

# ACOUSTIC ENERGY BALANCE DURING THE ONSET, GROWTH AND SATURATION OF THERMOACOUSTIC INSTABILITIES

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## ABSTRACT

*Thermoacoustic instabilities can occur in a wide range of combustors and are prejudicial since they can lead to increased mechanical fatigue or even catastrophic failure. A well-established formalism to predict the onset, growth and saturation of such instabilities is based on acoustic network models. This approach has been successfully employed to predict the frequency and amplitude of limit cycle oscillations in a variety of combustors. However, it does not provide any physical insight in terms of the acoustic energy balance of the system. On the other hand, Rayleigh's criterion may be used to quantify the losses, sources and transfers of acoustic energy within and at the boundaries of a combustor. However, this approach is cumbersome for most applications because it requires computing volume and surface integrals and averaging over an oscillation cycle. In this work, a new methodology for studying the acoustic energy balance of a combustor during the onset, growth and saturation of thermoacoustic instabilities is proposed. The two cornerstones of this new framework are the acoustic absorption coefficient  $\Delta$  and the cycle-to-cycle acoustic energy ratio  $\lambda$ , both of which do not require computing integrals. Used along with a suitable acoustic network model, where the flame frequency response is described using the weakly nonlinear Flame Describing Function (FDF) formalism, these two dimensionless numbers are shown to characterize: 1) the variation of acoustic energy stored within the combustor between two consecutive cycles, 2) the acoustic energy transfers occurring at the combustor's boundaries and 3) the sources and sinks of acoustic energy located within the combustor. The acoustic energy balance of the well-documented Palies burner is then analyzed during the onset, growth and saturation of thermoacoustic instabilities using this new methodology. It is demonstrated that this new approach allows a deeper understanding of the physical mechanisms at play. For instance, it is possible to determine when the flame acts as an acoustic energy source or sink, where acoustic damping is generated, and if acoustic energy is transmitted through the boundaries of the burner.*

## NOMENCLATURE

### Latin characters

|               |                                                       |
|---------------|-------------------------------------------------------|
| $A$           | Cross section area                                    |
| $\bar{c}$     | Sound speed                                           |
| $\mathcal{D}$ | Acoustic energy sink per unit volume and unit time    |
| $e_{ac}$      | Acoustic energy density                               |
| $E_{ac}$      | Instantaneous acoustic energy stored in the combustor |

|               |                                                        |
|---------------|--------------------------------------------------------|
| $f$           | Frequency                                              |
| $f_j$         | Riemann invariant in the flow direction at $x = x_j$   |
| $\mathcal{F}$ | Flame Describing Function (FDF)                        |
| $g_j$         | Riemann invariant opposite to the flow at $x = x_j$    |
| $G$           | Gain of the Flame Describing Function                  |
| $\mathbf{I}$  | Time-averaged acoustic energy flux                     |
| $\Im$         | Imaginary part                                         |
| $\mathbf{J}$  | Acoustic energy flux                                   |
| $k$           | (Complex) wave number                                  |
| $M$           | Mach number                                            |
| $\mathcal{M}$ | Longitudinal acoustic mode                             |
| $p$           | Gas pressure                                           |
| $\dot{q}$     | Local flame heat release rate per unit volume          |
| $\dot{Q}$     | Flame heat release rate                                |
| $\mathcal{R}$ | Acoustic reflection coefficient                        |
| $s$           | Entropy                                                |
| $S$           | Acoustic Scattering Matrix                             |
| $\mathcal{S}$ | Acoustic energy source per unit volume and unit time   |
| $t$           | Time variable                                          |
| $T$           | Gas temperature                                        |
| $\mathcal{T}$ | Acoustic Transfer Matrix                               |
| $u$           | Axial velocity                                         |
| $\mathbf{u}$  | Velocity vector                                        |
| $V$           | Volume of the combustor                                |
| $W$           | Time-averaged acoustic energy flux integrated over $A$ |
| $x$           | Axial coordinate                                       |
| $\mathbf{x}$  | Position vector                                        |
| $Z$           | Acoustic impedance                                     |

### Greek characters

|            |                                        |
|------------|----------------------------------------|
| $\alpha$   | Mean pressure ratio                    |
| $\gamma$   | Adiabatic index                        |
| $\Delta$   | Acoustic absorption coefficient        |
| $\theta$   | Cross section area ratio               |
| $\lambda$  | Cycle-to-cycle acoustic energy ratio   |
| $\Xi$      | Sound speed ratio                      |
| $\rho$     | Gas density                            |
| $\sigma$   | Growth rate                            |
| $\tau$     | Period of the oscillation cycle        |
| $\varphi$  | Phase of the Flame Describing Function |
| $\phi$     | Equivalence ratio                      |
| $\omega$   | Complex angular frequency              |
| $\omega_i$ | Damping rate                           |
| $\omega_r$ | Real angular frequency                 |

### Superscripts & diacritics

|             |                                                       |
|-------------|-------------------------------------------------------|
| $\bar{y}$   | Mean value of $y$                                     |
| $y'$        | Fluctuating value of $y$                              |
| $\tilde{y}$ | Complex amplitude of $y$ in the frequency domain      |
| $y^*$       | Complex conjugate of $y$                              |
| $y^+$       | Related to the wave propagating in the flow direction |

$y^-$  Related to the wave propagating opposite to the flow

## Subscripts

$y_u$  Value of  $y$  at the upstream reference location  
 $y_d$  Value of  $y$  at the downstream reference location  
 $y_f$  Value of  $y$  at the flame location

## INTRODUCTION

Thermoacoustic instabilities, also called combustion instabilities, occur in a wide variety of combustors, including industrial and domestic boilers [1, 2], rocket engines [3–5] and low- $\text{NO}_x$  gas turbines used either for power production or aircraft propulsion [5–9]. Such instabilities are undesirable as they lead to increased heat fluxes [10, 11], structural vibration and failure [5, 10, 11], flame extinction [5, 10] and significant sound levels at certain frequencies [7, 9]. Accordingly, a large research effort has been devoted to predicting the onset, growth and saturation of such instabilities since the 1950s [7, 12–17].

A well-established formalism to analyze such instabilities is based on acoustic network models, also called acoustic equivalents, where consecutive elements are related through jump conditions based on linearized conservation equations [16, 18, 19]. For instance, each element can be represented by a 2x2 matrix called Scattering matrix or S-matrix, which relates the ingoing and outgoing propagating waves across the element [20, 21]. Alternatively, Transfer Matrices relating the acoustic pressure and acoustic velocity across the element may be used [13, 14, 17, 20, 21]. For given acoustic boundary conditions, the combustor's thermoacoustic modes are then determined by searching for the complex roots of a characteristic equation [22, 23]. The real and imaginary parts of these solutions correspond to the frequencies and growth rates of the modes respectively [13, 15, 24]. This type of approach has been successfully employed to predict the frequency and amplitude of limit cycle oscillations (assuming that the flame frequency response can be represented using a weakly nonlinear description) in a variety of combustors such as slit burners [15], partially-premixed burners [16] or swirling flame burners [16, 23, 25] among others. However, these studies do not provide any information about the acoustic energy balance of the combustor during the onset, growth and saturation of these modes. Another approach is needed in order to characterize the evolution of the sources, sinks and transfers of acoustic energy inside and at the boundaries of the system.

Flames have been known to act as acoustic energy sources when the acoustic pressure and the flame heat release rate fluctuations have a phase difference smaller than  $90^\circ$  for more than a century [8, 26, 27]. Rayleigh introduced this criterion to discriminate between thermoacoustically stable and unstable combustors [26]. However, the presence of acoustic energy sources and sinks within the combustor as well as acoustic energy transfers at the combustor's boundaries are not accounted for with this original formulation. A more general analysis, sometimes called the generalized or modified Rayleigh's criterion, can be obtained by considering a local acoustic energy balance [8, 28–30]:

$$\frac{\partial e_{ac}}{\partial t} + \nabla \cdot \mathbf{J} = \mathcal{S} - \mathcal{D} \quad (1)$$

where  $e_{ac}$  is the acoustic energy density,  $\mathbf{J}$  is the acoustic energy flux and  $\mathcal{S}$  and  $\mathcal{D}$  are the acoustic energy sources and sinks per unit volume and unit time. Bold characters will be used to represent vectors throughout this work.

