

Poincaré type Kähler metrics and stability on toric varieties

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Declaration

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Abstract

In this thesis we study the relationship between the existence of extremal Kähler metrics and stability. We introduce a space of symplectic potentials for toric manifolds, which we show gives metrics with mixed Poincaré type and cone angle singularities. We show uniqueness and that existence implies stability for extremal metrics arising from these potentials. For quadrilaterals, we give a computable criterion for stability in certain cases, giving a definite log-stable region for generic quadrilaterals. We use computational tools to find new examples of stable and unstable toric manifolds. For Poincaré type manifolds with an S^1 -action, we prove a version of the LeBrun-Simanca openness theorem and Arezzo-Pacard blow-up theorem.

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Chapter 1

Introduction

One of the classical theorems of geometry is the uniformization theorem, which says that any Riemann surface admits a metric of constant curvature in any conformal class. In Kähler geometry, a natural generalization is to look for canonical Kähler metrics, like those of constant scalar curvature, in a given Kähler class. In [Cal82a], Calabi introduced the notion of an extremal Kähler metric as a critical point of the functional

$$g \mapsto \int_X S(g)^2 dV_g,$$

for Kähler metrics g in a fixed Kähler class. He showed that this is equivalent to the gradient of the scalar curvature being a holomorphic vector field and so this notion generalizes the notion of a constant scalar curvature metric, and is a good definition for a canonical metric in a Kähler class.

Levine showed in [Lev85] that there are complex manifolds which do not have extremal Kähler metrics in any of its Kähler classes. Unstable toric examples coupled with the theorem of Arezzo-Pacard-Singer in [APS11] shows that there are Kähler manifolds admitting extremal Kähler metrics in a non-empty proper subset of the Kähler cone. Since complex projective space with its Fubini-Study metric is an extremal Kähler metric, there are therefore examples showing a Kähler manifold may admit an extremal Kähler metric in all, some or none of its Kähler classes.

The main questions one would like to answer are if a Kähler manifold admits an extremal Kähler metric in a given Kähler class, and whether or not it is unique if it exists. Of these, the latter has been settled by the work of Berman-Berndtsson in [BB14], but finding when the former holds is one of the main open problems in complex geometry.

On a compact projective manifold X , when the Kähler class is the first Chern class of some line bundle L on X this should be related to a notion of algebro-geometric stability. The Yau-Tian-Donaldson conjecture, adapted to the extremal setting by Székelyhidi, states that X admits an extremal metric if and only if (X, L) is *relatively K-stable*. There are, however, examples, indicating that a stronger notion of stability may be needed, see [ACGTF08].

There are two main cases where this equivalence has been settled. The first is the Kähler-Einstein case. If X admits a Kähler-Einstein metric, then the first Chern class is either positive, negative or zero. If it is negative or zero, there is always a solution. This is Yau's resolution of the Calabi conjecture. However, for Fano manifolds, i.e. when the first Chern class is positive, there *is* an obstruction.

Recently, in [CDS15a], [CDS15b] and [CDS15c], Chen-Donaldson-Sun proved that the existence of a Kähler-Einstein metric on a Fano manifold was equivalent to K -stability. Their proof involved a continuity method where they used metrics with cone angle singularities along a divisor. Starting with a conical Kähler-Einstein metric of cone angle $2\pi\beta_0$ with $\beta_0 < 1$, one wants openness and closedness in $[\beta_0, 1]$. The main difficulty is closedness, i.e. that one can take limits for an increasing sequence $\beta_i \rightarrow \beta_\infty$ with $\beta_\infty \in (\beta_0, 1]$. They show that if this is not possible, one obtains a destabilising test configuration, and so K -stability is violated.

From the results of Chen-Donaldson-Sun we see that cone angle metrics have relevance to the smooth setting, but one can of course study such metrics in their own right. There is a notion of stability for such metrics, called log K -stability, or log stability. Here one has a parameter β_i for each irreducible component D_i of a simple normal crossings divisor D . Stability is then conjectured to be equivalent to a cscK or extremal metric with cone angle singularity $2\pi\beta_i$ along D_i . A natural problem in this context is to ask, given a simple normal crossings divisor D , what the set of stable parameters $(\beta_1, \dots, \beta_d)$ is.

The other case where the equivalence between relative K -stability and the existence of extremal metrics has been settled is the case of toric surfaces. For a toric manifold X , there is a very nice description of the space of T^n -invariant Kähler metrics in a class Ω from the work of Guillemin and Abreu in [Gui94a] and [Abr98], respectively. This is a space of convex functions on the polytope determining (X, Ω) . This allows one to express all the relevant quantities in terms of functions on a polytope in $\mathbb{R}^n$ and techniques from convex function theory can be employed. Using this framework, Donaldson proved in [Don09] that K -stability implies the existence of a constant scalar curvature metric on toric surfaces. It was extended to the extremal setting in [CLS12] by Chen, Li and Sheng.

The stability condition also becomes more tractable in this setting. One can define a T^n -invariant stability notion, involving piecewise linear functions on the polytope, and this works equally well in the log setting. This is particularly nice for surfaces, where Donaldson in [Don02] showed that it suffices to check stability on a finite dimensional space of such functions. This makes it more manageable to *prove* stability, which in general is a very hard problem.

In this thesis we investigate the relationship between the existence of canonical Kähler metrics and algebro-geometric stability. We in particular consider what happens when such a metric *does not* exist. Here there is a model type of behaviour that is expected to be significant, namely that of *Poincaré type metrics*.

Poincaré type metrics are complete metrics on the complement of a divisor that near the divisor are modelled on the product of the standard cusp or Poincaré type metric on the punctured unit disk with a smooth metric on the divisor. From examples of Székelyhidi in [Szé06], these metrics have been shown to

arise as limits of extremal smooth metrics. Also, Donaldson has in [Don09] given a model for what happens in the toric case when an extremal metric does not exist. The metrics arising in this model are of Poincaré type.

Poincaré type metrics have been studied extensively for many years. A central result is the existence of Kähler-Einstein metrics, analogous to Yau's theorem in the compact case, by Cheng-Yau, Kobayashi and Tian-Yau in [CY80], [Kob84] and [TY87], respectively. More recently, Auvray has in [Auv14c] and subsequent papers undertaken a study of the space of such metrics, and his results will be central to us. Heuristically, one can consider these metrics as "cone angle 0" metrics. Guenancia recently showed in [Gue15] that the Kähler-Einstein Poincaré type metrics are limits of Kähler-Einstein cone angle metrics when the cone angle goes to 0.

Since both verifying the stability conditions and solving the extremal equation are very difficult in general, finding sufficient conditions for the existence of an extremal metric is important. Two such results in the compact case are the openness theorem in [LS94] of LeBrun and Simanca and the blow-up theorem in [AP06] of Arezzo and Pacard and its generalizations. The former says that the set of Kähler classes admitting an extremal metric is open. The latter gives sufficient conditions for the blow-up of a Kähler manifold admitting an extremal metric to carry an extremal metric.

Most of the work of this thesis is on toric manifolds, but in the final part of the thesis we step a bit outside of the toric world and consider just an S^1 -action. Here we prove the main results of the thesis, which are extensions of these two theorems to the Poincaré type setting, with an S^1 -action. For the openness theorem we restrict to a certain subset of all $(1, 1)$ -classes.

Chapter outline

In chapter 2 we give some background on extremal Kähler metrics, stability and toric geometry. In this thesis we will study extremal Poincaré type metrics, and we recall their definition. We end the chapter with a more detailed discussion on the conjectural relationship between the existence of extremal Poincaré type metrics and the instability of a polarised compact complex manifold.

In chapter 3 we introduce a space of symplectic potentials on a polytope which we show induce metrics with mixed Poincaré type and cone angle singularities. We then show uniqueness of extremal metrics in this space of potentials, as well as proving log-stability with respect to toric degenerations whenever an extremal metric exists.

In chapter 4 we study log-stability and its dependence on weights in the particular case of toric surfaces whose moment polytope is a quadrilateral. We use constructions of Apostolov-Calderbank-Gauduchon and Legendre to show that stability is equivalent to the existence of an extremal symplectic potential in this case. We show that strictly semistable quadrilaterals are split into two pieces, each admitting a potential for an extremal Poincaré type metric. We also find examples indicating that there may be other types of

complete metrics on the complement of divisors that will arise in the unstable case.

In chapter 5 we take a computational approach to stability. We propose a method for proving stability that we show will always be conclusive in the case of vanishing Futaki invariant. We then apply the method to give new stable and unstable examples, both with vanishing and non-vanishing Futaki invariant. We apply this to Delzant polytopes, but our method works for any weighted polytope.

In chapter 6 we study the Fredholm theory of the Laplacian and Lichnerowicz operator on Poincaré type manifolds, under the assumption of an S^1 -action.

Following this, we in chapter 7 use the results of chapter 6 to prove an Arezzo-Pacard-type result for extremal Poincaré type manifolds with an S^1 -action, as well as showing that one can obtain extremal Kähler metric in certain classes close to a Kähler class admitting a Poincaré type extremal metric. We apply this to give new examples of extremal Poincaré type metrics.

Chapter 2

Background

In this chapter we discuss some of the background relating to this thesis. The metrics of interest to us are extremal Kähler metrics, and we recall their definition and some properties in section 2.1. In section 2.2 we discuss the Delzant theorem classifying compact symplectic toric manifolds by their moment polytope before continuing with the description of torus-invariant Kähler structures in section 2.3. The existence of extremal metrics is conjectured to be equivalent to a notion of stability, and we consider the definition of stability and this conjecture for toric manifolds in 2.4.

In this thesis we will be investigating extremal metrics defined on the complement of a divisor in a complex manifold. In section 2.5, we discuss the asymptotics of the metrics that we primarily will be considering, namely Poincaré type Kähler metrics. In the final section, 2.6, we motivate a lot of the work in this thesis by discussing how extremal Poincaré type metrics are expected to arise when a polarised compact complex manifold does not admit an extremal metric, both in the toric and non-toric setting.

Almost nothing in this chapter is new, but we do show in section 2.6 how some toric test configurations can be seen as arising from a construction generalising the deformation to the normal cone. We use this to give some indication on how the toric picture regarding what happens when an extremal metric does not exist could generalise to arbitrary surfaces.

2.1 Extremal Kähler metrics

Extremal Kähler metrics were introduced by Calabi in [Cal82b] as a generalisation of constant scalar curvature Kähler (cscK) metrics. We here recall some equivalent ways of considering these, special cases and some obstructions to existence.

Let (X, g, J, ω) denote a Kähler manifold X with metric g , complex structure J and symplectic form ω .

The Ricci form $\rho \in \Lambda^2(X)$ is the 2-form defined by

$$\rho(\mathfrak{X}_1, \mathfrak{X}_2) = \text{Ric}(\mathfrak{X}_1, J\mathfrak{X}_2),$$

where Ric is the Ricci tensor. Its expression in local holomorphic coordinates $(z^1, \dots, z^n)$ is

$$\rho = -i \sum_{\alpha, \beta=1}^n \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} (\log \det(g_{\gamma\bar{\delta}})) dz^\alpha \wedge d\bar{z}^\beta \quad (2.1)$$

$$= -i \partial \bar{\partial} (\log \det(g_{\gamma\bar{\delta}})). \quad (2.2)$$

Here $g_{\alpha\bar{\beta}} = g(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta})$ is the local expression for the Hermitian metric g on $TX \otimes_{\mathbb{R}} \mathbb{C}$ in the given coordinates, and $(g^{\alpha\bar{\beta}})$ is the inverse matrix of $(g_{\alpha\bar{\beta}})$. From equation 2.2, ρ is closed and its cohomology class can be shown to be $2\pi c_1(M)$ (see e.g. [Mor07, p.116 and 120]). We will denote by $S(g)$ the scalar curvature of g , the (complex) trace of the Ricci curvature, which in local complex coordinates then is given by

$$S(g) = - \sum_{\alpha, \beta=1}^n g^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} (\log \det(g_{\gamma\bar{\delta}})). \quad (2.3)$$

The volume element dV_g of g is

$$\begin{aligned} dV_g &= \frac{\omega^n}{n!} \\ &= \left(\frac{i}{2}\right)^n \det(g_{\gamma\bar{\delta}}) dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^n \end{aligned}$$

in local holomorphic coordinates.

We now assume X is compact. The total scalar curvature of g is

$$\bar{S}(g) = \int_X S(g) dV_g, \quad (2.4)$$

and the total volume is

$$V(g) = \int_X dV_g. \quad (2.5)$$

Both of these are independent of the particular metric in a given Kähler class. Indeed, in the latter case, $dV_g = \frac{\omega^n}{n!}$, so that

$$V(g) = \frac{1}{n!} \langle [\omega]^n, [X] \rangle.$$

For $\bar{S}(g)$, note the relation

$$n\alpha \wedge \omega^{n-1} = \text{tr}_\omega(\alpha) \omega^n \quad (2.6)$$

for a 2-form α . We also have that $\text{tr}_\omega = \Lambda|_{\Lambda^2(X)}$, the dual Lefschetz operator acting on 2-forms. If α is the Ricci form ρ , then, by definition,

$$\text{tr}_\omega(\rho) = S(g).$$

Hence we get that

$$\begin{aligned}\overline{S}(g) &= \int_X S(g) dV_g \\ &= n \int_X \rho \wedge \frac{1}{n!} \omega^{n-1} \\ &= 2\pi \frac{1}{(n-1)!} \langle c_1(X) \smile [\omega]^{n-1}, [X] \rangle\end{aligned}$$

which does not depend on the particular metric ω in the cohomology class of ω .

Our interest is in changing the metric g , keeping the complex structure fixed, i.e. we are interested in Kähler metrics on a fixed compact complex manifold (X, J) . The symplectic form $\omega(\cdot, \cdot) = g(J(\cdot), \cdot)$ defines a closed 2-form, so has an associated cohomology class. The space of Kähler metrics in a given cohomology class Ω is denoted

$$\mathcal{M}_\Omega = \{g \in \Gamma(S^2 T^* X) : g \text{ is a Riemannian metric which is Kähler with respect to } J, \text{ and } [\omega] = \Omega\}.$$

We shall often muddle the distinction between g and ω , as they determine one another. In particular, we may talk about the metric ω , when we really mean the metric associated to the symplectic form ω .

By the $\partial\bar{\partial}$ -lemma, the space of Kähler metrics in a fixed cohomology class can be viewed as an open set in an infinite-dimensional vector space. Specifically, we have that if $\omega_0, \omega_1 \in \mathcal{M}_\Omega$, then there exists a real function φ on X such that

$$\omega_1 = \omega_0 + i\partial\bar{\partial}\varphi.$$

φ is unique up to the addition of a constant. Conversely, given any $\varphi \in C^\infty(X, \mathbb{R})$ and $\omega_0 \in \mathcal{M}_\Omega$, the form $\omega_0 + i\partial\bar{\partial}\varphi$ defines a Kähler metric if and only if the associated bilinear form g is positive-definite, since $\omega_0 + i\partial\bar{\partial}\varphi$ is closed. We usually write $\omega_0 + i\partial\bar{\partial}\varphi > 0$ to mean that the associated symmetric tensor is positive-definite.

The space of such *Kähler potentials* is denoted

$$\widetilde{\mathcal{M}}_{\omega_0} = \{\varphi \in C^\infty(X, \mathbb{R}) : \omega_0 + i\partial\bar{\partial}\varphi > 0\}.$$

The condition that g is positive-definite is open, so this is an open subset of $C^\infty(X, \mathbb{R})$. $\mathcal{M}_\Omega$ is defined without specifying any particular element. However, we can view $\mathcal{M}_\Omega$ as sitting inside $\widetilde{\mathcal{M}}_{\omega_0}$, by e.g. choosing the representative function φ such that $\int_X \varphi dV_{g_0} = 0$.

An interesting question in Kähler geometry is on the existence of preferred or canonical metrics. This means that we consider the complex manifold (X, J) as fixed and vary the metric, equivalently the symplectic form, in a specific Kähler class and see if there is any preferred Kähler metric in this class. What is meant by preferred can vary, but for example constant Ricci curvature, i.e. Einstein, metrics or more generally constant scalar curvature metrics can be considered. Extremal Kähler metrics are a generalisation of these.

For a Kähler class Ω , define the *Calabi functional* $C = C_\Omega : \mathcal{M}_\Omega \rightarrow \mathbb{R}$ by

$$C(g) = \int_X S(g)^2 dV_g \quad (2.7)$$

for a metric g in $\mathcal{M}_\Omega$. Note that by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \overline{S}(g)^2 &= \left(\int_X S(g) \cdot 1 dV_g \right)^2 \\ &\leq \int_X S(g)^2 dV_g \cdot \int_X 1^2 dV_g \\ &= C(g) \cdot V(g) \end{aligned}$$

and so

$$C(g) \geq \frac{\overline{S}(g)^2}{V(g)}.$$

Equality holds if and only if $S(g)$ is constant, so the above lower bound is achieved if and only if Ω has a Kähler metric of constant scalar curvature. We are therefore interested in minimisers and so in critical points of the functional C . We have the following definition

Definition 2.1 ([Cal82b]). *Let (X, J) be a compact complex manifold and $\Omega \in H^2(X, \mathbb{R})$ a Kähler class. A metric which is a critical point of $C : \mathcal{M}_\Omega \rightarrow \mathbb{R}$ in the Kähler class Ω is called an extremal Kähler metric in the class Ω .*

The Euler-Lagrange equation for a critical point of C was originally calculated in [Cal82b].

Proposition 2.2 ([Cal82b]). *The Euler-Lagrange equation for C is equivalent to ω satisfying*

$$\bar{\partial} \circ \nabla^{1,0}(S(\omega)) = 0. \quad (2.8)$$

In fact, a critical point for the Calabi functional is a global minimiser, so Kähler metrics on a compact Kähler manifold satisfying 2.8 are global minimisers of the Calabi functional. See [Che09, Thm 1.5].

We will be interested in extremal metrics in the toric setting, and we will then consider torus-invariant metrics. In the search for extremal metrics this is justified by the following theorem.

Theorem 2.3 ([Cal85]). *An extremal Kähler metric on a complex manifold X is preserved under a maximal compact subgroup of the identity component of the group of all holomorphic transformations of X .*

2.2 Classification of toric manifolds

An important class of manifolds in this thesis are toric manifolds. These are compactifications of the complex n -torus $T_\mathbb{C}^n = (\mathbb{C}^*)^n$ admitting a holomorphic action of this torus extending the action on itself. We will

mainly adopt the symplectic point of view, where we instead consider the action of the compact group $T^n = (S^1)^n$ and consider the space as a fixed symplectic manifold. In this section we will define these manifolds and recall their classification in terms of certain polytopes, called *Delzant polytopes*. The theory readily extends to orbifolds, and we make some remarks on this as well.

Definition 2.4. *A toric symplectic orbifold of dimension $2n$ is a symplectic $2n$ -dimensional orbifold (M, ω) with a Hamiltonian action of the n -torus T^n .*

So, we have a moment map $\mu : M \rightarrow (\mathfrak{t}^n)^*$. For a torus, the image of the moment map μ is the convex hull of the fixed points of the action, provided M is compact. For manifolds, this is called the Atiyah-Guillemin-Sternberg convexity theorem and it has also been extended to orbifolds by Lerman-Tolman.

Theorem 2.5 ([Ati82], [GS82a],[LT97]). *Let T^n act in a Hamiltonian fashion on a compact symplectic orbifold (M, ω) of dimension $2n$, with moment map μ . Then the image $\mu(M)$ is convex, and moreover is the convex hull of the images of the fixed points of the T^n -action. Also, the sets $\mu^{-1}(x)$ for $x \in \mu(M)$ are connected.*

Only a certain type of images appear. The following definitions will capture precisely the type of image occurring. Recall that a half-space H in a vector space V is a set of the form $\{x \in V : l(x) \geq 0\}$ for some affine function $l : V \rightarrow \mathbb{R}$. Its boundary ∂H is the set $\{x \in V : l(x) = 0\}$.

Definition 2.6. *A convex polytope Δ in a finite dimensional vector space V is a non-empty intersection $\cap_{i=1}^k H_i$ of finitely many half-spaces H_i . A face of Δ is a non-trivial intersection*

$$F = \Delta \cap \partial H$$

for some half-space H such that $\Delta \subseteq H$. If H is unique, then F is called a facet.

Let $\langle \cdot, \cdot \rangle$ denotes the contraction $V \times V^* \rightarrow \mathbb{R}$. Given a lattice Λ in V^* , a polytope Δ is called Delzant, with respect to this lattice, if it is bounded and can be represented as

$$\Delta = \bigcap_{i=1}^k \{x \in V : \langle x, u_i \rangle \geq c_i\}$$

where each $u_i \in \Lambda$ and each $c_i \in \mathbb{R}$, and moreover that each vertex is an intersection of exactly n facets F_i such that the u_i form a basis of the lattice over $\mathbb{Z}$, where n is the dimension of V .

A rational Delzant polytope is a polytope satisfying the above but where the u_i associated to each vertex form a basis of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$ only, i.e. they are linearly independent, but may not span the lattice over $\mathbb{Z}$. A labelled rational Delzant polytope is a rational Delzant polytope with a positive integer label associated to each of its facets. A rational Delzant polytope (without labels) may be identified with the labelled rational Delzant polytope where all the labels are 1.

Two rational labelled Delzant polytopes Δ and Δ' are isomorphic if there is a translation τ and an $\mathrm{SL}_n\mathbb{Z}$ -transformation U taking Δ to Δ' , such that the integer label attached to a facet F of Δ equals the integer label attached to the facet $(U \circ \tau)(F)$ of Δ' .

We make some remarks and mention some language we will use. We will call the u_i appearing in the definition of a facet F_i the *conormal* to F_i . This is not unique, but we can fix it as follows. An element u of the lattice Λ is called *primitive* if $\lambda u \in \Lambda$ for some $|\lambda| \leq 1$ implies that $\lambda = \pm 1$. So up to sign, there is a unique multiple u of u_i which is primitive. We can fix the sign of u by requiring that $\Delta \subseteq \{x : \langle x, u \rangle \geq c_i\}$. We then say that u is *inward-pointing*. We can then take u_i to be $m_i u$, where m_i is the integer label associated to the facet F_i . In this way we incorporate all the data of a labelled rational Delzant polytope in a unique representation of Δ as an intersection

$$\Delta = \bigcap_{i=1}^k \{x \in V : \langle x, u_i \rangle \geq c_i\}$$

where each $\{x \in V : \langle x, u_i \rangle = c_i\}$ is a facet, and each u_i is an integer multiple of an inward-pointing primitive lattice vector.

Remark 2.7. If a rational Delzant polytope is in fact Delzant as in the above definition, it could be tempting to call such a polytope an *integral Delzant polytope*. However, in the literature the term *integral Delzant polytope* usually refers to the fact that the polytope has vertices lying on the dual lattice to Λ , which, under identifying (V, Λ^*) with $\mathbb{Z}^n$ by picking a basis of Λ^* over $\mathbb{Z}$, means that the vertices lie on the integer lattice $\mathbb{Z}^n$.

The classification theorem for symplectic toric manifolds, respectively orbifolds, says that they are classified by Delzant polytopes, respectively labelled rational Delzant polytopes.

Theorem 2.8 ([Del88] and [LT97]). *Let (M, ω) be a symplectic toric manifold and let μ be a moment map for the torus action. Then the image $\mu(M)$ of M is a Delzant polytope in $(\mathfrak{t}^n)^*$ with respect to the integer lattice in $\mathfrak{t}^n = \mathbb{R}^n$. Isomorphic symplectic toric manifolds give isomorphic Delzant polytopes and moreover, for each Delzant polytope P , there exists a toric symplectic manifold (M_P, ω_P) with a moment map whose image is P .*

For orbifolds, the image is a rational Delzant polytope which moreover can be endowed with a natural labelling depending on the orbifold structure. Isomorphic symplectic toric orbifolds give isomorphic labelled rational Delzant polytopes and for each labelled rational Delzant polytope P , there exists a toric symplectic orbifold (Q_P, ω_P) with a moment map whose image is P , and the natural labelling of the facets of $\mu(Q_P)$ agree with that of P .

Note here that μ maps to $(\mathfrak{t}^n)^*$, so in terms of the definition 2.6, we have $V = (\mathfrak{t}^n)^*$, $V^* = \mathfrak{t}^n$ and $\Lambda = \mathbb{Z}^n = \ker(\exp : \mathfrak{t}^n \rightarrow T^n)$.

We will not go into much of the proof of the classification theorem. We only remark that whenever one has a Hamiltonian action, one can do a quotient construction, called *symplectic reduction*. For a discussion of this which includes the orbifold setting (the original theorem was for manifolds), see [ACdSL03, Thm. III.40]. All toric orbifolds are obtained as such a quotient from a torus T^d acting on $\mathbb{C}^d$, for some d . One then takes the quotient by a subtorus $N = T^{d-n}$ and is left with a quotient space Q_P on which an n -torus $T^n = T^d/N$ acts in a Hamiltonian fashion. This gives a construction of (Q_P, ω_P) , the toric orbifold associated to a polytope P , which we will consequently use to give a very useful characterisation of torus-invariant metrics on Q_P in terms of data on the polytope P alone.

To construct the orbifold above (ignoring the symplectic form), we could instead have started with a complex point of view, where we would have everything complexified. That is, we would work with the complexified groups $N_{\mathbb{C}} \cong T_{\mathbb{C}}^{d-n}, T_{\mathbb{C}}^d$ and $T_{\mathbb{C}}^n$ and taken a quotient $\mathbb{C}^d // N_{\mathbb{C}}$, the GIT (Geometric Invariant Theory) quotient. As a smooth manifold, these are diffeomorphic, but the symplectic quotient comes with a symplectic structure and the GIT quotient comes with a complex structure.

In fact, one can show more. One can do a quotient construction called the Kähler reduction, where one obtains a quotient where the complex and symplectic structures are compatible and so they define a Kähler structure ([GS82b, thm 3.5, p. 522]) on the resulting manifold. All of this has been extended to the orbifold case in [LT97, ch.9], where they call on generalisations of the above theorems to orbifolds to prove it.

Remark 2.9. *The free points of the $T_{\mathbb{C}}^n = T_{\mathbb{C}}^d/N_{\mathbb{C}}$ -action on $\mathbb{C}^d // N_{\mathbb{C}}$ are the image of the free points of the $T_{\mathbb{C}}^d$ -action on $\mathbb{C}^d$. This is precisely $(\mathbb{C}^*)^d$. The open set on which the $T_{\mathbb{C}}^n$ -action is free is biholomorphic to $(\mathbb{C}^*)^d/N_{\mathbb{C}} = T_{\mathbb{C}}^d/N_{\mathbb{C}} = T_{\mathbb{C}}^n$. This is the open set we mentioned in the introduction to this section.*

Remark 2.10. *The complex quotient $\mathbb{C}^d // N_{\mathbb{C}}$ does not depend on which moment map we chose for the action on the resulting smooth manifold. That is, it does not depend on translations of P . In fact, more is true. Different polytopes can give rise to the same manifold (the complex quotient only depends on the “fan” of P , which in the compact case is the arrangement of the conormals of P in the lattice). The significance is that the polytope contains more information than the complex picture, we have also specified a cohomology class $\Omega = [\omega] \in H^2(M, \mathbb{R})$. This cohomology class turns out to be integral if and only if, after a translation, the vertices of P lie on the lattice.*

Remark 2.11. *It is important to note that theorem 2.8 is a classification of the symplectic structure only. M admits a canonical Kähler structure, but it is by no means the unique one. Theorem 2.13 below will show precisely how one can parametrize all the compatible complex structures. These are all biholomorphic as complex manifolds (the complex quotient above is unique), but are not Kähler isomorphic, i.e. the biholomorphisms do not preserve the symplectic form, only its cohomology class.*

2.3 Kähler metrics on toric manifolds

The upshot of the previous section is that one can construct a toric manifold both as a symplectic quotient (M, ω) and a complex quotient N . In both instances, we get coordinate charts on the open subsets M^0 and N^0 where the T^n -action, respectively the $T_{\mathbb{C}}^n$ -action, is free. So we have a holomorphic chart $N^0 \rightarrow (\mathbb{C}^*)^n$ and a Darboux chart $M^0 \rightarrow \mu(M^0) \times T^n$. Since M can be constructed as a Kähler reduction, we have a canonical complex structure on M which is biholomorphic to N . We can then describe the complex structure in symplectic coordinates and, via this biholomorphism, a symplectic structure in the complex coordinates. For this section, we will follow [Abr01] and [Abr03], where all of this is proved.

Suppose we are in complex coordinates w^j on $(\mathbb{C}^*)^n$. We can lift to complex coordinates z^j on the universal cover $\mathbb{C}^n$. So we get real coordinates $(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n)$ with $z^j = \xi^j + i\eta^j$. Then a Kähler form on N is given in these local coordinates by the complex Hessian of some real-valued function f ,

$$\omega = 2i\partial\bar{\partial}f \quad (2.9)$$

$$= i \sum_{\alpha, \beta=1}^n 2 \frac{\partial^2 f}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \wedge d\bar{z}^\beta. \quad (2.10)$$

If ω is T^n -invariant, it follows that f only depends on the real coordinates ξ_i . We then have that the complex Hessian above is in fact, up to scale, the *real* Hessian of f .

From the symplectic point of view we have $(T^n$ -equivariant) Darboux coordinates and so Darboux coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ on the universal cover $P^\circ \times \mathbb{R}^n$ such that $\omega = \sum_i dx^i \wedge dy^i$ is standard. Here $P^\circ = \mu(M^\circ)$ is the interior of the polytope corresponding to M . Finding the expression of the moment map ϕ in complex coordinates gives a way to identify the two charts (or rather, their universal covers) by mapping ξ_i to $\phi(\xi_i)$ and η_j to y_j . The moment map $\phi : (\mathbb{C}^n)^* \rightarrow (\mathbb{R}^n)^*$ turns out to be given by

$$\phi(z) = \left(\frac{\partial f}{\partial \xi^1}, \dots, \frac{\partial f}{\partial \xi^n} \right).$$

The metric in symplectic coordinates is also determined by a function u called the *symplectic potential*. One could also re-obtain the f in 2.9 from u . The maps ϕ and $x \mapsto (\frac{\partial u}{\partial x^1}, \dots, \frac{\partial u}{\partial x^n})$ are *Legendre transforms*, and f and u are *Legendre dual* to each other, so they satisfy

$$f(\xi) + u(x) = \sum_k \frac{\partial f}{\partial \xi^k} \frac{\partial u}{\partial x^k} \quad (2.11)$$

evaluated at $x = (\frac{\partial f}{\partial \xi^1}, \dots, \frac{\partial f}{\partial \xi^n})$ or $\xi = (\frac{\partial u}{\partial x^1}, \dots, \frac{\partial u}{\partial x^n})$. See [Gui94b, Appendix 1] for details about the Legendre transform.

There is a canonical u_P , which corresponds to taking the biholomorphism one obtains when constructing M as a Kähler quotient. Let P be a labelled rational Delzant polytope in $(\mathfrak{t}^n)^* = (\mathbb{R}^n)^*$. Let $F_1, \dots, F_d$ be the facets of P . Each F_i lies in a hyperplane

$$\{x \in (\mathbb{R}^n)^* : \langle x, u_i \rangle = c_i\},$$

where u_i is conormal to F_i and $c_i \in \mathbb{R}$. From the fact that P is Delzant we get that we can take u_i to lie in the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$. As explained above, we can choose the u_i uniquely so that they are primitive and inward-pointing. Having fixed such a u_i , define affine functions l_i on $(\mathbb{R}^n)^*$ by

$$l_i(x) = \langle x, m_i u_i \rangle - c_i \quad (2.12)$$

where m_i is the label of the facet F_i and c_i is chosen such that F_i lies in $l_i^{-1}(0)$. Then the polytope P is exactly given as the elements of $(\mathbb{R}^n)^*$ which are non-negative on l_i for all i . The interior P° of P is the set of all points for which l_i is positive for all i .

We have the following theorem, due to Guillemin in the manifold case and extended by Abreu to orbifolds. We include in the statement that J is Kähler, which we recall was due to [GS82b] for manifolds and [LT97] for orbifolds.

Theorem 2.12 ([Gui94a], [Abr01]). *Let (M, ω) be a symplectic toric orbifold. Then M has a canonical complex structure J_P coming from the representative (M_P, ω_P) of (M, ω) associated with a moment polytope P . This is Kähler, and is given by*

$$J_P = \begin{pmatrix} 0 & -U_P^{-1} \\ U_P & 0 \end{pmatrix} \quad (2.13)$$

where U_P is the Hessian of u_P , and $u_P : P^\circ \rightarrow \mathbb{R}$ is the smooth function, continuous on P , given by

$$u_P(x) = \frac{1}{2} \sum_{i=1}^d l_i(x) \log(l_i(x)) \quad (2.14)$$

where the l_i are the affine linear functions defining P , as in equation 2.12.

In our discussion above, one can start with any Kähler metric on N_P in Ω_P , and one can describe all toric Kähler structures on a given symplectic orbifold in a similar manner. They are also described by a strictly convex function on the interior of the polytope, satisfying a suitable condition. From now on, we will say that a function f on a non-open set A is *smooth* if it is the restriction to A of a smooth function defined on an open set containing A .

Theorem 2.13 ([Abr01, p.155]). *Let (M_P, ω_P) be the toric symplectic orbifold associated to the polytope P and J a compatible complex structure. Then J is determined on the interior P by a smooth strictly convex function u on P , so that in symplectic coordinates, J is given as*

$$J = \begin{pmatrix} 0 & -U^{-1} \\ U & 0 \end{pmatrix} \quad (2.15)$$

where U is the Hessian of u . The function u can be written as

$$u = u_P + h,$$

where u_P is given by 2.14 and h is a smooth function on P .

The converse also holds. Given a function u satisfying the above, there is a compatible complex structure J on (M_P, ω_P) such that J is given by equation 2.15.

Remark 2.14. In [Abr01], this result is stated with the additional requirement that the determinant satisfies

$$\det(U) = (\delta \cdot \prod_{i=1}^d l_i)^{-1} \quad (2.16)$$

where $\delta : P \rightarrow \mathbb{R}$ is a smooth, positive function on the full polytope P . However, Donaldson showed that this in fact a property, see [Don05a, Prop. 2].

Remark 2.15. The way we view the correspondence is that for each strictly convex function u on P which is smooth on P° , the Legendre transform gives a map $\psi_u : P^\circ \times T^n \rightarrow (\mathbb{C}^*)^n$, which we can then think of as a map between the free orbits in the symplectic quotient M_P and the complex quotient N_P , respectively. The function u then satisfies the above conditions precisely when ψ_u extends as a diffeomorphism $M_P \rightarrow N_P$ taking $[\omega_P] \in H^2(M_P, \mathbb{R})$ to $\Omega_P \in H^2(N_P, \mathbb{R})$. In particular, a compatible complex structure is then given by pulling back the complex structure of N_P via ψ_u , and a symplectic structure on N_P is given by pulling back the symplectic form on M_P via $(\psi_u)^{-1}$.

Definition 2.16. Given a polytope P and a choice of orbifold weights $\underline{m} = (m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$, a function u satisfying the conditions of theorem 2.13 will be called a symplectic potential for the orbifold corresponding to P . We will often simply say a symplectic potential for P .

We now want to consider the extremal condition in this setting. We will follow [Abr98]. Some inspiration has also been drawn from [Ros12].

First note that if (M, ω, J) is a compact toric manifold, then T^n is certainly contained in a maximal compact subgroup of the automorphism group of (M, J) . In particular, by theorem 2.3, if any Kähler metric in $[\omega]$ is extremal, it is T^n -invariant. Therefore, in the toric setting, it suffices to look at torus-invariant metrics considered above to decide whether or not an extremal metric exists.

We saw that the condition for a metric to be extremal was given in terms of the gradient of the scalar curvature. We start by giving a formula for the scalar curvature $S(g)$ of g in symplectic coordinates, due to Abreu. Let $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$ and let u^{ij} be the entries of its inverse matrix. We will later adopt the notation u_{ij}^{ij} for $\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x^i \partial x^j}$.

Theorem 2.17 ([Abr98, Thm. 4.1.i]). *Let $u \in \mathcal{S}$ be a symplectic potential on a polytope P . Then the scalar curvature of the Kähler metric g_u induced by u on the toric symplectic manifold (M_P, ω_P) is given by*

$$S(g_u) = \left(-\frac{1}{2} \sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x^i \partial x^j} \right) \circ \mu$$

where μ is the moment map $M_P \rightarrow P$.

Having this formula for the scalar curvature, the extremal condition in the toric setting follows in a straightforward manner. We note that since u is a function of the x^i variables only, all the partial derivatives with respect to the y^j variables of $S(u) = -\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x^i \partial x^j}$ vanish automatically. Thus the condition that the gradient of S is holomorphic becomes that all second derivatives with respect to the x^i variables vanish. In other words, we have

Proposition 2.18 ([Abr98, Thm 4.1.ii]). *Let $u \in \mathcal{S}$ be a symplectic potential on a polytope P . Then the Kähler metric induced by u on (M_P, ω_P) is extremal if and only if*

$$S(u) = -\frac{1}{2} \sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x^i \partial x^j}$$

is an affine linear function.

Invariant metrics with cone angle singularities along the torus-invariant divisors are as easy to describe as in the orbifold case. Below we will give a definition of the space of symplectic potentials that correspond to metrics with prescribed cone angle singularities along the torus-invariant divisors. This definition works equally well for any bounded convex polytope. Although these do not correspond to manifolds or orbifolds, it is natural to define such a space in this generality, and indeed allowing any such polytope features in Donaldson's continuity method for extremal metric on toric surfaces in [Don08]. There is also some geometric meaning for such polytopes, as they arise for toric Sasakian manifolds with irregular Reeb vector fields, see e.g. [MSY06] and [Abr10].

It will be convenient for us to encode the data of the boundary behaviour in a measure along the boundary. Let $d\lambda$ be the Lebesgue measure. We say a measure $d\sigma$ on the boundary of a bounded convex polytope P is a *positive boundary measure for P* if on the i^{th} facet F_i of P , $d\sigma$ satisfies

$$l_i \wedge d\sigma = \pm r_i d\lambda \tag{2.17}$$

where l_i is an affine function defining F_i and $r_i > 0$ is a constant. We say $d\sigma$ is *non-negative* if we relax the condition to $r_i \geq 0$, only. If $d\sigma$ is a non-negative boundary measure on P , we call the pair $(P, d\sigma)$ a *weighted polytope*.

Given a positive boundary measure there is a unique l_i such that $P \subseteq l_i^{-1}([0, +\infty))$ for all i , and that 2.17 is satisfied with $r_i = 1$. We call the collection $l_1, \dots, l_d$ the *canonical defining functions* of $(P, d\sigma)$. For any bounded convex polytope P , we then define a space of symplectic potentials.

Definition 2.19. *Let P be a bounded convex polytope and let $d\sigma$ be a positive boundary measure for P . Let l_i be the canonical defining functions for $(P, d\sigma)$. We define the space of symplectic potentials $\mathcal{S}_{P, d\sigma}$ to be the space of strictly convex functions $u \in C^\infty(P^\circ) \cap C^0(P)$ satisfying*

$$u = \frac{1}{2} \sum_i l_i \log l_i + h$$

, for some $h \in C^\infty(P)$.

In the case when P is Delzant, $\mathcal{S}_{P,d\sigma}$ then precisely describes metrics with cone angle singularities along the torus invariant divisors, the cone angle being prescribed by $d\sigma$.

Proposition 2.20 ([DGSW13, Prop. 2.1]). *Let P be a Delzant polytope with canonical measure $d\sigma^0$. Let $d\sigma$ be a positive boundary measure for P , so on each facet F_i of P , $d\sigma$ satisfies*

$$d\sigma|_{F_i} = r_i d\sigma^0|_{F_i}$$

for some $r_i > 0$. Then through the Legendre transform, symplectic potentials $u \in \mathcal{S}_{P,d\sigma}$ induce metrics with cone singularities along the torus invariant divisors D_i corresponding to the facets F_i . The cone angle singularity along D_i is $2\pi r_i$.

Remark 2.21. *The determinant of the Hessian of a symplectic potential satisfying 2.19 also satisfies 2.16.*

2.4 The stability condition

In this section we will define K -(semi)stability and in particular consider it for toric manifolds. The definition then makes sense for any weighted polytope, and we state the Yau-Tian-Donaldson conjecture in this setting. We prove some elementary properties of the set of stable weights for a polytope, as well as a lemma on parameterising certain toric test configurations that we will use several times in the thesis.

We begin with the stability notion in general. The definition of K -semistability, due to Donaldson in [Don02], involves associating a numerical invariant to certain degenerations of X and requiring this invariant to always be non-negative.

A *test configuration* $(\mathcal{X}, \mathcal{L})$ for a polarised variety (X, L) consists of the following data:

- A scheme $\mathcal{X}$ with a $\mathbb{C}^*$ -action,
- A $\mathbb{C}^*$ -equivariant holomorphic line bundle $\mathcal{L} \rightarrow \mathcal{X}$,
- A flat map $\pi : \mathcal{X} \rightarrow \mathbb{C}$ which is equivariant with respect to the $\mathbb{C}^*$ -action on $\mathcal{X}$ and the $\mathbb{C}^*$ -action on $\mathbb{C}$ by multiplication.

Moreover, one requires that for all non-zero t , $\mathcal{X}_t = \pi^{-1}(t)$ is isomorphic to X and $(\mathcal{X}_t, \mathcal{L}|_{\mathcal{X}_t})$ is isomorphic to (X, L^r) for some $r \geq 1$. r is called the exponent of the test configuration. The pair $(\mathcal{X}_0, \mathcal{L}_0)$ is called the *central fibre* of the test configuration.

The numerical invariant is the Donaldson-Futaki invariant, and is defined as follows (see [RT07, p. 210] for why this is equivalent to the original definition). The $\mathbb{C}^*$ -action on $(\mathcal{X}, \mathcal{L})$ fixes the central fibre $(\mathcal{X}_0, \mathcal{L}_0)$. Hence it induces an action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$ for all k . Moreover, when k is sufficiently large, the dimension $d(k)$ of $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$ and the weight $w(k)$ of the $\mathbb{C}^*$ -action are both polynomials in k , of degree $n = \dim(X)$

and $n + 1$, respectively. Thus we have expansions

$$\begin{aligned} d(k) &= a_0 k^n + a_1 k^{n-1} + O(k^{n-2}), \\ w(k) &= b_0 k^{n+1} + b_1 k^n + O(k^{n-1}). \end{aligned}$$

The *Donaldson-Futaki invariant* of $(\mathcal{X}, \mathcal{L})$ is given by

$$DF(\mathcal{X}, \mathcal{L}) = \frac{b_0 a_1 - b_1 a_0}{a_0}.$$

Definition 2.22. A polarised variety (X, L) is K -semistable if $DF(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$ for (X, L) .

K -(poly)stability says additionally that only certain trivial test configurations have vanishing Donaldson-Futaki invariant. However, here one must be careful about what exactly is meant by a trivial test configuration. This will not be too important for what we are considering, and so we will not go into this further. The Yau-Tian-Donaldson conjecture states that there is a cscK metric in $c_1(L)$ if and only if (X, L) is K -polystable. For the conjecture in the toric setting we use a condition we can express in terms of the polytope, which has been proven to be equivalent in the case of vanishing Futaki invariant in [CS16].

We first give the definition. This works for any weighted polytope, and we will make the definition in this context. We will then remark on its relation to K -stability and related stability notions when the polytope is Delzant.

Recall that we defined a *weighted polytope* $(P, d\sigma)$ to be a bounded convex polytope P together with a non-negative boundary measure $d\sigma$ for P , i.e. a measure on the boundary of P satisfying 2.17 for some constants $r_i \geq 0$. Note that for any P , given a fixed positive boundary measure $d\sigma^0$ on ∂P , any other $d\sigma$ satisfies

$$d\sigma|_{F_i} = r_i d\sigma^0,$$

for some $r_i \in \mathbb{R}_{\geq 0}$, where F_i is the i^{th} facet of P . In this way, once a positive background measure is fixed, we can identify any other non-negative boundary measure with a d -tuple $\underline{r} = (r_1, \dots, r_d) \in \mathbb{R}_{\geq 0}^d$. For a rational Delzant polytope there is a canonical $d\sigma^0$, but for general polytopes choosing a background measure is equivalent to choosing a particular scaling of the affine linear functions defining the polytope.

Let A be a bounded function on P . Also let $d\lambda$ be the Lebesgue measure on P . One can then define a functional $\mathcal{L}_A$ on the space of continuous convex functions on P by

$$\mathcal{L}_A(f) = \int_{\partial P} f d\sigma - \int_P A f d\lambda. \quad (2.18)$$

Note that there is a unique affine linear A such that $\mathcal{L}_A(f) = 0$ for all affine linear f .

Definition 2.23. Given a weighted polytope $(P, d\sigma)$, we call the affine linear function A such that $\mathcal{L}_A$ vanishes on all affine linear functions the affine linear function associated to the weighted polytope $(P, d\sigma)$. Also, we write $\mathcal{L} = \mathcal{L}_A$.

We say a function f on P is *piecewise linear* if it is the maximum of a finite number of affine linear functions. We say it is *rational* if the coefficients of the affine linear functions are all rational, up to multiplication by a common constant.

Definition 2.24. Let $(P, d\sigma)$ be a weighted polytope. We say P is *weighted polytope stable*, or more briefly *stable*, if

$$\mathcal{L}(f) \geq 0 \tag{2.19}$$

for all piecewise linear functions f , with equality if and only if f is affine linear. If $(P, d\sigma)$ is not stable, we say it is *unstable*. If 2.19 holds for all piecewise linear f , but there is a non-affine function f with $\mathcal{L}(f) = 0$, we say $(P, d\sigma)$ is *strictly semistable*. We say $(P, d\sigma)$ is *semistable* if it is either stable or strictly semistable.

Remark 2.25. One gets different notions of stability depending on what class of functions one tests on. For example, one could ask for 2.19 to hold for all smooth convex functions, or even all continuous convex functions. Though the latter will come up due to its relation with $\hat{K}$ -stability, the central notion for us will be testing on piecewise linear functions.

Definition 2.26. Let (N, Ω) be a complex toric manifold with a Kähler class Ω whose moment polytope is P . We say (N, Ω) is *analytically relatively K -stable* with respect to toric degenerations if P is a stable polytope with respect to its canonical boundary measure. We say (N, Ω) is *algebraically relatively K -stable* with respect to toric degenerations if the stability condition holds for all rational piecewise linear functions only. If the stability condition holds for all continuous convex functions, we say (N, Ω) is *relatively $\hat{K}$ -stable* with respect to toric degenerations.

If P is integral, so that Ω is $c_1(L)$ for some line bundle L , and f is piecewise linear and rational, then Donaldson showed in [Don02] that we can construct a test configuration $(\mathcal{X}, \mathcal{L})$ from f and $\mathcal{L}_a(f)$ is a constant multiple of the Donaldson-Futaki invariant for this test configuration, where a is the average scalar curvature of Kähler metrics in $c_1(L)$. Codogni and Stoppa have recently in [CS16] proved a result that implies that algebraic K -stability with respect to toric degenerations and K -stability are equivalent, i.e. on toric manifolds their results says that it suffices to test K -stability on the test configurations arising from rational piecewise linear functions via Donaldson's construction.

$\hat{K}$ -stability is a generalisation of K -stability, see [Szé15b] and [WN12]. Here one shows that test configurations give rise to a *filtration* of the homogeneous coordinate ring of a polarised complex manifold. Moreover, it turns out that filtrations coming from test configurations via this construction are given precisely by filtrations with finitely generated *Rees algebra*. In $\hat{K}$ -stability one associates to each filtration, with finitely generated Rees algebra or not, a Donaldson-Futaki invariant and requires this to always be positive, except for trivial filtrations. Székelyhidi showed in [Szé15b] that for a polarised toric variety, all continuous convex functions give rise to filtrations, and that of these it is precisely the rational piecewise linear functions that give test configurations.

Relative K -stability is a version of stability for when the classical Futaki invariant does not vanish, developed by Székelyhidi in [Szé06]. This involves checking stability on test configurations which are compatible with a maximal torus T and which also are orthogonal to T via a certain inner product. When one only has a test configuration which is compatible with T one can obtain one which is orthogonal, see [Szé14, p.116]. In the toric setting, all rational piecewise linear functions give test configurations which are compatible with the maximal torus. Székelyhidi showed in [Szé14, Prop. 6.25] that the Donaldson-Futaki invariant of the resulting test configuration orthogonal to the maximal torus is given by $\mathcal{L}_A(f)$, where A is the affine linear function associated to P . There is also the notion of relative $\hat{K}$ -stability where one tests on compatible filtrations that are orthogonal to the maximal torus. The same result holds here, i.e. the Donaldson-Futaki invariant of the orthogonal filtration induced by a convex function $f : P \rightarrow \mathbb{R}$ is $\mathcal{L}_A(f)$.

