

Streaks and Görtler vortices in boundary layer subject to pressure gradient: excitation by free-stream vortical disturbances, nonlinear evolution and secondary instability.

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This paper investigates streaks and Görtler vortices in a boundary layer over a flat or concave wall in a contracting or expanding stream, which provides a favorable or adverse pressure gradient, respectively. We consider first the excitation of streaks and Görtler vortices by free-stream vortical disturbances (FSVD), and their nonlinear evolution. The focus is on FSVD with sufficiently long wavelength, to which the boundary layer is most receptive. The formulation is directed at the general case where the Görtler number G_Λ (based on the spanwise wavelength Λ of FSVD) is of order one, and the intensity of FSVD is strong enough that the induced Görtler vortices acquire an $O(1)$ streamwise velocity in the region where the boundary-layer thickness becomes comparable with Λ , and the vortices are governed by the nonlinear boundary-region equations (NBRE). An important effect of a pressure gradient is that the oncoming FSVD are distorted by the non-uniform inviscid flow outside the boundary layer through convection and stretching. This process is accounted for by using the rapid distortion theory. The impact of the distorting FSVD is analysed to provide the appropriate initial and boundary conditions, which form, along with the NBRE, the appropriate initial-boundary-value problem describing the excitation and nonlinear evolution of the vortices. Numerical results show that in the presence of an adverse/favorable pressure gradient Görtler vortices saturate earlier/later, but the intensity of the saturated vortices is lower/higher than that in the case of zero pressure gradient. With the increase of the favorable pressure gradient, the nonlinearly generated harmonic component (0,2) dominates eventually, causing two mushrooms to appear within one spanwise period. For a very strong favorable pressure gradient, the saturation location of the Görtler vortices ceases to move downstream farther. It is also found that the increase of FSVD intensity reduces the effects of the Görtler number and pressure gradient on the growth of Görtler vortices. At a high turbulence level of $Tu = 14\%$, the curvature does not impact the evolution of vortices, and the pressure gradient only influences the saturation intensity. The unsteadiness of FSVD is found to reduce the boundary-layer response significantly at low turbulence levels, but that effect weakens as the turbulence level increases. A secondary instability analysis of vortices is performed for two moderate pressure gradients, one adverse and the other favorable. Three families of unstable modes have been identified, which may become dominant depending on the frequency and streamwise location. In the presence of an adverse pressure gradient, the secondary instability occurs earlier, but the unstable modes appear in a smaller band of

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streamwise wavenumber or frequency, and their growth rates are smaller. The opposite is true in the case of a favorable pressure gradient. The present theoretical framework, which accounts for the influence of the curvature, turbulence level and pressure gradient, allows for a detailed and integrated description of the key transition processes, and represents a useful step towards predicting the pre-transitional flow and transition itself of the boundary layer over a blade in turbo-machinery.

1. Introduction

It is of importance to understand and predict the transition process in the boundary layers over turbine blades, as it largely determines the load and heat transfer on the blade, and the overall efficiency of the turbine (Mayle 1991; Jahanmiri 2011). The key factors affecting instability and transition include high turbulence level, presence of pressure gradient and surface curvature (Schultz & Volino 2003; Volino & Simon 1995, 2000). Since the pressure gradient is favorable in an extended region from the leading edge, the flow there remains laminar, but exhibits strong three-dimensional and unsteady characteristics due to the influence of free-stream turbulence (FST) and centrifugal force. In the case of a concave wall, Görtler instability operates to cause earlier transition, the extent of which depends on the level of FST, but a convex curvature has little effect at all turbulence level (Volino & Simon 1995, 1997; Kim, Simon & Russ 1992). Currently, transition location is determined by using empirical correction relations deduced from limited experimental data (Walker 1993). In the case of flat-plate boundary layer, several relations between the transition Reynolds number and FST level have been proposed with or without a pressure gradient (Abu-Ghannam & Shaw 1980; Gostelow, Blunden & Walker 1994). However, the role of the spectra or length scales of FST has not been quantified despite the fact that they were found to be also relevant (Volino & Simon 1995), and neither have the combined effects of the pressure gradient and curvature been accounted for. Better understanding of transition process is crucial for the development of physics-based transition prediction method. In this paper, we investigate the fundamental process of transition in the boundary layer over a flat or concave wall subject to a pressure gradient. Specifically, the generation of Görtler vortices by FSVD, their evolution and secondary instability are to be considered in a unified framework, in which the pressure gradient, FSVD and curvature are all taken into account.

1.1. Görtler vortices

Görtler instability occurs in a boundary layer over a concave wall due to the imbalance between the centrifugal force and the wall-normal pressure gradient. Görtler (1941) started the investigation of this instability by using a local normal-mode approach based on a parallel-flow approximation. The key parameter controlling the instability was identified to be the Görtler number, defined as $G_\theta = (U_\infty \theta / \nu) \sqrt{\theta / r_0^*}$, where U_∞ is the free-stream velocity, θ and r_0^* denote the boundary-layer (momentum) thickness and radius of curvature respectively, and ν the kinematic viscosity. The instability causes the formation of spanwise periodic and streamwise elongated vortices. The analysis of Görtler was later extended by Smith (1955), Ragab & Nayfeh (1980) and Floryan & Saric (1982), who sought to include part of the non-parallel-flow effects, but continued to treat the instability as an eigenvalue problem. These studies showed that Görtler vortices with a fixed spanwise wavelength decay near the leading edge, but start to amplify from some distance downstream. However, without fully accounting for the streamwise variation of both the base flow and modal shape of the perturbations, the growth rate predicted by

the local theory is unsatisfactory in comparison with experimental results (Tani 1962; Finniss & Brown 1997).

Hall (1982, 1983) was the first to point out that in the general case where $G_\theta = O(1)$, Görtler instability should be formulated as an initial-value problem, which has to be solved by using a marching method. A local approach is justified only in the limit of $G_\theta \gg 1$, or equivalently sufficiently downstream for vortices with a fixed wavelength. These theoretical findings were confirmed by further studies of Day, Herbert & Saric (1990), Goulpié *et al.* (1996) and Bottaro & Luchini (1999), who compared the predictions by the two approaches. They found that different initial conditions led to different transient behaviour of Görtler vortices, and the local and marching methods gave the same results only sufficiently far downstream.

Experiments (Swearingen & Blackwelder 1987; Kottke 1988) and theoretical analyses indicate that the initial condition is vital for describing the evolution of Görtler vortices. For this reason, receptivity, i.e. the process of generating vortices by external disturbances, should be treated as an integrated part of the Görtler instability theory. Surface roughness elements and free-stream turbulence are two main external disturbances generating Görtler vortices. Denier, Hall & Seddougui (1991), Bassom & Hall (1994) and Sescu & Thompson (2015) studied the generation of Görtler vortices by roughness. They found that the shape and size (height and diameter) of roughness elements influence the excitation and evolution of Görtler vortices. Sescu & Afsar (2018) studied wall deformations to hamper the Görtler vortices excited by roughness using an optimal control method.

How free-stream turbulence excites precisely Görtler vortices and streak-like disturbances in the boundary layer is a topic of intense recent interest. Direct numerical simulations (DNS) were performed by Schrader, Brandt & Zaki (2011). In their work, the FSVD and inlet disturbances are represented by the continuous modes of the Orr-Sommerfeld (O-S) and Squire operators, in which non-parallelism is neglected completely. Unfortunately, these continuous modes are non-physical, and their eigenfunctions do not represent entrainment of FSVD into the boundary layer because the latter is fundamentally influenced by non-parallel-flow effects, as was pointed out by Dong & Wu (2013). For the flat-plate boundary layer, Leib, Wundrow & Goldstein (1999) showed that long-wavelength, low-frequency disturbances FSVD induce streaks, which are governed by linear unsteady boundary-region equations. Ricco, Luo & Wu (2011) extended their work to the case where FSVD are strong enough to generate nonlinear streaks. Generation of linear and nonlinear streaks in a compressible boundary layer has been considered by Ricco & Wu (2007) and Marensi *et al.* (2017) respectively. On the other hand, Wu, Zhao & Luo (2011) extended the work of Leib *et al.* (1999) to a concave wall and studied the excitation of Görtler vortices by FSVD and their linear evolution. Xu, Zhang & Wu (2017) (referred to as XZW hereafter) furthermore investigated the nonlinear evolution and secondary instability of Görtler vortices induced by FSVD. This theoretical work was based on the first principles, and presented a self-consistent framework to connect the intensity of turbulence with the transition process. The theoretical predictions were found to be in good agreement with experimental measurements.

1.2. The influence of pressure gradient

A strong streamwise pressure gradient always exists in turbo-machinery, and indeed influences the onset of transition (Schreiber, Steinert & Küsters 2002). However, previous studies of pressure-gradient effect have mostly restricted to (a) streak formation and bypass transition in boundary layers over flat walls, and (b) linear Görtler instability using the Falkner-Skan solutions as a model flow.

Experimental studies of bypass transition in the flat-plate boundary layer showed that transition onset location shifted downstream when the flow is subject to a streamwise acceleration (Blair 1992; Talamelli, Fornaciari & Westin 2002), and the location was found to depend on the history of the pressure gradient, as a result of which correlating it with a local pressure-gradient parameter is inadequate (Abu-Ghannam & Shaw 1980; Solomon, Walker & Gostelow 1996). Zaki & Durbin (2006) performed DNS of transition triggered by the continuous modes, which were presumed to represent FSVD (but see Dong & Wu (2013)). With the base flow being taken to be the Falkner-Skan boundary layer, they found that an adverse pressure gradient enhanced streaks and as a result transition occurred earlier. Johnson & Pinarbasi (2014) calculated the linear response to continuous modes of boundary layers with zero, adverse and favourable pressure gradients. The amplitude of streaks in the presence of an adverse pressure gradient was found to be greater than that in the favourable pressure-gradient case. Brinkerhoff & Yaras (2015) conducted a DNS of bypass transition in a boundary layer subjected to an adverse pressure gradient and elevated free-stream turbulence. They pointed out that two distinct processes, the development of varicose secondary instability and a rapid amplification of free stream disturbances in the inflectional boundary layer, caused rapid transition to turbulence.

For a Falkner-Skan boundary layer, the local linear stability analysis of Ragab & Nayfeh (1980) indicates that an adverse pressure gradient is destabilising, which is the same as its impact on Tollmien-Schlichting waves. This conclusion was confirmed by marching-method calculations with the initial (upstream) condition being taken to be an eigenmode or a more-or-less arbitrary disturbance velocity distribution (Goulpié *et al.* 1996; Matsson 2008). Rogenski, de Souza & Floryan (2016) performed simulations using the linearized N-S equations, which require an artificial ‘buffer region’ at the outlet of the computational domain. Spanwise periodic suction/blowing was introduced to excite Görtler vortices, and different forms of pressure gradient were considered. For all the cases, the destabilizing/stabilizing effect of an adverse/favorable pressure gradient was once again observed, but the variation of pressure gradient has a small effect once the vortices are established. Further calculations including nonlinearity showed that an adverse pressure gradient leads to earlier saturation and higher saturated amplitude (Rogenski *et al.* 2016). To the best of our knowledge, no existing work has considered the receptivity of streaks or Görtler vortices to physically realisable FSVD in the presence of a pressure gradient despite the experimental evidences indicating clearly that FSVD excite Görtler vortices (Aihara & Sonoda 1981; Volino & Simon 1995; Kim, Simon & Russ 1992). It should be pointed out that although a Falkner-Skan solution may, with a suitably chosen Hartree parameter, be used as a proxy of a local flow profile of a boundary layer with a streamwise pressure gradient, it neither represents nor approximates any boundary layer globally. This is because its far field is unbounded, different from the uniform and bounded far field of a typical flow past an aerodynamics body. It is impossible to specify physically meaningful disturbances in the oncoming flow, and thus a Falkner-Skan boundary layer cannot be a useful vehicle for studying receptivity. Moreover, unless the Görtler number is large, a Falkner-Skan boundary layer is also ill suited for analysing linear Görtler instability because vortices have a long scale, and are dependent on the entire base flow.

1.3. Secondary instability

Nonlinear Görtler vortices cause strong distortions of the streamwise velocity profile in both the wall-normal and spanwise directions. The resulting inflectional velocity profile may be highly unstable to high-frequency disturbances, which is referred to as secondary

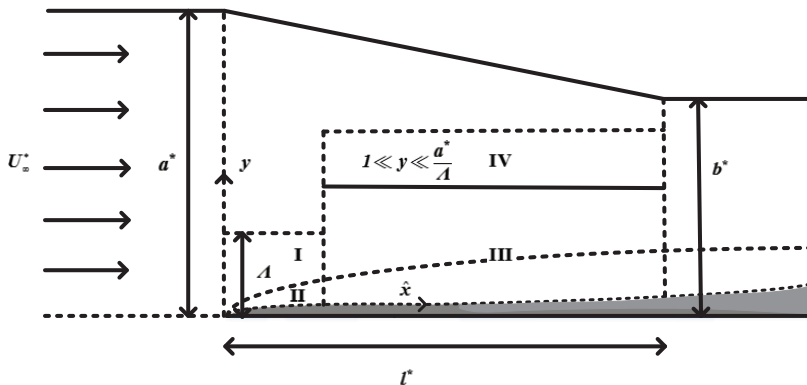


FIGURE 1. (Colour online) Schematic illustration of the physical problem and the associated asymptotic structure.

instability. Much effort has been made in the case of zero pressure gradient (Hall & Horseman (1991); Li & Malik (1995), XZW). The only study of pressure gradient effect on secondary instability of Görtler vortices is that of Aihara & Sonoda (1981). They found that the pressure gradient influences the secondary instability mainly through a change of the high shear layer where the secondary instability is amplified. The majority of experimental work focused primarily on correlating the transition location with the FST intensity on an empirical basis (Blair 1982). The influence of pressure gradient on the secondary instability of Görtler vortices remains to be investigated.

1.4. The aim of current work

In the present paper, we will study streaks and Görtler vortices in a boundary layer subjected to a streamwise pressure gradient, covering the aspects of excitation by high-intensity FSVD, nonlinear evolution and secondary instability of the vortices induced. To retain the key physics while keeping the problem mathematically tractable, we consider the boundary layer over a flat or curved plate inserted into a contracting or expanding channel. This setting is analogous to the use of a contoured ceiling to create a pressure gradient. In §2, we formulate the problem by describing the vortical disturbances in the oncoming flow and specifying the scaling relations. The flow is governed by an asymptotic structure consisting of four zones as is illustrated in figure 1. Two of those, referred to as regions I and IV, are outside the boundary layer, and at $O(\Lambda)$ and $O(\Lambda R_\Lambda)$ distances to the leading edge respectively. Different from the zero pressure gradient case, the inviscid base flow is now non-uniform. As a result, the vortical disturbances undergo distortion. The disturbance in these two regions is analysed to provide the appropriate boundary conditions. The response in region III, i.e. the boundary layer underneath region I, is obtained to provide the initial condition for the nonlinear boundary-region equations (NBRE) that govern the vortices induced in the boundary layer underneath region IV. In §3, the numerical procedure to solve NBRE is presented. Numerical results are presented in §4. The secondary instability of Görtler vortices is analysed in §5. Finally, a summary and conclusions are given in §6.

2. Formulation

We consider the incompressible boundary-layer flow over a concave plate with a characteristic radius of curvature r_0^* . The plate is placed in the middle of a contracting

or expanding stream between two contoured ceilings, which are often used to induce a pressure gradient in experiments (Abu-Ghannam & Shaw 1980; Gostelow *et al.* 1994). The ceilings far up- and down-stream of the contracting/expanding sector are parallel so that the oncoming flow is uniform with a speed U_∞ , superimposed on which is stationary, homogeneous turbulence. The flow is described in a curvilinear system $\mathbf{x} = (x_1, x_2, x_3) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, normalized by the spanwise integral scale of the turbulence, Λ , with x and y being along and normal to the plate respectively, and z along the span. The dimensionless time t and velocity $\mathbf{u}$ are introduced by

$$t = U_\infty t^* / \Lambda, \quad \mathbf{u} = \mathbf{u}^* / U_\infty, \quad (2.1)$$

where t^* and $\mathbf{u}^*$ are dimensional time and velocity respectively. Assume that free-stream turbulence consists of small-amplitude vortical fluctuations. Then the upstream velocity field can be written as

$$\mathbf{u} - \hat{\mathbf{i}} = \epsilon \mathbf{u}_\infty (x - t, y, z), \quad (2.2)$$

where $\epsilon \ll 1$ is a measure of the disturbance intensity. The Reynolds number based on Λ is defined as

$$R_\Lambda = U_\infty \Lambda / \nu, \quad (2.3)$$

where ν is the kinematic viscosity. Furthermore, we introduce the turbulent Reynolds number,

$$r_t = \epsilon R_\Lambda. \quad (2.4)$$

The global Görtler number G_Λ is defined as (Wu, Zhao & Luo 2011),

$$G_\Lambda = R_\Lambda^2 \Lambda / r_0^*. \quad (2.5)$$

The Reynolds number R_Λ is assumed asymptotically large, i.e. $R_\Lambda \gg 1$, but the Görtler number G_Λ is taken to be $O(1)$, so that the resulting formulation accommodates all regimes of Görtler instability.

The contraction ratio of the curved channel is denoted by

$$\sigma_c = \frac{a^*}{b^*} = \frac{U_-}{U_+}, \quad (2.6)$$

where a^* and b^* are the transverse dimensions of the upstream and downstream passages, and U_+ and U_- are the mean velocities there respectively (figure 1). The length of the nonparallel section is taken to be of $O(a^*)$, and the contraction ratio σ_c is of $O(1)$. The inviscid steady flow field $(\bar{U}, \bar{V})$ is given in appendix A. Upstream turbulence is assumed to have a small transverse length scale, namely $\Lambda/a^* \ll O(1)$ or $|\mathbf{k}^*|a^* \gg 1$, where $|\mathbf{k}^*| = (k_1^{*2} + k_2^{*2} + k_3^{*2})^{1/2}$ with $\mathbf{k}^*$ being a typical wavenumber vector of FSVD. The solution for the small disturbances in the inviscid flow can be determined by the rapid-distortion theory (Batchelor & Proudman 1954; Hunt 1973; Goldstein 1978).

In the present work, we focus on low-frequency, or long-streamwise-wavelength, components with $k_1^* a^* = O(1)$, and assume that $\Lambda/a^* = R_\Lambda^{-1}$ and $r_t = O(1)$, for which we introduce the slow time variable,

$$\hat{\tau} = t / R_\Lambda. \quad (2.7)$$

The flow divides itself into four asymptotic regions (figure 1). Over $O(\Lambda)$ distances surrounding the leading edge is an inviscid region I, in which the disturbance can be treated as a linear perturbation to the mean flow. The region II, which is beneath the linear region I, is a viscous boundary layer, where the perturbation is governed by the linearized unsteady boundary-layer equations. With the boundary-layer thickness growing with x , the solution becomes invalid at downstream distances $x \sim O(a^*/\Lambda) \sim$

254 $O(R_A)$, i.e. $\hat{x} \equiv x/R_A \sim O(1)$, since the spanwise and wall-normal viscous diffusion terms
 255 become comparable. A new solution in region III should be obtained for

$$\hat{x} = x/R_A = O(1). \quad (2.8)$$

256 The flow in this region is fully nonlinear and three-dimensional, and the streaks, which
 257 form as a response to the FSVD, take on the character of Görtler vortices. The streamwise
 258 velocity is greater than the normal and spanwise velocities by a factor of $O(R_A)$, while
 259 the pressure normalised by ρU_∞^2 is of $O(R_A^{-1})$ for the steady base flow, but of $O(R_A^{-2})$
 260 for the perturbation. Therefore, we can write the velocity and pressure fields, (u^*, v^*, w^*)
 261 and p^* , as

$$(u^*, v^*, w^*)/U_\infty = (u, R_A^{-1}v, R_A^{-1}w), \quad p^*/(\rho U_\infty^2) = R_A^{-1}(P_B + R_A^{-1}p), \quad (2.9)$$

262 where P_B is the steady mean pressure.

263 Substitution of (2.7)-(2.8) and (2.9) with the Lamé coefficients, $h_1 = (r_0^* - y^*)/r_0^*$,
 264 $h_2 = 1$ and $h_3 = 1$, into the Navier-Stokes (N-S) equations gives, at leading order, the
 265 equations for $\mathbf{u} \equiv (u, v, w)$ and p (Hall 1988),

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}_\tau + (\mathbf{u} \cdot \nabla)\mathbf{u} + G_A \chi_B u^2 \mathbf{j} = (-P_{B\hat{x}}, -p_y, -p_z) + (\partial_{yy}^2 + \partial_{zz}^2)\mathbf{u}, \quad (2.10)$$

266 which are the rigorous asymptotic limit of the N-S equations for $k_1 \ll k_3$. Here the term
 267 containing G_A reflects the essential influence of the wall curvature, and $\chi_B(\hat{x})$ is the
 268 scaled local radius of wall curvature.

269 The flow is decomposed as a sum of the base flow and the perturbation induced by
 270 FSVD, namely

$$\{u, v, w, p\} = \{U_B(\hat{x}, \eta), V_B(\hat{x}, \eta), 0, 0\} + (\hat{u}, \hat{v}, \hat{w}, \hat{p})(\hat{x}, \eta, z, t), \quad (2.11)$$

271 where the variable η is defined as

$$\eta = y/s \quad \text{with} \quad s = \sqrt{2\hat{x}/U_e}, \quad (2.12)$$

272 where $U_e(\hat{x})$ is the slip velocity given by the inviscid flow theory; see Appendix A.

273 The base flow is governed by the boundary-layer equations but there exist no similarity
 274 solutions because our slip velocity $U_e(\hat{x})$ is not one of the special cases $U_e(\hat{x}) = C\hat{x}^m$.
 275 Nevertheless, use of the variable η is convenient for specifying appropriate initial and
 276 boundary conditions, and presents a better numerical property for the computation of
 277 the base flow. Hence we rewrite the boundary-layer equations in the terms of $\hat{x}$ and η ,

$$\left. \begin{aligned} & \frac{\partial U_B}{\partial \hat{x}} + \eta \left(-\frac{1}{2\hat{x}} + \frac{U'_e}{2U_e} \right) \frac{\partial U_B}{\partial \eta} + \sqrt{\frac{U_e}{2\hat{x}}} \frac{\partial V_B}{\partial \eta} = 0, \\ & U_B \left[\frac{\partial U_B}{\partial \hat{x}} + \eta \left(-\frac{1}{2\hat{x}} + \frac{U'_e}{2U_e} \right) \frac{\partial U_B}{\partial \eta} \right] + V_B \sqrt{\frac{U_e}{2\hat{x}}} \frac{\partial U_B}{\partial \eta} = -P'_B + \frac{U_e}{2\hat{x}} \frac{\partial^2 U_B}{\partial \eta^2}, \end{aligned} \right\} \quad (2.13)$$

278 where the streamwise pressure gradient is given by $P'_B = -U_e U'_e$ with a prime denoting
 279 the differentiation with respect to $\hat{x}$.

280 In the outer region IV above region III, turbulence undergoes decay and distortion
 281 due to viscous effects and the strain field of the non-uniform inviscid background flow.
 282 The disturbance remains linear in all the regions if $r_t \ll 1$. However, for $r_t = O(1)$,
 283 nonlinearity comes into play in both regions III and IV. The Görtler vortices excited by
 284 FSVD evolve nonlinearly, saturate and may undergo secondary instability.