A macroscopic acoustic energy balance is then obtained by integrating Eq. (1) over the combustor's volume  $V$  [8, 30]:

$$\frac{dE_{ac}}{dt} + \oint_{\partial V} \mathbf{J} \cdot d\mathbf{S} = \iiint_V (\mathcal{S} - \mathcal{D}) dV \quad (2)$$

Equation (2) describes the evolution of the instantaneous acoustic energy  $E_{ac}$  stored within the combustor. This equation can be averaged over an arbitrary time interval [31]. In practice, this time interval is set to be the period  $\tau$  of an oscillation cycle [27, 30]:

$$\frac{1}{\tau} [E_{ac}]_t^{t+\tau} + \oint_{\partial V} \mathbf{I} \cdot d\mathbf{S} = \frac{1}{\tau} \int_t^{t+\tau} \iiint_V (S - \mathcal{D}) dV dt \quad (3)$$

where the brackets  $[\ ]$  represent a difference and  $\mathbf{I}$  is the time-averaged acoustic energy flux, given by [30]:

$$\mathbf{I} = \frac{1}{\tau} \int_t^{t+\tau} \mathbf{J} dt \quad (4)$$

Equation (3) shows that the acoustic energy stored within a combustor may increase or decrease depending on the acoustic energy transfers at its boundaries and the strength of the acoustic energy sources and sinks. In theory, this equation fully characterizes the evolution of the acoustic energy within and at the boundaries of a combustor during the onset, growth and saturation of a thermoacoustic mode of period  $\tau$ . However, the source and sink terms are integrals over the combustor's volume while the acoustic energy transfer terms are integrals over the boundaries' surfaces. Moreover, all contributions are averaged over an oscillation cycle. As a consequence, assessing the acoustic energy balance of an unstable combustor using the modified Rayleigh's criterion is a very challenging task and few studies have attempted to do so quantitatively [23, 27].

The objective of this work is to introduce a new framework that can be used to assess the contribution of each term in Eq. (3) without resorting to integral calculus. This new framework will be shown to be fully equivalent to Rayleigh's criterion and can be used alongside the usual acoustic network model analysis widely employed to predict the thermoacoustic stability of combustors. First, the acoustic energy balance of a generic non-compact element will be shown to be characterized by two dimensionless numbers: the acoustic absorption coefficient  $\Delta$  and the cycle-to-cycle acoustic energy ratio  $\lambda$ . This analysis will then be extended to a generic burner represented by a network of acoustic elements and a link with the generalized Rayleigh's criterion will be established. Finally, this new framework will be applied to a labscale combustor, namely the Palies burner, to gather information about the acoustic energy losses at its boundaries as well as the presence and strength of acoustic energy sources and sinks within it during the onset, growth and saturation of an instability.

## ANALYTICAL DESCRIPTION OF A GENERIC ELEMENT

**Assumptions** The first step is to consider a generic non-compact element, represented by a black box in Fig. 1. The reference plane located at  $x = x_u$  (respectively  $x = x_d$ ) corresponds to the inlet (respectively outlet) of the element. The fluid located between the inlet and outlet of the element is assumed to be a perfect gas flowing at a subsonic speed from left to right. The gas temperature  $T$ , density  $\rho$ , pressure  $p$ , velocity  $u$  and entropy  $s$  are assumed to be one-dimensional at these reference locations. Each of these physical variables can be decomposed into a mean  $\bar{y}$  and fluctuating  $y'$  variable at  $x = x_u$  and  $x = x_d$  such that  $y' \ll \bar{y}$ . The only exception is the fluid velocity  $u$  where the previous relation is replaced by  $u' \ll \bar{c}$  where  $\bar{c}$  is the sound speed. The subsonic regime assumption also implies  $\bar{u} < \bar{c}$ . The mean flow variables are assumed to be time-independent. Body forces, thermal conduction and thermal radiation are neglected. Entropy waves are allowed, but they are not considered explicitly in this work. The harmonic convention retained is  $\exp(+i\omega t)$  where  $\omega = \omega_r + i\omega_i$  is the complex angular frequency,  $\omega_r$  is the real angular frequency,  $\omega_i$  is the damping rate and  $t$  is the time variable. The growth rate, given by  $\sigma = -\omega_i$ , can be used alternatively.

**Acoustic flow field variables** Under these assumptions, the acoustic flow field variables at both reference locations are obtained by solving the wave equation in the presence of a subsonic mean flow [32, 33]. The acoustic pressure and acoustic velocity at the inlet and outlet of the system in the frequency domain are given by:

$$p'_u = (\tilde{p}_u^+ + \tilde{p}_u^-) e^{+i\omega t} \quad (5)$$

$$u'_u = \left( \frac{\tilde{p}_u^+}{\bar{\rho}_u \bar{c}_u} - \frac{\tilde{p}_u^-}{\bar{\rho}_u \bar{c}_u} \right) e^{+i\omega t} \quad (6)$$

$$p'_d = (\tilde{p}_d^+ + \tilde{p}_d^-) e^{+i\omega t} \quad (7)$$

$$u'_d = \left( \frac{\tilde{p}_d^+}{\bar{\rho}_d \bar{c}_d} - \frac{\tilde{p}_d^-}{\bar{\rho}_d \bar{c}_d} \right) e^{+i\omega t} \quad (8)$$

where the subscript  $u$  (respectively  $d$ ) refers to variables defined at the upstream (respectively downstream) reference location. The physical signals corresponding to the acoustic pressure and acoustic velocity at the inlet and outlet of the system are then obtained by considering the real part of Eqs. (5)-(8).  $\tilde{p}_j^+$  (respectively  $\tilde{p}_j^-$ ) is the complex amplitude of the wave propagating in the flow direction (respectively opposite to the flow) at reference location  $j = u$  or  $d$ , given by [32]:

$$\tilde{p}_j^+ = f_j e^{-ik_j^+ x_j} \quad (9)$$

$$\tilde{p}_j^- = g_j e^{+ik_j^- x_j} \quad (10)$$

where  $j = u$  or  $d$ ,  $f$  (respectively  $g$ ) is the Riemann invariant in the flow direction (respectively opposite to the flow) and  $k_j^\pm$  are the complex wave numbers. These wave numbers are given by [32]:

$$k_j^\pm = \frac{\omega}{\bar{c}_j \pm \bar{u}_j} \quad (11)$$

It should be noted that these wave numbers are complex quantities that spatially modulate  $\tilde{p}_j^+$  and  $\tilde{p}_j^-$ .

**Acoustic energy density and flux** The general expression for the acoustic energy density at any point inside the domain and in the presence of a mean flow is given by [28,34]:

$$e_{ac} = \frac{1}{2} \bar{\rho} (\mathbf{u}' \cdot \mathbf{u}') + \frac{1}{2} \frac{p' p'}{\bar{\rho} \bar{c}^2} + \frac{p'}{\bar{c}^2} (\bar{\mathbf{u}} \cdot \mathbf{u}') \quad (12)$$

where the symbol  $(\cdot)$  represents the dot product. Likewise, the acoustic energy flux at any point inside the domain and in the presence of a mean flow is given by [28,32,34]:

$$\mathbf{J} = p' \mathbf{u}' + \frac{p' p'}{\bar{\rho} \bar{c}^2} \bar{\mathbf{u}} + \frac{p'}{\bar{c}^2} (\bar{\mathbf{u}} \cdot \mathbf{u}') \bar{\mathbf{u}} + \bar{\rho} (\mathbf{u}' \cdot \mathbf{u}') \bar{\mathbf{u}} \quad (13)$$

Equations (12) and (13) state that the acoustic energy density and the acoustic energy flux are second-order terms that can be expressed solely in terms of first-order fluctuations [28,34]. If it is further assumed that there is no mean flow, the third term in the right-hand side of Eq. (12) and the second, third and fourth terms in the right-hand side of Eq. (13) cancel out and the classical expressions for the acoustic energy density and acoustic energy flux in a quiescent fluid are obtained [8,28–30].

