Nearly Kähler Geometry in Six Dimensions

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Declaration of Originality

I declare that the worked contained herein is my own and original, except where explicitly stated otherwise.

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Abstract

This thesis is an overview of the geometry of nearly Kähler six-manifolds. A nearly Kähler structure on a manifold M is a special kind of non-integrable Hermitian structure (g, J) quite different from a Kähler structure. Dimension six is particularly interesting due to connections with the exceptional holonomy group G_2 in dimension seven. Moreover, the classification in arbitrary dimension is reducible to that in dimension six, where there are only four known examples, homogeneous structures on S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and the variety of flags in $\mathbb{C}^3$.

With this background, we review the geometry of six dimensional nearly Kähler manifolds and explore examples. We prove that every nearly Kähler six-manifold (M, g, J) is Einstein, establish the connection with G_2 geometry and prove an important uniqueness theorem for complete nearly Kähler six-manifolds.

This final result is particularly important for the consideration of group actions: if (M, g, J) is a complete nearly Kähler six-manifold not isometric to a round sphere and G is a group of isometries of (M, g), then G preserves J also. Nearly Kähler six-manifolds with a large degree of symmetry are studied in some depth in this thesis. We review the works of Butruille and Podestà-Spiro where a group of isometries acts transitively and with codimension one, respectively. The former problem is solved completely, producing just the four examples alluded to above. In the latter situation, M must be one of S^6, $S^3 \times S^3$ or projective 3-space, and $\mathrm{SU}(3) \times \mathrm{SU}(3)$ is the only interesting group that can act. The classification of nearly Kähler structures on M is reduced to a system of non-linear ODE which, following Podestà-Spiro, we make the first steps in analysing. The work falls short of a classification of complete cohomogeneity one nearly Kähler six-manifolds and we conclude with a summary of the work remaining to be done in this direction.

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Chapter 1

Introduction

The subject of this thesis is a certain class of almost Hermitian manifolds. Recall that a triple (M, g, J) is called *almost Hermitian* if (M, g) is an even dimensional Riemannian manifold and J is an almost complex structure on M that is g-orthogonal,

$$g(X, Y) = g(JX, JY) \ , \ \forall \, X, Y \in \mathcal{X}(M).$$

Kähler geometry consists in making the most natural restriction on such structures, namely that J be g-parallel, defining a reduction of the holonomy of (M, g) to a subgroup of $\mathrm{U}(n)$. According to the classification of [GH80], however, the vanishing of ∇J, where ∇ is the Levi-Civita connection of g, is but one of 16 possible types of symmetry this tensor can possess. The next simplest is the nearly Kähler condition.

Definition 1.1. An almost Hermitian manifold (M^{2n}, g, J) is said to be *nearly Kähler* if ∇J is a skew TM-valued bilinear form,

$$(\nabla_X J)X = 0 \ , \ \forall \, X \in TM. \tag{1.1}$$

Such a triple (M, g, J) is said to be a *nearly Kähler manifold*. Two nearly Kähler manifolds (M_1, g_1, J_1), (M_2, g_2, J_2) are equivalent if there is a diffeomorphism $f : M_1 \to M_2$ such that $f^* g_2 = g_1$ and $f^* J_2 = J_2$.

Remark 1.2. Evidently, if f is a local isometry between (M_1, g_1) and (M_2, g_2), *e.g.* a Riemannian covering, and (g_2, J_2) is nearly Kähler, then $(g_1, f^* J_2)$ is nearly Kähler also.

For a nearly Kähler structure (g, J) one can compute that the Nijenhuis tensor N_J is given by the expression

$$N_J(X, Y) = 4J(\nabla_X J)Y, \ X, Y \in \mathcal{X}(M),$$

so J is integrable if and only if (g, J) is a Kähler structure.

Remark 1.3. Using the decomposition of the bundle of complex forms defined by J, the Nijenhuis tensor can be seen to define a map $\Lambda^{(2,0)}TM \to \Lambda^{(2,0)}TM$. Only in complex dimension three do the domain and range of this map have the same dimension. In this dimension, therefore, there exists the notion of a non-degenerate Nijenhuis tensor. This notion and its close relation to six-dimensional nearly Kähler geometry are discussed in [Ver08].

Definition 1.4. A nearly Kähler structure is said to be *strict* if for every $p \in M$ and non-zero $X \in T_p M$ the endomorphism $\nabla_X J$ is non-trivial. In particular, if (g, J) is strictly nearly Kähler then it is not Kähler and J is not integrable.

The latter definition only becomes relevant above complex dimension two.

Lemma 1.5. *If* (M, g, J) *is a strictly nearly Kähler manifold then* $\dim M \geq 6$.

Proof. For any almost Hermitian structure (g, J) one can prove the following, valid for any $X, Y \in \mathcal{X}(M)$,

$$(\nabla_X J)JY = -J(\nabla_X J)Y, \tag{1.2}$$
$$g((\nabla_X J)Y, Y) = 0, \tag{1.3}$$

The first of these follows by differentiating $J^2 = -\mathrm{Id}$, the second by differentiating $g(Y, JY) = 0$. Supposing now that (g, J) is nearly Kähler and M has dimension four, we show that (g, J) cannot be strictly nearly Kähler. The proof of this when $\dim M = 2$ is similar but shorter still.

Fix $p \in M$ and let X, JX, Y, JY be a unitary basis of T_pM. By (1.3) we see that $g((\nabla_X J)Y, Y) = 0$, and using (1.1) we have also

$$g((\nabla_X J)Y, X) = -g(Y, (\nabla_X J)X) = 0.$$

Using (1.2) we also have $g((\nabla_X J)Y, JX) = g((\nabla_X J)Y, JY) = 0$. Thus $(\nabla_X J)Y = 0$ and, by (1.2), $(\nabla_X J)JY = 0$. As (1.2) implies $(\nabla_X J)JX = 0$, we conclude $\nabla_X J = 0$. The nearly Kähler structure (g, J) cannot therefore be strict. □

The geometry of Kähler manifolds is very rich, and one can produce many examples, the complex projective spaces $\mathbb{CP}^n$and their complex submanifolds, for instance. There is also a simple local description: on a complex manifold, every Kähler metric is given locally by an expression of the form $\partial\bar{\partial}f$ for a real-valued function f. On the other hand, the curvature of a Kähler metric can have an arbitrary sign: there are abundant examples of Kähler manifolds with positive, negative and vanishing Ricci tensors. This is not the case for strictly nearly Kähler manifolds: according to Theorem 1.1 of [Nag02], every strictly nearly Kähler manifold has positive Ricci curvature. In particular, therefore, every complete strictly nearly Kähler manifold is compact with finite fundamental group. In dimension six, strictly nearly Kähler manifolds are also Einstein (Theorem 4.4 below). This and the paucity of examples of complete strictly nearly Kähler manifolds suggests that strictly nearly Kähler geometry is more rigid than Kähler geometry, at least globally (there is a brief discussion of the local problem in section 4.3 of [Bry06]).

On any almost Hermitian manifold (M, g, J) the following formula defines a connection preserving (g, J),

$$D_X Y = \nabla_X Y - \frac{1}{2}J(\nabla_X J)Y\ ,\ X, Y \in \mathcal{X}(M).$$

The condition that (g, J) be Kähler is equivalent to $D = \nabla$. In nearly Kähler geometry, where $D \neq \nabla$, it transpires that D is, for some purposes, the more useful connection. This is evidenced by recent work of Nagy (*e.g.* [Nag02]) and Dávila-Cabrera. The latter authors prove the following de Rham-type result (p 148, [DC12]). See section 1.1.2 for the concept of twistor space. For a definition of the notion of special algebraic torsion in (III) and (IV) see p. 152 of [DC12] – all that is important for the purposes of the discussion here is to know that such cases arise from a twistor construction.

Theorem 1.6. *Let (M^{2n}, g, J) be a complete simply connected nearly Kähler manifold. Then it is equivalent to a product of the following classes of nearly Kähler manifold*

(1) Kähler manifolds;

(2) Homogeneous strictly nearly Kähler manifolds of four definite types:

- *(I) The holonomy representation of D is real irreducible. If $n = 3$, then (M, g) is a round six-sphere;*
- *(II) The holonomy representation of D is complex irreducible and there is a D-parallel decomposition $TM = \mathcal{E} \oplus J\mathcal{E}$;*
- *(III) (M, g, J) has special algebraic torsion and its horizontal distribution is complex $\mathrm{Hol}(D)$-reducible. In this case (M, g, J) is the twistor space over a non-symmetric positive quaternion Kähler manifold;*
- *(IV) (M, g, J) has special algebraic torsion and its horizontal distribution is real $\mathrm{Hol}(D)$-irreducible. In this case (M, g, J) is the twistor space over a symmetric positive quaternion Kähler manifold;*

(3) Non-homogeneous twistor spaces over quaternion Kähler manifolds with positive scalar curvature.

(4) Non-homogeneous six dimensional strictly nearly Kähler manifolds.

No examples are yet known of nearly Kähler manifolds of type (3) or (4), and the problem of constructing examples is very pertinent. Related to this is a conjecture of LeBrun-Salamon (explored in their paper [LS94]) which asserts that every positive quaternion Kähler manifold is homogeneous (and therefore a Wolf space [Ale68]) – this would be true if (3) were vacuous. The problem of whether there exist non-homogeneous six dimensional nearly Kähler manifolds was a significant motivation for the author's original interest in nearly Kähler geometry.

The theorem strongly suggests that nearly Kähler geometry in dimension six has an exceptional character. This is due to a number of algebraic phenomena peculiar to dimension six elaborated in Chapter 4. In this dimension there are but four known examples: they are homogeneous strictly nearly Kähler structures on S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$. Their geometry is described in sections 1.1.1 and 3.2. Non-complete six dimensional examples will also be produced in section 4.5 and in Theorem 6.15: the former examples illustrate some of the special global features of nearly Kähler geometry in dimension six. Nearly Kähler geometry in dimension six is also intimately connected with the seven dimensional geometry of the exceptional Lie group G_2, and this connection constitutes one of the principal themes of this work.

The outline for this thesis is as follows. We present in 1.1 the first examples of nearly Kähler manifolds. As a means to understand these and other examples, the theory of 3-symmetric spaces is described in the chapter 3, concluding with a detailed discussion of the six-dimensional strictly nearly Kähler structures on$S^3 \times S^3$, $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$.

From chapter 4 onwards we specialise to dimension six: chapter 4 describe the special flavour nearly Kähler geometry takes in this dimension; chapters 5 and 6 review recent studies of nearly Kähler six manifolds with a large degree of symmetry: section 5.1 reviews the work of Butruille on homogeneous nearly Kähler manifolds; section 5.2 and chapter 6 that of Podestà and Spiro on the cohomogeneity one case. Although the full cohomogeneity one problem is so far unsolved, the interest of much of the work of Podestà and Spiro lies in their failure to produce new examples. Indeed, having shown that the only compact manifolds that can be strictly nearly Kähler and admit a cohomogeneity one isometric group action are S^6, $S^3 \times S^3$ and $\mathbb{CP}^3$. The anal-

ysis involved in proving that any such nearly Kähler structure is homogeneous seems at present rather difficult.

The outline above presupposes a large amount of material. In Chapter 2, we provide a short review of standard material on manifolds admitting groups actions with only one or two orbit types, *i.e.* homogeneous and cohomogeneity one group actions, respectively. This is included to fix the relevant concepts for later use and make the exposition in chapter 5 and 6 as streamlined as possible.

1.1 First Examples

1.1.1 The Round S^6

The first example of a strictly nearly Kähler manifold is the familiar round six-sphere. Here, the geometry of the exceptional Lie group G_2 is already manifest. An excellent introduction to G_2 geometry, including details of all the facts about G_2 used throughout this thesis, is the set of notes [SW10].

Recall that the exceptional group G_2 is defined as the group of automorphisms of the eight-dimensional normed algebra of octonions $\mathbb{O}$. As an inner product space $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$, and the algebra structure is defined by

$$(p, q) \cdot (r, s) = (pr - \bar{s}q, sp + q\bar{r}) \ , \ p, q, r, s \in \mathbb{H}.$$

It is a basic fact that G_2 acts by isometries on $\mathbb{O}$ fixing the identity $(1, 0)$. Thus G_2 is a subgroup of $\mathrm{SO}(7)$ and acts on the unit sphere in the orthogonal complement of 1 in $\mathbb{O}$. This action is transitive. Moreover, the isotropy subgroup K_x of any x in this S^6 is conjugate to $\mathrm{SU}(3)$. Indeed, K_x acts on the tangent plane $x^\perp$ and this latter space possesses a complex structure J_x defined by

$$J_x(y) = \mathrm{Im}(x \cdot y),$$

where Im is the orthogonal projection in $\mathbb{O}$ onto $1^\perp = \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$. Since K_x acts by automorphisms and isometries of $\mathbb{O}$, one sees that it acts by unitary transformations on the complex vector space $(x^\perp, J_x)$. A little more work shows that $K_x = \mathrm{SU}(x^\perp, J_x)$.

For $x, y \in \mathbb{R}^7 \subset \mathbb{O}$, the map appearing above,

$$\times : (x, y) \longmapsto \mathrm{Im}(x \cdot y),$$

is a bilinear operation called a *vector cross product.* As will be seen later, the notion of vector cross product is of some importance to nearly Kähler geometry. For now we prove the following.

Proposition 1.7. *Let n be the outward unit normal vector field of $S^6 \subset \mathbb{R}^7$ and define an almost complex structure on S^6 by*

$$JX = n \times X \ , \ X \in TS^6.$$

Then (g_{rd}, J) is a strict nearly Kähler structure on S^6.

Proof. Let ∇' be the the Levi-Civita connection of g_{rd}. For $X, Y \in \mathcal{X}(S^6)$, this is given by

$$\nabla'_X Y = \nabla_X Y - (n \cdot \nabla_X Y)\, n,$$

where $(\ ,\ )$ and ∇ are, respectively, the Euclidean metric and connection on $\mathbb{R}^7$. Now, for any orthonormal basis $e_1, \ldots, e_7$ of $\mathbb{R}^7$ and X a vector field orthogonal to n,

$$\nabla_X n = \sum_{i,j} X_i \nabla_i (\frac{x_j}{r} e_j) = \sum_{i,j} X_i e_j \left(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} \right) = \frac{X}{r} - \frac{X \cdot n}{r} n = \frac{X}{r}.$$

So for any X, Y vector fields orthogonal to n,

$$\begin{aligned}\nabla'_X(JY) &= (\nabla_X n) \times Y + n \times (\nabla_X Y) - n((\nabla_X n) \times Y + n \times (\nabla_X Y), n) \\ &= \frac{X}{r} \times Y + n \times \big(\nabla'_X Y + (\nabla_X Y, n)n\big) - n(\frac{X}{r} \times Y, n) \\ &= \frac{1}{r} X \times Y - n(\frac{1}{r} X \times Y, n) + J(\nabla'_X Y).\end{aligned}$$

Restricting to the unit sphere then, we see that

$$(\nabla'_X J)Y = \nabla'_X(JY) - J(\nabla'_X Y) = X \times Y - n(X \times Y, n),$$

which is a skew but non-trivial bilinear form in X, Y. □

How many nearly Kähler structures does (S^6, g_{rd}) possess? Consider an arbitrary orthonormal basis $E = \{e_1, \ldots, e_7\}$ of $\mathbb{R}^7$. Adjoin a symbol 1 to E, identify (with the obvious orderings) the resulting set with the standard basis of $\mathbb{H} \oplus \mathbb{H}$ and let $\mathbb{O}_E$ be the octonionic algebra these generate. Every such E therefore defines a vector cross product $\times_E$ on $\mathbb{R}^7$ and an almost complex structure J_E on S^6.

For two orthonormal bases E, E', when are the strictly nearly Kähler structures (g_{rd}, J_E) and $(g_{\mathrm{rd}}, J_{E'})$ equivalent, that is when does there exist $g \in \mathrm{Isom}(S^6, g_{\mathrm{rd}})$ such that $g^* J_{E'} = J_E$? Supposing such a $g \in \mathrm{SO}(7)$ exists, we have, for any $p \in S^6$ and $X \in T_p S^6$

$$n_p \times_E X = (J_E)_p X = (g^* J_{E'})_p(X) = n_{g \cdot p} \times_{E'} gX.$$

Setting $p = e_i$ and $X = e_j$, $j \neq i$, this becomes

$$e_i \times_E e_j = g e_i \times_{E'} g e_j.$$

Identifying $\mathbb{O}_E$ and $\mathbb{O}_{E'}$ with $\mathbb{O}$, this implies that $g \in \mathrm{G}_2$ and $gE = E'$. The set of equivalence classes of strict nearly Kähler structures (g_{rd}, J) on S^6 therefore contains the 7-dimensional coset space $\mathrm{SO}(7)/\mathrm{G}_2$ diffeomorphic to $\mathbb{RP}^7$. We shall later prove that every nearly Kähler structure on the round six-sphere is obtained in this way (Proposition 4.12). This description of the nearly Kähler structures on the round six-sphere paints a perhaps misleading picture of the abundance of nearly Kähler structures. Contrary to what one might expect, Theorem 4.5 proved below shows that the nearly Kähler geometry of (S^6, g_{rd}) is in fact very atypical.

1.1.2 Twistor Spaces over Positive Quaternion-Kähler Manifolds

The twistor spaces provide a particularly important class of examples of strictly nearly Kähler manifolds. While we confine ourselves here merely to statements of fact, our assertions will be borne out by later examples.

Let (M, g) be an even-dimensional oriented Riemannian manifold. The *twistor fibration*, $\tau : \mathcal{T}(M, g) \to M$, of M is the fibre bundle whose fibre over $p \in M$ is the set of orthogonal complex structures on the tangent plane inducing the correct orientation on $T_p M$,

$$\mathcal{T}(M, g)_p = \{J \in \mathrm{SO}(T_p M, g) \ : \ J^2 = -\mathrm{Id}_{T_p M}\}.$$

When $\dim M = 4$, it turns out that $\tau : \mathcal{T}(M, g) \to M$ is a 2-sphere bundle and can be identified with the bundle of unit anti-self-dual 2-forms on M. On the fibres of τ there is then an Hermitian structure, the

standard Kähler structure on $\mathbb{CP}^1$. In fact, it can be shown that there exists a canonical Kähler structure (h, J) on the fibres of the twistor fibration whatever the dimension of M and, given this, one can define a family of Hermitian structures $(g_s, J_\pm)$, $s > 0$, on the total space $\mathcal{T}(M, g)$ as follows.

First, fix a splitting

$$T_x\mathcal{T}(M,g) = T_x\tau^{-1}(\tau(x)) \oplus (\tau^* T_{\tau(x)}M)_x \ , \ x \in \mathcal{T}(M,g).$$

With respect to this, $J_\pm$ can be defined by

$$J_\pm|T_x\tau^{-1}(\tau(x)) = \pm J_x \ , \ J_\pm|(\tau^* T_{\tau(x)}M)_x = x,$$

and the metrics by

$$g_s = s\, h_\tau + \tau^* g \ , \ s > 0.$$

It turns out that if (M^{4n}, g) is quaternion-Kähler (self-dual when $n = 1$) with positive Ricci curvature, then (g_s, J_-) is strictly nearly Kähler for the unique value $s = \frac{1}{2}$ (see [ES85]). In fact, in four dimensions the only such manifolds are the round 4-sphere and $\mathbb{CP}^2$ with its Fubini-Study metric, and it is possible to identify the twistor spaces of these manifolds as $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$, respectively (Theorem 6.1, [Hit81]). We shall describe the strictly nearly Kähler structures on $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$ resulting from this construction more fully in section 3.2.

Chapter 2

Group Actions with Few Orbit Types

As advertised in the introduction, this is to be a short chapter reviewing for later use standard material on group actions. Throughout this thesis, if G denotes a Lie group, then its Lie algebra is written $\mathfrak{g}$. The adjoint representation of a Lie group G on its Lie algebra is denoted Ad_G.

Definition 2.1. A smooth action by a Lie group G on a smooth manifold M is a homomorphism $f : G \to \mathrm{Diff}(M)$. We refer to M as a *smooth G-space*, and for all $g \in G$ and $p \in M$ we abbreviate $f(g)(p)$ as $g \cdot p$. A diffeomorphism ϕ between G-spaces M_1 and M_2 is called a *G-diffeomorphism* or *G-equivalence*, if ϕ commutes with the given actions by G, *i.e.* $g \cdot \phi(p) = \phi(g \cdot p)$ for all $g \in G$ and $p \in M$.

G is said to act *transitively* on M if for some (and therefore any) $p \in M$ the orbit $G \cdot p$ coincides with M. For $p \in M$, the subgroup of elements of G fixing p, *i.e.* the *isotropy* or *stabiliser group* of p, is denoted H_p. For any $p \in M$ the G action defines a representation of H_p on T_pM, the *isotropy representation* of H_p.

G is said to act *effectively* if $\ker f$ is trivial, *almost effectively* if $\ker f$ is discrete. It can be assumed that any G-action is effective for as $\ker f$ is a closed normal subgroup, $\widetilde{G} = G/\ker f$ is a compact Lie group and acts effectively on M. If $\ker f$ is discrete, then $\widetilde{G}$ and G have isomorphic Lie algebras.

A tensor τ of rank (r, s) on M is said to be *G-invariant* if $g^*\tau = \tau$, *i.e.* for all vector fields $X_1, \ldots, X_r$ and 1-forms $\alpha_1, \ldots, \alpha_s$ we have

$$\alpha(g_*X_1, \ldots, g_*X_r, g^*\alpha_1, \ldots, g^*\alpha_s)|_p = \alpha(X_1, \ldots, X_r, \alpha_1, \ldots, \alpha_s)|_{g\cdot p}, \ \forall g \in G, \ p \in M.$$

In this thesis we shall be concerned firstly with G-invariant metrics on a manifold M, later with invariant almost complex structures and differential forms.

The basic result about group actions is the following (Chapter IV, Theorem 3.1, [Bre72]). See Chapter I, Theorem 3.1, [Bre72] for a list of properties of the quotient map $\pi : M \to M/G$.

Theorem 2.2. *Let G be a compact Lie group and M a connected smooth G-space with M/G connected. Then there exists a unique maximal orbit type G/H for the action, that is H is conjugate to a subgroup of each isotropy group H_p. The union M^* of orbits of this type is an open dense subset of M.*

Definition 2.3. If G and M are as in the theorem, then H is referred to as the *principal isotropy subgroup* and an orbit in M^* as a *principal orbit*. The set M^* is called the *regular part* of M. Orbits disjoint from M^* are referred to as *singular*: singular orbits correspond to points in $M/G \setminus \pi(M^*)$.

The following result is a general case of Theorem 9.3, Chapter I of [Bre72].

Theorem 2.4. *Let G be a connected Lie group and M a connected G-space. If the G-action is effective then there exists a connected Lie group $\widetilde{G}$ covering G that acts effectively on the universal covering space $\widetilde{M}$ of M, covers the action of G on M and that fits into the following short exact sequence*

$$1 \longrightarrow \pi_1(M) \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1.$$

In particular, if G is compact and $\pi_1(M)$ is finite, then $\widetilde{G}$ is compact.

2.1 Transitive Group Actions

The simplest G-spaces M are those for which G acts transitively. In this case there is only one principal orbit, namely the whole of M. We are particularly interested in the Riemannian situation.

Definition 2.5. A connected smooth G-space M is said to be *homogeneous* if G acts transitively. A Riemannian manifold (M, g) is said to be a *Riemannian homogeneous space*, or *homogeneous*, if there exists a connected Lie group G which acts transitively on M leaving g invariant.

In section 5.3 we shall classify all strictly nearly Kähler six-manifolds (M, g, J) admitting a transitive action by a group G preserving the structure (g, J). Homogeneous spaces are automatically complete, so Theorem 4.4 implies that any such space M is compact and $\pi_1(M)$ is finite. Moreover, the isometry group $\mathrm{Iso}(M, g)$ of a compact Riemannian manifold is a compact Lie group (see [MS39]), and so it follows that if G is a Lie group acting transitively and effectively on M, then G is compact. We shall therefore assume throughout this section that both G and M are compact and that $\pi_1(M)$ is finite.

The following theorem is the basic result on homogeneous spaces (*e.g.* chapter X of [KN69]).

Theorem 2.6. *Let (M, g) be a compact Riemannian homogeneous space for an action by a compact Lie group G. Fixing $p \in M$ and setting $H = H_p$, then*

(i) Each of the subgroups H_q for $q \in M$ is conjugate to H in G;

(ii) There is a metric $\hat{g}$ on G that is G-invariant and that induces an orthogonal splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with the property that $\mathfrak{m}$ is invariant under the adjoint action by H, $\mathrm{Ad}_G|_H \mathfrak{m} = \mathfrak{m}$;

(iii) Letting $\hat{\mathfrak{m}}$ denote the left-invariant distribution on G induced by $\mathfrak{m}$, then $\hat{g}|_{\hat{\mathfrak{m}}}$ defines a G-invariant metric $\bar{g}$ on G/H and there is an isometry between $(G/H, \bar{g})$ and (M, g) under which eH is identified with p, and $\mathfrak{m}$ identified with T_pM;

(iv) Under the above identification, there is a one-to-one correspondence between G-invariant tensor fields on M and $\mathrm{Ad}_G|_H$-invariant tensors on $\mathfrak{m}$.

Remark 2.7. A homogeneous space G/H for which there is an invariant decomposition of $\mathfrak{g}$ as in (ii) is said to be *reductive*. If G is not be compact, there may be no G-invariant metrics on G and G/H may not be reductive.

Remark 2.8. The above proposition provides a description of all G-invariant metrics on a compact homogeneous space G/H. In particular, if there is an H-invariant decomposition of the isotropy representation of G/H, $\mathfrak{m} = \bigoplus_{\alpha=1}^{r} \mathfrak{m}_\alpha$ where the $\mathfrak{m}_\alpha$ are mutually non-equivalent irreducible representations of H, then the space of G-invariant metrics on G/H is r-dimensional, for up to scale there exists a unique H-invariant metric on each of the $\mathfrak{m}_\alpha$ by Schur's lemma. In most cases of interest in this thesis this assumption on the decomposition of the isotropy representation will hold, but if any of the $\mathfrak{m}_\alpha$ appear with multiplicity greater than one, however, then the space of invariant metrics will be larger. The single case in which such considerations are of importance will be in section 5.2.

It is possible to present a homogeneous space G/H in a number of ways, for instance

$$\mathrm{SO}(2n)/\mathrm{SO}(2n-1) = S^{2n-1} = \mathrm{U}(n)/\mathrm{U}(n-1).$$

We describe now how a homogeneous space may be written in some sort of minimal form.

As we assume $\pi_1(M)$ to be finite, if G is a connected compact Lie group acting transitively and effectively on M, then there is by Theorem 2.4 a connected compact Lie group $\widetilde{G}$ acting transitively on the universal covering space $\widetilde{M}$. By this means we reduce to the case in which M is connected and simply connected.

The quotient map $G \to M$ is a principal H-bundle and has the following long exact sequence in homotopy,

$$\cdots \longrightarrow \pi_i(H) \longrightarrow \pi_i(G) \longrightarrow \pi_i(M) \longrightarrow \pi_{i-1}(H) \longrightarrow \cdots \longrightarrow \pi_0(M) \longrightarrow 1\,.$$

Since $\pi_1(M)$, $\pi_0(M)$ and $\pi_0(G)$ are assumed trivial, we see from this that H is connected. It is possible to reduce to the case in which G is semi-simple (Proposition 9, [Oni94]).

Lemma 2.9. *Let G/H be a compact almost effective homogeneous space with finite fundamental group. Let $\mathfrak{z} \oplus \mathfrak{g}_S$ be the decomposition of the Lie algebra of G into its centre $\mathfrak{z}$ and semi-simple part $\mathfrak{g}_S$, and let G_S be the connected subgroup of G with Lie algebra $\mathfrak{g}_S$. Then G_S acts transitively and almost effectively on G/H.*

Now, it is well known that if G is a compact connected semi-simple Lie group then $\pi_1(G)$ is finite (Remark 7.13, [BtD85]). The universal covering space $\widetilde{G}$ of G is therefore a compact semi-simple Lie group.

Lemma 2.10. *Let $M = G/H$ be a homogeneous space and $\pi : \widetilde{G} \to G$ the universal covering group of G. Then setting $\widetilde{H} = \pi^{-1}(H)$, the map*

$$\rho : \tilde{g}\widetilde{H} \mapsto \pi(\tilde{g})H$$

defines an equivalence between M and $\widetilde{G}/\widetilde{H}$ and the action of $\widetilde{G}$ on M is almost effective.

Proof. The map ρ is well defined and injective for if g_1 and g_2 belong to the same $\widetilde{G}$-orbit then $g_1 g_2^{-1} \in \widetilde{H}$, and this occurs if and only if $\pi(g_1)\pi(g_2)^{-1} \in H$, that is $\pi(g_1)$ and $\pi(g_2)$ belong to the same G-orbit. ρ is onto since π is onto. The action by $\widetilde{G}$ defines a homomorphism $f : \widetilde{G} \to \mathrm{Diff}(M)$, and, by construction, $\ker f$ is a subgroup of the discrete group of deck transformation of $\pi : \widetilde{G} \to G$. The action by $\widetilde{G}$ is therefore almost effective. □

By means of this fact and the preceding lemma, therefore, from the general case of a compact, connected homogeneous space M with finite fundamental group we may reduce to the case in which M is simply connected and $M = G/H$ where G is a compact, connected, simply-connected and semi-simple Lie group acting almost effectively on M and H is a connected compact subgroup.

Consider then G and H as above. G is simply connected and semi-simple, and so may be decomposed as a direct product of compact, connected, simply-connected and simple Lie groups $G_1, \ldots, G_r$. If for some p the projection $H \to G_p$ is onto then it follows from Theorem 1.4 of [BK06] that $\Pi_{i\neq p} G_p$ acts transitively on M with connected isotropy group $H \cap \Pi_{i\neq p} G_p$. We now recapitulate and formulate this reduction as a definition.

Definition 2.11. Let $M = G/H$ be an arbitrary compact, connected, simply connected homogeneous space. The presentation constructed above of M as $(\Pi_i G_i)/H$, where the G_i are compact, connected, simply-connected and simple Lie groups acting almost effectively on M and H is a connected subgroup of $\Pi_i G_i$ such that no projection $H \to G_i$ is onto, is called a *canonical presentation of G/H*.

Remark 2.12. The term canonical presentation is not used to suggest that this presentation is any sense unique, only that given the presentation of M as G/H the canonical presentation produces a particularly nice choice of group that acts transitively on M.

For a compact Riemannian manifold (M^n, g), the dimension of its isometry group is at most $\frac{1}{2}n(n+1)$. The classification of all connected, simply connected compact n-dimensional Riemannian homogeneous spaces is therefore reduced by the above construction to the classification of all simply connected, connected, compact semi-simple Lie groups of dimension at most $\frac{1}{2}n(n+1)$ and of their closed connected subgroups. This programme has been completed up to dimension eleven in [BK06], where it is further shown that every Riemannian homogeneous space of such dimension admits a homogeneous Einstein metric.

In section 5.1 we shall see that the hypothesis that G/H be strictly nearly Kähler further restricts the dimension of G: by considering the isotropy group we shall show that this dimension is at most 14. The list of connected simply connected compact semi-simple Lie groups up to this dimension is extremely short.

Theorem 2.13. *If G is a connected simply connected compact semi-simple Lie group with $\dim G \leq 14$, then G is isomorphic to a product of the groups $\mathrm{SU}(2)$, $\mathrm{SU}(3)$, $\mathrm{Sp}(2)$ or G_2.*

For G a compact Lie group, the homogeneous spaces G/H where H is a connected subgroup of maximal rank in G, *i.e.* H contains a maximal torus of G, are particularly important. Such a homogeneous space is referred to as a *partial G-flag*, a G-flag being a partial G-flag G/T where T is a maximal torus. There is a general theory of maximal subgroups H in a compact Lie group G due to Borel and de Siebenthal [BDS49]. The special nature of homogeneous spaces of this type allows one to count all G-invariant almost complex structures (Proposition 13.4, [BH58]).

Theorem 2.14. *Let G/H be a partial G-flag and let $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ be a decomposition of the isotropy representation into irreducible representations. If G/H possesses a G-invariant almost complex structure, then the $\mathfrak{m}_i$ are unique and G/H admits exactly 2^r G-invariant almost complex structures.*

Remark 2.15. Not every partial flag possesses an invariant almost complex structure, for instance $\mathbb{HP}^n = \mathrm{Sp}(n+1)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ does not (p. 35, [Yan87]). Theorem C of [Wan54] states that a partial flag G/H possesses an *integrable* G-invariant almost complex structure if and only if H is the centraliser of a maximal torus.

2.2 Group Actions of Cohomogeneity One

Definition 2.16. A compact connected Lie group G is said to act with *cohomogeneity one* on a manifold M if M^* consists of orbits of codimension one, equivalently M/G is one dimensional.

One can prove the following structure theorem for such spaces (Chapter IV, Theorem 8.2, [Bre72]).

Theorem 2.17. *Let G be a compact Lie group acting with cohomogeneity one on a compact manifold M with principal isotropy group H. There are two cases according to the topology of M/G:*

(i) M/G is diffeomorphic to S^1. Then M is a G/H-bundle over S^1.