2.1. The velocity fluctuation at large downstream distances in region I.

The flow in this region is divided into two parts, the base flow and disturbance, and in the upstream limit, is of the form:

$$u_i(\mathbf{x}, t) = \hat{\mathbf{i}} + \epsilon \mathbf{u}_\infty(x - t, y, z) \quad \text{as } x \rightarrow -\infty. \quad (2.14)$$

The perturbation upstream $\mathbf{u}_\infty$ may be represented as a supposition of individual Fourier components. For simplicity, only a pair of free-stream vortical modes with opposite (dimensional) spanwise wavenumbers $\pm k_3^*$ is considered in the present work. The Fourier component of the upstream perturbation is

$$u_{\infty i}(\mathbf{x}, t) = u_{\infty i}(\mathbf{x} - \hat{\mathbf{i}}t) = \hat{u}_i^\infty e^{i\mathbf{k} \cdot (\mathbf{x} - \hat{\mathbf{i}}t)}, \quad (2.15)$$

where the Fourier amplitudes $\hat{u}_i^\infty = \{\hat{u}_{1,\pm}^\infty, \hat{u}_{2,\pm}^\infty, \hat{u}_{3,\pm}^\infty\} = O(1)$ are perpendicular to $\mathbf{k} = \{k_1, k_2, \pm k_3\}$, since from the continuity equation, it follows that

$$k_1 \hat{u}_{1,\pm}^\infty + k_2 \hat{u}_{2,\pm}^\infty \pm k_3 \hat{u}_{3,\pm}^\infty = 0.$$

The perturbation is highly ‘anisotropic’ in the three dimensions, but is assumed to be isotropic in the transverse directions, that is, $k_2 = k_3 = 1$. The spanwise amplitude $\hat{u}_{3,\pm}^\infty$ is taken ± 1 , from which it follows that $\hat{u}_{2,\pm}^\infty \approx -1$.

The solution for the unsteady perturbation can be sought by applying the generalized rapid-distortion theory (Goldstein 1978). This is facilitated by introducing the Darwin-Lighthill ‘drift function’,

$$\Delta(x, y) = x + \int_{-\infty}^x \left\{ \frac{1}{\bar{U}(x', y(x', \Psi))} - 1 \right\} dx', \quad (2.16)$$

where $\bar{U}$ is the streamwise velocity of the inviscid base flow, Ψ is related to the stream-function $\psi(\bar{x}, \bar{y}_0)$ via $\Psi(x, y) = (a^*/\Lambda)\psi(\bar{x}, \bar{y}_0)$, which ensures that $(\bar{U}, \bar{V}) = (\Psi_y, -\Psi_x) = (\psi_{\bar{y}_0}, -\psi_{\bar{x}})$; Here $\psi(\bar{x}, \bar{y}_0)$ is given in appendix A. The integration in (2.16) is performed along a fixed streamline, along which $\Psi(x, y)$ is a constant. Goldstein (1978) showed that the perturbation velocity fluctuation is given by

$$u_i(\mathbf{x}, t) = u_i^H(\mathbf{x}, t) + \frac{\partial \phi}{\partial x_i}(\mathbf{x}, t), \quad (2.17)$$

where

$$u_i^H(\mathbf{x}, t) = \hat{u}_j^\infty T_{j,i}(x, y) e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{\mathbf{i}}t)} \quad (2.18)$$

is the homogeneous solution to the momentum equations, linearized about a potential flow, and $\phi(\mathbf{x}, t)$ is the potential function, with $\mathbf{X} = (X_1, X_2, X_3) = (\Delta, \Psi, z)$ being the Lagrangian co-ordinates. In this paper, Einstein’s summation convention is adopted to any Latin suffix occurring twice in a single-term expression. In (2.18),

$$T_{j,i}(x, y) = \frac{\partial \mathbf{X}(\mathbf{x})}{\partial x_i} \quad (2.19)$$

denotes the distortion tensor of the local Lagrangian co-ordinate $(\mathbf{X} - \hat{\mathbf{i}}t)$.

The irrotational part of the fluctuation, $\phi(\mathbf{x}, t)$, can further be split into the gust part $\phi^g(\mathbf{x}, t)$ and the scattered part $\phi^s(\mathbf{x}, t)$ due to the presence of the plate, namely,

$$\phi(\mathbf{x}, t) = \phi^g(\mathbf{x}, t) + \phi^s(\mathbf{x}, t). \quad (2.20)$$

This gust solution is governed by the Poisson equation,

$$\frac{\partial^2 \phi^g}{\partial x_j^2}(\mathbf{x}, t) = -\frac{\partial u_j^H}{\partial x_j}(\mathbf{x}, t) \quad (j = 1, 2, 3). \quad (2.21)$$

Goldstein & Durbin (1980) (referred to as GD hereafter) showed that with $|\mathbf{k}|a^*/\Lambda \gg 1$, the solution for ϕ^g may be sought in the WKB form,

$$\phi^g(\mathbf{x}, t) = \frac{1}{|\mathbf{k}|^2} D(x, y) e^{i\mathbf{k} \cdot (\mathbf{X} - t)} + O\left(\frac{\Lambda}{|\mathbf{k}|a^*}\right), \quad (2.22)$$

where the amplitude function $D(x, y)$ is a slowly varying function of x and y . After substituting (2.18) and (2.22) into (2.21), $\frac{1}{|\mathbf{k}|^2} D(x, y)$ is obtained as

$$\frac{1}{|\mathbf{k}|^2} D(x, y) = i \frac{\hat{u}_j^\infty \chi_i}{\chi^2} T_{j,i}(x, y) + O\left(\frac{\Lambda}{|\mathbf{k}|a^*}\right), \quad (2.23)$$

where $\chi_i = \partial(\mathbf{k} \cdot \mathbf{X})/\partial x_i$ and $\chi = (\chi_1^2 + \chi_2^2 + \chi_3^2)^{1/2}$ (see appendix D in GD). Since the mean flow is uniform upstream, $u_{\infty i}(\mathbf{x}, t) = \delta_{i1}$ and $T_{j,i} \rightarrow \delta_{i,j}$, so that the amplitude $D(x, y)$ tends to zero upstream, namely,

$$\frac{1}{|\mathbf{k}|^2} D(x, y) \rightarrow i \hat{u}_j^\infty k_j = 0 \quad \text{as } x \rightarrow -\infty. \quad (2.24)$$

The boundary condition on the plate surface has not been applied to determine the WKB form solution, and therefore (2.23) just represents a local solution to be used away from the wall (Hunt 1973). The boundary condition at the surfaces of the channel can be ignored at leading order for high-wavenumber disturbances because its effect is confined within a region of $O(\Lambda/|\mathbf{k}|a^*)$ surrounding the channel surfaces, as was pointed out by Goldstein (1979). GD also showed that

$$\frac{\partial \phi^g}{\partial x_i}(\mathbf{x}, t) = i \frac{\partial}{\partial x_i} \left\{ \frac{\hat{u}_j^\infty \chi_l}{\chi^2} T_{j,l}(x, y) e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{t})} \right\} + O\left(\frac{\Lambda}{|\mathbf{k}|a^*}\right) \quad (2.25)$$

reduces to

$$\frac{\partial \phi^g}{\partial x_i}(\mathbf{x}, t) = -\hat{u}_j^\infty \frac{\chi_i \chi_l}{\chi^2} T_{j,l}(x, y) e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{t})} \quad (2.26)$$

to leading order in $O(\Lambda/|\mathbf{k}|a^*)$.

The sum of $u_i^H + \partial \phi^g / \partial x_i$ will be referred to as the velocity of the distorted vortical disturbance, and denoted by $u_i^d(\mathbf{x}, t)$. It follows from (2.18) and (2.26) that

$$u_i^d(\mathbf{x}, t) \equiv u_i^H + \partial \phi^g / \partial x_i = \hat{u}_j^\infty \left[T_{j,i}(x, y) - \frac{\chi_i \chi_l}{\chi^2} T_{j,l}(x, y) \right] e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{t})}. \quad (2.27)$$

The pressure associated with the gust distortion by the strain field is given by the formula, $p^d = -\frac{\mathcal{D}}{\mathcal{D}t} \phi^g$, which follows from the linearized wall-normal momentum equation and is valid for an irrotational base flow (Goldstein 1978). Use of (2.22) with (2.23) gives

$$p^d = -i \hat{u}_j^\infty (\chi_l / \chi^2) \bar{U}_i \frac{\partial T_{j,l}}{\partial x_i} e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{t})}. \quad (2.28)$$

The scattered component of the irrotational solution, ϕ^s , is governed by the boundary-value problem consisting of the Laplace equation,

$$\frac{\partial^2 \phi^s}{\partial x_j^2}(\mathbf{x}, t) = 0, \quad (2.29)$$

and the boundary condition on the plate surface,

$$\begin{cases} \frac{\partial \phi^s(x, 0, z, t)}{\partial y} = -\frac{\partial \phi^g(x, 0, z, t)}{\partial y} - u_j^H(x, 0, z, t) & \text{for } x > 0, \\ \phi^s(x, 0, z, t) = 0 & \text{for } x < 0. \end{cases} \quad (2.30)$$

The solution in the $x = O(1)$ region of the leading edge has to be obtained by solving the full boundary-value problem (2.29)–(2.30) using the Wiener-Hopf technique. Fortunately, for the present purpose of investigating how FSVD excite Görtler vortices or streaks, it suffices to focus on the asymptotic behaviour of the solution at large downstream distances. Substitution of the slow variable $\bar{x} = x\Lambda/a^*$ into (2.29) suggests that $\partial^2/\partial x^2$ drops out of the equation. Noting that ϕ^s is proportional to $e^{ik_3 z}$, (as are u_i^H and ϕ^g), imposing the downstream boundary condition in (2.30), we obtain the solution of (2.29),

$$\begin{aligned}\phi^s(x, y, z, t) &= \frac{1}{|k_3|} \left[u_2^H(x, 0, z, t) + \frac{\partial \phi^g(x, 0, z, t)}{\partial y} \right] e^{-|k_3|y} \\ &= \frac{\hat{u}_j^\infty}{|k_3|} \left[T_{j,2}(x, 0) - \frac{\chi_2 \chi_l}{\chi^2} T_{j,l}(x, 0) \right] e^{i[k_1 \Delta(x, 0) + k_2 \Psi(x, 0) + k_3 z - t] - |k_3|y}. \quad (2.31)\end{aligned}$$

At $y = 0$, the corresponding velocity field is given by

$$u_i^s = \frac{i\hat{u}_j^\infty}{|k_3|} \chi_i \left[T_{j,2}(x, 0) - \frac{\chi_2 \chi_l}{\chi^2} T_{j,l}(x, 0) \right] e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{\mathbf{t}})}|_{y=0} + O\left(\frac{\Lambda}{|\mathbf{k}|a^*}\right) \quad (i = 1, 3); \quad (2.32)$$

$$u_2^s = -i\hat{u}_j^\infty \left[T_{j,2}(x, 0) - \frac{\chi_2 \chi_l}{\chi^2} T_{j,l}(x, 0) \right] e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{\mathbf{t}})}|_{y=0} + O\left(\frac{\Lambda}{|\mathbf{k}|a^*}\right), \quad (2.33)$$

and the pressure associated with ϕ^s is

$$p^s = -\frac{i\hat{u}_j^\infty}{|k_3|} \bar{U}_i \frac{\partial}{\partial x_i} \left[T_{j,2}(x, 0) - \frac{\chi_2 \chi_l}{\chi^2} T_{j,l}(x, 0) \right] e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{\mathbf{t}})}|_{y=0}. \quad (2.34)$$

The velocity consists of three parts: the homogeneous part, the gust part and the scattered part, namely,

$$u_i(x, y, z, t) = u_i^H(x, y, z, t) + \frac{\partial \phi^g(x, y, z, t)}{\partial x_i} + \frac{\partial \phi^s(x, y, z, t)}{\partial x_i}. \quad (2.35)$$

Substituting (2.18), (2.26) and (2.31) into (2.35), we have $u_2(x, 0, z, t) = 0$, and

$$\begin{aligned}u_1(x, 0, z, t) &= \left[\hat{u}_j^\infty T_{j,1}(x, 0) - \hat{u}_j^\infty \frac{\chi_1 \chi_l}{\chi^2} T_{j,l}(x, 0) \right. \\ &\quad \left. + \frac{i\chi_1}{|\chi_3|} \left(\hat{u}_j^\infty T_{j,2}(x, 0) - \hat{u}_j^\infty \frac{\chi_2 \chi_l}{\chi^2} T_{j,l}(x, 0) \right) \right] e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{\mathbf{t}})}|_{y=0}, \quad (2.36)\end{aligned}$$

$$\begin{aligned}u_3(x, 0, z, t) &= \left[\hat{u}_j^\infty T_{j,3}(x, 0) - \hat{u}_j^\infty \frac{\chi_3 \chi_l}{\chi^2} T_{j,l}(x, 0) \right. \\ &\quad \left. + \frac{i\chi_3}{|\chi_3|} \left(\hat{u}_j^\infty T_{j,2}(x, 0) - \hat{u}_j^\infty \frac{\chi_2 \chi_l}{\chi^2} T_{j,l}(x, 0) \right) \right] e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{\mathbf{t}})}|_{y=0}. \quad (2.37)\end{aligned}$$

The disturbance in the main bulk of the inviscid region can be calculated, but a numerical effort is required to evaluate the drift function Δ and the associated quantities $T_{j,i}$ and χ_l . For the present purpose of investigating the excitation of Görtler vortices and streaks, what is required is only the behaviour of the disturbance in the region where $y = O(1)$, corresponding to $\hat{y} = y/R_\Lambda = O(R_\Lambda^{-1})$. The streamwise velocity of the inviscid base flow $\bar{U} = U_e(\hat{x}) + \partial_{\hat{y}} \bar{U}(\hat{x}, 0) \hat{y} + O(\hat{y}^2)$, and $\Psi \approx U_e(\hat{x})y$, and so

$$\frac{1}{\bar{U}} \approx \frac{1}{U_e(\hat{x})} + R_\Lambda^{-1} \lambda_1 \Psi + O(R_\Lambda^{-2} \Psi^2) \quad \text{with} \quad \lambda_1(\hat{x}) = \partial_{\hat{y}} \bar{U}(\hat{x}, 0)/U_e^3,$$

an approximation uniformly valid since $U_e(\hat{x}) \neq 0$ for any $\hat{x}$. It follows that

$$\Delta(x, y) = x + \int_{-\infty}^x \left[\frac{1}{U_e(x)} - 1 \right] dx + R_\Lambda^{-1} \left(U_e(x) \int_{-\infty}^x \lambda_1(x) dx \right) y + O(R_\Lambda^{-2} y^2). \quad (2.38)$$

The carrier-wave factor of the disturbance is

$$\mathbf{k} \cdot \mathbf{X} = k_1 \Delta + k_2 \Psi + k_3 z = (\theta_0 + \varphi_1) + U_e(\hat{x}) \left[(k_2 + k_1 \int_{-\infty}^{\hat{x}} \lambda_1(\hat{x}) d\hat{x}) y + k_3 z \right], \quad (2.39)$$

where

$$\theta_0 = \hat{k}_1 \int_{-\infty}^0 \left[\frac{1}{U_e(\hat{x})} - 1 \right] d\hat{x}, \quad \varphi_1 = \hat{k}_1 \int_0^{\hat{x}} \frac{1}{U_e(\hat{x})} d\hat{x}, \quad (2.40)$$

with $\hat{k}_1 = R_A k_1$. The result (2.39) indicates that the local streamwise and wall-normal wavenumbers of the vortical disturbance in a nonuniform flow are both modified. The correction to the wall-normal wavenumber is proportional to k_1 and hence negligible since $k_1 \ll 1$; it actually vanishes because $\lambda_1 = 0$ in the present symmetric setting. The streamwise and wall-normal wavenumbers, k_1/U_e and $k_2 U_e$, are reduced and increased, respectively if $U_e > 1$, and the converse is true for $U_e < 1$. The local streamwise phase speed of the vortical disturbance is $U_e(x)$, consistent with Taylor's hypothesis.

For the steady base flow under consideration, the solution (A 3) shows that $\bar{U} = U_e(\hat{x}, 0) + O(\hat{y}^2)$ as $\hat{y} \rightarrow 0$, and it follows that

$$\Delta(x, y) = x + \int_{-\infty}^x \left(\frac{1}{U_e} - 1 \right) dx + O(\hat{y}^2), \quad (2.41)$$

$$T_{j,i}(x, y) = \begin{pmatrix} 1/U_e & U_e' y & 0 \\ 0 & U_e & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(\hat{y}^2), \quad (2.42)$$

where $U_e' = R_A^{-1} \partial U_e / \partial \hat{x}$. Furthermore,

$$e^{i\mathbf{k} \cdot (\mathbf{X} - \hat{\mathbf{t}})}|_{y=0} = e^{i[k_1 \Delta(\hat{x}, 0) + k_3 z - t]}, \quad (2.43)$$

$$\chi_1 = k_1/U_e(\hat{x}), \quad \chi_2(\hat{x}) = k_2 U_e(\hat{x}), \quad \chi_3 = k_3. \quad (2.44)$$

The expressions for u_i^d and p^d are rather complicated, but for $k_1 \ll 1$, which is the case of interest, we find from (2.27) and (2.28) that

$$(v^d, w^d) = (-k_3/(k_2 U_e), 1) \hat{u}_3^\infty (k_2^2 + k_3^2) (U_e^2/\chi^2) e^{i[k_1 \Delta(\bar{x}, 0) + k_2 U_e y + k_3 z - t]}, \quad (2.45)$$

$$p^d = 2(U_e^2/\chi^2) U_e' (k_2^2 + k_3^2) i k_3 \hat{u}_3^\infty e^{i[k_1 \Delta(\bar{x}, 0) + k_2 U_e y + k_3 z - t]}. \quad (2.46)$$

Similarly, from (2.33) and (2.34) the velocity and pressure fields of the scattered part are obtained as

$$(v^s, w^s) = (i, 1)(-i(k_3/|k_2|))(k_2^2 + k_3^2) \hat{u}_3^\infty U_e/\chi^2 e^{i[k_1 \Delta(\bar{x}, 0) - |k_3| y + k_3 z - t]}, \quad (2.47)$$

$$p^s = -U_e \frac{\partial \phi^s}{\partial x} = k_2^{-1} (k_2^2 + k_3^2) (U_e^2 k_2^2 - k_3^2) U_e U_e' \hat{u}_3^\infty / \chi^4 e^{i[k_1 \Delta(\bar{x}, 0) - |k_3| y + k_3 z - t]}. \quad (2.48)$$

The total streamwise and spanwise slip velocities, (2.36) and (2.37), simplify to

$$\hat{u}_s(0) = \hat{u}_1^\infty / U_e + (i k_1 / |k_3|) (k_2^2 + k_3^2) u_2^\infty / \chi^2, \quad (2.49)$$

$$\hat{w}_s(0) = (k_2^2 + k_3^2) \left[U_e - i(|k_3|/k_2) \right] \hat{u}_3^\infty U_e / \chi^2. \quad (2.50)$$

In the absence of a pressure gradient $U_e(\hat{x}, 0) \equiv 1$, the streamwise and spanwise slip velocities are

$$\hat{u}_s(0) = \hat{u}_1^\infty + (i k_1 / |k_3|) \hat{u}_2^\infty, \quad \hat{w}_s(0) = \hat{u}_3^\infty + (i k_3 / |k_3|) \hat{u}_2^\infty, \quad (2.51)$$

which are same as those given by Leib, Wundrow & Goldstein (1999).

The streamwise and spanwise slip velocities in (2.49) and (2.50) are reduced to zero across the viscous region II, where the disturbance is governed by the quasi-steady boundary-layer equations. An $O(\epsilon)$ response is driven by the streamwise and spanwise slip velocities. However, as in Leib, Wundrow & Goldstein (1999) and Wu, Zhao & Luo (2011), the response to the streamwise velocity $\hat{u}_s(0)$ is quasi two-dimensional, and remains bounded. Hence it is of no concern to us. Only the three-dimensional signature driven by the spanwise velocity $\hat{w}_s(0)$ develops into larger-amplitude streaks and eventually to Görtler vortices further downstream.

2.2. The inner region: nonlinear unsteady Görtler vortices

In the boundary region III corresponding to $\hat{x} = O(1)$ and $y = O(1)$, the disturbance is governed by the NBRE, which follow from substitution of (2.11) with (2.7) and (2.8) into (2.10) as

$$\left. \begin{aligned} \hat{u}_{\hat{x}} + \hat{v}_y + \hat{w}_z &= 0, \\ \hat{u}_{\hat{\tau}} + U_B \hat{u}_{\hat{x}} + \hat{u} U_{B\hat{x}} + \hat{v} U_{By} + V_B \hat{u}_y &= \hat{u}_{yy} + \hat{u}_{zz} + Q_1, \\ \hat{v}_{\hat{\tau}} + U_B \hat{v}_{\hat{x}} + \hat{u} V_{B\hat{x}} + V_B \hat{v}_y + \hat{v} V_{By} + 2G_{\Lambda} \chi_B U_B \hat{u} &= -\hat{p}_y + \hat{v}_{yy} + \hat{v}_{zz} + Q_2, \\ \hat{w}_{\hat{\tau}} + U_B \hat{w}_{\hat{x}} + V_B \hat{w}_y &= -\hat{p}_z + \hat{w}_{yy} + \hat{w}_{zz} + Q_3, \end{aligned} \right\} \quad (2.52)$$

where Q_1 , Q_2 and Q_3 represent the components in the nonlinear terms $-(\hat{\mathbf{u}} \cdot \nabla) \hat{\mathbf{u}} - G_{\Lambda} \chi_B \hat{u}^2 \mathbf{j}$.

In the present nonlinear regime, the disturbance consists of all harmonics and can be expressed as

$$(\hat{u}, \hat{v}, \hat{w}, \hat{p}) = r_t \sum_{m,n=-\infty}^{+\infty} \left(s^2 \hat{u}_{m,n}(\hat{x}, \eta), s \hat{v}_{m,n}(\hat{x}, \eta), \hat{w}_{m,n}(\hat{x}, \eta)/k_3, \hat{p}_{m,n}(\hat{x}, \eta) \right) E_{mn}. \quad (2.53)$$

where $E_{mn} = \exp(-im\hat{k}_1\hat{\tau} + ink_3z)$, $s = \sqrt{2\hat{x}/U_e}$ and $\hat{k}_1 = k_1 R_{\Lambda} = O(1)$. The factor s^2 in the streamwise velocity is introduced to offset the small divisor in numerical computations (Ricco *et al.* 2011). As the physical quantities are real, the Fourier coefficients are Hermitian, $\hat{q}_{-m,-n} = (\hat{q}_{m,n})_{cc}$, where $\hat{q}$ stands for any of $\{\hat{u}_{m,n}, \hat{v}_{m,n}, \hat{w}_{m,n}, \hat{p}_{m,n}\}$, and the subscript *cc* indicates the complex conjugate. Substituting (2.53) into (2.52) and using (2.12), we obtain the equations for the Fourier coefficients as follows.

