Discrete-time convergent nonlinear systems

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***Abstract*— The convergence property of discrete-time nonlinear systems is studied in this paper. The main result provides a Lyapunov-like characterization of the convergence property based on two distinct Lyapunov-like functions. These two functions are associated with the incremental stability property and the existence of a compact positively invariant set, which together guarantee the existence of a well-defined, bounded, and unique steady-state solution. The links with the conditions available in the recent literature are discussed. These generic results are subsequently used to derive constructive conditions for the class of discrete-time Lur'e-type systems. Such systems consist of an interconnection between a linear system and a static nonlinearity that satisfies cone-bounded (incremental) sector conditions. In this framework, the Lyapunov-like functions that characterize convergence are determined by solving a set of linear matrix inequalities. Several classes of Lyapunov-like functions are considered: both Lyapunov-Lur'e-type functions and quadratic functions. A numerical example illustrates the applicability of the results.**

***Index Terms*— Convergent systems, discrete-time systems, Lur'e systems, discrete-time Lyapunov Lur'e functions, Stability analysis, Linear matrix inequalities.**

I. Introduction

THE notion of *convergence* has been introduced for nonlinear continuous-time systems in [1] (see [2] for more details) to formalize the concept of nonlinear systems that "forget" their initial conditions after a transient and converge to a well-defined, bounded, and unique steady-state solution. Roughly speaking, a system is called convergent if two conditions hold: (i) there exists a (unique) solution that is bounded on the complete time axis; and (ii) this solution is globally asymptotically stable. This property is closely related to, but different from the ones of incremental stability [3]–[7] and contractivity [8], [9], see the comparative discussion in [10], [11]. These three stability properties have been used to tackle a wide variety of problems in (nonlinear) system theory [12]. Examples are: observer design [4], [8], stable inversion of nonlinear systems [13], [14], output regulation [15], [16], frequency response functions for nonlinear convergent systems [17], synchronization [18]–[20], extremum seeking control [21], [22], model reduction [23]–[25], and system identification [26]–[28]. In these problems, the notion of convergence provides uniqueness and boundedness of the steady-state response, which is a property that is not exhibited by other stability notions. For example, even if a non-convergent nonlinear system has a globally asymptotically stable equilibrium for a zero input, solutions of the system for a non-zero input may become unbounded or converge to one of the several bounded solutions depending on the initial condition, which is undesirable in the aforementioned problems.

The literature contains several contributions in the framework of convergent nonlinear continuous-time systems, see [12], [15] and the references therein. Nevertheless, to the best of the authors' knowledge, the work on the discrete-time case is rather limited. The papers [29], [30] provide generic sufficient conditions ensuring convergence, which are based on a quadratic (incremental) Lyapunov function. These sufficient conditions are derived in the form of linear matrix inequalities (LMIs) for the particular class of piecewise-affine systems. Recently, the paper [11] discusses in detail the three notions of convergence, incremental stability, and contraction analysis in the discrete-time case and identifies their similarities and differences. It is important to point out that, for discrete-time globally convergent systems, a characterization of convergence has been provided in [11, Theorem 10] by requiring a single time-varying Lyapunov function. The time dependency of that Lyapunov function comes from the explicit dependency on the time-varying steady-state solution. Based on the observation that, in practice, determining the steady-state solution of a nonlinear system can be a challenging task, the main result of this paper is to avoid this undesired dependency on the steady-state solution. We provide a Lyapunov-like characterization of convergent discrete-time nonlinear systems that does not require knowledge of the steady-state solution. The considered approach is based on the use of two Lyapunov-like functions: one Lyapunov function characterizes incremental stability and the other characterizes the existence of a compact positively invariant set. By searching independently for a Lyapunov function to prove incremental stability and to prove the existence of a positively invariant set, the condition is more generic than searching for a Lyapunov function to prove both simultaneously. This can be seen as the discrete-time counterpart of [15, Theorem 2.40] formulated for the continuous-time case.

In a second step, this paper uses this generic result to study the convergence of discrete-time Lur'e-type systems. Lur'e-type systems consist of an interconnection between

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linear dynamics and a static nonlinearity that satisfies sector conditions. These systems form an important system class because, on the one hand, these naturally capture physics, such as mechanical systems with nonlinear spring and damper characteristics, and electrical circuits with nonlinear elements such as nonlinear resistors or diodes. On the other hand, Lur'e-type systems are used by design, e.g., for nonlinear data-based control design [31], [32], data-driven modeling with neural network models [27], and identification of low-complexity nonlinear models [28]. Lur'e-type systems have been introduced in [33] and intensively studied in the literature either in the continuous-time domain [34], [35] or in the discrete-time domain [36], [37, Chapter 13]. This is still an active field of research [38]–[40]. In particular, for the Lyapunov function of Lur'e-Postnikov type, it should be emphasized that the positive definiteness condition on the matrix associated with the quadratic in the state Lyapunov function has recently been relaxed [41], [42], while the positive definiteness of the Lyapunov function is ensured by making use of the cone-bounded sector conditions. A suitable Lyapunov function of Lur'e-type form has been proposed for the stability analysis and the control design of discrete-time systems based on a Lyapunov function that consists of the sum of a quadratic term in the state and a cross term between the state and the nonlinearity [43], [44]. The latter term, which extends the class of Lyapunov functions that are quadratic in the state, can be viewed as an adequate discrete-time counterpart of the integral term of the Lyapunov-Lur'e-type function in the continuous-time domain. Based on quadratic and Lur'e-type Lyapunov functions, we present two sets of novel sufficient conditions in the form of LMIs for the convergence of discrete-time Lur'e-type systems. Furthermore, we show that if there exists a quadratic Lyapunov function that satisfies our LMI conditions, then we can also construct a parametrized Lur'e-type Lyapunov function that proves convergence of the Lur'e-type system. These novel LMI conditions are independent of the steady-state system response and thus straightforward to verify. Moreover, we show that a graphical frequency-domain condition can be formulated that implies the satisfaction of the LMI conditions, which is beneficial when the system dimensions are large.

The paper is organized as follows. Section II introduces the generic class of discrete-time nonlinear systems and the definitions of convergent and incrementally stable systems. Furthermore, it recalls several results from the literature on discrete-time convergent systems. Section III presents the main result dealing with the Lyapunov-like characterization of the convergence property based on two independent Lyapunov-like functions. Comparative discussions on the links between this result and those in the literature are presented. Section IV is devoted to the derivation of constructive conditions for the particular class of discrete-time Lur'e-type systems by considering several kinds of Lyapunov-like functions, based on the results of Section III. Section V provides a numerical illustration of the main results of the paper before concluding remarks are presented in Section VI. All the proofs of the theorems and associated technical lemmas are postponed to the appendix for the sake of readability.

Notation: the notation is standard. The sets $\mathbb{R}$ and $\mathbb{Z}$ denote the set of real numbers and the set of all integers, respectively. The set $\mathbb{R}^+$ is the set of nonnegative reals. For a positive definite symmetric matrix P and a vector x, $|x|_P$ is the weighted norm $\sqrt{x'Px}$ and $|x|$ is the Euclidean norm of the vector x. The Euclidean distance of a vector $x \in \mathbb{R}^n$ to a set $\mathcal{Z} \subset \mathbb{R}^n$ is denoted and defined as $\text{dist}(x, \mathcal{Z}) = d_{\mathcal{Z}}(x) = \inf_{\tilde{x}\in\mathcal{Z}} |x - \tilde{x}|$. For a positive definite symmetric matrix P and a nonnegative scalar c, $\mathcal{E}(P, c)$ denotes the ellipsoid $\{x \in \mathbb{R}^n,\ |x|_P^2 \leq c\}$. The ball centered at the origin with radius $\epsilon > 0$ is denoted by $\mathcal{B}(\epsilon) = \mathcal{E}(I_n, \epsilon^2)$. In matrices, $\star$ denotes a symmetric block. We recall the main comparison functions. A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is of class-$\mathcal{K}_\infty$, if it is of class-$\mathcal{K}$ and if $\lim_{t\to+\infty} \alpha(t) = +\infty$. A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is of class-$\mathcal{L}$ if it is continuous, strictly decreasing and $\lim_{t\to 0} \alpha(t) = 0$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class-$\mathcal{KL}$ if it is of class-$\mathcal{K}$ in the first argument and of class-$\mathcal{L}$ in the second argument. For a square matrix M, $\rho(M)$ denotes its spectral radius. The unitary complex number is denoted by $i \coloneqq \sqrt{-1}$.

II. Preliminaries for the generic case of nonlinear systems

This section recalls important results on discrete-time convergent systems. The relationship between, on the one hand, the convergence property and, on the other hand, incremental stability (Theorems 1 and 2 below, see [11]), is used in the current section to provide a Lyapunov-like characterization. Theorems 3 and 4 contain conditions for the convergence of discrete-time systems. These theorems are recalled from the literature and their similarities and differences with our results are discussed in Section III.

Let us consider a general discrete-time nonlinear system defined by

$$x_{k+1} = f(x_k, w_k, k), \quad \forall k \in \mathbb{Z}, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state and $w_k \in \mathbb{R}^m$ is the input, which is assumed to belong to a class of inputs $\mathcal{W}$: $w = \{w_k\}_{k\in\mathbb{Z}} \in \mathcal{W}$. We assume that the function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{Z} \to \mathbb{R}^n$ is continuous with respect to the first two arguments for any third argument. Note that the dynamics formally depend on the state x_k, the input w_k, and the time k.

We introduce the notation of the trajectory solution of system (1) with the input w as $\phi^w : \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ verifying

$$\begin{aligned}\phi^w(k+1, k_0, x_0) = f(\phi^w(k, k_0, x_0), w_k, k),\\ \forall (k, k_0, x_0) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^n,\ k \geq k_0.\end{aligned} \tag{2}$$

Based on ϕ^w, the following definition of a positively invariant set for a given input $w \in \mathcal{W}$ is introduced. A set $\mathcal{X} \in \mathbb{R}^n$ is said to be positively invariant (for a given input w) to the dynamics (1), if $\forall x_0 \in \mathcal{X}$, and $k, k_0 \in \mathbb{Z}$ such that $k \geq k_0$, we have $\phi^w(k, k_0, x_0) \in \mathcal{X}$.

The notion of convergence is defined as follows.

Definition 1: The discrete-time nonlinear system (1) is said to be uniformly convergent in a set $\mathcal{X} \subset \mathbb{R}^n$, for a class of inputs $\mathcal{W}$, if, for every input $w \in \mathcal{W}$:

- there exists a solution $\overline{x}_k^w$, called the *steady-state solution*, that is defined and bounded on $\mathbb{Z}$;
- $\overline{x}_k^w$ is uniformly asymptotically stable in $\mathcal{X}$.

Following this definition, there exists a function $\beta \in \mathcal{KL}$ such that $\forall x_0 \in \mathcal{X}$, $\forall k \geq k_0$,

$$|\phi^w(k, k_0, x_0) - \overline{x}_k^w| \leq \beta(|x_0 - \overline{x}_0^w|, k - k_0). \tag{3}$$

In the case that β in (3) is exponential in the second argument, the convergence property is called *exponential*. The set $\mathcal{X}$ is referred to as a convergence region. If $\mathcal{X} = \mathbb{R}^n$, then the convergence property is *global*.

The notion of convergence is closely related to the notion of incremental stability, which is defined as follows.

Definition 2: The system (1) is *uniformly asymptotically incrementally stable* in a set $\mathcal{X} \subset \mathbb{R}^n$, for a class of inputs $\mathcal{W}$, if, for every input $w \in \mathcal{W}$, there exists a $\mathcal{KL}$-function β such that

$$|\phi^w(k, k_0, x_0^a) - \phi^w(k, k_0, x_0^b)| \leq \beta(|x_0^a - x_0^b|, k - k_0),$$
$$\forall (k, k_0, x_0^a, x_0^b) \in \mathbb{Z} \times \mathbb{Z} \times \mathcal{X} \times \mathcal{X},\ k \geq k_0. \tag{4}$$

In the case where β in (4) is exponential in the second argument, the incremental stability property is *exponential*. If $\mathcal{X} = \mathbb{R}^n$, then the incremental stability property is *global*.