(ii) M/G is a connected closed interval of $\mathbb{R}$. Then M^* corresponds to the interior of M/G and there are two singular orbits with isotropy groups K_1, K_2 corresponding to the boundary points of M/G. Moreover, there are representations $K_i \to \mathrm{O}(V_i)$ of K_i on Euclidean spaces V_i such that the induced action on the unit sphere in V_i is transitive and has isotropy group H, and M is G-diffeomorphic to the identification space

$$G \times_{K_1} D_1 \bigcup_{\varphi} G \times_{K_2} D_2,$$

where D_i is the unit disk in V_i and φ:$G/H \to G/H$ is some G-equivalence of $\partial D_i = G/H$.

Remark 2.18. Using this description, it is in principle possible to give a complete classification of cohomogeneity one G-spaces. This has been done, for example, in dimensions five, six and seven in the thesis [Hoe07].

If M has finite fundamental group then (i) is not possible, for the long exact homotopy sequence gives

$$\pi_1(M) \longrightarrow \pi_1(S^1) \longrightarrow \pi_0(G/H),$$

and connectivity of G/H implies then that $\pi_1(M) \to \pi_1(S^1)$ is onto, a contradiction. All the cohomogeneity one G-spaces considered here then will be a union of disc bundles. For applying the theory of cohomogeneity one group actions to strictly nearly Kähler manifolds, therefore, we shall always assume that the topology is of the kind described in (ii) of the theorem.

Let V be a Euclidean space and $\rho : K \to \mathrm{O}(V)$ a representation of a compact Lie group that acts transitively on the unit sphere in V with isotropy group H. Setting $\hat{H} = \ker \rho \subset H$, the group $K/\hat{H}$ now acts effectively and transitively on the unit sphere in V with isotropy group $H/\hat{H}$. Borel has classified the compact Lie groups that act in this way. His results are summarised in Table 2.1.

$\hat{K}$	$\mathrm{SO}(n)$	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$\mathrm{Sp}(n)\mathrm{U}(1)$	$\mathrm{Sp}(n)$	G_2	$\mathrm{Spin}(7)$	$\mathrm{Spin}(9)$
$\hat{H}$	$\mathrm{SO}(n-1)$	$\mathrm{U}(n-1)$	$\mathrm{SU}(n-1)$	$\mathrm{Sp}(n-1)\mathrm{Sp}(1)$	$\mathrm{U}(n-1)\mathrm{Sp}(1)$	$\mathrm{Sp}(n-1)$	$\mathrm{SU}(3)$	G_2	$\mathrm{Spin}(7)$
$\hat{K}/\hat{H}$	S^{n-1}	S^{2n-1}	S^{2n-1}	S^{4n-1}	S^{4n-1}	S^{4n-1}	S^6	S^7	S^{15}

Table 2.1: Borel's list

Definition 2.19. The *group diagram* of a compact manifold admitting a cohomogeneity one almost effective action by a Lie group G, with principal isotropy group H and singular isotropy groups K_1, K_2, is the string of inclusions $G \supset K_1, K_2 \supset H$. We say that a sequence of compact groups $G \supset K_1, K_2 \supset H$ is *realisable* if it is the group diagram of a compact G-space.

The following result is taken from section 1 of [GWZ08].

Theorem 2.20. *A sequence of compact Lie groups $G \supset K_1, K_2 \supset H$ is realisable if and only if K_1/H and K_2/H are diffeomorphic to spheres in Euclidean spaces V_i, for i =1,2, respectively. All realisations are of the form*

$$G \times_{K_1} D_1 \bigcup_{\varphi} G \times_{K_2} D_2,$$

where D_i is the unit disk in V_i and $\varphi : G/H \to G/H$ is some G-equivalence of $\partial D_i = G/H$. Two such realisations are G-equivalent if and only if they are related by the following operations

(i) Interchanging K_1 and K_2;

(ii) Conjugating H, K_1 and K_2 by the same element of G;

(iii) Replacing one of the K_i with hK_ih^{-1} for $h \in N_G(H)$, where N_G denotes the connected component of the normaliser in G.

The set of G-equivalence classes of realisations of (G, H, K_1, K_2) is thus in one-to-one correspondence with the double quotient

$$N_0 \backslash N_G(H) / N_1,$$

where $N_i = N_G(H) \cap N_G(K_i)$, $i = 1, 2$.

In classifying cohomogeneity one nearly Kähler six-manifolds a small number of basic tools are used to eliminate the *a priori* groups that might act. The fundamental group is the main device to be used, but a useful trick is to enlarge the group which acts following the following construction (Proposition 2.1.16, [Hoe07]).

Proposition 2.21. *Let M be a compact simply connected manifold admitting a cohomogeneity one action with group diagram $G \supset K_1, K_2 \supset H$ and let L be a compact connected subgroup of $N_G(H) \cap N_G(K_1) \cap N_G(K_2)$. Define $G' = L/(L \cap H)$ and an action by $(g, [l]) \in G \times G'$ on M by the following action on each G-orbit $g_1 H_p \subset M$ (H_p=H or K_1, K_2) by*

$$(g, [l]) \cdot g_1 H_p = g g_1 l^{-1} H_p.$$

This action by $G \times G'$, the normal extension of G by L*, is smooth and almost effective, the $G \times G'$-orbits coincide with those of G and the group diagram is*

$$G \times G' \supset (K_1 \times 1) \cdot \Delta L, (K_2 \times 1) \cdot \Delta L \supset (H \times 1) \cdot \Delta L,$$

where $\Delta L = \{(l, [l]) : l \in L\}$.

The geometric picture of cohomogeneity one G-spaces is relatively straightforward.

Proposition 2.22. *If G acts with cohomogeneity one on a compact simply connected manifold M and M possesses a G-invariant metric g, then there is a unit vector field ξ on M, unique up to orientation, the integral curves of which are geodesics and such that*

(i) ξ is G-invariant and intersects all G-orbits orthogonally;

(ii) For $p \in M^*$, any integral curve γ_p of ξ through p can be extended to a minimal geodesic $\gamma_p : [0, l] \to M$ connecting the singular orbits.

Moreover, there is a G-diffeomorphism between M^* and $(0, l) \times G/H_p$. In these coordinates, g is of the form

$$dt^2 + g_t,$$

where $t \in (0, l)$ and $\{g_t\}_{t \in (0,l)}$ is a family of G-invariant metrics on the orbit G/H_p. Finally, for S a singular orbit with $\gamma_p(0) \in S$ and $H_{\gamma_p(0)}$=K, K acts transitively on the unit sphere in $V = (T_q S)^\perp$ with isotropy group H_p, and the normal bundle of S is G-equivalent to $G \times_K V$.

Chapter 3

3-Symmetric Spaces and Nearly Kähler Geometry

Like symmetric spaces, 3-symmetric spaces belong to the class of Riemannian homogeneous spaces defined by an automorphism of finite order, three in this case; they were first studied by Gray in [Gra72] and classified in [WG68a, WG68b] by Gray and Wolf. The role the concept plays in nearly Kähler geometry is not entirely clear, but Butruille has shown that all homogeneous strictly nearly Kähler manifolds are 3-symmetric (see the introduction to Chapter 5). The position adopted in this thesis is that the 3-symmetric space concept is mostly of interest as a means to generate and understand the geometry of examples.

The exposition in the proceeding section parallels the standard discussion of Riemannian symmetric spaces as Riemannian manifolds admitting a family of isometric involutions, working toward the formulation as certain algebraically special homogeneous spaces. We are able to characterise precisely when a 3-symmetric space is nearly Kähler (Proposition 3.5) and, using this result, in the next section describe the six dimensional examples.

3.1 3-Symmetric Spaces

Definition 3.1. An even dimensional Riemannian manifold (M, g) is said to be *locally 3-symmetric* if for each $p \in M$ there exists an isometry θ_p defined near p such that:

(i) $\theta_p^3 = \mathrm{Id}_M$;

(ii) p is an isolated fixed point of θ_p;

(iii) θ_p is holomorphic with respect to the *canonical almost complex* structure J of the family $\{\theta_p\}_{p\in M}$ defined by

$$(d\theta_p)_p = -\frac{1}{2}\mathrm{Id} + \frac{\sqrt{3}}{2}J_p.$$

In other words, $d\theta_p \circ J = J \circ d\theta_p$.

If the domain of the θ_p can be extended to the whole of M then (M, g) is said to be a *3-symmetric space*.

The assumption that the θ_p are local isometries implies that (g, J) is an almost Hermitian structure on M, and an almost Hermitian manifold (M, g, J) is said to be *(locally) 3-symmetric* if J is the canonical almost complex structure of a family of (local) isometries as above.

The following are some basic properties, proved in [Gra72]. The first requires the identities proved in section 4.2.

Proposition 3.2. *Let (M, g, J) be an almost Hermitian manifold and set $\Theta = -\frac{1}{2}\mathrm{Id} + \frac{\sqrt{3}}{2}J$, as before. Then*
(i) If (M, g, J) is nearly Kähler then Θ preserves all derivatives $\nabla^k J$;
(ii) (M, g, J) is Kähler and locally 3-symmetric if and only if it is locally Hermitian symmetric;
(iii) If (M, g, J) is simply connected, complete and locally 3-symmetric then it is 3-symmetric;
(iv) If (M, g, J) is 3-symmetric, then the group of isometries preserving J acts transitively. (M, g) is therefore a homogeneous space and, in particular, complete.

Remark 3.3. Statement (ii) shows that the 3-symmetric concept essentially excludes Kähler geometry. Moreover, given a family $\{\theta\}_{p\in M}$ as in (i) and (ii) of Definition 3.1, and defining $\Theta = -\frac{1}{2}\mathrm{Id} + \frac{\sqrt{3}}{2}J$ for J the canonical almost complex structure of the family, then theθ_p are holomorphic if and only if Θ preserves ∇J and $\nabla^2 J$ (Proposition 3.6 [Gra72]). Thus, part (i) of the proposition tells us that the meaning of the 3-symmetric concept for nearly Kähler geometry is that the infinitesimal isometries Θ integrate to local isometries. In fact, a strictly nearly Kähler six-manifold is 3-symmetric if and only if its curvature satisfies

$$\nabla R(X, X, JX, X, JX) = 0,\ \forall X \in \mathcal{X}(M).$$

This follows from Theorem 4.6 of [Gra72] and the observation that strictly nearly Kähler six-manifolds are Einstein (Theorem 4.4) and therefore analytic by the results of [DK81].

The classification of 3-symmetric spaces is analogous to Cartan's classification of Riemannian symmetric spaces. To describe how this goes, let (M, g) be a Riemannian 3-symmetric space and define G to be the largest connected group of isometries preserving the canonical almost complex structure, H the isotropy group in G of a fixed $p \in M$, and define $t \in \mathrm{Diff}(G)$ by $t(g) = \theta_p \circ g \circ \theta_p^{-1}$. Let G^t be the fixed point set of t, G_0^t its component containing the identity. The following properties are easily verified:

(a) t is an automorphism of order three;

(b) $G_0^t \subset H \subset G^t$;

(c) There are a G-invariant Riemannian metric $\bar{g}$ and a G-invariant almost complex structure $\bar{J}$ on G/H so that $(G/H, \bar{g}, \bar{J})$ and (M, g, J) are equivalent as almost Hermitian manifolds.

The following simple proposition initiates the classification.

Proposition 3.4. *If (G, H, t) satisfies (a)-(c) above and t_* is the induced map of the Lie algebra $\mathfrak{g}$ of G, then*
(i) $\mathfrak{h} = \{X \in \mathfrak{g} \mid t_* X = X\}$,
(ii) G/H is a reductive homogeneous space. Specifically, defining $\mathfrak{m} = \ker(1 + t + t^2)$, there is a vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{m}$ is $\mathrm{Ad}_G|_H$-invariant.

Given a reductive Lie algebra, when does it admit an automorphism of order three? If such a map t exists, there is a decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ where $T = t_*$ acts by $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ on $\mathfrak{m}^{\pm}$. As T is a Lie algebra automorphism the following hold

$$[\mathfrak{m}^+, \mathfrak{m}^+] \subset \mathfrak{m}^- \ ,\ [\mathfrak{m}^-, \mathfrak{m}^-] \subset \mathfrak{m}^+ \ ,\ [\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{h}^{\mathbb{C}}. \tag{3.1}$$

Conversely, given a reductive Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with a splitting $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ satisfying (3.1), we may define a linear map T on $\mathfrak{g}^{\mathbb{C}}$ in the obvious way,

$$T|\mathfrak{h} = \mathrm{Id}\ ,\ T|\mathfrak{m}^{\pm} = \tfrac{1}{2}(-1 \pm i\sqrt{3})\,\mathrm{Id}.$$

A 3-symmetric space is therefore equivalent to a reductive Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with a splitting $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ satisfying (3.1). In fact, the relations (3.1) imply that the canonical almost complex structure J of a 3-symmetric space is never integrable, for the i-eigenspace of $J \in \mathrm{End}(\mathfrak{m}^{\mathbb{C}})$ is, by definition,

$\mathfrak{m}^+$, and (3.1) implies that this is never a Lie algebra.

The next result provides our basic tool for building examples of nearly Kähler manifolds. By classification, Gray shows that the condition of natural reductivity is satisfied for all 3-symmetric spaces (G, H, t) (Theorem 6.1, [Gra72]). In fact, he shows that, up to scale, there is a unique metric making G/H naturally reductive.

Proposition 3.5. *Suppose that (G, H, t) satisfies (a)-(c) above and that there is an $\mathrm{Ad}_G(H)$- and t-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$. Then the induced 3-symmetric space structure on G/H is nearly Kähler if and only if it is a naturally reductive homogeneous space, that is*

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle = \langle X, [Y, Z]_{\mathfrak{m}}\rangle\,,\ \forall\, X, Y, Z \in \mathfrak{m}.$$

Here, $W_{\mathfrak{m}}$ denotes the orthogonal projection of $W \in \mathfrak{g}$ onto $\mathfrak{m}$.

3.2 3-symmetric Strictly Nearly Kähler Six-Manifolds

Three examples of homogeneous strictly nearly Kähler six-manifolds were described in section 1.1, namely the round S^6, and $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$ with non-standard almost Hermitian structures. In this section we describe in more detail the geometry of $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$ and the fourth and only further homogeneous example $S^3 \times S^3$ in greater detail, representing them as 3-symmetric spaces. The round six-sphere is also a 3-symmetric space but we shall not prove this. The fact that these four are the only 3-symmetric spaces in dimension six can be read from the lists given in [Gra72].

3.2.1 $S^3 \times S^3$

Let K be a connected and simply connected compact simple Lie group. Set $G = K \times K \times K$ and let $H = \Delta(K)$ be the diagonal. There is an obvious automorphism of G of order three fixing H, namely the cyclic permutation

$$(k_1, k_2, k_3) \in K \times K \times K \longmapsto (k_3, k_1, k_2).$$

The complement $\mathfrak{m}$ of Proposition 3.4 is easily seen to be

$$\{(X, Y, Z) \in \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k} \,|\, X + Y + Z = 0\}.$$

Given now a bi-invariant metric q on K, the Lie algebra of the isotropy group is always orthogonal to $\mathfrak{m}$ with respect to the product metric on G, and we thus satisfy the natural reductivity hypotheses of Proposition 3.5. To compute the resulting nearly Kähler structure on G/H, identify $K \times K \times K/\Delta(K)$ with $K \times K$ via

$$[k_1, k_2, k_3] \longmapsto (k_1 k_3^{-1}, k_2 k_3^{-1}).$$

Then it is straightforward to compute that, for left-invariant vector fields X, Y, X', Y' on K, the nearly Kähler structure on $K \times K$ is given by

$$\begin{aligned} g((X, Y), (X', Y')) &= q(X, X') + q(Y, Y') - \tfrac{1}{2}\left(q(X, Y') + q(Y, X')\right), \\ J(X, Y) &= \tfrac{2}{\sqrt{3}}(-Y + \tfrac{1}{2}X, X + \tfrac{1}{2}Y). \end{aligned} \tag{3.2}$$

If $K \times K$ is six dimensional then $K = \mathrm{SU}(2)$ and we obtain a strict nearly Kähler structure on $S^3 \times S^3$. This nearly Kähler structure is invariant also for the left action of $S^3 \times S^3$ on itself.

3.2.2 $\mathbb{CP}^3$

Identifying $(w,z) \in \mathbb{C}^4$ with $(w, zj) \in \mathbb{H} \oplus \mathbb{H}$, defines a transitive action by $G = \mathrm{Sp}(2)$ on $\mathbb{C}^4$. This descends to a transitive action on $\mathbb{CP}^3$. The isotropy group of $[1:0:0:0]$ is $H = S^1 \times \mathrm{Sp}(1)$.

Consider the following $\mathrm{Ad}(H)$-invariant subspaces of $\mathfrak{g}$,

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{pmatrix} \middle| \, \alpha \in \mathbb{H} \right\}, \; \mathfrak{p}^- = \left\{ \begin{pmatrix} zj & 0 \\ 0 & 0 \end{pmatrix} \middle| \, z \in \mathbb{C} \right\}.$$

The subspace $\mathfrak{m} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ is an $\mathrm{Ad}(H)$-invariant complement to $\mathfrak{h}$ in $\mathfrak{g}$. The representation of $\mathrm{Sp}(1)$ on $\mathfrak{p}^+$ is equivalent to that on $\mathbb{H}$ in which $\mathrm{Sp}(1)$ acts by multiplication on the left; $\lambda \in S^1$ acts on the right by $\overline{\lambda}$. On $\mathfrak{p}^-$, $\mathrm{Sp}(1)$ acts trivially and the representation of S^1 is equivalent to that on $\mathbb{C}$ in which λ acts as multiplication by λ^2. As the $\mathfrak{p}^\pm$ are irreducible, any invariant metric is of the form $g_{s,t} = sg + th$ for $s,t > 0$, where g and h are the standard Hermitian structures on $\mathbb{H}$ and $\mathbb{C}$, respectively.

Suppose now that $\sigma \in \mathrm{Aut}(G)$ fixes H: it induces an automorphism σ_* of $\mathfrak{m}$. As σ commutes with $\mathrm{Ad}(H)$ and the $\mathfrak{p}^\pm$ are irreducible and mutually non-equivalent, σ restricts to an automorphism of the $\mathfrak{p}^\pm$. The $\mathfrak{p}^\pm$ are actually complex representations of H: on $\mathfrak{p}^+$ there is a complex structure given by conjugation by $\mathrm{diag}(1,i)$; on $\mathfrak{p}^-$ the standard complex structure coincides with conjugation by $\mathrm{diag}(1, e^{i\frac{\pi}{4}})$. As both of these are elements of H, $\sigma_*|\mathfrak{p}_\pm$ must be complex linear. By Schur's lemma, $\sigma_*|\mathfrak{p}_\pm$ therefore acts as multiplication (on the right) by $\lambda_\pm \in \mathbb{C}^*$.

As an automorphism, σ_* must preserve the following Lie brackets

$$\left[\mathfrak{p}^+, \mathfrak{p}^+\right] \subset \mathfrak{h} \oplus \mathfrak{p}^-, \; \left[\mathfrak{p}^-, \mathfrak{p}^-\right] \subset \mathfrak{p}^-, \; \left[\mathfrak{p}^+, \mathfrak{p}^-\right] \subset \mathfrak{p}^+.$$

The first two relations are preserved if and only if $|\lambda_\pm| = 1$, and the final bracket if and only if $\lambda_+ = \overline{\lambda_-}$. If σ defines a 3-symmetric structure, therefore, it can be one of only two possibilities, $\lambda_\pm = \exp(\pm i\frac{2\pi}{3})$ or $\lambda_\pm = \exp(\mp i\frac{2\pi}{3})$. In fact, σ is nothing but the inner automorphism given by conjugation by $(1, e^{\pm i2\pi/3}) \in Z(H)$.

Let J_{NK} be the almost complex structure on $\mathbb{CP}^3$ defined by conjugation by $(1, e^{i2\pi/3})$. By construction, J_{NK} acts by multiplication by i on $\mathfrak{p}^+$, by $-i$ on $\mathfrak{p}^-$.

Using the above brackets it is not difficult to compute that $g_{s,t}$ is naturally reductive precisely when $t = 2s$. By Proposition 3.5, therefore, $(\mathbb{CP}^3, g_{1,2}, J_{\mathrm{NK}})$ is strictly nearly Kähler. This coincides with the twistor description of section 1.1, for it can be shown that the quotient fibration

$$S^2 = \frac{\mathrm{Sp}(1)}{\mathrm{U}(1)} \hookrightarrow \mathbb{CP}^3 = \frac{\mathrm{Sp}(2)}{\mathrm{U}(1) \times \mathrm{Sp}(1)} \xrightarrow{\tau} \mathbb{HP}^1 = \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1) \times \mathrm{Sp}(1)}, \tag{3.3}$$

is nothing but the twistor fibration of $S^4 = \mathbb{HP}^1$. The distribution generated by $\mathfrak{p}^-$ evidently generates the vertical distribution of τ and, with the family of metrics $g_{s,t}$ on its total space, τ is a Riemannian submersion with horizontal distribution $\mathfrak{p}^+$. It is now easy to see that the nearly Kähler structure $(g_{1,2}, J_{\mathrm{NK}})$ constructed here is precisely that described in section 1.1.2.

3.2.3 $\mathbb{F}_{1,2}$

The complex flag manifold $\mathbb{F}_{1,2}$ consists of pairs (π, l) where π is a complex 2-plane in $\mathbb{C}^3$ and l is a complex line in $\mathbb{C}^3$ lying in π. The flag inherits an action by $\mathrm{U}(3)$ from that on $\mathbb{C}^3$ and this is transitive with isotropy

subgroup conjugate to $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$. Indeed, $\mathrm{U}(3)$ is the set of unitary bases (u_1,u_2,u_3) of $\mathbb{C}^3$ and the map $(u_1,u_2,u_3)\in\mathrm{U}(3)\mapsto(\langle u_1,u_2\rangle,\langle u_1\rangle)\in\mathbb{F}_{1,2}$ is onto. By factoring out the determinant we can also view $\mathbb{F}_{1,2}$ as the homogeneous space $\mathrm{SU}(3)/T^2$ where T^2 is the standard maximal torus of $\mathrm{SU}(3)$, the set of elements of $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)\subset\mathrm{U}(3)$ with unit determinant.

There are three maps $\mathbb{F}_{1,2}\to\mathbb{CP}^2$, $\pi_1:(\pi,l)\mapsto l$, $\pi_1:(\pi,l)\mapsto l^\perp\cap\pi$ and $\pi_1:(\pi,l)\mapsto\pi^\perp$. These correspond to three splittings of the twistor fibration of $\mathbb{CP}^2$ and so give six complex structures $I_\pm,J_\pm,K_\pm$ as in section 1.1.2. It happens that I_+,J_+,K_+ are distinct while $I_-=J_-=K_-$. We construct these at the infinitesimal level.

The isotropy representation in this case has three summands,

$$\mathfrak{p}_1=\left\{\begin{pmatrix}0&-\bar z&0\\ z&0&0\\0&0&0\end{pmatrix}\;:\;z\in\mathbb{C}\right\},\quad \mathfrak{p}_2=\left\{\begin{pmatrix}0&0&0\\0&0&-\bar z\\0&z&0\end{pmatrix}\;:\;z\in\mathbb{C}\right\},$$
$$\mathfrak{p}_3=\left\{\begin{pmatrix}0&0&-\bar z\\0&0&0\\z&0&0\end{pmatrix}\;:\;z\in\mathbb{C}\right\}.$$

We identify $(u,v,w)\in\mathbb{C}^3$ with the following element of the isotropy representation $\mathfrak{m}=\mathfrak{p}_1\oplus\mathfrak{p}_2\oplus\mathfrak{p}_3$

$$\begin{pmatrix}0&-\bar u&v\\u&0&-\bar w\\-\bar v&w&0\end{pmatrix}.$$

The representations of the torus $H=\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$ on the $\mathfrak{p}_i$ are irreducible, so any invariant metric on $\mathbb{F}_{1,2}$ is of the form $g_{r,s,t}=rh_1+sh_2+th_3$ for $r,s,t>0$ where h_i are invariant metrics on the $\mathfrak{p}_i$ identified with the standard ones on $\mathbb{C}$. There is also an obvious invariant almost complex structure on $\mathfrak{m}$ through the identification with $\mathbb{C}^3$ and this restricts to an almost complex structure on each summand $\mathfrak{p}_i$. Let $\mathfrak{p}_i^{\mathbb{C}}=\mathfrak{p}_i^+\oplus\mathfrak{p}_i^-$ be the resulting decomposition of the complexification.

There are the following brackets

$$[(u,0,0),(0,v,0)]=(0,0,-\overline{ab}),\quad [(u,0,0),(0,0,w)]=(0,uw,0),$$
$$[(0,v,0),(0,0,w)]=(-\overline{vw},0,0),$$

and

$$[(u,0,0),(u',0,0)]=\mathrm{diag}(-\bar uu'+u\bar{u'},u\bar{u'}-u'\bar u,0)\quad etc.$$

Together these give

$$[\mathfrak{p}_1^+,\mathfrak{p}_2^+]\subset\mathfrak{p}_3^+,\quad [\mathfrak{p}_1^+,\mathfrak{p}_2^-]=\{0\},\quad [\mathfrak{p}_1^+,\mathfrak{p}_3^+]\subset\mathfrak{p}_2^+,$$
$$[\mathfrak{p}_1^+,\mathfrak{p}_3^-]=\{0\},\quad [\mathfrak{p}_1^+,\mathfrak{p}_1^+]=\{0\},\quad [\mathfrak{p}_1^+,\mathfrak{p}_1^-]\subset\mathfrak{h}^{\mathbb{C}}.$$

There are eight possible subspaces $\mathfrak{p}_1^\pm\oplus\mathfrak{p}_2^\pm\oplus\mathfrak{p}_3^\pm$ half of which are obtained from the other by complex conjugation. The brackets above show that three of these four, namely $\mathfrak{p}_1^+\oplus\mathfrak{p}_2^+\oplus\mathfrak{p}_3^-$, $\mathfrak{p}_1^-\oplus\mathfrak{p}_2^+\oplus\mathfrak{p}_3^+$ and $\mathfrak{p}_1^+\oplus\mathfrak{p}_2^-\oplus\mathfrak{p}_3^+$, are integrable – they represent the complex structures I_+,J_+,K_+ defined by the twistor construction above. The remaining subspace $\mathfrak{m}^+=\mathfrak{p}_1^+\oplus\mathfrak{p}_2^+\oplus\mathfrak{p}_3^+$ is not integrable but satisfies (3.1) with $\mathfrak{m}^-=\mathfrak{p}_1^-\oplus\mathfrak{p}_2^-\oplus\mathfrak{p}_3^-$, and so defines a 3-symmetric structure. It is easy to see that $g_{r,s,t}$ is naturally reductive if and only if $r=s=t$, and we obtain a strictly nearly Kähler structure on $\mathbb{F}_{1,2}$.

Chapter 4

Six-Dimensional Nearly Kähler Manifolds

Nearly Kähler geometry in dimension six has a peculiar flavour due to connections with the seven dimensional geometry associated to the exceptional Lie group G_2. The purpose of this section is to tease out this connection which seems to be amongst the most interesting aspects of nearly Kähler geometry and a significant motivation for its further study.

To make the transition from nearly Kähler geometry in dimension six to the seven dimensional geometry of G_2 we require the notion of the Riemannian cone.

Definition 4.1. The cone (M', g') over a Riemannian manifold (M, g) is the product manifold $M \times \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$, equipped with the metric $g' = dr^2 + r^2\pi^* g$. Here, we have canonical projections $r : M' \to \mathbb{R}_+ = (0, \infty)$, $\pi : M' \to M$, and we make the canonical identification of $T(M \times \mathbb{R}_+)$ with $\pi^* TM \oplus r^* T\mathbb{R}_+$. We refer to M as the *link* of its cone.

We shall prove the following theorem about cones over nearly Kähler manifolds.

Theorem 4.2. *(M^6, g) is strictly nearly Kähler if and only if there exists a constant $\mu > 0$ such that the cone over $(M, \mu^{-1} g)$ has holonomy a subgroup of G_2.*

Remark 4.3. Geometries defined on cones is an area that has received some attention in recent years. According to the holonomy theorem of Berger (*e.g.* chapter 3 of [Joy00]), there are three special geometries associated with irreducible non-symmetric Ricci flat metrics: $\mathrm{SU}(n)$ in dimension $2n$, G_2 in dimension 7 and $\mathrm{Spin}(7)$ in dimension 8. Riemannian cones with holonomy $\mathrm{SU}(n)$, *i.e.* Calabi-Yau cones, are well understood: their $(2n-1)$-dimensional links are Sasaki-Einstein manifolds. These are defined by a special kind of contact geometry, and there are a great many examples and a consequently well-developed theory (see *e.g.* the book [BG08]). The link of a cone with $\mathrm{Spin}(7)$ holonomy is a nearly G_2 manifold, and these also have been well studied (*e.g.* [FKMS97]).

The geometry of G_2 cones, *i.e.* nearly Kähler geometry, remains largely a mystery. Nearly Kähler manifolds with a high degree of symmetry have been studied in some detail but all attempts to date have failed to produce more than the four smooth compact examples presented above. This work is reviewed in later chapters. Non-complete examples of strictly nearly Kähler six-manifolds are easy to produce: we present an infinite family of these in section 4.5.

Theorem 4.2 is a local result, requiring a number of curvature identities for nearly Kähler manifolds. This argument is scattered over several papers by Gray [Gra69, Gra70, Gra76, Gra72] and appears not to have been brought together elsewhere in the literature. A corollary of Theorem 4.2 will be the following important fact.

Theorem 4.4. *If (M, g, J) is a strictly nearly Kähler six-manifold then it is Einstein with positive scalar curvature. In particular, if (M, g) is complete then it is compact and $\pi_1(M)$ is finite.*

The second statement of the theorem uses the Theorem of Bonnet-Myers. We shall see that the Einstein constant is proportional to the scale factor introduced in the statement of Theorem 4.2.

The objective in section 4.1 is to elaborate the meaning of the inclusion $\mathrm{Hol}(g) \subset \mathrm{G}_2$ and prove the following global result using Theorem 4.2. Our work on group actions in section 5 depends fundamentally on this theorem.

Theorem 4.5. *Let (M, g) be a complete oriented Riemannian six-manifold not isometric to a round sphere. Then there is at most one almost complex structure J on M inducing the correct orientation and such that (g, J) is strictly nearly Kähler.*

Remark 4.6. This is a genuinely global result. In section 4.5, non-complete examples of six-manifolds (M, g) are produced that violate the conclusion of Theorem 4.5.

4.1 G_2 Geometry and G_2 Cones

We commence this section by recalling some basic geometry associated to the Lie group G_2. General references for this section are [SW10] and [Bry12]. G_2 was defined in section 1.1.1 as $\mathrm{Aut}(\mathbb{O})$, the automorphism group of the non-associative algebra of octonions, but there are two closely related, and for our purposes more useful, equivalent definitions of this group. The first involves the notion of cross product introduced in section 1.1.1.

Definition 4.7. Let V be a seven dimensional inner product space. A *cross product* on V is a linear map $\times : V \otimes V \to V$ such that for all $v, w \in V$ the following identities hold

$$v \times w = -w \times v, \quad g(v \times w, w) = 0,$$
$$|v \times w|^2 = |v|^2|w|^2 - (v, w)^2.$$

Given such choices, one can define a multiplication on $\mathbb{R} \oplus V$ by

$$(s, v) \cdot (t, w) = st + sw + tv + (v, w) + v \times w \ , \ v, w \in V,$$

and it is straightforward to see that this makes $\mathbb{R} \oplus V$ into a normed division algebra. The resulting algebra must be isomorphic to $\mathbb{O}$, with V corresponding to the subspace of imaginary octonions, and it can be shown that the group of automorphisms of $\mathbb{O}$ is precisely the group of $g \in \mathrm{GL}(V)$ preserving the cross product,

$$g(v) \times g(w) = g(v \times w), \ \forall\, v, w \in V.$$

Consider now the trilinear form on V defined by

$$\phi_0(u, v, w) = (u, v \times w), \ u, v, w \in V. \tag{4.1}$$

It is not difficult to see that this is, in fact, an alternating form. We call it the *standard* G_2 *structure*. If one adopts the convention for octonionic multiplication of [Joy00], then setting $V = \mathbb{R}^7$ we have

$$\phi_0 = \sigma^{123} + \sigma^{145} + \sigma^{167} + \sigma^{246} - \sigma^{257} - \sigma^{347} - \sigma^{356}, \tag{4.2}$$

where $\sigma^{ijk} = \sigma^i \wedge \sigma^j \wedge \sigma^k$ and $\{\sigma^i\}_{i=1}^7$ is the basis of 1-forms dual to the standard orthonormal basis of $\mathbb{R}^7$. The interesting fact about ϕ_0 is that it determines both the metric and orientation vol on $\mathbb{R}^7$, these being related by the formula

$$(v \llcorner \phi_0) \wedge (w \llcorner \phi_0) \wedge \phi_0 = \frac{1}{6}(v, w)\mathrm{vol}, \ v, w \in \mathbb{R}^7.$$

We see then that the cross product can be recovered from ϕ_0 and the stabiliser of ϕ_0 in $\mathrm{GL}(V)$ is again G_2. The Euclidean metric is not uniquely determined by ϕ_0: for any $g \in \mathrm{SO}(7)$ the 3-form $g^*\phi_0$ preserves the previous formula and is of the same type as ϕ_0, in that it is determined by (4.1) for an isomorphic copy of $\mathbb{O}$.

