THE MODELLING AND OPTIMIZATION OF CHEMICAL PROCESSES WITH PLANT FAILURES

by

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Dedicated to my wife, Helen Fiona McGregor.

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ABSTRACT

The latest increases in size of chemical plants have had adverse effects upon their smooth operability. The larger and more complex a chemical plant is designed to be, the less reliable it becomes. Consequently, its design output specification is less likely to be met due to unexpected stoppages. One way of counteracting this effect is to overdesign equipment. One can then balance the extra expenditure needed for a more reliable design against the benefits attained by the decrease in plant downtime. The necessary analysis is called "Availability Analysis" and can be incorporated in the last stages of process design.

Two important aspects of availability analysis are discussed in this thesis. Firstly the problem of modelling of processes with intermediate storage tanks will be considered. An analytic expression for the availability of a system composed of two machines and an intermediate tank is derived using stochastic analysis. The availability of larger systems is approximated using a "blocking" technique whose performance is tested using Monte Carlo simulation.

Secondly, a branch and bound algorithm for the synthesis of optimal plant configurations with respect to overdesign is examined. Special consideration is given to the problem of optimum buffer storage size. Several problems are solved using the techniques developed here to demonstrate their effectiveness.

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C H A P T E R O N E

INTRODUCTION TO THE PROBLEMS OF UNRELIABLE PLANTS

1.1 Introduction

The field of reliability engineering and risk analysis is comparatively new, dating mainly from the technological developments of the Second World War and the U.S. Space program. A short summary of its progression through the last three decades can be found in [28].

The main bulk of the research so far has been carried out by the military and the aerospace, electronic and nuclear industries, with the chemical industry showing a delayed interest in the field. It was the increase in scale and complexity of chemical plants that gave the impetus to related research, as unexpected failures could lead to severe economic losses associated with shutdown, lost production and, perhaps, make processes uneconomic. As early as 1962 concern was being shown in the "discouraging frequency of failures of key process steps in the biochemical industry" [48].

During the last ten years, the problems have become more acute. Plants have become even larger and failure-prone. One can cite an example as an illustration: BOC air separation units have increased in size from an output of 12½ tons/day in the immediate post war years to 800 tons/day in the 1970's to units of 2000 tons/day under construction at the moment. As a consequence, BOC introduced reliability monitoring in their design and reduced plant downtimes to less than 1% through planned overhaul work [52].

The trend for energy conservation has also brought about a higher degree of integration within plants, resulting in a greater interdependence of plant sections and increasing the possibility of shutdowns.

The recent concern over environmental pollution has imposed statutory controls upon the chemical industry and the speedy introduction of new pollution-monitoring equipment. The latter has sometimes been inadequately tested and is itself the cause of stoppages [33]. Maintenance and overhaul costs have increased and supply shortages have forced plant managers to use less desirable feedstocks by modifying existing equipment, thereby causing "bottlenecks" [55].

Finally, automation and concern for safety have led to control, trip and alarm systems which, although improving safety and reliability by their nature, are themselves integrated parts of the plant and contribute to the problems caused by increased complexity.

All these developments in the chemical industry have motivated firms to look into the problems posed by plant stoppages, especially since existing research had not adequately covered chemical engineering interests.

1.2 The Development of Research in Plant Reliability

Previous research carried out by other industries (aerospace, electronic, nuclear) was naturally focused on the particular problems facing them: namely catastrophic failure of non-repairable systems. The analysis and data collection in this field is a lot less cumbersome than that required by the chemical industry, where one is concerned with non-catastrophic failure of repairable components. The very definition of "failure" itself may be subject to change, as a plant may be nominally "on" but producing off-spec product. The variety of processes, practices and operations involved, even in the production of the same chemical, makes modelling and data collection (especially repair data) a lot more difficult. Industrial competition and the sensitive and confidential nature of the data involved does not make their disclosure easier. It is not insignificant that the systematic collection of

reliability data has been pioneered by organizations which are not primarily profit-orientated such as NASA, UKAEA, U.S. Armed Forces, etc. [34].

However, large databases have appeared recently in Europe [1], [38], whereas in the U.S. the sheer size of companies has provided each with a significant pool of experience to rely upon. It comes therefore as no surprise that U.S. firms started research on this subject earlier than European ones.

The concern of the chemical industry about the problem of plant reliability is demonstrated by the number of published contributions from diverse firms: ICI [15], du Pont [13], [54], Dow Chemical Co. [16], Toyo Engineering Corporation [35], [36], Chemie GmbH [38], Nórsk Hydro [6], as well as contractors such as CJB [57].

There have been four main areas in the field of reliability engineering. The first two, involving product reliability and maintenance, fall in line with other industries and involve Operational Research techniques such as quality control, inventory control, job scheduling and component replacement strategies. A good introduction to this subject is [31].

There are, however, two areas where special techniques have to be employed. These are safety and environmental pollution and process availability analysis.

In the first of these, due to the dangerous nature of the chemicals involved and tighter worldwide government legislation, some firms have employed such techniques as Hazard & Operability studies and Fault Tree Analysis [39] to identify possible dangers during the lifetime of the plant.

Availability Analysis is concerned with the economic aspects of plant failures. It attempts to balance the incurred costs of plant

downtime against the extra costs needed to reduce such downtime. This thesis is concerned with problems arising in the latter area, and therefore a brief introduction to the subject follows.

1.3 Availability Analysis of Chemical Plants

This analysis is used in the last stages of process design and stems from the realization that unexpected stoppages may affect a plant's design output specification. One thus tries to balance the extra expenditure for a more reliable design against the expected benefits realized by the decrease in plant downtime.

We must now define the two basic concepts of reliability and availability, as some confusion often arises in the literature. Bellingham [8] distinguishes between the various reliability definitions:

(a) Designer's reliability: the probability that a system or device will operate as required for a given period of time. This is the definition we shall employ here, denoting reliability by R.

(b) Operator's reliability: this is a conditional probability used in place of (a) by many authors. It is the probability that a system or device will operate as required for a given future period of time under given operating conditions, given that it has performed satisfactorily up to the present.

Reliability, as defined above, is a useful concept for non-maintained systems, or systems which are difficult to repair while in operation (such as missiles, satellites, or in the chemical industry batch processes). However, chemical processes may be "on" or "off". Thus we are not so much interested in how long a process may perform without failure, but in how long the process is performing according to specifications. The concept of availability is equivalent to reliability in the case of maintained systems. As maintained systems are the norm in the chemical industry, studies which do not consider availability are not practically

useful for continuously operating chemical plants (see for example [46]).

There are three main definitions of availability [49]

(a) Steady state availability, A. This is the proportion of time the system is performing in the desired manner, in the long run.

(b) Instantaneous availability A(t). This is the probability that the system is at any operating state at time t.

(c) Finite or interval availability A(T). This is the fractional uptime for a process for some definite period of time T.

The relationship between (a), (b), (c) can be summarized by the relations

$$A(T) = \frac{1}{T} \int_0^T A(t)dt \quad \ldots\ldots \qquad (1.1)$$

$$A = \lim_{T\to\infty} A(T) \quad \ldots\ldots \qquad (1.2)$$

A convenient form of (1.2) is (1.3)

$$A = \frac{MTTF}{MTTF+MTTR} = \frac{1}{1+\rho} \; ; \; \rho = \frac{MTTR}{MTTF} \quad \ldots\ldots \qquad (1.3)$$

where

MTTF = mean time to failure for a system

MTTR = mean time to repair for a system

with no assumptions about failure or repair distributions involved.

A review of the work of various researchers on availability appears in [40].

During preliminary design, equipment is assumed to function according to specifications all the time (i.e. be fully available). Every piece of equipment is thus designed and rated assuming that equipment upstream is fully available, and so on.

Later on, allowance is made for stoppages which decrease the design (annual or daily) production rate. This is traditionally done by overdesigning equipment, i.e. increasing its capacity (and consequently its cost), so that the plant throughput is increased for the period the plant is "on" - keeping annual production capacity constant.

However, it has been recognized [3], [17], [35], [19], that traditional overdesign is not necessarily the most economical way of attaining design throughput targets. Other ways to achieve this are [17]:

(i) Purchase of more reliable - and more expensive - equipment of lower capacity.

(ii) Duplication of equipment on-line or use of spares (both options are called collectively "redundancy" options - "active" and "standby" respectively).

(iii) Inclusion of intermediate (buffer) storage between plant items and/or whole plant sections, so that effects of shutdowns upstream may be reduced.

The final design of a plant should be optimal with respect to an objective function (such as minimum capital cost or maximum discounted net present value) after all combinations of these alternatives have been considered.

There are two stages involved in the solution of the design problem presented above: one needs to be able to model the process in question and then one needs to have an optimization algorithm to pick out the best combinations between overdesign and its alternatives. These two aspects of the problem are going to concern us in this thesis. However, the results derived can be of help in tackling problems encountered in operational chemical plants as described in the next section.

1.4 Problems Arising in the Operation of Chemical Plants

Quantitative availability assessment of a process can be used for the improvement of the efficiency of existing plants. It may be that due to unforeseen stoppages a plant does not achieve its design throughput rate. Then, one is confronted with various options depending on whether money is going to be spent on labour or capital equipment.

The decision to employ labour instead of spending capital on improving equipment leads to the question of preventive maintenance or inspection/overhaul policies. These problems have been dealt with quite extensively in the literature [8], [31], and will not be discussed here.

If we decide to spend money on capital equipment, then the steps involved in our assessment follow the pattern of the steps involved in the design problem discussed in (1.3). However, there are two important differences between the operational and design problems. In the latter one could change the plant configuration (which we shall denote as Γ) at will. This is not always the case during operation, where Γ is more or less fixed. Such problems are likely to involve the introduction of intermediate storage at available sites.

Secondly, in the design problem one designs for a constant production rate. In operating problems one aims to increase that rate as much as possible.

If money is going to be spent for capital equipment, one is confronted with a variant of the design problem discussed above with a variable production rate. One can define another operating problem, if no money is to be spent at all. Then, one requires the operating levels of various plant departments or pieces of equipment so that the process profitability is maximized. This is a kind of production control/scheduling problem and seeks to find an operating policy for

the existing plant.

Another aspect of this problem is "debottlenecking" [55]. Here one requires to find the pieces of equipment that limit the desired production rate. It occurs mainly during plant "revamping" i.e. when feed or throughput is employed for which the plant was not originally designed.

A compact enumeration of the problems arising from unreliable equipment in chemical engineering and a list of references dealing with them is presented in Figures 1.1 and 1.2.

1.5 Scope of the Project

This thesis is concerned with obtaining analytical results for the modelling of unreliable processes with particular emphasis on the effects of intermediate storage on availability. An optimization algorithm for the design problem described in 1.3 is also presented. Finally some examples are used to test the effectiveness of the algorithm.

Chapter two tries to put the present work in perspective by examining previous research on this subject and highlighting the problem of modelling the effects of intermediate storage on availability.

Chapter three presents an analytical expression for the availability of a system composed of two machines connected by a tank using stochastic analysis. Chapter four uses the results of chapter three to produce an approximate expression for more complex systems which include intermediate storage. Despite the simplicity of the proofs of the results in these last two chapters, they do not appear to have been presented before. Chapter five proposes a branch and bound optimization algorithm for the solution of the design problem, and a simpler procedure to deal with the more specific problem of choosing optimum buffer size.

Chapter six presents some examples to demonstrate the use, effectiveness and limitations of the algorithms in chapter five.

Finally, Chapter seven summarizes the main conclusions derived from this work and suggests lines for further research.

const

DESIGN WITH

A = 1

IDEAL DESIGN

DESIGN WITH

A < 1

BASE CASE DESIGN

COST AND RELIABILITY DATA

[1]
[38]

OPTIMIZATION ALGORITHM CHOOSING BETWEEN ALTERNATIVES

DESIGN OBJECTIVES

[4], [18], [19], [25], [35], [46], [47], [49], [53]

EMPLOYMENT OF TECHNIQUES FOR AVAILABILITY ASSESSMENT OF ALTERNATIVES

[5], [13], [15], [17], [22], [23], [26], [27], [28], [40], [43], [44], [45], [49], [50], [54], [56].

CHOICE OF ALTERNATIVES

FINAL DESIGN

OVERDESIGN

MORE EXPENSIVE EQUIPMENT

REDUNDANCY

BUFFER STORAGE

Figure 1.1

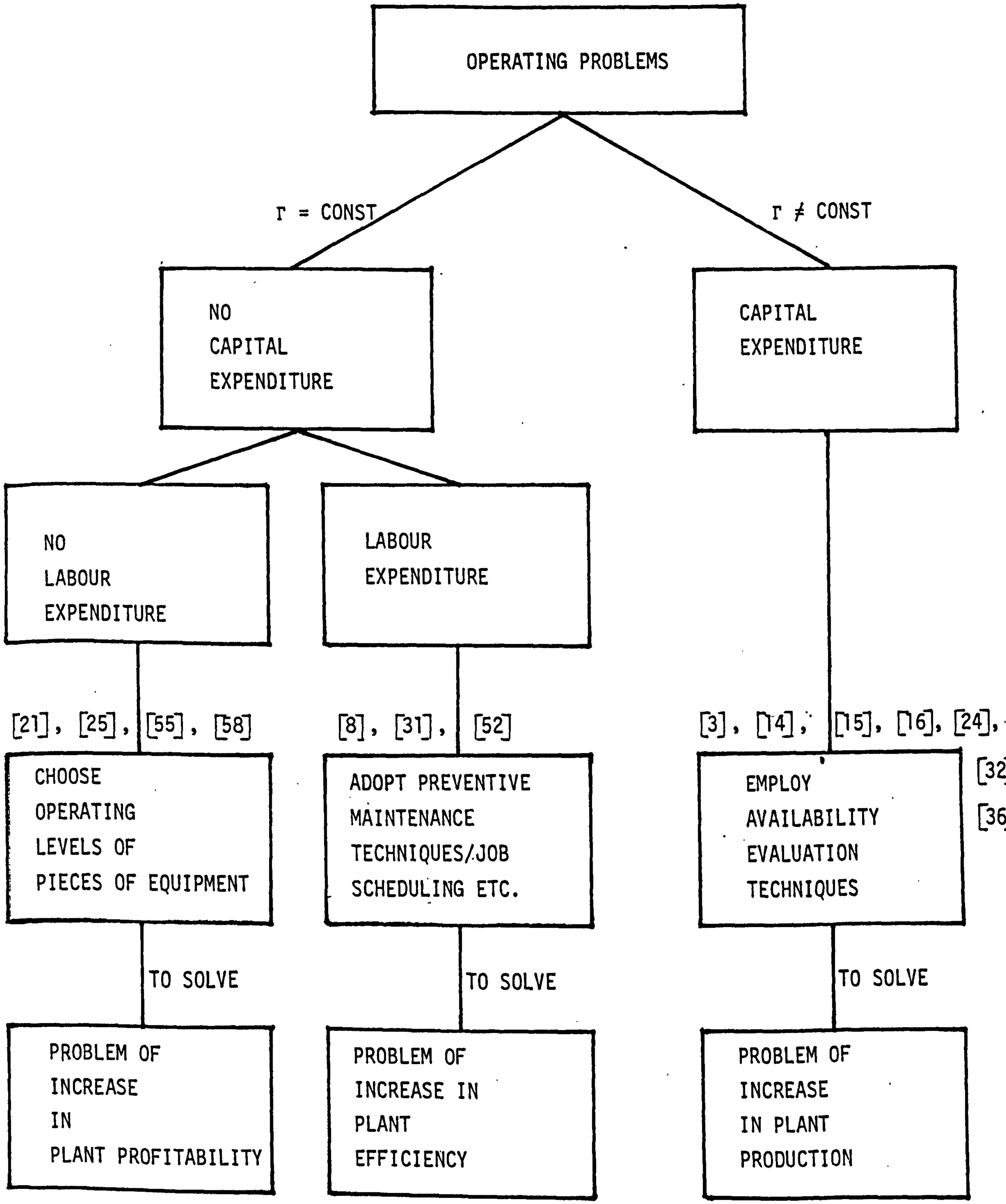

OPERATING PROBLEMS
Γ = CONST
Γ ≠ CONST
NO CAPITAL EXPENDITURE
CAPITAL EXPENDITURE
NO LABOUR EXPENDITURE
LABOUR EXPENDITURE
[21], [25], [55], [58]
[8], [31], [52]
[3], [14], [15], [16], [24], [32] [36]
CHOOSE OPERATING LEVELS OF PIECES OF EQUIPMENT
ADOPT PREVENTIVE MAINTENANCE TECHNIQUES/JOB SCHEDULING ETC.
EMPLOY AVAILABILITY EVALUATION TECHNIQUES
TO SOLVE
TO SOLVE
TO SOLVE
PROBLEM OF INCREASE IN PLANT PROFITABILITY
PROBLEM OF INCREASE IN PLANT EFFICIENCY
PROBLEM OF INCREASE IN PLANT PRODUCTION

Figure 1.2

C H A P T E R T W O

MATHEMATICAL MODELLING OF CHEMICAL PROCESSES WITH FAILURES

2.1 Introduction

It should be clear from the preceding chapter that the effect of alternative configurations on the availability, A, of a chemical plant needs to be assessed quantitatively, if availability analysis is to be used at the design stage.

Since A has been defined in probabilistic terms, the behaviour of a plant subject to unexpected failures is most easily modelled using probabilistic analysis. The way one proceeds depends on whether he decides to develop general analytical formulae or construct a computer simulation program. We shall examine these two approaches next.

2.2 Monte Carlo (MC) Simulation

2.2.1 General process modelling

Monte Carlo (MC) techniques involve the mathematical modelling of a plant using a computer. The plant's performance is then simulated over a sufficiently "long" period of time, by running this model for that time period. The computer generates equipment failures and repairs randomly, by sampling from given probability distributions and subsequently calculates statistically various measures of plant performance, such as A. It is not difficult to modify existing steady state chemical engineering simulation packages for such purposes; two such modifications have been reported in [14], [24], both on the CHESS simulation package.

In the first case Gaddy & Culbertson [24] use the modified CHESS package to determine the variations in process performance of ammonia plants. Their model uses 50 streams, 32 CHESS modules and a recycle.

They demonstrate the versatility of this approach by considering not only the effect of equipment failures on the final production rate, but also the effect of variations in process operating conditions. In the end they obtain the graph of a theoretical cumulative "production capability distribution", which is a very useful measure and preliminary design aid.

However, their results also show the disadvantages of MC simulation, namely that computer execution times were very large. Their model of the plant could only be run for a nominal time of 20 weeks, (<6 months) therefore the statistical significance of their results must be suspect. Contrast this time interval with that considered by other researchers in this field: Jenkins [32] for instance considers an interval of ten years, Ross [47] three years, Cherry et al [15] ten to twenty years and so on.

In [23] Freeman and Gaddy use MC simulation to calculate a very useful process performance measure called 'dependability' in what appears to be a more thorough application of the MC method. Their dependability measure takes into account the fact that plants can be run at various throughput rates without failing. They define dependability, D as

$$D = \int_{-\infty}^{\ell_1} \cdots \int_{-\infty}^{\ell_n} f(\underline{x})d\underline{x} \quad \underline{x} \in R^n$$

where $\underline{x}$ = process performance criterion vector

ℓ_i, i=1,n is a minimum specification for x_i and

$f(\underline{x})$ is the multivariate probability density function of $\underline{x}$.

D measures the percentage of time a process fails not only due to equipment breakdown, but also because of 'off-spec' performance. It is thus a much wider criterion of plant production than availability and very difficult to assess analytically since $f(\underline{x})$ is not known

analytically. Freeman & Gaddy proceed to show how D can be used to quantify overdesign in the design stages of a process.

It seems that MC simulation used in conjunction with computer packages is most useful where general non-series systems, non-exponential failure & repair distributions and complex performance measures (such as dependability) are involved. By deciding, however, on a Monte Carlo analysis of a process, one inevitably opts for an expensive computer bill. Kumamoto et al [37] have published a 'sophisticated' Monte Carlo method with an 'importance sampling technique' to render calculations more practical. For all their efforts, however, Monte Carlo simulation still remains the most expensive and time consuming method to evaluate plant performance.

Therefore, it is advisable to use it as a last resort, if no analytical expressions are available for the performance measures desired for a particular problem.

2.2.2 Buffer tank modelling and sizing

Because of a lack of analytical expressions for the effect of intermediate storage on plant availability, simulation has been extensively used in two problems related with the integration of buffer storage in a process. The first one involves the assessment of the performance of a plant with storage facilities and the second one involves the sizing of intermediate storage.

The first problem is the effect of buffer storage on plant availability. Obviously, if inserted between two units, it "protects" the unit downstream from upstream disturbances. That is, even if the upstream section of a plant is closed down because of some failure, the section downstream may continue working drawing its feed from the storage tank.

Wood, Muehl & Lyon [58] attempt to solve the operating problems outlined in chapter one via a combination of simulation and linear

programming techniques. Apart from performing availability calculations, their model can analyze the effects of redundancy and storage capacities and also size various units. They describe unit interdependencies by linear inequalities, i.e. if x_j is the "operating level" of unit j, then the inequality $x_i \leq x_j$ says that if unit j is not operating, unit i cannot operate (since $0 \leq x_i \leq 1$). The model is simple but requires as data five probability distributions for each unit and assumes that linear inequalities can adequately describe the system.

In another application, Golemanov & Blomberg [25] use MC simulation to optimize the performance of a pulp & paper mill. They calculate simultaneously the production rates from and the size of the buffer tanks by using a combined objective function. One of their main results highlights the fact that the ultimate profit function is very sensitive to the size of buffer storage. Edlund & Kallmén [21] are concerned with the same problem and arrive at similar, though less quantitative, results. Jenkins et al [32] and Ross [47] also point out the sensitivity of profit to tanks sizes. Sizing buffer storage tanks has therefore been one of the main aims of simulation studies. This second problem can be formulated as follows: How large should intermediate storage be, so that it includes enough hold-up to let the process continue even if upstream units have failed and are being repaired?

Cason [13] uses MC and reliability data and techniques to assess downtime. As we have seen, the usefulness of reliability techniques in conjunction with chemical plants is limited. Cason's model and sizing program is however simple and could be used for order-of-magnitude purposes. Ross [47] has also worked on these lines and has produced a graph of profit vs. tank capacity showing that traditional rule-of-thumb tank sizing decisions are above the optimal tank size. M.A. Malina [41] uses a simulation model to size product tanks and presents an easy

graphical method, illustrated by two worked examples. Unfortunately his method cannot be applied to intermediate storage sizing.

In this thesis we shall not use MC simulation but we shall try to assess plant performance and buffer tank sizes by concentrating on analytical techniques. Analytical availability expressions for processes with buffer storage are sadly lacking in the literature. Existing ones are reviewed in the relevant section of chapter three. One of the aims of this thesis is to fill this gap. This does not mean that MC techniques are not useful or valid. It is the author's opinion that they are best used for the analysis of specific plants in conjunction with analytical techniques. The best model for a particular process is a model written specifically for that process (as in [32], [25]) and some process measures such as dependability are almost impossible to calculate analytically. The final choice between simulation and analytical techniques (or between dependability and availability) is up to the designer, as long as the expense trade-off is clear. It is debatable though, whether MC simulation would be used at all in some cases (such as sizing storage tanks) if easy analytical expressions were present. In the next section we examine the mathematical tools available to develop such expressions.

2.3 Analytical Techniques

We have mentioned the drawbacks of Monte Carlo simulation and we have seen that its use arises mostly from the lack of analytical solutions of various systems, especially those involving tankage. As we are mainly interested in such analytical results, we shall have to choose a method to work with. There are three categories of analytical techniques.

(a) Approximate models: These are approximations of physical models which are normally difficult to solve and formulate if not simplified.

Cowan [17], Konoki [36] and Henley & Hoshino [27] use such models for availability calculations. Approximate models can be used as long as one is aware of the nature of the approximations involved and works in the region where they hold true.

In this thesis, we shall use certain approximations in chapter four to facilitate the calculation of availability for certain systems. In the course of assessing the effect of those approximations, we make use of a simulation program for comparison purposes.

(b) Graph theoretic models: These are rigorous models and normally consist of algorithms yielding the reliability of a structure. The main references on this subject are [12], [26], [51]. Allen and Pearson [5] have applied this technique to chemical engineering problems with multiple production rates and published computer codes of the methods involved.

Graph theoretic models are mainly useful for calculating the availability of a general structure if the product law in 2.4 is valid. Its main drawback for our purposes is the difficulty of dealing with buffer storage. Thankfully, chemical processes, even those with recycles, as we shall see later yield simple structures from the availability point of view, and thus complicated graph-theoretic algorithms are not necessary.

(c) Stochastic models: These models give accurate results, are simply formulated and theoretically sound. An introduction to stochastic processes is given in [20]. Their employment in modelling processes subject to random failures is demonstrated in [49] and [9], [10].

Stochastic modelling of chemical processes can be used to calculate both the availability and reliability of systems from the same initial set of equations, using different boundary conditions. As an added bonus, the most useful parameter in our case, availability, is the one most easily calculated. Their main drawback is that as the system

becomes larger, the number of (possibly differential) equations increases dramatically. The introduction of approximations then becomes necessary. In fact, this is the procedure we have followed to model the effect of intermediate storage on plant availability. We first calculate the availability of a single machine/tank/machine (M/T/M) system accurately using stochastic methods. Based on this analysis, we then use approximations to cope with larger, multitank systems.

Briefly, in the stochastic modelling of a system we follow the steps below:

(1) Identify all mutually exclusive states of the system in question and assign probability densities to each one.

(2) Derive the time-dependent probability differential equations by examining the transitions between states.

(3) Equate all time derivatives to zero for steady state.

(4) Solve the resulting equations for the probabilities.

(5) Group together all acceptable states which define the availability of a system and add.

This approach has been used in the literature [28], [49], [50], to derive expressions for the availability of easily recognizable subsystems ("blocks"), with the notable exception of subsystems which include buffer storage. In Fig. 2.3(a) we present the subsystems with analytical availability expressions readily available and in Fig. 2.3(b) the ones whose availability has not yet been expressed analytically.

We can combine the availability of such subsystems to yield the availability of a larger system. The technique used for this combination of subsystems is called 'blocking'. It is described in the next section, along with the equivalent concept to a flowsheet in availability analysis: the availability diagram (AD).

2.4 Availability Diagrams and Blocking

The availability diagram (AD) for a process is equivalent to a flowsheet in the simulation and/or design of chemical plants. As in a flowsheet one draws blocks for pieces of equipment that work together (e.g. a distillation column), similarly blocks in an AD represent pieces of equipment that fail together. Furthermore, different blocks in an AD are subject to different failure processes and may fail independently of each other. See [11] for an interesting discussion on this latter assumption.

The behaviour of the whole system as regards "failure" with respect to the failure of individual units in the system leads to two distinct types of systems:

(a) A series system is one that fails if any constituent part of the system fails.

(b) A redundant system is one where the system is "on" if any one unit is "on". We shall assume that for our purposes a systems "fails", if it does not produce output according to specifications.

There are two kinds of redundant systems:

(a) active redundant, parallel or 'hot' where all units operate 'on-line'.

(b) standby redundant systems. Here some units are kept as spare equipment (off-line) and are substituted 'on-line' if a similar unit has failed.

In this thesis, the blocks in an AD are drawn so that they represent parts of a plant that fail together. This justifies the analysis which follows in the next section, where independent failures for each block are assumed. Cowan [17] and Cherry et al [15] tend to draw blocks at the level at which data are available, but the above criterion should be adhered to, wherever possible, if the results are to be trusted. In

any case the level at which data are available is likely to be the level at which independent failures occur.

There are various combinations of blocks in an AD for which the availability has been calculated (see section 2.5). However for a large AD an approach has been used [15], [29] whereby the availability of individual blocks is combined in some way to yield the overall system availability. This procedure appears to be very useful, since it utilizes simple models (consisting of a few units) instead of larger and complicated ones. The "blocking" techniques developed so far [15], [29], follow the "product" rule of availabilities

$$A'_{series} = \prod_{i=1}^{N} A_i \quad \ldots\ldots \qquad (2.1)$$

for the availability of N machines arranged in series.

The model on which the above formula is based assumes that pieces of equipment may fail independently while 'idling' (i.e. waiting for other units to be repaired). In practice, if one machine fails, the others will be switched off until it has been repaired and are unlikely to fail while waiting (see, however, [3] for some cases to the contrary). Analysis of this system leads to the formula [7]

$$A_{series} = \left(1 + \sum_{i=1}^{N} \left(\frac{1-A_i}{A_i}\right)\right)^{-1} \quad \ldots\ldots \qquad (2.2)$$

which is independent of the failure and repair distributions involved in determining A_i.

Note that equations (2.1) and (2.2) give close results when units with high availability are considered.

For example, consider N=2

$$A'_{series} = \frac{1}{(1+\rho_1)(1+\rho_2)} = \frac{1}{1+\rho_1+\rho_2+\rho_1\rho_2} \quad \ldots\ldots \quad (2.3)$$

$$A_{series} = \frac{1}{1+\rho_1+\rho_2}$$

where ρ_1, ρ_2 are defined as in (1.3).

Thus, from (2.3) we can see that $A_{series} \approx A'_{series}$ provided that $\rho_1\rho_2 \approx 0$ or $A_i \approx 1$. This happens often, but is not always the case. See [3] and [32] for examples of low availabilities in practice.

Cowan [17] is aware of the approximations involved in (2.3) and works in the region $\rho_i << 1$ where errors are small.

In this work we shall use the approach in [7] and block units using (2.2) with the more realistic assumption that units on standby have a failure rate of zero.

It follows that our blocking approach will be:

(a) Reduce any series system to a series of recognizable subsystems subject to independent failures.

(b) Use (2.2) to block the relevant system.

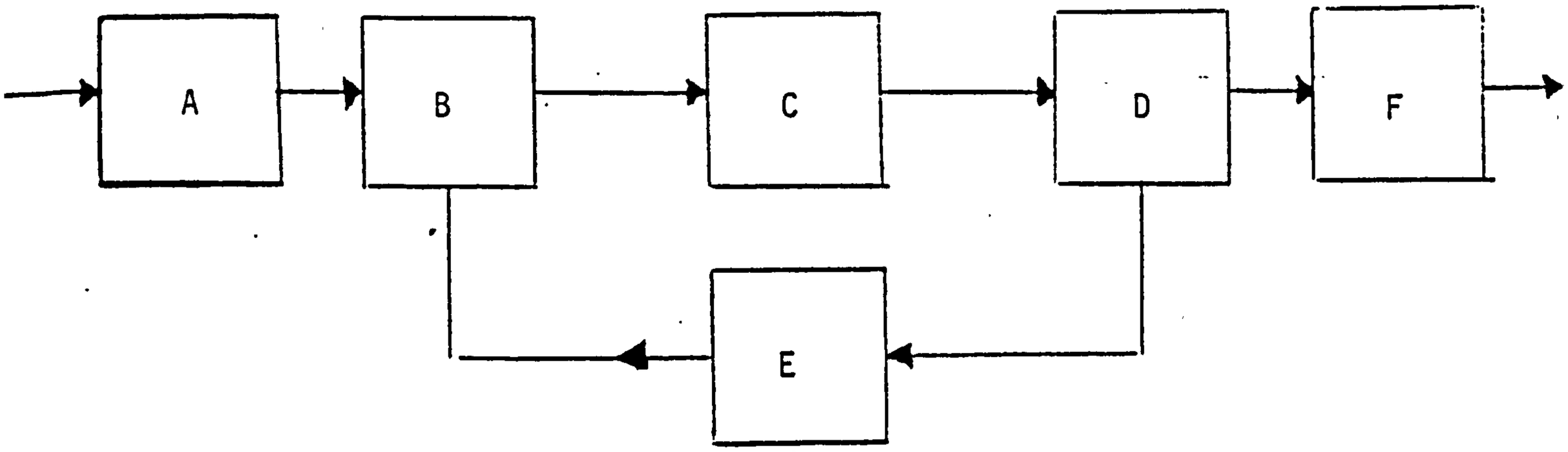
Two questions now arise: how to deal with intermediate storage tanks which destroy independence of failures and how to deal with recycles. The first question is dealt with extensively in chapters three and four. The question of how to deal with recycles can be answered immediately.

Recycles in chemical processes are easy to handle if one realizes that elements involved in a recycle are essentially elements in series according to our definition.

So, from the point of view of availability analysis the flowsheet in Fig. 2.1 is reduced to an equivalent AD in Fig. 2.2.

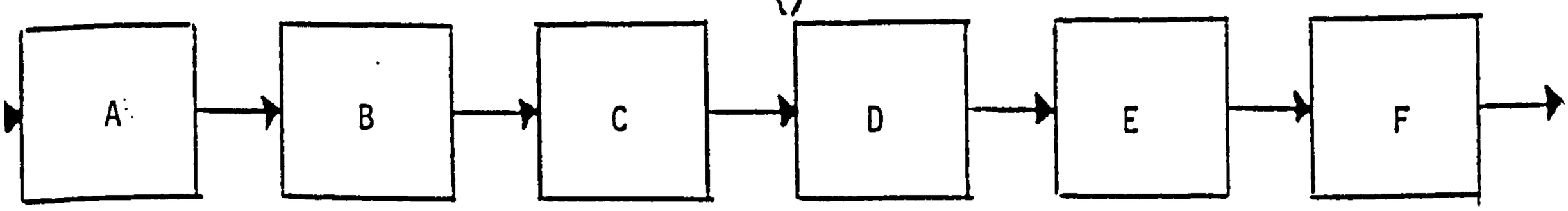
Figure 2.1

FLOWSHEET



AVAILABILITY DIAGRAM

Figure 2.2



Note that one can place the blocks B to E involved in the recycle in any order in the AD.

We can represent feed variations using a 'feed' block, F, with a 'failure' distribution corresponding to feed disturbances (or starvation). For example, if an oil refinery is being provided with feed subject to random disturbances (ships arriving late, pipe/rig failures) we can follow the previous analysis and suitably define availability etc. We can then use a feed block in series with the process, with similar properties, to other blocks.

To summarize: using AD's and the blocking formula (2.2) we can obtain the availability of large series systems. Most chemical processes

can be put into this form. What we need next is a method to obtain the individual availabilities A_i of the units in our AD required in (2.2) from failure and repair data. In the next section we shall present expressions for the availability of recognizable, commonly occurring subsystems.

2.5 Subsystem Availability

We have established that in order to calculate the overall system availability by blocking, one has to reduce the system in question to a series system from the availability point of view (provided that this is possible, of course). During this process of reduction, the availability of recognizable subsystems is calculated according to already derived formulae, using stochastic models. In this section we shall present the ones currently available in the literature.

The assumptions made in their derivation are:

(1) Failures occur independently (see previous section).

(2) The failure rate λ and the repair rate μ are constant with time. This assumption has experimental justifications presented in [6], [15], [32]. It is equivalent to saying that the failure and repair probability density functions are negative exponential. It also allows us to use the powerful Markov approach in stochastic analysis.

(3) Repair on failed equipment starts immediately. Repaired equipment performs as new.

(4) There is enough maintenance crew available for simultaneous repairs and there are no 'queues' of failed equipment in the workshop.

(5) During repairs non-failed pieces of equipment are idle or resting and no preventive maintenance is performed on them.

(6) When one component is repaired, the process is turned "on" immediately, if it has had to stop.

We can now present the expressions for those subsystems that have been solved in literature (see Fig. 2.3 for a pictorial representation).

(a) Single component subsystems

For a single component system with failure rate λ and repair rate μ, the availability is, according to (1.3)

$$\left.\begin{aligned} A &= \frac{\mu}{\lambda+\mu} = \frac{1}{1+\rho}, \quad \rho = \frac{\lambda}{\mu} \\ &\text{Note that } \lambda^{-1} = \text{MTTF}, \ \mu^{-1} = \text{MTTR} \end{aligned}\right\} \quad \ldots\ldots (2.3)$$

(b) Series system

We use Formula (2.2)

$$A_{series} = \frac{1}{1+\sum_{i=1}^{N} \frac{(1-A_i)}{A_i}} \quad \text{for N elements in series}$$

$$= \left(1+\sum_{i=1}^{N} \rho_i\right)^{-1} = (1+x)^{-1}, \quad x = \sum_{i=1}^{N} \rho_i$$

Note that x plays the same role as ρ for a blocked system.

(c) Active redundant system

For a definition of such a system see 2.4.

For 2 elements $A_p = A_1+A_2-A_1A_2$

For M elements in parallel

$$\left.\begin{aligned} A_p &= 1 - \prod_{i=1}^{M} (1-A_i) \\ \text{or } A_p &= 1 - \prod_{i=1}^{M} \left(\frac{\rho_i}{1+\rho_i}\right) \\ &\text{Normally } \rho_i = \rho, \ i=1,M \end{aligned}\right\} \quad \ldots\ldots\ldots (2.4)$$

(d) Standby redundant systems

For a definition of a standby redundant system see section 2.4.

In standby redundancy we have [49]:

(i) 'cold'standbys ; $\lambda=0$ when off, $\lambda=\lambda_{active}$ when "on"

(ii) 'hot' standbys ; $\lambda=\lambda_{active}$ whether "on" or "off"

(iii) 'warm'standbys ; $\lambda\neq\lambda_{active}$, non zero when "off"

$\lambda=\lambda_{active}$, when "on".