**Acoustic absorption coefficient** The acoustic absorption coefficient  $\Delta$  of a generic non-compact element, such as represented in Fig. 1, is defined as [32, 35]:

$$\Delta = 1 - \frac{|W_{\bar{p}_d^+}| + |W_{\bar{p}_u^-}|}{|W_{\bar{p}_u^+}| + |W_{\bar{p}_d^-}|} \quad (14)$$

where  $W_{\bar{p}_j^+}$  (respectively  $W_{\bar{p}_j^-}$ ) is the time-averaged acoustic energy flux integrated over the cross section area and associated with the wave propagating in the flow direction (respectively opposite to the flow) at reference location  $j$ . The acoustic absorption coefficient compares the average amount of acoustic energy entering and leaving the system over an oscillation cycle:

- $\Delta = 1$  : There is no acoustic energy leaving the system on average.
- $0 < \Delta < 1$  : There is more acoustic energy entering the system than leaving it on average. An increase in  $\Delta$  corresponds to proportionally less acoustic energy leaving the system.
- $\Delta = 0$  : The amount of acoustic energy leaving and entering the system is the same on average.
- $\Delta < 0$  : There is more acoustic energy leaving the system than entering it on average. A decrease in  $\Delta$  corresponds to proportionally more acoustic energy leaving the system.

The total time-averaged acoustic energy flux integrated over the cross section area  $W_u$  (respectively  $W_d$ ) at the upstream (respectively downstream) reference location is given by:

$$W_u = W_{\bar{p}_u^+} + W_{\bar{p}_u^-} = -I_u A_u \quad (15)$$

$$W_d = W_{\bar{p}_d^+} + W_{\bar{p}_d^-} = I_d A_d \quad (16)$$

where  $A_j$  is the cross section area and  $I_j$  is the time-averaged acoustic energy flux, both of which are assessed at reference location  $j$ . Based on Eq. (4) and Eq. (13), the time-averaged acoustic energy flux  $I_j$  is given by [28, 32, 34]:

$$I_j = \left\langle \left( \frac{p'_j}{\bar{\rho}_j} + \bar{u}_j u'_j \right) \left( \bar{p}_j u'_j + \frac{\bar{u}_j p'_j}{\bar{c}_j^2} \right) \right\rangle \quad (17)$$

where the quantity between the symbols  $\langle \rangle$  is averaged over an oscillation cycle. Using Eqs. (5)-(8) along with Eqs. (14)-(17), the expression for the acoustic absorption coefficient of a generic non-compact element under the previous assumptions is then given by [35, 36]:

$$\Delta = 1 - \frac{(1 + M_d)^2 |\bar{p}_d^+|^2 + \theta \Xi \alpha^{-1} (1 - M_u)^2 |\bar{p}_u^-|^2}{\theta \Xi \alpha^{-1} (1 + M_u)^2 |\bar{p}_u^+|^2 + (1 - M_d)^2 |\bar{p}_d^-|^2} \quad (18)$$

where  $M_u$  (respectively  $M_d$ ) is the Mach number at the upstream (respectively downstream) reference location,  $\theta = A_u/A_d$  is the cross section area ratio,  $\Xi = \bar{c}_u/\bar{c}_d$  is the sound speed ratio and  $\alpha = \bar{p}_u/\bar{p}_d$  is the mean pressure ratio. This expression holds for both compact and non-compact elements.

Equation (18) demonstrates that the acoustic absorption coefficient of an arbitrary non-compact element only depends on five dimensionless numbers characterizing the mean flow and on the complex amplitudes of the propagating waves on each side of the element. Equation (18) can also be expressed in terms of 1) the coefficients of the Scattering Matrix and 2) the acoustic reflection coefficient on the upstream or downstream side of the element. This alternative expression is now derived.

**S-Matrix and acoustic absorption coefficient** The Scattering Matrix, also called S-matrix, relates the complex amplitudes of the ingoing and outgoing propagating waves on each side of the non-compact element [20, 21, 35, 36]:

$$\begin{pmatrix} \tilde{p}_d^+ \\ \tilde{p}_u^- \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \times \begin{pmatrix} \tilde{p}_u^+ \\ \tilde{p}_d^- \end{pmatrix} \quad (19)$$

where the symbol  $\times$  represents the matrix product. An additional relation between the propagating waves on the upstream (respectively downstream) side is obtained using the acoustic reflection coefficient  $\mathcal{R}_u$  (respectively  $\mathcal{R}_d$ ), defined here as:

$$\mathcal{R}_u \tilde{p}_u^- = \tilde{p}_u^+ \quad (20)$$

$$\mathcal{R}_d \tilde{p}_d^+ = \tilde{p}_d^- \quad (21)$$

From Eqs. (18)-(21), the expressions of the acoustic absorption coefficient of a generic non-compact element given in terms of the S-matrix coefficients are given by [35, 36]:

$$\Delta = 1 - \frac{(1 + M_d)^2 |S_{12} + \mathcal{R}_u \det S|^2 + \theta \Xi \alpha^{-1} (1 - M_u)^2 |S_{22}|^2}{\theta \Xi \alpha^{-1} (1 + M_u)^2 |\mathcal{R}_u S_{22}|^2 + (1 - M_d)^2 |1 - \mathcal{R}_u S_{21}|^2} \quad (22)$$

$$\Delta = 1 - \frac{(1 + M_d)^2 |S_{11}|^2 + \theta \Xi \alpha^{-1} (1 - M_u)^2 |S_{21} + \mathcal{R}_d \det S|^2}{\theta \Xi \alpha^{-1} (1 + M_u)^2 |1 - \mathcal{R}_d S_{12}|^2 + (1 - M_d)^2 |\mathcal{R}_d S_{11}|^2} \quad (23)$$

Equation (22) (respectively Eq. (23)) should be used when the upstream (respectively downstream) acoustic reflection coefficient is known. These equations are equivalent and lead to the same predictions for the acoustic absorption coefficient if both reflection coefficients are known.

**Special case : the ingoing and outgoing time-averaged acoustic energy fluxes are equal** According to Eq. (14), the acoustic absorption coefficient  $\Delta$  cancels out when the time-averaged acoustic energy fluxes (integrated over the cross section area) in and opposite to the flow are equal at both reference locations:

$$\left. \begin{aligned} W_{\tilde{p}_u^+} &= W_{\tilde{p}_u^-} \\ W_{\tilde{p}_d^+} &= W_{\tilde{p}_d^-} \end{aligned} \right\} \Delta = 0 \quad (24)$$

Using Eqs. (15)-(17) along with Eqs. (20)-(21), this can be shown to be equivalent to:

$$\left. \begin{aligned} |\mathcal{R}_u| &= \frac{1 - M_u}{1 + M_u} \\ |\mathcal{R}_d| &= \frac{1 + M_d}{1 - M_d} \end{aligned} \right\} \Delta = 0 \quad (25)$$

As expected, Eq. (25) reduces to  $|\mathcal{R}_u| = |\mathcal{R}_d| = 1$  for a quiescent fluid: if both boundary conditions are fully reflective, then the acoustic absorption coefficient is equal to zero and there are no acoustic energy transfers at the boundaries of the element [30, 37]. However, in the presence of a subsonic mean flow, the acoustic absorption coefficient cancels out for  $|\mathcal{R}_u| < 1$  and  $|\mathcal{R}_d| > 1$ , as shown by Eq. (25). It should be noted that the conditions stated in Eq. (24) are not the only ones leading to an acoustic absorption coefficient equal to zero.