In the following, we recall strong, though nuanced, links between incremental stability and convergence, that are used in the main results of this paper in Section III.

Uniform global incremental stability and the existence of a compact and positively invariant set for (1) imply global uniform convergence, as stated in the next theorem. To prevent unbounded but globally incrementally stable solutions, the existence of a compact and positively invariant set is required. Furthermore, this compact set is independent of the incremental stability region, which is global here.

Theorem 1: [11, Theorem 13], proof available in [45]. Suppose that system (1) is *globally* uniformly asymptotically incrementally stable for a class of inputs $\mathcal{W}$ and that there exists a compact set $\mathcal{X} \subset \mathbb{R}^n$, which is positively invariant under (1), then system (1) is globally uniformly convergent.□

Uniform convergence on a compact set and local Lipschitz continuity of the dynamics imply uniform incremental stability on this convergence region, as formalized in the next theorem. It should be emphasized that the compactness of the convergence region and the regularity of the dynamics are required to design the function $\beta \in \mathcal{KL}$ in the relation (4) (see also [10] in the continuous-time framework for additional discussion).

Theorem 2: [11, Theorem 12], proof available in [45]. Suppose that the system (1) is uniformly convergent for a class of inputs $\mathcal{W}$ on a compact and positively invariant set $\mathcal{X} \subset \mathbb{R}^n$. Furthermore, assume that the right-hand side f of system (1) is locally Lipschitz continuous in the first argument on $\mathcal{X}$. Then the system (1) is uniformly asymptotically incrementally stable on $\mathcal{X}$. □

Next, we recall two results (with an adapted notation regarding the dependency of the dynamics on exogenous inputs and also with a slightly modified notation to ensure coherence) in the literature to set the stage for our main result presented in Section III and to allow for a comparative discussion later in the paper. Theorem 3 provides sufficient conditions for global exponential convergence thanks to a *single* quadratic Lyapunov-like function.

Theorem 3: [29, Theorem 1] Consider system (1). If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a constant ρ such that $0 < \rho < 1$ and

$$|f(x^a, w_k, k) - f(x^b, w_k, k)|_P \leq \rho |x^a - x^b|_P, \tag{5}$$

for all $x^a, x^b \in \mathbb{R}^n$ and all $k \in \mathbb{Z}$, and

$$\sup_{k \in \mathbb{Z}} |f(0, w_k, k)|_P = C < +\infty, \tag{6}$$

then system (1) is globally exponentially convergent for the class of inputs $\mathcal{W}$ that is defined as the set of inputs w verifying (6). Moreover the steady-state solution $\overline{x}_k^w$ satisfies

$$\sup_{k \in \mathbb{Z}} |\overline{x}_k^w|_P \leq \frac{C}{1 - \rho}. \tag{7}$$

□

Theorem 4 provides a characterization of global uniform convergence via a *single* Lyapunov-like function that explicitly depends on the steady-state solution $\overline{x}_k^w$. It is noteworthy that, since we deal here with a class of inputs $\mathcal{W}$, the Lyapunov-like function is consequently a control-Lyapunov function.

Theorem 4: [11, Theorem 10] Assume that system (1) is globally uniformly convergent for a class of inputs $\mathcal{W}$. Then, for every input $w \in \mathcal{W}$, there exists a smooth function $V : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^+$, a constant $\overline{c} \geq 0$, functions $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3 \in \mathcal{K}_\infty$ such that

$$\overline{\alpha}_1(|x - \overline{x}_k^w|) \leq V(k, x) \leq \overline{\alpha}_2(|x - \overline{x}_k^w|), \tag{8}$$

$$V(k+1, f(x, w_k, k)) - V(k, x) \leq -\overline{\alpha}_3(|x - \overline{x}_k^w|), \tag{9}$$

$$V(k, 0) \leq \overline{c} < +\infty, \tag{10}$$

for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Conversely, if a smooth function $V : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^+$, a constant $\overline{c} \geq 0$, functions $\overline{\alpha}_1, \overline{\alpha}_2 \in \mathcal{K}_\infty$, and $\overline{\alpha}_3 : \mathbb{R}^+ \to \mathbb{R}^+$ positive definite are given such that for some trajectory $\overline{x}_k^w : \mathbb{Z} \to \mathbb{R}^n$ of the nonlinear dynamics (1), the relations (8)–(10) hold, then the system (1) is globally uniformly convergent and the trajectory $\overline{x}_k^w$ is the unique bounded steady-state solution. □

III. Lyapunov-like characterization of convergence of generic nonlinear systems

This section is dedicated to the first main result of the paper, which provides a Lyapunov-like characterization of convergent nonlinear discrete-time systems exploiting two Lyapunov-like functions. Before it is formalized in Theorem 5, several assumptions are listed.

Assumption 1: There exist a continuous function $V_1 : \mathbb{R}^n \to \mathbb{R}^+$, a $\mathcal{K}$-function $\gamma(\cdot)$, $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, and a nonnegative scalar $c \in \mathbb{R}^+$ such that the following conditions are satisfied:

$$\alpha_1(|x|) \leq V_1(x) \leq \alpha_2(|x|), \tag{11}$$

$$V_1(f(x,w,k)) \leq c, \text{ if } V_1(x) \leq c \text{ and } \gamma(|w|) \leq 1, \quad (12)$$

for all $x \in \mathbb{R}^n$; $w \in \mathbb{R}^m$ and $k \in \mathbb{Z}$.

If Assumption 1 holds, then we can introduce the level set of $V_1(x)$ as follows:

$$\mathcal{S}_c = \{x \in \mathbb{R}^n,\ V_1(x) \leq c\}, \quad (13)$$

and the class of inputs based on $\gamma(\cdot)$ as follows:

$$\mathcal{W}_\gamma = \{w \in \mathbb{R}^m,\ |w| \leq \gamma^{-1}(1)\}. \quad (14)$$

Furthermore, a set $\mathcal{S} \subset \mathbb{R}^n$, which is positively invariant under (1) for the class of inputs $\mathcal{W}_\gamma$ and that is a superset of $\mathcal{S}_c$ can be selected. Such a set $\mathcal{S}$ always exists, e.g., natural choices are $\mathcal{S} = \mathcal{S}_c$ when considering the convergence property and $\mathcal{S} = \mathbb{R}^n$ when considering the global convergence property. The set $\mathcal{S}$, which is not necessarily compact, is the convergence region.

Assumption 2: Assume that there exist a continuous function $V_2 : \mathbb{Z} \times \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$, $\mathcal{K}_\infty$-functions $\alpha_3(\cdot)$, $\alpha_4(\cdot)$ and a $\mathcal{K}$-function $\alpha_5(\cdot)$ satisfying:

$$\alpha_3(|x^a - x^b|) \leq V_2(k, x^a, x^b) \leq \alpha_4(|x^a - x^b|), \quad (15)$$

$$V_2(k+1, f(x^a, w, k), f(x^b, w, k)) - V_2(k, x^a, x^b) \leq -\alpha_5(|x^a - x^b|), \quad (16)$$

for all x^a, $x^b \in \mathcal{S}$, $w \in \mathcal{W}_\gamma$ with $\mathcal{W}_\gamma$ defined by (14) and $k \in \mathbb{Z}$.

Assumption 3: Assume that

- there exist a $\mathcal{K}$-function γ and a compact set $\mathcal{X} \subset \mathbb{R}^n$ which is positively invariant with respect to the dynamics (1) for any input $w \in \mathcal{W}_\gamma$ with $\mathcal{W}_\gamma$ defined by (14) and which contains the origin in its interior.
- the right-hand side f of the system (1) is locally Lipschitz continuous with respect to the first argument for any $w \in \mathcal{W}_\gamma$ and any $k \in \mathbb{Z}$.

Assumption 4: Assume that the system (1) is uniformly convergent on $\mathcal{X}$, for the class of inputs $\mathcal{W}_\gamma$ defined by (14).

Theorem 5: Consider system (1).

Sufficient condition for convergence: from incremental stability to convergence.

Suppose Assumptions 1 and 2 hold.

Then, a set $\mathcal{S} \subset \mathbb{R}^n$, which is positively invariant under (1) for the class of inputs $\mathcal{W}_\gamma$ and which is a superset of $\mathcal{S}_c$ in (13) can be selected such that the system (1) is uniformly convergent on the set $\mathcal{S}$, for the class of inputs $\mathcal{W}_\gamma$, and has a unique steady-state solution $\overline{x}_k^w$, for every $w \in \mathcal{W}_\gamma$, which satisfies the implicit upper bound

$$V_1(\overline{x}_k^w) \leq c, \quad (17)$$

which implies the explicit upper bound:

$$|\overline{x}_k^w| \leq \alpha_1^{-1}(c), \quad (18)$$

When $\mathcal{S} = \mathbb{R}^n$, the system (1) is globally uniformly convergent. When α_3, α_4 and α_5 are quadratic functions, the system (1) is exponentially uniformly convergent.

Necessary condition for convergence: from convergence to incremental stability:

Suppose Assumptions 3 and 4 hold.

Then

- there exist a continuous function $V_1 : \mathbb{R}^n \to \mathbb{R}^+$, $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, and a nonnegative scalar $c \in \mathbb{R}^+$ such that conditions (11)–(12) are satisfied and also such that $\mathcal{S}_c = \{x \in \mathbb{R}^n,\ V_1(x) \leq c\} = \mathcal{X}$.
- There exist a local incremental Lyapunov function $V_2 : \mathbb{Z} \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$, $\mathcal{K}_\infty$-functions $\alpha_3(\cdot)$, $\alpha_4(\cdot)$ and a $\mathcal{K}$-function $\alpha_5(\cdot)$ satisfying conditions (15)–(16) and hence the system is uniformly incrementally stable in $\mathcal{X}$.

□

Proof: The proof of Theorem 5 and the necessary technical lemmas are presented in the appendix. □

Let us emphasize the motivation for and the positioning of the result in Theorem 5 with respect to the literature:

- The crucial point of our approach is the strong link between incremental stability and the convergence property. On the one hand, the existence of a positively invariant compact set (given by a Lyapunov function level set) and incremental stability (characterized by an incremental Lyapunov function, as defined in [11, Theorem 9]), (as formalized in Assumptions 1 and 2) imply convergence. On the other hand, the existence of a positively invariant compact set, a local Lipschitz continuous dynamics, and the convergence property (Assumptions 3 and 4) imply the existence of a Lyapunov function, of which a level set is the positively invariant compact set, and imply the existence of an incremental Lyapunov function.
- The quasi-equivalence between convergence and incremental stability is established under slightly different assumptions. For instance in Theorem 5, only one implication requires the local Lipschitz continuity of the dynamics, and not the dual implication, as in Theorem 2 (that is [11, Theorem 12]).
- The advantage of considering two generic Lyapunov-like functions is twofold and is beneficial from a computational (and conservatism) point of view since it allows an extra degree of freedom in the choice of the class of Lyapunov functions. First, we avoid a specific structure for the quadratic functions, which was needed to exploit the triangular inequality in Theorem 3 (that is [29, Theorem 1]). Second, these functions do not depend on the steady-state solution $\bar{x}^w$, unlike the characterization in Theorem 4 (that is [11, Theorem 10]), although the incremental Lyapunov function may still depend on time. The latter is important from an applicability point of view since the steady-state solution $\overline{x}_k^w$ is typically unknown.
- The domain of definition of the incremental Lyapunov function is more generic than the one defined in [11] and can be adapted to cope with local or global incremental stability/convergence properties.
- The class of inputs $\mathcal{W}$ considered here is the set of bounded inputs with a given bound. This is more useful in practice than trying to determine the inputs that satisfy the time-dependent conditions (10) or (6) as used

in [29] and [11]. Note, however, that the latter may admit unbounded inputs for specific dynamics f.