Most of the standby equipment pieces are similar (one is going to substitute a machine with a similar one) so, we can safely assume that $\rho_i = \rho$, all i=1, ..M for M pieces of equipment in standby. 'Hot' standbys are treated as in (c). 'Warm' standbys are difficult to model and their 'warm' failure rate is difficult to determine experimentally. We shall be solely concerned with 'cold' standbys with fully reliable failure detecting and switching mechanisms (i.e. there is no appreciable delay between failure and replacement).

Then, for one unit on-line and M-1 on cold standby [49]

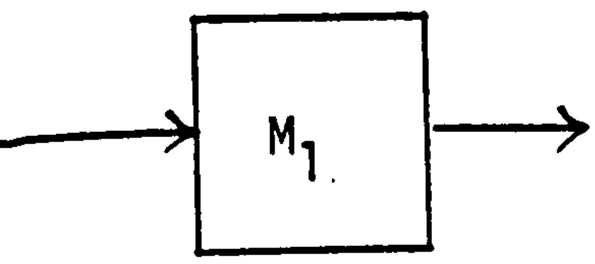
$$A_s = 1 - \frac{1}{\sum_{j=0}^{M} \frac{M}{j!} \rho^{j-M}} \quad \ldots\ldots \quad (2.5)$$

(e) Delay units

A delay unit (introduced in [15]) adds the effect of a failure upstream to the rest of the plant downstream after a constant delay time. It is equivalent to a tank which is kept always filled, even from some extraneous source. A delay unit "protects" the immediate downstream unit from disturbances upstream and increases the overall process availability.

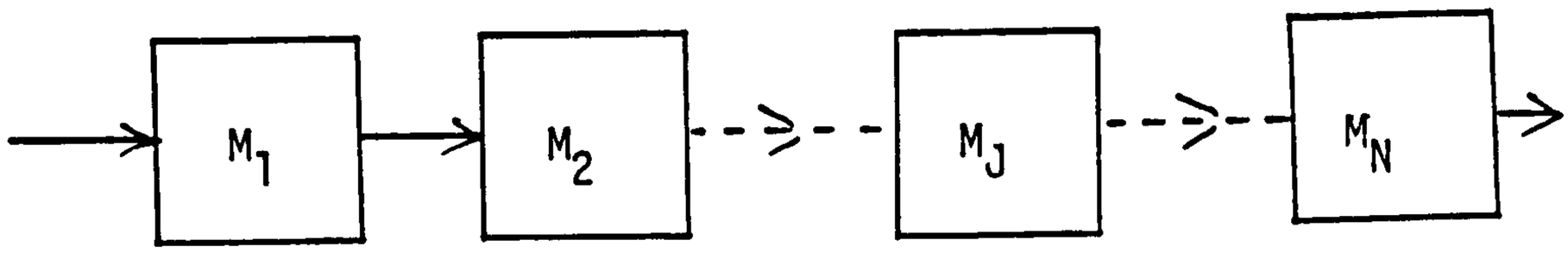
Fox and Zerbe [22] produced a formula for the availability of a delay unit/machine system, independent of failure distributions involved, and not subject to assumption (2) earlier in the section.

SINGLE UNIT

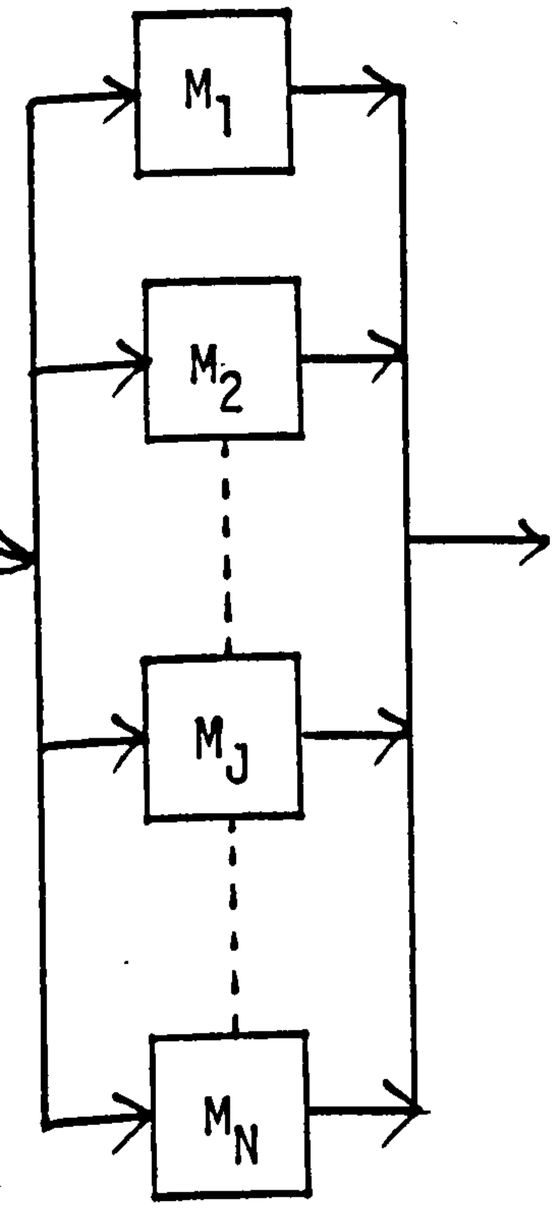


cf Equation (2.3)

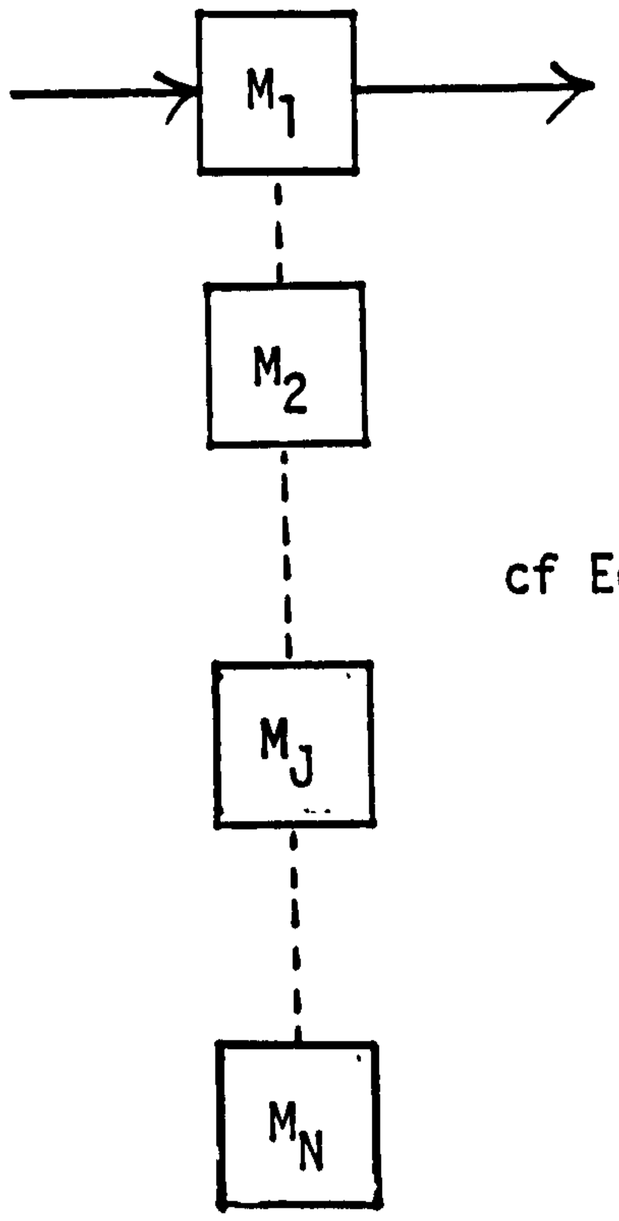
SERIES SYSTEM



cf Equation (2.2)



cf
Equation (2.4)



cf Equation (2.5)

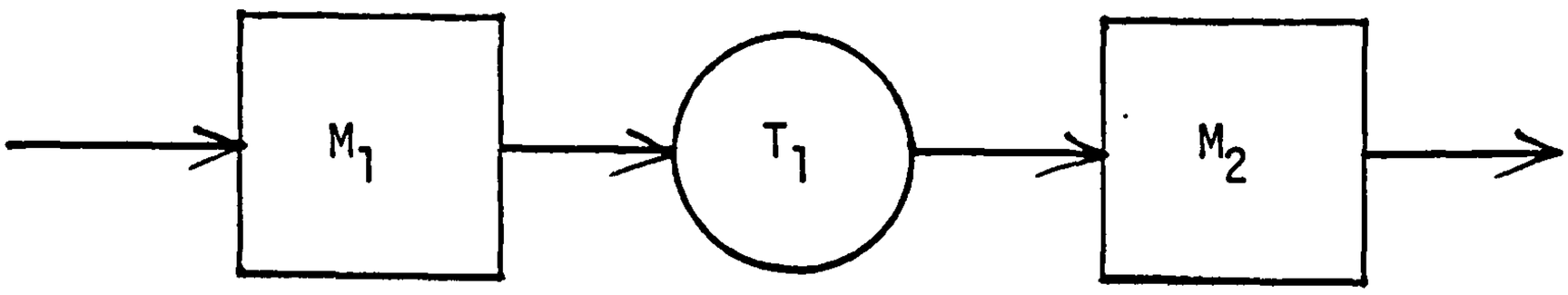
3. ACTIVE REDUNDANT SYSTEM

4. STANDBY REDUNDANT SYSTEM — HOT/WARM/COLD

Note: 3 and 4 can be hybrid redundant systems if not 100% rated.

SUBSYSTEMS WITH ANALYTICAL EXPRESSIONS FOR AVAILABILITY.

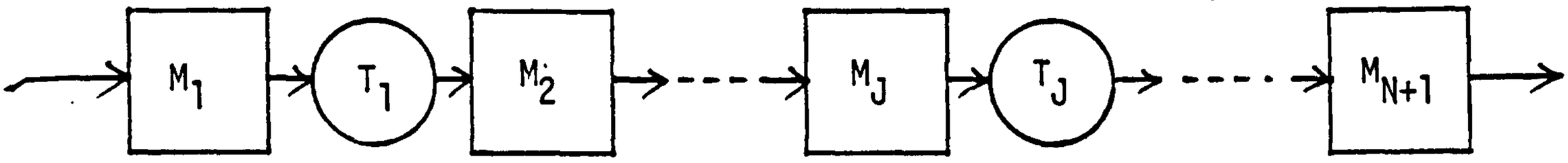
Figure 2.3(a)



5. SINGLE BUFFER TANK

M/T/M SYSTEM

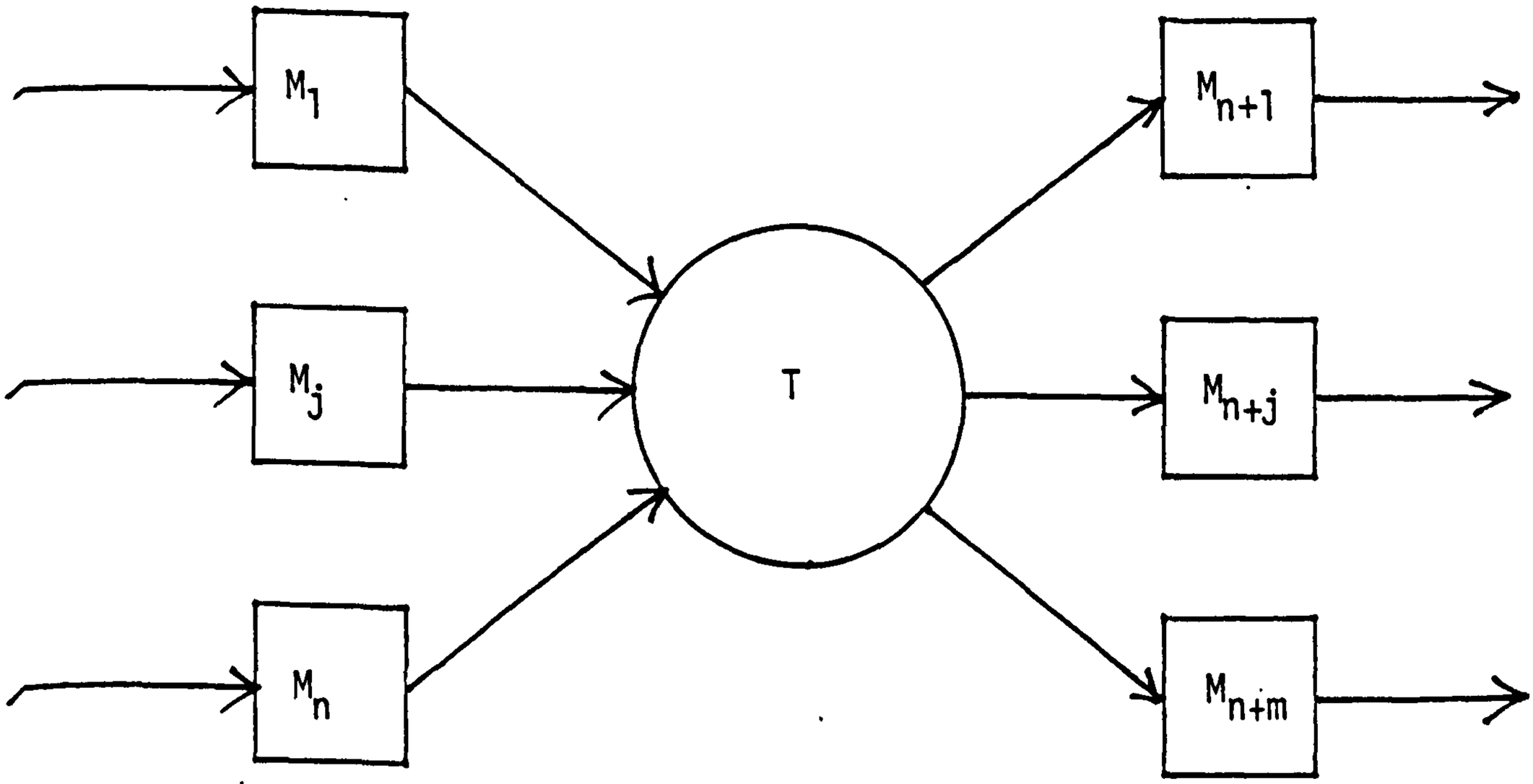
cf Equation (2.6) and Equation (3.13) in chapter three.



M/T/ .... M/T/M SYSTEM

6. SERIES SYSTEM WITH BUFFERS

cf section 4.2 and Equation (4.18) in chapter four



cf section 4.3 and Equation (4.25) in chapter four

7. PARALLEL REDUNDANCY AND BUFFER STORAGE nM/T/mM SYSTEM

SUBSYSTEMS WITHOUT ANALYTICAL EXPRESSIONS FOR AVAILABILITY

In the case of exponential repair distributions only, the formula reduces to the well-known equation [13], [27]:

$$A_{TOTAL} = 1-(1-A_{unit})e^{-\mu V/q} \quad \ldots\ldots \quad (2.6)$$

where V is the volume of the tank
and q the throughput flowrate.

(f) Hybrid redundancy

This is a concept introduced by Konoki [35], [36], and is particular to the chemical industry. It involves units that do not operate with 100% throughput and leads to a parameter which is more useful for chemical engineering purposes than availability, called efficiency, ξ. Because of its significance to chemical engineering problems we shall discuss hybrid redundancy and efficiency more extensively in the next section.

2.6 Hybrid Redundancy and Efficiency

We shall illustrate the meaning of plant efficiency using an example.

Consider one piece of equipment which has been designed to provide a production rate P_o over some period (say, a year). Because of stoppages and failures, it only produces $\bar{P}$ units in that period. Konoki [35] considers the "efficiency" of the unit

$$\xi = \frac{\bar{P}}{P_o} \quad \ldots\ldots \quad (2.7)$$

and this is clearly a parameter we are interested in. If the element in question is rated "100% capacity", then for the period it is 'on' it produces $P_o A$ units of output. So for 100% capacity units

$$\xi = A$$

However, when there are several acceptable output states, all making different contributions to the overall production rate $\xi \neq A$. To illustrate this let us use an example:

Consider two units in active redunancy, constituting a system as in Figure 2.4

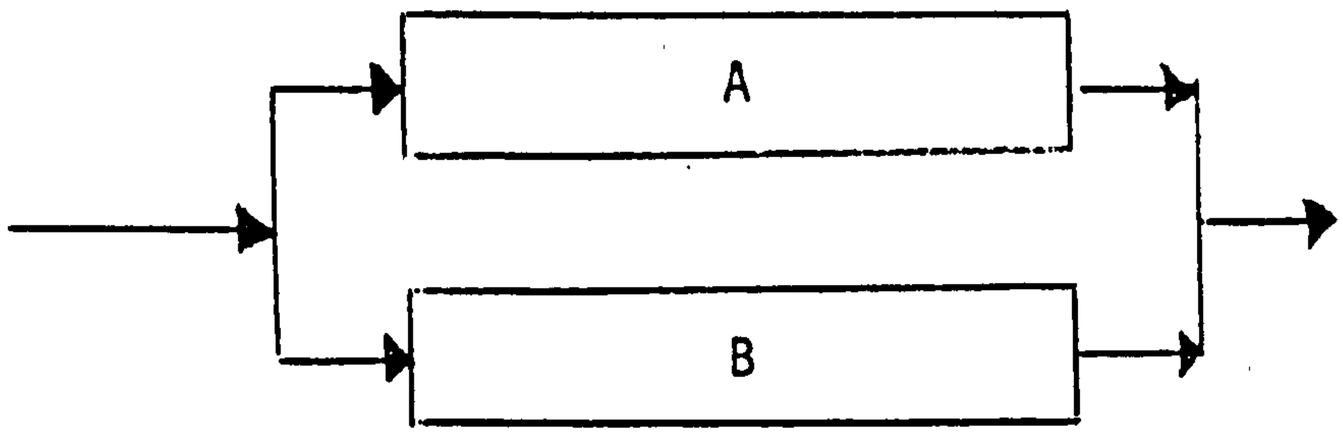


Figure 2.4

Suppose A,B are similar units of rated capacity 60%, but on-line they are contributing only 50% each. If one unit goes down, the other is not run at 50% but is taken to the limit of 60%

The expected production rate will be:

$$\bar{P} = P_{AB} \times 100\% P_o + P_A \times 60\% P_o + P_B \times 60\% P_o$$

where

P_{AB} = Probability of event "both units are up'

P_A = Probability of event "unit A is up"

P_B = Probability of event "unit B is up"

We have $P_A = P_B$ since A is a unit similar to B.

Therefore:

$$\bar{P} = (P_{AB} + 1.2 P_A) P_o$$

and

$$\xi = (P_{AB} + 1.2 P_A)$$

However, according to our definition of availability

$$A = \frac{\text{time up}}{\text{total time}} = P_{AB}+2P_A$$

Generally $\xi < A$ and A is an upper bound for ξ, corresponding to the case where all pieces of equipment are designed for 100% capacity.

In all previous subsystems ξ=A. Hybrid redundancy occurs only in the case of active or standby equipment.

Suppose we have M machines, each of capacity m_i, where $\sum_{i=1}^{M} m_i = 100M\%$ all in parallel, then if the system works at a flowrate P_o

ξ = A in equation (2.4)

If however $\sum_{i=1}^{M} m_i \neq 100M\%$, then

if up to r elements in a stage of M pieces of equipment in parallel can produce 100% capacity, then

$$\bar{P} = \left[\sum_{k=0}^{r-1} {}^{M}C_k \, \rho^{M-k} km_k\right] \frac{P_o}{(1+\rho)^M} + \left[\sum_{k-r}^{M} {}^{M}C_k \, \rho^{M-k}\right] \frac{P_o}{(1+\rho)^M}$$

and

$$\xi = \left[\sum_{k=0}^{r-1} {}^{M}C_k \, \rho^{M-k} \, km_k + \sum_{k=r}^{M} {}^{M}C_k \, \rho^{M-k}\right] / (1+\rho)^M \quad . \quad . \quad . \quad . \qquad (2.8)$$

If $m_k = 1/M$

$$\xi = \left[\sum_{k=0}^{M} {}^{M}C_k \, \rho^{M-k} \, k.\frac{1}{M}\right] /(1+\rho)^M = A$$

where ${}^{M}C_k = \frac{M!}{k!(M-k)!}$

Referring to the previous example,

m_i = 60% , r=M=2

$\xi = \frac{1+1.2\rho}{1+2\rho+\rho^2}$, as we found earlier.

We present below a series of expressions for the efficiency of some common hybrid redundant configurations.

$$3x50\% \qquad \xi = \frac{1.5\rho^2+3\rho+1}{(1+\rho)^3} \qquad \ldots\ldots \qquad (2.9)$$

$$4x50\% \qquad \xi = \frac{2\rho^3+6\rho^2+4\rho+1}{(1+\rho)^4} \qquad \ldots\ldots \qquad (2.10)$$

$$4x33^1/_3\% \qquad \xi = \frac{^4/_3\rho^3+4\rho^2+4\rho+1}{(1+\rho)^4} \qquad \ldots\ldots \qquad (2.11)$$

$$5x33^1/_3\% \qquad \xi = \frac{^5/_3\rho^4+^{20}/_3\rho^3+10\rho^2+5\rho+1}{(1+\rho)^5} \qquad \ldots \qquad (2.12)$$

$$5x25\% \qquad \xi = \frac{^5/_4\rho^4+10\rho^3+^{30}/_4\rho^2+5\rho+1}{(1+\rho)^5} \qquad \ldots \qquad (2.13)$$

Simmons [50] has derived also the following expressions for standby hybrid redundancy:

$$3x50\% \quad \xi = \frac{3R_1+2}{R_1^3+2(R_1+1)^3} \qquad \ldots\ldots \qquad (2.14)$$

$$4x50\% \quad \xi = \frac{2R_1+1}{(1+R_1)^4+2R_1^3(1+R_1)} \qquad \ldots\ldots \qquad (2.15)$$

where $R_1 = 1/\rho$

We shall employ these expressions again in Chapter five where we shall use our modelling results to try to find optimal configurations of chemical processes. However, there is a class of subsystems involving intermediate storage, which is very important to the chemical industry and which have not been solved satisfactorily so far ((e) being a special case). In the next two chapters we shall deal specifically with the problem of the effects of intermediate tankage on availability.

Only then can our modelling of a general process be adequate, so that we can proceed on to optimization questions.

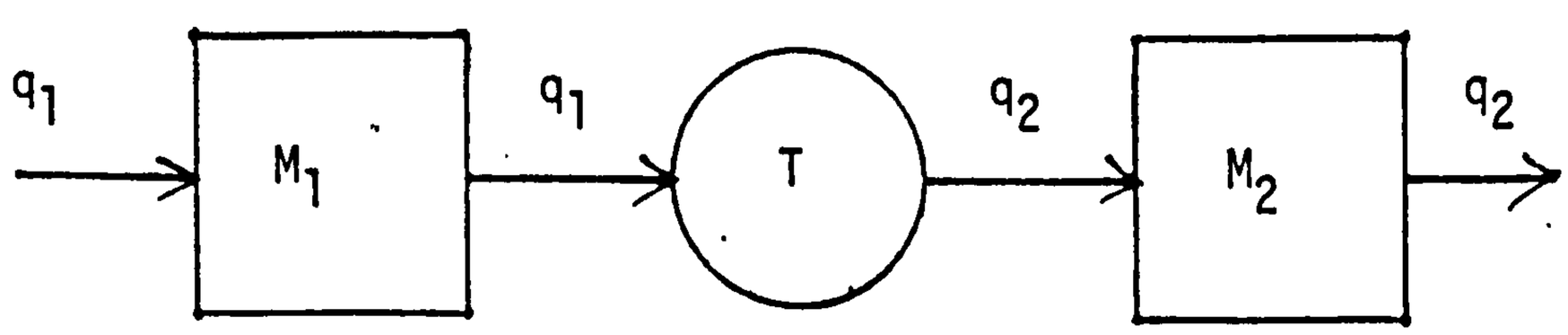
C H A P T E R T H R E E

THE SINGLE BUFFER SYSTEM

3.1 Introduction

One of the distinctive features of continuous flow chemical processes is the possibility of using buffer storage for increasing their availability. Since storage costs are as a rule high, there has always been the need to assess quantitatively their effect on production. Most of the relevant research has been carried out for production lines, where the items processed are discrete.

In the chemical industry, we are interested in the effect of buffer tanks on the output of a continuous flow process subject to failures. The simplest such system which we shall examine here consists of two machines with a single storage tank as depicted in Figure 3.1. Such a system we shall denote as M/T/M (Machine/Tank/Machine). Such systems have been examined before, but previous results have been calculated



q_1, q_2 flowrates through M_1, M_2 respectively

M_1, M_2 fail randomly

Tank never fails

under restrictive assumptions, either on the operating policy of the system or the parameters of the process. Thus, no analytical formulae have been published, except as special cases.

The differences of approach amongst various researchers who use analytic methods arise mainly for three reasons:

(A) The operating policy assumed for the system

A discussion of different policies can be found in [27]. Fox and Zerbe [22] assume a "tank-full" policy (policy A in [27]), i.e. the storage tank is always kept full, if necessary M_2 being shut down to allow it to fill. They arrive at an analytical formula for the system output described in 2.5 (e). Other papers are concerned with the"do nothing" policy (policy B in [27]) which can be summarized briefly as follows:

Both machines are run until one fails. If M_2 fails first, M_1 is run until either M_2 is repaired or the tank becomes full. If the tank becomes full before M_2 is repaired then M_1 is shut down (and hence cannot fail) until M_2 is repaired. Likewise, if M_1 fails first, M_2 is run until either M_1 is repaired or the tank becomes empty and in the latter case M_2 is shut down (and so cannot fail) until M_1 is repaired. If both machines are in the failed state, repair work can continue on both simultaneously. For the remainder of this paper, this is the operating policy considered. The policy produces a larger expected output than the "tank-full" policy ([27]) and is the more realistic of the two.

Murphy [43] has tried to extend the method of Fox and Zerbe [22] to include policy B. All other authors ([44], [45], [56]) are concerned with the latter "do nothing" policy.

(B) Definition of a balanced system

Referring to Figure 3.1, the system will be called "balanced" if

q_1 and q_2 are equal. Otherwise it will be called "unbalanced".

Cowan [17], Murphy [44], [45] and Wijngaard [56] discuss unbalanced systems. However, even for balanced systems the mathematical problems have not been satisfactorily solved; so the analysis in this chapter is restricted to the balanced case which provides the most useful model for steady state design purposes.

(C) Mathematical techniques and assumptions

Henley and Hoshino [27] use an approximate method based on certain assumptions on the relative magnitudes of the process parameters, Fox and Zerbe [22] and Murphy [44], [45] use renewal theory and Wijngaard [56] uses regeneration theory. Fox and Zerbe [22] and Murphy [43] examine first the case where M_1 never fails and then extend this to the policy where all work on M_2 (either operation or repair) ceases when M_1 fails. In subsequent analysis Murphy ([44], [45]) removes this rather restrictive assumption, but still assumes that repair work on M_2 is suspended when M_1 fails. Another approach for unbalanced systems is described in [3]. There, the flow is considered as a random walk between two reflecting barriers, with the tank level assuming a discrete number of predetermined levels, and being independent of the original level.

In this chapter we are also interested in analytical expressions and only Fox and Zerbe [22] and Wijngaard [56] have arrived at such expressions. We have already discussed the restrictions in the approach of Fox and Zerbe [22]. Wijngaard [56] assumes that the failure rates of M_1 and M_2 are the same when on standby as when running. We have seen the limitations of this assumption when we discussed equation (2.1) in chapter 2. Although he mentions how his approach might also be applied to the model we are considering (see his Remark page 44 in [56]) this is not developed further in his paper.

In this chapter we shall use the theory of stochastic processes to obtain an analytical formula for the availability of the balanced M/T/M system under the "do nothing" policy described in (A).

3.2 The Model

We shall assume that machines M_1 and M_2 have constant failure rates λ_1 and λ_2 respectively and constant repair rates, μ_1 and μ_2 respectively, independently of the past. This implies that the failure and repair times of the machines are mutually independent exponentially distributed random variables with means λ_1^{-1}, λ_2^{-1}, μ_1^{-1} and μ_2^{-1}. Under this assumption, if there were no tank the system would be a Markov chain and hence soluble by standard techniques. The presence of the tank destroys the Markov property but this problem can be overcome by treating the level of the tank as a "supplementary variable". For a more general discussion of the supplementary variable technique and its applications, see Cox and Miller ([20], section 6.3).

Let the random variable Y_t denote the fraction of the tank's volume which is filled at any time t; thus $0<Y_t<1$. We identify eight states.

1. Both machines down, $0<Y_t<1$
2. M_1 down, M_2 up , $0<Y_t<1$
3. M_1 up, M_2 down , $0<Y_t<1$
4. M_1 up, M_2 up , $0<Y_t<1$
5. M_1 down, M_2 up , $Y_t = 0$ (M_2 shut down)
6. M_1 up, M_2 up , $Y_t = 0$
7. M_1 up, M_2 down , $Y_t = 1$ (M_1 shut down)
8. M_1 up, M_2 up , $Y_t = 1$

We ignore the possibility M_2 down, $Y_t = 0$ (tank empty) since that case would only arise if M_2 went down at the precise moment the tank became empty, and the probability of that is zero. Similarly we ignore

the state M_1 down, $Y_t = 1$.

The system is therefore described by a state variable X_t, which takes one of the values 1 to 8, together with the supplementary variable Y_t.

For $0<y<1$, $t\geq 0$, $i = 1,2,3,4$ define the occupation density of state i by

$$P_i(t,y) = \lim_{h\downarrow 0} \frac{1}{h} P\{X_t = i,\ y \leq Y_t \leq y+h\}$$

assumed to exist and be continuous. For $t\geq 0$, $i = 5,6,7,8$ define

$$P_i(t) = P\{X_t = i\}$$

In Appendix 1 it is shown that these functions satisfy the following set of differential equations

$$\frac{\partial P_1}{\partial t} = -(\mu_1+\mu_2)P_1+\lambda_2 P_2+\lambda_1 P_3 \quad \ldots\ldots\ldots \quad (3.1)$$

$$\frac{\partial P_2}{\partial t} = \mu_2 P_1 -(\mu_1+\lambda_2)P_2 + \lambda_1 P_4 + c\frac{\partial P_2}{\partial y} \quad \ldots\ldots \quad (3.2)$$

$$\frac{\partial P_3}{\partial t} = \mu_1 P_1 -(\lambda_1+\mu_2)P_3 + \lambda_2 P_4 - c\frac{\partial P_3}{\partial y} \quad \ldots\ldots \quad (3.3)$$

$$\frac{\partial P_4}{\partial t} = \mu_1 P_2 + \mu_2 P_3 - (\lambda_1+\lambda_2)P_4 \quad \ldots\ldots\ldots \quad (3.4)$$

$$\frac{dP_5}{dt} = -\mu_1 P_5 + \lambda_1 P_6 + c\,P_2(t,0) \quad \ldots\ldots\ldots \quad (3.5)$$

$$\frac{dP_6}{dt} = \mu_1 P_5 - (\lambda_1+\lambda_2)P_6 \quad \ldots\ldots\ldots\ldots \quad (3.6)$$

$$\frac{dP_7}{dt} = -\mu_2 P_7 + \lambda_2 P_8 + cP_3(t,1) \quad \ldots\ldots\ldots \quad (3.7)$$

$$\frac{dP_8}{dt} = \mu_2 P_7 - (\lambda_1+\lambda_2)P_8 \quad \ldots\ldots\ldots\ldots \quad (3.8)$$

where $c = q/V$ and $P_2(t,0)$, $P_3(t,1)$ are defined by continuity in the second variable, i.e.

$$P_2(t,0) = \lim_{y\to 0} P_2(t,y)$$

$$P_3(t,1) = \lim_{y\to 1} P_3(t,y)$$

These are the forward equations of the system which, when the initial conditions are specified, may in principle be solved to determine the probability that the system is working at any time t, along with the normalising condition (3.9)

$$\int_0^1 \sum_{i=1}^{4} P_i(t,y)dy + \sum_{i=5}^{8} P_i(t) = 1 \quad \ldots\ldots \quad (3.9)$$

It is not immediately obvious that it is possible to satisfy (3.9) from (3.1) - (3.8). To do this we must have

$$\int_0^1 \left(\sum_{i=1}^{4} \frac{\partial P_i}{\partial t}(t,y)\right)dy + \sum_{i=5}^{8} \frac{dP_i}{dt}(t) = 0 \quad \ldots \quad (3.9)'$$

It may be seen, however that

$$\lambda_2 P_6(t) = cP_3(t,0)$$

since each side represents the expected rate of flow (at time t) from state 6 into state 3 with $Y_t = 0$. Similarly,

$$\lambda_1 P_8(t) = cP_2(t,1).$$

Substituting these relations into (3.1) - (3.8), equation (3.9)' follows. Thus the complete system (3.1) - (3.9) is consistent.

In practice, the equations are difficult to solve and in any case our main interest is in the equilibrium probabilities which are defined by:

$$\pi_i(y) = \lim_{t\to\infty} P_i(t,y) \quad (i=1,2,3,4, \quad 0<y<1)$$

$$\pi_i = \lim_{t\to\infty} P_i(t) \quad (i=5,6,7,8)$$

The π_i's also satisfy equations (3.1) - (3.8) with P_i replaced by π_i, $\frac{\partial P_i}{\partial y}$ by $\frac{\partial \pi_i}{\partial y}$ and $\frac{\partial P_i}{\partial t}$ (or $\frac{dP_i}{dt}$) by 0.

In addition we have the normalising condition

$$\int_0^1 \left(\sum_{i=1}^{4} \pi_i(y)\right)dy + \sum_{i=5}^{8} \pi_i = 1 \quad \ldots\ldots\ldots \quad (3.10)$$

and the boundary condition

$$\int_0^1 \pi_2(y)dy = \int_0^1 \pi_3(y)dy \quad \ldots\ldots\ldots \quad (3.11)$$

which corresponds to the fact that since the flow rate through the tank is constant, the proportion of time the level of the tank is rising equals the proportion of time the level is falling. (This is also suggested as a boundary condition in [15]).

The availability may now be defined as

$$A = \int_0^1 (\pi_2(y) + \pi_4(y))dy + \pi_6 + \pi_8 \quad \ldots \quad (3.12)$$

defining as system failure, the "failure to produce output".

In Appendix 2 it is shown that these equations may be solved explicitly to give

$$A(k) = \frac{(\rho_1-\rho_2 e^{-k})A_1A_2}{(\rho_1A_2-\rho_2A_1e^{-k})} \quad \ldots\ldots\ldots\ldots \quad (3.13)$$

where

$$\left.\begin{aligned} \rho_i &= \frac{\lambda_i}{\mu_i} \quad i=1,2 \\ A_i &= \text{availability of machine } i \\ k &= \frac{(\mu_1+\mu_2+\lambda_1+\lambda_2)(\lambda_1\mu_2-\lambda_2\mu_1)}{(\mu_1+\mu_2)\ (\lambda_1+\lambda_2)\ c} \end{aligned}\right\} \quad \ldots \quad (3.14)$$

An important special case occurs when $\rho_1 = \rho_2 = \rho$ so that the availabilities of M_1 and M_2 separately are equal. In this case $k = 0$ and (3.13) reduces to

$$A = \frac{1+c^*+\rho^{-1}}{\rho+\rho^{-1}+2+c^*(1+2\rho)} \quad \ldots\ldots\ldots\ldots \quad (3.15)$$

where

$$c^* = c(\lambda_1^{-1}+\lambda_2^{-1}) \quad \ldots\ldots\ldots\ldots \quad (3.16)$$

This can be derived by taking the limit as $\rho_2 \to \rho_1$ in (3.13).

The output of a balanced system with a variable flowrate q can easily be found from (3.13), (3.15). If q has a continuous distribution f(q) as it often does in chemical processes, then

$$\text{total output } \bar{P} = \int_0^\infty qA^*(q)f(q)dq \quad \ldots\ldots \quad (3.17)$$

where $A^*(q)$ is related to $A(k)$ as in (3.13) by

$$A^*(q) = A\ (\frac{\theta V}{q})$$

where $$\theta = \frac{(\lambda_1+\lambda_2+\mu_1+\mu_2)(\lambda_1\mu_2-\lambda_2\mu_1)}{(\lambda_1+\lambda_2)(\mu_1+\mu_2)}$$

Thus $$\xi = \frac{\overline{p}}{p_o} = A \text{ only if } q \text{ is constant.}$$

3.3 Asymptotic Behaviour

If equations (3.13) and (3.15) are correct, they should behave correctly in the limits of $V\to 0$ and $V\to\infty$.

(a) $V\to 0$

The availability of a system composed of two machines in series, with exponential failure and repair distributions is known to be from (2.2)

$$A_{series} = (1+\rho_1+\rho_2)^{-1}$$

If $V\to 0$ ($k\to 0$, $c\to\infty$)

$$A \to \frac{(\rho_1-\rho_2)(1+\rho_1)^{-1}\ (1+\rho_2)^{-1}}{\rho_1(1+\rho_2)^{-1} - \rho_2(1+\rho_1)^{-1}}$$

$$= (1+\rho_1+\rho_2)^{-1}$$

$$= A_{series} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.18)$$

Q.E.D.