Of further interest, it is elementary that the $\mathrm{GL}(V)$-orbit of ϕ_0 is open, for $\dim \mathrm{G}_2 = 14$ and

$$\dim \Lambda^3 V^* = 35 = \dim \mathrm{GL}(V) - \dim \mathrm{G}_2.$$

In fact, ϕ_0 is almost uniquely determined by this property: if φ is a 3-form on V with open $\mathrm{GL}(7)$-orbit and compact stabiliser, then φ belongs to the $\mathrm{GL}(7)$-orbit of ϕ_0. This property of ϕ_0 is referred to as *stability* (see [Hit01] for a discussion of this phenomenon). We shall see a manifestation of stability in dimension six in section 4.4. Any element in the $\mathrm{GL}(7)$-orbit of ϕ_0 is referred to as a G_2 *structure*.

The geometry described so far provides the local model for the theory of G_2 holonomy.

Definition 4.8. Let (M, g) be a Riemannian 7-manifold. A *cross product* on M is a bundle map $\times : TM \otimes TM \to TM$ such that for all $X, Y \in \mathcal{X}(M)$ the following identities hold

$$X \times Y = -Y \times X \,, \tag{4.3}$$

$$g(X \times Y, Y) = 0 \,, \tag{4.4}$$

$$|X \times Y|^2 = |X|^2|Y|^2 - g(X, Y)^2. \tag{4.5}$$

A G_2 *structure* φ on M is a 3-form such that φ_p is a stable 3-form on T_pM for every p in M; as described above φ induces a metric g_φ on M. These two notions are equivalent, being related by formula (4.1).

As for almost complex structures (section 1), there is a classification of G_2 structures φ due to Fernandez and Gray (Theorem 5.2, [FG82]) according to the symmetries of the torsion $\nabla\varphi$. The simplest case is the vanishing of this tensor.

Definition 4.9. A G_2 structure φ is said to be *torsion-free* if φ is g_φ-parallel, equivalently if $\mathrm{Hol}(g_\varphi) \subset \mathrm{G}_2$.

An important corollary of the classification of Fernandez and Gray is that the vanishing $\nabla\varphi = 0$ is equivalent to the equations $d\varphi = d{*}\varphi = 0$. An elementary consequence of the inclusion $\mathrm{Hol}(g) \subset \mathrm{G}_2$ is that g is Ricci flat (Proposition 7, [Bon66]). Both of these facts will be used below.

Sufficient preliminary material is now in place to return to our main line of inquiry. Denoting by ξ the radial vector field $\frac{\partial}{\partial r}$ on a cone, we have the following proposition, one half of Theorem 4.2.

Proposition 4.10. *If (M^6, g) is a Riemannian manifold such that the cone (M', g') possesses a parallel cross product $\times$, then the expression*

$$JX = \xi \times X \,, \ X \in \Gamma(\pi^* TM)$$

defines a strictly nearly Kähler structure on (M, g). Moreover, (M, g) is Einstein with constant scalar curvature 30.

Proof. By (4.4), for every $X \in \Gamma(\pi^*TM)$ JX is orthogonal to ξ and so defines a vector field on M. From this formula we also see that $g(JX, X) = 0$, and, polarising this, $g(JX, Y) = -g(X, JY)$. Equation (4.5) implies $|JX|^2 = |X|^2$, polarisation of which gives $g(JX, JY) = g(X, Y)$. These latter two formulae imply $J^2 = -\mathrm{Id}$, so J indeed defines an orthogonal almost complex structure on (M, g).

Let now ∇' be the Levi-Civita connection of g'. This is given by the formulae (*e.g.* Lemma 1.1 of [Gal79])

$$\nabla'_\xi \xi = 0\,, \qquad \nabla'_X \xi = \nabla'_\xi X = \frac{1}{r} X\,, \qquad \nabla'_X Y = \nabla_X Y - rg(X, Y)\xi,\ X, Y \in \Gamma(\pi^*TM). \tag{4.6}$$

Using these, (4.3) and (4.4), and the assumption that $\times$ is ∇'-parallel we compute

$$\begin{aligned}(\nabla_X J)X &= \nabla_X(JX) - J(\nabla_X X),\\ &= \nabla'_X(\xi \times X) + rg(\xi \times X, X)\xi - \xi \times \nabla_X X,\\ &= \nabla'_X e' \times X + \xi \times \nabla'_X X - \xi \times \nabla_X X,\\ &= \frac{1}{r} X \times X + \xi \times \nabla_X X - rg(X, X)\xi \times \xi - \xi \times \nabla_X X,\\ &= 0.\end{aligned}$$

The curvature tensor of g' is given by the formulae (Lemma 1.2, [Gal79])

$$R'(W, \xi) = 0 \quad , \quad R'(\cdot, \cdot)\xi = 0, \tag{4.7}$$

$$R'(X, Y)Z = R(X, Y)Z - (g(Y, Z)X - g(X, Z)Y). \tag{4.8}$$

Here $W \in \mathcal{X}(M')$, $X, Y \in \Gamma(\pi^*TM)$ and R is the curvature tensor of g. But $\mathrm{Hol}(g') \subset \mathrm{G}_2$ implies that g' is Ricci-flat (Definition 4.8). It is then easy to see from the curvature formulae above that g is Einstein with Einstein constant 5 and so $\mathrm{scal}_g = 30$. □

We conclude this section by proving the corollary to Theorem 4.2 quoted above.

Theorem. *Let (M, g) be a complete oriented Riemannian 6-manifold not isometric to a round sphere. Then there is at most one almost complex structure J on M inducing the correct orientation and such that (g, J) is strictly nearly Kähler.*

Proof. Let $(\widetilde{M}, \tilde{g})$ denote the universal Riemannian covering space of (M, g). It is strictly nearly Kähler, so its cone, $(\widetilde{M}', \tilde{g}')$, has holonomy a subgroup of G_2 by Theorem 4.2. If it can be shown that $(\widetilde{M}', \tilde{g}')$ is neither locally symmetric nor reducible then it will follow from Berger's theorem that $\mathrm{Hol}(\tilde{g}') = \mathrm{G}_2$. To this end we require the following result (Proposition 3.1, [Gal79]), the proof of which can be found in the Appendix to this work .

Lemma 4.11. *Let (M^n, g) be a complete Riemannian manifold. If the cone over (M, g) is reducible or locally symmetric, then it is flat and (M, g) is locally isometric to the standard n-sphere.*

Therefore, $(\widetilde{M}', \tilde{g}')$ is either isometric to a round six-sphere or $\mathrm{Hol}(\tilde{g}') = \mathrm{G}_2$. But the only isometric coverings $S^6 \to X$ are the double covering with $X = \mathbb{RP}^6$ and the trivial one with $X = S^6$. As $\mathbb{RP}^6$ is non-orientable, if $\mathrm{Hol}(\tilde{g}')$ is a proper subgroup of G_2, then (M, g) must be isometric to a round six-sphere. This contradicts our hypotheses.

Suppose then that $\mathrm{Hol}(\tilde{g}') = \mathrm{G}_2$ but there exist two almost complex structures J_1, J_2 such that (M, g, J_1) and (M, g, J_2) are both strictly nearly Kähler. This implies the existence of two parallel G_2 structures ϕ_1, ϕ_2 on $\widetilde{M}'$. But if $\mathrm{Hol}(\tilde{g}') = \mathrm{G}_2$ this is impossible unless $\phi_1 = \pm\phi_2$. Indeed, by a basic result in the representation theory of G_2 (Lemma 3.2, [FG82]), there is a unique 1-dimensional trivial subspace of $\Lambda^3(\mathbb{R}^7)^*$, namely that spanned by ϕ_0. It follows then that $J_1 = \pm J_2$. □

We can also count the number of nearly Kähler structures on the round sphere.

Proposition 4.12. *Let (S^6, g_{rd}) be a round six-sphere. Then the set of strictly nearly Kähler structures on S^6 compatible with g_{rd} and the standard orientation forms a manifold diffeomorphic to $\mathbb{RP}^7$.*

Proof. By Theorem 4.2 there is a one-to-one correspondence between almost complex structures J such that $(S^6, g_{\mathrm{rd}}, J)$ is strictly nearly Kähler and parallel G_2 structures on the cone $C(S^6, g_{\mathrm{rd}}) = \mathbb{R}^7 \setminus \{0\}$. Parallel G_2 structures on $\mathbb{R}^7$ are constant, so the question is how many constant G_2 structures compatible with the Euclidean metric are there? It is easy to see that the set of such is exactly the homogeneous space $\mathrm{SO}(7)/\mathrm{G}_2 \cong \mathbb{RP}^7$. □

4.2 Curvature Identities for Strictly Nearly Kähler Manifolds

In the previous section it was proven that for a six-manifold (M, g) to be strictly nearly Kähler it sufficed that the cone over (M, g) had holonomy a subgroup of G_2. To establish the necessity of this condition, we prove in this section several curvature identities of Gray. These will be used to conclude the proof of Theorem 4.2 in the next section and to show that strictly nearly Kähler manifolds are Einstein with positive scalar curvature. In this section we shall also be able to prove that any strictly nearly Kähler six-manifold is of constant type.

Definition 4.13. If (M, g, J) is an almost Hermitian manifold, the associated *Kähler form* is the 2-form defined by

$$\omega(X, Y) = g(X, JY),\ X, Y \in \mathcal{X}(M).$$

Define also a trilinear form by

$$\Omega(X, Y, Z) = g(Y, (\nabla_X J)Z),\ X, Y, Z \in \mathcal{X}(M).$$

For the mean time, no restriction is made on the dimension of M.

Lemma 4.14. *An almost Hermitian manifold (M, g, J) is nearly Kähler if and only if Ω is alternating.*

Proof. For any vector field X, $\nabla_X J$ is a skew-orthogonal endomorphism by (4.10), and so $\Omega(X, Y, Z)$ is alternating in Y, Z. To prove the lemma it therefore suffices to show that Ω is alternating in X, Z, which holds if and only if

$$g(Y, (\nabla_X J)Z) = -g(Y, (\nabla_Z J)X),\ \forall X, Y, Z \in \mathcal{X}(M).$$

As g is non-degenerate, we see that this is precisely the condition that $(\nabla_X J)Y$ be skew in X, Y. □

Recall the following formula from chapter 1

$$(\nabla_X J)JY = -J(\nabla_X J)Y, \tag{4.9}$$
$$g((\nabla_X J)Y, Y) = 0, \tag{4.10}$$

valid for any $X, Y \in \mathcal{X}(M)$. Using these we prove the first in a sequence of curvature identities.

Lemma 4.15. *If (M, g, J) is an almost Hermitian manifold, then the following identities hold*

$$\nabla\nabla\omega(W, X, Y, Z) - \nabla\nabla\omega(X, W, Y, Z) = \omega(R_{WX}Y, Z) + \omega(Y, R_{WX}Z), \tag{4.11}$$
$$|(\nabla_X J)Y|^2 = \nabla\nabla\omega(X, X, Y, JY), \tag{4.12}$$

valid for any $W, X, Y, Z \in \mathcal{X}(M)$.

Proof. To prove these, compute first that

$$\begin{aligned}\nabla\omega(X,Y,Z) &= X(\omega(Y,Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z)\\ &= g(\nabla_X Y, JZ) + g(Y, \nabla_X(JZ)) - g(\nabla_X Y, JZ) - g(Y, J(\nabla_X Z))\\ &= g(Y, (\nabla_X J)Z)\\ &= \Omega(X,Y,Z). \end{aligned} \tag{4.13}$$

The second derivative is then given by

$$\begin{aligned}\nabla\nabla\omega(W,X,Y,Z) &= W(\nabla\omega(X,Y,Z)) - \nabla\omega(\nabla_W X, Y, Z) - \nabla\omega(X, \nabla_W Y, Z) - \nabla\omega(X, Y, \nabla_W Z)\\ &= g(\nabla_W Y, (\nabla_X J)Z) + g(Y, \nabla_W((\nabla_X J)Z))) - g(Y, (\nabla_{\nabla_W X} J)Z)\\ &\qquad - g(\nabla_W Y, (\nabla_X J)Z) - g(Y, (\nabla_X J)(\nabla_W Z))\\ &= g(Y, (\nabla_W \nabla_X J - \nabla_{\nabla_W X} J)Z).\end{aligned}$$

Thus,

$$\nabla\nabla\omega(W,X,Y,Z) - \nabla\nabla\omega(X,W,Y,Z) = g(Y, ([\nabla_W, \nabla_X]J - \nabla_{[W,X]}J)Z).$$

This gives (4.11) if we recall the formula for the curvature of ∇ extended to $\mathrm{End}(TM)$.

To prove (4.12), observe that

$$g(Y, (\nabla_V J)JY) = -g((\nabla_V J)Y, JY) = g(J(\nabla_V J)Y, Y) = -g((\nabla_V J)JY, Y) = 0. \tag{4.14}$$

Therefore,

$$\begin{aligned}\nabla\nabla\omega(X,X,Y,JY) &= g(Y, \nabla_X((\nabla_X J)JY))) - g(Y, (\nabla_X J)(\nabla_X(JY))) - g(Y, (\nabla_{\nabla_X X} J)JY)\\ &= Xg(Y, (\nabla_X J)JY) - g(\nabla_X Y, (\nabla_X J)JY) + g((\nabla_X J)Y, \nabla_X(JY))\\ &= -g(J(\nabla_X Y), (\nabla_X J)Y) + g((\nabla_X J)Y, \nabla_X(JY))\\ &= g((\nabla_X J)Y, (\nabla_X J)Y).\end{aligned}$$

□

We proceed now to the case in which (g, J) is nearly Kähler. By (4.13) and Lemma 4.14, $\nabla\omega(X,Y,Z)$ is skew in X, Y, so that $\nabla\nabla\omega(W,X,Y,Z)$ is also skew in X, Y; then from (4.11) we see that

$$\begin{aligned}|(\nabla_X J)Y|^2 = \nabla\nabla\omega(X,X,Y,JY) &= -\nabla\nabla\omega(X,Y,X,JY)\\ &= -\nabla\nabla\omega(X,Y,X,JY) + \nabla\nabla\omega(Y,X,X,JY)\\ &= g(R_{XY}X, Y) - g(R_{XY}JX, JY).\end{aligned}$$

We can prove by a polarisation argument the following more general result,

$$g((\nabla_W J)X, (\nabla_Y J)Z) = g(R_{WX}Y, Z) - g(R_{WX}JY, JZ). \tag{4.15}$$

To see this, define elements of $(\Lambda^2 TM)^* \otimes (\Lambda^2 TM)^*$ by extending linearly the formulae

$$\alpha(W \wedge X, Y \wedge Z) = g((\nabla_W J)X, (\nabla_Y J)Z)),\ \beta((W \wedge X, Y \wedge Z) = g(R_{WX}Y, Z) - g(R_{WX}JY, JZ).$$

These are actually both elements of $S^2((\Lambda^2 TM)^*)$ and so are determined by their values on the diagonal. But

$$\alpha(X \wedge Y, X \wedge Y) = |(\nabla_X J)Y|^2 = g(R_{XY}X, Y) - g(R_{XY}JX, JY) = \beta(X \wedge Y, X \wedge Y),$$

so $\alpha \equiv \beta$, as required. Using this last identity the following can be proved.

Lemma 4.16. *If (M, g, J) is a nearly Kähler manifold, then the following identity holds*

$$\begin{aligned} 2g(Y, (\nabla\nabla J)(W, X, Z)) &= 2\nabla\nabla\omega(W, X, Y, Z) \\ &= -\mathop{\mathfrak{S}}_{X,Y,Z} g((\nabla_W J)X, J(\nabla_Y J)Z), \end{aligned}$$

valid for any $W, X, Y, Z \in \mathcal{X}(M)$. The symbol $\mathfrak{S}$ denotes a cyclic summation.

Proof. To prove this, combine (4.11) and (4.15) to obtain

$$\nabla\nabla\omega(W, X, Y, Z) - \nabla\nabla\omega(X, W, Y, Z) = -g((\nabla_W J)X, J(\nabla_Y J)Z). \tag{4.16}$$

As $\nabla\nabla\omega(W, X, Y, Z)$ is skew in the last three arguments, using (4.16) we have

$$\begin{aligned} \nabla\nabla\omega(W, W, Y, Z) = -\nabla\nabla\omega(W, Y, W, Z) &= -\nabla\nabla\omega(W, Y, W, Z) + \nabla\nabla\omega(Y, W, W, Z) \\ &= -g((\nabla_W J)Y, J(\nabla_W J)Z). \end{aligned}$$

Polarising this equation gives

$$\nabla\nabla\omega(W, X, Y, Z) + \nabla\nabla\omega(X, W, Y, Z) = -g((\nabla_W J)Y, J(\nabla_X J)Z) + g((\nabla_W J)Z, J(\nabla_X J)Y).$$

Adding this to equation (4.16) proves the lemma. □

Recall now the definition of the Ricci endomorphism: for an arbitrary local orthonormal frame $e_1, \ldots, e_{2n}$ of (M, g) this is defined by the trace

$$g(\mathrm{Ric}X, Y) = \sum_{i=1}^{2n} R_{Xe_iYe_i}\ ,\ X, Y \in \mathcal{X}(M).$$

We define a second endomorphism by

$$g(\mathrm{Ric}^*X, Y) = \sum_{i=1}^{2n} R_{Xe_iJYJe_i}\ ,\ X, Y \in \mathcal{X}(M).$$

Using (4.15), the difference of these tensors is given by

$$g((\mathrm{Ric} - \mathrm{Ric}^*)X, Y) = \sum_{i=1}^{2n} g((\nabla_X J)e_i, (\nabla_Y J)e_i). \tag{4.17}$$

Using this and Lemma 4.16 we can prove the next identity.

Lemma 4.17. *If (M, g, J) is a nearly Kähler manifold, then the following identity holds*

$$\begin{aligned} &2g(\nabla(\mathrm{Ric} - \mathrm{Ric}^*)(Z, X), Y) \\ &\qquad = g((\mathrm{Ric} - \mathrm{Ric}^*)JX, (\nabla_Z J)Y) + g((\mathrm{Ric} - \mathrm{Ric}^*)JY, (\nabla_Z J)X). \end{aligned} \tag{4.18}$$

valid for any , $X, Y, Z \in \mathcal{X}(M)$.

Proof. Differentiating (4.17) and employing Lemma 4.16 we obtain

$$\begin{aligned} &g(\nabla(\mathrm{Ric}-\mathrm{Ric}^*)(Z,X),X),\\ &\qquad = 2\sum_{i=1}^{2n} g((\nabla_X J)e_i,(\nabla\nabla J(Z,X,e_i))\\ &\qquad = -\sum_{i=1}^{2n}\{g((\nabla_Z J)X,(\nabla_{e_i}J)J(\nabla_X J)e_i)+g((\nabla_Z J)(\nabla_X J)e_i,(\nabla_X J)Je_i)\\ &\qquad\qquad +g((\nabla_Z J)e_i,(\nabla_{(\nabla_X J)e_i}J)JX)\}\\ &\qquad = -\sum_{i=1}^{2n}\{g((\nabla_{e_i}J)(\nabla_Z J)X,J(\nabla_{e_i}J)X)+g((\nabla_Z J)(\nabla_X J)e_i,(\nabla_X J)Je_i)\\ &\qquad\qquad +g((\nabla_X J)(\nabla_Z J)e_i,J(\nabla_X J)e_i)\}\,. \end{aligned} \tag{4.19}$$

Setting $W=(\nabla_X J)e_i$ the second term in brackets becomes $g((\nabla_z J)W,JW)$ and this is then seen to vanish by (4.14). Furthermore, equations (4.3) and (4.4) show that the matrix $A_{ij}=g(J(\nabla_Z J)e_i,e_j)$ is skew-symmetric,

$$A_{ij}=-g((\nabla_Z J)e_i,Je_j)=g(e_i,(\nabla_Z J)Je_j)=-g(e_i,J(\nabla_Z J)e_j)=-A_{ji},$$

and the third term in (4.19) is then seen to vanish for

$$\begin{aligned} \sum_{i=1}^{2n} g((\nabla_X J)(\nabla_Z J)e_i,J(\nabla_X J)e_i) &= \sum_{i,j} g((\nabla_X J)A_{ij}e_j,(\nabla_X J)e_i)\\ &= -\sum_{i,j} g((\nabla_X J)e_j,(\nabla_X J)(A_{ji}e_i)\\ &= -\sum_{i=1}^{2n} g((\nabla_X Je_i,J(\nabla_X J))(\nabla_Z J)e_i). \end{aligned}$$

Using (4.17), equation (4.19) becomes the identity

$$g(\nabla(\mathrm{Ric}-\mathrm{Ric}^*)(Z,X),X)=g((\mathrm{Ric}-\mathrm{Ric}^*)JX,(\nabla_Z J)X).$$

Polarising this we arrive at the required identity. □

We now assume that (M,g,J) is six-dimensional and strictly nearly Kähler. Fix $p\in M$ and let E_1,E_2 be orthonormal vector fields defined near p such that E_1,JE_1,E_2,JE_2 form an orthonormal set. Define a unit vector field E_3 and positive function μ by

$$(\nabla_{E_1}J)E_2=\mu E_3. \tag{4.20}$$

It is straightforward to see that μ cannot vanish if the nearly Kähler structure (g,J) is strict. The set $\mathcal{E}=\{E_1,JE_1,E_2,JE_2,E_3,JE_3\}$ is therefore a local unitary frame. Using this one obtains the following formula.

$$|(\nabla_X J)Y|^2=\mu^2\left\{|X|^2|Y|^2-g(X,Y)^2-g(X,JY)^2\right\},\ \forall X,Y\in\mathcal{X}(M).$$

The interesting fact is that μ is a constant function: employing the terminology of the paper [Gra70], (g,J) is said to be of *constant type*. We prove this using the identities derived above.

Polarising (4.21) gives

$$g((\nabla_X J)Y,(\nabla_X J)Z) = \mu^2\left\{|X|^2 g(Y,Z) - g(X,Y)g(X,Z) - g(X,JY)g(X,JZ)\right\}.$$

Substitute this into equation (4.17) to obtain

$$\begin{aligned} g((\mathrm{Ric}-\mathrm{Ric}^*)X,Y) &= \mu^2\sum_{i=1}^{6}\{g(X,Y) - g(e_i,X)g(e_i,Y) - g(e_i,JX)g(e_i,JY)\} \\ &= \mu^2\{6g(X,Y) - g(X,Y) - g(JX,JY)\} \\ &= 4\mu^2\, g(X,Y). \end{aligned}$$

We see then that $\mathrm{Ric}-\mathrm{Ric}^* = 4\mu^2\mathrm{Id}$. On the other hand, substituting this into (4.18) gives

$$2g(\nabla(\mathrm{Ric}-\mathrm{Ric}^*)(Z,X),Y) = \mu^2\{g(JX,(\nabla_Z J)Y) + g(JY,(\nabla_Z J)X)\} = 0.$$

This implies $\nabla(\mathrm{Ric}-\mathrm{Ric}^*) = 0$, and we conclude that μ is constant.

4.3 Conclusion of the Proof of Theorem 4.2

It was shown in the previous section that on a strictly nearly Kähler manifold (M^6,g,J) there exists a constant μ such that

$$|(\nabla_X J)(Y)|^2 = \mu^2\left\{|X|^2|Y|^2 - g(X,Y)^2 - g(X,JY)^2\right\}\ ,\ \forall\, X,Y\in\mathcal{X}(M). \tag{4.21}$$

Rescaling g by a factor of μ^{-1}, we may assume $\mu = 1$. Define now a product on the cone M' by the formulae

$$\xi\times X = -X\times\xi = JX\ ,\ \xi\times\xi = 0\ ,\ X\times Y = r(\nabla_X J)Y - r^2 g(X,JY)\xi.$$

Using (4.21), it is immediate that this product satisfies equations (4.3), (4.4) and (4.5), and so defines a cross product on (M',g'). The induced G_2 structure on M' is defined by

$$\varphi(X,Y,Z) = g'(X,Y\times Z)\ ,\ X,Y,Z\in\mathcal{X}(M').$$

We shall show that φ is closed and coclosed, so that, by the theorem of Fernandez and Gray quoted following Definition 4.8, (M',g') has holonomy contained in G_2. But first compute for $X,Y,Z\in\Gamma(\pi^*TM)$

$$\begin{aligned} \varphi(\xi,X,Y) &= g'(\xi, r(\nabla_X J)Y - r^2 g(X,JY)\xi) = -r^2\omega(X,Y), \\ \varphi(X,Y,Z) &= g'(X, r(\nabla_Y J)Z - r^2 g(Y,JZ)\xi) = -r^3\Omega(X,Y,Z). \end{aligned}$$

Thus

$$\varphi = -r^2 dr\wedge\omega - r^3\Omega. \tag{4.22}$$

To compute the dual $*'\varphi$ of this 3-form on (M',g'), recall first that the Riemannian volume on (M,g) is $\frac{1}{3!}\omega^3$. As $|\omega|^2 = 3$, we have then $*\omega = \frac{1}{2}\omega^2$. Returning to the unitary frame $\mathcal{E}$ defined by (4.20), we have

$$\Omega = \sigma^1\wedge\sigma^2\wedge\sigma^3 - \sigma^1\wedge\sigma^5\wedge\sigma^6 - \sigma^3\wedge\sigma^4\wedge\sigma^5 + \sigma^2\wedge\sigma^4\wedge\sigma^6, \tag{4.23}$$

where $\sigma^1,\ldots,\sigma^6$ is the basis dual to $\mathcal{E}$, so that $J^*\sigma^i = \sigma^{i+3}$ for $i\le 3$. But as $\mathrm{vol}_g = \sigma^1\wedge\cdots\wedge\sigma^6$, it is immediate that $*\Omega = J^*\Omega$, where

$$J^*\Omega(X,Y,Z) = \Omega(JX,JY,JZ)\ ,\ X,Y,Z\in\mathcal{X}(TM).$$

Orienting the cone M' using the volume form $rdr\wedge\frac{1}{3!}\omega^3$, we arrive at

$$*'\varphi = -r^4\frac{\omega^2}{2} + r^3 dr\wedge J^*\Omega.$$

We see then that $d\varphi = d*'\varphi = 0$ if and only if the following equations hold

$$d\omega = 3\Omega\ ,\ dJ^*\Omega = -2\omega^2.$$

Reinstating the original scale, these become

$$d\omega = 3\Omega\ ,\ dJ^*\Omega = -2\mu\,\omega^2.$$

It remains now to prove these. The first of these equations follows immediately from (4.13); the second requires the identities proven in the previous section. Now,

$$dJ\Omega(W,X,Y,Z) = \mathfrak{S}_{W,X,Y,Z}\sigma(W,X,Y,Z)\nabla J^*\Omega(W,X,Y,Z). \tag{4.24}$$

Where $\sigma(W,X,Y,Z) = 1$, $\sigma(Z,W,X,Y) = -1$ *etc.* As the tensor $\nabla J^*\Omega(W,X,Y,Z)$ is already skew in X,Y,Z, the main quantity to compute is

$$\beta(W,X,Y,Z) = \nabla J^*\Omega(W,X,Y,Z) - \nabla J^*\Omega(X,W,Y,Z).$$

Using (4.9) and (4.10) it is straightforward to show that

$$\Omega(JX,JY,JZ) = \Omega(X,Y,JZ).$$

Using this formula we then have

$$\begin{aligned}
&W(\Omega(X,Y,JZ)) - \Omega(\nabla_W X,Y,JZ) - \Omega(X,\nabla_W Y,JZ) - \Omega(X,Y,J\nabla_W Z) - \dots\\
&\quad = g(\nabla_W(JZ),(\nabla_X J)Y) + g(JZ,\nabla_W((\nabla_X J)Y))\\
&\qquad - g(JZ,(\nabla_{\nabla_W X}J)Y) - g(JZ,(\nabla_X J)(\nabla_W Y)) - g(J\nabla_W Z,(\nabla_X J)Y) - \dots\\
&\quad = g(JZ,([\nabla_W,\nabla_X]J - \nabla_{[W,X]}J)Y)\\
&\qquad + g((\nabla_W J)Z,(\nabla_X J)Y) - g((\nabla_X J)Z,(\nabla_W J)Y)\\
&\quad = g(JR_{WX}Y,JZ) - g(R_{WX}JY,JZ)\\
&\qquad + g((\nabla_W J)Z,(\nabla_X J)Y) - g((\nabla_X J)Z,(\nabla_W J)Y)\\
&\quad = g((\nabla_W J)X,(\nabla_Y J)Z) + g((\nabla_W J)Z,(\nabla_X J)Y) - g((\nabla_X J)Z,(\nabla_W J)Y).
\end{aligned} \tag{4.25}$$

We compute the components of β with reference to the framing $\mathcal{E} = \{E_i, JE_i\}_{i=1,2,3}$ defined above. We see from (4.25) that, in the first instance, only components of the form $\beta(E_i,E_j,JE_k,JE_i)$, $i,j,k \le 3$ can possibly be non-vanishing – certainly an index must appear twice and it is easy to see that if only one or if three Js appear then each term in (4.25) vanishes. Further inspection shows that the only non-zero components amongst these are $\beta(E_i,E_j,JE_i,JE_j) = 1$. We have then that

$$\beta = \sum_{i<j} E_i\wedge E_j\wedge JE_i\wedge JE_j = -\frac{1}{2}\omega^2.$$

Adding these up according to (4.24) gives $dJ^*\Omega = -4\cdot\frac{1}{2}\omega^2$ as required. This concludes the proof of Theorem 4.2. Proposition 4.10 then shows that g is Einstein with scalar curvature $30\mu > 0$, and we have also proven Theorem 4.4.

4.4 Half-flat $\mathrm{SU}(3)$ Structures

In the previous section it was shown that on a six-manifold M a strictly nearly Kähler structure (g, J) with scalar curvature $30\mu > 0$ is equivalent to a pair (ω, Ω), consisting of a 2- and a 3-form, satisfying the equations

$$d\omega = 3\Omega\ ,\ dJ^*\Omega = -2\mu\,\omega^2, \tag{4.26}$$

and such that the expression (4.22) defines a G_2 structure φ on the cone over (M, g). In fact, we see that

$$\omega = (\xi \llcorner \varphi)|_{\{r=1\}},\ \Omega = \varphi|_{\{r=1\}},$$

so the pair (ω, Ω) can be considered as being induced by M sitting as a hypersurface in the torsion-free G_2 manifold (M', φ). In this section we briefly explore the special geometry of hypersurfaces in G_2 manifolds. For this some algebraic preparation is necessary, much of it lifted from [Hit01].

Definition 4.18. Let $e_1, \ldots, e_6$ be the standard orthonormal basis of $V = \mathbb{R}^6$ and let J be the standard complex structure,

$$Je_i = e_{i+3},\ i = 1, 2, 3.$$

Denoting by $\{\sigma^i\}$ the basis of V^* dual to $\{e_i\}$ and setting $dz^i = \sigma^i + \sqrt{-1}\sigma^{i+3} \in (V \otimes \mathbb{C})^*$, the *standard* $\mathrm{SU}(3)$ *structure,* (ω_0, Ω_0)*, on* V is defined by

$$\omega_0 = \frac{\sqrt{-1}}{2}\sum_{i=1}^{3} dz^i \wedge d\bar{z}^i = \sigma^1 \wedge \sigma^4 + \sigma^2 \wedge \sigma^5 + \sigma^3 \wedge \sigma^6,$$
$$\Omega_0 = \mathrm{Re}\, dz^1 \wedge dz^2 \wedge dz^3 = \sigma^1 \wedge \sigma^2 \wedge \sigma^3 - \sigma^1 \wedge \sigma^5 \wedge \sigma^6 - \sigma^3 \wedge \sigma^4 \wedge \sigma^5 + \sigma^2 \wedge \sigma^4 \wedge \sigma^6.$$

An $\mathrm{SU}(3)$ structure on V is an element of the $\mathrm{GL}(6)$-orbit of the standard $\mathrm{SU}(3)$ structure.

Setting $\Psi_0 = dz^1 \wedge dz^2 \wedge dz^3$, the following are easily verified

$$\omega_0 \wedge \Omega_0 = 0,\ \mathrm{Im}\,\Psi_0 = J^*\mathrm{Re}\,\Psi_0,\ \omega^3 = \frac{2}{3}\Omega_0 \wedge J^*\Omega_0.$$

The stabiliser in $\mathrm{GL}(6)$ of ω_0 is $\mathrm{Sp}(3)$ and the stabiliser of Ω_0 is $\mathrm{SL}(3, \mathbb{C})$. The intersections of these groups, *i.e.* the stabiliser of the pair (ω_0, Ω_0), is well known to be $\mathrm{SU}(3)$. In fact, we have the following result (Proposition 12, [Bry06]). The final condition is only necessary to fix the scale of Ω.

