown in Fig. 6.6, which verifies the effectiveness of the design in a noisy setting.

Scenario 6.3 (Comparison with centralised estimation strategy)**.** In this scenario, we intend to employ a centralized estimation scheme such that the state estimation can be performed at each node. We assume that the communication graph of the nodes is jointly connected over a period of time as described in Scenario 6.1. Let us ignore the latency of the information exchange among nodes and pick one of the node as the “central node” (i.e., this node collects all the measurements from other nodes via the communication links and generates the estimated state vector). After adopting the conventional centralised estimation method, the dynamics of

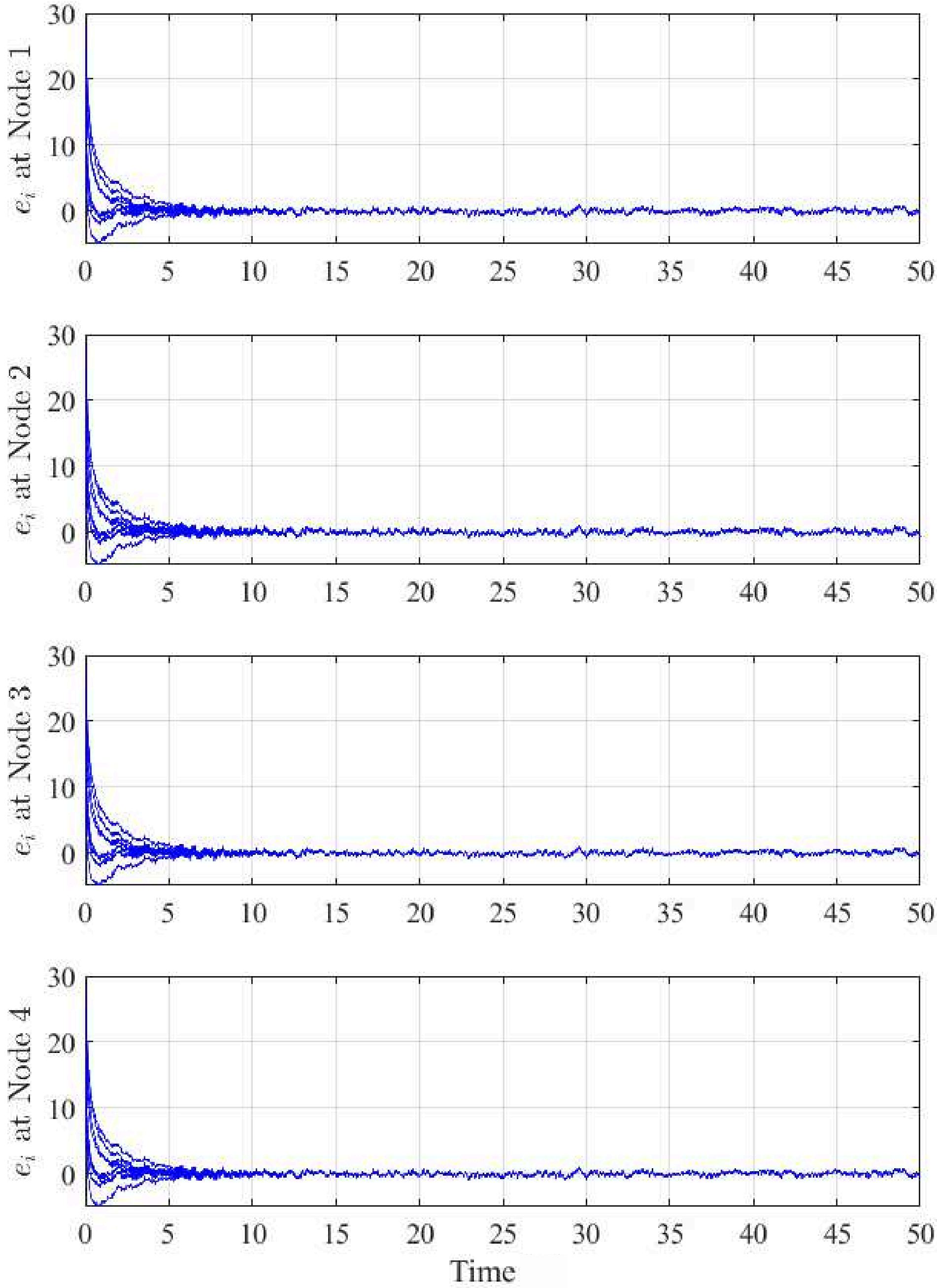


Figure 6.6: Estimation errors generated by the proposed distributed observer in the presence of disturbances and measurement noise in Scenario 6.2

the estimation of the central node is

$$\dot{x} = A\dot{x} + L(t)\left(C(t)\dot{x} - \mathbf{y}(t)\right))$$

where $\mathbf{y}(t)$ is the "lumped" measurements which is available for Node i at time t. Take Topology 3 and Topology 4 shown in Figure 6.3 for example. If we choose Node 2 as the central node, $\mathbf{y}(t)$ is $y_2(t)$ (for Topology 3) and $\begin{bmatrix} y_2^\top(t) & y_3^\top(t) \end{bmatrix}^\top$ (for Topology 4) respectively. Therefore, if the communication links is not steadily connected, the centralised observer gain $L(t)$ is a time varying parameter with changing matrix dimensions. Thus the conventional centralised observer is not applicable for scenarios with switching communication graphs.

6.2 Simulation Results of Distributed Observers With Unknown Inputs

In this section, we evaluate the accuracy of the DUIO proposed in Chapter 4. We assume that Room 5 is suffered from unpredictable influence on temperature, which is unknown to all observers. In this regard, w is set as the band-limited white noise with noise power 5, and D is given by

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$$

Moreover, due to the fault at the communication link, we assume that true input signal of HVAC in each room is only accessible to its corresponding node, and the third node does not have access to all inputs. Hence, we have B_1, B_2 and B_4 as given in Appendix D.4. Note that B_3 does not exist since Node 3 is assumed to have no reliable access to any input signals. Accordingly, we have

$$\begin{aligned} \bar{B}_1 &= \begin{bmatrix} B_2 & B_4 & D \end{bmatrix}, \ \bar{B}_2 = \begin{bmatrix} B_1 & B_4 & D \end{bmatrix}, \\ \bar{B}_3 &= \begin{bmatrix} B_1 & B_2 & B_4 & D \end{bmatrix}, \ \bar{B}_4 = \begin{bmatrix} B_1 & B_2 & D \end{bmatrix}. \end{aligned}$$

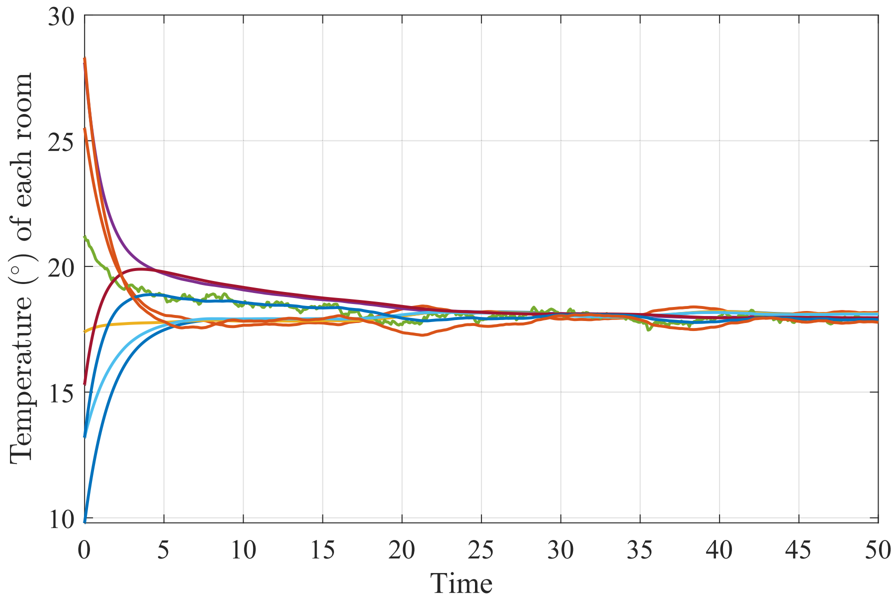
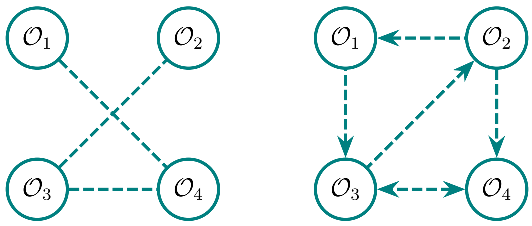


Figure 6.7: True temperature of each room in the presence of unknown disturbance.



(a) Scenario 6.4. (b) Scenario 6.5.

Figure 6.8: Network communication topologies in Scenarios 6.4 and 6.5.

In this section, the measurements y_i at each node are given by

$$y_1 = \begin{bmatrix} y_{r5} \\ y_{r6} \\ y_{r9} \end{bmatrix} \quad y_2 = \begin{bmatrix} y_{r2} \\ y_{r5} \\ y_{r9} \end{bmatrix} \quad y_3 = \begin{bmatrix} y_{r2} \\ y_{r5} \\ y_{r6} \\ y_{r9} \end{bmatrix} \quad y_4 = \begin{bmatrix} y_{r2} \\ y_{r5} \\ y_{r6} \end{bmatrix}.$$

Note that due to the unknown input disturbance w, the states of the system are different from those in Fig. 6.2, which are illustrated in Fig. 6.7.

The proposed distributed estimation schemes are applied in two scenarios, corresponding to the observers being connected (undirected) and strongly connected (directed) as shown in Fig. 6.8 respectively.

Scenario 6.4 (Undirected graph)**.** In the first scenario, the nodes are assumed to be connected via the unweighted undirected communication graph depicted in Fig. 6.8a implying that $\lambda_2(\mathcal{L}) = 0.5858$. Distributed state estimation is based on the distributed observers (4.3) where N_i, M_i, L_i, H_i, and P_i are obtained from the solution of the LMI (4.25) as provided in Appendix D.4. Moreover, following (4.26), χ is set to 654.1. Under these conditions, the estimation errors $e_i, i \in \mathbf{N}$ at all nodes are shown in Fig. 6.9.

According to the figure, the estimation errors of all state vectors at all the nodes converges to zero asymptotically.

Scenario 6.5 (Directed graph)**.** In the second scenario, the nodes are assumed to be connected via the unbalanced directed communication graph depicted in Fig. 6.8b. Distributed state estimation is based on the distributed observers given in (4.53) where N_i, M_i, L_i, H_i, and P_i still are the same as Scenario 6.4. According to Lemma 4.4, $R = \mathrm{diag}(1, 1.5, 1, 0.5)$, and following (4.54), χ is set as 400.8. Under these conditions, the estimation errors e_i at each node is illustrated in Fig. 6.11. According to the figure, the estimation errors of all the observers converge to zero asymptotically.