(b) $V\to\infty$

Case (i) $A_1<A_2$, $k>0$, $e^{-k}\to 0$.

Then

$$\lim_{k\to+\infty} A = \frac{\rho_1}{\rho_1 A_2} A_1A_2 = A_1 \qquad \ldots\ldots\ldots\ldots \quad (3.19)$$

as expected, see [26]

Case (ii) $A_1>A_2$, $k<0$, $e^{k}\to 0$.

Multiply by e^k

$$\lim_{k\to-\infty} A \frac{(\rho_1 e^k - \rho_2)}{\rho_1 A_2 e^k - \rho_2 A_1} A_1 A_2 = \frac{-\rho_2}{-\rho_2 A_1} A_1 A_2 = A_2 \quad \ldots\ldots \quad (3.20)$$

as expected, see [26]

When $V\to\infty$ units M_1 and M_2 tend to be "decoupled". Thus the availability tends to be equal to the minimum of A_1, A_2 (see also [26]).

Note that $\frac{dA(k)}{dk} = \frac{\rho_1 A_2 - \rho_1 A_1}{(\rho_1 A_2 - \rho_2 A_1 e^{-k})^2}$ is of

constant sign independent of k.

Therefore A(k) is monotonic and continuous in $k\epsilon[0,-\infty)$ and $k\epsilon(-\infty,0]$. For a particular system k will belong to either of these subsets or be zero in which (3.15) applies. Therefore we can say that

$$A_{series} \leq A(k) \leq \min\{A_1, A_2\} \quad \ldots\ldots\ldots \quad (3.21)$$

and the introduction of the tank is always an improvement to the system. This inequality is also examined in [26].

Note that it can be proved that neither the numerator nor the denominator of (3.13) can go to zero despite the form of the equation.

3.4 Extensions of the Method to Larger Systems

One is naturally inclined to try to extend the method presented here to solve more complex multi-tank systems. Unfortunately, the complexity of the equations and the number of states increases drastically when larger problems are considered.

As an example of such complexity, consider the states and equations for a two-tank, three-machine series system (M/T/M/T/M system). The difficulty involved in finding appropriate boundary conditions and solution methods is demonstrated in Appendix 3. In fact, even this M/T/M/T/M system has remained unsolved analytically, despite the efforts

of the author. It appears that numerical methods must be employed on large sets of differential equations in order to obtain the availability of larger systems.

This line of progress yields increasingly diminishing returns, as even writing down the equations for a two-tank system is a formidable task.

For a system with N tanks and N+1 machines the total number of states would be $3^N 2^{N+1}$. Applying the exclusion assumptions discussed in 3.2 (i.e. disallow states where a tank is empty and the machine downstream is 'off' and where a tank is full and the machine upstream is 'down') would lead to the following number of states.

$$2^{N+1} + 4 \sum_{k=1}^{N} \frac{N!}{(N-k)!k!} 2^{N-k} + 2(2^N-2) + 2(N-1)$$

A further simplification would be to exclude states whereby intermediate machines are "on" flanked by machines which are "off". These states would only involve a transfer of product from one tank to another. One could then dispense with the last 2(N-1) states and would be left with:

$$2^{N+1} + 4 \sum_{k=1}^{N} {}^{N}C_k 2^{N-k} + 4(2^{N-1}-1)$$

states to contend with. The number of equations for such systems is tabulated in Table 3.1

Table 3.1

No. of tanks	states and equations
1	8
2	32
3	104

We can see how sharply the size of the problem increases. This fact, along with the increase in equation complexity constitute the basis for our motivation to employ approximation techniques for larger systems. This is the subject of the next chapter.

C H A P T E R F O U R

THE AVAILABILITY OF COMPLEX TANK CONFIGURATIONS

4.1 Introduction

In the previous chapter, we obtained an expression for the availability of a single M/T/M system. We saw how it is possible to extend the method used to more complex systems composed of many machines and many tanks, and how the mathematics becomes prohibitively complicated. In this chapter, we shall try to use suitable approximations to analyze such multimachine-multitank systems.

The kinds of systems we shall be concerned with are depicted in Fig. 2.3(B). Subsystem 5 we have already dealt with in chapter three. Here we shall examine configurations 6 and 7.

An obvious way of approaching the problem of finding the availability of such systems is to attempt to use the blocking technique described in 2.4 recursively, starting from the derived expression of the M/T/M system. Any such recursive approximation must exhibit the following desirable features.

(1) The final availability must lie within the appropriate bounds for each system, i.e. it must behave asymptotically correctly for $V_i \to 0$ and $V_i \to \infty$.

(2) It must be consistently close to the actual availability, or at least, be a close underestimate of it.

(3) It should be independent of the blocking order.

(4) It should be easily calculable, in a form suitable for quick engineering calculations.

In the following sections we show how such expressions can be obtained for both kinds of multitank-multimachine systems: series or

parallel.

4.2 Series Systems

4.2.1 Types of series systems

We can split series systems into three categories. Note that in all three, we consider 'balanced' systems, as defined in chapter three.

Type I N tanks, K machines ; K=N+1 (Fig. 4.1)

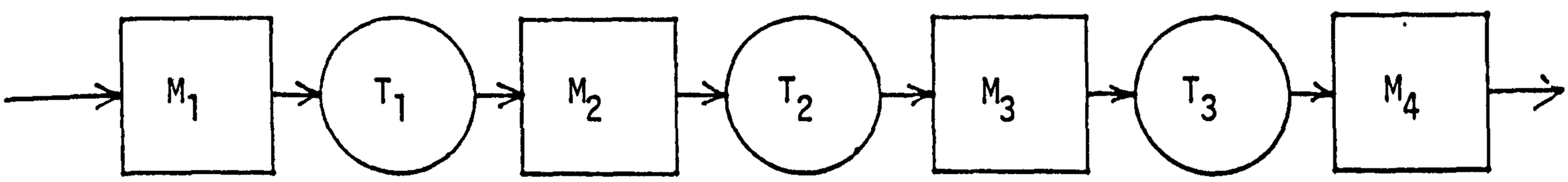


Figure 4.1

In Type I systems each tank is preceded and followed by a machine and intermediate machines are preceded and followed by a tank.

Type II N tanks, K machines ; K>N+1 (Fig. 4.2)

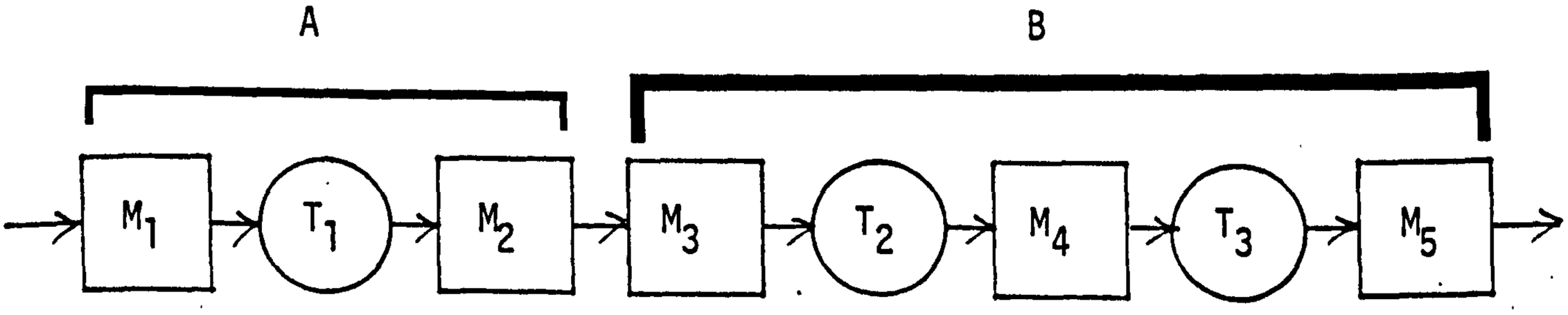


Figure 4.2

An example of such a system is shown in Fig. 4.2

In Type II systems, at least two intermediate machines are directly connected in series with no tanks in between.

In this case, we can split the N-tank/K-machine system into (K-N) systems of Type I which fail independently. We consider separately those availabilities of Type I systems which are formed when we divide the Type II system at the points where the two machines are directly connected. There are K-N+1 such points. We can then block the resulting Type I systems using the blocking formula (2.2).

As an example, consider the Type II system of Fig. 4.2. We can split it into two Type I systems; one is the system $M_1/T_1/M_2$ and the other is the system $M_3/T_2/M_4/T_3/M_5$. We have thus reduced the problem of analysis of such systems to the previous problem of analysis of Type I systems.

Type III N tanks, K machines ; K<N+1.

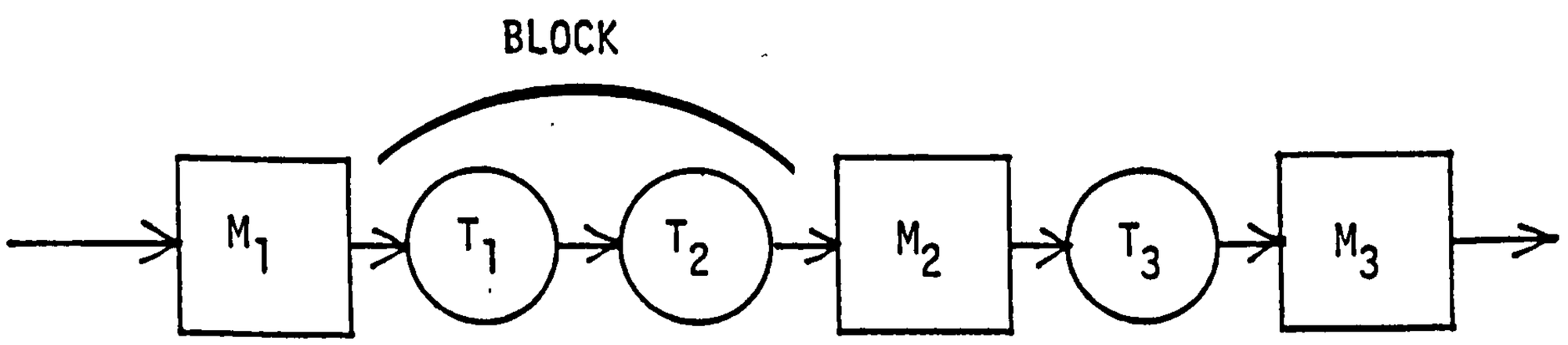


Figure 4.3

An example of such a system is shown in Fig. 4.3.

In Type III systems at least two tanks are connected in series with no intermediate machines. The tanks in series can be merged together into a single tank with a volume equal to the sum of the individual volumes, and produce again a Type I system. Note that this merger is not possible if we consider the possibility of tank failures, which, however, we ignore in this work. We can thus safely restrict ourselves to the analysis of Type I series systems as all other types can be reduced to a combination of Type I configurations.

4.2.2 The uncoupled recursion for series systems

In a Type I series system the appropriate bounds for the availability $A_{series}(V)$ are [27]:

$$A_{\substack{\text{no tanks}\\ \delta}}(V=0) < A_{series}(V) < \min_{i=1,N+1} \{A_i\} \text{ for N tanks} \quad \ldots(4.1)$$

Moreover, if individual $V_i \to 0$ (but not all) the system availability should tend to the availability of a Type II system formed if that tank with $V_i \to 0$ is omitted in the original Type I system.

Now, if we had a system such as the one depicted in Fig. 4.4

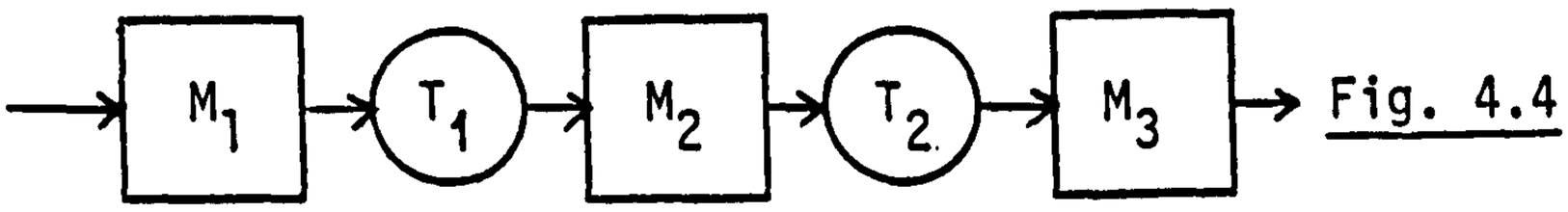


Fig. 4.4

and we let $V_1 \to 0$, then we should be able to use (2.2) to block the resulting Type II system $M_1/M_2/T_2/M_3$ in the following way:

$$A_{series} = (1 + \frac{1-A_1}{A_1} + \frac{1-A_{23}}{A_{23}})^{-1}$$

where A_{23} is the availability of the $M_2/T_2/M_3$ system. A similar formula can be obtained if $V_2 \to 0$ and if both V_1 and $V_2 \to 0$. In the latter case

$$A_{series} = (1 + \sum_{i=1}^{2} \frac{1-A_i}{A_i})^{-1} \qquad \text{(Formula 2.2)}$$

This observation leads us to the idea of expressing (2.2) recursively in such a way that larger M/T.../M systems can be calculated from smaller ones. In this way, their asymptotic behaviour, at least as $V_i \to 0$, should be correct.

Formula (2.2) can be recursively formulated as follows:

$$A_{s,k} = (1 + \sum_{i=1}^{\kappa} (\frac{1-A_i}{A_i}))^{-1} = \frac{A_{s,k-1}A_k}{A_{s,k-1}+A_k-A_{s,k-1}A_k} \quad \ldots \quad (4.2)$$

where:

$A_{s,k-1}$ = series availability of k-1 elements

A_k = availability of k^{th} element

We define $A_{s,0}=1$. It is easy to show the equivalence of the two expressions (2.2) and (4.2) by induction.

Remember that (2.2) can also be written

$$A_{s,k} = \frac{1}{1+\sum_{i=1}^{n}\rho_i} = (1+x)^{-1};\ x = \sum_{i=1}^{n}\rho_i \quad \ldots\ldots \quad (4.3)$$

We can now use recursion (4.2) to find a formula for a 3-component series system without tanks in a form that is suitable for extension to a system with tanks. For the remainder of this section we shall use such a 3-component Type I series system as an example to illustrate the approximations used.

Let us use an easier notation by using subscripts to indicate the units blocked in series into $A_{s,k}$ (thus $A_{s,2} = A_{12}$, say, $A_{s,3} = A_{123}$, etc.).

For the first 2 units 1 & 2 (see Fig. 4.5)

$$A_{12} = \frac{A_1A_2}{A_1+A_2-A_{12}} \rightarrow A_1 = \frac{A_{12}A_2}{A_2-A_{12}+A_{12}A_2} \quad \ldots\ldots \quad (4.4)$$

Then:

$$A_{123} = \frac{A_1A_{23}}{A_1+A_{23}-A_1A_{23}} \quad \ldots\ldots \quad (4.5)$$

Substituting (4.4) into (4.5) for A_1 we have:

$$A_{123} = \frac{A_{12}A_2A_{23}}{A_2A_{12}+A_2A_{23}-A_{12}A_{23}} \quad \ldots\ldots \quad (4.6)$$

In (4.6) we have an expression for the availability of a 3-component system which uses the combined availability of units 1 & 2, 2 & 3 and the availability of unit 2, as shown in Fig. 4.5.

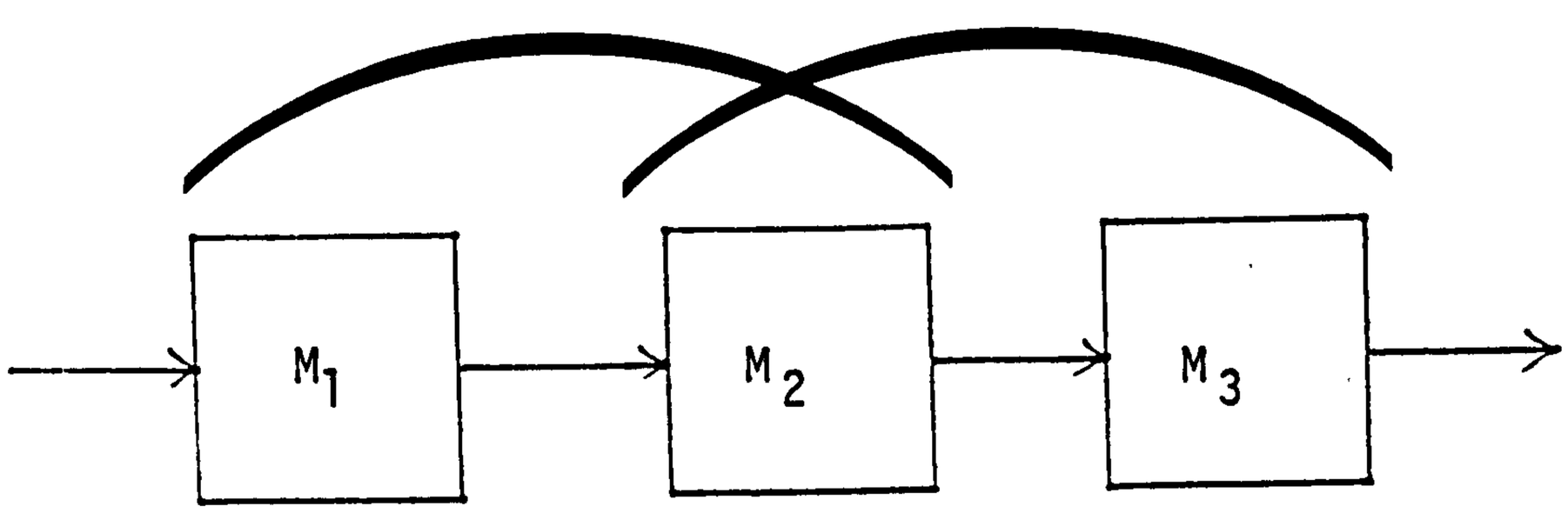


Figure 4.5

We now pose the following question: can we use (4.6) in the case of a $M_1/T_1/M_2/T_2/M_3$ system where we can calculate availabilities A_{12} (corresponding to the $M_1/T_1/M_2$ subsystem) and A_{23} (corresponding to the $M_2/T_2/M_3$ subsystem) from our knowledge of one-tank systems? Notice that the way we propose to block a two-tank Type I system in Fig. 4.6 is similar to the blocking of a 3-machine series system in Fig. 4.5.

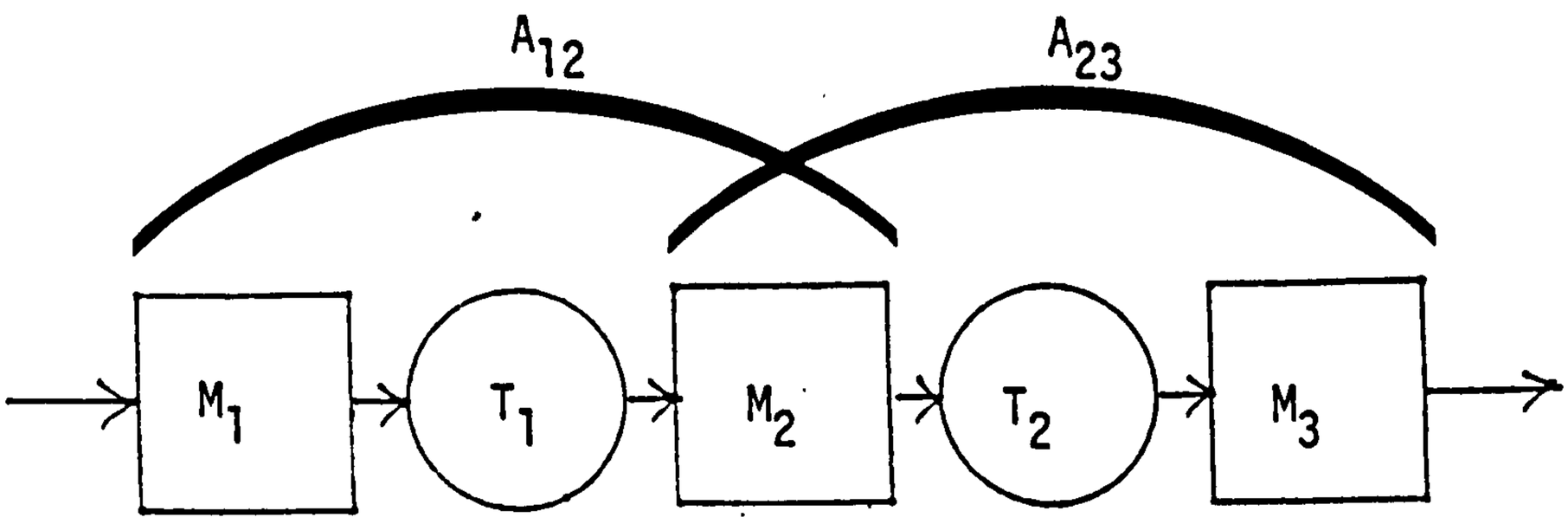


Figure 4.6

In order to answer this question we have to examine the assumptions under which (2.2) is valid. There are two such assumptions. Firstly, the failure and repair distributions associated with each

"block" must be independent and possess finite means (i.e. the MTTF and MTTR should be finite). Secondly, subsystems do not fail on 'standby', waiting for a failed subsystem to be repaired.

It appears that neither assumption is justified. Although it is difficult to assess the extent of the effect of tankage on the failure processes of the machines, it is clear that simultaneous breakdowns may indeed occur.

Let us then define a different availability A_k' for a M_k/T_k system based on a different definition of failure: "Failure of a M_k/T_k system is the event whereby M_k has failed and T_k is empty". Then (M_k/T_k) in series with M_{k+1} cannot 'fail' simultaneously as M_{k+1} is put on standby, when M_k/T_k is 'failed'. We can calculate A' for $M_1/T_1/M_2$ system.

$$A_{12} = \frac{A_1'A_2}{A_1'+A_2-A_1'A_2} \qquad \text{from (4.2)}$$

$$= \frac{\rho_1-\rho_2 e^{-k_1}}{\left(\frac{\rho_1}{A_1} - \frac{\rho_2}{A_2} e^{-k_1}\right)} \qquad \text{from (3.13)}$$

Solving for A_1' we obtain

$$A_1' = \frac{\rho_1-\rho_2 e^{-k_1}}{\rho_1-\rho_2 e^{-k_1}+\rho_1(\rho_1-\rho_2)} \quad \ldots\ldots (4.7)$$

or better

$$A_1' = (1+\rho_1')^{-1} \; ; \; \rho_1' = \frac{\rho_1(\rho_1-\rho_2)}{(\rho_1-\rho_2 e^{-k_1})} \quad \ldots\ldots (4.8)$$

We can use ρ_1' in (4.2) to obtain

$$A_{12} = (1+\rho_1'+\rho_2)^{-1} = (1+\rho_{12})^{-1} \quad \ldots\ldots (4.9)$$

where

$$\rho_{12} = \left(\frac{\rho_1^2 - \rho_2^2 e^{-k_1}}{\rho_1 - \rho_2 e^{-k_1}}\right) \quad \ldots\ldots \quad (4.10)$$

So, we consider the subsystem M_1/T_1 in series with M_2 with respective availabilities A_1' and A_2 to obtain (3.13).

Similarly, we can work from the other end. Define "failure" of the T_k/M_{k+1} system to be the event: M_{k+1} failed and T_k full. We have a corresponding availability A_2' for a $M_1/T_1/M_2$ system calculated in a similar way to A_1':

$$A_2' = \frac{1}{(1+\rho_2')} \quad ; \quad \rho_2' = \frac{\rho_2 e^{-k_1}(\rho_1 - \rho_2)}{(\rho_1 - \rho_2 e^{-k_1})} \quad \ldots\ldots \quad (4.11)$$

and

$$A_{12} = (\rho_1 + \rho_2' + 1)^{-1} \quad \ldots\ldots \quad (4.12)$$

From (4.9) and (4.12) we see that

$$\rho_{12} = \rho_1 + \rho_2' = \rho_1' + \rho_2 \quad \ldots\ldots\ldots \quad (4.13)$$

Consider now the $M_1/T_1/M_2/T_2/M_3$ system and use ρ_1', ρ_2 and ρ_3' as shown in Fig. 4.7

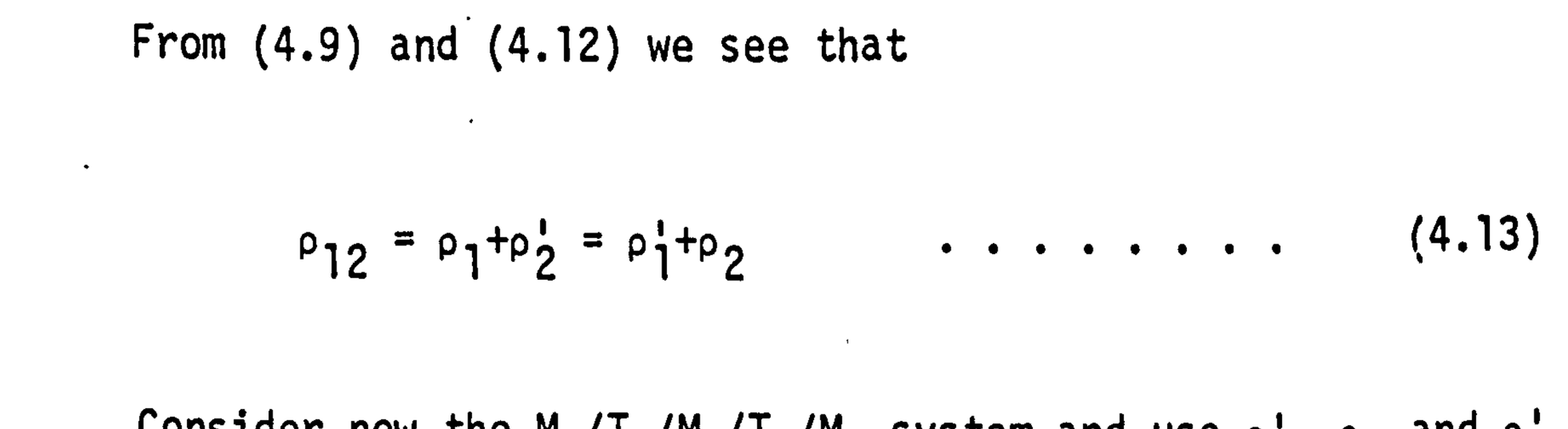


Figure 4.7

where ρ_1' , ρ_3' correspond to the M_1/T_1 and T_2/M_3 subsystems respectively. Then, adding and subtracting ρ_2:

$$A_{123} = (1+\rho_1'+\rho_2+\rho_3')^{-1} = (1+\rho_1'+\rho_3'+\rho_2+\rho_2-\rho_2)^{-1}$$

Using (4.13) we get:

$$A_{123} = (\rho_{12}+\rho_{23}-\rho_2+1)^{-1} =$$

$$= (1 + \frac{1-A_{12}}{A_{12}} + \frac{1-A_{23}}{A_{23}} - \frac{1-A_2}{A_2})^{-1} =$$

$$= \frac{A_{12}A_2A_{23}}{A_2A_{23}+A_2A_{12}-A_{12}A_{23}}$$

which is Equation (4.6) again!

It is easy to assume at this point that, via our new definitions of failure, we have arrived at a 'correct' - as distinct from 'approximate' - expression for availability.

However this is not the case.

The inherent assumption throughout is that the availability of the $M_k/T_k, T_k/M_{k+1}$ subsystems used in the blocking is a function only of the availability of the machine and the volume of the tank included in the subsystem and the availability of the immediately following (preceding) machine. We are therefore assuming "uncoupled" subsystems, namely that in Fig. 4.7.

$$\rho_1' = \rho_1' (A_1, A_2, V_1) \text{ and not } \rho_1' (A_1, A_2, A_3, A_4, V_1, V_2)$$

$$\rho_3' = \rho_3' (A_3, A_4, V_2) \text{ and not } \rho_3' (A_1, A_2, A_3, A_4, V_1\ V_2)$$

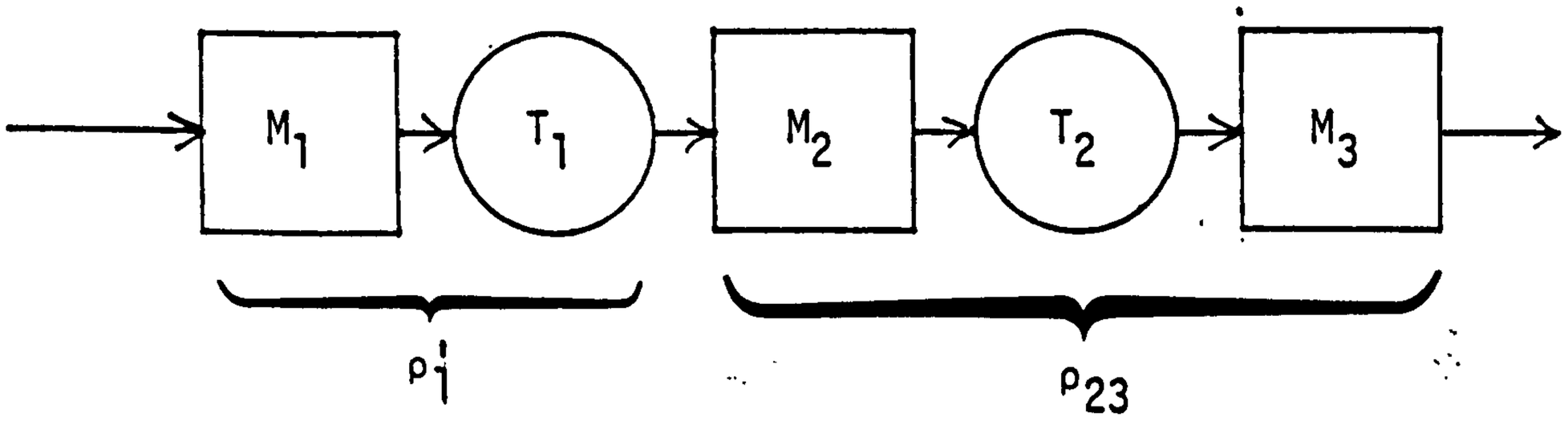
This simple assumption, however, provides us with an easy approximation which satisfies the conditions described in (4.1). In the next section we discuss its properties.

4.2.3. Properties of the approximation

(a) Order of blocking

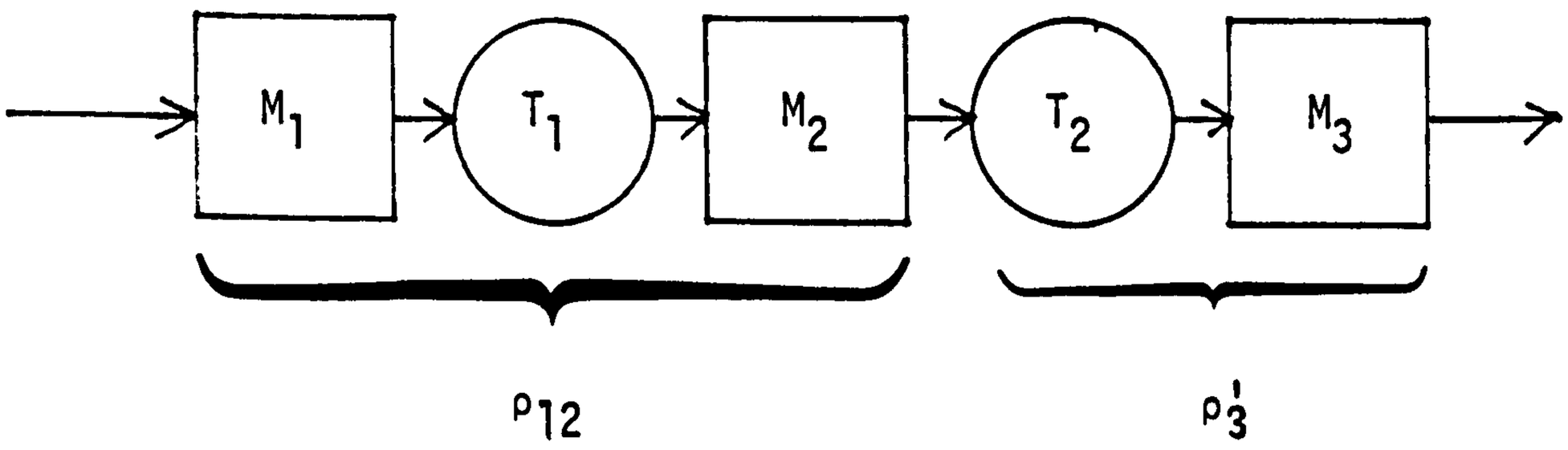
The following blockings are equivalent (using 4.13)

(i)



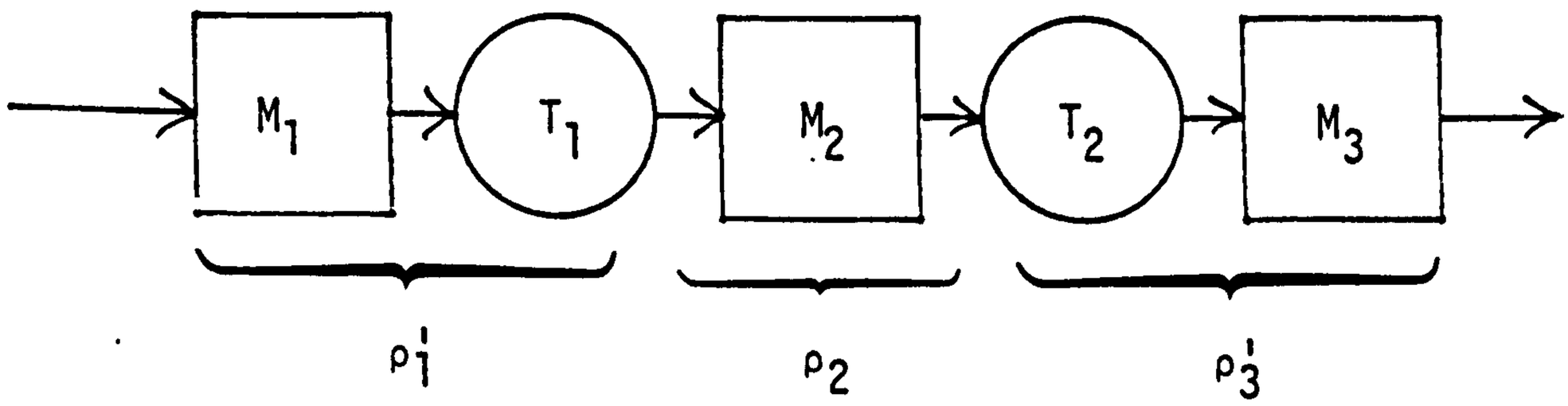
$$A_{123} = (1+\rho'_1+\rho_{23})^{-1}$$

(ii)



$$A_{123} = (1+\rho_{12}+\rho'_3)^{-1}$$

(iii)



$$A_{123} = (1+\rho'_1+\rho_2+\rho'_3)^{-1}$$

Figure 4.8

Thus, the blocking order which trouble Cherry et al [15] is irrelevant.

(b) Asymptotic behaviour

(i) Volumes go to zero

let $V_i \to 0$; $k_i \to 0$

Then:

$$\lim_{V_i \to 0} A_{123} = \lim_{V_i \to 0} = \frac{A_{12}A_2A_{23}}{A_{12}A_2+A_2A_{23}-A_{12}A_{23}}.$$

$$= \lim_{V_i \to 0} \frac{1}{\frac{1}{A_{23}} + \frac{1}{A_{12}} - \frac{1}{A_2}} = \frac{1}{\lim_{V_2 \to 0} \frac{1}{A_{23}} + \lim_{V_1 \to 0} \frac{1}{A_{12}} - \frac{1}{A_2}}$$

Since

$$\lim_{V_1 \to 0} A_{12} = \frac{1}{1+\rho_1+\rho_2} \quad \text{and} \quad \lim_{V_2 \to 0} A_{23} = \frac{1}{1+\rho_2+\rho_3}$$

Then

$$\lim_{V_i \to 0} A_{123} = \frac{1}{1+\rho_1+\rho_2+1+\rho_2+\rho_3-1-\rho_2} = \frac{1}{1+\sum_{i=1}^{3} \rho_i} \quad \ldots \quad (4.14)$$

(4.14) is the availability of a series system composed of three machines without tanks.

If k individual volumes (but not all) go to zero, the formula will correspond to the blocking according to (2.2) of K Type I systems. For instance, let $V_1 \to 0$, $V_2 \neq 0$ in Fig. 4.7.

Then:

$$\lim_{V_1 \to 0} A_{123} = \frac{1}{\frac{1}{A_{23}} + \lim_{V_1 \to 0} \frac{1}{A_{12}} - \frac{1}{A_2}} =$$

$$= \frac{1}{1+\rho_{23}+1+\rho_1+\rho_2-\rho_2-1} = \frac{1}{1+\rho_1+\rho_{23}} \quad \ldots . \quad (4.15)$$

(4.15) is indeed the expression for the availability of the system $M_1/M_2/T_2/M_3$ using (2.2), as required.

