

Supplementary Text S.3. Stability analysis of the blackfly population model based on the formula for the basic reproduction number derived in Supplementary text 2

By using the next generation matrix approach [S15], the basic reproduction number of the blackfly vector population R_0^{BF} was derived as follows:

$$\begin{aligned} R_0^{BF} &= \left(\frac{1}{2}\right) \left(\frac{\beta_N}{1/g + \mu_V} + \frac{1}{1 + \mu_V g} \frac{\beta_P}{\mu_V} \right) \left(\frac{1}{1 + \mu_E^0 \Delta_E} \right) \left(\frac{1}{1 + \mu_L \Delta_L} \right) \left(\frac{1}{1 + \mu_P \Delta_P} \right) \\ &= \left(\frac{1}{2}\right) \left(\beta_N g + \frac{\beta_P}{\mu_V} \right) \left(\frac{1}{1 + \mu_V g} \right) \left(\frac{1}{1 + \mu_E^0 \Delta_E} \right) \left(\frac{1}{1 + \mu_L \Delta_L} \right) \left(\frac{1}{1 + \mu_P \Delta_P} \right). \end{aligned}$$

Theorem 1 If $R_0^{BF} \leq 1$, then the no-flies steady state $\bar{X}(0,0,0,0,0)$ of the system is globally asymptotically stable and unstable otherwise.

Proof Construct a continuously differentiable and non-negative Lyapunov function

$$V = E + c_1 L + c_2 P + c_3 N + c_4 \Psi$$

where

$$\begin{aligned} c_1 &= \left(\frac{1}{2}\right) \left(\beta_N g + \frac{\beta_P}{\mu_V} \right) \left(\frac{1}{1 + \mu_V g} \right) \left(\frac{1}{1 + \mu_L \Delta_L} \right) \left(\frac{1}{1 + \mu_P \Delta_P} \right), \\ c_2 &= c_1 (1 + \mu_L \Delta_L), \quad c_3 = 2c_2 (1 + \mu_P \Delta_P), \quad c_4 = \frac{\beta_P}{\mu_V}, \end{aligned} \tag{S3.1}$$

then we obtain

$$\begin{aligned} \frac{dV}{dt} &= \frac{dE}{dt} + c_1 \frac{dL}{dt} + c_2 \frac{dP}{dt} + c_3 \frac{dN}{dt} + c_4 \frac{d\Psi}{dt} \\ &= \left[\beta_N N + \beta_P \Psi - \frac{E}{\Delta_E} - \mu_E^0 \left(1 + \frac{E}{K} \right) E \right] + c_1 \left(\frac{E}{\Delta_E} - \frac{L}{\Delta_L} - \mu_L L \right) + c_2 \left(\frac{L}{\Delta_L} - \frac{P}{\Delta_P} - \mu_P P \right) \\ &\quad + c_3 \left[\frac{0.5P}{\Delta_P} - \left(\frac{1}{g} + \mu_V \right) N \right] + c_4 \left(\frac{1}{g} N - \mu_V \Psi \right) \\ &= E \left[\frac{c_1}{\Delta_E} - \left(\frac{1}{\Delta_E} + \mu_E^0 \right) - \frac{\mu_E^0 E}{K} \right] + L \left[\frac{c_2}{\Delta_L} - c_1 \left(\frac{1}{\Delta_L} + \mu_L \right) \right] + P \left[\frac{c_3}{2\Delta_P} - c_2 \left(\frac{1}{\Delta_P} + \mu_P \right) \right] \\ &\quad + N \left[\frac{c_4}{g} + \beta_N - c_3 \left(\frac{1}{g} + \mu_V \right) \right] + \Psi (\beta_P - c_4 \mu_V) \\ &= \frac{E}{\Delta_E} \left(\frac{1}{1 + \mu_E^0 \Delta_E} \right) (R_0^{BF} - 1) - \frac{\mu_E^0}{K} E^2 \leq 0. \end{aligned}$$

The equality $\frac{dV}{dt} = 0$ holds if and only if $E = 0$. Then we get $L = P = N = \Psi = 0$ from the ODE model. Thus, by the LaSalle largest invariant set theorem, the no-flies steady state

$\bar{X} (0,0,0,0,0)$ is globally asymptotically stable when $R_0^{BF} \leq 1$. In this case, the flies go extinct.

Next, we prove that $\bar{X}$ is unstable when $R_0^{BF} > 1$. The Jacobian matrix $J(\bar{X})$ of the system at $\bar{X}$ is

$$J(\bar{X}) = \begin{pmatrix} -\left(\frac{1}{\Delta_E} + \mu_E^0\right) & 0 & 0 & \beta_N & \beta_P \\ \frac{1}{\Delta_E} & -\left(\frac{1}{\Delta_L} + \mu_L\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta_L} & -\left(\frac{1}{\Delta_P} + \mu_P\right) & 0 & 0 \\ 0 & 0 & \frac{1}{2\Delta_P} & -\left(\frac{1}{g} + \mu_V\right) & 0 \\ 0 & 0 & 0 & \frac{1}{g} & -\mu_V \end{pmatrix}.$$