Scenario 6.6 (Comparison with centralised estimation strategy)**.** In this scenario, we employ a centralised estimation strategy to achieve convergent state estimation where the nodes are connected via a time-invariant connected communication graph (as in Scenario 6.4). Since we ignore the effects of delay due to information exchange among the nodes, we assume the central node (Node 3 in this scenario) is accessible to all the distributed measurements in real-time. Hence, we can adopt the conventional UIO design for the centralised estimation scheme, which is given by:

$$\dot{z} = Nz + Ly$$
$$\hat{x} = z + Hy$$

where y is the lumped measurements composed of all the available measurements. N, L and

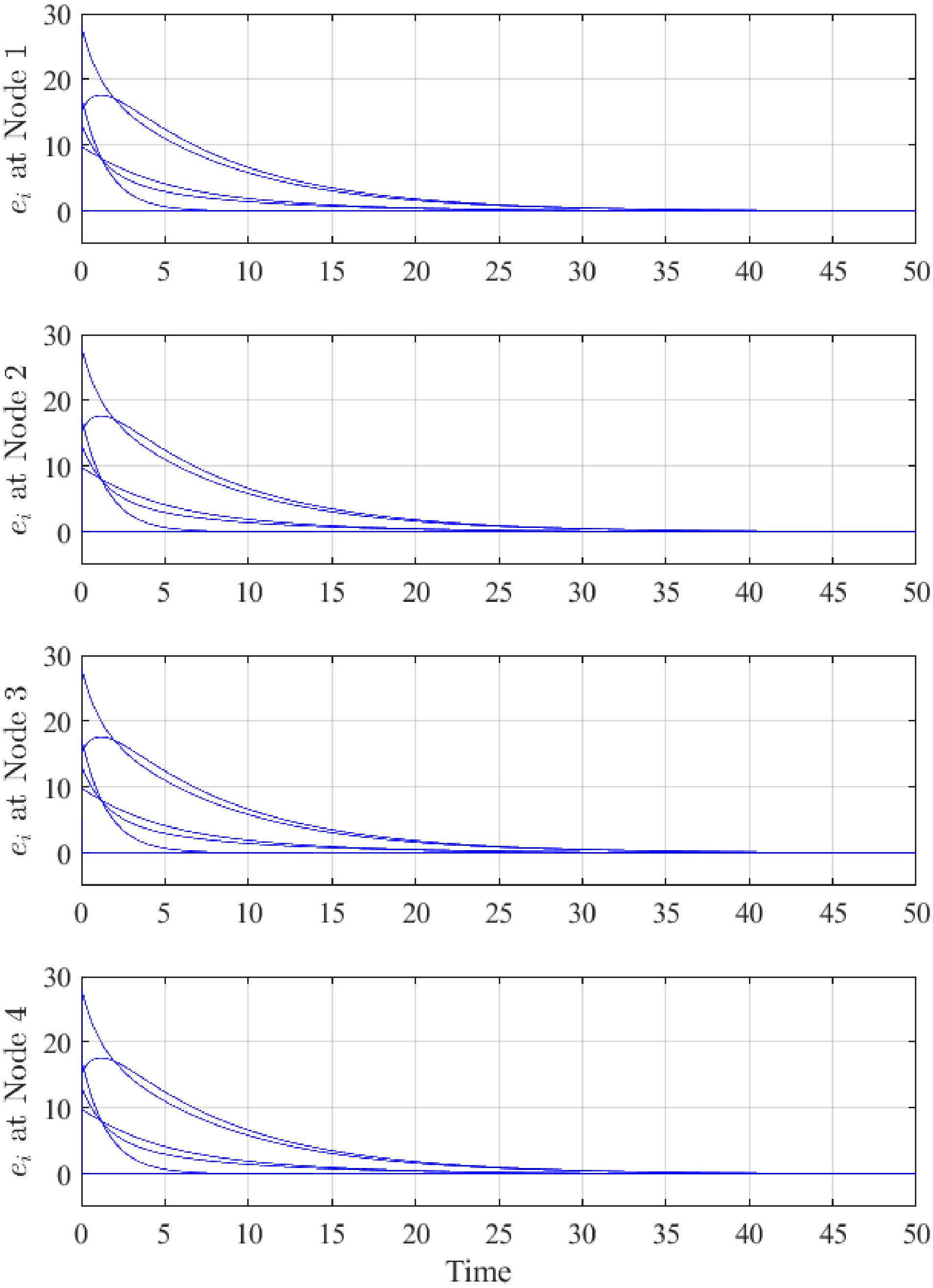


Figure 6.9: Estimation errors generated by the proposed distributed UIO under an undirected communication graph in the presence of unknown inputs for all the nodes in Scenario 6.4

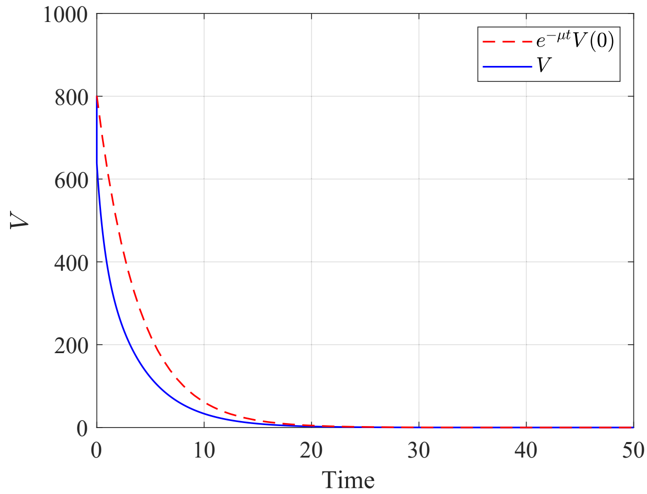


Figure 6.10: Lyapunov function V generated by the proposed distributed UIO in Scenario 6.4

H are chosen so that

$$M = I_n - HC, \quad N = MA - KC, \quad L = K + NH.$$

holds, where the corresponding numerical choices are illustrate in Appendix D.5. Under these conditions, the estimation errors at the central node (Node 3) are shown in Fig. 6.12.

Not surprisingly, from the figure, the estimation errors of all state vectors converge to zero asymptotically. However, in a real-life application scenario, the latency of the information transfer is not negligible and cannot be treated equally from all nodes to the central node. Take the topology depicted in Figure 6.8a for example. Even if we assume the information exchange time via each link is equal, it takes twice of the time to transfer signals (estimated state vectors) from Node 1 to Node 3 than transferring the signals from Node 4 to Node 3. Moreover, if the states of the system and the numbers of the nodes are enormous, the central nodes require very high bandwidth to receive the measurements from all other nodes and high computational resources to perform the calculation. Furthermore, suppose the estimated states are desired at each node locally. In that case, the central node has to send the generated state estimates back to the rest of the nodes (the same path as the central node collects all the measurements but the reverse direction for the undirected graph), which leads to an additional latency problem and non-synchronisation of the estimation at different nodes.

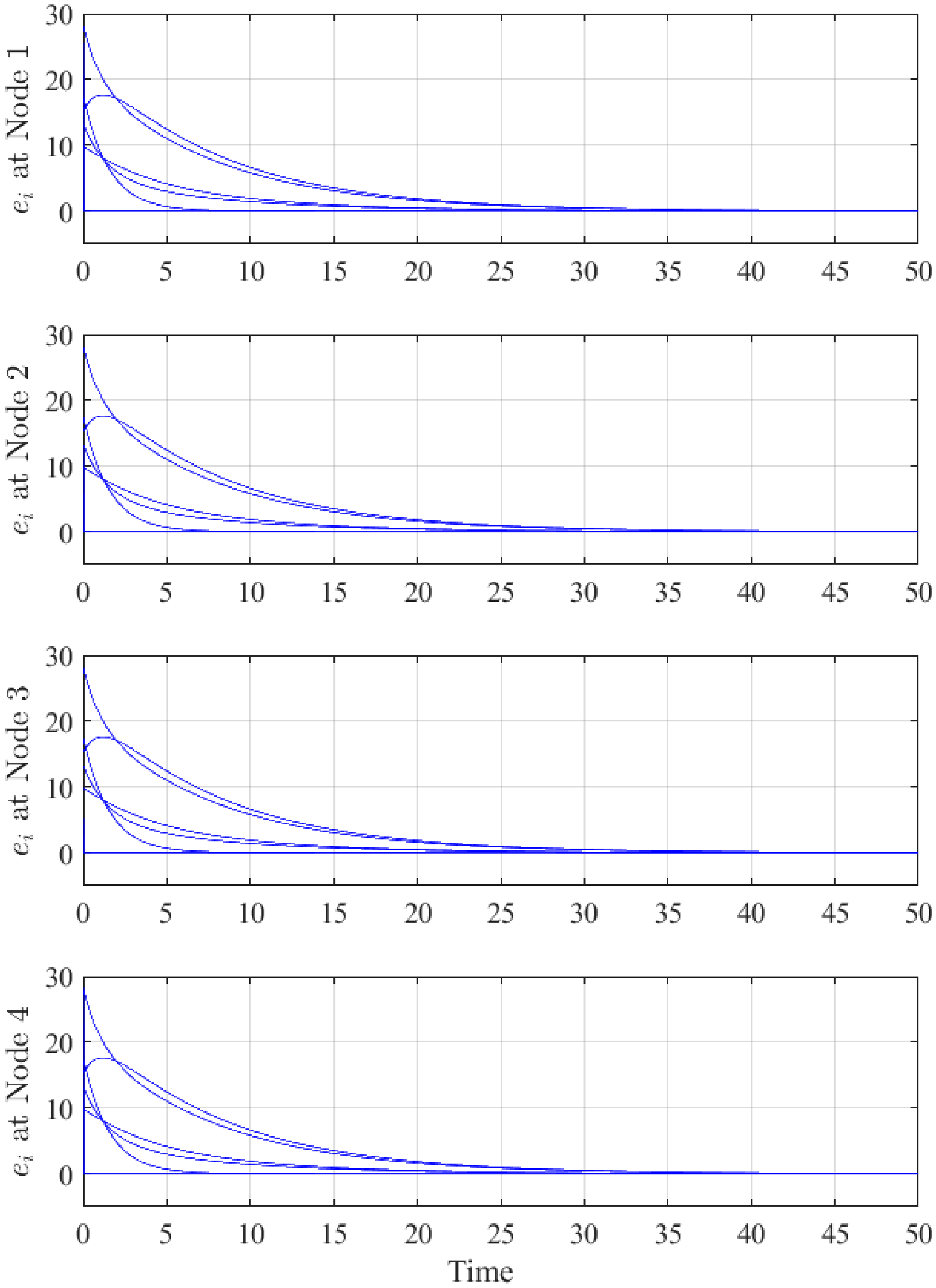


Figure 6.11: Estimation errors generated by the proposed distributed UIO with strongly directed connected communication graph in Scenario 6.5