(ii) Volumes go to infinity

Consider the cases:

case 1 $A_1>A_2>A_3$; $A_{12}\to A_2$; $A_{23}\to A_3$

Then

$$\lim_{V_i\to\infty} \frac{A_{12}A_2A_{23}}{A_{12}A_2+A_2A_{23}-A_{12}A_{23}} \to \frac{A_2A_2A_3}{A_2^2+A_2A_3-A_2A_3} \to A_3$$

case 2 $A_1>A_3>A_2$; $A_{12}\to A_2$, $A_{23}\to A_2$

Then

$$\lim_{V_i\to\infty} A_{123} \to \frac{A_2A_2A_2}{A_2^2+A_2^2-A_2^2} \to A_2$$

case 3 $A_3>A_1>A_2$; $A_{12}\to A_2$, $A_{23}\to A_2$

Then

$$\lim_{V_i\to\infty} A_{123} \to \frac{A_2A_2A_2}{A_2^2+A_2^2-A_2^2} \to A_2$$

case 4 $A_3>A_2>A_1$; $A_{23}\to A_2$, $A_{12}\to A_1$

$$\lim_{V_i\to\infty} A_{123} \to \frac{A_1A_2^2}{A_1A_2+A_2^2-A_1A_2} \to A_1$$

In cases 1-4 $\lim_{V_i\to\infty} A_{123} = \min_i \{A_i\}$ and the approximation works. This is not always true in the next two cases where $A_2 = \max_i \{A_i\}$.

There are still two cases to consider:

case 5 $A_2>A_1>A_3$; $A_{12}\to A_1$, $A_{23}\to A_3$

Then

$$\lim_{V_i\to\infty} A_{123} = \frac{1}{1/A_3+1/A_1-1/A_2}$$

Now if $A_3 << A_1, A_2$ or $A_1 \tilde{\approx} A_2$

$$\lim_{V_i \to \infty} A_{123} \leq A_3 = \min_i \{A_i\}$$

Similarly for

<u>case 6</u> $A_2 > A_3 > A_1$; $A_{12} \to A_1$, $A_{23} \to A_3$

$$\lim_{V_i \to \infty} = \frac{1}{\frac{1}{A_3} + \frac{1}{A_1} - \frac{1}{A_2}}$$

Now if $A_1 << A_3$, A_2 or $A_2 \tilde{\approx} A_3$

$$\lim_{V_i \to \infty} A_{123} = A_1 \leq \min_i \{A_i\}.$$

Thus (4.1) is satisfied in all cases. Compliance with (4.1) is very important since it ensures that (4.6) will always give values in the right range and behave asymptotically correctly, except in cases 5 and 6 above.

In those cases, where $A_2 = \max_i \{A_i\}$ we do not always get convergence to $\min_i\{A_i\}$. One can expect therefore the approximation to be unreliable in such cases. In section 4.4 we shall discuss this further.

(c) <u>Ease of use</u>

The fact that the blocking order of the tanks is irrelevant makes (4.6) particularly convenient for use in optimization. The recursion (4.2) can be adopted along with (4.6) to obtain a final recursive formula for k tanks. For instance, a 3-tank system would have an availability according to (4.2) of:

$$A_{1234} = \frac{A_{123}A_4'}{A_{123}+A_4'-A_{123}A_4'} \qquad \ldots\ldots \qquad (4.16)$$

A_4' is the availability of the final T_3/M_4 system as shown in Fig. (4.9).

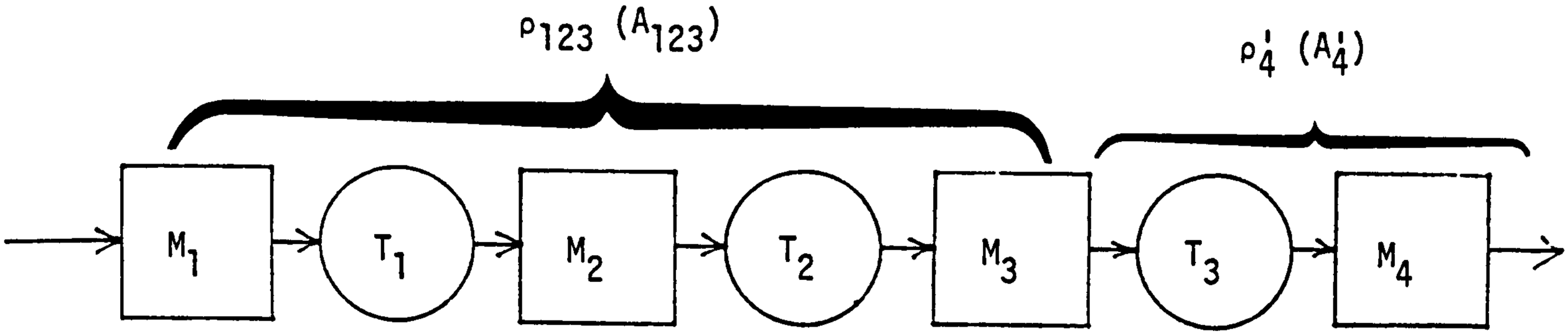


Figure 4.9

Different blockings involving smaller subsystems are also possible in line with Fig. (4.8). Now, if we substitute for A'_4 using A_3, A_4 and A_{34} we obtain

$$A_{1234} = \frac{A_{123}A_3A_{34}}{A_{123}A_3+A_3A_{34}-A_{123}A_{34}} \quad \ldots\ldots\ldots \quad (4.17)$$

The general recursion can be easily proved by induction. In general then for k tanks in a Type I system:

$$A_{s,k+1} = \frac{A_{s,k}A_kA_{k,k+1}}{A_{s,k}A_k+A_kA_{k,k+1}-A_{s,k}A_{k,k+1}} \quad \ldots\ldots\ldots\ldots \quad (4.18)$$

The notation in (4.18) is similar to that for (4.2); $A_{k,k+1}$ is the availability of the final $M_k/T_k/M_{k+1}$ subsystem.

(4.18) will from now on be used in this thesis as an approximation for the availability of a series Type I system. In section 4.4 we justify our adoption of (4.18) by comparing it with results obtained using simulation. Before we do this, however, let us consider suitable approximations for parallel tank systems.

4.3 Parallel Systems with Buffer Storage

4.3.1 Types of Systems

(a) Type I $nM/T/M_{n+1}$ system (see Fig. 4.10)

In this system we have one tank with n machines in parallel redun-

dancy upstream and one machine downstream.

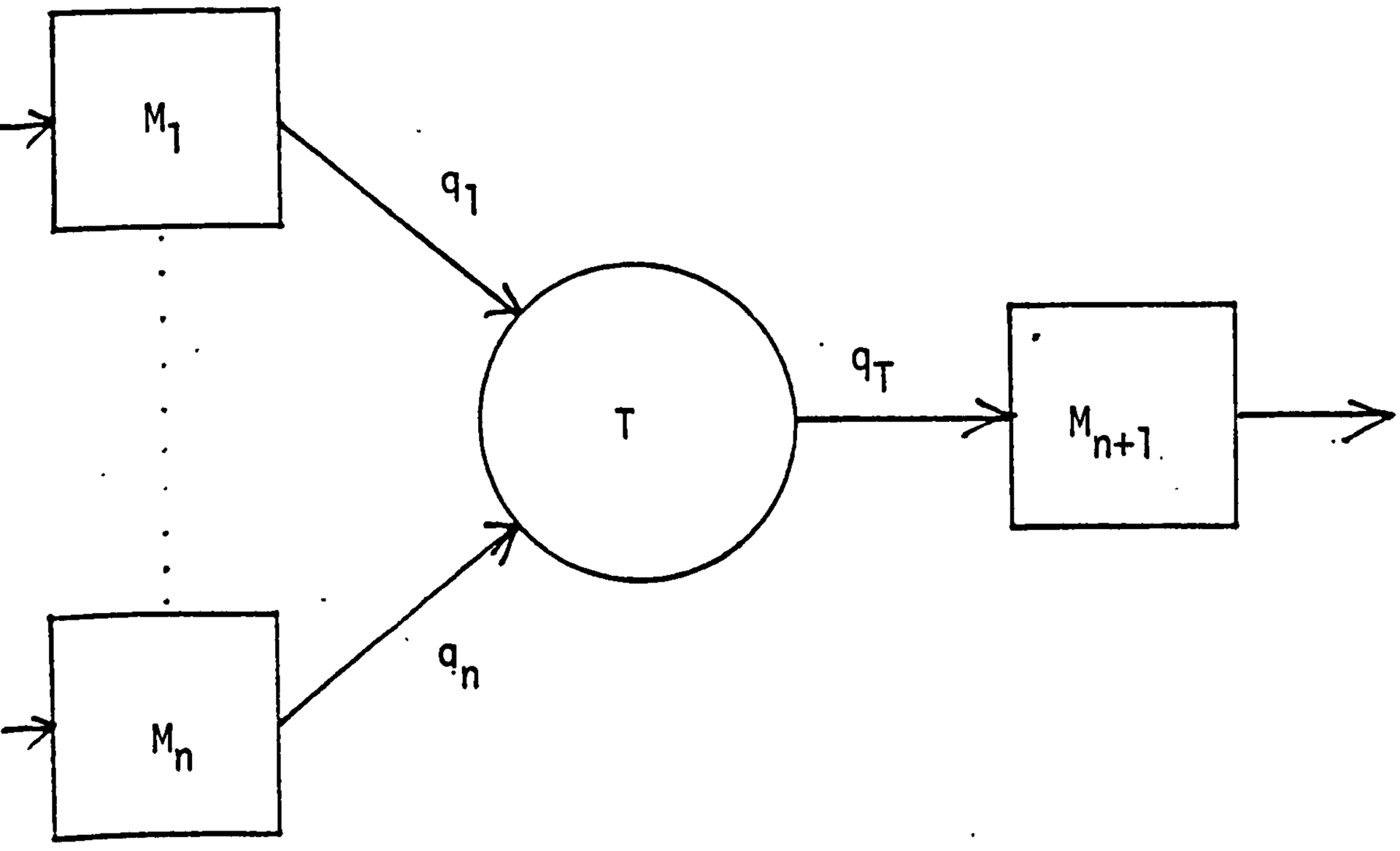


Figure 4.10

(b) Type II $\frac{nM/T/mM \text{ system}}{\text{(see Fig. 4.11)}}$

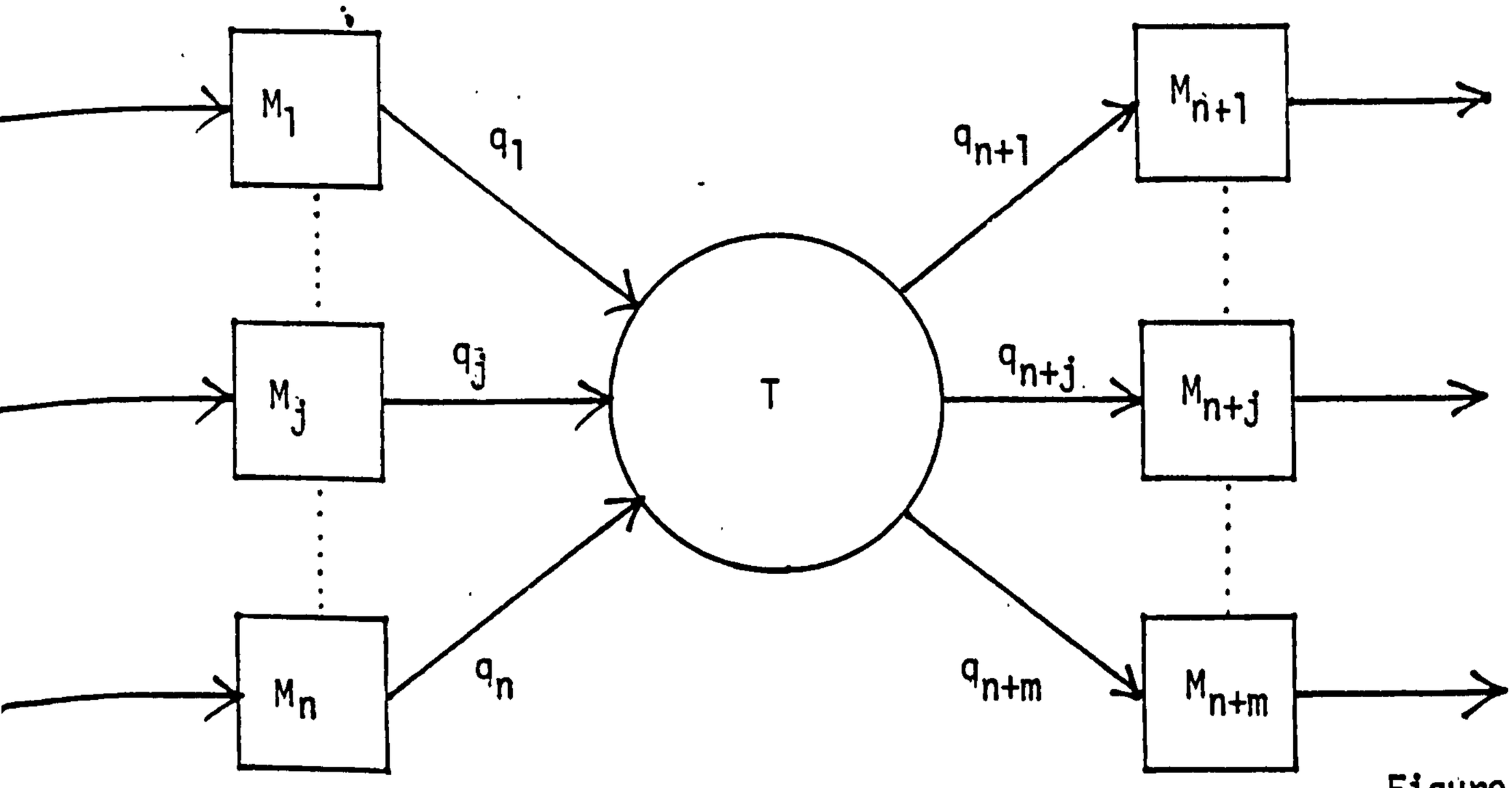


Figure 4.11

In this system we have one tank, n machines in active redundancy upstream and m machines in active redundancy downstream.

A Type II system can be considered as an m-fold combination of Type I systems. The total production downstream will be

$$\sum_{i=n+1}^{n+m} q_i A_i$$

where A_i = availability of each Type I system.

(c) Type III n(M/T/M) system (or higher)
(see Fig. 4.12)

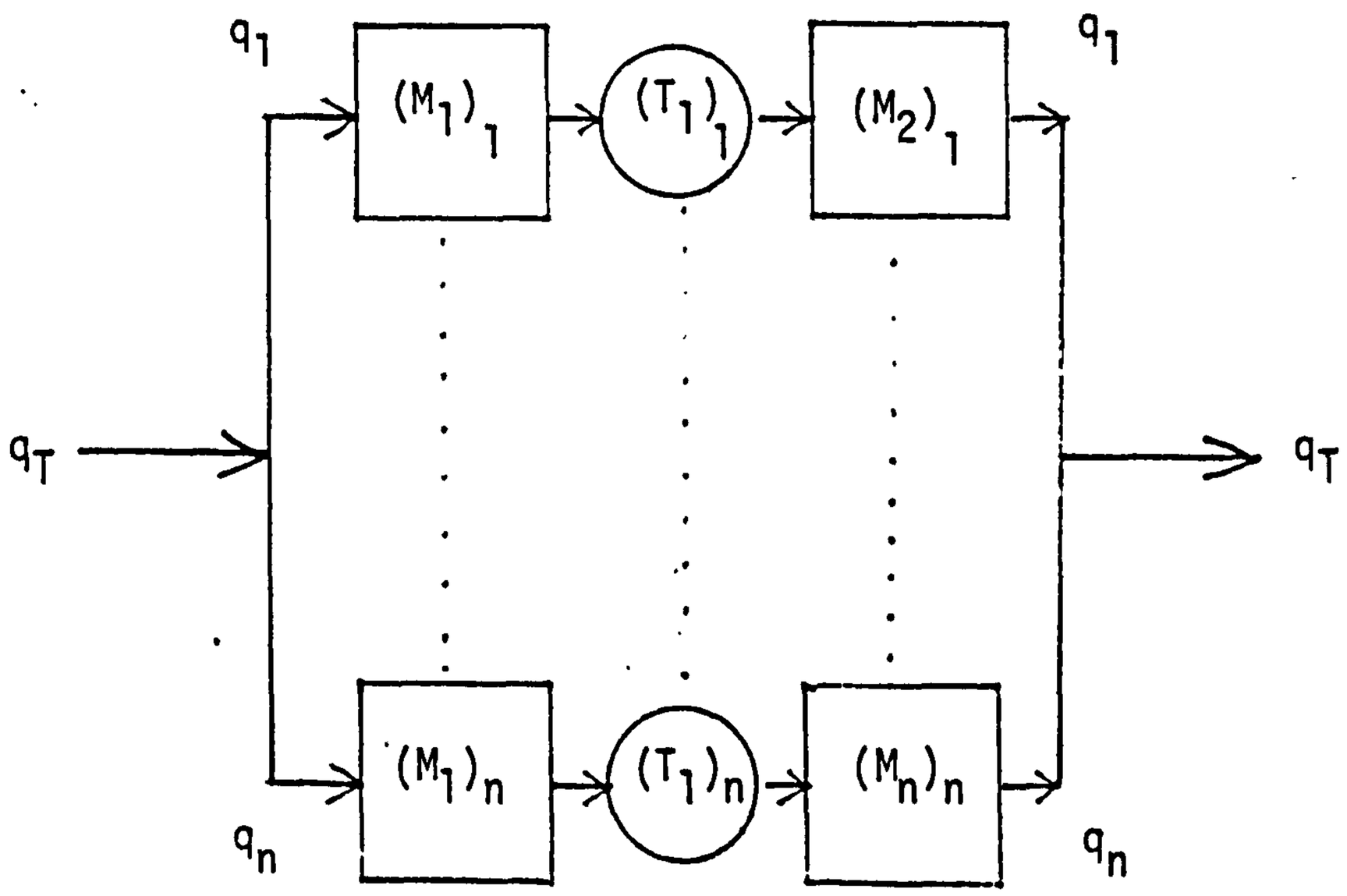


Figure 4.12

Here we have n(M/T/M) or higher Type I series systems in active redundancy.

All three configurations for parallel systems can be repeated by substituting standby redundancy in place of active. We shall use the subscript p to denote a system with such parallel redundancy in machines or tanks and s to denote standby redundancy. In this section we shall derive expressions for the availability of systems with active redundancy, but the method can be used to derive similar expressions

for standby redundancy. Note that all systems we consider are balanced, i.e.

$$q_T = \sum_{i=1}^{n} q_i \; (= \sum_{j=n+1}^{n+m} q_j \text{ , where appropriate})$$

As in series redundancy, we need only examine Type I parallel systems. A Type II system is M Type I systems and a Type III system can be dealt with using the techniques described in chapter two. The final availability of a Type III system is

$$A_{pT}(V) = 1 - \prod_{i=1}^{M} (1-A_i) \quad . \; . \; . \; . \; . \; . \qquad (4.19)$$

according to (2.4). A_i is the separate availability of each Type I series system in parallel. Thus we need only examine Type I parallel systems.

4.3.2 Approximation for Type I parallel systems

The requirements for an approximate formula for a Type I parallel system have been outlined in (4.1). The appropriate bounds for availability are now:

$$A_{pT}(V=0) \leq A_{pT}(V) \leq A_{n+1} \qquad . \; . \; . \; . \; . \; . \qquad (4.20)$$

where $A_{pT}(V=0)$ is the availability of the system with no tank, i.e. that of n machines in parallel, all in series with machine n+1. Contrast this with $A_p(0)$ which is the availability of the n machines in parallel only and $A_{pT}(0)$ the final system availability:

$$A_{pT}(0) = \left(1 + \frac{1-A_p(0)}{A_p(0)} + \frac{1-A_{n+1}}{A_{n+1}}\right)^{-1}$$

We begin by noting that the $nM/T/M'_{n+1}$ system can be approximated by n M/T/M systems as depicted in Fig. 4.13.

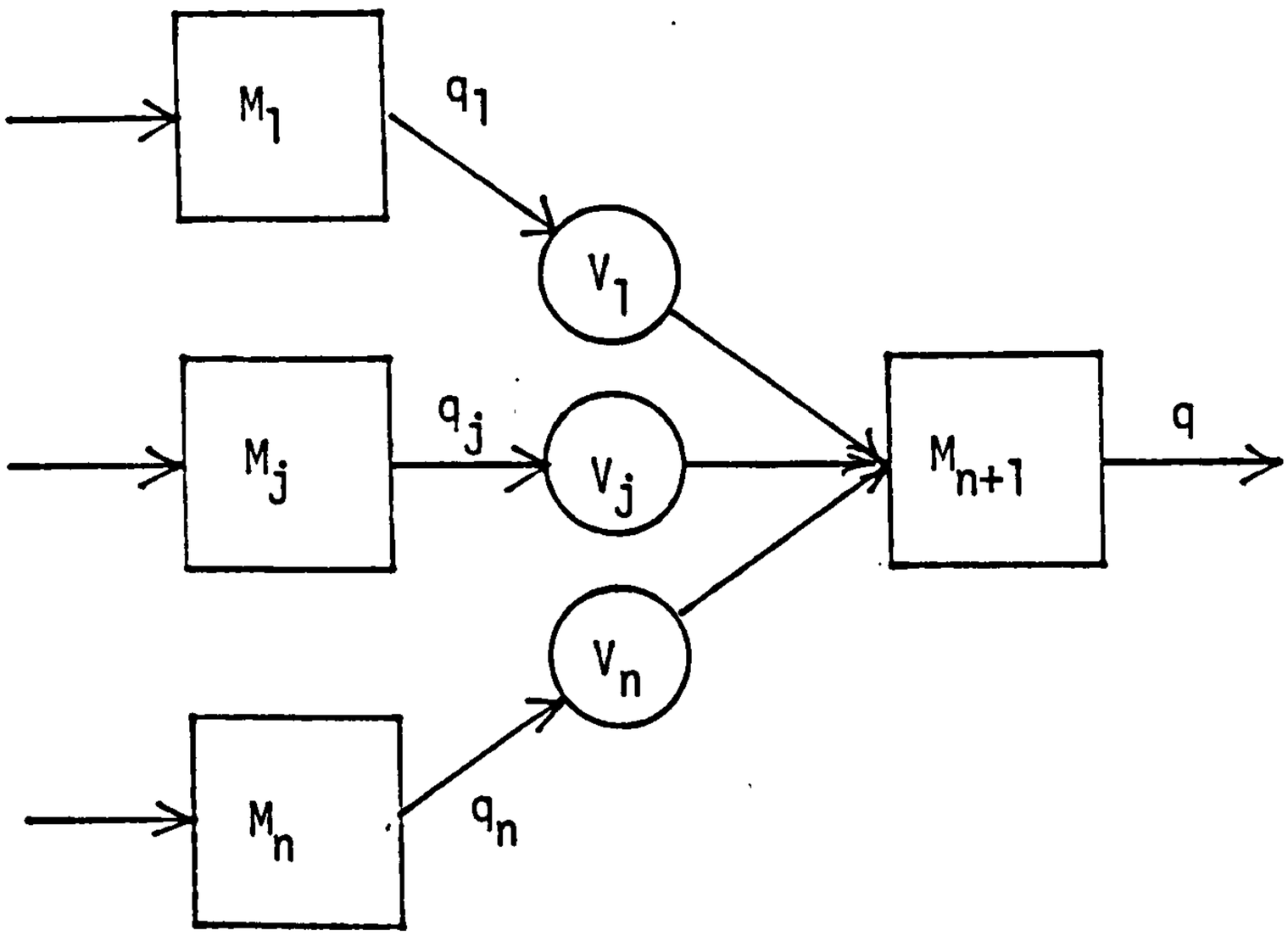


Figure 4.13

Each of the n tanks has a volume V_j equal to q_jV/q. (Normally $q_j=q/n$ since most machines in parallel have equal capacities). The approximation is justified by the behaviour of the system in practice. If, say, M_{n+1} is down for a period T, then all n upstream machines fill the tank and contribute q_jT units of volume each. If the tank is initially empty and $T>V/q$, then each machine would have filled a volume fraction q_j/q. Furthermore, if any one upstream machine j is down for a period T_j, the net flow from the tank is q_j.

The model breaks down in the following case. Suppose machines M_{n+1} and M_j are down for a combined period $\bar{T}_j$ and let $\tau_j = \sum_{i\neq j}^{n} \frac{V_i}{q_i}(1-y_i)$ where $0\leq y_i\leq 1$ is the fraction of the volume of the tank which is full. Then if $\bar{T}_j<\tau_j$ the system behaves as the approximate model. If, however, $\bar{T}_j>\tau_j$ then they exhibit different behaviour. In the real system the tank would be filled further by $(1-y_j)V_j$, even after the period τ_j had

elapsed. In the approximate model, tank j is separated from the other n-1 tanks and its empty space $(1-y_j)V_j$ cannot be filled after this period.

However, for such a state to occur we must have two machines down (M_{n+1} and M_j) and a large repair period $\bar{T}_j > \tau_j$. This probability of this state is

$$\sim A_j A_{n+1} \ (1-\exp(-\mu_j \bar{T}_j)) \quad . . . . . \quad (4.21)$$

which should be small.

There are more such deviations corresponding to failures of machine M_{n+1} and more than one M_j, and large repair times. The error should be larger, the smaller the volumes V_j are. (Then the probability that $\bar{T}_j > \tau_j$ becomes larger.) If, however, our approximation exhibits behaviour according to (4.1), namely as $V \to 0$ $A_p(V) \to A_p(0)$, the contribution to the final availability becomes smaller. In short, if (4.1) is valid, the more common the deviations of our model from the real system become, (as $V \to 0$), the less they contribute to the final availability (as $A_{pT}(V) \to A_{pT}(0)$).

Let us find an expression with such properties. By (4.19) or (2.4):

$$A_p(0) = 1 - \prod_{i=1}^{n} (1-A_i) = 1 - \prod_{i=1}^{n} \left(\frac{\rho_i}{1+\rho_i}\right) \quad . . . . \quad (4.22a)$$

and $$\rho_p(0) = \frac{1-A_p(0)}{A_p(0)} \quad . . . . \quad (4.22b)$$

Thus using (2.2):

$$Ap_T(0) = (1+\rho_p(0)+\rho_{n+1})^{-1} \quad . . . . . \quad (4.23)$$

let us use the same expressions for $V \neq 0$. Consider the availabilities

A_{vi} of the subsystems $M_i/T_i/M_{n+1}$ $i=1,n$ where $V_i=V(q_i/q)$

Then:

$$\rho_{vi} = \frac{1-A_{vi}}{A_{vi}} - \rho_{n+1} \quad (4.24a). \quad \text{(Given also by 4.8)}$$

$$A_p(v) = 1 - \prod_{i=1}^{n} \left(\frac{\rho_{vi}}{1+\rho_{vi}}\right) \qquad (4.24b)$$

$$\text{and } \rho_p(v) = \frac{1-A_p(v)}{A_p(v)} \qquad (4.24c)$$

Thus,

$$Ap_T(V) = (1+\rho_p(v)+\rho_{n+1})^{-1} \qquad \ldots\ldots \qquad (4.25)$$

which is of the same form as (4.23).

We can see immediately that (4.25) behaves asymptotically correctly:

(a) as $V\to 0$, $\frac{\rho_{vi}}{1+\rho_{vi}} \to \frac{\rho_i}{1+\rho_i}$,

$A_p(v)\to A_p(0)$ and $Ap_T(v)\to Ap_T(0)$.

(b) as $V\to\infty$ $A_{vi}\to 1$ and $\rho_{vi}\to 0$

so $Ap_T(\infty) \to \frac{1}{1+\rho_{n+1}} \to A_{n+1}$.

as required.

The same procedure can be followed in the case of standby redundancy. In the same way, cases (ii) and (iii) are not particular to active redundancy and their method of solution can be extended to standby redundancy. In Table 4.1 we present the availability expressions for all systems involving buffer storage.

TABLE 4.1

THE AVAILABILITY OF COMPLEX SYSTEMS INVOLVING BUFFER STORAGE

	TYPE I	TYPE II
SERIES	$A_{s,k+1} = \frac{A_{s,k}A_kA_{k,k+1}}{A_{s,k}A_k+A_kA_{k,k-1}-A_{s,k}A_{k,k+1}}$ Notation as in (4.18) Figure 4.1	$A_s = (1+\sum_{i=1}^{m}\frac{1-(A_{s,k})_i}{(A_{s,k})_i})^{-1}$ for m disjoint Type I systems Figure 4.2
TYPE I	ACTIVE $A_{pT}(V) = (1+\rho_p(V)+\rho_{n+1})^{-1}$ where $\rho_p(V) = \frac{1-A_p(V)}{A_p(v)}$ $A_p(V) = 1 - \prod_{i=1}^{n}(\frac{\rho_{vi}}{1+\rho_{vi}})$ $\rho_{vi} = \frac{1-A_{vi}}{A_{vi}} - \rho_{n+1}$ A_{vi} = availability of $M_i/T_i/M_{n+1}$ with T_i having volume $V_i = V(\frac{q_i}{q})$ Notation as in (4.24) Figure 4.10	STANDBY $A_{sT}(V) = (1+\rho_s(V)+\rho_{n+1})^{-1}$ where $\rho_s(V) = \frac{1-A_s(V)}{A_s(V)}$ $A_s(V) = 1-(\sum_{j=0}^{n}\frac{n}{j!}\frac{1}{\rho_{vj}^{n-j}})^{-1}$ $\rho_{vj} = \frac{1-A_{vj}}{A_{vj}} - \rho_{n+1}$ A_{vj} = availability of $M_i/T_i/M_{n+1}$ with T_i having volume V. Notation as in (4.24) Figure 4.10
TYPE II	$q_{TOT} = \sum_{i=n}^{n+m} q_iA_i$ A_i given by Type I expression Figure 4.11	$q_{TOT} = \sum_{i=n}^{m+n} q_iA_i$ A_i given by Type I expression Figure 4.11
TYPE III	$A_{pT}(V) = 1- \prod_{i=1}^{m}(1-A_{vi})$ A_{vj} given as in Type I Figure 4.12	$A_{sT}(V) = 1-(\sum_{j=0}^{n}\frac{n}{j!}\frac{1}{\rho_{vj}^{n-j}})^{-1}$ $\rho_{vj} = \frac{1-A_{vj}}{A_{vj}}$; A_{vj} as in Type I Figure 4.12

It remains to show how well approximations (4.18) and (4.25) perform compared to experimental results obtained by simulation. This comparison is made in the next section.

4.4 Performance of the Approximations

In this section, we shall examine the two main approximations discussed in this chapter: (4.18) for Type I series systems and (4.25) for Type I parallel systems. Since all other expressions are based on them, we need only assess the performance of these two.

Corresponding to the two systems we want to test, two computer simulation programs were written. In both cases, systems of machines and tanks were generated randomly and their behaviour simulated over a long period of time. The availability obtained by simulation, A_{sim}, was then compared with than given by (4.18) or (4.25).

The random systems in question were generated according to the following rules.

(1) Generate randomly $MTTF_i$ i=1,N for N machines, from a uniform distribution in the interval (200,2000). So, the maximum MTTF of any machine would be <2000 hrs.

(2) Generate availabilities A_i i=1,N uniformly in the interval (0.5, 1.0). Availabilities below 0.5 were not generated, since a machine with MTTR>MTTF is not often encountered in practice.

(3) From (1) and (2) calculate $MTTR_i$ i=1,N.

(4) Generate random numbers for constant k in equation (3.14) in the interval (-1,1).

(5) Generate tank volumes V_i corresponding to k_i for M tanks.

Since we have described the asymptotic behaviour of the approximations in sections 4.2.3(b) and 4.3.2, it is reasonable to try and test their behaviour around the 'middle' of the region of application,

that is around $-1<k<1$. This is the region where the exponential term e^{-k} is $O(1)$. (This is true even for $k<0$, because we then multiply by e^{k} as in (3.20) and then e^{k} is $O(1)$ for $k\approx -1$).

Using the data generated in (1) - (5) we employ simulation to obtain the value of A_{sim}. This is done over 1000 'cycles' (A 'cycle' was defined as a particular $TTF_i + TTR_i$ for a machine i). The figure of 1000 cycles was arrived at by experience, by observing the behaviour of A_{sim}. The number of cycles for parallel systems was set a bit higher at 1200. It is equivalent to $O(10^5)$ hours of operating time (around 10 years).

Using (4.18) or (4.25) according to the system we calculate the theoretical availability A of our approximation and compare it to A_{sim}. Since we want $A_{sim}-A\approx 0$, and because the systems were generated using random numbers, the difference $d=A_{sim}-A$ should be, for a large number of systems, distributed normally with mean zero. We then want to test the statistic

$$|z| = \left|\frac{\bar{x}}{\sigma/\sqrt{n}}\right| \qquad \bar{x} = \text{mean of differences}$$

$$\sigma = \text{standard deviation of differences}$$

to test the null hypothesis that $\bar{x}$ does not differ from zero significantly.

We have tried this test on a number of systems. Firstly we check the blocking formula we have decided to use, namely Equation (2.2). This is done by considering M/M/T/M and M/T/M/M systems.

As we have described before, we block these Type II series systems by considering the availability of the M/T/M subsystem first and then blocking it in series with the lone machine, unlike [15] or [17] who proceed by blocking the M-M subsystems first.

Then, we check approximation (4.18) using the two-tank, three-tank

and four-tank systems and (4.25) using a system with two, three and four machines on one side of the tank. The results are presented in Table 4.2.

In Table 4.2 we are looking for values of |z| higher than 2.58 for a 99% level of significance. We can readily see that the blocking formula employed performs satisfactorily for the 200 systems tested. This is not so with the Type I series and parallel systems. (4.18) performs well for 60 systems but has some bias for others. Thus, we check another quantity of interest: the average relative error of the approximation and the simulated value. We can see that the error is around 3-3.8% for all 120 systems and that it does not vary as much as z. The same is true for the Type I parallel approximation (4.25). For two upstream machines and 1000 cycles, the approximation has a small negative bias. For three and four upstream machines and 1200-1500 cycles it behaves better. Again, a look at the average relative errors (around 3%) highlights the fact that in general (4.25) even with a small bias does not differ much from A_{sim}. Graphs 4.1-4.4 show graphically the transient behaviour of some of the simulated systems discussed above.

We can accept an average relative error of around 3% as good enough for estimating the availability of such complex systems, since the alternative would be to resort to numerical solution of sets of equations such as the ones in Appendix 3. In fact, the errors in the failure and repair data themselves should normally be higher than 3% and therefore be a bigger source of inaccuracy than the approximations employed here.

Having thus modelled buffer storage subsystems to our satisfaction, we can turn our attention to the problem we set out to solve in Chapter

one: how to choose between alternative designs for a process subject to failing equipment. This is the subject of the next chapter.