We can now state the Yau-Tian-Donaldson conjecture in the toric setting. We will phrase this in terms of arbitrary bounded weighted polytopes. We say a symplectic potential $u \in \mathcal{S}_{P,d\sigma}$ is an *extremal potential* if u_{ij}^{ij} is an affine function. When P is Delzant, u then induces a Kähler metric and this is the condition for that metric to be extremal. A version of the Yau-Tian-Donaldson conjecture for polytopes then becomes:

Conjecture 2.27 ([Don02, Conj. 7.2.2]). *Let P be a bounded convex polytope with positive boundary measure $d\sigma$. Then there exists an extremal potential $u \in \mathcal{S}_{P,d\sigma}$ if and only if $(P, d\sigma)$ is a stable weighted polytope.*

For Delzant polytopes with integral vertices, so $\Omega_P = c_1(L_P)$ is an integral cohomology class, and with the canonical boundary measure, this becomes that X_P admits an extremal Kähler metric in Ω_P if and only if (X_P, L_P) is relatively $\hat{K}$ -stable with respect to degenerations arising from convex functions on P . This is the reason for the usage of the term *with respect to toric degenerations* in definition 2.26 above.

For surfaces, this conjecture has been proved when the associated affine linear function is constant in [Don09], and in the general extremal case when P is Delzant and $d\sigma$ is the standard boundary measure in [CLS12].

A natural question one could ask is given a polytope P , how does stability depend on the weight $d\sigma$? By specifying a positive background measure $d\sigma^0$, we identify the set of weights with $\mathbb{R}_{\geq 0}^d \setminus \{0\}$. We now give two elementary lemmas about the set of stable weights.

Lemma 2.28. *Let $(P, d\sigma^0)$ be a polytope with d facets $F_1, \dots, F_d$, and with $d\sigma^0$ an everywhere positive measure on the boundary ∂P of P as above. Then the set of weights $\underline{r} = (r_1, \dots, r_d) \in \mathbb{R}_{\geq 0}^d$ such that $(P, d\sigma_{\underline{r}})$ is stable is a convex subset of $\mathbb{R}_{\geq 0}^d$.*

Proof. Let $\underline{r}_0, \underline{r}_1$ be stable weights, and set $\underline{r}_t = (1-t)\underline{r}_0 + t\underline{r}_1$. Let A_t be the affine function associated to the weighted polytope $(P, d\sigma_{\underline{r}_t})$. Then $A_t = (1-t)A_0 + tA_1$, and so, for all convex functions f on P , we have

$$\begin{aligned} \mathcal{L}_{\underline{r}_t}(f) &= (1-t)\mathcal{L}_{\underline{r}_0}(f) + t\mathcal{L}_{\underline{r}_1}(f) \\ &\geq 0. \end{aligned}$$

Moreover, since $\mathcal{L}_{\underline{r}_t}(f) \geq 0$ with equality if and only if f is affine for $t = 0, 1$, it follows that this holds for all $t \in [0, 1]$ too. Hence $\underline{r}_t$ is a stable weight for all $t \in [0, 1]$, as required. $\square$

Lemma 2.29. *Let $\underline{r}$ be a stable weight for $(P, d\sigma^0)$. Then $c \cdot \underline{r}$ is a stable weight for all $c > 0$.*

Proof. $A_{c \cdot \underline{r}} = cA_{\underline{r}}$, and so $\mathcal{L}_{c \cdot \underline{r}} = c \cdot \mathcal{L}_{\underline{r}}$. The lemma follows immediately from this. $\square$

The stable set in $\mathbb{R}^d$ is thus a convex cone on the stable weights with $\sum_i r_i = 1$, and so to fully describe the stable set one can without loss of generality consider weights such that $\sum_i r_i = 1$. In chapter 4 we investigate the dependence of stability on the weights in the particular case of quadrilaterals.

Another question the definition of stability on polytopes raises is the following. If we have a function violating stability, can we assume it is in a certain nice class of functions? For example, if we knew $\mathcal{L}(f) = 0$ for some f which is rational piecewise linear, we could construct a toric degeneration and so we could realize this as an algebro-geometric degeneration violating relative K -stability in the usual sense. Moreover, if we only needed to check stability on a finite-dimensional space of test configurations it would make it easier to prove that a polytope is stable in given examples.

In complex dimension 2, there is a reduction to a finite-dimensional class of piecewise linear test functions. Moreover, in the case of vanishing Futaki invariant these can be taken to be rational. This will be important for the computational work in chapter 5. Specifically, the result is the following.

Let $\text{SPL}(P)$ denote the space of simple piecewise linear functions on $P \subseteq (\mathbb{R}^2)^*$, that is functions f of the form $x \mapsto \max\{0, h(x)\}$ for an affine linear function $h : (\mathbb{R}^2)^* \rightarrow \mathbb{R}$. We say f is rational if we can write it in the form $\lambda \cdot \max\{0, h(x)\}$, where $\lambda \in \mathbb{R}$ and h has rational coefficients. This is equivalent to the *crease* of f , i.e. the points where h is zero, being a line with rational slope in $(\mathbb{R}^2)^*$. Note also that we have a map

$$\text{SPL} : \text{Aff}(\mathbb{R}^2) \rightarrow \text{SPL}(P)$$

given by

$$h \mapsto \max\{0, h(x)\}.$$

Let C_1 denote the space of convex functions defined on union of the interior of P and the interior of its facets, such that $\int_{\partial P} f d\sigma < \infty$. In [Don02], Donaldson proved the following, assuming $A > 0$. It was extended by Wang-Zhou in [WZ11] to all A .

Theorem 2.30 ([Don02, Prop. 5.3.1], [WZ11, Thm. 4.1]). *Let (M, ω) be a toric symplectic 4-manifold with moment polytope P . If $\mathcal{L}(f) = 0$ for some convex function $f \in C_1$ which is not affine linear, then there exists a $g \in \text{SPL}(P)$ with crease in the interior of P such that $\mathcal{L}(g) = 0$.*

$\text{SPL}(P)$ is a 3-dimensional family of functions. Moreover, given $h \in \text{Aff}(\mathbb{R}^2)$ and $c \in \mathbb{R} \setminus \{0\}$,

$$\mathcal{L}(\text{SPL}(h)) > 0$$

if and only if

$$\mathcal{L}(\text{SPL}(c \cdot h)) > 0.$$

Thus for surfaces, it suffices to check stability on a 2-dimensional family of functions, a significant reduction from the infinite-dimensional space of functions one a-priori has to test stability on. Moreover, in the cscK case, Donaldson also went on to show

Proposition 2.31 ([Don02, Prop. 6.1]). *Let P have rational vertices and assume $A = a$ is constant. If there is a $g_0 \in \text{SPL}(P)$ such that its crease meets the interior of P , with $\mathcal{L}_a(g_0) \leq 0$, then there exists a rational simple piecewise linear function whose crease meets the interior of P with the same property.*

Thus we can actually realize the instability algebro-geometrically, i.e. if (M, ω) is not analytically K -stable with respect to toric degenerations, it is not K -stable in the usual sense.

In fact the picture is quite complete in this dimension. The toric stability criterion has been shown to be equivalent to the existence of an extremal metric for Delzant polytopes. The proof of stability assuming existence can be found in [ZZ08a] where they show that in any dimension, stability holds when one tests on functions which are given by piecewise linear functions. The existence of an extremal metric assuming stability is shown in [Don09] in the constant scalar curvature case, and [CLS12] for the general extremal case. The proof in the constant scalar curvature case actually holds for orbifolds too.

We end this section with a lemma that will be used several times in this thesis. Let P be a 2-dimensional polytope and fix two edges E_1 and E_2 of P with vertices v_1, w_1 and v_2, w_2 , respectively. Any point p on E_1 , respectively q on E_2 , can then be written as

$$\begin{aligned} p &= (1-s)v_1 + sw_1, \\ q &= (1-t)v_2 + tw_2 \end{aligned}$$

for some $s, t \in [0, 1]$. Let p_i , respectively q_i , be the i^{th} component of p , respectively q . This determines an affine linear function $l_{s,t}$ which vanishes on p, q and for which the coefficients for the non-constant terms are linear in s and t . Specifically, writing $l_{s,t} = ax + by + c$, let

$$\begin{aligned} a &= q_2 - p_2, \\ b &= p_1 - q_1, \\ c &= -ap_1 - bp_2. \end{aligned}$$

We then have

Lemma 2.32. *Let $\phi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be given by*

$$(s, t) \mapsto \mathcal{L}(\text{SPL}(l_{s,t})).$$

Then ϕ is a polynomial in (s, t) of bidegree $(3, 3)$ and total degree 5.

For the proof the integrals one has to perform, say the ones over the polytope, can be decomposed as the integral of $Al_{s,t}$ over some fixed region R , where this statement holds, and a quadrilateral region $Q_{s,t}$ bounded by $E_1, E_2, l_{0,0}^{-1}(0)$ and $l_{s,t}^{-1}(0)$. Direct computation, which we omit, then shows that this holds. Similarly for the boundary region.

Note that each simple piecewise linear function f with crease going meeting E_1 and E_2 can be rescaled so that its Donaldson-Futaki invariant $\mathcal{L}(f)$ is $\phi(s, t)$ for some (s, t) . It therefore suffices to check the positivity of ϕ on all points in $[0, 1] \times [0, 1]$ not corresponding to affine linear functions to conclude whether or not there is a simple piecewise linear function violating stability with crease meeting the boundary of P in E_1 and E_2 . Thus to check the stability of a 2-dimensional polytope P , one associates to each pair of edges of P the polynomial given by lemma 2.32 and then checks whether or not this has any non-trivial zeros.

2.5 Poincaré type metrics on the complement of a divisor

In this section we shall discuss cusp or Poincaré type metrics. These are modelled on a standard cusp metric that we describe first.

Consider the punctured unit (open) disk $B_1^* \subseteq \mathbb{C}$ with the metric

$$\frac{|dz|^2}{(|z| \log |z|)^2}. \quad (2.20)$$

Here we use the notation $|dz|^2 = dx^2 + dy^2$, where $z = x + iy$. This is the standard cusp metric on B_1^* . A computation shows that the associated symplectic form then is

$$\frac{idz \wedge d\bar{z}}{|z|^2 \log^2(|z|)} = 4i\partial\bar{\partial}(\log(-\log(|z|^2))). \quad (2.21)$$

We want to see why we call this a cusp metric. Using real coordinates r, θ , we see that the corresponding metric equals

$$\frac{dr^2 + r^2 d\theta^2}{r^2 (\log(r))^2}.$$

Putting $\varkappa = -\frac{1}{\log r}$, we can make a coordinate transformation to $(0, \infty) \times S^1$. In these coordinates, we have

$$\frac{|dz|^2}{(|z| \log |z|)^2} = \frac{1}{\varkappa^2} ds^2 + \varkappa^2 d\theta^2.$$

Putting $t = -\log \varkappa = \log(-\log(r))$, we then get

$$\frac{|dz|^2}{(|z| \log |z|)^2} = dt^2 + e^{-2t} d\theta^2.$$

Note that these two coordinate transformations are not holomorphic, so this only shows metrically how we can picture 2.20.

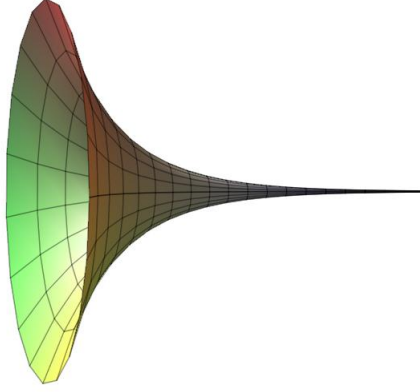


Figure 2.1: The standard cusp metric.

This metric can be obtained as follows. Consider the hyperbolic plane $\mathcal{H} = \{w \in \mathbb{C} : \text{Im}(w) > 0\}$. This has a metric of constant scalar curvature -1 given by $\frac{1}{v^2}(du^2 + dv^2) = \frac{|dw|^2}{(\text{Im}(w))^2}$, where $w = u + iv$. The group $\mathbb{Z}$ acts on $\mathcal{H}$ by translation in the real direction

$$w \mapsto w + n,$$

and this is an isometric action with respect to this metric. We get a quotient Riemann surface $\mathcal{H}/\mathbb{Z}$, which is diffeomorphic to a cylinder $S^1 \times \mathbb{R}_{>0}$, with a Kähler metric of constant curvature -1 . To see this is the same metric as above we must give a biholomorphism $\mathcal{H}/\mathbb{Z} \rightarrow B_1^*$. This can be seen by thinking of $\mathbb{C}$ as the universal cover of $\mathbb{C}^*$ via the map $z \mapsto e^{2\pi z}$. The pre-image of B_1^* via this map are the z with negative real part, which is simply $i\mathcal{H}$. The upshot is that the map $\mathcal{H} \rightarrow B_1^*$ given by

$$w \mapsto e^{2\pi iw}$$

is a holomorphic covering map which is invariant under the $\mathbb{Z}$ action on $\mathcal{H}$, and so induces a biholomorphism

$$\mathcal{H}/\mathbb{Z} \rightarrow B_1^*.$$

If we let z denote the coordinate on B_1^* , so that

$$z = e^{2\pi iw}$$

we get that

$$\begin{aligned} dz &= 2\pi izdw, \\ d\bar{z} &= -2\pi i\bar{z}d\bar{w}, \end{aligned}$$

so that

$$|dz|^2 = 4\pi^2 |z|^2 |dw|^2.$$

Now note that

$$\begin{aligned} \log(|z|^2) &= \log(e^{2\pi i(w-\bar{w})}) \\ &= 4\pi \operatorname{Im}(w). \end{aligned}$$

Putting all this together we get that

$$\begin{aligned} \frac{1}{(\operatorname{Im}(w))^2} |dw|^2 &= \frac{(4\pi)^2}{(\log(|z|^2))^2} \frac{|dz|^2}{4\pi^2 |z|^2} \\ &= \frac{4|dz|^2}{|z|^2 (\log(|z|^2))^2} \\ &= \frac{|dz|^2}{(|z| \log(|z|))^2}, \end{aligned}$$

as claimed.

We now make a general definition of a metric with such singularities along a simple normal crossings divisor D in a compact complex manifold X , due to Auvray. That D is simple normal crossings means that we can write $D = \sum_k D_k$, where each D_k is smooth, and the D_k intersect transversely in the sense that for each choice $k_1, \dots, k_l$ of distinct indices, we can around each point in $D_{k_1} \cap \dots \cap D_{k_l}$ find a holomorphic chart $(U, z_1, \dots, z_n)$ such that $D_{k_j} \cap U = \{z_j = 0\} \cap U$. Note that in particular l is at most the dimension of X . For the toric case later, the torus-invariant divisors appearing in our discussion will be simple normal crossings divisors. Note that on each such chart U , we have a standard locally defined cusp metric whose associated 2-form is given by

$$\omega_{cusp} = \sum_{j=1}^l \frac{idz_j \wedge d\bar{z}_j}{(|z_j| \log |z_j|)^2} + \sum_{j>l} idz_j \wedge d\bar{z}_j.$$

Given such a divisor, one can for each k define a model function f_k , which when patched together gives the model Kähler potential for a Poincaré type metric. More precisely, fix a holomorphic section σ_k of $\mathcal{O}(D_k)$ such that D_k is the zero set of σ_k . Also fix a Hermitian metric $|\cdot|_k$ on $\mathcal{O}(D_k)$, which we assume satisfies $|\sigma_k|_k \leq e^{-1}$. Thus, for each λ sufficiently large, the function $f_k = \log(\lambda - \log(|\sigma_k|_k^2))$ is defined on $X \setminus D_k$.

Let ω_0 be a Kähler metric on the whole of the compact manifold X . By the above we can, for sufficiently large λ , pick $A_k > 0$ such that if $f = \sum_k A_k f_k$, then $\omega_f = \omega_0 - i\partial\bar{\partial}f$ is a positive $(1,1)$ -form on $X \setminus D$. Poincaré type metrics are then metrics on $X \setminus D$ defined by a potential with similar asymptotics to f near D .

Definition 2.33 ([Auv14c, Def. 0.1], [Auv13, Def. 1.1]). *Let X be a compact complex manifold and let D be a simple normal crossings divisor in X . Let ω_0 be a Kähler metric on X in a class $\Omega \in H^2(X, \mathbb{R})$. A smooth, closed, real $(1,1)$ form on $X \setminus D$ is a Poincaré type Kähler metric if*

- ω is quasi-isometric to ω_{cusp} . That is, for every chart U as above, and every compact subset K of $B_{\frac{1}{2}} \cap U$, there exists a C such that throughout K , we have

$$C\omega_{cusp} \leq \omega \leq C^{-1}\omega_{cusp}.$$

Moreover, the class of ω is Ω if

- $\omega = \omega_0 + i\partial\bar{\partial}\varphi$ for a smooth function φ on $X \setminus D$ with $|\nabla_{\omega_f}^j \varphi|$ bounded for all $j \geq 1$ and $\varphi = O(f)$.

Remark 2.34. *The quasi-isometric property as stated here is equivalent to being quasi-isometric to ω_{cusp} in just a collection of charts covering D , or being quasi-isometric to the model ω_f above. The reason we do not explicitly mention f for this property (which is done in Auvray's definition), is because when we show the quasi-isometric property for some toric metrics later, it is more convenient to compare to the local metric in charts.*

2.6 The unstable case

One can ask what happens when a polarised manifold is relatively K -unstable and so does not admit an extremal metric. In this section we will discuss possible phenomena and this will also serve as motivation for studying the Poincaré type metrics of the previous section. We will focus on the case of surfaces. Many of the ideas come from the toric case, but we also give some indication on how this generalises to the non-toric case.

Consider the functional

$$\omega \mapsto \|S(\omega) - \hat{S}\|_{L^2(X)}^2,$$

for Kähler metrics ω in a given Kähler class Ω , where $\hat{S}$ is the average scalar curvature of Kähler metrics in Ω . This differs from the Calabi functional only by terms depending on the Kähler class Ω and $c_1(X)$.

One can introduce a norm $\|\cdot\|$ on test configurations, and Donaldson conjectured in [Don05b] that

$$\inf_{\omega \in \Omega} \|S(\omega) - \hat{S}\|_{L^2(X)} = \sup_{\{(\mathcal{X}, \mathcal{L}) : \|\mathcal{X}\| \neq 0\}} \left(-\frac{DF(\mathcal{X}, \mathcal{L})}{\|(\mathcal{X}, \mathcal{L})\|} \right). \quad (2.22)$$

This can be viewed as two inequalities, and he showed that

$$\inf_{\omega \in \Omega} \|S(\omega) - \hat{S}\|_{L^2(X)} \geq \sup_{\{(\mathcal{X}, \mathcal{L}) : \|\mathcal{X}\| \neq 0\}} \left(-\frac{DF(\mathcal{X}, \mathcal{L})}{\|(\mathcal{X}, \mathcal{L})\|} \right),$$

which in particular implied that if X admits a cscK metric in Ω , then (X, Ω) is K -semistable.

In [Szé06], Székelyhidi showed a version of this for the relative setting. He showed that for any non-trivial test configuration χ with $DF(\chi) < 0$ which is orthogonal to a maximal torus, and for any metric $\omega \in \Omega$, one has

$$\|S(\omega) - \hat{S}\|_{L^2(X)}^2 \geq \frac{DF^2(\mathcal{X}, \mathcal{L})}{\|(\mathcal{X}, \mathcal{L})\|^2} + \|\mathfrak{X}\|_{L^2}^2,$$

where $\mathfrak{X}$ is the extremal vector field. Since $\|S(\omega) - \hat{S}\|_{L^2(X)} = \|\mathfrak{X}\|_{L^2(X)}$ if ω is an extremal metric, this implied relative K -semistability whenever an extremal metric exists.

In the toric case, one can restrict to T^n -invariant metrics. A version of the conjecture in the non-relative setting then becomes

$$\inf_{u \in \mathcal{S}} \|S(u) - \hat{S}\|_{L^2} = \sup_{f \in \mathcal{C}_1 \cap L^2} \left(\frac{-\mathcal{L}_a(f)}{\|f\|_{L^2}} \right), \quad (2.23)$$

where $\mathcal{C}_1$ is the space of smooth convex functions defined on the union of the interior P° of P with the interior of its facets, such that $f \in L^1(\partial P)$. See [Szé08, Conj. 1].

If P is a polytope, then given a finite collection $f_1, \dots, f_k$ of affine linear functions, we can decompose P into subpolytopes $Q_1, \dots, Q_r$ such that the function $\max\{f_1, \dots, f_k\}$ on P is smooth precisely on the interior of the Q_i 's. We can endow the Q_i with a boundary measure $d\sigma_i$ by letting $d\sigma_i$ be $d\sigma$ on the facets of Q_i that lie in facets of P and letting $d\sigma_i$ be 0 on the remaining facets of Q_i . Donaldson conjectured (see [Don02, Conj. 7.2.3] and the following discussion) that if P is an unstable convex polytope with respect to a positive boundary measure $d\sigma$, then there should be such a decomposition of P into such subpolytopes $Q_1, \dots, Q_k$ such that each Q_i is semistable with respect to the boundary measure $d\sigma_i$ induced by $d\sigma$.

He conjectured that each of the semistable subpolytopes comes in one of two types. Either it is stable and admits an extremal potential inducing a complete extremal metric on the complement of the divisor corresponding to the facets with vanishing boundary measure, or it is a parallelogram which is strictly semistable. In the latter case this corresponds to collapsing behaviour under a minimising procedure for the Calabi functional.

Donaldson also gave in [Don09, Sect. 6.2] a model for the possible behaviour of the extremal metrics obtained in the first case, and these are metrics of Poincaré type, provided the edge is not an edge meeting a vertex of the original polytope P . In this case he conjectures that one should obtain more complicated behaviour involving metrics related to the Taub-NUT metric as blow-up limits near the vertex.

In [Szé08, Thm. 5], Székelyhidi showed if P is unstable, then there is always a convex function $\Phi \in \mathcal{C}_1 \cap L^2(P)$, unique up to scale, that attains the supremum on the right-hand side of 2.23. This is called the

optimal destabiliser of P . He also showed that if the optimal destabiliser Φ actually is piecewise linear, then the decomposition of P into subpolytopes given by the regions where Φ is affine linear gives the decomposition required in Donaldson's conjecture.

Assume now that the optimal destabiliser Φ is in fact simple piecewise linear, i.e. has only one crease. We then have a decomposition of P into two subpolytopes, Q_1 and Q_2 , say. Provided these are rational, that means we get a test configuration for N_P with central fibre X_0 that has two irreducible components N_1, N_2 with $N_i = N_{Q_i}$, and these irreducible components are glued along the divisor corresponding to the crease of Φ .

In the non-toric case there is a construction of test configurations where the central fibre has two irreducible components. This is the case of *deformation to the normal cone* and the definition of slope stability, see [RT06] and [RT07]. Here one considers the test configuration for (X, L) obtained from blowing up $Z \times \{0\} \subseteq X \times \mathbb{C}$ for a subscheme $Z \subseteq X$. One also needs to specify a line bundle on the total space. Let π denote the composition of the blow-down map with the projection to X , and let E be the exceptional divisor of this blow-up. We are then considering the line bundle $\pi^*L - cE$, for $c \in (0, \epsilon(Z)) \cap \mathbb{Q}$ and $\epsilon(Z)$ is the *Seshadri constant* of Z .

Supposing Z is a divisor in X , the central fibre is then $X \cup_Z E$. If Z is smooth, E is the projective completion $\mathbb{P}(N_Z \oplus \mathbb{C})$ of the normal bundle N_Z of Z . We can identify Z with $\mathbb{P}(N_Z)$ and this is how we glue E and X . If X in fact was a toric surface and Z one of the torus invariant divisors then this corresponds to the test configurations obtained from simple piecewise linear functions with crease parallel to an edge F and meeting two edges F_1, F_2 adjacent to F . The parameter c corresponds to how far away from F the crease is.

A more general procedure one could perform in order to obtain test configurations with central fibre having two irreducible components glued along a divisor is the following. One could begin by contracting some negative curve C on X meeting some fixed subscheme Z . One obtains a new surface X' containing a point p corresponding to the curve C and a subscheme Z' , the image of Z under the blow-down map.

One can now perform the deformation to the normal cone of Z' and then blow-up this in a manner such that the generic fibre is isomorphic to X again. Indeed, the central fibre of $Bl_{Z' \times \{0\}} X' \times \mathbb{C}$ contains an exceptional divisor E and a copy of X' which are glued along the exceptional divisor of $Bl_{Z'} X'$. Moreover, E contains a copy Z'' of Z' away from intersection $X' \cap E$, which meets the proper transform of any curve $\{q\} \times \mathbb{C}$ in the point of Z'' corresponding to q . In particular, applying this to p gives a subscheme of $Bl_{Z' \times \{0\}} X' \times \mathbb{C}$ and we blow up this subscheme and obtain a space $\mathcal{X}$. Note that this has a $\mathbb{C}^*$ -action and a map π to $\mathbb{C}$. In the generic fibre isomorphic to X' in the deformation to the normal cone, we are blowing up p , and hence the generic fibre $\pi^{-1}(t)$ of $\mathcal{X}$ is isomorphic to X again. In the central fibre $\tilde{X}$ we blow up at the point corresponding to p in $Z'' \subseteq E \setminus X'$.

We also need to specify a line bundle, requiring that its restriction to the generic fibre X is the original line bundle L . One consequence of C being a negative curve is that the Seshadri constant $\epsilon(Z')$ is *at least* that of Z . This can be seen in the toric case as follows.

Assume that we are blowing up in a torus-invariant curve, so that we are considering creases parallel to a fixed edge F . We can assume that $F \subseteq \{x_1 = 0\}$ and we are considering creases of the form $x_1 - c$ for $c > 0$. In this situation the Seshadri constant is essentially the maximum c such that the crease meets the two edges adjacent to F . We now contract a curve C meeting Z , corresponding to an edge F' meeting F . Let the other edge meeting F' be F'' . When we contract C , we contract F' to a vertex which is the intersection of two edges $\tilde{F}$ and $\tilde{F}''$ containing all the points of F and F'' , respectively. Note that $\tilde{Z}$ is then the curve corresponding to $\tilde{F}$. The upshot is that the maximum c such that $x_1 - c$ meets the edge $\tilde{F}''$, which is one of the edges adjacent to $\tilde{F} \subseteq \{x_1 = 0\}$, is the x_1 -coordinate of the vertex of F'' not meeting F' . In particular, this is greater than the maximum c such that $x_1 - c$ meets F' , which is simply the x_1 -coordinate of the vertex given by $F' \cap F''$. Thus the Seshadri constant of $\tilde{Z}$ cannot be smaller than that of Z .

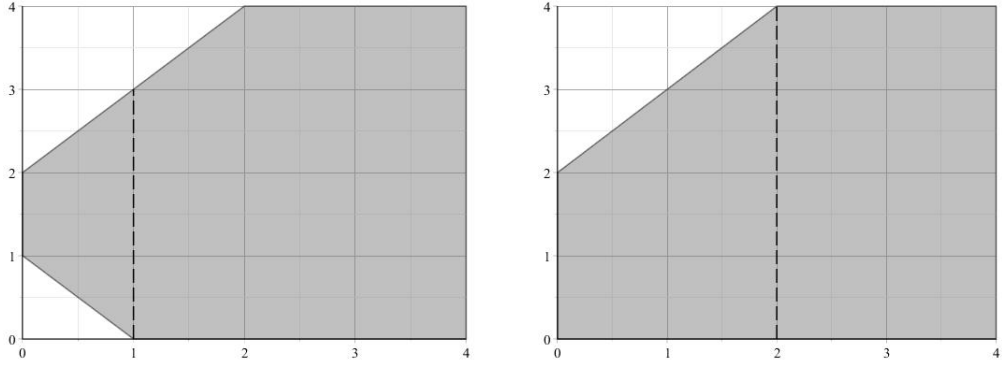


Figure 2.2: The Seshadri constant increases under contracting negative curves.

So we fix the line bundle in the following manner. The line bundle L can be written as $p^*L' - k[C]$, where $p : X \rightarrow X'$ is the blow-down map, L' is a line bundle on X' and $k > 0$. In the deformation to the normal cone of Z' in X' , we now choose the line bundle $\tilde{L} - c[E]$, where $c \in (\epsilon(Z), \epsilon(Z'))$ and we are suppressing the pull-back map. Finally, we pull this further back to $\mathcal{X}$ and choose the line bundle $\tilde{L} - c[E] - k[V]$ where V is the exceptional divisor in $\mathcal{X}$. The condition that $c \in (\epsilon(Z), \epsilon(\tilde{Z}))$ ensures that this is an ample line bundle. Note that it restricts to (X, L) on the generic fibre, since here V restricts to C and E is contained in the central fibre. Having done this once we can repeat the procedure by contracting some other curve.

For toric surfaces, the toric test configurations arising from this construction are those coming from simple piecewise linear functions with crease parallel to a given edge F , but not necessarily meeting the two edges parallel to F , it could meet other edges. In this case, the condition $c \in (\epsilon(Z), \epsilon(\tilde{Z}))$ ensures that the crease of the simple piecewise linear function on the polytope obtained from contracting a torus-invariant curve C is such that when we blow up again the crease does not meet the edge corresponding to C .

The result of doing this process a finite number of times for a fixed curve Z is a test configuration for (X, L) with central fibre which consists of two irreducible components glued along a divisor. One component

is obtained by contracting some negative curves $C_1, \dots, C_k$ on X and the other component is obtained by blowing up equally many points $p_1, \dots, p_k$ on a ruled surface.

We now go back to Donaldson's conjecture on the equality 2.22. If we have such a test configuration $\mathcal{X}$ as the optimal test configuration, i.e. $\mathcal{X}$ achieves the supremum, then there should be a sequence of metrics ω_k on X in $c_1(L)$ such that $\|S(\omega_k) - \hat{S}\|$ tends to $\frac{-DF(\mathcal{X})}{\|\mathcal{X}\|}$. One can then ask if this sequence of metrics tend to a limit.

In the toric case, where $\mathcal{X}$ divides the polytope into two pieces Q_1 and Q_2 via the crease of the corresponding simple piecewise linear function, the two pieces should then be semistable. If they in fact are stable, it then seems from Donaldson's conjectures in [Don02] and [Don09] that the expected generic behaviour is that the corresponding polarised manifolds should admit complete extremal metrics on the complement of the divisor corresponding to the crease, and that the predominant behaviour of these metrics are that of Poincaré type singularities. The metrics are moreover obtained as the limit of the ω_k .

In the general case, Donaldson's conjectures in the toric case then seem to suggest that one could expect that the generic behaviour should be that the two irreducible components of the optimal destabiliser $\mathcal{X}$ should admit complete extremal Kähler metrics on the complement of the divisor on which they intersect, and that moreover the behaviour of these metric generically is of that *Poincaré type behaviour*.

In the case of deformation to the normal cone, the two pieces are then a ruled surface N and the original X glued along a divisor D . One should expect to obtain Poincaré type extremal metrics on N and on X , but the extremal Poincaré type metric on X should belong to a *different Kähler class* than the original class on X , given by the parameter c in the deformation to the normal cone. In the more general construction above, we should then expect the same type of metrics on each irreducible piece, but now one piece is obtained by contracting some negative curves on X and the other is obtained by blowing up the ruled surface N in some points away from the common divisor D . This means that in the sequence ω_k , the volumes of the contracted curves shrink to 0.

Just as in the toric case one can envisage that one will have to repeat this procedure a finite number of times by choosing different divisors to obtain an optimally destabilising test configuration. One obtains a destabilising test configuration whose central fibre is a union of smooth irreducible components successively intersecting in divisors, and that the irreducible components admit complete extremal metrics away from these divisors, generically of Poincaré type. However, also as in the toric case, one should also expect some more complicated behaviour corresponding to collapsing behaviour in minimising procedures for the Calabi functional.

Remark 2.35. *It would be interesting to compute the Donaldson-Futaki invariant of the test configurations described above. One could hope the above procedure is sufficient for testing stability on surfaces. It is not clear in the toric case that all test configurations arising from simple piecewise linear functions arise from this construction, though. Therefore it might be that some modification is needed. However, in light*

of Donaldson's result on toric surfaces that says one only has to check stability on simple piecewise linear functions, it certainly seems that one should be able to test stability on surfaces via test configurations whose central fibre has no more than two irreducible components.

Chapter 3

Toric Poincaré type metrics

In this chapter we will define a space of symplectic potentials on a polytope P given a boundary measure $d\sigma$ that's allowed to be 0 along some facets. Section 3.1 is devoted to proving that in the case when P is Delzant, exactly these potentials induce metrics with a mix of Poincaré-type and cone singularities prescribed by $d\sigma$ along the divisors corresponding to the facets of P .

Having defined such potentials we prove some results when the corresponding Kähler metrics are extremal. In section 3.2, we prove that such extremal potentials are unique up to the addition of an affine linear function. In section 3.3, we show that if an extremal potential exists, then $(P, d\sigma)$ is stable as in definition 2.24.

3.1 The space of symplectic potentials

In this section we will define a space of symplectic potentials on a bounded convex polytope P . In the Delzant case we show that these potentials induce metrics with mixed Poincaré type and cone angle singularities along the torus invariant divisors on the complex manifold N_P corresponding to P .

We begin by proving that a certain model potential induces such metrics. Recall that a measure $d\sigma$ on the boundary of P is a *non-negative boundary measure for P* if on the i^{th} facet F_i of P , $d\sigma$ satisfies

$$l_i \wedge d\sigma = \pm r_i d\lambda \tag{3.1}$$

where l_i is an affine function defining F_i , $d\lambda$ is the Lebesgue measure on P and $r_i \in \mathbb{R}_{\geq 0}$ is a non-negative constant. The measure is positive if $r_i > 0$ for all i .

Let P be a Delzant polytope with facets $F_1, \dots, F_d$. We let (N_P, Ω_P) be the corresponding complex manifold and Kähler class associated to P , and let D_i be the divisor in N_P corresponding to the facet F_i . Suppose $d\sigma$ is a non-negative boundary measure for P . Let $\{i_1, \dots, i_k\}$ be the subset of $\{1, \dots, d\}$ on which r_i vanishes, which, after relabelling of the F_i , we will assume is $1, \dots, k$. Then we let D denote the divisor $D_1 + \dots + D_k$ corresponding to the facets on which $d\sigma$ vanishes.

Given a non-negative boundary measure $d\sigma$ for P , let $d\tilde{\sigma}$ be a positive boundary measure which agrees with $d\sigma$ on the facets where $d\sigma$ does not vanish. For a symplectic potential $v \in \mathcal{S}_{d\tilde{\sigma}}$ and positive real numbers $a_1, \dots, a_k > 0$, define $u_{\underline{a},v} : P^\circ \rightarrow \mathbb{R}$ by

$$u_{\underline{a},v} = v + \sum_{i=1}^k (-a_i \log l_i). \quad (3.2)$$

Proposition 3.1. *Let $(P, d\sigma)$ be a weighted Delzant polytope, where $d\sigma$ is a non-negative boundary measure. Then through the Legendre transform, $u_{\underline{a},v}$ defines a Kähler metric on $N_P \setminus D$ with mixed Poincaré and cone angle singularities in the class Ω_P . The Poincaré type singularity is along D , and the cone angle singularities are along the divisors D_i with $i > k$, the cone angle singularity along D_i being equal to that of the metric induced by v .*

Remark 3.2. *On the facets where the boundary measure does vanish, we do not actually have to require that the coefficient of $l_j \log l_j$ of v is positive. It could also vanish or be negative, as $-a_j \log l_j$ will always be the dominant term.*

The proof is rather long, and so we divide it up into a number of lemmas. We begin with a few of the essential calculations of the proof.

Lemma 3.3. *Let v be a function on a domain Ω in $\mathbb{R}^n$, whose Hessian (v_{ij}) is invertible. Then for any $a > 0$, and for x^1 positive and sufficiently small (depending on a and v), $u = -a \log x^1 + v$ has Hessian (u_{ij}) which is invertible. The inverse (u^{ij}) is given by*

$$u^{ij} = v^{ij} - \frac{av^{i1}v^{1j}}{(x^1)^2 + av^{11}} \quad (3.3)$$

where (v^{ij}) is the inverse matrix to (v_{ij}) .

Proof. First note that $u = -a \log x^1 + v$ has Hessian

$$u_{ij} = \begin{cases} v_{11} + \frac{a}{(x^1)^2} & \text{if } i = j = 1, \\ v_{ij} & \text{otherwise.} \end{cases}$$

A computation then shows that the inverse matrix is given by 3.3. □

Lemma 3.4. *Let V be a symmetric $n \times n$ -matrix. Then for any $j > 1$, we have*

$$\det(V_{11})\det(V_{jj}) - \det(V_{1j})^2 = \det((V_{jj})_{11})\det(V) \quad (3.4)$$

where V_{kl} is the $(n-1) \times (n-1)$ -matrix obtained by deleting the k^{th} row and l^{th} column of V , and $(V_{jj})_{11}$ is the matrix obtained by deleting the 1^{st} and j^{th} columns and rows of V .

The proof is by induction on n , the case $n = 3$ being the base step. For $n = 2$ we interpret $\det((V_{jj})_{11})$ as being 1 and then this becomes the usual formula for the determinant of a symmetric 2×2 matrix.

Let $F = \cup_{i=1}^k F_i$ denote the union of the facets along which we introduce a new behaviour for the symplectic potential. The steps in proving 3.1 are now to show that

- The Legendre transform gives a diffeomorphism $M_P \setminus \mu^{-1}(F) \rightarrow N_P \setminus D$,
- The metric on N_P obtained by pulling back ω_P via this diffeomorphism is quasi-isometric to the model Poincaré type metric,
- The metric has a potential which satisfies the required C^0 and higher order bounds.

We begin with the first of these. Note that the Legendre transform

$$x \mapsto \left(\frac{\partial u}{\partial x^1}, \dots, \frac{\partial u}{\partial x^n} \right)$$

of u gives rise to a map

$$\tilde{\psi}_u : P^\circ \times \mathbb{R}^n \rightarrow \mathbb{C}^n$$

by

$$(x, y) \mapsto \left(\frac{\partial u}{\partial x^1} + iy^1, \dots, \frac{\partial u}{\partial x^n} + iy^n \right).$$

This map is moreover the lifting of a map

$$\psi_u : P^\circ \times (S^1)^n \rightarrow (\mathbb{C}^*)^n.$$

We identify $P^\circ \times (S^1)^n$ and $(\mathbb{C}^*)^n$ with the free orbits M_P° and N_P° of the symplectic and complex quotient, respectively. We can then ask if ψ_u extends to any of the non-free orbits of the torus-action.

Lemma 3.5. *Through the Legendre transform u defines a diffeomorphism $\tilde{\psi}_u : P^\circ \times \mathbb{R}^n \rightarrow \mathbb{C}^n$ that arises from the restriction to M_P° and N_P° of a diffeomorphism $\psi_u : M_P \setminus \mu^{-1}(F) \rightarrow N_P \setminus D$.*

Proof. The map $\tilde{\psi}_u$ gives a diffeomorphism of $P^\circ \times \mathbb{R}^n$ onto its image, which is $\nabla u(P^\circ) + i\mathbb{R}^n$. This follows because u is smooth and strictly convex on the convex domain P . The strict convexity follows from the fact that v is strictly convex and that for all $w \in \mathbb{R}^n$,

$$w^T(u_{ij})w \geq w^T(v_{ij})w.$$

We must therefore show that the map is onto, i.e. the image of the gradient map of u is the whole of $\mathbb{R}^n$. Then, since u satisfies the usual boundary conditions near the other facets of P , the map extends to a diffeomorphism $\psi_u : M_P \setminus \mu^{-1}(F) \rightarrow N_P \setminus D$. To see this, it suffices to note that the gradient mapping

$$x \mapsto \left(\frac{\partial u}{\partial x^1}, \dots, \frac{\partial u}{\partial x^n} \right)$$

satisfies

$$\left| \left(\frac{\partial u}{\partial x^1}, \dots, \frac{\partial u}{\partial x^n} \right) \right| \rightarrow +\infty$$

as $x \rightarrow \partial P$. Indeed, we have that

$$\nabla u = \nabla v - \sum_i a_i \nabla \log(l_i),$$

and ∇v blows up as we approach the boundary. Also, $\nabla \log(l_i)$ blows up near $l_i = 0$, but note that

$$-a_i \nabla \log(l_i) = -\frac{a_i}{l_i} \nabla l_i$$

and

$$\nabla(l_i \log l_i) = (1 + \log l_i) \nabla l_i$$

so the contributions from the gradients of v and $u - v$ to ∇u do not cancel. $\square$

For the rest of the proof, we will consider only the case when one facet, which we assume is F_1 , has 0 boundary measure, and the rest are 1. Cone angle singularities do not give rise to any new difficulties in the proof, just more notation. Let $D = \mu^{-1}(F_1)$. After an $SL_n(\mathbb{Z})$ -transformation we may assume $F_1 \subseteq \{x^1 = 0\}$. We have

$$u = v - a \log x^1$$

for some $a > 0$ and symplectic potential v .

We wish to compute how ω_P transforms under the map from lemma 3.5. The complex coordinates $z_j = \xi_j + i\eta_j$ (on the universal cover of $(\mathbb{C}^*)^n$) are given by

$$\begin{aligned} \xi_j &= \frac{\partial u}{\partial x^j}, \\ \eta_j &= y^j. \end{aligned}$$

By lemma 3.5, this is a diffeomorphism $P^\circ \times \mathbb{R}^n \rightarrow \mathbb{C}^n$. The symplectic form ω_P is $\sum_j dx^j \wedge dy^j$ which becomes

$$\sum_{j \leq k} \frac{i}{2} u^{jk} (dz_j \wedge d\bar{z}_k + dz_k \wedge d\bar{z}_j).$$

Consider a vertex p of P , lying on F_1 . After an $SL_n(\mathbb{Z})$ -transformation and a translation we may assume $p = 0$ and the n facets meeting at p are $\{x : x^i = 0\}$ for $i = 1, \dots, n$. There is a chart around the point in N_P corresponding to p from $(\mathbb{C})^n$ such that the transition function $(\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n$ is given by $(z_1, \dots, z_n) \mapsto (e^{z_1}, \dots, e^{z_n})$ (when lifted to the universal cover, the inverse function is only defined locally). We will call the new coordinates w_j , i.e. $w_j = e^{z_j}$.

Note that $dw_j = e^{z_j} dz_j = w_j dz_j$ and $d\bar{w}_j = \bar{w}_j d\bar{z}_j$. Hence

$$dz_j \wedge d\bar{z}_j = \frac{1}{|w_j|^2} dw_j \wedge d\bar{w}_j.$$

The standard Poincaré type metric ω_{cusp} on $B_1^* \times (\mathbb{C})^{n-1}$ is

$$\omega_{\text{cusp}} = i \frac{dw_1 \wedge d\bar{w}_1}{|w_1|^2 (\log |w_1|)^2} + i \sum_{j=2}^n dw_j \wedge d\bar{w}_j.$$

This pulls back to the universal cover as

$$i \frac{dz_1 \wedge d\bar{z}_1}{(2\xi_1)^2} + i \sum_{j=2}^n e^{2\xi_j} dz_j \wedge d\bar{z}_j.$$

With all these preliminaries in place, we are now ready to show that the quasi-isometric property holds. In this proof we will not use our usual notation convention and so even if a formula has repeated indices, it does not mean we sum. We only sum when a summation sign is explicitly given.

Lemma 3.6. *In the above chart, $(\psi_u^{-1})^* \omega_P$ and ω_{cusp} are quasi-isometric near D .*

Proof. It suffices to show that the quasi-isometric property holds on the lifts to the universal cover in the region where the real part of all the z_i are smaller than some constant, say. We must show that in such a region

$$\left\| \sum_i \lambda_i \frac{\partial}{\partial z_i} \right\|_{g_u}$$

and

$$\left\| \sum_i \lambda_i \frac{\partial}{\partial z_i} \right\|_{g_{\text{cusp}}}$$

are mutually bounded, independent of the λ_i . Taking squares, these expressions become

$$\sum_{j,k} u^{jk} \lambda_j \bar{\lambda}_k$$

and

$$\frac{1}{4} |\lambda_1|^2 \left(\frac{\partial v}{\partial x^1} - \frac{a}{x^1} \right)^{-2} + \sum_{j>1} |\lambda_j|^2 e^{2 \frac{\partial v}{\partial x^j}}.$$

The latter sum equals

$$\frac{1}{4} |\lambda_1|^2 \left(\frac{\partial v}{\partial x^1} - \frac{a}{x^1} \right)^{-2} + \sum_{j>1} |\lambda_j|^2 x^j e^{1+2 \frac{\partial h}{\partial x^j}}.$$

Here we have used that $v = \frac{1}{2} \sum_k (x^k \log x^k) + h$ where h is a smooth function defined on the whole of P , so that $\xi_j = \frac{\partial v}{\partial x^j} = \frac{1}{2} \log x^j + \frac{1}{2} + \frac{\partial h}{\partial x^j}$ for $j > 1$. Note that $e^{1+2 \frac{\partial h}{\partial x^j}} \in [c_1, c_2]$ for some $c_i > 0$, since h is smooth on P .

For the proof it will be important what the boundary behaviour of the terms u^{ij} is. For $j = 1$, we have using 3.3 that

$$\begin{aligned} u^{11} &= v^{11} - \frac{a(v^{11})^2}{(x^1)^2 + av^{11}} \\ &= \frac{(x^1)^2 v^{11}}{(x^1)^2 + av^{11}}. \end{aligned} \quad (3.5)$$

Again using 3.3 and equation 3.4, we have that for $j > 1$,

$$u^{jj} = \frac{(x^1)^2 \det(V_{jj}) + k \det(V_{jj})_{11}}{(x^1)^2 \det(V) + k \det(V_{11})}.$$

Since v satisfies the usual boundary conditions, we have that there exists positive functions $\delta_0, \dots, \delta_3$ on the full polytope P such that

$$\begin{aligned} \det(V) &= \frac{\delta_0}{\prod_{k=1}^d l_k}, \\ \det(V_{11}) &= \frac{\delta_1}{\prod_{k \neq 1} l_k}, \\ \det(V_{jj}) &= \frac{\delta_2}{\prod_{k \neq j} l_k}, \\ \det((V_{jj})_{11}) &= \frac{\delta_3}{\prod_{k \neq 1, j} l_k}. \end{aligned}$$

Multiplying both the numerator and denominator by $\prod_{k \neq 1, j} l_k$, we get that

$$\frac{u^{jj}}{x^j} = \frac{x^1 \delta_2 + k \delta_3}{x^1 \delta_0 + k \delta_1}. \quad (3.6)$$

Similarly, using an analogous formula to 3.4, we have that for i, j not equal,

$$u^{ij} = \frac{(x^1)^2 \det(V_{ij}) + k \det(V_{ij})_{11}}{(x^1)^2 \det(V) + k \det(V_{11})}.$$

Using a reasoning like above, we get that when $i, j \neq 1$,

$$u^{ij} = x^i x^j \left(\frac{x^1 \delta_2 \pm k \delta_3}{x^1 \delta_0 + k \delta_1} \right), \quad (3.7)$$

where the δ_i are smooth functions defined on the whole polytope P and δ_0, δ_1 are everywhere positive. In the case when $i = 1$, we have that the δ_3 -term drops out.

We are now ready to obtain the required bounds. We begin by dealing with the diagonal terms. In the case $i = j = k = 1$, we have that

$$\left(\frac{\partial v}{\partial x^1} - \frac{a}{x^1} \right)^{-2} = (x^1)^2 \left(x^1 \frac{\partial v}{\partial x^1} - a \right)^{-2}$$

is mutually bounded with

$$(x^1)^2$$

near $x^1 = 0$, since $x^1 \frac{\partial v}{\partial x^1}$ equals $\frac{1}{2}x^1 \log x^1$ plus terms that are smooth on the whole polytope P .

Moreover, from 3.5, u^{11} is also mutually bounded with $(x^1)^2$ near $x^1 = 0$ since

$$\frac{u^{11}}{(x^1)^2} = \frac{v^{11}}{(x^1)^2 + av^{11}}$$

and $v^{11} = O(x^1)$.

In the case when $j > 1$, we have that

$$u^{jj} = O(x^j),$$

by 3.6, and so $u^{jj}|\lambda_j|^2$ and $|\lambda_j|^2 x^j e^{1+2\frac{\partial h}{\partial x^j}}$ are mutually bounded, independently of λ_j .

So far, we have shown that there is a $C > 0$ such that

$$C^{-1} \sum_j u^{jj} \lambda_j \bar{\lambda}_j \leq \frac{1}{4} |\lambda_1|^2 \left(\frac{\partial v}{\partial x^1} - \frac{a}{x^1} \right)^{-2} + \sum_{j>1} |\lambda_j|^2 e^{2\frac{\partial v}{\partial x^j}} \leq C \sum_j u^{jj} \lambda_j \bar{\lambda}_j, \quad (3.8)$$

independently of the λ_j . What remains is to deal with the off-diagonal terms.

If $i \neq j$ and both are greater than 1, we have by 3.7 that there is a bounded function f such that

$$\begin{aligned} |u^{ij} \lambda_i \bar{\lambda}_j| &= |x^i x^j \lambda_i \bar{\lambda}_j f| \\ &\leq C(|x^i \lambda_i|^2 + |x^j \lambda_j|^2) \\ &\leq C(|\lambda_i|^2 e^{2\frac{\partial v}{\partial x^i}} + |\lambda_j|^2 e^{2\frac{\partial v}{\partial x^j}}), \end{aligned}$$

since $e^{2\frac{\partial v}{\partial x^k}}$ is mutually bounded with x^k .

