

The work described in this thesis was carried out at Imperial College Mathematics Department under the supervision of Dr. R. S. Scorer, during the period September 1957 to December 1959. Alterations and additions have been made since then.

Ph.D. Degree Thesis

Title:

WAVE RESISTANCE IN STRATIFIED FLUIDS

by

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May, 1961.

ABSTRACT

The field of study of this thesis is the dynamics of continuous media, and, more precisely, the form and effect of gravity wave systems which are set up when fluids of variable density are disturbed from their state of rest.

In particular two types of fluid motion are considered. In one, the density is supposed to decrease exponentially upwards, as in Chapter I. In the other, the density is supposed to change abruptly from one value to another so that the media may be regarded as the superposition of two uniform fluids. This case is considered in Chapters II and III. In Chapters I and II the disturbance is imagined to be excited by a body which moves vertically through the fluid. The energy dispersed by the wave motions generated is drawn from the body itself, which experiences a resistance as a consequence: the wave resistance. In Chapter I, the resistance to motion for an axisymmetric body is found for a given speed and shape of body. Chapter II estimates the wave energy developed by the passage of a similar body through the fluid interface.

The subject of Chapter III is the ship wave pattern. A new approximation to this pattern is derived in a very simple manner and this approximation is valid for large

distances over the whole region of the pattern. The divergent and transverse wave systems are discussed and the nature of the surface displacement is explained in terms of the interference of the two systems.

Here I would like to express my very great thanks to Dr. R. S. Scorer for all the help and encouragement he has given me at all times in the preparation and presentation of this work. Without his guidance and criticisms, this thesis would never have been written.

I would also very much like to thank the Department of Scientific and Industrial Research, whose grant of a Research Studentship from September 1957 to December 1959 made this work possible.

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INTRODUCTION

The topics of Chapters I and II are the result of problems which have arisen in Meteorology. Books on this subject explain how the atmosphere is often heated by convection from below. An account of this may be found, for example, in

Natural Aerodynamics, 1958. R.S. Scorer,
Pergamon Press, London.

It is found that the convection very often takes the form of large volumes of air which rise from below through the agency of buoyancy forces. These forces come into being when temperature differences and the resulting density changes begin to disturb the hydrostatic equilibrium between the air mass and its surroundings. The rising mass of buoyant air is called a thermal.

The behaviour and general characteristics of this penetrative convection is clearly of great importance if we are to obtain insight into the mechanisms which are brought into play when convection is mainly responsible for the heating of the lower layers of the atmosphere. Obvious questions immediately pose themselves. What is the behaviour of this buoyant convection in its later history? What is responsible for the destruction of its kinetic energy? These and other questions have been

discussed by many writers in recent years¹; and at present the theory of atmospheric convection is incomplete.

One very pertinent question we may ask concerns the effect of a stable environment on a thermal which rises through it. The loss in buoyancy is offset by the release of latent heat when the thermal reaches its condensation level, and the upward motion persists. The disturbed surroundings, being stable, will take up wave-like motions. Thus a dispersion of kinetic energy takes place, and this energy is supplied by the thermal. This mechanism affords a means by which energy is lost to a much larger region than would be involved in the viscous dissipation of the thermal turbulence. Hence we ask: does this dispersion of energy by gravity waves in the perturbed stable environment significantly retard the thermal? It is this question which Chapter I sets out to answer.

Chapter II considers a similar question in which the thermal rises through neutral surroundings (adiabatic lapse rate) and meets an inversion, i.e. a thin layer of air where the temperature changes rapidly. This very stable layer will be thrown into wave motions when the thermal penetrates it, and energy is absorbed from the thermal and dispersed by the waves.

1. See, for example, editions of Quart. J. Roy. Meteorol. Soc. over the past ten years.

These problems can immediately be related to problems of wave motion in an incompressible stratified fluid. In Chapter I the density of the liquid decreases steadily upwards, and this decrease is taken to be an exponential one. In Chapter II there is a discontinuity in the density (corresponding to the inversion) and we have the case of two superposed fluids.

In contrast to the first two Chapters, Chapter III discusses the form of the gravity waves set up when the disturbance moves horizontally, viz. the wave train which accompanies a ship which moves on a steady course over deep water. Here the density change of the fluid is a discontinuity, with the upper fluid absent. (i.e. we neglect the density of the air in comparison with water.) This Chapter sets out to give in a straightforward and simple manner, a comprehensive formula for the wave pattern at large distances which is valid everywhere (i.e. including the critical region.) It describes the wave pattern in terms of the transverse and divergent waves, and explains the rapid disappearance of the system beyond the critical line in terms of the interference of the transverse and divergent waves.

Chapter I has been published in the form of a paper in *Journal of Fluid Mechanics*, and Chapter II has been published in paper form in *Quarterly Journal of the Royal*

Meteorological Society. Copies of these papers are included in this thesis.

Chapter III, in the form of a paper, has been accepted for publication by Journal of Fluid Mechanics.

CHAPTER I.WAVE RESISTANCE TO VERTICAL MOTION IN AN
INCOMPRESSIBLE AND INVISCID STRATIFIED FLUID

For the purposes of the problem we have in mind, it is sufficient to consider the thermal as having a fixed shape, and to be rising with a constant velocity. This Chapter, therefore, considers the effects of a gradual decrease in fluid density upwards when a body moves vertically through it at a steady speed. The body will experience a resistance to motion which is set up by wave motions generated within the fluid; the problem is to estimate this resistance. If the body is moving upwards through the fluid by virtue of its buoyancy, it may be substantially retarded by the waves before it comes to rest at its equilibrium level. A consideration of these and other effects is made here.

More specifically, we seek the effect of the excitation of gravity waves in a stable environment on the ascent of cloud thermals and growth of cumulus towers within it.

The accompanying reprint from Journal of Fluid Mechanics forms the substance of this Chapter. Some notes are appended which comment upon and discuss some points which arise from the printed text. Places where the notes

occur are marked by a raised number in brackets thus: (1).
The Chapter finishes with a section outlining the
conclusions drawn from the analysis and discussion.

NOTES TO THE PRINTED TEXT OF CHAPTER I.

These notes are made either to verify certain mathematical statements made in the text, or to comment on some physical aspects of the problem.

(1) The linearisation of the equations of motion may be carried out systematically as follows:

We assume that the perturbations β' , ρ' , u , w , and ψ may be expanded in terms of a parameter ε where ε is the thickness-length ratio of the body; e.g.

$$\psi = \varepsilon^2 \psi^{(2)} + o(\varepsilon^2).$$

Further, we assume that as $r_1 \rightarrow 0$, (where $r_1 = \frac{r}{b}$),

$$u = O(\varepsilon^2/r_1), \quad w = O(\varepsilon^2 \log r_1),$$

$$\frac{\partial u}{\partial r} = O(\varepsilon^2/r_1^2), \quad \frac{\partial w}{\partial r} = O(\varepsilon^2/r_1),$$

$$\frac{\partial \beta'}{\partial r} = O(\varepsilon^2/r_1), \quad \frac{\partial \rho'}{\partial r} = O(\varepsilon^2/r_1),$$

$$\beta' = O(\varepsilon^2 \log r_1) \text{ and } \rho' = O(\varepsilon^2 \log r_1).$$

The equations of motion (1), (2), and (3) of the text lead to the equations:

$$\begin{aligned} & \rho_0 \frac{\partial}{\partial z} \frac{d}{dt} \left(\frac{du}{dt} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) + \frac{\partial \rho_0}{\partial z} \frac{d}{dt} \left(\frac{du}{dt} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) \\ & - \rho_0 \frac{\partial}{\partial r} \frac{d}{dt} \left(\frac{dw}{dt} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) + g \frac{\partial}{\partial r} \left\{ \frac{1}{(1+\rho'/\rho_0)^2} \frac{d\rho'}{dt} \right\} \\ & = \frac{d}{dt} \left\{ \frac{\partial \beta'}{\partial z} \frac{\partial}{\partial r} \left(\frac{1}{1+\rho'/\rho_0} \right) \right\} - \frac{d}{dt} \left\{ \frac{\partial \beta'}{\partial r} \frac{\partial}{\partial z} \left(\frac{1}{1+\rho'/\rho_0} \right) \right\}, \end{aligned}$$

and
$$\frac{d\rho'}{dt} + \frac{\partial \rho}{\partial z} + u \frac{\partial \rho'}{\partial r} + w \frac{\partial \rho'}{\partial z} = 0,$$

where
$$\frac{d}{dt} = \frac{\partial}{\partial t} - W \frac{\partial}{\partial z}.$$

If we substitute for p' , ρ' , etc., their ϵ -expansions, and retain terms of lowest order, viz. ϵ or ϵ^2 only,⁺ we obtain equation (5) with the R.H.S. replaced by

$$\frac{d}{dt} \left\{ \frac{\partial}{\partial z} \left(u \frac{\partial u}{\partial r} \right) - \frac{\partial}{\partial r} \left(u \frac{\partial w}{\partial r} \right) - \beta u \frac{\partial u}{\partial r} \right\},$$

provided $r_1 \gg \epsilon$. This quantity is $O(\epsilon^4/r_1^3)$ when r_1 is $O(\epsilon)$ and is $O(\epsilon^4)$ if $r_1 \gg \epsilon$. If ϵ is small we may take equation (5) to hold everywhere, with an appreciable error in a region immediately surrounding the body. This region is small (if ϵ is small) and the streamlines within it will not differ greatly from the streamlines which obey equation (5) if both flows satisfy the boundary conditions at the surface of the body. Inside the separating streamline, equation (5) is no longer a good approximation to the motion; and the exact replacement of the body by sources and sinks is no simple matter since the density variation causes complications. However we may take equation (5) to hold everywhere, apply a boundary condition to ψ at $r=0$, and observe that for small ϵ we have

+ i.e. terms $O(\epsilon^2)$ when $r_1 \gg \epsilon$, and terms $O(\epsilon)$ otherwise, (i.e. when $r_1 \gg \epsilon$).

conditions which approximate to the problem we have in mind.

(2) The given conditions (8) represent a knowledge of both the velocity field and the surfaces of constant density at time $t=0$.

(3) If a viscosity term is included in the equation of motion, we have

$$\rho \frac{Dq}{Dt} = \rho E - \nabla p + \mu \nabla^2 q,$$

where E = body force due to gravity,

q = velocity of a fluid particle ($= (u, w)$),

and μ = coefficient of viscosity.

If we carry out a linearisation procedure we obtain the following equation of motion:-

$$\begin{aligned} & \frac{d^2}{dt^2} \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \psi}{\partial z} \right) - \frac{\beta}{r} \frac{\partial \psi}{\partial z} \right] + \alpha^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \\ &= \bar{\nu} \frac{d}{dt} \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right] \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \psi}{\partial z} \right) \right]; \end{aligned}$$

where $\frac{d}{dt}$ denotes $\frac{\partial}{\partial t} - W \frac{\partial}{\partial z}$, and where we assume

$\bar{\nu} = \frac{\mu}{\rho}$ is constant throughout the fluid, approximately.

Thermals have no wakes (see Scorer (1958), pp.125, 265, etc.)

and so we assume that the velocity shear at the boundary of the thermal is negligible, and write

$$r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right)_{r=0} = 0.$$

We now assume steady conditions, also that the wave amplitude is zero at large distances from the thermal, and that

$$\left| \frac{\bar{v} k_0}{W} \right| \ll 1.$$

We find that for a large range of k (i.e. where $\frac{k-k_0}{k_0}$ is not small compared with 1), β is modified in the formula for λ on p. 213 of the text as:

$$\beta \rightarrow \beta + \frac{k^2 k_0^4}{(k_0^2 - k^2)^2} \cdot \frac{\bar{v}}{W}$$

whilst k_0 is unchanged. Hence if $W > 0$ the strength of β is increased, while if $W < 0$ (descent) the strength of β is decreased. Hence the inclusion of the viscosity tends to bring the steady state solution into agreement with the solution in which the wave pattern is downstream.

(4) In the atmosphere, the potential temperature is defined as

$$\theta = \frac{1}{T} = T \left(\frac{p}{p_1} \right)^{\frac{\gamma-1}{\gamma}}$$

(γ = ratio of specific
heats of air)

where T is the temperature and p is the pressure. (The suffix 1 denotes some standard position). The modified pressure (Scorer, 1950) is defined as

$$\bar{\theta} = \frac{\gamma R}{\gamma-1} \cdot \left(\frac{p}{p_1} \right)^{\frac{\gamma-1}{\gamma}},$$

where R is the gas constant.

Hence $\frac{1}{\tau} \nabla \varpi = \frac{1}{\rho} \nabla p$. Also for adiabatic conditions $\frac{D\tau}{Dt} = 0$, and the equation of continuity gives $\text{div}(u, w) \approx 0$ provided there are no rapid fluctuations of the pressure (i.e. we exclude sound waves). Thus for the atmosphere we replace ρ and ρ by ϖ and τ respectively. The resistance to motion is found from the formula on line 25 of p. 227 of the printed text. The pressure p in the formula represents the perturbed pressure $= p' = \delta p$. Since $\frac{\delta p}{\rho_0} \rightarrow \frac{\delta \varpi}{\tau} = \frac{\delta p}{\rho_0}$, the resistance as found in the true wave resistance for the thermal.

- (5) The strict linearisation of the Euler equations in the horizontal direction does not permit the omission of the term

$$u \frac{\partial u}{\partial r}$$

in the vicinity of the body since when $r_1 = O(\varepsilon)$ this term is $O(\varepsilon)$, which is the same order as $\frac{\partial p}{\partial r}$ for small r_1 .

Hence strictly we have

$$p = W \rho_0 \int_r^\infty dr \frac{1}{r} \frac{\partial^2 \psi}{\partial z^2} + \frac{\rho_0}{2} u^2.$$

The extra contribution to the wave resistance term is

$$\begin{aligned} & \frac{\rho_0}{2} \int_{-b}^b dz f(z) f'(z) (1 - \beta z) u_a^2 \\ &= -\beta \frac{\rho_0}{2} \int_{-b}^b dz z f(z) f'(z) u_a^2, \end{aligned}$$

since $f(z) = f(-z)$ and $u_a^2(z) = u_a^2(-z)$.

Since this last expression has a factor β , it is omitted. ($= \beta \cdot O(\epsilon^4)$).

(6) The integral

$$\begin{aligned} & \int_0^\infty dk k^2 F(k) \int_0^b dz z f'(z) f(z) \cos kz H_0^{(1)}(r_a - z) \\ &= b^4 \epsilon^4 \int_0^1 dz z f_1'(z) f_1(z) \cos kz \{ \log \epsilon + \log(f_1(z)) + \dots \} \\ &= b^4 \epsilon^4 \log \epsilon \int_0^1 dz \{ f_1'(z) f_1(z) \}^2 + O(\epsilon^4) \dots \dots (A). \end{aligned}$$

From the linear theory of potential flow round a slender body we have that the kinetic energy of the surrounding fluid is

$$\frac{\rho_0}{2} \int_{\text{surface of body}} dS \phi \frac{\partial \phi}{\partial n} = \pi \rho_0 \int_{-b}^b dz f(z) \left[\phi \frac{\partial \phi}{\partial r} \right]_{r=a},$$

where dS = surface element
and n = inward normal
to surface, and

where ϕ is the potential, and is approximately

$$\frac{Wi}{2} \int_{-\infty}^{\infty} dk \cdot k K_0(r|k|) e^{-ikz} F(k),$$

by the linear theory. (e.g. put $\beta = 0$ in equation (5), and

write $\frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial z}$). It follows that the kinetic energy of the fluid for potential flow is

$$2\pi W^2 \rho_0 b^3 \varepsilon^4 |\log \varepsilon| \int_0^1 dz \left\{ f_1'(z) f_1(z) \right\}^2 + O(\varepsilon^4) \dots (B).$$

If we compare (A) and (B) we see that the last integral on p. 227 of the text may be interpreted as the effect of the change in virtual mass.

If we compare the change in kinetic energy resulting from the change in virtual mass with the work done by the wave resistance, we have for a spherical thermal of diameter $2a$

change in kinetic energy over a distance a
work done by the wave resistance over a distance a

$$= \frac{\pi B}{R_d(\nu)}, \text{ where } B = \text{buoyancy of thermal}$$

$$= \frac{\delta \rho}{\rho}.$$

For a slender body rising with the speed of a thermal of the same width and buoyancy B , this ratio is

$$\frac{2\pi B \varepsilon |\log \varepsilon|}{R_d(\nu)}.$$

Hence for the thermal we may neglect the effect of the change in virtual mass in comparison with the wave resistance.

CONCLUSIONS TO CHAPTER I.

Although the speed of ascent of a thermal in the atmosphere is low, the corresponding values of $v = \frac{b\sqrt{g\beta}}{\pi W} = \sqrt{\frac{a\beta}{\pi^2 B}}$ for atmospheric conditions are not large. This means that we are often working in a range of v in which the wave resistance increases as the speed decreases. The number $\frac{a\beta}{B}$ determines how much energy is spent in generating waves. In other words, the ratio of the density difference between points just above and just below the thermal to the density deficit of the thermal determines the wave energy. It is not until this ratio reaches a value of about 5 that the wave resistance will have an effect on the motion of the thermal. If only one third of the potential energy released is available for ordered kinetic motion, the wave resistance will begin to effect the motion when this ratio is about 2. However, as or before these conditions are reached, the buoyancy of the thermal has become negative and has a dominant effect on the motion.

As mentioned in the introduction, a cloud thermal may ascend through an environment whose lapse rate is the wet adiabatic and maintain its buoyancy by virtue of the release of latent heat. The buoyancy could then become quite small yet remain persistent during the ascent. Thus we could have a cloud thermal of diameter 10^4 cm. rising slowly with a buoyancy equivalent to $0.5 \cdot 10^{-4}$. If the

stability of the environment is 10^{-7} cm^{-1} , then the ratio of the wave resistance to the equivalent buoyancy is about 2.5 with $\frac{\alpha\beta}{\beta} = 10$. Thus under these conditions the motion would be very greatly effected by the surroundings. Moreover, the kinetic energy of ordered motion will be destroyed when the thermal has travelled a distance of approximately $1.5 \cdot 10^3 \text{ cm}$. i.e. a distance considerably less than its own diameter. The theory has assumed the speed constant throughout and the result of the retardation by the wave making process would invalidate this assumption. However this does not effect the qualitative nature of the result since we have shown that the velocity of ascent must be much less than supposed in order for the wave resistance to permit ascent. This in turn implies that a large percentage of energy is dissipated into waves.

Another objection which may be raised is that the slender body results have been applied to a sphere. In the following Chapter a figure of the slender body approximation to the sphere is shown. It shows that the radius a of the sphere is underestimated. Since the wave resistance R is proportional to a^4 (the change in ν has a comparatively small effect) this suggests that the resistance is underestimated. However allowances for non-linear terms in the equation of motion and for similar terms in the boundary conditions would also have to be made. No correction for the slender body approximation therefore is made. However

it seems likely that an underestimate rather than an overestimate has been made, because in neutral surroundings ($\beta \rightarrow 0$) the flow must reduce to the potential solution with the correct boundary conditions.

Summing up, we may expect the generation of waves to effect ordinary isolated thermals as they come to rest at their equilibrium position. Cloud thermals with low velocities of ascent (small equivalent buoyancy) will often be very greatly affected. In other circumstances the stable environment will have little effect on the motion of the thermal. The analysis shows that these results come about because of the importance of the ratio of the density deficit or equivalent deficit of the thermal to the environmental density difference at levels just above and below the thermal. The deficit determines the speed of ascent and the difference determines the relative buoyancy of the upper environment compared with the lower. When the difference is larger than the deficit the wave-making effect becomes comparable with the buoyancy.

CHAPTER II

THE GENERATION OF WAVE ENERGY AT A FLUID INTERFACE BY THE PASSAGE OF A VERTICALLY MOVING SLENDER BODY.

This Chapter considers the effect of a sudden decrease in density upwards as a body moves vertically through the fluid. This is in contrast to Chapter I, where the decrease in density upwards is gradual and continuous. There the vorticity is distributed throughout the fluid. Here the vorticity is confined to the interface (vortex sheet) and a velocity potential exists for all parts of the fluid. The vorticity is accounted for by a discontinuity in the potential at the interface. An estimate is made of the energy absorbed by wave motion at the interface when the body passes from one fluid to the other. The wave resistance will vary with distance from the interface and when this is large the resistance will be small. Hence in this Chapter we estimate the total wave energy rather than the wave resistance. In Chapter I the energy involved is infinite, since the motion is supposed steady, implying that the wave resistance has been acting for an infinite time.

The following reprint from Quarterly Journal of the Royal Meteorological Society forms most of this Chapter.

Some notes are appended which comment upon and discuss some points which arise in the printed text. Places where the notes occur are marked as in Chapter I. These notes are followed by a section enlarging on the conclusions.

NOTES TO THE PRINTED TEXT OF CHAPTER II.

These notes give a further discussion of some points which arise in the printed text.