Proposition 4.19. *A pair* $(\omega, \Omega) \in \Lambda^2 V^* \oplus \Lambda^3 V^*$ *lies in the same* $\mathrm{GL}(6)$*-orbit as* (ω_0, Ω_0) *if and only if the following hold:*

(i) ω *is non-degenerate,* $\omega^3 \neq 0$*;*
(ii) Ω *is* ω*-primitive,* $\omega \wedge \Omega = 0$*;*
(iii) The stabiliser in $\mathrm{GL}(6)$ *of* (ω, Ω) *is compact;*
(iv) The normalisation $\omega^3 = \frac{2}{3}\Omega \wedge J^*\Omega$ *holds.*

It is elementary that any non-degenerate 2-form has an open$\mathrm{GL}(6)$-orbit. The orbit of Ω_0 is also open since

$$\dim \Lambda^3 V^* = 20 = \dim \mathrm{GL}(6) - \dim \mathrm{SL}(3, \mathbb{C}).$$

As in dimension seven for 3-forms, there is therefore a notion of stability for 2- and 3-forms in dimension six. The stability of 3-forms is captured by the following algebraic construction.

Definition 4.20. Fix an orientation η on V. The wedge product defines a map $A : \Lambda^5 V^* \to \mathrm{Hom}(V^*, \Lambda^6 V^*)$, where

$$A(\psi)(\alpha) = \psi \wedge \alpha,\ \psi \in \Lambda^5 V^*, \alpha \in V^*.$$

This gives a map $A : \Lambda^5 V^* \to V \otimes \Lambda^6 V^*$. For every $\theta \in \Lambda^3 V^*$ we then define $S_\theta \in \mathrm{End}(V)$ by

$$S_\theta(v) \otimes \eta = A((v \llcorner \theta) \wedge \theta)\ ,\ v \in V.$$

It can be shown that there exists a degree 4 irreducible $\mathrm{SL}(V)$-invariant polynomial map $P : \Lambda^3 V^* \to \mathbb{R}$ such that

$$S_\theta^2(v) = P(\theta)\, v\ ,\ \forall\, v \in V.$$

There are precisely two disjoint open $\mathrm{GL}(6)$-orbits in $\Lambda^3 V^*$, the components of $P^{-1}(\mathbb{R} \setminus \{0\})$. The forms θ of interest to us are those which have $P(\theta) < 0$, for these define a complex structure J_θ on V by

$$J_\theta(v) = \frac{1}{\sqrt{-P(\theta)}} S_\theta(v),\ v \in V.$$

For ease of reference we call such forms stable and ignore the other orbit, where the stabiliser is $\mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$.

It turns out that $J_{\Omega_0} = J$. If Ω is a stable 3-form then for any $u, v, w \in V$ we have

$$J_\Omega^* \Omega(u, v, w) = \Omega(J_\Omega u, v, w). \tag{4.27}$$

Thus, with respect to the decomposition of complex forms $\Lambda^{(p,q)}(V^* \otimes \mathbb{C})$ defined by J_Ω, the following complex 3-form has bidegree $(3, 0)$,

$$\Psi = \Omega + i J_\Omega^* \Omega. \tag{4.28}$$

Moreover, if θ is a stable 3-form on V and ω is a 2-form, the equation $\omega \wedge \theta = 0$ is equivalent to the condition that ω be J_θ-invariant,

$$\omega(J_\theta v, J_\theta w) = \omega(v, w),\ v, w \in V.$$

For arbitrary Ω a stable 3-form and a stable (*i.e.* non-degenerate) 2-form ω satisfying $\omega \wedge \Omega = 0$, the J_Ω-invariant non-degenerate bilinear form defined by

$$g_{\omega,\Omega}(v, w) = \omega(J_\Omega v, w),\ v, w \in V,$$

will be have a definite signature or signature (p, p) for $p \in \mathbb{N}$. An $\mathrm{SU}(3)$ structure is therefore equivalent to a stable pair (ω, Ω) such that $g_{\omega,\Omega}$ is a positive form.

The generalisation of the above geometry to manifolds is straightforward.

Definition 4.21. On a six-dimensional manifold M, a pair $(\omega, \Omega) \in \Omega^2(M) \oplus \Omega^3(M)$ is called an $\mathrm{SU}(3)$ *structure* if (ω_p, Ω_p) is an $\mathrm{SU}(3)$ structure on $T_p M$ for every $p \in M$.

Given a strictly nearly Kähler structure (g, J) on a six-manifold M, our work in previous sections shows that the pair (ω, Ω) defined by

$$\omega(X, Y) = g(X, JY),\ \Omega(X, Y, Z) = g(Y, (\nabla_X J)Z),\ X, Y, Z \in \mathcal{X}(M), \tag{4.29}$$

is an $\mathrm{SU}(3)$ structure. However, it should be noted that the normalisation of Proposition (4.19) (iv) is not satisfied by the pair (ω, Ω) as defined here. This is somewhat in defiance of standard conventions, but makes no material difference.

Using the above facts we can thus provide the following description of nearly Kähler geometry. Note that the condition $\omega \wedge \Omega = 0$ follows automatically from (4.30) and so is not included as a hypothesis.

Theorem 4.22. *Let M be a six-manifold and suppose $(\omega, \Omega) \in \Omega^2(M) \oplus \Omega^3(M)$ are such that:*

(i) Ω is stable and the tensor $g_{\omega,\Omega}$ on M defined by

$$g_{\omega,\Omega}(X, Y) = \omega(J_\Omega X, Y),\ X, Y \in \mathcal{X}(M)$$

is a Riemannian metric;

(ii) There exists a $\mu > 0$ so that

$$d\omega = 3\Omega\ ,\ dJ^*_\Omega\Omega = -2\mu\,\omega^2. \tag{4.30}$$

Then $(g_{\omega,\Omega}, J_\Omega)$ is a strict nearly Kähler structure on M with scalar curvature 30μ. Conversely, given a strict nearly Kähler structure (g, J) on a six-manifold M, the induced pair defined by (4.29) satisfies (i) and (ii) with $\mu = \frac{1}{30}\mathrm{scal}_g$, and $J_\Omega = J$.

An important corollary of this formulation of strictly nearly Kähler geometry is the following non-existence result.

Lemma 4.23. *If (M, g, J) is a strictly nearly Kähler six-manifold, then there are no almost complex surfaces in M.*

Proof. Suppose that there exists an almost complex surface N in M. For $p \in N$, let X, JX, Y, JY and Z, JZ be local unitary framings of, respectively, TN and the normal bundle of N near p. Then it easy to see from (4.23) that $\Omega|N = 0$, which, from the second equation of (4.30), implies $\omega^2|N = 0$. However, since N is almost complex, $\omega^2|N$ is proportional to the metric volume form on N, a contradiction. □

The relevance of G_2 geometry to this discussion may now be revealed. Recall from section 1.1.1 that the stabiliser in G_2 of a unit vector $x \in V$ is conjugate to $\mathrm{SU}(3)$. It follows that the pair

$$\omega = (x \llcorner \phi_0)|_{x^\perp},\ \Omega = \phi_0|_{x^\perp},$$

is stabilised by $\mathrm{SU}(3)$. Furthermore, (ω, Ω) is an $\mathrm{SU}(3)$ structure on the plane $x^\perp$. Indeed, since G_2 is transitive on S^6 one can assume that $x = e_7$ and from the explicit expression (4.1) we can see that

$$\phi_0 = \sigma^7 \wedge \omega_0 + \Omega_0,\ *\phi_0 = -\frac{1}{2}\omega_0^2 + \sigma^7 \wedge J_0^*\Omega_0. \tag{4.31}$$

Therefore, if N is a seven-manifold and φ is a G_2 structure on N, any oriented hypersurface M in N inherits an $\mathrm{SU}(3)$ structure, $(\omega_{\varphi,M}, \Omega_{\varphi,M})$.

Proposition 4.24. *If φ a torsion-free G_2 structure on a seven manifold N, then for any oriented hypersurface M in N*

$$\omega_{\varphi,M} \wedge d\omega_{\varphi,M} = 0,\ d\Omega_{\varphi,M} = 0.$$

Proof. Recall that φ is parallel if and only if it is closed and coclosed. From (4.31) we then have

$$d\Omega = d(\varphi|_M) = (d\varphi)|_M,\ d\omega^2 = -2d(*\varphi|_M) = -2(d*\varphi)|_M.$$ □

This motivates the following.

Definition 4.25. An $\mathrm{SU}(3)$ structure (ω, Ω) on a six-manifold M is said to be *half-flat* if the following equations hold

$$\omega \wedge d\omega = 0, \ d\Omega = 0.$$

The relevance of the term half-flat is explained on p. 12 of [CS02]. In the case that N is a hypersurface in a torsion-free G_2 manifold (M, φ) and $(\omega, \Omega) = (\omega_\varphi, \Omega_\varphi)$, then one can show that half the components of the second fundamental form II of N in M vanish. Intrinsically, a half-flat $\mathrm{SU}(3)$ structure (ω, Ω) is one for which half of the components of the torsion of the $\mathrm{SU}(3)$ structure, a first order invariant of (ω, Ω), vanish identically. The whole torsion vanishes if and only if the $\mathrm{SU}(3)$ structure is Calabi-Yau.

Equations (4.26) show that the $\mathrm{SU}(3)$ structure of a strictly nearly Kähler six-manifold is half-flat. Moreover, any strictly nearly Kähler manifold can be realised as a hypersurface in a torsion-free G_2 manifold simply by passing to the cone. In general, however, not every half-flat $\mathrm{SU}(3)$ structure can be obtained in this way (Theorem 5, [Bry12]). On the other hand, if the $\mathrm{SU}(3)$ structure is real analytic, then the inverse problem can be solved uniquely (Theorem 4, [Bry06]). This result was originally stated as proven in [Hit01], but the proof is erroneous, as the examples in [Bry06] show.

In section 5.1 we classify all homogeneous strictly nearly Kähler six-manifolds. This requires classifying the invariant strictly nearly Kähler structures on $S^3 \times S^3$, and therefore falls into the larger classification of half-flat $\mathrm{SU}(3)$ structures on this Lie group proven in [MS13]. Before this paper the classification of half-flat $\mathrm{SU}(3)$ structures on an arbitrary six-dimensional Lie group was investigated in the PhD thesis [SH10].

4.5 Incomplete Nearly Kähler Six-Manifolds

We conclude this chapter by describing an infinite number of mildly singular strictly nearly Kähler six-manifolds. The basic ingredient for this is the notion of the sine cone.

Definition 4.26. The sine cone over a Riemannian manifold (M, g) is the manifold $M \times (0, \pi)$ equipped with the metric

$$d\theta^2 + \sin^2\theta \, g.$$

The sine cone over (M, g) is incomplete unless (M, g) is isometric to a round sphere. For general (M, g), its sine cone is somewhat degenerate, in a neighbourhood of the singularities at $\theta = 0, \pi$ approximating the cone (M', g').

The following simple result immediately provides us with strictly nearly Kähler six-manifolds.

Proposition 4.27. *For any Riemannian manifold (M, g), the cone over its sine cone is isometric to the Riemannian product $(\mathbb{R} \times M', dt^2 \oplus g')$.*

Proof. The cone over the sine cone of (M, g) is the space $M \times (0, \pi) \times (0, \infty)$ equipped with the metric

$$dr^2 + r^2(d\theta^2 + \sin^2\theta \, g), \ \theta \in (0, \pi), \ r \in (0, \infty).$$

The product $(\mathbb{R} \times M', dt^2 \oplus g')$ has the metric

$$dt^2 + ds^2 + s^2 \, g, \ s, t \in (0, \infty).$$

The formulae

$$s = r \sin\theta, \ t = r \cos\theta, \tag{4.32}$$

define an isometry between the two Riemannian manifolds under consideration. □

For a Riemannian five-manifold (M, g) then, the cone over its sine cone has holonomy contained in G_2 if and only if the cone over (M, g) has holonomy contained in $\mathrm{SU}(3)$, *i.e.* (M, g) is Sasaki-Einstein (see Remark 4.3). The simplest example of a Sasaki-Einstein manifold is the round S^5. There are two ways to see this: first, the cone over (S^5, g_{rd}) is $\mathbb{R}^6$ which has the standard Calabi-Yau structure; second (S^6, g_{rd}) is the sine cone over (S^5, g_{rd}), which is strictly nearly Kähler. There also exists a countably infinite family of non-isometric Sasaki-Einstein metrics $\{g_{p,q}\}_{p,q\in\mathbb{N}}$, on $S^2 \times S^3$ (Theorem 11.4.5, [BG08]). We have, therefore, an infinite family of incomplete strictly nearly Kähler manifolds.

Remark 4.28. Notice that the transformation $\iota : \theta \mapsto \pi - \theta$ is an isometry of the sine cone over (M, g) and fixes the hypersurface $\{\theta = \frac{\pi}{2}\}$, identifiable with (M, g). Thus (M, g) can be realised as a totally geodesic hypersurface in a strictly nearly Kähler six-manifold. In general, if N is an arbitrary strictly nearly Kähler six-manifold then any totally geodesic hypersurface carries an induced Sasaki-Einstein structure (Theorem 2.2, [FIMU08]). The proof of this parallels the discussion of half-flat $\mathrm{SU}(3)$ structures given above.

We explore now the geometry of the examples of nearly Kähler manifolds obtained by the foregoing construction. In dimension five we can give a special description of Sasaki-Einstein metrics. We begin with a definition.

Definition 4.29. Suppose (M, g) is a Sasaki-Einstein five-manifold, so that its cone is Calabi-Yau. Let (ω, Ω) be the $\mathrm{SU}(3)$ structure on the cone over M determined by the Calabi-Yau structure (see [Joy00]). Identify M with the hypersurface $\{r = 1\}$ in M' and set

$$\eta = (-\xi \lrcorner \omega)|_M,\ \omega_1 = \omega|M,\ \omega_2 = (\xi \lrcorner \Omega)|_M,\ \omega_3 = (-\xi \lrcorner J^*_\Omega \Omega)|_M,$$

where r is the radial coordinate on M' and $\xi = \frac{\partial}{\partial r}$. Thus

$$\omega = r^2\omega_1 + r\eta \wedge dr,\ \Omega = r^3\omega_2 \wedge \eta - r^2\omega_3 \wedge dr.$$

The following proposition is taken from the exposition of section 2 of [FIMU08]. It is of interest as a formulation of Sasaki-Einstein geometry independent of conical Calabi-Yau structures similar to the description of nearly Kähler geometry given in the previous section.

Proposition 4.30. *The quadruple $(\eta, \omega_1, \omega_2, \omega_3)$ defines an $\mathrm{SU}(2)$ structure on M, that is there exists a 4-form v on M such that*

$$\omega_i \wedge \omega_j = \delta_{ij} v,\ v \wedge \eta \neq 0, \tag{4.33}$$

and

$$X \lrcorner \omega_1 = Y \lrcorner \omega_2 \implies \omega_3(X, Y) \geq 0.$$

Moreover, the following differential relations are satisfied

$$d\eta = -2\omega_3,\ d\omega_1 = 3\eta \wedge \omega_2,\ d\omega_2 = -3\eta \wedge \omega_1.$$

We can now describe explicitly the nearly Kähler $\mathrm{SU}(3)$ structure on the sine cone over (M, g). The formulae are from Theorem 3.7 of [FIMU08]. Proposition 4.30 can be used to show that the $\mathrm{SU}(3)$ structure satisfies equations (4.30).

Proposition 4.31. *Let (M, g) be a Sasaki-Einstein five-manifold. Then for $(\eta, \omega_1, \omega_2, \omega_3)$ as above, the following defines a strictly nearly Kähler structure (ω, Ω) on $\mathbb{R} \times M$*

$$\omega = \sin^2\theta\,(-\sin\theta\,\omega_3 + \cos\theta\,\omega_1) + \sin\theta\,d\theta \wedge \eta, \tag{4.34}$$

$$\Omega = \sin^3\theta\,\eta \wedge \omega_2 + \sin^2\theta\,d\theta \wedge (\cos\theta\,\omega_3 - \sin\theta\,\omega_1) \wedge d\theta \tag{4.35}$$

and induces the metric $d\theta^2 + \sin^2\theta\,g$.

Proof. From the foregoing definition, the following defines a torsion-free G_2 structure on $\mathbb{R} \times M'$ inducing the product metric,

$$\varphi = dt \wedge (s^2\omega_1 + s\eta \wedge ds) + (s^3\omega_2 \wedge \eta - s^2\omega_3 \wedge ds). \tag{4.36}$$

We use the same convention for coordinates as Proposition 4.27, with s is the radial coordinate on M', t the cylindrical coordinate on $\mathbb{R} \times M'$. The isometry defined by (4.32) gives a conical torsion-free G_2 structure of the form

$$\varphi' = r^2 dr \wedge \omega + r^3\Omega.$$

Applying this transformation explicitly to (4.36) gives the desired formulae for (ω, Ω). □

Observe that the isometry $\iota : \theta \mapsto \pi - \theta$ of the sine cone over (M, g) does not preserve the $\mathrm{SU}(3)$ structure (4.34). Moreover, using (4.33) we obtain

$$\omega^3 = \sin^5\theta \, d\theta \wedge \eta \wedge v,$$

from which it is seen that ι preserves the orientation. There are therefore two distinct strictly nearly Kähler structures on $\mathbb{R} \times M$ inducing the same metric and orientation.

Moreover, if (M, g) is a homogeneous Sasaki-Einstein five-manifold with a compact group G acting transitively (in which case (M, g) is a quotient of (S^5, g_{rd}) or $(S^2 \times S^3, g_{1,1})$, Corollary 11.1.14, [BG08]), then its sine cone has a cohomogeneity one action by G. The existence of strictly nearly Kähler six-manifolds admitting a cohomogeneity one group action and for which there exist isometries failing to induce an equivalence of nearly Kähler structures is of relevance to chapters 5 and 6. See in particular remark 6.16.

Chapter 5

Nearly Kähler Six-Manifolds with Symmetry

In previous chapters we described four strictly nearly Kähler six-manifolds, namely S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$. These were all 3-symmetric and, in particular, homogeneous. A natural first attempt to produce further examples is to reduce the complexity of the PDEs (4.30) by introducing a high degree of symmetry. In section 5.1 we study strictly nearly Kähler manifolds with a transitive action by a group preserving the nearly Kähler structure, and, in section 5.2, by a cohomogeneity one Lie group action. In the homogeneous case the problem is reduced to pure algebra; in the cohomogeneity one case, once one has determined the groups that can act, the problem reduces to the analysis of a system of ordinary differential equations.

The homogeneous case is reviewed in full in section 5.1 and we present and prove a theorem of Butruille. The original exposition of Butruille is the French paper [But05], but we follow the later English summary of this article [But10]. The theorem (Theorem 2, [But05] or Theorem 1, [But10]) states that the four examples of strictly nearly Kähler structures on S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$ are the only homogeneous strictly nearly Kähler six-manifolds. By the results quoted in the introduction, this in fact proves that a homogeneous strictly nearly Kähler manifold of arbitrary dimension is 3-symmetric, an early conjecture of Gray (see Theorem 2 and the accompanying discussion in [But10]).

Section 5.2 reviews the work of Podestà and Spiro on simply connected compact strictly nearly Kähler six-manifolds admitting group actions of cohomogeneity one preserving the nearly Kähler structure. In that section we present the algebraic aspect of this problem taken from the first paper [PS10] of these authors, determining the possible group diagrams and the equivariant diffeomorphism types of the resulting six-manifolds. It turns out that only $\mathrm{SU}(3)$ and $\mathrm{SU}(2) \times \mathrm{SU}(2)$ can act in this way and the diffeomorphism types are S^6, $S^3 \times S^3$ and $\mathbb{CP}^3$. The $\mathrm{SU}(3)$ case is not very interesting, giving only the standard homogeneous strictly nearly Kähler structure on S^6; the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ case is further examined in chapter 6 but is not completely solved.

Before beginning this review, we establish the following important theorem (Proposition 3.1, [SNM04]).

Theorem 5.1. *Let (M, g, J) be a complete strictly nearly Kähler six-manifold not isometric to a round sphere. Then if $\alpha \in \mathrm{Iso}(M, g)$ is orientation preserving, α preserves J. In particular, if $G \subset \mathrm{Iso}(M, g)$ is a connected Lie group then G preserves J.*

Proof. Consider $\alpha^* J$. Since α preserves the metric it follows that $(g, \alpha^* J)$ is a strictly nearly Kähler structure on M. But by Theorem (4.5), the pair (g, J) is unique amongst strictly nearly Kähler structures

compatible with g and inducing the orientation of J. Since α is assumed orientation preserving, it follows that $\alpha^* J = J$. □

Remark 5.2. The discussion in section 4.5 shows that completeness is a necessary hypothesis in this theorem.

5.1 Homogeneous Nearly Kähler Six-Manifolds

The goal of this section is to classify all homogeneous strictly nearly Kähler manifolds of dimension six (Theorem 2 [But10]). From the discussion in section 2.1, the hypothesis that M be simply connected is not restrictive. Recall from Definition 2.11 that a canonical presentation G/H of a simply connected homogeneous space consists of a connected, simply-connected semi-simple compact Lie group $G = G_1 \times \cdots \times G_r$, where G_i are simple factors, and a closed subgroup H such that no projection $H \to G_i$ is onto.

Theorem 5.3. *Let M be a six-dimensional connected, simply connected homogeneous space, and suppose that G/H is a canonical presentation of M. If there exists a G-invariant strictly nearly Kähler structure on M, then G, H and M must be as given in Table 5.1. The nearly Kähler structure on M is, up to scaling, one of the 3-symmetric structures described in section 3.2 or one of those on the six-sphere described in section 1.1.1. The subgroup $T^2 \subset \mathrm{SU}(3)$ is the standard maximal torus.*

H	G	M
0	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$S^3 \times S^3$
T^2	$\mathrm{SU}(3)$	$\mathbb{F}_{1,2}$
$\mathrm{U}(1) \times \mathrm{SU}(2)$	$\mathrm{Sp}(2)$	$\mathbb{CP}^3$
$\mathrm{SU}(3)$	G_2	S^6

Table 5.1: Homogeneous strictly nearly Kähler manifolds

Remark 5.4. If $(G/H, g)$ is a homogeneous space not isometric to a round sphere, then by Theorem 5.1 any almost complex structure J such that (g, J) is strictly nearly Kähler is automatically G-invariant. In this case the hypothesis that G preserve J is superfluous in Theorem 5.3.

The proof of Theorem 5.3 is separated into two parts:

(I) It is shown that the smallest group G of isometries that can act transitively on a strictly nearly Kähler manifold of dimension six is one of those listed in Table 5.1. The proof of this consists of the observation that if G preserves a strictly nearly Kähler structure then it also preserves the $\mathrm{SU}(3)$ structure it defines; this implies $H \subset \mathrm{SU}(3)$. Using then the fact that $\dim G - \dim H = 6$, the classification of compact Lie groups given in Theorem 2.13 provides the list of groups that can act: a number of these *a priori* possibilities are incompatible with the nearly Kähler assumption; the remaining groups are those given in Table 5.1. This first part of the proof is completed by demonstrating that the inclusions $H \subset G$ are standard.

(II) It is next proven that the strictly nearly Kähler structures on S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$ constructed in previous sections are the unique G-invariant such structures for $G = \mathrm{G}_2$, $\mathrm{SU}(2) \times \mathrm{SU}(2)$, $\mathrm{Sp}(2)$ and $\mathrm{SU}(3)$, respectively. Notice that each of the three spaces S^6, $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$ are partial flag manifolds (see p. 10).

This dichotomy between a classification of partial flags G/H and the case $G/H = \mathrm{SU}(2) \times \mathrm{SU}(2)$ is reflected in the general theory of simply connected homogeneous Einstein six-manifolds [NR03]. As in the nearly Kähler case, the general classification of partial flags is tractable – in the nearly Kähler case this is accomplished relatively easily using Theorem 2.14 and the description of the invariant almost complex structures on $\mathbb{CP}^2$ and $\mathbb{F}_{1,2}$ provided in section 3.2. On the other hand, the classification of left-invariant Einstein metrics on $M = \mathrm{SU}(2) \times \mathrm{SU}(2)$ has not been resolved. This problem is soluble but non-trivial in the nearly Kähler situation.

5.1.1 Proof of Theorem 5.3: Part I

Let $(G/H, g, J)$ be a simply connected strictly nearly Kähler six-manifold, where G/H is a canonical presentation and G preserves (g, J). As in section 4.4, the nearly Kähler structure defines an $\mathrm{SU}(3)$ structure (ω, Ω) on G/H. By supposition G preserves g and J, and so fixes ω and, recalling (4.30), the equation $d\omega = 3\Omega$ implies that G preserves Ω also. We see then that H acts on $T_{eH}G/H$ fixing $(\omega_{eH}, \Omega_{eH})$, *i.e.* $H \subset \mathrm{SU}(3)$. This latter fact is the key observation. For the remainder of the proof, we make free use of the classification of semi-simple Lie algebras in low dimensions, Theorem 2.13.

Being a compact Lie algebra of dimension at most eight then, $\mathfrak{h}$ is a direct sum of factors isomorphic to $\mathfrak{su}(3)$, $\mathfrak{su}(2)$ or $\mathfrak{u}(1)$. As $\operatorname{rank} H \leq \operatorname{rank} \mathrm{SU}(3) = 2$, $\mathfrak{h}$ must be trivial or isomorphic to one of $\mathfrak{u}(1)$, $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$, $\mathfrak{su}(2)$, $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$, $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ or $\mathfrak{su}(3)$. It is shown now that none of the cases $\mathfrak{h} = \mathfrak{u}(1), \mathfrak{su}(2)$ or $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ are possible.

(i) The case $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ defines an embedding $\varphi : \mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathfrak{su}(3)$. This gives two mutually commuting three dimensional faithful representations ρ_1, ρ_2 of $\mathfrak{su}(2)$. By Schur's lemma, one of these representations is irreducible and the other trivial or there is a trivial 1 dimensional subspace of the ρ_i and an invariant complement V_i of dimension 2. In the first case there is a ρ_i with a non-zero kernel, a contradiction. In the second case, one of the V_i must be irreducible and the other trivial or both must be trivial. This again gives a ρ_i with a non-zero kernel. There can therefore be no such embedding φ.

(ii) Suppose that $\mathfrak{h} = \mathfrak{su}(2)$. As G is then nine dimensional, connected, simply connected and compact, we must have $G = \mathrm{SU}(2) \oplus \mathrm{SU}(2) \oplus \mathrm{SU}(2)$. Let $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ be the projections of $\mathfrak{h}$ to each of the three $\mathfrak{su}(2)$ factors in $\mathfrak{g}$. Since G/H is a canonical presentation of M, the $\mathfrak{h}_i$ have dimension at most two. But as all Lie algebras of dimension at most two are abelian and $\mathfrak{h} \subset \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ we obtain a contradiction. Thus one of the $\mathfrak{h}_i$ equals $\mathfrak{su}(2)$, and G/H cannot be canonical.

(iii) If $\mathfrak{h} = \mathfrak{u}(1)$ then $\mathfrak{g}$ has dimension seven. There is no compact semi-simple Lie group in this dimension.

We deduce the possible G in each of the remaining cases.

(iv) If $\mathfrak{h} = \{0\}$ then G is a connected simply connected compact semi-simple Lie algebra of dimension six. There is only one possibility, namely $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$.

(v) If $\mathfrak{h} = 2\mathfrak{u}(1)$ then G has dimension eight and can only be $\mathrm{SU}(3)$. Since H is then a maximal torus in G it is conjugate to T^2 and $G/H = \mathbb{F}_{1,2}$.

The remaining two cases require a little further work.

(vi) If $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{su}(2)$ then G has dimension ten and so is isomorphic to $\mathrm{Sp}(2)$. We determine which subalgebras of $\mathfrak{g}$ isomorphic to $\mathfrak{h}$ give rise to nearly Kähler structures.

The isotropy representation $\mathfrak{m}$ of $\mathfrak{h}$ is six dimensional and, by supposition, possesses an $\mathfrak{h}$-invariant complex structure. The action by the subalgebra $\mathfrak{su}(2)$ of $\mathfrak{h}$ must then be either the trivial representation

of complex dimension three, the sum of the trivial one dimensional complex representation $\mathbb{C}$ and the defining representation on $\mathbb{C}^2$ or the irreducible representation of complex dimension three on $\mathrm{Sym}^2(\mathbb{C}^2)$. The former case is not possible for an almost effective action. Schur's lemma implies that the complementary $\mathfrak{u}(1)$ factor acts by multiplication on these $\mathfrak{su}(2)$-irreducible summands, and so is determined by an integer weight. For W one of the three foregoing irreducible representations of $\mathfrak{su}(2)$, write W_p for the representation of $\mathfrak{u}(1)$ on W given by multiplication with weight $p \in \mathbb{Z}$. Thus $\mathfrak{m}$ is equivalent to either $\mathbb{C}_p \oplus \mathbb{C}^2_q$ or $\mathrm{Sym}^2(\mathbb{C}^2)_p$ for integers p, q.

If G/H is strictly nearly Kähler then there exists an invariant complex 3-form Ψ given by (4.28), so $(\Lambda^{(3,0)}\mathfrak{m})^H$ is non-trivial. Supposing $\mathfrak{m} = \mathrm{Sym}^2(\mathbb{C}^2)_p$, then $\mathfrak{u}(1)$ acts on $\Lambda^3\mathfrak{m}$ with weight $3p$, and so there are no H-invariant 3-forms unless $p = 0$, impossible if G acts almost effectively. If $\mathfrak{m} = \mathbb{C}_p \oplus \mathbb{C}^2_q$, then $\Lambda^{(3,0)}\mathfrak{m}$ decomposes as $\mathbb{C}_p \otimes \Lambda^2\mathbb{C}^2_q$, which has invariant elements if and only if $p = -2q$. This corresponds to the embedding

$$\mathrm{U}(1) \times \mathrm{Sp}(1) \longrightarrow \mathrm{Sp}(2);\ (z, \alpha) \longmapsto \mathrm{diag}(z^q, \alpha),$$

all of which have the same image in G.

(vii) If $\mathfrak{h} = \mathfrak{su}(3)$, then G is fourteen dimensional and so is isomorphic to G_2 or $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$. In the latter case, consider the projection of $\mathfrak{h}$ to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$: for dimensional reasons this has a non-trivial kernel, and as $\mathfrak{h}$ is simple this implies that this kernel equals $\mathfrak{h}$. The kernel of the projection $\mathfrak{h} \to \mathfrak{su}(3)$ must then be trivial. But this implies that $\mathfrak{h} \to \mathfrak{su}(3)$ is onto, contradicting the assumption that G/H is a canonical projection. Thus, $G = \mathrm{G}_2$. As above, the isotropy representation of $\mathfrak{h}$ is a complex representation of complex dimension three. There are only two such representations of $\mathfrak{su}(3)$, namely the trivial representation and the defining representation on $\mathbb{C}^3$. Since the action is almost effective we must be in the latter situation, in which case $\mathfrak{h}$ is the standard $\mathfrak{su}(3)$ subalgebra of $\mathfrak{g}_2$, and $G/H = S^6$.

This completes the proof of Part I.

5.1.2 Proof of Theorem 5.3: Part II

Proposition 5.5. *The strictly nearly Kähler structures on the spaces*

$$S^6 = \mathrm{G}_2/\mathrm{SU}(3)\ ,\ \mathbb{CP}^3 = \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)\ \text{ and }\ \mathbb{F}_{1,2} = \mathrm{U}(3)/\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$$

are unique up to scale for the given homogeneous structure.

Proof. The S^6 case is simplest since the isotropy representation is irreducible. There is then, up to scale, a unique invariant metric, the round metric, and, by Proposition 4.12, the almost complex structure must be one of those described in section 1.1.1.

Now consider $\mathbb{F}_{1,2} = \mathrm{SU}(3)/T^2$. The isotropy representation was described in section 3.2.3: there are three irreducible summands, $\mathfrak{p}_i$, $i = 1, 2, 3$, giving, by Theorem 2.14, at most eight $\mathrm{SU}(3)$-invariant almost complex structures on $\mathbb{F}_{1,2}$. This number is realised by the complex structures of the three distinct Kähler structures and that of the 3-symmetric strictly nearly Kähler structure described previously, together with their respective opposites. There cannot, therefore, be any further invariant almost complex structures, in particular there are no further invariant nearly Kähler structures.

Similarly, from the description of the isotropy representation of $\mathbb{CP}^3 = \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)$, the isotropy representation has two irreducible invariant subspaces giving at most four invariant almost complex structures.

This number is realised by the almost complex structures belonging to the standard Kähler structure and the nearly Kähler structure of section 3.2.2 together with their opposites. Again, there cannot, therefore, be any further invariant nearly Kähler structures.

For both $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$, there is up to scale a unique metric for which the 3-symmetric structures are naturally reductive and so the claimed uniqueness follows. □

Uniqueness of the left-invariant strictly nearly Kähler structure on $S^3 \times S^3$ requires a more involved argument. This algebraic calculation is carried out in [But10] pp. 10-13 and is essentially the new result of that paper. To perform this calculation we adopt the $\mathrm{SU}(3)$ structure description of nearly Kähler geometry, Theorem 4.22. Given, then, a left-invariant 2-form ω on $S^3 \times S^3$, we ask under what conditions does the pair $(\omega, \frac{1}{3}d\omega)$ define a strictly nearly Kähler structure.