If $i = 1$ and $j \neq 1$, then we obtain a power higher for the x^i -term, so that

$$\begin{aligned} |u^{1j} \lambda_1 \bar{\lambda}_j| &= |(x^1)^2 x^j \lambda_1 \bar{\lambda}_j f| \\ &\leq C((x^1)^2 |\lambda_1|^2 + |x^j \lambda_j|^2) \\ &\leq C\left(\frac{1}{4} |\lambda_1|^2 \left(\frac{\partial v}{\partial x^1} - \frac{a}{x^1} \right)^{-2} + |\lambda_j|^2 e^{2\frac{\partial v}{\partial x^j}}\right), \end{aligned}$$

using that $\left(\frac{\partial v}{\partial x^1} - \frac{a}{x^1} \right)^{-2}$ near p is mutually bounded with $(x^1)^2$.

This completes the proof of one part of the inequality, i.e. we have shown that there is a $C > 0$ such that

$$\left\| \sum_i \lambda_i \frac{\partial}{\partial z_i} \right\|_{g_u} \leq C \left\| \sum_i \lambda_i \frac{\partial}{\partial z_i} \right\|_{g_{cusp}}.$$

For the other inequality, in dealing with the off-diagonal terms, we use the fact that the dependence on x^i, x^j above was *better* than needed for the former inequality. That is, we have, for $i, j \neq 1$ and not equal to each other, that

$$\begin{aligned} u^{ij} \lambda_i \bar{\lambda}_j &\geq -C(|x^i \lambda_i|^2 + |x^j \lambda_j|^2) \\ &\geq -C(x^i |\lambda_i|^2 e^{2\frac{\partial v}{\partial x^i}} + x^j |\lambda_j|^2 e^{2\frac{\partial v}{\partial x^j}}), \end{aligned}$$

and so there is a neighbourhood of p , independently of the λ^i , such that

$$u^{ij} \lambda_i \bar{\lambda}_j \geq -\frac{C^{-1}}{N} (|\lambda_i|^2 e^{2\frac{\partial v}{\partial x^i}} + x^j |\lambda_j|^2 e^{2\frac{\partial v}{\partial x^j}}),$$

where the C now is the C satisfied by the diagonal terms in 3.8 and $N > 0$ is sufficiently large. In the case $i = 1$, say, the same holds with $(x^1)^2$ instead. Summing over all i, j we get that this does not cancel the contribution from the diagonal terms, for N sufficiently large, and so we have shown the required bound.

Thus we have shown that there is a neighbourhood of p and a $C > 0$, both independent of the λ_i , such that

$$C^{-1} \left\| \sum_i \lambda_i \frac{\partial}{\partial z_i} \right\|_{g_{cusp}} \leq \left\| \sum_i \lambda_i \frac{\partial}{\partial z_i} \right\|_{g_u} \leq C \left\| \sum_i \lambda_i \frac{\partial}{\partial z_i} \right\|_{g_{cusp}},$$

as required. $\square$

We have shown that g is quasi-isometric to g_{cusp} near D in the above chart. One can use the same reasoning on all the complex toric charts and this shows that g induces a metric quasi-isometric to the model Poincaré type metric.

Next, we show that there is a potential for the metric and that it satisfies the C^0 -estimate for potentials for Poincaré type metrics.

Lemma 3.7. *Let σ be a holomorphic section of $\mathcal{O}(D)$ defining D . Then $(\psi_u^{-1})^* \omega_P$ has a potential ϕ which satisfies*

$$|\phi| \leq C |\log(-\log(|\sigma|^2))|$$

near D .

Proof. The model potential $\log(-\log(|w_1|^2))$ equals

$$\log(f(x))$$

in symplectic coordinates, where $f(x) = \frac{2a}{x^1} - 2\frac{\partial v}{\partial x^1}$.

The potential ϕ of ω is the Legendre transform of u , so satisfies

$$\begin{aligned} \phi(z) &= u(x) - \sum_i x^i \xi_i \\ &= v(x) - a \log(x^1) + a - \sum_i x^i \frac{\partial v}{\partial x^i}. \end{aligned}$$

Note that all terms except $-a \log(x^1)$ are bounded. Thus ϕ is $O(\log(-\log(|w_1|^2)))$ if and only if $-a \log(x^1)$ is. But $\frac{\partial v}{\partial x^1}$ is $O(\log(x^1))$. In particular, for all x^1 sufficiently small, we have $|\frac{\partial v}{\partial x^1}| \leq \frac{a}{4x^1}$, say. Hence $\log(-\log(|w_1|^2))$ is mutually bounded with $\log(\frac{a}{x^1})$ near $x^1 = 0$, which in turn is mutually bounded with $-a \log(x^1)$ near $x^1 = 0$, as required. $\square$

The final step in proving 3.1 is to show that the derivatives of ϕ are bounded with respect to the Levi-Civita connection of the model metric, at any order.

Lemma 3.8. *Let ϕ be the above potential. Then ϕ satisfies the derivative bounds in definition 2.33.*

Proof. For $\nabla^{\omega_{cusp}}\phi$, a calculation similar to the above shows that we have

$$|\nabla^{\omega_{cusp}}\phi| = \left(\frac{a}{x^1} - \frac{\partial v}{\partial x^1}\right)^2 \left(u^{i1}x^j \frac{\partial^2 v}{\partial x^i \partial x^j} + \frac{au^{11}}{x^1}\right)^2 + \sum_{i>1} \left(\frac{\partial v}{\partial x^i}\right)^2 \left(u^{ij}x^l \frac{\partial^2 v}{\partial x^j \partial x^l}\right)^2.$$

But from 3.7, $\frac{u^{i1}}{x^1 x^i}$ is bounded for all i , and we also have that $x^i x^j \frac{\partial^2 v}{\partial x^i \partial x^j}$ is bounded, and so the first term is bounded. We also have that for all i , $\frac{1}{x^i} u^{ij} x^l \frac{\partial^2 v}{\partial x^j \partial x^l}$ is bounded, and so, as $x^i \frac{\partial v}{\partial x^i}$ is bounded, the remaining terms are bounded too.

For higher order derivatives, similar calculations and using induction show that the required bounds hold. This hinges on the fact that the norm of the covariant derivatives of the vector fields $\frac{\partial}{\partial w_i}$ always is finite at any order, and that taking successive derivatives of ϕ introduces a higher negative power of x^i 's that are cancelled by the contribution of a positive power of dividing the inverse Hessian terms, as in the previous paragraph and the formula for derivatives of the inverse of the Hessian of a function. $\square$

We will now define the space of symplectic potentials of interest to us. We introduce a bit of terminology. Given a non-negative boundary measure $d\sigma$ we call the $u_{a,v}$ in 3.2 arising from some positive measure $d\tilde{\sigma}$ agreeing with $d\sigma$ on the facets where $d\sigma$ does not vanish, a *model potential* for $(P, d\sigma)$.

Also, recall that associated to $d\tilde{\sigma}$ there is a canonical choice of defining functions l_i for P , whose zero sets intersect P in facets F_i . For a non-negative boundary measure $d\sigma$ we can get canonical defining functions for the i such that $d\sigma|_{F_i} \neq 0$ by the same requirement on these facets.

For the functions u and $u_{a,v}$ below we will let U and $U_{a,v}$ denote their respective Hessians. Given a non-negative boundary measure $d\sigma$, we let $d_{PT} : P \rightarrow \mathbb{R}$ be a positive function on P which is smaller than 1 everywhere, and which agrees with the distance function to the Poincaré type facets near these facets. The Poincaré type facets are the facets on which $d\sigma$ vanishes.

Definition 3.9. *Let P be a polytope with facets $F_1, \dots, F_d$ and let $d\sigma$ be a non-negative boundary measure. Let l_i be the canonical defining functions for the i such that $d\sigma$ does not vanish along F_i . Define $S_{P,d\sigma}$ to be the space of smooth strictly convex functions $u : P^\circ \rightarrow \mathbb{R}$ that can be written as*

$$u = \frac{1}{2} \sum_{i: d\sigma|_{F_i} \neq 0} l_i \log l_i + h, \quad (3.9)$$

for some $h \in C^\infty(P \setminus \cup_{i: d\sigma|_{F_i} = 0} F_i)$, and which moreover satisfy that there is a model potential $u_{a,v}$ for $(P, d\sigma)$ such that

- $|u| \leq C(-\log(d_{PT}))$ for some $C > 0$,

- there is a $c > 0$ such that

$$c^{-1}U_{\underline{a},v} \leq U \leq cU_{\underline{a},v}, \quad (3.10)$$

- for all $i \geq 1$, we have that $|\nabla^i u|_{u_{\underline{a},v}}$ and $|\nabla^i u_{\underline{a},v}|_{u_{\underline{a},v}}$ are mutually bounded.

Here $\nabla u = \nabla^1 u$ is the gradient of u with respect to $u_{\underline{a},v}$, ∇^i denotes the higher derivatives with respect to the Levi-Civita connection of $u_{\underline{a},v}$ and $|\cdot|_{u_{\underline{a},v}}$ denotes the norm on the higher tensor bundles of TP° with respect to $u_{\underline{a},v}$.

For a Delzant polytope, elements of $\mathcal{S}_{P,d\sigma}$ also give Kähler metrics with mixed Poincaré type and cone singularities. In the proof below, we will let F be the union of the facets F_i on which $d\sigma$ vanishes and D the divisor in the complex quotient N_P which corresponds to these facets.

Proposition 3.10. *Suppose P is a Delzant polytope and let $d\sigma$ be a non-negative boundary measure. Then for all $u \in \mathcal{S}_{P,d\sigma}$, u defines through the Legendre transform a Kähler metric on N_P in the class Ω_P with mixed Poincaré type and cone angle singularities, the singularity being prescribed by $d\sigma$.*

Conversely, if $\omega \in \Omega_P$ is the Kähler form of a T^n -invariant metric on N_P of Poincaré type along a torus-invariant divisor D , then it is induced by a function u on P° satisfying definition 3.9.

Proof. We first argue that functions in $\mathcal{S}_{P,d\sigma}$ induce the required kind of metrics. The strict convexity of u ensures that the Legendre transform of u induces a diffeomorphism of $P^\circ \times \mathbb{R}^n$ onto its image in $\mathbb{C}^n$. The gradient condition, i.e. the final condition of 3.9 applied to $i = 1$, then ensures that $|\nabla u| \rightarrow +\infty$ as $x \rightarrow \partial P$, since this holds for the model potentials for $(P, d\sigma)$. In particular, this implies that the map $P^\circ \times \mathbb{R}^n \rightarrow \mathbb{C}^n$ obtained from the Legendre transform of u is onto. The expansion 3.9 then ensures that the Legendre transform induces a map $P^\circ \times (S^1)^n \rightarrow (\mathbb{C}^*)^n$ which is actually the restriction of a diffeomorphism

$$\psi_u : M_P \setminus \mu^{-1}(F) \rightarrow N_P \setminus D,$$

as in the proof of lemma 3.5.

Equation 3.9 moreover implies that one has a Kähler potential on $N_P \setminus D$. This is mutually bounded with the model Poincaré type potential in the complex picture, by the C^0 -estimate in definition 3.9 and the fact that, in the Legendre transform formula, $\sum x^i \xi_i$ is bounded, again by the gradient condition.

The gradient condition also ensures that inequality 3.10 implies that the Kähler metric induced by u is quasi-isometric to the Kähler metric induced by $u_{\underline{a},v}$, and so to the model Poincaré type metric. Finally, it also follows that the bounds for the norm of the derivatives of the Kähler potential are equivalent to the bounds in the last point of definition 3.9.

For the converse, we have a potential for the T^n -invariant Poincaré type metric ω on the open set corresponding to the free points of the T^n -action, and this is induced by a function u satisfying 3.9. That $h \in C^\infty(P \setminus F)$ follows exactly as in the case of a positive boundary measure. By assumption ω is quasi-isometric to the model Poincaré type metric ω_f in definition 2.33, and so it quasi-isometric to the $\omega_{\underline{a},v}$ induced by a model symplectic potential $u_{\underline{a},v}$.

In the complex setting, for the general definition 2.33 of a Poincaré type metric, we could have used the gradient bounds with respect to any Poincaré type metric instead of the model Poincaré type metric ω_f . In particular, for a compact toric manifold, we can use $\omega_{\underline{a},v}$. Thus this definition implies a bound on the gradient of the complex potential, which implies that the quasi-isometric property carries over to the symplectic coordinates, i.e. 3.10 holds. For the C^0 -estimate, we again use the formula for the Legendre transform and C^0 -estimate in the complex picture together with this gradient bound. As before the higher order bounds are then also equivalent to the higher order bounds in the symplectic setting. $\square$

3.2 Uniqueness of extremal metrics

An important question is whether or not extremal Kähler metrics are unique in their Kähler class, up to the action of the automorphism group of the Kähler manifold. In the compact case, without assuming the manifold is toric, the answer is *yes*, as proved by Berman-Berndtsson ([BB14, Thm. 4.15]). For Poincaré type metrics, again not necessarily toric, Auvray showed in [Auv14c, Thm. 2] that this is true in the constant scalar curvature case, under the assumption that the line bundle $K_X \otimes \mathcal{O}(D)$ is ample. This generalised a proof of Chen ([Che00]) in the compact case, where K_X is assumed to be ample. We now show that uniqueness also holds for potentials satisfying definition 3.9.

Here in the toric case the proof is much simpler. In the general case the approach is to study geodesics in the space of Kähler potentials. The Mabuchi functional is a function on the space of Kähler metrics which is convex along geodesics and can be used to draw conclusions about uniqueness, if one knows the existence of geodesics of sufficiently high regularity. In our case geodesics in the space of Kähler metrics are straight lines in terms of the symplectic potential, so one always has a *smooth* geodesic. However, we do not actually need to mention geodesics at all. All we need is to introduce the Mabuchi functional, which we will show gives a well-defined smooth *convex* function on $\mathcal{S}_{P,d\sigma}$, and then the basic idea of the proof works, without the technicalities of the general case. This approach was used by Guan in [Gua99] to prove the analogous result for the compact toric case, but we will now show it applies in our situation too.

In the smooth case the Mabuchi functional is defined as follows. For a smooth, bounded function A on P , let $\mathcal{F}_A : \mathcal{S} \rightarrow \mathbb{R}$ be given by

$$\mathcal{F}_A(u) = -\frac{1}{2} \int_P \log(\det(u_{ij})) d\lambda + \mathcal{L}_A(u),$$

where $\mathcal{L}_A(u) = \int_{\partial P} u d\sigma - \int_P u A d\lambda$. Donaldson showed in [Don02] that this is, up to a constant depending only on P , not on u , the usual Mabuchi functional when $A = \frac{\text{Vol}(\partial P)}{\text{Vol}(P)}$ is half the average scalar curvature. It is up to a constant also only depending on P the modified Mabuchi functional when A is the affine linear function associated to the polytope P ([ZZ08b, p.1333]).

We would like to extend the domain of $\mathcal{F}_A$ to the Poincaré type symplectic potentials, then show that extremal metrics of Poincaré type are minima of $\mathcal{F}_A$ and use this to prove the uniqueness statement. To

show that the Mabuchi functional is actually defined for the symplectic potentials we are considering, we first prove how the determinant of the Hessians of these potentials behave.

Lemma 3.11. *Let $(P, d\tilde{\sigma})$ be a weighted polytope with $d\tilde{\sigma}$ being a positive measure. Suppose P has d edges. For $\underline{a} \in \mathbb{R}_{\geq 0}^d$ and $v \in \mathcal{S}$, let $u = u_{\underline{a}, v}$ be given by equation 3.2. Define $R \subseteq \{1, \dots, d\}$ to be the indices of the non-zero entries of $\underline{a}$ and let T be its complement. Then*

$$\det(u_{ij}) = \frac{\delta}{(\prod_{k \in R} l_k^2)(\prod_{k \in T} l_k)},$$

for some positive smooth function δ on the whole polytope P .

Proof. We do the proof in the case of one non-zero entry of $\underline{a}$. The same reasoning then works in an induction argument to give the general case.

So we assume the first entry is the non-zero one, and that $F_1 \subseteq \{x^1 = 0\}$, so that $l_1 = x^1$. Then there is an $a > 0$ and a v which is a symplectic potential for a positive measure such that

$$u = -a \log x^1 + v.$$

Now, there exists positive smooth functions δ_0, δ_1 on P , so that if $V = (v_{ij})$ and V_{11} is its $(1, 1)$ -minor, we have that

$$\begin{aligned} \det(V) &= \frac{\delta_0}{\prod_{k=1}^d l_k}, \\ \det(V_{11}) &= \frac{\delta_1}{\prod_{k>1} l_k}. \end{aligned}$$

Now $u_{11} = v_{11} + \frac{a}{(x^1)^2}$ and $u_{ij} = v_{ij}$ for all other i, j . Thus

$$\begin{aligned} \det(u_{ij}) &= \det(V) + \frac{a}{(x^1)^2} \det(V_{11}) \\ &= \frac{a\delta_1 + l_1\delta_0}{l_1^2 \prod_{k>1} l_k}. \end{aligned}$$

Putting $\delta = a\delta_1 + l_1\delta_0$, the result follows. $\square$

This implies the same sort of bound for any element of $\mathcal{S}_{P, d\sigma}$. As above, for a non-negative boundary measure $d\sigma$ on the boundary of a polytope P with facets $F_1, \dots, F_d$, we let R be the indices of the facets on which the boundary measure vanishes and T be its complement in $\{1, \dots, d\}$.

Lemma 3.12. *Let $u \in \mathcal{S}_{P, d\sigma}$ for a non-negative boundary measure $d\sigma$. Then the determinant of the Hessian of u is mutually bounded with that of a model potential $u_{\underline{a}, v} \in \mathcal{S}_{P, d\sigma}$, i.e.*

$$\det(u_{ij}) = \frac{\delta}{(\prod_{k \in R} l_k^2)(\prod_{k \in T} l_k)},$$

for some positive smooth function δ on P .

Proof. The Hessian in the symplectic picture is the inverse of the Hessian in the complex picture, and so the same holds for the determinants. This in particular implies that the $\det(u_{ij}) > 0$ on P° , so we need to determine what happens as one approaches the boundary of P , which corresponds to approaching the torus-invariant divisors in the complex case. The result follows by pulling back the volume form of the model Poincaré type metric to the universal cover, which the general case is mutually bounded with. Here we are using the quasi-isometric property and the gradient property, as in the proof of 3.10. $\square$

Proposition 3.13. *Let $u \in \mathcal{S}_{P,d\sigma}$ be a symplectic potential as in definition 3.9. Then $\log \det(u_{ij})$ is integrable on P .*

Proof. By 3.12, we have that

$$-\log \det(u_{ij}) = 2 \sum_{k \in R} \log(l_k) + \sum_{k \in T} \log(l_k) - \log(\delta)$$

for some δ bounded away from zero and bounded above. The integrability of this is then no different than the usual smooth case, which Donaldson showed is integrable in [Don02, Sect. 3.3]. $\square$

Having shown that $\log \det(u_{ij})$ is integrable for Poincaré type symplectic potentials, we can now define the Mabuchi functional on this space of functions. Recall from definition 2.23 that the affine linear function associated to a weighted polytope $(P, d\sigma)$ is the unique affine linear function A such that $\mathcal{L}_A$ vanishes on the affine linear functions.

Definition 3.14. *Let P be a polytope, let $d\sigma$ be a non-negative boundary measure and let A be the affine linear function associated to $(P, d\sigma)$. Then the Mabuchi functional $\mathcal{F}_A : \mathcal{S}_{P,d\sigma} \rightarrow \mathbb{R}$ is defined by*

$$\mathcal{F}_A(u) = -\frac{1}{2} \int_P \log(\det(u_{ij})) d\lambda + \mathcal{L}_A(u).$$

We will need that for any two symplectic potentials, the straight line between them is a smooth path in the space of symplectic potentials.

Lemma 3.15. *Let $u_0, u_1 \in \mathcal{S}_{P,d\sigma}$. Define u_t by*

$$u_t = (1 - t)u_0 + tu_1.$$

Then $u_t \in \mathcal{S}_{P,d\sigma}$ for all $t \in [0, 1]$.

We can assume u_0 and u_1 satisfy definition 3.9 with the same C , v and $\underline{a}$. Hence u_t satisfies the definition for the same C , v and $\underline{a}$, too.

Next we need an adaption of an integration by parts formula in [Don02, Lem. 3.3.5] to Poincaré type potentials¹. The proof follows Donaldson's argument, we are only working with a different function space.

¹Our formula is stated with a factor of 2 that does not appear in [Don02]. This is because Donaldson works with potentials modelled on $x \log x$ rather than $\frac{1}{2}x \log x$.

We phrase this closer to the statement found in [Don05a, Cor. 1]. For a Delzant polytope P with d facets, fix $i_1, \dots, i_k \in \{1, \dots, d\}$. Let Q be the complement in P of the facets $F_{i_1}, \dots, F_{i_k}$. For a point in P , let d_{PT} be the distance to the Poincaré type facets $F_{i_1} \cup \dots \cup F_{i_k}$ and let d_S be the distance to the smooth facets $\partial P \setminus (F_{i_1} \cup \dots \cup F_{i_k}) = \partial P \cap Q$. Let

$$\mathcal{C}_{P,d\sigma} = \{f \in C^\infty(P^\circ) \cap L^1(P, d\lambda) \cap L^1(\partial P, d\sigma) \cap C^0(Q) : f \text{ is convex, } d_{PT}f \in C^0(P) \\ \text{and } d_{PT}d_S\nabla f \in C^0(P, \mathbb{R}^n)\}.$$

First we note that this space contains the symplectic potentials.

Lemma 3.16. *Let P be a bounded polytope and $d\sigma$ a non-negative boundary measure for P . Then*

$$\mathcal{S}_{P,d\sigma} \subseteq \mathcal{C}_{P,d\sigma}.$$

We can now prove the integration by parts formula.

Lemma 3.17. *Let $u \in \mathcal{S}_{P,d\sigma}$ be a symplectic potential for a non-negative boundary measure $d\sigma$ and let $f \in \mathcal{C}_{P,d\sigma}$. Then $u^{ij}f_{ij}$ is integrable, and*

$$\int_P u^{ij}f_{ij}d\lambda = \int_P u^{ij}fd\lambda + 2 \int_{\partial P} fd\sigma.$$

Proof. For any $\delta > 0$, let P_δ be the polytope with faces parallel to P and a distance δ away from P . Let F denote the union $F_{i_1} \cup \dots \cup F_{i_k}$ of the Poincaré type facets and E the union of the remaining facets, and let F_δ, E_δ be the corresponding facets of P_δ . Since f is smooth over P_δ , we can integrate by parts to get that

$$\int_{P_\delta} u^{ij}f_{ij}d\lambda - \int_{P_\delta} u^{ij}fd\lambda = \int_{\partial P_\delta} (u^{ij}_i f - u^{ij}f_i).$$

The right hand side is a vector field, which we think of as an $(n-1)$ -form by contracting with $d\lambda$. We consider each term separately.

For a point $x \in \partial P_\delta$, define ν to be the unit normal to the face which x lies on. Then the term $u^{ij}f_i$ is just $\nabla_v f$, where v is the vector field $u^{ij}\nu_j$. Now divide the integral into two parts

$$\int_{\partial P_\delta} u^{ij}f_i = \int_{F_\delta} \nabla_v f + \int_{E_\delta} \nabla_v f.$$

To conclude that this integral vanishes, it suffices to prove that $|v| = O(d_{PT}^2 d_S)$. For then there is a $C > 0$ such that

$$\begin{aligned} \left| \int_{F_\delta} \nabla_v f \right| &\leq \int_{F_\delta} |v| |\nabla f| \\ &\leq C \int_{F_\delta} d_{PT}^2 d_S |\nabla f|. \end{aligned}$$

Since $d_{PT}\nabla f$ is continuous, it follows that $d_{PT}|\nabla f|$ is bounded from above on P , say by $D \in \mathbb{R}$. Thus

$$\begin{aligned} \left| \int_{F_\delta} \nabla_v f \right| &\leq CD(\text{Vol}(F_\delta)) \sup_{F_\delta} (d_{PT} d_S) \\ &\rightarrow 0. \end{aligned}$$

For $d_{PT} = \delta$ restricted to F_δ , and both d_S and $\text{Vol}(F_\delta)$ are bounded from above. We have $\text{Vol}(F_\delta) \leq \text{Vol}(F)$, where the volume is computed with respect to the standard measure given by contracting $d\lambda$ with the unit normal vector to F (not $d\sigma$).

For $\int_{E_\delta} \nabla_v f$, we use, like in [Don02], the convexity of f to obtain that

$$\left| \int_{E_\delta} \nabla_v f \right| \leq C \text{Vol}(\partial E_\delta) \sup_{x \in E_\delta} |d_{PT} f(y(x)) - d_{PT} f(x)|$$

where $y(x)$ is given as the closest point on F_δ to x , in the normal direction to E_δ at x . Since $d_{PT}f$ is continuous on the whole polytope P , and hence uniformly continuous, it follows that this goes to 0 as $\delta \rightarrow 0$.

To prove that $|v| = O(d_{PT}^2 d_S)$, we must consider the terms u^{ij} . These are $SL_n(\mathbb{Z})$ -invariant notions. So consider around a vertex, which we then may assume is 0 and that ∂P is given by the coordinate axes $x^1, \dots, x^n$, with P lying in the positive quadrant. Consider a point $x = (\delta, x_2, \dots, x_n) \in P$. Then $\nu = (1, 0, \dots, 0)$, and so the i^{th} coordinate of the vector field v is u^{ik} . Using a similar analysis as in the proof of proposition 3.1, we have that for each i , $u^{ik} = \pm \frac{\det(U_{ik})}{\det(U)}$ behaves, up to multiplication by a non-negative function, like $l_k^{\lambda_k}$, where $\lambda_k = 2$ if $x^k = 0$ is a Poincaré type facet and 1 if it is of smooth type. The upshot is that around p we can write $|\nu|$ has $\Pi_k(x^k)^{\lambda_k}$ multiplied with a non-negative function *on the whole polytope* P . Now $\Pi_k(x^k)^{\lambda_k} \leq C d_{PT}^2 d_S$, for some $C > 0$. Thus doing this over all vertices, of which it is only a finite number, we get that $|v| = O(d_{PT}^2 d_S)$.

For the second term, the analysis is similar. One gets that the in the model case, we have that $U^{-1} = (u^{ij})$ is diagonal with entries $u^{kk} = (ax^k)^2$ if $x^k = 0$ is of Poincaré type (for some $a > 0$) and $2x^k$ if $x^k = 0$ is of smooth type. Taking derivatives, one therefore sees that in the limit one gets either 0 or

$$2 \int_{\{x^k=0\} \cap P} f dx^k = 2 \int_{\{x^k=0\} \cap P} f d\sigma$$

in the Poincaré type and smooth type cases, respectively. Note that the definition of $d\sigma$ is $SL_n(\mathbb{Z})$ -invariant. When changing u to a general potential, all new terms are divided by x^k . Thus the integral of this over $x^k = \delta$ goes to 0 as $\delta \rightarrow 0$, and so the identity still holds. $\square$

Note that this forces $S(u)$ to be the affine linear function associated to $(P, d\sigma)$ if u is an extremal potential.

Corollary 3.18. *Suppose u is an extremal potential, i.e. $S(u)$ is affine linear. Then $S(u) = A$, where A is the affine linear function associated to $(P, d\sigma)$.*

Proof. Since $S(u) = -\frac{1}{2}u_{ij}^{ij}$, it follows by 3.17 that, for all affine linear f , we have

$$\begin{aligned}\int_P S(u)f d\lambda &= -\frac{1}{2} \int_P u_{ij}^{ij} f d\lambda \\ &= \int_{\partial P} f d\sigma.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{L}_{S(u)}(f) &= \int_{\partial P} f d\sigma - \int_P f S(u) d\lambda \\ &= 0\end{aligned}$$

for all affine linear f , and so $S(u)$ is the affine linear function associated to P . $\square$

The main calculations we need for the uniqueness result below is

Lemma 3.19. *Let $u_t = u + tv$ with $u \in \mathcal{S}_{P,d\sigma}$ and $v \in \mathcal{C}_{P,d\sigma}$ such that $u_t \in \mathcal{S}_{P,d\sigma}$ for all $|t| < \varepsilon$, for some $\varepsilon > 0$. Then for a smooth function A on P , we have that*

$$\frac{d}{dt}(\mathcal{F}_A(u_t)) = \frac{1}{2} \int_P (S(u_t) - A)v d\lambda \quad (3.11)$$

and

$$\frac{d^2}{dt^2}(\mathcal{F}_A(u_t)) = \frac{1}{2} \int_P u_t^{ik} u_t^{jl} v_{kl} v_{ij} d\lambda. \quad (3.12)$$

Proof. First note that

$$\frac{d}{dt}(\log \det(u_{ij} + tv_{ij})) = u_t^{ij} v_{ij}.$$

Since this is integrable over P , we can differentiate under the integral sign to obtain that

$$\frac{d}{dt}(\mathcal{F}_A(u_t)) = -\frac{1}{2} \int_P u_t^{ij} v_{ij} d\lambda + \mathcal{L}_A(v). \quad (3.13)$$

Since $v \in \mathcal{C}_{P,d\sigma}$, we have, by lemma 3.17, that

$$\begin{aligned}\frac{d}{dt}(\mathcal{F}_A(u_t)) &= -\frac{1}{2} \int_P u_t^{ij} v_{ij} - \frac{1}{2} \int_{\partial P} v d\sigma + \mathcal{L}_A(v) \\ &= \frac{1}{2} \int_P (S(u_t) - A)v d\lambda\end{aligned}$$

as required, since the scalar curvature $S(u_t)$ of the metric induced by u_t is $-\sum_{ij} u_t^{ij} u_t^{ij}$.

To obtain the second derivative, we differentiate equation 3.13 to obtain

$$\frac{d^2}{dt^2}(\mathcal{F}_A(u_t)) = -\frac{1}{2} \int_P \frac{d}{dt}(u_t^{ij}) v_{ij} d\lambda.$$

We have

$$u_t^{ij} u_{jk}^t = \delta_k^i.$$

Differentiating this and applying the inverse matrix of u_{ij}^t , we get that

$$\begin{aligned}\frac{d}{dt}(u_t^{ij}) &= -u_t^{ik}u_t^{jl}\frac{d}{dt}(u_{kl}^t) \\ &= -u_t^{ik}u_t^{jl}v_{kl}.\end{aligned}$$

Hence

$$\frac{d^2}{dt^2}(\mathcal{F}_A(u_t)) = \frac{1}{2} \int_P u_t^{ik}u_t^{jl}v_{kl}v_{ij}d\lambda,$$

as required. $\square$

Finally, our main result of this section is then

Theorem 3.20. *Let $u_0, u_1 \in \mathcal{S}_{P,d\sigma}$ for a bounded convex polytope P with non-negative boundary measure $d\sigma$. Suppose both u_0 and u_1 are extremal potentials. Then u_0 and u_1 differ by an affine linear function. In particular, if P is Delzant, extremal Kähler metrics in Ω_P with mixed Poincaré type and cone singularities along the torus invariant divisors arising from symplectic potentials are unique up to the action induced by a holomorphy potential.*

Proof. Consider $u_t = (1-t)u_0 + tu_1 = u_0 + tv$ with $v = u_1 - u_0$, which is a path in the space of symplectic potentials, by 3.15. Let A be the affine linear function associated to the weighted polytope $(P, d\sigma)$. Using equation 3.11, we have that

$$\frac{d}{dt}|_{t=t_0}(\mathcal{F}_A(u_t)) = \frac{1}{2} \int_P (S(u_{t_0}) - A)v d\lambda.$$

In particular, by 3.18,

$$\frac{d}{dt}|_{t=0,1}(\mathcal{F}_A(u_t)) = 0.$$

Since $\mathcal{F}_A$ is a convex function, $\frac{d}{dt}|_{t=t_0}(\mathcal{F}_A(u_t))$ is never negative, and so must be 0 for all $t_0 \in [0, 1]$. Hence $\mathcal{F}_A(u_t)$ is constant.

Thus $\frac{d^2}{dt^2}(\mathcal{F}_A(u_t)) = 0$. Using equation 3.12, we then get that

$$\int_P u_t^{ik}u_t^{jl}v_{kl}v_{ij}d\lambda = 0.$$

But u^{pq} is positive-definite, and so the only real-valued functions satisfying this are the functions with $v_{ij} \equiv 0$, i.e. the affine linear functions. So v is affine linear and the proof is complete. $\square$

3.3 Existence implies stability

In this section we show the following theorem.

Theorem 3.21. *Suppose $u \in \mathcal{S}_{P,d\sigma}$ is an extremal symplectic potential. Then $(P, d\sigma)$ is a stable polytope.*

This immediately gives the following corollary for extremal Poincaré type metrics.

Corollary 3.22. *Suppose $(N_P, c_1(L_P))$ is a polarised toric complex manifold with associated moment polytope P . Let $d\sigma$ be a non-negative boundary measure for P and $\underline{\beta} = (\beta_1, \dots, \beta_d)$ the associated weights for the divisors D_i corresponding to the facets F_i of P . Then if there exists an extremal potential $u \in \mathcal{S}_{P, d\sigma}$, we have that $(N_P, c_1(L_P))$ is relatively log K -stable of weight $\underline{\beta}$.*

Proof. From lemma 3.17, it follows that if f is a smooth function on the whole of the polytope P , then

$$\begin{aligned} \int_P u^{ij} f_{ij} d\lambda &= \int_P u^{ij} f d\lambda + 2 \int_{\partial P} f d\sigma \\ &= -2 \int_P S(u) f d\lambda + 2 \int_{\partial P} f d\sigma. \end{aligned}$$

From corollary 3.18, $S(u)$ is the affine function associated to $(P, d\sigma)$. Thus for all smooth f , we have

$$\int_P u^{ij} f_{ij} d\lambda = 2\mathcal{L}(f).$$

Given a piecewise linear function f , we can now argue as in [ZZ08a] and [ZZ08b], subdividing P into subpolytopes on which f is affine and applying the above, then taking the limit. Or we can argue as in [Ros12] using distributions that since f is convex and u^{ij} positive-definite, we have that certainly $\mathcal{L}(f) \geq 0$ for all f . Moreover, if f is not affine, then there exists at least one crease of f . Restricted to this crease, f_{ij} is a positive measure and so $\mathcal{L}(f) > 0$.

□

Chapter 4

Quadrilaterals

In this chapter we study in more detail stability and its relation to the existence of extremal metrics for toric surfaces. In particular, we consider how the weighted polytope stability of section 2.4 depends on weights in the case of quadrilaterals. We examine the case when the boundary measure vanishes on two edges and completely characterise the stable weights in this situation.

This gives some new indications about the metrics one should expect to arise in Donaldson's conjecture on the splitting of a polytope into semistable pieces, see section 2.6, and we also show what metrics arise for strictly semistable quadrilaterals. When allowing 0 boundary measure, we show that the set of stable weights along the boundary is not always open. This is unexpected, since this is an open condition when the boundary measure is everywhere positive. We relate this phenomenon to the singular behaviour the metrics have along the divisors corresponding to the edges with 0 boundary measure, and show that one should expect several types of behaviour. We also apply our results to show what the number of connected components of the set of unstable weights for any quadrilateral is.

The organisation of the chapter is as follows. In section 4.1, we state our main result, theorem 4.1, and some consequences.

In section 4.2, we recall the construction of Apostolov-Calderbank-Gauduchon in [ACG14] and [ACG15] and argue that we can still apply their construction for quadrilaterals of non-Delzant type, with arbitrary non-negative boundary measure.

Section 4.3 is devoted to proving theorem 4.1, using the framework of the previous section. A different construction of Legendre in [Leg11] must be used for some weighted trapezia, but we remark that her construction can be employed in the same way as the ACG construction. We find a definite stable region that generically splits the region that is unstable into 4 connected components. Moreover, in contrast to when the boundary measure is positive on all edges of the quadrilateral, we show that the stable region is in general *not open*.

In section 4.4, we relate our findings of the previous section to the question of existence of extremal metrics on the corresponding orbifold surface, whenever the quadrilateral is Delzant. In particular, we describe how the predominant behaviour of our solutions are of mixed cone singularity and Poincaré type singularities. However, the non-openness of the stable region on edges is related to the existence of an extremal metric with singularities along a divisor, but that this singular behaviour is neither conical nor of Poincaré-type. We also show that a strictly semistable quadrilateral which is not an *equipoised* trapezium (see definition 4.6) splits into two stable subquadrilaterals.

4.1 Statement of result

Let Q be the quadrilateral with vertices $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (1 + p, q)$ and $v_4 = (0, k)$, for some $q, k > 0$ and $p > \max \{-\frac{q}{k}, -1\}$. Then Q is a convex quadrilateral, and all quadrilaterals can be mapped to such a quadrilateral via a translation and a linear transformation. When the parameters are rational, this is a rational Delzant polytope, and so corresponds to a toric orbifold surface X_Q . Since it is a quadrilateral, $b_2(X_Q) = 2$.¹ The edges $E_1, \dots, E_4$ of Q are given as $l_i^{-1}(0)$, where

$$\begin{aligned} l_1(x, y) &= y, \\ l_2(x, y) &= -qx + py + q, \\ l_3(x, y) &= (q - k)x - (1 + p)y + k(1 + p), \\ l_4(x, y) &= x, \end{aligned}$$

and $Q = \bigcap_i l_i^{-1}([0, \infty))$. The canonical measure $d\sigma$ on ∂Q associated to these defining equations is thus given by

$$\begin{aligned} d\sigma|_{E_1} &= dx, \\ d\sigma|_{E_2} &= \frac{1}{q}dy, \\ d\sigma|_{E_3} &= (1 + p)dx, \\ d\sigma|_{E_4} &= dy. \end{aligned}$$

We will identify the weight $\underline{r} = (r_1, \dots, r_4)$, and so also the corresponding measure, with a formal sum $\sum_i r_i E_i$. Thus, for example, $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ is identified with $\frac{1}{2}E_1 + \frac{1}{2}E_2$.

The main result of the chapter is the following.

Theorem 4.1. *Let E_i, E_j be two different edges of Q that are not parallel. Then there exist explicit numbers $0 \leq r_0 < r_1 \leq 1$ such that $(1 - r)E_i + rE_j$ is*

¹In general, a two dimensional rational Delzant polytope with d edges is the moment polytope of a toric orbifold surface with $b_2 = d - 2$.

- stable if $r \in (r_0, r_1)$,
- not stable if $r \in [0, r_0)$ or $r \in (r_1, 1]$.

Moreover, $(1 - r)E_i + rE_j$ is

- stable at r_0 and r_1 if E_i and E_j are adjacent, unless $r_0 = 0$ or $r_1 = 1$, respectively,
- not stable at r_0 and r_1 if E_i and E_j are opposite.

If E_i and E_j are parallel, then $(1 - r)E_i + rE_j$ is unstable for all $r \in [0, 1]$.

We get the following corollary about the set of unstable weights for quadrilaterals. The vertices of $\sum r_i = 1$, corresponding to measures supported on one edge only, are always unstable. Thus the set of unstable weights can have at most four connected components. This is generically the case, but in the case of parallel sides there is different behaviour. Specifically, we have the following.

Corollary 4.2. *Let Q be a quadrilateral. Then the number of connected components of the unstable set is*

- 4 if Q has no parallel sides,
- 3 if Q is a trapezium which is not a parallelogram,
- 2 if Q is a parallelogram.

Proof. If Q has no parallel sides, then theorem 4.1 implies that there is a stable weight on each edge of the 3-simplex $\sum_i r_i = 1$. Since the stable set is a convex set, it follows that the stable set contains a sub-simplex whose complement has 4 connected components. Thus the unstable set does too.

If Q is a trapezium, but not a parallelogram, then the weights along the edge corresponding to weights which are non-zero only on the two parallel sides are all unstable. This reduces the number of connected components by one.

Finally, if Q is a parallelogram, the unstable set is precisely the two edges of the simplex $\sum r_i$ corresponding to having zero weights on two opposite edges of Q , which has two connected components. This follows because whenever $d\sigma$ does not vanish on two opposite edges, then one can use the product of the extremal potentials for $\mathbb{P}^1$ with Poincaré type singularity at one fixed point and cone angle singularity at the other fixed point to give an extremal potential for Q . Hence the unstable weights for a parallelogram are precisely the ones vanishing on opposite edges of Q . $\square$

The explicit numbers r_0 and r_1 of theorem 4.1 will be given in terms of the data k, p, q determining the quadrilateral in section 4.5. For now, we describe the property characterising them. Q has two pairs of opposite sides. Let $\phi(s, t)$ and $\psi(s, t)$ denote the polynomials of lemma 2.32 for these two pairs of edges. The domains of these functions each have two points which correspond to affine functions, i.e. the crease is exactly an edge of Q . These points are opposite vertices of $[0, 1] \times [0, 1]$, and, after possibly replacing e.g. $\phi(s, t)$ with $\phi(s, 1 - t)$, we can take these to be $(0, 0)$ and $(1, 1)$.

Similarly, we can also assume that in the case when $d\sigma$ vanishes on two adjacent sides, $(0, 0)$ in each domain is the point corresponding to the crease being on an edge with vanishing boundary measure. In the

case of $d\sigma$ vanishing on two opposite edges, we can assume ϕ is the function parameterising the Donaldson-Futaki invariant of simple piecewise linear functions with crease along the *other* pair of opposite edges. In particular, the points $(0,0)$ and $(1,1)$ in the domain of ϕ correspond to simple piecewise linear functions with crease on an edge where $d\sigma$ vanishes.

In fact, the points where the corresponding crease is an edge with vanishing boundary measure are critical points of ϕ or ψ .

Lemma 4.3. *Let P be a 2-dimensional polytope with non-negative boundary measure $d\sigma$. Let $\nu(s,t)$ be the polynomial in 2.32 for two edges F_1 and F_2 adjacent to an edge E along which $d\sigma$ vanishes. Then the point in $[0,1] \times [0,1]$ corresponding to a simple piecewise linear function with crease E is a critical point of ν .*

Proof. We may assume that $E = E_1$, $F_1 = E_4$ and $F_2 = E_2$, where E_1, E_2, E_4 are the edges specified at the beginning of this section. Note that P need not have 4 edges, but we can take the three edges we are considering to be of this form. The boundary measure vanishes along E , is $r_1 dy$ along F_1 and $r_2 dx$ along F_2 for some non-negative constants r_1, r_2 . The affine function $l_{s,t}$ that we integrate in 2.32 is

$$l_{s,t}(x,y) = (sq - tk)x - (1 + sp)y + tk(1 + sp)$$

and the point corresponding to the crease being the edge E is $(0,0)$.

By linearity it suffices to show that the directional derivative in two independent directions vanish. We first consider the partial derivative $\frac{\partial \phi}{\partial t}(0,0)$. So we are letting $s = 0$ and we would like to compute the derivative of

$$t \mapsto \int_{\partial P \cap \{l_t \geq 0\}} l_t d\sigma - \int_{P \cap \{l_t \geq 0\}} l_t A d\lambda$$

at 0, where $l_t = tk - y - tkx$. Here A is the affine linear function associated to the weighted polytope $(P, d\sigma)$.

In taking the integral over the polytope, the integrals of all the terms in lA is always divisible by t^2 , since the constant and x -term in l has a factor of t , and all integrals involving y will introduce an extra factor of t . Thus the derivative of $\int_{P \cap \{l_t > 0\}} lA d\lambda$ is 0 and we only need to consider the terms coming from the integral over the boundary.

For this part, we are then considering the derivative of

$$t \mapsto \int_{F_1 \cap \{l_t \geq 0\}} l_t d\sigma$$

since l_t is only positive on E and F_1 , and the boundary measure vanishes on E . But this equals

$$r_1 \int_0^{tk} (tk - y) dy$$

since $F_1 \subseteq \{x = 0\}$ and so $l_t = tk - y$ on F_1 . It follows that the derivative of this function vanishes at $t = 0$.

To complete the proof we need to check that the directional derivative in a linearly independent direction vanishes. One can consider $\frac{\partial \phi}{\partial s}(0,0)$. This case is similar. It then follows that $(0,0)$ is a critical point. $\square$

In particular, we get that the vanishing of the determinant at such a point is an invariant notion, i.e. it does not depend on the scale we used in defining ϕ and ψ .

Lemma 4.4. *Let $d\sigma_r$ be the boundary measure for Q corresponding to $rE_i + (1-r)E_j$ for edges E_i, E_j of Q , and let the polynomials of 2.32 for this boundary measure be ϕ_r and ψ_r . Then the determinant of the Hessian of ϕ_r or ψ_r at a point in $[0, 1] \times [0, 1]$ is quadratic in r .*

Proof. From their definition and lemma 2.28, ϕ_r and ψ_r are linear in r , and hence so are all their second derivatives with respect to s and t . Hence the determinant is of degree 2 in r . $\square$

We can then finally characterise what the r_0 and r_1 in theorem 4.1 are. They are given as the end-points of the intersection of the two regions where ϕ_r and ψ_r have *non-negative* determinant at the points corresponding to simple piecewise linear functions with crease an edge with 0 boundary measure. As remarked above, this does not depend on our choice of scale for ϕ and ψ .

The method of proof of theorem 4.1 is as follows. We first show that given any weights, there is a *formal ambitoric solution*, unless a simple condition necessary for stability is violated. A formal solution is a matrix-valued function H^{ij} with the correct boundary conditions associated to $(Q, d\sigma)$ and for which H^{ij}_{ij} is affine, but it may not be positive-definite everywhere in Q° . We then show that stability is equivalent to the positive-definiteness of the formal solution. We also show that in this case H^{ij} is in fact the inverse Hessian of a symplectic potential, so that in the case where Q is Delzant this is equivalent to the existence of a genuine extremal metric on the corresponding toric orbifold.

4.2 Formal solutions for quadrilaterals

We begin this section by reviewing the construction of Apostolov-Calderbank-Gauduchon, which we will refer to as the ACG construction. It will suffice for us to describe the construction only briefly. In particular, we will omit a lot of the formulae that are not directly used. However these can be found in [ACG15, Sect. 3.2].

Given a quadrilateral Q with no parallel edges, there is a 1-parameter family of conics $\mathcal{C}(Q)$ such that the edges of Q lie on tangent lines to $C(Q)$. Indeed, this condition just fixes four points on a dual conic $\mathcal{C}^*(Q)$, and there is a 1-parameter family of conics going through these four points. Given such a conic, we can sweep out the quadrilateral Q by taking the intersection of two tangent lines to $C(Q)$, provided we avoid having to use the tangent line to a point of $C(Q)$ at infinity.

Assuming this holds, we then get a new set of coordinates (x, y) on Q , by parameterising $C(Q)$ and identifying a point in $(x, y) \in C(Q) \times C(Q)$ with the intersection of the tangent lines to $C(Q)$ at x and y . The map is then well-defined away from the diagonal, and so to avoid any ambiguity we require $x > y$, so

that Q is the image under this map of a product of intervals $D = [\alpha_0, \alpha_\infty] \times [\beta_0, \beta_\infty]$ with

$$\alpha_0 < \alpha_\infty < \beta_0 < \beta_\infty. \quad (4.1)$$

This will be *positive ambitoric coordinates* for a quadrilateral Q .

Another way one could obtain new coordinates for a quadrilateral Q is the following. Take a line L with two marked points p_1, p_2 . One can then parameterise all the lines going through p_1 and p_2 , respectively, and take their intersections. This is well-defined provided we don't use the line L itself. For a given quadrilateral Q , there are two pairs (F_1, F'_1) and (F_2, F'_2) of opposite sides of Q . These coordinates are then obtained by letting p_i be the point corresponding to the intersection of F_i and F'_i . We call these coordinates *negative ambitoric coordinates*. This gives us a well-defined coordinate system provided the line containing p_1 and p_2 does not pass through the interior of the quadrilateral. Allowing one of the points p_i to be at infinity gives trapezia, whereas allowing the line to be the line at infinity gives parallelograms. Again, we can assume this map is defined on some product $D = [\alpha_0, \alpha_\infty] \times [\beta_0, \beta_\infty]$ of closed intervals satisfying 4.1.

Thus given the choice of such data, we get a map $\mu^\pm$, depending on whether we are considering positive or negative ambitoric coordinates. These send D to quadrilaterals $Q^\pm$. For rational parameters, [ACG15] showed that these were coordinates arising from what they call an ambitoric structure on a 4-orbifold. However, the maps can also be seen as simply giving new coordinates for quadrilaterals.

Remark 4.5. *Any given quadrilateral can admit multiple ambitoric coordinate systems, depending on the choice of data above, and it can also admit both positive and negative ambitoric coordinates.*

We now fix ambitoric coordinates as above, either positive or negative, and let A, B be *quartic* polynomials such that

$$\begin{aligned} A(\alpha_0) &= 0, A'(\alpha_0) = r_{\alpha_0}, \\ A(\alpha_\infty) &= 0, A'(\alpha_\infty) = r_{\alpha_\infty}, \\ B(\beta_0) &= 0, B'(\beta_0) = r_{\beta_0}, \\ B(\beta_\infty) &= 0, B'(\beta_\infty) = r_{\beta_\infty}, \end{aligned} \quad (4.2)$$

and

$$A + B = q\pi. \quad (4.3)$$

Here the r_γ are non-negative real numbers, $q(z) = q_0 z^2 + 2q_1 z + q_2$ is a quadratic, positive on $[\alpha_0, \alpha_\infty] \times [\beta_0, \beta_\infty]$, which is fixed by the choice of ambitoric coordinates for Q , and π is some other quadratic. This uniquely determines A and B , as these are 10 equations for 10 unknowns. It was shown in [ACG15] that these are in fact independent conditions.

