Navier-Stokes Equations with Stochastic Lie Transport: Well-Posedness and Inviscid Limit

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– Declaration of Originality –

Unless stated otherwise by reference or acknowledgement, this thesis is my own work.

– Copyright Statement –

– Acknowledgements –

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– Abstract –

The Navier-Stokes Equations are the fundamental model for viscous fluid dynamics, but no prediction is perfect. Introducing randomness into differential equations has long been an answer to uncertainty quantification; recent developments in the modelling literature point to the significance of *transport noise*, where the stochastic integral depends on the gradient of the solution. Analytically, this unbounded noise breaks classical frameworks built for studying nonlinear stochastic partial differential equations (SPDEs).

The primary goal of this thesis is to develop a rigorous solution theory for the Navier-Stokes Equations under *Stochastic Advection by Lie Transport*, in a general manner which extends to a wide class of nonlinear SPDEs with unbounded noise. Particular attention is given to the case of a domain with physical boundary, where existence results for strong solutions with transport noise were previously unknown. Our theory pertains to both the most classical *no-slip* boundary condition, as well as the *Navier* boundary conditions. In particular, strong solutions are obtained for the Navier boundary conditions.

Beyond well-posedness theory, we consider the inviscid limit of solutions under the no-slip and Navier boundary conditions. The deterministic theory is markedly different in these two scenarios, the core results of which are recovered in our stochastic setting. For the no-slip condition, the vanishing viscosity and noise limit is shown to provide a solution of the *deterministic* Euler equation *if and only if* the energy dissipation in a boundary layer of width approaching zero with viscosity likewise tends to zero. Whilst this property is unknown in general, for a special case of the Navier boundary conditions, the inviscid limit is demonstrated to exist and give a weak solution of the corresponding Stochastic Euler Equation.

– Contents –

Chapter 1

Introduction

The theoretical analysis of stochastic partial differential equations, abbreviated to SPDEs, has been recently established as one of mathematics' most exciting prospects. Attention has long been given to these equations due to their rich applications in areas such as turbulence, population dynamics, finance and neuroscience, of which more can be understood from the monographs [102, 105, 125] and references therein. The area's newfound spike in popularity can be attributed to developments across both analytical techniques and modelling. Indeed, Martin Hairer's Fields Medal winning work on *regularity structures*, [71], gave rigorous meaning to a drastically extended class of SPDEs which are ill-defined in standard function spaces due to spatial irregularities. At a similar time, Gubinelli, Imkeller and Perkowski developed a theory of *paracontrolled distributions*, [68], which again provided a toolkit for the seemingly intractable theory of distribution-valued SPDEs. These techniques are of a very different flavour to the more classical *variational approach* pioneered by Étienne Pardoux, [124], whereby the equation is formatted in a Gelfand Triple for a noise valued in the Hilbert Space.

From the modelling perspective, it is in applications to fluid dynamics and turbulence where the literature has been thriving. The choice of noise to best encapsulate physical properties of fluid equations is in constant study, now yielding a strong argument for transport type stochastic perturbations (where the stochastic integral depends on the gradient of the solution). The paper of Brzeźniak, Capinski and Flandoli [14] in 1992 was one of the first to bring attention to the significance of these equations with transport noise, whilst such ideas have newly been cemented through the specific stochastic transport scheme of [73]. In this paper Holm establishes a new class of stochastic equations driven by transport type noise which serve as fluid dynamics models by adding uncertainty in the transport of fluid parcels to reflect the unresolved scales. The physical significance of such equations in modelling, numerical analysis and data assimilation continues to be well documented, see [4, 19, 27, 28, 29, 31, 32, 40, 42, 50, 74, 75, 93, 95, 140] and in particular [49] for a comprehensive account of the topic. Towards their analysis, however, these equations find themselves in a structural limbo. The first-order noise operator introduces a singularity into the Hilbert Space of the variational framework, so such equations do not fit into this classical theory. The noise is, however, still well-defined in traditional function spaces, suggesting that the heavy machinery and distribution-tailored approaches of Hairer, Gubinelli et al. are not particularly appropriate.

With this outlook, a first goal of the thesis is to develop a rigorous solution theory to address a broad class of such equations. A setup for SPDEs along with tools for their analysis is presented in Chapter 2. We are keen to work in generality to facilitate applications, so our solution theory is developed in an abstract framework for the *Itô* SPDE

$$\boldsymbol{\Psi}_t = \boldsymbol{\Psi}_0 + \int_0^t \mathcal{A}(s, \boldsymbol{\Psi}_s)ds + \int_0^t \mathcal{G}(s, \boldsymbol{\Psi}_s)d\mathcal{W}_s, \tag{1.1}$$

where $\mathcal{W}$ is a Cylindrical Brownian Motion. These results comprise Chapter 3, whilst a complete introduction to Cylindrical Brownian Motion and infinite dimensional stochastic calculus is presented in Appendix A.1.

As the fundamental model of viscous fluid dynamics, our main object of study is the incompressible Navier-Stokes Equation. Although much of our analysis is set for an abstract noise, we centralise around perturbations through *Stochastic Advection by Lie Transport*: this is the transport noise scheme introduced by Holm [73]. The equation takes the form

$$u_t = u_0 - \int_0^t \mathcal{L}_{u_s} u_s \, ds + \nu \int_0^t \Delta u_s \, ds - \int_0^t B(u_s) \circ d\mathcal{W}_s - \nabla \rho_t, \tag{1.2}$$

where u represents the fluid velocity, ρ the pressure and $\nu > 0$ the viscosity. Here $\mathcal{L}$ denotes the nonlinear term, Δ the Laplacian, and this stochastic methodology is characterised by a *Stratonovich* integral driven by $\mathcal{W}$. The mapping B is defined along the components of $\mathcal{W}$ through a collection of vector fields (ξ_i), in N dimensions by

$$B_i : \phi \mapsto \sum_{j=1}^N \left(\xi_i^j \partial_j \phi + \phi^j \nabla \xi_i^j \right).$$

Physically the (ξ_i) represent spatial correlations, determined at coarse-grain resolutions from finely resolved numerical simulations; see [27, 28, 29]. A complete introduction to the equation (1.2) will come in Chapter 4, including a description of how (1.2) is posed in the form (1.1). Existence and uniqueness results for (1.2) are derived in Chapter 5, as a direct application of the theory developed in Chapter 3. The imposition of a physical boundary is central to the motivation and results of the thesis. In any case where the first-order noise operator is not demanded to be small relative to viscosity, existence results for strong solutions of any corresponding stochastic fluid equation were unknown. The necessity to extend well-posedness theory to the case of a bounded domain for equations modelling real world phenomena is clear, and constitutes one of the key contributions of the thesis. We equip (1.2) with two distinct, physically motivated boundary conditions. Firstly and most classically is the no-slip condition, given by

$$u = 0 \tag{1.3}$$

on the boundary $\partial\mathscr{O}$ of a smooth bounded domain $\mathscr{O}$ in two or three dimensions. This is typically considered to be the most physically reasonable boundary condition for viscous flow as well discussed in [34, 128, 132]. Other considerations are far from redundant though, and perhaps the most appreciated of them are the Navier boundary conditions, given in two dimensions by

$$u \cdot \mathbf{n} = 0, \qquad 2(Du)\mathbf{n} \cdot \iota + \alpha u \cdot \iota = 0 \tag{1.4}$$

where $\mathbf{n}$ is the unit outwards normal vector, ι the unit tangent vector, Du is the rate of strain tensor $(Du)^{k,l} := \frac{1}{2}\left(\partial_k u^l + \partial_l u^k\right)$ and $\alpha \in C^2(\partial\mathscr{O};\mathbb{R})$ represents a friction coefficient which determines the extent to which the fluid slips on the boundary relative to the tangential stress. These conditions were first proposed by Navier in [118, 119], and have been derived in [113] from the kinetic theory of gases and in [112] as a hydrodynamic limit. Furthermore, they have proven viable for modelling rough boundaries as seen in [7, 54, 126]. It should be noted that as one takes α to infinity the constraint $u \cdot \iota = 0$ dominates the second identity in (1.4) and as such, at least heuristically, gives some approximation to the no-slip condition (1.3); in the deterministic setting, the corresponding convergence of regular solutions is shown in [81]. A more thorough treatment of the conditions (1.3), (1.4) and their related function spaces arrives in Chapter 4, with well-posedness results in both cases the content of Chapter 5.

Having established well-posedness results, the remaining contributions of the thesis are concerned with the inviscid limit of our solutions. The effect of viscosity in the presence of a boundary is an extensively studied and physically meaningful phenomenon, which is well summarised in [108] and has seen developments across [5, 18, 21, 56, 66, 76, 80, 94, 109, 114, 115, 116, 135, 137, 139, 148, 152] to list only some contributions in the theory, observation and numerics of this analysis. A key observation is the production of vorticity through large gradients of velocity at the boundary ([107, 117, 136, 143, 149]), suggesting that one requires a control on the vorticity near the boundary to deduce that the zero viscosity limit exists and is a solution of the Euler Equation (that is, the Navier-Stokes Equation with ν set formally to zero). For Euler, the impermeability boundary condition $u \cdot \mathbf{n} = 0$ is the only reasonable option. In the classical setting of the no-slip boundary condition, it is completely unclear if such a control is possible outside of very specific cases regarding regularity of initial data or structure of the domain ([103, 104, 110, 133, 134]). The problem is in general wide open, with the leading contribution that of Kato in [79] who mathematically verifies the physical intuition that it is necessary and sufficient to control gradients of the velocity in a boundary strip of width approaching zero with viscosity. A positive result seems much more reasonable for the Navier boundary conditions, given that derivatives of velocity are tractable at the boundary. More than this, the theoretical attraction of the Navier boundary conditions (1.4) lies largely in their relation to vorticity, as if w is the vorticity of the fluid and $\kappa \in C^2(\partial\mathscr{O};\mathbb{R})$ denotes the curvature of the boundary then (1.4) is equivalent to

$$u \cdot \mathbf{n} = 0, \qquad w = (2\kappa - \alpha)u \cdot \iota. \tag{1.5}$$

This connection stands at the forefront of study into the inviscid limit, most clearly in the specific case of $\alpha = 2\kappa$ (the 'free boundary' condition) shown first in [97] though extended to general α in [22, 44, 81, 111] for example. Of course, the aforementioned results are in the deterministic setting; much less is known about the zero viscosity limit in the presence of stochasticity, and nothing is prior known for a transport type noise. Our contributions towards this literature form Chapter 6.

We now more precisely detail the structure of the thesis, including any already released papers upon which the chapters are based.

- Chapter 1 is the introduction, this chapter, which rounds off with some elementary notation to setup the thesis.

- Chapter 2 details the framework and toolkit used to understand and solve the SPDEs of this thesis. We explicitly pose Stratonovich SPDEs with unbounded noise, deducing a rigorous conversion to Itô form for strong solutions: this comes at the cost of a derivative from the first order noise, which we will typically avoid asking for, though a thorough conversion should be proved and we do not believe it has appeared. Additionally, we present techniques in the study of SPDEs which will be necessary in subsequent chapters. Of note, we first prove an existence and uniqueness result for an SPDE evolving in a finite dimensional Hilbert Space, under the classical Lipschitz and linear growth assumptions, driven by a Cylindrical Brownian Motion. If the Brownian Motion were finite dimensional then this would be classical, but an extension to the infinite dimensional noise seems absent. We further solidify criteria to obtain tightness of the laws of Hilbert Space valued processes, and extend a result of Glatt-Holtz and Ziane to deduce the existence of a limit process by a Cauchy argument.

- Chapter 3 presents an extended variational framework for nonlinear SPDEs with unbounded noise, used to obtain solutions of (1.1). Three different solution types of increasing strength are defined, along with criteria to establish their existence. The three notions can be understood as probabilistically and analytically weak, probabilistically strong and analytically weak, as well as probabilistically and analytically strong. The definitive novelty compared to prior well-posedness frameworks comes from the unbounded noise. This chapter is based on the author's paper [62].

- Chapter 4 offers a more comprehensive introduction to the equation (1.2), including a description of how we pose this equation in the form (1.1). Fundamental properties of the operators in (1.2) are detailed, as well as the function spaces relative to the boundary conditions (1.3), (1.4). The vorticity form of the equation is also presented, which will be necessary in considering the vanishing viscosity limit under Navier Boundary Conditions.

The bulk of this chapter can be found across the author's papers [61, 63].

- Chapter 5 illustrates applications of Chapter 3 to the equation (1.2), as set up in Chapter 4. Martingale weak solutions in 3D are recovered for the no-slip boundary, alongside weak solutions in 2D. Whilst both weak and strong solutions in 2D are verified under Navier boundary conditions, the existence problem for strong solutions in the no-slip case remains unsolved. To supplement the analysis we overview local strong existence results in 3D from the author's paper [63], only for the torus and the vorticity form of the equation with 'free boundary' $u \cdot \mathbf{n} = 0$, $\text{curl}u = 0$; no such results are currently available for the no-slip or general Navier boundary conditions.

- Chapter 6 consists of our inviscid limit results for (1.2) under the no-slip condition, as well as a special case of the Navier boundary conditions which imply a free boundary. For the former, we introduce an analogue to Kato's Criterion regarding the convergence of stochastic Navier-Stokes flows with a scaled noise to the strong solution of the deterministic Euler Equation. This is achieved firstly for the typical noise scaling of $\nu^{\frac{1}{2}}$, before considering a new parameter which approaches zero with viscosity but at a potentially different rate. We determine the implications of this for our criterion and clarify a sense in which the scaling by $\nu^{\frac{1}{2}}$ is optimal. The criterion holds in both two and three dimensions, with some technical simplifications in 2D. Under the latter Navier boundary conditions, as well as imposing a convex domain, the inviscid limit exists for the unscaled noise and is a probabilistically weak, analytically weak solution of the corresponding Stochastic Euler Equation. These results are proven in the author's papers [64, 61] respectively.

- Chapter 7 is the conclusion, summarising the key contributions of the thesis and what future work it may enable or inspire.

- Appendix A contains supplementary material for the core of the thesis. Of note, we present a short course on Stochastic Calculus in Infinite Dimensions, properly constructing the infinite dimensional stochastic integral as we interpret it, along with core results in this calculus used throughout our work and particularly in Chapter 2. These notes are given in the author's work [60], also containing the material of Chapter 2, which will be published by Springer with co-author Dan Crisan. We also state the key results from the paper [65] which are not proved in this thesis due to space constraints, regarding a local-in-time existence theory for strong solutions with applications to (1.2) in 3D. Other classical results from the literature are also given.

1.1 Preliminary Notation

1.1.1 Function Spaces

Here and throughout the thesis $\mathscr{O} \subset \mathbb{R}^N$ will be a smooth bounded domain equipped with Euclidean norm and Lebesgue measure λ, with $N = 2$ or 3. We shall refer to these cases as 2D and 3D respectively. We consider Banach Spaces as measure spaces equipped with their corresponding Borel σ-algebra. Let $(\mathcal{X}, \mu)$ denote a general topological measure space, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ and $(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ be separable Banach Spaces, and $(\mathcal{U}, \langle\cdot,\cdot\rangle_{\mathcal{U}})$, $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be general separable Hilbert spaces. We introduce the following spaces of functions.

- $L^p(\mathcal{X};\mathcal{Y})$ is the class of measurable $p-$integrable functions from $\mathcal{X}$ into $\mathcal{Y}$, $1 \leq p < \infty$, which is a Banach space with norm
$$\|\phi\|^p_{L^p(\mathcal{X};\mathcal{Y})} := \int_{\mathcal{X}} \|\phi(x)\|^p_{\mathcal{Y}}\, \mu(dx).$$
In particular $L^2(\mathcal{X};\mathcal{Y})$ is a Hilbert Space when $\mathcal{Y}$ itself is Hilbert, with the standard inner product
$$\langle \phi, \psi\rangle_{L^2(\mathcal{X};\mathcal{Y})} = \int_{\mathcal{X}} \langle \phi(x), \psi(x)\rangle_{\mathcal{Y}}\, \mu(dx).$$
In the case $\mathcal{X} = \mathscr{O}$ and $\mathcal{Y} = \mathbb{R}^N$ note that
$$\|\phi\|^2_{L^2(\mathscr{O};\mathbb{R}^N)} = \sum_{l=1}^{2} \left\|\phi^l\right\|^2_{L^2(\mathscr{O};\mathbb{R})}, \qquad \phi = \left(\phi^1, \dots, \phi^N\right), \quad \phi^l : \mathscr{O} \to \mathbb{R}.$$
We denote $\|\cdot\|_{L^p(\mathscr{O};\mathbb{R}^N)}$ by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^2(\mathscr{O};\mathbb{R}^N)}$ by $\|\cdot\|$, where the dimension N will be clear from the context.

- $L^\infty(\mathcal{X};\mathcal{Y})$ is the class of measurable functions from $\mathcal{X}$ into $\mathcal{Y}$ which are essentially bounded. $L^\infty(\mathcal{X};\mathcal{Y})$ is a Banach Space when equipped with the norm
$$\|\phi\|_{L^\infty(\mathcal{X};\mathcal{Y})} := \inf\{C \geq 0 : \|\phi(x)\|_Y \leq C \text{ for } \mu\text{-}a.e.\ x \in \mathcal{X}\}.$$

- $C(\mathcal{X};\mathcal{Y})$ is the space of continuous functions from $\mathcal{X}$ into $\mathcal{Y}$.

- $C_w(\mathcal{X};\mathcal{Y})$ is the space of 'weakly continuous' functions from $\mathcal{X}$ into $\mathcal{Y}$, by which we mean continuous with respect to the given topology on $\mathcal{X}$ and the weak topology on $\mathcal{Y}$.

- $C^m(\mathscr{O};\mathbb{R})$ is the space of $m \in \mathbb{N}$ times continuously differentiable functions from $\mathscr{O}$ to $\mathbb{R}$, that is $\phi \in C^m(\mathscr{O};\mathbb{R})$ if and only if for every N dimensional multi index $\alpha = \alpha_1, \dots, \alpha_N$ with $|\alpha| \leq m$, $D^\alpha\phi \in C(\mathscr{O};\mathbb{R})$ where D^α is the corresponding classical derivative operator $\partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_N}_{x_N}$.

- $C^\infty(\mathscr{O};\mathbb{R})$ is the intersection over all $m \in \mathbb{N}$ of the spaces $C^m(\mathscr{O};\mathbb{R})$.

- $C^m_0(\mathscr{O};\mathbb{R})$ for $m \in \mathbb{N}$ or $m = \infty$ is the subspace of $C^m(\mathscr{O};\mathbb{R})$ of functions which have compact support.

- $C^m(\mathscr{O};\mathbb{R}^N), C_0^m(\mathscr{O};\mathbb{R}^N)$ for $m\in\mathbb{N}$ or $m=\infty$ is the space of functions from $\mathscr{O}$ to $\mathbb{R}^N$ whose component mappings each belong to $C^m(\mathscr{O};\mathbb{R}), C_0^m(\mathscr{O};\mathbb{R})$.

- $W^{m,p}(\mathscr{O};\mathbb{R})$ for $1\le p<\infty$ is the sub-class of $L^p(\mathscr{O},\mathbb{R})$ which has all weak derivatives up to order $m\in\mathbb{N}$ also of class $L^p(\mathscr{O},\mathbb{R})$. This is a Banach space with norm

$$\|\phi\|^p_{W^{m,p}(\mathscr{O},\mathbb{R})} := \sum_{|\alpha|\le m} \|D^\alpha\phi\|^p_{L^p(\mathscr{O};\mathbb{R})},$$

where D^α is the corresponding weak derivative operator. In the case $p=2$ the space $W^{m,2}(\mathscr{O},\mathbb{R})$ is Hilbert with inner product

$$\langle\phi,\psi\rangle_{W^{m,2}(\mathscr{O};\mathbb{R})} := \sum_{|\alpha|\le m}\langle D^\alpha\phi, D^\alpha\psi\rangle_{L^2(\mathscr{O};\mathbb{R})}.$$

- $W^{m,\infty}(\mathscr{O};\mathbb{R})$ for $m\in\mathbb{N}$ is the sub-class of $L^\infty(\mathscr{O},\mathbb{R})$ which has all weak derivatives up to order $m\in\mathbb{N}$ also of class $L^\infty(\mathscr{O},\mathbb{R})$. This is a Banach space with norm

$$\|\phi\|_{W^{m,\infty}(\mathscr{O},\mathbb{R})} := \sup_{|\alpha|\le m}\|D^\alpha\phi\|_{L^\infty(\mathscr{O};\mathbb{R}^N)}.$$

- $W^{s,p}(\mathscr{O};\mathbb{R})$ for $0<s<1$ and $1\le p<\infty$ is the sub-class of functions $\phi\in L^p(\mathscr{O},\mathbb{R})$ such that

$$\int_{\mathscr{O}\times\mathscr{O}}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{sp+2}}d\lambda(x,y)<\infty.$$

This is a Banach space with respect to the norm

$$\|\phi\|^p_{W^{s,p}(\mathscr{O};\mathbb{R})} = \|\phi\|^p_{L^p(U;\mathbb{R})} + \int_{\mathscr{O}\times\mathscr{O}}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{sp+2}}d\lambda(x,y).$$

For $p=2$ this is a Hilbert space with inner product

$$\langle\phi,\psi\rangle_{W^{s,2}(\mathscr{O};\mathbb{R})} = \langle\phi,\psi\rangle_{L^2(\mathscr{O};\mathbb{R})} + \int_{\mathscr{O}\times\mathscr{O}}\frac{\big(\phi(x)-\phi(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{2s+2}}d\lambda(x,y).$$

- $W^{s,p}(\mathscr{O};\mathbb{R})$ for $1\le s<\infty$, $s\notin\mathbb{N}$ and $1\le p<\infty$ is, using the notation $\lfloor s\rfloor$ to mean the integer part of s, the sub-class of $W^{\lfloor s\rfloor,p}(\mathscr{O};\mathbb{R})$ such that the distributional derivatives $D^\alpha\phi$ belong to $W^{s-\lfloor s\rfloor,p}(\mathscr{O};\mathbb{R})$ for every multi-index α such that $|\alpha|=\lfloor s\rfloor$. This is a Banach space with norm

$$\|\phi\|^p_{W^{s,p}(\mathscr{O};\mathbb{R})} = \|\phi\|^p_{W^{\lfloor s\rfloor,p}(\mathscr{O};\mathbb{R})} + \sum_{|\alpha|=\lfloor s\rfloor}\int_{\mathscr{O}\times\mathscr{O}}\frac{|D^\alpha\phi(x)-D^\alpha\phi(y)|^p}{|x-y|^{(s-\lfloor s\rfloor)p+2}}dxdy$$

and a Hilbert space in the case $p=2$, with inner product

$$\langle\phi,\psi\rangle_{W^{s,2}(\mathscr{O};\mathbb{R})} = \langle\phi,\psi\rangle_{W^{\lfloor s\rfloor,2}(\mathscr{O};\mathbb{R})} + \int_{\mathscr{O}\times\mathscr{O}}\frac{\big(D^\alpha\phi(x)-D^\alpha\phi(y)\big)\big(D^\alpha\psi(x)-D^\alpha\psi(y)\big)}{|x-y|^{2(s-\lfloor s\rfloor)+2}}d\lambda(x,y).$$

- $W^{s,p}(\mathscr{O};\mathbb{R}^N)$ for $s>0$ and $1\leq p<\infty$ is the Banach space of functions $\phi:\mathscr{O}\to\mathbb{R}^N$ whose components (ϕ^l) are each elements of the space $W^{s,p}(\mathscr{O};\mathbb{R})$. The associated norm is
$$\|\phi\|^p_{W^{s,p}}=\sum_{l=1}^N\left\|\phi^l\right\|^p_{W^{s,p}(\mathscr{O};\mathbb{R})}$$
and similarly when $p=2$ this space is Hilbert with inner product
$$\langle\phi,\psi\rangle_{W^{s,2}}=\sum_{l=1}^N\left\langle\phi^l,\psi^l\right\rangle_{W^{s,2}(\mathscr{O},\mathbb{R})}.$$

- $W^{m,\infty}(\mathscr{O};\mathbb{R}^N)$ is the sub-class of $L^\infty(\mathscr{O},\mathbb{R}^N)$ which has all weak derivatives up to order $m\in\mathbb{N}$ also of class $L^\infty(\mathscr{O},\mathbb{R}^N)$. This is a Banach space with norm
$$\|\phi\|_{W^{m,\infty}}:=\sup_{l\leq N}\left\|\phi^l\right\|_{W^{m,\infty}(\mathscr{O};\mathbb{R})}.$$

- $W_0^{m,p}(\mathscr{O};\mathbb{R}),W_0^{m,p}(\mathscr{O};\mathbb{R}^N)$ for $m\in\mathbb{N}$ and $1\leq p\leq\infty$ is the closure of $C_0^\infty(\mathscr{O};\mathbb{R}),C_0^\infty(\mathscr{O};\mathbb{R}^N)$ in $W^{m,p}(\mathscr{O};\mathbb{R}),W^{m,p}(\mathscr{O};\mathbb{R}^N)$. $W_0^{m,p}(\mathscr{O};\mathbb{R})$ is characterised as the subspace of $W^{m,p}(\mathscr{O};\mathbb{R})$ consisting of zero-trace functions (see e.g. [43] pp.259).

- $\mathscr{L}(\mathcal{Y};\mathcal{Z})$ is the space of bounded linear operators from $\mathcal{Y}$ to $\mathcal{Z}$. This is a Banach Space when equipped with the norm
$$\|F\|_{\mathscr{L}(\mathcal{Y};\mathcal{Z})}=\sup_{\|y\|_{\mathcal{Y}}=1}\|Fy\|_{\mathcal{Z}}$$
and is simply the dual space $\mathcal{Y}^*$ when $\mathcal{Z}=\mathbb{R}$, with operator norm $\|\cdot\|_{\mathcal{Y}^*}$.

- $\mathscr{L}^2(\mathcal{U};\mathcal{H})$ is the space of Hilbert-Schmidt operators from $\mathcal{U}$ to $\mathcal{H}$, defined as the elements $F\in\mathscr{L}(\mathcal{U};\mathcal{H})$ such that for some basis (e_i) of $\mathcal{U}$,
$$\sum_{i=1}^\infty\|Fe_i\|^2_{\mathcal{H}}<\infty.$$
This is a Hilbert space with inner product
$$\langle F,G\rangle_{\mathscr{L}^2(\mathcal{U};\mathcal{H})}=\sum_{i=1}^\infty\langle Fe_i,Ge_i\rangle_{\mathcal{H}}$$
which is independent of the choice of basis (see e.g. [26]).

- For any $T>0$, $\mathscr{I}_T$ is the subspace of $C([0,T];[0,T])$ of strictly increasing functions.

- For any $T>0$, $\mathcal{D}([0,T];\mathcal{Y})$ is the space of cádlág functions from $[0,T]$ into $\mathcal{Y}$. It is a complete separable metric space when equipped with the metric
$$d(\phi,\psi):=\inf_{\eta\in\mathscr{I}_T}\left[\sup_{t\in[0,T]}|\eta(t)-t|\vee\sup_{t\in[0,T]}\|\phi(t)-\psi(\eta(t))\|_{\mathcal{Y}}\right]$$

which induces the so called Skorohod Topology (see [11] pp124 for details).

We will consider a partial ordering on $N-$dimensional multi-indices by $\alpha \leq \beta$ if and only if for all $l = 1, \ldots, N$ we have that $\alpha_l \leq \beta_l$. We extend this to notation $<$ by $\alpha < \beta$ if and only if $\alpha \leq \beta$ and for some $l = 1, \ldots, N$, $\alpha_l < \beta_l$.

1.1.2 Probabilistic Setup

Throughout the thesis, $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ will be a fixed filtered probability space satisfying the usual conditions of completeness and right continuity. We take $\mathcal{W}$ to be a Cylindrical Brownian Motion (Subsection A.1.3) over some Hilbert Space $\mathfrak{U}$ with orthonormal basis (e_i). Noting (A.23), $\mathcal{W}$ admits the representation $\mathcal{W}_t = \sum_{i=1}^{\infty} e_i W_t^i$ as a limit in $L^2(\Omega; \mathfrak{U}')$ whereby the (W^i) are a collection of independent standard real valued Brownian Motions and $\mathfrak{U}'$ is an enlargement of the Hilbert Space $\mathfrak{U}$ such that the embedding $J : \mathfrak{U} \to \mathfrak{U}'$ is Hilbert-Schmidt and $\mathcal{W}$ is a JJ^*-cylindrical Brownian Motion over $\mathfrak{U}'$. Given a process $F : [0, T] \times \Omega \to \mathscr{L}^2(\mathfrak{U}; \mathscr{H})$ progressively measurable and such that $F \in L^2\left(\Omega \times [0, T]; \mathscr{L}^2(\mathfrak{U}; \mathscr{H})\right)$, for any $0 \leq t \leq T$ we define the stochastic integral

$$\int_0^t F_s d\mathcal{W}_s := \sum_{i=1}^{\infty} \int_0^t F_s(e_i) dW_s^i,$$

where the infinite sum is taken in $L^2(\Omega; \mathscr{H})$ (Definition A.1.46). We can extend this notion to processes F which are such that $F(\omega) \in L^2\left([0, T]; \mathscr{L}^2(\mathfrak{U}; \mathscr{H})\right)$ for $\mathbb{P} - a.e.\ \omega$ via the traditional localisation procedure (Definition A.1.49). We shall always pose our SPDEs on a time interval $[0, T]$ where $T > 0$ is arbitrary but remains fixed; henceforth, we fix such an arbitrary T.

Chapter 2

A Primer on Stochastic Partial Differential Equations

2.1 Introduction

We will now describe the structure of and key contributions from this chapter: we build upon the stochastic calculus developed in Appendix A.1, from which some relevant notation is listed at the end of this section.

- Section 2.2 details a framework for the study of SPDEs, which are evolution equations involving integration of the form introduced in Appendix A.1. Through this framework we define notions of solutions for the abstract SPDE (1.1), motivated in particular by the recent attention given to transport type noise and Stratonovich equations (such as (1.2)). A rigorous mathematical understanding of these equations appears perilous for two key reasons. The first is the gradient dependency in the noise, taking us beyond the most general 'variational frameworks' seen in the literature as these are posed for a noise operator which is bounded on some Hilbert Space. The second is the Stratonovich integration, which we are likely to only understand as a corrected Itô integral, yet this conversion is highly nontrivial for a noise which is not bounded on a Hilbert Space. Of course the mere consideration of *nonlinear* SPDEs, such as the evolution equations of fluid dynamics, renders the well established linear theory insufficient.

  To be precise, we present a framework which shares its spirit with the variational approach to SPDEs pioneered by Pardoux in the 1970s and now best represented in the more recent books [102, 125, 127]. This classical framework considers an evolution equation with respect to a Gelfand Triple, say $V \hookrightarrow H \hookrightarrow V^*$, where solutions have paths which are square integrable in V, continuous in H and satisfying an identity in V^*. Here we recall our motivation of the Stochastic Navier-Stokes Equation; whilst analytically weak solutions fit this framework seamlessly, analytically strong solutions fit to the spaces $W^{2,2} \hookrightarrow W^{1,2} \hookrightarrow L^2$ which prompts our shift to a triplet of embedded Hilbert Spaces without any necessary duality structure (whilst L^2 could well be the dual of

$W^{2,2}$ with respect to the $W^{1,2}$ inner product, there is a finer structure to these spaces with boundary conditions to be imposed, which threaten the duality). Furthermore the aforementioned works allow only for a noise operator bounded in H (thus not of first order in applications), and do not consider Stratonovich equations. To include a Stratonovich transport type noise we in fact introduce a fourth Hilbert Space, necessary as the Itô-Stratonovich correction costs a derivative for transport type noise. We can then properly define weak, strong and local solutions of nonlinear PDEs with Stratonovich transport noise, alongside other more classical additive and multiplicative stochastic perturbations. We believe that presenting the technical details, in such generality, of these notions here facilitates the rigorous and free analysis of the equations in future works.

- Section 2.3 concludes this chapter and contains more advanced and novel techniques in the existence theory for nonlinear SPDEs. The beauty of the classical variational approach comes from the existence results, which certainly cannot be matched as elegantly in a framework built for 3D Navier-Stokes Equations and related stochastic fluid models. Instead we focus on techniques that can be used in this direction, centred around the *Galerkin Method* in which finite dimensional approximations of the SPDE are considered and some properties are used to deduce their limit. Immediately then an existence result for the finite dimensional equations is required, more precisely for where the Hilbert Space in which the equation evolves is finite dimensional but the driving Brownian Motion is still infinite dimensional. We assume standard Lipschitz and linear growth conditions, and to the best of our knowledge this result is not present in the literature. There are two predominant ways to deduce the existence of a limit of the finite dimensional approximations, which we detail now.

  The first is through *tightness*, which is the stochastic route to relative compactness arguments used in PDE theory. The idea is that from tightness we can deduce relative compactness of the laws of the processes over some suitable function space, at which point Skorohod's Representation Theorem enables the deduction of a limiting process almost surely on a new probability space. Criteria to deduce tightness in relevant function spaces are thus of great significance, and our criteria comes largely from the works of [2, 77, 131]. The second is through a Cauchy type argument in the relevant spaces, difficult to execute in the case of local solutions but recently this has been overcome to great effect due to Glatt-Holtz and Ziane in [59] and extended by us here. We defer a greater discussion of this highly technical result to Section 2.3.

2.1.1 Notation from Appendix A.1

In Appendix A.1, some notation is introduced for the spaces of integrable processes and martingales, as well as the quadratic variation and cross variation in Hilbert Spaces. This notation is summarised here.

- $\mathcal{I}^{\mathcal{H}}_T$ is defined in A.1.4;
- $\mathcal{I}^{\mathcal{H}}$ is defined in A.1.5;
- $\mathcal{M}^2, \mathcal{M}^2_c$ are defined in A.1.13;
- $\mathcal{M}^2_c(\mathcal{H})$ is defined in A.1.27;
- $\bar{\mathcal{M}}^2_c, \bar{\mathcal{M}}^2_c(\mathcal{H})$ are the corresponding semi-martingale spaces;
- $\mathcal{I}^{\mathcal{H},T}_M, \mathcal{I}^{\mathcal{H}}_M$ are defined in A.1.14;
- $\bar{\mathcal{I}}^{\mathcal{H},T}_N, \bar{\mathcal{I}}^{\mathcal{H}}_N$ are defined in A.1.15;
- $[\cdot]$ is defined in A.1.34;
- $[\cdot,\cdot]$ is defined in A.1.42;
- $\mathcal{I}^{\mathcal{H}}_T(\mathcal{W})$, $\mathcal{I}^{\mathcal{H}}(\mathcal{W})$ are defined in A.1.45;
- $\bar{\mathcal{I}}^{\mathcal{H}}_T(\mathcal{W})$, $\bar{\mathcal{I}}^{\mathcal{H}}(\mathcal{W})$ are defined in A.1.48;

The probabilistic notation for this chapter and throughout the thesis was set in Subsection 1.1.2.

2.2 Stochastic Differential Equations in Infinite Dimensions

2.2.1 The Stratonovich Integral

We look at first to define the Stratonovich Integral with respect to a one dimensional martingale, before doing so with respect to a Cylindrical Brownian Motion. The Itô stochastic integral is constructed in the Appendix, where we point to Definitions A.1.16, A.1.46 and A.1.49.

Definition 2.2.1. *For $M \in \bar{\mathcal{M}}^2_c$ and $\boldsymbol{\Psi} \in \mathcal{I}^{\mathcal{H}}_M \cap \bar{\mathcal{M}}^2_c(\mathcal{H})$, the Stratonovich stochastic integral is defined as*

$$\int_0^t \boldsymbol{\Psi}_s \circ dM_s := \int_0^t \boldsymbol{\Psi}_s dM_s + \frac{1}{2}[\boldsymbol{\Psi}, M]_t.$$

Definition 2.2.2. *For $B \in \mathcal{I}^{\mathcal{H}}(\mathcal{W})$ such that $B_{e_i} \in \bar{\mathcal{M}}^2_c(\mathcal{H})$ for every e_i and the limit*

$$\sum_{i=1}^{\infty}[B_{e_i}, W^i]_t \tag{2.1}$$

is well defined in $L^2(\Omega;\mathcal{H})$, the Stratonovich stochastic integral is defined as

$$\int_0^t B(s) \circ d\mathcal{W}_s := \sum_{i=1}^{\infty}\left(\int_0^t B_{e_i}(s)dW^i_s + \frac{1}{2}[B_{e_i}, W^i]_t\right)$$

where the limit is taken in $L^2(\Omega;\mathcal{H})$. The class of such processes will be denoted $\mathcal{I}^{\mathcal{H}}_\circ(\mathcal{W})$.

It will be necessary to extend this definition to processes $B \in \bar{\mathcal{I}}^{\mathcal{H}}(\mathcal{W})$, but we encounter more technical issues in trying to construct a sequence of stopping times such that the stopped process belongs to $\mathcal{I}^{\mathcal{H}}_{\circ}(\mathcal{W})$. We find it simplest to give the definition below.

Definition 2.2.3. *Suppose that there exists a sequence of stopping times (τ_n) which are $\mathbb{P}-a.s.$ monotone increasing and convergent to infinity such that:*

1. *For every n, the process*
$$B^n(\cdot) := B(\cdot)\mathbb{1}_{\cdot \leq \tau^n}$$
belongs to $\mathcal{I}^{\mathcal{H}}(\mathcal{W})$;
2. *For every n and i, the process*
$$B^{\tau_n}_{e_i}(\cdot) := B_{e_i}(\cdot \wedge \tau^n)$$
belongs to $\bar{\mathcal{M}}^2_c(\mathcal{H})$;
3. *The limit*
$$\sum_{i=1}^{\infty}[B^{\tau_n}_{e_i}, W^i]_t$$
is well defined in $L^2(\Omega; \mathcal{H})$.

Then the Stratonovich stochastic integral is defined at any $t \geq 0$ as
$$\int_0^t B(s) \circ d\mathcal{W}_s := \lim_{n\to\infty}\left(\sum_{i=1}^{\infty}\left(\int_0^t B^n_{e_i}(s)dW^i_s + \frac{1}{2}[B^{\tau_n}_{e_i}, W^i]_t\right)\right)$$
where the limit is taken $\mathbb{P}-a.s.$ in $\mathcal{H}$, and the infinite sum in $L^2(\Omega; \mathcal{H})$. The class of such processes will be denoted $\bar{\mathcal{I}}^{\mathcal{H}}_{\circ}(\mathcal{W})$.

This definition is of course completely analogous to the localisation procedure used in the previous constructions, for example Definition A.1.49, except we postulate in the first instance the existence of the localising sequence (τ_n). There does not appear to be a simple canonical way to construct such a sequence of localising times whereby each $B^{\tau_n}_{e_i} \in \bar{\mathcal{M}}^2_c(\mathcal{H})$ if one assumes only local martingality of these component processes, hence we assume the required properties explicitly.

2.2.2 Strong Solutions in the Abstract Framework

We now establish a framework in which to formulate SPDEs, posed for a quartet of embedded Hilbert Spaces
$$V \hookrightarrow H \hookrightarrow U \hookrightarrow X$$
where the embedding is meant as a continuous linear injection. Each Hilbert Space plays a role in defining solutions, and this structure of embeddings allows us to consider operators which are not bounded on a given Hilbert Space (particularly nonlinear and differential operators in

applications). We introduce at first the Itô SPDE

$$\boldsymbol{\Psi}_t = \boldsymbol{\Psi}_0 + \int_0^t \mathcal{Q}\boldsymbol{\Psi}_s ds + \int_0^t \mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s \tag{2.2}$$

where $\mathcal{W}$ continues to be a Cylindrical Brownian Motion over $\mathfrak{U}$ relative to our fixed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with representation (A.37). We impose now some conditions on the operators relative to these spaces. To do so we define the general map $\tilde{K} = \tilde{K}_{c,p,q} : H \to \mathbb{R}$ by

$$\tilde{K}(\phi) := c\left(1 + \|\phi\|_U^p + \|\phi\|_H^q\right)$$

for any constants c, p, q independent of ϕ.

Assumption 2.2.4. *$\mathcal{Q} : V \to U$ is measurable and there exists a map $\tilde{K} = \tilde{K}_{c,p,q}$ such that for any $\phi \in V$,*

$$\|\mathcal{Q}\phi\|_U \leq \tilde{K}(\phi)[1 + \|\phi\|_V^2].$$

Assumption 2.2.5. *$\mathcal{G}$ is understood as a measurable operator*

$$\mathcal{G} : U \to \mathscr{L}^2(\mathfrak{U}; X), \qquad \mathcal{G}|_H : H \to \mathscr{L}^2(\mathfrak{U}; U), \qquad \mathcal{G}|_V : V \to \mathscr{L}^2(\mathfrak{U}; H)$$

defined over $\mathfrak{U}$ by its action on the basis vectors

$$\mathcal{G}(\cdot, e_i) := \mathcal{G}_i(\cdot).$$

Each $\mathcal{G}_i$ is linear and there exists constants c_i such that for all $\phi \in V$, $\psi \in H$, $\eta \in U$:

$$\|\mathcal{G}_i\eta\|_X \leq c_i \|\eta\|_U, \qquad \|\mathcal{G}_i\psi\|_U \leq c_i \|\psi\|_H, \qquad \|\mathcal{G}_i\phi\|_H \leq c_i \|\phi\|_V$$
$$\sum_{i=1}^{\infty} c_i^2 < \infty.$$

Here $\mathcal{G}_i$ is assumed linear for the purposes of the Stratonovich conversion in Subsection 2.2.4, but would not otherwise be necessary. It is worth clarifying how $\mathcal{G}$ is defined over $\mathfrak{U}$: fix a $\phi \in V$ and consider $\mathcal{G}(\phi, \cdot) : \mathfrak{U} \to H$ (the arguments here apply for the larger spaces as well). Any $\alpha \in \mathfrak{U}$ has the representation

$$\sum_{i=1}^{\infty} \langle \alpha, e_i \rangle_{\mathfrak{U}} e_i$$

where

$$\sum_{i=1}^{\infty} \langle \alpha, e_i \rangle_{\mathfrak{U}}^2 < \infty.$$

Then

$$\mathcal{G}(\phi, \cdot) : \alpha \mapsto \sum_{i=1}^{\infty} \langle \alpha, e_i \rangle_{\mathfrak{U}} \mathcal{G}_i\phi$$

is well defined as an element of H. This is justified by showing that the sequence of partial sums is Cauchy in H: note that from Cauchy-Schwarz,

$$\left\|\sum_{i=m}^{n}\langle\alpha,e_i\rangle_{\mathfrak{U}}\mathcal{G}_i\phi\right\|_H^2\leq\left(\sum_{i=m}^{n}\langle\alpha,e_i\rangle_{\mathfrak{U}}^2\right)\left(\sum_{i=m}^{n}\|\mathcal{G}_i\phi\|_H^2\right)\leq\left(\sum_{i=m}^{\infty}\langle\alpha,e_i\rangle_{\mathfrak{U}}^2\right)\left(\sum_{i=m}^{\infty}c_i^2\|\phi\|_V^2\right)$$

which approaches zero as $m\to\infty$ as the sums are finite. We introduce now the first notion of our strong solutions.

Definition 2.2.6. *Let $\boldsymbol{\Psi}_0:\Omega\to H$ be $\mathcal{F}_0-$measurable. A pair $(\boldsymbol{\Psi},\tau)$ where τ is a $\mathbb{P}-a.s.$ positive stopping time and $\boldsymbol{\Psi}$ is a process such that for $\mathbb{P}-a.e.$ ω, $\boldsymbol{\Psi}_\cdot(\omega)\in C_w([0,T];H)$ and $\boldsymbol{\Psi}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)}\in L^2([0,T];V)$ with $\boldsymbol{\Psi}_\cdot\mathbb{1}_{\cdot\leq\tau}$ progressively measurable in V, is said to be a local strong solution of the equation (2.2) if the identity*

$$\boldsymbol{\Psi}_t=\boldsymbol{\Psi}_0+\int_0^{t\wedge\tau}\mathcal{Q}\boldsymbol{\Psi}_s ds+\int_0^{t\wedge\tau}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s \tag{2.3}$$

holds $\mathbb{P}-a.s.$ in U for all $t\in[0,T]$.

Remark 2.2.7. *Note that if $(\boldsymbol{\Psi},\tau)$ is a local strong solution of the equation (2.2), then $\boldsymbol{\Psi}_\cdot=\boldsymbol{\Psi}_{\cdot\wedge\tau}$ due to the integral representation (2.3). That is to say, the solution is automatically stopped at the local time of existence. Moreover the progressive measurability condition on $\boldsymbol{\Psi}_\cdot\mathbb{1}_{\cdot\leq\tau}$ may look a little suspect as $\boldsymbol{\Psi}_0$ itself may only belong to H and not V making it impossible for $\boldsymbol{\Psi}_\cdot\mathbb{1}_{\cdot\leq\tau}$ to be even adapted in V. We are mildly abusing notation here; what we really ask is that there exists a process $\boldsymbol{\Phi}$ which is progressively measurable in V and such that $\boldsymbol{\Phi}_\cdot=\boldsymbol{\Psi}_\cdot\mathbb{1}_{\cdot\leq\tau}$ almost surely over the product space $\Omega\times[0,T]$ with product measure $\mathbb{P}\times\lambda$. In particular, the processes $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ may disagree at time zero.*

Let us take a few moments to process this definition and ensure that the integrals make sense with the given regularity of the solution and the operators $\mathcal{Q}$, $\mathcal{G}$. The time integral in (2.3) is well defined in U as a Bochner Integral: first of all $\boldsymbol{\Psi}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)}\in L^2([0,T];V)$ for $\mathbb{P}-a.e.$ ω so $\boldsymbol{\Psi}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)}:[0,t]\to V$ is measurable hence $\mathcal{Q}(\boldsymbol{\Psi}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)}):[0,t]\to U$ is measurable from Assumption 2.2.4. Moreover the mapping $\mathcal{Q}(\boldsymbol{\Psi}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)})\mathbb{1}_{\cdot\leq\tau(\omega)}:[0,t]\to U$ is again measurable and we have that

$$\int_0^{t\wedge\tau(\omega)}\mathcal{Q}\left(\boldsymbol{\Psi}_s(\omega)\right)ds=\int_0^t\mathcal{Q}\left(\boldsymbol{\Psi}_s(\omega)\mathbb{1}_{s\leq\tau(\omega)}\right)\mathbb{1}_{s\leq\tau(\omega)}ds$$

so the required measurability in order to define the integral is satisfied. In this vein we have that

$$\begin{aligned}\int_0^{t\wedge\tau(\omega)}\|\mathcal{Q}\left(\boldsymbol{\Psi}_s(\omega)\right)\|_U ds&\leq\int_0^{t\wedge\tau(\omega)}\tilde{K}\left(\boldsymbol{\Psi}_s(\omega)\right)\left[1+\|\boldsymbol{\Psi}_s(\omega)\|_V^2\right]ds\\&\leq\sup_{s\in[0,t\wedge\tau(\omega)]}\left[\tilde{K}\left(\boldsymbol{\Psi}_s(\omega)\right)\right]\int_0^{t\wedge\tau(\omega)}1+\|\boldsymbol{\Psi}_s(\omega)\|_V^2 ds\\&<\infty\end{aligned}$$

employing Assumption 2.2.4 again and using the regularity specified by the local solution to deduce finiteness, which justifies that the integral is well defined.

As for the stochastic integral in (2.3), this is well defined in the sense of Definition A.1.49 in H (which then embeds into U), though it is done so formally via the process $\boldsymbol{\Phi}$ stipulated in Remark 2.2.7. We understand again that

$$\int_0^{t\wedge\tau}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s = \int_0^t \mathcal{G}(\boldsymbol{\Psi}_s\mathbb{1}_{s\leq\tau})\mathbb{1}_{s\leq\tau}d\mathcal{W}_s$$

should the integral be well defined, and so we actually define the integral by

$$\int_0^{t\wedge\tau}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s := \int_0^t \mathcal{G}\boldsymbol{\Phi}_s\mathbb{1}_{s\leq\tau}d\mathcal{W}_s.$$

The progressive measurability of $\mathcal{G}\boldsymbol{\Phi}\mathbb{1}_{\cdot\leq\tau} : [0,t]\times\Omega \to \mathscr{L}^2(\mathfrak{U};H)$ is immediate from the measurability assumption of 2.2.5, the same arguments above for the time integral and the progressive measurability of $\mathbb{1}_{\cdot\leq\tau}$ coming from the definition of the stopping time. Similarly we have that

$$\int_0^{t\wedge\tau}\sum_{i=1}^{\infty}\left\|\mathcal{G}_i\left(\boldsymbol{\Phi}_s(\omega)\right)\right\|_H^2 ds = \int_0^{t\wedge\tau}\sum_{i=1}^{\infty}\left\|\mathcal{G}_i\left(\boldsymbol{\Psi}_s(\omega)\mathbb{1}_{s\leq\tau}\right)\right\|_H^2 ds < \infty$$

$\mathbb{P}-a.s.$, using again Assumption 2.2.5, validating that the stochastic integral is well defined as a local martingale in H and thus in U from the continuous embedding. It should be observed that moment estimates are not assumed for our solution (or even the initial condition), so we cannot say that the stochastic integral is genuinely a square integrable martingale. We clarify another potential source of ambiguity with the following lemma, around whether there could be a meaningful dependence of the set of full probability on t used to satisfy (2.3) of Definition 2.2.6. The lemma justifies that it is sufficient to check the identity (2.3) at each $t\in[0,T]$ $\mathbb{P}-a.s.$.

Lemma 2.2.8. *Suppose that there exists a set $\bar{\Omega}\subset\Omega$, $\mathbb{P}(\bar{\Omega})=1$, and a pair $(\boldsymbol{\Psi},\tau)$ where τ is positive on $\bar{\Omega}$ and $\boldsymbol{\Psi}$ is a process such that for every $\omega\in\bar{\Omega}$, $\boldsymbol{\Psi}_\cdot(\omega)\in L^\infty([0,T];H)$ and $\boldsymbol{\Psi}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)}\in L^2([0,T];V)$. In addition, suppose that for every $t\in[0,T]$ there exists a set $\Omega_t\subset\Omega$, $\mathbb{P}(\Omega_t)=1$ whereby the identity (2.3) holds on Ω_t. Then there exists a set $\check{\Omega}\subset\Omega$, $\mathbb{P}(\check{\Omega})=1$, and a process $\check{\boldsymbol{\Psi}}$ such that for every $\omega\in\check{\Omega}$, $\check{\boldsymbol{\Psi}}_\cdot(\omega)\in L^\infty([0,T];H)$ and $\check{\boldsymbol{\Psi}}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)}\in L^2([0,T];V)$, where $\check{\boldsymbol{\Psi}}$ satisfies the identity*

$$\check{\boldsymbol{\Psi}}_t = \boldsymbol{\Psi}_0 + \int_0^{t\wedge\tau}\mathcal{Q}\check{\boldsymbol{\Psi}}_s ds + \int_0^{t\wedge\tau}\mathcal{G}\check{\boldsymbol{\Psi}}_s d\mathcal{W}_s \tag{2.4}$$

for all $\omega\in\check{\Omega}$ and $t\in[0,T]$. Furthermore, if we assume that for every $\omega\in\bar{\Omega}$, $\boldsymbol{\Psi}_\cdot(\omega)\in C([0,T];H)$, then we can choose $\check{\Omega}$ and $\check{\boldsymbol{\Psi}}$ such that $\boldsymbol{\Psi}_t(\omega)=\check{\boldsymbol{\Psi}}_t(\omega)$ for all $\omega\in\check{\Omega}$, $t\in[0,T]$.

Proof. We consider all rational times (t_k) in $[0,T]$, and construct

$$\check{\Omega} = \bar{\Omega} \cap \bigcap_k \Omega_{t_k}$$

as well as the process $\check{\boldsymbol{\Psi}}$ on $\check{\Omega} \times [0,T]$ by

$$\check{\boldsymbol{\Psi}}_t = \boldsymbol{\Psi}_0 + \int_0^{t\wedge\tau} \mathcal{Q}\boldsymbol{\Psi}_s ds + \int_0^{t\wedge\tau} \mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s.$$

Clearly $\mathbb{P}(\check{\Omega}) = 1$ and $\boldsymbol{\Psi}_{t_k}(\omega) = \check{\boldsymbol{\Psi}}_{t_k}(\omega)$ for all $\omega \in \check{\Omega}$, so $\boldsymbol{\Psi}_\cdot(\omega)$ and $\check{\boldsymbol{\Psi}}_\cdot(\omega)$ are equal $\lambda - a.s.$ on $[0,T]$ hence $\check{\boldsymbol{\Psi}}_\cdot(\omega) \in L^\infty([0,T];H)$ and $\check{\boldsymbol{\Psi}}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)} \in L^2([0,T];V)$. Due to this equivalence, then on $\check{\Omega}$,

$$\int_0^{t\wedge\tau} \mathcal{Q}\boldsymbol{\Psi}_s ds = \int_0^{t\wedge\tau} \mathcal{Q}\check{\boldsymbol{\Psi}}_s ds, \qquad \int_0^{t\wedge\tau} \mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s = \int_0^{t\wedge\tau} \mathcal{G}\check{\boldsymbol{\Psi}}_s d\mathcal{W}_s$$

so the identity (2.4) follows. With the additional assumption of continuity then we not only have that for all $\omega \in \check{\Omega}$, $\boldsymbol{\Psi}_\cdot(\omega)$ and $\check{\boldsymbol{\Psi}}_\cdot(\omega)$ are equal $\lambda - a.s.$ on $[0,T]$, but also that $\boldsymbol{\Psi}_\cdot(\omega) \in C([0,T];H)$ hence $C([0,T];U)$. From the identity (2.4) then $\check{\boldsymbol{\Psi}}_\cdot(\omega) \in C([0,T];U)$ as well, so $\boldsymbol{\Psi}_\cdot(\omega)$ and $\check{\boldsymbol{\Psi}}_\cdot(\omega)$ are equal $\lambda - a.s.$ on $[0,T]$ and are both continuous (into U) so must in fact be equal at all times $t \in [0,T]$. □

We treat the local solution here to deal with the additional technicalities which arise from accounting for the stopping time. Global solutions are similarly defined however.

Definition 2.2.9. *Let $\boldsymbol{\Psi}_0 : \Omega \to H$ be $\mathcal{F}_0-$measurable. A process $\boldsymbol{\Psi}$ such that for $\mathbb{P} - a.e.$ ω, $\boldsymbol{\Psi}_\cdot(\omega) \in C_w([0,T];H) \cap L^2([0,T];V)$ with $\boldsymbol{\Psi}$ progressively measurable in V, is said to be a strong solution of the equation (2.2) if the identity (2.2) holds $\mathbb{P} - a.s.$ in U for all $t \in [0,T]$.*

The solution is global in the sense that the identity holds on the given interval $[0,T]$, where T was arbitrary. One could certainly argue that a true global solution should be defined on $[0,\infty)$, inviting the question as to when these notions are equivalent; one answer is in the case where solutions are unique, ensuring that the solution on $[0,T+1]$ is an extension of that on $[0,T]$ and as such can be extended to $[0,\infty)$. Uniqueness is considered in the next subsection.

2.2.3 Uniqueness and Maximality

In this subsection we introduce the notion of uniqueness used in our solutions, as well as passage from local strong solutions to a stopping time which is *maximal*. We first state the key definitions.

Definition 2.2.10. *A local strong solution $(\boldsymbol{\Psi},\tau)$ of the equation (2.2) is said to be unique if for any other such local strong solution $(\boldsymbol{\Phi},\theta)$, then*

$$\mathbb{P}\left(\{\omega \in \Omega : \boldsymbol{\Psi}_t(\omega) = \boldsymbol{\Phi}_t(\omega) \quad \forall t \in [0, T\wedge\tau(\omega)\wedge\theta(\omega)]\}\right) = 1.$$

Remark 2.2.11. *This is in the sense of indistinguishability, Definition A.1.32, up until the minimum of the stopping times.*

Definition 2.2.12. *A pair* $(\boldsymbol{\Psi}, \Theta)$ *such that there exists a sequence of stopping times* (θ_j) *which are* $\mathbb{P}-a.s.$ *monotone increasing and convergent to* Θ*, whereby* $(\boldsymbol{\Psi}_{\cdot\wedge\theta_j}, \theta_j)$ *is a local strong solution of the equation (2.2) for each* j*, is said to be a maximal strong solution of the equation (2.2) if for any other pair* $(\boldsymbol{\Phi}, \Gamma)$ *with this property then* $\Theta \leq \Gamma$ $\mathbb{P}-a.s.$ *implies* $\Theta = \Gamma$ $\mathbb{P}-a.s.$.

Definition 2.2.13. *A maximal strong solution* $(\boldsymbol{\Psi}, \Theta)$ *of the equation (2.2) is said to be unique if for any other such solution* $(\boldsymbol{\Phi}, \Gamma)$*, then* $\Theta = \Gamma$ $\mathbb{P}-a.s.$ *and*

$$\mathbb{P}\left(\{\omega \in \Omega : \boldsymbol{\Psi}_t(\omega) = \boldsymbol{\Phi}_t(\omega) \quad \forall t \in [0, T \wedge \Theta(\omega))\}\right) = 1.$$

The remainder of this subsection is devoted to the following result.

Theorem 2.2.14. *Define a 'local regular solution' of the equation (2.2) to be a local strong solution* $(\boldsymbol{\Psi}, \tau)$*, with the alteration that the progressive measurability is either satisfied for the genuine process* $\boldsymbol{\Psi}_{\cdot}\mathbb{1}_{\cdot\leq\tau}$ *in* V *(and thus, not a version of it) or of* $\boldsymbol{\Psi}$ *in* H*. Suppose that there exists a local regular solution of the equation (2.2), and that any local regular solution is unique. Then there exists a unique maximal regular solution of the equation (2.2).*

To this end, we state and prove an important result in both this context and future ones.

Proposition 2.2.15. *Let* (A_k) *be a partition of* Ω*, which is to say that* $A_k \cap A_j = \emptyset$ *when* $k \neq j$ *and* $\bigcup_k A_k = \Omega$*, where each* $A_k \in \mathcal{F}_0$*. Consider a sequence* $(\boldsymbol{\Psi}_0^k)$*, each* $\boldsymbol{\Psi}_0^k : \Omega \to H$ $\mathcal{F}_0-$*measurable, and let* $((\boldsymbol{\Psi}^k, \tau^k))$ *be local strong solutions of the equation (2.2) corresponding to* $(\boldsymbol{\Psi}_0^k)$*. Then the pair* $(\boldsymbol{\Psi}, \tau)$ *defined by*

$$\boldsymbol{\Psi} = \sum_{k=1}^{\infty} \boldsymbol{\Psi}^k \mathbb{1}_{A_k}, \qquad \tau = \sum_{k=1}^{\infty} \tau^k \mathbb{1}_{A_k}$$

is a local strong solution of the equation (2.2) for the initial condition

$$\boldsymbol{\Psi}_0 = \sum_{k=1}^{\infty} \boldsymbol{\Psi}_0^k \mathbb{1}_{A_k}.$$

Proof. We first note that the infinite sums in defining $(\boldsymbol{\Psi}, \tau)$ and $\boldsymbol{\Psi}_0$ are given by a single term for any ω, so the limits are trivially defined $\mathbb{P}-a.s.$ in $C([0,T]; U)$, $\mathbb{R}$ and $\mathbb{R}$ respectively. As each $A_k \in \mathcal{F}_0$, then $\mathbb{1}_{A_k}$ is $\mathcal{F}_0-$measurable hence each $\tau^k\mathbb{1}_{A_k}$ remains a stopping time so their $\mathbb{P}-a.s.$ limit τ is again a stopping time. It is clear that for $\mathbb{P}-a.e.$ ω, $(\boldsymbol{\Psi}(\omega), \tau(\omega)) = (\boldsymbol{\Psi}^k(\omega), \tau^k(\omega))$ for some k, hence pathwise properties of the local strong solution are retained. That is, for $\mathbb{P}-a.e.$ ω, $\boldsymbol{\Psi}_{\cdot}(\omega) \in L^\infty([0,T]; H)$ and $\boldsymbol{\Psi}_{\cdot}(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)} \in L^2([0,T]; V)$. We next verify the identity (2.3). As the (A_k) partition Ω, it is sufficient to show that for every $t \in [0,T]$, the identity

$$\mathbb{1}_{A_k}\boldsymbol{\Psi}_t = \mathbb{1}_{A_k}\boldsymbol{\Psi}_0 + \mathbb{1}_{A_k}\int_0^{t\wedge\tau} \mathcal{Q}\boldsymbol{\Psi}_s ds + \mathbb{1}_{A_k}\int_0^{t\wedge\tau} \mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s \tag{2.5}$$

holds $\mathbb{P}-a.s.$ in U. We now look to reduce each term, seeing that

$$\mathbb{1}_{A_k}\boldsymbol{\Psi}_t = \mathbb{1}_{A_k}\boldsymbol{\Psi}^k_t$$
$$\mathbb{1}_{A_k}\boldsymbol{\Psi}_0 = \mathbb{1}_{A_k}\boldsymbol{\Psi}^k_0$$
$$\mathbb{1}_{A_k}\int_0^{t\wedge\tau}\mathcal{Q}\boldsymbol{\Psi}_s ds = \mathbb{1}_{A_k}\int_0^{t\wedge\tau^k}\mathcal{Q}\boldsymbol{\Psi}^k_s ds$$

holds $\mathbb{P}-a.s.$ in U, where the last equality is justified by

$$\begin{aligned}\mathbb{1}_{A_k}\int_0^{t\wedge\tau}\mathcal{Q}\boldsymbol{\Psi}_s ds &= (\mathbb{1}_{A_k})^2\int_0^{t\wedge\tau}\mathcal{Q}\boldsymbol{\Psi}_s ds = \mathbb{1}_{A_k}\int_0^{t}\mathbb{1}_{A_k}\mathbb{1}_{s\leq\tau}\mathcal{Q}\boldsymbol{\Psi}_s ds\\ &= \mathbb{1}_{A_k}\int_0^{t}\mathbb{1}_{A_k}\mathbb{1}_{A_k\cap\{s\leq\tau\}}\mathcal{Q}(\boldsymbol{\Psi}_s\mathbb{1}_{A_k})ds = \mathbb{1}_{A_k}\int_0^{t}\mathbb{1}_{A_k}\mathbb{1}_{A_k\cap\{s\leq\tau^k\}}\mathcal{Q}(\boldsymbol{\Psi}^k_s\mathbb{1}_{A_k})ds\\ &= \mathbb{1}_{A_k}\int_0^{t\wedge\tau^k}\mathcal{Q}(\boldsymbol{\Psi}^k_s)ds\end{aligned}$$

simply reversing the procedure to obtain the last line. We can apply the same arguments for the stochastic integral to obtain that

$$\mathbb{1}_{A_k}\int_0^{t\wedge\tau}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s = \mathbb{1}_{A_k}\int_0^{t\wedge\tau^k}\mathcal{G}\boldsymbol{\Psi}^k_s d\mathcal{W}_s$$

though we now have to be careful justifiying passage of $\mathbb{1}_{A_k}$ through the stochastic integral; as $\mathbb{1}_{A_k}$ is $\mathcal{F}_0-$measurable, then we can apply Proposition A.1.51. Therefore, to verify (2.5) it is sufficient to demonstrate that

$$\mathbb{1}_{A_k}\boldsymbol{\Psi}^k_t = \mathbb{1}_{A_k}\boldsymbol{\Psi}^k_0 + \mathbb{1}_{A_k}\int_0^{t\wedge\tau^k}\mathcal{Q}\boldsymbol{\Psi}^k_s ds + \mathbb{1}_{A_k}\int_0^{t\wedge\tau^k}\mathcal{G}\boldsymbol{\Psi}^k_s d\mathcal{W}_s,$$

but this follows from the fact that $(\boldsymbol{\Psi}^k,\tau^k)$ is a local strong solution for the initial condition $\boldsymbol{\Psi}^k_0$. To conclude that $(\boldsymbol{\Psi},\tau)$ is a local strong solution for the initial condition $\boldsymbol{\Psi}_0$ it only remains to show that $\boldsymbol{\Psi}_\cdot\mathbb{1}_{\cdot\leq\tau}$ is progressively measurable in V, which we deduce from

$$\boldsymbol{\Psi}_\cdot\mathbb{1}_{\cdot\leq\tau} = \left(\sum_{k=1}^\infty \boldsymbol{\Psi}^k_\cdot\mathbb{1}_{A_k}\right)\mathbb{1}_{\cdot\leq\tau} = \sum_{k=1}^\infty \boldsymbol{\Psi}^k_\cdot\mathbb{1}_{\cdot\leq\tau^k}\mathbb{1}_{A_k}$$

where the limit can be taken pointwise almost everywhere over the product space $\Omega\times[0,T]$ in V[1] (again for each fixed element of this product space, the infinite sum is in reality just a single term). From the progressive measurability of solutions we have that for each N, the process $\sum_{k=1}^N \boldsymbol{\Psi}^k_\cdot\mathbb{1}_{\cdot\leq\tau^k}\mathbb{1}_{A_k}$ is progressively measurable and hence for each fixed S is measurable as a mapping $\Omega\times[0,S]\to V$ where we equip $\Omega\times[0,S]$ with the sigma algebra $\mathcal{F}_S\times\mathcal{B}([0,S])$. The pointwise $\mathbb{P}\times\lambda-a.e.$ limit as $N\to\infty$ preserves the measurability which concludes the argument that $(\boldsymbol{\Psi},\tau)$ is a local strong solution of the equation (2.2) for the initial condition $\boldsymbol{\Psi}_0$. $\square$

[1] We note that strictly, this limit is taken for the representation $\boldsymbol{\Phi}$ described in Remark 2.2.7.

With this in place we set-up notation for proving Theorem 2.2.14. We define $\mathscr{X}$ as the set of all stopping times σ such that there exists a process $\boldsymbol{\Psi}$ for which $(\boldsymbol{\Psi},\sigma)$ is a local regular solution. We also define $\mathscr{Y}$ as the set of all stopping times given by the $\mathbb{P}-a.s.$ limit of monotone increasing elements of $\mathscr{X}$. We prove the existence of a maximal solution by showing that the maximum of any two elements of $\mathscr{X}$ is again in $\mathscr{X}$, a property which we use for the sequences in $\mathscr{X}$ to bound sequences in $\mathscr{Y}$ which then enables an application of Zorn's Lemma to deduce a maximal element. Of course, an assumption of Theorem 2.2.14 is that $\mathscr{X}$ is non-empty.

Lemma 2.2.16. *Under the assumptions of Theorem 2.2.14, for any $\sigma_1,\sigma_2\in\mathscr{X}$ we have that $\sigma_1\vee\sigma_2\in\mathscr{X}$.*

Proof. Our proof adapts the methodology of Proposition 2.2.15, but we establish the idea first. By definition of $\mathscr{X}$ we have that $(\boldsymbol{\Psi}^1,\sigma_1)$ and $(\boldsymbol{\Psi}^2,\sigma_2)$ are local regular solutions for some processes $\boldsymbol{\Psi}^1,\boldsymbol{\Psi}^2$, for the initial condition $\boldsymbol{\Psi}^1_0=\boldsymbol{\Psi}^2_0=\boldsymbol{\Psi}_0$. The uniqueness of solutions establishes that $\boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}$ and $\boldsymbol{\Psi}^2_{\cdot\wedge\sigma_1\wedge\sigma_2}$ are indistinguishable. Our approach is to construct a new solution $\boldsymbol{\Psi}$ which agrees with these indistinguishable processes up until $\sigma_1\wedge\sigma_2$, then extends to σ_1 as $\boldsymbol{\Psi}^1$ if $\sigma_1>\sigma_2$ and similarly for the reverse. One can construct the process $\boldsymbol{\Psi}$ and stopping time $\sigma=\sigma_1\vee\sigma_2$ by

$$\begin{aligned}\boldsymbol{\Psi} &:= \boldsymbol{\Psi}^1\mathbb{1}_{\sigma_1\geq\sigma_2}+\boldsymbol{\Psi}^2\mathbb{1}_{\sigma_1<\sigma_2}\\ \sigma &:= \sigma_1\mathbb{1}_{\sigma_1\geq\sigma_2}+\sigma_2\mathbb{1}_{\sigma_1<\sigma_2}.\end{aligned}$$

A comparison with Proposition 2.2.15 is immediately noted through the sets $A_1=\{\omega\in\Omega:\sigma_1(\omega)\geq\sigma_2(\omega)\}$ and $A_2=\{\omega\in\Omega:\sigma_1(\omega)<\sigma_2(\omega)\}$, though these sets are not $\mathcal{F}_0-$measurable hence we cannot apply the result directly. To show that $(\boldsymbol{\Psi},\sigma)$ is indeed a local regular solution we rewrite $\boldsymbol{\Psi}$ through

$$\begin{aligned}\boldsymbol{\Psi}_\cdot &= \left(\boldsymbol{\Psi}^1_\cdot-\boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}+\boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}\right)\mathbb{1}_{\sigma_1\geq\sigma_2}\\ &\qquad+\left(\boldsymbol{\Psi}^2_\cdot-\boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}+\boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}\right)\mathbb{1}_{\sigma_1<\sigma_2}\\ &= \boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}+\left(\boldsymbol{\Psi}^1_\cdot-\boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}\right)\mathbb{1}_{\sigma_1\geq\sigma_2}+\left(\boldsymbol{\Psi}^2_\cdot-\boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}\right)\mathbb{1}_{\sigma_1<\sigma_2}\\ &= \boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}+\left(\boldsymbol{\Psi}^1_\cdot-\boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}\right)\mathbb{1}_{\sigma_1\geq\sigma_2}+\left(\boldsymbol{\Psi}^2_\cdot-\boldsymbol{\Psi}^2_{\cdot\wedge\sigma_1\wedge\sigma_2}\right)\mathbb{1}_{\sigma_1<\sigma_2}.\end{aligned}\tag{2.6}$$

The purpose of this becomes clear when we look at the role of the $\mathcal{F}_0-$measurability in the proof of Proposition 2.2.15. First of all is for progressive measurability. We recall the definition of the stopped sigma algebra

$$\mathcal{F}_{\sigma_k}=\{A\in\mathcal{F}:\forall t\in[0,T],\, A\cap\{\sigma_k\leq t\}\in\mathcal{F}_t\}$$

and the property that $\mathbb{1}_{\sigma_1\geq\sigma_2}$ is $\mathcal{F}_{\sigma_1\wedge\sigma_2}-$measurable. The same is true of $\mathbb{1}_{\sigma_1<\sigma_2}$. Let us consider the case of progressive measurability in V, where the H case follows more simply. It is required to show that $\boldsymbol{\Psi}_\cdot\mathbb{1}_{\cdot\leq\sigma_1\vee\sigma_2}$ is progressively measurable in V. We show that this is true in each of the three terms in (2.6). In the first, we recall that a progressively measurable process stopped at a stopping time is again progressively measurable[2]; thus, $\boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}\mathbb{1}_{\cdot\wedge\sigma_1\wedge\sigma_2\leq\sigma_1} = \boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}$ is progressively measurable. So too, then, is its product with the real-valued progressively measurable $\mathbb{1}_{\cdot\leq\sigma_1\vee\sigma_2}$. For the following term in (2.6), note that

$$\begin{aligned}\left(\boldsymbol{\Psi}^1_\cdot - \boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}\right)\mathbb{1}_{\sigma_1\geq\sigma_2}\mathbb{1}_{\cdot\leq\sigma_1\vee\sigma_2} &= \left(\boldsymbol{\Psi}^1_\cdot - \boldsymbol{\Psi}^1_{\cdot\wedge\sigma_2}\right)\mathbb{1}_{\sigma_1\geq\sigma_2}\mathbb{1}_{\cdot\leq\sigma_1}\\ &= \left(\boldsymbol{\Psi}^1_\cdot\mathbb{1}_{\cdot\leq\sigma_1} - \boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}\mathbb{1}_{\cdot\leq\sigma_1}\right)\mathbb{1}_{\sigma_1\geq\sigma_2}.\end{aligned}$$

We know that the term inside of the brackets is progressively measurable and that the indicator function is $\mathcal{F}_{\sigma_2}-$measurable, so progressive measurability beyond σ_2 is assured. The danger is for times less than σ_2, however our construction ensures that for $s \leq \sigma_2$, $\boldsymbol{\Psi}^1_\cdot\mathbb{1}_{\cdot\leq\sigma_1} - \boldsymbol{\Psi}^1_{\cdot\wedge\sigma_1\wedge\sigma_2}\mathbb{1}_{\cdot\leq\sigma_1} = 0$. Thus progressive measurability is established, and similarly for the final term of (2.6). The only remaining argument where $\mathcal{F}_0-$measurability was used in Proposition 2.2.15 was the passage through the stochastic integral in the justification of

$$\mathbb{1}_{A_k}\int_0^{t\wedge\sigma}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s = \mathbb{1}_{A_k}\int_0^{t\wedge\sigma^k}\mathcal{G}\boldsymbol{\Psi}^k_s d\mathcal{W}_s.$$

We take the case $k=1$, with $k=2$ similar. We are required to justify that

$$\mathbb{1}_{\sigma_1\geq\sigma_2}\int_0^{t\wedge(\sigma_1\vee\sigma_2)}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s = \mathbb{1}_{\sigma_1\geq\sigma_2}\int_0^{t\wedge\sigma^1}\mathcal{G}\boldsymbol{\Psi}^1_s d\mathcal{W}_s. \tag{2.7}$$

We immediately note that

$$\begin{aligned}&\mathbb{1}_{\sigma_1\geq\sigma_2}\int_0^{t\wedge(\sigma_1\vee\sigma_2)}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s\\ &\qquad = \mathbb{1}_{\sigma_1\geq\sigma_2}\int_0^{t\wedge\sigma_1}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s + \mathbb{1}_{\sigma_1\geq\sigma_2}\int_{t\wedge\sigma_1}^{t\wedge\sigma_1\vee\sigma_2}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s\end{aligned}$$

where we argue that the final term is null $\mathbb{P}-a.s.$. We use that the events $\{t\leq\sigma_1\}$ and $\{t>\sigma_1\}$ are $\mathcal{F}_{t\wedge\sigma_1}-$measurable and partition Ω, so

$$\begin{aligned}&\int_{t\wedge\sigma_1}^{t\wedge\sigma_1\vee\sigma_2}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s\\ &\qquad = \int_{t\wedge\sigma_1}^{t\wedge\sigma_1\vee\sigma_2}\mathbb{1}_{t\leq\sigma_1}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s + \int_{t\wedge\sigma_1}^{t\wedge\sigma_1\vee\sigma_2}\mathbb{1}_{t>\sigma_1}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s\\ &\qquad = \int_{\sigma_1}^{t\wedge\sigma_1\vee\sigma_2}\mathbb{1}_{t>\sigma_1}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s.\end{aligned}\tag{2.8}$$

However as $\mathbb{1}_{\sigma_1\geq\sigma_2}$ is $\mathcal{F}_{\sigma_1}-$measurable, then through Proposition A.1.51 and the associated Remark A.1.52,

$$\mathbb{1}_{\sigma_1\geq\sigma_2}\int_{\sigma_1}^{t\wedge\sigma_1\vee\sigma_2}\mathbb{1}_{t>\sigma_1}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s = \int_{\sigma_1}^{t\wedge\sigma_1\vee\sigma_2}\mathbb{1}_{\sigma_1\geq\sigma_2}\mathbb{1}_{t>\sigma_1}\mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s = 0.$$

[2] This is why we had to ask that the process was genuinely progressively measurable, as a version may disagree at the stopped time hence the stopped progressively measurable process would not be a version of the stopped process itself.

Altogether and in light of (2.7), it is now sufficient to show

$$\mathbb{1}_{\sigma_1 \geq \sigma_2} \int_0^{t\wedge\sigma_1} \mathcal{G}\boldsymbol{\Psi}_s d\mathcal{W}_s = \mathbb{1}_{\sigma_1 \geq \sigma_2} \int_0^{t\wedge\sigma_1} \mathcal{G}\boldsymbol{\Psi}^1_s d\mathcal{W}_s. \tag{2.9}$$

We use the linearity of $\mathcal{G}$ and the representation (2.6) to consider the three corresponding integrals. Firstly,

$$\begin{aligned} &\mathbb{1}_{\sigma_1 \geq \sigma_2} \int_0^{t\wedge\sigma_1} \mathcal{G}(\boldsymbol{\Psi}^1_{s\wedge\sigma_1\wedge\sigma_2}) d\mathcal{W}_s \\ &= \mathbb{1}_{\sigma_1 \geq \sigma_2} \int_0^{t\wedge\sigma_1\wedge\sigma_2} \mathcal{G}(\boldsymbol{\Psi}^1_{s\wedge\sigma_1\wedge\sigma_2}) d\mathcal{W}_s + \mathbb{1}_{\sigma_1 \geq \sigma_2} \int_{t\wedge\sigma_1\wedge\sigma_2}^{t\wedge\sigma_1} \mathcal{G}(\boldsymbol{\Psi}^1_{s\wedge\sigma_1\wedge\sigma_2}) d\mathcal{W}_s \\ &= \mathbb{1}_{\sigma_1 \geq \sigma_2} \int_0^{t\wedge\sigma_1\wedge\sigma_2} \mathcal{G}(\boldsymbol{\Psi}^1_s) d\mathcal{W}_s + \mathbb{1}_{\sigma_1 \geq \sigma_2} \int_{t\wedge\sigma_1\wedge\sigma_2}^{t\wedge\sigma_1} \mathcal{G}(\boldsymbol{\Psi}^1_{s\wedge\sigma_1\wedge\sigma_2}) d\mathcal{W}_s \end{aligned} \tag{2.10}$$

Secondly, recalling that the integrand is null up to σ_2 and again using linearity of $\mathcal{G}$,

$$\begin{aligned} &\mathbb{1}_{\sigma_1 \geq \sigma_2} \int_0^{t\wedge\sigma_1} \mathcal{G}\left(\left(\boldsymbol{\Psi}^1_s - \boldsymbol{\Psi}^1_{s\wedge\sigma_1\wedge\sigma_2}\right) \mathbb{1}_{\sigma_1 \geq \sigma_2}\right) d\mathcal{W}_s \\ &= \mathbb{1}_{\sigma_1 \geq \sigma_2} \int_{t\wedge\sigma_1\wedge\sigma_2}^{t\wedge\sigma_1} \mathcal{G}\left(\left(\boldsymbol{\Psi}^1_s - \boldsymbol{\Psi}^1_{s\wedge\sigma_1\wedge\sigma_2}\right) \mathbb{1}_{\sigma_1 \geq \sigma_2}\right) d\mathcal{W}_s \\ &= \mathbb{1}_{\sigma_1 \geq \sigma_2} \int_{t\wedge\sigma_1\wedge\sigma_2}^{t\wedge\sigma_1} \mathcal{G}\left(\boldsymbol{\Psi}^1_s\right) d\mathcal{W}_s - \mathbb{1}_{\sigma_1 \geq \sigma_2} \int_{t\wedge\sigma_1\wedge\sigma_2}^{t\wedge\sigma_1} \mathcal{G}\left(\boldsymbol{\Psi}^1_{s\wedge\sigma_1\wedge\sigma_2}\right) d\mathcal{W}_s \end{aligned} \tag{2.11}$$

where we have pulled the $\mathcal{F}_{\sigma_1\wedge\sigma_2}-$measurable $\mathbb{1}_{\sigma_1 \geq \sigma_2}$ out of the stochastic integral, eliminating t from the lower integration point with a similar treatment to (2.8). Notice that summing (2.10) and (2.11), we achieve precisely

$$\mathbb{1}_{\sigma_1 \geq \sigma_2} \int_0^{t\wedge\sigma_1} \mathcal{G}\boldsymbol{\Psi}^1_s d\mathcal{W}_s$$

as desired in (2.9). Therefore, we would conclude the proof by showing that

$$\mathbb{1}_{\sigma_1 \geq \sigma_2} \int_0^{t\wedge\sigma_1} \mathcal{G}\left(\left(\boldsymbol{\Psi}^2_s - \boldsymbol{\Psi}^2_{s\wedge\sigma_1\wedge\sigma_2}\right) \mathbb{1}_{\sigma_1 < \sigma_2}\right) d\mathcal{W}_s = 0.$$

One can readily apply the techniques used already in this proof, using cancellation of the integrand up until $\sigma_1 \wedge \sigma_2$ then carrying $\mathbb{1}_{\sigma_1 \geq \sigma_2}$ inside this integral and obtaining complete cancellation due to the complement indicator function $\mathbb{1}_{\sigma_1 < \sigma_2}$. The proof is complete.

$\square$

Lemma 2.2.17. *Under the assumptions of Theorem 2.2.14, there exists a maximal regular solution of the equation (2.2).*

Proof. We wish to show that there exists a $\Theta \in \mathscr{Y}$ such that for any $\Gamma \in \mathscr{Y}$, $\Theta \leq \Gamma$ $\mathbb{P}-a.s.$ implies $\Theta = \Gamma$ $\mathbb{P}-a.s.$. This will be sufficient to conclude the proof, as for Θ given by the limit of (σ_j) with corresponding solutions $(\boldsymbol{\Psi}^j, \sigma_j)$, our process $\boldsymbol{\Psi}$ can be consistently defined on

$[0,\Theta)$ through the $\mathbb{P}-a.s.$ limit of $\mathbf{\Psi}^j$ from the uniqueness of local regular solutions.[3] Thus, $\mathbf{\Psi}$ satisfies

$$\mathbf{\Psi}(\omega) := \mathbf{\Psi}^j(\omega) \qquad \text{on } [0,\sigma_j(\omega)]. \tag{2.12}$$

We apply Zorn's Lemma on $\mathscr{Y}$, which we understand to be a partially ordered set for the relation $'\leq'$ defined by $\Gamma_1 \leq \Gamma_2$ if and only if for $\mathbb{P}-a.e.\ \omega$, $\Gamma_1(\omega) \leq \Gamma_2(\omega)$. The result would then follow from Zorn's Lemma if we can prove that for every sequence (Γ_k) in $\mathscr{Y}$ with $\Gamma_1 \leq \cdots \leq \Gamma_k \leq \Gamma_{k+1} \leq \dots$ there exists a $\Lambda \in \mathscr{Y}$ whereby $\Gamma_k \leq \Lambda$ for all k. Suppose now that each Γ_k is given by the increasing limit of (σ_j^k). Let's define the sequence (γ_n) as

$$\gamma_n := \bigvee_{k=1}^{n} \sigma_n^k$$

which by virtue of Lemma 2.2.16 is a sequence in $\mathscr{X}$. Inherited from each (σ_n^k), note that this is a $\mathbb{P}-a.s.$ monotone increasing sequence and therefore admits a limiting stopping time which we claim to be our Λ. By definition $\Lambda \in \mathscr{Y}$, and for each fixed k we see that $\gamma_n \geq \sigma_n^k$ for $n \geq k$. We thus have that the limit of the (γ_n) dominates the limit of the (σ_n^k), which proves the result. $\square$

Remark 2.2.18. *An alternative approach could be taken without appealing to Zorn's Lemma, following [39] Section 18 pp.71. We find our method to be simpler and more direct, hence of our preference.*

Lemma 2.2.19. *Let $(\mathbf{\Psi},\Theta)$ be a maximal regular solution of the equation (2.2). Under the assumptions of Theorem 2.2.14, $(\mathbf{\Psi},\Theta)$ is unique.*

Proof. We start by showing that $\Theta = \Gamma\ \mathbb{P}-a.s.$, for Γ as in Definition 2.2.13. Suppose that (θ_j), (γ_j) are the sequences in $\mathscr{X}$ convergent to Θ, Γ respectively as stipulated in Definition 2.2.12. From Lemma 2.2.16 we have that the sequence $(\theta_j \vee \gamma_j)$ lives in $\mathscr{X}$, and is clearly monotone increasing and convergent to $\Theta \vee \Gamma\ \mathbb{P}-a.s.$. But then $\Theta \vee \Gamma \in \mathscr{Y}$, which by the fact that $\Theta, \Gamma \leq \Theta \vee \Gamma\ \mathbb{P}-a.s.$ and the definition of the maximal solution, $\Theta = \Theta \vee \Gamma = \Gamma\ \mathbb{P}-a.s.$ The assumed uniqueness of the local regular solutions $((\mathbf{\Psi}, \theta_j \vee \gamma_j))$ concludes the result. $\square$

The way we chose to define the maximal time Θ did not exclude the possibility that some $\Gamma \in \mathscr{Y}$ could be such that $\Gamma > \Theta$ on a set of positive measure, which is often excluded from the definition: see [30, 70]. The following corollary shows that such a scenario cannot occur.

Corollary 2.2.19.1. *Let $(\mathbf{\Psi},\Theta)$ be the unique maximal regular solution of the equation (2.2). Then for any $\Gamma \in \mathscr{Y}$, $\Theta \geq \Gamma\ \mathbb{P}-a.s.$.*

Proof. Via the same arguments we have that $\Theta = \Theta \vee \Gamma\ \mathbb{P}-a.s.$ which concludes the proof. $\square$

Of course, we have demonstrated everything necessary to prove Theorem 2.2.14.

[3] This construction is completely analogous to the stochastic integral driven by a local martingale, see Section A.1.2.

Proof of 2.2.14. The proof follows from Lemmas 2.2.17, 2.2.19.

□

2.2.4 Stratonovich SPDEs in the Abstract Framework

We work now with the same initial condition $\mathbf{\Psi}_0$ and operators $\mathcal{Q}$, $\mathcal{G}$ but instead pose the question of how to understand the Stratonovich SPDE

$$\mathbf{\Psi}_t = \mathbf{\Psi}_0 + \int_0^t \mathcal{Q}\mathbf{\Psi}_s ds + \int_0^t \mathcal{G}\mathbf{\Psi}_s \circ d\mathcal{W}_s. \tag{2.13}$$

We would like to avoid introducing a strong solution for the equation (2.13) as understanding the Stratonovich integral is delicate given the necessary semi-martingale structure. If we can show that $\mathbf{\Psi}$ satisfies an evolution equation of an Itô stochastic integral plus a time integral then we can deduce the required semi-martingality and identify the martingale part, but we would need this assumption on semi-martingality *a priori* to define the Stratonovich integral in the sense of Definition 2.2.3 in order to show the desired representation. To this end, we identify a Stratonovich SPDE with an Itô one in the manner given here.

Theorem 2.2.20. *If $\mathbf{\Psi}$ is a strong solution of the equation*

$$\mathbf{\Psi}_t = \mathbf{\Psi}_0 + \int_0^t \left(\mathcal{Q}\mathbf{\Psi}_s + \frac{1}{2}\sum_{i=1}^\infty \mathcal{G}_i^2 \mathbf{\Psi}_s \right) ds + \int_0^t \mathcal{G}\mathbf{\Psi}_s d\mathcal{W}_s \tag{2.14}$$

then $\mathbf{\Psi}$ satisfies the identity (2.13) $\mathbb{P}-$ a.s. in X for all $t \in [0,T]$.

There is a little to unpack here before going on to the proof of this result. The first is how we understand the infinite sum of (2.14) and subsequently the SPDE. The operator

$$\sum_{i=1}^\infty \mathcal{G}_i^2 : V \to U$$

is defined as the pointwise limit of the partial sums, which is well defined as for any fixed $\phi \in V$,

$$\left\| \sum_{i=m}^n \mathcal{G}_i^2 \phi \right\|_U \leq \sum_{i=m}^n \left\| \mathcal{G}_i^2 \phi \right\|_U \leq \sum_{i=m}^n c_i \left\| \mathcal{G}_i \phi \right\|_H \leq \sum_{i=m}^n c_i^2 \left\| \phi \right\|_V, \tag{2.15}$$

which as seen before approaches zero as $m \to \infty$. To understand the strong solution as defined in Definition 2.2.9 we need to show that the new operator $\mathcal{Q} + \frac{1}{2}\sum_{i=1}^\infty \mathcal{G}_i^2$ satisfies the assumptions postulated in Assumption 2.2.4, but this is clear as $\mathcal{G}_i : V \to H$ and $\mathcal{G}_i : H \to U$ are bounded linear hence continuous so measurable, thus too is $\mathcal{G}_i^2 : V \to U$ and therefore the partial sums and the pointwise limit are as well. Moreover

$$\left\| \sum_{i=1}^\infty \mathcal{G}_i^2 \phi \right\|_U \leq \sum_{i=1}^\infty c_i^2 \left\| \phi \right\|_V$$

as seen in (2.15) so the boundedness is also satisfied so we can understand the SPDE (2.14) in the same manner as (2.2). It also remains to be checked that the Stratonovich integral of (2.13) is well defined for $\boldsymbol{\Psi}$ a strong solution of (2.14). We show this in the sense of Definition 2.2.3 for $\mathcal{H} = X$, the space in which the identity is satisfied. We show that the sequence of stopping times

$$\tau_n := n \wedge \inf\left\{s \geq 0 : \int_0^s \|\boldsymbol{\Psi}_r\|_V^2 \, dr \geq n\right\}$$

fit the requirements of Definition 2.2.3, noting immediately that the sequence is $\mathbb{P} - a.s.$ monotone increasing and convergent to infinity as the process

$$s \mapsto \int_0^s \|\boldsymbol{\Psi}_r\|_V^2 \, dr$$

is continuous. Moreover the process $\mathcal{G}\boldsymbol{\Psi}^n := \mathcal{G}(\boldsymbol{\Psi}.\mathbb{1}_{\cdot \leq \tau_n}) = \mathcal{G}\boldsymbol{\Psi}.\mathbb{1}_{\cdot \leq \tau_n}$ (using linearity of $\mathcal{G}$) is progressively measurable in H (we remark again that this is really $\mathcal{G}\boldsymbol{\Phi}^n$ as stipulated in Remark 2.2.7 but we identify the two) and satisfies the bound

$$\mathbb{E}\int_0^t \sum_{i=1}^\infty \|\mathcal{G}_i\boldsymbol{\Psi}_r^n\|_H^2 \, dr \leq \mathbb{E}\sum_{i=1}^\infty c_i^2 \int_0^t \|\boldsymbol{\Psi}_r^n\|_V^2 \leq \sum_{i=1}^\infty c_i^2 n < \infty$$

freely applying Tonelli's Theorem between the expectation, integral and sum. Thus $\mathcal{G}\boldsymbol{\Psi}^n \in \mathcal{I}^H(\mathcal{W})$ so the integral can be constructed in H and then embedded into X, but we note that the embedding $J : H \to X$ is a continuous linear operator and so from Proposition A.1.50 then $J(\mathcal{G}\boldsymbol{\Psi}^n) \in \mathcal{I}^X(\mathcal{W})$ and in particular

$$J\left(\int_0^t \mathcal{G}\boldsymbol{\Psi}_s^n d\mathcal{W}_s\right) = \int_0^t J(\mathcal{G}\boldsymbol{\Psi}_s^n) d\mathcal{W}_s$$

so there is no ambiguity in how we understand the integral as an element of X. Indeed we simply make the identification $\mathcal{G}\boldsymbol{\Psi}^n$ with $J(\mathcal{G}\boldsymbol{\Psi}^n)$ and will make no explicit reference to the embeddings in our analysis henceforth. It remains to show that:

1. Each $\mathcal{G}_i\boldsymbol{\Psi}^{\tau_n} \in \bar{\mathcal{M}}_c^2(X)$,
2. The infinite sum $\sum_{i=1}^\infty [\mathcal{G}_i\boldsymbol{\Psi}^{\tau_n}, W^i]_t$ converges in $L^2(\Omega; X)$.

For the first point we look at the evolution equation satisfied by $\boldsymbol{\Psi}^{\tau_n}$, which is

$$\boldsymbol{\Psi}_t^{\tau_n} = \boldsymbol{\Psi}_0^{\tau_n} + \int_0^t \left(\mathcal{Q} + \frac{1}{2}\sum_{i=1}^\infty \mathcal{G}_i^2\right)(\boldsymbol{\Psi}_s^n)\mathbb{1}_{s \leq \tau_n} ds + \int_0^t \mathcal{G}\boldsymbol{\Psi}_s^n d\mathcal{W}_s$$

$\mathbb{P} - a.s.$ in U, therefore from Proposition A.1.50 we have that

$$\mathcal{G}_i\boldsymbol{\Psi}_t^{\tau_n} = \mathcal{G}_i\boldsymbol{\Psi}_0^{\tau_n} + \int_0^t \mathcal{G}_i\left(\left(\mathcal{Q} + \frac{1}{2}\sum_{i=1}^\infty \mathcal{G}_i^2\right)(\boldsymbol{\Psi}_s^n)\mathbb{1}_{s \leq \tau_n}\right) ds + \int_0^t \mathcal{G}_i\mathcal{G}\boldsymbol{\Psi}_s^n d\mathcal{W}_s \tag{2.16}$$

$\mathbb{P} - a.s.$ in X ($\mathcal{G}_i : U \to X$ is bounded and linear). The time integral is of bounded-variation in X and from Proposition A.1.54 we have the result. The second condition to prove is that the infinite sum

$$\sum_{i=1}^{\infty}[\mathcal{G}_i\boldsymbol{\Psi}^{\tau_n}, W^i]_t \tag{2.17}$$

converges in $L^2(\Omega; X)$. From the identity (2.16) and the definition of the cross-variation for the semi-martingale,

$$[\mathcal{G}_i\boldsymbol{\Psi}^{\tau_n}, W^i]_t = \left[\int_0^{\cdot} \mathcal{G}_i\mathcal{G}\boldsymbol{\Psi}^n_s d\mathcal{W}_s, W^i\right]_t. \tag{2.18}$$

We can now use Lemma A.1.44 and the definition of the integral as an $L^2(\Omega; X)$ limit to see that

$$\left[\int_0^{\cdot} \mathcal{G}_i\mathcal{G}\boldsymbol{\Psi}^n_s d\mathcal{W}_s, W^i\right]_t = \left[\sum_{j=1}^{\infty}\int_0^{\cdot} \mathcal{G}_i\mathcal{G}_j\boldsymbol{\Psi}^n_s dW^j_s, W^i\right]_t = \lim_{m\to\infty}\left[\sum_{j=1}^{m}\int_0^{\cdot} \mathcal{G}_i\mathcal{G}_j\boldsymbol{\Psi}^n_s dW^j_s, W^i\right]_t,$$

where the limit is taken in $L^1(\Omega, X)$, should this limit exist and satisfy the conditions of Lemma A.1.44. We consider (a_k) as an orthonormal basis of X, so recalling Definition A.1.42,

$$\left[\sum_{j=1}^{m}\int_0^{\cdot} \mathcal{G}_i\mathcal{G}_j\boldsymbol{\Psi}^n_s dW^j_s, W^i\right]_t = \sum_{k=1}^{\infty}\left[\left\langle\sum_{j=1}^{m}\int_0^{\cdot} \mathcal{G}_i\mathcal{G}_j\boldsymbol{\Psi}^n_s dW^j_s, a_k\right\rangle_X, W^i\right]_t a_k.$$

Now we can first use Theorem A.1.9 to reduce this to

$$\sum_{k=1}^{\infty}\left[\sum_{j=1}^{m}\int_0^{\cdot} \left\langle\mathcal{G}_i\mathcal{G}_j\boldsymbol{\Psi}^n_s, a_k\right\rangle_X dW^j_s, W^i\right]_t a_k$$

from which the classical real valued theory informs us that for $m \geq i$, due to the independence of the Brownian Motions, this is simply

$$\sum_{k=1}^{\infty}\left(\int_0^t \left\langle\mathcal{G}_i^2\boldsymbol{\Psi}^n_s, a_k\right\rangle_X ds\right) a_k = \sum_{k=1}^{\infty}\left\langle\int_0^t \mathcal{G}_i^2\boldsymbol{\Psi}^n_s ds, a_k\right\rangle_X a_k = \int_0^t \mathcal{G}_i^2\boldsymbol{\Psi}^n_s ds.$$

Therefore the limit as $m \to \infty$ is well defined and we have the representation for (2.18). The convergence of the infinite sum in $L^2(\Omega, X)$, (2.17), will follow from our standard Cauchy argument, though we have to work a little harder here. The Cauchy argument requires showing that

$$\mathbb{E}\left\|\sum_{i=m}^{k}\int_0^t \mathcal{G}_i^2\boldsymbol{\Psi}^n_s ds\right\|_X^2 \longrightarrow 0$$

as $m, k \to \infty$ to which end we note that

$$\mathbb{E}\left\|\sum_{i=m}^{k}\int_0^t \mathcal{G}_i^2\boldsymbol{\Psi}^n_s ds\right\|_X^2 \leq \mathbb{E}\left(\sum_{i=m}^{k}\int_0^t \left\|\mathcal{G}_i^2\boldsymbol{\Psi}^n_s\right\|_X ds\right)^2$$

and

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=m}^{k}\int_0^t \left\|\mathcal{G}_i^2\mathbf{\Psi}_s^n\right\|_X ds\right)^2 &\leq \mathbb{E}\left(\sum_{i=m}^{k} c_i^2\int_0^t \left\|\mathbf{\Psi}_s^n\right\|_H ds\right)^2\\
&\leq \mathbb{E}\left(\sum_{i=m}^{k} c_i^2\int_0^t c\left\|\mathbf{\Psi}_s^n\right\|_V ds\right)^2\\
&\leq \left(c\sum_{i=m}^{k} c_i^2\right)^2 \mathbb{E}\left[t\int_0^t \left\|\mathbf{\Psi}_s^n\right\|_V^2 ds\right]\\
&\leq \left(c\sum_{i=m}^{k} c_i^2\right)^2 tn
\end{aligned}$$

where c is the constant from the embedding of $V \hookrightarrow H$. As $\sum_{i=1}^{\infty} c_i^2 < \infty$ then the Cauchy property follows, hence the Stratonovich integral is indeed well defined.

Proof of 2.2.20: We must show that

$$\int_0^t \mathcal{G}\mathbf{\Psi}_s \circ d\mathcal{W}_s = \int_0^t \mathcal{G}\mathbf{\Psi}_s d\mathcal{W}_s + \frac{1}{2}\int_0^t \sum_{i=1}^{\infty}\mathcal{G}_i^2\mathbf{\Psi}_s ds \tag{2.19}$$

$\mathbb{P}-a.s.$ in X for all $t \geq 0$. It is sufficient to show that for any n, the identity

$$\int_0^{t\wedge\tau_n} \mathcal{G}\mathbf{\Psi}_s \circ d\mathcal{W}_s = \int_0^{t\wedge\tau_n} \mathcal{G}\mathbf{\Psi}_s d\mathcal{W}_s + \frac{1}{2}\int_0^{t\wedge\tau_n} \sum_{i=1}^{\infty}\mathcal{G}_i^2\mathbf{\Psi}_s ds \tag{2.20}$$

holds $\mathbb{P}-a.s.$ in X for all $t \geq 0$. This can be seen as, defining the sets $A_n \subset \Omega$ to be those ω whereby (2.20) holds for all $t \geq 0$, then $\bigcap A_n$ is a set of full probability such that for every $\omega \in \bigcap A_n$, (2.20) holds for all n and t; in particular, for any $t \geq 0$ there is an n such that $\tau_n(\omega) \geq t$, for which (2.20) holds and is equivalent to (2.19) at this ω. We have that

$$\int_0^{t\wedge\tau_n} \mathcal{G}\mathbf{\Psi}_s \circ d\mathcal{W}_s = \int_0^{t\wedge\tau_n} \mathcal{G}_i\mathbf{\Psi}_s d\mathcal{W}_s + \frac{1}{2}\sum_{i=1}^{\infty}[\mathcal{G}_i\mathbf{\Psi}^{\tau_n}, W^i]_t$$

where the limit is taken in $L^2(\Omega; X)$. Of course we have just shown that

$$\sum_{i=1}^{\infty}[\mathcal{G}_i\mathbf{\Psi}^{\tau_n}, W^i]_t = \sum_{i=1}^{\infty}\int_0^t \mathcal{G}_i^2\mathbf{\Psi}_s^n ds$$

is well defined in this topology, but we must show that it is equal to

$$\int_0^t \sum_{i=1}^{\infty}\mathcal{G}_i^2\mathbf{\Psi}_s^n ds$$

for the limit $\mathbb{P}-a.s.$ in U as it was defined at (2.15). By an application of the Dominated Convergence Theorem with dominating function

$$\sum_{i=1}^{\infty} c_i^2 \left\|\mathbf{\Psi}^n\right\|_V$$

we can rewrite

$$\int_0^t \sum_{i=1}^{\infty} \mathcal{G}_i^2 \mathbf{\Psi}_s^n ds = \sum_{i=1}^{\infty} \int_0^t \mathcal{G}_i^2 \mathbf{\Psi}_s^n ds$$

as a limit $\mathbb{P}-a.s.$ in U, and thus $\mathbb{P}-a.s.$ in X. However the convergence in $L^2(\Omega; X)$ implies that of a subsequence $\mathbb{P}-a.s.$ in X, which agrees with the limit of the whole sequence $\mathbb{P}-a.s.$ in X thus giving the result. □

This theorem has been stated for the strong solution (Definition 2.2.9), though we note that all arguments follow in the corresponding local case by incorporating the stopping time as done in the justification that the integrals in Definition 2.2.6 are well defined. The result is stated below.

Corollary 2.2.20.1. *If $(\mathbf{\Psi}, \tau)$ is a local strong solution of the equation (2.14) then $\mathbf{\Psi}$ satisfies the identity*

$$\mathbf{\Psi}_t = \mathbf{\Psi}_0 + \int_0^{t\wedge\tau} \mathcal{Q}\mathbf{\Psi}_s ds + \int_0^{t\wedge\tau} \mathcal{G}\mathbf{\Psi}_s \circ d\mathcal{W}_s$$

$\mathbb{P}-$ a.s. in X for all $t \in [0, T]$.

Remark 2.2.21. *We should emphasise why the identity (2.13) holds only in X and not in U, the space in which (2.14) is satisfied. The evolution equation (2.16) allowed us to identify the semi-martingale structure in X as is this is where the integrals are constructed. We can, however, construct the stochastic integral in (2.16) in U, which allows us to conclude that the time integral is itself an element of U (as $\mathcal{G}_i\mathbf{\Psi}_t^n, \mathcal{G}_i\mathbf{\Psi}_0^n$ are as well). Unfortunately we cannot say that this is actually an integral in U (just an integral in X which is in turn an element of U) so pertinently we cannot say that this is of bounded-variation in U, which would be necessary when considering the cross-variation $[\mathcal{G}_i\mathbf{\Psi}^n, W^i]_t$ in U.*

We use Theorem 2.2.20 as a way of defining the Stratonovich SPDE, however as discussed, with a priori martingality assumptions then the Stratonovich integral is well defined and one can prove a converse of this theorem.

Theorem 2.2.22. *Let $\mathbf{\Psi}$ be such that for $\mathbb{P}-a.e.\ \omega$, $\mathbf{\Psi}_\cdot(\omega) \in C_w([0, T]; H) \cap L^2([0, T]; V)$ with $\mathbf{\Psi}$ progressively measurable in V. Assume in addition that $\mathcal{G}\mathbf{\Psi} \in \bar{\mathcal{I}}^U_\circ(\mathcal{W})$ and $\mathbf{\Psi}$ satisfies the identity (2.13) $\mathbb{P}-a.s.$ in U for all $t \geq 0$. Then $\mathbf{\Psi}$ satisfies the identity (2.14) $\mathbb{P}-a.s.$ in X for all $t \in [0, T]$.*

Proof. By assumption the Stratonovich integral is well defined so we can write

$$\mathbf{\Psi}_t = \mathbf{\Psi}_0 + \int_0^t \mathcal{Q}\mathbf{\Psi}_s ds + \sum_{i=1}^{\infty} \int_0^t \mathcal{G}_i\mathbf{\Psi}_s dW_s^i + \frac{1}{2}\sum_{i=1}^{\infty} \left[\mathcal{G}_i\mathbf{\Psi}, W^i\right]_t$$

for the cross variation taken in U. We wish to write out this cross variation process explicitly, so as in the proof of Theorem 2.2.20 we consider the evolution equation satisfied by $\mathcal{G}_i\mathbf{\Psi}$. We appreciate that

$$\mathcal{G}_i\left(\sum_{j=1}^{\infty}\left[\mathcal{G}_j\mathbf{\Psi},W^j\right]_t\right)=\sum_{j=1}^{\infty}\mathcal{G}_i\left(\left[\mathcal{G}_j\mathbf{\Psi},W^j\right]_t\right)$$

for the limit now in $L^2(\Omega;X)$, validated as $\mathcal{G}_i$ is bounded and linear from U into X. It is not clear if this if of finite-variation, so to proceed similarly to (2.18), we wish to show that

$$\left[\sum_{j=1}^{\infty}\mathcal{G}_i\left(\left[\mathcal{G}_j\mathbf{\Psi},W^j\right]\right),W^i\right]_t=0$$

$\mathbb{P}-a.s.$ for any $t\geq 0$. For this we again use Lemma A.1.44 to see that

$$\left[\sum_{j=1}^{\infty}\mathcal{G}_i\left(\left[\mathcal{G}_j\mathbf{\Psi},W^j\right]\right),W^i\right]_t=\lim_{m\to\infty}\left[\sum_{j=1}^{m}\mathcal{G}_i\left(\left[\mathcal{G}_j\mathbf{\Psi},W^j\right]\right),W^i\right]_t$$

for the limit in $L^1(\Omega;X)$. Each term in this sequence must be zero, though, as each $\left[\mathcal{G}_j\mathbf{\Psi},W^j\right]$ is of finite-variation in U so $\mathcal{G}_i\left(\left[\mathcal{G}_j\mathbf{\Psi},W^j\right]\right)$ is of finite-variation in X. Thus through considering the identity (2.13) in X, by the exact same process as Theorem 2.2.20, we prove the result.

□

2.2.5 Weak Solutions in the Abstract Framework

As in the study of PDEs, to expand the existence theory we shall also consider weaker notions of solution. We do this in two ways: *analytically* weak solutions and *probabilistically* weak solutions, the second of which we shall refer to as martingale solutions. We begin by giving a definition of analytically weak solutions in the established framework. For this, we extend our assumptions slightly.

Assumption 2.2.23. *$\mathcal{Q}:H\to X$ is measurable and there exists a map $\tilde{K}=\tilde{K}_{c,p,q}$ such that for any $\psi\in H$,*

$$\|\mathcal{Q}\psi\|_X\leq\tilde{K}(\phi)[1+\|\psi\|_H^2].$$

In addition, we now ask that there exists a bilinear form $\langle\cdot,\cdot\rangle_{X\times H}:X\times H\to\mathbb{R}$ such that for $\phi\in U$, $\psi\in H$,

$$\langle\phi,\psi\rangle_{X\times H}=\langle\phi,\psi\rangle_U\,. \tag{2.21}$$

We further require that the inclusions $V\hookrightarrow H\hookrightarrow U$ are dense. For the noise $\mathcal{G}$ we assume that there exists an adjoint operator $\mathcal{G}_i^*$ with the same boundedness properties as $\mathcal{G}_i$ satisfying

$$\langle\mathcal{G}_i\phi,\psi\rangle_U=\langle\phi,\mathcal{G}_i^*\psi\rangle_U \tag{2.22}$$

for all $\phi,\psi\in H$.

Remark 2.2.24. *A frequent setting is that X is given by the dual space H^*, and the bilinear from $\langle \cdot, \cdot \rangle_{X \times H}$ is the duality $\langle \cdot, \cdot \rangle_{H^* \times H}$ with respect to U. This is the induced Gelfand Triple.*

To simplify notation we consider $\mathcal{G}^*$ as an operator on $\mathfrak{U}$ in the same way as we do for $\mathcal{G}$. With this in place we can define such a solution.

Definition 2.2.25. *Let $\mathbf{\Psi}_0 : \Omega \to U$ be $\mathcal{F}_0-$measurable. A process $\mathbf{\Psi}$ such that for $\mathbb{P} - a.e.\ \omega$, $\mathbf{\Psi}_{\cdot}(\omega) \in C_w([0,T];U) \cap L^2([0,T];H)$ with $\mathbf{\Psi}_{\cdot}$ progressively measurable in H, is said to be a weak solution of the equation (2.2) if the identity*

$$\langle \mathbf{\Psi}_t, \phi \rangle_U = \langle \mathbf{\Psi}_0, \phi \rangle_U + \int_0^t \langle \mathcal{Q}\mathbf{\Psi}_s, \phi \rangle_{X \times H} ds + \int_0^t \langle \mathbf{\Psi}_s, \mathcal{G}^*\phi \rangle_U d\mathcal{W}_s \tag{2.23}$$

holds $\mathbb{P} - a.s.$ in $\mathbb{R}$ for all $\phi \in H$ and $t \in [0,T]$.

Immediately we observe that the stochastic integral is well defined through exactly the same justification as the integral for strong solutions. We also justify our use of the terminology 'weak'.

Proposition 2.2.26. *Let $\mathbf{\Psi}_0 : \Omega \to H$ be $\mathcal{F}_0-$measurable, and suppose that $\mathbf{\Psi}$ is a process whereby for $\mathbb{P} - a.e.\ \omega$, $\mathbf{\Psi}_{\cdot}(\omega) \in C_w([0,T];H) \cap L^2([0,T];V)$ with $\mathbf{\Psi}_{\cdot}$ progressively measurable in V. Then $\mathbf{\Psi}$ is a strong solution of the equation (2.2) if and only if it is a weak solution.*

Proof. We consider the two implications in turn, noting that we only need to show the equivalence of the two identities (2.2), (2.23):

$\Longrightarrow$: This direction is clear, as we simply take the inner product with ϕ in the identity (2.2) and then use Proposition A.1.51. The properties (2.21), (2.22) then conclude the implication.

$\Longleftarrow$: Via the identical process in reverse, we obtain that

$$\langle \mathbf{\Psi}_t, \phi \rangle_U = \langle \mathbf{\Psi}_0, \phi \rangle_U + \left\langle \int_0^t \mathcal{Q}\mathbf{\Psi}_s ds, \phi \right\rangle_U + \left\langle \int_0^t \mathcal{G}\mathbf{\Psi}_s d\mathcal{W}_s, \phi \right\rangle_U$$

for all $\phi \in H$. We then use the density of H in U to deduce this identity for all $\phi \in U$, from which the result follows.

$\square$

There is an important difference between the weak and strong solutions in terms of the Stratonovich Equation (2.13). Recall from the previous subsection that we experience a 'loss of a derivative' from $\mathcal{G}_i$ in the Itô-Stratonovich conversion, in the sense that our solutions had to have a degree of regularity greater than what was needed to define them. In the weak form this operator hits the test function instead, so it is there where some additional regularity is required. We have the following analogue of Theorem 2.2.20.

Theorem 2.2.27. *If $\mathbf{\Psi}$ is a weak solution of the equation (2.14) then $\mathbf{\Psi}$ satisfies the identity*

$$\langle \mathbf{\Psi}_t, \phi \rangle_U = \langle \mathbf{\Psi}_0, \phi \rangle_U + \int_0^t \langle \mathcal{Q}\mathbf{\Psi}_s, \phi \rangle_{X \times H} ds + \int_0^t \langle \mathbf{\Psi}_s, \mathcal{G}^*\phi \rangle_U \circ d\mathcal{W}_s \tag{2.24}$$

$\mathbb{P}-a.s.$ in $\mathbb{R}$ for all $\phi \in V$ and $t \in [0,T]$.

As we did for Theorem 2.2.20, we first make precise the meaning of a weak solution of (2.14) in terms of the 'Itô-Stratonovich Corrector'. The infinite sum is again taken as a pointwise limit, where $\boldsymbol{\Psi}$ satisfies the identity

$$\langle \boldsymbol{\Psi}_t, \phi\rangle_U = \langle \boldsymbol{\Psi}_0, \phi\rangle_U + \int_0^t \langle \mathcal{Q}\boldsymbol{\Psi}_s, \phi\rangle_{X\times H}\, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty \langle \mathcal{G}_i\boldsymbol{\Psi}_s, \mathcal{G}_i^*\phi\rangle_U\, ds + \int_0^t \langle \boldsymbol{\Psi}_s, \mathcal{G}^*\phi\rangle_U\, d\mathcal{W}_s.$$

Many of the technicalities of this result were addressed in Theorem 2.2.20, so in the proof here we only show directly that the Itô-Stratonovich Corrector is of the right form, where now we consider τ^n as the first hitting time in H and use the same notation $\boldsymbol{\Psi}^n$.

Proof of Theorem 2.2.27: Through the arguments of Theorem 2.2.20, it is sufficient to show that

$$\left[\langle \boldsymbol{\Psi}^n, \mathcal{G}_i^*\phi\rangle_U, W^i\right]_t = \int_0^t \langle \mathcal{G}_i\boldsymbol{\Psi}_s^n, \mathcal{G}_i^*\phi\rangle_U\, ds \tag{2.25}$$

for any given $\phi \in V$. For this we consider the evolution equation satisfied by $\langle \boldsymbol{\Psi}^n, \mathcal{G}_i^*\phi\rangle_U$. Because of this then $\mathcal{G}_i^*\phi \in H$ and can be used as a test function in the weak formulation, such that $\boldsymbol{\Psi}$ satisfies the identity

$$\begin{aligned}\langle \boldsymbol{\Psi}_t^n, \mathcal{G}_i^*\phi\rangle_U = \langle \boldsymbol{\Psi}_0^n, \mathcal{G}_i^*\phi\rangle_U + \int_0^t \langle \mathcal{Q}\boldsymbol{\Psi}_s^n, \mathcal{G}_i^*\phi\rangle_{X\times H}\, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty \langle \mathcal{G}_i\boldsymbol{\Psi}_s^n, \mathcal{G}_i^*\mathcal{G}_i^*\phi\rangle_U\, ds \\ + \int_0^t \langle \boldsymbol{\Psi}_s^n, \mathcal{G}_i^*\mathcal{G}^*\phi\rangle_U\, d\mathcal{W}_s.\end{aligned}$$

Of course we can rewrite $\langle \boldsymbol{\Psi}_s^n, \mathcal{G}_i^*\mathcal{G}^*\phi\rangle_U = \langle \mathcal{G}_i\boldsymbol{\Psi}_s^n, \mathcal{G}^*\phi\rangle_U$ so through the same process as in Theorem 2.2.20, the above identity implies (2.25) which concludes the proof. □

In accordance with the loss of regularity established in Theorem 2.2.20, we cannot say in Theorem 2.2.27 that

$$\int_0^t \langle \mathcal{G}\boldsymbol{\Psi}_s, \phi\rangle_U \circ d\mathcal{W}_s = \left\langle \int_0^t \mathcal{G}\boldsymbol{\Psi}_s \circ d\mathcal{W}_s, \phi\right\rangle_U$$

hence this should be considered a 'very weak' solution of the Stratonovich equation. To conclude this subsection we very briefly comment on the notion of a probabilistically weak solution, which we refer to as a martingale solution.

