PHASE TRANSITIONS DUE TO STRING-LIKE STRUCTURES

by

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Abstract

Systems which undergo phase transitions due to the proliferation of string-like structures are investigated and a model which can describe these transitions is developed.

Assuming static, non-rigid and non-interacting strings we show, from first principles, how a phase transition can take place. We examine both 'fundamental' and 'composite' strings and derive the order of the transition as well as the temperature at which it occurs. We concentrate on the role of topologically stable cosmic string in the restoration of spontaneously broken symmetries and show how the Ginzburg length scale arises in our description. For simple strings we find a second-order transition independent of the type of string. This is inconsistent with the experimental observation that different string mediated phase transitions are of different orders.

In order to improve the model, we include the vibrational and centre of mass degrees of freedom of the string. We find that our results remain unchanged. We discuss how the thermodynamic properties of strings are modified when stiffness is incorporated and examine the effect of the simplest self-interactions (adhesion). In both cases, the order of the transition does not change, although the temperature at which it occurs is shifted.

By including more general string interactions, appropriate to both global cosmic strings and crystal dislocations, we are able to show how the order of the phase transition depends on the system under investigation. We consider crystal melting due to dislocations and analyse the conditions for first-order melting for the case of iron. Furthermore, we examine rigid, interacting global cosmic strings. We find that there exists a possibility of a weakly first-order phase transition at a temperature which is just below that at which the usual

Hagedorn transition occurs.

Finally, we draw an analogy between the statistical mechanics of closed string loops and an equivalent gauge theory representation for both free and interacting strings.

Preface

This work was carried out in the Theoretical Physics Group of Imperial College under the supervision of Dr. R. J. Rivers, between 1987 and 1990. Financial support was provided by the Science Research Council. Except where otherwise stated, this work is original, and has not been previously submitted for a degree of this or any other university.

The work presented in Chapter 3 is unpublished. The work of Section 1.1 and that of Chapters 2, 4, 5 and 6 is based, in part, on six papers written in collaboration with Dr. E. Copeland, Dr. D. Haws and Dr. R. J. Rivers. In particular, the work of Section 1.1 appears in 'The Statistical Properties of Strings. I. Free Strings', proceedings of 'The Symposium on Cosmic Strings and Phase Transitions in the Early Universe', at Dampt (1989), to be published by the Cambridge University Press in 1990. The material of Chapter 2 appears in 'String Mediated Phase Transitions', Physica A158,460(1989), as well as in 'The Statistical Properties of Rigid Strings', Imperial College preprint TP/89-90/04, to be published by Physics Letters. The work of Chapters 4 and 6 appears in 'The Statistical Properties of Strings. II. Interacting Strings', again proceedings of the symposium at Dampt, to be published. The contents of Chapter 4 also appears in 'The Statistical Mechanics of Strings. I. Dislocation Melting', Imperial College preprint TP/89-90/05, submitted to Physical Review B. Finally, the work of Chapter 5 appears in 'The Statistical Mechanics of Strings. II. Interacting Cosmic Strings', Imperial College preprint TP/89-90/07.

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To Matthew

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Chapter 1

The Statistical Mechanics of Free Strings

In this chapter[1] we shall examine systems that undergo phase transitions caused by the proliferation of string-like structures (henceforth termed *strings*).

There is a diverse range of naturally occuring phenomena which can be understood in terms of such transitions. Examples can be found in polymer physics, nuclear physics, condensed matter physics as well as astrophysics. In particular, string-based phase transitions occur in:-

(a) synthetic macromolecules [1] (condensation)

(b) viral DNA [2] (helicity)

(c) iron [3, 4, 5, 6] (melting)

(d) superfluid ^4He [7, 8] (λ-transition)

(e) type-II superconductors [9] (superconducting/normal conducting transition)

(f) the early universe [10] (spontaneous symmetry breaking)

[1]The work of Section 1.1 appears in 'The Statistical Properties of Strings. I. Free Strings', proceedings of 'The Symposium on Cosmic Strings and Phase Transitions in the Early Universe', at Dampt (1989), to be published by the Cambridge University Press in 1990.

(g) quark-gluon plasmas [11] (deconfinement)

(h) glass [12] (glass transition).

The strings in (a) and (b) are the macromolecules themselves. The conformational phase transition for a single string can be used to examine the helix-coil transition for a one- or two-stranded macromolecule, as well as the functioning of bipolymers.

The melting of crystals can be understood by viewing a liquid as a highly faulted solid. The faults are the dislocation lines that disrupt the translational symmetry of the crystal. Melting can be considered to occur when the solid becomes unstable with respect to a state in which there is a very large number of faults.

^{4}He undergoes a transition (the λ-transition) at 2.18K from being a normal fluid above 2.18K to being a 'superfluid' below. The superfluid exhibits several remarkable properties. For example, it seems to have an infinite heat conductivity and zero viscosity. In the superfluid phase tubes of normal fluid can form around which the superfluid flows with non-zero vorticity. The transition between the two phases is thought to occur by the sudden proliferation in the number density of these vortex excitations.

A very similar picture has been proposed to explain the phase transition from the superconducting to the normal conducting phase. In this case, the string-like structures are the magnetic flux tubes.

The big bang model of the universe suggests that it was once very hot and dense. It has previously been demonstrated [13] that, at sufficiently high temperatures, it is possible to restore the full gauge symmetry of a spontaneously broken simple gauge group which could describe the interactions of particles. Therefore, when the universe was extremely hot we would expect the full symmetry to have been manifest. As the universe expanded and cooled it would have undergone a series of phase transitions at which this symmetry would have been broken. At these phase transitions, topologically stable defects may have been formed [10]. These may have been monopoles,

strings, domain walls or various combinations thereof. The presence of these defects in the early universe has intriguing consequences for cosmology. For example, it has been suggested (see Vilenkin [14] and references within) that strings may have provided the initial density perturbations about which galaxies and clusters of galaxies could form.

The quark deconfinement phase transition can also be explained using a string model. In this case, the strings are the colour flux tubes.

It is not just in the context of phase transitions that strings may be important. They also play a central role in our understanding of a variety of phenomena observed in condensed matter. For example, dislocations are used to examine phenomena such as plastic flow, fracture and cold working associated with the mechanical deformation of crystals. In addition, disclinations, line defects that disrupt rotational symmetry, may be important in understanding the physical properties of glass.

In this chapter we shall present a very simple model which can describe some of the string mediated phase transitions mentioned above. This model was initially developed by Copeland et al. [15] in the context of 'composite' strings. We shall start by displaying the thermodynamic properties of simple fundamental strings and shall show, from first principles, how a phase transition can come about. Initially, we shall look at the case of static non-interacting strings. As a next step, we shall demonstrate how the model can be improved by including vibrational and centre of mass degrees of freedom for realistic elementary strings. Furthermore, we shall consider the effect of associating an excluded volume with each string segment. This is perhaps the simplest string interaction that can be considered.

Finally, we shall present the work of Copeland et al. [15] who examined 'composite' strings such as cosmic strings or vortex lines. As opposed to fundamental strings, 'composite' strings are collective excitations of an underlying theory. We shall show how the description of the phase transition is modified in this case. In particular, we

shall consider Nielen-Olesen strings [16] and derive Kibble's prediction of the string formation temperature [10].

1.1 Thermodynamics of classical strings

1.1.1 Simple strings

As already mentioned, there are many phase transitions which can be understood in terms of the condensation of string-like structures. Therefore, in order to keep the following analysis as general as possible, we shall start by looking at the most simple string model in which the strings are assumed to be characterized by:-

(1) A constant intrinsic energy per unit length, denoted by σ, *independent* of the string configuration.

(2) A constant non-zero scale a below which a string is unable to bend. It is assumed that this scale is much less than the typical string length.

(3) A string interaction energy which is weak enough such that its only effect is to allow the system to obtain thermal equilibrium.

(4) Negligible vibrational and centre of mass degrees of freedom.

The thermodynamics of a gas of such strings has been discussed in a large number of articles [17, 18, 19, 20]. Historically, the problem was first addressed by Frautchi [21] and Carlitz [22] in an attempt to understand hadronic matter at high density. Although we no longer believe the string model to be a good description of this, their work is still of interest because they reached some important conclusions about the statistical properties of free strings. For example, they pointed out that the statistical properties of free strings are somewhat different to those of 'normal' systems and that care is needed to derive their thermodynamic properties correctly. Therefore, we shall start by briefly reviewing some of the basics of statistical mechanics.

The principle idea behind statistical mechanics is that an isolated macroscopic system, specified by its energy E and its volume V, has a large number of different microstates, $\Omega(E,V)$, *i.e.* ways of distributing the energy of the system amoungst its 'particles'. The fundamental postulate of equilibrium statistical mechanics is that the state of the system is equally likely to be found in any one of these microstates. Thus the most probable state of the system will be that which maximizes the density of states. This is the the microcanonical ensemble approach.

However, for large systems, it is often easier to use the less fundamental canonical ensemble approach. In this case, it is assumed that the system under investigation is in thermal contact with a large 'heat bath' and that it is this combination which forms the isolated system. Hence, assuming homogeneity, the properties derived for the system under investigation are the same as those of the system as a whole. Furthermore, it is assumed that $T_{bath}C_v^{bath}V_{bath} \gg \epsilon_i$ (in units where Boltzmann's constant, $k_B = 1$) for all states i of the system of interest with energy ϵ_i. T is the temperature of the heat bath, V is its volume and C_v is its heat capacity.

The fundamental quantity from which all thermodynamic quantities can be derived is the partition function Z, defined by

$$Z(V) := k\int dE\,\Omega(E,V)\,e^{-\beta E} \tag{1.1}$$

$$\beta := \frac{1}{T} \tag{1.2}$$

for some constant k. In order to evaluate Z we must first obtain an expression for $\Omega(E,V)$.

Let us start by considering a system consisting of a single string of length L. From assumption (1), the energy of the string depends solely on its length and hence $\Omega(E,V) \to \Omega(L,V)$, the number of configurations of such a string. Classically, an infinitely thin string (of any length) has an infinite number of different configurations because it is able to bend on any scale. In order to count the number of configurations it is necessary to impose a lower cut-off on the scale at which the string can bend.

This can be done by quantizing the string, in which case the lower cut-off is related to the string tension. (For strings of a finite thickness quantization is unnecessary as the width of the string provides the lower cut-off.) Alternatively, an arbitrary cut-off scale can be introduced which then has to be determined by experiment. As stated in assumption (2), it is this approach that we shall take, the arbitrary cut-off scale being equal to a. In order to count the number of configurations, the string is considered to be a polymer of hinged segments each of length a. Furthermore, the string segments are restricted to lie on a cubic lattice of coordination number z, with each segment joining two nearest neighbour lattice sites (*i.e.* the lattice spacing is a).

Since it is assumed that the only role of the string interactions is to enable the phase space to be explored (see assumption (3)), it is immediately apparent that the string trajectories are Brownian. Thus the number of configurations of a closed string of length L is given by

$$\Omega(L,V) = \frac{AV}{2a^3}\left(\frac{L}{a}\right)^{-3/2-1} z^{L/a} \tag{1.3}$$

(assuming that $\frac{L}{a} \gg 1$). The factor $z^{L/a}$ is the number of possible configurations of a string of length L starting at a given position on the lattice, and V/a^3 is the number of possible points from which it can start. $A\,(L/a)^{-3/2}$ is the probability of a Brownian walk returning to its origin after L/a steps. The factor of $1/2$ is included as free strings are non-orientable and the extra divisor of L/a arises from the fact that an L/a step loop has L/a possible starting points. For open strings the factor $A\,(L/a)^{-3/2-1}$ is excluded.

Whether the strings are open or closed depends on the string type. For example, QCD flux tubes are typically open whilst cosmic strings are either closed or infinite in length. For non-interacting strings the difference between the two cases is not particularly significant as only the fine details of the thermodynamics depend on whether or not the strings are closed. Therefore, for the rest of this section we shall assume that the strings are closed.

From assumption (1), the energy of a single string of length L is $E(L) = \sigma L$. Thus the partition function describing a system containing a single closed string of arbitrary length is

$$Z_{1,\text{closed}} = \frac{AV}{2a^3} \sum_{L/a=1}^{\infty} \left(\frac{L}{a}\right)^{-5/2} e^{-\beta L \sigma_{eff}} \tag{1.4}$$

where

$$\begin{aligned} \sigma_{eff} &= \sigma - \frac{\ln z}{\beta a} \\ &= \sigma\left(1 - \frac{T}{T_{st}}\right) \end{aligned} \tag{1.5}$$

In (1.4), σ_{eff} plays the role of an effective string tension. It is the vanishing of σ_{eff} that causes the system to undergo a phase transition. This occurs at a temperature

$$T_{st} = \frac{\sigma a}{\ln z} \tag{1.6}$$

The significance of this temperature will be discussed shortly.

First, let us consider a system containing a 'gas' of string loops. For many types of string, interactions enable a string to split up and then rejoin to form smaller loops. Hence, the number of strings and the length of each string are not conserved. If this is the only effect of the interactions then the appropriate partition function is that for a system containing an arbitrary number of strings, namely

$$Z = 1 + Z_1 + \frac{1}{2!}Z_1^2 + \frac{1}{3!}Z_1^3 + \ldots \tag{1.7}$$

$$= \exp Z_1 \tag{1.8}$$

where the factor of $1/n!$ accounts for the indistinguishability of the loops.

From this partition function a number of interesting thermodynamic quantities can be derived. For example, the mean energy density is given by

$$\begin{aligned} \frac{\overline{E}}{V} &:= -\frac{1}{V}\frac{\partial}{\partial\beta}\ln Z \\ &= \frac{A\sigma}{2a^2} \sum_{L/a=1}^{\infty} \left(\frac{L}{a}\right)^{-3/2} e^{-\beta L \sigma_{eff}} \end{aligned} \tag{1.9}$$

and the pressure by

$$\begin{aligned} P &:= T\left(\frac{\partial}{\partial V}\ln Z\right)_T \\ &= \frac{A}{2\beta a^3}\sum_{L/a=1}^{\infty}\left(\frac{L}{a}\right)^{-5/2} e^{-\beta L\sigma_{eff}} \end{aligned} \tag{1.10}$$

From (1.7) we can evaluate the specific heat of our system. It is defined by

$$C_v = \frac{1}{T^2}\frac{\partial^2}{\partial\beta^2}\ln Z \tag{1.11}$$

from which it can be easily shown that, for a system of closed strings, the specific heat diverges as T_{st} is approached like

$$C_v \sim (T - T_{st})^{-1/2} \tag{1.12}$$

Further thermodynamic quantities may be determined by an appropriate application of Maxwell's relations.

There are other potentially interesting quantities which may be determined from the partition finction. For example, we can calculate the average number of loops:

$$\begin{aligned} \overline{N}_{\text{loop}} &= \frac{1}{Z}\sum_{n=0}^{\infty}\frac{n(Z_1)^n}{n!} \\ &= Z_1 \end{aligned} \tag{1.13}$$

Furthermore, the mean number of loops of size L can be determined:

$$R(L) = \frac{AV}{2a^3}\left(\frac{L}{a}\right)^{-5/2} e^{-\beta L\sigma_{eff}} \tag{1.14}$$

From (1.14), it can be seen that for $T \ll T_{st}$ loops of long length are exponentially suppressed. For $T \approx T_{st}$

$$\frac{dR}{R} \propto \frac{dL}{L} \tag{1.15}$$

and hence the loops have a scale-invariant distribution. These results have been confirmed for non-interacting Nambu strings by Vilenkin and Smith [23] and by Vilenkin and Sakellariadou [24].

The unusual property of a system of free strings that is not shared by more 'normal' systems is that the partition function diverges for all temperatures $T > T_{st}$. Thus canonical thermodynamics does not make sense at temperatures above T_{st}. At T_{st}, the loops have a scale-invariant distribution and the system is unstable to loop energy densities higher than those present. Hence, increasing the energy density further no longer causes the energy density in loops to increase. Instead the extra energy density floods into an infinite string [25] (where we are considering the thermodynamic limit as the volume of the system tends to infinity) and in this way the energy density in the box can become arbitrarily large. If string interactions for fundamental strings can be ignored at high densities, then the temperature T_{st} is the maximum temperature of a system in thermal equilibrium with a string network. This temperature is often referred to as the Hagedorn temperature [26]. It can be considered to be the transition temperature of the system as it marks the point at which the energy density of the system changes from being dominated by loops to being dominated by infinite strings.

As we shall see later, in the case of 'composite' strings, interactions can *not* be ignored. As the temperature of the system approaches T_{st}, the strings become fat [15] interactions between strings become important and the free string picture breaks down. The result is that the number of available states is reduced which, in turn, means that the entropy can no longer balance the energy and consequently the system does not possess a maximum temperature. However, at temperatures above T_{st} the system can exist in a new phase which is no longer describable in terms of 'free' strings.

Obviously, the model presented so far is rather simplistic. Therefore, we shall now try to improve its accuracy by including the possibility of vibrational and centre of mass degrees of freedom (*i.e.* we shall relax assumption (4)) and then by imposing an excluded volume on the string segments (*i.e.* we shall relax assumption (3)).

1.1.2 Including vibrational and centre of mass degrees of freedom

The most obvious drawback of the simple model presented so far is that it does not take into account the vibrational and centre of mass degrees of freedom. For 'composite' strings this is plausible since these degrees of freedom can be traded for a temperature dependent string tension and thickness. For fundamental strings this is not so. To demonstrate the effect of incorporating these degrees of freedom we shall examine the case of Nambu strings.

The standard way to describe a Nambu string is by a vector function $\underline{x}(s,t)$ that satisfies the two gauge conditions:

$$\left.\begin{array}{rcl} \underline{x}'.\,\dot{\underline{x}} & = & 0 \\ \underline{x}'^2 + \dot{\underline{x}}^2 & = & 1 \end{array}\right\} \tag{1.16}$$

Here s is a parameter describing the position along the string and primes and dots denote derivatives with respect to s and t respectively. Physically, the first gauge condition means that each segment of the string moves perpendicular to itself and the second that the energy in a segment ds is σds. In this gauge, the general solution to the equation of motion in the string's centre of mass reference frame can be written as

$$\underline{x} = \frac{1}{2}\left[\underline{a}(s+t) + \underline{b}(s-t)\right] \tag{1.17}$$

$$x^0 = t \tag{1.18}$$

where, by virtue of the gauge conditions (1.16), $\underline{a}'$ and $\underline{b}'$ must have unit magnitude. Here, primes denote derivatives with respect to $s+t$ for $\underline{a}$ and $s-t$ for $\underline{b}$. To evaluate the number of configurations of a string of centre of mass energy E_{cm} we proceed as before and consider the string to be a hinged polymer. As we are taking the lengths of the segments to equal the lower cut-off scale a, the lengths will vary in the centre of mass frame due to the different velocities of the segments. This is because the lower cut-off scale plays the role of an effective string width (in the case of 'composite'

strings it is the string width) and hence, from the first gauge condition of (1.16), it is parallel to the direction of motion of the string. It is only when each scale is measured in its 'proper' rest frame that the string segments are of equal lengths. Therefore, the string is described by a finite number of points (the ends of each segment) which are equally spaced in the parameter s, by a separation a, where a is now the 'proper' lower cut-off on the scale of string bends.

Each segment of the string between two adjacent points is straight and has a constant perpendicular velocity. Hence we have that

$$\underline{x}'_i = \frac{1}{2}[\underline{a}'_i(s_i + t_i) + \underline{b}'_i(s_i - t_i)] = \text{constant} \tag{1.19}$$

$$\dot{\underline{x}}_i = \frac{1}{2}[\underline{a}'_i(s_i + t_i) - \underline{b}'_i(s_i - t_i)] = \text{constant} \tag{1.20}$$

for all i, where i represents the ith segment. Thus, as suggested by Albrecht and Turok [20], the configuration of a string of energy E_{cm} can be described by a discrete set of n $(= E_{cm}/\sigma a)$ $\underline{a}'_i$ and n $\underline{b}'_i$ vectors. The direction of each of these vectors is arbitrary. However, as already mentioned, their magnitude must be unity in order to satisfy the gauge conditions (1.16). There are two further restrictions that must be placed on the set of $\underline{a}'_i s$ and $\underline{b}'_i s$, namely that

$$\sum_{i=1}^{n} \underline{a}'_i = 0 \qquad \sum_{i=1}^{n} \underline{b}'_i = 0 \tag{1.21}$$

These restrictions arise because we are considering *closed* loops in their *centre of mass* reference frame.

Thus a given string configuration of energy E_{cm} can be described in terms of *two* closed random walks, each consisting of n steps of unit length. The total number of configurations of a string of energy E_{cm} is therefore

$$\rho(E_{cm}) = Cn^{-4}e^{n \ln z} \tag{1.22}$$

(assuming that $E_{cm}/\sigma a \gg 1$). The extra divisor of n accounts for the n possible starting points for the string.

The partition function describing a single Nambu string is then

$$Z_{1,\text{closed}} = C \int d\underline{p} \sum_{E_{cm}/\sigma a} \left(\frac{E_{cm}}{\sigma a}\right)^{-4} \exp\left[\frac{E_{cm}}{\sigma a} \ln z - \beta E(\underline{p}, E_{cm})\right] \tag{1.23}$$

To a good approximation $E(\underline{p}, E_{cm})$ can be taken to be [18, 19, 21, 22]

$$E(\underline{p}, E_{cm}) = E_{cm} + \frac{\underline{p}^2}{2E_{cm}} \tag{1.24}$$

Performing the momentum integral, we finally obtain that

$$Z_{1,\text{closed}} \propto \beta^{-3/2} \sum_{E_{cm}/\sigma a} \left(\frac{E_{cm}}{\sigma a}\right)^{-5/2} e^{-\beta E_{cm} \sigma_{eff}/\sigma} \tag{1.25}$$

which, except for an unimportant normalization, is the same as (1.4). Therefore, including the vibrational and the centre of mass degrees of freedom does not alter the qualitative results obtained previously.

1.1.3 Excluded volume

Let us now revert to static strings, but allow for rudimentary string interactions. Perhaps the most simple type of string-string force to consider is an infinite repulsive force which prevents two string segments having centre-to-centre separations of less than a. This is particularly natural when the strings are, for example, macromolecules.

There are two simple changes that we can make in order to get a flavour of the effect of imposing an excluded volume. The first is to eliminate self-intersection from the string trajectories which up to now have been Brownian. The effect of this is to change the exponent in (1.3) and (1.4) from $-5/2$ to -q, where $q \approx 2.5 \rightarrow 3$, and σ_{eff} to

$$\sigma_{eff} = \sigma - \frac{\ln(z-1)}{\beta a} \tag{1.26}$$

We note that the value of q is obtained numerically and, unlike the case of random walks, cannot be evaluated analytically.

The second change arises when we consider multiple strings. In equation (1.7) we assumed that the volume available to a given string is the same regardless of the

string density. If the strings are not to overlap this assumption can no longer be made. At high densities a reasonable approximation may be to consider the string gas to be a gas of open strings each of length a. In this case, the partition function for the system would be given by

$$Z_{\text{high string density}} \approx \sum_n \left(\begin{array}{c} V/a^3 \\ n \end{array} \right) e^{-\beta n \sigma a} \tag{1.27}$$

where the approximation is only valid when $< n >= O(V/a^3)$. This system no longer possesses a Hagedorn temperature.

At lower string densities a better approximation (proposed by Rivier and Duffy [12]) may be to take

$$Z = \sum_N \left(\begin{array}{c} V/a^3 \\ N \end{array} \right) \frac{Z_N}{(V/a^3)^N} \tag{1.28}$$

where

$$Z_N = \left[\frac{AV}{2a^3} \sum_{n=1}^{\infty} n^{-q} e^{-\beta \sigma_{eff} n a} \right]^N \tag{1.29}$$

This gives that

$$Z = \exp \left[\frac{V}{a^3} \ln \left(1 + \frac{A}{2} \sum_n n^{-q} e^{-\beta \sigma_{eff} n a} \right) \right] \tag{1.30}$$

The combinatorial factor $\left(\begin{array}{c} V/a^3 \\ N \end{array} \right)$ is the number of ways of putting N strings down on a lattice consisting of V/a^3 boxes. This is an excluded volume in the sense that no two strings can start at the same point. This partition function is very similar to that discussed previously, especially in the limit of small Z_1 when the logarithm can be expanded about $Z_1 = 0$ to give that

$$Z \sim \exp \left[\frac{AV}{2a^3} \sum_n n^{-q} e^{-\beta \sigma_{eff} n a} \right] \tag{1.31}$$

in agreement with equation (1.8).

1.2 Thermodynamics of 'composite' strings

So far we have only considered the strings as fundamental objects. In order to understand the λ-transition, the phase transition from the superconducting to the normal conducting phase and the restoration of a spontaneously broken symmetry we need to take into account the fact that the defects present in these systems are not fundamental.

In this section we shall restrict our discussion to phase transitions in the early universe. Assuming the hot big bang model of the universe and that the universe can be correctly described by a spontaneously broken gauge theory, we expect the universe to have undergone a series of phase transitions during the course of its early history. As already mentioned in the introduction to this chapter, there is a variety of topologically stable defects that may have been formed during these transitions. The phase transition is characterised by the order parameter $<\phi>$ obtaining a non-zero value as the universe expands and cools. This value lies on the surface M of minima of the effective potential of the theory. It is the homotopy groups, $\Pi_k(M)$, of the manifold M of degenerate vacuum states which determine the structures of the defects [10]. Domain walls can form if $\Pi_0(M)$ is non-trivial, cosmic strings if $\Pi_1(M)$ is non-trivial and monopoles if $\Pi_2(M)$ is non-trivial. We shall concentrate on the case when the defects are cosmic strings. The reason for this is that theories which produce domain walls and those that give rise to monopoles can be ruled out as inconsistent with present day observations. For example, domain walls would produce large anisotropies in the microwave background [44]. Furthermore, monopoles would be produced in large numbers and, as they can only disappear through annihilation, it is extremely difficult to avoid an overbundance of monopoles in the universe now [45].

We shall present the work developed by Copeland et al. [15] who consider scalar QED, the simplest theory with vortex solutions such as cosmic strings. Analogous

methods can be used for other theories (for example, see Wiegel [8] for an analysis of the λ-transition). We shall see how strings arise during the symmetry restoring phase transition and how the presence of the underlying theory alters the description of the transition as compared to that obtained in the case of fundamental strings. In particular, we shall see how the interpretation of the Hagedorn temperature [26] changes.

Before we start to review the model proposed by Copeland et al. [15], we should clearly state their assumptions. In order to use the results of statistical mechanics they assume that the strings are formed in thermal equilibrium. Although this is still open to debate, it would appear that cosmic strings could well be formed in such a state. This is because the effect of gravity at the scales under consideration is not strong, and hence the relaxation time of the string network is small in comparison with the expansion time of the universe. The second assumption is that the strings exist in flat space-time.

Scalar QED is defined by the Lagrange density:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\,|(\partial_\mu + ieA_\mu)\phi|^2 + \frac{1}{2}m_o^2\,|\phi|^2 - \frac{g}{4!}\,|\phi|^4 \tag{1.32}$$

where ϕ is a complex scalar field, A_μ is the gauge field with charge e and $m_o^2 \geq 0$. The partition function for the theory takes the form

$$Z \propto \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A\, e^{-I_\beta[\phi,A]} \tag{1.33}$$

where

$$I_\beta[\phi, A] = -\int_0^\beta d\tau \int d^3x\, \mathcal{L}_E[\phi, A] \tag{1.34}$$

and $\mathcal{L}_E$ is the Euclidean form of the Lagrangian (1.32). The sum over the configurations of $\phi(\tau, \underline{x})$ and $A(\tau, \underline{x})$ is restricted to fields periodic in τ, with period β, and the signature of the Euclidean space is taken to be -4.

The standard approach to evaluating this partition function is to adopt the mean field approximation. One restricts the path integral to periodic configurations whose

Euclidean space-time average is specified in advance to be

$$\frac{1}{\beta v}\int_0^\beta d\tau \int d^3x \mid \phi(\tau,\underline{x}) \mid = \overline{\phi} \tag{1.35}$$

say (where v is the spatial volume of the system).The resulting expression [27]:

$$Z = \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A \left[\delta\left(\overline{\phi} - \frac{1}{\beta v}\int_0^\beta d\tau \int d^3x \mid \phi(\tau,\underline{x}) \mid\right)\right] e^{-I_\beta[\phi,A]} \tag{1.36}$$

has the following interpretation. The field $\phi(\tau,\underline{x})$ is coupled to a *constant* source j. The effect of this source is to enable the thermal average $<\phi>$ to take almost any desired value ($<\phi>$ will be constant by virtue of the translational invariance of j for large v). If j is chosen such that $<\phi>=\overline{\phi}$, where $\overline{\phi}$ satisfies

$$\frac{\partial V}{\partial \overline{\phi}} = 0 \tag{1.37}$$

in the absence of external sources, the thermal average $<\phi>$ for the original system is the value of $\overline{\phi}$ for which V is minimized. In principle it is straight forward to compute $<\phi>$ by performing a saddle point expansion for Z [28]. This is equivalent to performing a loop expansion for $V(\phi)$, the generating function for zero momentum 1PI Green functions. From its physical interpretation, $V(\phi)$ is necessarily a gauge-variant quantity. However physical conclusions drawn from it should be gauge-invariant. It is most convenient to calculate $V(\phi)$ in the covariant gauge, for which, at high temperatures and to $O(g,e^2)$, [13],

$$V(\phi) = -\frac{1}{2}m_o^2 \mid \phi \mid^2 \left(1 - \frac{T^2}{T_c^2}\right) + \frac{g}{4!} \mid \phi \mid^4 \tag{1.38}$$

where T_c^2 includes gauge fixed loops and takes the form

$$T_c^2 = \frac{m_o^2}{\frac{g}{18} + \frac{e^2}{4}} \tag{1.39}$$

Since the fields $\phi(\tau,\underline{x})$ and $A(\tau,\underline{x})$ are periodic in τ they permit the Taylor expansion:

$$\phi(\tau,\underline{x}) = \sum_n \phi_n(\underline{x})\, e^{2\pi i n\tau/\beta} \qquad A^\mu(\tau,\underline{x}) = \sum_n A_n^\mu(\underline{x})\, e^{2\pi i n\tau/\beta} \tag{1.40}$$

in terms of denumerable sets of three dimensional fields. Substituting these expansions into $\mathcal{L}_E$ one can see that the mass of the $\phi_n(n \neq 0)$ and $A_n(n \neq 0)$ modes, (termed ϕ' and A') are large and positive. (ϕ_o, A_o) and (ϕ', A') are referred to as the light and heavy modes respectively. The partition function Z can then be rewritten as

$$Z \propto \int D\phi_o DA_o \, e^{-\beta \bar{I}[\phi_o, A_o]} \tag{1.41}$$

where

$$e^{-\beta \bar{I}[\phi_o, A_o]} \propto \int D\phi' DA' \, e^{-\bar{I}_\beta[\phi, A]} \tag{1.42}$$

The mass corrections are obtained by keeping terms quadratic in ϕ_o and A_{o_μ}. The heavy modes give rise to a temperature-dependent mass term, $\frac{\Delta m^2}{2} \mid \phi \mid^2$, for the scalar field. The contributions to $\frac{\Delta m^2}{2}$ are shown diagrammatically in Figure 1.1. The effect of this term is to replace m_o^2 by $m_o^2 \left(1 - \frac{T^2}{T_c^2}\right)$, in the Euclidean time-independent effective action obtained from (1.32).

The heavy scalar modes induce a vector mass $\frac{\Delta \mu^2}{2} A_\mu A^\mu$ in the tadpole approximation, (Figure 1.2). This induced mass is of order e^2T^2. Unfortunately, the tadpole diagram does not describe the total effect of the temperature-dependent self mass to $O(e^2T^2)$. For large T, non-local contributions, like the photon self-energy, also have to be included as they give terms of the same order. All the relevant one-loop diagrams which contribute to $O(e^2T^2)$ are shown in Figure 1.3. Further problems arise because the heat bath gives a preferential inertial frame which leads to the temporal and spatial components of A_μ being decoupled. This gives rise to two independent self-mass terms $\Pi_L(k)$ and $\Pi_T(k)$ for momentum k [29]. The same preferential reference frame makes the Π's non-analytic in k_o. The result is that different masses are obtained depending on how one takes the zero momentum limit in the inverse fourier transforms of the masses [29]. Only the tadpole term, with no momentum dependence, is immune from this uncertainty. Yet another complication arises because the three-dimensional gauge theory is infra-red singular. The effect of these difficulties is to make it hard to explicitly evaluate the gauge mass, even to $O(e^2T^2)$.

