

An Application of Malliavin Calculus to Hedging Exotic Barrier Options

A thesis presented for the degree of
Doctor of Philosophy of Imperial College, London
by

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Signed:

A handwritten signature in black ink, consisting of the Chinese characters '李鸿宾' (Li Hongbin) in a cursive style.

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Abstract

The thesis uses Malliavin's Stochastic Calculus of Variations to identify the hedging strategies for Barrier style derived securities. The thesis gives an elementary treatment of this calculus which should be accessible to the non-specialist. The thesis deals also with extensions of the calculus to the composition of a Generalized Function and a Stochastic Variable which makes it applicable to the discontinuous payoffs encountered with Barrier Structures. The thesis makes a mathematical contribution by providing an elementary calculus for the composition of a Generalized function with a Stochastic Variable in the presence of a conditional expectation.

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Introduction

Malliavin calculus, which is also called the stochastic calculus of variations, is named after Paul Malliavin. It was originally created as a tool for studying the regularity of densities of solutions of stochastic differential equations. See, for example, [30] and [31]. Essentially, it is an infinite-dimensional differential calculus on Wiener space. Then in 1984, Ocone published the paper [37], and then in 1991, the paper [27] was published; subsequently Malliavin calculus was recognized to have an important role in the field of mathematical finance. Today, more and more applications in mathematical finance are based on Malliavin calculus. For example, they include Greeks calculation by simulation in [6], [7], [20], [21],[22] and [29], hedging portfolio in [4] and [5], and not just for systems driven by Brownian motion, but for systems driven by the general Lévy process [14].

In fact, much of the theory of Malliavin calculus is based upon the well known Itô stochastic calculus [25], named after the Japanese mathematician K. Itô. During the middle of the 20th century, Itô gave the martingale representation theorem which states that a random variable $G \in L^2(\Omega)$ is $\mathcal{F}_T$ -measurable with respect to the filtration generated by a Brownian motion W_t , $t \in [0, T]$, can be written in terms of an Itô integral with respect to this Brownian motion, i.e., there exists a unique square-integrable adapted process $\varphi(t, \omega)$ such that

$$G(\omega) = E[G] + \int_0^T \varphi(t, \omega) dW_t.$$

This result is important in mathematical finance because it asserts the existence of the representation, and this leads to the existence of a replicating trading strategy. However, it does not help with the finding of $\varphi(t, \omega)$ explicitly.

It would be quite interesting to see whether there is an explicit expression for $\varphi(t, \omega)$ in the martingale representation. After comprehensive consideration of the fundamental

theorem of classic calculus, intuitively, people think the $\varphi(t, \omega)$ term should be associated with a differentiation type of operation in a probabilistic setting. Then, in the late 20th century, mathematicians J.M.C Clark and Daniel Ocone gave the Clark-Ocone theorem of stochastic analysis [37], which is a generalization of the Itô representation theorem, in the sense that it gives an explicit expression for the integrand $\varphi(t, \omega)$ (under some extra conditions):

$$\varphi(t, \omega) = E [D_t G | \mathcal{F}_t],$$

where $D_t G$ is the Malliavin derivative of G and is the bridge between Malliavin calculus and Itô stochastic calculus. In addition, the Clark-Ocone formula can be used to obtain explicit formula for replicating portfolios of contingent claims in the complete market [27].

There are many ways of introducing the Malliavin derivatives, like the original way, via Chaos expansion, see [35] and [39] for examples. Another way is via the directional derivative, see [46] and [47] for examples. In this thesis, we follow the approach in [5] by using the definition of directional derivative and subsequently extend the classical Malliavin Calculus and the Clark-Ocone formula. In [5], H.P. Bermin used the Riesz representation theorem to give the relationship between the directional derivative and the Malliavin derivative:

$$\mathbf{D}_\gamma F(\omega) = \int_0^T D_t F(\omega) \dot{\gamma}(t) dt .$$

Using this definition, we develop a lot of useful propositions for the Malliavin derivative. However, Bermin [5] does not give a complete treatment in his paper, so in this thesis, we will give a more detailed treatment.

In this thesis, we follow the result in [37] and present the Clark-Ocone formula in $\mathbb{D}_{1,2}$. However, the limitation of the Malliavin derivative is demonstrated by giving an example and we find that the space $\mathbb{D}_{1,2}$ is not large enough for our application. So we follow [46], [47] and describe the extension of the Clark-Ocone formula such that it is true for the space $\mathbb{D}_{-\infty}$ whose elements are to be interpreted as generalized stochastic variables, that is, as composites of distributions and stochastic processes. So the extension will involve the composition of distributions with stochastic variables, say $T(S_t)$, where T is a distribution, and S_t is the usual log-normal variable. This kind of “calculus” has been developed by Watanabe [47]. If T is infinitely differentiable, then $T(S_t)$ is a nice

object and we can do Malliavin Calculus easily. But if $T(x) = 1_{(K,\infty)}(S_t)$, for example, then there are problems. This occurs in practice because barrier options contain indicator function, $1_{\{\cdot > K\}}$, for example, which are discontinuous functions, and the derivative of this indicator involves the Dirac Delta function δ_K in the sense of distributions. However, we also make a mathematical work by providing an elementary calculus for the composition of a generalized function with stochastic variable in the presence of a conditional expectation.

Subsequently we arrive at a hedging portfolio formula by using the extended Malliavin calculus and Clark-Ocone formula.

Then we follow the definition of the directional derivative to extend Malliavin calculus to the two-dimensional case and extend the propositions and proofs in detail. Also, we derive the hedging portfolio formula in the multi-dimensional case.

Our main application using Malliavin calculus is to two exotic barrier options [12], the rainbow barrier option and the protected barrier option, which are driven by one Brownian motion in one asset and two Brownian motions in multi-assets, respectively. According to [12], the protected barrier option is an up-and-out call option with a protection period for a fixed period of time, $[0, t^*]$, at the start of the option's life during which the up-and-out call cannot be knocked out. At the end of this fixed period, the call is knocked out and a rebate C_1 is paid if the underlying stock price at time t^* , satisfies $S_{t^*} \geq B > K$, where B is the given barrier level and K the strike level. Otherwise, the up-and-out call remains alive until the first time after the protection period ends that the underlying stock price hits the barrier, or until expiration, whichever comes first. That is, if after the protection period has elapsed, the underlying stock price hits the barrier prior to expiration, the up-and-out call is knocked out and a constant rebate C_2 is paid at time τ . We denote by τ the first time after t^* that S hits B conditional on $S_{t^*} < B$. If, on the other hand, the stock price has not hit the barrier by the expiration date T , the up-and-out call becomes a standard call, and consequently is exercised if the underlying stock price S_T finishes above the strike K , and becomes worthless otherwise. So the payoff of a protected barrier option at time T is

$$C_1 e^{r(T-t^*)} 1_{\{S_{t^*} \geq B\}} + C_2 e^{r(T-\tau)} 1_{\{\tau < T, S_{t^*} < B\}} + (S_T - K)^+ 1_{\{\tau \geq T, S_{t^*} < B\}}.$$

We couldn't find any examples of this in the literature to apply Malliavin calculus to find

the Malliavin derivative of a random time. Although we follow [5] to look at the so-called protected barrier option or partial barrier option, we find the barrier options described in [5] are not exactly the same as the protected barrier option in [12]. One way in which they differ is that the protected barrier option in [12] has a rebate amount C_1 which is paid if S is greater than the barrier at time t^* and a rebate amount C_2 which is paid if the time τ occurs before time T . However, the barrier options in [5] are still partial or protected, because the term “ τ ” appearing in [5] is the length of the monitoring period, like the one t^* in [12], but it is different from the random time τ in [12]. So our application of Malliavin Calculus is an extension of the ones in [5] and include the calculation of Malliavin derivative of a random time.

The rainbow barrier option in [12] is another extension to a European up-and-out call option. There are two assets in this option, denoted by S^1 and S^2 . Like a standard European up-and-out call, this option is knocked out if the value of a “trigger” asset S^1 rises to hit some barrier $B > S_0^1$ before the option expires at T . In contrast to a European up-and-out call, if the barrier is not hit prior to T , then the payoff at T is $(S_T^2 - K)^+$, where S_T^2 is the terminal spot price of some other asset. So the payoff of a rainbow barrier option is

$$(S_T^2 - K)^+ 1_{\left\{ \max_{0 \leq t \leq T} S_t^1 < B \right\}}.$$

The first objective of the thesis is to calculate the Malliavin derivative of a random time occurring in the protected barrier option; then the multi-dimensional Malliavin Calculus is applied to the rainbow barrier option. Finally the replicating portfolios of these two exotic barrier options are formulated and a comparison of the results between the Malliavin approach and the traditional delta-hedging approach is made.

This thesis is organized as follows. In Chapter 1, we summarize some preliminaries, like the Black and Scholes framework [9], and state the necessary conditions for the possibility of hedging different options. In Chapter 2, the concepts of Malliavin calculus are stated and some propositions and proofs via the definition of directional derivative are given in detail. In Chapter 3, we describe the extension of the Malliavin derivative and the Clark-Ocone formula in a bigger space. The hedging procedure is explained by means of a self-financing portfolio and is thereafter related to the Malliavin Calculus. Then, in

Chapter 4, we describe the extension of the Malliavin derivative to the multi-dimensional case, some propositions and proofs are given and the hedging portfolio strategy is derived in the two-dimensional case. Finally, in Chapter 5, we will apply Malliavin calculus to show how to hedge three types of barrier options: the digital barrier option with a random time, and the protected barrier option and the rainbow barrier option as studied in [12]. We find the Malliavin derivative of $g(\tau)$, where g is some nice functions and τ is the time described above. We could not find any examples of this in the literature so we took a simplistic approach. Nonetheless, our treatment may shed light upon the case of a more general time when applied to the protected barrier option and by using an extension of multi-dimensional Malliavin calculus, we find a replicating portfolio of the rainbow barrier option. At the end, I compare the hedging results of the three exotic barrier options by using the Malliavin calculus approach with the traditional delta-hedging approach.

Chapter 1

The Preliminaries

In this chapter, we first recall some basic concepts from functional analysis, stochastic calculus and review some concepts about market model in finance. All the definitions and theorems are quoted from text books; for more detail, we refer to [10], [16], [39] and [44].

1.1 Basic Concepts from Functional Analysis

Let us recall some basic concepts from functional analysis, and quote some definitions and theorems from, see for example, [10], [16], [39] and [44].

Definition 1.1. A map T from a non-empty subset $\mathcal{D}(T)$ (the domain of T) of a linear space X to a linear space Y is linear if for all $\alpha, \beta \in \mathbb{R}$, $x, y \in \mathcal{D}(T)$,

$$\alpha x + \beta y \in \mathcal{D}(T)$$

and

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Notation 1.2. Let X be a Banach space, that is a complete, normed vector space over $\mathbb{R}$, and let $\|x\|$ denote the norm of the element $x \in X$.

Definition 1.3. A linear functional T is called bounded (or continuous), if

$$\|T\| := \sup_{\|x\| \leq 1} |T(x)| < \infty.$$

Sometimes we write $\langle T, x \rangle$ or Tx instead of $T(x)$ and call $\langle T, x \rangle$ “the action of T on x ”.

Definition 1.4. The set of all bounded linear functionals is called the dual of X and is denoted by X^* . Equipped with the norm $\|\cdot\|$, the space X^* is a Banach space.

Example 1.5. Let $X = C_0([0, T])$ be the space of continuous real functions ω on $[0, T]$ such that $\omega(0) = 0$. Then

$$\|\omega\|_\infty := \sup_{t \in [0, T]} |\omega(t)|$$

is a norm on X , called the uniform norm. With this norm, X is a Banach space and its dual X^* can be identified with the space $M([0, T])$ of all signed measures ν on $[0, T]$, with the norm

$$\|\nu\| = \sup_{|f| \leq 1} \int_0^T f(t) d\nu(t) = |\nu|([0, T]).$$

Definition 1.6. Let $T : X \mapsto Y$ be a continuous map of a Banach space X into a Banach space Y . X and Y have dual spaces X^* and Y^* . Let T^* be defined as follows:

$$T^*f(x) = f(Tx) \quad \text{for } f \in Y^* \text{ and } x \in X,$$

and

$$|T^*f(x)| \leq \|f\| \|T\| \|x\|,$$

so T^*f is a bounded linear functional on X , i.e., $T^*f \in X^*$. So

$$T^* : Y^* \mapsto X^*,$$

and T^* is called the adjoint of T .

Notation 1.7. For a set, E , in a Banach space, X , we write $\bar{E}$ for the closure of E .

Definition 1.8. Let H be a Hilbert space. By an operator in H we mean a linear mapping T whose domain $\mathcal{D}(T)$ is a subspace of H and whose range $\mathcal{R}(T)$ lies in H .

Definition 1.9. The graph $\mathcal{G}(T)$ of an operator T is the subspace of $H \times H$ that consists of the ordered pairs $\{x, Tx\}$, where x ranges over $\mathcal{D}(T)$.

Definition 1.10. An operator S is an extension of T , if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $Sx = Tx$ for $x \in \mathcal{D}(T)$.

Remark 1.11. If the operator T is continuous, then T has a continuous extension to the closure of $\mathcal{D}(T)$, i.e., $\overline{\mathcal{D}(T)}$, hence to H , since $\overline{\mathcal{D}(T)}$ is complemented in H .

Definition 1.12. A closed operator in H is one whose graph is closed subspace of $H \times H$. That is, a linear operator $T : \mathcal{D}(T) \subset H \rightarrow H$ is closed, i.e., if $\{x_n\} \subset \mathcal{D}(T)$ and

$$\lim_{n \rightarrow \infty} x_n = x \in H$$

and

$$\lim_{n \rightarrow \infty} Tx_n = y \in H,$$

then one has

$$x \in \mathcal{D}(T) \quad \text{and} \quad Tx = y.$$

Definition 1.13. An operator T is closable, if $\overline{\mathcal{G}(T)}$, the closure of the graph of T in $H \times H$, is the graph of a linear operator, which we call $\overline{T}$. Note that $\overline{T}$ extends T .

Definition 1.14. An operator T in H is densely defined if and only if $\mathcal{D}(T)$ is dense in H , i.e.

$$\overline{\mathcal{D}(T)} = H.$$

Notation 1.15. Let $\mathcal{D}(T^*)$ be the domain of the adjoint T^* to T . That is, $\mathcal{D}(T^*)$ consists of all $y \in H$ for which the linear functional

$$x \rightarrow (Tx, y) \tag{1.1}$$

is continuous on $\mathcal{D}(T)$.

If $y \in \mathcal{D}(T^*)$, then the Hahn-Banach theorem extends functional (1.1) to a continuous linear functional on H , and therefore there exists an elements $T^*y \in H$ that satisfies

$$(Tx, y) = (x, T^*y), \quad x \in \mathcal{D}(T). \tag{1.2}$$

Obviously, T^*y will be uniquely determined by equation (1.2) if and only if T is densely defined.

Definition 1.16. Let U be an open subset of a Banach space X and let f be a function from U to $\mathbb{R}^m$.

- We say that f has a directional derivative (or Gâteaux derivative) $D_y f(x)$ at the point $x \in U$ in the direction $y \in X$ if

$$D_y f(x) := \frac{d}{d\epsilon} [f(x + \epsilon y)]|_{\epsilon=0} \in \mathbb{R}^m$$

exists.

- We say that f is Fréchet-differentiable at $x \in U$ if there exists a bounded linear map $A : X \rightarrow \mathbb{R}^m$, that is, $A = (A_1, \dots, A_m)^T$, with $A_i \in X^*$ for $i = 1, \dots, m$, such that

$$\lim_{\substack{h \rightarrow 0 \\ h \in X}} \frac{|f(x+h) - f(x) - Ah|}{\|h\|} = 0.$$

We write

$$f'(x) = \begin{bmatrix} f'(x)_1 \\ \vdots \\ f'(x)_m \end{bmatrix} = A \in (X^*)^m$$

for the Fréchet derivative of f at x .

Proposition 1.17. [See Proposition 4.6 of [39]]

- If f is Fréchet-differentiable at $x \in U \subset X$, then f has a directional derivative at x in all directions $y \in X$ and

$$D_y f(x) = \langle f'(x), y \rangle \in \mathbb{R}^m,$$

where $\langle f'(x), y \rangle = (\langle f'_1(x), y \rangle, \dots, \langle f'_m(x), y \rangle)^T$ is the m -vector whose i th component is the action of the i th component $f'_i(x)$ of $f'(x)$ on y .

- Conversely, if f has a directional derivative at all $x \in U$ in all the directions $y \in X$, and the linear map

$$y \rightarrow D_y f(x), \quad y \in X$$

is continuous for all $x \in U$, then there exists an element $\nabla f(x) \in (X^*)^m$ such that

$$D_y f(x) = \langle \nabla f(x), y \rangle.$$

If this map $x \rightarrow \nabla f(x) \in (X^*)^m$ is continuous on U , then f is Fréchet-differentiable and

$$f'(x) = \nabla f(x).$$

1.2 Distributions

In this section, we review some concepts of distributions, which, also known as generalized functions, extend the concept of derivative to all integrable functions. See, for example, [44] and [49].

Definition 1.18. Let $S(\mathbb{R}^n)$ denote the Schwartz space, which is the space of all infinitely differentiable rapidly decreasing functions. We say $\varphi \in S(\mathbb{R}^n)$, if any derivative of φ , multiplied with any power of $|x|$, converges towards 0 for $|x| \rightarrow \infty$.

Definition 1.19. Let $S'(\mathbb{R}^n)$ denote the Schwartz distribution space, which is the dual of the Schwartz space. We say a distribution $T \in S'(\mathbb{R}^n)$, if T is a continuous linear functional on $S(\mathbb{R}^n)$ with values in $\mathbb{R}$.

Remark 1.20. From the above definition, we see $T \in S'(\mathbb{R}^n)$ if and only if $T : S(\mathbb{R}^n) \rightarrow \mathbb{R}$ is linear and $f_n \rightarrow f$ in $S(\mathbb{R}^n)$ implies that $T(f_n) \rightarrow T(f)$ in $\mathbb{R}$.

Definition 1.21. If T is a distribution, we define its derivative T' by

$$\langle T', f \rangle = - \langle T, f' \rangle. \quad (1.3)$$

Remark 1.22. This definition extends the ordinary definition of derivative, every distribution becomes infinitely differentiable and the usual properties of derivative hold.

Proposition 1.23. [See Theorem 2.2 of [42]] If φ is a continuous and piecewise differentiable, then its distributional derivative is given by

$$(T_\varphi)' = T_{\varphi'}. \quad (1.4)$$

Definition 1.24. A sequence T_n in $S'(\mathbb{R}^n)$ is said to converge in $S'(\mathbb{R}^n)$, if $T_n(f) \rightarrow T(f)$ for each $f \in S(\mathbb{R}^n)$.

Remark 1.25. Lebesgue's Dominated Convergence Theorem implies that if

- g_n is measurable for all n ,
- $g_n(x) \rightarrow g(x)$ pointwise,
- $|g_n(x)| \leq h(x)$ for some integrable function h ,

then

$$\int g_n(x)f(x)dx \rightarrow \int g(x)f(x)dx \quad (1.5)$$

for each $f \in S(\mathbb{R}^n)$.

Proposition 1.26. [See Theorem 5.19 of [49]] For any $T \in S'(\mathbb{R}^n)$, there is a sequence (φ_n) in $S(\mathbb{R}^n)$, such that $\varphi_n \rightarrow T$ in $S'(\mathbb{R}^n)$. That is, for every $f \in S(\mathbb{R}^n)$,

$$\langle T, f \rangle = \lim_n \int \varphi_n(x)f(x)dx.$$

Remark 1.27. The above proposition tells us that every distribution can be approximated by a sequence in $S(\mathbb{R}^n)$.

Definition 1.28. The so-called Dirac delta function on $\mathbb{R}$, $\delta_x(\cdot)$

- Obeys $\delta_x(s) = 0$ for all $s \neq x$,
- Satisfies $\int_{-\infty}^{\infty} \delta_x(s) ds = 1$.

Remark 1.29. For $f \in S(\mathbb{R})$,

$$\begin{aligned}
 \langle (1_{\{\cdot > x\}})', f \rangle &= -\langle 1_{\{\cdot > x\}}, f' \rangle \\
 &= -\int_{-\infty}^{\infty} 1_{\{s > x\}} f'(s) ds \\
 &= -\int_x^{\infty} f'(s) ds \\
 &= f(x) - \lim_{s \rightarrow \infty} f(s) \\
 &= f(x).
 \end{aligned}$$

Notice that $\lim_{s \rightarrow \infty} f(s) = 0$, because $f \in S(\mathbb{R})$ and is a rapidly decreasing function.