TABLE 4.2

(A) CHECK ON THE BLOCKING FORMULA EMPLOYED

(i) M_1-M_2/T/M_3 SYSTEM

RUN	$\bar{x}$	σ	z
1	.0117	.02794	2.255
2	.00911	.019728	2.487
3	.00794	.019408	2.20
4	.01192	.02648	2.424

30 TRIALS/RUN

1000 CYCLES.

(ii) M_1/T/M_2-M_3 SYSTEM

RUN	$\bar{x}$	σ	z
1	.01067	.018231	2.551
2	.00899	.022704	1.72
3	.00307	.026539	.5042
4	.00121	.023062	.220

20 TRIALS/RUN

1000 CYCLES.

(B) CHECK ON THE APPROXIMATION (4.18)

TYPE I SERIES SYSTEM M/T/M/T/M

RUN	$\bar{x}$	σ	z	AVG. ERROR
1	.01116	.020314	3.0	3.4%
2	.00423	.0234	0.99	3.5%
3	.01385	.021636	3.5	3.8%
4	.00558	.018725	1.632	2.95%

30 TRIALS/RUN

1000 CYCLES.

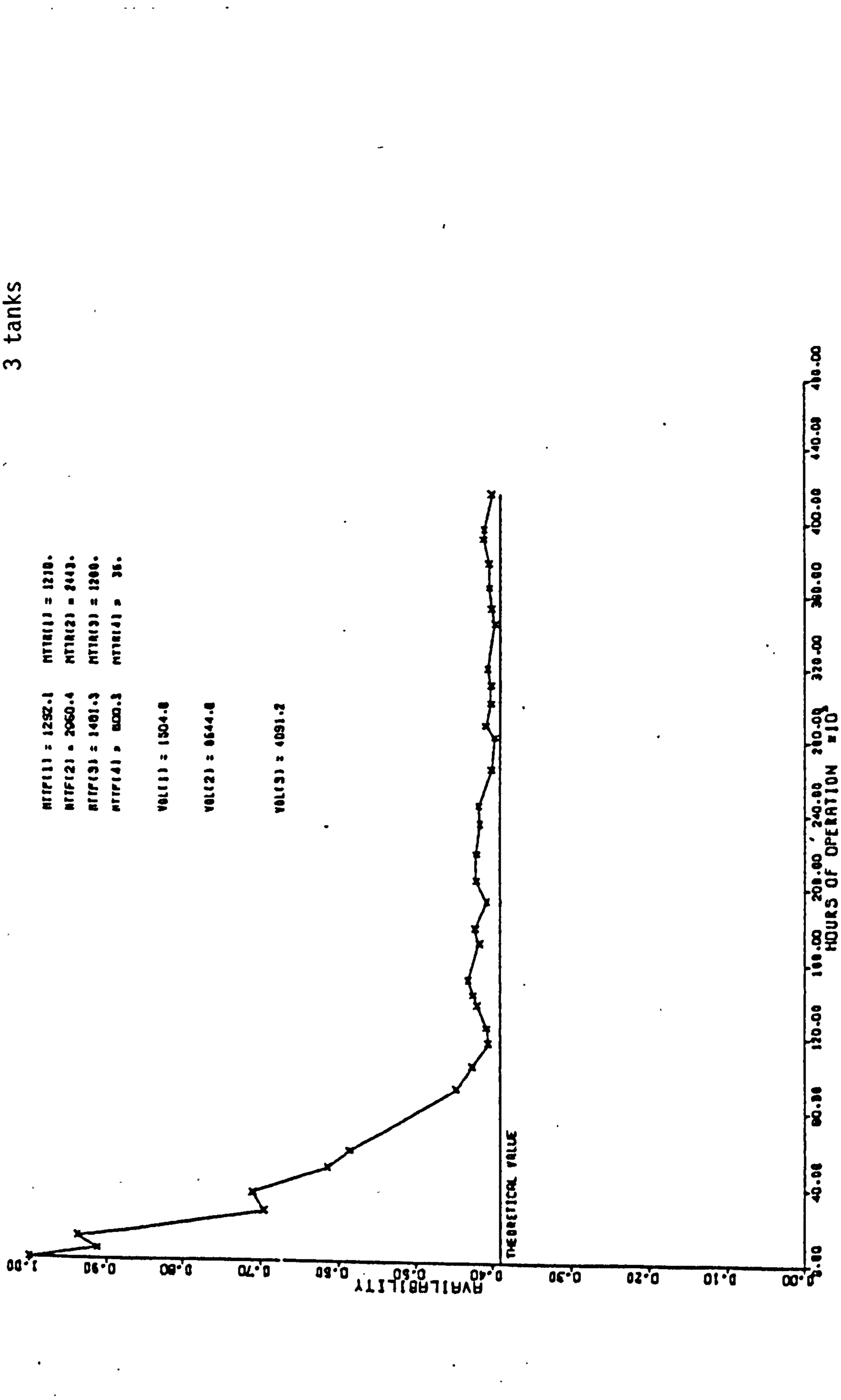
(C) CHECK ON THE APPROXIMATION (4.25)

TYPE I PARALLEL SYSTEM $nM/T/M_{n+1}$

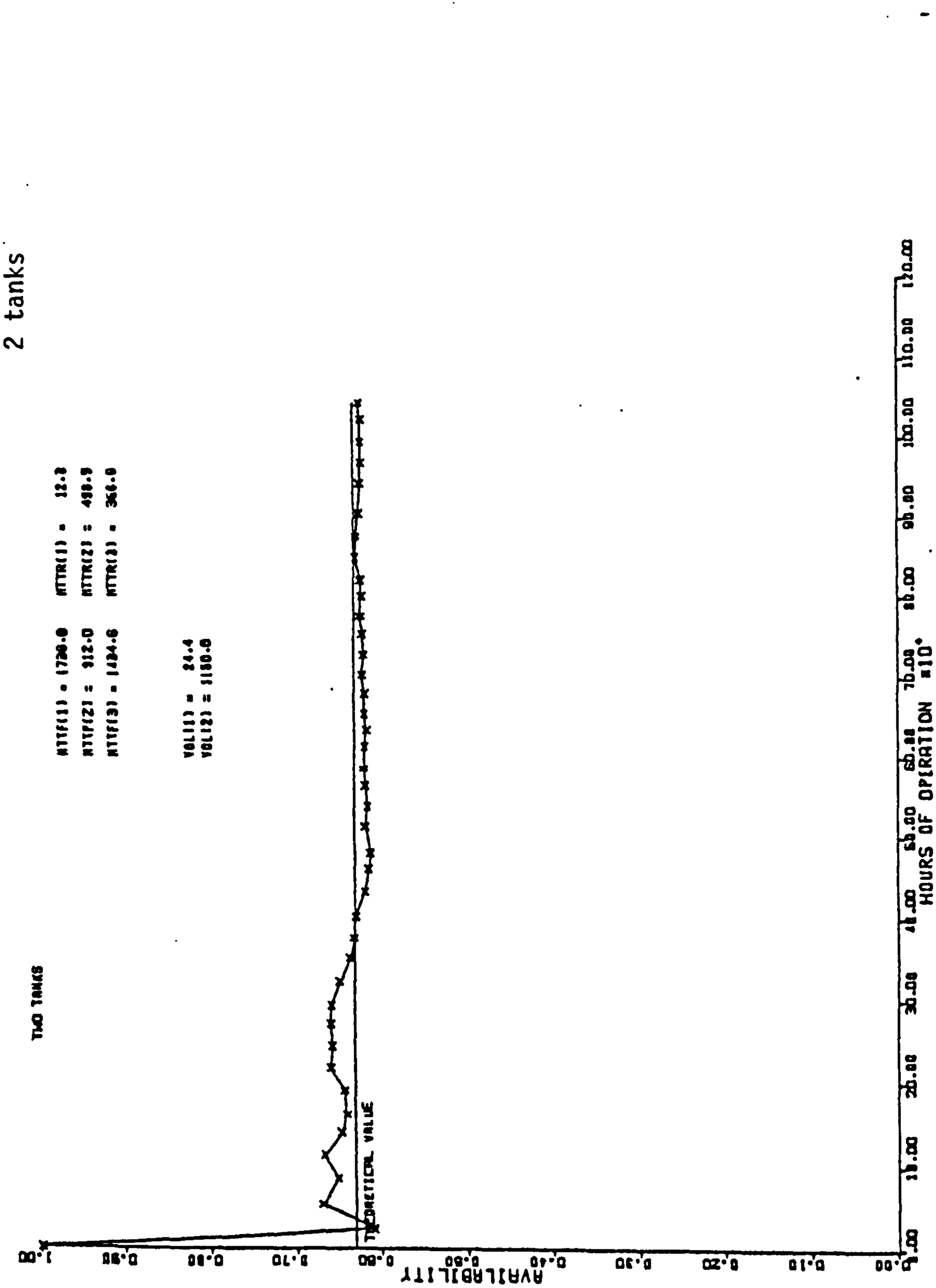
N*	RUN	$\bar{x}$	σ	z	AVG. ERROR	CYCLES
2	1	-.01826	.02514	3.97	3.607%	1000
2	2	-.01235	.01942	3.48	2.69%	1000
3	3	-.00414	.020376	1.113	2.272%	1200
4	4	-.01041	.02257	2.53	2.87%	1500

* NO. OF MACHINES IN PARALLEL UPSTREAM OF TANK.

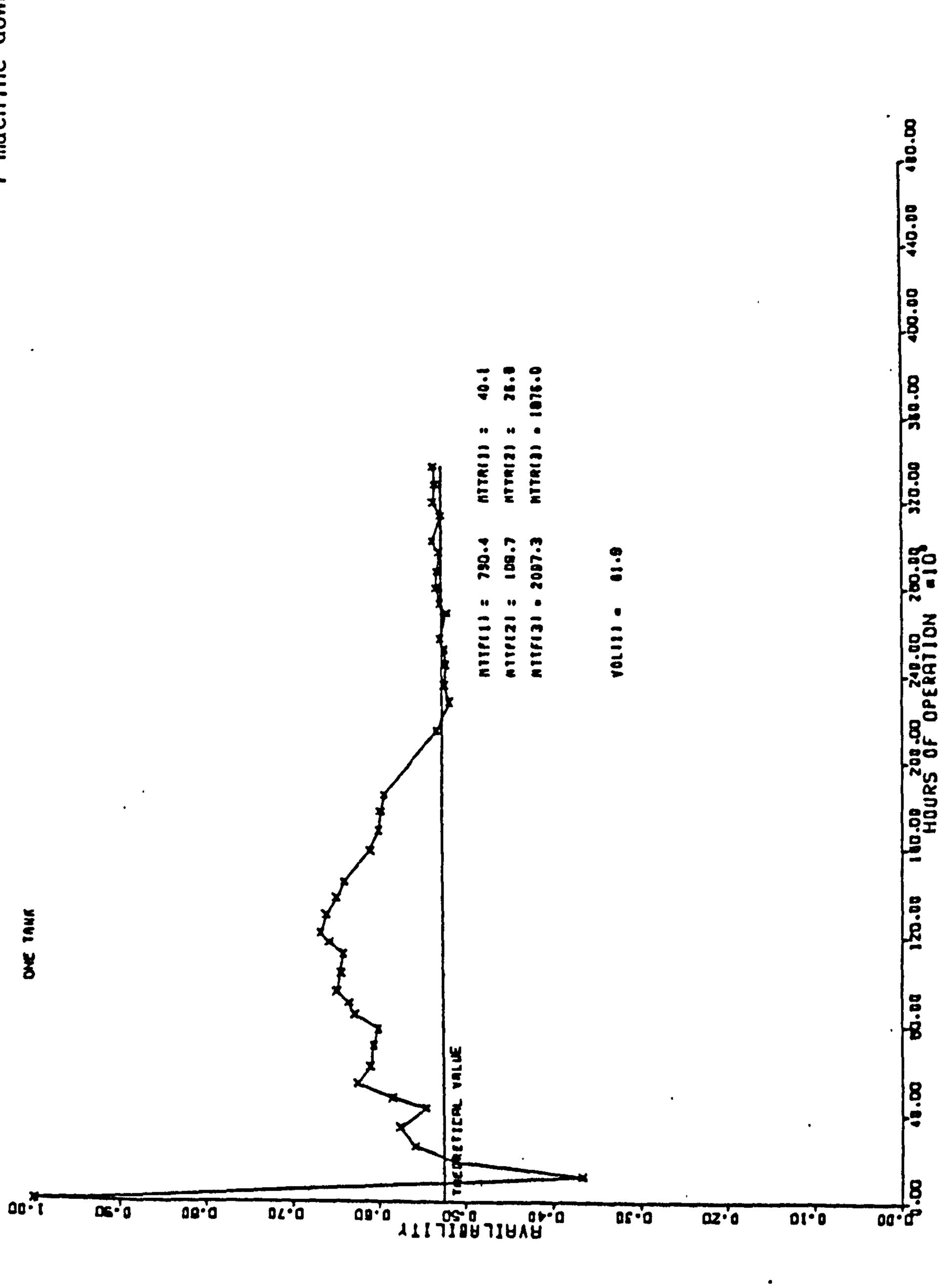
Graph 4.1
(series system)
4 machines
3 tanks



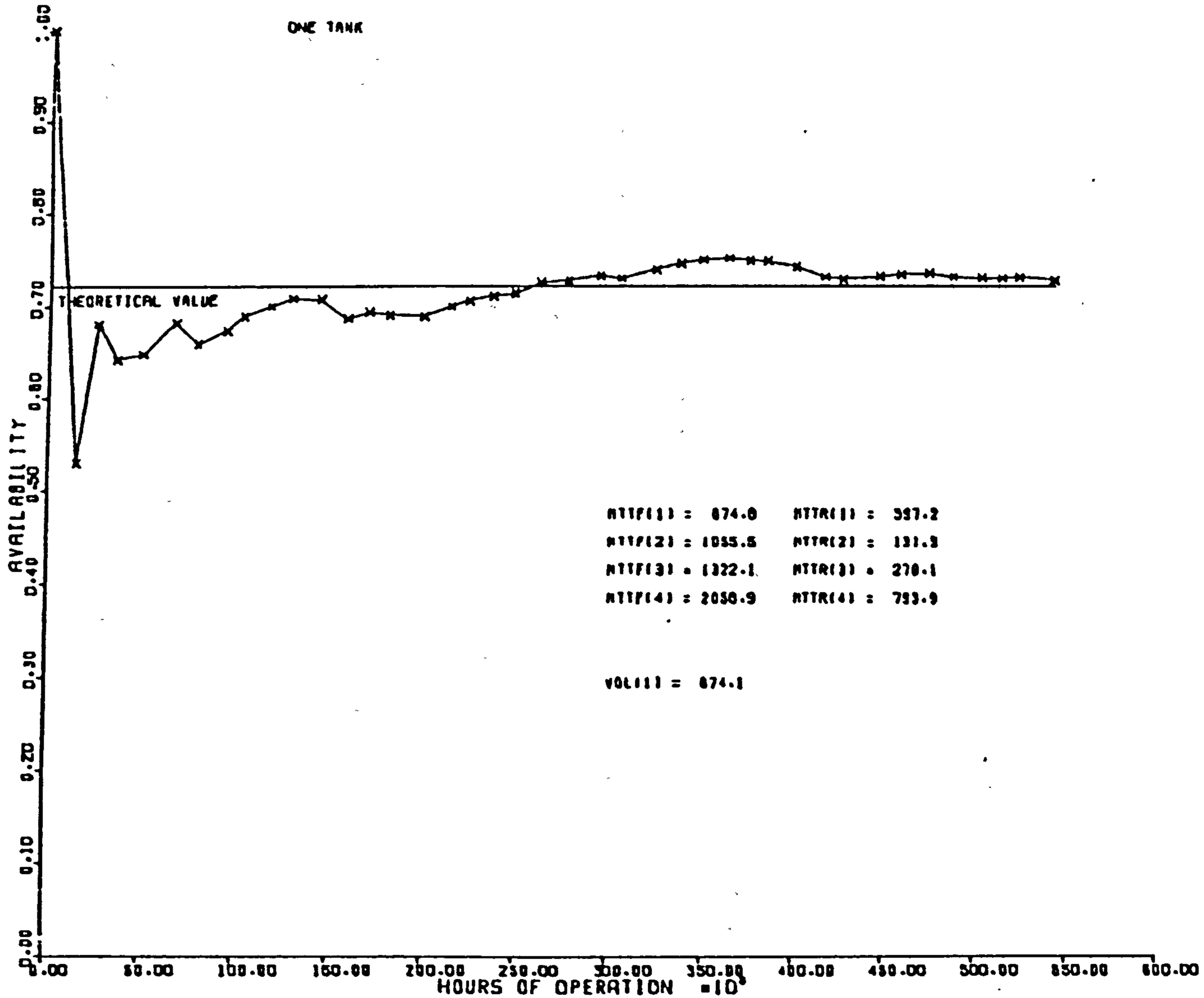
Graph 4.2
(series system)
3 machines
2 tanks



Graph 4.3
(parallel system)
2 machines upstream
1 machine downstream



Graph 4.4
(parallel system)
3 machines upstream
1 machine downstream



C H A P T E R F I V E

OPTIMUM DESIGN USING AVAILABILITY ANALYSIS

5.1 Introduction

In previous chapters we obtained expressions for the availability of subsystems (some including buffer storage) which can be used in conjunction with the blocking formula (2.2) to find the availability of a large series system. In this chapter we shall assume that the overall plant availability A_T, a function of the plant configuration r, can be calculated using the techniques described previously, so that we can work on the design problem we set out to solve in chapter one.

During the design stage, different configurations r_i of a unit in the process may be considered; each results in a plant configuration r, a plant availability $A_T(r)$ and a corresponding cost $C_T(r,A_T)$.

Eventually, we want to choose one configuration r^* which optimizes $C_T(r^*,A_T)$.

The choice of the objective function is determined by the problem we wish to solve. Various problems have been outlined in chapter one and Figure 1.2. In the design problem, to which we shall devote our attention in this thesis, we are interested in reducing the overall capital cost of overdesign. Ideally, the design of a plant should be such that the annual production rate P* is always met. In a fully available process, we design for a rated capacity $P_o = P^*$ and we are certain to achieve our annual production rate. However, no process is fully available. Account of process breakdowns is taken during the design stage by overdesigning equipment, so that during the time

the process is "on", the plant can work at a higher rate in order to meet the target P*. Usually, equipment with a rating $P_o > P^*$ will cost more than one with a rating P*. The question now arises: how do we choose $\Delta P = P_o - P^*$, the amount of overdesign?

The simplest method is to assume a pattern of failures on the plant resulting in a steady state availability A_T. Then, if we design for:

$$P_o = \frac{P^*}{A_T} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.1)$$

we should expect an average output over a year of P*. The use of steady state availability for design purposes can be justified if steady state is achieved in a relatively short period. This is a reasonable assumption for most systems [28]. If the transient period is long, design by availability is not suitable anyway, since reliability then becomes the important parameter. We can see from (5.1) that by increasing A_T, one reduces overdesign ΔP and consequently capital cost. As cost increases with increasing A_T, we have an optimization problem that can be stated as follows:

> "Given ϕ configurations r_i, i=1, ... ϕ for each of N elements in a series system, find the overall configuration r* so that the capital cost $C_T^* (r^*, A_T (r^*))$ is minimised".

Note that if some of the options considered are hybrid redundancy options ξ should be employed instead of A.

The form of the cost functions C_T we shall use in this thesis is the well-known exponential form [30]

$$\text{cost} = K' (P_o)^{\beta} \qquad \ldots\ldots\ldots\ldots \quad (5.2)$$

where

K', β are constants and $\beta \simeq 0.7$

let us use an example to illustrate the optimization problem: suppose we want to install active redundancy for a unit. Then, we can perform the following calculations:

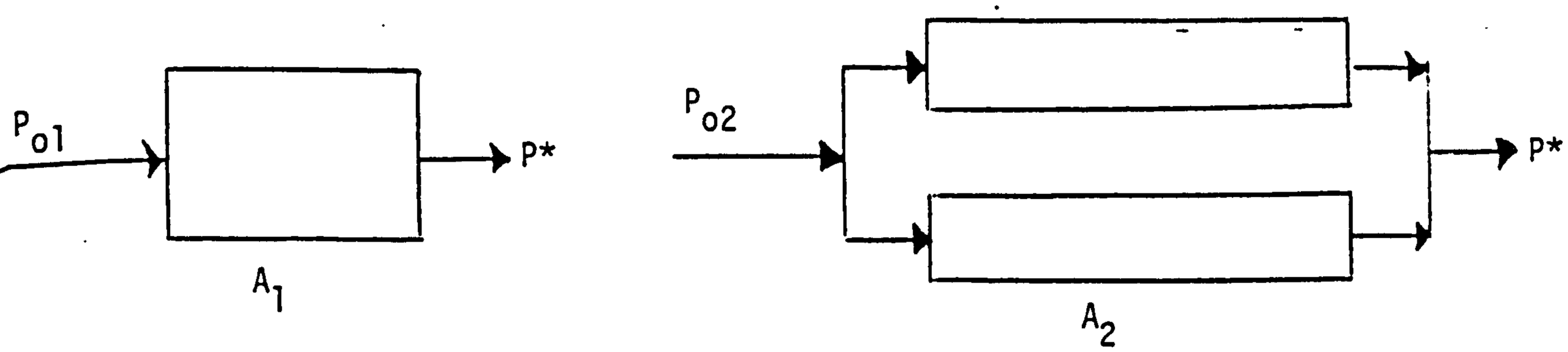


Figure 5.1

<u>single equipment</u>

rated capacity P_{o1}

design for

$P_{o1} = (P^*/A_1)^{\beta}$

costing

$C_{o1} = K_1' \left(\frac{P^*}{A_1}\right)^{\beta}$

<u>2x100% active redundancy</u>

rated capacity P_{o2}

design for

$P_{o2} = (P^*/A_2)^{\beta}$; $A_2 > A_1$

costing

$C_{o2} = 2K_1' \left(\frac{P^*}{A_2}\right)^{\beta}$

and $$\left(\frac{C_{o1}}{C_{o2}}\right) = \frac{1}{2} \left(\frac{P^*}{A_1}\right)^{\beta} \left(\frac{A_2}{P^*}\right)^{\beta} = \frac{1}{2} \left(\frac{A_2}{A_1}\right)^{\beta}$$

(Fig. 5.1)

We can see from Fig. 5.1 that by improving the availability of a system, we are actually decreasing the capital cost of overdesign

if (in this case)

$$(\frac{A_2}{A_1})^{\beta}>2$$

If we used hybrid redundancy (2x50% say) the equivalent condition would now be

$$(\frac{2\xi_2}{\xi_1})^{\beta}>2$$

Of course, in the process we alter other costs such as maintenance and operating costs. However Cowan [17] and de la Mare [42] have found that the major contribution to the final cost at the design stage is that of capital cost. This observation, and the difficulty of obtaining maintenance and operating cost data, leads us to an objective function containing capital costs only.

So, the statement of our problem can be expressed as:

$$\left.\begin{aligned} \min_{\Gamma_i} C_T = \min_{i} \sum_{i=1}^{N} C_i' &= \min_{\underline{x}} \sum_{i=1}^{N} \sum_{j=1}^{\phi} b_{ij} C_{ij} \left(A_T(b_{ij})\right) \\ \text{subject to } \sum_{j=1}^{\phi} b_{ij} &= 1 \; ; \; b_{ij} \in \{0,1\} \\ \text{and } \quad C_{ij} &= K_i' \left(\frac{P^*}{A_T(b_{ij})}\right)^{\beta_i} \end{aligned}\right\} \quad \ldots (5.3)$$

where

C_{ij} = cost of equipment i under option j

b_{ij} = 1 if option j is chosen for equipment;

= 0 otherwise

Problem (5.3) is a mixed-integer non-linear programming problem

with binary integer variables b_{ij} and real variables C_{ij}. These latter real variables depend on other integer variables (the number of pieces of equipment in position i, n_i) and real variables (the volumes of buffer tanks, V_i). If we decide to enumerate all possible combinations and choose the optimum one, we would have to calculate the cost of N^{ϕ} configurations. For a reasonable number of options (active and standby redundancy with and without storage and hybrid active and standby redundancy) ϕ=6. Even for two pieces of equipment the enumerative approach leads us to 2^6 plant configurations and consequently 2^6 multivariable optimizations with respect to n_i and V_i. One algorithm which tries to avoid exhaustive enumeration is that due to Cowan [17], [18].

5.2 The Work of J.L. Cowan

5.2.1 Cowan's optimization algorithm

J.L. Cowan described the following heuristic algorithm which attempts to solve (5.3) avoiding exhaustive enumeration [18].

STEP 1 Number the elements 1 to N. Consider a 'base case' option B_0 consisting of all elements with no redundancy or buffer storage. Thus B_0 corresponds to the simple overdesign option. Let k=1.

STEP 2 Consider all options for element k. Choose option j (j=1,..ϕ), which is optimal with respect to B_{k-1}.

STEP 3 Call B_k the configuration with option j substituted for the original configuration of element k and all other configurations constant. Let k=k+1 and go back to step 2 until k=N.

STEP 4 Repeat steps 2-3 (which constitutes an "iteration") until configuration B'_k is reached for which $B'_{k-1}=B'_k$. This terminates the algorithm, since for n>k', $B_n=B'_k$ and no further improvement is made. Then B'_k is the proposed final configuration r.

Cowan was unable to prove that this algorithm would produce an

optimal Γ. However the algorithm always improves upon the initial cost.

Cowan dealt with buffer storage in the following way. The process was split in two stages: upstream and downstream of the tank site. The elements in these two stages were 'blocked' using a blocking algorithm similar to [29]. The optimum buffer volume was then computed using certain approximations. She did not deal with more than one tank per process. Her previous algorithm was then modified, so that for each option j for element k a new optimum volume was computed and each change of base B_k involved a new optimum tank volume. If there was more than one tank site, the process cost was evaluated for each site separately and the best site chosen.

If one can calculate analytically the effect of storage on availability, then the above procedure is not necessary. Buffer storage becomes then one amongst the options ϕ considered for each element in step 2. Allen and Coker [4] have extended Cowan's algorithm in this way, and used it to improve the production performance of an existing plant. In chapter six we shall use Cowan's algorithm as modified by Allen and Coker [4] to include intermediate storage, using our expressions for the effects of storage developed in chapters three and four.

5.2.2 Cowan's algorithm and one option

It is instructive to study Cowan's algorithm before we try to solve (5.3) by other means, because in this way we can introduce certain ideas which lead naturally to a new algorithm.

Cowan's algorithm is essentially a discrete form of the well-known successive minimization algorithm along orthogonal directions. We shall show this as follows:

Suppose we have N elements in series and that we have the single option of using just parallel active redundancy (including over-design

at n_i=1) to improve the availability of the system and hopefully reduce its cost. We require n_i^*, i=1,..N, the optimum number of units installed in stage i.

The availability of the series system is by (2.2)

$$A_T = (1+\sum_{i=1}^{N} x_i\ (n_i))^{-1} \qquad i=1,N \quad .....(5.4)$$

where $$x_i(n_i) = \frac{1-A_{pi}(n_i)}{A_{pi}(n_i)} \qquad .....(5.5)$$

$$A_{pi}(n_i) = 1-(1-A_i)^{n_i} \text{ according to (2.4)}$$

The redundancy is of the form nx100%. Consider the exponential form of the cost function

$$C_i = K_i'\ \text{x (capacity)}^{\beta_i} \text{ (from 5.2)}$$

Thus the cost for stage i is

$$C_i = n_i K_i'\ (\frac{P^*}{A_T})^{\beta_i} \qquad n_i = 1,2..N \quad .....(5.6)$$

Incorporating $(P^*)^{\beta}$ into K_i' and using (5.4) we can obtain

$$C_i = n_i K_i\ (1+x_i(n_i) + \sum_{j\neq i} x_j(n_j))^{\beta_i} \quad ..... \quad (5.7)$$

The total cost of the system is

$$C_T = \sum_{i=1}^{N} C_i = \sum_{i=1}^{N} n_i K_i\ (1+x_i(n_i) + \sum_{j\neq i} x_j\ (n_j))^{\beta_i} \quad ..(5.8)$$

(5.8) is an integer form of the general equation

$$C_T = \sum_{i=1}^{N} v_i K_i \left(1+x_i(v_i) + \sum_{j\neq i} x_j\,(v_j)\right)^{\beta_i} \qquad \text{with } x_i = x_i(v_i) = \frac{(1-A_i)^{v_i}}{1-(1-A_i)^{v_i}} \qquad \ldots \quad (5.9)$$

(5.8) is obtained from (5.9) by integer collocation at the points $v_i=n_i$ (since $n_i \epsilon N^+ CR^+ \ni v_i$). We distinguish between x_i and $\sum_{j\neq i} x_j$ in (5.7) - (5.9) for reasons which will become obvious later on.

We can examine Cowan's algorithm by looking at (5.9). (5.9) is convex with respect to v_i (see Appendix 4) and may look like the function in Figure 5.2.

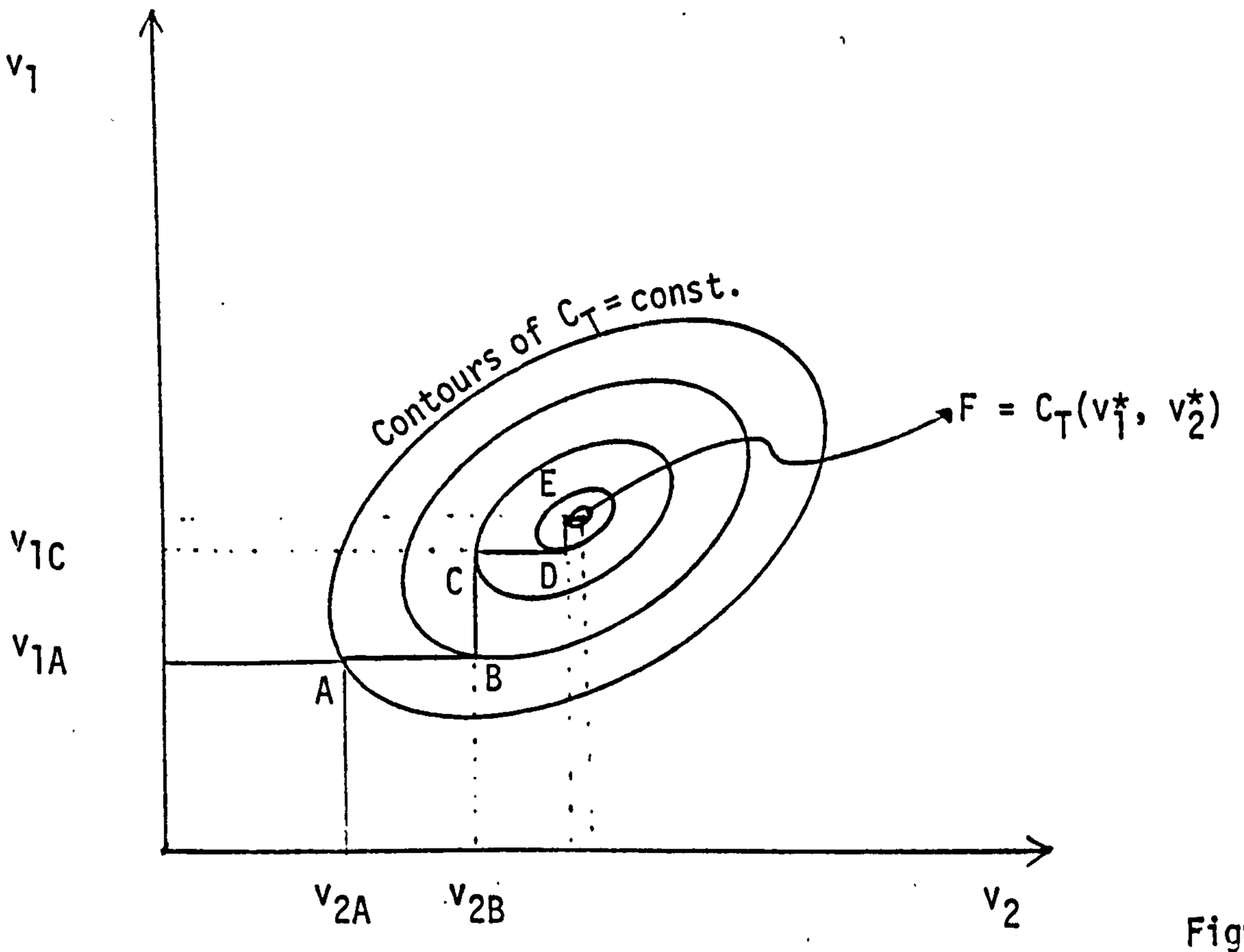


Figure 5.2

Let the overdesign option be point A(1,1). Then, starting from A we keep v_1 constant and optimize v_2 by moving along the line v_1=1. The optimum along this line is point B. Let B_1 be the "configuration of stage 1 optimal relative to B_o (i.e. point A)". Point B defines the new base B_1. We then keep V_{B2} constant and optimize along BC. Thus, at C = (V_{2B}, V_{2C}) we have a configuration B_2 optimal with respect

to B_1. Point D defines B_3 and so on, until we reach point F = (v_1^*, v_2^*) where the algorithm stops, because the optimum configuration along the v_2-axis (B_6) will be the same as the one along the v_1-axis (B_7), i.e. k'=7 in Cowan's algorithm.

The difference between the above procedure and Cowan's original algorithm lies only in the vector space involved. Cowan works in the space of integer numbers and therefore paths such as ABCDEF and contours of F=const will be non-continuous and only defined at the lattice points v_i=1,2,n (i=1,2). However, it is clear that Cowan's algorithm is a discrete form of a successive minimization algorithm which optimizes along directions parallel to the axes. Like all such algorithms, it works well if the general continuous function (C_T in Fig. 5.2) is smooth and convex for the particular option considered. Even if strict optimality cannot be guaranteed with discrete steps, a very close approach to the optimum should usually be achieved. However, the picture changes considerably when we consider many options, as we do in the next section.

5.2.3. The multiple option problem

Cowan's algorithm performs well if one is considering alternatives within one option only. If more options are allowed, then we have a number of local optima, each one formed by a combination of options. One cannot then be certain that the final choice will be optimal, although the algorithm will lead to improved designs.

However, Cowan's algorithm has been used successfully [3], [17], [18] to improve the performance of plants. Experience with Cowan's algorithm seems to indicate firstly that the optimum in question is a flat one and secondly that a close approach is obtained [17]. This has also been our experience.

Examination of the progress of Cowan's algorithm reveals the fact

that it seems to use a lot more enumerations than necessary. Since it tries all ϕ options for each element it uses up at least $\phi N+\phi$ enumerations (multiple optimizations if storage is considered) before it stops. However, sometimes it is obvious to an engineer that certain options which are expensive will surely lead to suboptimal configurations. The question we shall deal with in the next section is whether we can find an algorithm which will intelligently avoid such superfluous enumerations and, therefore, be more efficient than Cowan's.

5.3 The MOP (Multiple Option) Algorithm

In this section we present an alternative algorithm to Cowan's, which we shall show to be more efficient in practice. First, we have to transform problem (5.3) to a form convenient for our purposes.

We can express (5.6) and (5.7) in terms of x's.

From (5.9), letting $Q_i=1-A_i$

$$n_i = \left[\ln \left(\frac{x_i}{1+x_i}\right) \frac{1}{\ln Q_i}\right] \qquad \ldots\ldots \qquad (5.10)$$

where the brackets [ ] indicate the operation "taking the next largest integer" to

(5.7) becomes for parallel redundancy

$$C_{pi} = \frac{1}{\ln Q_i} \ln\left(\frac{x_i}{1+x_i}\right) K_i\left(1+x_i+\sum_{j\neq i} x_j\right)^{\beta_i} \ldots\ldots \qquad (5.11)$$

Optimizing with respect to the x_i's is equivalent to optimizing with respect to the n_i's in parallel redundancy, though we have to round off our results to the next integer using the operator in (5.10).

For standby redundancy we have a similar expression to (5.11). The general form is:

$$C_{si} = f_s(x_i)K_i \left(1+x_i+\sum_{j\neq i} x_j\right)^{\beta_i} \quad \ldots \quad (5.12)$$

In fact, for each of the options we are considering (including hybrid redundancy and buffer storage) we have an expression of the form:

$$n_i = f_j(x_i) \quad \text{or} \quad V_i = \bar{f}_j(x_i) \qquad \begin{matrix} j=1 \ldots \phi \\ i=1 \ldots N \end{matrix}$$

The problem is according to (5.3)

$$\min C_T = \min \sum_{j=1}^{\phi} \sum_{i=1}^{N} b_{ij} C_{ij} \quad \text{so that} \quad \sum_{j=1}^{\phi} b_{ij}=1, \text{ all } i.$$

We can convert this to the following optimization problem taking account of the form of the functions C_{ij}.

Find b_{ij}, x_{ij} for

$$\min C_T = \min_{ij} \sum_{j=1}^{\phi} \sum_{i=1}^{N} b_{ij} f_j(x_j) K_i \left(1+\sum_{i=1}^{N}\sum_{j=1}^{\phi} b_{ij}x_{ij}\right)^{\beta_i} + \sum_{i=1}^{N}\sum_{j=1}^{\phi} b_{ij} K_{Ti} V_{ij}^{\gamma_i} \quad \ldots (5.13)$$

where $\sum_{j=1}^{\phi} b_{ij}=1$, all i $\quad b_{ij}\varepsilon\{0,1\}$

$$x_{ij}\varepsilon R^{+} \quad , \quad x_{ij}=x_{ij}(b_{ij})$$

$$n_{ij}=f_j(x_{ij}),\ V_{ij}=\bar{f}_j(x_{ij})$$

The problem involves both integer optimization (on the b_{ij}) and continuous optimization (on the x_{ij}). Note that by optimizing with respect to x_{ij} only, we can use the same variable for optimizing with

respect to redundancy and buffer storage. (Otherwise we would have to optimize with respect to n_i and V_i separately).

The MOP algorithm presented next, reduces the search amongst the N^{ϕ} nodes to a search of $O(\phi N)$.

MOP Algorithm

STEP 1 Relax (5.13) to (5.14)

$$\Sigma \min C_{ij}(b_{ij},x_{ij}) + b_{ij}K_T V_{ij}^{\gamma_i} \quad \ldots\ldots \quad (5.14)$$

let $C_T^* = \infty$ (numerical infinity in the computer)

let k=1

STEP 2 Ignore b_{ij} and transform (5.14)

to the ϕN problems:

$$\sum_{i=1}^{N} \min C_{ij}(x_{ij}) \text{ for each option j and stage i } \ldots \quad (5.15)$$

STEP 3 Transform (5.15) to the $N\phi$ problems

(adding $K_T V_i^{\gamma_i}$ for the appropriate options)

$$C_{ij}=f_j(x_{ij})\ (1+x_{ij}+z_{ij})^{\beta_i}\ ;\ z_{ij} = \sum_{k\neq j}^{N} x_{ik} \quad \ldots\ldots \quad (5.16)$$

STEP 4 Perform ϕN optimizations with respect to x_{ij},z_{ij} to give x_{ij}^*,z_{ij}^*. Calculate the optima

$$C_{ij}^* = C_{ij}\ (x_{ij}^*,z_{ij}^*)$$

STEP 5 Arrange the optima C_{ij}^* in a matrix $C=\{C_{ij}^*\}$ and pick out the smallest element in each row. (Choose an optimal option j for each element i with a corresponding cost C_{ij}^*). Keep a record of i and j.

STEP 6 Check $C_T = \sum_{i=1}^{N} C_{ij}^*$. If $C_T > C_T^*$ go to 10

If $C_T < C_T^*$ continue.

STEP 7 Perform a multiple optimization on

$C_T = \Sigma C^*_{ij}\ (x_{ij})$ with respect to x_{ij}

let the minimum value of C_T be $\bar{C}_T$

and the optimum value of x_{ij} be x^{**}_{ij}

STEP 8 If $\bar{C}_T < C^*_T$, then $\bar{C}_T \rightarrow C^*_T$, go to 9

If $\bar{C}_T > C^*_T$, then backtrack and

restore the option disregarded in step 9, for $k \neq 1$.