Given A, B satisfying the above, we can define $\mathfrak{t}$ -invariant metrics on $D^\circ \times \mathfrak{t}$ by

$$g_\pm = \left(\frac{x-y}{q(x,y)} \right)^{\pm 1} \left(\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x) \left(\frac{y^2 d\tau_0 + 2y d\tau_1 + d\tau_2}{(x-y)q(x,y)} \right)^2 + B(y) \left(\frac{x^2 d\tau_0 + 2x d\tau_1 + d\tau_2}{(x-y)q(x,y)} \right)^2 \right),$$

provided A, B are positive throughout D° . Here $q(x, y)$ denotes $q_0xy + q_1(x + y) + q_2$ and (τ_0, τ_1, τ_2) are coordinates on the torus $\mathfrak{t}$ that satisfy

$$2q_1\tau_1 = q_2\tau_0 + q_0\tau_2.$$

Regardless of whether or not A and B are positive, the projection of this to the $\mathfrak{t}$ -fibres of the tangent bundle of $D^\circ \times \mathfrak{t}$ comes from a map $D^\circ \rightarrow S^2\mathfrak{t}^*$, which moreover is actually the restriction of a smooth map $D \rightarrow S^2\mathfrak{t}^*$.

We can then use one of the maps $\mu^\pm$ to consider this as a map on $Q^\pm$ instead. From the formulae of [ACG15], the $\mu^\pm$ are defined on an open subset containing D , and so it takes smooth functions on D to smooth functions on $Q^\pm$. Let $H_\pm : Q^\pm \rightarrow S^2\mathfrak{t}^*$ be the function

$$(x, y) \mapsto \left(\frac{x - y}{q(x, y)} \right)^{\pm 1} \left(A(x) \left(\frac{y^2 d\tau_0 + 2y d\tau_1 + d\tau_2}{(x - y)q(x, y)} \right)^2 + B(y) \left(\frac{x^2 d\tau_0 + 2x d\tau_1 + d\tau_2}{(x - y)q(x, y)} \right)^2 \right).$$

Then $H_\pm$ is smooth on $Q^\pm$. We then also have, as in [ACG15], that $H^\pm$ satisfies the boundary conditions required in 4.8 below for $Q^\pm$ with a boundary measure determined by the r_k and a choice of lattice, which we take to be generated by the normals to two adjacent sides of $Q^\pm$. In [ACG15], it was also shown that H_{ij}^{ij} is affine if and only if in equation 4.3, the quadratic π is orthogonal to the quadratic q under a suitable inner product.

Given a boundary measure $d\sigma$ on ∂Q , there is an associated affine function, see definition 2.23. In this chapter we will follow [ACG15] and call this affine function ζ , as A is used in the definition of an ambitoric metric above. We will need the following definition.

Definition 4.6 ([Leg11, Defn. 1.2]). *Let Q be a quadrilateral with vertices $v_1, \dots, v_4$ ordered such that v_1 and v_3 do not lie on a common edge of Q . An affine function f on a quadrilateral Q is equiposed on Q if*

$$\sum_i (-1)^i f(v_i) = 0.$$

A weighted quadrilateral $(Q, d\sigma)$ is an equiposed quadrilateral if its associated affine function ζ is equiposed.

There are many choices of ambitoric coordinates for a given quadrilateral. However, in the search for extremal potentials on weighted quadrilaterals, there is a preferred such coordinate system. In [ACG15], it was shown that almost all weighted rational Delzant quadrilaterals with rational weights admits ambitoric coordinates of the form above in which the solution H^{ij} to the system 4.2 has π orthogonal to q , under a necessary condition for stability. However, their argument did not use the rationality of the weights nor of the quadrilateral and so holds in the setting where we consider irrational parameters, and non-negative boundary measures.

Lemma 4.7 ([ACG15, Lem. 4]). *Let $(Q, d\sigma)$ be a weighted quadrilateral. Then provided $(Q, d\sigma)$ is not an equiposed trapezium, Q admits ambitoric coordinates such that the matrix H solving the system 4.2 has π is orthogonal to q if and only if $\phi(1, 0)$ and $\phi(0, 1)$ are positive.*

Here ϕ is the polynomial described in section 4.1. The points $(1,0)$ and $(0,1)$ correspond to the two simple piecewise linear functions with crease along a diagonal of Q .

We will call these coordinates *preferred ambitoric coordinates* for $(Q, d\sigma)$, and to obtain extremal potentials from the ambitoric ansatz we necessarily have to work in these coordinates. For the case of equiposed trapezia, we will require a different construction of Calabi type toric metrics due to Legendre in [Leg11, Sect. 4] that we describe in the next section.

The key in the argument of [ACG15] to show that relative K -stability is equivalent to the existence of an ambitoric extremal metric, goes back to Legendre in [Leg11]. The idea is to use the formal solution H^{ij} in preferred coordinates for $(Q, d\sigma)$, even though this is not necessarily positive-definite. One then shows that the positive-definiteness of H^{ij} is equivalent to stability.

The crucial lemma for this argument in the case of positive boundary measure is a version of Donaldson's integration by parts formula, that we generalised to the vanishing boundary measure setting in 3.17. In the current situation, the formula is applied to matrices that may not be the inverse Hessian of a function. In 3.17, the f are allowed to blow-up near the boundary at a certain rate. However, we will only need to consider smooth functions, so we only include these in our statement. In this case the proof is easier, as it is a direct application of Stokes's theorem, and so we omit it. This lemma has been used also in several other works such as in [Leg11]. The only difference is that we are allowing the r_i to be 0, which does not affect the proof.

Lemma 4.8. *Let P be a polytope in $\mathfrak{t}^*$, with facets $F_i = l_i^{-1}(0)$ for some affine functions l_i that are non-negative on P . Let $u_i = dl_i$ be the conormal to F_i , and define a measure $d\sigma$ on ∂P by $d\sigma|_{F_i} \wedge u_i = \pm d\lambda$, where $d\lambda$ is the Lebesgue measure on $\mathfrak{t}^*$. Suppose $H : P \rightarrow S^2 \mathfrak{t}^*$ is a smooth function on P such that on ∂P ,*

$$\begin{aligned} H(u_i, v) &= 0 \text{ for all } i \text{ and for all } v, \\ dH(u_i, u_i) &= r_i u_i \text{ for all } i, \end{aligned}$$

for non-negative numbers r_i . Then for any smooth function f on P ,

$$\int_P H^{ij} f_{ij} d\lambda = \int_P H_{ij}^{ij} f d\lambda + \int_{\partial P} f d\sigma_{\mathbb{R}},$$

where f_{ij} is the Hessian of f computed with respect to a basis of $\mathfrak{t}^*$ whose volume form is $d\lambda$, H^{ij} is the matrix obtained by evaluating H on the dual basis for $\mathfrak{t}$ and H_{kl}^{ij} is the Hessian of the function H^{ij} computed in these coordinates.

The formal solutions from the preferred ambitoric coordinates will give functions satisfying these boundary conditions, and with H_{ij}^{ij} affine. We will then show that stability is equivalent to H^{ij} being positive-definite. In the next section we will also show that if H^{ij} is positive-definite, then it is the inverse of the Hessian of a symplectic potential.

We are now ready to prove that stability is equivalent to the existence of positive formal solutions. The proof is exactly as in [ACG15]. The difference in the statement is that we have shown that one can interpret this for quadrilaterals that are not necessarily Delzant, and with weights that can be any non-negative real number, not necessarily positive and rational.

Proposition 4.9. *Let $H_{A,B}$ be the formal extremal solution associated to a weight $d\sigma$ of a quadrilateral Q admitting preferred ambitoric coordinates for this weight. Then $d\sigma$ is a stable weight if and only if A, B are positive functions on $(\alpha_0, \alpha_\infty)$ and (β_0, β_∞) , respectively.*

Proof. From 4.8 and that $H = H_{A,B}$ solves $H_{ij}^{ij} = \zeta$, it follows that

$$\mathcal{L}(f) = \int_P H^{ij} f_{ij} d\lambda$$

for all smooth f . This can also be applied in the sense of distributions to piecewise linear functions, and one obtains as in [ACG15, p. 6], that for simple piecewise linear functions with crease I ,

$$\mathcal{L}(f) = \int_I H(u_f, u_f) d\nu_f, \quad (4.4)$$

where u_f is a conormal to I suitably scaled and $d\nu_f$ satisfies $u_f \wedge d\nu_f = d\lambda$. For a general piecewise linear function f , one gets a positive combination of such contributions over all creases of f .

In particular, if A, B are positive on the interior regions, then $H_{A,B}$ is positive-definite and so this is positive for all piecewise linear functions. Thus $(Q, d\sigma)$ is stable.

Conversely, suppose A, B are not both positive on the interior regions. Assume first that $A(\alpha) \leq 0$ with $\alpha \in (\alpha_0, \alpha_\infty)$. Then letting f be a simple piecewise linear function with crease $I = \mu(\{\alpha\} \times [\beta_0, \beta_\infty])$, one gets in 4.4 that $H(u_f, u_f)$ is a positive multiple of $A(\alpha)$, and in particular $\mathcal{L}(f)$ is a positive multiple of $A(\alpha)$, and hence non-negative. Thus $(Q, d\sigma)$ is not stable. The argument for B is identical, using a simple piecewise linear function with crease of the form $\mu([\alpha_0, \alpha_\infty] \times \{\beta\})$ instead. $\square$

4.3 The stable region

In this section we will apply the extension of the ACG construction to arbitrary quadrilaterals with non-negative boundary measure to analyse the set of weights for which a quadrilateral is stable, and in particular prove theorem 4.1.

We begin with a lemma giving a sufficient condition for a weighted quadrilateral to admit preferred ambitoric coordinates. Given two edges E, F , let ϕ, ψ be the functions $[0, 1] \times [0, 1] \rightarrow \mathbb{R}$ parameterising the Donaldson-Futaki invariant of simple piecewise linear functions with crease meeting the two edges adjacent to E and F , respectively. We can suppose $(0, 0)$ is the vertex of $[0, 1] \times [0, 1]$ corresponding to the affine function vanishing exactly along E and similarly for ψ and F . Then $(1, 0)$ and $(0, 1)$ correspond to the two simple piecewise linear functions with crease a diagonal of Q , both for ψ and ϕ .

Lemma 4.10. *Let $(Q, d\sigma)$ be a weighted quadrilateral with $d\sigma$ vanishing on two edges E and F . If the Hessians of ϕ and ψ at $(0, 0)$ are both positive semi-definite, then ϕ and ψ are positive at $(1, 0)$ and $(0, 1)$.*

Proof. The proof uses direct computation. Consider the one-parameter family of boundary measures $d\sigma_r$ as in the statement of 4.1, and so we have corresponding polynomials ϕ_r and ψ_r . Note that $\phi_r(1, 0)$ is linear in r , and similarly for $\phi_r(0, 1)$. Let r_1, r_2 be the values for which $\phi_r(1, 0) = 0$ and $\phi_r(0, 1) = 0$, respectively.

A calculation shows the key property for our purposes, namely that the sign of the determinant of the Hessian of ϕ_{r_i} at $(0, 0)$ is the *opposite* of the sign of the determinant of the Hessian of ψ_{r_i} at $(0, 0)$. Thus the set of r for which these determinants are both positive is contained in the region where $\phi_r(1, 0)$ and $\phi_r(0, 1)$ have the same sign. Moreover, when $r = 0, 1$ at most one of the diagonals can correspond to a destabilising simple piecewise linear function. In particular, the region in which $\phi_r(1, 0)$ and $\phi_r(0, 1)$ have the same sign must intersect $[0, 1]$ and necessarily be such that this sign is positive. Then the region where the determinant condition holds must be contained in this region and the result follows. $\square$

We now use proposition 4.9 to give an easily computable criterion for stability on weighted quadrilaterals with no parallel sides and with 0 boundary measure on two adjacent sides. For a quadrilateral Q with edges $E_1, \dots, E_4$, let ϕ and ψ be the two functions corresponding to evaluating $\mathcal{L}$ on simple piecewise linear functions with crease meeting two opposite edges of Q . Also, given edges E_i, E_j , let $d\sigma_r$ be the measure corresponding to the formal sum $rE_i + (1 - r)E_j$, as in the statement of theorem 4.1.

Proposition 4.11. *Let E_i, E_j be adjacent sides of Q . Then $(Q, d\sigma_r)$ for $r \neq 0, 1$ is stable if and only if the Hessians of the functions ϕ and ψ are positive semi-definite at the points corresponding to the SPL function whose crease is an edge with 0 boundary measure.*

Proof. First note that under these conditions $(Q, d\sigma)$ is never an equiposed trapezium. Now, if the Hessians of ϕ and ψ at the points p, q corresponding to the simple piecewise linear function with crease an edge with 0 boundary are not positive semi-definite, then $(Q, d\sigma_r)$ is not stable. Indeed, from lemma 4.10, p and q are critical points of ϕ and ψ . Thus if the positive semi-definiteness does not hold, then either ϕ or ψ decreases in some ray away from p or q . Since ϕ and ψ are 0 at p and q , respectively, it follows that $\mathcal{L}_{\underline{r}}$ is negative on some simple piecewise linear function, hence $\underline{r}$ is not a stable weight for Q .

Conversely, suppose the Hessians are positive semi-definite. From lemma 4.10, $(Q, d\sigma_r)$ admits preferred ambitoric coordinates. So we must show that the formal solution $H_{A,B}$ has A and B positive in $(\alpha_0, \alpha_\infty)$ and (β_0, β_∞) , respectively.

We first show that A is positive if the Hessian of $\phi(s, t)$ is positive semi-definite at p , where ϕ is the Donaldson-Futaki invariant of simple piecewise linear functions with crease along the edges corresponding to $y = \beta_0$ and $y = \alpha_\infty$. Consider the Donaldson-Futaki invariant of functions f_c with crease $x = c$. From the proof of 4.9 we have that this is given by

$$\mathcal{L}(f_c) = A(c)h_c,$$

where h_c is a function obtained from integrating a smooth positive function over $I_c = \{(c, t) : t \in [\beta_0, \beta_\infty]\}$. In particular, if for simplicity the edge with 0 boundary measure is $x = \alpha_0$, we have that $A(\alpha_0) = A'(\alpha_0) = 0$, and so

$$\frac{d^2}{ds^2} \Big|_{s=0} (\mathcal{L}(f_{\alpha_0+s})) = A''(\alpha_0)h_{\alpha_0}.$$

It follows that $A''(\alpha_0) \geq 0$, since $\phi_{s,t}$ is positive semi-definite.

Now, A is a polynomial of degree 4 with a double zero at α_0 and a simple zero at α_∞ . Moreover, the condition on $A'(\alpha_\infty)$ implies that A is positive near α_∞ . If $A''(\alpha_0) > 0$, then A is positive near α_0 , too, and so this means that A must have two more zeros, counted with multiplicities, if A is not positive in $(\alpha_0, \alpha_\infty)$. But this means that A has five zeros, counted with multiplicities, and so A has degree at least 5, a contradiction. If $A''(\alpha_0) = 0$, then A can have no zeros in $(\alpha_0, \alpha_\infty)$, since it has degree 4 and we have 4 zeros at α_0 and α_∞ counted with multiplicity. In particular, A has constant sign in this interval. Since A is positive near α_∞ , it therefore follows that A is positive in $(\alpha_0, \alpha_\infty)$.

Similarly, one obtains the result for the case when the 0 boundary measure occurs at α_∞ . The same argument also works to show that B is positive in (β_0, β_∞) if ψ has positive semi-definite Hessian at q . $\square$

We are now ready to prove an analogous result for the case when opposite sides have 0 boundary measure. We analogously get a criterion that is easy to compute, but note that in this case it is an *open* condition. Note also that we only need to check this for *one* of the functions ϕ, ψ . Lemma 4.14 below, which forms part of the proof, will show that if ϕ is positive-definite at $(0, 0)$, then ψ is automatically positive.

Proposition 4.12. *Let $(Q, d\sigma)$ be a weighted quadrilateral where $d\sigma$ vanishes exactly on two opposite edges and such that $(Q, d\sigma)$ is not an equipoised trapezium. Then $(Q, d\sigma)$ is stable if and only if the Hessian of the function ϕ is positive-definite at the point $(0, 0)$.*

Remark 4.13. *We will see below that in the case of equipoised trapezia the same conclusion holds, but for now we will consider the cases where we can apply the ACG construction.*

Proof. As before, the points corresponding to an affine function are critical points of ϕ . Therefore, if the Hessian is not positive semi-definite at these points, then ϕ decreases in some direction. Since ϕ is zero at these points, it follows that if the Hessian is not positive semi-definite, then $(Q, d\sigma)$ is not stable.

Now assume the determinant condition holds. By lemma 4.10, $(Q, d\sigma)$ admits preferred ambitoric coordinates. Without loss of generality, assume that in these coordinates, the edges with 0 boundary measure correspond to $x = \alpha_0$ and $x = \alpha_\infty$, respectively. Let A, B be the quartics for the formal solution $H_{A,B}$.

To show that the B is positive, it suffices to show that all B 's in a region of weights containing the weights for which the Hessian condition holds, are positive. This follows from lemma 4.14 below and the positivity of the A on $(\alpha_0, \alpha_\infty)$ for all weights satisfying the Hessian condition that we will now show.

We do this by a similar argument as in the adjacent case, considering the second derivative of the Donaldson-Futaki invariant

$$\mathcal{L}(f_c) = A(c)h_c$$

of the family f_c of simple piecewise linear functions with crease $x = c$. The sign of this is the same as $A''(\alpha_0)$. Now, since A is a quartic with a double zero at both α_0 and α_∞ , it follows that $A(z) = \lambda(z - \alpha_0)^2(z - \alpha_\infty)^2$. Thus

$$A''(\alpha_0) = \lambda(\alpha_0 - \alpha_\infty)^2.$$

In particular, the sign of λ , which is positive if and only if A is positive on $(\alpha_0, \alpha_\infty)$, equals the sign of the second derivative of $\mathcal{L}(f_c)$. Since ϕ is positive-definite at the critical point, it follows that $\mathcal{L}(f_c) > 0$. Thus $\lambda > 0$ and so A is positive throughout $(\alpha_0, \alpha_\infty)$.

Finally we must consider the borderline case when the Hessian is strictly positive semi-definite. The above also shows that if ϕ is only positive semi-definite, then $\mathcal{L}$ vanishes on functions with crease $x = \alpha$ for all $\alpha \in [\alpha_0, \alpha_\infty]$. So in this case positive semi-definiteness is not sufficient, one needs ϕ to be positive-definite. $\square$

To complete the proof of proposition 4.12, we must show the following lemma.

Lemma 4.14. *Let $(Q, d\sigma)$ be a quadrilateral admitting preferred ambitoric coordinates and for which $d\sigma$ vanishes on two opposite edges of Q . Moreover, suppose $d\sigma$ is such that the Hessian of ϕ is strictly positive semi-definite at $(0, 0)$. Then for the formal solution $H_{A,B}$ associated to $d\sigma$, B is positive on (β_0, β_∞) .*

Proof. Under this Hessian condition, it follows that in the formal solution A is identically zero. Thus the formal solution satisfies

$$\begin{aligned} \pi(z)q(z) &= A(z) + B(z) \\ &= B(z). \end{aligned}$$

Thus q divides B and so the zeros of B are β_0, β_∞ and the zeros of q . However, recall that q must be chosen so that it does not have any zeros in $[\beta_0, \beta_\infty]$. Thus B has no zeros in (β_0, β_∞) , and so has constant sign in this interval. Since the boundary conditions imply that B increases from β_0 , it therefore follows that B is positive throughout (β_0, β_∞) , as required. $\square$

We have now found an easily computable criterion for stability for all weighted quadrilaterals with boundary measure vanishing on two edges, apart from equiposed trapezia, where either the boundary measure vanishes on two non-parallel sides or the boundary measure vanishes on the two parallel sides and is equal on the two non-parallel sides, using a normalisation as in [Leg11, Eqn. 4.7].

Of these two cases, the former are always unstable by an example due to Székelyhidi.

Proposition 4.15 ([Szé08, Prop. 15]). *Suppose Q has parallel sides. Then for any boundary measure which is supported on the two parallel sides, $(Q, d\sigma)$ is strictly semistable.*

For the latter case, let E and F be the two sides that are not parallel. We can apply a simple argument using the ACG construction in almost all situations to determine the stability of the boundary measure $d\sigma_r$ corresponding to $rE + (1-r)F$. The construction applies to all but one value of r , say r' . Doing this we get from proposition 4.12 that for $r \in (0, 1) \setminus \{r'\}$, $d\sigma_r$ is a stable weight for all $r \in (r_0, r_1) \setminus \{r'\}$, for some r_0, r_1 . Since the stable set is connected it follows that provided r' is neither r_0 nor r_1 , the stable set is (r_0, r_1) .

To rule out that r' can be one of the r_i and be a stable weight, we briefly mention the construction of Legendre in [Leg11] for equipoised trapezia. From this it will also be clear that the arguments of the next section will apply to this construction, so that stability for such trapezia are equivalent to the existence of an extremal metric with Poincaré type singularities along the two divisors corresponding to the opposite edges.

In this case, one can realize the moment polytope as the image of $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$ under the map

$$(x, y) \mapsto (x, xy),$$

for some $\alpha_2 > \alpha_1 > 0$ and $\beta_2 > \beta_1 \geq 0$. Let t_1, t_2 be the angle coordinates corresponding to these coordinates. One obtains metrics from two functions $A : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ and $B : [\beta_1, \beta_2] \rightarrow \mathbb{R}$, positive on the interiors of their domains, as

$$\frac{xdx^2}{A(x)} + \frac{xdy^2}{B(y)} + \frac{A(x)}{x}(dt_1 + ydt_2)^2 + xB(y)dt_2^2,$$

whenever A, B vanish at the end-points, and the derivatives of A and B at the end-points are determined by $d\sigma$, for positive weights.

The extremal condition is that A is a polynomial of degree at most 4, B is a polynomial of degree 2 with leading term $-a_2$, where a_2 is the coefficient of x^2 in A . This determines A and B uniquely and puts a condition on the conormals, only involving those along the edges $y = \beta_1$ and $y = \beta_2$. As before we can let A have double zeros at either end-points, which correspond to the boundary measure vanishing at $x = \alpha_1$ or $x = \alpha_2$. In the case when the boundary measure vanishes on both sides, one can then use exactly the same arguments as before to determine that stability is equivalent to this formal solution being positive on the interior, and that this in turn is equivalent to the Hessian condition of 4.12.

We now analyse the stable region, with the goal of proving that the region in which the Hessian condition is satisfied is non-empty, unless $(Q, d\sigma)$ satisfies the conditions of Székelyhidi's example. We first consider measures supported on adjacent sides.

Proposition 4.16. *Let Q be a quadrilateral and fix two adjacent sides E and F . Then there exists a boundary measure $d\sigma$ supported on E and F such that $(Q, d\sigma)$ is stable.*

The key step to proving this is the following lemma.

Lemma 4.17. *Let $(Q, d\sigma)$ be a weighted quadrilateral, and let Δ_1, Δ_2 be the two triangles obtained by splitting Q in two via a diagonal. Define a boundary measure $d\tau_i$ on Δ_i to be $d\sigma$ on the edges shared with Q and 0 on the edge corresponding to the diagonal of Q . Then the associated affine linear functions A_i of $(\Delta_i, d\tau_i)$ are never equal.*

Proof. We can assume that Δ_1 has vertices $(-1, 0)$, $(0, 1)$ and $(0, c)$ with $c > 0$ and that Δ_2 has vertices $(-1, 0)$, $(0, 1)$ and $(p, -q)$ with $q > 0$. The edges E_i of Q have defining functions

$$\begin{aligned} l_1(x, y) &= c - cx - y, \\ l_2(x, y) &= c + cx - y, \\ l_3(x, y) &= q + qx + (1 + p)y, \\ l_4(x, y) &= q - qx + (1 - p)y. \end{aligned}$$

We must also assume that $l_1(p, q)$ and $l_2(p, q)$ are positive to ensure that $\Delta_1 \cup \Delta_2$ is a convex quadrilateral. The boundary measure along E_i can be written as $r_i dy$, for some $r_i \geq 0$, but not all 0. Let $d\sigma_1$ be the boundary measure for Δ_1 and similarly for Δ_2 .

A long but elementary calculation shows that the affine linear function A_i associated to $(\Delta_i, d\sigma_i)$ is given by

$$\begin{aligned} A_1(x, y) &= 3(r_1 - r_2)x + \frac{3(r_1 + r_2)}{c}y, \\ A_2(x, y) &= 3(r_4 - r_3)x - \frac{3(r_3 + r_4 + r_3p - r_4p)}{q}y. \end{aligned}$$

We must show that they never can be equal provided Q is convex.

First of all, if $p \in (-1, 1)$, then the coefficient of y for A_2 is negative. Since the coefficient of y for A_1 is always non-negative, this means we must have $p \notin (-1, 1)$. By symmetry it suffices to check all the cases where $p \geq 1$, so we need to check that there is no solution for $p \in [1, 1 + \frac{q}{c}]$, the end-point $1 + \frac{q}{c}$ coming from the condition that Q is convex.

For this one can check that the general solution to $A_1 = A_2$ giving r_3 and r_4 in terms of r_1 and r_2 is affine linear in p . In particular, if r_3 is negative for $p = 1$ and for $p = 1 + \frac{q}{c}$ whenever r_1 and r_2 are positive, then this holds for all $p \in [1, 1 + \frac{q}{c}]$ and we are done.

This is indeed the case as one can check that at $p = 1$, the solution is

$$r_3 = -\frac{q(r_1 + r_2)}{c},$$

and at $p = 1 + \frac{q}{c}$, the solution is

$$r_3 = -\frac{r_2 q}{c},$$

both of which are negative. □

We can now prove proposition 4.16.

Proof. If there were no such weights, then there would have to exist a strictly semistable $(Q, d\sigma)$ with unique destabilising simple piecewise linear function given by a diagonal of Q . Indeed, there would certainly have to be one such boundary with crease going through one vertex of Q . If this was the case and this was not a diagonal, then $(Q, d\sigma)$ would admit preferred ambitoric coordinates. But then the simple piecewise linear functions with crease the ambitoric coordinate lines would have positive Donaldson-Futaki invariant. In particular, the formal solution $H_{A,B}$ would be positive-definite and so $(Q, d\sigma)$ would be stable, a contradiction.

If there was such a strictly semistable polytope whose unique destabilising function had crease a diagonal, it would follow from an argument similar to one given in [Don02], that the corresponding weighted subpolytopes $(\Delta_i, d\tau_i)$ would then have equal associated affine linear function A_i . But this violates lemma 4.17. $\square$

Finally, we consider the case when the boundary measure is supported on opposite sides.

Proposition 4.18. *Let Q be a quadrilateral and fix two opposite sides E and F . Then there exists a boundary measure $d\sigma$ supported on E and F such that $(Q, d\sigma)$ is stable if and only if E and F are not parallel.*

We already know one direction of this proposition due to Székelyhidi's example, so to prove 4.18, we thus have to show that if E and F are opposite sides that are not parallel, then there exists a stable weight. However, it will also be transparent in the proof that both directions are true.

Proof. We use the determinant condition of proposition 4.11. This determinant is a quadratic in r . Using similar calculations as in section 4.5 below, one can show that at the critical point of this quadratic, the value is

$$\frac{p^2 k^4 q (kp + k + q)^2}{4(k^2 p^2 + 2k^2 p + 2kpq + k^2 + 2kp + 2kq + q^2 + 2k)(k^2 p^2 + 2k^2 pq + 2k^2 p + 2kpq + 2kq^2 + k^2 + 2kq + q^2)}.$$

Here we are using the formulae given for the quadrilateral Q and the boundary measure as in section 4.1.

The numerator of this is always positive unless $p = 0$, which is the case when E and F are parallel. Note that $kp + k + q \neq 0$, it is in fact positive, since $p > -1$ and $q > 0$.

In the case of the denominator, we consider each factor separately. These are both quadratics in p , so it suffices to show that there are no zeros of these quadratics for the allowed values of p , and that at some point they are positive.

For

$$k^2 p^2 + 2k^2 p + 2kpq + k^2 + 2kp + 2kq + q^2 + 2k$$

the roots are

$$p = \frac{-k - q - 1 \pm \sqrt{2q + 1}}{k}.$$

Since $k > 0$, the larger of these roots is the one taking the positive sign, and so we must show that such a root is smaller than either -1 or $-\frac{q}{k}$. But if

$$\frac{-k - q - 1 + \sqrt{2q + 1}}{k} > -1,$$

then

$$\sqrt{2q + 1} > q + 1.$$

Since both sides are greater than 0, this inequality is preserved when squaring, and so this implies

$$q^2 < 0,$$

a contradiction. Thus all roots of the first factor satisfy that $p < -1$, hence it is positive for any convex quadrilateral.

For

$$k^2 p^2 + 2k^2 p q + 2k^2 p + 2k p q + 2k q^2 + k^2 + 2k q + q^2$$

the roots are

$$p = \frac{-kq - k - q \pm \sqrt{k^2 q^2 + 2k^2 q}}{k}.$$

The greater of these is again taking the positive sign, and if

$$\frac{-kq - k - q + \sqrt{k^2 q^2 + 2k^2 q}}{k} > -\frac{q}{k},$$

then

$$\sqrt{k^2 q^2 + 2k^2 q} > kq + k.$$

Squaring, this would imply that

$$k^2 < 0,$$

again a contradiction. Thus both terms are positive whenever $p > \max\{-\frac{q}{k}, -1\}$.

We have shown that the critical weight is a stable weight unless E and F are parallel. What remains is to show that the critical weight is a valid weight, i.e. lies in $(0, 1)$. Since the determinant condition is violated at both $r = 0$ and $r = 1$, it therefore suffices to show that the determinant increases at $r = 0$ to conclude that the critical r must lie in $(0, 1)$.

A computation shows that the derivative of the determinant at $r = 0$ being positive is equivalent to

$$\begin{aligned} & k^4 p^4 + 4k^4 p^3 + 4k^3 p^3 q + 6k^4 p^2 + 4k^3 p^3 + 16k^3 p^2 q + 6k^2 p^2 q^2 + 4k^4 p \\ & + 12k^3 p^2 + 16k^3 p q + 8k^2 p^2 q + 16k^2 p q^2 + 4k p q^3 + k^4 + 12k^3 p + 4k^3 q \\ & + 16k^2 p q + 10k^2 q^2 + 4k p q^2 + 4k q^3 + q^4 + 4k^3 + 8k^2 q + 4k q^2 \end{aligned} \quad (4.5)$$

being positive. Now, if $q \geq k$, then $p > -1$. In this case, we make the substitution $p = -1 + a$ above, so $a > 0$. We then have that 4.5 becomes

$$a^4 k^4 + 4 a^3 k^3 q + 4 a^3 k^3 + 4 a^2 k^3 q + 6 a^2 k^2 q^2 + 8 a^2 k^2 q - 4 k^3 q a + 4 k^2 q^2 a + 4 k q^3 a + 4 k q^2 a + q^4.$$

Since a, k and q are positive, the only negative term above is $-4k^3qa$. However, since we are assuming $q \geq k$, this is dominated by the term $+4kq^3a$. Hence this is always positive for all $a, k, q > 0$.

In the case when $q \leq k$, one can use the substitution $p = -\frac{q}{k} + a$ instead and use a similar argument to obtain the same conclusion. Thus the derivative of the determinant is positive at $r = 0$, and this completes the proof. $\square$

From the above results we thus get the following characterisation of the stable weights for a quadrilateral, which is simply theorem 4.1 with the numbers in the statement explicitly given. Recall from section 4.1 that fixing two edges E_i and E_j we have two associated polynomials ϕ and ψ .

Corollary 4.19. *Let Q be a quadrilateral and fix two edges E_i, E_j of Q . Let c_0, c_1 denote weights $(1-r)E_i + rE_j$ for which the determinant of ϕ vanishes at $(0,0)$ and similarly define c_2, c_3 for ψ . Then the weights of this form which are stable weights for Q are precisely given by*

- the intersection of $[c_0, c_1]$ and $[c_2, c_3]$ with $(0,1)$ if E_i and E_j are adjacent,
- $(c_0, c_1) \cap (0,1)$ if E_i and E_j are opposite.

This is always non-empty unless E_i and E_j are parallel edges, and in this case all such weights are unstable.

It follows from the examples of unstable positive weights for quadrilaterals with no parallel sides in [ACG15, Prop. 6] that for a quadrilateral with no parallel sides, and with boundary measure $d\sigma$ supported on one edge only, $(Q, d\sigma)$ is unstable and not strictly semistable, i.e. $\mathcal{L}(f) < 0$ for some simple piecewise linear function f . From this and our characterisation of the stable set along edges of the simplex $\sum_i r_i = 1$, it follows that given two adjacent edges E and F on such a quadrilateral, the r such that $rE + (1-r)F$ is stable is a *closed* non-empty interval, contained in $(0,1)$. Thus we get the following corollary.

Corollary 4.20. *Let Q be a quadrilateral without parallel sides. Then the set of weights $d\sigma$ for which $(Q, d\sigma)$ is stable, identified with a subset of $\mathbb{R}_{\geq 0}^4 \setminus \{0\}$, is neither open nor closed.*

This is surprising. When looking at positive weights the set of weights in $\mathbb{R}_{>0}^4$ for which $(Q, d\sigma)$ is stable is open. Indeed, in Donaldson's continuity method for extremal metrics on toric surfaces in [Don08], he in particular showed that for any polytope with positive weights, the set of weights which admits an extremal potential is open. The openness of the stable set then follows as we will show in the next section that when $d\sigma$ is positive on each edge of Q and $(Q, d\sigma)$ is stable, then the formal solution is the inverse Hessian of a symplectic potential. In the next section we will also discuss how the points on the boundary of the stable region can be explained by the formation of metrics with different sort of asymptotics than the Poincaré type metrics.

Remark 4.21. *If the optimal destabiliser of an unstable polytope is a simple piecewise linear function, then the polytope splits into two semistable pieces, and these pieces have vanishing boundary measure on only one edge. It is therefore interesting to see if this sort of behaviour can happen when one allows only one edge with zero boundary measure. The answer to this is yes. This follows from the above and the convexity of the stable set.*

Pick an edge E of Q and let the two edges adjacent to it be F_1, F_2 . Then there exists minimal r_1, r_2 such that $d\sigma_1 = (1 - r_1)E + r_1F_1$ and $d\sigma_2 = (1 - r_2)E + r_2F_2$ are stable weights, respectively. It follows that all convex combinations $(1 - r)d\sigma_1 + rd\sigma_2$ of these two weights are stable, and these vanish only on the edge opposite E when $r \in (0, 1)$.

4.4 Relation to the existence of extremal metrics

In the previous section, we showed how the existence of a positive-definite formal solution is equivalent to stability. In this section we will show that if the formal solutions are positive-definite then the H^{ij} is the inverse Hessian of a symplectic potential. These (generically) correspond to Poincaré type metrics on the edges with weight 0, however we show that a different behaviour occurs too. This will explain the non-openness of the stable set in the previous section. This corresponds to symplectic potential having the behaviour $u^{11} = O(x^3)$ near an edge lying in $x = 0$. This in turn correspond to the metric being modelled on

$$\frac{|dz|^2}{|z|^2(-\log(|z|))^{\frac{3}{2}}}$$

near the divisor corresponding to this edge.

The section has three parts. First we consider edges where the boundary measure is positive. Next, we take the case when the A, B have exactly double roots on edges where the boundary measure vanishes, and finally we consider the case of triple root. We emphasize that in this section we consider all metrics coming from the ACG construction. In other words, A, B can be arbitrary positive functions on $(\alpha_0, \alpha_\infty)$ and (β_0, β_∞) , respectively, satisfying the boundary conditions 4.2 and 4.3, not necessarily extremal potentials.

At edges with non-vanishing boundary measure, we get metrics with cone angles determined by $d\sigma$. Below we let F be the union of the edges where $d\sigma$ vanishes.

Lemma 4.22. *Let $(Q, d\sigma)$ be a weighted quadrilateral admitting positive or negative ambitoric coordinates defined on $[\alpha_0, \alpha_\infty] \times [\beta_0, \beta_\infty]$. Let A and B be positive on $(\alpha_0, \alpha_\infty)$ and (β_0, β_∞) , respectively, satisfying 4.2 and 4.3, but which do not have to be quartics. Let $H_{A,B}$ be the corresponding positive-definite map $Q \rightarrow S^2\mathfrak{t}^*$. Then $H_{A,B}$ is the inverse Hessian of a symplectic potential on Q° which satisfies the Guillemin boundary conditions at each edge with non-zero boundary measure. In particular, the $H_{A,B}$ equals the inverse Hessian*

of a function u which can be written as

$$\frac{1}{2} \sum_{\{i: d\sigma|_{E_i} \neq 0\}} l_i \log l_i + h,$$

where $h \in C^\infty(P^\circ) \cap C^0(P \setminus F)$ and l_i is the affine linear function defining l_i determined by $d\sigma$.

Proof. The proof follows from the analogous statement for rational weights, proved in [ACG15]. Indeed, by changing basis by a transformation which is not necessarily in $\mathrm{SL}_2(\mathbb{Z})$, one gets that $(Q, d\sigma)$ gets mapped to a quadrilateral where the boundary measure along our given edge E is the standard one. Thus we get that the open polytope $Q^\circ \cup E^\circ$ and the composition of the previous ambitoric coordinates with these transformations are ambitoric coordinates for this polytope with the new weight. Since this is rational data, it follows that it comes from an ambitoric structure on $\mathbb{C} \times \mathbb{C}^*$, and in particular by a symplectic potential satisfying the standard boundary conditions along E . It therefore follows that the original positive-definite matrix $H_{A,B}$ also comes from a symplectic potential satisfying the Guillemin boundary conditions along E determined by $d\sigma$. $\square$

Next, we consider the edges with 0 boundary measure, where the corresponding function vanishes exactly to second order.

Proposition 4.23. *Let $H_{A,B}$ be the function associated to an ambitoric structure on a weighted quadrilateral $(Q, d\sigma)$ as in lemma 4.22. Suppose A, B vanish exactly to second order at the points corresponding to the edges with 0 boundary measure. Then $u \in \mathcal{S}_{Q, d\sigma}$.*

Proof. We do the proof in the case of positive ambitoric coordinates. The proof in the negative case is similar.

Let $x = \alpha_0$ be an edge with 0 boundary measure. Let the symplectic coordinates be χ and η , which turn out to be given by

$$\begin{aligned} \chi &= \frac{(x - \alpha_0)(y - \alpha_0)}{q(x, y)}, \\ \eta &= \frac{(\beta_0 - x)(y - \beta_0)}{q(x, y)}. \end{aligned}$$

We then have that, for example

$$\frac{\partial \chi}{\partial x} = \frac{(y - \alpha_0)(q(x, y) - (x - \alpha_0) \frac{\partial q}{\partial x}(x, y))}{q^2(x, y)}.$$

Recall the inequalities 4.1, so that e.g. $y - \alpha_0$ is positive and bounded away from zero. Since $q(x, y)$ is smooth and positive in a neighbourhood of $[\alpha_0, \alpha_\infty] \times [\beta_0, \beta_\infty]$, it follows from this and similar calculations for the other entries in the Jacobian of this coordinate change that taking derivatives with respect to the (x, y) and (χ, η) variables are mutually bounded.

Taking n derivatives of the Hessian u_{ij} of the symplectic potential u in the χ direction is therefore mutually bounded with taking n derivatives in the x -direction of

$$\frac{q(x, y)}{(x - y)A(x)}.$$

This is in turn mutually bounded with taking n derivatives of

$$\frac{1}{(x - \alpha_0)^2}$$

with respect to x , since A vanishes exactly to order 2 at α_0 . Hence it is mutually bounded with

$$\frac{1}{(x - \alpha_0)^{2+n}}.$$

Taking n derivatives of the Hessian of the model symplectic potential in the χ -direction is mutually bounded with $\frac{1}{\chi^{2+n}}$, which in turn near $x = \alpha_0$ is mutually bounded with

$$\frac{1}{(x - \alpha_0)^{2+n}}$$

as well. Thus the symplectic potential u is mutually bounded with the model for derivatives to any order. Hence $u \in \mathcal{S}_{P,d\sigma}$. $\square$

Proposition 4.24. *Let $H_{A,B}$ be the function associated to an ambitoric structure on a weighted quadrilateral $(Q, d\sigma)$ as in lemma 4.22. Suppose A or B vanish to third order at a point corresponding to an edge $E = l^{-1}(0)$ with 0 boundary measure. Then u has the asymptotics of the model potential where one exchanges the term $-a \log(l)$ with*

$$\frac{a}{l},$$

with $a > 0$.

Proof. The proof is exactly the same as 4.23. One now instead obtains one higher power of $\frac{1}{(x - \alpha_0)}$ for both the model and the symplectic potential coming from the ambitoric framework. $\square$

Remark 4.25. *One could consider higher order vanishing as well and obtain metrics with different asymptotics near an edge with 0 boundary measure. However, for the purposes of extremal metrics, these are the only possibilities we have to consider. In that case the A and B are quartics with at least two distinct zeros, and so can at most vanish to third order at one of these zeros. Note also that a third order zero can only occur in the case when two adjacent sides have 0 boundary measure, as otherwise both zeros of A or B are double zeros.*

The model potential $\frac{1}{x}$ on $[0, \infty)$ induces the metric

$$\omega = \frac{idz \wedge d\bar{z}}{|z|^2 (-\log(|z|^2))^{\frac{3}{2}}} \quad (4.6)$$

on the unit punctured disk via the Legendre transform. Thus if one defines a space analogous to $\mathcal{S}_{P,d\sigma}$ for which the boundary behaviour is modeled on $\frac{a}{l}$ near a facet E contained in the zero set of an affine linear function l , one obtains by similar arguments as in the Poincaré type case a metric with the behaviour of ω near the divisor corresponding to E . In the case when there are several facets with 0 boundary measure, one can define spaces where one chooses either this or the Poincaré type behaviour on each such facet to get metrics with mixed cone singularities, Poincaré type and the behaviour of 4.6 along torus invariant divisors.

Applying the results of the previous section together with this immediately gives the following result regarding extremal metrics.

Corollary 4.26. *Suppose $(Q, d\sigma)$ is a stable weighted Delzant quadrilateral. Then X_Q admits an extremal metric in Ω_Q on the complement of the torus invariant divisors corresponding to edges with 0 weight and with cone angle singularities along the torus invariant divisors corresponding to edges with positive weight, the cone angle being prescribed by $d\sigma$.*

Let K be the Hessian of the function computing the Donaldson-Futaki invariant of simple piecewise linear functions with crease meeting two adjacent edges to an edge E with 0 boundary measure. If K is positive-definite at the point corresponding to an affine linear function with zero set containing E , then the metric has Poincaré type singularities along the torus invariant divisor corresponding to E , whereas if K is strictly positive semi-definite at this point, then the singularity along the corresponding divisor is modeled on

$$\frac{idz \wedge d\bar{z}}{|z|^2(-\log(|z|^2))^{\frac{3}{2}}}.$$

Our final result is an application to the conjecture about what happens when an extremal metric does not exist. It follows from lemma 4.17 that there are no strictly semistable weighted quadrilaterals whose unique destabilising function is a diagonal of Q . In fact, we get the following corollary, which shows that the conjecture of Donaldson holds in this case.

Corollary 4.27. *Let Q be a quadrilateral and suppose $d\sigma$ is a strictly semistable weight for Q . Then the crease of f splits Q into two subpolytopes $(Q_i, d\sigma_i)$, both of which are quadrilaterals and which admit an extremal potential $u_i \in \mathcal{S}_{Q_i, d\sigma_i}$.*

Proof. As remarked above, $(Q, d\sigma)$ admits preferred ambitoric coordinates in this case. In the formal solution $H_{A,B}$, we cannot have that both A and B are positive, as then $(Q, d\sigma)$ would be stable. Since $\mathcal{L}(h)$ is never negative for any h and A, B at any interior point is a positive multiple of the Donaldson-Futaki invariant of a simple piecewise linear function, it follows that either A or B has a zero in the interior of their domains of definition, but that they are not negative anywhere. Say A has a zero at $\alpha \in (\alpha_0, \alpha_\infty)$. Since $A \geq 0$, it follows that A must have a double zero at $x = \alpha$.

Restricting the ambitoric structure to $[\alpha_0, \alpha] \times [\beta_0, \beta_\infty]$ and $[\alpha, \alpha_\infty] \times [\beta_0, \beta_\infty]$ then gives two subpolytopes of Q . These are quadrilaterals as the crease of f is $x = \alpha$, which meets two opposite edges of Q . Moreover,

the restriction of A and B to these subpolytopes give extremal potentials u_i for $(Q_i, d\sigma_i)$. Since the order of vanishing at $x = \alpha$ is exactly 2, proposition 4.23 implies that $u_i \in \mathcal{S}_{Q_i, d\sigma_i}$. $\square$

Remark 4.28. *Note that while we have shown that metrics with singularities modelled on 4.6 can arise as solutions of the extremal equation when the boundary measure vanishes on at least one side, suggesting that these types of potentials could arise in the decomposition of a polytope into semistable subpolytopes, the above corollary shows that this does not occur for quadrilaterals.*

4.5 Formulae for stable weights for quadrilaterals

We will use the notation of section 4.1, so the k, p, q below are parameters determining Q . The formulae have been found by explicitly calculating ϕ and ψ in terms of the data of Q and $d\sigma_r$ alone using Maple, and then solving the condition determinant condition of corollary 4.19 for r .

We first consider the case when two adjacent edges have vanishing boundary measure, and without loss of generality we can take $d\sigma_r$ to vanish along E_2 and E_3 .

Let $c_0, \dots, c_3$ be given as

$$\begin{aligned} c_0 &= \frac{a_1 + \sqrt{b_1}}{2d_1}, \\ c_1 &= \frac{a_1 - \sqrt{b_1}}{2d_1}, \\ c_2 &= \frac{a_2 + \sqrt{b_2}}{2d_2}, \\ c_3 &= \frac{a_2 - \sqrt{b_2}}{2d_2}, \end{aligned}$$

where the a_i, b_i and d_i will be defined below. Then r_0, r_1 are the end-points of the intersection

$$[c_0, c_1] \cap [c_2, c_3] \cap (0, 1).$$

Here we swap the order of c_0 and c_1 if $c_1 < c_0$, and similarly for c_2, c_3 and ignore any c_i that is not real.