Definition 2.2.28. *Let $\boldsymbol{\Psi}_0 : \Omega \to H$ be $\mathcal{F}_0-$measurable. If there exists a filtered probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, a cylindrical Brownian Motion $\tilde{\mathcal{W}}$ over $\mathfrak{U}$ with respect to $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, an $\tilde{\mathcal{F}}_0-$measurable $\tilde{\boldsymbol{\Psi}}_0 : \tilde{\Omega} \to H$ with the same law as $\boldsymbol{\Psi}_0$, a process $\tilde{\boldsymbol{\Psi}}$ such that for $\tilde{\mathbb{P}}-a.e.$ ω, $\tilde{\boldsymbol{\Psi}}_\cdot(\omega) \in C_w([0,T];H) \cap L^2([0,T];V)$ with $\tilde{\boldsymbol{\Psi}}$ progressively measurable in V, is said to be a martingale strong solution of the equation (2.2) if the identity*

$$\tilde{\boldsymbol{\Psi}}_t = \tilde{\boldsymbol{\Psi}}_0 + \int_0^t \mathcal{Q}\tilde{\boldsymbol{\Psi}}_s ds + \int_0^t \mathcal{G}\tilde{\boldsymbol{\Psi}}_s d\mathcal{W}_s$$

holds $\tilde{\mathbb{P}}-a.s.$ *in* U *for all* $t \in [0,T]$.

2.2.6 Time-Dependent Operators

We did not facilitate time dependence in the operators $\mathcal{Q}, \mathcal{G}$ as solely for the fact that if $\mathcal{G}$ was time dependent then the conversion from Stratonovich to Itô Form would be more troublesome. There is no real additional difficulty in establishing a framework for the Itô Form for time-dependent operators, so we briefly do so now. We only consider the Itô form here so there is no longer a need for the space X, thus we work with the triple

$$V \hookrightarrow H \hookrightarrow U$$

and now the SPDE

$$\boldsymbol{\Psi}_t = \boldsymbol{\Psi}_0 + \int_0^t \mathcal{Q}(s, \boldsymbol{\Psi}_s)ds + \int_0^t \mathcal{G}(s, \boldsymbol{\Psi}_s)d\mathcal{W}_s. \tag{2.26}$$

We require the assumptions now as:

Assumption 2.2.29. *The operators* $\mathcal{Q} : [0,T] \times V \to U$ *and* $\mathcal{G} : [0,T] \times V \to \mathscr{L}^2(\mathfrak{U}; H)$ *are measurable.*

Assumption 2.2.30. *There exists a* $C_{\cdot} : [0,T] \to \mathbb{R}$ *bounded, and constants* c_i *such that for every* $\phi \in V$ *and* $t \in [0,T]$,

$$\begin{aligned}
\|\mathcal{Q}(t, \boldsymbol{\phi})\|_U &\leq C_t \tilde{K}(\phi)\left[1 + \|\boldsymbol{\phi}\|_V^2\right] \\
\|\mathcal{G}_i(t, \boldsymbol{\phi})\|_H^2 &\leq C_t c_i (1 + \|\phi\|_V^2) \\
\sum_{i=1}^\infty c_i &< \infty
\end{aligned}$$

Definitions of solutions in this framework, and a justification that the integrals are well-defined, then follows largely in the same way as for (2.2) so we omit the details here. What is slightly more delicate is the progressive measurability of the process $\mathcal{G}(\cdot, \boldsymbol{\Phi}_\cdot)\mathbb{1}_{\cdot \leq \tau}$ in $\mathscr{L}^2(\mathfrak{U}; H)$. From the measurability of $\mathcal{G}$ and the progressive measurability of $\boldsymbol{\Phi}$ we have that for any fixed t, the mapping

$$\mathcal{G}(\cdot, \boldsymbol{\Phi}_\cdot) : [0,t] \times [0,t] \times \Omega \to \mathscr{L}^2(\mathfrak{U}; H)$$

defined by

$$(s, r, \omega) \mapsto \mathcal{G}(s, \boldsymbol{\Phi}_r(\omega))$$

is $\mathcal{B}([0,t]) \times \mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable, and hence the mapping

$$(s, \omega) \mapsto \mathcal{G}(s, \boldsymbol{\Phi}_s(\omega))$$

is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable as is the indicator up to the stopping time, so the product retains the required measurability. For completeness we define the notion of a strong solution here.

Definition 2.2.31. *A process $\boldsymbol{\Psi}$ such that for $\mathbb{P}-a.e.\ \omega$, $\boldsymbol{\Psi}_\cdot(\omega) \in C_w([0,T];H) \cap L^2([0,T];V)$ with $\boldsymbol{\Psi}$ progressively measurable in V, is said to be a strong solution of the equation (2.26) if the identity (2.26) holds $\mathbb{P}-a.s.$ in U for all $t \in [0,T]$.*

2.3 Preliminary Results in the Existence Theory for Nonlinear SPDEs

2.3.1 Existence and Uniqueness in Finite Dimensions

As suggested in the introduction of this chapter, our techniques in the existence theory centre around taking finite dimensional approximations and using somewhat familiar theory. This scheme is referred to as a *Galerkin Approximation*, used traditionally in the analysis for highly non-trivial PDEs so offers a very reasonable first suggestion for the study of their stochastic counterparts. This approach will only work if we can quickly deduce the existence and uniqueness of solutions of the finite dimensional system, which we do in this subsection. It should be noted however that we still work with the Cylindrical Brownian Motion $\mathcal{W}$, over the same infinite dimensional Hilbert Space $\mathfrak{U}$; it is only the space in which the equation satisfies its identity that is assumed finite dimensional. To avoid concerns around solutions only existing until T when we use the Galerkin Approximation, we state this result globally.

Theorem 2.3.1. *Fix a finite-dimensional Hilbert Space $\mathcal{H}$. Suppose the following:*

1: For any $S > 0$, the operators $\mathscr{A} : [0,S] \times \mathcal{H} \to \mathcal{H}$ and $\mathscr{G} : [0,S] \times \mathcal{H} \to \mathscr{L}^2(\mathfrak{U};\mathcal{H})$ are measurable;

2: There exists a $C_\cdot : [0,\infty) \to \mathbb{R}$ bounded on $[0,S]$ for every S, and constants c_i such that for every $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathcal{H}$ and $t \in [0,\infty)$,

$$\|\mathscr{A}(t,\boldsymbol{\phi})\|_{\mathcal{H}}^2 \leq C_t \left[1 + \|\boldsymbol{\phi}\|_{\mathcal{H}}^2\right]$$
$$\|\mathscr{G}_i(t,\boldsymbol{\phi})\|_{\mathcal{H}}^2 \leq C_t c_i \left[1 + \|\boldsymbol{\phi}\|_{\mathcal{H}}^2\right]$$
$$\sum_{i=1}^\infty c_i < \infty$$
$$\|\mathscr{A}(t,\boldsymbol{\phi}) - \mathscr{A}(t,\boldsymbol{\psi})\|_{\mathcal{H}}^2 + \sum_{i=1}^\infty \|\mathscr{G}_i(t,\boldsymbol{\phi}) - \mathscr{G}_i(t,\boldsymbol{\psi})\|_{\mathcal{H}}^2 \leq C_t \|\boldsymbol{\phi} - \boldsymbol{\psi}\|_{\mathcal{H}}^2$$

3: $\boldsymbol{\Phi}_0 \in L^2(\Omega;\mathcal{H})$.

Then there exists a process $\boldsymbol{\Phi} : [0,\infty) \times \Omega \to \mathcal{H}$ such that for $\mathbb{P}-a.e.\ \omega$, $\boldsymbol{\Phi}_\cdot(\omega) \in C([0,S];\mathcal{H})$ for every $S > 0$, $\boldsymbol{\Phi}$ is progressively measurable in $\mathcal{H}$ and the identity

$$\boldsymbol{\Phi}_t = \boldsymbol{\Phi}_0 + \int_0^t \mathscr{A}(s,\boldsymbol{\Phi}_s)ds + \int_0^t \mathscr{G}(s,\boldsymbol{\Phi}_s)d\mathcal{W}_s \tag{2.27}$$

holds $\mathbb{P}-a.s.$ *in* $\mathcal{H}$ *for every* $t \geq 0$.

We remark that the operators satisfy the assumptions of 2.2.29, 2.2.30 for the spaces $V = H = U := \mathcal{H}$ and that the conclusion of this theorem is the existence of a strong solution of (2.27) in the sense of Definition 2.2.31, taken beyond T.

Proof. With $\mathcal{H}$ finite dimensional, we first restrict ourselves to finitely many Brownian Motions in our stochastic integral to make things classical. Fixing any $S > 0$, let us define the operator $\mathscr{G}^k : [0,S] \times \mathcal{H} \times \mathfrak{U} \to \mathcal{H}$ on the basis vectors (e_i) of $\mathfrak{U}$ by $\mathscr{G}^k(\cdot,\cdot,e_i) = \mathscr{G}(\cdot,\cdot,e_i)$ for $i \leq k$, and zero otherwise. We consider at first the equation

$$\mathbf{\Phi}_t^k = \mathbf{\Phi}_0^k + \int_0^t \mathscr{A}(s,\mathbf{\Phi}_s^k)ds + \int_0^t \mathscr{G}^k(s,\mathbf{\Phi}_s^k)d\mathcal{W}_s$$

or equivalently,

$$\mathbf{\Phi}_t^k = \mathbf{\Phi}_0^k + \int_0^t \mathscr{A}(s,\mathbf{\Phi}_s^k)ds + \sum_{i=1}^k \int_0^t \mathscr{G}_i(s,\mathbf{\Phi}_s^k)dW_s^i$$

for $\mathbf{\Phi}_0^k := \mathbf{\Phi}_0$. The existence and uniqueness of solutions to this finite-dimensional system is then classical (for solutions defined as in the theorem), see e.g. [78] Theorems 5.2.5 and 5.2.9. An extension to infinite dimensional noise remains somewhat standard so we omit full details here, deferring to [60] and classical works such as [89].

□

Similarly we have uniqueness, where the proof is once more deferred.

Theorem 2.3.2. *Suppose* $\mathbf{\Psi}$ *is another strong solution of (2.27). Then for every* $S \geq 0$,

$$\mathbb{P}\left(\{\omega \in \Omega : \mathbf{\Phi}_t(\omega) = \mathbf{\Psi}_t(\omega) \quad \forall t \in [0,S]\}\right) = 1.$$

2.3.2 Tightness Criteria

Unsurprisingly one route into the relative compactness methods of PDE theory is through tightness, owing to Prokhorov's Theorem. Notably we can connect this weak limit of measures with a genuine process through Skorohod's Representation Theorem, see for example [11] pp.70. Simple criteria through which we can establish tightness in the space of solutions to SPDEs will prove useful. Recalling our notions of solution, for example Definitions 2.2.9 and 2.2.25, we consider tightness criteria in both the spaces $L^2([0,T];\mathcal{H})$ and $C([0,T];\mathcal{H})$ for suitably chosen $\mathcal{H}$

We first give two results for tightness in $C([0,T];\mathcal{H})$. This method was applied in [131] but was not established into a result, so we give a full proof here.

Lemma 2.3.3. *Let $\mathcal{Y}$ be a reflexive Banach Space and $\mathcal{H}$ a Hilbert Space such that $\mathcal{Y}$ is compactly and densely embedded into $\mathcal{H}$, and consider the induced Gelfand Triple*

$$\mathcal{Y} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{Y}^*.$$

For some fixed $T > 0$ let $\mathbf{\Psi}^n : \Omega \to C([0,T];\mathcal{H})$ be a sequence of measurable processes such that for every $t \in [0,T]$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left(\sup_{t\in[0,T]} \|\mathbf{\Psi}^n_t\|_{\mathcal{H}}\right) < \infty \tag{2.28}$$

and for any sequence of stopping times (γ_n) with $\gamma_n : \Omega \to [0,T]$, and any $\varepsilon > 0$, $y \in \mathcal{Y}$,

$$\lim_{\delta\to 0^+} \sup_{n\in\mathbb{N}} \mathbb{P}\left(\left\{\omega \in \Omega : \left|\left\langle \mathbf{\Psi}^n_{(\gamma_n+\delta)\wedge T} - \mathbf{\Psi}^n_{\gamma_n}, y\right\rangle_{\mathcal{H}}\right| > \varepsilon\right\}\right) = 0. \tag{2.29}$$

Then the sequence of the laws of $(\mathbf{\Psi}^n)$ is tight in the space of probability measures over $C([0,T];\mathcal{Y}^*)$. If, in addition, we have that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left(\sup_{t\in[0,T]} \|\mathbf{\Psi}^n_t\|_{\mathcal{Y}}\right) < \infty \tag{2.30}$$

then the sequence of the laws of $(\mathbf{\Psi}^n)$ is tight in the space of probability measures over $C([0,T];\mathcal{H})$.

Proof. We essentially combine the tightness criteria of Theorems A.4.11 and A.4.12, in the specific case outlined here.[4] Firstly in reference to Theorem A.4.12 we may take E to be $\mathcal{Y}^*$ and $\mathbb{F}$ to be $(\mathcal{Y}^*)^*$, which is well known to separate points in $\mathcal{Y}^*$ from a corollary of the Hahn-Banach Theorem which asserts that for every $\phi \in \mathcal{Y}^*$ there exists a $\psi \in (\mathcal{Y}^*)^*$ such that $\langle \phi, \psi\rangle_{\mathcal{Y}^*\times(\mathcal{Y}^*)^*} = \|\phi\|_{\mathcal{Y}^*}$. We also note that condition 2 in Theorem A.4.12 is satisfied for (μ_n) taken to be the sequence of laws of $(\mathbf{\Psi}^n)$ over $C([0,T];\mathcal{Y}^*)$, owing to the property (2.28). Indeed as $\mathcal{Y}$ is compactly embedded into $\mathcal{H}$ then $\mathcal{H}$ is compactly embedded into $\mathcal{Y}^*$, so one only needs to take a bounded subset of $\mathcal{H}$ for this condition. Considering the closed ball of radius M in $\mathcal{H}$, $\tilde{B}_M$, we have that

$$\begin{aligned}
\mathbb{P}\left(\left\{\omega \in \Omega : \mathbf{\Psi}^n(\omega) \notin C\left([0,T];\tilde{B}_M\right)\right\}\right) &\leq \mathbb{P}\left(\left\{\omega \in \Omega : \mathbf{\Psi}^n(\omega) \notin C\left([0,T];\tilde{B}_M\right)\right\}\right)\\
&\leq \mathbb{P}\left(\left\{\omega \in \Omega : \sup_{t\in[0,T]} \|\mathbf{\Psi}^n_t(\omega)\|_{\mathcal{H}} > M\right\}\right)\\
&\leq \frac{1}{M}\mathbb{E}(\sup_{t\in[0,T]} \|\mathbf{\Psi}^n_t\|_{\mathcal{H}})\\
&\leq \frac{1}{M}\sup_{n\in\mathbb{N}}\mathbb{E}(\sup_{t\in[0,T]} \|\mathbf{\Psi}^n_t\|_{\mathcal{H}})
\end{aligned}$$

from which we see an arbitrarily large choice of M will justify condition 2 of Theorem A.4.12. By applying the theorem, it only remains to show that for every $\psi \in (\mathcal{Y}^*)^*$ the sequence of the laws of

[4] Whilst these criteria are presented in the Skorohod space, it is known that the Skorohod Topology is equivalent to the uniform topology when restricted to continuous functions, see e.g. [11] pp.124. Thus, the results also hold in the space of continuous functions.

$\langle \mathbf{\Psi}^n, \psi \rangle_{\mathcal{Y}^* \times (\mathcal{Y}^*)^*}$ is tight in the space of probability measures over $C([0,T];\mathbb{R})$. By the reflexivity of $\mathcal{Y}$ for every $\psi \in (\mathcal{Y}^*)^*$ there exists a $y \in \mathcal{Y}$ such that $\langle \mathbf{\Psi}^n, \psi \rangle_{\mathcal{Y}^* \times (\mathcal{Y}^*)^*} = \langle \mathbf{\Psi}^n, y \rangle_{\mathcal{Y}^* \times \mathcal{Y}}$ and as $\mathbf{\Psi}^n_t \in \mathcal{H}$ $\mathbb{P} - a.s.$, then this is furthermore just $\langle \mathbf{\Psi}^n, y \rangle_{\mathcal{H}}$. The problem is now reduced to showing tightness in $C([0,T];\mathbb{R})$, which by Theorem A.4.11 is satisfied if we can show that for for any sequence of stopping times (γ_n), $\gamma_n : \Omega \to [0,T]$, and constants (δ_n), $\delta_n \geq 0$ and $\delta_n \to 0$ as $n \to \infty$:

1. For every $t \in [0,T]$, the sequence of the laws of $\langle \mathbf{\Psi}^n_t, y \rangle_{\mathcal{H}}$ is tight in the space of probability measures over $\mathbb{R}$,
2. For every $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\left(\left\{\omega \in \Omega : \left|\left\langle \mathbf{\Psi}^n_{(\gamma_n+\delta_n)\wedge T} - \mathbf{\Psi}^n_{\gamma_n}, y\right\rangle_{\mathcal{H}}\right| > \varepsilon\right\}\right) = 0.$

We address each item in turn: as for 1, we are required to show that for every $\varepsilon > 0$ and $t \in [0,T]$, there exists a compact $K_\varepsilon \subset \mathbb{R}$ such that for every $n \in \mathbb{N}$,

$$\mathbb{P}\left(\{\omega \in \Omega : \langle \mathbf{\Psi}^n_t(\omega), y \rangle_{\mathcal{H}} \notin K_\varepsilon\}\right) < \varepsilon.$$

To this end define B_M as the closed ball of radius M in $\mathbb{R}$, then

$$\begin{aligned}
\mathbb{P}\left(\{\omega \in \Omega : \langle \mathbf{\Psi}^n_t(\omega), y \rangle_{\mathcal{H}} \notin B_M\}\right) &= \mathbb{P}\left(\{\omega \in \Omega : |\langle \mathbf{\Psi}^n_t(\omega), y \rangle_{\mathcal{H}}| > M\}\right) \\
&\leq \frac{1}{M}\mathbb{E}(|\langle \mathbf{\Psi}^n_t, y \rangle_{\mathcal{H}}|) \\
&\leq \frac{\|y\|_{\mathcal{H}}}{M} \sup_{n\in\mathbb{N}} \mathbb{E}\left(\|\mathbf{\Psi}^n_t\|_{\mathcal{H}}\right)
\end{aligned}$$

so setting

$$M := \frac{\varepsilon}{2\,\|y\|_{\mathcal{H}} \sup_{n\in\mathbb{N}} \mathbb{E}\left(\|\mathbf{\Psi}^n_t\|_{\mathcal{H}}\right)}$$

justifies item 1. As for 2, note that for each fixed $j \in \mathbb{N}$ we have that

$$\left|\left\langle \mathbf{\Psi}^j_{(\gamma_j+\delta_j)\wedge T} - \mathbf{\Psi}^j_{\gamma_j}, y\right\rangle_{\mathcal{H}}\right| \leq \sup_{n\in\mathbb{N}} \left|\left\langle \mathbf{\Psi}^n_{(\gamma_n+\delta_j)\wedge T} - \mathbf{\Psi}^n_{\gamma_n}, y\right\rangle_{\mathcal{H}}\right|$$

so in particular

$$\lim_{j\to\infty} \mathbb{P}\left(\left\{\left|\left\langle \mathbf{\Psi}^j_{(\gamma_j+\delta_j)\wedge T} - \mathbf{\Psi}^j_{\gamma_j}, y\right\rangle_{\mathcal{H}}\right| > \varepsilon\right\}\right) \leq \lim_{j\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}\left(\left\{\left|\left\langle \mathbf{\Psi}^n_{(\gamma_n+\delta_j)\wedge T} - \mathbf{\Psi}^n_{\gamma_n}, y\right\rangle_{\mathcal{H}}\right| > \varepsilon\right\}\right).$$

As (δ_j) was an arbitrary sequence of non-negative constants approaching zero, we can generically take $\delta \to 0^+$ and 2 is implied by (2.29). The proof of the first statement is complete.

To apply Theorem A.4.12 in the second statement we may take E to be $\mathcal{H}$ and $\mathbb{F}$ to be the collection of functions defined for each $y \in \mathcal{Y}$ by $\langle \cdot, y \rangle_{\mathcal{H}}$, which separates points in $\mathcal{H}$ from the density of $\mathcal{Y}$. The remainder of the proof follows as above. $\square$

For completeness, we state a criterion from the literature for tightness in the $L^2([0,T];\mathcal{H})$ space as well. A brief proof is given here, which is taken from [131] Lemma 5.2.

Lemma 2.3.4. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert Spaces such that $\mathcal{H}_1$ is compactly embedded into $\mathcal{H}_2$, and for some fixed $T > 0$ let $(\mathbf{\Psi}^n) : \Omega \times [0,T] \to \mathcal{H}_1$ be a sequence of measurable processes such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T \|\mathbf{\Psi}^n_s\|^2_{\mathcal{H}_1}\, ds < \infty \tag{2.31}$$

and for any $\varepsilon > 0$,

$$\lim_{\delta\to 0^+} \sup_{n\in\mathbb{N}} \mathbb{P}\left(\left\{\omega\in\Omega : \int_0^{T-\delta} \left\|\mathbf{\Psi}^n_{s+\delta}(\omega) - \mathbf{\Psi}^n_s(\omega)\right\|^2_{\mathcal{H}_2} ds > \varepsilon\right\}\right) = 0. \tag{2.32}$$

Then the sequence of the laws of $(\mathbf{\Psi}^n)$ is tight in the space of probability measures over $L^2([0,T];\mathcal{H}_2)$.

Proof. Observe that from Chebyshev's Inequality and condition (2.31),

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}\left(\left\{\omega\in\Omega : \int_0^T \|\mathbf{\Psi}^n_s\|^2_{\mathcal{H}_1}\, ds > R\right\}\right) \leq \lim_{R\to\infty}\frac{1}{R}\left(\sup_{n\in\mathbb{N}}\mathbb{E}\int_0^T \|\mathbf{\Psi}^n_s\|^2_{\mathcal{H}_1}\, ds\right) = 0$$

so in particular for any given $\varepsilon > 0$, there exists an R such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left(\left\{\omega\in\Omega : \int_0^T \|\mathbf{\Psi}^n_s\|^2_{\mathcal{H}_1}\, ds > R\right\}\right) < \frac{\varepsilon}{2}. \tag{2.33}$$

Define B_R to be the closed ball of radius R in $L^2([0,T];\mathcal{H}_2)$. From (2.32) for any $k\in\mathbb{N}$ there exists a $\delta_k > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left(\left\{\omega\in\Omega : \int_0^{T-\delta_k} \left\|\mathbf{\Psi}^n_{s+\delta_k}(\omega) - \mathbf{\Psi}^n_s(\omega)\right\|^2_{\mathcal{H}_2} ds > \frac{1}{k}\right\}\right) \leq \frac{\varepsilon}{2^{k+1}}. \tag{2.34}$$

Defining

$$\Gamma_k := \left\{\phi\in L^2([0,T];\mathcal{H}_2) : \int_0^{T-\delta_k} \|\phi_{s+\delta_k}(\omega) - \phi_s(\omega)\|^2_{\mathcal{H}_2}\, ds \leq \frac{1}{k}\right\}$$

then the set

$$A := B_R \cap \bigcap_{k=1}^{\infty}\Gamma_k$$

is shown to be relatively compact in $L^2([0,T];\mathcal{H}_2)$, and

$$\sup_{n\in\mathbb{N}}\mathbb{P}(\mathbf{\Psi}^n\notin A) \leq \sup_{n\in\mathbb{N}}\mathbb{P}(\mathbf{\Psi}^n\notin B_R) + \sum_{k=1}^{\infty}\sup_{n\in\mathbb{N}}\mathbb{P}(\mathbf{\Psi}^n\notin\Gamma_k) \leq \frac{\varepsilon}{2} + \sum_{k=1}^{\infty}\frac{\varepsilon}{2^{k+1}} \leq \varepsilon$$

which concludes the proof.

$\square$

For clarity, we lay out some of the steps that one may to take to use this theory in order to deduce the existence of solutions. We refer to strong solutions of the equation (2.2), defined in 2.2.6.

- Consider a Galerkin Approximation with solutions $(\boldsymbol{\Psi}^n)$ existing at least locally in V, as described in Subsection 2.3.1 and applying Theorem 2.3.1.
- Use the tightness criteria to show that the $(\boldsymbol{\Psi}^n)$ are tight in the spaces $C([0,T];H)$ and $L^2([0,T];V)$, hence in their intersection endowed with the sum metric, which is again a complete separable metric space.
- Apply Prokhorov's Theorem to deduce that the sequence of the laws of $(\boldsymbol{\Psi}^n)$ is relatively compact in the space of probability measures over $C([0,T];H) \cap L^2([0,T];V)$ endowed with the topology of weak convergence.
- Apply Skorohod's Representation Theorem to deduce the existence of a new probability space, a sequence of $C([0,T];H) \cap L^2([0,T];V)$ valued random variables on this new probability space with the same laws as a subsequence of the $(\boldsymbol{\Psi}^n)$, and a random variable $\tilde{\boldsymbol{\Psi}}$ admitted as an almost sure limit on this space.
- Use the powerful almost sure convergence to deduce that $\tilde{\boldsymbol{\Psi}}$ is a strong solution of the equation (2.2) posed on the new probability space; this is a *probabilistically weak* or *martingale* solution.
- Prove the uniqueness of solutions on this space, and apply a Yamada-Watanabe type argument (e.g. [130]) to obtain the existence of a strong solution on the original probability space.

Of course, when dealing with highly non-trivial SPDEs the approach has to be fine tuned with potential truncations and appeals to alternate spaces. This is simply a sketch of the rough idea; we shall see a concrete approach in Section 3.2.

2.3.3 Cauchy Criteria

A more direct way to deduce the existence of a limiting process from the Galerkin Approximations comes from the Cauchy Property. In practice due to potential nonlinearities one requires some truncation to get sufficient control on the approximating sequence. More precisely for the n^{th} term of the sequence one must work up to a first hitting time of this process, giving a stopping time τ_n. The question is then whether we can deduce a limiting process up to some time τ, where $0 < \tau \leq \tau_n$ for all n. Such a result was proven by Glatt-Holtz and Ziane in [58] Lemma 5.1.

The result presented here is an extension of this, and has important applications in the deduction of *maximal* and *global* solutions. The result of Glatt-Holtz and Ziane asserts that, under assumptions of a Cauchy property of the sequence of processes up until their first hitting times and some weak equicontinuity at *time zero*, then a limiting process and positive stopping time exist (which are then argued to be a local strong solution, as a limit of the Galerkin Approximation). No characterisation of this stopping time is given though, hence completely separate arguments are required to consider what interval the solution exists upon. We demonstrate that if instead one imposes that the processes satisfy a weak equicontinuity assumption at *all* times then the limiting stopping time can be taken as a first hitting time of the limiting process for an arbitrarily large hitting parameter. Application of this result *immediately* yields that solutions exist up until they blow up, removing the need for further analysis towards the interval on which solutions exist.

Proposition 2.3.5. *Fix $T > 0$. For $t \in [0, T]$ let X_t denote a Banach Space with norm $\|\cdot\|_{X,t}$ such that for all $s > t$, $X_s \hookrightarrow X_t$ and $\|\cdot\|_{X,t} \leq \|\cdot\|_{X,s}$. Suppose that $(\boldsymbol{\Psi}^n)$ is a sequence of processes $\boldsymbol{\Psi}^n : \Omega \mapsto X_T$, $\|\boldsymbol{\Psi}^n\|_{X,\cdot}$ is adapted and $\mathbb{P}-a.s.$ continuous, $\boldsymbol{\Psi}^n \in L^2(\Omega; X_T)$, and such that $\sup_n \|\boldsymbol{\Psi}^n\|_{X,0} \in L^\infty(\Omega; \mathbb{R})$. For any given $M > 1$ define the stopping times*

$$\tau_n^{M,T} := T \wedge \inf\left\{s \geq 0 : \|\boldsymbol{\Psi}^n\|_{X,s}^2 \geq M + \|\boldsymbol{\Psi}^n\|_{X,0}^2\right\}. \tag{2.35}$$

Furthermore suppose

$$\lim_{m\to\infty} \sup_{n\geq m} \mathbb{E}\left[\|\boldsymbol{\Psi}^n - \boldsymbol{\Psi}^m\|_{X,\tau_m^{M,t}\wedge\tau_n^{M,t}}^2\right] = 0 \tag{2.36}$$

and that for any stopping time γ and sequence of stopping times (δ_j) which converge to 0 $\mathbb{P}-a.s.$,

$$\lim_{j\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}\left(\|\boldsymbol{\Psi}^n\|_{X,(\gamma+\delta_j)\wedge\tau_n^{M,T}}^2 - \|\boldsymbol{\Psi}^n\|_{X,\gamma\wedge\tau_n^{M,T}}^2\right) = 0. \tag{2.37}$$

Then there exists a stopping time $\tau_\infty^{M,T}$, a process $\boldsymbol{\Psi} : \Omega \mapsto X_{\tau_\infty^{M,T}}$ whereby $\|\boldsymbol{\Psi}\|_{X,\cdot\wedge\tau_\infty^{M,T}}$ is adapted and $\mathbb{P}-a.s.$ continuous, and a subsequence indexed by (m_j) such that

- $\tau_\infty^{M,T} \leq \tau_{m_j}^{M,T}$ $\mathbb{P}-a.s.$,
- $\lim_{j\to\infty} \|\boldsymbol{\Psi} - \boldsymbol{\Psi}^{m_j}\|_{X,\tau_\infty^{M,T}} = 0$ $\mathbb{P}-a.s.$.

Moreover for any $R > 0$ we can choose M to be such that the stopping time

$$\tau^{R,T} := T \wedge \inf\left\{s \geq 0 : \|\boldsymbol{\Psi}\|_{X,s\wedge\tau_\infty^{M,T}}^2 \geq R\right\} \tag{2.38}$$

satisfies $\tau^{R,T} \leq \tau_\infty^{M,T}$ $\mathbb{P}-a.s.$. Thus $\tau^{R,T}$ is simply $T \wedge \inf\left\{s \geq 0 : \|\boldsymbol{\Psi}\|_{X,s}^2 \geq R\right\}$.

Remark 2.3.6. *A consequence of the properties that $\sup_n \|\boldsymbol{\Psi}^n\|_{X,0} \in L^\infty(\Omega; \mathbb{R})$ and $\lim_{j\to\infty} \|\boldsymbol{\Psi} - \boldsymbol{\Psi}^{m_j}\|_{X,\tau_\infty^{M,T}} = 0$ $\mathbb{P}-a.s.$ is that $\|\boldsymbol{\Psi}\|_{X,0} \in L^\infty(\Omega; \mathbb{R})$. Therefore for $R > \left\|\|\boldsymbol{\Psi}\|_{X,0}\right\|_{L^\infty(\Omega;\mathbb{R})}$ we have that $\tau^{R,T}$ is $\mathbb{P}-a.s.$ positive, hence so too is $\tau_\infty^{M,T}$ for appropriately chosen M.*

Proof. Property (2.36) implies that for any given $j \in \mathbb{N}$ we can choose an $n_j \in \mathbb{N}$ such that for all $k \geq n_j$,

$$\mathbb{E}\left(\left\|\boldsymbol{\Psi}^k - \boldsymbol{\Psi}^{n_j}\right\|_{X,\tau_{n_j}^{M,t}\wedge\tau_k^{M,t}}^2\right) \leq 2^{-4j}. \tag{2.39}$$

We shall make use of delicate manipulations of the subsequence indexed by (n_j), and for this we introduce a new sequence of stopping times. We now impose that

$$M > 2 + \left\|\sup_{n\in\mathbb{N}} \|\boldsymbol{\Psi}^n\|_{X,0}^2\right\|_{L^\infty(\Omega;\mathbb{R})}$$

and define

$$\tilde{M}^2 := \frac{M - \left\|\sup_{n\in\mathbb{N}} \|\mathbf{\Psi}^n\|_{X,0}^2\right\|_{L^\infty(\Omega;\mathbb{R})}}{2} > 1.$$

The purpose of this is to define

$$\sigma_j^M := T \wedge \inf\left\{s > 0 : \|\mathbf{\Psi}^{n_j}\|_{X,s} \geq (\tilde{M} - 1 + 2^{-j}) + \|\mathbf{\Psi}^{n_j}\|_{X,0}\right\}$$

and ensure that $\sigma_j^M \leq \tau_{n_j}^{M,T}$ at every ω. Note the key difference in not squaring the norm, and also that $\tilde{M} > 1$ so each σ_j^M is necessarily positive. To demonstrate the inequality, it is sufficient to show that, $\mathbb{P} - a.s.$,

$$\left((\tilde{M} - 1 + 2^{-j}) + \|\mathbf{\Psi}^{n_j}\|_{X,0}\right)^2 \leq M + \|\mathbf{\Psi}^{n_j}\|_{X,0}^2 \tag{2.40}$$

or more easily

$$\left(\tilde{M} + \|\mathbf{\Psi}^{n_j}\|_{X,0}\right)^2 \leq M + \|\mathbf{\Psi}^{n_j}\|_{X,0}^2.$$

This is possible as

$$\begin{aligned}\left(\tilde{M} + \|\mathbf{\Psi}^{n_j}\|_{X,0}\right)^2 &\leq 2\tilde{M}^2 + 2\|\mathbf{\Psi}^{n_j}\|_{X,0}^2 = M - \left\|\sup_n \|\mathbf{\Psi}^n\|_{X,0}^2\right\|_{L^\infty(\Omega;\mathbb{R})} + 2\|\mathbf{\Psi}^{n_j}\|_{X,0}^2\\ &\leq M + \|\mathbf{\Psi}^{n_j}\|_{X,0}^2.\end{aligned}$$

The property (2.40) is thus verified, so $\sigma_j^M \leq \tau_{n_j}^{M,T}$ and hence the subsequence $(\mathbf{\Psi}^{n_j})$ enjoys the same properties up until the corresponding σ_j^M. In particular from (2.39),

$$\mathbb{E}\left(\|\mathbf{\Psi}^{n_{j+1}} - \mathbf{\Psi}^{n_j}\|_{X,\sigma_j^M\wedge\sigma_{j+1}^M}\right) \leq \left[\mathbb{E}\left(\|\mathbf{\Psi}^{n_{j+1}} - \mathbf{\Psi}^{n_j}\|_{X,\sigma_j^M\wedge\sigma_{j+1}^M}^2\right)\right]^{\frac{1}{2}} \leq 2^{-2j}, \tag{2.41}$$

hence in defining the sets

$$\Omega_j := \left\{\omega \in \Omega : \|\mathbf{\Psi}^{n_{j+1}}(\omega) - \mathbf{\Psi}^{n_j}(\omega)\|_{X,\sigma_j^M(\omega)\wedge\sigma_{j+1}^M(\omega)} < 2^{-(j+2)}\right\} \tag{2.42}$$

we have, by Chebyshev's Inequality and (2.41),

$$\mathbb{P}\left(\Omega_j^C\right) \leq 2^{j+2}\mathbb{E}\left(\|\mathbf{\Psi}^{n_{j+1}} - \mathbf{\Psi}^{n_j}\|_{X,\sigma_j^M\wedge\sigma_{j+1}^M}\right) \leq 2^{-j+2}.$$

We have, therefore, that

$$\sum_{j=1}^\infty \mathbb{P}\left(\Omega_j^C\right) < \infty$$

from which we see

$$\mathbb{P}\left(\bigcap_{K=1}^\infty \bigcup_{j=K}^\infty \Omega_j^C\right) = 0$$

courtesy of Borel-Cantelli. It then follows that the set

$$\hat{\Omega} := \bigcup_{K=1}^{\infty}\bigcap_{j=K}^{\infty}\Omega_j$$

is such that $\mathbb{P}(\hat{\Omega}) = 1$ so that in verifying $\mathbb{P}-a.s.$ properties, we can in fact simply show that they hold *everywhere* on $\hat{\Omega}$. More precisely, we also take $\hat{\Omega}$ to be such that every $\|\mathbf{\Psi}^{n_j}\|_{X,\cdot}$ is continuous on $\hat{\Omega}$, which is only a further countable intersection of full measure sets. We proceed by considering the sets

$$\hat{\Omega}_K := \bigcap_{j=K}^{\infty}\Omega_j$$

with the idea to just show such properties on $\hat{\Omega}_K$ for all K (as their union makes up $\hat{\Omega}$). We look to construct a new stopping time σ^M_∞ (which will prove to be the desired $\tau^{M,T}_\infty$) given as the $\mathbb{P}-a.e.$ limit of (σ^M_j), built from demonstrating that (σ^M_j) is monotone decreasing everywhere on $\hat{\Omega}_K$ for all K. In other words we show that for sufficiently large j (in fact, just $j \geq K$) that the set

$$\Gamma := \{\sigma^M_j < \sigma^M_{j+1}\} \cap \hat{\Omega}_K \tag{2.43}$$

is empty. Firstly we observe from the strict inequality $\sigma^M_j < \sigma^M_{j+1}$ on this set that $\sigma^M_j < T$, implying that

$$\sigma^M_j = \inf\left\{s > 0 : \|\mathbf{\Psi}^{n_j}\|_{X,s} \geq (\tilde{M} - 1 + 2^{-j}) + \|\mathbf{\Psi}^{n_j}\|_{X,0}\right\}$$

so by the continuity of $\|\mathbf{\Psi}^{n_j}\|_{X,\cdot}$,

$$\|\mathbf{\Psi}^{n_j}\|_{X,\sigma^M_j} = (\tilde{M} - 1 + 2^{-j}) + \|\mathbf{\Psi}^{n_j}\|_{X,0}\,. \tag{2.44}$$

Using the definition of Ω_j, (2.42), for $j \geq K$, we have that

$$\|\mathbf{\Psi}^{n_j}\|_{X,\sigma^M_j\wedge\sigma^M_{j+1}} - \|\mathbf{\Psi}^{n_{j+1}}\|_{X,\sigma^M_j\wedge\sigma^M_{j+1}} \leq \|\mathbf{\Psi}^{n_{j+1}} - \mathbf{\Psi}^{n_j}\|_{X,\sigma^M_j\wedge\sigma^M_{j+1}} < 2^{-(j+2)} \tag{2.45}$$

and also

$$\|\mathbf{\Psi}^{n_{j+1}}\|_{X,0} - \|\mathbf{\Psi}^{n_j}\|_{X,0} \leq \|\mathbf{\Psi}^{n_{j+1}} - \mathbf{\Psi}^{n_j}\|_{X,0} < 2^{-(j+2)}. \tag{2.46}$$

Combining (2.44), (2.45) and (2.46), whilst using that $\sigma^M_j < \sigma^M_{j+1}$, we see that

$$\begin{aligned}
\|\mathbf{\Psi}^{n_{j+1}}\|_{X,\sigma^M_j\wedge\sigma^M_{j+1}} &> \|\mathbf{\Psi}^{n_j}\|_{X,\sigma^M_j\wedge\sigma^M_{j+1}} - 2^{-(j+2)}\\
&= \|\mathbf{\Psi}^{n_j}\|_{X,\sigma^M_j} - 2^{-(j+2)}\\
&= \|\mathbf{\Psi}^{n_j}\|_{X,0} + (\tilde{M} - 1 + 2^{-j}) - 2^{-(j+2)}\\
&> \|\mathbf{\Psi}^{n_{j+1}}\|_{X,0} - 2^{-(j+2)} + (\tilde{M} - 1 + 2^{-j}) - 2^{-(j+2)}\\
&= \|\mathbf{\Psi}^{n_{j+1}}\|_{X,0} + (\tilde{M} - 1 + 2^{-(j+1)}),
\end{aligned} \tag{2.47}$$

where in the last line we have used the manipulation

$$-2^{-(j+2)} - 2^{-(j+2)} + 2^{-j} = -2^{-j-1} + 2^{-j} = 2^{-j}\left(-2^{-1}+1\right) = 2^{-j}\left(2^{-1}\right) = 2^{-(j+1)}.$$

The hard work is done in showing that the set Γ defined in (2.43) is empty, as on this set note that

$$\|\boldsymbol{\Psi}^{n_{j+1}}\|_{X,\sigma_j^M\wedge\sigma_{j+1}^M} \leq \|\boldsymbol{\Psi}^{n_{j+1}}\|_{X,\sigma_{j+1}^M} \leq \|\boldsymbol{\Psi}^{n_{j+1}}\|_{X,0} + (\tilde{M} - 1 + 2^{-(j+1)}),$$

which contradicts (2.47), hence Γ must be empty. Thus on every $\hat{\Omega}_K$, and furthermore the whole of $\hat{\Omega}$, the sequence (σ_j^M) is eventually monotone decreasing (and bounded below by 0). Furthermore we define σ_∞^M as the pointwise limit $\lim_{j\to\infty}\sigma_j^M$ on $\hat{\Omega}$, which must itself be a stopping time as the $\mathbb{P}-a.s.$ limit of stopping times. As mentioned this shall prove to be our $\tau_\infty^{M,T}$, and for the existence of $\boldsymbol{\Psi}$ we show that on $\hat{\Omega}$ the subsequence $(\boldsymbol{\Psi}^{n_j})$ is Cauchy in $X_{\sigma_\infty^M}$. Every $\omega\in\hat{\Omega}$ belongs to $\hat{\Omega}_K$ for some K, and furthermore to $\hat{\Omega}_L$ for all $L>K$. We fix arbitrary $\omega\in\hat{\Omega}$ and select an associated K. At this ω, for any $j>k\geq K$, observe that

$$\begin{aligned}
\|\boldsymbol{\Psi}^{n_j} - \boldsymbol{\Psi}^{n_k}\|_{X,\sigma_\infty^M} &= \|\boldsymbol{\Psi}^{n_j} - \boldsymbol{\Psi}^{n_{k+1}} + \boldsymbol{\Psi}^{n_{k+1}} - \boldsymbol{\Psi}^{n_k}\|_{X,\sigma_\infty^M}\\
&\leq \|\boldsymbol{\Psi}^{n_j} - \boldsymbol{\Psi}^{n_{k+1}}\|_{X,\sigma_\infty^M} + \|\boldsymbol{\Psi}^{n_{k+1}} - \boldsymbol{\Psi}^{n_k}\|_{X,\sigma_\infty^M}\\
&\leq \|\boldsymbol{\Psi}^{n_j} - \boldsymbol{\Psi}^{n_{k+1}}\|_{X,\sigma_\infty^M} + 2^{-(k+2)}\\
&\leq \sum_{l=k}^{j} 2^{-(l+2)}\\
&\leq 2^{-(k+1)}
\end{aligned}$$

having carried out an inductive argument in the penultimate step. We are thus free to take K large enough so that this difference is arbitrarily small; therefore there exists a limit in the Banach Space $X_{\sigma_\infty^M}$, which we call $\boldsymbol{\Psi}$. The process $\|\boldsymbol{\Psi}\|_{X,\cdot\wedge\sigma_\infty^M}$ is adapted and $\mathbb{P}-a.s.$ continuous, as

$$\sup_{r\in[0,T]}\left|\|\boldsymbol{\Psi}\|_{X,r\wedge\sigma_\infty^M} - \|\boldsymbol{\Psi}^{n_j}\|_{X,r\wedge\sigma_\infty^M}\right| \leq \sup_{r\in[0,T]}\left|\|\boldsymbol{\Psi} - \boldsymbol{\Psi}^{n_j}\|_{X,r\wedge\sigma_\infty^M}\right| = \|\boldsymbol{\Psi} - \boldsymbol{\Psi}^{n_j}\|_{X,\sigma_\infty^M},$$

which has $\mathbb{P}-a.s.$ limit as $j\to\infty$ equal to zero. Thus $\|\boldsymbol{\Psi}\|_{X,\cdot\wedge\sigma_\infty^M}$ is given, $\mathbb{P}-a.s.$, as the uniform in time limit of adapted and continuous processes, verifying the result. Moving on, it is now that we make use of (2.37) much in the same way as we did for (2.36). This will be done in the context of $\gamma := \sigma_\infty^M$ and $\delta_j := \sigma_j^M - \sigma_\infty^M$. Indeed for any $j\in\mathbb{N}$ we can choose an $m_j\in\mathbb{N}$ (where $m_j = n_l$ some l) such that for all $k\geq m_j$,

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left(\|\boldsymbol{\Psi}^n\|^2_{X,\sigma_k^M\wedge\tau_n^{M,T}} - \|\boldsymbol{\Psi}^n\|^2_{X,\sigma_\infty^M\wedge\tau_n^{M,T}}\right) \leq 2^{-2j}.$$

In particular, through a relabelling of $\sigma_{m_j}^M = \sigma_l^M$,

$$\mathbb{E}\left(\|\boldsymbol{\Psi}^{m_j}\|^2_{X,\sigma_{m_j}^M} - \|\boldsymbol{\Psi}^{m_j}\|^2_{X,\sigma_\infty^M}\right) \leq 2^{-2j}$$

by choosing n as m_j and using that $\sigma^M_\infty \leq \sigma^M_k \leq \sigma^M_{m_j} \leq \tau^{M,T}_{m_j}$. In a familiar way we define

$$\Omega'_j := \left\{ \|\mathbf{\Psi}^{m_j}\|^2_{X,\sigma^M_{m_j}} - \|\mathbf{\Psi}^{m_j}\|^2_{X,\sigma^M_\infty} < 2^{-(j+2)} \right\}$$

so that, just as we showed for (2.42),

$$\check{\Omega}_K := \bigcap_{j=K}^{\infty} \Omega'_j, \qquad \check{\Omega} := \bigcup_{K=1}^{\infty} \check{\Omega}_K, \qquad \mathbb{P}\left(\check{\Omega}\right) = 1.$$

For arbitrary given $R > 0$, the plan now is to find a constant M such that at every $\omega \in \hat{\Omega} \cap \check{\Omega}$, either $\sigma^M_\infty = T$ or $\|\mathbf{\Psi}\|^2_{X,\sigma^M_\infty} \geq R$. In both instances it is clear that $\tau^{R,T} \leq \sigma^M_\infty$, thus proving the proposition. To this end we fix an $\omega \in \hat{\Omega} \cap \check{\Omega}$ such that $\sigma^M_\infty < T$. As σ^M_∞ is the decreasing limit of $(\sigma^M_{m_j})$ then for sufficiently large m_j we must also have that $\sigma^M_{m_j} < T$. Exactly as in (2.44),

$$\|\mathbf{\Psi}^{m_j}\|_{X,\sigma^M_{m_j}} = (\tilde{M} - 1 + 2^{-j}) + \|\mathbf{\Psi}^{m_j}\|_{X,0} . \tag{2.48}$$

From the proven convergence we also have that for sufficiently large m_j,

$$\|\mathbf{\Psi} - \mathbf{\Psi}^{m_j}\|_{X,\sigma^M_\infty} < 1 \tag{2.49}$$

which implies that $\|\mathbf{\Psi}\|_{X,\sigma^M_\infty} > \|\mathbf{\Psi}^{m_j}\|_{X,\sigma^M_\infty} - 1$, and likewise as $\omega \in \check{\Omega}_K$ for some K,

$$\|\mathbf{\Psi}^{m_j}\|^2_{X,\sigma^M_{m_j}} - \|\mathbf{\Psi}^{m_j}\|^2_{X,\sigma^M_\infty} < 1. \tag{2.50}$$

We fix an m_j large enough so that (2.48), (2.49) and (2.50) all hold. Substituting (2.48) into (2.50) gives that

$$\|\mathbf{\Psi}^{m_j}\|^2_{X,\sigma^M_\infty} > \left((\tilde{M} - 1 + 2^{-j}) + \|\mathbf{\Psi}^{m_j}\|_{X,0}\right)^2 - 1 > (\tilde{M} - 1)^2 - 1.$$

If $\tilde{M} > 2$ then the expression on the right is positive and

$$\|\mathbf{\Psi}^{m_j}\|_{X,\sigma^M_\infty} > \left((\tilde{M} - 1)^2 - 1\right)^{\frac{1}{2}} .$$

Furthermore

$$\|\mathbf{\Psi}\|_{X,\sigma^M_\infty} > \left((\tilde{M} - 1)^2 - 1\right)^{\frac{1}{2}} - 1$$

where the right hand side is of course monotone increasing and unbounded in $\tilde{M}$ and hence M. By choosing M large enough such that

$$\left[\left((\tilde{M} - 1)^2 - 1\right)^{\frac{1}{2}} - 1\right]^2 > R$$

we complete the proof. □

Chapter 3

Weak and Strong Solutions to Nonlinear SPDEs with Unbounded Noise

3.1 Introduction

Having established a basic framework through which we understand (3.1), criteria to deduce the existence and uniqueness of solutions are presented in this chapter. We shall rely on the tools developed in Section 2.3. Our approach is to extend techniques of the classical variational framework in order to treat nonlinear SPDEs with potentially unbounded noise in their optimal spaces. More precisely, we recall (1.1) to be

$$\boldsymbol{\Psi}_t = \boldsymbol{\Psi}_0 + \int_0^t \mathcal{A}(s, \boldsymbol{\Psi}_s)ds + \int_0^t \mathcal{G}(s, \boldsymbol{\Psi}_s)d\mathcal{W}_s \tag{3.1}$$

and frame this in a triplet of embedded Hilbert Spaces

$$V \hookrightarrow H \hookrightarrow U,$$

where $\mathcal{W}$ is a Cylindrical Brownian Motion over some auxiliary Hilbert Space $\mathfrak{U}$, and $\mathcal{G}$ (over $\mathfrak{U}$) is unbounded on H but maps H into U. The complete setup is given in Subsection 3.1.1. We consider three different notions of solution, increasing in strength and hence in the assumptions imposed. In brief, these are:

- **Martingale Weak Solutions**: these solutions are *probabilistically weak*, as well as *analytically weak* in the sense that the identity (3.1) is not satisfied in U but rather in the dual space H^* of H[1], with U embedded into H^*. Solutions belong, pathwise, to the space $L^\infty([0,T];U) \cap L^2([0,T];H)$. Martingale weak solutions are introduced in Definition 3.2.6. Their existence is the content of Theorem 3.2.7.

- **Weak Solutions**: these solutions are *probabilistically strong*, whilst *analytically weak* in the above sense. Solutions again belong, pathwise, to the space $L^\infty([0,T];U) \cap$

[1] This definition of a weak solution and the one presented in Subsection 2.2.5 are explicitly connected for (1.2) in Section 4.4.

$L^2([0,T];H)$. Weak solutions are introduced in Definition 3.3.3. Their existence and uniqueness is the content of Theorem 3.3.5.

- **Strong Solutions**: these solutions are *probabilistically strong*, as well as *analytically strong* in the sense that the identity (3.1) is satisfied in U. Solutions belong, pathwise, to the space $L^\infty([0,T];H) \cap L^2([0,T];V)$. Strong solutions are introduced in Definition 3.4.3. Their existence and uniqueness is the content of Theorem 3.4.5.

The variational approach to nonlinear SPDEs with additive and multiplicative noise has long been studied, initially in the works [69, 84, 124] and more recently [35, 99, 100, 101, 120] to name just a few. For an unbounded noise operator, necessary in applications to transport noise models, much less has been developed. Let us first mention the paper of Röckner, Shang and Zhang, [131], where the norm of $\mathcal{G}(s,f)$ in U can depend on the H norm of f but only up to a small constant relative to the coercivity of $\mathcal{A}(s,\cdot)$. Such an assumption is removed in this work. Tang and Wang, [142], consider solutions for a pair of embedded Hilbert Spaces, $H \hookrightarrow U$, with pathwise continuity in H and satisfying the evolution equation in U. Contrasting the more traditional triplet of spaces, their result is built for *inviscid* fluid equations in which the additional $L^2([0,T],V)$ regularity is not expected. *Viscous* fluid equations could only be solvable in this framework by imposing a more regular initial condition, required for solutions of the inviscid counterpart, and the additional expected $L^2([0,T],V)$ property would not be immediately recovered. Forthwith, this suggests a different role for the current chapter in applications. We illustrate how one can use this criteria, as well as that of Section A.3 to obtain solutions to stochastic Navier-Stokes equations, in D dimensions with initial condition u_0; a more complete description comes in Chapter 5:

- 2D, $u_0 \in L^2$: existence and uniqueness of probabilistically strong, analytically weak solutions is expected; one can apply Theorem 3.3.5, see Theorems 5.2.2, 5.3.1.
- 2D, $u_0 \in W^{1,2}$: existence and uniqueness of probabilistically strong, analytically strong solutions is expected; one can apply Theorem 3.4.5, see Theorem 5.3.3.
- 3D, $u_0 \in L^2$: existence of probabilistically weak, analytically weak solutions is expected; one can apply Theorem 3.2.7, see Theorem 5.2.1. We appreciate, however, that the emergence of convex integration techniques in SPDEs suggests that probabilistically strong, analytically weak solutions could be obtainable, even in the absence of uniqueness: see [72], for example.
- 3D, $u_0 \in W^{1,2}$: existence and uniqueness of probabilistically strong, analytically strong *local* solutions is expected; one can apply the results of Section A.3, for which complete details of the application are given in the author's paper [63], and are summarised in Subsection 5.4.

We now explain the plan of the chapter along with the overarching methods and contributions at each stage:

- Section 3.1 is the introduction, this section, which concludes with the core functional framework used throughout this chapter.

- Section 3.2 treats martingale weak solutions. The necessary assumptions on the functions $\mathcal{A}$, $\mathcal{G}$ are stated as Assumption Set 1 in Subsection 3.2.1. The definition of a solution and the statement of existence is given in Subsection 3.2.2, with the remaining subsections dedicated to its proof. Our approach is to consider a finite-dimensional approximating sequence of solutions, known as a *Galerkin System*, which is shown to exhibit a limiting solution through tightness and relative compactness arguments. The Galerkin System is constructed in Subsection 3.2.3, where uniform moment estimates in the energy norm of solutions are also obtained. The tightness is verified in Subsection 3.2.4, whilst passage to the limit occurs in Subsection 3.2.5.

- Section 3.3 concerns weak solutions. Assumption Set 1 is expanded upon in order to obtain uniqueness, giving rise to Assumption Set 2 of Subsection 3.3.1. The definition of a solution and the statement of existence and uniqueness is given in Subsection 3.3.2. The proof is given in Subsection 3.3.3, where uniqueness facilitates passage from martingale weak solutions to weak solutions through a classical Yamada-Watanabe result. We draw attention to the fact that martingale weak solutions are only shown to exist for an initial condition belonging to $L^\infty(\Omega; U)$; weak solutions are initially shown to exist only for such an initial condition as well, though this is then relieved to the unbounded case. The method involves chopping up the unbounded initial condition onto intervals on which it is bounded, obtaining the relevant solutions, and then piecing these solutions together to obtain one for the original unbounded initial condition. We are unable to execute this approach for probabilistically weak solutions, as for each interval on which the initial condition is bounded one may obtain a solution on a distinct probability space.

- Section 3.4 addresses strong solutions. Our assumptions are again expanded with Assumption Set 3 in Subsection 3.4.1. The result boils down to showing that the weak solution has the additional regularity of a strong solution. The first step towards this end is taken in Subsection 3.4.3 where uniform estimates at the level of the Galerkin System are shown, for the energy norm of strong solutions, up until first hitting times of the processes in the energy norm of weak solutions. Relative compactness arguments ensure that our weak solution gains this regularity up until any stopping time which is less than at least a subsequence of the aforementioned first hitting times. Two questions now present themselves in looking to quantify these times:

  1. Does such a $\mathbb{P}-a.s.$ positive stopping time exist?
  2. If such a stopping time does exist, can it be taken to infinity as we increase the threshold for the first hitting times?

  We recall the significance of Proposition 2.3.5 in addressing these concerns. The first question was answered by Glatt-Holtz and Ziane in [58], Lemma 5.1. The abstract lemma asserts that, under assumptions of a Cauchy property of the Galerkin System up until their first hitting times and some weak equicontinuity at the *initial* time, then a limiting process and $\mathbb{P}-a.s.$ positive stopping time smaller than a subsequence of the first hitting times exist (which are then argued to be a local strong solution, as a limit of the Galerkin

Approximation). No characterisation of this limiting stopping time is given though, hence completely separate arguments are required to pass to a *maximal* solution and further still to show that the maximal solution is global. Complete, and heavy, arguments to give rise to a maximal strong solution, and further to characterise the maximal time by the blow-up in the energy norm of strong solutions, are given in the author's paper [65] Subsection 3.6. Further still, global solutions would then be deduced by directly controlling the energy of the solution, which requires analysis in a space in which the evolution equation may not hold; this is certainly true of the example which we consider in Subsection 5.2.

Proposition 2.3.5 greatly simplifies the quoted procedure for maximality and blow-up, and enables global solutions to be deduced through an estimate of the energy norm of *weak* solutions, requiring analysis in a space in which the evolution equation certainly does hold. The extension is such that if instead one imposes that the processes satisfy a weak equicontinuity assumption at *all* times then the limiting stopping time can be taken as a first hitting time of the limiting process for an arbitrarily large hitting threshold. Application of this result *immediately* yields that solutions exist up until blow-up, pertinently as blow-up *in the energy norm of weak solutions* in our application; we emphasise again the advantage that this has, as the previous approaches discussed would only produce a characterisation of the maximal time in the energy norm of the strong solution. Our result demands a higher level estimate only for the Galerkin System, which is clearer as the evolution equation is satisfied in all relevant spaces. Towards the application of Proposition 2.3.5, the Cauchy property is validated in Subsection 3.4.4 with the equicontinuity following in Subsection 3.4.5. This is all tied together in Subsection 3.4.6, before a discussion around the potential continuity of the process in Subsection 3.4.7.

3.1.1 Functional Framework

Recall that our object of study is the Itô SPDE (3.1),

$$\boldsymbol{\Psi}_t = \boldsymbol{\Psi}_0 + \int_0^t \mathcal{A}(s, \boldsymbol{\Psi}_s)ds + \int_0^t \mathcal{G}(s, \boldsymbol{\Psi}_s)d\mathcal{W}_s$$

which we pose for a triplet of embedded Hilbert Spaces

$$V \hookrightarrow H \hookrightarrow U$$

whereby the embeddings are continuous linear injections, and $H \hookrightarrow U$ is compactly embedded. The equation (3.1) continues to be posed on a time interval $[0, T]$ for arbitrary but fixed $T \geq 0$. The mappings $\mathcal{A}, \mathcal{G}$ are such that $\mathcal{A} : [0, T] \times V \to U, \mathcal{G} : [0, T] \times H \to \mathscr{L}^2(\mathfrak{U}; U)$ are measurable. Understanding $\mathcal{G}$ as a mapping $\mathcal{G} : [0, T] \times H \times \mathfrak{U} \to U$, we use the notation $\mathcal{G}_i(\cdot, \cdot) := \mathcal{G}(\cdot, \cdot, e_i)$ introduced in Subsection 2.2.2. We further impose the existence of a system of elements (a_k) of V which form an orthogonal basis of U and a basis of H. Let us define the spaces $V_n := \operatorname{span}\{a_1, \dots, a_n\}$ and $\mathcal{P}_n$ as the orthogonal projection to V_n in U, that is

$$\mathcal{P}_n : f \mapsto \sum_{k=1}^{n} \langle f, a_k \rangle_U \, a_k.$$

It is required that the $(\mathcal{P}_n)$ are uniformly bounded in H, which is to say that there exists a constant c independent of n such that for all $f \in H$,

$$\|\mathcal{P}_n f\|_H \leq c \|f\|_H . \tag{3.2}$$

Moreover, our setup can be expanded by considering the induced Gelfand Triple

$$H \hookrightarrow U \hookrightarrow H^*$$

defined relative to the inclusion mapping $i : H \to U$; indeed, the embedding of U into H^* is given by the composition of the isomorphism mapping U into U^* with the adjoint $i^* : U^* \to H^*$. In particular, the duality pairing between H and H^*, $\langle \cdot, \cdot \rangle_{H^* \times H}$, is compatible with $\langle \cdot, \cdot \rangle_U$ in the sense that for for any $f \in U$, $g \in H$,

$$\langle f, g \rangle_{H^* \times H} = \langle f, g \rangle_U .$$

We assume that $\mathcal{A} : [0,T] \times H \to H^*$ is measurable. Specific bounds on the mappings $\mathcal{A}$ and $\mathcal{G}$ will be imposed in Assumption Sets 1, 2 and 3. In order to make the assumptions we introduce some more notation here: we shall let $c_\cdot : [0,T] \to \mathbb{R}$ denote any bounded function, and for any constant $p \in \mathbb{R}$ we define the functions $K_U : U \to \mathbb{R}$, $K_H : H \to \mathbb{R}$, $K_V : V \to \mathbb{R}$ by

$$K_U(\phi) = 1 + \|\phi\|_U^p , \quad K_H(\phi) = 1 + \|\phi\|_H^p , \quad K_V(\phi) = 1 + \|\phi\|_V^p .$$

We may also consider these mappings as functions of two variables, e.g. $K_U : U \times U \to \mathbb{R}$ by

$$K_U(\phi, \psi) = 1 + \|\phi\|_U^p + \|\psi\|_U^p .$$

Our assumptions will be stated for 'the existence of a K such that' where we really mean 'the existence of a p such that, for the corresponding K'.

3.2 Martingale Weak Solutions

We first introduce the necessary assumptions for the main result of this section.

3.2.1 Assumption Set 1

Recall the setup and notation of Subsection 3.1.1. We assume that there exists a $c_\cdot$, K and $\gamma > 0$ such that for all $\phi, \psi \in V$, $f \in H$ and $t \in [0,T]$:

Assumption 3.2.1.

$$\|\mathcal{A}(t,f)\|_{H^*} + \sum_{i=1}^\infty \|\mathcal{G}_i(t,f)\|_U^2 \leq c_t K_U(f)\left[1 + \|f\|_H^2\right], \tag{3.3}$$

$$\|\mathcal{A}(t,\phi) - \mathcal{A}(t,\psi)\|_U^2 \leq c_t K_V(\phi,\psi)\|\phi-\psi\|_V^2, \tag{3.4}$$

$$\sum_{i=1}^\infty \|\mathcal{G}_i(t,\phi) - \mathcal{G}_i(t,\psi)\|_U^2 \leq c_t K_V(\phi,\psi)\|\phi-\psi\|_H^2. \tag{3.5}$$

Assumption 3.2.2.

$$2\langle \mathcal{A}(t,\phi),\phi\rangle_U + \sum_{i=1}^\infty \|\mathcal{G}_i(t,\phi)\|_U^2 \leq c_t\left[1+\|\phi\|_U^2\right] - \gamma\|\phi\|_H^2, \tag{3.6}$$

$$\sum_{i=1}^\infty \langle \mathcal{G}_i(t,\phi),\phi\rangle_U^2 \leq c_t\left[1+\|\phi\|_U^4\right]. \tag{3.7}$$

Assumption 3.2.3. [2]

$$\langle \mathcal{A}(t,\phi), f\rangle_U \leq c_t\|f\|_H\left[K_U(\phi) + \|\phi\|_H^{\frac{3}{2}}\right], \tag{3.8}$$

$$\sum_{i=1}^\infty \langle \mathcal{G}_i(t,\phi), f\rangle_U^2 \leq c_t\|f\|_H^2 K_U(\phi). \tag{3.9}$$

Assumption 3.2.4.

$$\langle \mathcal{A}(t,\phi) - \mathcal{A}(t,f),\psi\rangle_{H^*\times H} \leq c_t\|\psi\|_V\left[1+\|\phi\|_H + \|f\|_H\right]\|\phi-f\|_U, \tag{3.10}$$

$$\sum_{i=1}^\infty \langle \mathcal{G}_i(t,\phi) - \mathcal{G}_i(t,f),\psi\rangle_U^2 \leq c_t\|\psi\|_V^2\|\phi-f\|_U^2. \tag{3.11}$$

Remark 3.2.5. *By taking negative f in Assumption 3.2.3, the conditions are equivalent to asking for the absolute value of the left hand side. The same holds with ψ in Assumption 3.2.4.*

We briefly comment on the purpose of each assumption:

- Assumption 3.2.1 ensures that the integrals in the definition of a martingale weak solution (Definition 3.2.6) are well-defined. Additionally, it ensures that the Galerkin System is well-posed (Definition 3.2.9).
- Assumption 3.2.2 is used to show uniform estimates for the Galerkin System (Proposition 3.2.12).
- Assumption 3.2.3 is employed to demonstrate tightness of the Galerkin System (Propositions 3.2.13, 3.2.15).
- Assumption 3.2.4 facilitates passage to the limit for a martingale weak solution (Proposition 3.2.19).

[2] In fact in (3.8), the exponent $3/2$ could be replaced by any $q<2$.

3.2.2 Definitions and Main Result

We now state the definition and main result for martingale weak solutions.

Definition 3.2.6. *Let $\boldsymbol{\Psi}_0 : \Omega \to U$ be $\mathcal{F}_0-$measurable. If there exists a filtered probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, a Cylindrical Brownian Motion $\tilde{\mathcal{W}}$ over $\mathfrak{U}$ with respect to $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, an $\mathcal{F}_0-$measurable $\tilde{\boldsymbol{\Psi}}_0 : \tilde{\Omega} \to U$ with the same law as $\boldsymbol{\Psi}_0$, and a progressively measurable process $\tilde{\boldsymbol{\Psi}}$ in H such that for $\tilde{\mathbb{P}} - a.e.$ $\tilde{\omega}$, $\tilde{\boldsymbol{\Psi}}_\cdot(\omega) \in L^\infty([0,T];U) \cap C_w([0,T];U) \cap L^2([0,T];H)$ and*

$$\tilde{\boldsymbol{\Psi}}_t = \tilde{\boldsymbol{\Psi}}_0 + \int_0^t \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}_s)ds + \int_0^t \mathcal{G}(s, \tilde{\boldsymbol{\Psi}}_s)d\tilde{\mathcal{W}}_s \tag{3.12}$$

holds $\tilde{\mathbb{P}} - a.s.$ in H^* for all $t \in [0,T]$, then $\tilde{\boldsymbol{\Psi}}$ is said to be a martingale weak solution of the equation (3.1).[3]

Theorem 3.2.7. *Let Assumption Set 1 hold. For any given $\mathcal{F}_0-$measurable $\boldsymbol{\Psi}_0 \in L^\infty(\Omega;U)$, there exists a martingale weak solution of the equation (3.1).*

The remainder of this section is dedicated to the proof of Theorem 3.2.7.

3.2.3 The Galerkin System

We assume that

$$\boldsymbol{\Psi}_0 \in L^\infty(\Omega;U). \tag{3.13}$$

as in Theorem 3.2.7, and consider the Galerkin Equations

$$\boldsymbol{\Psi}^n_t = \boldsymbol{\Psi}^n_0 + \int_0^t \mathcal{P}_n\mathcal{A}(s, \boldsymbol{\Psi}^n_s)ds + \int_0^t \mathcal{P}_n\mathcal{G}(s, \boldsymbol{\Psi}^n_s)d\mathcal{W}_s \tag{3.14}$$

for the initial condition $\boldsymbol{\Psi}^n_0 := \mathcal{P}_n\boldsymbol{\Psi}_0$ and $\mathcal{P}_n\mathcal{G}(e_i,\cdot) := \mathcal{P}_n\mathcal{G}_i(\cdot)$. Note that $\|\boldsymbol{\Psi}^n_0\|_U \leq \|\boldsymbol{\Psi}_0\|_U$ as each $\mathcal{P}_n$ is an orthogonal projection in U, so in particular

$$\sup_{n\in\mathbb{N}} \|\boldsymbol{\Psi}^n_0\|^2_{L^\infty(\Omega,U)} < \infty. \tag{3.15}$$

Remark 3.2.8. *We may consider the finite dimensional V_n as a Hilbert Space equipped with any of the equivalent V, H, U inner products.*

Definition 3.2.9. *A pair $(\boldsymbol{\Psi}^n, \tau)$ where τ is a $\mathbb{P} - a.s.$ positive stopping time and $\boldsymbol{\Psi}^n$ is an adapted process in V_n such that for $\mathbb{P} - a.e.$ ω, $\boldsymbol{\Psi}^n_\cdot(\omega) \in C([0,T];V_n)$, is said to be a local strong solution of the equation (3.14) if the identity*

$$\boldsymbol{\Psi}^n_t = \boldsymbol{\Psi}^n_0 + \int_0^{t\wedge\tau} \mathcal{P}_n\mathcal{A}(s, \boldsymbol{\Psi}^n_s)ds + \int_0^{t\wedge\tau} \mathcal{P}_n\mathcal{G}(s, \boldsymbol{\Psi}^n_s)d\mathcal{W}_s \tag{3.16}$$

holds $\mathbb{P} - a.s.$ in V_n for all $t \in [0,T]$.

We note that composition with $\mathcal{P}_n$ does not disturb the measurability or boundedness of the mappings. As $\mathcal{P}_n$ is continuous in U then it is measurable, so $\mathcal{P}_n\mathcal{A} : [0,T] \times V \to U$ is

[3] A detailed justification that the terms in this definition are well defined is given in Subsection 2.2.2.

measurable and similarly for $\mathcal{P}_n\mathcal{G}$. Moreover $\mathcal{P}_n$ is bounded with respect to the H^* norm, as a linear map on finite dimensional space. Therefore the terms in Definition 3.2.9 make sense, where we again refer to Subsections 2.2.2 and 2.2.6. In looking to deduce the existence of such a solution, we first consider a truncated version of the equation. For any fixed $R > 0$, we introduce the function $f_R : [0, \infty) \to [0, 1]$ constructed such that

$$f_R \in C^\infty([0,\infty);[0,1]), \qquad f_R(x) = 1, \ \forall x \in [0, R], \qquad f_R(x) = 0, \ \forall x \in [2R, \infty).$$

We consider now the equation

$$\boldsymbol{\Psi}^{n,R}_t = \boldsymbol{\Psi}^{n,R}_0 + \int_0^t f_R\left(\left\|\boldsymbol{\Psi}^{n,R}_s\right\|_H^2\right)\mathcal{P}_n\mathcal{A}(s,\boldsymbol{\Psi}^{n,R}_s)ds + \int_0^t f_R\left(\left\|\boldsymbol{\Psi}^{n,R}_s\right\|_H^2\right)\mathcal{P}_n\mathcal{G}(s,\boldsymbol{\Psi}^{n,R}_s)d\mathcal{W}_s \tag{3.17}$$

for $\boldsymbol{\Psi}^{n,R}_0 := \boldsymbol{\Psi}^n_0$ which we use as an intermediary step to deduce the existence of solutions as in Definition 3.2.9. Solutions of the truncated equation are defined in the sense of Theorem 2.3.1, and due to Assumption 3.2.1 we can apply this theorem in the case of $\mathcal{H} := V_n$, $\mathscr{A} := \mathcal{P}_n\mathcal{A}$, $\mathscr{G} := \mathcal{P}_n\mathcal{G}$ to deduce the existence of solutions to (3.17) for all $n \in \mathbb{N}, R > 0$.

The motivation for considering (3.17) is to prove the existence of local strong solutions to (3.14) by considering local intervals of existence on which the truncation threshold isn't reached. More than this, for any first hitting time in the fundamental $L^\infty([0,T];U) \cap L^2([0,T];H)$ norm, we show that a large enough truncation can be chosen so that solutions to (3.14) exist up until this first hitting time.

Lemma 3.2.10. *For any $t > 0$, $M > 1$ and fixed $n \in N$, there exists an $R > 0$ such that the following holds; letting $\boldsymbol{\Psi}^{n,R}$ be the solution of (3.17) and*

$$\tau^{M,t}_{n,R}(\omega) := t \wedge \inf\left\{s \geq 0 : \sup_{r\in[0,s]}\left\|\boldsymbol{\Psi}^{n,R}_r(\omega)\right\|_U^2 + \int_0^s \left\|\boldsymbol{\Psi}^{n,R}_r(\omega)\right\|_H^2 dr \geq M + \left\|\boldsymbol{\Psi}^n_0(\omega)\right\|_U^2\right\}$$

then $(\boldsymbol{\Psi}^{n,R}_{\cdot\wedge\tau^{M,t}_{n,R}}, \tau^{M,t}_{n,R})$ is a local strong solution of (3.14).

Proof. It is clear that for any $R > \|\boldsymbol{\Psi}^n_0\|^2_{L^\infty(\Omega,H)}$ and the stopping time

$$\tau^t_{n,R}(\omega) := t \wedge \inf\left\{s \geq 0 : \sup_{r\in[0,s]}\left\|\boldsymbol{\Psi}^{n,R}_r(\omega)\right\|_H^2 \geq R\right\}$$

that $(\boldsymbol{\Psi}^{n,R}, \tau^t_{n,R})$ is a local strong solution of (3.14). We would therefore be done if there exists an $R > \|\boldsymbol{\Psi}^n_0\|^2_{L^\infty(\Omega,H)}$ such that $\tau^{M,t}_{n,R} \leq \tau^t_{n,R}$ $\mathbb{P} - a.s.$. Due to the norm equivalence on V_n, there exists some constant c_n such that the stopping time

$$\tilde{\tau}^t_{n,R}(\omega) := t \wedge \inf\left\{s \geq 0 : \sup_{r\in[0,s]}\left\|\boldsymbol{\Psi}^{n,R}_r(\omega)\right\|_U^2 \geq c_n R\right\}$$

satisfies $\tilde{\tau}^t_{n,R} \leq \tau^t_{n,R}$ $\mathbb{P} - a.s.$. Thus if we choose

$$R > \max\left\{ \|\boldsymbol{\Psi}^n_0\|^2_{L^\infty(\Omega,H)}, \frac{M + \|\boldsymbol{\Psi}^n_0\|^2_{L^\infty(\Omega,U)}}{c_n} \right\}$$

then for any $s \in [0, \tau^{M,t}_{n,R}(\omega)]$ we have that

$$\begin{aligned}
\sup_{r\in[0,s]} \left\|\boldsymbol{\Psi}^{n,R}_r(\omega)\right\|^2_U &\leq \sup_{r\in[0,s]} \left\|\boldsymbol{\Psi}^{n,R}_r(\omega)\right\|^2_U + \int_0^s \left\|\boldsymbol{\Psi}^{n,R}_r(\omega)\right\|^2_H dr \\
&\leq M + \|\boldsymbol{\Psi}^n_0(\omega)\|^2_U \\
&\leq c_n R
\end{aligned}$$

and hence $\tau^{M,t}_{n,R} \leq \tilde{\tau}^t_{n,R} \leq \tau^t_{n,R}$ $\mathbb{P} - a.s.$, which implies the result. □

With the small technicality of introducing stopping times, Theorem 2.3.2 gives us the uniqueness of such solutions as well. From here, one can deduce the existence of a unique *maximal* strong solution $(\boldsymbol{\Psi}^n, \Theta^n)$ of the equation (3.14) as defined in Definition 2.2.12, by applying Theorem 2.2.14 with all spaces taken as V_n. Introducing the stopping times

$$\tau^{M,t}_n(\omega) := t \wedge \inf\left\{ s \geq 0 : \sup_{r\in[0,s]} \|\boldsymbol{\Psi}^n_r(\omega)\|^2_U + \int_0^s \|\boldsymbol{\Psi}^n_r(\omega)\|^2_H\, dr \geq M + \|\boldsymbol{\Psi}^n_0(\omega)\|^2_U \right\} \quad (3.18)$$

then we claim that $(\boldsymbol{\Psi}^n_{\cdot\wedge\tau^{M,t}_n}, \tau^{M,t}_n)$ is a local strong solution of (3.14). By uniqueness of local strong solutions then $\boldsymbol{\Psi}^n_{\cdot\wedge\tau^{M,t}_{n,R}}$ and $\boldsymbol{\Psi}^{n,R}_{\cdot\wedge\tau^{M,t}_{n,R}}$ are indistinguishable for R chosen as in Lemma 3.2.10, which implies that $\tau^{M,t}_n = \tau^{M,t}_{n,R}$ as well.

Remark 3.2.11. *The stopping time $\tau^{M,t}_n$ defined in (3.18) is the first hitting time with respect to a norm which is central to the arguments of the chapter. As such for a function $\boldsymbol{\Phi} \in C([0,t];U) \cap L^2([0,t];H)$ we define the norm*

$$\|\boldsymbol{\Phi}\|^2_{UH,t} := \sup_{r\in[0,t]} \|\boldsymbol{\Phi}_r\|^2_U + \int_0^t \|\boldsymbol{\Phi}_r\|^2_H\, dr \quad (3.19)$$

making explicit the dependence on the time t.

We wish to remove the need for localisation in the strong solution of (3.14), by showing that our maximal time must be greater than T. That is, we want to show that

$$\mathbb{P}\left(\{\omega \in \Omega : \Theta^n(\omega) \leq T\}\right) = 0. \quad (3.20)$$

Moreover,

$$\begin{aligned}\mathbb{P}\left(\{\omega \in \Omega : \Theta^n(\omega) \leq T\}\right) &\leq \mathbb{P}\left(\left\{\omega \in \Omega : \sup_{M\in\mathbb{N}} \tau_n^{M,T+1}(\omega) \leq T\right\}\right)\\ &= \mathbb{P}\left(\bigcap_{M\in\mathbb{N}} \{\omega \in \Omega : \tau_n^{M,T+1}(\omega) \leq T\}\right)\\ &= \lim_{M\to\infty} \mathbb{P}\left(\left\{\omega \in \Omega : \tau_n^{M,T+1}(\omega) \leq T\right\}\right)\end{aligned}$$

from the property that each $\tau_n^{M,T+1}$ corresponds to a local strong solution so is bounded by Θ^n from Corollary 2.2.19.1, and the fact that $\tau_n^{M,T+1}$ is increasing in M. From the characterisation of $\tau_n^{M,T+1}$ note that

$$\left\{\omega \in \Omega : \tau_n^{M,T+1}(\omega) \leq T\right\} = \left\{\omega \in \Omega : \|\boldsymbol{\Psi}^n(\omega)\|^2_{UH,T\wedge\tau_n^{M,T+1}(\omega)} \geq M + \|\boldsymbol{\Psi}^n_0(\omega)\|^2_U\right\}$$

so a simple application of Chebyshev's Inequality informs us that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \tau_n^{M,T+1}(\omega) \leq T\right\}\right) \leq \frac{1}{M}\mathbb{E}\left[\|\boldsymbol{\Psi}^n\|^2_{UH,T\wedge\tau_n^{M,T+1}} - \|\boldsymbol{\Psi}^n_0\|^2_U\right]. \tag{3.21}$$

Therefore, to show (3.20) we want a control on this expectation independent of M. Let us introduce the notation

$$\boldsymbol{\Psi}^{n,M}_{\cdot} := \boldsymbol{\Psi}^n_{\cdot\wedge\tau_n^{M,T+1}}, \qquad \check{\boldsymbol{\Psi}}^n_{\cdot} := \boldsymbol{\Psi}^n_{\cdot}\mathbb{1}_{\cdot\leq\tau_n^{M,T+1}} \tag{3.22}$$

where we note dependence on M is implicit in $\check{\boldsymbol{\Psi}}^n$, and by construction

$$\|\boldsymbol{\Psi}^n\|^2_{UH,T\wedge\tau_n^{M,T+1}} = \left\|\check{\boldsymbol{\Psi}}^n\right\|^2_{UH,T}. \tag{3.23}$$

This uniform control comes in the following proposition.

Proposition 3.2.12. *Let $(\boldsymbol{\Psi}^n, \Theta^n)$ be the maximal strong solution of equation (3.14). There exists a constant C independent of M, n such that*

$$\mathbb{E}\left(\|\boldsymbol{\Psi}^n\|^2_{UH,T\wedge\tau_n^{M,T+1}}\right) \leq C\left[\mathbb{E}\left(\|\boldsymbol{\Psi}^n_0\|^2_U\right) + 1\right]. \tag{3.24}$$

Proof. By equipping V_n with the U inner product we can apply an Itô Formula, Proposition A.2.7 in the case that all spaces are taken as V_n, to see that for any $0 \leq r \leq T$, the identity

$$\begin{aligned}\left\|\boldsymbol{\Psi}^{n,M}_r\right\|^2_U &= \|\boldsymbol{\Psi}^n_0\|^2_U + 2\int_0^{r\wedge\tau_n^{M,t}} \langle \mathcal{P}_n\mathcal{A}(s, \boldsymbol{\Psi}^n_s), \boldsymbol{\Psi}^n_s\rangle_U\, ds\\ &+ \int_0^{r\wedge\tau_n^{M,t}} \sum_{i=1}^\infty \|\mathcal{P}_n\mathcal{G}_i(s, \boldsymbol{\Psi}^n_s)\|^2_U\, ds + 2\sum_{i=1}^\infty \int_0^{r\wedge\tau_n^{M,t}} \langle \mathcal{P}_n\mathcal{G}_i(s, \boldsymbol{\Psi}^n_s), \boldsymbol{\Psi}^n_s\rangle_U\, dW^i_s\end{aligned}$$

holds $\mathbb{P}-a.s.$. Recalling (3.22) we write this as

$$\left\|\boldsymbol{\Psi}^{n,M}_r\right\|_U^2 = \|\boldsymbol{\Psi}^n_0\|_U^2 + 2\int_0^r \langle \mathcal{P}_n\mathcal{A}(s,\boldsymbol{\Psi}^n_s), \boldsymbol{\Psi}^n_s\rangle_U \mathbb{1}_{s\leq\tau_n^{M,T+1}}ds$$
$$+\int_0^r\sum_{i=1}^\infty\|\mathcal{P}_n\mathcal{G}_i(s,\boldsymbol{\Psi}^n_s)\|_U^2\mathbb{1}_{s\leq\tau_n^{M,T+1}}ds + 2\sum_{i=1}^\infty\int_0^r\left\langle\mathcal{P}_n\mathcal{G}_i(s,\check{\boldsymbol{\Psi}}^n_s),\check{\boldsymbol{\Psi}}^n_s\right\rangle_U dW^i_s. \tag{3.25}$$

Note that we have left the indicator function outside of the time integral terms as we have no linearity assumption to take it through the $\|\mathcal{P}_n\mathcal{G}_i(s,\boldsymbol{\Psi}^n_s)\|_U^2$ term: more precisely, it may be the case that $\mathcal{P}_n\mathcal{G}_i(s,0)\neq 0$. To do this for the stochastic integral we are just relying on the linearity of the inner product. As $\mathcal{P}_n$ is an orthogonal projection, we use the self-adjoint property and that $\|\mathcal{P}_n\mathcal{G}_i(s,\boldsymbol{\Psi}^n_s)\|_U^2 \leq \|\mathcal{G}_i(s,\boldsymbol{\Psi}^n_s)\|_U^2$ to reduce this to

$$\left\|\boldsymbol{\Psi}^{n,M}_r\right\|_U^2 \leq \|\boldsymbol{\Psi}^n_0\|_U^2 + 2\int_0^r \langle \mathcal{A}(s,\boldsymbol{\Psi}^n_s), \boldsymbol{\Psi}^n_s\rangle_U \mathbb{1}_{s\leq\tau_n^{M,T+1}}ds + \int_0^r\sum_{i=1}^\infty\|\mathcal{G}_i(s,\boldsymbol{\Psi}^n_s)\|_U^2\mathbb{1}_{s\leq\tau_n^{M,T+1}}ds$$
$$+2\sum_{i=1}^\infty\int_0^r\left\langle\mathcal{G}_i(s,\check{\boldsymbol{\Psi}}^n_s),\check{\boldsymbol{\Psi}}^n_s\right\rangle_U dW^i_s.$$

We then apply Assumption 3.2.2, (3.6), and incorporate the indicator function into the norm to obtain that

$$\left\|\boldsymbol{\Psi}^{n,M}_r\right\|_U^2 \leq \|\boldsymbol{\Psi}^n_0\|_U^2 + c\int_0^r 1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_U^2 ds - \gamma\int_0^r\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^2 ds + 2\sum_{i=1}^\infty\int_0^r\left\langle\mathcal{G}_i(s,\check{\boldsymbol{\Psi}}^n_s),\check{\boldsymbol{\Psi}}^n_s\right\rangle_U dW^i_s$$

still $\mathbb{P}-a.s.$. We now look to take the supremum over all terms, which we can do by controlling the stochastic integral with the absolute value, and further taking expectation. It should be appreciated that

$$\left\|\check{\boldsymbol{\Psi}}^n(\omega)\right\|_{UH,T}^2 \leq M + \|\boldsymbol{\Psi}_0\|_{L^\infty(\Omega,U)}^2 \tag{3.26}$$

which ensures the integrability of all terms, and that the stochastic integral is a genuine square integrable martingale. By taking the supremum in time, expectation, and applying the Burkholder-Davis-Gundy Inequality all in one step, we arrive at

$$\mathbb{E}\left(\sup_{r\in[0,T]}\left\|\boldsymbol{\Psi}^{n,M}_r\right\|_U^2\right) + \gamma\mathbb{E}\int_0^T\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^2 ds \leq 2\mathbb{E}\left(\|\boldsymbol{\Psi}^n_0\|_U^2\right) + c$$
$$+c\int_0^T\mathbb{E}\left(\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_U^2\right)ds + c\mathbb{E}\left(\int_0^T\sum_{i=1}^\infty\left\langle\mathcal{G}_i\left(s,\check{\boldsymbol{\Psi}}^n_s\right),\check{\boldsymbol{\Psi}}^n_s\right\rangle_U^2 ds\right)^{\frac{1}{2}}. \tag{3.27}$$

Note that we formally have to take the supremum individually for each term on the left hand side and then sum them, hence the appearance of a 2 in front of the initial condition. We will freely use Tonelli's Theorem to interchange between expectation and integration in time. The constant c now depends on T, which is not meaningful. Turning to the final term, with (3.7) we see that

$$
\begin{aligned}
c\mathbb{E}\left(\int_0^T \sum_{i=1}^{\infty} \left\langle \mathcal{G}_i\left(s, \check{\mathbf{\Psi}}^n_s\right), \check{\mathbf{\Psi}}^n_s \right\rangle_U^2 ds\right)^{\frac{1}{2}} &\leq c\mathbb{E}\left(\int_0^T 1 + \left\|\check{\mathbf{\Psi}}^n_s\right\|_U^4 ds\right)^{\frac{1}{2}}\\
&\leq c + c\mathbb{E}\left(\int_0^T \left\|\check{\mathbf{\Psi}}^n_s\right\|_U^4 ds\right)^{\frac{1}{2}}\\
&\leq c + c\mathbb{E}\left(\sup_{r\in[0,T]} \left\|\check{\mathbf{\Psi}}^n_r\right\|_U^2 \int_0^T \left\|\check{\mathbf{\Psi}}^n_s\right\|_U^2 ds\right)^{\frac{1}{2}}\\
&\leq c + \frac{1}{2}\mathbb{E}\left(\sup_{r\in[0,T]} \left\|\check{\mathbf{\Psi}}^n_r\right\|_U^2\right) + c\mathbb{E}\int_0^T \left\|\check{\mathbf{\Psi}}^n_s\right\|_U^2 ds \quad (3.28)
\end{aligned}
$$

having applied Young's Inequality. Substituting into (3.27), and scaling with γ as needed, then we have that

$$
\mathbb{E}\left(\left\|\check{\mathbf{\Psi}}^n\right\|_{UH,T}^2\right) \leq c\left[\mathbb{E}\left(\|\mathbf{\Psi}^n_0\|_U^2\right) + 1\right] + c\int_0^T \mathbb{E}\left(\left\|\check{\mathbf{\Psi}}^n_s\right\|_U^2\right) ds.
$$

One can control $\left\|\check{\mathbf{\Psi}}^n_s\right\|_U^2 \leq \left\|\check{\mathbf{\Psi}}^n\right\|_{UH,s}^2$ which is integrable due to (3.26), from which the standard Grönwall Inequality gives the result, recalling (3.23).

□

The expectation in (3.21) is thus finite, so taking the limit $M \to \infty$ achieves (3.20). It is further evident from our calculations that for any τ such that $(\mathbf{\Psi}^n, \tau)$ is a local strong solution of the equation (3.14), the inequality (3.24) holds (that is, with τ replacing $\tau_n^{M,T+1}$) where C is independent of the choice of τ. Therefore we can choose a $\mathbb{P}-a.s.$ increasing sequence of stopping times which approach Θ^n by definition of the maximal time, and apply the Monotone Convergence Theorem through the expectation to yield that

$$
\mathbb{E}\left(\|\mathbf{\Psi}^n\|_{UH,T}^2\right) \leq C\left[\mathbb{E}\left(\|\mathbf{\Psi}^n_0\|_U^2\right) + 1\right] \tag{3.29}
$$

where to be precise, we obtain $\mathbb{E}\left(\|\mathbf{\Psi}^n\|_{UH,T\wedge\Theta^n}^2\right)$ on the left hand side, which is simply $\mathbb{E}\left(\|\mathbf{\Psi}^n\|_{UH,T}^2\right)$ due to (3.20). Moreover for $\mathbb{P}-a.e.$ $\omega \in \Omega$, we can choose a $\tau(\omega) > T$ such that $(\mathbf{\Psi}^n, \tau)$ is a local strong solution of the equation (3.14), hence $\mathbf{\Psi}^n$ satisfies the identity (3.14) $\mathbb{P}-a.s.$ for all $t \in [0,T]$. Furthermore, $\mathbf{\Psi}^n \in C([0,T]; V_n)$; we thus call $\mathbf{\Psi}^n$ the unique strong solution of the equation (3.14). Of course we can bound $\|\mathbf{\Psi}^n_0\|_U^2 \leq \|\mathbf{\Psi}_0\|_U^2 \leq \|\mathbf{\Psi}_0\|_{L^\infty(\Omega;U)}^2$ which is finite independent of n and can be substituted in to (3.24), (3.29). Combining this with (3.21) achieves that

$$
\lim_{M\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}\left(\left\{\omega \in \Omega : \tau_n^{M,T+1}(\omega) \leq T\right\}\right) = 0. \tag{3.30}
$$

3.2.4 Tightness

We now look to deduce the existence of a process taken as the limit of $\boldsymbol{\Psi}^n$ in some sense, which is done through a tightness argument. We pursue this with Lemma 2.3.4, with the spaces $\mathcal{H}_1 := H$, $\mathcal{H}_2 := U$. Having already demonstrated (2.31) we now justify (2.32), so for any $\varepsilon > 0$ define the set

$$A^{\delta,n} := \left\{ \int_0^{T-\delta} \left\|\boldsymbol{\Psi}^n_{s+\delta} - \boldsymbol{\Psi}^n_s\right\|_U^2 ds > \varepsilon \right\}$$

where we remove the explicit dependence on ω for brevity, and the required condition (2.32) is that

$$\lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{P}\left(A^{\delta,n}\right) = 0. \tag{3.31}$$

We look to show this condition by taking advantage of the control granted to us with the stopping times $\tau_n^{M,T+1}$, and the property (3.30). For any $M > 1$,

$$\begin{aligned}
A^{\delta,n} &= A^{\delta,n} \cap \left[\left\{\tau_n^{M,T+1} > T\right\} \cup \left\{\tau_n^{M,T+1} \leq T\right\}\right] \\
&= \left[A^{\delta,n} \cap \left\{\tau_n^{M,T+1} > T\right\}\right] \cup \left[A^{\delta,n} \cap \left\{\tau_n^{M,T+1} \leq T\right\}\right] \\
&\subset \left[A^{\delta,n} \cap \left\{\tau_n^{M,T+1} > T\right\}\right] \cup \left\{\tau_n^{M,T+1} \leq T\right\}.
\end{aligned}$$

In particular,

$$\mathbb{P}\left(A^{\delta,n}\right) \leq \mathbb{P}\left(A^{\delta,n} \cap \left\{\tau_n^{M,T+1} > T\right\}\right) + \mathbb{P}\left(\left\{\tau_n^{M,T+1} \leq T\right\}\right) \tag{3.32}$$

where

$$A^{\delta,n,M} := A^{\delta,n} \cap \left\{\tau_n^{M,T+1} > T\right\} = \left\{ \int_0^{T-\delta} \left\|\boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_s\right\|_U^2 ds > \varepsilon \right\} \tag{3.33}$$

continuing to use the notation (3.22). From (3.32),

$$\lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{P}\left(A^{\delta,n}\right) \leq \lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{P}\left(A^{\delta,n,M}\right) + \lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{P}\left(\left\{\tau_n^{M,T+1} \leq T\right\}\right)$$

holds for any $M > 1$, so it must also hold in the limit as $M \to \infty$. As there is no dependency on δ in the final probability, we obtain

$$\begin{aligned}
\lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{P}\left(A^{\delta,n}\right) &\leq \lim_{M \to \infty} \lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{P}\left(A^{\delta,n,M}\right) + \lim_{M \to \infty} \sup_{n \in \mathbb{N}} \mathbb{P}\left(\left\{\tau_n^{M,T+1} \leq T\right\}\right) \\
&= \lim_{M \to \infty} \lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{P}\left(A^{\delta,n,M}\right).
\end{aligned} \tag{3.34}$$

Recalling the definition (3.33), we can apply Chebyshev's Inequality to see that

$$\mathbb{P}\left(A^{\delta,n,M}\right) \leq \frac{1}{\varepsilon} \mathbb{E} \int_0^{T-\delta} \left\|\boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_s\right\|_U^2 ds.$$

Therefore using (3.34), to justify (3.31) one may simply prove that

$$\lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{E} \int_0^{T-\delta} \left\|\boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_s\right\|_U^2 ds = 0$$

for any fixed $M > 1$. This prompts the following result, which follows a similar method to [131] Lemma 2.12.

Proposition 3.2.13. *For any $M > 1$,*

$$\lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{E} \int_0^{T-\delta} \left\| \boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_{s} \right\|_U^2 ds = 0. \tag{3.35}$$

Therefore, by Lemma 2.3.4, the sequence of the laws of $(\boldsymbol{\Psi}^n)$ is tight in the space of probability measures over $L^2([0,T];U)$.

Just ahead of proving the result, we note that from the definition of K in Subsection 3.2.1 and the property (3.26), that

$$\sup_{r \in [0,t]} K_U(\check{\boldsymbol{\Psi}}^n_r(\omega)) \leq c, \tag{3.36}$$

for a constant c dependent on M and $\boldsymbol{\Psi}_0$, a dependence which is not meaningful in the following arguments as these remain fixed.

Proof. Similarly to the proof of Proposition 3.2.12, observe that for any $s \in [0,T]$,

$$\boldsymbol{\Psi}^{n,M}_{s+\delta} = \boldsymbol{\Psi}^n_0 + \int_0^{s+\delta} \mathcal{P}_n \mathcal{A}(r, \boldsymbol{\Psi}^n_r) \mathbb{1}_{r \leq \tau_n^{M,T+1}} dr + \int_0^{s+\delta} \mathcal{P}_n \mathcal{G}(r, \boldsymbol{\Psi}^n_r) \mathbb{1}_{r \leq \tau_n^{M,T+1}} d\mathcal{W}_r.$$

Therefore

$$\boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_{s} = \int_s^{s+\delta} \mathcal{P}_n \mathcal{A}(r, \boldsymbol{\Psi}^n_r) \mathbb{1}_{r \leq \tau_n^{M,T+1}} dr + \int_s^{s+\delta} \mathcal{P}_n \mathcal{G}(r, \boldsymbol{\Psi}^n_r) \mathbb{1}_{r \leq \tau_n^{M,T+1}} d\mathcal{W}_r \tag{3.37}$$

which for any fixed s is just an evolution equation in parameter δ, so we can apply the Itô Formula (e.g. Proposition A.2.7) to deduce that

$$\begin{aligned} \left\| \boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_{s} \right\|_U^2 &= 2 \int_s^{s+\delta} \left\langle \mathcal{P}_n \mathcal{A}(r, \check{\boldsymbol{\Psi}}^n_r), \check{\boldsymbol{\Psi}}^n_r - \check{\boldsymbol{\Psi}}^n_s \right\rangle_U \mathbb{1}_{r \leq \tau_n^{M,T+1}} dr \\ &+ \int_s^{s+\delta} \sum_{i=1}^\infty \left\| \mathcal{P}_n \mathcal{G}_i(r, \check{\boldsymbol{\Psi}}^n_r) \right\|_U^2 \mathbb{1}_{r \leq \tau_n^{M,T+1}} dr + 2 \int_s^{s+\delta} \left\langle \mathcal{P}_n \mathcal{G}(r, \check{\boldsymbol{\Psi}}^n_r), \check{\boldsymbol{\Psi}}^n_r - \check{\boldsymbol{\Psi}}^n_s \right\rangle_U d\mathcal{W}_r \end{aligned}$$

where we have incorporated the indicator function into the $\boldsymbol{\Psi}^n$ and $\boldsymbol{\Psi}^{n,M}$. Just as we proceeded from (3.25) in Proposition 3.24, we can use properties of the projection $\mathcal{P}_n$ and apply the assumption (3.6), to obtain the inequality

$$\begin{aligned} \left\| \boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_{s} \right\|_U^2 \leq c \int_s^{s+\delta} 1 + \left\| \check{\boldsymbol{\Psi}}^n_r \right\|_U^2 dr - 2 \int_s^{s+\delta} \left\langle \mathcal{A}(r, \check{\boldsymbol{\Psi}}^n_r), \check{\boldsymbol{\Psi}}^n_s \right\rangle_U dr \\ + 2 \int_s^{s+\delta} \left\langle \mathcal{G}(r, \check{\boldsymbol{\Psi}}^n_r), \check{\boldsymbol{\Psi}}^n_r - \check{\boldsymbol{\Psi}}^n_s \right\rangle_U d\mathcal{W}_r \end{aligned}$$

where we have simply dropped the helpful $-\gamma \left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^2$ contribution from consideration. Using (3.36) in the first integral, c.f. (3.26), we can actually pass to the bound

$$\left\|\boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_s\right\|_U^2 \leq c\delta - 2\int_s^{s+\delta} \left\langle \mathcal{A}(r, \check{\boldsymbol{\Psi}}^n_r), \check{\boldsymbol{\Psi}}^n_s \right\rangle_U dr + 2\int_s^{s+\delta} \left\langle \mathcal{G}(r, \check{\boldsymbol{\Psi}}^n_r), \check{\boldsymbol{\Psi}}^n_r - \check{\boldsymbol{\Psi}}^n_s \right\rangle_U d\mathcal{W}_r$$

integrating the constant over the time interval of size δ. To this we can invoke Assumption 3.2.3, (3.8), and then take expectation to reach

$$\mathbb{E}\left(\left\|\boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_s\right\|_U^2\right) \leq c\delta + c\mathbb{E}\int_s^{s+\delta} \left(1 + \left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^{\frac{3}{2}}\right)\left(1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^{\frac{3}{2}}\right) dr \tag{3.38}$$

where the stochastic integral of null expectation drops out. Of course, in aiming to show (3.35), we are actually interested in

$$\lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{E}\int_0^{T-\delta} \left\|\boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_s\right\|_U^2 ds.$$

With (3.38), use of the Fubini-Tonelli Theorem and considering the iterated integral as an integral over the product space, we obtain

$$\begin{aligned}
&\lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{E}\int_0^{T-\delta} \left\|\boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_s\right\|_U^2 ds \\
&\qquad = \lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \int_0^{T-\delta} \mathbb{E}\left\|\boldsymbol{\Psi}^{n,M}_{s+\delta} - \boldsymbol{\Psi}^{n,M}_s\right\|_U^2 ds \\
&\qquad \leq \lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \int_0^{T-\delta} \left[c\delta + c\mathbb{E}\int_s^{s+\delta} \left(1 + \left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^{\frac{3}{2}}\right)\left(1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^{\frac{3}{2}}\right) dr\right] ds \\
&\qquad = c\lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{E}\int_0^{T-\delta}\int_s^{s+\delta} \left(1 + \left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^{\frac{3}{2}}\right)\left(1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^{\frac{3}{2}}\right) drds \\
&\qquad = c\lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} \mathbb{E}\int_0^{T}\int_{0 \vee (r-\delta)}^{r \wedge T-\delta} \left(1 + \left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^{\frac{3}{2}}\right)\left(1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^{\frac{3}{2}}\right) dsdr.
\end{aligned} \tag{3.39}$$

Note that for each fixed r,

$$\begin{aligned}
&\int_{0 \vee (r-\delta)}^{r \wedge T-\delta} \left(1 + \left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^{\frac{3}{2}}\right)\left(1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^{\frac{3}{2}}\right) ds \\
&\qquad = \left(1 + \left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^{\frac{3}{2}}\right)\int_{0 \vee (r-\delta)}^{r \wedge T-\delta} \left(1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^{\frac{3}{2}}\right) ds \\
&\qquad \leq \left(1 + \left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^{\frac{3}{2}}\right)\left[\left(\int_{0 \vee (r-\delta)}^{r \wedge T-\delta} 1 ds\right)^{\frac{1}{4}}\left(\int_{0 \vee (r-\delta)}^{r \wedge T-\delta} \left(1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^{\frac{3}{2}}\right)^{\frac{4}{3}} ds\right)^{\frac{3}{4}}\right] \\
&\qquad \leq c\delta^{\frac{1}{4}}\left(1 + \left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^2\right)\left(\int_{0 \vee (r-\delta)}^{r \wedge T-\delta} 1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^2 ds\right)^{\frac{3}{4}} \\
&\qquad \leq c\delta^{\frac{1}{4}}\left(1 + \left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^2\right)
\end{aligned} \tag{3.40}$$

having used (3.26) in the final line, and Hölder's Inequality beforehand. Substituting this back into (3.39),

$$\begin{aligned}\lim_{\delta\to 0^+}\sup_{n\in\mathbb{N}}\mathbb{E}\int_0^{T-\delta}\left\|\boldsymbol{\Psi}^{n,M}_{s+\delta}-\boldsymbol{\Psi}^{n,M}_{s}\right\|^2 ds \leq c\lim_{\delta\to 0^+}\sup_{n\in\mathbb{N}}\mathbb{E}\int_0^{T} c\delta^{\frac{1}{4}}\left(1+\left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^2\right)dr \leq \lim_{\delta\to 0^+} c\delta^{\frac{1}{4}}\\ = 0\end{aligned}$$

which concludes the proof.

□

Remark 3.2.14. *Reversing the order of integration in (3.39) may appear superfluous, as similarly to (3.40) one could directly obtain*

$$\int_s^{s+\delta}\left(1+\left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H^{\frac{3}{2}}\right)\left(1+\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^{\frac{3}{2}}\right)dr \leq c\delta^{\frac{1}{4}}\left(1+\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^2\right)$$

which leads to the result without change. This additional step would facilitate the proof if there were a full exponent of 2 on $\left\|\check{\boldsymbol{\Psi}}^n_r\right\|_H$, so although not necessary here, it may be useful in other scenarios.

To achieve a characterisation of the limit process at each time, we will need to show tightness in $C\left([0,T];H^*\right)$. The idea is to apply Lemma 2.3.3, for $\mathcal{Y}=H$ and $\mathcal{H}=U$. The condition (2.28) has already been shown from the stronger (3.29) so to apply the Lemma we only need to verify (2.29). This is reminiscent of the condition (2.32) just verified, so just as we saw for Proposition 3.2.13 it is sufficient to verify the following.

Proposition 3.2.15. *For any sequence of stopping times (γ_n) with $\gamma_n:\Omega\to[0,T]$, and any $M>1$, $f\in H$,*

$$\lim_{\delta\to 0^+}\sup_{n\in\mathbb{N}}\mathbb{E}\left(\left|\left\langle\boldsymbol{\Psi}^{n,M}_{(\gamma_n+\delta)\wedge T}-\boldsymbol{\Psi}^{n,M}_{\gamma_n},f\right\rangle_U\right|\right)=0. \tag{3.41}$$

Therefore, by Lemma 2.3.3, the sequence of the laws of $(\boldsymbol{\Psi}^n)$ is tight in the space of probability measures over $C\left([0,T];H^*\right)$.

Proof. Recalling (3.37), substituting in γ_n for s and stopping the process at T, we see that

$$\boldsymbol{\Psi}^{n,M}_{(\gamma_n+\delta)\wedge T}-\boldsymbol{\Psi}^{n,M}_{\gamma_n}=\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T}\mathcal{P}_n\mathcal{A}(r,\check{\boldsymbol{\Psi}}^n_r)\mathbb{1}_{r\leq\tau_n^{M,T+1}}dr+\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T}\mathcal{P}_n\mathcal{G}(r,\check{\boldsymbol{\Psi}}^n_r)\mathbb{1}_{r\leq\tau_n^{M,T+1}}d\mathcal{W}_r$$

holds $\mathbb{P}-a.s.$, to which we take the inner product with arbitrary $f\in H$ and absolute value to see that

$$\begin{aligned}\left|\left\langle\boldsymbol{\Psi}^{n,M}_{(\gamma_n+\delta)\wedge T}-\boldsymbol{\Psi}^{n,M}_{\gamma_n},f\right\rangle_U\right| &\leq \left|\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T}\left\langle\mathcal{A}(r,\check{\boldsymbol{\Psi}}^n_r),\mathcal{P}_nf\right\rangle_U\mathbb{1}_{r\leq\tau_n^{M,T+1}}dr\right|\\ &+\left|\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T}\left\langle\mathcal{G}(r,\check{\boldsymbol{\Psi}}^n_r),\mathcal{P}_nf\right\rangle_U\mathbb{1}_{r\leq\tau_n^{M,T+1}}d\mathcal{W}_r\right|\end{aligned}$$

having also carried over the projection $\mathcal{P}_n$. Similarly to Proposition 3.2.13, we invoke the assumption (3.8) to achieve the inequality

$$\left|\left\langle \boldsymbol{\Psi}^{n,M}_{(\gamma_n+\delta)\wedge T} - \boldsymbol{\Psi}^{n,M}_{\gamma_n}, f\right\rangle_U\right| \leq c\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T}\left[K_U(\check{\boldsymbol{\Psi}}^n_r) + \|\check{\boldsymbol{\Psi}}^n_r\|_H^{\frac{3}{2}}\right]\left[K_U(f) + \|f\|_H^{\frac{3}{2}}\right]dr$$
$$+ \left|\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T}\left\langle \mathcal{G}(r, \check{\boldsymbol{\Psi}}^n_r), \mathcal{P}_n f\right\rangle_U \mathbb{1}_{r\leq \tau_n^{M,T+1}}d\mathcal{W}_r\right|$$

where (3.2) has also been employed. We again use (3.36), and also allowing our constant c to depend on f, we have that

$$\left|\left\langle \boldsymbol{\Psi}^{n,M}_{(\gamma_n+\delta)\wedge T} - \boldsymbol{\Psi}^{n,M}_{\gamma_n}, f\right\rangle_U\right| \leq c\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T} 1 + \|\check{\boldsymbol{\Psi}}^n_r\|_H^{\frac{3}{2}}\,dr$$
$$+ \left|\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T}\left\langle \mathcal{G}(r, \check{\boldsymbol{\Psi}}^n_r), \mathcal{P}_n f\right\rangle_U \mathbb{1}_{r\leq \tau_n^{M,T+1}}d\mathcal{W}_r\right| \tag{3.42}$$

still $\mathbb{P}-a.s.$. Now taking expectation and applying the Burkholder-Davis-Gundy Inequality to the stochastic integral, and then using the assumption (3.9),

$$\mathbb{E}\left(\left|\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T}\left\langle \mathcal{G}(r, \check{\boldsymbol{\Psi}}^n_r), \mathcal{P}_n f\right\rangle_U \mathbb{1}_{r\leq \tau_n^{M,T+1}}d\mathcal{W}_r\right|\right)$$
$$\leq c\mathbb{E}\left(\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T}\left\langle \mathcal{G}_i(r, \check{\boldsymbol{\Psi}}^n_r), \mathcal{P}_n f\right\rangle_U^2 \mathbb{1}_{r\leq \tau_n^{M,T+1}}dr\right)^{\frac{1}{2}}$$
$$\leq c\mathbb{E}\left(\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T} K_U(\check{\boldsymbol{\Psi}}^n_r)K_H(f)dr\right)^{\frac{1}{2}}$$
$$\leq c\mathbb{E}\left(\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T} 1dr\right)^{\frac{1}{2}}$$
$$\leq c\delta^{\frac{1}{2}}.$$

Putting all of this back into (3.42), and reducing the constant integral to a δ as just seen,

$$\mathbb{E}\left(\left|\left\langle \boldsymbol{\Psi}^{n,M}_{(\gamma_n+\delta)\wedge T} - \boldsymbol{\Psi}^{n,M}_{\gamma_n}, f\right\rangle_U\right|\right) \leq c\mathbb{E}\int_{\gamma_n}^{(\gamma_n+\delta)\wedge T}\|\check{\boldsymbol{\Psi}}^n_r\|_H^{\frac{3}{2}}\,dr + c\delta + c\delta^{\frac{1}{2}}.$$

The remaining integral is treated exactly as was seen in (3.40), from which taking the supremum over n and limit as $\delta \to 0^+$ gives the result.

□

3.2.5 Existence of Martingale Weak Solutions

With tightness achieved, it is now a standard procedure to apply the Prohorov and Skorohod Representation Theorems to deduce the existence of a new probability space on which a sequence of processes with the same distribution as a subsequence of $(\boldsymbol{\Psi}^n)$ have some almost sure convergence to a limiting process. For notational simplicity we take this subsequence and keep it simply indexed by n. We state the precise result in the below theorem, following e.g. [121] Proposition 4.9 and Theorem 4.10. In particular we reference [121] Theorem A.1 to justify the use of a singular Cylindrical Brownian Motion $\tilde{\mathcal{W}}$, independent of n.

Theorem 3.2.16. *There exists a filtered probability space* $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, *a cylindrical Brownian Motion* $\tilde{\mathcal{W}}$ *over* $\mathfrak{U}$ *with respect to* $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, *a sequence of random variables* $(\tilde{\boldsymbol{\Psi}}^n_0)$, $\tilde{\boldsymbol{\Psi}}^n_0 : \tilde{\Omega} \to U$ *and a* $\tilde{\boldsymbol{\Psi}}_0 : \tilde{\Omega} \to U$, *a sequence of processes* $(\tilde{\boldsymbol{\Psi}}^n)$, $\tilde{\boldsymbol{\Psi}}^n : \tilde{\Omega} \times [0,T] \to H$ *is progressively measurable and a process* $\tilde{\boldsymbol{\Psi}} : \tilde{\Omega} \times [0,T] \to U$ *such that:*

1. *For each* $n \in \mathbb{N}$, $\tilde{\boldsymbol{\Psi}}^n_0$ *has the same law as* $\boldsymbol{\Psi}^n_0$;
2. *For* $\tilde{\mathbb{P}} - a.e.\ \omega$, $\tilde{\boldsymbol{\Psi}}^n_0(\omega) \to \tilde{\boldsymbol{\Psi}}_0(\omega)$ *in* U, *and thus* $\tilde{\boldsymbol{\Psi}}_0$ *has the same law as* $\boldsymbol{\Psi}_0$;
3. *For each* $n \in \mathbb{N}$ *and* $t \in [0,T]$, $\tilde{\boldsymbol{\Psi}}^n$ *satisfies the identity*
$$\tilde{\boldsymbol{\Psi}}^n_t = \tilde{\boldsymbol{\Psi}}^n_0 + \int_0^t \mathcal{P}_n \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}^n_s) ds + \int_0^t \mathcal{P}_n \mathcal{G}(s, \tilde{\boldsymbol{\Psi}}^n_s) d\tilde{\mathcal{W}}_s$$
$\tilde{\mathbb{P}} - a.s.$ *in* V_n;
4. *For* $\tilde{\mathbb{P}} - a.e\ \omega$, $\tilde{\boldsymbol{\Psi}}^n(\omega) \to \tilde{\boldsymbol{\Psi}}(\omega)$ *in* $L^2([0,T];U)$ *and* $C([0,T];H^*)$.

We now have our candidate martingale weak solution, and to prove that this is such a solution we need only to verify that $\tilde{\boldsymbol{\Psi}}$ is progressively measurable in H, for $\tilde{\mathbb{P}} - a.e.\ \omega$ $\tilde{\boldsymbol{\Psi}}_\cdot(\omega) \in L^\infty([0,T];U) \cap C_w([0,T];U) \cap L^2([0,T];H)$ and the identity (3.12). In fact from item 3, we can deduce that

$$\tilde{\mathbb{E}}\left(\left\|\tilde{\boldsymbol{\Psi}}^n\right\|^2_{UH,T}\right) \leq C\left[\mathbb{E}\left(\left\|\tilde{\boldsymbol{\Psi}}^n_0\right\|^2_U\right) + 1\right] \leq C\left[\left\|\tilde{\boldsymbol{\Psi}}_0\right\|^2_{L^\infty(\tilde{\Omega};U)} + 1\right] < \infty \tag{3.43}$$

in the same manner as we showed (3.29), without any need for localisation. The fact that $\left\|\tilde{\boldsymbol{\Psi}}^n_0\right\| \leq \left\|\tilde{\boldsymbol{\Psi}}_0\right\|$ $\tilde{\mathbb{P}} - a.s.$ and $\left\|\tilde{\boldsymbol{\Psi}}_0\right\|^2_{L^\infty(\tilde{\Omega};U)} < \infty$ is inherited from $\boldsymbol{\Psi}^n_0$, $\boldsymbol{\Psi}_0$ of the same law in U. This prompts the following results.