**Variation of acoustic energy inside the element** Up to this point, the focus has been on transfers of acoustic energy through the boundaries of a generic non-compact element. The next step is to consider the variation of acoustic energy stored inside the element between two consecutive cycles.

The acoustic flow field variables at a point  $\mathbf{x}$  located inside the generic element depicted in Fig. 1 are assumed to be given by:

$$p'(\mathbf{x}, t) = \tilde{p}(\mathbf{x})e^{+i\omega t} \quad (26)$$

$$\mathbf{u}'(\mathbf{x}, t) = \tilde{\mathbf{u}}(\mathbf{x})e^{+i\omega t} \quad (27)$$

These flow field variables need not be one-dimensional. Using Eq. (12) along with Eqs. (26)-(27), the acoustic energy density  $e_{ac}$  can be expressed as:

$$e_{ac}(\mathbf{x}, t) = e^{2i\omega t} \chi(\mathbf{x}) \quad (28)$$

where  $\chi(\mathbf{x})$  is a function accounting for the spatial variations of the acoustic flow field variables. The instantaneous acoustic energy  $E_{ac}$  stored within the combustor at any time  $t$  is then obtained by integrating Eq. (28) over the combustor's volume. The cycle-to-cycle acoustic energy ratio  $\lambda$ , defined as the ratio of acoustic energy contained inside the element at time  $t + \tau$  and at time  $t$ , is then given by:

$$\lambda = \frac{E_{ac}(t + \tau)}{E_{ac}(t)} = e^{-2\omega_i \tau} \quad (29)$$

Equation (29) becomes  $\lambda = \exp(2\omega_i \tau)$  for the  $\exp(-i\omega t)$  harmonic convention. The cycle-to-cycle acoustic energy ratio  $\lambda$  characterizes the evolution of the acoustic energy stored inside a non-compact element:

- $\lambda > 1$  : The acoustic energy stored inside the element increases with each oscillation cycle. The larger  $\lambda$ , the larger the growth.
- $\lambda = 1$  : The acoustic energy stored inside the element is constant.
- $0 < \lambda < 1$  : The acoustic energy stored inside the element decreases with each oscillation cycle. The smaller  $\lambda$ , the larger the drop.

**Conclusion** The transfers of acoustic energy at the boundaries of an arbitrary element were shown to be characterized by the acoustic absorption coefficient  $\Delta$ . On the other hand, the variation of acoustic energy stored within the element between two consecutive cycles was shown to be characterized by the cycle-to-cycle acoustic energy ratio  $\lambda$ . This approach will now be extended to analyze the acoustic energy balance of a generic burner comprising several non-compact elements.

## APPLICATION TO A GENERIC BURNER

**Thermoacoustic stability prediction** A generic combustor can be described as a network of connected non-compact elements. The ingoing and outgoing waves downstream of element  $j$  correspond to the outgoing and ingoing waves upstream of element  $j + 1$ . At the upstream and downstream limits, knowledge of the acoustic boundary conditions is needed in order to close the problem.

An example of such a network, also called acoustic equivalent, is represented in Fig. 2 for a generic combustor described using five elements. Each of these elements corresponds to a given part of the burner, such as the plenum, the injector or the flame for instance. Some of these elements contain fresh gases, represented in blue in Fig. 2, while others are located after the flame and thus contain hot gases, represented in red in Fig. 2. The number of elements needed to represent a given combustor depends on its geometry and on the desired accuracy of the acoustic network model.

Predicting the thermoacoustic stability of a combustor using an acoustic equivalent such as the one depicted in Fig. 2 is always based on a variant of the following approach:

1. The Scattering Matrix  $S$  or Transfer Matrix  $\mathcal{T}$  of the entire system (e.g. the combination of Elements A, B, C, D and E in Fig. 2) is assessed using experiments, numerical simulations or analytical models.
2. The acoustic reflection coefficient  $\mathcal{R}$  or acoustic impedance  $\mathcal{Z}$  are determined at the inlet and outlet of the system.
3. A characteristic equation is solved. If the acoustic response of the system is described using the Scattering Matrix  $S$  and its boundary conditions are described using reflection coefficients  $\mathcal{R}$ , this equation is given by:

$$S_{11}S_{22}\mathcal{R}_u\mathcal{R}_d = (1 - S_{21}\mathcal{R}_u)(1 - S_{12}\mathcal{R}_d) \quad (30)$$

Equation (30) can be derived from Eqs. (19)-(21) for instance. The roots  $\omega = \omega_r + i\omega_i$  of Eq. (30) then provide the frequencies  $\omega_r$  and damping rates  $\omega_i$  of the thermoacoustic modes of the combustor. Limit cycle oscillations, which are inherently nonlinear, can be predicted with (linear) acoustic network models by using a weakly nonlinear description of the flame frequency response [15–17, 25].

Once the roots  $\omega$  are known, the cycle-to-cycle acoustic energy ratio  $\lambda$  can be computed for each mode and each oscillation level using Eq. (29). It should be noted that  $\lambda$  is a function of the period  $\tau$  and damping rate  $\omega_i$  only and thus, it is the same for the entire combustor and for each of its subsystems (Elements A, B, C, D and E in Fig. 2 for instance).

The values taken by the cycle-to-cycle acoustic energy ratio  $\lambda$  and damping rate  $\omega_i$  clearly indicate the nature of a given thermoacoustic mode at each oscillation level. For the  $\exp(+i\omega t)$  harmonic convention, a linearly stable thermoacoustic mode is characterized by  $\omega_i > 0$  and  $0 < \lambda < 1$ . A linearly unstable thermoacoustic mode (i.e. of growing amplitude) is characterized by  $\omega_i < 0$  and  $\lambda > 1$ . Finally, a thermoacoustic mode featuring periodic oscillations, i.e. a limit cycle, is characterized by  $\omega_i = 0$  and  $\lambda = 1$ .

Even though there is a direct link, given by Eq. (29), between the cycle-to-cycle acoustic energy ratio  $\lambda$  and the damping rate  $\omega_i$ , these two quantities are not fully equivalent. The former provides quantitative information about the evolution of the acoustic energy stored within the combustor. This information is not readily accessible when considering the latter. Moreover, for a given damping rate  $\omega_i$ ,  $\lambda$  will be different depending on the frequency of the oscillation cycle.

**Acoustic energy transfers** The acoustic absorption coefficient can be computed for the entire combustor and for each of its subsystems (Elements A, B, C, D and E in Fig. 2 for instance). As a consequence, the acoustic absorption coefficient can be used to gain more insight into the acoustic energy balance of a combustor during the onset, growth and saturation of thermoacoustic instabilities. For each element of interest, the acoustic absorption coefficient depends on the specific mode and oscillation level considered.

The acoustic absorption coefficient  $\Delta$  of the entire combustor provides information about the overall acoustic energy transfers at the system's boundaries. If  $\Delta = 0$ , then the amount of acoustic energy entering and leaving the combustor during an oscillation cycle is identical. For instance, this situation arises for a quiescent fluid (or a very low-Mach flow) for fully reflective boundary conditions, as demonstrated above. In this specific case, the combustor is completely isolated: no acoustic energy transfers occur on average at the upstream and downstream boundaries. On the other hand, if  $\Delta < 0$ , then the combustor loses energy to its surroundings. This is observed for combustors with a closed boundary on one side and an open boundary (with acoustic radiation losses) on the other side for instance. Finally, if  $\Delta > 0$ , there is a net transfer of acoustic energy from the surroundings to the combustor. This may happen if a loudspeaker is present at one of the boundaries for instance.