(1) A discussion of the accuracy of the slender body theory is given, for example, in Temple (1958), where the particular case of a prolate spheroid is worked out in detail. It is shown that for small values of a/b , the slender body approximation is a good one. On the other hand, the slender body approximation to a sphere of unit radius actually realises a prolate spheroid whose major axis is about 1.28 and whose minor axis is about 0.62. Thus the error made in the approximation to a sphere is large, but not enormous. A better approximation is possible if the value of a is increased and the value of b decreased, i.e. if we 'aim' at an oblate spheroid. Fig. 4 of this Chapter shows three spheroids and the actual shapes realised by their slender body approximations. For example, to obtain a true sphere we would require a doublet on the z -axis, and this could be obtained by a limiting process by writing

$$\left. \begin{aligned} a &= \left(\frac{3}{2}\right)^{1/2} A^{3/2} \delta^{-1}, \\ b &= \delta^2, \\ f(z) &= a \left(1 - \left(\frac{z}{b}\right)^2\right)^{1/2}, \quad \left|\frac{z}{b}\right| \leq 1 \end{aligned} \right\} \begin{array}{l} \text{where } A = \\ \text{radius of} \\ \text{the sphere,} \end{array}$$

and with $\delta \rightarrow 0$. Thus we could define the boundary conditions (3) of the text to hold, and proceed as in paragraph 3. The problem would then remain to find the shape of the body which corresponds to the boundary condition (3). There are, however, two points to consider in this approach. Firstly, the sources emit a fluid of different density as they cross the interface. This complicates the problem of finding the separating streamlines of the flow defined by (3), and the shape of these streamlines would not remain fixed as the interface is crossed. Secondly, there are linear approximations made elsewhere - notably in equation (5) where velocity squared terms are neglected - and so a spurious accuracy is obtained if we refine the slender body approximation. For these reasons no attempt is made to correct the slender body approximation to the sphere.

(2) We assume that as $r \rightarrow 0$, then

$$u = O(\epsilon^2/r_1), \quad w = O(\epsilon^2 \log r_1) \\ \frac{\partial u}{\partial r} = O(\epsilon^2/r_1^2) \text{ and } \frac{\partial w}{\partial r} = O(\epsilon^2/r_1), \quad (C)$$

where ϵ = thickness-length ratio of the body = a/b
and $r_1 = r/b$.

If we retain terms of lowest order in ϵ only, Euler's equation of motion in the horizontal direction reads

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) + \frac{\partial p}{\partial r} = 0, \quad (D)$$

since the term $u \frac{\partial u}{\partial t}$ may not be neglected in comparison with the other terms if $r_1 = 0$ (ϵ). However the equations (C) hold only for the potential ϕ_1 . For $\phi_{2,3,4}$, equations (C) read: $u, w, \frac{\partial u}{\partial t}$, and $\frac{\partial w}{\partial t} = O(\epsilon)$ as $r \rightarrow 0$. Hence for the discussion of the wave resistance we may omit the term $u \frac{\partial u}{\partial t}$ in (D).

(3) In view of the remarks made in notes (1) and (2) it is interesting to note that it is possible to deduce the expression for the wave energy (19) directly from the expression for the potential for the waves ϕ_4 , without further appeal to the slender body theory. $2\phi_4$ expresses the potential for the waves developed during the passage of the body across the interface, and since the energy carried by these waves is constant, we may find the wave energy from the formula

$$E_{\text{waves}} = \text{potential energy}_{\text{waves}} + \text{kinetic energy}_{\text{waves}}$$

where the potential and kinetic energies are calculated at $t = 0$.

For the potential energy we have

$$E_P = \lim_{L \rightarrow \infty} \frac{\rho_2 - \rho_1}{2} \int_0^L dr \, 2\pi r \, \zeta_{t=0}^2,$$

where $\zeta(r, t)$ is the displacement of the interface from its rest position. In conjunction with (16), this leads to the

expression

$$E_p = (\rho_2 - \rho_1) g a^4 v^2 \int_0^\infty dk \int_0^\infty dl \frac{k l F_1(\sqrt{v}k) F_1(\sqrt{v}l)}{(k+v)(l+v)} \lim_{L \rightarrow \infty} \int_0^L dr_1 \pi r_1 J_0(kr_1) J_0(lr_1).$$

But as $L \rightarrow \infty$,

$$\begin{aligned} \int_0^L dr_1 \pi r_1 J_0(kr_1) J_0(lr_1) &\sim \frac{\sin L(k-l)}{\sqrt{k l} (k-l)} - \frac{\cos L(k+l)}{\sqrt{k l} (k+l)} \\ &\rightarrow \pi \frac{\delta(k-l)}{\sqrt{k l}}, \end{aligned}$$

where δ denotes the Dirac delta function.

Hence we have

$$\begin{aligned} E_p &= \pi (\rho_2 - \rho_1) g a^4 v^2 \int_0^\infty dk \frac{k \{F_1(\sqrt{v}k)\}^2}{(k+v)^2} \\ &= 2\pi a^4 g \Delta \rho v^2 \int_0^\infty ds \frac{s^3 \{F_1(s)\}^2}{(s^2+v^2)^2}. \end{aligned}$$

The kinetic energy may be found from the formula

$$\text{kinetic energy of surrounding fluid} = \frac{\rho}{2} \int_{\text{surface of body}} dS \phi \frac{\partial \phi}{\partial n}$$

where dS denotes a surface element and n denotes the inward normal.

In conjunction with (16) this leads to the expression

$$\begin{aligned}
E_K &= (\rho_2 + \rho_1) \frac{\pi a^4 W^2 v^2 \sigma^2}{b} \int_0^\infty dk \int_0^\infty dl \frac{k l^2 F_1(\sqrt{2}k) F_1(\sqrt{2}l)}{(k+v)(l+v)} \\
&\quad \cdot \lim_{L \rightarrow \infty} \int_0^L dr_1 \pi r_1 J_0(kr_1) J_0(lr_1) \\
&= \pi g (\rho_2 - \rho_1) a^4 \sigma^2 v \int_0^\infty dk \frac{k^2 [F_1(\sqrt{2}k)]^2}{(k+v)^2} \\
&= 2\pi a^4 g \Delta \rho \sigma^2 \int_0^\infty da \frac{a^5 [F_1(a)]^2}{(a^2 + v^2)^2}
\end{aligned}$$

If we now add $E_K + E_P$ we obtain an expression for E_{waves} which leads to (19). However this method gives no estimate of the rate of growth and decay of the wave resistance $R(= R_3 + R_4)$ as the thermal passes through the inversion.

(4) The resistance to motion which arises from the potential expressed by $\phi_1 + \phi_2$ is taken into account if we consider the change in virtual mass as the body passes from one fluid to the other, since $\phi_3 = \phi_4 = 0$ when $g = 0$. We again use the formula

$$\text{K.E. of surrounding fluid} = \frac{\rho}{2} \int_{\text{surface of body}} dS \phi \frac{\partial \phi}{\partial n}$$

where n denotes the inward normal to the surface of the body and dS a surface element. The slender body approximation (1) in conjunction with the expression for ϕ_1 on page 46 of the text then gives

K.E. of surrounding fluid =

$$2\pi W^2 \rho b^3 \varepsilon^4 |\log \varepsilon| \int_0^1 dz \left\{ f_1(z) f_1'(z) \right\}^2 + O(\varepsilon^4) \quad (E)$$

The change in kinetic energy E_S of the surrounding fluid as the body passes from a position well above the interface to one well below it is given by replacing ρ by $\Delta\rho$ in equation (E). In this case there is a gain of energy by the surrounding fluid which is supplied by the body. If however the body passes from the lower fluid to the upper, there is a loss of energy by the surrounding fluid, and this offsets the energy lost to the waves at the interface. For small values of ϵ , the effect of the change in kinetic energy of the surrounding fluid is more important than the effect of the energy absorbed by the waves, since $E_S = O(\epsilon^4 |\log \epsilon|)$ and $E_{\text{waves}} = O(\epsilon^4)$. However E_{waves} has a factor $g (\approx 10^3)$ which is not present in E_S , and unless ϵ is very small or the speed $|W|$ is large, E_S is small compared with E_{waves} . In fact for a spherical thermal

$$\frac{E_S}{E_{\text{waves}}} = \frac{B}{e_1(v)} ; \text{ and for a slender body moving at a thermal speed of } W = \sqrt{gBa}, \text{ then}$$

$$\frac{E_S}{E_{\text{waves}}} = \frac{2 B \epsilon |\log \epsilon|}{e_1(v)} .$$

CONCLUSIONS TO CHAPTER II.

If rising convection currents meet an inversion and pass through it, three ratios may be considered. These are

$$\frac{W}{V} , \quad \frac{W}{K} , \quad \text{and} \quad \frac{K}{W} ,$$

where W = wave energy,

K = kinetic energy of ordered motion,

and V = work done by the buoyancy.

The wave energy is calculated on the assumption that the thermal is successful in its first attempt to pass through the inversion although subsequently there may be a considerable change in its motion. The work done by the buoyancy is taken to be the work done from when the thermal is just below the inversion to when it is just above. The kinetic energy K is the kinetic energy of the thermal moving as a solid body + the kinetic energy of the non-turbulent surroundings.

For a 2% inversion and thermal buoyancies ranging from 2% down to .5%, the ratios are approximately as follows:-

Thermal Buoyancy.	$\frac{W}{V}$	$\frac{W}{K}$	$\frac{K}{V}$
2%	- .25	.32	- .78
1.7%	- .62	.61	-1.01
1.0%	(V=0)	1.28	(V=0)
.85%	2.21	1.21	1.82
.75%	1.26	1.14	1.10
.5%	.13	.64	.20

Thus for a thermal whose buoyancy is one half the strength of the inversion, the wave energy generated can bring the thermal to a halt, whereas the buoyancy forces, in comparison, do no work at all. On the other hand, the buoyancy of the thermal may be nil. If then the rising air mass has a velocity equivalent to that of a buoyancy of between 1.7% and .85%, the ratio of

$$\begin{array}{lll} \frac{W}{V} & \text{lies between } .4 \text{ and } .5, \\ \frac{W}{K} & " & " \quad .6 \text{ and } 1.2, \\ \text{and } \frac{K}{V} & " & " \quad .3 \text{ and } .8. \end{array}$$

In these cases it is still possible for the wave motions to have a significant effect on the motion. These results show that very definite changes due to wave motions may be brought about when the thermal meets an inversion above

which the buoyancy changes sign. Even if the thermal would not rise far above the inversion in the absence of waves, the result remains valid. For the loss in buoyancy will have most effect only when the thermal has crossed the interface, by which time the waves have developed.

Objections may be made that the slender body results have been applied to the sphere, and that a constant velocity W of thermal ascent has been assumed. These points have been considered in the conclusions to Chapter I, and similar arguments apply here.

An important result, in contrast to the result of Chapter I, is that the above statements and figures are independent of the size of the thermal. Hence the small thermals are equally likely to be halted at the interface by waves as are the larger ones. It is the ratio of the inversion strength to the density deficit of the thermal which determines the importance of the wave making effect. The vorticity is concentrated at the interface (inversion) and it is easier for waves to develop here than in a thick layer where the density variations of the environment are smoothed out.

Finally, we note that the results of this Chapter and of Chapter I have many similarities, as would be expected. R_d and ν of Chapter I are comparable with E and ν of this Chapter, respectively, and similarities between fig. 16 (Chap.I) and fig. 2 (this Chapter) are not surprising.

CHAPTER III.A STATIONARY PHASE APPROXIMATION TO THE
SHIP WAVE PATTERN.

This Chapter again considers an abrupt change in the density of the fluid: in fact the density of the upper fluid is taken to be zero and the body is supposed to move horizontally across the surface of the lower fluid. In particular we consider a ship moving on a steady course over the sea. Michell (1898)⁺ found an integral expression for the wave resistance of a ship and previously Kelvin (1887) invented his method of stationary phase to describe the wave pattern which accompanies the ship as it moves over the sea. However the usual method of stationary phase fails near the edge of the pattern. This Chapter uses a method described by Scorer (1950) to derive in a simple manner an expression for the wave pattern (at large distances from the ship) which remains valid in the vicinity of the critical angle. The nature of the disturbance in this region is readily seen. The divergent and transverse wave systems in a critical region are shown, together with their variation of amplitude and phase.

The figures and their captions appear at the end of the Chapter. (pp. 46-53.)

+ In this Chapter, the references are listed on p. 45.

1. Introduction.

The pattern of gravity waves set up behind a ship on a steady course has been discussed by many writers. Hogner (1923) has given a first approximation to this pattern, and most recently Ursell (1960) has given an asymptotic series solution for the critical region for a point disturbance. In this Chapter a first approximation is derived in a much simpler manner.

An integral expression for the wave motion caused by a ship in deep water is given for example by Hogner (1923), equation (51). The form which we will use here is

$$\zeta = \text{im} \int_{-\rho}^{\infty} \psi(k) e^{ix\phi(k)} dk,$$

where $\phi(k) = \cosh k + \frac{\tan \Theta}{2} \sinh 2k$, and where $\psi(k)$ is some function which depends on the shape of the ship. For this purpose we may assume that ρ is infinite. ζ is the surface displacement at the point $(x,y) = (r,\Theta)$, where the x-axis is drawn on the surface against the direction of motion. The unit of distance is U^2/g , where U is the steady velocity of the ship. Terms of order r^{-1} have been omitted in this representation.

The usual method of stationary phase gives a first approximation to ζ in the region $|\Theta| < \tan^{-1} \frac{1}{2\sqrt{2}} = \Theta_c$, but it fails in the vicinity of $\Theta = \Theta_c$ since $\phi'(k)$ and $\phi''(k)$ vanish together along this line. (The dashes denote

differentiation with respect to k). Scorer's method overcomes this difficulty by replacing $\phi(k)$ with parabolas of the third order applied at the stationary values of ϕ . This leads to an approximation in terms of the Airy integral $Ai(z)$ and the conjugate integral $Gi(z)$:

$$Ai(z) + i Gi(z) = \frac{1}{\pi} \int_0^{\infty} du \exp\left(zu + \frac{u^3}{3}\right).$$

$Ai(z)$ has been tabulated by Miller (1946). Scorer (1950) tabulates $Gi(z)$ for $|z| \leq 10$, and there is an asymptotic expression valid to the accuracy of the tables for larger values.

2. The approximation.

For this purpose the range of integration is suitably divided at k_I where $\phi(k_I) = 0$. The function $\phi(k)$ has two turning points when $|\Theta| < \Theta_c$ and these are denoted by the suffixes T and D, where $k_D < k_I < k_T$. The approximation to $\phi(k)$ is then as follows :-

$$(i) \quad \phi(k) \simeq \phi_T + \frac{1}{2}\phi_T''(k-k_T)^2 + \frac{1}{6}\phi_T'''(k-k_T)^3, \text{ if } k > k_I,$$

$$\text{and (ii) } \phi(k) \simeq \phi_D + \frac{1}{2}\phi_D''(k-k_D)^2 + \frac{1}{6}\phi_D'''(k-k_D)^3, \text{ if } k < k_I.$$

Figure 1 (p.48) illustrates this. The expression for ζ is written $\zeta = \zeta_T + \zeta_D$, where $\zeta_T = \int_{k_I}^{\infty}$ and $\zeta_D = \int_{-\infty}^{k_I}$.

It is assumed that $\psi(k)$ is sensibly constant in the region

of stationary phase and we have for large values of x

$$\begin{aligned}\zeta_T &= im \int_{k_I}^{\infty} \psi(k) e^{ix\phi(k)} dk \\ &\approx im \psi_T e^{i\phi_T x} \int_{k_I - k_T}^{\infty} dk \exp ix \left(\frac{1}{2} \phi_T'' k^2 + \frac{1}{6} \phi_T''' k^3 \right).\end{aligned}$$

Following Scorer (1950) we now replace the lower limit of integration by $-\frac{\phi_T''}{\phi_T'''} .$ This does not effect the result because the contribution from the interval $(k_I - k_T, -\frac{\phi_T''}{\phi_T'''})$ is negligible when x is large. We may now write ζ in the form

$$\zeta_S \approx im\pi \psi_S \left(\frac{2}{\phi_S'''} \right)^{\frac{1}{3}} \exp ix \left(\phi_S + \frac{1}{3} \frac{(\phi_S'')^3}{(\phi_S''')^2} \right) \cdot \left\{ Ai(-q_S)^{\pm 1} Gi(-q_S) \right\},$$

where $q_S = (\phi_S'')^2 \cdot \left(\frac{x}{2(\phi_S''')^2} \right)^{\frac{2}{3}} .$ Here and after where an

alternative sign is attached to a symbol the upper and lower sign correspond to $S = T, D$ respectively. This approximation also holds when T, I and D coincide for when $|\Theta| = \Theta_c .$ The expression is simplified if we write

$$\psi(k) = A(k) e^{ia(k)}$$

$$\text{and } Ai(z) + i Gi(z) = K(z) e^{i\kappa(z)} .$$

Scorer has tabulated $K(z)$ and $\kappa(z)$ for $|z| \leq 6 .$

We may then write

$$\zeta_S \approx \pi A_S \left(\frac{2}{\phi_S'''} \right)^{\frac{1}{3}} K(-q_S) \sin \left\{ \phi_S \cdot x \pm \frac{2}{3} q_S^{\frac{2}{3}} \pm \kappa(-q_S) + a_S \right\} . \quad (1)$$

As usual, ζ_T and ζ_D are called the transverse and divergent wave systems respectively. If the stationary phase approximations to $K(-q_S)$ and $\kappa(-q_S)$ are substituted in this formula the usual stationary phase approximations to ζ_T and ζ_D are obtained. For $K(-q_S) \sim (\pi^2 q_S)^{-\frac{1}{2}}$ and $\kappa(-q_S) \sim -\frac{2}{3}q_S^{\frac{2}{3}} + \frac{\pi}{4}$ as $q_S \rightarrow \infty$. By (1) we then have

$$\zeta_S \approx A_S \left(\frac{2\pi}{x|\rho_S''|} \right)^{\frac{1}{2}} \sin(\rho_S x \pm \frac{\pi}{4} + a_S)$$

for large values of q_S .

The method may be justified briefly as follows. One of two conditions is to be satisfied. The first is that the ρ_S'' should be sufficiently small so that both

$$\exp ix \left\{ \frac{\rho_S^{(v)}}{4!} \left(-\frac{\rho_S''}{\rho_S'''} \right)^4 + \frac{\rho_S^{(v)}}{5!} \left(-\frac{\rho_S''}{\rho_S'''} \right)^5 + \dots \right\} \approx 1 \quad (a)$$

$$\text{and } \left| (k_I - k_S) + \frac{\rho_S''}{\rho_S'''} \right| \ll \left\{ |\rho_S'' x|^{-\frac{1}{2}}, |\rho_S''' x|^{-\frac{1}{3}} \right\}_{\min} \quad (b)$$

hold. (a) expresses the condition that the cubic approximation to ρ is sufficiently accurate in the neighbourhood of stationary phase. (b) expresses the condition that the contribution from the intervals $(k_I - k_S, -\frac{\rho_S''}{\rho_S'''})$ are negligible. It is easily found that the length of this interval is $O(\rho_S'')^2$ when the ρ_S'' are small so that condition (b) is weaker than (a). Hence the first condition may be written

$$(\rho_S'')^4 x \ll 1. \quad (a')$$

The alternative, which is the condition for the validity of the usual method of stationary phase, is that the ϕ_S'' should be sufficiently large so that

$$|(\phi_S'')^3 x| \gg 1. \quad (c)$$

The regions in which the alternatives (a') and (c) hold overlap if x is sufficiently large, and the approximation (1) is then valid for all $|\Theta| \leq \Theta_c$.

For values of $|\Theta| > \Theta_c$, $\phi(k)$ has no turning point and we write

$$(iii) \quad \phi(k) \simeq \phi_I + \phi_I'(k-k_I) + \frac{1}{6}\phi_I'''(k-k_I)^3,$$

for all values of k . Figure 2 (p. 49) shows this.