Lemma 3.2.17. $\tilde{\boldsymbol{\Psi}}^n \to \tilde{\boldsymbol{\Psi}}$ *in* $L^2\left(\tilde{\Omega}; L^2([0,T];U)\right)$.

Proof. This is immediate from an application of the Dominated Convergence Theorem, using the convergence in item 4 and the uniform boundedness (3.43). □

Proposition 3.2.18. $\tilde{\boldsymbol{\Psi}}$ *is progressively measurable in* H *and for* $\tilde{\mathbb{P}} - a.e.\ \omega$, $\tilde{\boldsymbol{\Psi}}_\cdot(\omega) \in L^\infty([0,T];U) \cap L^2([0,T];H)$.

Proof. From (3.43) we have that the sequence $(\tilde{\boldsymbol{\Psi}}^n)$ is uniformly bounded in $L^2\left(\tilde{\Omega}; L^2([0,T];H)\right)$ and $L^2\left(\tilde{\Omega}; L^\infty([0,T];U)\right)$. Firstly then we can deduce the existence of a subsequence $(\tilde{\boldsymbol{\Psi}}^{n_k})$ which is weakly convergent in the Hilbert Space $L^2\left(\tilde{\Omega}; L^2([0,T];H)\right)$ to some $\boldsymbol{\Phi}^1$, but we may also identify $L^2\left(\tilde{\Omega}; L^\infty([0,T];U)\right)$ with the dual space of $L^2\left(\tilde{\Omega}; L^1([0,T];U)\right)$ and as such from the Banach-Alaoglu Theorem we can extract a further subsequence $(\tilde{\boldsymbol{\Psi}}^{n_l})$ which is convergent to some $\boldsymbol{\Phi}^2$ in the weak* topology. These limits imply that $(\tilde{\boldsymbol{\Psi}}^{n_l})$ is convergent to both $\boldsymbol{\Phi}^1$ and $\boldsymbol{\Phi}^2$ in the weak topology of $L^2\left(\tilde{\Omega};L^2\left([0,T];U\right)\right)$, but from Lemma 3.2.17 then $(\tilde{\boldsymbol{\Psi}}^{n_l})$ converges to $\tilde{\boldsymbol{\Psi}}$ strongly (hence weakly) in this space as well. By uniqueness of limits in the weak topology then $\tilde{\boldsymbol{\Psi}} = \boldsymbol{\Phi}^1 = \boldsymbol{\Phi}^2$ as elements of $L^2\left(\tilde{\Omega};L^2\left([0,T];U\right)\right)$, so they agree $\tilde{\mathbb{P}}\times\lambda - a.s.$. Thus for $\tilde{\mathbb{P}} - a.e.\ \omega$, $\tilde{\boldsymbol{\Psi}}_\cdot(\omega) \in L^\infty\left([0,T];U\right)\cap L^2\left([0,T];H\right)$.

The progressive measurability is justified similarly; for any $t\in[0,T]$, we can use the progressive measurability of $(\tilde{\boldsymbol{\Psi}}^{n_k})$ to instead deduce $\tilde{\boldsymbol{\Psi}}$ as the weak limit in $L^2\left(\tilde{\Omega}\times[0,t];H\right)$ where $\tilde{\Omega}\times[0,t]$ is equipped with the $\tilde{\mathcal{F}}_t\times\mathcal{B}\left([0,t]\right)$ sigma-algebra. Therefore $\tilde{\boldsymbol{\Psi}}:\tilde{\Omega}\times[0,t]\rightarrow H$ is measurable with respect to this product sigma-algebra which justifies the progressive measurability. □

Proposition 3.2.19. *$\tilde{\boldsymbol{\Psi}}$ satisfies the identity (3.12). Moreover for $\tilde{\mathbb{P}} - a.e.\ \omega$, $\tilde{\boldsymbol{\Psi}}_\cdot(\omega)\in C_w\left([0,T];U\right)$.*

Proof. As a consequence of the Hahn-Banach Theorem we know that the dual space separates points, from which it is immediate that $\tilde{\boldsymbol{\Psi}}$ satisfies the identity (3.12) if and only if for every $f\in H$, $\tilde{\boldsymbol{\Psi}}$ satisfies

$$\left\langle \tilde{\boldsymbol{\Psi}}_t, f\right\rangle_{H^*\times H} = \left\langle \tilde{\boldsymbol{\Psi}}_0, f\right\rangle_{H^*\times H} + \left\langle \int_0^t \mathcal{A}(s,\tilde{\boldsymbol{\Psi}}_s)ds, f\right\rangle_{H^*\times H} + \left\langle \int_0^t \mathcal{G}(s,\tilde{\boldsymbol{\Psi}}_s)d\tilde{\mathcal{W}}_s, f\right\rangle_{H^*\times H}$$

$\tilde{\mathbb{P}} - a.s.$ in $\mathbb{R}$ for all $t\in[0,T]$[4]. In fact, as the system (a_k) forms a basis of H, then it is sufficient to show the identity for any $\psi\in\bigcup_n V_n$ replacing f. Using the continuity and linearity of the duality pairing, it is thus sufficient to show the new identity

$$\left\langle \tilde{\boldsymbol{\Psi}}_t, \psi\right\rangle_{H^*\times H} = \left\langle \tilde{\boldsymbol{\Psi}}_0, \psi\right\rangle_{H^*\times H} + \int_0^t \left\langle \mathcal{A}(s,\tilde{\boldsymbol{\Psi}}_s), \psi\right\rangle_{H^*\times H} ds + \int_0^t \left\langle \mathcal{G}(s,\tilde{\boldsymbol{\Psi}}_s), \psi\right\rangle_{H^*\times H} d\tilde{\mathcal{W}}_s \tag{3.44}$$

for any $\psi\in\bigcup_n V_n$, $\mathbb{P} - a.s.$ for all $t\in[0,T]$. Of course $\tilde{\boldsymbol{\Psi}}^n$ satisfies a corresponding identity

$$\left\langle \tilde{\boldsymbol{\Psi}}^n_t, \psi\right\rangle_{H^*\times H} = \left\langle \tilde{\boldsymbol{\Psi}}^n_0, \psi\right\rangle_{H^*\times H} + \int_0^t \left\langle \mathcal{P}_n\mathcal{A}(s,\tilde{\boldsymbol{\Psi}}^n_s), \psi\right\rangle_{H^*\times H} ds + \int_0^t \left\langle \mathcal{P}_n\mathcal{G}(s,\tilde{\boldsymbol{\Psi}}^n_s), \psi\right\rangle_{H^*\times H} d\tilde{\mathcal{W}}_s$$

so we look to show convergence in each of the corresponding terms. Fixing any such ψ and t, we appreciate that the required identity (3.44) would follow from only showing either $\mathbb{P} - a.s.$ or $L^p\left(\tilde{\Omega};\mathbb{R}\right)$ convergence of a subsequence for the corresponding terms, as a $\mathbb{P} - a.s.$ convergent subsequence can be extracted from the $L^p\left(\tilde{\Omega};\mathbb{R}\right)$ limit. Henceforth, we are concerned with

[4] We need not concern ourselves with whether the set of full probability depends on f, just as was true for the dependence on t, by separability of H: see Lemma 2.2.8.

the limits

$$\left\langle \tilde{\boldsymbol{\Psi}}^n_t, \psi \right\rangle_{H^* \times H} \longrightarrow \left\langle \tilde{\boldsymbol{\Psi}}_t, \psi \right\rangle_{H^* \times H} \tag{3.45}$$

$$\left\langle \tilde{\boldsymbol{\Psi}}^n_0, \psi \right\rangle_{H^* \times H} \longrightarrow \left\langle \tilde{\boldsymbol{\Psi}}_0, \psi \right\rangle_{H^* \times H} \tag{3.46}$$

$$\int_0^t \left\langle \mathcal{P}_n \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}^n_s), \psi \right\rangle_{H^* \times H} ds \longrightarrow \int_0^t \left\langle \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}_s), \psi \right\rangle_{H^* \times H} ds \tag{3.47}$$

$$\int_0^t \left\langle \mathcal{P}_n \mathcal{G}(s, \tilde{\boldsymbol{\Psi}}^n_s), \psi \right\rangle_{H^* \times H} d\tilde{\mathcal{W}}_s \longrightarrow \int_0^t \left\langle \mathcal{G}(s, \tilde{\boldsymbol{\Psi}}_s), \psi \right\rangle_{H^* \times H} d\tilde{\mathcal{W}}_s \tag{3.48}$$

of a subsequence, $\tilde{\mathbb{P}} - a.s.$ or in $L^p\left(\tilde{\Omega}; \mathbb{R}\right)$. The first convergence, (3.45), holds $\tilde{\mathbb{P}} - a.s.$ whilst the same convergence is true of (3.46) as $\tilde{\boldsymbol{\Psi}}^n_0 \to \tilde{\boldsymbol{\Psi}}_0$ in U, from the fact that the system (a_k) forms an orthogonal basis of U. We look to show (3.47) in $L^1\left(\tilde{\Omega}; \mathbb{R}\right)$, and do so by first observing that as $\psi \in \bigcup_n V_n$ then it is certainly in V_k for some k, so without loss of generality in the limit as $n \to \infty$ we can assume that $n > k$. Moreover,

$$\left\langle \mathcal{P}_n \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}^n_s), \psi \right\rangle_{H^* \times H} = \left\langle \mathcal{P}_n \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}^n_s), \psi \right\rangle_U = \left\langle \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}^n_s), \psi \right\rangle_U = \left\langle \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}^n_s), \psi \right\rangle_{H^* \times H}$$

by construction of the Gelfand Triple and that $\mathcal{P}_n$ is self-adjoint in U and is the identity on $V_k \subset V_n$. Revisiting (3.47),

$$\begin{aligned}\left| \left\langle \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}^n_s), \psi \right\rangle_{H^* \times H} - \left\langle \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}_s), \psi \right\rangle_{H^* \times H} \right| &= \left| \left\langle \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}^n_s) - \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}_s), \psi \right\rangle_{H^* \times H} \right| \\ &\leq c_t K_V(\psi) \left[1 + \left\| \tilde{\boldsymbol{\Psi}}^n_s \right\|_H + \left\| \tilde{\boldsymbol{\Psi}}_s \right\|_H \right] \left\| \tilde{\boldsymbol{\Psi}}^n_s - \tilde{\boldsymbol{\Psi}}_s \right\|_U\end{aligned}$$

where we have invoked Assumption 3.2.4, (3.10). Therefore, incorporating the $K_V(\psi)$ into the constant,

$$\begin{aligned}&\tilde{\mathbb{E}} \left| \int_0^t \left\langle \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}^n_s), \psi \right\rangle_{H^* \times H} - \left\langle \mathcal{A}(s, \tilde{\boldsymbol{\Psi}}_s), \psi \right\rangle_{H^* \times H} ds \right| \\ &\qquad \leq c \tilde{\mathbb{E}} \int_0^t \left[1 + \left\| \tilde{\boldsymbol{\Psi}}^n_s \right\|_H + \left\| \tilde{\boldsymbol{\Psi}}_s \right\|_H \right] \left\| \tilde{\boldsymbol{\Psi}}^n_s - \tilde{\boldsymbol{\Psi}}_s \right\|_U ds \\ &\qquad \leq c \left(\tilde{\mathbb{E}} \int_0^t 1 + \left\| \tilde{\boldsymbol{\Psi}}^n_s \right\|_H^2 + \left\| \tilde{\boldsymbol{\Psi}}_s \right\|_H^2 ds \right)^{\frac{1}{2}} \left(\tilde{\mathbb{E}} \int_0^t \left\| \tilde{\boldsymbol{\Psi}}^n_s - \tilde{\boldsymbol{\Psi}}_s \right\|_U^2 ds \right)^{\frac{1}{2}}\end{aligned}$$

where we have used Hölder's Inequality over the product integral, recalling that the progressive measurability ensures no issues in applying Fubini-Tonelli for the order of integration. This approaches zero, owing to both the uniform boundedness of the first expectation due to (3.43) and Proposition 3.2.18, as well as the convergence in Lemma 3.2.17 for the second expectation. The convergence (3.47) is justified, to which we move on to (3.48), to be shown in $L^2\left(\tilde{\Omega}; \mathbb{R}\right)$. The duality pairing is simply the U inner product in this case, and by again passing over the

$\mathcal{P}_n$, we can apply the Burkholder-Davis-Gundy Inequality to obtain

$$\begin{aligned}\tilde{\mathbb{E}}\left|\int_0^t \left\langle \mathcal{G}(s,\tilde{\boldsymbol{\Psi}}^n_s),\psi\right\rangle_U - \left\langle \mathcal{G}_i(s,\tilde{\boldsymbol{\Psi}}_s),\psi\right\rangle_U d\tilde{\mathcal{W}}_s\right|^2 &\leq \tilde{\mathbb{E}}\int_0^t \sum_{i=1}^\infty \left\langle \mathcal{G}_i(s,\tilde{\boldsymbol{\Psi}}^n_s) - \mathcal{G}(s,\tilde{\boldsymbol{\Psi}}_s),\psi\right\rangle_U^2 ds \\ &\leq c\tilde{\mathbb{E}}\int_0^t \left\|\tilde{\boldsymbol{\Psi}}^n_s - \tilde{\boldsymbol{\Psi}}^s\right\|_U^2 ds\end{aligned}$$

having applied the assumed (3.11) and contained the $K_V(\psi)$ in our constant. This again approaches zero due to Lemma 3.2.17, concluding the justification that $\tilde{\boldsymbol{\Psi}}$ satisfies (3.12). It now only remains to determine that for $\tilde{\mathbb{P}}-a.e.\ \omega$, $\tilde{\boldsymbol{\Psi}}_\cdot(\omega) \in C_w\left([0,T];U\right)$. By the identity just shown it is clear that $\left\langle \tilde{\boldsymbol{\Psi}}_\cdot(\omega), f\right\rangle \in C\left([0,T];\mathbb{R}\right)$ where $f \in H$ was arbitrary, but to conclude the weak continuity we must instead show this for any $\eta \in U$. Furthermore we fix such an ω and $\eta \in U$, any $t \in [0,T]$ and sequence of times (t_k) in $[0,T]$ such that $t_k \to t$. To demonstrate the continuity let's fix $\varepsilon > 0$, and choose an $f \in H$ such that

$$\|\eta - f\|_U < \frac{\varepsilon}{4}\sup_{s\in[0,T]}\left\|\tilde{\boldsymbol{\Psi}}_s(\omega)\right\|_U$$

where the right hand side is of course finite from Proposition 3.2.18. Note that there exists a $K \in \mathbb{N}$ such that for all $k \geq K$,

$$\left|\left\langle \tilde{\boldsymbol{\Psi}}_{t_k}(\omega) - \tilde{\boldsymbol{\Psi}}_t(\omega), f\right\rangle_U\right| < \frac{\varepsilon}{2}.$$

Then for all $k \geq K$ we have that

$$\begin{aligned}\left|\left\langle \tilde{\boldsymbol{\Psi}}_{t_k}(\omega) - \tilde{\boldsymbol{\Psi}}_t(\omega), \eta\right\rangle_U\right| &\leq \left|\left\langle \tilde{\boldsymbol{\Psi}}_{t_k}(\omega) - \tilde{\boldsymbol{\Psi}}_t(\omega), \eta - f\right\rangle_U\right| + \left|\left\langle \tilde{\boldsymbol{\Psi}}_{t_k}(\omega) - \tilde{\boldsymbol{\Psi}}_t(\omega), f\right\rangle_U\right| \\ &< 2\sup_{s\in[0,T]}\left\|\tilde{\boldsymbol{\Psi}}_s(\omega)\right\|_U \|\eta - f\|_U + \frac{\varepsilon}{2} \\ &< \varepsilon\end{aligned}$$

demonstrating the weak continuity and finishing the proof. □

3.3 Weak Solutions

We first introduce the necessary extension of Assumption Set 1 for the main result of this section.

3.3.1 Assumption Set 2

Recall the setup and notation of Subsection 3.1.1. We assume that there exists a $c_\cdot$, K and $\gamma > 0$ such that for all $f, g \in H$ and $t \in [0,T]$:

Assumption 3.3.1.

$$\|\mathcal{A}(t,f)\|_{H^*}^2 \leq c_t K_U(f)\left[1 + \|f\|_H^2\right]. \tag{3.49}$$

Assumption 3.3.2.

$$2\left\langle \mathcal{A}(t,f)-\mathcal{A}(t,g), f-g\right\rangle_{H^*\times H}+\sum_{i=1}^{\infty}\left\|\mathcal{G}_i(t,f)-\mathcal{G}_i(t,g)\right\|_U^2 \\ \leq c_t K_U(f,g)\left[1+\|f\|_H^2+\|g\|_H^2\right]\|f-g\|_U^2-\gamma\|f-g\|_H^2, \tag{3.50}$$

$$\sum_{i=1}^{\infty}\left\langle \mathcal{G}_i(t,f)-\mathcal{G}_i(t,g), f-g\right\rangle_U^2 \leq c_t K_U(f,g)\left[1+\|f\|_H^2+\|g\|_H^2\right]\|f-g\|_U^4. \tag{3.51}$$

We briefly comment on the purpose of each assumption:

- Assumption 3.3.1 ensures sufficient regularity in the drift to apply Proposition A.2.7 (Lemma 3.3.6).
- Assumption 3.3.2 is used to show uniqueness of weak solutions (Proposition 3.3.7).

3.3.2 Definitions and Results

We now state the definitions and main result for weak solutions.

Definition 3.3.3. *Let $\mathbf{\Psi}_0 : \Omega \to U$ be $\mathcal{F}_0-$measurable. A process $\mathbf{\Psi}$ which is progressively measurable in H and such that for $\mathbb{P}-a.e.\ \omega$, $\mathbf{\Psi}_\cdot(\omega) \in C([0,T];U) \cap L^2([0,T];H)$, is said to be a weak solution of the equation (3.1) if the identity (3.1) holds $\mathbb{P}-a.s.$ in H^* for all $t \in [0,T]$.*

Definition 3.3.4. *A weak solution $\mathbf{\Psi}$ of the equation (3.1) is said to be the unique solution if for any other such solution $\mathbf{\Phi}$,*

$$\mathbb{P}\left(\{\omega \in \Omega : \mathbf{\Psi}_t(\omega) = \mathbf{\Phi}_t(\omega) \quad \forall t \geq 0\}\right) = 1.$$

Theorem 3.3.5. *Let Assumption Sets 1 and 2 hold. For any given $\mathcal{F}_0-$measurable $\mathbf{\Psi}_0 : \Omega \to U$, there exists a unique weak solution of the equation (3.1).*

The rest of this section is devoted to the proof of Theorem 3.3.5.

3.3.3 Existence and Uniqueness of Weak Solutions

We continue to work with $\mathbf{\Psi}_0 \in L^\infty(\Omega;U)$, and take $\tilde{\mathbf{\Psi}}$ to be a martingale weak solution of the equation (3.1) as given in Theorem 3.2.7. Our approach to uniqueness is to consider the energy of the difference of two solutions, by applying Proposition A.2.7. For this, improved regularity is needed.

Lemma 3.3.6. *For $\tilde{\mathbb{P}}-a.e.\ \omega$, $\mathcal{A}\left(\cdot, \tilde{\mathbf{\Psi}}_\cdot(\omega)\right) \in L^2([0,T];H^*)$. Moreover, $\tilde{\mathbf{\Psi}}_\cdot(\omega) \in C([0,T];U)$.*

Proof. This additional regularity comes from Assumption 3.3.1, (3.49), as

$$\begin{aligned}\int_0^T \left\|\mathcal{A}\left(s, \tilde{\boldsymbol{\Psi}}_s(\omega)\right)\right\|_{H^*}^2 ds &\leq c\int_0^T K\left(\tilde{\boldsymbol{\Psi}}_s(\omega)\right)\left[1+\left\|\tilde{\boldsymbol{\Psi}}_s(\omega)\right\|_H^2\right] ds\\ &\leq c \sup_{r\in[0,T]} K\left(\tilde{\boldsymbol{\Psi}}_r(\omega)\right)\int_0^T \left[1+\left\|\tilde{\boldsymbol{\Psi}}_s(\omega)\right\|_H^2\right] ds\\ &< \infty\end{aligned}$$

using that $\tilde{\boldsymbol{\Psi}}_\cdot(\omega) \in L^\infty([0,T];U) \cap L^2([0,T];H)$ as a martingale weak solution. The continuity now follows as an application of Proposition A.2.7. □

Proposition 3.3.7. *Suppose that $\tilde{\boldsymbol{\Phi}}$ is another martingale weak solution of (3.1) with respect to the same filtered probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, cylindrical Brownian Motion $\tilde{\mathcal{W}}$ and initial condition $\tilde{\boldsymbol{\Phi}}_0 = \tilde{\boldsymbol{\Psi}}_0$ $\tilde{\mathbb{P}} - a.s.$. In addition assume that for $\tilde{\mathbb{P}} - a.e.$ ω, $\tilde{\boldsymbol{\Phi}}_\cdot(\omega) \in C([0,T];U)$. Then*

$$\tilde{\mathbb{P}}\left(\left\{\omega \in \tilde{\Omega} : \tilde{\boldsymbol{\Psi}}_t(\omega) = \tilde{\boldsymbol{\Phi}}_t(\omega) \quad \forall t \in [0,T]\right\}\right) = 1.$$

Proof. We make our argument by considering the expectation of the difference of the solutions $\tilde{\boldsymbol{\Psi}}, \tilde{\boldsymbol{\Phi}}$, and to do so we need to manufacture an increased regularity through stopping times once more. To this end let's define the stopping times (α_R) by

$$\alpha_R := T \wedge \inf\left\{r \geq 0 : \left\|\tilde{\boldsymbol{\Psi}}\right\|_{UH,r}^2 \geq R\right\} \wedge \inf\left\{r \geq 0 : \left\|\tilde{\boldsymbol{\Phi}}\right\|_{UH,r}^2 \geq R\right\}$$

and subsequent processes

$$\tilde{\boldsymbol{\Psi}}_\cdot^R := \tilde{\boldsymbol{\Psi}}_\cdot \mathbb{1}_{\cdot \leq \alpha_R}, \qquad \tilde{\boldsymbol{\Phi}}_\cdot^R := \tilde{\boldsymbol{\Phi}}_\cdot \mathbb{1}_{\cdot \leq \alpha_R}, \qquad \boldsymbol{\Pi} = \tilde{\boldsymbol{\Psi}}^R - \tilde{\boldsymbol{\Phi}}^R.$$

Moreover the difference process satisfies

$$\tilde{\boldsymbol{\Psi}}_{t\wedge\alpha_R} - \tilde{\boldsymbol{\Phi}}_{t\wedge\alpha_R} = \int_0^t \mathbb{1}_{s\leq\alpha_R}\left[\mathcal{A}\left(s, \tilde{\boldsymbol{\Psi}}_s^R\right) - \mathcal{A}\left(s, \tilde{\boldsymbol{\Phi}}_s^R\right)\right] ds + \int_0^t \mathbb{1}_{s\leq\alpha_R}\left[\mathcal{G}\left(s, \tilde{\boldsymbol{\Psi}}_s^R\right) - \mathcal{G}\left(s, \tilde{\boldsymbol{\Phi}}_s^R\right)\right] d\tilde{\mathcal{W}}_s$$

and we can apply the Energy Equality of Proposition A.2.7 to see that

$$\begin{aligned}\left\|\tilde{\boldsymbol{\Psi}}_{t\wedge\alpha_R} - \tilde{\boldsymbol{\Phi}}_{t\wedge\alpha_R}\right\|_U^2 &= 2\int_0^t \mathbb{1}_{s\leq\alpha_R}\left\langle \mathcal{A}\left(s, \tilde{\boldsymbol{\Psi}}_s^R\right) - \mathcal{A}\left(s, \tilde{\boldsymbol{\Phi}}_s^R\right), \boldsymbol{\Pi}_s\right\rangle_{H^*\times H} ds\\ &+ \int_0^t \mathbb{1}_{s\leq\alpha_R}\sum_{i=1}^\infty \left\|\mathcal{G}_i\left(s, \tilde{\boldsymbol{\Psi}}_s^R\right) - \mathcal{G}_i\left(s, \tilde{\boldsymbol{\Phi}}_s^R\right)\right\|_U^2 ds\\ &+ 2\int_0^t \left\langle \mathcal{G}\left(s, \tilde{\boldsymbol{\Psi}}_s^R\right) - \mathcal{G}\left(s, \tilde{\boldsymbol{\Phi}}_s^R\right), \boldsymbol{\Pi}_s\right\rangle_U d\tilde{\mathcal{W}}_s.\end{aligned}$$

Motivated by the use of Lemma A.4.16, we consider arbitrary stopping times $0 \leq \theta_j \leq \theta_k \leq T$ and substitute θ_j into the above, then subtract this from the identity for any $\theta_j \leq r \leq T$, to give that

$$\begin{aligned}\left\|\tilde{\mathbf{\Psi}}_{t\wedge\alpha_R}-\tilde{\mathbf{\Phi}}_{t\wedge\alpha_R}\right\|_U^2 = \left\|\tilde{\mathbf{\Psi}}_{\theta_j\wedge\alpha_R}-\tilde{\mathbf{\Phi}}_{\theta_j\wedge\alpha_R}\right\|_U^2 &+ 2\int_{\theta_j}^t \mathbb{1}_{s\leq\alpha_R}\left\langle \mathcal{A}\left(s,\tilde{\mathbf{\Psi}}_s^R\right)-\mathcal{A}\left(s,\tilde{\mathbf{\Phi}}_s^R\right),\mathbf{\Pi}_s\right\rangle_{H^*\times H} ds\\
&+\int_{\theta_j}^t \mathbb{1}_{s\leq\alpha_R}\sum_{i=1}^\infty\left\|\mathcal{G}_i\left(s,\tilde{\mathbf{\Psi}}_s^R\right)-\mathcal{G}_i\left(s,\tilde{\mathbf{\Phi}}_s^R\right)\right\|_U^2 ds\\
&+2\int_{\theta_j}^t\left\langle \mathcal{G}\left(s,\tilde{\mathbf{\Psi}}_s^R\right)-\mathcal{G}\left(s,\tilde{\mathbf{\Phi}}_s^R\right),\mathbf{\Pi}_s\right\rangle_U d\tilde{\mathcal{W}}_s.\end{aligned}$$

We next employ Assumption 3.3.2, (3.50), and combine a few steps into one. Firstly we use the boundedness coming from the stopping time α_R to control the K_U term in (3.50), much like we saw from (3.36); it should be noted that without the continuity of the processes in U, we could not guarantee such a control. Secondly, we simply ignore the helpful γ term. All of this leads to the new inequality

$$\begin{aligned}\left\|\tilde{\mathbf{\Psi}}_{r\wedge\alpha_R}-\tilde{\mathbf{\Phi}}_{r\wedge\alpha_R}\right\|_U^2 \leq \left\|\tilde{\mathbf{\Psi}}_{\theta_j\wedge\alpha_R}-\tilde{\mathbf{\Phi}}_{\theta_j\wedge\alpha_R}\right\|_U^2 &+ c\int_{\theta_j}^r\left(1+\left\|\tilde{\mathbf{\Psi}}_s^R\right\|_H^2+\left\|\tilde{\mathbf{\Phi}}_s^R\right\|_H^2\right)\|\mathbf{\Pi}_s\|_U^2 ds\\
&+2\int_{\theta_j}^t\left\langle \mathcal{G}\left(s,\tilde{\mathbf{\Psi}}_s^R\right)-\mathcal{G}\left(s,\tilde{\mathbf{\Phi}}_s^R\right),\mathbf{\Pi}_s\right\rangle_U d\tilde{\mathcal{W}}_s.\end{aligned}$$

We take the absolute value on the right hand side, followed by the supremum over $r\in[\theta_j,\theta_k]$, then the expectation and immediately apply the Burkholder-Davis-Gundy Inequality to achieve that

$$\begin{aligned}\tilde{\mathbb{E}}\left(\sup_{r\in[\theta_j,\theta_k]}\left\|\tilde{\mathbf{\Psi}}_{r\wedge\alpha_R}-\tilde{\mathbf{\Phi}}_{r\wedge\alpha_R}\right\|_U^2\right) \leq \tilde{\mathbb{E}}&\left(\left\|\tilde{\mathbf{\Psi}}_{\theta_j\wedge\alpha_R}-\tilde{\mathbf{\Phi}}_{\theta_j\wedge\alpha_R}\right\|_U^2\right)\\
&+c\tilde{\mathbb{E}}\int_{\theta_j}^{\theta_k}\left(1+\left\|\tilde{\mathbf{\Psi}}_s^R\right\|_H^2+\left\|\tilde{\mathbf{\Phi}}_s^R\right\|_H^2\right)\|\mathbf{\Pi}_s\|_U^2 ds\\
&+c\tilde{\mathbb{E}}\left(\int_{\theta_j}^{\theta_k}\sum_{i=1}^\infty\left\langle \mathcal{G}_i\left(s,\tilde{\mathbf{\Psi}}_s^R\right)-\mathcal{G}_i\left(s,\tilde{\mathbf{\Phi}}_s^R\right),\mathbf{\Pi}_s\right\rangle_U^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$

We now use the assumption (3.51) and follow the same process as (3.28) to obtain that

$$\begin{aligned}&c\tilde{\mathbb{E}}\left(\int_{\theta_j}^{\theta_k}\sum_{i=1}^\infty\left\langle \mathcal{G}_i\left(s,\tilde{\mathbf{\Psi}}_s^R\right)-\mathcal{G}_i\left(s,\tilde{\mathbf{\Phi}}_s^R\right),\mathbf{\Pi}_s\right\rangle_U^2 ds\right)^{\frac{1}{2}}\\
&\qquad\leq c\tilde{\mathbb{E}}\left(\int_{\theta_j}^{\theta_k}\left(1+\left\|\tilde{\mathbf{\Psi}}_s^R\right\|_H^2+\left\|\tilde{\mathbf{\Phi}}_s^R\right\|_H^2\right)\|\mathbf{\Pi}_s\|_U^4 ds\right)^{\frac{1}{2}}\\
&\qquad\leq \frac{1}{2}\tilde{\mathbb{E}}\left(\sup_{r\in[\theta_j,\theta_k]}\|\mathbf{\Pi}_r\|_U^2\right)+c\tilde{\mathbb{E}}\int_{\theta_j}^{\theta_k}\left(1+\left\|\tilde{\mathbf{\Psi}}_s^R\right\|_H^2+\left\|\tilde{\mathbf{\Phi}}_s^R\right\|_H^2\right)\|\mathbf{\Pi}_s\|_U^2 ds.\end{aligned}$$

We now use that $\|\mathbf{\Pi}_r\|_U^2 \leq \left\|\tilde{\mathbf{\Psi}}_{r\wedge\alpha_R} - \tilde{\mathbf{\Phi}}_{r\wedge\alpha_R}\right\|_U^2$ and rearrange to give

$$\tilde{\mathbb{E}}\left(\sup_{r\in[\theta_j,\theta_k]}\left\|\tilde{\mathbf{\Psi}}_{r\wedge\alpha_R} - \tilde{\mathbf{\Phi}}_{r\wedge\alpha_R}\right\|_U^2\right) \leq 2\tilde{\mathbb{E}}\left(\left\|\tilde{\mathbf{\Psi}}_{\theta_j\wedge\alpha_R} - \tilde{\mathbf{\Phi}}_{\theta_j\wedge\alpha_R}\right\|_U^2\right)$$
$$+ c\tilde{\mathbb{E}}\int_{\theta_j}^{\theta_k}\left(1 + \left\|\tilde{\mathbf{\Psi}}_s^R\right\|_H^2 + \left\|\tilde{\mathbf{\Phi}}_s^R\right\|_H^2\right)\left\|\tilde{\mathbf{\Psi}}_{s\wedge\alpha_R} - \tilde{\mathbf{\Phi}}_{s\wedge\alpha_R}\right\|_U^2 ds.$$

We can now apply Lemma A.4.16 to deduce that

$$\tilde{\mathbb{E}}\left(\sup_{r\in[0,T]}\left\|\tilde{\mathbf{\Psi}}_{r\wedge\alpha_R} - \tilde{\mathbf{\Phi}}_{r\wedge\alpha_R}\right\|_U^2\right) = 0$$

as of course $\tilde{\mathbf{\Psi}}_0 = \tilde{\mathbf{\Phi}}_0$ $\tilde{\mathbb{P}} - a.s.$. We note that $\left(\sup_{r\in[0,T]}\left\|\tilde{\mathbf{\Psi}}_{r\wedge\alpha_R} - \tilde{\mathbf{\Phi}}_{r\wedge\alpha_R}\right\|^2\right)$ is a monotone increasing sequence in R, hence we take the limit as $R \to \infty$ and apply the Monotone Convergence Theorem to obtain

$$\tilde{\mathbb{E}}\left(\sup_{r\in[0,T]}\left\|\tilde{\mathbf{\Psi}}_r - \tilde{\mathbf{\Phi}}_r\right\|_U^2\right) = 0$$

which gives the result. □

It is now immediate that Theorem 3.3.5 holds in this case of the bounded initial condition.

Corollary 3.3.7.1. *There exists a unique weak solution* $\mathbf{\Psi}$ *of the equation (3.1).*

Proof. This follows from a classical Yamada-Watanabe type result, proven rigorously in this setting in [130]. □

To prove Theorem 3.3.5 it thus only remains to extend the result to an arbitrary $\mathcal{F}_0-$measurable $\mathbf{\Psi}_0 : \Omega \to U$, which we now fix.

Proof of Theorem 3.3.5: We first show the existence of such a solution. The idea is as in [65] Theorem 3.40 where we use the fact that for each $k \in \mathbb{N}\cup\{0\}$ there exists a weak solution $\mathbf{\Psi}^k$ of the equation (3.1) for the bounded initial condition $\mathbf{\Psi}_0\mathbb{1}_{k\leq\|\mathbf{\Psi}_0\|_U<k+1}$. We argue that the process $\mathbf{\Psi}$ defined by

$$\mathbf{\Psi} := \sum_{k=1}^{\infty}\mathbf{\Psi}^k\mathbb{1}_{k\leq\|\mathbf{\Psi}_0\|_U<k+1}$$

is a weak solution. In fact this comes identically to Proposition 2.2.15[5], for the partition

$$A_k := \{\omega\in\Omega : k \leq \|\mathbf{\Psi}_0(\omega)\|_U < k+1\}$$

with stopping times τ^k trivially taken as T, and $\mathbf{\Psi}_0^k = \mathbf{\Psi}_0\mathbb{1}_{k\leq\|\mathbf{\Psi}_0\|_U<k+1}$. One can show uniqueness identically to Proposition 3.3.7.

[5] We have to shift our spaces to exactly fit the framework of Proposition 2.2.15, but the proof is the same.

□

3.4 Strong Solutions

We first introduce the necessary extension of Assumption Sets 1 and 2 for the main result of this section.

3.4.1 Assumption Set 3

Recall the setup and notation of Subsection 3.1.1. We now impose the existence of a new Banach Space $\bar{H}$[6] which is an extension of H, or precisely, $H \subseteq \bar{H} \subseteq U$ and for every $f \in \bar{H}$, $\|f\|_{\bar{H}} = \|f\|_H$. In addition, $\mathcal{G} : [0,T] \times V \to \mathscr{L}^2\left(\mathfrak{U}; \bar{H}\right)$ is assumed measurable. We also suppose that there exists a real valued sequence (μ_n) with $\mu_n \to \infty$ such that for any $f \in \bar{H}$,

$$\|(I - \mathcal{P}_n)f\|_U \leq \frac{1}{\mu_n} \|f\|_{\bar{H}} \tag{3.52}$$

where I represents the identity operator in U. Furthermore we assume that there exists a $\gamma > 0$ such that for any $\varepsilon > 0$, there exists a $c_\cdot$, K (dependent on ε) such that for any $\phi \in V$, $\phi^n \in V_n$ and $t \in [0,T]$:

Assumption 3.4.1.

$$\|\mathcal{A}(t,\phi)\|_U^2 + \sum_{i=1}^{\infty} \|\mathcal{G}_i(t,\phi)\|_{\bar{H}}^2 \leq c_t K_U(\phi)\left[1 + \|\phi\|_H^4 + \|\phi\|_V^2\right] \tag{3.53}$$

Assumption 3.4.2.

$$2\left\langle \mathcal{P}_n\mathcal{A}(t,\phi^n), \phi^n\right\rangle_H + \sum_{i=1}^{\infty} \|\mathcal{P}_n\mathcal{G}_i(t,\phi^n)\|_H^2 \leq c_t K_U(\phi^n)\left[1 + \|\phi^n\|_H^4\right] - \gamma\|\phi^n\|_V^2, \tag{3.54}$$

$$\sum_{i=1}^{\infty} \left\langle \mathcal{P}_n\mathcal{G}_i(t,\phi^n), \phi^n\right\rangle_H^2 \leq c_t K_U(\phi^n)\left[1 + \|\phi^n\|_H^6\right] + \varepsilon\|\phi^n\|_V^2. \tag{3.55}$$

We briefly comment on the purpose of each assumption:

- The property (3.52) is used to show the Cauchy result (Lemma 3.4.8).
- Assumption 3.4.1 ensures that the integrals in the definition of a strong solution (Definition 3.4.3) are well-defined. It also plays a role in the Cauchy result (Lemma 3.4.8).
- Assumption 3.4.2 is employed to demonstrate higher uniform estimates for the Galerkin System (Proposition 3.4.7).

3.4.2 Definitions and Results

We now state the definitions and main result for strong solutions.

[6] The need for this extension comes from applications where $\mathcal{G}$ would not map from V into H, but would map into the enlarged $\bar{H}$, see Subsection 4.4.2.

Definition 3.4.3. *Let $\boldsymbol{\Psi}_0 : \Omega \to H$ be $\mathcal{F}_0-$measurable. A process $\boldsymbol{\Psi}$ which is progressively measurable in V and such that for $\mathbb{P}-a.e.$ ω, $\boldsymbol{\Psi}_\cdot(\omega) \in L^\infty([0,T];H) \cap L^2([0,T];V)$, is said to be a strong solution of the equation (3.1) if the identity (3.1) holds $\mathbb{P}-a.s.$ in U for all $t \in [0,T]$.*

Note that a strong solution necessarily has continuous paths in U, from the evolution equation satisfied in this space.

Definition 3.4.4. *A strong solution $\boldsymbol{\Psi}$ of the equation (3.1) is said to be unique if for any other such solution $\boldsymbol{\Phi}$,*

$$\mathbb{P}\left(\{\omega \in \Omega : \boldsymbol{\Psi}_t(\omega) = \boldsymbol{\Phi}_t(\omega) \quad \forall t \geq 0\}\right) = 1.$$

Theorem 3.4.5. *Let Assumption Sets 1, 2 and 3 hold. For any given $\mathcal{F}_0-$measurable $\boldsymbol{\Psi}_0 : \Omega \to H$, there exists a unique strong solution of the equation (3.1).*

Our approach to proving this comes from the following lemma.

Lemma 3.4.6. *Let $\boldsymbol{\Phi}_0 : \Omega \to H$ be $\mathcal{F}_0-$measurable and suppose that $\boldsymbol{\Phi}$ is a unique weak solution of the equation (3.1) such that $\boldsymbol{\Phi}$ is progressively measurable in V, and for $\mathbb{P}-a.e.$ ω, $\boldsymbol{\Phi}_\cdot(\omega) \in L^\infty([0,T];H) \cap L^2([0,T];V)$. Then $\boldsymbol{\Phi}$ is a unique strong solution of the equation (3.1).*

Proof. We first address the existence. It only needs to be justified that $\boldsymbol{\Phi}$ satisfies the required identity (3.1) in U. By definition of the weak solution, this identity is satisfied in H^* and in fact the only term failing to belong to U is $\int_0^t \mathcal{A}(s,\boldsymbol{\Phi}_s)ds$, $\mathbb{P}-a.s.$. However, the additional regularity in $\boldsymbol{\Phi}$ and Assumption 3.4.1 ensure that $\mathcal{A}(\cdot,\boldsymbol{\Phi}_\cdot)$ belongs to $L^2([0,T];U)$ $\mathbb{P}-a.s.$; see, for example, [60] Subsection 2.2, which justifies that the terms are well defined. This concludes the fact that $\boldsymbol{\Phi}$ is a strong solution. For uniqueness, suppose that $\boldsymbol{\Psi}$ is another strong solution. Then in particular $\boldsymbol{\Psi}$ is a weak solution, from which uniqueness of weak solutions gives the result. □

The rest of this section is dedicated to the proof of Theorem 3.4.5.

3.4.3 An Improved Uniform Estimate for the Galerkin System

We first assume that

$$\boldsymbol{\Psi}_0 \in L^\infty(\Omega;H) \tag{3.56}$$

similarly to (3.13). In order to match the regularity of a strong solution, better estimates are required for the Galerkin System introduced in Subsection 3.2.3. We recall the notation (3.18), (3.22), for the unique strong solution $\boldsymbol{\Psi}^n$ of the equation (3.14), as well as introducing the corresponding notation to (3.19) for the higher norm: for a function $\boldsymbol{\Phi} \in C([0,t];H) \cap L^2([0,t];V)$ we define the norm

$$\|\boldsymbol{\Phi}\|_{HV,t}^2 := \sup_{r\in[0,t]} \|\boldsymbol{\Phi}_r\|_H^2 + \int_0^t \|\boldsymbol{\Phi}_r\|_V^2 \, dr \tag{3.57}$$

making explicit the dependence on the time t.

Proposition 3.4.7. *There exists a constant C, dependent on M but independent of n, such that*

$$\mathbb{E}\left\|\boldsymbol{\Psi}^n\right\|^2_{HV,\tau_n^{M,T+1}} \leq C\left[\mathbb{E}\left(\|\boldsymbol{\Psi}^n_0\|^2_H\right)+1\right]. \tag{3.58}$$

Proof. By equipping V_n with the H inner product, exactly as we saw for (3.25), we obtain the identity

$$\begin{aligned}\left\|\boldsymbol{\Psi}^{n,M}_r\right\|^2_H &= \|\boldsymbol{\Psi}^n_0\|^2_H + 2\int_0^r \langle\mathcal{P}_n\mathcal{A}(s,\boldsymbol{\Psi}^n_s),\boldsymbol{\Psi}^n_s\rangle_H \mathbb{1}_{s\leq\tau_n^{M,T+1}}ds\\ &+\int_0^r\sum_{i=1}^\infty\|\mathcal{P}_n\mathcal{G}_i(s,\boldsymbol{\Psi}^n_s)\|^2_H\mathbb{1}_{s\leq\tau_n^{M,T+1}}ds + 2\sum_{i=1}^\infty\int_0^r\left\langle\mathcal{P}_n\mathcal{G}_i(s,\check{\boldsymbol{\Psi}}^n_s),\check{\boldsymbol{\Psi}}^n_s\right\rangle_H dW^i_s.\end{aligned}$$

$\mathbb{P}-a.s.$ in $\mathbb{R}$ for all $r\in[0,T]$. In the direction of Lemma A.4.16, similarly to Proposition 3.3.7, let $0\leq\theta_j<\theta_k\leq t$ be two arbitrary stopping times. By substituting in θ_j to the above, and then subtracting this from the identity for any $\theta_j\leq r\leq t$ $\mathbb{P}-a.s.$, then we also have the equality

$$\begin{aligned}\left\|\boldsymbol{\Psi}^{n,M}_r\right\|^2_H &= \left\|\boldsymbol{\Psi}^{n,M}_{\theta_j}\right\|^2_H + 2\int_{\theta_j}^r\left(\langle\mathcal{P}_n\mathcal{A}(s,\boldsymbol{\Psi}^n_s),\boldsymbol{\Psi}^n_s\rangle_H + \sum_{i=1}^\infty\|\mathcal{P}_n\mathcal{G}_i(s,\boldsymbol{\Psi}^n_s)\|^2_H\right)\mathbb{1}_{s\leq\tau_n^{M,T+1}}ds\\ &+2\sum_{i=1}^\infty\int_{\theta_j}^r\left\langle\mathcal{P}_n\mathcal{G}_i(s,\check{\boldsymbol{\Psi}}^n_s),\check{\boldsymbol{\Psi}}^n_s\right\rangle_H dW^i_s\end{aligned}$$

$\mathbb{P}-a.s.$. Invoking Assumption 3.4.2, (3.54), and immediately applying (3.36), we deduce the inequality

$$\begin{aligned}\left\|\boldsymbol{\Psi}^{n,M}_r\right\|^2_H \leq \left\|\boldsymbol{\Psi}^{n,M}_{\theta_j}\right\|^2_H + c\int_{\theta_j}^r 1+\left\|\check{\boldsymbol{\Psi}}^n_s\right\|^4_H ds - \gamma\int_{\theta_j}^r\left\|\check{\boldsymbol{\Psi}}^n_s\right\|^2_V ds\\ +2\sum_{i=1}^\infty\int_{\theta_j}^r\left\langle\mathcal{P}_n\mathcal{G}_i(s,\check{\boldsymbol{\Psi}}^n_s),\check{\boldsymbol{\Psi}}^n_s\right\rangle_H dW^i_s\end{aligned} \tag{3.59}$$

having now assimilated the indicator function through the norms into the $\check{\boldsymbol{\Psi}}^n$, and noting that c again depends on M (and the initial condition). We now take the absolute value of the stochastic integral, followed by the supremum over $r\in[\theta_j,\theta_k]$, then the expectation and immediately apply the Burkholder-Davis-Gundy Inequality to see that

$$\begin{aligned}\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_k]}\left\|\boldsymbol{\Psi}^{n,M}_r\right\|^2_H\right)+\gamma\mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|^2_V ds \leq 2\mathbb{E}\left(\left\|\boldsymbol{\Psi}^{n,M}_{\theta_j}\right\|^2_H\right) + c\mathbb{E}\int_{\theta_j}^{\theta_k}1+\left\|\check{\boldsymbol{\Psi}}^n_s\right\|^4_H ds\\ + c\mathbb{E}\left(\int_{\theta_j}^{\theta_k}\sum_{i=1}^\infty\left\langle\mathcal{P}_n\mathcal{G}_i(s,\check{\boldsymbol{\Psi}}^n_s),\check{\boldsymbol{\Psi}}^n_s\right\rangle^2_H ds\right)^{\frac{1}{2}}.\end{aligned}$$

Now is the time to use the assumed (3.55), which gives us that for any choice of $\varepsilon > 0$,

$$\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_k]}\left\|\boldsymbol{\Psi}^{n,M}_r\right\|_H^2\right) + \gamma\mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_V^2 ds \leq 2\mathbb{E}\left(\left\|\boldsymbol{\Psi}^{n,M}_{\theta_j}\right\|_H^2\right) + c\mathbb{E}\int_{\theta_j}^{\theta_k} 1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^4 ds$$
$$+ c_\varepsilon\mathbb{E}\left(\int_{\theta_j}^{\theta_k} 1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^6 ds\right)^{\frac{1}{2}} + \varepsilon\mathbb{E}\left(\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_V^2 ds\right)^{\frac{1}{2}} \tag{3.60}$$

having used the simple property that $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$. We now apply Young's Inequality to see that

$$\left(\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_V^2 ds\right)^{\frac{1}{2}} \leq \frac{1}{2}\left[\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_V^2 ds + 1\right] \tag{3.61}$$

which can be plugged back into (3.60) for the choice $\varepsilon := \gamma$ to see that

$$\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_k]}\left\|\boldsymbol{\Psi}^{n,M}_r\right\|_H^2\right) + \frac{\gamma}{2}\mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_V^2 ds \leq 2\mathbb{E}\left(\left\|\boldsymbol{\Psi}^{n,M}_{\theta_j}\right\|_H^2\right) + c\mathbb{E}\int_{\theta_j}^{\theta_k} 1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^4 ds$$
$$+ c\mathbb{E}\left(\int_{\theta_j}^{\theta_k} 1 + \left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^6 ds\right)^{\frac{1}{2}}$$

where c is dependent on γ, which is not meaningful. Taking out the constant integrals as a constant for simplicity, we reach the inequality

$$\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_k]}\left\|\boldsymbol{\Psi}^{n,M}_r\right\|_H^2\right) + \frac{\gamma}{2}\mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_V^2 ds \leq 2\mathbb{E}\left(\left\|\boldsymbol{\Psi}^{n,M}_{\theta_j}\right\|_H^2\right) + c + c\mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^4 ds$$
$$+ c\mathbb{E}\left(\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^2\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^4 ds\right)^{\frac{1}{2}}.$$

We handle the last term just as we did in (3.28), also using that $\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^2 \leq \left\|\boldsymbol{\Psi}^{n,M}_s\right\|_H^2$, to obtain

$$\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_k]}\left\|\boldsymbol{\Psi}^{n,M}_r\right\|_H^2\right) + \mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_V^2 ds \leq c\mathbb{E}\left(\left\|\boldsymbol{\Psi}^{n,M}_{\theta_j}\right\|_H^2\right) + c + c\mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\check{\boldsymbol{\Psi}}^n_s\right\|_H^4 ds. \tag{3.62}$$

where we have also scaled γ out of the inequality, through a dependence in c. Now we may apply the Stochastic Grönwall lemma, Lemma A.4.16, for the processes

$$\boldsymbol{\phi} = \left\|\check{\boldsymbol{\Psi}}^n\right\|_H^2, \qquad \boldsymbol{\psi} = \left\|\check{\boldsymbol{\Psi}}^n\right\|_V^2, \qquad \boldsymbol{\eta} = \left\|\check{\boldsymbol{\Psi}}^n\right\|_H^2$$

noting that the bound (A.81) owes to $\tau_n^{M,T+1}$ defined in (3.18). The application of this result conclude the proof.

$\square$

3.4.4 The Cauchy Property

Towards an application of Proposition 2.3.5 for the Galerkin System and $\tau_n^{M,T}$ as in (3.18) and used throughout the paper, we verify (2.36) in this subsection.

Lemma 3.4.8. *For arbitrary $m < n$, $\phi^n \in V_n, \psi^m \in V_m$, define*

$$A := 2\left\langle \mathcal{P}_n\mathcal{A}(s,\phi^n) - \mathcal{P}_m\mathcal{A}(s,\psi^m), \phi^n - \psi^m \right\rangle_U + \sum_{i=1}^{\infty}\left\|\mathcal{P}_n\mathcal{G}_i(s,\phi^n) - \mathcal{P}_m\mathcal{G}_i(s,\psi^m)\right\|_U^2$$

$$B := \sum_{i=1}^{\infty}\left\langle \mathcal{P}_n\mathcal{G}_i(s,\phi^n) - \mathcal{P}_m\mathcal{G}_i(s,\psi^m), \phi^n - \psi^m \right\rangle_U^2.$$

Then there exists a sequence (λ_m) with $\lambda_m \to \infty$ such that

$$A \leq c_s\left[K_U(\phi^n,\psi^m) + \|\phi^n\|_H^2 + \|\psi^m\|_H^2\right]\|\phi^n - \psi^m\|_U^2 - \frac{\gamma}{2}\|\phi^n - \psi^m\|_H^2 + \frac{c_s}{\lambda_m}K_U(\phi^n,\psi^m)\left[1 + \|\phi^n\|_H^4 + \|\psi^m\|_H^4 + \|\phi^n\|_V^2 + \|\psi^m\|_V^2\right]; \tag{3.63}$$

$$B \leq c_s\left[K_U(\phi^n,\psi^m) + \|\phi^n\|_H^2 + \|\psi^m\|_H^2\right]\|\phi^n - \psi^m\|_U^4 + \frac{c_s}{\lambda_m}K_U(\phi^n,\psi^m)\left[1 + \|\phi^n\|_H^4 + \|\psi^m\|_H^4\right]. \tag{3.64}$$

Proof. We show the inequalities independently, starting with A through

$$\begin{aligned}
A &= 2\left\langle \mathcal{P}_n\left[\mathcal{A}(s,\phi^n) - \mathcal{A}(s,\psi^m)\right] + \left[\mathcal{P}_n - \mathcal{P}_m\right]\mathcal{A}(s,\psi^m), \phi^n - \psi^m \right\rangle_U \\
&\qquad + \sum_{i=1}^{\infty}\left\|\mathcal{P}_n\left[\mathcal{G}_i(s,\phi^n) - \mathcal{G}_i(s,\psi^m)\right] + \left[\mathcal{P}_n - \mathcal{P}_m\right]\mathcal{G}_i(s,\psi^m)\right\|_U^2 \\
&\leq 2\left\langle \mathcal{P}_n\left[\mathcal{A}(s,\phi^n) - \mathcal{A}(s,\psi^m)\right], \phi^n - \psi^m \right\rangle_U + \sum_{i=1}^{\infty}\left\|\mathcal{P}_n\left[\mathcal{G}_i(s,\phi^n) - \mathcal{G}_i(s,\psi^m)\right]\right\|_U^2 \\
&\qquad + 2\left\langle \left[\mathcal{P}_n - \mathcal{P}_m\right]\mathcal{A}(s,\psi^m), \phi^n - \psi^m \right\rangle_U + \sum_{i=1}^{\infty}\left\|\left[\mathcal{P}_n - \mathcal{P}_m\right]\mathcal{G}_i(s,\psi^m)\right\|_U^2 \\
&\qquad + 2\sum_{i=1}^{\infty}\left\|\mathcal{P}_n\left[\mathcal{G}_i(s,\phi^n) - \mathcal{G}_i(s,\psi^m)\right]\right\|_U\left\|\left[\mathcal{P}_n - \mathcal{P}_m\right]\mathcal{G}_i(s,\psi^m)\right\|_U \\
&\leq 2\left\langle \mathcal{A}(s,\phi^n) - \mathcal{A}(s,\psi^m), \phi^n - \psi^m \right\rangle_U + \sum_{i=1}^{\infty}\left\|\mathcal{G}_i(s,\phi^n) - \mathcal{G}_i(s,\psi^m)\right\|_U^2 \\
&\qquad + 2\left\langle \mathcal{A}(s,\psi^m), [I - \mathcal{P}_m]\phi^n - \psi^m \right\rangle_U + \sum_{i=1}^{\infty}\left\|\mathcal{P}_n\left[I - \mathcal{P}_m\right]\mathcal{G}_i(s,\psi^m)\right\|_U^2 \\
&\qquad + 2\sum_{i=1}^{\infty}\left\|\mathcal{G}_i(s,\phi^n) - \mathcal{G}_i(s,\psi^m)\right\|_U\left\|\mathcal{P}_n\left[I - \mathcal{P}_m\right]\mathcal{G}_i(s,\psi^m)\right\|_U \\
&=: \alpha + \beta + \kappa
\end{aligned}$$

having used that $\mathcal{P}_n$ is an orthogonal projection on U and $\mathcal{P}_n\mathcal{P}_m = \mathcal{P}_m$. We look to show the appropriate bounds on α, β and κ. For α we simply apply Assumption 3.3.2, (3.50). Moving on to β, we use again that $\mathcal{P}_n$ is an orthogonal projection and the property (3.52) to see that

$$\beta \leq \frac{2}{\mu_m}\left\|\mathcal{A}(s,\psi^m)\right\|_U\left\|\phi^n-\psi^m\right\|_H + \sum_{i=1}^\infty \frac{1}{\mu_m^2}\left\|\mathcal{G}_i(s,\psi^m)\right\|_{\bar{H}}^2.$$

Through Young's Inequality with a constant c dependent on γ, we can bound this further by

$$\frac{c}{\mu_m^2}\left(\left\|\mathcal{A}(s,\psi^m)\right\|_U^2 + \sum_{i=1}^\infty\left\|\mathcal{G}_i(s,\psi^m)\right\|_{\bar{H}}^2\right) + \frac{\gamma}{2}\left\|\phi^n-\psi^m\right\|_H^2$$

to which we apply Assumption 3.4.1, (3.53), to the bracketed term. As for κ, we have that

$$\begin{aligned}
\kappa &\leq \sum_{i=1}^\infty\left(\left\|\mathcal{G}_i(s,\phi^n)\right\|_U + \left\|\mathcal{G}_i(s,\psi^m)\right\|_U\right)\left\|\left[I-\mathcal{P}_m\right]\mathcal{G}_i(s,\psi^m)\right\|_U\\
&\leq c_s\sum_{i=1}^\infty\left(\left\|\mathcal{G}_i(s,\phi^n)\right\|_{\bar{H}} + \left\|\mathcal{G}_i(s,\psi^m)\right\|_{\bar{H}}\right)\left\|\left[I-\mathcal{P}_m\right]\mathcal{G}_i(s,\psi^m)\right\|_U\\
&\leq \frac{c_s}{\mu_m}\sum_{i=1}^\infty\left(\left\|\mathcal{G}_i(s,\phi^n)\right\|_{\bar{H}} + \left\|\mathcal{G}_i(s,\psi^m)\right\|_{\bar{H}}\right)\left\|\mathcal{G}_i(s,\psi^m)\right\|_{\bar{H}}\\
&\leq \frac{c_s}{\mu_m}\sum_{i=1}^\infty\left(\left\|\mathcal{G}_i(s,\phi^n)\right\|_{\bar{H}}^2 + \left\|\mathcal{G}_i(s,\psi^m)\right\|_{\bar{H}}^2\right)
\end{aligned}$$

which we handle through (3.53) once more. Altogether then, with notation $\lambda_m := \min\{\mu_m, \mu_m^2\}$, we have that

$$\begin{aligned}
A \leq{}& c_sK_U(\phi^n,\psi^m)\left[1+\left\|\phi^n\right\|_H^2+\left\|\psi^m\right\|_H^2\right]\left\|\phi^n-\psi^m\right\|_U^2 - \gamma\left\|\phi^n-\psi^m\right\|_H^2\\
&+\frac{c_s}{\lambda_m}K_U(\psi^m)\left[1+\left\|\psi^m\right\|_H^4+\left\|\psi^m\right\|_V^2\right] + \frac{\gamma}{2}\left\|\phi^n-\psi^m\right\|_H^2\\
&\quad+\frac{c_s}{\lambda_m}\left(K_U(\phi^n)\left[1+\left\|\phi^n\right\|_H^4+\left\|\phi^n\right\|_V^2\right] + c_sK_U(\psi^m)\left[1+\left\|\psi^m\right\|_H^4+\left\|\psi^m\right\|_V^2\right]\right)
\end{aligned}$$

which compresses into (3.63). It remains to show the bound on B, which we approach in a similar manner:

$$\begin{aligned}
B &= \sum_{i=1}^\infty\left\langle\mathcal{P}_n\left[\mathcal{G}_i(s,\phi^n)-\mathcal{G}_i(s,\psi^m)\right] + \left[\mathcal{P}_n-\mathcal{P}_m\right]\mathcal{G}_i(s,\psi^m), \phi^n-\psi^m\right\rangle_U^2\\
&\leq 2\sum_{i=1}^\infty\left(\left\langle\mathcal{G}_i(s,\phi^n)-\mathcal{G}_i(s,\psi^m),\phi^n-\psi^m\right\rangle_U^2 + \left\langle\left[I-\mathcal{P}_m\right]\mathcal{G}_i(s,\psi^m),\phi^n-\psi^m\right\rangle_U^2\right).
\end{aligned}$$

The first term here is precisely what we have in (3.51). For the second we use that $I - \mathcal{P}_m$ is itself an orthogonal projection in U, as well as the assumed (3.3) and (3.52):

$$\begin{aligned}\langle [I-\mathcal{P}_m]\,\mathcal{G}_i(s,\psi^m), \phi^n - \psi^m\rangle_U^2 &= \langle \mathcal{G}_i(s,\psi^m), [I-\mathcal{P}_m]\,(\phi^n-\psi^m)\rangle_U^2\\ &\leq \frac{c_s}{\lambda_m}K_U(\psi^m)\left[1+\|\psi^m\|_H^2\right]\|\phi^n-\psi^m\|_H^2\\ &\leq \frac{c_s}{\lambda_m}K_U(\psi^m)\left[1+\|\phi^n\|_H^4+\|\psi^m\|_H^4\right]\end{aligned}$$

which in total provides (3.64). We note that this is a coarse bound, though sufficient for our purposes.

□

Proposition 3.4.9. *For any $m,n \in \mathbb{N}$ with $m < n$, define the process $\boldsymbol{\Psi}^{m,n}$ by*

$$\boldsymbol{\Psi}^{m,n}_r(\omega) := \boldsymbol{\Psi}^n_r(\omega) - \boldsymbol{\Psi}^m_r(\omega).$$

Then for the sequence (λ_j) proposed in Lemma 3.4.8 and m sufficiently large such that $\lambda_m > 1$, there exists a constant C dependent on M but independent of m,n such that

$$\mathbb{E}\,\|\boldsymbol{\Psi}^{m,n}\|^2_{UH,\tau_m^{M,T+1}\wedge\tau_n^{M,T+1}} \leq C\left[\mathbb{E}\,\|\boldsymbol{\Psi}^{m,n}_0\|_U^2 + \frac{1}{\sqrt{\lambda_m}}\right] \tag{3.65}$$

and in particular,

$$\lim_{m\to\infty}\sup_{n\geq m}\left[\mathbb{E}\,\|\boldsymbol{\Psi}^{m,n}\|^2_{UH,\tau_m^{M,T+1}\wedge\tau_n^{M,T+1}}\right] = 0. \tag{3.66}$$

Proof. The proof uses very similar methods to those used in Proposition 3.4.7, this time relying on Lemma 3.4.8 instead of Assumption 3.4.2. For any $0 \leq r \leq t$, $\boldsymbol{\Psi}^{m,n}$ satisfies the identity

$$\begin{aligned}\boldsymbol{\Psi}^{m,n}_{r\wedge\tau_m^{M,T+1}\wedge\tau_n^{M,T+1}} = \boldsymbol{\Psi}^{m,n}_0 &+ \int_0^{r\wedge\tau_m^{M,T+1}\wedge\tau_n^{M,T+1}} \mathcal{P}_n\mathcal{A}(s,\boldsymbol{\Psi}^n_s) - \mathcal{P}_m\mathcal{A}(s,\boldsymbol{\Psi}^m_s)ds\\ &+ \sum_{i=1}^\infty\int_0^{r\wedge\tau_m^{M,T+1}\wedge\tau_n^{M,T+1}} \mathcal{P}_n\mathcal{G}_i(s,\boldsymbol{\Psi}^n_s) - \mathcal{P}_m\mathcal{G}_i(s,\boldsymbol{\Psi}^m_s)dW^i_s\end{aligned}$$

$\mathbb{P}-a.s.$ in V_n. We thus apply Proposition A.2.7 to this difference process, reaching the equality

$$\begin{aligned}\left\|\boldsymbol{\Psi}^{m,n}_{r\wedge\tau_m^{M,T+1}\wedge\tau_n^{M,T+1}}\right\|_U^2 &= \|\boldsymbol{\Psi}^{m,n}_0\|_U^2 + 2\int_0^{r\wedge\tau_m^{M,T+1}\wedge\tau_n^{M,T+1}} \langle \mathcal{P}_n\mathcal{A}(s,\boldsymbol{\Psi}^n_s) - \mathcal{P}_m\mathcal{A}(s,\boldsymbol{\Psi}^m_s), \boldsymbol{\Psi}^{m,n}_s\rangle_U\, ds\\ &+ \int_0^{r\wedge\tau_m^{M,T+1}\wedge\tau_n^{M,T+1}} \sum_{i=1}^\infty \|\mathcal{P}_n\mathcal{G}_i(s,\boldsymbol{\Psi}^n_s) - \mathcal{P}_m\mathcal{G}_i(s,\boldsymbol{\Psi}^m_s)\|_U^2\, ds\\ &+ 2\sum_{i=1}^\infty\int_0^{r\wedge\tau_m^{M,T+1}\wedge\tau_n^{M,T+1}} \langle \mathcal{P}_n\mathcal{G}_i(s,\boldsymbol{\Psi}^n_s) - \mathcal{P}_m\mathcal{G}_i(s,\boldsymbol{\Psi}^m_s), \boldsymbol{\Psi}^{m,n}_s\rangle_U\, dW^i_s.\end{aligned}$$

We introduce notation similar to (3.22), that is

$$\begin{aligned}
\hat{\mathbf{\Psi}}^m_\cdot &:= \mathbf{\Psi}^m_\cdot \mathbb{1}_{\cdot \leq \tau_m^{M,T+1} \wedge \tau_n^{M,T+1}} & \hat{\mathbf{\Psi}}^n_\cdot &:= \mathbf{\Psi}^n_\cdot \mathbb{1}_{\cdot \leq \tau_m^{M,T+1} \wedge \tau_n^{M,T+1}}\\
\hat{\mathbf{\Psi}}^{m,n} &:= \hat{\mathbf{\Psi}}^n - \hat{\mathbf{\Psi}}^m & \mathbf{\Psi}^{m,n,M}_\cdot &= \mathbf{\Psi}^{m,n}_{\cdot \wedge \tau_m^{M,T+1} \wedge \tau_n^{M,T+1}}
\end{aligned}$$

and thus rewrite our equality as

$$\begin{aligned}
\left\|\mathbf{\Psi}^{m,n,M}_r\right\|_U^2 = \|\mathbf{\Psi}^{m,n}_0\|_U^2 &+ 2\int_0^r \langle \mathcal{P}_n\mathcal{A}(s,\mathbf{\Psi}^n_s) - \mathcal{P}_m\mathcal{A}(s,\mathbf{\Psi}^m_s), \mathbf{\Psi}^{m,n}_s\rangle_U \mathbb{1}_{s \leq \tau_m^{M,T+1} \wedge \tau_n^{M,T+1}} ds\\
&+ \int_0^r \sum_{i=1}^\infty \|\mathcal{P}_n\mathcal{G}_i(s,\mathbf{\Psi}^n_s) - \mathcal{P}_m\mathcal{G}_i(s,\mathbf{\Psi}^m_s)\|_U^2 \mathbb{1}_{s \leq \tau_m^{M,T+1} \wedge \tau_n^{M,T+1}} ds\\
&+ 2\sum_{i=1}^\infty \int_0^r \left\langle \mathcal{P}_n\mathcal{G}_i(s,\hat{\mathbf{\Psi}}^n_s) - \mathcal{P}_m\mathcal{G}_i(s,\hat{\mathbf{\Psi}}^m_s), \hat{\mathbf{\Psi}}^{m,n}_s\right\rangle_U dW^i_s.
\end{aligned}$$

Identically to Proposition 3.4.7, we fix arbitrary stopping times $0 \leq \theta_j < \theta_k \leq t$ and have that for any $\theta_j \leq r \leq t$ $\mathbb{P} - a.s.$,

$$\begin{aligned}
\left\|\mathbf{\Psi}^{m,n,M}_r\right\|_U^2 = \left\|\mathbf{\Psi}^{m,n,M}_{\theta_j}\right\|_U^2 &+ 2\int_{\theta_j}^r \langle \mathcal{P}_n\mathcal{A}(s,\mathbf{\Psi}^n_s) - \mathcal{P}_m\mathcal{A}(s,\mathbf{\Psi}^m_s), \mathbf{\Psi}^{m,n}_s\rangle_U \mathbb{1}_{s \leq \tau_m^{M,T+1} \wedge \tau_n^{M,T+1}} ds\\
&+ \int_{\theta_j}^r \sum_{i=1}^\infty \|\mathcal{P}_n\mathcal{G}_i(s,\mathbf{\Psi}^n_s) - \mathcal{P}_m\mathcal{G}_i(s,\mathbf{\Psi}^m_s)\|_U^2 \mathbb{1}_{s \leq \tau_m^{M,T+1} \wedge \tau_n^{M,T+1}} ds\\
&+ 2\sum_{i=1}^\infty \int_{\theta_j}^r \left\langle \mathcal{P}_n\mathcal{G}_i(s,\hat{\mathbf{\Psi}}^n_s) - \mathcal{P}_m\mathcal{G}_i(s,\hat{\mathbf{\Psi}}^m_s), \hat{\mathbf{\Psi}}^{m,n}_s\right\rangle_U dW^i_s.
\end{aligned}$$

holds $\mathbb{P} - a.s.$. Combining the time integrals and applying Lemma 3.4.8, (3.63), we deduce the inequality

$$\begin{aligned}
\left\|\mathbf{\Psi}^{m,n,M}_r\right\|_U^2 \leq \left\|\mathbf{\Psi}^{m,n,M}_{\theta_j}\right\|_U^2 &+ c\int_{\theta_j}^r \left[1 + \left\|\hat{\mathbf{\Psi}}^m_s\right\|_H^2 + \left\|\hat{\mathbf{\Psi}}^n_s\right\|_H^2\right]\left\|\hat{\mathbf{\Psi}}^{m,n}_s\right\|_U^2 - \frac{\gamma}{2}\left\|\hat{\mathbf{\Psi}}^{m,n}_s\right\|_H^2 ds\\
&+ \frac{c}{\lambda_m}\int_{\theta_j}^r \left[1 + \left\|\hat{\mathbf{\Psi}}^m_s\right\|_H^4 + \left\|\hat{\mathbf{\Psi}}^n_s\right\|_H^4 + \left\|\hat{\mathbf{\Psi}}^m_s\right\|_V^2 + \left\|\hat{\mathbf{\Psi}}^n_s\right\|_V^2\right] ds\\
&+ 2\sum_{i=1}^\infty \int_{\theta_j}^r \left\langle \mathcal{P}_n\mathcal{G}_i(s,\hat{\mathbf{\Psi}}^n_s) - \mathcal{P}_m\mathcal{G}_i(s,\hat{\mathbf{\Psi}}^m_s), \hat{\mathbf{\Psi}}^{m,n}_s\right\rangle_U dW^i_s
\end{aligned}$$

again applying (3.36). All in one step we take the expectation, supremum over $r \in [\theta_j, \theta_k]$, apply the Burkholder-Davis-Gundy Inequality and employ Lemma 3.4.8, (3.64), to deduce that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_k]}\left\|\boldsymbol{\Psi}^{m,n,M}_r\right\|_U^2\right) + \frac{\gamma}{2}\mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\hat{\boldsymbol{\Psi}}^{m,n}_s\right\|_H^2 \\
&\qquad\leq 2\mathbb{E}\left(\left\|\boldsymbol{\Psi}^{m,n,M}_{\theta_j}\right\|_U^2\right) + c\mathbb{E}\int_{\theta_j}^{\theta_k}\left[1+\left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_H^2+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_H^2\right]\left\|\hat{\boldsymbol{\Psi}}^{m,n}_s\right\|_U^2 ds \\
&\qquad\quad + \frac{c}{\lambda_m}\mathbb{E}\int_{\theta_j}^{\theta_k} 1+\left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_H^4+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_H^4+\left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_V^2+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_V^2 ds \\
&\qquad\quad + c\mathbb{E}\left(\int_{\theta_j}^{\theta_k}\left[1+\left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_H^2+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_H^2\right]\left\|\hat{\boldsymbol{\Psi}}^{m,n}_s\right\|_U^4 ds\right)^{\frac{1}{2}} \\
&\qquad\quad + \frac{1}{\sqrt{\lambda_m}}\mathbb{E}\left(\int_{\theta_j}^{\theta_k} 1+\left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_H^4+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_H^4 ds\right)^{\frac{1}{2}}
\end{aligned}$$

where we have used that $(a+b)^{\frac{1}{2}} \leq a^{\frac{1}{2}} + b^{\frac{1}{2}}$ in the last term. Applying Young's Inequality as we did for (3.61), and using that m is large enough so that $\lambda_m \geq 1$ hence $\frac{1}{\lambda_m} \leq \frac{1}{\sqrt{\lambda_m}}$, we have that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_k]}\left\|\boldsymbol{\Psi}^{m,n,M}_r\right\|_U^2\right) + \frac{\gamma}{2}\mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\hat{\boldsymbol{\Psi}}^{m,n}_s\right\|_H^2 \\
&\qquad\leq 2\mathbb{E}\left(\left\|\boldsymbol{\Psi}^{m,n,M}_{\theta_j}\right\|_U^2\right) + c\mathbb{E}\int_{\theta_j}^{\theta_k}\left[1+\left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_H^2+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_H^2\right]\left\|\hat{\boldsymbol{\Psi}}^{m,n}_s\right\|_U^2 ds \\
&\qquad\quad + \frac{c}{\sqrt{\lambda_m}} + \frac{c}{\sqrt{\lambda_m}}\mathbb{E}\int_{\theta_j}^{\theta_k} \left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_H^4+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_H^4+\left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_V^2+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_V^2 ds \\
&\qquad\quad + c\mathbb{E}\left(\int_{\theta_j}^{\theta_k}\left[1+\left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_H^2+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_H^2\right]\left\|\hat{\boldsymbol{\Psi}}^{m,n}_s\right\|_U^4 ds\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.67}$$

The property that

$$\int_0^T \left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_H^2+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_H^2 ds \leq c, \tag{3.68}$$

owing to the first hitting times and as used in Proposition 3.4.7, is of vital importance here. We firstly control

$$\frac{c}{\sqrt{\lambda_m}}\mathbb{E}\int_{\theta_j}^{\theta_k} \left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_H^4+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_H^4+\left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_V^2+\left\|\hat{\boldsymbol{\Psi}}^{n}_s\right\|_V^2 ds \tag{3.69}$$

where we note that

$$\frac{c}{\sqrt{\lambda_m}}\mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\hat{\boldsymbol{\Psi}}^{m}_s\right\|_H^4 ds \leq \frac{c}{\sqrt{\lambda_m}}\mathbb{E}\left(\sup_{r\in[0,T]}\left\|\hat{\boldsymbol{\Psi}}^{m}_r\right\|_H^2\int_{\theta_j}^{\theta_k}\left\|\tilde{\boldsymbol{\Psi}}^{m}_s\right\|_H^2 ds\right) \leq \frac{c}{\sqrt{\lambda_m}}\mathbb{E}\left(\sup_{r\in[0,T]}\left\|\hat{\boldsymbol{\Psi}}^{m}_r\right\|_H^2\right)$$

where this expectation is bounded courtesy of Proposition 3.4.7. Using the same proposition for the $\|\cdot\|_V^2$ terms, we ultimately bound (3.69) by $\frac{c}{\sqrt{\lambda_m}}$. We estimate the final term in (3.67) as we did in (3.28), which combined with the control on (3.69), reduces (3.67) to

$$\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_k]}\left\|\boldsymbol{\Psi}^{m,n,M}_r\right\|_U^2\right)+\mathbb{E}\int_{\theta_j}^{\theta_k}\left\|\hat{\boldsymbol{\Psi}}^{m,n}_s\right\|_H^2$$
$$\leq c\mathbb{E}\left(\left\|\boldsymbol{\Psi}^{m,n,M}_{\theta_j}\right\|_U^2\right)+c\mathbb{E}\int_{\theta_j}^{\theta_k}\left[1+\left\|\hat{\boldsymbol{\Psi}}^m_s\right\|_H^2+\left\|\hat{\boldsymbol{\Psi}}^n_s\right\|_H^2\right]\left\|\hat{\boldsymbol{\Psi}}^{m,n}_s\right\|_U^2 ds+\frac{c}{\sqrt{\lambda_m}}$$

with similar seen in Proposition 3.4.7. An application of Lemma A.4.16 then provides that

$$\mathbb{E}\left(\sup_{r\in[0,T]}\left\|\boldsymbol{\Psi}^{m,n,M}_r\right\|_U^2\right)+\mathbb{E}\int_0^T\left\|\hat{\boldsymbol{\Psi}}^{m,n}_s\right\|_H^2\leq C\left[\|\boldsymbol{\Psi}^{m,n}_0\|_U^2+\frac{1}{\sqrt{\lambda_m}}\right]$$

which gives (3.65). The next result, (3.66), follows from this as

$$\|\boldsymbol{\Psi}^{m,n}_0\|_U^2=\|(\mathcal{P}_n-\mathcal{P}_m)\boldsymbol{\Psi}_0\|_U^2=\|\mathcal{P}_n\left[(I-\mathcal{P}_m)\boldsymbol{\Psi}_0\right]\|_U^2\leq\|(I-\mathcal{P}_m)\boldsymbol{\Psi}_0\|_U^2$$

and the fact that the system (a_k) forms an orthogonal basis of U.

□

3.4.5 Weak Equicontinuity

In this subsection, the remaining condition of Proposition 2.3.5, (2.37), is verified.

Lemma 3.4.10. *Let θ be a stopping time and (δ_j) a sequence of stopping times which converge to 0 $\mathbb{P}-a.s.$. Then*

$$\lim_{j\to\infty}\sup_{n\in\mathbb{N}}\mathbb{E}\left(\|\boldsymbol{\Psi}^n\|^2_{UH,(\theta+\delta_j)\wedge\tau_n^{M,T}}-\|\boldsymbol{\Psi}^n\|^2_{UH,\theta\wedge\tau_n^{M,T}}\right)=0.$$

Proof. We look at the energy identity satisfied by $\boldsymbol{\Psi}^n$ up until the stopping time $\theta\wedge\tau_n^{M,T}$ and then $(\theta+r)\wedge\tau_n^{M,T}$ for some $r\geq 0$. We have that

$$\left\|\boldsymbol{\Psi}^n_{\theta\wedge\tau_n^{M,T}}\right\|_U^2=\|\boldsymbol{\Psi}^n_0\|_U^2+2\int_0^{\theta\wedge\tau_n^{M,T}}\left\langle\mathcal{P}_n\mathcal{A}\left(s,\boldsymbol{\Psi}^n_s\right),\boldsymbol{\Psi}^n_s\right\rangle_U ds+\int_0^{\theta\wedge\tau_n^{M,T}}\sum_{i=1}^\infty\|\mathcal{P}_n\mathcal{G}_i\left(s,\boldsymbol{\Psi}^n_s\right)\|_U^2 ds$$
$$+2\int_0^{\theta\wedge\tau_n^{M,T}}\left\langle\mathcal{P}_n\mathcal{G}\left(s,\boldsymbol{\Psi}^n_s\right),\boldsymbol{\Psi}^n_s\right\rangle_U d\mathcal{W}_s$$

and similarly for $(\theta+r)\wedge\tau_n^{M,T}$, from which the difference of the equalities gives

$$\left\|\boldsymbol{\Psi}^n_{(\theta+r)\wedge\tau_n^{M,T}}\right\|_U^2=\left\|\boldsymbol{\Psi}^n_{\theta\wedge\tau_n^{M,T}}\right\|_U^2+2\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+r)\wedge\tau_n^{M,T}}\left\langle\mathcal{P}_n\mathcal{A}\left(s,\boldsymbol{\Psi}^n_s\right),\boldsymbol{\Psi}^n_s\right\rangle_U ds$$
$$+\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+r)\wedge\tau_n^{M,T}}\sum_{i=1}^\infty\|\mathcal{P}_n\mathcal{G}_i\left(s,\boldsymbol{\Psi}^n_s\right)\|_U^2 ds+2\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+r)\wedge\tau_n^{M,T}}\left\langle\mathcal{P}_n\mathcal{G}\left(s,\boldsymbol{\Psi}^n_s\right),\boldsymbol{\Psi}^n_s\right\rangle_U d\mathcal{W}_s.$$

Handling the projections as we did in (3.25), and invoking (3.6), we reduce to

$$\left\|\boldsymbol{\Psi}^n_{(\theta+r)\wedge\tau_n^{M,T}}\right\|_U^2 - \left\|\boldsymbol{\Psi}^n_{\theta\wedge\tau_n^{M,T}}\right\|_U^2 + \gamma\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+r)\wedge\tau_n^{M,T}}\|\boldsymbol{\Psi}^n_s\|_H^2\,ds$$
$$\leq c\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+r)\wedge\tau_n^{M,T}} 1ds + \int_{\theta\wedge\tau_n^{M,T}}^{(\theta+r)\wedge\tau_n^{M,T}} \left\langle \mathcal{G}\left(s,\boldsymbol{\Psi}^n_s\right), \boldsymbol{\Psi}^n_s\right\rangle_U d\mathcal{W}_s.$$

where the constant c again depends on M, through (3.36). We must be somewhat careful before taking the supremum over $r\in[0,\delta_j]$ as the term $\left\|\boldsymbol{\Psi}^n_{(\theta+r)\wedge\tau_n^{M,T}}\right\|_U^2 - \left\|\boldsymbol{\Psi}^n_{\theta\wedge\tau_n^{M,T}}\right\|_U^2$ could be negative, so we cannot necessarily bound $\gamma\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+r)\wedge\tau_n^{M,T}}\|\boldsymbol{\Psi}^n_s\|_H^2\,ds$ by the supremum of the right hand side. Therefore, we first treat this integral directly. We rely on both Proposition 3.4.7 and the control from $\tau_n^{M,T}$. Indeed,

$$\mathbb{E}\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}\|\boldsymbol{\Psi}^n_s\|_H^2\,ds$$
$$\leq \mathbb{E}\left(\sup_{r\in[0,\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_r\|_H\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}\|\boldsymbol{\Psi}^n_s\|_H\,ds\right)$$
$$\leq \left[\mathbb{E}\left(\sup_{r\in[0,\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_r\|_H^2\right)\right]^{\frac{1}{2}}\left[\mathbb{E}\left(\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}\|\boldsymbol{\Psi}^n_s\|_H\,ds\right)^2\right]^{\frac{1}{2}}$$
$$\leq c\left[\mathbb{E}\left(\left(\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}1+\|\boldsymbol{\Psi}^n_s\|_H^2\,ds\right)\left(\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}\|\boldsymbol{\Psi}^n_s\|_H\,ds\right)\right)\right]^{\frac{1}{2}}$$
$$\leq c\left[\mathbb{E}\left(\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}\|\boldsymbol{\Psi}^n_s\|_H\,ds\right)\right]^{\frac{1}{2}}$$
$$\leq c\left[\left[\mathbb{E}\left(\sup_{r\in[0,\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_r\|_H^2\right)\right]^{\frac{1}{2}}\left[\mathbb{E}\left(\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}1ds\right)^2\right]^{\frac{1}{2}}\right]^{\frac{1}{2}}$$
$$\leq c\left[\mathbb{E}(\delta_j^2)\right]^{\frac{1}{4}}.$$

Noting that δ_j is $\mathbb{P}-a.s.$ monotone decreasing (as $j\to\infty$) and convergent to 0, the Monotone Convergence Theorem thus justifies that

$$c\left[\mathbb{E}(\delta_j^2)\right]^{\frac{1}{4}} = o_j$$

where o_j represents a constant independent of n which goes to zero as $\delta_j\to 0$. We can now employ this bound directly as we take expectation, the supremum over $r\in[0,\delta_j]$ and then use the Burkholder-Davis-Gundy Inequality,

$$\mathbb{E}\left[\sup_{r\in[0,\delta_j]}\left\|\boldsymbol{\Psi}^n_{(\theta+r)\wedge\tau_n^{M,T}}\right\|_U^2-\left\|\boldsymbol{\Psi}^n_{\theta\wedge\tau_n^{M,T}}\right\|_U^2+\gamma\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}\|\boldsymbol{\Psi}^n_s\|_H^2\,ds\right]$$
$$\leq o_j+c\mathbb{E}\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}1ds+c\mathbb{E}\left(\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}\sum_{i=1}^{\infty}\left\langle\mathcal{G}_i\left(s,\boldsymbol{\Psi}^n_s\right),\boldsymbol{\Psi}^n_s\right\rangle_U^2ds\right)^{\frac{1}{2}}.$$

We then use (3.7) and again control the U norm by a constant, achieving that

$$\mathbb{E}\left[\sup_{r\in[0,\delta_j]}\left\|\boldsymbol{\Psi}^n_{(\theta+r)\wedge\tau_n^{M,T}}\right\|_U^2-\left\|\boldsymbol{\Psi}^n_{\theta\wedge\tau_n^{M,T}}\right\|_U^2+\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}\|\boldsymbol{\Psi}^n_s\|_H^2\,ds\right]\leq o_j \tag{3.70}$$

having also scaled out the γ. We now need to relate the expression on the left hand side with what we are interested in, which is $\|\boldsymbol{\Psi}^n\|^2_{UH,(\theta+\delta_j)\wedge\tau_n^{M,T}}-\|\boldsymbol{\Psi}^n\|^2_{UH,\theta\wedge\tau_n^{M,T}}$. We have that

$$\|\boldsymbol{\Psi}^n\|^2_{UH,(\theta+\delta_j)\wedge\tau_n^{M,T}}-\|\boldsymbol{\Psi}^n\|^2_{UH,\theta\wedge\tau_n^{M,T}}$$
$$=\sup_{s\in[0,(\theta+\delta_j)\wedge\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_s\|_U^2-\sup_{s\in[0,\theta\wedge\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_s\|_U^2+\int_{\theta\wedge\tau_n^{M,T}}^{(\theta+\delta_j)\wedge\tau_n^{M,T}}\|\boldsymbol{\Psi}^n_s\|_H^2\,ds$$

and claim

$$\sup_{s\in[0,(\theta+\delta_j)\wedge\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_s\|_U^2-\sup_{s\in[0,\theta\wedge\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_s\|_U^2\leq\sup_{r\in[0,\delta_j]}\left\|\boldsymbol{\Psi}^n_{(\theta+r)\wedge\tau_n^{M,T}}\right\|_U^2-\left\|\boldsymbol{\Psi}^n_{\theta\wedge\tau_n^{M,T}}\right\|_U^2. \tag{3.71}$$

Indeed, we have that

$$\sup_{s\in[0,(\theta+\delta_j)\wedge\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_s\|_U^2\leq\sup_{s\in[0,\theta\wedge\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_s\|_U^2+\sup_{s\in[\theta\wedge\tau_n^{M,T},(\theta+\delta_j)\wedge\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_s\|_U^2-\left\|\boldsymbol{\Psi}^n_{\theta\wedge\tau_n^{M,T}}\right\|_U^2$$

as the left hand side must equal either $\sup_{s\in[0,\theta\wedge\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_s\|_U^2$ or $\sup_{s\in[\theta\wedge\tau_n^{M,T},(\theta+\delta_j)\wedge\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_s\|_U^2$, both of which are greater than the subtracted term $\left\|\boldsymbol{\Psi}^n_{\theta\wedge\tau_n^{M,T}}\right\|_U^2$. Appreciating that

$$\sup_{s\in[\theta\wedge\tau_n^{M,T},(\theta+\delta_j)\wedge\tau_n^{M,T}]}\|\boldsymbol{\Psi}^n_s\|_U^2=\sup_{r\in[0,\delta_j]}\left\|\boldsymbol{\Psi}^n_{(\theta+r)\wedge\tau_n^{M,T}}\right\|_U^2$$

then yields the claim (3.71), which in combination with (3.70) grants that

$$\mathbb{E}\left[\|\boldsymbol{\Psi}^n\|^2_{UH,(\theta+\delta_j)\wedge\tau_n^{M,T}}-\|\boldsymbol{\Psi}^n\|^2_{UH,\theta\wedge\tau_n^{M,T}}\right]\leq o_j.$$

This proves the result.

□

3.4.6 Existence and Uniqueness of Strong Solutions

We now have all of the ingredients needed to apply Proposition 2.3.5. In this subsection, we fix $\mathbf{\Psi}$ to be the unique weak solution of the equation (3.1) for the initial condition fixed in (3.56), known to exist from Theorem 3.3.5.

Proposition 3.4.11. *For any $R > 0$, there exists a subsequence indexed by (m_j) and a constant $M > 1$ such that the stopping time*

$$\tau^{R,T} := T \wedge \inf \left\{ s \geq 0 : \sup_{r \in [0,s]} \|\mathbf{\Psi}_r\|_U^2 + \int_0^s \|\mathbf{\Psi}_r\|_H^2 \, dr \geq R \right\}$$

satisfies the property $\tau^{R,T} \leq \tau_{m_j}^{M,T}$ for all m_j $\mathbb{P} - a.s.$.

Proof. This comes from a direct application of Lemma 2.3.5, for the Banach Space $X_t := C([0,t];U) \cap L^2([0,t];H)$ equipped with norm $\|\cdot\|_{UH,t}$. The conditions (2.36) and (2.37) are given in Proposition 3.4.9 and Lemma 3.4.10. The only component in need of proof is that the limiting process $\mathbf{\Phi}$ we achieve from Lemma 2.3.5 is in fact $\mathbf{\Psi}$. We are given that $(\mathbf{\Psi}^{m_j})$ converges to $\mathbf{\Psi}$ in $L^2\left([0, \tau_\infty^{M,T}]; H\right)$ which is a stronger convergence (up to $\tau_\infty^{M,T}$) than what we had in Theorem 3.2.16, which was used to show that the limit process was a weak solution of the equation (3.1). Hence in the same manner as Proposition 3.2.19, we see that $\mathbf{\Phi}$ must be a *local* weak solution as well, up until the stopping time $\tau_\infty^{M,T}$, so from the uniqueness of weak solutions we must have that $\mathbf{\Phi}$ is in fact equal to $\mathbf{\Psi}$ on $[0, \tau_\infty^{M,T}]$, which is sufficient for the result. $\square$

This allows us to deduce the additional regularity required for $\mathbf{\Psi}$ to be a strong solution.

Proposition 3.4.12. *The process $\mathbf{\Psi}$ is progressively measurable in V and such that for $\mathbb{P} - a.e.$ ω, $\mathbf{\Psi}_\cdot(\omega) \in L^\infty([0,T];H) \cap L^2([0,T];V)$.*

Proof. Propositions 3.4.7 and 3.4.11 together imply that, for any $R > 0$,

$$\mathbb{E}\left[\sup_{r \in [0,\tau^{R,T}]} \left\|\mathbf{\Psi}_r^{m_j}\right\|_1^2 + \int_0^{\tau^{R,T}} \left\|\mathbf{\Psi}_r^{m_j}\right\|_2^2 dr\right] \leq C_R$$

where C_R now incorporates $\|\mathbf{\Psi}_0\|_{L^\infty(\Omega;H)}^2$ and shows the dependency on M in terms of R. We also note use of (3.2) in this deduction. In consequence the sequence of processes $(\mathbf{\Psi}_\cdot^{m_j} \mathbb{1}_{\cdot \leq \tau^{R,T}})$ is uniformly bounded in both $L^2(\Omega; L^\infty([0,T];H))$ and $L^2\left(\Omega; L^2([0,T];V)\right)$. We can deduce the existence of a subsequence which is weakly convergent in the Hilbert Space $L^2\left(\Omega; L^2([0,T];V)\right)$ to some $\check{\mathbf{\Phi}}$, but we may also identify $L^2(\Omega; L^\infty([0,T];H))$ with the dual space of $L^2\left(\Omega; L^1([0,T];H)\right)$ and as such from the Banach-Alaoglu Theorem we can extract a further subsequence which is convergent to some $\hat{\mathbf{\Phi}}$ in the weak* topology. These limits imply convergence to both $\check{\mathbf{\Phi}}$ and $\hat{\mathbf{\Phi}}$ in the weak topology of $L^2\left(\Omega; L^2([0,T];H)\right)$. However, we know that $(\mathbf{\Psi}_\cdot^{m_j} \mathbb{1}_{\cdot \leq \tau^{R,T}})$ converges to $\mathbf{\Psi}_\cdot \mathbb{1}_{\cdot \leq \tau^{R,T}}$ $\mathbb{P} - a.s.$ in $L^2([0,T];H)$ from Lemma 2.3.5, and from the Dominated Convergence Theorem (domination comes easily from $\mathbb{1}_{\cdot \leq \tau^{R,T}}$, $\tau^{R,T} \leq \tau_{m_j}^{M,T}$) then in fact the convergence holds in $L^2\left(\tilde{\Omega}; L^2([0,T];H)\right)$. Of course the same convergence then holds in the weak topology, and by uniqueness of limits in the weak topology then $\mathbf{\Psi}_{\cdot}\mathbb{1}_{\cdot\leq\tau^{R,T}} = \check{\mathbf{\Phi}} = \hat{\mathbf{\Phi}}$ as elements of $L^2\left(\Omega; L^2\left([0,T];H\right)\right)$, so they agree $\mathbb{P}\times\lambda - a.s.$. Thus for $\mathbb{P} - a.e.\ \omega$, $\mathbf{\Psi}_{\cdot}(\omega)\mathbb{1}_{\cdot\leq\tau^{R,T}(\omega)} \in L^\infty\left([0,T];H\right)\cap L^2\left([0,T];V\right)$. At each such ω the regularity of $\mathbf{\Psi}$ as a weak solution ensures that for sufficiently large R (dependent on ω), $\tau^{R,T}(\omega) = T$. Therefore for $\mathbb{P} - a.e.\ \omega$, $\mathbf{\Psi}_{\cdot}(\omega) \in L^\infty\left([0,T];H\right)\cap L^2\left([0,T];V\right)$.

The progressive measurability is justified similarly; for any $t\in[0,T]$, we can use the progressive measurability of $(\mathbf{\Psi}^{m_j}_{\cdot})$ and hence $(\mathbf{\Psi}^{m_j}_{\cdot}\mathbb{1}_{\cdot\leq\tau^{R,T}})$ to instead deduce $\mathbf{\Psi}_{\cdot}\mathbb{1}_{\cdot\leq\tau^{R,T}}$ as the weak limit in $L^2\left(\Omega\times[0,t];V\right)$ where $\Omega\times[0,t]$ is equipped with the $\mathcal{F}_t\times\mathcal{B}\left([0,t]\right)$ sigma-algebra. Therefore $\mathbf{\Psi}_{\cdot}\mathbb{1}_{\cdot\leq\tau^{R,T}} : \Omega\times[0,t]\to V$ is measurable with respect to this product sigma-algebra which justifies the progressive measurability of the truncated process. To carry this property over to $\mathbf{\Psi}$, we recognise $\mathbf{\Psi}$ as the $\mathbb{P}\times\lambda$ almost everywhere limit of the sequence $(\mathbf{\Psi}_{\cdot}\mathbb{1}_{\cdot\leq\tau^{R,T}})$ as $R\to\infty$ over the product space $\Omega\times[0,t]$ equipped with product sigma-algebra $\mathcal{F}_t\times\mathcal{B}\left([0,t]\right)$. Such a limit preserves the measurability in this product sigma-algebra, justifying the progressive measurability of $\mathbf{\Psi}$.

□

Applying Lemma 3.4.6 in conjunction with Proposition 3.4.12 now proves Theorem 3.4.5 in the case of a bounded initial condition (3.56). Extending this result to a general $\mathcal{F}_0-$measurable $\mathbf{\Psi}_0 : \Omega\to H$ is done exactly as we did for the weak solution in Subsection 3.3.3, so we do not repeat these steps and conclude the proof of Theorem 3.4.5 here.

3.4.7 Pathwise Continuity

We close this section with a somewhat more informal discussion around the pathwise continuity of the strong solution $\mathbf{\Psi}$ in H. One may well have expected this to appear in the definition of the strong solution, though it seems challenging to verify and additional assumptions that could facilitate it are perhaps too restrictive. A first result is given now.

Lemma 3.4.13. *Let Assumption Sets 1, 2 and 3 hold. Suppose that there exists a continuous bilinear form $\langle\cdot,\cdot\rangle_{U\times V} : U\times V\to\mathbb{R}$ such that for every $f\in H$, $\phi\in V$,*

$$\langle f,\phi\rangle_{U\times V} = \langle f,\phi\rangle_H .$$

Then for $\mathbb{P} - a.e.\ \omega$, $\mathbf{\Psi}_{\cdot}(\omega)\in C_w\left([0,T];H\right)$.

Proof. Fixing arbitrary $\phi\in V$, $\mathbf{\Psi}$ satisfies the identity

$$\langle\mathbf{\Psi}_t,\phi\rangle_{U\times V} = \langle\mathbf{\Psi}_0,\phi\rangle_{U\times V} + \left\langle\int_0^t \mathcal{A}(s,\mathbf{\Psi}_s)ds,\phi\right\rangle_{U\times V} + \left\langle\int_0^t \mathcal{G}(s,\mathbf{\Psi}_s)d\mathcal{W}_s,\phi\right\rangle_{U\times V}$$

$\mathbb{P} - a.s.$ for all $t\in[0,T]$, and by continuity of the bilinear form,

$$\langle\mathbf{\Psi}_t,\phi\rangle_{U\times V} = \langle\mathbf{\Psi}_0,\phi\rangle_{U\times V} + \int_0^t \langle\mathcal{A}(s,\mathbf{\Psi}_s),\phi\rangle_{U\times V}\, ds + \int_0^t \langle\mathcal{G}(s,\mathbf{\Psi}_s),\phi\rangle_{U\times V}\, d\mathcal{W}_s.$$

We want to use that $\langle \mathbf{\Psi}_t, \phi\rangle_{U\times V} = \langle \mathbf{\Psi}_t, \phi\rangle_H$, but it is not yet clear that $\mathbf{\Psi}_t \in H$ $\mathbb{P}-a.s.$ for all $t \in [0,T]$. It can, in fact, be verified by a very similar process to the proof of Proposition 3.4.12. To show this let us fix any $t\in[0,T]$. For $\mathbb{P}-a.e.$ ω, $\mathbf{\Psi}_\cdot(\omega) \in L^\infty([0,T];H)$ thus for a countable dense subset (t_n) of $[0,T]$, $\mathbf{\Psi}_{t_n}(\omega)\in H$ and such that $\|\mathbf{\Psi}_{t_n}(\omega)\|_H \leq c$ independent of n. We choose now any sequence (t_k) which converges to t. The sequence $(\mathbf{\Psi}_{t_k}(\omega))$ is uniformly bounded in H, so admits a weakly convergent subsequence to a limit ψ. However, as $\mathbf{\Psi}_\cdot(\omega)\in C([0,T];U)$ then $(\mathbf{\Psi}_{t_k}(\omega))$ is strongly hence weakly convergent to $\mathbf{\Psi}_t(\omega)$ in U, so by uniqueness of limits in the weak topology then $\mathbf{\Psi}_t(\omega)=\psi$ and in particular $\mathbf{\Psi}_t(\omega)\in H$. Thus, $\mathbb{P}-a.s.$ for all $t\in[0,T]$,

$$\langle \mathbf{\Psi}_t, \phi\rangle_H = \langle \mathbf{\Psi}_0, \phi\rangle_{U\times V} + \int_0^t \langle \mathcal{A}(s,\mathbf{\Psi}_s), \phi\rangle_{U\times V}\, ds + \int_0^t \langle \mathcal{G}(s,\mathbf{\Psi}_s), \phi\rangle_{U\times V}\, d\mathcal{W}_s$$

so that $\langle \mathbf{\Psi}_t, \phi\rangle_H$ is $\mathbb{P}-a.s.$ continuous. This is not quite enough for weak continuity, as we instead require that $\langle \mathbf{\Psi}_t, f\rangle_H$ is $\mathbb{P}-a.s.$ continuous for any $f\in H$, not just any $\phi\in V$. From this point, though, we can extend the continuity to f by the density of V in H, identically to the end of the proof of Proposition 3.2.19. □

Pathwise continuity of $\mathbf{\Psi}$ in H would now follow from only pathwise continuity of the norm $\|\mathbf{\Psi}_\cdot\|_H$, due to Lemma A.1.28. A popular way to show such a property is through Kolmogorov's Continuity Theorem, by verifying that there exists a constant C such that for all $s<t\in[0,T]$,

$$\mathbb{E}\left(\left|\|\mathbf{\Psi}_t\|_H^2 - \|\mathbf{\Psi}_s\|_H^2\right|^4\right) \leq C(t-s)^2$$

which is the typical choice of parameters to apply Kolmogorov's Continuity Theorem for a Brownian Motion and related SDEs. Our assumptions don't quite allow us to show this, even at the level of the Galerkin System: the issue arises in the V-norm dependence in (3.55), which is necessary to have for applications in the case of Navier Boundary Conditions as seen in Theorem 5.3.3. There is, however, a simpler case where continuity can be deduced.

Lemma 3.4.14. *Let Assumption Sets 1, 2 and 3 hold. Suppose that there exists a continuous bilinear form $\langle\cdot,\cdot\rangle_{U\times V}: U\times V\to\mathbb{R}$ such that for every $f\in H$, $\phi\in V$,*

$$\langle f, \phi\rangle_{U\times V} = \langle f,\phi\rangle_H .$$

In addition, suppose that $\bar{H}$ can be taken as H. Then for $\mathbb{P}-a.e.$ ω, $\mathbf{\Psi}_\cdot(\omega)\in C([0,T];H)$.

Proof. This is now a direct application of Proposition A.2.7. □

Chapter 4

An Introduction to the Stochastic Navier-Stokes Equations

4.1 Introduction

In this chapter, we prepare our analysis of the incompressible Stochastic Navier-Stokes Equations by posing the problems in the framework of Subsection 3.1.1. As described in Chapter 1, we are concerned with the existence and uniqueness of solutions of the equation (1.2) under the no-slip boundary condition (1.3) and Navier boundary conditions (1.4) on a smooth bounded domain $\mathscr{O} \subset \mathbb{R}^N$. For $N = 2$ or 3, and for the no-slip condition, the problem is summarised by the system of equations

$$\left.\begin{array}{ll} u_t = u_0 - \int_0^t \mathcal{L}_{u_s} u_s \, ds + \nu \int_0^t \Delta u_s \, ds - \int_0^t B(u_s) \circ d\mathcal{W}_s - \nabla \rho_t & \text{in } \mathscr{O} \\ \sum_{j=1}^N \partial_j u^j = 0 & \text{in } \mathscr{O} \\ u = 0 & \text{on } \partial\mathscr{O} \end{array}\right\} \tag{4.1}$$

where the middle restriction is the statement that u is divergence-free, with superscript notation denoting the component mapping, corresponding to incompressibility. The operators are properly defined in Section 4.3. The Navier boundary conditions are given in two dimensions, for which the problem is

$$\left.\begin{array}{ll} u_t = u_0 - \int_0^t \mathcal{L}_{u_s} u_s \, ds + \nu \int_0^t \Delta u_s \, ds - \int_0^t B(u_s) \circ d\mathcal{W}_s - \nabla \rho_t & \text{in } \mathscr{O} \\ \sum_{j=1}^2 \partial_j u^j = 0 & \text{in } \mathscr{O} \\ u \cdot \mathbf{n} = 0, \qquad 2(Du)\mathbf{n} \cdot \iota + \alpha u \cdot \iota = 0 & \text{on } \partial\mathscr{O}. \end{array}\right\} \tag{4.2}$$

The structure of this chapter is as follows:

- Section 4.2 introduces new function spaces to incorporate the incompressibility and boundary conditions of problems (4.1) and (4.2). Fundamental properties of these spaces are also presented.
- Section 4.3 demonstrates essential bounds on the classical and stochastic operators in (1.2), acting as the cornerstone of our analysis in the discussed function spaces.

- Section 4.4 concludes the chapter with a transposition of (1.2) to the form (1.1), totalling a complete description of the problems (4.1), (4.2) in terms of the framework from Subsection 3.1.1. As a brief supplement, we also pass from the velocity form presented in (1.2) to the vorticity form which is often seen in the literature. This is less relevant to the thesis, but is revisited in Subsections 5.4.2 and 6.3.2.

4.2 Functional Analytic Setup

Let us first recall the notation from Subsection 1.1.1, where $\mathscr{O}$ is a smooth bounded domain in $\mathbb{R}^N$ for $N = 2$ or 3. In the following spaces that we introduce the dependence on N is suppressed, where the wider context will inform the choice of N.

Definition 4.2.1. *We define $C^\infty_{0,\sigma}$ as the subset of $C^\infty_0(\mathscr{O};\mathbb{R}^N)$ of functions which are divergence-free, that is the elements $\phi \in C^\infty_0(\mathscr{O};\mathbb{R}^N)$ such that $\sum_{j=1}^N \partial_j \phi^j = 0$. L^2_σ is defined as the completion of $C^\infty_{0,\sigma}$ in $L^2(\mathscr{O};\mathbb{R}^N)$, whilst we introduce $\bar{W}^{1,2}_\sigma$ as the intersection of $W^{1,2}(\mathscr{O};\mathbb{R}^N)$ with L^2_σ.*

The spaces L^2_σ, $\bar{W}^{1,2}_\sigma$ will prove relevant to both problems (4.1) and (4.2). So too will the Leray Projector and associated Stokes Operator, introduced here:

Definition 4.2.2. *The Leray Projector $\mathcal{P}$ is defined as the orthogonal projection in $L^2(\mathscr{O};\mathbb{R}^N)$ onto L^2_σ. The Stokes Operator $A : W^{2,2}(\mathscr{O};\mathbb{R}^N) \to L^2_\sigma$ is defined as $-\mathcal{P}\Delta$.*

Certainly the Stokes Operator A is well-defined on $W^{2,2}(\mathscr{O};\mathbb{R}^N)$; extensions of A will be considered in Subsection 4.3.1. For now, we give important properties of the Leray Projector $\mathcal{P}$ and the structure of L^2_σ.

Lemma 4.2.3. *For any $m \in \mathbb{N}$, $\mathcal{P}$ is continuous as a mapping $\mathcal{P} : W^{m,2}(\mathscr{O};\mathbb{R}^N) \to W^{m,2}(\mathscr{O};\mathbb{R}^N)$.*

Proof. See [147] Remark 1.6. □

Proposition 4.2.4. *Define the space*

$$L^{2,\perp}_\sigma := \{\psi \in L^2(\mathscr{O};\mathbb{R}^N) : \psi = \nabla g \text{ for some } g \in W^{1,2}(\mathscr{O};\mathbb{R})\}.$$

Then indeed $L^{2,\perp}_\sigma$ is the orthogonal complement to L^2_σ in $L^2(\mathscr{O};\mathbb{R}^N)$, i.e. for any $\phi \in L^2_\sigma$ and $\psi \in L^{2,\perp}_\sigma$ we have that

$$\langle \phi, \psi \rangle = 0,$$

and every $f \in L^2(\mathscr{O};\mathbb{R}^N)$ has the unique decomposition

$$f = \phi + \psi \tag{4.3}$$

for some $\phi = \mathcal{P}f \in L^2_\sigma$, $\psi = \nabla g \in L^{2,\perp}_\sigma$.

Proof. See [147] Theorems 1.4, 1.5 and [129] Theorem 2.6. □

Proposition 4.2.4 is known as the Helmholtz-Weyl Decomposition. The remainder of this section is split into the cases of the no-slip and Navier boundary conditions.

4.2.1 No-Slip Boundary

We introduce the spaces which shall play the role of H and V from Subsection 3.1.1 in the study of (4.1).

Definition 4.2.5. *We define $W^{1,2}_\sigma$ as the intersection of $W^{1,2}_0(\mathscr{O};\mathbb{R}^N)$ with L^2_σ, and $W^{2,2}_\sigma$ as the intersection of $W^{2,2}(\mathscr{O};\mathbb{R}^N)$ with $W^{1,2}_\sigma$.*

The fact that these spaces embed the divergence-free and boundary condition, interpreted in the sense of weak derivatives and trace, is given in Lemma 4.2.6 below. For this, we recall the definition of trace. We first define the restriction mapping on functions $f \in W^{1,2}(\mathscr{O};\mathbb{R}) \cap C(\bar{\mathscr{O}};\mathbb{R})$ by the restriction of f to the boundary $\partial\mathscr{O}$, which is then shown to be a bounded operator into $W^{\frac{1}{2},2}(\partial\mathscr{O};\mathbb{R})$ (see Lemma **??** and more classical sources of e.g. [43]). By the density of $C(\bar{\mathscr{O}};\mathbb{R})$ then the trace operator is well defined on the whole of $W^{1,2}(\mathscr{O};\mathbb{R})$ as a continuous linear extension of the restriction mapping, and furthermore on $W^{1,2}(\mathscr{O};\mathbb{R}^N)$ by the trace of the components. Further recall, e.g. [43] pp.259, that $W^{1,2}_0(\mathscr{O};\mathbb{R})$ is characterised as the subspace of $W^{1,2}(\mathscr{O};\mathbb{R})$ consisting of zero-trace functions.

Lemma 4.2.6. *$W^{1,2}_\sigma$ is precisely the subspace of $W^{1,2}_0(\mathscr{O};\mathbb{R}^N)$ consisting of divergence-free functions. Moreover, $W^{1,2}_\sigma$ is the completion of $C^\infty_{0,\sigma}$ in $W^{1,2}(\mathscr{O};\mathbb{R}^N)$.*

Proof. We claim at first that any given $f \in W^{1,2}_\sigma$ is divergence-free. In [129] Lemma 2.11 it is shown that f satisfies the property

$$\langle f, \nabla\phi\rangle = 0$$

for every $\phi \in C^\infty_0(\mathscr{O};\mathbb{R})$, or equivalently that

$$\sum_{j=1}^N \left\langle f^j, \partial_j\phi\right\rangle_{L^2(\mathscr{O};\mathbb{R})} = 0$$

for such ϕ. We can use the compact support of ϕ and $W^{1,2}(\mathscr{O};\mathbb{R}^N)$ regularity of f to carry out an integration by parts, taking the sum through the inner product to see that

$$\left\langle \sum_{j=1}^N \partial_j f^j, \phi\right\rangle_{L^2(\mathscr{O};\mathbb{R})} = 0.$$

Observing that $\phi \in C^\infty_0(\mathscr{O};\mathbb{R})$ was arbitrary and using the density of this space in $L^2(\mathscr{O};\mathbb{R})$, we deduce that the divergence of f is zero as an element of $L^2(\mathscr{O};\mathbb{R})$ and so f is divergence-free. Working in the reverse direction now, we suppose that $g \in W^{1,2}_0(\mathscr{O};\mathbb{R}^N)$ is divergence-free with the intent to show that $g \in W^{1,2}_\sigma$, which is to say $g \in L^2_\sigma$. For this we simply refer to Lemma 2.14 again of [129], noting that g satisfies all 'weak divergence-free' notions of [129]. The final statement to prove is the characterisation of $W^{1,2}_\sigma$ as the completion of $C^\infty_{0,\sigma}$ in $W^{1,2}(\mathscr{O};\mathbb{R}^N)$. We defer this to [147] Theorem 1.6, using the representation of $W^{1,2}_\sigma$ just proved. $\square$

It will be convenient to work with different, but equivalent, inner products on these spaces.

Proposition 4.2.7. *The bilinear form*

$$\langle f, g \rangle_1 := \sum_{j=1}^N \langle \partial_j f, \partial_j g \rangle$$

defines an inner product on $W^{1,2}_\sigma$ *equivalent to* $\langle \cdot, \cdot \rangle_{W^{1,2}(\mathscr{O};\mathbb{R}^N)}$. *Additionally, the bilinear form*

$$\langle \phi, \psi \rangle_2 := \langle A\phi, A\psi \rangle$$

defines an inner product on $W^{2,2}_\sigma$ *equivalent to* $\langle \cdot, \cdot \rangle_{W^{2,2}(\mathscr{O};\mathbb{R}^N)}$.

Proof. The equivalence of $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_{W^{1,2}(\mathscr{O};\mathbb{R}^N)}$ is immediate from Poincaré's Inequality for zero-trace functions (see e.g. [129] Theorem 1.9). For the second statement, see [25] Proposition 4.12, [129] Exercises 2.12, 2.13 and the discussion in Subsection 2.3. □

Henceforth, we shall consider $W^{1,2}_\sigma$, $W^{2,2}_\sigma$ as Hilbert Spaces equipped with the $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ inner products unless otherwise stated. The convenience is in part due to the following duality relation.

Lemma 4.2.8. *For every* $f \in W^{1,2}_\sigma$, $g \in W^{2,2}(\mathscr{O};\mathbb{R}^N)$,

$$\langle f, g \rangle_1 = \langle f, Ag \rangle .$$

Proof. The proof is direct, using at first the zero-trace property of f to conduct the integration by parts with null boundary integral, and then that $f = \mathcal{P}f$:

$$\langle f, g \rangle_1 = \sum_{j=1}^N \langle \partial_j f, \partial_j g \rangle = -\sum_{j=1}^N \left\langle f, \partial_j^2 g \right\rangle = -\sum_{j=1}^N \langle \mathcal{P}f, \Delta g \rangle = \langle f, Ag \rangle .$$

□

We shall work with a basis belonging to these spaces.

Proposition 4.2.9. *There exists a collection of functions* (a_k), $a_k \in W^{1,2}_\sigma \cap C^\infty(\overline{\mathscr{O}};\mathbb{R}^N)$ *such that the* (a_k) *are eigenfunctions of* A, *are an orthonormal basis in* L^2_σ *and an orthogonal basis in* $W^{1,2}_\sigma$. *The eigenvalues* (λ_k) *are strictly positive and approach infinity as* $k \to \infty$.

Proof. See [129] Theorem 2.24. □

As a consequence, every $f \in L^2_\sigma$ admits the representation

$$f = \sum_{k=1}^\infty f_k a_k \tag{4.4}$$

where $f_k = \langle f, a_k \rangle$, as a limit in $L^2(\mathscr{O};\mathbb{R}^N)$. Moreover, for each $n \in \mathbb{N}$ we define $\mathcal{P}_n$ as the orthogonal projection in $L^2(\mathscr{O};\mathbb{R}^N)$ onto $V_n := \operatorname{span}\{a_1, \ldots, a_n\}$, that is

$$\mathcal{P}_n : f \mapsto \sum_{k=1}^{n} \langle f, a_k \rangle \, a_k. \tag{4.5}$$

The projections $(\mathcal{P}_n)$ are not just self-adjoint in L^2_σ, but also $W^{1,2}_\sigma$.