STEP 9 Find the next smallest element in the matrix C (irrespective of row and column). Let this element be $C^*_{\ell k}$ $1 \leq \ell \leq n$, $1 \leq k \leq \phi$.

Calculate $C_T = \sum_{i=1}^{k} C^*_{ij} + C^*_{\ell k} - C^*_{\ell j}$

i.e.

Form the sum of the elements C^*_{ij} disregarding the previous option j for element ℓ in favour of the new option k.

If all C^*_{ij} have been enumerated go to 10

Otherwise, let k=k+1 and go back to 6.

STEP 10 C^*_T is the optimum cost, x^{**}_{ij} , b^{**}_{ij} the optimum variables, where $b^{**}_{ij} = 1$ if option j has been selected for C^*_T and 0 otherwise.

What the algorithm does is this:

In steps 1-4 it relaxes and decomposes the original problem, from a 2N-multiple optimization problem to ϕN 2-variable optimization problems. For every option j and every element i we calculate the optimum partial cost C^*_{ij} with respect to x_{ij} and the dummy variable z_{ij}.

In step 5 we arrange C^*_{ij} in a matrix $C = \{C^*_{ij}\}$ and pick out the smallest element in each row. Let us use an example with i=j=4.

$$\begin{bmatrix} (C_{11}) & C_{12} & C_{13} & C_{14} \\ C_{21} & (C_{22}) & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & (C_{34}) \\ C_{41} & (C_{42}) & C_{43} & C_{44} \end{bmatrix}$$

In matrix C we have cirled the minimum elements in each row

Suppose $C_{11} = \min_j C_{1j}$, $C_{22} = \min_j C_{2j}$, $C_{34} = \min_j C_{3j}$ and $C_{42} = \min C_{4j}$, $j=1,4$. Thus $C_T = C_{11}+C_{22}+C_{34}+C_{42}$.

This means that we have chosen option 1 for element 1, option 2 for elements 2 and 4 and option 4 for element 3. Picking out the minimum elements in each row, we have picked out the variables b_{ij}. In step 8, if C_T is greater than the current minimum cost C_T^* we terminate. Otherwise, we calculate $\bar{C}_T = \min C_T$ using some multivariable optimization procedure (and the new constraints $z_i = \sum_{j\neq i} x_j$). $\bar{C}_T$ is always greater than C_T, since it is the optimum of the same objective function with more constraints on the z_i. We let $\bar{C}_T \rightarrow C_T^*$ and choose the next smallest element in the matrix, say C_{21} to take the place of C_{22} (step 9). Our new option for element 2 is going to be option 1. We then repeat the above procedure with one exception: if the new $\bar{C}_T$ is larger than C_T^*, then we restore the option we discarded in step 9 previously. Thus in our example above we would return to the option set {1,2,4,2} which produced a better cost. If then, the next smallest element in the matrix is C_{44}, we perform an optimization on $C_T = C_{11}+C_{22}+C_{34}+C_{44}$ and <u>not</u> on $C_{11}+C_{21}+C_{34}+C_{44}$. We thus keep the option for each element that produces the best result at every stage. We repeat this procedure until all matrix elements have been examined or until $C_T > C_T^*$.

In such branch and bound algorithms there are normally two questions to be answered: will the algorithm terminate in a manageable number of steps? And, will it find the optimum? In the next section we seek to answer these questions.

5.4 Convergence of the Algorithm

5.4.1 Refinements to the MOP algorithm

Before we discuss convergence, let us try to simplify some stages in the algorithm, namely the ones requiring some sort of optimization.

Up to now we have treated x as a real variable. Since eventually we require V_i and n_i we must convert back to those variables using a function $n_i = [f_s(x_i)]$ or $V_i = f_v(x_i)$. However, since the cost functions $C_i = C_i(x_i)$ are convex (see Appendix 4) we can optimize C_i treating x_i directly as a discrete variable, taking values only at $n_i = 1,2,3 \ldots$ and as a real variable only when we consider x_i as a function of V_i. Because $C_i = C_i(x_i)$ is convex, the optimum value produced by enumeration of x_i at integer n_i will be the optimum integer value closest to the continuous optimum $\underline{x}^*$.

Thus in steps 4 and 7 we optimize $C_i(x_i)$ by a simple enumeration. Essentially, then step 7 is a "Cowan step", where we optimize C_T for a particular set of options. We have seen in section 5.2 that when there is no choice among options (as in this case) Cowan's algorithm performs well.

The next simplification comes when we consider hybrid redundancy options. In the hybrid redundancy formula (2.8) the efficiency ξ is a function of r and N, and so are the coefficients ${}^{N}C_{r}$. Thus, we cannot express $C_{ij}(x_{ij})$ as a function of x_{ij} for all hybrid redundant options collectively, as we did for standby and parallel redundancy.

Therefore we consider each hybrid redundancy option on its own. Then, there is no optimization involved with respect to x_{ij}, since x_{ij} is fixed by the choice of option, e.g. for the 3x50 hybrid redundancy option:

$$x_{ij} = \frac{1-A_h}{A_h} \quad ; \quad A_h = \frac{1.5\rho_i^2+3\rho_i+1}{(1+\rho_i)^3}$$

and

$$C_{ij} = 3K_i((1+x_{ij}+z_{ij})/2)^{\beta_i}$$

The only variable in the above expression for C_{ij} is z_{ij}. Thus in step 4 we merely calculate this expression for x_{ij}.

This is a way to treat more than one non-convex option: namely consider each one separately at a fixed n_i (or other parameter) and add more rows to the matrix C in step 5.

The final simplification to the MOP algorithm has to do with z*. The function to be optimized in step 4 is of the general form

$$f_{ij}(x_{ij})(1+x_{ij}+z_{ij})^{\beta_i} + K_T V_{ij}^{\gamma_i}$$

where, as we have seen $f_{ij}(x_{ij})$ and x_{ij} may be constant for hybrid redundancy.

Because $z_{ij} \geq 0$, it is easy to see (using the Lagrangian) that z^*_{ij} must attain its lower bound at the optimum. Thus, $z^*_{ij} = 0$ and the only optimizations we have to perform in step 4 are the ones with respect to x_{ij}, when x_{ij} is variable. Since the buffer tank options will produce a slightly lower cost than the corresponding redundancy options (see Appendix 4), we can simply let the cost of the buffer options be slightly less than the cost of the corresponding redundancy options.

To recap:

In step 4 we enumerate the costs C_{ij} for the options of active

and standby redundancy for various n_i at $z^*_{ij} = 0$, and choose the minimum. We subtract a small number from these costs and obtain an estimate of the costs of the corresponding "tankage plus redundancy" options. For the hybrid redundancy options we simply calculate all costs at $z^*_{ij} = 0$, since x_{ij} does not vary. And finally, in step 7 we use a "Cowan step" to obtain the optimum cost of the particular configuration. These refinements of the algorithm have the following two advantages: they speed up the initial step of the algorithm with no extra computational cost, and they help optimize directly in the integer space.

5.4.2 Termination

Since the algorithm is a search through a matrix of $N\phi$ elements, after step 5 it should terminate after K searches where

$$K \leq N\phi - N+1$$

Even if it takes the maximum number of steps to terminate, it will have performed K N-variable optimizations and ϕN 2-variable ones. We have seen that the work involved for the initial ϕN optimizations is small. The main computational costs arise from the K "Cowan" optimizations in step 7.

We can compare the work of the MOP algorithm theoretically with that of Cowan's own as modified by Allen and Coker [4]. We saw in 5.2.3 that when only one iteration is necessary the number of nodes, K_c, examined is $N\phi+N$. This is the minimum number of searches needed for Cowan's algorithm to converge. The maximum K for the MOP algorithm is $N\phi-N+1$. We can than see that theoretically:

$$\max K < \min K_c$$

However, K should be a lot less than $N\phi-N+1$. Examination of the individual cost functions reveals the following fact:

Each cost function is of the form (5.17) for a particular configuration

$$C_{ij} = f_{ij}(x_i^*)(1+x_i^*+z_i^*)^{\beta_i} + K_T V_i^{\gamma_i} \quad \ldots\ldots\ldots (5.17)$$

and $x_i, z_i < 1$

Thus C_T from step 4 is approximately equal to

$$C_T \simeq \sum_{i=1}^{N} f_i(x_i^*)(1+\beta_i x_i^*) + K_T V_i^{\gamma_i}$$

and $$\bar{C}_T \simeq \sum_{i=1}^{N} f_i(x_i^*)(1+\beta_i x_i^{**}+\beta_i z_i^{**}) + K_T V_i^{\gamma_i}$$

Because z_i^{**} is small (<1) the optimum from step 4 C_T and the optimum from step 10 C_T^* are quite close numerically and the condition $\bar{C}_T < C_T^*$ which terminates the algorithm is met early on in the enumeration. This is in fact the case in practice as we shall see in chapter six.

5.4.3 Optimality

It can not be strictly proved that the MOP algorithm will find the optimum of the problem. However in the instances it has been used it has converged to the same solution as Cowan's algorithm using a lot less computer time. There are a few things we can discuss about the algorithm, however:

(a) The Stopping Criterion

This is constructed using the observation that, if an objective function $F(\underline{x})$ has a minimum at $\underline{x}^*$ and the same function $F(\underline{x})$ is subject to constraints on $\underline{x}$ giving rise to a new minimum $\hat{\underline{x}}$, then

$$F(\hat{\underline{x}}) \geq F(\underline{x}^*) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5.18)$$

We observe now that a multivariable optimization on $C_T(\underline{x}) = \Sigma\, C_i(\underline{x})$ resulting in $\bar{C}_T$ is equivalent to a constrained optimization on $C_T(\underline{x},\underline{z}) = \Sigma\, C_i(\underline{x},\underline{z})$ with N constraints of the form $z_i = \sum_{j\neq i} x_j$. We also know that the C_i are convex, non-negative functions (see Appendix 4).

Thus

$$\min \Sigma C_i \geq \Sigma \min C_i \quad \ldots\ldots\ldots\ldots\ldots (5.19)$$

Using (5.18) and (5.19) we can see that

$$C_T = \sum_{i=1}^{N} \min C_i^*(\underline{x},\underline{z}) \leq \min \sum_{i=1}^{N} C_i(\underline{x},\underline{z}) \leq \min \sum_{i=1}^{N} C_i(\underline{x},\underline{z}) = \bar{C}_T$$

STEP 4 by (5.19) $z_i = \sum_{j\neq i} x_j$ by (5.18)

So, if at some stage K in the iteration, during which C_{Tk} progressively increases, we have the conditions that

$$C_T^* \leq C_{Tk}$$

then, we are certain that

$$C_T^* < \bar{C}_{Tk+1} \quad \text{for all subsequent k.}$$

Thus, our stopping criterion is justified.

(b) Progress Heuristics

During the progress of the algorithm, we increase C_T by considering the next smallest element in the matrix C, backtracking only if $\bar{C}_T$ new $> \bar{C}_T$ old $= C_T^*$ in the manner of step 8.

This heuristic was arrived at using some engineering common sense: For each element in the process we choose initially the cheapest option available. By optimizing the cost corresponding to those cheapest options we obtain C_T^*. Next, we proceed by substituting the next cheapest option for the cheapest element in the process.

However, in step 8 if this substitution leads to an overall $\bar{C}_T$ greater than the previous $\bar{C}_T$, we revert to the configuration which provided us with the cheaper alternative. Thus, during the progress of the algorithm we always keep that option for each element that leads to the cheapest overall process cost.

This heuristic is based on the observation that redundancy is likely to be installed for the cheaper elements in the process.

Although we can not prove that the MOP algorithm will lead to the overall optimum configuration, it has been observed [17], that the optimum in question is a flat one. This fact is confirmed by our observations during the progress of the MOP algorithm, namely that after a certain stage, the various configurations have costs quite close to each other.

Furthermore, as the MOP algorithm always converges to the same answer as Cowan's algorithm, there is a strong hint that the optimum may have been found. (See section 6.5 for further discussion).

5.5 Optimum Buffer Storage

The insertion of buffer storage makes by far the most significant contribution to the improvement of overdesign cost. This observation due to many researchers [2], [25], [32], [47] is confirmed by our work. Therefore, it is worth trying to produce an algorithm which calculates optimum buffer sizes. This has been the aim of many workers in this field who have been using mainly MC simulation for sizing intermediate tanks (see chapter two).

If the only option we choose to adopt is insertion of intermediate storage, then our problem is simplified. The cost function becomes:

$$C_T = \sum_{i=1}^{N} K_i \left(1+\sum_{i=1}^{N} x_i\right)^{\beta_i} + \sum_{i=1}^{N-1} K_{Ti} V_i^{\gamma_i} \quad \ldots\ldots \quad (5.20)$$

where

V_i is given by

$$V_i = -\frac{q}{\theta_i} \ln \frac{\rho_i x_i - U_i}{\rho_{i+1} x_i} \qquad i = 1,N-1 \quad \ldots\ldots \quad (5.21)$$

q is the flowrate through the tank and θ_i, U_i are constants (see

Appendix 4). As x_i, V_i are real variables we can optimize (5.20) using an optimization algorithm for continuous variables. If we do this, we observe from Appendix 4 that the dependence of x_i on V_i leads to a flat minimum for V_i. Thus, there is a range of tank sizes one may use, without affecting seriously the cost of the process. Also, the constant θ_i from chapter three is given by:

$$\theta_i = \delta(\lambda_1\mu_2 - \lambda_2\mu_1)$$

and is normally very small. We observed in chapter three that if $\theta_i = 0$, then the availabilities of the machines on either side of the tank are equal ($\rho_1 = \rho_2$).

In that case, the resulting availability of a M/T/M system is given by (see chapter three)

$$\left.\begin{aligned} A &= \frac{1+\rho^{-1}+c^*}{\rho^{-1}+2+\rho+c^*(1+2\rho)} \quad ; \quad c^* = \frac{q}{V}\sigma \\ \sigma &= (\lambda_1^{-1}+\lambda_2^{-1}) \end{aligned}\right\} \quad \ldots\ldots(5.22)$$

If we let $\bar{c} = {}^{1}/_{c^*}$, then the equivalent optimization problem becomes

$$\min \sum_{i=1}^{2} K_i(1+x+\rho)^{\beta} + K_T V^{\gamma} \qquad \ldots\ldots\ldots\ldots\ldots (5.23)$$

where
$$x = \frac{\rho}{(1+\rho^{-1})\bar{c}+1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5.24)$$

If we divide (5.23) through by $\sum_{i=1}^{2} K_i$ and convert V to $\bar{c}$ we obtain

$$\left.\begin{aligned} &\min_{\bar{c}} \{(1+x+\rho)^{\beta} + s\bar{c}^{\gamma}\} \\ &\text{or} \quad x = \frac{\rho}{(1+\rho^{-1})\bar{c}+1} \\ &\qquad s = \frac{(q\sigma)^{\gamma} K_T}{\sum_{i=1}^{2} K_i} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5.25)$$

The optimum of (5.25) is given by the solution of (5.26)

$$\left.\begin{aligned} \gamma(1+x+\rho)^{\gamma-1} + s\gamma\bar{c}^{\gamma-1}\frac{\partial\bar{c}}{\partial x} &= 0 \\ \frac{\partial\bar{c}}{\partial x} &= -\frac{\rho}{x(1+\rho^{-1})} \end{aligned}\right\} \quad \ldots\ldots\ldots (5.26)$$

It can also be shown that (5.25) is convex. (5.26) can be easily solved for x for various values of s and ρ.

Although, as we have pointed out, (5.25) is a formulation of the special problem of finding the optimum buffer storage of a $M_1/T_1/M_2$ system with $\rho_1=\rho_2=\rho$, it can be used to provide approximations to the general problem $\rho_1\neq\rho_2$ as long as the availabilities are close enough. This happens for the reasons we have explained before: namely, that det Λ for availabilities numerically close is almost zero and that (5.20) has a flat minimum with respect to V.

In graph (5.1) we have plotted the availability of a process vs. $\bar{c}$ on the same graph as availability vs. optimum $\bar{c}(=\bar{c}^*)$. We have assumed that $\beta=\gamma=0.7$.

This graph can be used in the following way:

PROCEDURE FOR OBTAINING OPTIMUM BUFFER STORAGE

STEP 1 Calculate the availability A_T of the two stage process where tankage is to be installed.

STEP 2 Calculate $$s = \frac{(MTTF_1+MTTF_2)^{0.7} * K_T * (q)^{0.7}}{\sum_{i=1}^{2} K_i}$$

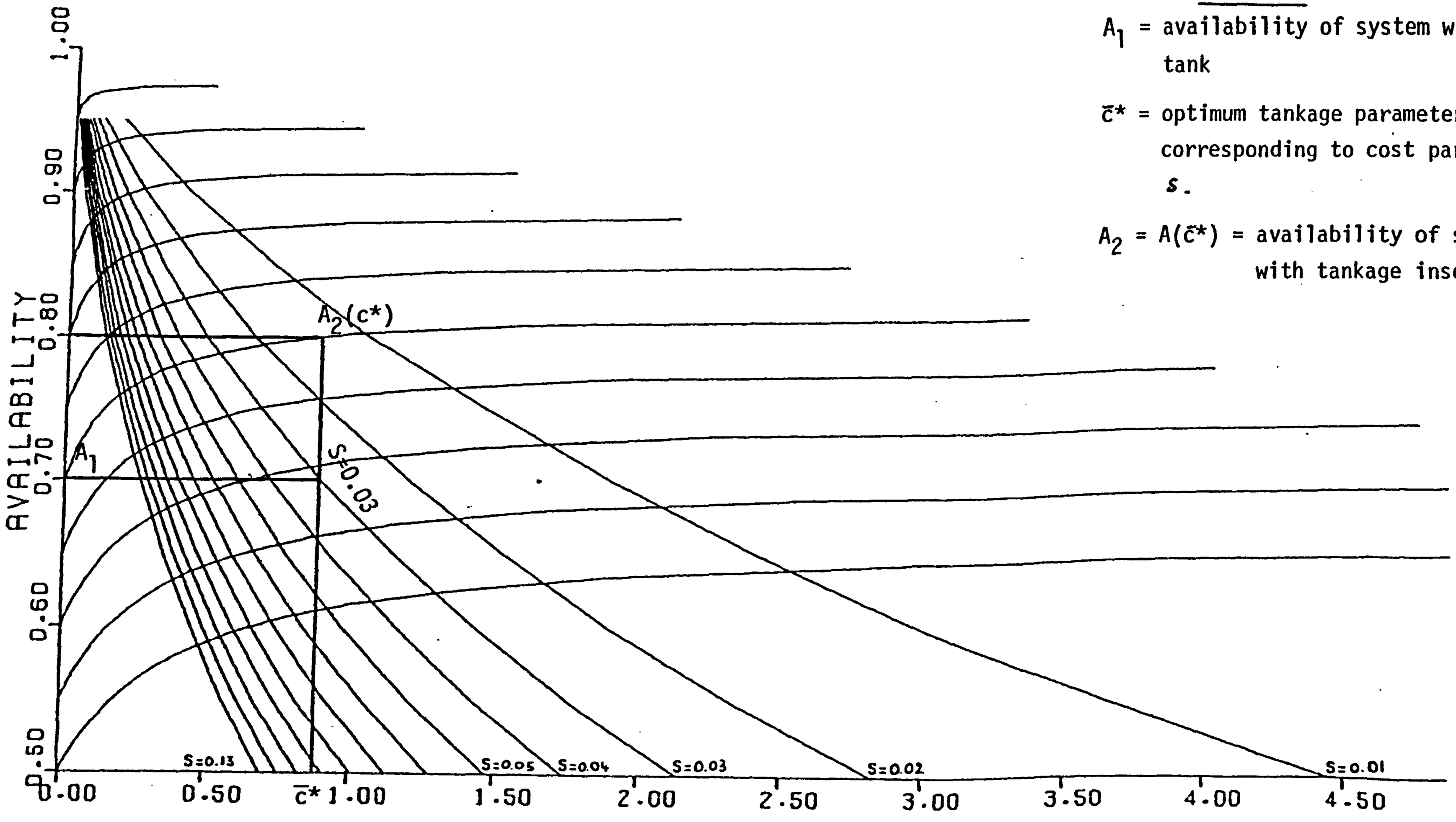
Remember that K_i = cost constant of equipment times the annual production rate to the power β (see Equations 5.6 - 5.7). Also note that q, P*, V and MTTR, MTTF must be in consistent units. K_T is the cost constant of tankage according to Equation (5.2).

Graph 5.1

A_1 = availability of system without tank

$\bar{c}^*$ = optimum tankage parameter corresponding to cost parameter s.

$A_2 = A(\bar{c}^*)$ = availability of system with tankage inserted.



STEP 3 Read off $\bar{c}^*$ from the appropriate parametric curve of s.

STEP 4 Read back the new availability of the process.

One can repeat steps 1 - 4 as many times as desired, for as many tanks as are required. Each time only the relevant M/T/M system will be considered separately. This is possible because of the following decomposition.

The original problem (5.20) is

$$\min C_T = \min \sum_{i=1}^{N} K_i \left(1+ \sum_{i=1}^{N} x_j\right)^{\beta_i} + \sum_{i=1}^{N-1} K_{Ti} V_i^{\gamma_i} \quad \ldots (5.27)$$

and $$\min C_T \cong \min \sum_{i=1}^{N-1} K_i \left(1+\beta_i \sum_{i=1}^{N} x_i\right) + \sum_{i=1}^{N-1} K_{Ti} V_i^{\gamma_i}$$

We can add the constant $N \sum_{i=1}^{N} K_i \beta_i \rho_i$ without affecting the optimum.

We can then write (5.27) as:

$$\min \left[\sum_{i=1}^{N} K_i (1+\beta_1 (x_1+\rho_1)) + K_{T1} V_1^{\gamma_1} \right]$$

$$+\min \left[\sum_{i=1}^{N} K_i (1+\beta_2 (\rho_2+x_2)) + K_{T2} V_2^{\gamma_2} \right] + \ldots$$

$$+ \ldots \min \left[\sum_{i=1}^{N} K_i (1+\beta_{n-1} (\rho_{n-1}+x_{n-1})) + K_{Tn-1} V_{n-1}^{\gamma_{n-1}} \right]$$

$$+ \sum_{i=1}^{N} K_i (1+\beta_n (\rho_n+x_n)) \quad \ldots\ldots\ldots (5.28)$$

Each of the terms in (5.28) is a function only of one independent variable x_i. This is why we can optimize the sum (5.27) by optimizing each term in (5.28) with respect to x_i.

C H A P T E R S I X

EXAMPLES

6.1 Introduction

In this chapter we shall apply the optimizing procedures of the last one to some examples. Firstly we illustrate the use of Graph 5.1 and its limitations using a few simple cases of a M/T/M system optimization. Then, we use some examples to show how the MOP algorithm can be used to lead to better designs of some hypothetical processes. Finally, an example taken from Cowan [17] corresponding to a plant in operation is examined. The MOP algorithm is also compared to Cowan's optimization algorithm both in terms of nodes searched and computer time expended.

For all examples the number of options considered for each element in a process were 10, as follows:

(1) nx100% active parallel redundancy n=1,2,3,4

(2) nx100% standby redundancy n=2,3,4,5

(3) 3x50% standby hybrid redundancy

(4) 4x50% standby hybrid redundancy

(5) 3x50% active hybrid redundancy

(6) $4x33^{1}/_{3}\%$ active hybrid redundancy

(7) 4x50% active hybrid redundancy

(8) Insertion of buffer storage after element i

(9) nx100% active parallel redundancy (n=2,3,4) followed by buffer storage

(10) 2x100% standby redundancy followed by buffer storage.

If we make use of hybrid redundancy options the relevant quantity we should use is ξ instead of A.

6.2 Use of Graph 5.1

The examples in Table 6.1 correspond to cases of M/T/M systems with unequal availabilities. The difference in availabilities ranges from (0.02) to (0.10). Volumes V_1^* and costs C_{T1}^* have been obtained via a direct optimization of (5.20). Volumes V_2^* and costs C_{T2}^* have been calculated using the approximate procedure explained in 5.6.

If one compares the costs, one sees that they do not vary significantly, even for big differences in numerical value of the availabilities. However, the tank volumes may differ by large amounts. The difference is larger for cheaper tanks (as expected).

One can see from Table 6.1 that there is a wide range of volumes for every process for which the cost remains almost constant. The reason for this is the form of the cost function and the dependence of availability on volume. For the M/T/M systems in Table 6.1:

$$C_T = \sum_{i=1}^{2} K_i(1+\rho_i'+\rho_2)^\gamma + K_T V^\gamma$$

and

$$\rho_1' = \frac{\rho_1(\rho_1-\rho_2)}{\rho_1-\rho_2\exp(-K_c V)}$$

} Notation as in chapter four

Since $K_c \sim O(10^{-3})$, $\gamma \sim 0.7$ and $\rho_1, \rho_2 < 1$ one can check directly from the above equations that large changes in volume produce small changes in cost.

The option of buffer storage is indeed instrumental in reducing process costs. (We discuss this in more detail in 6.5). However, the volume of intermediate storage itself is of less importance. In other words, the graph of process cost versus intermediate storage volume shows a sharp decrease, followed by a flat minimum.

TABLE 6.1

RUN.	MTTF	MTTR	A	s	V_1^*	V_2^*	C_{T1}^*	C_{T2}^*
1 UNIT 1 UNIT 2	970 990	30 10	.97 .99	.01 .10	84.3 0	90 0	20458 20571	20459 20571
2 UNIT 1 UNIT 2	500 600	43.3 38.0	.92 .94	.01 .10	480 89.5	605 110	21344 21864	21349 21868
3 UNIT 1 UNIT 2	880 900	120 100	.88 .90	.01 .10	1118.2 0	1860 285	22242 23348	22278 23669
4 UNIT 1 UNIT 2	830 850	170 150	.83 .85	.01 .10	2480 466	2760 489	23152 24338	23154 24340
5 UNIT 1 UNIT 2	750 730	250 270	.75 .73	.01 .10	4308 8404	4625 858	25531 27440	25532 27441
6 UNIT 1 UNIT 2	700 730	300 270	.70 .73	.01 .10	4360 916	5148 922	26270 28331	26277 28331
7 UNIT 1 UNIT 2	800 850	200 150	.80 .85	.01 .10	2017 483	2970 515	23673 24844	23695 24846
8 UNIT 1 UNIT 2	950 500	50 55.5	.95 .90	.01 .10	407.5 103	720 217	21631 22067	21965 22496
9 UNIT 1 UNIT 2	850 950	150 50	.85 .95	.01 .10	439 132	1674 594	22499 22921	22599 23612
10 UNIT 1 UNIT 2	700 800	300 200	.70 .80	.01 .10	2169 659	4500 825	25986.5 27433	26107 27461

Each unit has a cost constant of 10000

and an exponent of 0.7.

The tank cost constants are determined by s

The tank cost exponents are all 0.7

All costs in pounds (£)

This may be the cause of the difficulty in devising rules to size storage tanks, and why in most cases quite arbitrary decisions are made [54]. Thus it may be that other factors (such as site availability, size, maintainability, etc.) may be more instrumental in determining storage size than the objective function used in 5.6.

Next, we shall use three hypothetical cases, both to illustrate the use of Graph 5.1 in practice and study the performance of the MOP algorithm.

6.3 Example 1

Here we have three elements (stages of a hypothetical process) in series. Details of the process are given in Table 6.2. The recommended configuration of the system is shown in Table 6.3.

Example 1 has been chosen so that the availabilities of the units are numerically close. We should thus expect the results from Table 6.3 and the procedure outlined in 5.6 to be close for the option of optimal tankage.

We can try to calculate the volumes of the buffer tanks using Graph 5.1. Proceeding with calculations:

$$\sum_{i=1}^{2} K_i = (10+20+40)(8760)^{0.7} = 40250$$

For the first tank $\sigma_{12} = 300$

$$\therefore s_1 = \frac{(300)^{0.7}\ 50}{40250} \simeq 0.07$$

The availability of the first two machines without the tank is 0.74. The value for c* corresponding to A = 0.74 and s=0.07 is 0.35. The optimum volume is thus $105m^3$.

For the second tank $\sigma_{g3} = 400.$

$$s_2 = \frac{(400)^{0.7}\ 50}{40250} \simeq 0.08$$

The availability of the second and third machines together is 0.78. c* corresponding to A = 0.78 and s = 0.08 is 0.25. The optimum volume is thus 100 m^3.

TABLE 6.2

EXAMPLE 1

	MTTF	MTTR	Cost	Exponent	Availability
UNIT 1	100 hrs	20 hrs	£10/m^3yr	0.7	0.832
UNIT 2	200 hrs	30 hrs	£20/m^3yr	0.7	0.8696
UNIT 3	200 hrs	25 hrs	£40/m^3yr	0.7	0.889

$P^* = 8760 m^3/yr$ $\quad q = 1 m^3/hr$

$C_{TANK} = £50/m^3$ $\quad$ exponent: 0.7

EXAMPLE 2

	MTTF	MTTR	Cost	Exponent	Availability
UNIT 1	60 hrs	40 hrs	£10/m^3yr	0.7	0.6
UNIT 2	80 hrs	20 hrs	£20/m^3yr	0.7	0.8
UNIT 3	95 hrs	5 hrs	£40/m^3yr	0.7	0.95

$P^* = 8760 m^3/yr$ $\quad q = 1 m^3/hr$

$C_{TANK} = £50/m^3$ $\quad$ exponent: 0.7

TABLE 6.3

Optimum Configurations from MOP Algorithm

EXAMPLE 1

Stage	Option	Details	Availability
1	buffer storage	V_1=102m^3	0.9248
2	buffer storage	V_2=88m^3	0.9416
3	1x100%	-	0.8889

TOTAL COST = £49956

TOTAL AVAILABILITY = 0.7884

EXAMPLE 2

Stage	Option	Details	Availability
1	2x100 standby redundancy + storage	V_1=51.2m^3	0.9311
2	buffer and storage only	V_2=23.2m^3	0.8318
3	1x100%	-	0.95

TOTAL COST = £57367

TOTAL AVAILABILITY = 0.7526

The approximate solution to the problem is then: $V_1 = 105m^3$ and $V_2 = 100m^3$ with a total availability of 0.8 and a cost of £49,966. Compare this with the exact solution $V_1 = 102m^3$, $V_2 = 88m^3$ and a cost of £49,956 corresponding to an availability of 0.79. Thus the approximate and exact values differ by only 0.02%.

6.4 Example 2

Here, we have again three elements in a hypothetical process and we seek its best configuration. The process details are the same as for Example 1, but the difference in availabilities is considerable (see Table 6.1).

We can repeat the approximate procedure described in 5.6 and 6.3 to arrive at the following approximate volumes: $V_1 = 210m^3$, $V_2 = 70m^3$. The cost is now £60,648.

The exact answer is $V_1 = 98m^3$, $V_2 = 21m^3$ with a cost of £59,700. One can see that the answers are very different as far as the volumes are concerned. However, since the original cost of the process was £64,683 (just overdesign options), even these volume overestimates obtained using Graph 5.1 will produce an improved design.

Unlike example 1, choice of buffer storage is not the best option. The MOP algorithm arrives at a better design still as shown in Table 6.3. This new design improves the cost further to £57,367.

6.5 Example 3

This is again an example of a hypothetical process with five units in series. The numbers for the MTTF's and MTTR's were generated randomly, and so were the costs. (The latter have been rounded off to the next £500). All availabilities are far apart numerically, so one is not recommended to use Graph 5.1. The MOP algorithm instead, converges at the configuration shown in Table 6.4.

TABLE 6.4

EXAMPLE 3

	MTTF(hrs)	MTTR(hrs)	K_i	Exponent	AVAILABILITY
UNIT 1	1231.4	246.4	500	0.7	0.8333
UNIT 2	1318.8	640.3	2000	0.7	0.6732
UNIT 3	1051.5	615.1	5000	0.7	0.6309
UNIT 4	282.3	66.6	2000	0.7	0.8091
UNIT 5	501.8	154.3	15000	0.7	0.7648

K_{TANK} = 5 exponent = 0.7

RECOMMENDED CONFIGURATION

Stage	Option	Details	Availability
1	nx100% standby red.	2x100%	0.9836
2	buffer storage	3639 prod.hrs	0.9463
3	buffer storage	1042 prod.hrs	0.7348
4	buffer storage	911 prod.hrs	0.9844
5	overdesign	1x100%	0.7648

TOTAL COST 39894.8 units

TOTAL AVAILABILITY 0.5689

These first three hypothetical examples were used for two purposes. Firstly to show how Graph 5.1 can be used successfully in certain cases and secondly to provide some comparison between the alternative algorithms considered in this chapter (Cowan's and MOP). We defer such discussion until after we have examined the next example, which is very different from the above, in that it represents a real plant.

6.6 Ammonia Plant Example

This example is taken from Cowan [17]. It is a section of an ammonia plant in current operation. All availability and cost data has been worked out by Cowan [17]. Unfortunately the representation of costs in [17] is different from the one we use here. In [17] the cost ratios with respect to the B_o basis are listed (see 5.2 for definition of B_o). For our purposes, we have converted ratios to absolute numbers. If one seeks the correspondence between our numbers and Cowan's own, he will have to use the following conversion formula:

$$K_i = \left(\frac{P^*}{C_o}\right)^{\beta_i} * 100\ rc \quad \ldots\ldots \qquad (6.1)$$

where rc is Cowan's cost ratio

and K_i is the cost coefficient used in this thesis and β_i the exponent.

P^* and C_o are the annual and daily production rates of the plant used by Cowan (365000 tons/yr and 1380 tons/day respectively). Note that C_o is the overdesign daily rate corresponding to a P^*/A_p annual production. Cowan calculated A_p to be 0.72 using Formula (2.1) (the product rule).

The tank coefficient is given by:

$$K_T = \frac{100\ rc}{(C_o)^{\gamma}} \times \Sigma \text{ costs of plant} \quad \ldots\ldots \qquad (6.2)$$

After this necessary introduction, we show the plant data, as

presented by Cowan, after the conversions (6.1) and (6.2) have been made, in Table 6.5. Remember that the actual values are meaningless and only the ratios matter.

We have examined this plant for a variety of cases:

Case (i) corresponds to the recommended configuration of the plant with no buffer options considered.

Case (ii) corresponds to the recommended configuration of the plant with buffer options considered in three tank sites: after elements 3, 6 and 11. As Cowan [17] could only handle one tank per process, her choice of tank sites was limited. In [17] tank options for all three sites are never considered together.

Case (iii) corresponds to the recommended configuration of the plant with buffer options in more tank sites: after elements 1, 2, 3, 4, 6, 11, 12.

Cases (iv) and (v) illustrate the effects of increasing tank costs.

The recommended configurations for cases (i) - (v) are shown in Tables 6.6 - 6.9. The recommended configuration for the more general case (iii) is also shown pictorially in Figure 6.1.

From examination of Tables 6.6 - 6.9, one can readily see that the basis configuration of the plant remains unchanged. The effect of dearer tank costs converts the recommended process configuration back to the one where no tankage is considered. (cf. cases (i) and (v)). Also, the introduction of buffer storage in each site does not affect the optimal configurations of other elements.

One can also see (by examining Examples 1, 2, 3 and the ammonia plant) that standby redundancy options are in general favoured over parallel redundancy options. This is in line with [50].