Cf. Hogner (1923), equation (33). This approximation may be regarded as a further extension of the principle of stationary phase. For although here no turning point exists, the rate of change of phase is sufficiently small if Θ is sufficiently close to Θ_c . We have

$$\begin{aligned} \zeta_T &= \text{im} \int_{k_I}^{\infty} dk \psi(k) e^{ix\phi(k)} \\ &\simeq \text{im} \psi_I e^{ix\phi_I} \int_0^{\infty} dk \exp ix(\phi_I' k + \frac{1}{6} \phi_I''' k^3) \\ &= \text{im} \pi \psi_I \left(\frac{2}{\phi_I'''} \right)^{\frac{1}{3}} e^{ix\phi_I} \left\{ A_1(p) + iG_1(p) \right\}, \end{aligned}$$

where $p = \rho'_I \left(\frac{2x^2}{\rho_I^m} \right)^{\frac{1}{3}}$. A similar expression for ζ_D holds except that the term in $G_1(p)$ is prefixed by a minus sign. These results for ζ_T and ζ_D may be added to eliminate the term in $G_1(p)$ and we obtain

$$\zeta = 2\pi A_I \left(\frac{2}{\rho_I^m x} \right)^{\frac{1}{3}} A_1(p) \sin \{ \rho_I x + a_I \}.$$

This expression rapidly tends to zero as the angle $|\theta|$ increases. However it is of interest to extend the two wave systems beyond the critical line, so that the manner in which the transverse and divergent wave system interfere with each other may be seen. The expressions for ζ_T and ζ_D when $|\theta| > \theta_c$ may be written

$$\zeta_S \approx \pi A_I \left(\frac{2}{\rho_I^m x} \right)^{\frac{1}{3}} K(p) \sin \{ \rho_I x \pm \kappa(p) + a_I \}. \quad (2)$$

3. The Wave pattern.

The surface displacement given by (1) and (2) is calculated in the vicinity of $\theta = \theta_c$ with $x \leq 10^4$. It is assumed that $\psi(k)$ in the region is a constant, which for convenience we take to be π^{-1} . The parameter used for the calculations is Δ , where $\Delta^2 = 1 - 8 \tan^2 \theta$. This gives

$$\phi_S = \frac{3}{4} \sqrt{\frac{3}{2}} \left(1 \pm \frac{\Delta}{3}\right) \left(\frac{1 \pm \frac{\Delta}{3}}{1 \pm \Delta}\right)^{\frac{1}{2}}, \quad \phi_S'' = \pm \frac{3\Delta}{\sqrt{2}} \left(\frac{1 \pm \Delta}{1 \pm \Delta}\right)^{\frac{1}{2}},$$

$$\phi_I = \frac{3}{4} \sqrt{\frac{3}{2}} \left(\frac{1 - \frac{2}{3}\Delta^2}{1 - \Delta^2}\right)^{\frac{1}{2}}, \quad \phi_I' = \frac{-1}{2\sqrt{2}} \cdot \frac{\Delta^2}{(1 - \Delta^2)^{\frac{1}{2}}},$$

$$q_S x^{-\frac{2}{3}} = \frac{\Delta^2}{2 \cdot 3^{\frac{2}{3}}} \cdot \frac{1 \pm \frac{\Delta}{3}}{1 \pm \Delta} \cdot \left(\frac{1 \pm \Delta}{1 \pm \Delta}\right)^{\frac{1}{2}} \text{ and } p x^{-\frac{2}{3}} = \frac{-\Delta^2}{2 \cdot 3^{\frac{1}{3}} (1 - \frac{2}{3}\Delta^2)(1 - \Delta^2)^{\frac{1}{2}}}.$$

The substitution $y = \frac{x}{2} - \eta$ is also made, so that for small Δ , $\frac{\eta}{x} \approx \frac{\Delta^2(1 + \Delta^2/4)}{4\sqrt{2}}$; and for small η , p and q_S are proportional to $x^{-\frac{1}{3}}$. It is interesting that although there is an optimum choice of axes by which the approximation is most accurate, a change of origin shows that the critical line is fixed only in direction. This is illustrated by the extent of the region where the surface displacement is $O(x^{-\frac{1}{3}})$. Ursell (1960) (pp. 429, 430) has shown that this region extends to a distance $O(x^{\frac{1}{3}})$ on both sides of the critical line. Here it is shown by the form of p and q . For example outside the critical line, for values of p less than about 1 (or for values of η in the range $0 > \eta > -\frac{1}{2}x^{\frac{1}{3}}$), ζ_T, ζ_D and ζ are comparable with ζ_c . For small values of Δ a series expansion is made, and for points just within the critical angle

$$\zeta_S \approx x^{-\frac{1}{3}} \left(1 \pm \Delta + \frac{\Delta^2}{18}\right) K \left[\frac{\Delta^2 x^{\frac{2}{3}}}{2 \cdot 3^{\frac{1}{3}}} \left(1 \pm \frac{2}{3}\Delta + \frac{2}{3}\Delta^2\right) \right] \\ \cdot \sin \left\{ \left(\frac{3}{2}\right)^{\frac{3}{2}} \left(1 + \frac{\Delta^2}{6} + \frac{61}{216} \Delta^4\right) x \pm K \left[\frac{\Delta^2 x^{\frac{2}{3}}}{2 \cdot 3^{\frac{1}{3}}} \left(1 \pm \frac{2}{3}\Delta + \frac{2}{3}\Delta^2\right) \right] \right\}.$$

The variation of amplitude and phase with η for the divergent and transverse wave systems is shown in figure 3 (p. 50). ζ_T, ζ_D and ζ may be written in the forms

$$\zeta_S \propto B_S \sin \left\{ \frac{1}{2}(\phi_T + \phi_D) \pm \epsilon_S \right\}$$

and $\zeta \propto B \sin \left\{ \frac{1}{2}(\phi_T + \phi_D) + \epsilon \right\},$

so that ϵ is the phase of the combined wave system with respect to the curve $\phi_T + \phi_D = \text{constant}$. B has a maximum of about .09 at $\eta \approx 13$, a minimum of about 0.06 at $\eta \approx 28$, and at $\eta = 0, B \approx 0.71$. Since the amplitudes B_S and the phases ϵ_S of the two wave systems are comparable with each other in the vicinity of Θ_c , the phase of the resultant wave system changes rapidly in regions where ϵ_T and ϵ_D are opposed, that is in regions where $\epsilon_T + \epsilon_D = \pi$. Ursell (1960) (figures 2,3,4 and p. 431) shows this and explains this in terms of the zeros of the Airy integral. Here in figure 4 (p. 51) it is shown by a bend in the curve of zero surface displacement in the region of $\eta = 28$. Figure 4 also shows the two wave systems extended beyond the critical line. Here the amplitude of both waves steadily decrease and the crests of one system rapidly approach the troughs of the other.

The general character of the results is similar to those of Hogner (1923) and Ursell (1960). However differences in detail are to be expected since a different

form of ϕ is used here. There is a corresponding difference in ψ but this is not taken into account since the nature of ψ depends on the ship, as does the nature of the waves. This partly accounts for the difference between Hogner's curves of the divergent and transverse wave amplitudes and those shown in figure 3. There is also another reason for this difference. Hogner defines ζ_1 and ζ_2 differently from their counterparts in this paper. In effect the values ζ_T and ζ_D are augmented and diminished respectively by an integral taken along a path in the complex k -plane from the point k_I on the real axis to $-i\infty$. The contours of the system, not shown here, are similar to those given by Ursell (1960), figure 4. Ursell (1960) derives for ζ a full asymptotic development whose form remains unchanged across the critical line. However the simplicity and directness of the method described here have obvious advantages if a first approximation is required.

Further, when $\psi(k)$ is variable the method quickly gives an idea of the magnitude of the transverse and divergent waves over the whole wave pattern. Thus for ships of broad beam moving at slow speeds the transverse waves dominate the pattern, but for slender high speed craft the divergent waves are most prominent. For example

$$\psi = \exp \left[-\frac{ag}{u^2} \cosh k - \frac{bg}{u^2} \cosh k \sinh |k| \right] \quad (3)$$

represents a 'forcive' whose strength at any point is

proportional to

$$(a^2 + x^2)^{-1} (b^2 + x^2)^{-1}.$$

This forcive is of infinite extent, but beyond certain points its effect is negligible, and expression (3) therefore represents a disturbance whose beam-length ratio is b/a .

If the ship is moving at an angle of , (3) becomes

$$\psi(k) = \exp\left[-\frac{ag}{U^2} \cosh k |\cos\beta - \sin\beta \sinh k|\right] \cdot \exp\left[-\frac{bg}{U^2} \cosh k |\sin\beta + \cos\beta \sinh k|\right].$$

In particular for points at θ ($\theta > 0$) well within the critical angle, this gives expressions for the amplitudes of the transverse and divergent waves which are proportional to

$$\exp\left[-a' \sqrt{\frac{1 \pm \Delta/3}{1 \pm \Delta}} \left|\cos\beta + \frac{\sin\beta}{\sqrt{2}} \sqrt{\frac{1 \mp \Delta}{1 \pm \Delta}}\right|\right] \exp\left[-b' \sqrt{\frac{1 \pm \Delta/3}{1 \pm \Delta}} \left|\sin\beta - \frac{\cos\beta}{\sqrt{2}} \sqrt{\frac{1 \mp \Delta}{1 \pm \Delta}}\right|\right] \cdot \left(\frac{1 \pm \Delta/3}{1 \pm \Delta}\right)^{5/3} (\Delta x)^{-1/2}$$

where

$$a' = \frac{ag}{U^2} \sqrt{\frac{3}{2}} \quad \text{and} \quad b' = \frac{bg}{U^2} \sqrt{\frac{3}{2}}.$$

For corresponding points at $-\theta$, we obtain the same expressions for the wave amplitudes except that β is replaced by $-\beta$. The relative amplitudes for $\beta = 0$, $\beta = \pi/2$ are shown in figure 5.(p. 52) where values of $a' = 1$ and $b' = 0.1$ have been taken. It shows that for such a craft moving ahead at high speed, the divergent waves are most prominent, whereas a broadside current

produces mostly transverse waves.

An interesting case arises when $\beta = \tan^{-1} \sqrt{2} \approx 54\frac{3}{4}^\circ$, i.e. when the ship lies parallel to the wave crests at the critical line, for then the largest wave amplitudes occur. For negative Θ very large divergent waves occur which dominate the pattern. When $|\Theta|$ is smaller or when Θ is positive, however, transverse waves of smaller amplitude are most apparent. Figure 6 (p.53) illustrates this. These results are similar to those of Scorer (1956) (No.351), and Scorer (1956) (No.354), where lee waves caused by direct and oblique airflow over an isolated hill are described.

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CHAPTER III.CAPTIONS TO FIGURES

- Fig.1 An illustration of the approximation to $\phi(k)$ when
(p.48) $|\Theta| \leq \Theta_c$. The approximation is most accurate where
the rate of change of phase is zero.
- Fig.2 The approximation to $\phi(k)$ when $|\Theta| > \Theta_c$. The
(p.49) approximation is most accurate where the rate of
change of phase is least.
- Fig.3 Curves of amplitude and phase for the divergent
(p.50) and transverse wave systems in the critical region
for $x = 10^4$. The amplitude of both systems are
comparable with each other, but the variation of
amplitude and phase is more rapid for the transverse
waves.
- Fig.4 Crests and troughs of the two wave systems in a
(p.51) critical region in the vicinity of $x = 10^4$. Curves
of zero surface displacement are shown as broken
lines. These curves turn sharply in the region of
 $\eta = 28$, where the two wave systems are opposed,
because the two systems have nearly equal amplitudes.
Beyond the critical line the amplitudes of both

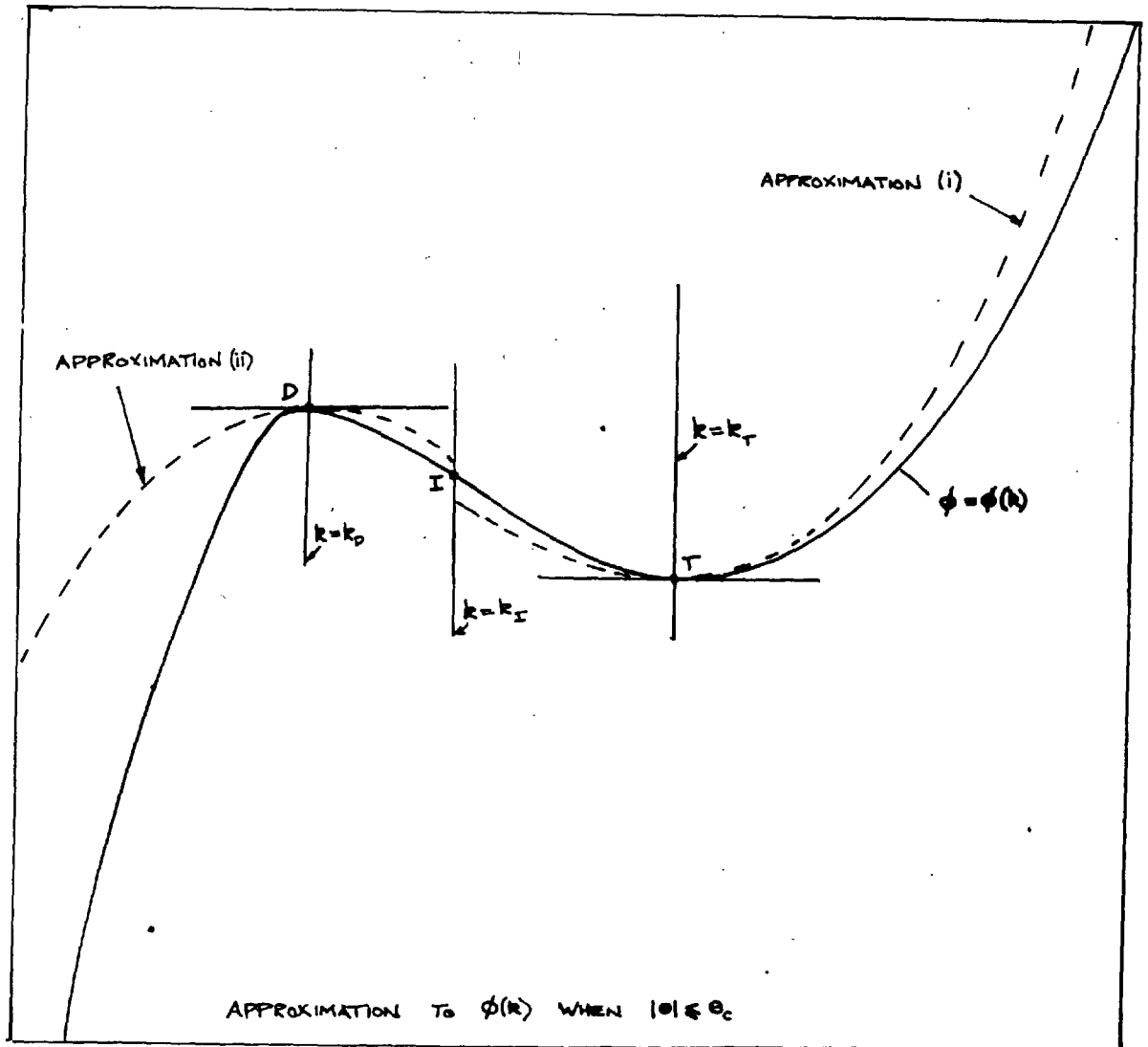
Fig.4 systems are equal and decrease like $\frac{1}{\eta}$. The troughs
 cont'd of one system rapidly approach the crests of the
 other and the resultant surface displacement dies
 away extremely rapidly, like $(\eta x)^{-\frac{1}{2}} \exp(-c \frac{\eta^3}{x})$.
 But for values of η in the range $0 > \eta > -\frac{1}{2}x^{\frac{1}{3}}$, ζ_T , ζ_D
 and ζ are comparable with ζ_c .

Fig.5 Variation of the amplitude of the transverse and
 (p.52) divergent waves for a slender high speed craft,
 whose beam-length ratio is 1:10.

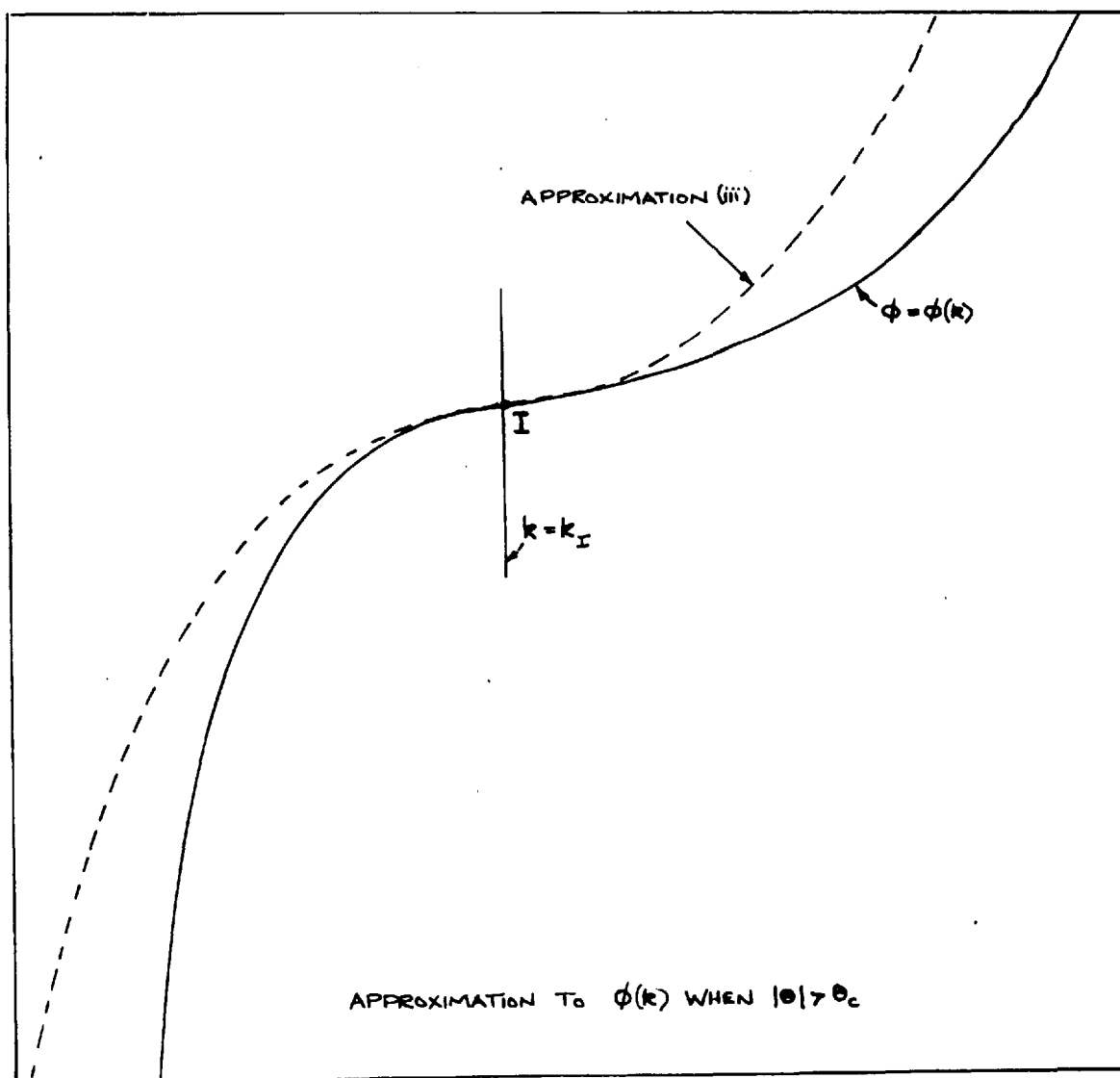
(a) For no yaw. $\beta = 0$.

(b) Lateral motion. $\beta = \pi/2$. This also illustrates
 the nature of the relative wave amplitudes for a
 slow moving broad beamed ship.

Fig.6 Variation of the amplitude of the transverse and
 (p.53) divergent waves for a slender high speed craft whose
 beam-length ratio is 1:10, for an angle of yaw
 $\beta = \tan^{-1} \sqrt{2} \approx 54\frac{3}{4}^\circ$. Here the centre line of
 the ship is parallel to the wave crests at the
 critical line on the after port side. (i.e. for
 negative Θ).