Let $\{\sigma_i\}_{i\in\mathbb{Z}_3}$ and $\{\tau_i\}_{i\in\mathbb{Z}_3}$ be bases of left-invariant 1-forms on S^3 satisfying the Maurer-Cartan equations

$$d\sigma_i = \sigma_{i+1} \wedge \sigma_{i+2},\ d\tau = \tau i + 1 \wedge \tau_{i+2},\ i \in \mathbb{Z}_3. \tag{5.1}$$

Trivialising $\Lambda^2(S^3 \times S^3)$ by the frame $\{\sigma_i \wedge \tau_j\}_{i,j}$, it is straightforward to show that any invariant 2-form can be written in the form

$$\sum_{i\in\mathbb{Z}_3} a_i\, \sigma_{i+1} \wedge \sigma_{i+2} + \sum_{i\in\mathbb{Z}_3} b_i\, \tau_{i+1} \wedge \tau_{i+2} + \sum_{i,j\in\mathbb{Z}_3} c_{ij}\, \sigma_i \wedge \tau_j,$$

for some $a_i, b_i, c_{ij} \in \mathbb{R}$. Let ω be of this form and abbreviate $A = (a_i), B = (b_i) \in \mathbb{R}^3$, $C = (c_{ij}) \in \mathrm{M}_3(\mathbb{R})$.

Lemma 5.6. *If ω is a left-invariant 2-form on $S^3 \times S^3$, then ω is non-degenerate and $\omega \wedge d\omega = 0$ if and only if there are bases $\{\sigma_i\} i \in \mathbb{Z}_3, \{\tau_i\} i \in \mathbb{Z}_3$ of left invariant 1-forms on S^3 and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ so that*

$$\omega = \lambda_1\, \sigma_1 \wedge \tau_1 + \lambda_2\, \sigma_2 \wedge \tau_2 + \lambda_3\, \sigma_3 \wedge \tau_3.$$

Proof. Given an arbitrary pair of bases $\{\sigma_i\}_{i\in\mathbb{Z}_3}, \{\tau_i\}_{i\in\mathbb{Z}_3}$, compute

$$\begin{aligned}\omega^3 = 3! \sum_{i,j,k,l} a_i b_j c_{kl}\, \sigma_{i+1} \wedge \sigma_{i+2} \wedge \tau_{j+1} \wedge \tau_{j+2} \wedge \sigma_k \wedge \tau_l \\ + \sum_{i,j,k,l,m,n} c_{ij} c_{kl} c_{mn}\, \sigma_i \wedge \tau_j \wedge \sigma_k \wedge \tau_l \wedge \sigma_n \wedge \tau_n.\end{aligned}$$

In the first sum, the only non-vanishing terms have $k = i$ and $l = j$. The sum is then the matrix product $(A^t C B)\,\mathrm{vol}$, where $\mathrm{vol} = \prod_i \sigma_i \wedge \tau_i$. The second sum evaluates to $3! \det C\,\mathrm{vol}$, so ω is non-degenerate if and only if

$$A^t C B + \det C \neq 0. \tag{5.2}$$

Using (5.1), we have

$$d\omega = \sum_i c_{ij}\, \sigma_{i+1} \wedge \sigma_{i+2} \wedge \tau_j - \sum_i c_{ij}\, \sigma_i \wedge \tau_{j+1} \wedge \tau_{j+2}.$$

Thus

$$\begin{aligned}\omega \wedge d\omega = &\sum_{i,j,k} a_i c_{jk}\, \sigma_{i+1} \wedge \sigma_{i+2} \wedge \sigma_j \wedge \tau_{k+1} \wedge \tau_{k+2} \\ &+ \sum_{i,j,k} b_i c_{jk}\, \tau_{i+1} \wedge \tau_{i+2} \wedge \tau_k \wedge \sigma_{j+1} \wedge \sigma_{j+2} \\ &+ \sum_{i,j,k,l} c_{ij} c_{kl}\, \sigma_i \wedge \sigma_{k+1} \wedge \sigma_{k+2} \wedge \tau_j \wedge \tau_l \\ &+ \sum_{i,j,k,l} c_{ij} c_{kl}\, \sigma_i \wedge \sigma_k \wedge \tau_j \wedge \tau_{l+1} \wedge \tau_{l+2}.\end{aligned}$$

The first two sums vanish if and only if $A^tC = CB = 0$. In the third, the only non-zero summands have $k = i$, and the whole sum then vanishes as $c_{ij}c_{il}$ is symmetric in k, l. Likewise for the fourth sum. If $A^tC = CB = 0$ then from (5.2) we must have $\det C \neq 0$ and $A = B = 0$. We diagonalise C as follows. As $C \in \mathrm{GL}(\mathbb{R}^3)$ we may write $C = S \cdot O$, for S symmetric and $O \in \mathrm{SO}(3)$. Now let $P \in \mathrm{SO}(3)$ diagonalise S, $S = P^tDP$ for $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$; then $C = P^tDPO$. But $(P^t, PO) \in \mathrm{SO}(3) \times \mathrm{SO}(3)$, so we can act by an element of $S^3 \times S^3$ to obtain $C = D$, and the lemma now follows. □

We can now characterise when ω defines an $\mathrm{SU}(3)$ structure.

Lemma 5.7. *If $\omega = \sum_i \lambda_i\, \sigma_i \wedge \tau_i$ is non-degenerate and satisfies $\omega \wedge d\omega = 0$, then the 3-form $\Omega = \frac{1}{3} d\omega$ is stable if and only if*

$$(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 + \lambda_3)(\lambda_1 - \lambda_2 - \lambda_3) < 0. \tag{5.3}$$

The resulting bilinear form $g_{\omega,\Omega}$ is a positive definite metric if and only if

$$\lambda_1\lambda_2\lambda_3 > 0.$$

Proof. Denoting by $\{\sigma^i\}_{i\in\mathbb{Z}_3}$ and $\{\tau^i\}_{i\in\mathbb{Z}_3}$ the bases of vector fields dual to $\{\sigma_i\}$ and $\{\tau_i\}$, we have

$$\sigma^i \llcorner \Omega = \frac{1}{3}\left(\lambda_{k+2}\, \sigma_{k+1} \wedge \tau_{k+2} - \lambda_{k+1}\, \sigma_{k+2} \wedge \tau_{k+1} - \lambda_k \tau_{k+1} \wedge \tau_{k+2}\right).$$

Further calculation shows that

$$\begin{aligned}9(\sigma^k \llcorner \Omega) \wedge \Omega = &-2\lambda_{k+2}\lambda_{k+1}\, \mathrm{vol}_\sigma \wedge \tau_{k+1} \wedge \tau_{k+2} \\ &+ (\lambda_{k+2}^2 + \lambda_{k+1}^2 - \lambda_k^2)\, \sigma_{k+1} \wedge \sigma_{k+2} \wedge \mathrm{vol}_\tau,\end{aligned}$$

where $\mathrm{vol}_\sigma = \prod_i \sigma_i$, $\mathrm{vol}_\tau = \prod_i \tau_i$. Thus

$$\begin{aligned}9(\sigma^k \llcorner \Omega) \wedge \Omega \wedge \sigma_l &= -\delta_{kl}(\lambda_{k+2}^2 + \lambda_{k+1}^2 - \lambda_k^2)\, \mathrm{vol}, \\ 9(\sigma^k \llcorner \Omega) \wedge \Omega \wedge \tau_l &= -2\delta_{kl}\lambda_{k+2}\lambda_{k+1}\, \mathrm{vol},\end{aligned}$$

so then

$$3S_\Omega(\sigma^k) = -(\lambda_{k+2}^2 + \lambda_{k+1}^2 - \lambda_k^2)\, \sigma^k - 2\lambda_{k+2}\lambda_{k+1}\, \tau^k.$$

A similar calculation shows that

$$3S_\Omega(\tau^k) = (\lambda_{k+2}^2 + \lambda_{k+1}^2 - \lambda_k^2)\, \tau^k - 2\lambda_{k+2}\lambda_{k+1}\, \sigma^k.$$

Then

$$\begin{aligned}81S^2_\Omega(\sigma^k) &= \left\{(\lambda_{k+2}^2+\lambda_{k+1}^2-\lambda_k^2)^2-(2\lambda_{k+2}\lambda_{k+1})^2\right\}\sigma^k\\ &= \left(\lambda_k^4+\lambda_{k+1}^4+\lambda_{k+2}^4-2\lambda_k^2\lambda_{k+1}^2-2\lambda_k^2\lambda_{k+2}^2-2\lambda_{k+1}^2\lambda_{k+2}^2\right)\sigma^k\\ &= (\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2-\lambda_3)(\lambda_1-\lambda_2+\lambda_3)(\lambda_1-\lambda_2-\lambda_3)\sigma^k.\end{aligned}$$

It is seen then that Ω is stable if and only if (5.3) holds. We now compute $g_{\omega,\Omega}$. The almost complex structure defined by Ω is given by the expressions

$$J_\Omega(\sigma^k)=\alpha_k\sigma^k+\beta^k\tau^k,\quad J_\Omega(\tau^k)=-\alpha_k\tau^k+\beta_k\sigma^k, \tag{5.4}$$

where

$$\begin{gathered}\alpha_k=\kappa^{-1}(\lambda_k^2-\lambda_{k+2}^2-\lambda_{k+1}^2),\ \beta_k=-2\kappa^{-1}\lambda_{k+2}\lambda_{k+1},\\ \kappa=\left[-(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2-\lambda_3)\ (\lambda_1-\lambda_2+\lambda_3)(\lambda_1-\lambda_2-\lambda_3)\right]^{\frac{1}{2}}.\end{gathered} \tag{5.5}$$

Now compute

$$g_{\omega,\Omega}(\sigma^j,\sigma^k)=-\delta_{jk}\lambda_j\beta_j,\ g_{\omega,\Omega}(\tau^j,\tau^k)\qquad =-\delta_{jk}\lambda_j\beta_j,\ g_{\omega,\Omega}(\sigma^j,\tau^k)=\delta_{jk}\lambda_j\beta_j.$$

Thus $g_{\omega,\Omega}$ is the direct sum of the three quadratic forms defined by the matrices

$$\begin{pmatrix}-\lambda_j\beta_j & \lambda_j\alpha_j\\ \lambda_j\alpha_j & -\lambda_j\beta_j\end{pmatrix},\quad j=1,2,3,$$

and so is positive definite if and only if

$$-\lambda_j\beta_j,\ (\lambda_j\beta_j)^2-(\lambda_j\alpha_j)^2>0.$$

The second expression equals $\lambda_j^2\kappa^2$, and so is positive. The first expression is the second inequality of the lemma. □

Lemma 5.8. *Let $\omega=\sum_i\lambda_i\,\sigma_i\wedge\tau_i$ be non-degenerate, satisfy $\omega\wedge d\omega=0$ and be such that the 3-form $\Omega=\frac{1}{3}d\omega$ is stable. Then (ω,Ω) defines a strictly nearly Kähler structure on $S^3\times S^3$ if and only if there is a $\mu>0$ such that*

$$\lambda_i\alpha_i=-\frac{1}{3}\mu\kappa^2\beta_i, \tag{5.6}$$

where α_i,β_i and κ are given in (5.5).

Proof. From equation (4.27) it follows that

$$\begin{aligned}-3J^*_\Omega\Omega &= \sum_i\lambda_i\left\{J^*\sigma_{i+1}\wedge\sigma_{i+2}\wedge\tau_i+\sigma_{i+1}\wedge J^*\sigma_{i+2}\wedge\tau_i+\sigma_{i+1}\wedge\sigma_{i+2}\wedge J^*\tau_i\right.\\ &\qquad\left.-J^*\sigma_i\wedge\tau_{i+1}\wedge\tau_{i+2}-\sigma_i\wedge J^*\tau_{i+1}\wedge\tau_{i+2}-\sigma_i\wedge\tau_{i+1}\wedge J^*\tau_{i+2}\right\},\\ &= \sum_i\lambda_i\left\{-\beta_i(\mathrm{vol}_\sigma+\mathrm{vol}_\tau)+(\alpha_{i+1}+\alpha_{i+2}-\alpha_i)(\sigma_{i+1}\wedge\sigma_{i+2}\wedge\tau_i+\sigma_i\wedge\tau_{i+1}\wedge\tau_{i+2})\right.\\ &\qquad\left.+\beta_{i+1}(\sigma_{i+2}\wedge\tau_i\wedge\tau_{i+1}-\sigma_i\wedge\sigma_{i+1}\wedge\tau_{i+2})+\beta_{i+2}(\sigma_{i+1}\wedge\tau_{i+2}\wedge\tau_i-\sigma_i\wedge\tau_{i+1}\wedge\sigma_{i+2})\right\}.\end{aligned}$$

The first term is obviously closed, while the third and fourth terms are exact. We have then

$$\begin{aligned}-3dJ^*_\Omega\Omega &= 2\sum_i \lambda_i(\alpha_{i+1}+\alpha_{i+2}-\alpha_i)\,\sigma_{i+1}\wedge\sigma_{i+2}\wedge\tau_{i+1}\wedge\tau_{i+2}\\ &= -2\sum_i \lambda_i\alpha_i\,\sigma_{i+1}\wedge\sigma_{i+2}\wedge\tau_{i+1}\wedge\tau_{i+2}\\ &= -2\left(\lambda_0\alpha_0\,\sigma_1\wedge\sigma_2\wedge\tau_1\wedge\tau_2+\lambda_1\alpha_1\,\sigma_2\wedge\sigma_0\wedge\tau_2\wedge\tau_0+\lambda_2\alpha_2\,\sigma_0\wedge\sigma_1\wedge\tau_0\wedge\tau_1\right).\end{aligned}$$

But

$$\begin{aligned}\omega^2 &= \sum_{i,j}\lambda_i\lambda_j\sigma_i\wedge\sigma_j\wedge\tau_i\wedge\tau_j,\\ &= 2\left(\lambda_0\lambda_1\,\sigma_0\wedge\sigma_1\wedge\tau_0\wedge\tau_1+\lambda_0\lambda_2\,\sigma_0\wedge\sigma_2\wedge\tau_0\wedge\tau_2+\lambda_1\lambda_2\,\sigma_1\wedge\sigma_2\wedge\tau_1\wedge\tau_2\right).\end{aligned}$$

Comparing coefficients gives (5.6). □

We can now prove the required uniqueness for nearly Kähler structures on $S^3\times S^3$ (Proposition 2.5, [But10]).

Proposition 5.9. *Up to scaling and sign, there exists a unique left-invariant strictly nearly Kähler structure on $S^3\times S^3$, namely that given in (3.2).*

Proof. Define $c=-\frac{1}{3}\mu\lambda_1\lambda_2\lambda_3$ and $\Lambda=\lambda_1^2+\lambda_2^2+\lambda_3^2$. Then (5.6) is the statement that each λ_i^2 solves the quadratic equation

$$2x^2-\Lambda x-c=0. \tag{5.7}$$

This possesses at most two solutions, so $\lambda_2^2=\lambda_3^2$, say, and we suppose that $\lambda_1^2\neq\lambda_2^2$. The sum of the roots is then equal to half the coefficient of the linear term in (5.7), *i.e.* $\lambda_1^2+\lambda_2^2=\frac{1}{2}\Lambda$. But $\Lambda=\lambda_1^2+2\lambda_2^2$, so $\lambda_1=0$. This contradicts the second inequality of Lemma 5.7. Thus $\lambda_1^2=\lambda_2^2=\lambda_3^2$. Since $\lambda_1\lambda_2\lambda_3>0$, the λ_i must coincide or two coincide and the third differs by a sign. Any of the latter solutions can be transformed into one with $\lambda_1=\lambda_2=\lambda_3$: if $\lambda_1=\lambda_2=-\lambda_3<0$, say, then acting on one factor of $S^3\times S^3$ by

$$\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}\in\mathrm{SO}(3)$$

has the desired effect. It is easy to see that the strictly nearly Kähler structure (3.2) has $\lambda_1=\lambda_2=\lambda_3$, and so the theorem is proven. □

5.2 Nearly Kähler Six-Manifolds of Cohomogeneity One

In this section we consider strictly nearly Kähler six-manifolds admitting a cohomogeneity one action by a connected compact Lie group G, classifying the G that can act and fixing the diffeomorphism type of the underlying manifolds. The main result of this section is Theorem 1.1 of [PS10].

For M a compact strictly nearly Kähler six-manifold with a G-action, $\pi_1(M)$ is finite by Theorem 4.4, and so by Theorem 2.4 the G-action may be lifted to an action by a compact group on the compact universal covering space of M. The action on the latter space is evidently of cohomogeneity one, so there is therefore no loss of generality in the assumption that M be simply connected. We now state the theorem.

Theorem 5.10. *Let $G \supset K_1, K_2 \supset H$ be the group diagram of a cohomogeneity one almost effective group action preserving the nearly Kähler structure of a compact connected simply connected strictly nearly Kähler six-manifold (M, g, J). Then $(\mathfrak{g}, \mathfrak{h}, \mathfrak{k}_1, \mathfrak{k}_2)$ and the G-diffeomorphism type of M are contained in Table 5.2. The quadruple (G, H, K_1, K_2) consists of connected groups and G may be assumed simply connected. The G-actions in column five are those described below and Δ denotes the diagonal map of a standard inclusion.*

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{k}_1$	$\mathfrak{k}_2$	M
$\mathfrak{su}(3)$	$\mathfrak{su}(2)$	$\mathfrak{su}(3)$	$\mathfrak{su}(3)$	S^6
$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\Delta(\mathfrak{u}(1))$	$\Delta(\mathfrak{su}(2))$	$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	S^6
$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\Delta(\mathfrak{u}(1))$	$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\mathfrak{u}(1) \oplus \mathfrak{su}(2)$	$\mathbb{CP}^3$
$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\Delta(\mathfrak{u}(1))$	$\Delta(\mathfrak{su}(2))$	$\Delta(\mathfrak{su}(2))$	$S^3 \times S^3$

Table 5.2: Group diagrams of nearly Kähler six-manifolds

Remark 5.11. If $\mathrm{SU}(3)$ acts with cohomogeneity one on a compact strictly nearly Kähler six-manifold (M, g), then it is shown in Theorem 1.2, [PS10] that (M, g) has constant sectional curvature. But as the isotropy representation on the principal orbit $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$ is irreducible we see that in a neighbourhood of a given principal orbit $g = dt^2 + f(t)g_{\mathrm{rd}}$ for some positive function f. Thus if g is complete $f = \sin^2 \lambda t$ for some $\lambda > 0$ and the metric is round. The more interesting situation is therefore when $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts with cohomogeneity one. This is further explored in chapter 6.

The proof of Theorem 5.10 is separated into three parts:

(I) We prove that the pairs of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ that can arise are $(\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathfrak{u}(1))$ and $(\mathfrak{su}(3), \mathfrak{su}(2))$, and the inclusions $\mathfrak{h} \subset \mathfrak{g}$ are determined. We use the facts that G preserves (g, J) and that $\pi_1(M)$ is trivial together with Lemma 4.23;

(II) Using Table 2.1, we compute the singular isotropy groups compatible with the above $(\mathfrak{g}, \mathfrak{h})$;

(III) Finally, we show that any admissible quadruple (G, H, K_1, K_2) consists of connected groups and show that the topology of the resulting cohomogeneity one G-spaces is unique up to G-equivalence.

Before commencing the proof, we describe cohomogeneity one group actions on S^6, $S^3 \times S^3$ and $\mathbb{CP}^3$ preserving their unique homogeneous nearly Kähler structures. The groups that act are all subgroups of the relevant transitive groups given earlier.

5.2.1 S^6 as a Space of Cohomogeneity One

The round S^6 stands out in having two distinct cohomogeneity one group actions. It was already seen that (S^6, g_{rd}) is the homogeneous space $\mathrm{G}_2/\mathrm{SU}(3)$. Recall that the subgroup of G_2 fixing a unit vector x is identified with $\mathrm{SU}(3)$ via the identification between $\mathbb{C}^3$ and T_xS^6 defined by the almost complex structure J_x of section 1.1.1. This inclusion of $\mathrm{SU}(3)$ in G_2 thus provides an action on S^6 preserving the nearly Kähler structures of section 1.1.1.

If $y \in S^6$ is perpendicular to x, then the subgroup H_y of $\mathrm{SU}(3)$ fixing y fixes J_xy also and so is conjugate to $\mathrm{SU}(2)$. The $\mathrm{SU}(3)$-orbit through y is therefore the equatorial 5-sphere. On the other hand if $y = \pm x$, H_y is the whole of $\mathrm{SU}(3)$. The six-sphere thus admits a cohomogeneity one action by $\mathrm{SU}(3)$, whose principal orbits form a family of great spheres and whose singular orbits are the poles $\pm x$.

The second action is by $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Identifying $\mathrm{SU}(2) \times \mathrm{SU}(2)$ with pairs of unit quaternions (x, y), define an action on $(p, q) \in \mathbb{O}$ by

$$(x, y) \cdot (p, q) = (xp\bar{x}, yq\bar{x}). \tag{5.8}$$

It is straightforward to check that this action preserves octonionic multiplication and so induces an inclusion into G_2. In particular, we can restrict to an action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$.

Suppose then that $(x, y) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ fixes $(p, q) \in \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$,

$$xp\bar{x} = p\ ,\ yq\bar{x} = q.$$

If p and q are both non-zero, their $\mathrm{SU}(2) \times \mathrm{SU}(2)$-orbit contains $(i, 1)$. Then $x = y$ and $x \in \mathrm{U}(1)$. Orbits of such (p, q) therefore have cohomogeneity one, are diffeomorphic to $S^3 \times S^2$, and have isotropy group conjugate to the diagonal in $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of the subgroup $\mathrm{U}(1) \subset \mathrm{SU}(2)$. If $q = 0$, the orbit contains $(i, 0)$ so $x \in \mathrm{U}(1)$, but y is arbitrary. If $p = 0$, the orbit contains $(0, 1)$ so that $y = x$ with x arbitrary. The two singular isotropy groups are, therefore, $\mathrm{U}(1) \times \mathrm{SU}(2)$ and $\Delta(\mathrm{SU}(2))$, giving singular orbits S^2 and S^3, respectively.

5.2.2 $S^3 \times S^3$ as a Space of Cohomogeneity One

Define an isometric action of $(x, y) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ on $(p, q) \in S^3 \times S^3 \subset \mathbb{H} \oplus \mathbb{H}$ by

$$(x, y) \cdot (p, q) = (xp\bar{x}, yq\bar{x}).$$

The orbit of a point (p, q) contains $(r, 1)$ for $r \in S^3$. If (x, y) stabilises $(r, 1)$ then $x = y$. If $r = \pm 1$ then x is arbitrary, and the isotropy group here is conjugate to $\Delta(\mathrm{SU}(2))$. If r has non-trivial imaginary part s, then $x = a + \frac{s}{|r|}b$ where a, b are real and $a^2 + b^2 = 1$, so the isotropy group is conjugate to $\Delta(\mathrm{U}(1))$.

This action is therefore of cohomogeneity one, with principal orbit type $S^3 \times S^2$ and two singular orbits diffeomorphic to S^3.

5.2.3 $\mathbb{CP}^3$ as a Space of Cohomogeneity One

The action by $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{CP}^3$ is by the inclusion into $\mathrm{Sp}(2)$ corresponding to the splitting $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$. Recall the twistor fibration $\tau : \mathbb{CP}^3 \to \mathbb{HP}^1$. The 2-spheres $\tau^{-1}([1\colon 0])$ and $\tau^{-1}([0\colon 1])$ are $\mathrm{SU}(2) \times \mathrm{SU}(2)$-orbits with isotropy groups $\mathrm{U}(1) \times \mathrm{SU}(2)$ and $\mathrm{SU}(2) \times \mathrm{U}(1)$, respectively. The isotropy of any complex line $[z_0 : z_1 : z_2 : z_3]$ with both $z_0 + z_1 j$ and $z_2 + z_3 j$ non-trivial, however, is evidently conjugate to $\Delta(\mathrm{U}(1))$.

From these calculations it is seen that $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts with cohomogeneity one on $\mathbb{CP}^3$. The principal orbit type is $S^3 \times S^2$ and the two singular orbits are 2-spheres.

5.2.4 Proof of Theorem 5.10: Part I

As in the statement of Theorem 5.10, let $G \supset K_1, K_2 \supset H$ be the group diagram of (M, g, J). Fix a point $p \in M^*$ with isotropy group H. As in Proposition 2.22, we let ξ be the unit geodesic vector field on M^*.

As in the proof of Theorem 5.3, the isotropy representation on T_pM induces an inclusion $\mathfrak{h} \subset \mathfrak{su}(3)$. Furthermore, the G-orbit through p has codimension one, so $\mathfrak{h}$ must fix a unit vector normal to $G \cdot p$, say $u \in T_pM$. As $\mathfrak{h}$ also fixes J, $\mathfrak{h}$ fixes Ju and so $\mathfrak{h} \subset \mathfrak{su}(2)$. This limits $\mathfrak{h}$ to be one of $\{0\}$, $\mathfrak{u}(1)$ and $\mathfrak{su}(2)$.

Using now $\dim \mathfrak{g} - \dim \mathfrak{h} = 5$, we obtain the following list.

(i) $\mathfrak{h} = \{0\}$ and $\mathfrak{g}$ is isomorphic to $5\mathfrak{u}(1)$ or $\mathfrak{su}(2) \oplus 2\mathfrak{u}(1)$;

(ii) $\mathfrak{h} = \mathfrak{u}(1)$ and $\mathfrak{g}$ is isomorphic to $6\mathfrak{u}(1)$, $3\mathfrak{u}(1) \oplus \mathfrak{su}(2)$ or $2\mathfrak{su}(2)$;

(iii) $\mathfrak{h} = \mathfrak{su}(2)$ and $\mathfrak{g}$ is isomorphic to $5\mathfrak{u}(1) \oplus \mathfrak{su}(2)$, $2u(1) \oplus 2\mathfrak{su}(2)$ or $\mathfrak{su}(3)$.

Almost all of these can be ruled out, as we now show.

(i) $\mathfrak{h} = \{0\}$:

(a) Suppose $\mathfrak{g} = 5\mathfrak{u}(1)$ and let S be a singular orbit in M with isotropy group K. By Theorem 2.17, K/H is a sphere, *i.e.* $\mathfrak{k}$ is isomorphic to $\mathfrak{u}(1)$ or $\mathfrak{su}(2)$. But K must be abelian since G is, so $K = S^1$ and S is a 4-torus. The normal bundle of S is $G \times_K \mathbb{R}^2$, where K acts on $\mathbb{R}^2$ via the isomorphism $K \cong \mathrm{SO}(2)$. As G is abelian, this bundle is nothing but the trivial bundle $S \times \mathbb{R}^2$. Gluing two copies of this along the boundaries of the respective unit disc bundles identifies M as $S \times S^2$, which has infinite fundamental group.

(b) Suppose $\mathfrak{g} = 2\mathfrak{u}(1) \oplus \mathfrak{su}(2)$. Let S_1 be a singular orbit with isotropy group K_1. By Theorem 2.17, $\mathfrak{k}_1$ is either $\mathfrak{u}(1)$ or $\mathfrak{su}(2)$. Suppose that $\mathfrak{k}_1 = \mathfrak{su}(2)$ so that S_1 has dimension two, and let S_2 be the other singular orbit of M. But then as S_1 has codimension larger than two, we have $\pi_1(M \backslash S_1) = \pi_1(M) = 1$, while $M \backslash S_1$ retracts onto the second singular orbit S_2, so $\pi_1(S_2)$ is trivial. Whichever the isotropy group of the singular orbit S_2, this is not possible, so the $\mathfrak{k}_i$ are isomorphic to $\mathfrak{u}(1)$.

Supposing now that $\mathfrak{k}_i$ is in the centre of $\mathfrak{g}$, if $q_i \in S_i$ and $K_i = H_{q_i}$ then $T_{q_i} S_i = (T_{q_i} M)^{\mathfrak{k}_i}$. The latter space is J_{q_i}-invariant since $\mathfrak{k}_i$ commutes with J_{q_i} and also four dimensional. This contradicts Lemma 4.23. By conjugating in G, we may therefore assume that both projections of the $\mathfrak{k}_i$ to the $\mathfrak{su}(2)$ factor in $\mathfrak{g}$ coincide with the standard Cartan subalgebra $\mathfrak{l}$. Now consider $N_G(H) \cap N_G(K_1) \cap N_G(K_2)$ – its Lie algebra is abelian and contains $\mathfrak{l}$. Let L be the connected subgroup of G generated by $\mathfrak{l}$. By Proposition 2.21, L defines a normal extension of the G-action to a cohomogeneity one group action by $G \times G'$ where $G' = L/(L \cap H)$. From the description of the group diagram given in Proposition 2.21, the principal isotropy group has Lie algebra $\mathfrak{l} = \mathfrak{u}(1)$. We therefore reduce to the case $(\mathfrak{g}, \mathfrak{h}) = (3\mathfrak{u}(1) \oplus \mathfrak{su}(2), \mathfrak{u}(1))$ dealt with below.

(ii) $\mathfrak{h} = \mathfrak{u}(1)$:

(a) If $\mathfrak{g} = 6\mathfrak{u}(1)$, then H is an ideal in G and the action cannot be almost effective.

(b) Suppose $\mathfrak{g} = 3\mathfrak{u}(1) \oplus \mathfrak{su}(2)$, let S be a singular orbit and fix $q \in S$ with isotropy group K. K acts transitively on the unit sphere in a Euclidean space $\mathbb{R}^n$ with isotropy group H. Let $\hat{H}$ be the kernel of the action on $\mathbb{R}^n$, $\hat{\mathfrak{h}}$ its Lie algebra: as $\mathfrak{h}$ has dimension one, $\hat{\mathfrak{h}} = \mathfrak{h}$ or $\hat{\mathfrak{h}} = \{0\}$.

Suppose that $\hat{\mathfrak{h}} = \mathfrak{h}$. Then $\mathfrak{k}/\hat{\mathfrak{h}}$ acts effectively with trivial isotropy group, so from Table 2.1 is isomorphic to either $\mathfrak{u}(1)$ or $\mathfrak{su}(2)$. But $\hat{\mathfrak{h}} = \mathfrak{h}$ is an ideal, so if the latter case obtains then $\mathfrak{h}$ is in the centraliser of a copy of $\mathfrak{su}(2)$ in $\mathfrak{g}$. This implies that $\mathfrak{h}$ is in the centre of $\mathfrak{g}$, which is impossible if G acts almost effectively.

It remains then that $\mathfrak{k}/\mathfrak{h} = \mathfrak{u}(1)$. As $\mathfrak{h}$ has a non-trivial projection to the $\mathfrak{su}(2)$ factor in $\mathfrak{g}$ (for else $\mathfrak{h}$ lies in the centre and the action is not almost effective), we have $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{k}_0$ for a sub-algebra $\mathfrak{k}_0$ lying in the centre of $\mathfrak{g}$. As $\mathfrak{k}_0$ is contained in the centre of $\mathfrak{g}$, we have $T_q S = (T_q M)^{\mathfrak{k}_0}$. However, the space $(T_q M)^{\mathfrak{k}_0}$ is J_q-invariant and four dimensional, contradicting Lemma 4.23.

We conclude that $\hat{\mathfrak{h}} = \{0\}$. From Table 2.1, $(\mathfrak{k}, \mathfrak{h})$ is either $(\mathfrak{so}(3), \mathfrak{so}(2))$ or $(\mathfrak{u}(2), \mathfrak{u}(1))$, and S is a

torus of dimension two or three. However, as in (ib), $\operatorname{codim} S > 2$ implies that $\pi_1(S) = 1$.

(iii) $\mathfrak{h} = \mathfrak{su}(2)$:

We saw above that the isotropy representation of $\mathfrak{h}$ on T_pM decomposes as $\langle u, J_p u\rangle \oplus V$ where V is a two dimensional complex representation of $\mathfrak{h}$. From basic representation theory of $\mathfrak{su}(2)$, V must be either trivial or the defining representation on $\mathbb{C}^2$. The former case corresponds to $\mathfrak{g} = 5\mathfrak{u}(1) \oplus \mathfrak{su}(2)$ or $\mathfrak{g} = 2\mathfrak{u}(1) \oplus 2\mathfrak{su}(2)$ with $\mathfrak{h}$ equal to an $\mathfrak{su}(2)$ factor. Since $\mathfrak{h}$ is then an ideal the action cannot be almost effective. The case in which $V = \mathbb{C}^2$ corresponds to the isotropy representation of the standard inclusion $\mathfrak{su}(2) \subset \mathfrak{su}(3)$.

The list of pairs $(\mathfrak{g}, \mathfrak{h})$ is thus reduced to two possibilities, $(\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathfrak{u}(1))$ and $(\mathfrak{su}(3), \mathfrak{su}(2))$. In the latter case, it has been shown that $\mathfrak{su}(2) \subset \mathfrak{su}(3)$ is the standard inclusion. It remains to determine the inclusion $\mathfrak{u}(1) \subset \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

Lemma 5.12. *Up to conjugation, the inclusion $\mathfrak{h} \subset \mathfrak{g}$ is the diagonal of the standard Cartan sub-algebra.*

Proof. Denote by $(\mathfrak{su}_2)_i$, $i = 1, 2$, the summands in $\mathfrak{g}$, and let $p_i : \mathfrak{g} \to (\mathfrak{su}_2)_i$, $i = 1, 2$, be the orthogonal projections. It is first shown that $p_i(\mathfrak{h}) \neq \{0\}$ for $i = 1, 2$.

Suppose not, $p_2(\mathfrak{h}) = \{0\}$, say, and consider the set $N = (M^*)^H$. N is evidently an almost complex submanifold of M. Since H fixes ξ, ξ takes values in TN; the remaining directions in N come from those generated by the centraliser of $\mathfrak{h}$ in $\mathfrak{g}$ mod $\mathfrak{h}$. By assumption, this is precisely $(\mathfrak{su}_2)_2$. Thus N has dimension four, contradicting Lemma 4.23.