- A corollary of the sufficient part could be obtained by considering a single function V_1 instead of two separate functions. A natural choice is thus $\mathcal{S} = \mathcal{S}_c$ and $V_2(k, x^a, x^b) = V_1(x^a - x^b)$. The conditions to satisfy are (11), (12), and (16).
- This result is a stepping stone to the constructive results for Lur'e-type systems presented in Section IV.

The following section derives sufficient conditions based on Theorem 5 and proposes suitable Lyapunov functions to ensure the convergence property of discrete-time Lur'e-type systems.

IV. Convergent discrete-time Lur'e-type systems

Let us consider the discrete-time Lur'e-type system, defined as the interconnection between linear dynamics and a static nonlinearity, described by

$$\begin{aligned} x_{k+1} &= Ax_k + B\varphi(y_k) + Fw_k =: f(x_k, w_k), \quad (19)\\ y_k &= Cx_k, \quad \forall k \in \mathbb{Z}, \quad (20)\end{aligned}$$

where $x_k \in \mathbb{R}^n$ is the state of the system, $w_k \in \mathbb{R}^m$ is the input, $y_k \in \mathbb{R}^p$ is the output, and $\varphi : \mathbb{R}^p \to \mathbb{R}^p$ is a static decentralized nonlinearity [46]. Note that the mapping f does not explicitly depend on time k and that this dependency is dropped in the sequel. We assume here that $\varphi(\cdot)$ verifies two cone-bounded sector conditions (in a component-wise manner): the first relates to the output of the nonlinearity characterized by a positive definite diagonal matrix $\Omega \in \mathbb{R}^{p\times p}$:

$$s_\Omega(R, y) := \varphi(y)'R(\varphi(y) - \Omega y) \leq 0, \quad \forall y \in \mathbb{R}^p, \quad (21)$$

for any positive definite diagonal matrix $R \in \mathbb{R}^{p\times p}$, and the second relates to its slope characterized by a positive definite diagonal matrix $\overline{\Omega} \in \mathbb{R}^{p\times p}$:

$$\begin{aligned} &s_{\overline{\Omega}}(S, y^a, y^b) := (\varphi(y^a) - \varphi(y^b))'S\\ &\times (\varphi(y^a) - \varphi(y^b) - \overline{\Omega}(y^a - y^b)) \leq 0, \ \forall y^a, y^b \in \mathbb{R}^p, \quad (22)\end{aligned}$$

for any positive definite diagonal matrix $S \in \mathbb{R}^{p\times p}$. When no ambiguity occurs, we will denote $\varphi_k := \varphi(y_k) = \varphi(Cx_k)$.

The conditions (21) and (22) call for some comments. First of all, setting $y^b = 0$ in (22) results in a condition of type (21), roughly speaking $s_{\overline{\Omega}}(S, y, 0) = s_{\overline{\Omega}}(S, y)$. We can thus assume that $\Omega \leq \overline{\Omega}$. Thanks to condition (22), the nonlinearity φ is continuous and induces the continuity with respect to x_k and w_k of the map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, which defines the dynamics (19). Moreover, condition (22) ensures that the nonlinearity φ is monotonic, which avoids the existence of multiple fixed points of the system (19).

The main result of this section provides numerically tractable sufficient conditions for the construction of Lyapunov-like functions in the sufficiency part of Theorem 5 (from incremental stability to convergence), required for the class of Lur'e-type systems in (19)–(20). Here, we consider a specific class of candidate functions as follows. Let us introduce the extended state, or graph function, $\xi(x_k) = \begin{bmatrix} x_k \\ \varphi(y_k)\end{bmatrix} = \begin{bmatrix} x_k \\ \varphi(Cx_k)\end{bmatrix} \in \mathbb{R}^{n+p}$ and consider functions with the structure:

$$\tilde{V}(x_k) = \xi(x_k)'\tilde{P}\xi(x_k), \ \tilde{P} = \begin{bmatrix} \tilde{P}_a & \tilde{P}_b \\ \star & \tilde{P}_c \end{bmatrix}, \quad (23)$$

with $0_n < \tilde{P}_a = \tilde{P}_a' \in \mathbb{R}^{n\times n}$, but with the indefinite symmetric matrix $\tilde{P}_c = \tilde{P}_c' \in \mathbb{R}^{p\times p}$ and matrix $\tilde{P}_b \in \mathbb{R}^{n\times p}$. Notice that the matrix $\tilde{P}$ can thus be indefinite.

This function $\tilde{V}$, which is quadratic in the extended state $\xi(x_k)$, motivates the following discussion. Obviously, by imposing $\tilde{P}_b = 0_{n\times p}$ and $\tilde{P}_c = 0_p$, we retrieve Lyapunov functions that are quadratic in the state x_k: $\tilde{V}(x_k) = |x_k|^2_{\tilde{P}_a}$. By restricting the choice of functions to $\tilde{P}_b = C'R\Omega$, with a positive semi-definite diagonal matrix $R \in \mathbb{R}^{p\times p}$, and $\tilde{P}_c = 0_p$, we recover the class of discrete-time Lur'e-type Lyapunov functions introduced in [43]: $\tilde{V}(x_k) = |x_k|^2_{\tilde{P}_a} + 2y_k'R\Omega\varphi_k$. The generic function $\tilde{V}$ as in (23) has already been introduced in the context of Generalized Lyapunov functions for discrete-time Lur'e-type systems in [47], but requires the matrix $\tilde{P}$ to be symmetric and positive definite. The idea to relax the positive definiteness condition is inspired by the results in [41], [42], which deal with the continuous-time case of Lur'e-type systems. Next, tractable sufficient conditions for the convergence property are presented that make use of two classes of quadratic Lyapunov functions, namely quadratic in the extended state ξ_k and quadratic in the state x_k.

A. Quadratic Lyapunov function in the extended state

To clarify the notation and improve the readability of the sequel of the paper, we introduce the following notation:

$$G_\Omega(S) = \begin{bmatrix} 0_n & -C'S\Omega \\ \star & 2S \end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+p)}$$

with $S \in \mathbb{R}^{p\times p}$ diagonal and positive definite. This allows us to write $2s_\Omega(S, Cx) = \xi(x)'G_\Omega(S)\xi(x) \leq 0$ and $2s_{\overline{\Omega}}(S, Cx^a, Cx^b) = (\xi(x^a) - \xi(x^b))'G_{\overline{\Omega}}(S)(\xi(x^a) - \xi(x^b)) \leq 0$. Finally, we introduce the matrix $M_0 \in \mathbb{R}^{2(n+p)\times(n+2p)}$, which depends only on the matrices of the system (19)–(20):

$$M_0 = \begin{bmatrix} I_n & 0_{n\times p} & 0_{n\times p} \\ 0_{p\times n} & I_p & 0_p \\ A & B & 0_{n\times p} \\ 0_{p\times n} & 0_p & I_p \end{bmatrix}.$$

Building on the generic result of Theorem 5, the following theorem provides sufficient conditions for convergence in the form of bilinear matrix inequalities (BMIs) that can be interpreted as linear matrix inequalities (LMIs) by fixing a single scalar (or hyper-parameter) $0 < \tau_1 < 1$.

Theorem 6: Consider symmetric matrices $P_1, P_2 \in \mathbb{R}^{(n+p)\times(n+p)}$, diagonal positive definite matricse S_0, S_1, S_2, S_3, S_4, $S_5 \in \mathbb{R}^{p\times p}$, and a positive scalar $\tau_1 \in (0, 1)$ such that the following inequalities hold:

$$P_1 + G_\Omega(S_0) > 0_{n+p}, \quad (24)$$

$$M_0' \begin{bmatrix} -\tau_1 P_1 - G_\Omega(S_1) & 0_{n+p} \\ \star & P_1 - G_\Omega(S_2) \end{bmatrix} M_0 < 0_{n+2p}, \tag{25}$$

$$P_2 + G_{\overline{\Omega}}(S_3) > 0_{n+p}, \tag{26}$$

$$M_0' \begin{bmatrix} -P_2 - G_{\overline{\Omega}}(S_4) & 0_{n+p} \\ \star & P_2 - G_{\overline{\Omega}}(S_5) \end{bmatrix} M_0 < 0_{n+2p}. \tag{27}$$

Then, for any positive scalar σ, the assumptions of the sufficiency part of Theorem 5 are satisfied by taking $V_1(x) = \xi(x)'P_1\xi(x)$ and $V_2(k, x^a, x^b) = V_2(x^a, x^b) = (\xi(x^a) - \xi(x^b))'P_2(\xi(x^a) - \xi(x^b))$ and there exist quadratic functions α_3, α_4, and α_5, and a sufficiently large scalar $c \geq 0$ defining the level set $\mathcal{S}_c$. In conclusion, the Lur'e-type system (19)–(20) is globally exponentially uniformly convergent (GEUC), for the class of inputs $\mathcal{W} = \{w_k \in \mathbb{R}^m,\ |w_k| \leq \sqrt{\sigma}\}$. □

Proof: The proof of Theorem 6 is presented in Appendix V. □

The dimension of the LMIs in Theorem 6 is $4n + 6p$. The following corollary to Theorem 6 uses two Lyapunov functions of the same form to provide LMIs of dimension $2n + 3p$. Hence, it enables a more efficient check for convergence at the expense of conservatism, since these conditions depend on a reduced set of decision variables.

Corollary 1: Suppose $\bar{\Omega} \succeq \Omega$. Consider the symmetric matrix $P \in \mathbb{R}^{(n+p)\times(n+p)}$, the diagonal positive-definite matrices S_0, S_1, $S_2 \in \mathbb{R}^{p\times p}$, and the scalar $\tau_1 \in (0, 1)$ such that the following inequalities hold:

$$P + G_{\bar{\Omega}}(S_0) > 0_{n+p}, \tag{28}$$

$$M_0' \begin{bmatrix} -\tau_1 P - G_{\bar{\Omega}}(S_1) & 0_{n+p} \\ \star & P - G_{\bar{\Omega}}(S_2) \end{bmatrix} M_0 < 0_{n+2p}, \tag{29}$$

Then, there exist symmetric matrices $P_1, P_2 \in \mathbb{R}^{(n+p)\times(n+p)}$ and diagonal positive definite matrices $S_3, S_4 \in \mathbb{R}^{p\times p}$ such that the matrix inequalities in Theorem 6 are satisfied. □

Proof: By $\bar{\Omega} \succeq \Omega$, the satisfaction of (28) ensures the satisfaction of (24) for $P_1 = P$ and the same S_0; and (26) for $P_2 = P$ and $S_3 = S_0$. Furthermore, (29) ensures the satisfaction of (25) for $P_1 = P$, and the same S_1 and S_2; and (27) for $P_2 = P, S_4 = S_1$, and $S_5 = S_2$. □

Remark 1: Note that the input matrix F does not appear in inequalities (24)–(27). Consequently, the convergence property of system (19)–(20) is independent of the input matrix F. From the proof of the theorem, it can be concluded that the input matrix F defines a constant η which relates the 'size' σ of the input to the 'size' of the level set $\mathcal{S}_c$ by (13) according to $\sigma \geq \eta c$, where this set can be used to formulate a bound on the steady-state solution.

B. Quadratic Lyapunov function in x_k

In this subsection, we consider Lyapunov functions that are quadratic in the state x, which is a special case of (23) where

$$\tilde{P} = \begin{bmatrix} \tilde{P}_a & 0_{n\times p} \\ \star & 0_p \end{bmatrix}.$$

Sufficient conditions for the construction of the quadratic Lyapunov functions V_1 and V_2, which guarantee the satisfaction of the conditions in Theorem 5, and thereby also the convergence of the Lur'e-type system, are provided in the following theorem.