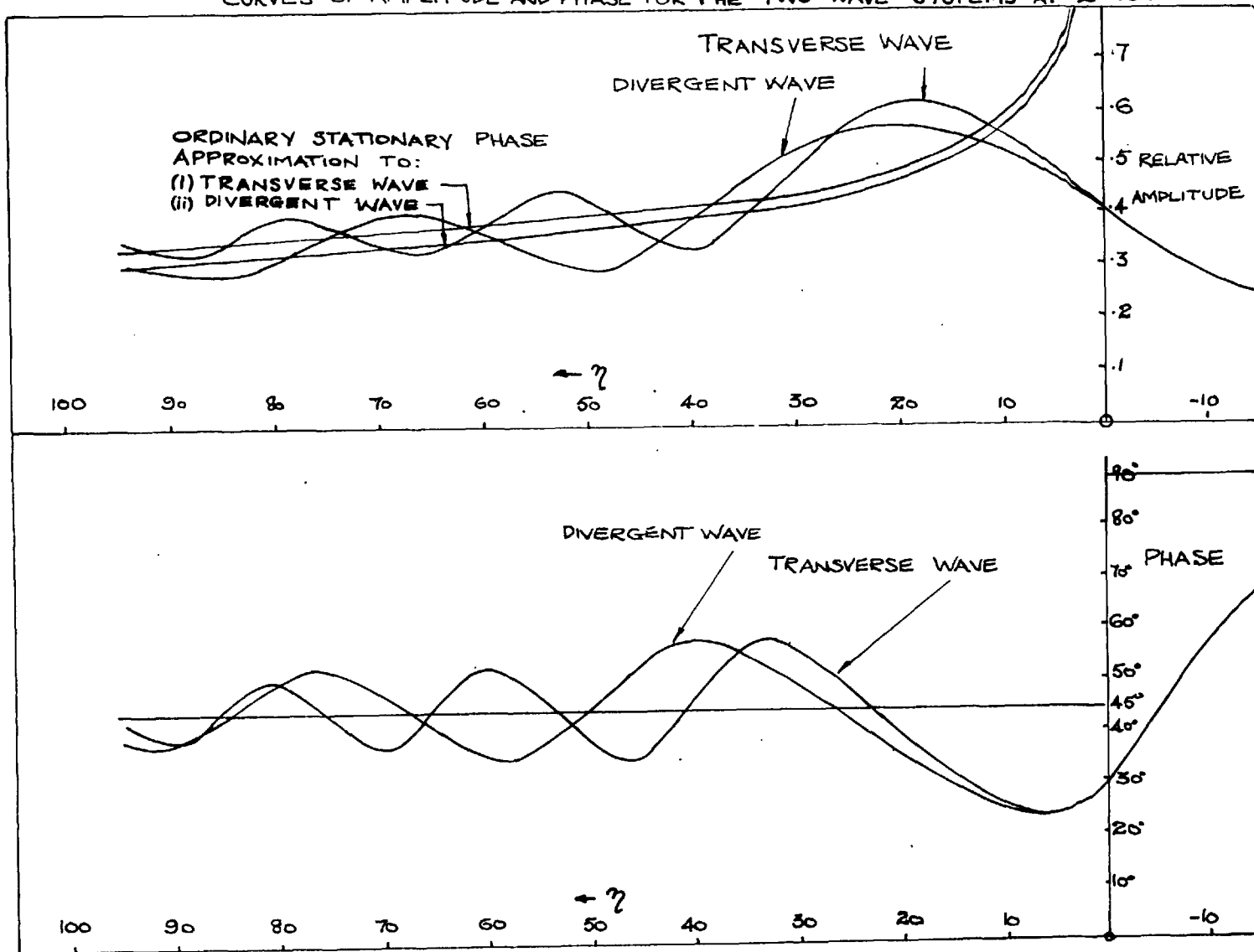


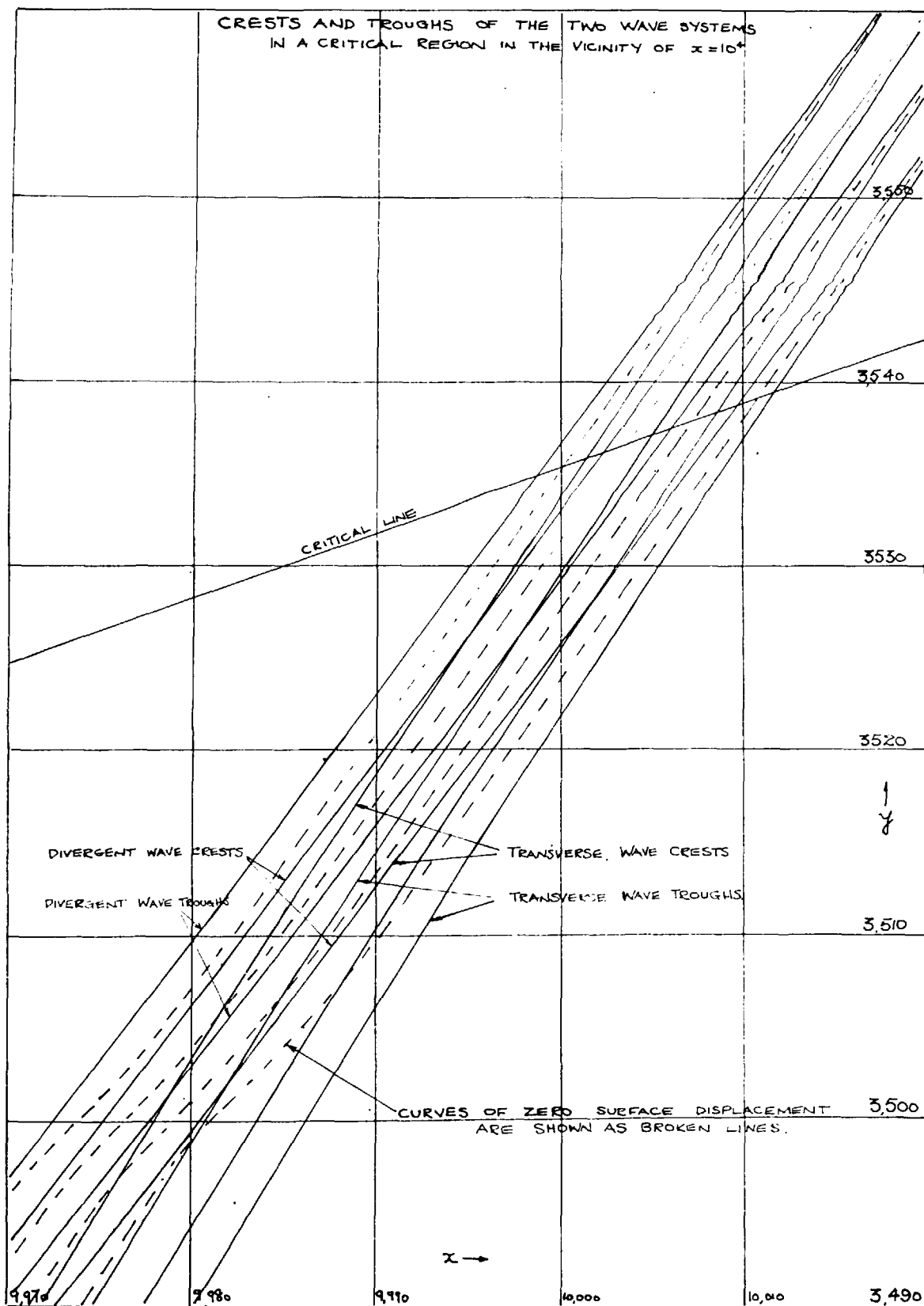
CHAPTER III. FIG. 1.



CHAPTER III FIG. 2.

CURVES OF AMPLITUDE AND PHASE FOR THE TWO WAVE SYSTEMS AT $x=10^4$





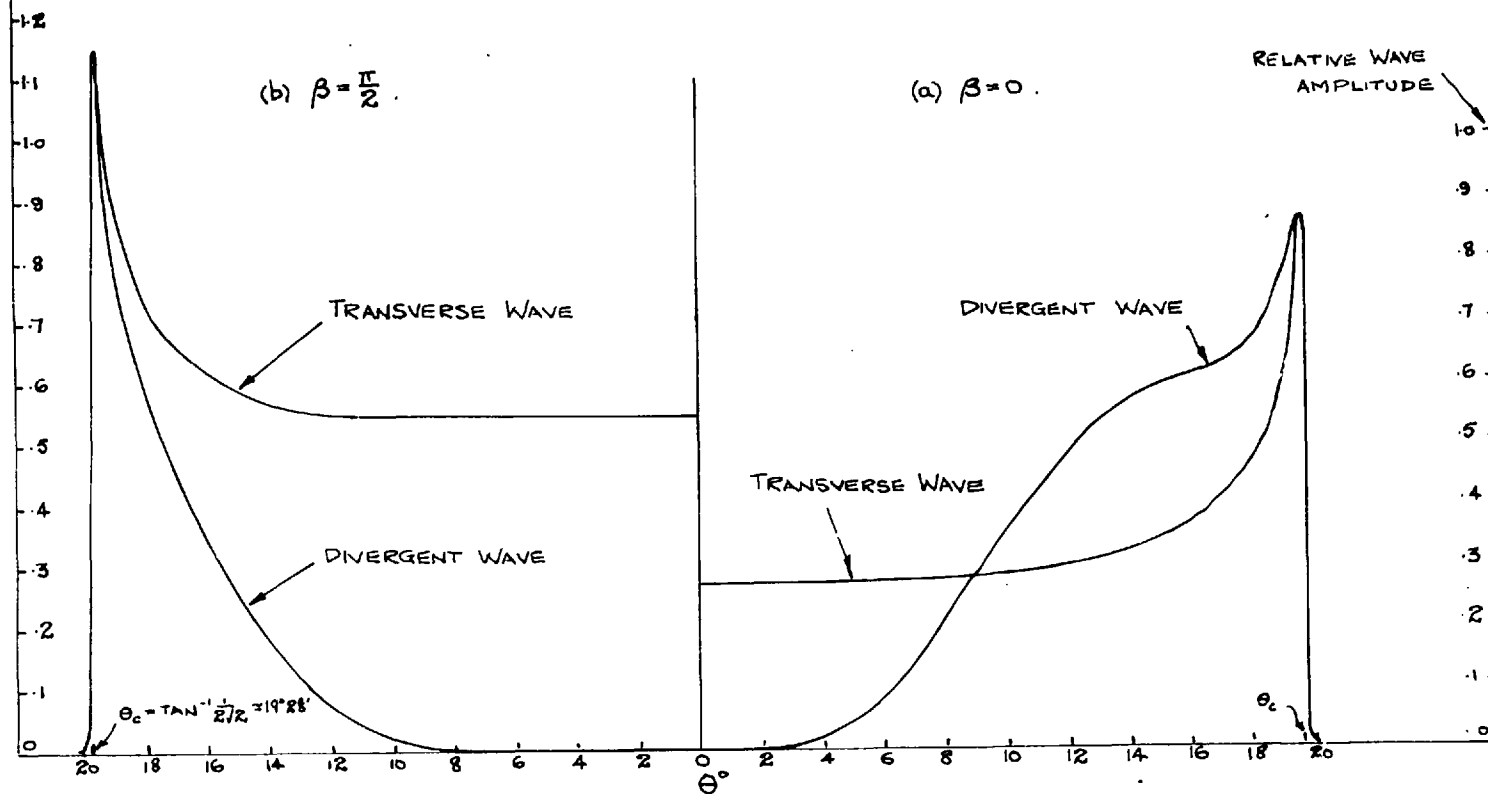
CHAPTER III FIG. 4

CHAPTER III. FIG. 5.

WAVE AMPLITUDES FOR SLENDER HIGH SPEED CRAFT.

ANGLE OF YAW $\beta = 0$. (CASE (a)).

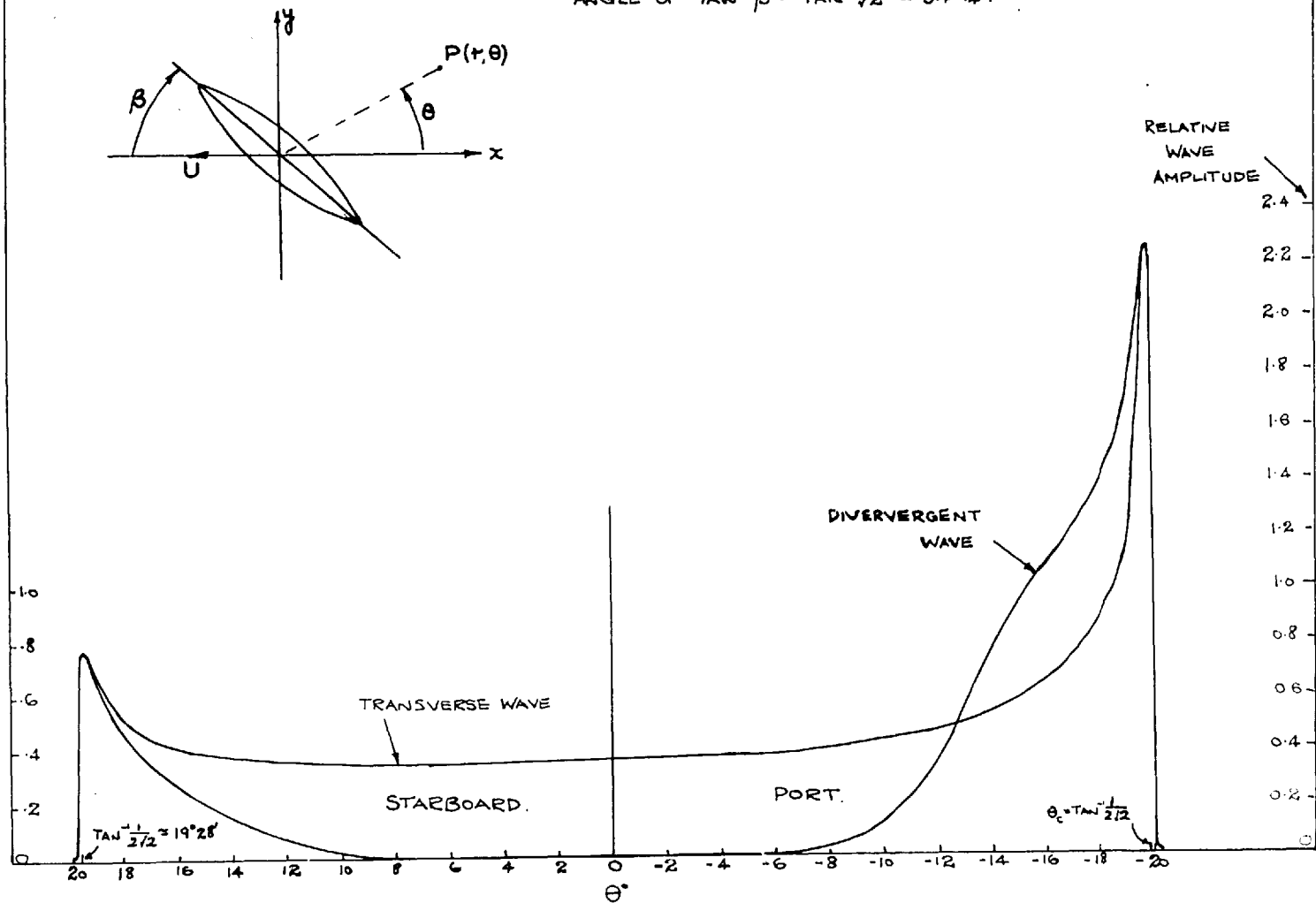
$= \frac{\pi}{2}$. (CASE (b)).



CHAPTER III. FIG. 6.

WAVE AMPLITUDES FOR SLENDER HIGH SPEED CRAFT.

ANGLE OF YAW $\beta = \tan^{-1}/2 \approx 54^{\circ}34'$.



The generation of wave energy at a fluid interface by the passage of a vertically moving slender body

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(Manuscript received 21 March 1960; in revised form 8 July 1960. Communicated by R. S. Scorer)

SUMMARY

In this paper an estimate is made of the energy absorbed by wave motion at the interface of two superposed fluids when a body passes from one fluid to the other. The fluids are supposed perfect, incompressible, and bounded only by the interface, and axi-symmetric and symmetrical two-dimensional cases are considered. The body is supposed to move vertically and approach the interface from a great distance, and after crossing the interface to recede to a great distance, the speed being kept steady throughout. It is assumed that there were no waves originally present at the interface and that no splashing occurs during the crossing. The linear approximation to irrotational waves is used at the interface, and the slender body approximation is used at the boundary of the body. Transform methods are used to solve the Laplace equation for the velocity potential. Expressions are found for the waves set up as the body crosses the interface, and the energy used to generate the waves is found from a calculation of the work done against the wave resistance. The wave energy may be regarded as the sum of the wave energies for two cases: one in which the density difference between the two fluids is small, and another, in which the density of the upper fluid is zero. For the special cases of a spheroid and a spindle-shaped body, curves are shown of the wave energy against a dimensionless form of the inverse speed. Criteria are given for when maximum wave energy is developed for a thermal, and an estimate of this energy is made. This is done in Section 6.

1. INTRODUCTION

The problem arises in the study of the ascent of thermals through the atmosphere. During the ascent, the thermal acquires kinetic energy derived from its potential energy by the mechanism of the buoyancy, and subsequently much of this energy is dissipated into turbulent eddies within the thermal itself. However, if the surrounding air layers are disturbed into wave motions by the passage of the thermal, energy is radiated away from it by them and this provides another means by which the thermal loses energy. This action involves a dispersion rather than a dissipation of energy. Although eventually all the kinetic energy is dissipated by viscosity, the wave motions spread the energy over a much larger region than that of the thermal and its immediate surroundings. Here the eddy motion is predominant.

An estimate of the amount of energy dispersed into waves enables an estimate to be made of the effect on the thermal. For this purpose the thermal is regarded as a local region of turbulence such that if we move with the thermal during its ascent, this region is bounded by some closed surface, outside of which the motion of the air is well behaved. Both a steady speed of ascent and a fixed shape of the surface which confines the turbulence are assumed. In this paper we are interested in the case when the thermal rises through an inversion. An inversion is taken here to mean a thin layer of air where the temperature changes rapidly, so that the temperature may be regarded as discontinuous at the height of the layer. For these purposes the atmosphere may be regarded as an incompressible fluid whose density is equal to the reciprocal of the potential temperature, and an inversion is represented by a discontinuity in density, i.e., by a fluid interface. This analogy is well known and is discussed, for example, by Scorer (1952). The problem of how much wave energy is developed as a thermal ascends through an inversion corresponds to the case when the density difference between the two superposed fluids is small. The two-dimensional case of the problem occurs when a bent-over, buoyant chimney plume meets an inversion. The case which occurs when the upper fluid is absent or has negligible density has no application to the problem of the thermal rising through an inversion,

but it is included since this case is as readily handled. The theory assumes that the thermal crosses the interface without appreciable loss of speed. Subsequently the speed may change considerably, but this will not affect the result since most of the wave energy is developed when the thermal crosses the interface. Once an estimate of the wave energy is obtained, a comparison may be made with the kinetic energy of the thermal or with the work done in crossing the interface against the hydrostatic pressure. This comparison then gives a measure of the extent to which the generation of waves affects the motion of the thermal.

~~Fig. 1 (Plate I) is a photograph taken by Mr. J. M. Richards during one of his experiments at Imperial College on the behaviour of thermals. It shows a wave crest at an interface which has been induced by the passage of a (two-dimensional) thermal.~~

2. THE AXI-SYMMETRIC CASE

The boundary conditions

For the axisymmetric case, cylindrical coordinates are used, with the following notation :

- r — horizontal coordinate
- z — height. The z axis is coincident with that of the body
- (u, w) — velocity of a fluid particle
- ϕ — velocity potential, where

$$u = -\frac{\partial\phi}{\partial r} \text{ and } w = -\frac{\partial\phi}{\partial z}.$$

The velocity potential ϕ describes the motion in both upper and lower fluids.

Thus $\phi = \phi$ (lower fluid) if $z < 0$, and $\phi = \phi$ (upper fluid) if $z > 0$.

W — steady vertical velocity of the body, positive upwards

a — half-width of the body

b — half-length of the body

ρ_2 — density of the lower fluid

$\rho_1 = \rho_2 - \Delta\rho$ = density of the upper fluid

$\rho_2 - \delta\rho$ = density of the thermal

$\sigma = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$

$\nu = gb\sigma/W^2$

The undisturbed interface is taken to be the surface $z = 0$, and the centre of the body is supposed to cross the surface at $t = 0$. The body is replaced by line sources and sinks distributed along the z -axis, and the strength of these sources and sinks may be written as

$$-2\pi W f(z - Wt) f'(z - Wt). \quad . \quad . \quad . \quad . \quad (1)$$

The accent denotes partial differentiation with respect to z . If we travel along the z -axis with a velocity W the sources have a relatively steady strength in a stream of velocity W and the separating stream lines represent the surface of a body in the usual way. At points near the z -axis

$$u \simeq -\frac{W}{r} f(z - Wt) f'(z - Wt), \quad . \quad . \quad . \quad . \quad (2)$$

or in terms of the velocity potential

$$\lim_{r \rightarrow 0} \left(r \frac{\partial\phi}{\partial r} \right) = W f(z - Wt) f'(z - Wt) \quad . \quad . \quad . \quad (3)$$

If the axisymmetric body whose shape is given by $r_a = f(z)$ is a slender one so that $r_a = f(z)$ is small, then Eq. (2) gives

$$-\left(\frac{u}{W}\right)_{r=r_a} \approx f'(z - Wt)$$

Hence, if we travel along the z -axis with a velocity W

$$\left(\frac{u}{w - W}\right)_{r=r_a} = f'(z) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

approximately if $|w| \ll |W|$. But Eq. (4) is the condition that $r_a = f(z)$ is a fixed boundary in a stream of velocity $-W$. Hence the source and sink distribution whose strength is given by Eq. (1) is equivalent to a body which moves along the z -axis with a velocity W and whose shape is given approximately by $r_a = f(z)$ if it is slender. The accuracy of this approximation is discussed, for example, by Temple (1958).⁶⁾

At the interface the pressure is continuous, and hence, with the usual linear approximations,

$$\rho \left(\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right) \text{ is continuous at } z = 0 \quad . \quad . \quad . \quad . \quad (5)$$

as is shown by Stoker (1957, pp. 21-22). To the same degree of approximation, the vertical velocity of the fluid particles at $z = 0$ is continuous, and hence

$$\frac{\partial \phi}{\partial z} \text{ is continuous at } z = 0 \quad . \quad . \quad . \quad . \quad (6)$$

Since Eqs. (5) and (6) involve linear approximations, no attempt is made to refine the slender-body approximation (1).

3. CALCULATION OF THE VELOCITY POTENTIAL

The equation satisfied by $\phi(r, z, t)$ is the Laplace equation in cylindrical polar coordinates:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad . \quad . \quad . \quad . \quad (7)$$

This equation is transformed by means of the Hankel transform

$$\bar{\phi}(k, z, t) = \int_0^\infty dr \cdot r J_0(kr) \phi(r, z, t) \quad . \quad . \quad . \quad . \quad (8)$$

If we multiply Eq. (7) by $r J_0(kr)$ and integrate with respect to r from ϵ to ∞ and let ϵ tend to zero we obtain

$$-k^2 \bar{\phi} + \frac{\partial^2 \bar{\phi}}{\partial z^2} = \lim_{r \rightarrow 0} \left(r \frac{\partial \phi}{\partial r} \right)$$

Here we assume that $\lim_{r \rightarrow \infty} (r \partial \phi / \partial r) = 0$ and that $\lim_{r \rightarrow \infty} (r^{\frac{1}{2}} \phi) = 0$. Therefore

$$-k^2 \bar{\phi} + \frac{\partial^2 \bar{\phi}}{\partial z^2} = W f(z - Wt) f'(z - Wt), \quad . \quad . \quad . \quad . \quad (9)$$

by Eq. (3).