Remark 1.30. For $f \in S(\mathbb{R})$, since $(f(s) - f(x)) \delta_x(s) \equiv 0$ on $\mathbb{R}$, we have

$$\begin{aligned}
 \int_{-\infty}^{\infty} \delta_x(s) f(s) ds &= \int_{-\infty}^{\infty} (f(s) - f(x)) \delta_x(s) ds + f(x) \int_{-\infty}^{\infty} \delta_x(s) ds \\
 &= f(x).
 \end{aligned}$$

We cannot interpret $\int_{-\infty}^{\infty} \delta_x(s) f(s) ds$ as an integral in the usual sense, because there is no such function $\delta_x(s)$ with these properties. The $\delta_x(s)$ function is thought of as a generalized function.

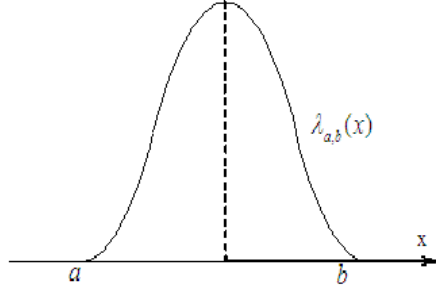
Let us recall the following functions which can be used for the construction of some test functions, see [42]. Now consider the function

$$h(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

The key property of $h(x)$ is that, at the transition point $x = 0$, all of its derivatives exist and are zero. Now consider the function

$$\lambda_{a,b}(x) = h\left(\frac{x-a}{b-a}\right) h\left(\frac{b-x}{b-a}\right). \quad (1.6)$$

This is a test function, i.e., $\lambda_{a,b}(x) \in S(\mathbb{R})$, whose graph is a smooth “pulse” with support $[a, b]$. We will often assume our smooth pulse to have integral 1. See Figure 1.1.

Figure 1.1: A pulse function supported on $[a, b]$ 

1.3 Stochastic Calculus

In this section, we review some basic concepts from stochastic calculus, see, for example, [41].

Definition 1.31. A filtration is a family of σ -algebras $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ that is increasing, i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$.

For convenience, we will write $\mathcal{F}$ for the filtration $(\mathcal{F}_t)_{0 \leq t \leq \infty}$.

Definition 1.32. A filtered complete probability space $(\Omega, \mathcal{F}, P)$ is said to satisfy the usual hypotheses if

- $\mathcal{F}_0$ contains all the P -null sets.
- $\mathcal{F}_t = \bigcap_{u > t} \mathcal{F}_u$, for all t , $0 \leq t < \infty$; that is, the filtration $\mathcal{F}$ is right continuous.
- $\mathcal{F}_\infty = \sigma \left(\bigcup_{t \geq 0} \mathcal{F}_t \right)$.

We always assume that the usual hypotheses hold.

Definition 1.33. A random variable $\tau : \Omega \rightarrow [0, \infty]$ is a stopping time if the event $\{\tau \leq t\} \in \mathcal{F}_t$, for every t , $0 \leq t < \infty$.

Proposition 1.34 (See [41]). τ is a stopping time if and only if the event $\{\tau < t\} \in \mathcal{F}_t$, for each $0 \leq t \leq \infty$.

Definition 1.35. A stochastic process X on $(\Omega, \mathcal{F}, P)$ is a collection of $\mathbb{R}$ -valued or $\mathbb{R}^d$ -valued random variables $(X_t)_{0 \leq t \leq \infty}$. The process X is said to be adapted if $X_t \in \mathcal{F}_t$ (that is, is $\mathcal{F}_t$ measurable) for each t . The function $t \mapsto X_t(\omega)$ mapping $[0, \infty)$ into $\mathbb{R}$ are called the sample paths of the stochastic process X .

Definition 1.36. Let X be a stochastic process and let Λ be a Borel set in $\mathbb{R}$. Define

$$\tau(\omega) = \inf \{t > 0 : X_t \in \Lambda\}.$$

Then τ is called the hitting time of Λ for X .

Proposition 1.37 (See [41]). Let X be an adapted stochastic process with almost surely sample paths which are right continuous, with left limits. Then the hitting time of Λ is a stopping time.

Definition 1.38. Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space and W_t be a stochastic process. W_t is called a Wiener process with respect to $\mathcal{F}_t$, if

- $W_0 = 0$ almost surely,
- W_t is $\mathcal{F}_t$ -measurable for every t ,
- $P(\omega \in \Omega : t \mapsto W_t(\omega) \text{ is continuous function in } t) = 1$,
- $W_t - W_s$ is independent to $\mathcal{F}_s$ for all $t > s$ and $W_t - W_s \sim N(0, t - s)$.

Definition 1.39. The Banach space $\Omega = C_0([0, T])$ is called Wiener space, which is naturally equipped with the Borel σ -algebra generated by the topology of the uniform norm.

Remark 1.40. Because we can regard each path $t \rightarrow W(t, \omega)$ of the Wiener process starting at 0 as an element ω of $C_0([0, T])$, so we call it Wiener space. Thus we may identify $W(t, \omega)$ with the value $\omega(t)$ at time t of an element $\omega \in C_0([0, T])$: $W(t, \omega) = \omega(t)$.

Remark 1.41. This measurable space is equipped with the probability measure P , which is given by the probability law of the Wiener process:

$$\begin{aligned} & P \{W_{t_1} - W_{t_0} \in F_1, W_{t_2} - W_{t_1} \in F_2, \dots, W_{t_k} - W_{t_{k-1}} \in F_k\} \\ &= \int_{F_1 \times \dots \times F_k} \rho(t_1, x, x_1) \rho(t_2 - t_1, x_1, x_2) \cdots \rho(t_k - t_{k-1}, x_{k-1}, x_k) dx_1 \cdots dx_k \end{aligned}$$

where $F_i \subset \mathbb{R}$, $0 \leq t_1 < t_2 < \dots < t_k \leq T$, and

$$\rho(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}|x-y|^2}, \quad t \in [0, T], \quad x, y \in \mathbb{R}.$$

The measure P is called Wiener measure on Ω .

Definition 1.42. Let m be a real number, and define the first passage time to level m

$$\tau = \min\{t \geq 0; W_t = m\}.$$

This is the first time the Brownian motion W reaches the level m . If the Brownian motion never reaches the level m , we set $\tau = \infty$.

Proposition 1.43 (Reflection principle, see [41]). Let W_t be standard Brownian motion starting at zero. Then

$$P\{\tau \leq t, W_t \leq x\} = P\{W_t \geq 2m - x\},$$

for $x \leq m, m > 0$.

Definition 1.44. An Itô process is a stochastic process of the form

$$X_t = X_0 + \int_0^t \theta_s ds + \int_0^t \vartheta_s dW_s,$$

where X_0 is nonrandom and θ_s, ϑ_s are adapted stochastic processes.

Proposition 1.45 (Itô formula, see [41]). Let $X_t, t \geq 0$, be an Itô process and let $f(t, x)$ be $C^{1,2}$ function (i.e., $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ are defined and continuous). Then, for

every $T \geq 0$,

$$df(t, X_t) = f_t(t, X_t) dt + f_x(t, X_t) dX_t + \frac{1}{2} f_{xx}(t, X_t) d\langle X, X \rangle_t.$$

Definition 1.46. Let $F : \Omega \rightarrow \mathbb{R}$ be a random variable, choose $g \in L^2([0, T])$, and consider

$$\gamma(t) = \int_0^t g(s) ds \in \Omega. \quad (1.7)$$

Then we define the directional derivative of F at the point $\omega \in \Omega$ in the direction $\gamma \in \Omega$ by

$$D_\gamma F(\omega) = \frac{d}{d\epsilon} [F(\omega + \epsilon\gamma)]|_{\epsilon=0},$$

if the derivative exists in some sense (to be made precise later).

Remark 1.47. The set of $\gamma \in \Omega$ which can be written in equation (1.7) for some $g \in L^2([0, T])$, is called the Cameron-Martin space and denoted by $\mathcal{H}$.

1.4 Black-Scholes Model

In this section, we review some concepts in the Black-Scholes model, see for example, [2], [8], [9] and [45].

On the complete Wiener measure space $(\Omega, \mathcal{F}, P)$ live two assets, one locally risk-free asset, R_t (i.e. a bank account where money grows at the short interest rate r), and one risky asset, S_t . They satisfy the stochastic differential equations

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \in [0, T], \quad (1.8)$$

$$dR_t = r R_t dt, \quad R_0 = 1, \quad t \in [0, T]. \quad (1.9)$$

Here the mean rate of return μ , the volatility σ and r are positive constants.

Definition 1.48. Let h_t^0 be the number of risk-free asset and h_t^1 the number of stocks owned by the investor at time t . The couple $h_t = (h_t^0, h_t^1)$, $t \in [0, T]$, is called a portfolio or

trading strategy. Assume h_t^0 and h_t^1 are measurable and adapted processes such that

$$\int_0^T |h_t^0| r dt < \infty, \int_0^T |h_t^1 \mu| dt < \infty, \int_0^T (h_t^1 \sigma)^2 dt < \infty$$

almost surely. Then the value of the portfolio at time t is

$$V_t^h = h_t^0 R_t + h_t^1 S_t. \quad (1.10)$$

For a trading strategy, (h_t^0, h_t^1) , the gain G_t^h made via the portfolio (h_t^0, h_t^1) up to time t is defined to be

$$G_t^h = \int_0^t h_s^0 dR_s + \int_0^t h_s^1 dS_s.$$

Note that both of integrals are understood to be the limit over partitions of Riemann-Stieltjes sums, the essential point is these are stochastic integrals.

Definition 1.49. The portfolio h is said to be self-financing if

$$V_t^h = V_0^h + G_t^h,$$

i.e., there is no fresh investment and there is no consumption. An alternative definition is given by

$$dV_t^h = h_t^0 dR_t + h_t^1 dS_t. \quad (1.11)$$

Definition 1.50. The discounted value of an asset, (A_t) , say, is defined by

$$\tilde{A}_t = \frac{A_t}{R_t}.$$

So $\tilde{A}_t$ is simply the value of A_t in units of the bond R .

Now we look at the discounted prices of stock S_t . By Itô's formula,

$$d\tilde{S}_t = d\left(\frac{S_t}{R_t}\right) = (\mu - r) \tilde{S}_t dt + \sigma \tilde{S}_t dW_t,$$

and the discounted value of a portfolio is

$$\begin{aligned}
 d\tilde{V}_t^h &= d\left(\frac{V_t^h}{R_t}\right) = -r\frac{V_t^h}{R_t}dt + \frac{dV_t^h}{R_t} \\
 &= -rh_t^1\tilde{S}_t dt + h_t^1 R_t^{-1}dS_t \\
 &= h_t^1 d\tilde{S}_t.
 \end{aligned} \tag{1.12}$$

Notice that σ , μ and r are all constants and not stochastic processes, so if $\mu = r$, then $\tilde{S}_t$ is a P -martingale.

To avoid arbitrage, we restrict the class of trading strategy that we can adopt, specifically we require that there is a non-negative, integrable, $\mathcal{F}_T$ measurable random variable ξ such that

$$\tilde{V}_t^h = \frac{V_t^h}{R_t} \geq -E^P[\xi|\mathcal{F}_t], \tag{1.13}$$

and write $h_t \in SF(\xi)$. Such strategies have discounted values which cannot lose more than an amount $\xi(\omega)$. A strategy in $SF(\xi)$ will be called *admissible*. One obvious choice for ξ is $\xi = 0$.

Definition 1.51. An arbitrage opportunity is a self-financing portfolio h_t such that

$$V_0^h \leq 0, V_T^h \geq 0 \quad a.s.$$

and

$$P\{\omega : V_T^h(\omega) > 0\} > 0.$$

Proposition 1.52 (See [45]). Let h_t^0, h_t^1 be (progressively) measurable adapted processes satisfying the integrability conditions for R_t and S_t respectively. Then (h_t^0, h_t^1) is a self-financing trading strategy if and only if

$$\tilde{V}_t^h = \tilde{V}_0^h + \int_0^t h_s^1 d\tilde{S}_s.$$

Proposition 1.53 (See [45]). Suppose that under the measure P , $\tilde{S}_t$ is a martingale and h_t^1 is (partial) strategy such that

$$\int_0^t h_s^1 d\tilde{S}_s$$

is a local martingale. Define a portfolio value by setting

$$\tilde{V}_t = x_0 + \int_0^t h_s^1 d\tilde{S}_s, \quad x_0 \in \mathbb{R}$$

and a partial strategy h_t^0 by

$$h_t^0 = \tilde{V}_t - h_t^1 \tilde{S}_t.$$

Then (h_t^0, h_t^1) is a self-financing strategy with $\tilde{V}_t \equiv \tilde{V}_t^h$. If h_t^1 is chosen so that $h_t^1 \in SF(\xi)$, then $\tilde{V}_t^h$ is a super-martingale.

Proposition 1.54 (See [45]). With h_t^0, h_t^1, V_t^h as in the previous lemma, then (h_t^0, h_t^1) cannot be an arbitrage opportunity.

Proposition 1.55 (See [45]). If the strategy (h_t^0, h_t^1) such that $\tilde{S}_t$ and $\int_0^t h_s^1 d\tilde{S}_s$ are P -martingale, then $\tilde{V}_t^h$ is a P -martingale too.

We refer to [2] and [45] for the proofs of these propositions in detail.

Definition 1.56. A derivative is a contract on the risky asset that produces a payoff G at maturity time T . The payoff is an $\mathcal{F}_T$ -measurable nonnegative random variable.

Definition 1.57. A nonnegative $\mathcal{F}_T$ -measurable payoff G can be replicated if there exists a self-financing portfolio (h_t^0, h_t^1) in R_t and S_t such that $V_T^h = G$.

We present some famous theorems as follows.

Proposition 1.58 (Martingale Representation Theorem, see [45]). Let $M(t)$, $0 \leq t \leq T$, be a martingale with respect to the filtration $\mathcal{F}_t$, i.e., for every t , $M(t)$ is $\mathcal{F}_t$ -measurable and for $0 \leq s \leq t \leq T$, $E^P[M(t)|\mathcal{F}_t] = M(s)$. Then there is an adapted process $\Gamma(s)$, $0 \leq s \leq T$, such that

$$M(t) = M(0) + \int_0^t \Gamma(s) dW_s, \quad 0 \leq t \leq T. \quad (1.14)$$

Proposition 1.59 (Itô Representation Theorem, see [45]). If $G \in L^2(\Omega, \mathcal{F}_T, P)$ is $\mathcal{F}_T$ -measurable, then there exists a unique adapted process $\varphi = \varphi(t)$, $0 \leq t \leq T$, such that

$$G = E^P[G] + \int_0^T \varphi(t) dW_t. \quad (1.15)$$

Proposition 1.60 (Change of Measure, see [45]). *Let P and Q be measures on $(\Omega, \mathcal{F})$ such that Q is equivalent with P with Radon-Nikodým derivative of Q with respect to P*

$$Z = \frac{dQ}{dP},$$

where Z is non-negative random variable with $E^P[Z] = 1$. Let s and t satisfying $0 \leq s \leq t \leq T$ be given and let Y be an $\mathcal{F}_t$ -measurable random variable. Then

$$E^Q[Y(t)|\mathcal{F}_s] = \frac{E^P[Z(t)Y(t)|\mathcal{F}_s]}{E^P[Z(t)|\mathcal{F}_s]}. \quad (1.16)$$

Proposition 1.61 (Girsanov Theorem, see [45]). *Let $\theta(t)$, $0 \leq t \leq T$, be an adapted process. Define*

$$Z(t) = \exp \left\{ - \int_0^t \theta(s) dW_s - \frac{1}{2} \int_0^t \theta^2(s) ds \right\}$$

and

$$\tilde{W}_t = W_t + \int_0^t \theta(s) ds.$$

Assume $E^P[\int_0^T (\theta(s)Z(s))^2 ds] < \infty$, and suppose that $E^P[Z(T)] = 1$ and define a measure Q on $(\Omega, \mathcal{F})$ by

$$\frac{dQ}{dP} = Z(T).$$

Then the process $\tilde{W}_t$, $0 \leq t \leq T$ is a Q -Brownian motion.

Proposition 1.62 (Cameron-Martin Theorem, see [45]). *For any $F \in L^2(\Omega)$, and $\gamma \in \mathcal{H}$, we have*

$$E[F(\omega + \epsilon\gamma)] = E \left[F(\omega) \exp \left\{ \epsilon \int_0^T g(t) dW_t - \frac{1}{2} \epsilon^2 \int_0^T g^2(t) dt \right\} \right], \quad (1.17)$$

where $\gamma(t) = \int_0^t g(s) ds$, for some $g \in L^2([0, T])$.

Remark 1.63. Consider $E^P[F(\omega + \epsilon\gamma)]$ where ω is an element of $C_0([0, T])$, and $\gamma(t) = \int_0^t g(s) ds$, $g \in L^2[0, T]$. The measure P sees ω as a path of a Brownian motion. Rewrite

$E^P [F(\omega + \epsilon\gamma)]$ as

$$E^P [F(\omega + \epsilon\gamma)] = E^P \left[F(\omega + \epsilon\gamma) \exp \left\{ \epsilon \int_0^T g(t) dW_t + \frac{1}{2} \epsilon^2 \int_0^T g^2(t) dt \right\} \right. \\ \left. \exp \left\{ -\epsilon \int_0^T g(t) dW_t - \frac{1}{2} \epsilon^2 \int_0^T g^2(t) dt \right\} \right].$$

By Girsanov's theorem, let us define $dQ/dP = \exp \left\{ -\epsilon \int_0^T g(t) dW_t - \frac{1}{2} \epsilon^2 \int_0^T g^2(t) dt \right\}$ and $\tilde{W}_t = W_t + \epsilon \int_0^t g(s) ds$, i.e., $\tilde{W}_t = W_t + \epsilon\gamma$ is Q -Brownian motion. Then

$$E^P [F(\omega + \epsilon\gamma)] \\ = E^Q \left[F(\tilde{\omega}) \exp \left\{ \epsilon \int_0^T g(t) d \left(\tilde{W}_t - \epsilon \int_0^t g(s) ds \right) + \frac{1}{2} \epsilon^2 \int_0^T g^2(t) dt \right\} \right] \\ = E^Q \left[F(\tilde{\omega}) \exp \left\{ \epsilon \int_0^T g(t) d\tilde{W}_t - \frac{1}{2} \epsilon^2 \int_0^T g^2(t) dt \right\} \right].$$

This expression has the same value what ever Q and $\tilde{W}_t$ are chosen, so long as Q sees $\tilde{W}$ as a Brownian motion and $\tilde{\omega}$ as the paths of $\tilde{W}$.

Definition 1.64. The probability measure Q defined in Remark 1.63 is called the risk-neutral probability measure if $\tilde{S}_t$ is a Q -martingale on $(\Omega, \mathcal{F}_T, Q)$.

Hereafter, let us assume we are in the risk-neutral probability measure Q . Then the evolution of the risk-neutralized process for the stock price is geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad t \in [0, T], \quad (1.18)$$

where $\{W_t, t \in [0, T]\}$ is standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, Q)$, and r, S_0, σ with $\sigma \neq 0$ are all positive constants. By letting the coefficients be constants we know for sure that there exists a unique continuous solution to the stochastic differential equation, SDE, 1.18 and 1.18, see e.g. [18] for further details about regularity conditions. The solution to the stochastic differential equation (1.18) is:

$$S_t = S_0 \exp \left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}, \quad t \in [0, T].$$

Now let us value portfolio process under the risk-neutral measure Q . First, we know under the measure Q , the discounted stock price is a martingale. Then

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t ,$$

and from equation (1.12), we know the discounted value of a portfolio is

$$d\tilde{V}_t^h = h_t^1 d\tilde{S}_t = \sigma \tilde{S}_t h_t^1 dW_t .$$

From here we confirm again that the discounted value of the portfolio is a Q -martingale. Then using the martingale property, if we have

$$E^Q[\tilde{V}_T^h] = V_0^h = 0 , \quad (1.19)$$

then $V_T^h = 0$, Q -a.s., if $V_T^h \geq 0$. So, we cannot have $P(V_T^h > 0) = 0$. This implies there are no arbitrage opportunities resulting from admissible portfolios in S_t and R_t .

$$\tilde{V}_t^h = E^Q [\tilde{V}_T^h | \mathcal{F}_t] ,$$

or

$$V_t^h = E^Q [e^{-r(T-t)} V_T^h | \mathcal{F}_t] . \quad (1.20)$$

Definition 1.65. The basic market is complete if every contingent claim G in $L^2(\Omega, \mathcal{F}_T, Q)$, a $\mathcal{F}_T$ -measurable stochastic variable, is attainable by a self-financing portfolio. Let $\mathcal{G}$ be the set of such contingent claims.