The continuity equation:

$$2\mathcal{B}_v \hat{u}_{m,n} + s^2 \frac{\partial \hat{u}_{m,n}}{\partial \hat{x}} - \mathcal{B}_v \eta \frac{\partial \hat{u}_{m,n}}{\partial \eta} + \frac{\partial \hat{v}_{m,n}}{\partial \eta} + in \hat{w}_{m,n} = 0. \quad (2.54)$$

The momentum equations:

$$\begin{aligned} & \left[s^2(-im\hat{k}_1 + n^2 k_3^2) + 2\mathcal{B}_v U_B + s^2 U_{B\hat{x}} - \mathcal{B}_v \eta U_{B\eta} \right] \hat{u}_{m,n} + U_B s^2 \frac{\partial \hat{u}_{m,n}}{\partial \hat{x}} \\ & + (-\mathcal{B}_v \eta U_B + s V_B) \frac{\partial \hat{u}_{m,n}}{\partial \eta} - \frac{\partial^2 \hat{u}_{m,n}}{\partial \eta^2} + U_{B\eta} \hat{v}_{m,n} = -r_t s^2 \hat{l}_{m,n}, \end{aligned} \quad (2.55)$$

$$\begin{aligned} & (s^3 V_{B\hat{x}} - s \mathcal{B}_v \eta V_{B\eta} + 2s^3 G_{\Lambda} \chi_B U_B) \hat{u}_{m,n} + \left[s^2(-im\hat{k}_1 + n^2 k_3^2) + \mathcal{B}_v U_B + s V_{B\eta} \right] \hat{v}_{m,n} \\ & + s^2 U_B \frac{\partial \hat{v}_{m,n}}{\partial \hat{x}} + (s V_B - \mathcal{B}_v \eta U_B) \frac{\partial \hat{v}_{m,n}}{\partial \eta} - \frac{\partial^2 \hat{v}_{m,n}}{\partial \eta^2} + \frac{\partial \hat{p}_{m,n}}{\partial \eta} = -r_t s^2 \hat{e}_{m,n}, \end{aligned} \quad (2.56)$$

$$s^2(-im\hat{k}_1 + n^2k_3^2)\hat{w}_{m,n} + s^2U_B\frac{\partial\hat{w}_{m,n}}{\partial\hat{x}} + (-\mathcal{B}_v\eta U_B + sV_B)\frac{\partial\hat{w}_{m,n}}{\partial\eta} - \frac{\partial^2\hat{w}_{m,n}}{\partial\eta^2} + s^2(in\hat{k}_3^2)\hat{p}_{m,n} = -r_t s^2\hat{h}_{m,n}, \quad (2.57)$$

where $\mathcal{B}_v = (U_e - \hat{x}U_e')/U_e^2$, with a prime denoting the differentiation with respect to $\hat{x}$. In the above equations, we have put

$$\hat{l}_{m,n} = \left\{ s^2 \frac{\partial(\widehat{u_a u_a})}{\partial\hat{x}} + 4\mathcal{B}_v \widehat{u_a u_a} - \mathcal{B}_v \eta \frac{\partial(\widehat{u_a u_a})}{\partial\eta} + \frac{\partial(\widehat{u_a v_a})}{\partial\eta} + \frac{1}{k_3} \frac{\partial(\widehat{u_a w_a})}{\partial z} \right\}_{m,n}, \quad (2.58)$$

$$\hat{e}_{m,n} = \left\{ s^2 \frac{\partial(\widehat{v_a v_a})}{\partial\hat{x}} + 3\mathcal{B}_v \widehat{v_a v_a} - \mathcal{B}_v \eta \frac{\partial(\widehat{v_a v_a})}{\partial\eta} + \frac{\partial(\widehat{v_a v_a})}{\partial\eta} + \frac{1}{k_3} \frac{\partial(\widehat{v_a w_a})}{\partial z} + s^3 G_{\Lambda} \chi_B \widehat{u_a^2} \right\}_{m,n}, \quad (2.59)$$

$$\hat{h}_{m,n} = \left\{ s^2 \frac{\partial(\widehat{w_a u_a})}{\partial\hat{x}} + 2\mathcal{B}_v \widehat{w_a u_a} - \mathcal{B}_v \eta \frac{\partial(\widehat{w_a u_a})}{\partial\eta} + \frac{\partial(\widehat{w_a v_a})}{\partial\eta} + \frac{1}{k_3} \frac{\partial(\widehat{w_a w_a})}{\partial z} \right\}_{m,n}, \quad (2.60)$$

where the $\widehat{}$ symbol denotes the double Fourier transform with respect to $\hat{\tau}$ and z , and for convenience of computation the new dependent variables (u_a, v_a, w_a) are introduced,

$$(u_a, v_a, w_a) = \sum_{m,n=-\infty}^{+\infty} \left(\hat{u}_{m,n}(\hat{x}, \eta), \hat{v}_{m,n}(\hat{x}, \eta), \hat{w}_{m,n}(\hat{x}, \eta) \right) \exp(-im\hat{k}_1\hat{\tau} + in\hat{k}_3z). \quad (2.61)$$

The viscous motion in region III is to influence the outer region IV through the displacement effect. By taking the spanwise average of the continuity equation in (2.13) and (2.52), and integrating with respect to y , it can be shown that the normal velocity has the asymptote

$$\bar{V} = V_B + \hat{v} \rightarrow \bar{\delta}_{\hat{x}}(\hat{x}, \hat{\tau}) \quad \text{as } y \rightarrow \infty, \quad (2.62)$$

where $\bar{\delta}$ is the spanwise averaged displacement thickness,

$$\bar{\delta}(\hat{x}, \hat{\tau}) = \frac{k_3}{2\pi} \int_0^{2\pi/k_3} \int_0^\infty \left[1 - \frac{u(y, z, \hat{x}, \hat{\tau})}{U_e(\hat{x})} \right] dy dz. \quad (2.63)$$

2.3. Disturbances at the outer edge of the boundary layer

As was shown by Ricco, Luo & Wu (2011) and XZW, the perturbation in the outer region IV is composed of two parts: the two-dimensional disturbance $\bar{\mathbf{u}} = (\bar{u}_0, \bar{v}_0)$ and $\bar{p}_0$, induced by the viscous motion within the boundary layer through the displacement (2.63), and the three-dimensional disturbance convected from upstream. Considering that the outer region corresponds to $x = O(a^*/\Lambda) \sim R_\Lambda$ and $1 \ll y \ll O(a^*/\Lambda)$, the disturbances are governed by the boundary-region equations (2.52) in the limit $y \rightarrow \infty$. The flow field in this region can be decomposed as

$$\left. \begin{aligned} u &= U_e(\hat{x}) + \epsilon[\bar{u}_0(\hat{x}, \hat{y}, \hat{\tau}) + \hat{u}_0(\hat{x}, y, z, \hat{\tau})] + \dots, \\ v &= -U_e'(\hat{x})y + \epsilon R_\Lambda[\bar{v}_0(\hat{x}, \hat{y}, \hat{\tau}) + \hat{v}_0(\hat{x}, y, z, \hat{\tau})] + \dots, \\ w &= \epsilon R_\Lambda \hat{w}_0(\hat{x}, y, z, \hat{\tau}) + \dots, \\ p &= -\frac{1}{2} + \epsilon R_\Lambda^2[\bar{p}_0(\hat{x}, y, \hat{\tau}) + \epsilon \hat{p}_1(\hat{x}, y, z, \hat{\tau})] + \dots, \end{aligned} \right\} \quad (2.64)$$

where we have introduced the far field variable

$$\hat{y} = y/R_\Lambda. \quad (2.65)$$

Here the factor R_Λ in v and w , and R_Λ^2 in p , arise in order to undo the normalization introduced in (2.9). The terms $\bar{u}_0$, $\bar{v}_0$ and $\bar{p}_0$ represent the two-dimensional part and are governed by the linearised unsteady Euler equations,

$$\bar{u}_{0\hat{x}} + \bar{v}_{0\hat{y}} = 0, \quad \bar{u}_{0\hat{\tau}} + U_e \bar{u}_{0\hat{x}} = -\bar{p}_{0\hat{x}}, \quad \bar{v}_{0\hat{\tau}} + U_e \bar{v}_{0\hat{x}} = -\bar{p}_{0\hat{y}}, \quad (2.66)$$

subject to the boundary condition,

$$(\bar{u}_0, \bar{v}_0) \rightarrow (0, 0) \quad \text{as} \quad \hat{x}, \hat{y} \rightarrow \infty. \quad (2.67)$$

The other boundary condition follows from matching $\epsilon R_\Lambda \bar{v}_0$ with (2.62), the spanwise averaged normal velocity at the outer edge of the boundary layer, and it reads

$$\bar{v}_0 = (1/\epsilon R_\Lambda) \bar{\delta}_{\hat{x}}(\hat{x}, \hat{\tau}) \quad \text{at} \quad \hat{y} = 0. \quad (2.68)$$

Substitution of (2.64) with (2.65) into the N-S equations shows that $\bar{v}_0$ appears in the governing equations of the three-dimensional component $(\hat{u}_0, \hat{v}_0, \hat{w}_0, \hat{p}_1)$, indicating that the latter is also influenced by the viscous displacement effect. The coupling with $\bar{v}_0$ can be removed by the generalized Prandtl transformation,

$$\bar{y} = U_e(\hat{x})y - \hat{\delta}(\hat{x}, \hat{\tau}), \quad (2.69)$$

where $\hat{\delta}$ is to be chosen. Note that the coordinate is stretched as well as being translated, unlike the conventional Prandtl transformation, which involves just translation. By using (2.69) and the linearised Euler equations (2.66), the governing equations for $(\hat{v}_0, \hat{w}_0)$ and $\hat{p}_1$ can be written as

$$U_e \hat{v}_{0\bar{y}} + \hat{w}_{0z} = 0, \quad (2.70)$$

$$\mathcal{L} \begin{pmatrix} \hat{v}_0 \\ \hat{w}_0 \end{pmatrix} - \begin{pmatrix} U'_e \hat{v}_0 \\ 0 \end{pmatrix} = -r_t \begin{pmatrix} U_e \hat{p}_{1\bar{y}} \\ \hat{p}_{1z} \end{pmatrix} + \nabla^2 \begin{pmatrix} \hat{v}_0 \\ \hat{w}_0 \end{pmatrix}, \quad (2.71)$$

where $\nabla^2 = U_e^2 \partial_{\bar{y}}^2 + \partial_z^2$, and $\mathcal{L}$ is the nonlinear differential operator,

$$\mathcal{L} = \partial_{\hat{\tau}} + U_e \partial_{\hat{x}} + r_t (U_e \hat{v}_0 \partial_{\bar{y}} + \hat{w}_0 \partial_z) + (-\hat{\delta}_{\hat{\tau}} - U_e \hat{\delta}_{\hat{x}} + U_e \bar{\delta}_{\hat{x}}(\hat{x}, \hat{\tau})) \partial_{\bar{y}}. \quad (2.72)$$

Note that the fast variable y , or $\bar{y}$ defined in (2.69), describes the phase variation of FSVD, and so its presence should not obscure the fact that we are presently considering the far field, where the base flow is non-uniform; see the expansion (2.64). As (2.72) indicates, the dependence on $\bar{v}_0$ is removed if we choose $\hat{\delta}$ to satisfy the equation and the ‘boundary condition’,

$$\hat{\delta}_{\hat{\tau}} + U_e \hat{\delta}_{\hat{x}} = U_e \bar{\delta}_{\hat{x}}(\hat{x}, \hat{\tau}), \quad \hat{\delta}(0, \hat{\tau}) = 0 \quad \text{for all} \quad \hat{\tau} > 0. \quad (2.73)$$

Let $\bar{\delta} = \sum \bar{\delta}_m(\hat{x}) \exp[imk_1(\Delta(x, 0) - \hat{\tau})]$. Then the solution for $\hat{\delta}$ can be written as

$$\hat{\delta}(\hat{x}, \hat{\tau}) = \sum \hat{\delta}_m(\hat{x}) \exp[imk_1(\Delta(x, 0) - \hat{\tau})] \quad \text{with} \quad \hat{\delta}_m = \bar{\delta}_m + im \int_0^{\hat{x}} \bar{\delta}_m(\hat{x}) d\hat{x}. \quad (2.74)$$

In order to seek the solution to (2.71), we introduce the stream function $\hat{\psi}(\bar{y}, z)$ such that $\hat{v}_0 = \hat{\psi}_z / U_e$ and $\hat{w} = -\hat{\psi}_{\bar{y}}$. Then the wall-normal and spanwise momentum equations are simplified to a transport equation for the longitudinal vorticity $\nabla^2 \hat{\psi} = (U_e^2 \partial_{\bar{y}}^2 + \partial_z^2) \hat{\psi}$,

$$(\partial_{\hat{\tau}} + U_e \partial_{\hat{x}}) \nabla^2 \hat{\psi} - 2U'_e \nabla^2 \hat{\psi} - \nabla^4 \hat{\psi} = r_t (\hat{\psi}_{\bar{y}} \partial_z - \hat{\psi}_z \partial_{\bar{y}}) \nabla^2 \hat{\psi} \equiv r_t N(\hat{x}, \bar{y}, z, \hat{\tau}), \quad (2.75)$$

where $2U'_e \nabla^2 \hat{\psi}$ represents the effect of vortex stretching, which is absent for the zero-pressure-gradient boundary layer. The pressure is found to satisfy the equation

$$\nabla^2 \hat{p}_1 = -2\hat{\psi}_{\bar{y}z}^2 + 2\hat{\psi}_{\bar{y}\bar{y}} \hat{\psi}_{zz} + 2(U'_e(\hat{x})/r_t) \hat{\psi}_{\bar{y}z} \equiv S + (2U'_e(\hat{x})/r_t) \hat{\psi}_{\bar{y}z}. \quad (2.76)$$

Equation (2.75) can be solved using a spectral method, where the solution is written as

$$\hat{\psi} = \sum_{m,n,j} \hat{\psi}_{m,n}^{(j)}(\hat{x}) e^{-im\hat{k}_1\hat{\tau} + i(jk_2\bar{y} + nk_3z)}. \quad (2.77)$$

Inserting (2.77) into (2.75) leads to a system of coupled equations for $\hat{\psi}_{m,n}^{(j)}$,

$$\left[U_e \frac{\partial}{\partial \hat{x}} - im\hat{k}_1 - \frac{2U'_e n^2 k_3^2}{n^2 k_3^2 + U_e^2 j^2 k_2^2} + (n^2 k_3^2 + U_e^2 j^2 k_2^2) \right] \hat{\psi}_{m,n}^{(j)}(\hat{x}) = -\frac{r_t \hat{N}_{m,n}^j(\hat{x})}{n^2 k_3^2 + U_e^2 j^2 k_2^2}, \quad (2.78)$$

where $\hat{N}_{m,n}^j$ denotes the Fourier transform of the nonlinear term $N(\hat{x}, \hat{y}, z, \hat{\tau})$. The initial condition for $\hat{\psi}_{m,n}^{(j)}$ follows from matching with (2.45), the distorted disturbance upstream,

$$\hat{\psi}_{m,n}^{(j)} \rightarrow i(jk_2)^{-1} \left[(nk_3)^2 + (jk_2)^2 \right] U_e^2 / (n^2 k_3^2 + j^2 k_2^2 U_e^2) \hat{u}_3^\infty(jk_2, nk_3) \text{ as } \hat{x} \rightarrow 0. \quad (2.79)$$

The solution for the pressure can be expressed as

$$\hat{p}_1 = \sum_{m,n,j} \hat{p}_{m,n}^{(j)}(\hat{x}) e^{-im\hat{k}_1\hat{\tau} + i(jk_2\bar{y} + nk_3z)} \quad \text{with} \quad \hat{p}_{m,n}^{(j)}(\hat{x}) = \frac{\hat{S}_{m,n}^j + (2U'_e/r_t) j n k_2 k_3 \hat{\psi}_{m,n}^j}{U_e^2 j^2 k_2^2 + n^2 k_3^2}, \quad (2.80)$$

where $\hat{S}_{m,n}^j$ denotes the Fourier transform of the nonlinear term S in (2.76).

In order to facilitate the matching with the solution in the boundary layer, the generalised Prandtl transformation (2.69) is applied to (2.77) and (2.80), and we rewrite

$$\sum_m \left(\hat{\psi}_{m,n}^{(j)}(\hat{x}), \hat{p}_{m,n}^{(j)}(\hat{x}) \right) e^{-im\hat{k}_1\hat{\tau} - ijk_2\hat{\delta}(\hat{x}, \hat{\tau})} = \sum_m \left(\phi_{m,n}^{(j)}(\hat{x}), \pi_{m,n}^{(j)}(\hat{x}) \right) e^{-im\hat{k}_1\hat{\tau}}, \quad (2.81)$$

where the operations involved amount to Fourier transform, at each $\hat{x}$, the left-hand side with respect to $\hat{\tau}$ for all n and j . In terms of $(\phi_{m,n}^{(j)}, \pi_{m,n}^{(j)})$, the solution for $\hat{v}_0$, $\hat{w}_0$ and $\hat{p}_1$ can be written as

$$(\hat{v}_0, \hat{w}_0, \hat{p}_1) = \sum_{m,n} \left(\hat{v}_{m,n}^\dagger(\hat{x}, y), \hat{w}_{m,n}^\dagger(\hat{x}, y), \hat{p}_{m,n}^\dagger(\hat{x}, y) \right) e^{-im\hat{k}_1\hat{\tau} + ink_3z}, \quad (2.82)$$

where

$$(\hat{v}_{m,n}^\dagger, \hat{w}_{m,n}^\dagger) = i \sum_j (nk_3/U_e, -jk_2) \phi_{m,n}^{(j)}(\hat{x}) e^{ij k_2 U_e y}, \quad \hat{p}_{m,n}^\dagger = \sum_j \pi_{m,n}^{(j)}(\hat{x}) e^{ij k_2 U_e y}, \quad (2.83)$$

with y being related to η via $y = \sqrt{2\hat{x}/U_e} \eta = s\eta$. The flow field in region III is decomposed as (2.53), and so the matching condition becomes

$$(\hat{u}_{m,n}, \hat{v}_{m,n}, \hat{w}_{m,n}, \hat{p}_{m,n}) \rightarrow (0, s^{-1} \hat{v}_{m,n}^\dagger, k_3 \hat{w}_{m,n}^\dagger, \epsilon R_A \hat{p}_{m,n}^\dagger) \text{ as } \eta \rightarrow \infty. \quad (2.84)$$

It is worth pointing out that the effect of the interaction between the boundary-region III and the outer region IV through the displacement $\hat{\delta}$ (see (2.63)) is accounted for by the transformation $(\hat{\psi}_{m,n}^{(j)}, \hat{p}_{m,n}^{(j)}) \rightarrow (\phi_{m,n}^{(j)}, \pi_{m,n}^{(j)})$, defined in (2.81).

Equation (2.78) has to be solved numerically in general. An exact solution can be obtained for the special case where the oncoming FSVD consist of a pair of oblique components with the same frequency and amplitude, but opposite spanwise wavenumbers $\pm k_3$. In this case, the nonlinear terms in (2.78) vanish, and so $\hat{\psi}_{m,n}^{(j)} = 0$ except the seeded components, $\hat{\psi}_{1,\pm 1}^{(1)}$, which satisfy

$$\left\{ U_e (U_e^2 k_2^2 + k_3^2) \frac{\partial}{\partial \hat{x}} + \left[-i\hat{k}_1 (U_e^2 k_2^2 + k_3^2) - 2U'_e k_3^2 + (U_e^2 k_2^2 + k_3^2)^2 \right] \right\} \hat{\psi}_{1,\pm 1}^{(1)} = 0. \quad (2.85)$$

The solution satisfies the initial condition (2.79) is found as

$$\hat{\psi}_{1,\pm 1}^{(1)}(\hat{x}) = \mp i c_\infty (U_e^2/\chi^2) A_v e^{i\varphi_1},$$

where $c_\infty = -\hat{u}_3^\infty (k_2^2 + k_3^2)/k_2$ with $\hat{u}_3^\infty = \hat{u}_{3+}^\infty = -\hat{u}_{3-}^\infty$ being the amplitude of the spanwise velocity of the FSVD, φ_1 is defined in (2.40), and we also introduced the notation

$$\mathcal{A}_v = \exp \left\{ - \int_0^{\hat{x}} \frac{1}{U_e} (U_e^2 k_2^2 + k_3^2) d\hat{x} \right\}. \quad (2.86)$$

It follows that

$$\hat{\psi} = 2c_\infty (U_e^2/\chi^2) \sin(k_3 z) A_v \exp(i\varphi_1 + ik_2 \bar{y}) + c.c., \quad (2.87)$$

and

$$(\hat{v}_0, \hat{w}_0) = \left(2k_3/U_e \cos(k_3 z), -2ik_2 \sin(k_3 z) \right) \hat{c}_\infty \mathcal{A}_v e^{i(\varphi_1 + k_2 \bar{y})} + c.c., \quad (2.88)$$

where we have put $\hat{c}_\infty = c_\infty U_e^2/\chi^2$. Then rewriting $\hat{v}_0, \hat{w}_0$ and $\hat{p}_1$ in terms of y by using (2.69), we find that for $1 \ll y \ll R_A$,

$$(\hat{v}_0, \hat{w}_0, \hat{p}_1) = \sum_{m,n} (\hat{v}_{m,n}^\dagger, \hat{w}_{m,n}^\dagger, \hat{p}_{m,n}^\dagger) \exp(-im\hat{k}_1 \hat{\tau} + ik_3 n z), \quad (2.89)$$

where

$$\left. \begin{aligned} \hat{v}_{m,\pm 1}^\dagger &= (k_3/U_e) \hat{c}_\infty \mathcal{A}_v \left[e^{i(\varphi_1 + k_2 U_e y)} \phi_m + e^{-i(\varphi_1 + k_2 U_e y)} \phi_{-m}^* \right], \\ \hat{w}_{m,\pm 1}^\dagger &= \mp k_2 \hat{c}_\infty \mathcal{A}_v \left[e^{i(\varphi_1 + k_2 U_e y)} \phi_m - e^{-i(\varphi_1 + k_2 U_e y)} \phi_{-m}^* \right], \\ \hat{p}_{m,\pm 1}^\dagger &= \frac{1}{r_t} \hat{p}_\infty \mathcal{A}_v \left[e^{i(\varphi_1 + k_2 U_e y)} \phi_m + e^{-i(\varphi_1 + k_2 U_e y)} \phi_{-m}^* \right], \\ \hat{p}_{m,0}^\dagger &= -2(k_3/U_e)^2 \hat{c}_\infty^2 \mathcal{A}_v^2 \left[e^{2i(\varphi_1 + k_2 U_e y)} \pi_m + e^{-2i(\varphi_1 + k_2 U_e y)} \pi_{-m}^* \right], \\ \hat{p}_{0,\pm 2}^\dagger &= 2k_2^2 \hat{c}_\infty^2 \mathcal{A}_v^2, \end{aligned} \right\} \quad (2.90)$$

where $\hat{p}_\infty = -2ik_2 k_3 U_e' \hat{c}_\infty$, and ϕ_m and π_m are computed according to

$$e^{i\hat{k}_1 \hat{\tau} - ik_2 \hat{\delta}(\hat{x}, \hat{\tau})} = \sum_m \phi_m(\hat{x}) e^{-im\hat{k}_1 \hat{\tau}}, \quad e^{2i\hat{k}_1 \hat{\tau} - 2ik_2 \hat{\delta}(\hat{x}, \hat{\tau})} = \sum_m \pi_m(\hat{x}) e^{-im\hat{k}_1 \hat{\tau}}$$

with $\phi_m = \phi_{m,1}^{(1)}$ and $\pi_m = \pi_{m,1}^{(1)}$. All other components $\hat{v}_{m,n}^\dagger = \hat{w}_{m,n}^\dagger = 0$ ($n \neq \pm 1$), and $\hat{p}_{m,n}^\dagger = 0$ ($n \neq 0, \pm 1, \pm 2$).