For convenience we replace the r.h.s. of Eq. (9) by

$$\frac{W}{4\pi i} m F(m) e^{-im(z - Wt)}$$

solve for $\bar{\phi}$, and then integrate with respect to m from $-\infty$ to ∞ . Here

$$F(m) = \int_{-\infty}^{\infty} dz \{f(z)\}^2 e^{imz} = \int_{-b}^b dz \{f(z)\}^2 e^{imz}, \quad (10)$$

so that

$$\int_{-\infty}^{\infty} dm \frac{W}{4\pi i} m F(m) e^{-im(z-Wt)} = W f(z-Wt) f'(z-Wt).$$

For example, for a spheroid

$$f(z) = a \sqrt{1 - \left(\frac{z}{b}\right)^2} \text{ and } F(m) = \frac{4a^2}{b^2} \left(\frac{\sin mb - mb \cos mb}{m^3} \right).$$

The solution of Eq. (9) which satisfies the condition Eq. (6) may then be written

$$\bar{\phi} = \frac{iW}{4\pi} \cdot \frac{mF(m)}{k^2 + m^2} e^{-ik(z-Wt)} + (\text{sgn } z) G(k, m, t) e^{-k|z|}, \quad (11)$$

where $\text{sgn } z = +1$ if $z > 0$, and $\text{sgn } z = -1$, if $z < 0$. $G(m, k, t)$ is some function of k, m , and t , and is determined from the boundary condition (5) which gives

$$\rho_1 \left(\frac{\partial^2 \bar{\phi}}{\partial t^2} + g \frac{\partial \bar{\phi}}{\partial z} \right)_{z=+0} = \rho_2 \left(\frac{\partial^2 \bar{\phi}}{\partial t^2} + g \frac{\partial \bar{\phi}}{\partial z} \right)_{z=-0}.$$

The substitution in this equation for $\bar{\phi}$ as given by Eq. (11) gives

$$\frac{\partial^2 G}{\partial t^2} + g \sigma k G = \frac{W}{4\pi} \cdot \frac{m^2 (g - iW^2 m)}{k^2 + m^2} F(m) e^{imWt}.$$

Hence

$$G = \frac{W\sigma}{4\pi} \cdot \frac{m^2 (g - iW^2 m) e^{imWt}}{(k^2 + m^2)(gk - W^2 m^2)} + d_1(k, m) \cos \sqrt{g\sigma k} t \\ + d_2(k, m) \sin \sqrt{g\sigma k} t,$$

where d_1 and d_2 are functions of k and m and may be determined from the initial condition that no waves are originally present. If $G = G_0$, say, when $g = 0$, then

$$G_0 = \frac{iW\sigma}{4\pi} \cdot \frac{mF(m) e^{imWt}}{k^2 + m^2}.$$

G may be written $G = G_0 + (G - G_0)$, where $(G - G_0)$ represents a term due to gravity. If we now integrate w.r.t. m from $-\infty$ to ∞ and then use the inverse of transform (8), we obtain

$$\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4,$$

where

$$\phi_1 = \frac{iW}{4\pi} \int_0^\infty dk \int_{-\infty}^\infty dm \frac{k J_0(\tau k) m F(m) e^{-im(z-Wt)}}{k^2 + m^2}, \\ \phi_2 = (\text{sgn } z) \cdot \frac{iW\sigma}{4\pi} \int_0^\infty dk \int_{-\infty}^\infty dm \frac{k J_0(\tau k) m F(m) e^{imWt - |z|k}}{k^2 + m^2}, \\ \phi_3 = (\text{sgn } z) \cdot \frac{g\sigma}{4\pi W} \int_0^\infty dk \int_{-\infty}^\infty dm \frac{k J_0(\tau k) (i\sigma k - m) m F(m) e^{imWt - |z|k}}{\left((k^2 + m^2) \left(m^2 - \frac{g\sigma k}{W^2} \right) \right)},$$

and

$$\phi_4 = (\text{sgn } z) \cdot \frac{g\sigma}{4\pi W} \int_0^\infty dk \cdot k J_0(\tau k) e^{-|z|k} \cdot \{ d_1(k) \cos \sqrt{g\sigma k} t \\ + d_2(k) \sin \sqrt{g\sigma k} t \} \quad (12)$$

(The principal value of the expression for ϕ_3 is understood). $\phi_1 + \phi_2$ represents the solution when gravity is absent, and the remaining terms $\phi_3 + \phi_4$ represent the effect of gravity which is sought.

From Eq. (10) it follows that

$$|F(m)| < C e^{b|im(m)|}$$

where C is a fixed constant. It will also be assumed that the body is symmetrical about its centre so that $f(z) = f(-z)$ and $F(m) = F(-m)$. If the body is not in the interface, so that $|Wt| > b$, it is possible to integrate round an infinite semi-circle in the m -plane and so evaluate the m -integral which occurs in the integral expression for ϕ_3 . The residues from the poles at $m = \pm ik$, $m = \pm \sqrt{g\sigma k}/W$ give a contribution to this integral, and we obtain

$$\begin{aligned} \phi_3 = & -(\operatorname{sgn} Wt) \frac{g\sigma}{4W} \int_0^\infty dk \frac{J_0(rk) e^{-|z|k} F\left(\frac{\sqrt{g\sigma k}}{W}\right)}{k + g\sigma/W^2} \left\{ \sigma k \cos \sqrt{g\sigma k} t \right. \\ & \left. - \frac{\sqrt{g\sigma k}}{W} \sin \sqrt{g\sigma k} t \right\} \\ & - \frac{g\sigma}{4W} \int_0^\infty dk \frac{k J_0(rk) e^{-|z|k} F(ik)}{k + g\sigma/W^2} \cdot (1 - (\operatorname{sgn} Wt z) \sigma^2) e^{-|Wt|k}. \end{aligned}$$

The positive root is taken throughout. The first integral is the contribution from the residues and represents a wave train on the interface. If no waves are present in the initial condition when $|Wt| > b$, $t < 0$, it follows that $d_1(k)$ and $d_2(k)$ are such that

$$\begin{aligned} \phi_4 = & -(\operatorname{sgn} Wz) \frac{g\sigma}{4W} \int_0^\infty dk \frac{J_0(rk) e^{-|z|k} F\left(\frac{\sqrt{g\sigma k}}{W}\right)}{k + \frac{g\sigma}{W^2}} \cdot \left\{ k \cos \sqrt{g\sigma k} t \right. \\ & \left. - \frac{\sqrt{g\sigma k}}{W} \sin \sqrt{g\sigma k} t \right\}, \quad (13) \end{aligned}$$

since then the residues from ϕ_3 cancel with ϕ_4 and all wave expressions such as (12) are absent. If we put $t > 0$, $|Wt| > b$, the potential representing the waves generated during the passage of the body through the interface is obtained from ϕ_4 plus the residues from ϕ_3 . In this case the residues reinforce ϕ_4 , and so

$$\phi_{\text{waves}} = 2\phi_4,$$

where ϕ_4 is given by expression (13). The velocity potential which represents the effect of gravity is given by $\phi_3 + \phi_4$ where ϕ_3 is given by expression (12) and ϕ_4 by expression (13). If the shape of the body is written as

$$r_a = a f_1\left(\frac{z}{b}\right) \text{ where } f_1\left(\frac{z}{b}\right) = 0 \text{ if } |z| > b,$$

then Eq. (10) may be written

$$F(m) = 2a^2 b F_1(mb)$$

where

$$F_1(m) = \int_0^1 d\theta \{f_1(\theta)\}^2 \cos m\theta.$$

If we substitute this expression for $F(m)$ in the expressions for ϕ_3 and ϕ_4 , and write

$$\frac{Wt}{b} = \tau; \quad \frac{z}{b} = z_1; \quad \frac{r}{b} = r_1; \quad (14)$$

we obtain

$$\phi_3 = (\operatorname{sgn} z_1) \frac{a^2 W \nu}{2\pi b} \int_0^\infty dk \int_{-\infty}^\infty dm \frac{k J_0(r_1 k) m F_1(m) (i\sigma k - m) e^{im\tau - |z_1|k}}{(k^2 + m^2)(m^2 - \nu k)}, \quad (15)$$

and

$$\phi_4 = -(\operatorname{sgn} W z_1) \frac{a^2 W \nu}{2b} \int_0^\infty dk \frac{J_0(r_1 k) e^{-|z_1|k} F_1(\sqrt{\nu k})}{k + \nu} \{ \sigma k \cos \sqrt{\nu k} \tau - \sqrt{\nu k} \sin \sqrt{\nu k} \tau \} \quad (16)$$

From Eq. 13 we see that the amplitude of the wave system is given by

$$W \propto \frac{a^2}{b} \times \text{some function of } \nu \text{ and } \tau.$$

4. THE WAVE ENERGY

The upthrust R on the body, owing to the wave resistance, is found from a consideration of the pressure forces on its surface. Euler's equation of motion in the horizontal direction is

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} = 0$$

if second-order terms are omitted.⁽²⁾ Hence if the static thrust is omitted, the pressure p is given approximately by $p = \rho \partial \phi / \partial t$, and the upthrust R is given by

$$R = \int_{-\infty}^{\infty} dz \{ p_a f'(z - Wt) \} 2\pi f(z - Wt)$$

where p_a denotes the pressure at the surface of the body. The wave resistance is found by the substitution for p_a in terms of ϕ_3 and ϕ_4 :

$$R \simeq 2\pi \int_{-\infty}^{\infty} dz f(z - Wt) f'(z - Wt) \rho \frac{\partial}{\partial t} (\phi_3 + \phi_4),$$

where ϕ_3 and ϕ_4 are given by (12) and (13), with $r = r_a$. If we write

$$f(z - Wt) f'(z - Wt) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} ds s F(s) e^{-is(z - Wt)},$$

we obtain

$$R = \frac{1}{2i} \left\{ \rho_2 \int_{-\infty}^0 dz e^{-isz} + \rho_1 \int_0^\infty dz e^{-isz} \right\} \left\{ \int_{-\infty}^{\infty} ds s F(s) e^{isWt} \frac{\partial}{\partial t} (\phi_3 + \phi_4) \right\}.$$

We assume that $J_0(r_a k) \simeq 1$, and carry out the differentiation with respect to t and the integration with respect to z under the integral signs $\int ds$, $\int dk$ and $\int dm$. Expressions for R_3 and R_4 are then obtained, where R_3 corresponds to ϕ_3 , R_4 corresponds to ϕ_4 , and $R_3 + R_4 = R$. In terms of the notation (14) these expressions are:

$$R_3 = \frac{2a^4 W^2 (\rho_1 + \rho_2) \nu}{\pi b^2} \int_0^\infty ds \int_0^\infty dk \int_{-\infty}^\infty dm \frac{F_1(s) F_1(m) s k m^2 (\sigma k \cos m\tau - m \sin m\tau) (s \cos s\tau + \sigma k \sin s\tau)}{(k^2 + m^2)(k^2 + s^2)(m^2 - \nu k)}, \quad (17)$$

and

$$R_4 = -(\operatorname{sgn} W) \cdot \frac{a^4 W^2 (\rho_1 + \rho_2) \nu^{3/2}}{b^2} \int_0^\infty ds \int_0^\infty dk \frac{F_1(s) F_1(\sqrt{\nu k}) s \sqrt{k}}{(k^2 + s^2)(k + \nu)} \cdot (\sigma k \sin \sqrt{\nu k} \tau + \sqrt{\nu k} \cos \sqrt{\nu k} \tau) (s \cos s\tau + \sigma k \sin s\tau). \quad (18)$$

If $|\tau| > 1$, the integration with respect to s yields a factor $k e^{-|\tau|k}$ and the integration

with respect to m yields a bounded integrand for the remaining integration with respect to k . Therefore

$$|R_{3,4}| < c \int_0^\infty dk \cdot k^2 e^{-|\tau|k} = 0 \quad (\tau^{-3}).$$

It follows that the wave resistance to motion when the body is not crossing the interface is of the order (distance) $^{-3}$, confirming that most of the wave energy is developed as the body crosses the interface.

The work done by the body against the wave resistance is E , say, where

$$\begin{aligned} E &= -(\operatorname{sgn} W) \int_{-\infty}^{\infty} dz (R_3 + R_4) \\ &= -(\operatorname{sgn} W) \cdot 2b \cdot \lim_{T \rightarrow \infty} \int_0^T d\tau (R_3^+ + R_4^+) \\ &= E_3 + E_4, \text{ say,} \end{aligned}$$

and where R_3^+ and R_4^+ denote those parts of expressions (17) and (18) which are even functions of τ . Now the integrand of R_3^+ contains the factor

$$(s + m) \cos(s + m) \tau - (s - m) \cos(s - m) \tau.$$

If the integration with respect to τ is carried out under the integral signs $\int ds$, $\int dk$ and $\int dm$, and the limit as $T \rightarrow \infty$ is taken, this factor becomes

$$\frac{1}{2} \{ (s + m) \delta(s + m) + (s - m) \delta(s - m) \},$$

where δ denotes the Dirac delta function. The integration with respect to s then shows that E_3 is zero. To evaluate E_4 , we again integrate with respect to τ under the integral signs, and obtain

$$E_4 = \frac{2a^4 W^2 (\rho_1 + \rho_2) \nu^{3/2}}{b^2} \int_0^\infty ds \int_0^\infty dk \frac{F_1(s) F_1(\sqrt{\nu k}) s \sqrt{k}}{(k + \nu)(k^2 + s^2)} \lim_{T \rightarrow \infty} \int_0^T d\tau (\sigma^2 k \sin s\tau \sin \sqrt{\nu k} \tau + s \sqrt{\nu k} \cos s\tau \cos \sqrt{\nu k} \tau).$$

If we write $k = \lambda^2/\nu$, integrate with respect to τ , and take the limit as $T \rightarrow \infty$, this becomes

$$\begin{aligned} E_4 &= \frac{2a^4 W^2 (\rho_1 + \rho_2) \nu}{b} \int_0^\infty ds \int_0^\infty d\lambda \frac{F_1(s) F_1(\lambda) \lambda^3}{(\lambda^2 + s^2)(\lambda^4 + s^2 \nu^2)} \{ (\nu^2 s - \sigma^2 \lambda^3) \delta(s + \lambda) \\ &\quad + (\nu^2 s + \sigma^2 \lambda^3) \delta(s - \lambda) \} \\ &= \frac{2a^4 g \Delta \rho}{b} \int_0^\infty ds \frac{s^3 [F_1(s)]^2}{(s^2 + \nu^2)^2} (\sigma^2 s^2 + \nu^2). \end{aligned}$$

This result may be written

$$\pi \frac{E}{a^4 g \Delta \rho} = (1 - \sigma^2) e_1(\nu) + \sigma^2 e_2(\nu), \quad (19)$$

$$\text{where} \quad e_1(\nu) = 2\nu^2 \int_0^\infty ds \cdot \frac{s^3 [F_1(s)]^2}{(\nu^2 + s^2)^2}, \quad (20)$$

$$\text{and} \quad e_2(\nu) = 2 \int_0^\infty ds \cdot \frac{s^3 [(F_1(s))]^2}{\nu^2 + s^2}. \quad (21)$$

The expression (19) may also be derived directly from a consideration of the wave energy from ϕ_4 , Eq. (16). Approximate methods were used to evaluate the integral expressions for $e_1(\nu)$ and $e_2(\nu)$. If we write $e_1(\nu)$ in the form

$$e_1(\nu) = -\nu \int_0^\infty ds \cdot \tan^{-1} \frac{s}{\nu} \cdot \frac{d^2}{ds^2} \{ s F_1(s) \}^2$$

The Laplace equation for the velocity potential is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad . \quad . \quad . \quad . \quad (7)'$$

and we write
$$\bar{\phi}(r, k, t) = \int_0^\infty dr \cos kr \cdot \phi(r, z, t) \quad . \quad . \quad . \quad . \quad (8)'$$

If we multiply (7)' by $\cos kr$ and integrate with respect to r from 0 to ∞ , we obtain

$$-k^2 \bar{\phi} + \frac{\partial^2 \bar{\phi}}{\partial z^2} = W f'(z - Wt) \quad . \quad . \quad . \quad . \quad (9)'$$

Here we assume that $r \cdot \phi(r, z, t) \rightarrow 0$ as $r \rightarrow 0$, and that $\phi \rightarrow 0$ as $r \rightarrow \infty$. The r.h.s. of this equation is replaced by

$$\frac{W}{2\pi i} m F(m) e^{-im(z-Wt)}, \quad . \quad . \quad . \quad . \quad (10)'$$

where
$$F(m) = \int_{-\infty}^\infty dz \cdot f(z) e^{imz} \quad . \quad . \quad . \quad . \quad (10)'$$

Eq. (10)' may be written

$$F(m) = 2ab F_1(mb),$$

where $F_1(m) = \int_0^1 d\theta f_1(\theta) \cos m\theta$, and where the cross-section of the cylinder is given by

$$\begin{aligned} r_a &= a f_1\left(\frac{z}{b}\right), \quad \left|\frac{z}{b}\right| < 1 \\ &= 0, \quad \left|\frac{z}{b}\right| > 1. \end{aligned}$$

An analysis exactly similar to that used previously gives

$$\phi_{\text{waves}} = 2\phi_4,$$

where now

$$\begin{aligned} \phi_4 &= -(\text{sgn } W z_1) a W \nu \int_0^\infty dk \frac{\cos(r_1 k) e^{-|z_1|k} F_1(\sqrt{\nu k})}{k(\nu + k)} \\ &\quad \cdot (\sigma k \cos \sqrt{\nu k} \tau - \sqrt{\nu k} \sin \sqrt{\nu k} \tau) \quad . \quad (16)' \end{aligned}$$

Expressions for the wave resistance and therefore for the energy E per unit length of the cylinder may be found as before. Then $p \simeq \rho \partial \phi / \partial t$ and

$$R = 2 \int_{-\infty}^\infty dz p_a \cdot f'(z - Wt).$$

If similar linear approximations are made as before, then it follows that

$$\frac{E}{\pi a^4 b g \Delta \rho} = (1 - \sigma^2) \eta_1(\nu) + \sigma^2 \eta_2(\nu) \quad . \quad . \quad . \quad (19)'$$

where
$$\eta_1(\nu) = \frac{8\nu^3}{\pi} \int_0^\infty ds \frac{s \{F_1(s)\}^2}{(s^2 + \nu^2)^2}, \quad . \quad . \quad . \quad (20)'$$

and
$$\eta_2(\nu) = \frac{8\nu}{\pi} \int_0^\infty ds \cdot \frac{s \{F_1(s)\}^2}{s^2 + \nu^2} \quad . \quad . \quad . \quad (21)'$$

by use of (22). Some comparison for this wave energy is required. The internal kinetic energy of the thermal will be at least as great as the kinetic energy of the thermal moving as a solid homogeneous sphere. To this energy is to be added the kinetic energy of the surroundings. This is accounted for by the virtual mass coefficient, which for the sphere is $\frac{1}{2}$. Hence the kinetic energy of translation is at least $\frac{3}{4} \cdot \frac{4}{3} \pi a^3 W^2 (\rho_2 - \delta\rho)$

$$\simeq \pi a^4 g \delta\rho. \quad (23)$$

The effect of the change in kinetic energy of the surrounding fluid as the thermal moves to less dense surroundings is neglected. The work done by the buoyancy forces as the sphere crosses the interface is to be taken into account. (The work done by the buoyancy forces when the sphere is not crossing the interface is not relevant since most of the wave energy is developed during the time of crossing). This work is easily shown to be

$$\begin{aligned} & 2a \left(\frac{4}{3} \pi a^3 \right) g \left\{ \frac{\rho_2 + \rho_1}{2} - (\rho_2 - \delta\rho) \right\} \\ &= \frac{8\pi a^4}{3} \cdot g \delta\rho \left(1 - \frac{\Delta\rho}{2\delta\rho} \right) = \frac{8\pi a^4}{3} g \delta\rho \cdot (1 - \nu_0), \end{aligned} \quad (24)$$

where
$$\nu_0 = \frac{\Delta\rho}{2\delta\rho}.$$

The wave energy is $\frac{\pi}{2} a^4 g \Delta\rho e_1(\nu_0)$, and this is to be compared with expressions (23) and (24). Thus.

$$\frac{\text{wave energy}}{\text{energy of translation}} = \frac{2}{3} \cdot \nu_0 e_1(\nu_0), \quad (25)$$

and

$$\frac{\text{wave energy}}{\text{work done by the buoyancy}} = \frac{3}{4} \cdot \frac{\nu_0 \cdot e_1(\nu_0)}{1 - \nu_0} \quad (26)$$

The curve of $e_1(\nu)$ for a sphere shows that a maximum occurs at $\nu = 1$ approximately, and that $e_1(\nu)$ is then about 0.64. Hence when $\Delta\rho = 2\delta\rho$, the work done by the buoyancy is zero, and the ratio of the wave energy to the kinetic energy of motion is about ~~0.25~~ 1.28. On the other hand, if $\Delta\rho = \delta\rho$, the ratios (25) and (26) become about ~~0.25~~ and ~~0.18~~ respectively. Typical values for the atmosphere are

$$\Delta\rho = 2 \text{ per cent}, \quad \delta\rho = 5 \text{ per cent},$$

and the ratios (25) and (26) are then ~~0.007~~ and ~~0.003~~ approximately. Thus the energy lost by the generation of waves as the sphere crosses the interface will substantially affect the ensuing motion only in cases where $\Delta\rho \simeq 2\delta\rho$; that is, in cases where the average density of the two fluids is equal to the density of the sphere. Only in the case where the buoyancy changes sign as the thermal passes through the inversion will the energy spent in making waves have a significant effect on the motion: in this case the change in buoyancy will subsequently completely alter the character of the motion, and hence the thermal itself. If the inversion is such that the change in buoyancy as the thermal penetrates it will not radically change the motion of the thermal, the energy dispersed into wave motions of the thin inversion layer may be neglected.