Proposition 1.66 (See [45]). Let G be a nonnegative $L^2(\Omega, \mathcal{F}_T, Q)$ random variable. Then there exists a self-financing admissible portfolio $h_t = (h_t^0, h_t^1)$ in S_t and R_t such that $G = V_T^h$.

Proof. We follow [45] to give the proof. Since $G \in L^2(\Omega, \mathcal{F}_T, Q)$, then $\frac{G}{R_T} \in L^2(\Omega, \mathcal{F}_T, Q)$. By the Itô representation theorem, there exists an adapted and measurable process $\varphi =$

$\{\varphi_t, 0 \leq t \leq T\}$ such that

$$\frac{G}{R_T} = E^Q \left[\frac{G}{R_T} \right] + \int_0^T \varphi_s dW_s .$$

Set

$$M_t = E^Q \left[\frac{G}{R_T} | \mathcal{F}_t \right] = E^Q \left[\frac{G}{R_T} \right] + \int_0^t \varphi_s dW_s ,$$

so $G = R_T M_T$. Define the portfolio $h_t = (h_t^0, h_t^1)$ by

$$\begin{aligned} h_t^1 &= \frac{\varphi_t}{\sigma \tilde{S}_t} , \\ h_t^0 &= M_t - h_t^1 \tilde{S}_t . \end{aligned}$$

The discounted value of this portfolio is

$$\tilde{V}_t^h = h_t^0 + h_t^1 \tilde{S}_t = M_t .$$

So its final value will be

$$V_T^h = R_T \tilde{V}_T^h = R_T M_T = G .$$

Finally, let us show this portfolio is self-financing.

$$\begin{aligned} dV_t^h &= d(R_t M_t) \\ &= R_t dM_t + M_t dR_t \\ &= R_t \varphi dW_t + M_t r R_t dt \\ &= h_t^1 \sigma \tilde{S}_t R_t dW_t + \left(h_t^0 + h_t^1 \tilde{S}_t \right) r R_t dt \\ &= h_t^0 dR_t + h_t^1 dS_t . \end{aligned}$$

□

So the market modeled by equation (1.9) and equation (1.18) is complete. So we know any $G \in L^2(\Omega, \mathcal{F}_T, Q)$ is attainable by a self-financing portfolio. Hence, we can write the

value at time t of a derivative with payoff G as follows:

$$V_t^h = E^Q [e^{-r(T-t)} G | \mathcal{F}_t]. \quad (1.21)$$

However, we cannot see the explicit form of the portfolio from the equation above, then we want to find the self-financing portfolio h satisfying

$$dV_t^h = [h_t^0 r R_t + h_t^1 r S_t] dt + h_t^1 \sigma S_t dW_t, \quad (1.22)$$

$$V_T^h = G \quad Q - a.s.. \quad (1.23)$$

The initial value of the portfolio that replicates G is

$$V_0^h = E^Q [e^{-rT} G].$$

Finally, let us state the first and second fundamental theorems of asset pricing. See, [15] and [23] for detail.

Proposition 1.67 (First Fundamental Theorem of Asset Pricing, see [15]). *No arbitrage if and only if there is a risk neutral measure Q .*

Proposition 1.68 (Second Fundamental Theorem of Asset Pricing, see [23]). *An arbitrage-free market is complete if and only if there is a unique equivalent martingale measure Q .*

1.5 Multi-dimensional Black-Scholes Model

In this section, we recall some definitions and theorems from multiple stocks driven by multi-dimensional Brownian motions, see for example, [33], [38] and [45].

Let $W_t = (W_t^1, \dots, W_t^d)$ be a multi-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.

Proposition 1.69 (Girsanov Theorem in higher dimensions, see [45]). *Let T be a fixed positive time, and let $\theta_t = (\theta_t^1, \dots, \theta_t^d)$ be a d -dimensional adapted process. Define*

$$Z(t) = \exp \left\{ - \int_0^t \theta_s \cdot dW_s - \frac{1}{2} \int_0^t \|\theta_s\|^2 ds \right\},$$

$$\tilde{W}_t = W_t + \int_0^t \theta_s ds, \quad (1.24)$$

and assume that

$$E^P \left[\int_0^T \|\theta_s\|^2 Z^2(s) ds \right] < \infty.$$

Suppose $E[Z(T)] = 1$ and define a measure Q by $\frac{dQ}{dP} = Z(T)$. Then the process $\tilde{W}_t$ is a d -dimensional Q -Brownian motion.

Proposition 1.70 (Martingale Representation in higher dimensions, see [45]). *Let T be a fixed positive time, and assume $\mathcal{F}_t$, $0 \leq t \leq T$, is the filtration generated by the d -dimensional Brownian motion W_t , $0 \leq t \leq T$. Let M_t , $0 \leq t \leq T$ be a martingale with respect to this filtration under P . Then there is an adapted, d -dimensional process $\Gamma_s = (\Gamma_s^1, \dots, \Gamma_s^d)$, $0 \leq s \leq T$, such that*

$$M_t = M_0 + \int_0^t \Gamma_s \cdot dW_s, \quad 0 \leq t \leq T.$$

Definition 1.71. A probability measure Q is said to be risk-neutral if

- Q and P are equivalent (i.e. for every $A \in \mathcal{F}$, $P(A) = 0$ if and only if $Q(A) = 0$);
- under Q , the discounted stock price $\tilde{S}_i(t)$ is a martingale for every $i = 1, \dots, m$.

We assume there are m stocks, each with stochastic differential equation

$$dS_t^i = \alpha_t^i S_t^i dt + S_t^i \sum_{j=1}^d \sigma_{ij}(t) dW_t^j, \quad i = 1, \dots, m. \quad (1.25)$$

where the mean vector $(\alpha_t^i)_{i=1, \dots, m}$ and the volatility matrix $(\sigma_{ij}(t))_{i=1, \dots, m, j=1, \dots, d}$ are adapted processes. These stocks are typically correlated. Set $\sigma_t^i = \sqrt{\sum_{j=1}^d \sigma_{ij}^2(t)}$, which is assumed never zero, and define processes

$$W_t^{*i} = \sum_{j=1}^d \int_0^t \frac{\sigma_{ij}(s)}{\sigma_s^i} dW_s^j, \quad i = 1, \dots, m.$$

By Lévy's theorem, W_t^{*i} is a Brownian motion and we rewrite equation (1.25) in terms of

the Brownian motion W_t^{*i} as

$$dS_t^i = \alpha_t^i S_t^i dt + S_t^i \sigma_t^i dW_t^{*i}.$$

Then the discounted stock prices are

$$\begin{aligned} d\tilde{S}_t^i &= (\alpha_t^i - r) \tilde{S}_t^i dt + \sigma_t^i \tilde{S}_t^i dW_t^{*i}, \quad i = 1, \dots, m. \\ &= (\alpha_t^i - r) \tilde{S}_t^i dt + \tilde{S}_t^i \sum_{j=1}^d \sigma_{ij}(t) dW_t^j. \end{aligned} \quad (1.26)$$

By using the multi-dimensional Girsanov's theorem, $\tilde{W}_t$ given by equation (1.24) is a d -dimensional Brownian motion under an equivalent probability measure Q , if and only if

$$\alpha_t^i - r = \sum_{j=1}^d \sigma_{ij}(t) \theta_t^j, \quad i = 1, \dots, m. \quad (1.27)$$

i.e.,

$$\alpha_t - rI = \Sigma(t)\underline{\theta},$$

where $\underline{\theta} = (\theta_t^1, \dots, \theta_t^d)^{tr}$ and $\Sigma(t)$ is the matrix of covariances. We call these the *market price of risk equations*. There are m equations in the d unknown processes $\theta_t^1, \dots, \theta_t^d$. If there is a solution to the market price of risk equations, then there is no arbitrage.

Assume the initial capital value is V_0^h and choose adapted portfolio processes $h_t = (h_t^0, h_t^1, \dots, h_t^m)$. Then

$$\begin{aligned} dV_t^h &= \sum_{i=1}^m h_t^i dS_t^i + h_t^0 dR_t \\ &= \sum_{i=1}^m h_t^i dS_t^i + \left(V_t^h - \sum_{i=1}^m h_t^i S_t^i \right) r dt \\ &= r V_t^h dt + \sum_{i=1}^m h_t^i ((\alpha_t^i - r) S_t^i dt + S_t^i \sigma_t^i dW_t^{*i}) \\ &= r V_t^h dt + \sum_{i=1}^m h_t^i R_t d\tilde{S}_t^i, \end{aligned}$$

and the process of the discounted portfolio value is

$$d\tilde{V}_t^h = \sum_{i=1}^m h_t^i d\tilde{S}_t^i. \quad (1.28)$$

Then we have the following proposition.

Proposition 1.72 (See [45]). *Let Q be a risk-neutral measure, and let V_t^h be the value of a portfolio. Under the measure Q , the discounted portfolio value $\tilde{V}_t^h$ is a martingale.*

Proposition 1.73 (See [45]). *Suppose we have a market model with a filtration generated by a d -dimensional Brownian motion and with a risk-neutral measure Q . Let G be a $\mathcal{F}_T$ -measurable random variable, which is the payoff of some derivative security. Then, there exists a self-financing admissible portfolio $h_t = (h_t^1, \dots, h_t^m)$ such that $G = V_T^h$.*

Proof. We follow [45] to give the proof. Let us define $\Pi(t)$ by

$$\Pi(t) = E^Q [e^{-r(T-t)} G | \mathcal{F}_t]$$

so that $\Pi(t)/R_t$ satisfies

$$\frac{\Pi(t)}{R_t} = E^Q \left[\frac{G}{R_T} | \mathcal{F}_t \right],$$

i.e. $\Pi(t)/R_t$ is Q martingale. Then according to the Martingale representation theorem, there are process $\Gamma_s = (\Gamma_s^1, \dots, \Gamma_s^d)$ such that

$$\frac{\Pi(t)}{R_t} = \Pi(0) + \sum_{j=1}^d \int_0^t \Gamma_s^j d\tilde{W}_s^j, \quad 0 \leq t \leq T. \quad (1.29)$$

Consider a portfolio value process V_t^h that begins at V_0^h . According to equation (1.28) and equation (1.26),

$$d\tilde{V}_t^h = \sum_{i=1}^m h_t^i d\tilde{S}_t^i = \sum_{i=1}^m \sum_{j=1}^d h_t^i \tilde{S}_t^i \sigma_{ij}(t) d\tilde{W}_t^j,$$

or equivalently,

$$\frac{V_t^h}{R_t} = V_0^h + \sum_{j=1}^d \int_0^t \sum_{i=1}^m h_s^i \tilde{S}_s^i \sigma_{ij}(s) d\tilde{W}_s^j. \quad (1.30)$$

Comparing equation (1.29) and equation (1.30), we see that in order to hedge G , we should take

$$\Pi(0) = V_0^h$$

and choose the portfolio process $h_t^1, \dots, h_t^m$ so that the hedging equations

$$\Gamma_t^j = \sum_{i=1}^m h_t^i e^{-rt} S_t^i \sigma_{ij}(t), \quad j = 1, 2, \dots, d.$$

are satisfied. There are d equations in m unknown processes $h_t^1, \dots, h_t^m$. □

Chapter 2

Concepts of Malliavin Calculus

Malliavin calculus is about a probabilistic differential stochastic calculus over an infinite-dimensional space. Let $(\Omega, \mathcal{F}, Q)$ be a complete probability space, and $L^2(\Omega, \mathcal{F}, Q)$ be the set of square-integrable random variables; we write $L^2(\Omega)$ for short.

Roughly speaking, Malliavin calculus deals with quantities such as $dF/d\omega$, for F belonging to $L^2(\Omega)$ and $\omega \in \Omega$. It is not too difficult to define such a term over a finite-dimensional subspace. Obviously, it comes down to classical functional calculus. However, we would like to extend this theory to an infinite-dimensional space like $L^2(\Omega)$.

For the Malliavin calculus developed in the next sections, we recommend our reader to refer to papers listed in the Bibliography. The paper [39] gives a basic introduction to Malliavin calculus, while [32] and [35] give a detailed discussion of the theory. The reader can also see more applications of Malliavin calculus in [4] and [5]. They provide a good insight into how Malliavin calculus connects with mathematical finance.

We mainly follow the method of [35], [39], [46] and [47], which developed Malliavin calculus in a number of alternative ways.

We let $[0, T]$ be a fixed finite time-interval and let $W_t = W_t(\omega)$, $t \in [0, T]$, $\omega \in \Omega$, be a *one-dimensional* Wiener process, or equivalently Brownian motion, on a complete probability space $(\Omega, \mathcal{F}, Q)$ such that $W_0 = 0$, Q -a.s.. For any t , let $\mathcal{F}_t$ be the σ -algebra generated by W_s , $0 \leq s \leq t$, augmented by all the Q -zero measure events. We denote the corresponding filtration by $\mathcal{F} = \{\mathcal{F}_t : t \in [0, T]\}$.

There are a number of alternative ways to introduce the Malliavin derivatives. In this

thesis, we follow the original construction to identify the probability space $(\Omega, \mathcal{F}, Q)$ with $(C_0[0, T], \mathcal{B}(C_0[0, T]), \mu)$ such that $W_t(\omega) = \omega(t)$ for all $t \in [0, T]$. We call this concrete infinite-dimensional space $C_0[0, T]$ the “classical Wiener space”. Readers are suggested to look at Example 1.5. Here ω is its sample path and $C_0[0, T]$ denotes the Wiener space - that is, the space of all continuous real-valued functions ω on $[0, T]$ such that $\omega(0) = 0$,

$$\|\omega\|_\infty := \sup_{t \in [0, T]} |\omega(t)|$$

and

$$W_t(\alpha\omega_1 + \beta\omega_2) = \alpha\omega_1(t) + \beta\omega_2(t);$$

$\mathcal{B}(C_0[0, T])$ denotes the corresponding Borel σ -algebra and μ denotes the unique *Wiener measure*.

In [16] and [39], they use an approach based on Wiener-Itô chaos expansion to introduce the Malliavin derivatives. The chaos expansion theorem concerns the representation of square-integrable random variables in terms of an infinite orthogonal sum. The theorem was first proved by Wiener [48] in 1938. Later, Itô [25] showed that the expansion could be expressed in terms of iterated Itô integrals in the Wiener space setting. Here we give some related definitions and state the Wiener-Itô chaos expansion.

Definition 2.1. A real function $g : [0, T]^n \rightarrow R$ is called symmetric if

$$g(t_{\sigma_1}, \dots, t_{\sigma_n}) = g(t_1, \dots, t_n)$$

for all permutations $\sigma = (\sigma_1, \dots, \sigma_n)$ of $(1, 2, \dots, n)$.

Let $L^2([0, T]^n)$ be the standard space of square integrable Borel real functions on $[0, T]^n$ such that

$$\|g\|_{L^2([0, T]^n)}^2 := \int_{[0, T]^n} g^2(t_1, \dots, t_n) dt_1 \cdots dt_n < \infty.$$

Let $\tilde{L}^2([0, T]^n) \subset L^2([0, T]^n)$ be the space of symmetric square-integrable Borel real functions on $[0, T]^n$. Let us consider the set

$$S_n = \{(t_1, \dots, t_n) \in [0, T]^n : 0 \leq t_1 \leq \dots \leq t_n \leq T\}.$$

Definition 2.2. Let f be a deterministic function defined on S_n for $(n \geq 1)$ such that

$$\|f\|_{L^2(S_n)}^2 := \int_{S_n} f^2(t_1, \dots, t_n) dt_1 \cdots dt_n < \infty. \quad (2.1)$$

Then we can form the n -fold iterated Itô integral as

$$\int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, \dots, t_n) dW_{t_1} \cdots dW_{t_n} \in L^2(\Omega), \quad (2.2)$$

due to the construction of the Itô integral.

Definition 2.3. If $g \in \tilde{L}^2([0, T]^n)$ we define

$$\begin{aligned} I_n(g) &:= \int_{[0, T]^n} g(t_1, \dots, t_n) dW_{t_1} \cdots dW_{t_n} \\ &= n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} g(t_1, \dots, t_n) dW_{t_1} \cdots dW_{t_n}. \end{aligned} \quad (2.3)$$

We also call iterated n -fold Itô integrals the $I_n(g)$ as above.

Proposition 2.4. [Wiener Chaos Expansion, see Theorem 1.1 in [39]] Let F be an $\mathcal{F}_T$ -measurable stochastic variable such that $F \in L^2(\Omega)$. Then there exists a unique sequence $\{f_n\}_{n=0}^\infty$ of deterministic functions $f_n \in \tilde{L}^2([0, T]^n)$ such that

$$F = \sum_{n=0}^{\infty} I_n(f_n) = E^Q[F] + \sum_{n=1}^{\infty} I_n(f_n), \quad (2.4)$$

where the convergence is in $L^2(\Omega)$. Moreover, we define $J_n(F)$ to be the orthogonal projection of the stochastic variable F on the n -th Wiener chaos, and as a consequence, $J_n(F) = I_n(f_n)$ and

$$F = \sum_{n=0}^{\infty} J_n(F). \quad (2.5)$$

For a complete proof, refer to [39].

We see that the Wiener chaos expansion is closely related to the Itô representation theorem and the Clark-Ocone formula. We see many proofs of theorems and propositions are based on Wiener chaos expansion from [16], but in this thesis we follow the classical approach of the Malliavin derivative on the Wiener space.

Definition 2.5. Let us define Cameron-Martin space, $\mathcal{H}$, which is a subspace of Wiener Space, by:

$$\mathcal{H} = \left\{ \gamma : [0, T] \rightarrow \mathbb{R}; \gamma(t) = \int_0^t \dot{\gamma}(s) ds; \|\gamma\|_{\mathcal{H}}^2 = \int_0^T (\dot{\gamma}(s))^2 ds < \infty \right\}.$$

Remark 2.6. The elements of $\mathcal{H}$ are the absolutely continuous functions on $[0, T]$. For $\gamma \in \mathcal{H}$ the derivative of γ exists almost everywhere on $[0, t]$ and $\gamma(t) = \int_0^t \dot{\gamma}(s) ds$, see, [19]. Note that $\mathcal{H}$ is a Hilbert space with inner product

$$\langle \gamma_1(t), \gamma_2(t) \rangle_{\mathcal{H}} = \int_0^t \dot{\gamma}_1(s) \dot{\gamma}_2(s) ds.$$

Definition 2.7. Let $\mathcal{P}$ be the space of random variables $F : \Omega \rightarrow \mathbb{R}$ of the form

$$F(\omega) = p(W_{t_1}(\omega), \dots, W_{t_n}(\omega)), \quad \text{for all } \omega \in \Omega, \quad (2.6)$$

where the deterministic function $p : \mathbb{R}^n \rightarrow \mathbb{R}$ is a real polynomial in n variables, i.e. p is a finite linear combination of finite products of powers of the variables x_i and $p(x_1, \dots, x_n) = \sum_{\alpha} a_{\alpha} x^{\alpha}$, with $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\alpha \in \{(\alpha_1, \dots, \alpha_n) : \alpha_i \in \mathbb{N} \cup \{0\}\}$ and $t_1, \dots, t_n$ are any choice of “ t ’s” from $[0, T]$, $n \in \mathbb{N}$.

We now define the directional derivative of a stochastic variable in all the directions in the Cameron-Martin space.

Definition 2.8. For $F \in \mathcal{P}$, the directional derivative $\mathbf{D}_{\gamma}F(\omega)$ at the point $\omega \in \Omega$ in all the directions $\gamma \in \mathcal{H}$ is defined by

$$\mathbf{D}_{\gamma}F(\omega) := \lim_{\varepsilon \rightarrow 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon} = \frac{d}{d\varepsilon} F(\omega + \varepsilon\gamma)|_{\varepsilon=0}, \quad (2.7)$$

where ε is real.

Remark 2.9. Note that for all $t \in [0, T]$, we can write $W_t(\omega + \varepsilon\gamma) = \omega(t) + \varepsilon\gamma(t)$.

We start our treatment from the definition of the directional derivatives. Using this definition, we will develop a lot of useful propositions for the Malliavin derivative and give a full treatment, providing our own proofs in detail.

Proposition 2.10. *For $F \in \mathcal{P}$, the directional derivative of F can be expressed as*

$$\mathbf{D}_\gamma F(\omega) = \sum_{i=1}^n \frac{\partial p}{\partial x_i} (W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \gamma(t_i), \quad (2.8)$$

where p is a polynomial and the notation $\partial p / \partial x_i$ means the partial derivative with respect to the i -th variable. In particular, if $\gamma(t_i) = 0$, for all i , then $\mathbf{D}_\gamma F(\omega) = 0$.