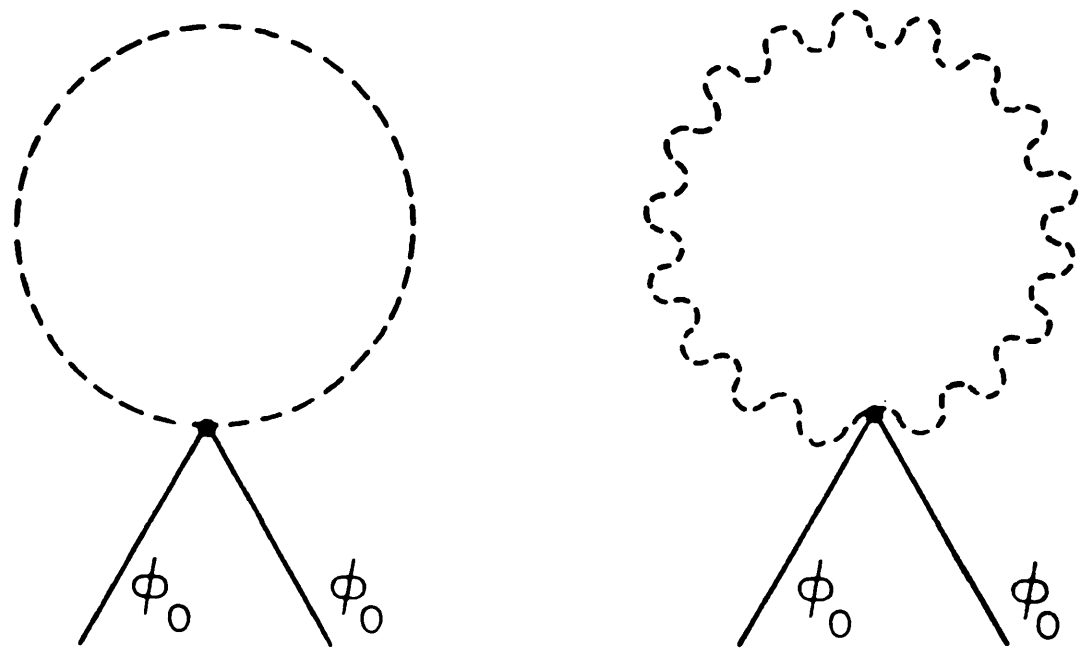

Figure 1.1: *Feynman graphs for the tadpole corrections to the scalar mass. Dashed lines denote the heavy modes, solid lines the light modes and wavy lines the gauge field modes.*

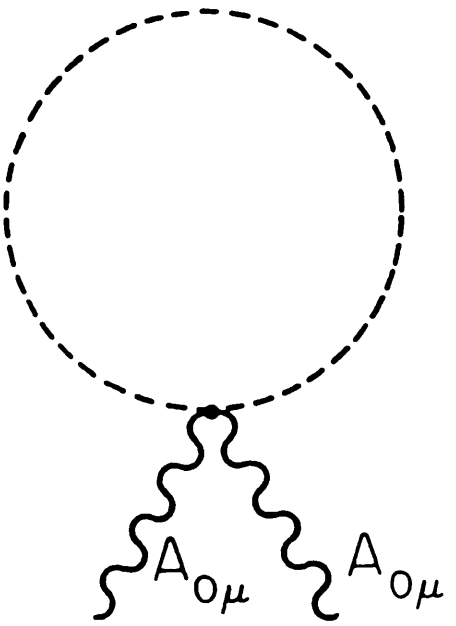

Figure 1.2: *Feynman graph for the tadpole correction to the gauge field mass.*

There is no easy way to resolve these problems. In order to overcome these complications, Copeland et al. [15] restrict their calculation to the regime where $g \gg e^2$, in which the gauge field contributions cannot be large. (The qualitative details of their discussion do not appear to change when a gauge mass is introduced.) Terms of order e^2T^2 are then constrained by $e^2\eta^2(e^2m_o^2/g)$ and the vector mass is approximately unchanged. At the same time the vector loop gives a small contribution to the effective scalar mass. The effect is to replace $\overline{I}$ of (1.42) by

$$\overline{I}[\phi_o, A_{o_\mu}] = -\int d^3x \mathcal{L}(\phi_o(x), A_{o_\mu}(x)) \tag{1.43}$$

where $\mathcal{L}$ is derived from (1.32) by: i) going Euclidean, ii) removing Euclidean time dependence, iii) deleting the massive modes, iv) replacing m_o^2 by $m_o^2(1 - \frac{T^2}{T_c^2})$, and v) implementing gauge fixing.

Thus, at high temperatures and working to order g, the heavy modes can be integrated out to obtain a spatially local effective potential for the massless modes.

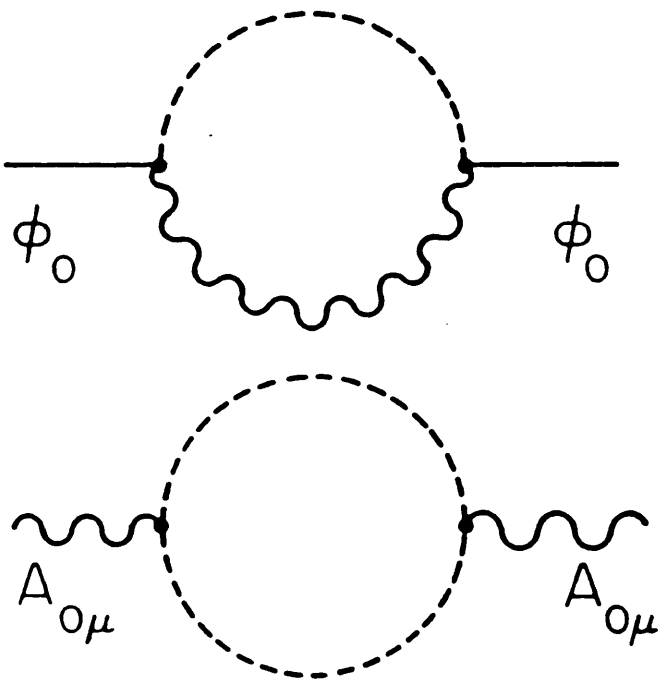

Figure 1.3: *The remaining Feynman graphs of* $O(e^2)$.

The result is that

$$Z = \int \mathcal{D}\phi_o \mathcal{D}A_{o_\mu}\, e^{-\beta \bar{I}[\phi_o, A_{o_\mu}]} \tag{1.44}$$

where in the covariant gauge

$$\begin{aligned} \bar{I}[\phi_o, A_{o_\mu}] &= \int d^3x \left[\frac{1}{4} F_{o_{ij}} F_o^{ij} - \frac{1}{2} \left(\partial_i \phi_o\right) \left(\partial^i \phi_o^\star\right) \right. \\ &- \frac{1}{2} \imath e A_{o_i} \left[\phi_o \partial^i \phi_o^\star - \phi_o^\star \partial^i \phi_o\right] - \frac{1}{2} m_o^2 \left(1 - \frac{T^2}{T_c^2}\right) |\phi_o|^2 \\ &- \left. \frac{1}{2} e^2 |\phi_o|^2 A_{o_i} A_o^i + \frac{g}{4!} |\phi_o|^4 + \frac{1}{\xi} (\partial_i A_o^i)^2 \right] \end{aligned} \tag{1.45}$$

and the term $\frac{1}{\xi}(\partial_i A_o^i)$ describes the gauge fixing.

To evaluate the partition function (1.44), Copeland et al. [15] employ the saddle point method. The dominant contributions to the functional integral come from the field configurations that satisfy the stationary equations:

$$\frac{\delta I}{\delta \phi} \Big|_{\phi=\phi_{saddle}, A^\mu = A^\mu_{saddle}} = 0, \qquad \frac{\delta I}{\delta A_\mu} \Big|_{\phi=\phi_{saddle}, A^\mu = A^\mu_{saddle}} = 0 \tag{1.46}$$

that is, from the field configurations that satisfy the equations of motion:

$$\left.\begin{aligned} \partial^i F_{ji} &= \tfrac{1}{2} \imath e (\phi^* \partial_j \phi - \phi \partial_j \phi^*) - e^2 A_j |\phi|^2 \\ \left(\partial_i + \imath e A_i\right)^2 \phi &= -m_o^2 \left(1 - \tfrac{T^2}{T_c^2}\right) \phi + \tfrac{g}{3!} |\phi|^2 \phi \end{aligned}\right\} \tag{1.47}$$

The contribution of any solution of these equations to the partition function can be found by substitution into (1.44). The solution ϕ =constant, $A = 0$ is the minimum energy solution and therefore gives the maximum contribution. The approach in the early 70s to the evaluation of (1.44) [13] was to assume that all other field configurations make a negligible contribution to the functional allowing the partition function to be written as

$$Z_{MF} = \exp\left[\beta V \left(\frac{1}{2} m_s^2(T) < \phi >^2 - \frac{g}{4!} < \phi >^4\right)\right] \tag{1.48}$$

where

$$m_s^2(T) = m_o^2 \left(1 - \frac{T^2}{T_c^2}\right) \quad \text{and} \quad < \phi >^2 = \frac{6 m_s^2}{g} \tag{1.49}$$

to order $g, e^2, \frac{e^2T}{m_s^2}$ and assuming that $T \leq T_c$. From this it can be seen that the system undergoes a second-order phase transition at a temperature T_c (as the order parameter $<\phi>$ decreases smoothly to zero, reaching zero at T_c).

As pointed out by Copeland et al. [15] this mean field approximation is rather crude. It assumes that the only important configuration near the critical point is the one of uniform density, whereas we know that there are large fluctuations in $<\phi>$ and $<A>$ near the critical temperature. In principle one should find all the maxima of the functional and sum their various contributions to Z. Copeland et al. [15] do not do this but instead they assume (ad hoc) that string-like field configurations make the dominant contribution to the partition function and that the contribution from other non-constant field configurations can be neglected. This assumption does not appear to be as bad as it sounds. In fact, the independent vortex model of the λ-transition in superfluid helium includes the same assumptions (with no further justification) and yet gives a good quantitative agreement with experiment.

The simplest string solution to the saddle point equations is an infinite straight string running, for example, along the z-axis. This can be expressed as [16]:

$$\left.\begin{array}{rcl} \phi & = & |\phi(r)|\, e^{i\theta} \\ \underline{A} & = & \frac{r \wedge \underline{k}}{r} |\underline{A}(r)| \end{array}\right\} \tag{1.50}$$

where $\underline{k}$ is a unit vector in the z-direction and θ is an arbitrary phase denoting the winding around the minimum of the potential. Imposing the gauge conditions $A_{o,o}(x) = 0$, $\underline{\nabla}.\underline{A}_o(x) = 0$, and substituting (1.50) into (1.47) Copeland et al. [15] obtain that

$$\left.\begin{array}{rcl} -\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}|\phi_o|\right) + \left[\left(\frac{1}{r} - e|\underline{A}_o|\right)^2 - m_o^2\left(1 - \frac{T^2}{T_c^2}\right) + \frac{g}{3!}|\phi_o|^2\right]|\phi_o| & = & 0 \\ -\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r|\underline{A}_o|\right)\right) - e\frac{1}{r}|\phi_o|^2 + e^2|\underline{A}_o||\phi_o|^2 & = & 0 \end{array}\right\} \tag{1.51}$$

The solutions to the equations of motion (1.51) are shown schematically in Figure 1.4. At large distances from the string

$$\lim_{r\to\infty} |\phi(r)| \to \eta(T) \tag{1.52}$$

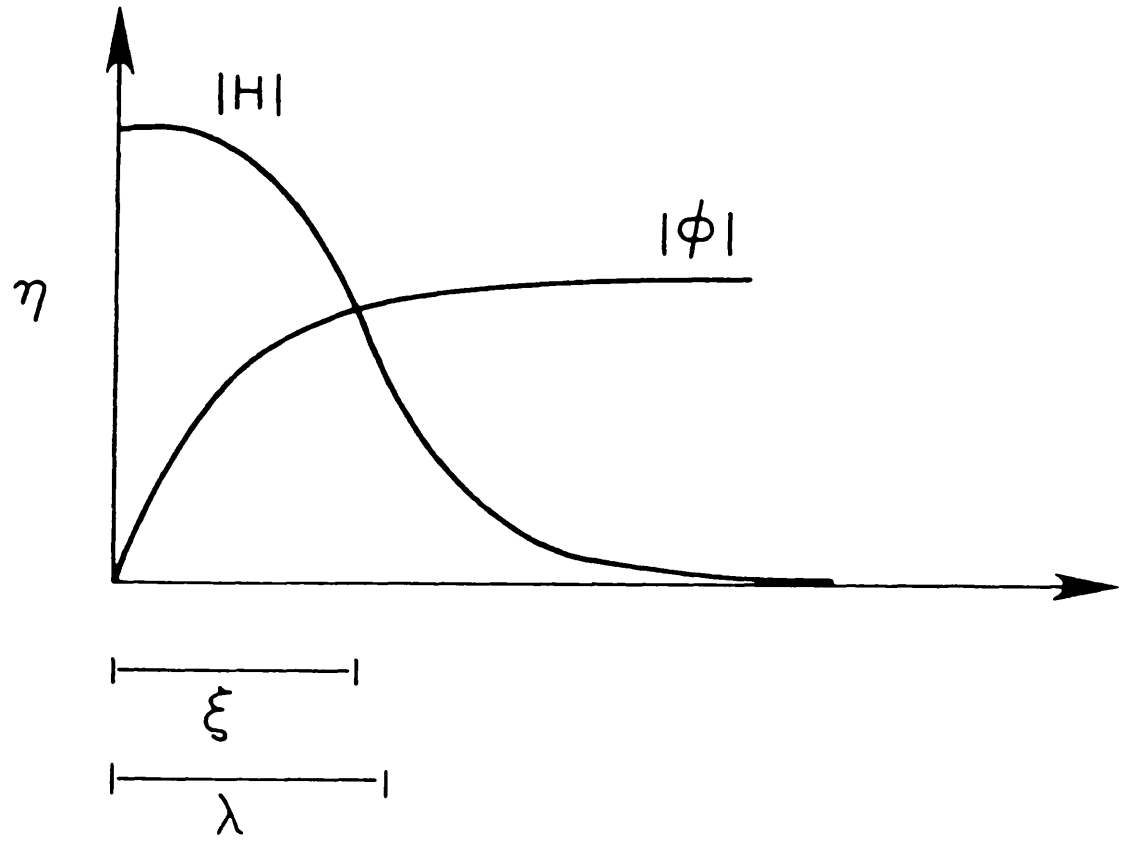

Figure 1.4: *An example of the field configurations for a vortex solution at a temperature T.*

$$\xi = O(m_s^{-1}) \qquad \lambda = O(m_v^{-1})$$

where

$$\eta^2(T) = \frac{6m_s^2(T)}{g} \tag{1.53}$$

At the core $\mid \phi \mid$ vanishes. The thickness of the core is determined by m_s^{-1}, the Compton wavelength of the Higgs particle. The magnetic field is restricted to the core, the skin depth being determined by m_v^{-1}, the inverse of the vector mass, m_v

$$m_v(T) = e\eta(T) = \sqrt{\frac{6}{g}} e m_s(T) \tag{1.54}$$

The energy per unit length of the vortex has two separate components, one due to the scalar field

$$\sigma_s(T) = O(\eta^2(T)) \tag{1.55}$$

and the other due to the vector field

$$\sigma_v(T) = O\left(\frac{e^2\eta^4}{m_v^2}\right) = O(\eta^2(T)) \tag{1.56}$$

Note that if an additional $\underline{A}_o$ mass term of the form $\Delta\mu^2 = \alpha e^2 T^2$ had been included, its effect would be to replace σ_v of (1.56) by

$$\sigma_v(T) = O\left(\frac{\eta^4(T)}{\eta^2(T) + \alpha T^2}\right) \tag{1.57}$$

Increasing the temperature causes σ_s and σ_v to decrease, reaching zero at $T = T_c$. Furthermore, the energy per unit length of the string is reduced (this would be true even if the $\Delta\mu^2$ correction had been included) and its width, m_s^{-1}, is increased.

We shall now return to the evaluation of the contribution of the strings to the partition function (1.44). Copeland et al. [15] assume that the strings are curved smoothly so that any segment of length of order the width is straight. In this case, the strings are, to a good approximation, also solutions to (1.51), and have an energy per unit length approximately equal to that of a completely straight string. The total partition function, Z, can be written as

$$Z = \exp(Z_1) \tag{1.58}$$

where

$$Z_1 = \sum_L \Omega(L)\, e^{-\beta E(L)} \tag{1.59}$$

The interaction energy of the strings is neglected when the strings are more than a distance $a(T)$ (a=the width of the string) apart, and is included as an infinite repulsive/attractive force when the strings are within a distance $a(T)$[2]. This is done by restricting the strings to be non-self-intersecting random walks[3].

As the interest lies in the effect of topologically stable strings on the phase transition, the strings must be either in the form of loops or infinite in length (in the absence of monopoles). For the case of loops, the analysis of Section 1.1 applies. For infinite strings the canonical ensemble cannot be used to derive their statistical properties [25, 30]; the more fundamental microcanonical ensemble has to be used (see [10, 25, 30]).

The partition function $Z_{\rm loop}$ diverges at temperatures greater than T_{st} (the temperature at which $\sigma_{eff} = 0$). This temperature is given by

$$T_{st} = \frac{\sigma(T_{st})a(T_{st})}{\ln z} \approx \gamma\eta^2(T_{st})m^{-1}(T_{st}) \tag{1.60}$$

for some $\gamma = O(1)$ and $m = \min(m_s, m_v)$ the inverse width of the string (in our case, $g \gg e^2$ and so $m = m_v$). Since the right hand side of (1.60) vanishes at $T = T_c$, it follows that

$$T_{st} < T_c \tag{1.61}$$

as it must be. The difference between T_{st} and T_c is small. Explicitly,

$$1 - \frac{T_{st}^2}{T_c^2} = O(e^2) \qquad (\text{for } m = m_v) \tag{1.62}$$

[2]This is effectively what occurs in the numerical simulations. When two strings come within a distance a of each other, they intercommute. Effectively, the exponential force of repulsion is being replaced by a step function type force which acts over a distance a. More sophisticated forces will be discussed in later Chapters.

[3]Modelling the strings by Brownian walks would correspond to neglecting string-string interactions completely.

The width of the strings (which is of the same order as their mean separation) can also be calculated at this temperature. By substitution

$$\left.\begin{array}{rcl} m_s(T_{st}) & = & O(em_s(T=0)) \\ m_v(T_{st}) & = & O(em_v(T=0)) \end{array}\right\} \tag{1.63}$$

for $m = m_v$. That is, at the phase transition, the separation of the centres of the flux tubes is scaled up by a factor of $O(1/e)$ as compared to the closest packing of cold strings. Note that, unlike the case of fundamental strings, the 'composite' strings considered here have a finite width. Hence, there is no need to impose a lower cut-off on the scale of bending as this is provided naturally by the string width.

Finally, we shall look at the interpretation of T_{st}. It only makes sense to discuss string solutions when fluctuations that could remove them are improbable. The free energy associated with such a fluctuation is

$$(\Delta f)a^3 \approx \frac{\eta^2(T)}{m_v} \tag{1.64}$$

where a is a measure of the mean string separation. This fluctuation has a high probability when the free energy required is substantially less than the thermal energy T. The two are equal when

$$T \sim \eta^2(T)m_v^{-1} \tag{1.65}$$

i.e. when $T \sim T_{st}$. Above T_{st} there are large fluctuations in the ϕ field and it is no longer appropriate to describe the field in terms of string-like structures. Thus, T_{st} is interpreted as the string formation temperature, as opposed to the limiting temperature obtained in the case of fundamental strings. Above T_{st} the system is in a phase for which strings cannot exist. At T_{st} the string loops have a scale invariant distribution (see (1.14)), and below T_{st} it is thermodynamically more favourable to have small loops.

From a cosmological point of view a more interesting question to address is "what happens as the universe cools through T_{st}?". The calculations of Copeland et al. [15], presented in this section, are for flat space, but still they seem to suggest that we

would be unable to recognize any string configurations when the expanding universe was very hot. As it cools through T_{st} strings would be formed. Initially most of the string length would be in infinite strings. As the universe cooled further strings would quickly break up and then rejoin to form loops of the smallest possible sizes. Eventually the strings would no longer be able to chop themselves up fast enough to remain in thermal equilibrium. Still, most of the string length is expected to be in the form of small loops. This picture seems to be confirmed by the simulations of Albrecht and Turok [31] as well as by those of Bennett and Bouchett [32]. Furthermore, if correct, it makes the string domination scenario of Kibble [33] and Bennett [34] seem unlikely.

1.3 Discussion

In this chapter we have seen that the most notable characteristic of a system containing free fundamental strings is that it has a finite maximum temperature. In four space-time dimensions this limiting temperature is reached while the energy of the system is still dominated by finite loops. The system is unstable to the formation of infinite strings at high energy densities.

The existence of a limiting temperature has interesting consequences. If string interactions for fundamental strings can be ignored at high densities, the temperature T_{st} would be the maximum equilibrium temperature of a system containing fundamental strings. This might have interesting cosmological consequences [35, 36].

Furthermore, we pesented a model which was developed by Copeland et. al. [15] who looked at 'composite' strings and, in particular, at Nielsen-Olesen type vortices. We demonstrated how these strings develop an effective temperature dependent tension, σ_{eff}, and a temperature dependent width. We then went on to examine the contribution of these string solutions to the partition function of a scalar gauge theory. The result was that the partition function diverges at a temperature T_{st} which

is just below that found using the mean field approach. The model pictures the restoration of symmetry as being due to overlapping strings filling the whole space.

One might be worried about the use of the canonical ensemble in a description of a phase transition because there are large fluctuations in some of the thermodynamic quantities. For example, the mean energy density due to loops of string is finite right up to the phase transition. However, the r.m.s. fluctuations in this quantity diverge like $(T-T_{st})^{-1/2}$ for Brownian strings. Therefore, the mean energy density in loops is not a sensible quantity to discuss at temperatures close to T_{st}. An interesting result is obtained when the partition function is evaluated for an arbitrary number of space dimensions d. For the case of Brownian strings in d space dimensions, $q = d/2$ [19] in (1.29). Hence, when evalauting C_v, the condition for it to remain finite in the limit $T \rightarrow T_{st}$ is that $d \geq 4$, (*i.e.* for strings in four space dimensions and above the fluctuations in the energy do not diverge in the canonical ensemble). Of course the actual value of T_{st} also depends on the number of dimensions. However, even in four dimensions not all interesting quantities have such large r.m.s. fluctuations about their mean values. For example, the r.m.s. fluctuations in the mean number of loops of size nl, $(R(nl))$, is proportional to $R(nl)^{1/2}$ which means that $R(nl)$ can be sensibly discussed even at temperatures very close to T_{st}. The point to note is that if one uses the canonical ensemble to describe a phase transition, one must check that the mean quantities discussed do not have large fluctuations about their mean values.

It is important to distinguish between the meaning of T_{st} for systems containing fundamental strings and those containing 'composite' strings. For the former it represents a maximum temperature for the system whilst for the latter it describes the maximum temperature at which it is sensible to discuss string-like field configurations. At temperatures above T_{st}, the equilibrium statistical properties of the field theory can be discussed but not in terms of strings. In this case T_{st} represents the string formation temperature. It is interesting to note that $T_{st} = O(T_{\text{Ginzburg}})$ the

string formation temperature suggested by Kibble [10].

From the partition function for the vortex solutions we examined the equilibrium distribution of strings. As T_{st} is approached most of the string length goes into infinite strings and the distribution of loops is scale invariant. Both string trajectories are approximately Brownian, with them being exactly Brownian if string-string interactions are completely neglected. This agrees with the results obtained by Mitchell and Turok [19], who used a phase space calculation to evaluate the string network configuration, as well as with the computer simulations of Vilenkin and Smith [23] and Vilenkin and Vachaspati [37]. Furthermore, a recent simulation of string formation in two dimensions has confirmed that the trajectories are only approximately Brownian [38]. At low temperatures the string network is no longer in thermal equilibrium, and most of the string is in the form of loops with an exponentially suppressed distribution of large loop sizes. Again, this is in agreement with [19] and the simulations [23, 37]. It also makes the string domination scenario of Kibble [33] and Bennett [34] seem unlikely, but only detailed simulations of string dynamics in an expanding universe could rule it out altogether. Hodges has recently numerically analysed the distribution of global strings as they are formed in a second-order phase transition. He takes the strings to be in an expanding universe and finds that at equilibrium, for a horizon size ten times larger than the correlation length at the Ginzburg temperature, a small fraction of the string length is in infinite strings [39]. This appears to be in contradiction with the results of Vilenkin and Smith [23]. However, Hodges allows many time steps before looking at the string distribution, whereas the authors of [23] immediately freeze the Higgs fields and then obtain the string distribution. The difference between the two methods lies in the time and length scales that are used in the problem. The result obtained, that below T_{st} the smallest loops dominate the partition function, still appears to hold.

The values of the masses m_s and m_v at T_{st} are important in determining the rele-

vant length scales at the phase transition. Copeland et al. [15] demonstrate that both masses become scaled down by a factor of $O(e)$ relative to those at zero temperature. This means that at the phase transition the network of strings has the separation of the centres of the flux tubes scaled up by a factor of $O(1/e)$ over that of cold strings. This is just the Ginzburg length scale that was used by Kibble [10] to determine the initial scale for the distribution of the Higgs fields at the transition. Hence the results of Copeland et al. [15] appear to be in agreement Kibble's estimates of initial string densities etc.

As we have seen, at high temperatures Copeland et al. [15] investigate τ independent solutions for the fields. This is sensible because in this regime the width of the string, m_s^{-1}, is much greater than β and so, to a good approximation, the variation of the fields over the interval β can be taken to be zero. However, at low temperatures neglecting the τ dependence of the solutions is no longer justifiable. It is not clear what the interpretation of these τ dependent solutions would be in terms of real time t. This is because of the difficulty in performing a Wick rotation in order to return to real time.

In this chapter, we also looked at the effect of eliminating self-intersection from the string trajectories. At high densities this improvement is irrelevant because non-self-intersecting walks are well approximated by Brownian walks [40]. The reason is that the excluded volume of the walk becomes essentially homogeneous at high densities and so, to a good approximation, there is an equal probability of a string going in any direction. However, at lower densities the difference between neglecting string interactions completely and incorporating them to some degree by modelling the string trajectories by non-self-intersecting random walks becomes more noticable. Non-self-intersecting walks have been investigated in polymer physics (see Wiegel [8] and references therein). For example, at low densities they find that $q = 11/4$ in (1.29).

In this discussion we have concentrated on closed string configurations. We could have extended it to include open strings. As pointed out by Copeland et al. [15] the result would then have been that the number of string configurations is maximized by having an exponentially suppressed distribution of open strings (*i.e.* the number of long open strings is exponentially suppressed). In a GUT model, where the monopole mass is of the same order of magnitude as the string tension, this would be a good estimate for the distribution of monopoles connected by strings in the early universe. This result is in good agreement with numerical simulations of such a transition [41].

In their analysis of 'composite' strings, Copeland et al. [15] looked at cosmic strings and, in particular, at local rather than global strings. It is the former type which are expected to be important in galaxy forming scenarios. With this in mind, it is features like the scaling at high temperatures and the exponential suppression of large loops at low temperatures which are of most interest.

As mentioned in the beginning of this chapter, there are many types of string mediated phase transitions which can be described using the model developed by Copeland et al. [15]. Of course, not all of these strings behave like topological defects in systems with local symmetry breaking. For example, superconducting flux-lines and 'odd' lines in amorphous materials do, whilst superfluid vortices and crystal dislocations behave like topological defects in systems with global symmetry breaking.

However, whatever the type of string, the calculations presented in this chapter predict that all phase transitions due to the proliferation of string-like structures are *second-order*. We know from experiment that this is not the case. For example, melting is a first-order phase transition. Furthermore, a laboratory system that undergoes an analogous phase transition to the restoration of a spontaneously broken U(1) gauge symmetry is the superconductor. The mean field approach predicts that the phase transition from the superconducting to the normal conducting state is second-order. However, in practice it is probably weakly first-order, the first-order transition being

induced by the large fluctuations present near the phase transition.

Obviously, the model presented in this chapter describing phase transitions due to string-like structures is just too simple. For example, the posssibility of the strings being rigid or antirigid has been ignored. QCD flux tubes most likely possess rigidity [42] and the same appears to be true for cosmic strings [43]. In addition, self-interactions of the string as well as mutual interactions between strings have been neglected. In the subsequent chapters, we shall try to improve the model by including such effects. We shall see how, for some systems, the possibility of a *first-order* phase transition can arise.

Chapter 2

The Statistical Properties of Rigid Strings

In the previous chapter, we reviewed a simple model of string mediated phase transitions [15, 19]. In this model it was assumed that the strings are characterized by: i) a constant intrinsic energy per unit length *independent* of the configuration of the string, ii) a constant scale a below which the string cannot bend, and iii) a negligible string interaction energy. With these assumptions, the conclusion was that all string phase transitions are second-order and share the same critical exponents. Experimentally, we know that this is not the case. The discrepancies between the model and experiment are due to its naivety. By taking into account the varying nature of the strings it is possible to produce a more accurate model. Specifically, we shall, in this chapter[1], relax the first assumption and allow the strings to be rigid (or antirigid). This is necessary because it has been suggested that cosmic strings incur significant energy changes on bending [43], and that QCD flux tubes most likely possess rigidity [42]. In later chapters we shall relax the third assumption by including self-interactions of the strings as well as mutual interactions between strings.

In general, we would expect stiffness to have a significant effect on strings. Previous calculations [42, 43] have shown that in a string theory the Nambu action has

[1]The material of this chapter appears in 'String Mediated Phase Transitions', Physica A158,460(1989), and in 'The Statistical Properties of Rigid Strings', Imperial College preprint TP/89-90/04, to be published by Physics Letters.

geometrical corrections which are just the rigidity (the bending) of the string. These terms, although usually small as compared to the Nambu action, can be important in regimes where the radius of curvature becomes of order the string width. In fact, these terms can be shown to prevent the formation of 'cusps' in strings. Strings with rigidity have a natural length scale which is different from that of strings without rigidity. It has been suggested [25] that it is this length scale, and in particular the distance between the kinks of the string, which determines the size of the loops which can break off from long strings. The shorter this distance, the smaller the loops and the faster they radiate away.

In this chapter, we shall see that at low temperatures the effect of rigidity is to make the strings straighter. Furthermore, we shall see how, at high temperatures, the effects of rigidity are rather different for 'fundamental' and 'composite' strings. In particular, we shall see how the temperature T_{st} can be dramatically altered for 'fundamental' strings and how, in the case of cosmic strings (which are 'composite'), T_{st} remains essentially unchanged, even when significant stiffness is present.

2.1 The thermodynamics of rigid strings

The energy of a single rigid string of length L in a configuration C can be written as

$$E(L,C) = \sigma L + \epsilon(L,C) \tag{2.1}$$

where $\epsilon(L,C)$ is the energy of the string configuration due to its stiffness. If $N(L,C)$ is the number of string configurations with energy $E(L,C)$ then the partition function for a system consisting of a single string of arbitrary length, in thermal equilibrium with a heat bath at temperature $T = 1/\beta$ (in units where $k_B = 1$), is

$$Z_1 = \sum_L \zeta(L)\, e^{-\beta\sigma L} \tag{2.2}$$

where

$$\zeta(L) = \sum_C N(L,C)\, e^{-\beta\epsilon(L,C)} \tag{2.3}$$

In (2.3), $\zeta(L)$ is the conformational partition function for strings of length L and the sum is over all possible string configurations.