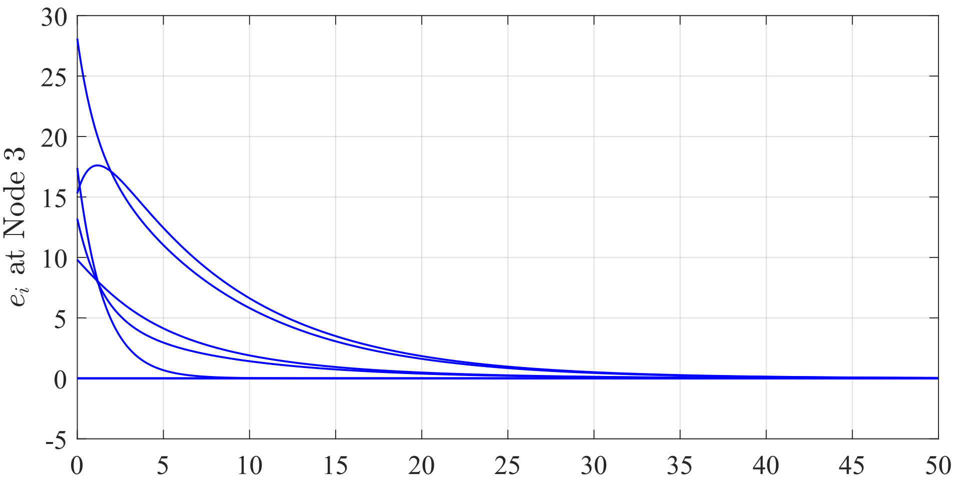


Figure 6.12: Estimation errors of the central node by employing the described centralised UIO in Scenario 6.6

Even though we do not consider all those pitfalls for a centralised estimation strategy, we observe that its state estimation performance (shown in Figure 6.9) is almost the same as the scenario when employing our proposed DUIO scheme (shown in Figure 6.12). In this sense, we reach to the conclusion that we benefit from the distributed estimation structure from the perspective of signal synchronisation and computation efficiency without compromising the estimation performance.

6.3 Simulation Results of Sensor Fault-Tolerant Distributed Observers

In this section, we apply the robust distributed state estimation strategies presented in Chapter 5. Let measurement at each node as

$$y_1 = y_{r1}, \quad y_2 = 0, \quad y_3 = 0, \quad y_4 = y_{r8}.$$

Moreover, five repeated measurements at each node are considered, i.e., $r_i = 5, \forall i \in \{1, 2, \cdots, 6\}$. The nodes are assumed to be connected by the unweighted undirected communication graph depicted in Fig. 6.13.

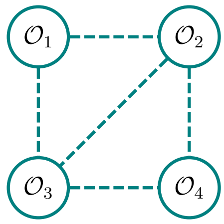


Figure 6.13: Network topology for Scenarios 6.7, 6.8 and 6.9

By considering Assumption 5.3 we assume that three sensors are considered healthy and all the diagonal elements of $\boldsymbol{\beta}_i, i \in \{1, 2, \cdots, 6\}$, are selected as 0.5. Other sensors are assumed to be suffered from fault with magnitude over 1000. To model the noises and faults, various sinusoidal waves, square waves, and uniformly random signals are applied.

Based on the system matrix A and the measurement at each node, the similarity transformation matrix T_i are given as in Appendix D.6. It is worth mentioning that there are no T_{id} for Node 2 and Node 3 since they are completely undetectable.

Under those settings, we consider three scenarios for simulation, which correspond to no fault or noise case, exploitation of sensor redundancy case and no sensor redundancy case. The simulation details are presented as follows.

Scenario 6.7 (No sensor fault)**.** In this first scenario, we show that if all the sensors are healthy, it is possible to estimate the state vector of the system without sensor redundancy. We consider the estimation strategy given in (5.4) when just one of the measurements is chosen among the redundant measurements set $\{y_{i,k}\}, k \in \{1, 2, \cdots, 5\}$. By considering the corresponding parameters as in Appendix D.6, the simulated estimation errors is demonstrated in Fig. 6.14. We observe that it is possible to estimate the state vector of the system without sensor redundancy when all the sensors are healthy.

Scenario 6.8 (With sensor redundancy)**.** In this scenario, we evaluate the performance of estimation strategy in Theorem 5.2 to tolerate the effects of sensor faults by exploiting sensor redundancy. In simulation, we set the fixed-time delay for Algorithm 1 as $\tau = 2 \times 10^{-4}$ s. According to the simulated estimation errors in Fig. 6.15, all the local observers are capable of estimating the state vector of the entire system. Moreover, from the simulation results we observe that the influence of the sever faults with large magnitude are rejected without any

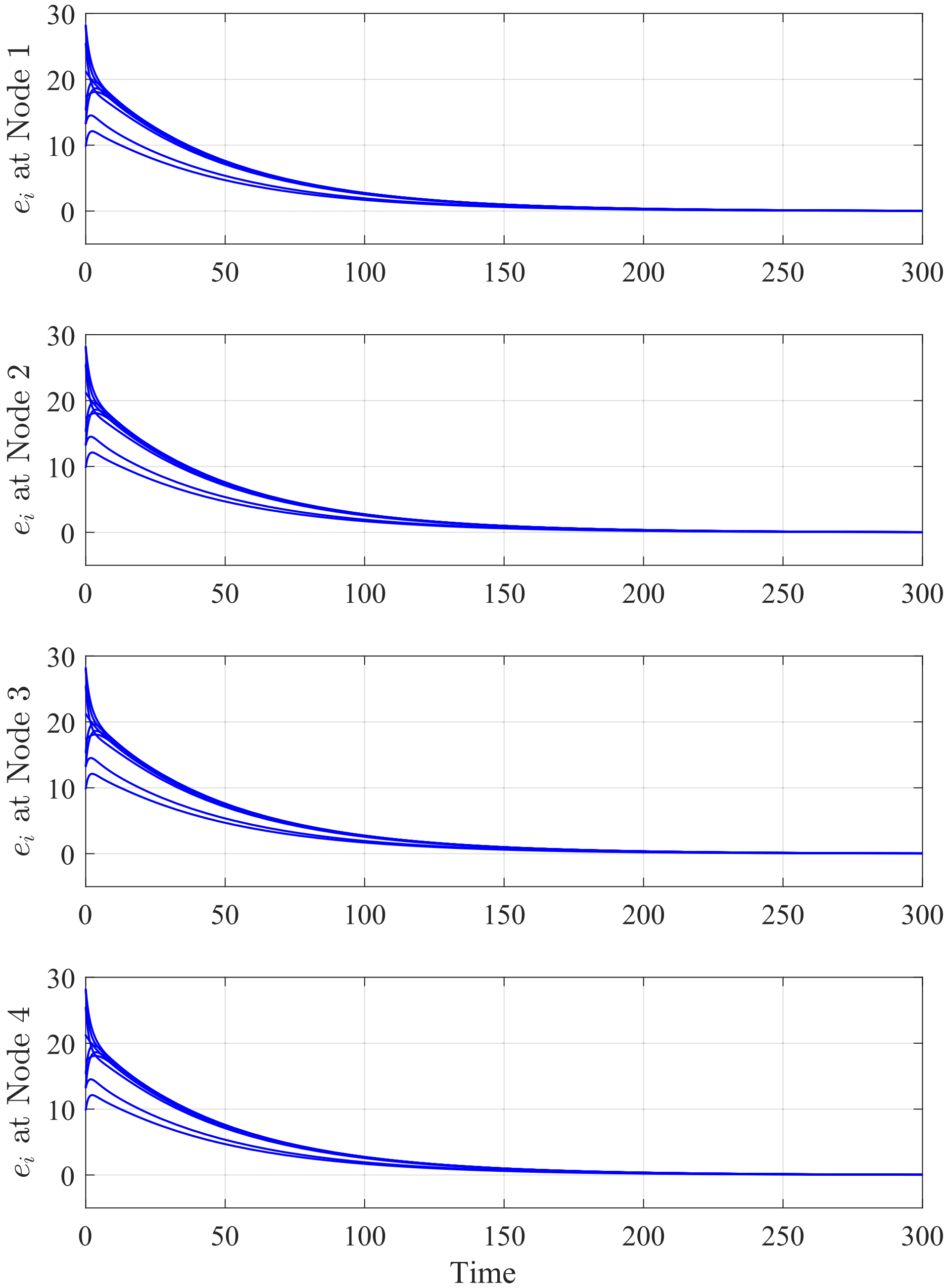


Figure 6.14: Estimation errors generated by the proposed distributed observer with no measurement disturbance or noise in Scenario 6.7