**Sources and sinks of acoustic energy** The acoustic absorption coefficient can also be computed for every subsystem of the combustor (Elements A, B, C, D and E in Fig. 2 for instance). For each element, the corresponding acoustic absorption coefficient provides information about the balance between the ingoing and outgoing acoustic energy fluxes at its boundaries. For a given mode and a given oscillation level, each element of the burner can thus be determined to act as a source or a sink of acoustic energy if  $\Delta < 0$  or  $\Delta > 0$  respectively. If  $\Delta = 0$ , the corresponding element does not affect the acoustic energy balance of the combustor.

**Link with Rayleigh's criterion** There is a strong link between the generalized Rayleigh's criterion and the formalism introduced in the previous sections. This link is now explored.

One form of the generalized Rayleigh's criterion is an acoustic energy balance performed over the combustor's volume  $V$  and averaged over an oscillation period  $\tau$ , such as given by Eq. (3) and reproduced here for the sake of readability:

$$\frac{1}{\tau} [E_{ac}]_t^{t+\tau} + \oint_{\partial V} \mathbf{I} \cdot d\mathbf{S} = \frac{1}{\tau} \int_t^{t+\tau} \iiint_V (S - \mathcal{D}) dV dt \quad (31)$$

where  $E_{ac}$  is the instantaneous acoustic energy contained inside the combustor,  $\mathbf{I}$  is the time-averaged acoustic energy flux, and  $S$  and  $\mathcal{D}$  are the acoustic energy sources and sinks per unit volume and unit time.

The first term in Eq. (31) corresponds to the variation of acoustic energy contained inside the combustor. This first term is characterized by the cycle-to-cycle acoustic energy ratio  $\lambda$  in the previous analysis. The second term in Eq. (31) corresponds to acoustic energy transfers at the boundaries of the combustor. This second term is characterized by the acoustic absorption coefficient  $\Delta$  of the entire combustor in the previous analysis. Finally, the last term in Eq. (31) corresponds to the contribution of the acoustic energy sources and sinks within the combustor. This term is characterized for each subsystem of the combustor by the corresponding acoustic absorption coefficient.

For limit cycle oscillations, the first term in Eq. (31) cancels out, which means that the acoustic energy sources and sinks located within the combustor are exactly balanced by the acoustic energy transfers at the combustor's boundaries. On the other hand, if the second and third terms in Eq. (31) are not balanced, then the acoustic energy stored within the combustor cannot be a constant. An important example, corresponding to the acoustic behavior of a lean premixed flame, is now investigated.

**Equivalence between Rayleigh's criterion and the acoustic absorption coefficient for a flame element** In this subsection, a perfectly premixed lean flame stabilized in a constant cross section area duct ( $\theta = 1$ ), as sketched in Fig. 3, is investigated. The flame is assumed to be a non-moving heat source and the Mach number at both reference locations is assumed to be negligible  $M_u \sim M_d \sim 0$ . These typical assumptions lead to an apparent contradiction between the conservation equations [13, 38, 39]. However, this paradox is out of the scope of this work. Using the mean conservation of momentum, it is further shown that  $\alpha = 1$ . Finally, the mean sound speed ratio  $\Xi$  can be determined by considering the mean conservation of energy.

The acoustic flow field variables at  $x = x_u$  and  $x = x_d$  are given by Eqs. (5)-(8). The nonlinear flame frequency response to incoming acoustic waves is described using a Flame Describing Function (FDF) defined as [15, 25]:

$$\mathcal{F}(\omega, |u'_u|/\bar{u}_u) = \frac{\dot{Q}'/\bar{Q}}{u'_u/\bar{u}_u} \quad (32)$$

where  $\dot{Q}$  is the flame heat release rate. This description of the flame frequency response is weakly nonlinear because it only generates a response at the frequency  $\omega$  even though it depends on the amplitude of the acoustic oscillations. The FDF can be further decomposed into a gain  $G$  and a phase lag  $\varphi$  such that  $\mathcal{F} = Ge^{i\varphi}$ . The Scattering Matrix of the corresponding flame element, depicted in Fig. 3, can be determined from the literature [21, 23]. By further using Eq. (22) (respectively Eq. (23)) along with Eq. (32), the expression of the

acoustic absorption coefficient  $\Delta_f$  of the flame element for a given upstream (respectively downstream) acoustic reflection coefficient is obtained. It is then possible to show that:

$$\Delta_f > 0 \Leftrightarrow (|\mathcal{R}_u|^2 - 1) \cos \varphi < 2\Im(\mathcal{R}_u) \sin \varphi \quad (33)$$

$$\Delta_f < 0 \Leftrightarrow (|\mathcal{R}_u|^2 - 1) \cos \varphi > 2\Im(\mathcal{R}_u) \sin \varphi \quad (34)$$

where  $\Im$  represents the imaginary part of a complex quantity. On the other hand, the source term  $\mathcal{S}_f$  corresponding to the flame contribution in Rayleigh's criterion is given by [8, 23, 27]:

$$\begin{aligned} \mathcal{S}_f &= \frac{\gamma-1}{\gamma\bar{p}} \frac{1}{\tau} \int_t^{t+\tau} \iiint_V p' \dot{q}' dV dt \\ &= \frac{\gamma-1}{\gamma\bar{p}} \frac{1}{\tau} \int_t^{t+\tau} p'_u \dot{Q}' dt \end{aligned} \quad (35)$$

where  $\gamma$  is the adiabatic index and  $\dot{q}$  is the local flame heat release rate per unit volume. The final expression in Eq. (35) is obtained using the fact that the acoustic pressure is a constant across the flame [13, 30]. Equations (32) and (35) then yield:

$$\mathcal{S}_f = K \langle p'_u \mathcal{F} u'_u \rangle = K [(p'_u)^* \mathcal{F} u'_u + p'_u (\mathcal{F} u'_u)^*] \quad (36)$$

where the symbol  $()^*$  denotes the complex conjugate and  $K$  is a positive constant. Using Eqs. (5)-(6) along with Eq. (36) and the definition of the upstream acoustic reflection coefficient, given by Eq. (20), inequalities analogous to Eqs. (33)-(34) are obtained and it is thus shown that:

$$\mathcal{S}_f > 0 \Leftrightarrow \Delta_f < 0 \quad (37)$$

$$\mathcal{S}_f < 0 \Leftrightarrow \Delta_f > 0 \quad (38)$$

As a conclusion, the source term in Rayleigh's criterion and the acoustic absorption coefficient of the corresponding element are fully equivalent for perfectly premixed lean flames. When the source term is positive, corresponding to a net transfer of energy from the flame to the acoustic flow field variables, the acoustic absorption coefficient is negative, and vice versa. This result was obtained for a perfectly premixed lean flame. Its extension to a wider range of flames, including partially-premixed and non-premixed flames, will be explored in a future work.

## EXAMPLE OF APPLICATION : THE PALIES BURNER

**Description of the setup** The Palies burner, designed and operated at EM2C laboratory, contains a turbulent premixed swirling flame confined within a cylindrical tube [25, 40], as sketched in Fig. 4. The thermoacoustic stability of this combustor was characterized experimentally for various flames and for increasing lengths of the combustion chamber [25]. The Palies burner was also scrutinized using a wide array of diagnostics, including velocity measurements inside the combustion chamber [40], Flame Describing Functions measurements for various operating conditions [25, 40, 41], and several flame imaging techniques [40, 42], making it a good candidate for testing the analytical framework introduced in this work.