2.4. Initial conditions

The nonlinear boundary-region equations (2.54)-(2.57) must be solved subject to appropriate outer boundary and upstream conditions that characterise the impact of FSVD on the boundary layer. The former conditions are provided in §2.3 by the larger- η asymptotic solution that matches with the solution in region IV. In this section, we shall derive the upstream (or initial) conditions by considering the small- $\hat{x}$ limits of the boundary-region equations (2.52) in the regions corresponding to $\bar{y} = O(1)$ and $\eta \gg O(1)$.

When $R_A^{-1/2} \ll \hat{x} \ll 1$ and $\bar{y} = O(1)$, $\hat{u} \rightarrow 0$, and the nonlinear terms in (2.52), or equivalently in (2.71), become negligible so that the disturbance is governed by the linearized version of (2.70)-(2.71), and consists primarily of the seeded fundamental components $(1, \pm 1)$, each comprising of two parts: the disturbance advected/distorted by the inhomogeneous flow and the complementary part induced by the presence of the

492 wall as well as by the viscous motion in boundary layer. The wall-normal and spanwise
 493 velocities of the former is given by (2.88), which may be rewritten as

$$\left(v_{1,\pm 1}^d, w_{1,\pm 1}^d\right) = \left(\mp |k_3|/(k_2 U_e), 1\right)(k_2^2 + k_3^2)(U_e^2/\chi^2)\hat{u}_{3,\pm}^\infty \mathcal{A}_v e^{i(\varphi_1 + k_2 \bar{y})}, \quad (2.91)$$

494 and the associated pressure is

$$(\hat{p}_1^d)_{1,\pm} = \pm(2U_e'/r_t)\mathbf{i}|k_3|(k_2^2 + k_3^2)(U_e^2/\chi^4)u_3^\infty \mathcal{A}_v e^{i(\varphi_1 + k_2 \bar{y})}. \quad (2.92)$$

495 The streamfunction corresponding to the velocities (2.91) is found as

$$\hat{\psi}_{1,\pm 1}^d = \mathbf{i}k_2^{-1}(k_2^2 + k_3^2)(U_e/\chi^2)\hat{u}_{3,\pm}^\infty \mathcal{A}_v e^{i(\varphi_1 + k_2 \bar{y})}.$$

496 In order to match with the solution in the boundary layer as well as with the solution
 497 upstream, the complementary solution to (2.70)-(2.71), which attenuates exponentially
 498 in the wall-normal direction, has also to be included. The velocities for this part of the
 499 solution are denoted by $v_{1,\pm 1}^c$ and $w_{1,\pm 1}^c$, and they take the form,

$$\left(v_{1,\pm 1}^c, w_{1,\pm 1}^c\right) = b(\hat{x})\left(1, \mp \mathbf{i}\right)e^{i\varphi_1 - (|k_3|/U)\bar{y}}, \quad (2.93)$$

500 and the corresponding the pressure has the expression,

$$p^c = (U_e'/U_e)b(\hat{x})\bar{y}e^{i\varphi_1 - (k_3/U)\bar{y}} + \bar{g}(\hat{x})e^{i\varphi_1 - (k_3/U)\bar{y}}, \quad (2.94)$$

501 where $b(\hat{x})$ and $\bar{g}(\hat{x})$ are functions of $\hat{x}$ yet to be determined. Substitution of (2.93)- (2.94)
 502 into the linearized version of (2.70)-(2.71) shows that they are satisfied provided that

$$\bar{g}(\hat{x}) = Ub'/|k_3|. \quad (2.95)$$

503 The above result is in fact a displacement-pressure relation, showing how the displace-
 504 ment induces the pressure.

505 It follows from (2.91)-(2.92) and (2.93)-(2.94) that the total velocities and pressure of
 506 the disturbance are given by

$$\hat{v}_{1,\pm 1} = \mp(|k_3|/k_2)(k_2^2 + k_3^2)(U_e/\chi^2)\hat{u}_3^\infty \mathcal{A}_v e^{i(\varphi_1 + k_2 \bar{y})} + b(\hat{x})e^{i\varphi_1 - (|k_3|/U_e)\bar{y}}, \quad (2.96)$$

$$\hat{w}_{1,\pm 1} = (k_2^2 + k_3^2)(U_e^2/\chi^2)\hat{u}_3^\infty \mathcal{A}_v e^{i(\varphi_1 + k_2 \bar{y})} \mp \mathbf{i}b(\hat{x})e^{i\varphi_1 - (|k_3|/U_e)\bar{y}}, \quad (2.97)$$

$$\begin{aligned} (\hat{p}_1)_{1,\pm 1} &= \pm(2U_e'/r_t)\mathbf{i}|k_3|(k_2^2 + k_3^2)(U_e^2/\chi^4)u_3^\infty \mathcal{A}_v e^{i(\varphi_1 + k_2 \bar{y})} \\ &+ \left[(U_e'(\hat{x})/U_e)b(\hat{x})\bar{y} + \bar{g}(\hat{x})\right]e^{i\varphi_1 - (|k_3|/U_e)\bar{y}}. \end{aligned} \quad (2.98)$$

507 The unknown function $b(\hat{x})$ is to be determined by matching the outer solution (2.96)-
 508 (2.97) with the inner solution. For this purpose, the asymptote of the former in the limit
 509 $\hat{x} \ll 1$ with $\eta = O(1)$ is worked out as follows,

$$\begin{aligned} \hat{v}_{1,\pm 1} &\rightarrow \mp(|k_3|/k_2)(k_2^2 + k_3^2)(U_e^2/\chi^2)\hat{u}_3^\infty \mathcal{A}_v e^{i\varphi_1} \left[1 + \mathbf{i}k_2 s U_e \bar{\eta} - k_2^2 s^2 U_w^2 \bar{\eta}^2 + \dots\right] \\ &+ b e^{i\varphi_1} \left[1 - s|k_3|\bar{\eta} + (s k_3)^2 \bar{\eta}^2 + \dots\right], \end{aligned} \quad (2.99)$$

$$\begin{aligned} \hat{w}_{1,\pm 1} &\rightarrow (k_2^2 + k_3^2)(U_e^2/\chi^2)\hat{u}_3^\infty \mathcal{A}_v e^{i\varphi_1} \left[1 + \mathbf{i}k_2 s U \bar{\eta} + \dots\right] \\ &\mp \mathbf{i}b(\hat{x}) \left[1 - s|k_3|\bar{\eta} + \dots\right]; \end{aligned} \quad (2.100)$$

510 here use has been made of the relation $\bar{y} = sU_e(\eta - \beta) = sU_e \bar{\eta}$, which holds for $\hat{x} \ll 1$,

511 where $\beta = \int_0^\infty (1 - F')d\eta$. Inspection of the terms to be matched with the inner expansion

512 suggests that

$$b(\hat{x}) = b_0(\hat{x}) + sb_1(\hat{x}) + s^2b_2(\hat{x}) + \dots \quad (2.101)$$

513 In the region corresponding to $\hat{x} \ll 1$ but $\eta = O(1)$, the solution to the boundary-region
514 equations (2.54)- (2.57) can be expanded as power series,

$$\left(\hat{u}_{1,\pm 1}, \hat{v}_{1,\pm 1}, \hat{w}_{1,\pm 1}, (\hat{p}_1)_{1,\pm 1} \right) = q_{\pm} \sum_{n=0}^{\infty} s^n \left(2\hat{x}U_n, V_n, \mp iW_n, \mp is^{-1}P_n \right). \quad (2.102)$$

515 Substitution into (2.54)- (2.57) gives the governing equations of the first two terms in
516 the power series (2.102) as

$$\left. \begin{aligned} 2U_0 - \eta U'_0 + V'_0 + W_0 &= 0, \\ U''_0 + FU'_0 + (\eta F'' - 2F')U_0 - F''V_0 &= 0, \\ P'_0 &= 0, \\ W''_0 + FW'_0 &= 0, \end{aligned} \right\} \quad (2.103)$$

517 and

$$\left. \begin{aligned} 3U_1 - \eta U'_1 + V'_1 + W_1 &= 0, \\ U''_1 + FU'_1 + (\eta F'' - 3F')U_1 - F''V_1 &= 0, \\ V''_0 + FV'_0 - (\eta F')'V_0 + [\eta(\eta'F)' - F]U_0 &= P'_1, \\ W''_1 + FW'_1 - F'W_1 &= ik_3^2P_0. \end{aligned} \right\} \quad (2.104)$$

518 Equations (2.103)-(2.104) are to be solved subject to the no-slip condition at $\eta = 0$.
519 The boundary conditions as $\eta \rightarrow \infty$ are obtained by matching with (2.99)- (2.100). The
520 solutions at the first and second orders are formally the same as those in Leib *et al.*
521 (1999), but the present variable η is different from theirs. The solution to (2.103) is

$$U_0 = \frac{1}{4}\eta F'', \quad V_0 = \frac{1}{4}(\eta^2 F'' - 3\eta F' - F), \quad W_0 = F', \quad (2.105)$$

522 of which V_0 has the asymptote,

$$V_0 \rightarrow \frac{1}{4}[-3\eta - (\eta - \beta)] = -\bar{\eta} - \frac{3}{4}\beta \quad \text{as } \eta \rightarrow \infty.$$

523 Matching at $O(1)$ and $O(s)$ the $\bar{\eta}$ -independent terms in the present boundary-layer
524 solution and its outer counterpart (2.99), with the rescaling factor in (2.53) being taken
525 into account, we obtain b_0 and b_1 ,

$$b_0(\hat{x}) = \pm(|k_3|/k_2)(k_2^2 + k_3^2)(U_e/\chi^2)\hat{u}_3^\infty \mathcal{A}_v, \quad b_1(\hat{x}) = q_{\pm}(-\frac{3}{4}\beta), \quad (2.106)$$

526 whilst matching of the terms proportional to $\bar{\eta}$ yields

$$-q_{\pm} = \mp(|k_3|/k_2)(k_2^2 + k_3^2)(U_e/\chi^2)\hat{u}_3^\infty \mathcal{A}_v(ik_2U_e) - |k_3|b_0. \quad (2.107)$$

527 Use of (2.106) in (2.107) gives

$$q_{\pm} = \pm i(|k_3|/k_2)(k_2^2 + k_3^2)(U_e/\chi^2)\hat{u}_3^\infty \mathcal{A}_v(k_2U_e - i|k_3|). \quad (2.108)$$

528 Matching the $O(1)$ terms in the spanwise velocity (2.100) and (2.102) yields

$$(k_2^2 + k_3^2)(U_e^2/\chi^2)\hat{u}_3^\infty \mathcal{A}_v e^{i\varphi_1} \mp ib_0(\hat{x}) = \mp iq_{\pm}/|k_3|,$$

529 which is identical to (2.107), confirming the consistency of the ongoing analysis. Further-
530 more, matching the $O(s)$ terms in (2.100) and (2.102) gives the boundary condition for
531 W_1 ,

$$W_1 \rightarrow i(Uk_2 + i|k_3|)\bar{\eta} - \frac{3}{4}\beta|k_3| \quad \text{as } \eta \rightarrow \infty. \quad (2.109)$$

Since $U_1 \rightarrow 0$ as $\eta \rightarrow \infty$, from the first equation in (2.104) follows the far-field behaviour for V_1 ,

$$V_1 \rightarrow -i(Uk_2 + i|k_3|)(\frac{1}{2}\eta - \beta\eta) + \frac{3}{4}\beta|k_3|\eta + c_1 \quad \text{as } \eta \rightarrow \infty, \quad (2.110)$$

where the constant c_1 is determined by numerical solution for U_1 and V_1 . Finally carrying out the matching of the wall-normal velocity at $O(s^2)$ in $\hat{s}v_{1,\pm 1}$ and (2.99), we find that

$$b_2 = q_{\pm}c_1. \quad (2.111)$$

Substitution of (2.106) and (2.111) along with (2.101) completes the construction of the outer solution. Using the latter and the inner solution (2.102), a composite solution, which is uniformly valid for $y = O(1)$ and the boundary layer, can be constructed as

$$\hat{u}_{1,\pm 1} \rightarrow q_{\pm}(2\hat{x}/s^2)(U_0 + sU_1), \quad (2.112)$$

$$\begin{aligned} \hat{v}_{1,\pm 1} \rightarrow q_{\pm} \left\{ V_0 + sV_1 + \frac{i\mathcal{A}_v e^{i\varphi_1 - iU_e k_2 s\beta}}{(U_e k_2 - i|k_3|)s} \left(e^{iU_e k_2 s\eta} - e^{-|k_3|s\eta} \right) \right. \\ \left. + e^{-|k_3|s\eta} \left(\frac{\beta}{4} + sc_1 \right) - \bar{v}_c \right\}, \end{aligned} \quad (2.113)$$

$$\begin{aligned} \hat{w}_{1,\pm 1} \rightarrow \mp i q_{\pm} \left\{ W_0 + sW_1 + \frac{U_e k_2 \mathcal{A}_v e^{i\varphi_1 - iU_e k_2 s\beta}}{(U_e k_2 - i|k_3|)s} \left(e^{iU_e k_2 s\eta} - e^{-|k_3|s\eta} \right) \right. \\ \left. + e^{-|k_3|s\eta} \left[1 - \beta \left(iU_e k_2 - \frac{|k_3|}{4} \right) s \right] - \bar{w}_c \right\}, \end{aligned} \quad (2.114)$$

as $\hat{x} \rightarrow 0$, where $q_{\pm}$ is defined by (2.108), and $\bar{v}_c$ and $\bar{w}_c$, which represent the common parts between the inner and outer solutions for the wall-normal and spanwise velocities respectively, have the expressions

$$\bar{v}_c = -\eta + \frac{\beta}{4} + s \left[-\frac{i}{2} (U_e k_2 + i|k_3|) \eta^2 + \beta \left(iU_e k_2 - \frac{|k_3|}{4} \right) \eta + c_1 \right], \quad (2.115)$$

$$\bar{w}_c = 1 + s \left[i(U_e k_2 + i|k_3|) \eta - \beta \left(iU_e k_2 - \frac{|k_3|}{4} \right) \right]. \quad (2.116)$$

In summary, the analysis in this section shows that the excitation and nonlinear evolution of streaks and Görtler vortices are governed by the initial-boundary-value problem, consisting of the governing equations (2.54)-(2.57), the upstream conditions (2.112)-(2.114) and the far-field conditions (2.90).

3. Numerical methods

An explicit method was employed to solve the boundary-layer equations (2.13) (Schetz 1993). The no-slip conditions are prescribed at the wall, and the Blasius profiles are specified at the inlet. The top boundary condition is consistent with the slip velocity given by the inviscid solution. The details are given in appendix A.

The initial-boundary-value problem describing the excitation and nonlinear evolution of the vortices is solved numerically using the methods to be described below. We define $\hat{f} = \hat{u}_{\eta}$ and $\hat{g} = \hat{w}_{\eta}$, and recast equations (2.54)-(2.57) into a system of first-order

equations,

$$\left. \begin{aligned} \hat{u}_\eta &= \hat{f}, \quad \hat{v}_\eta = -2\mathcal{B}_v\hat{u} - in\hat{w} + \mathcal{B}_v\eta\hat{f} - s^2\hat{u}_{\hat{x}}, \quad \hat{w}_\eta = \hat{g}, \\ \hat{f}_\eta &= \left[s^2(-im\hat{k}_1 + n^2k_3^2) + 2\mathcal{B}_vU_B + s^2U_{B\hat{x}} - \mathcal{B}_v\eta U_{B\eta} \right] \hat{u} + U_{B\eta}\hat{v} \\ &\quad + (-\mathcal{B}_v\eta U_B + sV_B)\hat{f} + s^2U_B\hat{u}_{\hat{x}} + r_t s^2\hat{l}_{m,n}, \\ \hat{p}_\eta &= \left[-s^3V_{B\hat{x}} + s\mathcal{B}_v\eta V_{B\eta} + 2s\mathcal{B}_vV_B + \mathcal{B}_v\eta(s^2U_{B\hat{x}} - B_v\eta U_{B\eta}) \right. \\ &\quad \left. - 2s^3G_{A\chi_B}U_B + \mathcal{B}_v\eta s^2(-im\hat{k}_1 + n^2k_3^2) \right] \hat{u} \\ &\quad + \left[-s^2(-im\hat{k}_1 + n^2k_3^2) - \mathcal{B}_vU_B - sV_{B\eta} + \mathcal{B}_v\eta U_{B\eta} \right] \hat{v} \\ &\quad + (sV_B - \mathcal{B}_v\eta U_B)in\hat{w} - \mathcal{B}_v\hat{f} - in\hat{g} + s^3V_Bu_{\hat{x}} - s^2U_B\hat{v}_{\hat{x}} - s^2\hat{f}_{\hat{x}} \\ &\quad + r_t s^2(\mathcal{B}_v\eta\hat{l}_{m,n} - \hat{e}_{m,n}), \\ \hat{g}_\eta &= \left[s^2(-im\hat{k}_1 + n^2k_3^2) \right] \hat{w} + s^2(in\hat{k}_3^2)\hat{p} + (-\mathcal{B}_v\eta U_B + sV_B)\hat{g} + s^2U_B\hat{w}_{\hat{x}} + r_t s^2\hat{h}_{m,n}. \end{aligned} \right\} \quad (3.1)$$

They can be rewritten in a vector form,

$$\frac{\partial \mathbf{U}}{\partial \eta} = C_0 \mathbf{U} + C_1 \frac{\partial \mathbf{U}}{\partial \hat{x}} + f(\mathbf{U}), \quad (3.2)$$

where $\mathbf{U} = (\hat{u}, \hat{v}, \hat{w}, \hat{f}, \hat{p}, \hat{g})$ is the solution vector, C_0 and C_1 are the coefficient matrices, and $f(\mathbf{U})$ stands for nonlinear terms. For brevity, the subscripts ' m, n ' are omitted. The coefficient matrices are

$$\left. \begin{aligned} C_0(1, 4) &= 1; C_0(2, 1) = -2\mathcal{B}_v, C_0(2, 3) = -in, C_0(2, 4) = \mathcal{B}_v\eta; C_0(3, 6) = 1; \\ C_0(4, 1) &= s^2(-im\hat{k}_1 + n^2k_3^2) + 2\mathcal{B}_vU_B + s^2U_{B\hat{x}} - \mathcal{B}_v\eta U_{B\eta}, \\ C_0(4, 2) &= U_{B\eta}, C_0(4, 4) = -\mathcal{B}_v\eta U_B + sV_B; \\ C_0(5, 1) &= -s^3V_{B\hat{x}} + \mathcal{B}_v\eta sV_{B\eta} - 2s^3G_{A\chi_B}U_B + 2\mathcal{B}_v sV_B \\ &\quad + \mathcal{B}_v\eta s^2(-im\hat{k}_1 + n^2k_3^2) + \mathcal{B}_v\eta s^2U_{B\hat{x}} - \mathcal{B}_v^2\eta^2U_\eta, \\ C_0(5, 2) &= -s^2(-im\hat{k}_1 + n^2k_3^2) - \mathcal{B}_vU_B - sV_{B\eta} + \mathcal{B}_v\eta U_{B\eta}, \\ C_0(5, 3) &= (sV_B - \mathcal{B}_v\eta U_B)in, C_0(5, 4) = -\mathcal{B}_v, C_0(5, 6) = -in; \\ C_0(6, 3) &= s^2(-im\hat{k}_1 + n^2k_3^2), C_0(6, 5) = s^2(in\hat{k}_3^2), C_0(6, 6) = -\mathcal{B}_v\eta U_B + sV_B, \end{aligned} \right\} \quad (3.3)$$

and

$$\left. \begin{aligned} C_1(2, 1) &= -s^2; \\ C_1(4, 1) &= s^2U_B; \\ C_1(5, 1) &= s^3V_B, C_1(5, 2) = -s^2U_B, C_1(5, 4) = -s^2; \\ C_1(6, 3) &= s^2U_B. \end{aligned} \right\} \quad (3.4)$$

For components with $n = 0$, $\hat{u}$ and $\hat{v}$ are decoupled from $\hat{w}$, and the system (3.1) can be simplified to three equations for $\hat{u}$, $\hat{v}$ and $\hat{f}$, and two equations for $\hat{w}$ and $\hat{g}$.

The NBRE (3.2) are parabolic in the streamwise direction, and hence can be solved by a marching procedure in $\hat{x}$ -direction. The Malik scheme (Malik 1990) is employed in η -direction, and the second-order backward finite-difference scheme is applied in the streamwise direction, namely

$$\left. \begin{aligned} \mathbf{U}_{j+1} - \mathbf{U}_j &= \frac{h}{2} (\mathbf{U}'_{j+1} + \mathbf{U}'_j), \\ \frac{\partial \mathbf{U}^{n+1}}{\partial \hat{x}} &= \left[\frac{3}{2} \mathbf{U}^{n+1} - 2\mathbf{U}^n + \frac{1}{2} \mathbf{U}^{n-1} \right] / \Delta \hat{x}, \end{aligned} \right\} \quad (3.5)$$

where j and n indicate the position of mesh points in the transverse and streamwise directions respectively. Grouping all terms at the streamwise location $n + 1$ to the left-

U_∞	R_A	G_A	l^*/A	Streamwise locations x^* ($\hat{x}$)								
				20	40	60	70	80	90	100	110	
5	1145	1501	666	0.048	0.096	0.144	0.168	0.191	0.215	0.238	0.262	

TABLE 1. The parameters and the streamwise locations in the experiments of Swearingen & Blackwelder (1987).

hand side (except those in the nonlinear terms), we end up with the discrete equations,

$$\left[I - \frac{h}{2} c_{j+1} \right] \mathbf{U}_{j+1}^{n+1} + \left[-I - \frac{h}{2} c_j \right] \mathbf{U}_j^{n+1} = \frac{h}{2} [g_{j+1} + g_j] + \frac{h}{2} [f(\mathbf{U}_{j+1}^{n+1}) + f(\mathbf{U}_j^{n+1})], \quad (3.6)$$

where I is the unit matrix, and we have put

$$f(\mathbf{U}_j^{n+1}) = \begin{pmatrix} 0 & 0 & 0 & r_t s^2 \hat{l}, & r_t s^2 (\mathcal{B}_v \eta \hat{l} - \hat{e}), & r_t s^2 \hat{h} \end{pmatrix}_j^{n+1}, \quad (3.7)$$

$$c_{j+1} = C_{0,j+1} + \frac{3}{2\Delta\hat{x}} C_{1,j+1}, \quad g_{j+1} = C_{1,j+1} (-2\mathbf{U}_{j+1}^n + \frac{1}{2}\mathbf{U}_{j+1}^{n-1}) / \Delta\hat{x}. \quad (3.8)$$

On introducing the discrete solution vector,

$$\Phi_{n+1} = \left(\mathbf{U}_J^{n+1}, \mathbf{U}_{J-1}^{n+1}, \dots, \mathbf{U}_1^{n+1}, \mathbf{U}_0^{n+1} \right)^T, \quad (3.9)$$

with J being the number of mesh points in the wall-normal direction, the system of nonlinear algebraic equations, (3.6), is written in the matrix form,

$$L\Phi_{n+1} = A + N_{n+1}, \quad (3.10)$$

where L is the linear operator, A stands for the terms whose values are evaluated at upstream positions, and N_{n+1} denotes the nonlinear terms. An underrelaxion iteration method, which was described in XZW, is applied in order to obtain the convergent solution. The so-called 3/2-rule is followed in order to eliminate the so-called aliasing error (Kim, Moin & Moser 1987). For adverse, zero and weak favorable pressure gradients, seventeen Fourier modes are retained to capture nonlinear effects. For the strong pressure gradients, thirty-three Fourier modes are found to be sufficient. The domain size in the η -direction is 30, within which 1000 grid points are deployed. The marching starts from $\hat{x} = 0.001$. It is found that the adverse pressure gradient case requires a smaller streamwise marching step $\Delta\hat{x}$. In the present work, $\Delta\hat{x} = 0.0001$ suffices for both the adverse and favorable cases, and $\Delta\hat{x} = 0.001$ for the reference zero gradient case, which is same as that used in XZW.