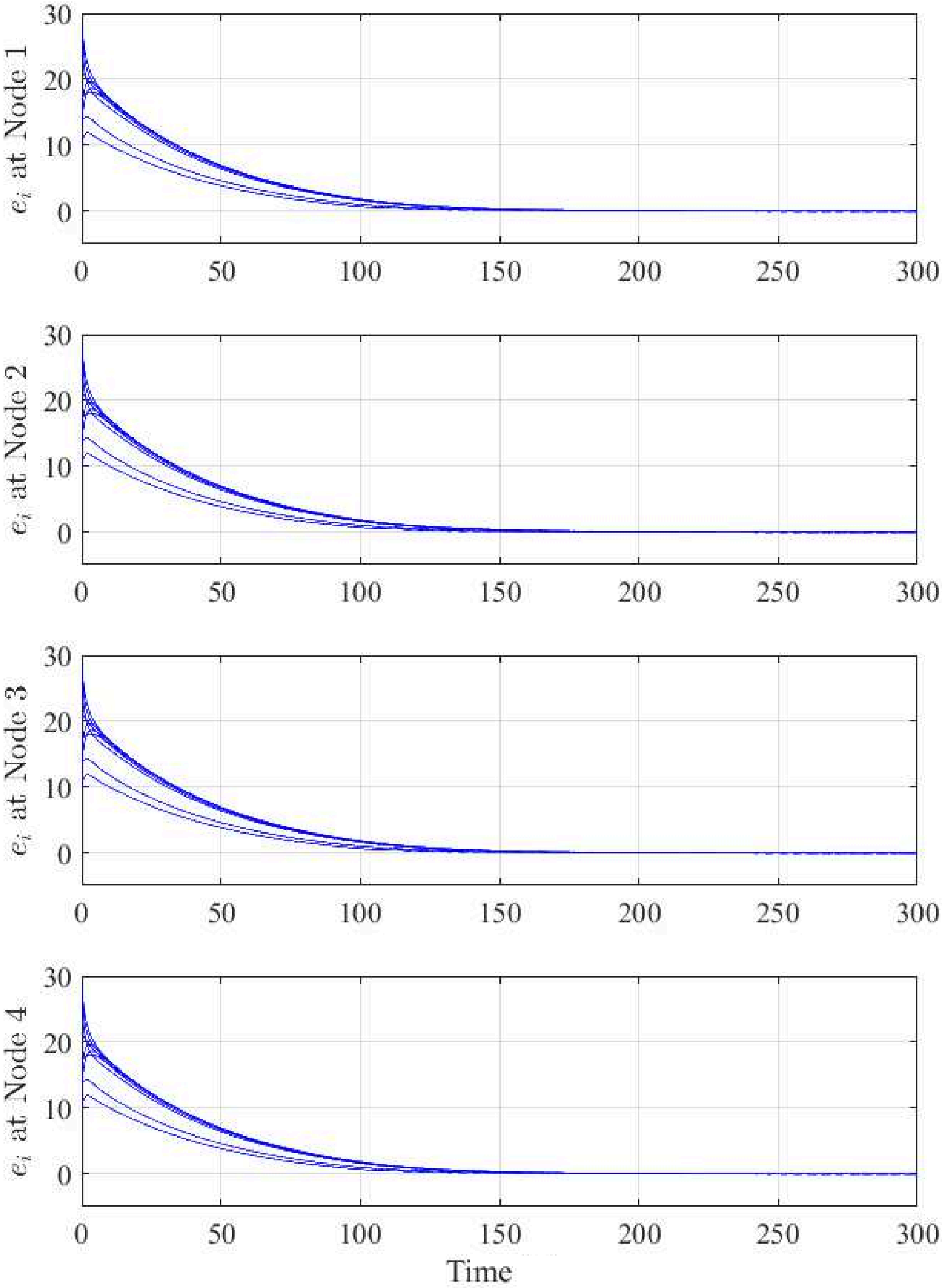


Figure 6.15: Estimation errors generated by the proposed distributed observer exploiting sensor redundancy in the presence of measurement noise and faults in Scenario 6.8

transient abnormal behaviour.

Scenario 6.9 (Without sensor redundancy)**.** In this scenario, we deliberately do not consider sensor redundancy when applying distributed estimation scheme, which may cause the state estimation being sensitive to sensor faults. To illustrate this issue, we repeat the first scenario when just the unhealthy sensors are employed for the distributed observers. The estimation errors are demonstrated in Fig. 6.16. The simulation results indicate that without redundancy in the measurements, the performance of the distributed observers in state estimation is significantly deteriorated.

6.4 Summary

This chapter evaluated the proposed distributed estimation schemes based on a simplified heat exchange model. We verified the effectiveness of all the presented designs by achieving desired state estimates (either asymptotic or with bounded errors) at all nodes. Moreover, we compared our distributed designs with the conventional centralised estimation strategies and concluded that the state estimation benefits from distributed design concerning wider application scenarios, information synchronisation and computation efficiency. Furthermore, for the scenario that faults and noises may occur in measurements, we compared the performance of the proposed scheme using sensor redundancy with the results generated by a conventional design, which addresses the merits of our design.

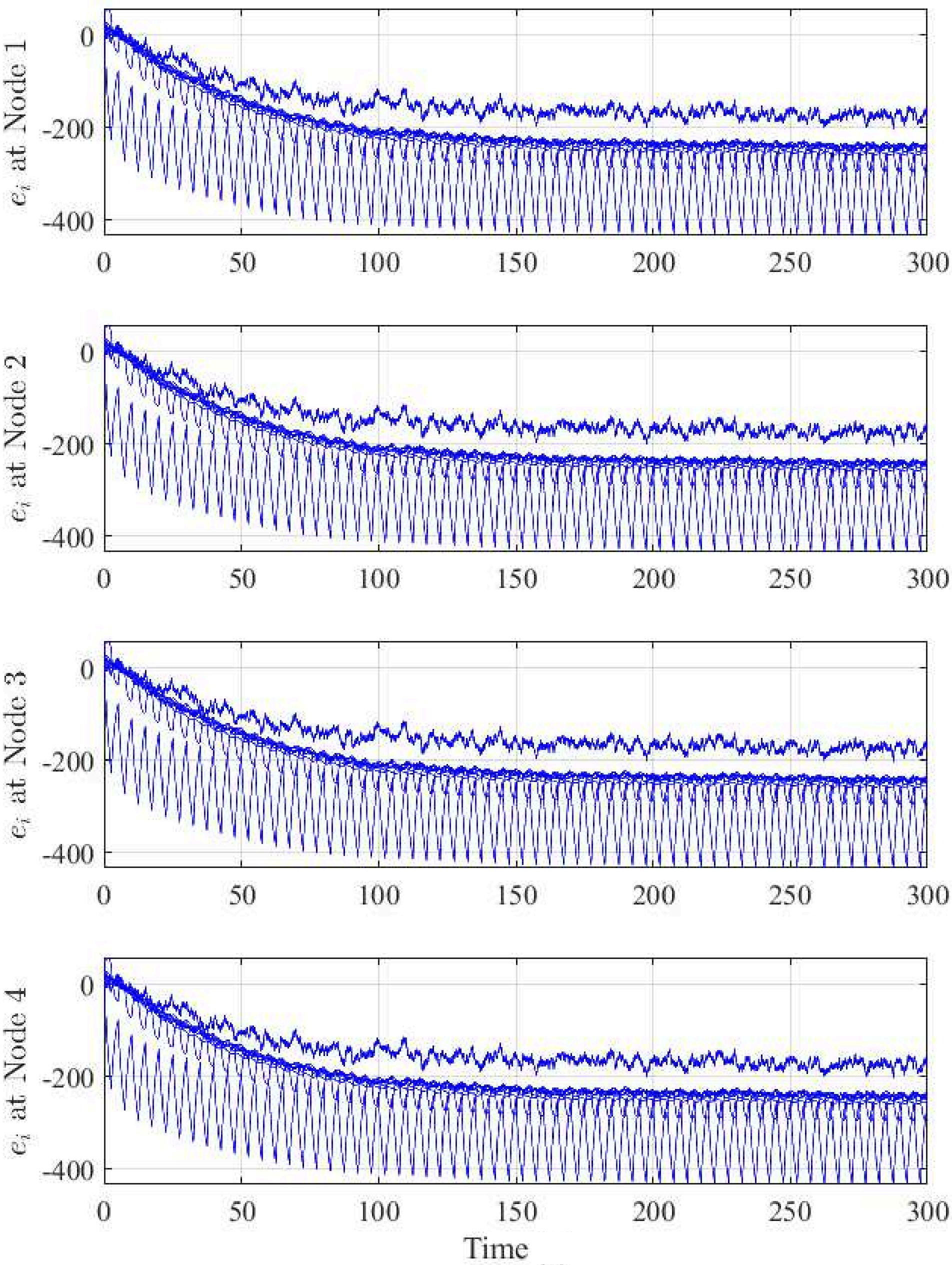


Figure 6.16: Estimation errors generated by the proposed distributed observer without sensor redundancy in the presence of measurement noise and faults in Scenario 6.9

Chapter 7

Conclusion

The main contributions of this thesis are concluded in this chapter. After that, we present some directions for future research.

7.1 Summary of Thesis Achievements

Throughout this thesis, we focused our effort on *distributed state estimation problem.* Motivated by ensuring proper operation of the LSSs, we aimed to estimate the precise state vector of such systems. Indeed, the sensors of an LSS are often distributed over the entire system plant, and the local measurement at each node may not guarantee the detectability of the whole system. Therefore, we assumed a communication network exchanging state estimates among nodes, letting the estimation strategy operating in a distributed way such that the entire system state vector can be estimated at each node.

Moreover, we considered the fact that LSSs are susceptible to faults and disturbances. In this work, we considered three different aspects of faults meanwhile the asymptotic state estimate was desired to obtain, which shaped the main achievements of this work.

First we addressed the problem of distributed state estimation in the presence of communication links failure. We formulated this problem in Chapter 3, and proposed a systematical method to design a network of distributed observers where the communication network is not steadily connected. We showed that the communication among nodes is not required to be continuous to guarantee the performance of the state estimation under the proposed strategy, and distributed state estimation can be performed if the joint of communication links over finite time intervals makes the network communication topology connected. We also illustrated that the proposed distributed observer design has a guaranteed solution under the condition that the ensemble of all measurements ensures the detectability of the entire systems. Moreover, the results were extended to another scenario when the LTI system is subject to $\mathscr{L}_2$ external disturbance and measurement noises. Accordingly, observer gains were chosen such that a desired $\mathscr{H}_\infty$ performance to attenuate the effect of input and output disturbances (noise) is guaranteed. In this condition, we proved that our distributed state estimation scheme is guaranteed for all LTI systems in the presence of connected switching topologies.

The other considered problem was distributed estimation in the presence of unknown inputs. We proposed a DUIO design where each observer just had access to a part of the system inputs available locally, whereas the other inputs remained unknown, and accordingly some rank conditions were assumed to be satisfied to construct such DUIO. The feasible solution of an LMI provides sufficient choices of observer parameters that guarantee convergence of the estimated states under some joint detectability conditions. We also discussed the necessary and sufficient existence conditions, and illustrated the necessity of our conditions under the framework of our design. Moreover, we showed that the proposed design can be extended to a broader scenario that the communication network is strongly directed connected.

Finally we considered a distributed state estimation problem when the measurements of the LSS are contaminated, i.e., there are some unknown and unbounded additive noise and faults at some sensor measurements. To deal with this problem, we resorted to the sensor redundancy at each node, and we exploited this redundancy to achieve state estimates with desired performance. Based on the availability of the redundant sensors at each node, we proposed a robust estimation strategy such that under certain conditions, the complete rejection of a range of additive faults in measurements without detecting the unhealthy ones has been shown to be possible. We also proved that the solutions of the LMIs to generate the proposed observer design were guaranteed for LTI systems. Finally, we illustrated that it was possible to extend the main results to nonlinear systems with globally Lipschitz property.

After the main results, we constructed a simplified heat exchange model, and applied the proposed distributed estimation strategies to this model. The results verified that in each simulation scenario, the desired performance was obtained by deploying the corresponding distributed estimation strategies.

7.2 Future Work

One of the limitations of the proposed estimation strategies is that they are mostly developed based on linear systems. Indeed, some LSSs has nonlinear dynamic model such as water systems [61], and they are also crucial hence desired to be monitored. Although we extend the results of sensor fault-tolerant observer design to a nonlinear scenario in Chapter 5, however the estimated system is assumed to be globally Lipschitz, which narrows the application scenarios. Therefore, developing distributed estimation strategies for nonlinear systems is an important future work that worth focusing on.