Proof. We offer the following proof. For $F \in \mathcal{P}$, by Definition 2.7, F is in the form of

$$F(\omega) = p(W_{t_1}(\omega), \dots, W_{t_n}(\omega)) = p(\omega(t_1), \dots, \omega(t_n)),$$

for some polynomial function p . For $\gamma \in \mathcal{H}$, we have

$$\begin{aligned} F(\omega + \varepsilon\gamma) &= p(W_{t_1}(\omega + \varepsilon\gamma), \dots, W_{t_n}(\omega + \varepsilon\gamma)) \\ &= p(\omega(t_1) + \varepsilon\gamma(t_1), \dots, \omega(t_n) + \varepsilon\gamma(t_n)). \end{aligned}$$

By adding and subtracting each term, we have

$$\begin{aligned} &F(\omega + \varepsilon\gamma) - F(\omega) \\ &= p(\omega(t_1) + \varepsilon\gamma(t_1), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) - p(\omega(t_1), \dots, \omega(t_n)) \\ &= p(\omega(t_1) + \varepsilon\gamma(t_1), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) - p(\omega(t_1), \omega(t_2) + \varepsilon\gamma(t_2), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) \\ &+ p(\omega(t_1), \omega(t_2) + \varepsilon\gamma(t_2), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) - p(\omega(t_1), \omega(t_2), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) \\ &+ p(\omega(t_1), \omega(t_2), \omega(t_3) + \varepsilon\gamma(t_3), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) - \dots \\ &\quad \vdots \\ &+ p(\omega(t_1), \dots, \omega(t_{n-1}), \omega(t_n) + \varepsilon\gamma(t_n)) - p(\omega(t_1), \dots, \omega(t_n)). \end{aligned}$$

By Definition 2.8, we have

$$\begin{aligned}
& \mathbf{D}_\gamma F(\omega) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{F(\omega + \varepsilon \gamma) - F(\omega)}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{p(\omega(t_1) + \varepsilon \gamma(t_1), \dots, \omega(t_n) + \varepsilon \gamma(t_n)) - p(\omega(t_1), \omega(t_2) + \varepsilon \gamma(t_2), \dots, \omega(t_n) + \varepsilon \gamma(t_n))}{\varepsilon \gamma(t_1)} \gamma(t_1) \\
&+ \lim_{\varepsilon \rightarrow 0} \frac{p(\omega(t_1), \omega(t_2) + \varepsilon \gamma(t_2), \dots, \omega(t_n) + \varepsilon \gamma(t_n)) - p(\omega(t_1), \omega(t_2), \dots, \omega(t_n) + \varepsilon \gamma(t_n))}{\varepsilon \gamma(t_2)} \gamma(t_2) \\
&\vdots \\
&+ \lim_{\varepsilon \rightarrow 0} \frac{p(\omega(t_1), \dots, \omega(t_{n-1}), \omega(t_n) + \varepsilon \gamma(t_n)) - p(\omega(t_1), \dots, \omega(t_n))}{\varepsilon \gamma(t_n)} \gamma(t_n) \\
&= \frac{\partial p}{\partial x_1}(W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \gamma(t_1) + \dots + \frac{\partial p}{\partial x_n}(W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \gamma(t_n).
\end{aligned}$$

□

Remark 2.11. In particular, by taking $p(x) = x$ and $t \in [0, T]$, the directional derivative of the Wiener process $W_t(\omega)$ is given by $\gamma(t)$, i.e., $\mathbf{D}_\gamma W_t(\omega) = \gamma(t)$.

Proposition 2.12. For $F \in \mathcal{P}$, the map $\gamma \mapsto \mathbf{D}_\gamma F(\omega)$ is a continuous linear functional on $\mathcal{H}$ for each $\omega \in \Omega$.

Proof. We offer the following proof. For each $\omega \in \Omega$ and any $t \in [0, T]$,

$$\begin{aligned}
|\gamma(t) - \gamma_m(t)| &= \left| \int_0^t (\dot{\gamma}(s) - \dot{\gamma}_m(s)) ds \right| \\
&\leq \int_0^t |\dot{\gamma}(s) - \dot{\gamma}_m(s)| ds \\
&\leq \left(\int_0^t |\dot{\gamma}(s) - \dot{\gamma}_m(s)|^2 ds \right)^{1/2} (t)^{1/2} \\
&\leq \|\gamma(s) - \gamma_m(s)\|_{\mathcal{H}} \sqrt{T}.
\end{aligned}$$

Hence, when $\gamma_m \rightarrow \gamma$ in $\mathcal{H}$, then we have $\gamma_m(t) \rightarrow \gamma(t)$ for any $t \in [0, T]$, as $m \rightarrow \infty$.

Moreover, from Proposition 2.10, we have

$$\begin{aligned}
 \mathbf{D}_{\gamma_m} F(\omega) &= \sum_{i=1}^n \frac{\partial p}{\partial x_i} (W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \gamma_m(t_i) \\
 &\rightarrow \sum_{i=1}^n \frac{\partial p}{\partial x_i} (W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \gamma(t_i) \\
 &= \mathbf{D}_{\gamma} F(\omega) \quad \text{as } m \rightarrow \infty.
 \end{aligned}$$

□

Proposition 2.13. [Product Rule for Directional Derivatives] For $F, G \in \mathcal{P}$, the directional derivative of the product FG is given by

$$\mathbf{D}_{\gamma}(F(\omega)G(\omega)) = F(\omega)\mathbf{D}_{\gamma}G(\omega) + G(\omega)\mathbf{D}_{\gamma}F(\omega). \quad (2.9)$$

Proof. We offer the following proof. For $F \in \mathcal{P}$, and $G \in \mathcal{P}$, by Definition 2.7, we can write

$$F(\omega) = p_F(W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \text{ and } G(\omega) = p_G(W_{s_1}(\omega), \dots, W_{s_k}(\omega))$$

for some polynomial functions p_G and p_F . Observe that the product $p = p_G p_F$ is a polynomial in the variables $W_{t_1}(\omega), \dots, W_{t_n}(\omega), W_{s_1}(\omega), \dots, W_{s_k}(\omega)$. For $\gamma \in \mathcal{H}$, by adding and subtracting each term, we have

$$\begin{aligned}
 &F(\omega + \varepsilon\gamma)G(\omega + \varepsilon\gamma) - F(\omega)G(\omega) \\
 &= p_F(\omega(t_1) + \varepsilon\gamma(t_1), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) p_G(\omega(s_1) + \varepsilon\gamma(s_1), \dots, \omega(s_k) + \varepsilon\gamma(s_k)) \\
 &\quad - p_F(\omega(t_1), \dots, \omega(t_n)) p_G(\omega(s_1), \dots, \omega(s_k)) \\
 &= p_F(\omega(t_1) + \varepsilon\gamma(t_1), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) \\
 &\quad \{p_G(\omega(s_1) + \varepsilon\gamma(s_1), \dots, \omega(s_k) + \varepsilon\gamma(s_k)) - p_G(\omega(s_1), \dots, \omega(s_k))\} \\
 &+ p_G(\omega(s_1), \dots, \omega(s_k)) \\
 &\quad \{p_F(\omega(t_1) + \varepsilon\gamma(t_1), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) - p_F(\omega(t_1), \dots, \omega(t_n))\}.
 \end{aligned}$$

Taking the limit and by Definition 2.8, we have

$$\begin{aligned}
& \mathbf{D}_\gamma F(\omega)G(\omega) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{F(\omega + \varepsilon\gamma)G(\omega + \varepsilon\gamma) - F(\omega)G(\omega)}{\varepsilon} \\
&= p_G(\omega(s_1), \dots, \omega(s_k)) \cdot \\
&\quad \lim_{\varepsilon \rightarrow 0} \frac{p_F(\omega(t_1) + \varepsilon\gamma(t_1), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) - p_F(\omega(t_1), \dots, \omega(t_n))}{\varepsilon} \\
&+ \lim_{\varepsilon \rightarrow 0} p_F(\omega(t_1) + \varepsilon\gamma(t_1), \dots, \omega(t_n) + \varepsilon\gamma(t_n)) \cdot \\
&\quad \lim_{\varepsilon \rightarrow 0} \frac{p_G(\omega(s_1) + \varepsilon\gamma(s_1), \dots, \omega(s_k) + \varepsilon\gamma(s_k)) - p_G(\omega(s_1), \dots, \omega(s_k))}{\varepsilon} \\
&= F(\omega)\mathbf{D}_\gamma G(\omega) + G(\omega)\mathbf{D}_\gamma F(\omega).
\end{aligned}$$

□

2.1 Malliavin Derivative for $F(\omega)$ belonging to $\mathcal{P}$

In this section, we introduce the Malliavin derivative via the relationship with the directional derivative. From Proposition 2.12 we see the map $\gamma \mapsto \mathbf{D}_\gamma F(\omega)$ is a continuous linear functional on $\mathcal{H}$ for each $\omega \in \Omega$. Then by the Riesz representation theorem [44], there exists a unique stochastic variable $\nabla F(\omega) \in \mathcal{H}$ such that

$$\mathbf{D}_\gamma F(\omega) = \langle \gamma, \nabla F(\omega) \rangle_{\mathcal{H}} := \int_0^T \nabla \dot{F}(\omega) \dot{\gamma} dt. \quad (2.10)$$

Moreover, $\nabla \dot{F}(\omega)$ is unique in $L^2([0, T])$, and since $\nabla F(\omega)$ is an $\mathcal{H}$ -valued stochastic variable, the map $t \mapsto \nabla F(t, \omega)$ is absolutely continuous with respect to Lebesgue measure on $[0, T]$. Now, let the Malliavin derivative $D_t F(\omega)$ denote the Radon-Nikodym derivative of $\nabla F(\omega)$ with respect to the Lebesgue measure, that is $D_t F(\omega) = \nabla \dot{F}(\omega)$. Then we have

$$\mathbf{D}_\gamma F(\omega) = \int_0^T D_t F(\omega) \dot{\gamma}(t) dt. \quad (2.11)$$

Identifying equation (2.11) with equation (2.8) we have the following result, which is taken as a definition in [5].

Proposition 2.14. [See Definition 3.2 in [5]] For $F \in \mathcal{P}$, the Malliavin derivative of F is the stochastic process $\{D_t F : t \in [0, T]\}$ given by

$$D_t F(\omega) = \sum_{i=1}^n \frac{\partial p}{\partial x_i} (W_{t_1}(\omega), \dots, W_{t_n}(\omega)) 1_{[0, t_i]}(t). \quad (2.12)$$

Proof. We offer the following proof. Rewrite $\gamma(t_i)$ as $\int_0^T 1_{[0, t_i]}(t) \dot{\gamma}(t) dt$. By Proposition 2.10, we have

$$\begin{aligned} \mathbf{D}_\gamma F(\omega) &= \sum_{i=1}^n \frac{\partial p}{\partial x_i} (W_{t_1}, \dots, W_{t_n}) \int_0^T 1_{[0, t_i]}(t) \dot{\gamma}(t) dt \\ &= \int_0^T \sum_{i=1}^n \frac{\partial p}{\partial x_i} (W_{t_1}, \dots, W_{t_n}) 1_{[0, t_i]}(t) \dot{\gamma}(t) dt. \end{aligned} \quad (2.13)$$

Identifying equation (2.11) and equation (2.13), we conclude that

$$D_t F(\omega) = \sum_{i=1}^n \frac{\partial p}{\partial x_i} (W_{t_1}, \dots, W_{t_n}) 1_{[0, t_i]}(t).$$

□

Remark 2.15. Note that $F \mapsto D_t F$ is a linear operation from $\mathcal{P}$ into $L^2([0, T] \times \Omega)$.

Proposition 2.16. [Product Rule for Malliavin Derivatives] For $F, G \in \mathcal{P}$, the Malliavin derivative of the product FG is given by

$$D_t (F(\omega)G(\omega)) = F(\omega)D_t G(\omega) + G(\omega)D_t F(\omega). \quad (2.14)$$

Proof. We offer the following proof. From the relationship given by equation (2.11), we know

$$\mathbf{D}_\gamma F(\omega) = \int_0^T D_t F(\omega) \dot{\gamma}(t) dt \quad \text{and} \quad \mathbf{D}_\gamma G(\omega) = \int_0^T D_t G(\omega) \dot{\gamma}(t) dt.$$

Inserting the above two equations into equation (2.9), we get

$$\begin{aligned}
 \mathbf{D}_\gamma(F(\omega)G(\omega)) &= F(\omega)\mathbf{D}_\gamma G(\omega) + G(\omega)\mathbf{D}_\gamma F(\omega) \\
 &= F(\omega) \int_0^T D_t G(\omega) \dot{\gamma}(t) dt + G(\omega) \int_0^T D_t F(\omega) \dot{\gamma}(t) dt \\
 &= \int_0^T (F(\omega) D_t G(\omega) + G(\omega) D_t F(\omega)) \dot{\gamma}(t) dt. \tag{2.15}
 \end{aligned}$$

We also have

$$\mathbf{D}_\gamma(F(\omega)G(\omega)) = \int_0^T D_t(F(\omega)G(\omega)) \dot{\gamma}(t) dt. \tag{2.16}$$

Comparing the above two equations, we get

$$D_t(F(\omega)G(\omega)) = F(\omega)D_t G(\omega) + G(\omega)D_t F(\omega).$$

□

Remark 2.17. Note that D_t is a derivation on $\mathcal{P}$.

Now we consider the Malliavin derivative of $F(\omega)$ when $F(\omega)$ has the form of $\int_0^T g(s) dW_s$ with respect to a *simple* deterministic integrand. In the following proposition, we show that the Malliavin derivative of $F(\omega)$ is the integrand itself.

Proposition 2.18. If $F(\omega) = \int_0^T g(s) dW_s$ and the integrand $g : [0, T] \rightarrow \mathbb{R}$ is a simple function, then $F(\omega) \in \mathcal{P}$ and the Malliavin derivative of F is given by

$$D_t F(\omega) = D_t \left(\int_0^T g(s) dW_s \right) = g(t) 1_{[0, T]}(t). \tag{2.17}$$

Proof. We offer the following proof. Let $g(s) = \sum_{j=1}^k \alpha_j 1_{[t_{j-1}, t_j)}(s)$ and $\alpha_j \in \mathbb{R}$, and $0 = t_0 < t_1 < \dots < t_k = T$. Then the stochastic integral of g with respect to Brownian

motion W can be written as $\int_0^T g(s) dW_s = \sum_{j=1}^k \alpha_j (W_{t_j} - W_{t_{j-1}})$. Think of

$$\begin{aligned} \sum_{j=1}^k \alpha_j (x_j - x_{j-1}) &= \alpha_1 (x_1 - x_0) + \alpha_2 (x_2 - x_1) + \cdots + \alpha_k (x_k - x_{k-1}) \\ &= (\alpha_1 - \alpha_2)x_1 + \cdots + (\alpha_{k-1} - \alpha_k)x_{k-1} + \alpha_k x_k - \alpha_1 x_0 \\ &= p(x_0, x_1, \dots, x_k), \end{aligned}$$

where p is a polynomial in the variables $x_0, x_1, \dots, x_k$. So we get

$$\int_0^T g(s) dW_s = p(W_{t_0}, W_{t_1}, \dots, W_{t_k}). \quad (2.18)$$

We note the integral of simple functions with respect to Brownian motion can be written as a polynomial function in Brownian motion, i.e. $F(\omega) = \int_0^T g(s) dW_s \in \mathcal{P}$. Then by Proposition 2.14, we conclude that

$$\begin{aligned} D_t \left(\int_0^T g(s) dW_s \right) &= D_t (p(W_{t_0}, W_{t_1}, \dots, W_{t_k})) \\ &= \sum_{j=0}^k \frac{\partial p}{\partial x_j} (W_{t_0}, W_{t_1}, \dots, W_{t_k}) 1_{[0, t_j]}(t). \end{aligned}$$

Moreover, note that $\partial p / \partial x_j = (\alpha_j - \alpha_{j+1})$ for $j = 1, \dots, k-1$ and $\partial p / \partial x_k = \alpha_k$ and $\partial p / \partial x_0 = -\alpha_1$, then we have

$$\begin{aligned} &D_t \left(\int_0^T g(s) dW_s \right) \\ &= (\alpha_1 - \alpha_2) 1_{[0, t_1]}(t) + \cdots + (\alpha_{k-1} - \alpha_k) 1_{[0, t_{k-1}]}(t) + \alpha_k 1_{[0, t_k]}(t) - \alpha_1 1_{[0, t_0]}(t) \\ &= \alpha_1 1_{[0, t_1]}(t) + \cdots + \alpha_k 1_{[t_{k-1}, t_k]}(t) \\ &= g(t) 1_{[0, T]}(t). \end{aligned}$$

□

Proposition 2.19. *Let the stochastic variable $F(\omega)$ be in the form of*

$$F(\omega) = p(\theta_1, \dots, \theta_n), \quad (2.19)$$

where p is polynomial, $\theta_i = \int_0^T g_i(s) dW_s$ and $g_i(s)$ are simple functions for $i = 1, 2, \dots, n$. Then the Malliavin derivative of $F(\omega)$ is given by

$$D_t F(\omega) = \sum_{i=1}^n \frac{\partial p}{\partial \theta_i}(\theta_1, \dots, \theta_n) g_i(t) 1_{[0, T]}(t). \quad (2.20)$$

Proof. We offer the following proof. By Proposition 2.18, we have $\theta_i \in \mathcal{P}$ and write θ_i as

$$\theta_i = \int_0^T g_i(s) dW_s = p_i(W_{t_1}, \dots, W_{t_k})$$

for some polynomials p_i . Then we rewrite $p(\theta_1, \dots, \theta_n)$ as

$$p(\theta_1, \dots, \theta_n) = p(p_1(W_{t_1}, \dots, W_{t_k}), \dots, p_n(W_{t_1}, \dots, W_{t_k})).$$

Notice that p is polynomial in the variables $W_{t_1}, \dots, W_{t_k}$, so $F(\omega) \in \mathcal{P}$. By Proposition 2.14, we have

$$D_t F(\omega) = \sum_{j=1}^k \frac{\partial p}{\partial x_j}(W_{t_1}, \dots, W_{t_k}) 1_{[0, t_j]}(t). \quad (2.21)$$

But by the standard derivative chain rule

$$\frac{\partial p}{\partial x_j} = \sum_{i=1}^n \frac{\partial p}{\partial p_i} \frac{\partial p_i}{\partial x_j}. \quad (2.22)$$

Then inserting equation (2.22) into equation (2.21), we get

$$\begin{aligned} D_t F(\omega) &= \sum_{j=1}^k \sum_{i=1}^n \frac{\partial p}{\partial p_i} \frac{\partial p_i}{\partial x_j}(W_{t_1}, \dots, W_{t_k}) 1_{[0, t_j]}(t) \\ &= \sum_{i=1}^n \frac{\partial p}{\partial p_i} \left(\sum_{j=1}^k \frac{\partial p_i}{\partial x_j}(W_{t_1}, \dots, W_{t_k}) 1_{[0, t_j]}(t) \right) \\ &= \sum_{i=1}^n \frac{\partial p}{\partial \theta_i} D_t(\theta_i) \\ &= \sum_{i=1}^n \frac{\partial p}{\partial \theta_i} g_i(t) 1_{[0, T]}(t). \end{aligned}$$

□

Now let $g_{im}(s)$ be a sequence of the simple $L^2([0, T])$ functions converging to g_i in $L^2([0, T])$ for $i = 1, 2, \dots, n$. For a polynomial p , we define

$$F_m(\omega) : = p \left(\int_0^T g_{1m}(s) dW_s, \dots, \int_0^T g_{nm}(s) dW_s \right) \quad (2.23)$$

and

$$F(\omega) : = p \left(\int_0^T g_1(s) dW_s, \dots, \int_0^T g_n(s) dW_s \right). \quad (2.24)$$

By Lemma A.1, we have $F(\omega) \in L^2(\Omega)$. By Lemma A.2, we know

$$F_m(\omega) \rightarrow F(\omega), \quad \text{in } L^2(\Omega), \quad \text{as } m \rightarrow \infty. \quad (2.25)$$

We will use these facts to calculate the Malliavin derivative of $F(\omega)$ later.

Remark 2.20. By Hölder's inequality, we know the space $L^p(\Omega)$ are “nested”, i.e.

$$L^\infty(\Omega) \subseteq L^{p_1}(\Omega) \subseteq L^{p_2}(\Omega) \subseteq L^1(\Omega)$$

for $1 \leq p_1 \leq p_2 \leq \infty$. Besides $\mathcal{P}$ is dense in $L^p(\Omega)$ for all $1 \leq p < \infty$. See Problem 1.1.7 in [35] for detail.

Now we state the Integration by parts formula for $F, G \in \mathcal{P}$.