TABLE 6.5

Element No.	Type of Equipment	MTTF hrs	MTTR hrs	A -	K_i £/ton yr	β_i -
1	Air separation section	756	34.3	.9567	2480	0.6
2	Air compressor	2729	56	.9799	53.7	0.35
3	Steam boiler	2167	106.4	.9532	13285	0.7
4	Electrostatic precipitators	2500	13.4	.9947	1784.4	0.75
5	Raw process gas ($Co\&H_2$)compressor	1008	31	.9702	17336	0.82
6	N_2 gas compressor	1718.4	43.3	.9754	4815.6	0.82
7	N_2 and H_2 compressor	793.4	28.6	.9652	9248.3	0.82
8	O_2 gas blower	3370	40	.9883	6.4	0.68
9	N_2 gas blower	3245	40	.9878	1.4	0.68
10	N_2 gas blower	2746	39.5	.9858	2.7	0.68
11	Dust removal section	3370	40	.9883	76	0.72
12	Coal gasification section	4608	29.4	.9937	11659	0.9
13	Remaining plant section	513.6	23.0	.9571	39947	0.7

Tank sites available:

Between 3 & 4, 6 & 7, 11 & 12

Tank data: K_T = £15.9/ton

γ = 0.65

TABLE 6.6

OPTIMUM CONFIGURATION OF NH_3 PLANT

Case (i) - No Tankage Options

Element No.	Option	Details	Efficiency
1	nx100 standby redundancy	2x100%	.9990
2	nx100 standby redundancy	2x100%	.9998
3	simple overdesign	1x100%	.9532
4	simple overdesign	1x100%	.9947
5	simple overdesign	1x100%	.9702
6	simple overdesign	1x100%	.9754
7	simple overdesign	1x100%	.9652
8	hybrid standby redundancy	4x50%	~1
9	hybrid standby redundancy	4x50%	~1
10	hybrid standby redundancy	4x50%	~1
11	hybrid standby redundancy	3x50%	.9999
12	simple overdesign	1x100%	.9937
13	simple overdesign	1x100%	.9571

Total cost: 119284.4 units
Total efficiency : .8341

For this example we use efficiency instead of availability, since we make use of hybrid redundancy options

TABLE 6.7

OPTIMUM CONFIGURATION OF NH_3 PLANT

Case (ii) - Limited Tankage Option after Elements 3, 6 and 11

Element No.	Option	Details	Efficiency
1	nx100 standby redundancy	2x100%	0.9990
2	nx100 standby redundancy	2x100%	0.9998
3	insertion of buffer storage	V=44.5 hrs production	0.9572
4	1x100%	-	0.9947
5	1x100%	-	0.9702
6	insertion of buffer storage	V_2=228 hrs production	0.9968
7	1x100%	-	0.9652
8	hybrid standby redundancy	4x50%	~1
9	hybrid standby redundancy	4x50%	~1
10	hybrid standby redundancy	4x50%	~1
11	hybrid standby redundancy	3x50%	0.9999
12	1x100%	-	0.9937
13	1x100%	-	0.9571

Total cost of process = 118029.5

Total efficiency = 0.8528

TABLE 6.8

OPTIMUM CONFIGURATION OF NH_3 PLANT

Case (iii) - Extensive Tankage Options after Elements 1,2,3,4,6,11,12

Element No.	Option	Details	Efficiency
1	insertion of buffer storage	V_1=201.3 hrs production	0.9734
2	nx100 standby redundancy	2x100%	0.9998
3	insertion of buffer storage	V_2=43.7 hrs production	0.9572
4	insertion of buffer storage	V_3 = 37.3 hrs production	0.9991
5	overdesign	1x100%	0.9702
6	insertion of buffer storage	V_4=223.9 hrs production	0.9967
7	overdesign	1x100%	0.9652
8	hybrid standby redundancy	4x50%	~1
9	hybrid standby redundancy	4x50%	~1
10	hybrid standby redundancy	4x50%	~1
11	hybrid standby redundancy	3x50%	0.9999
12	insertion of buffer storage	V_5=56.6 hrs production	0.9988
13	overdesign	1x100%	0.9571

Total cost = 117020.5 units

Total efficiency = 0.8408

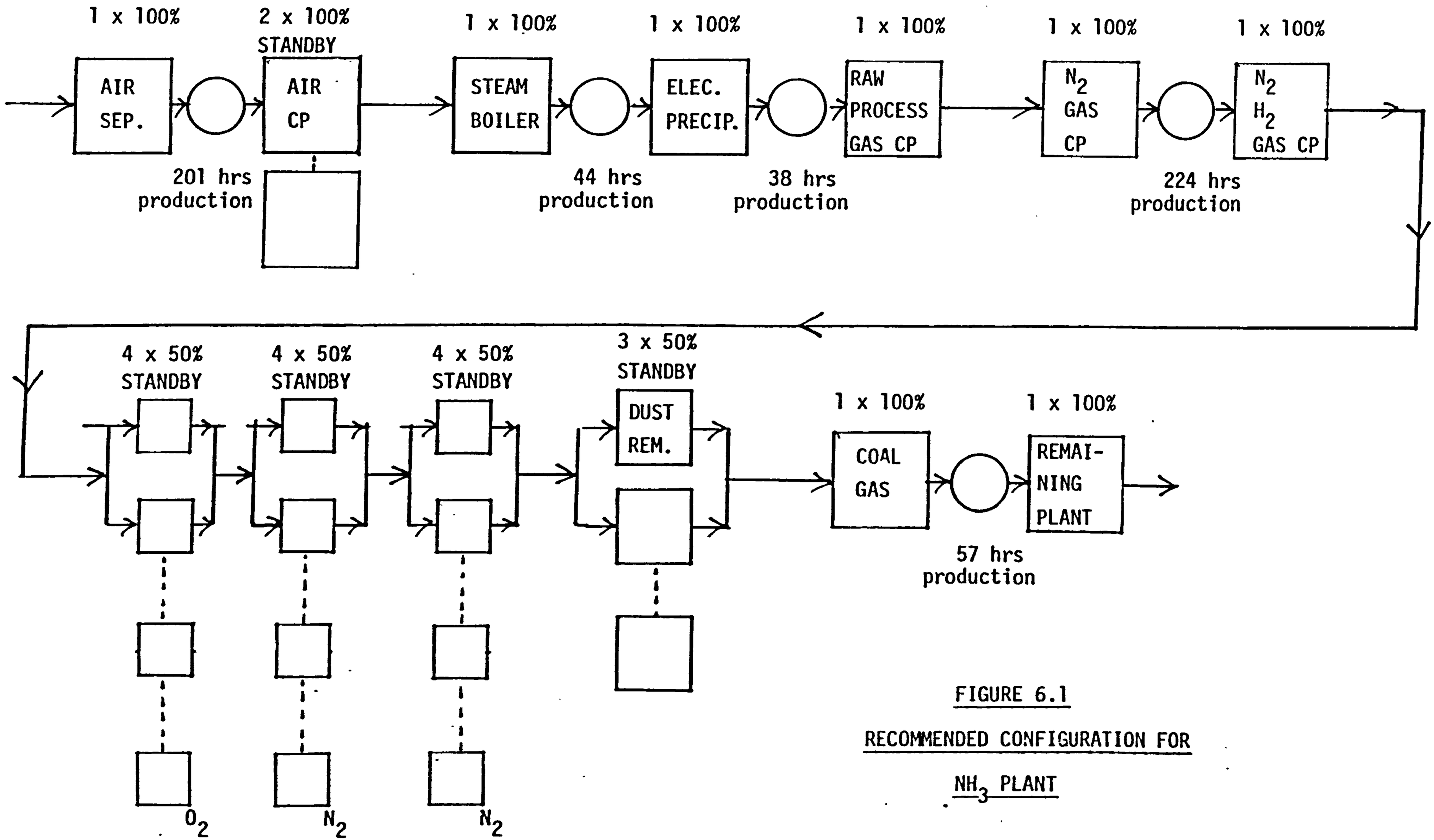


FIGURE 6.1

RECOMMENDED CONFIGURATION FOR

NH_3 PLANT

TABLE 6.9

EFFECT OF COST OF INTERMEDIATE TANKAGE

Tank cost constant	Element	Option	Details	Process cost
K_T=15.9 case(ii)	3	buffer storage insertion	V=44.5 hrs	118029.5 units
	6	buffer storage insertion	V=228 hrs	
	11	hybrid standby redundancy	3x50%	
K_T=31.8 case(iv)	3	overdesign	1x100%	118619.6 units
	6	insertion of buffer storage	V=129 hrs	
	11	hybrid standby redundancy	3x50%	
K_T=47.7 case(v)	3	overdesign	1x100%	119284.8 units
	6	overdesign	1x100%	
	11	overdesign	1x100%	

All other elements keep the options chosen by the algorithm as in case (ii).

6.7 Comparison Between Cowan's Algorithm and the MOP Algorithm

We have tried all examples using both Cowan's algorithm (adapted to include buffer storage options as in [4]) and the MOP algorithm. In all cases they both converged to the same answer. As expected from the analysis in section 5.5.2 the MOP algorithm consistently searched through fewer nodes than Cowan's algorithm. It also used less CP time. The results are presented in Table 6.10.

It should be noted that although Cowan's algorithm may use many more nodes, the MOP algorithm uses fewer but more expensive searches. The reason for this is that it normally starts with the option of buffer storage for a first guess, if this is allowed. Thus, tank volume optimizations are carried out for each node it searches. On the other hand, many of Cowan's algorithm's searches (especially in the beginning) do not involve such tank volume optimizations and are consequently a lot less costly. This can be seen clearly for case (iii) in Table 6.10.

We can reduce the time needed for a tank volume optimization by improving the numerical method used. The method used here is a simple bisection method for the solution of equation (15) in Appendix 4. The bisection method was used for its guaranteed convergence, because equation (15) proved very difficult to solve with other more sophisticated methods (such as secant). However, there is certainly room for improvement. Such improvement would affect the performance of both methods, but the MOP algorithm would benefit the most.

Even so, we can see that the proposed algorithm performs consistently better than the best available algorithm so far, both in terms of searches and CP time. It is therefore recommended for use in design purposes in place of Cowan's algorithm. Although we have not proved that the recommended design is optimal, the fact that both algorithms arrive

TABLE 6.10

COMPARISON BETWEEN MOP & COWAN'S ALGORITHM

Example	MOP		Cowan		Total No. of Nodes
	Nodes	CP Time (sec)	Nodes	CP Time (sec)	
1	9	0.829	37	2.925	7168
2	11	1.012	37	3.405	7168
3	29	7.211	86	20.891	458752_8
NH_3 Plant case(i)	62	16.912	189	19.235	$\sim 8 \times 10^8$
case(ii)	59	30.94	195	66.413	$\sim 8 \times 10^8$
case(iii)	69	59.376	106	64.885	$\sim 8 \times 10^8$
case(iv)	64	31.080	203	41.919	$\sim 8 \times 10^8$
case(v)	64	16.47	193	19.25	$\sim 8 \times 10^8$

TABLE 6.11

EFFECT OF INTERMEDIATE TANKAGE ON COST REDUCTION

Example	Cost Overdesign only	Cost Buffer storage only	Optimal Cost
1	52834.8	49956.7	49956.7
2	60900.6	59699.2	57366.9
3	49357	39934	39894
NH_3 Plant case(ii)	124375.1	118865.3	118029.4
case(iii)	124375.1	118479.25	117020.5
case(iv)	124375.1	118727.4	118619.6
case(v)	124375.1	119388.8	119284.8

Costs in £

at the same result is an encouraging sign.

It is also instructive to study the effect of the intermediate storage option on the improvement of the final process cost. In Table 6.11 we present the costs of the processes examined when only intermediate storage is considered and compare them with the initial costs and the final costs when all options have been considered. It can be seen that insertion of intermediate storage makes by far the most drastic cost reduction in a process. Consideration of all other alternatives leads to only marginally better designs and after a lot of time consumed. Thus, insertion of buffer storage should be the first choice of any designer (or operator) who wants to improve either the availability (or efficiency) of the process.

C H A P T E R S E V E N

CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

The aims of this work have been to model satisfactorily the effects of intermediate storage on the availability of a process and to provide an optimization algorithm to deal with the design of a process when plant failures are taken into account.

The simple machine-tank-machine system was modelled rigorously using stochastic analysis and an analytical expression was derived for its availability. This can be found as equation (3.13) on page 46. Larger systems were approximated using the blocking formula (4.2) in chapter four. The results for various systems are tabulated in Table 4.1, page 71 The approximate results were tested by randomly generating multitank-multimachine systems and calculating their availability using simulation. The difference between the calculated and simulated values was then compared in Table 4.2,pp76-77. As the average relative error was small (3 - 4%) in the region where one would expect substantial errors, the approximations were considered satisfactory.

Ideally, these results should be taken up by researchers in this field and tested with other simulation programs. It would also be interesting if they could be applied to model real processes involving storage. One could then check the applicability of such assumptions as the use of exponential failure and repair distributions, the operating policy adopted for the tanks and the use of balanced systems. One could also test the approximations involved in modelling larger systems.

The question of rigorous models for larger systems remains open. A starting point for their analytic solution could be provided by Appendix 3. If suitable simple boundary conditions could be found,

then one could attempt to solve them directly. One could also envisage the automatic generation of states and equations for larger systems, as the probability differential equations can be seen to follow a pattern. These equations could then be solved numerically. However, work in this area is very complex, and one might have to wait until enough reliability and availability data have been collected in order to make higher accuracy worthwhile.

As regards optimization a branch and bound algorithm was presented on page 93 (the MOP algorithm) which has been compared with the best available algorithm so far. (Cowan's algorithm [16] as modified by Allen and Coker [4]). The results in chapter six are encouraging in that they show a consistent reduction in the number of nodes searched and computer time expended (see Table 6.10, p123).

One of the most striking aspects of this comparison has been the convergence of both algorithms to the same answer in all cases examined. One has to remember that neither algorithm has been proved to be truly optimal, although both lead to improved designs. However, these two algorithms, starting from different nodes, using different stepping rules and different stopping criteria are seen to converge to identical answers in spaces sometimes of the order of 10^8 nodes. There is a strong hint, therefore, that the final result is, indeed, optimal.

Some insight into the nature of the functions involved and a better understanding of how these algorithms work, may lead to further algorithms dealing with more general mixed integer non-linear programming problems. Since this field is notoriously difficult, any contribution to solve even limited classes of problems must be welcome.

The problem of sizing buffer tanks was also examined. It was concluded that other objective functions than the ones used here may provide better answers to this problem. The objective function used

here involves minimizing the capital cost of overdesign. However, one can envisage further costs being added to this objective function: operating costs, maintenance costs and costs of downtime, perhaps discounted over a period of years along with the capital costs used here. The difficulty with inclusion of such costs seems to be in quantifying costs of downtime and the lack of adequate data.

This search for data has been the most frustrating aspect of this work. We would like to have tested more real-life examples in chapter six, if more were available. Cost data have not been generally the problem. It seems, however, that reliability and availability data are both unreliable and unavailable at this point in time. However, there are encouraging signs that databanks may solve this problem in the future. The author has observed in private conversations that the chemical industry is increasingly concerned about this lack of data and is working towards their collection. Let us hope that in the future such data will be available and that methods such as those employed here will be more widely used to design more reliable and cost effective plants.

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L I S T O F S Y M B O L S

A = steady state availability

$A(t)$ = instantaneous availability

$A(T)$ = interval availability

A_{series} = availability of a system in series form

A_T = total plant availability

A_h = availability of a system in hybrid redundancy

A_p = availability of a system in parallel redundancy

A_s = availability of a system in standby redundancy

B_k = base case design at k^{th} stage of algorithm

C_T = cost of a process

C_i = cost of a stage i of a process

C_{ij} = cost of a stage i of a process under option j

C^* = optimal cost of a process

K = cost constant for a piece of equipment

K_T = cost constant for a tank

N = number of stages in series

M = number of stages in parallel

$(MTTF)_i$ = mean time to fail for stage i

$(MTTR)_i$ = mean time to repair for stage i

$P(\ )$ = probability of event ( )

P^o = rated capacity of equipment

P^* = target annual production rate of a plant

$\bar{P}$ = real production rate of a plant

Q_i = $1 - A_i$ = unavailability of stage i

R = reliability

R_1 = $1/\rho$

T = time interval

V = volume of tank

X_t = state variable for Markov chain (defined in 3.1)

Y_t = state variable for Markov chain (defined in 3.1)

b_{ij} = binary variable (defined in equation (5.3))

c = q/V

$\bar{c}$ = defined in (5.22)

c* = defined in (3.16)

k = stage in an algorithm

k_c = defined in Appendix 4

m_i = % throughput of machine i

m, n = used in shorthand configurations of machines/tanks denoting number of machines in series or redundancy (e.g. mM/T/nM system). Also used as an index

q = flowrate through a tank

s = cost parameter defined in (5.25)

t = time

v = defined in equation (5.9)

x = (MTTR/MTTF) for a stage

y = fraction of a tank which is full

z = $\sum_{j \neq i} x_j$ defined in (5.16)

Greek letters

β_i = cost exponent for machine i

γ_i = cost exponent for tank i

Γ = system configuration

θ = defined in (3.17)

λ_i = failure rate of machine i

μ_i = repair rate of machine i

ξ = efficiency (defined in (2.7))

π = steady state probability

ρ = (MTTR/MTTF) for a unit

σ = defined in (5.22)

τ = defined in 4.3.2

ϕ = number of options

Subscripts and Superscripts

i = normally refers to stage or machine

j = normally refers to option

k = normally refers to stage of an algorithm

p = normally indicates parallel redundant system

s = normally indicates standby redundant system

v = normally refers to volume

* = normally refers to optimal

$\uparrow$ = means machine "up"

$\downarrow$ = means machine "down"

Note that auxiliary variables are defined locally for proof purposes especially in the Appendices.

A P P E N D I X 1

DERIVATION OF THE DIFFERENTIAL EQUATIONS

Fix $t \geq 0$, $0 < y < 1$ and consider some infinitesimal time span δt.

To calculate $P_1(t+\delta t, y)$ we consider three possibilites:

(i) $X_t = 1$ and no machine is repaired in $(t, t+\delta t)$

(ii) $X_t = 2$ and M_2 fails in $(t, t+\delta t)$

(iii) $X_t = 3$ and M_1 fails in $(t, t+\delta t)$.

All other cases have probability $o(\delta t)$. Using the assumption of continuity we then have

$$P_1(t+\delta t, y) = P_1(t,y)\{1-\mu_1\delta t-\mu_2\delta t\} + P_2(t,y)\lambda_2\delta t + P_3(t,y)\lambda_1\delta t + o(\delta t).$$

Subtracting $P_1(t,y)$ from both sides, dividing by δt and letting $\delta t \to 0$ we obtain (3.1).

For state 2 writing $\delta y = c\delta t$ we have

$$P_2(t+\delta t, y) = P_2(t, y+\delta y)(1-\mu_1\delta t-\lambda_2\delta t) + P_1(t,y)\mu_2\delta t + P_4(t,y)\lambda_1\delta t + o(\delta t) \to$$

$$\to \frac{P_2(t+\delta t, y) - P_2(t,y)}{\delta t} = \frac{P_2(t, y+\delta y) - P_2(t,y)}{\delta y / c}$$

$$- (\mu_1+\lambda_2)P_2(t, y+\delta y) + \mu_2 P_1(t,y) + \lambda_1 P_4(t,y) + o\,\frac{(\delta t)}{\delta t}$$

and (3.2) is obtained by letting $\delta t \to 0$.

Equations (3.3) and (3.4) are derived similarly.

Now let $y = 0$ and calculate $P_5(t+\delta t)$. Again there are three possibilities, all others having negligible probability.

(i) $X_t = 5$ and M_1 is not repaired in $(t, t+\delta t)$ - note that by assumption M_2 cannot fail when $X_t = 5$.

(ii) $X_t = 6$ and M_1 fails in $(t, t+\delta t)$.

(iii) $X_t=2$, $Y_t<c\delta t$ and no failure or repair occurs in $(t,t+\delta t)$.

Thus we obtain

$$P_5(t+\delta t) = P_5(t)(1-\mu_1\delta t) + P_6(t)\lambda_1\delta t + \int_0^{c\delta t} P_2(t,y)dy + o(\delta t)$$

From continuity $\int_0^{c\delta t} P_2(t,y)dy = c\delta t P_2(t,0) + o(\delta t)$. Subtracting $P_5(t)$ from both sides; dividing by δt and letting $\delta t\to 0$, yields (3.5). $P_6(t+\delta t)$ can be derived in a much simpler way. If both machines are working at time $(t+\delta t)$ the tank can become empty in interval $(t+\delta t)$ only if, in addition to the tank becoming empty, M_1 was repaired.

Hence we have

$$P_6(t+\delta t) = P_6(t)(1-\lambda_1\ t-\lambda_2\delta t) + P_5(t)\mu_1\delta t + o(\delta t)$$

which gives (3.6) by the usual limiting process. The derivation of (3.7) is similar to that of (3.5) and the derivation of (3.8) is similar to that of (3.6).

A P P E N D I X 2

DERIVATION OF THE EQUILIBRIUM PROBABILITIES

It is required to find continuous functions $\pi_i(y)$ $(0\leqslant y\leqslant 1,\ i=1,2,3,4)$ and numbers π_5,π_6,π_7,π_8 to satisfy

$$0 = -(\mu_1+\mu_2)\pi_1+\lambda_2\pi_2+\lambda_1\pi_3 \tag{1a}$$

$$0 = \mu_2\pi_1-(\mu_1+\lambda_2)\pi_2+\lambda_1\pi_4 + c\frac{\partial\pi_2}{\partial y} \tag{2a}$$

$$0 = \mu_1\pi_1-(\lambda_1+\mu_2)\pi_3+\lambda_2\pi_4 - c\frac{\partial\pi_3}{\partial y} \tag{3a}$$

$$0 = \mu_1\pi_2+\mu_2\pi_3-(\lambda_1+\lambda_2)\pi_4 \tag{4a}$$

$$0 = -\mu_1\pi_5+\lambda_1\pi_6+c\pi_2(0) \tag{5a}$$

$$0 = \mu_1\pi_5-(\lambda_1+\lambda_2)\pi_6 \tag{6a}$$

$$0 = -\mu_2\pi_7+\lambda_2\pi_8 + c\pi_3(1) \tag{7a}$$

$$0 = \mu_2\pi_7-(\lambda_1+\lambda_2)\pi_8 \tag{8a}$$

together with boundary conditions 3.10 and 3.11.

Combining (1a) and (4a) we obtain

$$\pi_1 = \frac{\lambda_2\pi_2}{(\mu_1+\mu_2)} + \frac{\lambda_1\pi_3}{(\mu_1+\mu_2)} \quad ; \quad \pi_4 = \frac{\mu_1\pi_2}{(\lambda_1+\lambda_2)} + \frac{\mu_2\pi_3}{(\lambda_1+\lambda_2)} \tag{9a}$$

Substituting for π_1 and π_4 from (9a) in (2a) and (3a) we get

$$\left.\begin{aligned} c\frac{\partial\pi_2}{\partial y} &= \lambda_2\mu_1\delta\pi_2-\lambda_1\mu_2\delta\pi_3 \\ c\frac{\partial\pi_3}{\partial y} &= \lambda_2\mu_1\delta\pi_2-\lambda_1\mu_2\delta\pi_3 \end{aligned}\right\} \tag{10a}$$

where $$\delta = \frac{(\lambda_1+\lambda_2+\mu_1+\mu_2)}{(\lambda_1+\lambda_2)(\mu_1+\mu_2)} \qquad (11a)$$

From (10a) we get that $\frac{\partial \pi_2}{\partial y} = \frac{\partial \pi_3}{\partial y}$ which, in conjunction with boundary condition 3.11 yields

$$\pi_2(y) = \pi_3(y)$$

Combining this result with (10a) we get

$$\frac{\partial \pi_3}{\partial y} = -k\pi_3 \rightarrow \pi_3(y) = \pi_2(y) = \pi_0\, e^{-ky} \qquad (12a)$$

where k has been defined in 3.14 and π_0 is some constant of integration.

From (9a)

$$\pi_1(y) = \frac{(\lambda_1+\lambda_2)\pi_0 e^{-ky}}{(\mu_1+\mu_2)} \quad ; \quad \pi_4 = \frac{(\mu_1+\mu_2)}{(\lambda_1+\lambda_2)}\ \pi_0 e^{-ky} \qquad (13a)$$

Note that
$$\left.\begin{aligned} &\lim_{y\to 0}\pi_2 = \pi_0 \\ &\lim_{y\to 1}\pi_3 = \pi_0 e^{-k} \end{aligned}\right\} \qquad (14a)$$

Adding pairwise (5a) to (6a) and (7a) to (8a) and using (14a) we can solve for π_5, π_6, π_7 & π_8 yielding (15a)

$$\left.\begin{aligned} &\pi_5 = \frac{(\lambda_1+\lambda_2)c\pi_0}{\mu_1\lambda_2} \quad ; \quad \pi_6 = \frac{c}{\lambda_2}\ \pi_0 \\ &\pi_7 = \frac{(\lambda_1+\lambda_2)c\pi_0 e^{-k}}{\lambda_1\mu_2} \quad ; \quad \pi_8 = \frac{c\pi_0}{\lambda_1}\, e^{-k} \end{aligned}\right\} \qquad (15a)$$

Use (3.10) to find π_0 and after some manipulation we arrive at

$$\pi_0 = \left[\frac{(1-e^{-k})}{k} \beta\delta + c\left[\frac{1}{\mu_1} + \frac{e^{-k}}{\mu_2}\right] + c\left[\frac{\rho_1}{\lambda_2} + \frac{\rho_2 e^{-k}}{\lambda_1}\right] + c\left[\frac{1}{\lambda_2} + \frac{e^{-k}}{\lambda_1}\right] \right]^{-1} \qquad (16a)$$

and by using the definition of availability

$$A_k = \frac{\left(\frac{\mu_1+\mu_2}{\lambda_1+\lambda_2} + 1\right)\left(\frac{1-e^{-k}}{k}\right) + c\left(1/\lambda_2 + e^{-k}/\lambda_1\right)}{\left(\frac{1-e^{-k}}{k}\right)\beta\delta + c\left(\frac{1}{\mu_1} + \frac{e^{-k}}{\mu_2}\right) + c\left(\frac{\rho_1}{\lambda_2} + \frac{\rho_2 e^{-k}}{\lambda_1}\right) + c\left(\frac{1}{\lambda_2} + \frac{e^{-k}}{\lambda_1}\right)}$$

where

$$\beta = (\lambda_1+\lambda_2+\mu_1+\mu_2).$$

Multiplying out by k, grouping e^{-k} and constant terms together, and after considerable manipulation we arrive at

$$A_k = \frac{(\rho_1-\rho_2 e^{-k})A_1A_2}{(\rho_1A_2-\rho_2A_1e^{-k})}$$

which is a fairly simple formula for the availability of the system.

A P P E N D I X 3

THE TWO TANK SYSTEM

The problem considered here is:

Given the configuration below

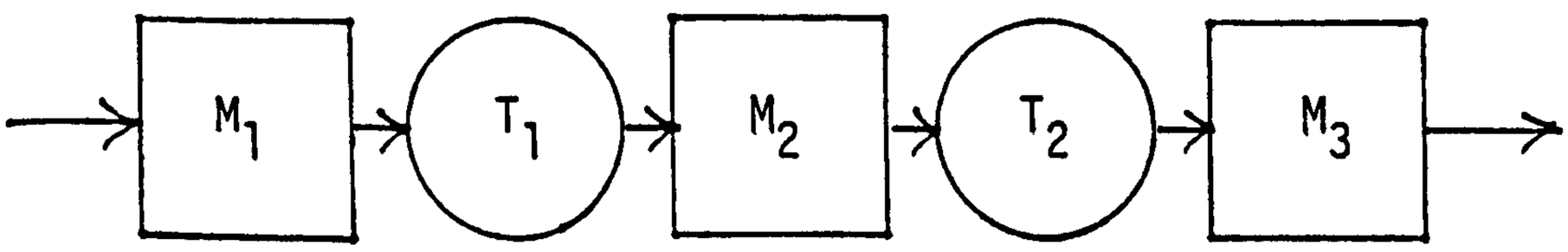


Figure 1

where q = flowrate (= constant)

V_i, $i=1,2$ Volumes of T_1, T_2, $V_1 \neq V_2$

$c_i = q/V_i$, $0<y_i<1$ fraction of V_i which is full

Exponential failure and repair distributions for machine M_i, with failure and repair rates λ_i, μ_i respectively $(i=1,2,3)$

Find the availability of the system.

We adopt the 'do nothing' policy as in chapter three and the same stochastic approach

STATE SPACE (letting ↑ for machine 'on' and ↓ for machine 'off")

state	1	M_1↓	M_2↓	M_3↓	$0<y_1<1$	$0<y_2<1$
	2	M_1↑	M_2↓	M_3↓	"	(T_1 filling)
	3	M_1↓	M_2↑	M_3↓	"	(T_1 emptying/T_2 filling)
	4	M_1↓	M_2↓	M_3↑	"	(T_2 emptying)
	5	M_1↑	M_2↑	M_3↓	"	(T_2 filling)
	6	M_1↑	M_2↓	M_3↑	"	(T_1 filling/T_2 emptying)
	7	M_1↓	M_2↑	M_3↑	"	(T_1 emptying)
	8	M_1↑	M_2↑	M_3↑	"	
state	9	M_1↑	M_2↑	M_3↓	T_1 empty	$0<y_2<1$
	10	M_1↓	M_2↑	M_3↓	"	
	11	M_1↑	M_2↑	M_3↑	"	
	12	M_1↓	M_2↑	M_3↑	"	
state	13	M_1↑	M_2↓	M_3↓	T_1 full	$0<y_2<1$
	14	M_1↑	M_2↑	M_3↓	"	
	15	M_1↑	M_2↓	M_3↑	"	
	16	M_1↑	M_2↑	M_3↑	"	
state	17	M_1↓	M_2↓	M_3↑	$0<y_1<1$	T_2 empty
	18	M_1↓	M_2↑	M_3↑	"	
	19	M_1↑	M_2↓	M_3↑	"	
	20	M_1↑	M_2↑	M_3↑	"	
state	21	M_1↑	M_2↑	M_3↓	$0<y_1<1$	T_2 full
	22	M_1↑	M_2↑	M_3↑	"	
	23	M_1↓	M_2↑	M_3↓	"	
	24	M_1↓	M_2↑	M_3↑	"	
state	25	M_1↑	M_2↑	M_3↑	T_1, T_2 empty	

state	26	$M_1\downarrow$	$M_2\uparrow$	$M_3\uparrow$	T_1, T_2 empty
	27	$M_1\downarrow$	$M_2\uparrow$	$M_3\uparrow$	T_1 empty T_2 full
	28	$M_1\downarrow$	$M_2\uparrow$	$M_3\downarrow$	"
	29	$M_1\uparrow$	$M_2\uparrow$	$M_3\uparrow$	"
	30	$M_1\uparrow$	$M_2\uparrow$	$M_3\downarrow$	"
	31	$M_1\uparrow$	$M_2\uparrow$	$M_3\downarrow$	T_1 full T_2 full
	32	$M_1\uparrow$	$M_2\uparrow$	$M_3\uparrow$	"
	33	$M_1\uparrow$	$M_2\uparrow$	$M_3\uparrow$	T_1 full T_2 empty
	34	$M_1\uparrow$	$M_2\downarrow$	$M_3\uparrow$	

If a tank is empty the state tank empty/next machine down is instaneous with a pdf = 0. The same occurs with the state tank full/previous machine down. These states are not considered.

In fact although states 27 - 30 are allowed, they conflict with the policies adopted. Therefore, they are not considered when we derive their pdf's.

There is a pdf P_i associated with each state i (i=1,34). The differential equations associated with all pdf's are:

<u>states 1 - 8</u>

$$\frac{\partial P_1}{\partial t} = -(\mu_1+\mu_2+\mu_3)P_1 + \lambda_1 P_2 + \lambda_2 P_3 + \lambda_3 P_4$$

$$\frac{\partial P_2}{\partial t} = \mu_1 P_1 -(\lambda_1+\mu_2+\mu_3)P_2 + \lambda_2 P_5 + \lambda_3 P_6 - c_1 \frac{\partial P_2}{\partial y_1}$$

$$\frac{\partial P_3}{\partial t} = \mu_2 P_1 -(\mu_1+\lambda_2+\mu_3)P_3 + \lambda_1 P_5 + \lambda_3 P_7 + c_1 \frac{\partial P_3}{\partial y_1} - c_2 \frac{\partial P_3}{\partial y_2}$$

$$\frac{\partial P_4}{\partial t} = \mu_3 P_1 -(\mu_1+\mu_2+\lambda_3)P_4 + \lambda_1 P_6 + \lambda_2 P_7 + c_2 \frac{\partial P_4}{\partial y_2}$$

$$\frac{\partial P_5}{\partial t} = \mu_2 P_2 + \mu_1 P_3 - (\lambda_1+\lambda_2+\mu_3)P_5 + \lambda_3 P_8 - c_2 \frac{\partial P_5}{\partial y_2}$$

$$\frac{\partial P_6}{\partial t} = \mu_3 P_2 + \mu_1 P_4 - (\lambda_1+\mu_2+\lambda_3)P_6 + \lambda_2 P_8 - c_1 \frac{\partial P_6}{\partial y_1} + c_2 \frac{\partial P_6}{\partial y_2}$$

$$\frac{\partial P_7}{\partial t} = \mu_3 P_3 + \mu_2 P_4 - (\mu_1+\lambda_2+\lambda_3)P_7 + \lambda_1 P_8 + c_1 \frac{\partial P_7}{\partial y_1}$$

$$\frac{\partial P_8}{\partial t} = \mu_3 P_5 + \mu_2 P_6 + \mu_1 P_7 - (\lambda_1+\lambda_2+\lambda_3)P_8$$

states 9 - 12

$$\frac{\partial P_9}{\partial t} = \lambda_3 P_{11} + \mu_1 P_{10} - c_2 \frac{\partial P_9}{\partial y_2} - (\lambda_1+\lambda_2+\mu_3)P_9$$

$$\frac{\partial P_{10}}{\partial t} = \lambda_1 P_9 + \lambda_3 P_{12} - (\mu_1+\mu_3)P_{10} + c_1 P_3(0,y_2)$$

$$\frac{\partial P_{11}}{\partial t} = \mu_1 P_{12} + \mu_3 P_9 - (\lambda_1+\lambda_2+\lambda_3)P_{11}$$

$$\frac{\partial P_{12}}{\partial t} = \mu_3 P_{10} + \lambda_1 P_{11} - (\mu_1+\lambda_3)P_{12} + c_1 P_7(0,y_2) + c_2 \frac{\partial P_{12}}{\partial y_2}$$

states 13 - 16

$$\frac{\partial P_{13}}{\partial t} = -(\mu_2+\mu_3)P_{13} + \lambda_2 P_{14} + \lambda_3 P_{15} + c_1 P_2(1,y_2)$$

$$\frac{\partial P_{14}}{\partial t} = \mu_2 P_{13} + \lambda_3 P_{16} - (\lambda_1+\lambda_2+\mu_3)P_{14} - c_2 \frac{\partial P_{14}}{\partial y_2}$$

$$\frac{\partial P_{15}}{\partial t} = \lambda_2 P_{16} + \mu_3 P_{13} - (\mu_2+\lambda_3)P_{15} + c_2 \frac{\partial P_{15}}{\partial y_2} + c_1 P_6(1,y_2)$$

$$\frac{\partial P_{16}}{\partial t} = \mu_2 P_{15} + \mu_3 P_{14} - (\lambda_1+\lambda_2+\lambda_3)P_{16}$$

states 17 - 20

$$\frac{\partial P_{17}}{\partial t} = \lambda_1 P_{19} + \lambda_2 P_{18} - (\mu_1+\mu_2)P_{17} + c_2 P_4(y_1,0)$$

$$\frac{\partial P_{18}}{\partial t} = \lambda_1 P_{20} + \mu_2 P_{17} - (\mu_1+\lambda_2+\lambda_3)P_{18} + c_1 \frac{\partial P_{18}}{\partial y_1}$$

$$\frac{\partial P_{19}}{\partial t} = \mu_1 P_{17} + \lambda_2 P_{20} - (\lambda_1+\mu_2)P_{19} - c_1 \frac{\partial P_{19}}{\partial y_1} + c_2 P_6(y_1,0)$$

$$\frac{\partial P_{20}}{\partial t} = \mu_1 P_{18} + \mu_2 P_{19} - (\lambda_1+\lambda_2+\lambda_3)P_{20}$$

states 21 - 24

$$\frac{\partial P_{21}}{\partial t} = \mu_1 P_{23} + \lambda_3 P_{22} + c_2 P_5(y_1,1) - (\lambda_1+\mu_3)P_{21} - c_1 \frac{\partial P_{21}}{\partial y_1}$$

$$\frac{\partial P_{22}}{\partial t} = \mu_3 P_{21} + \mu_1 P_{24} - (\lambda_1+\lambda_2+\lambda_3)P_{22}$$

$$\frac{\partial P_{23}}{\partial t} = \lambda_1 P_{21} + \lambda_3 P_{24} - (\mu_1+\mu_3)P_{23} + c_2 P_3(y_1,1)$$

$$\frac{\partial P_{24}}{\partial t} = \lambda_1 P_{22} + \mu_3 P_{23} - (\mu_1+\lambda_2+\lambda_3)P_{24} + c_1 \frac{\partial P_{24}}{\partial y_1}$$

states 25 - 34

$$\frac{dP_{25}}{dt} = \mu_1 P_{26} - (\lambda_1+\lambda_2+\lambda_3)P_{25}$$

$$\frac{dP_{26}}{dt} = \lambda_1 P_{25} - \mu_1 P_{26} + c_1 P_{18}(0) + c_2 P_{12}(0)$$

$$\frac{dP_{31}}{dt} = \lambda_3 P_{32} - \mu_3 P_{31} + c_2 P_{14}(0) + c_1 P_{21}(1)$$

$$\frac{dP_{32}}{dt} = \mu_3 P_{31} - (\lambda_1+\lambda_2+\lambda_3)P_{32}$$

$$\frac{dP_{33}}{dt} = \mu_2 P_{34} - (\lambda_1+\lambda_2+\lambda_3)P_{33}$$

$$\frac{dP_{34}}{dt} = \lambda_2 P_{33} - \mu_2 P_{34} + c_1 P_{19}(1) + c_2 P_{15}(0)$$

The equations are derived in the pattern of Appendix 1.