Since the distributed estimation scheme requires the information exchange among observers, communication graph among nodes is a vital aspect to pay attention to. In this work, we mainly focus on the undirected connected graph though we have extended the DUIO design to the directed strongly connected case. The next step can be extending our design to more practical topology cases. For example, state estimation with directed but switching graph is a challenging problem to investigate. Moreover, throughout this work, we utilise the property that the Laplacian of a connected graph has positive eigenvalues with a simple zero eigenvalue. Therefore, it is worth exploiting other properties of the communication graph to reduce the observer gain or to obtain the convergence rate of the estimation error with an arbitrary but pre-assigned value.

We deal with the distributed state estimation problem assuming that there are three specific types of faults at the monitored system. However, there are other categories of faults or disturbances that are worth considering. For example, the system may suffer from a model disturbance with $A + \varDelta A$, which would be an interesting direction. Furthermore, fault or attack can occur on the observer side as well. For example, when some of the nodes are under attack and send malicious information (such as wrong state estimates) to their neighbours, the distributed observers should be re-designed to obtain acceptable state estimates.

Finally, the estimation strategies we propose in this work aim to monitor the operation status of LSSs. Hence, it is meaningful to utilise the estimated state to obtain further objectives. One of the common targets is to detect, isolate and accommodate the faults or attacks to the system. Another direction that worth taking is to generate the control signal using the estimated state for the system plant.

Acronyms

DUIO Distributed Unknown Input Observer.

LMI Linear Matrix Inequality.

LSS Large-Scale System.

LTI Linear Time-Invariant.

UIO Unknown Input Observer.

Bibliography

[1] J. Lunze, *Feedback control of large-scale systems.* Prentice Hall New York, 1992.

[2] D. D. Siljak, *Decentralized control of complex systems.* Courier Corporation, 2011.

[3] C. Lin, W. Wu, B. Zhang, B. Wang, W. Zheng, and Z. Li, "Decentralized reactive power optimization method for transmission and distribution networks accommodating large-scale dg integration," *IEEE Transactions on Sustainable Energy*, vol. 8, no. 1, pp. 363–373, January 2017.

[4] G. Sanz, R. Pérez, and R. Sánchez, "Pressure control of a large scale water network using integral action," *IFAC Proceedings Volumes*, vol. 45, no. 3, pp. 270–275, March 2012.

[5] L. Stadelmann, T. Sandy, A. Thoma, and J. Buchli, "End-effector pose correction for versatile large-scale multi-robotic systems," *IEEE Robotics and Automation Letters*, vol. 4, no. 2, pp. 546–553, April 2019.

[6] W. M. Wonham, *Linear Multivariable Control a Geometric Approach*, 3rd ed. New York, NY, USA: Springer, 1985.

[7] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Systems Magazine*, vol. 27, no. 2, pp. 71–82, April 2007.

[8] W. Chen, J. Liu, Y. Chen, S. Z. Khong, D. Wang, T. Başar, L. Qiu, and K. H. Johansson, "Characterizing the positive semidefiniteness of signed Laplacians via effective resistances," in *Proceedings of the IEEE 55th Conference on Decision and Control*, Las Vegas, NV, USA, December 2016, pp. 985–990.

[9] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, September 2004.

[10] M. Pirani and S. Sundaram, "On the smallest eigenvalue of grounded Laplacian matrices," *IEEE Transactions on Automatic Control*, vol. 61, no. 2, pp. 509–514, February 2016.

[11] F. R. Chung, *Spectral Graph Theory.* Providence, RI, USA: American Mathematical Society, 1997.

[12] A. Mitra and S. Sundaram, "Distributed observers for LTI systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 11, pp. 3689–3704, November 2018.

[13] R. Olfati-Saber, "Distributed Kalman filtering for sensor networks," in *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans, LA, USA, December 2007, pp. 5492–5498.

[14] M. Kamgarpour and C. Tomlin, "Convergence properties of a decentralized Kalman filter," in *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, December 2008, pp. 3205–3210.

[15] R. Olfati-Saber, "Kalman-consensus filter: Optimality, stability, and performance," in *Proceedings of the 48th IEEE Conference on Decision and Control held jointly with the 28th Chinese Control Conference*, Shanghai, China, December 2009, pp. 7036–7042.

[16] B. Shen, Z. Wang, and Y. S. Hung, "Distributed $\mathcal{H}_\infty$-consensus filtering in sensor networks with multiple missing measurements: The finite-horizon case," *Automatica*, vol. 46, no. 10, pp. 1682–1688, October 2010.

[17] M. Farina, G. Ferrari-Trecate, and R. Scattolini, "Distributed moving horizon estimation for linear constrained systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 11, pp. 2462–2475, November 2010.

[18] I. Matei and J. S. Baras, "Consensus-based linear distributed filtering," *Automatica*, vol. 48, no. 8, pp. 1776–1782, August 2012.

[19] T. Kim, H. Shim, and D. D. Cho, "Distributed Luenberger observer design," in *Proceedings of the 55th IEEE Conference on Decision and Control*, Las Vegas, NV, USA, December 2016, pp. 6928–6933.

[20] W. Han, H. L. Trentelman, Z. Wang, and Y. Shen, "A simple approach to distributed observer design for linear systems," *IEEE Transactions on Automatic Control*, vol. 64, no. 1, pp. 329–336, January 2019.

[21] S. Park and N. C. Martins, "Design of distributed linear time-invariant observers for state omniscience," *IEEE Transactions on Automatic Control*, vol. 62, no. 2, pp. 561–576, February 2017.

[22] L. Wang and A. S. Morse, "A distributed observer for a time-invariant linear system," *IEEE Transactions on Automatic Control*, vol. 63, no. 7, pp. 2123–2130, July 2018.

[23] L. Wang, J. Liu, and A. S. Morse, "A distributed observer for a continuous-time linear system," in *2019 American Control Conference (ACC)*. IEEE, 2019, pp. 86–91.

[24] B. Aclkmese, D. P. Scharf, J. M. Carson III, and F. Y. Hadaegh, "Distributed estimation for spacecraft formations over time-varying sensing topologies," in *Proceedings of the 17th IFAC World Congress*, Seoul, Korea, July 2008, pp. 2123–2130.

[25] V. Ugrinovskii, "Distributed robust estimation over randomly switching networks using $\mathcal{H}_\infty$ consensus," *Automatica*, vol. 49, no. 1, pp. 160–168, January 2013.

[26] Q. Zhang and J. Zhang, "Distributed parameter estimation over unreliable networks with Markovian switching topologies," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2545–2560, October 2012.

[27] L. Wang, J. Liu, A. S. Morse, and B. D. Anderson, "A distributed observer for a discrete-time linear system," in *Proceedings of the 58th IEEE Conference on Decision and Control*, Nice, France, December 2019, pp. 367–372.

[28] H. Xu, J. Wang, H. Wang, S. Zhao, and H. Lin, "Distributed observer design for achieving asymptotical omniscience over time-variant disconnected networks," in *Proceedings of the 21st IFAC World Congress*, Berlin, Germany, July 2020, pp. 3565–3570.

[29] S. Bhattacharyya, "Observer design for linear systems with unknown inputs," *IEEE transactions on Automatic Control*, vol. 23, no. 3, pp. 483–484, June 1978.

[30] J. Chen, R. J. Patton, and H.-Y. Zhang, "Design of unknown input observers and robust fault detection filters," *International Journal of control*, vol. 63, no. 1, pp. 85–105, 1996.

[31] M. Hou and P. C. Müller, "Design of decentralized linear state function observers," *Automatica*, vol. 30, no. 11, pp. 1801–1805, 1994.

[32] A. Chakrabarty, S. Sundaram, M. J. Corless, G. T. Buzzard, S. H. Żak, and A. E. Rundell, "Distributed unknown input observers for interconnected nonlinear systems," in *Proceedings of the American Control Conference*, Boston, MA, USA, July 2016, pp. 101–106.

[33] A. Mitra, J. A. Richards, S. Bagchi, and S. Sundaram, "Resilient distributed state estimation with mobile agents: Overcoming Byzantine adversaries, communication losses, and intermittent measurements," *Autonomous Robots*, vol. 43, pp. 743–768, March 2019.

[34] A. Mitra and S. Sundaram, "Byzantine-resilient distributed observers for LTI systems," *Automatica*, vol. 108, pp. 1–12, October 2019.

[35] D. Ding, Z. Wang, H. Dong, and H. Shu, "Distributed $\mathcal{H}_\infty$ state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case," *Automatica*, vol. 48, no. 8, pp. 1575–1585, August 2012.

[36] J. Hu, Z. Wang, J. Liang, and H. Dong, "Event-triggered distributed state estimation with randomly occurring uncertainties and nonlinearities over sensor networks: a delay-fractioning approach," *Journal of the Franklin Institute*, vol. 352, no. 9, pp. 3750–3763, September 2015.

[37] M. S. Chong, H. Sandberg, and J. P. Hespanha, "A secure state estimation algorithm for nonlinear systems under sensor attacks," *arXiv:2008.12697*, 2020.

[38] J. K. Hunter, *An Introduction to Real Analysis.* Davis, CA, USA: Department of Mathematics, University of California at Davis, 2014.

[39] L. Bieberbach, *Analytische fortsetzung.* Springer-Verlag, 2013, vol. 3.

[40] G. Tao, "A simple alternative to the Barbalat lemma," *IEEE Transactions on Automatic Control*, vol. 42, no. 5, p. 698, May 1997.

[41] T. J. Laffey and H. Smigoc, "Common solution to the Lyapunov equation for 2×2 complex matrices," *Linear Algebra and its Applications*, vol. 420, no. 2–3, pp. 609–624, January 2007.

[42] A. J. van der Schaft, "$\mathcal{L}_2$ gain analysis of nonlinear systems and nonlinear state-feedback $\mathcal{H}_\infty$," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 770–784, June 1992.

[43] S. P. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory.* Philadelphia, PA, USA: SIAM, 1994.

[44] S. Mondal, G. Chakraborty, and K. Bhattacharyy, "LMI approach to robust unknown input observer design for continuous systems with noise and uncertainties," *International Journal of control, Automation and systems*, vol. 8, no. 2, pp. 210–219, 2010.

[45] H. K. Khalil and J. W. Grizzle, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice Hall, 2002.

[46] P. J. Antsaklis and A. N. Michel, *Linear Systems.* New York, NY, USA: Springer Science & Business Media, 2006.

[47] Z. Li and Z. Duan, *Cooperative Control of Multi-Agent Systems: A Consensus Region Approach.* Boca Raton, FL: CRC Press, 2017.

[48] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, September 2004.