Lemma 4.2.10. *For every $n \in \mathbb{N}$, $\|a_n\|_1^2 = \lambda_n$ and $\mathcal{P}_n$ is self-adjoint in $W^{1,2}_\sigma$.*

Proof. A proof of the first statement uses Lemma 4.2.8,

$$\|a_n\|_1^2 = \langle a_n, a_n \rangle_1 = \langle a_n, A a_n \rangle = \langle a_n, \lambda_n a_n \rangle = \lambda_n \|a_n\|^2 = \lambda_n.$$

For the second, we consider a representation of elements $f \in W^{1,2}_\sigma$ not by (4.4) but instead as

$$f = \sum_{k=1}^{\infty} \frac{\langle f, a_k \rangle_1}{\|a_k\|_1^2} a_k \tag{4.6}$$

which is convergent in $W^{1,2}_\sigma$, given that (a_k) form an orthogonal basis of $W^{1,2}_\sigma$. Therefore

$$\langle f, a_k \rangle = \left\langle \sum_{j=1}^{\infty} \frac{\langle f, a_j \rangle_1}{\|a_j\|_1^2} a_j, a_k \right\rangle = \sum_{j=1}^{\infty} \frac{\langle f, a_j \rangle_1}{\|a_j\|_1^2} \langle a_j, a_k \rangle = \frac{\langle f, a_k \rangle_1}{\|a_k\|_1^2}$$

holds. Consequently, for $f \in W^{1,2}_\sigma$, the representation (4.4) is in fact equivalent to (4.6) so the limit in (4.4) holds in $W^{1,2}_\sigma$. Using the orthogonality of (a_k) in $W^{1,2}_\sigma$, the result is now clear. □

A particular consequence is that for $f \in W^{1,2}_\sigma$, $\|\mathcal{P}_n f\|_1 \leq \|f\|_1$.

4.2.2 Navier Boundary

We recall the Navier boundary conditions (1.4)

$$u \cdot \mathbf{n} = 0, \qquad 2(Du)\mathbf{n} \cdot \iota + \alpha u \cdot \iota = 0$$

where $\mathbf{n}$ is the unit outwards normal vector, ι the unit tangent vector, Du is the rate of strain tensor $(Du)^{k,l} := \frac{1}{2}\left(\partial_k u^l + \partial_l u^k\right)$ and $\alpha \in C^2(\partial\mathscr{O};\mathbb{R})$. In expanded form, D is a mapping $D : W^{1,2}(\mathscr{O};\mathbb{R}^2) \to L^2(\mathscr{O};\mathbb{R}^{2\times 2})$ defined by

$$f \mapsto \begin{bmatrix} \partial_1 f^1 & \frac{1}{2}\left(\partial_1 f^2 + \partial_2 f^1\right) \\ \frac{1}{2}\left(\partial_2 f^1 + \partial_1 f^2\right) & \partial_2 f^2. \end{bmatrix}$$

Note that if $f \in W^{2,2}(\mathscr{O};\mathbb{R}^2)$ then $Df \in W^{1,2}(\mathscr{O};\mathbb{R}^{2\times 2})$ so the trace of its components is well defined and henceforth we can understand the boundary condition

$$2(Du)\mathbf{n} \cdot \iota + \alpha u \cdot \iota = 0$$

on $\partial\mathscr{O}$ to be in this trace sense. The same is true for $u \cdot \mathbf{n} = 0$. We look to impose these conditions by incorporating them into the function spaces where our solution takes value, and will always assume that $\alpha \in C^2(\partial\mathscr{O};\mathbb{R})$ so as to match the regularity required in [22]. Recall $\bar{W}^{1,2}_\sigma := L^2_\sigma \cap W^{1,2}(\mathscr{O};\mathbb{R}^N)$ from Definition 4.2.1; in the case of Navier boundary conditions, we only consider $N = 2$.

Definition 4.2.11. *We define $\bar{W}^{2,2}_\alpha$ by*

$$\bar{W}^{2,2}_\alpha := \left\{ f \in W^{2,2}(\mathscr{O};\mathbb{R}^2) \cap \bar{W}^{1,2}_\sigma : 2(Df)\mathbf{n} \cdot \iota + \alpha f \cdot \iota = 0 \text{ on } \partial\mathscr{O} \right\}.$$

The space $\bar{W}^{2,2}_\alpha$ directly addresses part of the Navier boundary conditions. The role of $\bar{W}^{1,2}_\sigma$ is made explicit below.

Lemma 4.2.12. *$\bar{W}^{1,2}_\sigma$ is precisely the subspace of $W^{1,2}(\mathscr{O};\mathbb{R}^2)$ consisting of divergence-free functions f such that $f \cdot \mathbf{n} = 0$ on $\partial\mathscr{O}$.*

Proof. The proof is the same as Lemma 4.2.6, using that [129] Lemma 2.11 characterises L^2_σ as the subspace of $L^2(\mathscr{O};\mathbb{R}^2)$ of 'weakly' divergence-free functions f also satisfying $f \cdot \mathbf{n} = 0$ 'weakly' on $\partial\mathscr{O}$. □

Remark 4.2.13. *As both $D : W^{2,2}(\mathscr{O};\mathbb{R}^2) \to W^{1,2}(\mathscr{O};\mathbb{R}^{2\times 2})$ and the trace mapping $W^{1,2}(\mathscr{O};\mathbb{R}) \to L^2(\partial\mathscr{O};\mathbb{R})$ are continuous, then $\bar{W}^{1,2}_\sigma$, $\bar{W}^{2,2}_\alpha$ are closed in the $W^{1,2}(\mathscr{O};\mathbb{R}^2)$, $W^{2,2}(\mathscr{O};\mathbb{R}^2)$ norms respectively.*

Once more it will be convenient to work in the $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ inner products introduced in Proposition 4.2.7.

Proposition 4.2.14. *The bilinear form $\langle\cdot,\cdot\rangle_1$ defines an inner product on $\bar{W}^{1,2}_\sigma$ equivalent to $\langle\cdot,\cdot\rangle_{W^{1,2}(\mathscr{O};\mathbb{R}^2)}$.*

Proof. The proof relies on showing that the Poincaré Inequality holds for the component mappings of functions in $\bar{W}^{1,2}_\sigma$. The inequality (see e.g. [129] Theorem 1.9) holds for the component mapping f^j of $f \in \bar{W}^{1,2}_\sigma$ if

$$\int_{\mathscr{O}} f^j d\lambda = 0.$$

Observe that via an approximation with the set $C^\infty_{0,\sigma}$ which is dense in L^2_σ, one can conclude that for all $g \in L^2_\sigma$ and $\psi \in W^{1,2}(\mathscr{O};\mathbb{R})$,

$$\langle g, \nabla\psi\rangle = 0$$

(this is the statement of [129] Lemma 2.11). Moreover by choosing ϕ as the function $\phi(x^1, x^2) := x^j$, then

$$\int_{\mathscr{O}} f^j d\lambda = \langle f, \nabla\phi\rangle = 0$$

so the Poincaré Inequality holds. □

We shall also endow $W_0^{1,2}(\mathscr{O};\mathbb{R})$ with the corresponding one dimensional inner product, and label it $\langle\cdot,\cdot\rangle_{W_0^{1,2}}$.

Proposition 4.2.15. *The bilinear form $\langle\cdot,\cdot\rangle_2$ defines an inner product on $\bar{W}^{2,2}_{\alpha}$ equivalent to $\langle\cdot,\cdot\rangle_{W^{2,2}(\mathscr{O};\mathbb{R}^2)}$.*

Proof. We must show the existence of constants c_1, c_2 such that for all $f \in \bar{W}^{2,2}_{\alpha}$,

$$c_1 \left\|f\right\|_{W^{2,2}}^2 \leq \left\|f\right\|_2^2 \leq c_2 \left\|f\right\|_{W^{2,2}}^2 .$$

The constant c_2 can in fact be taken as 2, as

$$\left\|f\right\|_2^2 \leq \left\|\Delta f\right\|^2 \leq 2 \left\|f\right\|_{W^{2,2}}^2 .$$

The existence of such a c_1 is much more challenging and relies on estimates of the Stokes Equation with the Navier boundary conditions, which has been proven in [145] Theorem 5.10. We use, in their notation, that $\mathbf{f} = A\mathbf{u}$ is a solution of the Stokes problem with $\pi = 0$, which gives the result. □

Henceforth we consider $\bar{W}^{1,2}_{\sigma}$, $\bar{W}^{2,2}_{\alpha}$ as Hilbert Spaces equipped with the $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ inner products, unless otherwise stated. The following lemma will further facilitate our work in these spaces.

Proposition 4.2.16. *There exists a collection of functions $(\bar{a}_k)$, $\bar{a}_k \in W^{3,2}(\mathscr{O};\mathbb{R}^2) \cap \bar{W}^{2,2}_{\alpha}$, such that the $(\bar{a}_k)$ are eigenfunctions of A, are an orthonormal basis in L^2_{σ} and a basis in $\bar{W}^{1,2}_{\sigma}$. Moreover the eigenvalues $(\bar{\lambda}_k)$ are strictly positive and approach infinity as $k \to \infty$.*

Proof. This is the content of [22] Lemma 2.2, where $(\bar{a}_k)$ is (v_k) in their notation. The fact that this system consists of eigenfunctions of A follows from (2.10) by taking $\mathcal{P}$ of the top line: that is,

$$\bar{\lambda}_k\bar{a}_k = \mathcal{P}\left(\bar{\lambda}_k\bar{a}_k\right) = \mathcal{P}\left(-\Delta\bar{a}_k + \nabla\pi_k\right) = A\bar{a}_k$$

using that $\mathcal{P}\nabla\pi_k = 0$ from Proposition 4.2.4. □

One can see that we are building a framework to parallel that of Subsection 4.2.1, though a significant difference comes in the presence of a boundary integral for Green's type identities. What we achieve now is the following, recalling $\kappa \in C^2(\partial\mathscr{O};\mathbb{R})$ to be the curvature of $\partial\mathscr{O}$.

Lemma 4.2.17. *For $f \in \bar{W}^{2,2}_{\alpha}$, $\phi \in \bar{W}^{1,2}_{\sigma}$, we have that*

$$\langle\Delta f, \phi\rangle = -\left\langle f, \phi\right\rangle_1 + \left\langle(\kappa - \alpha)f, \phi\right\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)} .$$

Proof. This is demonstrated in [81] equation (5.1). □

Due to this boundary integral we will make great use of Lebesgue spaces and fractional Sobolev spaces defined on the boundary $\partial\mathscr{O}$. The spaces $L^p(\partial\mathscr{O};\mathbb{R}^2)$ can be defined precisely as in Subsection 1.1.1 for $\partial\mathscr{O}$ equipped with its surface measure, and in fact the same is true for

$W^{s,2}(\partial\mathscr{O};\mathbb{R})$ with $0 < s < 1$ and hence $W^{s,2}(\partial\mathscr{O};\mathbb{R}^2)$ in the same manner. Indeed this definition is given in [67] pp.20, where it is shown to be equivalent to the often used definition locally as the space $W^{s,2}(\mathbb{R};\mathbb{R})$ via a coordinate transformation and partition of unity. The stability under a change of variables ensures results of Hölder's Inequality and (one dimensional) Sobolev Embeddings hold. The relation to the trace of functions in $\mathscr{O}$ is stated now.

Lemma 4.2.18. *For $\frac{1}{2} < s < \frac{3}{2}$ the trace operator is bounded and linear from $W^{s,2}(\mathscr{O};\mathbb{R})$ into $W^{s-\frac{1}{2},2}(\partial\mathscr{O};\mathbb{R})$.*

Proof. See [38] Theorem 1. □

We shall make use of this result for the characterisation of the fractional Sobolev spaces as interpolation spaces. In fact the renowned book of Adams [1] defines these spaces in this way (7.57, pp.250), and their equivalence is well understood (see for example [146] pp.83). A proof of this equivalence relies on a norm preserving extension operator for the space as defined by interpolation (which follows from the classical integer valued case, see [1] Theorem 5.24 and [138] chapter 6), the equivalence of the interpolation space on $\mathbb{R}^2$ with that defined by Fourier transformations (see [1] 7.63 pp.252), the further equivalence of this space with our definition on $\mathbb{R}^2$ ([36] Proposition 4.17), and finally a norm preserving extension for this fractional Sobolev space ([37] Theorem 5.4). Therefore there exists a constant c such that for $f \in W^{1,2}(\mathscr{O};\mathbb{R}^2)$ and $0 < s < 1$, we have that

$$\|f\|_{W^{s,2}} \leq c\,\|f\|^{1-s}\,\|f\|_{W^{1,2}}^{s}. \tag{4.7}$$

In particular for $s = \frac{1}{2}$, if the result of Lemma 4.2.18 were true in this limiting case (that is, an embedding of $W^{\frac{1}{2},2}(\mathscr{O};\mathbb{R}^2)$ into $L^2(\partial\mathscr{O};\mathbb{R}^2)$) then combined with (4.7) we would obtain

$$\|f\|_{L^2(\partial\mathscr{O};\mathbb{R}^2)}^2 \leq c\,\|f\|\,\|f\|_{W^{1,2}}. \tag{4.8}$$

This inequality is in fact true and is classical in the study of our problem (see for example [98] pp.130, [81] equation (2.5)) though some additional machinery is required to prove it. In short the result can be achieved by showing that the trace is a continuous linear operator from an appropriate Besov space which similarly interpolates between $L^2(\mathscr{O};\mathbb{R}^2)$ and $W^{1,2}(\mathscr{O};\mathbb{R}^2)$ (see [1] Theorem 7.43 and Remark 7.45 for the trace embedding, and the Besov Spaces subchapter for the interpolation). Moving on we introduce the finite dimensional projections $(\bar{\mathcal{P}}_n)$, where $\bar{\mathcal{P}}_n$ is the orthogonal projection onto $\bar{V}_n := \operatorname{span}\{\bar{a}_1, \dots, \bar{a}_n\}$ in $L^2(\mathscr{O};\mathbb{R}^2)$. That is, $\bar{\mathcal{P}}_n$ is given by

$$\bar{\mathcal{P}}_n : f \mapsto \sum_{k=1}^{n} \langle f, \bar{a}_k \rangle\, \bar{a}_k.$$

One may ask if we have the corresponding result to Lemma 4.2.10, which would be the self-adjoint property of $\bar{\mathcal{P}}_n$ in $\bar{W}^{1,2}_\sigma$. The answer is unfortunately no, as the system $\bar{a}_k$ is not orthogonal in $\bar{W}^{1,2}_\sigma$. However, an equivalent inner product will be defined in Subsection 5.3.2 in which this property holds.

It will also be of use to us to consider the vorticity, which we do by introducing the operator curl : $W^{1,2}(\mathscr{O};\mathbb{R}^2) \to L^2(\mathscr{O};\mathbb{R})$ by

$$\text{curl} : f \to \partial_1 f^2 - \partial_2 f^1.$$

The vorticity is given by the curl of velocity. A significant property in the study of vorticity is the following.

Lemma 4.2.19. *For all $f \in W^{1,2}(\mathscr{O};\mathbb{R}^2)$, $\text{curl}(\mathcal{P}f) = \text{curl}f$.*

Proof. If $\phi = \nabla g \in W^{1,2}(\mathscr{O};\mathbb{R}^2)$ then $\text{curl}\phi = \partial_1\partial_2 g - \partial_2\partial_1 g = 0$, which from Proposition 4.2.4 establishes that $\text{curl}\,(\mathcal{P}f) = \text{curl}\left([\mathcal{P} + \mathcal{P}^{\perp}]f\right) = \text{curl}f$ where $\mathcal{P}^{\perp}$ is the complement projection $I - \mathcal{P}$ on $L^2(\mathscr{O};\mathbb{R}^2)$. $\square$

The curl is intrinsically related to the Navier boundary conditions through κ, by the relation proven as Lemma 2.1 in [22] which we state here.

Lemma 4.2.20. *For all $f \in W^{2,2}(\mathscr{O};\mathbb{R}^2)$ with $f \cdot \mathbf{n} = 0$ on $\partial\mathscr{O}$, we have that*

$$2(Df)\mathbf{n} \cdot \iota + 2\kappa f \cdot \iota - \text{curl}f = 0$$

on $\partial\mathscr{O}$.

Moreover we can make the condition (1.5) discussed in the introduction precise.

Corollary 4.2.20.1. *Suppose that $f \in W^{2,2}(\mathscr{O};\mathbb{R}^2)$ with $f \cdot \mathbf{n} = 0$ on $\partial\mathscr{O}$. Then f satisfies*

$$2(Df)\mathbf{n} \cdot \iota + \alpha f \cdot \iota = 0$$

on $\partial\mathscr{O}$ if and only if it satisfies

$$\text{curl}f = (2\kappa - \alpha)f \cdot \iota$$

on $\partial\mathscr{O}$.

4.3 Operator Bounds

This section presents fundamental properties of the operators in (1.2), divided into Subsection 4.3.1 for the classical operators $\mathcal{L}$ and A (the Stokes Operator, Definition 4.2.2) and Subsection 4.3.2 for the operator B appearing from the stochastic term.

4.3.1 Nonlinear and Stokes Operators

Firstly we briefly comment on the pressure term $\nabla\rho$, which will not play any role in our analysis. ρ does not come with an evolution equation and is simply chosen to ensure the incompressibility (divergence-free) condition; moreover in Subsection 4.4.1 we shall project the equation through $\mathcal{P}$ which as established in Proposition 4.2.4 nullifies gradient terms. The procedure of solving the projected equating and finding a pressure later is well discussed in [129] Section 5, and an explicit form for the semi-martingale pressure for the corresponding Stochastic Euler Equation is given in [140] Subsection 3.3.

The mapping $\mathcal{L}$ is defined for sufficiently regular functions $f, g : \mathscr{O} \to \mathbb{R}^N$ by

$$\mathcal{L}_f g := \sum_{j=1}^{N} f^j \partial_j g.$$

Here and throughout the text we make no notational distinction between differential operators acting on a vector valued function or a scalar valued one; that is, we understand $\partial_j g$ by its component mappings $(\partial_j g)^l := \partial_j g^l$. We can immediately give some clarification as to 'sufficiently regular'.

Lemma 4.3.1. *There exists a constant c such that for any $f, g \in W^{k,2}(\mathscr{O};\mathbb{R}^N)$ as specified in the right hand side of the below, we have the bounds*

$$\|\mathcal{L}_f g\| + \|\mathcal{L}_g f\| \leq c \|g\|_{W^{1,2}} \|f\|_{W^{2,2}}; \tag{4.9}$$

$$\|\mathcal{L}_g f\|_{W^{1,2}} \leq c \|g\|_{W^{1,2}} \|f\|_{W^{3,2}}; \tag{4.10}$$

$$\|\mathcal{L}_g f\|_{W^{1,2}} \leq c \|g\|_{W^{2,2}} \|f\|_{W^{2,2}}. \tag{4.11}$$

In the proof of this result and throughout the thesis, we shall frequently use Sobolev Embeddings according to Theorem A.4.13. Furthermore, in this proof we precisely deconstruct the norms of the vector fields into components; having established that understanding here, we will omit such details in future proofs.

Proof. For (4.9) starting with the term $\mathcal{L}_f g$:

$$\begin{aligned}
\|\mathcal{L}_f g\| &\leq \sum_{j=1}^{N} \|f^j \partial_j g\| \leq \sum_{j=1}^{N}\sum_{l=1}^{N} \left\|f^j \partial_j g^l\right\|_{L^2(\mathscr{O};\mathbb{R})} \leq \sum_{j=1}^{N}\sum_{l=1}^{N} \|f^j\|_{L^\infty(\mathscr{O};\mathbb{R})} \left\|\partial_j g^l\right\|_{L^2(\mathscr{O};\mathbb{R})} \\
&\leq c\sum_{j=1}^{N}\sum_{l=1}^{N} \|f^j\|_{W^{2,2}(\mathscr{O};\mathbb{R})} \left\|\partial_j g^l\right\|_{L^2(\mathscr{O};\mathbb{R})} \leq c\|f\|_{W^{2,2}} \|g\|_{W^{1,2}}
\end{aligned}$$

using that $\left(\sum_{l=1}^{N} |a_l|^2\right)^{1/2} \leq \sum_{l=1}^{N} |a_l|$ and the Sobolev Embedding $W^{2,2}(\mathscr{O};\mathbb{R}) \hookrightarrow L^\infty(\mathscr{O};\mathbb{R})$. Similarly comes the bound

$$\begin{aligned}
\|\mathcal{L}_g f\| &\leq \sum_{j=1}^{N}\sum_{l=1}^{N} \left\|g^j \partial_j f^l\right\|_{L^2(\mathscr{O};\mathbb{R})} \leq \sum_{j=1}^{N}\sum_{l=1}^{N} \|g^j\|_{L^4(\mathscr{O};\mathbb{R})} \left\|\partial_j f^l\right\|_{L^4(\mathscr{O};\mathbb{R})} \\
&\leq c\sum_{j=1}^{N}\sum_{l=1}^{N} \|g^j\|_{W^{1,2}(\mathscr{O};\mathbb{R})} \left\|\partial_j f^l\right\|_{W^{1,2}(\mathscr{O};\mathbb{R})} \leq c\sum_{j=1}^{N}\sum_{l=1}^{N} \|g^j\|_{W^{1,2}(\mathscr{O};\mathbb{R})} \left\|f^l\right\|_{W^{2,2}(\mathscr{O};\mathbb{R})} \\
&\leq c\|g\|_{W^{1,2}} \|f\|_{W^{2,2}}
\end{aligned}$$

applying the embedding of $W^{1,2}(\mathscr{O};\mathbb{R}) \hookrightarrow L^4(\mathscr{O};\mathbb{R})$. This justifies (4.9). As for (4.10),

$$\|\mathcal{L}_g f\|_{W^{1,2}}^2 = \|\mathcal{L}_g f\|^2 + \sum_{k=1}^N \|\partial_k \mathcal{L}_g f\|^2$$

where

$$\|\mathcal{L}_g f\|^2 \leq c\,\|g\|_{W^{1,2}}^2 \|f\|_{W^{2,2}}^2 \leq c\,\|g\|_{W^{1,2}}^2 \|f\|_{W^{3,2}}^2$$

having applied (4.9), and

$$\begin{aligned}
\sum_{k=1}^N \|\partial_k \mathcal{L}_g f\|^2 &\leq c \sum_{k=1}^N \sum_{j=1}^N \left(\left\|\partial_k g^j \partial_j f\right\|^2 + \left\|g^j \partial_k \partial_j f\right\|^2 \right) \\
&= c \sum_{k=1}^N \sum_{j=1}^N \sum_{l=1}^N \left(\left\|\partial_k g^j \partial_j f^l\right\|_{L^2(\mathscr{O};\mathbb{R})}^2 + \left\|g^j \partial_k \partial_j f^l\right\|_{L^2(\mathscr{O};\mathbb{R})}^2 \right) \\
&\leq c \sum_{k=1}^N \sum_{j=1}^N \sum_{l=1}^N \left(\left\|\partial_k g^j\right\|_{L^2(\mathscr{O};\mathbb{R})}^2 \left\|\partial_j f^l\right\|_{L^\infty(\mathscr{O};\mathbb{R})}^2 + \left\|g^j\right\|_{L^4(\mathscr{O};\mathbb{R})}^2 \left\|\partial_k \partial_j f^l\right\|_{L^4(\mathscr{O};\mathbb{R})}^2 \right) \\
&\leq c \sum_{k=1}^N \sum_{j=1}^N \sum_{l=1}^N \left(\left\|\partial_k g^j\right\|_{L^2(\mathscr{O};\mathbb{R})}^2 \left\|\partial_j f^l\right\|_{W^{2,2}(\mathscr{O};\mathbb{R})}^2 + \left\|g^j\right\|_{W^{1,2}(\mathscr{O};\mathbb{R})}^2 \left\|\partial_k \partial_j f^l\right\|_{W^{1,2}(\mathscr{O};\mathbb{R})}^2 \right) \\
&\leq c\,\|g\|_{W^{1,2}}^2 \|f\|_{W^{3,2}}^2 .
\end{aligned}$$

Combining these justifies (4.10). Indeed (4.11) is attained by the same procedure, instead passing to the bounds

$$\begin{aligned}
&\left\|\partial_k g^j \partial_j f^l\right\|_{L^2(\mathscr{O};\mathbb{R})}^2 + \left\|g^j \partial_k \partial_j f^l\right\|_{L^2(\mathscr{O};\mathbb{R})}^2 \\
&\qquad \leq \left\|\partial_k g^j\right\|_{L^4(\mathscr{O};\mathbb{R})}^2 \left\|\partial_j f^l\right\|_{L^4(\mathscr{O};\mathbb{R})}^2 + \left\|g^j\right\|_{L^\infty(\mathscr{O};\mathbb{R})}^2 \left\|\partial_k \partial_j f^l\right\|_{L^2(\mathscr{O};\mathbb{R})}^2
\end{aligned}$$

and continuing as above.

□

In fact, $\mathcal{L}$ can be extended to $W^{1,2}(\mathscr{O};\mathbb{R}^N)$. Similarly to the above, for $f, g \in W^{1,2}(\mathscr{O};\mathbb{R}^N)$,

$$\|\mathcal{L}_f g\|_{L^{6/5}} \leq c \sum_{k=1}^N \|f\|_{L^3} \|\partial_k g\| \leq c\,\|f\|_1 \|g\|_1 \tag{4.12}$$

having applied a general Hölder Inequality, noting that $\frac{5}{6} = \frac{1}{3} + \frac{1}{2}$, and using the coarse Sobolev Embedding of $W^{1,2}(\mathscr{O};\mathbb{R}^N)$ into $L^3(\mathscr{O};\mathbb{R}^N)$. Using that $L^6(\mathscr{O};\mathbb{R}^N)$ is the Hölder Conjugate of $L^{6/5}(\mathscr{O};\mathbb{R}^N)$, and that $W^{1,2}(\mathscr{O};\mathbb{R}^N) \hookrightarrow L^6(\mathscr{O};\mathbb{R}^N)$, then for $\phi, f, g \in W^{1,2}(\mathscr{O};\mathbb{R}^N)$ the trilinear form

$$\langle \mathcal{L}_\phi f, g \rangle = \langle \mathcal{L}_\phi f, g \rangle_{L^{6/5} \times L^6} \tag{4.13}$$

is well-defined.

Lemma 4.3.2. *For every $\phi \in \bar{W}^{1,2}_{\sigma}$, $f, g \in W^{1,2}(\mathscr{O};\mathbb{R}^N)$ we have that*

$$\langle \mathcal{L}_\phi f, g\rangle = -\langle f, \mathcal{L}_\phi g\rangle . \tag{4.14}$$

Moreover,

$$\langle \mathcal{L}_\phi f, f\rangle = 0. \tag{4.15}$$

Proof. Of course (4.15) follows from (4.14) with the choice $g := f$ and symmetry of the inner product, so we just show (4.14). It is classical that the result holds in the case that ϕ has zero trace (see e.g. [129] Lemma 3.2) by an approximation in $W^{1,2}(\mathscr{O};\mathbb{R}^N)$ of compactly supported functions, though without that we have to do the integration by parts directly:

$$\begin{aligned}
\langle \mathcal{L}_\phi f, g\rangle &= \sum_{j=1}^N\sum_{l=1}^N \left\langle \phi^j\partial_j f^l, g^l\right\rangle_{L^2(\mathscr{O};\mathbb{R})}\\
&= \sum_{j=1}^N\sum_{l=1}^N \left(\left\langle \phi^j\partial_j f^l, g^l\right\rangle_{L^2(\mathscr{O};\mathbb{R})} + \left\langle \partial_j\phi^j f^l, g^l\right\rangle_{L^2(\mathscr{O};\mathbb{R})}\right)\\
&= \sum_{j=1}^N\sum_{l=1}^N \left\langle \partial_j(\phi^j f^l), g^l\right\rangle_{L^2(\mathscr{O};\mathbb{R})}\\
&= -\sum_{j=1}^N\sum_{l=1}^N \left\langle \phi^j f^l, \partial_j g^l\right\rangle_{L^2(\mathscr{O};\mathbb{R})} + \sum_{j=1}^N\sum_{l=1}^N \left\langle \phi^j f^l, g^l\mathbf{n}^j\right\rangle_{L^2(\partial\mathscr{O};\mathbb{R})}\\
&= -\langle f, \mathcal{L}_\phi g\rangle
\end{aligned}$$

where we have used that $\sum_{j=1}^N \partial_j\phi^j = 0$ (divergence-free) in $\mathscr{O}$ and $\sum_{j=1}^N \phi^j\mathbf{n}^j = 0$ ($\phi\cdot\mathbf{n} = 0$) on $\partial\mathscr{O}$, from Lemma 4.2.12.

□

Moving on to the Stokes Operator A, most of our analysis hinges on Lemmas 4.2.8 and 4.2.17. We note just one further property here.

Lemma 4.3.3. *$A\mathcal{P}$ is equal to A on $W^{2,2}(\mathscr{O};\mathbb{R}^N)$.*

Proof. We call upon the representation (4.3), noting from Lemma 4.2.3 that for $f \in W^{2,2}(\mathscr{O};\mathbb{R}^N)$, $\mathcal{P}f \in W^{2,2}(\mathscr{O};\mathbb{R}^N)$ hence $\psi = \nabla g$ from (4.3) also belongs to $W^{2,2}(\mathscr{O};\mathbb{R}^N)$. Evidently,

$$Af = A(\mathcal{P}f + \nabla g) = A\mathcal{P}f + A\nabla g.$$

The result would thus be true if $A\nabla g = 0$. This is not difficult to see however, as

$$A\nabla g = -\mathcal{P}\Delta(\nabla g) = -\mathcal{P}\nabla(\Delta g) = 0$$

observing that $\Delta g \in W^{1,2}(\mathscr{O};\mathbb{R})$ given $\nabla g \in W^{2,2}(\mathscr{O};\mathbb{R}^N)$. □

4.3.2 Diffusion Operator

Continuing the notation used in Subsection 3.1.1 for B defined over the basis vectors (e_i) by $B(e_i) = B_i$, recall from the introduction that B is defined relative to vector fields $(\xi_i) : \mathscr{O} \to \mathbb{R}^N$ for sufficiently regular $f : \mathscr{O} \to \mathbb{R}^N$ by

$$B_i : f \mapsto \sum_{j=1}^{N} \left(\xi_i^j \partial_j f + f^j \nabla \xi_i^j \right).$$

To distinguish between the roles of the above terms, we rewrite

$$B_i : f \mapsto \mathcal{L}_{\xi_i} f + \mathcal{T}_{\xi_i} f$$

where $\mathcal{L}$ is as before, and $\mathcal{T}$ is a new operator that we introduce defined by

$$\mathcal{T}_g f := \sum_{j=1}^{N} f^j \nabla g^j.$$

We shall assume throughout that each ξ_i belongs at least to the space $\bar{W}^{1,2}_\sigma$. If for some fixed $m \in \mathbb{N}$ we have $\xi_i \in W^{m+2,\infty}(\mathscr{O};\mathbb{R}^N)$ then for all $k = 0, \dots, m+1$,[1], $f \in W^{j,2}(\mathscr{O};\mathbb{R}^N)$ as prescribed by the right hand side,

$$\|\mathcal{T}_{\xi_i} f\|^2_{W^{k,2}} \leq c \|\xi_i\|^2_{W^{k+1,\infty}} \|f\|^2_{W^{k,2}} \tag{4.16}$$
$$\|\mathcal{L}_{\xi_i} f\|^2_{W^{k,2}} \leq c \|\xi_i\|^2_{W^{k,\infty}} \|f\|^2_{W^{k+1,2}} \tag{4.17}$$
$$\|B_i f\|^2_{W^{k,2}} \leq c \|\xi_i\|^2_{W^{k+1,\infty}} \|f\|^2_{W^{k+1,2}}. \tag{4.18}$$

We assume throughout this section that each $\xi_i \in \bar{W}^{1,2}_\sigma \cap W^{m+2,\infty}(\mathscr{O};\mathbb{R}^N)$. Moreover $\mathcal{T}_{\xi_i}$ is a bounded linear operator on $L^2(\mathscr{O};\mathbb{R}^N)$ so has adjoint $\mathcal{T}^*_{\xi_i}$ satisfying the same boundedness. In conjunction with property (4.14), $\mathcal{L}_{\xi_i}$ is a densely defined operator in $L^2(\mathscr{O};\mathbb{R}^N)$ with domain of definition $W^{1,2}(\mathscr{O};\mathbb{R}^N)$, which is anti-symmetric with same dense domain of definition. Likewise then B_i^* is the densely defined $-\mathcal{L}_{\xi_i} + \mathcal{T}^*_{\xi_i}$. Our techniques centre around energy estimates, where the key idea as to how we preserve these estimates in the case of a transport type noise owes to the following proposition. The result has been understood on the torus but perhaps not on a bounded domain, and I have not seen a complete proof for arbitrarily high orders. The cancellation of derivatives is truly the crux of recent analytical developments for transport noise.

Proposition 4.3.4. *There exists a constant c such that, for each ξ_i and for all $f \in W^{k+2,2}(\mathscr{O};\mathbb{R}^N)$ with $k = 0, \dots, m$, we have the bounds*

$$\left\langle B_i^2 f, f \right\rangle_{W^{k,2}} + \|B_i f\|^2_{W^{k,2}} \leq c \|\xi_i\|^2_{W^{k+2,\infty}} \|f\|^2_{W^{k,2}}, \tag{4.19}$$
$$\langle B_i f, f \rangle^2_{W^{k,2}} \leq c \|\xi_i\|^2_{W^{k+1,\infty}} \|f\|^4_{W^{k,2}}. \tag{4.20}$$

[1] We understand $W^{0,p}$ as the corresponding L^p space.

Proof. We begin with showing (4.19), tasked to control

$$\left\langle B_i^2 f, f\right\rangle_{W^{k,2}} + \|B_i f\|^2_{W^{k,2}} \tag{4.21}$$

and do so with each derivative in the sum for the inner product: that is, we are looking at

$$\left\langle D^\alpha B_i^2 f, D^\alpha f\right\rangle + \left\langle D^\alpha B_i f, D^\alpha B_i f\right\rangle. \tag{4.22}$$

We consider B more generally as $B_\phi = \mathcal{L}_\phi + \mathcal{T}_\phi$, such that $B_{\xi_i} = B_i$. With this notation, we simplify the derivatives in (4.22) by writing

$$\begin{aligned} D^\alpha B_{\xi_i} f &= \sum_{\alpha' \leq \alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f \\ &= \sum_{\alpha' < \alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f + B_{\xi_i} D^\alpha f. \end{aligned} \tag{4.23}$$

Plugging this result in, we also see that

$$\begin{aligned} D^\alpha B_{\xi_i}^2 f &= D^\alpha B_{\xi_i}\left(B_{\xi_i} f\right) \\ &= \sum_{\alpha' < \alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} B_{\xi_i} f + B_{\xi_i} D^\alpha B_{\xi_i} f \end{aligned}$$

which we will use in our analysis of (4.22), reducing the expression to

$$\left\langle \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} B_{\xi_i} f + B_{\xi_i} D^\alpha B_{\xi_i} f, D^\alpha f\right\rangle + \left\langle D^\alpha B_{\xi_i} f, D^\alpha B_{\xi_i} f\right\rangle$$

which we further break up in terms of the adjoint $B^*_{\xi_i}$:

$$\left\langle \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f\right\rangle + \left\langle D^\alpha B_{\xi_i} f, B^*_{\xi_i} D^\alpha f\right\rangle + \left\langle D^\alpha B_{\xi_i} f, D^\alpha B_{\xi_i} f\right\rangle \tag{4.24}$$

requiring that $D^\alpha f \in W^{1,2}(\mathscr{O};\mathbb{R}^N)$, which is satisfied by the assumption $f \in W^{k+2,2}(\mathscr{O};\mathbb{R}^N)$. By summing the second and third inner products and using (4.23), this becomes

$$\left\langle \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f\right\rangle + \left\langle D^\alpha B_{\xi_i} f, B^*_{\xi_i} D^\alpha f + \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f + B_{\xi_i} D^\alpha f\right\rangle$$

which we look to simplify by combining $B^*_{\xi_i}$ and B_{ξ_i}, noting that

$$B_i^* + B_i = \mathcal{L}^*_{\xi_i} + \mathcal{T}_i^* + \mathcal{L}_{\xi_i} + \mathcal{T}_i = \mathcal{T}_i^* + \mathcal{T}_i.$$

Indeed we arrive at the expression

$$\left\langle \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f\right\rangle + \left\langle D^\alpha B_{\xi_i} f, \left(\mathcal{T}_{\xi_i} + \mathcal{T}^*_{\xi_i}\right) D^\alpha f + \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f\right\rangle.$$

As we are looking to achieve control with respect to the $W^{k,2}(\mathscr{O};\mathbb{R}^N)$ norm of f, then it is the terms with differential operators of order greater than k that concern us. Of course this was the motivating factor behind combining B_{ξ_i} and its adjoint, nullifying the additional derivative coming from $\mathcal{L}_{\xi_i}$. There are more higher order terms to go though, and the strategy will be to write these in terms of commutators with a differential operator of controllable order. This will involve considering different aspects of our sum in tandem, which will be helped with (4.23) reducing our expression again to

$$\Big\langle \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f\Big\rangle + \Big\langle \sum_{\beta<\alpha} B_{D^{\alpha-\beta}\xi_i} D^\beta f + B_{\xi_i} D^\alpha f, (\mathcal{T}_{\xi_i} + \mathcal{T}^*_{\xi_i}) D^\alpha f + \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f\Big\rangle.$$

Ultimately the terms in the summand are split up into

$$\big\langle B_{\xi_i} D^\alpha f, (\mathcal{T}_{\xi_i} + \mathcal{T}^*_{\xi_i}) D^\alpha f\big\rangle \tag{4.25}$$
$$+\Big\langle \sum_{\beta<\alpha} B_{D^{\alpha-\beta}\xi_i} D^\beta f, (\mathcal{T}_{\xi_i} + \mathcal{T}^*_{\xi_i}) D^\alpha f + \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f\Big\rangle \tag{4.26}$$
$$+\sum_{\alpha'<\alpha}\left(\Big\langle B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f\Big\rangle + \Big\langle B_{\xi_i} D^\alpha f, B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f\Big\rangle\right) \tag{4.27}$$

with the intention of controlling each one individually. Firstly for a treatment of (4.25),

$$\begin{aligned}
(4.25) &= \big\langle (\mathcal{L}_{\xi_i} + \mathcal{T}_{\xi_i}) D^\alpha f, (\mathcal{T}^*_{\xi_i} + \mathcal{T}_{\xi_i}) D^\alpha f\big\rangle \\
&= \big\langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}^*_{\xi_i} D^\alpha f\big\rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f\rangle + \big\langle \mathcal{T}_{\xi_i} D^\alpha f, \mathcal{T}^*_{\xi_i} D^\alpha f\big\rangle + \langle \mathcal{T}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f\rangle \\
&= \Big(\langle \mathcal{T}_{\xi_i}\mathcal{L}_{\xi_i} D^\alpha f, D^\alpha f\rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f\rangle\Big) + \Big(\big\langle \mathcal{T}^2_{\xi_i} D^\alpha f, D^\alpha f\big\rangle + \langle \mathcal{T}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f\rangle\Big).
\end{aligned}$$

We now bound the brackets in terms of $\|D^\alpha f\|^2$ separately, starting with the latter one as

$$\big\langle \mathcal{T}^2_{\xi_i} D^\alpha f, D^\alpha f\big\rangle \leq \big\|\mathcal{T}^2_{\xi_i} D^\alpha f\big\| \, \|D^\alpha f\| \leq c\,\|\xi_i\|_{W^{1,\infty}} \|\mathcal{T}_{\xi_i} D^\alpha f\| \, \|D^\alpha f\| \leq c\,\|\xi_i\|^2_{W^{1,\infty}} \|D^\alpha f\|^2$$

and similarly

$$\langle \mathcal{T}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f\rangle \leq \|\mathcal{T}_{\xi_i} D^\alpha f\| \, \|\mathcal{T}_{\xi_i} D^\alpha f\| \leq c\,\|\xi_i\|^2_{W^{1,\infty}} \|D^\alpha f\|^2 .$$

Now for the first bracket, we add and subtract a term to have an expression through the commutator of the operators:

$$\begin{aligned}
&\langle \mathcal{T}_{\xi_i}\mathcal{L}_{\xi_i} D^\alpha f, D^\alpha f\rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f\rangle \\
&= \langle (\mathcal{T}_{\xi_i}\mathcal{L}_{\xi_i} - \mathcal{L}_{\xi_i}\mathcal{T}_{\xi_i}) D^\alpha f, D^\alpha f\rangle + \langle \mathcal{L}_{\xi_i}\mathcal{T}_{\xi_i} D^\alpha f, D^\alpha f\rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f\rangle \\
&= \langle (\mathcal{T}_{\xi_i}\mathcal{L}_{\xi_i} - \mathcal{L}_{\xi_i}\mathcal{T}_{\xi_i}) D^\alpha f, D^\alpha f\rangle + \big\langle \mathcal{T}_{\xi_i} D^\alpha f, \mathcal{L}^*_{\xi_i} D^\alpha f\big\rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f\rangle \\
&= \langle (\mathcal{T}_{\xi_i}\mathcal{L}_{\xi_i} - \mathcal{L}_{\xi_i}\mathcal{T}_{\xi_i}) D^\alpha f, D^\alpha f\rangle .
\end{aligned}$$

The commutator term is given explicitly through

$$\mathcal{T}_{\xi_i}\mathcal{L}_{\xi_i}D^\alpha f = \mathcal{T}_{\xi_i}\Big(\sum_{j=1}^N \xi_i^j\partial_j D^\alpha f\Big) = \sum_{k=1}^N\Big(\sum_{j=1}^N \xi_i^j\partial_j D^\alpha f\Big)^k \nabla\xi_i^k = \sum_{k=1}^N\sum_{j=1}^N \xi_i^j\partial_j D^\alpha f^k\nabla\xi_i^k$$

and

$$\begin{aligned}\mathcal{L}_{\xi_i}\mathcal{T}_{\xi_i}D^\alpha f &= \mathcal{L}_{\xi_i}\Big(\sum_{k=1}^N D^\alpha f^k\nabla\xi_i^k\Big) = \sum_{j=1}^N \xi_i^j\partial_j\Big(\sum_{k=1}^N D^\alpha f^k\nabla\xi_i^k\Big) = \sum_{j=1}^N\sum_{k=1}^N \xi_i^j\partial_j\big(D^\alpha f^k\nabla\xi_i^k\big)\\ &= \sum_{j=1}^N\sum_{k=1}^N\Big(\xi_i^j\partial_j D^\alpha f^k\nabla\xi_i^k + \xi_i^j D^\alpha f^k\partial_j\nabla\xi_i^k\Big)\end{aligned}$$

such that

$$(\mathcal{T}_{\xi_i}\mathcal{L}_{\xi_i} - \mathcal{L}_{\xi_i}\mathcal{T}_{\xi_i})D^\alpha f = -\sum_{j=1}^N\sum_{k=1}^N \xi_i^j D^\alpha f^k\partial_j\nabla\xi_i^k.$$

Therefore

$$\begin{aligned}\|(\mathcal{T}_{\xi_i}\mathcal{L}_{\xi_i} - \mathcal{L}_{\xi_i}\mathcal{T}_{\xi_i})D^\alpha f\| &\leq c\sum_{j=1}^N\sum_{k=1}^N\left\|\xi_i^j D^\alpha f^k\partial_j\nabla\xi_i^k\right\| \leq c\sum_{j=1}^N\sum_{k=1}^N\sum_{l=1}^N\left\|\xi_i^j D^\alpha f^k\partial_j\partial_l\xi_i^k\right\|_{L^2(\mathscr{O};\mathbb{R})}\\ &\leq c\sum_{j=1}^N\sum_{k=1}^N\sum_{l=1}^N\left\|\xi_i^j\partial_j\partial_l\xi_i^k\right\|_{L^\infty(\mathscr{O};\mathbb{R})}\left\|D^\alpha f^k\right\|_{L^2(\mathscr{O};\mathbb{R})}\\ &\leq c\,\|\xi_i\|_{W^{2,\infty}}^2\sum_{j=1}^N\sum_{k=1}^N\sum_{l=1}^N\left\|D^\alpha f^k\right\|_{L^2(\mathscr{O};\mathbb{R})}\\ &\leq c\,\|\xi_i\|_{W^{2,\infty}}^2\,\|D^\alpha f\|\end{aligned}$$

and

$$\langle(\mathcal{T}_{\xi_i}\mathcal{L}_{\xi_i} - \mathcal{L}_{\xi_i}\mathcal{T}_{\xi_i})D^\alpha f, D^\alpha f\rangle \leq \|(\mathcal{T}_{\xi_i}\mathcal{L}_{\xi_i} - \mathcal{L}_{\xi_i}\mathcal{T}_{\xi_i})D^\alpha f\|\,\|D^\alpha f\| \leq c\,\|\xi_i\|_{W^{2,\infty}}^2\,\|D^\alpha f\|^2.$$

Combining these inequalities we determine the bound

$$(4.25) \leq c\,\|\xi_i\|_{W^{2,\infty}}^2\,\|D^\alpha f\|^2.$$

As for (4.26) we look to use Cauchy-Schwarz and bound each item in the inner product. Indeed straight from (4.18) in the $L^2(\mathscr{O};\mathbb{R}^N)$ setting, by simply replacing ξ_i with $D^{\alpha-\beta}\xi_i$, we have that

$$\left\|B_{D^{\alpha-\beta}\xi_i}D^\beta f\right\|^2 \leq c\left\|D^{\alpha-\beta}\xi_i\right\|_{W^{1,\infty}}^2\left\|D^\beta f\right\|_{W^{1,2}}^2 \leq c\,\|\xi_i\|_{W^{k+1,\infty}}^2\,\|f\|_{W^{k,2}}^2.$$

Moreover,

$$\left\|\sum_{\beta<\alpha}B_{D^{\alpha-\beta}\xi_i}D^\beta f\right\|^2 \leq c\sum_{\beta<\alpha}\left\|B_{D^{\alpha-\beta}\xi_i}D^\beta f\right\|^2 \leq c\,\|\xi_i\|_{W^{k+1,\infty}}^2\,\|f\|_{W^{k,2}}^2.$$

In addition to this,

$$\left\|(\mathcal{T}_{\xi_i}+\mathcal{T}_{\xi_i}^*)D^\alpha f\right\| \leq \left\|\mathcal{T}_{\xi_i}D^\alpha f\right\| + \left\|\mathcal{T}_{\xi_i}^*D^\alpha f\right\| \leq c\left\|\xi_i\right\|_{W^{k+1,\infty}}\left\|D^\alpha f\right\|$$

using the equivalence in operator norm of the adjoint. Together this amounts to

$$\begin{aligned}
(4.26) &\leq \left\|\sum_{\beta<\alpha}B_{D^{\alpha-\beta}\xi_i}D^\beta f\right\|\cdot\left\|(\mathcal{T}_{\xi_i}+\mathcal{T}_{\xi_i}^*)D^\alpha f+\sum_{\alpha'<\alpha}B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f\right\| \\
&\leq \left\|\sum_{\beta<\alpha}B_{D^{\alpha-\beta}\xi_i}D^\beta f\right\|\left(\left\|(\mathcal{T}_{\xi_i}+\mathcal{T}_{\xi_i}^*)D^\alpha f\right\|+\left\|\sum_{\alpha'<\alpha}B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f\right\|\right) \\
&\leq c\left\|\xi_i\right\|_{W^{k+1,\infty}}\left\|f\right\|_{W^{k,2}}\left(c\left\|\xi_i\right\|_{W^{k+1,\infty}}\left\|D^\alpha f\right\|+c\left\|\xi_i\right\|_{W^{k+1,\infty}}\left\|f\right\|_{W^{k,2}}\right) \\
&\leq c\left\|\xi_i\right\|_{W^{k+1,\infty}}^2\left\|f\right\|_{W^{k,2}}^2.
\end{aligned}$$

Let's now turn our attentions to (4.27), which for each α' in the sum we rewrite as

$$\left\langle D^\alpha f, B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}B_{\xi_i}f+B_{\xi_i}^*B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f\right\rangle \tag{4.28}$$

and employing (4.23) again we see this becomes

$$\begin{aligned}
&\left\langle D^\alpha f, B_{D^{\alpha-\alpha'}\xi_i}\left(\sum_{\beta<\alpha'}B_{D^{\alpha'-\beta}\xi_i}D^\beta f+B_{\xi_i}D^{\alpha'}f\right)+B_{\xi_i}^*B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f\right\rangle \\
=&\left\langle D^\alpha f,\sum_{\beta<\alpha'}B_{D^{\alpha-\alpha'}\xi_i}B_{D^{\alpha'-\beta}\xi_i}D^\beta f\right\rangle+\left\langle D^\alpha f, B_{D^{\alpha-\alpha'}\xi_i}B_{\xi_i}D^{\alpha'}f+B_{\xi_i}^*B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f\right\rangle.
\end{aligned}$$

We have split up these terms to make our approach clearer, as the two right hand sides of the inner products will be considered separately. For the first inner product, two applications of (4.18) give that

$$\begin{aligned}
\left\|B_{D^{\alpha-\alpha'}\xi_i}B_{D^{\alpha'-\beta}\xi_i}D^\beta f\right\|^2 &\leq c\left\|D^{\alpha-\alpha'}\xi_i\right\|_{W^{1,\infty}}^2\left\|B_{D^{\alpha'-\beta}\xi_i}D^\beta f\right\|_{W^{1,2}}^2 \\
&\leq c\left\|D^{\alpha-\alpha'}\xi_i\right\|_{W^{1,\infty}}^2\left(c\left\|D^{\alpha'-\beta}\xi_i\right\|_{W^{2,\infty}}^2\left\|D^\beta f\right\|_{W^{2,2}}^2\right) \\
&\leq c\left\|\xi_i\right\|_{W^{k+1,\infty}}^4\left\|f\right\|_{W^{k,2}}^2.
\end{aligned}$$

Moreover,

$$\left\|\sum_{\beta<\alpha'}B_{D^{\alpha-\alpha'}\xi_i}B_{D^{\alpha'-\beta}\xi_i}D^\beta f\right\|^2 \leq c\left\|B_{D^{\alpha-\alpha'}\xi_i}B_{D^{\alpha'-\beta}\xi_i}D^\beta f\right\|^2 \leq c\left\|\xi_i\right\|_{W^{k+1,\infty}}^4\left\|f\right\|_{W^{k,2}}^2.$$

As for the second inner product, we rewrite the right side as

$$B_{D^{\alpha-\alpha'}\xi_i}\left((\mathcal{L}_{\xi_i}+\mathcal{T}_{\xi_i})D^{\alpha'}f\right)+(\mathcal{L}_{\xi_i}^*+\mathcal{T}_{\xi_i}^*)B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f$$

and further

$$\left(B_{D^{\alpha-\alpha'}\xi_i}\mathcal{L}_{\xi_i}-\mathcal{L}_{\xi_i}B_{D^{\alpha-\alpha'}\xi_i}\right)D^{\alpha'}f+B_{D^{\alpha-\alpha'}\xi_i}\mathcal{T}_{\xi_i}D^{\alpha'}f+\mathcal{T}_{\xi_i}^*B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f. \tag{4.29}$$

Starting with the latter two terms,

$$\begin{aligned}\left\|B_{D^{\alpha-\alpha'}\xi_i}\mathcal{T}_{\xi_i}D^{\alpha'}f\right\|^2 &\leq c\left\|D^{\alpha-\alpha'}\xi_i\right\|_{W^{1,\infty}}^2\left\|\mathcal{T}_{\xi_i}D^{\alpha'}f\right\|_{W^{1,2}}^2\\ &\leq c\left\|D^{\alpha-\alpha'}\xi_i\right\|_{W^{1,\infty}}^2\left(c\left\|\xi_i\right\|_{W^{2,\infty}}^2\left\|D^{\alpha'}f\right\|_{W^{1,2}}^2\right)\\ &\leq c\left\|\xi_i\right\|_{W^{k+1,\infty}}^4\left\|f\right\|_{W^{k,2}}^2\end{aligned}$$

and likewise

$$\begin{aligned}\left\|\mathcal{T}_{\xi_i}^*B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f\right\|^2 &\leq c\left\|\xi_i\right\|_{W^{1,\infty}}^2\left\|B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f\right\|^2\\ &\leq c\left\|\xi_i\right\|_{W^{1,\infty}}^2\left(c\left\|D^{\alpha-\alpha'}\xi_i\right\|_{W^{1,\infty}}^2\left\|D^{\alpha'}f\right\|_{W^{1,2}}^2\right)\\ &\leq c\left\|\xi_i\right\|_{W^{k+1,\infty}}^4\left\|f\right\|_{W^{k,2}}^2.\end{aligned}$$

Now we show explicitly that the commutator in

$$\left(B_{D^{\alpha-\alpha'}\xi_i}\mathcal{L}_{\xi_i}-\mathcal{L}_{\xi_i}B_{D^{\alpha-\alpha'}\xi_i}\right)D^{\alpha'}f \tag{4.30}$$

from (4.29) is of first order (so of k^{th} order when composed with $D^{\alpha'}$), through the expressions

$$\begin{aligned}B_{D^{\alpha-\alpha'}\xi_i}\mathcal{L}_{\xi_i}D^{\alpha'}f &= \sum_{j=1}^N\left(D^{\alpha-\alpha'}\xi_i^j\partial_j\Big(\sum_{k=1}^N\xi_i^k\partial_kD^{\alpha'}f\Big)+\Big(\sum_{k=1}^N\xi_i^k\partial_kD^{\alpha'}f\Big)^j\nabla D^{\alpha-\alpha'}\xi_i^j\right)\\ &= \sum_{j=1}^N\sum_{k=1}^N\Bigg(D^{\alpha-\alpha'}\xi_i^j\partial_j\xi_i^k\partial_kD^{\alpha'}f\\ &\qquad\qquad + D^{\alpha-\alpha'}\xi_i^j\xi_i^k\partial_j\partial_kD^{\alpha'}f+\xi_i^k\partial_kD^{\alpha'}f^j\nabla D^{\alpha-\alpha'}\xi_i^j\Bigg)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{\xi_i}B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f &= \sum_{k=1}^N\xi_i^k\partial_k\left(\sum_{j=1}^N D^{\alpha-\alpha'}\xi_i^j\partial_jD^{\alpha'}f+D^{\alpha'}f^j\nabla D^{\alpha-\alpha'}\xi_i^j\right)\\ &= \sum_{j=1}^N\sum_{k=1}^N\Bigg(\xi_i^k\partial_kD^{\alpha-\alpha'}\xi_i^j\partial_jD^{\alpha'}f+\xi_i^kD^{\alpha-\alpha'}\xi_i^j\partial_k\partial_jD^{\alpha'}f\\ &\qquad\qquad +\xi_i^k\partial_kD^{\alpha'}f^j\nabla D^{\alpha-\alpha'}\xi_i^j+\xi_i^kD^{\alpha'}f^j\partial_k\nabla D^{\alpha-\alpha'}\xi_i^j\Bigg)\end{aligned}$$

such that

$$(4.30)=\sum_{j=1}^N\sum_{k=1}^N\left(D^{\alpha-\alpha'}\xi_i^j\partial_j\xi_i^k\partial_kD^{\alpha'}f-\xi_i^k\partial_kD^{\alpha-\alpha'}\xi_i^j\partial_jD^{\alpha'}f-\xi_i^kD^{\alpha'}f^j\partial_k\nabla D^{\alpha-\alpha'}\xi_i^j\right)$$

and in particular

$$\begin{aligned}
\|(4.30)\|^2 &\leq c\sum_{j=1}^N\sum_{k=1}^N\Bigg(\left\|D^{\alpha-\alpha'}\xi_i^j\partial_j\xi_i^k\partial_k D^{\alpha'}f\right\|^2 \\
&\qquad\qquad + \left\|\xi_i^k\partial_k D^{\alpha-\alpha'}\xi_i^j\partial_j D^{\alpha'}f\right\|^2 + \left\|\xi_i^k D^{\alpha'}f^j\partial_k\nabla D^{\alpha-\alpha'}\xi_i^j\right\|^2\Bigg) \\
&= c\sum_{j=1}^N\sum_{k=1}^N\sum_{l=1}^N\Bigg(\left\|D^{\alpha-\alpha'}\xi_i^j\partial_j\xi_i^k\partial_k D^{\alpha'}f^l\right\|^2_{L^2(\mathscr{O};\mathbb{R})} \\
&\qquad\qquad + \left\|\xi_i^k\partial_k D^{\alpha-\alpha'}\xi_i^j\partial_j D^{\alpha'}f^l\right\|^2_{L^2(\mathscr{O};\mathbb{R})} + \left\|\xi_i^k D^{\alpha'}f^j\partial_k\partial_l D^{\alpha-\alpha'}\xi_i^j\right\|^2_{L^2(\mathscr{O};\mathbb{R})}\Bigg) \\
&\leq c\left\|\xi_i\right\|^4_{W^{k+2,\infty}}\sum_{j=1}^N\sum_{k=1}^N\sum_{l=1}^N\Bigg(\left\|\partial_k D^{\alpha'}f^l\right\|^2_{L^2(\mathscr{O};\mathbb{R})} \\
&\qquad\qquad + \left\|\partial_j D^{\alpha'}f^l\right\|^2_{L^2(\mathscr{O};\mathbb{R})} + \left\|D^{\alpha'}f^l\right\|^2_{L^2(\mathscr{O};\mathbb{R})}\Bigg) \\
&\leq c\left\|\xi_i\right\|^4_{W^{k+2,\infty}}\left(\sum_{j=1}^N\|f\|^2_{W^{k,2}} + \sum_{k=1}^N\|f\|^2_{W^{k,2}} + \sum_{j=1}^N\sum_{k=1}^N\left\|D^{\alpha'}f\right\|^2\right) \\
&\leq c\left\|\xi_i\right\|^4_{W^{k+2,\infty}}\|f\|^2_{W^{k,2}}.
\end{aligned}$$

Finally now we can piece these four inequalities together to produce a bound on (4.28):

$$\begin{aligned}
|(4.28)| &\leq \|D^\alpha f\|\left\|\sum_{\beta<\alpha'}B_{D^{\alpha-\alpha'}\xi_i}B_{D^{\alpha'-\beta}\xi_i}D^\beta f + (4.29)\right\| \\
&\leq \|D^\alpha f\|\Bigg(\left\|\sum_{\beta<\alpha'}B_{D^{\alpha-\alpha'}\xi_i}B_{D^{\alpha'-\beta}\xi_i}D^\beta f\right\| + \left\|B_{D^{\alpha-\alpha'}\xi_i}\mathcal{T}_{\xi_i}D^{\alpha'}f\right\| \\
&\qquad\qquad + \left\|\mathcal{T}^*_{\xi_i}B_{D^{\alpha-\alpha'}\xi_i}D^{\alpha'}f\right\| + \|(4.30)\|\Bigg) \\
&\leq c\|D^\alpha f\|\Bigg(\|\xi_i\|^2_{W^{k+1,\infty}}\|f\|_{W^{k,2}} + \|\xi_i\|^2_{W^{k+1,\infty}}\|f\|_{W^{k,2}} \\
&\qquad\qquad + \|\xi_i\|^2_{W^{k+1,\infty}}\|f\|_{W^{k,2}} + \|\xi_i\|^2_{W^{k+2,\infty}}\|f\|_{W^{k,2}}\Bigg) \\
&\leq c\|D^\alpha f\|\left(\|\xi_i\|^2_{W^{k+2,\infty}}\|f\|_{W^{k,2}}\right) \\
&\leq c\|\xi_i\|^2_{W^{k+2,\infty}}\|f\|^2_{W^{k,2}}
\end{aligned}$$

and subsequently of (4.27):

$$(4.27) = \sum_{\alpha'<\alpha}(4.28) \leq c\sum_{\alpha'<\alpha}\|\xi_i\|^2_{W^{k+2,\infty}}\|f\|^2_{W^{k,2}} \leq c\|\xi_i\|^2_{W^{k+2,\infty}}\|f\|^2_{W^{k,2}}.$$

We can now conclude the proof of (4.19) by observing that

$$
\begin{aligned}
(4.21) &= \sum_{|\alpha|\leq k} (4.22) \\
&= \sum_{|\alpha|\leq k} (4.25) + (4.26) + (4.27) \\
&\leq c \sum_{|\alpha|\leq k} \left(\|\xi_i\|_{W^{2,\infty}}^2 \|D^\alpha f\|^2 + \|\xi_i\|_{W^{k+1,\infty}}^2 \|f\|_{W^{k,2}}^2 + \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 \right) \\
&\leq c \sum_{|\alpha|\leq k} \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 \\
&= c \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 .
\end{aligned}
$$

As for (4.20), using (4.23) once more, we see that for each α in the sum for the inner product,

$$
\begin{aligned}
|\langle D^\alpha B_i f, D^\alpha f\rangle| &= \left| \left\langle \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f + B_{\xi_i} D^\alpha f, D^\alpha f \right\rangle \right| \\
&= \left| \left\langle \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f, D^\alpha f \right\rangle + \langle B_{\xi_i} D^\alpha f, D^\alpha f\rangle \right| \\
&\leq \left| \left\langle \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f, D^\alpha f \right\rangle \right| + |\langle \mathcal{T}_{\xi_i} D^\alpha f, D^\alpha f\rangle|
\end{aligned}
$$

using the cancellation from (4.15) to dispose of the order $k+1$ term. In our treatment of (4.26) in (4.19), we showed the bound

$$
\left\| \sum_{\beta<\alpha} B_{D^{\alpha-\beta}\xi_i} D^\beta f \right\| \leq c \|\xi_i\|_{W^{k+1,\infty}} \|f\|_{W^{k,2}}
$$

and therefore

$$
\left| \left\langle \sum_{\alpha'<\alpha} B_{D^{\alpha-\alpha'}\xi_i} D^{\alpha'} f, D^\alpha f \right\rangle \right| \leq c \|\xi_i\|_{W^{k+1,\infty}} \|f\|_{W^{k,2}} \|D^\alpha f\| \leq c \|\xi_i\|_{W^{k+1,\infty}} \|f\|_{W^{k,2}}^2
$$

whilst a simple control on the second term yields the result. $\square$

The penultimate result of this subsection is completely analogous to Lemma 4.3.3.

Lemma 4.3.5. *We have that*

$$
B_i : L^{2,\perp}_\sigma(\mathscr{O};\mathbb{R}^N) \cap W^{1,2}(\mathscr{O};\mathbb{R}^N) \to L^{2,\perp}_\sigma(\mathscr{O};\mathbb{R}^N)
$$

and moreover that $\mathcal{P}B_i = \mathcal{P}B_i\mathcal{P}$ on $W^{1,2}(\mathscr{O};\mathbb{R}^N)$.

Proof. For $\nabla g \in L^{2,\perp}_\sigma(\mathscr{O};\mathbb{R}^N) \cap W^{1,2}(\mathscr{O};\mathbb{R}^N)$,

$$\begin{aligned} B_i(\nabla g) = \mathcal{L}_{\xi_i}(\nabla g) + \mathcal{T}_{\xi_i}(\nabla g) &= \sum_{j=1}^N \xi_i^j \partial_j(\nabla g) + \sum_{j=1}^N \partial_j g \nabla \xi_i^j = \sum_{j=1}^N \xi_i^j(\nabla \partial_j g) + \sum_{j=1}^N (\nabla \xi_i^j)\partial_j g \\ &= \nabla \sum_{j=1}^N \xi_i^j \partial_j g \in L^{2,\perp}_\sigma(\mathscr{O};\mathbb{R}^N) \end{aligned}$$

using in the last line the assumption that $\nabla g \in W^{1,2}(\mathscr{O};\mathbb{R}^N)$ and that $\xi_i \in W^{1,\infty}(\mathscr{O};\mathbb{R}^N)$. The property $\mathcal{P}B_i = \mathcal{P}B_i\mathcal{P}$ follows identically to Lemma 4.3.3. $\square$

We note that this result holds true only in the presence of the additional $\mathcal{T}_{\xi_i}$ term in the operator, highlighting the significance of considering a noise which is not purely transport. In the case of strong solutions, we shall also use Proposition 4.3.6 below.

Proposition 4.3.6. *There exists a constant c such that for every $f \in W^{3,2}(\mathscr{O};\mathbb{R}^N)$,*

$$\|[\Delta, B_i]f\|^2 \leq c\|\xi_i\|^2_{W^{3,\infty}}\|f\|^2_{W^{2,2}}$$

where $[\Delta, B_i]$ is the commutator

$$[\Delta, B_i] := \Delta B_i - B_i\Delta.$$

Proof. Recalling (4.23), we have that

$$\partial_k^2 B_i = B_{\partial_k^2\xi_i} + 2B_{\partial_k\xi_i}\partial_k + B_i\partial_k^2$$

so

$$[\Delta, B_i] = \sum_{k=1}^N B_{\partial_k^2\xi_i} + 2B_{\partial_k\xi_i}\partial_k$$

hence the bound follows from (4.18).

$\square$

4.4 Formulation of Solutions

This section draws together the previous two in establishing how the addressed function spaces and operators allow us to format the problems (4.1), (4.2) in terms of the framework from Subsection 3.1.1. Subsection 4.4.1 converts (1.2) into the desired form (1.1). The framework of problem (4.1) is described in Subsection 4.4.2, and for problem (4.2) in Subsection 4.4.3.

4.4.1 Projection and Itô Conversion

Recall the equation (1.2),

$$u_t = u_0 - \int_0^t \mathcal{L}_{u_s}u_s\,ds + \nu\int_0^t \Delta u_s\,ds - \int_0^t Bu_s \circ d\mathcal{W}_s - \nabla\rho_t \tag{4.31}$$

as well as the abstract form (1.1),

$$\boldsymbol{\Psi}_t = \boldsymbol{\Psi}_0 + \int_0^t \mathcal{A}(s, \boldsymbol{\Psi}_s)ds + \int_0^t \mathcal{G}(s, \boldsymbol{\Psi}_s)d\mathcal{W}_s \tag{4.32}$$

treated in Chapter 3. There are two key differences between (4.31) and (4.32). The first of those is the pressure, which is not directly given as a function of u; as described at the start of Subsection 4.3.1, it is commonplace to solve the equation under the Leray Projector $\mathcal{P}$ and subsequently append a pressure to it. Following this, we first consider the projected equation

$$u_t = u_0 - \int_0^t \mathcal{P}\mathcal{L}_{u_s}u_s\, ds - \nu \int_0^t Au_s\, ds - \int_0^t \mathcal{P}Bu_s \circ d\mathcal{W}_s \tag{4.33}$$

where A is the Stokes Operator $-\mathcal{P}\Delta$, and we assume that $u_t, u_0 \in L^2_\sigma$. The second difference owes to the Stratonovich Integration as opposed to Itô. A rigorous connection between the Itô and Stratonovich forms was made in Subsection 2.2.4, and we note that (4.33) is of the form (2.13) for the operators

$$\mathcal{Q} := -\left(\mathcal{P}\mathcal{L} + \nu A\right), \qquad \mathcal{G} := -\mathcal{P}B.$$

Theorem 2.2.20 thus motivates the corresponding Itô form as

$$u_t = u_0 - \int_0^t \mathcal{P}\mathcal{L}_{u_s}u_s\ ds - \nu \int_0^t Au_s\, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty (\mathcal{P}B_i)^2 u_s ds - \int_0^t \mathcal{P}Bu_s d\mathcal{W}_s.$$

Courtesy of Lemma 4.3.5, we observe that $(\mathcal{P}B_i)^2 = \mathcal{P}B_i\mathcal{P}B_i = \mathcal{P}B_i^2$ so we may rewrite this equation as

$$u_t = u_0 - \int_0^t \mathcal{P}\mathcal{L}_{u_s}u_s\ ds - \nu \int_0^t Au_s\, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty \mathcal{P}B_i^2 u_s ds - \int_0^t \mathcal{P}Bu_s d\mathcal{W}_s \tag{4.34}$$

which completes the transposition of (4.31) into the form (4.32). The equivalence between (4.33) and (4.34) is dependent on the nature of solutions considered, and whilst we primarily work with (4.34) hereafter, explicit solutions to (4.33) will be revisited.

4.4.2 No-Slip Boundary

Problem (4.1) fits the framework of Subsection 3.1.1 in the sense of equation (4.34), the spaces

$$V := W^{2,2}_\sigma, \qquad H := W^{1,2}_\sigma, \qquad U := L^2_\sigma$$

and operators

$$\mathcal{A} = -\left(\mathcal{P}\mathcal{L} + \nu A\right) + \frac{1}{2}\sum_{i=1}^\infty \mathcal{P}B_i^2, \qquad \mathcal{G} := -\mathcal{P}B.$$

We use the basis (a_k) specified in Proposition 4.2.9, and assume that each $\xi_i \in \bar{W}^{1,2}_\sigma \cap W^{2,\infty}(\mathscr{O};\mathbb{R}^N)$ with $\sum_{i=1}^\infty \|\xi_i\|^2_{W^{2,\infty}} < \infty$. The remaining aspect of the framework not addressed is how the operator $\mathcal{A}$ extends to $\mathcal{A}: W^{1,2}_\sigma \to \left(W^{1,2}_\sigma\right)^*$. For $f \in W^{2,2}_\sigma$, $g \in W^{1,2}_\sigma$, from (4.13), Lemma 4.2.8 and the definition of the adjoint B_i^*,

$$\begin{aligned}\langle \mathcal{P}\mathcal{L}_f f, g\rangle &= \langle \mathcal{L}_f f, g\rangle_{L^{6/5}\times L^6}\\ \langle Af, g\rangle &= \langle f, g\rangle_1\\ \left\langle \frac{1}{2}\sum_{i=1}^\infty \mathcal{P}B_i^2 f, g\right\rangle &= \frac{1}{2}\sum_{i=1}^\infty \langle B_i f, B_i^* g\rangle\end{aligned}$$

where the right hand sides are well defined for $f \in W^{1,2}_\sigma$. Thus we extend $\mathcal{A}$ from the dense $W^{2,2}_\sigma$ to $W^{1,2}_\sigma$ as a mapping into $\left(W^{1,2}_\sigma\right)^*$ by the following duality pairings, for $f, g \in W^{1,2}_\sigma$:

$$\begin{aligned}\langle \mathcal{P}\mathcal{L}_f f, g\rangle_{\left(W^{1,2}_\sigma\right)^*\times W^{1,2}_\sigma} &= \langle \mathcal{L}_f f, g\rangle_{L^{6/5}\times L^6}\\ \langle Af, g\rangle_{\left(W^{1,2}_\sigma\right)^*\times W^{1,2}_\sigma} &= \langle f, g\rangle_1\\ \left\langle \frac{1}{2}\sum_{i=1}^\infty \mathcal{P}B_i^2 f, g\right\rangle_{\left(W^{1,2}_\sigma\right)^*\times W^{1,2}_\sigma} &= \frac{1}{2}\sum_{i=1}^\infty \langle B_i f, B_i^* g\rangle .\end{aligned}$$

Finally, it should be noted that the noise operator $\mathcal{P}B_i$ does not necessarily map from $W^{2,2}_\sigma$ into $W^{1,2}_\sigma$, but only into the larger space $\bar{W}^{1,2}_\sigma$ given that $\mathcal{P}$ does not preserve the zero-trace property in general. This motivates the need for the extension $\bar{H}$ used in Assumption Set 3, Subsection 3.4.1.

4.4.3 Navier Boundary

Problem (4.2) fits the framework of Subsection 3.1.1 in the sense of equation (4.34), the spaces

$$V := \bar{W}^{2,2}_\alpha, \qquad H := \bar{W}^{1,2}_\sigma, \qquad U := L^2_\sigma$$

and operators

$$\mathcal{A} = -\left(\mathcal{P}\mathcal{L} + \nu A\right) + \frac{1}{2}\sum_{i=1}^\infty \mathcal{P}B_i^2, \qquad \mathcal{G} := -\mathcal{P}B.$$

We use the basis $(\bar{a}_k)$ specified in Proposition 4.2.16, and assume that each $\xi_i \in \bar{W}^{1,2}_\sigma \cap W^{2,\infty}(\mathscr{O};\mathbb{R}^N)$ with $\sum_{i=1}^\infty \|\xi_i\|^2_{W^{2,\infty}} < \infty$. We finally extend $\mathcal{A} : \bar{W}^{1,2}_\sigma \to \left(\bar{W}^{1,2}_\sigma\right)^*$ from the dense domain $\bar{W}^{2,2}_\alpha$. The only difference from Subsection 4.4.2 is in the Stokes Operator, as from Lemma 4.2.17 for $f \in \bar{W}^{2,2}_\alpha$, $g \in \bar{W}^{1,2}_\sigma$,

$$\langle Af, g\rangle = \langle f, g\rangle_1 - \langle (\kappa - \alpha) f, g\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)} .$$

Therefore, the extension is prescribed by the following duality pairings for $f, g \in \bar{W}^{1,2}_\sigma$:

$$\begin{aligned}
\langle \mathcal{P}\mathcal{L}_f f, g\rangle_{(\bar{W}^{1,2}_\sigma)^* \times \bar{W}^{1,2}_\sigma} &= \langle \mathcal{L}_f f, g\rangle_{L^{6/5}\times L^6}\\
\langle Af, g\rangle_{(\bar{W}^{1,2}_\sigma)^* \times \bar{W}^{1,2}_\sigma} &= \langle f, g\rangle_1 - \langle (\kappa-\alpha) f, g\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)}\\
\left\langle \frac{1}{2}\sum_{i=1}^\infty \mathcal{P}B_i^2 f, g\right\rangle_{(\bar{W}^{1,2}_\sigma)^* \times \bar{W}^{1,2}_\sigma} &= \frac{1}{2}\sum_{i=1}^\infty \langle B_i f, B_i^* g\rangle .
\end{aligned}$$

4.4.4 Vorticity Form

An alternative way to consider the equation (4.31) is through the *vorticity* form, where the vorticity w is given by curl$u$. Use of the vorticity form will prove important in Subsection 6.3.2, and strong solutions will be directly considered in Subsection 5.4.2. Passage from (4.31) to the vorticity form arises from taking the curl of the equation. The results differ in the cases of two and three spatial dimensions; we begin with the case $N = 2$. Recalling Lemma 4.2.19, as well as the classical result that $\mathrm{curl}(\mathcal{L}_\phi \phi) = \mathcal{L}_\phi(\mathrm{curl}\phi)$ (e.g. [129] Subsection 12.4), then it is clear that

$$\mathrm{curl}\left(\mathcal{P}\mathcal{L}_u u + \nu A u\right) = \mathcal{L}_u w - \nu\Delta w.$$

For the stochastic contribution, we demonstrate the following.

Lemma 4.4.1. *For $\phi \in W^{2,2}(\mathscr{O};\mathbb{R}^2)$, $\mathrm{curl}\left(B_i\phi\right) = \mathcal{L}_{\xi_i}(\mathrm{curl}\phi)$.*

Proof. The approach is direct:

$$\begin{aligned}
\mathrm{curl}\left(B_i\phi\right) &= \partial_1\left(\sum_{j=1}^2\left[\xi_i^j\partial_j\phi^2 + \phi^j\partial_2\xi_i^j\right]\right) - \partial_2\left(\sum_{j=1}^2\left[\xi_i^j\partial_j\phi^1 + \phi^j\partial_1\xi_i^j\right]\right)\\
&= \sum_{j=1}^2\left(\partial_1\xi_i^j\partial_j\phi^2 + \xi_i^j\partial_1\partial_j\phi^2 + \partial_1\phi^j\partial_2\xi_i^j + \phi^j\partial_1\partial_2\xi_i^j\right)\\
&\qquad - \sum_{j=1}^2\left(\partial_2\xi_i^j\partial_j\phi^1 + \xi_i^j\partial_2\partial_j\phi^1 + \partial_2\phi^j\partial_1\xi_i^j + \phi^j\partial_2\partial_1\xi_i^j\right)\\
&= \sum_{j=1}^2\left(\partial_1\xi_i^j\partial_j\phi^2 + \xi_i^j\partial_1\partial_j\phi^2 + \partial_1\phi^j\partial_2\xi_i^j - \partial_2\xi_i^j\partial_j\phi^1 - \xi_i^j\partial_2\partial_j\phi^1 - \partial_2\phi^j\partial_1\xi_i^j\right)\\
&= \sum_{j=1}^2 \xi_i^j\partial_j\left(\partial_1\phi^2 - \partial_2\phi^1\right) + \sum_{j=1}^2\left(\partial_1\xi_i^j\partial_j\phi^2 + \partial_1\phi^j\partial_2\xi_i^j - \partial_2\xi_i^j\partial_j\phi^1 - \partial_2\phi^j\partial_1\xi_i^j\right).
\end{aligned}$$

Observe that the first sum is simply $\mathcal{L}_{\xi_i}(\mathrm{curl}\phi)$, so it is sufficient to show that the second is zero. We expand out the summation:

$$\begin{aligned}
&\left(\partial_1\xi_i^1\partial_1\phi^2 + \partial_1\phi^1\partial_2\xi_i^1 - \partial_2\xi_i^1\partial_1\phi^1 - \partial_2\phi^1\partial_1\xi_i^1\right)\\
&\qquad\qquad + \left(\partial_1\xi_i^2\partial_2\phi^2 + \partial_1\phi^2\partial_2\xi_i^2 - \partial_2\xi_i^2\partial_2\phi^1 - \partial_2\phi^2\partial_1\xi_i^2\right)
\end{aligned}$$

for which the only remaining terms are

$$\partial_1\xi_i^1\partial_1\phi^2 - \partial_2\phi^1\partial_1\xi_i^1 + \partial_1\phi^2\partial_2\xi_i^2 - \partial_2\xi_i^2\partial_2\phi^1$$

or simply

$$(\mathrm{curl}\phi)\sum_{j=1}^{2}\partial_j\xi_i^j$$

which is zero given that $\xi_i \in \bar{W}^{1,2}_{\sigma}$ is divergence-free. $\square$

Moreover, $\mathrm{curl}\,(\mathcal{P}B_i u) = \mathcal{L}_{\xi_i} w$ and $\mathrm{curl}\left(\mathcal{P}B_i^2 u\right) = \mathcal{L}_{\xi_i}\left[\mathrm{curl}\,(B_i u)\right] = \mathcal{L}_{\xi_i}^2 w$. Taking the curl of (4.34) thus leads to the equation

$$w_t = w_0 - \int_0^t \mathcal{L}_{u_s} w_s\, ds + \nu\int_0^t \Delta w_s\, ds + \frac{1}{2}\int_0^t \sum_{i=1}^{\infty}\mathcal{L}_{\xi_i}^2 w_s ds - \sum_{i=1}^{\infty}\int_0^t \mathcal{L}_{\xi_i} w_s\, dW_s^i. \tag{4.35}$$

Let us move on to the case $N = 3$. The curl takes on a new definition here, which is the true definition; for the curl of a two dimensional vector field, one can simply identify this three dimensional vector field with a scalar function. We defer further explanation to [8], and introduce the curl of a function $f \in W^{1,2}(\mathscr{O};\mathbb{R}^3)$ by

$$\mathrm{curl}f := \begin{pmatrix} \partial_2 f^3 - \partial_3 f^2 \\ \partial_3 f^1 - \partial_1 f^3 \\ \partial_1 f^2 - \partial_2 f^1 \end{pmatrix}.$$

We proceed by taking this three dimensional curl of the equation (4.34). There is no change for the Stokes Operator, though to denote the effect on the nonlinear term we introduce notation of the Lie Derivative $\mathscr{L}$ defined on sufficiently regular functions $f, g : \mathscr{O} \to \mathbb{R}^3$ by

$$\mathscr{L}_f g := \mathcal{L}_f g - \mathcal{L}_g f. \tag{4.36}$$

In [129] Subsection 12.1 it is shown that, with notation $\psi := \mathrm{curl}f$, $\mathrm{curl}\,(\mathcal{L}_f f) = \mathscr{L}_f\psi$. We now consider how the curl operation interacts with the stochastic term.

Lemma 4.4.2. *For $f \in W^{2,2}(\mathscr{O};\mathbb{R}^3)$ and $\psi := \mathrm{curl}f$, we have that*

$$\mathrm{curl}(B_i f) = \mathscr{L}_{\xi_i}\psi. \tag{4.37}$$

Proof. We shall show only that the identity (4.37) holds in its first component, with the others following similarly. To this end we calculate the first component of the left hand side of (4.37):

$$
\begin{aligned}
[\mathrm{curl}(B_i f)]^1 &= \partial_2 [B_i f]^3 - \partial_3 [B_i f]^2 \\
&= \partial_2 \left(\sum_{j=1}^{3} \xi_i^j \partial_j f^3 + f^j \partial_3 \xi_i^j \right) - \partial_3 \left(\sum_{j=1}^{3} \xi_i^j \partial_j f^2 + f^j \partial_2 \xi_i^j \right) \\
&= \sum_{j=1}^{3} \left(\partial_2 \xi_i^j \partial_j f^3 + \xi_i^j \partial_2 \partial_j f^3 + \partial_2 f^j \partial_3 \xi_i^j + f^j \partial_2 \partial_3 \xi_i^j \right) \\
&\qquad - \sum_{j=1}^{3} \left(\partial_3 \xi_i^j \partial_j f^2 + \xi_i^j \partial_3 \partial_j f^2 + \partial_3 f^j \partial_2 \xi_i^j + f^j \partial_3 \partial_2 \xi_i^j \right) \\
&= \sum_{j=1}^{3} \left(\partial_2 \xi_i^j \partial_j f^3 + \xi_i^j \partial_2 \partial_j f^3 + \partial_2 f^j \partial_3 \xi_i^j - \partial_3 \xi_i^j \partial_j f^2 - \xi_i^j \partial_3 \partial_j f^2 - \partial_3 f^j \partial_2 \xi_i^j \right) \\
&= \sum_{j=1}^{3} \xi_i^j \partial_j (\partial_2 f^3 - \partial_3 f^2) + \sum_{j=1}^{3} \left(\partial_2 \xi_i^j \partial_j f^3 + \partial_2 f^j \partial_3 \xi_i^j - \partial_3 \xi_i^j \partial_j f^2 - \partial_3 f^j \partial_2 \xi_i^j \right) \\
&= [\mathcal{L}_{\xi_i} \psi]^1 + \sum_{j=1}^{3} \left(\partial_2 \xi_i^j \partial_j f^3 + \partial_2 f^j \partial_3 \xi_i^j - \partial_3 \xi_i^j \partial_j f^2 - \partial_3 f^j \partial_2 \xi_i^j \right).
\end{aligned}
$$

Therefore it remains to show that

$$
\sum_{j=1}^{3} \left(\partial_2 \xi_i^j \partial_j f^3 + \partial_2 f^j \partial_3 \xi_i^j - \partial_3 \xi_i^j \partial_j f^2 - \partial_3 f^j \partial_2 \xi_i^j \right) = -[\mathcal{L}_\psi \xi_i]^1. \tag{4.38}
$$

We expand the sum in (4.38) to

$$
\begin{aligned}
&\left(\partial_2 \xi_i^1 \partial_1 f^3 + \partial_2 f^1 \partial_3 \xi_i^1 - \partial_3 \xi_i^1 \partial_1 f^2 - \partial_3 f^1 \partial_2 \xi_i^1 \right) \\
&\qquad + \left(\partial_2 \xi_i^2 \partial_2 f^3 + \partial_2 f^2 \partial_3 \xi_i^2 - \partial_3 \xi_i^2 \partial_2 f^2 - \partial_3 f^2 \partial_2 \xi_i^2 \right) \\
&\qquad\qquad + \left(\partial_2 \xi_i^3 \partial_3 f^3 + \partial_2 f^3 \partial_3 \xi_i^3 - \partial_3 \xi_i^3 \partial_3 f^2 - \partial_3 f^3 \partial_2 \xi_i^3 \right)
\end{aligned}
$$

achieving some immediate cancellation in the second two brackets to

$$
\begin{aligned}
&\left(\partial_2 \xi_i^1 \partial_1 f^3 + \partial_2 f^1 \partial_3 \xi_i^1 - \partial_3 \xi_i^1 \partial_1 f^2 - \partial_3 f^1 \partial_2 \xi_i^1 \right) \\
&\qquad + \left(\partial_2 \xi_i^2 \partial_2 f^3 - \partial_3 f^2 \partial_2 \xi_i^2 \right) + \left(\partial_2 f^3 \partial_3 \xi_i^3 - \partial_3 \xi_i^3 \partial_3 f^2 \right).
\end{aligned}
$$

We now simply rewrite the above by combining like terms, into

$$
\partial_2 \xi_i^1 (\partial_1 f^3 - \partial_3 f^1) + \partial_3 \xi_i^1 (\partial_2 f^1 - \partial_1 f^2) + (\partial_2 \xi_i^2 + \partial_3 \xi_i^3)(\partial_2 f^3 - \partial_3 f^2)
$$

or more succinctly as

$$
-\partial_2 \xi_i^1 \psi^2 - \partial_3 \xi_i^1 \psi^3 + (\partial_2 \xi_i^2 + \partial_3 \xi_i^3) \psi^1
$$

to which we add and subtract $\partial_1\xi_i^1\psi^1$ to arrive at

$$-\sum_{j=1}^{3}\psi^j\partial_j\xi_i^1+\sum_{j=1}^{3}\left(\partial_j\xi_i^j\right)\psi^1.$$

The first term is precisely $-[\mathcal{L}_\psi\xi_i]^1$ as we wished to show, appreciating that the second term is zero given the divergence-free condition on $\xi_i\in\bar{W}^{1,2}_\sigma$.

□

In three dimensions, the vorticity form is thus motivated as

$$w_t=w_0-\int_0^t\mathscr{L}_{u_s}w_s\,ds+\nu\int_0^t\Delta w_s\,ds+\frac{1}{2}\int_0^t\sum_{i=1}^{\infty}\mathscr{L}_{\xi_i}^2w_sds-\sum_{i=1}^{\infty}\int_0^t\mathscr{L}_{\xi_i}w_s\,dW_s^i. \tag{4.39}$$

Of course we have arrived at this equation from the velocity form, where the velocity u persists in the formulation. Consequently the equation is not closed in w, nor is a path from the vorticity of the fluid to its velocity prescribed. These questions are addressed in the following result.

Theorem 4.4.3. *There exists a mapping $K:\mathscr{O}\times\mathscr{O}\to\mathbb{R}^3$ such that for every $\phi\in W^{1,2}_\sigma\cap W^{k,p}(\mathscr{O};\mathbb{R}^3)$ where $k\in\mathbb{N}\cup\{0\}$, $1<p<\infty$, the function $BS_K\phi:\mathscr{O}\to\mathbb{R}^3$ defined $\lambda-a.e.$ by*

$$(BS_K\phi)(x)=\int_{\mathscr{O}}K(x,y)\phi(y)dy \tag{4.40}$$

is such that:

1. *$BS_K\phi\in L^2_\sigma(\mathscr{O};\mathbb{R}^3)\cap W^{k+1,p}(\mathscr{O};\mathbb{R}^3)$;*
2. *$\operatorname{curl}(BS_K\phi)=\phi$;*
3. *There exists a constant C independent of ϕ (but dependent on k,p) such that*

$$\|BS_K\phi\|_{W^{k+1,p}}\leq C\,\|\phi\|_{W^{k,p}}.$$

Proof. See [41] Theorem 1. □

It should immediately be noted that such a K is not claimed to be unique, and in [41] is explicitly shown to be non-unique, however it does allow us to identify *a* velocity from a given vorticity satisfying the divergence-free and zero-trace boundary conditions. Furthermore we can fix a specific K from the class of admissible integral kernels postulated in Theorem 4.4.3. We thus understand the nonlinear term as a mapping

$$\phi\mapsto\mathscr{L}_{BS_K\phi}\phi.$$

This mapping shall at times be simply denoted by $\mathscr{L}_{BS_K}$, and the equation (4.39) is thus closed in w by the form

$$w_t = w_0 - \int_0^t \mathscr{L}_{BS_K w_s} w_s \, ds + \nu \int_0^t \Delta w_s \, ds + \frac{1}{2} \int_0^t \sum_{i=1}^{\infty} \mathscr{L}_{\xi_i}^2 w_s ds - \sum_{i=1}^{\infty} \int_0^t \mathscr{L}_{\xi_i} w_s \, dW_s^i. \quad (4.41)$$

Absence of the Leray Projector in (4.41) is critical in allowing us to prove the existence of local strong solutions in three dimensions on a bounded domain, which may not be a core result of the thesis but we still find worthy of some exposition in Subsection 5.4.2.

Chapter 5

Well-Posedness for the Stochastic Navier-Stokes Equations

5.1 Introduction

Having established various formulations of solutions of the equation (1.2) in Section 4.4, we now state and prove precise existence and uniqueness results under the different conditions. To underline the contributions of this chapter, we mention again that no existence result for analytically strong solutions of a fluid equation under a transport type noise on a bounded domain was previously known, aside from the case where the gradient dependency in the noise is assumed small relative to the viscosity as seen in [16] (which is necessary in the Itô case). Such an assumption requires little to no extensions of the proof in the case of a general non-differential multiplicative noise, as the additional derivative can be directly controlled with the viscosity in energy estimates (see [58] for a bounded multiplicative noise in the Navier-Stokes Equations with no-slip boundary condition). For a Stratonovich transport noise the assumption may not be necessary, as conversion to Itô form yields a corrector term which may provide cancellation of the top order derivative arising from the noise in energy estimates; the mechanism of cancellation was demonstrated in Proposition 4.3.4. Whilst this control has facilitated the existence of analytically strong solutions of Stratonovich transport equations on the torus or whole space in many settings ([3, 30, 32, 33, 91, 92, 141]), the situation is very different for analytically strong solutions with a boundary. The difficulty lies in the energy norm for the solutions: for strong solutions this is produced from a $W^{1,2}$ inner product, a space in which the Leray Projector is not an orthogonal projection and indeed does not commute with the derivatives entering into consideration from this inner product. The Leray Projector fails to preserve the zero trace property, and in general destroys our boundary estimates in the tangent direction. The key property which we exploit in the present chapter is that the Leray Projector maps not into a zero trace space, but instead a space satisfying a specific choice of Navier boundary conditions; even though this space does not need to match the Navier boundary conditions imposed, it is only in the Navier boundary case and not the no-slip case where strong solutions are proven to exist. To avoid excessive technical detail here, we defer a more thorough inspection of this difference and its role in our proof to the conclusion of this chapter. Whilst we also prove new existence results for analytically weak solutions on bounded domains, they were most likely expected and do not require much novelty to solve. The jump in technical difficulty for strong solutions is explained in the conclusion. The structure of this chapter is as follows:

- Section 5.2 is concerned with the no-slip boundary condition, established as problem (4.1) and precisely formulated in Subsection 4.4.2. Martingale weak solutions are obtained in 3D (Subsection 5.2.1), whilst unique weak solutions are achieved in 2D (Subsection 5.2.2). The method of proof is an application of the theory developed in Chapter 3.

- Section 5.3 addresses the Navier boundary conditions, established as problem (4.2) and precisely formulated in Subsection 4.4.3. Only two spatial dimensions are considered, in which unique weak solutions are attained (Subsection 5.3.1), with unique strong solutions following under the technical assumption that $\alpha \geq \kappa$ (Subsection 5.3.2). Once more, the method of proof is an application of the theory developed in Chapter 3.

- Section 5.4 deals, in less detail, with other variants of the problem in which unique local strong solutions are shown to exist in 3D. For the no-slip condition, such solutions are not yet known to exist. In Subsection 5.4.1 results on the torus $\mathbb{T}^3$ are presented, where in fact additional regularity on the solutions is shown in order to truly solve the Stratonovich Equation in L^2_σ. In Subsection 5.4.2 we do return to a bounded domain, however we can only obtain solutions for the vorticity form presented in Subsection 4.4.4 under the free boundary condition. These problems are tangential to the core thesis so we do not give complete proofs, deferring to the author's paper [63]; instead we state the results and describe how they are accomplished as an application of the theory presented in Subsection A.3.

- Section 5.5 is the conclusion of this chapter, discussing in some more detail how we have overcome the critical challenges in our novel contributions and what problems still remain. More precisely, we explain why the existence for strong solutions is of a technically greater challenge than for weak solutions and how this has been resolved, the necessity of constraining $\alpha \geq \kappa$ for strong solutions under Navier boundary conditions, and why we could only prove the existence of strong solutions under variant conditions to the traditional no-slip boundary.

5.2 No-Slip Boundary

Recall that $\mathscr{O} \subset \mathbb{R}^N$, where N will be chosen as either two or three in the results of this section. Solutions of problem (4.1) are presented in this section: martingale weak solutions in 3D in Subsection 5.2.1 as well as weak solutions in 2D in Subsection 5.2.2. The solutions are obtained through an application of the theory developed in Chapter 3, for which we recall from Subsection 4.4.2 that the framework of Subsection 3.1.1 is satisfied for the spaces

$$V := W^{2,2}_\sigma, \qquad H := W^{1,2}_\sigma, \qquad U := L^2_\sigma$$

and operators

$$\mathcal{A} = -\left(\mathcal{P}\mathcal{L} + \nu A\right) + \frac{1}{2}\sum_{i=1}^{\infty}\mathcal{P}B_i^2, \qquad \mathcal{G} := -\mathcal{P}B.$$

5.2.1 Martingale Weak Solutions

This subsection is dedicated to the proof of Theorem 5.2.1 below.

Theorem 5.2.1. *Let $N = 2$ or 3, $u_0 \in L^\infty\left(\Omega; L^2_\sigma\right)$ be $\mathcal{F}_0-$measurable, $(\xi_i) \in \bar{W}^{1,2}_\sigma \cap W^{2,\infty}$ with $\sum_{i=1}^\infty \|\xi_i\|^2_{W^{2,\infty}} < \infty$. Then there exists a filtered probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, a Cylindrical Brownian Motion $\tilde{\mathcal{W}}$ over $\mathfrak{U}$ with respect to $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, an $\mathcal{F}_0-$measurable $\tilde{u}_0 : \tilde{\Omega} \to U$ with the same law as u_0, and a progressively measurable process $\tilde{u}$ in $W^{1,2}_\sigma$ such that for $\tilde{\mathbb{P}} - a.e.\ \omega$, $\tilde{u}_\cdot(\omega) \in C_w\left([0,T]; L^2_\sigma\right) \cap L^2\left([0,T]; W^{1,2}_\sigma\right)$ and*

$$\tilde{u}_t = \tilde{u}_0 - \int_0^t \mathcal{P}\mathcal{L}_{\tilde{u}_s}\tilde{u}_s\ ds - \nu\int_0^t A\tilde{u}_s\, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty \mathcal{P}B_i^2\tilde{u}_s ds - \int_0^t \mathcal{P}B(\tilde{u}_s)d\tilde{\mathcal{W}}_s$$

holds $\tilde{\mathbb{P}} - a.s.$ in $\left(W^{1,2}_\sigma\right)^*$ for all $t \in [0,T]$.

Theorem 5.2.1 would be proven as a direct application of Theorem 3.2.7 if we verify Assumption Set 1 (Subsection 3.2.1). We demonstrate the assumptions in turn, where $\phi, \psi \in W^{2,2}_\sigma$ and $f \in W^{1,2}_\sigma$. We shall frequently use, without explicit reference on each occasion, that $\mathcal{P}$ as an orthogonal projection in $L^2(\mathscr{O};\mathbb{R}^N)$ is a bounded linear operator on this space with operator norm bounded by 1.

3.2.1: Firstly for (3.3),

$$\|\mathcal{P}\mathcal{L}_f f\|_{(W^{1,2}_\sigma)^*} = \sup_{\eta \in W^{1,2}_\sigma}\langle \mathcal{L}_f f, \eta\rangle \leq \sup_{\eta \in W^{1,2}_\sigma}\|\mathcal{L}_f f\|_{L^{6/5}}\|\eta\|_{L^6} \leq c\|\mathcal{L}_f f\|_{L^{6/5}} \leq c\|f\|_1^2$$

recalling (4.12). The remaining terms of (3.3) are immediate, using (4.18). For (3.4), we have that

$$\begin{aligned}\|\mathcal{P}\mathcal{L}_\phi\phi - \mathcal{P}\mathcal{L}_\psi\psi\|^2 &= \|\mathcal{P}\mathcal{L}_{\phi-\psi}\phi + \mathcal{P}\mathcal{L}_\psi(\phi-\psi)\|^2\\ &\leq c\left(\|\mathcal{L}_{\phi-\psi}\phi\|^2 + \|\mathcal{L}_\psi(\phi-\psi)\|^2\right)\\ &\leq c\left(\|\phi-\psi\|_1^2\|\phi\|_2 + \|\phi-\psi\|_1^2\|\psi\|_2^2\right)\end{aligned}$$

having applied (4.9). The remaining verification of (3.4) and (3.5) is clear, using linearity of A, B_i and the bound (4.18).

3.2.2: Immediate from Lemmas 4.2.8, 4.3.2 and Proposition 4.3.4, setting $\gamma := \nu$. We note use of the bound

$$\left\langle \mathcal{P}B_i^2\phi, \phi\right\rangle + \|\mathcal{P}B_i\phi\|^2 \leq \left\langle B_i^2\phi, \phi\right\rangle + \|B_i\phi\|^2$$

before applying Proposition 4.3.4.

3.2.3: The difficult part is again the nonlinear term, but employing Lemma 4.3.2 and (4.12),

$$\begin{aligned}\langle \mathcal{P}\mathcal{L}_\phi \phi, f\rangle = -\langle \phi, \mathcal{L}_\phi f\rangle \leq \|\phi\|_{L^6} \|\mathcal{L}_\phi f\|_{L^{6/5}} &\leq c\sum_{k=1}^N \|\phi\|_{L^6} \|\phi\|_{L^3} \|\partial_k f\| \\ &\leq c\|\phi\|_1 \|\phi\|_{L^3} \|f\|_1\end{aligned}$$

to which we shift our attentions to the control on $\|\phi\|_{L^3}$. We apply the Gagliardo-Nirenberg Inequality (Theorem A.4.14) to deduce that

$$\|\phi\|_{L^3} \leq c\|\phi\|^{1/2} \|\phi\|_1^{1/2}$$

which is achieved directly in $N = 3$ with the values $p = 3$, $\alpha = 1/2$, $q = 2$ and $m = 1$, whilst for $N = 2$ we take $p = 4$ and use that $\|\psi\|_{L^3} \leq c\|\psi\|_{L^4}$. This is a sufficient control for the nonlinear term in (3.8). The remainder of Assumption 3.2.3 is again clear, using the adjoint B_i^* which satisfies the same boundedness (4.18).

3.2.4: Starting with the nonlinear term, similarly to what we've just seen and using (4.9),

$$\begin{aligned}\langle \mathcal{P}\mathcal{L}_\phi\phi - \mathcal{P}\mathcal{L}_f f, \psi\rangle_{(W^{1,2}_\sigma)^*\times W^{1,2}_\sigma} &= \langle \mathcal{L}_{\phi-f}\phi + \mathcal{L}_f(\phi - f), \psi\rangle \\ &= \langle \phi, \mathcal{L}_{\phi-f}\psi\rangle + \langle \phi - f, \mathcal{L}_f\psi\rangle \\ &\leq c\sum_{k=1}^N \|\phi\|_{L^6} \|\phi - f\| \|\partial_k\psi\|_{L^3} + \|\phi - f\| \|\mathcal{L}_f\psi\| \\ &\leq c\left(\sum_{k=1}^N \|\phi\|_1 \|\phi - f\| \|\partial_k\psi\|_{W^{1,2}} + \|\phi - f\| \|f\|_1 \|\psi\|_2\right) \\ &\leq c\|\psi\|_2 (\|\phi\|_1 + \|f\|_1) \|\phi - f\|.\end{aligned}$$

There is little else to prove for this assumption as the remaining terms are linear; we pass A and $(B_i^*)^2$ onto ψ for (3.10), using Lemma 4.2.8, and B_i^* onto ψ for (3.11).

Therefore, Assumption Set 1 holds and Theorem 5.2.1 is proven.

5.2.2 Weak Solutions

As expected we can do better in 2D, through an application of Theorem 3.3.5.

Theorem 5.2.2. *Let $N = 2$, $u_0 : \Omega \to L^2_\sigma$ be $\mathcal{F}_0-$measurable, $(\xi_i) \in \bar{W}^{1,2}_\sigma \cap W^{2,\infty}$ with $\sum_{i=1}^\infty \|\xi_i\|^2_{W^{2,\infty}} < \infty$. Then there exists a progressively measurable process u in $W^{1,2}_\sigma$ such that for $\mathbb{P}-a.e.$ ω, $u_\cdot(\omega) \in C\left([0,T]; L^2_\sigma\right) \cap L^2\left([0,T]; W^{1,2}_\sigma\right)$ and*

$$u_t = u_0 - \int_0^t \mathcal{P}\mathcal{L}_{u_s}u_s\, ds - \nu\int_0^t Au_s\, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty \mathcal{P}B_i^2 u_s ds - \int_0^t \mathcal{P}Bu_s d\mathcal{W}_s$$

holds $\mathbb{P}-a.s.$ *in* $\left(W^{1,2}_\sigma\right)^*$ *for all* $t \in [0,T]$*. Moreover if* v *is any other such process, then* u *and* v *are indistinguishable.*[1]

Theorem 5.2.2 would be proven as a direct application of Theorem 3.3.5 if we verify Assumption Set 2 (Subsection 3.3.1). We demonstrate the assumptions in turn, where $f, g \in W^{1,2}_\sigma$.

3.3.1: The validity of Theorem 5.2.2 in two dimensions, but not three, hinges upon satisfaction of this assumption for the nonlinear term. From two instances of Hölder's Inequality as well as Theorem A.4.14 with $p=4, q=2, \alpha=1/2$ and $m=1$, giving that $\|f\|_{L^4} \leq c\|f\|^{\frac{1}{2}}\|f\|_1^{\frac{1}{2}}$,

$$\begin{aligned}\|\mathcal{P}\mathcal{L}_f f\|_{(W^{1,2}_\sigma)^*} &= \sup_{\|\eta\|_1=1}|\langle \mathcal{L}_f f, \eta\rangle| = \sup_{\|\eta\|_1=1}|\langle f, \mathcal{L}_f \eta\rangle| \leq \sup_{\|\eta\|_1=1}\|f\|_{L^4}\|\mathcal{L}_f\eta\|_{L^{4/3}}\\ &\leq c \sup_{\|\eta\|_1=1}\sum_{k=1}^{2}\|f\|_{L^4}\|f\|_{L^4}\|\partial_k\eta\| \leq c\sup_{\|\eta\|_1=1}\|f\|\|f\|_1\|\eta\|_1\\ &\leq c\|f\|\|f\|_1 \end{aligned} \tag{5.1}$$

so the contribution from the nonlinear term is confirmed. Bounding the remaining operators comes from the definition of their norms, using (4.18).

3.3.2: First of all, note that

$$\begin{aligned}\langle \mathcal{P}\mathcal{L}_f f - \mathcal{P}\mathcal{L}_g g, f-g\rangle_{(W^{1,2}_\sigma)^*\times W^{1,2}_\sigma} &= \langle \mathcal{L}_{f-g}f + \mathcal{L}_g(f-g), f-g\rangle\\ &= \langle \mathcal{L}_{f-g}f, f-g\rangle \leq c\|f-g\|\|f-g\|_1\|f\|_1\\ &\leq c\|f-g\|^2\|f\|_1^2 + \frac{\nu}{2}\|f-g\|_1^2\end{aligned}$$

having used Lemma 4.3.2 to achieve cancellation, following an identical control to the previous assumption, and using Young's Inequality. The constant c is dependent on ν, which is not meaningful. Due to linearity of A, B_i and using both results of Proposition 4.3.4, (3.50) and then (3.51) hold.

Therefore, Assumption Set 2 holds and Theorem 5.2.2 is proven. Having motivated this notion of solution from the Stratonovich form, we provide an explicit result.

Proposition 5.2.3. *Let* u *be the weak solution postulated in Theorem 5.2.2. Then* u *satisfies the identity*

$$\langle u_t, \phi\rangle = \langle u_0, \phi\rangle - \int_0^t \langle \mathcal{L}_{u_s}u_s, \phi\rangle\, ds - \nu\int_0^t \langle u_s, \phi\rangle_1\, ds - \int_0^t \langle Bu_s, \phi\rangle \circ d\mathcal{W}_s$$

$\mathbb{P}-a.s.$ *in* $\mathbb{R}$ *for every* $\phi \in W^{2,2}_\sigma$*, for all* $t \in [0,T]$.

Proof. This is a two step proof. In the first, for any $\phi \in W^{2,2}_\sigma$ we hit the identity satisfied by u by $\langle \cdot, \phi\rangle_{(W^{1,2}_\sigma)^*\times W^{1,2}_\sigma}$, and with Proposition A.1.50 and reducing to $\langle\cdot,\phi\rangle$, we pass to the

[1] Recall Definition A.1.32 for indistinguishability, equivalent to uniqueness as in Definition 3.3.4.

equality

$$\begin{aligned}\langle u_t, \phi \rangle = \langle u_0, \phi \rangle - \int_0^t \langle \mathcal{P}\mathcal{L}_{u_s} u_s, \phi \rangle_{(W^{1,2}_\sigma)^* \times W^{1,2}_\sigma} ds - \int_0^t \langle \nu A u_s, \phi \rangle_{(W^{1,2}_\sigma)^* \times W^{1,2}_\sigma} ds \\ + \frac{1}{2} \int_0^t \left\langle \sum_{i=1}^\infty \mathcal{P}B_i^2 u_s, \phi \right\rangle_{(W^{1,2}_\sigma)^* \times W^{1,2}_\sigma} ds - \int_0^t \langle \mathcal{P}B u_s, \phi \rangle \, d\mathcal{W}_s.\end{aligned}$$

The second step is to apply Theorem 2.2.27 to deduce

$$\begin{aligned}\langle u_t, \phi \rangle = \langle u_0, \phi \rangle - \int_0^t \langle \mathcal{P}\mathcal{L}_{u_s} u_s, \phi \rangle_{(W^{1,2}_\sigma)^* \times W^{1,2}_\sigma} ds - \int_0^t \langle \nu A u_s, \phi \rangle_{(W^{1,2}_\sigma)^* \times W^{1,2}_\sigma} ds \\ - \int_0^t \langle \mathcal{P}B u_s, \phi \rangle \circ d\mathcal{W}_s.\end{aligned}$$

$\mathbb{P} - a.s.$ in $\mathbb{R}$ for every $\phi \in W^{2,2}_\sigma$, for all $t \in [0,T]$. The result now follows by definition of the terms. □

We remark that the same holds true for the martingale weak solutions of Theorem 5.2.1.

5.3 Navier Boundary

In this section $\mathscr{O} \subset \mathbb{R}^2$. Solutions of problem (4.2) are presented: weak solutions are given in Subsection 5.3.1 whilst strong solutions are presented in Subsection 5.3.2. The solutions are obtained through an application of the theory developed in Chapter 3, for which we recall from Subsection 4.4.3 that the framework of Subsection 3.1.1 is satisfied for the spaces

$$V := \bar{W}^{2,2}_\alpha, \qquad H := \bar{W}^{1,2}_\sigma, \qquad U := L^2_\sigma$$

and operators

$$\mathcal{A} = -(\mathcal{P}\mathcal{L} + \nu A) + \frac{1}{2}\sum_{i=1}^\infty \mathcal{P}B_i^2, \qquad \mathcal{G} := -\mathcal{P}B.$$

5.3.1 Weak Solutions

The main result of this subsection is the following:

Theorem 5.3.1. *Let $\alpha \in C^2(\partial\mathscr{O};\mathbb{R})$, $u_0 : \Omega \to L^2_\sigma$ be $\mathcal{F}_0-$measurable, $(\xi_i) \in \bar{W}^{1,2}_\sigma \cap W^{2,\infty}$ with $\sum_{i=1}^\infty \|\xi_i\|^2_{W^{2,\infty}} < \infty$. Then there exists a progressively measurable process u in $\bar{W}^{1,2}_\sigma$ such that for $\mathbb{P} - a.e.$ ω, $u_\cdot(\omega) \in C\left([0,T];L^2_\sigma\right) \cap L^2\left([0,T];\bar{W}^{1,2}_\sigma\right)$ and*

$$u_t = u_0 - \int_0^t \mathcal{P}\mathcal{L}_{u_s} u_s \, ds - \nu \int_0^t A u_s \, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty \mathcal{P}B_i^2 u_s ds - \int_0^t \mathcal{P}B u_s d\mathcal{W}_s$$

holds $\mathbb{P} - a.s.$ in $\left(\bar{W}^{1,2}_\sigma\right)^*$ for all $t \in [0,T]$. Moreover if v is any other such process, then u and v are indistinguishable.

Just as we have seen in the previous section, Theorem 5.3.1 would be proven as a direct application of Theorem 3.3.5 if we verify Assumption Sets 1 and 2 (Subsections 3.2.1, 3.3.1). Much of the proof overlaps with Section 5.2, and in fact the only difference lies in the Stokes Operator. This owes to the boundary integral in the Green's type identity Lemma 4.2.17, which isn't present in the corresponding no-slip result of Lemma 4.2.8. Thus, for $f \in \bar{W}^{1,2}_{\sigma}$,

$$\begin{aligned}
\|Af\|_{(\bar{W}^{1,2}_{\sigma})^*} &= \sup_{\|\eta\|_1=1} \left|\langle f,\eta\rangle_1 - \langle(\kappa-\alpha)f,\eta\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)}\right| \\
&\leq \sup_{\|\eta\|_1=1} \left(\|f\|_1\|\eta\|_1 + \|\kappa-\alpha\|_{L^\infty(\partial\mathscr{O};\mathbb{R})}\|f\|_{L^2(\partial\mathscr{O};\mathbb{R}^2)}\|\eta\|_{L^2(\partial\mathscr{O};\mathbb{R}^2)}\right) \\
&\leq \sup_{\|\eta\|_1=1} \left(\|f\|_1\|\eta\|_1 + c\|f\|_1\|\eta\|_1\right) \\
&\leq c\|f\|_1
\end{aligned}$$

where the constant c is dependent on α and κ which is not meaningful, and we have used (a coarser version of) the inequality (4.8). Through the same arguments, with the exact version of (4.8),

$$\begin{aligned}
-2\nu\langle Af,f\rangle_{(\bar{W}^{1,2}_{\sigma})^*\times\bar{W}^{1,2}_{\sigma}} \leq 2\nu\left(-\|f\|_1^2 + c\|f\|\|f\|_1\right) &\leq 2\nu\left(-\|f\|_1^2 + c\|f\|^2 + \frac{1}{2}\|f\|_1^2\right) \\
&\leq -\nu\|f\|_1^2 + c\|f\|^2
\end{aligned}$$

having applied Young's Inequality, where c depends on κ,α,ν. With these bounds, all of the assumptions from Assumption Sets 1 and 2 follow exactly as seen in Section 5.2. This proves Theorem 5.3.1. The analogy to the Proposition 5.2.3 also holds.

Proposition 5.3.2. *Let u be the weak solution postulated in Theorem 5.3.1. Then u satisfies the identity*

$$\begin{aligned}
\langle u_t,\phi\rangle = \langle u_0,\phi\rangle - \int_0^t \langle\mathcal{L}_{u_s}u_s,\phi\rangle\,ds - \nu\int_0^t\langle u_s,\phi\rangle_1\,ds \\
+ \nu\int_0^t\langle(\kappa-\alpha)u_s,\phi\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)}\,ds - \int_0^t\langle Bu_s,\phi\rangle\circ d\mathcal{W}_s
\end{aligned}$$

$\mathbb{P}-a.s.$ in $\mathbb{R}$ for every $\phi \in \bar{W}^{2,2}_{\alpha}$, for all $t\in[0,T]$.

Proof. Identical to Proposition 5.2.3. □

5.3.2 Strong Solutions

This subsection is dedicated to the proof of Theorem 5.3.3.

Theorem 5.3.3. *Let $\alpha \in C^2(\partial\mathscr{O};\mathbb{R})$ be such that $\alpha \geq \kappa$, $u_0 : \Omega \to \bar{W}^{1,2}_{\sigma}$ be $\mathcal{F}_0-$measurable, $(\xi_i) \in L^2_{\sigma}\cap W^{3,2}_0 \cap W^{3,\infty}$ with $\sum_{i=1}^\infty \|\xi_i\|^2_{W^{3,\infty}} < \infty$. Then there exists a progressively measurable* *process u in $\bar{W}^{2,2}_{\alpha}$ such that for $\mathbb{P}-a.e.$ $\tilde{\omega}$, $u_{\cdot}(\omega) \in C\left([0,T];\bar{W}^{1,2}_{\sigma}\right) \cap L^2\left([0,T];\bar{W}^{2,2}_{\alpha}\right)$ and*

$$u_t = u_0 - \int_0^t \mathcal{P}\mathcal{L}_{u_s}u_s\, ds - \nu\int_0^t Au_s\, ds + \frac{1}{2}\int_0^t\sum_{i=1}^{\infty}\mathcal{P}B_i^2u_s ds - \int_0^t \mathcal{P}Bu_s d\mathcal{W}_s$$

holds $\mathbb{P}-a.s.$ in L^2_σ for all $t \in [0,T]$. Moreover if v is any other such process, then u and v are indistinguishable.