4. Numerical results

The excitation and nonlinear evolution of Görtler vortices are controlled by several parameters. Two of them are ϵ , measuring the turbulence level, and G_A , characterizing the strength of the centrifugal force; their roles have been demonstrated by XZW for the boundary layer with zero pressure gradient. They remain important in the contracting flow here. The influence of the pressure gradient, whose intensity is controlled by σ_c , will also be discussed in this paper. Figure 2 presents the downstream development of the slip velocity for different σ_c . For all the cases with $\sigma_c \neq 1$, the slip velocity changes appreciably before $\hat{x} \approx 0.35$ and then tends to a constant. The pressure gradient thus

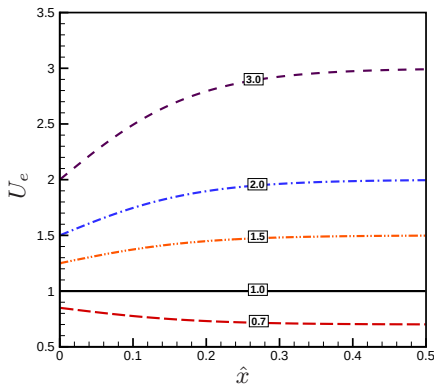


FIGURE 2. (Colour online) The downstream development of the slip velocity for different σ_c .

affects the boundary-layer flow in the range of $\hat{x} = 0 \sim 0.35$, which is the main domain of our interest.

The computations are performed pertaining to the experimental conditions of Swearingen & Blackwelder (1987), the key parameters in which are given in Table 1. We shall first investigate the response of a curved plate boundary layer to weak FSVD with $\epsilon = 0.0007$. A more common measure is turbulence level Tu . These two are related by the equation: $Tu = 2\epsilon(\hat{u}_1^{\infty 2} + \hat{u}_2^{\infty 2} + \hat{u}_3^{\infty 2})^{1/2}/\sqrt{3}$. Hence the intensity of the turbulence is $Tu = 0.14\%$. Four selected cases are explored with the same G_A and R_A but different σ_c . The two cases with $\sigma_c > 1$ correspond to favorable pressure gradients, whereas the case with $\sigma_c < 1$ has an adverse pressure gradient. The intensity of the boundary-layer response is represented by $u_{1,rms}$, the r.m.s of all harmonic components defined as

$$u_{1,rms} \equiv r_t \left[\sum_{n \neq 0} |s^2 \hat{u}_n|^2 \right]^{1/2}. \quad (4.1)$$

The results are displayed in figure 3(a). The predicted $u_{1,rms}$ experiences, in sequence, a non-modal growth, linear growth and then nonlinear saturation in all four cases. The four curves overlap as $\hat{x} \rightarrow 0$, indicating that the pressure gradient does not affect the initial non-modal growth. It strongly influences the linear evolution of Görtler vortices; an adverse/favorable pressure gradient enhances/reduces the linear amplification, suggesting that a decelerating/accelerating flow is more/less unstable, respectively. This trend is consistent with previous linear stability analysis (e.g. Ragab & Nayfeh (1980)). After the linear growth stage, the perturbation saturates, which occurs earlier in the adverse case. In the case of a strong favorable pressure gradient ($\sigma_c = 3.0$), after a weak linear growth from $\hat{x} \approx 0.04$ to $\hat{x} \approx 0.14$, $u_{1,rms}$ grows exponentially at a fairly large rate until $\hat{x} \approx 0.28$, where saturation starts. Interestingly, for a larger σ_c , $u_{1,rms}$ saturates at a higher level, indicating that an accelerating flow would provide more energy to the vortices downstream. The flat-plate case ($G_A = 0$) is also computed for comparison. The results show that after the non-modal growth stage, streaks in a decelerating flow are more energetic at the beginning, whereas the streaks in an accelerating flow are weaker and even experience decay, but eventually regain growth and saturate at a higher level than that in the case with zero or an adverse pressure gradient. The present result that a favorable pressure gradient leads to a higher amplitude of the saturated vortices agrees with the experimental observation of Crane & Umur (1990) and computational finding of

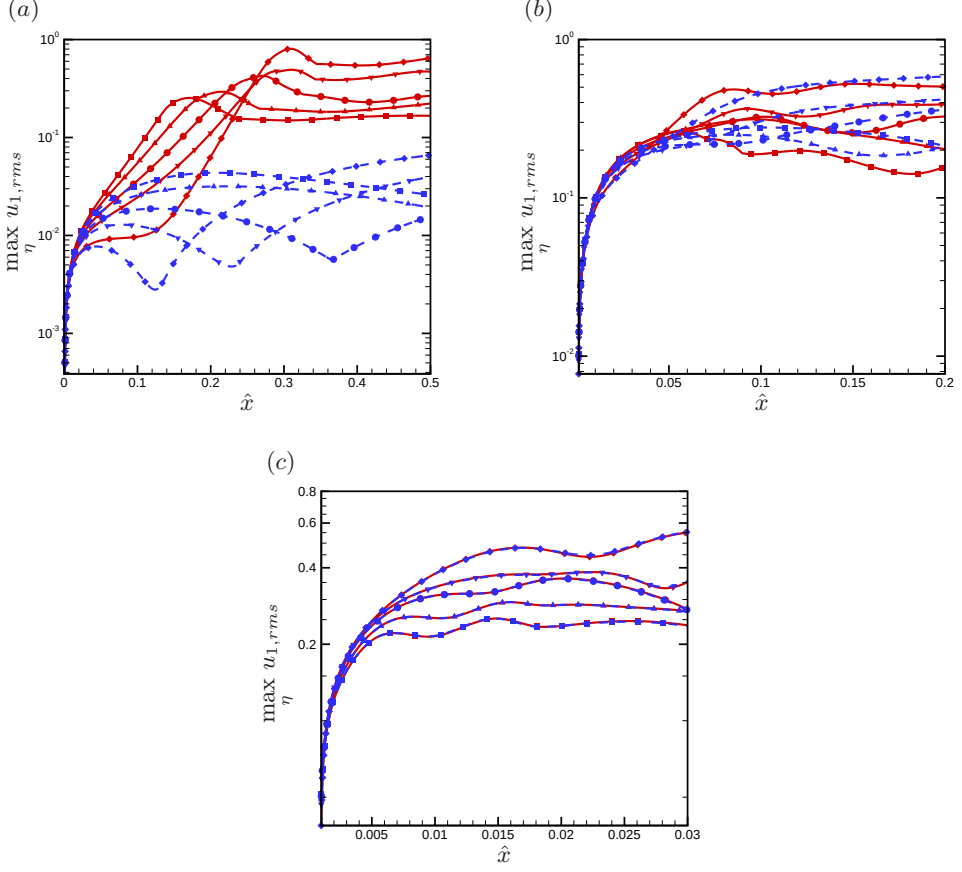


FIGURE 3. (Colour online) The downstream development of $\max u_{1,rms}$ for different σ_c at $R_A = 1145$. (a) $\epsilon = 0.0007$; (b) $\epsilon = 0.014$; (c) $\epsilon = 0.07$. The Görtler numbers are $G_A = 1501$ (solid lines) and $G_A = 0$ (dashed lines). The symbols indicate different σ_c . Squares: $\sigma_c = 0.7$; deltas: $\sigma_c = 1.0$; circles: $\sigma_c = 1.5$; gradients: $\sigma_c = 2.0$; diamonds: $\sigma_c = 3.0$.

Zaki & Durbin (2006) and Matsson (2008). On the other hand, figures 3(a) and (b) show that for the same pressure gradient and at lower or moderate level of FSVD, the Görtler vortices resulting from concave curvature have larger amplitude and saturate earlier than streaks in the flat-plate case, and thus stronger and earlier secondary instability are expected. This explains earlier transition observed in experiments (Volino & Simon 1995; Kim, Simon & Russ 1992).

It has been demonstrated by XZW that the linear growth of Görtler vortices can be bypassed with sufficient strong FSVD for the zero-pressure gradient case. A similar behaviour is noted in figure 3(b), where at the turbulence level $\epsilon = 0.014$ corresponding to $Tu = 2.8\%$, the linear amplification stage of Görtler vortices almost disappears. This result implies that it would be inappropriate to use local eigenmodes as initial conditions for integrating NBRE when FSVD is strong. Similar to the case of $\epsilon = 0.0007$, $u_{1,rms}$ for a larger σ_c saturates later but at a higher level again. The evolution of $u_{1,rms}$ exhibits less difference between the curved and flat-plate boundary layers, indicating a diminished role of centrifugal instability at elevated levels of FSVD.

The boundary layer responses to even stronger FSVD ($\epsilon = 0.07, Tu = 14\%$) are displayed in figure 3(c). The curved and flat-plate cases are both computed again and

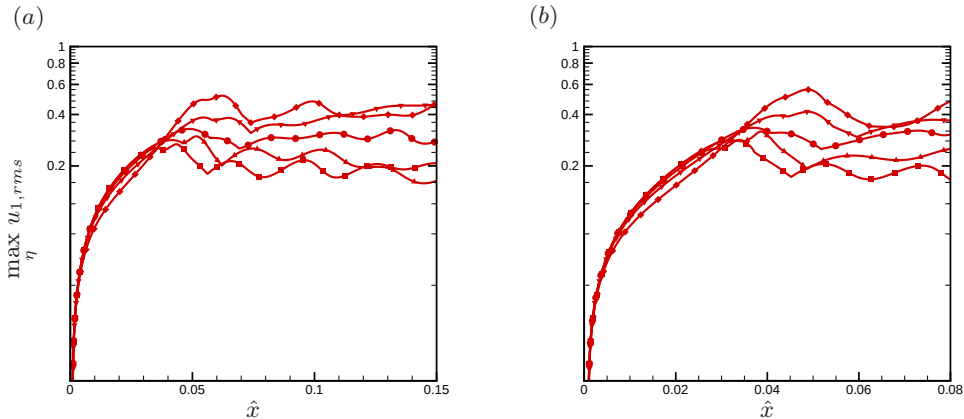


FIGURE 4. (Colour online) The downstream development of $\max u_{1,rms}$ for different σ_c at $R_A = 1145$ and $\epsilon = 0.014$. The Görtler numbers are $G_A = 7505$ (a) and 15010 (b). The symbols indicate different σ_c . Squares: $\sigma_c = 0.7$; deltas: $\sigma_c = 1.0$; circles: $\sigma_c = 1.5$; gradients: $\sigma_c = 2.0$; diamonds: $\sigma_c = 3.0$.

their results overlapped completely, indicating that the centrifugal instability plays no role at such a high turbulence level. This theoretical result is in agreement with the experimental finding; see Volino & Simon (1995) and references therein. For all FSVD levels considered, $\epsilon = 0.0007, 0.014$ and 0.07 , the pressure gradient has a significant effect on the evolution of the disturbances.

Larger Görtler number and acceleration are often concurrent in many applications, including gas turbines, where free-stream turbulence level is also high (Volino & Simon 2000). Hence two larger Görtler numbers, 7505 and 15010, are chosen to represent such flow conditions, and the evolution of the vortex intensity is shown in figures 4(a) and 4(b) respectively. As noted earlier, the linear growth stage of the accelerating flow is longer than that of the decelerating flow. The increase of the Görtler number leads to a larger linear growth rate of Görtler vortices. As a result, the saturation location moves forward, but the saturated amplitude is slightly lower.

In order to probe more deeply into nonlinear interactions, the development of the maximum amplitudes of the fundamental and harmonic components is shown in figure 5. It is noted that in all the five cases, the nonlinearly generated $(0,0)$ mode acquires the largest amplitude. For the adverse, zero and weak favorable pressure cases ($\sigma_c \leq 2.0$), the amplitude of the seeded fundamental modes $(0, \pm 1)$ is the second largest. However, once the favorable pressure is strong enough ($\sigma_c > 2.0$), the nonlinearly generated harmonic component $(0,2)$ grows quickly to take the second place instead, whereas the fundamentals are submerged among nonlinearly generated higher-order harmonic components ($n = 4, 5 \dots$), which acquire rather small amplitudes.

The streamwise velocity profiles at different streamwise locations are shown in figure 6(a) for the case of $\sigma_c = 0.7$. As a reference, the streamwise velocity profiles for $\sigma_c = 1.0$ are presented in figure 6(b). The two spanwise positions, $z = 0$ and $z = \pi$, will be referred to as peak and valley, as they correspond to the high- and low-speed regions, respectively. The velocity profiles in the linear modal growth phase ($\hat{x} = 0.02 \sim 0.15$) and nonlinear saturation stage ($\hat{x} > 0.15$) are plotted. For $\sigma_c = 0.7$, the streamwise velocity at the outer edge of the boundary layer decreases because of the adverse pressure gradient. At the valley ($z = \pi$), the perturbations are in the negative streamwise direction, causing a velocity deficit, similar to the zero pressure gradient case (XZW). The streamwise

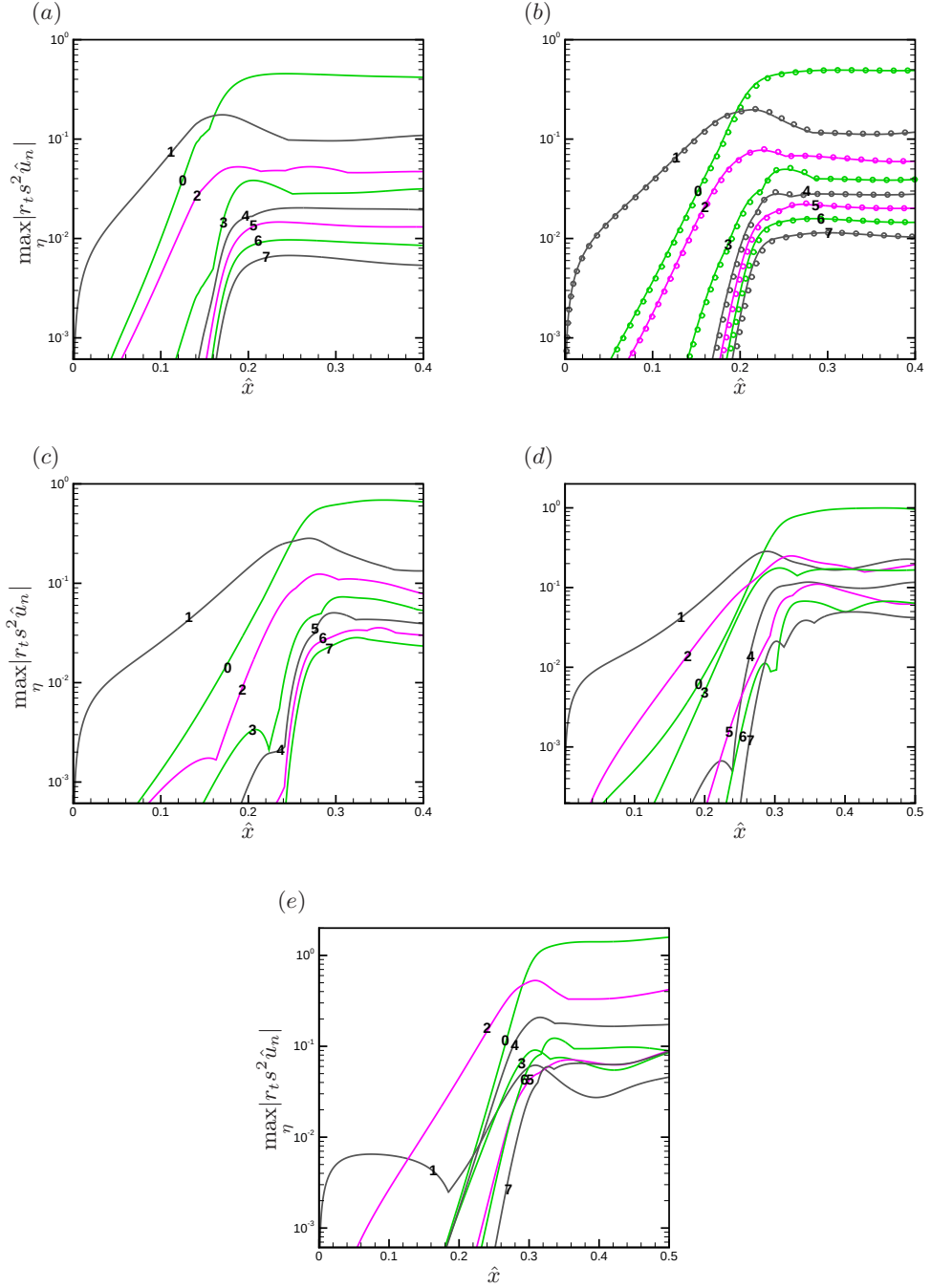


FIGURE 5. (Colour online) Development of the fundamental mode $\max_{\eta} |r_t s^2 \hat{u}_1|$ and harmonic components $\max_{\eta} |r_t s^2 \hat{u}_n|$ ($n = 0, 1, 2, \dots, 7$). (a) $\sigma_c = 0.7$. (b) $\sigma_c = 1.0$. (c) $\sigma_c = 1.5$. (d) $\sigma_c = 2.0$. (e) $\sigma_c = 3.0$. The parameters are $G_A = 1501$, $R_A = 1145$ and $\epsilon = 0.0007$.

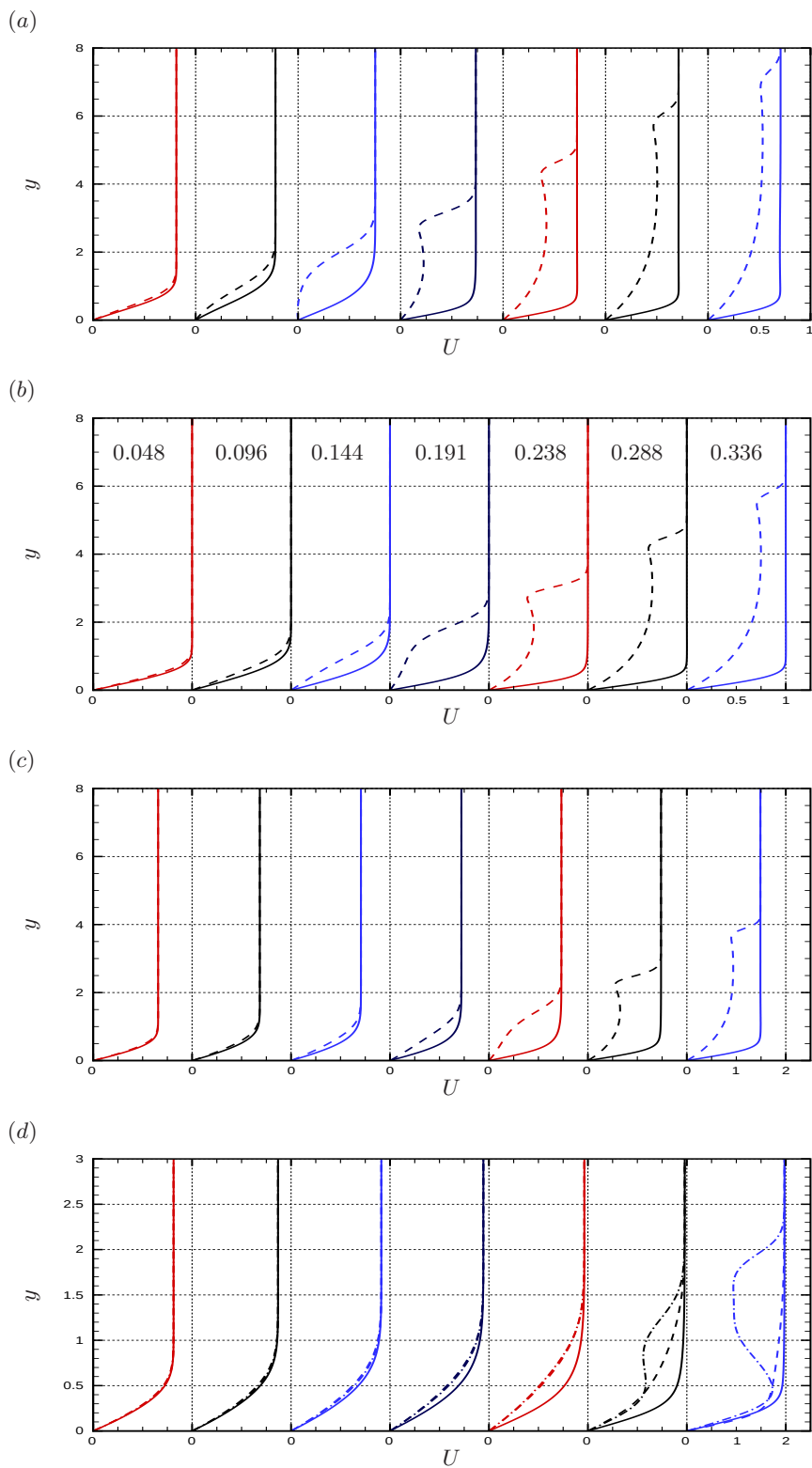


FIGURE 6. (a,b,c,d) For caption see facing page.