The curves of $e_1(\nu)$ and $e_2(\nu)$ agree qualitatively with what might be expected from physical arguments. At high speeds and low speeds the wave energy is small, for at high speeds there is little time for the waves to develop, and at low speeds there is little or no deformation of the interface outside the region of penetration. At intermediate speeds, however, the front of the body will generate a wave crest which will subside into a trough at the same time as the rear of the body leaves the interface. The trough then subsides still further. Thus there is a condition which is most favourable to the generation of

waves. From expressions (20a) and (20b) it follows that for small ν , $e_1(\nu) \propto \nu^2$ and for large ν , $e_1(\nu) \propto \nu^{-2}$, so there is an optimum condition for the maximum generation of waves for any body. This condition occurs at some intermediate speed given by $W^2 = n bg \Delta \rho / 2 \rho_2$ say, where n is some number which depends only on the shape of the body. For the thermal, this criterion becomes $\nu_0 = n$, on the use of expression (22).

Similar remarks for the case of bent-over, buoyant, chimney plume apply as in the case of the thermal: the plume is regarded as a horizontal cylinder which rises through an inversion, and this case corresponds to the symmetrical two-dimensional case. The outcome is that the wave-making process as the plume passes through an inversion may be neglected if the buoyancy change does not markedly affect the subsequent motion of the plume.

ACKNOWLEDGMENTS

I would like to express my thanks to Dr. R. S. Scorer for his encouragement and for his many helpful suggestions throughout this work. ~~I would also like to thank Mr. J. Richards of the Meteorology Department of Imperial College, London, for his kind permission to reproduce his photograph.~~

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Wave resistance to vertical motion in a stratified fluid

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(Received 14 March 1959, and in revised form 12 July 1959)

When a body moves through a stratified fluid, i.e. one whose density decreases upwards, gravity waves are set up and this causes a resistance to motion. An axisymmetric case is considered in which a body moves steadily and vertically through a fluid whose density decreases exponentially upwards. The fluid is supposed perfect, incompressible, and unbounded in all directions. The equations of motion are linearized, and with a fairly general initial motion of the surrounding fluid, the limit of the solution as $t \rightarrow \infty$ is evaluated. Transform methods are used to solve the equation of motion, and the methods of steepest descents and stationary phase are used to obtain approximate solutions.

Streamlines and the distortion of the constant density levels for a spindle-shaped body are shown. The curves of resistance against a function of the velocity for the circular cylinder, the sphere, and a spindle-shaped body are also given. A criterion is given for when the maximum wave resistance for a sphere may be expected, and an estimate of this maximum resistance is made.

1. Introduction

The problem of a body rising through a stratified fluid has applications in the study of buoyant convection currents rising through stable surroundings. This type of motion frequently occurs in the atmosphere when a thermal rises through a cumulus cloud. In this case the expansion of the thermal during its ascent causes a condensation of the water vapour within it. The release of latent heat warms the thermal, and this causes a further expansion and a further decrease in density of the rising air mass. This offsets the loss in buoyancy as the thermal rises to higher, less dense regions, and the upward motion is maintained (Scorer 1958). For the purposes of this problem, the atmosphere may be regarded as an incompressible fluid whose density decreases upwards, and whose static stability parameter is the same as that of the atmosphere (Scorer 1950, 1958); and the thermal can be taken to have a fixed, spherical shape. What is required is a measure of the energy used to generate the gravity waves set up in the stable surroundings of the cumulus cloud as the thermal rises, so that the importance of this effect on the motion and growth of the thermal can be estimated. This is obtained in §10.

The problem considered in this paper may also be regarded as a first approach to the study of the dispersion of stirring motions into gravity waves. A criterion is suggested for when the maximum amount of energy is used to make waves, and hence for when the maximum amount of energy is absorbed and dispersed by gravity waves.

2. The equation of motion

Perturbation methods are applied to the basic equations of flow to obtain a linear equation governing the stream function of the perturbation.

The notation employed, is, using, cylindrical co-ordinates:

z = height (the z -axis is coincident with that of the body),

r = horizontal co-ordinate,

(u, w) = velocity of the fluid,

W = steady vertical velocity of the body (positive upwards),

β = static stability parameter $= -\frac{1}{\rho_0} \frac{d\rho_0}{dz} = \text{constant} > 0$,

where $\rho_0(z)$ = undisturbed density,

$$\alpha^2 = g\beta,$$

and

$$k_0 = \left| \frac{\alpha}{W} \right|.$$

If we write

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + (w - W) \frac{\partial}{\partial z},$$

the equations of motion relative to axes fixed in the body are

$$\frac{Du}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \quad (1)$$

and

$$\frac{Dw}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial z} = -g. \quad (2)$$

Since the fluid is incompressible

$$\frac{D\rho}{Dt} = 0, \quad (3)$$

and the equation of continuity reduces to

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0. \quad (4)$$

Equation (4) implies the existence of a stream function, $\psi(r, z, t)$, such that

$$u = \frac{1}{r} \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{1}{r} \frac{\partial \psi}{\partial r}.$$

We now write $\rho = \rho_0 + \rho'$ and $p = p_0 + p'$, where ρ' and p' are small perturbations of the undisturbed density and pressure, ρ_0 and p_0 , respectively. If we substitute $\rho_0 + \rho'$ for ρ , and $p_0 + p'$ for p in equations (1), (2) and (3) and neglect products of u , w , p' , ρ' and their derivatives, these three equations become

$$\rho_0 \left(\frac{\partial}{\partial t} - W \frac{\partial}{\partial z} \right) u + \frac{\partial p'}{\partial r} = 0, \quad (1)'$$

$$\rho_0 \left(\frac{\partial}{\partial t} - W \frac{\partial}{\partial z} \right) w + \frac{\partial p'}{\partial z} = -\rho' g \quad (2)'$$

and

$$\left(\frac{\partial}{\partial t} - W \frac{\partial}{\partial z} \right) \rho' + w \frac{\partial \rho_0}{\partial z} = 0. \quad (3)'$$

The elimination of ρ' and p' from these equations and the substitution for u and w in terms of ψ leads to the following equation of motion:⁽¹⁾

$$\left(\frac{\partial}{\partial t} - W \frac{\partial}{\partial z}\right)^2 \left\{ \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \psi}{\partial z} \right) - \frac{\beta}{r} \frac{\partial \psi}{\partial z} \right\} + \alpha^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) = 0. \quad (5)$$

The first two terms in the curly brackets represent the vorticity, the term containing β is an inertia term which arises because the density decreases upwards, and the last term represents the effect of gravity.

If it is assumed that a steady motion exists for which $\partial/\partial t \equiv 0$, then equation (5) becomes

$$\frac{\partial^2}{\partial z^2} \left\{ \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \psi}{\partial z} \right) - \frac{\beta}{r} \frac{\partial \psi}{\partial z} \right\} + k_0^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) = 0. \quad (5')$$

Equation (5)' has the same form for both W and $-W$. Thus the equation of steady motion makes no distinction between ascent and descent at equal speeds. The assumption that there is no perturbation of the fluid at large distances from the body, together with the boundary condition at the surface of the body, leads to a solution of equation (5)' in which a wave pattern appears below the body. This seems unreasonable if the body is in descent. On the other hand, if the assumption of no perturbation at infinity is dropped, the solution of equation (5)' is indeterminate. In some cases it may be argued that the term containing β may be neglected because it is small compared with the other terms of equations (5) and (5)'. These equations then become

$$\left(\frac{\partial}{\partial t} - W \frac{\partial}{\partial z}\right)^2 \left\{ \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \psi}{\partial z} \right) \right\} + \alpha^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) = 0 \quad (6)$$

and

$$\frac{\partial^2}{\partial z^2} \left\{ \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \psi}{\partial z} \right) \right\} + k_0^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) = 0 \quad (6')$$

respectively. The same equations are obtained if β and g are regarded as variable parameters, and $\beta \rightarrow 0$, $g \rightarrow \infty$, in such a way that α^2 remains constant.

However, the solution of the equation of steady motion (6)' is indeterminate, even if the assumption of no perturbation at infinity is made. Also the circumstances in which the term in equation (5) containing β may be neglected do not seem obvious. For these reasons it is better to use the equation of unsteady motion (5), and with a general type of initial motion to look for a limiting value of the solution as $t \rightarrow \infty$.

3. The boundary conditions

If line sources and sinks are placed on the z -axis, the separating streamlines may be regarded, in the usual manner, as the outline of a body. The strength of these line sources may be written conveniently as

$$-2\pi W f'(z) f(z),$$

the accent denoting differentiation with respect to z . At points near the axis,

$$u = \frac{1}{r} \frac{\partial \psi}{\partial z} = -\frac{W}{r} f'(z) f(z),$$

and hence

$$\psi(0, z, t) = -\frac{W}{2} \{f(z)\}^2 = \psi_a, \quad \text{say}, \quad (7)$$

where ψ is defined so that the constant of integration is zero. If the axisymmetric body whose shape is given by $r_a = f(z)$ is a slender one, so that r_a is small, then

$$\frac{\partial}{\partial z} \psi(r_a, z, t) \simeq \frac{\partial}{\partial z} \psi(0, z, t),$$

and
$$\frac{u(r_a, z, t)}{W - w} \simeq \frac{1}{W f(z)} \frac{\partial}{\partial z} \psi(0, z, t) \quad \text{if } |w| \ll |W|.$$

However, if $r_a = f(z)$ is a boundary of the fluid, then

$$-\frac{u(r_a, z, t)}{W - w} = f'(z).$$

Hence equation (7) holds for a body whose shape is given approximately by $r_a = f(z)$, if the body is a slender one.

The flow pattern at $t = 0$ is given (compatibly) as ⁽²⁾

$$\left. \begin{aligned} \psi(r, z, 0) &= \psi_0 \\ \frac{\partial}{\partial t} \psi(r, z, 0) &= \dot{\psi}_0 \end{aligned} \right\} \quad \text{and} \quad (8)$$

The initial motion described by ψ_0 and $\dot{\psi}_0$ is taken to be of a general nature and may contain waves. Alternatively, the body may be started suddenly from rest in a still fluid. The solution of the equation of impulsive motion (i.e. equation (5)' with $g = 0$) is equivalent to a potential flow solution multiplied by a factor $\cos \frac{1}{2} \beta r$. Thus the motion in the horizontal direction has a 'wavelength' of $2/\beta$; but this seems of theoretical interest only since the amplitude of the motion dies away rapidly in an exponential manner.

4. The transformation of the equation of motion

Equation (5) is transformed by means of a Fourier transform in the vertical direction and a Hankel-like transform in the horizontal direction:

$$\Psi(r, k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz e^{ikz} \psi(r, z, t), \quad (9)$$

$$\chi(m, k, t) = \int_0^{\infty} dr J_1(mr) \Psi(r, k, t). \quad (10)$$

The transforms corresponding to ψ_0 , $\dot{\psi}_0$ and ψ_a are

$$\Psi_0, \chi_0; \dot{\Psi}_0, \dot{\chi}_0 \quad \text{and} \quad \Psi_a(k) = \Psi(0, k, t)$$

respectively. If we multiply equation (5) by e^{ikz} and integrate with respect to z from $-\infty$ to $+\infty$, and assume ψ vanishes at $\pm\infty$, we obtain

$$\left(\frac{\partial}{\partial t} + iWk \right)^2 \left\{ \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) - k^2 \frac{\Psi}{r} + i\beta k \frac{\Psi}{r} \right\} + \alpha^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) = 0.$$

If this equation is multiplied by $rJ_1(mr)$ and integrated with respect to r from 0 to ∞ , it becomes

$$\left[\left(\frac{\partial}{\partial t} + iWk \right)^2 (k^2 - i\beta k + m^2) + \alpha^2 m^2 \right] \chi = m(\alpha^2 - W^2 k^2) \Psi_a(k). \quad (11)$$

To derive equation (11), it has been assumed that $J_1(mr) (\partial/\partial r) \Psi(r, k, t)$ vanishes at $r = 0$ and $r = \infty$ and that $\Psi(r, k, t)$ vanishes at $r = \infty$; it then follows that

$$\int_0^\infty dr r J_1(mr) \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) = m \Psi(0, k, t) - m^2 \chi(m, k, t).$$

Equation (11) is an ordinary differential equation with t as the independent variable. Its solution may be written as the sum of a particular integral and a complementary function: $\chi = \chi_1 + \chi_2$. Correspondingly, $\Psi = \Psi_1 + \Psi_2$ and $\psi = \psi_1 + \psi_2$.

5. The particular integral

The particular integral of equation (11) is obtained by putting $\partial/\partial t \equiv 0$, since the right-hand side is independent of time. This gives $\chi_1 = m \Psi_a / (m^2 - \lambda^2)$, where

$$\lambda^2 = \frac{k^4 - i\beta k^3}{k_0^2 - k^2}.$$

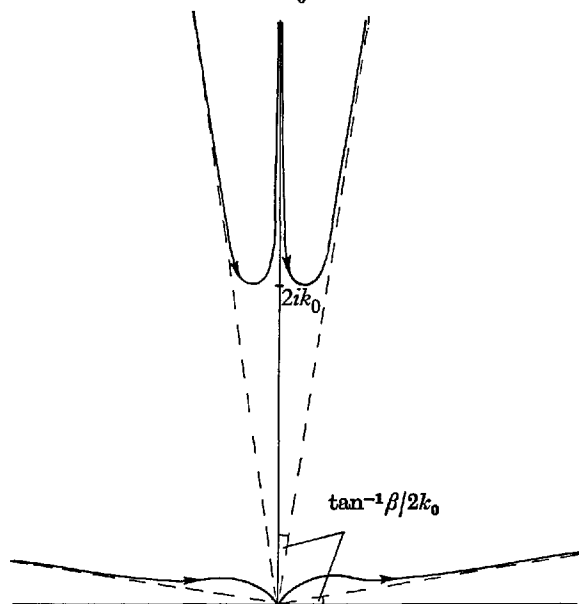


FIGURE 1. Path for λ in the complex λ -plane as k varies.

The inverse of the transformation (10) is

$$\Psi(r, k, t) = r \int_0^\infty dm m J_1(rm) \chi(m, k, t).$$

Hence,

$$\Psi_1(r, k, t) = r \Psi_a(k) \int_0^\infty dm \frac{m^2 J_1(rm)}{m^2 - \lambda^2}.$$

This integral is evaluated in Watson (1944, p. 424).† Substituting its value, we obtain

$$\Psi_1(r, k, t) = (\frac{1}{2}\pi i) r \lambda H_1^{(1)}(r\lambda) \Psi_a(k),$$

where

$$\text{Im}(\lambda) > 0.$$

† The integral $\int \frac{z^2 H_1^{(1)}(rz) dz}{z^2 - \lambda^2}$ taken round an infinite semicircle in the upper half-plane gives the results immediately.

The inverse of the transform (9) then gives

$$\psi_1(r, z, t) = \frac{\pi i}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{-ikz} r \Lambda H_1^{(1)}(r\lambda) \Psi_a(k),$$

where $\text{Im}(\lambda) > 0$. The path of λ , as k varies from $-\infty$ to $+\infty$, is shown in figure 1. If $\beta/k_0 \ll 1$, this path lies approximately on the real and imaginary axes. If it is assumed that the body is symmetrical about its centre so that $r_a = f(z) = f(-z)$, then $\Psi_a(k)$ is an even function of k . ψ_1 may then be written

$$\psi_1 = -\sqrt{\frac{\pi}{2}} \text{Im} \int_0^{\infty} dk \Psi_a(k) e^{ikz} r \Lambda H_1^{(1)}(r\lambda), \quad (12)$$

where

$$\Lambda = \begin{cases} \lambda_1 & \text{if } k < k_0, \\ i\lambda_2 & \text{if } k > k_0 \end{cases}$$

and

$$\lambda_1^2 = \frac{k^4}{k_0^2 - k^2} \quad (\lambda_1 > 0),$$

$$\lambda_2^2 = \frac{k^4}{k^2 - k_0^2} \quad (\lambda_2 > 0).$$

6. The complementary function

Since the particular integral is independent of time, the behaviour of ψ as $t \rightarrow \infty$ depends solely upon the behaviour of ψ_2 as $t \rightarrow \infty$. The complementary function is obtained if we write $\Psi_a(k) = 0$ in equation (11). This gives

$$\chi_2 = A_1(k, m) e^{\gamma_1 t} + A_2(k, m) e^{\gamma_2 t},$$

where A_1 and A_2 are functions of k and m , and γ_1 and γ_2 are the roots of

$$(\gamma + iWk)^2 (k^2 - i\beta k + m^2) + \alpha^2 m^2 = 0. \quad (13)$$

At $t = 0$, χ and $\partial\chi/\partial t$ have the given values χ_0 and $\dot{\chi}_0$, respectively, and A_1 and A_2 are found from these conditions. We thus obtain

$$A_1 = [\gamma_2(\chi_0 - \chi_1) - \dot{\chi}_0]/[\gamma_2 - \gamma_1],$$

$$A_2 = [\gamma_1(\chi_0 - \chi_1) - \dot{\chi}_0]/[\gamma_1 - \gamma_2].$$

The inverse of the transforms (8) and (9) gives

$$\psi_2 = \frac{r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \int_0^{\infty} dm e^{-ikz} m J_1(rm) [A_1 e^{\gamma_1 t} + A_2 e^{\gamma_2 t}].$$

A transformation to polar co-ordinates is now made, so that one range of integration is finite. We write $k = \rho \sin \theta$, $m = \rho \cos \theta$ and thus obtain

$$\psi_2 = \frac{r}{\sqrt{2\pi}} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_0^{\infty} d\rho \rho^2 \cos \theta \exp[-iz\rho \sin \theta] \\ \times J_1(r\rho \cos \theta) [A_1(\rho, \theta) e^{\gamma_1 t} + A_2(\rho, \theta) e^{\gamma_2 t}],$$

where

$$\gamma_{1,2} = -iW\rho \sin \theta \pm i\alpha \cos \theta \sqrt{\frac{\rho}{\rho - i\beta \sin \theta}}.$$

We write

$$I_1^{(+)} = \frac{r}{\sqrt{2\pi}} \int_0^{\frac{1}{2}\pi} d\theta \int_0^\infty d\rho \rho^2 \cos \theta \exp[-iz\rho \sin \theta] J_1(r\rho \cos \theta) A_1(\rho, \theta) e^{\gamma_1 t},$$

$$I_1^{(-)} = \frac{r}{\sqrt{2\pi}} \int_{-\frac{1}{2}\pi}^0 d\theta \int_0^\infty d\rho \rho^2 \cos \theta \exp[-iz\rho \sin \theta] J_1(r\rho \cos \theta) A_1(\rho, \theta) e^{\gamma_1 t}.$$

Similarly the notations $I_2^{(+)}$ and $I_2^{(-)}$ are used for integrals containing A_2 .

To find the limit of the integrals as $t \rightarrow \infty$, we consider the ρ -integration first, and deform the paths of integration in the complex ρ -plane to regions where $\text{Re}(\gamma_{1,2}) < 0$.† Two cases arise, corresponding to ascent and descent of the body. For ascent, it is found that if certain assumptions about the initial conditions are made, no poles are crossed in the deformation in the ρ -plane. The whole path may be deformed to regions where $\text{Re}(\gamma_{1,2}) < 0$. Hence in this case the integrals tend to zero as $t \rightarrow \infty$. For the case of descent, the same assumptions are made, but poles of χ_1 are crossed at points where $\gamma = 0$, and the residues give rise to a steady term as $t \rightarrow \infty$. A difficulty arises for the integrals $I_1^{(-)}$ and $I_2^{(+)}$ because it is not possible to deform the whole of the path in the ρ -plane to regions where

$$\text{Re}(\gamma_{1,2}) < 0.$$

The second integration with respect to θ then shows that these integrals do not have limits as $t \rightarrow \infty$. However, if the assumption is made that $\beta/k_0 \ll 1$ this difficulty is partly overcome in the sense that the integrals $I_1^{(-)}$ and $I_2^{(+)}$ taken along the deformed paths remain negligible for a certain interval of time. During this time the integrals are approximately equal to the residue terms from the poles crossed by the deformations. From these residues the limit of ψ_2 is obtained. Thus it may be shown that

$$\lim_{t \rightarrow \infty} \psi_2 = \begin{cases} 0, & \text{if } W > 0 \\ \sqrt{2\pi} \int_0^{k_0} dk \Psi_a(k) r \Lambda J_1(r \Lambda) \sin kz & \text{if } W < 0. \end{cases} \quad (14)$$

The remainder of this section gives a proof of the expression (14) and discusses some points which arise from the preceding paragraph.