The considered configuration of the Palies burner is now briefly described. A perfect premixture of  $\text{CH}_4$  and air, corresponding to a lean flame of equivalence ratio  $\phi = 0.7$ , is injected at the bottom of the setup, as depicted in Fig. 4. These fresh gases are then accelerated through a water-cooled converging nozzle placed above a plenum of length 224 mm. The average and fluctuating velocities

are measured using a hot-wire probe placed on top of the converging nozzle [25]. The bulk velocity at the hot-wire probe location is  $\bar{u}_b = 4.13$  m/s. A swirler then generates a flow of Swirl number  $\sim 0.55$  [25,41]. The corresponding swirling flame is stabilized using a conical bluff body protruding into a combustion chamber of length 400 mm. More details about this specific configuration of the Palies burner and the associated diagnostics can be found in the literature [25,40–42].

**Thermoacoustic stability prediction using OSCILOS** The Open Source Combustion Instability Low-Order Simulator (OSCILOS) is an acoustic network model tool that can be used for thermoacoustic stability prediction [16, 19, 43]. It has been successfully employed to predict the frequencies, growth/damping rates and limit cycle amplitudes of thermoacoustic modes in a variety of combustors, including a partially-premixed turbulent confined combustor [16], a pressurized gas turbine [44] and a premixed swirling combustor [16, 19], corresponding to the configuration of the Palies burner described in the previous section. This study is based on the exact same acoustic equivalent that was characterized and validated in one of those earlier works [19]. The flame frequency response is described using a polynomial fit of the corresponding experimental FDF [16, 19, 25]. The acoustic equivalent consists of a large number of gray-boxes describing the acoustic response of small subsets of the combustor. For instance, the converging nozzle represented in Fig. 4 is divided into 60 ducts of length 1 mm and of constant cross section area [19]. The upstream (respectively downstream) acoustic reflection coefficient is set to  $\mathcal{R}_u = 0.98$  (respectively  $\mathcal{R}_d = -1$ ). More information about this acoustic equivalent can be found in [16, 19]. A characteristic equation similar to Eq. (30) is then solved for each forcing level, thus providing the frequencies and damping rates of all modes in the range 0 – 400 Hz.

For the considered configuration of the Palies burner, OSCILOS predicts the presence of two thermoacoustic modes in this frequency range. The evolution of the frequencies and damping rates of these two modes, denoted by  $\mathcal{M}_1$  and  $\mathcal{M}_2$  respectively, is represented in Fig. 5. The first thermoacoustic mode  $\mathcal{M}_1$ , represented in blue in Fig. 5, has a frequency in the range  $f = 96 - 100$  Hz depending on the oscillation level  $|u'|/\bar{u}$ . This first mode is linearly stable since its damping rate  $\omega_i$  remains positive for all  $|u'|/\bar{u}$ . The second thermoacoustic mode  $\mathcal{M}_2$ , represented in red in Fig. 5, has a frequency in the range  $f = 121 - 130$  Hz depending on the oscillation level. This mode is first linearly unstable, since its damping rate  $\omega_i$  is negative at limited oscillation levels, until a limit cycle is reached at  $|u'|/\bar{u} = 0.63$ . The corresponding frequency of the limit cycle oscillations is  $f = 125.7$  Hz. These predictions are in excellent agreement with the frequency  $f = 126$  Hz and peak-to-peak oscillation level  $|u'|/\bar{u} = 0.68$  obtained from the experimental data [25].

This classical thermoacoustic stability analysis provides information about the presence of thermoacoustic modes, along with their frequencies and growth/damping rates (if applicable). If a linearly unstable thermoacoustic mode is present, then the frequency and amplitude of the corresponding limit cycle oscillations can also be obtained from such an analysis. However, this approach does not provide any information regarding the acoustic behavior of the flame (source versus sink), the acoustic energy transfers occurring at the boundaries of the burner, or the damping and generation of acoustic energy within the burner. All of this information will now be obtained using the new framework introduced in this study.

**Acoustic energy stored inside the Palies burner** The first step is to quantify the variation of the acoustic energy contained inside the Palies burner between two consecutive oscillation cycles. This can be achieved by computing the cycle-to-cycle acoustic energy ratio  $\lambda$ , given by Eq. (29), which only depends on the damping rate  $\omega_i$  and period  $\tau$  of the oscillation cycle. These ratios  $\lambda$  are then plotted as functions of the oscillation level in Fig. 6 for modes  $\mathcal{M}_1$  (blue dots) and  $\mathcal{M}_2$  (red dots).

For all oscillation levels investigated,  $\lambda < 1$  for the first mode  $\mathcal{M}_1$ , as shown in Fig. 6. This is consistent with the previous conclusion that  $\mathcal{M}_1$  was a linearly stable mode at all oscillation levels. For instance, at an oscillation level  $|u'|/\bar{u} = 0.269$ ,  $\lambda = 0.162$  which indicates that the amount of acoustic energy stored within the combustor is equal to 16.2% of that stored within the combustor one cycle earlier. Moreover,  $\lambda$  increases for increasing oscillation levels in Fig. 6, thus indicating that this mode becomes less stable as the oscillation level is increased.

On the other hand,  $\lambda > 1$  at limited oscillation levels for the second mode  $\mathcal{M}_2$ , which implies that the amount of acoustic energy associated with mode  $\mathcal{M}_2$  and stored within the Palies burner increases at each oscillation cycle. This is expected since mode  $\mathcal{M}_2$  was shown to be linearly unstable at low oscillation levels. For instance, at an oscillation level  $|u'|/\bar{u} = 0.269$ ,  $\lambda = 1.33$ , corresponding to a 33% cycle-to-cycle increase in the amount of acoustic energy contained inside the burner. The cycle-to-cycle acoustic energy ratio  $\lambda$  decreases as the oscillation level increases, until  $\lambda$  becomes unity at  $|u'|/\bar{u} = 0.63$ , corresponding to a constant amount of acoustic energy stored within the burner. This level corresponds to limit cycle oscillations, in accordance with the earlier predictions.

**Acoustic energy transfers at the boundaries of the Palies burner** Now that the variation of acoustic energy stored within the Palies burner has been characterized, it is worth focusing on the acoustic energy transfers occurring at its boundaries. This is achieved by computing the acoustic absorption coefficient of the entire Palies burner using Eq. (18). The complex amplitudes of the propagating waves at the inlet and outlet of the burner are extracted from OSCILOS. Likewise, the Mach number  $M_u$  (respectively  $M_d$ ) at the inlet (respectively outlet) of the combustor, the mean pressure ratio  $\alpha$ , the mean sound speed ratio  $\Xi$  and the cross section area ratio  $\theta$  are reconstructed from the mean flow analysis performed in OSCILOS. The acoustic absorption coefficient might also be computed using either Eq. (22) or Eq. (23) if the Scattering matrix  $S$  of the entire burner as well as one of the boundary acoustic reflection coefficients are known.

The acoustic absorption coefficient  $\Delta$  of the entire Palies burner is represented as a function of the oscillation level  $|u'|/\bar{u}$  in Fig. 7- (Bottom Right) for modes  $\mathcal{M}_1$  (blue dots) and  $\mathcal{M}_2$  (red dots). For both modes,  $\Delta < 0$  at all oscillation levels, thus indicating that the Palies burner loses acoustic energy to its surroundings in all cases explored. This observation is consistent with the boundary acoustic reflection coefficients used in the acoustic equivalent. Moreover, the acoustic absorption coefficient of the Palies burner is quite similar for both modes and only slightly varies with the oscillation level. On average,  $\Delta \sim -0.022$  with a standard deviation of about 10 %, which means that the average amount of acoustic energy leaving the Palies burner through its boundaries is 2.2% larger than that entering the burner.

Since the Palies burner constantly loses acoustic energy to its surroundings while being unstable (mode  $\mathcal{M}_2$ ), there is at least one source of acoustic energy present within it. In fact, it is now demonstrated that there is a collection of acoustic energy sources and sinks located within the Palies burner that behave differently depending on the specific mode and oscillation level.