The a_i, b_i and d_i are given as follows.

$$\begin{aligned} a_1 &= k^6 - 18k^4p^5q - 82k^4p^4q - 16k^3p^4q^2 - 136k^4p^3q - 88k^3p^3q^2 - 96k^4p^2q - 156k^3p^2q^2 \\ &\quad - 24k^2p^2q^3 + 6kp^2q^4 - 22k^4pq - 112k^3pq^2 - 52k^2pq^3 + 8kpq^4 + 2q^5 + 2k^5 - 28k^3q^2 \\ &\quad - 28k^2q^3 + 6k^5p^2 + 8k^5p + 2k^4q + 2q^5p + 2q^4k - 24k^5p^5 - 34k^5p^4 - 16k^5p^3 \\ &\quad + 4k^5q + 3k^4q^2 + 8k^3q^3 + 3k^2q^4 + 4kq^5 + 36k^4p^2q^2 - 4k^3p^2q^3 + 23k^5pq + 26k^4pq^2 \\ &\quad + 10k^3pq^3 + 3k^2pq^4 + 3kpq^5 + 24k^5p^4q - 3k^4p^4q^2 + 54k^5p^3q + 10k^4p^3q^2 - 6k^3p^3q^3 \\ &\quad + 52k^5p^2q + 3k^5p^5q + 25p^4k^6 + 30p^3k^6 + 20p^2k^6 + 7pk^6 + 2p^6k^6 - 6p^6k^5 + 11p^5k^6 + q^6, \end{aligned}$$

$$\begin{aligned}
b_1 = & 4p^{12}k^{12} + 44p^{11}k^{12} + 36k^{11}p^{11}q + 221p^{10}k^{12} + 330k^{11}p^{10}q + 141k^{10}p^{10}q^2 + 670p^9k^{12} \\
& + 1390k^{11}p^9q + 1094k^{10}p^9q^2 + 310k^9p^9q^3 + 1365p^8k^{12} + 3560k^{11}p^8q + 3914k^{10}p^8q^2 \\
& + 2088k^9p^8q^3 + 405k^8p^8q^4 + 1968p^7k^{12} + 6168k^{11}p^7q + 8488k^{10}p^7q^2 + 6408k^9p^7q^3 \\
& + 2472k^8p^7q^4 + 288k^7p^7q^5 + 2058p^6k^{12} + 7588k^{11}p^6q + 12334k^{10}p^6q^2 + 11784k^9p^6q^3 \\
& + 6606k^8p^6q^4 + 1764k^7p^6q^5 + 42k^6p^6q^6 + 1572p^5k^{12} + 6748k^{11}p^5q + 12532k^{10}p^5q^2 \\
& + 14252k^9p^5q^3 + 10268k^8p^5q^4 + 4324k^7p^5q^5 + 588k^6p^5q^6 - 108k^5p^5q^7 + 870p^4k^{12} \\
& + 4320k^{11}p^4q + 9000k^{10}p^4q^2 + 11680k^9p^4q^3 + 10244k^8p^4q^4 + 5728k^7p^4q^5 + 1704k^6p^4q^6 \\
& - 96k^5p^4q^7 - 90k^4p^4q^8 + 340p^3k^{12} + 1940k^{11}p^3q + 4488k^{10}p^3q^2 + 6456k^9p^3q^3 \\
& + 6672k^8p^3q^4 + 4640k^7p^3q^5 + 2136k^6p^3q^6 + 360k^5p^3q^7 - 132k^4p^3q^8 - 20k^3p^3q^9 + 89p^2k^{12} \\
& + 578k^{11}p^2q + 1469k^{10}p^2q^2 + 2312k^9p^2q^3 + 2730k^8p^2q^4 + 2380k^7p^2q^5 + 1418k^6p^2q^6 \\
& + 648k^5p^2q^7 + 77k^4p^2q^8 + 2k^3p^2q^9 + 9k^2p^2q^{10} + 14pk^{12} + 102k^{11}pq + 278k^{10}pq^2 + 478k^9pq^3 \\
& + 652k^8pq^4 + 700k^7pq^5 + 556k^6pq^6 + 380k^5pq^7 + 198k^4pq^8 + 62k^3pq^9 + 30k^2pq^{10} + 6kpq^{11} \\
& + k^{12} + 8k^{11}q + 22k^{10}q^2 + 40k^9q^3 + 79k^8q^4 + 80k^7q^5 + 116k^6q^6 + 80k^5q^7 + 79k^4q^8 \\
& + 40k^3q^9 + 22k^2q^{10} + 8kq^{11} + q^{12},
\end{aligned}$$

$$\begin{aligned}
d_1 = & 2p^6k^6 + 2k^6p^5q + 11p^5k^6 + 6k^6p^4q - 3p^6k^5 + 3k^5p^5q + 6k^5p^4q^2 + 25p^4k^6 + 6k^6p^3q \\
& - 12k^5p^5 + 24k^5p^4q + 12k^5p^3q^2 - 9k^4p^5q - 3k^4p^4q^2 + 6k^4p^3q^3 + 30p^3k^6 + 2k^6p^2q \\
& - 17k^5p^4 + 54k^5p^3q + 6k^5p^2q^2 - 41k^4p^4q + 10k^4p^3q^2 + 6k^4p^2q^3 - 8k^3p^4q^2 - 6k^3p^3q^3 \\
& + 2k^3p^2q^4 + 20p^2k^6 - 8k^5p^3 + 52k^5p^2q - 68k^4p^3q + 36k^4p^2q^2 - 44k^3p^3q^2 - 4k^3p^2q^3 \\
& + 7pk^6 + 3k^5p^2 + 23k^5pq - 48k^4p^2q + 26k^4pq^2 - 78k^3p^2q^2 + 10k^3pq^3 - 12k^2p^2q^3 \\
& + 3k^2pq^4 + 3kp^2q^4 + 3kpq^5 + k^6 + 4k^5p + 4k^5q - 11k^4pq + 3k^4q^2 - 56k^3pq^2 + 8k^3q^3 \\
& - 26k^2pq^3 + 3k^2q^4 + 4kpq^4 + 4kq^5 + q^5p + q^6 + k^5 + k^4q - 14k^3q^2 - 14k^2q^3 + q^4k + q^5,
\end{aligned}$$

$$\begin{aligned}
a_2 = & -8k^4p^4q + 4k^4pq - 12k^4p^2q + 12k^2p^2q^3 + 16kpq^4 - 36k^3p^2q^2 - 8k^2pq^3 - 8k^3p^3q^2 \\
& + 4k^5 + 4q^5 + 2k^6 + 2q^6 + p^6k^6 + 7p^5k^6 + 20p^4k^6 + 30p^3k^6 + 25p^2k^6 + 11pk^6 \\
& - 20k^4p^3q - 32k^3pq^2 + 4k^3p^4q^2 + 12k^2p^3q^3 + 12kp^2q^4 + 3k^5p^5q + 2k^5q + 6k^4q^2 \\
& + 4k^3q^3 + 6k^2q^4 + 2kq^5 + 4pq^5 + 4k^5p^4 + 16k^5p^3 + 24k^5p^2 + 16k^5p + 4k^4q - 8k^3q^2 \\
& - 8k^2q^3 + 4kq^4 + 18k^5p^4q + 42k^5p^3q + 14k^4p^3q^2 - 6k^3p^3q^3 + 44k^5p^2q + 30k^4p^2q^2 \\
& + 4k^3p^2q^3 - 3k^2p^2q^4 + 19k^5pq + 22k^4pq^2 + 14k^3pq^3 + 3k^2pq^4 + 3kpq^5,
\end{aligned}$$

$$\begin{aligned}
b_2 = & k^{12}p^{12} + 14k^{12}p^{11} + 6k^{11}p^{11}q + 89k^{12}p^{10} + 78k^{11}p^{10}q + 9k^{10}p^{10}q^2 + 340k^{12}p^9 + 448k^{11}p^9q \\
& + 152k^{10}p^9q^2 - 20k^9p^9q^3 + 870k^{12}p^8 + 1520k^{11}p^8q + 896k^{10}p^8q^2 + 48k^9p^8q^3 - 90k^8p^8q^4 \\
& + 1572k^{12}p^7 + 3420k^{11}p^7q + 2848k^{10}p^7q^2 + 768k^9p^7q^3 - 276k^8p^7q^4 - 108k^7p^7q^5 + 2058k^{12}p^6 \\
& + 5404k^{11}p^6q + 5710k^{10}p^6q^2 + 2712k^9p^6q^3 - 18k^8p^6q^4 - 420k^7p^6q^5 + 42k^6p^6q^6 + 1968k^{12}p^5 \\
& + 6160k^{11}p^5q + 7744k^{10}p^5q^2 + 5120k^9p^5q^3 + 1136k^8p^5q^4 - 464k^7p^5q^5 + 288k^5p^5q^7 + 1365k^{12}p^4 \\
& + 5088k^{11}p^4q + 7308k^{10}p^4q^2 + 6016k^9p^4q^3 + 2414k^8p^4q^4 + 64k^7p^4q^5 + 12k^6p^4q^6 + 672k^5p^4q^7 \\
& + 405k^4p^4q^8 + 670k^{12}p^3 + 2990k^{11}p^3q + 4800k^{10}p^3q^2 + 4560k^9p^3q^3 + 2556k^8p^3q^4 + 524k^7p^3q^5 \\
& + 240k^6p^3q^6 + 672k^5p^3q^7 + 918k^4p^3q^8 + 310k^3p^3q^9 + 221k^{12}p^2 + 1190k^{11}p^2q + 2129k^{10}p^2q^2 \\
& + 2168k^9p^2q^3 + 1554k^8p^2q^4 + 484k^7p^2q^5 + 242k^6p^2q^6 + 504k^5p^2q^7 + 737k^4p^2q^8 + 614k^3p^2q^9 \\
& + 141k^2p^2q^{10} + 44k^{12}p + 288k^{11}pq + 584k^{10}pq^2 + 580k^9pq^3 + 520k^8pq^4 + 208k^7pq^5 + 64k^6pq^6 \\
& + 248k^5pq^7 + 300k^4pq^8 + 368k^3pq^9 + 216k^2pq^{10} + 36kpq^{11} + 4k^{12} + 32k^{11}q + 76k^{10}q^2 + 64k^9q^3 \\
& + 76k^8q^4 + 32k^7q^5 + 8k^6q^6 + 32k^5q^7 + 76k^4q^8 + 64k^3q^9 + 76k^2q^{10} + 32kq^{11} + 4q^{12},
\end{aligned}$$

$$\begin{aligned}
d_2 = & p^6k^6 + k^6p^5q + 7p^5k^6 + 3k^6p^4q + 3k^5p^5q + 3k^5p^4q^2 + 20p^4k^6 + 18k^5p^4q + 30p^3k^6 - 8k^6p^2q \\
& + 2k^5p^4 + 42k^5p^3q - 12k^5p^2q^2 - 4k^4p^4q + 14k^4p^3q^2 - 12k^4p^2q^3 + 2k^3p^4q^2 - 6k^3p^3q^3 - 8k^3p^2q^4 \\
& + 25p^2k^6 - 9k^6pq + 8k^5p^3 + 44k^5p^2q - 12k^5pq^2 - 10k^4p^3q + 30k^4p^2q^2 - 18k^4pq^3 - 4k^3p^3q^2 \\
& + 4k^3p^2q^3 - 12k^3pq^4 + 6k^2p^3q^3 - 3k^2p^2q^4 - 9k^2pq^5 + 11pk^6 - 3k^6q + 12k^5p^2 + 19k^5pq - 3k^5q^2 \\
& - 6k^4p^2q + 22k^4pq^2 - 6k^4q^3 - 18k^3p^2q^2 + 14k^3pq^3 - 6k^3q^4 + 6k^2p^2q^3 + 3k^2pq^4 - 3k^2q^5 + 6kp^2q^4 \\
& + 3kpq^5 - 3kq^6 + 2k^6 + 8k^5p + 2k^5q + 2k^4pq + 6k^4q^2 - 16k^3pq^2 + 4k^3q^3 - 4k^2pq^3 + 6k^2q^4 \\
& + 8kpq^4 + 2kq^5 + 2pq^5 + 2q^6 + 2k^5 + 2k^4q - 4k^3q^2 - 4k^2q^3 + 2kq^4 + 2q^5.
\end{aligned}$$

In the case of opposite edges, we take $d\sigma_r$ to vanish along E_1 and E_3 . Here we only have to check the positivity of the determinant of the Hessian of ϕ at $(0,0)$. The solutions now have a much nicer expression. They are

$$\begin{aligned}
r_0 &= \frac{2(p+1)k}{k^2p^2 + 2k^2p + 2kpq + k^2 + 2kp + 2kq + q^2 + 2k}, \\
r_1 &= \frac{(kp + k + q)^2}{k^2p^2 + 2k^2pq + 2k^2p + 2kpq + 2kq^2 + k^2 + 2kq + q^2}.
\end{aligned}$$

Chapter 5

Computational results

In this chapter we present a computational approach for checking stability on toric surfaces. This applies equally well for any weighted polytope $(P, d\sigma)$ as in section 2.4, and we work in this generality. We propose an approach that is not conclusive in general, but we argue its merits for two reasons. First of all, in many cases it can be successfully employed to prove stability and we give a few examples of this here using a computer program in Maple by the author. Secondly, in the notable case when the polytope has constant associated affine linear function, the method *is conclusive*. In particular, this gives the main result of the chapter.

Theorem 5.1. *There is an explicit algorithm which allows one to determine whether or not a polarised toric surface (X, L) is K -stable and this always concludes after a finite number of steps, depending quadratically only on $b_2(X)$.*

This always concludes in finite (short) time, and the author has from this made a computer program that takes as input a Delzant polytope and returns whether or not it is K -stable.

The organisation of the chapter is as follows. In section 5.1, we describe the computer program written by the author as well as basic properties of the problem that allow us to give an approach to proving stability. In section 5.2 we consider the case of vanishing classical Futaki invariant, and prove theorem 5.1. Finally, in section 5.3 we apply the approach to give new examples of stable and unstable toric surfaces, both in the case of vanishing Futaki invariant and in the relative case. We also apply our results to a family of polytopes of Wang and Zhou, and this gives a good approximation to the point in the family at which stability changes.

5.1 The computer program and the approach

The computer program is written in Maple. One inputs the data of the weighted polytope by specifying the vertices of P in coordinates x, y and specifying what multiple of the standard boundary measure one would

like. The vertices must also be in a specific order, starting with the vertex whose x -coordinate is lowest and going anti-clockwise. In the case there are two vertices v, w satisfying this, one starts with the one which has the lower y -coordinate of the two, v say, and so the final vertex will be w .

The program then can compute all the relevant quantities. It determines the associated affine linear function A . Also, given two pairs (E_i, p_i) of an edge E_i of P and a point p_i on E_i , it determines $\mathcal{L}(f)$ of a simple piecewise linear function with crease intersecting the edges E_i at the points p_i , for a scale we will make explicit later. One can also input an affine linear function and obtain the value $\mathcal{L}(f)$ of the corresponding simple piecewise linear function.

To compute all these quantities, one needs to integrate polynomials of at most order 2 over subpolytopes of P , and linear polynomials over the boundary of P . This is achieved by having explicitly computed such quantities over triangles, quadrilaterals and the relevant part of their boundaries, and then subdividing P (or a subpolytope) into such basic components.

We will now describe how the proposed approach works. The main idea is to use the parameterisation of lemma 2.32 to consider the restriction of $\mathcal{L}$ to simple piecewise linear functions with crease going through two fixed edges of P as a polynomial $p(s, t)$ in two variables $(s, t) \in [0, 1] \times [0, 1]$ and then applying Sturm's theorem for counting the number of zeros of a polynomial in one variable on each strip $t = c$.

Using lemma 2.32 we can define a family p_t of polynomials in one variable of degree at most 3 simply by setting

$$p_t(s) = p(s, t).$$

By computing at sufficiently many points, one can find what the polynomial is using the computational tools described above.

The approach now goes as follows. For polynomials q_1 and q_2 in s , let $\text{rem}(q_1, q_2)$ be the remainder when dividing q_1 by q_2 . We then let $p_t^0 = p_t$ and let $p_t^1 = p_t'$ and define polynomials p_t^2 and p_t^3 in s by

$$\begin{aligned} p_t^2(s) &= -\text{rem}(p_t^0, p_t^1), \\ p_t^3(s) &= -\text{rem}(p_t^1, p_t^2). \end{aligned}$$

Since each p_t is a polynomial of degree at most 3 in s , the sequence $p_t^0, \dots, p_t^3$ is the Sturm sequence for p_t . Sturm's theorem then says that the number of (distinct) zeros in $(0, 1]$ of p_t is given by the number of sign changes in the sequence

$$p_t^0(0), p_t^1(0), p_t^2(0), p_t^3(0)$$

minus the number of sign changes in the sequence

$$p_t^0(1), p_t^1(1), p_t^2(1), p_t^3(1).$$

Here if one of p^1 or p^2 is 0 at some point, the sign is counted as opposite of that of the following in the sequence. Zeros of p^3 count as having the opposite sign of the sign of the previous term in the sequence. Thus to determine if there are any simple piecewise linear functions with crease meeting our two fixed edges violating stability, we need to check if the number of sign changes is different for any choice of t .

The first step is then to determine if either $p_t(0)$ or $p_t(1)$ are zero for any $t \in [0, 1]$. Since this is a polynomial this is easily determinable by using Sturm's theorem again. If this is not the case, then this already violates stability. So assume now that we have $p_t(0)$ and $p_t(1)$ positive for each $t \in [0, 1]$.

The next thing to note is that $p_t^3(0) = p_t^3(1)$ for all t . Indeed, this is the constant term in the Sturm sequence and so is independent of s . Also, the number of sign changes in the $s = 1$ sequence has to be less than or equal to that of the $s = 0$ sequence, simply because there cannot be a negative number of sign changes.

Now $p_t^1(s) = \frac{\partial p}{\partial s}(s, t)$ is a polynomial of degree 3 in t for any given s . Hence we can explicitly determine its roots in the interval of interest at $s = 0$ and $s = 1$, say these are $\alpha_1 < \dots < \alpha_k$ where $\alpha_i \in (0, 1)$ and $k \leq 6$. Letting $\alpha_0 = 0$ and $\alpha_{k+1} = 1$, we then have that in (α_i, α_{i+1}) the signs of $p_1^t(0)$ and $p_1^t(1)$ do not change. The possibilities are:

- $p_1^t(0) > 0$ and $p_1^t(1) < 0$ for all $t \in (\alpha_i, \alpha_{i+1})$, in which case the $s = 0$ sequence changes sign at most two times and the $s = 1$ sequence changes sign at least once,
- $p_1^t(0) < 0$ and $p_1^t(1) > 0$ for all $t \in (\alpha_i, \alpha_{i+1})$, in which case the $s = 0$ sequence changes sign at least once and $s = 1$ sequence changes sign at most two times,
- $p_1^t(0), p_1^t(1) > 0$ for all $t \in (\alpha_i, \alpha_{i+1})$, in which case the $s = 0$ and $s = 1$ sequences change sign at most two times,
- $p_1^t(0), p_1^t(1) < 0$ for all $t \in (\alpha_i, \alpha_{i+1})$, in which case the $s = 0$ and $s = 1$ sequences change sign at least once.

In the first of these cases we can immediately conclude that there is no zero.

Lemma 5.2. *Suppose that for all $t \in (\alpha, \beta)$ we have that p_t is positive at $s = 0$ and $s = 1$, and that $p_t^1(0) > 0$ and $p_t^1(1) < 0$. Then p does not have any zeros in $[0, 1] \times (\alpha, \beta)$.*

Proof. In this interval, the number of sign changes of the Sturm sequence at $s = 0$ is at most two, since its first two terms are positive. Since the number of sign changes of the $s = 1$ is never greater than that of the $s = 0$ sequence, it follows that there are at most two sign changes for the $s = 1$ sequence and at least one sign change for the $s = 0$ sequence. But then the number of sign changes of both sequences is 1 if p_t^3 is negative and 2 if p_t^3 is positive. In either case, it is equal. Note that if either $p_t^2(s)$ or $p_t^3(s)$ for some t at $s = 0$ or $s = 1$, then the terms $p_t^2(s)$ and $p_t^3(s)$ count as having opposite signs, so the same conclusion holds in this case. $\square$

For the remaining three cases, it is possible that p can have zeros, and it is here that the inconclusiveness of the approach comes in. The issue is that the two remaining terms in the Sturm sequence are rational

functions whose numerator and denominators in general have higher degree than four, so we cannot determine their zeros exactly.

However, in many cases one can conclude that the sign of the required terms have a particular sign in the given intervals (α_i, α_{i+1}) and so one is able to carry out the rest of the analysis. The key is to use that the final term of the sequence is always equal.

For example, in the final case above, the only possibility that would ensure a zero of p in $[0, 1] \times (\alpha_i, \alpha_{i+1})$ is if the $s = 0$ sequence changed sign three times and the $s = 1$ sequence changed sign once. For this to be the case, we would need to find a subinterval of (α_i, α_{i+1}) in which $p_t^2(0)$ is positive, but $p_t^2(1)$ and $p_t^3(0)$ are negative. If we can show that no such interval exists, then there are no zeros of p in the given region.

In fact, 2 or 3 sign changes in the $s = 0$ sequence is necessary for p to have a zero.

Lemma 5.3. *If $p_t^0(0)$ and $p_t^0(1)$ are positive and there is a region (α, β) such that the Sturm sequence at $s = 0$ has either 0 or 1 sign change, then there is no zero of p in $[0, 1] \times (\alpha, \beta)$.*

A final simple case which allows one to conclude that there are no zeros of p , is in regions where both the p_t^2 terms are negative.

Lemma 5.4. *If $p_t^0(0)$ and $p_t^0(1)$ are positive and there is a region (α, β) such that both $p_t^2(0)$ and $p_t^2(1)$ are positive for all $t \in (\alpha, \beta)$, then p does not have any zeros in $[0, 1] \times (\alpha, \beta)$.*

Proof. In this case, there is one change in sign of $p_t^0(s)$, $p_t^1(s)$ and $p_t^2(s)$ for every t for both the $s = 0$ and the $s = 1$ sequence, regardless of what p_t^1 is. The final term is equal in both cases, and so it follows that the change in sign will be the same for every t . $\square$

Remark 5.5. *Some special care needs to be taken when dealing with the case when some of the points in $[0, 1] \times [0, 1]$ correspond to affine functions. This happens either when the two sides are adjacent, or that another edge is adjacent to both of them. In the former case, all functions with $s = 1$ or $t = 0$ will correspond to affine functions. But then one can divide p by $(1 - s)t$. This polynomial is of bidegree $(2, 2)$ and so one can apply the Sturm method to this polynomial instead. In the case when the edges share an adjacent edge, one of the vertices of $[0, 1] \times [0, 1]$ will be a trivial zero that one has to disregard. However, one can apply the Sturm sequence anyway as it counts the number of distinct zeros, and so simply disregard the one given by the trivial zero.*

5.2 The case of vanishing Futaki invariant

A notable special case is when the associated affine linear function is constant. When P is Delzant and equipped with its standard boundary measure, this corresponds to the case when the classical Futaki invariant vanishes, and an extremal metric on X_P in the class Ω_P is then necessarily of constant scalar curvature. In general we say that $(P, d\sigma)$ has vanishing Futaki invariant in this case, even if P is not Delzant.

When P has vanishing Futaki invariant, the degree of the polynomial $p(s, t)$ drops by one in each variable and so it is of bidegree $(2, 2)$. In terms of the Sturm sequence approach, this means that the p_2^t terms now are equal. Hence once $p(s, t)$ is guaranteed to be positive if $s = 0$ or 1 , we have that if $p_1^t(s)$ has the same sign at $s = 0$ and $s = 1$, or if $p_1^t(1) \leq 0$, then there cannot be a zero of p for that value of t . Thus we need to consider the case when $p_1^t(1) > 0$ and $p_1^t(0) \leq 0$. Note that $p_1^t(0)$ and $p_1^t(1)$ are quadratics and so we can determine explicitly the values of t for which this holds.

If $p_1^t(0) < 0$, then the only possibility for p to have a zero at this value of t is if $p_2^t(0) \geq 0$. If this does not hold, then the number of sign changes is 1 in each case. However, if this does hold then the number of sign changes of the $s = 0$ sequence is two and for the $s = 1$ sequence it is 0 if $p_2^t(0) > 0$ and 1 if $p_2^t(1) = 0$. Special care needs to be taken for t_0 such that $p_1^{t_0}(0) = 0$. However, by applying the Sturm method to the polynomial $p(s, t_0)$ in s one can easily deal with this separately.

As in the general case, when the two edges are adjacent we must above take special care. The polynomial obtained by dividing by $t(1 - s)$ is of bidegree $(1, 1)$ and so it suffices to check its positivity on the boundary to conclude that it is positive throughout $[0, 1] \times [0, 1]$, which is equivalent to there being no non-trivial zeros. In the case when the edges share an adjacent edge, one of the vertices of $[0, 1] \times [0, 1]$ will be a trivial zero that one has to disregard. However, this can just be accounted for when doing the Sturm sequence as in the general case.

Thus, we have a conclusive algorithm that can determine if a polytope with vanishing Futaki invariant is stable. In particular, since [CS16] has recently shown that it suffices to check K -stability for toric varieties on toric degenerations, we can always determine if a polarised toric surface is K -stable by determining if the classical Futaki invariant vanishes and if it does, apply the following steps for each pair of edges of P :

- Check that the associated polynomial $p(s, t)$ is positive at $s = 0$ and $s = 1$,
- Determining the values of t for which $p_1^t(1) > 0$ and $p_1^t(0) \leq 0$,
- In each such region determine if the constant term $p_2^t(0)$ is non-negative,
- Check separately at the zeros of $p_1^t(0)$.

The maximum number of steps required to check this is the same for every pair of edges. Thus the number of steps required when implementing the algorithm depends only on the number of pairs of distinct edges of P and this dependence is linear. In particular, it depends only on the number of vertices, which equals $b_2(X) + 2$, and so depends only on $b_2(X)$, and this dependence is quadratic. This concludes the proof of 5.1.

5.3 Applications

We will now apply the above computer program and algorithm to produce unstable and stable examples. Unstable examples are easier to show are unstable, as one simply needs to exhibit one simple piecewise linear

function violating stability. However, if they all seem to have positive Donaldson-Futaki invariant, we invoke the algorithm in order to prove stability.

We will only apply this to actual Delzant polytopes with their usual boundary measure, so that this means that there is a corresponding polarised toric manifold which we verify is relatively K -stable or unstable with respect to toric degenerations. We will do both the case of vanishing and non-vanishing Futaki invariant. We will also apply the approach to a family of polytopes due to [WZ11], and approximate at what point this family goes from being stable to unstable.

5.3.1 Unstable examples

In this section we present some unstable examples. First we give an example with vanishing Futaki invariant. A general way to obtain such polytopes is by starting with one with vanishing Futaki invariant and doing some cutting of the corners in a symmetric fashion. This will then preserve the vanishing Futaki invariant condition. This method has been used many times before, for example in [Don02, Sec. 7.2] and [WZ11, Sec. 2.2].

Example 5.6. *Take P to have 16 vertices, where*

$$\begin{aligned} v_1 &= (-100009, -2), \\ v_2 &= (-100008, -3), \\ v_3 &= (-100006, -4), \\ v_4 &= (-100000, -6), \end{aligned}$$

and the remaining 12 are given by reflecting in the x and y axes and the diagonal $y = -x$. This then has vanishing Futaki invariant with $a = \frac{200012}{1200079}$. This corresponds to the complex manifold X obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ 12 times. Each fixed point of $\mathbb{P}^1 \times \mathbb{P}^1$ is first blown up once. Each of these is either on the divisor $\{0\} \times \mathbb{P}^1$ or $\{\infty\} \times \mathbb{P}^1$ and one blows up successively at the fixed point which is the intersection of the previous exceptional divisor and one of these two divisors twice. The class is $c_1(L)$ for some line bundle one can determine from the polytope.

Let E_i be the edge connecting vertex v_i and v_{i+1} (with $v_{17} = v_1$). Then the graph of simple piecewise linear functions with crease going through edges E_4 and E_{12} in the region $[0, \frac{1}{5000}] \times [\frac{4999}{5000}, 1]$ is given in figure 5.1.

It is clear from this that $(X, c_1(L))$ is K -unstable. For example, the simple piecewise linear function f corresponding to the point $(\frac{1}{5000}, \frac{4999}{5000})$ has Donaldson-Futaki invariant

$$\mathcal{L}(f) = -\frac{272635012}{1200079},$$

and so this violates stability. Note that this example is only a minor modification of an example of [WZ11], where we consider a slightly different Kähler class and also specific values instead of asymptotics.

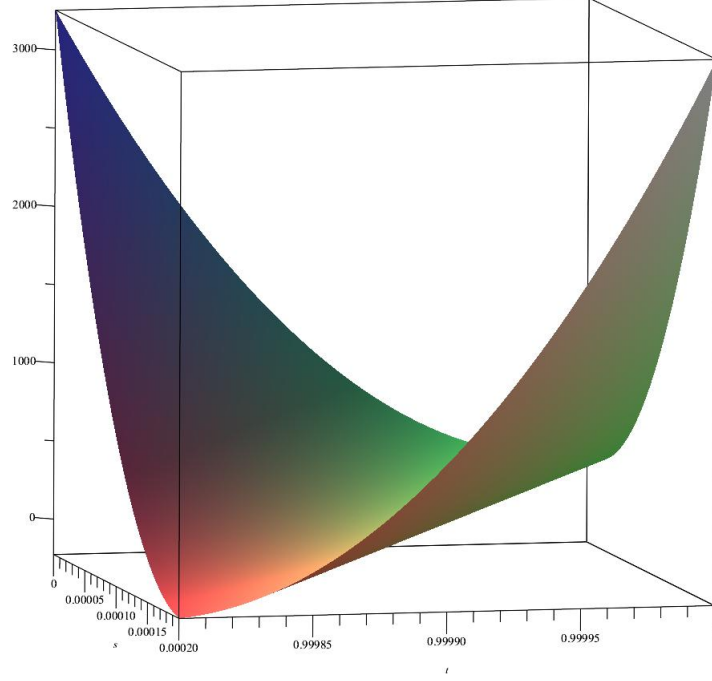


Figure 5.1: Graph of $p(s, t)$ for example 5.6 in the region $[0, \frac{1}{5000}] \times [\frac{4999}{5000}, 1]$.

The next example has non-vanishing Futaki invariant.

Example 5.7. Take P to have vertices

$$\begin{aligned}
 v_1 &= (0, 15), v_2 = (1, 14), \\
 v_3 &= (7, 11), v_4 = (10, 10), \\
 v_5 &= (14, 9), v_6 = (19, 8), \\
 v_7 &= (25, 7), v_8 = (32, 6), \\
 v_9 &= (40, 5), v_{10} = (49, 4), \\
 v_{11} &= (89, 0), v_{12} = (1034, 0), \\
 v_{13} &= (1000, 17), v_{14} = (0, 17).
 \end{aligned}$$

The associated affine linear function is

$$\frac{403073373791035632}{32209103321825561266313}x + \frac{26451417138778253376}{32209103321825561266313}y + \frac{3349671044343621439158}{32209103321825561266313},$$

which in particular is non-constant. The polytope corresponds to the complex manifold X obtained by blowing

up the 2nd Hirzebruch surface 10 times, each time at the point which is at the intersection of the proper transform of a fixed divisor and the previous exceptional divisor. The Kähler class is $c_1(L)$ for some line bundle determinable from P . Then the graph of $p(s, t)$ for simple piecewise linear functions with crease going through edges 10 and 14 is given in figure 5.2.

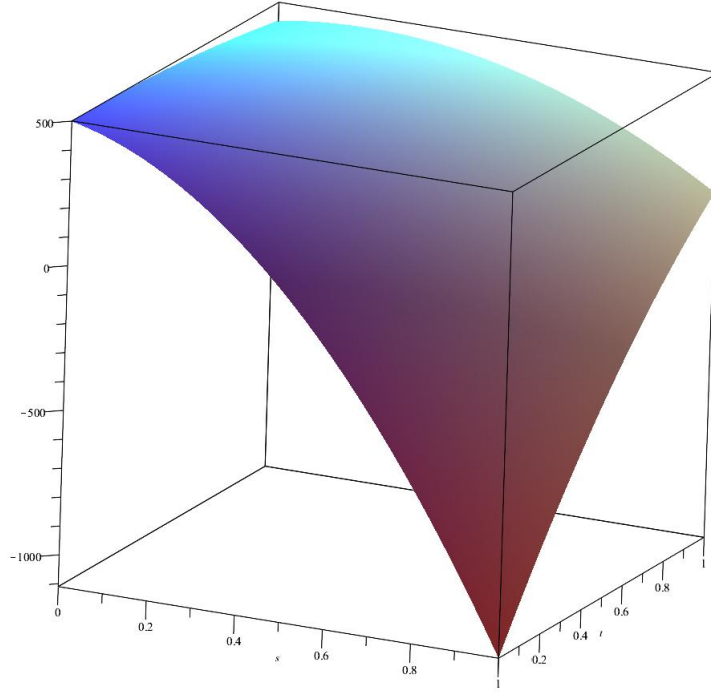


Figure 5.2: Graph of $p(s, t)$ for example 5.7.

It is clear from this that $(X, c_1(L))$ is relatively K -unstable. For example, the simple piecewise linear function corresponding to the point $(1, 0)$, which has crease going through vertices v_{11} and v_{13} , has Donaldson-Futaki invariant

$$\mathcal{L}(f) = -\frac{35749948793469237237251630}{32209103321825561266313},$$

and so violates stability.

5.3.2 The family of Wang-Zhou

In this section we will consider a family of Delzant polytopes P_k constructed by Wang-Zhou in [WZ11]. They showed that for large k , P_k is unstable. The P_k have 9 vertices and it is currently not known whether or not there exists any Delzant polytopes with fewer vertices that are unstable, when using the usual boundary measure.

Let P_k be the polytope with vertices

$$\begin{aligned} v_1 &= (0, -k), \\ v_2 &= (7, -k), \\ v_3 &= (7, k), \\ v_4 &= (5, k + 10), \\ v_5 &= (4, k + 14), \\ v_6 &= (3, k + 17), \\ v_7 &= (2, k + 19), \\ v_8 &= (1, k + 20), \\ v_9 &= (0, k + 20). \end{aligned}$$

The complex manifold X whose moment polytope is P_k is $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up 5 times, first at a fixed point lying on a torus invariant divisor D and then blowing up successively at the fixed point which is the intersection of the proper transform of D and the exceptional divisor obtained from the previous blow-up. The Kähler class is $\Omega_k = \pi^*(k[D]) + \alpha$, where α is some fixed cohomology class on X .

It was shown in [WZ11] that for large enough k there are simple piecewise linear functions that violate stability with crease going through edges E_2 and E_9 (i.e. the two edges parallel to the y -axis). When changing the value of k , the Ω_k changes along a straight line path in the Kähler cone, and the space of (relatively) K -stable classes is a subcone of this which moreover is convex. Thus there is a critical value k_0 for which P_k is stable for all $k < k_0$ and unstable for $k > k_0$. We will now approximate this to a high order, and our method can be applied to approximate k_0 to any desired order.

We will only consider simple piecewise linear functions with crease going through the edges E_2 and E_9 . If one applies the algorithm to the other pairs of edges one can show that there are no violating functions with crease going through those edges around the critical k_0 , but we will not show that. We will first graph the behaviour for very small and very large values of k to see exactly what behaviour in the Sturm sequence corresponds to the change in stability. First let k be 1. The graph of the polynomial $p(s, t)$ over $[0, 1] \times [0, 1]$ is given in figure 5.3 below.

This seems to be positive everywhere except at the point $(0, 1)$ corresponding to an affine linear function. However, we will prove this is the case by applying the algorithm. The graphs of the elements in the Sturm sequence is given in figure 5.5 below.

For $s = 0$, we have that p_t^0 is always positive, and p_t^1, p_t^2, p_t^3 all have one zero, say at t_1, t_2, t_3 , respectively. The key to notice is that

$$t_1 < t_2 < t_3$$

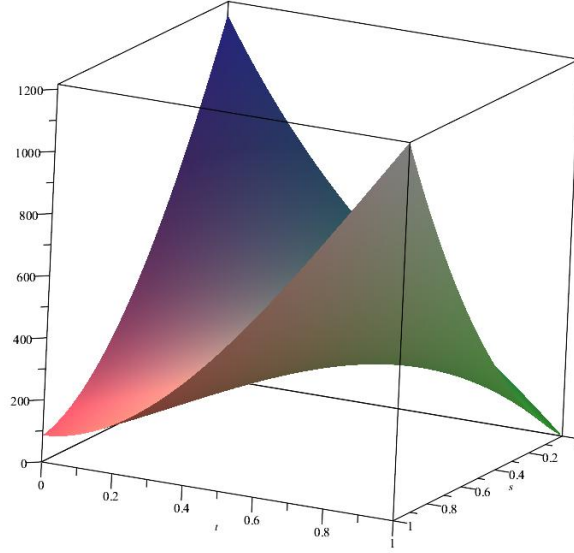


Figure 5.3: Graph of $p(s, t)$ for P_k when $k = 1$.

and that $p_t^i < 0$ for $t < t_i$. Hence for all $t < t_3$, the sign changes once in the Sturm sequence of $p_t(s)$, and for all $t \geq t_3$, it does not change. Hence by lemma 5.3, p does not have any non-trivial zeros.

Next, we consider the case when $k = 10000$. The graph of $p(s, t)$ is given in figure 5.4.

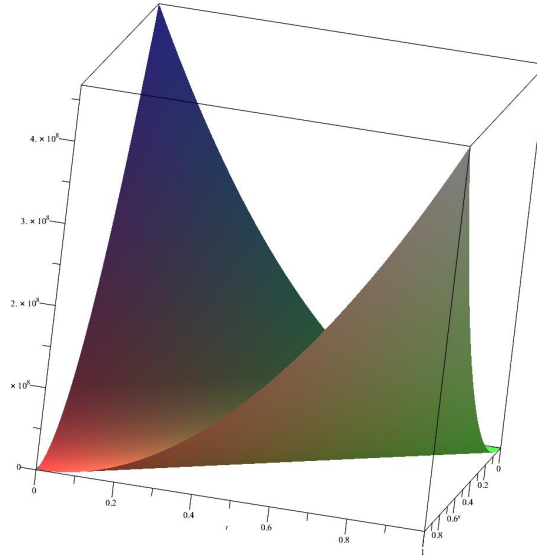


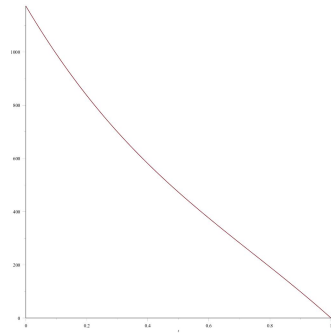
Figure 5.4: Graph of $p(s, t)$ for P_k when $k = 10000$.

It is now unclear whether or not $p(s, t)$ has a non-trivial zero. The graph of the Sturm sequences is given in figure 5.6. The behaviour for the first three terms is similar to before. That is, we have that p_t^0 is positive for both $s = 0$ and $s = 1$, and we have numbers $t_1 < t_2$ and $\tilde{t}_1 < \tilde{t}_2$ such that p_t^i is negative at $s = 0$, respectively $s = 1$, if and only if $t < t_i$, respectively, $t < \tilde{t}_i$. We also have a number $t_3 < 1$ which is greater than all four of these numbers such that p_t^3 is positive for $t > t_3$.

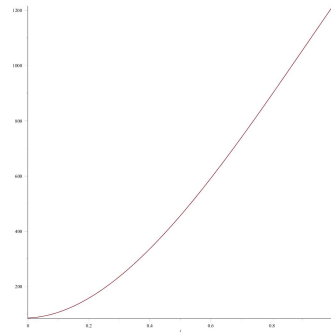
However, p_t^3 now has two additional zeros in $(0, 1)$, see figure 5.7. Say these are $u < v$. These satisfy that

$$\begin{aligned} v &< t_2, \\ \tilde{t}_2 &< u. \end{aligned}$$

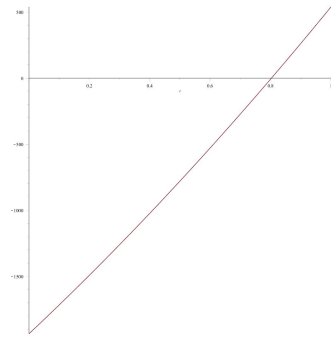
In the region (u, v) , p_t^3 is positive. This means that the number of sign changes of p_t^i when $s = 0$ and $t \in (u, v)$ is now 2, since $v < t_2$. However, at $s = 0$, the number of sign changes is now 0, as $\tilde{t}_2 < u$. Thus the number of zeros of $p_t(s)$ for $t \in (u, v)$ is now 2. In particular, $p(s, t)$ has a non-trivial zero and P_k is unstable.



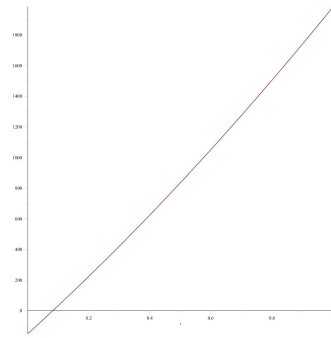
(a) Graph of p_t^0 at $s = 0$



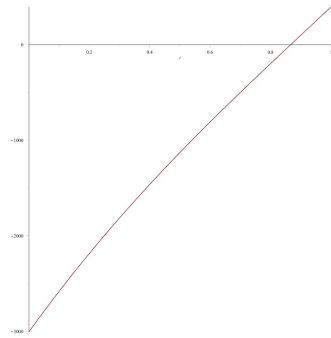
(b) Graph of p_t^0 at $s = 1$



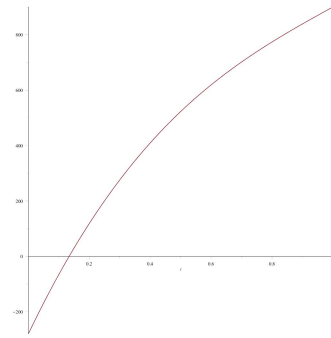
(c) Graph of p_t^1 at $s = 0$



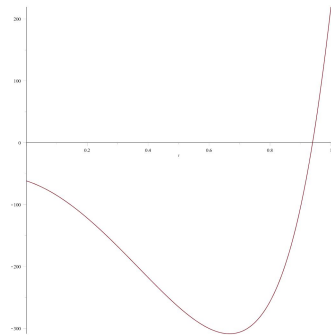
(d) Graph of p_t^1 at $s = 1$



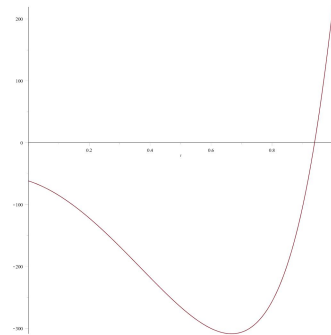
(e) Graph of p_t^2 at $s = 0$



(f) Graph of p_t^2 at $s = 1$

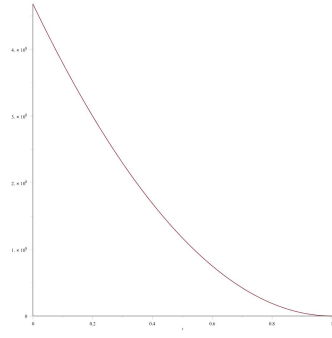


(g) Graph of p_t^3 at $s = 0$

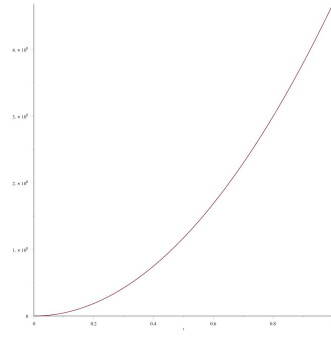


(h) Graph of p_t^3 at $s = 1$

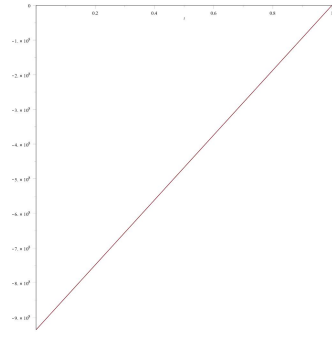
Figure 5.5: The Sturm sequence when $k = 1$.



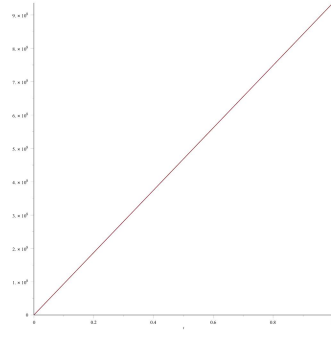
(a) Graph of p_t^0 at $s = 0$



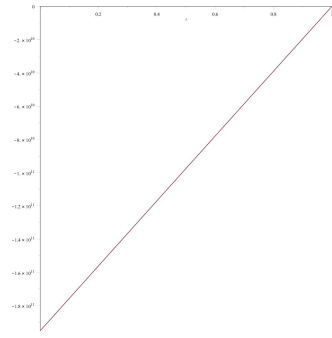
(b) Graph of p_t^0 at $s = 1$



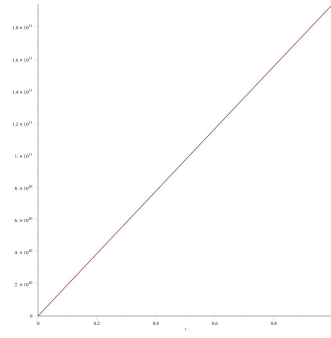
(c) Graph of p_t^1 at $s = 0$



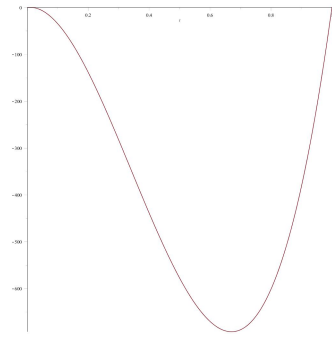
(d) Graph of p_t^1 at $s = 1$



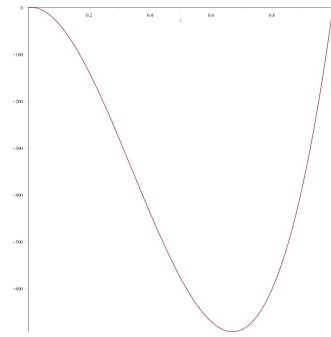
(e) Graph of p_t^2 at $s = 0$



(f) Graph of p_t^2 at $s = 1$



(g) Graph of p_t^3 at $s = 0$



(h) Graph of p_t^3 at $s = 1$

Figure 5.6: The Sturm sequence when $k = 10000$.

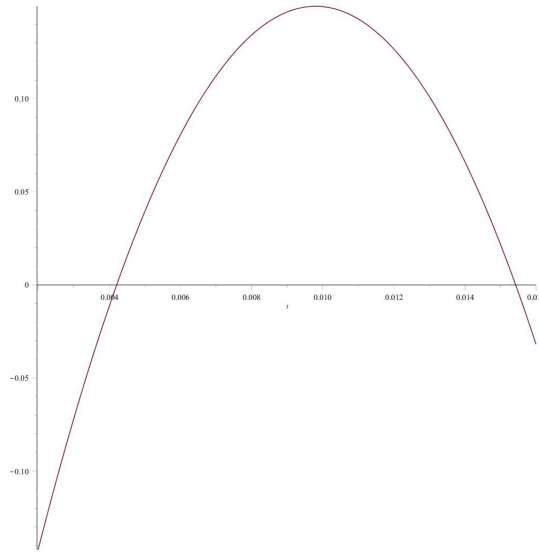


Figure 5.7: Graph of p_t^3 in the region $[0.002, 0.016]$.

The additional zeros of the final term in the Sturm sequence is the essential thing that happens when one passes from a stable to an unstable polarisation in this case. When $k_1 = 6861.50274$ one can use exactly the same steps to show that one has a similar behaviour as in the first case. However, when $k_2 = 6861.50275$, the third polynomial in the Sturm sequence gets additional zeros satisfying analogous inequalities to the second case above. Thus the polarisation corresponding to this choice of k is unstable.

Hence the critical value k_0 where stability changes lies in $(6861.50274, 6861.50275)$, and so

$$k_0 \approx 6861.5027.$$

5.3.3 New stable examples

In this section we give two examples of stable polytopes, one with vanishing and the other with non-vanishing Futaki invariant. As far as the author is aware, these are new examples of stable polytopes.

Example 5.8. Take P to have 12 vertices given by

$$v_1 = (-100009, -2),$$

$$v_2 = (-100008, -3),$$

$$v_3 = (-100002, -6),$$

and the remaining given by reflecting in the x and y axes and the diagonal $y = -x$. The manifold Y corresponding to P is the manifold obtained by the same procedure as in example 5.6, but ending the process

before blowing up on the exceptional divisors a second time. In other words, we have that

$$\pi : X \rightarrow Y$$

is the blow-up at Y in four fixed points. Moreover, if we let $E_1, \dots, E_4$ be the corresponding exceptional divisors, we have

$$L = \pi^*(\tilde{L}) \otimes [E_1]^{-\otimes 2} \otimes \dots \otimes [E_4]^{-\otimes 2},$$

and so

$$c_1(L) = \pi^*(c_1(\tilde{L})) - 2(c_1([E_1]) + \dots + c_1([E_4])).$$

Applying the algorithm from theorem 5.1 shows that this polytope is stable, and so $(Y, c_1(\tilde{L}))$ is K -stable. In particular, it follows from Donaldson's resolution of the YTD conjecture for toric surfaces in [Don09] that Y admits a cscK metric in $c_1(\tilde{L})$.

Also note the following. In the blow-up theorems of Arezzo-Pacard and Arezzo-Pacard-Singer in [AP09] and [APS11], there is a simple upper bound ε_1 for the maximum ε_0 for which $(\tilde{M}, \pi^*\Omega - \varepsilon(\sum_i a_i[E_i]))$ admits a cscK/extremal metric for all $\varepsilon \in (0, \varepsilon_0)$, given by

$$\sup\{\varepsilon > 0 : \pi^*\Omega - \varepsilon(\sum_i a_i[E_i]) \text{ is a Kähler class}\}.$$

It is not a new fact that the greatest ε_0 such that this holds in general has to be taken to be less than ε_1 . Indeed, the Hirzebruch surfaces admit extremal metrics in all Kähler classes, and so by the fact that there are unstable toric surfaces and that all toric surfaces are obtained by successively blowing up a Hirzebruch surface, one concludes that there must exist some toric surface and some Kähler class for which this bound is not obtained. However, this example explicitly shows this fact. The example of $\text{Bl}_p\mathbb{P}^2$ shows that $\varepsilon_0 = \varepsilon_1$ can occur too.

In the final example we take a polytope corresponding to a polarised complex surface with non-vanishing Futaki invariant.

Example 5.9. Take P to have vertices

$$\begin{aligned} v_1 &= (0, 5), \\ v_2 &= (1, 4), \\ v_3 &= (3, 3), \\ v_4 &= (6, 2), \\ v_5 &= (10, 1), \\ v_6 &= (15, 0), \\ v_7 &= (16, 0), \\ v_8 &= (4, 6), \\ v_9 &= (0, 6). \end{aligned}$$

The polytope is given in figure 5.8.