Case I. Ascent $W > 0$

It is assumed that A_1 and A_2 are $O(\exp[b|\text{Im} \rho|]\rho^{-3-\Delta})$ as $|\rho| \rightarrow \infty$, where $\Delta > 0$ and $b \geq 0$. Equation (13) shows that if $\gamma = 0$, then $m^2 = \lambda^2$. Hence χ_1 has poles at points where $\gamma = 0$, A_1 has a pole at $\gamma_1 = 0$ and A_2 has a pole at $\gamma_2 = 0$. Figure 2 shows the ρ -plane. Regions where $\text{Re}(\gamma) < 0$ are shaded, and the poles of $A_{1,2}$ at $\gamma_{1,2} = 0$ are shown at the points marked Γ . γ has a pole at $\rho = i\beta \sin \theta$ and a branch point at the origin. A cut is made along the imaginary ρ -axis to join these points, so that γ is single valued in the cut plane.

We consider the upper bounds of the integrands of the integrals $I_1^{(+)}$, etc., taken along the deformed paths. Upper bounds for the integrals taken along the deformed paths may then be found. The range of the θ -integration is divided into the ranges $(0, \pm \epsilon)$ and $(\pm \epsilon, \pm \frac{1}{2}\pi)$, where ϵ is a small positive number.

† The transformation $\rho \rightarrow \rho(\gamma)$ is not a simple one and it is better to keep to the independent variables (ρ, θ) .

Discussion of the integral $I_1^{(+)}$

(i) No deformation is made in the interval $0 < \theta < \epsilon$, and it may be shown that $\left| \int_0^\infty d\rho \right| < M$, for some fixed M .

(ii) In $\epsilon < \theta < \frac{1}{2}\pi$, the path of integration in the ρ -plane is deformed as shown in figure 3.

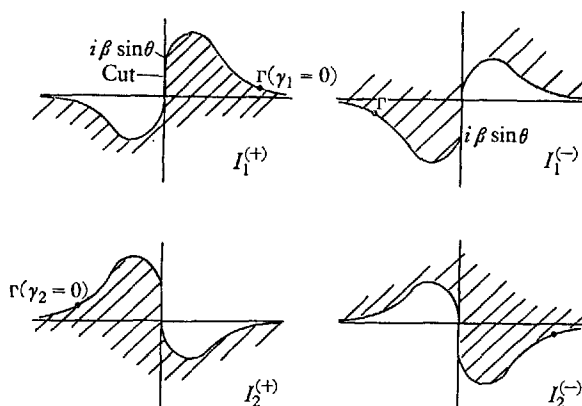


FIGURE 2. Case I. Paths of $\text{Re } (\gamma) = 0$ in the complex ρ -plane.

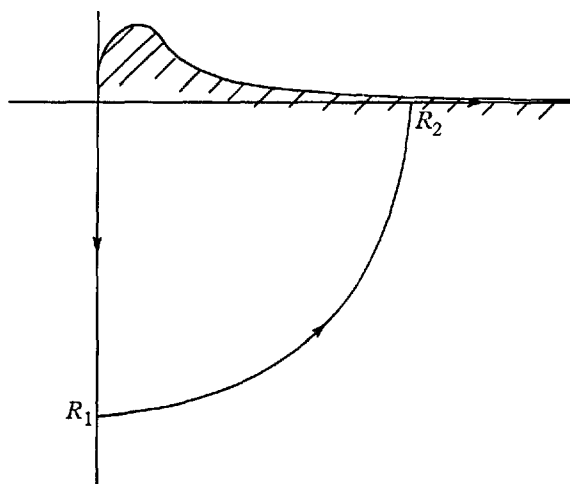


FIGURE 3. Case I. Deformation for $I_1^{(+)}$ in $\epsilon < \theta < \frac{1}{2}\pi$.

It follows that

$$\left| \int_{R_1 R_2 \infty} d\rho \right| < K\epsilon$$

by suitable choice of R_1 , and that

$$\left| \int_{0 R_1} d\rho \right| < \frac{K}{Wt \sin \theta - 2(r + |z| + b)}$$

if $Wt \sin \theta > 2(r + |z| + b)$, where K is a fixed number.

Discussion of the integral $I_1^{(-)}$.

(i) In the interval $-\epsilon < \theta < 0$ the path is deformed as shown in figure 4.

It may be shown that $\left| \int_{0, \infty} d\rho \right| < M e^{ck_0(r+|z|+b)}$, where c is some constant which depends on β and k_0 only.

(ii) In $-\frac{1}{2}\pi < \theta < -\epsilon$, the deformation is similar to that used in $I_1^{(+)}$, see figure 5.

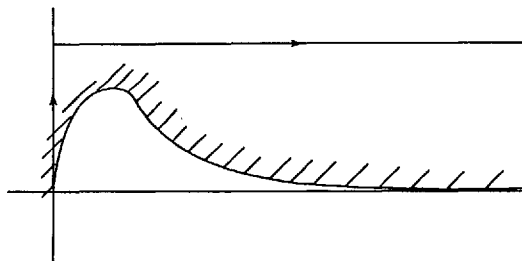


FIGURE 4. Case I. Deformation for $I_1^{(-)}$ in $-\epsilon < \theta < 0$.

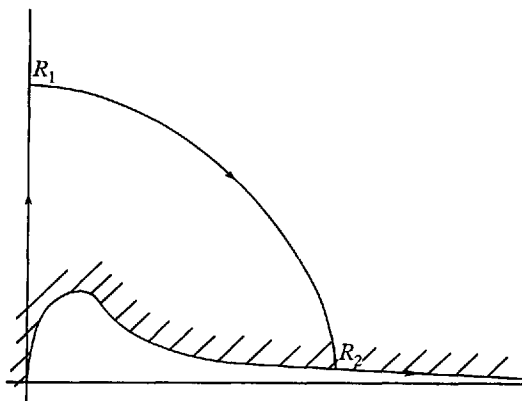


FIGURE 5. Case I. Deformation for $I_1^{(-)}$ in $-\frac{1}{2}\pi < \theta < -\epsilon$.

In this case it may be shown that

$$\left| \int_{0R_1R_2\infty} d\rho \right| < K\epsilon + \frac{K}{Wt \sin \theta - 2(r+|z|+b)}.$$

Similar results hold for the integrals $I_2^{(+)}$ and $I_2^{(-)}$, when the paths are deformed in a similar manner. If we collect these results for the integrals $I_1^{(+)}$, etc., and consider upper bounds for varying θ , we obtain the following upper limit for the θ -integration:

$$\left| \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{\text{paths}} d\rho \right| < K \left\{ (1 + \exp[ck_0(r+|z|+b)]) \epsilon + \frac{1}{Wt_0 \sin \epsilon - 2(r+|z|+b)} \right\},$$

where $Wt_0 \sin \epsilon > 2(r+|z|+b)$.

For poles crossed by the deformations there will be contributions from the residues, which give terms like

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \ominus e^{\gamma r t}. \quad (15)$$

Here Θ is some function of θ and a pole occurs at $\rho = \rho_p(\theta)$, say, where

$$\gamma = \gamma(\rho_p, \theta) = \gamma_p(\theta).$$

If $\text{Re}(\gamma_p) < 0$, the expression (15) will decrease with time, but if $\text{Re}(\gamma_p) > 0$ (e.g. for the deformation used for $I_2^{(-)}$), this may not be so. In general, if it is possible to deform the path of integration in the θ -plane to regions where $\text{Re}(\gamma_p) < 0$ (where γ_p is such that $\text{Re}(\gamma_p(\pm \frac{1}{2}\pi)) \leq 0$), the expression (15) will decrease with time. However, this deformation is not always possible, and, for example, we may have $\text{Im}(\gamma_p) = \text{constant} = Wl$, say, and $\text{Re}(\gamma_p) = 0$. There is then a steady fluctuation present throughout the motion. This type of motion is easier to see if we use equation (6). The assumption that

$$\psi \propto e^{-ikz} r J_1(mr) e^{iWt}$$

then gives

$$(l+k)^2(k^2+m^2) = k_0^2 m^2. \quad (13')$$

Hence equation (6) has a solution

$$\psi = r e^{iWt} \int_0^\infty dk \Psi_a(k) e^{-ikz} m J_1(rm),$$

where m satisfies equation (13') and l is fixed.

This fluctuating type of motion does not seem to be of practical interest since it must be present in the initial conditions and it is not easy to see how this motion could arise in the first instance, unless a fluctuating type of disturbance were present.

With regard to the θ -integration for the residues in the case where

$$\text{Im}(\gamma_p) = \text{constant}, \quad \text{Re}(\gamma_p) > 0,$$

the fluctuation would increase with time. In this case the linear theory is inadequate to say what eventually happens.

If it is now assumed that poles of the above description do not occur in A_1 and A_2 and hence in χ_0 and $\dot{\chi}_0$, i.e. that certain types of waves are absent from the initial condition, then

$$|\psi_2| < K \left\{ (1 + \exp[ck_0(r + |z| + b)])e + \frac{1}{Wt \sin \epsilon - 2(r + |z| + b)} \right\}.$$

This holds for all (r_1, z_1, t) if $r_1 + |z_1| < r + |z|$ and $t > t_0$. Hence $\psi_2 \rightarrow 0$ as $t \rightarrow \infty$ if $W > 0$ for all regions of the flow. This result holds for any $\beta > 0$.

Case II. Descent $W < 0$

The same methods are used as in Case I, but it is no longer possible to deform the paths of integrations for the integrals $I_1^{(-)}$ and $I_2^{(+)}$ in the ρ -plane so that they lie completely in regions where $\text{Re}(\gamma) < 0$. The singularities of γ prevent this. Figure 6, which corresponds to figure 2 of Case I, shows the ρ -plane.

The paths for the integrals $I_1^{(-)}$ and $I_2^{(+)}$ are deformed as shown in figures 8, 9 and 10. In the regions where $\text{Re}(\gamma) > 0$, the paths are deformed so as to pass through a col as in the method of steepest descent. An upper bound for the contributions from segments of the paths where $\text{Re}(\gamma) > 0$ is then made. The track of the cols for varying θ is shown in figure 7.

The position of the cols is given by $\rho = \rho_c(\theta)$, say, and at these points

$$\gamma = \gamma(\rho_c, \theta) = \gamma_c(\theta).$$

We write

$$\operatorname{cosec} \theta^* \cot \theta^* = \frac{3\sqrt{3}}{8} \frac{\beta}{k_0} \quad (\tfrac{1}{2}\pi > \theta^* > 0)$$

and

$$\operatorname{cosec} \theta^{**} \cot \theta^{**} = \frac{1}{2} \frac{\beta}{k_0} \quad (\tfrac{1}{2}\pi > \theta^{**} > 0).$$

Then if $|\theta| > \theta^*$, it may be shown that two cols lie on the cut; and that if $|\theta| > \theta^{**}$ both zeros of γ lie on the cut. We suppose that β and g are variable parameters

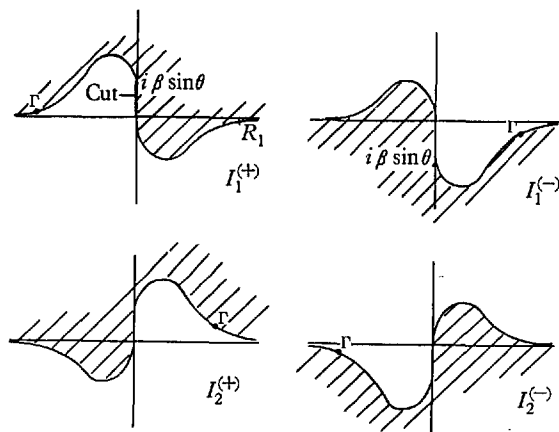


FIGURE 6. Case II. Paths of $\operatorname{Re}(\gamma) = 0$ in the complex ρ -plane.

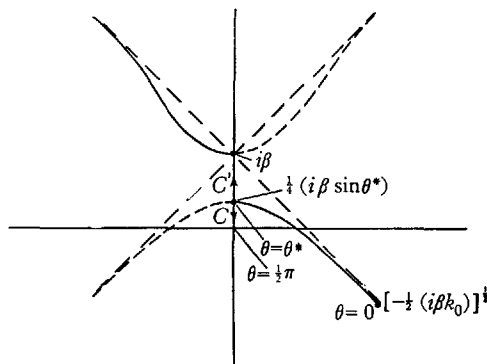


FIGURE 7. Case II. Path of the cols for varying θ in the complex ρ -plane.

with α , k_0 and W held fixed, and that $0 < \beta/k_0 \ll 1$. The range of the θ -integration is divided into the ranges $(0, \pm \epsilon)$; $(\pm \epsilon \pm \tfrac{1}{2}\pi \mp \delta)$; and $(\pm \tfrac{1}{2}\pi \mp \delta, \pm \tfrac{1}{2}\pi)$; where ϵ and δ are small positive numbers. β is chosen so that $\tfrac{1}{2}\pi - \theta^* < \delta$ in order to restrict the contributions from the paths in regions where $\operatorname{Re}(\gamma) > 0$.

Discussion of the integral $I_1^{(+)}$

(i) There is no deformation of the path in $0 < \theta < \epsilon$ and it may be shown that

$$\left| \int_0^\infty d\rho \right| < M.$$

(ii) In $\epsilon < \theta < \frac{1}{2}\pi$, the path is deformed to $\text{Re}(\gamma) = 0$. On this path it may be shown that

$$\left| \frac{d\gamma/d\rho}{W \sin \theta} \right| > 1.$$

With the assumption that $A(\rho, \theta)$ is of bounded variation on any finite length of the path, it follows that

$$\left| \int_{0, \infty} d\rho \right| < \frac{K \exp [ck_0(r + |z| + b)]}{Wt \sin \theta} + M\epsilon.$$

Similar results hold for $I_2^{(-)}$.

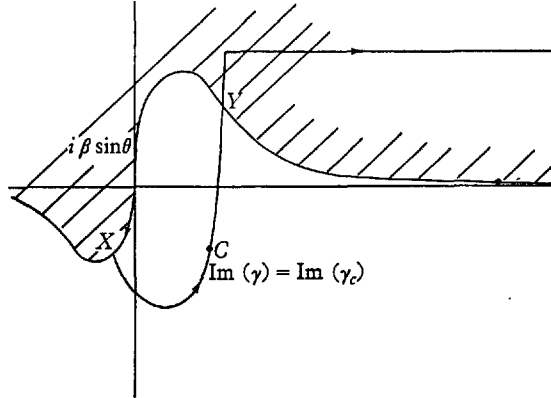


FIGURE 8. Case II. Deformation for $I_2^{(+)}$ in $0 < \theta < \epsilon$.

Discussion of the integral $I_2^{(+)}$

(a) Contributions from the paths which lie in regions where $\text{Re}(\gamma) \leq 0$

(i) In $0 < \theta < \epsilon$, the path is deformed as shown in figure 8.

It may be shown that

$$\left| \int_{OX+Y\infty} d\rho \right| < M \exp [ck_0(r + |z| + b)].$$

(ii) In $\epsilon < \theta < \frac{1}{2}\pi - \delta$, the path is deformed as in figure 9. Here

$$\left| \int_{OX} d\rho \right| < \frac{K \exp [ck_0(r + |z| + b)]}{Wt \sin \theta},$$

as may be shown by a discussion similar to that used above for the integral $I_1^{(+)}$, (ii).

If $\beta/k_0 \ll \sin \delta$, it may be shown that

$$\left| \frac{d\gamma/d\rho}{W \sin \theta} \right| > \frac{1}{2}$$

and that the integrand is uniformly bounded with respect to β and θ on the deformed path.

It then follows that

$$\left| \int_{YZ} d\rho \right| < \frac{K \exp [ck_0(r + |z| + b)]}{Wt \sin \theta},$$

where $|OZ| = k_0$, say.

As in Case I, integral $I_1^{(+)}$, it may also be shown that

$$\left| \int_{Z_\infty} d\rho \right| < \frac{K}{Wt \sin \theta - 2(r + |z| + b)} + M\epsilon.$$

(iii) In $\frac{1}{2}\pi - \delta < \theta < \frac{1}{2}\pi$, the paths are deformed in the three subintervals

$$\frac{1}{2}\pi - \delta < \theta < \theta^*; \quad \theta^* < \theta < \theta^{**}; \quad \theta^{**} < \theta < \frac{1}{2}\pi;$$

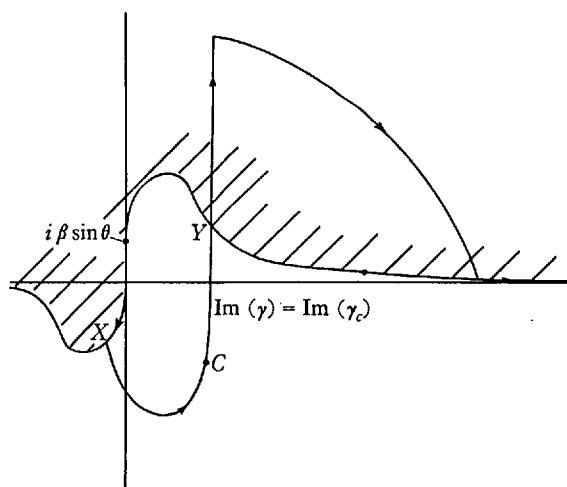


FIGURE 9. Case II. Deformation for $I_2^{(+)}$ in $\epsilon < \theta < \frac{1}{2}\pi$.

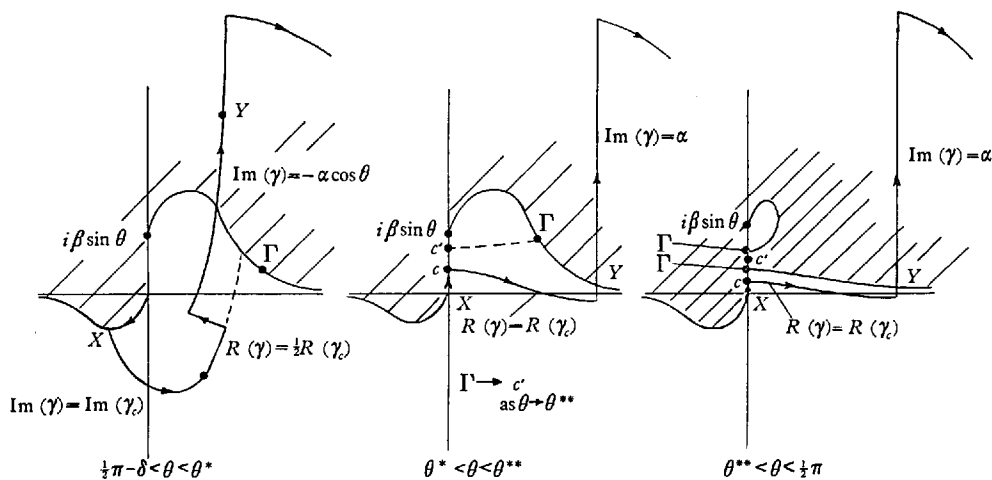


FIGURE 10. Case II. Various deformations for $I_2^{(+)}$ in $\frac{1}{2}\pi - \delta < \theta < \frac{1}{2}\pi$.

as shown in figure 10. For points on the segment $Y\infty$ of the deformed path, where $|OY| = k_0$, it may be shown that

$$\left| \frac{d\gamma/d\rho}{W \sin \theta} \right| > \frac{1}{2}$$

Similar arguments hold for $\left| \int_{Y\infty} d\rho \right|$ as in (ii) above.

(b) Contributions from the paths which lie in regions where $\text{Re}(\gamma) \geq 0$

If we consider the path of the col, we may show that in the interval

$$0 < \theta < \theta^*, \quad \text{Re}(\gamma_c/W) < 5\sqrt{(\beta k_0)};$$

and that in

$$\theta^* < \theta < \frac{1}{2}\pi, \quad \text{Re}(\gamma_c/W) < \frac{1}{8}\beta.$$

(i) In $0 < \theta < \frac{1}{2}\pi - \delta$, it may be shown that the length of the path XY is less than

$$K\sqrt{(\beta k_0)}.$$

(ii) In $\frac{1}{2}\pi - \delta < \theta < \frac{1}{2}\pi$, the length of the path in the region where

$$\text{Re}(\gamma) \geq 0 \quad \text{is} \quad O(k_0).$$

It may also be shown that the integrand is uniformly bounded with respect to β and θ on the deformed paths. From these results it may then be shown that

$$\left| \int_0^{\frac{1}{2}\pi} d\theta \int_{XY} d\rho \right| < K[\sqrt{(\beta k_0)} + \delta] \exp[ck_0(r + |z| + b)] \exp[5W\sqrt{(\beta k_0)}t].$$

The integral $I_1^{(-)}$ may be discussed in a manner similar to $I_2^{(+)}$.

From the above results for the integrals $I_1^{(+)}$, etc., we obtain the following upper limit for the θ -integration:

$$\left| \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{\text{paths}} d\rho \right| < K \exp[ck_0(r + |z| + b)] \left\{ \epsilon + \frac{1}{Wt \sin \epsilon} + \frac{1}{Wt \sin \epsilon - 2(r + |z| + b)} \right. \\ \left. + [\sqrt{(\beta k_0)} + \delta] \exp[5W\sqrt{(\beta k_0)}t] \right\}.$$

This holds for all $0 < \beta < \bar{\beta}$, for some $\bar{\beta} > 0$. (c does not increase if β decreases.)

It also holds for all (r_1, z_1, t) , where $r_1 + |z_1| < r + |z|$, and $t > t_0$.