In order to simplify the evaluation of $\zeta(L)$ we make two approximations. Firstly, we approximate the string to a polymer of hinged straight segments of length a, and secondly we restrict the string segments to lie along the edges of a cubic lattice of lattice spacing a. Shortly, we shall relax the second condition. To start with we shall also assume that the energy required to bend a string through $0°, 90°$ and $180°$ is $0, u$, and ∞ (*i.e.* no backtracking) respectively. With these assumptions it is easy to show that

$$\zeta(L) = \sum_{n=0}^{\infty} N(L,n) e^{-\beta n u} \tag{2.4}$$

where $N(L,n)$ is the number of configurations of a string of length L, with $(n+1)$ straight segments ($0 \leq n \leq \frac{L}{a} - 1$). That is, for open strings

$$N(L,n) = \left(\frac{V}{a^3}\right) \frac{\left(\frac{L}{a}-1\right)! h^n}{n!\left(\frac{L}{a}-n-1\right)!} \theta\left(\frac{L}{a}-n-1\right) \tag{2.5}$$

where V is the volume of the system, $h = 2d-2$ and d is the dimension of the lattice (for a cubic lattice $d = 3$). Substituting (2.5) into (2.4) and performing the sum, ζ is found to be

$$\zeta(L) = \frac{V}{a^3}(1 + h e^{-\beta u})^{L/a-1} \tag{2.6}$$

We shall now lift the restriction on the angle that each segment of the string is allowed to make with its neighbour. Firstly, we shall consider the case of a string that is restricted to lie on a two dimensional triangular lattice, for which three bending angles are possible. For such a lattice let p_1, p_2 and p_3 be the probability of a string bending through an angle of $60°, 120°$ and $180°$ respectively, and let the energy cost associated with these bends be u_1, u_2 and u_3. If there is a total of n bends in the string of which k are of type 1, j are of type 2 and $(n-k-j)$ are of type 3 then

$$N(n,k,j) = \left(\frac{V}{a^3}\right) \frac{\left(\frac{L}{a}-1\right)! h^n}{n!\left(\frac{L}{a}-n-1\right)!} \frac{n!}{k!j!(n-k-j)!} p_1^k p_2^j p_3^{n-k-j} \theta\left(\frac{L}{a}-n-1\right) \tag{2.7}$$

and therefore

$$\zeta(L) = \sum_{n,k,j} N(n,k,j) e^{-\beta u_1 k} e^{-\beta u_2 j} e^{-\beta u_3 (n-k-j)} \tag{2.8}$$

$$= \left(\frac{V}{a^3}\right) \left[1 + h\left(p_1 e^{-\beta u_1} + p_2 e^{-\beta u_2} + p_3 e^{-\beta u_3}\right)\right]^{L/a-1} \tag{2.9}$$

The results (2.6) and (2.9) permit a generalization to the case where adjacent segments make arbitrary angles with each other. For such string configurations

$$\zeta(L) = \left(\frac{V}{a^3}\right) \left[1 + h \int d\theta\, p(\theta)\, e^{-\beta u(\theta)}\right]^{L/a-1} \tag{2.10}$$

where $p(\theta)$ is the probability (at $\beta = 0$) of a segment of string bending through an angle θ and $u(\theta)$ is the energy cost to do so. As the strings are no longer on a lattice, $h = 2d - 2$ where $2d$ is no longer the lattice coordination number, but instead an *effective* coordination number which need not be specified here. If we neglect string-string interactions, we expect, in three dimensions, that

$$p(\theta) = \frac{1}{2} \sin\theta \tag{2.11}$$

Furthermore, it has been shown, for example, that bending a segment of cosmic string to form an arc of radius R, where $R \gg a$, costs an energy u

$$|u| = O\left(\sigma a \left(\frac{a}{R}\right)^2\right) \tag{2.12}$$

(there is still some controversy over the sign of u [42, 43, 46]). In this case we expect, for small θ, that

$$|u(\theta)| = 4\kappa\sigma a \tan^2\left(\frac{\theta}{2}\right) \tag{2.13}$$

$$\simeq \kappa\sigma a\theta^2, \quad \text{where} \quad \kappa \simeq O(1) \tag{2.14}$$

From (2.4) we can evaluate the mean number of folds per unit length, $n(T)$, for an open string of length L at a temperature T. We have that

$$n(T) = \frac{1}{L} \frac{\sum_n N(L,n) n e^{-\beta n u}}{\zeta(L)} \tag{2.15}$$

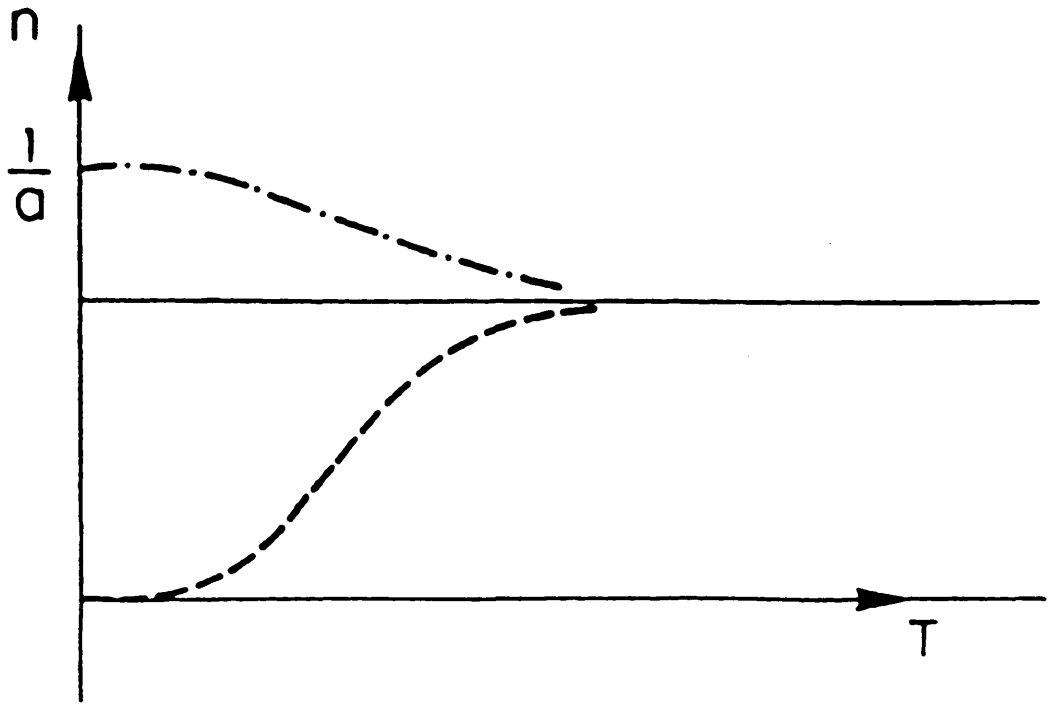

Figure 2.1: *The number of folds/unit length, n, as a function of temperature for a stiff string. The dashed, solid, and dot-dashed lines correspond to $u > 0$, $u = 0$, $u < 0$, respectively.*

$$= -\frac{1}{\beta L}\frac{d}{du}\left[\ln \zeta\left(L\right)\right] \tag{2.16}$$

$$= \frac{\left(\frac{L}{a}-1\right)}{L}\frac{h\int d\theta p(\theta)e^{-\beta u(\theta)}}{\left(1+h\int d\theta p(\theta)e^{-\beta u(\theta)}\right)} \tag{2.17}$$

Therefore, for large L a string will, on average, be straight on scales up to

$$l(T) \simeq a\left[\left(h\int d\theta p(\theta)e^{-\beta u(\theta)}\right)^{-1}+1\right] \tag{2.18}$$

The variation of $n(T)$ with temperature is shown qualitatively in Figure 2.1. As would be expected, for low temperatures and positive rigidity ($u > 0$) the string has very few folds per unit length. Specifically, on substituting (2.11) and (2.14) into (2.18) we see that

$$l(T) \sim a\left(\frac{4\beta\kappa\sigma a}{h}+1\right) \tag{2.19}$$

Let us now restrict our discussion to the phase transition in the early universe where cosmic strings are thought to have been formed. The simplest candidates for cosmic strings are Nielsen-Olesen vortices [16]. At low temperatures, such strings are characterised by

$$\sigma \sim \frac{m_o^2}{g} \tag{2.20}$$

$$a \sim \frac{\sqrt{g}}{em_o} \tag{2.21}$$

(assuming that $e^2 \ll g \ll 1$) where we have used (1.49) and (1.53-1.55) in the limit $T \to 0$. Here m_o is the zero temperature scalar mass, g is the scalar self coupling and e is the gauge coupling. Thus, for rigid cosmic strings at low temperatures we have that

$$l(T) \sim \beta\sigma a^2 \sim \beta\frac{1}{e^2} \tag{2.22}$$

which implies that such strings in thermal equilibrium are typically very straight. This is in contrast with the result obtained for antirigid strings ($u < 0$), where, at low temperatures, the number of folds is maximized.

At low temperatures, the leading factor in determining the most probable string configuration is the energy associated with a configuration. Whereas, at high temperatures, it is no longer the energy that is the dominant factor in determining the string configuration but the entropy. The most probable configuration is that which is most numerous. Hence, the difference in the distribution of rigid and antirigid strings in thermal equilibrium becomes less as the temperature is increased.

For the systems that we are considering the length and number of strings is arbitrary. String interactions enable a string to split up and then rejoin to form smaller loops. Therefore, we now need to lift the restrictions on the length and number of strings. This is trivial provided string interactions are weak. The partition function when the number of strings is arbitrary is [15]

$$Z = \exp Z_1 \tag{2.23}$$

(see (1.8)).

So far we have assumed that the strings are open. At finite energies (and in the large V limit) cosmic strings, vortex lines and dislocation lines are closed. This reduces the number of configurations $N(L,C)$. In the absence of rigidity (and assuming a free random walk with $L \gg a$), the fraction of non-orientable string configurations of length L which are closed is given by

$$f(L,u=0) = \frac{A}{2}\left(\frac{L}{a}\right)^{-d/2-1} \tag{2.24}$$

where A is a constant (see (1.3)). The effect of rigidity is to change the normalization of this function but *not* the L dependence, (see Ziman [47]) so that

$$f(L,u\neq 0) = \frac{A(\beta)}{2}\left(\frac{L}{a}\right)^{-d/2-1} \tag{2.25}$$

It follows that

$$Z_{1,\text{closed}} = \frac{A(\beta)}{2}\sum_{L/a=1}^{\infty}\left(\frac{L}{a}\right)^{-d/2-1}\zeta(L)\,e^{-\beta\sigma L} \tag{2.26}$$

$$= \frac{A(\beta)V}{2a^3}\sum_{L/a=1}^{\infty}\left(\frac{L}{a}\right)^{-d/2-1}e^{-\beta L\sigma_{eff}(\beta,L)} \tag{2.27}$$

where

$$\sigma_{eff} = \sigma - \frac{\eta(\beta)}{\beta a} \tag{2.28}$$

and

$$\eta(\beta) = \ln\left(1 + h\int d\theta\, p(\theta)e^{-\beta u(\theta)}\right). \tag{2.29}$$

The variation of σ_{eff} with temperature is shown in Figure 2.2. Note that we have assumed that $L \gg a$ and then extrapolated the power behaviour of (2.25) down to short lengths. To the accuracy at which we are working, there is no loss in doing so.

In (2.27), σ_{eff} plays the role of an effective string tension for the string. It is the vanishing of σ_{eff} that causes the phase transition to occur. As is shown in Figure 2.2, the effect of rigidity (antirigidity) is to increase (decrease) the phase transition temperature, T_{st} (which, from our previous comments, is unaffected by

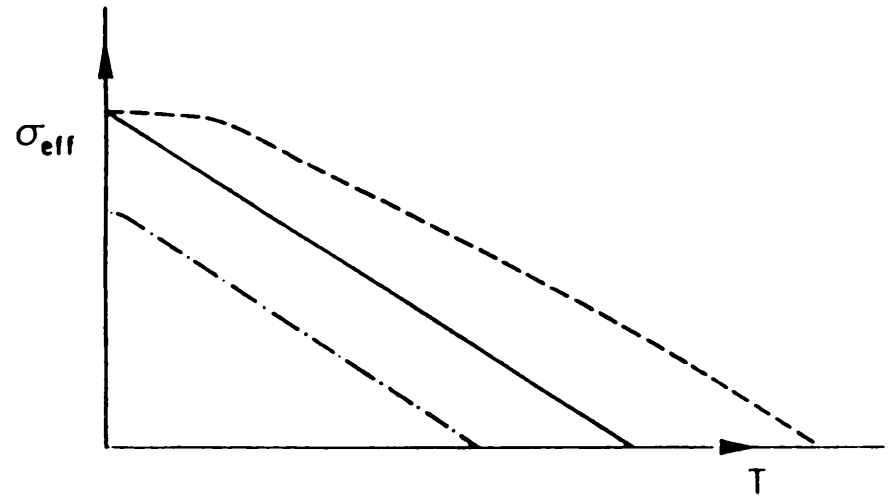

Figure 2.2: *The variation of σ_{eff} with temperature. The dashed, solid, and dot-dashed lines correspond to $u > 0$, $u = 0$, $u < 0$, respectively.*

closure). Explicitly, if, for simplicity, we only allow the string to bend through 90° we see that, at T_{st},

$$e^{\beta_{st}\sigma a} = 1 + he^{-\beta_{st}u} \tag{2.30}$$

For small $\beta_{st}u$, the effect of stiffness is to change the phase transition temperature by a fractional amount

$$\frac{\Delta T}{T_{st}} \approx \frac{|u|h}{(1+h)\sigma a} \tag{2.31}$$

As might be anticipated, for large $u \sim \sigma a$, the effect of rigidity is dramatic, giving $\Delta T/T_{st} \sim O(1)$.

For cosmic strings we saw, in (2.12), that $u \sim \sigma a$. Therefore, it would appear, a priori, that rigidity plays an important role and hence that the results of Copeland et al. [15], who neglect string stiffness, are incorrect. However, cosmic strings are composite and so, treated as a static three dimensional ensemble, the string parameters

are temperature dependent. For temperatures $T \sim T_{st}$ [15]

$$\sigma \sim \frac{m_o^2}{g}\left(1-\frac{T^2}{T_c^2}\right) \tag{2.32}$$

$$a \sim \frac{\sqrt{g}}{em_o}\left(1-\frac{T^2}{T_c^2}\right)^{-1/2} \tag{2.33}$$

(see (1.49) and (1.53-1.55)). As a result of this temperature dependence $\beta_{st}\sigma a$ is *small* near the original (non-rigid) string phase transition. We can then rewrite (2.30) as

$$\beta a(T_{st})\sigma(T_{st}) = he^{-k\beta a(T_{st})\sigma(T_{st})} \tag{2.34}$$

for some constant k. Defining c to be the solution to the equation $c = he^{-kc}$, we see that

$$\sigma^2(T_{st})a^2(T_{st}) = T_{st}^2c^2 \tag{2.35}$$

and thus $T_{st} < T_c$ for cosmic strings. More explicitly, using (2.32) and (2.33) we find that

$$\left(1-\frac{T_{st}^2}{T_c^2}\right) = O(c^2e^2) \tag{2.36}$$

However, c depends very weakly on T, e^2 and g and is of $O(1)$. Hence we have that

$$\left(1-\frac{T_{st}^2}{T_c^2}\right) = O(e^2) \tag{2.37}$$

which is the same result as that obtained in the absence of rigidity (see (1.62)).

As for free strings, we observe that the mean number of loops is Z_1 (see (1.13)) and that the mean number of loops of length L is

$$R(L) \propto L^{-q-1}\, e^{-\beta L\sigma_{eff}} \tag{2.38}$$

(see (1.14)). For temperatures $T < T_{st}$ long strings are again exponentially suppressed. For temperatures close to T_{st}, $\sigma_{eff} \approx 0$ and

$$R(L) \propto L^{-q-1} \tag{2.39}$$

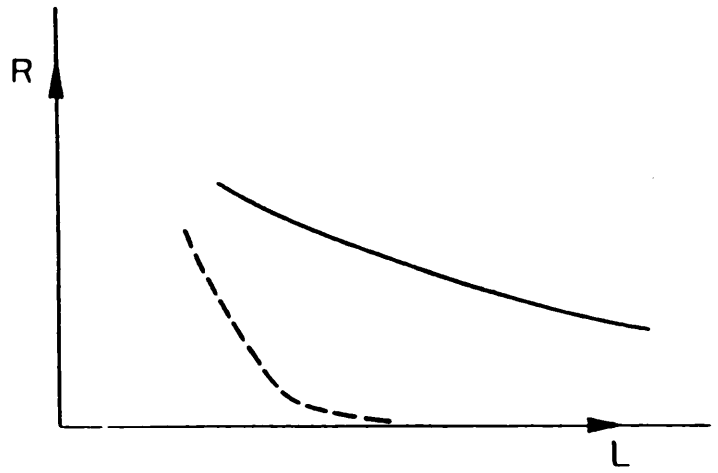

Figure 2.3: *The mean number of loops, $R(L)$, as L varies for $T = T_{st}$ (solid line) and $T < T_{st}$ (dashed line). The solid line shows a scale invariant distribution and the dashed line an exponential suppression of long loops.*

and hence, as before, the string distribution is scale invariant with $dR/R \propto dL/L$ (see (1.15)). The function $R(L)$ is depicted qualitatively in Figure 2.3. The effect of rigidity is to change the normalization of the loop distribution and not its functional dependence on L.

From (2.23) and (2.27) (and using the definition (1.11)) we can evaluate the specific heat of the system. We find that, in less than four dimensions, the specific heat diverges like

$$C_v \approx (T - T_{st})^{d/2-2} \tag{2.40}$$

as T_{st} is approached. Comparing (2.40) in three space dimensions with (1.12), we see that rigidity does not effect the exponent.

2.2 Discussion

The simple string model of Copeland et al. [15] predicts that all the string related phase transitions are second-order sharing the same critical exponents. However, as

we have already stated, experimental results show that this is not the case. Rigidity alone has proved insufficient to account for the experimental observations, changing neither the critical exponent of the specific heat nor the order of the phase transition.

Rigidity does, however, have two interesting effects. The first is to change the critical temperature. For 'elementary' strings with small $u \ll \sigma a$ the fractional change is $\frac{\Delta T}{T_{st}(u=0)} \sim \frac{|u|}{\sigma a}$. For large $u \sim \sigma a$ the change is more significant, namely $\frac{\Delta T}{T_{st}(u=0)} \sim O(1)$. For composite strings, like cosmic strings, the effect of rigidity on the critical temperature is negligibly small. Composite strings have a natural cut-off scale (of $O(m_v^{-1})$ for $e^2 \ll g$) which varies with temperature and prevents the strings from bending on scales which are larger than this cut-off scale. This resticts the effect of stiffness due to the temperature dependence of the parameters making $\beta_{st}\sigma a \ll 1$ in the transition region. Secondly, rigidity changes the effective size of the string segments. If the one-scale model [25] describing the evolution of cosmic strings is correct, then the net result of rigidity (antirigidity) would be to increase (decrease) the correlation length which governs the sizes of the large loops that are chopped off the string network. This, in turn, would effect the bounds on σ obtained from gravitaional radiation constraints.

In the above analysis we have ignored the interactions of the strings, which bring about a much more complicated modification of the string statistics. Therefore, in the next chapter we shall include simple self-interactions of the strings for specific string configurations. Later, we shall also incorporate mutual interactions between strings for general string configurations.

Chapter 3

The Statistical Properties of Self-interacting Strings with Zip-like Configurations

In the previous chapter we included, in the simple model of Copeland et al. [15], the possibility of strings having rigidity. We found that the main effect of rigidity is to increase the critical temperature. For 'elementary' strings with a substantial rigidity, such that the energy required to bend a segment of string is of the same order as its intrinsic energy, the fractional change in the critical temperature is of $O(1)$ (assuming that $\beta_{st}u \ll 1$). However, for 'composite' strings, such as cosmic strings, the effect of rigidity is essentially washed out by the temperature dependence of the parameters and the result $\left(1 - \frac{T_{st}^2}{T_c^2}\right) = O(e^2)$ (for $e^2 \ll g$) still holds (see (2.37)). Furthermore, we found that rigidity does not alter the order of the transition and hence the model still predicts all string phase transitions to be second-order.

In an attempt to account for the experimental observation that different string mediated transitions are of different orders we shall try to include interactions of the string. For 'composite' strings we expect, from our results so far, the transition temperature to be essentially unchanged when interactions are included. However, our results concerning the order of the phase transition may be modified.

In order to investigate this we shall start by looking at a very extreme case of string interactions, namely that of adhesion. The idea is to obtain a flavour of the

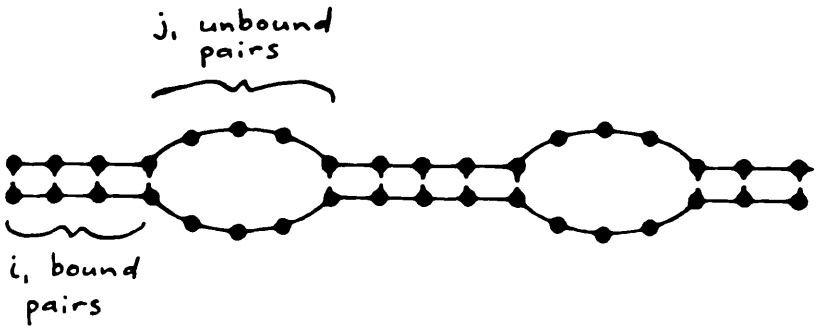

Figure 3.1: *Unknotted simple zips. The broken circles correspond to non-interacting string loops and the double lines to interacting string segments.*

effect of interactions by examining a system which consists of strings with specific configurations, such that the string forces are always attractive. It is for such systems that we expect interactions to play the most dramatic role. For systems consisting of strings with general configurations, one expects the effects of interactions to largely cancel out, as interacting strings are orientable. Since such systems are much harder to analyse, we shall start by looking at the simpler case which, as it is so extreme, should give us an idea of the effect of interactions on the phase transition. (Interactions for general string configurations will be examined in Chapters 4 and 5.) In fact, although restricting the string configurations is unrealistic in the context of cosmic strings, this is not so if the strings are polymers.

In this chapter we shall look at adhesive self-interactions for two kinds of string configurations. Firstly, we shall solve for unknotted configurations (*e.g.* those that simulate the double-helix in DNA (for reviews see [2, 48])) like the zips shown in Figure 3.1, and then for generalisations of zips [49] like those of Figure 3.2. Much of our discussion is based on work performed by Wiegel [1] who examines the helix-coil transition for several models of a one- or two-stranded macromolecule.

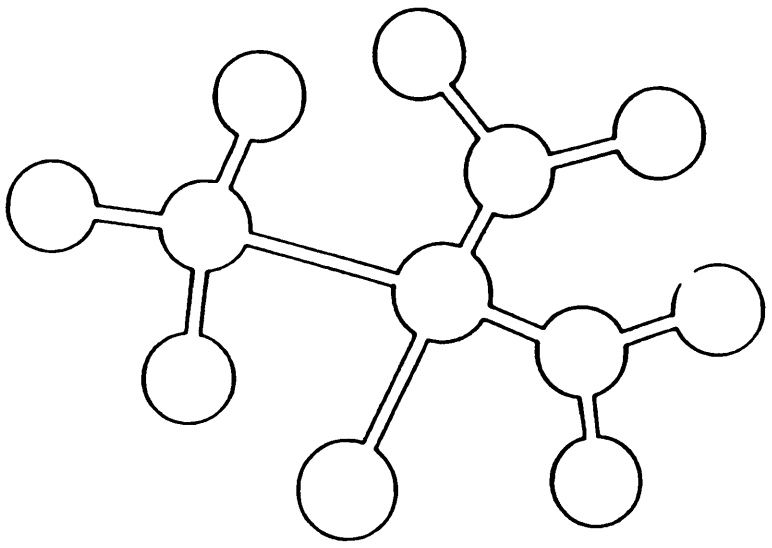

Figure 3.2: *Unknotted branching zips. The broken circles correspond to non-interacting string loops and the double lines to interacting string segments.*

3.1 String transitions for simple (unknotted) zip configurations

In this section we shall consider string phase transitions where we restrict the string configurations to a specific class, $\mathcal{C}$, as shown in Figure 3.1 (see Wiegel [1]).

For the sake of simplicity, we approximate the string to a polymer of hinged straight segments each of length a. We take each string to consist of an alternation of stretches of bound and unbound pairs of string segments. For the bound stretches, we assume that the pairs of segments are each bound with a binding energy ϵ_o, and that only nearest neighbours interact. Whereas, we take each of the unbound stretches to be made up of a single non-interacting string loop.

Suppose that this configuration starts with a straight stretch of i_1 bound pairs, followed by a loop consisting of j_1 unbound pairs, and so on. If this configuration ends

with j_s unbound pairs, then the string is characterised by all sequences of natural numbers

$$\mathcal{C} = (i_1, j_1, i_2, j_2, \ldots, i_s, j_s) \tag{3.1}$$

where $s = 1, 2, 3, \ldots$ and where

$$\sum_{l=1}^{s} (i_l + j_l) = N \tag{3.2}$$

for a string consisting of N pairs of segments. We note that the configuration $\mathcal{C}$, given in (3.1), corresponds to a string which starts with at least one bound pair and ends with at least one unbound pair. There are, of course, three alternative boundary conditions, but in the limit of large N they give the same results.

For a single string of configuration $\mathcal{C}$, in thermal equilibrium at a temperature $T = 1/\beta$ (in units where $k_B = 1$), the partition function Z_1 is given by

$$Z_1 = \sum_{N=1}^{\infty} Z(N)\, e^{-2N\beta\sigma a} \tag{3.3}$$

In (3.3), $Z(N)$ is the conformational partition function for a string of total length $2Na$.

Let the conformational partition function for two parallel sections of string consisting of i bound pairs be u_i and that for a loop of string consisting of j unbound pairs be v_j. The conformational partition function for the configuration $\mathcal{C}$ is then given by

$$Z(N) = \sum_{s=1}^{\infty} \sum_{\{i_l, j_l\}} \prod_{l=1}^{s} u_{i_l} v_{j_l} \tag{3.4}$$

where the summation over the set $\{i_l, j_l\}$ is restricted so that (3.2) is always satisfied. To eliminate this restriction, we introduce the generating functions

$$U(x) = \sum_{i=1}^{\infty} u_i x^i \tag{3.5}$$

$$V(x) = \sum_{j=1}^{\infty} v_j x^j \tag{3.6}$$

$$\tilde{Z}(x) = \sum_{N=1}^{\infty} Z(N) x^N \tag{3.7}$$

If we then multiply both sides of (3.4) by x^N and sum over N, we find that

$$\begin{aligned} \tilde{Z}(x) &= U(x)V(x) + [U(x)V(x)]^2 + \ldots \\ &= \frac{U(x)V(x)}{1 - U(x)V(x)} \end{aligned} \tag{3.8}$$

For the interacting sections of the string the conformational partition function is taken to be given by the product of i factors w:

$$u_i = \alpha w^i = \alpha e^{\beta|\epsilon_o|i} \tag{3.9}$$

For anti-parallel segments, as is the case here (since we are looking at a single string), we assume that the force is attractive and hence that ϵ_o is negative. (Note that for cosmic strings this is indeed the case.) The extra factor $\alpha \ll 1$ has been included in (3.9) to account for all destabilizing effects associated with the finite length of the interacting string sections. Its smallness favours a single long stretch of bound pairs over a sequence of short stretches containing the same number of bound pairs.

From (3.4), we are able to evaluate the average fraction of bound segment pairs, $\theta(T)$, at a temperature T:

$$\theta(T) = \frac{1}{N} \frac{\sum_{s=1}^{\infty} \sum_{\{i_l, j_l\}} \prod_{l=1}^{s} i_l u_{i_l} v_{j_l}}{Z(N)} \tag{3.10}$$

$$= \frac{1}{N} \frac{w}{Z(N)} \frac{\partial Z(N)}{\partial w} \tag{3.11}$$

$$= \frac{1}{N} \frac{\partial(\ln Z(N))}{\partial(\ln w)} \tag{3.12}$$

For string loops with no self-interactions, as is the case for the unbound sections, the conformational partition function v_j is given by

$$v_j \simeq A z^{2j} (2j)^{-q} \tag{3.13}$$

The factor v_j measures the volume in configurational space available to a random walk which returns to its origin after $2j$ steps. As stated in Chapter 1, for non-rigid strings, A is a temperature independent constant and q is a number dependent on

the type of random walk. For example, $q = 3/2$ for free random walks and $q = 7/4$ for self-avoiding random walks. In contrast with the model presented in the previous chapters, the string has not been restricted to lie on a lattice and hence here z is *not* the lattice coordination number. As before, in order to denumerate the number of string configurations, we must impose a lower cut-off, a, on the scale of bending (for 'composite' strings, a is the string width). In that case, z is a temperature independent constant such that z^{2j} is proportional to the total volume in configurational space available to a random walk of $2j$ steps. To simplify (3.13) we absorb the factor 2^{-q} (which is due to the two-stranded nature of the string configuration) in the constant A and substitute $b = z^2$ to find that

$$v_j \simeq A b^j j^{-q} \tag{3.14}$$

To obtain an expression for the generating function $U(x)$ we substitute (3.9) into (3.5) and sum over i. This gives that

$$U(x) = \frac{\alpha w x}{1 - wx} \tag{3.15}$$

Similarly, on inserting (3.14) into (3.6) we find that the generating function $V(x)$ is given by

$$V(x) = A \sum_{j=1}^{\infty} j^{-q} (bx)^j \tag{3.16}$$

Inverting (3.7) and substituting for $\tilde{Z}(x)$ using (3.8), we find that the conformational partition function, $Z(N)$, is obtained by performing the contour integral

$$Z(N) = \frac{1}{2\pi\imath} \oint_C \frac{U(x)V(x)}{(1 - U(x)V(x))} \frac{1}{x^{N+1}} \, dx \tag{3.17}$$

where C encircles the origin in an anticlockwise direction. There are four singularities in the integrand: (1) a pole of order $N + 1$ at $x = 0$, due to the factor x^{-N-1}; (2) a branch line along the positive real x-axis starting at $x = b^{-1}$, as $V(x) = +\infty$ for $x > b^{-1}$ (assuming that $q > 1$); (3) another branch line starting at $x = w^{-1}$, as

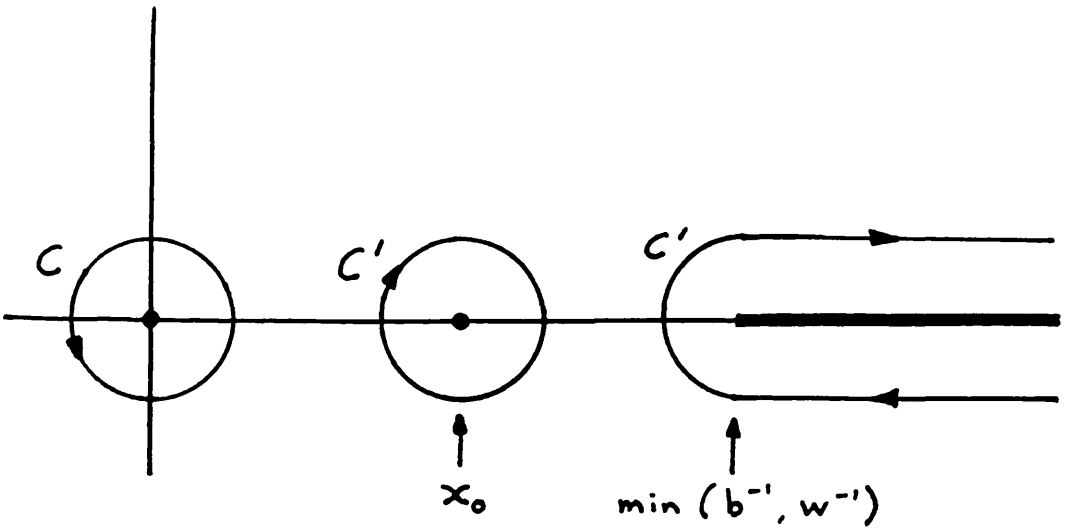

Figure 3.3: *The contour of integration in equation (3.17) and the singularities of the integrand. The contour C can be deformed into C'. The respective positions of the singularities depend on the temperature of the system.*

$U(x) = +\infty$ for $x \geq w^{-1}$; and (4) a pole at each solution to the equation

$$V(x_o) = U^{-1}(x_o) = \frac{1 - wx_o}{\alpha w x_o} \tag{3.18}$$

These singularities are depicted in Figure 3.3. The contour of integration C can be deformed into the contour C', where C' consists of a small circle around x_o and a loop around the branch line which starts at $\min(b^{-1}, w^{-1})$. From (3.17), the conformational partition function is then given, to leading order, by the contribution of the singularity nearest to the origin, x^*. Hence

$$Z(N) \simeq K(x^*) x^{*-N-1} \qquad (N \gg 1) \tag{3.19}$$

where $K(x^*)$ is some constant which depends on x^*.

The partition function for a single string of configuration $\mathcal{C}$ is then obtained by substituting (3.19) into (3.3). We find that

$$Z_1 \simeq K(x^*) \sum_{N=1}^{\infty} x^{*-N-1} e^{-2N\beta\sigma a} \tag{3.20}$$

$$= \frac{K(x^\star)}{x^\star} \sum_{N=1}^{\infty} e^{-2N\beta\sigma_{eff}a} \tag{3.21}$$

where

$$\begin{aligned} \sigma_{eff} &= \sigma + \frac{\ln x^\star}{2\beta a} \\ &= \sigma + \frac{T \ln x^\star}{2a} \end{aligned} \tag{3.22}$$

Note that, strictly speaking, we cannot use the expression for $Z(N)$ given in (3.19) for small values of N since the other singularities of the integrand are expected to give comparable contributions in this regime. It is the divergence of Z_1 that marks the phase transition and hence it is reasonable to neglect these extra contributions near the transition. However, at low temperatures this approximation is less valid. As the aim of this chapter is to get a flavour of the effect of self-interactions rather than to perform accurate calculations, we shall nevertheless persist with (3.20). From (3.21) and (3.22), we can immediately see that the system undergoes a phase transition at $T = T_{st}$ where T_{st} satisfies

$$T_{st} = -\frac{2\sigma a}{\ln x^\star(T_{st})} \tag{3.23}$$

At this temperature σ_{eff} vanishes and the whole space becomes filled with string. We note that here we are taking the strings to be 'elementary' strings with temperature independent parameters.

So far we have just considered the conformational phase transition. The partition function for a system containing an arbitrary number of strings is

$$Z = \exp Z_1 \tag{3.24}$$

(see (1.8) and (2.23)) if mutual interactions between strings can be neglected [15] (*i.e.* in the 'free gas approximation'). From (3.24) it can be shown that as T_{st} is approached the number density of strings rapidly increases. T_{st} defines a maximum temperature for the system (the Hagedorn temperature [26]). As the temperature of the system

approaches T_{st} an increasing amount of energy must be put into the system in order to cause a given temperature rise. This occurs because the energy goes into creating more string length rather than causing the temperature to increase.

From (3.24) and the definition (1.11) we can evaluate the specific heat of our system. It diverges as T_{st} is approached like

$$C_v \sim (T - T_{st})^{-3} \tag{3.25}$$

In order to determine the single string partition function from (3.21) we need to solve for x^*. As already mentioned, there are three possible non-zero singularities that may dominate the conformational partition function, $Z(N)$. Which one dominates is determined by the temperature of the system. Specifically, in the interval $0 \leq x \leq b^{-1}$ the function $V(x)$ increases from zero at, $x = 0$, to its maximum finite value of

$$V = A \sum_{j=1}^{\infty} j^{-q} = A\zeta(q) \tag{3.26}$$

at $x = b^{-1}$ (where ζ denotes Riemann's zeta function). In the same interval, $U^{-1}(x)$ decreases from infinity at, $x = 0$, to the value

$$U^{-1} = \frac{b}{\alpha w} - \frac{1}{\alpha} \tag{3.27}$$

at $x = b^{-1}$.