Theorem 7: Consider positive definite matrices P_{11}, $P_{21} \in \mathbb{R}^{n\times n}$, diagonal positive definite matrices $S_1, S_4 \in \mathbb{R}^{p\times p}$, and a positive scalar $\tau_1 \in (0, 1)$ such that the following inequalities hold:

$$\begin{bmatrix} A' \\ B' \end{bmatrix} P_{11} \begin{bmatrix} A' \\ B' \end{bmatrix}' - \begin{bmatrix} \tau_1 P_{11} & -C'S_1\Omega \\ \star & 2S_1 \end{bmatrix} < 0_{n+p}, \tag{30}$$

$$\begin{bmatrix} A' \\ B' \end{bmatrix} P_{21} \begin{bmatrix} A' \\ B' \end{bmatrix}' - \begin{bmatrix} P_{21} & -C'S_4\overline{\Omega} \\ \star & 2S_4 \end{bmatrix} < 0_{n+p}. \tag{31}$$

Then the assumptions of the sufficiency part of Theorem 5 are verified by taking $V_1(x) = x'P_{11}x$ and $V_2(k, x^a, x^b) = V_2(x^a, x^b) = (x^a - x^b)'P_{21}(x^a - x^b)$ and there exist quadratic functions α_3, α_4, and α_5, and a sufficiently large nonnegative scalar c defining the level set $\mathcal{S}_c$. As a conclusion of Theorem 5, the Lur'e-type system (19)–(20) is globally exponentially uniformly convergent, for the class of inputs $\mathcal{W} = \{w_k \in \mathbb{R}^m,\ |w_k| \leq \sqrt{\sigma}\}$. Furthermore the steady-state solution $\overline{x}_k^w$ belongs to the ellipsoid $\mathcal{E}(P_{11}, c)$. □

Proof: The proof of Theorem 7 follows a similar line of reasoning as the proof of Theorem 6 and is therefore omitted. □

The following corollary of Theorem 7 is the counterpart of Corollary 1 and provides a more efficient check for convergence at the expense of conservatism compared to the conditions in Theorem 7.

Corollary 2: Suppose $\bar{\Omega} \succeq \Omega$. Consider the positive definite matrix $P \in \mathbb{R}^{n\times n}$, the diagonal positive definite matrix $S \in \mathbb{R}^{p\times p}$, and the scalar $\tau \in (0, 1)$ such that the following inequality holds:

$$\begin{bmatrix} A' \\ B' \end{bmatrix} P \begin{bmatrix} A' \\ B' \end{bmatrix}' - \begin{bmatrix} \tau_1 P & -C'S\bar{\Omega} \\ \star & 2S \end{bmatrix} < 0_{n+p}, \tag{32}$$

Then, there exist positive definite matrices $P_{11}, P_{21} \in \mathbb{R}^{n\times n}$ and diagonal positive definite matrices $S_1, S_4 \in \mathbb{R}^{p\times p}$ such that the matrix inequalities in Theorem 7 are satisfied. □

Proof: The proof follows a similar line of reasoning as the proof of Corollary 1 and is therefore omitted. □

The remainder of the subsection shows that the conditions of Theorem 6 are less conservative than those of Theorem 7. The class of Lyapunov functions in Theorem 6 includes the quadratic ones used in Theorem 7, nevertheless, the conditions in both theorems are only sufficient and based on multiple S-procedures. Theorem 8 states that based on a solution of the conditions in Theorem 7, we can construct a parameterized solution of the conditions in Theorem 6 which leads to a smaller set in the state-space containing the steady-state solution.

Theorem 8: Assume that the conditions of Theorem 7 are satisfied. Then there exist additional matrices P_{12}, $P_{22} \in \mathbb{R}^{n\times p}$, symmetric matrices P_{13}, $P_{23} \in \mathbb{R}^{p\times p}$, diagonal positive definite matrices S_0, S_2, S_3, and $S_5 \in \mathbb{R}^{p\times p}$ such that

1) conditions (24)–(27) are verified;
2) $\mathcal{S}_c = \{x \in \mathbb{R}^n,\ \xi_k'P_1\xi_k = V_1(x) \leq c\} \subset \mathcal{E}(P_{11}, c)$.

□

Proof: The proofs of Theorem 8 and an auxiliary lemma can be found in Appendix VI. □

C. Graphical frequency-domain condition

Let us denote by $H(z) : \mathbb{C} \mapsto \mathbb{C}^{p\times p}$ the transfer-function matrix of the LTI part of (19) - (20) from input $\varphi(\cdot)$ to output y, defined as follows:

$$H(z) := C(zI - A)^{-1}B. \tag{33}$$

Furthermore, consider the following transfer-function matrix:

$$\tilde{H}(z) \coloneqq C\left(zI - \left(\tilde{A}\right)\right)^{-1} B, \quad \tilde{A} \coloneqq A + \frac{1}{2}B\bar{\Omega}C. \tag{34}$$

Let us consider the case where all elements of the static decentralized nonlinearity $\varphi(\cdot)$ are scaled so that all elements have the same maximum slope $\bar{\gamma} > 0$, i.e., $\bar{\Omega} = \bar{\gamma}I_p$. If the elements of the nonlinearity do not share the same maximum slope, then a scaling should be performed to obtain a nonlinearity whose elements have the same maximum slope, see [46] for more details.

For a real, rational transfer-function matrix $\tilde{H}$ with all poles inside the unit circle, the infinity system norm is denoted by $\|\tilde{H}\|_\infty$ and is defined by

$$\left\|\tilde{H}\right\|_\infty \coloneqq \sup_{\omega\in[0,\pi)} \zeta_{\max}\left(\tilde{H}\left(e^{i\omega}\right)\right), \tag{35}$$

where $\zeta_{\max}(\cdot)$ denotes the maximum singular value. Now, the following theorem formulates an easy-to-check, graphical condition for convergence for the class of Lur'e-type systems under study.

Theorem 9: Suppose that $\bar{\Omega} = \bar{\gamma}I_p \succeq \Omega$, the matrix $\tilde{A}$ has all its eigenvalues inside the unit disc, and $\|\tilde{H}\|_\infty < 2\bar{\gamma}^{-1}$, where $\tilde{A}$ and $\tilde{H}$ are defined in (34). Then, system (19)–(20) is globally exponentially uniformly convergent, for the class of bounded inputs $\mathcal{W} = \{w_k \in \mathbb{R}^m\}$.

Proof: The proof can be found in Appendix VII. □

The conditions of Theorem 9 provide the insight that the infinity norm of the transfer-function matrix of the LTI block must remain below $2\bar{\gamma}^{-1}$, where the factor 2 comes from a loop transformation that makes the nonlinearity satisfy a symmetric sector condition.

The frequency-domain condition of Theorem 9 can be checked graphically by plotting the maximum singular values of the transfer-function matrix $\tilde{H}(e^{i\omega})$ over a sufficiently dense grid for $\omega \in [0, \pi)$ and by checking whether the matrix $\tilde{A}$ has all its eigenvalues inside the unit disc. This graphical check is demonstrated by a numerical example in the next section. The benefit of this graphical check is that it can be applied to large-scale systems, whereas the LMI implementations of Theorems 6 and 7 are limited to $2n + 4p$ being in the order of hundreds and Corollaries 1-2 are limited to $n + 2p$ in the order of hundreds. However, the drawback of this graphical check is that the Lyapunov functions are not explicitly found, thereby, disallowing defining the level set $\mathcal{S}_c$ and formulating explicit bounds on the steady-state solution.

V. Illustrative example

In this section, we demonstrate the application of the results in Theorems 6, 7, 8, and 9 to an example Lur'e-type system.

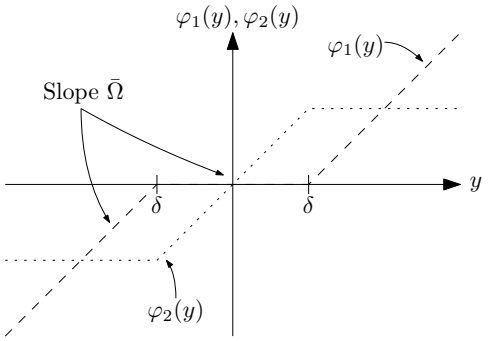


Fig. 1. Graph of $\boldsymbol{\varphi_1}$ (dashed) and $\boldsymbol{\varphi_2}$ (dotted) with deadzone/saturation length $\boldsymbol{\delta}$ and slope $\bar{\boldsymbol{\Omega}}$.

Consider system (19)-(20) with $n = 2$ states, $p = 2$ outputs, and $m = 1$ input characterized by the following matrices:

$$\begin{aligned} A &= \begin{bmatrix} 0.5 & 0.1 \\ 0.3 & -0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.1 \\ 0 & -0.5 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \end{aligned} \tag{36}$$

and the following nonlinear function $\varphi(y)$:

$$\varphi(y) = \begin{bmatrix} \varphi_1(x_1) \\ \varphi_2(x_1 + x_2) \end{bmatrix} \tag{37}$$

where

$$\begin{aligned} \varphi_1(y) &:= \bar{\Omega}\text{sign}(y)\max(0, |y| - \delta), \\ \varphi_2(y) &:= \bar{\Omega}\left(y - \text{sign}(y)\max(0, |y| - \delta)\right) \end{aligned}$$

The graphs of φ_1 and φ_2 are depicted in Figure 1, where it can be observed that φ_1 admits a deadzone characteristic, whereas φ_2 admits a saturation characteristic. The variable δ represents the deadzone and saturation length and the variable $\bar{\Omega}$ represents the maximum slope of both nonlinearities. For this example, we select $\delta = 0.05$ and $\bar{\Omega} = 0.6$. implying that $\Omega = \bar{\Omega}$. This system can be thought of as the discretized dynamics of a mechanical system with nonlinear spring and damper characteristics or, alternatively, the discretized dynamics of an electrical circuit with nonlinear elements such as nonlinear resistors.

Let us first turn to Theorem 7, which characterizes convergence by searching for Lyapunov functions that are quadratic in the state x. The conditions of Theorem 7 lead to linear matrix inequalities by fixing the hyper-parameter τ_1. It can be verified that for $\tau_1 = 0.9$, the linear matrix inequalities (30)–(31) are satisfied by the following matrices:

$$\begin{aligned} P_{11} &= \begin{bmatrix} 1.4598 & 0.1084 \\ \star & 1.1063 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} 1.3431 & 0.0990 \\ \star & 1.1030 \end{bmatrix}, \\ S_1 &= \begin{bmatrix} 0.8977 & 0 \\ 0 & 0.5855 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 0.8291 & 0 \\ 0 & 0.6275 \end{bmatrix}. \end{aligned}$$

Next, we turn to Theorem 8, which states that there are additional matrix variables such that the conditions of Theorem 6 are satisfied. In turn, this implies the existence of a Lyapunov function that is quadratic in the extended state ξ.

It can be verified that (24)–(27) are satisfied by the following additional matrices:

$$P_{12} = \begin{bmatrix} 0.7495 & -0.0308 \\ * & 0.2604 \end{bmatrix}, \quad P_{13} = \begin{bmatrix} -0.2260 & -0.0348 \\ * & 0.0819 \end{bmatrix},$$
$$P_{22} = \begin{bmatrix} 0.6628 & -0.0161 \\ * & 0.2730 \end{bmatrix}, \quad P_{23} = \begin{bmatrix} -0.1259 & -0.0446 \\ * & 0.0668 \end{bmatrix},$$
$$S_0 = \begin{bmatrix} 0.9079 & 0 \\ 0 & 0.7065 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.8156 & 0 \\ 0 & 0.6309 \end{bmatrix},$$
$$S_3 = \begin{bmatrix} 0.8590 & 0 \\ 0 & 0.7071 \end{bmatrix}, \quad S_5 = \begin{bmatrix} 0.7895 & 0 \\ 0 & 0.6667 \end{bmatrix},$$

where

$$P_1 = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{13} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} P_{21} & P_{22} \\ * & P_{23} \end{bmatrix}.$$

Let us now analyze the positively invariant set $\mathcal{S}_c$ in (13) in which the steady-state solution $\bar{x}_k^w$ resides. To compute a value for c, we take the following sequence of calculations:

1) Compute a sufficiently large τ_2 such that (i) τ_2 is strictly greater than $\rho\left(\begin{bmatrix} F' & 0_{m\times p} \end{bmatrix} P_1 \begin{bmatrix} F \\ 0_{p\times m} \end{bmatrix}\right)$; and (ii) such that inequality (49) holds.
2) Take a constant $c \geq \dfrac{\tau_2 \sigma}{1-\tau_1}$.