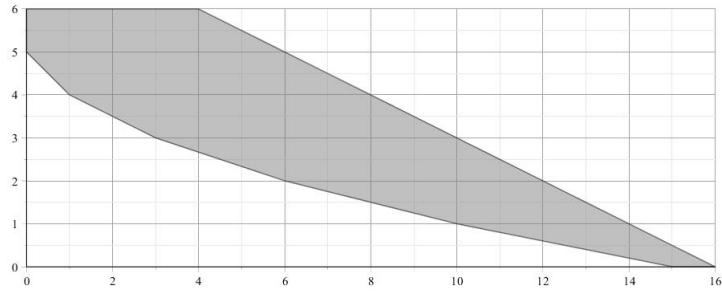


Figure 5.8: The polytope P .

This corresponds to the complex manifold X obtained by blowing up the 2nd Hirzebruch surface H_2 repeatedly at the intersection of the previous exceptional divisor and a fixed torus invariant divisor 5 times.

This is a stable polytope. We will here verify that one of the polynomials satisfies the required positivity condition. We consider the polynomial associated to simple piecewise linear functions with creasing going through the 4th and 7th edge. Its graph is given in figure 5.9.

It seems pretty clear that the minimum of p over $[0, 1] \times [0, 1]$ is at $(1, 0)$, where the value is positive, but we shall use the Sturm method to prove this. First, $p_t^0(0)$ and $p_t^1(1)$ are positive for each t . The graphs of the remaining terms are given in figure 5.10.

The key features are that the third term has two zeros, say at $t_3 < \tilde{t}_3$, and that each of $p_t^1(0), p_t^2(0)$ and

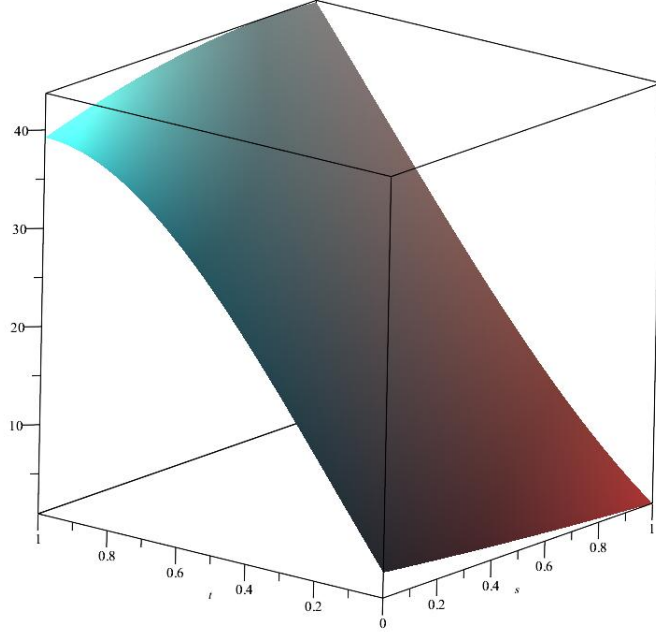


Figure 5.9: The graph of $p(s, t)$.

$p_t^2(1)$ has one zero, at $t_1, t_2, \tilde{t}_2$, say, while $p_t^1(1)$ is always negative. We have the following inequalities:

$$t_3 < t_2,$$

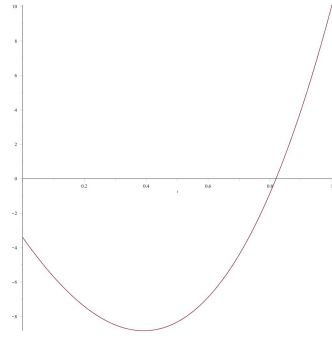
$$t_2 < \tilde{t}_2,$$

$$\tilde{t}_2 < t_1,$$

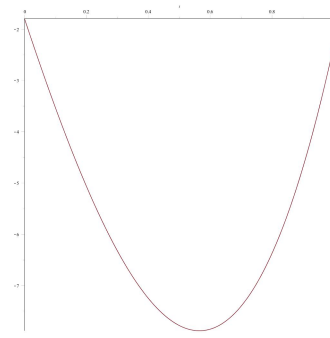
$$t_1 < \tilde{t}_3.$$

Thus at $p_t^2(0)$ we have that for all $t > t_2$ the sign has changed once and is negative, whereas for all $t < t_2$, the sign has changed twice and is now positive. Thus at $s = 2$, for all $t < t_3$ the sign changes three times as $t_3 < t_2$, for all $t \in (t_3, \tilde{t}_3)$ the sign changes twice as $t_2 < \tilde{t}_3$, and for all $t > \tilde{t}_3$ the sign changes once. In particular, by lemma 5.3, there is no zero in $(\tilde{t}_3, 1]$.

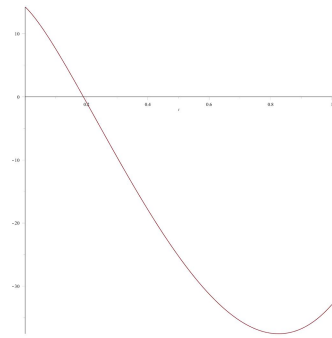
At $p_t^2(1)$ we have that for all $t > \tilde{t}_2$ the sign has changed once and is negative, and for all $t < \tilde{t}_2$ the sign has changed twice and is positive. Thus for all $t < t_3$ the sign changes three times as $t_3 < \tilde{t}_2$ and the sign changes twice for all $t \in (t_3, \tilde{t}_3)$ as $\tilde{t}_2 < \tilde{t}_3$. This is the same as for the $s = 0$ sequence, and so there is no zero of p in $[0, 1] \times [0, 1]$.



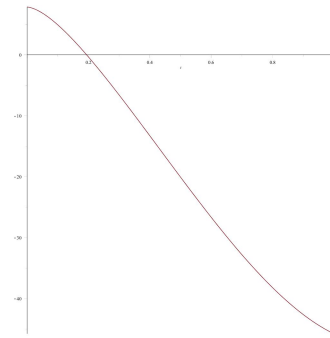
(a) Graph of p_t^1 at $s = 0$



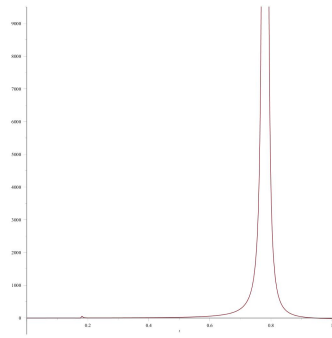
(b) Graph of p_t^1 at $s = 1$



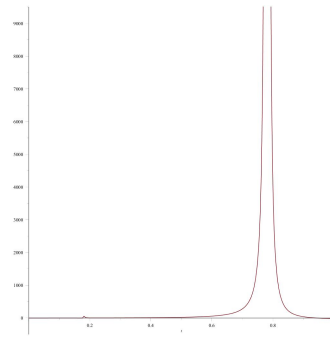
(c) Graph of p_t^2 at $s = 0$



(d) Graph of p_t^2 at $s = 1$



(e) Graph of p_t^3 at $s = 0$



(f) Graph of p_t^3 at $s = 1$

Figure 5.10: The last 3 terms of the Sturm sequence for edges 4 and 7.

Chapter 6

Weighted spaces on Poincaré type manifolds with an S^1 -action

In this chapter we will investigate the Fredholm properties of the Laplacian and Lichnerowicz operators associated to a Poincaré type metric on the complement of a smooth divisor D in a complex manifold X , under the assumptions of an S^1 -action fixing D and on the asymptotics of the metric near D . The main simplification these assumptions lead to are that the operators near D then become comparable to the model operators on a cylinder, and one can apply the theory of [LM85], which we recall. Auvray has considered related topics already, see [Auv14a]. Here he considers conjugated Lichnerowicz operators near the divisor, in order to prove that extremal Poincaré type metrics induce extremal metrics on the divisor along which the Poincaré type behaviour occurs. The novelty of what is done in this chapter is a global result.

The assumptions we make on the asymptotics of the Poincaré type metric ω are the following. We will be using the discussions [Auv14c] and [Auv13, Sec. 3] of Auvray on a tubular neighbourhood around D . Since we have an S^1 -action preserving D , we can choose an S^1 -invariant section $\sigma \in H^0(X, [D])$ defining D . Then the function

$$t = \log(-\log(|\sigma|^2))$$

is a smooth S^1 -invariant function on $X \setminus D$, bounded from below and going to $+\infty$ as one approaches D .

In general, without assuming X admits an holomorphic S^1 -action, one can use the exponential map obtained from a smooth metric on X to get an S^1 -fibration π over $[c, \infty) \times D$ for some c . However, in our case, since our global S^1 -action is via isometries when we take ω_0 to be an invariant metric and D is an invariant divisor, the local action coincides with the global one. When no global action is assumed, one needs to perturb $\log(-\log(|\sigma|^2))$ in order to get a function on $X \setminus D$ which is S^1 -invariant and behaves like the local S^1 -invariant model in the tubular neighbourhood. No such perturbation is needed for us, though.

We can simply work with t .

Using σ to define local coordinates in a trivialisation of the tubular neighbourhood, the upshot is that we can decompose X as a union of a compact region M and a region N which is the total space of an S^1 -fibration over $[c, +\infty) \times D$, and the S^1 -action coincides with the global S^1 -action on $X \setminus D$. Moreover, the first factor of π is t . Also, S^1 -invariant functions on N are precisely the pull-backs via π of functions on $[c, +\infty) \times D$ that are invariant under the product action which is trivial on the first component and the restriction of the S^1 -action on X to D .

Auvray showed that the model Poincaré type metric $\tilde{\omega}$ in a class $[\omega_0]$ has an expansion

$$\tilde{g} = dt^2 + e^{-2t}\vartheta^2 + \pi^*h_0 + O(e^{-t}),$$

which is valid for derivatives at any order. Here h_0 is the metric on D associated to the Kähler form ω_0 on X restricted to D and ϑ is a connection 1-form associated to the normal bundle of D . Moreover, in trivialising charts for N , ϑ satisfies

$$\vartheta = d\theta + O(1)$$

for derivatives to any order. Also, ϑ integrates to 2π on each S^1 -fibre.

We will consider the case when the metric g associated to ω satisfies

$$g = dt^2 + e^{-2t}\vartheta^2 + \pi^*g_D + O(e^{\eta t}), \tag{6.1}$$

for some Kähler metric g_D on D and $\eta < 0$, again for derivatives at any order. Crucially for us, Auvray has in [Auv14b, Thm. 4.8] shown that when D is smooth, i.e. has no intersecting irreducible components, this expansion holds for extremal Poincaré type metrics, where h is in fact an extremal metric on D .

Remark 6.1. *In this chapter and the following there will be a few different Kähler metrics discussed. We therefore make the following convention that applies to this and the next chapter.*

- ω_0 is an S^1 -invariant Kähler form on the whole of the compact complex manifold X ;
- $\omega = \omega_0 + i\partial\bar{\partial}\varphi$ is the Kähler form associated to an S^1 -invariant metric of Poincaré type on $X \setminus D$;
- ω_D is the Kähler form of a Kähler metric g_D on D such that ω satisfies 6.1.

In section 6.1 we recall the local result of Lockhart-McOwen. This forms the basis for global estimates, which we prove in 6.2. Using these estimates, we prove in 6.3 the Fredholm results. We end the chapter by commenting on the additional issues arising when we do not assume there is an S^1 -action in section 6.4.

Remark 6.2. *For the reader familiar with the cylindrical ends theory, there will be no surprises in this chapter. For such a reader, we point out the main steps, from which it will be clear that the main result, theorem 6.16, holds. For the Sobolev spaces, it is immediate from the definitions that the norm is equivalent to the model norm on a half-cylinder near the divisor D . Lemma 6.7 says that this holds also for the Hölder*

spaces. Thus the Lockhart-McOwen theory can be applied, and the Fredholm properties of the operators follow. The regularity result lemma 6.10 is then the crucial component to prove the characterisation in theorem 6.16 of the cokernel in terms of the kernel of a different weight.

6.1 Estimates in the model case

Starting from a model result on a full cylinder $\mathbb{R} \times D$ in [LM85], we follow the approach of [Pac08] to establish a basic estimate for the local model of a Poincaré type metric.

Consider $B_{e^{-1}}^* \times D$, the product of the punctured disk of radius e^{-1} with D , with the product metric

$$\omega_{\text{mod}} = \omega_{\text{cusp}} + \omega_D,$$

where

$$\omega_{\text{cusp}} = \frac{idz \wedge d\bar{z}}{|z|^2 \log^2(|z|^2)},$$

and ω_D is the Kähler form of a smooth metric g_D on D .

Let $t = \log(-\log(|z|^2))$ and θ be the usual angular coordinate on B_1^* . In these coordinates, the metric g associated to ω is given by

$$g = dt^2 + e^{-2t} d\theta^2 + g_D^2.$$

The Laplacian operator Δ_{mod} of ω_{mod} is given by

$$f \mapsto \frac{\partial^2 f}{\partial t^2} - \frac{\partial f}{\partial t} + \Delta_D(f) \quad (6.2)$$

for S^1 -invariant functions on $B_{e^{-1}}^* \times D$, where Δ_D is the Laplacian associated to ω_D . Our convention is that the Laplacian is given by $g^{i\bar{j}} \partial_i \partial_{\bar{j}} f$ in local complex coordinates.

More important to us will be the Lichnerowicz operator, whose definition we now recall. For a general complex manifold Y , the complex vector bundle $T^{1,0}Y$ is a holomorphic vector bundle and so has a $\bar{\partial}$ -operator. Given a Kähler metric ω_Y on Y , we can then get a map $\mathcal{D}$ sending functions to sections of $T^{1,0}Y \otimes \Lambda^{0,1}Y$ by

$$\mathcal{D}_{\omega_Y}(f) = \bar{\partial} \circ \nabla^{1,0}(f).$$

The associated Lichnerowicz operator is the fourth order operator $\mathcal{D}_{\omega_Y}^* \mathcal{D}_{\omega_Y}$, where $\mathcal{D}_{\omega_Y}^*$ is the formal adjoint to $\mathcal{D}_{\omega_Y}$. It can be written as

$$\mathcal{D}_{\omega_Y}^* \mathcal{D}_{\omega_Y}(f) = \frac{1}{2} \Delta^2(f) + \langle i\partial\bar{\partial}(f), \rho \rangle + \frac{1}{2} \langle df, dS(\omega_Y) \rangle.$$

Here ρ is the Ricci form associated to ω_Y .

To compute the Lichnerowicz operator $\mathcal{D}^*\mathcal{D}_{\text{mod}}$ of ω_{mod} when $Y = B_{e^{-1}}^* \times D$, note that since this is a product metric, the only mixed terms we get come from the Laplacian term. Also, ω_{cusp} is Einstein with constant -2 , and so the term involving the Ricci form in the t -direction simply becomes $-\frac{1}{2}\Delta_{\text{cusp}}(f)$. The operator is thus given by

$$f \mapsto \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\right)^2(f) - \left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\right)(\Delta_D f) - \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\right)(f) + \mathcal{D}_D^*\mathcal{D}_D f, \quad (6.3)$$

where again Δ_D and $\mathcal{D}_D^*\mathcal{D}_D$ denote the Laplacian and Lichnerowicz operators of ω_D .

We will sometimes write L or L_{mod} to refer to either of these operators, in particular in statements that apply to both. We also let $l = 2$ when L is the Laplace operator and $l = 4$ when it is the Lichnerowicz operator.

Note that S^1 -invariant functions on $B_{e^{-1}} \times D$ are equivalent to functions on $C = [0, \infty) \times D$, and the operators 6.2 and 6.3 are then translation invariant elliptic operators on a (half) cylinder, precisely as considered in [LM85].

We will first recall the basic result on the full cylinder $\tilde{C} = \mathbb{R} \times D$ from which we will obtain estimates for our operators on $X \setminus D$. We will define weighted Sobolev spaces on both C and $\tilde{C}$ as follows. Let t be the coordinate as above, and define the L^2 -norm by

$$\|f\|_{L_\delta^2(C)} = \int_C |f|^2 e^{-2\delta t} dV_g \quad (6.4)$$

$$= \int_C |f|^2 e^{-2\delta t - t} dt dV_h, \quad (6.5)$$

and similarly for $\tilde{C}$. We define the weighted Sobolev norm by

$$\|f\|_{W_\delta^{2,k}} = \sum_{i=0}^k \|\nabla^i f\|_{L_\delta^2}. \quad (6.6)$$

We then define the space $W_\delta^{2,k}(\tilde{C})$ to be the completion of the smooth functions with compact support under this norm.

Using the formulae 6.2 and 6.3, we can consider these as linear operators $\Delta : W_\delta^{2,k} \rightarrow W_\delta^{2,k-2}$ or $\mathcal{D}^*\mathcal{D} : W_\delta^{2,k} \rightarrow W_\delta^{2,k-4}$. We would like to recall the Fredholm properties of these operators. The fundamental result (see [LM85, Sec. 2] and the references therein) is then that Δ and $\mathcal{D}^*\mathcal{D}$ are Fredholm and on the full cylinder in fact an isomorphism for each δ which is not an *indicial root*. δ is an indicial root if there is a complex number λ with $\text{Im}(\lambda) = \delta$, and for which there is a non-trivial solution to $L(f) = 0$ of the form

$$e^{i\lambda t} p(t, x)$$

where $x \in D$ and p is a polynomial in t .

The set of indicial roots is a discrete set only. Also, whenever δ is not an indicial root, we have that there is a $c > 0$ such that for all $f \in W_\delta^{2,k}(\tilde{C})$,

$$\|f\|_{W_\delta^{2,k}(\tilde{C})} \leq c \|\Delta f\|_{W_\delta^{2,k-2}(\tilde{C})}, \quad (6.7)$$

and similarly for $\mathcal{D}^*\mathcal{D}$.

The inequality 6.7 implies a similar bound for the half cylinder C , but now involving a bound on the function away from the divisor.

Proposition 6.3. *Let $\delta \in \mathbb{R}$ not be an indicial root. Then there is a $c > 0$ and an $A > 0$, such that for all $f \in W_\delta^{2,k}(C)$*

$$\|f\|_{W_\delta^{2,k}(C)} \leq c(\|Lf\|_{W_\delta^{2,k-l}(C)} + \|f\|_{W^{2,k}([0,A] \times D)}). \quad (6.8)$$

Proof. Let χ be a smooth cut-off function on $[0, \infty)$ with support in $[1, \infty)$ and equal to 1 in $[2, \infty)$. Then we have that for some $c_1 > 0$ depending on χ and δ , we have

$$\begin{aligned} \|f\|_{W_\delta^{2,k}(C)} &\leq \|\chi f\|_{W_\delta^{2,k}(C)} + \|(1-\chi)f\|_{W_\delta^{2,k}(C)} \\ &\leq \|\chi f\|_{W_\delta^{2,k}(C)} + c_1 \|f\|_{W^{2,k}([0,2] \times D)}. \end{aligned}$$

Now,

$$\|\chi f\|_{W_\delta^{2,k}(C)} = \|\chi f\|_{W_\delta^{2,k}(\tilde{C})}$$

and so, by 6.7, there is a $c_2 > 0$ such that

$$\|\chi f\|_{W_\delta^{2,k}(C)} \leq c_2 \|L(\chi f)\|_{W_\delta^{2,k-l}(\tilde{C})}.$$

Finally, there is a $c_3 > 0$ such that

$$\|L(\chi f)\|_{W_\delta^{2,k-l}(\tilde{C})} \leq c_3(\|\chi L(f)\|_{W_\delta^{2,k-l}(\tilde{C})} + \|f\|_{W^{2,k-l}([1,2] \times D)}).$$

Putting all these together gives the required inequality. $\square$

Remark 6.4. *It was in no way essential that we worked with $t \in [0, +\infty)$. An analogous result holds for any $[\lambda, +\infty)$ with $\lambda \in \mathbb{R}$.*

We end this section by noting that the above results for C give corresponding results for S^1 -invariant functions on $Y = Y_\lambda = B_{e^{-\lambda}}^* \times D$ for any $\lambda > 0$. Letting $t = \log(-\log(|z|^2))$, where z is the usual complex coordinate on B_1^* , we can define $W_\delta^{2,k}$ -spaces just as in 6.4 and 6.6. Let $p : Y \rightarrow C$ denote the projection map and let dV_D be the volume form associated to the metric g_D on D . For $f \in W_\delta^{2,k}(\tilde{C})$, we have that $p^*f \in W_\delta^{2,k}(Y)$ and moreover this is a continuous map, since

$$\int_Y |p^*f|^2 e^{2\delta t} dV_g = 2\pi \int_C |f|^2 e^{2\delta t} e^{-t} dV_D,$$

and all the derivatives in the θ -directions vanish. This is a closed subspace $\tilde{W}_\delta^{2,k}(Y)$ of $W_\delta^{2,k}(Y)$ and so a Banach space. It is the completion of the S^1 -invariant functions with compact support on Y under the $W_\delta^{2,k}$ -norm. Proposition 6.3 and remark 6.4 then immediately gives

Corollary 6.5. *If δ is not an indicial root for L , then for any $\lambda \in (0, +\infty)$ there exists a $c > 0$ and an $r \in (0, e^{-\lambda})$ such that for all $f \in \tilde{W}_\delta^{2,k}(Y)$, we have*

$$\|f\|_{W_\delta^{2,k}(Y)} \leq c(\|Lf\|_{W_\delta^{2,k-l}(Y)} + \|f\|_{W^{2,k}((B_{e^{-\lambda}} \setminus B_r) \times D)}). \quad (6.9)$$

6.2 Estimates in the global case

We will now discuss how we can use the results of the previous section to give similar results for the Laplace and Lichnerowicz operator in the Poincaré type setting, under the assumption of a global S^1 -action and on the behaviour near D . We largely follow the lines of [Pac08], where similar results were obtained in the case of operators on manifolds with cylindrical ends. Although we are not working on a (metric) cylinder, the estimate 6.8 is what is required to use precisely the same line of argument as in [Pac08].

We begin by defining the Sobolev spaces of interest to us. Recall that we have an S^1 -invariant section $\sigma \in H^0(X, [D])$ defining D . The function

$$t = \log(-\log(|\sigma|^2))$$

is a smooth S^1 -invariant function on $X \setminus D$, bounded from below and going to $+\infty$ as one approaches D .

We then define the L^2 -norm with weight δ on $X \setminus D$ by

$$\|f\|_{L_\delta^2(X \setminus D)}^2 = \int_{X \setminus D} |f|^2 e^{-2\delta t} \omega^n, \quad (6.10)$$

where we recall that ω is our Poincaré type metric on $X \setminus D$.

We define the weighted $W^{2,k}$ -norm by

$$\|f\|_{W_\delta^{2,k}(X \setminus D)}^2 = \sum_{i=0}^k \|\nabla^i f\|_{L_\delta^2(X \setminus D)}^2. \quad (6.11)$$

Here we are using the Levi-Civita connection of ω and the pointwise norm on the tensor bundles to get a function $|\nabla^i f|$ that we take the L_δ^2 -norm of.

We define the $W_\delta^{2,k}$ -spaces to be the completion of the S^1 -invariant compactly supported smooth functions with respect to this norm. Note that one does not have to restrict to S^1 -invariant functions to define these spaces, but we will only be interested in the S^1 -invariant case. However, to not make the notation too cumbersome, we have chosen to not highlight this S^1 -invariance in our notation. Note also that one could develop an analogous theory for any $W^{p,k}$ -norm, but we will only require the $W^{2,k}$ -version.

The first goal of the section is to prove the following estimate. In the statement and the sequel, Δ_ω and $\mathcal{D}_\omega^* \mathcal{D}_\omega$ refers to the Laplacian and Lichnerowicz operators associated to ω .

Proposition 6.6. *Let S^1 act on X holomorphically and suppose $D \subseteq X$ is a divisor invariant under this action. Suppose that ω is a Poincaré type Kähler metric in the class of $[\omega_0] \in H^2(X)$ satisfying 6.1. Suppose*

δ is not an indicial root for Δ_ω , respectively for $\mathcal{D}_\omega^* \mathcal{D}_\omega$. Then there exists a compact subset $K \subseteq X \setminus D$ and a $c > 0$ such that

$$\|f\|_{W_\delta^{2,k}(X \setminus D)} \leq c(\|\Delta_\omega f\|_{W_\delta^{2,k-2}(X \setminus D)} + \|f\|_{L^2(K)}) \quad (6.12)$$

or, respectively,

$$\|f\|_{W_\delta^{2,k}(X \setminus D)} \leq c(\|\mathcal{D}_\omega^* \mathcal{D}_\omega f\|_{W_\delta^{2,k-4}(X \setminus D)} + \|f\|_{L^2(K)}). \quad (6.13)$$

For this we need a neighbourhood around D , in which our operator is comparable to the model operator on $B_r^* \times D$ for some $r < 1$. This is provided by the tubular neighbourhood, as discussed in the introduction to this chapter. From this we will be able to obtain the above estimates from the local results of the previous section.

The first property we have of this decomposition is that the restriction of the $L_\delta^2(X \setminus D)$ -norms to S^1 -invariant functions on N gives a norm which is equivalent to the $L_\delta^2([c, +\infty) \times D)$ -norm. Indeed, since ϑ integrates to 2π on each fibre, the volume form of $dt^2 + e^{-2t}\vartheta^2 + \pi^*h$ is precisely the volume form in the local definition of the L_δ^2 -spaces (up to a factor of 2π), and the volume form of g is mutually bounded with this.

Next, the $W^{2,k}$ -norms are also equivalent. By the above, it suffices to note that $|\nabla f|_\omega$ and $|\nabla_{\text{mod}} f|_{\omega_{\text{mod}}}$ are mutually bounded (similar statements for higher derivatives then follow).

Finally, since all the identities hold *at any order* in the derivatives, it follows that for a Poincaré type metric there is a $c > 0$ such that the estimates

$$\|\Delta_\omega f - \Delta_{\text{mod}} f\|_{W_\delta^{2,k-2}(N)} \leq c\|e^{-\eta' t} f\|_{W_\delta^{2,k}(N)}, \quad (6.14)$$

$$\|\mathcal{D}_\omega^* \mathcal{D}_\omega f - \mathcal{D}^* \mathcal{D}_{\text{mod}} f\|_{W_\delta^{2,k-4}(N)} \leq c\|e^{-\eta' t} f\|_{W_\delta^{2,k}(N)}, \quad (6.15)$$

hold.

With all these preliminaries in place, we are now ready to prove proposition 6.6.

Proof of Proposition 6.6. We only do the proof for the Laplace operator. The case of the Lichnerowicz follows exactly the same steps.

Let χ_1 be bump function supported in N equal to 1 for all p such that $t(p) \geq a + 1$, where a is chosen such that $t(p) \geq a$ implies that $p \in N$, and let $\chi_0 = 1 - \chi_1$. By possibly increasing M , we may suppose that χ_0 is supported in M° . Note that both the $d\chi_i$ has equal and compact support in $X \setminus D$. Then there is a $c > 0$, depending on χ and δ , such that for $i = 0, 1$, we have

$$\|\Delta_\omega(\chi_i f)\|_{W_\delta^{2,k-2}(X \setminus D)} \leq c(\|\Delta_\omega f\|_{W_\delta^{2,k-2}(X \setminus D)} + \|f\|_{W^{2,k}(\text{supp}(d\chi_i))}). \quad (6.16)$$

Next, since $\chi_0 f$ has support in M° , at a distance from ∂M independent of f , it follows from the usual Schauder theory and that the δ -norm is equivalent to the usual Sobolev norm on M , that there is a $c > 0$

$$\begin{aligned}\|\chi_0 f\|_{W_\delta^{2,k}(X \setminus D)} &= \|\chi_0 f\|_{W_\delta^{2,k}(M)} \\ &\leq c(\|\Delta_\omega(\chi_0 f)\|_{W_\delta^{2,k}(M)} + \|\chi_0 f\|_{W_\delta^{2,k-2}(M)}) \\ &\leq c(\|\Delta_\omega(\chi_0 f)\|_{W_\delta^{2,k}(X \setminus D)} + \|f\|_{L_\delta^2(M)}).\end{aligned}$$

By choosing K to contain M , what is left to prove is a similar bound for $\|\chi_1 f\|_{W_\delta^{2,k}(X \setminus D)}$.

To prove this we have that by corollary 6.5 and the fact that the $W_\delta^{2,k}$ -norms defined with respect to ω and ω_{mod} are equivalent on N_i , there is a $c > 0$ such that for each $i = 1, \dots, d$,

$$\begin{aligned}\|\chi_1 f\|_{W_\delta^{2,k}(N_i)} &\leq c(\|\Delta_{\text{mod}}(\chi_1 f)\|_{W_\delta^{2,k-2}(N_i)} + \|\chi_1 f\|_{W_\delta^{2,k}(N_i \cap K')}) \\ &\leq c(\|\Delta_{\text{mod}}(\chi_1 f)\|_{W_\delta^{2,k-2}(N_i)} + \|\chi_1 f\|_{W_\delta^{2,k}(K')}),\end{aligned}$$

for some compact subset $K' \subseteq X \setminus D$. It follows that there is a $c > 0$ such that

$$\begin{aligned}\|\chi_1 f\|_{W_\delta^{2,k}(X \setminus D)} &\leq \sum_i \|\chi_1 f\|_{W_\delta^{2,k}(N_i)} \\ &\leq c(\sum_i \|\Delta_{\text{mod}}(\chi_1 f)\|_{W_\delta^{2,k-2}(N_i)} + \|\chi_1 f\|_{W_\delta^{2,k}(K')}).\end{aligned}$$

Moreover, if we let K be a compact subset of $X \setminus D$ containing K' (and from the above, M) in its interior, then by the Schauder estimates there is a $c > 0$ such that

$$\|\chi_1 f\|_{W_\delta^{2,k}(K')} \leq c(\|\Delta_{\text{mod}}(\chi_1 f)\|_{W_\delta^{2,k-2}(X \setminus D)} + \|\chi_1 f\|_{L^2(K)}).$$

Hence by 6.14 and 6.16, there is a $\tilde{c} > 0$ such that

$$\|\chi_1 f\|_{W_\delta^{2,k}(X \setminus D)} \leq \tilde{c}(\|e^{-\eta' t} f\|_{W_\delta^{2,k}(X \setminus D)} + \|\Delta_\omega(f)\|_{W_\delta^{2,k-2}(X \setminus D)} + \|f\|_{L^2(K)}).$$

The proof will be complete if we can show (after possibly increasing K independently of f) that a bound of the form

$$\|e^{-\eta' t} f\|_{W_\delta^{2,k}(X \setminus D)} \leq c(\|\Delta_\omega f\|_{W_\delta^{2,k-2}(X \setminus D)} + \|f\|_{L^2(K)}) + \frac{\varepsilon}{\tilde{c}} \|f\|_{W_\delta^{2,k}(X \setminus D)} \quad (6.17)$$

holds for some $\varepsilon < 1$.

So, let

$$U_s = \{p \in X \setminus D : t(p) \geq s\}.$$

Then

$$\begin{aligned}\|e^{-\eta' t} f\|_{W_\delta^{2,k}(X \setminus D)} &= \|e^{-\eta' t} f\|_{W_\delta^{2,k}(U_s^c)} + \|e^{-\eta' t} f\|_{W_\delta^{2,k}(U_s)} \\ &\leq c_s \|f\|_{W_\delta^{2,k}(U_s^c)} + e^{-\eta' s} \|f\|_{W_\delta^{2,k}(U_s)},\end{aligned}$$

for some c_s depending on s . Set s so that $e^{-\eta's} \leq \frac{\tilde{c}}{2}$ and increase K to include U_{s+1}^c . Note that s , and hence c_s and K , only depends on $\tilde{c}$, not f . By the interior Schauder estimates and the fact that on each U_s^c the $W_\delta^{2,k}$ -norm is equivalent to the usual Sobolev norm, there is a $c > 0$ such that

$$\begin{aligned} \|f\|_{W_\delta^{2,k}(U_s^c)} &\leq c(\|\Delta_\omega f\|_{W_\delta^{2,k-2}(U_{s+1}^c)} + \|f\|_{L^2(U_{s+1}^c)}) \\ &\leq c(\|\Delta_\omega f\|_{W_\delta^{2,k-2}(X \setminus D)} + \|f\|_{L^2(K)}). \end{aligned}$$

Combining this with the above gives 6.17 with $\varepsilon = \frac{1}{2}$, and this completes the proof. $\square$

Next, we turn our attention to weighted Hölder spaces. The usual definition of Hölder norm on compact Riemannian manifolds is given by e.g.

$$\|f\|_{0,\alpha} = |f|_{C^0} + \sup_{x \neq y} \left(\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \right),$$

where d is the distance function associated to g . When the metric is of Poincaré type, however, the injectivity radius goes to 0 as one approaches the divisor. It has therefore been found (see e.g. [TY87] and [Kob84]) that the concept of quasi-coordinates works better in this case.

In the one-dimensional case, one constructs a 1-parameter family ϕ_η of maps from a ball of fixed size $B_R(0)$ in the unit disk in $\mathbb{C}$ as

$$\phi_\eta(z) = e^{\frac{1+\eta}{1-\eta} \frac{1+z}{1-z}}.$$

The union of the image of these maps for $\eta \in (0, 1)$ cover an open neighbourhood of 0 in the unit punctured disk. The model Poincaré type metric pulls back to a metric on $B_R(0)$, *independently of η* , and this metric is quasi-isometric to the Euclidean metric. The $C^{k,\alpha}$ -norm of f is then defined to be

$$\sup_{\eta \in (0,1)} (\|\phi_\eta^* f\|_{C^{k,\alpha}(B_R(0))}).$$

In general dimensions, one takes a finite cover of a neighbourhood of D with charts U_i such that D is given by $z_1 = 0$. One can then for each chart define a map $\Phi_\eta : B_R(0) \times V_i$, where V_i is an open set in $\mathbb{C}^{n-1}$, which is simply the product of ϕ_η and the identity on the last components. The union of the image of the Φ_η 's for $\eta \in (0, 1)$ is then $U \setminus D$ for some neighbourhood U of D in X . The $C^{k,\alpha}$ -norm of a function on $X \setminus D$ is defined as the supremum of the $C^{k,\alpha}$ -norm of the pull-back of the function via these maps, for all the charts in the cover, plus the usual $C^{k,\alpha}$ -norm on a compact subset containing the complement of $U \setminus D$ in $X \setminus D$ in its interior. There is also a similarly defined norm on the exterior bundles.

For the S^1 -invariant case, however, the quasi-coordinates definition simplifies. Indeed, the problem with the injectivity radius going to 0 does not come into play because the functions are constant on the shrinking S^1 -fibres. This is highlighted by the following lemma.

Lemma 6.7. *The $C^{k,\alpha}$ -norm on N defined with respect to quasi-coordinates is equivalent to the norm defined by identifying such S^1 -invariant functions with S^1 -invariant functions on $[\lambda, \infty) \times D$ and using the Hölder*

norm with respect to

$$dt^2 + g_D,$$

on each tube of the form $\{p : t(p) \in [s - c, s + c]\}$ for some, and hence any, c .

Proof. It suffices to prove the $C^{0,\alpha}$ -version, because the C^k -norm with respect to either norm is simply $\sup_{X \setminus D} |\nabla_\omega^i f|_\omega$.

Recall that σ is a holomorphic section of the line bundle associated to D , whose zero set is D and which we used to define t . We first note that in any chart where σ is the first coordinate, which we can use to cover a neighbourhood of D , we have

$$t(\Phi_\eta(z_1, \dots, z_n)) = \log 2 + \log\left(\frac{1+\eta}{1-\eta}\right) + \log\left(\frac{|z_1|^2 - 1}{|z_1^2 - 1|}\right).$$

Hence $\Phi_\eta(B_R(0) \times U_i) \subseteq [\log 2 + \log(\frac{1+\eta}{1-\eta}) - c_1, \log 2 + \log(\frac{1+\eta}{1-\eta}) + c_2] \times S^1 \times U_i$ for some constants $c_1, c_2 > 0$ independent of $\eta \in (0, 1)$. Now, $dt^2 + e^{-2t}d\theta + g_D$ is the model metric in t -coordinates, so pulls back to a metric, independent of η , whose derivatives are mutually bounded to any order with the Euclidean metric via Φ_η . Thus computing the $C^{k,\alpha}$ -norm with respect to $g_{\text{euc}} + g_D$ on $B_R(0) \times U_i$ or $dt^2 + e^{-2t}d\theta + g_D$ on $\Phi_\eta(B_R(0) \times U_i)$ are equivalent. The latter is also the same as the norm computed with respect to $dt^2 + g_D$ on S^1 -invariant functions. Hence the $C^{k,\alpha}$ -norm computed in quasi-coordinates is dominated by that computed with respect to $dt^2 + g_D$ on $[\log 2 + \log(\frac{1+\eta}{1-\eta}) - c_1, \log 2 + \log(\frac{1+\eta}{1-\eta}) + c_2] \times U_i$.

By taking c_1, c_2 to be the minimum possible, we note that while the image of Φ_η may not cover all of $[\log 2 + \log(\frac{1+\eta}{1-\eta}) - c_1, \log 2 + \log(\frac{1+\eta}{1-\eta}) + c_2] \times S^1 \times U_i$, it covers at least a neighbourhood of some point in every fibre of the projection away from the S^1 -factor (in the interior). Moreover, the $C^{k,\alpha}$ -norm is independent of the point chosen in any such fibre. So then by the above the quasi-coordinate norm dominates that computed with respect to $dt^2 + g_D$ on $[\log 2 + \log(\frac{1+\eta}{1-\eta}) - c_1, \log 2 + \log(\frac{1+\eta}{1-\eta}) + c_2]$ as well. $\square$

We now add weights to our discussion, in the following definition, see [Auv14c, Defn. 3.1].

Definition 6.8. Let $\delta \in \mathbb{R}$ be a real parameter. We define the $C_\delta^{k,\alpha}$ -space to be

$$C_\delta^{k,\alpha}(X \setminus D) = e^{\delta t} C^{k,\alpha}(X \setminus D)$$

equipped with the norm

$$\|f\|_{C_\delta^{k,\alpha}(X \setminus D)} = \|e^{-\delta t} f\|_{C^{k,\alpha}(X \setminus D)}.$$

We first have an elementary lemma.

Lemma 6.9. Fix δ, η, k and α . Then

$$C_\delta^{k,\alpha}(X \setminus D) \subseteq W_\eta^{2,k}(X \setminus D)$$

if and only if $\eta > \delta - \frac{1}{2}$.

Proof. First note that $e^{\delta t} \in C_{\delta}^{k,\alpha}$ and has finite L_{η}^2 norm if and only if $\eta > \delta - \frac{1}{2}$, showing one direction of the claim.

In the other direction, for the case $\delta = 0$, this follows because for all $f \in C^{k,\alpha}(X \setminus D)$, $|\nabla^i f|$ is bounded and $X \setminus D$ has finite volume with respect to the volume form given by $e^{-2\eta t} \omega^n$, where ω is a Poincaré type metric, if $\eta > -\frac{1}{2}$. For the case of other values of δ , note that because $e^{\delta t} \in C_{\delta}^{k,\alpha}(X \setminus D)$ for all k, α , we have that $\nabla^i f \in C_{\delta}^{k-i,\alpha}(X \setminus D)$ for all i . It then follows that $|e^{-\delta t} \nabla^i f|$ is bounded for every i , and so we can apply the previous argument. $\square$

A crucial result that will allow us to prove Fredholm properties for our operators on weighted Hölder spaces is the following regularity result. This follows [Pac08, Lem. 12.1.1]

Proposition 6.10. *Suppose $f \in L_{\delta-\frac{1}{2}}^2$ and suppose that $\Delta_{\omega} f \in C_{\delta}^{k-2,\alpha}$ or, respectively, $\mathcal{D}_{\omega}^* \mathcal{D}_{\omega} f \in C_{\delta}^{k-4,\alpha}$ in the sense of distributions for a weight δ . Then $f \in C_{\delta}^{k,\alpha}$. Moreover, there is a $c > 0$ such that*

$$\|f\|_{C_{\delta}^{k,\alpha}} \leq c(\|\Delta_{\omega} f\|_{C_{\delta}^{k-2,\alpha}} + \|f\|_{L_{\delta-\frac{1}{2}}^2})$$

or, respectively,

$$\|f\|_{C_{\delta}^{k,\alpha}} \leq c(\|\mathcal{D}_{\omega}^* \mathcal{D}_{\omega} f\|_{C_{\delta}^{k-4,\alpha}} + \|f\|_{L_{\delta-\frac{1}{2}}^2}).$$

Proof. The proof is identical for both operators. We do it for $\mathcal{D}^* \mathcal{D} = \mathcal{D}_{\omega}^* \mathcal{D}_{\omega}$.

From the usual elliptic theory, it follows that $f \in C_{\text{loc}}^{k,\alpha}(X \setminus D)$, so we need to estimate the $C_{\delta}^{k,\alpha}$ -norm. Using the Schauder estimates and that the δ -norms are equivalent to the unweighted norm on any compact subset K of $X \setminus D$, we get immediately that there is a $c > 0$, depending on K , such that

$$\|f\|_{C_{\delta}^{k,\alpha}(K)} \leq c(\|\mathcal{D}^* \mathcal{D} f\|_{C_{\delta}^{k-4,\alpha}(X \setminus D)} + \|f\|_{L_{\delta-\frac{1}{2}}^2(X \setminus D)}).$$

Thus we have to show that the required bound holds near the divisor. Thinking of f as a function on $[\lambda, +\infty) \times D$ for some fixed λ , there is $c > 0$, independent of s , such that for all $s > \lambda + 2$, we have

$$\|f\|_{C^{k,\alpha}([s-1, s+1] \times D)} \leq c(\|\mathcal{D}^* \mathcal{D} f\|_{C^{k-4,\alpha}([s-2, s+2] \times D)} + \|f\|_{L^2([s-2, s+2] \times D)}).$$

Next, we multiply this by $e^{-\delta s}$. We have

$$\|e^{-\delta s} f\|_{L^2([s-2, s+2] \times D)}^2 \leq \max\{e^{-4\delta}, e^{4\delta}\} \|e^{-\delta t} f\|_{L^2([s-2, s+2] \times D)},$$

since $e^{-2\delta t} \leq e^{-2\delta(s+2)}$ in $[s-2, s+2] \times D$ if $\delta < 0$ and $e^{-2\delta t} \leq e^{-2\delta(s-2)}$ if $\delta > 0$. Here the L^2 -norm is computed with respect to the volume form $dt \wedge \omega_D^{n-1}$ and ω_D is the smooth metric ω_0 restricted to D . Since the volume form of the Poincaré type metric ω is mutually bounded with $e^{-t} dt \wedge d\theta \wedge \omega_D^{n-1}$ and f is S^1 -invariant, it follows that

$$\|e^{-\delta s} f\|_{L^2([s-2, s+2] \times D)}^2 \leq \frac{\max\{e^{-4\delta}, e^{4\delta}\}}{2\pi} \|f\|_{L_{\delta-\frac{1}{2}}^2(X \setminus D)}^2.$$

So by possibly increasing c we get an inequality of the form

$$\|e^{-\delta s} f\|_{C^{k,\alpha}([s-1,s+1]\times D)} \leq c(\|e^{-\delta s} \mathcal{D}^* \mathcal{D} f\|_{C^{k-4,\alpha}([s-2,s+2]\times D)} + \|f\|_{L^2_{\delta-\frac{1}{2}}(X\setminus D)}).$$

Now, by a similar argument as above, one can show that $\|e^{-\delta s} f\|_{C^{k,\alpha}([s-1,s+1]\times D)}$ is mutually bounded with $\|e^{-\delta t} f\|_{C^{k,\alpha}([s-1,s+1]\times D)}$, independently of s . Similarly for $\|e^{-\delta s} \mathcal{D}^* \mathcal{D} f\|_{C^{k-4,\alpha}([s-2,s+2]\times D)}$. Thus there is a $c > 0$ such that for all $s > \lambda + 2$, we have

$$\|e^{-\delta t} f\|_{C^{k,\alpha}([s-1,s+1]\times D)} \leq c(\|e^{-\delta t} \mathcal{D}^* \mathcal{D} f\|_{C^{k-4,\alpha}([s-2,s+2]\times D)} + \|f\|_{L^2_{\delta-\frac{1}{2}}(X\setminus D)}).$$

Thus by taking the supremum over all $s > \lambda + 2$, we get the required result. $\square$

6.3 Fredholm properties

In this section we will use the estimates of the previous section to prove that Δ_ω and $\mathcal{D}_\omega^* \mathcal{D}_\omega$ are Fredholm operators whenever δ is not an indicial root. The main goal is to prove the statements for the $C_\delta^{k,\alpha}$ -spaces, but we begin by considering the $W_\delta^{2,k}$ -spaces.

The proof that the kernel is finite dimensional follows exactly the same line of argument as the corresponding statement on cylinders as in [Pac08, Thm. 9.1.1].

Proposition 6.11. *The Laplacian, respectively the Lichnerowicz operator, has finite-dimensional kernel both on the $W_\delta^{2,k}$ and $C_\delta^{k,\alpha}$ -spaces.*

Proof. We only do the case of the Laplacian, the argument is the same for the Lichnerowicz operator. It suffices to consider the case when δ is not an indicial root. Also note that it suffices to prove it for the $W_\delta^{2,k}$ -spaces, since by the regularity result 6.10, the kernel on $C_\delta^{k,\alpha}$ and $W_{\delta-\frac{1}{2}}^{2,k}$ coincide.

The proof is by contradiction. If this is not the case, we can take a sequence $f_i \in W_\delta^{2,k}(X \setminus D)$ which are linearly independent. Using Gram-Schmidt, we can assume that

$$\|f_i\|_{L^2_\delta(X\setminus D)} = 1$$

and that

$$\int_{X\setminus D} f_i f_j e^{-2\delta t} dV_g = 0$$

for all $i \neq j$.

Note that by 6.6, since the f_i are in the kernel of Δ_ω , the former implies that the f_i is a bounded sequence in $W_\delta^{2,k}(X \setminus D)$, too. In particular, it is bounded in $W^{2,k}(K)$. We can therefore extract a subsequence which converges in $L^2(K)$, by the Rellich Compactness theorem.

Now, 6.6 and that the f_i are in the kernel of Δ_ω gives that

$$\|f_i - f_j\|_{L^2_\delta(X\setminus D)} \leq c\|f_i - f_j\|_{L^2(K)},$$

and so since f_i converge in $L^2(K)$, they converge in $L^2_\delta(X \setminus D)$, too, say to f .

Since all the f_i are orthogonal, f is orthogonal to all the f_i , i.e.

$$\int_{X \setminus D} f_i f e^{-2\delta t} dV_g = 0$$

for all i . Taking the limit of this again gives that

$$\|f\|_{L^2_\delta(X \setminus D)} = 0.$$

But

$$\|f_i\|_{L^2_\delta(X \setminus D)} = 1$$

for all i , so we must have

$$\|f\|_{L^2_\delta(X \setminus D)} = 1,$$

too. This gives the desired contradiction. $\square$

Next, we show that the image of our operators on the $W^{2,k}_\delta$ -spaces are closed whenever δ is not an indicial root. We follow [Pac08, Thm. 9.2.1].

Proposition 6.12. *Suppose δ is not an indicial root for Δ_ω , respectively $\mathcal{D}_\omega^* \mathcal{D}_\omega$. Then the image of Δ_ω , respectively $\mathcal{D}_\omega^* \mathcal{D}_\omega$, is closed.*

Proof. We do the proof for $\Delta = \Delta_\omega$ only. Take a sequence $f_i \in W^{2,k}_\delta(X \setminus D)$ such that Δf_i converges in $W^{2,k-2}_\delta$, say too g . We want to show that g is in the image of Δ , too.

We can assume the f_i are orthogonal to the kernel of Δ with respect to the $W^{2,k}_\delta$ inner product, since this is a finite-dimensional space by 6.11. We first claim that then f_i has to be bounded. If this was not the case, we could rescale and extract a subsequence h_i so that

$$\|h_i\|_{W^{2,k}_\delta(X \setminus D)} = 1$$

for all i and that

$$\Delta h_i \rightarrow 0,$$

as $i \rightarrow \infty$, since the Δf_i form a bounded sequence in $W^{2,k-2}_\delta(X \setminus D)$. As in the proof of 6.11, we can apply Rellich's theorem to conclude that the h_i converge in $L^2(K)$. Since h_i is Cauchy in $L^2(K)$ and Δh_i is Cauchy in $W^{2,k-2}_\delta$, we can use the estimate 6.6 to conclude that h_i is Cauchy in $W^{2,k}_\delta$.

Say the limit is h . Then h is both in the kernel of Δ as well as satisfying $\|h_i\|_{W^{2,k}_\delta} = 1$. But we assumed the f_i and hence the h_i are orthogonal to the kernel of Δ , so $\|h\|_{W^{2,k}_\delta} = 0$, a contradiction.

Thus the f_i form a bounded sequence. We can therefore in this sequence use Rellich's theorem and 6.6 to conclude that the f_i in fact are Cauchy, and so has a limit in $W^{2,k}_\delta$. It follows that $\Delta f_i = g$, and the proposition is proved. $\square$

Finally, we want to show that the cokernel of Δ_ω and $\mathcal{D}_\omega^* \mathcal{D}_\omega$ are finite dimensional too. This will follow from the identification of the adjoint of our operators with operators going between $W^{2,k}$ -spaces of other weights, similarly to [Pac08, Ch.10]. The key to this is that both operators are formally self-adjoint, i.e. self-adjoint when restricted to functions of compact support.