From (3.26) and (3.27) (and assuming that $q > 1$), there is a temperature T_o above which (3.18) does not have a solution and hence x_o is not a singularity of the integrand in (3.17). This temperature is obtained from

$$\frac{b}{\alpha} e^{-|\epsilon_o|/T_o} = \frac{1}{\alpha} + A\zeta(q) \tag{3.28}$$

Consequently, for $T \geq T_o$ the singularity nearest to the origin is fixed at b^{-1} and hence, from (3.19), the conformational partition function is given by

$$Z(N) \simeq K(b^{-1})\, b^{N+1} \qquad (N \gg 1) \tag{3.29}$$

In this case we have, from (3.22), that the effective string tension is given by

$$\sigma_{eff} = \sigma - \frac{T \ln b}{2a} \tag{3.30}$$

and, from (3.23), that

$$T_{st} = \frac{2\sigma a}{\ln b} \qquad \text{where} \qquad T_{st} \geq T_o \tag{3.31}$$

The maximum temperature that the system can reach is thus the same as that for non-self-interacting, non-rigid, closed strings. Furthermore, on substituting (3.29) into (3.12), we find that the average fraction of bound pairs is

$$\theta(T) = 0 \qquad \text{for} \qquad T \geq T_o \tag{3.32}$$

This is what we expect since for high temperatures it is the entropy that is the dominant factor in determining the string configuration. If the temperature is high enough, the attractive force which binds the string segments can be overcome, and hence the configuration is that of a non-self-interacting string.

For lower temperatures such that $T < T_o$, (3.18) has a solution and hence the singularity nearest to the origin is at x_o, where $x_o < b^{-1}$. In this case the conformational partition function is given by

$$Z(N) \simeq K(x_o)\, x_o^{-N-1} \qquad (N \gg 1) \tag{3.33}$$

We now wish to obtain an expression for x_o. However, we can only do so in two temperature regimes, namely at very low temperatures and at temperatures close to T_o.

Firstly, in the limit $T \to 0$ (which implies that $w^{-1} \to 0$), the pole at x_o approaches that at w^{-1}. The conformational partition function then becomes

$$Z(N) \simeq K(w^{-1})\, w^{N+1} \qquad (N \gg 1) \tag{3.34}$$

and hence, from (3.22), the effective string tension is given by

$$\sigma_{eff} = \sigma - \frac{|\epsilon_o|}{2a} \tag{3.35}$$

As already mentioned, (3.35) should not be taken too seriously as the expression for the single string partition function, (3.21), is not accurate for low temperatures.

As $T \to 0$ we have, from substituting (3.34) into (3.12), that the average fraction of bound pairs is

$$\theta(T) = 1 \qquad (N \gg 1) \tag{3.36}$$

Again, this is as expected, since it is now the energy that is the dominant factor in determining the string configuration. The factor w, which is present due to the attractive forces between the string segments, is extremely large at low temperatures and hence the configuration consists solely of a sequence of bound string segment pairs. This is consistent with results obtained previously (in the context of 'composite' strings) by the authors of [15, 19, 23, 37]. Using different methods, they all show that at low temperatures loops of large length are exponentially supressed.

Secondly, we take T to be slightly lower than T_o, so that

$$w = w_o + \delta \qquad (\delta \ll 1) \tag{3.37}$$

where $w_o = w(T_o)$. This, in turn, implies that x_o is slightly smaller than b^{-1}:

$$x_o = b^{-1} - \varepsilon \qquad (\varepsilon \ll 1) \tag{3.38}$$

In the vicinity of the branch point, $V(x)$ may be replaced by its asymptotic expansion found by combining (3.16) with (3.38):

$$V(x_o) = V(b^{-1}) - A\sum_{j=1}^{\infty} j^{-q}\left[1 - (1-\varepsilon b)^j\right] \tag{3.39}$$

For $\varepsilon \ll 1$ we can write $(1-\varepsilon b)^j \simeq e^{-\varepsilon b j}$ and replace the summation over j by an integration over $x = \varepsilon b j$. This gives that

$$V(x_o) \simeq V(b^{-1}) - Ab^{q-1}(q-1)^{-1}\Gamma(2-q)\varepsilon^{q-1} \tag{3.40}$$

(assuming that $q > 1$) where Γ denotes the gamma function. Furthermore, from (3.37) and (3.38), the product $(wx_o)^{-1}$ has the expansion

$$(wx_o)^{-1} \simeq \frac{b}{w_o}\left(1 - \frac{\delta}{w_o} + \varepsilon b + \ldots\right) \tag{3.41}$$

If we now substitute (3.40) and (3.41) into (3.18) we find a relation between ε and δ given by

$$\varepsilon^{q-1} \simeq \frac{(q-1)b^{2-q}}{\alpha w_o^2 A\Gamma(2-q)}\delta \tag{3.42}$$

The behaviour of x_o then follows from (3.38) by substituting for ε using (3.42), and for δ using (3.37). The result is that

$$x_o \simeq b^{-1} - C(w - w_o)^{1/(q-1)} \tag{3.43}$$

where

$$C = \left[\frac{(q-1)b^{2-q}}{\alpha w_o^2 A\Gamma(2-q)}\right]^{1/(q-1)} \tag{3.44}$$

From (3.43) and the definition of the effective string tension given in (3.22), we obtain that

$$\sigma_{eff} \simeq \sigma - \frac{T}{2a}\ln b - \frac{CbT}{2a}\left(e^{|\epsilon_o|/T} - e^{|\epsilon_o|/T_o}\right)^{1/(q-1)} \tag{3.45}$$

The system thus undergoes a phase transition at $T = T_{st}$ (assuming that $T_{st} < T_o$) where T_{st} is given by

$$T_{st} \simeq \frac{2\sigma a}{\ln b + Cb\left(e^{|\epsilon_o|/T_{st}} - e^{|\epsilon_o|/T_o}\right)^{1/(q-1)}} \tag{3.46}$$

In this temperature regime, the conformational partition function $Z(N)$ is obtained by taking $x^* = x_o$ in (3.19), where x_o is given by (3.43). From this we can then derive an expression for the fraction of bound segment pairs, $\theta(T)$. The result is that

$$\theta(T) \sim (T - T_o)^{(2-q)/(q-1)} \tag{3.47}$$

Consequently, as $T \to T_o$ the fraction of bound segment pairs approaches zero in a smooth fashion. The variation of $\theta(T)$ with temperature is depicted in Figure 3.4.

As we have seen, the behaviour of the conformational partition function depends on the temperature of the system. In particular, it depends on the ratio T/T_o where T_o is the solution to (3.28). From this equation we have that

$$T_o = \frac{|\epsilon_o|}{\ln\left(\frac{b}{1+\alpha A\zeta(q)}\right)} \simeq \frac{|\epsilon_o|}{\ln b} \tag{3.48}$$

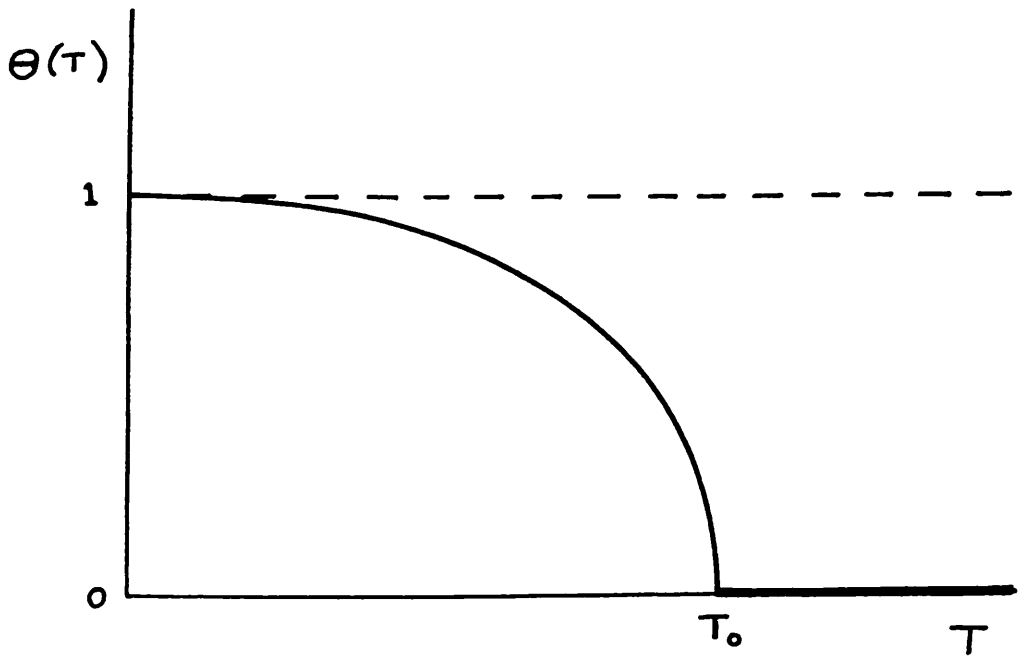

Figure 3.4: *The temperature dependence of the fraction of bound pairs, $\theta(T)$, for simple string structures.*

and hence that T_o depends on the binding energy, $|\epsilon_o|$, for a pair of anti-parallel string segments each of length a.

For weak attractive forces, such that $|\epsilon_o| < 2\sigma a$, we have that $T_o \leq T_{st}(|\epsilon_o| = 0)$, where $T_{st}(|\epsilon_o| = 0) = \frac{2\sigma a}{\ln b}$ is the transition temperature for a simple string with no rigidity or self-interactions [15]. The variation of the effective string tension with temperature for strings which experience weak adhesion is illustrated in Figure 3.5. As shown, the transition temperature, T_{st}, for closed strings is the same as that obtained in the absence of self-interactions. When the transition occurs, the binding energy $|\epsilon_o|$ plays no role and the string configuration is that of a non-self-interacting closed loop. Cooling the system causes sections of bound string to form giving rise to configuration as shown in Figure 3.1.

For larger binding energies such that $|\epsilon_o| \geq 2\sigma a$, we have that $T_o \geq T_{st}(|\epsilon_o|) = 0)$. For such strings, the effective string tension is shown as a function of temperature

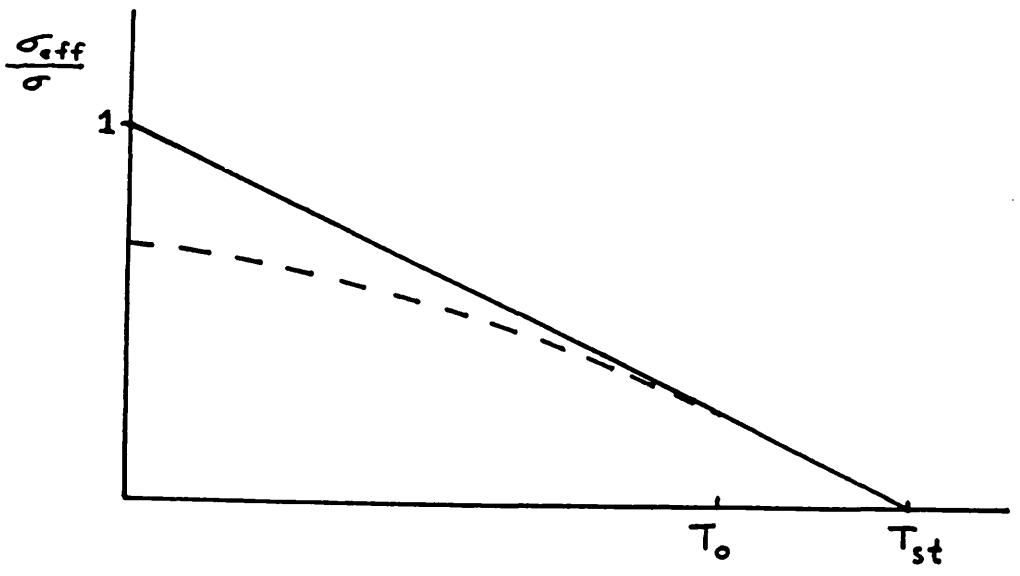

Figure 3.5: *The variation of σ_{eff}/σ with T for elementary strings with weak self-interactions such that $T_o \leq T_{st}$. The solid line corresponds to $|\epsilon_o| = 0$ and the dashed line to small $|\epsilon_o|$ with $\epsilon_o < 0$ (i.e. attractive forces).*

in Figure 3.6. In this case, the transition temperature $T_{st} < T_{st}(|\epsilon_o| = 0)$ and hence sections of bound string are already present at the transition. Again, cooling the system results in an increase in the fraction of bound segment pairs.

In this section we examined string phase transitions for self-interacting strings with configurations like those shown in Figure 3.1. We found that for $|\epsilon_o| < 2\sigma a$ the effect of adhesion is negligible. Conversly, for $|\epsilon_o| \geq 2\sigma a$ we obtained the result that adhesion reduces the transition temperature by a small amount. In both cases, the inclusion of self-interactions has proved insufficient to account for the experimental observation that different string phase transitions are of different orders. Again, we predict a second-order transition independent of the string type.

As a next step, we shall include the possibility of more complicated string configurations, namely those of branching zips, as shown in Figure 3.2. We shall examine how the transition temperature is affected and see if we can accommodate the experimen-

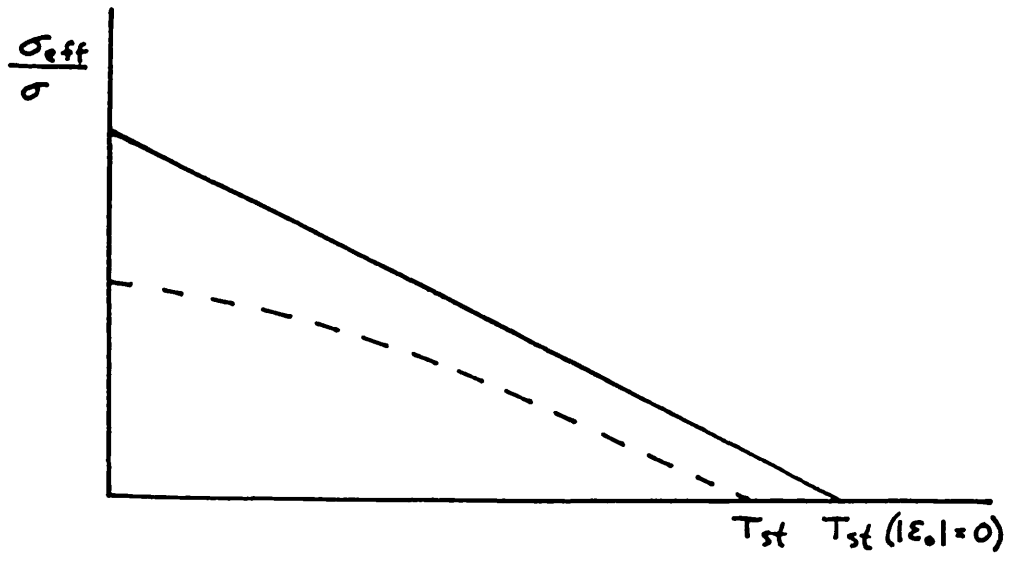

Figure 3.6: *The variation of σ_{eff}/σ with T for elementary strings with string self-interactions such that $T_o > T_{st}$. The solid line corresponds to $|\epsilon_o| = 0$ and the dashed line to large $|\epsilon_o|$ with $\epsilon_c < 0$ (i.e. attractive forces).*

tal observation mentioned above. Again, we shall follow the discussion presented by Wiegel [1], who examines transitions in synthetic polynucleotides which form branching structures.

3.2 String transitions for branching (unknotted) zip configurations

In this section we shall examine branching zip structures, like those of Figure 3.2, which consist of repeated units (see Wiegel [1]). We shall take a unit to be composed of two sections: one made up of a set of bound string segments and the other a closed loop which consists of a set of unbound string segments. Such a unit is depicted in Figure 3.7, and we shall call the loop a loop of order zero.

If, as before, we denote the generating function of a bound section by $U(x)$ and that of an unbound loop by $V(x)$, then the summation over the configurations corre-

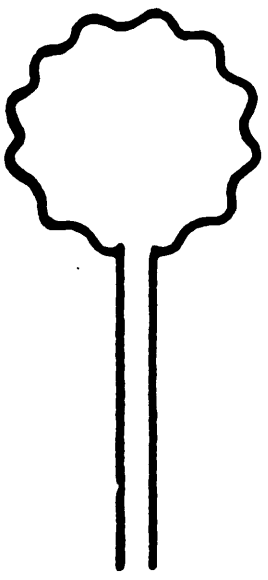

Figure 3.7: *Diagram representing a unit of order zero. The parallel lines represent interacting string segments and the wavy circle a non-interacting loop.*

sponding to Figure 3.7 leads to a generating function

$$\tilde{Z}_o(x) = U(x)V(x) \tag{3.49}$$

Next, we consider all configurations indicated by Figure 3.8, in which the string consists of a bound section followed by a loop, to which a number of units of the type in Figure 3.7 are attached. We shall call the central loop of these configurations a loop of first order. The generating function for such configurations is found from (3.49) by replacing the x in $V(x)$ by $x + \tilde{Z}_o(x)$. This gives that

$$\tilde{Z}_1(x) = U(x)V(x + \tilde{Z}_o(x)) \tag{3.50}$$

Generally, we can consider all diagrams which start with a bound section followed by a loop of order k to which are attached bound sections ending in loops of orders at most equal to $k - 1$. Summing over all such configurations results in the generating function

$$\tilde{Z}_k(x) = U(x)V(x + \tilde{Z}_{k-1}(x)) \tag{3.51}$$

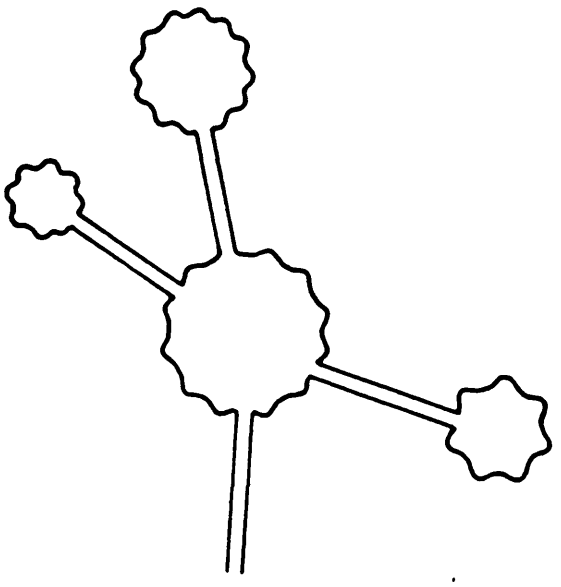

Figure 3.8: *A loop of first order to which loops of order zero are attached.*

Taking the limit $k \to \infty$ in (3.51) includes all possible branching structures and gives a functional equation

$$\tilde{Z}(x) = U(x)V(x + \tilde{Z}(x)) \tag{3.52}$$

from which the generating function, $\tilde{Z}(x)$, has to be solved.

Once determined, we use, as in the simpler case considered previously, (3.17) to obtain the conformational partition function, $Z(N)$ for large N. Here again, it is the singularity of $\tilde{Z}(x)$ which is nearest to the origin that gives the main contribution to $Z(N)$.

In order to solve for $\tilde{Z}(x)$ we introduce a new complex variable

$$t = x + \tilde{Z}(x) \tag{3.53}$$

in terms of which (3.52) reads

$$t = x + U(x)V(t) \tag{3.54}$$

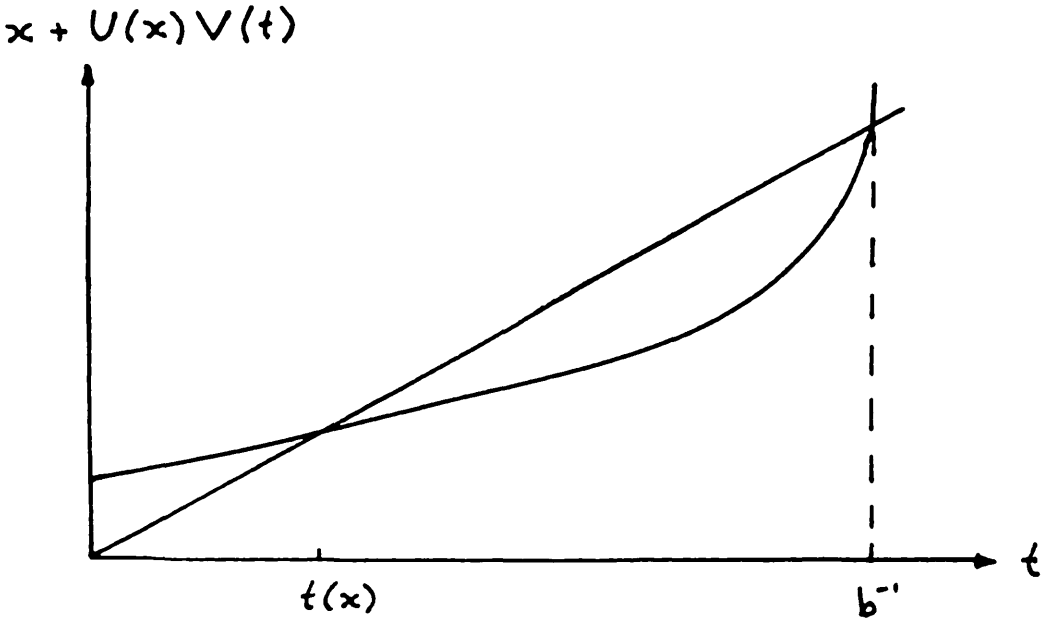

Figure 3.9: *Both sides of equation (3.54) as a function of t; the first point of intersection defines $t(x)$.*

If we denote the singularity nearest to the origin by x^* we have that

$$t^* = x^* + U(x^*)V(t^*) \tag{3.55}$$

where $t^* = t(x^*)$ and that

$$1 = U(x^*)V'(t^*) \tag{3.56}$$

where the prime denotes differentiation with respect to the argument.

The behaviour of the generating function, $\tilde{Z}(x)$, near to its singularity is determined by the behaviour of $t(x)$ for values of x which are slightly smaller than x^* (see Figure 3.9). [We note that (3.54) also has another solution which is slightly greater than x^*, but in this case $t(x)$ is a decreasing function of x (see Figure 3.9). The singularity x^* is the value of x when the two solutions coincide (see Figure 3.10).] Expanding all functions up till second order in the small quantities $t^* - t$ and $x^* - x$

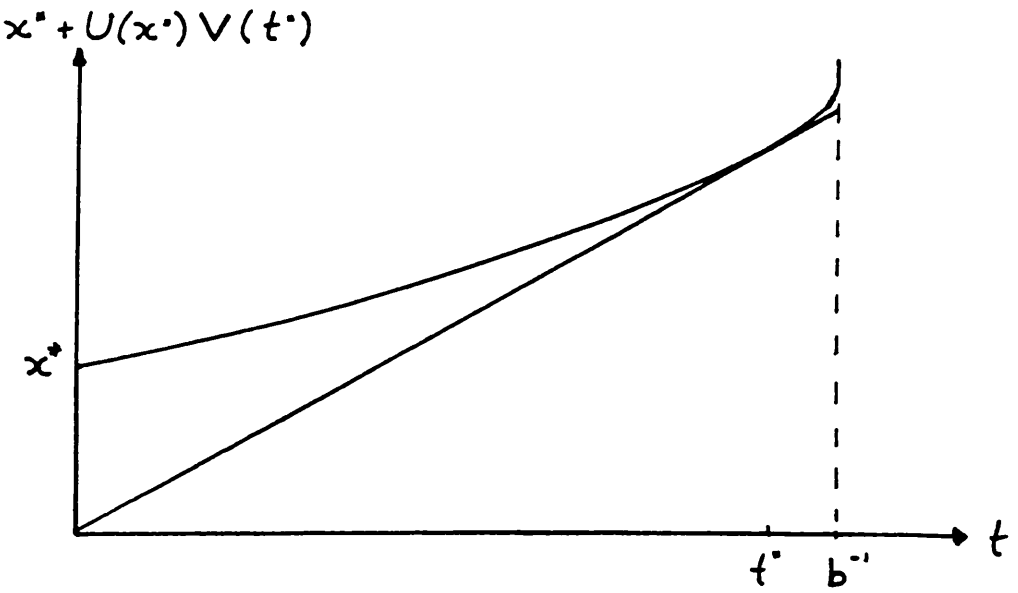

Figure 3.10: *Graphic representation of both sides of equations (3.55) and (3.56).*

we obtain that

$$t = t^* - (t^* - t) \tag{3.57}$$

$$x = x^* - (x^* - x) \tag{3.58}$$

and that

$$U(x) = U(x^*) - U'(x^*)(x^* - x) + \frac{1}{2}U''(x^*)(x^* - x)^2 - \ldots \tag{3.59}$$

$$V(t) = V(t^*) - V'(t^*)(t^* - t) + \frac{1}{2}V''(t^*)(t^* - t)^2 - \ldots \tag{3.60}$$

If we then substitute these expressions into (3.54), use (3.55) and (3.56), and solve for $t(x)$, we find the result that

$$t(x) \simeq t^* - C(x^* - x)^{1/2} \tag{3.61}$$

where the constant C is given by

$$C = \left[\frac{2[1 + U'(x^*)V(t^*)]}{U(x^*)V''(t^*)}\right]^{1/2} \tag{3.62}$$

Inserting the expression for $t(x)$ given in (3.61) into (3.53), we obtain the asymptotic expansion for the generating function

$$\tilde{Z}(x) = t^\star - x^\star - C(x^\star - x)^{1/2} \tag{3.63}$$

Hence, $\tilde{Z}(x)$ has a branch point at $x^\star$ and a branch line from $x^\star$ to $+\infty$.

The conformational partition function then follows by substituting for $\tilde{Z}(x)$ in (3.17), performing the contour integral and using Stirling's formula for $N \gg 1$ to give that

$$Z(N) \simeq C' N^{-3/2} x^{\star -N} \qquad (N \gg 1) \tag{3.64}$$

where C' is a constant. Substituting (3.64) into (3.3) we find that the partition function for a single string of length $2Na$ with a 'tree-like' structure is given by

$$Z_1 \simeq C' \sum_{N=1}^{\infty} N^{-3/2} x^{\star -N} e^{-2N\beta\sigma a} \tag{3.65}$$

From this expression we can, as in the previous case, use (3.24) and (1.11) to evaluate the specific heat for a system containing an arbitrary number of strings. We find that it diverges as T_{st} is approached like

$$C_v \sim (T - T_{st})^{-3/2} \tag{3.66}$$

We now need to solve for $x^\star$ in order to obtain the conformational partition function, $Z(N)$. Using (3.23), we can then obtain an expression for T_{st}, the temperature at which σ_{eff} vanishes. We notice that for small values of the complex variable t, the function on the right hand side of (3.54) increases slowly with t, whereas for larger values of t the increase is rather more rapid, with the function diverging when $t > b^{-1}$. Increasing x thus causes the first intersection of Figure 3.9, $t(x)$, to increase at a greater rate than that at which the second intersection decreases. From this we deduce that the value $t^\star$ is only slightly smaller than b^{-1}. Hence, we can write

$$t^\star = b^{-1} - \varepsilon \qquad (\varepsilon \ll 1) \tag{3.67}$$

In order to express ε in terms of $x^\star$ we use (3.55) and substitute for $U(x^\star)$ and $t^\star$ using (3.56) and (3.67) respectively. This gives that

$$b^{-1} - \varepsilon = x^\star + \frac{V(t^\star)}{V'(t^\star)} \tag{3.68}$$

where we now need to express $V(t^\star)$ and $V'(t^\star)$ in terms of ε.

From the definition of the generating function $V(t)$, given in (3.16), we have that, at the branch point,

$$V(t^\star) = A\sum_{j=1}^{\infty} j^{-q}(1-\varepsilon b)^j \tag{3.69}$$

For $\varepsilon \ll 1$ we proceed as we did in the previous section by writing $(1-\varepsilon b)^j \simeq e^{-\varepsilon b j}$ and replacing the summation over j by an integration over $x = \varepsilon b j$. We find that

$$V(t^\star) \simeq \frac{A}{(q-1)}\left[e^{-\varepsilon b} - (\varepsilon b)^{q-1}\Gamma(2-q,\varepsilon b)\right] \tag{3.70}$$

where $\Gamma(2-q,\varepsilon b)$ denotes the incomplete gamma function (for $\varepsilon b \ll 1$ we have that $\Gamma(2-q,\varepsilon b) \simeq \Gamma(2-q)$). Furthermore, differentiating the generating function $V(t)$ with repect to its argument, and using (3.67) to evaluate its value at the branch point gives

$$V'(t^\star) \simeq Abe^{\varepsilon b}(\varepsilon b)^{q-2}\Gamma(2-q,\varepsilon b) \tag{3.71}$$

If we then substitute for $V(t^\star)$ and $V'(t^\star)$ in (3.68) we obtain, to leading order, that

$$\varepsilon = \frac{1}{b}\left[(1-bx^\star)(q-1)\Gamma(2-q,\varepsilon b)\right]^{1/(2-q)} \tag{3.72}$$

(assuming that $q > 1$ and that $\varepsilon b \ll 1$) and hence that $x^\star \simeq b^{-1}$.

The singularity $x^\star$ can now be found. To do so we use (3.72) to substitute for ε in the expression for $V'(t^\star)$ given in (3.71). Then, we take the resulting equation in conjunction with the expression for $U(x^\star)$, given in (3.18), to solve (3.56). The final result is that

$$x^\star \simeq \frac{1}{b}\frac{1}{\left(1+\frac{A\alpha w}{(q-1)}\right)} \tag{3.73}$$

In the limit $T \rightarrow 0$, we have that $\alpha w \gg 1$ and hence, from (3.73), that $x^{\star}b \ll 1$. From (3.72), this is inconsistent with the assumption that $\varepsilon b \ll 1$ and hence the expression for the singularity, $x^{\star}$, in not valid in this temperature regime. However, from (3.52) and (3.53) we have that

$$\tilde{Z}(x) = U(x)V(t) \tag{3.74}$$

and hence the generating function has a branch line starting at $x = w^{-1}$, due to the divergence of $U(x)$ for $x \geq w^{-1}$. Therefore, in the limit of very low temperatures, where $w^{-1} \ll 1$, it is this singularity which dominates and so we have that

$$x^{\star} = \frac{1}{w} \qquad \text{for} \qquad T \rightarrow 0 \tag{3.75}$$

The effective string tension is then given by

$$\sigma_{eff} = \sigma - \frac{|\epsilon_o|}{2a} \tag{3.76}$$

where we have used (3.22) and (3.75). From (3.11), we can evaluate the average fraction of bound pairs, $\theta(T)$. We find that

$$\theta(T) = 1 \qquad \text{for} \qquad T \rightarrow 0 \tag{3.77}$$

in agreement with the more simple case considered previously.

At high temperatures, such that $|\epsilon_o|/T \ll 1$, we have that $w \simeq 1$ and hence $x^{\star}$ can be approximated by

$$x^{\star} \simeq \frac{1}{b}\left(1 - \frac{A\alpha w}{(q-1)}\right) \tag{3.78}$$

giving an effective string tension

$$\sigma_{eff} = \sigma - \frac{T}{2a}\left(\ln b + \frac{A\alpha w}{(q-1)}\right) \tag{3.79}$$

The system then experiences a phase transition, when $\sigma_{eff} = 0$, at a temperature which is given by

$$T_{st} \simeq \frac{2\sigma a}{\ln b} \frac{1}{\left[1 + \frac{A\alpha e^{|\epsilon_o|/T_{st}}}{(q-1)\ln b}\right]} \tag{3.80}$$

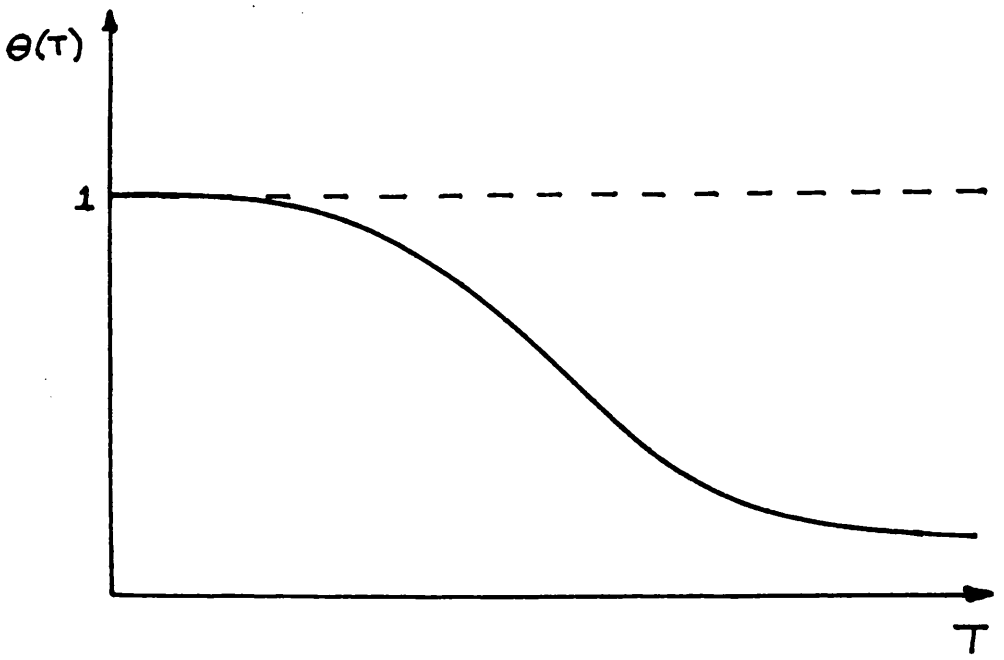

Figure 3.11: *The temperature dependence of the fraction of bound pairs, $\theta(T)$, for branching string structures.*

As in the case considered previously, it is immediately apparent, from (3.80), that the transition temperature is reduced when weak attractive self-interactions are included. Furthermore, in order to obtain $Z(N)$ we insert the expression for x^*, given in (3.78), into (3.64). From the resulting equation in conjunction with (3.11), we can then evaluate the fractional number of bound pairs at high temperatures. We find that

$$\theta(T) \simeq \frac{1}{1 + \frac{(q-1)e^{-|\epsilon_0|/T}}{A\alpha}} \tag{3.81}$$

This result is depicted in Figure 3.11. As expected, the fraction $\theta(T)$ decreases as the temperature of the system is increased. If we compare this fraction to that obtained in the case of simple zips (see (3.47)), we observe that here the transition between the structures at low temperatures, which are totally bound, to those at high temperatures, which are unbound, is more gradual. This is because of the very large number of 'tree-like' configurations.