Proposition 2.21. [Integration by Parts Formula, see Lemma 4.12 in [39]] Suppose $F, G \in \mathcal{P}$ and $\gamma \in \mathcal{H}$, with $\gamma(t) = \int_0^t \dot{\gamma}(s) ds$ and $\dot{\gamma}(s) \in L^2([0, T])$. Then

$$E [\mathbf{D}_\gamma F \cdot G] = E \left[F \cdot G \cdot \int_0^T \dot{\gamma}(t) dW_t \right] - E [F \cdot \mathbf{D}_\gamma G]. \quad (2.26)$$

Proof. We follow [39] to give a proof. By Definition 2.8, we have

$$E [\mathbf{D}_\gamma F \cdot G] = E \left[\lim_{\epsilon \rightarrow 0} \frac{F(\omega + \epsilon \gamma) - F(\omega)}{\epsilon} \cdot G \right] = \lim_{\epsilon \rightarrow 0} \frac{E [F(\omega + \epsilon \gamma) \cdot G - F(\omega) \cdot G]}{\epsilon}.$$

Now let us recall the Cameron-Martin theorem (see, [11]). We have

$$\begin{aligned} & E[F(\omega + \epsilon\gamma)G(\omega)] \\ &= E\left[F(\omega)G(\omega - \epsilon\gamma) \exp\left\{\epsilon \int_0^T \dot{\gamma}(t)dW_t - \frac{1}{2}\epsilon^2 \int_0^T (\dot{\gamma}(s))^2 dt\right\}\right]. \end{aligned}$$

Then

$$\begin{aligned} E[\mathbf{D}_\gamma F \cdot G] &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} E\left[F(\omega)G(\omega - \epsilon\gamma) e^{\epsilon \int_0^T \dot{\gamma}(t)dW_t - \frac{1}{2}\epsilon^2 \int_0^T (\dot{\gamma}(s))^2 dt} - F(\omega) \cdot G\right] \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} E\left[F(\omega)G(\omega - \epsilon\gamma) \left(e^{\epsilon \int_0^T \dot{\gamma}(t)dW_t - \frac{1}{2}\epsilon^2 \int_0^T (\dot{\gamma}(s))^2 dt} - 1\right)\right] \\ &\quad - E\left[F(\omega) \lim_{\epsilon \rightarrow 0} \frac{G(\omega) - G(\omega - \epsilon\gamma)}{\epsilon}\right] \\ &= E\left[F \cdot G \cdot \int_0^T \dot{\gamma}(t)dW_t\right] - E[F \cdot \mathbf{D}_\gamma G]. \end{aligned}$$

□

Proposition 2.22. [Closability of the Operator, see Theorem 4.11 in [39]] The Malliavin derivative D_t is closable from $L^2(\Omega)$ to $L^2([0, T] \times \Omega)$.

Proof. We follow [39] to give the proof. For $\{F_n\}_{n=1}^\infty \in \mathcal{P}$ such that $F_n \rightarrow 0$ in $L^2(\Omega)$ and $D_t F_n$ converges in $L^2([0, T] \times \Omega)$, we want to show $D_t F_n$ converges to zero. Now suppose the limit is η , and prove $\eta = 0$. By Proposition 2.21, for any $G \in \mathcal{P}$, we have

$$E[\mathbf{D}_\gamma F_n \cdot G] = E\left[F_n \cdot G \cdot \int_0^T \dot{\gamma}(t)dW_t\right] - E[F_n \cdot \mathbf{D}_\gamma G].$$

Since $G, \int_0^T \dot{\gamma}(t)dW_t$ and $\mathbf{D}_\gamma G$ are elements of $L^2(\Omega)$, and $F_n \rightarrow 0$ in $L^2(\Omega)$ as $n \rightarrow \infty$. Then

$$E[\mathbf{D}_\gamma F_n \cdot G] \rightarrow 0, \quad n \rightarrow \infty,$$

for any $G \in \mathcal{P}$. Note $\mathbf{D}_\gamma F_n = \int_0^T D_t F_n \dot{\gamma}(t)dt$, and we have

$$\begin{aligned} \mathbf{D}_\gamma F_n - \mathbf{D}_\gamma F_m &= \int_0^T D_t F_n \dot{\gamma}(t)dt - \int_0^T D_t F_m \dot{\gamma}(t)dt \\ &= \int_0^T (D_t F_n - D_t F_m) \dot{\gamma}(t)dt. \end{aligned}$$

Then take the absolute value on the both side of the above equality,

$$\begin{aligned} |\mathbf{D}_\gamma F_n - \mathbf{D}_\gamma F_m| &\leq \int_0^T |D_t F_n - D_t F_m| |\dot{\gamma}(t)| dt \\ &\leq \left(\int_0^T |D_t F_n - D_t F_m|^2 dt \right)^{1/2} \left(\int_0^T |\dot{\gamma}(t)|^2 dt \right)^{1/2}. \end{aligned}$$

Then

$$\begin{aligned} \|\mathbf{D}_\gamma F_n - \mathbf{D}_\gamma F_m\|_2^2 &\leq E \left[\int_0^T |D_t F_n - D_t F_m|^2 dt \right] \int_0^T |\dot{\gamma}(t)|^2 dt \\ &= \|D_t F_n - D_t F_m\|_{L^2(\Omega \times [0, T])}^2 \|\gamma\|_{\mathcal{H}}^2 \\ &\rightarrow 0, \end{aligned}$$

since by hypothesis $D_t F_n$ converge in $L^2([0, T] \times \Omega)$. So $\mathbf{D}_\gamma F_n$ is Cauchy sequence in $L^2(\Omega)$. Consequently,

$$E[\mathbf{D}_\gamma F_n \cdot \xi] \rightarrow 0 \quad \text{as } n \rightarrow \infty, \text{ for } \forall \xi \in L^2(\Omega),$$

because $\mathcal{P}$ is dense in $L^2(\Omega)$. Then conclude $\mathbf{D}_\gamma F_n \rightarrow 0$ in $L^2(\Omega)$ as $n \rightarrow \infty$. Since this holds for all $\gamma \in \mathcal{H}$, we obtain that $D_t F_n \rightarrow 0$ in $L^2([0, T] \times \Omega)$. $\square$

Proposition 2.23. *Let the stochastic variable $F(\omega)$ be in the form of*

$$F(\omega) = p(\theta_1, \dots, \theta_n),$$

where $\theta_i = \int_0^T g_i(s) dW_s$ and deterministic functions $g_i(s) \in L^2([0, T])$ for all i . Then the Malliavin derivative of $F(\omega)$ is given by

$$D_t F(\omega) = \sum_{i=1}^n \frac{\partial p}{\partial x_i}(\theta_1, \dots, \theta_n) g_i(t) 1_{[0, T]}(t). \quad (2.27)$$

Proof. We offer the following proof. Let us consider a sequence $F_m(\omega)$ defined by equation (2.23) with

$$F_m(\omega) \rightarrow F(\omega) \quad \text{in } L^2(\Omega)$$

as $m \rightarrow \infty$. By Proposition 2.19, the Malliavin derivative of $F_m(\omega)$ is given by

$$\begin{aligned} D_t F_m(\omega) &= \sum_{i=1}^n \frac{\partial p}{\partial x_i} \left(\int_0^T g_{1m}(s) dW_s, \dots, \int_0^T g_{nm}(s) dW_s \right) \cdot g_{im}(t) 1_{[0,T]}(t) \\ &\rightarrow \sum_{i=1}^n \frac{\partial p}{\partial x_i} \left(\int_0^T g_1(s) dW_s, \dots, \int_0^T g_n(s) dW_s \right) \cdot g_i(t) 1_{[0,T]}(t). \end{aligned}$$

By Proposition 2.22, we get

$$D_t F(\omega) = \sum_{i=1}^n \frac{\partial p}{\partial x_i} \left(\int_0^T g_1(s) dW_s, \dots, \int_0^T g_n(s) dW_s \right) \cdot g_i(t) 1_{[0,T]}(t).$$

□

Hereafter, we redefine $\mathcal{P}$ from Definition 2.7 as the set of random variables $F : \Omega \rightarrow \mathbb{R}$ of the form

$$F(\omega) = p \left(\int_0^T g_1(s) dW_s, \dots, \int_0^T g_n(s) dW_s \right). \quad (2.28)$$

where $g_i(s) \in L^2([0, T])$ are some *deterministic* functions $i = 1, 2, \dots, n$.

2.2 Malliavin Derivative for $F(\omega)$ belonging to $\mathbb{D}_{1,2}$

Following [35], we define the *iterated derivative* $D^k F = D_{t_1} \cdots D_{t_k} F$ such that $D^k F$ is defined almost everywhere $dt^k \times dQ$. Then for every $p > 1$ and any natural number $k \geq 1$, we introduce the semi-norm on the set $\mathcal{P}$ by

$$|||F|||_{k,p} = \left[E^Q [|F|^p] + \sum_{j=1}^k E^Q \left[\|D^j F\|_{L^2([0,T]^j)}^p \right] \right]^{1/p}. \quad (2.29)$$

Here, we set $|||F|||_{0,p}^p = E^Q [|F|^p]$. Since the operator D_t is closable by Proposition 2.22, we define by $\mathbb{D}_{k,p}$ the Banach space which is the closure of $\mathcal{P}$ under $|||\cdot|||_{k,p}$. The family of semi-norms verifies the following properties, see [35].

- **Monotonicity:** $|||F|||_{k,p} \leq |||F|||_{k',p'}$, for any $F \in \mathcal{P}$, if $k \leq k'$ and $p \leq p'$.
- **Closability:** The operator D^k is closable from $\mathcal{P}$ into $L^p([0, T] \times \Omega)$.

- **Compatibility:** Let k, k' be natural numbers, $p, p' > 1$ be real numbers. Suppose that the sequence of random variables $F_n \in \mathcal{P}$ is such that $\|F_n\|_{k,p} \rightarrow 0$, as $n \rightarrow \infty$ and $\|F_n - F_m\|_{k',p'} \rightarrow 0$, as $n, m \rightarrow \infty$. Then $\|F_n\|_{k',p'} \rightarrow 0$ as $n \rightarrow \infty$.

Remark 2.24. By the property of monotonicity, it follows that $\mathbb{D}_{k',p'} \subset \mathbb{D}_{k,p}$ if $k \leq k'$ and $p < p'$. For $k = 0$, we set $\|F\|_{0,p} = \|\cdot\|_p$ and $\mathbb{D}_{0,p} = L^p(\Omega)$.

Remark 2.25. In particular, set $k = 1$ and $p = 2$ and introduce the norm $\|\cdot\|_{1,2}$ on the set $\mathcal{P}$ and we get the norm

$$\|F\|_{1,2} = \left[E^Q [|F|^2] + E^Q \left[\|D_t F\|_{L^2([0,T])}^2 \right] \right]^{1/2}. \quad (2.30)$$

For each $\omega \in \Omega$, $t \mapsto D_t F(\omega) \in L^2([0, T])$. That is, $\|D_t F(\omega)\|_{L^2([0,T])}^2 = \int_0^T (D_t F(\omega))^2 ds < \infty$, and

$$E^Q \left[\|D_t F\|_{L^2([0,T])}^2 \right] = E^Q \left[\int_0^T (D_t F(\omega))^2 ds \right].$$

So this is the L^2 norm squared of $(\omega, t) \mapsto D_t F(\omega)$ as an element of $L^2(\Omega \times [0, T])$.

Remark 2.26. The norm $\|\cdot\|_{1,2}$ is exactly that defined in equation (4.18) of [39]. But the difference with the approach of Bermin [5] is that Øksendal defines differentiable random variables as in Definition 4.7 of his notes and then defines the norm $\|\cdot\|_{1,2}$. Bermin shows that every F in $\mathcal{P}$ is differentiable in the sense of Definition 4.7, then defines $\|\cdot\|_{1,2}$ on $\mathcal{P}$ and then defines $\mathbb{D}_{1,2}$ as the Banach space which is the completion of $\mathcal{P}$ under $\|\cdot\|_{1,2}$.

Remark 2.27. In fact, the Banach space $\mathbb{D}_{1,2}$ is a Hilbert space and consists of all stochastic variables $F \in L^2(\Omega)$ such that there exists a sequence of stochastic variables $F_n \in \mathcal{P}$ such that $\{F_n\}_{n=1}^\infty \rightarrow F$ in $L^2(\Omega)$ and $\{D_t F_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^2([0, T] \times \Omega)$.

In fact, the space $\mathbb{D}_{1,2}$ have some nice properties and will play an important role in the sequel. For example, we have the following proposition.

Proposition 2.28. If F is $\mathcal{F}_s$ -measurable and in $\mathbb{D}_{1,2}$ with $s < t$, then $D_t F = 0$.

Proof. For $F \in \mathcal{P}$, and F is $\mathcal{F}_s$ -measurable, we have

$$D_t F(\omega) = \sum_{i=1}^n \frac{\partial p}{\partial x_i} \left(\int_0^s g_1(r) dW_r, \dots, \int_0^s g_n(r) dW_r \right) \cdot g_i(t) 1_{[0,s]}(t).$$

For $s < t$, we have $D_t F(\omega) = 0$. It also true for $F \in \mathbb{D}_{1,2}$ by approximation. $\square$

Before ending with this section, we state two important chain rules [35] for the Malliavin derivative.

Proposition 2.29. *[See Proposition 1.2.3 in [35]] Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F_1, \dots, F_n)$ be a stochastic vector and $F_i \in \mathbb{D}_{1,2}$ for $i = 1, \dots, n$. Then $\varphi(F) \in \mathbb{D}_{1,2}$, and*

$$D_t \varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F) D_t F_i. \quad (2.31)$$

From [35], we have the following result, which we will use in the later section.

Proposition 2.30. *[See Lemma 1.2.3 in [35]] Suppose that there exists a sequence of stochastic variables $F_n \in \mathbb{D}_{1,2}$ such that $F_n \rightarrow F$ in $L^2(\Omega)$ and*

$$\sup_n E^Q \left[\|D_t F_n\|_{L^2([0,T])}^2 \right] < \infty.$$

Then $F \in \mathbb{D}_{1,2}$ and $\{D_t F_n\}_{n=1}^\infty \rightarrow D_t F$ in the weak topology of $L^2([0, T] \times \Omega)$.

Moreover, if G is a Lipschitz function of a stochastic vector process belonging to $\mathbb{D}_{1,2}$, then there is a chain-rule formula on which we can calculate the Malliavin derivative of G , as follows:

Proposition 2.31. *[See Proposition 1.2.4 in [35]] Let φ be a Lipschitz function, i.e., $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$|\varphi(x) - \varphi(y)| \leq K |x - y|$$

for any $x, y \in \mathbb{R}^m$ and some constant K , and $F = (F_1, \dots, F_n)$ be a stochastic vector and $F_i \in \mathbb{D}_{1,2}$ for $i = 1, \dots, n$, and the law of F be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. Then $\varphi(F) \in \mathbb{D}_{1,2}$, and

$$D_t \varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F) D_t F_i. \quad (2.32)$$

In the next two small sections, we will give the proof of the Malliavin derivative of S defined by equation (1.18), and its maximum in detail, which we need to use frequently in the application to exotic barrier options.

2.2.1 Malliavin Derivative of S_t

In this section, let us calculate the Malliavin derivative of S_t defined by equation (1.18).

Proposition 2.32. [See Corollary 9 in [4]] Fix $s \in [0, T]$ and let S_s denote the time s price of the stock following the stochastic equation (1.18). Then $S_s \in \mathbb{D}_{1,2}$ and

$$D_t(S_s) = \sigma S_s 1_{[0,s]}(t). \quad (2.33)$$

Proof. We offer the following proof. By Taylor expansion, $e^{\sigma W_s} = \sum_{k=0}^{\infty} \frac{(\sigma W_s)^k}{k!}$ for every ω and also in $L^2(\Omega)$. Moreover $\sum_{k=0}^n \frac{(\sigma W_s)^k}{k!} \in \mathcal{P}$ and $\sum_{k=0}^n \frac{(\sigma W_s)^k}{k!} \rightarrow e^{\sigma W_s}$ in $L^2(\Omega)$. Define

$$P_n(s) = S_0 e^{(r - \frac{\sigma^2}{2})s} \sum_{k=0}^n \frac{(\sigma W_s)^k}{k!}. \quad (2.34)$$

As

$$\begin{aligned} D_t \left(\sum_{k=0}^n \frac{(\sigma W_s)^k}{k!} \right) &= \sum_{k=0}^n D_t \left(\frac{(\sigma W_s)^k}{k!} \right) \\ &= \sum_{k=0}^n \frac{\sigma^k}{k!} D_t ((W_s)^k) \\ &= \sum_{k=0}^n \frac{\sigma^k}{k!} k (W_s)^{k-1} 1_{[0,s]}(t) \\ &= \sigma \sum_{k=0}^{n-1} \frac{(\sigma W_s)^k}{k!} 1_{[0,s]}(t) \rightarrow \sigma e^{\sigma W_s} 1_{[0,s]}(t) \quad \text{as } n \rightarrow \infty, \end{aligned}$$

we have

$$\begin{aligned}
D_t(P_n(s)) &= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)s} \sum_{k=0}^{n-1} \frac{(\sigma W_s)^k}{k!} \sigma 1_{[0,s]}(t) \\
&\rightarrow S_0 e^{\left(r - \frac{\sigma^2}{2}\right)s} \sigma e^{\sigma W_s} 1_{[0,s]}(t) \quad \text{in } L^2(\Omega) \\
&= \sigma S_s 1_{[0,s]}(t).
\end{aligned}$$

By Proposition 2.22, we know D_t is a closable operator, so we get $D_t(S_s) = \sigma S_s 1_{[0,s]}(t)$.

Moreover, by the norm $\|\cdot\|_{1,2}$ on the set $\mathcal{P}$, we have

$$\|P_n(s) - S_s\|_{1,2}^2 = E^Q[|P_n(s) - S_s|^2] + E^Q[\|D_t(P_n(s)) - D_t(S_s)\|_{L^2([0,T])}^2].$$

Since $P_n(s) \rightarrow S_s$ in $L^2(\Omega)$ and $D_t(P_n(s)) \rightarrow D_t(S_s)$ in $L^2(\Omega \times [0, T])$, we complete our proof by showing $\|P_n(s) - S_s\|_{1,2}^2 \rightarrow 0$, i.e. $S_s \in \mathbb{D}_{1,2}$. $\square$

Remark 2.33. Since $\mathcal{P}$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$, we can follow the proof above and get $S_s \in \mathbb{D}_{k,p}$, for every $k \geq 1$ and $p > 1$.

2.2.2 Malliavin Derivative of M_{t_1, t_2}^S

Let us first introduce the following stochastic variables for later use:

$$M_{t_1, t_2}^S : = \sup_{t \in [t_1, t_2]} S_t$$

for $0 \leq t_1 \leq t_2 \leq T$, where S_t is the solution to (1.18).

In this section, we find the Malliavin derivative of M_{t_1, t_2}^S in detail. In fact, Bermin [4] gives a proof of this result, but not fully treated and we supply more mathematical detail.

Definition 2.34. Let the random variable $\alpha(t_1, t_2)$ denote the first time that S achieves its maximum in the interval $[t_1, t_2]$ with $0 \leq t_1 \leq t_2$, where S_t follows the process in equation (1.18). That is

$$\alpha(t_1, t_2) = \inf \{t \in [t_1, t_2] : S_t = M_{t_1, t_2}^S\}. \quad (2.35)$$

Let us consider for each $n \in \mathbb{N}$, a partition $\pi_n = \{t_1 = s_1 < \cdots < s_n = t_2\}$, and require $\pi_n \subseteq \pi_{n+1}$ and $\cup_n \pi_n$ is dense in $[t_1, t_2]$. Define φ_n by

$$\varphi_n(\underline{x}) = \max_{1 \leq i \leq n} x_i ;$$

here $\underline{x} = (x_1, \cdots, x_n)$ and note $\varphi_n(\underline{x})$ is a Lipschitz function. Define a sequence of sets

$$\begin{aligned} E_1 &= \left\{ \underline{x} \in \mathbb{R}^n : x_1 = \max_{1 \leq i \leq n} x_i \right\}, \\ &\vdots \\ E_k &= \left\{ \underline{x} \in \mathbb{R}^n : x_k = \max_{1 \leq i \leq n} x_i \text{ and } x_1 < x_k, \cdots, x_{k-1} < x_k \right\}. \end{aligned}$$

We list some properties of this sequence

- (i) $E_i \cap E_j = \emptyset$ if $i < j$.

For should $\underline{x} \in E_i \cap E_j$, then $\underline{x} \in E_j$ and $x_1 < \max_{1 \leq r \leq n} x_r, \cdots, x_i < \max_{1 \leq r \leq n} x_r = x_j$. But $\underline{x} \in E_i$, means that $x_i = \max_{1 \leq r \leq n} x_r = x_j$, which is contradictory.