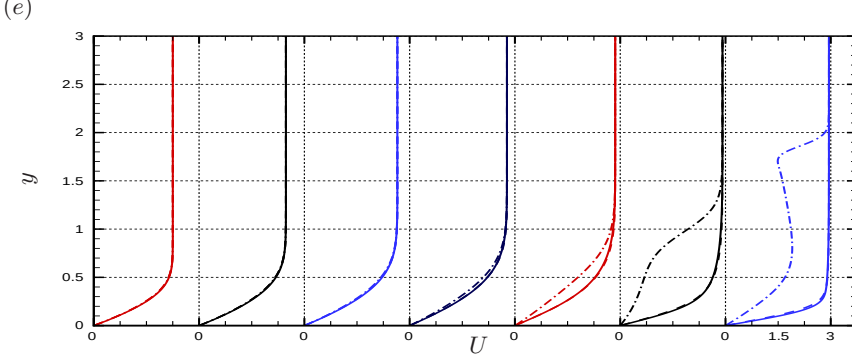


FIGURE 6. (Colour online) The profiles of the streamwise velocity U at various $\hat{x}$ -locations for $\sigma_c = 0.7$ (a), 1.0 (b), 1.5 (c), 2.0 (d) and 3.0 (e). Solid lines: at the peak ($z = 0$); dashed lines: at the valley ($z = \pi$); dashed-dotted lines in (d) and (e): half way between the peak and valley ($z = \pi/2$). The parameters are $G_A = 1501$, $R_A = 1145$ and $\epsilon = 0.0007$.

velocity becomes inflectional in the normal direction at $x^* = 80$ cm, earlier than in the zero and favorable pressure gradient cases. Additionally, it is worth noting that near $x^* = 60$ cm ($\hat{x} = 0.144$) and in the vicinity of $z = \pi$, the streamwise velocity becomes negative near the wall, i.e. a small zone of reverse flow forms, but the solution can be marched downstream without encountering any singularity, possibly because of the spanwise ellipticity. At the peak ($z = 0$), the streamwise velocity near the wall ($0 < y < 0.8$) decreases slightly but eventually increases.

Figure 6(c) displays the streamwise velocity profiles in the case of $\sigma_c = 1.5$, for which the streamwise velocity at the outer edge of the boundary layer is increasing because of the favorable pressure gradient. A favorable pressure gradient inhibits the linear modal growth and delays nonlinear saturation, which starts from $\hat{x} \approx 0.24$, later than in the flat-plate case ($\hat{x} = 0.18$). At $x^* = 100$ cm ($\hat{x} = 0.238$), an inflectional point appears on the profile at $z = \pi$, where the width of the boundary layer is thickening.

Figures 6(d) and (e) show the streamwise velocity profiles for $\sigma_c = 2.0$ and 3.0 respectively. The larger favorable pressure gradient furthermore restrains the growth of both the base flow and vortices. The Görtler vortices develop nearer to the wall, in the regions $y < 2.5$ and $y < 2.2$, respectively. Interestingly, there is no inflectional point in the peak ($z = 0$) or in the valley ($z = \pi$) at all streamwise positions. Instead, half way between the peak and valley ($z = \pi/2$), an inflectional point appears on the profile at $x^* = 120$ cm. It is the growth of (0,2) mode shown in figure 5(d) that leads to this result. The onset location of inflectional point moves obviously downstream as σ_c increases until $\sigma_c < 2.0$, but remains almost at the same streamwise location for $\sigma_c = 2.0$ and $\sigma_c = 3.0$. On the other hand, the vortices become more concentrated in the wall region as σ_c increases.

The local boundary-layer displacement thickness, defined as

$$\delta^*(x, z) = \int_0^\infty \left[1 - \frac{u(x, y, z)}{U_e(x)} \right] dy^* = \left(\frac{2\nu x^*}{U_e U_\infty} \right)^{1/2} \int_0^\infty \left[1 - \frac{u(x, \eta, z)}{U_e(x)} \right] d\eta, \quad (4.2)$$

becomes spanwise dependent due to the presence of Görtler vortices. It is significantly influenced by the streamwise pressure gradient. Figure 7(a) displays the predicted boundary-layer thickness δ^* at typical spanwise positions for FSVD intensity $\epsilon = 0.0007$ and different σ_c . For the three cases of $\sigma_c = 0.7, 1.0$ and 1.5, the displacement thickness δ^* in the peak region decreases somewhat in the streamwise direction. In contrast, in the

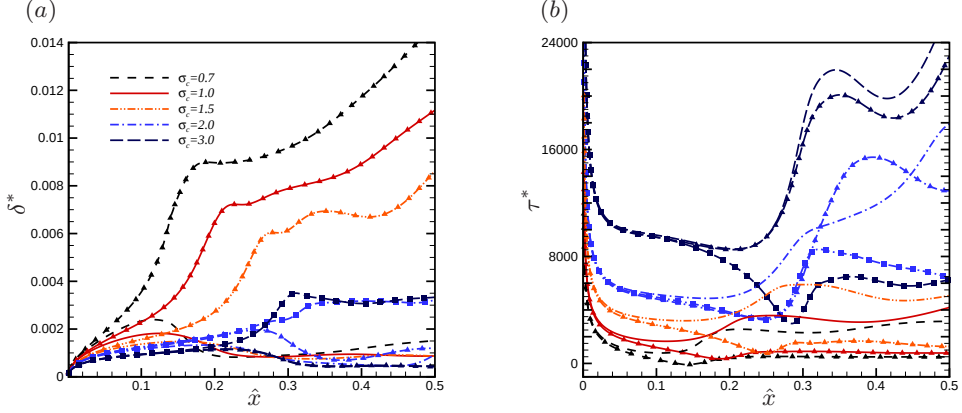


FIGURE 7. (Colour online) Streamwise development of the displacement thickness (a) and the skin friction (b). Solid lines: at the peak ($z = 0$); lines with deltas: at the valley ($z = \pi$); lines with squares: half way between peak and valley ($z = \pi/2$). The parameters are $G_A = 1501$, $R_A = 1145$ and $\epsilon = 0.0007$.

valley region δ^* varies significantly. It increases quickly because of the rapid exponential amplification of the vortices in the linear stage. The variation becomes moderate entering the nonlinear stage and then another phase of rapid increase occurs. For the stronger favorable pressure gradient cases of $\sigma_c = 2.0$ and 3.0 , the displacement thickness drops somewhat at both the peak and valley, in between which δ^* increases and then saturates. For the strongest favorable pressure gradient case $\sigma_c = 3.0$, the displacement thicknesses at the peak ($z = 0$) and valley ($z = \pi$) overlap. This is due to the growth and dominance of the $(0, 2)$ mode. Figure 7(b) plots the predicted skin friction $\tau^* = \frac{\partial u^*}{\partial y^*}|_{y^*=0}$. For the cases with larger values of σ_c , in the non-modal and linear growth stages, the skin frictions are all significant higher at both the peak and valley, indicating that a favorable pressure gradient enhances the skin friction, whereas an adverse one attenuates it. For all the cases, there is a relatively abrupt increase of the skin friction near the nonlinear saturation location at the valley. It is significant that the nonlinear development of Görtler vortices plays a key role in the drastic change of the skin friction. As is expected with the growth of $(0, 2)$ mode in the case of strong favorable pressure gradient $\sigma_c = 3.0$, the skin frictions at the peak and valley differ slightly, and the skin friction at the middle of the peak and valley ($z = \pi/2$) becomes the smallest.

Figures 8(a)-(e) present the evolution of Görtler vortices for different σ_c . The contours of the streamwise velocity at ten streamwise locations are plotted to illustrate the vortical structure. The spanwise extent is one spanwise wavelength of FSVD. In the cases of adverse, zero and weak favorable pressure gradient, a single mushroom structure appears. However, strong favorable pressure gradients boost the growth of $(0, 2)$ mode, which may become comparable with the fundamental modes. As a result, a structure consisting of two anti-symmetric mushrooms appears as is shown in figure 8(d) for $\sigma_c = 2.0$. For the even stronger favorable pressure, $\sigma_c = 3.0$, two identical mushrooms can be seen within one wavelength, because the harmonic component $(0, 2)$ dominates. Figures 8(a)-(e) indicate that in the presence of an adverse/favorable pressure gradient, the mushroom is taller/shorter at the same streamwise location compared with the zero pressure gradient case. This relation turns out to hold for other parameter values as will be shown below, and is consistent with the experimental measurement of Aihara & Sonoda (1981) and

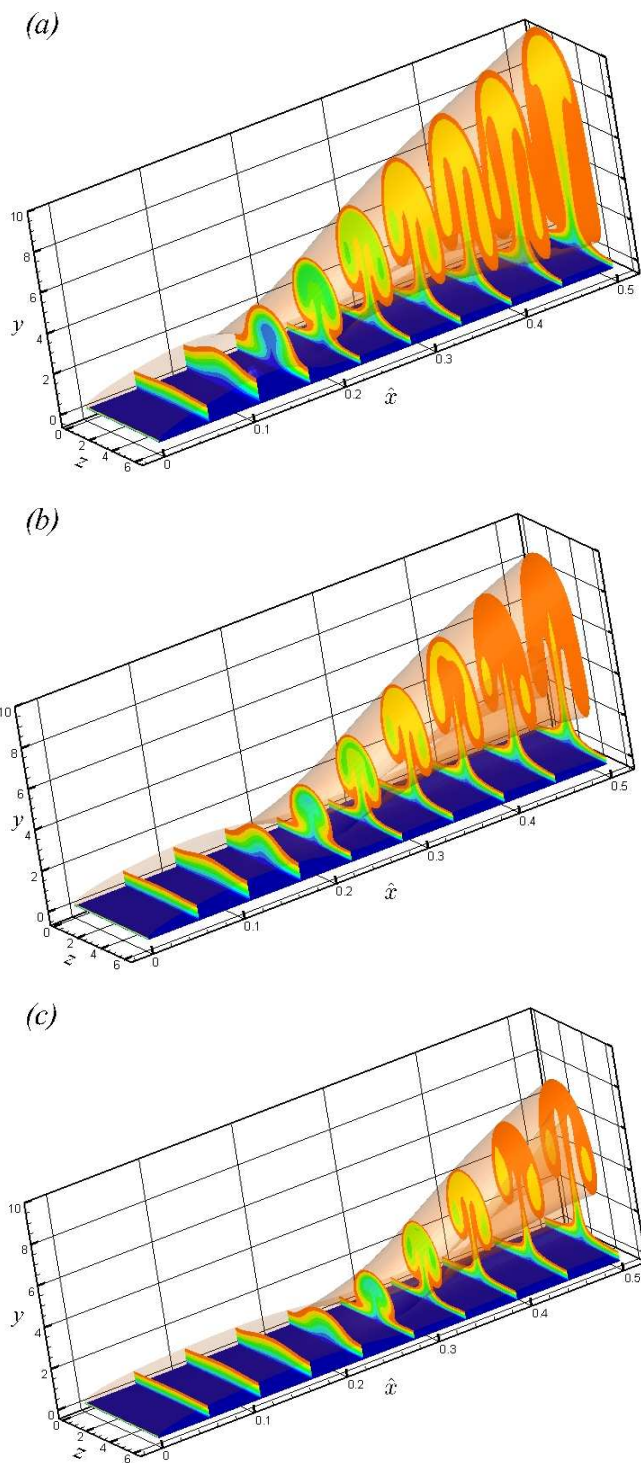


FIGURE 8. (a,b,c) For caption see facing page.

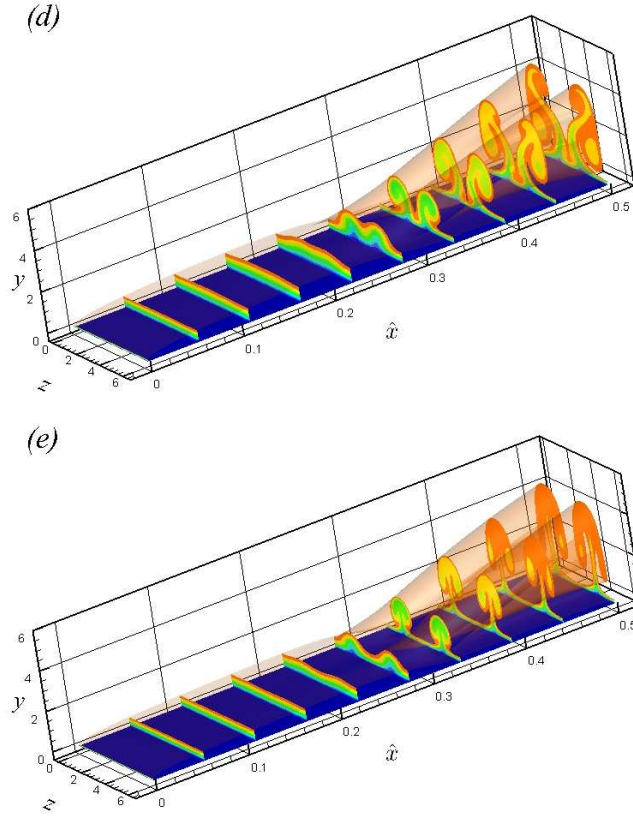


FIGURE 8. (Colour online) Nonlinear evolution of Görtler vortices as shown by contours of $u(y, z)/U_e(\hat{x})$ in the (y, z) plane at different streamwise locations for $\sigma_c = 0.7$ (a), 1.0 (b), 1.5 (c), 2.0 (d) and 3.0 (e). The parameters are $G_A = 1501$, $R_A = 1145$ and $\epsilon = 0.0007$. The increment of the contour values is $0.1U_e(\hat{x})$.

the numerical results of Matsson (2008) and Zaki & Durbin (2006) despite the fact that the initial disturbances used in the computations are somewhat arbitrary.

The development of the maximum amplitudes of the fundamental and harmonic components are presented in figure 9 for $\epsilon = 0.014$. For such a high intensity of FSVD, the saturation position for different pressure gradients exhibits small differences. Again, only the two fundamental modes $(0, \pm 1)$, the mean-flow distortion $(0, 0)$ and the harmonic $(0, 2)$ are predominant, while higher harmonics $(n = 3, 4, \dots)$ have much smaller amplitudes. For the cases of adverse, zero and weak favorable pressure gradient, the fundamental modes are the second strongest. However, once the favorable pressure gradient is large enough, the amplitude of the nonlinearly generated $(0, 2)$ mode takes the second place. It is worth noting that $(0, 2)$ mode starts to overtake the fundamental modes $(0, \pm 1)$ when $\sigma_c = 1.5$ as opposed to $\sigma_c = 2.0$ with the lower turbulence level of $\epsilon = 0.0007$.

Figure 10(a) displays the boundary-layer displacement thickness δ^* at representative spanwise positions for different σ_c . For all these five cases, the displacement thickness at the peak remains small throughout the entire evolution. The largest δ^* appears at the spanwise position where the top of the mushroom is. Comparing with figure 7, one may note that the difference between δ^* at the peak and valley becomes smaller with an

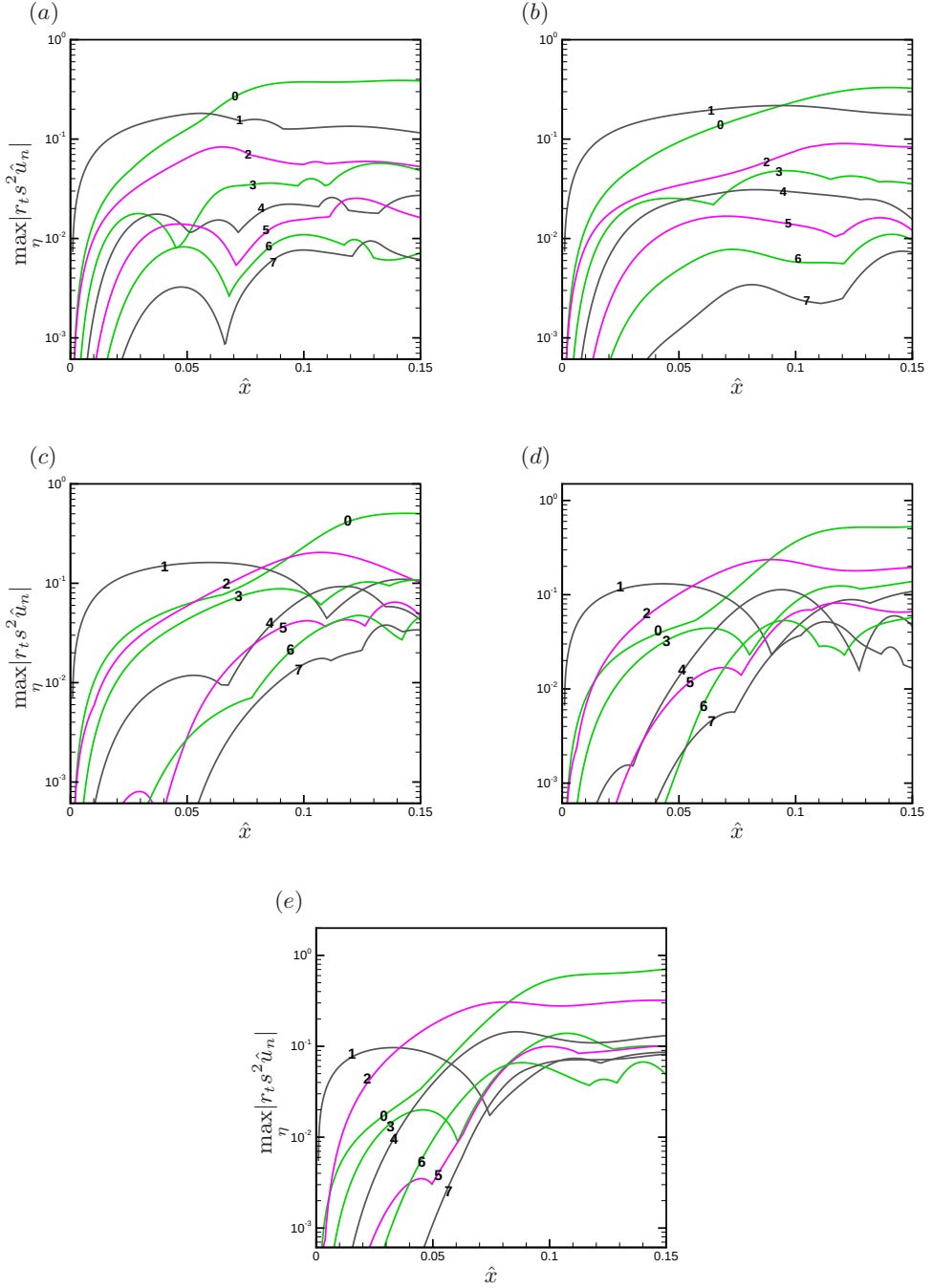


FIGURE 9. (Colour online) Development of the fundamental mode $\max_{\eta} |r_t s^2 \hat{u}_1|$ and harmonic components $\max_{\eta} |r_t s^2 \hat{u}_n|$ ($n = 0, 1, 2, \dots, 7$). (a) $\sigma_c = 0.7$. (b) $\sigma_c = 1.0$. (c) $\sigma_c = 1.5$. (d) $\sigma_c = 2.0$. (e) $\sigma_c = 3.0$. The parameters are $G_A = 1501$, $R_A = 1145$ and $\epsilon = 0.014$.

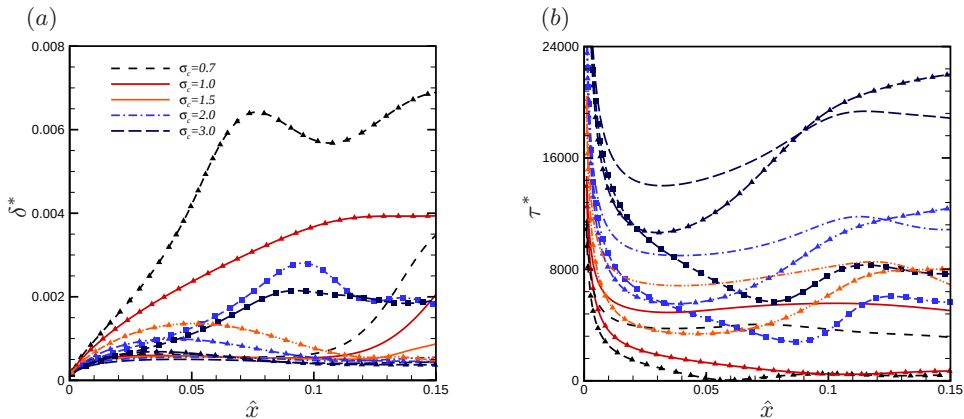


FIGURE 10. (Colour online) Streamwise development of the displacement thickness (a) and the skin friction (b) in the contracting and expanding flows with $\epsilon = 0.014$. Lines with no symbols: at the peak ($z = 0$); lines with deltas: at the valley ($z = \pi$); lines with squares: between the peak and valley ($z = \pi/2$).

increased FSVD level. Figure 10(b) shows the skin friction τ^* . With a larger σ_c , the skin frictions at both the peak and valley are significantly higher. For a fixed σ_c , the location of abrupt increase moves forward upstream to $\hat{x} = 0.008$ compared with $\hat{x} = 0.028$ in the lower FSVD case of $\epsilon = 0.0007$ shown in figure 7.

Figure 11 presents the structure of the vortices. For the cases of $\sigma_c = 0.7, 1.0$ and 1.5 , the vortices do not exhibit the typical mushroom shape, and look more like streaks in the flat-plate boundary layer (cf. Ricco *et al.* (2011)). At the early stage, a plateau forms near the centre, which elongates and then distorts into a cat-faced shape. This not unexpected because there is no linear growth stage where the curvature plays a destabilising role. For the strong favorable cases ($\sigma_c = 2.0$ and 3.0), a linear Görtler instability operates, but the saturation location moves downstream (see figure 3(b)), where two mushrooms form within a wavelength. Figure 12 shows the evolution of the vortices for a much larger Görtler number $G_A = 7505$. The familiar mushroom structure appears again for $\sigma_c = 3.0$. This explains the observation of mushroom structure in a boundary layer over highly curved wall subject to a strong favorable pressure gradient and intense FSVD (Crane & Umur 1990).