3.3 Discussion

The simple string model of Copeland et al. [15] predicts that all string related phase transitions are second-order. However, as already stated, experimental results show that this is not the case. In the previous chapter we found that rigidity alone is insufficient to account for the experimental observations. In this chapter, we have examined how the string phase transition is affected by the inclusion of attractive self-interactions for particular (unknotted) string configurations. For both simple zips and branching zips adhesion alone has proved inadequate to explain the fact that different string transitions are of different orders.

However, including attractive self-interactions does affect the transition temperature, T_{st}, reducing it by a small fractional amount as compared to that obtained using the very simple model of Copeland et al. [15]. Furthermore, we found that at low temperatures the fraction of string in loops is small. This is consistent with the model of [15] which predicts that large loops are exponentially suppressed at low temperatures. At high temperatures we found that most of the string is in the form of unbound loops. Again, this is what we would expect as at high temperatures it is the entropy which is the dominant factor in determining the string configuration and not the the energy. The attractive forces can be overcome in this regime.

So far we have assumed that the strings are 'elementary' *i.e.* that they have temperature independent parameters. For 'composite' strings, such as cosmic strings, with temperature dependent parameters we found, in Chapter 2, that rigidity plays an unimportant role, leaving the results of Copeland et al. [15] essentially unchanged. This is due to $\beta\sigma a$ being *small* near the original (non-self-interacting) string phase transition. As this is also the case here, we expect the effect of adhesion on the critical temperature to be negligibly small too.

Assuming that the binding energy $|\epsilon_o| \sim \sigma a$ we have that, near the transition, $w \sim 1$. Inserting this result into the two expressions for the transition temperature

(given in (3.46) and (3.80)) and using the results for cosmic strings that

$$\sigma(T_{st}) \sim \frac{m_o^2}{g}\left(1 - \frac{T_{st}^2}{T_c^2}\right) \tag{3.82}$$

and

$$a(T_{st}) \sim \frac{\sqrt{g}}{em_o}\left(1 - \frac{T_{st}^2}{T_c^2}\right)^{-1/2} \tag{3.83}$$

(see (1.49), (1.53-1.55), (2.32) and (2.33)) we find, for both simple and branching zip structures, that

$$1 - \frac{T_{st}^2}{T_c^2} = O(e^2) \qquad \text{for } e^2 \ll g \tag{3.84}$$

This is the same result as that obtained in the absence of interactions (see (1.62) and (2.37)).

As a next step, we shall include string-string interactions in addition to self-interactions for *general* closed string configurations. We shall test the model on iron and shall show that melting is a first-order phase transition, in agreement with experimental observations. We shall then apply the same formalism to global cosmic strings in order to determine the order of the phase transition at which they are formed.

Chapter 4

Dislocation Melting

In the previous chapter we examined how the simple model of Copeland et al. [15] is effected by the inclusion of attractive string self-interactions for particular (unknotted) string configurations. We found that the main effect of these specific interactions is to reduce the temperature at which the transition occurs. For 'composite' strings, with temperature dependent parameters, even this effect turned out to be negligible. Specifically, for Nielsen-Olesen vortices [16] we found that the critical temperature satisfies $1 - \frac{T_{st}^2}{T_c^2} = O(e^2)$ (assuming that $e^2 \ll g$), which is the same result as that obtained in the absence of adhesion (see (1.62)). Furthermore, for strings with zip-like configurations, we found that attractive self-interactions did not change the order of the transition. The model, as it stands, still predicts all string mediated phase transitions to be second-order.

However, restricting the string configurations, as we did in the previous chapter, is unrealistic for most systems containing strings. (Note that some macromolecules are naturally found with zip-like structures and the analysis of Chapter 3 describes transitions like the helix-coil transition very well [1].) For example, cosmic strings in the early universe and dislocation lines in crystals experience *both* repulsive and attractive forces. The force is repulsive (attractive) for parallel (antiparallel) strings [50, 51]. As the string orientations are in general random, we would expect the string configurations to be rather more complicated than those described in Chapter 3. For

this reason, we shall now proceed by incorporating string interactions for *general* string configurations[1]. In doing so, we shall include mutual interactions between strings as well as self-interactions of a particular string.

We shall start by looking at the simpler case where we take the strings to be fundamental objects with temperature independent parameters. (In the next chapter we shall extend the analysis to include 'composite' strings, such as global cosmic strings.) We shall see how the inclusion of string interactions can give rise to a first-order phase transition, and how it is the strength of the string interactions which determines the order of the transition.

Specifically, in this chapter, we shall re-examine and extend a model of crystal melting proposed by Edwards and Warner [5]. In this model it is the sudden proliferation in the number density of dislocations, which behave as interacting random walks on an underlying lattice, that drives the transition. This is not the only model of melting. At the other extreme, Kleinert [52] has proposed a model for two-dimensional melting in which the presence of *disclinations* is crucial in order for a first-order transition to occur. Disclinations are rotation dislocations that disrupt the rotational symmetry and which induce curvature into the crystal. In his analysis, Kleinert suggests that the disclinations form a resevoir of entropy which is frozen into the crystal. When the dislocations proliferate, this entropy is released which, in turn, favours the formation of further dislocations. In this case, dislocations still provide a trigger for melting but, for example, would not be the mechanism for describing the latent heat. In our model, by downgrading the role of the lattice and not invoking a lattice field theory [52, 53, 54], we capture the essential physics of melting while also developing a formalism that is applicable to the melting of iron and glass, as well as

[1]The work of this chapter appears in 'The Statistical Properties of Strings. II. Interacting Strings', proceedings of 'The Symposium on Cosmic Strings and Phase Transitions in the Early Universe', at Dampt (1989), to be published by the Cambridge University Press in 1990. It also appears in 'The Statistical Mechanics of Strings. I. Dislocation Melting', Imperial College preprint TP/89-90/05, submitted to Physical Review B.

the λ-transition of ^4He. In this chapter, we shall limit our discussion to the process of iron melting.

In dislocation melting, a liquid at the melting temperature T_M is considered to be a highly faulted solid. First-order melting is characterised by a discrete jump in ρ, the density of dislocations, at a temperature T_M. Edwards and Warner [5] take a rather simplified approach to the statistics. Although their calculations lead to a prediction that melting is a first-order transition they would also predict that all other related string phase transitions are too. This is known experimentally not to be the case. Our more careful calculations do not make these approximations and the condition for a first-order transition becomes more subtle, enabling us to correctly predict the order of the phase transition in a wider variety of systems.

The chapter is structured as follows. In Section 4.1 we discuss the string statistics and give an analytic saddle-point approximation to the model. Section 4.2 examines the interaction-free approximation, and Section 4.3 comments on the effects of making dislocations self-avoiding (*i.e.* the excluded volume problem). In Sections 4.4 and 4.5 we present the theoretical and numerical analysis of the melting of iron.

The extension of these ideas to glass, helium and cosmic strings is straightforward (see [55] for glass and helium and Chapter 5 for cosmic strings).

4.1 The model

The melting of crystals can be understood by viewing a liquid as a highly faulted solid. The faults are the dislocation lines that disrupt the translational symmetry of the crystal. Melting can be considered to occur when the solid becomes unstable with respect to a state in which there is a very large number of faults. There have been many attempts to implement this idea [3, 4, 5, 52, 53, 54, 56] over the years but we shall focus on one of the simplest [5]. In this model a dislocation is considered to be an *orientable* (lattice) string which forms a *closed* path within the lattice, having

constant intrinsic energy/length, independent of the orientation of the dislocation. Initially, for simplicity, we consider the underlying lattice to be a simple cubic lattice of spacing a_o. In the latter sections, when we consider the melting of iron in detail, for better accuracy we will take the underlying lattice to be body-centered cubic.

On a lattice a string consists of a sequence of hinged straight segments, each joining two lattice sites in one of the crystal directions. For a cubic crystal each segment has minimum length a_o . However, as we saw in Chapter 2, because of stiffness the effective step length $l_o > a_o$. The configuration of a string of length $L > l_o$ can be specified by enumerating the lattice sites $\underline{r}^{(i)}$ $(i = 1, 2, \ldots, L/l_o)$ of the segment ends, with respect to a fixed origin. At thermal equilibrium (temperature $T = 1/\beta$ in units where $k_B = 1$), the partition function for a system containing N strings of lengths $L_1, L_2, \ldots, L_N$ is

$$Z_N(L_1, \ldots, L_N) = \sum_{\{\underline{r}_p\}} e^{-\beta E[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]} \tag{4.1}$$

where $p = 1, 2, \ldots, N$. Here $\underline{r}_p = (\underline{r}_p^{(1)}, \underline{r}_p^{(2)}, \ldots, \underline{r}_p^{(n)}; n = L_p/l_o)$ denotes the configuration of the pth string, length L_p. The sum is taken over all N-string configurations, the energy for the configuration $\{\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N\}$ being $E[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]$. The general N-string partition function is obtained by summing over all string lengths,

$$Z_N = \sum_{\{L_p\}} Z_N(L_1, L_2, \ldots, L_N) \tag{4.2}$$

When the string $\underline{r} = (\underline{r}^{(1)}, \underline{r}^{(2)}, \ldots, \underline{r}^{(n)}; n = L/l_o)$ has $n >> 1$ steps it can be approximated by a continuous curve, parameterised by $\underline{r}(s)$, $0 \leq s \leq L = nl_o$. In this approximation the sum (4.1) is re-expressible as a functional integral over paths $\underline{r}_p(s_p)$ $(0 \leq s_p \leq L_p)$ as

$$Z_N(L_1, \ldots, L_N) = \oint \prod_1^N d\mu_{L_p}[\underline{r}_p]\, e^{-\beta E[\underline{r}_1, \ldots, \underline{r}_N]} \tag{4.3}$$

where $d\mu_L[\underline{r}]$ is a measure for the number density of configurations in which the lattice string is close to the continuum string $\underline{r}(s)$. In this continuum approximation

the lattice mainly serves to discretise an otherwise infinite number of configurations. By definition

$$\oint d\mu_L[\underline{r}] = n(L) \tag{4.4}$$

where $n(L)$ is the number of closed loops of length L on the lattice.

There are two separate contributions to $E[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]$. The first is the intrinsic core energy of the strings

$$E_o[\underline{r}_1, \ldots, \underline{r}_N] = \sigma \sum_{p=1}^{N} L_p \tag{4.5}$$

The core energy density σ takes the form

$$\sigma = \frac{\delta \mu a_o^2}{4\pi} \tag{4.6}$$

where μ is the shear modulus, and $\delta \sim 1$ takes into account non-linear effects (assumed constant), and the choice of the Burgers vector as a_o. The dislocations in our model are thus more primitive than real dislocations but, as we shall see, they still cause crystals to melt.

The second, and much more complicated term, describes string interactions. Assuming string interactions are pairwise it can be written as

$$E_{int}^{(2)}[\underline{r}_1, \ldots, \underline{r}_N] = \frac{1}{2} \sum_{\alpha,\beta} E_{int}[\underline{r}_\alpha, \underline{r}_\beta] \tag{4.7}$$

The general form for $E_{int}[\underline{r}_1, \underline{r}_2]$ has been given by Blin [51]:

$$\begin{aligned} E_{int}[\underline{r}_1, \underline{r}_2] &= \frac{\mu}{4\pi} \int \frac{(\underline{b}_1 . d\underline{l}_1)(\underline{b}_2 . d\underline{l}_2)}{r_{12}} - \frac{\mu}{2\pi} \int \frac{(\underline{b}_1 \wedge \underline{b}_2) . (d\underline{l}_1 \wedge d\underline{l}_2)}{r_{12}} \\ &+ \frac{\mu}{4\pi(1-\nu)} \int (\underline{b}_1 \wedge d\underline{l}_1)(\underline{\nabla}\,\underline{\nabla} r_{12})(\underline{b}_2 \wedge d\underline{l}_2) \end{aligned} \tag{4.8}$$

where r_{12} is the separation between the two incremental dislocation sections $d\underline{l}_1$ and $d\underline{l}_2$, with associated Burgers vectors $\underline{b}_1$ and $\underline{b}_2$, and ν is the Poisson ratio. Blin's expression is only valid in the dilute string approximation, and this should be kept in mind throughout. We follow Edwards and Warner [5] in making the simplifying assumption that E_{int} can be approximated by

$$E_{int}[\underline{r}_1, \underline{r}_2] = \frac{\gamma \mu}{4\pi} \int \frac{(\underline{b}_1 . d\underline{l}_1)(\underline{b}_2 . d\underline{l}_2)}{r_{12}} \tag{4.9}$$

The factor γ is introduced in an attempt to take into account the other terms in Blin's description. We note that (4.9) gives a result very similar to that of choosing

$$E_{int}[\underline{r}_1, \underline{r}_2] = \frac{\gamma\mu b^2}{4\pi} \int \frac{d\underline{l}_1 . d\underline{l}_2}{r_{12}} \tag{4.10}$$

The form (4.10) is, for example, relevant for dilute *global cosmic strings* and for dilute vortices in liquid helium. Since we are concentrating in this chapter on melting we persist with (4.9). In the next chapter, where we examine global cosmic strings, we shall use an expression similar to (4.10) for the form of the interactions.

In terms of the string parameters, s_1 and s_2, (4.9) becomes

$$E_{int}[\underline{r}_1, \underline{r}_2] = \frac{\gamma\mu}{4\pi} \int ds_1\, ds_2 \frac{(\underline{b}_1 . \dot{\underline{r}}_1(s_1))\,(\underline{b}_2 . \dot{\underline{r}}_2(s_2))}{|\underline{r}_1(s_1) - \underline{r}_2(s_2)|} \tag{4.11}$$

where $\dot{\underline{r}} = d\underline{r}/ds$. Introducing the scalar source term

$$\rho(\underline{r}) = \sum_{p=1}^{N} \int ds_p\, \underline{b}_p . \dot{\underline{r}}_p(s_p)\, \delta(\underline{r} - \underline{r}_p(s_p)) \tag{4.12}$$

the total interaction energy for N strings can be expressed more simply as

$$E^{(2)}_{int}[\underline{r}_1, \ldots, \underline{r}_N] = \frac{\gamma\mu}{8\pi} \int d\underline{u}\, d\underline{v}\, \rho(\underline{u}) \frac{1}{|\underline{u} - \underline{v}|} \rho(\underline{v}) \tag{4.13}$$

where, by virtue of the diagonal terms, we have taken self-interactions into account with the same force-law.

The form (4.13) permits a great simplification in the construction of Z_N of (4.3). Firstly, Z_N factors as

$$Z_N(L_1, L_2, \ldots, L_N) = e^{-(\beta\sigma \sum_p L_p)}\, Z^{int}_N(L_1, L_2, \ldots, L_N) \tag{4.14}$$

where

$$Z^{int}_N(L_1, L_2, \ldots, L_N) = \oint \prod_1^N d\mu_{L_p}[\underline{r}_p]\ \exp\left[-\frac{\beta\gamma\mu}{8\pi} \int d\underline{u}\, d\underline{v}\, \rho(\underline{u}) \frac{1}{|\underline{u} - \underline{v}|} \rho(\underline{v}) \right] \tag{4.15}$$

contains all the interaction effects. This, in turn, can be written as a path integral over a scalar field $A(\underline{r})$ in three dimensions,

$$Z^{int}_N(L_1, \ldots, L_N) = \int DA\, e^{-\frac{1}{2}\int d\underline{r}\ (\underline{\nabla} A)^2} \oint \prod_1^N d\mu_{L_p}[\underline{r}_p]\, e^{i\sqrt{\beta\gamma\mu} \int A(\underline{r})\, \rho(\underline{r})\, d\underline{r}} \tag{4.16}$$

as becomes obvious upon completing the square in the exponent.

There are now two ways in which to proceed. We choose to integrate over $d\mu[\underline{r}]$ first, then to integrate over A, and finally to integrate over the $\{L_p\}$ to obtain Z_N of (4.2). [The alternative, in which we first integrate over the $\{L_p\}$, then over $d\mu[\underline{r}]$ and, finally, over A will be discussed in Chapter 6. It leads to a second-quantized Euclidean field theory of dislocations]. From the form (4.12) for $\rho(\underline{r})$, $Z_N^{int}(L_1,\ldots,L_N)$ becomes

$$Z_N^{int}(L_1,\ldots,L_N) = \int DA\, e^{-\frac{1}{2}\int d\underline{r}\,(\nabla A)^2} \prod_1^N I_{L_p}[A] \tag{4.17}$$

where

$$I_{L_p}[A] = \oint d\mu_{L_p}[\underline{r}]\; e^{i\sqrt{\beta\gamma\mu}\int ds_p\, \underline{b}_p\cdot\underline{\dot{r}}_p(s_p)\, A(\underline{r}_p(s_p))} \tag{4.18}$$

$I_L[A]$ was calculated by Edwards and Warner [5], and we shall quote the result later. However, since we part company from them after this point, we shall examine (4.18) carefully.

Let $p(\underline{r}_1,\underline{r}_2;L)$ be the probability density that a random walk of L/l_o steps of length l_o begins at $\underline{r}_1$ and ends at $\underline{r}_2$. Then p can be written as the path integral over paths $\underline{r}(s)$ giving

$$p(\underline{r}_1,\underline{r}_2;L) = \int_{\underline{r}(0)=\underline{r}_1}^{\underline{r}(L)=\underline{r}_2} D\underline{r}\; e^{-\frac{3}{2l_o}\int_0^L \underline{\dot{r}}^2\, ds} \tag{4.19}$$

Thus, on the lattice, $I_L[A]$ can be decomposed as

$$I_L[A] = \frac{z^{L/l_o}}{(L/l_o)} \sum_{\underline{u}} \oint_{\underline{r}(0)=\underline{u}}^{\underline{r}(L)=\underline{u}} D\underline{r}\; e^{-\int_0^L ds\left[\frac{3}{2l_o}\underline{\dot{r}}^2 - i\sqrt{\beta\gamma\mu}\,\underline{b}\cdot\underline{\dot{r}}\,A(\underline{r})\right]} \tag{4.20}$$

where we have taken $\underline{r}_1 = \underline{r}_2 = \underline{u}$ as dislocations are generally closed. In (4.20) the loop integral describes the probability that a path of length L will return to its starting point $\underline{u}$. The sum over $\underline{u}$ permits that point to be anywhere on the lattice. The factor $(L/l_o)^{-1}$ removes the overcounting of orientable closed loops that could have begun at L/l_o starting points. Finally, z denotes the number of possible directions in which the string can move at each step. For a cubic crystal $z = 2d-1 = 5$ in $d = 3$ dimensions if the string is not permitted to backtrack. For a *b.c.c.* crystal with no backtracking $z = 7$.

When $A = 0$, $I_L[0]$ equals $n(L)$, the number of closed paths of L/l_o steps on the lattice. For later use, it is helpful to factor out the power of z as

$$I_L[0] = z^{L/l_o}\,\bar{n}(L/l_o) \tag{4.21}$$

$$= \frac{z^{L/l_o}}{(L/l_o)} \sum_{\underline{u}} \oint D\underline{r}\, e^{-\frac{3}{2l_o}\int_0^L ds\, \dot{\underline{r}}^2} \tag{4.22}$$

Returning to $I_L[A]$ of (4.20) Edwards and Warner show that, on averaging over $\underline{b}$, the effect of the string-string interactions is to give

$$I_L[A] = I_L[0] e^{-\frac{l_o b^2 \beta\gamma\mu L}{6V}\int d\underline{r}\, A^2(\underline{r})} \tag{4.23}$$

$$= n(L)\, e^{-\frac{l_o b^2 \beta\gamma\mu L}{6V}\int d\underline{r}\, A^2(\underline{r})} \tag{4.24}$$

The result (4.24) is most easily seen by using the closure of the string $\underline{r}(s) = \underline{r}(s+L)$ to Fourier expand $\underline{r}(s)$ as $\underline{r}(s) = \sum_n \underline{r}_n\, e^{2\pi i n s/L}$. Inserting (4.24) into (4.17) gives

$$Z_N^{int}(L_1, L_2, \ldots, L_N) = \int DA\, e^{-\frac{1}{2}\int d\underline{r}\left[(\underline{\nabla} A)^2 + M^2 A^2\right]} \prod_p n(L_p) \tag{4.25}$$

with

$$M^2 = \frac{1}{3} l_o b^2 \beta\gamma\mu \left(\sum_p L_p/V\right) \tag{4.26}$$

The A-field integral is a Gaussian and can be performed to obtain

$$Z_N^{int}(L_1, L_2, \ldots, L_N) = \left(\prod_{p=1}^N n\,(L_p)\right) \det\left[-\nabla^2 + M^2\right]^{-1/2}$$

$$= \left(\prod_{p=1}^N n\,(L_p)\right) \exp\left[-\frac{1}{2} tr \ln\left(-\nabla^2 + M^2\right)\right]$$

$$= \left(\prod_{p=1}^N n\,(L_p)\right) \exp\left[-\frac{1}{2} V \int \frac{d\underline{k}}{(2\pi)^3} \ln(\underline{k}^2 + M^2)\right] \tag{4.27}$$

where M^2 is given by (4.26).

If we now return to $Z_N(L_1, L_2, \ldots, L_N)$ of (4.14), it becomes, after using the definition (4.21),

$$Z_N(L_1, \ldots, L_N) = \left[\prod_{p=1}^N \bar{n}(L_p/l_o)\right] e^{-\beta V G\left(\sum L_p/V\right)} \tag{4.28}$$

where, in terms of the dislocation density $\rho = \sum L_p/V$

$$\beta G(\rho) = \beta\left(\sigma - \frac{\ln z}{l_o\beta}\right)\rho + \frac{1}{2}\int \frac{d\underline{k}}{(2\pi)^3}\ln(\underline{k}^2 + \frac{1}{3}l_o b^2\beta\gamma\mu\rho) \tag{4.29}$$

$$= \beta\sigma_{eff}\rho + \frac{1}{2}\int \frac{d\underline{k}}{(2\pi)^3}\ln\left(1 + \frac{l_o b^2\beta\gamma\mu\rho}{3\underline{k}^2}\right) + \bar{c} \tag{4.30}$$

In (4.30) the constant $\bar{c}$ can be absorbed in the normalization of Z_N, and

$$\sigma_{eff} = \sigma - \frac{\ln z}{l_o\beta} \tag{4.31}$$

is the effective core energy per unit length.

The effect of our rudimentary way to incorporate stiffness (replacing a_o by fixed $l_o > a_o$) is to increase σ_{eff} at fixed T, as we know to be the case. A more realistic description of stiffness was given in Chapter 2, where we incorporated a temperature-dependent effective step-length $l_o(T)$ as

$$\sigma_{eff} = \sigma - \frac{\ln(1 + (z-1)e^{-\beta u})}{a_o\beta} \tag{4.32}$$

(see (2.28) and (2.29)) where u is the energy cost in turning the dislocation through a right angle. Given the nature of the approximations that we have already made, it would be overcomplicating the issue to introduce this further temperature dependence and we let (4.31) stand.

The sums (integrals) in (4.2) over string lengths can be performed to obtain

$$Z_N = \int \prod_1^N \left(dL_p\,\bar{n}\left(L_p/l_o\right)/l_o\right)\, e^{-\beta VG(\sum L_p/V)} \tag{4.33}$$

where $\bar{n}$ was defined in (4.22). Introducing the decomposition of unity,

$$1 = \int dL\,\delta\left(L - \sum_p L_p\right) \tag{4.34}$$

enables Z_N to be written as

$$Z_N = \int dL\, e^{-\beta VG(L/V)} \int \prod_1^N \left(dL_p\,\bar{n}\left(L_p/l_o\right)/l_o\right)\,\delta(L - \sum L_p) \tag{4.35}$$

Rather than work in terms of ρ and L_p, it is convenient to work with the dimensionless parameters $l = L/l_o$, $l_p = L_p/l_o$, $v = V/l_o^3$, $\lambda = l/v$. Then

$$Z_N = \int dl\, e^{-\beta v \bar{G}(\lambda)} \int \prod_1^N dl_p\, \bar{n}(l_p)\, \delta(l - \sum l_p) \tag{4.36}$$

$$= \int dl\, e^{-\beta v \bar{G}(\lambda)} \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha l} \left(\int_{\eta_o}^{\infty} d\eta\, \bar{n}\,(\eta)\; e^{-i\alpha\eta} \right)^N \tag{4.37}$$

where

$$\beta \bar{G}(\lambda) = \beta \sigma_{eff} l_o \lambda + \frac{1}{2} l_o^3 \int \frac{d\underline{k}}{(2\pi)^3} \ln \left(1 + \frac{b^2 \beta \gamma \mu \lambda}{3 l_o \underline{k}^2} \right) \tag{4.38}$$

and $\eta_o \sim O(1)$ measures the shortest string in units of l_o.

Finally, the integrand of Z_N exponentiates in the grand canonical partition function Z, giving

$$Z = \sum_N \frac{1}{N!} Z_N \tag{4.39}$$

$$= v \int d\lambda\, e^{-\beta v \bar{G}(\lambda)} \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha\lambda v} \exp \left(\int_{\eta_o}^{\infty} d\eta\, \bar{n}(\eta)\, e^{-i\alpha\eta} \right) \tag{4.40}$$

It is this equation that contains all the information about melting.

We note that if we took $\bar{n}(\eta) = \delta(\eta - \eta_o)$ in (4.37) we would recover the result of [5], provided *we did not sum over N*, that

$$Z_{EW} = e^{-\beta V G(L/V)} \tag{4.41}$$

This corresponds to a situation in which all dislocations are assumed to have the same length and where the fact that they are closed has not been properly accounted for. It is this approximation that Edwards and Warner make. Since they do not sum over N, the closure factor $\bar{n}(\eta)$, given in (4.22), plays no role in obtaining a first-order phase transition. As we shall soon see, in our analysis this is not the case. The closure factor $\bar{n}(\eta)$ is crucial for first-order melting to occur.

The integral (4.40) cannot be evaluated analytically but, from (4.22)

$$\bar{n}(\eta) = v x^3 p(\eta)/\eta \tag{4.42}$$

where $p(\eta)$ is the probability that a path of η steps will close on a given starting-point, and x^3 is the number of starting points for a path in a cell of side l_o. Thus, in the large-V limit Z permits an approximate saddle-point solution

$$Z = \exp \bar{Z}_1 \tag{4.43}$$

where

$$\bar{Z}_1 = -v \left[\beta \bar{G}(\lambda_o) - \imath \alpha_o \lambda_o - x^3 \int_{\eta_o}^{\infty} d\eta \, (p(\eta)/\eta) \, e^{-\imath \alpha_o \eta} \right] \tag{4.44}$$

in which α_o, λ_o are the simultaneous solutions to the equations

$$\beta \bar{G}'(\lambda_o) = \imath \alpha_o \tag{4.45}$$

and

$$\lambda_o = x^3 \int_{\eta_o}^{\infty} d\eta \, p(\eta) \, e^{-\imath \alpha_o \eta} \tag{4.46}$$

Inserting (4.45) in (4.44) to eliminate α_o gives

$$\bar{Z}_1 = -v \left[\beta \left(\bar{G}(\lambda_o) - \lambda_o \bar{G}'(\lambda_o) \right) - x^3 \int_{\eta_o}^{\infty} d\eta \, (p(\eta)/\eta) \, e^{-\beta \bar{G}'(\lambda_o)\eta} \right] \tag{4.47}$$

where

$$\lambda_o = x^3 \int_{\eta_o}^{\infty} d\eta \, p(\eta) \, e^{-\beta \bar{G}'(\lambda_o)\eta} \tag{4.48}$$

In order to make any further progress we need an analytic form for $p(\eta)$ or $\bar{n}(L/l_o)$. For $L >> l_o$, the probability distribution $p(\underline{u}, \underline{v}; L)$ of (4.19) satisfies the diffusion equation

$$\left(\frac{1}{6} l_o \nabla_{\underline{u}}^2 - \frac{\partial}{\partial L} \right) p(\underline{u}, \underline{v}; L) = 0 \tag{4.49}$$

subject to the boundary condition

$$p(\underline{u}, \underline{v}; 0) = \delta(\underline{u} - \underline{v}) \tag{4.50}$$

This has solution

$$p(\underline{u}, \underline{v}; L) = \int \frac{d\underline{k}}{(2\pi)^3} \, e^{\imath \underline{k} \cdot (\underline{u} - \underline{v}) - \frac{1}{6} L l_o \underline{k}^2} \tag{4.51}$$

from which $\bar{n}(L/l_o)$ is constructed, via (4.22), as

$$\bar{n}(L/l_o) = \frac{vx^3}{(L/l_o)} \int \frac{d\underline{k}}{(2\pi)^3} e^{-\frac{1}{6}Ll_o\underline{k}^2} \tag{4.52}$$

$$= Avx^3(L/l_o)^{-5/2} \tag{4.53}$$

where $A = (3/2\pi)^{3/2}$. Equivalently,

$$p(\eta) = A\eta^{-3/2} \tag{4.54}$$

To the accuracy at which we are working there is no loss in extrapolating the power behaviour (4.53) and (4.54) down to short lengths, and we shall henceforth do so. Note that (4.53) is the same statistical factor which was used in the previous chapters (see (1.4), (2.27) and (3.14)) except for a factor of 1/2. We do not include this factor here as interacting strings *are* orientable.

4.2 The no-interaction approximation

As a simple test of the formalism we start by considering the trivial case in which string interactions are neglected ($\gamma = 0$). In this case $\bar{G}(\lambda)$ of (4.38) becomes

$$\bar{G}^{(0)}_{(\lambda)} = \sigma_{eff} l_o \lambda \tag{4.55}$$

thus $\lambda_o \bar{G}'(\lambda_o) - \bar{G}(\lambda_o)$ vanishes in (4.47), to give $\bar{Z}_1 = Z_1$,

$$Z_1 = \int_{\eta_o}^{\infty} d\eta\, \bar{n}(\eta)\, e^{-\beta\sigma_{eff} l_o \eta} \tag{4.56}$$

After minimal rewriting Z_1 becomes (via (4.53))

$$Z_1 = \frac{1}{l_o} \int dL\, Z_1(L) \tag{4.57}$$

$$= \frac{1}{l_o} \int dL\, \bar{n}(L/l_o)\, e^{-\beta\sigma_{eff} L}$$

$$\simeq \frac{Avx^3}{l_o} \int dL\, (L/l_o)^{-q-1}\, e^{-\beta\sigma_{eff} L} \tag{4.58}$$

where σ_{eff} was defined in (4.31) and $q = 3/2$. Note that Z_1 given in (4.58) is the continuum limit of the single free string partition function of (1.4). It is the partition function for a system containing a single closed string of arbitrary length. Thus

$$Z = \exp Z_1 \tag{4.59}$$

is the partition function for a non-interacting 'gas' of strings (see (1.8), (2.23) and (3.24)).

For such independent strings a phase transition occurs at the Hagedorn temperature [26] T_H, at which

$$\sigma_{eff}(T_H) = \sigma - T_H \left(\ln z\right)/l_o = 0 \tag{4.60}$$

(see (1.5)). Above T_H, Z_1 diverges, and T_H is understood as a *maximum* temperature. As T_H is approached the heat applied to the system is converted into the creation of yet more string, as the effective energy cost/unit length diminishes. At T_H the dislocation density becomes, from (4.48)

$$\lambda_o^H = x^3 \int_{\eta_o}^{\infty} d\eta \, p(\eta) \tag{4.61}$$

corresponding to the maximum fraction of cell edges having dislocations along them. [The integral is just the probability that a path leaving the origin will return].

The Hagedorn temperature is reached while the energy density in strings is finite. Increasing the energy density further does not cause the energy density in loops to increase. Instead, the extra energy density floods into an infinite string and, in this way, the energy density in the crystal can be made arbitrarily large. Of course, in reality, a point is reached when the strings start to overlap , the interactions between the strings become important and the free string picture no longer applies. The effect of interactions is to reduce the number of available states to a value which is less than that of free strings. Hence the entropy can no longer balance the energy and consequently the system does not possess a maximum temperature. The system can

exist in a new phase that is no longer describable at temperatures above T_{st} in terms of 'free' strings.

The 'free' string picture provides a crude description of the λ-transition in ^4He [7, 8], in which the strings are superfluid vortex loops [2]. The transition is pictured as occurring when the cores of the vortices overlap, corresponding to the system entering the normal fluid. Similarly, with some modifications, it can also describe the GUT transition in the early universe [15]. However, it is not a good candidate for a first-order melting transition, for which the interactions are all-important in bringing about the necessary cooperative behaviour.

The Hagedorn transition seems to be a generic feature of the thermodynamic properties of strings. As we shall see shortly, the existence of a first-order transition requires that the string-string interaction strength exceeds a certain threshold.