We shall prove the result through a direct application of Theorem 3.4.5, and Lemma 3.4.14 for the continuity. We assume that $\alpha \geq \kappa$ in accordance with the statement of Theorem 5.3.3. To prove the theorem it is sufficient to verify Assumption Set 3, given in Subsection 3.4.1, as well as the conditions of Lemma 3.4.14: namely that we can take $\bar{H} = H$ and that there exists a continuous bilinear form $\langle\cdot,\cdot\rangle_{U\times V} : L^2_\sigma \times \bar{W}^{2,2}_{\alpha} \to \mathbb{R}$ such that for $f \in \bar{W}^{1,2}_{\sigma}$ and $\psi \in \bar{W}^{2,2}_{\alpha}$,

$$\langle f, \psi\rangle_{U\times V} = \langle f, \psi\rangle_H . \tag{5.2}$$

To this end we will actually endow $\bar{W}^{1,2}_{\sigma}$ with a new inner product, which we denote as

$$\langle f, \psi\rangle_H := \langle f, \psi\rangle_1 - \langle(\kappa-\alpha)f, \psi\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)} . \tag{5.3}$$

Lemma 5.3.4. *The bilinear form $\langle\cdot,\cdot\rangle_H$ defines an inner product on $\bar{W}^{1,2}_{\sigma}$ equivalent to $\langle\cdot,\cdot\rangle_1$.*

Proof. The result follows from the observation that

$$\|f\|_1^2 \leq \|f\|_1^2 - \langle(\kappa-\alpha)f, f\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)} \leq \|f\|_1^2 + c\,\|f\|\,\|f\|_1 \leq c\,\|f\|_1^2$$

having used that $-(\kappa-\alpha)$ is non-negative and (4.8). Once more the constant c is dependent on κ, α through their supremums. $\square$

Remark 5.3.5. *This result is exactly why we require $\alpha \geq \kappa$. The necessity of this requirement is considered in the conclusion.*

Accordingly for $\phi \in L^2_\sigma$ and $\psi \in \bar{W}^{2,2}_{\alpha}$ we define

$$\langle\phi, \psi\rangle_{U\times V} := \langle\phi, A\psi\rangle \tag{5.4}$$

from which the property (5.2) follows as

$$\langle f, A\psi\rangle = -\langle f, \Delta\psi\rangle = \langle f, \psi\rangle_1 - \langle(\kappa-\alpha)f, \psi\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)} = \langle f, \psi\rangle_H$$

through Lemma 4.2.17. Furthermore with the choice of $\bar{H} = H = \bar{W}^{1,2}_{\sigma}$ endowed with the $\langle\cdot,\cdot\rangle_H$ inner product (5.3), Theorem 5.3.3 will follow from validating Assumption Set 3. Towards property (3.52) we consider the relation between $(\bar{a}_k)$ and the newly defined H inner product:

$$\langle\bar{a}_j, \bar{a}_k\rangle_H = \langle\bar{a}_j, A\bar{a}_k\rangle = \bar{\lambda}_k\langle\bar{a}_j, \bar{a}_k\rangle$$

which is equal to zero if $j \neq k$ and $\bar{\lambda}_k$ if $j = k$. Therefore the system $(\bar{a}_k)$ is orthogonal in the H inner product, so each $\bar{\mathcal{P}}_n$ is self-adjoint for this inner product identically to Lemma 4.2.10. Moreover for $f \in \bar{W}^{1,2}_\sigma$,

$$\begin{aligned}\left\|\left(I-\bar{\mathcal{P}}_n\right)f\right\|^2 &= \sum_{k=n+1}^{\infty}\langle f,\bar{a}_k\rangle^2 = \frac{1}{\bar{\lambda}_n}\sum_{k=n+1}^{\infty}\langle f,\bar{a}_k\rangle^2\bar{\lambda}_n \leq \frac{1}{\bar{\lambda}_n}\sum_{k=n+1}^{\infty}\langle f,\bar{a}_k\rangle^2\bar{\lambda}_k\\ &\leq \frac{1}{\bar{\lambda}_n}\sum_{k=1}^{\infty}\langle f,\bar{a}_k\rangle^2\bar{\lambda}_k = \frac{1}{\bar{\lambda}_n}\sum_{k=1}^{\infty}\frac{\langle f,\bar{a}_k\rangle_H^2}{\bar{\lambda}_k} = \frac{1}{\bar{\lambda}_n}\sum_{k=1}^{\infty}\frac{\langle f,\bar{a}_k\rangle_H^2}{\|\bar{a}_k\|_H^2}\\ &= \frac{1}{\bar{\lambda}_n}\|f\|_H^2.\end{aligned}$$

For Assumption 3.4.1, control on the Stokes Operator is clear and for the diffusion B we again have (4.18). The nonlinear term requires more precision, for which the same application of Gagliardo-Nirenberg used in (5.1) achieves that

$$\begin{aligned}\|\mathcal{P}\mathcal{L}_\phi\phi\|^2 &\leq c\sum_{k=1}^{2}\|\phi\|_{L^4}^2\|\partial_k\phi\|_{L^4}^2 \leq c\sum_{k=1}^{2}\|\phi\|\,\|\phi\|_1\,\|\partial_k\phi\|\,\|\partial_k\phi\|_{W^{1,2}}\\ &\leq c\|\phi\|\,\|\phi\|_1^2\,\|\phi\|_2 \leq c\|\phi\|\left(\|\phi\|_1^4+\|\phi\|_2^2\right) \end{aligned}\tag{5.5}$$

as required. It is in Assumption 3.4.2 where the difficulty lies, precisely in the noise contribution, so we verify this assumption in full. Firstly we note that as $\mathcal{P}_n := \bar{\mathcal{P}}_n$ is an orthogonal projection then

$$2\langle\mathcal{P}_n\mathcal{A}(t,\phi^n),\phi^n\rangle_H + \sum_{i=1}^{\infty}\|\mathcal{P}_n\mathcal{G}_i(t,\phi^n)\|_H^2 \leq 2\langle\mathcal{A}(t,\phi^n),\phi^n\rangle_H + \sum_{i=1}^{\infty}\|\mathcal{G}_i(t,\phi^n)\|_H^2, \tag{5.6}$$

$$\sum_{i=1}^{\infty}\langle\mathcal{P}_n\mathcal{G}_i(t,\phi^n),\phi^n\rangle_H^2 \leq \sum_{i=1}^{\infty}\langle\mathcal{G}_i(t,\phi^n),\phi^n\rangle_H^2 \tag{5.7}$$

so it is sufficient to bound the right hand side of each inequality. We have used that $\mathcal{A}(t,\phi^n) \in H$, which is true in our case ($\bar{V}_n \subset W^{3,2}(\mathscr{O};\mathbb{R}^2), H = \bar{W}^{1,2}_\sigma$) but not assumed in the abstract setup. For clarity, we give distinct proofs for each inequality.

Verification of (3.54): In the following we drop the superscript notation for ϕ^n to simply ϕ, and again use superscripts for the components. We break down $\mathcal{A}$ into its functions and recall $\phi \in \bar{V}_n = \operatorname{span}\{\bar{a}_1,\dots,\bar{a}_n\}$. The Stokes Operator is clearest as

$$-2\nu\langle A\phi,\phi\rangle_H = -2\nu\langle A\phi,A\phi\rangle = -2\nu\|\phi\|_2^2 \tag{5.8}$$

using the representation (5.2) with (5.4). Moving on to the nonlinear term, first note that in (5.5) we demonstrated

$$\|\mathcal{L}_\phi\phi\| \leq c\|\phi\|^{\frac{1}{2}}\|\phi\|_1\|\phi\|_2^{\frac{1}{2}}.$$

This inequality is now used in the relevant term $-2\langle\mathcal{P}\mathcal{L}_\phi\phi,\phi\rangle_H = -2\langle\mathcal{P}\mathcal{L}_\phi\phi,A\phi\rangle$, controlled by

$$2\left|\langle\mathcal{P}\mathcal{L}_\phi\phi,A\phi\rangle\right| \leq 2\|\mathcal{L}_\phi\phi\|\,\|\phi\|_2 \leq c\|\phi\|^{\frac{1}{2}}\|\phi\|_1\|\phi\|_2^{\frac{3}{2}} \leq c\|\phi\|^2\|\phi\|_1^4 + \frac{\nu}{2}\|\phi\|_2^2 \tag{5.9}$$

through another Young's Inequality, with conjugate exponents 4 and $\frac{4}{3}$. It remains to consider the noise contribution

$$\sum_{i=1}^{\infty}\left(\left\langle \mathcal{P}B_i^2\phi, \phi\right\rangle_H + \|\mathcal{P}B_i\phi\|_H^2\right). \tag{5.10}$$

Fixing any i, we use (5.2) in the first term and (5.3) in the second term to observe that

$$\left\langle \mathcal{P}B_i^2\phi, \phi\right\rangle_H + \|\mathcal{P}B_i\phi\|_H^2 = \left\langle \mathcal{P}B_i^2\phi, A\phi\right\rangle + \|\mathcal{P}B_i\phi\|_1^2 - \langle(\kappa-\alpha)\mathcal{P}B_i\phi, \mathcal{P}B_i\phi\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)}. \tag{5.11}$$

We are faced with the issue of how to obtain cancellation of the derivative coming from B_i. The idea is to use a Green's Identity type result in $\|\mathcal{P}B_i\phi\|_1^2$ to produce an L^2 inner product with Stokes Operator on one side, carry over an adjoint of B_i (that is, including a $-\mathcal{L}_{\xi_i}$ term) in $\left\langle \mathcal{P}B_i^2\phi, A\phi\right\rangle$, commute it with the Stokes Operator and combine with $\|\mathcal{P}B_i\phi\|_1^2$ to achieve cancellation. This remains some way away, so let us take it one step at a time. To apply a Green's type result to $\|\mathcal{P}B_i\phi\|_1^2$ we need information on $\mathcal{P}B_i\phi$ at the boundary, which has proven greatly problematic as $\mathcal{P}$ does not preserve a null boundary even for compactly supported functions. This is where the Navier boundary conditions prove pivotal. We can utilise Corollary 4.2.20.1, given that $\mathcal{P}B_i\phi\cdot\mathbf{n}$ is zero on $\partial\mathscr{O}$ as $\mathcal{P}B_i\phi \in \bar{W}^{1,2}_{\sigma}$, and Lemma 4.2.19 establishes that $\operatorname{curl}(\mathcal{P}B_i\phi) = \operatorname{curl}(B_i\phi)$. Lemma 4.4.1 informs us that $\operatorname{curl}(B_i\phi) = \mathcal{L}_{\xi_i}(\operatorname{curl}\phi)$.

In summary then we have that $\operatorname{curl}(\mathcal{P}B_i\phi) = \mathcal{L}_{\xi_i}(\operatorname{curl}\phi)$. We can now use the property $\xi_i \in W^{2,2}_0(\mathscr{O};\mathbb{R}^2)$ to deduce that $\mathcal{L}_{\xi_i}(\operatorname{curl}\phi) \in W^{1,2}_0(\mathscr{O};\mathbb{R})$. Indeed by assumption there exists a sequence of functions (ψ^k) in $C_0^\infty(\mathscr{O};\mathbb{R}^2)$ which converge to ξ_i in $W^{2,2}(\mathscr{O};\mathbb{R}^2)$, so $\mathcal{L}_{\psi^k}(\operatorname{curl}\phi) \in W^{1,2}_0(\mathscr{O};\mathbb{R})$ and converges to $\mathcal{L}_{\xi_i}(\operatorname{curl}\phi)$ in $W^{1,2}(\mathscr{O};\mathbb{R})$ as

$$\begin{aligned}\left\|\mathcal{L}_{\xi_i}(\operatorname{curl}\phi) - \mathcal{L}_{\psi^k}(\operatorname{curl}\phi)\right\|_{W^{1,2}(\mathscr{O};\mathbb{R})} &= \left\|\mathcal{L}_{\xi_i-\psi^k}(\operatorname{curl}\phi)\right\|_{W^{1,2}(\mathscr{O};\mathbb{R})}\\ &\leq c\left\|\xi_i - \psi^k\right\|_{W^{2,2}}\|\operatorname{curl}\phi\|_{W^{2,2}(\mathscr{O};\mathbb{R})}\end{aligned}$$

owing to (4.11). Therefore by Corollary 4.2.20.1, $\mathcal{P}B_i\phi$ satisfies the boundary condition

$$2(D[\mathcal{P}B_i\phi])\mathbf{n}\cdot\iota + 2\kappa f\cdot\iota = 0.$$

In particular $\mathcal{P}B_i\phi$ belongs to $\bar{W}^{2,2}_{2\kappa}$, so from Lemma 4.2.17 we have that

$$\|\mathcal{P}B_i\phi\|_1^2 = -\langle\mathcal{P}B_i\phi, \Delta\mathcal{P}B_i\phi\rangle - \langle\kappa\mathcal{P}B_i\phi, \mathcal{P}B_i\phi\rangle. \tag{5.12}$$

The expression (5.11) is now further equal to

$$\left\langle \mathcal{P}B_i^2\phi, A\phi\right\rangle - \langle\mathcal{P}B_i\phi, \Delta\mathcal{P}B_i\phi\rangle - \langle(2\kappa-\alpha)\mathcal{P}B_i\phi, \mathcal{P}B_i\phi\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)}. \tag{5.13}$$

The boundary integral is managed with (4.8), (4.18) and Young's Inequality:

$$\begin{aligned}\left|\langle(2\kappa-\alpha)\mathcal{P}B_i\phi,\mathcal{P}B_i\phi\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)}\right| &\leq c\left\|\mathcal{P}B_i\phi\right\|\left\|\mathcal{P}B_i\phi\right\|_1\\ &\leq c\left\|\xi_i\right\|_{W^{2,\infty}}^2\left\|\phi\right\|_1\left\|\phi\right\|_2\\ &\leq C_\varepsilon\left\|\xi_i\right\|_{W^{2,\infty}}^2\left\|\phi\right\|_1^2+\varepsilon\left\|\xi_i\right\|_{W^{2,\infty}}^2\left\|\phi\right\|_2^2 \end{aligned}\tag{5.14}$$

for any choice of $\varepsilon>0$. The first two terms of (5.13) are considered in tandem. We use the properties that $\mathcal{P}B_i=\mathcal{P}B_i\mathcal{P}$ and $A=A\mathcal{P}$ (Lemmas 4.3.5, 4.3.3). Therefore,

$$\begin{aligned}\left\langle\mathcal{P}B_i^2\phi,A\phi\right\rangle-\left\langle\mathcal{P}B_i\phi,\Delta\mathcal{P}B_i\phi\right\rangle&=\left\langle(\mathcal{P}B_i)^2\phi,A\phi\right\rangle+\left\langle\mathcal{P}B_i\phi,AB_i\phi\right\rangle\\&=\left\langle\mathcal{P}B_i\phi,B_i^*A\phi\right\rangle-\left\langle\mathcal{P}B_i\phi,\Delta B_i\phi\right\rangle\end{aligned}$$

for the adjoint $B_i^*=-\mathcal{L}_{\xi_i}+\mathcal{T}_{\xi_i}^*$. We look to commute the Laplacian and B_i, using Proposition 4.3.6 to bound the commutator, and subsequently the cancellation of the derivative in B_i when considered with the adjoint. Indeed,

$$\begin{aligned}-\left\langle\mathcal{P}B_i\phi,\Delta B_i\phi\right\rangle&=-\left\langle\mathcal{P}B_i\phi,([\Delta,B_i]+B_i\Delta)\phi\right\rangle\\&=-\left\langle\mathcal{P}B_i\phi,[\Delta,B_i]\phi\right\rangle-\left\langle\mathcal{P}B_i\phi,\mathcal{P}B_i\Delta\phi\right\rangle\\&=-\left\langle\mathcal{P}B_i\phi,[\Delta,B_i]\phi\right\rangle+\left\langle\mathcal{P}B_i\phi,\mathcal{P}B_iA\phi\right\rangle\\&=-\left\langle\mathcal{P}B_i\phi,[\Delta,B_i]\phi\right\rangle+\left\langle\mathcal{P}B_i\phi,B_iA\phi\right\rangle.\end{aligned}$$

Our interests have reduced to

$$\left\langle\mathcal{P}B_i\phi,B_i^*A\phi\right\rangle-\left\langle\mathcal{P}B_i\phi,[\Delta,B_i]\phi\right\rangle+\left\langle\mathcal{P}B_i\phi,B_iA\phi\right\rangle$$

or simply

$$\left\langle\mathcal{P}B_i\phi,(\mathcal{T}_{\xi_i}+\mathcal{T}_{\xi_i}^*)A\phi-[\Delta,B_i]\phi\right\rangle\tag{5.15}$$

which we bound through Cauchy-Schwarz, (4.16), (4.18) and Proposition 4.3.6 to see that

$$\begin{aligned}\left\langle\mathcal{P}B_i\phi,(\mathcal{T}_{\xi_i}+\mathcal{T}_{\xi_i}^*)A\phi-[\Delta,B_i]\phi\right\rangle&\leq\left\|\mathcal{P}B_i\phi\right\|\left(\left\|(\mathcal{T}_{\xi_i}+\mathcal{T}_{\xi_i}^*)A\phi\right\|+\left\|[\Delta,B_i]\phi\right\|\right)\\&\leq c\left\|\xi_i\right\|_{W^{1,\infty}}\left\|\phi\right\|_{W^{1,2}}\left(\left\|\xi_i\right\|_{W^{1,\infty}}\left\|A\phi\right\|+\left\|\xi_i\right\|_{W^{3,\infty}}\left\|\phi\right\|_{W^{2,2}}\right)\\&\leq c\left\|\xi_i\right\|_{W^{1,\infty}}\left\|\phi\right\|_1\left(\left\|\xi_i\right\|_{W^{1,\infty}}\left\|\phi\right\|_2+\left\|\xi_i\right\|_{W^{3,\infty}}\left\|\phi\right\|_2\right)\\&\leq C_\varepsilon\left\|\xi_i\right\|_{W^{3,\infty}}^2\left\|\phi\right\|_1^2+\varepsilon\left\|\xi_i\right\|_{W^{3,\infty}}^2\left\|\phi\right\|_2^2.\end{aligned}\tag{5.16}$$

Through (5.14) and (5.16) we obtain a control on the required expression given in (5.10),

$$\sum_{i=1}^\infty\left(\left\langle\mathcal{P}B_i^2\phi,\phi\right\rangle_H+\left\|\mathcal{P}B_i\phi\right\|_H^2\right)\leq\sum_{i=1}^\infty\left(C_\varepsilon\left\|\xi_i\right\|_{W^{3,\infty}}^2\left\|\phi\right\|_1^2+2\varepsilon\left\|\xi_i\right\|_{W^{3,\infty}}^2\left\|\phi\right\|_2^2\right).\tag{5.17}$$

A choice of

$$\varepsilon=\frac{\nu}{2\sum_{i=1}^\infty\left\|\xi_i\right\|_{W^{3,\infty}}^2}$$

and combining (5.8), (5.9) and (5.17) concludes the verification.

□

Verification of (3.55): We wish to bound

$$\sum_{i=1}^{\infty} \langle \mathcal{P}B_i\phi, \phi\rangle_H^2 . \tag{5.18}$$

It should be noted that some level of cancellation of the derivatives is needed, as simply using the representation (5.2), Cauchy-Schwarz and Young's Inequality would lead to too high a power on the $\|\phi\|_V$ term. By definition,

$$\langle \mathcal{P}B_i\phi, \phi\rangle_H = \langle \mathcal{P}B_i\phi, \phi\rangle_1 - \langle (\kappa-\alpha)\mathcal{P}B_i\phi, \phi\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)}$$

and as we saw in the verification of (3.54) above, $\mathcal{P}B_i\phi \in \bar{W}^{2,2}_{2\kappa}$ and

$$\langle \mathcal{P}B_i\phi, \phi\rangle_1 = -\langle \Delta\mathcal{P}B_i\phi, \phi\rangle - \langle \kappa\mathcal{P}B_i\phi, \phi\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)} .$$

Combining the two and incorporating $\mathcal{P}$, then in particular

$$\langle \mathcal{P}B_i\phi, \phi\rangle_H^2 \leq 2\langle A\mathcal{P}B_i\phi, \phi\rangle^2 + 2\langle (2\kappa-\alpha)\mathcal{P}B_i\phi, \phi\rangle^2_{L^2(\partial\mathscr{O};\mathbb{R}^2)} . \tag{5.19}$$

For the boundary integral we again apply (4.8) and absorb the boundedness of α and κ into a constant:

$$\langle (2\kappa-\alpha)\mathcal{P}B_i\phi, \phi\rangle^2_{L^2(\partial\mathscr{O};\mathbb{R}^2)} \leq c\|\mathcal{P}B_i\phi\|\|\mathcal{P}B_i\phi\|_1\|\phi\|\|\phi\|_1 \leq c\|\xi_i\|^2_{W^{2,\infty}}\|\phi\|\|\phi\|_1^2\|\phi\|_2$$

and through Young's Inequality,

$$c\|\xi_i\|^2_{W^{2,\infty}}\|\phi\|\|\phi\|_1^2\|\phi\|_2 \leq C_\varepsilon\|\xi_i\|^2_{W^{2,\infty}}\|\phi\|^2\|\phi\|_1^4 + \varepsilon\|\xi_i\|^2_{W^{2,\infty}}\|\phi\|_2^2 . \tag{5.20}$$

In the first term of (5.19) we again use that $A = A\mathcal{P}$ and take projection over to see that

$$\langle A\mathcal{P}B_i\phi, \phi\rangle^2 = \langle \Delta B_i\phi, \phi\rangle^2 .$$

Owing to the $W^{3,2}_0(\mathscr{O};\mathbb{R}^2)$ regularity of ξ_i, we can conduct an integration by parts with null boundary integral to obtain that

$$\langle \Delta B_i\phi, \phi\rangle^2 = \left(\sum_{k=1}^{2}\langle \partial_k B_i\phi, \partial_k\phi\rangle\right)^2 \leq 2\sum_{k=1}^{2}\langle \partial_k B_i\phi, \partial_k\phi\rangle^2 .$$

From (4.20) we have that

$$\sum_{k=1}^{2}\langle \partial_k B_i\phi, \partial_k\phi\rangle^2 \leq c\|\xi_i\|^2_{W^{2,\infty}}\|\phi\|_1^4 . \tag{5.21}$$

Together (5.20) and (5.21) give the result.

□

Therefore, Assumption Set 3 is verified which proves Theorem 5.3.3. Once more, we connect this solution to the original Stratonovich equation.

Proposition 5.3.6. *Let u be the strong solution postulated in Theorem 5.3.3. Then u satisfies the identity*

$$u_t = u_0 - \int_0^t \mathcal{P}\mathcal{L}_{u_s}u_s\, ds - \nu\int_0^t Au_s\, ds - \int_0^t \mathcal{P}Bu_s \circ d\mathcal{W}_s$$

$\mathbb{P}-a.s.$ *in* $\left(\bar{W}^{1,2}_\sigma\right)^*$ *for all* $t \in [0,T]$.

Proof. A direct application of Theorem 2.2.20. □

5.4 Strong Solutions in 3D

Well-posedness results for strong solutions in 3D are stated in this section, obtained as an application of the theory presented in Subsection A.3. Complete proofs are given in the author's paper [63].

5.4.1 The Torus

To pose the problem on the three dimensional torus $\mathbb{T}^3$, we briefly introduce some new notation. The divergence-free condition remains in the problem, but the corresponding constraint to a boundary condition is the zero-average restriction; thus, we introduce $\dot{L}^2$ as the sub-class of $L^2(\mathbb{T}^3;\mathbb{R}^3)$ of functions ϕ such that

$$\int_{\mathbb{T}^3} \phi\, d\lambda = 0.$$

Recall that any function $f \in L^2(\mathbb{T}^3;\mathbb{R}^3)$ admits the representation

$$f(x) = \sum_{k\in\mathbb{Z}^3} f_k e^{ik\cdot x} \tag{5.22}$$

whereby each $f_k \in \mathbb{C}^3$ is such that $f_k = \overline{f_{-k}}$, see e.g. [129] Subsection 1.5 for details. Let us further introduce $\check{L}^2_\sigma$ the subset of $\dot{L}^2$ of functions f whereby for all $k \in \mathbb{Z}^3$, $k \cdot f_k = 0$ with f_k as in (5.22). This enforces the divergence-free condition. For general $m \in \mathbb{N}$ we introduce $\check{W}^{m,2}_\sigma$ as the intersection of $W^{m,2}(\mathbb{T}^3;\mathbb{R}^3)$ respectively with $L^2_\sigma(\mathbb{T}^3;\mathbb{R}^3)$. To apply the general solution theory, we establish the spaces

$$V := \check{W}^{3,2}_\sigma, \qquad H := \check{W}^{2,2}_\sigma, \qquad U := \check{W}^{1,2}_\sigma, \qquad X := \check{L}^2_\sigma$$

and by an application of Theorem A.3.17, we have the following:

Theorem 5.4.1. *Let $u_0 : \Omega \to \check{W}^{1,2}_\sigma$ be $\mathcal{F}_0-$measurable, $(\xi_i) \in \check{L}^2_\sigma \cap W^{3,\infty}$ with $\sum_{i=1}^\infty \|\xi_i\|^2_{W^{3,\infty}} < \infty$. Then there exists a unique $U-$valued maximal strong solution (u,Θ) of the equation (4.34) in the sense of Definitions A.3.14, A.3.15, A.3.16. Moreover at $\mathbb{P}-a.e.$ ω for which $\Theta(\omega) < \infty$,* *we have that*

$$\sup_{r\in[0,\Theta(\omega))}\|u_r(\omega)\|_{W^{1,2}}^2+\int_0^{\Theta(\omega)}\|u_r(\omega)\|_{W^{2,2}}^2\,dr=\infty. \tag{5.23}$$

Proof. See [63] Theorem 3.6. □

The case of the $H-$valued solution, coming from Theorem A.3.13, is particularly interesting. The additional degree of regularity obtained in these solutions is pertinent for the Itô-Stratonovich conversion. The following result is obtained via Theorem A.3.13, followed by Theorem 2.2.20.

Theorem 5.4.2. *Let $u_0:\Omega\to\check{W}^{2,2}_\sigma$ be $\mathcal{F}_0-$measurable, $(\xi_i)\in\check{L}^2_\sigma\cap W^{3,\infty}$ with $\sum_{i=1}^\infty\|\xi_i\|_{W^{3,\infty}}^2<\infty$. Then there exists a pair (u,τ) such that: τ is a $\mathbb{P}-a.s.$ positive stopping time and u is a process whereby for $\mathbb{P}-a.e.$ ω, $u_\cdot(\omega)\in C\left([0,T];\check{W}^{2,2}_\sigma\right)$ and $u_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)}\in L^2\left([0,T];\check{W}^{3,2}_\sigma\right)$ with $u_\cdot\mathbb{1}_{\cdot\leq\tau}$ progressively measurable in $\check{W}^{3,2}_\sigma$, and moreover satisfying the identity*

$$u_t=u_0-\int_0^{t\wedge\tau}\mathcal{P}\mathcal{L}_{u_s}u_s\,ds-\nu\int_0^{t\wedge\tau}Au_s\,ds-\int_0^{t\wedge\tau}\mathcal{P}Bu_s\circ d\mathcal{W}_s$$

$\mathbb{P}-a.s.$ in $\check{L}^2_\sigma$ for all $t\in[0,T]$.

Proof. See [63] Theorem 3.1. □

Note that this is the first result considered in the thesis, and to my knowledge the first rigorously proven in the literature, of the existence of a genuinely strong solution to a Stratonovich fluid equation with unbounded noise. The corresponding global existence and uniqueness result in 2D also holds, as a simpler proof to the one of Theorem 5.3.3.

5.4.2 The Vorticity Form with Boundary

Recall from Subsection 4.4.4 a derivation of the vorticity form of the equation, in three spatial dimensions as

$$w_t=w_0-\int_0^t\mathscr{L}_{BS_Kw_s}w_s\,ds+\nu\int_0^t\Delta w_s\,ds+\frac{1}{2}\int_0^t\sum_{i=1}^\infty\mathscr{L}_{\xi_i}^2w_s ds-\sum_{i=1}^\infty\int_0^t\mathscr{L}_{\xi_i}w_s\,dW^i_s. \tag{5.24}$$

We impose the free boundary condition $w=0$ on $\partial\mathscr{O}$. Through the choice of spaces

$$V:=W^{2,2}_\sigma,\qquad H:=W^{1,2}_\sigma,\qquad U:=L^2_\sigma$$

we can apply Theorem A.3.13 to obtain the following:

Theorem 5.4.3. *Let $w_0:\Omega\to W^{1,2}_\sigma$ be $\mathcal{F}_0-$measurable, $(\xi_i)\in L^2_\sigma\cap W^{2,2}_0\cap W^{3,\infty}$ with $\sum_{i=1}^\infty\|\xi_i\|_{W^{3,\infty}}^2<\infty$. Then there exists a unique $H-$valued maximal strong solution (w,Θ) of the equation (5.24) in the sense of Definitions A.3.9, A.3.10, A.3.12. Moreover at $\mathbb{P}-a.e.$ ω for which $\Theta(\omega)<\infty$, we have that*

$$\sup_{r\in[0,\Theta(\omega))}\|w_r(\omega)\|_1^2+\int_0^{\Theta(\omega)}\|w_r(\omega)\|_2^2\,dr=\infty. \tag{5.25}$$

Proof. See [63] Theorem 4.3. □

5.5 Conclusion

We now address some key questions surrounding this chapter.

- How do the Navier boundary conditions solve the problems faced by the no-slip condition for the existence of strong solutions (and, why is this existence much more difficult than for weak solutions)?

The critical question is around satisfaction of Assumption Set 3 (Subsection 3.4.1), and we point in particular to Assumption 3.4.2, necessary in showing uniform estimates of the Galerkin Approximation in the energy norm of the strong solution. For the Navier boundary conditions, a critical step, see (5.7), was that the projection $\bar{\mathcal{P}}_n$ is self-adjoint in $\bar{W}^{1,2}_\sigma$. It may at first appear that the corresponding result in the no-slip case is Lemma 4.2.10, that $\mathcal{P}_n$ is self-adjoint in $W^{1,2}_\sigma$. The difficulty is the fact that for $\phi^n \in V_n$, $\mathcal{P}B_i^2\phi^n$ and $\mathcal{P}B_i\phi^n$ do not themselves belong to $W^{1,2}_\sigma$, as the Leray Projector destroys the zero trace property, and $\mathcal{P}_n$ is *only self-adjoint on* $W^{1,2}_\sigma$ *and not* $\bar{W}^{1,2}_\sigma$. This owes to the fact that the system of eigenfunctions in $W^{1,2}_\sigma$ (Proposition 4.2.9) is not a basis for $\bar{W}^{1,2}_\sigma$, whilst the system in $\bar{W}^{2,2}_\alpha$ is (Proposition 4.2.16). In fact, we can explicitly show that $\mathcal{P}_n$ is not self-adjoint in $\bar{W}^{1,2}_\sigma$.

Lemma 5.5.1. *For at least some $n \in \mathbb{N}$, $\mathcal{P}_n$ as defined in (4.5) is not self-adjoint on $\bar{W}^{1,2}_\sigma$.*

Proof. Consider $f \in \bar{W}^{1,2}_\sigma$, such that $f \notin W^{1,2}_\sigma$. We can decompose $\bar{W}^{1,2}_\sigma$ as a direct sum

$$\bar{W}^{1,2}_\sigma = W^{1,2}_\sigma \oplus \left(W^{1,2}_\sigma\right)^\perp$$

as $W^{1,2}_\sigma$ is a closed subspace of $\bar{W}^{1,2}_\sigma$, so has orthogonal complement $\left(W^{1,2}_\sigma\right)^\perp$. We introduce the corresponding orthogonal projections in $\bar{W}^{1,2}_\sigma$, $\mathcal{Q}$ onto $W^{1,2}_\sigma$ and $\mathcal{Q}^\perp$ onto $\left(W^{1,2}_\sigma\right)^\perp$. Then

$$f = \mathcal{Q}f + \mathcal{Q}^\perp f$$

where $\mathcal{Q}^\perp f$ is non-zero as $f \notin W^{1,2}_\sigma$. Of course $\mathcal{Q}^\perp f \in \bar{W}^{1,2}_\sigma \subset L^2_\sigma$, and as (a_k) form a basis of L^2_σ, then there exists some $n \in \mathbb{N}$ such that $\mathcal{P}_n\mathcal{Q}^\perp f \neq 0$. If we assume for a contradiction that $\mathcal{P}_n$ is self-adjoint in $\bar{W}^{1,2}_\sigma$, then in particular it is the orthogonal projection onto V_n in $\bar{W}^{1,2}_\sigma$. However $\mathcal{Q}^\perp f$ is orthogonal to the entirety of $W^{1,2}_\sigma$, and in particular V_n, so $\mathcal{P}_n\mathcal{Q}^\perp f = 0$ which is our contradiction. □

Therefore we cannot justify a bound

$$\left\langle \mathcal{P}_n\mathcal{P}B_i^2\phi^n, \phi^n \right\rangle_1 + \|\mathcal{P}_n\mathcal{P}B_i\phi^n\|_1^2 \leq \left\langle \mathcal{P}B_i^2\phi^n, \phi^n \right\rangle_1 + \|\mathcal{P}B_i\phi^n\|_1^2. \tag{5.26}$$

The direct presence of the $\mathcal{P}_n$ disables our commutator arguments used in the Proof of Theorem 5.3.3, as good commutator properties between $\mathcal{P}_n$ and B_i are not apparent. We note the task was still uphill even after removing the $\mathcal{P}_n$, as $\mathcal{P}$ remains which prohibits direct use of

Proposition 4.3.4. The property $\mathcal{P}B_i = \mathcal{P}B_i\mathcal{P}$, Lemma 4.3.5, was key in proving Theorem 5.3.3, so that we can cope with the presence of $\mathcal{P}$ and still achieve some cancellation of the top order derivative. We see, therefore, how the problem of strong solutions is vastly more difficult than of weak solutions; orthogonality of all considered projections in L^2 means a simple application of Proposition 4.3.4 handles the noise contributions.

- Why is the condition $\alpha \geq \kappa$ necessary in Theorem 5.3.3?

This property facilitated use of Lemma 3.4.14 to obtain continuity of the strong solution, by obtaining the relation (5.2), though it may be unclear as to whether or not this condition was necessary to attain solutions of just $L^\infty\left([0,T];\bar{W}^{1,2}_\sigma\right)$ regularity for example. The alternative route would be to work directly with the $\langle\cdot,\cdot\rangle_1$ inner product on $\bar{W}^{1,2}_\sigma$. The issue is the same term discussed in the previous point, showing (3.54) required for uniform estimates of the Galerkin System in the energy norm of the strong solution. Given the presence of $\mathcal{P}$ our only option is to introduce the Stokes operator into these terms with Lemma 4.2.17 and then use the commutator estimate of Proposition 4.3.6. Doing this we obtain that

$$\left\langle \mathcal{P}B_i^2\phi, \phi\right\rangle_1 = \left\langle \mathcal{P}B_i^2\phi, A\phi\right\rangle + \left\langle (\kappa-\alpha)\mathcal{P}B_i^2\phi, \phi\right\rangle_{L^2(\partial\mathscr{O};\mathbb{R})}$$

where we see that the issue is in the control of the $\mathcal{P}B_i^2\phi$ term on the boundary, as with the usual employment of (4.8) we would require the $W^{3,2}$ norm of ϕ which is too much. The assumption that $\alpha \geq \kappa$ allows us use of the H inner product in the Itô Formula (Proposition A.2.7), removing the need to manage this problematic boundary condition but of course this isn't without a cost. Given that $\mathcal{P}B_i\phi \notin \bar{W}^{2,2}_\alpha$ then we cannot say that

$$\|\mathcal{P}B_i\phi\|_H^2 = \langle \mathcal{P}B_i\phi, A\mathcal{P}B_i\phi\rangle$$

where the right hand side is the term that we need to reach to apply our commutator estimates. Thus we have to treat the boundary integral $-\langle(\kappa-\alpha)\mathcal{P}B_i\phi, \mathcal{P}B_i\phi\rangle_{L^2(\partial\mathscr{O};\mathbb{R})}$ directly, as we saw in the estimate of (5.10), which would not appear if we applied the Itô Formula for $\langle\cdot,\cdot\rangle_1$. The significance is that this boundary integral is controllable, as the derivatives are shared so the use of (4.8) now only demands the $W^{2,2}$ norm of ϕ in an acceptable way. There is the more hidden additional cost of the boundary integrals in the Galerkin System u^n for $\|u^n_t\|_H^2$, $\|u^n_0\|_H^2$ which are immediately dealt with once we have established the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_H$ (and *only* because of this equivalence). Thus the assumption that $\alpha \geq \kappa$ is necessary.

- For which noise are these results still valid? That is, what types of noise satisfy Assumption Sets 1, 2 and 3?

It is clear that the assumptions hold for any additive noise in $W^{1,2}(\mathscr{O};\mathbb{R}^2)$ or any linear noise bounded as an operator $L^2(\mathscr{O};\mathbb{R}^2) \to L^2(\mathscr{O};\mathbb{R}^2)$ and $W^{1,2}(\mathscr{O};\mathbb{R}^2) \to W^{1,2}(\mathscr{O};\mathbb{R}^2)$. Such a Nemystkii type noise has proven popular in the study of fluid SPDEs, see for instance ([58, 59]). In addition an Itô type transport noise with sufficiently small gradient dependency relative to viscosity is also valid. A very interesting question is whether or not the assumptions hold for a purely transport Stratonovich noise. In the author's paper [64] Assumption Sets 1 and 2 are verified in the case of $\mathcal{P}\mathcal{L}_{\xi_i}$ for ξ_i satisfying the same assumptions as in the SALT Noise, so results pertaining to weak solutions carry through, though Assumption Set 3 remains to be verified. Most critically is (3.54) pertaining to the key term that we have been considering, (5.10). For the SALT noise the property that $\mathcal{P}B_i = \mathcal{P}B_i\mathcal{P}$ is vital; our estimate, as in (5.13), boils down to controlling

$$\left\langle \mathcal{P}B_i^2\phi, A\phi\right\rangle + \left\langle \mathcal{P}B_i\phi, A\mathcal{P}B_i\phi\right\rangle$$

which relied heavily on this property that $\mathcal{P}B_i = \mathcal{P}B_i\mathcal{P}$. The question is whether or not we could achieve the necessary bound for

$$\left\langle (\mathcal{P}\mathcal{L}_{\xi_i})^2\phi, A\phi\right\rangle + \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, A\mathcal{P}\mathcal{L}_{\xi_i}\phi\right\rangle.$$

By taking the $\mathcal{P}$ over, using that $A = A\mathcal{P}$ and $\mathcal{L}_{\xi_i}^* = -\mathcal{L}_{\xi_i}$ there is no issue in reducing this to

$$-\left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{L}_{\xi_i}A\phi\right\rangle + \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, A\mathcal{L}_{\xi_i}\phi\right\rangle \tag{5.27}$$

or simply

$$\left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, [A, \mathcal{L}_{\xi_i}]\phi\right\rangle$$

where $[A, \mathcal{L}_{\xi_i}]$ represents the commutator $A\mathcal{L}_{\xi_i} - \mathcal{L}_{\xi_i}A$. It is sufficient to show that $[A, \mathcal{L}_{\xi_i}]$ is of second order, but although that would be possible for $[\Delta, \mathcal{L}_{\xi_i}]$ the presence of $\mathcal{P}$ means this is not clear. We hope to make a compelling point regarding the importance of SALT noise here. To simply illustrate why the property that $\mathcal{P}B_i = \mathcal{P}B_i\mathcal{P}$ is so important, *assume that* $\mathcal{P}\mathcal{L}_{\xi_i} = \mathcal{P}\mathcal{L}_{\xi_i}\mathcal{P}$ *(which is untrue!)*. Then we could write (5.27) as

$$\begin{aligned}
-\left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{P}\mathcal{L}_{\xi_i}A\phi\right\rangle + \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, A\mathcal{L}_{\xi_i}\phi\right\rangle &= \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{P}\mathcal{L}_{\xi_i}\Delta\phi\right\rangle - \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \Delta\mathcal{L}_{\xi_i}\phi\right\rangle \\
&= \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{L}_{\xi_i}\Delta\phi\right\rangle - \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \Delta\mathcal{L}_{\xi_i}\phi\right\rangle
\end{aligned}$$

or simply

$$-\left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, [\Delta, \mathcal{L}_{\xi_i}]\phi\right\rangle$$

which as suggested is controllable. The proof for B_i is more complicated, but this is the essential idea. It seems, therefore, that the best method to show the necessary bound for $\mathcal{L}_{\xi_i}$ is to actually write

$$\mathcal{L}_{\xi_i} = B_i - \mathcal{T}_{\xi_i}$$

then use the fact that $\mathcal{T}_{\xi_i}$ is of zeroth order (so unproblematic) and then treat the B_i as seen. More precisely, for the first term in (5.27), we would write

$$\begin{aligned}
-\left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{P}\mathcal{L}_{\xi_i}A\phi\right\rangle &= -\left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{P}B_iA\phi\right\rangle + \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{P}\mathcal{T}_{\xi_i}A\phi\right\rangle \\
&= \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{P}B_i\Delta\phi\right\rangle + \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{P}\mathcal{T}_{\xi_i}A\phi\right\rangle \\
&= \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{L}_{\xi_i}\Delta\phi\right\rangle + \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{T}_{\xi_i}\Delta\phi\right\rangle + \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{T}_{\xi_i}A\phi\right\rangle.
\end{aligned}$$

Therefore the entirety of (5.27) is

$$-\left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, [\Delta, \mathcal{L}_{\xi_i}]\phi\right\rangle + \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{T}_{\xi_i}\Delta\phi\right\rangle + \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{T}_{\xi_i}A\phi\right\rangle$$

or more compactly
$$\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, (\mathcal{T}_{\xi_i}\Delta + \mathcal{T}_{\xi_i}A - [\Delta, \mathcal{L}_{\xi_i}])\,\phi\rangle\,.$$

This is now easy to control as the operator in the brackets is of second order, so the assumption (3.54) is indeed verified. The proof for (3.55) sees no change to the one for B_i. In summary, a purely transport Stratonovich noise does satisfy the assumptions, but to prove it one has to revert to the SALT noise hence we emphasise how significant the structure of the SALT noise is for the analysis.

- Do the strong solutions with Navier boundary conditions converge to the corresponding no-slip weak solutions?

As discussed in the introduction, the idea is that as α approaches infinity in (1.4) then the second term dominates the identity giving that $u \cdot \iota = 0$ on $\partial\mathscr{O}$, hence $u = 0$ on $\partial\mathscr{O}$. In the deterministic case this result is proven in [81] Theorem 9.2, though we appreciate that it is in fact done for the strong solution with no-slip condition and this a priori additional regularity is necessary in the approach. In lieu of this regularity, tightness arguments on the sequence of strong solutions with α tending to infinity have proven fruitless.

In fact this method would be successful if we could show that the solutions have $L^\infty\left([0,T];\bar{W}^{1,2}_\sigma\right)$ $\cap L^2\left([0,T];W^{2,2}(\mathscr{O};\mathbb{R}^2)\right)$ regularity uniformly in α. Such a result is exceptionally appealing as with the limit inheriting this regularity, we would prove that *strong solutions* of the equation exist for the no-slip condition. The difficulty in this problem has already been explicated, so naturally this approach is of great interest. This regularity would enter at the level of the Galerkin Approximation. Scanning through where the dependency on α would lie in Proposition 3.4.7, the first instance is in the $\|u^n_r\|^2_H$ term, though as $\|u^n_r\|^2_1 \leq \|u^n_r\|^2_H$ then this is irrelevant. Importantly is the initial condition $\|u^n_0\|^2_H$, where the boundary integral $\langle(\kappa-\alpha)u^n_0, u^n_0\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)}$ is problematic. This is surely to be expected as strong solutions for the no-slip condition will only exist for $u_0 \in W^{1,2}_0(\mathscr{O};\mathbb{R}^2)$. Indeed if we impose this condition, then we can use that $\|u^n_0\|^2_H \leq \|u_0\|^2_H$ and then the boundary integral $\langle(\kappa-\alpha)u_0, u_0\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)}$ is null. We see that there is no α dependency in treating the nonlinear term, and in the Stokes Operator it is only in the equivalence of the $\|\cdot\|_2$ and $\|\cdot\|_{W^{2,2}}$ norms which does require some nuance but can be dealt with. Inevitably it is the transport type noise where this unravels, and we see the dependency on α in the term

$$-\left\langle(\kappa-\alpha)\mathcal{P}\mathcal{G}_i\phi^n, \mathcal{P}\mathcal{G}_i\phi^n\right\rangle_{L^2(\partial\mathscr{O};\mathbb{R}^2)}$$

which appears to be impossible to handle. We bring attention to this as it is again the issue of the Leray Projector not preserving the zero trace property which prevents progress.

- How did the cases considered for strong solutions in 3D enable their well-posedness results?

Firstly for the torus setting of Subsection 5.4.1, the pivotal point is that *the Leray Projector commutes with derivatives* so the corresponding finite-dimensional projections (and indeed, the

Leray Projector) are self-adjoint in all $\check{W}^{m,2}_{\sigma}$. Comparing to the first point of this conclusion, this is where the difference lies. This trouble is negated by the vorticity form of the equation used in Subsection 5.4.2, with the free boundary condition, as the Leray Projector is not present in the formulation. This owes to its derivation as a curl, thus divergence-free, and with with the (ξ_i) approaching zero quickly at the boundary then the noise already belongs to the desired $W^{1,2}_{\sigma}$ space. Without the projections then cancellation of the top order derivative is straightforwards, similarly to Proposition 4.3.4. Complete details of both cases are given in [63].

Chapter 6

The Inviscid Limit of Stochastic Navier-Stokes Flows

6.1 Introduction

This chapter centres around the zero viscosity limit problem for solutions of the Stochastic Navier-Stokes Equations given in Chapter 5. We recall from Chapter 1 the contrasting states of the deterministic problem dependent on the boundary conditions imposed. To most simply present the known results, let us for now consider the two dimensional case and an initial condition $u_0 \in L^2_\sigma \cap W^{m,2}(\mathscr{O};\mathbb{R}^2)$ where $m > 2$. Then there exists a strong solution $\bar{u}$ to the Euler Equation, and (at least weak) solutions u^ν to the Navier Stokes Equation with viscosity ν under the no-slip and Navier boundary conditions. The question is whether u^ν converges to $\bar{u}$ in $L^\infty([0,T];L^2_\sigma)$ as ν approaches zero. Under Navier boundary conditions the answer is affirmative, quoting [22, 44, 81, 111] for example. Without further assumption on the structure of the domain or analyticity of the initial condition, the answer is unknown for the no-slip case. What is known is a necessary and sufficient condition for the convergence, established by Kato in [79]. Kato's work shows the convergence to be true if and only if

$$\lim_{\nu\to 0} \nu \int_0^T \|\nabla u^\nu_s\|^2_{L^2(\Gamma_{c\nu})}\, ds = 0,$$

where $\Gamma_{c\nu}$ is a boundary strip of width $c\nu$ for $c > 0$ fixed but arbitrary. This mathematically reflects the observed production of vorticity through large gradients of velocity at the boundary. Forty years later, Kato's criterion remains the most fundamental result available in this area having seen only minor extensions such as [82] and [150]. Moreover we still have little to no understanding regarding the validity of this criterion, and whether general flows converge in the zero viscosity limit is one of the fundamental open problems of fluid mechanics.

Our primary goal is to establish some stochastic counterpart to these results, valid for Lie transport noise amongst other choices. In a continuation of the thesis, the physical relevance of extending this theory to the stochastic setting is evident. However, a second motivation stands at the fore of this work in the no-slip case. There has been a plethora of recent developments on the regularising effect of noise, particularly transport noise, largely on well-posedness and stability: see, for example, [6, 24, 45, 46, 47, 48, 49, 51, 52 55, 96]. Furthermore by establishing a stochastic Kato criterion valid for general transport noise, we hope to initiate considerations for a specific choice of (regularising) noise under which the criterion is satisfied. This would constitute a prodigious development in the fundamental open problem of resolving the boundary layer criterion.

Towards this end, for the no-slip condition our main object of study will be a Navier-Stokes Equation perturbed by a general noise which enters either in Itô form

$$u_t = u_0 - \int_0^t \mathcal{L}_{u_s} u_s \, ds + \nu \int_0^t \Delta u_s \, ds - \nu^{\frac{1}{2}} \int_0^t \mathcal{G}(u_s) d\mathcal{W}_s + \nabla \rho_t = 0 \tag{6.1}$$

or Stratonovich

$$u_t = u_0 - \int_0^t \mathcal{L}_{u_s} u_s \, ds + \nu \int_0^t \Delta u_s \, ds - \nu^{\frac{1}{2}} \int_0^t \mathcal{G}(u_s) \circ d\mathcal{W}_s + \nabla \rho_t = 0. \tag{6.2}$$

The noise shall be such that the corresponding existence for martingale weak and weak solutions given in Theorems 5.2.1, 5.2.2 holds. This stochastic integral is scaled by a parameter that must go to zero with viscosity, which as given is traditionally taken to be $\nu^{\frac{1}{2}}$ having been motivated in [86] as the only noise scaling which leads to non-trivial limiting measures (in the limit $t \to \infty$ and $\nu \to 0$) for an additive noise in two dimensions in the absence of a boundary. The significance of this scaling for energy balance is further underlined in [85, 87, 88] and has been used to study the inviscid limit problem in [57, 106]. We firstly establish our stochastic Kato's Criterion for this choice of scaling, demonstrating the equivalence of the conditions taken in expectation. Furthermore a new criterion is presented dependent on an abstract scaling choice, and is specifically analysed in the case where this parameter is some exponent of viscosity. In particular we show that with the choice ν^{β}, then for $\frac{1}{4} < \beta < \frac{1}{2}$ one requires the condition

$$\lim_{\nu \to 0} \nu^{4\beta - 1} \mathbb{E} \int_0^T \|\nabla u_s^{\nu}\|_{L^2(\Gamma_{c\nu})}^2 \, ds = 0$$

in order to deduce the convergence to Euler. For $\beta \geq \frac{1}{2}$ then the anticipated condition

$$\lim_{\nu \to 0} \nu \mathbb{E} \int_0^T \|\nabla u_s^{\nu}\|_{L^2(\Gamma_{c\nu})}^2 \, ds = 0$$

is both necessary and sufficient for the convergence.

For the Navier boundary we revert to SALT noise, and instead of scaling the stochastic integral we consider the inviscid limit towards the corresponding SALT Euler Equation

$$u_t = u_0 - \int_0^t \mathcal{L}_{u_s} u_s \, ds - \int_0^t B(u_s) \circ d\mathcal{W}_s + \nabla \rho_t = 0. \tag{6.3}$$

Our principal result is the existence of a global, probabilistically weak, analytically weak solution of (6.3) as a zero viscosity limit of solutions of (1.2) under Navier boundary conditions (1.4) where $\alpha = 2\kappa$ and $\kappa \geq 0$. We comment that the existence of any solution of the Euler Equation with a transport noise on a bounded domain was completely open. The closest result that we have seen comes from [15], where the authors proved the existence of a global, probabilistically weak, analytically weak solution of the Euler Equation with a somewhat general multiplicative noise, though one which does not allow for gradient dependency. The solution is constructed as an inviscid limit of linearised and regularised Navier-Stokes Equations satisfying the same free boundary condition. For linear multiplicative noise the vanishing viscosity limit of the full Navier-Stokes system was established in [9], again with the free boundary condition. The zero viscosity limit for the general Navier boundary conditions has thus far only been determined with additive noise, courtesy of [20]. The core property of any approach to this problem lies in showing uniform in time $W^{1,2}$ estimates on the Navier-Stokes Equations which are independent of the viscosity. The classical deterministic method relies on a maximum principle (see the fundamental works of [22, 44]), which extends to the case of additive noise as the difference of the solution of the equation with the noise satisfies a deterministic parabolic PDE pathwise. Lacking the ability to do this for multiplicative noise, one instead looks to show an energy estimate of the vorticity directly and in doing so requires the free boundary condition for the Laplacian term. Our method is the same, and so we impose that $\alpha = 2\kappa$, and the original requirement of Theorem 5.3.3 that $\alpha \geq \kappa$ implies that one must take $\kappa \geq 0$.

Elsewhere, the zero viscosity limit of solutions of the Stochastic Navier-Stokes Equations has so far received little treatment; the works of [10, 57] determine measure-theoretic results for the problem posed in two dimensions with periodic boundary conditions and an additive noise. In the classical case of a no-slip boundary condition, the only results of which we are aware are given in [106, 151]. A stochastic Kato type result is proven in each paper, for the limit to deterministic Euler in [106] and stochastic Euler in [151], though once more only in 2D and with additive noise.

The structure of this chapter is as follows:

- Section 6.2 develops our theory under the no-slip boundary condition. The main result is presented in Subsection 6.2.5.
- Section 6.3 presents our work for the Navier boundary conditions in the case where $\alpha = 2\kappa$, referred to as the free boundary condition. The primary result is given in Subsection 6.3.1.
- Section 6.4 concludes this chapter and discusses potential extensions to the results given.

6.2 No-Slip Boundary

In order to address both the Itô form (6.1) and Stratonovich form (6.2), we shall study the inviscid limit of

$$u_t = u_0 - \int_0^t \mathcal{P}\mathcal{L}_{u_s}u_s\, ds - \nu \int_0^t Au_s\, ds + \frac{\nu}{2}\int_0^t \sum_{i=1}^{\infty} \mathcal{P}\mathcal{Q}_i^2 u_s ds - \nu^{\frac{1}{2}} \int_0^t \mathcal{P}\mathcal{G}u_s d\mathcal{W}_s \tag{6.4}$$

where $\mathcal{Q}_i$ is either $\mathcal{P}\mathcal{G}_i$ or 0, satisfying assumptions to be stated in Subsection 6.2.1. The case $\mathcal{Q}_i = 0$ leaves us with the projected form of (6.1) whilst $\mathcal{Q}_i = \mathcal{P}\mathcal{G}_i$ corresponds to (6.2) via taking the Leray Projection and then converting to Itô Form, as was seen in Subsection 4.4.1. In the case where $\mathcal{P}\mathcal{G}_i^2 = (\mathcal{P}\mathcal{G}_i)^2$ then we can instead take $\mathcal{Q}_i = \mathcal{G}_i$ as the resulting equation (6.4) is the same; we know this to be the case for SALT noise from Lemma 4.3.5. Examples of noise satisfying these assumptions is briefly discussed in Subsection 6.2.2.

As we are interested in the vanishing viscosity limit, we assume that $0 < \nu < 1$. We shall consider the problem in $N = 2$ or 3 dimensions, for a deterministic $u_0 \in W^{m,2}\left(\mathscr{O};\mathbb{R}^N\right) \cap L^2_\sigma$ with $m > 1 + \frac{N}{2}$, and wish to characterise the convergence of selected martingale weak solutions of (6.4) to a solution of the Euler Equation. This selection is made precise in Subsection 6.2.4, and a solution of the Euler Equation is defined now. We no longer consider an arbitrary $T > 0$ but in this section fix T as specified in the following: recall a result proved by many authors, specifically referring to [13] Theorem 1 and the near immediate identity (3.4) in [79], which is that there exists a $T > 0$ and a unique $\bar{u} \in C\left([0,T]; W^{m,2}(\mathscr{O};\mathbb{R}^N) \cap L^2_\sigma\right) \cap C^1\left([0,T] \times \bar{\mathscr{O}};\mathbb{R}^N\right)$ such that the identity

$$\partial_t \bar{u} = -\mathcal{P}\mathcal{L}_{\bar{u}}\bar{u} \tag{6.5}$$

holds on $\mathscr{O} \times [0,T]$, and $\bar{u}_t|_{t=0} = u_0$ holds on $\mathscr{O}$. Moreover for every $t \in [0,T]$,

$$\|\bar{u}_t\|^2 = \|u_0\|^2 . \tag{6.6}$$

Notions of spatially weak and space-time weak solutions of (6.4) are introduced in Subsection 6.2.3, shown to be equivalent to the weak solution of Subsection 5.2.2, and necessary in order to use $\bar{u}$ as a test function in the weak formulation. Where only martingale weak solutions are available in 3D, and the probability space could change dependent on the viscosity considered, a unified construction and specific selection of solutions is required, which occurs in Subsection 6.2.4. Our Stochastic Kato Criterion is introduced in Subsection 6.2.5, with the key implication proven in Subsection 6.2.6. In Subsection 6.2.7 we consider a new parameter in the noise which approaches zero at a (possibly) different rate, determining the implications of this for our criterion and clarifying a sense in which the scaling of $\nu^{\frac{1}{2}}$ is optimal.

6.2.1 Assumptions on the Noise

We impose the existence of some $p,q,r \in \mathbb{R}$ and constants (c_i) such that for all $f,g \in \bar{W}^{1,2}(\mathscr{O};\mathbb{R}^N)$, $\phi,\psi \in W^{2,2}_\sigma$, defining $K(f,g) := 1 + \|f\|^p + \|g\|^q + \|f\|^2_{W^{1,2}} + \|g\|^2_{W^{1,2}}$,

$$\|\mathcal{G}_i f\|^2 \leq c_i\left(1 + \|f\|^2_{W^{1,2}}\right) \tag{6.7}$$
$$\|\mathcal{G}_i f - \mathcal{G}_i g\|^2 \leq c_i\left[1 + \|f\|^p_{W^{1,2}} + \|g\|^q_{W^{1,2}}\right]\|f-g\|^2_{W^{1,2}} \tag{6.8}$$
$$\|\mathcal{Q}_i\phi\|^2_{W^{1,2}} \leq c_i\|\phi\|^2_2 \tag{6.9}$$
$$\left\langle \mathcal{Q}_i^2\phi,\phi\right\rangle + \|\mathcal{G}_i\phi\|^2 \leq c_i\left(1 + \|\phi\|^2\right) + k_i\|\phi\|^2_1 \tag{6.10}$$
$$\langle \mathcal{G}_i f, f\rangle^2 \leq c_i\left(1 + \|f\|^4\right) \tag{6.11}$$
$$\langle \mathcal{G}_i f, g\rangle^2 \leq c_i\left[1 + \|f\|^2 + \|g\|^p\right]\|g\|^2_{W^{1,2}} \tag{6.12}$$
$$\langle \mathcal{G}_i f - \mathcal{G}_i g, \phi\rangle^2 \leq c_i\left[1 + \|\phi\|^p_2\right]\|f-g\|^2 \tag{6.13}$$
$$\langle \mathcal{G}_i f - \mathcal{G}_i g, f-g\rangle^2 \leq c_i K(f,g)\|f-g\|^4, \tag{6.14}$$

where $\sum_{i=1}^\infty c_i < \infty$ and $\sum_{i=1}^\infty k_i \leq 1$[1]. In fact we require that these bounds hold on any measurable subset of $\mathscr{O}$ with smooth boundary. For each $i \in \mathbb{N}$, $\mathcal{Q}_i$ must be linear and possess a densely defined adjoint $\mathcal{Q}_i^*$ in $L^2(\mathscr{O};\mathbb{R}^N)$ with domain of definition $W^{1,2}(\mathscr{O};\mathbb{R}^N)$ where for every $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that, if f,g also belong to L^2_σ,

$$\|\mathcal{Q}_i^* f\|^2 \leq c_i\|f\|^2_{W^{1,2}} \tag{6.15}$$
$$\langle \mathcal{Q}_i(f-g), \mathcal{Q}_i^*(f-g)\rangle + \|\mathcal{G}_i f - \mathcal{G}_i g\|^2 \leq c_i K(f,g)\|f-g\|^2 + k_i\|f-g\|^2_{W^{1,2}}. \tag{6.16}$$

Moreover we assume that $\mathcal{Q}_i^*$ has structure $\mathcal{Q}_i^* = \mathcal{A}_i + \hat{\mathcal{A}}_i$ where if $f \in W^{1,2}(\mathscr{O};\mathbb{R}^N)$ has support in a set $\mathcal{U} \subset \bar{\mathscr{O}}$, then $\mathcal{A}_i f$ again has support in $\mathcal{U}$ and $\left\|\hat{\mathcal{A}}_i f\right\|^2 \leq c_i\|f\|^2$.

6.2.2 Examples

We consider examples of noise for which the assumptions imposed in Subsection 6.2.1 are satisfied. First of all it should be stated that the assumptions cover SALT noise used in (1.2), for $\xi_i \in \bar{W}^{1,2}_\sigma \cap W^{2,\infty}(\mathscr{O};\mathbb{R}^N)$ with $\sum_{i=1}^\infty \|\xi_i\|^2_{W^{2,\infty}} < \infty$, all of which have been demonstrated across Subsections 4.3.2 and 5.2.1. It is immediate that our setting covers the additive noise used in the works of [10, 57, 106], whilst also enabling linear multiplicative noise as seen in [53] and Nemytskii operators as present in [58, 144]. Using the property (4.14) it is largely straightforwards to see that the usual transport noise $\mathcal{G}_i = \mathcal{P}\mathcal{L}_{\xi_i}$, $\mathcal{Q}_i = \mathcal{G}_i$ for $\xi_i \in W^{1,2}_\sigma \cap W^{1,\infty}(\mathscr{O};\mathbb{R}^N)$ with $\sum_{i=1}^\infty \|\xi_i\|^2_{W^{1,\infty}} < \infty$ also satisfies our assumptions. We note that incorporating $\mathcal{P}$ into $\mathcal{G}_i$ makes no difference to the equation (6.4) however it will be necessary to verify the assumptions. We only draw attention to the condition (6.10), and by extension (6.16), as there are no subtleties in the other inequalities. The argument is that

$$\left\langle (\mathcal{P}\mathcal{L}_{\xi_i})^2\phi,\phi\right\rangle = \langle \mathcal{L}_{\xi_i}\mathcal{P}\mathcal{L}_{\xi_i}\phi,\phi\rangle = -\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{L}_{\xi_i}\phi\rangle = -\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{P}\mathcal{L}_{\xi_i}\phi\rangle$$

[1] Actually, we only need that $\sum_{i=1}^\infty k_i < 2$. Our choice just avoids additional the technical details of having to reference this sum explicitly.

so

$$\left\langle (\mathcal{P}\mathcal{L}_{\xi_i})^2\phi, \phi \right\rangle + \|\mathcal{P}\mathcal{L}_{\xi_i}\phi\|^2 = -\left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{P}\mathcal{L}_{\xi_i}\phi \right\rangle + \left\langle \mathcal{P}\mathcal{L}_{\xi_i}\phi, \mathcal{P}\mathcal{L}_{\xi_i}\phi \right\rangle = 0$$

hence the assumption certainly holds. We also note that the assumptions hold for $\mathcal{G}_i = \mathcal{L}_{\xi_i}$ and $\mathcal{Q}_i = 0$ if $\sum_{i=1}^\infty \|\xi_i\|_{W^{1,\infty}}^2 \leq 1$, which is an Itô transport noise with sufficiently small gradient dependency.

6.2.3 Notions of Weak Solutions

We give two definitions for weak solutions of the equation (6.4).

Definition 6.2.1. *Let $u_0 : \Omega \to L^2_\sigma$ be $\mathcal{F}_0-$measurable. A process u which is progressively measurable in $W^{1,2}_\sigma$ and such that for $\mathbb{P}-a.e.$ ω, $u_\cdot(\omega) \in L^\infty\left([0,T]; L^2_\sigma\right) \cap C_w\left([0,T]; L^2_\sigma\right) \cap L^2\left([0,T]; W^{1,2}_\sigma\right)$, is said to be a spatially weak solution of the equation (6.4) if the identity*

$$\begin{aligned}\langle u_t, \phi \rangle = \langle u_0, \phi \rangle - \int_0^t \langle \mathcal{L}_{u_s} u_s, \phi \rangle \, ds - \nu \int_0^t \langle u_s, \phi \rangle_1 \, ds \\ + \frac{\nu}{2} \int_0^t \sum_{i=1}^\infty \langle \mathcal{Q}_i u_s, \mathcal{Q}_i^* \phi \rangle \, ds - \nu^{\frac{1}{2}} \int_0^t \langle \mathcal{G} u_s, \phi \rangle \, d\mathcal{W}_s \end{aligned} \tag{6.17}$$

holds for every $\phi \in W^{1,2}_\sigma$, $\mathbb{P}-a.s.$ in $\mathbb{R}$ for all $t \in [0,T]$.

As touched upon in Proposition 5.2.3, this formulation is equivalent to u satisfying the identity (6.4) in $(W^{1,2}_\sigma)^*$, c.f. Subsection 4.4.2. It enables a simpler connection to the alternative stated below.

Definition 6.2.2. *Let $u_0 : \Omega \to L^2_\sigma$ be $\mathcal{F}_0-$measurable. A process u which is progressively measurable in $W^{1,2}_\sigma$ and such that for $\mathbb{P}-a.e.$ ω, $u_\cdot(\omega) \in C_w\left([0,T]; L^2_\sigma\right) \cap L^2\left([0,T]; W^{1,2}_\sigma\right)$, is said to be a space-time weak solution of the equation (6.4) if the identity*

$$\begin{aligned}\langle u_t, \phi_t \rangle = \langle u_0, \phi_0 \rangle + \int_0^t \langle u_s, \partial_s \phi_s \rangle \, ds - \int_0^t \langle \mathcal{L}_{u_s} u_s, \phi_s \rangle \, ds - \nu \int_0^t \langle u_s, \phi_s \rangle_1 \, ds \\ + \frac{\nu}{2} \int_0^t \sum_{i=1}^\infty \langle \mathcal{Q}_i u_s, \mathcal{Q}_i^* \phi_s \rangle \, ds - \nu^{\frac{1}{2}} \int_0^t \langle \mathcal{G} u_s, \phi_s \rangle \, d\mathcal{W}_s \end{aligned} \tag{6.18}$$

holds for every $\phi \in C^1\left([0,t] \times \overline{\mathscr{O}}; \mathbb{R}^N\right)$ such that $\phi_s \in W^{1,2}_\sigma$ for every $s \in [0,t]$, $\mathbb{P}-a.s.$ in $\mathbb{R}$ for all $t \in [0,T]$.

The difference in the two definitions comes from whether or not there is time-dependency in the test function. We will see in Subsection 6.2.6 that the time-dependency is necessary for us to characterise the zero viscosity limit, as we shall use a corrected solution of the nonstationary Euler Equation as a test function in this weak formulation. On the other hand this formulation is impractical to demonstrate and to work with on the whole in the stochastic setting, as we do not have differentiability in time for our approximate solutions. Therefore both representations serve a purpose, and we must show their equivalence.

Proposition 6.2.3. *A process u is a spatially weak solution of the equation (6.4) if and only if it is a space-time weak solution.*

Proof. We fix a u with the regularity specified in the definitions, and wish to show the relation (6.17) $\iff$ (6.18). We consider the two implications:

$\Longleftarrow$: We fix a $\phi \in W^{1,2}_\sigma$, and note that for any constant function in time $\psi_s = \psi$, $\psi \in C^1\left(\mathscr{O};\mathbb{R}^N\right) \cap W^{1,2}_\sigma$, we do indeed have the identity (6.17) for ψ as the time derivative is null. Given that even $C^\infty_{0,\sigma}$ is dense in $W^{1,2}_\sigma$ then most certainly too is $C^1\left(\overline{\mathscr{O}};\mathbb{R}^N\right) \cap W^{1,2}_\sigma$ so we can take a sequence (ψ^n) in this space convergent to ϕ in $\|\cdot\|_1$. The result is now straightforwards in the limit.

$\Longrightarrow$: We fix a $\phi \in C^1\left([0,t] \times \overline{\mathscr{O}};\mathbb{R}^N\right)$ such that $\phi_s \in W^{1,2}_\sigma$ for every $s \in [0,t]$, and shall consider simple approximations of this ϕ and use the relation (6.17) on the constant time steps. To this end for every stopping time τ with $u.\mathbb{1}_{\cdot\leq\tau} \in L^2\left(\Omega \times [0,t]; W^{1,2}_\sigma\right)$, we introduce a sequence of partitions

$$I_l \subset I_{l+1}, \quad I_l := \left\{0 = t^l_0 < t^l_1 < \cdots < t^l_{k_l} = t\right\}, \quad \max_j |t^l_j - t^l_{j-1}| \to 0 \text{ as } l \to \infty$$

which are such that the process defined by

$$\tilde{u}^l_s(\omega) := \sum_{j=1}^{k_l-1} \mathbb{1}_{[t^l_{j-1},t^l_j]}(s) u_{t^l_j}(\omega) \mathbb{1}_{t^l_j \leq \tau(\omega)}$$

belongs to $L^2\left(\Omega \times [0,t]; W^{1,2}_\sigma\right)$ and converges to $u.\mathbb{1}_{\cdot\leq\tau}$ in this space. Such a partition is available to us due to Lemma A.2.1. Now for each l, j from the identity (6.17) we have that

$$\begin{aligned}\left\langle u_{t^l_j}, \phi_{t^l_j}\right\rangle = \left\langle u_{t^l_{j-1}}, \phi_{t^l_j}\right\rangle &- \int_{t^l_{j-1}}^{t^l_j} \left\langle \mathcal{L}_{u_s} u_s, \phi_{t^l_j}\right\rangle ds - \nu \int_{t^l_{j-1}}^{t^l_j} \left\langle u_s, \phi_{t^l_j}\right\rangle_1 ds \\ &+ \frac{\nu}{2} \int_{t^l_{j-1}}^{t^l_j} \sum_{i=1}^\infty \left\langle \mathcal{Q}_i u_s, \mathcal{Q}_i^* \phi_{t^l_j}\right\rangle ds - \nu^{\frac{1}{2}} \int_{t^l_{j-1}}^{t^l_j} \left\langle \mathcal{G} u_s, \phi_{t^l_j}\right\rangle d\mathcal{W}_s \end{aligned} \tag{6.19}$$

which we shall utilise in the context of

$$\begin{aligned}\langle u_t, \phi_t\rangle - \langle u_0, \phi_0\rangle &= \sum_{j=1}^{k_l} \left(\left\langle u_{t^l_j}, \phi_{t^l_j}\right\rangle - \left\langle u_{t^l_{j-1}}, \phi_{t^l_{j-1}}\right\rangle\right) \\ &= \sum_{j=1}^{k_l} \left(\left\langle u_{t^l_j}, \phi_{t^l_j}\right\rangle - \left\langle u_{t^l_{j-1}}, \phi_{t^l_j}\right\rangle\right) + \sum_{j=1}^{k_l} \left(\left\langle u_{t^l_{j-1}}, \phi_{t^l_j}\right\rangle - \left\langle u_{t^l_{j-1}}, \phi_{t^l_{j-1}}\right\rangle\right). \end{aligned} \tag{6.20}$$

To address the first sum we introduce some familiar notation in

$$\tilde{\phi}^l_s := \sum_{j=1}^{k_l-1} \mathbb{1}_{[t^l_{j-1},t^l_j]}(s) \phi_{t^l_j},$$

allowing us to deduce from (6.19) that

$$\sum_{j=1}^{k_l}\left(\left\langle u_{t_j^l},\phi_{t_j^l}\right\rangle - \left\langle u_{t_{j-1}^l},\phi_{t_j^l}\right\rangle\right) = -\int_0^t \left\langle \mathcal{L}_{u_s}u_s, \tilde{\phi}_s^l\right\rangle ds - \nu\int_0^t \left\langle u_s, \tilde{\phi}_s^l\right\rangle_1 ds$$
$$+\frac{\nu}{2}\int_0^t\sum_{i=1}^\infty \left\langle \mathcal{Q}_i u_s, \mathcal{Q}_i^*\tilde{\phi}_s^l\right\rangle ds - \nu^{\frac{1}{2}}\int_0^t \left\langle \mathcal{G}u_s, \tilde{\phi}_s^l\right\rangle d\mathcal{W}_s. \tag{6.21}$$

As for the second sum of (6.20), observe that for each j we can rewrite this as

$$\begin{aligned}\left\langle u_{t_{j-1}^l},\phi_{t_j^l}\right\rangle - \left\langle u_{t_{j-1}^l},\phi_{t_{j-1}^l}\right\rangle &= \int_{\mathscr{O}} u_{t_{j-1}^l}(x)\left[\phi_{t_j^l}(x)-\phi_{t_{j-1}^l}(x)\right]dx\\ &= \int_{\mathscr{O}} u_{t_{j-1}^l}(x)\left[\int_{t_{j-1}^l}^{t_j^l}\partial_s\phi_s(x)ds\right]dx\\ &= \int_{t_{j-1}^l}^{t_j^l}\left\langle \tilde{u}_s^l,\partial_s\phi_s\right\rangle ds\end{aligned}$$

using the $C^1\left([0,t]\times\overline{\mathscr{O}};\mathbb{R}^N\right)$ regularity of ϕ and Fubini's Theorem. Summation then gives the integral over $[0,t]$ hence we deduce that

$$\langle u_t,\phi_t\rangle = \langle u_0,\phi_0\rangle + \int_0^t\left\langle \tilde{u}_s^l,\partial_s\phi_s\right\rangle ds - \int_0^t \left\langle \mathcal{L}_{u_s}u_s, \tilde{\phi}_s^l\right\rangle ds - \nu\int_0^t \left\langle u_s, \tilde{\phi}_s^l\right\rangle_1 ds$$
$$+\frac{\nu}{2}\int_0^t\sum_{i=1}^\infty \left\langle \mathcal{Q}_i u_s, \mathcal{Q}_i^*\tilde{\phi}_s^l\right\rangle ds - \nu^{\frac{1}{2}}\int_0^t \left\langle \mathcal{G}u_s, \tilde{\phi}_s^l\right\rangle d\mathcal{W}_s \tag{6.22}$$

holds for all $l\in\mathbb{N}$. We look to analyse the limit, noting that the convergence $\tilde{u}^l_\cdot \to u_\cdot \mathbb{1}_{\cdot\leq\tau}$ in $L^2\left(\Omega\times[0,t];W^{1,2}_\sigma\right)$ implies that in $L^2\left(\Omega;L^2\left([0,t];W^{1,2}_\sigma\right)\right)$ so we can extract a subsequence (relabelled again as $\tilde{u}^l$ for simplicity) which is convergent $\mathbb{P}-a.e.$ in $L^2\left([0,t];W^{1,2}_\sigma\right)$, from which we see that

$$\begin{aligned}\left|\int_0^t\left\langle \tilde{u}_s^l,\partial_s\phi_s\right\rangle ds - \int_0^t\langle u_s,\partial_s\phi_s\rangle ds\right| &\leq \int_0^t\left\|\tilde{u}_s^l-u_s\right\|\|\partial_s\phi_s\|ds\\ &\leq \left\|\tilde{u}_s^l-u_s\right\|_{L^2([0,t];L^2(\mathscr{O};\mathbb{R}^N))}\|\partial_s\phi_s\|_{L^2([0,t];L^2(\mathscr{O};\mathbb{R}^N))}\\ &\longrightarrow 0.\end{aligned} \tag{6.23}$$

Addressing the convergence in the other terms, we need to establish in what sense $\tilde{\phi}^l\to\phi$. We claim that the convergence holds in $L^2\left([0,t];W^{1,2}_\sigma\right)$. As $\phi\in C^1\left([0,t]\times\overline{\mathscr{O}};\mathbb{R}^N\right)$ then for each $m=1,\dots,N$ and every fixed $x\in\mathscr{O}$ then $\partial_m\phi(x)\in C^1\left([0,t];\mathbb{R}^N\right)$ so is in particular Lipschitz Continuous. Of course

$$\partial_m\tilde{\phi}_s^l := \sum_{j=1}^{k_l-1}\mathbb{1}_{[t_{j-1}^l,t_j^l]}(s)\partial_m\phi_{t_j^l}$$

so

$$\partial_m\phi_s - \partial_m\tilde{\phi}^l_s := \sum_{j=1}^{k_l-1}\mathbb{1}_{[t^l_{j-1},t^l_j]}(s)\left[\partial_m\phi_s - \partial_m\phi_{t^l_j}\right]$$

thus if K_x is the constant of Lipschitz Continuity, then

$$\left\|\partial_m\phi(x) - \partial_m\tilde{\phi}^l(x)\right\|_{L^\infty([0,t];\mathbb{R}^N)} \leq K_x \max_j|t^l_j - t^l_{j-1}|$$

which converges to zero as $l\to\infty$. Now then we have that

$$\begin{aligned}
\left\|\phi - \tilde{\phi}^l\right\|^2_{L^2\left([0,t];W^{1,2}_\sigma\right)} &= \int_0^t\sum_{m=1}^N\left\|\partial_m\phi_s - \partial_m\tilde{\phi}^l_s\right\|^2 ds\\
&= \sum_{m=1}^N\int_0^t\int_{\mathscr{O}}|\partial_m\phi_s(x) - \partial_m\tilde{\phi}^l_s(x)|^2 dxds\\
&= \sum_{m=1}^N\int_{\mathscr{O}}\left\|\partial_m\phi(x) - \partial_m\tilde{\phi}^l(x)\right\|^2_{L^2([0,t];\mathbb{R}^N)}dx\\
&\leq c\sum_{m=1}^N\int_{\mathscr{O}}\left\|\partial_m\phi(x) - \partial_m\tilde{\phi}^l(x)\right\|^2_{L^\infty([0,t];\mathbb{R}^N)}dx
\end{aligned}$$

where c continues to represent a generic constant. The claim then follows from the Dominated Convergence Theorem, with dominating function $4\|\partial_m\phi\|^2_{L^\infty([0,t];\mathbb{R}^N)}$ which is continuous on $\overline{\mathscr{O}}$ hence square integrable. In fact with the observation that the supremum of the integral is bounded by the integral of the supremum, we have shown the stronger convergence in $L^\infty\left([0,t];W^{1,2}_\sigma\right)$. With the claim established we again look to take limits in (6.22), and using the linearity of $\mathcal{Q}_i^*$ and (6.15) then the convergence of the time integrals follows exactly as in (6.23). As for the stochastic integral we apply the Stochastic Dominated Convergence Theorem, Lemma A.1.53, using that

$$\sum_{i=1}^\infty\left\langle\mathcal{G}_iu_s,\tilde{\phi}^l_s - \phi_s\right\rangle^2 \leq \sum_{i=1}^\infty c_i\|u_s\|_1^2\left\|\tilde{\phi}^l_s - \phi_s\right\|^2 \leq \sum_{i=1}^\infty c_i\|u_s\|_1^2\left\|\tilde{\phi}^l - \phi\right\|^2_{L^\infty([0,t];L^2_\sigma)} \longrightarrow 0$$

establishing $\mathbb{P}-a.s.$ convergence, and domination from

$$\sum_{i=1}^\infty c_i\|u_s\|^2_{L^\infty([0,t];W^{1,2}_\sigma)}\|\phi\|^2_{L^\infty([0,t];L^2_\sigma)}$$

for example. Application of Lemma A.1.53 justifies convergence of the integrals, which concludes the proof.

□

We can now refer to such a solution as simply a weak solution of the equation (6.4), equivalent again to the notion demonstrated in Theorem 5.2.2. *Uniqueness* is once more meant in the sense of indistinguishability (Definition A.1.32), and a *martingale weak* solution corresponds to

Theorem 5.2.1 in the obvious way. Identically to these two theorems, the same well-posedness results carry through to the new noise.

Theorem 6.2.4. *For any given $\mathcal{F}_0-$measurable $u_0 \in L^\infty\left(\Omega; L^2_\sigma\right)$, there exists a martingale weak solution of the equation (6.4).*

Theorem 6.2.5. *If $N = 2$ then for any given $\mathcal{F}_0-$measurable $u_0 : \Omega \to L^2_\sigma$, there exists a unique weak solution of the equation (6.4).*

6.2.4 Selection of Martingale Weak Solutions

We introduce the notation o_ν to represent any constant dependent on ν such that $\lim_{\nu\to 0} o_\nu = 0$, and build upon the existence result Theorem 6.2.4 in the case of equation (6.4) for some more precise energy estimates. Henceforth in this section we fix $u_0 \in W^{m,2}\left(\mathscr{O};\mathbb{R}^N\right) \cap L^2_\sigma$ with $m > 1 + \frac{N}{2}$ and the corresponding strong solution $\bar{u}$ of the Euler Equation introduced at the start of this section.

Proposition 6.2.6. *There exists a martingale weak solution $\tilde{u}$ of the equation (6.4) which satisfies*

$$\tilde{\mathbb{E}}\left(\sup_{r\in[0,T]} \|\tilde{u}_r\|^2\right) \leq (1 + o_\nu)\, \|u_0\|^2 + o_\nu \tag{6.24}$$

and for every $t \in [0, T]$,

$$\tilde{\mathbb{E}}\left[\|\tilde{u}_t\|^2 + \nu \int_0^t \|\tilde{u}_s\|_1^2\, ds\right] \leq (1 + o_\nu)\, \|u_0\|^2 + o_\nu. \tag{6.25}$$

Proof. Of course the existence of a martingale weak solution is established, so it is only the estimates (6.28) and (6.29) which must be justified and can be done so for the $\tilde{u}$ constructed in the proof of Theorem 3.2.7. We start with (6.28): recall in Proposition 3.2.18 how the regularity of the limit process was obtained from the uniform bounds (3.29). In the same manner it is sufficient to show that

$$\mathbb{E}\left(\sup_{r\in[0,T]} \|u^n_r\|^2\right) \leq (1 + o_\nu)\, \|u^n_0\|^2 + o_\nu \tag{6.26}$$

for every $n \in \mathbb{N}$, where u^n is the strong solution of

$$u^n_t = u^n_0 - \int_0^t \mathcal{P}_n\mathcal{P}\mathcal{L}_{u^n_s}u^n_s\, ds - \nu\int_0^t \mathcal{P}_nAu^n_s\, ds + \frac{\nu}{2}\int_0^t\sum_{i=1}^\infty \mathcal{P}_n\mathcal{P}\mathcal{Q}_i^2u^n_s ds - \nu^{\frac{1}{2}}\int_0^t \mathcal{P}_n\mathcal{P}\mathcal{G}u^n_s d\mathcal{W}_s$$

in analogy with (3.14), and o_ν is independent of n. Identically to Proposition 3.2.12 but simply ignoring the contribution from the Stokes Operator in the inequality and using that the initial condition is deterministic, we arrive at

$$\mathbb{E}\left(\sup_{r\in[0,T]} \|u^n_r\|^2\right) \leq \|u^n_0\|^2 + c\nu + c\nu\int_0^T \mathbb{E}\left(\|u^n_s\|^2\right) ds + c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T\sum_{i=1}^\infty \langle\mathcal{G}_i\hat{u}^n_s, \hat{u}^n_s\rangle^2\, ds\right)^{\frac{1}{2}}.$$

The final term is controlled similarly, just a little more precisely:

$$c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T\sum_{i=1}^{\infty}\langle\mathcal{G}_i\hat{u}_s^n,\hat{u}_s^n\rangle^2\,ds\right)^{\frac{1}{2}} \leq c\nu^{\frac{1}{2}} + c\nu^{\frac{1}{2}}\mathbb{E}\left(\sup_{r\in[0,T]}\|\hat{u}_r^n\|^2\int_0^T\|\hat{u}_s^n\|^2\,ds\right)^{\frac{1}{2}}$$
$$\leq c\nu^{\frac{1}{2}} + \nu^{\frac{1}{2}}\mathbb{E}\left(\sup_{r\in[0,T]}\|\hat{u}_r^n\|^2\right) + c\nu^{\frac{1}{2}}\mathbb{E}\int_0^T\|\hat{u}_s^n\|^2\,ds$$

so that

$$\left(1-\nu^{\frac{1}{2}}\right)\mathbb{E}\left(\sup_{r\in[0,T]}\|u_r^n\|^2\right) \leq \|u_0^n\|^2 + c\nu + c\left(\nu+\nu^{\frac{1}{2}}\right)\int_0^T\mathbb{E}\left(\|u_s^n\|^2\right)ds$$

and furthermore through dividing by $\left(1-\nu^{\frac{1}{2}}\right)$ and rewriting with the o_ν notation,

$$\mathbb{E}\left(\sup_{r\in[0,T]}\|u_r^n\|^2\right) \leq (1+o_\nu)\|u_0^n\|^2 + o_\nu + o_\nu\int_0^T\mathbb{E}\left(\|u_s^n\|^2\right)ds$$

having rewritten $\frac{1}{1-\nu^{1/2}}$ as $1+\frac{\nu^{1/2}}{1-\nu^{1/2}}$. We now apply the standard Grönwall Inequality to deduce that

$$\mathbb{E}\left(\sup_{r\in[0,T]}\|u_r^n\|^2\right) \leq e^{o_\nu}\left[(1+o_\nu)\|u_0^n\|^2 + o_\nu\right] \leq (1+o_\nu)\|u_0^n\|^2 + o_\nu$$

using the convergence $\lim_{\nu\to 0}e^{o_\nu}=1$, demonstrating (6.26) to conclude the justification of (6.28). Similarly for (6.29) it is sufficient to demonstrate that

$$\mathbb{E}\left[\|u_t^n\|^2 + \nu\int_0^t\|u_s^n\|_1^2\,ds\right] \leq (1+o_\nu)\|u_0^n\|^2 + o_\nu. \tag{6.27}$$

The proof is near identical to (6.26), though as we do not take the supremum then we take the expectation of the stochastic integral directly so this term is null, and we maintain the term coming from the Stokes Operator. Thus we arrive directly at

$$\mathbb{E}\left(\|u_t^n\|^2\right) + \nu\mathbb{E}\int_0^t\|u_s^n\|_1^2\,ds \leq \|u_0^n\|^2 + c\nu + c\nu\int_0^T\mathbb{E}\left(\|u_s^n\|^2\right)ds$$

from which the remainder of the proof follows as above, noting that

$$\int_0^T\mathbb{E}\left(\|u_s^n\|^2\right)ds \leq \int_0^T\left[\mathbb{E}\left(\|u_s^n\|^2\right) + \nu\mathbb{E}\int_0^s\|u_r^n\|_1^2\,dr\right]ds$$

so we can apply the Grönwall Inequality bounding the entirety of the left hand side as is required.

□

Remark 6.2.7. *Estimates (6.24) and (6.25) were stated in this way to make the dependency on the initial condition explicit, though we shall henceforth keep the initial condition constant* *in ν and as such we may refer to the more direct inequalities*

$$\tilde{\mathbb{E}}\left(\sup_{r\in[0,T]}\|\tilde{u}_r\|^2\right) \leq \|u_0\|^2 + o_\nu \tag{6.28}$$

$$\tilde{\mathbb{E}}\left[\|\tilde{u}_t\|^2 + \nu\int_0^t \|\tilde{u}_s\|_1^2\,ds\right] \leq \|u_0\|^2 + o_\nu \tag{6.29}$$

where o_ν depends on this fixed u_0.

We now clarify how the martingale weak solutions of (6.4) are selected and some notation for the following subsection. Formally we must work with an arbitrary sequence of viscosities (ν_k) such that $\nu_k \to 0$ as $k \to \infty$. For each such k we then choose a martingale weak solution $\tilde{u}^k$ as specified in Proposition 6.2.6, which we recall is defined with respect to a filtered probability space $\left(\tilde{\Omega}^k, \tilde{\mathcal{F}}^k, (\tilde{\mathcal{F}}_t^k), \tilde{\mathbb{P}}^k\right)$, a Cylindrical Brownian Motion $\tilde{\mathcal{W}}^k$ over $\mathfrak{U}$ with respect to $\left(\tilde{\Omega}^k, \tilde{\mathcal{F}}^k, (\tilde{\mathcal{F}}_t^k), \tilde{\mathbb{P}}^k\right)$, and an $\mathcal{F}_0-$measurable $\tilde{u}_0^k : \tilde{\Omega} \to L^2_\sigma$ with the same law as u_0. Immediately we note that as u_0 is deterministic then $\tilde{u}_0^k$ of the same law must simply be u_0 itself. It is less obvious how we can consider the limiting properties of this sequence of solutions where each $\tilde{u}^k$ is defined on a different probability space. We rectify this with the following:

- The standard infinite dimensional product space

$$\tilde{\Omega} := \bigtimes_{k=0}^{\infty}\tilde{\Omega}^k, \quad \tilde{\mathcal{F}} := \bigotimes_{k=0}^{\infty}\tilde{\mathcal{F}}^k, \quad \tilde{\mathcal{F}}_t := \bigotimes_{k=0}^{\infty}\tilde{\mathcal{F}}_t^k, \quad \tilde{\mathbb{P}} := \bigtimes_{k=0}^{\infty}\tilde{\mathbb{P}}^k$$

  such that $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$ is a filtered probability space;

- The component projections $(\mathcal{P}^k)$, $\mathcal{P}^k : \tilde{\Omega} \to \tilde{\Omega}^k$ and subsequently defined $(\hat{u}^k)$, $(\hat{\mathcal{W}}^k)$ by

$$\hat{u}^k := \tilde{u}^k\mathcal{P}^k, \qquad \hat{\mathcal{W}}^k = \tilde{\mathcal{W}}^k\mathcal{P}^k.$$

By construction for each k, $\hat{u}^k$ is a martingale weak solution of (6.4) relative to the Cylindrical Brownian Motion $\hat{\mathcal{W}}^k$ and filtered probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$. We can now make sense of taking the limit as $\nu \to 0$ in expectation for martingale weak solutions of (6.4), by choosing an arbitrary sequence of viscosities (ν_k) convergent to zero and constructing the new solutions relative to a single probability space as above. We now fix this sequence and the related constructions.

6.2.5 The Main Result

We state the main result of this section, and assess what needs to be proved. We introduce notation for $f \in W^{1,2}(\mathscr{O};\mathbb{R}^N)$,

$$\|\nabla f\|_{\Gamma_c}^2 = \sum_{k=1}^N \|\partial_k f\|_{L^2(\Gamma_c;\mathbb{R}^N)}^2,$$

where for a constant c, Γ_c is the boundary strip of width radius c, defined by the set of all points $x \in \mathscr{O}$ such that there exists a y on the boundary with the distance from x to y less than c. Following the construction in Subsection 6.2.4, for notational simplicity we consider an arbitrary $\hat{u}^k$ and relabel it as u, understanding that there is an implicit dependency on ν and that it is of course still defined over the new filtered probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$. We shall use $\mathbb{E}$ to represent the expectation taken with respect to $\tilde{\mathbb{P}}$ on this probability space. We formally consider the process $\bar{u}$ representing the solution of the Euler Equation defined at (6.5) to reside on this space as a constant.

Theorem 6.2.8. *The following conditions are equivalent:*

1. $\mathbb{E}\left(\sup_{r\in[0,T]} \|u_r - \bar{u}_r\|^2\right) = o_\nu$,
2. *For every* $t \in [0,T]$ *and* $\phi \in L^2(\Omega \times \mathscr{O}; \mathbb{R}^N)$, $\mathbb{E}\left(\langle u_t - \bar{u}_t, \phi\rangle\right) = o_\nu$,
3. $\nu\mathbb{E}\int_0^T \|u_s\|_1^2\, ds = o_\nu$,
4. *For any constant* $\tilde{c} > 0$, $\nu\mathbb{E}\int_0^T \|\nabla u_s\|_{\Gamma_{\tilde{c}\nu}}^2\, ds = o_\nu$.

Remark 6.2.9. *This is a stochastic parallel of Kato's result, [79] Theorem 1.*

We shall prove the theorem by demonstrating that 1 $\Longrightarrow$ 2 $\Longrightarrow$ 3 $\Longrightarrow$ 4 $\Longrightarrow$ 1, and now identify what needs proving in this. Of course 1 $\Longrightarrow$ 2 is trivial, as is 3 $\Longrightarrow$ 4. For 2 $\Longrightarrow$ 3, from (6.29) we have that

$$\nu\mathbb{E}\int_0^T \|u_s\|_1^2\, ds \leq \|u_0\|^2 + o_\nu - \mathbb{E}\left(\|u_T\|^2\right).$$

To show that the limit exists and is zero it is sufficient to demonstrate that the limit supremum is zero, and as

$$\begin{aligned}\limsup_{\nu\to 0}\left[\nu\mathbb{E}\int_0^T \|u_s\|_1^2\, ds\right] &\leq \limsup_{\nu\to 0}\left[\|u_0\|^2 - \mathbb{E}\left(\|u_T\|^2\right) + o_\nu\right]\\ &\leq \|u_0\|^2 - \liminf_{\nu\to 0}\mathbb{E}\left(\|u_T\|^2\right)\end{aligned}$$

then we only need to verify that

$$\|u_0\|^2 \leq \liminf_{\nu\to 0}\mathbb{E}\left(\|u_T\|^2\right). \tag{6.30}$$

Item 2 is the statement that for every $t \in [0,T]$, (u_t) converges to $\bar{u}_t$ weakly in $L^2(\Omega \times \mathscr{O}; \mathbb{R}^N)$. With the known result that norms are weakly lower semicontinuous[2], we employ the assumed Item 2 for time T to see that

$$\|\bar{u}_T\|^2 \leq \liminf_{\nu\to 0}\mathbb{E}\left(\|u_T\|^2\right).$$

The property (6.30) then follows from the energy identity (6.6).

[2] Observe that if (x_n) is weakly convergent to x, then $\|x\|^2 = \langle x, x\rangle = \lim_{n\to\infty}\langle x_n, x\rangle = \liminf_{n\to\infty}\langle x_n, x\rangle \leq \liminf_{n\to\infty}\|x_n\|\,\|x\|$ which implies $\|x\| \leq \liminf_{n\to\infty}\|x_n\|$.

6.2.6 Proof of the Remaining Implication

This subsection is dedicated to proving the final implication 4 $\implies$ 1 of Theorem 6.2.8. We recall a result proved in Kato's paper [79], stated for a fixed $\nu < 1$.

Lemma 6.2.10. *There exists a function $v \in C^1\left([0,T] \times \bar{\mathscr{O}}; \mathbb{R}^N\right)$ and a constant c (which may depend on $\tilde{c}$) such that:*

1. *For every $t \in [0,T]$, $v_t \in L^2_\sigma$, $v_t = \bar{u}_t$ on $\partial\mathscr{O}$ and v_t is supported on $\Gamma_{\tilde{c}\nu}$,*
2. *v satisfies the estimates*

$$\sup_{r\in[0,T]} \|v_r\| \leq c\nu^{\frac{1}{2}} \tag{6.31}$$

$$\sup_{r\in[0,T]} \|\partial_t v_r\| \leq c\nu^{\frac{1}{2}} \tag{6.32}$$

$$\sup_{r\in[0,T]} \|v_r\|_{W^{1,2}} \leq c\nu^{-\frac{1}{2}} \tag{6.33}$$

3. *For any $f \in W^{1,2}_\sigma$,*

$$\sup_{r\in[0,T]} |\langle \mathcal{L}_f f, v_r\rangle| \leq c\nu \|\nabla f\|^2_{\Gamma_{\tilde{c}\nu}}. \tag{6.34}$$

We explain the significance of the lemma now: with the identity

$$\|u_r - \bar{u}_r\|^2 = \|u_r\|^2 + \|\bar{u}_r\|^2 - 2\langle u_r, \bar{u}_r\rangle$$

then we want to make use of the formulation (6.18), which we cannot immediately do as $\bar{u}$ does not necessarily vanish on the boundary. This is where we introduce v from the lemma, as v is prescribed to equal $\bar{u}$ on the boundary so that $\bar{u} - v$ satisfies the regularity of ϕ required in Definition 6.2.2. The idea is that the terms involving $\bar{u}$ can be well controlled using the smoothness of $\bar{u}$, and the terms involving v require only an assumption on the energy dissipation within the boundary strip as this is where v is supported. As v is small in L^2_σ with low viscosity, then the excess terms in v will be small as well. Thus we rewrite

$$\|u_r - \bar{u}_r\|^2 = \|u_r\|^2 + \|\bar{u}_r\|^2 + 2\langle u_r, v_r\rangle - 2\langle u_r, \bar{u}_r - v_r\rangle \tag{6.35}$$

where

$$\begin{aligned}\langle u_r, \bar{u}_r - v_r\rangle = \langle u_0, u_0 - v_0\rangle &+ \int_0^r \langle u_s, \partial_s(\bar{u}_s - v_s)\rangle\, ds - \int_0^r \langle \mathcal{L}_{u_s} u_s, \bar{u}_s - v_s\rangle\, ds\\ &- \nu\int_0^r \langle u_s, \bar{u}_s - v_s\rangle_1\, ds + \frac{\nu}{2}\int_0^r \sum_{i=1}^\infty \langle \mathcal{Q}_i u_s, \mathcal{Q}_i^*(\bar{u}_s - v_s)\rangle\, ds - \nu^{\frac{1}{2}}\int_0^t \langle \mathcal{G}u_s, \bar{u}_s - v_s\rangle\, d\mathcal{W}_s.\end{aligned}$$

Before taking the supremum and then expectation in the direction of Item 1, we appreciate that

$$\langle u_0, u_0 - v_0\rangle = \|u_0\|^2 - \langle u_0, v_0\rangle$$

and with (6.5), (4.14):

$$\begin{aligned}
\int_0^r \langle u_s, \partial_s(\bar{u}_s - v_s)\rangle\, ds &- \int_0^r \langle \mathcal{L}_{u_s}u_s, \bar{u}_s - v_s\rangle\, ds \\
&= \int_0^r \langle u_s, \partial_s\bar{u}_s\rangle - \langle u_s, \partial_s v_s\rangle - \langle \mathcal{L}_{u_s}u_s, \bar{u}_s\rangle + \langle \mathcal{L}_{u_s}u_s, v_s\rangle\, ds \\
&= \int_0^r -\langle u_s, \mathcal{P}\mathcal{L}_{\bar{u}_s}\bar{u}_s\rangle - \langle u_s, \partial_s v_s\rangle - \langle \mathcal{L}_{u_s}u_s, \bar{u}_s\rangle + \langle \mathcal{L}_{u_s}u_s, v_s\rangle\, ds \\
&= \int_0^r -\langle u_s, \mathcal{L}_{\bar{u}_s}\bar{u}_s\rangle - \langle u_s, \partial_s v_s\rangle + \langle u_s, \mathcal{L}_{u_s}\bar{u}_s\rangle + \langle \mathcal{L}_{u_s}u_s, v_s\rangle\, ds \\
&= \int_0^r \langle u_s, \mathcal{L}_{u-\bar{u}_s}\bar{u}_s\rangle\, ds + \int_0^r \langle \mathcal{L}_{u_s}u_s, v_s\rangle\, ds - \int_0^r \langle u_s, \partial_s v_s\rangle\, ds.
\end{aligned}$$

Substituting all of this into (6.35) gives

$$\begin{aligned}
\|u_r - \tilde{u}_r\|^2 &= \|u_r\|^2 + \|\bar{u}_r\|^2 + 2\langle u_r, v_r\rangle - 2\|u_0\|^2 + 2\langle u_0, v_0\rangle \\
&- 2\int_0^r \langle u_s, \mathcal{L}_{u-\bar{u}_s}\bar{u}_s\rangle\, ds - 2\int_0^r \langle \mathcal{L}_{u_s}u_s, v_s\rangle\, ds + 2\int_0^r \langle u_s, \partial_s v_s\rangle\, ds \\
&+ 2\nu\int_0^r \langle u_s, \bar{u}_s - v_s\rangle_1\, ds - \nu\int_0^r \sum_{i=1}^\infty \langle \mathcal{Q}_i u_s, \mathcal{Q}_i^*(\bar{u}_s - v_s)\rangle\, ds + 2\nu^{\frac{1}{2}}\int_0^t \langle \mathcal{G}u_s, \bar{u}_s - v_s\rangle\, d\mathcal{W}_s.
\end{aligned}$$

We now take the supremum followed by the expectation, considering in the first line

$$\begin{aligned}
\mathbb{E}\left[\sup_{r\in[0,T]}\left(\|u_r\|^2 + \|\bar{u}_r\|^2\right) - 2\|u_0\|^2\right] &+ \mathbb{E}\left[\sup_{r\in[0,T]} 2\langle u_r, v_r\rangle + 2\langle u_0, v_0\rangle\right] \\
&\leq \|u_0\|^2 + o_\nu + \|u_0\|^2 - 2\|u_0\|^2 + 2\mathbb{E}\left[\sup_{r\in[0,T]}\|u_r\|\,\|v_r\|\right] + 2\|u_0\|\,\|v_0\|
\end{aligned}$$

having used (6.28) and (6.6). Through another application of (6.28) and employing (6.31), then this entire expression is bounded by o_ν. Overall

$$\begin{aligned}
\mathbb{E}\left(\sup_{r\in[0,T]}\|u_r - \bar{u}_r\|^2\right) &\leq o_\nu + 2\mathbb{E}\int_0^T |\langle u_s, \mathcal{L}_{u_s-\bar{u}_s}\bar{u}_s\rangle| ds \\
&+ 2\mathbb{E}\int_0^T |\langle \mathcal{L}_{u_s}u_s, v_s\rangle| ds + 2\mathbb{E}\int_0^T |\langle u_s, \partial_s v_s\rangle| ds \\
&+ 2\nu\mathbb{E}\int_0^T |\langle u_s, \bar{u}_s - v_s\rangle_1| ds + \nu\mathbb{E}\int_0^T \sum_{i=1}^\infty |\langle \mathcal{Q}_i u_s, \mathcal{Q}_i^*(\bar{u}_s - v_s)\rangle| ds \\
&+ 2\nu^{\frac{1}{2}}\mathbb{E}\left(\sup_{r\in[0,T]}\left|\int_0^r \langle \mathcal{G}u_s, \bar{u}_s - v_s\rangle\, d\mathcal{W}_s\right|\right) \\
&:= o_\nu + \sum_{k=1}^6 J_k
\end{aligned}$$

and we now treat the integrals individually. For J_1, in the first line we use cancellation of the nonlinear term from (4.15) and recall that $m > 1 + N/2$ is fixed from the start of this section:

$$\begin{aligned}
|\langle u_s, \mathcal{L}_{u_s-\bar{u}_s}\bar{u}_s\rangle| &= |\langle u_s, \mathcal{L}_{u_s-\bar{u}_s}\bar{u}_s\rangle - \langle \bar{u}_s, \mathcal{L}_{u_s-\bar{u}_s}\bar{u}_s\rangle| \\
&= |\langle u_s - \bar{u}_s, \mathcal{L}_{u_s-\bar{u}_s}\bar{u}_s\rangle| \\
&\leq \sum_{k=1}^{N}\left|\left\langle u_s - \bar{u}_s, (u_s-\bar{u}_s)^k\partial_k\bar{u}_s\right\rangle\right| \\
&\leq \sum_{k=1}^{N}\sum_{l=1}^{N}\left|\left\langle (u_s - \bar{u}_s)^l, (u_s-\bar{u}_s)^k\partial_k\bar{u}_s^l\right\rangle_{L^2(\mathscr{O};\mathbb{R})}\right| \\
&\leq \sum_{k=1}^{N}\sum_{l=1}^{N}\left\|(u_s - \bar{u}_s)^l(u_s-\bar{u}_s)^k\right\|_{L^1(\mathscr{O};\mathbb{R})}\left\|\partial_k\bar{u}_s^l\right\|_{L^\infty(\mathscr{O};\mathbb{R})} \\
&\leq \sum_{k=1}^{N}\sum_{l=1}^{N}\left\|(u_s - \bar{u}_s)^l\right\|_{L^2(\mathscr{O};\mathbb{R})}\left\|(u_s-\bar{u}_s)^k\right\|_{L^2(\mathscr{O};\mathbb{R})}\left\|\bar{u}_s^l\right\|_{W^{1,\infty}(\mathscr{O};\mathbb{R})} \\
&\leq c\left\|u_s - \bar{u}_s\right\|^2\left\|\bar{u}_s\right\|_{W^{m,2}}.
\end{aligned}$$

Just as we did with the initial condition, we will now freely assimilate finite norms of $\bar{u}$ into our constants. As $\bar{u} \in C\left([0,T]; W^{m,2}(\mathscr{O};\mathbb{R}^N)\right)$, then we bound the above by simply $c\left\|u_s - \bar{u}_s\right\|^2$ so

$$J_1 := 2\mathbb{E}\int_0^T |\langle u_s, \mathcal{L}_{u_s-\bar{u}_s}\bar{u}_s\rangle| ds \leq c\mathbb{E}\int_0^T \left\|u_s - \bar{u}_s\right\|^2 ds.$$

With (6.34) then we have

$$J_2 := 2\mathbb{E}\int_0^T |\langle \mathcal{L}_{u_s}u_s, v_s\rangle| ds \leq c\nu\mathbb{E}\int_0^T \left\|\nabla u_s\right\|_{\Gamma_{\tilde{c}\nu}}^2 ds = o_\nu$$

critically applying the assumption 4, and likewise

$$J_3 := 2\mathbb{E}\int_0^T |\langle u_s, \partial_s v_s\rangle| ds \leq 2\mathbb{E}\int_0^T \left\|u_s\right\|\left\|\partial_s v_s\right\| ds \leq c\nu\mathbb{E}\int_0^T \left\|u_s\right\| ds = o_\nu$$

using (6.32) then (6.28). Next we treat

$$J_4 := 2\nu\mathbb{E}\int_0^T |\langle u_s, \bar{u}_s - v_s\rangle_1| ds \leq 2\nu\mathbb{E}\int_0^T\sum_{k=1}^{N}|\langle \partial_k u_s, \partial_k\bar{u}_s\rangle| ds + 2\nu\mathbb{E}\int_0^T\sum_{k=1}^{N}|\langle \partial_k u_s, \partial_k v_s\rangle| ds$$

with the integrals individually:

$$\begin{aligned}
2\nu\mathbb{E}\int_0^T \sum_{k=1}^N |\langle \partial_k u_s, \partial_k \bar{u}_s\rangle| ds &\leq c\nu\mathbb{E}\int_0^T \|u_s\|_1 \|\bar{u}_s\|_{W^{1,2}} ds \\
&\leq c\nu\mathbb{E}\int_0^T \|u_s\|_1 ds \\
&\leq c\nu^{\frac{1}{2}}\left(\nu\mathbb{E}\int_0^T \|u_s\|_1^2 ds\right)^{\frac{1}{2}} \\
&= c\nu^{\frac{1}{2}} = o_\nu
\end{aligned}$$

having used (6.29). With the fact that v has support in $\Gamma_{\tilde{c}\nu}$ and property (6.34) then

$$\begin{aligned}
2\nu\mathbb{E}\int_0^T \sum_{k=1}^N |\langle \partial_k u_s, \partial_k v_s\rangle| ds &= 2\nu\mathbb{E}\int_0^T \sum_{k=1}^N |\langle \partial_k u_s, \partial_k v_s\rangle_{L^2(\Gamma_{\tilde{c}\nu};\mathbb{R}^N)}| ds \\
&\leq c\nu\mathbb{E}\int_0^T \|\nabla u_s\|_{\Gamma_{\tilde{c}\nu}} \|v_s\|_{W^{1,2}} ds \\
&\leq c\nu^{\frac{1}{2}}\mathbb{E}\int_0^T \|\nabla u_s\|_{\Gamma_{\tilde{c}\nu}} ds \\
&= c\left(\nu\mathbb{E}\int_0^T \|\nabla u_s\|_{\Gamma_{\tilde{c}\nu}}^2 ds\right)^{\frac{1}{2}} \\
&= o_\nu
\end{aligned}$$

using the assumption 4 once more, hence $J_4 = o_\nu$. We now move on to the noise terms, the first of which is dealt with near identically:

$$\begin{aligned}
J_5 &:= \nu\mathbb{E}\int_0^T \sum_{i=1}^\infty |\langle \mathcal{Q}_i u_s, \mathcal{Q}_i^*(\bar{u}_s - v_s)\rangle| ds \\
&\leq \nu\mathbb{E}\int_0^T \sum_{i=1}^\infty |\langle \mathcal{Q}_i u_s, \mathcal{Q}_i^*\bar{u}_s\rangle| ds + \nu\mathbb{E}\int_0^T \sum_{i=1}^\infty |\langle \mathcal{Q}_i u_s, \mathcal{Q}_i^* v_s\rangle| ds
\end{aligned}$$

having used the linearity of $\mathcal{Q}_i^*$. Then from (6.7) and (6.15),

$$\nu\mathbb{E}\int_0^T \sum_{i=1}^\infty |\langle \mathcal{Q}_i u_s, \mathcal{Q}_i^*\bar{u}_s\rangle| ds \leq c\nu\mathbb{E}\int_0^T (1+\|u_s\|_1)\|\bar{u}_s\|_{W^{1,2}} ds \leq c\nu^{\frac{1}{2}}\left(\nu\mathbb{E}\int_0^T (1+\|u_s\|_1)^2 ds\right)^{\frac{1}{2}}$$

which we write as o_ν using (6.29) again. For the second integral we must use the assumed structure $\mathcal{Q}_i^* = \mathcal{A}_i + \hat{\mathcal{A}}_i$ with $\mathcal{A}_i$ preserving the support on $\Gamma_{\tilde{c}\nu}$,

$$\begin{aligned}
\nu\mathbb{E}\int_0^T \sum_{i=1}^\infty |\langle \mathcal{Q}_i u_s, \mathcal{Q}_i^* v_s\rangle| ds &= \nu\mathbb{E}\int_0^T \sum_{i=1}^\infty |\langle \mathcal{Q}_i u_s, \mathcal{A}_i v_s\rangle_{L^2(\Gamma_{\tilde{c}\nu};\mathbb{R}^N)} + \left\langle \mathcal{Q}_i u_s, \hat{\mathcal{A}}_i v_s\right\rangle| ds \\
&\leq c\nu\mathbb{E}\int_0^T \left(1 + \|u_s\|_{W^{1,2}(\Gamma_{\tilde{c}\nu};\mathbb{R}^N)}\right)\|v_s\|_{W^{1,2}} + (1+\|u_s\|_1)\|v_s\|\, ds \\
&= c\nu\mathbb{E}\int_0^T \left(1 + \|u_s\|_{W^{1,2}(\Gamma_{\tilde{c}\nu};\mathbb{R}^N)}\right)\|v_s\|_{W^{1,2}}\, ds + o_\nu \\
&\leq c\nu^{\frac{1}{2}}\mathbb{E}\int_0^T 1 + \|u_s\| + \|\nabla u_s\|_{\Gamma_{\tilde{c}\nu}}\, ds + o_\nu \\
&= o_\nu + c\left(\nu\mathbb{E}\int_0^T \|\nabla u_s\|^2_{\Gamma_{\tilde{c}\nu}}\, ds\right)^{\frac{1}{2}} \\
&= o_\nu
\end{aligned}$$

with (6.28) and the assumption 4. Therefore $J_5 = o_\nu$. Only one term now remains:

$$\begin{aligned}
J_6 &:= 2\nu^{\frac{1}{2}}\mathbb{E}\left(\sup_{r\in[0,T]}\left|\int_0^r \langle \mathcal{G}u_s, \bar{u}_s - v_s\rangle\, d\mathcal{W}_s\right|\right) \\
&\leq c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T \sum_{i=1}^\infty \langle \mathcal{G}_i u_s, \bar{u}_s - v_s\rangle^2 ds\right)^{\frac{1}{2}} \\
&\leq c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T \sum_{i=1}^\infty \langle \mathcal{G}_i u_s, \bar{u}_s\rangle^2 ds\right)^{\frac{1}{2}} + c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T \sum_{i=1}^\infty \langle \mathcal{G}_i u_s, v_s\rangle^2 ds\right)^{\frac{1}{2}}
\end{aligned}$$

having used the BDG Inequality. Using (6.12) we see that

$$\begin{aligned}
c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T \sum_{i=1}^\infty \langle \mathcal{G}_i u_s, \bar{u}_s\rangle^2 ds\right)^{\frac{1}{2}} &\leq c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T \left[1 + \|u_s\|^2 + \|\bar{u}_s\|^p\right]\|\bar{u}_s\|^2_{W^{1,2}}\, ds\right)^{\frac{1}{2}} \\
&= c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T 1 + \|u_s\|^2\, ds\right)^{\frac{1}{2}}
\end{aligned}$$

which is just o_ν again from (6.28), and with (6.7) then

$$\begin{aligned}
c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T \sum_{i=1}^\infty \langle \mathcal{G}_i u_s, v_s\rangle^2 ds\right)^{\frac{1}{2}} &= c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T \sum_{i=1}^\infty \langle \mathcal{G}_i u_s, v_s\rangle^2_{L^2(\Gamma_{\tilde{c}\nu};\mathbb{R}^N)}\, ds\right)^{\frac{1}{2}} \\
&\leq c\nu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T \left[1 + \|u_s\|^2_{W^{1,2}(\Gamma_{\tilde{c}\nu};\mathbb{R}^N)}\right]\|v_s\|^2\, ds\right)^{\frac{1}{2}} \\
&\leq c\nu\mathbb{E}\left(\int_0^T 1 + \|u_s\|^2_{W^{1,2}(\Gamma_{\tilde{c}\nu};\mathbb{R}^N)}\, ds\right)^{\frac{1}{2}} \\
&\leq c\nu\mathbb{E}\left(\int_0^T 1 + \|u_s\|^2 + \|\nabla u_s\|^2_{\Gamma_{\tilde{c}\nu}}\, ds\right)^{\frac{1}{2}} \\
&= o_\nu.
\end{aligned}$$

Therefore

$$\mathbb{E}\left(\sup_{r\in[0,T]}\|u_r-\bar{u}_r\|^2\right)\leq o_v+\sum_{k=1}^{6}J_k\leq o_\nu+\int_0^T\mathbb{E}\left(\|u_s-\bar{u}_s\|^2\right)ds$$

so from the standard Grönwall Inequality, $\mathbb{E}\left(\sup_{r\in[0,T]}\|u_r-\bar{u}_r\|^2\right)\leq o_\nu$ which gives the result.

6.2.7 Optimal Noise Scaling

To further motivate the scaling rate of $\nu^{\frac{1}{2}}$ in the stochastic integral, we consider a different parameter μ in the equation

$$u_t=u_0-\int_0^t\mathcal{P}\mathcal{L}_{u_s}u_s\,ds-\nu\int_0^t Au_s\,ds+\frac{\mu}{2}\int_0^t\sum_{i=1}^{\infty}\mathcal{P}\mathcal{Q}_i^2u_sds-\mu^{\frac{1}{2}}\int_0^t\mathcal{P}\mathcal{G}u_sd\mathcal{W}_s \tag{6.36}$$

where we shall consider the limit $\mu\to 0$ with ν formally as in Subsection 6.2.4 by a sequence (μ_k) such that $\mu_k\to 0$ as $k\to\infty$. Corresponding notation $o_{\nu,\mu}$ is introduced to mean any constant dependent on ν and μ such that $\lim_{\nu,\mu\to 0}o_{\nu,\mu}=0$ for the limit taken jointly, which again is formally understood by $\lim_{k\to\infty}o_{\nu_k,\mu_k}=0$. We shall similarly use o_μ. Our assumptions on the noise must now be tweaked to accommodate this difference, and by replacing $\mathcal{G}$ with $(\frac{\mu}{\nu})^{\frac{1}{2}}\mathcal{G}$ then we see that the only necessary change of assumption is in (6.10) and (6.16) where we now impose that $\sum_{i=1}^{\infty}k_i\leq\frac{\nu}{\mu}$. The following lemma is deduced exactly as in Proposition 6.2.6.