- (ii) $\cup_i E_i = \mathbb{R}^n$.

For if $\underline{x} \in \mathbb{R}^n$, then $\max_{1 \leq r \leq n} x_r$ exists and there is a first index j , say, such that $x_j = \max_{1 \leq r \leq n} x_r$. For this index, $x_1 < x_j, \cdots, x_{j-1} < x_j$, so $\underline{x} \in E_j$. So $\forall \underline{x} \in \mathbb{R}^n, \exists j \in [1, n] \cap \mathbb{N}$, such that $\underline{x} \in E_j \subseteq \cup_i E_i$, that is $\mathbb{R}^n \subseteq \cup_i E_i$.

So we can write

$$\varphi_n(\underline{x}) = \sum_{i=1}^n x_i 1_{E_i}(\underline{x})$$

and

$$\frac{\partial \varphi_n}{\partial x_i}(x_1, \cdots, x_n) = 1_{E_i}(\underline{x}),$$

in the sense of distributions. Now we compose φ_n with the values of S at points of a partition, π_n . That is, $M_{\pi_n}^S = \varphi_n(S_{s_1}, \cdots, S_{s_n}) = \max_i S_{s_i}$. We define

$$\begin{aligned} A_i &= \{ \omega : \underline{S}_{\pi_n}(\omega) \in E_i \} \\ &= \{ \omega : S_{s_1}(\omega) < S_{s_i}(\omega), \cdots, S_{s_{i-1}}(\omega) < S_{s_i}(\omega) = M_{\pi_n}^S(\omega) \}, \end{aligned} \tag{2.36}$$

where $\underline{S}_{\pi_n}(\omega) = (S_{s_1}(\omega), \dots, S_{s_n}(\omega)) \in \mathbb{R}^n$. So

$$M_{\pi_n}^S = \sum_{i=1}^n S_{s_i} 1_{E_i}(\underline{S}_{\pi_n}) = \sum_{i=1}^n S_{s_i} 1_{A_i}, \quad (2.37)$$

and $M_{\pi_n}^S$ coincides with S_{s_i} on the set A_i .

Definition 2.35. Let us define a measurable function

$$\alpha_n = \sum_{i=1}^n s_i 1_{A_i}, \quad (2.38)$$

where $s_i \in \pi_n$ and A_i is defined by equation (2.36).

Note that α_n is not a stopping time, but is a measurable function. It is the first time that the finite sequence $(S_{s_1}, \dots, S_{s_n})$ achieves its maximum, i.e.,

$$\alpha_n = \min_{\pi_n} \{t_j : S_{t_j} = M_{\pi_n}^S\} \text{ and } M_{\pi_n}^S = S_{\alpha_n}(\omega).$$

and we provide the following lemma.

Lemma 2.36. Let α_n be defined by equation (2.38) and $\alpha(t_1, t_2)$ be defined by equation (2.35). We show that

$$\alpha_n \rightarrow \alpha(t_1, t_2) \quad Q - a.s.. \quad (2.39)$$

Proof. First, we notice that for index $n < m$, we have the partition $\pi_n \subset \pi_m$, and clearly there exists a sequence n_k such that we have $\bigcup_k \pi_{n_k} \subseteq \bigcup_n \pi_n$. On the other hand, for fixed n , there exists k such that $n_k \geq n$ and $\pi_n \subseteq \pi_{n_k} \subseteq \bigcup_k \pi_{n_k}$, so $\bigcup_n \pi_n \subseteq \bigcup_k \pi_{n_k}$. Hence, we have

$$\bigcup_n \pi_n = \bigcup_k \pi_{n_k}.$$

Now given $\epsilon > 0$, there exists $\delta > 0$ such that if $|t - \alpha(t_1, t_2)(\omega)| < \delta$, then

$$|S_t(\omega) - S_{\alpha(t_1, t_2)}(\omega)| < \epsilon.$$

Hence there exists $t_i \in \pi_n$ inside $\{s : |s - \alpha(t_1, t_2)(\omega)| < \delta\}$ for all fine enough π_n such that $|S_{t_i}(\omega) - S_{\alpha(t_1, t_2)}(\omega)| < \epsilon$, i.e., $S_{\alpha(t_1, t_2)}(\omega) - S_{t_i}(\omega) < \epsilon$. But $M_{\pi_n}^S = S_{\alpha_n}(\omega)$ lies

between $S_{t_i}(\omega)$ and $S_{\alpha(t_1, t_2)}(\omega)$, so

$$S_{\alpha_n}(\omega) = M_{\pi_n}^S \rightarrow M_{t_1, t_2}^S(\omega).$$

Therefore considering $\lim_n \sup \alpha_n$, there is a subsequence $\alpha_{n_k} \rightarrow \lim_n \sup \alpha_n \in [t_1, t_2]$, i.e.

$$M_{\pi_{n_k}}^S = S_{\alpha_{n_k}} \rightarrow S_{\lim_n \sup \alpha_n},$$

and

$$S_{\alpha_{n_k}}(\omega) = M_{\pi_{n_k}}^S \rightarrow M_{t_1, t_2}^S(\omega).$$

Hence

$$S_{\lim_n \sup \alpha_n} = M_{t_1, t_2}^S(\omega) = S_{\alpha(t_1, t_2)}(\omega).$$

Similarly for $\lim_n \inf \alpha_n$, we have

$$S_{\lim_n \inf \alpha_n} = M_{t_1, t_2}^S(\omega) = S_{\alpha(t_1, t_2)}(\omega).$$

Therefore

$$S_{\lim_n \sup \alpha_n} = S_{\lim_n \inf \alpha_n} = M_{t_1, t_2}^S(\omega) = S_{\alpha(t_1, t_2)}(\omega),$$

and

$$\lim_n \sup \alpha_n = \lim_n \inf \alpha_n = \alpha(t_1, t_2).$$

We finish the proof with the remark stated in 8.16 Remark in [28], for $\omega \in \Omega$, the point $t_\omega \in [t_1, t_2]$ at which $S_{t_\omega}(\omega) = M_{t_1, t_2}^S(\omega)$ is unique Q -a.s. $\square$

Now we can prove the following proposition due to Bermin [4].

Proposition 2.37. [See Corollary 9 in [4]] For $0 \leq t_1 \leq t_2 \leq T$, we have $M_{t_1, t_2}^S \in \mathbb{D}_{1,2}$ and

$$D_t M_{t_1, t_2}^S = M_{t_1, t_2}^S \sigma 1_{(-\infty, t_1)}(t) + M_{t_1, t_2}^S 1_{\{M_{t_1, t}^S \leq M_{t_1, t_2}^S\}} \sigma 1_{[t_1, t_2]}(t), \quad (2.40)$$

or

$$D_t M_{t_1, t_2}^S = \sigma M_{t_1, t_2}^S 1_{[0, \alpha(t_1, t_2)]}(t). \quad (2.41)$$

Proof. By Proposition 2.31, and plugging in our φ_n ,

$$D_t \varphi_n(\underline{S}) = \sum_{i=1}^n \frac{\partial \varphi_n}{\partial x_i}(\underline{S}) D_t S_{s_i} = \sum_{i=1}^n 1_{A_i} \sigma S_{s_i} 1_{[0, s_i]}(t), \quad (2.42)$$

and $M_{\pi_n}^S \in \mathbb{D}_{1,2}$.

Since $\cup_n \pi_n$ is dense in $[t_1, t_2]$ and for Q -a.e., $\omega \in \Omega$, $s \mapsto S_s(\omega)$ is a continuous function. So it is uniformly continuous on $[t_1, t_2]$, and we can arrange for our partitions such that for all $s \in [t_1, t_2]$, there exists i , $1 \leq i \leq n$ such that $|S_s(\omega) - S_{s_i}(\omega)| < \epsilon$ for all large enough n . So

$$M_{t_1, t_2}^S(\omega) - M_{\pi_n}^S(\omega) \leq \epsilon$$

for all large enough n . Also by Lemma 2.36, we have

$$M_{\pi_n}^S \rightarrow M_{t_1, t_2}^S, \quad Q - a.s.$$

Then since S is a sub-martingale it follows from Doob's maximal inequality * that

$$\|M_{t_1, t_2}^S\|_{L^2(\Omega)}^2 = E^Q \left[\sup_{t_1 \leq t \leq t_2} |S_t|^2 \right] \leq \left(\frac{2}{2-1} \right)^2 E^Q [|S_{t_2}|^2] < \infty.$$

Hence $M_{t_1, t_2}^S \in L^2(\Omega)$. Moreover,

$$E^Q \left[(M_{\pi_n}^S - M_{t_1, t_2}^S)^2 \right] = E^Q \left[(M_{t_1, t_2}^S)^2 \right] - 2E^Q [M_{\pi_n}^S M_{t_1, t_2}^S] + E^Q \left[(M_{\pi_n}^S)^2 \right],$$

and we know $0 \leq M_{\pi_n}^S \leq M_{t_1, t_2}^S$, Q -a.s., whence $0 \leq M_{\pi_n}^S M_{t_1, t_2}^S \leq (M_{t_1, t_2}^S)^2$ and each term lies in $L^1(\Omega)$ and $M_{\pi_n}^S M_{t_1, t_2}^S \nearrow (M_{t_1, t_2}^S)^2$, Q -a.s. Then the monotone convergence theorem tells us that $E^Q [M_{\pi_n}^S M_{t_1, t_2}^S] \nearrow E^Q [(M_{t_1, t_2}^S)^2]$. Similarly, $E^Q [(M_{\pi_n}^S)^2] \nearrow$

* See e.g. 3.8 Theorem in [28], For any $p > 1$, we have

$$E \left[\sup_{0 \leq t \leq T} |M_t|^p \right] \leq \left(\frac{p}{p-1} \right)^p E [|M_T|^p]$$

Actually, the inequality holds if we replace $|M_t|$ by any continuous non-negative sub-martingale.

$E^Q [(M_{t_1, t_2}^S)^2]$. Since all integrals are finite, this shows

$$M_{\pi_n}^S \rightarrow M_{t_1, t_2}^S \text{ in } L^2(\Omega).$$

Moreover,

$$D_t M_{\pi_n}^S = \sigma S_{\alpha_n} 1_{[0, \alpha_n]}(t) = \sigma M_{\pi_n}^S 1_{[0, \alpha_n]}(t). \quad (2.43)$$

Taking $L^2([0, T])$ -norms in the above equation, we get

$$\begin{aligned} \|D_t M_{\pi_n}^S\|_{L^2([0, T])}^2 &= \int_0^T |D_t M_{\pi_n}^S|^2 dt \\ &= \int_0^T \sigma^2 (M_{\pi_n}^S)^2 1_{[0, \alpha_n]}(t) dt \\ &\leq \sigma^2 (M_{\pi_n}^S)^2 T \\ &\leq \sigma^2 (M_{t_1, t_2}^S)^2 T. \end{aligned}$$

Taking the expectation on both sides of the above inequality, we get

$$E^Q [\|D_t M_{\pi_n}^S\|_{L^2([0, T])}^2] \leq \sigma^2 T E^Q [(M_{t_1, t_2}^S)^2] = \sigma^2 T \|M_{t_1, t_2}^S\|_{L^2(\Omega)}^2 < \infty.$$

By Proposition 2.30, we show $\sup_n E^Q [\|D_t M_{\pi_n}^S\|_{L^2([0, T])}^2] < \infty$ and $M_{t_1, t_2}^S \in \mathbb{D}_{1,2}$.

On the other hand, since $\alpha_n \rightarrow \alpha(t_1, t_2)$ a.s. it follows that

$$D_t M_{\pi_n}^S = \sigma S_{\alpha_n} 1_{[0, \alpha_n]}(t) \rightarrow \sigma S_{\alpha(t_1, t_2)} 1_{[0, \alpha(t_1, t_2)]}(t).$$

By Proposition 2.30 again, $D_t M_{\pi_n}^S$ converges weakly to $D_t M_{t_1, t_2}^S$ in $L^2([0, T] \times \Omega)$ and $S_{\alpha(t_1, t_2)} = M_{t_1, t_2}^S \in L^2(\Omega)$, hence

$$D_t M_{t_1, t_2}^S = \sigma M_{t_1, t_2}^S 1_{[0, \alpha(t_1, t_2)]}(t).$$

It is also noticed that the indicator function can be written as

$$1_{[0, \alpha(t_1, t_2)]}(t) = 1_{(-\infty, t_1)}(t) + 1_{\{M_{t_1, t}^S \leq M_{t, t_2}^S\}} 1_{[t_1, t_2]}(t).$$

Now consider a function

$$J(t, \omega) = 1_{(-\infty, t_1)}(t) + 1_{\{M_{t_1, t}^S \leq M_{t, t_2}^S\}}(\omega) 1_{[t_1, t_2]}(t). \quad (2.44)$$

We observe when one of these two terms is non-zero, the other is zero. So

$$J(t, \omega) = \begin{cases} 1 & \text{if } t < t_1 \text{ and } \omega \in \Omega, \\ 1 & \text{if } t_1 \leq t \leq t_2 \text{ and } M_{t_1, t}^S \leq M_{t, t_2}^S, \\ 0 & \text{if } t_1 \leq t \leq t_2 \text{ and } M_{t_1, t}^S > M_{t, t_2}^S. \end{cases}$$

Note that in the case $t_1 \leq t \leq t_2$ and $M_{t_1, t}^S \leq M_{t, t_2}^S$, then $M_{t_1, t_2}^S = M_{t, t_2}^S$. But, as soon as M_{t_1, t_2}^S is achieved, ω is no longer in the set $\{M_{t_1, t}^S \leq M_{t, t_2}^S\}$ for t greater than the first time S hits its maximum over $[t_1, t_2]$. So $J(t, \omega)$ switches off the (first) time S achieves its maximum over $[t_1, t_2]$. So $J(t, \omega) = 1$ for points (t, ω) such that $0 \leq t \leq \alpha(t_1, t_2)$. $\square$

Remark 2.38. Notice that if we set $t_1 = 0$ and $t_2 = t$, then we have

$$D_t M_{0, t}^S = \sigma M_{0, t}^S 1_{[0, \alpha(0, t)]}(t). \quad (2.45)$$

Notice that for Brownian motion, there is a result, (e.g. 8.15 Proposition in [28]) which states that Brownian motion on $[0, t]$, Q -almost surely hits its running maximum strictly before t , and another result, (e.g. 8.17 Problem in [28]), which says the last time Brownian motion is equal to its running maximum has a distribution obeying an arc-sine law, which shows that the probability that the last time Brownian motion hits its maximum is strictly before t is equal to one. We then can extend these two results to the process S , and therefore

$$1_{[0, \alpha(0, t)]}(t) = 0 \quad Q - a.s. \quad (2.46)$$

That is,

$$D_t M_{0, t}^S = 0, \quad Q - a.s. \quad (2.47)$$

Chapter 3

The Clark-Ocone formula and its extension

We will present the Clark-Ocone formula and its extension in this chapter and show how the self-financing portfolio generating a contingent claim can be formally derived by using Malliavin Calculus.

3.1 The Clark-Ocone formula and its limitations

Let us begin this chapter by recalling the well-known Itô representation theorem, which states that if $G \in L^2(\Omega)$ is $\mathcal{F}_T$ measurable, then there exists a unique $\mathbb{F}$ -adapted process $\varphi(t)$, for $0 \leq t \leq T$, such that

$$G = E^Q[G] + \int_0^T \varphi(t) dW_t. \quad (3.1)$$

This result only provides the existence of the integrand, but does not give us a way to calculate the integrand explicitly. However, from the point of view of applications it is important also to be able to find the integrand more explicitly. This can be achieved by the Clark-Ocone formula, which says that, under some suitable conditions, we have an explicit form for the integrand given by $\varphi(t) = E^Q[D_t F | \mathcal{F}_t]$, $0 \leq t \leq T$, where $D_t F$ is the Malliavin derivative of F . This is one reason why the Clark-Ocone formula makes

Malliavin calculus useful in mathematical finance.

We only present the formula as a theorem as a generalization of the Itô representation theorem, and for details on how the Clark-Ocone formula was established, we refer to [39].

Proposition 3.1. *[Theorem 5.8 in [39]] Let $G \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$ -measurable. Then we have*

$$G = E^Q[G] + \int_0^T E^Q[D_t G | \mathcal{F}_t] dW_t.$$

Note that the Clark-Ocone formula presented here is only valid for those contingent claims $G \in \mathbb{D}_{1,2} \subset \mathbb{D}_{0,2} = L^2(\Omega)$. Therefore, there are some restrictions when we use the Malliavin calculus approach. We follow Example 4.1 of [5] in detail to show how the Malliavin Calculus approach has its limitations.

Example 3.2. *Let us assume $A \in \mathcal{F}_T$ such that $1_A \in \mathbb{D}_{1,2}$. Then there exists a sequence $(F_n) \subset \mathcal{P}$ converging to 1_A in $\|\cdot\|_{1,2}$. By Proposition 2.31, if we take $\varphi(x) = x^2$ on $[0, 1]$ and $F = (1_A)$, i.e., a one-dimensional vector, then $D_t 1_A = 2 \cdot 1_A \cdot D_t 1_A$. So $D_t 1_A = 0$ on $\Omega \setminus A$; for $\omega \in A$, $D_t 1_A(\omega) = 2D_t 1_A(\omega)$, so that $D_t 1_A = 0$ on A also. So $D_t 1_A = 0$, for every $A \in \mathcal{F}_T$. Now by Proposition 3.1 for 1_A ,*

$$\begin{aligned} 1_A &= E^Q[1_A]1_\Omega + \int_0^T E^Q[D_t 1_A | \mathcal{F}_t] dW_t \\ &= E^Q[1_A]1_\Omega \\ &= Q(A)1_\Omega, \end{aligned}$$

which cannot be true for every $A \in \mathcal{F}_T$. If $\emptyset \neq A \neq \Omega$, then the assumption $1_A \in \mathbb{D}_{1,2}$ leads to $1_A = Q(A)1_\Omega$, which is a contradiction. Note $L^2(\Omega)$ is equivalent classes and sets which differ by a Q null set are regarded as the same set. So, in $L^2(\Omega)$, $\emptyset \neq A \neq \Omega$, means $0 < Q(A) < 1$. If $A = \emptyset$ or Ω , then no contradiction occurs. So we can conclude that when $A \in \mathcal{F}_T$, then $1_A \notin \mathbb{D}_{1,2}$. So for example, the indicator of the set, $\{S_T \leq K\}$, does not have a Malliavin derivative in the usual sense, although $1_{\{S_T \leq K\}} \in L^2(\Omega)$.

In order to use the Clark-Ocone formula in applications, we see the space $\mathbb{D}_{1,2}$ is not big enough. However, in the next section, we will describe how to extend the Clark-Ocone formula to a larger space.

3.2 The Extension of the Clark-Ocone formula

From the previous section, we see there are some limitations to calculating the Malliavin derivative of G if the claim G is not in $\mathbb{D}_{1,2}$, hence we need to extend the Clark-Ocone formula. As mentioned before, for any $\mathcal{F}_T$ -measurable claim $G \in L^2(\Omega)$, we can find a self-financing portfolio to replicate G , and in our thesis, our purpose is to find the self-financing portfolio via the Clark-Ocone formula. Hence, from our point of view, for applications we need Clark-Ocone formula to be valid for any $\mathcal{F}_T$ -measurable stochastic variable in $L^2(\Omega)$. Note that the space $\mathbb{D}_{1,2}$ is a dense subspace of $L^2(\Omega)$. One's first reaction might be that the extension of the Clark-Ocone formula seems trivial. However, we will see the key point is that the *Malliavin derivative* cannot be defined in the usual way for all stochastic variables in $L^2(\Omega)$, and we have to define the Malliavin derivative in the sense of distributions in order to extend the Clark-Ocone formula.

First let us follow [47] and extend some definitions and summarize some results, then characterize the new space for which the Clark-Ocone formula is valid. Note that in [47], Watanabe deals with abstract Wiener space, but to see the connection with our concrete case observe that W_t is an element of the dual of our concrete Wiener Space.

Now we describe the Ornstein-Uhlenbeck semi-group and Ornstein-Uhlenbeck operator by following [47].

Definition 3.3. For $F(\omega) \in \mathcal{P}$ and $t \geq 0$, define $T_t(F)$ as follows:

$$T_t(F)(\omega) \triangleq \int_{\Omega} F\left(e^{-t}\omega + \sqrt{1 - e^{-2t}}u\right) Q(du).$$

The operator T_t satisfies the following properties. See [47] for a complete proof.