For convenience of notation, let

$$h_1 = \frac{1}{\Delta_E} + \mu_E^0, h_2 = \frac{1}{\Delta_L} + \mu_L, h_3 = \frac{1}{\Delta_P} + \mu_P, h_4 = \frac{1}{g} + \mu_V, h_5 = \mu_V.$$

Then we obtain the characteristic equation at $\bar{X}$ as follows:

$$\begin{aligned} & (\lambda + h_1)(\lambda + h_2)(\lambda + h_3)(\lambda + h_4)(\lambda + h_5) - \beta_N \frac{1}{\Delta_E} \frac{1}{\Delta_L} \frac{1}{2\Delta_P} (\lambda + h_5) - \beta_P \frac{1}{\Delta_E} \frac{1}{\Delta_L} \frac{1}{2\Delta_P} \frac{1}{g} \\ & = \lambda^5 + k_1\lambda^4 + k_2\lambda^3 + k_3\lambda^2 + k_4\lambda + k_5 = 0, \end{aligned} \quad (S3.2)$$

where

$$k_1 = h_1 + h_2 + h_3 + h_4 + h_5,$$

$$k_2 = h_1(h_2 + h_3 + h_4 + h_5) + h_2(h_3 + h_4 + h_5) + h_3(h_4 + h_5) + h_4h_5,$$

$$k_3 = h_1(h_2h_3 + h_2h_4 + h_2h_5 + h_3h_4 + h_3h_5 + h_4h_5) + h_2(h_3h_4 + h_3h_5 + h_4h_5) + h_3h_4h_5,$$

$$k_4 = h_1(h_2h_3h_4 + h_2h_3h_5 + h_2h_4h_5 + h_3h_4h_5) + h_2h_3h_4h_5 - \beta_N \frac{1}{\Delta_E} \frac{1}{\Delta_L} \frac{1}{2\Delta_P},$$

$$k_5 = h_1 h_2 h_3 h_4 h_5 - \beta_N \frac{1}{\Delta_E} \frac{1}{\Delta_L} \frac{1}{2\Delta_P} h_5 - \beta_P \frac{1}{\Delta_E} \frac{1}{\Delta_L} \frac{1}{2\Delta_P} \frac{1}{g}.$$

Denote the eigenvalues of the characteristic equation (S3.2) as λ_i , $i = 1, 2, \dots, 5$. When $R_0^{BF} > 1$, we get

$$\begin{aligned} k_5 &= \left(\frac{1}{\Delta_E} + \mu_E^0\right) \left(\frac{1}{\Delta_L} + \mu_L\right) \left(\frac{1}{\Delta_P} + \mu_P\right) \left(\frac{1}{g} + \mu_V\right) \mu_V - \beta_N \frac{1}{\Delta_E} \frac{1}{\Delta_L} \frac{1}{2\Delta_P} \mu_V - \beta_P \frac{1}{\Delta_E} \frac{1}{\Delta_L} \frac{1}{2\Delta_P} \frac{1}{g} \\ &= \left(\frac{1}{\Delta_E} + \mu_E^0\right) \left(\frac{1}{\Delta_L} + \mu_L\right) \left(\frac{1}{\Delta_P} + \mu_P\right) \left(\frac{1}{g} + \mu_V\right) \mu_V (1 - R_0^{BF}) \\ &< 0, \end{aligned}$$

which means that $\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 = -k_5 > 0$. This indicates that at least one of $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ has a positive real part. Thus, $\bar{X}$ is unstable if $R_0^{BF} > 1$. This completes the proof.

Theorem 2 When $R_0^{BF} > 1$, then the unique positive equilibrium $X^*(E^*, L^*, P^*, N^*, \Psi^*)$ is globally asymptotically stable.

Proof Construct a continuously differentiable and non-negative Lyapunov function

$$\begin{aligned} W &= E - E^* - E^* \ln \frac{E}{E^*} + c_1 \left(L - L^* - L^* \ln \frac{L}{L^*} \right) + c_2 \left(P - P^* - P^* \ln \frac{P}{P^*} \right) \\ &\quad + c_3 \left(N - N^* - N^* \ln \frac{N}{N^*} \right) + c_4 \left(\Psi - \Psi^* - \Psi^* \ln \frac{\Psi}{\Psi^*} \right), \end{aligned}$$

where c_1, c_2, c_3, c_4 are given by equation S3.1. The global minimum of W occurs at the positive equilibrium X^* , and the function W takes the value $W = 0$ at the positive equilibrium X^* .