[49] F. R. S. Sevilla, I. M. Jaimoukha, B. Chaudhuri, and P. Korba, “A semidefinite relaxation procedure for fault-tolerant observer design,” *IEEE Transactions on Automatic Control*, vol. 60, no. 12, pp. 3332–3337, December 2015.

[50] C. Zhang, I. M. Jaimoukha, and F. R. S. Sevilla, “Fault-tolerant observer design with a tolerance measure for systems with sensor failures,” in *Proceedings of the American Control Conference*, Boston, MA, USA, July 2016, pp. 7523–7528.

[51] S. S. Nasrolahi and F. Abdollahi, “Sensor fault detection and recovery in satellite attitude control,” *Acta Astronautica*, vol. 145, pp. 275–283, April 2018.

[52] K. Rothenhagen and F. W. Fuchs, “Current sensor fault detection, isolation, and reconfiguration for doubly fed induction generators,” *IEEE Transactions on Industrial Electronics*, vol. 56, no. 10, pp. 4239–4245, October 2009.

[53] Y. Peng, W. Qiao, L. Qu, and J. Wang, “Sensor fault detection and isolation for a wireless sensor network-based remote wind turbine condition monitoring system,” *IEEE Transactions on Industry Applications*, vol. 54, no. 2, pp. 1072–1079, March-April 2018.

[54] H. Behzad, A. Casavola, F. Tedesco, and M. A. Sadrnia, “Fault-tolerant sensor reconciliation schemes based on unknown input observers,” *International Journal of Control*, DOI: 10.1080/00207179.2018.1484568.

[55] J.-J. E. Slotine, J. K. Hedrick, and E. A. Misawa, “On Sliding Observers for Nonlinear Systems,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 109, no. 3, pp. 245–252, 9 1987.

[56] T. Ito, “A Filippov solution of a system of differential equations with discontinuous right-hand sides,” *Economics Letters*, vol. 4, no. 4, pp. 349–354, 1979.

[57] A. F. Filippov, *Differential Equations With Discontinuous Righthand Sides.* Dordrecht, The Netherland: Springer, 1988.

[58] I. Karafyllis, "Lyapunov theorems for systems described by retarded functional differential equations," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 64, no. 3, pp. 590–617, February 2006.

[59] E. Witrant, S. Mocanu, and O. Sename, "A hybrid model and MIMO control for intelligent buildings temperature regulation over WSN," in *Proceedings of the 8th IFAC Workshop on Time-Delay Systems*, Sinaia, Romania, September 2009, pp. 420–425.

[60] S. Boyd and L. Vandenberghe, *Convex Optimization.* Cambridge, UK: Cambridge University Press, 2004.

[61] S. G. Vrachimis, D. G. Eliades, and M. M. Polycarpou, "Leak detection in water distribution systems using hydraulic interval state estimation," in *Proceedings of the Conference on Control Technology and Applications*, Copenhagen, Denmark, August 2018, pp. 565–570.

Appendix A

Derivation of Equation (4.9)

The equation to be proved is obtained by expanding the error along with the system dynamics (4.1), the output equation (as in (2.1)), and the local observer (4.3). We start by noting that

$$e_i = x - z_i - H_i y_i = (I_n - H_i C_i)x - z_i. \tag{A.1}$$

Taking the time derivative of (A.1) yields

$$\begin{aligned}\dot{e}_i =& (I_n - H_i C_i)(Ax + B_i u_i + \bar{B}_i \acute{w}_i) \\ & - N_i z_i - M_i B_i u_i - L_i y_i - \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i),\end{aligned}$$

which by adding and subtracting the term $(I_n - H_i C_i)A\hat{x}_i$ to the right-hand side and since $\hat{x}_i = z_i + H_i y_i$, can be restated as follows:

$$\begin{aligned}\dot{e}_i =& (I_n - H_i C_i)Ae_i + (I_n - H_i C_i - M_i)B_i u_i \\ & + (I_n - H_i C_i)\bar{B}_i \acute{w}_i + (I_n - H_i C_i)A(z_i + H_i y_i) \\ & - N_i z_i - L_i y_i - \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i).\end{aligned} \tag{A.2}$$

According to the definition of e_i, one gets

$$-K_iC_ie_i + K_iC_i(x - \hat{x}_i) = \mathbf{0}_{n\times 1}. \tag{A.3}$$

Since $\hat{x}_i = z_i + H_iy_i$, from (A.3), it follows that

$$-K_iC_ie_i + K_iy_i - K_iC_iz_i - K_iC_iH_iy_i = \mathbf{0}_{n\times 1}. \tag{A.4}$$

We note that

$$\hat{x}_j - \hat{x}_i = x - \hat{x}_i - (x - \hat{x}_j) = e_i - e_j. \tag{A.5}$$

Now, by adding the zero term (A.4) to the right-hand side of (A.2) and by considering (A.5), after grouping similar terms, we finally obtain (4.9).

Appendix B

Proof of Theorem 5.3

From (5.47), (5.1b), and (5.48), it follows that the dynamics of $e_i = x - \hat{x}_i$, $i \in \mathbf{N}$, are given by

$$\dot{e}_i = (A + L_i C_i)e_i + f(\hat{x}_i) - f(x) + \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i), \quad i \in \mathbf{N},$$

which, since $-(\hat{x}_j - \hat{x}_i) = e_j - e_i$, can be restated as follows:

$$\dot{e}_i = (A + L_i C_i)e_i + f(\hat{x}_i) - f(x) + \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(e_j - e_i), \quad i \in \mathbf{N}. \tag{B.1}$$

We should show that along (B.1), the estimation errors converge to zero. Accordingly, consider the following Lyapunov candidate:

$$V = \sum_{i=1}^{N} e_i^\top P_i e_i \tag{B.2}$$

which is a positive definite function of the estimation error, since $P_{id} \succ 0, i \in \mathbf{N}$. The derivative of V along (B.1) reads as

$$\begin{aligned} \dot{V} = &\sum_{i=1}^{N} e_i^\top \left((A + L_i C_i)^\top P_i + P_i(A + C_i L_i)\right) e_i + 2\chi \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \left(e_j - e_i\right)^\top e_i \\ &+ 2 \sum_{i=1}^{N} \left(f(\hat{x}_i) - f(x)\right)^\top P_i e_i. \end{aligned} \tag{B.3}$$

According to Assumption 5.4, one obtains

$$\sum_{i=1}^{N} \left(f(\hat{x}_i) - f(x)\right)^\top P_i e_i \leq \sum_{i=1}^{N} \gamma \|\hat{x}_i - x\| \|P_i e_i\|.$$

Since $e_i := x - \hat{x}_i$, it follows that

$$2\sum_{i=1}^{N} \left(f(f(x) - \hat{x}_i)\right)^\top P_i e_i \leq 2\gamma \sum_{i=1}^{N} \|e_i\| \|P_i e_i\|. \tag{B.4}$$

Since $2\|e_i\|\|P_i e_i\| \leq e_i^\top e_i + e_i^\top P_i^2 e_i$, by considering (B.4) and according to the definition of $\bar{P}$ in (5.52), one obtains

$$2\sum_{i=1}^{N} \left(f(f(x) - \hat{x}_i)\right)^\top P_i e_i \leq \gamma e^\top \bar{P} e \tag{B.5}$$

where

$$e = \begin{bmatrix} e_1^\top & e_2^\top & \dots & e_N^\top \end{bmatrix}^\top.$$

Moreover,

$$2\chi \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \left(e_j - e_i\right)^\top e_i = -2\chi e^\top \left(\mathcal{L} \otimes I_n\right) e. \tag{B.6}$$

From (B.3), (B.5), and (B.6), and according to the definition of $\varLambda$ in (5.9), it follows that

$$\dot{V} \leq e^\top (\varLambda + \gamma \bar{P}) e - 2\chi e^\top \left(\mathcal{L} \otimes I_n\right) e. \tag{B.7}$$

Note that, since the pair (C_i, A) may not be detectable, $\varLambda$ may not be negative definite. Then we decompose e the same way as in Theorem 5.1. Hence, using (5.15), the inequality (B.7) yields

$$\dot{V} \leq e_c^\top (\varLambda + \gamma \bar{P}) e_c + 2 e_r^\top (\varLambda + \gamma \bar{P}) e_c + e_r^\top (\varLambda + \gamma \bar{P}) e_r - 2\chi e_r^\top (\mathcal{L} \otimes I_n) e_r. \tag{B.8}$$

To guarantee negative definiteness of $\dot{V}$, one must show that $e_c^\top (\varLambda + \gamma \bar{P}) e_c$ is negative definite. Since $e_c = \mathbf{1}_N \otimes \omega$, according to the definition of $\varLambda$ and $\bar{P}$ in (5.9) and (5.52) respectively, one obtains

$$e_c^\top (\varLambda + \gamma \bar{P}) e_c = \omega^\top \left(\sum_{i=1}^{N} \left(\varLambda_i + \gamma (P_i^2 + I_n)\right) \right) \omega. \tag{B.9}$$

In what follows, we show that the right-hand side of (B.9) is negative definite. According to (5.21) we have

$$-\sum_{i=1}^{N}\big(\varLambda_i+\gamma(P_i^2+I_n)\big)=T_o\,(K-M)\,T_o^\top-\gamma NI_n-\gamma\tilde{P}\tilde{P}^\top \tag{B.10}$$

where $\tilde{P}$ is as defined in (5.50). According to (5.49) and the Schur Complement [43] one obtains

$$T_o\,(K-M)\,T_o^\top-\gamma NI_n-\gamma\tilde{P}\tilde{P}^\top\succ 0. \tag{B.11}$$

From (B.10) and (B.11), it follows that the right-hand side of (B.9) is negative definite, and as a result,

$$e_c^\top(\varLambda+\gamma\bar{P})e_c=\omega^\top\left(\sum_{i=1}^{N}\big(\varLambda_i+\gamma(P_i^2+I_n)\big)\right)\omega<0,\forall\omega\neq\mathbf{0}_{n\times 1}. \tag{B.12}$$

Now, we investigate the other terms in the right-hand side of the inequality (B.8). As $e_c=\mathbf{1}_N\otimes\omega$, one has

$$e_r^\top(\varLambda+\gamma\bar{P})e_c=e_r^\top\varLambda_P^\top\omega=\omega^\top\varLambda_P e_r \tag{B.13}$$

where $\varLambda_P$ is defined in (5.52). According to (B.12), (B.13), and (5.25), (B.8) is satisfied if

$$\begin{aligned}&\dot{V}\leq\\&-\begin{bmatrix}\omega\\ e_r\end{bmatrix}^\top\begin{bmatrix}Q & -\varLambda_P\\ -\varLambda_P^\top & 2\chi\lambda_2(\mathcal{L})I_{Nn}-\big(\varLambda+\gamma\bar{P}\big)\end{bmatrix}\begin{bmatrix}\omega\\ e_r\end{bmatrix}\end{aligned} \tag{B.14}$$

where $Q\in\mathbb{R}^{n\times n}\succ 0$ is defined in (5.52). From (5.51), it follows that

$$2\chi\lambda_2(\mathcal{L})I_{Nn}-\varLambda-\gamma\bar{P}-\varLambda_P^\top Q^{-1}\varLambda_P\succ 0,$$

and thus by using the Schur Complement we get

$$\begin{bmatrix}Q & -\varLambda_P\\ -\varLambda_P^\top & 2\chi\lambda_2(\mathcal{L})I_{Nn}-\big(\varLambda+\gamma\bar{P}\big)\end{bmatrix}\succ 0. \tag{B.15}$$

According to (B.14) and (B.15), there exists a $k \in \mathbb{R}_{>0}$ such that

$$\dot{V} \leq -k \left\| \begin{bmatrix} \omega \\ e_r \end{bmatrix} \right\|^2 \tag{B.16}$$

where

$$k = \lambda_{\min} \left(\begin{bmatrix} Q & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\lambda_2(\mathcal{L}) I_{Nn} - \left(\Lambda + \gamma \bar{P}\right) \end{bmatrix} \right). \tag{B.17}$$

By using the same analysis as stated in the end of proof of Theorem 5.1 (from (5.30) to (5.31)), we show that V asymptotically converges to zero, which implies that the estimation errors $e_i, \forall i \in \mathbf{N}$, converge to zero.