It can be verified that both conditions of Step 1) are satisfied for $\tau_2 = 0.3811$. Finally, via Step 2), we compute $c = 3.8108$ for $\sigma = 1$, which defines the set $\mathcal{S}_c$. As a result, the system is globally exponentially convergent for the class of inputs $\mathcal{W} = \{w_k \in \mathbb{R}^m, |w_k| \leq \sqrt{\sigma}\}$ for $\sigma = 1$, i.e., any input that admits $|w_k| \leq 1$ for all k. By Theorem 6, we have the property that the steady-state solution $\bar{x}_k^w \in \mathcal{S}_c$, where the set $\mathcal{S}_c$ corresponds to the Lyapunov function which is quadratic in the extended state ξ. Consider also the set $\mathcal{E}(P_{11}, c)$, which is not necessarily positively invariant. The set $\mathcal{E}(P_{11}, c)$ corresponds to the Lyapunov function which is quadratic in the state x. By Theorem 8, we have the property that $\mathcal{S}_c \subset \mathcal{E}(P_{11}, c)$, which is consistent with the observations in Figure 4. Consequently, one of the benefits of the convergence characterization via Lyapunov functions that are quadratic in the extended state ξ is that the positively invariant set $\mathcal{S}_c$ is a subset of $\mathcal{E}(P_{11}, c)$.

Alternatively, the condition

$$\|\tilde{H}\|_\infty = \sup_{\omega\in[0,\pi)} \zeta_{\max}(\tilde{H}(e^{i\omega})) < 2\bar{\gamma}^{-1} \tag{38}$$

in Theorem 9 can be checked for $\bar{\gamma} = \bar{\Omega} = 0.6$ to conclude on the GEUC property. Since the conditions of Theorems 6–7 are verified in the above analysis, the condition (38) is expected to hold. Figure 2 shows that, indeed, the maximum singular value $\zeta_{\max}(\tilde{H}(e^{i\omega}))$ remains below the threshold $2\bar{\gamma}^{-1}$ for any $\omega \in [0, \pi)$. This check, in addition to the eigenvalues of $\tilde{A} + \frac{1}{2}\bar{\gamma}BC$ being inside the unit disc, implies that the considered system is GEUC for the class of bounded inputs. This alternative, graphical method of checking the GEUC property can also be efficiently applied to large-scale systems.

Next, we demonstrate the convergence property by means of forward simulation of the Lur'e-type system in (19)-(20) characterized by (36) and (37). Hereto, we perform two

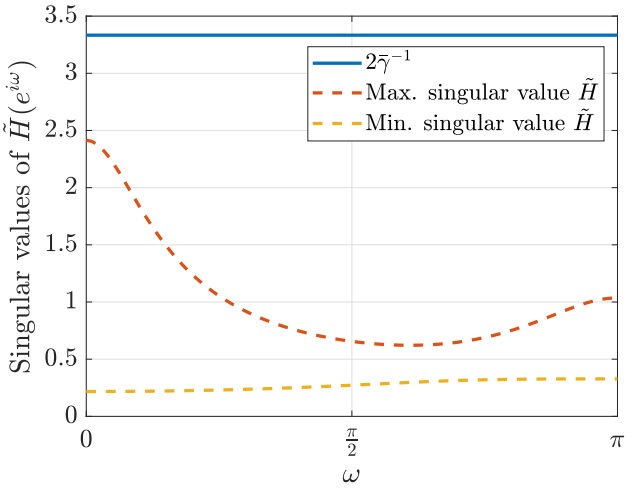


Fig. 2. The singular values of the transfer function matrix $\tilde{\boldsymbol{H}}(\boldsymbol{e^{i\omega}})$ as a function of the frequency $\boldsymbol{\omega}$ (dashed lines) and the threshold $\boldsymbol{2\bar{\gamma}^{-1}}$ (solid line).

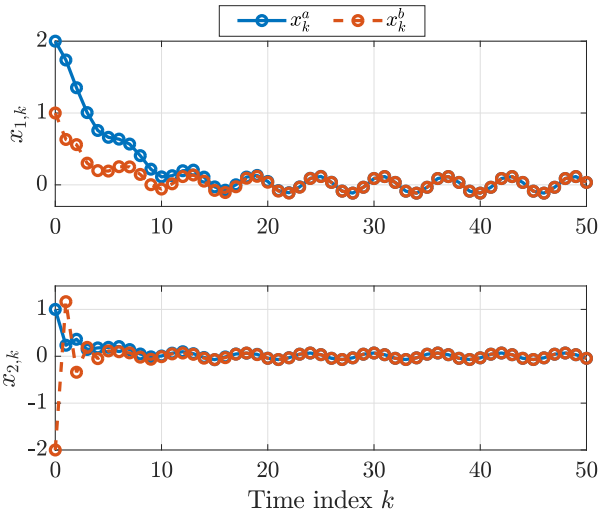


Fig. 3. Time response for initial conditions $\boldsymbol{x_0^a}$ (solid blue) and $\boldsymbol{x_0^b}$ (dashed red) for the input $\boldsymbol{w_k = \cos(k\pi/3)}$.

simulations, one with the initial condition x_0^a and the other with the initial condition x_0^b, where

$$x_0^a = \begin{bmatrix} 2 & 1 \end{bmatrix}^\top \text{ and } x_0^b = \begin{bmatrix} 1 & -2 \end{bmatrix}^\top.$$

For both simulations, we take the same periodic input $w_k = \cos(k\pi/3)$, which admits to $|w_k| \leq \sqrt{\sigma}$ for $\sigma = 1$. The state trajectory for the first 50 time steps is presented in Figure 3. This figure shows that both trajectories converge to each other, which is a property of convergent systems. Furthermore, it can be observed that the period time of the state trajectory is equal to the period time of the input, which is another property of convergent systems, see [29, Lemma 1], and also [48] and [8] in the context of *contracting* systems. Moreover, Figure 4 depicts the steady-state state trajectory $\bar{x}_k^w$ in the phase plane, where it can be seen that the state $\bar{x}_k^w$ lies indeed inside $\mathcal{S}_c$.

Finally, Figure 5 shows that the values of the Lyapunov functions $V_1(x^a)$ and $V_1(x^b)$ decrease to a value below the

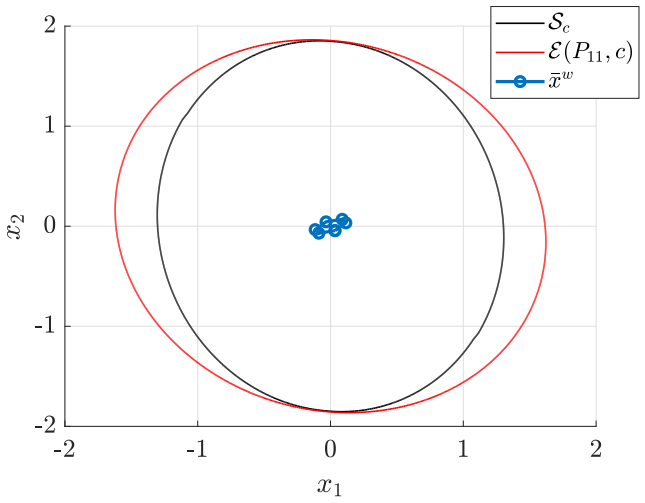


Fig. 4. The steady-state state trajectory $\bar{\boldsymbol{x}}^{\boldsymbol{w}}$ (blue), the level set $\boldsymbol{V_1(x) = c}$ (black), and the ellipsoid $\boldsymbol{\mathcal{E}(P_{11}, c)}$.

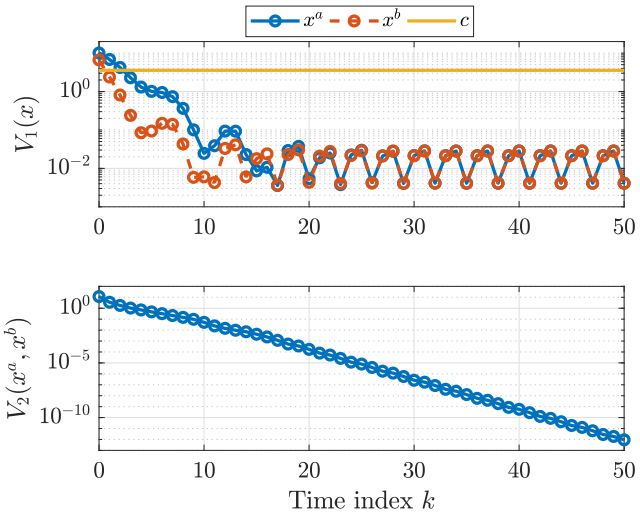


Fig. 5. Top: Lyapunov function $\boldsymbol{V_1}$ along the trajectories $\boldsymbol{x^a}$ (solid blue) and $\boldsymbol{x^b}$ (dashed red). Bottom: Incremental Lyapunov function $\boldsymbol{V_2}$ evaluated along $\boldsymbol{x^a - x^b}$.

level c and then remain below the level c (without decreasing further). It can also be observed that the incremental Lyapunov function $V_2(x^a - x^b)$ decreases exponentially to zero.

The example firstly shows a sequence of computations to assess the convergence property for Lur'e-type systems and to compute a positively invariant set in which the steady-state solution resides thereby linking the Theorem 6, 7, and 8 together. Furthermore, the example shows that the convergence property can also be checked graphically via Figure 2. Moreover, the example highlights important properties of convergent systems by means of simulation results in Figures 3, 4, and 5.

VI. Conclusion

A Lyapunov-like characterization of the convergence property for discrete-time nonlinear systems based on two distinct Lyapunov-like functions is proposed. One of the advantages of this characterization is that it does not require knowledge

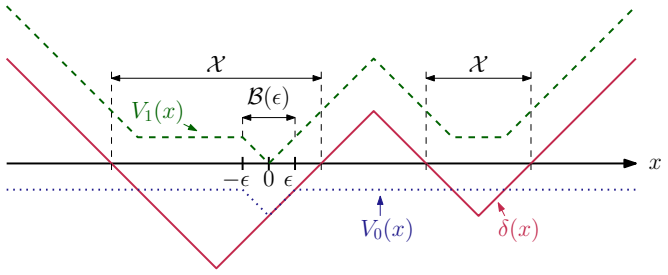


Fig. 6. Illustration of the functions $\boldsymbol{\delta(x)}$ (red solid), $\boldsymbol{V_0(x)}$ (blue dotted) and $\boldsymbol{V_1(x)}$ (green dashed).

of the steady-state solution as opposed to the characterization provided in the literature using a single Lyapunov function. The conditions are further specialized to the case of discrete-time Lur'e-type systems and conditions based on several classes of Lyapunov-like functions are formulated in terms of linear matrix inequalities and a graphical frequency-domain check.

Appendix I
Technical Lemma 1

Lemma 1 is used below in the proof of Theorem 5 as given in Appendix IV.

Lemma 1: Let us consider a set $\mathcal{X} \subset \mathbb{R}^n$, which contains the origin, is compact, and is positively invariant with respect to the dynamics (1) for any input $w \in \mathcal{W}_\gamma$. Then there exists a continuous function $V_1 : \mathbb{R}^n \to \mathbb{R}^+$, $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, and a nonnegative scalar $c \in \mathbb{R}^+$ such that conditions (11)–(12) are verified and such that $\mathcal{S}_c = \{x \in \mathbb{R}^n,\ V_1(x) \leq c\} = \mathcal{X}$. □

Proof: [Proof of Lemma 1] The proof constructs the function V_1 with the desired properties. The function V_1 is based on the auxiliary functions $\delta(x) : \mathbb{R}^n \to \mathbb{R}$ and $V_0 : \mathbb{R}^n \to \mathbb{R}$. Let us denote the signed distance function $\delta(x)$ of the boundary $\partial\mathcal{X}$ of the closed set $\mathcal{X}$ as follows:

$$\delta(x) = \begin{cases} d_{\partial\mathcal{X}}(x), & \text{if } x \notin \mathcal{X}, \\ 0, & \text{if } x \in \partial\mathcal{X}, \\ -d_{\partial\mathcal{X}}(x), & \text{if } x \in \mathcal{X}. \end{cases} \tag{39}$$

For illustration, a disconnected set $\mathcal{X}$ and its corresponding distance function $\delta(x)$ are depicted in Figure 6. Since the origin is inside the interior of the set $\mathcal{X}$, denoted by $\mathcal{X}^o$, there exists $\epsilon > 0$ such that $\mathcal{B}(\epsilon) \subset \mathcal{X}^o$, see Figure 6. Consequently, we define $V_0 : \mathbb{R}^n \to \mathbb{R}$ such that

$$V_0(x) := \begin{cases} \delta(0) + |x|, & \text{if } x \in \mathcal{B}(\epsilon), \\ \delta(0) + \epsilon, & \text{elsewhere}, \end{cases} \tag{40}$$

and, finally, the function $V_1 : \mathbb{R}^n \to \mathbb{R}$ given by

$$V_1(x) := \max(\delta(x), V_0(x)) - \delta(0). \tag{41}$$

Both $V_0(x)$ and $V_1(x)$ are depicted in Figure 6.