Thus, $\mathfrak{t} = p_1(\mathfrak{h}) \oplus p_2(\mathfrak{h})$ is a Cartan subalgebra in $\mathfrak{g}$ containing $\mathfrak{h}$. Let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{t}$ and $\mathfrak{n}_i$ the orthogonal complement of $p_i(\mathfrak{h})$ in $(\mathfrak{su}_2)_i$, $i = 1, 2$. We claim that $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is equivalent to the diagonal of the standard inclusion if and only if the $\mathfrak{n}_i$ are equivalent as representation of $\mathfrak{h}$.

To see this recall that by standard representation theory there are generators H_i of $p_i(\mathfrak{h})$, generators X_i, Y_i of $\mathfrak{n}_i$ and integers p_i such that

$$[H_i, X_i] = p_i Y_i \ , \ [H_i, Y_i] = -p_i X_i, \ i = 1, 2.$$

$\mathfrak{h}$ is the subspace spanned by (H_1, H_2), and the statement that the $\mathfrak{n}_i$ are equivalent as representations of $\mathfrak{h}$ is precisely the equality of the p_i, for then the vector space isomorphism θ sending X_1 to X_2 and Y_1 to Y_2 commutes with $\mathfrak{h}$. Extending this to $\mathfrak{g}$ by $\theta : H_1 \mapsto H_2$, we obtain $\theta \in \mathfrak{aut}(\mathfrak{g})^{\mathfrak{k}}$ such that $\theta|p_1(\mathfrak{h}) = p_2(\mathfrak{k})$. The image of $\mathfrak{t}$ under $1 \oplus \theta$ is then a Cartan sub-algebra of $\mathfrak{g}$ in which $\mathfrak{h}$ sits diagonally. Since $\mathfrak{aut}(\mathfrak{su}_2) = \mathfrak{inn}(\mathfrak{su}_2)$, $1 \oplus \theta$ is inner.

It is proven now that if the $\mathfrak{n}_i$ are not equivalent then $\nabla J = 0$, contradicting the hypothesis that the nearly Kähler structure is strict.

The 3-form Ω associated to the strictly nearly Kähler structure (g, J) is H-invariant and so is a linear form on the space of invariants $(\Lambda^3 T_pM)^K$. To compute this space, first remark that as a H-space we have

$$\begin{aligned} \Lambda^3 T_pM \quad = \big\{(\mathfrak{a} \oplus \langle \xi_p \rangle) \otimes (\Lambda^2 \mathfrak{n}_1 \oplus \Lambda^2 \mathfrak{n}_2)\big\} \oplus \big\{(\mathfrak{n}_1 \otimes \Lambda^2 \mathfrak{n}_2) \oplus (\Lambda^2 \mathfrak{n}_1 \oplus \mathfrak{n}_2)\big\} \\ \oplus (\mathfrak{a} \oplus \langle \xi_p, \rangle) \otimes (\mathfrak{n}_1 \otimes \mathfrak{n}_2). \end{aligned} \tag{5.9}$$

Both $\mathfrak{a}$ and ξ_p are trivial factors. As the $\mathfrak{n}_i$ are assumed non-equivalent, the final factor is irreducible. Being one-dimensional the $\Lambda^2 \mathfrak{n}_i$ are trivial. In all we have then that

$$(\Lambda^3 T_pM)^K = (\mathfrak{a} \oplus \langle \xi_p \rangle) \otimes (\Lambda^2 n_1 \oplus \Lambda^2 \mathfrak{n}_2),$$

and Ω_p is determined by the components $\Omega_p(\xi_p, \mathfrak{n}_i, \mathfrak{n}_i)$ and $\Omega_p(\mathfrak{a}, \mathfrak{n}_i, \mathfrak{n}_i)$.

Since $J_p \in \mathrm{End}(T_pM)$ commutes with the K-action, our assumption on the $\mathfrak{n}_i$ and Schur's lemma imply that $J|\mathfrak{n}_i = \mathfrak{n}_i$ and $J|\mathfrak{a} = \langle \xi_p \rangle$. Let γ be the integral curve of ξ passing through p and choose a vector field v along γ such that $v_p \in \mathfrak{n}_i$. Setting $w = Jv$, then $w_p \in \mathfrak{n}_i$, and we have (extending v and w arbitrarily after the first line)

$$\begin{aligned}\Omega_p(\xi_p, v_p, w_p) = g((\nabla_\xi J)v, w)_p = g(\nabla_\xi(Jv) - J(\nabla_\xi v), w)_p &= g(\nabla_\xi w, w)_p - g(\nabla_\xi v, v)_p \\ &= \frac{1}{2}\xi\left(g(w,w) - g(v,v)\right)_p \\ &= 0.\end{aligned}$$

But $\mathfrak{n}_i = \langle v_p, w_p \rangle$, so $\Omega_p(\xi_p, \mathfrak{n}_i, \mathfrak{n}_i) = 0$. Since $J_p|\mathfrak{a} = \langle \xi_p \rangle$ and $J|\mathfrak{n}_i = \mathfrak{n}_i$, the vanishing $\Omega_p(\mathfrak{a}, \mathfrak{n}_i, \mathfrak{n}_i) = 0$ follows from (4.27). For every $p \in M^*$, therefore, $\Omega_p = 0$ and thus $(\nabla J)_p = 0$. As M^* is dense, $\nabla J \equiv 0$. □

5.2.5 Proof of Theorem 5.10: Part II

The above arguments have whittled down the possible pairs $(\mathfrak{g}, \mathfrak{h})$ to just the two cases $(\mathfrak{su}(2)\oplus\mathfrak{su}(2), \Delta(\mathfrak{u}(1)))$ and $(\mathfrak{su}(3), \mathfrak{su}(2))$. Now let S be a singular orbit in M and fix $q \in S$ with isotropy group K.

Suppose first that $\mathfrak{h}{=}\mathfrak{su}(2)$. This is simple, so the ideal $\hat{\mathfrak{h}}$ is trivial or equal to $\mathfrak{su}(2)$. In the former case the action of $\mathfrak{k}$ on the unit sphere in the normal isotropy representation is effective with isotropy sub-algebra $\mathfrak{su}(2)$, so $\mathfrak{k}{=}\mathfrak{su}(3)$ or $\mathfrak{so}(4)$ by Table 2.1. As established in (ii) of section 5.1.1, there is no inclusion of $\mathfrak{so}(4){=}\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ into $\mathfrak{su}(3)$, so we must have $\mathfrak{k}{=}\mathfrak{su}(3)$.

If instead $\hat{\mathfrak{h}}{=}\mathfrak{h}$, then $\mathfrak{k}/\hat{\mathfrak{h}}{=}\mathfrak{su}(2)$ or $\mathfrak{u}(1)$. But then $\mathfrak{k}{=}\mathfrak{h}\oplus\mathfrak{p}$ for $\mathfrak{p}$ a non-zero sub-algebra of $\mathfrak{k}$ commuting with $\mathfrak{h}$. However, the only such commuting algebra in $\mathfrak{su}(3)$ is the trivial one, a contradiction. This gives the first four columns of the first row of Table 5.2.

Consider next the case $(\mathfrak{g}, \mathfrak{k}){=}(\mathfrak{su}(2) \oplus \mathfrak{su}(2), \Delta(\mathfrak{u}(1)))$. Again, $\hat{\mathfrak{h}}$ is $\{0\}$ or $\mathfrak{h}$. In the former case $\mathfrak{k}$ is $\mathfrak{so}(3){=}\mathfrak{su}(2)$ or $\mathfrak{u}(2){=}\mathfrak{u}(1)\oplus\mathfrak{su}(2)$. When $\hat{\mathfrak{h}} = \mathfrak{h}$, we have $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}$ for $\mathfrak{p}$ a sub-algebra of $\mathfrak{k}$ commuting with $\mathfrak{h}$ and isomorphic to $\mathfrak{su}(2)$ or $\mathfrak{u}(1)$. The latter cannot occur for as $T_pS{=}(T_pM)^{\mathfrak{p}}$, if $\mathfrak{p}{=}\mathfrak{u}(1)$ then S is a complex surface, contradicting Lemma 4.23.

It remains to determine the second inclusion in the following sequence

$$\mathfrak{u}(1) \subset \mathfrak{k} \subset \mathfrak{su}(2) \oplus \mathfrak{su}(2), \tag{5.10}$$

for $\mathfrak{k}{=}\mathfrak{su}(2)$ or $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$. The composition is the diagonal of the standard Cartan subalgebra $\mathfrak{u}(1) \subset \mathfrak{su}(2)$.

If $\mathfrak{k}{=}\mathfrak{su}(2)$ the first inclusion in (5.10) is of the standard torus. The representation of $\mathfrak{k}$ on the normal fibre $(T_qS)^\perp$ at q is equivalent to the adjoint action of $\mathfrak{su}(2)$. Since $\mathfrak{k}$ commutes with J_q and this representation is irreducible, we must have $J_q(T_qS)^\perp = T_qS$ or $J_q(T_qS)^\perp = (T_qS)^\perp$. Since $(T_qS)^\perp$ has dimension three the latter is not possible, so the isotropy representation of $S = G/K$ is the adjoint representation. The second inclusion in (5.10) may therefore be identified with the diagonal.

Suppose now that $\mathfrak{k} = \mathfrak{su}(2)\oplus\mathfrak{u}(1)$. This algebra comes from the homogeneous space $S^3 = \mathrm{U}(2)/\mathrm{U}(1)$, where $\mathrm{U}(1)$ is identified with the subgroup of matrices of the form $\mathrm{diag}(1, z)$ for $z \in \mathrm{U}(1)$. The Lie algebra

$\mathfrak{u}(2)$ is the set of 2×2 skew-Hermitian matrices, and the splitting $\mathfrak{u}(2)=\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ is given by identifying $\mathfrak{su}(2)$ with the subspace of trace-free elements and $\mathfrak{u}(1)$ with the span of $\mathrm{diag}(i,i)$. The projection of the inclusion $\mathfrak{h}\subset\mathfrak{k}$ to each factor is therefore the standard torus.

Now, as the representation of K on $(T_qS)^q$ is irreducible and K preserves J, we have $JT_qS = T_qS$. As a representation of complex dimension one, the restriction of the isotropy representation to $\mathfrak{su}(2)\subset\mathfrak{k}$ must therefore be trivial. It follows then that this factor coincides with one of those in $\mathfrak{g}=\mathfrak{su}(2)\oplus\mathfrak{su}(2)$, and the $\mathfrak{u}(1)$ factor lies in the complementary factor. Conjugating so that this $\mathfrak{u}(1)$ is the standard Cartan subalgebra, we complete the relevant parts of Table 5.2.

5.2.6 Proof of Theorem 5.10: Part III

If G acts with cohomogeneity one on a nearly Kähler six-manifold (M,g,J) preserving (g,J), then we have shown that $\mathfrak{g}$ is $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ or $\mathfrak{su}(3)$. In particular G is semi-simple, so $\pi_1(G)$ is finite and the universal covering group $\widetilde{G}$ of G is compact. Moreover, if $\pi:\widetilde{G}\to G$ is the universal covering then $g\in\widetilde{G}$ acts on $p\in M$ by $g\cdot p=\pi(g)\cdot p$. Since the orbits of the $\widetilde{G}$-action are identical to those of the G-action, $\widetilde{G}$ acts with cohomogeneity one. We may therefore assume that G is simply connected, *i.e.* G=SU(3) or $\mathrm{SU}(2)\times\mathrm{SU}(2)$.

Lemma 5.13. *For the $(\mathfrak{g},\mathfrak{k},\mathfrak{h}_1,\mathfrak{h}_2)$ obtained above the groups H, K_1 and K_2 are connected.*

Proof. Let S_i be the singular orbit corresponding to K_i. For all three possible algebras $\mathfrak{k}_i$, S_i has codimension greater than two. As in (i)(b) above, this implies that the S_i are simply connected. Having assumed G and M to be connected $S_i=G/K_i$ must be connected, so the long exact homotopy sequence then implies that the K_i are connected. Finally, as the spheres $\hat{H}_i/\hat{K}$ have dimension greater than 1, connectivity of the H_i implies that of K. □

Having reduced the possible quadruples (G,K,H_1,H_2) to those of the three examples S^6, $S^3\times S^3$ and $\mathbb{CP}^3$, it remains to show that up to equivalence there is a unique way to identify the disc bundles defined by the quadruples (G,H,K_1,K_2). According to Theorem 2.20, all we need show is that the following double quotient is trivial

$$N_0\backslash N_G(H)/N_1, \tag{5.11}$$

where $N_i=N_G(H)\cap N_G(K_i)$, and N_G denotes the normaliser.

If $(G,H)=(\mathrm{SU}(3),\mathrm{SU}(2))$, then the normaliser of H equals H. Since the only singular isotropy group in this case equals G, (5.11) is trivial as required. When $(G,H)=(\mathrm{SU}(2)\times\mathrm{SU}(2),\Delta(\mathrm{U}(1)))$, the normaliser of H again equals H. The normaliser of each of the possible singular isotropy sub-algebras $\Delta(\mathrm{SU}(2))$ and $\mathrm{SU}(2)\oplus\mathrm{U}(1)$ is also trivial, so in all cases (5.11) is trivial.

Chapter 6

The ODEs Defined by Strictly Nearly Kähler Six-Manifolds of Cohomogeneity One

In section 5.2 we completed the algebraic part of the classification of compact strictly nearly Kähler six-manifolds admitting a cohomogeneity one action by a compact group G. To progress further in this problem we derive and make a detailed study of the system of ODEs such an action defines when $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Our exposition follows closely that of [PS12] throughout.

We are not able to classify completely the solutions to the ODEs obtained. The best result so far obtained is Theorem 6.15, which states that there exists a one-parameter family of mutually non-equivalent strictly nearly Kähler structures on the tangent bundle TS^3 invariant under the cohomogeneity one action by G. Thus if M is a six-manifold on which G acts with cohomogeneity one with a singular orbit S diffeomorphic to S^3, then there exists a one-parameter strictly nearly Kähler structure in a neighbourhood of S. This family interpolates between the two examples arising from neighbourhoods of the singular orbits of this type in $S^3 \times S^3$ and S^6.

6.1 Cohomogeneity One Actions by $\mathrm{SU}(2) \times \mathrm{SU}(2)$

Let (M, g, J) be a strictly nearly Kähler six-manifold and suppose that $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ acts with cohomogeneity one preserving (g, J). The principal isotropy group of any point in the regular part M^* of M is conjugate in G to $H = \Delta(\mathrm{U}(1))$ (Lemma 5.12). For $X \in \mathfrak{g}$ we let $\hat{X}$ be the corresponding Killing vector field on M. We make the following assumptions throughout this section.

(i) Fix $p \in M^*$ and let γ be a unit speed geodesic through p meeting every orbit in M^* orthogonally. Denote $t \in I = (-a, a)$, $a > 0$, the arc-length parameter, and set $\xi = \dot{\gamma}$.

(ii) This section is only immediately concerned with what occurs away from any singular orbits. We shall therefore abuse notation and write $M = M^* = I \times G/H$ where $p = \{0\} \times H/H$. Singular orbits will be considered in the next section. In these coordinates g is of the form

$$dt^2 + g_t,$$

for a family of G-invariant metrics $\{g_t\}_{t\in(a,b)}$ on G/H. Also, $\xi = \frac{\partial}{\partial t}$ and

$$g(\xi, \xi) = 1,\ g(\xi, \hat{X}) = 0,\ X \in \mathfrak{g}. \tag{6.1}$$

(iii) We fix a basis H, E, V of $\mathfrak{su}(2)$ having the following Lie brackets

$$[H, E] = V,\ [H, V] = -E,\ [E, V] = -\frac{1}{2}H. \tag{6.2}$$

If we were to identify $\mathfrak{su}(2)$ with $\operatorname{Im}\mathbb{H}$ then the identifications

$$H = \frac{1}{2}i,\ E = \frac{1}{2\sqrt{2}}j,\ V = \frac{1}{2\sqrt{2}}k, \tag{6.3}$$

are consistent with these brackets. The normalisations are chosen such that E and V are unit vectors with respect to the Killing form on $\mathfrak{su}(2)$. The vector H spans the standard Cartan sub-algebra of $\mathfrak{su}(2)$ and we let $\mathfrak{n}$ be the orthogonal complement of this sub-algebra with respect to the Killing form.

Define now a basis of $\mathfrak{g}$ by

$$\begin{array}{ccccc} U = (H, H) & , & E_1 = (E, 0) & , & V_1 = (V, 0), \\ A = (H, -H) & , & E_2 = (0, E) & , & V_2 = (0, V). \end{array} \tag{6.4}$$

then $\mathfrak{k}$ is spanned by U. For $i = 1, 2$, we let $\mathfrak{su}(2)_i$ be, respectively, the first and second factor in $\mathfrak{g}$ and $\mathfrak{n}_i$ the complement of the projection of $\mathfrak{k}$ to $\mathfrak{su}(2)_i$, *i.e.* $\mathfrak{n}_i = \langle E_i, V_i\rangle$. We thus obtain a trivialisation of γ^*TM

$$\gamma^*TM = \langle\, \xi, \hat{A}, \hat{E}_1, \hat{V}_1, \hat{E}_2, \hat{V}_2 \,\rangle. \tag{6.5}$$

The basis dual to this trivialisation we denote by $dt, a, e_1, v_1, e_2, v_2$, orienting M by

$$\tau = dt \wedge a \wedge e_1 \wedge v_1 \wedge e_2 \wedge v_2.$$

We follow the same procedure as in the sequence of lemmas culminating in Proposition 5.9. We first determine the spaces of G-invariant 2- and 3-forms.

Lemma 6.1. *The space of G-invariant 2-forms on M is five dimensional, generated over $C^\infty(I)$ by $\omega^1, \ldots, \omega^5$, where*

$$\begin{array}{l} \omega^1|_\gamma = dt \wedge a,\quad \omega^2|_\gamma = e_1 \wedge v_1,\quad \omega^3|_\gamma = e_2 \wedge v_2, \\ \omega^4|_\gamma = e_1 \wedge e_2 + v_1 \wedge v_2\ ,\quad \omega^5|_\gamma = e_1 \wedge v_2 - v_1 \wedge e_2\ . \end{array} \tag{6.6}$$

The space of G-invariant 3-forms on M is eight dimensional, generated over $C^\infty(I)$ by ψ^{ai}, $a = 1, 2$, $i = 2, 3, 4, 5$, where

$$\psi^{1i}|_\gamma = dt \wedge \omega^i|_\gamma,\ \psi^{2i} = a \wedge \omega^i|_\gamma,\ i = 2, 3, 4, 5. \tag{6.7}$$

Proof. By G-invariance, it suffices to compute $(\gamma^*\Lambda^2T^*M)^K$. As a representation of K the space γ^*T^*M splits as

$$\langle dt\rangle \oplus \langle a\rangle \oplus \langle e_1, v_1\rangle \oplus \langle e_2, v_2\rangle.$$

We then get

$$\begin{aligned} \gamma^*\Lambda^2T^*M = {} & \Lambda^2\langle e_1, v_1\rangle \oplus \Lambda^2\langle e_2, v_2\rangle \oplus (\langle dt\rangle \otimes \langle a\rangle) \\ & \oplus (\langle dt\rangle \otimes \langle e_1, v_1\rangle) \oplus (\langle dt\rangle \otimes \langle e_2, v_2\rangle) \\ & \oplus (\langle a\rangle \otimes \langle e_1, v_1\rangle) \oplus (\langle a\rangle \otimes \langle e_2, v_2\rangle) \\ & \oplus (\langle e_1, v_1\rangle \otimes \langle e_2, v_2\rangle). \end{aligned}$$

Since the $\mathfrak{n}_i$ are irreducible, none of the invariant subspaces in lines two and three can contain trivial subspaces. The third subspace in the first line is evidently trivial, and it was shown in the proof of Lemma 5.12 that the $\Lambda^2\mathfrak{n}_i$ are also trivial. These three trivial subspaces give ω^1, ω^2 and ω^3.

Since the $\mathfrak{n}_i$ are equivalent as K-modules to $\mathfrak{n}$, we can make the following K-invariant decomposition

$$\mathfrak{n}_1 \otimes \mathfrak{n}_2 = \Lambda^2\mathfrak{n} \oplus S_0^2(\mathfrak{n}) \oplus T$$

where $S_0^2(\mathfrak{n})$ consists of symmetric trace-free elements of $\mathfrak{n}^{2\otimes}$ and T consists of elements of pure trace, *i.e.* those proportional to $e \otimes e + v \otimes v$. Both T and $\Lambda^2\mathfrak{n}$ are trivial, spanned by ω^4 and ω^5, respectively. As $S_0^2(\mathfrak{n})$ is two dimensional, were it to contain a trivial subspace it would be a trivial representation, and this cannot happen for then $\mathfrak{n}_1 \otimes \mathfrak{n}_2$ must be trivial, which it is not. The possible trivial subspaces have therefore been exhausted.

From equation 5.9 in the proof of Lemma 5.12, we know that

$$(\Lambda^3\gamma^*TM)^K = \left\{\langle \hat{A}, \xi\rangle \otimes (\Lambda^2\hat{\mathfrak{n}}_1 \oplus \Lambda^2\hat{\mathfrak{n}}_2)\right\} \oplus \left\{\langle \hat{A}, \xi\rangle \otimes (\hat{\mathfrak{n}}_1 \otimes \hat{\mathfrak{n}}_2)^K\right\}. \tag{6.8}$$

The lemma now follows from the computation of $(\mathfrak{n}_1 \otimes \mathfrak{n}_2)^K$. □

The following identities are easily proved

$$\omega_4^2 = \omega_5^2 = -2\omega_2 \wedge \omega_3,\ \omega_4 \wedge \omega_2 = \omega_4 \wedge \omega_3 = \omega_5 \wedge \omega_2 = \omega_5 \wedge \omega_3 = 0. \tag{6.9}$$

Lemma 6.2. *The derivatives of the ω^i are given as follows*

$$\begin{array}{cc} d\omega^1 = \frac{1}{4}(\psi^{12} - \psi^{13}), & d\omega^2 = d\omega^3 = 0, \\ d\omega^4 = -2\psi^{25}, & d\omega^5 = 2\psi^{24}. \end{array}$$

Proof. By the Cartan formula, if α is a G-invariant 1-form and X, Y are G-invariant vector fields then,

$$d\alpha(X, Y) = -\alpha([X, Y]). \tag{6.10}$$

We use this to compute the derivatives of a and the e_i, v_i. First,

$$[\hat{E}_1, \hat{V}_1] = \widehat{[E_1, V_1]} = -\frac{1}{2}\widehat{(H, 0)} = -\frac{1}{4}\hat{A},$$

and similarly $[\hat{E}_2, \hat{V}_2] = \frac{1}{4}\hat{A}$. It follows then that

$$da = \frac{1}{4}(\omega^2 - \omega^3). \tag{6.11}$$

It is clear that $de_1 = a \wedge v_1$ and $dv_1 = -a \wedge e_1$. On the other hand, $de_2 = -a \wedge v_2$ and $dv_2 = a \wedge e_2$, the position of the minus signs being reversed since now $[\hat{A}, E_2] = -\hat{V}_2$. We can now compute

$$d\omega^1 = -dt \wedge da = -\frac{1}{4}dt \wedge (\omega^2 - \omega^3),$$

and

$$d\omega^2 = de_1 \wedge dv_1 - e_1 \wedge dv_1 = a \wedge v_1 \wedge v_1 + e_1 \wedge a \wedge e_1 = 0.$$

A similar calculation shows that $d\omega^3 = 0$. Next,

$$\begin{aligned} d\omega^4 &= de_1 \wedge e_2 - e_1 \wedge de_2 + dv_1 \wedge v_2 - v_1 \wedge dv_2 \\ &= a \wedge v_1 \wedge e_2 + e_1 \wedge a \wedge v_2 - a \wedge e_1 \wedge v_2 - v_1 \wedge a \wedge e_2 \\ &= -2a \wedge (e_1 \wedge v_2 - v_1 \wedge e_2) \\ &= -2a \wedge \omega^5. \end{aligned}$$

The derivative of ω^5 is obtained in the same manor. □

Lemma 6.3. *The derivatives of the $\psi^{\alpha i}$ are given as follows*

$$\begin{aligned} d\psi^{12} = d\psi^{13} = 0, \quad d\psi^{14} = 2dt \wedge a \wedge \omega^5, \quad d\psi^{15} = -2dt \wedge a \wedge \omega^4, \\ d\psi^{22} = -\tfrac{1}{4}\omega^2 \wedge \omega^3, \quad d\psi^{23} = -\tfrac{1}{4}\omega^2 \wedge \omega^3, \quad d\psi^{24} = d\psi^{25} = 0. \end{aligned}$$

Proof. All of these are obvious from Lemma 6.2 except perhaps the closure of ψ^{24} and ψ^{25}: use Lemma 6.2, and equations (6.11) and (6.9) to obtain

$$d\psi^{24} = da \wedge \omega^4 - a \wedge d\omega^4 = \frac{1}{4}(\omega^2 - \omega^3) \wedge \omega^4 = 0.$$ □

Lemma 6.4. *If $\omega = \sum_i f_i \omega^i$ is an invariant 2-form on M, then, writing the invariant 3-form $\frac{1}{3}d\omega$ as $\sum_{a,i} p_{ai}\psi^{ai}$, we have*

$$\begin{aligned} p_{22} = p_{23} = 0, \quad p_{12} = \tfrac{1}{3}\left(f_2' + \tfrac{f_1}{4}\right), \quad p_{13} = \tfrac{1}{3}\left(f_3' - \tfrac{f_1}{4}\right), \\ p_{14} = \tfrac{f_4'}{3}, \quad p_{15} = \tfrac{f_5'}{3}, \quad p_{24} = \tfrac{2}{3}f_5, \quad p_{25} = -\tfrac{2}{3}f_4. \end{aligned}$$

Proof. Using Lemma 6.2 we compute

$$\begin{aligned} d\omega &= \sum_i (f_i' dt \wedge \omega^i + f_i d\omega^i) \\ &= \sum_{i=2}^{5} f_i' \psi^{1i} + \frac{1}{4} f_1(\psi^{12} - \psi^{13}) - 2f_4\psi^{25} + 2f_5\psi^{24}, \end{aligned}$$

from which the claimed formulae follow. □

Lemma 6.5. *Suppose $\omega = \sum_i f_i \omega^i$ is an invariant 2-form on M and set $\Omega = \frac{1}{3}d\omega$. Then Ω is a stable 3-form and $g_{\omega,\Omega} = \omega(\cdot, J_\Omega \cdot)$ satisfies (6.1) if and only if the following conditions are satisfied*

(i) $f_1 < 0$ and there exists $\theta_0 \in \mathbb{R}$ such that

$$f_4 = \mathfrak{f}_4 \cos\theta_0 \ , \ f_5 = \mathfrak{f}_4 \sin\theta_0 \ , \tag{6.12}$$

where $\mathfrak{f}_4 = \sqrt{f_4^2 + f_5^2}$;

(ii) $4\mathfrak{f}_4^2 - (f_1)^2\left\{(\mathfrak{f}_4')^2 - \left(f_2' + \tfrac{f_1}{4}\right)\left(f_3' - \tfrac{f_1}{4}\right)\right\} = 0$;

(iii) $(\mathfrak{f}_4')^2 - \left(f_2' + \tfrac{f_1}{4}\right)\left(f_3' - \tfrac{f_1}{4}\right) > 0$.

With respect to the basis (6.5), $J_\Omega = K \oplus L$ where

$$K = \begin{pmatrix} 0 & f_1 \\ -\frac{1}{f_1} & 0 \end{pmatrix},$$

$$L = \tfrac{f_1}{\mathfrak{f}_4^2}\begin{pmatrix} 0 & -p_{15}p_{24} + p_{14}p_{25} & p_{13}p_{25} & -p_{13}p_{24} \\ p_{15}p_{24} - p_{14}p_{25} & 0 & p_{13}p_{24} & p_{13}p_{25} \\ p_{12}p_{25} & p_{12}p_{24} & 0 & p_{15}p_{24} - p_{14}p_{25} \\ -p_{12}p_{24} & p_{12}p_{25} & -p_{15}p_{24} + p_{14}p_{25} & 0 \end{pmatrix}$$

and where $\Omega = \sum_{\alpha,i} p_{\alpha i}\psi^{\alpha i}$, with the coefficients given by Lemma 6.4.

Proof. Using (6.9) and the expressions for ω_4^2, ω_5^2 given in (6.9), we have

$$\begin{aligned}(\xi \llcorner \Omega) \wedge \Omega &= \sum_{i,j} p_{1i}\omega^i \wedge (p_{1j}dt + p_{2j}a) \wedge \omega^j, \\ &= 2p_{12}p_{13}dt \wedge \omega^2 \wedge \omega^3 + p_{14}(p_{14}dt + p_{24}a)(\omega^4)^2 + p_{15}(p_{15}dt + p_{24}a)(\omega^5)^2 \\ &= (\xi \llcorner \Omega) \wedge \Omega \\ &= 2\left\{(p_{12}p_{13} - p_{14}^2 - p_{15}^2)dt - (p_{14}p_{24} + p_{15}p_{25})a\right\} \wedge \omega^2 \wedge \omega^3.\end{aligned}$$

This gives

$$S_\Omega(\xi) = 2(p_{14}p_{24} + p_{15}p_{25})\xi + 2(p_{12}p_{13} - p_{14}^2 - p_{15}^2)\hat{A}.$$

Similarly,

$$S_\Omega(\hat{A}) = -2(p_{14}p_{24} + p_{15}p_{25})\hat{A} + 2(p_{24}^2 + p_{25}^2)\xi.$$

Supposing now that Ω is stable and (6.1) is satisfied, then $g_{\omega,\Omega}(S_\Omega(\xi), \xi) = 0$, which occurs if and only if

$$p_{14}p_{24} + p_{15}p_{25} = 0.$$

Using the formulae from the Lemma 6.4, this equation becomes

$$\frac{2}{9}(f_4' f_5 - f_5' f_4) = 0.$$

Therefore, if $f_4 \neq 0$ the ratio f_5/f_4 is constant; if $f_5 \neq 0$, then f_4/f_5 is constant. The expressions (6.12) for f_4 and f_5 now follow.

Now,

$$S_\Omega^2(\xi) = 4(p_{12}p_{13} - p_{14}^2 - p_{15}^2)(p_{24}^2 + p_{25}^2)\xi.$$

We recognise from this that

$$P(\Omega) = 4(p_{12}p_{13} - p_{14}^2 - p_{15}^2)(p_{24}^2 + p_{25}^2).$$

Thus, Ω is stable if and only if

$$p_{12}p_{13} - p_{14}^2 - p_{15}^2 < 0.$$

Using Lemma 6.4 and (6.12) this inequality is easily seen to provide condition (iii), and also the expression for K.

The first condition in (6.1) is

$$1 = \omega(\xi, J_\Omega \xi) = -f_1\sqrt{\frac{p_{14}^2 + p_{15}^2 - p_{12}p_{15}}{p_{24}^2 + p_{25}^2}}. \tag{6.13}$$

This gives $f_1 < 0$ and, using Lemma 6.4, also (ii).