Appendix C

Proof of Theorem 5.4

From (5.47), it follows that

$$\dot{x}(t-\tau) = Ax(t-\tau) + f(x(t-\tau)) + Bu(t-\tau). \tag{C.1}$$

Since $-(\hat{x}_j(t) - \hat{x}_i(t)) = e_j(t) - e_i(t)$, from (5.2), (5.53) and (C.1), and on the basis of Algorithm 1, the differential equation describing $e_i(t) = x(t-\tau) - \hat{x}_i(t), i \in \mathbf{N}$, is given by

$$\begin{aligned}\dot{e}_i(t) = {} & (A + L_iC_i)e_i(t) + f(\hat{x}_i(t)) - f(x(t-\tau)) + \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(e_j(t) - e_i(t)) \\ & - L_i\left(\varpi_{i,m}(t-\tau) + \delta_{i,m}(t-\tau)\right) \\ & - L_i\boldsymbol{\eta_i}\mathrm{msgn}\big(\boldsymbol{\beta_i}^{-1}\mathrm{sgn}(H_i)(C_ie_i(t) - \varpi_{i,m}(t-\tau) - \delta_{i,m}(t-\tau))\big).\end{aligned} \tag{C.2}$$

To analyze the evolution of $e_i(t)$ along (C.2), we consider the same Lyapunov candidate as (5.11). Thus, following a procedure similar to that in the proof of Theorem 5.3, the derivation of V along (C.2) yields

$$\begin{aligned}\dot{V}(t) \le {} & e^\top(t)(\varLambda + \gamma\bar{P})e(t) - 2\chi e^\top(t)(\boldsymbol{\mathcal{L}} \otimes I_n)e(t) - \sum_{i=1}^{N} 2e_i^\top P_iL_i\Big(\varpi_{i,m}(t-\tau) + \delta_{i,m}(t-\tau) \\ & + \boldsymbol{\eta_i}\mathrm{msgn}\big(\boldsymbol{\beta_i}^{-1}\mathrm{sgn}(H_i)(C_ie_i(t) - \varpi_{i,m}(t-\tau) - \delta_{i,m}(t-\tau))\big)\Big).\end{aligned} \tag{C.3}$$

Now, from (C.3) and (5.42), and following a procedure similar to that in the proof of Theorem 5.3 for (B.7), one obtains

$$\begin{aligned}\dot{V}(t) \leq & -\mu V(t) - 2\sum_{i=1}^{N}(H_iC_ie_i(t))^\top\Big(\varpi_{i,m}(t-\tau)+\delta_{i,m}(t-\tau) \\ & + \boldsymbol{\eta_i}\mathrm{msgn}\big(\boldsymbol{\beta_i}^{-1}\mathrm{sgn}(H_i)(C_ie_i(t)-\varpi_{i,m}(t-\tau)-\delta_{i,m}(t-\tau))\big)\Big).\end{aligned} \tag{C.4}$$

Now, we use the same analysis as in the proof of Theorem 5.2, we can finally get

$$\dot{V}(t) \leq -\mu V(t) + 4\sum_{i=1}^{N}\sum_{q=1}^{p_i}|H_{i,q}|\beta_{i,q}^2. \tag{C.5}$$

Similarly, we derive that

$$\lim_{t\to\infty}\sup\|e(t)\| \leq 2\sqrt{\frac{\sum_{i=1}^{N}\sum_{q=1}^{p_i}|H_{i,q}|\beta_{i,q}^2}{\mu\min\limits_{i\in\mathbf{N}}(\lambda_{\min}(P_i))}},$$

which completes the proof.

Appendix D

Parameters for Simulation in Chapter 6

In this appendix, the parameters that have been used for building the distributed observers in Chapter 7 are presented. We keep 4 significant digits for non-integer elements in all matrices.

D.1 Parameters for the Heat Exchange Model

$$A = 10^{-2} \times \begin{bmatrix} -30 & 25 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 25 & -65 & 25 & 0 & 15 & 0 & 0 & 0 & 0 \\ 0 & 25 & -65 & 0 & 0 & 40 & 0 & 0 & 0 \\ 5 & 0 & 0 & -60 & 15 & 0 & 40 & 0 & 0 \\ 0 & 15 & 0 & 15 & -75 & 10 & 0 & 35 & 0 \\ 0 & 0 & 40 & 0 & 10 & -70 & 0 & 0 & 20 \\ 0 & 0 & 0 & 40 & 0 & 0 & -50 & 10 & 0 \\ 0 & 0 & 0 & 0 & 35 & 0 & 10 & -60 & 15 \\ 0 & 0 & 0 & 0 & 0 & 20 & 0 & 15 & -35 \end{bmatrix},$$

$$B = 10^{-2} \times \begin{bmatrix} 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}^{\top}$$

$$F = \begin{bmatrix} 2.735 & -2.699 & 2.450 & -4.311 & 8.629 & -1.066 & 4.208 & -4.912 & 2.001 \\ -0.04437 & -1.378 & 3.134 & -0.7507 & 1.658 & -3.093 & 0.5353 & -0.9277 & 1.945 \\ 0.3186 & 5.958 & -0.5059 & 10.13 & -12.724 & 6.247 & -6.028 & 13.10 & 0.5874 \end{bmatrix}$$

D.2 Parameters for Scenario 6.1 (Switching Graph, No Noise)

$$L_1 = 10^{-2} \times \begin{bmatrix} -17.26 & 2.124 \\ -12.08 & -4.770 \\ -9.349 & -6.006 \\ -4.872 & -10.38 \\ -3.478 & -11.63 \\ -7.541 & -7.440 \\ -4.547 & -10.73 \\ -4.006 & -11.25 \\ -6.331 & -8.909 \end{bmatrix}, \quad L_2 = 10^{-2} \times \begin{bmatrix} -7.898 & -5.593 \\ -9.541 & -5.299 \\ -9.879 & -3.579 \\ -6.318 & -7.047 \\ -6.146 & -7.066 \\ -4.270 & -8.972 \\ -6.226 & -7.158 \\ -6.099 & -7.264 \\ -5.018 & -8.406 \end{bmatrix}$$

$$L_3 = 10^{-2} \times \begin{bmatrix} -6.103 & -4.811 \\ -6.565 & -5.459 \\ -6.094 & -4.836 \\ -6.305 & -4.473 \\ -5.642 & -5.011 \\ -5.861 & -4.775 \\ -1.879 & -8.906 \\ -4.941 & -5.829 \\ -5.657 & -5.140 \end{bmatrix}, \quad L_4 = 10^{-2} \times \begin{bmatrix} -5.268 & -5.174 \\ -5.204 & -6.299 \\ -3.446 & -7.007 \\ -7.720 & -2.597 \\ -6.390 & -3.805 \\ -2.685 & -7.485 \\ -8.153 & -2.178 \\ -7.407 & -2.904 \\ -0.7696 & -9.550 \end{bmatrix}$$

$P_1 = P_2 = P_3 = P_4 = 10^{-1} \times I_9.$

D.3 Parameters for Scenario 6.2 (Switching Graph, $\mathscr{L}_2$ Noise)