If we choose $t_1 \gg t_0$ and β such that $\sqrt{(\beta k_0)} \ll 1/5Wt_1$ (i.e. $\sqrt{(\beta/k_0)} \ll 1/5\alpha t_1$), then

$$\left| \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{\text{paths}} d\rho \right| < K \exp[ck_0(r + |z| + b)] (3\epsilon + 4\delta) \quad (16)$$

during some interval of time (t_1, t_2) , $t_2 > t_1$. If (ϵ, δ) is chosen so that the expression on the right-hand side of (16) is small compared with the average value of

$$|\psi_1 + \text{residue terms}|^\dagger$$

in the region $r_1 + |z_1| < r + |z|$, then

$$\left| \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{\text{paths}} d\rho \right|$$

may be neglected in the interval $t_1 < t < t_2$. This interval may be called a 'quasi-limit' interval. An estimate of $\psi_2 = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_0^\infty d\rho$ for large t may be found from

the asymptotic expression of $\int_{XY} d\rho$. The method of steepest descent through the col gives the first n -terms in the form†

$$\sum_{r=1}^n \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \Theta_r \exp[\gamma_c(\theta)t] t^{-r-\frac{1}{2}},$$

† ψ_1 and the residue terms change little as $\beta \rightarrow 0$.

‡ A modification is needed at $\theta = \theta^*$ since $d\gamma/d\rho = 0$ there.

where the Θ_r are functions of θ , and $\text{Re}(\gamma_c(\pm \frac{1}{2}\pi)) = \text{Re}(\gamma_c(0)) = 0$. A different form of deformation of the path in the ρ -plane occurs at $\theta = \theta^*$ and hence not all the Θ_r are holomorphic at this point. Hence complete deformation of the path of integration in the θ -plane to regions where $\text{Re}(\gamma_c) \leq 0$ is not possible,[†] and

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{XY} d\rho$$

gives terms

$$O\left(\frac{\exp[\frac{1}{8}(\beta|W|t)]}{t^{r+\frac{1}{2}}}\right) + \dots,$$

where r is some integer. Hence $|\psi_2|$ diverges as $t \rightarrow \infty$, and eventually the motion increases without limit.

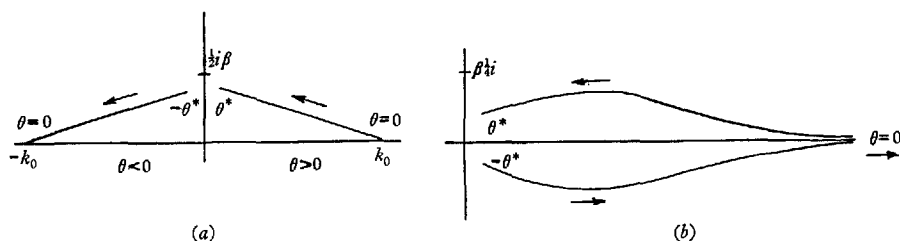


FIGURE 11. Paths of integration for k , (a), and λ , (b).

The assumption that second-order terms are small no longer holds at this stage and the linear theory ^{is} seems inadequate. Physically, the medium is a dispersive one in which local disturbances are propagated outwards in all directions. The gradually increasing inertia forces retard the propagation to deeper levels, and waves ahead cannot disperse so rapidly in descent as in ascent.

Similar assumptions concerning the poles of χ_0 and $\dot{\chi}_0$ are made as in Case I. In the case where the motion is started suddenly from rest, a pole of χ_0 occurs at $\rho = i\beta \sin \theta$. It ^{is} seems better to treat this pole separately, since the arguments used earlier would not hold in this case.

With the above assumptions, the limit of the complementary function in the 'quasi-limit' interval is found from the residues of the poles of χ_1 at $\gamma = 0$, which are crossed by the deformations used for $I_1^{(-)}$ and $I_2^{(+)}$. If we substitute for the residues, after a little manipulation we obtain

$$-\sqrt{\frac{\pi}{2}} i r \int_L dk \Psi_a(k) e^{-ikz} \lambda J_1(r\lambda) \text{sgn } \theta,$$

where L is the path in the k -plane shown in figure 11. Figure 11 also shows the corresponding path for λ .

If $\beta r \ll 1$ and $\beta z \ll 1$, then these paths may be taken to lie approximately on the real axes. With the assumption that $\Psi_a(k)$ is even, this leads to the expression

$$\sqrt{2\pi} \int_0^{k_0} dk \Psi_a(k) r \Lambda J_1(r\Lambda) \sin kz. \quad (17)$$

[†] Other paths may be chosen, but a different form of deformation occurs at some stage because the contours of γ change their form at $\theta = \theta^*$ and at $\theta = \theta^{**}$.

In conclusion, for Case II, if $\beta/k_0 \ll 1$, there is a region around the body where, for a given time interval, ψ_2 is given by the expression (17). The length of this interval becomes infinite as $\beta/k_0 \rightarrow 0$.

7. The complete solution

The results of §§ 5 and 6 give the following expression for ψ as $t \rightarrow \infty$:

$$\psi = \frac{W}{2} \operatorname{Im} \int_0^\infty dk F(k) r \Lambda H_1^{(1)}(r \Lambda) \exp [i (\operatorname{sgn} W) kz], \quad (18)$$

where
$$F(k) = \int_0^\infty dz \cos kz \{f(z)\}^2.$$

Hence the wave pattern produced by a descending body is the same as that produced by an ascending one. The waves are downstream in each case. When the initial conditions are those of a steady, numerically greater velocity, and ψ_0 is given by expression (18), it may be shown that the same wave pattern eventually emerges.

Methods similar to those used in §§ 4–6, may be employed to solve equation (6). There is no essential singularity of the integrand in the corresponding double integral expression for ψ_2 . The paths of $\operatorname{Re}(\gamma) = 0$ coincide with the real ρ -axis. The poles of χ_1 also occur on the real ρ -axis, and the double integral expressions for ψ_1 and ψ_2 are improper integrals. If principal values of the integrals are taken we obtain after one integration

$$\psi_1 = \frac{W}{2} \operatorname{Im} \int_0^\infty dk F(k) r \Lambda H_1^{(1)}(r \Lambda) \cos kz$$

and
$$\lim_{t \rightarrow \infty} \psi_2 = \frac{W}{2} \operatorname{Im} \int_0^{k_0} dk F(k) r \Lambda H_1^{(1)}(r \Lambda) i (\operatorname{sgn} W) \sin kz.$$

These two results differ from the corresponding expressions when $\beta \neq 0$. However, the sum $\psi = \psi_1 + \psi_2$ is the same in both cases, and in this sense the limit of the solution of equation (5) as $\beta \rightarrow 0$ is the same as the solution of the limiting form of equation (5) as $\beta \rightarrow 0$.⁽³⁾

8. Asymptotic approximation to the motion

An asymptotic expression for the streamlines and the distortion of the constant density levels can be obtained by the method of steepest descents. The displacement of the streamlines is given approximately by

$$\Delta_s = \int_z^\infty dz \frac{u}{W} = -\frac{\psi}{Wr},$$

and the displacement Δ_{ρ_0} of the constant density levels by

$$\Delta_{\rho_0} = \int_z^\infty dz \frac{w}{W}.$$

If we write $k = k_0 \sin \xi$ and use the asymptotic expression for $H_1^{(1)}(x)$

$$H_1^{(1)}(x) \simeq \sqrt{\frac{2}{\pi x}} \exp [i(x - \frac{3}{4}\pi)],$$

then expression (18) yields

$$\Delta_s \simeq - \sqrt{\frac{k_0^3}{2\pi r}} \operatorname{Im} \int_M d\xi F(k_0 \sin \xi) \sin \xi \cos^{\frac{1}{2}} \xi \exp [ik_0 \sin \xi (z + r \tan \xi) - \frac{3}{4}\pi i],$$

when $W > 0$. M is the path shown in figure 12. It is deformed to pass through the col at C . This col contributes an exponentially small term to the asymptotic approximation to Δ_s and is omitted. It may be regarded as the potential flow displacement.

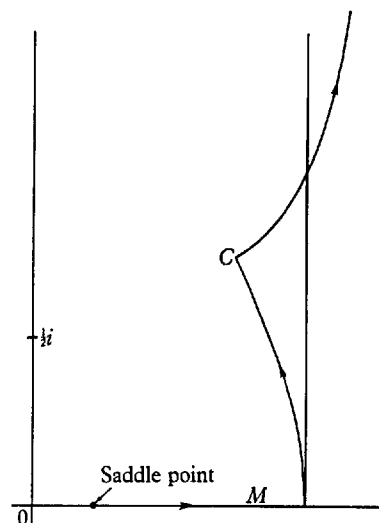


FIGURE 12. Path of integration for ξ .

When $z < 0$ a saddle point appears on the real axis between the origin and $(\frac{1}{2}\pi, 0)$. The method of stationary phase then gives

$$\begin{aligned} \Delta_s &\simeq -\frac{1}{r} \frac{k_0 F(k_0 \sin \xi^*)}{\sqrt{(2 + \sin^2 \xi^*)}} \sin \xi^* \cos^2 \xi^* \cos \left(\frac{zk_0 \sin \xi^*}{1 + \cos^2 \xi^*} \right) \quad \text{if } z < 0 \\ &\simeq 0 \quad \text{if } z > 0 \end{aligned}$$

for the asymptotic value of Δ_s . Here ξ^* satisfies the equation $dG/d\xi = 0$, where

$$G(\xi) = irk_0 \sin \xi \left(\tan \xi + \frac{z}{r} \right),$$

and where $z/r = \tan \phi$, say. This gives

$$\tan^3 \xi^* + 2 \tan \xi^* + z/r = 0$$

or

$$\sin 2\xi^* + 2 \tan (\xi^* - \phi) = 0.$$

A similar result holds for Δ_{ρ_0} :

$$\begin{aligned} \Delta_{\rho_0} &\simeq -\frac{1}{r} \frac{k_0 F(k_0 \sin \xi^*)}{\sqrt{(2 + \sin^2 \xi^*)}} \sin^2 \xi^* \cos \xi^* \cos (rk_0 \tan^2 \xi^* \sec \xi^*) \quad \text{if } z < 0 \\ &\simeq 0 \quad \text{if } z > 0. \end{aligned}$$

If $W < 0$ the reverse holds with Δ_s and Δ_{ρ_0} asymptotically zero if $z < 0$.

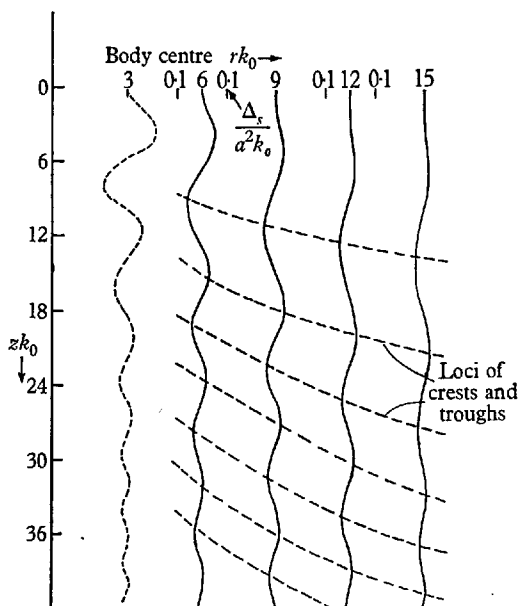


FIGURE 13. Streamlines for a spindle-shaped body.

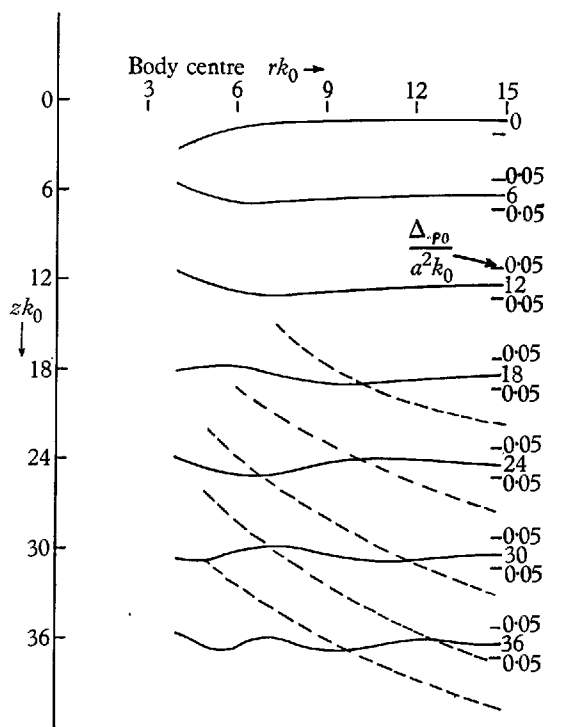


FIGURE 14. Deformation of the constant density levels for a spindle-shaped body.

If ξ^* is used as a parameter we may write the displacements as

$$\frac{1}{r} \times \text{amplitude factor} \times \cos \{(r \text{ or } z) \times \text{phase factor}\}.$$

The results for a spindle-shaped body whose shape is given by

$$\begin{aligned} f(z) &= a \cos \frac{\pi z}{2b} & \text{if } |z| < b \\ &= 0 & \text{if } |z| > b \end{aligned}$$

have been plotted and are shown in the figures 13 and 14. a and b are the half-breadth and half-length of the body and k_0^{-1} has been used as a unit length to measure distances from the body. $a^2 k_0$ has been used as a unit displacement to draw the streamlines and the constant density levels. It may be shown that the slopes of the curves joining the crests and troughs (shown dotted in the figures) are asymptotically $-\cot \xi^*$ so that the approximate rate at which the disturbance spreads outwards is $\frac{1}{2}rW$. A value of $\frac{3}{2}\pi$ for bk_0 was used in the calculations.

9. The wave resistance

The wave resistance to the motion of the body is found from a consideration of the pressures on its surface. If the static thrust is neglected the upthrust is

$$R = 2\pi \int_{-b}^b dz p \sin \eta \frac{ds}{dz} f(z),$$

as shown in figure 15. p is the perturbed pressure.⁽⁴⁾ Euler's equation of motion relative to the body for horizontal motion under steady conditions is

$$W \frac{\partial u}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial r}$$

when products of small quantities have been dropped.⁽⁵⁾ The assumption that $1/\rho \simeq 1/\rho_0$ gives

$$p = W\rho_0 \int_r^\infty \frac{dr}{r} \frac{\partial^2 \psi}{\partial z^2} = \frac{W^2 \rho_0}{2} \int_0^\infty dk k^2 F(k) H_0^{(1)}(r\Lambda) \exp[i(\text{sgn } W) kz].$$

Further, we assume that the body is slender so that $ds/dz \simeq 1$ and $\sin \eta \simeq f'(z)$. If we write $\rho_0(z) \simeq \rho_0(1 - \beta z)$, where ρ_0 = density at $z = 0$, then we obtain

$$\frac{R}{\pi \rho_0 W^2} = \text{Im} \int_{-b}^b dz (1 - \beta z) f(z) f'(z) \int_0^\infty dk k^2 F(k) \exp[i(\text{sgn } W) kz] H_0^{(1)}(r_a \Lambda).$$

Since $f(z)$ is even, this becomes

$$\begin{aligned} \frac{R}{2\pi \rho_0 W^2} &= (\text{sgn } W) \int_0^{k_0} dk k^2 F(k) \int_0^b dz f'(z) f(z) \sin kz J_0(r_a \lambda_1) \\ &\quad - \beta \text{Im} \int_0^\infty dk k^2 F(k) \int_0^b dz f'(z) f(z) \cos kz H_0^{(1)}(r_a \Lambda). \end{aligned}$$

The latter integral represents the effect of the change in virtual mass, and is

neglected because it is multiplied by $\beta^{(6)}$. If $r_a \lambda_1$ is small over a large part of the range $0 < k < k_0$, so that $J_0(r_a \lambda_1) \simeq 1$, then approximately

$$R = -(\text{sgn } W) \pi \rho_0 W^2 \int_0^{k_0} dk k^3 \{F(k)\}^2,$$

since
$$\int_0^b dz f'(z) f(z) \sin kz = -\frac{k}{2} \int_0^b dz \cos kz \{f(z)\}^2 = -\frac{k}{2} F(k).$$

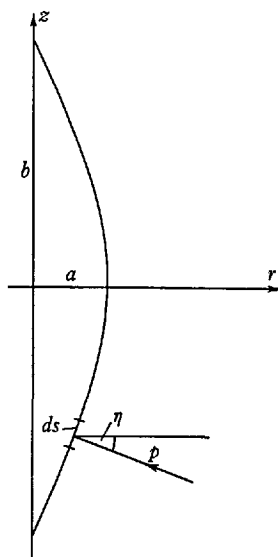


FIGURE 15. Calculation of the wave resistance.

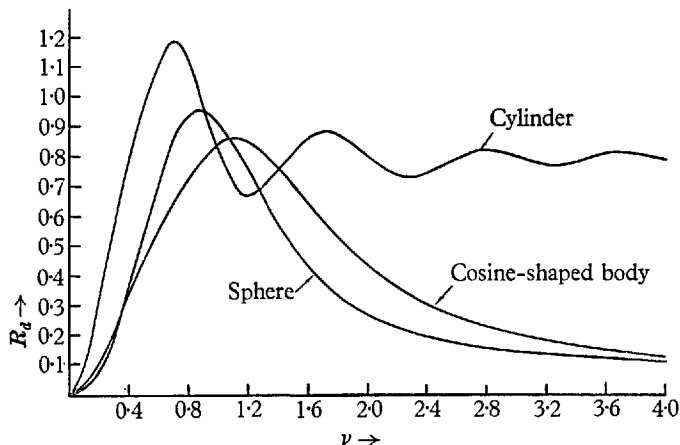


FIGURE 16. Curves of R_d against ν .

$$R_d = \frac{\text{Wave resistance}}{\beta a^4 g \rho_0},$$

$$\nu = \frac{b \sqrt{(g\beta)}}{\pi W}.$$

If the shape of the body is written as $r_a = a f_1(z/b)$, where a = half-breadth and b = half-width of the body, the wave resistance is given by

$$\frac{R}{\rho_0 a^4 g \beta} = \frac{1}{\pi \nu^2} \int_0^{\pi \nu} dk k^3 \{F_1(k)\}^2 = R_d, \quad \text{say,} \quad (19)$$

where $\nu = b k_0 / \pi$ and

$$F_1(k) = \int_0^1 dz \cos kz \{f_1(z)\}^2.$$

For small ν , $R_d \propto \nu^2$. In many cases the right-hand side of equation (19) will tend to zero as $\nu \rightarrow \infty$, and hence, in general there will be some condition of maximum resistance.

Wave resistance curves for the spindle-shaped body, the sphere and the circular cylinder (whose shape is given by $f(z) = a$, if $|z| < b$, $f(z) = 0$ if $|z| > b$), are shown plotted against ν , that is against a function of velocity, in figure 16. Many assumptions break down for the circular cylinder and its resistance curve is of theoretical interest only.

10. Applications and conclusions

The rise of a thermal through the atmosphere has a velocity given, approximately, by Scorer (1957)

$$W = \sqrt{(gBa)},$$

where $B = \text{buoyancy} = \text{density difference/density}$, and $2a = \text{width of the thermal}$. To estimate the wave resistance it is assumed that the thermal is a sphere whose diameter is the width of the thermal. The ratio of the wave resistance to the buoyancy is then

$$\frac{\text{resistance}}{\text{buoyancy}} = \frac{3}{4\pi} \frac{a\beta}{B} R_a \left(\sqrt{\frac{a\beta}{\pi^2 B}} \right) = 0.05 \left(\frac{a\beta}{B} \right)^2,$$

roughly, if $a\beta/B < 3$. For the following values of the constants (which may occur in the atmosphere)

$$a = 10^4 \text{ cm.}, \quad B = 10^{-2}, \quad \beta = 10^{-7} \text{ cm}^{-1},$$

this ratio is about 0.0005. Thus in these cases the wave resistance to motion is negligible. This remains true over a large range of values of B and β . On the other hand, the following values of the constants:

$$a = 10^5 \text{ cm.}, \quad B = 3 \cdot 10^{-3}, \quad \beta = 10^{-7} \text{ cm}^{-1},$$

give a ratio of about 0.5.

As a rough guide, a sphere will experience most wave resistance at a speed given by

$$\frac{a\sqrt{g\beta}}{W} = 3.$$

The ratio of the wave resistance to the weight of the displaced fluid is then about $\frac{1}{4}a\beta$, where a is the radius of the sphere.

Hence in general the dispersion of energy due to wave motion is small. However, when the situation is specially favourable to the surrounding fluid, and the forces involved are small, this dispersion may play a major part in the behaviour of the fluid, as in the case of the larger thermal with a smaller buoyancy (e.g. towards the end of the ascent).

In a similar manner, it may also be possible for these waves to affect the behaviour of turbulent eddies. For instance, if the eddies are large, and if their outer regions move slowly, a considerable proportion of energy may be dispersed by gravity waves. A criterion for this energy dispersion to be a maximum would be

$$\frac{a\sqrt{g\beta}}{W} = n,$$

where the number n depends on the shape of the eddy, a is a typical dimension, say the eddy size, and W a velocity typical of its outer regions.

I would like to express my thanks to Dr R. S. Scorer for the very many helpful discussions and guidance he has given me throughout this work.

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