The continuity of $V_1(\cdot)$ is inherited by the continuity of $\delta(\cdot)$ and $V_0(\cdot)$. Let us introduce $\kappa = \sup_{x\in\mathcal{X}} |x|$ which is finite thanks to the compactness of $\mathcal{X}$. In addition, we have $\kappa > \epsilon > 0$ due to the inclusion $\mathcal{B}(\epsilon) \subset \mathcal{X}^o$. To prove that V_1 satisfies the conditions (11)–(12), let us prove that

$$\frac{\epsilon}{\kappa}|x| \leq V_1(x) \leq |x|. \tag{42}$$

Proof of the upper bound of V_1 in (42): We show that irrespectively of the two expressions that can be taken by V_1 in (41), it is upper bounded by $|x|$. Let us first study $\delta(x)$ in (39). If $x \in (\mathcal{X} \cup \partial\mathcal{X})$, then

$$|\delta(x) - \delta(0)| = |d_{\partial\mathcal{X}}(0) - d_{\partial\mathcal{X}}(x)| \leq |x|.$$

If $x \notin \mathcal{X}$, then the intersection between the boundary $\partial\mathcal{X}$ and the line segment connecting $0 \in \mathcal{X}$ to x is non-empty and contains at least one point z. This implies that

$$\begin{aligned} |x| = |x-0| &= |x-z| + |0-z|, \\ &\geq d_{\partial\mathcal{X}}(x) + d_{\partial\mathcal{X}}(0), \\ &\geq \delta(x) - \delta(0). \end{aligned}$$

Let us now turn to relation (40). If $x \notin \mathcal{B}(\epsilon)$, then $|x| \geq \epsilon$ and we find $V_0 - \delta(0) = \epsilon \leq |x|$. If $x \in \mathcal{B}(\epsilon)$, then $V_0 - \delta(0) = |x|$. We conclude that $V_1(x) \leq |x|$ for any $x \in \mathbb{R}^n$.

Proof of the lower bound of V_1 in (42): We first show that for $x \in \mathcal{B}(\kappa)$, the function V_1 in (41) is lower bounded by $\frac{\epsilon}{\kappa}|x|$. After that, we show that the same lower bound also holds for $x \notin \mathcal{B}(\kappa)$. If $x \in \mathcal{B}(\kappa)$, we have, thanks to the relation (40)

$$\begin{aligned} V_1(x) &= \max(\delta(x), V_0(x)) - \delta(0), \\ &\geq V_0(x) - \delta(0), \\ &\geq \min(|x|, \epsilon). \end{aligned}$$

Because $x \in \mathcal{B}(\kappa)$, we have $|x| \leq \kappa$ and because $\mathcal{B}(\epsilon) \subset \mathcal{B}(\kappa)$, it holds that $\epsilon < \kappa$, resulting in $V_1(x) \geq \frac{\epsilon}{\kappa}|x|$. If $x \notin \mathcal{B}(\kappa) \supset \mathcal{X}$ (note that $x \notin \mathcal{X}$ in this case), then $\delta(x) = \text{dist}(x, \partial\mathcal{X}) \geq \text{dist}(x, \mathcal{B}(\kappa)) \geq |x| - \kappa \geq 0$, thanks to a geometric argument. Then,

$$\begin{aligned} V_1(x) &= \max(\delta(x), V_0(x)) - \delta(0), \\ &\geq \delta(x) - \delta(0), \\ &\geq |x| - \kappa - \delta(0). \end{aligned}$$

Furthermore, due to the inclusion $\mathcal{B}(\epsilon) \subset \mathcal{X}^{\text{o}}$, $-\delta(0) = \text{dist}(0, \partial\mathcal{X}) > \epsilon$. By emphasizing that $\frac{|x|}{\kappa} \geq 1$, and by recalling that $\kappa > \epsilon$, we conclude that

$$V_1(x) \geq \left(\frac{|x|}{\kappa} - 1\right)\kappa + \epsilon \geq \left(\frac{|x|}{\kappa} - 1\right)\epsilon + \epsilon = \frac{\epsilon}{\kappa}|x|.$$

To end the proof, we have to determine the constant c such that $\mathcal{S}_c = \{x \in \mathbb{R}^n,\ V_1(x) \leq c\} = \mathcal{X}$. By recalling that $-\delta(0) > \epsilon$, we can deduce that $V_0(x) - \delta(0) < 0$, $\forall x \in \mathbb{R}^n$ and finally, by selecting $c = -\delta(0) > 0$,

$$\mathcal{S}_c := \{x \in \mathbb{R}^n,\ V_1(x) \leq c\} = \{x \in \mathbb{R}^n,\ \delta(x) \leq 0\} = \mathcal{X}$$

defines the level set of V_1. □

APPENDIX II
TECHNICAL LEMMA 2

Lemma 2 is used below in the proof of Theorem 5 given in Appendix IV.

Lemma 2: Consider system (1) and assume that it is uniformly incrementally stable on the set $\mathcal{X}$, which is compact and positively invariant with respect to the dynamics (1) for any input $w \in \mathcal{W}_\gamma$. Then, there exist a continuous function $V_2 : \mathbb{Z} \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$, $\mathcal{K}_\infty$-functions $\alpha_3(\cdot)$, $\alpha_4(\cdot)$, and $\mathcal{K}$-function $\alpha_5(\cdot)$ such that (15)–(16) hold. □

Proof: [Proof of Lemma 2] Lemma 2 is an adaptation in the local context of the sufficiency part of [11, Theorem 9] (see the detailed proof in [45], which is inspired by the continuous-time framework in [4]). To make the paper self-contained, we detail the proof as follows. Consider the extended system

$$\begin{cases} x_{k+1}^a = f(x_k^a, w_k, k), \\ x_{k+1}^b = f(x_k^b, w_k, k), \end{cases} \tag{43}$$

for a given sequence of inputs $w \in \mathcal{W}_\gamma$, and the diagonal set $\Delta := \{(x', x')' \in \mathbb{R}^{2n}, x \in \mathbb{R}^n\}$.

Let $z_k = ((x_k^a)', (x_k^b)')' \in \mathbb{R}^{2n}$, then the distance from z_k to the diagonal set Δ is given by

$$d_\Delta(z_k) = \inf_{\tilde{z} \in \Delta} |\tilde{z} - z_k| = \frac{1}{\sqrt{2}}|x_k^a - x_k^b|. \tag{44}$$

By denoting $F(k, w_k, z_k) = \begin{bmatrix} f(x_k^a, w_k, k) \\ f(x_k^b, w_k, k) \end{bmatrix}$, we have the extended dynamics

$$z_{k+1} = F(k, w_k, z_k). \tag{45}$$

Starting from $x_k^a \in \mathcal{X}$ and $x_k^b \in \mathcal{X}$, and the set $\mathcal{X}$ being positively invariant for the dynamics (1) for inputs $w \in \mathcal{W}_\gamma$, we have $z_{k+1} \in \mathcal{X}^2$. The system (1) is uniformly incrementally stable on $\mathcal{X}$, which implies that the extended system (45) is uniformly asymptotically stable on the domain $\mathcal{X}^2$, with respect to the diagonal set Δ.

That is, there exists a $\mathcal{KL}$-function β such that the relation (4) holds $\forall (k, k_0, x_0^a, x_0^b) \in \mathbb{Z}^2 \times \mathcal{X}^2$, $k \geq k_0$. Thanks to [49, Proposition 7], there exist two $\mathcal{K}_\infty$-functions θ_1 and θ_2 such that $\beta(s, k - k_0) \leq \theta_1(\theta_2(s)e^{k-k_0})$, $\forall (s, k) \in \mathbb{R}^+ \times \mathbb{Z}$, $k \geq k_0$. Let $\theta_3(s) = \theta_1^{-1}(s)$, $\forall s \in \mathbb{R}^+$ be the inverse function of θ_1, which is also a $\mathcal{K}_\infty$-function. We infer that

$$\begin{aligned} &\theta_3(|\phi^w(k, k_0, x_0^a) - \phi^w(k, k_0, x_0^b)|) \\ &\quad \leq \theta_3(\beta(|x_0^a - x_0^b|, k - k_0)) \leq \theta_2(|x_0^a - x_0^b|)e^{k-k_0}, \\ &\qquad \forall (k, k_0, x_0^a, x_0^b) \in \mathbb{Z}^2 \times \mathcal{X}^2,\ k \geq k_0. \end{aligned}$$

Define $W_w : \mathbb{Z} \times \mathcal{X}^2$ as

$$\begin{aligned} W_w(k_0, x_0^a, x_0^b) &= \sum_{k=k_0}^{\infty} \theta_3(|\phi^w(k, k_0, x_0^a) - \phi^w(k, k_0, x_0^b)|) \\ = \theta_3(|x_0^a - x_0^b|) &+ \sum_{k=k_0+1}^{\infty} \theta_3(|\phi^w(k, k_0, x_0^a) - \phi^w(k, k_0, x_0^b)|). \end{aligned}$$

W_w is a convergent series of nonnegative scalars and satisfies the following bounds:

$$\begin{aligned} \theta_3(|x_0^a - x_0^b|) &\leq W_w(k_0, x_0^a, x_0^b) \\ &\leq \theta_2(|x_0^a - x_0^b|) \sum_{k=k_0}^{\infty} e^{k-k_0} = \frac{e}{e-1}\theta_2(|x_0^a - x_0^b|). \end{aligned}$$

Note that the latter bound does not depend on the sequence of inputs. Since the set of inputs $\mathcal{W}_\gamma$ is compact, as is the set $\mathcal{X}^2$, we can apply [50, Lemma 4.2], which implies that

the trajectories $\phi^w(k,k_0,x_0^a)$ and $\phi^w(k,k_0,x_0^b)$ are uniformly continuous with respect to $(x_0^a,x_0^b)\in\mathcal{X}^2$ and to the sequence of inputs w. It is thus possible to define the desired function $V_2:\mathbb{Z}\times\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ as

$$V_2(k_0,x_0^a,x_0^b)=\sup_{\gamma(|w_k|)\leq 1} W_w(k_0,x_0^a,x_0^b). \tag{46}$$

By selecting $\alpha_3(s)=\theta_3(s)$ and $\alpha_4(s)=\dfrac{e}{e-1}\theta_2(s)$, the condition (15) is verified. The continuity of V_2 with respect to the last two arguments is obtained thanks to [50, Lemma 4.4], restricted on $\mathcal{X}^2$. By following the proof of [50, Theorem 1], selecting $\alpha_5(s)=\theta_3(s)$, we can verify condition (16). □

Appendix III
Technical Lemma 3

Lemma 3 is used below in the proof of Theorem 8 given in Appendix VI.