The first to note is that the dual of L_δ^2 can be identified with $L_{-\delta}^2$.

Lemma 6.13. *For all δ we have that*

$$L_\delta^2(X \setminus D)^* = L_{-\delta}^2,$$

where an element of $L_{-\delta}^2$ is identified with an element of $(L_\delta^2)^*$ by the L^2 inner product.

Proof. This is a direct consequence of the Riesz representation theorem. If $l \in L_\delta^2(X \setminus D)^*$, then

$$\tilde{l}(f) = l(e^{-\delta t} f)$$

is an element of $L^2(X \setminus D)^*$. Thus there exists a $h \in L^2$ such that

$$\tilde{l}(f) = \int_{X \setminus D} f h dV_g.$$

Hence

$$\begin{aligned} l(f) &= \tilde{l}(e^{\delta t} f) \\ &= \int_{X \setminus D} f e^{\delta t} h dV_g, \end{aligned}$$

and $e^{\delta t} h \in L_{-\delta}^2(X \setminus D)$. □

The operators Δ_ω and $\mathcal{D}_\omega^* \mathcal{D}_\omega$ have adjoint operators $(L_\delta^2)^* \rightarrow (W_\delta^{2,2})^*$ and $(L_\delta^2)^* \rightarrow (W_\delta^{2,4})^*$, respectively. This is defined on a dense open subset only. Since our operators are formally self-adjoint, we can use this to show that on the domains of their adjoints, the adjoint equals the original operator.

Lemma 6.14. *Suppose $f \in L_{-\delta}^2 = (L_\delta^2)^*$ is in the kernel of Δ_ω^* , respectively $(\mathcal{D}_\omega^* \mathcal{D}_\omega)^*$. Then $f \in C_{\frac{1}{2}-\delta}^{2,\alpha}$, respectively, $C_{\frac{1}{2}-\delta}^{4,\alpha}$.*

Proof. Let L denote either Δ_ω or $\mathcal{D}_\omega^* \mathcal{D}_\omega$. That f is in the kernel of L^* means that

$$\int_{X \setminus D} f L u dV_g = 0$$

for all u the domain of L . In particular, this holds for all $u \in C_c^\infty(X \setminus D)$. Hence for such u , since L is formally self-adjoint, we have

$$\int_{X \setminus D} L(f) u dV_g = 0.$$

This means $L(f) = 0$ in the sense of distributions. Therefore, since $0 \in C_{-\delta}^{k,\alpha}$, we have, by proposition 6.10, that $f \in C_{\frac{1}{2}-\delta}^{k+2,\alpha}$ when $L = \Delta_\omega$ and $f \in C_{\frac{1}{2}-\delta}^{k+4,\alpha}$ when $L = \mathcal{D}_\omega^* \mathcal{D}_\omega$. □

Proposition 6.15. *Let δ not be an indicial root for $\Delta = \Delta_\omega$, respectively $\mathcal{D}^*\mathcal{D} = \mathcal{D}_\omega^*\mathcal{D}_\omega$. Then*

$$\text{Im } \Delta_{C_\delta^{k+2,\alpha}} = (\text{Ker}(\Delta_{C_{1-\delta}^{k+2,\alpha}}))^\perp,$$

respectively

$$\text{Im } \mathcal{D}^*\mathcal{D}_{C_\delta^{k+4,\alpha}} = (\text{Ker}(\mathcal{D}^*\mathcal{D}_{C_{1-\delta}^{k+4,\alpha}}))^\perp,$$

where $\perp$ denotes the L^2 -inner product and subscripts denote the domains of the operators.

Proof. As before, let L denote either Δ or $\mathcal{D}^*\mathcal{D}$. For $\varepsilon > 0$, let L_ε be L with the domain $W_{\frac{1}{2}+\delta+\varepsilon}^{2,2}$ or $W_{\frac{1}{2}+\delta+\varepsilon}^{2,4}$. It follows from general functional analysis that

$$\text{Im } L_\varepsilon = (\text{Ker}(L_\varepsilon^*))^*.$$

Now from lemma 6.9, $C_\delta^{k,\alpha} \subseteq L_{\delta-\frac{1}{2}+\varepsilon}^2$ if and only if $\varepsilon > 0$. By lemma 6.14 and that dual of $L_{\delta-\frac{1}{2}+\varepsilon}^2$ can be identified with $L_{\frac{1}{2}-\delta-\varepsilon}^2$ via the L^2 -inner product, we have that for any k ,

$$(\text{Ker}(L_\varepsilon^*))^* = (\text{Ker } L_{C_{1-\delta-\varepsilon}^{k+l}})^\perp,$$

where $l = 2$ in the case of the Laplacian and 4 in the case of the Lichnerowicz operator. But for $|\varepsilon|$ sufficiently small and not necessarily positive, $\text{Ker } L_{C_{1-\delta-\varepsilon}^{k+l}}$ does not change, as δ is not an indicial root. Hence

$$(\text{Ker}(L_0^*))^* = (\text{Ker } L_{C_{1-\delta}^{k+l}})^\perp,$$

too. Finally, by proposition 6.10, it follows that

$$\text{Im } L_0 \cap C_\delta^{k,\alpha} = \text{Im } L_{C_\delta^{k+l}}.$$

This completes the proof. □

From propositions 6.11, 6.12 and 6.15 together with lemma 6.10 it follows that we have the following theorem

Theorem 6.16. *Let X be a Kähler manifold on which S^1 acts holomorphically. Let D be a smooth irreducible divisor preserved by this action and suppose ω is a Poincaré type metric on $X \setminus D$ that satisfies 6.1. Suppose δ is not an indicial root for $\Delta = \Delta_\omega$, respectively $\mathcal{D}^*\mathcal{D} = \mathcal{D}_\omega^*\mathcal{D}_\omega$. Then Δ , respectively $\mathcal{D}^*\mathcal{D}$, is a Fredholm operator on the Hölder spaces of S^1 -invariant functions. Moreover, we have*

$$\text{Im } \Delta_{C_\delta^{k+2,\alpha}} = (\text{Ker}(\Delta_{C_{1-\delta}^{k+2,\alpha}}))^\perp,$$

respectively

$$\text{Im } \mathcal{D}^*\mathcal{D}_{C_\delta^{k+4,\alpha}} = (\text{Ker}(\mathcal{D}^*\mathcal{D}_{C_{1-\delta}^{k+4,\alpha}}))^\perp,$$

where $\perp$ denotes the L^2 -inner product and subscripts denote the domains of the operators.

Remark 6.17. *The result holds also in the case when D is not irreducible (though still smooth, so the irreducible components are smooth and do not intersect). Let us say that the irreducible components of D are $D_1, \dots, D_r$. One can decompose $X \setminus D$ into a union of a compact piece and a finite number of tubular neighbourhoods N_i that do not intersect. Then in precisely the same fashion one can define weighted spaces, with a weight for each component of D .*

The basic estimates 6.14 and 6.15 hold in this setting, too. Indeed, now one just has to use a partition of unity supported in a compact subset of $X \setminus D$ and the neighbourhoods N_i of each component of D . One can then follow exactly the same steps as above, considering the behaviour on each N_i separately.

The conclusion is that theorem 6.16 holds for all smooth D , not just irreducible ones.

6.4 Remark on new issues arising in the case when no S^1 -action exists

A natural question to ask given the results of this chapter is whether one can extend the results to the case when there is no S^1 -invariance. The answer should almost certainly be *yes*. There should be a Fredholm theory for the Laplace and Lichnerowicz operators in this setting, with some weights. In this section we outline some of the issues one would have to deal with in order to be able to obtain such results.

The strategy that has been taken above is to start with a local estimate that we then use to prove a global estimate, and in turn use this to prove the Fredholm properties. The strategy for proving the Fredholm properties from the global estimates seem like they would work in many different settings, given the regularity result 6.10. Since the proof of 6.10 follows directly from the definition of the weighted spaces, it seems that to prove Fredholm properties for our operators in the general setting it should suffice to give a local estimate and from this obtain a global estimate.

The local estimates from the Lockhart-McOwen theory do not immediately apply in this setting and so extending an estimate such as 6.7 to the case when there is no S^1 -invariance would be a crucial step in order to deduce a similar Fredholm result.

An additional issue comes when trying to follow the same line of argument to give global estimates 6.6. Suppose we have managed to show an estimate along the lines of 6.7 in the setting where there is no S^1 -action. We could now try the same approach to give a global estimate 6.6. However, when we do not consider S^1 -invariant functions, the non-triviality of the normal bundle to D comes into play. In the S^1 -invariant setting one chooses an S^1 -invariant bump function χ_1 which is supported in the tubular neighbourhood N around D . S^1 -invariant functions are then identified with functions on the model $[a, +\infty) \times D$ and the Lichnerowicz operator of the metric is comparable to the model Lichnerowicz operator in this neighbourhood.

However, if the normal bundle is non-trivial, then N may not be a product. Hence in the case when there is no S^1 -invariance, one has to further subdivide and use a partition of unity subordinate to a cover

of $X \setminus D$ consisting of an open set U whose closure in $X \setminus D$ is compact, along with open sets coming from a cover U_i of D contained in N of the form $\Delta \times U_i$. Call this partition of unity χ_i for $i = 0, 1, \dots$ and with $\text{supp}(\chi_i) \subseteq U_i$ for all $i \geq 1$.

The issue that now comes up is that there is no neighbourhood of D on which the derivatives of all the χ_i vanish, they are supported on the sets $\Delta \times U_i \cap U_j$ in a neighbourhood of the origin. Thus in the analogue of the estimate 6.16 above, the term

$$\|f\|_{W^{2,k}(\text{supp}(d\chi_i))}$$

is no longer the $W^{2,k}$ -norm of some compact subset of $X \setminus D$. The author does not at the moment see how one can deal with this term.

Chapter 7

Perturbing extremal Poincaré type manifolds with an S^1 -action

In this chapter we will prove two perturbation results for extremal Poincaré type metrics, based on two such results in the compact setting. We will use the Fredholm theory proved in the previous section to show that one gets an analogous result in the Poincaré type setting, under the assumption of an S^1 -action on X fixing D .

The first theorem we prove is the following. Let X be a compact complex manifold, $D \subseteq X$ a smooth, irreducible divisor, and let $H^{1,1}(X)$ denote the space of real $(1,1)$ -classes on X . Let $H_0^{1,1}(X) \subseteq H^{1,1}(X)$ denote the subspace of classes which vanish on D .

Theorem 7.1. *Let ω be an extremal Poincaré type metric on $X \setminus D$ in $\Omega = [\omega]$. Then the set of classes A in $H_0^{1,1}(X)$ such that there is an extremal Kähler metric of Poincaré type in the class $\Omega + A$ is open.*

The proof follows closely the proof in the compact case, due to LeBrun and Simanca in [LS93]. Note that there no assumption is made on the class one perturbs by, i.e. one does not restrict to $H_0^{1,1}$.

The second result shows that one can obtain extremal metrics in some Kähler classes after blowing up extremal Poincaré type manifolds, again with an S^1 -action. In the compact case, these are the results of Arezzo-Pacard in [AP06] and [AP09], Arezzo-Pacard-Singer in [APS11] and Székelyhidi in [Szé12] and [Szé15a]. Arezzo and Pacard give sufficient conditions for the blow-up of a Kähler manifold X admitting a cscK metric in a class Ω to admit a cscK metric in perturbations of the pullback of Ω via the blow-down map. The extensions to the case of extremal metrics are by Arezzo-Pacard-Singer and Székelyhidi.

We will now state the result. Given a holomorphy potential f , there is an associated holomorphic vector field $\mathfrak{X}_f$. The condition that a metric is extremal is that its scalar curvature $S(\omega)$ is a holomorphy potential, and so there is an associated holomorphic vector field $\mathfrak{X}_{S(\omega)}$, called the *extremal vector field*. Moreover, let G denote the group of Hamiltonian isometries of $(X \setminus D, \omega)$ coming from automorphisms of X that fix D .

This has an associated moment map

$$\mu : X \setminus D \rightarrow \mathfrak{g}^*,$$

where $\mathfrak{g}$ is the Lie algebra of G . We normalise this so that

$$\int_{X \setminus D} \mu(\mathfrak{X}) \omega^n = 0$$

for all $\mathfrak{X} \in \mathfrak{g}$. We also fix an inner product on $\mathfrak{g}$ by

$$\langle \mathfrak{X}_1, \mathfrak{X}_2 \rangle = \int_{X \setminus D} \mu(\mathfrak{X}_1) \mu(\mathfrak{X}_2) \omega^n.$$

This allows us to identify elements of $\mathfrak{g}^*$ with elements of $\mathfrak{g}$, and so a vector field on X . In particular, we can consider the element $\sum_i a_i^{n-1} \mu(p_i)$ in the statement below as a vector field.

Theorem 7.2. *Let X be a compact complex manifold admitting a holomorphic S^1 -action that preserves a smooth irreducible divisor D , and suppose $\omega \in \Omega$ is an extremal Poincaré type Kähler metric on $X \setminus D$. Let $p_1, \dots, p_k \in X \setminus D$ and $a_1, \dots, a_k > 0$ be chosen such that both $\mathfrak{X}_{S(\omega)}$ and*

$$\sum_i a_i^{n-1} \mu(p_i)$$

vanish at p_i for all i . Then there is a constant $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the blow-up of X at the points p_i admits an extremal Poincaré type Kähler metric with Poincaré type singularity along $\pi^{-1}(D) \cong D$ in the class

$$\pi^*(\Omega) - \varepsilon^2 \left(\sum_i a_i [E_i] \right),$$

where π is the blow-down map and $E_i = \pi^{-1}(p_i)$ are the exceptional divisors.

Remark 7.3. *One expects theorems 7.1 and 7.2 to hold also when there is no S^1 -action. Moreover, one should also be able to avoid the requirement in theorem 7.1 that A vanishes on D . We discuss this and other directions these theorems lead to in section 7.7.*

We end the introduction to this chapter by describing the key new issue that we need to deal with. In the previous chapter we developed the Fredholm theory for the Lichnerowicz operator acting on weighted Hölder spaces. We wish to use the results of the previous section to write our metric as a perturbation of a metric on $X \setminus D$. For the spaces of chapter 6, the elements of the $C_\eta^{k,\alpha}$ -spaces give potentials for Poincaré type Kähler metrics precisely when $\eta \leq 0$. However, recall that 0 is an indicial root, so we can only apply the Fredholm theory for $\eta < 0$. Moreover, the $C_0^{k,\alpha}$ -spaces do not contain functions that grow like $O(t)$ near D .

In light of the results by Auvray in [Auv14b, Thm. 4.8], it is clear that perturbing by functions in the $C_\eta^{4,\alpha}$ -spaces with $\eta < 0$ is doomed. Recall from the beginning of chapter 6 that in a tubular neighbourhood

around D we have an S^1 -fibration over $[c, \infty) \times D$. Composing this with the projection to the D -factor, we obtain a map Π mapping a neighbourhood of D in X to D . Auvray shows that if ω is an extremal Poincaré type metric on $X \setminus D$, then near D we can write

$$g = a(dt^2 + e^{-2t}d\theta^2) + \Pi^*g_D + O(e^{\eta t}), \quad (7.1)$$

where g_D is an extremal metric on D and $\eta < 0$. The constant a is given by

$$\frac{1}{a} = \hat{S}(g_D) + \frac{1}{4\pi \text{Vol}(D)} \left(\int_{X \setminus D} S(g) \Delta(e^t) dV_g - \int_{X \setminus D} \nabla^g S(g) \cdot e^t dV_g \right).$$

Here $\hat{S}(g_D)$ is the average scalar curvature of the metric g_D on D .

This quantity, however, is not in general preserved when changing the Kähler class or blowing-up. In other words, if there is an extremal Kähler metric $\tilde{g}$ on $(Bl_p X \setminus D, \pi^*\Omega - \varepsilon[E])$ or $(X, \Omega + A)$, then

$$\tilde{g} = b(dt^2 + e^{-2t}d\theta^2) + \Pi^*g_D + O(e^{\eta t}),$$

for some $b > 0$ defined by a similar equation, and in general $b \neq a$. Thus we have to allow for some functions which grow like $O(t)$ near D , too, since t is the Kähler potential for $dt^2 + e^{-2t}d\theta^2$.

The organisation of the chapter is as follows. In section 7.1, we apply the linear theory of the previous chapter to explicitly find the kernel and cokernel of the Lichnerowicz operator on the space of potentials that will allow us to prove theorems 7.1 and 7.2.

In section 7.2, we prove theorem 7.1, following the same argument as that of LeBrun and Simanca in the compact case.

The argument for theorem 7.2 is more involved, however. Here we follow the proof of [Szé12], and we outline this in section 7.3. The additional linear theory needed involving doubly weighted spaces is discussed in 7.4, before we solve the non-linear equation in 7.5.

In section 7.6 we apply our results to produce new examples of extremal Poincaré type metrics. We end the chapter in section 7.7 by discussing possible future directions leading on from theorem 7.1 and 7.2.

7.1 Applying the linear theory

In this section we will identify the kernel and cokernel of the Lichnerowicz operator for the weights we are considering. We will also do a finite-dimensional adjustment of the domain and range to allow for the functions we need to be able to perturb the approximate metrics we make to extremal metrics. The main goal of the section is to prove proposition 7.13. This explicitly gives the kernel and cokernel for the Lichnerowicz operator acting on some spaces related to the $C_\eta^{k,\alpha}$ -spaces, for small negative η .

We will continue to use the assumptions of remark 6.1. Further, we will be discussing several Lichnerowicz operators, so we will use the following conventions. Note in particular the last point, which says that $\mathcal{D}_0^*\mathcal{D}_0$ denotes an operator on functions on D , not on X .

- $\mathcal{D}^*\mathcal{D} = \mathcal{D}_\omega^*\mathcal{D}_\omega$ is the Lichnerowicz operator associated to the metric ω on $X \setminus D$;
- $\mathcal{D}^*\mathcal{D}_D = \mathcal{D}_{\omega_D}^*\mathcal{D}_{\omega_D}$ is the Lichnerowicz operator associated to the metric ω_D on D ;
- $\mathcal{D}_0^*\mathcal{D}_0$ is the Lichnerowicz operator associated to the metric ω_0 **restricted to D** .

The first goal is to find the kernel and cokernel for small negative weights, proposition 7.7. We will need the following definition. In the statement $\mathfrak{h}$ is the space of real holomorphic vector fields on X .

Definition 7.4 ([Auv14b, Defn. 1.1]). *Let $\mathfrak{X}$ be a real holomorphic vector field on X . We say $\mathfrak{X}$ is parallel to D if, when writing $\mathfrak{X}$ as the real part of $\sum_i f_i \frac{\partial}{\partial z_i}$ in coordinates near D where D is given by $z_1 = 0$, we have $f_1 = 0$ on D . We write $\mathfrak{h}_{//}^D$ for the subspace of $\mathfrak{h}$ of such vector fields. The space of potentials for vector fields in $\mathfrak{h}_{//}^D$ with zeros is denoted $\overline{\mathfrak{h}_{//}^D}$.*

Vector fields parallel to D as in definition 7.4 enter in our discussion of weighted spaces because of the following result of Auvray.

Lemma 7.5 ([Auv14c, Lem. 5.2]). *Let $\mathfrak{X} \in \mathfrak{h}$ be a real holomorphic vector field on X . Then its L^2 -norm $\|\mathfrak{X}\|_{L^2(X \setminus D, \omega)}$ with respect to a Poincaré type Kähler metric ω on $X \setminus D$ is finite if and only if $\mathfrak{X} \in \mathfrak{h}_{//}^D$.*

We will also need the following integration by parts result.

Lemma 7.6. *Let $f \in C_\eta^{4,\alpha}$ and $g \in C_{\eta'}^{4,\alpha}$ with $\eta + \eta' < 1$. Then*

$$\int_{X \setminus D} \mathcal{D}^*\mathcal{D}(f)g\omega^n = \int_{X \setminus D} \langle \mathcal{D}(f), \mathcal{D}(g) \rangle \omega^n.$$

Proof. For simplicity we consider the case when $\eta' = \eta$, so in particular $\eta < \frac{1}{2}$. It will however be clear that the same argument goes through for any choice of η and η' satisfying $\eta + \eta' < 1$.

Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a bump function supported on $(-\infty, 1]$ and equal to 1 in $(-\infty, 0]$ and let $\chi_a(x) = \chi(x - a)$. We can consider χ as a function on $X \setminus D$ by composing with the function t . We then have that

$$\lim_{a \rightarrow \infty} \int_{X \setminus D} \chi_a \mathcal{D}^*\mathcal{D}(f)g\omega^n = \int_{X \setminus D} \mathcal{D}^*\mathcal{D}(f)g\omega^n, \quad (7.2)$$

$$\lim_{a \rightarrow \infty} \int_{X \setminus D} \chi_a \langle \mathcal{D}(f), \mathcal{D}(g) \rangle = \int_{X \setminus D} \langle \mathcal{D}(f), \mathcal{D}(g) \rangle \omega^n. \quad (7.3)$$

Since $\chi_a g$ has compact support, it follows that

$$\int_{X \setminus D} \chi_a \mathcal{D}^*\mathcal{D}(f)g\omega^n = \int_{X \setminus D} \langle \mathcal{D}(f), \mathcal{D}(\chi_a g) \rangle \omega^n.$$

This differs from

$$\int_{X \setminus D} \chi_a \langle \mathcal{D}(f), \mathcal{D}(g) \rangle \omega^n$$

by terms involving at least one derivative of χ_a , hence is an integral over $K_a = \{x \in X \setminus D : t(x) \in [a, a+1]\}$.

Since $f \in C_\eta^{4,\alpha}$, we have that $|\mathcal{D}(f)| \leq ce^{\eta t}$ for some $c > 0$. Also, the derivative of χ_a is bounded on $[a, a+1]$ independently of a . Finally, we have that by possibly increasing c , g and the norm of its gradient is bounded by $ce^{\eta t}$ as well. Thus

$$|\int_{X \setminus D} \chi_a \mathcal{D}^* \mathcal{D}(f) g \omega^n - \int_{X \setminus D} \langle \mathcal{D}(f), \mathcal{D}(\chi_a g) \rangle \omega^n| \leq C \int_{K_a} e^{2\eta t} \omega^n$$

for some $C > 0$. This latter integral is mutually bounded with

$$\int_a^{a+1} e^{2\eta t - t} dt,$$

which goes to zero as $a \rightarrow \infty$ precisely if $\eta < \frac{1}{2}$. Thus the limits in 7.2 and 7.3 agree, and the proof is complete. $\square$

We are now ready to characterise the kernel and cokernel of the Lichnerowicz operator for the weights of interest to us. Let $\mathfrak{h}^D$ be the space of real holomorphic vector fields on D and let s be the codimension of the subspace consisting of vector fields induced by a vector field parallel to D . Also, let $\mathfrak{k}$ denote the vector fields in $\mathfrak{h}_{//}^D$ whose induced vector field on D vanishes, and let $\bar{\mathfrak{k}}$ denote the potentials for such vector fields.

For a function f on D , we can extend f to an S^1 -invariant function $\Pi^* f$ near D by using the projection $\Pi : N_D \setminus D \rightarrow D$. Using an S^1 -invariant bump function supported on N_D and only depending on t , we can then consider $\chi \Pi^* f$ as a globally defined function.

Recall that φ was the Kähler potential of the Poincaré type metric ω . Given a function $f \in \ker \mathcal{D}_0^* \mathcal{D}_0$, we can pull back the corresponding real holomorphic vector field $\mathfrak{X}_f$ to the tubular neighbourhood around D . Recall also that $\mathcal{D}_0^* \mathcal{D}_0$ denotes the Lichnerowicz operator on D associated to the metric $\omega_{0|D}$. Given a bump function χ depending only t and supported near D , we let ψ_f be given by

$$\psi_f = \chi(\Pi^* f + d\varphi(\Pi^* \mathfrak{X}_f)).$$

Given a basis $f_1, \dots, f_r$ for $\ker \mathcal{D}_0^* \mathcal{D}_0$, we let $\psi_i = \psi_{f_i}$. We will assume $f_{s+1}, \dots, f_r$ form a basis for the subspace of $\ker \mathcal{D}_0^* \mathcal{D}_0$ of potentials for vector fields on D induced by vector fields on X .

Lemma 7.7. *Consider the Lichnerowicz operator*

$$\mathcal{D}^* \mathcal{D} = \mathcal{D}_\omega^* \mathcal{D}_\omega : C_\eta^{4,\alpha}(X \setminus D) \rightarrow C_\eta^{0,\alpha}(X \setminus D)$$

on the Poincaré type weighted spaces. Then there is a $\kappa > 0$ such that

$$\begin{aligned} \ker \mathcal{D}^* \mathcal{D}_{C_\eta^{4,\alpha}} &= \overline{\mathfrak{h}_{//}^D} \text{ if } \eta \in (0, 1), \\ \ker \mathcal{D}^* \mathcal{D}_{C_\eta^{4,\alpha}} &\subseteq \bar{\mathfrak{k}} \text{ and is of codimension 1 if } \eta \in (-\kappa, 0), \\ C_\eta^{0,\alpha} \cap \overline{\mathfrak{h}_{//}^D}^\perp &= \text{Im}(\mathcal{D}^* \mathcal{D}_{C_\eta^{4,\alpha}}) \oplus \langle \mathcal{D}^* \mathcal{D}(\psi_i) : i \in \{1, \dots, s\} \rangle \text{ if } \eta \in (-\kappa, 0). \end{aligned}$$

Proof. First note that 0 is an indicial root. Indeed, the constant functions are in the kernel of $\mathcal{D}^*\mathcal{D}$ and are in $C_\delta^{k,\alpha}$ precisely when $\delta \geq 0$, so the kernel changes at $\delta = 0$. By the duality between the kernel and cokernel for weights δ and $1 - \delta$, it follows that 1 is also an indicial root. Moreover, Auvray showed in [Auv14a] that there are no indicial roots in $(0, 1)$. Thus there exists a $\kappa > 0$ such that the kernel and cokernel of $\mathcal{D}^*\mathcal{D}$ is constant in the intervals stated.

We first establish the claim for $\eta \in (0, 1)$. If $f \in C_\eta^{4,\alpha}(X \setminus D)$ with $\eta < \frac{1}{2}$, we may apply the integration by parts lemma 7.6 to conclude that

$$\int_{X \setminus D} \mathcal{D}^*\mathcal{D}(f)f = \int_{X \setminus D} |\mathcal{D}f|^2.$$

Thus if $f \in \ker \mathcal{D}^*\mathcal{D}$, we have that $f \in \ker \mathcal{D}$. This choice of weights means that the holomorphic vector field $\mathfrak{X}_f$ associated to f then is in L^2 . Thus it follows from lemma 7.5 that $\mathfrak{X}_f \in \mathfrak{h}_{//}^D$ and hence $f \in \overline{\mathfrak{h}_{//}^D}$. Since the elements of $\overline{\mathfrak{h}_{//}^D}$ are in $C_\eta^{4,\alpha}$ for any $\eta > 0$, it follows that the kernel is as stated for $\eta \in (0, \frac{1}{2})$. Since there are no indicial roots in $(0, 1)$, the same conclusion then holds for all $\eta \in (0, 1)$.

For $\eta \in (-\kappa, 0)$ the kernel is strictly smaller, since the constants are in $\overline{\mathfrak{h}_{//}^D}$, but not in the domain of $\mathcal{D}^*\mathcal{D}$ for these weights. For these weights the associated holomorphic vector field has to have norm in the order of $e^{\eta' t}$ with $\eta' \leq \eta$. Hence if $f \in \ker \mathcal{D}_{C_\eta^{4,\alpha}}$ with $\eta < 0$, we have to have that $\mathfrak{X}_f^D$, the induced vector field on D , is trivial, i.e. $f \in \bar{\mathfrak{k}}$. Indeed, in taking the norm of a vector field

$$\sum_i \sigma_i \frac{\partial}{\partial z_i}$$

with $\sigma_1(0) = 0$, we have, in the model case, that the contribution from $\sigma_1 \frac{\partial}{\partial z_1}$ is

$$\begin{aligned} g_{1\bar{1}}|\sigma_1|^2 &= O\left(\frac{|z_1|^2}{|z_1|^2 \log^2(|z_1|^2)}\right) \\ &= O(e^{-2t}). \end{aligned}$$

For $\sigma_i \frac{\partial}{\partial z_i}$ with $i > 1$, the contribution to the norm is $O(1)$. Since the general case is mutually bounded with this it follows that for f to lie in $C_\eta^{4,\alpha}$ with $-1 \leq \eta < 0$, one necessarily has to have $\sigma_i = 0$ for all $i > 1$, and then

$$\|\mathfrak{X}_f\| = O(e^{-t}),$$

as required. Note that since $\bar{\mathfrak{k}}$ also contains the constants, the codimension of $\ker \mathcal{D}^*\mathcal{D}$ in $\bar{\mathfrak{k}}$ is at least one.

By [LM85, Thm. 1.4], the index in this range equals the index in the local case, which Auvray showed in [Auv14a, Lem. 3.10] is $-\dim \ker \mathcal{D}^*\mathcal{D}_D = -r$. Note that the dimension of potentials for holomorphic vector fields on D induced by a vector field parallel to D on X is $r - s$. Also,

$$\text{Im}(\mathcal{D}^*\mathcal{D}_{C_\eta^{4,\alpha}}) \subseteq \overline{\mathfrak{h}_{//}^D}^\perp$$

Thus if we can exhibit at least s linearly independent elements in $C_\eta^{0,\alpha} \cap \overline{\mathfrak{h}_{//}^D}^\perp$ that are not in the image of $\mathcal{D}^*\mathcal{D}$ on $C_\eta^{4,\alpha}$ and which are linearly independent of the image as well, then we have found the full cokernel of $\mathcal{D}^*\mathcal{D}$, because then our reduction of the kernel above

$$\begin{aligned} \text{ind } (\mathcal{D}^*\mathcal{D}_{C_\eta^{4,\alpha}}) &= \dim \ker \mathcal{D}_{C_\eta^{4,\alpha}} - \dim \text{coker } \mathcal{D}_{C_\eta^{4,\alpha}} \\ &\leq (\dim \overline{\mathfrak{h}_{//}^D} - (r - s)) - (\dim \overline{\mathfrak{h}_{//}^D} + s) \\ &= -r, \end{aligned}$$

and so the kernel cannot be smaller, nor can the cokernel be any larger.

In [Auv14a], Auvray showed that

$$\mathcal{D}^*\mathcal{D}(\psi_f) \in C_{-1}^{0,\alpha},$$

for any $f \in \ker \mathcal{D}_0^*\mathcal{D}_0$. If this is in the image of $\mathcal{D}^*\mathcal{D}$ on $C_\eta^{4,\alpha}$ with $\eta < 0$, say

$$\mathcal{D}^*\mathcal{D}(\psi) = \mathcal{D}^*\mathcal{D}(v),$$

then

$$\psi - v \in \ker \mathcal{D}^*\mathcal{D}_{C_{\eta'}^{4,\alpha}}$$

for any $\eta' > 0$. By the previous part, this implies $\psi - v = h \in \overline{\mathfrak{h}_{//}^D}$.

We now invoke lemma 7.8 below which says that $\psi - h \in C_\eta^{4,\alpha}$ for some weight $\eta < 0$ if and only if f is a potential for the vector field on D induced by h , under a suitable normalisation. This completes the proof, because then for each ψ coming from an f inducing a holomorphic vector field on D which also is induced by a holomorphic vector parallel to D , we can choose a $h \in \overline{\mathfrak{h}_{//}^D}$ such that $\psi - h \in C_\eta^{4,\alpha}$ with $\eta < 0$ and $\mathcal{D}^*\mathcal{D}(\psi - h) = \mathcal{D}^*\mathcal{D}(\psi)$. Hence

$$\mathcal{D}^*\mathcal{D}(\psi) \in \text{Im } \mathcal{D}^*\mathcal{D}_{C_\eta^{4,\alpha}}$$

if and only if f induces a holomorphic vector field on D also induced by a holomorphic vector field on X parallel to D . $\square$

Lemma 7.8. *For $f \in \ker \mathcal{D}_0^*\mathcal{D}_0$, let $\psi = \psi_f = \chi(\Pi^*f + d\varphi(\Pi^*\mathfrak{X}_f))$. Then there is a $h \in \overline{\mathfrak{h}_{//}^D}$ such that $\psi - h \in C_\eta^{4,\alpha}$ for some $\eta < 0$ if and only if the associated vector field $\mathfrak{X}_f$ of f on D is induced by a vector field in $\mathfrak{h}_{//}^D$.*

The reader may find the case of the constants illustrative. Then we are considering $\mathcal{D}^*\mathcal{D}(\chi)$ and $\chi \notin C_\eta^{4,\alpha}$ for any $\eta < 0$. However, since χ is 1 in a neighbourhood of D , we have $\chi - 1 \in C_\eta^{4,\alpha}$ for any $\eta < 0$ and $\mathcal{D}^*\mathcal{D}(\chi) = \mathcal{D}^*\mathcal{D}(\chi - 1)$. This in particular shows that one has to choose the correct normalisation of h .

Proof. Let $Z \in \mathfrak{h}_{//}^D$ be a vector field on X such that $Z|_D = \mathfrak{X}_f$. Let h be the potential for Z with respect to the Poincaré type metric $\omega = \omega_0 + i\partial\bar{\partial}\varphi$. Let h_0 be the corresponding potential with respect to the smooth background metric ω_0 . Using [Auv14b, Prop. 1.2], we then have that

$$\begin{aligned} h &= h_0 + d\varphi(Z) \\ &= h_0 + (\nabla^{\omega_0} h_0) \cdot \varphi. \end{aligned}$$

Up to a constant, f is the restriction of h_0 to D , and so we can renormalise h to assume this is true. It then follows that

$$\chi\Pi^*f_0 - h_0 \in C_\eta^{4,\alpha}$$

for some $\eta < 0$. Hence

$$\psi_f - h \in C_\eta^{4,\alpha}$$

for some $\eta < 0$, too.

Conversely, suppose $\psi_f - h \in C_\eta^{4,\alpha}$ for some $\eta < 0$. Then the associated vector fields are also equal to order $e^{\eta t}$ and so their restrictions to D must be equal. $\square$

Remark 7.9. *The functions ψ_i above are only supported in a neighbourhood of D , depending on a choice of χ . For estimates later for the blow-up theorem, the behaviour of these functions near the blown up point does not really play a role. Thus after choosing the point p to blow up, we choose χ so that it supported away from $B_{2r_\varepsilon}(p)$ for all $\varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$, where $B_{2r_\varepsilon}(p)$ is a ball around p of a radius that will be specified later. We can make ε_0 as large as we want for this. In particular it has no impact on the largest possible ε_0 for which theorem 7.2 holds. The discussion also applies when there are several points, just making sure χ is supported away from such balls around all the points blown up.*

We now turn our attention to the finite dimensional adjustment of the $C_\eta^{4,\alpha}$ -spaces mentioned in the introduction to this section. As remarked at the end of the introduction to this chapter, if $\tilde{\omega}$ is the approximate solution to the extremal equation in either of the two main theorems of this chapter, then trying to solve our equations with a metric of the form $\tilde{\omega} + i\partial\bar{\partial}\phi$ for some $\phi \in C_\eta^{4,\alpha}$ with $\eta < 0$ will in general not be possible. We would thus like to include functions of the form $h + \sigma$ for $h \in \overline{\mathfrak{h}_{//}^D}$ and $\sigma \in C_\eta^{4,\alpha}$ in our range. Since $\overline{\mathfrak{h}_{//}^D} \subseteq \cap_{\delta>0} C_\delta^{4,\alpha}$, it follows from the linear theory that by choosing σ so that $h - \sigma$ is orthogonal to $\overline{\mathfrak{h}_{//}^D}$, there is for each h an element of $\phi \in \cap_{\delta>0} C_\delta^{4,\alpha}$ such that $\mathcal{D}^*\mathcal{D}(\phi) - h \in C_\eta^{4,\alpha}$. However, this intersection contains for example the functions t^n for any n , so we need to ensure that such an element is bounded by Ct for some $C > 0$ to be a valid potential for a Poincaré type metric.

We note that in chapter 6, we got different functions t depending on the Hermitian metric $|\cdot|$ we chose on $\mathcal{O}(D)$. We now want to fix this function, but allow metrics which satisfy 6.1 for a different choice of this Hermitian metric. This is reflected in the parameter a in 7.4 below. For a metric with these asymptotics,

which includes the extremal case, the asymptotics of the ψ_f only depend on the metric on D and a up to leading order. The first step to this is the following. For a metric g_D on D we let $\mathcal{D}^*\mathcal{D}_D$ denote the corresponding Lichnerowicz operator.

Lemma 7.10. *Let ω be a Poincaré type metric on $X \setminus D$ which is asymptotically a product, i.e. satisfies*

$$\omega = a\omega_{cusp} + \Pi^*g_D + O(e^{-\eta't}) \quad (7.4)$$

for some $a, \eta' > 0$ and metric g_D on D . Then there exists $\eta < 0$ such that for all $\tilde{f} \in \ker \mathcal{D}^*\mathcal{D}_D$ there exists $\sigma \in C_\eta^{0,\alpha}(X \setminus D)$, $\phi \in C^{4,\alpha}(D)$ and $f \in \ker \mathcal{D}^*\mathcal{D}_D$ such that

$$\mathcal{D}_\omega^*\mathcal{D}_\omega(\chi\Pi^*(\phi) + t\chi\Pi^*f) = \chi\Pi^*\tilde{f} + \sigma. \quad (7.5)$$

Moreover, f is unique and ϕ is unique up to an element of $\ker \mathcal{D}^*\mathcal{D}_D$.

Proof. We begin with the case of the model metric on $\Delta^* \times D$. Recall from 6.3 that the Lichnerowicz operator L_{mod} then is given by

$$f \mapsto \frac{1}{2} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right)^2 (f) - \left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right) (\Delta_D f) - \left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right) (f) + \mathcal{D}^*\mathcal{D}_D f.$$

Then for the function $t \cdot \Pi^*f$, with f a function on D , we get that

$$L_{\text{mod}}(t \cdot f) = \Pi^*(f + \Delta_D f + t\mathcal{D}^*\mathcal{D}_D f).$$

In particular, for all $f \in \text{Ker } \mathcal{D}^*\mathcal{D}_D$, we have

$$L_{\text{mod}}(t \cdot \Pi^*f) = \Pi^*(f + \Delta_D(f)).$$

Now, note that since

$$\int_D (f + \Delta_D(f)) \cdot f = \int_D |f|^2 + |df|^2,$$

we have that $f + \Delta_D(f) \notin (\text{Ker } \mathcal{D}^*\mathcal{D}_D)^\perp = \text{Im } \mathcal{D}^*\mathcal{D}_D$.

In fact, if $f_1, \dots, f_r$ is a basis of $\ker \mathcal{D}^*\mathcal{D}_D$, then $f_i + \Delta_D f_i$ form a basis of a complementary space to $\text{Im } \mathcal{D}^*\mathcal{D}_D$. This follows by integrating by parts again, since if $\mathcal{D}^*\mathcal{D}_D(\phi) = f + \Delta_D f$, then

$$\begin{aligned} \int_D |f|^2 + |df|^2 &= \int_D (f + \Delta_D(f)) \cdot f \\ &= \int_D \mathcal{D}^*\mathcal{D}_D(\phi) f \\ &= \int_D \phi \mathcal{D}^*\mathcal{D}_D(f) \\ &= 0, \end{aligned}$$

as $f \in \ker \mathcal{D}^* \mathcal{D}_D$. In particular, given any $\psi \in C^{0,\alpha}(D)$, there exists $f \in \ker \mathcal{D}^* \mathcal{D}_D$ and $\phi \in C^{4,\alpha}(D)$ such that

$$\mathcal{D}^* \mathcal{D}_D(\phi) + f + \Delta_D f = \psi.$$

Applying this to $\tilde{f}$ in the model case, we have that there are ϕ and f such that

$$\begin{aligned} \Pi^* \tilde{f} &= \Pi^*(\mathcal{D}^* \mathcal{D}_D(\phi) + f + \Delta_D f) \\ &= \mathcal{D}^* \mathcal{D}(\Pi^* \phi + t \cdot \Pi^* f). \end{aligned}$$

We can then let σ be given by

$$\sigma = \chi \mathcal{D}^* \mathcal{D}(\Pi^* \phi + t \cdot \Pi^* f) - \mathcal{D}^* \mathcal{D}(\chi \Pi^* \phi + t \cdot \chi \Pi^* f),$$

which lies in $C_\eta^{0,\alpha}$ for any η , since σ vanishes identically in a neighbourhood of D .

In the general case, we use the asymptotics 7.4. It then follows that there is an $\eta < 0$ such that the estimates 6.15 hold, using the model operator which is $a\omega_{cusp}$ instead of ω_{cusp} . This affects the coefficients of f_i and $\Delta_1 f_i$ above, but it does not affect the conclusion, because we still obtain a positive combination of f_i and $\Delta_D f_i$. In particular $\mathcal{D}^* \mathcal{D}(\chi \Pi^*(\phi) + t \chi \Pi^* f)$ agrees with $\mathcal{D}^* \mathcal{D}_{\text{mod}}(\chi \Pi^*(\phi) + t \chi \Pi^* f)$ up to an element of order $e^{\eta t}$. Hence we can always solve our equation up to an error of order $e^{\eta t}$ with $\eta < 0$, as required. $\square$

The metric g_D might not be equal to $\omega_{0|D}$. The final piece we need is the following lemma.

Lemma 7.11. *Let ω be a Poincaré type metric satisfying 7.4. Then there is an $\eta < 0$ such that for each $h \in \overline{\mathfrak{h}_{//}^D}$ there is a (unique) $f \in \ker \mathcal{D}^* \mathcal{D}_D$ with*

$$h - \chi \Pi^* f \in C_\eta^{4,\alpha}.$$

Also, h and f induce the same vector field on D .

Proof. Here we use the local result concerning the Lichnerowicz operator of Auvray, see [Auv14a, Lem. 3.8]. For each $f_0 \in \ker \mathcal{D}_0^* \mathcal{D}_0$ on D , we get an element $\mathcal{D}_\omega^* \mathcal{D}_\omega(\chi \Pi^* f_0) \in C_\eta^{4,\alpha}$ and given a basis for $\ker \mathcal{D}_0^* \mathcal{D}_0$, these elements span the cokernel of the Lichnerowicz operator for small weights. Calculations similar to those in the proof of 7.10 shows that the same holds for the elements $\chi \Pi^* f$ with $f \in \ker \mathcal{D}_D^* \mathcal{D}_D$. This means that given $f \in \ker \mathcal{D}_D^* \mathcal{D}_D$ we can choose $f_0 \in \ker \mathcal{D}_0^* \mathcal{D}_0$ such that

$$\mathcal{D}_D^* \mathcal{D}_D(\chi \Pi^*(f - f_0)) = \mathcal{D}^* \mathcal{D}(\phi),$$

for some $\phi \in C_\eta^{4,\alpha}$. But then

$$\chi \Pi^*(f - f_0) - \phi \in \ker \mathcal{D}^* \mathcal{D}_{C_{\eta'}^{4,\alpha}}$$

for any $\eta' > 0$, and so equals an element of $\overline{\mathfrak{h}_{//}^D}$. In particular, the asymptotics of $\chi \Pi^* f$ equals that of a function in $\psi_{f'}$ for $f' \in \ker \mathcal{D}_0^* \mathcal{D}_0$, using lemma 7.8. Since the $\chi \Pi^* f$ and $\psi_{f'}$ asymptotically span a vector

space of the same dimension, this means that for any functions which differ from an element $\psi_{f'}$ by a function of order $e^{\eta t}$, we can find an element f of $\ker \mathcal{D}_D^* \mathcal{D}_D$ such that the same holds for $\chi \Pi^* f$. In particular, this holds for all elements of $\overline{\mathfrak{h}_{//}^D}$. It is also necessary that these induce the same vector field on D .

The uniqueness follows because $\chi \Pi^*(f) \in C_\eta^{4,\alpha}$ if and only if $f = 0$. $\square$

We will now define some new spaces related to the $C_\eta^{k,\alpha}$ -spaces. We will let $\oplus$ denote the *internal direct sum*. We will also use $\times$, for which we mean the *external direct sum*.

We can decompose $\ker \mathcal{D}^* \mathcal{D}_D$ as $V' \oplus V''$, where the elements of V' are potentials for vector fields induced by vector fields on X and V'' is a complementary subspace. We can then define a map

$$\lambda : \ker \mathcal{D}^* \mathcal{D}_D \rightarrow \chi \Pi^* C^\infty(D) \oplus t \chi \Pi^* \ker \mathcal{D}^* \mathcal{D}_D$$

as follows. Here the internal sum is as subspaces of $C_{\eta'}^{4,\alpha}$ for some fixed $\eta' > 0$.

For $f \in V'$ we map to the element $\chi \Pi^* \phi + t \chi \Pi^* \tilde{f}$ given by lemma 7.10 such that

$$\mathcal{D}^* \mathcal{D}(\Pi^* \phi + t \chi \Pi^* \tilde{f}) - \chi \Pi^* f \in C_\eta^{0,\alpha}.$$

To make this well-defined, we choose the unique ϕ which is orthogonal to the kernel of $\mathcal{D}^* \mathcal{D}_D$ via the L^2 -inner product on D associated to g_D . On V'' we simply map f to $\chi \Pi^* f$.

This in turn gives us a map

$$\ker \mathcal{D}^* \mathcal{D}_D \rightarrow C_\eta^{0,\alpha} \oplus \chi \Pi^* \ker \mathcal{D}^* \mathcal{D}_D$$

given by

$$f \mapsto \mathcal{D}_\omega^* \mathcal{D}_\omega(\lambda(f)).$$

Fixing some norm on $\ker L_D$, this is bounded.

The range of the map $C_\eta^{4,\alpha} \times \ker \mathcal{D}^* \mathcal{D}_D \rightarrow C_\eta^{0,\alpha} \oplus \chi \Pi^* \ker \mathcal{D}^* \mathcal{D}_D$ given by

$$(\phi, f) \mapsto \mathcal{D}_\omega^* \mathcal{D}_\omega(\phi + \lambda(f)) \tag{7.6}$$

now contains $\overline{\mathfrak{h}_{//}^D}$, by lemma 7.11, and $\overline{\mathfrak{h}_{//}^D}$ is in fact contained in the subspace of $C_{\eta'}^{0,\alpha}$, with $\eta' > 0$, given by

$$C_\eta^{0,\alpha} \oplus \chi \Pi^* V'.$$

These spaces will be important for us in what follows, so we make the following definition.

Definition 7.12. *Let ω be a Poincaré type metric on $X \setminus D$ which satisfies 7.4. We then let*

$$\tilde{C}_\eta^{k,\alpha} = C_\eta^{k,\alpha} \oplus \chi \Pi^* V'.$$

The Fredholm properties for the map 7.6 will be crucial for proving theorems 7.1 and 7.2.

Proposition 7.13. *Suppose ω is a Poincaré metric satisfying 7.4 for some $a > 0$ and a metric g_D on D . Then there exists a $\kappa > 0$ such that for all $\eta \in (-\kappa, 0)$,*

$$\mathcal{D}_\omega^* \mathcal{D}_\omega : C_\eta^{4,\alpha}(X \setminus D) \times \ker \mathcal{D}^* \mathcal{D}_D \rightarrow \tilde{C}_\eta^{0,\alpha}(X \setminus D) \quad (7.7)$$

given by

$$(\phi, f) \mapsto \mathcal{D}_\omega^* \mathcal{D}_\omega(\phi + \lambda(f))$$

is a Fredholm operator. Moreover,

$$\ker \mathcal{D}_\omega^* \mathcal{D}_\omega = \ker(\mathcal{D}_\omega^* \mathcal{D}_\omega)|_{C_\eta^{4,\alpha} \times \{0\}}$$

and

$$\text{im } \mathcal{D}_\omega^* \mathcal{D}_\omega = \overline{\mathfrak{h}_{//}^D}^\perp.$$

Proof. The fact that this map is Fredholm follows from the fact that the map

$$\mathcal{D}_\omega^* \mathcal{D}_\omega : C_\eta^{4,\alpha} \rightarrow C_\eta^{0,\alpha}$$

is Fredholm.

For the image, note that for any (ϕ, f) , we have that

$$\phi + \lambda(f) \in C_{\eta'}^{4,\alpha}$$

for any $\eta' > 0$. Hence by lemma 7.7

$$\text{im}(\mathcal{D}_\omega^* \mathcal{D}_\omega) \subseteq \overline{\mathfrak{h}_{//}^D}^\perp.$$

Conversely, suppose $\psi \in \tilde{C}_\eta^{0,\alpha} \cap \overline{\mathfrak{h}_{//}^D}^\perp$. Then there exists $f' \in \ker \mathcal{D}^* \mathcal{D}_D$ such that

$$\psi - \mathcal{D}_\omega^* \mathcal{D}_\omega(\lambda(f')) \in C_\eta^{0,\alpha}.$$

Moreover, this element remains orthogonal to $\overline{\mathfrak{h}_{//}^D}$. Thus by 7.7 and the analogous result to 7.11 for the elements $\psi_1, \dots, \psi_s$ in 7.7, there exists $\phi \in C_\eta^{4,\alpha}$ and $f'' \in \ker \mathcal{D}^* \mathcal{D}_D$ such that

$$\mathcal{D}_\omega^* \mathcal{D}_\omega(\phi + \lambda(f'')) = \psi - \mathcal{D}_\omega^* \mathcal{D}_\omega(\lambda(f')).$$

The result follows with $f = f' + f''$.