Finally, in this section, we present the results on the response of the boundary layer to unsteady FSVD ($\hat{k}_1 \neq 0$). The particular interest is in the nonlinear evolution of Görtler vortices subject to adverse pressure gradients. Two different frequencies $f^* = 5$ Hz and 10 Hz, corresponding to $\hat{k}_1 = 26.35$ and 52.69 , are selected. The turbulence levels are $\epsilon = 0.0007$ and 0.014 . The amplitude of the unsteady streaks or Görtler vortices is represented by the root mean square (r.m.s.) of the streamwise velocity, defined as

$$u_{rms} \equiv r_t \left[\sum_{m,n} |s^2 \hat{u}_{m,n}|^2 \right]^{1/2}.$$

Figure 13 shows the development of $\max_{\eta} u_{rms}$ for the two frequencies with the steady case being included for comparison. The effect of unsteadiness is found to stabilise the Görtler vortices in both $\epsilon = 0.0007$ and 0.014 cases. When $\epsilon = 0.0007$ (figure 13(a)), a linear stage may be identified, and vortices with a higher frequency has a smaller growth rate. In the case of $\hat{k}_1 = 52.69$, there appears a rather sudden change of the slope at

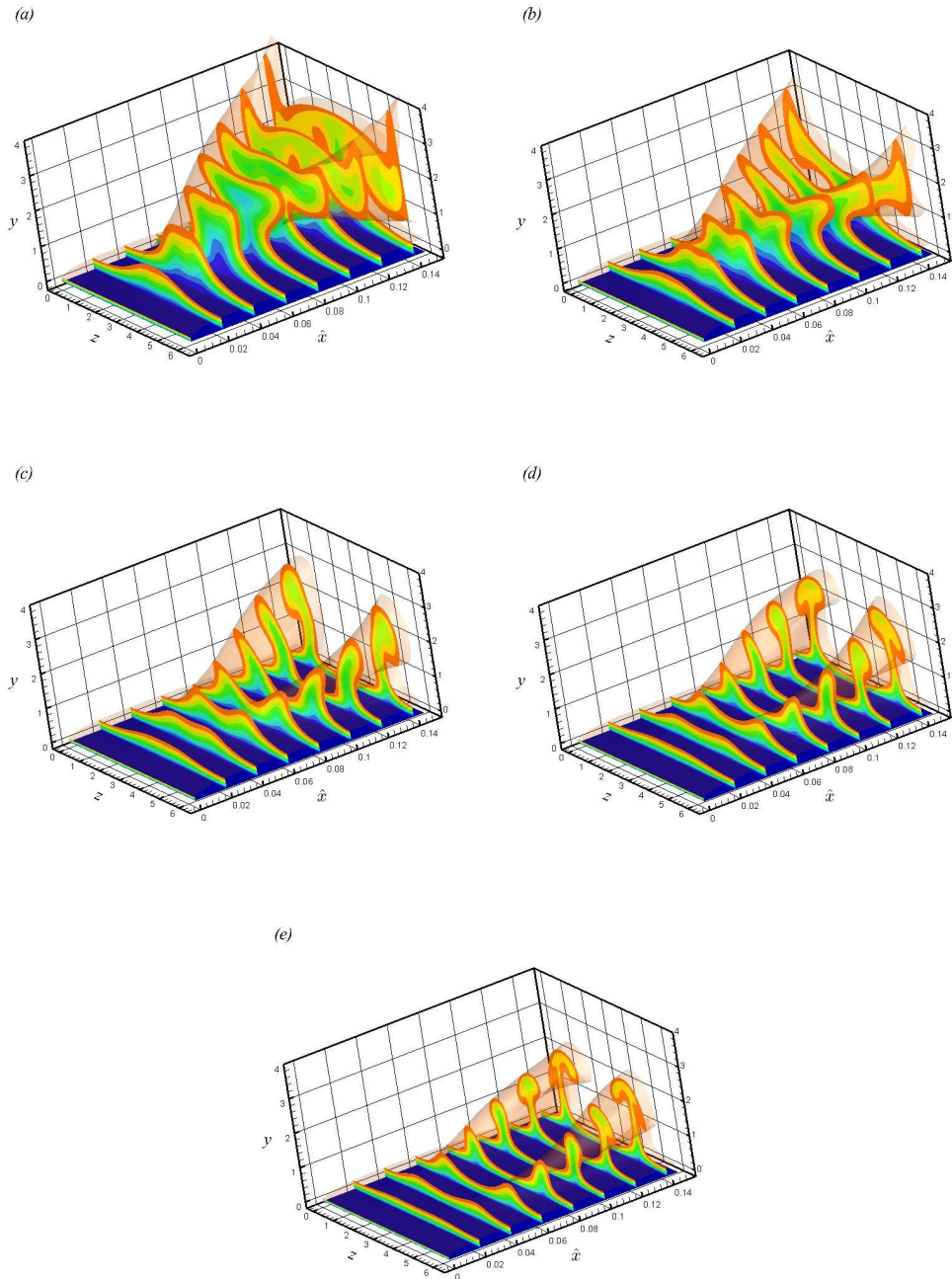


FIGURE 11. (Colour online) Nonlinear evolution of Görtler vortices for $\sigma_c = 0.7$ (a), 1.0 (b), 1.5 (c), 2.0 (d) and 3.0 (e). The parameters are $G_A = 1501$, $R_A = 1145$ and $\epsilon = 0.014$. Eight contours are plotted from $\hat{x} = 0.001$ to $\hat{x} = 0.14$. The increment of the contour values is $0.1U_e(\hat{x})$.

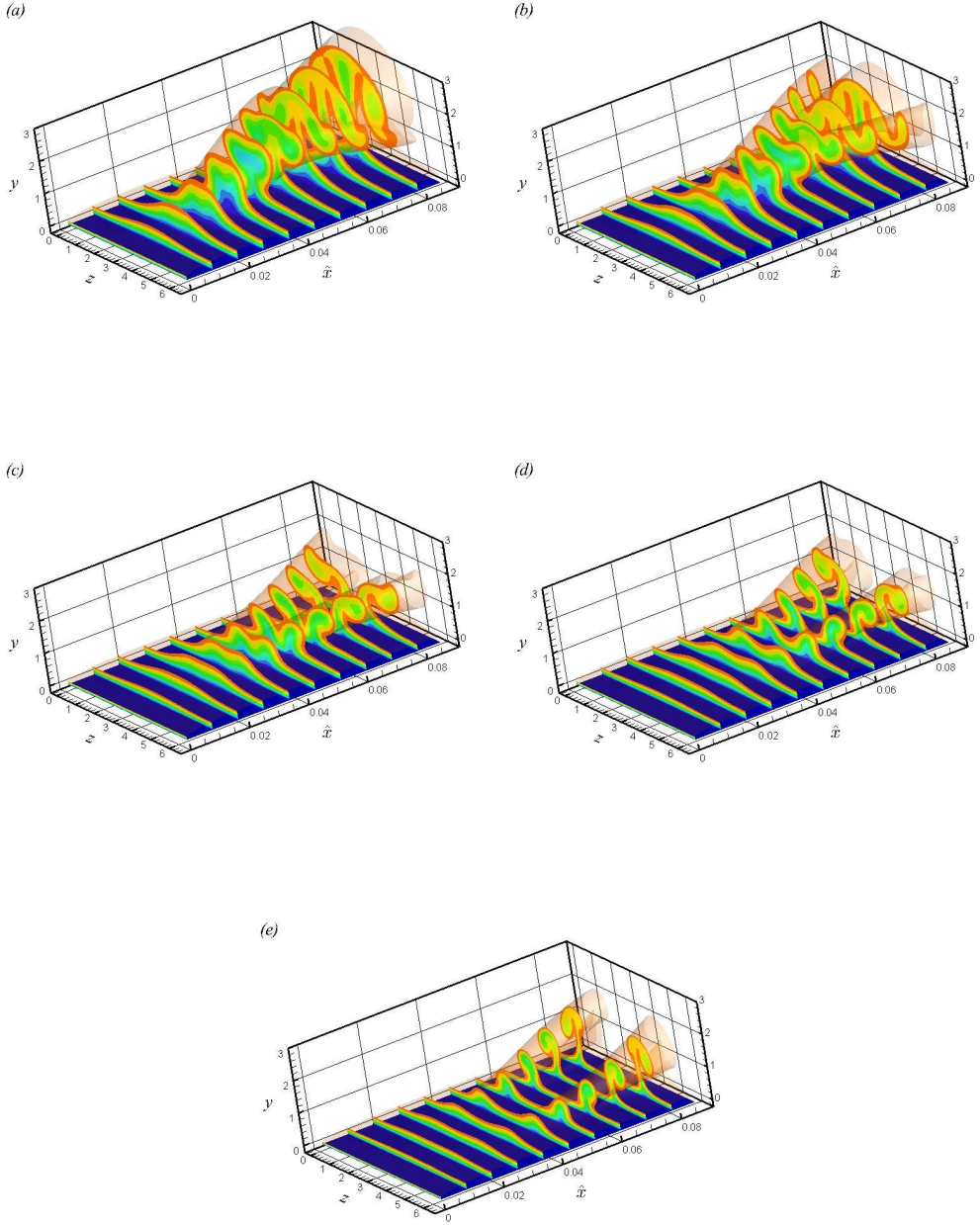


FIGURE 12. (Colour online) Nonlinear evolution of Görtler vortices for $\sigma_c = 0.7$ (a), 1.0 (b), 1.5 (c), 2.0 (d) and 3.0 (e). The parameters are $G_\Lambda = 7505$, $R_\Lambda = 1145$ and $\epsilon = 0.014$. Ten contours are plotted from $\hat{x} = 0.001$ to $\hat{x} = 0.08$. The increment of the contour values is $0.1U_e(\hat{x})$.

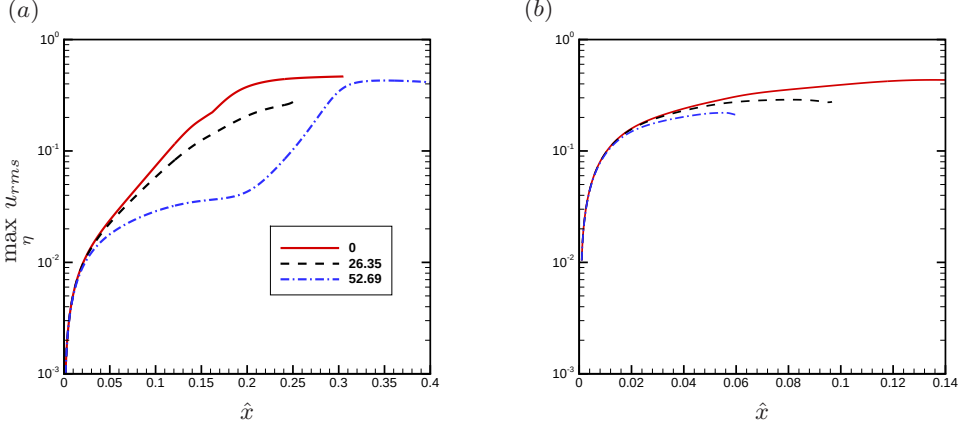


FIGURE 13. (Colour online) Comparison of downstream development of $\max u_{rms}$ for different frequencies with $\sigma_c = 0.8$. The parameters are $G_A = 1501$ and $R_A = 1145$. (a) $\epsilon = 0.0007$; (b) $\epsilon = 0.014$.

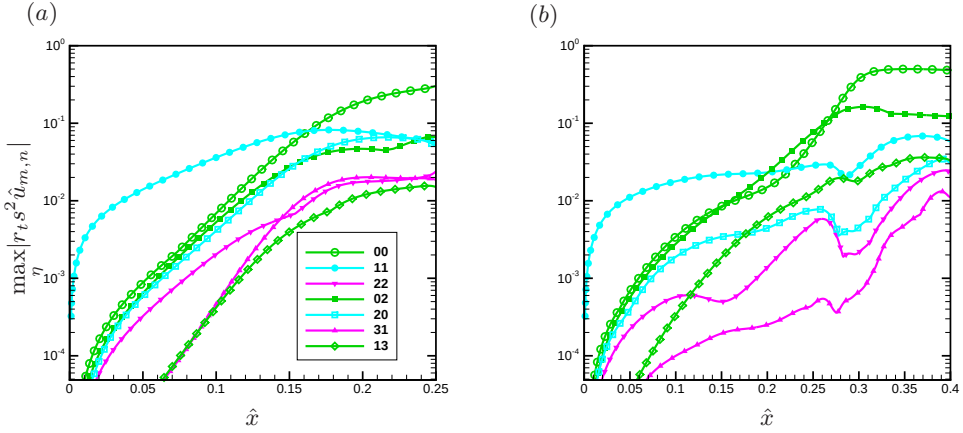


FIGURE 14. (Colour online) Development of the fundamental mode $\max |r_t s^2 \hat{u}_{11}|$ and harmonic components $\max |r_t s^2 \hat{u}_{m,n}|$. (a) $\hat{k}_1 = 26.35$. (b) $\hat{k}_1 = 52.69$. The parameters are $G_A = 1501$, $R_A = 1145$ and $\epsilon = 0.0007$.

$\hat{x} \approx 0.21$, and eventually u_{rms} saturates at almost the same level as that of zero- or low-frequency case. At the higher turbulence level $\epsilon = 0.014$ (figure 13(b)), no linear stage appears, and the vortices are practically streaks, and saturate much earlier (at $\hat{x} \approx 0.03$) compared with the case of $\epsilon = 0.014$.

In order to investigate the nature of nonlinear vortices, figure 14 presents the development of the maximum amplitudes of the fundamental and harmonic components for the cases of $\hat{k}_1 = 26.35$ and 52.69 . The mean-flow distortion $(0,0)$ acquires a magnitude appreciably greater than that of the seeded fundamental modes $(1,\pm 1)$. For the case $\hat{k}_1 = 52.69$, the nonlinearly generated harmonic component $(0,2)$ grows quickly from $\hat{x} \approx 0.22$ to overtake the fundamental modes $(1,\pm 1)$ and acquire the second largest saturation amplitude, unlike the steady case, where the fundamental modes acquire the second largest saturation amplitude.

5. Secondary instability of Görtler vortices

5.1. Formulation and methodology

When Görtler vortices have acquired a sufficiently large amplitude and saturated nonlinearly, the distorted velocity profile $U(y, z; x, t)$ becomes inflectional in both the streamwise and spanwise directions and hence may be susceptible to inviscid instability. The pressure gradient influences the high shear layer in the three-dimensional velocity field, and hence does the same to the secondary instability of Görtler vortices. However, to the best of our knowledge there has not been any theoretical study of secondary instability of the Görtler vortices that are excited by FSVD in a boundary layer subject to pressure gradient. We shall carry out a secondary instability analysis, treating it as part of our integrated approach that aims at predicting transition caused by FSVD.

Since the flow field of Görtler vortices varies with $\hat{x}$ and $\hat{\tau}$ ($\hat{x}$ only for steady FSVD) very slowly, the dependence on these two variables can be treated as being parametric (Ricco *et al.* 2011; XZW). The total velocity field $\mathbf{u}_s(x, y, z, t)$ and pressure $p_s(x, y, z, t)$ can be decomposed into a base flow $(U_G(y, z), P_G(y, z))$ and a secondary perturbation $\epsilon_s(\mathbf{u}'_s(x, y, z, t), p'_s(x, y, z, t))$, where $\epsilon_s \ll 1$. The transverse velocities of the vortex base flow, V_G and W_G , are of $O(R_A^{-1})$ and hence negligible, and so $\mathbf{U}_G = (U, 0, 0)$ with U being given by

$$U(\eta, z; \hat{x}, \hat{\tau}) \equiv U_B(\hat{x}, \eta) + r_r s^2 \sum_{m,n} \hat{u}_{m,n}(\hat{x}, \eta) \exp(-im\hat{k}_1\hat{\tau} + ink_3z). \quad (5.1)$$

Substituting $\mathbf{u}_s$ and p_s into the N-S equations and neglecting the $O(\epsilon_s^2)$ nonlinear terms, we obtain the linearized N-S equations,

$$\left. \begin{aligned} \partial_t \mathbf{u}'_s &= -(\mathbf{U}_G \cdot \nabla) \mathbf{u}'_s - (\mathbf{u}'_s \cdot \nabla) \mathbf{U}_G - \nabla p'_s + \frac{1}{R_A} \nabla^2 \mathbf{u}'_s, \\ \nabla \cdot \mathbf{u}'_s &= 0. \end{aligned} \right\} \quad (5.2)$$

The secondary disturbance satisfies the no-slip condition at the wall, and attenuates in the far field,

$$\mathbf{u}'_s = 0 \quad \text{at } y = 0, \quad (\mathbf{u}'_s, p'_s) \rightarrow 0 \quad \text{as } y \rightarrow \infty. \quad (5.3)$$

Given that the coefficients in (5.2) depend on y and z only, the solution for the secondary perturbation $\phi'_s(x, y, z, t) \equiv (\mathbf{u}'_s, p'_s)$ can be sought of the normal-mode form,

$$\phi'_s(x, y, z, t) = \phi_s(y, z) e^{i(\alpha x - \omega t)} + c.c., \quad (5.4)$$

where α and ω denote the non-dimensional streamwise wavenumber and frequency of the secondary disturbance respectively. The shape function $\phi_s(y, z)$ is governed by a system of partial differential equations, $\mathcal{L}_E(\partial_y, \partial_z, U_G; \alpha, \omega) \phi_s = 0$, which forms together with the homogeneous boundary conditions, $\mathbf{u}_s = 0$ at $y = 0$ and $(\mathbf{u}_s, p_s) \rightarrow 0$ as $y \rightarrow \infty$, a so-called bi-global instability problem (Theofilis 2011). In the temporal formulation, α is real, whereas ω is complex valued and obtained as an eigenvalue.

Since the governing equations for ϕ_s have periodic coefficients, according to Floquet theory, the solution for $\phi_s(y, z)$ takes the form

$$\phi_s(y, z) = e^{iq\beta z} \sum_{n=-\infty}^{+\infty} \phi_s^{(n)}(y) e^{in\beta z}, \quad (5.5)$$

where β is the spanwise wavenumber and $0 \leq q \leq \frac{1}{2}$. Only fundamental modes ($q = 0$) will be presented in the present paper. With the discretization in the wall-normal direction by a finite-difference scheme on variable grids, an algebraic system of the form $\mathbf{A}(\omega) \mathbf{a} = \mathbf{B} \mathbf{a}$

is established, where $\mathbf{A}$ and $\mathbf{B}$ are matrices. This generalized eigenvalue problem is solved by the Arnoldi approach. Our code was designed for the secondary instability analysis of compressible flows, but is adopted to the incompressible case by assigning a small Mach number. The code is validated by comparing with results in XZW, which used a code specifically for incompressible flows.

5.2. Numerical results on secondary instability

The left column of figure 15 displays the growth rates and phase speeds of secondary modes in the region of $x^* = 60 - 100$ cm for the adverse pressure gradient case $\sigma_c = 0.7$. As Görtler vortices grow to a larger amplitude and acquire more complex spatial structure, the growth rates of secondary instability become larger, and an increasing number of instability modes appears. However, of interest are the most amplified modes. Three families of dominant modes, odd modes I (OM-I), odd modes II (OM-II) and even modes I (EM-I) are identified, which have the same structures respectively as those in the zero pressure gradient case shown by XZW. The OM-I modes are the dominant ones before $x^* = 80$ cm. They start to appear at $x^* = 60$ cm, which is farther upstream than $x^* = 80$ cm in the zero pressure gradient case. The most dangerous modes are in the EM-I family from $x^* = 80$ cm. Both OM-I and EM-I are amplifying from $x^* = 60$ cm to $x^* = 90$ cm, but their growth rates start to decrease from $x^* = 100$ cm because of the over saturation of Görtler vortices shown in figure 3(a). The wavenumbers of OM-I and EM-I modes are less than 3.5, and their growth rates are exceeded by those of another family of modes, OM-II, existing in the larger wavenumber region. The growth rates of OM-II modes at different streamwise locations almost overlap in the wavenumber range $\alpha = 0 - 1.3$ at $x^* = 90$ cm and 100 cm, indicating that smaller-streamwise-wavenumber modes vary slowly. Figure 15(c) shows that the local phase speed, which is defined by $C_r = \omega_r / (\alpha U_e)$. It appears that C_r is weakly dependent on α for EM-I and OM-II, but the phase speed of the OM-I varies with α appreciably, especially when $\alpha < 0.8$. For all three families of unstable modes, the phase speeds increase with $\hat{x}$.

For the favorable pressure gradient case $\sigma_c = 1.5$, the onset of secondary instability appears further downstream and the most dangerous modes are also OM-I, OM-II and EM-I. The right column of figure 15 shows the growth rates and the local phase speeds of the modes in the region of $x^* = 110 - 140$ cm. The OM-I modes are dominant for α less than approximately 0.8. The most dangerous modes are EM-I for $\alpha = 0.8 - 4.0$. The maximum temporal growth rate of EM-I modes is attained almost at $x^* = 120$ cm, while it appears at $x^* = 90$ cm and $x^* = 100$ cm for the adverse pressure ($\sigma_c = 0.7$) and zero pressure cases respectively. OM-II modes become dominant when $x^* > 130$ and their wavenumber $\alpha > 4.0$. A comparison of figure 15(a) and (b) indicates that as the pressure gradient becomes favorable, the overall growth rates increase significantly, and the band of instability also becomes broader for each of all three families of unstable modes. As figure 15(d) indicates, the phase speeds of the three unstable modes increase with $\hat{x}$. The phase speeds of OM-II modes undergo the most significant change, and those of the EM-I and OM-I modes change less so in that order.

When a temporal secondary mode (α, ω) is found, the corresponding spatial growth rate can be obtained using Gaster transformation (Gaster 1962), according to which the complex wavenumber $\tilde{\alpha}$ is related to α and ω via

$$\tilde{\alpha} = \tilde{\alpha}_r + i\tilde{\alpha}_i = \left(\alpha - \frac{\omega_i}{|C_g|} \sin\chi\right) - i\left(\frac{\omega_i}{|C_g|} \cos\chi\right), \quad (5.6)$$

where $C_g = \frac{\partial\omega}{\partial\alpha} \equiv |C_g|e^{i\chi}$ is the (complex-valued) group velocity with χ being its phase.

Figure 15(e) shows the spatial growth rates of the three dominant modes obtained

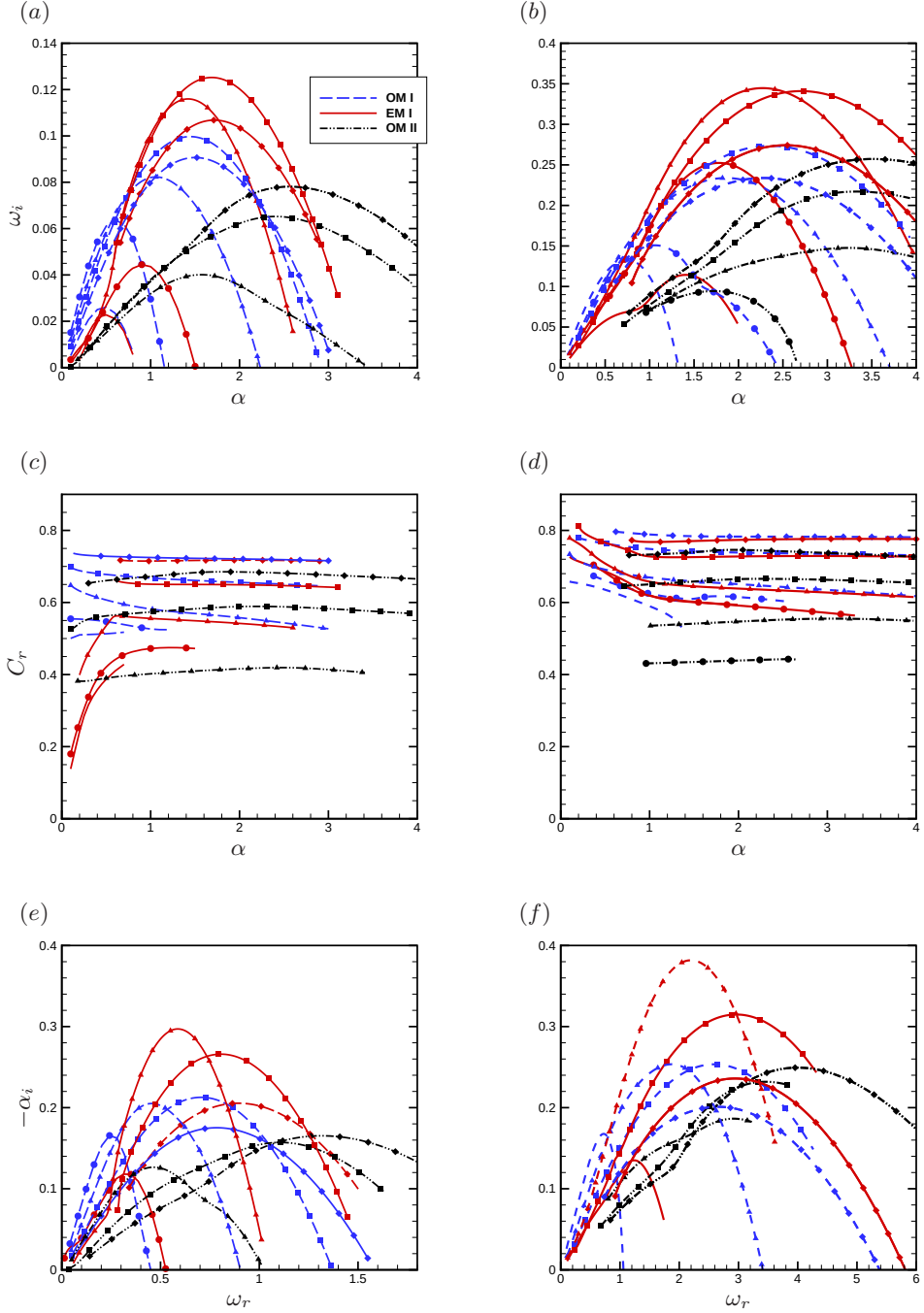


FIGURE 15. (Colour online) Secondary instability characteristics for $\sigma_c = 0.7$ (Left column) and 1.5 (Right column): (a,b) the temporal growth rate; (c,d) the phase speed; (e,f) the spatial growth rate. The streamwise locations analysed for $\sigma_c = 0.7$ are at $x^* = 60$ (no symbols), 70 (circles), 80 (deltas), 90 (squares) and 100 cm (diamonds), corresponding to $\hat{x} = 0.144, 0.168, 0.191, 0.215$ and 0.238 , respectively. The streamwise locations analysed for $\sigma_c = 1.5$ are at $x^* = 110$ (no symbols), 115 (circles), 120 (deltas), 130 (squares) and 140 cm (diamonds), corresponding to $\hat{x} = 0.264, 0.276, 0.288, 0.312$ and 0.336 , respectively.