Lemma 3: Consider a symmetric positive definite matrix $M>0$ and a symmetric positive semidefinite matrix $N(\kappa)$ of the same dimension and depending on a parameter $\kappa>0$. If the function $\kappa\mapsto N(\kappa)$ is continuous at $\kappa=0$ and $\lim_{\kappa\to 0}N(\kappa)=0$, then there exists a sufficiently small scalar $\kappa^*>0$, such that for any scalar $\kappa\in]0;\kappa^*]$, $M+N(\kappa)>0$. □

Proof: [Proof of Lemma 3] Since the inequality $M>0$ is strict, there exists a scalar $\kappa_1>0$ such that $M>\kappa_1 I$. Thanks to the limit for $N(\kappa)$, we have that there exists $\kappa^*(\kappa_1)>0$, such that $\forall\kappa\in]0;\kappa^*]$, $\rho(N(\kappa))<\dfrac{\kappa_1}{2}$, which gives $N(\kappa)\geq -\rho(N(\kappa))I>-\dfrac{\kappa_1}{2}I$. By summing the two inequalities, we obtain $M+N(\kappa)>\dfrac{\kappa_1}{2}I>0$. □

Appendix IV
Proof of Theorem 5

Proof: [Proof of Theorem 5]

From incremental stability to convergence property: This implication is proved thanks to two main points: incremental stability related to conditions (15) and the existence of a positively invariant compact set related to conditions (11).

For $|w_k|\leq\gamma^{-1}(1)$, the level set $\mathcal{S}_c$, defined by (13), is invariant for the dynamics (1), thanks to the condition in (12). It is the preimage of the closed interval $[0;c]$ via a continuous map $V_1(\cdot)$ and is thus closed. Furthermore, the fact that $V_1(\cdot)$ is radially unbounded thanks to (11) implies that the set $\mathcal{S}_c$ is bounded, and hence compact.

The input w_k in the mapping f can be incorporated in the time-dependency k such that f has only two arguments, namely the state x_k and the time k. By assumption, this mapping f is continuous with respect to its first argument (the state) and there exists a compact positively invariant set $\mathcal{S}_c$. Then, Lemma 2 in [29] applies and we conclude that there exists a steady-state solution $\overline{x}_k^w\in\mathcal{S}_c$, $\forall k\in\mathbb{Z}$, for any $w\in\mathcal{W}_\gamma$. Relation (11) implies that the system has the following uniformly bounded steady-state property:

$$|\overline{x}_k^w|\leq\alpha_1^{-1}(c). \tag{47}$$

Substituting x^b by the steady-state $\overline{x}_k^w$ in the relations (15) leads to uniform global asymptotic stability of the solution $\overline{x}_k^w$. Consequently, the system is globally uniformly convergent for the class of inputs verifying $\gamma(|w_k|)\leq 1$.

From convergence property to incremental stability: The proof of this implication can be divided into two steps. The first step ensures the construction of a suitable function $V_1:\mathbb{R}^n\to\mathbb{R}^+$. The properties of the set $\mathcal{X}$ are sufficient to apply Lemma 1, which guarantees that the function V_1 and the functions α_1 and α_2 verify (11)–(12). The second step is possible thanks to [11, Theorem 12] leading to the fact that the system is uniformly asymptotically incrementally stable on the compact set $\mathcal{X}$. Then, Lemma 2 guarantees the existence of V_2, α_3, α_4, and α_5 such that inequalities (15)–(16) hold. □

Appendix V
Proof of Theorem 6

Proof: [Proof of Theorem 6] Assume that the LMI (24) holds. Thanks to the strict inequality, there exists a positive scalar $\tilde{\alpha}_1>0$, such that pre- and post-multiplying (24) by $\xi(x_k)'$ and $\xi(x_k)$ leads to

$$\xi(x_k)'P_1\xi(x_k)+2s_\Omega(S_0,Cx_k)>\tilde{\alpha}_1x_k'x_k,\ \forall x_k\neq 0.$$

We have $V_1(x_k)=\xi(x_k)'P_1\xi(x_k)\geq\tilde{\alpha}_1x_k'x_k$, because $s_\Omega(S_0,Cx_k)\leq 0$. By definition, we have $V_1(x)\leq\rho(P_1)|\xi(x)|^2$. Next, we consider the equality $|\xi(x)|^2=|x|^2+|\varphi(Cx)|^2$. When the condition (21) is satisfied, the components of the nonlinearity have the same sign as the corresponding components of the output. Then we also have the condition $y'\Omega(\varphi(y)-\Omega y)\leq 0$, which implies $|\varphi(y)|^2=\varphi(y)'\varphi(y)\leq y'\Omega\varphi(y)\leq y'\Omega\Omega y\leq\rho(C'\Omega\Omega C)|x|^2$. By taking $\tilde{\alpha}_2=\rho(P_1)(1+\rho(C'\Omega\Omega C))$, condition (11) is satisfied with $\alpha_1(r)=\tilde{\alpha}_1r^2$ and $\alpha_2(r)=\tilde{\alpha}_2r^2$, $\forall r\in\mathbb{R}^+$.

Let us now prove condition (12). For a positive scalar τ_2 strictly greater than $\rho\left(\begin{bmatrix}F'&0_{m\times p}\end{bmatrix}P_1\begin{bmatrix}F\\0_{p\times m}\end{bmatrix}\right)$, the matrix

$$\begin{aligned}M_2(\tau_2)&:=\begin{bmatrix}F'&0_{m\times p}\end{bmatrix}(P_1-G_\Omega(S_2))\begin{bmatrix}F\\0_{p\times m}\end{bmatrix}-\tau_2I_m\\&=\begin{bmatrix}F'&0_{m\times p}\end{bmatrix}P_1\begin{bmatrix}F\\0_{p\times m}\end{bmatrix}-\tau_2I_m\end{aligned} \tag{48}$$

is invertible. For τ_2 tending to infinity, the matrix $M_2(\tau_2)$, given in (48) tends continuously to the null matrix. Thanks to Lemma 3, for τ_2 large enough, we have

$$\begin{aligned}&M_0'\begin{bmatrix}-\tau_1P_1-G_\Omega(S_1)&0_{n+p}\\0_{n+p}&P_1-G_\Omega(S_2)\end{bmatrix}M_0\\&\qquad+[\star]M_2^{-1}(\tau_2)\times\\&\left(\begin{bmatrix}F'&0_{m\times p}\end{bmatrix}(P_1-G_\Omega(S_2))\begin{bmatrix}A&B&0_{n\times p}\\0_{p\times n}&0_p&I_p\end{bmatrix}\right)\\&\qquad<0_{n+2p}.\end{aligned}$$

By applying a Schur complement to the latter, we obtain:

$$\begin{aligned}&M_1'\begin{bmatrix}-\tau_1P_1-G_\Omega(S_1)&0_{n+p}&0_{(n+p)\times m}\\\star&P_1-G_\Omega(S_2)&0_{(n+p)\times m}\\\star&\star&-\tau_2I_m\end{bmatrix}M_1\\&\qquad<0_{n+2p+m}\end{aligned} \tag{49}$$

with $M_1 \in \mathbb{R}^{(2(n+p)+m)\times(n+2p+m)}$ defined as follows:

$$M_1 = \left[\begin{array}{c|c} M_0 & \begin{matrix} 0_{(n+p)\times m} \\ F \\ 0_{p\times m} \end{matrix} \\ \hline 0_{m\times(n+2p)} & I_m \end{array}\right].$$

Define

$$\eta_k \coloneqq \begin{bmatrix} \xi(x_k) \\ \varphi(CAx_k + B\varphi(Cx_k) + Fw_k) \\ w_k \end{bmatrix},$$

such that

$$\begin{bmatrix} \xi(x_k) \\ \xi(x_{k+1}) \\ w_k \end{bmatrix} = M_1\eta_k.$$

Pre- and post-multiplying inequality (25) by η_k' and η_k respectively leads to

$$\begin{aligned} &\xi(x_{k+1})'P_1\xi(x_{k+1}) - \tau_1\xi(x_k)'P_1\xi(x_k) - \tau_2 w_k'w_k \\ &- 2s_\Omega(S_1, Cx_k) - 2s_\Omega(S_2, Cx_{k+1}) < 0,\ \forall x_k \neq 0. \end{aligned} \quad (50)$$

Let us now set $c \geq \rho\sigma = \dfrac{\tau_2\sigma}{1-\tau_1}$, which allows to write:

$$-c(1-\tau_1) + \tau_2\sigma \leq 0. \quad (51)$$

Summing inequalities (50) and (51) concludes that

$$\begin{aligned} &[\xi(x_{k+1})'P_1\xi(x_{k+1}) - c] - \tau_1[\xi(x_k)'P_1\xi(x_k) - c] \\ &\quad - \tau_2[w_k'w_k - \sigma] - 2s_\Omega(S_1, Cx_k) \\ &\quad - 2s_\Omega(S_2, Cx_{k+1}) < 0,\ \forall x_k \neq 0. \end{aligned} \quad (52)$$

Thanks to the sector bounded condition (21) and the use of the S-procedure, we obtain the condition (12), with $\gamma(r) = \dfrac{1}{\sigma}r^2$, $\forall r \in \mathbb{R}^+$.

Consider in the sequel the time-independent incremental Lyapunov function $V_2(x^a, x^b) = (\xi(x^a) - \xi(x^b))'P_2(\xi(x^a) - \xi(x^b))$. First, we verify conditions (15). By pre- and post-multiplying inequality (26) by $(\xi(x_k^a) - \xi(x_k^b))'$ and its transpose respectively, and, using the fact that the inequality is strict, there exists a positive scalar $\tilde{\alpha}_3$ such that the following inequality holds:

$$\begin{aligned} &(\xi(x_k^a) - \xi(x_k^b))'P_2(\xi(x_k^a) - \xi(x_k^b)) + 2s_{\overline{\Omega}}(S_3, Cx_k^a, Cx_k^b) \\ &\qquad > \tilde{\alpha}_3|x_k^a - x_k^b|^2,\ \forall x_k^a - x_k^b \neq 0, \end{aligned}$$

inducing $V_2(x_k^a, x_k^b) \geq \tilde{\alpha}_3|x_k^a - x_k^b|^2$.

In the same way as the upper bound for V_1 and thanks to Condition (22), we have $V_2(x_k^a, x_k^b) \leq \rho(P_2)(1 + \rho(C'\Omega\Omega C))|x_k^a - x_k^b|^2$.

To conclude the proof, let us prove that condition (16) is satisfied. We can check that

$$\begin{bmatrix} \xi(x_k^a) - \xi(x_k^b) \\ \xi(x_{k+1}^a) - \xi(x_{k+1}^b) \end{bmatrix} = M_0 \begin{bmatrix} \xi(x_k^a) - \xi(x_k^b) \\ \varphi(Cx_{k+1}^a) - \varphi(Cx_{k+1}^b) \end{bmatrix},$$

which is independent of the input w_k. Pre- and post-multiplying inequality (27) by $\begin{bmatrix} \xi(x_k^a) - \xi(x_k^b) \\ \varphi(Cx_{k+1}^a) - \varphi(Cx_{k+1}^b) \end{bmatrix}'$ and its transpose, respectively, and thanks to the strict inequality, there exists a positive scalar $\tilde{\alpha}_5$, such that we obtain:

$$\begin{aligned} &(\xi(x_{k+1}^a) - \xi(x_{k+1}^b))'P_2(\xi(x_{k+1}^a) - \xi(x_{k+1}^b)) \\ &\quad - (\xi(x_k^a) - \xi(x_k^b))'P_2(\xi(x_k^a) - \xi(x_k^b)) \\ &- 2s_{\overline{\Omega}}(S_4, Cx_k^a, Cx_k^b) - 2s_{\overline{\Omega}}(S_5, Cx_{k+1}^a, Cx_{k+1}^b) \\ &\qquad < -\tilde{\alpha}_5|x_k^a - x_k^b|^2,\ \forall x_k^a - x_k^b \neq 0. \end{aligned} \quad (53)$$

Thanks to the sector bounded condition (22), the latter inequality reduces into

$$\begin{aligned} &(\xi(x_{k+1}^a) - \xi(x_{k+1}^b))'P_2(\xi(x_{k+1}^a) - \xi(x_{k+1}^b)) \\ &\quad - (\xi(x_k^a) - \xi(x_k^b))'P_2(\xi(x_k^a) - \xi(x_k^b)) \\ &\qquad < -\tilde{\alpha}_5|x_k^a - x_k^b|^2,\ \forall x_k^a - x_k^b \neq 0, \end{aligned} \quad (54)$$

which satisfies condition (16) and finalizes the proof. □

APPENDIX VI
PROOF OF THEOREM 8

Proof: [Proof of Theorem 8] We decompose the matrices P_1 and P_2 used in the conditions of Theorem 6 as follows:

$$P_1 = \begin{bmatrix} P_{11} & P_{12} \\ \star & P_{13} \end{bmatrix}, \quad P_2 = \begin{bmatrix} P_{21} & P_{22} \\ \star & P_{23} \end{bmatrix},$$

where $P_{11}, P_{21} \in \mathbb{R}^{n\times n}$ satisfy the conditions of Theorem 7. The idea of the proof is to provide a specific solution that is parameterized in a single parameter κ for the additional matrices $P_{12}, P_{13}, P_{22}, P_{23}$ such that the conditions (24)–(27) of Theorem 6 are satisfied.