Finally, for the kernel, the index of this operator compared to that of $\mathcal{D}^* \mathcal{D} : C_\eta^{4,\alpha} \rightarrow C_\eta^{0,\alpha}$ now changes by $\dim \ker \mathcal{D}^* \mathcal{D}_D - \dim V'$, which exactly equals the reduction in cokernel dimension above. Thus the kernel remains unchanged. $\square$

The final piece that we will need for the proof of both of the two main theorems of the chapter is to make sure that the scalar curvature of an S^1 -invariant extremal metric of Poincaré type lies in the space we defined above.

Lemma 7.14. *Let $\omega \in \Omega$ be Poincaré type metric on $X \setminus D$. Then $S(\omega) \in C^{k,\alpha}(X \setminus D)$ for any k and α . In particular, if ω is an extremal metric invariant under the S^1 -action above, then $S(\omega) \in \overline{\mathfrak{h}_{//}^D}$.*

Proof. Auvray showed in [Auv14c, Prop. 1.6] that the Ricci form ρ_ω associated to ω is bounded at any order, i.e. lies in the space $C^\infty(\Lambda^{1,1}, X \setminus D)$. Similarly, so does ω , by definition of a Poincaré type metric. Hence both $\rho_\omega \wedge \omega^{n-1}$ and ω^n lie in $C^\infty(\Lambda^{n,n}, X \setminus D)$, and so the scalar curvature function

$$S(\omega) = \frac{n\rho_\omega \wedge \omega^{n-1}}{\omega^n}$$

lies in $C^\infty(X \setminus D)$, as required.

For the final part, since ω is extremal, $S(\omega)$ lies in the kernel of the Lichnerowicz operator and by the above also in $C_\eta^{k,\alpha}$ for any $\eta \geq 0$. Taking e.g. $\eta = \frac{1}{2}$, lemma 7.7 then implies that $S(\omega) \in \overline{\mathfrak{h}_{//}^D}$. $\square$

7.2 Perturbing the Kähler class

In this section we prove theorem 7.1. Let G be the connected component of the group of isometries of $(X \setminus D, \omega)$ that come from automorphisms of X fixing D . Let ω be a G -invariant extremal Kähler metric on $X \setminus D$ in the class Ω , which we write as $\omega_0 + i\partial\bar{\partial}\varphi$ for some G -invariant Kähler metric ω_0 on X . Let $\mathcal{H}^{1,1}$ denote ω_0 -harmonic $(1,1)$ -forms, and let $\mathcal{H}_0^{1,1}$ be the subspace of those ς in $\mathcal{H}^{1,1}$ that vanish along D .

Let $U \subseteq \mathcal{H}_0^{1,1} \times C_\eta^{4,\alpha} \times \ker \mathcal{D}^* \mathcal{D}_D$, where $\eta < 0$, be the open subset on which

$$\omega + \varsigma + i\partial\bar{\partial}(\phi + \lambda(f)) \tag{7.8}$$

is a Kähler form. We consider the scalar curvature operator

$$U \rightarrow \tilde{C}_\eta^{0,\alpha}$$

given by

$$(\varsigma, \phi, f) \mapsto S(\omega + \varsigma + i\partial\bar{\partial}(\phi + \lambda(f))).$$

Note that $S(\omega)$ lies in $\tilde{C}_\eta^{0,\alpha}$, by the extremality of ω . When we are changing the metric as above it follows that the scalar curvature remains in this space.

The condition that a metric $\tilde{\omega}$ is extremal is that the scalar curvature lies in the finite-dimensional space given by the kernel of the Lichnerowicz operator associated to this metric, i.e. lies in $\overline{\mathfrak{h}_{//}^D}$. Following [LS93], it will be convenient to express this as follows.

Let $p_0, p_1, \dots, p_r$, where p_0 is constant, denote a basis for the kernel of the Lichnerowicz operator, acting on G -invariant functions, which we assume to be an orthonormal basis with respect to the L^2 -inner product. There is then an associated projection operator $\Pi_{\tilde{\omega}}$ given by

$$\phi \mapsto \sum \langle p_i, \phi \rangle p_i. \quad (7.9)$$

The extremal condition is that

$$(1 - \Pi_{\tilde{\omega}})(S(\tilde{\omega})) = 0. \quad (7.10)$$

In the Poincaré type case, it will be important that if $\tilde{\omega}$ is a perturbation of ω of the form 7.8, then the image of $\Pi_{\tilde{\omega}}$ lies in $\tilde{C}_\eta^{0,\alpha}$. We will show this below.

It is also important that the kernel of the operator $1 - \Pi_{\tilde{\omega}}$ depends smoothly on $\tilde{\omega}$ when perturbing by G -invariant functions. One obtains a not-necessarily orthonormal basis $p_0, \dots, p_r$ as follows. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, the Lie algebra of G . Let $\mathfrak{g}_0 \subseteq \mathfrak{g}$ denote the ideal consisting of vector fields with zero that are Killing and let $\mathfrak{z}_0 = \mathfrak{z} \cap \mathfrak{g}_0$ be its intersection with the centre $\mathfrak{z}$ of $\mathfrak{g}$. Let $\xi_1, \dots, \xi_r$ be a basis for $\mathfrak{z}_0$.

We can then get a basis for the subspace of the kernel for the smooth metric $\omega_0 + \varsigma$ consisting of functions whose vector field lie in $\overline{\mathfrak{h}}_{//}^D$ by letting

$$\begin{aligned} p_0(\omega_0 + \varsigma) &= 1, \\ p_j(\omega_0 + \varsigma) &= i\mathbf{G}\bar{\partial}_{g_0}^*(\iota_{\omega_0 + \varsigma}(J\xi_j + i\xi_j)), \end{aligned}$$

for $j = 1, \dots, r$, where $\mathbf{G}$ is the Green's operator of g_0 .

We can then use the result [Auv14b, Prop. 1.2] of Auvray to perturb this to a basis for the Poincaré type metric $\tilde{\omega} = \omega + \varsigma + i\partial\bar{\partial}\phi$. Recall that $\omega = \omega_0 + i\partial\bar{\partial}\varphi$. We then let p_0 be unchanged and set

$$p_j(\tilde{\omega}) = p_j(\omega_0 + \varsigma) + \xi_j \cdot (\varphi + \phi).$$

Finally, we can then use Gram-Schmidt to obtain an orthonormal basis. Since all of the steps involved in obtaining this basis depend smoothly on the parameters (ς, ϕ) , it follows that the kernel of $1 - \Pi_{\tilde{\omega}}$ depends smoothly on (ς, ϕ) , too.

We now show that $\Pi_{\tilde{\omega}}$ and $1 - \Pi_{\tilde{\omega}}$ map to the correct spaces. This is the point at which we use the assumption that $\varsigma|_D = 0$.

Lemma 7.15. *Let $p_0, \dots, p_r$ be the orthonormal basis obtained from the above procedure. We then have that if $(\varsigma, \phi, f) \in U \subseteq \mathcal{H}_0^{1,1} \times C^{4,\alpha} \times \ker \mathcal{D}^*\mathcal{D}_D$ then*

$$p_j(\varsigma + \omega + i\partial\bar{\partial}(\phi + \lambda(f))) \in \tilde{C}_\eta^{0,\alpha}$$

for all j . In particular, the operators $\Pi_{\tilde{\omega}}$ and $(1 - \Pi_{\tilde{\omega}})$ map $\tilde{C}_\eta^{0,\alpha}$ to itself.

Proof. The metrics we obtain are all asymptotically products for *the same metric on D* , using that $\alpha|_D = 0$ and the asymptotic properties of extremal Poincaré type metrics. It therefore follows by 7.11 that a basis for the kernel of the Lichnerowicz operator associated to $\varsigma + \omega + i\partial\bar{\partial}(\phi + \lambda(f))$ lies in $\tilde{C}_\eta^{0,\alpha}$. $\square$

As in [LS93], it follows from the smooth dependence of the basis that there is a neighbourhood V of $(0, 0)$ in U in which the determinant of the matrix whose $(i, j)^{\text{th}}$ entry is

$$\langle p_i(\omega), p_j(\tilde{\omega}) \rangle_{L^2(X \setminus D, \omega)}$$

is non-zero. Thus in V the kernel of the composite operator $(1 - \Pi_\omega)(1 - \Pi_{\tilde{\omega}})$ equals the kernel of $(1 - \Pi_{\tilde{\omega}})$.

We would like to use the inverse function theorem to prove theorem 7.1. Since the linearised operator has a kernel and cokernel, we need to argue that this does not cause a problem. This is where we use that the kernel of $(1 - \Pi_\omega)(1 - \Pi_{\tilde{\omega}})$ equals the kernel of $(1 - \Pi_{\tilde{\omega}})$, i.e. $\overline{\mathfrak{h}_{//}^D}$, on V . We can then use the operator $(1 - \Pi_\omega)(1 - \Pi_{\tilde{\omega}})$ which gives us a map between two fixed spaces (if we did not do the second projection, the space would vary with ς and ϕ) and the extremal condition remains that the scalar curvature is in the kernel of this latter operator.

So let $V' = V \cap (\mathcal{H}_0^{1,1} \times (\ker \mathcal{D}^* \mathcal{D}_{C_\eta^{4,\alpha}})^\perp \times \ker \mathcal{D}^* \mathcal{D}_D)_G$ and let $W = (\tilde{C}_\eta^{0,\alpha})_G \cap (\overline{\mathfrak{h}_{//}^D})^\perp$. Here the G indicates that we are considering G -invariant functions. Consider the map

$$\mathbf{S} : V' \rightarrow \mathcal{H}_0^{1,1} \times W$$

given by

$$(\varsigma, \phi, f) \mapsto (\varsigma, (1 - \Pi_\omega)(1 - \Pi_{\tilde{\omega}})(S(\tilde{\omega}))),$$

where we recall $\tilde{\omega} = \omega + \varsigma + i\partial\bar{\partial}(\phi + \lambda(f))$. The linearisation of this map was computed in [LS93, Prop. 7]. The key properties of this are that

- The projection to the first factor $\mathcal{H}_0^{1,1}$ is simply equal to ς ,
- The projection to the second factor $\tilde{C}_\eta^{0,\alpha}$ is $(1 - \Pi_\omega)(P(\varsigma)) - 2\mathcal{D}_\omega^* \mathcal{D}_\omega(\phi + \lambda(f))$ for some operator $P : \mathcal{H}_0^{1,1} \rightarrow \tilde{C}_\eta^{0,\alpha}$.

$D\mathbf{S}_{(0,0,0)}$ is a Fredholm operator, since $\mathcal{D}_\omega^* \mathcal{D}_\omega$ is. We want to show it is an isomorphism.

To see that the kernel is trivial, we first note that if (ς, ϕ, f) is in the kernel, then $\varsigma = 0$ since the projection to first factor of $D\mathbf{S}_{(0,0,0)}$ is ς . Thus we have to consider when $(1 - \Pi_\omega)(\mathcal{D}^* \mathcal{D}(\phi + \lambda(f))) = 0$. But then it follows that actually $\mathcal{D}^* \mathcal{D}(\phi + \lambda(f)) = 0$ and so $f = 0$ and $\phi \in \ker \mathcal{D}^* \mathcal{D}_{C_\eta^{4,\alpha}}$. But since all functions in the domain are orthogonal to $\ker \mathcal{D}^* \mathcal{D}_{C_\eta^{4,\alpha}}$, it follows that $\phi = 0$, too.

As for the image, if $(\varsigma, \psi) = D\mathbf{S}_{(0,0,0)}(\varsigma', \phi, f)$, then certainly $\varsigma' = \varsigma$ by the first property of $D\mathbf{S}_{(0,0,0)}$. So we need to show that for all $\varsigma \in \mathcal{H}_0^{1,1}$ and $\psi \in W$ there exists $(\phi, f) \in C^{4,\alpha} \times \ker \mathcal{D}^* \mathcal{D}_D$ such that

$$\begin{aligned} (\mathcal{D}^* \mathcal{D}(\phi + \lambda(f))) &= (1 - \Pi_\omega)(\mathcal{D}^* \mathcal{D}(\phi + \lambda(f))) \\ &= \frac{1}{2}(1 - \Pi_\omega)(P(\varsigma)) - \frac{1}{2}\psi. \end{aligned}$$

But the former term is orthogonal to $\overline{\mathfrak{h}_{//}^{\mathcal{D}}}$ since we are using the projection $(1 - \Pi_\omega)$. Thus by the linear theory, we can solve the equation if and only if ψ is orthogonal to $\overline{\mathfrak{h}_{//}^{\mathcal{D}}}$, which holds by definition of W . Hence the linearised operator is onto too.

Thus we have shown that $D\mathbf{S}_{(0,0,0)}$ is an isomorphism and so by the inverse function theorem, $\mathbf{S}$ is a diffeomorphism in a neighbourhood of $(0, 0, 0)$. The extremal condition 7.10 in V is that $\mathbf{S}(\varsigma, \phi, f) = (\varsigma, 0)$. It therefore follows that for all $\varsigma \in \mathcal{H}_0^{1,1}$ sufficiently small we can solve the extremal equation. This completes the proof of theorem 7.1.

7.3 Outline of the blow-up argument

We will follow the argument of Székelyhidi in [Szé12] and adopt much of his notation. Many of the details will remain the same. The subsequent sections will then be devoted to proving the steps outlined here.

The result applies when blowing up several points. However, most of the key features of the proof are present already in the case of blowing up one point. In the proof of Székelyhidi the points where additional arguments are needed when blowing up more points are explained. Since the key new feature here is the Poincaré type behaviour, and the new arguments deal with what happens near the divisor, and so away from the blown up points, we will therefore focus only on the case of one point.

Let M denote $X \setminus D$. The basic idea is as follows. Consider $Bl_p M$ as being the union of a neighbourhood of the exceptional divisor of $Bl_0 \mathbb{C}^n$ and $M \setminus \{p\}$ identified over an annular region. One can use the *Burns-Simanca* metric on $Bl_0 \mathbb{C}^m$ and the extremal metric on M to construct an approximate solution ω_ε in the class $\pi^*([\omega]) - c_m \varepsilon^2 [E]$ to the extremal equation. The goal is then to rephrase the equation as a fixed point problem and show that in a neighbourhood of 0, this becomes a contraction. Applying the contraction mapping theorem one obtains a fixed point and so a solution to the equation.

One feature of the proof is that one first solves a slightly more general equation and then apply a second argument to show that this means that one obtains an extremal metric. For this equation, we will need to lift potentials of holomorphic vector fields on M to $Bl_p M$, so we begin by recalling this discussion from [Szé12]. The only difference between this and our case is that we have to make sure the divisor is fixed by the isometries we consider.

Let T be a maximal torus of the elements fixing the point p in the group G of Hamiltonian isometries of (M, ω) that fix the divisor D , where ω is the extremal Poincaré type metric on M . Let H be the commutator of T in G , and let now $\overline{\mathfrak{h}_{//}^{\mathcal{D}}}$ be the space of Hamiltonian functions for vector fields in $\text{Lie}(H)$, and similarly write $\bar{\mathfrak{t}}$ for the functions in $\text{Lie}(T)$.

Define a map $\mathbf{l} : \overline{\mathfrak{h}_{//}^{\mathcal{D}}} \rightarrow C^\infty(Bl_p M)$ as follows. Let $\mathfrak{h}'$ be a complementary subspace to $\bar{\mathfrak{t}}$, chosen such that all functions in $\mathfrak{h}'$ vanish at p . For $f \in \bar{\mathfrak{t}}$, define $\mathbf{l}(f)$ to be the Hamiltonian function, with respect to ω_ε , of the holomorphic lift of the vector field $\mathfrak{X}_f$ to $Bl_p M$. This is well defined up to a constant, and we normalise so that $\mathbf{l}(f) = f$ on $M \setminus B_1$. Letting γ_1 be a cut-off function as in the construction of the approximate metric

below, we let $\mathbf{l}(f) = \gamma_1 f$ for $f \in \mathfrak{h}'$. We then define $\mathbf{l}$ on $\overline{\mathfrak{h}_{//}^D} = \bar{\mathfrak{t}} \oplus \mathfrak{h}'$ by linearity.

The more general equation we wish to solve is then given by the following, which is the analogue of proposition 15 of [Szé12] in our setting.

Proposition 7.16. *Let $p \in M$ be chosen such that the extremal vector field vanishes at p . Then there exists $c, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there is a smooth $u : Bl_p M \rightarrow \mathbb{R}$ satisfying $u = O(|t|)$ and an $h \in \overline{\mathfrak{h}_{//}^D}$ such that*

$$s(\omega_\varepsilon + i\partial\bar{\partial}(u)) - \frac{1}{2}\nabla(\mathbf{l}(h)) \cdot \nabla(u) = \mathbf{l}(h). \quad (7.11)$$

Moreover, h has an expansion

$$h = s(\omega) + \varepsilon^{2n-2}c_n(\mu(p) + \frac{1}{\text{Vol}(X)}) + h_\varepsilon \quad (7.12)$$

for some constant c_n depending only on the dimension and h_ε satisfying $|h_\varepsilon| \leq c\varepsilon^{2n-2+\delta}$ for some $c, \delta > 0$.

In the compact case, Székelyhidi showed in [Szé12] that this implies theorem 7.2, the key being the expansion 7.12 of h and a proposition regarding deformations of moment maps, see [Szé12, Prop. 8]. Since Székelyhidi showed that this proposition holds in the non-compact setting, the argument applies also in our case. Thus we will focus on proving 7.16.

The main steps are the following. For each $\varepsilon > 0$, we obtain a metric ω_ε on $Bl_p M$ that is an approximate solution to the equation 7.11 under a norm on $Bl_p M$ which is weighted near the exceptional divisor. One then constructs a one-sided inverse to the Lichnerowicz operator of ω_ε . By expanding the terms in equation 7.11 as a constant term, a linear and non-linear term, one then rephrases the equation as a fixed point problem, by using this one-sided inverse.

The key to showing that the operator we then obtain is a contraction for all $\varepsilon > 0$ sufficiently small is to have control over the norm of this inverse as $\varepsilon \rightarrow 0$ as well as having control over the non-linear part of 7.11. However, the control on the non-linear part will actually follow from control of the linear part. The core of the argument is therefore to show that we can define a new norm on $Bl_p M$ weighted near the exceptional divisor so that the Lichnerowicz operators for the metrics ω_ε remain Fredholm under this norm and we have control on the norm of related operators, which behaves correctly as $\varepsilon \rightarrow 0$.

The approximate metric ω_ε is obtained by interpolating the extremal metric on M with the Burns-Simanca metric on $Bl_0 \mathbb{C}$. The inverse to its Lichnerowicz operator will then be a perturbation of the Lichnerowicz operator of ω on M away from the exceptional divisor and the inverse to the Lichnerowicz operator near the exceptional divisor. Since the mapping properties of the Lichnerowicz operator associated to the Burns-Simanca metric is known to us already from the results in the compact case, the key to finding this one-sided inverse is therefore to analyse the Lichnerowicz operator of ω away from the blown-up point. This will be the content of the next section.

7.4 Doubly weighted spaces

The goal of this section is to prove the results of the linear theory needed to apply the contraction mapping theorem to solve equation 7.11. The following proposition is the key to this. The crucial component of this result is the control on the norm of the inverse operator. In the statement ω_ε is a metric on $Bl_p M$ we will construct below and L_{ω_ε} is the corresponding linearisation of the scalar curvature operator. Also $\mathfrak{X} = \nabla \mathbf{l}(S(\omega))$.

Proposition 7.17. *If $n > 2$, there exists $\varepsilon_0, \kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\eta \in (-\kappa, 0)$ and $\delta \in (4-2n, 0)$, the operator*

$$C_{\delta, \eta}^{4, \alpha}(Bl_p M) \times \ker \mathcal{D}^* \mathcal{D}_D \times \overline{\mathfrak{h}}_{//}^D \rightarrow \widetilde{C}_{\delta-4, \eta}^{0, \alpha}(Bl_p M)$$

given by

$$(\phi, f, h) \mapsto L_{\omega_\varepsilon}(\phi + \lambda(f)) - \frac{1}{2} \mathfrak{X}(\phi + \lambda(f)) - \mathbf{l}(h) \quad (7.13)$$

has a right inverse P whose operator norm is bounded by a constant independent of ε .

If $n = 2$, if we then let $\delta < 0$ with $|\delta|$ sufficiently small, the operator also has a right inverse P for all sufficiently small ε , but now with operator norm bounded by $C\varepsilon^\delta$ for some $C > 0$.

Here the $C_{\delta, \eta}^{k, \alpha}(Bl_p M)$ -spaces are weighted Hölder spaces to be defined below. They consist of exactly the same functions as the Poincaré type Hölder spaces $C_\eta^{k, \alpha}(Bl_p M)$, but with a norm weighted near the exceptional divisor as well. The reason for introducing these spaces is to obtain the bound on the operator norm of P . If we had not added the extra weight the operator would still have had a right inverse, but we would not have the same control on its norm.

We first discuss the approximate solution, i.e. the metric ω_ε . Around the point p to be blown up, we use T -invariant normal coordinates to write the metric as

$$\omega = i\partial\bar{\partial}\left(\frac{|z|^2}{2} + \phi(z)\right)$$

for some ϕ which is $O(|z|^4)$. After scaling ω , we can assume the normal coordinates are defined for $|z| \leq 1$.

The Burns-Simanca metric¹ ζ is a metric on the blow-up $Bl_0 \mathbb{C}^n$ of $\mathbb{C}^n$ at the origin which is scalar flat and asymptotically Euclidean. We write

$$\zeta = i\partial\bar{\partial}\left(\frac{|z|^2}{2} + \psi(z)\right).$$

Let $r_\varepsilon = \varepsilon^{\frac{2n-1}{2n+1}}$. We consider $Bl_p M$ as the manifold obtained by gluing the complement of a ball around p in M with a neighbourhood of the exceptional divisor in the blow-up of $\mathbb{C}^n$ in the origin. This is achieved by identifying the annulus $B_{2r_\varepsilon} \setminus B_{r_\varepsilon}$ with a corresponding annulus around the exceptional divisor on $Bl_0 \mathbb{C}^n$,

¹In [Szé12] this symplectic form is denoted η , but we use this symbol to denote a weight parameter.

using the coordinate transformation $w = \varepsilon^{-1}z$. The approximate solution ω_ε will be constructed by gluing ω and ζ on this annulus.

Let γ be a cut-off function $\mathbb{R} \rightarrow [0, 1]$ with

$$\begin{aligned}\gamma(x) &= 0, \quad x < 1, \\ \gamma(x) &= 1, \quad x > 2.\end{aligned}$$

Define γ_1 to be

$$\gamma_1(r) = \gamma\left(\frac{r}{r_\varepsilon}\right),$$

and let $\gamma_2 = 1 - \gamma_1$. We define ω_ε to be ω on the complement of B_1 and

$$i\partial\bar{\partial}\left(\frac{|z|^2}{2} + \gamma_1(|z|)\phi(z) + \varepsilon^2\gamma_2(|z|)\psi(\varepsilon^{-1}z)\right)$$

on $B_1 \setminus B_\varepsilon$.

Since $\frac{2n-1}{2n+1} < 1$, we have that $r_\varepsilon > \varepsilon$ and so $B_\varepsilon \subseteq B_{r_\varepsilon}$. On B_{r_ε} , we have $\gamma_1 = 0$ and $\gamma_2 = 1$, so that

$$\begin{aligned}\omega_\varepsilon &= i\varepsilon^2\partial\bar{\partial}\left(\frac{|\varepsilon^{-1}z|^2}{2} + \psi(\varepsilon^{-1}z)\right) \\ &= \varepsilon^2\zeta.\end{aligned}$$

So in the pre-image of B_ε in $Bl_p M$ under the blow-down map, we let ω_ε equal the scaled Burns-Simanca metric $\varepsilon^2\zeta$.

When rewriting equation 7.11 as a fixed point problem, it will be important to introduce a new norm on $C_\eta^{4,\alpha}(Bl_p M)$ such that the map becomes a contraction near the origin. In the definition below, we will be **restricting all spaces to T -invariant functions**. This is *not* highlighted in the notation, but it is important for what follows that we are not using the spaces consisting of just the S^1 -invariant functions.

Recall that we have fixed T -invariant normal coordinates at p , defined for $|z| \leq 1$. We define the doubly weighted Hölder norm $C_{\delta,\eta}^{k,\alpha}$ on $M_p = M \setminus \{p\}$ and $Bl_p M$ by

$$\|\phi\|_{C_{\delta,\eta}^{k,\alpha}(M_p)} = \|\phi\|_{C_\eta^{k,\alpha}(M \setminus B_{\frac{1}{2}}, \omega)} + \sup_{r \in (0, \frac{1}{2})} r^{-\delta} \|\phi\|_{C^{k,\alpha}(B_{2r} \setminus B_r, r^{-2}\omega)}$$

and

$$\|\phi\|_{C_{\delta,\eta}^{k,\alpha}(Bl_p M)} = \|\phi\|_{C_\eta^{k,\alpha}(M \setminus B_1, \omega)} + \sup_{r \in [\varepsilon, \frac{1}{2})} r^{-\delta} \|\phi\|_{C^{k,\alpha}(B_{2r} \setminus B_r, r^{-2}\omega_\varepsilon)} + \varepsilon^{-\delta} \|\phi\|_{C^{k,\alpha}(B_\varepsilon, \zeta)}.$$

Here in e.g. $C_\eta^{k,\alpha}(B_{2r} \setminus B_r, r^{-2}\omega_\varepsilon)$, the second entry denotes the metric we are using to compute norms with. Also recall $r_\varepsilon = \varepsilon^{\frac{2n-1}{2n+1}}$. We similarly define spaces $\tilde{C}_{\delta,\eta}^{k,\alpha}$.

From the Fredholm theory of the previous chapter and section 7.1 together with that of weighted spaces of punctured compact manifolds, it follows that there is a similar Fredholm result for the doubly weighted

spaces, with the Lichnerowicz operator being Fredholm for all indices (δ, η) except those where either δ is an indicial root of Δ^2 on $\mathbb{C}^n \setminus \{0\}$, i.e. $\mathbb{Z} \setminus (4 - 2n, 0)$, or η is an indicial root of $\mathcal{D}^*\mathcal{D}$ on the Poincaré type weighted spaces of chapter 6.

We can now apply this to the weights relevant for us to construct a one-sided inverse to the Lichnerowicz operator, once the cokernel is taken care of.

Proposition 7.18. *Suppose ω is an extremal metric on M . Let $\delta \in (4 - 2n, 0)$, $\eta \in (-\kappa, 0)$ and $\alpha \in (0, 1)$. Then the linear operator*

$$C_{\delta, \eta}^{4, \alpha}(M_p) \times \ker \mathcal{D}^*\mathcal{D}_D \times \overline{\mathfrak{h}_{//}^D} \rightarrow \tilde{C}_{\delta-4, \eta}^{0, \alpha}(M_p)$$

given by

$$(\phi, f, h) \mapsto \mathcal{D}_\omega^*\mathcal{D}_\omega(\phi + \lambda(f)) - h,$$

has a bounded right inverse $P : \tilde{C}_{\delta-4, \eta}^{0, \alpha} \rightarrow C_{\delta, \eta}^{4, \alpha}(M_p) \times \ker \mathcal{D}^*\mathcal{D}_D \times \overline{\mathfrak{h}_{//}^D}$.

Proof. The weights are not indicial roots, and hence we can apply the Fredholm theory to conclude that the image of

$$\mathcal{D}^*\mathcal{D} : C_{\delta, \eta}^{4, \alpha}(M_p) \rightarrow C_{\delta-4, \eta}^{0, \alpha}(M_p)$$

is the orthogonal complement of the kernel of

$$\mathcal{D}^*\mathcal{D} : C_{4-2n-\delta, 1-\eta}^{4, \alpha}(M_p) \rightarrow C_{-2n-\delta, 1-\eta}^{0, \alpha}(M_p).$$

Moreover, since there are no indicial roots in $(4 - 2n, 0) \times (-\kappa, 0)$, the kernel and cokernel of this operator coincides with the kernel and cokernel for the Poincaré type weighted spaces $C_\eta^{4, \alpha}$. The result therefore follows by 7.13. $\square$

The linearisation of the scalar curvature operator is not quite the Lichnerowicz operator, but rather the operator L_ω which in local coordinates is given by

$$\Delta_\omega^2(\phi) + \text{Ric}^{i\bar{j}}\phi_{i\bar{j}}.$$

We then have that for any ω ,

$$\mathcal{D}_\omega^*\mathcal{D}_\omega(\phi) = L_\omega(\phi) - \frac{1}{2}\nabla S(\omega) \cdot \nabla \phi.$$

We now apply this to show that small perturbations in the potential give small perturbations of the corresponding linear operator, and that this can be achieved independently of ε . Let ω_ε be the approximate solution to the extremal equation on $Bl_p M$. For any δ' , we have that for all $\phi = (u, f) \in C_{\delta', 0}^{4, \alpha} \times \ker \mathcal{D}^*\mathcal{D}_D$, the metric

$$\omega_\phi = \omega_\varepsilon + i\partial\bar{\partial}(u + \lambda(f))$$

is then another metric of Poincaré type on $Bl_p M$.

We then have the following. The proof is identical to that of [Szé12, Prop. 20] and [Szé12, Prop. 25]. The essential thing is that provided

$$\|\phi\|_{C_{2,0}^{4,\alpha}(Bl_p M) \times \ker \mathcal{D}^* \mathcal{D}_D}$$

is sufficiently small, the metrics ω and ω_ϕ are quasi-isometric of some fixed constant (say $\frac{1}{2}$).

Proposition 7.19. *Let $\delta < 0$. Then there are $c, C > 0$ such that if $\|\phi\|_{C_{2,0}^{4,\alpha}(Bl_p M) \times \ker \mathcal{D}^* \mathcal{D}_D} \leq c$ then for all η , we have*

$$\|L_{\omega_\phi}(f) - L_{\omega_\varepsilon}(f)\|_{\tilde{C}_{\delta-4}^{0,\alpha}} \leq C \|\phi\|_{C_{2,0}^{2,\alpha} \times \ker \mathcal{D}^* \mathcal{D}_D} \|f\|_{C_{\delta,\eta}^{4,\alpha} \times \ker \mathcal{D}^* \mathcal{D}_D}.$$

The result also holds on subsets U of $Bl_p M$.

Since the scalar curvature of ω_ϕ can be written as

$$s(\omega_\phi) = s(\omega_\varepsilon) + L_{\omega_\varepsilon}(\phi) + Q_\varepsilon(\phi)$$

and L_{ω_ε} is the linearisation of the scalar curvature, we get from this the following estimate for the non-linear operator Q_{ω_ε} .

Proposition 7.20. *Let $U \subseteq Bl_p M$. Then there exists $c, C > 0$ such that if $\phi, \psi \in C_\eta^{4,\alpha} \times \ker \mathcal{D}^* \mathcal{D}_D$ for some $\eta < 0$ with $\|\phi\|_{C_{2,0}^{4,\alpha}(U) \times \ker \mathcal{D}^* \mathcal{D}_D}, \|\psi\|_{C_{2,0}^{4,\alpha}(U) \times \ker \mathcal{D}^* \mathcal{D}_D} \leq c$, then*

$$\|Q_{\omega_\varepsilon}(\phi) - Q_{\omega_\varepsilon}(\psi)\|_{\tilde{C}_{\delta-4,\eta}^{0,\alpha}(U)} \leq C (\|\phi\|_{C_{2,0}^{4,\alpha}(U) \times \ker \mathcal{D}^* \mathcal{D}_D} + \|\psi\|_{C_{2,0}^{4,\alpha}(U) \times \ker \mathcal{D}^* \mathcal{D}_D}) \|\phi - \psi\|_{C_{\delta,\eta}^{4,\alpha}(U) \times \ker \mathcal{D}^* \mathcal{D}_D}.$$

Proof. As in [Szé12, Lem. 21], this follows from 7.19 by the mean value theorem. $\square$

The proof of 7.17 now follows as in [Szé12, Prop. 22], by first constructing an approximate inverse. On $Bl_0 \mathbb{C}^n$ with the Burns-Simanca metric ζ one can also introduce weighted spaces, and obtain similar results on the mapping properties of the Lichnerowicz operator associated to ζ on these weighted spaces, see [Szé12, Prop. 18]. The approximate inverse is then constructed by using the inverse of proposition 7.18 and the inverse on the weighted spaces on $Bl_0 \mathbb{C}^n$ with the Burns-Simanca metric ζ .

The next step is then to perturb to obtain the required inverse. In addition to the bounds obtained above and those for the Lichnerowicz operator of the Burns-Simanca metric, this also uses 7.19 and a bound on the lifts $\mathbf{I}(h)$ for functions h in the kernel of the Lichnerowicz operator associated to the metric ω on M . See [Szé12, Lem. 19]. Note that the bound being dependent on ε for the case $n = 2$ comes from the mapping properties of the Lichnerowicz operator of the Burns-Simanca metric on $Bl_0 \mathbb{C}^2$, and so the same dependence holds in our case too.

7.5 Solving the non-linear problem

In this section we will finish the proof and solve the non-linear problem using the contraction mapping theorem. This follows [Szé12, Ch. 5.4].

The first step is to make a better approximating metric than ω_ε , in the sense that it matches up with the Burns-Simanca metric to higher order. For this, we want a T -invariant function $\Gamma \in C_{\delta,\eta}^{4,\alpha}$ which near the blown-up point p satisfies

$$\Gamma(z) = -|z|^{4-2n} + O(|z|^{5-2n}).$$

For $n = 2$, one requires the leading term to be $\log(|z|)$ instead. This implies that Γ solves

$$\mathcal{D}^*\mathcal{D}(\Gamma + \lambda(f)) = h_0 - c_n\delta_p,$$

for some dimensional constant c_n , in the sense of distributions, where $h_0 \in C_{\delta,\eta}^{0,\alpha}$, $f \in \ker \mathcal{D}^*\mathcal{D}_D$ and δ_p is the Dirac delta function at p . Here we have $\delta \in (4 - 2n, 0)$ and $\eta \in (-\kappa, 0)$. Recall that $\lambda(f)$ vanishes near p for all $f \in \ker \mathcal{D}^*\mathcal{D}_D$, so that $\Gamma + \lambda(f) = \Gamma$ near p .

From our linear theory of chapter 6 and section 7.1, this has a solution if and only if $h_0 - c_n\delta_p$ is orthogonal to the kernel of $\mathcal{D}^*\mathcal{D}$ on $C_{\delta,\eta}^{4,\alpha} \times \ker L_D$. As in the compact case, this implies that

$$h_0 = c_n(\mu(p) + \frac{1}{\text{Vol}(X)}).$$

In solving equation 7.11, we now write

$$\begin{aligned} u &= \varepsilon^{2n-2}\gamma_1\Gamma + v, \\ f &= \varepsilon^{2n-2}f_0 + g, \\ h &= \varepsilon^{2n-2}h_0 + k. \end{aligned}$$

Rearranging 7.11 to separate out the linear term as in [Szé12], we can write our equation in the form

$$L_{\omega_\varepsilon}(v + \lambda(g)) - \frac{1}{2}\mathfrak{X}(v + \lambda(g)) - \mathbf{l}(k) = \tilde{Q}(v, g, k)$$

for some non-linear operator $\tilde{Q} : C_{\delta,\eta}^{4,\alpha} \times \ker \mathcal{D}^*\mathcal{D}_D \times \overline{\mathfrak{h}_{//}^D} \rightarrow \tilde{C}_{\delta,\eta}^{0,\alpha}$. The full expression is

$$\begin{aligned} \tilde{Q}(u, g, v) &= \mathbf{l}(S(\omega)) - S(\omega_\varepsilon) - Q_{\omega_\varepsilon}(u + \lambda(f)) + \frac{1}{2}\nabla \mathbf{l}(h) \cdot \nabla(u + \lambda(f)) \\ &\quad - L_{\omega_\varepsilon}(\varepsilon^{2n-2}\gamma_1\Gamma + \lambda(f_0)) + \frac{1}{2}\mathfrak{X}(\varepsilon^{2n-2}\gamma_1\Gamma + \lambda(f_0)) + \varepsilon^{2n-2}\mathbf{l}(h_0). \end{aligned}$$

Applying P from 7.17 to the right-hand side, we get a map

$$\mathcal{N} = P \circ \tilde{Q} : C_{\eta}^{4,\alpha} \times \ker \mathcal{D}^*\mathcal{D}_D \times \overline{\mathfrak{h}_{//}^D} \rightarrow \tilde{C}_{\eta}^{0,\alpha}.$$

Now if (v, g, k) solves the fixed-point problem

$$(v, g, k) = \mathcal{N}(v, g, k), \quad (7.14)$$

then applying the map 7.13 to both sides gives a solution to our original equation, since P is a right inverse to 7.13.

The remaining steps of the proof are now to show that $\mathcal{N}$ is a contraction on some set and that $(0, 0, 0)$ is a good approximate solution for all $\varepsilon > 0$ sufficiently small. The contraction mapping theorem then implies that we get a fixed point of $\mathcal{N}$ and so a solution to our original equation. To obtain the expansion 7.12 of the element h in 7.16, we will let the set W_ε on which $\mathcal{N}$ is a contraction depend on ε , so that being an element of W_ε automatically implies this expansion.

To achieve this, we first obtain that $\mathcal{N}$ is a contraction with a specific constant on a set V for all $\varepsilon > 0$ sufficiently small. The proof of this relies only on 7.20 and a bound on the lifts $\mathbf{l}(h)$ for $h \in \overline{\mathfrak{h}_{//}^D}$, see [Szé12, Lem. 19]. This bound is a local result depending on the behaviour near p , so goes over in our setting too.

Lemma 7.21. *Suppose $n > 2$. Then there exists $c_0 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mathcal{N}$ is a contraction with constant $\frac{1}{2}$ on*

$$V = \{(v, g, k) \in C_\eta^{4, \alpha} \times \ker \mathcal{D}^* \mathcal{D}_D \times \overline{\mathfrak{h}_{//}^D} : \|v\|_{C_{2,0}^{4, \alpha}}, |g|, |k| < c_0\}.$$

When $n = 2$, the same holds true on

$$V = V_\varepsilon = \{(v, g, k) \in C_\eta^{4, \alpha} \times \ker \mathcal{D}^* \mathcal{D}_D \times \overline{\mathfrak{h}_{//}^D} : \|v\|_{C_{2,0}^{4, \alpha}}, |g|, |k| < c_0 \varepsilon^\theta\},$$

for some $\theta > 0$ that can be taken to be as small as one likes, by taking $\delta < 0$ sufficiently close to 0.

See [Szé12, Lem. 23] for the proof.

To show that $(0, 0, 0)$ gives a small enough approximate solution for all ε sufficiently small, the only difference between our case and the compact case of [Szé12, Lem. 24] is away from the exceptional divisor of the blow-up. Let F denote the element such that $P(F) = \mathcal{N}(0, 0, 0)$, where P is the inverse operator of proposition 7.17, i.e. $F = \tilde{Q}(0, 0, 0)$. Note that bounding F is sufficient to bound $\mathcal{N}(0, 0, 0)$, by the boundedness of P .

Lemma 7.22. *Let $\delta, \eta < 0$ with δ be close to $4 - 2n$, and greater than $4 - 2n$ in the case $n > 2$. Let $r_\varepsilon = \varepsilon^{\frac{2n-1}{2n+1}}$. Then there is a $c > 0$ such that*

$$\|F\|_{\tilde{C}_{\delta-4, \eta}^{0, \alpha}(Bl_p M)} \leq c r_\varepsilon^{4-\delta}.$$

Proof. To deal with the contribution away from the exceptional divisor, we must estimate

$$-Q_\omega(\varepsilon^{2n-2}(\Gamma + \lambda(f_0))) + \frac{1}{2} \varepsilon^{4n-4} \nabla h_0 \cdot \nabla(\Gamma + \lambda(f_0))$$

on $M \setminus B_{2r_\varepsilon}$ (see [Szé12, Eqn. 29]).

For the first term, we use proposition 7.20, which applies because $\omega = \omega_\varepsilon$ on $M \setminus B_{2r_\varepsilon}$, to conclude that

$$\begin{aligned} \|Q_\omega(\varepsilon^{2n-2}(\Gamma + \lambda(f_0)))\|_{\tilde{C}_{\delta-4,\eta}^{0,\alpha}(M \setminus B_{2r_\varepsilon})} &\leq C \cdot \|\varepsilon^{2n-2}(\Gamma, f_0)\|_{C_{2,0}^{4,\alpha}(M \setminus B_{2r_\varepsilon}) \times \ker \mathcal{D}^* \mathcal{D}_D} \\ &\quad \cdot \|\varepsilon^{2n-2}(\Gamma, f_0)\|_{C_{\delta,\eta}^{4,\alpha}(M \setminus B_{2r_\varepsilon}) \times \ker \mathcal{D}^* \mathcal{D}_D} \\ &\leq C' \varepsilon^{4n-4} \|\Gamma\|_{C_{2,0}^{4,\alpha}(M \setminus B_{2r_\varepsilon})} \cdot \|\Gamma\|_{C_{\delta,\eta}^{4,\alpha}(M \setminus B_{2r_\varepsilon})}, \end{aligned}$$

for all $\varepsilon > 0$ sufficiently small, where $C' = C|f_0|^2$ and $|\cdot|$ is the fixed norm on $\ker \mathcal{D}^* \mathcal{D}_D$.

We now use that $\|\gamma_1 \Gamma\|_{C_{\delta',\eta}^{4,\alpha}}$ is bounded by c if $\delta' < 4 - 2n$ and $c r_\varepsilon^{4-2n-\delta'}$ if $\delta' > 0$. This only depends on the behaviour near the blown-up point, so applies as in [Szé12]. We then have

$$\|\Gamma\|_{C_{2,0}^{4,\alpha}(M \setminus B_{2r_\varepsilon})} \leq c_1 r_\varepsilon^2$$

and

$$\|\Gamma\|_{C_{\delta,\eta}^{4,\alpha}(M \setminus B_{2r_\varepsilon})} \leq c_2 r_\varepsilon^{4-2n-\delta}.$$

Hence

$$\begin{aligned} \|Q_\omega(\varepsilon^{2n-2}(\Gamma + \lambda(f_0)))\|_{\tilde{C}_{\delta-4,\eta}^{0,\alpha}(M \setminus B_{2r_\varepsilon})} &\leq c \varepsilon^{4n-4} r_\varepsilon^{6-4n-\delta}, \\ &\leq c r_\varepsilon^{4-\delta}, \end{aligned}$$

where we are using that $n \geq 2$.

For the latter term, its $\tilde{C}_{\delta-4,\eta}^{0,\alpha}$ norm is, as in [Szé12], bounded by

$$c \varepsilon^{4n-4} \|(\Gamma, f_0)\|_{C_{\delta-3,\eta}^{1,\alpha} \times \ker \mathcal{D}^* \mathcal{D}_D}$$

for some $c > 0$, using the multiplicative property of the norm and that h_0 is bounded in the unweighted spaces. Again, the $\tilde{C}_{\delta',\eta}^{4,\alpha}$ -norm of Γ is bounded provided $\delta' < 4 - 2n$, and so for δ sufficiently close to $4 - 2n$ it follows that $\|(\Gamma, f_0)\|_{C_{\delta-3,\eta}^{1,\alpha} \times \ker \mathcal{D}^* \mathcal{D}_D}$ is bounded. Hence

$$\|\varepsilon^{4n-4} \nabla h_0 \cdot \nabla(\Gamma + \lambda(f_0))\|_{\tilde{C}_{\delta-4,\eta}^{0,\alpha}} \leq c r_\varepsilon^{4-\delta},$$

again if δ is sufficiently close to $4 - 2n$. □

The proof of proposition 7.16 now follows just as in the compact case, see [Szé12, pp.1444-1445]. One defines a new subset W_ε which contains $\mathcal{N}(0, 0, 0)$ and which is contained in V for all $\varepsilon > 0$ sufficiently small. Moreover, $\mathcal{N}$ maps W_ε to itself. One then obtains a fixed point $(v_\varepsilon, g_\varepsilon, h_\varepsilon)$ in W_ε and this set is defined such that such a solution implies the bound $|h_\varepsilon| < c \varepsilon^{2n-2+\delta'}$ for some $\delta' > 0$.

The case $n = 2$ needs special care because the bounds now depend on ε . However, the argument for this is the same as in the compact case, since the bounds depend on ε in the same manner in either case.

7.6 Application of results

In this section we show how theorem 7.2 gives examples of new Poincaré type extremal metrics, on toric varieties.

Let P be a Delzant polytope and take D to be one of the torus-invariant divisors corresponding to a facet F of P defined by an affine linear function l . Take p_i to be fixed points of the torus-actions. We now have that

$$\begin{aligned}\overline{\mathfrak{h}_{//}^D} &= \bar{\mathfrak{t}} \\ &= \text{Aff}(\mathbb{R}^n).\end{aligned}$$

and that the kernel on the negative weighted spaces (of small negative weight) is the span of l .

The condition of theorem 7.2 on the points now becomes that we blow-up on fixed points of the torus action that do not lie on D . Thus we have

Proposition 7.23. *Let (X, Ω) be a toric manifold with Kähler class Ω and let D be a torus-invariant divisor. Let $p_1, \dots, p_k$ be fixed points of the torus action with $p_i \notin D$ for all i . Suppose there exists a torus-invariant extremal Poincaré type metric in Ω with Poincaré type singularity along D . Then for any $a_1, \dots, a_k > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $Bl_{p_1, \dots, p_k} X$ admits a T^n -invariant extremal Poincaré type metric in the class*

$$\pi^* \Omega - \varepsilon \left(\sum_i a_i [E_i] \right)$$

with Poincaré type singularity along $\pi^{-1}(D) \cong D$, where π is the blow-down map and $E_i = \pi^{-1}(p_i)$ are the exceptional divisors of the blow-up.

In particular, we can start with the extremal Poincaré type metric on $\mathbb{P}^2 \setminus \mathbb{P}^1$ found by Abreu [Abr01] to obtain an extremal metric of Poincaré type in some class on any toric variety that is obtained by successively blowing up $\mathbb{P}^2$ in fixed points of the torus-action that do not meet the (pullback of) the line at infinity.

We can also apply theorem 7.1 to any class that vanishes on D . In the toric case, in terms of the polytope picture, this means that we can get a solution when we slightly change the vertices of polytope but fix the vertices on the facet with the Poincaré type singularity. In the surface case, however, we can do better, as then the requirement is simply that the area of the curve along which the Poincaré type singularity occurs, is fixed. Given any class we can rescale so that this holds, and so in this case one can perturb with *any* Kähler class.

7.7 Further directions

We end the chapter by discussing some related results one could hope to prove.

Naturally one would like to remove the S^1 -assumption in theorems 7.1 and 7.2. An extension of the linear theory of chapter 6, as discussed in section 6.4, and the extension of the discussion in section 7.1 to this setting seems to be the key to this. Once such a result is in place, one should expect the proof to follow in more or less the same manner. A resolution of this would lead to a wealth of new examples, as one could then apply this to the metrics of [TY87].

In theorem 7.1, we assume that $\alpha|_D = 0$. In light of the asymptotics 7.1 for an extremal Poincaré type metric, this ensures that we do not need to change the induced metric on D . However, one should be able to get rid of this assumption. One would then need to allow the metric on D to change, and so we would need to consider spaces of the form

$$\widehat{C}_\eta^{4,\alpha} = C_\eta^{4,\alpha} \oplus \chi \Pi^* C^{4,\alpha}(D).$$

In other words, one would need to consider all pullback functions from D , not just those of $\ker \mathcal{D}^* \mathcal{D}_D$. Some such spaces have already been considered by Biquard in [Biq97] and one could hope to extend the discussion to the Lichnerowicz operator.

In Auvray's work, the divisor D is taken to be a simple normal crossings divisor, not just a smooth divisor. One could hope to prove the analogous results in this case and this would involve weighted spaces with different weight functions near each irreducible component of D . That is, one has a weight function t_i which behaves like $\log(-\log(|\sigma_i|^2))$ near D_i , where σ_i is a holomorphic section of $\mathcal{O}(D_i)$ cutting out D_i . One could probably use the same weight parameter η near each irreducible component, though.

It would be very interesting to give a similar result to theorem 7.2 when one blows up a point *on* the divisor D . This would be more involved and incorporate some new elements from that of the compact case. One would need a new local model, gluing in not a copy of the Burns-Simanca metric, but some other metric. A key step in this extension would then require one to develop the linear theory of the Lichnerowicz operator for the new model metric.

Finally, another interesting direction would be to explore similar results for the metrics 4.6 encountered in chapter 4. One would expect this to take on a different flavour to the Poincaré type case, as there seems to be a new *obstruction* to solving the equation in this setting. That is, in the toric case, one expects the determinant condition found in chapter 4 to be preserved. From the work of Székelyhidi, see [Szé06], this should be the fact that one preserves the condition that the Donaldson-Futaki invariant $F(c)$ for the pair (X, D) associated to taking the deformation to the normal cone of D with line bundle $\mathcal{L}_c$ has

$$F''(0) = 0.$$

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