**Acoustic energy sources and sinks located within the Palies burner** The strengths and locations of the various sources and sinks of acoustic energy located within the Palies burner are now investigated. In order to characterize them, the geometry of the burner is divided into five elements, as represented in Fig. 4. These elements correspond to physical components of the Palies burner and are defined as:

- Element A: Tranquilization box and plenum
- Element B: Water-cooled converging nozzle
- Element C: Injector (including the swirler)
- Element D: Flame and bottom of the combustion chamber
- Element E: Remainder of the combustion chamber

The complex amplitudes of the propagating waves and the Mach numbers at the inlet and outlet of each of these elements are then extracted using OSCILOS, as are the corresponding mean pressure ratios, mean sound speed ratios and cross section area ratios. The acoustic absorption coefficients of Elements A, B, C, D and E in Fig. 4 are computed using Eq. (18), and are represented as functions of the oscillation level in Fig. 7 for modes  $\mathcal{M}_1$  (blue dots) and  $\mathcal{M}_2$  (red dots).

Element D, corresponding to the portion of the combustion chamber where the flame is located, is first investigated. As shown earlier, the source term associated with the flame in Rayleigh's criterion and the acoustic absorption coefficient of the corresponding flame element are equivalent quantities. The acoustic absorption coefficient corresponding to the flame element of the Palies burner is always positive for mode  $\mathcal{M}_1$ , as shown in Fig. 7-(Middle Right). This demonstrates that the flame acts as an acoustic energy sink at all oscillation levels for mode  $\mathcal{M}_1$ , which explains why this mode is always linearly stable. On the other hand,  $\Delta < 0$  at low oscillation levels for mode  $\mathcal{M}_2$ , as depicted in Fig. 7-(Middle Right), which means that the flame acts as an acoustic energy source in that case. As the oscillation level increases, the acoustic absorption coefficient increases until it reaches  $\Delta = 0$  at  $|u'|/\bar{u} = 0.77$  (obtained using linear interpolation). At this oscillation level, the acoustic energy balance of the system is unaffected by the flame. Furthermore, at the limit cycle oscillation level  $|u'|/\bar{u} = 0.63$ ,  $\Delta = -0.012$  for mode  $\mathcal{M}_2$ , which means that the amount of acoustic energy leaving the flame element is still 1.2% larger than that entering the flame element. In other words, the flame element still acts as an acoustic energy source during limit cycle oscillations. This is explained by the presence of acoustic energy losses at the boundaries of the burner, as discussed later.

Figure 7 also demonstrates that for mode  $\mathcal{M}_1$ , Elements A and E act as strong acoustic energy sources at all forcing levels while Elements B and C act as weak acoustic energy sources at all forcing levels. Moreover, the strengths of these acoustic energy sources

decrease for increasing oscillation levels. In the end, all five elements of the Palies burner affect its acoustic energy balance to a certain extent for mode  $\mathcal{M}_1$ . For mode  $\mathcal{M}_2$ , Fig. 7 shows that  $|\Delta| < 8 \times 10^{-3}$  for Elements B and C at all forcing levels. As a consequence, the acoustic energy balance of the combustor is unaffected by the converging nozzle and the injector in that case. On the other hand, the acoustic absorption coefficient of Elements A and E are positive at low oscillation levels  $|u'|/\bar{u}$ , thus indicating that these elements act as acoustic energy sinks at low oscillation levels. As  $|u'|/\bar{u}$  is increased, the acoustic absorption coefficients for these elements decrease until they reach  $\Delta = 0$  at  $|u'|/\bar{u} = 0.63$  corresponding to limit cycle oscillations. At oscillation levels  $|u'|/\bar{u} > 0.63$ ,  $\Delta < 0$  for Elements A and E, thus indicating that these elements now act as acoustic energy sources.

It is now possible to describe what happens in terms of acoustic energy balance during the onset, growth and saturation of mode  $\mathcal{M}_2$  in the considered configuration of the Palies burner. Initially, the flame acts as a strong source of acoustic energy that is not counterbalanced by the weak acoustic energy sinks appearing in the plenum and combustion chamber and by the limited acoustic energy losses at the boundaries of the burner. As a consequence, the amount of acoustic energy stored within the burner increases with each oscillation cycle, thus leading to an increase in the oscillation level. The strength of the main source of acoustic energy (i.e. the flame) decreases as the oscillation level increases, thus decreasing the rate at which a surplus of acoustic energy is stored within the burner. A balance is reached when  $|u'|/\bar{u} = 0.63$ , corresponding to limit cycle oscillations. At this specific oscillation level, the flame still acts as a weak acoustic energy source but all of the other elements in the burner do not affect the acoustic energy balance. All of the acoustic energy produced by the flame is lost at the boundaries of the burner thus leading to a constant amount of acoustic energy stored within the burner.

This previous analysis demonstrates that the framework introduced in this work is able to accurately describe the evolution of the acoustic energy balance of an unsteady combustor at a system and subsystem scale without resorting to integral calculus.

## CONCLUSION

Thermoacoustic instabilities are an important issue in the propulsion and power production industries. The frequencies and growth/damping rates of thermoacoustic modes can be predicted by an acoustic network model used along with a suitable description of the flame frequency response. However, this approach does not provide any information about the production, dissipation or storage of acoustic energy within and at the boundaries of the combustor. Another approach, known as Rayleigh's criterion, is based on a local or macroscopic acoustic energy balance of the combustor. Though indisputably effective, this approach is rarely used in practice because it requires computing volume and surface integrals and averaging the results over an oscillation cycle.

A new framework building on acoustic network models but providing information about each term in the acoustic energy balance was introduced in this work. The cornerstones of this new framework were shown to be the following two dimensionless numbers: the cycle-to-cycle acoustic energy ratio  $\lambda$  and the acoustic absorption coefficient  $\Delta$ . The cycle-to-cycle acoustic energy ratio  $\lambda$  was shown to quantify the variation of acoustic energy stored within the combustor between two consecutive cycles:  $\lambda > 1$  (respectively  $\lambda < 1$ ) corresponding to an increase (respectively decrease) in the total acoustic energy stored within the system. Likewise, the acoustic absorption coefficient  $\Delta$  of the entire combustor was shown to quantify the acoustic energy transfers occurring at the boundaries of the combustor:  $\Delta < 0$  (respectively  $\Delta > 0$ ) corresponding to a net (respectively negative) transfer of acoustic energy from the combustor to its surroundings.

Finally, it was shown that the acoustic absorption coefficient  $\Delta$  of a given subsystem of the burner could be used to determine its acoustic behavior:  $\Delta < 0$  corresponding to an acoustic energy source and  $\Delta > 0$  to an acoustic energy sink. It was shown that for a lean premixed flame, this new criterion was fully equivalent to Rayleigh's criterion.

This new approach was then applied to the well-documented Palies burner. For each mode and each oscillation level, the strengths and locations of the various sources and sinks of acoustic energy were determined for the first time. For instance, the flame was determined to act as an acoustic energy sink for the linearly stable thermoacoustic mode and as an acoustic energy source for the linearly unstable thermoacoustic mode, even during limit cycle oscillations. It was also determined that the relative amount of acoustic energy lost through the boundaries of the Palies burner was comparable for all modes and all oscillation levels.

This study demonstrates that the generation, damping and transfer of acoustic energy within and at the boundaries of a combustor can be fully characterized by this novel approach. Since it only requires basic calculus, it can be easily implemented in acoustic network model codes, thus providing additional information about the acoustic energy balance of combustors during the onset, growth and satu-

ration of thermoacoustic instabilities.