4.3 Excluded volume

Before attempting to solve equations (4.44) onwards we observe that the result (4.41) of Edwards and Warner [5], obtained by setting $\alpha_o = 0$ in (4.44), is

$$\bar{Z}_{EW} = -v\beta\bar{G}(\lambda_o) \tag{4.62}$$

where $\bar{G}'(\lambda_o) = 0$. So that a first-order transition could arise from (4.62), it was necessary to introduce by hand a term

$$\Delta E = Ga_o^4(\sum L_p/V)^2 \tag{4.63}$$

in the energy $E[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]$ for a N-string configuration. This was argued to be a consequence of the excluded volume of the dislocations, which causes the crystal to expand.

We shall see, when considering iron in the next Section, that there is no need to introduce any such term in $\bar{G}(\lambda)$ in order to get a first-order transition. However, we

[2] although this must not be taken too literally as interactions need to be accounted for

shall now show that new non-linear terms are automatically induced by giving the dislocations an excluded volume *i.e.* making it impossible for dislocations to overlap.

In principle we could modify the forces between dislocations to take this into account. As already stated in Section 1.1.3, there are two simple changes that we can make which give a flavour of the effect of imposing non-intersection. The first is to eliminate self-intersection by taking

$$p(\eta) = A\eta^{-q} \tag{4.64}$$

where $q = 7/4$, at low dislocation density, rather than $3/2$, as in (4.54). [In the no-interaction approximation q is similarly changed in (4.58).] The second change is to take

$$Z = \sum_N \binom{x^3 v}{N} \frac{Z_N}{(x^3 v)^N} \tag{4.65}$$

rather than (4.39), a proposal made by Rivier and Duffy [12]. The combination $\binom{x^3 v}{N}$ is the number of ways of putting N dislocations down on a lattice consisting of v boxes, in each of which there are x^3 places from which a dislocation can start. This is an excluded volume in the sense that no dislocation can start at the same point as any other. Note that (4.65) is a generalization of (1.28), allowing for more complicated lattices than the simple cubic lattice for which $x = 1$.

With Z_N as given in (4.37), the sum (4.65) can be performed, to give

$$Z = v \int d\lambda\, e^{-\beta v \tilde{G}(\lambda)} \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha\lambda v} \exp\left[v x^3 \ln\left(1 + A \int_{\eta_o} d\eta\, \eta^{-q-1}\, e^{-i\alpha\eta}\right)\right] \tag{4.66}$$

where the power-behaviour (4.64) has already been used.

After eliminating the α_o saddle-point, the counterpart to (4.48) becomes

$$\lambda_o = \frac{A x^3 \int_{\eta_o} d\eta\, \eta^{-q}\, e^{-\beta \tilde{G}'(\lambda_o)\eta}}{1 + A \int_{\eta_o} d\eta\, \eta^{-q-1}\, e^{-\beta \tilde{G}'(\lambda_o)\eta}} \tag{4.67}$$

On integrating by parts this becomes

$$\lambda_o = \frac{A x^3}{\beta \tilde{G}'(\lambda_o)} \left(\frac{\eta_o^{-q}\, e^{-\beta \tilde{G}'(\lambda_o)\eta_o} - q \int_{\eta_o} d\eta\, \eta^{-q-1}\, e^{-\beta \tilde{G}'(\lambda_o)\eta}}{1 + A \int_{\eta_o} d\eta\, \eta^{-q-1}\, e^{-\beta \tilde{G}'(\lambda_o)\eta}} \right) \tag{4.68}$$

Inserting this equation in (4.66) gives $Z = \exp \tilde{Z}_1$, where

$$\tilde{Z}_1 = -v\left[\beta\bar{G}(\lambda_o) - \beta\lambda_o\bar{G}'(\lambda_o) - x^3 \ln\left(\frac{1 + A\, e^{-\beta\bar{G}'(\lambda_o)\eta_o/(q\eta_o^q)}}{1 + \beta\bar{G}'(\lambda_o)\lambda_o/(qx^3)}\right)\right] \tag{4.69}$$

where, again, we have integrated by parts. This is in contrast with

$$\bar{Z}_1 = -v\left[\beta\bar{G}(\lambda_o) - \beta\lambda_o\bar{G}'(\lambda_o) - \frac{Ax^3\, e^{-\beta\bar{G}'(\lambda_o)\eta_o}}{q\eta_o^q} + \frac{\beta\bar{G}'(\lambda_o)\lambda_o}{q}\right] \tag{4.70}$$

that we get on inserting the simplified (4.48) into (4.44).

Let us now consider the simplest case, in which we neglect long-range interactions ($\gamma = 0$), for which $\bar{G}'(\lambda_o) = \sigma_{eff} l_o$. Then, for $\sigma_{eff} \geq 0$

$$\begin{aligned} \tilde{Z}_1 &= -vx^3\left[\ln\left(1 + \frac{\beta\sigma_{eff} l_o\lambda_o}{qx^3}\right) - \ln\left(1 + \frac{A\, e^{-\beta\sigma_{eff} l_o\eta_o}}{q\eta_o^q}\right)\right] \\ &= -vx^3\left[\frac{1}{qx^3}\beta\sigma_{eff} l_o\lambda_o - \frac{1}{2}\left(\frac{\beta\sigma_{eff} l_o\lambda_o}{qx^3}\right)^2 + \ldots\right] \end{aligned} \tag{4.71}$$

We see that the effect of self-avoidance has been to generate terms $O(\lambda^2)$ (albeit of the opposite sign from that of [5]). However, the general result persists that, in this case, there is yet again a phase transition at the Hagedorn temperature. When $\gamma \neq 0$ the situation is more complicated but, in practice, we find that the relevant saddle-point occurs for λ_o so small that the terms $O(\lambda^2)$ are negligible, and we shall have no need to worry about an excluded volume. Similarly, the results are relatively insensitive as to whether $q = 3/2$ or $7/4$, and we remain with the former.

4.4 The melting of iron

The discussion of Section 4.1 assumed a simple cubic crystal. Some slight modifications have to be made when considering iron which has a body-centered cubic crystalline structure.

Let a_o be the half-diagonal length of the lattice cell, thus $\tilde{a} = 2a_o/\sqrt{3}$ measures the cell size. The step-length $l_o = xa_o$. To be concrete we again adopt the power

behaviour of (4.53), even for small L/l_o, and thus take

$$n(L) = \frac{2AV}{\tilde{a}^3}\left(\frac{L}{l_o}\right)^{-5/2} z^{L/l_o} \tag{4.72}$$

$$= \frac{3\sqrt{3}AVx^3}{4l_o^3}\left(\frac{L}{l_o}\right)^{-5/2} z^{L/l_o} \tag{4.73}$$

For volume V the number of 'big' boxes with half-diagonal length l_o is

$$v = \frac{3\sqrt{3}}{8}\frac{V}{l_o^3} \tag{4.74}$$

The analysis of Section 4.1 goes through, as before, to give (in place of (4.40))

$$Z = v\int d\lambda\, e^{-\beta v\bar{G}(\lambda)} \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha\lambda v} \exp\left[2Avx^3 \int_{\eta_o}^{\infty} d\eta\, \eta^{-5/2}\, e^{-i\alpha\eta}\right] \tag{4.75}$$

where

$$\beta\bar{G}(\lambda) = \beta\sigma_{eff} l_o\lambda + \frac{4l_o^3}{3\sqrt{3}}\int \frac{d\underline{k}}{(2\pi)^3} \ln\left(1 + \frac{\sqrt{3}b^2\beta\gamma\mu\lambda}{8l_o\underline{k}^2}\right) \tag{4.76}$$

The dimensionless λ in (4.76) is related to L/V by

$$\lambda = \frac{L}{l_o}\frac{1}{v} = \frac{8}{3\sqrt{3}} l_o^2 \frac{L}{V} \tag{4.77}$$

We now introduce y as

$$y = \theta\lambda = \left(\frac{\sqrt{3}b^2\beta\gamma\mu d^2}{8l_o}\right)\lambda \tag{4.78}$$

where d^{-1} is the ultra-violet cut-off, related to the effective dislocation core radius. Integrating over $k \leq d^{-1}$ in (4.76) gives

$$\bar{G}(\lambda) = \frac{8l_o^3}{3\sqrt{3}} g(y) \tag{4.79}$$

where

$$g(y) = \frac{y}{6\pi^2\beta d^3}\left[\frac{9\pi\delta d}{2\gamma l_o} - \frac{18\pi^2 d\ln z}{b^2 l_o^2\beta\gamma\mu} + 1 + \frac{\ln(1+y)}{2y} - \sqrt{y}\arctan\frac{1}{\sqrt{y}}\right] \tag{4.80}$$

On rescaling α, and running the saddle-point contour along a path parallel to the *Im*-α axis, $\bar{Z}_1$ finally becomes

$$\bar{Z}_1 = -v\left[\frac{8}{3\sqrt{3}}\beta l_o^3 g(y_o) - x^3\alpha_o y_o - 2Ax^3\int_{\eta_o}^{\infty} d\eta\, \eta^{-5/2}\, e^{-x^3\theta\alpha_o\eta}\right] \tag{4.81}$$

in which y_o,α_o are the solutions to

$$y(\alpha) = 2A\theta x^3 \int_{\eta_o}^{\infty} d\eta\, \eta^{-3/2}\, e^{-\alpha\theta x^3 \eta} \tag{4.82}$$

and

$$\alpha(y) = \frac{8}{3\sqrt{3}x^3}\beta l_o^3 g'(y) \tag{4.83}$$

$$= \frac{4}{9\sqrt{3}\pi^2}\frac{a_o^3}{d^3}\left[\frac{9\pi d\delta}{2\gamma l_o}\left(1 - \frac{4\pi \ln z}{b^2 l_o \delta\beta\mu}\right) + \frac{3}{2}\left(1 - \sqrt{y}\arctan\frac{1}{\sqrt{y}}\right)\right] \tag{4.84}$$

Equations (4.81) to (4.84) replace (4.44) to (4.46) and are the basis for our subsequent calculations.

From (4.75) and (4.76) we see that the Hagedorn transition, which occurs for non-interacting dislocations, also takes place when interactions are included. The partition function given in (4.75) diverges when $\sigma_{eff} < 0$, due to the integration over dislocation lengths. This means that there is a Hagedorn transition at a temperature T_H (the temperature at which $\sigma_{eff} = 0$ (see (4.60)), where T_H has the same value as that obtained in the case of no interactions. Above this temperature the partition function diverges and hence the saddle-point approximation no longer makes sense in this regime. From (4.82) and (4.84), the Hagedorn dislocation density for iron is given by the solution to the equation

$$y_H = 2A\theta x^3 \int_{\eta_o}^{\infty} d\eta\, \eta^{-3/2} \exp\left[-\frac{2\theta}{3\sqrt{3}\pi^2}\left(\frac{a_o x}{d}\right)^3\left(1 - \sqrt{y_H}\arctan\frac{1}{\sqrt{y_H}}\right)\eta\right] \tag{4.85}$$

Therefore, the effect of interactions is to reduce the density of dislocations at the Hagedorn transition, while leaving the Hagedorn temperature unchanged.

Prior to inserting numbers for a_o, b, d etc. it is possible to see, qualitatively, how a first-order melting transition can come about. Let us plot (4.82) and (4.84) as graphs of α against y. Equation (4.82) implies that $\alpha \to \infty$ as $y \to 0$, and $\alpha \to 0$ as $y \to y_{max}$, where

$$y_{max} = 2A\theta x^3 \int_{\eta_o}^{\infty} d\eta\, \eta^{-3/2} \tag{4.86}$$

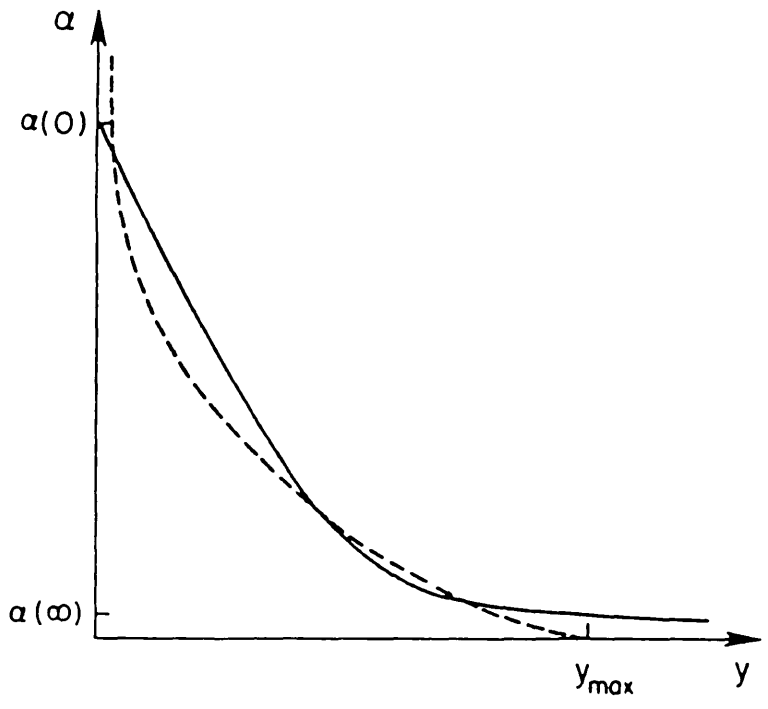

Figure 4.1: *A sketch of $y(\alpha)$ and $\alpha(y)$ where these functions are defined by equations (4.82) and (4.84).*

From (4.82) the value of y is constrained to lie in the range $0 \leq y \leq y_{max}$ (see Figure 4.1). The graph of α against y given by (4.84) is rather different. For small y, $\alpha \to \alpha(0)$,

$$\alpha(0) = \frac{4}{9\sqrt{3}\pi^2}\frac{a_o^3}{d^3}\left[\frac{9\pi d\delta}{2\gamma l_o}\left(1 - \frac{4\pi \ln z}{b^2 l_o \delta\beta\mu}\right) + \frac{3}{2}\right] \tag{4.87}$$

whereas, for large y, $\alpha \to \alpha(\infty)$,

$$\alpha(\infty) = \frac{4}{9\sqrt{3}\pi^2}\frac{a_o^3}{d^3}\left[\frac{9\pi d\delta}{2\gamma l_o}\left(1 - \frac{4\pi \ln z}{b^2 l_o \delta\beta\mu}\right)\right] \tag{4.88}$$

Again see Figure 4.1. From this figure, it is apparent that it is possible for the two functions $\alpha(y)$ to have *three* intersections. This describes a situation in which the free energy $-\bar{Z}_1/\beta$ has *two* minima. As T increases, the global minimum can flip from one to the other, giving rise to a discrete jump in y, which is what signals a *first-order* transition. Obviously, we are only interested in $y \leq y_H$ as our equations are not

valid for higher densities. Therefore, the occurence of a first-order phase transition depends on whether or not there are three intersections of the two functions $\alpha(y)$ in the region $0 \leq y \leq y_H$.

Associated with the discontinuous jump in y is a latent heat of disorder, L, the energy required to increase the dislocation density from the lower value of y_o to the higher one. If y_{o_1} and y_{o_2} are the two values of y at the melting point such that $y_{o_2} > y_{o_1}$, then the latent heat is given by

$$L = U(y_{o_2}) - U(y_{o_1}) \tag{4.89}$$

where U is the internal energy as a function of the dislocation density. The easiest way to obtain an expression for U is to present the partition function as

$$Z = -\imath v x^3 \int_o^\infty dy\, e^{-\beta v F(y)} \tag{4.90}$$

where

$$e^{-\beta v F(y)} = e^{-\frac{8}{3\sqrt{3}} v \beta l_o^3 g(y)} \int_{-\imath\infty}^{\imath\infty} d\alpha \exp\left[v x^3 \left(\alpha y + 2A \int_{\eta_o}^\infty d\eta\, \eta^{-5/2}\, e^{-x^3 \theta \alpha \eta} \right) \right] \tag{4.91}$$

Using (4.90) and the standard definition of U gives

$$U = -\frac{\partial \ln Z}{\partial \beta} \tag{4.92}$$

$$= \frac{v \int dy\, \frac{\partial}{\partial \beta} (\beta F(y))\, e^{-\beta v F(y)}}{\int dy\, e^{-\beta v F(y)}} \tag{4.93}$$

Since the integrand of (4.91) is independent of temperature

$$\frac{\partial}{\partial \beta} (\beta F(y)) = \frac{\partial}{\partial \beta} \left(\frac{8}{3\sqrt{3}} \beta l_o^3 g(y) \right) \tag{4.94}$$

To evaluate U we insert (4.94) into (4.93) and then approximate (4.93) by taking the large-V limit and performing a saddle-point in y. This gives an energy density

$$\frac{U}{V} = \frac{\partial}{\partial \beta} (\beta g(y)) \,|_{y_o} \tag{4.95}$$

where y_o is the solution to (4.82,4.84). Inserting the expression for $g(y)$ defined in (4.80) into (4.95) gives an energy density

$$\frac{U(y_o)}{V} = \frac{y_o}{6\pi^2 d^3 \beta}\left[\frac{9\pi\delta d}{2\gamma l_o} + \frac{3}{2}\left(1 - \sqrt{y_o}\arctan\frac{1}{\sqrt{y_o}}\right)\right] \tag{4.96}$$

and hence we can determine the latent heat of melting by using (4.89).

As we shall soon see, whether the functions $\alpha(y)$ and $y(\alpha)$ have three intersections or not depends highly on the values of the parameters. Our results contrast with those of Edwards and Warner [5], in which the free energy is taken to be $\bar{G}(\lambda)$ (or, up to normalization, $g(y)$). As we have observed, this corresponds to taking $\alpha_o = 0$ in (4.81) and inevitably gives a *first-order* transition, once $g(y)$ has been modified to allow for excluded volume effects of the dislocations. [In the absence of such terms only the Hagedorn transition arises.]

4.5 Numerical analysis

We shall now attempt to apply the ideas discussed in the previous sections to the process of melting iron. We shall concentrate on the simplest case, namely that of non-self-avoiding dislocations with no excluded volume. The idea is to choose reasonable values for the free parameters of the theory and then, using numerical analysis, to see if a first-order transition occurs. If it does, by fine tuning of the parameters, we shall try to find a set that gives a value for the melting temperature close to experiment.

The free parameters of the theory are the lattice spacing a_o, the length of the Burgers vector b, the ultra-violet cut-off d^{-1}, the step length l_o, the shear modulus μ, as well as the numbers δ and γ which narrow some of the approximations of the theory. We choose values for these parameters, and then, for a range of temperatures, we look for three intersections of the two functions $\alpha(y)$ and $y(\alpha)$ given by equations (4.82) and (4.84). Once we have found a set of parameters for which this is so, we examine the free energy and look for a flip in the global minimum to determine the melting temperature T_M (see Figure 4.2). Then, using equations (4.89) and (4.96) we

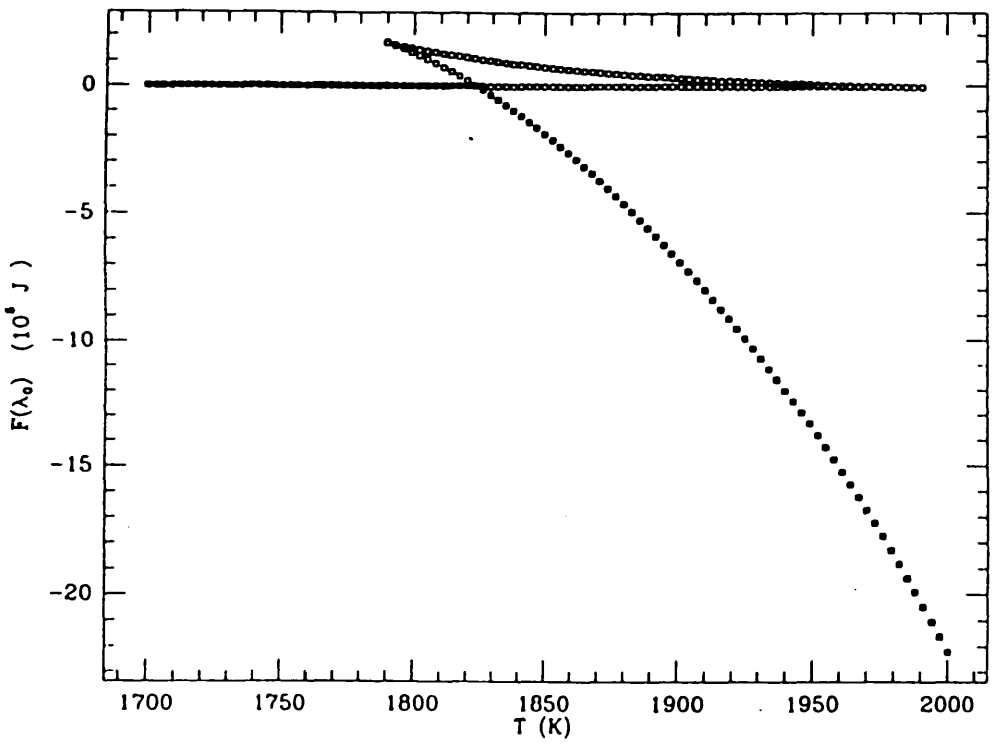

Figure 4.2: *A graph of the free energy determined at its stationary points versus temperature.*

calculate a value for the latent heat.

The parameters a_o, b, d^{-1}, l_o and μ are clearly dependent on temperature, but for the sake of simplicity we shall take them to be constant. We choose $a_o = 2.48\mathring{A}$, the lattice spacing for iron at room temperature. The Burgers vector $b \sim a_o$ and we take it to equal a_o. As mentioned by Edwards and Warner [5], the ultra-violet cut-off d^{-1} is of the magnitude $2\pi/a_o$. They took $d = \frac{qa_o}{2\pi}$ with $q = 15$, and it is this value for q that we shall use. The step length $l_o = xa_o$ and it appears that our best fit to experimental data occurs for $x = 2$ which, as we shall soon see, is not too different from an approximate estimate. We take the shear modulus $\mu = 4.2 \times 10^{10}$ N/m^2, $\delta \sim 1$ and $\gamma \sim 12$. We note that our values for δ and γ are in effect equivalent to those of Edwards and Warner [5], to the nearest whole number. In their final expression for the free energy density Edwards and Warner have wrongly taken the dislocation 'core energy density' to be $\sigma = \frac{\delta G a_o^2}{4\pi}$ where the bulk modulus $G = 10^{10}$ N/m^2 and $\delta = 3$. If instead of the bulk modulus Edwards and Warner had used the shear modulus, as we have done in (4.6) (and as, indeed, they have done in their initial expression for σ), with $\mu = 5 \times 10^{10}$ N/m^2, a value of $\delta = 3/5$ would be required to obtain their value for the 'core energy density'. Furthermore, when evaluating the interaction energy by summing over all possible pairs of strings, as is done in (4.7), Edwards and Warner have omitted the explicit factor of 1/2 which takes account of double-counting. This makes our choice of $\gamma = 12$ equivalent to their $\gamma = 6$.

For our choice of parameters we find a melting temperature for iron $T_M = 1824$ K (the experimental value is $T_{M_{exp}} = 1808$ K) (see Figure 4.2) and a latent heat $L \sim 10^7$ J/m^3 (see Figure 4.3). Note, from (4.6) and (4.60), that the Hagedorn temperature $T_H = 3797$ K. As $T_M < T_H$, the partition function is finite at the transition and hence our results are valid. The jump in the dislocation density is from $\lambda_{o_1} = 1.2 \times 10^{-5}$ to $\lambda_{o_2} = 2 \times 10^{-2}$, which corresponds to 1 in 200 lattice sites being dislocated just after melting (see Figure 4.4).

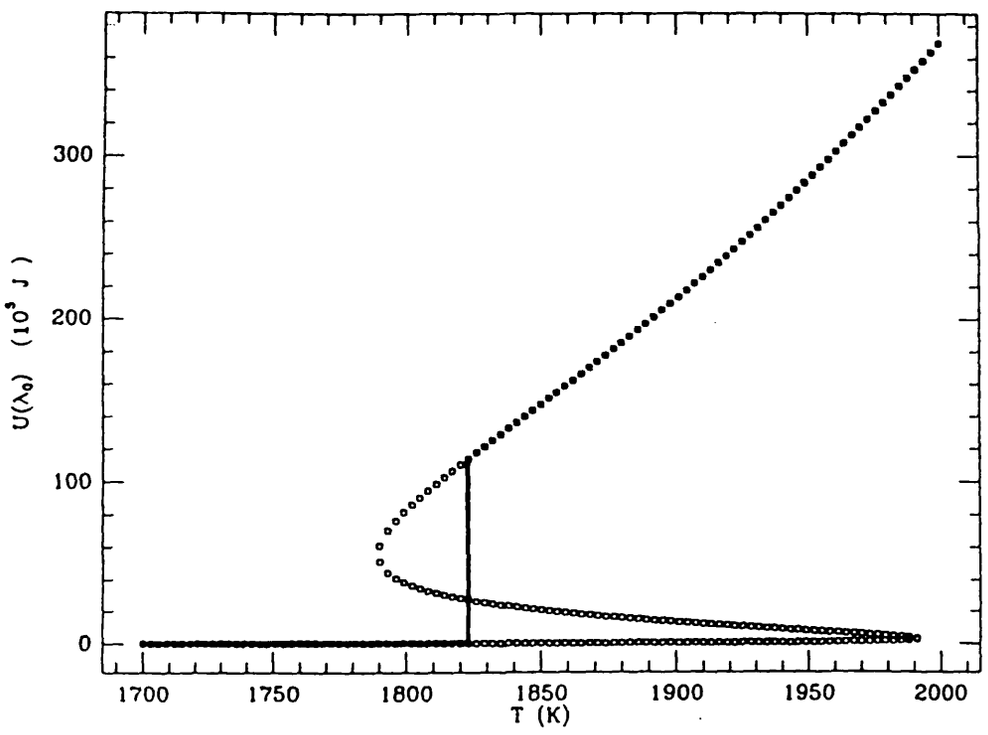

Figure 4.3: *A graph of $U(\lambda_0)$ versus temperature.*

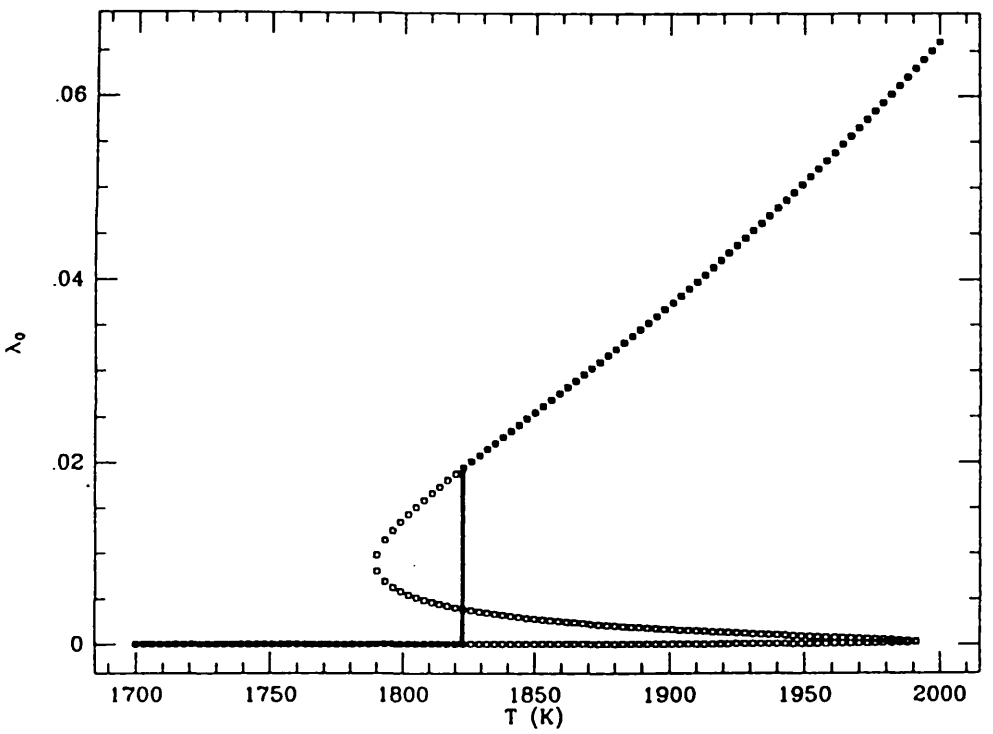

Figure 4.4: *A graph of* λ_0 *versus temperature.*

In Section 4.1 we mentioned that a very approximate way to incorporate stiffness is to replace the step length by l_o, where $l_o > a_o$. If we compare equations (4.31) and (4.32), we can make a rough estimate for l_o at the melting temperature,

$$l_o = \frac{a_o \ln z}{\ln[1 + (z-1)\, e^{-\beta_M u}]} \tag{4.97}$$

where we take the energy required to bend a dislocation to be $u \sim \sigma a_o$ and $\beta_M = 1/T_M$. Substituting our parameter values into (4.97) and using $T_M = 1824$ K, we find that

$$l_o \sim 3a_o \tag{4.98}$$

which is of the same order as our chosen value of $l_o = 2a_o$. Of course, this is a very approximate estimate because for a *b.c.c.* lattice there are two angles through which a dislocation can bend. A more accurate evaluation of the step length would need to allow for the fact that the probability of turning through a given angle and the energy cost to do so both depend on the angle itself (see Chapter 2).

As the equations of our theory are non-linear, the results are very sensitive to the choice of parameters. We now propose to examine the effect of each parameter on the melting temperature, the latent heat as well as the dislocation density.

- The lattice spacing increases as the temperature does, so at the melting point it is larger than the value we have taken. If instead we choose a larger value for a_o we find that melting occurs at a higher temperature and that the latent heat and the dislocation density λ_{o_2} remain unchanged. This follows directly from equations (4.80,4.81), which give the free energy, (4.82,4.84), which determine the solutions y_o and α_o, and (4.89,4.96), which give the latent heat, where a_o and T only occur in the combination a_o^3/T. Hence, increasing T such that a_o^3/T remains constant leaves all other quantities unaffected.

- The shear modulus decreases as the temperature increases. Reducing its value reduces both the melting temperature and the latent heat but leaves λ_{o_2} un-

changed. Again, this follows from the equations mentioned above, where μ always occurs in the combination μ/T, whereas T does not. It is this T dependence that causes the latent heat to decrease because $L = L(T, T_\mu = T/\mu)$ with $L \propto T$.

- The effect of increasing q is to decrease T_M, L and λ_{o_2}. In this case the dependence on q is rather more complicated, which is why λ_{o_2} is also changed.

- Stiffer dislocations, which corresponds to a larger value for x, cause T_M, L and λ_{o_2} to become larger. Part of the increase in λ_{o_2} is due to the fact that by definition $\lambda \propto x^2$. Stiffer dislocations also undergo a first-order transition more readily *i.e.* a first-order transition occurs for a larger range of parameters.

- Decreasing γ causes T_M, L and λ_{o_2} all to decrease. However, here the main effect is to make a first-order transition less likely. As discussed in Section 4.1, switching γ off completely results in only the Hagedorn transition persisting.

- A larger value for δ results in an increase in the melting temperature, a decrease in the latent heat as well as in the dislocation density. The order of the transition is also slightly affected, a first-order transition being more difficult for large values of δ.

Finally let us compare our findings with those of Edwards and Warner. In their analysis of crystal melting, Edwards and Warner develop an expression for the free energy by examining in detail the energy due to dislocations in crystals. Two of the terms that they include are the intrinsic energy of the dislocation cores and the energy due to mutual interactions between dislocations. They also introduce an extra term, which they argue to be a consequence of the excluded volume of the dislocations. However, in evaluating the free energy, Edwards and Warner assume that all dislocations have the same length, and that their number of dislocations,

N, is fixed. We feel that our model in which the dislocations are allowed to have arbitrary length and to be arbitrary in number is physically more plausible.

At first glance, Edwards' and Warner's results for the melting of iron seem to be rather more impressive than ours. On closer examination, however, they appear to contain a number of numerical errors. For example, the graph of the latent heat, L, against the dimensionless dislocation density, y, appears inconsistent with their equations. In fact, with their set of parameters and equations the latent heat of iron does not reach the experimental value of 2.1×10^9 J/m^3. Hence they cannot use this graph to determine y, and consequently they cannot predict the values for δ and γ using the method they propose. There are also errors in their interaction energy calculation resulting in an interaction energy of twice the correct value. If we amend the errors in their calculation, but retain their assumptions the free energy density would be given by $g(y)$ of (4.80). Thus, using the parameters for iron suggested by Edwards and Warner: $a_o = b = 2.53\mathring{A}$, $q = 15$, $l_o = a_o$ (which is equivalent to neglecting stiffness), $\mu = 5 \times 10^{10}$ N/m^2, $\delta = 3$, $\gamma = 6$ and $z = 8$ (with z in $g(y)$ substituted by $z - 1$) we find a melting temperature $T_M = 7480$ K, a latent heat $L = 2 \times 10^8$ J/m^3 and a dislocation density $y = 2 \times 10^{-2}$ when the crystal has just melted.

4.6 Discussion

In the previous sections we have attempted to demonstrate how the inclusion of particular forms of string interactions could, in principle, account for the diverse range of phase transitions that string like systems experience.