Proposition 3.4. [See Properties of $T_t F$ on page 13 in [47]]

- If $F \in \mathcal{P}$, then $T_t(F) \in \mathcal{P}$.
- For $F \in \mathcal{P}$ and $F = \sum_{n=0}^m J_n(F)$, then $T_t(F) = \sum_n e^{-nt} J_n(F)$.
- T_t is a contraction on L^p , $1 \leq p < \infty$, i.e., $\|T_t F\|_p \leq \|F\|_p$.

Definition 3.5. [Ornstein-Uhlenbeck Operator] Let $F \in \mathcal{P}$ be a square-integrable random variable. Define the generator L of the semi-group T_t as follows:

$$L(F) = \sum_{n=0}^{\infty} (-n) J_n(F), \quad (3.2)$$

provided this series converges in $L^2(\Omega)$. We call L the Ornstein-Uhlenbeck operator. The domain of this operator is the set

$$\text{Dom} L = \left\{ F \in \mathcal{P}, F = \sum_{n=0}^{\infty} J_n(F) : \sum_{n=1}^{\infty} n^2 \|J_n F\|_2^2 < \infty \right\}.$$

Remark 3.6. Note that L maps $\mathcal{P}$ to $\mathcal{P}$.

Definition 3.7. [See Definition 1.10 in [47]] For $F \in \mathcal{P}$, $-\infty < k < \infty$ and $1 < p < \infty$. Define the norm $\|\cdot\|_{k,p}$ as follows:

$$\|F\|_{k,p} \triangleq \left\| (I - L)^{k/2} F \right\|_p, \quad (3.3)$$

where

$$(I - L)^{k/2} F \triangleq \sum_{n=0}^{\infty} (1 + n)^{k/2} J_n(F) \in \mathcal{P}. \quad (3.4)$$

Remark 3.8. In [47], it is shown that the norm $\|\cdot\|_{k,p}$ defined in equation (2.29) and the norm $\|\cdot\|_{k,p}$ defined in equation (3.3) are equivalent whenever $p > 1$ and $k \in \mathbb{N}$. This is in fact given by the Meyer inequalities. For more details, see, for example, [46] and [47].

Example 3.9. By Definition 3.5, on the orthogonal subspaces $J_n(L^2)$, $0 \leq n < \infty$, the operator L acts as a scalar multiplier by

$$L(J_k(F)) = \sum_n (-n) J_n(J_k(F)) = -k J_k(F),$$

since J_n is an orthogonal projection and $J_n J_k = 0$ for $n \neq k$. Therefore,

$$(I - L)(F) = \sum_n (1 + n) J_n(F).$$

Moreover,

$$\begin{aligned}
 (I - L)^2(F) &= (I - L) \{(I - L)(F)\} \\
 &= \sum_n (1 + n) J_n((I - L)(F)) \\
 &= \sum_n (1 + n) J_n \left(\sum_k (1 + k) J_k(F) \right) \\
 &= \sum_n (1 + n) \sum_k (1 + k) J_n(J_k(F)) \\
 &= \sum_n (1 + n)^2 J_n(F).
 \end{aligned}$$

So we can take polynomial function of $I - L$, and it follows that if h is a polynomial (in one variable), then

$$h \{(I - L)\}(F) = \sum_n h(1 + n) J_n(F).$$

Proposition 3.10. [See Proposition 1.7 in [47]] For $k \leq k'$ and $p \leq p'$, we have

- 1) $\|F\|_{k,p} \leq \|F\|_{k',p'}, \forall F \in \mathcal{P}$.
- 2) For any $-\infty < k < \infty$ and $1 < p < \infty$, $\|\cdot\|_{k,p}$ are compatible. i.e., for any (k, p) , (k', p') if the sequence $F_n \in \mathcal{P}$, $\|F_n\|_{k,p} \rightarrow 0$ as $n \rightarrow \infty$ and $\|F_n - F_m\|_{k',p'} \rightarrow 0$ as $n, m \rightarrow \infty$, then

$$\|F_n\|_{k',p'} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. We follow [47] to give the proof, but in more detail. For fixed k , by Definition 3.7, we know $\|F\|_{k,p} \leq \|F\|_{k,p'}$, when $p \leq p'$. It is enough to prove $\|F\|_{k,p} \leq \|F\|_{k',p}$ for $k \leq k'$. To do so, we need to show

$$\begin{aligned}
 \int_0^\infty e^{-t} t^{\alpha-1} T_t F dt &= \int_0^\infty e^{-t} t^{\alpha-1} \left(\sum_n e^{-nt} J_n(F) \right) dt \\
 &= \sum_n \left(\int_0^\infty e^{-t} t^{\alpha-1} e^{-nt} dt \right) J_n(F) \\
 &= \sum_n \left(\int_0^\infty e^{-(1+n)t} t^{\alpha-1} dt \right) J_n(F).
 \end{aligned}$$

Changing variables $r = (1 + n)t$,

$$\int_0^\infty e^{-(1+n)t} t^{\alpha-1} dt = \frac{1}{(1+n)^\alpha} \int_0^\infty e^{-r} r^{\alpha-1} dr = \frac{1}{(1+n)^\alpha} \Gamma(\alpha),$$

where $\Gamma(\alpha)$ represents the Gamma function with $\alpha > 0$. Therefore

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} T_t F dt &= \sum_n \left(\frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(1+n)t} t^{\alpha-1} dt \right) J_n(F) \\ &= \sum_n (1+n)^{-\alpha} J_n(F) \\ &= (I - L)^{-\alpha} F. \end{aligned}$$

From above the equation and by Proposition 3.4, T_t is an L^p contraction, so we get

$$\|(I - L)^{-\alpha} F\|_p \leq \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} \|T_t F\|_p dt \leq \|F\|_p.$$

Then $(I - L)^{-\alpha}$ maps from $\mathcal{P}$ to $\mathcal{P}$, for all $\alpha > 0$. We know $\|F\|_{k,p} \leq \|F\|_{k',p}$, when $k \leq k'$. By Definition 3.7, that means

$$\|(I - L)^{k/2} F\|_p \leq \|(I - L)^{k'/2} F\|_p, \text{ when } k \leq k'.$$

So

$$\|(I - L)^{-\alpha} ((I - L)^{k'/2} F)\|_p \leq \|(I - L)^{k'/2} F\|_p.$$

Finally, let us choose $-\alpha = \frac{k-k'}{2} > 0$, then we finish the proof. $\square$

Proof. 2) Let $G_n = (I - L)^{k'/2} F_n \in \mathcal{P}$. Then

$$\begin{aligned} \|G_n - G_m\|_{p'} &= \|(I - L)^{k'/2} F_n - (I - L)^{k'/2} F_m\|_{p'} \\ &= \|(I - L)^{k'/2} (F_n - F_m)\|_{p'} \\ &= \|F_n - F_m\|_{k',p'} \rightarrow 0 \text{ as } n, m \rightarrow \infty. \end{aligned}$$

Therefore, there exists $G \in L^{p'}$, such that $\|G_n - G\|_{p'} \rightarrow 0$. But by Definition 3.7,

$$\|F_n\|_{k,p} = \|(I - L)^{k/2} F_n\|_p = \|(I - L)^{(k-k')/2} G_n\|_p \rightarrow 0.$$

It is enough to show $G = 0$. If $H \in \mathcal{P}$, then $(I - L)^{(k'-k)/2} H \in \mathcal{P}$. Noting that $\mathcal{P} \subset L^q$, for every $1 < q < \infty$, we have

$$\begin{aligned} E[G \cdot H] &= \lim_{n \rightarrow \infty} E[G_n \cdot H] \\ &= \lim_{n \rightarrow \infty} E\left[(I - L)^{(k-k')/2} G_n \cdot (I - L)^{(k'-k)/2} H\right] \\ &= 0, \end{aligned}$$

since $\mathcal{P}$ is dense in L^q , for all q . Then $G = 0$.

Finally, we get

$$\|F_n\|_{k',p'} = \|(I - L)^{k'/2} F_n\|_{p'} = \|G_n\|_{p'} \rightarrow 0.$$

□

Now, we move forward to extend the previous definition of $\mathbb{D}_{k,p}$ in Chapter 2.

Definition 3.11. For $-\infty < k < \infty$ and $1 < p < \infty$, define $\mathbb{D}_{k,p}$ as the completion of $\mathcal{P}$ by the norm $\|\cdot\|_{k,p}$.

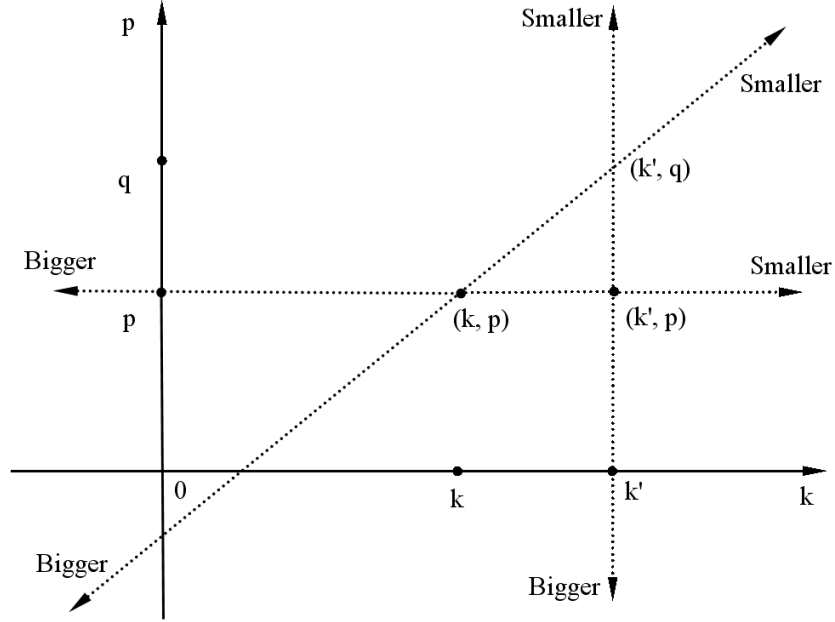
Remark 3.12. Note that by this definition, we have $\mathbb{D}_{0,p} = L^p$.

Remark 3.13. By Proposition 3.10, we know $\|F\|_{k,p} \leq \|F\|_{k',p'}$, when $k \leq k'$ and $p \leq p'$. So a Cauchy sequence in $\|\cdot\|_{k',p'}$, is also a Cauchy sequence in $\|\cdot\|_{k,p}$. Hence we get

$$\mathbb{D}_{k',p'} \subset \mathbb{D}_{k,p} \quad \text{if } k \leq k' \text{ and } p \leq p'.$$

Moreover, if we let $0 < k < k'$ and $1 < p < q$, we have the following inclusions:

$$\begin{array}{ccccccc} \mathbb{D}_{k',p} & \subset & \mathbb{D}_{k,p} & \subset & \mathbb{D}_{0,p} & = & L^p(\Omega) & \subset & \mathbb{D}_{-k,p} & \subset & \mathbb{D}_{-k',p} \\ \cup & & \cup & & \cup & & & & \cup & & \cup \\ \mathbb{D}_{k',q} & \subset & \mathbb{D}_{k,q} & \subset & \mathbb{D}_{0,q} & = & L^q(\Omega) & \subset & \mathbb{D}_{-k,q} & \subset & \mathbb{D}_{-k',q} \end{array} \quad (3.5)$$

Figure 3.1: Any point in this cone gives a $\mathbb{D}_{k',q}$ smaller than $\mathbb{D}_{k,p}$ 

Proposition 3.14. [See page 26 in [47]] Define an operator A as follows:

$$A = (I - L)^{-k/2} . \quad (3.6)$$

Then the maps $A : L^p \mapsto \mathbb{D}_{k,p}$ and $A : \mathbb{D}_{-k,q} \mapsto L^q$ are isometric isomorphisms. Hence $A^* : (\mathbb{D}_{k,p})' \mapsto L^q$ is also an isometric isomorphism, if $1/p + 1/q = 1$.

Proof. If $G \in L^p$, then

$$\|AG\|_{k,p} = \left\| (I - L)^{-k/2} G \right\|_{k,p} = \left\| (I - L)^{k/2} (I - L)^{-k/2} G \right\|_p = \|G\|_p .$$

If $G \in \mathbb{D}_{-k,q}$, then

$$\|AG\|_q = \left\| (I - L)^{-k/2} G \right\|_q = \|G\|_{-k,q} .$$

□

Remark 3.15. Since $L^p \stackrel{A}{\cong} \mathbb{D}_{k,p}$, then $\mathbb{D}'_{k,p} \stackrel{A^*}{\cong} (L^p)' \equiv L^q$. We also know that $\mathbb{D}_{-k,q} \stackrel{A}{\cong} L^q$, therefore $L^q \stackrel{A^{-1}}{\cong} \mathbb{D}_{-k,q}$. Hence we get $\mathbb{D}'_{k,p} \stackrel{A^{-1}A^*}{\cong} \mathbb{D}_{-k,q}$.

Remark 3.16. We know $f \mapsto E[fg]$ is a linear functional on $\mathcal{P}$ and

$$\langle f, g \rangle_{L^2(\Omega)} = E[fg]. \quad (3.7)$$

For $F, G \in \mathcal{P}$, taking the absolute value of $\langle F, G \rangle_{L^2(\Omega)}$, we have

$$\begin{aligned} \left| \langle F, G \rangle_{L^2(\Omega)} \right| &= \left| \langle (I - L)^{-k/2} (I - L)^{k/2} F, G \rangle_{L^2(\Omega)} \right| \\ &= \left| \langle (I - L)^{k/2} F, (I - L)^{-k/2} G \rangle_{L^2(\Omega)} \right| \\ &\leq \left\| (I - L)^{k/2} F \right\|_p \left\| (I - L)^{-k/2} G \right\|_q \\ &= \|F\|_{k,p} \|G\|_{-k,q}. \end{aligned} \quad (3.8)$$

This shows that any $G \in (\mathcal{P}, \|\cdot\|_{-k,q})$ defines a continuous linear functional on $(\mathcal{P}, \|\cdot\|_{k,p})$ and this extends to elements of the completion of $\mathcal{P}$ in $\|\cdot\|_{k,p}$ and $\|\cdot\|_{-k,q}$. So $\mathbb{D}'_{k,p} = \mathbb{D}_{-k,q}$ where $1/p + 1/q = 1$.

Definition 3.17. Define

$$\mathbb{D}_\infty = \cap_{k,p} \mathbb{D}_{k,p}; \quad (3.9)$$

then $\mathbb{D}_\infty$ is a complete countably normed space, see Proposition 2.4 in [3]. The dual of $\mathbb{D}_\infty$ is

$$\mathbb{D}'_\infty = \cup_{k,p} \mathbb{D}'_{k,p} \stackrel{\Delta}{=} \mathbb{D}_{-\infty}; \quad (3.10)$$

see Proposition 2.9 in [3]. Since $\mathbb{D}'_{k,p} = \mathbb{D}_{-k,q}$ where $\frac{1}{p} + \frac{1}{q} = 1$, we conclude that

$$\mathbb{D}_{-\infty} = \cup_{k,q} \mathbb{D}_{-k,q}. \quad (3.11)$$

Remark 3.18. By Remark 2.33 we know $S_s \in \mathbb{D}_\infty$ and it is quite natural to think $M_{t_1, t_2}^S \in \mathbb{D}_\infty$. But it is shown in [36] that $M_{t_1, t_2}^S \notin \mathbb{D}_\infty$, although $M_{t_1, t_2}^S \in \mathbb{D}_{1,2}$, as we prove in Proposition 2.37.

3.2.1 Composition of a Distribution with a Stochastic Variable

In this section, we follow [47] and understand the elements in the spaces $\mathbb{D}_\infty$, $\mathbb{D}_{-\infty}$, $S(\mathbb{R})$ and $S'(\mathbb{R})$ and their relationship clearly.

First, we follow [47] and consider only the one-dimensional case. Now suppose that $F: \Omega \rightarrow \mathbb{R}$ belongs to $\mathbb{D}_\infty$, while $\langle D_t F, D_t F \rangle_{\mathcal{H}} \in \mathbb{D}_\infty$ also and

$$E \left[\frac{1}{\langle D_t F, D_t F \rangle_{\mathcal{H}}^p} \right] < \infty, \quad \forall \ 1 < p < \infty. \quad (3.12)$$

Watanabe shows that if $\varphi \in S(\mathbb{R})$, then $\varphi(F(\omega))$ belongs to $\mathbb{D}_\infty$. The following spaces are introduced: for $\varphi \in S(\mathbb{R})$,

$$\|\varphi\|_{T_{2k}} := \left\| (1 + x^2 - \Delta)^k \varphi \right\|_\infty, \quad (3.13)$$

where Δ is the Laplacian and k is an integer. The completion of $S(\mathbb{R})$ in this norm is denoted by $\mathbb{T}_{2k}$. The following relations hold;

- $S(\mathbb{R}) \subset \cdots \subset \mathbb{T}_{2k} \subset \cdots \subset \mathbb{T}_2 \subset \mathbb{T}_0 \subset \mathbb{T}_{-2} \cdots \subset \mathbb{T}_{-2k}$,
- $\bigcap_k \mathbb{T}_k = S(\mathbb{R})$,
- $\bigcup_k \mathbb{T}_k = S'(\mathbb{R})$.

Theorem 1.12 of section 1.4 of [47] shows that for any $k \in \mathbb{N}$ and $p \in (1, \infty)$, there exists a constant $C_{k,p} > 0$ such that Meyer's inequality holds:

$$\|\varphi(F(\omega))\|_{-2k,p} \leq C_{k,p} \|\varphi\|_{T_{-2k}}, \quad (3.14)$$

for all $\varphi \in S(\mathbb{R})$. So the map $\varphi \mapsto \varphi(F(\omega))$ from $S(\mathbb{R})$ into $\mathbb{D}_\infty$ can be extended, uniquely, to a map from $\mathbb{T}_{-2k}$ into $\mathbb{D}_{-2k,p}$. Since $p \in (1, \infty)$ and $k \in \mathbb{N}$ are arbitrary, the map $\varphi \mapsto \varphi(F(\omega))$ extends a map of $\bigcup_k \mathbb{T}_k \mapsto \bigcup_{k,p} \mathbb{D}_{-2k,p} = \mathbb{D}_{-\infty}$. Consequently, the map $\varphi \mapsto \varphi(F(\omega))$ from $S(\mathbb{R}) \mapsto \mathbb{D}_\infty$ has a continuous extension to $S'(\mathbb{R}) \mapsto \mathbb{D}_{-\infty}$.

By Theorem 1.12 of [47], we know an element, $T \in \mathbb{T}_{-2k}$, is the limit in $\|\cdot\|_{T_{-2k}}$ of a sequence, $(\varphi_n) \subset S(\mathbb{R})$. So (φ_n) is Cauchy in $\|\cdot\|_{T_{-2k}}$ and therefore $\varphi_n(F)$ is Cauchy in

$\mathbb{D}_{-2k,p}$. Hence one defines $T(F)$ as

$$T(F) = \lim_n \varphi_n(F) \quad \text{in } \|\cdot\|_{-2k,p} . \quad (3.15)$$

This is well-defined. So we have

Definition 3.19. Let $F : \Omega \rightarrow \mathbb{R}$ and $F \in \mathbb{D}_\infty$, while $\langle D_t F, D_t F \rangle_{\mathcal{H}} \in \mathbb{D}_\infty$ and

$$E \left[\frac{1}{\langle D_t F, D_t F \rangle_{\mathcal{H}}^p} \right] < \infty, \quad \forall \ 1 < p < \infty . \quad (3.16)$$

For $T \in S'(\mathbb{R})$, we call $T(F)$ the composition of the distribution T with a stochastic variable F as an element in $\mathbb{D}_{-\infty}$, i.e., $T(F) \in \mathbb{D}_{-\infty}$.

Before we finish this section, let us state a general chain rule for the Malliavin derivative of $T(F)$ belonging to $\mathbb{D}_{-\infty}$. For the proof, we refer to [5].

Proposition 3.20. [See Corollary 5.1 in [5]] Let T be the element of the space of $\mathbb{R}^n$ valued Schwartz distribution $S'(\mathbb{R}^n)$. Suppose that $F = (F_1, \dots, F_n)$ is a stochastic vector with $F_i \in \mathbb{D}_\infty$ for all i and suppose that the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. Then the composite $T(F) \in \mathbb{D}_{-\infty}$, and

$$D_t T(F) = \sum_{i=1}^n \frac{\partial T}{\partial x_i}(F) D_t F_i, \quad (3.17)$$

where $\frac{\partial T}{\partial x_i}$ is the derivative of T as a distribution and $\frac{\partial T}{\partial x_i}(F)$ the composition of a distribution with a stochastic variable.