$$L_1 = \begin{bmatrix} -3.895 & -2.514 \\ -3.983 & -2.847 \\ -3.738 & -2.988 \\ -3.170 & -2.856 \\ -2.578 & -2.150 \\ -3.321 & -2.832 \\ -3.227 & -3.020 \\ -2.516 & -2.307 \\ -2.856 & -2.735 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -5.699 & -6.649 \\ -5.614 & -6.827 \\ -4.968 & -6.230 \\ -5.288 & -6.714 \\ -5.503 & -6.993 \\ -4.931 & -6.337 \\ -5.285 & -6.783 \\ -5.611 & -7.243 \\ -6.102 & -7.935 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} -5.551 & -4.498 \\ -5.454 & -4.446 \\ -5.351 & -4.361 \\ -5.324 & -4.495 \\ -5.320 & -4.438 \\ -5.280 & -4.351 \\ -5.401 & -4.658 \\ -5.295 & -4.484 \\ -5.215 & -4.434 \end{bmatrix}, \quad L_4 = \begin{bmatrix} -2.547 & -1.980 \\ -2.592 & -2.140 \\ -2.792 & -2.436 \\ -2.805 & -1.969 \\ -2.394 & -1.913 \\ -2.726 & -2.403 \\ -2.860 & -1.984 \\ -2.187 & -1.765 \\ -2.324 & -2.048 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 13.48 & -13.44 & 2.269 & -1.459 & -2.642 & 1.275 & -0.2206 & 2.249 & -0.7619 \\ -13.44 & 29.23 & -10.98 & -2.721 & -4.888 & -2.345 & 2.817 & -1.810 & 1.465 \\ 2.269 & -10.98 & 25.71 & 3.731 & -2.770 & -18.51 & -3.035 & -0.05503 & 2.566 \\ -1.459 & -2.721 & 3.731 & 30.35 & -5.606 & -2.650 & -22.87 & 1.008 & 0.5261 \\ -2.642 & -4.888 & -2.770 & -5.606 & 39.55 & -2.069 & 0.4047 & -17.38 & 2.034 \\ 1.275 & -2.345 & -18.51 & -2.650 & -2.069 & 34.14 & 2.297 & 1.494 & -12.69 \\ -0.2206 & 2.817 & -3.035 & -22.87 & 0.4047 & 2.2970 & 24.82 & -5.476 & -0.3804 \\ 2.249 & -1.810 & -0.05503 & 1.008 & -17.38 & 1.494 & -5.476 & 29.28 & -7.208 \\ -0.7619 & 1.465 & 2.566 & 0.5261 & 2.034 & -12.69 & -0.3804 & -7.208 & 14.64 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 10.70 & -9.439 & -0.9582 & -0.8536 & -4.334 & 3.343 & 0.3384 & 1.588 & 0.05300 \\ -9.439 & 21.41 & -5.836 & -3.178 & 1.815 & -5.788 & 1.814 & -3.291 & 1.357 \\ -0.9582 & -5.836 & 25.47 & 4.572 & -6.188 & -13.10 & -2.517 & 3.303 & -2.922 \\ -0.8536 & -3.178 & 4.572 & 26.35 & -3.179 & -3.396 & -19.49 & -1.073 & 0.6530 \\ -4.334 & 1.815 & -6.188 & -3.179 & 28.40 & 0.2406 & -2.610 & -12.06 & -2.230 \\ 3.343 & -5.788 & -13.10 & -3.396 & 0.2406 & 30.17 & 3.406 & -2.310 & -9.575 \\ 0.3384 & 1.814 & -2.517 & -19.49 & -2.610 & 3.406 & 22.58 & -3.240 & -0.1344 \\ 1.588 & -3.291 & 3.303 & -1.073 & -12.06 & -2.310 & -3.240 & 23.11 & -5.887 \\ 0.05300 & 1.357 & -2.922 & 0.6530 & -2.230 & -9.575 & -0.1344 & -5.887 & 15.87 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 11.77 & -12.33 & 1.951 & -1.522 & -1.793 & 0.02748 & 0.3533 & 0.7992 & 0.6785 \\ -12.33 & 29.86 & -12.26 & -0.6936 & -7.869 & 0.3330 & 0.6869 & 2.253 & -0.08826 \\ 1.951 & -12.26 & 22.62 & -0.8919 & 8.699 & -16.11 & 1.252 & -9.359 & 4.130 \\ -1.522 & -0.6936 & -0.8919 & 34.44 & -5.878 & 2.350 & -25.91 & -0.8155 & -0.1303 \\ -1.793 & -7.869 & 8.699 & -5.878 & 35.40 & -11.25 & -1.645 & -13.09 & -2.293 \\ 0.02748 & 0.3330 & -16.11 & 2.350 & -11.25 & 31.62 & -1.606 & 8.233 & -13.42 \\ 0.3533 & 0.6869 & 1.252 & -25.91 & -1.645 & -1.606 & 28.39 & -3.076 & 1.111 \\ 0.7992 & 2.253 & -9.359 & -0.8155 & -13.09 & 8.233 & -3.076 & 20.72 & -5.534 \\ 0.6785 & -0.08826 & 4.130 & -0.1303 & -2.293 & -13.42 & 1.111 & -5.534 & 15.53 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 10.72 & -10.93 & 1.083 & 1.179 & 0.9668 & -0.6149 & -2.811 & 0.1843 & 0.9920 \\ -10.93 & 28.17 & -11.40 & -6.196 & -8.183 & 0.8334 & 6.022 & 3.009 & -0.8923 \\ 1.083 & -11.40 & 25.03 & 1.950 & 1.682 & -16.94 & -1.365 & -4.378 & 3.110 \\ 1.179 & -6.196 & 1.950 & 22.70 & -4.634 & 1.474 & -16.31 & 0.2174 & -0.7374 \\ 0.9668 & -8.183 & 1.682 & -4.634 & 33.87 & -6.940 & -1.952 & -14.84 & 1.434 \\ -0.6149 & 0.8334 & -16.94 & 1.474 & -6.940 & 30.17 & -0.4827 & 2.719 & -11.85 \\ -2.811 & 6.022 & -1.365 & -16.31 & -1.952 & -0.4827 & 19.46 & -3.665 & 0.2781 \\ 0.1843 & 3.009 & -4.378 & 0.2174 & -14.84 & 2.719 & -3.665 & 25.81 & -6.162 \\ 0.9920 & -0.8923 & 3.110 & -0.7374 & 1.434 & -11.85 & 0.2781 & -6.162 & 14.96 \end{bmatrix}$$

D.4 Parameters for Scenarios 6.4 and 6.5 (Unknown Input)

$$B_1 = \begin{bmatrix} 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.12 & 0 & 0 & 0 \end{bmatrix}^\top$$

$$B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.07 \end{bmatrix}^\top$$

$P_1 = P_2 = P_3 = P_4 = 0.1000 \times I_9$

$$N_1 = \begin{bmatrix} -0.3000 & 0.2500 & 0 & 0.05000 & 0 & 0 & 0 & 0 & 0 \\ 0.2500 & -0.6500 & 0.2500 & 0 & 0.1286 & 0 & 0 & 0 & 0 \\ 0 & 0.2500 & -0.6500 & 0 & 0 & -0.2667 & 0 & 0 & 0 \\ 0.05000 & 0 & 0 & -0.6000 & 0 & 0 & 0.4000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2873 & -0.01429 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.01429 & -3831 & 0 & 0 & -0.06667 \\ 0 & 0 & 0 & 0.4000 & 0 & 0 & -0.5000 & 0.1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1000 & -0.6000 & 0.1000 \\ 0 & 0 & 0 & 0 & 0 & -0.06667 & 0 & 0 & -3831 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} -0.3000 & -0.1667 & 0 & 0.05000 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3831 & 0 & 0 & -0.02143 & 0 & 0 & 0 & 0 \\ 0 & -0.1667 & -0.6500 & 0 & 0 & 0.4000 & 0 & 0 & 0 \\ 0.05000 & 0 & 0 & -0.6000 & 0 & 0 & 0.4000 & 0 & 0 \\ 0 & -0.02143 & 0 & 0 & -2873 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4000 & 0 & 0.08571 & -0.7000 & 0 & 0 & 0.1333 \\ 0 & 0 & 0 & 0.4000 & 0 & 0 & -0.5000 & 0.1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1000 & -0.6000 & -0.1000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3831 \end{bmatrix}$$

$$N_3 = \begin{bmatrix} -0.3000 & -0.1667 & 0 & 0.05000 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3831 & 0 & 0 & -0.02143 & 0 & 0 & 0 & 0 \\ 0 & -0.1667 & -0.6500 & 0 & 0 & 0.2667 & 0 & 0 & 0 \\ 0.05000 & 0 & 0 & -0.6000 & -0.1000 & 0 & 0.4000 & 0 & 0 \\ 0 & -0.02143 & 0 & 0 & -2873 & -0.01429 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.01429 & -3831 & 0 & 0 & -0.06667 \\ 0 & 0 & 0 & 0.4000 & 0 & 0 & -0.5000 & 0.1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1000 & -0.6000 & -0.1000 \\ 0 & 0 & 0 & 0 & 0 & -0.06667 & 0 & 0 & -3831 \end{bmatrix}$$

$$N_4 = \begin{bmatrix} -0.3000 & -0.1667 & 0 & 0.05000 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3831 & 0 & 0 & -0.02143 & 0 & 0 & 0 & 0 \\ 0 & -0.1667 & -0.6500 & 0 & 0 & -0.2667 & 0 & 0 & 0 \\ 0.05000 & 0 & 0 & -0.6000 & 0 & 0 & 0.4000 & 0 & 0 \\ 0 & -0.02143 & 0 & 0 & -2873 & -0.01429 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.01429 & -3831 & 0 & 0 & 0.1500 \\ 0 & 0 & 0 & 0.4000 & 0 & 0 & -0.5000 & 0.1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1000 & -0.6000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1333 & 0 & 0.1500 & -0.3500 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix},\ M_2 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix},\ M_4 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 0&0&0\\ 0.1500&0&0\\ 0&0.4000&0\\ 0.1500&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0.3500&0&0.1500\\ 0&0&0 \end{bmatrix},\ L_2 = \begin{bmatrix} 0.2500&0&0\\ 0&0&0\\ 0.2500&0&0\\ 0&0.1500&0\\ 0&0&0\\ 0&0.1000&0.2000\\ 0&0&0\\ 0&0.3500&0.1500\\ 0&0&0 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0.2500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.2500 & 0 & 0.4000 & 0 \\ 0 & 0.1500 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.3500 & 0 & 0.1500 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; L_4 = \begin{bmatrix} 0.2500 & 0 & 0 \\ 0 & 0 & 0 \\ 0.2500 & 0 & 0.4000 \\ 0 & 0.1500 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.3500 & 0 \\ 0 & 0 & 0.2000 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; H_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; H_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

D.5 Parameters for Scenarios 6.6 (Centralised)

D.6 Parameters for Scenarios 6.7, 6.8 and 6.9 (Sensor Redundancy)

$$P_1 = \begin{bmatrix} 9.242 & 3.441 & 0.4665 & 0.1442 & -0.05692 & -0.3808 & -0.4614 & -0.7075 & -0.8864 \\ 3.441 & 7.164 & 2.890 & -0.2475 & 1.186 & 1.051 & -0.6135 & -0.0754 & -0.2479 \\ 0.4665 & 2.890 & 8.129 & -0.7340 & 0.4154 & 4.539 & -0.8998 & -0.2544 & 1.014 \\ 0.1442 & -0.2475 & -0.7340 & 8.893 & 1.594 & -0.6235 & 6.072 & 0.8115 & -0.5993 \\ -0.05692 & 1.186 & 0.4154 & 1.594 & 6.012 & 0.8630 & 0.9268 & 3.066 & 0.5955 \\ -0.3808 & 1.051 & 4.539 & -0.6235 & 0.8630 & 7.508 & -0.7580 & 0.3469 & 3.076 \\ -0.4614 & -0.6135 & -0.8998 & 6.072 & 0.9268 & -0.7580 & 10.31 & 1.249 & -0.5090 \\ -0.7075 & -0.07543 & -0.2544 & 0.8115 & 3.066 & 0.3469 & 1.249 & 5.896 & 1.984 \\ -0.8864 & -0.2479 & 1.014 & -0.5993 & 0.5955 & 3.076 & -0.5090 & 1.984 & 10.79 \end{bmatrix}$$

$P_2 = P_3 = I_9,\ P_4 = P_1$

$H_1 = -2.888,\ H_2 = H_3 = 0,\ H_4 = -2.524$

$$L_1 = \begin{bmatrix} -0.3949 & 0.2125 & -0.04244 & 0.03010 & -0.03158 & -0.01388 & -0.02231 & -0.02439 & -0.01276 \end{bmatrix}^\top$$

$L_2 = L_3 = \mathbf{0}_{9\times 1}$

$$L_4 = \begin{bmatrix} -0.02131 & -0.03244 & -0.01256 & -0.04980 & 0.3348 & -0.03960 & 0.07663 & -0.6504 & 0.1119 \end{bmatrix}^\top$$