In particular, we applied the model to a theory of melting driven by dislocations [3, 56] and obtained encouraging results. For example, in the case of iron, we found that there is a small range of parameter values for which first-order melting is possible. We chose a set that gives a first-order transition at 1824 K with a corresponding latent

heat of $\sim 10^7$ J/m^3. Both our model and the one presented by Edwards and Warner predict values for the latent heat that are much smaller than the experimental value $L_{exp} = 2.1 \times 10^9$ J/m^3. We have only included the configurational entropy in the free energy. There is, in addition, also a contribution from the vibrational entropy which is due to the vibrations of the dislocation lines and the surrounding lattice. Vibrational degrees of freedom probably contribute an amount to the latent heat comparable to that of the configurational degrees of freedom. Incorporating this extra contribution to the latent heat is probably *not* enough to account for the discrepancy between our result and experiment. A possible explanation was given earlier in that the jump in the dislocation density could trigger the start of another process (Kleinert [52]). An alternative, and in our view a more plausible explanation, has been presented by Cotterill [57]. He suggested that the excess entropy of the liquid compared to the crystal at the same density is due mainly to the liquid's fluidity rather than its disorder. A small increase in the number density of dislocations can greatly increase the mobility of the atoms of the crystal. If this is correct it serves no purpose to fit the latent heat we have determined to the experimental value.

Qualitatively, our results are very different from those of Edwards and Warner. In their picture, the fact that dislocation lines are closed plays no role in obtaining a first-order phase transition. Also, it is essential that they include both the mutual interactions between dislocations *and* an excluded volume effect in the free energy in order to obtain first-order melting. Furthermore, they claim that a first-order phase transition is inevitable, the parameter values playing no role in determining the order of the transition. Our picture is very different. For us, closure is crucial for first-order melting to take place and, in our view, an excluded volume is not necessary to obtain a first-order phase transition but neither is such a transition inevitable. In fact, the order of the transition is extremely sensitive to the parameter values. From this we deduce that the order of the transition depends very closely on the system

under investigation. This is true even if the string interactions are of the same form as those experienced by dislocations, due to the sensitivity on the parameter values. For example, preliminary calculations show that the λ-transition of ^4He, is a *second-order* phase transition. In this case, the strings are superfluid vortices which behave as dislocations in their interactions, like topological defects in systems with *global* symmetry-breaking (with global string counterparts [50]).

There are, however, many caveats to the model. For example, the form for the interactions, given in (4.9), is only valid in the dilute string approximation. We have assumed that we are dealing with the 'classical regime' in our use of the $N!$ in (4.39) and we have assumed configurations in which the overlap between strings and anti-strings is negligible. Thus the results are not accurate for high string densities. We have also neglected the fact that the energy of a dislocation depends on its orientation, a screw dislocation having a lower energy than if it were an edge dislocation. Furthermore, we have restricted our dislocations to have the smallest allowable Burgers' vectors and we have not included the temperature dependence of the parameters.

Finally, we should mention that dislocation models are just one description of melting. There are many more (for details and references see [58]). For example, we can start from the free energy density functional theory of non-uniform systems, where the solid is viewed as a strongly non-uniform liquid. Freezing is pictured to result from the competition between the configurational entropy, which favours disorder, and the correlational entropy, which favours localization.

Chapter 5

The Statistical Mechanics of Interacting Cosmic Strings

In Chapter 1 we presented a simple model of string mediated phase transitions which was developed by Copeland et al. [15]. In this model the strings are assumed to be characterised by a constant intrinsic energy per unit length σ, a constant non-zero thickness a (which is much less than their length), a rigidity u, that measures the energy required to bend the string and a string interaction energy which includes both mutual and self-interactions. In their analysis, Copeland et al. [15] took the energy due to rigidity and interactions to be negligible, as compared to σa, and found that all string phase transitions are second-order and share the same critical exponents. In particular, Copeland et al. [15] examined the GUT transition due to the creation of cosmic string, which (for *local* $U(1)$ cosmic string) occurs at a Hagedorn temperature T_H [26] which satisfies

$$1 - \frac{T_H^2}{T_c^2} = O(g) \tag{5.1}$$

(see (1.62)). In (5.1), T_c is the mean-field temperature (that temperature at which a structureless transition would occur, see Chapter 1 or [15]), and g is the scalar self coupling. It has been assumed that $g \sim e^2 \ll 1$, where e is the gauge coupling. An identical result to (5.1) would have been obtained for the case of global strings with an effective cut-off [50]. The implication of (5.1) is that, at the phase transition, the separation between the flux tubes is scaled up by a factor of $O\left(\frac{1}{\sqrt{g}}\right)$ as compared to

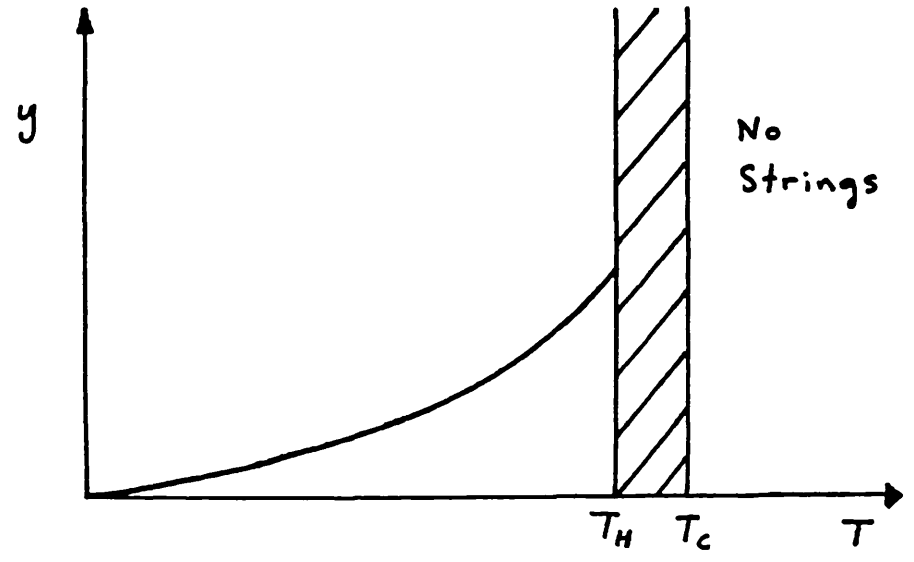

Figure 5.1: *The string density y as a function of temperature for the simplest case of no first-order transition. Above T_c no strings are possible. In the shaded region the effective strings are so wide and their strength so weak that it is not useful to use a string picture.*

the separation at zero temperature. This is just the Ginzburg length scale that was used by Kibble [10] to determine the initial length scale for the distribution of the Higgs fields at the phase transition.

Above the temperature T_c there is no global symmetry-breaking in the effective theory and strings are *not* possible. The heat-bath is just populated by the underlying field quanta. The interval between T_H and T_c is more complicated. For elementary strings (*e.g.* superstrings) that also undergo a Hagedorn transition [26] at a temperature T_H, the fluctuations are so large above T_H that the temperature is ill-defined, and the energy goes into creating an infinite string. This is not the case for composite vortex-like cosmic strings. Above T_H the effective string width is so large, and the string energy density so small that the dilute-string approximation used in (5.1) breaks down. A description based on strings is inappropriate here (see Figure 5.1).

Experimentally, we know that the order of the phase transition varies for different string-like structures. For example, melting (where the strings are dislocations) is a first-order transition. The model proposed in [15] is just too simple to accomodate this. Consequently, in Chapter 2, we proposed a more realistic model in which the known rigidity of cosmic strings was taken into account. The effect of this was to marginally increase the critical temperature, with (5.1) still holding, while increasing the effective size of the string segments. However, the order of the transition remained unchanged. We then sought to improve the model even further by including string interactions. Firstly, we examined the effect of simple self-interactions (adhesion) and found that their inclusion left the order of the transition unaltered. We then went on to look at more general interactions. In particular, we re-examined and extended a dislocation model of crystal melting proposed by Edwards and Warner [5]. When this was applied to the case of *iron* we obtained a *first-order* phase transition.

In this chapter[1] we shall extend these ideas to examine the phase transition at which the simpler *global* cosmic strings [26] are formed. Such strings experience interactions of approximately the same strength, as compared to the core energy density, as those experienced by the dislocation lines of iron. Therefore, if global cosmic strings were fundamental objects, with temperature independent parameters, we would expect a first-order phase transition with a significant latent heat. This would have observable cosmological consequences. However, cosmic strings are 'composite' objects, with temperature dependent parameters, and hence it is not obvious what effect interactions have on the order of the transition.

Initially, we shall include long-range interactions but ignore rigidity and allow the string to backtrack. Only later shall we improve the model by examining the effect of imposing no backtracking and incorporating rigidity.

[1]The work of this chapter appears in 'The Statistical Mechanics of Strings. II. Interacting Cosmic Strings', Imperial College preprint TP/89-90/07.

5.1 Non-self-avoiding interacting strings

As in the previous chapters, we require the strings to lie on the links of a lattice in order to make string configurations denumerable. Specifically, a global string is an *orientable* string which forms a *closed* path within the lattice. We assume a cubic lattice of spacing a (the temperature-dependent width scale of the string).

We follow the analysis of the previous chapter where we examined dislocation melting. As before, we present the partition function for a system containing N strings of lengths $L_1, L_2, \ldots, L_N$ as a functional integral over string paths

$$Z_N(L_1, \ldots, L_N) = \oint \prod_{p=1}^{N} d\mu_{L_p}[\underline{r}_p]\, e^{-\beta E[\underline{r}_1, \ldots, \underline{r}_N]} \tag{5.2}$$

(see (4.3)). In (5.2), $\underline{r}_p = (\underline{r}_p^{(1)}, \underline{r}_p^{(2)}, \ldots, \underline{r}_p^{(n)}; n = L_p/a)$ denotes the configuration of the pth string, length L_p, and $E[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]$ is the energy for the configuration $\{\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N\}$. For strings consisting of a large number of steps, $d\mu_L[\underline{r}]$ is a measure for the number density of configurations in which the lattice string is close to a continuum string $\underline{r}(s)$.

There are two separate contributions to $E[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]$. (Later, when we shall incorporate rigidity in a more sophisticated manner, there will be a third contibution to the energy). The first is the intrinsic core energy of the strings

$$E_o[\underline{r}_1, \ldots, \underline{r}_N] = \sigma \sum_{p=1}^{N} L_p \tag{5.3}$$

where σ is the core energy density (see (4.5)). The simplest candidates for global cosmic strings are $U(1)$ vortices arising from the Lagrangian density [15, 50]

$$\mathcal{L} = \frac{1}{2} |\partial_\mu \varphi|^2 + \frac{1}{2} m_s^2 |\varphi|^2 - \frac{g}{4!} |\varphi|^4 \tag{5.4}$$

for a complex scalar field φ. Note that (5.4) is equivalent to (1.32) without the gauge field. In thermal equilibrium the effective scalar theory acquires temperature-dependent parameters, as in Landau-Ginsberg superconductivity. Specifically, in the

one-loop approximation the effective scalar mass becomes

$$m_s(T) = m_o \left(1 - \frac{T^2}{T_c^2}\right)^{1/2} \tag{5.5}$$

where

$$T_c = \sqrt{\frac{18 m_o^2}{g}} \tag{5.6}$$

(see (1.39) and (1.49) in the limit $e^2 \to 0$). T_c is the mean-field critical temperature, at which the scalar field becomes massless. In terms of T_c the core energy density takes the form

$$\sigma \simeq \eta^2(T) \tag{5.7}$$

with

$$\eta^2(T) = \frac{6 m_s^2(T)}{g} = \frac{6 m_o^2}{g}\left(1 - \frac{T^2}{T_c^2}\right) \tag{5.8}$$

on imposing a cut-off at $a = m_s(T)^{-1}$ (*c.f.* (1.53)).

The second term describes string interactions. As for dislocations, we asssume that the interactions are pairwise, in which case this term can be written as

$$E_{int}^{(2)}[\underline{r}_1, \ldots, \underline{r}_N] = \frac{1}{2}\sum_{\alpha,\beta} E_{int}[\underline{r}_\alpha, \underline{r}_\beta] \tag{5.9}$$

(see (4.7)). The interaction energy for two parallel global U(1)-strings has been previously studied [50, 59]. For parallel strings with large separations, R, the force per unit length between them is given by [50]

$$F = \frac{2\pi\sigma}{R} \tag{5.10}$$

where the force is attractive (repulsive) for a string/antistring (string/string) pair. This result can be extended to include strings of all orientations. This has been done by Duffy [59] in the context of vortices in superfluid helium. Consider the simplest theory, containing global string solutions, with Lagrange density

$$\mathcal{L} = |\nabla\phi|^2 - V(\phi) \tag{5.11}$$

where $\phi = \eta e^{i\theta}$ is a complex scalar field. The interaction energy then takes the approximate form

$$E_{int} = \eta^2 \int d^3\underline{r}\,(\nabla\theta)^2 \tag{5.12}$$

Furthermore, the phase θ satisfies

$$\oint \underline{\nabla}\theta.d\underline{l} = 2\pi n \tag{5.13}$$

for any closed contour around the string and where n is the winding number of the string. From (5.13) we deduce that $\nabla\theta = n/r$ and hence, using this result together with the divergence theorem, we have that

$$\nabla^2\theta = 2\pi \sum_\alpha n_\alpha \delta^{(2)}(\underline{r} - \underline{r}_\alpha) \tag{5.14}$$

where we have generalised to a system of multiple parallel strings. In (5.14) α labels the strings, n_α is the winding number of the αth string and the δ-function is present as $\nabla\theta$ is undefined at the string core. The solution to (5.14) is

$$\theta(\underline{r}) = -\frac{1}{2}\sum_\alpha \int n_\alpha \frac{\delta^{(2)}(\underline{r}' - \underline{r}_\alpha)}{|\,\underline{r} - \underline{r}'\,|}\, d^3\underline{r}' \tag{5.15}$$

subject to the boundary condition that $\theta = 0$ at infinity. Integrating (5.12) by parts and using (5.14) and (5.15) with the same boundary condition gives that

$$E_{int} = \pi\eta^2 \sum_{\alpha,\beta} \int n_\alpha n_\beta \frac{dl_\alpha dl_\beta}{|\,\underline{r}_\alpha - \underline{r}_\beta\,|} \tag{5.16}$$

where here dl_α is an incremental string section of the αth string. On comparing (5.10) with (5.16) in the case of two interacting strings, we infer the simple form for closed strings

$$E_{int}[\underline{r}_1, \underline{r}_2] = 2\pi\sigma \oint \frac{d\underline{l}_1.d\underline{l}_2}{r_{12}} \tag{5.17}$$

where r_{12} is the separation between the two incremental string sections $d\underline{l}_1, d\underline{l}_2$ (see (4.10)). We note that (5.17) is only valid in the dilute string approximation where the mean separation between strings is much greater than a.

The form of the interaction given in (5.17) is very similar to an approximate expression for the energy due to interactions between dislocations in crystals (see [5] and references within). Therefore, we can proceed as we did in the case of dislocations (see Chapter 4) to evaluate the partition function in terms of a *vector* source term

$$\underline{\rho}(\underline{r}) = \sum_{p=1}^{N} \int ds_p \, \underline{r}'_p(s_p) \, \delta(\underline{r} - \underline{r}_p(s_p)) \tag{5.18}$$

where $\underline{r}' = d\underline{r}/ds$ (for dislocations we had a *scalar* source term, see (4.12)). For N strings we find that

$$Z_N(L_1, L_2, \ldots, L_N) = e^{-(\beta\sigma \sum_p L_p)} \, Z_N^{int}(L_1, L_2, \ldots, L_N) \tag{5.19}$$

where

$$Z_N^{int}(L_1, L_2, \ldots, L_N) = \oint \prod_{p=1}^{N} d\mu_{L_p}[\underline{r}_p] \, \exp\left[-\beta\pi\sigma \int d\underline{u} \, d\underline{v} \, \frac{\underline{\rho}(\underline{u}) \cdot \underline{\rho}(\underline{v})}{|\underline{u} - \underline{v}|} \right] \tag{5.20}$$

contains self-interactions of the strings as well as mutual interactions between strings (*c.f.* (4.15)). The interaction partition function $Z_N^{int}(L_1, \ldots, L_N)$ can be written as a path integral over a *vector* field $\underline{A}(\underline{r})$ in three dimensions, [5]

$$Z_N^{int}(L_1, \ldots, L_N) = \int D\underline{A} \, e^{-\frac{1}{2}\int d\underline{r} \, (\underline{\nabla} \cdot \underline{A})^2} \oint \prod_{p=1}^{N} d\mu_{L_p}[\underline{r}_p] \, e^{i\pi\sqrt{8\beta\sigma} \int d\underline{r} \, \underline{A}(\underline{r}) \cdot \underline{\rho}(\underline{r})} \tag{5.21}$$

(for dislocations it was a *scalar* field, see (4.16)). We proceed in an identical fashion as in the case of crystal melting by using the closure of the string $\underline{r}(s) = \underline{r}(s + L)$ to Fourier expand $\underline{r}(s)$ as $\underline{r}(s) = \sum_n \underline{r}_n \, e^{2\pi i n s/L}$, and find that

$$Z_N^{int}(L_1, L_2, \ldots, L_N) = \int D\underline{A} \, e^{-\frac{1}{2}\int d\underline{r} \left[(\underline{\nabla} \cdot \underline{A})^2 + M^2 \underline{A}^2 \right]} \prod_p n(L_p) \tag{5.22}$$

where

$$M^2 = 8\pi^2 a \beta\sigma \left(\sum_p L_p / V \right) \tag{5.23}$$

and

$$n(L) = \frac{z^{L/a}}{(L/a)} \sum_{\underline{u}} \oint_{\underline{r}(0)=\underline{u}}^{\underline{r}(L)=\underline{u}} D\underline{r} \, e^{-\frac{3}{2a}\int_0^L ds \, \underline{r}'^2} \tag{5.24}$$

$$= z^{L/a} \, \bar{n}(L) \tag{5.25}$$

is the number of closed paths of L/a steps on the lattice (*c.f.* (4.21), (4.22), (4.25) and (4.26)). As before, the factor $z^{L/a}$ in (5.24) is the total number of paths of length L starting at a given point, the loop integral describes the probability that such a path is closed, the sum permits that point to be anywhere on the lattice and the factor $(L/a)^{-1}$ removes the overcounting of closed loops. In (5.22) the $\underline{A}$-field integral is a Gaussian and can be performed to obtain that

$$\begin{aligned} Z_N^{\rm int}(L_1, L_2, \ldots, L_N) &= \left(\prod_{p=1}^{N} n(L_p)\right) \det\left[-\nabla^2 + M^2\right]^{-1/2} \\ &= \left(\prod_{p=1}^{N} n(L_p)\right) \exp\left[-\frac{1}{2} tr \ln\left(-\nabla^2 + M^2\right)\right] \\ &= \left(\prod_{p=1}^{N} n(L_p)\right) \exp\left[-\frac{3}{2} V \int \frac{d\underline{k}}{(2\pi)^3} \ln(\underline{k}^2 + M^2)\right] \end{aligned} \tag{5.26}$$

(*c.f.* (4.27)). In (5.26), the factor of 3 arises from the fact that $\underline{A}(\underline{r})$ is a three dimensional vector field.

If we now return to $Z_N(L_1, L_2, \ldots, L_N)$ of (5.19), it becomes, after using (5.25),

$$Z_N(L_1, \ldots, L_N) = \left[\prod_{p=1}^{N} \bar{n}(L_p/a)\right] e^{-\beta V G\left(\sum_p L_p/V\right)} \tag{5.27}$$

where, in terms of the string density $\rho = \sum_p L_p/V$

$$\beta G(\rho) = \beta \sigma_{eff} \rho + \frac{3}{2} \int \frac{d\underline{k}}{(2\pi)^3} \ln\left(1 + \frac{M^2(\rho)}{\underline{k}^2}\right) + \bar{c} \tag{5.28}$$

for some $\bar{c}$, and

$$\sigma_{eff} = \sigma - \frac{\ln z}{a\beta} \tag{5.29}$$

is the effective core energy per unit length of string (*c.f.* (4.28) and (4.31)).

As in the case of crystal melting, we now insert the decomposition of unity $1 = \int dL\, \delta\left(L - \sum_p L_p\right)$, sum over string lengths and the number, N, of strings to give the grand canonical partition function

$$Z = \sum_N \frac{1}{N!} Z_N \tag{5.30}$$

$$= v \int d\lambda\, e^{-\beta v \bar{G}(\lambda)} \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha\lambda v} \exp\left(\int_{\eta_o}^{\infty} d\eta\, \bar{n}(\eta)\, e^{-i\alpha\eta}\right) \tag{5.31}$$

where

$$\beta\bar{G}(\lambda) = \beta\sigma_{eff}a\lambda + \frac{3}{2}a^3\int\frac{d\underline{k}}{(2\pi)^3}\ln\left(1+\frac{8\pi^2\beta\sigma\lambda}{a\underline{k}^2}\right) \tag{5.32}$$

In (5.32) $l = L/a$, $l_p = L_p/a$, $v = V/a^3$, $\lambda = l/v$ are all dimensionless parameters, and $\bar{c}$ has been absorbed into the normalisation.

As in Chapter 4, we introduce the dimensionless parameter y as

$$y = \theta\lambda = \left(\frac{8\pi^2\beta\sigma d^2}{a}\right)\lambda \tag{5.33}$$

where d^{-1} is the ultra-violet cut-off which we shall take to be equal to the inverse of the core radius, a^{-1}. Integrating over $k \leq a^{-1}$ in (5.32) gives that

$$\beta\bar{G}(y) = \frac{y}{2\pi^2}\left[\frac{1}{4} - \frac{\ln z}{4a\beta\sigma} + 1 + \frac{\ln(1+y)}{2y} - \sqrt{y}\arctan\frac{1}{\sqrt{y}}\right] \tag{5.34}$$

The integral (5.31) cannot be evaluated analytically but in the large-V limit we can obtain an approximate expression for Z by using the saddle-point method. The result is that

$$Z = \exp\bar{Z}_1 \tag{5.35}$$

(see (4.43)) where, on rescaling α, and running the saddle-point contour along a path parallel to the Im-α axis, $\bar{Z}_1$ becomes

$$\bar{Z}_1 = -v\left[\beta\bar{G}(y_o) - \alpha_o y_o - A\int_{\eta_o}^{\infty} d\eta\,\eta^{-q-1}\,e^{-\alpha_o\theta\eta}\right] \tag{5.36}$$

where y_o,α_o are the solutions to the equations

$$y(\alpha) = A\theta\int_{\eta_o}^{\infty} d\eta\,\eta^{-q}\,e^{-\alpha\theta\eta} \tag{5.37}$$

and

$$\alpha(y) = \beta\bar{G}'(y) \tag{5.38}$$

$$= \frac{1}{2\pi^2}\left[\frac{1}{4}\left(1 - \frac{\ln z}{\sigma a\beta}\right) + \frac{3}{2}\left(1 - \sqrt{y}\arctan\frac{1}{\sqrt{y}}\right)\right] \tag{5.39}$$

(*c.f.* (4.81-4.84)). In (5.36) and (5.37), the form $A\eta^{-q}$ is given by the loop integral of (5.24) with $\eta = L/a$. Again, we extrapolate down to short lengths even though this expression for the probability density is only valid for a large number of steps, η.

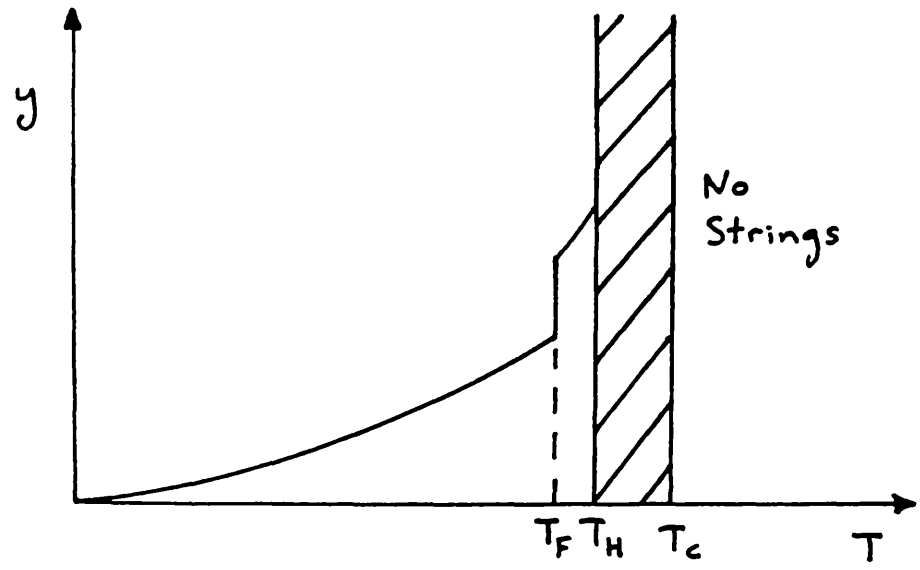

Figure 5.2: *The string density y as a function of temperature when a weak first-order transition is possible at T_F. The remainder of the diagram is interpreted as in Figure 5.1.*

From the expression for the partition function given in (5.31) and (5.32), we notice that the saddle-point solutions are only valid for $\sigma_{eff} \geq 0$. As soon as σ_{eff} goes negative the partition function diverges and a Hagedorn [26] transition takes place. This transition is unaffected by interactions as the interaction energy has decoupled from the effective core energy. The temperature for which $\sigma_{eff} = 0$ is the Hagedorn temperature, T_H (see Figure 5.1). Below T_H, the possibility of a first-order transition exists. Specifically, if the curves $y(\alpha)$ and $\alpha(y)$ given in equations (5.37) and (5.39) have *three* intersections then the free energy $-\bar{Z}_1/\beta$ has *two* minima. As T approaches T_H from below, the global minimum can flip from one to the other, giving rise to a discrete jump in y, *i.e.* a *first-order* transition (see Figure 5.2).

We shall now insert values for the temperature dependent parameters $\sigma(T)$ and $a(T)$ for global cosmic strings, evaluate the Hagedorn temperature and look for a first-order transition below it. The core energy density was given in (5.7,5.8). We

take $a(T) = m_s(T)^{-1}$. Working in units where the mean field temperature $T_c = 1$, substituting for $\sigma(T)$ and $a(T)$ in (5.29) and solving for $\sigma_{eff} = 0$ gives a Hagedorn temperature

$$T_H = \left(1 + \frac{1}{\bar{T}^2}\right)^{-1/2} \quad \text{where} \quad \bar{T} = \frac{1}{\ln z}\sqrt{\frac{2}{g}} \tag{5.40}$$

and hence, as in the case of no interactions, (5.1) holds. Furthermore, from the saddle-point conditions (5.37) and (5.39) we find that

$$y(\alpha, T) = A\theta(T) \int_{\eta_o}^{\infty} d\eta\, \eta^{-q}\, e^{-\alpha\theta(T)\eta} \tag{5.41}$$

where

$$\theta(T) = \frac{8\pi^2}{T}\left(\frac{2}{g}\left(1 - T^2\right)\right)^{1/2} \tag{5.42}$$

and

$$\alpha(y(T)) = \frac{1}{2\pi^2}\left[\frac{1}{4}\left(1 - \frac{1}{\bar{T}}\frac{T}{(1-T^2)^{1/2}}\right) + \frac{3}{2}\left(1 - \sqrt{y(T)}\arctan\frac{1}{\sqrt{y(T)}}\right)\right] \tag{5.43}$$

We are now in a position to evaluate the latent heat

$$L = U(y_{o_2}) - U(y_{o_1}) \tag{5.44}$$

where U is the internal energy as a function of the string densities y_{o_1} and y_{o_2} ($y_{o_2} > y_{o_1}$) at the phase transition (see (4.89)). Using the standard definition of U given in (4.92) we find, after some tedious algebra, an energy density (in units where $\hbar = c = 1$)

$$\begin{aligned} \frac{U(y_o)}{V} &= g\sigma_o^2 y_o(T_F)\left(1 - T_F^2\right)\left[\frac{1}{6}\left[1 + 3T_F^2\left(1 - \frac{1}{\bar{T}}\frac{T_F}{(1-T_F^2)^{1/2}}\right)\right]\right. \\ &+ \left.\left(1 - \sqrt{y_o(T_F)}\arctan\frac{1}{\sqrt{y_o(T_F)}}\right)\left(1 + 2T_F^2\right) + T_F^2\frac{\ln(1 + y_o(T_F))}{y_o(T_F)}\right] \end{aligned} \tag{5.45}$$

where σ_o is the core energy density at zero temperature and T_F is the first-order transition temperature. Hence, we can determine the latent heat by using (5.44). Note that the expression for internal energy given in (5.45) is much more complicated

than the equivalent equation for dislocations, (4.96). This is because we have now included the temperature dependence of the parameters.

For the case in hand, namely that of strings that can backtrack and are non-self-avoiding with no excluded volume, we shall approximate the probability of a random walk being closed by that obtained in the continuum limit. This means that we shall take $A = (3/2\pi)^{3/2}$ and $q = 3/2$. Still ignoring stiffness, the only undetermined parameter of the theory is g.

Early universe phenomenology requires that g be small [60]. We look for solutions to the saddle-point conditions for $g = 10^{-1}, 10^{-2}, 10^{-3}$ and 10^{-4}. We find that, in all cases, a first-order transition occurs, such that T_F is only just less than T_H and the jump in the string density is from $\lambda_{o_1} \sim 0$ to $\lambda_{o_2} \sim 0.2$. The value of λ_{o_2} is small and hence the dilute approximation is valid at the transition, justifying the force law given in (5.17). Numerically, we obtain the result that

$$1 - T_F^2 = O(g) \tag{5.46}$$

which is the same equation as that satisfied by T_H (see (5.1)).

Substituting (5.46) into (5.43) and (5.45) to determine the behaviour of the latent heat gives

$$L = O(g^2) \tag{5.47}$$

Hence, for strings that can backtrack and have no excluded volume associated with them, the transition is weakly first-order for all physically acceptable values of the scalar coupling, g.

5.2 Interacting strings with rigidity and no backtracking

As the next step, we try to examine the effect of imposing no backtracking on the strings. To do this we take $z = 2d - 1$ (instead of $2d$), and the minimum number of

steps for a closed path to be four (instead of two). When applied to global cosmic strings we now find that a first-order transition *does not occur* even when stiffness is absent. Imposing no backtracking reduces the number of available states, and hence the entropy, which, in turn, requires an increase in the transition temperature in order for the entropy to balance the energy at the transition. As $T_F \sim T_H$ for walks where backtracking is allowed, this means that only the Hagedorn transition survives. Some caution is necessary since, by imposing the strings to lie on a cubic lattice we have reduced the number of configurations to a value which is less than that in the continuum limit. If we were to lift this restriction, the effective coordination number, z, would increase and a first-order transition would become more likely.

However, we still have to incorporate rigidity (see Chapter 2). Although we have a good understanding of the rigidity of local cosmic strings [43], the situation is not so clear for global strings with an effective cut-off of the type that we are considering here. Fortunately, as we shall soon see, the details of string stiffness turn out to be immaterial. If the pth string consists of N_p staight segments and the energy required to bend a string through $0°, 90°$ and $180°$ is $0, u$, and ∞ (*i.e.* no backtracking) respectively, then the partition function becomes

$$Z_N(L_1, L_2, \ldots, L_N) = e^{-(\beta\sigma \sum_p L_p)}\, Z_N^{int,rig}(L_1, L_2, \ldots, L_N) \tag{5.48}$$

where

$$Z_N^{int,rig}(L_1, L_2, \ldots, L_N) = \int D\underline{A}\, e^{-\frac{1}{2}\int d\underline{r}\, (\underline{\nabla}.\underline{A})^2} \prod_{p=1}^{N} \sum_{N_p=1}^{L_p/a} I_{L_p,N_p}[\underline{A}] \tag{5.49}$$

and

$$I_{L,N}[\underline{A}] = \oint d\mu_{L,N}[\underline{r}]\, e^{-\beta(N-1)u + i\pi\sqrt{8\beta\sigma}\int ds\, \underline{r}'(s).\underline{A}(\underline{r}(s))} \tag{5.50}$$

instead of (5.19)-(5.21).

The number of configurations, $\oint d\mu_{L,N}$, of a closed string is approximately

$$\oint d\mu_{L,N} \simeq \frac{(z-1)^{N-1}}{(L/a)} \frac{(L/a-1)!}{(N-1)!\,(L/a-N)!} \sum_{\underline{u}} \oint_{\underline{r}(0)=\underline{u}}^{\underline{r}(L)=\underline{u}} D\underline{r}\, e^{-\frac{3}{2a}\int_0^L ds\, \underline{r}'^2} \tag{5.51}$$

(see (2.5) for open strings). The closure factor is an estimate as it does not account for rigidity. As pointed out by Ziman [47], the effect of rigidity is to leave the critical exponent q unchanged and to convert the constant A into a parameter $A(\beta)$ which depends weakly on temperature (see (2.25)).

Substituting (5.51) into (5.50) and summing over the number of bends we find that

$$\begin{aligned} I_L[\underline{A}] &= \sum_{N=1}^{L/a} I_{L,N}[\underline{A}] \\ &= \frac{\left(1+(z-1)\,e^{-\beta u}\right)^{L/a}}{L/a} \sum_{\underline{u}} \oint_{\underline{r}(0)=\underline{u}}^{\underline{r}(L)=\underline{u}} D\underline{r}\, \exp\left[-\frac{3}{2a}\int_0^L ds\, \underline{r}'^2 \right. \\ &\qquad \left. +\imath\pi\sqrt{8\beta\sigma}\int ds\, \underline{r}'(s).\underline{A}(\underline{r}(s))\right] \end{aligned} \tag{5.52}$$

The interaction term has decoupled from the rigidity term and hence we can use our previous results with

$$z \to 1+(z-1)e^{-\beta u} \tag{5.53}$$

Thus, all equations remain unaltered except those containing σ_{eff} which becomes

$$\sigma_{eff} = \sigma - \frac{\ln\left(1+(z-1)\,e^{-\beta u}\right)}{a\beta} \tag{5.54}$$

Consequently, the equation for the Hagedorn temperature (5.40) no longer applies. Instead, we use (5.54) and solve for $\sigma_{eff}(T_H) = 0$.