Finally, even though the focus of this article has been on acoustic network models, this novel framework is perfectly adapted to purely experimental or numerical approaches where the combustor is not described by an acoustic equivalent (as long as the flame response can be assumed to be weakly nonlinear). For instance, the complex amplitudes of the waves propagating in and opposite to the flow direction can be determined experimentally at various reference locations using the multi-microphone method. If the mean flow parameters at these reference locations are known (or measured), one can easily compute the corresponding acoustic absorption coefficients  $\Delta$  and characterize the acoustic behavior of each subsystem (acoustic energy source/sink). This can also be performed for the entire combustor, thus characterizing the acoustic energy transfers at its boundaries. Similarly, the damping rate  $\omega_i$  can be assessed from a variety of physical signals, such as the acoustic pressure inside the combustion chamber, and the cycle-to-cycle acoustic energy ratio  $\lambda$  subsequently computed. The same approach can be followed in numerical studies.

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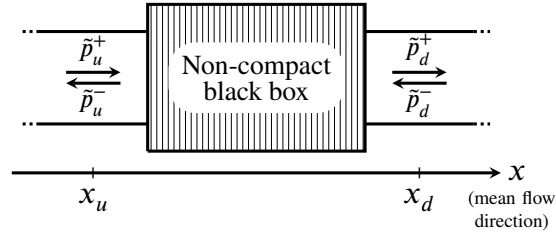


**FIGURE 1.** A GENERIC NON-COMPACT ELEMENT.

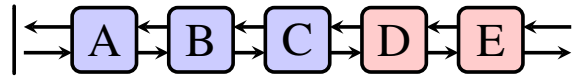


**FIGURE 2.** ACOUSTIC EQUIVALENT OF A GENERIC BURNER COMPRISING FIVE ELEMENTS. BLUE AND RED ELEMENTS CONTAIN COLD AND HOT GASES RESPECTIVELY.

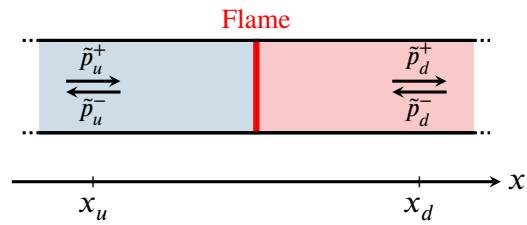


**FIGURE 3.** A PERFECTLY PREMIXED LEAN FLAME STABILIZED IN A DUCT.

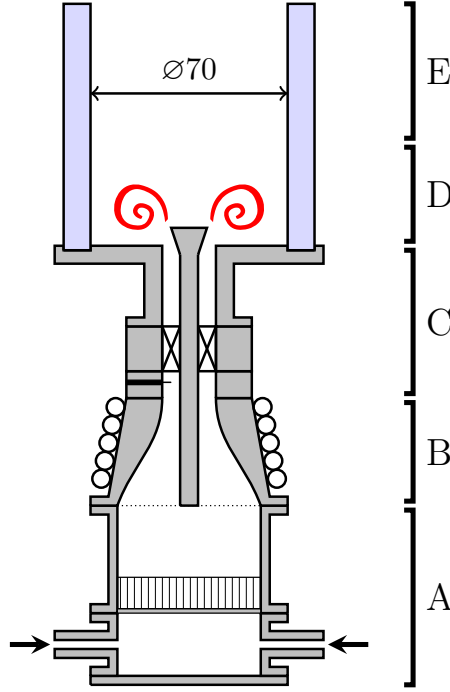


**FIGURE 4.** SKETCH REPRESENTING THE PALIES BURNER AND THE RETAINED SUBDIVISION INTO FIVE ELEMENTS.

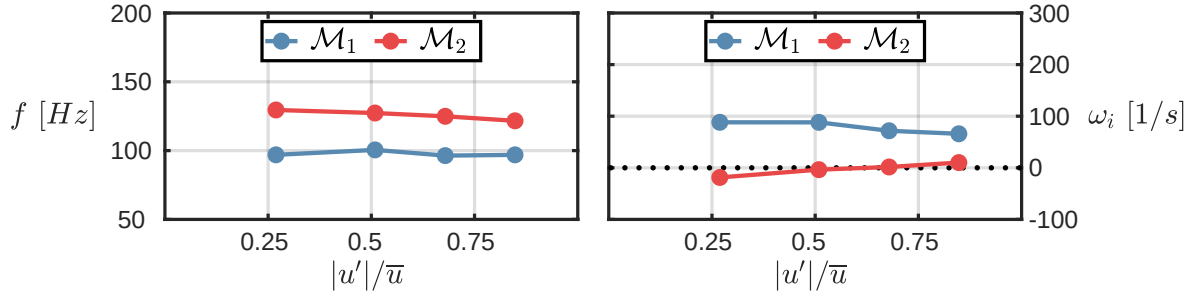


**FIGURE 5.** FREQUENCIES AND DAMPING RATES OF THE THERMOACOUSTIC MODES APPEARING IN THE PALIES BURNER AS PREDICTED BY A LOW-ORDER NETWORK TOOL (OSCILOS).

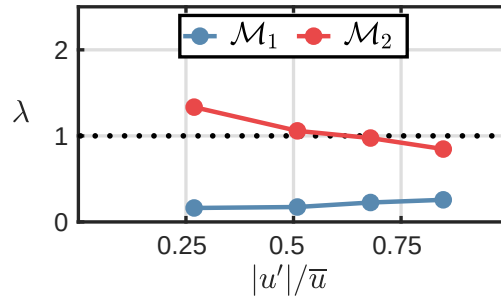


**FIGURE 6.** EVOLUTION OF THE ACOUSTIC ENERGY STORED INSIDE THE PALIES BURNER OBTAINED USING A LOW-ORDER NETWORK TOOL (OSCILOS) AND THE ANALYSIS ABOVE.

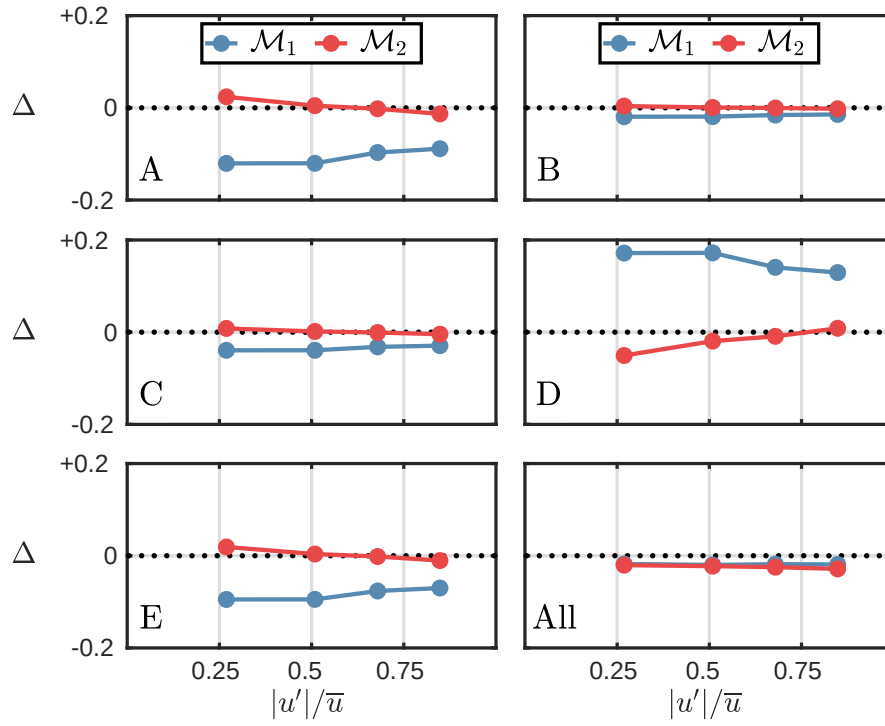


**FIGURE 7.** EVOLUTION OF THE ACOUSTIC ABSORPTION COEFFICIENT OF VARIOUS ELEMENTS OF THE PALIES BURNER OBTAINED USING A LOW-ORDER NETWORK TOOL (OSCILOS) AND THE ANALYSIS ABOVE.