Inequality (24): Inequality (24) reads $\begin{bmatrix} P_{11} & P_{12} - C'\Omega S_0 \\ \star & P_{13} + 2S_0 \end{bmatrix} > 0$. Consider a positive scalar κ and choose $S_0 = \kappa I_p$, $P_{13} = -\kappa I_p$ and $P_{12} = \kappa C'\Omega$, such that the latter inequality becomes the block diagonal inequality $\begin{bmatrix} P_{11} & 0_{n\times p} \\ \star & \kappa I_p \end{bmatrix} > 0$, which satisfies condition (24) thanks to $P_{11} > 0$ and $\kappa > 0$.

Inequality (25): Let us rewrite inequality (25) as follows. By inverting rows and columns and splitting the matrix to separate the additional variables, we obtain that inequality (25) is equivalent to:

$$\begin{aligned} &\left[\begin{array}{c} A' \\ B' \\ \hdashline 0 \end{array}\right] P_{11} \left[\begin{array}{c} A' \\ B' \\ \hdashline 0 \end{array}\right]' - \left[\begin{array}{cc:c} \tau_1 P_{11} & -C'\Omega S_1 & 0 \\ \star & 2S_1 & 0 \\ \hdashline \star & \star & 0 \end{array}\right] \\ &\quad + \left[\begin{array}{cc:c} 0 & -\tau_1 P_{12} & A'P_{12} + A'C'\Omega S_2 \\ \star & -\tau_1 P_{13} & B'P_{12} + B'C'\Omega S_2 \\ \hdashline \star & \star & P_{13} - 2S_2 \end{array}\right] < 0. \end{aligned} \quad (55)$$

With the specific choice $S_2 = \kappa I_p$, we have $P_{13} - 2S_2 = -3\kappa I_p < 0_p$, which ensures its invertibility. Applying a Schur complement to (55) leads to the equivalent inequality:

$$\begin{aligned} &\left[\begin{array}{c} A' \\ B' \end{array}\right] P_{11} \left[\begin{array}{c} A' \\ B' \end{array}\right]' - \left[\begin{array}{cc} \tau_1 P_{11} & -C'\Omega S_1 \\ \star & 2S_1 \end{array}\right] \\ &+ \kappa \left[\begin{array}{cc} 0 & -\tau_1 C'\Omega \\ \star & +\tau_1 I_p \end{array}\right] + \frac{\kappa}{3} \left[\begin{array}{c} 2A'C'\Omega \\ 2B'C'\Omega \end{array}\right] \left[\begin{array}{c} 2A'C'\Omega \\ 2B'C'\Omega \end{array}\right]' < 0. \end{aligned} \quad (56)$$

We recognize in the first terms inequality (30) and, thanks to Lemma 3, there exists a sufficiently small κ such that inequality (25) holds.

Inequality (26): By choosing $P_{22} = \kappa C'\overline{\Omega}$, $P_{23} = -\kappa I_p$ and $S_3 = \kappa I_p$, we have

$$\left[\begin{array}{cc} P_{21} & P_{22} - C'\overline{\Omega}S_0 \\ \star & P_{23} + 2S_3 \end{array} \right] = \left[\begin{array}{cc} P_{21} & 0_{n\times p} \\ \star & \kappa I_p \end{array} \right] > 0, \quad (57)$$

which implies inequality (26) thanks to $P_{21} > 0_n$ and $\kappa > 0$.

Inequality (27): By rewriting Inequality (27) and separating terms that depend only on the additional variables, we obtain the following inequality:

$$\left[\begin{array}{c} A' \\ B' \\ \hdashline 0 \end{array} \right] P_{21} \left[\begin{array}{c} A' \\ B' \\ \hdashline 0 \end{array} \right]' - \left[\begin{array}{cc:c} P_{21} & -C'\overline{\Omega}S_3 & 0 \\ \star & 2S_3 & 0 \\ \hdashline \star & \star & 0 \end{array} \right] + \left[\begin{array}{cc:c} 0 & -P_{22} & A'(P_{22} + C'\overline{\Omega}S_5) \\ \star & -P_{23} & B'(P_{22} + C'\overline{\Omega}S_5) \\ \hdashline \star & \star & P_{23} - 2S_5 \end{array} \right] < 0. \quad (58)$$

By choosing $S_5 = \kappa I_p$, the latter inequality is equivalent to the following Schur complement:

$$\left[\begin{array}{c} A' \\ B' \end{array} \right] P_{21} \left[\begin{array}{c} A' \\ B' \end{array} \right]' - \left[\begin{array}{cc} P_{21} & -C'\overline{\Omega}S_3 \\ \star & 2S_3 \end{array} \right] \kappa \left[\begin{array}{cc} 0 & -C'\overline{\Omega} \\ \star & \kappa I_p \end{array} \right] - \frac{\kappa}{3} \left[\begin{array}{c} 2A'C'\overline{\Omega} \\ 2B'C'\overline{\Omega} \end{array} \right] \left[\begin{array}{c} 2A'C'\overline{\Omega} \\ 2B'C'\overline{\Omega} \end{array} \right]' < 0. \quad (59)$$

We recognize inequality (31) in the first terms of (59). By Lemma 3, we can find a sufficiently small κ such that inequality (27) holds.

To end the proof, we only have to derive a lower bound for the Lyapunov function $V_1(x)$ of Theorem 6:

$$\begin{aligned} V_1(x) &= \xi(x)' \left[\begin{array}{cc} P_{11} & \kappa C'\Omega \\ \star & -\kappa I_p \end{array} \right] \xi(x) \\ &= x'P_{11}x + 2\kappa\varphi(y)'\Omega y - \kappa\varphi(y)'\varphi(y), \quad (60) \\ &= x'P_{11}x + \kappa\varphi(y)'\varphi(y) - 2s_\Omega(\kappa I_p, y). \quad (61) \end{aligned}$$

Thanks to the sector condition (21), we have

$$V_1(x) \geq \kappa|\varphi(y)|^2 + x'P_{11}x \geq x'P_{11}x. \quad (62)$$

The value of the level set c imposed by Theorem 7, is a lower bound on the value of the level set imposed by Theorem 6. Therefore, this allows the inclusion $\mathcal{S}_c \subset \mathcal{E}(P_{11}, c)$ and finalizes the proof. □

Appendix VII
Proof of Theorem 9

Proof: The proof is structured as follows. First, we recall the bounded real lemma inequalities and rewrite these for our use. We then show that satisfaction of the bounded real lemma inequalities guarantees the satisfaction of the conditions of Theorems 6–7. Note that we only need to prove the case in which $\bar{\gamma} > 0$, since the Lur'e-type system (19)–(20) boils down to an LTI system in the case $\bar{\gamma} = 0$. In the LTI case, the condition $\|\tilde{H}\|_\infty < 2\bar{\gamma}^{-1}$ together with the assumption that all the eigenvalues of $\tilde{A}$ are inside the unit disc implies global exponential stability, where $\tilde{A}$ and $\tilde{H}$ are defined in (34). Any globally exponentially stable LTI system is also GEUC.

We recall the discrete-time bounded real lemma from [51]. The bounded real lemma states that $\|\tilde{H}\|_\infty < 2\bar{\gamma}^{-1}$ and $\tilde{A}$ has all its eigenvalues inside the unit disc if and only if there exists a symmetric, positive definite matrix P such that the following matrix inequality is satisfied:

$$\begin{bmatrix} \tilde{A}^\top P\tilde{A} - P & \tilde{A}^\top PB & C^\top \\ \star & B^\top PB - 2\bar{\gamma}^{-1}I_p & 0_p \\ \star & \star & -2\bar{\gamma}^{-1}I_p \end{bmatrix} \prec 0. \quad (63)$$

Using a Schur complement on (63), we find

$$\begin{bmatrix} \tilde{A}^\top \\ B^\top \end{bmatrix} P \begin{bmatrix} \tilde{A} & B \end{bmatrix} - \begin{bmatrix} P - \frac{1}{2}C^\top\bar{\gamma}C & 0_{n\times p} \\ \star & 2\bar{\gamma}^{-1}I_p \end{bmatrix} \prec 0. \quad (64)$$

Thanks to the strict inequality in (64), there exists a $\tau_1 \in (0, 1)$ such that the following inequality is satisfied:

$$\begin{bmatrix} \tilde{A}^\top \\ B^\top \end{bmatrix} P \begin{bmatrix} \tilde{A} & B \end{bmatrix} - \begin{bmatrix} \tau_1 P - \frac{1}{2}C^\top\bar{\gamma}C & 0_{n\times p} \\ \star & 2\bar{\gamma}^{-1}I_p \end{bmatrix} \prec 0. \quad (65)$$

By the choice $\bar{\gamma}^{-1}I_p = S$ with S the matrix in (32), this inequality is equivalent to

$$\begin{bmatrix} \tilde{A}^\top \\ B^\top \end{bmatrix} P \begin{bmatrix} \tilde{A} & B \end{bmatrix} - \begin{bmatrix} \tau_1 P - \frac{1}{2}C^\top S\bar{\gamma}^2C & 0_{n\times p} \\ \star & 2S \end{bmatrix} \prec 0. \quad (66)$$

Next, we pre- and post-multiply inequality (66) with the matrix

$$\begin{bmatrix} I_n & -\frac{1}{2}\bar{\gamma}C^\top \\ 0_{n\times p} & I_p \end{bmatrix}, \quad (67)$$

and its transpose, respectively. This congruence transformation gives the following matrix inequality:

$$\begin{bmatrix} A^\top \\ B^\top \end{bmatrix} P \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} \tau_1 P & -\bar{\gamma}C^\top S \\ \star & 2S \end{bmatrix} \prec 0, \quad (68)$$

where the equality $\tilde{A} = A + \frac{1}{2}B\bar{\Omega}C$ is used. Since the matrix in (67) is invertible, inequality (68) is equivalent to (66).

Now, notice that (68) is equivalent to (32) for $\bar{\Omega} = \bar{\gamma}I_p$. Therefore, satisfaction of (63) guarantees the existence of a $\tau_1 \in (0, 1)$ such that (32) is satisfied for $S = \bar{\gamma}I_p$. Moreover, the satisfaction of the inequality $\|\tilde{H}\|_\infty < 2\bar{\gamma}^{-1}$ and the condition that the eigenvalues of $\tilde{A}$ are inside the unit disc, is equivalent to the existence of a $P \succ 0$ such that (63) holds. Thus, the satisfaction of $\|\tilde{H}\|_\infty < 2\bar{\gamma}^{-1}$ and the condition that the eigenvalues of $\tilde{A}$ are inside the unit disc guarantees the satisfaction of the conditions of Corollary 2, i.e., the existence of a $P \succ 0, \tau_1 \in (0, 1)$, and S such that inequality (32) is satisfied. In turn, Corollary 2 guarantees the satisfaction of the conditions of Theorem 7. Finally, by Theorem 8, we conclude that the conditions of Theorem 6 are also satisfied. The conditions of Theorems 6 and 7 conclude that the system is GEUC for the class of bounded inputs as given in the theorem statement. □

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