Furthermore, the equation for the latent heat (5.45) now has an extra term due to the temperature dependence of $u(T)/T$. Assuming that strings are rigid with

$$u(T) = k\sigma(T)a(T) \quad \text{where} \quad k << 1 \tag{5.55}$$

(see (2.12)) and inserting our previous definitions of $\sigma(T)$ and $a(T)$ we find that

$$\begin{aligned} \left.\frac{U(y_o)}{V}\right|_{u\neq 0} &= \left.\frac{U(y_o)}{V}\right|_{u=0} + \frac{g\sigma_o^2}{6} y_o(T_F)(1-T_F^2)\left[\frac{kr(T_F)}{1+r(T_F)}\right. \\ &+ \left.\frac{3T_F^3}{(1-T_F^2)^{1/2}}\sqrt{\frac{g}{2}}\left(\ln z - \ln\left(1+r\left(T_F\right)\right)\right)\right] \end{aligned} \tag{5.56}$$

where

$$r(T_F) = (z-1)\, e^{-u(T_F)/T_F} \tag{5.57}$$

The free parameters of the theory are now k and g. For no backtracking, we find that only a Hagedorn transition occurs for all parameter values. This is as expected since $k = 0$ (corresponding to the case examined previously with no rigidity) gives only a Hagedorn transition and increasing k makes a first-order transition less likely. To verify this, we examine the transition where strings are allowed to backtrack. In this case, a first-order transition does occur at a temperature T_F just below T_H, with $1 - T_F^2 = O(g)$. Increasing k, which corresponds to the string being more rigid, increases T_H and decreases the value of the latent heat *i.e.* the transition becomes less first-order. This would be equally true if we were to choose a more realistic description of string stiffness.

5.3 Discussion

In summary, the results for global cosmic strings are highly dependent on the statistics, a first-order transition becoming more likely as the number of allowed configurations is increased. Unfortunately, the counting of configurations of closed paths on a lattice cannot be performed analytically and hence we have had to use rather coarse numerical approximations. Unlike the melting of iron (see Chapter 4), the nature of the phase transition for global cosmic strings turns out to be highly dependent on these approximations. One reason for this is that if a first-order transition does occur, it takes place at a temperature T_F which is extremely close to the Hagedorn temperature, T_H. Hence, only a slight variation in the parameters is necessary for the Hagedorn transition to dominate. Therefore, we are unable to say whether or not there exists a first-order phase transition at a temperature below that at which the Hagedorn transition occurs, although it does seem most unlikely. If it does exist it is very weakly first-order, the latent heat being of order g^2. Furthermore, it would

occur at a temperature T_F which is just below T_H and satisfies the same equation as T_H, namely that $1 - T_F^2 = O(g)$. Such a weak transition would have no observable effects.

Chapter 6

String Statistical Mechanics as a Gauge Field Theory

In this chapter[1] we shall draw an analogy between the statistical mechanics of closed string loops and an equivalent gauge theory representation. Similar calculations have been performed previously by Kosterlitz and Thouless [53], Thomas and Stone [54], and Kleinert [52]. It is on these that our work is based.

In the first section we shall examine free strings (like those described in the first chapter [15]) with no rigidity or interactions. We shall see how a free string theory can be expressed in terms of a free massive real scalar field theory. In the second section we shall extend our work to include string interactions. We shall look at two types of interactions, namely those relevant to cosmic strings (see Chapter 5) as well as those experienced by dislocation lines (see Chapter 4). We shall show that introducing string interactions with a finite range induces a QED gauge field into the field theory, and that the scalar field becomes a complex field due to the strings being orientable.

Furthermore, we shall examine how, by taking a limiting case of the field theory, we can extract the model of dislocation melting presented by Edwards and Warner [5]. In the absence of a dilation term, we shall demonstrate that the free energy density

[1]The work of this chapter appears in 'The Statistical Properties of Strings. II. Interacting Strings', proceedings of 'The Symposium on Cosmic Strings and Phase Transitions in the Early Universe', at Dampt (1989), to be published by the Cambridge University Press in 1990.

obtained by Edwards and Warner can be written as a one-loop effective potential. Finally, we shall see how the introduction of an excluded volume of the form $b\rho^2$ in the free energy of Edwards and Warner corresponds to the inclusion of a $g \mid \phi \mid^4$ self-interaction term in the analogue field theory.

6.1 Free strings

As in the previous chapters, we require the strings to lie on the links of a cubic lattice in order to make the string configurations denumerable. The total number of paths that begin at $\underline{u}$, end at $\underline{v}$ and consist of L/a steps is then given by

$$\Gamma[\underline{u},\underline{v};L/a] = \frac{\mathcal{N}}{a^3}\int_{\underline{r}(0)=\underline{u}}^{\underline{r}(L)=\underline{v}} \mathcal{D}\underline{r}\,\exp\left[-\frac{1}{a}\int_0^L ds\left(\frac{3}{2}\underline{r}'^2 - \ln z\right)\right] \tag{6.1}$$

where $\mathcal{N}$ is a dimensionless constant and where we have used (4.19) for the probability distribution. For free strings the total number of closed paths on the lattice is

$$\begin{aligned} n(L) &= \frac{1}{2La^2}\int d\underline{u}\,\Gamma[\underline{u},\underline{u};L/a] &(6.2)\\ &= \oint d\mu_L[\underline{r}] &(6.3)\end{aligned}$$

(see (4.4) and (4.22)), where the factor of $1/2$ is present because, for non-interacting strings, the paths are non-orientable.

We wish to calculate the single string partition function

$$Z_1 = \sum_L n(L)\,e^{-\beta\sigma L} \tag{6.4}$$

for which

$$\begin{aligned} -\frac{dZ_1}{d(\beta\sigma)} &= \sum_L n(L)\,L\,e^{-\beta\sigma L}\\ &= \frac{1}{2a^3}\int_0^\infty dL\int d\underline{u}\,\Gamma[\underline{u},\underline{u};L/a]\,e^{-\beta\sigma L} &(6.5)\end{aligned}$$

where we have used (6.2) and have taken the continuum limit (as most of what we wish to study takes place on a length scale which is much greater than that of the

lattice). From (6.1), Γ is a solution to the diffusion equation

$$\left(\frac{a^2}{6}\nabla_u^2 + \ln z\right)\Gamma[\underline{u},\underline{v};L/a] = a\frac{\partial}{\partial L}\Gamma[\underline{u},\underline{v};L/a] \tag{6.6}$$

(*c.f.* (4.49)) subject to the boundary condition that

$$\Gamma[\underline{u},\underline{v};0] = a^3\,\delta(\underline{u}-\underline{v}) \tag{6.7}$$

(*c.f.* (4.50)). This type of Green function permits the decomposition

$$\Gamma[\underline{u},\underline{v};L/a] = \sum_n \Phi_n(\underline{u})\Phi_n(\underline{v})\,e^{-\omega_n^2 L/a} \tag{6.8}$$

where

$$\left.\begin{array}{rcl} \int d\underline{u}\,\Phi_n(\underline{u})\Phi_m(\underline{u}) & = & a^3\delta_{nm} \\ \sum_n \Phi_n(\underline{u})\Phi_n(\underline{v}) & = & a^3\delta(\underline{u}-\underline{v}) \end{array}\right\} \tag{6.9}$$

and hence $\Phi_n(\underline{u})$ satisfies the equation

$$\left(-\frac{a^2}{6}\nabla_u^2 - \ln z\right)\Phi_n(\underline{u}) = \omega_n^2\Phi_n(\underline{u}) \tag{6.10}$$

If we now substitute (6.8) into (6.5), use (6.9) and perform the L, $\underline{u}$ and $(\beta\sigma)$ integrations we obtain that

$$\begin{aligned} Z_1 &= -\frac{1}{2}\sum_n \ln(\omega_n^2 + \beta\sigma a) \\ &= -\frac{1}{2}\ln\det\left[-\frac{a^2}{6}\nabla_u^2 - \ln z + \beta\sigma a\right] \end{aligned} \tag{6.11}$$

It can be shown that this expression for Z_1 is equivalent to that given in (1.4). This is done by using a suitable regularisation procedure to evaluate the determinant in (6.11) (*e.g.* ζ-function regularisation), and by taking the continuum limit in (1.4).

For free strings the partition function for a 'gas' of loops is

$$Z = e^{Z_1} = \det\left[-\frac{a^2}{6}\nabla_u^2 - \ln z + \beta\sigma a\right]^{-1/2} \tag{6.12}$$

which, when written in terms of a real scalar field, permits a functional integral representation

$$\begin{aligned} Z &\sim \int \mathcal{D}\phi\,\exp\left[-\frac{1}{a^3}\int d\underline{r}\,\phi\left(-\frac{a^2}{6}\nabla_u^2 - \ln z + \beta\sigma a\right)\phi\right] \\ &\sim \int \mathcal{D}\phi\,\exp\left[-\frac{1}{a}\int d\underline{r}\,\left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}M^2\phi^2\right)\right] \end{aligned} \tag{6.13}$$

with

$$M^2 = \frac{6}{a^2}(\beta\sigma a - \ln z) \tag{6.14}$$

Thus we have seen how to rewrite a free string theory as a free massive field theory.

6.2 Interacting strings

6.2.1 Cosmic strings

We start by considering global strings with an effective mass cut-off (not put in by hand) which experience interactions of the form

$$E_{int}[\underline{r}_1, \underline{r}_2] = 2\pi\sigma \oint (d\underline{l}_1 . d\underline{l}_2) \frac{e^{-mr_{12}}}{r_{12}} \tag{6.15}$$

for some constant m. This is a generalisation of the interaction energy for global strings given in (5.17), and is equivalent to imposing a finite range on the force between the strings. The form given in (6.15) could be used as a crude approximation to the interaction energy for *local* cosmic strings. The correct expression for such strings is a 2×2 matrix due to interactions between the gauge fields, the Higgs fields as well as interactions between the two types of fields.

As in Chapter 5, we introduce a vector source term $\underline{\rho}(\underline{r})$ (given in (5.18)) and write the N-string interaction energy as

$$E^{(2)}_{int}[\underline{r}_1, \ldots, \underline{r}_N] = \pi\sigma \int d\underline{u}\, d\underline{v}\, \underline{\rho}(\underline{u}).\underline{\rho}(\underline{v}) \frac{e^{-m|\underline{u}-\underline{v}|}}{|\,\underline{u}-\underline{v}\,|} \tag{6.16}$$

(*c.f.* (5.20)). Now, in three dimensions,

$$G(\underline{u}, \underline{v}) = \frac{1}{4\pi} \frac{e^{-m|\underline{u}-\underline{v}|}}{|\,\underline{u}-\underline{v}\,|} \tag{6.17}$$

is the solution of the Helmholtz equation

$$(\nabla^2_u - m^2)\, G(\underline{u}, \underline{v}) = -\delta(\underline{u}-\underline{v}) \tag{6.18}$$

Substituting G into (6.16), we obtain that

$$\exp(-\beta E^{(2)}_{int}[\underline{r}_1,\ldots,\underline{r}_N]) = \exp\left(-4\pi^2\beta\sigma\int d\underline{u}\,d\underline{v}\,\underline{\rho}(\underline{u}).G(\underline{u},\underline{v})\underline{\rho}(\underline{v})\right) \tag{6.19}$$

which can then be re-expressed as a functional integral over a vector field $\underline{A}(\underline{r})$ in three dimensions. For the general N-string case we find that

$$\exp(-\beta E^{(2)}_{int}[\underline{r}_1,\ldots,\underline{r}_N]) = \mathcal{N}^{-1}\int \mathcal{D}\underline{A}\,\exp\left[-\frac{1}{2a^3}\int d\underline{r}\,\underline{A}(\underline{r}).(-\nabla^2+m^2)\underline{A}(\underline{r}) + \imath\pi\sqrt{\frac{8\beta\sigma}{a^3}}\int d\underline{r}\,\underline{A}(\underline{r}).\underline{\rho}\right] \tag{6.20}$$

where here

$$\mathcal{N} = \int \mathcal{D}\underline{A}\,\exp\left[-\frac{1}{2a^3}\int d\underline{r}\,\underline{A}(\underline{r}).(-\nabla^2+m^2)\underline{A}(\underline{r})\right] \tag{6.21}$$

As for the case when $m = 0$ (see (5.20) and (5.21)), we have that for N strings

$$\begin{aligned} Z_N(L_1,...,L_N) &= \oint\prod_{p=1}^{N} d\mu_{L_p}[\underline{r}_p]\,e^{-\beta E[\underline{r}_1,...,\underline{r}_N]} \\ &= \mathcal{N}^{-1}\int \mathcal{D}\underline{A}\exp\left[-\frac{1}{2a^3}\int d\underline{r}\,\underline{A}(\underline{r}).(-\nabla^2+m^2)\underline{A}(\underline{r})\right] \\ &\times \oint\prod_{p=1}^{N} d\mu_{L_p}[\underline{r}_p]\exp\left[-\beta\sigma L_p + \imath\pi\sqrt{\frac{8\beta\sigma}{a^3}}\int d\underline{r}\,\underline{A}(\underline{r}).\underline{\rho}\right] \end{aligned} \tag{6.22}$$

If we now substitute for $\underline{\rho}(\underline{r})$ using (5.18), take the continuum limit and integrate over all possible string lengths we obtain the general N-string partition function

$$\begin{aligned} Z_N &= \sum_{\{L_p\}} Z_N(L_1,...,L_N) \\ &= \mathcal{N}^{-1}\int \mathcal{D}\underline{A}\exp\left[-\frac{1}{2a^3}\int d\underline{r}\,\underline{A}(\underline{r}).(-\nabla^2+m^2)\underline{A}(\underline{r})\right] \\ &\times \left(\int\frac{dL}{a}\oint d\mu_L[\underline{r}]\,\exp\left[-\beta\sigma L + \imath\pi\sqrt{\frac{8\beta\sigma}{a^3}}\int ds\,\underline{r}'(s).\underline{A}(\underline{r}(s))\right]\right)^N \end{aligned} \tag{6.23}$$

Thus, the full partition function $Z = \sum_N \frac{Z_N}{N!}$ is given by

$$Z = \frac{\int \mathcal{D}\underline{A}\exp\left[-\frac{1}{2a^3}\int d\underline{r}\,\underline{A}(\underline{r}).(-\nabla^2+m^2)\underline{A}(\underline{r})\right]\,I[\underline{A}]}{\int \mathcal{D}\underline{A}\exp\left[-\frac{1}{2a^3}\int d\underline{r}\,\underline{A}(\underline{r}).(-\nabla^2+m^2)\underline{A}(\underline{r})\right]} \tag{6.24}$$

where

$$I[\underline{A}] = \exp\left[\int \frac{dL}{a} \oint d\mu_L[\underline{r}] \exp\left(-\beta\sigma L + \imath\pi\sqrt{\frac{8\beta\sigma}{a^3}} \int ds\, \underline{r}'(s).\underline{A}(\underline{r}(s))\right)\right] \quad (6.25)$$

We note that taking $\underline{A} = 0$ in (6.25), to give $I[0]$, corresponds to switching off the string interactions. In this case we have from (6.24) that

$$Z = I[0] = e^{Z_1} \quad (6.26)$$

where Z_1 is given by (6.4). This is the free string result, up to the factor of $1/2$ mentioned earlier, since interacting strings are orientable (see (1.8)). The effect of eliminating this factor for free strings is to replace the real scalar field in (6.13) by a complex scalar field to give that

$$Z \sim \int \mathcal{D}\phi\, \mathcal{D}\phi^\star \exp\left[-\frac{1}{a}\int d\underline{r}\left(|\nabla\phi|^2 + M^2\,|\phi|^2\right)\right] \quad (6.27)$$

where M^2 is given by (6.14).

In order to evaluate $I[\underline{A}]$, and hence the partition function for global interacting strings with a finite range over which the string interactions can act, we introduce a generalisation of (6.1) for open strings, namely

$$\Gamma_{\underline{A}}[\underline{u},\underline{v};L/a] = \frac{\mathcal{N}}{a^3}\int_{\underline{r}(0)=\underline{u}}^{\underline{r}(L)=\underline{v}} \mathcal{D}\underline{r} \exp\left[-\frac{1}{a}\int_0^L ds\left(\frac{3}{2}\underline{r}'(s)^2 - \ln z + \imath e\underline{r}'.\underline{A}(\underline{r}(s))\right)\right] \quad (6.28)$$

where $e = \pi\sqrt{8\beta\sigma/a}$. Now, $\Gamma_{\underline{A}}$ satisfies the equation

$$\left(\frac{1}{6}(a\nabla_u + \imath e\underline{A})^2 + \ln z\right)\Gamma_{\underline{A}}[\underline{u},\underline{v};L/a] = a\frac{\partial}{\partial L}\Gamma_{\underline{A}}[\underline{u},\underline{v};L/a] \quad (6.29)$$

which in the limit $\underline{A} \to 0$ becomes the equation for free strings given in (6.6). Hence, following the same proceedure as we did for free strings we obtain that

$$\begin{aligned} I[\underline{A}] &= \det\left[-\frac{1}{6}(a\nabla_u + \imath e\underline{A})^2 + \beta\sigma a - \ln z\right]^{-1} \\ &\sim \int \mathcal{D}\phi\mathcal{D}\phi^\star \exp\left[-\frac{1}{a^3}\int d\underline{r}\left(\frac{1}{6}\,|(a\nabla + \imath e\underline{A})\phi|^2 + (\beta\sigma a - \ln z)\,|\phi|^2\right)\right] \end{aligned} \quad (6.30)$$

in terms of a complex scalar field. (It is complex because of the $\det^{-1}$, rather than the $\det^{-1/2}$ as in the previous case.)

Inserting (6.30) into (6.24) we obtain, up to an overall normalisation, that

$$Z = \int \mathcal{D}\phi \mathcal{D}\phi^* \mathcal{D}\underline{A}\, e^{-S[\phi,\phi^*,\underline{A}]} \tag{6.31}$$

where

$$S[\phi,\phi^*,\underline{A}] = \frac{1}{a^3}\int d\underline{r}\, \left[\frac{1}{2}\underline{A}.(-\nabla^2 + m^2)\underline{A} + |(a\nabla + \imath e\underline{A})\phi|^2 + a^2M^2\,|\phi|^2\right] \tag{6.32}$$

(after rescaling the ϕ fields , $\frac{1}{6}\phi^2 \rightarrow \phi^2$). Here m^{-1} is the range of the string-string force, $e^2 = 8\pi^2\beta\sigma/a$ is a measure of its strength, and $M^2 = 6(\beta\sigma a - \ln z)/a^2 = 6\beta\sigma_{eff}/a$. If we choose the gauge $\underline{\nabla}.\underline{A} = 0$, we have that

$$S[\phi,\phi^*,\underline{A}] = \frac{1}{a^3}\int d\underline{r}\, \left[\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}m^2\underline{A}^2 + |(a\nabla + \imath e\underline{A})\phi|^2 + a^2M^2\,|\phi|^2\right] \tag{6.33}$$

where

$$F_{ij} = \nabla_i A_j - \nabla_j A_i \tag{6.34}$$

and hence S is the action for massive scalar QED with no explicit self-interaction. Thus the introduction of string interactions is equivalent to the incorporation of the QED gauge field. Furthermore, requiring the strings to be orientable changes the scalar field from a real field to a complex field.

6.2.2 Dislocations

Finally, we shall look at string interactions of the form

$$E_{int}[\underline{r}_1,\underline{r}_2] = \frac{\gamma\mu}{4\pi}\oint ds_1 ds_2\, (\underline{b}_1.\underline{r}_1'(s_1))\,(\underline{b}_2.\underline{r}_2'(s_2))\frac{e^{-m|\underline{r}_1(s_1)-\underline{r}_2(s_2)|}}{|\,\underline{r}_1(s_1) - \underline{r}_2(s_2)\,|} \tag{6.35}$$

which is a generalisation of the interaction energy for dislocations given in (4.11).

For constant $\underline{b}$, the effect is nominally to replace $\underline{A}$ in (6.20)-(6.34) by $\underline{b}A$, where A now is a scalar field. The result, up to an overall normalisation, is then that

$$Z = \int \mathcal{D}\phi \mathcal{D}\phi^* \mathcal{D}A\, e^{-S[\phi,\phi^*,A]} \tag{6.36}$$

where

$$S[\phi,\phi^*,A]=\frac{1}{a}\int d\underline{r}\,\left[\frac{1}{2}(\nabla A)^2+\frac{1}{2}m^2A^2+\mid\nabla\phi\mid^2+e^2\underline{b}^2A^2\mid\phi\mid^2+M^2\mid\phi\mid^2\right] \quad (6.37)$$

and

$$e^2=\frac{\beta\gamma\mu}{a} \quad (6.38)$$

As in the previous case there is no explicit $\mid\phi\mid^4$ term.

We are now in a position to make a direct connection between the partition function of (6.36,6.37) and that presented by Edwards and Warner [5], given in (4.30,4.41). To do this we integrate out the A field in (6.37) to obtain that

$$Z = \int\mathcal{D}\phi\mathcal{D}\phi^*\frac{\exp\left[-\frac{1}{a}\int d\underline{r}\,(\mid\nabla\phi\mid^2+M^2\mid\phi\mid^2)\right]}{\det\left[a\,(-\nabla^2+m^2+2e^2b^2\mid\phi\mid^2)^{1/2}\right]} \quad (6.39)$$

$$= \int\mathcal{D}\phi\mathcal{D}\phi^*\exp\left(-\Theta[\phi,\phi^*]\right) \quad (6.40)$$

where

$$\Theta[\phi,\phi^*]=\frac{1}{a}\int d\underline{r}\,\left(\mid\nabla\phi\mid^2+M^2\mid\phi\mid^2\right)+\frac{1}{2}tr\,\ln\left[a^2\left(-\nabla^2+m^2+2e^2b^2\mid\phi\mid^2\right)\right] \quad (6.41)$$

As pointed out by Thomas and Stone [54], the string density $\rho\propto\mid\phi\mid^2$. This can be seen from (6.39), from which we find that

$$\frac{1}{Z}\frac{\partial Z}{\partial M^2}=-\frac{V}{a}<\mid\phi\mid^2> \quad (6.42)$$

But from the definition of M^2, given in (6.14), and using (6.24,6.25), we have that

$$\frac{1}{Z}\frac{\partial Z}{\partial M^2}=\frac{a}{6Z}\frac{\partial Z}{\partial(\beta\sigma)}=-\frac{a}{6}<L> \quad (6.43)$$

Hence, on comparing (6.42) and (6.43), we obtain the result

$$<\rho>=\frac{<L>}{V}=\frac{6}{a^2}<\mid\phi\mid^2> \quad (6.44)$$

Let us now *assume* that ϕ and ϕ^* are constant. This corresponds to neglecting the effect due to closed loops and to taking a fixed number of strings, N, each of the

same length, L/N. In this case, we obtain that the free energy density is given by

$$\frac{\Theta}{\beta V} = \frac{M^2 \mid \phi \mid^2}{a\beta} + \frac{1}{2\beta}\int \frac{d\underline{k}}{(2\pi)^3} \ln\left[a^2\left(\underline{k}^2 + m^2 + 2e^2b^2 \mid \phi \mid^2\right)\right] \qquad (6.45)$$

which, on using the result (6.44), becomes

$$\frac{\Theta}{\beta V} = \frac{M^2 a}{6\beta}\rho + \frac{1}{2\beta}\int \frac{d\underline{k}}{(2\pi)^3} \ln\left[a^2\left(\underline{k}^2 + m^2 + \frac{e^2b^2a^2}{3}\rho\right)\right] \qquad (6.46)$$

$$= \left(\sigma - \frac{\ln z}{\beta a}\right)\rho + \frac{1}{2\beta}\int \frac{d\underline{k}}{(2\pi)^3} \ln\left(1 + \frac{b^2\beta\gamma\mu a\rho}{3(\underline{k}^2 + m^2)}\right) + \text{const.} \qquad (6.47)$$

Thus, in the absence of a $\mid \phi \mid^4$ term, the free energy density of (6.47) is the *one-loop* effective potential. Furthermore, for $m = 0$, this expression is the same as that obtained by Edwards and Warner [5] in the absence of a dilation term (see (4.30)).

For $M^2 < 0$ (*i.e.* $\sigma_{eff} < 0$) the free energy per unit length is negative and hence the space becomes filled with string. At this point $< \phi >$ obtains a finite value. Thomas and Stone [54] interpret $< \phi >$ as the number density of infinite strings. This is consistent with the interpretation of the Hagedorn temperature [26], given in Chapter 1, as the temperature above which all string goes into an infinite string thus allowing the energy density in the box to become arbitrarily large. To prevent the density from becoming infinite, we need a $g \mid \phi \mid^4$ term to produce a short-distance repulsion in (6.37). Such a term could be generated by renormalisation. However, we can also put such a term in by hand in order to try and reduce the number of allowed string configurations, which is necessary if backtracking and self-intersection are to be eliminated. Edwards and Warner [5] introduce a dilation term (see (4.63)) of the form $b\rho^2$. Since $\rho \propto \mid \phi \mid^2$, this corresponds to adding a $g \mid \phi \mid^4$ self-interaction term in the analogue field theory. Up to a normalisation of g, S of (6.37) would then read

$$S[\phi, \phi^\star, A] = \frac{1}{a}\int d\underline{r}\left[\frac{1}{2}(\underline{\nabla} A)^2 + \frac{1}{2}m^2A^2 + \mid \nabla\phi \mid^2 + e^2\underline{b}^2A^2 \mid \phi \mid^2 + M^2 \mid \phi \mid^2 + \frac{g}{a^2} \mid \phi \mid^4\right] \qquad (6.48)$$

which is just a conventional model for the symmetry breaking of O(2). Potentially this could change the form of the interactions. The presence of the $g \mid \phi \mid^4$ term

induces a ϕ loop contribution to the one-loop approximation, in addition to the A field loop. We then have, instead of (6.46), a free energy density of the form

$$\frac{\Theta}{\beta V} = \frac{M^2 a}{6\beta}\rho + \frac{ga}{36\beta}\rho^2 + \frac{1}{2\beta}\int \frac{d\underline{k}}{(2\pi)^3} \ln\left[a^2\left(\underline{k}^2 + m^2 + \frac{e^2b^2a^2}{3}\rho\right)\right] + \frac{1}{\beta}\int \frac{d\underline{k}}{(2\pi)^3} \ln\left[\left(a^2\underline{k}^2 + M^2 + \frac{2g}{3}\rho\right)\right] \tag{6.49}$$

where the last term is the ϕ loop contribution. We should note that (6.49) does not make sense for $M^2 < 0$. In this case it is necessary to shift the fields. Thus the excluded volume effect may not be as straight forward as it initially seemed and hence further investigation is required.

6.3 Discussion

In Chapter 1 we presented an analytic description of a phase transition which leads to the formation of strings. The simplest theory that possesses vortex solutions is scalar QED with Lagrange density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\,|(\partial_\mu + ieA_\mu)\phi|^2 + \frac{1}{2}m_o^2\,|\phi|^2 - \frac{g}{4!}\,|\phi|^4$$

(see (1.32)). Starting from this theory and using the mean-field approximation, we obtained a three-dimensional temperature dependent effective field theory. From this theory we derived the classical statistical mechanics of strings and showed how to obtain an expression for the partition function which describes a system of closed string loops [15]. In this chapter we started from the partition function and demonstrated how the statistical mechanics of strings can be represented in terms of a three-dimensional field theory of string variables ϕ. There is a duality between the underlying field quanta of the temperature dependent effective theory (obtained in Chapter 1), and the field theory of string variables (given in this Chapter). Such dualities are well known in lattice field theories, where a Villain transformation is used to relate one theory to the other.

For free strings we found that the analogue theory is a free massive field theory. Introducing interactions of the type relevant to global cosmic strings with an effective mass cut-off turns out to be equivalent to introducing a QED gauge field into the theory. We also examined interactions of the type relevant to dislocations. In this case we assumed that the complex scalar field is constant and thus were able to reproduce the model presented by Edwards and Warner [5] with an energy density equal to the one-loop effective potential. Furthermore, we imposed an excluded dislocation volume and found that this corresponds to the inclusion of a $g\,|\phi|^4$ self-intersection term in the action of the analogue field theory.

In the previous chapters, where we used the statistical mechanics approach to examine the string phase transitions, we found that a Hagedorn transition always occured when the effective string tension $\sigma_{eff} = 0$. In the analogue field theory this corresponds to $M^2 = 0$, the point at which the group symmetry is spontaneously broken.

The analogy described in this chapter is extremely interesting and obviously there is scope for work beyond that which we have presented here. For example, the role of the self-interaction term requires further investigation. It would also be interesting to show how such a term could be generated through renormalisation. Furthermore, using this analogy, it would be interesting to examine the role played by both closed and open strings.

Chapter 7

Conclusion and Ideas for Further Work

In this investigation we developed a model of phase transitions caused by the proliferation of string-like structures. We concentrated on two specific phase transitions, namely crystal melting and the formation of cosmic strings in the early universe.

As a first step, we assumed the strings to be free static objects with no vibrational degrees of freedom, no rigidity and no interactions. For systems containing fundamental strings, we found that a phase transition occurs at a temperature T_{st}, where we interpreted T_{st} as the maximum temperature of the system. At this temperature, the energy of the system is dominated by finite loops and the system is unstable to the formation of infinite strings at high energy densities.

We then looked at free 'composite' strings and specifically at Nielsen-Olesen type vortices. Such strings develop a temperature dependent width and effective string tension. We examined the contribution of these string solutions to the partition function of a scalar gauge theory and saw that a Hagedorn transition occurs, with the partition function diverging at a temperature T_{st}. As this temperature is approached from below, most of the string length goes into infinite strings and the distribution of loops is scale-invariant. We showed that T_{st} satisfies $1 - \frac{T_{st}^2}{T_c^2} = O(e^2)$ (for $e^2 \ll g$) and, using this result, we demonstrated how the Ginzburg length scale arises in this description.

For free strings, we found that our model predicts all string mediated phase transitions to be second-order. Experimentally, we know that this is incorrect and that the order of the transition varies for different systems.

Consequently, we included in our model the possibility of strings being rigid. We found that this does not alter the order of the phase transition. However, for fundamental strings we showed that the effect of stiffness is to increase the critical temperature. For 'composite' strings this effect is negligible, due to the temperature dependence of the parameters, and T_{st} still satisfies the equation given previously for free strings. Rigidity changes the effective size of the string segments, making the strings straighter at low temperatures. As for free strings, we obtained that the distribution of loops is scale-invariant at the transition.

Our next step was to incorporate string interactions into the model. Initially, we looked at the simplest form, namely that of attractive self-interactions for specific string configurations. Again, the order of the transition remained unaltered. We found, for fundamental strings, that the critical temperature is reduced and, for 'composite' strings, that it still satisfies the equation given previously.

More complicated interactions were then examined. In particular, we examined interactions appropriate to dislocations as well as to global cosmic strings for general string configurations. Our description included both self-interactions of the strings and mutual interactions between strings, where we took the force to be repulsive for parallel strings and attractive for antiparallel strings. We found that the order of the transition is highly dependent on the parameter values and hence on the system under investigation. When we applied this model to the dislocation melting of iron, we obtained, for a given set of parameter values, a first-order phase transition in agreement with experimental observations.

Furthermore, we used the same model to describe the thermodynamic properties of interacting global cosmic strings. We found that the usual Hagedorn transition occurs

at a temperature which is unaffected by the presence of interactions. In addition to this transition, we showed that there exists the possibility of another transition which, if present, is weakly first-order with a latent heat of $O(g^2)$. The existence of this transition is highly dependent on the counting of string configurations, which can only be performed using rather coarse numerical approximations. Hence, our results for global cosmic strings were inconclusive.

Finally, we demonstrated how the statistical mechanics of strings can be represented in terms of a quantum field theory for both free and interacting strings. For the former, we obtained that the analogue field theory is a free massive field theory. Introducing interactions of the type relevant to dislocations or global cosmic strings with an effective mass cut-off, turned out to be equivalent to adding a QED gauge field into the theory. We also showed how imposing an excluded volume on the strings corresponds to the inclusion of a $g \mid \phi \mid^4$ self-interaction term in the action of the analogue field theory. Using this picture, the Hagedorn transition occurs when the group symmetry is spontaneously broken and the effective scalar mass vanishes. This process is unaffected by the presence of the gauge field, which is what we demonstrated previously when examining the statistical mechanics of interacting strings.

The model presented in this thesis can be applied to a wider variety of systems. In particular, it would be interesting to investigate the glass transition and the λ-transition in superfluid ^{4}He. Dislocation melting is just one description of melting. It would be worth examining other models of melting and seeing if our model can be incorporated into these.

Throughout our analysis of cosmic strings we have assumed flat space-time. An obvious improvement to our model would be to include the effects of curvature. Furthermore, the transition at which *local* cosmic strings are formed requires further investigation, as it is these string which are expected to be important in galaxy forming scenarios. Such strings are rather more complicated due to the gauge field playing

an active part, in addition to the Higgs field.

Finally, further work is necessary in order to understand the analogy between the statistical mechanics of strings and the equivalent gauge theory representation. The manifestation of the excluded volume effect as a self-interaction term in the action of the field theory needs investigation, as well as how such a term can be generated through renormalisation. The duality relation between the equivalent field theory and the finite temperature effective field theory of Chapter 1 is extremely interesting and requires further examination.

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