It remains to compute the rest of J_Ω. As we do not require the explicit formula for L and the calculation is rather tedious, we leave the details to the reader. □

Lemma 6.6. *Suppose $\omega = \sum_i f_i\omega^i$ is an invariant 2-form on M, $\Omega = \frac{1}{3}d\omega$ is a stable 3-form, ω is J_Ω-invariant and $g_{\omega,\Omega} = \omega(\cdot, J_\Omega\cdot)$ satisfies (6.1). Then, for $\mu > 0$, the equation*

$$dJ_\Omega^*\Omega = -2\mu\,\omega^2$$

is equivalent to the following ODE system

$$[(f_2' + \frac{1}{4}f_1)f_1]' + 12\mu f_1 f_2 = 0, \tag{6.14}$$

$$[(f_3' - \frac{1}{4}f_1)f_1]' + 12\mu f_1 f_3 = 0, \tag{6.15}$$

$$(\mathfrak{f}_4' f_1)' - 4\frac{\mathfrak{f}_4}{f_1} + 12\mu f_1\mathfrak{f}_4 = 0, \tag{6.16}$$

$$f_1(f_2' - f_3' + \frac{1}{2}f_1) + 48\mu(f_2 f_3 - \mathfrak{f}_4^2) = 0. \tag{6.17}$$

If this is satisfied then equation (ii) of Lemma 6.5 can be replaced with the following algebraic condition satisfied for some $t_0 \in I$,

$$\left\{4\mathfrak{f}_4^2 - \left((\mathfrak{f}_4')^2 - \left(f_2' + \frac{f_1}{4}\right)\left(f_3' - \frac{f_1}{4}\right)\right)(f_1)^2\right\}\Big|_{t=t_0} = 0. \tag{6.18}$$

Proof. Let us write $J_\Omega^*\Omega = \sum_{a,i}\hat{p}_{ai}\psi^{ai}$. By (4.27) and Lemma 6.5, for $X, Y \in \mathfrak{g}$ we have

$$\begin{aligned} J_\Omega^*\Omega(\hat{A}, \hat{X}, \hat{Y}) &= \Omega(J_\Omega\hat{A}, \hat{X}, \hat{Y})\,, \\ &= \sum_i p_{2i} f_1\,\omega^i(X, Y) \end{aligned}$$

so $\hat{p}_{2i} = f_1 p_{2i}$. Similarly, $\hat{p}_{1i} = -\frac{1}{f_1}p_{1i}$. Using Lemma 6.3, we have

$$\begin{aligned} dJ_\Omega^*\Omega &= \sum_i \left(\hat{p}_{1i}\,d\psi^{1i} + (\hat{p}_{2i}')\,dt\wedge\psi^{2i} + \hat{p}_{2j}d\psi^{2j}\right), \\ &= -\frac{1}{4}(\hat{p}_{22} + \hat{p}_{23})\,\omega^2\wedge\omega^3 \\ &\qquad + dt\wedge a\wedge\left\{(2\hat{p}_{14} + \hat{p}_{25}')\,\omega^5 + (\hat{p}_{24}' - \hat{p}_{15})\,\omega^4 + \hat{p}_{22}'\omega^2 + \hat{p}_{23}'\,\omega^3\right\}. \end{aligned}$$

Now, as $(\omega^4)^2 = (\omega^5)^2 = -2\omega^2\wedge\omega^3$, we have

$$\omega^2 = 2\sum_{i=1}^5 f_1 f_i\,dt\wedge a\wedge\omega^i + 2(f_2 f_3 - \mathfrak{f}_4^2)\,\omega^2\wedge\omega^3.$$

The equation $dJ_\Omega^*\Omega = -2\mu\,\omega^2$ is therefore equivalent to the following system of ODE

$$\begin{aligned} \hat{p}_{22}' &= -4\mu f_1 f_2, \\ \hat{p}_{23}' &= -4\mu f_1 f_3, \\ \hat{p}_{24}' - 2p_{15} &= -4\mu f_1 f_4, \\ \hat{p}_{25}' + 2\hat{p}_{14} &= -4\mu f_1 f_5, \\ \hat{p}_{22} + \hat{p}_{23} &= 16\mu(f_2 f_3 - \mathfrak{f}_4^2). \end{aligned}$$

Finally, using the expressions for the $\hat{p}_{ai}$ obtained above and those for the p_{ai} from Lemma 6.4, these five equations give the four (6.14)-(6.17), one equation being lost since f_4 and f_5 are proportional by Lemma 6.5.

It remains to prove (6.18). Define

$$A = 4\mathfrak{f}_4^2 - f_1^2\left((\mathfrak{f}_4')^2 - \left(f_2' + \frac{f_1}{4}\right)\left(f_3' - \frac{f_1}{4}\right)\right).$$

Then by equations (6.14), (6.15) and (6.16) we have

$$\begin{aligned} A' &= 8\mathfrak{f}_4\mathfrak{f}_4' - \left\{2\mathfrak{f}_4' f_1(\mathfrak{f}_4' f_1)' - \left[f_1\left(f_2' + \frac{f_1}{4}\right)\right]' f_1\left(f_3' - \frac{f_1}{4}\right) - f_1\left(f_2' + \frac{f_1}{4}\right)\left[f_1\left(f_3' - \frac{f_1}{4}\right)\right]'\right\}, \\ &= 8\mathfrak{f}_4\mathfrak{f}_4' - \left\{2\mathfrak{f}_4' f_1\left(4\mathfrak{f}_4 f_1^{-1} + 12\mu f_1\mathfrak{f}_4\right) + 12\mu f_1^2 f_2\left(f_3' - \frac{f_1}{4}\right) + 12\mu f_1^2 f_3\left(f_2' + \frac{f_1}{4}\right)\right\}, \\ &= 12\mu f_1^2\left\{2\mathfrak{f}_4\mathfrak{f}_4' - (f_2 f_3)' + \frac{1}{4}f_1(f_2' - f_3')\right\}. \end{aligned}$$

But notice that the expression in brackets here is proportional to the derivative of (6.17) and so we obtain $A' = 0$. If the system (6.14)-(6.17) is satisfied, therefore, the equation $A = 0$ of Lemma 6.5 is satisfied if and only if $A(t_0) = 0$ for some t_0. □

Corollary 6.7. *Let ω be a G-invariant 2-form on M with $\omega|_\gamma = \sum_i f_i\omega^i$. Then if $(\omega, \frac{1}{3}d\omega)$ defines a strictly nearly Kähler structure on M the f_i are analytic functions.*

Proof. We know by Theorem 4.4 and the results of [DK81] that $g_{\omega,\frac{1}{3}d\omega}$ is analytic in any normal coordinate system. Consider then the Killing vector field $\hat{A}$: in a normal coordinate system it satisfies a first order linear PDE with analytic coefficients, and the Cauchy-Kowalevski theorem implies then that $\hat{A}$ is analytic in these coordinates, uniquely determined by $\hat{A}_{t=0}$, $\nabla\hat{A}|_{t=0}$. It follows then that $f_1 = g_{\omega,\frac{1}{3}d\omega}(\hat{A}|_\gamma, \hat{A}|_\gamma)$ is an analytic function. Analyticity of the remaining f_i follows from (6.14), (6.15) and (6.16) using basic regularity properties of linear ODEs. □

6.2 The Space of Solutions

The description of cohomogeneity one strictly nearly Kähler structures given in in Lemma 6.6 is not entirely satisfactory, there being three equations of second order, one of first order and an algebraic condition. The role of f_1 is also not entirely clear. In this section we reformulate Lemma 6.6 by change of variables as a regular system of second order ODE with a number of first integrals.

First, for $f_1, f_2, f_3, \mathfrak{f}_4 : (a, b) \to \mathbb{R}$ smooth functions with $f_1 < 0$, choose $t_0 \in (a, b)$ and define

$$s(t) = \int_{t_0}^t \frac{1}{f_1(u)}du\,,\ g(t) = \frac{1}{2}\int_{t_0}^t f_1(u)du.$$

As $s'(t) < 0$, there is a well-defined change of parameterisation from t to s and we set

$$h_1(s) = g(t(s)),\ h_2(s) = f_2(t(s)) + f_3(t(s)),\ h_3(s) = f_2(t(s)) - f_3(t(s)),\ h_4 = 2\mathfrak{f}_4(t(s)). \qquad (6.19)$$

Proposition 6.8. *The change of variables (6.19) gives a one-to-one correspondence between functions $f_1, f_2, f_3, \mathfrak{f}_4 : (a, b) \to \mathbb{R}$ such that $f_1 < 0$ and such that the system of ODE 6.14-6.17 together with the algebraic constraint 6.18 are satisfied and solutions h_1, h_2, h_3, h_4 to the regular system of ODE*

$$\begin{aligned} &h_1'' + \tfrac{2(h_1')^2 h_3 + \frac{4}{9\mu} h_4' h_4}{h_2^2 - h_3^2 - h_4^2} = 0, \\ &h_2'' + 24\mu\, h_1' h_2 = 0, \\ &h_3'' - \tfrac{2(h_1')^2 h_3 + \frac{4}{9\mu} h_4' h_4}{h_2^2 - h_3^2 - h_4^2} + 24\mu\, h_1' h_3 = 0, \\ &h_4'' + 24\mu\, h_1' h_4 = 0, \end{aligned} \tag{6.20}$$

with initial conditions $a_i = h_i(0)$, $b_i = h_i'(0)$ satisfying

$$a_1 = 0,\ b_1 > 0,\ b_2^2 - b_3^2 - b_4^2 - b_1^2 - 2b_1 b_3 < 0,\ \mathcal{I}(a_2, a_3, a_4, b_1, b_2, b_3, b_4) = 0,$$

where $\mathcal{I} = (\mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^3, \mathcal{I}^4) : \mathbb{R}^7 \to \mathbb{R}^4$ is defined by

$$\begin{aligned} \mathcal{I}^1 &= 12\mu(a_2^2 - a_3^2 - a_4^2) + b_1 + b_3, \\ \mathcal{I}^2 &= 4a_4^2 + b_2^2 - b_3^2 - b_4^2 - b_1^2 - 2b_3 b_1, \\ \mathcal{I}^3 &= a_2 b_2 - a_3 b_3 - a_4 b_4 - a_3 b_1, \\ \mathcal{I}^4 &= \frac{9\mu}{2} b_1(a_2^2 - a_3^2 - a_4^2) + a_4^2. \end{aligned}$$

Proof. Equations (6.14) and (6.15) are equivalent to

$$[(f_2' + \frac{1}{4} f_1) f_1]' \pm [(f_3' - \frac{1}{4} f_1) f_1]' + 12\mu f_1 (f_2 \pm f_3) = 0.$$

Making the change of variables (6.19) and using $\frac{ds}{dt} = f_1(t)$ and $h_1'(s) = \frac{1}{2}(f_1(t(s)))^2$, these become

$$h_2'' + 24\mu\, h_1' h_2 = 0\,,\ h_3'' + h_1'' + 24\mu\, h_1' h_3 = 0. \tag{6.21}$$

We recognise the first of these as the second equation in (6.20) and we work now to find h_1''. First, the ODEs (6.16) and (6.17) become, respectively,

$$h_4'' + 24\mu\, h_1' h_4 - 4h_4 = 0\,,\ h_3' + h_1' + 12\mu\,(h_2^2 - h_3^2 - h_4^2) = 0. \tag{6.22}$$

From the proof of Lemma 6.6, equation (6.18) is satisfied at $t = t_0$ if and only if it is satisfied at all t. This latter condition is equivalent to the following equation in the new variables.

$$(h_2')^2 - (h_3')^2 - (h_4')^2 - (h_1')^2 - 2h_1' h_3' + 4h_4^2 = 0. \tag{6.23}$$

Differentiate the second equation of (6.22) and subtract the second equation of (6.21) to obtain

$$h_2 h_2' - h_3 h_3' - h_4 h_4' - h_1' h_3' = 0. \tag{6.24}$$

Now differentiate this and replace the expressions for the second derivatives, h_i'', $i = 2, 3, 4$, given above, to give

$$(h_2')^2 - (h_3')^2 - (h_4')^2 - h_1' h_3' - 24\mu\, h_1'(h_2^2 - h_3^2 - h_4^2) = 0.$$

Now subtract (6.23) from this to give

$$(h_1')^2 + h_1'h_3' - 8h_4^2 - 24\mu\, h_1'(h_2^2 - h_3^2 - h_4^2) = 0,$$

and use the second equation of (6.22) to obtain

$$h_1'(h_2^2 - h_3^2 - h_4^2) + \frac{2}{9\mu}h_4^2 = 0. \tag{6.25}$$

Finally, differentiating this and using (6.23) we arrive at the first equation of (6.20).

The second equation in (6.22) gives the equation $\mathcal{I}^1 = 0$, (6.23) gives $\mathcal{I}^2 = 0$, (6.24) gives $\mathcal{I}^3 = 0$, and (6.25) gives the component $\mathcal{I}^4 = 0$. Conversely, if h_1, h_2, h_3, h_4 satisfy the system (6.20) and the algebraic conditions $\mathcal{I} = 0$ at $t = t_0$, then working back along the sequence of differentiations above one arrives at the system (6.14)-(6.17) and the condition (6.18) for $f_1, f_2, f_3, \mathfrak{f}_4$. □

6.3 Local Homogeneity

The three homogeneous nearly Kähler spaces S^6, $S^3 \times S^3$ and $\mathbb{CP}^3$ admit cohomogeneity one actions by the group $\mathrm{SU}(2)\times\mathrm{SU}(2)$ preserving the nearly Kähler structure. In this section we characterise nearly Kähler spaces of cohomogeneity one that are locally homogeneous. We then determine the functions $f_1, \ldots, f_5$ of these three examples.

Definition 6.9. A Riemannian manifold (M, g) is *locally homogeneous* if for every $p \in M$ the pseudo-group of local isometries generated by the algebra of germs of Killing vector fields at p acts transitively on a neighbourhood of p.

For a strictly nearly Kähler structure (g, J) we denote by $\mathfrak{aut}_p(g, J)$ the Lie algebra of germs of Killing vectors fields at p preserving J.

Proposition 6.10. *Let (M, g, J) be a six-dimensional strictly nearly Kähler manifold admitting a cohomogeneity one action by $\mathrm{SU}(2) \times \mathrm{SU}(2)$ preserving (g, J). Then (M, g) is locally homogeneous if and only if it is locally equivalent to one of the compact homogeneous strictly nearly Kähler six-manifolds S^6, $S^3 \times S^3$ and $\mathbb{CP}^3$. Moreover, this occurs if and only if for every $p \in M$, $\mathfrak{aut}_p(g, J)$ is strictly larger than $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.*

Proof. Fix $p \in M$ and let $\mathfrak{g}$ be the Lie algebra of germs of Killing vector fields at p that preserve J and let $\mathfrak{h}$ be the isotropy algebra of vector fields in $\mathfrak{g}$ vanishing at p. Let G, H denote the connected, simply connected Lie groups generated by $\mathfrak{g}$ and $\mathfrak{h}$ – by assumption G acts transitively on a neighbourhood of p. To prove the lemma it suffices to show that H is a closed subgroup of G, for then (M, g) is locally isometric to a homogeneous space (Theorem 5.1 [Tri92]) and we can apply the classification of Theorem 5.3.

Suppose that H is not closed in G, then the Lie algebra $\bar{\mathfrak{h}}$ of the closure of H is strictly larger than $\mathfrak{h}$ and contains $\mathfrak{h}$ as an ideal (Lemma 2, p. 612, [Mos50]). We obtain then a decomposition $\bar{\mathfrak{h}} = \mathfrak{h} \oplus \mathfrak{p}$ with $\mathfrak{p}$ non-trivial and commuting with $\mathfrak{h}$.

The cohomogeneity one action defines an inclusion $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \subset \mathfrak{g}$, and $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \cap \mathfrak{h}$ is the isotropy algebra of this, equal to $\mathfrak{u}(1)$ by Theorem 5.10. Moreover, the isotropy representation of $(\mathfrak{g}, \mathfrak{h})$ defines an embedding of $\mathfrak{h}$ into $\mathfrak{su}(3)$, and so $\mathfrak{h}$ is isomorphic to one of $\mathfrak{u}(1), 2\mathfrak{u}(1), \mathfrak{su}(2), \mathfrak{u}(1) \oplus \mathfrak{su}(2)$ and $\mathfrak{su}(3)$. Since there exists a non-trivial subspace $\mathfrak{p}$ complementary to and commuting with $\mathfrak{h}$, the isotropy representation of $\mathfrak{h}$ must act reducibly with a trivial subspace; this restricts $\mathfrak{h}$ to be one of $\mathfrak{u}(1)$ and $\mathfrak{su}(2)$. But since H is

not closed it cannot be semi-simple (Corollary 1 p. 615 [Mos50]), so $\mathfrak{h}$ is isomorphic to $\mathfrak{u}(1)$ and therefore coincides with the isotropy subalgebra of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ action. But this generates a closed subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, and $\mathrm{SU}(2) \times \mathrm{SU}(2)$, being semi-simple, is closed in G, contradicting the hypothesis that H is not closed in G.

The final statement follows from Theorem 5.10 as follows. Since G contains $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, a generic point has an orbit of codimension at most one. If there is a G-orbit of codimension one then G acts with cohomogeneity one, but this is not possible if G is strictly larger than $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ by the classification of groups that can act with cohomgeneity one. Thus G acts transitively. □

Remark 6.11. Let (M, g, J) be a strictly nearly Kähler six-manifold admitting a cohomgeneity one action by $\mathrm{SU}(2) \times \mathrm{SU}(2)$. This defines four functions $h_1(s), h_2(s), h_3(s), h_4(s)$ on M^* as in Proposition 6.20. By Corollary 6.7 each of the functions $h_i(s)$ is analytic in s, and the second order problem (6.20) is thus determined by the eight parameters $a_i = h_i(0)$ and $b_i = h_i'(0)$, the first of which, a_1, is automatically zero. The four independent constraints $\mathcal{I} = 0$ leave three independent parameters.

The one parameter group $\mathcal{A} = \{e^{s\hat{A}}\}_s$ on M^* further reduces the number of parameters characterising the local existence problem for nonequivalent strictly nearly Kähler six manifolds that are of cohomogeneity one for an action by G from three to two. A simple computation also shows that the induced action of $\mathcal{A}$ on 2-forms restricts to an endomorphism of $\langle \omega^4, \omega^5 \rangle$ acting there by two dimensional rotations, removing the apparent freedom to choose the parameter θ_0 in (6.12) arbitrarily. However, if the solution extends from the principal part to a singular orbit of type S^3 then it is shown below that one must fix $\theta_0 = \frac{\pi}{2}$.

The argument given for the proof of Proposition 4.2 of [PS12] shows that, in fact, the maximal dimension of a local family of non-equivalent G-invariant strictly nearly Kähler structures is precisely two. Contrary to the claim made there, however, it does not show that the metrics contained in this family are mutually non-isometric. This claim relies upon the mistaken assumption that if (g, J) is a strictly nearly Kähler structure and φ is an isometry of g then φ preserves J also: in the absence of completeness we have seen that this is not the case (section 4.5).

6.3.1 The f_i of S^6

As before we consider $S^6 \subset \mathrm{Im}\,\mathbb{O}$. The singular orbits are $G \cdot (i, 0) \cong S^3$ and $G \cdot (0, 1) \cong S^2$, and these are connected by the following curve

$$\gamma(t) = (i \cos t, \sin t),\ t \in (0, \tfrac{\pi}{2}).$$

With the identifications (6.3) and the action (5.8), we see that

$$\begin{aligned} \hat{E}_1|_{\gamma(t)} &= -\tfrac{1}{\sqrt{2}}(k \cos t, -\tfrac{j}{2} \sin t), \quad \hat{E}_2|_{\gamma(t)} = \tfrac{1}{2\sqrt{2}}(0, j \sin t), \\ \hat{V}_1|_{\gamma(t)} &= \tfrac{1}{\sqrt{2}}(j \cos t, \tfrac{k}{2} \sin t), \qquad \hat{V}_2|_{\gamma(t)} = \tfrac{1}{2\sqrt{2}}(0, k \sin t), \\ \hat{A}|_{\gamma(t)} &= (0, -i \sin t). \end{aligned}$$

From these it is evident that γ meets all principal orbits orthogonally. Using $JX|_{\gamma(t)} = \mathrm{Im}(\gamma_t \cdot X)$ for X a vector field on S^6, we obtain

$$\begin{aligned} J\hat{E}_1|_{\gamma(t)} &= \tfrac{1}{\sqrt{2}}(j(\cos t^2 t - \tfrac{1}{2} \sin^2 t), -\tfrac{k}{2} \sin t \cos t), \quad J\hat{E}_2|_{\gamma(t)} = \tfrac{1}{2\sqrt{2}}(j \sin^2 t, -k \sin t \cos t), \\ J\hat{V}_1|_{\gamma(t)} &= \tfrac{1}{\sqrt{2}}(k(\cos^2 t + \tfrac{1}{2} \sin^2 t), -\tfrac{j}{2} \sin t \cos t), \qquad J\hat{V}_2|_{\gamma(t)} = \tfrac{1}{2\sqrt{2}}(k \sin^2 t, j \sin t \cos t), \\ J\hat{A}|_{\gamma(t)} &= (-i \sin^2 t, \sin t \cos t). \end{aligned}$$

Then using the Euclidean metric on $\operatorname{Im}\mathbb{O}$, we obtain

$$f_1(t) = -\sin t, \quad f_2(t) = -\tfrac{1}{8}\cos t(2\cos^2 t - \sin^2 t),$$
$$f_3(t) = -\tfrac{1}{8}\sin^2 t\cos t, \quad f_4(t) = 0, \quad f_5(t) = -\tfrac{1}{8}\sin^2 t\cos t.$$

6.3.2 The f_i of $\mathbb{CP}^3$

The necessary computations are most easily made in the total space of the quotient fibration $\mathbb{H}^2 \to \mathbb{CP}^3$. Thus, for $p = [1:0:0:0] \in \mathbb{CP}^3$, let $\tilde{p} = (1,0) \in \mathbb{H}^2$ be a lift. We identify the basis (6.4) with the following elements of $\mathfrak{sp}(2)$.

$$E_1 = \tfrac{1}{2\sqrt{2}}\begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \quad V_1 = \tfrac{1}{2\sqrt{2}}\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \tfrac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix},$$
$$V_2 = \tfrac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}, \; A = \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

The following curve interpolates between the two S^2 singular orbits at $t = 0$ and $t = \frac{\pi}{2}$,

$$\gamma(t) = \exp\left[t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] \cdot p = (\cos t, \sin t).$$

The curve $\tilde{\gamma}(t) = (\cos t, \sin t)$ is a geodesic and a lift of γ. As vector fields on $\mathbb{H}^2$,

$$\hat{E}_1|_{\tilde{\gamma}(t)} = \tfrac{1}{2\sqrt{2}}(j\cos t, 0), \quad \hat{V}_1|_{\tilde{\gamma}(t)} = \tfrac{1}{2\sqrt{2}}(k\cos t, 0), \quad \hat{E}_2|_{\tilde{\gamma}(t)} = \tfrac{1}{2\sqrt{2}}(0, j\sin t),$$
$$\hat{V}_2|_{\tilde{\gamma}(t)} = \tfrac{1}{2\sqrt{2}}(0, k\sin t), \; \hat{A}|_{\tilde{\gamma}(t)} = \tfrac{1}{2}(i\cos t, -i\sin t).$$

All of these but $\hat{A}$ are orthogonal to the Hopf fibre through $\tilde{p}$. Discarding then that component of $\hat{A}$, it is easy to see that γ meets the principal orbits orthogonally. Recall that $T_p\mathbb{CP}^3 = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{pmatrix} \middle| \; \alpha \in \mathbb{H} \right\}, \; \mathfrak{p}^- = \left\{ \begin{pmatrix} zj & 0 \\ 0 & 0 \end{pmatrix} \middle| \; z \in \mathbb{C} \right\}.$$

These determine distributions on $\mathbb{CP}^3$ and, for a vector field X, we let z_X denote the projection to the distribution generated by $\mathfrak{p}^-$, α_X the component in that generated by $\mathfrak{p}^+$. We find then

$$z_{\hat{E}_1|_{\gamma(t)}} = \tfrac{1}{2\sqrt{2}}\cos^2 t, \quad \alpha_{\hat{E}_1|_{\gamma(t)}} = -\tfrac{j}{2\sqrt{2}}\cos t\sin t, \quad z_{\hat{E}_2|_{\gamma(t)}} = \tfrac{1}{2\sqrt{2}}\sin^2 t, \quad \alpha_{\hat{E}_2|_{\gamma(t)}} = \tfrac{j}{2\sqrt{2}}\cos t\sin t,$$
$$z_{\hat{V}_1|_{\gamma(t)}} = \tfrac{i}{2\sqrt{2}}\cos^2 t, \quad \alpha_{\hat{V}_1|_{\gamma(t)}} = -\tfrac{k}{2\sqrt{2}}\cos t\sin t, \quad z_{\hat{V}_2|_{\gamma(t)}} = \tfrac{i}{2\sqrt{2}}\sin^2 t, \quad \alpha_{\hat{V}_2|_{\gamma(t)}} = \tfrac{k}{2\sqrt{2}}\cos t\sin t,$$
$$z_{\hat{A}|_{\gamma(t)}} = 0\,, \quad \alpha_{\hat{A}|_{\gamma(t)}} = -\tfrac{i}{2}\cos t\sin t\,, \quad z_{\dot{\gamma}(t)} = 0 \quad \alpha_{\dot{\gamma}(t)} = 1.$$

Now using the metric

$$g((z,\alpha),(z',\alpha')) = \frac{1}{2}(z,z') + (\alpha,\alpha')\,, \; z \in \mathbb{C}\,, \; \alpha \in \mathbb{H},$$

and $J|_{\mathfrak{p}^\pm} = \pm i\mathrm{Id}_{\mathfrak{p}^\pm}$ we get

$$f_1 = -\sin t\cos t, \; f_2 = -\tfrac{1}{16}(2\sin^2 t - \cos^2 t)\cos^2 t,$$
$$f_3 = -\tfrac{1}{16}(2\cos^2 t - \sin^2 t)\sin^2 t, \quad f_4 = 0, \quad f_5 = \tfrac{3}{16}\sin^2\cos^2 t.$$

6.3.3 The f_i of $S^3 \times S^3$

The following curve connects the singular orbits through $(\pm 1, 1) \in S^3 \times S^3$

$$\gamma(t) = \exp(tN) \cdot (1,1),\ N = \frac{\sqrt{6}}{2}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, 0, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right),\ t \in (0, \tfrac{\pi}{2\sqrt{6}}).$$

Using this curve it is straightforward using the formula (3.2) to compute that

$$f_1 = -\tfrac{\sqrt{2}}{3},\ f_2 = -\tfrac{\sqrt{3}}{36}\sin(2\sqrt{6}t),$$
$$f_3 = 0, \quad f_4 = 0, \quad f_5 = -\tfrac{\sqrt{3}}{36}\sin(\sqrt{6}t).$$

6.4 Solutions Extending over an S^3 Singular Orbit

If $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ acts with cohomogeneity one on (M, g) with one singular orbit S diffeomorphic to S^3, then the isotropy subgroup of that orbit can be identified with $K = \Delta(\mathrm{SU}(2))$ and M with the vector bundle $G \times_{\mathrm{Ad}} \mathfrak{k}$, where Ad is the adjoint action of K on its Lie algebra. In section 6.1 we derived the equations satisfied by strictly nearly Kähler structures on the principal part $M^* = G \times_K (\mathfrak{k} \setminus \{0\})$, and in this section we analyse the problem of extending any solution to these equations over the zero section $G \times_K \{0\} = S$.

Defining then

$$E_\pm = E_1 \pm E_2\ ,\ V_\pm = V_1 \pm V_2,$$

we have $\mathfrak{k}{=}\langle U, E_+, V_+\rangle$, and $\mathfrak{m} = \langle A, E_-, V_-\rangle$ is the isotropy representation of $S{=}G/K$. Let $dt, a, e_\pm, v_\pm$ be the basis of 1-forms dual to $\xi, \hat{A}, \hat{E}_\pm, \hat{V}_\pm$. The representations of K on $\mathfrak{k}$ and $\mathfrak{m}$ are equivalent, the isomorphism being simply $U \mapsto A$, $E_+ \mapsto E_-$ and $V_+ \mapsto V_-$.

The principal orbits are the sphere bundles of vectors of fixed length in $G \times_K \mathfrak{k}$, tangent to which are $A, E_\pm, V_\pm$. The curve $\gamma : t \in \mathbb{R}_+ \mapsto [e, tU]$ is a geodesic and meets all the orbits in M^* orthogonally.

Lemma 6.12. *Let ω be a G-invariant 2-form on $M \setminus S$, with $\omega|_\gamma = \sum_i f_i \omega^i$. Then ω admits a smooth extension to M if and only if $f_1, \ldots, f_5$ extend smoothly at $t = 0$ and the following hold:*

(i) f_1 and f_4 are even functions and f_2, f_3 and f_5 are odd;
(ii) $f_3'(0) = \frac{1}{2}f_1(0) + f_2'(0)$, $f_5'(0) = -\frac{1}{4}f_1(0) - f_2'(0)$, $f_4(0) = 0$.

Moreover, if ω extends smoothly, $(\omega|_S)^3$ is non-zero if and only if $\alpha_1 \neq 0$.

Proof. We adopt the Cartesian coordinate system (t, x, y) for $\mathfrak{k}$ which identifies $\mathfrak{k}$ with $\mathrm{Im}\,\mathbb{H}$, hence $\frac{\partial}{\partial t} = i$, $\frac{\partial}{\partial x} = j$, $\frac{\partial}{\partial y} = k$. The geodesic γ can then be represented by the integral curve $t \mapsto it$ of $\frac{\partial}{\partial t}$. As representations of K we can identify $E_\pm$ with $2\sqrt{2}\frac{\partial}{\partial x}$, $V_\pm$ with $2\sqrt{2}\frac{\partial}{\partial y}$. However, as Killing vector fields on M^*

$$\hat{E}_+|_{\gamma(t)} = \frac{d}{ds}\left(e^{\frac{1}{2\sqrt{2}}js}\gamma(t)e^{-\frac{1}{2\sqrt{2}}js}\right)\Big|_{s=0} = -\frac{t}{\sqrt{2}}\frac{\partial}{\partial y}\Big|_{\gamma(t)},$$
$$\hat{V}_+|_{\gamma(t)} = \frac{d}{ds}\left(e^{\frac{1}{2\sqrt{2}}ks}\gamma(t)e^{-\frac{1}{2\sqrt{2}}ks}\right)\Big|_{s=0} = \frac{t}{\sqrt{2}}\frac{\partial}{\partial x}\Big|_{\gamma(t)}.$$

We have then that

$$e_1|_{\gamma(t)} = \left(-\frac{\sqrt{2}}{t}dy + 2e_-\right)\Big|_{\gamma(t)}, v_1|_{\gamma(t)} = \left(\frac{\sqrt{2}}{t}dx + 2v_-\right)\Big|_{\gamma(t)},$$
$$e_2|_{\gamma(t)} = \left(-\frac{\sqrt{2}}{t}dy - 2e_-\right)\Big|_{\gamma(t)}, v_2|_{\gamma(t)} = \left(\frac{\sqrt{2}}{t}dx + 2v_-\right)\Big|_{\gamma(t)},$$

and so, for clarity suppressing restrictions on the right hand side,

$$\begin{aligned}
\omega|_{\gamma(t)} &= f_1 dt\wedge a + f_2\left(-\frac{\sqrt{2}}{t}dy + 2e_-\right)\wedge\left(\frac{\sqrt{2}}{t}dx + 2v_-\right)\\
&\quad + f_3\left(-\frac{\sqrt{2}}{t}dy - 2e_-\right)\wedge\left(\frac{\sqrt{2}}{t}dx + 2v_-\right) + \frac{2\sqrt{2}}{2}f_4\left(-e_-\wedge dy + v_-\wedge dx\right)\\
&\quad + f_5\left(-2e_-\wedge v_- + \frac{4}{t^2}dx\wedge dy\right)\\
&= 2\frac{f_2+f_3+2f_5}{t^2}dx\wedge dy + (f_2+f_3-2f_5)e_-\wedge v_-\\
&\quad + f_1 dt\wedge a + \sqrt{2}\frac{f_3-f_2}{t}(dx\wedge e_- + dy\wedge v_-)\\
&\quad + \frac{2\sqrt{2}}{t}f_4(-dx\wedge v_- + dy\wedge e_-).
\end{aligned}$$

The restriction of ω to $\mathfrak{k}\setminus\{0\}$ defines a K-equivariant map $\mathfrak{k}\setminus\{0\}\longrightarrow\Lambda^2(\mathfrak{k}\oplus\mathfrak{m})$. As $\Lambda^2(\mathfrak{k}\oplus\mathfrak{m})\cong\Lambda^2\mathfrak{k}\oplus\Lambda^2\mathfrak{m}\oplus(\mathfrak{k}\otimes\mathfrak{m})$, then, ω extends over S if and only if each of the respective components $\omega^{\mathfrak{k}}$, $\omega^{\mathfrak{m}}$ and $\omega^{\mathfrak{k}\otimes\mathfrak{m}}$ extend. Now,

$$\omega^{\Lambda^2\mathfrak{k}}|_{\gamma(t)} = 2\frac{f_2+f_3+2f_5}{t^2}dx\wedge dy|_\gamma,\ \omega^{\Lambda^2\mathfrak{m}}|_{\gamma(t)} = (f_2+f_3-2f_5)e_-\wedge v_-.$$

Let $R:(t,x,y)\mapsto(-t,x,-y)$ then $R\in\mathrm{SO}(3)$ and $\omega|_{\gamma((-t,0)} = R^*\omega_{\gamma((0,t))}$, and we see then that ω extends if and only if the following coefficients are odd in t

$$\frac{f_2+f_3+2f_5}{t^2},\ f_2+f_3-2f_5.$$

In particular, f_2+f_3 and f_5 are odd and

$$f_2'(0)+f_3'(0)+2f_5'(0) = 0\,. \tag{6.26}$$

We consider now the remaining component,

$$\omega^{\mathfrak{k}\otimes\mathfrak{m}}|_{\gamma(t)} = f_1 dt\wedge a + \sqrt{2}\frac{f_3-f_2}{t}(dx\wedge e_- + dy\wedge v_-) + \frac{2\sqrt{2}}{t}f_4(-dx\wedge v_- + dy\wedge e_-).$$

From this we see that f_1 and $\lambda = 2\frac{f_3-f_2}{t}$ must be even and $\frac{f_4}{t}$ odd. Together with our previous deductions about the parity of f_2, f_3 and f_5, we arrive then at (i) of the lemma.