Lemma 6.2.11. *There exists a martingale weak solution $\tilde{u}$ of the equation (6.36) which satisfies*

$$\tilde{\mathbb{E}}\left(\sup_{r\in[0,T]}\|\tilde{u}_r\|^2\right)\leq(1+o_\mu)\|u_0\|^2+o_\mu \tag{6.37}$$

and for every $t\in[0,T]$,

$$\tilde{\mathbb{E}}\left[\|\tilde{u}_t\|^2+\nu\int_0^t\|\tilde{u}_s\|_1^2\,ds\right]\leq(1+o_\mu)\|u_0\|^2+o_\mu. \tag{6.38}$$

The selection of martingale weak solutions is now as in Subsection 6.2.4, for the solution $\tilde{u}^k$ corresponding to parameters ν_k,μ_k. We can now state the main result in this context.

Proposition 6.2.12. *Suppose that $\mu\nu^{-\frac{1}{2}}=o_{\nu,\mu}$ and for any constant $\tilde{c}>0$,*

$$\left(1+\frac{\mu^2}{\nu^2}\right)\nu\mathbb{E}\int_0^T\|\nabla u_s\|^2_{\Gamma_{\tilde{c}\nu}}\,ds=o_{\nu,\mu}.$$

Then $\mathbb{E}\left(\sup_{r\in[0,T]}\|u_r-\bar{u}_r\|^2\right)=o_{\nu,\mu}$.

Remark 6.2.13. *We do not achieve an equivalence of conditions as in Theorem 6.2.8, as $\frac{\mu^2}{\nu}\mathbb{E}\int_0^T\|u_s\|_1^2\,ds=o_{\nu,\mu}$ may not be necessary for the weak convergence in item 2.*

Proof. Identically to Theorem 6.2.8, we obtain that

$$\begin{aligned}
\mathbb{E}\left(\sup_{r\in[0,T]}\|u_r-\bar{u}_r\|^2\right) &\leq o_{\nu,\mu}+2\mathbb{E}\int_0^T|\langle u_s,\mathcal{L}_{u_s-\bar{u}_s}\bar{u}_s\rangle|ds\\
&+2\mathbb{E}\int_0^T|\langle\mathcal{L}_{u_s}u_s,v_s\rangle|ds+2\mathbb{E}\int_0^T|\langle u_s,\partial_s v_s\rangle|ds\\
&+2\nu\mathbb{E}\int_0^T|\langle u_s,\bar{u}_s-v_s\rangle_1|ds+\mu\mathbb{E}\int_0^T\sum_{i=1}^\infty|\langle\mathcal{Q}_iu_s,\mathcal{Q}_i^*(\bar{u}_s-v_s)\rangle|ds\\
&+2\mu^{\frac{1}{2}}\mathbb{E}\left(\sup_{r\in[0,T]}\left|\int_0^r\langle\mathcal{G}u_s,\bar{u}_s-v_s\rangle\,d\mathcal{W}_s\right|\right)\\
&:=o_{\nu,\mu}+\sum_{k=1}^6 J_k.
\end{aligned}$$

The integrals J_1 to J_4 are controlled in the same way as the previous proof, so we begin with J_5, again writing

$$J_5\leq\mu\mathbb{E}\int_0^T\sum_{i=1}^\infty|\langle\mathcal{Q}_iu_s,\mathcal{Q}_i^*\bar{u}_s\rangle|ds+\mu\mathbb{E}\int_0^T\sum_{i=1}^\infty|\langle\mathcal{Q}_iu_s,\mathcal{Q}_i^*v_s\rangle|ds$$

and further

$$\mu\mathbb{E}\int_0^T\sum_{i=1}^\infty|\langle\mathcal{Q}_iu_s,\mathcal{Q}_i^*\bar{u}_s\rangle|ds\leq c\mu\mathbb{E}\int_0^T(1+\|u_s\|_1)\,ds\leq c\mu\nu^{-\frac{1}{2}}\left(\nu\mathbb{E}\int_0^T(1+\|u_s\|_1)^2\,ds\right)^{\frac{1}{2}}$$

so from (6.38) and the first assumption, this is just $o_{\nu,\mu}$. In addition

$$\begin{aligned}
\mu\mathbb{E}\int_0^T\sum_{i=1}^\infty|\langle\mathcal{Q}_iu_s,\mathcal{Q}_i^*v_s\rangle|ds&\leq c\mu\nu^{-\frac{1}{2}}\mathbb{E}\int_0^T 1+\|u_s\|+\|\nabla u_s\|_{\Gamma_{\tilde{c}\nu}}\,ds\\
&\leq c\mu\nu^{-\frac{1}{2}}+c\left(\frac{\mu^2}{\nu}\mathbb{E}\int_0^T\|\nabla u_s\|^2_{\Gamma_{\tilde{c}\nu}}\,ds\right)^{\frac{1}{2}}\\
&=o_{\nu,\mu}
\end{aligned}$$

from (6.37) and the assumptions. This demonstrates that $J_5=o_{\nu,\mu}$ so it only remains to consider J_6, which we again estimate with

$$J_6\leq c\mu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T\sum_{i=1}^\infty\langle\mathcal{G}_iu_s,\bar{u}_s\rangle^2\,ds\right)^{\frac{1}{2}}+c\mu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T\sum_{i=1}^\infty\langle\mathcal{G}_iu_s,v_s\rangle^2\,ds\right)^{\frac{1}{2}}.$$

The first term is controlled as before, and for the second

$$\begin{aligned}
c\mu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T\sum_{i=1}^\infty \langle\mathcal{G}_i u_s, v_s\rangle^2\,ds\right)^{\frac{1}{2}} &\leq c\mu^{\frac{1}{2}}\mathbb{E}\left(\int_0^T \left[1+\|u_s\|^2_{W^{1,2}(\Gamma_{\tilde{c}\nu};\mathbb{R}^N)}\right]\|v_s\|^2\,ds\right)^{\frac{1}{2}}\\
&\leq c\mu^{\frac{1}{2}}\left(\nu\mathbb{E}\int_0^T 1+\|u_s\|^2_{W^{1,2}(\Gamma_{\tilde{c}\nu};\mathbb{R}^N)}\,ds\right)^{\frac{1}{2}}\\
&\leq c\mu^{\frac{1}{2}}\left(\nu\mathbb{E}\int_0^T 1+\|u_s\|^2+\|\nabla u_s\|^2_{\Gamma_{\tilde{c}\nu}}\,ds\right)^{\frac{1}{2}}\\
&= o_{\nu,\mu}
\end{aligned}$$

which completes the proof. □

An interesting corollary of this result comes from considering $\mu = \nu^\alpha$ for different values of α.

Corollary 6.2.13.1. *Suppose that $\mu = \nu^\alpha$ in (6.36). If $\frac{1}{2} < \alpha < 1$ then the condition*

$$\nu^{2\left(\alpha-\frac{1}{2}\right)}\mathbb{E}\int_0^T\|\nabla u_s\|^2_{\Gamma_{\tilde{c}\nu}}\,ds = o_\nu$$

implies that $\mathbb{E}\left(\sup_{r\in[0,T]}\|u_r-\bar{u}_r\|^2\right) = o_\nu$. If $\alpha \geq 1$ then all four conditions of Theorem 6.2.8 are equivalent.

Proof. For $\mu = \nu^\alpha$ and $\frac{1}{2} < \alpha < 1$ it is sufficient to note that $\mu\nu^{-\frac{1}{2}} = \nu^{\alpha-\frac{1}{2}} = o_\nu$ as well as that

$$\frac{\mu^2}{\nu} = \frac{\nu^{2\alpha}}{\nu} = \nu^{2\left(\alpha-\frac{1}{2}\right)}$$

which is also greater than or equal to ν (recall $\nu < 1$), so the assumptions of Proposition 6.2.12 are satisfied which proves the result in this case. In the case $\alpha \geq 1$ then we have $\nu \geq \nu^{2\left(\alpha-\frac{1}{2}\right)}$ so this implication again holds assuming just 4 in Theorem 6.2.8. The proof of the remaining implications is unchanged from Subsection 6.2.5. □

Remark 6.2.14. *For $\frac{1}{2} < \alpha < 1$ we cannot show the four conditions are equivalent as in order to demonstrate that 2 $\implies$ 3 in the same way we would need an estimate*

$$\nu^{2\left(\alpha-\frac{1}{2}\right)}\mathbb{E}\int_0^T\|\nabla u_s\|_1^2\,ds \leq C$$

for some C independent of ν, which we cannot achieve.

This corollary highlights the critical cases of $\alpha = \frac{1}{2}$ and $\alpha = 1$.

Case $\alpha = \frac{1}{2}$: We lose the key property that $\nu^{\alpha-\frac{1}{2}} = o_\nu$ which was used in control of J_5, so even the limiting condition that

$$\mathbb{E}\int_0^T\|\nabla u_s\|^2_{\Gamma_{\tilde{c}\nu}}\,ds = o_\nu$$

is insufficient for the result. Inspecting the proof though, one can estimate J_5 and hence show the result with the stronger assumptions

$$\mathbb{E}\int_0^T \|\nabla u_s\|^2_{W^{1,2}(\Gamma_{\tilde{c}\nu};\mathbb{R}^N)}\, ds = o_\nu \qquad \text{and} \qquad \nu\mathbb{E}\int_0^T \|u_s\|_1^2\, ds = o_\nu$$

noting that the second assumption is item 3 of Theorem 6.2.8.

Case $\alpha = 1$: This is the smallest α in which we can show the equivalence of all conditions as well as the smallest α in which we do not need to impose a stronger condition than Kato's (4) to determine the convergence. The scaling rate of $\mu = \nu$ is thus considered optimal.

6.3 Navier Boundary

Here we return to SALT noise. Let us refer to the solution provided by Theorem 5.3.3 as the strong solution of problem (4.2), which is the SALT Navier-Stokes Equation under Navier boundary conditions. The SALT Euler Equation, introduced as (6.3), follows the same projection and conversion laid out in Subsection 4.4.1 to the equation

$$u_t = u_0 - \int_0^t \mathcal{P}\mathcal{L}_{u_s}u_s\, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty \mathcal{P}B_i^2 u_s ds - \int_0^t \mathcal{P}Bu_s d\mathcal{W}_s. \tag{6.39}$$

In Subsection 6.3.1 we state the main result for the existence of martingale weak solutions of (6.39) as an inviscid limit of strong solutions of problem (4.2) in the special case of the free boundary condition. In Subsection 6.3.2 we establish estimates on the strong solutions of (4.2) uniformly in viscosity. In Subsection 6.3.3 we show that the laws of the sequence of solutions with viscosities approaching zero is tight in the space of probability measures over $C([0,T];L^2_\sigma)$, using Lemma **??**. Passage to the SALT Euler Equation on a new probability space in Subsection 6.3.4 is then straightforwards.

6.3.1 The Main Result

We state the key definitions and results.

Definition 6.3.1. *Let $u_0 : \Omega \to \bar{W}^{1,2}_\sigma$ be $\mathcal{F}_0-$measurable. If there exists a filtered probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, a cylindrical Brownian Motion $\tilde{\mathcal{W}}$ over $\mathfrak{U}$ with respect to $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}\right)$, an $\tilde{\mathcal{F}}_0-$measurable $\tilde{u}_0 : \tilde{\Omega} \to \bar{W}^{1,2}_\sigma$ with the same law as u_0, and a progressively measurable process $\tilde{u}$ in $W^{1,2}_\sigma$ such that for $\tilde{\mathbb{P}} - a.e.$ $\tilde{\omega}$, $\tilde{u}.(\omega) \in L^\infty\left([0,T];\bar{W}^{1,2}_\sigma\right) \cap C\left([0,T];L^2_\sigma\right)$ and*

$$\langle \tilde{u}_t, \phi\rangle = \langle \tilde{u}_0, \phi\rangle - \int_0^t \langle \mathcal{L}_{\tilde{u}_s}\tilde{u}_s, \phi\rangle\, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty \langle B_i\tilde{u}_s, B_i^*\phi\rangle\, ds - \int_0^t \langle B\tilde{u}_s, \phi\rangle\, d\tilde{\mathcal{W}}_s \tag{6.40}$$

holds for every $\phi \in \bar{W}^{1,2}_\sigma$ $\tilde{\mathbb{P}} - a.s.$ in $\mathbb{R}$ for all $t \in [0,T]$, then $\tilde{u}$ is said to be a martingale weak solution of the equation (6.39).

Theorem 6.3.2. *Let $\kappa \geq 0$ everywhere on $\partial\mathscr{O}$. For any given $\mathcal{F}_0-$measurable $u_0 \in L^2\left(\Omega; \bar{W}^{1,2}_\sigma\right)$, there exists a martingale weak solution $\tilde{u}$ of the equation (6.39). Moreover for any sequence of viscosities (ν^k) monotonically decreasing to zero, with corresponding strong solutions of the problem (4.2) for $\alpha := 2\kappa$ given by $(\tilde{u}^k)$, then as $k \to \infty$, there is a subsequence indexed by n_k such that $\tilde{u}^{n_k} \longrightarrow \tilde{u}$ $\mathbb{P}-a.s.$ in $C\left([0,T]; L^2_\sigma\right)$.*

Remark 6.3.3. *For each viscosity ν^k and for any given probability space and cylindrical Brownian Motion, there is a strong solution of the problem (4.2) (for $\alpha := 2\kappa$) given by Theorem 5.3.3. In the statement of Theorem 6.3.2, $\tilde{u}^k$ is taken to be the strong solution relative to the probability space and cylindrical Brownian Motion specified in the definition of the weak martingale solution $\tilde{u}$. Thus, $\mathbb{P}$ in Theorem 6.3.2 is taken as the $\tilde{\mathbb{P}}$ from Definition 6.3.1.*

Throughout this section we assume the conditions required for Theorem 6.3.2, namely $u_0 \in L^2\left(\Omega; \bar{W}^{1,2}_\sigma\right)$ and $\kappa \geq 0$ on $\partial\mathscr{O}$. This ensures that $2\kappa \geq \kappa$ so Theorem 5.3.3 applies for $\alpha := 2\kappa$.

6.3.2 Estimates Independent of the Viscosity

For any given ν we know that there exists a unique strong solution u of the equation (4.34) given by Theorem 5.3.3 for $\alpha := 2\kappa$. In this subsection we look to obtain an estimate on u independent of ν, to facilitate the required regularity in the limit as $\nu \to 0$. To achieve this we shall consider the vorticity of u, and denote again $w := \text{curl}u$. Recall from Lemma 4.4.1 that $\text{curl}(B_i f) = \mathcal{L}_{\xi_i}(\text{curl}f)$. We denote $\mathcal{L}_{\xi_i}$ by the simple $\mathcal{L}_i$.

Lemma 6.3.4. *The process w is progressively measurable in $W^{1,2}_0(\mathscr{O};\mathbb{R})$ and such that for $\mathbb{P}-a.e.$ ω, $w_\cdot(\omega) \in C\left([0,T]; L^2(\mathscr{O};\mathbb{R})\right) \cap L^2\left([0,T]; W^{1,2}_0(\mathscr{O};\mathbb{R})\right)$. Moreover w satisfies the identity*

$$\begin{aligned}\langle w_t, \phi\rangle_{L^2(\mathscr{O};\mathbb{R})} &= \langle w_0, \phi\rangle_{L^2(\mathscr{O};\mathbb{R})} - \int_0^t \langle \mathcal{L}_{u_s} w_s, \phi\rangle_{L^2(\mathscr{O};\mathbb{R})}\,ds - \nu\int_0^t \langle w_s, \phi\rangle_{W^{1,2}_0(\mathscr{O};\mathbb{R})}\,ds\\ &\quad - \frac{1}{2}\int_0^t \sum_{i=1}^\infty \langle \mathcal{L}_i w_s, \mathcal{L}_i\phi\rangle_{L^2(\mathscr{O};\mathbb{R})}\,ds - \sum_{i=1}^\infty \int_0^t \langle \mathcal{L}_i w_s, \phi\rangle_{L^2(\mathscr{O};\mathbb{R})}\,dW^i_s \qquad (6.41)\end{aligned}$$

for every $\phi \in W^{1,2}_0(\mathscr{O};\mathbb{R})$, $\mathbb{P}-a.s.$ in $\mathbb{R}$ for all $t \in [0,T]$.

Proof. We first comment on the regularity of w, noting that the curl is a continuous linear operator from $W^{2,2}(\mathscr{O};\mathbb{R}^2) \to W^{1,2}(\mathscr{O};\mathbb{R})$ and $W^{1,2}(\mathscr{O};\mathbb{R}^2) \to L^2(\mathscr{O};\mathbb{R})$. This justifies the progressive measurability and pathwise regularity for the general $W^{1,2}(\mathscr{O};\mathbb{R})$ space, whilst the fact that w belongs to $W^{1,2}_0(\mathscr{O};\mathbb{R})$ is a direct consequence of Corollary 4.2.20.1 in the choice of $\alpha = 2\kappa$. It only remains to show the identity (6.41), which we look to do by first considering $\phi \in C^\infty_0(\mathscr{O};\mathbb{R})$ and defining $\psi \in C^\infty_0(\mathscr{O};\mathbb{R}^2)$ by $\psi^1 := \partial_2\phi$, $\psi^2 := -\partial_1\phi$. The similarity with the curl is swiftly noted, and the idea is to take the inner product of the identity satisfied by u with ψ and pass over the 'curl' through integration by parts. Proceeding with this, u satisfies the identity

$$\langle u_t, \psi \rangle = \langle u_0, \psi \rangle - \int_0^t \langle \mathcal{P}\mathcal{L}_{u_s} u_s, \psi \rangle \, ds - \nu \int_0^t \langle A u_s, \psi \rangle \, ds + \frac{1}{2} \int_0^t \sum_{i=1}^{\infty} \langle \mathcal{P} B_i^2 u_s, \psi \rangle \, ds - \sum_{i=1}^{\infty} \int_0^t \langle \mathcal{P} B_i u_s, \psi \rangle \, dW_s^i$$

and we look to simplify each term down to its respective one in (6.41). Expanding the first inner product into its components,

$$\langle u_t, \psi \rangle = \left\langle u_t^1, \partial_2 \phi \right\rangle_{L^2(\mathscr{O};\mathbb{R})} - \left\langle u_t^2, \partial_1 \phi \right\rangle_{L^2(\mathscr{O};\mathbb{R})} = - \left\langle \partial_2 u_t^1, \phi \right\rangle_{L^2(\mathscr{O};\mathbb{R})} + \left\langle \partial_1 u_t^2, \phi \right\rangle_{L^2(\mathscr{O};\mathbb{R})} = \langle w_t, \phi \rangle_{L^2(\mathscr{O};\mathbb{R})}$$

having used that ϕ is compactly supported. The same integration by parts process will be used for the remaining terms. There is nothing to note for the initial condition, so moving on to the nonlinear term, we recall the continuity of $\mathcal{L} : W^{2,2}(\mathscr{O};\mathbb{R}^2) \to W^{1,2}(\mathscr{O};\mathbb{R}^2)$, so that $\mathcal{P}\mathcal{L}_{u_s} u_s \in W^{1,2}(\mathscr{O};\mathbb{R}^2)$ hence $\mathrm{curl}\,(\mathcal{P}\mathcal{L}_{u_s} u_s) \in L^2(\mathscr{O};\mathbb{R})$. We use Lemma 4.2.19 to remove $\mathcal{P}$ from consideration, from which the well established result that $\mathrm{curl}\,(\mathcal{L}_{u_s} u_s) = \mathcal{L}_{u_s} w_s$ (see e.g. [129] Subsection 12.4) provides the required expression. We have thus far achieved that

$$\langle w_t, \phi \rangle_{L^2(\mathscr{O};\mathbb{R})} = \langle w_0, \phi \rangle_{L^2(\mathscr{O};\mathbb{R})} - \int_0^t \langle \mathcal{L}_{u_s} w_s, \phi \rangle_{L^2(\mathscr{O};\mathbb{R})} \, ds - \nu \int_0^t \langle A u_s, \psi \rangle \, ds + \frac{1}{2} \int_0^t \sum_{i=1}^{\infty} \langle \mathcal{P} B_i^2 u_s, \psi \rangle \, ds - \sum_{i=1}^{\infty} \int_0^t \langle \mathcal{P} B_i u_s, \psi \rangle \, dW_s^i.$$

In the stochastic integral, Lemmas 4.2.19 and 4.4.1 verify that $\mathrm{curl}(\mathcal{P} B_i u_s) = \mathcal{L}_i w_s$. For the remaining terms we do not have sufficient regularity to conduct the integration by parts. To account for this we take an approximation of u_s by functions (f^n), $f^n \in W^{3,2}(\mathscr{O};\mathbb{R}^2)$ and $f^n \longrightarrow u_s$ in $W^{2,2}(\mathscr{O};\mathbb{R}^2)$. Firstly for the Stokes Operator, we have that

$$\langle A f^n, \psi \rangle = \langle \mathrm{curl}\,(A f^n)\,, \phi \rangle_{L^2(\mathscr{O};\mathbb{R})} = - \langle \mathrm{curl}\,(\Delta f^n)\,, \phi \rangle_{L^2(\mathscr{O};\mathbb{R})} = - \langle \Delta \mathrm{curl}\,(f^n)\,, \phi \rangle_{L^2(\mathscr{O};\mathbb{R})} = \langle \mathrm{curl}\,(f^n)\,, \phi \rangle_{W_0^{1,2}(\mathscr{O};\mathbb{R})}$$

hence

$$\langle A u_s, \psi \rangle = \lim_{n\to\infty} \langle A f^n, \psi \rangle = \lim_{n\to\infty} \langle \mathrm{curl}\,(f^n)\,, \phi \rangle_{W_0^{1,2}(\mathscr{O};\mathbb{R})} = \langle w_s, \phi \rangle_{W_0^{1,2}(\mathscr{O};\mathbb{R})}\,.$$

Similarly we have that $\mathrm{curl}\,\left(\mathcal{P} B_i^2 f^n\right) = \mathcal{L}_i \left[\mathrm{curl}\,(B_i f^n)\right] = \mathcal{L}_i^2 \left[\mathrm{curl}(f^n)\right]$. Thus

$$\left\langle \mathcal{P} B_i^2 f^n, \psi \right\rangle = \left\langle \mathcal{L}_i^2[\mathrm{curl}(f^n)], \phi \right\rangle_{L^2(\mathscr{O};\mathbb{R})} = - \left\langle \mathcal{L}_i[\mathrm{curl}(f^n)], \mathcal{L}_i \phi \right\rangle_{L^2(\mathscr{O};\mathbb{R})}$$

having used that the $L^2(\mathscr{O};\mathbb{R})$ adjoint of the densely defined $\mathcal{L}_i$ is $-\mathcal{L}_i$ (which follows from Lemma 4.3.2 in the one dimensional case too). Via the same approximation just used, we conclude that the identity (6.41) holds for $\phi \in C_0^\infty(\mathscr{O};\mathbb{R})$. We use the density of this space in $W_0^{1,2}(\mathscr{O};\mathbb{R})$ to finish the proof. □

We shall use this identity to consider estimates on the vorticity, which then translate to estimates on the velocity.

Proposition 6.3.5. *For any $t \in [0,T]$, w satisfies the identity*

$$\|w_t\|^2_{L^2(\mathscr{O};\mathbb{R})} + 2\nu \int_0^t \|w_s\|^2_{W^{1,2}_0(\mathscr{O};\mathbb{R})} ds = \|w_0\|^2_{L^2(\mathscr{O};\mathbb{R})}$$

$\mathbb{P} - a.s.$.

Proof. Through the same equivalence discussed in Subsection 6.2.3, the weak solution w can be viewed as satisfying an identity in $\left(W^{1,2}_0(\mathscr{O};\mathbb{R})\right)^*$ through which Proposition A.2.7 can be applied. That is w satisfies, for every $t \in [0,T]$, the identity

$$w_t = w_0 - \int_0^t \mathcal{L}_{u_s} w_s \, ds + \nu \int_0^t \Delta w_s \, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty \mathcal{L}_i^2 w_s ds - \sum_{i=1}^\infty \int_0^t \mathcal{L}_i w_s dW^i_s$$

$\mathbb{P} - a.s.$ in $\left(W^{1,2}_0(\mathscr{O};\mathbb{R})\right)^*$, with sufficient regularity to apply Proposition A.2.7 for the spaces $\mathcal{H}_1 := W^{1,2}_0(\mathscr{O};\mathbb{R})$, $\mathcal{H}_2 := L^2(\mathscr{O};\mathbb{R})$ and $\mathcal{H}_3 := \left(W^{1,2}_0(\mathscr{O};\mathbb{R})\right)^*$. Therefore w satisfies

$$\begin{aligned} \|w_t\|^2_{L^2(\mathscr{O};\mathbb{R})} = \|w_0\|^2_{L^2(\mathscr{O};\mathbb{R})} &- 2\int_0^t \langle \mathcal{L}_{u_s} w_s, w_s\rangle_{\left(W^{1,2}_0(\mathscr{O};\mathbb{R})\right)^* \times W^{1,2}_0(\mathscr{O};\mathbb{R})} ds \\ &+ 2\nu \int_0^t \langle \Delta w_s, w_s\rangle_{\left(W^{1,2}_0(\mathscr{O};\mathbb{R})\right)^* \times W^{1,2}_0(\mathscr{O};\mathbb{R})} ds \\ &+ \int_0^t \sum_{i=1}^\infty \left(\left\langle \mathcal{L}_i^2 w_s, w_s\right\rangle_{\left(W^{1,2}_0(\mathscr{O};\mathbb{R})\right)^* \times W^{1,2}_0(\mathscr{O};\mathbb{R})} + \|\mathcal{L}_i w_s\|^2_{L^2(\mathscr{O};\mathbb{R})}\right) ds \\ &- 2\sum_{i=1}^\infty \int_0^t \langle \mathcal{L}_i w_s, w_s\rangle_{L^2(\mathscr{O};\mathbb{R})} dW^i_s. \end{aligned} \tag{6.42}$$

By definition

$$\langle \mathcal{L}_{u_s} w_s, w_s\rangle_{\left(W^{1,2}_0(\mathscr{O};\mathbb{R})\right)^* \times W^{1,2}_0(\mathscr{O};\mathbb{R})} = \langle \mathcal{L}_{u_s} w_s, w_s\rangle_{L^{4/3}(\mathscr{O};\mathbb{R}) \times L^4(\mathscr{O};\mathbb{R})}$$

which is simply zero again from Lemma 4.3.2, whilst

$$\langle \Delta w_s, w_s\rangle_{\left(W^{1,2}_0(\mathscr{O};\mathbb{R})\right)^* \times W^{1,2}_0(\mathscr{O};\mathbb{R})} = -\|w_s\|^2_{W^{1,2}_0(\mathscr{O};\mathbb{R})}$$

is also directly from the definition. Similarly

$$\left\langle \mathcal{L}_i^2 w_s, w_s\right\rangle_{\left(W^{1,2}_0(\mathscr{O};\mathbb{R})\right)^* \times W^{1,2}_0(\mathscr{O};\mathbb{R})} = -\langle \mathcal{L}_i w_s, \mathcal{L}_i w_s\rangle_{L^2(\mathscr{O};\mathbb{R})} = -\|\mathcal{L}_i w_s\|^2_{L^2(\mathscr{O};\mathbb{R})}$$

hence

$$\left\langle \mathcal{L}_i^2 w_s, w_s\right\rangle_{\left(W^{1,2}_0(\mathscr{O};\mathbb{R})\right)^* \times W^{1,2}_0(\mathscr{O};\mathbb{R})} + \|\mathcal{L}_i w_s\|^2_{L^2(\mathscr{O};\mathbb{R})} = 0$$

whilst we also have that
$$\langle \mathcal{L}_i w_s, w_s \rangle_{L^2(\mathscr{O};\mathbb{R})} = 0$$
again from Lemma 4.3.2. Putting this all back into (6.42) gives the result. $\square$

Corollary 6.3.5.1. *There exists a constant C independent of ν, ω, such that*

$$\sup_{r\in[0,T]} \|u_r\|_1^2 \leq C \|w_0\|_{L^2(\mathscr{O};\mathbb{R})}^2$$

$\mathbb{P}-a.s.$

Proof. This now follows immediately from the classical control on the gradient of functions in $\bar{W}^{1,2}_{\sigma}$ by their curl (see for example [81] Lemma 3.1, a result attributed to Yudovich). $\square$

6.3.3 Tightness

We now consider a sequence of viscosities (ν^k) monotonically decreasing to zero as in Theorem 6.3.2, and corresponding strong solutions of (4.2) for $\alpha := 2\kappa$ given by u^k. Without loss of generality we assume $\nu^k < 1$ for all k.

Proposition 6.3.6. *For any sequence of stopping times (γ_k) with $\gamma_k : \Omega \to [0,T]$, and any $\phi \in \bar{W}^{1,2}_{\sigma}$,*
$$\lim_{\delta\to 0^+} \sup_{k\in\mathbb{N}} \mathbb{E}\left(\left|\left\langle u^k_{(\gamma_k+\delta)\wedge T} - u^k_{\gamma_k}, \phi\right\rangle\right|\right) = 0.$$

Proof. We use the identity satisfied by the inner product, which is

$$\begin{aligned}\left\langle u^k_{(\gamma_k+\delta)\wedge T} - u^k_{\gamma_k}, \phi\right\rangle = &-\int_{\gamma_k}^{(\gamma_k+\delta)\wedge T} \left\langle \mathcal{L}_{u^k_s} u^k_s, \phi\right\rangle ds - \nu^k \int_{\gamma_k}^{(\gamma_k+\delta)\wedge T} \left\langle A u^k_s, \phi\right\rangle ds \\ &+ \frac{1}{2}\int_{\gamma_k}^{(\gamma_k+\delta)\wedge T} \sum_{i=1}^{\infty} \left\langle B_i^2 u^k_s, \phi\right\rangle ds - \sum_{i=1}^{\infty}\int_{\gamma_k}^{(\gamma_k+\delta)\wedge T} \left\langle B_i u^k_s, \phi\right\rangle dW^i_s\end{aligned}$$

having immediately passed $\mathcal{P}$ over to ϕ. Inspecting each term, as seen in (5.1) we have that

$$\left|\left\langle \mathcal{L}_{u^k_s} u^k_s, \phi\right\rangle\right| \leq c \left\|u^k_s\right\| \left\|u^k_s\right\|_1 \|\phi\|_1 \leq c \|w_0\|^2_{L^2(\mathscr{O};\mathbb{R})},$$

where we have used Corollary 6.3.5.1 and will continue to absorb norms of ϕ into our constant. Similarly we have that $\left\langle A u^k_s, \phi\right\rangle = \left\langle u^k_s, \phi\right\rangle_H$, recalling $\langle\cdot,\cdot\rangle_H$ from (5.3) and Lemma 5.3.4, and as $\nu^k < 1$,
$$\nu^k \left|\left\langle A u^k_s, \phi\right\rangle\right| \leq c \left\|u^k_s\right\|_1 \|\phi\|_1 \leq c \|w_0\|_{L^2(\mathscr{O};\mathbb{R})}$$
and using that $\left\langle B_i^2 u^k_s, \phi\right\rangle = \left\langle B_i u^k_s, B_i^* \phi\right\rangle$ we achieve the same bound there. Using that $\|w_0\|_{L^2(\mathscr{O};\mathbb{R})} \leq 1 + \|w_0\|^2_{L^2(\mathscr{O};\mathbb{R})}$, we have so far shown that

$$\left|\left\langle u^k_{(\gamma_k+\delta)\wedge T} - u^k_{\gamma_k}, \phi\right\rangle\right| \leq \int_{\gamma_k}^{(\gamma_k+\delta)\wedge T} c(1 + \|w_0\|^2_{L^2(\mathscr{O};\mathbb{R})}) ds + \left|\sum_{i=1}^{\infty}\int_{\gamma_k}^{(\gamma_k+\delta)\wedge T} \left\langle B_i u^k_s, \phi\right\rangle dW^i_s\right|.$$

Taking expectation and applying the Burkholder-Davis-Gundy Inequality, observe firstly that

$$\begin{aligned}\mathbb{E}\left(\left|\sum_{i=1}^{\infty}\int_{\gamma_k}^{(\gamma_k+\delta)\wedge T}\left\langle B_iu_s^k,\phi\right\rangle dW_s^i\right|\right) &\leq c\mathbb{E}\left(\int_{\gamma_k}^{(\gamma_k+\delta)\wedge T}\sum_{i=1}^{\infty}\left\langle B_iu_s^k,\phi\right\rangle^2 ds\right)^{\frac{1}{2}}\\ &\leq c\mathbb{E}\left(\int_{\gamma_k}^{(\gamma_k+\delta)\wedge T}\|w_0\|_{L^2(\mathscr{O};\mathbb{R})}^2\,ds\right)^{\frac{1}{2}}\\ &\leq c\left[\mathbb{E}\int_{\gamma_k}^{(\gamma_k+\delta)\wedge T}\|w_0\|_{L^2(\mathscr{O};\mathbb{R})}^2\,ds\right]^{\frac{1}{2}}.\end{aligned}$$

Using δ as an upper bound for the length of the integrals, we have that

$$\mathbb{E}\left(\left|\left\langle u_{(\gamma_k+\delta)\wedge T}^k-u_{\gamma_k}^k,\phi\right\rangle\right|\right)\leq c\delta\mathbb{E}\left(\|w_0\|_{L^2(\mathscr{O};\mathbb{R})}^2\right)+c\left[\delta\mathbb{E}\left(\|w_0\|_{L^2(\mathscr{O};\mathbb{R})}^2\right)\right]^{\frac{1}{2}}$$

where the result now follows from the fact that $u_0\in L^2\left(\Omega;\bar{W}_\sigma^{1,2}\right)$ implies $w_0\in L^2\left(\Omega;L^2(\mathscr{O};\mathbb{R})\right)$. □

Corollary 6.3.6.1. *The sequence of the laws of (u^k) is tight in the space of probability measures over $C\left([0,T];L^2_\sigma\right)$.*

Proof. We apply the second statement of Lemma 2.3.3 in the case of $\mathcal{Y}:=\bar{W}_\sigma^{1,2}$ and $\mathcal{H}:=L^2_\sigma$. Condition (2.30) is satisfied from Corollary 6.3.5.1. As for (2.29), we use Chebyshev's Inequality to see that

$$\mathbb{P}\left(\left\{\omega\in\Omega:\left|\left\langle u_{(\gamma_k+\delta)\wedge T}^k-u_{\gamma_k}^k,\phi\right\rangle\right|>\varepsilon\right\}\right)\leq\frac{1}{\varepsilon}\mathbb{E}\left(\left|\left\langle u_{(\gamma_k+\delta)\wedge T}^k-u_{\gamma_k}^k,\phi\right\rangle\right|\right)$$

from which the condition follows courtesy of Proposition 6.3.6. □

6.3.4 Passage to the Stochastic Euler Equation

Using the uniform boundedness from Corollary 6.3.5.1 and the tightness due to Corollary 6.3.6.1, as well as the progressive measurability of the processes (u^k) in $\bar{W}_\alpha^{2,2}$, we obtain the following theorem as seen equivalently in Subsection 3.2.5. One requires taking a subsequence, as in the statement of Theorem 6.3.2, though for simplicity we index it again by k here.

Theorem 6.3.7. *There exists a filtered probability space $\left(\tilde{\Omega},\tilde{\mathcal{F}},(\tilde{\mathcal{F}}_t),\tilde{\mathbb{P}}\right)$, a cylindrical Brownian Motion $\tilde{\mathcal{W}}$ over $\mathfrak{U}$ with respect to $\left(\tilde{\Omega},\tilde{\mathcal{F}},(\tilde{\mathcal{F}}_t),\tilde{\mathbb{P}}\right)$, a random variable $\tilde{u}_0:\tilde{\Omega}\to\bar{W}_\sigma^{1,2}$, a sequence of processes $(\tilde{u}^k)$, $\tilde{u}^k:\tilde{\Omega}\times[0,T]\to\bar{W}_\alpha^{2,2}$ is progressively measurable and a progressively measurable process $\tilde{u}:\tilde{\Omega}\times[0,T]\to\bar{W}_\sigma^{1,2}$ such that:*

1. *$\tilde{u}_0$ has the same law as u_0;*
2. *For each $k\in\mathbb{N}$ and $t\in[0,T]$, $\tilde{u}^k$ satisfies the identity*

$$\tilde{u}_t^k=\tilde{u}_0-\int_0^t\mathcal{P}\mathcal{L}_{\tilde{u}_s^k}\tilde{u}_s^k\,ds-\nu^k\int_0^tA\tilde{u}_s^k\,ds+\frac{1}{2}\int_0^t\sum_{i=1}^{\infty}\mathcal{P}B_i^2\tilde{u}_s^kds-\int_0^t\mathcal{P}B\tilde{u}_s^kd\tilde{\mathcal{W}}_s$$

$\tilde{\mathbb{P}}-a.s.$ in L^2_σ;

3. *For $\tilde{\mathbb{P}}-a.e$ ω, $\tilde{u}^k(\omega) \to \tilde{u}(\omega)$ in $C\left([0,T];L^2_\sigma\right)$;*
4. *For $\tilde{\mathbb{P}}-a.e$ ω, $\tilde{u}(\omega) \in L^\infty\left([0,T];\bar{W}^{1,2}_\sigma\right)$.*

It is immediate that $\tilde{u}$ has the necessary regularity to be a martingale weak solution of the equation (6.39). Indeed each $\tilde{u}^k$ is a strong solution of problem (4.2) relative to this new probability space and Cylindrical Brownian Motion (thus is *the unique* strong solution), so in order to prove Theorem 6.3.2 it only remains to show that $\tilde{u}$ satisfies the identity (6.40).

Proposition 6.3.8. *The process $\tilde{u}$ satisfies the identity (6.40): that is*

$$\langle \tilde{u}_t, \phi\rangle = \langle \tilde{u}_0, \phi\rangle - \int_0^t \langle \mathcal{L}_{\tilde{u}_s}\tilde{u}_s, \phi\rangle\, ds + \frac{1}{2}\int_0^t \sum_{i=1}^\infty \langle B_i\tilde{u}_s, B_i^*\phi\rangle\, ds - \int_0^t \langle B\tilde{u}_s, \phi\rangle\, d\tilde{\mathcal{W}}_s$$

holds for every $\phi \in \bar{W}^{1,2}_\sigma$ $\tilde{\mathbb{P}}-a.s.$ in $\mathbb{R}$ for all $t\in[0,T]$.

Proof. We fix arbitrary $\psi \in \bar{W}^{2,2}_\alpha$ and $t\in[0,T]$, with the aim to show the result for ψ and pass to the limit for any $\phi\in\bar{W}^{1,2}_\sigma$. Taking the inner product with ψ in item 2 of Theorem 6.3.7, we have that

$$\begin{aligned}\left\langle \tilde{u}^k_t, \psi\right\rangle = \langle \tilde{u}_0,\psi\rangle - \int_0^t \left\langle \mathcal{L}_{\tilde{u}^k_s}\tilde{u}^k_s, \psi\right\rangle\, ds - \nu^k\int_0^t \left\langle A\tilde{u}^k_s,\psi\right\rangle ds \\ + \frac{1}{2}\int_0^t\sum_{i=1}^\infty \left\langle B_i^2\tilde{u}^k_s,\psi\right\rangle ds - \int_0^t \left\langle B\tilde{u}^k_s,\psi\right\rangle d\tilde{\mathcal{W}}_s\end{aligned}$$

holds $\tilde{\mathbb{P}}-a.s.$ in $\mathbb{R}$, so we look to show the convergence of each term in turn $\tilde{\mathbb{P}}-a.s.$. It is immediate that $\left\langle \tilde{u}^k_t,\psi\right\rangle$ converges to $\langle\tilde{u}_t,\psi\rangle$ from the convergence in $C\left([0,T];L^2_\sigma\right)$. For the nonlinear term, we have that

$$\begin{aligned}\left|\left\langle \mathcal{L}_{\tilde{u}^k_s}\tilde{u}^k_s,\psi\right\rangle - \langle\mathcal{L}_{\tilde{u}_s}\tilde{u}_s,\psi\rangle\right| &= \left|\left\langle \mathcal{L}_{\tilde{u}^k_s-\tilde{u}_s}\tilde{u}^k_s + \mathcal{L}_{\tilde{u}_s}(\tilde{u}^k_s-\tilde{u}_s),\psi\right\rangle\right| \\ &\leq \left|\left\langle \mathcal{L}_{\tilde{u}^k_s-\tilde{u}_s}\tilde{u}^k_s,\psi\right\rangle\right| + \left|\left\langle \mathcal{L}_{\tilde{u}_s}(\tilde{u}^k_s-\tilde{u}_s),\psi\right\rangle\right|.\end{aligned}$$

We treat both cases with Lemma 4.3.2 and Hölder's Inequality:

$$\begin{aligned}\left|\left\langle \mathcal{L}_{\tilde{u}^k_s-\tilde{u}_s}\tilde{u}^k_s,\psi\right\rangle\right| = \left|\left\langle \tilde{u}^k_s, \mathcal{L}_{\tilde{u}^k_s-\tilde{u}_s}\psi\right\rangle\right| \leq \left\|\tilde{u}^k_s\right\|_{L^6}\left\|\mathcal{L}_{\tilde{u}^k_s-\tilde{u}_s}\psi\right\|_{L^{6/5}} &\leq c\sum_{l=1}^2\left\|\tilde{u}^k_s\right\|_{L^6}\left\|\tilde{u}^k_s-\tilde{u}_s\right\|\left\|\partial_l\psi\right\|_{L^3} \\ &\leq c\left\|\tilde{u}^k_s\right\|_1\left\|\tilde{u}^k_s-\tilde{u}_s\right\|\left\|\psi\right\|_{W^{2,2}}\end{aligned}$$

whilst

$$\begin{aligned}\left|\left\langle \mathcal{L}_{\tilde{u}_s}(\tilde{u}^k_s-\tilde{u}_s),\psi\right\rangle\right| = \left|\left\langle \tilde{u}^k_s-\tilde{u}_s, \mathcal{L}_{\tilde{u}_s}\psi\right\rangle\right| &\leq c\sum_{l=1}^2\left\|\tilde{u}^k_s-\tilde{u}_s\right\|\left\|\tilde{u}_s\right\|_{L^4}\left\|\partial_l\psi\right\|_{L^4} \\ &\leq c\left\|\tilde{u}^k_s-\tilde{u}_s\right\|\left\|\tilde{u}_s\right\|_1\left\|\psi\right\|_{W^{2,2}}.\end{aligned}$$

We now comment on the regularity of $\tilde{u}^k$; the estimates shown in Subsection 6.3.2 were done so for the unique strong solution of the problem (4.2) on the original probability space, though are identically obtained for the unique strong solution on this new probability space. Furthermore from Corollary 6.3.5.1 we have that there exists a constant C such that for every k and $\tilde{\mathbb{P}}-a.e.$ $\omega \in \tilde{\Omega}$,

$$\sup_{r\in[0,T]} \left\|\tilde{u}^k_r(\omega)\right\|_1^2 \leq C \left\|\tilde{w}_0(\omega)\right\|^2_{L^2(\mathscr{O};\mathbb{R})}$$

where $\tilde{w}_0$ is curl$\tilde{u}_0$. With the same argument as Proposition 3.2.18, taking a weak* convergent subsequence in $L^\infty\left([0,T];\bar{W}^{1,2}_\sigma\right)$ and using Banach-Alaoglu, one also has that

$$\left\|\tilde{u}(\omega)\right\|^2_{L^\infty\left([0,T];\bar{W}^{1,2}_\sigma\right)} \leq C \left\|\tilde{w}_0(\omega)\right\|^2_{L^2(\mathscr{O};\mathbb{R})}.$$

As $\tilde{u}_0$ has the same law as u_0 in $\bar{W}^{1,2}_\sigma$ then $\tilde{u}_0 \in L^2\left(\tilde{\Omega};\bar{W}^{1,2}_\sigma\right)$ so $\tilde{w}_0 \in L^2\left(\tilde{\Omega};L^2(\mathscr{O};\mathbb{R})\right)$. One therefore obtains that

$$\begin{aligned}\left|\int_0^t \left\langle \mathcal{L}_{\tilde{u}^k_s}\tilde{u}^k_s,\psi\right\rangle ds - \int_0^t \left\langle \mathcal{L}_{\tilde{u}_s}\tilde{u}_s,\psi\right\rangle ds\right| &\leq c\int_0^t \left(\left\|\tilde{u}^k_s\right\|_1 + \left\|\tilde{u}_s\right\|_1\right)\left\|\tilde{u}^k_s - \tilde{u}_s\right\| \left\|\psi\right\|_{W^{2,2}} ds\\ &\leq c\left\|\tilde{w}_0\right\|_{L^2(\mathscr{O};\mathbb{R})}\left\|\psi\right\|_{W^{2,2}}\int_0^t \left\|\tilde{u}^k_s - \tilde{u}_s\right\| ds\end{aligned}$$

which we know converges to zero $\tilde{\mathbb{P}}-a.s.$, again using the convergence in $C\left([0,T];L^2_\sigma\right)$. For the Stokes Operator term we have that

$$\left|\left\langle A\tilde{u}^k_s,\psi\right\rangle\right| = \left|\left\langle \tilde{u}^k_s,\psi\right\rangle_H\right| \leq c\left\|\tilde{u}^k_s\right\|_1 \left\|\psi\right\|_1 \leq c\left\|\tilde{w}_0\right\|_{L^2(\mathscr{O};\mathbb{R})}\left\|\psi\right\|_1$$

thus

$$\left|\nu^k\int_0^t \left\langle A\tilde{u}^k_s,\psi\right\rangle ds\right| \leq ct\left\|\tilde{w}_0\right\|_{L^2(\mathscr{O};\mathbb{R})}\left\|\psi\right\|_1 \nu^k$$

which is convergent to zero $\tilde{\mathbb{P}}-a.s.$. Next observe that

$$\left|\left\langle B_i^2\tilde{u}^k_s,\psi\right\rangle - \left\langle B_i^2\tilde{u}_s,\psi\right\rangle\right| = \left|\left\langle B_i^2(\tilde{u}^k_s-\tilde{u}_s),\psi\right\rangle\right| = \left|\left\langle \tilde{u}^k_s-\tilde{u}_s,(B_i^*)^2\psi\right\rangle\right| \leq c_i\left\|\tilde{u}^k_s-\tilde{u}_s\right\|\left\|\psi\right\|_{W^{2,2}}$$

from which we again see the desired convergence in this term. In the stochastic integral we have to pass to a subsequence. Indeed we can only show the convergence in expectation; to this end we have that

$$\begin{aligned}\tilde{\mathbb{E}}\left(\left|\int_0^t \left\langle B\tilde{u}^k_s,\psi\right\rangle d\tilde{\mathcal{W}}_s - \int_0^t \left\langle B\tilde{u}_s,\psi\right\rangle d\tilde{\mathcal{W}}_s\right|\right) &\leq c\tilde{\mathbb{E}}\left(\int_0^t\sum_{i=1}^\infty \left\langle B_i(\tilde{u}^k_s-\tilde{u}_s),\psi\right\rangle^2 ds\right)^{\frac{1}{2}}\\ &\leq c\tilde{\mathbb{E}}\left(\int_0^t \left\|\tilde{u}^k_s-\tilde{u}_s\right\|^2\left\|\psi\right\|_1^2 ds\right)^{\frac{1}{2}}\\ &\leq c\left\|\psi\right\|_1\left[\tilde{\mathbb{E}}\left(\int_0^t\left\|\tilde{u}^k_s-\tilde{u}_s\right\|^2 ds\right)\right]^{\frac{1}{2}}.\end{aligned}$$

The fact that this quantity approaches zero as $k\to\infty$ now follows from the Dominated Convergence Theorem, as in Lemma 3.2.17. Thus we can take a subsequence, indexed by (k_j),

such that
$$\int_0^t \left\langle B\tilde{u}_s^{k_j}, \psi \right\rangle d\tilde{\mathcal{W}}_s \longrightarrow \int_0^t \langle B\tilde{u}_s, \psi \rangle \, d\tilde{\mathcal{W}}_s$$
$\tilde{\mathbb{P}} - a.s.$. Moreover, taking the $\tilde{\mathbb{P}} - a.s.$ convergence along this subsequence for all terms, we realise that $\tilde{u}$ satisfies

$$\langle \tilde{u}_t, \psi \rangle = \langle \tilde{u}_0, \psi \rangle - \int_0^t \langle \mathcal{L}_{\tilde{u}_s} \tilde{u}_s, \psi \rangle \, ds + \frac{1}{2} \int_0^t \sum_{i=1}^{\infty} \langle B_i \tilde{u}_s, B_i^* \psi \rangle \, ds - \int_0^t \langle B\tilde{u}_s, \psi \rangle \, d\tilde{\mathcal{W}}_s$$

$\tilde{\mathbb{P}} - a.s.$ in $\mathbb{R}$. It only remains to show this identity for $\phi \in \bar{W}^{1,2}_\sigma$, for which we choose a sequence of such ψ approximating ϕ in $\bar{W}^{1,2}_\sigma$, and the result soon follows by taking the corresponding limit in each term. This completes the proof. □

6.4 Conclusion

We briefly address some questions that have arisen around the results of this chapter.

- Could we prove a corresponding result to Theorem 6.2.8 for the zero viscosity convergence to the Stochastic Euler Equation?

Proof. Following the original proof of Kato in [79], our method relies on the a priori existence of the solution of the Euler Equation and its differentiability in time to be (nearly) used as a test function in the weak formulation of Navier-Stokes. Such regularity eludes us in the stochastic version, so a new approach would be required; the successful result of Wang and Zhao in [151] is for additive noise, which enables transformation into a random PDE where the same method can be applied. More thought would be needed on how to extend this result to the case of multiplicative noise. □

- Do we have uniqueness for the SALT Euler Equation?

The question of uniqueness of weak solutions of the Euler Equation is certainly of interest, where both uniqueness and non-uniqueness have been established dependent on the regularity imposed. The classical (positive) uniqueness result is for initial vorticities in $L^\infty(\mathscr{O};\mathbb{R})$ proven by Yudovich in [153], which he then extends to a wider class in [154]. Simply talking about the $L^\infty(\mathscr{O};\mathbb{R})$ case here, we note that on the torus this result was proven in [91]. Success hinges on showing an energy estimate of the vorticity in $L^\infty([0,T];L^\infty(\mathscr{O};\mathbb{R}))$, and in fact it was proven in [91] (22) that the $L^\infty(\mathscr{O};\mathbb{R})$ norm of the vorticity is constant in time via use of the Itô-Wentzel formula on the flow map. This was done for a sequence of strong solutions with more regular initial conditions approximating the desired $w_0 \in L^\infty(\mathscr{O};\mathbb{R})$, which is an approach we couldn't take here given that only weak solutions have been established. Any ideas for us would rely on showing additional uniform regularity at the level of the Galerkin Approximations.

- Could we prove the existence of strong solutions to the SALT Euler Equation?

In a similar vein, we would look to show additional uniform regularity for the approximating sequence of solutions, which now is the SALT Navier-Stokes Equations with free boundary condition and viscosity approaching zero. The regularity of each of these solutions again comes from the Galerkin Approximation. The difficulty lies in the fact that as we pass up to a higher order Sobolev Space, we are faced with the $\bar{\mathcal{P}}_n$ no longer being self-adjoint, which as established in Subsection 5.5 for the no-slip case, appears to block our estimates.

- For which noise does the free boundary inviscid limit still apply?

The choice to state and prove Theorem 6.3.2 for SALT noise specifically, instead of a general noise as in Theorem 6.2.8, comes down to several reasons. It is clear from our approach with the vorticity form that precise assumptions would be needed on $\mathrm{curl}\mathcal{Q}_i$ which would be somewhat cumbersome. Moreover, we want to highlight the power of SALT noise; we see this in Proposition 6.3.5, as we can achieve pathwise bounds given the complete cancellation in the stochastic integral. This does not exclude the result for additive and Nemystkii type noise though: we instead can only demonstrate Proposition 6.3.5 as an inequality in expectation, however the tightness and hence passage to the Stochastic Euler Equation is still valid. Additional technicalities are needed to introduce a pathwise bound into the system, which is done through the familiar first hitting times. Proposition 6.3.6 is then shown up until these first hitting times, which still gives the tightness result exactly as seen in Proposition 3.2.15. One would also have to take expectations in the proof of Proposition 6.3.8, but the result follows with only these minor technical adjustments. The scenario of a transport noise is much more delicate, as the correct Itô-Stratonovich corrector in the weak formulation of the Euler Equation is $-\left\langle \mathcal{P}\mathcal{L}_{\xi_i}u_s, \mathcal{L}_{\xi_i}u_s\right\rangle$ where this different structure retaining the Leray Projector (as opposed to for B_i) requires some different considerations which we do not detail here. The idea, however, is to again introduce $\mathcal{T}_{\xi_i}$ and use the good commutative properties of $\mathcal{P}$ and B_i.

Chapter 7

Conclusion

The final commentary of this thesis will be a review of its main contributions, as well as a survey of future work that could answer open questions by building upon the theory presented here. Firstly we list our results, in what I consider a descending order of significance:

- The existence and uniqueness of strong solutions to the Navier-Stokes Equations with Stochastic Lie Transport under general Navier boundary conditions, Theorem 5.3.3. This is the first strong existence result for a fluid equation perturbed by unbounded noise with a physical boundary, where the size of the noise is not restricted relative to viscosity. We have also proven local existence and uniqueness for the less challenging vorticity form on a bounded domain in 3D, Theorem 5.4.3.
- The existence of a martingale weak solution to the Euler Equation with Stochastic Lie Transport on a bounded convex domain, Theorem 3.2.7, as an inviscid limit of strong solutions of the Stochastic Navier-Stokes Equations with free boundary. This is the first existence result for an inviscid fluid equation perturbed by unbounded noise with a physical boundary.
- Criteria for the existence, and in some cases uniqueness, of a variety of solutions to a general SPDE with unbounded noise: Theorem 3.2.7 for martingale weak solutions, Theorem 3.3.5 for weak solutions, Theorem 3.4.5 for strong solutions, and Theorems A.3.13, A.3.17 for local strong solutions. These extend previously known results to the unbounded noise case, applicable to the Navier-Stokes Equations with Stochastic Lie Transport amongst many other equations and noise structures.
- A version of Kato's necessary and sufficient conditions to deduce the convergence of (martingale) weak solutions of the Stochastic Navier-Stokes Equation with no-slip boundary to the strong solution of the deterministic Euler Equation, Theorem 6.2.8. Our conditions apply for Lie Transport alongside other unbounded noise schemes, extending results known only for additive noise and in 2D. Novel conditions dependent on the scaling rate of the stochastic integral are given in Subsection 6.2.7, in particular Proposition 6.2.12.
- A rigorous conversion between the Itô and Stratonovich forms of an SPDE with unbounded noise under very general assumptions, Theorem 2.2.20. Whilst the converted form had

been known heuristically, a proper proof and perhaps an understanding of the cost of regularity appeared absent thus far. A conversion in some 'very weak' form does not pay the same cost, Theorem 2.2.27.

- The existence of solutions to the Navier-Stokes Equation with genuine Stratonovich Stochastic Lie Transport: Propositions 5.2.3, 5.3.2 for very weak solutions under the no-slip and Navier boundary conditions, Proposition 5.3.6 for weak solutions under Navier boundary conditions, and Theorem 5.4.2 for local strong solutions on $\mathbb{T}^3$. Aside from a recent result from Galeati and Luo in [52], Proposition 2.14, for only the existence of very weak solutions to some Stratonovich Stochastic fluid PDE, I am unaware of any concretely demonstrated existence to such equations.

- A Cauchy and weak equicontinuity criterion for the existence of a limiting stochastic process and stopping time, whereby the life of the stopping time can be characterised by the first hitting time of the limit process. By imposing a weak equicontinuity at all times instead of just the initial time, this extends the previous result from Glatt-Holtz and Ziane in [58], Lemma 5.1, in which the stopping time was only shown to be $\mathbb{P} - a.s.$ positive with no information on its lifespan.

- Existence (and uniqueness) results for weak solutions of the Navier-Stokes Equation with Stochastic Lie Transport: Theorems 5.2.1, 5.2.2 for the no-slip boundary and Theorem 5.3.1 for Navier boundary conditions. These results were certainly expected and do not pose particular difficulty over the known torus case, yet constitute a necessary addition to the sparse literature.

- The existence and uniqueness of solutions to an SPDE on a finite-dimensional Hilbert Space driven by a Cylindrical Brownian Motion, under Lipschitz and linear growth constraints, Theorems 2.3.1, 2.3.2. Of course for a finite-dimensional driving Brownian Motion the result is classical, whilst surely being expected in the infinite-dimensional case. Nevertheless I could not find the result, and having also seen papers work with a projected Cylindrical Brownian Motion to compensate, I believe it is a worthy addition.

- The existence and uniqueness of a maximal solution to an abstract SPDE when the existence and uniqueness of a local solution is known, Theorem 2.2.14. The result is most likely considered standard, however I haven't seen it presented in generality and we offer a direct and novel proof with Zorn's Lemma.

- An inequality on the Lie Transport operator used for energy estimates in arbitrarily high order Sobolev Spaces, Proposition 4.3.4. The result has been understood on the torus but perhaps not on a bounded domain, and I have not seen a complete proof for arbitrarily high orders.

- Explicit proofs of tightness criteria, Lemmas 2.3.3, **??**, combining classical results from Aldous ([2]) and Jakubowski ([77]). The method was used by Röckner, Shang and Zhang in [131] though not formatted into a material result.

- An energy equality for local solutions to SPDEs without assumed moment estimates and within a variational framework, Proposition A.2.7. It is precisely in allowing for local solutions and omitting the need for moment estimates that extends the result of Prevôt and Röckner, [127] Lemma 4.2.5. We also enjoy more freedom over the largest space U by not imposing it to be V^*, i.e. a Gelfand Triple.

- Various properties and constructions for stochastic calculus in infinite dimensions, Section A.1. In particular I shall mention a thorough treatment of the cross-variation between a Hilbert Space valued martingale and a real valued martingale, from Definition A.1.40 to Proposition A.1.43, necessary in the Itô-Stratonovich conversion and a concept that I haven't seen established.

To draw the thesis to a close, we list some open problems and future work:

- The existence and uniqueness of strong solutions to the Navier-Stokes Equation with Stochastic Lie Transport under the no-slip boundary condition, see Subsection 5.5.

- The existence and uniqueness of strong solutions to the Euler Equation with Stochastic Lie Transport on a bounded domain, see Subsection 6.4.

- A version of Kato's necessary and sufficient conditions to deduce the convergence of weak solutions of the Stochastic Navier-Stokes Equation with no-slip boundary to the martingale weak solution of the Stochastic Euler Equation, see Subsection 6.4.

- The existence of a martingale weak solution to the Euler Equation with Stochastic Lie Transport on a bounded (not necessarily convex) domain, as an inviscid limit of strong solutions of the Stochastic Navier-Stokes Equations with general Navier boundary conditions, see Section 6.1.

- The uniqueness or non-uniqueness of martingale weak solutions to the Euler Equation with Stochastic Lie Transport on a bounded domain dependent on the regularity of the initial condition, see Subsection 6.4.

- The convergence of strong solutions of the Navier-Stokes Equations with Stochastic Lie Transport under Navier boundary conditions to the weak solution with a no-slip boundary as α tends to infinity, see Subsection 5.5.

- Various analytical properties of the solutions considered in the thesis, such as long-time behaviour through stability, ergodicity and invariant measures. We can also explore these concepts in the inviscid limit.

- Potential regularisation-by-noise results pertaining to turbulent behaviour at the boundary. I shall briefly discuss an approach. The recent work of Lucio Galeati in [51] has established an enhanced dissipation mechanism from transport noise, used to show improved well-posedness properties in [47, 48] amongst others. The idea is that a well chosen Stratonovich transport noise produces an Itô-Stratonovich corrector of a Laplacian, and an Itô integral which is very small in a suitably weak sense. In particular we can choose coefficients such that the Itô integral approaches zero but the Laplacian remains fixed, hence the enhanced dissipation. In [48] this approach was used to show that for any given $\varepsilon > 0$, there exists a noise such that the corresponding Stochastic 3D Navier-Stokes Equation has a global solution with probability greater than $1 - \varepsilon$. This is proven as the smallness of the Itô integral makes the SPDE close to a deterministic Navier-Stokes with as large 'viscosity' as we like (coming from the Itô-Stratonovich corrector) which thus has a global solution. Given the unknown state of the vanishing viscosity limit problem under the no-slip boundary condition, as discussed in Section 6.1, a very interesting question is the following:

For any $\varepsilon > 0$, does there exist a noise and a viscosity ν such that for all $\mu < \nu$, the weak solution of the corresponding Stochastic Navier-Stokes Equation with viscosity μ and no-slip boundary condition is 'within ε' of the weak solution of the Stochastic Euler Equation?

This idea of Galeati has only been proven on the torus, but *if similar also holds on the bounded domain*, then we would hope to find success in three steps: closeness of the Stochastic Navier-Stokes to deterministic Navier-Stokes with additional Laplacian; closeness of the deterministic Navier-Stokes with additional Laplacian and small viscosity to deterministic Euler with additional Laplacian (in other words, closeness of deterministic Navier-Stokes with viscosity $1 + \mu$ to Navier-Stokes with viscosity 1); closeness of deterministic Euler with additional Laplacian to Stochastic Euler. Carrying the techniques over to a bounded domain conjures great challenge, largely again due to the Leray Projector through the absence of an explicit form in this case. Other exciting questions to ponder, though lacking any clear path to their answer, could be in finding a noise to directly verify a Stochastic Kato Criterion, as well as a noise that could restore well-posedness for the famously ill-posed Prandtl Equations.

– Bibliography –

[1] Adams, R.A., Fournier, J.J.F.: Sobolev spaces, *Pure and Applied Mathematics (Amsterdam)*, vol. 140, second edn. Elsevier/Academic Press, Amsterdam (2003)

[2] Aldous, D.: Stopping times and tightness. The Annals of Probability pp. 335–340 (1978)

[3] Alonso-Orán, D., Bethencourt de León, A.: On the well-posedness of stochastic boussinesq equations with transport noise. Journal of Nonlinear Science **30**(1), 175–224 (2020)

[4] Alonso-Orán, D., Bethencourt de León, A., Holm, D.D., Takao, S.: Modelling the climate and weather of a 2d lagrangian-averaged euler–boussinesq equation with transport noise. Journal of Statistical Physics **179**(5), 1267–1303 (2020)

[5] Arakeri, J.H., Shankar, P.: Ludwig prandtl and boundary layers in fluid flow: How a small viscosity can cause large effects. Resonance **5**(12), 48–63 (2000)

[6] Attanasio, S., Flandoli, F.: Renormalized solutions for stochastic transport equations and the regularization by bilinear multiplicative noise. Communications in Partial Differential Equations **36**(8), 1455–1474 (2011)

[7] Basson, A., Gérard-Varet, D.: Wall laws for fluid flows at a boundary with random roughness. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences **61**(7), 941–987 (2008)

[8] Bertozzi, A.L., Majda, A.J.: Vorticity and incompressible flow, *Cambridge Texts in Applied Mathematics*, vol. 27. Cambridge University Press, Cambridge (2002)

[9] Bessaih, H.: Martingale solutions for stochastic euler equations. Stochastic analysis and applications **17**(5), 713–725 (1999)

[10] Bessaih, H., Ferrario, B.: Inviscid limit of stochastic damped 2d navier–stokes equations. Nonlinearity **27**(1), 1 (2013)

[11] Billingsley, P.: Convergence of probability measures. John Wiley & Sons (2013)

[12] Black, F., Scholes, M.: The pricing of options and corporate liabilities. Journal of political economy **81**(3), 637–654 (1973)

[13] Bourguignon, J.P., Brezis, H.: Remarks on the euler equation. Journal of functional analysis **15**(4), 341–363 (1974)

[14] Brzeźniak, Z., Capiński, M., Flandoli, F.: Stochastic navier-stokes equations with multiplicative noise. Stochastic Analysis and Applications **10**(5), 523–532 (1992)

[15] Brzezniak, Z., Peszat, S.: Stochastic two dimensional euler equations. Annals of probability pp. 1796–1832 (2001)

[16] Brzeźniak, Z., Slavik, J.: Well-posedness of the 3d stochastic primitive equations with multiplicative and transport noise. Journal of Differential Equations **296**, 617–676 (2021)

[17] Burkholder, D.L., Davis, B.J., Gundy, R.F.: Integral inequalities for convex functions of operators on martingales. In: Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, vol. 2, pp. 223–240. Univ. California Press Berkeley, Calif. (1972)

[18] Cebeci, T., Bradshaw, P.: Momentum transfer in boundary layers. Washington (1977)

[19] Chapron, B., Crisan, D., Holm, D., Mémin, E., Radomska, A.: Stochastic Transport in Upper Ocean Dynamics II: STUOD 2022 Workshop, London, UK, September 26–29. Springer Nature (2024)

[20] Cipriano, F., Torrecilla, I.: Inviscid limit for 2d stochastic navier–stokes equations. Stochastic Processes and their Applications **125**(6), 2405–2426 (2015)

[21] Clauser, F.H.: The turbulent boundary layer. Advances in applied mechanics **4**, 1–51 (1956)

[22] Clopeau, T., Mikelic, A., Robert, R.: On the vanishing viscosity limit for the 2d incompressible navier-stokes equations with the friction type boundary conditions. Nonlinearity **11**(6), 1625 (1998)

[23] Coffey, W., Kalmykov, Y.P.: The Langevin equation: with applications to stochastic problems in physics, chemistry and electrical engineering, vol. 27. World Scientific (2012)

[24] Coghi, M., Maurelli, M.: Existence and uniqueness by kraichnan noise for 2d euler equations with unbounded vorticity. arXiv preprint arXiv:2308.03216 (2023)

[25] Constantin, P., Foias, C.: Navier-Stokes Equations. University of Chicago Press (1988)

[26] Conway, J.B.: A course in operator theory. American Mathematical Soc. (2000)

[27] Cotter, C., Crisan, D., Holm, D., Pan, W., Shevchenko, I.: Data assimilation for a quasi-geostrophic model with circulation-preserving stochastic transport noise. Journal of Statistical Physics **179**(5), 1186–1221 (2020)

[28] Cotter, C., Crisan, D., Holm, D.D., Pan, W., Shevchenko, I.: Modelling uncertainty using stochastic transport noise in a 2-layer quasi-geostrophic model. arXiv preprint arXiv:1802.05711 (2018)

[29] Cotter, C., Crisan, D., Holm, D.D., Pan, W., Shevchenko, I.: Numerically modeling stochastic lie transport in fluid dynamics. Multiscale Modeling & Simulation **17**(1), 192–232 (2019)

[30] Crisan, D., Flandoli, F., Holm, D.D.: Solution properties of a 3d stochastic euler fluid equation. Journal of Nonlinear Science **29**(3), 813–870 (2019)

[31] Crisan, D., Holm, D., Korn, P.: An implementation of hasselmann's paradigm for stochastic climate modelling based on stochastic lie transport. Nonlinearity **36**(9), 4862 (2023)

[32] Crisan, D., Holm, D.D., Luesink, E., Mensah, P.R., Pan, W.: Theoretical and computational analysis of the thermal quasi-geostrophic model. arXiv preprint arXiv:2106.14850 (2021)

[33] Crisan, D., Lang, O.: Well-posedness properties for a stochastic rotating shallow water model. Journal of Dynamics and Differential Equations pp. 1–31 (2023)

[34] Day, M.A.: The no-slip condition of fluid dynamics. Erkenntnis **33**(3), 285–296 (1990)

[35] Debussche, A., Glatt-Holtz, N., Temam, R.: Local martingale and pathwise solutions for an abstract fluids model. Phys. D **240**(14), 1123–1144 (2011)

[36] Demengel, F., Demengel, G., Erné, R.: Functional spaces for the theory of elliptic partial differential equations. Springer (2012)

[37] Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhikers guide to the fractional sobolev spaces. Bulletin des sciences mathématiques **136**(5), 521–573 (2012)

[38] Ding, Z.: A proof of the trace theorem of sobolev spaces on lipschitz domains. Proceedings of the American Mathematical Society **124**(2), 591–600 (1996)

[39] Doob, J.L.: Measure theory, vol. 143. Springer Science & Business Media (2012)

[40] Dufée, B., Mémin, E., Crisan, D.: Stochastic parametrization: an alternative to inflation in ensemble kalman filters. Quarterly Journal of the Royal Meteorological Society **148**(744), 1075–1091 (2022)

[41] Enciso, A., Garcia-Ferrero, M.A., Peralta-Salas, D.: The biot–savart operator of a bounded domain. Journal de Mathématiques Pures et Appliquées **119**, 85–113 (2018)

[42] Ephrati, S.R., Cifani, P., Luesink, E., Geurts, B.J.: Data-driven stochastic lie transport modeling of the 2d euler equations. Journal of Advances in Modeling Earth Systems **15**(1), e2022MS003,268 (2023)

[43] Evans, L.C.: Partial differential equations, vol. 19. American Mathematical Soc. (2010)

[44] Filho, M.L., Lopes, H.N., Planas, G.: On the inviscid limit for two-dimensional incompressible flow with navier friction condition. SIAM journal on mathematical analysis **36**(4), 1130–1141 (2005)

[45] Flandoli, F.: Topics on regularization by noise. Lecture Notes, University of Pisa (2013)

[46] Flandoli, F.: An open problem in the theory of regularization by noise for nonlinear pdes. In: Workshop Classic and Stochastic Geometric Mechanics, pp. 13–29. Springer (2015)

[47] Flandoli, F., Galeati, L., Luo, D.: Delayed blow-up by transport noise. Communications in Partial Differential Equations **46**(9), 1757–1788 (2021)

[48] Flandoli, F., Luo, D.: High mode transport noise improves vorticity blow-up control in 3d navier–stokes equations. Probability Theory and Related Fields **180**(1), 309–363 (2021)

[49] Flandoli, F., Luongo, E.: Stochastic partial differential equations in fluid mechanics, vol. 2328. Springer Nature (2023)

[50] Flandoli, F., Pappalettera, U.: 2d euler equations with stratonovich transport noise as a large-scale stochastic model reduction. Journal of Nonlinear Science **31**(1), 1–38 (2021)

[51] Galeati, L.: On the convergence of stochastic transport equations to a deterministic parabolic one. Stochastics and Partial Differential Equations: Analysis and Computations **8**(4), 833–868 (2020)

[52] Galeati, L., Luo, D.: Weak well-posedness by transport noise for a class of 2d fluid dynamical equations. arXiv preprint arXiv:2305.08761 (2023)

[53] Gao, X., Gao, H.: Existence and uniqueness of weak solutions to stochastic 3d navier–stokes equations with delays. Applied Mathematics Letters **95**, 158–164 (2019)

[54] Gérard-Varet, D., Masmoudi, N.: Relevance of the slip condition for fluid flows near an irregular boundary. Communications in Mathematical Physics **295**(1), 99–137 (2010)

[55] Gess, B., Yaroslavtsev, I.: Stabilization by transport noise and enhanced dissipation in the kraichnan model. arXiv preprint arXiv:2104.03949 (2021)

[56] Gie, G.M., Kelliher, J.P.: Boundary layer analysis of the navier–stokes equations with generalized navier boundary conditions. Journal of Differential Equations **253**(6), 1862–1892 (2012)

[57] Glatt-Holtz, N., Šverák, V., Vicol, V.: On inviscid limits for the stochastic navier–stokes equations and related models. Archive for Rational Mechanics and Analysis **217**(2), 619–649 (2015)

[58] Glatt-Holtz, N., Ziane, M., et al.: Strong pathwise solutions of the stochastic navier-stokes system. Advances in Differential Equations **14**(5/6), 567–600 (2009)

[59] Glatt-Holtz, N.E., Vicol, V.C.: Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise. Ann. Probab. **42**(1), 80–145 (2014). DOI 10.1214/12-AOP773. URL https://doi-org.univaq.clas.cineca.it/10.1214/12-AOP773

[60] Goodair, D.: Stochastic calculus in infinite dimensions and spdes. arXiv preprint arXiv:2203.17206 (2022)

[61] Goodair, D.: Navier-stokes equations with navier boundary conditions and stochastic lie transport: Well-posedness and inviscid limit. arXiv preprint arXiv:2308.04290 (2023)

[62] Goodair, D.: Weak and strong solutions to nonlinear spdes with unbounded noise. arXiv preprint arXiv:2401.10076 (2024)

[63] Goodair, D., Crisan, D.: On the 3d navier-stokes equations with stochastic lie transport. In: Stochastic Transport in Upper Ocean Dynamics II: STUOD 2022 Workshop, London, UK, September 26–29, vol. 11, p. 53. Springer Nature (2023)

[64] Goodair, D., Crisan, D.: The zero viscosity limit of stochastic navier-stokes flows. arXiv preprint arXiv:2305.18836 (2023)

[65] Goodair, D., Crisan, D., Lang, O.: Existence and uniqueness of maximal solutions to spdes with applications to viscous fluid equations. Stochastics and Partial Differential Equations: Analysis and Computations pp. 1–64 (2023)

[66] Gorbushin, A., Zametaev, V.: Asymptotic analysis of viscous fluctuations in turbulent boundary layers. Fluid Dynamics **53**, 9–20 (2018)

[67] Grisvard, P.: Elliptic problems in nonsmooth domains. SIAM (2011)

[68] Gubinelli, M., Imkeller, P., Perkowski, N.: Paracontrolled distributions and singular pdes. In: Forum of Mathematics, Pi, vol. 3, p. e6. Cambridge University Press (2015)

[69] Gyöngy, I., Krylov, N.V.: On stochastic equations with respect to semimartingales i. Stochastics: An International Journal of Probability and Stochastic Processes **4**(1), 1–21 (1980)

[70] Hairer, M.: An introduction to stochastic pdes. arXiv preprint arXiv:0907.4178 (2009)

[71] Hairer, M.: A theory of regularity structures. Inventiones mathematicae **198**(2), 269–504 (2014)

[72] Hofmanová, M., Lange, T., Pappalettera, U.: Global existence and non-uniqueness of 3d euler equations perturbed by transport noise. Probability Theory and Related Fields pp. 1–73 (2023)

[73] Holm, D.D.: Variational principles for stochastic fluid dynamics. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **471**(2176), 20140,963 (2015)

[74] Holm, D.D., Luesink, E.: Stochastic wave–current interaction in thermal shallow water dynamics. Journal of Nonlinear Science **31**(2), 1–56 (2021)

[75] Holm, D.D., Luesink, E., Pan, W.: Stochastic circulation dynamics in the ocean mixed layer. arXiv preprint arXiv:2006.05707 (2020)

[76] Iftimie, D., Sueur, F.: Viscous boundary layers for the navier–stokes equations with the navier slip conditions. Archive for rational mechanics and analysis **199**(1), 145–175 (2011)

[77] Jakubowski, A.: On the skorokhod topology. In: Annales de l'IHP Probabilités et statistiques, 3, pp. 263–285 (1986)

[78] Karatzas, I., Shreve, S.: Brownian motion and stochastic calculus, vol. 113. Springer Science & Business Media (1991)

[79] Kato, T.: Remarks on zero viscosity limit for nonstationary navier-stokes flows with boundary. In: Seminar on nonlinear partial differential equations, pp. 85–98. Springer (1984)

[80] Keller, H.B.: Numerical methods in boundary-layer theory. Annual Review of Fluid Mechanics **10**(1), 417–433 (1978)

[81] Kelliher, J.P.: Navier–stokes equations with navier boundary conditions for a bounded domain in the plane. SIAM journal on mathematical analysis **38**(1), 210–232 (2006)

[82] Kelliher, J.P.: On kato's conditions for vanishing viscosity. Indiana University Mathematics Journal pp. 1711–1721 (2007)

[83] Kreyszig, E.: Introductory functional analysis with applications, vol. 17. John Wiley & Sons (1991)

[84] Krylov, N.V., Rozovskii, B.L.: Stochastic evolution equations. In: Stochastic Differential Equations: Theory And Applications: A Volume in Honor of Professor Boris L Rozovskii, pp. 1–69. World Scientific (2007)

[85] Kuksin, S., Penrose, O.: A family of balance relations for the two-dimensional navier–stokes equations with random forcing. Journal of statistical physics **118**(3-4), 437–449 (2005)

[86] Kuksin, S.B.: The eulerian limit for 2d statistical hydrodynamics. Journal of statistical physics **115**, 469–492 (2004)

[87] Kuksin, S.B.: Remarks on the balance relations for the two-dimensional navier–stokes equation with random forcing. Journal of statistical physics **122**(1), 101–114 (2006)

[88] Kuksin, S.B.: On distribution of energy and vorticity for solutions of 2d navier-stokes equation with small viscosity. Communications in mathematical physics **284**, 407–424 (2008)

[89] Kunita, H., Kunita, H.: Stochastic flows and stochastic differential equations, vol. 24. Cambridge university press (1990)

[90] Kunita, H., Watanabe, S.: On square integrable martingales. Nagoya Mathematical Journal **30**, 209–245 (1967)

[91] Lang, O., Crisan, D.: Well-posedness for a stochastic 2d euler equation with transport noise. Stochastics and Partial Differential Equations: Analysis and Computations **11**(2), 433–480 (2023)

[92] Lang, O., Crisan, D., Mémin, É.: Analytical properties for a stochastic rotating shallow water model under location uncertainty. Journal of Mathematical Fluid Mechanics **25**(2), 29 (2023)

[93] Lang, O., Pan, W.: A pathwise parameterisation for stochastic transport. arXiv preprint arXiv:2202.10852 (2022)

[94] Lee, J., Jung, S.Y., Sung, H.J., Zaki, T.A.: Effect of wall heating on turbulent boundary layers with temperature-dependent viscosity. Journal of Fluid Mechanics **726**, 196–225 (2013)

[95] van Leeuwen, P.J., Crisan, D., Lang, O., Potthast, R.: Bayesian inference for fluid dynamics: A case study for the stochastic rotating shallow water model. arXiv preprint arXiv:2112.15216 (2021)

[96] Bethencourt-de León, A., Takao, S.: Transport noise restores uniqueness and prevents blow-up in geometric transport equations. arXiv preprint arXiv:2211.14695 (2022)

[97] Lions, J.L.: " quelques méthodes de résolution des problèmes aux limites non-linéaires,". Dunod (1969)

[98] Lions, P.L.: Mathematical topics in fluid mechanics. Vol. 1, *Oxford Lecture Series in Mathematics and its Applications*, vol. 3. The Clarendon Press, Oxford University Press, New York (1996). Incompressible models, Oxford Science Publications

[99] Liu, W.: Well-posedness of stochastic partial differential equations with lyapunov condition. Journal of Differential Equations **255**(3), 572–592 (2013)

[100] Liu, W., Röckner, M.: Spde in hilbert space with locally monotone coefficients. Journal of Functional Analysis **259**(11), 2902–2922 (2010)

[101] Liu, W., Röckner, M.: Local and global well-posedness of spde with generalized coercivity conditions. Journal of differential equations **254**(2), 725–755 (2013)

[102] Liu, W., Röckner, M.: Stochastic partial differential equations: an introduction. Springer (2015)

[103] Lopes Filho, M.C., Mazzucato, A.L., Lopes, H.N.: Vanishing viscosity limit for incompressible flow inside a rotating circle. Physica D: Nonlinear Phenomena **237**(10-12), 1324–1333 (2008)

[104] Lopes Filho, M.C., Mazzucato, A.L., Nussenzveig Lopes, H., Taylor, M.: Vanishing viscosity limits and boundary layers for circularly symmetric 2d flows. Bulletin of the Brazilian Mathematical Society, New Series **39**(4), 471–513 (2008)

[105] Lototsky, S.V., Rozovsky, B.L., et al.: Stochastic partial differential equations. Springer (2017)

[106] Luongo, E.: Inviscid limit for stochastic navier-stokes equations under general initial conditions. arXiv preprint arXiv:2111.14189 (2021)

[107] Lyman, F.: Vorticity production at a solid boundary. Appl. Mech. Rev **43**(8), 157–158 (1990)

[108] Maekawa, Y., Mazzucato, A.: The inviscid limit and boundary layers for navier-stokes flows. arXiv preprint arXiv:1610.05372 (2016)

[109] Malik, M.R.: Numerical methods for hypersonic boundary layer stability. Journal of computational physics **86**(2), 376–413 (1990)

[110] Masmoudi, N.: The euler limit of the navier-stokes equations, and rotating fluids with boundary. Archive for rational mechanics and analysis **142**, 375–394 (1998)

[111] Masmoudi, N., Rousset, F.: Uniform regularity for the navier–stokes equation with navier boundary condition. Archive for Rational Mechanics and Analysis **203**, 529–575 (2012)

[112] Masmoudi, N., Saint-Raymond, L.: From the boltzmann equation to the stokes-fourier system in a bounded domain. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences **56**(9), 1263–1293 (2003)

[113] Maxwell, J.C.: Vii. on stresses in rarified gases arising from inequalities of temperature. Philosophical Transactions of the royal society of London **7**(170), 231–256 (1879)

[114] Métivier, G.: Small viscosity and boundary layer methods: theory, stability analysis, and applications. Springer Science & Business Media (2004)

[115] Moore, D.: The boundary layer on a spherical gas bubble. Journal of Fluid Mechanics **16**(2), 161–176 (1963)

[116] Morris, S., Canright, D.: A boundary-layer analysis of benard convection in a fluid of strongly temperature-dependent viscosity. Physics of the Earth and planetary interiors **36**(3-4), 355–373 (1984)

[117] Morton, B.R.: The generation and decay of vorticity. Geophysical & Astrophysical Fluid Dynamics **28**(3-4), 277–308 (1984)

[118] Navier, C.: Mémoire sur les lois du mouvement des fluides. éditeur inconnu (1822)

[119] Navier, C.: Sur les lois de l'équilibre et du mouvement des corps élastiques. Mem. Acad. R. Sci. Inst. France **6**(369), 1827 (1827)

[120] Neelima, Šiška, D.: Coercivity condition for higher moment a priori estimates for nonlinear spdes and existence of a solution under local monotonicity. Stochastics **92**(5), 684–715 (2020)

[121] Nguyen, P., Tawri, K., Temam, R.: Nonlinear stochastic parabolic partial differential equations with a monotone operator of the ladyzenskaya-smagorinsky type, driven by a lévy noise. Journal of Functional Analysis **281**(8), 109,157 (2021)

[122] Nirenberg, L.: On elliptic partial differential equations. In: Il principio di minimo e sue applicazioni alle equazioni funzionali, pp. 1–48. Springer (2011)

[123] Oksendal, B.: Stochastic differential equations: an introduction with applications. Springer Science & Business Media (2013)

[124] Pardoux, E.: Equations aux dérivées partielles stochastiques monotones, these, univ (1975)

[125] Pardoux, É., et al.: Stochastic partial differential equations: An introduction. Springer (2021)

[126] Paré; s, C.: Existence, uniqueness and regularity of solution of the equations of a turbulence model for incompressible fluids. Applicable Analysis **43**(3-4), 245–296 (1992)

[127] Prévôt, C., Röckner, M.: A concise course on stochastic partial differential equations, *Lecture Notes in Mathematics*, vol. 1905. Springer, Berlin (2007)

[128] Richardson, S.: On the no-slip boundary condition. Journal of Fluid Mechanics **59**(4), 707–719 (1973)

[129] Robinson, J.C., Rodrigo, J.L., Sadowski, W.: The three-dimensional Navier–Stokes equations: Classical theory, vol. 157. Cambridge university press (2016)

[130] Röckner, M., Schmuland, B., Zhang, X.: Yamada-watanabe theorem for stochastic evolution equations in infinite dimensions. Condensed Matter Physics (2008)

[131] Röckner, M., Shang, S., Zhang, T.: Well-posedness of stochastic partial differential equations with fully local monotone coefficients. arXiv preprint arXiv:2206.01107 (2022)

[132] Ruckenstein, E., Rajora, P.: On the no-slip boundary condition of hydrodynamics. Journal of colloid and interface science **96**(2), 488–491 (1983)

[133] Sammartino, M., Caflisch, R.E.: Zero viscosity limit for analytic solutions, of the navier-stokes equation on a half-space.¶ i. existence for euler and prandtl equations. Communications in mathematical physics **192**, 433–461 (1998)

[134] Sammartino, M., Caflisch, R.E.: Zero viscosity limit for analytic solutions of the navier-stokes equation on a half-space.¶ ii. construction of the navier-stokes solution. Communications in mathematical physics **192**, 463–491 (1998)

[135] Schetz, J.A., Bowersox, R.D.: Boundary layer analysis. American Institute of Aeronautics and Astronautics (2011)

[136] Serpelloni, E., Faccenna, C., Spada, G., Dong, D., Williams, S.D.: Vertical gps ground motion rates in the euro-mediterranean region: New evidence of velocity gradients at different spatial scales along the nubia-eurasia plate boundary. Journal of Geophysical Research: Solid Earth **118**(11), 6003–6024 (2013)

[137] Shaing, K.C., Cahyna, P., Becoulet, M., Park, J.K., Sabbagh, S., Chu, M.: Collisional boundary layer analysis for neoclassical toroidal plasma viscosity in tokamaks. Physics of Plasmas **15**(8), 082,506 (2008)

[138] Stein, E.M.: Singular integrals and differentiability properties of functions, vol. 2. Princeton university press (1970)

[139] Stevenson, P., Thorpe, R., Davidson, J.: Incipient motion of a small particle in the viscous boundary layer at a pipe wall. Chemical Engineering Science **57**(21), 4505–4520 (2002)

[140] Street, O.D., Crisan, D.: Semi-martingale driven variational principles. Proceedings of the Royal Society A **477**(2247), 20200,957 (2021)

[141] Tang, H.: On stochastic euler-poincaré equations driven by pseudo-differential/multiplicative noise. Journal of Functional Analysis p. 110075 (2023)

[142] Tang, H., Wang, F.Y.: A general framework for solving singular spdes with applications to fluid models driven by pseudo-differential noise. arXiv preprint arXiv:2208.08312 (2022)

[143] Tani, I.: Production of longitudinal vortices in the boundary layer along a concave wall. Journal of Geophysical Research **67**(8), 3075–3080 (1962)

[144] Taniguchi, T.: Global existence of a weak solution to 3d stochastic navier–stokes equations in an exterior domain. Nonlinear Differential Equations and Applications NoDEA **21**, 813–840 (2014)

[145] Tapia, P.A., Amrouche, C., Conca, C., Ghosh, A.: Stokes and navier-stokes equations with navier boundary conditions. Journal of Differential Equations **285**, 258–320 (2021)

[146] Tartar, L.: An introduction to Sobolev spaces and interpolation spaces, vol. 3. Springer Science & Business Media (2007)

[147] Temam, R.: Navier-Stokes equations: theory and numerical analysis, vol. 343. American Mathematical Soc. (2001)

[148] Temam, R., Wang, X.: Boundary layers associated with incompressible navier–stokes equations: the noncharacteristic boundary case. Journal of Differential Equations **179**(2), 647–686 (2002)

[149] Wallace, J.M., Vukoslavčević, P.V.: Measurement of the velocity gradient tensor in turbulent flows. Annual review of fluid mechanics **42**, 157–181 (2010)

[150] Wang, X.: A kato type theorem on zero viscosity limit of navier-stokes flows. Indiana University Mathematics Journal pp. 223–241 (2001)

[151] Wang, Y.G., Zhao, M.: On kato's conditions for the inviscid limit of the two-dimensional stochastic navier-stokes equation. arXiv preprint arXiv:2303.06578 (2023)

[152] Weinan, E.: Boundary layer theory and the zero-viscosity limit of the navier-stokes equation. Acta Mathematica Sinica **16**, 207–218 (2000)

[153] Yudovich, V.I.: Non-stationary flows of an ideal incompressible fluid. Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki **3**(6), 1032–1066 (1963)

[154] Yudovich, V.I.: Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. Mathematical Research Letters **2**(1), 27–38 (1995)

Appendix A

Supplementary Material

A.1 Stochastic Calculus in Infinite Dimensions

We start by giving more explicit motivation for this calculus; that is, why work in infinite dimensions? Finite dimensional stochastic differential equations have rich applications in physics and finance, for example in Langevin Equations modelling the movement of a particle in space [23] or the Black-Scholes options pricing model for the dynamics of the price of a stock [12]. These are applications of classical Itô calculus, where the integral of a process takes values in Euclidean spaces. Whilst this theory is adequate in such applications, mathematical models for physical phenomena far exceed those for the position of a particle or that of a tradable stock price. The extension to infinite dimensions is necessary for stochastic differential equations modelling functions of both space and time, such as the velocity or temperature of a fluid. It is therefore necessary to define the stochastic integral

$$\int_0^t \mathbf{\Psi}_s dW_s \tag{A.1}$$

for a class of stochastic processes $\mathbf{\Psi} : \Omega \times [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^d$. We regard $\mathbf{\Psi}$ not as a pointwise defined function but rather an element of a function space, which is our motivating context for stochastic integration of Hilbert Space valued processes.

A.1.1 A Classical Construction for Hilbert Space Valued Processes

The construction of the stochastic integral for Hilbert Space valued processes precisely mirrors the standard one dimensional Itô integral; as such we start from simple processes. $\mathcal{H}$ will denote a general separable Hilbert space, with norm and inner product given by $\|\cdot\|_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ respectively. W represents a standard one dimensional Brownian Motion with respect to the fixed filtered probability space $(\Omega; \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Definition A.1.1. *Let $0 = t_0 < \dots < t_i < t_{i+1} < \dots$ be a partition of the time interval $[0, \infty)$ such that (t_i) approaches infinity. A simple $\mathcal{H}$-valued process is one that for $\mathbb{P} - a.e.$ ω takes the form*

$$\mathbf{\Psi}_t(\omega) = a_0(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} a_i(\omega)\mathbb{1}_{(t_i, t_{i+1}]}(t),$$

where each $a_i \in L^2(\Omega;\mathcal{H})$ *and is* $\mathcal{F}_{t_i}-$*measurable, with respect to the Borel sigma-algebra on* $\mathcal{H}$.

Definition A.1.2. *The Itô integral of a simple* $\mathcal{H}$*-valued process* $\mathbf{\Psi}$*, with respect to Brownian motion, is defined as*

$$\int_0^t \mathbf{\Psi}_s dW_s := \sum_{i=0}^{\infty} a_i\big(W_{t_{i+1}\wedge t} - W_{t_i\wedge t}\big).$$

Note that in reality the above is a finite sum, so the right hand side is well defined. Indeed it can alternatively be expressed as

$$\sum_{i=0}^{k-1} a_i\big(W_{t_{i+1}} - W_{t_i}\big) + a_k\big(W_t - W_{t_k}\big), \tag{A.2}$$

where k is such that $t \in (t_k, t_{k+1}]$. We define the integral for a more general class of integrands, using approximations by simple processes. For this we introduce the notion of *progressive measurability.*

Definition A.1.3. *An* $\mathcal{H}$*-valued process* $\mathbf{\Psi}$ *is said to be progressively measurable if for every* $T > 0$*, the restricted process* $\mathbf{\Psi} : \Omega \times [0,T] \to \mathcal{H}$ *is measurable with respect to the product measure* $\mathcal{F}_T \times \mathcal{B}([0,T])$.

Definition A.1.4. *We use* $\mathcal{I}_T^{\mathcal{H}}$ *to denote the class of* $\mathcal{H}$*-valued processes* $\mathbf{\Psi}$ *which are progressively measurable and satisfy the square integrability condition*

$$\mathbb{E}\left(\int_0^T \|\mathbf{\Psi}_s\|_{\mathcal{H}}^2\, ds\right) < \infty. \tag{A.3}$$

In other words, $\mathbf{\Psi} \in L^2\big(\Omega \times [0,T];\mathcal{H}\big)$ *where the domain space* $\Omega \times [0,T]$ *is a measure space equipped with the product measure* $\mathbb{P} \times \lambda$.[1]

We have given the definition for progressively measurable processes, not *previsible* processes[2] as will commonly be seen in the literature. Progressive measurability is a weaker condition than previsibility, but thankfully most reasonably behaved processes (adapted and left continuous for example) are both progressively measurable and previsible. We make the definition here for the more general class of integrands *in the cases where the integrator is continuous.* Other authors (e.g. [90]) may opt for previsible processes as these become necessary in retaining nice properties (e.g. martingality) when defining the stochastic integral with respect to discontinuous integrators.

Definition A.1.5. *The class of processes* $\mathbf{\Psi}$ *such that* $\mathbf{\Psi} \in \mathcal{I}_T^{\mathcal{H}}$ *for all* $T > 0$ *is denoted by* $\mathcal{I}^{\mathcal{H}}$.

$\mathcal{I}^{\mathcal{H}}$ represents our class of integrands for all times, though there will be nothing wrong with defining the integral for times $t \leq T$ in the class $\mathcal{I}_T^{\mathcal{H}}$. The construction comes as a limit of simple integrals, for which we need the following proposition which holds no differently to [123] Lemma 3.1.5 for example.

[1] The progressive measurability of $\mathbf{\Psi}$ ensures that it is measurable over this product space, and Tonelli's Theorem justifies exchanging the order of integration.

[2] An $\mathcal{H}$-valued process is said to be previsible if it is measurable with respect to the sigma-algebra generated by all left-continuous adapted processes.

Proposition A.1.6. *For any $\mathbf{\Psi} \in \mathcal{I}^{\mathcal{H}}_T$, there exists a sequence of simple processes $(\mathbf{\Psi}^n)$ which converge to $\mathbf{\Psi}$ in $L^2\big(\Omega \times [0,T];\mathcal{H}\big)$.*

This approximation is the final piece of the constructive jigsaw, allowing us to define the stochastic integral below.

Definition A.1.7. *We define the Itô stochastic integral for processes $\mathbf{\Psi} \in \mathcal{I}^{\mathcal{H}}$ by*

$$\int_0^t \mathbf{\Psi}_s dW_s := \lim_{n\to\infty} \int_0^t \mathbf{\Psi}^n_s dW_s, \tag{A.4}$$

where $(\mathbf{\Psi}^n)$ is the sequence of simple processes postulated in Proposition A.1.6 which approach $\mathbf{\Psi}$ in $L^2\big(\Omega \times [0,t];\mathcal{H}\big)$, and the limit is taken in $L^2\big(\Omega;\mathcal{H}\big)$.

The fact that this is the natural topology in which to take the limit of simple stochastic integrals falls from the Itô Isometry for simple processes, which further justifies that the construction is independent of the choice of simple approximation.

Proposition A.1.8. *For a simple process $\mathbf{\Psi}^n$ and any time $t > 0$,*

$$\mathbb{E}\left(\left|\left|\int_0^t \mathbf{\Psi}^n_s dW_s\right|\right|^2_{\mathcal{H}}\right) = \mathbb{E}\Big(\int_0^t \left\|\mathbf{\Psi}^n_s\right\|^2_{\mathcal{H}} ds\Big). \tag{A.5}$$

Proof. Let's suppose that $\mathbf{\Psi}^n$ takes the form

$$\mathbf{\Psi}^n_t(\omega) = a^n_0(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} a^n_i(\omega)\mathbb{1}_{(t^n_i, t^n_{i+1}]}(t) \tag{A.6}$$

as outlined in Definition A.1.1. Then applying Definition A.1.2, we deconstruct the LHS of (A.5):

$$\begin{aligned}
\mathbb{E}\left(\left|\left|\int_0^t \mathbf{\Psi}^n_s dW_s\right|\right|^2_{\mathcal{H}}\right) &= \mathbb{E}\left(\left|\left|\sum_{i=0}^{\infty} a^n_i\big(W_{t^n_{i+1}\wedge t} - W_{t^n_i\wedge t}\big)\right|\right|^2_{\mathcal{H}}\right)\\
&= \mathbb{E}\left(\Big\langle \sum_{i=0}^{\infty} a^n_i\big(W_{t^n_{i+1}\wedge t} - W_{t^n_i\wedge t}\big), \sum_{j=0}^{\infty} a^n_j\big(W_{t^n_{j+1}\wedge t} - W_{t^n_j\wedge t}\big)\Big\rangle_{\mathcal{H}}\right)\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\big(\langle a^n_i, a^n_j\rangle_{\mathcal{H}}\,(W_{t^n_{i+1}\wedge t} - W_{t^n_i\wedge t})(W_{t^n_{j+1}\wedge t} - W_{t^n_j\wedge t})\big)
\end{aligned}$$

recalling once more that the infinite sum is actually a finite sum (A.2) so there is no difficulty in extracting it from the inner product and expectation. For $i \neq j$, and without loss of generality $i < j$, the random inner product is $\mathcal{F}_{t_j}-$measurable as the continuity of the inner product preserves measurability, and therefore $\left\langle a^n_i, a^n_j\right\rangle_{\mathcal{H}}(W_{t^n_{i+1}\wedge t} - W_{t^n_i\wedge t})$ and $(W_{t^n_{j+1}\wedge t} - W_{t^n_j\wedge t})$ are independent from the independent increments of Brownian Motion. The terms thus vanish and we are left with

$$\sum_{i=0}^{\infty} \mathbb{E}\Big(\left\|a^n_i\right\|^2_{\mathcal{H}}\,(W_{t^n_{i+1}\wedge t} - W_{t^n_i\wedge t})^2\Big)$$

to which we note independence again and assert that this is just

$$\sum_{i=0}^{\infty} \mathbb{E}\big(\|a_i^n\|_{\mathcal{H}}^2 \big)\big(t_{i+1}^n \wedge t - t_i^n \wedge t\big),$$

which is precisely the integral

$$\int_0^t \sum_{i=0}^{\infty} \mathbb{E}\big(\|a_i^n\|_{\mathcal{H}}^2 \big) \mathbb{1}_{(t_i^n, t_{i+1}^n]}(s) ds.$$

We can write

$$\|a_i^n\|_{\mathcal{H}}^2 \, \mathbb{1}_{(t_i^n, t_{i+1}^n]}(s) = \left\| a_i^n \mathbb{1}_{(t_i^n, t_{i+1}^n]}(s) \right\|_{\mathcal{H}}^2,$$

the infinite sum of which is a single non-zero term, equal to

$$\left\| a_0^n(\omega) \mathbb{1}_{\{0\}}(s) + \sum_{i=0}^{\infty} a_i^n(\omega) \mathbb{1}_{(t_i^n, t_{i+1}^n]}(s) \right\|_{\mathcal{H}}^2$$

at every s except for zero which is a set of Lebesgue measure zero in $[0, t]$. Again the infinite sum is truly a single non-zero term, justifying its exchange with expectation, and the above is of course $\|\boldsymbol{\Psi}_s^n\|_{\mathcal{H}}^2$, proving the result.

□

So, why is this useful in terms of the limit in (A.4)? First and foremost it ensures that the limit is uniquely defined; given that $L^2\big(\Omega; \mathcal{H}\big)$ is complete we need only show that the sequence of stochastic integrals is Cauchy in this space. The Itô Isometry tells us that $\big(\int_0^t \boldsymbol{\Psi}_s^n dW_s \big)$ is Cauchy in $L^2\big(\Omega; \mathcal{H}\big)$ if and only if $(\boldsymbol{\Psi}^n)$ is Cauchy in $L^2\big(\Omega \times [0, t]; \mathcal{H}\big)$, which is of course true as by definition the $(\boldsymbol{\Psi}^n)$ are convergent (to $\boldsymbol{\Psi}$) in this space. Furthermore the isometry extends to the general integral defined in Definition A.1.7, as a trivial corollary of the discussion here.

Corollary A.1.8.1. *The Itô Isometry (A.5) holds for all processes $\boldsymbol{\Psi} \in \mathcal{I}^{\mathcal{H}}$.*

Without direct appeal to the formal construction, we may also understand the integral (A.1) as a random element of the dual space $\mathcal{H}^*$ and identify the functional with its counterpart in $\mathcal{H}$ in the usual sense.

Theorem A.1.9. *The Itô stochastic integral defined in Definition A.1.7 is the unique element of $\mathcal{H}$ satisfying the duality relation*

$$\left\langle \int_0^t \boldsymbol{\Psi}_s dW_s, \phi \right\rangle_{\mathcal{H}} = \int_0^t \langle \boldsymbol{\Psi}_s, \phi \rangle_{\mathcal{H}} \, dW_s \tag{A.7}$$

for all $\phi \in \mathcal{H}$. The above are random inner products, defined by

$$\langle \boldsymbol{\Psi}_s, \phi \rangle_{\mathcal{H}} (\omega) := \langle \boldsymbol{\Psi}_s(\omega), \phi \rangle_{\mathcal{H}}$$

and similarly for the LHS.

Remark A.1.10. *As a corollary, by the Riesz-Representation Theorem, it is consistent to define the Itô stochastic integral as an $\mathcal{H}^*$ valued random variable via the mapping*

$$\phi \mapsto \Big(\int_0^t \langle \mathbf{\Psi}_s, \phi \rangle_{\mathcal{H}} \, dW_s \Big)(\omega).$$

Proof. Given that we have defined (A.1) as a limit of simple processes, it will come as no surprise that we use this approach to prove the relation (A.7). We will prove that the result holds for simple processes $\mathbf{\Psi}^n$, and later that it is preserved in the $L^2\big(\Omega \times [0,t]; \mathcal{H}\big)$ limit. Firstly though we ought to verify that the RHS of (A.7) makes sense, that is to say $\langle \mathbf{\Psi}, \phi \rangle_{\mathcal{H}}$ is a valid (one dimensional) integrand. Thus we must show the standard progressive measurability and square integrability conditions: for the former, note that the progressive measurability of $\mathbf{\Psi}$ is preserved under composition with the continuous mapping $\langle \cdot, \phi \rangle_{\mathcal{H}}$. The latter is straightforward, as

$$\mathbb{E}\Big(\int_0^T \langle \mathbf{\Psi}_s, \phi \rangle_{\mathcal{H}}^2 \, ds \Big) \leq \mathbb{E}\Big(\int_0^T \|\mathbf{\Psi}_s\|_{\mathcal{H}}^2 \, \|\phi\|_{\mathcal{H}}^2 \, ds \Big) = \|\phi\|_{\mathcal{H}}^2 \, \mathbb{E}\Big(\int_0^T \|\mathbf{\Psi}_s\|_{\mathcal{H}}^2 \, ds \Big), \tag{A.8}$$

which is finite by (A.3). Let's suppose that $\mathbf{\Psi}^n$ takes the form (A.6). Then applying Definition A.1.2, we deconstruct the LHS of (A.7):

$$\begin{aligned}
\Big\langle \int_0^t \mathbf{\Psi}_s^n dW_s, \phi \Big\rangle_{\mathcal{H}} &= \Big\langle \sum_{i=0}^{\infty} a_i^n \big(W_{t_{i+1}^n \wedge t} - W_{t_i^n \wedge t} \big), \phi \Big\rangle_{\mathcal{H}} \\
&= \sum_{i=0}^{\infty} \Big\langle a_i^n \big(W_{t_{i+1}^n \wedge t} - W_{t_i^n \wedge t} \big), \phi \Big\rangle_{\mathcal{H}} \\
&= \sum_{i=0}^{\infty} \langle a_i^n, \phi \rangle_{\mathcal{H}} \big(W_{t_{i+1}^n \wedge t} - W_{t_i^n \wedge t} \big)
\end{aligned}$$

and proceed similarly for the RHS, observing that the integrand

$$\begin{aligned}
\langle \mathbf{\Psi}_s, \phi \rangle_{\mathcal{H}} &= \Big\langle a_0^n \big(\mathbb{1}_{\{0\}}(s) \big) + \sum_{i=0}^{\infty} a_i^n \big(\mathbb{1}_{(t_i^n, t_{i+1}^n]}(s) \big), \phi \Big\rangle_{\mathcal{H}} \\
&= \big\langle a_0^n \big(\mathbb{1}_{\{0\}}(s) \big), \phi \big\rangle_{\mathcal{H}} + \sum_{i=0}^{\infty} \Big\langle a_i^n \big(\mathbb{1}_{(t_i^n, t_{i+1}^n]}(s) \big), \phi \Big\rangle_{\mathcal{H}} \\
&= \langle a_0^n, \phi \rangle_{\mathcal{H}} \big(\mathbb{1}_{\{0\}}(s) \big) + \sum_{i=0}^{\infty} \langle a_i^n, \phi \rangle_{\mathcal{H}} \big(\mathbb{1}_{(t_i^n, t_{i+1}^n]}(s) \big)
\end{aligned}$$

is again simple. This is completely analogous to showing that $\langle \mathbf{\Psi}, \phi \rangle_{\mathcal{H}}$ was a valid integrand. Applying Definition A.1.2 to the above proves the result for simple processes, so all that remains to show is preservation in the limit. By definition

$$\Big\langle \int_0^t \mathbf{\Psi}_s dW_s, \phi \Big\rangle_{\mathcal{H}} = \Big\langle \lim_{n \to \infty} \int_0^t \mathbf{\Psi}_s^n dW_s, \phi \Big\rangle_{\mathcal{H}}$$

and a reminder that this limit is taken in $L^2\big(\Omega; \mathcal{H}\big)$. We would like to take the limit outside of the inner product, in some appropriate topology, and use the result for simple functions: the steps would be

$$\begin{aligned}\Big\langle \lim_{n\to\infty}\int_0^t \mathbf{\Psi}_s^n dW_s, \phi\Big\rangle_{\mathcal{H}} &= \lim_{n\to\infty}\Big\langle \int_0^t \mathbf{\Psi}_s^n dW_s, \phi\Big\rangle_{\mathcal{H}} \\ &= \lim_{n\to\infty}\int_0^t \langle \mathbf{\Psi}_s^n, \phi\rangle_{\mathcal{H}}\, dW_s\end{aligned}$$

so it should be clear that the topology we want to take this limit in is that of $L^2\big(\Omega;\mathbb{R}\big)$, as the last line would be precisely the RHS of (A.7) by definition if we can show that the simple real valued process $\langle \mathbf{\Psi}^n, \phi\rangle_{\mathcal{H}}$ converges to $\langle \mathbf{\Psi}, \phi\rangle_{\mathcal{H}}$ in $L^2\big(\Omega\times[0,t];\mathbb{R}\big)$. Thankfully it is straightforwards to justify taking this limit outside of the inner product: if (f_n) converges to f in $L^2\big(\Omega;\mathcal{H}\big)$ then

$$\begin{aligned}\mathbb{E}\Big(\langle f_n,\phi\rangle_{\mathcal{H}} - \langle f,\phi\rangle_{\mathcal{H}}\Big)^2 = \mathbb{E}\Big(\langle f_n - f,\phi\rangle_{\mathcal{H}}\Big)^2 &\leq \mathbb{E}\Big(\|f_n - f\|_{\mathcal{H}}^2\,\|\phi\|_{\mathcal{H}}^2\Big) \\ &= \|\phi\|_{\mathcal{H}}^2\,\mathbb{E}\Big(\|f_n - f\|_{\mathcal{H}}^2\Big) \longrightarrow 0\end{aligned}$$

so $(\langle f_n,\phi\rangle_{\mathcal{H}})$ converges to $\langle f,\phi\rangle_{\mathcal{H}}$ in $L^2\big(\Omega;\mathbb{R}\big)$, as required to justify the interchange. To show the convergence of $\langle \mathbf{\Psi}^n, \phi\rangle_{\mathcal{H}}$ to $\langle \mathbf{\Psi}, \phi\rangle_{\mathcal{H}}$ in $L^2\big(\Omega\times[0,t];\mathbb{R}\big)$ we apply the same argument:

$$\begin{aligned}\|\langle \mathbf{\Psi},\phi\rangle_{\mathcal{H}} - \langle \mathbf{\Psi}^n,\phi\rangle_{\mathcal{H}}\|_{L^2\left(\Omega\times[0,t];\mathbb{R}\right)} &= \|\langle \mathbf{\Psi} - \mathbf{\Psi}^n,\phi\rangle_{\mathcal{H}}\|_{L^2\left(\Omega\times[0,t];\mathbb{R}\right)} \\ &= \mathbb{E}\Big(\int_0^t \langle \mathbf{\Psi}_s - \mathbf{\Psi}_s^n,\phi\rangle_{\mathcal{H}}^2\, ds\Big) \\ &\leq \mathbb{E}\Big(\int_0^t \|\mathbf{\Psi}_s - \mathbf{\Psi}_s^n\|_{\mathcal{H}}^2\,\|\phi\|_{\mathcal{H}}^2\, ds\Big) \\ &= \|\phi\|_{\mathcal{H}}^2\,\mathbb{E}\Big(\int_0^t \|\mathbf{\Psi}_s - \mathbf{\Psi}_s^n\|_{\mathcal{H}}^2\, ds\Big) \\ &= \|\phi\|_{\mathcal{H}}^2\,\|\mathbf{\Psi} - \mathbf{\Psi}^n\|_{L^2\left(\Omega\times[0,t];\mathcal{H}\right)} \\ &\longrightarrow 0,\end{aligned}$$

where convergence to 0 is by definition of the approximating sequence $\mathbf{\Psi}^n$.

$\square$

We provide two applications of this result below, both of which will be fundamental to our SPDE framework.

Proposition A.1.11. *The Itô Isometry holds for a multi-dimensional driving Brownian motion, in the sense that if $(\mathbf{\Psi}^i)_{i=1}^n$ are a collection of processes in $\mathcal{I}^{\mathcal{H}}$, and $(W^i)_{i=1}^n$ are independent Brownian Motions, then*

$$\mathbb{E}\left(\Big\|\sum_{i=1}^n \int_0^t \mathbf{\Psi}_s^i dW_s^i\Big\|_{\mathcal{H}}^2\right) = \sum_{i=1}^n \mathbb{E}\Big(\int_0^t \|\mathbf{\Psi}_s^i\|_{\mathcal{H}}^2\, ds\Big).$$

Proof. We look to simplify the left hand side of the required equality, applying Parseval's identity (e.g. [83] pp.170) for a basis (e_k) of $\mathcal{H}$:

$$\begin{aligned}\mathbb{E}\bigg(\Big|\Big|\sum_{i=1}^{n}\int_0^t \mathbf{\Psi}_s^i dW_s^i\Big|\Big|_{\mathcal{H}}^2\bigg) &= \mathbb{E}\sum_{k=1}^{\infty}\Big\langle \sum_{i=1}^{n}\int_0^t \mathbf{\Psi}_s^i dW_s^i, e_k\Big\rangle_{\mathcal{H}}^2 \\ &= \mathbb{E}\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{n}\int_0^t \left\langle \mathbf{\Psi}_s^i, e_k\right\rangle_{\mathcal{H}} dW_s^i\Big)^2\end{aligned}$$

having used linearity of the inner product to pull out the sum and Theorem A.1.9. We can now regard the infinite sum as an integral with respect to the counting measure and apply Tonelli's Theorem, also expanding the square to obtain

$$\sum_{k=1}^{\infty}\mathbb{E}\Big(\sum_{i=1}^{n}\int_0^t \left\langle \mathbf{\Psi}_s^i, e_k\right\rangle_{\mathcal{H}} dW_s^i\Big)^2 = \sum_{k=1}^{\infty}\mathbb{E}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\int_0^t \left\langle \mathbf{\Psi}_s^i, e_k\right\rangle_{\mathcal{H}} dW_s^i\Big)\Big(\int_0^t \left\langle \mathbf{\Psi}_s^j, e_k\right\rangle_{\mathcal{H}} dW_s^j\Big).$$

For the cross terms $i \neq j$ we make use of the independence of the Brownian Motions and hence the respective stochastic integrals, as well as the standard property that the Itô integral has zero expectation to nullify these terms. Our expression reduces to

$$\sum_{k=1}^{\infty}\sum_{i=1}^{n}\mathbb{E}\Big(\int_0^t \left\langle \mathbf{\Psi}_s^i, e_k\right\rangle_{\mathcal{H}} dW_s^i\Big)^2$$

to which we can apply the Itô Isometry (Corollary A.1.8.1 for the Hilbert Space $\mathbb{R}$, which is of course the standard Isometry) giving us

$$\sum_{k=1}^{\infty}\sum_{i=1}^{n}\mathbb{E}\int_0^t \left\langle \mathbf{\Psi}_s^i, e_k\right\rangle_{\mathcal{H}}^2 ds.$$

From this, we apply Tonelli twice more to take the infinite sum all the way through,

$$\sum_{i=1}^{n}\mathbb{E}\int_0^t \sum_{k=1}^{\infty}\left\langle \mathbf{\Psi}_s^i, e_k\right\rangle_{\mathcal{H}}^2 ds.$$

A final application of Parseval's identity gives the result.

□

Whilst we chose to prove Proposition A.1.8 and subsequently Corollary A.1.8.1 from first principles in the Hilbert Space setting, the method of proof here touches upon a fundamental aspect of this theory: with a good understanding of the standard setting in $\mathbb{R}$, we can apply Theorem A.1.9 to straightforwardly deduce key properties here. Indeed if we accepted the Itô Isometry in $\mathbb{R}$, we could have just proven Corollary A.1.8.1 in the simple vein of Proposition A.1.11. We take this approach in extending the result of Theorem A.1.9.