Differentiating W along the solutions of the system and using the equilibrium relations and equation S3.1, we obtain

$$\begin{aligned} \frac{dW}{dt} &= \left(1 - \frac{E}{E^*}\right) \left[\beta_N N + \beta_P \Psi - \frac{E}{\Delta_E} - \mu_E^0 \left(1 + \frac{E}{K}\right) E \right] + c_1 \left(1 - \frac{L}{L^*}\right) \left(\frac{E}{\Delta_E} - \frac{L}{\Delta_L} - \mu_L L \right) \\ &\quad + c_2 \left(1 - \frac{P}{P^*}\right) \left(\frac{L}{\Delta_L} - \frac{P}{\Delta_P} - \mu_P P \right) + c_3 \left(1 - \frac{N}{N^*}\right) \left[\frac{0.5P}{\Delta_P} - \left(\frac{1}{g} + \mu_V\right) N \right] \\ &\quad + c_4 \left(1 - \frac{\Psi}{\Psi^*}\right) \left(\frac{1}{g} N - \mu_V \Psi \right) \end{aligned}$$

$$\begin{aligned}
&= \beta_N N + \beta_P \Psi - \left(\frac{1}{\Delta_E} + \mu_E^0 \right) E - \frac{\mu_E^0}{K} E^2 - \beta_N E^* \frac{N}{E} - \beta_P E^* \frac{\Psi}{E} + \left(\frac{1}{\Delta_E} + \mu_E^0 \right) E^* + \frac{\mu_E^0 E^*}{K} E \\
&\quad + c_1 \left[\frac{E}{\Delta_L} - \left(\frac{1}{\Delta_L} + \mu_L \right) L - \frac{L^*}{\Delta_L} \frac{E}{L} + \left(\frac{1}{\Delta_L} + \mu_L \right) L^* \right] \\
&\quad + c_2 \left[\frac{L}{\Delta_P} - \left(\frac{1}{\Delta_P} + \mu_P \right) P - \frac{P^*}{\Delta_P} \frac{L}{P} + \left(\frac{1}{\Delta_P} + \mu_P \right) P^* \right] \\
&\quad + c_3 \left[\frac{P}{2\Delta_P} - \left(\frac{1}{g} + \mu_V \right) N - \frac{N^*}{2\Delta_P} \frac{P}{N} + \left(\frac{1}{g} + \mu_V \right) N^* \right] \\
&\quad + c_4 \left(\frac{1}{g} N - \mu_V \Psi - \frac{\Psi^*}{g} \frac{N}{\Psi} + \mu_V \Psi^* \right) \\
&= E \left[\frac{c_1}{\Delta_E} - \left(\frac{1}{\Delta_E} + \mu_E^0 \right) + \frac{\mu_E^0 E^*}{K} \right] - \frac{\mu_E^0}{K} E^2 + L \left[\frac{c_2}{\Delta_L} - c_1 \left(\frac{1}{\Delta_L} + \mu_L \right) \right] \\
&\quad + P \left[\frac{c_3}{2\Delta_P} - c_2 \left(\frac{1}{\Delta_P} + \mu_P \right) \right] + N \left[\frac{c_4}{g} + \beta_N - c_3 \left(\frac{1}{g} + \mu_V \right) \right] + \Psi (\beta_P - c_4 \mu_V) \\
&\quad - \beta_N E^* \frac{N}{E} - \beta_P E^* \frac{\Psi}{E} - c_1 \frac{L^*}{\Delta_L} \frac{E}{L} - c_2 \frac{P^*}{\Delta_P} \frac{L}{P} - c_3 \frac{N^*}{2\Delta_P} \frac{P}{N} - c_4 \frac{\Psi^*}{g} \frac{N}{\Psi} \\
&\quad + \left(\frac{1}{\Delta_E} + \mu_E^0 \right) E^* + c_1 \left(\frac{1}{\Delta_L} + \mu_L \right) L^* + c_2 \left(\frac{1}{\Delta_P} + \mu_P \right) P^* + c_3 \left(\frac{1}{g} + \mu_V \right) N^* + c_4 \mu_V \Psi^* \\
&= -\frac{\mu_E^0}{K} \left[E - \frac{K}{\mu_E^0} \left(\frac{1}{\Delta_E} + \mu_E^0 \right) (R_0^{BF} - 1) \right]^2 - \beta_N N^* \left(\frac{E^*}{N^*} \frac{N}{E} + \frac{L^*}{E^*} \frac{E}{L} + \frac{P^*}{L^*} \frac{L}{P} + \frac{N^*}{P^*} \frac{P}{N} - 4 \right) \\
&\quad - \beta_P \Psi^* \left(\frac{E^*}{\Psi^*} \frac{\Psi}{E} + \frac{L^*}{E^*} \frac{E}{L} + \frac{P^*}{L^*} \frac{L}{P} + \frac{N^*}{P^*} \frac{P}{N} + \frac{\Psi^*}{N^*} \frac{N}{\Psi} - 5 \right).
\end{aligned}$$

Because the geometric mean is always less than or equal to the arithmetic mean, we have

$\frac{dW}{dt} \leq 0$ and the equality holds if and only if (E, L, P, N, Ψ) take the equilibrium values

$(E^*, L^*, P^*, N^*, \Psi^*)$. Therefore, by the LaSalle largest invariant set theorem, it follows that the

positive equilibrium X^* is globally asymptotically stable.

Reference

[S15] van den Driessche P, Watmough, J. 2002 Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences* **180**, 29-48. ([doi:10.1016/S0025-5564\(02\)00108-6](https://doi.org/10.1016/S0025-5564(02)00108-6))