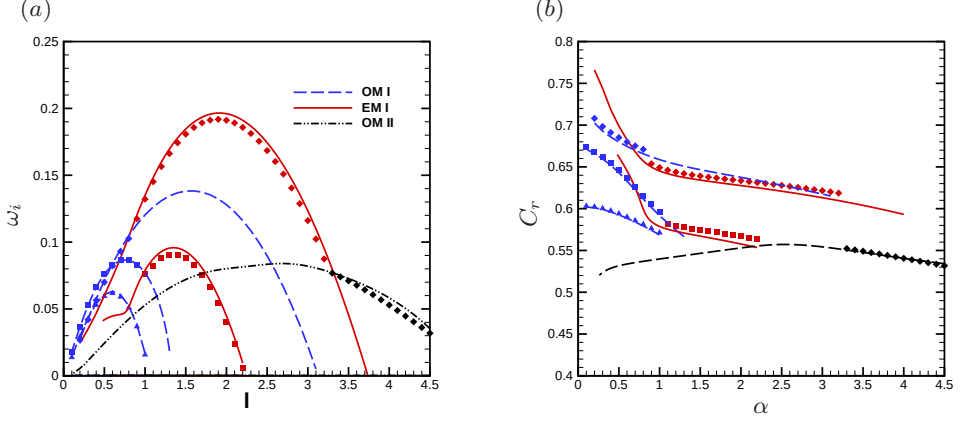


FIGURE 16. (Colour online) Secondary instability characteristics for $\sigma_c = 1.0$: (a) the temporal growth rate and (b) the phase speed versus the streamwise wavenumber α at $x^* = 80$ (deltas), 90 (squares) and 100 cm (diamonds) corresponding to $\hat{x} = 0.191$, 0.215 and 0.238.

for $\sigma_c = 0.7$ at different streamwise locations in the range of $x^* = 70 - 100$ cm. At $x^* = 70$ cm, the maximum growth rate of OM-I is about 0.17, attained at $\omega_r = 0.25$. In the range of 80 – 100 cm, the maximum growth rate of OM-I increases moderately to 0.2. This slow-varying characteristic is also possessed by OM-II, whose maximum growth rate is almost 0.15 both at $x^* = 90$ cm and 100 cm. EM-I modes experience an abrupt growth from $x^* = 70$ cm to $x^* = 80$ cm, with its maximum growth rate being 0.3. It then decreases to 0.26 at $x^* = 90$ cm and plummets further to 0.16 at $x^* = 100$ cm. It should be noted that the intensity of the nonlinear Görtler vortices gradually decreased from $x^* = 70$ cm ($\hat{x} = 0.168$) to $x^* = 100$ cm ($\hat{x} = 0.238$), as is shown in figure 3(b), suggesting the nonlinear spatial structure rather than the amplitude of Görtler vortices alone controls the secondary instability. The adverse pressure gradient narrows the band width of unstable modes as can be noted by comparing our present results with those in figure 22 of XZW. For instance, before $x^* = 100$ cm, no EM-I mode with $\alpha > 1.6$ appears in the adverse pressure gradient case but such modes exist in the zero pressure gradient case.

Figure 15(f) shows the spatial growth rates versus the frequency at different streamwise locations for $\sigma_c = 1.5$. The frequency band of instability becomes broader for each of three families of unstable modes. The most unstable frequencies of the OM-I, EM-I and OM-II are found to be 2.0 (200 Hz), 2.2 (240 Hz) and 4.0 (400 Hz), respectively, higher than those of their counterparts in zero and adverse pressure gradient cases. The structure of these unstable modes is smaller than that in the adverse and zero pressure cases. The most dominant modes overall are still EM-I. Interestingly, the spatial growth rates only moderately larger in the favorable pressure gradient case.

Figure 16 displays the characteristics of secondary instability for the zero pressure gradient case. The results of XZW are also plotted and a good agreement is observed, which validates our code. At $x^* = 80$ cm, only OM-I modes start to grow, since the Görtler vortices in the zero pressure gradient case saturate later. However, the maximum growth rate, attained by an EM-I mode, is about 0.2 at $x^* = 100$ cm, which is larger than that in the the adverse pressure gradient case.

In order to demonstrate the influence of an adverse/favorable pressure gradient on the linear growth of a single mode, we traced the downstream development of unstable modes each with a fixed frequency. The variation of their growth rates with the streamwise

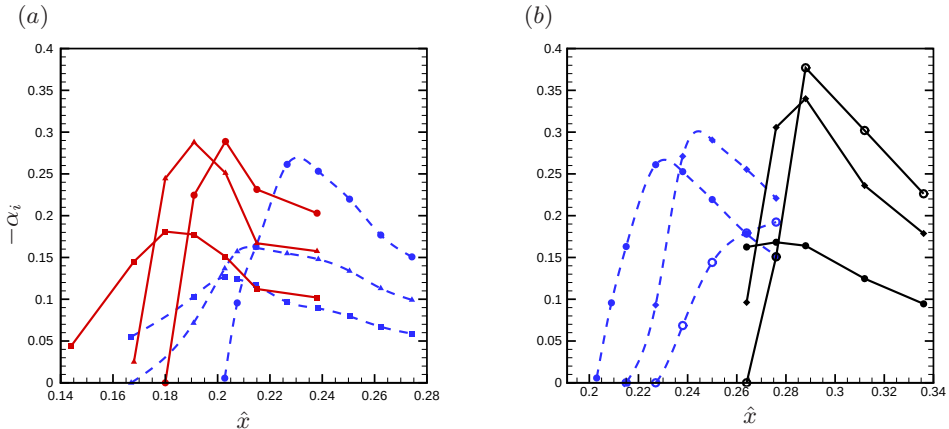


FIGURE 17. (Colour online) The local spatial growth rates of modes, each with a fixed frequency. The results are shown for three different frequencies in each figure. (a) solid lines: $\sigma_c = 0.7$; dashed lines: $\sigma_c = 1.0$ (data of XZW). Squares: $\omega = 0.3$ (65 Hz); deltas: 0.5 (107 Hz); circles: 0.8 (174 Hz). (b) solid lines: $\sigma_c = 1.5$; dashed lines: $\sigma_c = 1.0$ (data of XZW). Circles: 0.8 (174 Hz); gradients: 1.6 (348 Hz); hollow circles: 2.4 (522 Hz).

distance is displayed in figure 17. Three different frequencies are traced in each subfigure. As is illustrated, the lower-frequency modes start to grow first, and higher-frequency ones amplify later but more strongly. For all three frequencies ($\omega = 0.3, 0.5$ and 0.8), an adverse pressure gradient causes the secondary instability to occur earlier. For the modes with frequencies $\omega = 0.3$ and 0.5 , the adverse pressure gradient also lends them a larger maximum growth rate. In the presence of a favorable pressure gradient, the onset of secondary instability is delayed, as Figure 17(b) shows. The growth rates of lower-frequency modes (e.g. $\omega = 0.8$) are reduced, but those of higher-frequency modes are enhanced. The moderately larger growth rates are unlikely to be able to compensate the later occurrence of the secondary instability, and so a favorable/adverse pressure gradient postpones/promotes transition as was observed in experiments (Volino & Simon 1995; Kim, Simon & Russ 1992).

Experimental measurements indicate that the overall FST intensity T_u alone is inadequate to characterise transition location, and it has been speculated that the spectra or length scales of FST appear to be relevant (Abu-Ghannam & Shaw 1980; Volino & Simon 1995). Since transition involves two fundamental physical processes on distinct scales, formation of streaks or Görtler vortices and secondary instability, steady and low-frequency components as well as components with high frequencies in the band of the secondary instability must be introduced in an improved correlation.

6. Summary and conclusions

In this paper, we have investigated the excitation of streaks and Görtler vortices by FSVD, as well as their nonlinear evolution and secondary instability. The problem is of crucial importance for understanding and predicting transition in the boundary-layer flows on turbine blades. The special case of zero pressure gradient was studied by XZW. Motivated by the fact that a streamwise pressure gradient is usually present in many applications especially in turbo-machinery, in this paper we considered further the influence of pressure gradient on the excitation, evolution and secondary instability of Görtler vortices. For definitiveness and mathematical tractability, we formulated the

problem for a concave plate placed in a contracting or expanding stream. The pressure gradient is controlled by the contraction ratio σ_c , with the adverse, zero and favorable pressure gradients corresponding to $\sigma_c < 1.0$, $\sigma_c = 1.0$ and $\sigma_c > 1.0$, respectively. The Görtler number G_A is assumed to be of order one, for which the Görtler instability in the generic regime where the spanwise wavelength and the local boundary-layer thickness are comparable, should be treated as an initial-value rather than an eigenvalue problem (Hall 1983). The required initial condition can only be properly specified by considering the receptivity to ambient disturbances, an important type of which are FSVD. The latter are taken to be strong enough to generate Görtler vortices of $O(1)$ magnitude so that nonlinearity is significant in the majority of flow field, affecting the receptivity and evolution. A significant effect due to the presence of a pressure gradient is that the inviscid steady flow outside the boundary layer is non-uniform. The oncoming FSVD must undergo distortion first before their entrainment into the boundary layer. Both were analysed in order to specify the appropriate initial (upstream) and boundary (far-field) conditions. These and the NBRE governing the response of the boundary layer to FSVD furnishes a nonlinear initial-boundary-layer problem, which describes both the excitation and evolution of Görtler vortices.

The resulting nonlinear initial-value-boundary problem was solved numerically first for steady components of FSVD, which are modelled by a pair of oblique modes with opposite spanwise wavenumbers $\pm k_3$. It was found that as in the zero pressure gradient case, the Görtler vortices are excited in an adverse or favorable pressure gradient boundary layer over a concave wall. They all undergo non-modal growth, quasi-exponential amplification before nonlinear saturation for a moderate level of FSVD, but the linear stage is bypassed if FSVD are strong enough. The influence of pressure gradient on Görtler vortices is two-fold, reflected in their amplification and saturated intensity. For weak or moderate FSVD (e.g. $\epsilon = 0.0007$), an adverse pressure gradient boost the linear amplification rate of Görtler vortices, leading to earlier saturation than in the case of zero pressure gradient, but the saturated intensity of the vortices is smaller. For fairly weak favorable pressure gradients, the amplification is inhabited, and the saturation is decayed, but as the favorable pressure gradient increases, the saturation location ceases to move downstream any further. This is because the seeded steady modes $(0, \pm 1)$ decay, whereas the nonlinearly generated harmonic component $(0, 2)$ acquires a larger amplitude and eventually becomes dominant. For sufficiently strong FSVD (e.g. $\epsilon = 0.014$), the modal growth stage is bypassed, and the pressure gradient only has a marked impact on the saturated intensity of Görtler vortices and influences the saturation location weakly. Interestingly, with the level of FSVD being increased to $\epsilon = 0.07$, the curvature plays no role in the excitation and evolution of the vortices, which are then equivalent to streaks arising in the flat-plate case.

We have also investigated nonlinear evolution of Görtler vortices excited by unsteady components of FSVD in the boundary layer with an adverse pressure gradient. The unsteadiness is found to stabilize the flow significantly at a weak turbulence level (e.g. $\epsilon = 0.0007$), but the stabilization effect diminishes as the turbulence level is elevated, suggesting that unsteady vortices are as important as steady ones in the presence of strong FSVD. As the frequency increases, the nonlinearly generated harmonic component $(0, 2)$ attains a larger amplitude and eventually becomes dominant.

Görtler vortices in their nonlinear stage cause a strong distortion of the streamwise velocity profile in both the wall-normal and spanwise directions, and thus may be susceptible to essentially inviscid secondary instability. A secondary instability analysis with viscosity included was performed for nonlinearly developing Görtler vortices in an adverse case $\sigma_c = 0.7$ and a favorable case $\sigma_c = 1.5$. Three families of the most dangerous

unstable modes, odd modes I, II and even modes I, were identified in our calculations. The temporal growth rates were converted to spatial ones by using Gaster transformation. A pressure gradient was found to have an intriguing two-fold effect on the secondary instability. In the presence of an adverse pressure gradient, secondary instability takes place earlier. However, the band of instability becomes narrower for each of all three families of modes and the overall growth rates are smaller too. The opposite is true for a favorable pressure gradient. Whether favorable or adverse, a pressure gradient enhances preferentially secondary instability modes with different frequencies. It is worth noting that our secondary instability analysis was performed for fairly weak adverse/favorable cases, yet the pressure gradient can have a significant impact on the secondary instability characteristics.

The present theoretical framework allows quantitative relations to be established between the free-stream vortical disturbances and the key transition processes, excitation, nonlinear evolution and secondary instability, in a boundary layer subject to an adverse or favorable pressure gradient. Although FSVD are modelled by a pair of oblique modes for simplicity in this paper, the formulation may be applied to the case where a continuum of low-frequency components are present in FSVD. Our framework thus forms a basis for developing an efficient method to correlate, on the physical basis, the transition location with the characteristics of free-stream turbulence, wall curvature and pressure gradient for flows of practical interest, such as the boundary layers over turbo-machinery blades. A major next step is to extend the analysis to take blade geometry into account.

Appendix A. Mean flow calculations

A.1. Inviscid flow

GD considered the incompressible flow through a two-dimensional converging passage, for which the complex potential of the inviscid flow, $\bar{w} = \phi + i\psi$, is

$$\bar{w} = \left(\frac{\sigma_c + 1}{2}\right)\bar{z} + \frac{\sigma_c - 1}{2\pi} \left[\ln \sinh \pi \left(\bar{z} + \frac{i}{2}\right) - \frac{\pi i}{2} \right], \quad (\text{A } 1)$$

where $\bar{z} = \bar{x} + i\bar{y}_0$ is a complex variable with $\bar{y}_0 = yA/a^*$. The stream function, which corresponds to the imaginary part of $w(\bar{z})$, is

$$\psi(\bar{x}, \bar{y}_0) = \left(\frac{\sigma_c + 1}{2}\right)\bar{y}_0 - \frac{\sigma_c - 1}{2\pi} \left[\arctan\left(\frac{\cot \pi \bar{y}_0}{\tanh \pi \bar{x}}\right) + \frac{\pi}{2} \right]. \quad (\text{A } 2)$$

Hence the inviscid velocity components, $\bar{U}(\bar{x}, \bar{y}_0)$ and $\bar{V}(\bar{x}, \bar{y}_0)$, can be obtained as

$$\bar{U}(\bar{x}, \bar{y}_0) = \text{Re}\left(\frac{d\bar{w}}{d\bar{z}}\right) = \left(\frac{\sigma_c + 1}{2}\right) + \frac{(\sigma_c - 1) \sinh 2\pi \bar{x}}{2(\cosh 2\pi \bar{x} + \cos 2\pi \bar{y}_0)}, \quad (\text{A } 3)$$

$$\bar{V}(\bar{x}, \bar{y}_0) = \text{Im}\left(\frac{d\bar{w}}{d\bar{z}}\right) = -\frac{\sigma_c - 1}{2} \frac{\sin 2\pi \bar{y}_0}{\cosh 2\pi \bar{x} + \cos 2\pi \bar{y}_0}. \quad (\text{A } 4)$$

It follows that the inviscid streamwise slip velocity on the wall is

$$U_e(\bar{x}) = \bar{U}(\bar{x}, 0) = \left(\frac{\sigma_c + 1}{2}\right) + \frac{(\sigma_c - 1) \sinh 2\pi \bar{x}}{2(\cosh 2\pi \bar{x} + 1)}, \quad (\text{A } 5)$$

which provides the matching condition for the boundary-layer equations (2.13).

Although $\bar{x}$ is the natural streamwise variable describing the mean flow, we shall use $\hat{x}$ instead so that the velocity field can readily be used for the computation of G rlter

vortices. After switching $\bar{x}$ to $\hat{x}$, the slip velocity (A 5) reads

$$U_e(\hat{x}) = \left(\frac{\sigma_c + 1}{2}\right) + \frac{(\sigma_c - 1)}{2} \frac{\sinh(2\pi r \hat{x})}{\cosh(\pi r \hat{x}) + 1} \quad (r = R_A \Lambda / a^*). \quad (\text{A } 6)$$

In terms of $\hat{x}$ and η , the boundary-layer equations are rewritten as (2.13), which can be solved by a downstream marching method.

A.2. Finite-difference schemes for the two-dimensional boundary layer

Equations (2.13) are discretised by a first-order forward difference scheme for the derivative with respect to $\hat{x}$, and a central difference for the derivatives with respect to η . The explicit finite-difference scheme converts the momentum equation to

$$\begin{aligned} U_B^{i,j} \left(\frac{U_B^{i+1,j} - U_B^{i,j}}{\Delta \hat{x}} \right) + \eta_j \left(-\frac{1}{2\hat{x}_i} + \frac{U_e^{i+1} - U_e^i}{\Delta \hat{x}} \frac{1}{2U_e^i} \right) U_B^{i,j} \left(\frac{U_B^{i,j+1} - U_B^{i,j-1}}{2\Delta \eta} \right) \\ + \frac{V_B^{i,j}}{\sqrt{2\hat{x}_i/U_e^i}} \left(\frac{U_B^{i,j+1} - U_B^{i,j-1}}{2\Delta \eta} \right) = U_e^i \frac{U_e^{i+1} - U_e^i}{\Delta \hat{x}} \\ + \frac{1}{2\hat{x}_i/U_e^i} \left(\frac{U_B^{i,j+1} - 2U_B^{i,j} + U_B^{i,j-1}}{(\Delta \eta)^2} \right), \end{aligned} \quad (\text{A } 7)$$

where (i, j) denotes a mesh point, and the quantity at which is denoted by $q^{i,j}$. Hence the streamwise velocity $U_B^{i+1,j}$ at $(i+1, j)$ can be calculated using the known information at the previous streamwise location,

$$\begin{aligned} U_B^{i+1,j} = E^{i,j} (U_B^{i,j+1} + U_B^{i,j-1}) - (2E^{i,j} - 1) U_B^{i,j} + \frac{U_e^i}{U_B^{i,j}} (U_e^{i+1} - U_e^i) \\ - \Delta \hat{x} \left[\eta_j \left(-\frac{1}{2\hat{x}_i} + \frac{U_e^{i+1} - U_e^i}{\Delta \hat{x}} \frac{1}{2U_e^i} \right) + \frac{V_B^{i,j}}{U_B^{i,j} \sqrt{2\hat{x}_i/U_e^i}} \right] \frac{U_B^{i,j+1} - U_B^{i,j-1}}{2\Delta \eta}, \end{aligned} \quad (\text{A } 8)$$

where

$$E^{i,j} = \frac{U_e^i \Delta \hat{x}}{2\hat{x}_i U_{i,j} (\Delta \eta)^2}. \quad (\text{A } 9)$$

The continuity equation is discretised at the point $(i+1, j-1/2)$, leading to

$$\begin{aligned} \frac{1}{2} \left(\frac{U_B^{i+1,j} - U_B^{i,j}}{\Delta \hat{x}} + \frac{U_B^{i+1,j-1} - U_B^{i,j-1}}{\Delta \hat{x}} \right) + \eta_j \left(-\frac{1}{2\hat{x}_i} + \frac{\partial U_e^i}{\partial \hat{x}} \frac{1}{2U_e^i} \right) \frac{U_B^{i+1,j} - U_B^{i+1,j-1}}{\Delta \eta} \\ + \frac{V_B^{i+1,j} - V_B^{i+1,j-1}}{\sqrt{2\hat{x}/U_e(\hat{x})} \Delta \eta} = 0, \end{aligned} \quad (\text{A } 10)$$

from which the wall-normal velocity $V_B^{i+1,j}$ is obtained,

$$V_B^{i+1,j} = V_B^{i+1,j-1} - \frac{1}{2} \sqrt{2\hat{x}/U_e(\hat{x})} F^{i,j}, \quad (\text{A } 11)$$

where

$$F^{i,j} = \frac{\Delta \eta}{\Delta \hat{x}} \left(U_B^{i+1,j} - U_B^{i,j} + U_B^{i+1,j-1} - U_B^{i,j-1} \right) - \eta_j \left(\frac{1}{\hat{x}_i} - \frac{\partial U_e^i}{\partial \hat{x}} \frac{1}{U_e^i} \right) (U_B^{i+1,j} - U_B^{i+1,j-1}). \quad (\text{A } 12)$$

To provide the appropriate initial conditions, we now exam the boundary layer near the leading edge. As $\hat{x} \rightarrow 0$, $U_e \rightarrow \frac{1}{2}(\sigma_c + 1) + \frac{\sigma_c - 1}{2}\pi r\hat{x}$. Thus U_B and V_B expand as

$$U_B \rightarrow U_{B,0} + \hat{x}U_{B1} + \dots, \quad V_B \rightarrow \sqrt{\frac{U_e}{2\hat{x}}}(V_{B,0} + \hat{x}V_{B1} + \dots). \quad (\text{A } 13)$$

Introduce the stream function Ψ_B such that $U_B = \Psi_{B,y}$ and $V_B = -\Psi_{B,\hat{x}}$. It can be shown that

$$\Psi_B = U_e s F(\hat{x}, \eta), \quad U_{B,0} = U_e F', \quad V_{B,0} = s^{-1}(\eta F' - F), \quad (\text{A } 14)$$

where $F(\eta)$ satisfies the Blasius equation, $F'''' + FF'' = 0$, subject to the boundary and matching conditions: $F(0) = F'(0) = 0$ and $F' \rightarrow 1$ as $\eta \rightarrow \infty$.

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