As $\mathfrak{k}$ and $\mathfrak{m}$ are equivalent there is an invariant decomposition $\mathfrak{k}\otimes\mathfrak{m} = \Lambda^2\mathfrak{k}\oplus S^2\mathfrak{k}$, and we can identify the components of $\omega^{\mathfrak{k}\otimes\mathfrak{m}}$ with respect to this decompositions as

$$\omega^{S^2\mathfrak{k}}|_{\gamma(t)} = f_1 dt\wedge a + \sqrt{2}\frac{f_3-f_2}{t}(dx\wedge e_- + dy\wedge v_-)\,,\ \omega^{\Lambda^2\mathfrak{k}}|_\gamma = \frac{2\sqrt{2}}{t}f_4(-dx\wedge v_- + dy\wedge e_-).$$

For $(t,x,y)\in\mathfrak{k}$ with $x^2+t^2\neq 0$, define $\rho=\sqrt{t^2+x^2+y^2}$ and

$$\sin\theta = \frac{y}{\rho}\,,\ \cos\theta = \frac{\sqrt{t^2+x^2}}{\rho}\,,\ \sin\phi = \frac{x}{\sqrt{t^2+x^2}}\,,\ \cos\phi = \frac{t}{\sqrt{t^2+x^2}}.$$

Then the transformation

$$Q = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & \sin\theta \\ -\sin\phi & \cos\phi & 0 \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & \cos\theta \end{pmatrix} \in \mathrm{SO}(3)$$

sends (t,x,y) to $(\rho,0,0)$, so that $\omega_p = Q^*\omega_{(\rho,0,0)}$. Recalling that A, E_-, V_- transforms in the same manner as dt, dx, dy, this implies

$$\begin{aligned}
\omega_p = {}& \frac{1}{\sqrt{2}}f_1(\rho)(\cos\theta\cos\phi\,dt + \cos\theta\sin\phi\,dx + \sin\theta\,dy) \\
&\qquad\wedge(\cos\theta\cos\phi\,a + \cos\theta\sin\phi\,e_- + \sin\theta\,v_-) \\
&+\frac{\lambda(\rho)}{\sqrt{2}}(-\sin\phi\,dt+\cos\phi\,dx)\wedge(-\sin\phi\,a+\cos\phi\,e_-) \\
&+\frac{\lambda(\rho)}{\sqrt{2}}(-\sin\theta\cos\phi\,dt-\sin\theta\sin\phi\,dx+\cos\theta\,dy) \\
&\qquad\wedge(-\sin\theta\cos\phi\,a-\sin\theta\sin\phi\,e_-+\cos\theta\,v_-) \\
= {}& \frac{1}{\sqrt{2}}\left(f_1(\rho)\frac{t^2}{\rho^2}+\lambda(\rho)\left(1-\frac{t^2}{\rho^2}\right)\right)dt\wedge a + \frac{1}{\sqrt{2}}\left(f_1(\rho)\frac{x^2}{\rho^2}+\lambda(\rho)\left(1-\frac{x^2}{\rho^2}\right)\right)dx\wedge e_- \\
&+\frac{1}{\sqrt{2}}\left(f_1(\rho)\frac{y^2}{\rho^2}+\lambda(\rho)\left(1-\frac{y^2}{\rho^2}\right)\right)dy\wedge v_- + \sqrt{2}\frac{tx}{\rho^2}\left(f_1(\rho)-\lambda(\rho)\right)dt\wedge e_- \\
&+\sqrt{(2)}\frac{ty}{\rho}\left(f_1(\rho)-\lambda(\rho)\right)dt\wedge e_- + \sqrt{2}\frac{xy}{\rho^2}\left(f_1(\rho)-\lambda(\rho)\right)dx\wedge v_-.
\end{aligned}$$

From this we see that

$$0 = \lim_{\rho\to 0}(f_1(\rho)-\lambda(\rho)) = f_1(0) - 2(f_3'(0)-f_2'(0)).$$

This and (6.26) above give (ii) of the lemma.

Finally, from (i) and (ii) of the lemma we have

$$\omega_{\gamma(0)} = \alpha_1\left(dt\wedge a + \frac{\sqrt{2}}{2}(dx\wedge e_- + dy\wedge v_-)\right), \tag{6.27}$$

and therefore $(\omega_{\gamma(0)})^3\neq 0$ if and only if $f_1(0)\neq 0$. □

Lemma 6.13. *Let ω be a G-invariant 2-form on $M\setminus S$ with $\omega|_\gamma = \sum_i f_i\omega^i$. If $(\omega, \frac{1}{3}d\omega)$ defines a strictly nearly Kähler structure on $M\setminus S$, then $\frac{1}{3}d\omega$ extends over M as a stable 3-form if and only if $f_1(0)\neq 0$, $f_4 = 0$ and $f_5'(0)\neq 0$.*

Proof. The condition $f_1(0) \neq 0$ is necessary from Lemma 6.12. If $(\omega, \frac{1}{3}d\omega)$ defines a strictly nearly Kähler structure on M then $\mathfrak{f}_4$ satisfies the non-singular ODE (6.16) for all $t \geq 0$. By the proof of Corollary 6.7, f_1 is analytic on the whole of M, so $\mathfrak{f}_4$ is analytic. Now, from (i) of Lemma 6.12, f_4 and f_5 have opposite parity, so it follows that either $f_4 = 0$ or $f_5 = 0$. If $f_5 = 0$, then $\mathfrak{f}_4 = f_4$ is even and $\mathfrak{f}_4'(0) = 0$. But from (ii) of Lemma 6.12, by differentiating (6.16) one sees then that $\mathfrak{f}_4^{(k)}(0) = 0$ for all $k \geq 0$, so if $\mathfrak{f}_4$ is an analytic function we must have $\mathfrak{f}_4 = 0$, a contradiction. It must then be the case that $\mathfrak{f}_4 = f_5$ and $f_5'(0) \neq 0$.

In Lemma 6.5, M^* was oriented by $\tau = dt \wedge E_1 \wedge V_1 \wedge E_2 \wedge V_2$ which does not extends to a volume form over M. By assumption, however, ω^3 is a volume form on M and

$$\omega^3|_{\gamma(t)} = 6f_1(f_2f_3 - f_5^2)\tau.$$

Using this volume form, the polynomial of Definition 4.20 differs on M^* by a factor of $(6f_1(f_2f_3 - f_5^2))^2$. Ignoring positive numerical factors, therefore, on M^* we have

$$P(\tfrac{1}{3}d\omega|_{\gamma(t)}) = -\frac{f_5^2}{f_1^2(f_2f_3 - f_5^2)^2}\left((f_5')^2 - \left(f_2' + \frac{f_1}{4}\right)\left(f_3' - \frac{f_1}{4}\right)\right).$$

Since $f_1(0) \neq 0$, we see then that $\frac{1}{3}d\omega$ is stable if and only if

$$\lim_{t\to 0}\frac{f_5^2}{(f_2f_3 - f_5^2)^2}\left((f_5'f_1)^2 - f_1\left(f_2' + \frac{f_1}{4}\right)\left(f_3' - \frac{f_1}{4}\right)\right) > 0. \tag{6.28}$$

Define

$$g = f_2f_3 - f_5^2\,,\ h = (f_5'f_1)^2 - f_1\left(f_2' + \frac{f_1}{4}\right)\left(f_3' - \frac{f_1}{4}\right).$$

Then $g(0) = 0$ as $f_2(0) = f_3(0) = f_5(0) = 0$. Moreover, from (ii) of Lemma 6.12 we see that

$$f_2'(0) + \frac{1}{4}f_1(0) = f_3'(0) - \frac{1}{4}f_1(0) = -f_5'(0)$$

and so $h(0) = 0$. We apply L'Hôpital's rule to compute (6.28).

Firstly, from (6.17), and (6.14) and (6.15, for $k \geq 1$ we have

$$g^{(k)} = -\frac{1}{48\mu}\left[f_1(f_2' - f_3' + \frac{1}{2}f_1)\right]^{(k)} = \frac{1}{4}\left[f_1(f_3 - f_2)\right]^{(k-1)}.$$

Thus $g'(0) = 0$ and $g''(0) = \frac{1}{4}f_1(0)(f_3'(0) - f_2'(0)) = \frac{1}{8}f_1(0)^2 \neq 0$. The product rule shows that $(g^2)^{(k)}(0) = 0$ for $k < 4$ and $(g^2)^{(4)}(0) = 6g''(0)^2$.

Now, by (6.14), (6.15) and (6.16)

$$\begin{aligned} h'(0) &= \left\{2f_5'f_1(f_5'f_1)' - \left[f_1\left(f_2' + \frac{f_1}{4}\right)\right]' f_1\left(f_3' - \frac{f_1}{4}\right) - f_1\left(f_2' + \frac{f_1}{4}\right)\left[f_1\left(f_3' - \frac{f_1}{4}\right)\right]'\right\}\Big|_{t=0} \\ &= \left\{2f_5'f_1\left(\frac{4}{f_1}f_5 - 12\mu f_1f_5\right) + 12\mu f_1^2f_2\left(f_3' - \frac{f_1}{4}\right) + 12\mu f_1^2f_3\left(f_2' + \frac{f_1}{4}\right)\right\}\Big|_{t=0} \\ &= 0. \end{aligned}$$

Similarly, using also (ii) of Lemma 6.12,

$$\begin{aligned} h''(0) &= \left\{ 2f_5' f_1 \left(\frac{4}{f_1} f_5 - 12\mu f_1 f_5 \right)' + 12\mu (f_1 f_2)' f_1 \left(f_3' - \frac{f_1}{4} \right) + 12\mu (f_1 f_3)' f_1 \left(f_2' + \frac{f_1}{4} \right) \right\} \Big|_{t=0} \\ &= 2f_1(0) f_5'(0) \left(\frac{4}{\alpha_1} f_5'(0) - 12\mu f_1(0) f_5'(0) \right) - 12\mu f_1(0)^2 f_2'(0) f_5'(0) - 12\mu f_1(0)^2 f_3'(0) f_5'(0) \\ &= 8 f_5'(0)^2. \end{aligned}$$

Then $(f_5^2 h)^{(k)} = 0$ for $k < 4$ and the fourth derivative is $6 f_5'(0)^2 h''(0) = 48 f_5'(0)^4$. Thus (6.28) equals

$$\frac{8^2 \cdot 48 f_5'(0)^4}{6 f_1(0)^4},$$

which is positive as required. □

As a result of these lemmas we can prove the following. As a check on this result, note that near the singular orbit of type S^3 in S^6 and near both S^3 singular orbits in $S^3 \times S^3$, the f_i satisfy (i), (ii) and (iii).

Proposition 6.14. *Let $f_1, f_2, f_3, \mathfrak{f}_4$ be smooth functions on some interval $(-a, a)$, $a > 0$ such that*

(i) f_1 is even and strictly negative;

(ii) f_2, f_3 and $\mathfrak{f}_4$ are odd and

$$f_3'(0) = \frac{1}{2} f_1(0) + f_2'(0)\,, \quad \mathfrak{f}_4'(0) = -\frac{1}{4} f_1(0) - f_2'(0) \neq 0;$$

(iii) $f_1, f_2, f_3, \mathfrak{f}_4$ satisfy the differential system (6.14)-(6.17) and (6.18).

Then there exists an $\varepsilon \leq a$ and a G-invariant strictly nearly Kähler structure (ω, Ω) on a tubular neighbourhood $G \cdot \gamma([0, \varepsilon))$ of $S \cong S^3$ such that $\omega|_\gamma = \sum_i f_i \omega^i$.

Proof. We let ω be the G-invariant 2-form on M^* defined by $\omega|_\gamma = \sum_i f_i \omega^i$ and set $\Omega = \frac{1}{3} d\omega$. By the previous lemma, (ω, Ω) extends over M and Ω is stable. It remains to show that $g = g_{\omega,\Omega}$ is a positive definite metric near S. Now, $T_{\gamma(0)} M = \mathfrak{h} \oplus \mathfrak{m}$. As $\gamma'(0) \in \mathfrak{h}$ and

$$J\gamma'(0) = \lim_{t \to 0} J\gamma'(t) = \lim_{t \to 0} \frac{1}{f_1} \hat{A} = \frac{1}{f_1(0)} A \in \mathfrak{m}$$

we see that $J\mathfrak{h} \cap \mathfrak{m} \neq \{0\}$ so $J\mathfrak{h} = \mathfrak{m}$ since $\mathfrak{h}$ and $\mathfrak{m}$ are irreducible. But then, $g_{\gamma(0)}(\mathfrak{h}, \mathfrak{m}) = \omega_{\gamma(0)}(\mathfrak{m}, \mathfrak{m})$ and this vanishes by (6.27). Finally, $g(\gamma'(0), \gamma'(0)) = 1$ so by G-invariance

$$g(E_+, E_+) = g(E_+, [U, V_+]) = -g([V_+, E_+], U) = \frac{1}{4} g(U, U) = \frac{1}{4}.$$

Similarly $g(E_+, V_+) = 0$ and $g(V_+, V_+) = \frac{1}{4}$. In this basis, therefore, $g_{\gamma(0)}$ is proportional to the standard metric. In particular, $g_{\gamma(0)}$ is positive definite and so by continuity g is positive definite in a neighbourhood of S. □

Our final result represents the extent of the analysis of the cohomogeneity one problem that has so far been carried out. It shows that functions as in Proposition 6.14 may always be found. Indeed, we see that nonequivalent non-homgeneous solutions extending over an S^3 singular orbit come in a one-dimensional family, which should be compared with the situation on the principal part where there is at most a two-parameter family of nonequivalent non-locally homogeneous solutions (Remark 6.11).

Theorem 6.15. *For every value of $\mu > 0$, there exists a one-parameter family of non-equivalent strictly nearly Kähler structures on*

$$TS^3 = (\mathrm{SU}(2) \times \mathrm{SU}(2)) \times_{\Delta(\mathrm{SU}(2))} \mathbb{R}^3$$

that are not locally homogeneous, have scalar curvature 30μ and are preserved by the cohomogeneity one action by $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

Remark 6.16. More than this is asserted in [PS12]. There it is claimed that the family constructed above consists not only of non-equivalent nearly Kähler structures but that all the metrics in the family are non-isometric. Their proof appears to rely on the assumption that any isometry preserving the metric in a strictly nearly Kähler structure preserves the whole structure. For non-complete spaces we saw in section 4.5 that this was not the case.

Proof. We fix $\mu = 2$ for convenience. For odd functions h_1, h_2, h_3, h_4 satisfying

$$h_1'(0) > 0\,,\ h_3'(0) = -h_1'(0)\,,\ h_2'(0) = -h_4'(0) \neq 0, \tag{6.29}$$

we have $\mathcal{I}(\{h_i(0), h_i'(0)\}) = 0$, so by the previous proposition and Proposition 6.8 it suffices to solve the system (6.20) with $\mu = 2$ for odd functions h_1, h_2, h_3, h_4 satisfying (6.29). Given such functions we have in particular

$$h_1''(h_2^2 - h_3^2 - h_4^2) + 2(h_1')^2 h_3 + \frac{2}{9} h_4' h_4 = 0.$$

Differentiating this, taking the limit $s \to 0$ and using (6.29) gives

$$2\left(b_1^3 - \frac{1}{9} b_4^2\right) = 0\,.$$

The problem is therefore to solve (6.20) for odd functions h_1, h_2, h_3, h_4 satisfying

$$h_1'(0) = \alpha > 0\,,\ h_3'(0) = -\alpha\,,\ h_4'(0) = -h_2'(0) = 3\alpha\sqrt{\alpha}. \tag{6.30}$$

Choosing the negative square root, $h_4'(0) = -3\alpha\sqrt{\alpha}$, gives an isometric solution.

To solve this problem it is slightly easier to solve for the even functions p_1, p_2, p_3, p_4 defined by $h_i = sp_i$, $i = 1, 2, 4$, and $h_4 = s(p_3 - p_1)$. The initial conditions (6.29) then become

$$p_1(0) = \alpha\,,\ p_3(0) = 0\,, p_4(0) = -p_2(0) = 3\alpha\sqrt{\alpha},$$

and the system (6.20) becomes

$$\begin{aligned}
&p_1'' + \tfrac{2}{s}\left(p_1' + \tfrac{18p_1p_1'(p_3-p_1)+p_4p_4'}{9(p_2^2-(p_3-p_1)^2-p_4^2)}\right) + \tfrac{2}{s^2}\tfrac{9p_1^2(p_3-p_1)+p_4^2}{9(p_2^2-(p_3-p_1)^2-p_4^2)} + \tfrac{2(p_1')^2(p_3-p_1)}{p_2^2-(p_3-p_1)^2-p_4^2} = 0,\\
&p_2'' + \tfrac{2}{s}p_2' + 48p_1p_2 + 48s\,p_1'p_2 = 0,\\
&p_3'' + \tfrac{2}{s}p_3' + 48p_1(p_3-p_1) + 48s\,p_1'(p_3-p_1) = 0,\\
&p_4'' + \tfrac{2}{s}p_4' + 4p_4(12p_1 - 1) + 48s\,p_1'p_4 = 0.
\end{aligned} \tag{6.31}$$

We can now use the following theorem (Théorème 7.1 [Mal74]) to solve this initial value problem. For a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, $\widetilde{F}_{\mathbf{x}}$ denotes the formal Taylor expansion of F about $\mathbf{x} \in \mathbb{R}^n$.

Theorem. *Let k be an integer, $\mathbf{y}_0 \in \mathbb{R}^m$ and Φ a smooth $\mathbb{R}^m$-valued function of the $m+1$ variables $x, y_1, \ldots, y_m$ defined in a neighbourhood of $\mathbf{x}_0 = (0, \mathbf{y}_0)$. Then if there exists a formal power series $H = \sum_{i=0}^{\infty} c_i x^i$ with coefficients in $\mathbb{R}^m$ and $c_0 = \mathbf{y}_0$ satisfying*

$$x^k \frac{dH}{dx} = \widetilde{\Phi}_{\mathbf{x}_0}(x, H),$$

there exists a smooth $\mathbb{R}^m$-valued function $F(x)$ satisfying $\widetilde{F}_0 = H$ and

$$x^k \frac{dF}{dx} = \Phi(x, F).$$

Defining $\mathcal{P} = (p_1, p_2, p_3, p_4)$ and $\mathcal{Q} = \mathcal{P}'$ the problem at hand has the following schematic form,

$$\mathcal{P}' = Q,\ \mathcal{Q}' = \frac{1}{s^2}A(\mathcal{P}) + \frac{1}{s}B(\mathcal{P}, \mathcal{Q}) + C(s, P, Q),\ \mathcal{P}(0) = \mathcal{P}_0,\ \mathcal{Q}(0) = 0, \tag{6.32}$$

where where A, B and C are smooth $\mathbb{R}^4$-valued functions defined on suitable open neighbourhoods of $\mathcal{P}_0 \in \mathbb{R}^4$, $(\mathcal{P}_0, 0) \in \mathbb{R}^8$ and $(0, \mathcal{P}_0, 0) \in \mathbb{R}^9$, respectively. We seek formal solutions of the form

$$\widetilde{\mathcal{P}} = \sum_{n\geq 0} \frac{\mathcal{P}_{2n}}{(2n)!} s^{2n},\ \widetilde{\mathcal{Q}} = \sum_{n\geq 1} \frac{\mathcal{P}_{2n}}{(2n-1)!} s^{2n-1}.$$

There are then the following Taylor expansions

$$\widetilde{A}(\widetilde{\mathcal{P}}(s)) = \sum_{n\geq 0} \frac{A_{2n}}{(2n)!} s^{2n},\ \widetilde{B}(\widetilde{\mathcal{P}}(s), \widetilde{\mathcal{Q}}(s)) = \sum_{k\geq 0} \frac{B_{2n+1}}{(2n+1)!} s^{2n+1},\ \widetilde{C}(s, \widetilde{\mathcal{P}}(s), \widetilde{\mathcal{Q}}(s)) = \sum_{n\geq 0} \frac{C_{2n}}{(2n)!} s^{2n},$$

so that (6.32) becomes

$$\mathcal{P}_{2n+2} = \frac{A_{2n+2}}{(2n+2)(2n+1)} + \frac{B_{2n+1}}{2n+1} + C_{2n}. \tag{6.33}$$

Now,

$$A_{2n+2} = \frac{d^{2n+2}}{ds^{2n+2}} A(\widetilde{\mathcal{P}}(s))\Big|_{s=0} = dA|_{\mathcal{P}(0)} \cdot \mathcal{P}_{2n+2} + \cdots,$$

where the ellipsis denotes terms involving the lower order coefficients P_i, $i \leq 2n$. Similarly,

$$B_{2n+1} = \frac{d^{2n+1}}{ds^{2n+1}} B(\widetilde{\mathcal{P}}(s), \widetilde{\mathcal{Q}}(s))\Big|_{s=0} = \frac{\partial B}{\partial \mathcal{Q}} \cdot \mathcal{P}_{2n+2}\Big|_{(\mathcal{P}(0),0)} + \cdots.$$

We can then write (6.33) as the following recursion relation

$$\mathcal{P}_{2n+2} = \frac{1}{(2n+2)(2n+1)} dA|_{\mathcal{P}(0)} \cdot \mathcal{P}_{2n+2} + \frac{1}{2n+1} \frac{\partial B}{\partial \mathcal{Q}}\Big|_{(\mathcal{P}(0),0)} \cdot \mathcal{P}_{2n+2} + D_{2n}, \tag{6.34}$$

where D_{2n} is a fixed function of $\mathcal{P}_0, \mathcal{P}_2, \mathcal{P}_4, \ldots, \mathcal{P}_{2n}$. Defining then

$$\mathcal{L}_{2n} = \mathrm{Id} - \frac{1}{(2n+2)(2n+1)} dA|_{\mathcal{P}(0)} - \frac{1}{2n+1} \frac{\partial B}{\partial \mathcal{Q}}\Big|_{(\mathcal{P}(0),0)},$$

we show that $\det \mathcal{L}_{2n} \neq 0$ so that $\widetilde{\mathcal{P}}$ is uniquely determined. First, compute

$$dA|_{\mathcal{P}(0)} = \begin{pmatrix} 6 & 0 & -2 & -\frac{4}{3\sqrt{\alpha}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \frac{\partial B}{\partial \mathcal{Q}}\Big|_{(\mathcal{P}(0),0)} = \begin{pmatrix} 6 & 0 & 0 & -\frac{2}{3\sqrt{\alpha}} \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

We find then that

$$\det \mathcal{L}_{2n} = -\frac{2n^2 - 3n - 8}{(2n+1)(n+1)} \left(\frac{2n-1}{2n+1}\right)^3,$$

non-zero for all positive integral n. There therefore exists a genuine smooth solution, $\mathcal{P}$, to (6.31) with Taylor expansion $\widetilde{\mathcal{P}}$ at $s = 0$.

Since $\widetilde{\mathcal{P}}$ has only even powers of s, the even function $\mathcal{P}(|s|)$ is smooth at $s = 0$. It also satisfies 6.31 and so defines a nearly Kähler structure of the kind claimed in the statement of the theorem. Moreover, by Corollary 6.7, any such solution is analytic and so is uniquely determined by the initial parameter$\alpha = p_1(0)$. We thus obtain a unique family of invariant strictly nearly Kähler structures $\{(g_\alpha, J_\alpha)\}_{\alpha>0}$ on TS^3.

It remains to show that the nearly Kähler structures (g_α, J_α) and (g_β, J_α) are locally equivalent if and only if $\alpha = \beta$. To this end let p be a point in the singular orbit S and $\varphi : U \to U$ a local equivalence on a neighbourhood U of p. Defining $\mathfrak{g}_\alpha = \mathfrak{aut}(g_\alpha, J_\alpha)$, and, *mutatis mutandis*, $\mathfrak{g}_\beta$, we see that φ defines an isomorphism $\varphi_* : \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$.

By Proposition 6.10, a solution g_α is locally homogeneous if and only if α equals the value of $\frac{1}{2}f_1^2$ at the S^3 singular orbit in S^6 or $S^3 \times S^3$ equipped with their unique homogeneous nearly Kähler metrics with scalar curvature 60. Thus, if α, β are distinct from these two values, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are isomorphic to $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. This implies that φ preserves the orbits of the action by $G{=}\mathrm{SU}(2){\times}\mathrm{SU}(2)$, and so, in particular, preserves S. Moreover, as φ preserves principal orbits, φ_* must preserve the isotropy algebra $\mathfrak{h} \subset \mathfrak{g}$. As a Lie algebra automorphism, φ_* also preserves the commutator subgroup of $\mathfrak{h}$, *i.e.* $\varphi_*\hat{A}_p = \hat{A}_{\varphi(p)}$. With these facts we conclude that

$$2\alpha = g_\alpha(\hat{A}_p, \hat{A}_p) = g_\beta(\hat{A}_{\varphi(p)}, \hat{A}_{\varphi(p)}) = 2\beta. \qquad \square$$

Chapter 7

Conclusion

The analysis of cohomogeneity one strictly nearly Kähler six manifolds presented in the previous chapter terminates rather abruptly, leaving several problems unanswered. While we have analysed the problem of extending solutions from the principal part to a singular orbit of type S^3, what are the corresponding conditions for singular orbits of type S^2? Furthermore, is there an existence result analogous to Theorem 6.15 for $\mathrm{SU}(2)\times\mathrm{SU}(2)$-invariant strictly nearly Kähler structures on the relevant vector bundle over S^2? Given the generality of the method employed in the proof of Theorem 6.15, an assertion to the positive does not seem unreasonable. As to the more interesting problem presented by compact strictly nearly Kähler six-manifolds of cohomogeneity one, only numerical analysis of the ODE system (6.20) or the power series solution of Theorem 6.15 would seem to provide a hope of understanding the behavior of solutions.

Appendix

Proof of Gallot's Lemma

The purpose of this appendix is to prove the following result of Gallot quoted in the proof of Theorem 4.5.

Lemma. *Let (M^n, g) be a complete Riemannian manifold. If the cone over (M, g) is locally reducible or locally symmetric, then (M, g) is locally isometric to the round n-sphere.*

Proof. Recall from the proof of Proposition 4.10 the formulae for the Levi-Civita connection ∇' and curvature R' of the cone (M', g') over (M, g),

$$\nabla'_\xi \xi = 0\,, \nabla'_X \xi = \nabla'_\xi X = \frac{1}{r} X\,,\ \nabla'_X Y = \nabla_X Y - r g(X,Y)\xi\ ,$$

$$R'(X',\xi) = 0\,, R'(\cdot,\cdot)\xi = 0\,,\ R'(X,Y)Z = R(X,Y)Z - (g(Y,Z)X - g(X,Z)Y)\,,$$

where X' is tangent to M', X, Y are tangent to M and $\xi = \frac{\partial}{\partial r}$. From this it is clear that the cone on (M, g) is flat precisely when (M, g) has sectional curvature equal to one, i.e. is locally isometric to the n-sphere. To establish the lemma, then, it suffices to prove that if the cone is either locally reducible or locally symmetric then it is flat.

It is well known that a Riemannian manifold is locally symmetric if and only if its Riemann curvature tensor is parallel. In particular the following component vanishes

$$\begin{aligned} 0 = (\nabla'_X R')(\xi, Y)Z &= \nabla'_X(R'(\xi,Y)Z) - R'(\nabla'_X\xi, Y)Z - R'(\xi, \nabla'_X Y)Z - R'(\xi,Y)\nabla'_X Z\,, \\ &= -R'(\nabla'_X\xi, Y)Z\,, \\ &= -\frac{1}{r} R'(X,Y)Z\,. \end{aligned}$$

If the cone is locally symmetric, therefore, it is flat.

Suppose now that (M', g') is locally reducible. There therefore exist $\mathrm{Hol}_0(g')$-invariant orthogonal subbundles V_1, V_2 of TM' such that $TM' = V_1 \oplus V_2$. Define then the following subsets

$$C_i = \left\{ m \in M' \,:\, \xi_m \in V_i \right\}\ ,\ i = 1, 2\,.$$

For a point $m_1 \in C_1$ let $M_2(m_1)$ be an integral submanifold for the distribution V_2 through m_1. Similarly, for a point $m_2 \in C_2$ let $M_1(m_2)$ be an integral submanifold for the distribution V_1 through m_2.

Lemma .1. *For any $m_i \in C_i$, $i = 1, 2$, $M_1(m_2)$ and $M_2(m_1)$ are totally geodesic.*

Proof. Consider vector fields X, Y on M_2: these take values in V_2, so parallel transport along fibres of the normal bundle of M_2 defines local extensions of X and Y taking values in V_2, and the derivative $\nabla'_X Y$ is also a section of V_2. But the second fundamental form of M_2 is

$$\mathrm{II}(X,Y) = (\nabla'_X Y)^\perp \,,$$

which component lies in V_1 and so vanishes. □

Lemma .2. *The set $M' \setminus (C_1 \cup C_2)$ is dense.*

Proof. If the statement of the lemma were false then there would exist a point m with an open neighbourhood U disjoint from $M' \setminus (C_1 \cup C_2)$. The C_i are disjoint so $\xi|_U$ lies wholly in, say, V_1.

Consider now, as above, the integral manifold $M_2(m)$. The intersection $W = U \cap M_2(m)$ is a relatively open subset in $M_2(m)$ contained in C_1. Choose a point $w \in W$ distinct from m and which can be connected to m by a geodesic γ in M_2. Consider now the following vector field along γ

$$X'(t) = r\xi_{\gamma(t)} - t\dot\gamma(t)\,.$$

Then $X'(0) = r\xi_{m'} \in V_1$ and by the formulae for ∇',

$$\nabla'_{\dot\gamma} X' = r\nabla'_{\dot\gamma}\xi - \dot\gamma = r\frac{\dot\gamma}{r} - \dot\gamma = 0\,.$$

As a parallel vector field along γ, $X'(t)$ therefore lies in V_1 for all t. Notice now that

$$r\xi_{\gamma(1)} = X'(1) + \dot\gamma(1)\,,$$

in which the first factor on the right is an element of V_1 while the second of V_2. But $W \subset C_1$, that is $\xi_{\gamma(1)} \in V_1$, and, therefore, $\dot\gamma(1) = 0$, a contradiction. □

Lemma .3. *For any $m_i \in C_i$, the integral manifolds $M_1(m_2)$ and $M_2(m_1)$ are flat.*

Proof. Consider a geodesic γ in M_2 with $\gamma(0) = m_1$, and define, as in the previous lemma, the following parallel vector field along it

$$X'(t) = r\xi_{\gamma(t)} - t\dot\gamma(t)\,.$$

Again, $X'(0) = r\xi_{m'_1} \in V_1$, so X' takes values in V_1. However, observe now that as the curvatures $R'(\cdot,\cdot)$ annihilate ξ we have

$$R'(\cdot,\cdot)X' = -tR'(\cdot,\cdot)\dot\gamma\,.$$

But the endomorphisms $R'(\cdot,\cdot)$ are elements of $\mathfrak{hol}(g')$ and so preserve the splitting $V_1 \oplus V_2$ of the holonomy representation. In consequence, the left hand side of the above equality lies in V_2, while the right is in V_1, so $R'(\cdot,\cdot)\dot\gamma = 0$.

Looking now at the Jacobi equation, we see that for every geodesic γ emanating from m_1, any Jacobi field J along γ satisfies

$$\frac{D^2}{dt^2}J = 0\,.$$

Thus if $J(0) = 0$ and $\dot J(0) = w$, then $J(t) = tW(t)$ where W is the parallel translate of w along γ. But any Jacobi field satisfying $J(0) = 0$ and $\dot J(0) = w$ can be written as

$$J(t) = (d\exp_{m_1})_{t\dot\gamma(0)}(tw)\,,$$

where $\exp_p$ is the exponential map for M_2 at $p \in M_2$ with its induced metric. Given then two Jacobi fields J_1, J_2 with $J_i(0) = 0$ and $\dot{J}_i(0) = w_i$, we have

$$w_1 \cdot w_2 = g'(J_1(1), J_2(1))_{\gamma(1)} = g'((d\exp_{m_1'})_{\dot{\gamma}(0)}(w_1), (d\exp_{m_1'})_{\dot{\gamma}(0)}(w_2))\,.$$

As this holds for all $\dot{\gamma}(0), w_1, w_2 \in T_{m_1}M_2$, we see that $\exp_{m_1}$ is an isometry and $M_2(m_1)$ is flat. □

Consider now a point $m \in M \setminus (C_1 \cup C_2)$. By the choice of m, the components ξ_i of ξ_m in V_i are non-vanishing and we define geodesics γ_i so as to satisfy

$$\gamma_i(0) = m\ ,\ \dot{\gamma}_i(0) = -\xi_i\ ,\ i = 1, 2\,.$$

Define along γ_i the following parallel vector field

$$X_i' = r\xi_{\gamma_i(t)} - (t-1)\dot{\gamma}_i(t)\,.$$

Then $X_i'(0) = \xi_j$, where $i \neq j$, and so $X_i'(t) \in V_j$ for all t.

Since $\gamma_i'(0) \neq \pm\xi$, we claim that is defined for all t, in particular for $t = 1$. Given this then $X_i'(1) = r\xi_{\gamma_i(1)} \in V_j$, that is $\gamma_i(1) \in C_j$. Were this the case then by the previous lemma m lies at the intersection of two orthogonal totally geodesic flat submanifolds, namely $M_1(\gamma_2(1))$ and $M_2(\gamma_1(1))$, and R' vanishes on the dense subset $M' \setminus (C_1 \cup C_2)$, and so is trivial by continuity.

It remains to show that $\gamma(t)$ exists for all values of the arc-length parameter t. Indeed, let γ be an arbitrary unit speed geodesic in M' with $\gamma(0) = (r_0, p)$ and $\gamma'(0) = (a\xi_{p_0}, X)$ with $X \in T_{p_0}M \setminus \{0\}$. Let c be the geodesic in M with $c(0) = p$ and $c'(0) = X$. Then the following map is a local isometry onto its image

$$f : \mathbb{C} \setminus \mathbb{R}_- \longrightarrow M';\ (re^{i\theta}) = (r, c(\theta)).$$

It follows then that γ is complete.

□

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