Theorem A.1.12. *Suppose that $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces such that $\boldsymbol{\Psi} \in \mathcal{I}^{\mathcal{H}_1}$ and $T \in \mathscr{L}(\mathcal{H}_1; \mathcal{H}_2)$. Then the process $T\boldsymbol{\Psi}$ defined by*

$$(T\boldsymbol{\Psi})_s(\omega) = T\big(\boldsymbol{\Psi}_s(\omega)\big)$$

belongs to $\mathcal{I}^{\mathcal{H}_2}$ and is such that

$$T\Big(\int_0^t \boldsymbol{\Psi}_s dW_s\Big) = \int_0^t T\boldsymbol{\Psi}_s dW_s. \tag{A.9}$$

Proof. We shall prove first that $T\boldsymbol{\Psi} \in \mathcal{I}^{\mathcal{H}_2}$. The progressive measurability is preserved under the continuity of T, and for C the (square of the) boundedness constant associated to T we have that at any time t,

$$\mathbb{E}\Big(\int_0^t \|T\boldsymbol{\Psi}_s\|_{\mathcal{H}_2}^2 \, ds\Big) \leq C\mathbb{E}\Big(\int_0^t \|\boldsymbol{\Psi}_s\|_{\mathcal{H}_1}^2 \, ds\Big) < \infty$$

as $\boldsymbol{\Psi} \in \mathcal{I}^{\mathcal{H}_1}$, showing that $T\boldsymbol{\Psi} \in \mathcal{I}^{\mathcal{H}_2}$. To commute T with the integral we shall use the characterisation from Theorem A.1.9, having now established that the right hand side of (A.9) is well defined in $\mathcal{H}_2$. We introduce $T^* \in \mathscr{L}\left(\mathcal{H}_2; \mathcal{H}_1\right)$ as the adjoint of T, and observe that for any $\phi \in \mathcal{H}_2$,

$$\begin{aligned}
\Big\langle T\Big(\int_0^t \boldsymbol{\Psi}_s dW_s\Big), \phi\Big\rangle_{\mathcal{H}_2} &= \Big\langle \int_0^t \boldsymbol{\Psi}_s dW_s, T^*\phi\Big\rangle_{\mathcal{H}_1} \\
&= \int_0^t \langle \boldsymbol{\Psi}_s, T^*\phi\rangle_{\mathcal{H}_1} dW_s \\
&= \int_0^t \langle T\boldsymbol{\Psi}_s, \phi\rangle_{\mathcal{H}_2} dW_s \\
&= \Big\langle \int_0^t T\boldsymbol{\Psi}_s dW_s, \phi\Big\rangle_{\mathcal{H}_2}
\end{aligned}$$

applying Theorem A.1.9 twice. As this equality holds for arbitrary $\phi \in \mathcal{H}_2$ then we have proven (A.9), which is of course an identity $\mathbb{P} - a.e.$ in $\mathcal{H}_2$.

□

A.1.2 Martingale and Local Martingale Integrators

As expected, we can extend the definition to integrators beyond Brownian motion, in the same manner as the standard Itô integral. We begin the extension to continuous square integrable martingales, and then to continuous local martingales.

Definition A.1.13. *We shall denote the class of real valued martingales M such that $M_t \in L^2\big(\Omega; \mathbb{R}\big)$ for every $t \geq 0$ by $\mathcal{M}^2$. The subclass of such martingales with $\mathbb{P} - a.s.$ continuous paths will be represented by $\mathcal{M}_c^2$.*

Definition A.1.14. *For any $M \in \mathcal{M}_c^2$, define $\mathcal{I}_M^{\mathcal{H},T}$ to be the class of $\mathcal{H}$-valued processes $\mathbf{\Psi}$ which are progressively measurable on $[0,T] \times \Omega$ and satisfy the square integrability condition*

$$\mathbb{E}\Big(\int_0^T \|\mathbf{\Psi}_s\|_{\mathcal{H}}^2 \, d[M]_s \Big) < \infty,$$

where $[M]_.$ is the quadratic variation of $M_.$. We similarly denote by $\mathcal{I}_M^{\mathcal{H}}$ the class of processes $\mathbf{\Psi}$ such that $\mathbf{\Psi} \in \mathcal{I}_M^{\mathcal{H},T}$ for all $T > 0$.

Constructing the integral

$$\int_0^t \mathbf{\Psi}_s dM_s$$

for $\mathbf{\Psi} \in \mathcal{I}_M^{\mathcal{H}}$ now falls from what we have already done for (A.1). We use simple processes $\mathbf{\Psi}^n$ as in Definition A.1.1 to approximate $\mathbf{\Psi}$, in the sense that

$$\lim_{n \to 0} \mathbb{E}\Big(\int_0^T \|\mathbf{\Psi}_s - \mathbf{\Psi}_s^n\|_{\mathcal{H}}^2 \, d[M]_s \Big) = 0.$$

Simply replacing W by M in Definitions A.1.2 and A.1.7 completes the construction, though we do not give the details here. Let's now move on to the more delicate matter of integration with respect to a local martingale. This begins again with notation for our set of integrands.

Definition A.1.15. *For a continuous local martingale N, define $\bar{\mathcal{I}}_N^{\mathcal{H},T}$ to be the class of progressively measurable processes $\mathbf{\Psi}$ such that*

$$\int_0^T \|\mathbf{\Psi}_s\|_{\mathcal{H}}^2 \, d[N]_s < \infty \quad \mathbb{P} - a.e. \tag{A.10}$$

Also define $\bar{\mathcal{I}}_N^{\mathcal{H}}$ to be those $\mathbf{\Psi}$ in $\bar{\mathcal{I}}_N^{\mathcal{H},T}$ for every T.

Suppose that N is localised by the stopping times (T_n). Without loss of generality this sequence of stopping times can be chosen such that the stopped processes N^{T_n} defined by

$$N_t^{T_n} := N_{t \wedge T_n}$$

are bounded; if (T_n') are localising stopping times, then we can simply set

$$T_n = T_n' \wedge \inf\{0 \le t < \infty : |N_t| \ge n\}$$

so that for each n, N^{T_n} is a bounded continuous martingale and hence in $\mathcal{M}_c^2$. Note of course that the new stopping times (T_n) are still non-decreasing and approach infinity $\mathbb{P} - a.s.$ by the pathwise continuity of N. Continuing in this manner, for a process $\mathbf{\Psi} \in \bar{\mathcal{I}}_N^{\mathcal{H}}$ let's define some more non-decreasing random times (R_n) by

$$R_n := n \wedge \inf\{0 \le t < \infty : \int_0^t \|\mathbf{\Psi}_s\|_{\mathcal{H}}^2 \, d[N]_s \ge n\} \tag{A.11}$$

taking the convention that the infimum of the empty set is infinite. The random variables (R_n) are stopping times as they are simply first hitting times of the continuous and adapted processes

$$\int_0^t \|\mathbf{\Psi}_s\|^2_{\mathcal{H}}\, d[N]_s.$$

Again these times tend to infinity $\mathbb{P}-a.s.$ by condition (A.10). Now define τ_n by

$$\tau_n = R_n \wedge T_n \tag{A.12}$$

and the truncated processes $\mathbf{\Psi}^n$ as

$$\mathbf{\Psi}^n_t := \mathbf{\Psi}_t \mathbb{1}_{t\leq\tau_n}.$$

Understanding the fact that for $m \leq n$, and $t \leq \tau_m$, we have $\mathbf{\Psi}_t\mathbb{1}_{t\leq\tau_n} = \mathbf{\Psi}_t\mathbb{1}_{t\leq\tau_m}$ and also that $N_t^{\tau_n} = N_t^{\tau_m}$, then eventually the sequence

$$\left(\int_0^t \mathbf{\Psi}^n_s dN_s^{\tau_n}\right)$$

is constant $\mathbb{P}-a.s.$. Thus, we can make the following definition.

Definition A.1.16. *In the setting described, we define*

$$\int_0^t \mathbf{\Psi}_s dN_s := \lim_{n\to\infty}\int_0^t \mathbf{\Psi}^n_s dN_s^{\tau_n} \tag{A.13}$$

for the limit taken $\mathbb{P}-a.s.$ in $\mathcal{H}$.

Of course to do this we require that at any n, $\mathbf{\Psi}^n \in \mathcal{I}^{\mathcal{H}}_{N^{\tau_n}}$: the process $(\mathbb{1}_{t\leq\tau_n})$ is progressively measurable, as it is both left continuous and adapted (adaptedness becomes clear when for each fixed t, we write the random variable $\mathbb{1}_{t\leq\tau_n}$ as $1-\mathbb{1}_{t>\tau_n}$). The square integrability in A.1.14 comes from the fact that the random variable

$$\int_0^t \|\mathbf{\Psi}^n_s\|^2_{\mathcal{H}}\, d[N^{\tau_n}]_s$$

is bounded by n $\mathbb{P}-a.s.$ (owing to (A.11)), hence the expectation satisfies the same bound. Where N is itself a genuine martingale, this procedure defines the stochastic integral for processes with only the regularity (A.10). In this case we do not have to stop the integrator, we just truncate the integrand.

Definition A.1.17. *In the special case where the continuous local martingale is a Brownian Motion, we denote $\bar{\mathcal{I}}^{\mathcal{H}}_W$ by simply $\bar{\mathcal{I}}^{\mathcal{H}}$. This class of processes differs from $\mathcal{I}^{\mathcal{H}}$ due to the assumption $\mathbb{P}-a.s.$, i.e. (A.10), as opposed to (A.3) in expectation.*

We extend properties of the stochastic integral to this class of processes.

Proposition A.1.18. *Let $\mathbf{\Psi} \in \bar{\mathcal{I}}^{\mathcal{H}}$ and $\phi \in L^\infty(\Omega;\mathcal{H})$ be $\mathcal{F}_0-$measurable. Then $\langle \mathbf{\Psi}, \phi\rangle \in \bar{\mathcal{I}}^{\mathcal{H}}$ and for every $t>0$ we have that*

$$\left\langle \int_0^t \mathbf{\Psi}_r dW_r, \phi \right\rangle_{\mathcal{H}} = \int_0^t \langle \mathbf{\Psi}_r, \phi\rangle_{\mathcal{H}}\, dW_r \tag{A.14}$$

$\mathbb{P}-a.s.$. *The above are random inner products defined by*

$$\langle \mathbf{\Psi}_s, \phi\rangle_{\mathcal{H}}(\omega) := \langle \mathbf{\Psi}_s(\omega), \phi(\omega)\rangle_{\mathcal{H}}$$

and similarly for the left hand side.

Proof. We should first justify that $\langle \mathbf{\Psi}_\cdot, \phi\rangle_{\mathcal{H}} \in \bar{\mathcal{I}}^{\mathbb{R}}$. The progressive measurability follows as for every $T > 0$ the mapping

$$\langle \mathbf{\Psi}_\cdot, \phi\rangle_{\mathcal{H}} : t \times \omega \times \tilde{\omega} \mapsto \langle \mathbf{\Psi}_t(\omega), \phi(\tilde{\omega})\rangle_{\mathcal{H}}$$

is $\mathcal{B}([0,T]) \times \mathcal{F}_T \times \mathcal{F}_0$ measurable, so in particular it is $\mathcal{B}([0,T]) \times \mathcal{F}_T \times \mathcal{F}_T$ measurable and as such

$$\langle \mathbf{\Psi}_\cdot, \phi\rangle_{\mathcal{H}} : t \times \omega \mapsto \langle \mathbf{\Psi}_t(\omega), \phi(\omega)\rangle_{\mathcal{H}}$$

is $\mathcal{B}([0,T]) \times \mathcal{F}_T$ measurable as required. Note that we have used the progressive measurability requirement on $\mathbf{\Psi}$. We also appreciate that for $\mathbb{P}-a.e.\ \omega$,

$$\int_0^T \langle \mathbf{\Psi}_r(\omega), \phi(\omega)\rangle_{\mathcal{H}}^2\, dr \leq \|\phi(\omega)\|_{\mathcal{H}}^2 \int_0^T \|\mathbf{\Psi}_r(\omega)\|_{\mathcal{H}}^2\, dr < \infty$$

again by assumption on $\mathbf{\Psi} \in \bar{\mathcal{I}}^{\mathcal{H}}$. Thus $\langle \mathbf{\Psi}, \phi\rangle_{\mathcal{H}} \in \bar{\mathcal{I}}^{\mathbb{R}}$. To compute the integrals we introduce the stopping times

$$\tau_j := j \wedge \inf\left\{0 \leq t < \infty : (1 + \|\phi\|_{\mathcal{H}}^2) \int_0^t \|\mathbf{\Psi}_r\|_{\mathcal{H}}^2\, dr \geq j\right\}$$

such that for every $j \in \mathbb{N}$, $\mathbf{\Psi}.\mathbb{1}_{\cdot \leq \tau_j} \in \mathcal{I}^{\mathcal{H}}$, $\left\langle \mathbf{\Psi}.\mathbb{1}_{\cdot\leq\tau_j}, \phi\right\rangle_{\mathcal{H}} \in \mathcal{I}^{\mathbb{R}}$. It is sufficient to show that

$$\left\langle \int_0^t \mathbf{\Psi}_r \mathbb{1}_{r \leq \tau^j} dW_r, \phi \right\rangle_{\mathcal{H}} = \int_0^t \left\langle \mathbf{\Psi}_r \mathbb{1}_{r\leq\tau^j}, \phi\right\rangle_{\mathcal{H}} dW_r \tag{A.15}$$

holds $\mathbb{P}-a.e.$ for every j. We now fix arbitrary $j \in \mathbb{N}$. Our plan is as follows: we consider a sequence of simple processes $(\mathbf{\Phi}^n)$ which approximate $\mathbf{\Psi}.\mathbb{1}_{\cdot\leq\tau_j}$ in $L^2(\Omega \times [0,t]; \mathcal{H})$ as postulated in Proposition A.1.6. We then claim that $(\langle \mathbf{\Phi}^n, \phi\rangle_{\mathcal{H}})$ is a sequence of $\mathbb{R}$ valued simple processes which converge to $\left\langle \mathbf{\Psi}.\mathbb{1}_{\cdot\leq\tau_j}, \phi\right\rangle_{\mathcal{H}}$ in $L^2(\Omega \times [0,t]; \mathbb{R})$. Following this, we prove (A.14) for this simple case and show the identity holds in the limit.

We first show that for each $n \in \mathbb{N}$, $\langle \mathbf{\Phi}^n, \phi\rangle_{\mathcal{H}}$ is a simple process. Let $\mathbf{\Phi}^n$ have representation as in Definition A.1.1. Then

$$\begin{aligned}\langle \mathbf{\Phi}^n, \phi\rangle_{\mathcal{H}} &= \left\langle a_0^n \mathbb{1}_{\{0\}} + \sum_{i=0}^{\infty} a_i^n \mathbb{1}_{(t_i^n, t_{i+1}^n]}, \phi \right\rangle_{\mathcal{H}} \\ &= \langle a_0^n, \phi\rangle_{\mathcal{H}} \mathbb{1}_{\{0\}} + \sum_{i=0}^{\infty} \langle a_i^n, \phi\rangle_{\mathcal{H}} \mathbb{1}_{(t_i^n, t_{i+1}^n]}\end{aligned}$$

so this would satisfy the requirements of an $\mathbb{R}$ valued simple process if for each $i \in \mathbb{N}$, $\langle a_i^n, \phi\rangle_{\mathcal{H}} \in L^2(\Omega;\mathbb{R})$ and is $\mathcal{F}_{t_i}-$measurable. For the square integrability constraint, observe that

$$\mathbb{E}\left(\langle a_i^n, \phi\rangle_{\mathcal{H}}^2\right) \leq \mathbb{E}\left(\|a_i^n\|_{\mathcal{H}}^2 \|\phi\|_{\mathcal{H}}^2\right) \leq \|\phi\|_{L^\infty(\Omega;\mathcal{H})}^2 \mathbb{E}\left(\|a_i^n\|_{\mathcal{H}}^2\right) < \infty$$

by the assumptions of $a_i \in L^2(\Omega;\mathcal{H})$ and $\phi \in L^\infty(\Omega;\mathcal{H})$. The $\mathcal{F}_{t_i}$ measurability follows in the same way as the progressive measurability of $\langle \mathbf{\Psi}, \phi\rangle_{\mathcal{H}}$. Indeed the required $L^2(\Omega \times [0,t];\mathbb{R})$ convergence follows similarly as

$$\begin{aligned}
\left\|\left\langle \mathbf{\Psi}_\cdot \mathbb{1}_{\cdot \leq \tau_j}, \phi\right\rangle_{\mathcal{H}} - \langle \mathbf{\Phi}^n_\cdot, \phi\rangle_{\mathcal{H}}\right\|_{L^2(\Omega\times[0,t];\mathbb{R})} &= \left\|\left\langle \mathbf{\Psi}_\cdot \mathbb{1}_{\cdot \leq \tau_j} - \mathbf{\Phi}^n, \phi\right\rangle_{\mathcal{H}}\right\|_{L^2(\Omega\times[0,t];\mathbb{R})} \\
&\leq \left\|\left\|\mathbf{\Psi}_\cdot \mathbb{1}_{\cdot \leq \tau_j} - \mathbf{\Phi}^n\right\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}}\right\|_{L^2(\Omega\times[0,t];\mathbb{R})} \\
&\leq \|\phi\|_{L^\infty(\Omega;\mathcal{H})} \left\|\left\|\mathbf{\Psi}_\cdot \mathbb{1}_{\cdot \leq \tau_j} - \mathbf{\Phi}^n\right\|_{\mathcal{H}}\right\|_{L^2(\Omega\times[0,t];\mathbb{R})}
\end{aligned}$$

and by assumption

$$\left\|\left\|\mathbf{\Psi}_\cdot \mathbb{1}_{\cdot \leq \tau_j} - \mathbf{\Phi}^n\right\|_{\mathcal{H}}\right\|_{L^2(\Omega\times[0,t];\mathbb{R})} \longrightarrow 0$$

as $n \to \infty$, so the convergence is proved. To show the identity (A.14) in the case of the simple process $\mathbf{\Phi}^n$, observe that

$$\begin{aligned}
\left\langle \int_0^t \mathbf{\Phi}^n_r dW_r, \phi\right\rangle_{\mathcal{H}} &= \left\langle \sum_{i=0}^\infty a_i^n (W_{t^n_{i+1}\wedge t} - W_{t^n_i \wedge t}), \phi\right\rangle_{\mathcal{H}} \\
&= \sum_{i=0}^\infty \left\langle a_i^n \left(W_{t^n_{i+1}\wedge t} - W_{t^n_i \wedge t}\right), \phi\right\rangle_{\mathcal{H}} \\
&= \sum_{i=0}^\infty \langle a_i^n, \phi\rangle_{\mathcal{H}} \left(W_{t^n_{i+1}\wedge t} - W_{t^n_i \wedge t}\right) \\
&= \int_0^t \langle \mathbf{\Phi}^n_r, \phi\rangle_{\mathcal{H}} dW_r
\end{aligned}$$

as required. In order to conclude the argument, by definition of the integral we have that

$$\begin{aligned}
\int_0^t \mathbf{\Psi}_r \mathbb{1}_{r \leq \tau^j} dW_r &= \lim_{n\to\infty} \int_0^t \mathbf{\Phi}^n_r dW_r \\
\int_0^t \left\langle \mathbf{\Psi}_r \mathbb{1}_{r \leq \tau^j}, \phi\right\rangle_{\mathcal{H}} dW_r &= \lim_{n\to\infty} \int_0^t \langle \mathbf{\Phi}^n_r, \phi\rangle_{\mathcal{H}} dW_r,
\end{aligned}$$

where the first limit is taken in $L^2(\Omega;\mathcal{H})$ and the second one in $L^2(\Omega;\mathbb{R})$. For each we can thus extract a $\mathbb{P}-a.s.$ convergent subsequence in the appropriate space, so by taking successive subsequences we can find one common subsequence indexed by (n_k) such that the above limits hold $\mathbb{P}-a.s.$. Thus

$$\begin{aligned}\left\langle \int_0^t \mathbf{\Psi}_r \mathbb{1}_{r\leq\tau^j} dW_r, \phi \right\rangle_{\mathcal{H}} &= \left\langle \lim_{n_k\to\infty} \int_0^t \mathbf{\Phi}_r^{n_k} dW_r, \phi \right\rangle_{\mathcal{H}} \\ &= \lim_{n_k\to\infty} \left\langle \int_0^t \mathbf{\Phi}_r^{n_k} dW_r, \phi \right\rangle_{\mathcal{H}} \\ &= \lim_{n_k\to\infty} \int_0^t \langle \mathbf{\Phi}_r^n, \phi\rangle_{\mathcal{H}} dW_r \\ &= \int_0^t \left\langle \mathbf{\Psi}_r \mathbb{1}_{r\leq\tau^j}, \phi\right\rangle_{\mathcal{H}} dW_r\end{aligned}$$

so (A.15) is justified and the proof is complete.

□

We note that the $\mathcal{F}_0-$measurability requirement on ϕ really comes into play in showing the $\mathcal{F}_{t_i}-$measurability of $\langle a_i^n, \phi\rangle_{\mathcal{H}}$. If we were to consider the integral over some $[s,t]$ interval instead, then one could relax ϕ to only being $\mathcal{F}_s-$measurable. In fact the result can also be extended to unbounded ϕ. To do this we shall prove a Stochastic Dominated Convergence Theorem.

Lemma A.1.19. *Let $(\mathbf{\Psi}^n)$ be a sequence in $\bar{\mathcal{I}}^{\mathcal{H}}$ such that there exists processes $\mathbf{\Psi}: \Omega\times[0,\infty)\to\mathcal{H}$ and $\mathbf{\Phi}\in L^2([0,\infty);\mathbb{R})$ $\mathbb{P}-a.s.$, with the properties that for every $T>0$, $\mathbb{P}\times\lambda-a.e.$ $(\omega,t)\in\Omega\times[0,T]$:*

1. *$\|\mathbf{\Psi}_t^n(\omega)\|_{\mathcal{H}} \leq |\mathbf{\Phi}_t(\omega)|$ for all $n\in\mathbb{N}$;*
2. *$(\mathbf{\Psi}_t^n(\omega))$ is convergent to $\mathbf{\Psi}_t(\omega)$ in $\mathcal{H}$.*

Then $\mathbf{\Psi}\in\bar{\mathcal{I}}^{\mathcal{H}}$ and for every $t>0$, there exists a subsequence indexed by (n_k) such that

$$\lim_{n_k\to\infty}\int_0^t \mathbf{\Psi}_r^{n_k} dW_r = \int_0^t \mathbf{\Psi}_r dW_r \tag{A.16}$$

$\mathbb{P}-a.s.$.

Proof. Immediately we note that $\mathbf{\Psi}$ inherits the progressive measurability from $\mathbf{\Psi}^n$ from the almost everywhere limit in the product space $\Omega\times[0,T]$ when equipped with product sigma-algebra $\mathcal{F}_T\times\mathcal{B}([0,T])$. Similarly we must have that for $\mathbb{P}\times\lambda-a.e.$ (ω,t), $\|\mathbf{\Psi}_t(\omega)\|_{\mathcal{H}}\leq|\mathbf{\Phi}_t(\omega)|$ so $\mathbf{\Psi}\in L^2([0,\infty);\mathcal{H})$ $\mathbb{P}-a.s.$ hence belongs to $\bar{\mathcal{I}}^{\mathcal{H}}$. We look to find a common sequence of localising times for the stochastic integrals, and then demonstrate (A.16) by showing the identity holds true when stopped at each localising time. To this end we introduce the stopping times

$$\tau_j := j\wedge\inf\left\{0\leq t<\infty : \int_0^t|\mathbf{\Phi}_r|^2dr\geq j\right\}$$

which from the assumption 1 serve as a sequence of localising times for every $\mathbf{\Psi}^n$, and too for $\mathbf{\Psi}$. Thus for any fixed $t > 0$ and $j \in \mathbb{N}$ we wish to show that

$$\lim_{n_k \to \infty} \int_0^t \mathbf{\Psi}_r^{n_k} \mathbb{1}_{r \leq \tau_j} dW_r = \int_0^t \mathbf{\Psi}_r \mathbb{1}_{r \leq \tau_j} dW_r \tag{A.17}$$

for a subsequence (n_k) $\mathbb{P} - a.s.$, or equivalently that

$$\lim_{n_k \to \infty} \int_0^t (\mathbf{\Psi}_r^{n_k} - \mathbf{\Psi}_r) \mathbb{1}_{r \leq \tau_j} dW_r = 0.$$

We first assess the convergence in $L^2(\Omega; \mathcal{H})$, applying Corollary A.1.8.1 for each fixed n to see that

$$\mathbb{E} \left\| \int_0^t (\mathbf{\Psi}_r^n - \mathbf{\Psi}_r) \mathbb{1}_{r \leq \tau_j} dW_r \right\|_{\mathcal{H}}^2 = \mathbb{E} \left(\int_0^t \left\| (\mathbf{\Psi}_r^n - \mathbf{\Psi}_r) \mathbb{1}_{r \leq \tau_j} \right\|_{\mathcal{H}}^2 dr \right).$$

Observing that for $\mathbb{P} \times \lambda - a.e.$ (ω, t),

$$\begin{aligned} \left\| (\mathbf{\Psi}_r^n(\omega) - \mathbf{\Psi}_r(\omega)) \mathbb{1}_{r \leq \tau_j}(\omega) \right\|_{\mathcal{H}}^2 &\leq \left(\left\| \mathbf{\Psi}_r^n(\omega) \mathbb{1}_{r \leq \tau_j(\omega)} \right\|_{\mathcal{H}} + \left\| \mathbf{\Psi}_r(\omega) \mathbb{1}_{r \leq \tau_j}(\omega) \right\|_{\mathcal{H}} \right)^2 \\ &\leq 4 |\mathbf{\Phi}_r(\omega) \mathbb{1}_{r \leq \tau_j}(\omega)|^2. \end{aligned}$$

Then with dominating function $4|\mathbf{\Phi}.\mathbb{1}_{\cdot \leq \tau_j}|^2$ we can apply the standard Dominated Convergence Theorem for the integral over the product space (we face no problems with the order of integration from Tonelli's Theorem given the progressive measurability) to deduce that

$$\lim_{n \to \infty} \mathbb{E} \left(\int_0^t \left\| (\mathbf{\Psi}_r^n - \mathbf{\Psi}_r) \mathbb{1}_{r \leq \tau_j} \right\|_{\mathcal{H}}^2 dr \right) = 0$$

and therefore

$$\lim_{n \to \infty} \mathbb{E} \left\| \int_0^t (\mathbf{\Psi}_r^n - \mathbf{\Psi}_r) \mathbb{1}_{r \leq \tau_j} dW_r \right\|_{\mathcal{H}}^2 = 0.$$

Thus we have justified the convergence (A.17) in the sense of $L^2(\Omega; \mathcal{H})$, from which we can deduce a $\mathbb{P} - a.s.$ convergent subsequence and the result is proved.

□

Proposition A.1.20. *Let $\mathbf{\Psi} \in \bar{\mathcal{I}}^{\mathcal{H}}$ and $\phi : \Omega \to \mathcal{H}$ be $\mathcal{F}_0-$measurable. Then $\langle \mathbf{\Psi}, \phi \rangle \in \bar{\mathcal{I}}^{\mathcal{H}}$ and for every $t > 0$ we have that*

$$\left\langle \int_0^t \mathbf{\Psi}_r dW_r, \phi \right\rangle_{\mathcal{H}} = \int_0^t \langle \mathbf{\Psi}_r, \phi \rangle_{\mathcal{H}} dW_r \tag{A.18}$$

$\mathbb{P} - a.s.$.

Proof. A justification that $\langle \mathbf{\Psi}_r, \phi \rangle_{\mathcal{H}} \in \bar{\mathcal{I}}^{\mathbb{R}}$ is precisely as in Proposition A.1.18. To apply this result, we rewrite ϕ in a trivial way as

$$\phi := \sum_{k=1}^{\infty} \phi \mathbb{1}_{k \leq \|\phi\|_{\mathcal{H}} < k+1}$$

where the limit is taken $\mathbb{P}-a.s.$ in $\mathcal{H}$ (similarly to (A.2) this is just a finite sum at each fixed ω, or more precisely just a single element of the sum). Introducing the notation

$$\phi^n := \sum_{k=1}^{n} \phi \mathbb{1}_{k \leq \|\phi\|_{\mathcal{H}} < k+1}$$

then clearly $\phi^n \in L^\infty(\Omega;\mathcal{H})$ and is still $\mathcal{F}_0-$measurable, so we can apply Proposition A.1.18 to see that

$$\left\langle \int_0^t \mathbf{\Psi}_r dW_r, \phi^n \right\rangle_{\mathcal{H}} = \int_0^t \langle \mathbf{\Psi}_r, \phi^n \rangle_{\mathcal{H}} dW_r.$$

We can take the $\mathbb{P}-a.s.$ limit in $\mathcal{H}$ outside of the inner product on the left hand side, so it is sufficient to show that

$$\lim_{n\to\infty} \int_0^t \langle \mathbf{\Psi}_r, \phi^n \rangle_{\mathcal{H}} dW_r = \int_0^t \langle \mathbf{\Psi}_r, \phi \rangle_{\mathcal{H}} dW_r \tag{A.19}$$

or at least that this is true for a subsequence. This is an immediate application of Lemma A.1.19, with dominating function simply the limit $\langle \mathbf{\Psi}, \phi \rangle_{\mathcal{H}}$.

□

The same is true for multiplication by real valued random variables, where the proof is identical. We state the result here.

Proposition A.1.21. *Let $\mathbf{\Psi} \in \bar{\mathcal{I}}^{\mathcal{H}}$ and $\eta : \Omega \to \mathbb{R}$ be $\mathcal{F}_0-$measurable. Then $\eta\mathbf{\Psi} \in \bar{\mathcal{I}}^{\mathcal{H}}$ and for every $t > 0$ we have that*

$$\eta \int_0^t \mathbf{\Psi}_r dW_r = \int_0^t \eta \mathbf{\Psi}_r dW_r$$

$\mathbb{P}-a.s.$.

A.1.3 Cylindrical Brownian Motion

Having now addressed the question of how to integrate a Hilbert Space valued process with respect to a finite dimensional local martingale, we look to extend this theory to the case of an infinite dimensional martingale. The aforementioned construction then arises from the one dimensional projections of the driving process. Our notion of infinite dimensional Brownian Motion is that of a Cylindrical Brownian Motion, which we explore in this subsection. Additional operator theory is used in this subsection to better understand the background of the process, however it is not necessary in the resulting integral constructed in Subsection A.1.5. We will denote by Q a bounded positive self-adjoint operator on $\mathcal{H}$, that is $Q \in \mathscr{L}(\mathcal{H};\mathcal{H})$, and $t \wedge s$ to be the minimum of $t, s \geq 0$.

Definition A.1.22. *A Q-Cylindrical Brownian Motion over $\mathcal{H}$ is a process $\mathcal{W}^Q$ taking values in the space of functions from $\mathcal{H}$ to $\mathbb{R}$, that is*

$$\mathcal{W}^Q : \Omega \times [0,\infty) \to \mathbb{R}^{\mathcal{H}}$$

such that for each $h \in \mathcal{H}$ and $t \geq 0$, $\mathcal{W}_h^Q(t) : \Omega \to \mathbb{R}$ is a zero mean Gaussian random variable, and for every $g, h \in \mathcal{H}$ and all times $t, s \geq 0$,

$$\mathbb{E}\big(\mathcal{W}_g^Q(t)\mathcal{W}_h^Q(s)\big) = \langle Qg, h\rangle_{\mathcal{H}}\,(t \wedge s).$$

Remark A.1.23. *We note that this definition is not restricted to a Brownian Motion; one may consider any zero-mean Gaussian random variable with a correlation function $R(t,s)$ replacing that of the Brownian Motion $t \wedge s$. We consider only the Brownian Motion case here to maintain a direct approach of the key principles, though the general case is defined in [105] Subsection 3.2.2.*

To make this abstract definition workable, we look to how such processes can be represented. This motivates the notion of a regular process.

Definition A.1.24. *A Q-Cylindrical Brownian Motion $\mathcal{W}^Q$ is said to be regular if there exists a square integrable $\mathcal{H}$-valued process $\widetilde{\mathcal{W}}^Q$ (that is, $\widetilde{\mathcal{W}}_t^Q \in L^2\big(\Omega;\mathcal{H}\big)$ for all $t \geq 0$) such that for every $h \in \mathcal{H}$, $\mathcal{W}_h^Q$ has the same law as the process $\left\langle \widetilde{\mathcal{W}}^Q, h\right\rangle_{\mathcal{H}}$ defined by*

$$\left\langle \widetilde{\mathcal{W}}^Q, h\right\rangle_{\mathcal{H}}(t,\omega) := \left\langle \widetilde{\mathcal{W}}^Q(t,\omega), h\right\rangle_{\mathcal{H}}.$$

We will not hesitate to identify the functional valued process $\mathcal{W}^Q$ with the Hilbert space one $\widetilde{\mathcal{W}}^Q$. In the next theorem however, we keep the distinction for clarity.

Theorem A.1.25. *A Q-Cylindrical Brownian Motion $\mathcal{W}^Q$ is regular if and only if $Q \in \mathscr{L}^1(\mathcal{H};\mathcal{H})$, that is, Q is trace-class. In this case $\mathcal{W}^Q$ admits the regular representation*

$$\widetilde{\mathcal{W}}^Q(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}e_i W_t^i \tag{A.20}$$

for $t \geq 0$, where the (e_i) are an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of the self-adjoint trace-class (hence compact) operator Q, (λ_i) are the corresponding eigenvalues, and (W^i) are independent one dimensional Brownian Motions. The infinite sum is taken as a limit in $L^2\big(\Omega;\mathcal{H}\big)$.

Proof. We consider the two implications. In the first one we show that if Q is trace-class then (A.20) is a regular representation of $\mathcal{W}^Q$. In the second we show that if $\mathcal{W}^Q$ is regular then Q is trace-class, so by the first implication, (A.20) is again a regular representation of $\mathcal{W}^Q$.

$\Longleftarrow$: A sensible place to start would be to verify that for trace-class Q, (A.20) does indeed define an element of $L^2\big(\Omega;\mathcal{H}\big)$. We will rely on completeness of the space and show that the sequence of partial sums is Cauchy. Observe that

$$\mathbb{E}\Big(\Big\|\sum_{i=m}^{n}\sqrt{\lambda_i}e_iW_t^i\Big\|_{\mathcal{H}}^2\Big) = \mathbb{E}\Big(\sum_{i=m}^{n}|\lambda_i||W_t^i|^2\Big) = \mathbb{E}\big(|W_t|^2\big)\sum_{i=m}^{n}\lambda_i.$$

All we then require to conclude the Cauchy property is that $\sum_{i=1}^{\infty}\lambda_i<\infty$ which it is given that Q is trace class. To conclude that (A.20) is a regular representation of $\mathcal{W}^Q$, it is sufficient to show that the one dimensional processes $\left\langle \widetilde{\mathcal{W}}^Q, h\right\rangle_{\mathcal{H}}$ satisfy the conditions postulated of the $\mathcal{W}_h^Q$ in Definition A.1.22 as these characterise the distribution.

First of all for each $h\in\mathcal{H}$ and time t, we must verify that the random variable

$$\omega \mapsto \Big\langle \sum_{i=1}^{\infty}\sqrt{\lambda_i}e_i W_t^i(\omega), h\Big\rangle_{\mathcal{H}} \tag{A.21}$$

is zero mean Gaussian. Note that

$$\Big\langle \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\lambda_i}e_i W_t^i, h\Big\rangle_{\mathcal{H}} = \lim_{n\to\infty}\Big\langle \sum_{i=1}^{n}\sqrt{\lambda_i}e_i W_t^i, h\Big\rangle_{\mathcal{H}},$$

where the second limit is in $L^2\big(\Omega;\mathbb{R}\big)$ just as we did in Theorem A.1.9. The random variable (A.21) is thus an L^2 limit of zero mean Gaussian random variables, so is itself zero mean Gaussian (convergence in L^2 implies that in distribution so we have Gaussianity, and L^2 convergence implies L^1 from which we readily deduce the zero mean property). It remains to show that for each $g,h\in\mathcal{H}$ and $s,t>0$ that

$$\mathbb{E}\bigg(\Big\langle \sum_{i=1}^{\infty}\sqrt{\lambda_i}e_i W_t^i, g\Big\rangle_{\mathcal{H}}\Big\langle \sum_{j=1}^{\infty}\sqrt{\lambda_j}e_i W_s^j, h\Big\rangle_{\mathcal{H}}\bigg) = \langle Qg,h\rangle_{\mathcal{H}}\,(t\wedge s).$$

We take the limit through the first inner product on the LHS as above, that is

$$\mathbb{E}\Bigg(\bigg(\lim_{n\to\infty}\Big\langle \sum_{i=1}^{n}\sqrt{\lambda_i}e_i W_t^i, g\Big\rangle_{\mathcal{H}}\bigg)\Big\langle \sum_{j=1}^{\infty}\sqrt{\lambda_j}e_i W_s^j, h\Big\rangle_{\mathcal{H}}\Bigg) \tag{A.22}$$

and argue that for a sequence of functions (f_n) convergent to f in $L^2\big(\Omega;\mathbb{R}\big)$, and $a\in L^2\big(\Omega;\mathbb{R}\big)$, that

$$(\lim_{L^2} f_n)(a) = \lim_{L^1}(f_n a).$$

The right side is well defined as the limit of a Cauchy sequence:

$$\mathbb{E}\big(|f_n a - f_m a|\big) = \mathbb{E}\big(|(f_n-f_m)a|\big) \le \mathbb{E}\big(|f_n-f_m|^2\big)\mathbb{E}\big(|a|^2\big)$$

and a similar calculation shows that this element of $L^1\big(\Omega;\mathbb{R}\big)$ is the left side. Applying to (A.22) and pulling the L^1 limit through the expectation produces

$$\lim_{n\to\infty}\mathbb{E}\bigg(\Big\langle \sum_{i=1}^{n}\sqrt{\lambda_i}e_i W_t^i, g\Big\rangle_{\mathcal{H}}\Big\langle \sum_{j=1}^{\infty}\sqrt{\lambda_j}e_j W_s^j, h\Big\rangle_{\mathcal{H}}$$

Playing the same game, this is

$$\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{E}\bigg(\Big\langle \sum_{i=1}^{n}\sqrt{\lambda_i}e_i W_t^i, g\Big\rangle_{\mathcal{H}}\Big\langle \sum_{j=1}^{m}\sqrt{\lambda_j}e_j W_s^j, h\Big\rangle_{\mathcal{H}}\bigg)$$

and further

$$\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{E}\Big(\sum_{i=1}^{n}\sum_{j=1}^{m}\langle\sqrt{\lambda_i}e_i,g\rangle_{\mathcal{H}}\langle\sqrt{\lambda_j}e_j,h\rangle_{\mathcal{H}}W_t^iW_s^j\Big).$$

Independence of the Brownian Motions (W^i) and the fact that $t\wedge s$ is the correlation function of Brownian Motion gives that this is equal to

$$\lim_{n\to\infty}\sum_{i=1}^{n}\langle\sqrt{\lambda_i}e_i,g\rangle_{\mathcal{H}}\langle\sqrt{\lambda_i}e_i,h\rangle_{\mathcal{H}}(t\wedge s)=\lim_{n\to\infty}\sum_{i=1}^{n}\langle\lambda_ie_i,g\rangle_{\mathcal{H}}\langle e_i,h\rangle_{\mathcal{H}}(t\wedge s)$$

which is just

$$\Big\langle\sum_{i=1}^{\infty}\lambda_i\,\langle e_i,g\rangle\,e_i,h\Big\rangle_{\mathcal{H}}(t\wedge s)=\langle Qg,h\rangle_{\mathcal{H}}\,(t\wedge s)$$

as required.

$\Longrightarrow$: For the reverse direction, assume that $\mathcal{W}^Q$ is regular with corresponding process $\widetilde{\mathcal{W}}^Q$ (which is not necessarily of the form (A.20)). We want an expression in terms of the trace of Q, so we exploit Definition A.1.22:

$$\sum_{i=1}^{\infty}\mathbb{E}\big(|\mathcal{W}^Q_{e_i}(t)|^2\big)=\sum_{i=1}^{\infty}\langle Qe_i,e_i\rangle_{\mathcal{H}}\,t$$

and use an alternative expression from the assumed regular representation:

$$\begin{aligned}\sum_{i=1}^{\infty}\mathbb{E}\big(|\mathcal{W}^Q_{e_i}(t)|^2\big)&=\sum_{i=1}^{\infty}\mathbb{E}\big(\big\langle\widetilde{\mathcal{W}}^Q(t),e_i\big\rangle_{\mathcal{H}}^2\big)\\&=\mathbb{E}\Big(\sum_{i=1}^{\infty}\big\langle\widetilde{\mathcal{W}}^Q(t),e_i\big\rangle_{\mathcal{H}}^2\Big)=\mathbb{E}\big(\big\|\widetilde{\mathcal{W}}^Q(t)\big\|_{\mathcal{H}}^2\big)<\infty\end{aligned}$$

where the infinite sum is pulled inside the expectation from the Monotone Convergence Theorem, and finiteness is by assumption on $\widetilde{\mathcal{W}}^Q\in L^2\big(\Omega;\mathcal{H}\big)$. Hence Q is trace-class, and by the first implication, (A.20) is a regular representation of $\mathcal{W}^Q$.

□

Remark A.1.26. *A standard Cylindrical Brownian Motion, that is where Q is the identity, is thus not regular.*

In light of this remark, it would be convenient to have such a representation for Cylindrical Brownian Motion (denoted simply by $\mathcal{W}$); we would like this to be something along the lines of

$$\mathcal{W}_t=\sum_{i=1}^{\infty}e_iW_t^i,\tag{A.23}$$

where the (e_i) form an orthonormal basis of $\mathcal{H}$, and (W^i) are real-valued independent Brownian Motions. We can in fact explicitly construct a larger Hilbert space $\mathcal{H}'$ such that the inclusion mapping $J:\mathcal{H}\hookrightarrow\mathcal{H}'$ is Hilbert-Schmidt. The composition $Q:=JJ^*$ is then trace-class on $\mathcal{H}'$, and indeed $\mathcal{W}$ is a Q-Cylindrical Brownian Motion on $\mathcal{H}$': we defer the details to e.g.

[105] Problem 3.2.6. To be precise, that is $J(e_i) = \sqrt{\lambda_i}\eta_i$ for each i, where the (η_i) form an orthonormal basis (of $\mathcal{H}'$) of eigenfunctions of JJ^* with eigenvalues λ_i, and for any $h \in \mathcal{H}$,

$$\mathcal{W}_h(t) = \left\langle \sum_{i=1}^{\infty} J(e_i)W_t^i, J(h) \right\rangle_{\mathcal{H}'}.$$

In this spirit, we consider (A.23) as a formal representation of Cylindrical Brownian Motion.

A.1.4 Martingale Theory in Hilbert Spaces

We first define the notion of a Hilbert space valued martingale, before addressing martingale properties of the stochastic integral.

Definition A.1.27. *A process M taking values in a Hilbert space $\mathcal{H}$ is said to be a martingale if for every $h \in \mathcal{H}$, the process $\langle M, h\rangle_{\mathcal{H}}$ is a real valued martingale. The martingale is said to be continuous if for $\mathbb{P}-a.e.$ ω and every $T > 0$, $M(\omega) : [0, T] \to \mathcal{H}$ is continuous. The martingale is said to be square integrable if for every $t \geq 0$, $\mathbb{E}\big(\|M_t\|_{\mathcal{H}}^2\big) < \infty$. The class of continuous square integrable martingales will be denoted $\mathcal{M}_c^2(\mathcal{H})$.*[3]

We look to show that $\mathcal{M}_c^2(\mathcal{H})$ is closed in a relevant topology, for which we shall use a sufficient condition for conitnuity proven now.

Lemma A.1.28. *Let $\psi \in C_w([0,T]; \mathcal{H})$ and $\|\psi\|_{\mathcal{H}}^2 \in C([0,T]; \mathbb{R})$, that is, ψ is weakly continuous with continuous norm. Then $\psi \in C([0,T]; \mathcal{H})$.*

Proof. We fix some $t \in [0,T]$ and look to show that ψ is continuous at t. To this end consider arbitrary $s \in [0,T]$. Then

$$\begin{aligned}
\|\psi_t - \psi_s\|_{\mathcal{H}}^2 &= \langle \psi_t - \psi_s, \psi_t - \psi_s\rangle_{\mathcal{H}} \\
&= \langle \psi_t - \psi_s, \psi_t\rangle_{\mathcal{H}} - \langle \psi_t - \psi_s, \psi_s\rangle_{\mathcal{H}} \\
&= \langle \psi_t - \psi_s, \psi_t\rangle_{\mathcal{H}} + \|\psi_s\|_{\mathcal{H}}^2 - \langle \psi_t, \psi_s\rangle_{\mathcal{H}} \\
&= \langle \psi_t - \psi_s, \psi_t\rangle_{\mathcal{H}} + \|\psi_s\|_{\mathcal{H}}^2 + \langle \psi_t, \psi_t - \psi_s\rangle_{\mathcal{H}} - \|\psi_t\|_{\mathcal{H}}^2 \\
&= 2\langle \psi_t - \psi_s, \psi_t\rangle_{\mathcal{H}} + \Big(\|\psi_s\|_{\mathcal{H}}^2 - \|\psi_t\|_{\mathcal{H}}^2\Big).
\end{aligned}$$

For any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $s \in [0 \vee (t-\delta), (t+\delta) \wedge T]$,

$$\langle \psi_t - \psi_s, \psi_t\rangle_{\mathcal{H}} < \frac{\varepsilon}{3},$$
$$\|\psi_s\|_{\mathcal{H}}^2 - \|\psi_t\|_{\mathcal{H}}^2 < \frac{\varepsilon}{3}$$

from each of the two assumptions, which gives the result. ☐

[3] Recall that $\mathcal{M}_c^2 := \mathcal{M}_c^2(\mathbb{R})$, see Definition A.1.13.

Proposition A.1.29. *Let (M^n) be a sequence in $\mathcal{M}_c^2(\mathcal{H})$ and M a process with values in $\mathcal{H}$ such that at every time $t \geq 0$,*

$$\lim_{n\to\infty} \mathbb{E}\left(\|M_t^n - M_t\|_{\mathcal{H}}^2\right) = 0.$$

Then $M \in \mathcal{M}_c^2(\mathcal{H})$.

Proof. The square integrability is trivial, so we just consider the martingale property and continuity. For every $\phi \in \mathcal{H}$, at every $t \geq 0$, we have that

$$\begin{aligned}\mathbb{E}\left(|\langle M_t^n, \phi\rangle_{\mathcal{H}} - \langle M_t, \phi\rangle_{\mathcal{H}}|\right) = \mathbb{E}\left(|\langle M_t^n - M_t, \phi\rangle_{\mathcal{H}}|\right) &\leq \|\phi\|_{\mathcal{H}}\, \mathbb{E}\left(\|M_t^n - M_t\|_{\mathcal{H}}\right)\\ &\leq \|\phi\|_{\mathcal{H}} \left(\mathbb{E}\left(\|M_t^n - M_t\|_{\mathcal{H}}^2\right)\right)^{\frac{1}{2}}\end{aligned}$$

which converges to zero as $n \to \infty$. In particular, $(\langle M_t^n, \phi\rangle_{\mathcal{H}})$ converges to $\langle M_t, \phi\rangle_{\mathcal{H}}$ in $L^1(\Omega; \mathbb{R})$, and as the former is a sequence of real-valued martingales by definition, then it is standard that $\langle M, \phi\rangle_{\mathcal{H}}$ is again a martingale. As ϕ was arbitrary, we deduce the martingality of M. It remains to show pathwise continuity. For this we introduce the finite dimensional projections $(\mathcal{P}_k)$, defined by

$$\mathcal{P}_k : \phi \mapsto \sum_{i=1}^k \langle \phi, e_i\rangle_{\mathcal{H}}\, e_i$$

for (e_i) a fixed orthonormal basis of $\mathcal{H}$. The goal is to first show that $\mathcal{P}_k M$ has continuous paths for each k; as $\mathcal{P}_k$ is an orthogonal projection in $\mathcal{H}$, it is clear that at every $t \geq 0$,

$$\lim_{n\to\infty} \mathbb{E}\left(\|\mathcal{P}_k\left(M_t^n - M_t\right)\|_{\mathcal{H}}^2\right) = 0$$

and in particular

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{i=1}^k \langle M_t^n - M_t, e_i\rangle_{\mathcal{H}}^2\right) = 0.$$

We can now exploit equivalent results from the finite dimensional theory: note that

$$\lim_{n\to\infty} \mathbb{E}\left(\left|\sum_{i=1}^k \langle M_t^n, e_i\rangle - \sum_{i=1}^k \langle M_t, e_i\rangle_{\mathcal{H}}\right|^2\right) \leq k \lim_{n\to\infty} \mathbb{E}\left(\sum_{i=1}^k \langle M_t^n - M_t, e_i\rangle_{\mathcal{H}}^2\right) = 0,$$

so the sequence in n, $\left(\sum_{i=1}^k \langle M^n, e_i\rangle\right)$, of real valued martingales is convergent in $L^2(\Omega; \mathbb{R})$ at every $t \geq 0$. From Proposition A.4.1 we have that $\mathcal{M}_c^2$ is closed in this topology hence $\sum_{i=1}^k \langle M, e_i\rangle$ is continuous. As this is true for every $k \in \mathbb{N}$, we may take differences to see that $\langle M, e_i\rangle$ is continuous for every i, and therefore $\mathcal{P}_k M$ is continuous in $\mathcal{H}$ for each k (of course, still pathwise $\mathbb{P} - a.s.$). In fact we take the same approach as the referenced proposition, working directly with the sequence of submartingales $(\sum_{i=1}^k \langle M, e_i\rangle_{\mathcal{H}}^2)$ indexed by k. For any

$\varepsilon > 0$, by Proposition A.4.2, we have that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \sup_{t\in[0,T]} \sum_{i=k+1}^{j} \langle M_t(\omega), e_i\rangle_{\mathcal{H}}^2 > \varepsilon\right\}\right) \leq \frac{1}{\varepsilon}\mathbb{E}\left(\sum_{i=k+1}^{j} \langle M_T, e_i\rangle_{\mathcal{H}}^2\right), \tag{A.24}$$

but we have already established that $\mathbb{E}\left(\|M_T\|_{\mathcal{H}}^2\right) < \infty$, or equivalently

$$\mathbb{E}\left(\sum_{i=1}^{\infty} \langle M_t(\omega), e_i\rangle_{\mathcal{H}}^2\right) < \infty.$$

Therefore in the limit as $k \to \infty$, (A.24) approaches zero uniformly in $j > k$. One may choose a subsequence such that for all $k_l < k_m$,

$$\mathbb{P}\left(\left\{\omega \in \Omega : \sup_{t\in[0,T]} \sum_{i=k_l+1}^{k_m} \langle M_t(\omega), e_i\rangle_{\mathcal{H}}^2 > \frac{1}{l}\right\}\right) \leq \frac{1}{2^l}.$$

By the Borel Cantelli Lemma the subsequence $\sum_{i=1}^{k_l} \langle M, e_i\rangle_{\mathcal{H}}^2$ is Cauchy in $C([0,T];\mathbb{R})$ $\mathbb{P}-a.s.$. It thus admits a limit $\mathbb{P}-a.s.$ in $C([0,T];\mathbb{R})$, which agrees with the limit at each t, given by $\sum_{i=1}^{\infty} \langle M_t, e_i\rangle_{\mathcal{H}}^2$ which is of course $\|M_t\|_{\mathcal{H}}^2$. Therefore, the process $\|M\|_{\mathcal{H}}^2$ is pathwise continuous; from Lemma A.1.28, it is now sufficient to just show weak continuity. This is clear however, as for any given $\phi \in \mathcal{H}$ and $t \geq 0$, we have again that

$$\lim_{n\to\infty} \mathbb{E}\left(\langle M_t^n - M_t, \phi\rangle_{\mathcal{H}}^2\right) = 0,$$

so $\langle M, \phi\rangle_{\mathcal{H}}$ is shown to belong to $\mathcal{M}_c^2$ exactly as was done for $\sum_{i=1}^{k} \langle M, e_i\rangle_{\mathcal{H}}$, thus concluding the proof.

□

Proposition A.1.30. *For a standard real valued Brownian Motion W and $\mathbf{\Psi} \in \mathcal{I}^{\mathcal{H}}$, the Itô stochastic integral*

$$\int_0^{\cdot} \mathbf{\Psi}_s dW_s$$

belongs to $\mathcal{M}_c^2(\mathcal{H})$.

Proof. Recalling Definition A.1.7, the integral is defined at each time t as a limit in $L^2(\Omega;\mathcal{H})$ of simple integrals. From Proposition A.1.29 it is sufficient to show that these approximating integrals all belong to $\mathcal{M}_c^2(\mathcal{H})$, which is clear referring to the representation (A.2) and the definition of $\mathcal{M}_c^2(\mathcal{H})$.

□

This result extends to the case of a general martingale integrator as in the finite dimensional setting. Local martingality is then defined as we would expect, and we have the following result.

Proposition A.1.31. *For a continuous local martingale N and $\mathbf{\Psi} \in \bar{\mathcal{I}}_N^{\mathcal{H}}$, the Itô stochastic integral*

$$\int_0^t \mathbf{\Psi}_s dN_s$$

is itself a continuous local martingale.

Proof. We claim that the localising stopping times are given simply by the (τ_n) defined in (A.12). We have already seen that these tend to infinity almost surely, so it just remains to show that at any fixed $n \in \mathbb{N}$ the stopped process is a continuous martingale. This is now exactly as in Proposition A.1.30. □

The next ingredient would be a definition of quadratic variation, which we look to do via a Doob-Meyer decomposition for $M \in \mathcal{M}_c^2(\mathcal{H})$ (Theorem A.4.3). For this however, we must first define a notion of uniqueness in the decomposition.

Definition A.1.32. *Two $\mathcal{H}$-valued processes $\mathbf{\Psi}$, $\mathbf{\Phi}$ are said to be indistinguishable if there exists a set $A \in \mathcal{F}$ with $\mathbb{P}(A) = 1$ such that at all $t \geq 0$ and all $\omega \in A$, $\mathbf{\Psi}_t(\omega) = \mathbf{\Phi}_t(\omega)$.*

As we have become accustomed to, this notion can be built up from the finite dimensional projections and one-dimensional theory.

Lemma A.1.33. *Let $\mathbf{\Psi}$ and $\mathbf{\Phi}$ be $\mathcal{H}$-valued processes. Then $\mathbf{\Psi}$ is indistinguishable from $\mathbf{\Phi}$ if and only if for every basis vector e_i, $\langle \mathbf{\Psi}, e_i \rangle_{\mathcal{H}}$ is indistinguishable from $\langle \mathbf{\Phi}, e_i \rangle_{\mathcal{H}}$.*

Proof. The first implication is trivial so we consider only the reverse one. That is, assume that for every e_i there exists a set $A_i \in \mathcal{F}$ with $\mathbb{P}(A_i) = 1$ and for all $t \geq 0$ and $\omega \in A_i$,

$$\langle \mathbf{\Psi}_t(\omega), e_i \rangle_{\mathcal{H}} = \langle \mathbf{\Phi}_t(\omega), e_i \rangle_{\mathcal{H}} .$$

We now define $A := \bigcap_i A_i$, which is again of full probability and in $\mathcal{F}$, and for any $\omega \in A$, $t \geq 0$,

$$\|\mathbf{\Psi}_t(\omega) - \mathbf{\Phi}_t(\omega)\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} \langle \mathbf{\Psi}_t(\omega) - \mathbf{\Phi}_t(\omega), e_i \rangle_{\mathcal{H}}^2 = 0$$

which completes the proof. □

Revisiting the quadratic variation, our question is can we show that $\|M\|_{\mathcal{H}}^2$ defines a sub-martingale? Well we have integrability by definition, and

$$\|M_t\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} \langle M_t, e_i \rangle_{\mathcal{H}}^2 ,$$

where the limit is defined $\mathbb{P} - a.s.$. Again by definition the above projections are martingales and so the squares are sub-martingales. The process $\|M\|_{\mathcal{H}}^2$ is adapted as each $\|M_t\|_{\mathcal{H}}^2$ is the $\mathbb{P} - a.s.$ limit of $\mathcal{F}_t-$measurable random variables (on the complete measure space), and it is a sub-martingale as we can apply the Monotone Convergence Theorem to take the limit through the expectation for the defining sub-martingale property. As such, we have the following:

Definition A.1.34. *For $M \in \mathcal{M}_c^2(\mathcal{H})$, the quadratic variation $[M]$ of M is defined to be the unique*[4] *continuous, adapted, non-decreasing process with $[M]_0 = 0$ ($\mathbb{P} - a.s.$) specified in the Doob-Meyer decomposition (Theorem A.4.3) such that*

$$\|M\|_{\mathcal{H}}^2 - [M]$$

is a real valued martingale.

Proposition A.1.35. *Suppose that $\mathbf{\Psi} \in \mathcal{I}^{\mathcal{H}}$, then*

$$\left[\int_0^{\cdot} \mathbf{\Psi}_r dW_r\right]_t = \int_0^t \|\mathbf{\Psi}_r\|_H^2 \, dr. \tag{A.25}$$

Proof. The fact that the process in (A.25) is continuous, adapted, non-decreasing and starting from zero is clear. It simply remains to show the required martingality. To this end observe that at each time t,

$$\begin{aligned}
\left\|\int_0^t \mathbf{\Psi}_r dW_r\right\|_{\mathcal{H}}^2 - \int_0^t \|\mathbf{\Psi}_r\|_H^2 \, dr &= \sum_{i=1}^{\infty} \left\langle \int_0^t \mathbf{\Psi}_r dW_r, e_i \right\rangle_{\mathcal{H}}^2 - \int_0^t \sum_{i=1}^{\infty} \langle \mathbf{\Psi}_r, e_i\rangle_H^2 \, dr \\
&= \sum_{i=1}^{\infty} \left(\int_0^t \langle \mathbf{\Psi}_r, e_i\rangle_{\mathcal{H}} \, dW_r\right)^2 - \sum_{i=1}^{\infty} \int_0^t \langle \mathbf{\Psi}_r, e_i\rangle_H^2 \, dr \\
&= \sum_{i=1}^{\infty} \left(\left(\int_0^t \langle \mathbf{\Psi}_r, e_i\rangle_{\mathcal{H}} \, dW_r\right)^2 - \int_0^t \langle \mathbf{\Psi}_r, e_i\rangle_H^2 \, dr\right)
\end{aligned}$$

having applied Theorem A.1.9 to the first term and the Monotone Convergence Theorem to the second term, where the infinite sum is a limit taken $\mathbb{P} - a.s.$. From the standard one dimensional theory, for each n,

$$\sum_{i=1}^{n} \left(\left(\int_0^t \langle \mathbf{\Psi}_r, e_i\rangle_{\mathcal{H}} \, dW_r\right)^2 - \int_0^t \langle \mathbf{\Psi}_r, e_i\rangle_H^2 \, dr\right)$$

is a real valued martingale so to conclude the proof we only need to justify that the $\mathbb{P} - a.e.$ limit also holds in $L^1(\Omega; \mathbb{R})$ as convergence in this space preserves martingality. This is a straightforwards application of the Monotone Convergence Theorem applied to each integral separately, which concludes the proof.

□

We also look to reconcile this definition with the one often stated in the one-dimensional case, as a limit in probability over any time partition with mesh approaching zero.

[4] Uniqueness here is ‘up to indistinguishability’, as defined in Definition A.1.32.

Proposition A.1.36. *Let $\mathbf{\Psi} \in \mathcal{I}_T^{\mathcal{H}}$ and consider any sequence of partitions*

$$I_l := \left\{0 = t_0^l < t_1^l < \cdots < t_{k_l}^l = T\right\}$$

with $\max_j |t_j^l - t_{j-1}^l| \to 0$ *as* $l \to \infty$*. Then for all* $t \in [0,T]$*, for any* $\varepsilon > 0$*,*

$$\lim_{l\to\infty} \mathbb{P}\left(\left\{\left|\sum_{t_{j+1}^l \le t} \left\|\int_{t_j^l}^{t_{j+1}^l} \mathbf{\Psi}_r dW_r\right\|_{\mathcal{H}}^2 - \int_0^t \|\mathbf{\Psi}_r\|_{\mathcal{H}}^2\, dr\right| > \varepsilon\right\}\right) = 0.$$

We prove this result with an intermediary lemma, stated in the setting of Proposition A.1.36.

Lemma A.1.37. *Define the sequence of stopping times (τ^n) at every $n \in \mathbb{N}$ by*

$$\tau^n := n \wedge \inf\left\{t \in [0,T] : \left\|\int_0^t \mathbf{\Psi}_r dW_r\right\|_{\mathcal{H}}^2 + \int_0^t \|\mathbf{\Psi}_r\|_{\mathcal{H}}^2\, dr \ge n\right\}$$

and the process $\mathbf{\Psi}^n_{\cdot} := \mathbf{\Psi}_{\cdot}\mathbb{1}_{\cdot \le \tau^n}$*. Suppose that for every n and all $t \in [0,T]$,*

$$\lim_{l\to\infty} \mathbb{E}\left(\left|\sum_{t_{j+1}^l \le t} \left\|\int_{t_j^l}^{t_{j+1}^l} \mathbf{\Psi}_r^n dW_r\right\|_{\mathcal{H}}^2 - \int_0^t \|\mathbf{\Psi}_r^n\|_{\mathcal{H}}^2\, dr\right|\right) = 0. \tag{A.26}$$

Then for any $\varepsilon > 0$,

$$\lim_{l\to\infty} \mathbb{P}\left(\left\{\left|\sum_{t_{j+1}^l \le t} \left\|\int_{t_j^l}^{t_{j+1}^l} \mathbf{\Psi}_r dW_r\right\|_{\mathcal{H}}^2 - \int_0^t \|\mathbf{\Psi}_r\|_{\mathcal{H}}^2\, dr\right| > \varepsilon\right\}\right) = 0.$$

Proof. Define

$$A^l := \left\{\left|\sum_{t_{j+1}^l \le t} \left\|\int_{t_j^l}^{t_{j+1}^l} \mathbf{\Psi}_r dW_r\right\|_{\mathcal{H}}^2 - \int_0^t \|\mathbf{\Psi}_r\|_{\mathcal{H}}^2\, dr\right| > \varepsilon\right\}$$

then for any n,

$$\begin{aligned} A^l = A^l \cap [\{\tau^n > T\} \cup \{\tau^n \le T\}] &= \left[A^l \cap \{\tau^n > T\}\right] \cup \left[A^l \cap \{\tau^n \le T\}\right] \\ &\subset \left[A^l \cap \{\tau^n > T\}\right] \cup \{\tau^n \le T\}. \end{aligned}$$

In particular,

$$\mathbb{P}\left(A^l\right) \le \mathbb{P}\left(A^l \cap \{\tau^n > T\}\right) + \mathbb{P}\left(\{\tau^n \le T\}\right) \tag{A.27}$$

where

$$A^{l,n} := A^l \cap \{\tau^n > T\} = \left\{\left|\sum_{t_{j+1}^l \le t} \left\|\int_{t_j^l}^{t_{j+1}^l} \mathbf{\Psi}_r^n dW_r\right\|_{\mathcal{H}}^2 - \int_0^t \|\mathbf{\Psi}_r^n\|_{\mathcal{H}}^2\, dr\right| > \varepsilon\right\}.$$

We then take the limit as $l \to \infty$ in (A.27), which holds for all n so too must hold in the limit as $n \to \infty$, providing that

$$\begin{aligned}\lim_{l\to\infty} \mathbb{P}\left(A^l\right) &\leq \lim_{n\to\infty}\lim_{l\to\infty} \mathbb{P}\left(A^l \cap \{\tau^n > T\}\right) + \lim_{n\to\infty} \mathbb{P}\left(\{\tau^n \leq T\}\right)\\ &= \lim_{n\to\infty}\lim_{l\to\infty} \mathbb{P}\left(A^l \cap \{\tau^n > T\}\right) \qquad (A.28)\end{aligned}$$

given that the (τ^n) approach infinity $\mathbb{P}-a.s.$. From Chebyshev's Inequality and the assumption (A.26),

$$\begin{aligned}&\lim_{n\to\infty}\lim_{l\to\infty} \mathbb{P}\left(A^l \cap \{\tau^n > T\}\right)\\ &\qquad \leq \frac{1}{\varepsilon}\lim_{n\to\infty}\lim_{l\to\infty} \mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\left\|\int_{t^l_j}^{t^l_{j+1}} \boldsymbol{\Psi}^n_r dW_r\right\|_{\mathcal{H}}^2 - \int_0^t \|\boldsymbol{\Psi}^n_r\|^2_{\mathcal{H}}\, dr\right|\right) = 0\end{aligned}$$

which combined with (A.28) gives the result.

□

Proof of Proposition A.1.36: We once again look to prove this by considering the finite dimensional projections on which the result is known to be true, before showing that it is preserved in the limit. This approach is ultimately taken for the localised sequence by applying Lemma A.1.37, looking to verify (A.26). Identically to the proof of Proposition A.1.35 we have that

$$\begin{aligned}&\mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\left\|\int_{t^l_j}^{t^l_{j+1}} \boldsymbol{\Psi}^n_r dW_r\right\|_{\mathcal{H}}^2 - \int_0^t \|\boldsymbol{\Psi}^n_r\|^2_{H}\, dr\right|\right)\\ &\qquad = \mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\sum_{i=1}^{\infty}\left(\int_{t^l_j}^{t^l_{j+1}} \langle \boldsymbol{\Psi}^n_r, e_i\rangle_{\mathcal{H}}\, dW_r\right)^2 - \int_0^t \langle \boldsymbol{\Psi}^n_r, e_i\rangle^2_{H}\, dr\right|\right)\\ &\qquad \leq \mathbb{E}\left(\sum_{i=1}^{\infty}\left|\sum_{t^l_{j+1}\leq t}\left(\int_{t^l_j}^{t^l_{j+1}} \langle \boldsymbol{\Psi}^n_r, e_i\rangle_{\mathcal{H}}\, dW_r\right)^2 - \int_0^t \langle \boldsymbol{\Psi}^n_r, e_i\rangle^2_{H}\, dr\right|\right)\\ &\qquad = \sum_{i=1}^{\infty}\mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\left(\int_{t^l_j}^{t^l_{j+1}} \langle \boldsymbol{\Psi}^n_r, e_i\rangle_{\mathcal{H}}\, dW_r\right)^2 - \int_0^t \langle \boldsymbol{\Psi}^n_r, e_i\rangle^2_{H}\, dr\right|\right)\end{aligned}$$

where on the last line we have applied the Monotone Convergence Theorem. From Theorem A.4.8 we know that for each $i \in \mathbb{N}$,

$$\lim_{l\to 0}\mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\left[\left(\int_{t^l_j}^{t^l_{j+1}} \langle \boldsymbol{\Psi}^n_r, e_i\rangle_{\mathcal{H}}\, dW_r\right)^2 - \int_0^t \langle \boldsymbol{\Psi}^n_r, e_i\rangle^2_{H}\, dr\right]\right|\right) = 0$$

so we would be done if we can justify the interchange of infinite sum and limit in l. We look to apply the Dominated Convergence Theorem to proceed, noting that for each fixed $i, l \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}&\left(\left|\sum_{t^l_{j+1}\leq t}\left[\left(\int_{t^l_j}^{t^l_{j+1}}\langle \mathbf{\Psi}^n_r, e_i\rangle_{\mathcal{H}}\, dW_r\right)^2 - \int_0^t \langle \mathbf{\Psi}^n_r, e_i\rangle_H^2\, dr\right]\right|\right)\\
&\leq \sum_{t^l_{j+1}\leq t}\mathbb{E}\left[\left(\int_{t^l_j}^{t^l_{j+1}}\langle \mathbf{\Psi}^n_r, e_i\rangle_{\mathcal{H}}\, dW_r\right)^2\right] + \mathbb{E}\left[\int_0^t \langle \mathbf{\Psi}^n_r, e_i\rangle_H^2\, dr\right]\\
&= \sum_{t^l_{j+1}\leq t}\mathbb{E}\left[\int_{t^l_j}^{t^l_{j+1}}\langle \mathbf{\Psi}^n_r, e_i\rangle_H^2\, dr\right] + \mathbb{E}\left[\int_0^t \langle \mathbf{\Psi}^n_r, e_i\rangle_H^2\, dr\right]\\
&= 2\mathbb{E}\left[\int_0^t \langle \mathbf{\Psi}^n_r, e_i\rangle_H^2\, dr\right], \qquad \text{(A.29)}
\end{aligned}$$

which is a bound uniform in l and summable in i. Thus, $2\mathbb{E}\left[\int_0^t \langle \mathbf{\Psi}^n_r, e_i\rangle_H^2\, dr\right]$ is our dominating function which justifies the application of the Dominated Convergence Theorem, concluding the verification of (A.26) and hence the proof by Lemma A.1.37.

□

Lemma A.1.38. *Suppose that (M^n) is a sequence of martingales in $\mathcal{M}^2_c(\mathcal{H})$ which at every time $t \geq 0$, converges in $L^2\big(\Omega;\mathcal{H}\big)$ to some M_t. Suppose in addition that at any time $t \geq 0$, the sequence $([M^n]_t)$ converges to some ψ_t in $L^1\big(\Omega;\mathbb{R}\big)$ where ψ is continuous, adapted and non-decreasing ($\mathbb{P}-a.s.$). Then $M \in \mathcal{M}^2_c(\mathcal{H})$ and $[M]$ is indistinguishable from ψ.*

Proof. The fact that $M \in \mathcal{M}^2_c(\mathcal{H})$ is immediate from Proposition A.1.29, so we move on to the quadratic variation. Observe that for each n, by definition

$$\|M^n\|^2_{\mathcal{H}} - [M^n]$$

is a real valued martingale, so the $L^1\big(\Omega;\mathbb{R}\big)$ limit at each time t,

$$\lim_{n\to\infty}\big(\|M^n\|^2_{\mathcal{H}} - [M^n]\big)$$

is again a real valued martingale if it exists. But this is clear as it is just

$$\lim_{n\to\infty}\|M^n\|^2_{\mathcal{H}} - \lim_{n\to\infty}[M^n] = \|M\|^2_{\mathcal{H}} - \psi$$

by definition of the limit. From Definition A.1.34 then if $\psi_0 = 0$ $\mathbb{P}-a.s.$ we have that ψ satisfies the conditions of the quadratic variation of M, so is indistinguishable from it. This property is immediate as ψ_0 is the limit in $L^1(\Omega;\mathbb{R})$ of $([M^n]_0)$ by assumption, whilst this is just a sequence of zeros $\mathbb{P}-a.s.$, hence the result. □

In part due to this result, we provide an alternative characterisation of the quadratic variation under some additional boundedness assumptions. We reintroduce the projections $(\mathcal{P}_k)$ used in Proposition A.1.29.

Proposition A.1.39. *For any $M \in \mathcal{M}_c^2(\mathcal{H})$, $[M]$ has the representation*

$$[M] = \sum_{i=1}^{\infty} [\langle M, e_i \rangle_{\mathcal{H}}]$$

$\mathbb{P} - a.s.$ for the limit taken in $C([0,T];\mathbb{R})$ for any $T \geq 0$.

Proof. Observe that

$$\|\mathcal{P}_k M\|_{\mathcal{H}}^2 = \sum_{i=1}^{k} \langle M, e_i \rangle_{\mathcal{H}}^2$$

which is the sum of k one dimensional submartingales, and in particular

$$\sum_{i=1}^{k} \left(\langle M, e_i \rangle_{\mathcal{H}}^2 - [\langle M, e_i \rangle_{\mathcal{H}}] \right)$$

is a martingale. From the definition of the quadratic variation it is thus clear that

$$[\mathcal{P}_k M] = \sum_{i=1}^{k} [\langle M, e_i \rangle_{\mathcal{H}}] . \tag{A.30}$$

Moreover we have that at each $t \geq 0$, $\mathcal{P}_k M_t$ is convergent to M_t $\mathbb{P} - a.s.$ in $\mathcal{H}$, and furthermore in $L^2(\Omega;\mathcal{H})$ from an application of the Monotone Convergence Theorem to the difference process $\|M - \mathcal{P}_k M\|_{\mathcal{H}}^2 = \sum_{i=k+1}^{\infty} \langle M, e_i \rangle_{\mathcal{H}}^2$. Reminiscent of Lemma A.1.38 we look to show convergence of the sequence $([\mathcal{P}_k M])$ to some ψ at each $t \geq 0$ in $L^1(\Omega;\mathbb{R})$, and in fact we show the stronger convergence in $L^1(\Omega; C([0,T];\mathbb{R}))$ for each $T \geq 0$. We proceed by showing the Cauchy property in this Banach Space. To do this we consider, for $j < k$,

$$[\mathcal{P}_k M] - [\mathcal{P}_j M] = \sum_{i=j+1}^{k} [\langle M, e_i \rangle_{\mathcal{H}}] . \tag{A.31}$$

From this identity and the preceding work, it is clear that

$$[\mathcal{P}_k M] - [\mathcal{P}_j M] = [\mathcal{P}_k M - \mathcal{P}_j M] \tag{A.32}$$

and therefore

$$\sup_{t \in [0,T]} |[\mathcal{P}_k M]_t - [\mathcal{P}_j M]_t| = \sup_{t \in [0,T]} [\mathcal{P}_k M - \mathcal{P}_j M]_t = [\mathcal{P}_k M - \mathcal{P}_j M]_T. \tag{A.33}$$

Furthermore we are concerned with a control in expectation of this term. From the property that real valued martingales have constant expectation,

$$\begin{aligned}\mathbb{E}\left(\|\mathcal{P}_kM_T-\mathcal{P}_jM_T\|_{\mathcal{H}}^2-[\mathcal{P}_kM-\mathcal{P}_jM]_T\right)&=\mathbb{E}\left(\|\mathcal{P}_kM_0-\mathcal{P}_jM_0\|_{\mathcal{H}}^2-[\mathcal{P}_kM-\mathcal{P}_jM]_0\right)\\&=\mathbb{E}\left(\|\mathcal{P}_kM_0-\mathcal{P}_jM_0\|_{\mathcal{H}}^2\right)\end{aligned}$$

so in particular

$$\begin{aligned}\mathbb{E}\left([\mathcal{P}_kM-\mathcal{P}_jM]_T\right)&=\mathbb{E}\left(\|\mathcal{P}_kM_T-\mathcal{P}_jM_T\|_{\mathcal{H}}^2-\|\mathcal{P}_kM_0-\mathcal{P}_jM_0\|_{\mathcal{H}}^2\right)\\&=\mathbb{E}\left(\sum_{i=j+1}^{k}\left(\langle M_T,e_i\rangle_{\mathcal{H}}^2-\langle M_0,e_i\rangle_{\mathcal{H}}^2\right)\right)\\&\leq\mathbb{E}\left(\sum_{i=j+1}^{k}\langle M_T,e_i\rangle_{\mathcal{H}}^2\right)+\mathbb{E}\left(\sum_{i=j+1}^{k}\langle M_0,e_i\rangle_{\mathcal{H}}^2\right).\end{aligned}$$

By the square integrability assumption on M we have that at every $t\geq 0$, $\mathbb{E}\left(\sum_{i=1}^{\infty}\langle M_t,e_i\rangle_{\mathcal{H}}^2\right)<\infty$ which justifies that the right hand side of the above approaches zero as $j\to\infty$, uniformly in k. With (A.33) we deduce that the sequence $([\mathcal{P}_kM])$ is Cauchy in $L^1\left(\Omega;C\left([0,T];\mathbb{R}\right)\right)$ for each $T\geq 0$ so admits a limit in this space which we call ψ. Through the $L^1\left(\Omega;C\left([0,T];\mathbb{R}\right)\right)$ convergence we can deduce the existence of a subsequence which is $\mathbb{P}-a.s.$ convergent in $C\left([0,T];\mathbb{R}\right)$. In fact we can upgrade this to convergence over the whole sequence, as was note that $\mathbb{P}-a.s.$ at each $t\geq 0$ the real valued sequence $([\mathcal{P}_kM]_t)$ not only has a convergent subsequence but is non-decreasing in k, which implies convergence of the whole sequence. Convergence of the whole sequence $\mathbb{P}-a.s.$ in $C\left([0,T];\mathbb{R}\right)$ can then be deduced by the Cauchy property with (A.33) and (A.32). In summary thus far we have that

$$\psi=\sum_{i=1}^{\infty}[\langle M,e_i\rangle_{\mathcal{H}}]\quad\mathbb{P}-a.s.$$

for the limit (of the sequence of finite sums) taken in $C\left([0,T];\mathbb{R}\right)$ for any $T\geq 0$. It only remains to show that ψ is indistinguishable from $[M]$, which we do by verifying the conditions of Lemma A.1.38. The convergence has already been established hence we need only the regularity on ψ. It is continuous by construction, and must be adapted as ψ_t is the $\mathbb{P}-a.s.$ limit of the $\mathcal{F}_t-$measurable $[\mathcal{P}_kM]_t$ on the complete measure space. This limit similarly preserves the non-decreasing property, so ψ satisfies the required conditions and must be indistinguishable from $[M]$ due to Lemma A.1.38.

□

Following on from the quadratic variation we look to introduce the cross-variation between martingales. Guided by the motivation to use the Stratonovich integral, introduced in Subsection 2.2.1, we will need to consider the cross-variation between elements of $\mathcal{M}_c^2(\mathcal{H})$ and $\mathcal{M}_c^2$. Defining this akin to a polarisation identity is out of the question as one cannot take sums of these martingales (there is no canonical way to sum an element of a Hilbert Space with an element of $\mathbb{R}$), so we look to use the characterisation in terms of their product.

Indeed from the classical theory, Theorem A.4.4, for any given $\mathbf{\Psi} \in \mathcal{M}_c^2(\mathcal{H})$, $Y \in \mathcal{M}_c^2$ and e_i a basis vector of $\mathcal{H}$, there exists a unique continuous, adapted, bounded-variation process $[\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]$ with $[\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]_0 = 0$ ($\mathbb{P} - a.s.$) such that

$$\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}} Y - [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]$$

is a real valued martingale. We would love to immediately have the existence and uniqueness of a corresponding $\mathcal{H}$-valued process which gives a martingale when subtracted from $\mathbf{\Psi}Y$, but that isn't clear in the same way that the quadratic variation was as in that case $\|\mathbf{\Psi}\|_{\mathcal{H}}^2$ was a genuine real valued submartingale. Our approach, therefore, comes from the characterisation in Proposition A.1.39. We make a first definition for the projected process.

Definition A.1.40. *For $\mathbf{\Psi} \in \mathcal{M}_c^2(\mathcal{H})$ and $Y \in \mathcal{M}_c^2$, for any $k \in \mathbb{N}$, we define the cross-variation process $[\mathcal{P}_k\mathbf{\Psi}, Y]$ by*

$$[\mathcal{P}_k\mathbf{\Psi}, Y] := \sum_{i=1}^{k} [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y] e_i.$$

Indeed we can verify that such a process gives us the desired properties of a cross-variation in $\mathcal{H}$.

Proposition A.1.41. *For $\mathbf{\Psi} \in \mathcal{M}_c^2(\mathcal{H})$ and $Y \in \mathcal{M}_c^2$, for any $k \in \mathbb{N}$, $[\mathcal{P}_k\mathbf{\Psi}, Y]$ is the unique continuous, adapted, bounded-variation $\mathcal{H}$-valued process satisfying $[\mathcal{P}_k\mathbf{\Psi}, Y]_0 = 0$ $\mathbb{P} - a.s.$ such that*

$$(\mathcal{P}_k\mathbf{\Psi})Y - [\mathcal{P}_k\mathbf{\Psi}, Y]$$

is an $\mathcal{H}$-valued martingale.

Proof. It falls from the corresponding properties of the real valued cross-variations that $[\mathcal{P}_k\mathbf{\Psi}, Y]$ is continuous, adapted and of bounded-variation (one can apply the triangle inequality for the norm $\|\cdot\|_{\mathcal{H}}$) satisfying $[\mathcal{P}_k\mathbf{\Psi}, Y]_0 = 0$ $\mathbb{P} - a.s.$. In addition observe that

$$(\mathcal{P}_k\mathbf{\Psi})Y - [\mathcal{P}_k\mathbf{\Psi}, Y] = \sum_{i=1}^{k} \left(\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}} Y - [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]\right) e_i,$$

which we look to show is an $\mathcal{H}$-valued martingale. To this end consider arbitrary $\phi \in \mathcal{H}$. Then

$$\begin{aligned}
\langle (\mathcal{P}_k\mathbf{\Psi})Y - [\mathcal{P}_k\mathbf{\Psi}, Y], \phi\rangle_{\mathcal{H}} &= \left\langle \sum_{i=1}^{k} \left(\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}} Y - [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]\right) e_i, \sum_{j=1}^{\infty} \langle \phi, e_j\rangle_{\mathcal{H}} e_j \right\rangle_{\mathcal{H}} \\
&= \sum_{i=1}^{k}\sum_{j=1}^{\infty} \left\langle \left(\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}} Y - [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]\right) e_i, \langle \phi, e_j\rangle_{\mathcal{H}} e_j \right\rangle_{\mathcal{H}} \\
&= \sum_{i=1}^{k} \left(\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}} Y - [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]\right) \langle \phi, e_i\rangle_{\mathcal{H}},
\end{aligned}$$

where we recall that each $\langle \mathbf{\Psi}, e_i \rangle_{\mathcal{H}} Y - [\langle \mathbf{\Psi}, e_i \rangle_{\mathcal{H}}, Y]$ is a martingale, hence too is $(\langle \mathbf{\Psi}, e_i \rangle_{\mathcal{H}} Y - [\langle \mathbf{\Psi}, e_i \rangle_{\mathcal{H}}, Y]) \langle \phi, e_i \rangle_{\mathcal{H}}$ and therefore the finite sum is a martingale.

To comment on the uniqueness of $[\mathcal{P}_k \mathbf{\Psi}, Y]$ we invoke Lemma A.1.33. Suppose that Π is an $\mathcal{H}$-valued process which is continuous, adapted and of bounded-variation satisfying $\Pi_0 = 0$ $\mathbb{P} - a.s.$. Moreover suppose that

$$(\mathcal{P}_k \mathbf{\Psi})Y - \Pi$$

is an $\mathcal{H}$-valued martingale. Take any basis vector e_j. Then

$$\langle (\mathcal{P}_k \mathbf{\Psi})Y - \Pi, e_j \rangle_{\mathcal{H}}$$

is a real valued martingale, but this is just

$$\sum_{i=1}^{k} \langle \mathbf{\Psi}, e_i \rangle_{\mathcal{H}} \langle e_i, e_j \rangle_{\mathcal{H}} Y - \langle \Pi, e_j \rangle_{\mathcal{H}}. \tag{A.34}$$

From the regularity of Π we have again that $\langle \Pi, e_j \rangle_{\mathcal{H}}$ is continuous, adapted and of bounded-variation satisfying $\langle \Pi_0, e_j \rangle_{\mathcal{H}} = 0$ $\mathbb{P} - a.s.$. There are two cases here: $j \leq k$ and $j > k$. In the latter case we have that the first term in (A.34) is null so in particular $\langle \Pi, e_j \rangle_{\mathcal{H}}$ is a martingale. Thus $\langle \Pi, e_j \rangle_{\mathcal{H}} \in \mathcal{M}_c$ and is of bounded-variation, so by Lemma A.4.5 it is $\mathbb{P} - a.s.$ constant. As the process starts from zero then it is indistinguishable from the null process. Therefore $\langle \Pi, e_j \rangle_{\mathcal{H}}$ is indistinguishable from $\langle [\mathcal{P}_k \mathbf{\Psi}, Y], e_j \rangle$ which is similarly zero (recall Definition A.1.40).

In the alternative case $j \leq k$, the real valued martingale (A.34) is given by

$$\langle \mathbf{\Psi}, e_j \rangle_{\mathcal{H}} Y - \langle \Pi, e_j \rangle_{\mathcal{H}}.$$

Therefore $\langle \Pi, e_j \rangle_{\mathcal{H}}$ satisfies the requirements of the cross-variation process $[\langle \mathbf{\Psi}, e_j \rangle_{\mathcal{H}}, Y]$ hence is indistinguishable from it. This, however, is simply $\langle [\mathcal{P}_k \mathbf{\Psi}, Y], e_k \rangle$ from its definition. Combining with Lemma A.1.33, the theorem is proven.

□

It remains for us to define the cross-variation process $[\mathbf{\Psi}, Y]$. Referencing Lemma A.4.6, we know that $\mathbb{P} - a.s.$ for any $t \geq 0$,

$$[\langle \mathbf{\Psi}, e_i \rangle_{\mathcal{H}}, Y]_t^2 \leq [\langle \mathbf{\Psi}, e_i \rangle_{\mathcal{H}}]_t [Y]_t,$$

hence, using the non-decreasing property of quadratic variation,

$$\begin{aligned}\sup_{t\in[0,T]} [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]_t^2 &\leq \sup_{t\in[0,T]} \left([\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}]_t [Y]_t\right)\\ &\leq \left(\sup_{t\in[0,T]} [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}]_t\right)\left(\sup_{t\in[0,T]} [Y]_t\right)\\ &= [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}]_T [Y]_T.\end{aligned}$$

Moreover for $j < k$,

$$\begin{aligned}\sup_{t\in[0,T]} \left\|[\mathcal{P}_k\mathbf{\Psi}, Y]_t - [\mathcal{P}_j\mathbf{\Psi}, Y]_t\right\|_{\mathcal{H}}^2 &= \sup_{t\in[0,T]} \left\|\sum_{i=j+1}^{k} [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]_t e_i\right\|_{\mathcal{H}}^2\\ &= \sup_{t\in[0,T]} \sum_{i=j+1}^{k} [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]_t^2\\ &\leq \sum_{i=j+1}^{k} \sup_{t\in[0,T]} [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]_t^2\\ &\leq \sum_{i=j+1}^{k} [\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}]_T [Y]_T\\ &= [\mathcal{P}_k\mathbf{\Psi} - \mathcal{P}_j\mathbf{\Psi}]_T [Y]_T\end{aligned}$$

using (A.31) and (A.32) for the last line. We look to follow a similar approach to Proposition A.1.39, taking the expectation and showing a Cauchy property. It is unknown if this resulting term is integrable, so in order to take the expectation we introduce the localising times

$$\tau_n := n \wedge \inf\{0 \leq t < \infty : [Y]_t \geq n\}.$$

Then

$$\begin{aligned}&\mathbb{E}\left(\left(\sup_{t\in[0,T]} \left\|[\mathcal{P}_k\mathbf{\Psi}, Y]_t \mathbb{1}_{t\leq\tau_n} - [\mathcal{P}_j\mathbf{\Psi}, Y]_t \mathbb{1}_{t\leq\tau_n}\right\|_{\mathcal{H}}\right)^2\right)\\ &\qquad= \mathbb{E}\left(\sup_{t\in[0,T]} \left\|[\mathcal{P}_k\mathbf{\Psi}, Y]_t \mathbb{1}_{t\leq\tau_n} - [\mathcal{P}_j\mathbf{\Psi}, Y]_t \mathbb{1}_{t\leq\tau_n}\right\|_{\mathcal{H}}^2\right)\\ &\qquad\leq \mathbb{E}\left([\mathcal{P}_k\mathbf{\Psi} - \mathcal{P}_j\mathbf{\Psi}]_T [Y]_T \mathbb{1}_{t\leq\tau_n}\right)\\ &\qquad\leq n\mathbb{E}\left([\mathcal{P}_k\mathbf{\Psi} - \mathcal{P}_j\mathbf{\Psi}]_T\right)\end{aligned}$$

which was shown to approach zero as $j \to \infty$, uniformly in k, in Proposition A.1.39. We have thus demonstrated that for every $n \in \mathbb{N}$ and $T \geq 0$, the sequence $([\mathcal{P}_k\mathbf{\Psi}, Y].\mathbb{1}_{\cdot\leq\tau_n})$ is Cauchy in the Banach Space $L^2(\Omega; L^\infty([0,T];\mathcal{H}))$. We can therefore extract a subsequence which converges $\mathbb{P} - a.s.$ in $L^\infty([0,T];\mathcal{H})$. To remove the truncation we introduce the sets

$$A_n := \{\omega \in \Omega : \tau_n(\omega) \geq T\}.$$

Then on every A_n, there exists a subsequence of $([\mathcal{P}_k\boldsymbol{\Psi}, Y])$ which is $\mathbb{P}-a.s.$ (within A_n) convergent in $L^\infty([0,T];\mathcal{H})$. It should be noted that the choice of subsequence may be dependent on n, hence the separation, and also that this convergence is now of continuous processes so can be taken in $C([0,T];\mathcal{H})$. We look to upgrade this convergence to be of the whole sequence, by another Cauchy argument. Fix any $\varepsilon > 0$, and we look to show the existence of a $J \in \mathbb{N}$ such that for all $k \geq J$,

$$\sup_{t\in[0,T]} \left\|[\mathcal{P}_k\boldsymbol{\Psi}, Y]_t - [\mathcal{P}_J\boldsymbol{\Psi}, Y]_t\right\|_{\mathcal{H}}^2 < \varepsilon$$

or, equivalently, as already shown in the proof,

$$\sup_{t\in[0,T]} \sum_{i=J+1}^{k} \left[\langle \boldsymbol{\Psi}, e_i\rangle_{\mathcal{H}}, Y\right]_t^2 < \varepsilon.$$

Let the convergent subsequence be indexed by k_m. This convergence implies that the subsequence is Cauchy, so there exists a J_m such that for all $k_m > J_m$,

$$\sup_{t\in[0,T]} \left\|[\mathcal{P}_{k_m}\boldsymbol{\Psi}, Y]_t - [\mathcal{P}_{J_m}\boldsymbol{\Psi}, Y]_t\right\|_{\mathcal{H}}^2 < \varepsilon$$

or equivalently

$$\sup_{t\in[0,T]} \sum_{i=J_m+1}^{k_m} \left[\langle \boldsymbol{\Psi}, e_i\rangle_{\mathcal{H}}, Y\right]_t^2 < \varepsilon.$$

We now set $J := J_m$ and argue that for every $k > J$ there exists a $k_m > k$ such that

$$\sup_{t\in[0,T]} \sum_{i=J+1}^{k} \left[\langle \boldsymbol{\Psi}, e_i\rangle_{\mathcal{H}}, Y\right]_t^2 \leq \sup_{t\in[0,T]} \sum_{i=J+1}^{k_m} \left[\langle \boldsymbol{\Psi}, e_i\rangle_{\mathcal{H}}, Y\right]_t^2 < \varepsilon$$

which proves the Cauchy property. Hence on each A_n, the sequence $([\mathcal{P}_k\boldsymbol{\Psi}, Y])$ is $\mathbb{P}-a.s.$ convergent in $C([0,T];\mathcal{H})$. We use that

$$\begin{aligned}\mathbb{P}\left(\bigcup_n A_n\right) = 1 - \mathbb{P}\left(\bigcap_n A_n^c\right) &= 1 - \mathbb{P}\left(\bigcap_n \{\omega \in \Omega : \tau_n(\omega) < T\}\right)\\ &= 1 - \lim_{n\to\infty} \mathbb{P}\left(\{\omega \in \Omega : \tau_n(\omega) < T\}\right)\end{aligned}$$

owing to the fact that the (τ_n) are $\mathbb{P}-a.s.$ increasing, which is zero given that they approach infinity. Thus we obtain the $\mathbb{P}-a.s.$ convergence on Ω.

Definition A.1.42. *For $\boldsymbol{\Psi} \in \mathcal{M}_c^2(\mathcal{H})$ and $Y \in \mathcal{M}_c^2$, we define the cross-variation process $[\boldsymbol{\Psi}, Y]$ by*

$$[\boldsymbol{\Psi}, Y] := \sum_{i=1}^{\infty} \left[\langle \boldsymbol{\Psi}, e_i\rangle_{\mathcal{H}}, Y\right] e_i$$

$\mathbb{P}-a.s.$ for the limit taken in $C([0,T];\mathcal{H})$.

Our next question is then very natural: do we have the corresponding characterisation as in Proposition A.1.41? The fact that this cross-variation is continuous, adapted and starting from zero is proven as in Proposition A.1.39. In fact the martingality of

$$\mathbf{\Psi}Y - [\mathbf{\Psi}, Y] \tag{A.35}$$

is again proven near identically, as we use that

$$(\mathcal{P}_k\mathbf{\Psi})Y - \sum_{i=1}^{k}[\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]e_i$$

is a martingale, and the convergence of this at each $t \geq 0$ in $L^1(\Omega; \mathcal{H})$ to (A.35). It is the bounded-variation which proves problematic. Possibly the most logical approach is to show that the finite sums are of uniformly bounded total variation, through an argument like

$$V_{\mathcal{H}}^T([\mathcal{P}_k\mathbf{\Psi}, Y]) = V_{\mathcal{H}}^T\left(\sum_{i=1}^{k}[\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]e_i\right) \leq \sum_{i=1}^{k} V_{\mathcal{H}}^T([\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]e_i) = \sum_{i=1}^{k} V_{\mathbb{R}}^T([\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]).$$

Then from Lemma A.4.7, we have that

$$V_{\mathbb{R}}^T([\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}, Y]) \leq \frac{1}{2}([\langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}}]_T + [Y]_T),$$

but this means taking $\sum_{i=1}^{k}[Y]_T$ which will explode as $k \to \infty$. In lieu of this bounded-variation, we do of course have the weaker property that every projection is of bounded variation. In fact, this is enough for uniqueness.

Suppose that Π is an $\mathcal{H}$-valued process which is continuous, adapted, satisfying $\Pi_0 = 0$ and such that for every basis vector e_j, $\langle \Pi, e_j\rangle$ is of bounded-variation $\mathbb{P} - a.s.$. Moreover suppose that

$$\mathbf{\Psi}Y - \Pi$$

is an $\mathcal{H}$-valued martingale. Take any basis vector e_j. Then

$$\langle \mathbf{\Psi}Y - \Pi, e_j\rangle_{\mathcal{H}}$$

is a real valued martingale, but this is just

$$\sum_{i=1}^{\infty} \langle \mathbf{\Psi}, e_i\rangle_{\mathcal{H}} \langle e_i, e_j\rangle_{\mathcal{H}} Y - \langle \Pi, e_j\rangle_{\mathcal{H}}$$

or simply

$$\langle \mathbf{\Psi}, e_j\rangle_{\mathcal{H}} Y - \langle \Pi, e_j\rangle_{\mathcal{H}}.$$

Thus $\langle \Pi, e_j\rangle_{\mathcal{H}}$ is indistinguishable from $[\langle \mathbf{\Psi}, e_j\rangle_{\mathcal{H}}, Y]$ which is simply $\langle [\mathbf{\Psi}, Y], e_k\rangle$ from its definition. Combining with Lemma A.1.33 we deduce the uniqueness in this class, so have proven the following:

Proposition A.1.43. *For $\mathbf{\Psi} \in \mathcal{M}_c^2(\mathcal{H})$ and $Y \in \mathcal{M}_c^2$, $[\mathbf{\Psi}, Y]$ is the unique continuous, adapted $\mathcal{H}$-valued process satisfying $[\mathbf{\Psi}, Y]_0 = 0$ $\mathbb{P}-a.s.$ such that for every basis vector e_j, $\langle [\mathbf{\Psi}, Y], e_j \rangle_{\mathcal{H}}$ is of bounded-variation $\mathbb{P}-a.s.$ and*

$$\mathbf{\Psi} Y - [\mathbf{\Psi}, Y]$$

is an $\mathcal{H}$-valued martingale.

We now state and prove the analogous result to Lemma A.1.38, which will be necessary in the Itô-Stratonovich conversion.

Lemma A.1.44. *Suppose that $(\mathbf{\Psi}^n)$ is a sequence of martingales in $\mathcal{M}_c^2(\mathcal{H})$ which at every time $t \geq 0$, converges in $L^2\big(\Omega; \mathcal{H}\big)$ to some $\mathbf{\Psi}_t$. Let $Y \in \mathcal{M}_c^2$. Suppose in addition that at any time $t \geq 0$, the sequence $([\mathbf{\Psi}^n, Y]_t)$ converges to some L_t in $L^1\big(\Omega; \mathbb{R}\big)$ where L is continuous, adapted and for every basis vector e_j, $\langle L, e_j \rangle_{\mathcal{H}}$ is of bounded variation $\mathbb{P}-a.s.$. Then $\mathbf{\Psi} \in \mathcal{M}_c^2(\mathcal{H})$ and $[\mathbf{\Psi}, Y]$ is indistinguishable from L.*

Proof. For each n the martingale in question is the $\mathcal{H}$-valued one

$$\mathbf{\Psi}^n Y - [\mathbf{\Psi}^n, Y].$$

We use again that the $L^1\big(\Omega; \mathcal{H}\big)$ limit preserves martingality and that by Hölder's inequality the $L^1\big(\Omega; \mathcal{H}\big)$ limit of $\mathbf{\Psi}_t^n Y_t$ is $\mathbf{\Psi}_t Y_t$. With the same arguments as in Lemma A.1.38 we conclude the proof. □

If the given processes were only (continuous) local martingales, then we can make a slightly modified version of the definition. Assuming without loss of generality that $\mathbf{\Psi}$ and Y are locally square integrable (see the discussion after Definition A.1.15), localised by stopping times (R_n) and (T_n) respectively, then for a new sequence of stopping times defined by $\tau_n = R_n \wedge T_n$, the stopped processes $\mathbf{\Psi}^{\tau_n}$ and Y^{τ_n} are genuine square integrable martingales (in their respective spaces), so the cross variation $[\mathbf{\Psi}^{\tau_n}, Y^{\tau_n}]$ can be defined. The canonical localisation procedure is evident once more, as the (τ_n) tend to infinity almost surely, the consistency conditions that for $m \leq n$ and $t \leq \tau_m$ we have

$$\mathbf{\Psi}_t^{\tau_m} = \mathbf{\Psi}_t^{\tau_n} \qquad \text{and} \qquad Y_t^{\tau_m} = Y_t^{\tau_n}$$

allow us once more to define the process at any t by

$$[\mathbf{\Psi}, Y]_t := \lim_{n \to \infty} [\mathbf{\Psi}^{\tau_n}, Y^{\tau_n}]_t \tag{A.36}$$

for the limit taken $\mathbb{P}-a.s.$ in $\mathcal{H}$. Then the process

$$\mathbf{\Psi} Y - [\mathbf{\Psi}, Y]$$

is itself a local martingale, localised by the stopping times (τ_n). The argument justifying this is identical to Proposition A.1.31, from which it is similarly clear that

$$[\mathbf{\Psi}, Y]^{\tau_n} = [\mathbf{\Psi}^{\tau_n}, Y^{\tau_n}].$$

In the traditional way, these notions can all be extended to semi-martingales (that is, a martingale plus a bounded-variation process). The quadratic and cross variation of such semi-martingales is then simply the quadratic/cross variation of the corresponding martingale parts. To this end we introduce the notation $\bar{\mathcal{M}}_c^2$ and $\bar{\mathcal{M}}_c^2(\mathcal{H})$ to be the corresponding spaces of square integrable continuous semi-martingales, and similarly $\bar{\mathcal{M}}_c, \bar{\mathcal{M}}_c(\mathcal{H})$ to be the spaces of continuous semi-martingales.

A.1.5 Integration Driven by Cylindrical Brownian Motion

For our analysis now we will need to make reference to two distinct Hilbert Spaces; one over which $\mathcal{W}$ is a Cylindrical Brownian Motion, and the other in which our integrand maps to. Henceforth we introduce $\mathfrak{U}$ as the Hilbert Space over which $\mathcal{W}$ is a Cylindrical Brownian Motion. We shall take (e_i) as an orthonormal basis over $\mathfrak{U}$ and (a_i) an orthonormal basis over $\mathcal{H}$.

Definition A.1.45. *Denote by $\mathcal{I}^{\mathcal{H}}_T(\mathcal{W})$ the class of progressively measurable operator valued processes B belonging to the set $L^2\big(\Omega \times [0,T]; \mathscr{L}^2(\mathfrak{U};\mathcal{H})\big)$. Measurability here is again defined with respect to the Borel sigma-algebra on $\mathscr{L}^2(\mathfrak{U};\mathcal{H})$. The class of processes B such that $B \in \mathcal{I}^{\mathcal{H}}_T(\mathcal{W})$ for all T will be denoted by $\mathcal{I}^{\mathcal{H}}(\mathcal{W})$.*

Note that we make no explicit reference to $\mathfrak{U}$, the space on which $\mathcal{W}$ is a cylindrical Brownian Motion. This is because, in practice, the space $\mathfrak{U}$ will be arbitrarily chosen; this shall be discussed later. Recall (A.23), that if $\mathcal{W}$ is a Cylindrical Brownian motion over $\mathfrak{U}$, it can be formally represented by

$$\mathcal{W}(t) = \sum_{i=1}^{\infty} e_i W^i_t, \tag{A.37}$$

where the (W^i) are standard independent one-dimensional Brownian motions.

Definition A.1.46. *For $B \in \mathcal{I}^{\mathcal{H}}(\mathcal{W})$ we define the Itô stochastic integral*

$$\int_0^t B(s)d\mathcal{W}_s \tag{A.38}$$

as the $\mathcal{H}$-valued random variable

$$\sum_{i=1}^{\infty}\int_0^t B_{e_i}(s)dW^i_s, \tag{A.39}$$

where each integral is defined as in Definition A.1.7 and the infinite sum is taken in $L^2\big(\Omega;\mathcal{H}\big)$.

The immediate response to this definition is to prove that (*A.39*) is well defined; that is the integrals are well defined, as is the limit. Firstly for each i, B_{e_i} is trivially in $L^2\big(\Omega \times [0,T];\mathcal{H}\big)$ as this norm is bounded by the $L^2\big(\Omega \times [0,T]; \mathscr{L}^2(\mathfrak{U};\mathcal{H})\big)$ norm of B. The progressive measurability is inherited from that of B.

In order to show that the limit of partial sums is well defined, we proceed similarly to the method applied for (A.20) and argue that the sequence of partial sums is Cauchy. Observe that

$$\begin{aligned}\Big|\Big|\sum_{i=m}^{n}\int_0^t B_{e_i}(s)dW_s^i\Big|\Big|^2_{L^2\left(\Omega;\mathcal{H}\right)} &= \mathbb{E}\Big|\Big|\sum_{i=m}^{n}\int_0^t B_{e_i}(s)dW_s^i\Big|\Big|^2_{\mathcal{H}}\\ &= \sum_{i=m}^{n}\mathbb{E}\int_0^t \|B_{e_i}(s)\|^2_{\mathcal{H}}\,ds\end{aligned}$$

having applied the Itô Isometry A.1.11 to the above. But by the assumption that $B \in L^2\big(\Omega\times[0,t];\mathscr{L}^2(\mathfrak{U};\mathcal{H})\big)$ we know

$$\mathbb{E}\int_0^t\sum_{i=1}^{\infty}\|B_{e_i}(s)\|^2_{\mathcal{H}}\,ds < \infty$$

and thus, by Tonelli's theorem regarding the infinite sum as an integral with respect to the counting measure,

$$\sum_{i=1}^{\infty}\mathbb{E}\int_0^t\|B_{e_i}(s)\|^2_{\mathcal{H}}\,ds < \infty$$

demonstrating that the sequence of partial sums is indeed Cauchy in $L^2\left(\Omega;\mathcal{H}\right)$. Of course the $L^2(\Omega;\mathcal{H})$ norm of the limit is the limit of the $L^2(\Omega;\mathcal{H})$ norms, so we have justified the following.

Proposition A.1.47. *For $B \in \mathcal{I}^{\mathcal{H}}(\mathcal{W})$, we have*

$$\mathbb{E}\left(\Big|\Big|\int_0^t B(s)d\mathcal{W}_s\Big|\Big|^2_{\mathcal{H}}\right) = \mathbb{E}\left(\int_0^t\|B(s)\|^2_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})}\,ds\right).$$

It is worth noting that whilst we impose the condition

$$\mathbb{E}\left(\int_0^t\sum_{i=1}^{\infty}\|B_{e_i}(s)\|^2_{\mathcal{H}}\,ds\right) < \infty$$

one may instead require the looser condition

$$\int_0^t\sum_{i=1}^{\infty}\|B_{e_i}(s)\|^2_{\mathcal{H}}\,ds < \infty \qquad \mathbb{P}-a.s. \tag{A.40}$$

or equivalently that $B : \Omega \to L^2\big([0,t];\mathscr{L}^2(\mathfrak{U};\mathcal{H})\big)$ for $\mathbb{P}-a.e.\ \omega$. Our formulation follows the classical construction as laid out in Subsection A.1.1, ensuring that the integral is a genuine square integrable martingale, which one anticipates would be lost with just the assumption (A.40). We can just as straightforwardly follow the arguments from Definition A.1.15, which are laid out here. The extension is pertinent in applications, and this brief will closely monitor the necessity of localisation in the abstract frameworks of Sections 2.2 and 2.3.

Definition A.1.48. *Denote by $\bar{\mathcal{I}}^{\mathcal{H}}_T(\mathcal{W})$ the class of progressively measurable operator valued processes B such that $B(\omega)$ belongs to the set $L^2\big([0,T];\mathscr{L}^2(\mathfrak{U};\mathcal{H})\big)$ for $\mathbb{P}-a.e.\ \omega$. The class of processes B such that $B \in \bar{\mathcal{I}}^{\mathcal{H}}_T(\mathcal{W})$ for all T will be denoted by $\bar{\mathcal{I}}^{\mathcal{H}}(\mathcal{W})$.*

Using the template for Definition A.1.16, for a process $B \in \bar{\mathcal{I}}^{\mathcal{H}}(\mathcal{W})$ let us introduce

$$\tau_n := n \wedge \inf\{0 \leq t < \infty : \int_0^t \|B(s)\|^2_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})}\, ds \geq n\}$$

taking the convention that the infimum of the empty set is infinite. The (τ_n) are stopping times as they are simply first hitting times of the continuous and adapted random variable $\int_0^t \|B(s)\|^2_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})}\, ds$. These times tend to infinity $\mathbb{P}-a.s.$ by condition (A.40). Now define the truncated processes B^n as

$$B^n(t) := B(t)\mathbb{1}_{t\leq\tau_n}$$

and using the fact that for $m \leq n$, and $t \leq \tau_m$, we have

$$B(t)\mathbb{1}_{t\leq\tau_n} = B(t)\mathbb{1}_{t\leq\tau_m}$$

we can make the following consistent definition.

Definition A.1.49. *In the setting described, we define*

$$\int_0^t B(s)d\mathcal{W}_s := \lim_{n\to\infty}\int_0^t B^n(s)d\mathcal{W}_s \tag{A.41}$$

$\mathbb{P}-a.s.\ \omega$ in $\mathcal{H}$.

We will have no reservations in writing (A.41) as a formal expression

$$\sum_{i=1}^{\infty}\int_0^t B_{e_i}(s)dW^i_s. \tag{A.42}$$

We say the expression is only formal, as the infinite sum in (A.42) is *not* the $L^2\big(\Omega;\mathcal{H}\big)$ limit of the partial sums of the local martingales as presented. We understand (A.42) only by (A.41), that is by the limit as $n\to\infty$ of the infinite sum in $L^2\big(\Omega;\mathcal{H}\big)$ of the genuine square integrable martingales given by stopping the local martingales at τ_n.

We now look to show a series of properties of this integral which were shown for a one dimensional Brownian Motion across the earlier subsections. The first is the corresponding result of Theorem A.1.12.

Proposition A.1.50. *Suppose that $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces such that $B \in \bar{\mathcal{I}}^{\mathcal{H}_1}(\mathcal{W})$ and $T \in \mathscr{L}(\mathcal{H}_1;\mathcal{H}_2)$. Then the process TB defined by*

$$TB_{e_i}(s,\omega) = T\big(B_{e_i}(s,\omega)\big)$$

belongs to $\bar{\mathcal{I}}^{\mathcal{H}_2}(\mathcal{W})$. *In addition, we have that*

$$T\Big(\int_0^t B(s)d\mathcal{W}_s\Big) = \int_0^t TB(s)d\mathcal{W}_s.$$

Moreover the two integrals are defined as limits $\mathbb{P}-a.s.$ *with respect to the same stopping times, and if* $B \in \mathcal{I}^{\mathcal{H}_1}(\mathcal{W})$ *then* $TB \in \mathcal{I}^{\mathcal{H}_2}(\mathcal{W})$

Proof. Assume at first that $B \in \bar{\mathcal{I}}^{\mathcal{H}_1}(\mathcal{W})$. We shall prove first that $TB \in \bar{\mathcal{I}}^{\mathcal{H}_2}(\mathcal{W})$. The progressive measurability is preserved under the continuity of T, and letting C be such that for any $\phi \in \mathcal{H}_1$,

$$\|T\phi\|_{\mathcal{H}_2}^2 \leq C\,\|\phi\|_{\mathcal{H}_1}^2\,,$$

which exists owing to the boundedness of T, we have

$$\int_0^t \sum_{i=1}^\infty \|TB_{e_i}(s)\|_{\mathcal{H}_2}^2\,ds \leq C\int_0^t \sum_{i=1}^\infty \|B_{e_i}(s)\|_{\mathcal{H}_1}^2\,ds < \infty \tag{A.43}$$

holding $\mathbb{P}-a.s.$ as $B \in \bar{\mathcal{I}}^{\mathcal{H}_1}(\mathcal{W})$. In addition for any stopping time τ_n as in Definition A.1.49,

$$\begin{aligned}\mathbb{E}\Big(\int_0^t \sum_{i=1}^\infty \big\|\big(TB_{e_i}(s)\big)\mathbb{1}_{s\leq\tau_n}\big\|_{\mathcal{H}_2}^2\,ds\Big) &= \mathbb{E}\Big(\int_0^t \sum_{i=1}^\infty \big\|T\big(B_{e_i}(s)\mathbb{1}_{s\leq\tau_n}\big)\big\|_{\mathcal{H}_2}^2\,ds\Big)\\ &\leq C\mathbb{E}\Big(\int_0^t \sum_{i=1}^\infty \|B_{e_i}(s)\mathbb{1}_{s\leq\tau_n}\|_{\mathcal{H}_1}^2\,ds\Big)\\ &< \infty.\end{aligned}$$

So the new stochastic integral

$$\int_0^t TB(s)d\mathcal{W}_s$$

can be constructed using the same sequence of stopping times. We will freely use the linearity of T to commute it with the indicator function. To carry T through the integral it is sufficient to show that for B^n as in Definition A.1.49, then

$$T\Big(\sum_{i=1}^\infty \int_0^t B_{e_i}^n(s)dW_s^i\Big) = \sum_{i=1}^\infty \int_0^t TB_{e_i}^n(s)dW_s^i, \tag{A.44}$$

where the left hand side limit is taken in $L^2\big(\Omega;\mathcal{H}_1\big)$ and the right side in $L^2\big(\Omega;\mathcal{H}_2\big)$. From the $L^2\big(\Omega;\mathcal{H}_1\big)$ limit there exists a subsequence convergent $\mathbb{P}-a.e.$ in $\mathcal{H}_1$. Working with this subsequence, we can pass the $\mathbb{P}-a.s.$ limit through the continuous T such that it is now the $\mathbb{P}-a.s.$ limit in $\mathcal{H}_2$. Linearity of T allows us to pass through the summation as well, so we have now that

$$T\Big(\sum_{i=1}^\infty \int_0^t B_{e_i}^n(s)dW_s^i\Big) = \lim_{n_k\to\infty}\sum_{i=1}^{n_k} T\Big(\int_0^t B_{e_i}^n(s)dW_s^i\Big) \tag{A.45}$$

for the limit $\mathbb{P}-a.e.$ of the subsequence indexed by (n_k). Applying Theorem A.1.12, we can commute T with the integral on the right side of (A.45). However we have justified already that the limit over the whole sequence in $L^2\big(\Omega;\mathcal{H}_2\big)$ exists, agreeing with the $L^2\big(\Omega;\mathcal{H}_2\big)$ limit of the subsequence, which in turn agrees with the $\mathbb{P}-a.s.$ limit. Thus (A.44) is verified, completing the proof. In the case where $B \in \mathcal{I}^{\mathcal{H}_1}(\mathcal{W})$, it is clear from (A.43) that $TB \in \mathcal{I}^{\mathcal{H}_2}(\mathcal{W})$

$\square$

Proposition A.1.51. *Let $B \in \bar{\mathcal{I}}^{\mathcal{H}}(\mathcal{W})$ and $\phi : \Omega \to \mathcal{H}$ be $\mathcal{F}_0-$measurable. Then for every $t > 0$ we have that*

$$\left\langle \int_0^t B_r d\mathcal{W}_r, \phi \right\rangle_{\mathcal{H}} = \int_0^t \langle B_r, \phi \rangle_{\mathcal{H}} d\mathcal{W}_r \tag{A.46}$$

$\mathbb{P}-a.s.$. Moreover if $\eta : \Omega \to \mathbb{R}$ is $\mathcal{F}_0-$measurable, then $\eta B \in \bar{\mathcal{I}}^{\mathcal{H}}(\mathcal{W})$ and for every $t > 0$ we have that

$$\eta \int_0^t B_r d\mathcal{W}_r = \int_0^t \eta B_r d\mathcal{W}_r \tag{A.47}$$

$\mathbb{P}-a.s.$.