To check the consistency of the system of equations from the flow point of view we need to show that

$$\sum_{i=1}^{8} \int_0^1 \int_0^1 \frac{\partial P_i}{\partial t}(y_1,y_2)dy_1dy_2 + \sum_{9}^{16} \int_0^1 \frac{\partial P_i}{\partial t}(y_2)dy_2 + \sum_{17}^{24} \int_0^1 \frac{\partial P_i}{\partial t}(y_1)dy_1 +$$

$$+ \sum_{25}^{34} \frac{dP_i}{dt} = 0$$

In fact after adding all equations, the above condition is true because

$$\lambda_2 P_{25} = c_1 P_{19}(0) \quad ; \quad \lambda_1 P_{32} = c_1 P_{24}(1) \quad ; \quad \lambda_2 \int_0^1 P_9(y_2)dy_2 =$$

$$= c_1 \int_0^1 P_2(0,y_2)dy_2 \quad ; \quad \lambda_2 \int_0^1 P_{11}(y_2)dy_2 = c_1 \int_0^1 P_6(0,y_2)dy_2 \quad ;$$

$$\lambda_1 \int_0^1 P_{14}(y_2)dy_2 = c_1 \int_0^1 P_3(1,y_2)dy_2 \quad ; \quad \lambda_1 \int_0^1 P_{16}(y_2)dy_2 =$$

$$= c_1 \int_0^1 P_7(1,y_2)dy_2 \quad ; \quad \lambda_3 \int_0^1 P_{18}(y_1)dy_1 = c_2 \int_0^1 P_3(y_1,0)dy_1 \quad ;$$

$$\lambda_3 \int_0^1 P_{20}(y_1)dy_1 = c_2 \int_0^1 P_5(y_1,0)dy_1 \quad ; \quad \lambda_2 \int_0^1 P_{24}(y_1)dy_1 =$$

$$= c_2 \int_0^1 P_4(y_1,1)dy_1 \quad ; \quad \lambda_2 \int_0^1 P_{22}(y_1)dy_1 = c_2 \int_0^1 P_6(y_1,1)dy_1 \quad ;$$

$$\lambda_1 P_{33} = c_1 P_{18}(1) \quad ; \quad \lambda_3 P_{25} = c_1 P_9(0) \quad ; \quad \lambda_2 P_{32} = c_2 P_{15}(1) \quad ;$$

$$\lambda_3 P_{33} = c_2 P_{14}(0) \quad ; \quad c_2 P_9(1) = c_1 P_{21}(0) \quad ; \quad c_2 P_{12}(0) = c_1 P_{24}(0)$$

The real difficulty of the problem is to find the relevant boundary conditions. Clearly for a steady state

$$\frac{\partial P_i}{\partial t} = 0 \ , \quad \text{all } i$$

and from probability considerations

$$\int_0^1 \int_0^1 \sum_{i=1}^{8} \pi_i(y_1,y_2)dy_1dy_2 + \int_0^1 \sum_{9}^{16} \pi_1(y_2)dy_2 +$$

$$+ \int_0^1 \sum_{17}^{24} \pi_i(y_1)dy_1 + \sum_{i=25}^{32} \pi_i = 1 \qquad \text{(B.C.1)}$$

(2 conditions on y_1, y_2)

The availability function is:

$$A = \int_0^1 \int_0^1 (\pi_4+\pi_6+\pi_7+\pi_8)dy_1dy_2 + \int_0^1 (\pi_{11}+\pi_{12}+\pi_{15}+\pi_{16})dy_2$$

$$+ \int_0^1 (\pi_{18}+\pi_{20}+\pi_{22}+\pi_{24})dy_1 + \pi_{25} + \pi_{32} + \pi_{33}$$

Suppose we approach the problem as before, namely that as $t\to\infty$ all material through M_1 will pass through M_2 and the time the level any tank is rising = the time the level is falling.

A pointwise equality of probabilities would yield

$$P_7 + P_3 = P_2 + P_6 \qquad \text{BC2} \quad (2 \text{ conditions on } y_1, y_2)$$

$$P_{18} + P_{24} = P_{19} + P_{21} \qquad \text{BC3} \quad (1 \text{ condition on } y_1)$$

$$P_4 + P_6 = P_3 + P_5 \qquad \text{BC4} \quad (2 \text{ conditions on } y_1, y_2)$$

$$P_9 + P_{14} = P_{12} + P_{15} \qquad \text{BC5} \quad (1 \text{ condition on } y_2)$$

During P_3 and P_6 there is no accumulation over the two tanks:

$$c_1 \frac{\partial P_3}{\partial y_1} = c_2 \frac{\partial P_3}{\partial y_2} \; ; \; c_1 \frac{\partial P_6}{\partial y_1} = c_2 \frac{\partial P_6}{\partial y_2} \qquad \text{BC6} \quad (2 \text{ conditions})$$

we also have no accumulation in the system as $t\to\infty$, so:

$$c_1 \frac{\partial P_7}{\partial y_1} = c_2 \frac{\partial P_5}{\partial y_2} \quad \text{and} \quad c_1 \frac{\partial P_2}{\partial y_1} = c_2 \frac{\partial P_4}{\partial y_2} \qquad \text{BC7} \quad (2 \text{ conditions})$$

Overall we have 16 conditions for the 16 remaining differential equations on y_1, y_2. We could attempt to solve the problem.

If we do that we obtain a set of solutions in the form

$$\text{const} \times \exp(-k_1y_1) \exp(-k_2y_2)$$

for probabilities $1\to8$.

Unfortunately, these solutions that satisfy this set of boundary

conditions and equations 1→8, do not satisfy the consistency equations.

It seems therefore that integral equations have to be employed as boundary conditions. We would need the 8 consistency equations involving integrals as 8 boundary conditions plus boundary conditions 1 to 5 (8 conditions). We can not use BC6 and BC7 as one would expect. Thus, the problem becomes extremely complicated, and approximations are indeed a more attractive proposition.

A P P E N D I X 4

FORM OF COST FUNCTIONS

In this Appendix, we shall try to examine whether the cost functions corresponding to the different options considered in chapters five and six are convex. If they are, the stopping criterion in section 5.4.3 is valid.

A function F(x), $x \epsilon I \subset R$ is convex if its second derivative is positive for all $x \epsilon I$. It can be easily shown that a function which is a linear combination of convex functions is itself a convex function. Since the overall cost function C_T in chapter five is the sum of cost functions corresponding to different options, we need only cosider whether the cost functions corresponding to different options are convex.

As a convex function plus a constant yields again a convex function, we need not consider constant terms in C_T such as the cost of options of hybrid redundancy. Thus, the only options we need to examine are: active nx100% redundancy, standby nx100% redundancy, buffer storage insertion, active redundancy followed by buffer storage and standby redundancy followed by buffer storage. The notation we use here is consistent with that employed in chapters two to six. Since the cost functions are the same for all stages i=1, .. N for ease of notation we shall drop the subscript i from each variable.

(1) Active Redundancy nx100%

For M pieces in parallel, according to (2.4)

$$Ap(M) = 1 - Q^M$$

Q = unavailability of each piece of equipment in parallel (assuming that they are all similar)

Then

$$Ap = \frac{1}{1+x} \quad ; \quad x = \frac{Q^M}{1-Q^M} \Rightarrow M = \frac{1}{\ln Q} \ln\left(\frac{x}{1+x}\right)$$

Thus

$$C = M\ K\ Ap^{-\beta} = \frac{K}{\ln Q} \ln\left(\frac{x}{1+x}\right) (1+x+z)^{\beta}$$

$$\simeq \frac{K}{\ln Q} \ln\left(\frac{x}{1+x}\right) (1+\beta(x+z)) \qquad \ldots\ldots\ldots (1)$$

since $x,z \ll 1$

Min C(x,z) is at z=0, as we have mentioned in 5.4.1, a result easy $z \geq 0$
to prove using Lagrange multipliers.

Thus, we look at the behaviour of the function with respect to the variable we are optimising, namely x.

$$\frac{dC}{dx} = \frac{K}{\ln Q} \left[\beta \ln\left(\frac{x}{1+x}\right) + \frac{1+\beta(x+z)}{x(1+x)} \right] \qquad \ldots\ldots\ldots (2)$$

negative positive

and

$$\frac{d^2C}{dx^2} = \frac{K}{\ln Q} \left[- \frac{2x-1-2\beta zx-\beta z}{x^2(1+x)^2} \right] \qquad \ldots\ldots\ldots (3)$$

Since $\ln Q < 0$ $(Q<1)$ and x,z positive, we have that

$$\frac{d^2C}{dx^2} > 0$$

Thus C is convex with respect to x and C exhibits a minimum.

(2) <u>Standby Redundancy nx100%</u>

For M-1 pieces in standby redundancy, we have again

$$C = M\ K\ A_s^{-\beta} \text{ , M integer} \qquad \ldots\ldots\ldots (4)$$

where

$$A_s = 1 - \left(\sum_{j=0}^{M} \frac{M}{j!} \frac{1}{\rho^{M-j}} \right)^{-1} \quad ; \quad \rho = \frac{1-A}{A} \quad \text{for each unit.}$$

We need to show that the discrete function C exhibits a unique minimum with respect to M. To show this we shall use the following lemma.

Lemma 1

If a continuous convex function $\phi(x)$, $x \epsilon I \subset R$ approximates a discrete function $f(x_n)$, $x_n \epsilon Z \subset I$ in such a way that the following conditions are satisfied

a) $\phi(x_n) = f(x_n) - \epsilon(x_n)$, all n, $\epsilon_n(x_n) > 0$.

b) If for some m

$$f(x_m) < f(x_{m+1}),$$

then f is monotonic for m'>m.

Then:

The function $f(x_n)$ exhibits a unique global minimum with no local minima.

Proof

We shall first show that a minimum exists. If $f(x_n)$ is monotonically increasing, clearly $f(x_1)$ is the global minimum. Suppose $f(x_n)$ does not have a minimum. Then it is monotonically decreasing for all x_n. Since ϕ has a minimum, say at x*, $\phi(x) > \phi(x^*)$, for $x \neq x^*$. However, if $f(x_n)$ is monotonically decreasing, at some n, say n*, we shall have $f(x_n^*) < \phi(x^*)$. So, for $n>n^*$ we shall have $\phi(x_n) = f(x_n) + \epsilon(x_n)$. This means that $\epsilon(x_n)$ for $n>n^*$ is negative, something which contradicts condition (a). So $f(x_n)$ has a minimum.

Let this minimum be n_1. Then $f(n_1+1) > f(n_1)$. So, from condition (b) $f(n_1+k) > f(n_1+k-1)$ for $k>1$, and $f(x_n)$ is monotonic from then on. So n_1 is a unique minimum with no other local minima in existence. (QED).

We shall try to find a function $\phi(x)$ that satisfies the conditions of lemma 1. First we shall show that condition (b) is satisfied for our discrete function C.

From (4)

$$C = M\,K\,A_s^{-\beta} = M\,K\left(1 - \frac{1}{\sum_{j=0}^{M} \frac{M}{j!}\,\rho^{j-M}}\right)^{-\beta}$$

$$\cong M\,K\left(1 + \frac{\beta}{\sum_{j=0}^{M} \frac{M}{j!}\,\rho^{j-M}}\right)$$

Suppose at some M, $C(M) < C(M+1)$. Let $\Sigma(M) = \sum_{j=0}^{M} \frac{\rho^j}{j!}$ Then:

$$M\,K\left(1 + \frac{\beta\rho^M}{M\Sigma(M)}\right) < (M+1)\,K\left(1 + \frac{\beta\rho^{M+1}}{(M+1)\,\Sigma(M+1)}\right)$$

or

$$\frac{\beta\rho^M}{\Sigma(M)} < 1 + \frac{\beta\rho^{M+1}}{\Sigma(M+1)} \qquad \ldots\ldots\ldots (5)$$

We want to show that

$$\frac{\beta\rho^{M+1}}{\Sigma(M+1)} < 1 + \frac{\beta\rho^{M+2}}{\Sigma(M+2)} \qquad \ldots\ldots\ldots (6)$$

Multiplying (5) by ρ and noting that $\rho<1$ we have

$$\frac{\beta\rho^{M+1}}{\Sigma(M)} < 1 + \frac{\beta\rho^{M+2}}{\Sigma(M+1)} \qquad \ldots\ldots\ldots (7)$$

We also note that $\frac{\Sigma(M)}{\Sigma(M+1)} < 1$. Thus (7) becomes

$$\frac{\beta\rho^{M+1}}{\Sigma(M+1)} < 1 + \frac{\beta\rho^{M+2}}{\Sigma(M+1)} < 1 + \frac{\beta\rho^{M+2}}{\cancel{\Sigma(M+1)}} \cdot \frac{\cancel{\Sigma(M+1)}}{\Sigma(M)} \rightarrow$$

$$\rightarrow \frac{\beta\rho^{M+1}}{\Sigma(M+1)} < 1 + \frac{\beta\rho^{M+2}}{\Sigma(M+2)} \qquad \text{(QED).}$$

So, our cost function satisfies condition (b). Thus, we need only find a continuous function $\phi(x)$ which is convex and satisfies condition (a) in order to show that our discrete function C has a unique minimum (and no local ones).

We have that

$$\sum_{j=0}^{M} \frac{M}{j!} \rho^{j-M} \cong \frac{M}{\rho^M} e^{\rho} \quad \text{for } \rho<1 \text{ and n large.}$$

Consider the function $A' = 1 - \frac{\rho^M e^{-\rho}}{M}$ $_{M\epsilon R}$ which approximates A_s. From (4) we get

$$C = M K A_s^{-\beta}$$

$$\frac{dC}{dA} = -\beta M K A_s^{-(\beta+1)} = -\beta \frac{C}{A}$$

So $\delta C = \frac{dC}{dA} \delta A$; $\frac{\delta C}{C} = -\beta \frac{\delta A}{A}$

Now $\delta A = A' - A_s > 0$ for all n. Thus $\frac{\delta C}{C} < 0$, all n. This is what we require in condition (a), since δC = C(approx) - C(discrete) or in the notation of the lemma $\phi_n < f_n$, so that $\varepsilon_n > 0$. Note that in this case, the error is $\frac{\delta C}{C} < -\beta \frac{\delta A}{A}$ and since $\frac{\delta A}{A} \to 0$ as $n \to \infty$, $\frac{\delta C}{C} \to 0$ as n increases, which is an extra bonus. We now need to show that C"approximate" is convex.

Let $\frac{1}{1+x} = 1 - \frac{e^{-\rho}\rho^M}{M}$; $x = \frac{e^{-\rho}\rho^M}{M - e^{-\rho}\rho^M}$; $M = \left(\frac{1+x}{x}\right) e^{-\rho}\rho^M$ . . . (8)

Thus x is an implicit function of M and vice versa (we are now considering <u>real</u> M as the approximation). We can then find the derivatives of M with respect to x.

So:

$$C \cong \left(\frac{1+x}{x}\right) e^{-\rho}\rho^M K(1+\beta(x+z))$$

$$\frac{dx}{dM} = \frac{e^{\rho}\rho^M(M\ln\rho - 1)}{(Me^{\rho} - \rho^M)^2} \qquad \text{(from (8))}$$

$$\text{Thus} \quad \frac{dM}{dx} = \frac{(Me^{\rho} - \rho^{M})^2}{e^{\rho}\rho^{M}(M\ln\rho - 1)} < 0 \quad \text{as } \ln\rho < 0 \qquad \ldots\ldots\ldots (9)$$

So,

$$\frac{dC}{dx} = -\frac{1}{x^2} e^{-\rho}\rho^{M} K(1+(x+z)\beta) + \left(\frac{1+x}{x}\right) e^{-\rho} K \rho^{M} \ln\rho \frac{dn}{dx} (1+\beta(x+z))$$

$$+ \beta\left(\frac{1+x}{x}\right) e^{-\rho}\rho^{M} K \qquad \ldots\ldots\ldots\ldots (10)$$

(10) can be written as:

$$\frac{dC}{dx} = e^{-\rho}\rho^{M} K \left[-\frac{(1+\beta(x+z))}{x^2} + \left(\frac{1+x}{x}\right) \ln\rho \frac{dM}{dx} (1+\beta(x+z)) + \beta\left(\frac{1+x}{x}\right) \right] \quad \ldots (11)$$

So:

$$\frac{d^2C}{dx^2} = e^{-\rho}\rho^{M} K \left[\frac{2}{x^3} (1+\beta(x+z)) - \frac{\beta}{x^2} - \frac{1}{x^2} \ln\rho \frac{dM}{dx} (1+\beta(x+z)) \right.$$

$$\left. + \left(\frac{1+x}{x}\right)\ln\rho \frac{d^2n}{dx^2} (1+\beta(x+z)) + \left(\frac{1+x}{x}\right)\ln\rho \frac{dM}{dx} \beta - \frac{\beta}{x^2} \right]$$

$$+ e^{-\rho}\rho^{M} K \frac{dM}{dx} \ln\rho \left[-\frac{(1+\beta(x+z))}{x^2} + \left(\frac{1+x}{x}\right)\ln\rho \frac{dM}{dx} (1+\beta(x+z)) \right.$$

$$\left. + \beta\left(\frac{1+x}{x}\right) \right] \qquad \ldots\ldots\ldots\ldots (12)$$

(12) can be written as

$$\frac{d^2C}{dx^2} = e^{-\rho}\rho^{M} K(A + B + C)$$

where

$$A = \frac{2}{x^3} (1+\beta(x+z)) - 2\frac{\beta}{x^2} = 2\frac{(1+\beta z)}{x^3} > 0$$

$$B = \left(\frac{1+x}{x}\right)\ln\rho \frac{d^2M}{dx^2}(1+\beta(x+z)) + \left(\frac{1+x}{x}\right)\ln\rho \frac{dM}{dx}(1+\beta(x+z)) + \beta\left(\frac{1+x}{x}\right)$$

B is positive since $\ln\rho$, $\frac{dM}{dx} < 0$ and

$$\frac{d^2M}{dx^2} = -\frac{e^{-\rho}\rho^{M}(M\ln\rho - 1)\ln\rho}{\rho^{2\rho}\rho^{2M}(M\ln\rho - 1)^2} < 0.$$

$$C = -\frac{2}{x^2}\ln\rho\,\frac{dM}{dx}\,(1+(x+z)\beta) + (\frac{1+x}{x})\ln\rho\,\frac{dM}{dx}\,\beta$$

C is negative. However A+C is positive, since

$$A+C = \frac{2\ln\rho x\rho^M(\beta x^2-1-\beta z) + 2(1+\beta z)(M\ln\rho-1)}{x^3} \quad \ldots\ldots (13)$$

We have made use of the fact that (9) can be written as

$$\frac{dM}{dx} = \frac{\rho^M}{x^2(M\ln\rho-1)}$$

(13) is equal to:

$$\frac{2\rho^M\ln\rho\beta x^3 - 2\ln\rho x\rho^M - 2\beta z\ln\rho x\rho^M + 2M\ln\rho + 2\beta zM\ln\rho - 2 - 2\beta z}{x^3(M\ln\rho-1)}$$

Neglecting terms of order $x^n\rho^M$ since $x,\rho << 0$ we get

$$\frac{2(M\ln\rho+\beta zM\ln\rho-1-\beta z)}{x^3(M\ln\rho-1)} > 0$$

So

$$\frac{d^2C}{dx^2} = e^{-\rho}\rho^M\,K(A+B+C) > 0$$

So, the continuous approximation to the discrete cost function is convex. Since all conditions to lemma 1 have been fulfilled, we have shown that the discrete cost function has a unique minimum with no local minima (which amounts to convexity in the case of discrete functions).

(3) <u>Insertion of tank after equipment i, with a single equipment in place i+1</u>

We have that (from chapter 3)

$$\frac{1}{1+x+\rho_2} = \frac{(\rho_1-\rho_2e^{-k})A_1A_2}{(\rho_1A_2-\rho_2A_1e^{-k})} \qquad (x = \rho_i' \text{ in chapter 4})$$

$$x = \frac{\rho_1(\rho_1-\rho_2)}{(\rho_1-\rho_2e^{-k})} \qquad \text{Let } U = \rho_1(\rho_1-\rho_2) \;;\; k_c = k/V$$

$$x = \frac{U}{\rho_1 - \rho_2 e^{-k_c V}} \quad \text{or} \quad V = -\frac{1}{k_c} \ln\left(\frac{\rho_1 x - U}{\rho_2 x}\right) \quad \ldots\ldots(14)$$

and $\rho_1 x - U > 0$

The cost of the tank is of the form $K_T V^\gamma$ so the cost of both machines up and downstream is

$$C = K(1+x+z)^\beta + K_T V^\gamma \simeq K(1+\beta(x+z)) + K_T V^\gamma$$

$$\frac{dC}{dx} = K\beta + K_T \gamma V^{\gamma-1} \left(\frac{dV}{dx}\right) = 0 \quad \ldots\ldots\ldots\ldots(15)$$

$$\frac{dV}{dx} = -\frac{1}{k_c} \frac{U}{(\rho_1 x - U)x} < 0 \quad \text{since } \frac{U}{k_c} > 0 \text{ always}$$

and

$$\frac{dC}{dx} = K\beta - \frac{K_T}{k_c} \gamma V^{\gamma-1} \frac{U}{(\rho_1 x - U)x} \quad \ldots\ldots\ldots\ldots(16)$$

$$\frac{d^2C}{dx^2} = \frac{K_T}{k_c} \frac{U}{x^2} \gamma \frac{V^{\gamma-1}}{(\rho_1 x - U)^2} \left(\frac{(\gamma-1)U}{k_c V} + (2\rho_1 x - U) \right) \quad \ldots\ldots\ldots(17)$$

The term outside the brackets is positive. The term inside the brackets is

$$2\rho_1 x - U + \frac{(\gamma-1)U}{k_c V} \quad \ldots\ldots\ldots\ldots(18)$$

$(2\rho_1 x - U) > 0$ from (14) but $\frac{(\gamma-1)U}{k_c V} < 0$.

V is of course a function of x. As $V \to 0$ the second term in (18) becomes very large and negative. So $\frac{d^2C}{dx^2}$ becomes negative for $(2\rho_1 x - U) < \frac{(1-\gamma)U}{k_c V}$ or $V < \frac{(1-\gamma)U}{k_c(2\rho_1 x - U)}$.

However, an order of magnitude analysis shows the following

$$(1-\gamma) = o(10^{-1})$$

$$U = o(\rho_1{}^2) \qquad \rho_1 \sim o(10^{-2}, 10^{-3})$$

$$2\rho_1 x - U = o(\rho_1)$$

$$k_c = o(1/\lambda) \qquad \lambda \sim o(10^{-3}, 10^{-2})$$

So for $\frac{d^2C}{dx^2}$ to turn negative we must have

$$V < \frac{(1-\gamma)U}{k_c(2\rho_1 x - U)} \simeq o(10^{-1} \cdot 10^{-2} 10^{-3}) \simeq o(10^{-4})$$

Thus $\frac{d^2C}{dx^2}$ is positive for the whole range of $V>10^{-4}$, and turns negative around 10^{-4}. So, for all practical purposes $V>0$ we can work with it as if it were convex all through, since a value of $V=0.1$ would indicate that the option of buffer storage is not a good alternative, anyway.

Note that corresponding to the limits $V\to 0$ and $V\to\infty$, there are limits for x. From (3.21):

$$\frac{1}{1+\rho_1+\rho_2} \leqslant \frac{1}{1+x+\rho_2} \leqslant \min\left\{ \frac{1}{1+\rho_1}, \frac{1}{1+\rho_2} \right\}$$

or

$$\max\left\{ 0, (\rho_1-\rho_2) \right\} \leqslant x \leqslant \rho_1 \qquad \ldots\ldots\ldots\ldots (19)$$

$$(\lim V\to\infty) \qquad\qquad (\lim V\to 0)$$

(4) Parallel redundancy followed by buffer storage

Here, we have two parameters determing x: M, the pieces of equipment in parallel and V the volume of the tank following them. From chapter four:

$$\rho_{vi} = \frac{U}{\rho_i - \rho_{M+1}} e^{-k_c V}$$

Assuming that all pieces of equipment in parallel are similar:

$$Ap = 1 - \left(\frac{\rho_{vi}}{1+\rho_{vi}}\right)^{M} = 1 - \left(\frac{U}{U+\rho_i-\rho_{M+1}}e^{-k_c V}\right)^{M}$$

$$x = \frac{1-Ap}{Ap} \rightarrow \left(\frac{x}{1+x}\right)^{1/M} = \frac{U}{U+\rho_i-\rho_{M+1}}e^{-k_c V}$$

$$V = -\frac{I}{k_c} \ln\left[\frac{\rho_i}{\rho_{M+1}} - \left[\left(\frac{1+x}{x}\right)^{1/M} - 1\right]\frac{U}{\rho_{M+1}}\right] \quad \ldots\ldots (20)$$

(20) reduces to (14) for M=1 as expected. The final cost of the system is:

$$C = M\,K(1+\beta x+\beta z) + K_T V^{\gamma}$$

$$\frac{dC}{dx} = M\,K\beta + K_T\,\gamma\,V^{\gamma-1}\,\frac{dV}{dx} \quad \ldots\ldots\ldots\ldots (21)$$

where from (20)

$$V = -\frac{1}{k_c} \ln\left(b - c\left(\frac{1+x}{x}\right)^{1/M}\right)$$

with

$$b = \frac{\rho_i}{\rho_{M+1}} + \frac{U}{\rho_{M+1}} \quad ; \quad c = \frac{U}{\rho_{M+1}}$$

Thus

$$\frac{dV}{dx} = -\frac{1}{k_c\left(b - c\left(\frac{1+x}{x}\right)^{1/M}\right)}\,\frac{c}{Mx^2}\left(\frac{1+x}{x}\right)^{\frac{1-M}{M}} < 0$$

as $\frac{c}{k_c}$ is always positive.

In differentiating we have kept M constant. We are only interested in the behaviour of the function with respect to V. This happens, because in practice we optimize with respect to M and V, by enumerating M=2,3,4,5 and finding V opt at each stage. Thus, we are only interested in the convexity of the function as regards V.

$$\frac{d^2C}{dx^2} = K_T\gamma(\gamma-1)V^{\gamma-2}\left(\frac{dV}{dx}\right)^2 + K_T\gamma V^{\gamma-1}\frac{d^2V}{dx^2} \quad \ldots\ldots (22)$$

$$\frac{d^2V}{dx^2} = \frac{c}{k_c(b-c(\frac{1+x}{x})^{1/M})^2}\left(b-c(\frac{1+x}{x})^{\frac{1}{M}}\right)'\left(\frac{1+x}{x}\right)^{\frac{1-M}{M}}\cdot\frac{1}{Mx^2} +$$

$$+\frac{c}{k_c(b-c(\frac{1+x}{x})^{\frac{1}{M}})}\frac{2}{Mx^3}\left(\frac{1+x}{x}\right)^{\frac{1-M}{M}}$$

$$+\frac{c}{k_c(b-c(\frac{1+x}{x})^{\frac{1}{M}})}\frac{1}{Mx^4}\left(\frac{1-M}{M}\right)\left(\frac{1+x}{x}\right)^{\frac{1-2M}{M}} \quad \ldots\ldots\ldots(23)$$

Substituting (19) in (18) we get.

$$\frac{d^2C}{dx^2} = \frac{K_T\gamma(\gamma-1)V^{\gamma-2}}{k_c^2(b-c(\frac{1+x}{x})^{\frac{1}{M}})^2}\frac{c^2}{M^2x^2}\left(\frac{1+x}{x}\right)^{2(\frac{1-M}{M})} \qquad (= A_1)$$

$$+\frac{K_T\gamma V^{\gamma-1}c}{k_c(b-c(\frac{1+x}{x})^{\frac{1}{M}})^2}\left(b-c(\frac{1+x}{x})^{\frac{1}{M}}\right)'\left(\frac{1+x}{x}\right)^{\frac{1-M}{M}}\frac{1}{Mx^2} \qquad (= A_2)$$

$$+\frac{K_T\gamma V^{\gamma-1}c}{k_c(b-c(\frac{1+x}{x})^{\frac{1}{M}})}\frac{2}{Mx^3}\left(\frac{1+x}{x}\right)^{\frac{1-M}{M}} \qquad (= A_3)$$

$$+\frac{K_T\gamma V^{\gamma-1}c}{k_c(b-c(\frac{1+x}{x})^{\frac{1}{M}})}\frac{1}{Mx^4}\left(\frac{1-M}{M}\right)\left(\frac{1+x}{x}\right)^{\frac{1-2M}{M}} \quad (= A_4) \quad \ldots\ldots(24)$$

$$A_3 + A_4 = \frac{K_T\gamma V^{\gamma-1}c}{k_c(b-c(\frac{1+x}{x})^{\frac{1}{M}})}\cdot\frac{1}{Mx^3}\cdot\left(\frac{1+x}{x}\right)^{\frac{1-M}{M}}\left[\frac{3x+xM}{Mx(1+x)}\right] > 0$$

$$A_1 + A_2 = \frac{K_T\gamma V^{\gamma-1}c^2}{M^2x^2(b-c(\frac{1+x}{x})^{\frac{1}{M}})^2}\left(\frac{1+x}{x}\right)^{2(\frac{1-M}{M})}\left[\frac{\gamma-1}{Vk_c} + \frac{1}{x^2}\right] \quad \ldots(25)$$

In (25) the term outside the square brackets is positive. The term inside the brackets is negative, and as $V\to 0$ it becomes large. In

fact, it will turn negative when

$$V < \frac{(\gamma-1)x^2}{k_c}$$

or $$V < \frac{o(10^{-1})\, o(10^{-4})\, o(10^{-3})}$$

i.e. $V < o(10^{-7}, 10^{-8})$

Even then $A_1 + A_2 < 0$ and $A_3 + A_4 > 0$ and it may be that $\frac{d^2C}{dx^2}$ will still be positive even for V as low as that. As $V \to 0$ $\frac{d^2C}{dx^2}$ will eventually turn negative, but for all practical purposes (V>0) we can consider $\frac{d^2C}{dx^2}$ as positive over the region of interest.

The limits for x corresponding to $V \to 0$ and $V \to \infty$ are:

$$\max\left\{ \frac{1}{((\frac{b}{c})^M - 1)} , 0 \right\} \leq x \leq \left(\frac{1}{(\frac{b-1}{c})^M - 1} \right) \quad \ldots\ldots(26)$$

lim $V \to \infty$ lim $V \to 0$

which for M=1 reduce to equation (19) as expected.

(5) Standby redundancy followed by a buffer storage

As in the previous section, we obtain from chapter 4:

For M pieces in standby redundancy:

$$\rho_s = \frac{U}{\rho_i - \rho_{M+1}} e^{-k_c V} \quad ; \quad x = (\frac{1-A_{sv}}{A_{sv}}) \simeq \frac{1}{Me^{\rho_s} \rho_s^{-M}} - 1$$

$$x = \left[\frac{M \exp(\frac{U}{\rho_1 - \rho_{M+1}} e^{-k_c V})}{(\frac{U}{\rho_1 - \rho_{M+1}} e^{-k_c V})^M} - 1 \right]^{-1} \quad \ldots\ldots\ldots(27)$$

we have that

$$(\frac{1+x}{Mx}) = e^{\rho_s} \rho_s^{-M} \quad \ldots\ldots\ldots\ldots\ldots(28)$$

For M=1

$$\left(\frac{1+x}{x}\right) = e^{\rho_s}\rho_s^{-1} \approx \frac{1+\rho_s}{\rho_s} \Rightarrow x = \rho_s$$

and

$$x = \frac{U}{\rho_1 - \rho_2 e^{-k_c V}} \quad \text{as in section (3).}$$

For M=2

$$\frac{1+x}{2x} = \frac{1+\rho_s+\rho_s^{2}/2}{\rho_s^{2}}$$

Call $R_1 = {}^{1}/_{\rho_s}$

Then

$$\left(\frac{1+x}{2x}\right) = R_1^2 + R_1 + {}^{1}/_{2} \quad ; \quad R_1 = \frac{\rho_1 - \rho_{M+1} e^{-k_c V}}{U}$$

R_1 is in fact the positive real solution of

$$R_1^2 + R_1 - \frac{1}{2x}$$

namely

$$R_1 = -\frac{1+\sqrt{1+\frac{2}{x}}}{2} \quad R_1>0 \quad \text{for } \sqrt{1+\frac{2}{x}} > 1 \text{ , i.e. } x>0.$$

Thus

$$V = -\frac{1}{k_c} \ln\left[\frac{\rho_1 - UR_1}{\rho_{M+1}}\right] \qquad \ldots\ldots\ldots\ldots\ldots(29)$$

(29) is a general equation for all M. However, in the general case, R is the solution of (25)

$$\left(\frac{1+x}{Mx}\right) = e^{{}^{1}/_{R}} R_1^{M} \qquad \ldots\ldots\ldots\ldots\ldots(30)$$

and here we have approximated the series in ${}^{1}/_{R_1}$ with e^{1/R_1} as in section 2.

For simplicity we shall deal with M=2 because in chapter six we only consider that case, as it is more common. For M=2, the results

up to now have not been subjected to the approximation used in (30).

Now:

$$C = 2K(1+\beta(x+z)) + K_T V^{\gamma} \qquad \ldots\ldots\ldots\ldots(31)$$

$$\frac{dC}{dx} = 2K\beta + K_T\gamma V^{\gamma-1}\frac{dV}{dx} \qquad \ldots\ldots\ldots\ldots(32)$$

$$\frac{dV}{dx} = \frac{dV}{dR_1}\frac{dR_1}{dx}$$

$$\frac{dV}{dR_1} = \frac{1}{k_c}\frac{U\rho_2}{(\rho_1 - UR_1)} > 0$$

$$\frac{dR_1}{dx} = -\frac{1}{2x^2}\left(1+\frac{2}{x}\right)^{-\frac{1}{2}} < 0$$

Thus

$$\frac{dV}{dx} = -\frac{U\rho_2}{k_c(\rho_1 - UR)}\frac{1}{2x^2}\left(1+\frac{2}{x}\right)^{-\frac{1}{2}} < 0$$

$$\frac{d^2C}{dx^2} = K_T\gamma V^{\gamma-2}(\gamma-1)\left(\frac{dV}{dx}\right)^2 + K_T\gamma V^{\gamma-1}\left(\frac{d^2V}{dx^2}\right) \qquad \ldots\ldots\ldots(33)$$

(33) is equal to (34) after some lengthy manipulation:

$$\frac{d^2C}{dx^2} = K_T\gamma V^{\gamma-1}\frac{U^2}{k_c(\rho_1 - UR_1)^2}\frac{1}{4x^4}\left(1+\frac{2}{x}\right)^{-1}\left[\frac{(\gamma-1)}{k_c V}\rho_2^2 + 1\right]$$

$$+ K_T\gamma V^{\gamma-1}\frac{U\rho_2}{k_c(\rho_1 - UR_1)}\frac{1}{x^3}\left(1+\frac{2}{x}\right)^{-\frac{1}{2}}\left(1 - \frac{1}{2x+4}\right) \qquad \ldots\ldots\ldots(34)$$

The second term in (34) is positive. In the first term the quantity outside the brackets is positive, but the quantity inside the brackets may be negative if

$$V < \frac{(\gamma-1)\rho^2}{k_c}$$

Again, order of magnitude arguments show that for (35) to be true, $V \sim o(10^{-8})$. So, we can consider $\frac{d^2C}{dx^2} > 0$ for all practical purposes $(V>0)$.

The limits for x are now:

$$\frac{1}{2e^{\Delta\rho}\Delta\rho^2-1} \lesssim x \lesssim \frac{1}{2e^{\rho}\rho^{-2}-1} \qquad \ldots\ldots\ldots(36)$$

where $\Delta\rho = (\rho_1-\rho_3)$

Up to now we have considered only the cost functions resulting from options considered in chapter six. We have seen that they are for all practical purposes convex and therefore the analysis of chapter five and the MOP algorithm stopping criterion are valid.

In section 5.6 we needed a proof that equation (5.25) is convex. This equation was used to construct Figure 5.1. In the next section we examine this equation.

(6) <u>Insertion of buffer storage between two machines with equal availabilities</u>

From (5.25)

$$F(x) \doteq (1+x+\rho)^{\gamma} + s\bar{c}^{\gamma} \simeq 1 + \gamma x + \gamma\rho + s\bar{c}^{\gamma}$$

$$x = \frac{\rho}{(1+\rho^{-1})\bar{c} + 1} \quad ; \quad \bar{c} = \frac{\rho - x}{x(1+\rho^{-1})}$$

$$\frac{dF}{dx} = \gamma + \gamma\bar{c}^{\gamma-1}s\frac{d\bar{c}}{dx}$$

$$\frac{d\bar{c}}{dx} = -\frac{\rho}{x^2(1+\rho^{-1})} \qquad \ldots\ldots\ldots\ldots\ldots(37)$$

$$\frac{d^2F}{dx^2} = \gamma(\gamma-1)\bar{c}^{\gamma-2}s\left(\frac{d\bar{c}}{dx}\right)^2 + \gamma\bar{c}^{\gamma-1}s\frac{d^2\bar{c}}{dx^2}$$

$$\frac{d^2\bar{c}}{dx^2} = \frac{2\rho}{x^3(1+\rho^{-1})} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(38)$$

So:

$$\frac{d^2F}{dx^2} = \gamma \frac{\bar{c}^{\gamma-1} s\,\rho}{x^3(1+\rho^{-1})} \left[\frac{(\gamma-1)\rho}{x\bar{c}(1+\rho^{-1})} + 2 \right] \qquad \ldots\ldots\ldots\ldots(39)$$

The quantity outside the square brackets in (39) is positive. The quantity inside the brackets may be positive or negative if

$$\bar{c} < \frac{(\gamma-1)\rho}{2x(1+\rho^{-1})} \quad \text{or} \quad \bar{c} \sim o(\rho) \sim o(10^{-2})$$

From Graph 5.1 one can see that the region of application of the results is in fact for $\bar{c} > 10^{-1}$, where $\frac{d^2\bar{c}}{dx^2}$ is positive and $\bar{c}$ has a minimum.