Proof. Firstly we make clear that $\langle B, \phi \rangle_{\mathcal{H}}$ is understood as a process defined by the mapping

$$e_i \times \omega \times t \to \langle B_t(e_i, \omega), \phi(\omega) \rangle_{\mathcal{H}}.$$

The fact that $\langle B, \phi \rangle_{\mathcal{H}} \in \bar{\mathcal{I}}^{\mathbb{R}}(\mathcal{W})$ is completely analogous to Proposition A.1.18, where the progressive measurability follows as the mapping

$$\langle B_\cdot, \phi \rangle_{\mathcal{H}} : t \times \omega \times \tilde{\omega} \to \langle B_t(\omega), \phi(\tilde{\omega}) \rangle_{\mathcal{H}}$$

is $\mathcal{B}([0,T]) \times \mathcal{F}_T \times \mathcal{F}_0$ measurable as a mapping into $\mathscr{L}^2(\mathfrak{U}; \mathbb{R})$. Similarly we have that

$$\left\| \langle B_t(\omega), \phi(\omega) \rangle_{\mathcal{H}} \right\|_{\mathscr{L}^2(\mathfrak{U};\mathbb{R})} \leq \|\phi(\omega)\|_{\mathcal{H}} \|B_t(\omega)\|_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})},$$

which is sufficient to justify that $\langle B, \phi \rangle_{\mathcal{H}} \in \bar{\mathcal{I}}^{\mathbb{R}}(\mathcal{W})$. The extension of Proposition A.1.20 to this result is then identical to the extension of Theorem A.1.12 to A.1.50, so we conclude the proof of (A.46) here. The property (A.47) follows identically in analogy with Proposition A.1.21. We make explicit that ηB is defined by the mapping

$$e_i \times \omega \times t \to \eta(\omega) B_t(e_i, \omega).$$

$\square$

Remark A.1.52. *Although we have not explicitly addressed the construction of the integral over a time interval $[s,t]$ where $s > 0$, this can be done without any extra difficulty just as in the standard real valued case. If we were to just consider the integral over $[s,t]$ in Proposition A.1.51, then the results extends to any $\mathcal{F}_s-$measurable ϕ, η in Proposition A.1.51. To show this we of course revisit Proposition A.1.18, appreciating that the $\mathcal{F}_s-$measurability does not disturb the measurability requirements of the simple process.*

We also extend the Stochastic Dominated Convergence Theorem to this setting.

Lemma A.1.53. *Let (B^n) be a sequence in $\bar{\mathcal{I}}^{\mathcal{H}}(\mathcal{W})$ such that there exists processes $B: \Omega \times [0,\infty) \to \mathscr{L}^2(\mathfrak{U};\mathcal{H})$ and $Q \in L^2([0,\infty);\mathbb{R})$ $\mathbb{P}-a.s.$, with the properties that for every $T > 0$, $\mathbb{P} \times \lambda - a.e.$ $(\omega, t) \in \Omega \times [0,T]$:*

1. $\|B^n_t(\omega)\|_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})} \leq |Q_t(\omega)|$ *for all* $n \in \mathbb{N}$*;*
2. $(B^n_t(\omega))$ *is convergent to* $B_t(\omega)$ *in* $\mathscr{L}^2(\mathfrak{U};\mathcal{H})$*.*

Then $B \in \bar{\mathcal{I}}^{\mathcal{H}}(\mathcal{W})$ and for every $t > 0$, there exists a subsequence indexed by (n_k) such that

$$\lim_{n_k \to \infty} \int_0^t B^{n_k}_r d\mathcal{W}_r = \int_0^t B_r d\mathcal{W}_r \tag{A.48}$$

$\mathbb{P}-a.s.$.

Proof. The proof is identical to that of Lemma A.1.19, simply replacing $\mathcal{H}$ by $\mathscr{L}^2(\mathfrak{U};\mathcal{H})$ when dealing with the integrands and using the appropriate Itô Isometry A.1.47. □

We now shift attentions to results regarding the martingale properties of the integral.

Proposition A.1.54. *For $B \in \mathcal{I}^{\mathcal{H}}(\mathcal{W})$, the Itô stochastic integral*

$$\int_0^t B(s) d\mathcal{W}_s$$

belongs to $\mathcal{M}^2_c(\mathcal{H})$. If $B \in \mathcal{I}^{\mathcal{H}}(\mathcal{W})$ only, then the integral is a continuous local martingale

Proof. This follows immediately from Propositions A.1.29, A.1.30 and subsequently A.1.31. □

The martingality of the integral also allows us to consider the quadratic variation as defined in Definition A.1.34.

Proposition A.1.55. *For $B \in \mathcal{I}^{\mathcal{H}}(\mathcal{W})$, we have that*

$$\left[\int_0^{\cdot} B_r d\mathcal{W}_r\right]_t = \int_0^t \|B_r\|^2_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})} dr. \tag{A.49}$$

Proof. At each time t, the integral

$$\int_0^t B_r d\mathcal{W}_r$$

is defined to be the $L^2(\Omega;\mathcal{H})$ limit of the sequence

$$\sum_{i=1}^n \int_0^t B_r(e_i) dW^i_r.$$

We look to infer the quadratic variation of this sequence of processes using Proposition A.1.35, to then apply Lemma A.1.38. We claim that

$$\left[\sum_{i=1}^n \int_0^{\cdot} B_r(e_i) dW^i_r\right]_t = \int_0^t \sum_{i=1}^n \|B_r(e_i)\|^2_{\mathcal{H}} dr$$

which is to say

$$\left\|\sum_{i=1}^{n}\int_0^t B_r(e_i)dW_r^i\right\|_{\mathcal{H}}^2 - \int_0^t \sum_{i=1}^{n}\|B_r(e_i)\|_{\mathcal{H}}^2\,dr \tag{A.50}$$

is a martingale. For the orthonormal basis (a_k) of $\mathcal{H}$,

$$\begin{aligned}
&\left\|\sum_{i=1}^{n}\int_0^t B_r(e_i)dW_r^i\right\|_{\mathcal{H}}^2 - \int_0^t \sum_{i=1}^{n}\|B_r(e_i)\|_{\mathcal{H}}^2\,dr\\
&\quad= \sum_{k=1}^{\infty}\left\langle \sum_{i=1}^{n}\int_0^t B_r(e_i)dW_r^i, a_k\right\rangle_{\mathcal{H}}^2 - \int_0^t \sum_{i=1}^{n}\|B_r(e_i)\|_{\mathcal{H}}^2\,dr\\
&\quad= \sum_{k=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\langle \int_0^t B_r(e_i)dW_r^i, a_k\right\rangle_{\mathcal{H}}\left\langle \int_0^t B_r(e_j)dW_r^j, a_k\right\rangle_{\mathcal{H}} - \int_0^t \sum_{i=1}^{n}\|B_r(e_i)\|_{\mathcal{H}}^2\,dr\\
&\quad= \left(\sum_{k=1}^{\infty}\sum_{i=1}^{n}\left\langle \int_0^t B_r(e_i)dW_r^i, a_k\right\rangle_{\mathcal{H}}^2 - \int_0^t \sum_{i=1}^{n}\|B_r(e_i)\|_{\mathcal{H}}^2\,dr\right)\\
&\qquad\qquad + \left(\sum_{k=1}^{\infty}\sum_{i\neq j}\left\langle \int_0^t B_r(e_i)dW_r^i, a_k\right\rangle_{\mathcal{H}}\left\langle \int_0^t B_r(e_j)dW_r^j, a_k\right\rangle_{\mathcal{H}}\right)\\
&\quad= \sum_{i=1}^{n}\left(\left\|\int_0^t B_r(e_i)dW_r^i\right\|_{\mathcal{H}}^2 - \int_0^t \|B_r(e_i)\|_{\mathcal{H}}^2\,dr\right)\\
&\qquad\qquad + \left(\sum_{k=1}^{\infty}\sum_{i\neq j}\left\langle \int_0^t B_r(e_i)dW_r^i, a_k\right\rangle_{\mathcal{H}}\left\langle \int_0^t B_r(e_j)dW_r^j, a_k\right\rangle_{\mathcal{H}}\right).
\end{aligned}$$

Inspecting the last equality, by Proposition A.1.35 we have that

$$\sum_{i=1}^{n}\left(\left\|\int_0^t B_r(e_i)dW_r^i\right\|_{\mathcal{H}}^2 - \int_0^t \|B_r(e_i)\|_{\mathcal{H}}^2\,dr\right)$$

is a finite sum of martingales, so the process defined in (A.50) will itself be a martingale as well if we show that the same is true of

$$\sum_{k=1}^{\infty}\sum_{i\neq j}\left\langle \int_0^t B_r(e_i)dW_r^i, a_k\right\rangle_{\mathcal{H}}\left\langle \int_0^t B_r(e_j)dW_r^j, a_k\right\rangle_{\mathcal{H}}. \tag{A.51}$$

We consider the above at first for each fixed k, rewriting it as

$$\sum_{i\neq j}\left(\int_0^t \langle B_r(e_j), a_k\rangle_{\mathcal{H}}\,dW_r^j\right)\left(\int_0^t \langle B_r(e_i), a_k\rangle_{\mathcal{H}}\,dW_r^i\right). \tag{A.52}$$

Note that for each $i \neq j$,

$$\left(\int_0^t \langle B_r(e_j), a_k\rangle_{\mathcal{H}}\,dW_r^j\right)\left(\int_0^t \langle B_r(e_i), a_k\rangle_{\mathcal{H}}\,dW_r^i\right)$$

is the product of two independent real-valued square-integrable martingales, so is itself a martingale, hence the finite sum given in (A.52) retains the martingale property. We emphasise again that the martingale property is always considered with respect to the fixed filtration $(\mathcal{F}_t)$ of our probability space.

We wish to show that the martingale property remains true in the limit of the infinite sum for (A.51). Convergence of the infinite sum is defined $\mathbb{P}-a.s.$, and it is sufficient to show that the convergence also holds in $L^1(\Omega;\mathbb{R})$ as this topology retains the martingale property. For this we show that the sequence is Cauchy in $L^1(\Omega;\mathbb{R})$, taking the difference of the l^{th} and m^{th} terms to see that

$$\begin{aligned}
\mathbb{E}&\left|\sum_{k=m+1}^{l}\sum_{i\neq j}\left(\int_0^t \langle B_r(e_j),a_k\rangle_{\mathcal{H}}\,dW_r^j\right)\left(\int_0^t \langle B_r(e_i),a_k\rangle_{\mathcal{H}}\,dW_r^i\right)\right|\\
&\leq \sum_{k=m+1}^{l}\sum_{i\neq j}\mathbb{E}\left|\left(\int_0^t \langle B_r(e_j),a_k\rangle_{\mathcal{H}}\,dW_r^j\right)\left(\int_0^t \langle B_r(e_i),a_k\rangle_{\mathcal{H}}\,dW_r^i\right)\right|\\
&\leq \frac{1}{2}\sum_{k=m+1}^{l}\sum_{i\neq j}\mathbb{E}\left[\left(\int_0^t \langle B_r(e_j),a_k\rangle_{\mathcal{H}}\,dW_r^j\right)^2+\left(\int_0^t \langle B_r(e_i),a_k\rangle_{\mathcal{H}}\,dW_r^i\right)^2\right]\\
&\leq n^2\sum_{k=m+1}^{l}\sup_i\mathbb{E}\left(\int_0^t \langle B_r(e_i),a_k\rangle_{\mathcal{H}}\,dW_r^i\right)^2\\
&\leq n^2\sum_{k=m+1}^{l}\sup_i\mathbb{E}\int_0^t \langle B_r(e_i),a_k\rangle_{\mathcal{H}}^2\,dr\\
&\leq n^2\sum_{k=m+1}^{\infty}\sup_i\mathbb{E}\int_0^t \langle B_r(e_i),a_k\rangle_{\mathcal{H}}^2\,dr
\end{aligned}$$

having used the Itô Isometry. This is a monotone decreasing sequence in m, convergent to zero, hence the Cauchy property is shown so there exists a limit in $L^1(\Omega;\mathbb{R})$ which must agree with the $\mathbb{P}-a.s.$ limit (we can take a $\mathbb{P}-a.s.$ convergent subsequence from the $L^1(\Omega;\mathbb{R})$ convergence) and the martingale property of the process defined in (A.51) and hence (A.50) is shown. By applying Lemma A.1.38, we deduce that $\left[\int_0^{\cdot} B_r d\mathcal{W}_r\right]_t$ is the $L^1(\Omega;\mathbb{R})$ limit of the sequence

$$\int_0^t\sum_{i=1}^{n}\|B_r(e_i)\|_{\mathcal{H}}^2\,dr$$

in n. Similarly to the analysis just conducted we can show that this sequence is Cauchy in $L^1(\Omega;\mathbb{R})$ and it's limit agrees with the $\mathbb{P}-a.s.$ limit, which is of course

$$\int_0^t\|B_r\|_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})}^2\,dr$$

taking the infinite sum through the integral with either Tonelli's Theorem (identifying the infinite sum as a integral with respect to the counting measure) or the Monotone Convergence Theorem. The proof is concluded. $\square$

We also have the analogous result to Proposition A.1.36.

Proposition A.1.56. *Let $B \in \mathcal{I}^{\mathcal{H}}_T(\mathcal{W})$ and consider any sequence of partitions*

$$I_l := \left\{0 = t^l_0 < t^l_1 < \cdots < t^l_{k_l} = T\right\}$$

with $\max_j |t^l_j - t^l_{j-1}| \to 0$ *as* $l \to \infty$. *Then for all* $t \in [0,T]$, *for any* $\varepsilon > 0$,

$$\lim_{l\to\infty} \mathbb{P}\left(\left\{\left|\sum_{t^l_{j+1}\leq t}\left\|\int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r\right\|^2_{\mathcal{H}} - \int_0^t \|B_r\|^2_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})}\,dr\right| > \varepsilon\right\}\right) = 0. \tag{A.53}$$

Proof. Following the method used in Proposition A.1.36 we again would like to reduce this to a familiar case and extrapolate the result to the limit. We introduce a sequence of stopping times (τ^n) defined at every $n \in \mathbb{N}$ by

$$\tau^n := n \wedge \inf\left\{t \in [0,T] : \left\|\int_0^t B_r d\mathcal{W}_r\right\|^2_{\mathcal{H}} + \int_0^t \|B_r\|^2_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})}\,dr \geq n\right\}.$$

For every n we define the process

$$B^n_\cdot := B_\cdot \mathbb{1}_{\cdot\leq\tau^n}$$

and now look to show that

$$\lim_{l\to\infty} \mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\left\|\int_{t^l_j}^{t^l_{j+1}} B^n_r d\mathcal{W}_r\right\|^2_{\mathcal{H}} - \int_0^t \|B^n_r\|^2_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})}\,dr\right|\right) = 0. \tag{A.54}$$

This is precisely in line with the method of Proposition A.1.36. We have that

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\left\|\int_{t^l_j}^{t^l_{j+1}} B^n_r d\mathcal{W}_r\right\|^2_{\mathcal{H}} - \int_0^t \|B^n_r\|^2_{\mathscr{L}^2(\mathfrak{U};\mathcal{H})}\,dr\right|\right)\\
&\quad= \mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\left\|\int_{t^l_j}^{t^l_{j+1}} B^n_r d\mathcal{W}_r\right\|^2_{\mathcal{H}} - \int_0^t \sum_{i=1}^\infty \|B^n_r(e_i)\|^2_{\mathcal{H}}\,dr\right|\right)\\
&\quad= \mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\sum_{k=1}^\infty\left\langle\int_{t^l_j}^{t^l_{j+1}} B^n_r d\mathcal{W}_r, a_k\right\rangle^2_{\mathcal{H}} - \int_0^t \sum_{i=1}^\infty\sum_{k=1}^\infty \langle B^n_r(e_i), a_k\rangle^2_{\mathcal{H}}\,dr\right|\right)\\
&\quad= \mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\sum_{k=1}^\infty\left(\int_{t^l_j}^{t^l_{j+1}} \langle B^n_r, a_k\rangle_{\mathcal{H}}\, d\mathcal{W}_r\right)^2 - \int_0^t \sum_{k=1}^\infty\sum_{i=1}^\infty \langle B^n_r(e_i), a_k\rangle^2_{\mathcal{H}}\,dr\right|\right)\\
&\quad= \mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\sum_{k=1}^\infty\left(\int_{t^l_j}^{t^l_{j+1}} \langle B^n_r, a_k\rangle_{\mathcal{H}}\, d\mathcal{W}_r\right)^2 - \int_0^t \sum_{k=1}^\infty \|\langle B^n_r(e_i), a_k\rangle_{\mathcal{H}}\|^2_{\mathscr{L}^2(\mathfrak{U};\mathbb{R})}\,dr\right|\right)\\
&\quad\leq \sum_{k=1}^\infty \mathbb{E}\left(\left|\sum_{t^l_{j+1}\leq t}\left(\int_{t^l_j}^{t^l_{j+1}} \langle B^n_r, a_k\rangle_{\mathcal{H}}\, d\mathcal{W}_r\right)^2 - \int_0^t \|\langle B^n_r(e_i), a_k\rangle_{\mathcal{H}}\|^2_{\mathscr{L}^2(\mathfrak{U};\mathbb{R})}\,dr\right|\right)
\end{aligned}$$

having applied Proposition A.1.50 and the Dominated Convergence Theorem to take the infinite sum in k through the time integral and expectation. From Proposition A.1.50 and Proposition A.1.54 then $\int_0^{\cdot} \langle B_r^n, a_k\rangle_{\mathcal{H}}\, d\mathcal{W}_r$ belongs to $\mathcal{M}_c^2$, with quadratic variation $\int_0^t \|\langle B_r(e_i), a_k\rangle_{\mathcal{H}}\|^2_{\mathscr{L}^2(\mathfrak{U};\mathbb{R})}\, dr$ coming from Proposition A.1.55. Just as we used in Proposition A.1.36, we have that for each fixed $k \in \mathbb{N}$,

$$\lim_{l\to\infty} \mathbb{E}\left(\left|\sum_{t_{j+1}^l \leq t}\left(\int_{t_j^l}^{t_{j+1}^l} \langle B_r^n, a_k\rangle_{\mathcal{H}}\, d\mathcal{W}_r\right)^2 - \int_0^t \|\langle B_r^n(e_i), a_k\rangle_{\mathcal{H}}\|^2_{\mathscr{L}^2(\mathfrak{U};\mathbb{R})}\, dr\right|\right) = 0$$

so it is sufficient to justify the interchange of limit in l and summation in k. This follows identically to the justification in Proposition A.1.36, appealing this time to the Itô Isometry A.1.47. □

A.2 Enhanced Regularity and an Energy Equality

Having established a framework for SPDEs in Section 2.2, we introduce techniques to facilitate our analysis in it. The Itô Formula is well regarded as one of the most useful tools in stochastic analysis, and we establish what can be considered as a specific case of this result in our framework. We shall introduce a new setting in which our established solution framework falls, with the understanding that we would like to apply this to solutions whilst also using the results to deduce the existence of solutions when they are not a priori known. This is well understood in the typical variational framework, for which we again refer to [102, 127], but we take care in addressing subtle changes. The first could be loss of the duality structure, though for this result we do assume a bilinear form relation which behaves similarly. The second is that we conduct the proof for *local* solutions, necessary for our motivating class of equations, so it is important for us to explicitly address how the localisation affects the proof. Indeed the consideration of local solutions, as well as the related localisation in the construction of the integral, martingale theory and analytical techniques, is an important extension of the framework of [102, 125, 127]. Similarly the final key change is that we do not assume any integrability over the probability space of our processes, demanding again another source of localisation which we find worthy of detailing.

To prove this energy equality we shall rely on looking at partitions in time over which some nice properties are satisfied, before taking the limit as the increments go to zero. Towards this we recall the following lemma from [127], Lemma 4.2.6. This section follows the ideas of [127] Lemma 4.2.5.

Lemma A.2.1. *Let $\mathcal{X}_1, \mathcal{X}_2$ be two Banach Spaces with continuous embedding $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ and suppose that for some $T > 0$ and a stopping time τ, $\boldsymbol{\Phi} : \Omega \times [0,T] \to \mathcal{X}_2$ is such that for $\mathbb{P} - a.e.$ ω, $\boldsymbol{\Phi}_\cdot(\omega) \in C([0,T];\mathcal{X}_2)$ and $\boldsymbol{\Phi}_\cdot \mathbb{1}_{\cdot\leq\tau} \in L^2(\Omega \times [0,T];\mathcal{X}_1)$. Then for any $A \subset (0,T)$ with $\lambda(A) = 0$ there exists a sequence of partitions (I_l) such that*

1. *$I_l := \left\{0 = t_0^l < t_1^l < \cdots < t_{k_l}^l = T\right\}$, $\max_j |t_j^l - t_{j-1}^l| \to 0$ as $l \to \infty$;*

2. *$I_l \subset I_{l+1}$;*

3. *$I_l \cap A = \emptyset$;*

4. *For $\mathbb{P}-a.e.\ \omega$ and every t^l_j with $1 \leq j \leq k_{l-1}$, $\mathbf{\Phi}_{t^l_j}(\omega)\mathbb{1}_{t^l_j \leq \tau(\omega)} \in \mathcal{X}_1$;*

5. *The processes $\hat{\mathbf{\Phi}}^l, \tilde{\mathbf{\Phi}}^l$ defined at each $t \in [0,T]$ and $\omega \in \Omega$ by*

$$\hat{\mathbf{\Phi}}^l_t(\omega) := \sum_{j=2}^{k_l} \mathbb{1}_{[t^l_{j-1}, t^l_j)}(t)\mathbf{\Phi}_{t^l_{j-1}}(\omega)\mathbb{1}_{t^l_{j-1} \leq \tau(\omega)}, \qquad \tilde{\mathbf{\Phi}}^l_t(\omega) := \sum_{j=1}^{k_{l-1}} \mathbb{1}_{[t^l_{j-1}, t^l_j)}(t)\mathbf{\Phi}_{t^l_j}(\omega)\mathbb{1}_{t^l_j \leq \tau(\omega)}$$

   belong to $L^2(\Omega \times [0,T]; \mathcal{X}_1)$ and both converge to $\mathbf{\Phi}.\mathbb{1}_{\cdot \leq \tau}$ in this space.

Before moving on we take a moment to discuss this result. In the statement of [127] Lemma 4.2.6, there is no continuity assumption on $\mathbf{\Phi}$ and indeed this is surplus to requirement, however we want to make explicit that it is genuinely the process $\mathbf{\Phi}$ taken in item 5 and not some other representative of an equivalence class. The assumptions are of course reminiscent of Definition 2.2.6 and just as was stressed there that the progressively measurable process in V was not necessarily the continuous process itself but just a $\mathbb{P} \times \lambda - a.s.$ equivalent representation, we are reminded again that elements of $L^2(\Omega \times [0,T]; \mathcal{X}_1)$ are only an equivalence class of $\mathbb{P} \times \lambda - a.s.$ equal functions. Therefore the representations $\hat{\mathbf{\Phi}}^l, \tilde{\mathbf{\Phi}}^l$ are significant as they are piecewise constant in the space $\mathcal{X}_1$.

We impose some additional structure on the established framework to conduct the analysis of this subsection. We work with a triple of embedded Hilbert Spaces

$$V \hookrightarrow H \hookrightarrow U,$$

where the embeddings are continuous, V is assumed dense in H, and there exists a continuous bilinear form $\langle \cdot, \cdot \rangle_{U \times V} : U \times V \to \mathbb{R}$ such that for every $\phi \in H, \psi \in V$,

$$\langle \phi, \psi \rangle_{U \times V} = \langle \phi, \psi \rangle_H .$$

We suppose that for some $T > 0$ and stopping time τ:

1. $\mathbf{\Psi}_0 \in L^2(\Omega; H)$ is $\mathcal{F}_0-$measurable;

2. $\eta \in L^2(\Omega \times [0,T]; U))$;

3. $B \in \mathcal{I}^H(\mathcal{W})$;

4. $\mathbf{\Psi}.\mathbb{1}_{\cdot \leq \tau} \in L^2(\Omega \times [0,T]; V))$ and is progressively measurable in V;

5. The identity
$$\mathbf{\Psi}_t = \mathbf{\Psi}_0 + \int_0^{t \wedge \tau} \eta_s ds + \int_0^{t \wedge \tau} B_s d\mathcal{W}_s \tag{A.55}$$
   holds $\mathbb{P} - a.s.$ in U for all $t \in [0,T]$.

Remark A.2.2. *Once more, it is clear from assumption 5 that for $\mathbb{P}-a.e.$ ω, $\boldsymbol{\Psi}(\omega) \in C([0,T];U)$ but the progressively measurable process assumed in item 4 is only equivalent to this $\boldsymbol{\Psi}.\mathbb{1}_{\cdot\leq\tau}$ $\mathbb{P}\times\lambda-a.s.$. It is a slight abuse of notation commonplace in the literature and we shall follow suit without excessive clarification at each use.*

With this structure in place, we first look to deduce some improved regularity on $\boldsymbol{\Psi}$. We are motivated by applications to the existence theory of solutions, and if η, B were given by functions of $\boldsymbol{\Psi}$ then note that $\boldsymbol{\Psi}$ is *nearly* a local strong solution of the corresponding SPDE. The missing ingredient is pathwise continuity in H, which we look to deduce from only the given properties.

For this we fix an application of Lemma A.2.1 relative to the assumptions laid out above. We take T and τ as in the assumptions, $\mathcal{X}_1 = V$, $\mathcal{X}_2 = U$, $\boldsymbol{\Phi} = \boldsymbol{\Psi}$. From item 4 we have that for $\lambda-a.e.$ $t \in [0,T]$, $\mathbb{E}\left(\|\boldsymbol{\Psi}_t\mathbb{1}_{t\leq\tau}\|_V^2\right) < \infty$ and we take A to be the $\lambda-$zero set on which this does not hold. Then $\hat{\boldsymbol{\Psi}}^l, \tilde{\boldsymbol{\Psi}}^l, I^l$ are defined as in Lemma A.2.1 and we define

$$I := \bigcup_{l\in\mathbb{N}} I^l.$$

Lemma A.2.3. *We have that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|\boldsymbol{\Psi}_t\|_H^2\right) < \infty.$$

Proof. For $\mathbb{P}-a.e.$ ω and every $s<t$ with $s,t \in I\cap[0,\tau]/\{T\}$, observe that

$$\begin{aligned}
&\left\|\int_s^t B_r d\mathcal{W}_r\right\|_H^2 - \left\|\boldsymbol{\Psi}_t - \boldsymbol{\Psi}_s - \int_s^t B_r d\mathcal{W}_r\right\|_H^2 + 2\left\langle \boldsymbol{\Psi}_s, \int_s^t B_r d\mathcal{W}_r\right\rangle_{\mathcal{H}}\\
&= \left\|\int_s^t B_r d\mathcal{W}_r\right\|_H^2 - \|\boldsymbol{\Psi}_t - \boldsymbol{\Psi}_s\|_H^2 - \left\|\int_s^t B_r d\mathcal{W}_r\right\|_H^2\\
&\qquad + 2\left\langle \boldsymbol{\Psi}_t - \boldsymbol{\Psi}_s, \int_s^t B_r d\mathcal{W}_r\right\rangle_{\mathcal{H}} + 2\left\langle \boldsymbol{\Psi}_s, \int_s^t B_r d\mathcal{W}_r\right\rangle_{\mathcal{H}}\\
&= 2\left\langle \boldsymbol{\Psi}_t, \int_s^t B_r d\mathcal{W}_r\right\rangle_{\mathcal{H}} - \|\boldsymbol{\Psi}_t - \boldsymbol{\Psi}_s\|_H^2\\
&= 2\left\langle \boldsymbol{\Psi}_t, \boldsymbol{\Psi}_t - \boldsymbol{\Psi}_s - \int_s^t \eta_r dr\right\rangle_{\mathcal{H}} - \|\boldsymbol{\Psi}_t\|_H^2 - \|\boldsymbol{\Psi}_s\|_H^2 + 2\langle \boldsymbol{\Psi}_t, \boldsymbol{\Psi}_s\rangle_H\\
&= \|\boldsymbol{\Psi}_t\|_H^2 - \|\boldsymbol{\Psi}_s\|_H^2 - 2\left\langle \boldsymbol{\Psi}_t, \int_s^t \eta_r dr\right\rangle_{\mathcal{H}}\\
&= \|\boldsymbol{\Psi}_t\|_H^2 - \|\boldsymbol{\Psi}_s\|_H^2 - 2\int_s^t \langle \eta_r, \boldsymbol{\Psi}_t\rangle_{U\times V}\, dr
\end{aligned}$$

which we rewrite as the equality

$$\begin{aligned}\|\mathbf{\Psi}_t\|_H^2 - \|\mathbf{\Psi}_s\|_H^2 &= 2\int_s^t \langle \eta_r, \mathbf{\Psi}_t\rangle_{U\times V}\, dr + 2\left\langle \mathbf{\Psi}_s, \int_s^t B_r d\mathcal{W}_r\right\rangle_{\mathcal{H}} \\ &\quad + \left\|\int_s^t B_r d\mathcal{W}_r\right\|_H^2 - \left\|\mathbf{\Psi}_t - \mathbf{\Psi}_s - \int_s^t B_r d\mathcal{W}_r\right\|_H^2 .\end{aligned}$$

Note that we have had to exclude T from this set, as one does not know if $\mathbf{\Psi}_T \in V$ $\mathbb{P}-a.s.$. Using this equality, for any $l \in \mathbb{N}$ and $t = t_i^l \in I_l \cap (0,\tau]/\{T\}$,

$$\begin{aligned}\|\mathbf{\Psi}_t\|_H^2 - \|\mathbf{\Psi}_0\|_H^2 &= \sum_{j=0}^{i-1}\left(\left\|\mathbf{\Psi}_{t_{j+1}^l}\right\|_H^2 - \left\|\mathbf{\Psi}_{t_j^l}\right\|_H^2\right) \\ &= \sum_{j=0}^{i-1}\Bigg(2\int_{t_j^l}^{t_{j+1}^l}\left\langle \eta_r, \mathbf{\Psi}_{t_{j+1}^l}\right\rangle_{U\times V} dr + 2\left\langle \mathbf{\Psi}_{t_j^l}, \int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\rangle_{\mathcal{H}} \\ &\qquad + \left\|\int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2 - \left\|\mathbf{\Psi}_{t_{j+1}^l} - \mathbf{\Psi}_{t_j^l} - \int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2\Bigg) \\ &= 2\sum_{j=0}^{i-1}\left(\int_{t_j^l}^{t_{j+1}^l}\left\langle \eta_r, \mathbf{\Psi}_{t_{j+1}^l}\right\rangle_{U\times V} dr + \int_{t_j^l}^{t_{j+1}^l}\left\langle B_r, \mathbf{\Psi}_{t_j^l}\right\rangle_{\mathcal{H}} d\mathcal{W}_r\right) \\ &\qquad + \sum_{j=0}^{i-1}\left(\left\|\int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2 - \left\|\mathbf{\Psi}_{t_{j+1}^l} - \mathbf{\Psi}_{t_j^l} - \int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2\right) \\ &= 2\sum_{j=0}^{i-1}\left(\int_{t_j^l}^{t_{j+1}^l}\left\langle \eta_r, \tilde{\mathbf{\Psi}}_r^l\right\rangle_{U\times V} dr\right) + 2\sum_{j=1}^{i-1}\left(\int_{t_j^l}^{t_{j+1}^l}\left\langle B_r, \hat{\mathbf{\Psi}}_r^l\right\rangle_{\mathcal{H}} d\mathcal{W}_r\right) \\ &\qquad + 2\int_0^{t_1^l}\langle B_r, \mathbf{\Psi}_0\rangle_{\mathcal{H}}\, d\mathcal{W}_r \\ &\qquad + \sum_{j=0}^{i-1}\left(\left\|\int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2 - \left\|\mathbf{\Psi}_{t_{j+1}^l} - \mathbf{\Psi}_{t_j^l} - \int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2\right) \\ &= 2\int_0^t\left\langle \eta_r, \tilde{\mathbf{\Psi}}_r^l\right\rangle_{U\times V} dr + 2\int_0^t\left\langle B_r, \hat{\mathbf{\Psi}}_r^l\right\rangle_H d\mathcal{W}_r + 2\int_0^{t_1^l}\langle B_r, \mathbf{\Psi}_0\rangle_H\, d\mathcal{W}_r \\ &\qquad + \sum_{j=0}^{i-1}\left(\left\|\int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2 - \left\|\mathbf{\Psi}_{t_{j+1}^l} - \mathbf{\Psi}_{t_j^l} - \int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2\right) \end{aligned} \tag{A.56}$$

where we have applied Proposition A.1.51 and Remark A.1.52. In particular we have that

$$\begin{aligned}\|\mathbf{\Psi}_t\|_H^2 \le \|\mathbf{\Psi}_0\|_H^2 &+ 2\int_0^t\left\langle \eta_r, \tilde{\mathbf{\Psi}}_r^l\right\rangle_{U\times V} dr + 2\int_0^t\left\langle B_r, \hat{\mathbf{\Psi}}_r^l\right\rangle_H d\mathcal{W}_r \\ &+ 2\int_0^{t_1^l}\langle B_r, \mathbf{\Psi}_0\rangle_H\, d\mathcal{W}_r + \sum_{j=0}^{i-1}\left(\left\|\int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2\right).\end{aligned}$$

Our goal is to show that

$$\mathbb{E}\left(\max_{t\in I_l\cap(0,\tau]/\{T\}}\|\mathbf{\Psi}_t\|_H^2\right) \le c$$

for some constant c independent of l. To this end, observe that

$$\begin{aligned}
\mathbb{E}\left(\max_{t\in I_l\cap(0,\tau]/\{T\}}\|\boldsymbol{\Psi}_t\|_H^2\right) \leq \mathbb{E}\left(\|\boldsymbol{\Psi}_0\|_H^2\right)& \\
+ 2\mathbb{E}\left(\int_0^{T\wedge\tau}\left|\left\langle \eta_r, \tilde{\boldsymbol{\Psi}}^l_r\right\rangle_{U\times V}\right| dr\right) &+ 2\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^{t\wedge\tau}\left\langle B_r, \hat{\boldsymbol{\Psi}}^l_r\right\rangle_H d\mathcal{W}_r\right|\right)\\
+ 2\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^{t\wedge\tau}\left\langle B_r, \boldsymbol{\Psi}_0\right\rangle_H d\mathcal{W}_r\right|\right)&\\
+ \mathbb{E}\left[\max_{t\in I_l\cap(0,\tau]/\{T\}}\sum_{j=0}^{i-1}\left(\left\|\int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r\right\|_H^2\right)\right]&.
\end{aligned}$$

We shall treat each term individually. Firstly we have that

$$\begin{aligned}
2\mathbb{E}\left(\int_0^{T\wedge\tau}\left|\left\langle \eta_r, \tilde{\boldsymbol{\Psi}}^l_r\right\rangle_{U\times V}\right| dr\right) &\leq \mathbb{E}\left(\int_0^{T\wedge\tau}\|\eta_r\|_U^2 + \left\|\tilde{\boldsymbol{\Psi}}^l_r\right\|_V^2 dr\right)\\
&\leq \mathbb{E}\left(\int_0^{T\wedge\tau}\|\eta_r\|_U^2 dr\right) + \max_{m\leq L}\mathbb{E}\left(\int_0^{T\wedge\tau}\left\|\tilde{\boldsymbol{\Psi}}^m_r\right\|_V^2 dr\right) + 1
\end{aligned}$$

where L is taken sufficiently large so that for all $m \geq L$,

$$\mathbb{E}\left(\int_0^{T\wedge\tau}\left|\left\|\tilde{\boldsymbol{\Psi}}^m_r\right\|_V^2 - \|\boldsymbol{\Psi}_r\|_V^2\right| dr\right) \leq \frac{1}{2}.$$

For the first stochastic integral we apply the classical Burkholder-Davis-Gundy Inequality, Theorem A.4.10, seeing that

$$\begin{aligned}
2\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^{t\wedge\tau}\left\langle B_r, \hat{\boldsymbol{\Psi}}^l_r\right\rangle_H d\mathcal{W}_r\right|\right) &\leq c\mathbb{E}\left(\int_0^{T\wedge\tau}\left\|\left\langle B_r, \hat{\boldsymbol{\Psi}}^l_r\right\rangle_H\right\|^2_{\mathscr{L}^2(\mathfrak{U};\mathbb{R})} dr\right)^{\frac{1}{2}}\\
&\leq c\mathbb{E}\left(\int_0^{T\wedge\tau}\|B_r\|^2_{\mathscr{L}^2(\mathfrak{U};H)}\left\|\hat{\boldsymbol{\Psi}}^l_r\right\|_H^2 dr\right)^{\frac{1}{2}}\\
&\leq c\mathbb{E}\left(\sup_{r\in[0,T\wedge\tau]}\left\|\hat{\boldsymbol{\Psi}}^l_r\right\|_H^2\int_0^{T\wedge\tau}\|B_r\|^2_{\mathscr{L}^2(\mathfrak{U};H)} dr\right)^{\frac{1}{2}}\\
&= c\mathbb{E}\left(\sup_{r\in[0,T\wedge\tau]}\left\|\hat{\boldsymbol{\Psi}}^l_r\right\|_H^2\right)^{\frac{1}{2}}\left(\int_0^{T\wedge\tau}\|B_r\|^2_{\mathscr{L}^2(\mathfrak{U};H)} dr\right)^{\frac{1}{2}}\\
&= \frac{1}{2}\mathbb{E}\left(\sup_{r\in[0,T\wedge\tau]}\left\|\hat{\boldsymbol{\Psi}}^l_r\right\|_H^2\right) + c\mathbb{E}\left(\int_0^{T\wedge\tau}\|B_r\|^2_{\mathscr{L}^2(\mathfrak{U};H)} dr\right)\\
&= \frac{1}{2}\mathbb{E}\left(\sup_{t\in I_l\cap(0,\tau]/\{T\}}\|\boldsymbol{\Psi}_t\|_H^2\right)\\
&\qquad + c\mathbb{E}\left(\int_0^{T\wedge\tau}\|B_r\|^2_{\mathscr{L}^2(\mathfrak{U};H)} dr\right),
\end{aligned}$$

where c here is a generic constant changing from line to line, independent of l. Putting this together we now see that

$$\begin{aligned}\frac{1}{2}\mathbb{E}\left(\max_{t\in I_l\cap(0,\tau]/\{T\}}\|\boldsymbol{\Psi}_t\|_H^2\right) &\leq \mathbb{E}\left(\|\boldsymbol{\Psi}_0\|_H^2\right)+\mathbb{E}\left(\int_0^{T\wedge\tau}\|\eta_r\|_U^2\,dr\right)\\ &+\max_{m\leq L}\mathbb{E}\left(\int_0^{T\wedge\tau}\left\|\tilde{\boldsymbol{\Psi}}_r^m\right\|_V^2 dr\right)+1+c\mathbb{E}\left(\int_0^{T\wedge\tau}\|B_r\|_{\mathscr{L}^2(\mathfrak{U};H)}^2\,dr\right)\\ &+2\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t\langle B_r,\boldsymbol{\Psi}_0\rangle_H\,d\mathcal{W}_r\right|\right)\\ &+\mathbb{E}\left[\max_{t\in I_l\cap(0,\tau]/\{T\}}\sum_{j=0}^{i-1}\left(\left\|\int_{t_j^l}^{t_{j+1}^l}B_r d\mathcal{W}_r\right\|_H^2\right)\right].\end{aligned}$$

For the stochastic integral involving the initial condition we can treat this identically to generate the bound

$$2\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^{t\wedge\tau}\langle B_r,\boldsymbol{\Psi}_0\rangle_H\,d\mathcal{W}_r\right|\right)\leq\mathbb{E}\left(\|\boldsymbol{\Psi}_0\|_H^2\right)+c\mathbb{E}\left(\int_0^{T\wedge\tau}\|B_r\|_{\mathscr{L}^2(\mathfrak{U};H)}^2\,dr\right).$$

As for the final term, it is clear that the supremum over all partitions can be bounded by taking the partition for $t=t_{k_l}^l=T$. Thus

$$\begin{aligned}\mathbb{E}\left[\max_{t\in I_l\cap(0,\tau]/\{T\}}\sum_{j=0}^{i-1}\left(\left\|\int_{t_j^l}^{t_{j+1}^l}B_r d\mathcal{W}_r\right\|_H^2\right)\right] &\leq \mathbb{E}\left[\sum_{j=0}^{k_l-1}\left(\left\|\int_{t_j^l}^{t_{j+1}^l}B_r d\mathcal{W}_r\right\|_H^2\right)\right]\\ &=\sum_{j=0}^{k_l-1}\mathbb{E}\left(\left\|\int_{t_j^l}^{t_{j+1}^l}B_r d\mathcal{W}_r\right\|_H^2\right)\\ &=\sum_{j=0}^{k_l-1}\mathbb{E}\left(\int_{t_j^l}^{t_{j+1}^l}\|B_r\|_{\mathscr{L}^2(\mathfrak{U};H)}^2\,dr\right)\\ &=\mathbb{E}\left(\int_0^T\|B_r\|_{\mathscr{L}^2(\mathfrak{U};H)}^2\,dr\right)\end{aligned}$$

having applied Proposition A.1.47. In total then we have that

$$\begin{aligned}\mathbb{E}\left(\max_{t\in I_l\cap(0,\tau]/\{T\}}\|\boldsymbol{\Psi}_t\|_H^2\right) &\leq 4\mathbb{E}\left(\|\boldsymbol{\Psi}_0\|_H^2\right)+2\mathbb{E}\left(\int_0^{T\wedge\tau}\|\eta_r\|_U^2\,dr\right)\\ &+2\max_{m\leq L}\mathbb{E}\left(\int_0^{T\wedge\tau}\left\|\tilde{\boldsymbol{\Psi}}_r^m\right\|_V^2 dr\right)+2+c\mathbb{E}\left(\int_0^T\|B_r\|_{\mathscr{L}^2(\mathfrak{U};H)}^2\,dr\right),\end{aligned}$$

which is a finite bound independent of l. As $I_l\subset I_{l+1}$ then the sequence

$$\max_{t\in I_l\cap(0,\tau]/\{T\}}\|\boldsymbol{\Psi}_t\|_H^2$$

is $\mathbb{P}-a.s.$ monotone increasing in l, so we can apply the Monotone Convergence Theorem to see that

$$\mathbb{E}\left(\sup_{t\in I\cap(0,\tau]/\{T\}}\|\boldsymbol{\Psi}_t\|_H^2\right)=\lim_{l\to\infty}\mathbb{E}\left(\max_{t\in I_l\cap(0,\tau]/\{T\}}\|\boldsymbol{\Psi}_t\|_H^2\right)<\infty. \tag{A.57}$$

Thus for $\mathbb{P}-a.e.\ \omega$, we have that

$$\sup_{t\in I\cap(0,\tau(\omega)]/\{T\}}\|\boldsymbol{\Psi}_t(\omega)\|_H^2=\tilde{c}<\infty$$

and $\boldsymbol{\Psi}_\cdot(\omega)\in C([0,T];U)$. We fix such an ω and any $t\in[0,\tau(\omega)]$. As the mesh of the partitions go to zero then there there is a sequence of times (t_n) in $I\cap(0,\tau(\omega)]/\{T\}$ such that $t_n\longrightarrow t$. The sequence $(\boldsymbol{\Psi}_{t_n}(\omega))$ is uniformly bounded in H so admits a weakly convergent subsequence in this space, to a limit which we call ψ. From the continuous embedding of H into U this weak convergence also holds in U, but from the continuity of $\boldsymbol{\Psi}_\cdot(\omega)$ in U we have that $(\boldsymbol{\Psi}_{t_n}(\omega))$ converges strongly and therefore weakly to $\boldsymbol{\Psi}_t(\omega)$ in U. By the uniqueness of limits in the weak topology, we conclude that $\boldsymbol{\Psi}_t(\omega)=\psi$ and thus belongs to H. Moreover the weak limit preserves the boundedness in H, so $\|\boldsymbol{\Psi}_t(\omega)\|_H\leq\tilde{c}$. Therefore

$$\sup_{t\in I\cap(0,\tau(\omega)]/\{T\}}\|\boldsymbol{\Psi}_t(\omega)\|_H^2=\sup_{t\in[0,\tau(\omega)]}\|\boldsymbol{\Psi}_t(\omega)\|_H^2$$

for our fixed ω in a full measure set, and thus $\mathbb{P}-a.s.$. We also know that $\boldsymbol{\Psi}_\cdot=\boldsymbol{\Psi}_{\cdot\wedge\tau}$ from the identity (A.55), so this equality extends to

$$\sup_{t\in I\cap(0,\tau(\omega)]/\{T\}}\|\boldsymbol{\Psi}_t(\omega)\|_H^2=\sup_{t\in[0,T]}\|\boldsymbol{\Psi}_t(\omega)\|_H^2.$$

Combining this with (A.57) concludes the proof. □

Having now justified that for $\mathbb{P}-a.e.\ \omega$ and all $t\in[0,T]$ that $\boldsymbol{\Psi}_t(\omega)\in H$, we move on to prove weak continuity in this space.

Lemma A.2.4. *For $\mathbb{P}-a.e.\ \omega$, $\boldsymbol{\Psi}_\cdot(\omega)$ is weakly continuous in H.*

Proof. We fix $t\in[0,T]$ but now take any sequence of times (t_n) such that $t_n\to t$. For $\mathbb{P}-a.e.$ ω and any given $\phi\in H$ we must justify that

$$\lim_{n\to\infty}\langle\boldsymbol{\Psi}_{t_n}(\omega)-\boldsymbol{\Psi}_t(\omega),\phi\rangle_H=0. \tag{A.58}$$

For any $\varepsilon>0$, from the density of V in H there exists $\phi^k\in V$ such that

$$\left\|\phi-\phi^k\right\|_H<\frac{\varepsilon}{4\|\boldsymbol{\Psi}(\omega)\|_{L^\infty([0,T];H)}}$$

and then from the continuity in U there exists an $N\in\mathbb{N}$ sufficiently large such that for all $n\geq N$,

$$\|\boldsymbol{\Psi}_{t_n}(\omega)-\boldsymbol{\Psi}_t(\omega)\|_U<\frac{\varepsilon}{2\|\phi^k\|_V}.$$

Putting this together,

$$\begin{aligned}
|\langle \boldsymbol{\Psi}_{t_n}(\omega) - \boldsymbol{\Psi}_t(\omega), \phi\rangle_H| &\leq |\left\langle \boldsymbol{\Psi}_{t_n}(\omega) - \boldsymbol{\Psi}_t(\omega), \phi - \phi^k \right\rangle_H| + |\left\langle \boldsymbol{\Psi}_{t_n}(\omega) - \boldsymbol{\Psi}_t(\omega), \phi^k \right\rangle_H| \\
&\leq \|\boldsymbol{\Psi}_{t_n}(\omega) - \boldsymbol{\Psi}_t(\omega)\|_H \left\|\phi - \phi^k\right\|_H + |\left\langle \boldsymbol{\Psi}_{t_n}(\omega) - \boldsymbol{\Psi}_t(\omega), \phi^k \right\rangle_{U\times V}| \\
&\leq 2\,\|\boldsymbol{\Psi}(\omega)\|_{L^\infty([0,T];H)} \left\|\phi - \phi^k\right\|_H + \|\boldsymbol{\Psi}_{t_n}(\omega) - \boldsymbol{\Psi}_t(\omega)\|_U \left\|\phi^k\right\|_V \\
&< \varepsilon
\end{aligned}$$

as required.

$\square$

Guided by the goal to show that $\boldsymbol{\Psi} \in C([0,T];H)$ $\mathbb{P}-a.s.$, and recalling Lemma A.1.28, then continuity of the norm process of $\boldsymbol{\Psi}$ is now desirable. We show this with an explicit evolution equation, which is the energy equality. We first prove this at times in I:

Lemma A.2.5. *The equality*

$$\|\boldsymbol{\Psi}_t\|_H^2 = \|\boldsymbol{\Psi}_0\|_H^2 + \int_0^t \left(2\,\langle \eta_s, \boldsymbol{\Psi}_s\rangle_{U\times V} + \|B_s\|^2_{\mathscr{L}^2(\mathcal{U};H)}\right) ds + 2\int_0^t \langle B_s, \boldsymbol{\Psi}_s\rangle_H \, d\mathcal{W}_s \tag{A.59}$$

holds $\mathbb{P}-a.s.$ in $\mathbb{R}$ for all $t \in I \cap (0,\tau]/\{T\}$.

Proof. We recall (A.56), that is,

$$\begin{aligned}
\|\boldsymbol{\Psi}_t\|_H^2 - \|\boldsymbol{\Psi}_0\|_H^2 = 2\int_0^t \left\langle \eta_r, \tilde{\boldsymbol{\Psi}}^l_r \right\rangle_{U\times V} dr + 2\int_0^t \left\langle B_r, \hat{\boldsymbol{\Psi}}^l_r \right\rangle_H d\mathcal{W}_r + 2\int_0^{t_1^l} \langle B_r, \boldsymbol{\Psi}_0\rangle_H \, d\mathcal{W}_r \\
+ \sum_{j=0}^{i-1}\left(\left\|\int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2 - \left\|\boldsymbol{\Psi}_{t_{j+1}^l} - \boldsymbol{\Psi}_{t_j^l} - \int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2\right)
\end{aligned}$$

holds $\mathbb{P}-a.s.$ for all $t = t_i^l \in I_l \cap (0,\tau]/\{T\}$. The idea from now is to fix any $t \in I \cap (0,\tau]/\{T\}$ and consider the limit of this identity (as $l \to \infty$) $\mathbb{P}-a.s.$. In fact limits of the whole sequence will not be possible, but for each term it is sufficient to take a subsequence convergent to what we desire, and to iterate the procedure of taking successive subsequences. For all sufficiently large l the above identity holds at any $t \in I \cap (0,\tau]/\{T\}$, hence we can consider any such t. We first of all argue that in this limit, the following convergence holds:

$$2\int_0^t \left\langle \eta_r, \tilde{\boldsymbol{\Psi}}^l_r \right\rangle_{U\times V} dr \longrightarrow 2\int_0^t \langle \eta_r, \boldsymbol{\Psi}_r\rangle_{U\times V} \, dr \tag{A.60}$$

$$2\int_0^t \left\langle B_r, \hat{\boldsymbol{\Psi}}^l_r \right\rangle_H d\mathcal{W}_r \longrightarrow 2\int_0^t \langle B_r, \boldsymbol{\Psi}_r\rangle_H \, d\mathcal{W}_r \tag{A.61}$$

$$2\int_0^{t_1^l} \langle B_r, \boldsymbol{\Psi}_0\rangle_H \, d\mathcal{W}_r \longrightarrow 0. \tag{A.62}$$

The first term is shown by the convergence in $L^1(\Omega;\mathbb{R})$,

$$\begin{aligned}\mathbb{E}\left|\int_0^t \left\langle \eta_r, \tilde{\mathbf{\Psi}}^l_r - \mathbf{\Psi}_r\right\rangle_{U\times V} dr\right| &\leq \mathbb{E}\int_0^t \|\eta_r\|_U \left\|\tilde{\mathbf{\Psi}}^l_r - \mathbf{\Psi}_r\right\|_V dr\\ &\leq \left(\mathbb{E}\int_0^t \|\eta_r\|_U^2 \, dr\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^t \left\|\tilde{\mathbf{\Psi}}^l_r - \mathbf{\Psi}_r\right\|_V^2 dr\right)^{\frac{1}{2}},\end{aligned}$$

which approaches zero from the construction of the sequence $\tilde{\mathbf{\Psi}}^l$ (recall Lemma A.2.1), thus exhibiting a $\mathbb{P}-a.s.$ convergent subsequence. We verify the limit in the next terms with the Stochastic Dominated Convergence Theorem, Lemma A.1.53. In particular, as $\hat{\mathbf{\Psi}}^l \to \mathbf{\Psi}$ in $L^2(\Omega\times[0,T];V)$, then there exists a $\mathbb{P}\times\lambda - a.s.$ convergent subsequence in V hence in H. Consequently, observe that

$$\sum_{i=1}^\infty \left|\langle B(e_i), \mathbf{\Psi}\rangle_H - \left\langle B(e_i), \hat{\mathbf{\Psi}}^l\right\rangle_H\right|^2 = \sum_{i=1}^\infty \left|\left\langle B(e_i), \mathbf{\Psi} - \hat{\mathbf{\Psi}}^l\right\rangle_H\right|^2 \leq \left(\sum_{i=1}^\infty \|B(e_i)\|_H^2\right)\left\|\mathbf{\Psi} - \hat{\mathbf{\Psi}}^l\right\|_H^2$$

so in fact the corresponding subsequence of $\left(\left\langle B, \hat{\mathbf{\Psi}}^l\right\rangle_H\right)$ is $\mathbb{P}\times\lambda - a.s.$ convergent to $\langle B, \mathbf{\Psi}\rangle_H$ in $\mathscr{L}^2(\mathfrak{U};\mathbb{R})$. By the construction of $\hat{\mathbf{\Psi}}^l$ from Lemma A.2.1, it is evident that

$$\sup_{r\in[0,T]}\left\|\hat{\mathbf{\Psi}}^l_s\right\|_H^2 \leq \sup_{r\in[0,T]}\|\mathbf{\Psi}_s\|_H^2$$

which we know to be $\mathbb{P}-a.s.$ finite from Lemma A.2.3, so one can choose the dominating function

$$\sup_{r\in[0,T]}\|\mathbf{\Psi}_r\|_H^2 \sum_{i=1}^\infty \|B(e_i)\|_H^2$$

as Q in Lemma A.1.53, facilitating passage to the limit of a subsequence $\mathbb{P}-a.s.$ in (A.61). The same is true of (A.61), by an application of Lemma A.1.53 to the subsequence of $\left(\langle B_r, \mathbf{\Psi}_0\rangle_H \mathbb{1}_{r\leq t_1^l}\right)$ which approaches zero by continuity of the integral (Proposition A.1.54). We next inspect the limit of

$$\sum_{j=0}^{i-1}\left\|\int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2,$$

which in Proposition A.1.36 was shown to be given, in probability, by

$$\int_0^t \|B_s\|_{\mathscr{L}^2(\mathfrak{U};H)}^2 \, ds$$

so once more we can extract a subsequence along which $\mathbb{P}-a.s.$ convergence holds. Furthermore, comparing what we have shown to (A.59), it remains to demonstrate that

$$\sum_{j=0}^{i-1}\left\|\mathbf{\Psi}_{t_{j+1}^l} - \mathbf{\Psi}_{t_j^l} - \int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\|_H^2 \longrightarrow 0. \tag{A.63}$$

To this end, we wish to use that

$$\mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r = \int_{t^l_j}^{t^l_{j+1}} \eta_r dr \tag{A.64}$$

in U, from the identity (A.55), which appears difficult as the H norm is above the regularity of this integral. We rewrite

$$\begin{aligned}
&\left\|\mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r\right\|_H^2 \\
&\quad = \left\langle \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r, \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r \right\rangle_H \\
&\quad = \left\langle \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r, \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} \right\rangle_H \\
&\qquad - \left\langle \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r, \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r \right\rangle_H .
\end{aligned} \tag{A.65}$$

Let us consider the first of these terms. Indeed,

$$\begin{aligned}
&\sum_{j=0}^{i-1} \left\langle \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r, \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} \right\rangle_H \\
&\quad = \sum_{j=0}^{i-1} \left\langle \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r, \mathbf{\Psi}_{t^l_{j+1}} \right\rangle_H - \sum_{j=1}^{i-1} \left\langle \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r, \mathbf{\Psi}_{t^l_j} \right\rangle_H \\
&\qquad - \left\langle \mathbf{\Psi}_{t^l_1} - \mathbf{\Psi}_0 - \int_0^{t^l_1} B_r d\mathcal{W}_r, \mathbf{\Psi}_0 \right\rangle_H \\
&\quad = \sum_{j=0}^{i-1} \left\langle \int_{t^l_j}^{t^l_{j+1}} \eta_r dr, \mathbf{\Psi}_{t^l_{j+1}} \right\rangle_{U\times V} - \sum_{j=1}^{i-1} \left\langle \int_{t^l_j}^{t^l_{j+1}} \eta_r dr, \mathbf{\Psi}_{t^l_j} \right\rangle_{U\times V} \\
&\qquad - \left\langle \mathbf{\Psi}_{t^l_1} - \mathbf{\Psi}_0 - \int_0^{t^l_1} B_r d\mathcal{W}_r, \mathbf{\Psi}_0 \right\rangle_H ,
\end{aligned} \tag{A.66}$$

where we have excluded the case for $\mathbf{\Psi}_0$ as this may not belong to V, in order to apply (A.64) as we have done. Note that

$$\begin{aligned}
\left\langle \int_{t^l_j}^{t^l_{j+1}} \eta_r dr, \mathbf{\Psi}_{t^l_{j+1}} \right\rangle_{U\times V} &= \int_{t^l_j}^{t^l_{j+1}} \left\langle \eta_r, \mathbf{\Psi}_{t^l_{j+1}} \right\rangle_{U\times V} dr \\
&= \int_0^t \left\langle \eta_r, \mathbf{\Psi}_{t^l_{j+1}} \right\rangle_{U\times V} \mathbb{1}_{[t^l_j, t^l_{j+1}]}(r) dr
\end{aligned} \tag{A.67}$$

and similarly for t^l_j. Moreover, we can rewrite

$$\tilde{\mathbf{\Psi}}^l_r \mathbb{1}_{r\leq t} = \sum_{j=0}^{k_{l-2}} \mathbb{1}_{[t^l_j, t^l_{j+1}]}(r) \mathbf{\Psi}_{t^l_{j+1}} \mathbb{1}_{r\leq t}.$$

Therefore, if $i \leq k_{l-1}$ then

$$\sum_{j=0}^{i-1} \mathbb{1}_{[t_j^l, t_{j+1}^l]}(r)\boldsymbol{\Psi}_{t_{j+1}^l}\mathbb{1}_{r\leq t} = \tilde{\boldsymbol{\Psi}}_r^l \mathbb{1}_{r\leq t},$$

but this is certainly the case as $t = t_i^l \in I \cap (0,\tau]/\{T\}$ is excluded from being $T = t_{k_l}^l$. Meanwhile, we have that

$$\hat{\boldsymbol{\Psi}}_r^l \mathbb{1}_{r\leq t} = \sum_{j=1}^{k_{l-1}} \mathbb{1}_{[t_j^l, t_{j+1}^l]}(r)\boldsymbol{\Psi}_{t_j^l}\mathbb{1}_{r\leq t}$$

which implies

$$\sum_{j=1}^{i-1} \mathbb{1}_{[t_j^l, t_{j+1}^l]}(r)\boldsymbol{\Psi}_{t_j^l}\mathbb{1}_{r\leq t} = \hat{\boldsymbol{\Psi}}_r^l \mathbb{1}_{r\leq t}.$$

Together with (A.67), we achieve that

$$\begin{aligned}\sum_{j=0}^{i-1}\left\langle \int_{t_j^l}^{t_{j+1}^l} \eta_r dr, \boldsymbol{\Psi}_{t_{j+1}^l}\right\rangle_{U\times V} &= \int_0^t \left\langle \eta_r, \tilde{\boldsymbol{\Psi}}_r^l\right\rangle_{U\times V} dr\\ \sum_{j=1}^{i-1}\left\langle \int_{t_j^l}^{t_{j+1}^l} \eta_r dr, \boldsymbol{\Psi}_{t_j^l}\right\rangle_{U\times V} &= \int_0^t \left\langle \eta_r, \hat{\boldsymbol{\Psi}}_r^l\right\rangle_{U\times V} dr.\end{aligned}$$

Revisiting (A.66), it is apparent that

$$\begin{aligned}&\sum_{j=0}^{i-1}\left\langle \boldsymbol{\Psi}_{t_{j+1}^l} - \boldsymbol{\Psi}_{t_j^l} - \int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r, \boldsymbol{\Psi}_{t_{j+1}^l} - \boldsymbol{\Psi}_{t_j^l}\right\rangle_H\\ &\qquad = \int_0^t \left\langle \eta_r, \tilde{\boldsymbol{\Psi}}_r^l - \hat{\boldsymbol{\Psi}}_r^l\right\rangle_{U\times V} dr - \left\langle \boldsymbol{\Psi}_{t_1^l} - \boldsymbol{\Psi}_0 - \int_0^{t_1^l} B_r d\mathcal{W}_r, \boldsymbol{\Psi}_0\right\rangle_H,\end{aligned}$$

which we argue approaches zero along a subsequence $\mathbb{P} - a.s.$. We construct the first term as

$$\int_0^t \left\langle \eta_r, \tilde{\boldsymbol{\Psi}}_r^l - \hat{\boldsymbol{\Psi}}_r^l\right\rangle_{U\times V} dr = \int_0^t \left\langle \eta_r, \tilde{\boldsymbol{\Psi}}_r^l - \boldsymbol{\Psi}_r\right\rangle_{U\times V} dr - \int_0^t \left\langle \eta_r, \hat{\boldsymbol{\Psi}}_r^l - \boldsymbol{\Psi}_r\right\rangle_{U\times V} dr,$$

to be treated exactly as we did for (A.60). The remaining term is split up,

$$\left\langle \boldsymbol{\Psi}_{t_1^l} - \boldsymbol{\Psi}_0 - \int_0^{t_1^l} B_r d\mathcal{W}_r, \boldsymbol{\Psi}_0\right\rangle_H = \left\langle \boldsymbol{\Psi}_{t_1^l} - \boldsymbol{\Psi}_0, \boldsymbol{\Psi}_0\right\rangle_H - \left\langle \int_0^{t_1^l} B_r d\mathcal{W}_r, \boldsymbol{\Psi}_0\right\rangle_H.$$

The first inner product must approach zero from the weak continuity shown in Proposition A.2.4, whereas in the second we may use Proposition A.1.51 to take $\boldsymbol{\Psi}_0$ inside the integral, which was treated in (A.62). So returning to (A.65), with the goal of (A.63) still in mind, our attentions turn to

$$\sum_{j=0}^{i-1}\left\langle \boldsymbol{\Psi}_{t_{j+1}^l} - \boldsymbol{\Psi}_{t_j^l} - \int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r, \int_{t_j^l}^{t_{j+1}^l} B_r d\mathcal{W}_r\right\rangle_H. \tag{A.68}$$

For this we cannot use a time approximation in order to utilise (A.64) as we have just done, so we look to consider an approximation in the Hilbert Space instead. From the density of V in H, there exists an orthonormal basis of H consisting of elements from V, which we call (a_k), and let $\mathcal{P}_n$ be the corresponding orthogonal projection in H onto span $\{a_1, \ldots, a_n\}$. Then (A.68) can itself be rewritten as

$$\sum_{j=0}^{i-1}\left\langle \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r, \mathcal{P}_n \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r \right\rangle_H \tag{A.69}$$

$$+\sum_{j=0}^{i-1}\left\langle \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r, (I-\mathcal{P}_n) \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r \right\rangle_H \tag{A.70}$$

for any $n \in \mathbb{N}$, where I is the identity operator in H. Of course, the purpose of this is to use (A.64) and rewrite the first term as

$$\sum_{j=1}^{i-1}\left\langle \int_{t^l_j}^{t^l_{j+1}} \eta_r dr, \mathcal{P}_n \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r \right\rangle_{U\times V},$$

but we can be more astute here. In fact we appreciate that $\mathcal{P}_n \in \mathscr{L}(H;V)$, so by applying Proposition A.1.50 we can take the $\mathcal{P}_n$ through the stochastic integral and see that $\mathcal{P}_n B \in \bar{\mathcal{I}}^V(\mathcal{W})$. Thus if we define the $V-$valued process $M_s := \int_0^s \mathcal{P}_n B_r d\mathcal{W}_r$, then this term is

$$\sum_{j=0}^{i-1}\left\langle \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r, M_{t^l_{j+1}} - M_{t^l_j} \right\rangle_H$$

which can be treated identically to (A.66) (in fact a little more simply, as $M_0 = 0$). Therefore it only remains to consider (A.70), which we control by

$$\begin{aligned}
&\left|\sum_{j=0}^{i-1}\left\langle \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r, (I-\mathcal{P}_n) \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r \right\rangle_H\right| \\
&\qquad \leq \sum_{j=0}^{i-1}\left\| \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r\right\|_H \left\|(I-\mathcal{P}_n) \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r\right\|_H \\
&\qquad \leq \left(\sum_{j=0}^{i-1}\left\| \mathbf{\Psi}_{t^l_{j+1}} - \mathbf{\Psi}_{t^l_j} - \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r\right\|_H^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{i-1}\left\|(I-\mathcal{P}_n) \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r\right\|_H^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Here we have that $I - \mathcal{P}_n \in \mathscr{L}(H;H)$, so this operator can be taken through the integral again by Propostion A.1.50, and we once more understand that

$$\sum_{j=0}^{i-1}\left\|(I-\mathcal{P}_n) \int_{t^l_j}^{t^l_{j+1}} B_r d\mathcal{W}_r\right\|_H^2 \tag{A.71}$$

is, due to Proposition A.1.36, converges in probability (and hence $\mathbb{P}-a.s.$ along a subsequence) to

$$\int_0^t \|(I-\mathcal{P}_n)B_s\|^2_{\mathscr{L}^2(\mathfrak{U};H)}\,ds.$$

By the Monotone Convergence Theorem, this approaches zero as $n\to\infty$. It is not necessarily clear how to utilise this, so we pause and collect the full picture of everything that we have achieved in the aim of (A.63). We have established that

$$\begin{aligned}
&\sum_{j=0}^{i-1}\left\|\boldsymbol{\Psi}_{t^l_{j+1}}-\boldsymbol{\Psi}_{t^l_j}-\int_{t^l_j}^{t^l_{j+1}}B_r d\mathcal{W}_r\right\|_H^2\\
&\leq\left(\sum_{j=0}^{i-1}\left\|\boldsymbol{\Psi}_{t^l_{j+1}}-\boldsymbol{\Psi}_{t^l_j}-\int_{t^l_j}^{t^l_{j+1}}B_r d\mathcal{W}_r\right\|_H^2\right)^{\frac{1}{2}}\left(\sum_{j=0}^{i-1}\left\|\int_{t^l_j}^{t^l_{j+1}}(I-\mathcal{P}_n)B_r d\mathcal{W}_r\right\|_H^2\right)^{\frac{1}{2}}+c_n o_l
\end{aligned}$$

where $c_n o_l$ is a random variable which for every fixed n, is convergent to zero $\mathbb{P}-a.s.$ along a subsequence as $l\to\infty$. Denoting

$$Y_l:=\sum_{j=0}^{i-1}\left\|\boldsymbol{\Psi}_{t^l_{j+1}}-\boldsymbol{\Psi}_{t^l_j}-\int_{t^l_j}^{t^l_{j+1}}B_r d\mathcal{W}_r\right\|_H^2$$

then the above is written as

$$Y_l\leq Y_l^{\frac{1}{2}}\left(\sum_{j=0}^{i-1}\left\|\int_{t^l_j}^{t^l_{j+1}}(I-\mathcal{P}_n)B_r d\mathcal{W}_r\right\|_H^2\right)^{\frac{1}{2}}+c_n o_l.$$

To consider the $\mathbb{P}-a.s.$ limit, again along the iterative taking of subsequences including that required in the limit of (A.71), we first note that we already know this limit of Y_l is well-defined; indeed, it is immediate from the equality (A.56) stated at the start of the proof, given that the limit in all other terms has now been established. So, for any fixed n, for the limit $\mathbb{P}-a.s.$ along the subsequence,

$$\begin{aligned}
\lim_{l\to\infty}Y_l&\leq\lim_{l\to\infty}\left(Y_l^{\frac{1}{2}}\left(\sum_{j=0}^{i-1}\left\|\int_{t^l_j}^{t^l_{j+1}}(I-\mathcal{P}_n)B_r d\mathcal{W}_r\right\|_H^2\right)^{\frac{1}{2}}\right)\\
&=\left(\lim_{l\to\infty}Y_l^{\frac{1}{2}}\right)\left(\lim_{l\to\infty}\left(\sum_{j=0}^{i-1}\left\|\int_{t^l_j}^{t^l_{j+1}}(I-\mathcal{P}_n)B_r d\mathcal{W}_r\right\|_H^2\right)^{\frac{1}{2}}\right)\\
&=\left(\lim_{l\to\infty}Y_l\right)^{\frac{1}{2}}\int_0^t\|(I-\mathcal{P}_n)B_s\|^2_{\mathscr{L}^2(\mathfrak{U};H)}\,ds
\end{aligned}$$

implying that

$$\left(\lim_{l\to\infty}Y_l\right)^{\frac{1}{2}}\leq\int_0^t\|(I-\mathcal{P}_n)B_s\|^2_{\mathscr{L}^2(\mathfrak{U};H)}\,ds$$

for any n, which can be made arbitrarily small for sufficiently large n, hence (A.63) is justified and the proof is concluded.

□

It remains to extend Lemma A.2.5 to all times in $t \in [0,\tau]$.

Proposition A.2.6. *The equality*

$$\|\boldsymbol{\Psi}_t\|_H^2 = \|\boldsymbol{\Psi}_0\|_H^2 + \int_0^{t\wedge\tau}\left(2\langle \eta_s, \boldsymbol{\Psi}_s\rangle_{U\times V} + \|B_s\|_{\mathscr{L}^2(\mathcal{U};H)}^2\right)ds + 2\int_0^{t\wedge\tau}\langle B_s, \boldsymbol{\Psi}_s\rangle_H \, d\mathcal{W}_s \tag{A.72}$$

holds $\mathbb{P}-a.s.$ *in* $\mathbb{R}$ *for all* $t\in[0,T]$*. Moreover for* $\mathbb{P}-a.e.$ ω, $\boldsymbol{\Psi}_\cdot(\omega)\in C([0,T];H)$.

Proof. We look to fill in the gaps of the time interval $[0,\tau]$ outside of I in Lemma A.2.5 by showing that for any given $t\in[0,\tau]$, and for a sequence of times (t_l) in I convergent to t from below, the corresponding sequence $(\boldsymbol{\Psi}_{t_l})$ converges to $\boldsymbol{\Psi}_t$ in H, $\mathbb{P}-a.s.$. Let us fix such a t and construct the required increasing sequence of times (t_l) in I. For each $l\in\mathbb{N}$, there is a unique t^l_{j-1} with the property that $t\in[t^l_{j-1},t^l_j)$. We consider the sequence of $t_l := t^l_{j-1}$, which is non-decreasing and convergent to t. Let t_m be any element of this sequence. We can consider the process

$$\boldsymbol{\Phi}_\cdot = \boldsymbol{\Psi}_\cdot - \boldsymbol{\Psi}_{t_m}$$

which satisfies the same assumptions as $\boldsymbol{\Psi}$ and holds the same relation with I. Then by Lemma A.2.5, for any t_n with $t_m < t_n$,

$$\|\boldsymbol{\Phi}_{t_n}\|_H^2 = \|\boldsymbol{\Phi}_0\|_H^2 + \int_0^{t_n}\left(2\langle \eta_s, \boldsymbol{\Phi}_s\rangle_{U\times V} + \|B_s\|_{\mathscr{L}^2(\mathcal{U};H)}^2\right)ds + 2\int_0^{t_n}\langle B_s, \boldsymbol{\Phi}_s\rangle_H \, d\mathcal{W}_s$$

holds $\mathbb{P}-a.s.$ in $\mathbb{R}$, and subtracting the same equality for t_m,

$$\|\boldsymbol{\Phi}_{t_n}\|_H^2 = \|\boldsymbol{\Phi}_{t_m}\|_H^2 + \int_{t_m}^{t_n}\left(2\langle \eta_s, \boldsymbol{\Phi}_s\rangle_{U\times V} + \|B_s\|_{\mathscr{L}^2(\mathcal{U};H)}^2\right)ds + 2\int_{t_m}^{t_n}\langle B_s, \boldsymbol{\Phi}_s\rangle_H \, d\mathcal{W}_s.$$

More explicitly, this is

$$\begin{aligned}\|\boldsymbol{\Psi}_{t_n} - \boldsymbol{\Psi}_{t_m}\|_H^2 = \int_{t_m}^{t_n}&\left(2\langle \eta_s, \boldsymbol{\Psi}_s - \boldsymbol{\Psi}_{t_m}\rangle_{U\times V} + \|B_s\|_{\mathscr{L}^2(\mathcal{U};H)}^2\right)ds\\ &+ 2\int_{t_m}^{t_n}\langle B_s, \boldsymbol{\Psi}_s - \boldsymbol{\Psi}_{t_m}\rangle_H \, d\mathcal{W}_s.\end{aligned} \tag{A.73}$$

If we can show that

$$\lim_{m\to\infty}\sup_{n\geq m}\|\boldsymbol{\Psi}_{t_n} - \boldsymbol{\Psi}_{t_m}\|_H^2 = 0 \tag{A.74}$$

$\mathbb{P}-a.s.$ then we would prove that the sequence $(\boldsymbol{\Psi}_{t_k})$ is ($\mathbb{P}-a.s.$) Cauchy in H thus exhibits a limit in H. This limit also holds in U, a space in which $\boldsymbol{\Psi}$ is continuous from the assumed identity (A.55), thus we know that the limit exists in U and is given by $\boldsymbol{\Psi}_t$. This must agree with the limit in the finer topology of H, so the convergence of $(\boldsymbol{\Psi}_{t_k})$ to $\boldsymbol{\Psi}_t$ in H would be verified through the property (A.74). We recall that $\hat{\boldsymbol{\Psi}}^m$ is constant on every interval of the form $[t^m_{j-1},t^m_j)$, given by $\boldsymbol{\Psi}_{t^m_{j-1}}$, and by construction of the times (t_l) then for some j, $t_m = t^m_{j-1}$

and $t_n \in [t^m_{j-1}, t^m_j)$. In particular, $\hat{\mathbf{\Psi}}^m$ is constant on $[t_m, t_n)$ and is given by $\mathbf{\Psi}_{t_m}$. Therefore we can rewrite (A.73) as

$$\|\mathbf{\Psi}_{t_n} - \mathbf{\Psi}_{t_m}\|_H^2 = \int_{t_m}^{t_n} \left(2\left\langle \eta_s, \mathbf{\Psi}_s - \hat{\mathbf{\Psi}}^m_s\right\rangle_{U\times V} + \|B_s\|^2_{\mathscr{L}^2(\mathcal{U};H)}\right)ds + 2\int_{t_m}^{t_n} \left\langle B_s, \mathbf{\Psi}_s - \hat{\mathbf{\Psi}}^m_s\right\rangle_H d\mathcal{W}_s.$$

The required convergence of each of the terms on the right hand side for (A.74) is now conducted exactly as we did for (A.60), (A.61) and is simple by continuity of the integral for the $\int_{t_m}^{t_n} \|B_s\|^2_{\mathscr{L}^2(\mathcal{U};H)}\, ds$ term. Moreover, for a subsequence at least, we have that $(\mathbf{\Psi}_{t_l})$ converges to $(\mathbf{\Psi}_t)$ in H $\mathbb{P}-a.s.$, hence the norms in H converge as well. Continuity of the integrals in the corresponding equalities (A.59) establish that

$$\|\mathbf{\Psi}_t\|_H^2 = \|\mathbf{\Psi}_0\|_H^2 + \int_0^t \left(2\left\langle \eta_s, \mathbf{\Psi}_s\right\rangle_{U\times V} + \|B_s\|^2_{\mathscr{L}^2(\mathcal{U};H)}\right)ds + 2\int_0^t \left\langle B_s, \mathbf{\Psi}_s\right\rangle_H d\mathcal{W}_s$$

where we recall t was arbitrarily chosen in $[0, \tau]$. Passage to (A.72) is now trivial as $\mathbf{\Psi}_t = \mathbf{\Psi}_\tau$ for $t \geq \tau$. The final part of the result which needs to be proven is the continuity of $\mathbf{\Psi}$ in H. From the equality (A.72) we know that the $\|\mathbf{\Psi}\|_H^2$ is $\mathbb{P}-a.s.$ continuous in $\mathbb{R}$, and from Lemma A.2.4 that $\mathbf{\Psi}$ is $\mathbb{P}-a.s.$ weakly continuous in H. The required continuity thus follows from Lemma A.1.28.

□

We wish to relax the integrability constraints over Ω, in accordance with Definition 2.2.6. In the same setting of the Hilbert Spaces V, H, U, we impose the new assumptions for some $T > 0$ and stopping time τ:

1. $\mathbf{\Psi}_0 : \Omega \to H$ is $\mathcal{F}_0-$measurable;
2. For $\mathbb{P}-a.e.$ ω, $\eta(\omega) \in L^2([0,T]; U)$;
3. $B \in \bar{\mathcal{I}}^H(\mathcal{W})$;
4. For $\mathbb{P}-a.e.$ ω, $\mathbf{\Psi}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)} \in L^2([0,T]; V)$ and $\mathbf{\Psi}_\cdot\mathbb{1}_{\cdot\leq\tau}$ is progressively measurable in V;
5. The identity
$$\mathbf{\Psi}_t = \mathbf{\Psi}_0 + \int_0^{t\wedge\tau} \eta_s ds + \int_0^{t\wedge\tau} B_s d\mathcal{W}_s \tag{A.75}$$
holds $\mathbb{P}-a.s.$ in U for all $t \in [0,T]$.

We restate Proposition A.2.6 for the new setting.

Proposition A.2.7. *The equality*

$$\|\mathbf{\Psi}_t\|_H^2 = \|\mathbf{\Psi}_0\|_H^2 + \int_0^{t\wedge\tau} \left(2\left\langle \eta_s, \mathbf{\Psi}_s\right\rangle_{U\times V} + \|B_s\|^2_{\mathscr{L}^2(\mathcal{U};H)}\right)ds + 2\int_0^{t\wedge\tau} \left\langle B_s, \mathbf{\Psi}_s\right\rangle_H d\mathcal{W}_s \tag{A.76}$$

holds $\mathbb{P}-a.s.$ in $\mathbb{R}$ for any $t \in [0,T]$. Moreover for $\mathbb{P}-a.e.$ ω, $\mathbf{\Psi}_\cdot(\omega) \in C([0,T]; H)$.

Proof. The idea is simply to apply Proposition A.2.6 for some truncated versions of the processes. We consider the stopping times

$$\begin{aligned}
\tau_n^1 &:= n \wedge \inf\{0 \leq t < \infty : \int_0^t \|\eta_s\|_U^2 \, ds \geq n\} \\
\tau_n^2 &:= n \wedge \inf\{0 \leq t < \infty : \int_0^t \|B_s\|_{\mathscr{L}^2(\mathfrak{U};H)}^2 \, ds \geq n\} \\
\tau_n^3 &:= n \wedge \inf\{0 \leq t < \infty : \int_0^t \|\mathbf{\Psi}_s \mathbb{1}_{s \leq \tau}\|_V^2 \, ds \geq n\} \\
\tau_n &:= \tau_n^1 \wedge \tau_n^2 \wedge \tau_n^3.
\end{aligned}$$

Then for every n, we have that

$$\begin{aligned}
\mathbf{\Psi}_0 \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n}, &\qquad \eta_\cdot \mathbb{1}_{\cdot \leq \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} \\
B_\cdot \mathbb{1}_{\cdot \leq \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n}, &\qquad \mathbf{\Psi}_{\cdot \wedge \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n}
\end{aligned}$$

satisfy the previous assumptions of 1, 2, 3, 4. Moreover from (A.75) we have that for any $t \in [0,T]$

$$\begin{aligned}
\mathbf{\Psi}_{t \wedge \tau_n} &= \mathbf{\Psi}_0 + \int_0^{t \wedge \tau \wedge \tau_n} \eta_s ds + \int_0^{t \wedge \tau \wedge \tau_n} B_s d\mathcal{W}_s \\
&= \mathbf{\Psi}_0 + \int_0^{t \wedge \tau} \eta_s \mathbb{1}_{s \leq \tau_n} ds + \int_0^{t \wedge \tau} B_s \mathbb{1}_{s \leq \tau_n} d\mathcal{W}_s
\end{aligned}$$

$\mathbb{P} - a.s.$ and moreover

$$\begin{aligned}
\mathbf{\Psi}_{t \wedge \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} &= \mathbf{\Psi}_0 \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} + \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} \int_0^{t \wedge \tau} \eta_s \mathbb{1}_{s \leq \tau_n} ds + \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} \int_0^{t \wedge \tau} B_s \mathbb{1}_{s \leq \tau_n} d\mathcal{W}_s \\
&= \mathbf{\Psi}_0 \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} + \int_0^{t \wedge \tau} \eta_s \mathbb{1}_{s \leq \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} ds + \int_0^{t \wedge \tau} B_s \mathbb{1}_{s \leq \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} d\mathcal{W}_s
\end{aligned}$$

having applied Proposition A.1.51. Therefore we can apply Proposition A.2.6 to see that the equality

$$\begin{aligned}
\left\| \mathbf{\Psi}_{t \wedge \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} \right\|_H^2 &= \left\| \mathbf{\Psi}_0 \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} \right\|_H^2 + \int_0^{t \wedge \tau} 2 \left\langle \eta_s \mathbb{1}_{s \leq \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n}, \mathbf{\Psi}_{s \wedge \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} \right\rangle_{U \times V} ds \\
&+ \int_0^{t \wedge \tau} \left\| B_s \mathbb{1}_{s \leq \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n} \right\|_{\mathscr{L}^2(\mathcal{U};H)}^2 ds \\
&+ 2 \int_0^{t \wedge \tau} \left\langle B_s \mathbb{1}_{s \leq \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H \leq n}, \mathbf{\Psi}_{s \wedge \tau_n} \mathbb{1}_{\|\mathbf{\Psi}_0\|_H} \right\rangle_H d\mathcal{W}_s
\end{aligned}$$

holds for all $t \in [0,T]$ $\mathbb{P} - a.s.$. We rewrite this as

$$\begin{aligned}\mathbb{1}_{\|\boldsymbol{\Psi}_0\|_H \leq n} \|\boldsymbol{\Psi}_{t\wedge\tau_n}\|_H^2 = \mathbb{1}_{\|\boldsymbol{\Psi}_0\|_H \leq n} \|\boldsymbol{\Psi}_0\|_H^2 + \mathbb{1}_{\|\boldsymbol{\Psi}_0\|_H \leq n} \int_0^{t\wedge\tau\wedge\tau_n} 2 \langle \eta_s, \boldsymbol{\Psi}_s \rangle_{U\times V} ds \\ + \mathbb{1}_{\|\boldsymbol{\Psi}_0\|_H \leq n} \int_0^{t\wedge\tau\wedge\tau_n} \|B_s\|^2_{\mathscr{L}^2(\mathcal{U};H)} ds \\ + \mathbb{1}_{\|\boldsymbol{\Psi}_0\|_H \leq n} 2 \int_0^{t\wedge\tau\wedge\tau_n} \langle B_s, \boldsymbol{\Psi}_s \rangle_H d\mathcal{W}_s.\end{aligned}$$

Therefore for any $t \in [0,T]$, $\mathbb{P} - a.e.$ ω with n sufficiently large so that $\|\boldsymbol{\Psi}_0(\omega)\|_H \leq n$ and $t \leq \tau_n(\omega)$, the identity (A.75) holds. We can always find such a large enough n, which completes the justification of this identity. The continuity then follows identically as we have again from Proposition A.2.6 that for every n and $\mathbb{P} - a.e.$ ω, $\boldsymbol{\Psi}_{\cdot\wedge\tau_n}(\omega) \in C([0,T];H)$, and we conclude the proof. □

A.3 Local Theory for SPDEs

A.3.1 Functional Framework

Our object of study is the Itô SPDE (1.1),

$$\boldsymbol{\Psi}_t = \boldsymbol{\Psi}_0 + \int_0^t \mathcal{A}(s, \boldsymbol{\Psi}_s) ds + \int_0^t \mathcal{G}(s, \boldsymbol{\Psi}_s) d\mathcal{W}_s$$

which we pose for a triplet of embedded, separable Hilbert Spaces

$$V \hookrightarrow H \hookrightarrow U$$

whereby the embeddings are continuous linear injections. We ask that there is a continuous bilinear form $\langle \cdot, \cdot \rangle_{U\times V} : U \times V \to \mathbb{R}$ such that for $f \in H$ and $\psi \in V$,

$$\langle f, \psi \rangle_{U\times V} = \langle f, \psi \rangle_H .$$

The equation (1.1) is posed on a time interval $[0,T]$ for arbitrary but henceforth fixed $T \geq 0$. The mappings $\mathcal{A}, \mathcal{G}$ are such that $\mathcal{A} : [0,T] \times V \to U, \mathcal{G} : [0,T] \times V \to \mathscr{L}^2(\mathfrak{U}; H)$ are measurable. Understanding $\mathcal{G}$ as a mapping $\mathcal{G} : [0,T] \times V \times \mathfrak{U} \to H$, we introduce the notation $\mathcal{G}_i(\cdot,\cdot) := \mathcal{G}(\cdot,\cdot,e_i)$. We further impose the existence of a system of elements (a_k) of V with the following properties. Let us define the spaces $V_n := \operatorname{span}\{a_1, \ldots, a_n\}$ and $\mathcal{P}_n$ as the orthogonal projection to V_n in U. It is required that the $(\mathcal{P}_n)$ are uniformly bounded in H, which is to say that there exists a constant c independent of n such that for all $\phi \in H$,

$$\|\mathcal{P}_n f\|_H \leq c \|f\|_H .$$

We also suppose that there exists a real valued sequence (μ_n) with $\mu_n \to \infty$ such that for any $f \in H$,

$$\left\|(I - \mathcal{P}_n)f\right\|_U \leq \frac{1}{\mu_n}\left\|f\right\|_H$$

where I represents the identity operator in U. Specific bounds on the mappings $\mathcal{A}$ and $\mathcal{G}$ will be imposed in Assumption Sets A and B. In order to make the assumptions we introduce some more notation here: we shall let $c_\cdot : [0,T] \to \mathbb{R}$ denote any bounded function, and for any constant $p \in \mathbb{R}$ we define the functions $K_U : U \to \mathbb{R}$, $K_H : H \to \mathbb{R}$, $K_V : V \to \mathbb{R}$ by

$$K_U(\phi) = 1 + \left\|\phi\right\|_U^p, \quad K_H(\phi) = 1 + \left\|\phi\right\|_H^p, \quad K_V(\phi) = 1 + \left\|\phi\right\|_V^p.$$

We may also consider these mappings as functions of two variables, e.g. $K_U : U \times U \to \mathbb{R}$ by

$$K_U(\phi,\psi) = 1 + \left\|\phi\right\|_U^p + \left\|\psi\right\|_U^p.$$

Our assumptions will be stated for 'the existence of a K such that...' where we really mean 'the existence of a p such that, for the corresponding K, ...'.

A.3.2 Assumption Set A

Recall the setup and notation of Subsection A.3.1. We assume that there exists a $c_\cdot$, K and $\gamma > 0$ such that for all $\phi, \psi \in V$, $\phi^n \in V_n$, $f \in H$ and $t \in [0,T]$:

Assumption A.3.1.

$$\begin{aligned}
\left\|\mathcal{A}(t,\phi)\right\|_U^2 + \sum_{i=1}^\infty \left\|\mathcal{G}_i(t,\phi)\right\|_H^2 &\leq c_t K_U(\phi)\left[1 + \left\|\phi\right\|_V^2\right],\\
\left\|\mathcal{A}(t,\phi) - \mathcal{A}(t,\psi)\right\|_U^2 &\leq c_t K_V(\phi,\psi)\left\|\phi-\psi\right\|_V^2,\\
\sum_{i=1}^\infty \left\|\mathcal{G}_i(t,\phi) - \mathcal{G}_i(t,\psi)\right\|_U^2 &\leq c_t K_U(\phi,\psi)\left\|\phi-\psi\right\|_H^2.
\end{aligned}$$

Assumption A.3.2.

$$\begin{aligned}
2\left\langle \mathcal{P}_n\mathcal{A}(t,\phi^n), \phi^n\right\rangle_H + \sum_{i=1}^\infty \left\|\mathcal{P}_n\mathcal{G}_i(t,\phi^n)\right\|_H^2 &\leq c_t K_U(\phi^n)\left[1 + \left\|\phi^n\right\|_H^4\right] - \gamma\left\|\phi^n\right\|_V^2,\\
\sum_{i=1}^\infty \left\langle \mathcal{P}_n\mathcal{G}_i(t,\phi^n), \phi^n\right\rangle_H^2 &\leq c_t K_U(\phi^n)\left[1 + \left\|\phi^n\right\|_H^6\right].
\end{aligned}$$

Assumption A.3.3.

$$2\left\langle \mathcal{A}(t,\phi)-\mathcal{A}(t,\psi),\phi-\psi\right\rangle_U+\sum_{i=1}^{\infty}\|\mathcal{G}_i(t,\phi)-\mathcal{G}_i(t,\psi)\|_U^2$$
$$\leq c_tK_U(\phi,\psi)\left[1+\|\phi\|_H^2+\|\psi\|_H^2\right]\|\phi-\psi\|_U^2-\gamma\|\phi-\psi\|_H^2,$$
$$\sum_{i=1}^{\infty}\left\langle \mathcal{G}_i(t,\phi)-\mathcal{G}_i(t,\psi),\phi-\psi\right\rangle_U^2\leq c_tK_U(\phi,\psi)\left[1+\|\phi\|_H^2+\|\psi\|_H^2\right]\|\phi-\psi\|_U^4.$$

Assumption A.3.4.

$$2\left\langle \mathcal{A}(t,\phi),\phi\right\rangle_U+\sum_{i=1}^{\infty}\|\mathcal{G}_i(t,\phi)\|_U^2\leq c_tK_U(\phi)\left[1+\|\phi\|_H^2\right],$$
$$\sum_{i=1}^{\infty}\left\langle \mathcal{G}_i(t,\phi),\phi\right\rangle_U^2\leq c_tK_U(\phi)\left[1+\|\phi\|_H^4\right].$$

Assumption A.3.5.

$$\left\langle \mathcal{A}(t,\phi)-\mathcal{A}(t,\psi),f\right\rangle_U\leq c_tK_U(\phi,\psi)(1+\|f\|_H)\left[1+\|\phi\|_V+\|\psi\|_V\right]\|\phi-\psi\|_H.$$

A.3.3 Assumption Set B

Recall the setup and notation of Subsection A.3.1. Suppose now that X is a separable Hilbert Space with continuous embedding $U\hookrightarrow X$. We ask that there is a continuous bilinear form $\langle\cdot,\cdot\rangle_{X\times H}:X\times H\to\mathbb{R}$ such that for $\phi\in U$ and $f\in H$,

$$\langle\phi,f\rangle_{X\times H}=\langle\phi,f\rangle_U.$$

Moreover it is now necessary that the system (a_k) forms an orthogonal basis of U. The operators $\mathcal{A}$ and $\mathcal{G}$ must now be extended to the larger spaces, and are such that for any $T>0$, $\mathcal{A}:[0,T]\times H\to X$ and $\mathcal{G}:[0,T]\times H\to\mathscr{L}^2(\mathfrak{U};U)$ are measurable. We assume that there exists a $c_\cdot$, K and $\gamma>0$ such that for all $\phi\in V$, $f,g\in H$ and $t\in[0,T]$:

Assumption A.3.6.

$$\|\mathcal{A}(t,f)\|_X^2+\sum_{i=1}^{\infty}\|\mathcal{G}_i(t,f)\|_U^2\leq c_tK_U(f)\left[1+\|f\|_H^2\right],$$
$$\|\mathcal{A}(t,f)-\mathcal{A}(t,g)\|_X^2\leq c_tK_U(f,g)\left[1+\|f\|_H^2+\|g\|_H^2\right]\|f-g\|_H^2.$$

Assumption A.3.7.

$$2\left\langle \mathcal{A}(t,f)-\mathcal{A}(t,g),f-g\right\rangle_X+\sum_{i=1}^{\infty}\left\|\mathcal{G}_i(t,f)-\mathcal{G}_i(t,g)\right\|_X^2$$
$$\leq c_tK_U(f,g)\left[1+\|f\|_H^2+\|g\|_H^2\right]\|f-g\|_X^2,$$
$$\sum_{i=1}^{\infty}\left\langle \mathcal{G}_i(t,f)-\mathcal{G}_i(t,g),f-g\right\rangle_X^2\leq c_tK_U(f,g)\left[1+\|f\|_H^2+\|g\|_H^2\right]\|f-g\|_X^4.$$

Assumption A.3.8.

$$2\left\langle \mathcal{A}(t,\phi),\phi\right\rangle_U+\sum_{i=1}^{\infty}\left\|\mathcal{G}_i(t,\phi)\right\|_U^2\leq c_tK_U(\phi)-\gamma\|\phi\|_H^2,$$
$$\sum_{i=1}^{\infty}\left\langle \mathcal{G}_i(t,\phi),\phi\right\rangle_U^2\leq c_tK_U(\phi).$$

A.3.4 $H-$Valued Solutions

We state the definitions and main result for $H-$valued solutions.

Definition A.3.9. *Let $\boldsymbol{\Psi}_0:\Omega\to H$ be $\mathcal{F}_0-$ measurable. A pair $(\boldsymbol{\Psi},\tau)$ where τ is a $\mathbb{P}-a.s.$ positive stopping time and $\boldsymbol{\Psi}$ is a process such that for $\mathbb{P}-a.e.$ ω, $\boldsymbol{\Psi}_\cdot(\omega)\in C([0,T];H)$ and $\boldsymbol{\Psi}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)}\in L^2([0,T];V)$ with $\boldsymbol{\Psi}_\cdot\mathbb{1}_{\cdot\leq\tau}$ progressively measurable in V, is said to be an $H-$valued local strong solution of the equation (1.1) if the identity*

$$\boldsymbol{\Psi}_t=\boldsymbol{\Psi}_0+\int_0^{t\wedge\tau}\mathcal{A}(s,\boldsymbol{\Psi}_s)ds+\int_0^{t\wedge\tau}\mathcal{G}(s,\boldsymbol{\Psi}_s)d\mathcal{W}_s$$

holds $\mathbb{P}-a.s.$ in U for all $t\in[0,T]$.

Definition A.3.10. *A pair $(\boldsymbol{\Psi},\Theta)$ such that there exists a sequence of stopping times (θ_j) which are $\mathbb{P}-a.s.$ monotone increasing and convergent to Θ, whereby $(\boldsymbol{\Psi}_{\cdot\wedge\theta_j},\theta_j)$ is an $H-$valued local strong solution of the equation (1.1) for each j, is said to be an $H-$valued maximal strong solution of the equation (1.1) if for any other pair $(\boldsymbol{\Phi},\Gamma)$ with this property then $\Theta\leq\Gamma$ $\mathbb{P}-a.s.$ implies $\Theta=\Gamma$ $\mathbb{P}-a.s.$.*

Remark A.3.11. *We do not require Θ to be finite in this definition, in which case we mean that the sequence (θ_j) is monotone increasing and unbounded for such ω.*

Definition A.3.12. *An $H-$valued maximal strong solution $(\boldsymbol{\Psi},\Theta)$ of the equation (1.1) is said to be unique if for any other such solution $(\boldsymbol{\Phi},\Gamma)$, then $\Theta=\Gamma$ $\mathbb{P}-a.s.$ and*

$$\mathbb{P}\left(\{\omega\in\Omega:\boldsymbol{\Psi}_t(\omega)=\boldsymbol{\Phi}_t(\omega)\quad\forall t\in[0,\Theta)\}\right)=1.$$

Theorem A.3.13. *Let Assumption Set A hold. For any given $\mathcal{F}_0-$ measurable $\boldsymbol{\Psi}_0:\Omega\to H$, there exists a unique $H-$valued maximal strong solution $(\boldsymbol{\Psi},\Theta)$ of the equation (1.1). Moreover* *at $\mathbb{P}-a.e.$ ω for which $\Theta(\omega)<\infty$, we have that*

$$\sup_{r\in[0,\Theta(\omega))}\|\mathbf{\Psi}_r(\omega)\|_H^2+\int_0^{\Theta(\omega)}\|\mathbf{\Psi}_r(\omega)\|_V^2\,dr=\infty.$$

Proof. See [65] Theorem 3.15. □

A.3.5 $U-$Valued Solutions

We state the definitions and main result for $U-$Valued Solutions.

Definition A.3.14. *Let $\mathbf{\Psi}_0:\Omega\to U$ be $\mathcal{F}_0-$measurable. A pair $(\mathbf{\Psi},\tau)$ where τ is a $\mathbb{P}-a.s.$ positive stopping time and $\mathbf{\Psi}$ is a process such that for $\mathbb{P}-a.e.$ ω, $\mathbf{\Psi}_\cdot(\omega)\in C\left([0,T];U\right)$ and $\mathbf{\Psi}_\cdot(\omega)\mathbb{1}_{\cdot\leq\tau(\omega)}\in L^2\left([0,T];H\right)$ with $\mathbf{\Psi}_\cdot\mathbb{1}_{\cdot\leq\tau}$ progressively measurable in H, is said to be a $U-$valued local strong solution of the equation (1.1) if the identity*

$$\mathbf{\Psi}_t=\mathbf{\Psi}_0+\int_0^{t\wedge\tau}\mathcal{A}(s,\mathbf{\Psi}_s)ds+\int_0^{t\wedge\tau}\mathcal{G}(s,\mathbf{\Psi}_s)d\mathcal{W}_s$$

holds $\mathbb{P}-a.s.$ in X for all $t\in[0,T]$.

Definition A.3.15. *A pair $(\mathbf{\Psi},\Theta)$ such that there exists a sequence of stopping times (θ_j) which are $\mathbb{P}-a.s.$ monotone increasing and convergent to Θ, whereby $(\mathbf{\Psi}_{\cdot\wedge\theta_j},\theta_j)$ is a $U-$valued local strong solution of the equation (1.1) for each j, is said to be a $U-$valued maximal strong solution of the equation (1.1) if for any other pair $(\mathbf{\Phi},\Gamma)$ with this property then $\Theta\leq\Gamma$ $\mathbb{P}-a.s.$ implies $\Theta=\Gamma$ $\mathbb{P}-a.s.$.*

Definition A.3.16. *A $U-$valued maximal strong solution $(\mathbf{\Psi},\Theta)$ of the equation (1.1) is said to be unique if for any other such solution $(\mathbf{\Phi},\Gamma)$, then $\Theta=\Gamma$ $\mathbb{P}-a.s.$ and*

$$\mathbb{P}\left(\{\omega\in\Omega:\mathbf{\Psi}_t(\omega)=\mathbf{\Phi}_t(\omega)\quad\forall t\in[0,\Theta)\}\right)=1.$$

Theorem A.3.17. *Let Assumption Sets A and B hold. For any given $\mathcal{F}_0-$ measurable $\mathbf{\Psi}_0:\Omega\to U$, there exists a unique $U-$valued maximal strong solution $(\mathbf{\Psi},\Theta)$ of the equation (1.1). Moreover at $\mathbb{P}-a.e.$ ω for which $\Theta(\omega)<\infty$, we have that*

$$\sup_{r\in[0,\Theta(\omega))}\|\mathbf{\Psi}_r(\omega)\|_U^2+\int_0^{\Theta(\omega)}\|\mathbf{\Psi}_r(\omega)\|_H^2\,dr=\infty.$$

Proof. See [65] Theorem 4.9. □

A.4 Accessories from Analysis and Probability

A.4.1 Classical Results from Real Valued Stochastic Analysis

We collect some results from the finite dimensional theory, which we employ in our constructions.

Proposition A.4.1. *Let (M^n) be a sequence in $\mathcal{M}_c^2$ and M a process with values in $\mathbb{R}$ such that at every time $t \geq 0$,*

$$\lim_{n\to\infty} \mathbb{E}\left(|M_t^n - M_t|^2\right) = 0.$$

Then $M \in \mathcal{M}_c^2$.

Proof. See [78] Proposition 1.5.23. □

Proposition A.4.2 (First Submartingale Inequality)**.** *Let M be a real valued continuous submartingale such that for every $t \geq 0$, $M_t \geq 0$ $\mathbb{P}-a.s.$. Then*

$$\mathbb{P}\left(\left\{\omega \in \Omega : \sup_{t\in[0,T]} M_t(\omega) > \varepsilon\right\}\right) \leq \frac{1}{\varepsilon}\mathbb{E}\left(M_T\right)$$

Proof. See [78] Theorem 1.3.8. □

Theorem A.4.3 (Doob-Meyer Decomposition)**.** *Let $M \in \mathcal{M}_c^2$. Then there exists a unique (up to indistinguishability) continuous, adapted, non-decreasing real valued process $[M]$ with $[M]_0 = 0$ ($\mathbb{P}-a.s.$) such that*

$$M^2 - [M]$$

is a real valued martingale.

Proof. See [78] Theorem 1.4.10. □

Theorem A.4.4. *Let $M, N \in \mathcal{M}_c$. Then there exists a unique (up to indistinguishability) continuous, adapted, bounded-variation process $[M,N]$ with $[M,N]_0 = 0$ ($\mathbb{P}-a.s.$) such that*

$$MN - [M,N]$$

is a real valued martingale.

Proof. See [78] Theorem 1.5.13. □

Lemma A.4.5. *Let $M \in \mathcal{M}_c^2$ be of bounded-variation. Then M is constant $\mathbb{P}-a.s.$*

Proof. This constitutes part of the proof of [78] Theorem 1.5.13. □

Lemma A.4.6. *Let $M, N \in \mathcal{M}_c^2$. Then $\mathbb{P}-a.s.$ for any $t \geq 0$,*

$$[M,N]_t^2 \leq [M]_t[N]_t.$$

Proof. See [78] Problem 1.5.7. □

Lemma A.4.7. *Let $M, N \in \mathcal{M}_c^2$. Then $\mathbb{P}-a.s.$ for any $T \geq 0$,*

$$V_{\mathbb{R}}^T\left([M,N]\right) \leq \frac{1}{2}\left([M]_t + [N]_t\right)$$

Proof. See [78] Problem 1.5.7. □

Theorem A.4.8. *Let $M \in \mathcal{M}_c^2$ and consider any sequence of partitions*

$$I_l := \left\{0 = t_0^l < t_1^l < \cdots < t_{k_l}^l = T\right\}$$

with $\max_j |t_j^l - t_{j-1}^l| \to 0$ *as* $l \to \infty$*. Then for all* $t \in [0, T]$*, for any* $\varepsilon > 0$*,*

$$\lim_{l\to\infty} \mathbb{P}\left(\left\{\left|\sum_{t_{j+1}^l \le t} \left|M_{t_{j+1}^l} - M_{t_j^l}\right|^2 - [M]_t\right| > \varepsilon\right\}\right) = 0.$$

If, in addition, $|M|, [M] \in L^\infty(\Omega \times [0,T]; \mathbb{R})$ *then*

$$\lim_{l\to\infty} \mathbb{E}\left(\left|\sum_{t_{j+1}^l \le t} \left|M_{t_{j+1}^l} - M_{t_j^l}\right|^2 - [M]_t\right|\right) = 0.$$

Proof. See [78] Theorem 1.5.8. □

Theorem A.4.9 (Lévy's Characterisation of Brownian Motion)**.** *Let M be a real-valued continuous local martingale with $M_0 = 0$ $\mathbb{P} - a.s.$, and $[M]_t = t$. Then M is a Brownian Motion.*

Proof. See [78] Theorem 1.3.16 □

Theorem A.4.10 (Burkholder-Davis-Gundy Inequality)**.** *For every $p \geq 1$, there exists a constant C_p such that, for every real-valued continuous local martingale M with $M_0 = 0$ $\mathbb{P} - a.s.$, and for any stopping time $\tau \geq 0$,*

$$\mathbb{E}\left(\sup_{t\in[0,\tau]} |M_t|^p\right) \leq C_p \mathbb{E}\left([M]_\tau^{\frac{p}{2}}\right).$$

Proof. See [17]. □

A.4.2 Classical Tightness Criteria

Theorem A.4.11. *Let $(\mathbf{\Psi}^n)$ be a sequence of processes in $\mathcal{D}([0,T]; \mathbb{R})$. Suppose that for any sequence of stopping times (γ_n), $\gamma_n : \Omega \to [0,T]$, and constants (δ_n), $\delta_n \geq 0$ and $\delta_n \to 0$ as $n \to \infty$:*

1. *For every $t \in [0,T]$, the sequence of the laws of $(\mathbf{\Psi}_t^n)$ is tight in the space of probability measures over $\mathbb{R}$,*
2. *For every $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\left(\left\{\omega \in \Omega : \left|\mathbf{\Psi}^n_{(\gamma_n+\delta_n)\wedge T} - \mathbf{\Psi}^n_{\gamma_n}\right| > \varepsilon\right\}\right) = 0.$*

Then the sequence of the laws of $(\mathbf{\Psi}^n)$ is tight in the space of probability measures over $\mathcal{D}([0,T]; \mathbb{R})$.

Proof. See [2] Theorem 1. □

Theorem A.4.12. *Let E be a metric space and $\mathbb{F}$ a collection of functions in $C(E;\mathbb{R})$ with the property that $\mathbb{F}$ separates points in E and is closed under addition. Let (μ^n) be a sequence of probability measures on $\mathcal{D}([0,T];E)$ satisfying the following:*

1. *For each $\varepsilon > 0$ there exists a compact $K \subset E$ such that for every $n \in \mathbb{N}$, $\mu^n(\mathcal{D}([0,T];K)) > 1-\varepsilon$,*
2. *For every $f \in \mathbb{F}$, defining the mapping $\tilde{f} : \mathcal{D}([0,T];E) \to \mathcal{D}([0,T];\mathbb{R})$ by $[\tilde{f}(\phi)](t) = f[\phi(t)]$, then the sequence $(\mu^n \circ \tilde{f}^{-1})$ is tight in the space of probability measures over $\mathcal{D}([0,T];\mathbb{R})$.*

Then the sequence (μ^n) is tight in the space of probability measures over $\mathcal{D}([0,T];E)$.

Proof. See [77] Theorem 3.1. □

A.4.3 Classical Results from Functional Analysis

Theorem A.4.13. *Letting $k \in \{0\} \cup \mathbb{N}$ be arbitrary and $' \hookrightarrow '$ denote a continuous linear inclusion operator, we have the following results:*

- *For $mp > N$, or $m = N$ and $p = 1$,*
$$W^{m+k,p}(\mathcal{O};\mathbb{R}) \hookrightarrow C_b^k(\mathcal{O};\mathbb{R}) \tag{A.77}$$
and further if $p \leq r \leq \infty$, then
$$W^{m+k,p}(\mathcal{O};\mathbb{R}) \hookrightarrow W^{k,r}(\mathcal{O};\mathbb{R}) \tag{A.78}$$
and in particular
$$W^{m,p}(\mathcal{O};\mathbb{R}) \hookrightarrow L^r(\mathcal{O};\mathbb{R}). \tag{A.79}$$
- *For $mp = N$, if $p \leq r < \infty$ then (A.78) and (A.79) again hold.*
- *For $mp < N$, if $p \leq r \leq Np/(N-mp)$ then (A.78) and (A.79) again hold.*

The boundedness constant from the embeddings is dependent only on the choices of m, p, N, r, k.

Proof. See [1] Section 4. □

Theorem A.4.14 (Gagliardo-Nirenberg Inequality)**.** *Let $p, q, \alpha \in \mathbb{R}$, $m \in \mathbb{N}$, $\mathscr{O} \subset \mathbb{R}^N$ be such that $p > q \geq 1$, $m > N(\frac{1}{2} - \frac{1}{p})$ and $\frac{1}{p} = \frac{\alpha}{q} + (1-\alpha)(\frac{1}{2} - \frac{m}{N})$. Then there exists a constant c (dependent on the given parameters) such that for any $f \in L^p(\mathscr{O};\mathbb{R}) \cap W^{m,2}(\mathscr{O};\mathbb{R})$, we have*
$$\|f\|_{L^p(\mathscr{O};\mathbb{R})} \leq c\,\|f\|^{\alpha}_{L^q(\mathscr{O};\mathbb{R})}\,\|f\|^{1-\alpha}_{W^{m,2}(\mathscr{O};\mathbb{R})}. \tag{A.80}$$

Proof. See [122] pp.125-126. □

Remark A.4.15. *In the original paper [122], the inequality is stated for only the* m^{th} *order derivative and with an additional* $\|f\|_{L^r}$ *term on the bounded domain, for any* $r > 0$*. By considering the full* $W^{m,2}(\mathscr{O};\mathbb{R}^N)$ *norm, one can remove this additional term through interpolation.*

A.4.4 Stochastic Grönwall Lemma

Continuing to look at techniques from PDE theory, a Stochastic Grönwall Lemma will prove of great significance in applications. Whilst in some situations we can apply the classical Grönwall Lemma to the expectation of the process, this is complicated when we have control by the expectation of a product of processes. To overcome this Glatt-Holtz and Ziane proved the following, see [58] Lemma 5.3.

Lemma A.4.16. *Fix* $t > 0$ *and suppose that* $\boldsymbol{\phi}, \boldsymbol{\psi}, \boldsymbol{\eta}$ *are real-valued, non-negative stochastic processes. Assume, moreover, that there exists constants* $c', \hat{c}$ *(allowed to depend on* t*) such that for* $\mathbb{P}-a.e.$ ω*,*

$$\int_0^t \boldsymbol{\eta}_s(\omega)ds \leq c' \tag{A.81}$$

and for all stopping times $0 \leq \theta_j < \theta_k \leq t$*,*

$$\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_k]} \boldsymbol{\phi}_r\right) + \mathbb{E}\left(\int_{\theta_j}^{\theta_k} \boldsymbol{\psi}_s ds\right) \leq \hat{c}\mathbb{E}\left(\left(\boldsymbol{\phi}_{\theta_j} + 1\right) + \int_{\theta_j}^{\theta_k} \boldsymbol{\eta}_s\boldsymbol{\phi}_s ds\right) < \infty.$$

Then there exists a constant C *dependent only on* $c', \hat{c}, t$ *such that*

$$\mathbb{E}\left(\sup_{r\in[0,t]} \boldsymbol{\phi}_r\right) + \mathbb{E}\left(\int_0^t \boldsymbol{\psi}_s ds\right) \leq C\left[\mathbb{E}(\boldsymbol{\phi}_0) + 1\right].$$

Proof. We shall make explicit reference to this constant $\hat{c}$, in defining a sequence of stopping times

$$\theta_j(\omega) := t \wedge \inf\left\{r \geq 0 : \int_0^r \boldsymbol{\eta}_s(\omega)ds \leq \frac{j}{2\hat{c}}\right\}$$

for $j = 0, 1, \ldots$. Clearly $\theta_0 = 0$ $(\mathbb{P}-a.s.)$. From the boundedness (A.81) uniformly over $\mathbb{P}-a.e.$ ω, there exists some finite N such that $\theta_N = t$ $(\mathbb{P}-a.s.)$. Moreover for $0 \leq j < j+1 \leq N$, from the time continuity of the integral and characterisation of the first hitting times we have that

$$\int_{\theta_j}^{\theta_{j+1}} \boldsymbol{\eta}_s ds \leq \frac{1}{2\hat{c}}$$

$\mathbb{P}-a.s.$ (and is in fact an equality for $j+1 < N$). From the assumed inequality we see that for any such j,

$$\begin{aligned}\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_{j+1}]} \boldsymbol{\phi}_r\right) + \mathbb{E}\left(\int_{\theta_j}^{\theta_{j+1}} \boldsymbol{\psi}_s ds\right) &\leq \hat{c}\mathbb{E}\left(\left(\boldsymbol{\phi}_{\theta_j} + 1\right) + \int_{\theta_j}^{\theta_{j+1}} \boldsymbol{\eta}_s\boldsymbol{\phi}_s ds\right)\\ &\leq \hat{c}\mathbb{E}\left(\left(\boldsymbol{\phi}_{\theta_j} + 1\right) + \frac{1}{2\hat{c}}\sup_{r\in[\theta_j,\theta_{j+1}]} \boldsymbol{\phi}_r\right)\end{aligned}$$

and therefore

$$\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_{j+1}]}\boldsymbol{\phi}_r\right)+\mathbb{E}\left(\int_{\theta_j}^{\theta_{j+1}}\boldsymbol{\psi}_s ds\right)\leq 2\hat{c}\mathbb{E}\left(\boldsymbol{\phi}_{\theta_j}+1\right). \tag{A.82}$$

For $j=0$ then

$$\mathbb{E}\left(\sup_{r\in[0,\theta_1]}\boldsymbol{\phi}_r\right)+\mathbb{E}\left(\int_{0}^{\theta_1}\boldsymbol{\psi}_s ds\right)\leq 2\hat{c}\mathbb{E}\left(\boldsymbol{\phi}_0+1\right)$$

and we use this along with (A.82) to make an inductive argument. Suppose that for some such j,

$$\mathbb{E}\left(\sup_{r\in[0,\theta_j]}\boldsymbol{\phi}_r\right)+\mathbb{E}\left(\int_{0}^{\theta_j}\boldsymbol{\psi}_s ds\right)\leq c\mathbb{E}\left(\boldsymbol{\phi}_0+1\right) \tag{A.83}$$

for a general constant c as seen throughout this proof, dependent on $c', \hat{c}, t$. Then

$$\begin{aligned}
&\mathbb{E}\left(\sup_{r\in[0,\theta_{j+1}]}\boldsymbol{\phi}_r\right)+\mathbb{E}\left(\int_{0}^{\theta_{j+1}}\boldsymbol{\psi}_s ds\right)\\
&\qquad\leq\left[\mathbb{E}\left(\sup_{r\in[0,\theta_j]}\boldsymbol{\phi}_r\right)+\mathbb{E}\left(\int_{0}^{\theta_j}\boldsymbol{\psi}_s ds\right)\right]+\left[\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_{j+1}]}\boldsymbol{\phi}_r\right)+\mathbb{E}\left(\int_{\theta_j}^{\theta_{j+1}}\boldsymbol{\psi}_s ds\right)\right]\\
&\qquad\leq c\mathbb{E}\left(\boldsymbol{\phi}_0+1\right)+2\hat{c}\mathbb{E}\left(\boldsymbol{\phi}_{\theta_j}+1\right)\\
&\qquad\leq c\mathbb{E}\left(\boldsymbol{\phi}_0+1\right)+2\hat{c}\mathbb{E}\sup_{r\in[0,\theta_j]}\left(\boldsymbol{\phi}_r+1\right)\\
&\qquad\leq c\mathbb{E}\left(\boldsymbol{\phi}_0+1\right)+2\hat{c}c\mathbb{E}\left(\boldsymbol{\phi}_0+1\right)\\
&\qquad= c\mathbb{E}\left(\boldsymbol{\phi}_0+1\right)
\end{aligned}$$

thanks to (A.82) and two applications of (A.83). Hence, by induction, we can conclude that (A.83) holds for all $j=0,\dots,N$ and in particular for θ_N which is $\mathbb{P}-a.s.$ equal to t. This completes the proof. □