Remark 3.21. Note that when the stochastic variable F_i is restricted to $\mathbb{D}_\infty$, one can compute the Malliavin derivative of $T(F)$ without specifying the restriction for $T(\cdot)$ as long as $T \in S'(\mathbb{R})$. Of course, we can use this proposition to compute the Malliavin derivative of $(S_T - K)^+$ due to $S_T \in \mathbb{D}_\infty$.

Later we present a more elementary way of dealing with the composition of a distribution and a stochastic variable, but only when a conditional expectation is present.

3.2.2 Extending the Clark-Ocone formula to a larger space

From the inclusions stated in the previous section, we know that $\mathbb{D}_\infty \subset \mathbb{D}_{1,2}$, hence the Clark-Ocone formula is obviously valid for any element of $\mathbb{D}_\infty$. However, in this section, we describe the extension of the Clark-Ocone formula to be valid in $\mathbb{D}_{-\infty}$ in the sense of distribution.

We introduce new spaces to assist our extension of the Clark-Ocone formula. We adapt the treatment in [47]. Let $\{e_i : 1 \leq i < \infty\}$ be an orthonormal basis of $L^2([0, T])$. For polynomial variables $F_1, \dots, F_n \in \mathcal{P}$ we can form an $L^2([0, T])$ -valued stochastic variable, $F(\omega)$, by defining

$$F(\omega) := \sum_{i=1}^n F_i(\omega) e_i \in L^2([0, T]), \quad (3.18)$$

where $n \in \mathbb{N}$ is arbitrary. The operator, $I - L$, defined previously in equation (3.4), can be extended to stochastic variables such as $F(\omega)$ by

$$(I - L)F(\omega) = \sum_{i=1}^n (I - L)F_i(\omega) e_i \in L^2([0, T]). \quad (3.19)$$

Note that $I - L$ maps $\mathcal{P}$ into $\mathcal{P}$. This allows us to define

$$(I - L)^{k/2} F(\omega) := \sum_{i=1}^n ((I - L)^{k/2} F_i)(\omega) e_i. \quad (3.20)$$

The stochastic variable

$$\omega \mapsto \|(I - L)^{k/2} F(\omega)\|_2 \quad (3.21)$$

is measurable and an element of $L^p(\Omega)$ for $1 \leq p < \infty$. We define $\|\cdot\|_{k,p}^{L^2([0,T])}$ on the linear space of such F by

$$\|F\|_{k,p}^{L^2([0,T])} = \left(\int_{\Omega} \|(I - L)^{k/2} F\|_2^p \right)^{1/p}. \quad (3.22)$$

We denote the completion of the linear space of such F with respect to $\|\cdot\|_{k,p}^{L^2([0,T])}$ by $\mathbb{D}_{k,p}(L^2([0, T]))$. Clearly, $\mathbb{D}_{k,p}(L^2([0, T]))$ is a generalization of $\mathbb{D}_{k,p}$. The extension of L to $\mathbb{D}_{k,p}(L^2([0, T]))$ is possible, as is the extension of D_t to $\mathbb{D}_{k,p}(L^2([0, T]))$, by a procedure identical to that we just employed for $I - L$. The relation between $\mathbb{D}_{k,p}(L^2([0, T]))$ for

various values of k and p is identical to those for $\mathbb{D}_{k,p}$ in equation (3.5). Accordingly, we can define

$$\mathbb{D}_\infty (L^2([0, T])) = \bigcap_{k,p} \mathbb{D}_{k,p}(L^2([0, T])), \quad (3.23)$$

and it turns out that the dual of $\mathbb{D}_\infty(L^2[0, T])$ can be identified:

$$\begin{aligned} \mathbb{D}'_\infty (L^2([0, T])) &= \bigcup_{k,p} \mathbb{D}'_{k,p} (L^2([0, T])) \\ &= \bigcup_{k,q} \mathbb{D}_{-k,q} (L^2([0, T])) \\ &= \mathbb{D}_{-\infty} (L^2([0, T])). \end{aligned} \quad (3.24)$$

We prove the following Theorem about the construction in [47].

Theorem 3.22. *Let F and G be the stochastic variables with values in $L^2([0, T])$, and of the form given by equation (3.18) and define,*

$$[F, G] = E \left[\langle F, G \rangle_{L^2[0,T]} \right]. \quad (3.25)$$

For F with $[F, F] < \infty$, define an operator

$$\bar{A}F = (I - L)^{-k/2} F. \quad (3.26)$$

Then $\bar{A} = (I - L)^{-k/2}$ is self-adjoint operator.

Proof. Consider

$$\begin{aligned} F(\omega) &= \sum_{i=1}^n F_i(\omega) e_i \in L^2([0, T]), \\ G(\omega) &= \sum_{j=1}^n G_j(\omega) e_j \in L^2([0, T]). \end{aligned}$$

Then

$$\begin{aligned}\langle F, G \rangle_{L^2([0, T])} &= \left\langle \sum_{i=1}^n F_i(\omega) e_i, \sum_{j=1}^n G_j(\omega) e_j \right\rangle_{L^2([0, T])} \\ &= \sum_{i=1}^n F_i(\omega) G_i(\omega) \in \mathcal{P}.\end{aligned}$$

By equation (3.25), we have

$$\begin{aligned}[\bar{A}F, G] &= E \left[\langle (I - L)^{-k/2} F, G \rangle_{L^2([0, T])} \right] \\ &= E \left[\left\langle \sum_{i=1}^n (I - L)^{-k/2} (F_i(\omega)) e_i, \sum_{j=1}^n G_j(\omega) e_j \right\rangle_{L^2([0, T])} \right] \\ &= E \left[\sum_{i=1}^n ((I - L)^{-k/2} F_i) G_i \right] \\ &= \sum_i E \left[((I - L)^{-k/2} F_i) G_i \right] \\ &= \sum_i \langle (I - L)^{-k/2} F_i, G_i \rangle_{L^2(\Omega)}.\end{aligned}$$

We know the operator $A = (I - L)^{-k/2}$ is a self-adjoint operator for $F_i \in \mathcal{P}$. Then we have

$$\langle (I - L)^{-k/2} F_i, G_i \rangle_{L^2(\Omega)} = \langle F_i, (I - L)^{-k/2} G_i \rangle_{L^2(\Omega)}.$$

Hence, we have

$$[\bar{A}F, G] = \sum_i \langle F_i, (I - L)^{-k/2} G_i \rangle_{L^2(\Omega)} = [F, \bar{A}G]. \quad (3.27)$$

□

The reason for introducing these spaces is for their use in the next theorem, which holds by the Meyer inequalities. See, for example, [47].

Proposition 3.23. [See Proposition 1.9 in [47]] For every $p > 1$ and $k \in \mathbb{R}$, the Malliavin

derivative operator D_t is a continuous operator from $\mathbb{D}_{k,p}$ to $\mathbb{D}_{k-1,p}(L^2([0, T]))$, that is

$$\|D_t G\|_{k-1,p}^{L^2([0,T])} \leq \|G\|_{k,p}.$$

In fact, if $F \in \mathbb{D}_\infty = \bigcap_{k,p} \mathbb{D}_{k,p}$, then $F \in \mathbb{D}_{k,p}$ for every $p > 1$ and $k \in \mathbb{R}$, and by the theorem above, $D_t F \in \mathbb{D}_{k-1,p}(L^2([0, T]))$, so by equation (3.23), we get $D_t F \in \mathbb{D}_\infty(L^2([0, T]))$. On the other hand, for $G \in \mathbb{D}_{-\infty} = \bigcup_{-k,q} \mathbb{D}_{-k,q}$, then there exists $q > 1$ and $k \in \mathbb{R}$, such that $G \in \mathbb{D}_{-k,q}$, and by the theorem above again, we get $D_t G \in \mathbb{D}_{-k-1,q}(L^2([0, T]))$. Using equation (3.24), we have $D_t G \in \mathbb{D}_{-\infty}(L^2([0, T]))$. Consequently, the Malliavin derivative D_t is a continuous map from $\mathbb{D}_\infty$ to $\mathbb{D}_\infty(L^2([0, T]))$ and extends continuously as a map $\mathbb{D}_{-\infty} \rightarrow \mathbb{D}_{-\infty}(L^2([0, T]))$.

Hence $D_t G$ is well-defined if $G \in \mathbb{D}_{-\infty}$, and we need to interpret $D_t G$ in the sense of distributions. However, we already know $L^2(\Omega) \subset \mathbb{D}_{-\infty}$, and the space that we are interested in for the Clark-Ocone formula to hold is $L^2(\Omega)$ rather than all of $\mathbb{D}_{-\infty}$. If $G \in L^2(\Omega)$, we can understand the Clark-Ocone formula in the usual way, although we may have to understand $D_t G$ in the sense of distributions. The next proposition shows this.

Proposition 3.24. [See Lemma 5.1 in [5]] For $G \in L^2(\Omega)$, the map $G \mapsto E^Q[D_t G | \mathcal{F}_t]$ is continuous from $L^2(\Omega)$ to $L^2([0, T] \times \Omega)$, i.e., for some constant K_1 ,

$$\|E^Q[D_t G | \mathcal{F}_t]\|_{L^2(\Omega \times [0,T])} \leq K_1 \|G\|_{L^2(\Omega)}.$$

Also, the map $G \mapsto D_t(E^Q[G | \mathcal{F}_t])$ is continuous from $L^2(\Omega)$ to $L^2([0, T] \times \Omega)$, and

$$E^Q[D_t G | \mathcal{F}_t] = D_t(E^Q[G | \mathcal{F}_t]). \quad (3.28)$$

Proof. We follow [5] to give the proof. Notice that if this relation holds on a dense linear subspace of $L^2(\Omega)$ then it will hold on all of $L^2(\Omega)$, because one can extend it from the dense subspace to the whole space. So consider a dense subset of $L^2(\Omega)$, the linear span of the martingale exponentials

$$\left\{ \exp \left(\int_0^T h(s) dW(s) - \frac{1}{2} \int_0^T h^2(s) ds \right); \quad h \in L^2([0, T]) \right\}.$$

Let us denote by $\varphi_i(T) = \exp \left(\int_0^T h_i(s) dW(s) - \frac{1}{2} \int_0^T h_i^2(s) ds \right)$. Then the following three properties hold; see, for example [46]:

- a) $E^Q [\varphi_i(T) \varphi_j(T)] = \exp \left(\int_0^T h_i(s) h_j(s) ds \right)$,
- b) $E^Q [\varphi_i(T) | \mathcal{F}_t] = \varphi_i(t)$,
- c) $\varphi_i(T) \in \mathbb{D}_{1,2}$ with $D_t \varphi_i(T) = h_i(t) \varphi_i(T)$.

Now consider a real-valued linear combination of these martingale exponentials:

$$\Phi^n(T) = \sum_{i=1}^n c_i \varphi_i(T).$$

From a), we get

$$\|\Phi^n(T)\|_{L^2(\Omega)}^2 = E^Q \left[\left(\sum_{i=1}^n c_i \varphi_i(T) \right)^2 \right] = \sum_{i=1}^n \sum_{j=1}^n c_i c_j \exp \left(\int_0^T h_i(s) h_j(s) ds \right).$$

Moreover, from b) and c), we get

$$E^Q [D_t (\Phi^n(T)) | \mathcal{F}_t] = E^Q \left[\sum_{i=1}^n c_i h_i(t) \varphi_i(T) | \mathcal{F}_t \right] = \sum_{i=1}^n c_i h_i(t) \varphi_i(t), \quad (3.29)$$

and

$$D_t (E^Q [\Phi^n(T) | \mathcal{F}_t]) = D_t \left(E^Q \left[\sum_{i=1}^n c_i \varphi_i(T) | \mathcal{F}_t \right] \right) = \sum_{i=1}^n c_i h_i(t) \varphi_i(t). \quad (3.30)$$

Hence, we have

$$E^Q [D_t \Phi^n(T) | \mathcal{F}_t] = D_t (E^Q [\Phi^n(T) | \mathcal{F}_t]).$$

Hence, inserting the results, we get

$$\begin{aligned} \|E^Q [D_t \Phi^n(T) | \mathcal{F}_t]\|_{L^2([0,T] \times \Omega)}^2 &= \left\| \sum_{i=1}^n c_i h_i(t) \varphi_i(t) \right\|_{L^2([0,T] \times \Omega)}^2 \\ &= \int_{[0,T] \times \Omega} \left(\sum_{i=1}^n c_i h_i(t) \varphi_i(t) \right)^2 d(Q \times \lambda). \end{aligned}$$

So by Fubini's theorem,

$$\|E^Q [D_t \Phi^n(T) | \mathcal{F}_t]\|_{L^2([0,T] \times \Omega)}^2 = \int_0^T \sum_{i=1}^n \sum_{j=1}^n c_i c_j h_i(t) h_j(t) \left(\int_{\Omega} \varphi_i(t) \varphi_j(t) dQ \right) dt.$$

By property a) again, we have

$$\begin{aligned} \|E^Q [D_t \Phi^n(T) | \mathcal{F}_t]\|_{L^2([0,T] \times \Omega)}^2 &= \int_0^T \sum_{i=1}^n \sum_{j=1}^n c_i c_j h_i(t) h_j(t) \exp \left(\int_0^t h_i(s) h_j(s) ds \right) dt \\ &= \sum_{i=1}^n \sum_{j=1}^n c_i c_j \left[\exp \left(\int_0^t h_i(s) h_j(s) ds \right) \right]_{t=0}^T \\ &= \|\Phi^n(T)\|_{L^2(\Omega)}^2 - \left(\sum_{i=1}^n c_i \right)^2, \end{aligned}$$

from which we finish the proof. $\square$

Remark 3.25. So, for $G \in L^2(\Omega)$, the Malliavin derivative, $D_t G$, might only make sense in terms of distributions, as a composition of a distribution with a stochastic variable, but the process $E^Q [D_t G | \mathcal{F}_t]$ is a stochastic process in $L^2(\Omega \times [0, T])$ in the usual sense and $\int_0^T E^Q [D_t G | \mathcal{F}_t] dW_t$ is an ordinary Itô integral.

Remark 3.26. Note that for given a sequence $G_n \rightarrow G$ in $L^2(\Omega)$ and $D_t G$ exists, then we have

$$E^Q [D_t G_n | \mathcal{F}_t] \rightarrow E^Q [D_t G | \mathcal{F}_t] \text{ in } L^2([0, T] \times \Omega). \quad (3.31)$$

We can use this proposition to calculate $E^Q [D_t G | \mathcal{F}_t]$ by approximation, as we will show in Chapter 5.

Hence the Clark-Ocone formula actually make sense in the usual way when restricted

to $L^2(\Omega)$. In Chapter V part 1, Theorem 1 of [46], Üstünel describes the extension of the Clark-Ocone theorem to $\mathbb{D}_{-\infty}$. Since we are only interested in $L^2(\Omega)$, we finish our discussion of the Clark-Ocone theorem here.

Proposition 3.27. *For $G \in L^2(\Omega)$, we have the representation formula*

$$G = E^Q[G] + \int_0^T E^Q[D_t G | \mathcal{F}_t] dW_t. \quad (3.32)$$

The replicating portfolio h of any contingent claim G in $L^2(\Omega)$, is represented as follows

Proposition 3.28. *[See Theorem 5.2 in [5]] Any contingent claim $G \in L^2(\Omega)$ can be replicated by the self-financing portfolio $h = (h^0, h^1)$ defined by*

$$h_t^0 = e^{-rt} (V_t^h - h_t^1 S_t), \quad (3.33)$$

$$h_t^1 = e^{-r(T-t)} \sigma^{-1} S_t^{-1} E^Q[D_t G | \mathcal{F}_t], \quad (3.34)$$

where h_t^0 is the number of units to be held at time t in the locally risk-free asset R_t , and h_t^1 is the number of units to be held in the stock S_t .

Proof. We offer the following proof. First recall that the discounted value process V^h/R_t is a Q -local martingale, and

$$\begin{aligned} d(e^{-rt} V_t^h) &= e^{-rt} dV_t^h - r V_t^h e^{-rt} dt \\ &= (e^{-rt} h_t^0 r R_t + e^{-rt} h_t^1 r S_t - r V_t^h e^{-rt}) dt + e^{-rt} h_t^1 \sigma S_t dW_t \\ &= e^{-rt} r (h_t^0 R_t + h_t^1 S_t - V_t^h) dt + e^{-rt} h_t^1 \sigma S_t dW_t \\ &= e^{-rt} h_t^1 \sigma S_t dW_t. \end{aligned}$$

Thus, by using the Itô formula and the Girsanov kernel for Radon-Nikodym derivative, we have

$$V_T^h = e^{rT} V_0^h + \int_0^T e^{r(T-t)} h_t^1 \sigma S_t dW_t. \quad (3.35)$$

We can identify the coefficients in the Clark-Ocone formula (3.32) and equation (3.35). We see that the initial amount of money required to replicate the contingent claim G by the

self-financing portfolio h is given by

$$V_0^h = e^{-rT} E^Q[G], \quad (3.36)$$

and the number of units to be held at time t in the stock S by

$$h_t^1 = e^{-r(T-t)} \sigma^{-1} S_t^{-1} E^Q[D_t G | \mathcal{F}_t].$$

So the number of units to be held at time t in the locally risk-free asset R_t is given by

$$h_t^0 = e^{-rt} (V_t^h - h_t^1 S_t).$$

□

3.3 Generalization of Conditional Expectation of Distributions

In this section, we will develop our way of generalizing the conditional expectation of a distribution composed with a stochastic variable. Note that in the previous section, F is restricted by the conditions stated at the outset, say $F \in \mathbb{D}_\infty$. However, our treatment of this matter is elementary and direct and leads to a formula for, for example, $E^Q[T(S_T) | \mathcal{F}_t]$ where T is an element of $S'(\mathbb{R})$. It does not construct $T(S_T)$, and only suggests that it may exist independently of equation (3.15). Although it would be a pleasant piece of mathematics to demonstrate how our approach is consistent with Watanabe's, the thesis has another direction.

Definition 3.29. Let $X = (X_t)$ be an adapted L^1 process on $[0, T]$. For B a real number, random variable, $E^Q[1_{\{X_T \leq B\}} | \mathcal{F}_t]$ is a function of B . The function

$$F_{t, X_T}^Q(B, \omega) := E^Q[1_{\{X_T \leq B\}} | \mathcal{F}_t] \quad (3.37)$$

is called the $\mathcal{F}_t$ -conditional distribution function of X_T under the measure Q .

Remark 3.30. Sometimes, $F_{t, X_T}^Q(B, \omega)$ can be expressed as a deterministic function of B , X_t and other real variables, some of which are functions of ω .

Example 3.31. For our stock S following equation (1.18), it has the $\mathcal{F}_t$ -conditional distribution function under the risk-neutral measure Q and is given by equation (C.14). Note $F_{t,S_T}^Q(B, \omega)$ is a deterministic function of several real variables and one stochastic variable. We suppress the dependence on r, σ, T, t as these remain fixed in our context, and observe that $F_{t,S_T}^Q(B, \omega)$ is the composition of the smooth function

$$\varphi(B, x) = N \left(\frac{\ln \frac{B}{x} - \mu(T-t)}{\sigma \sqrt{T-t}} \right), \quad x > 0, \quad (3.38)$$

with (B, S_t) and μ is given by equation (C.2).

Example 3.32. For $0 \leq t \leq T$ and $0 < x = S_t \leq B$, the $\mathcal{F}_t$ -conditional distribution function of $M_{t,T}^S$ under the measure Q is given by equation (C.22). Note $F_{t,M_{t,T}^S}^{x,Q}(B, \omega)$ is a deterministic function of several real variables and one stochastic variable. Again we suppress the dependence on r, σ, T, t as these remain fixed in our context, and observe that $F_{t,M_{t,T}^S}^{x,Q}(B, \omega)$ is the composition of the smooth function

$$\psi(B, x) = N \left(\frac{\ln \frac{B}{x} - \mu(T-t)}{\sigma \sqrt{T-t}} \right) - e^{\frac{2\mu}{\sigma^2} \ln \frac{B}{x}} N \left(\frac{-\ln \left(\frac{B}{x} \right) - \mu(T-t)}{\sigma \sqrt{T-t}} \right), \quad 0 < x \leq B,$$

with (B, S_t) and μ is given by equation (C.2).

When $F_{t,X_T}^Q(B, \omega)$ is the composition of a smooth function of several real variables with B, X_t and, possibly, other real variables, say $F_{t,X_T}^Q(B, \omega) = \rho(B, X_t)$, $\rho \equiv \rho(y, x)$, clearly, then the process (X_t) is Markovian. The range of the variables, B, X_t may be restricted; for example, if (X_t) is a non-negative process, it makes no sense to have $B < 0$. Later when we consider $F_{t,M_{t,T}^S}^{x,Q}(B, \omega)$, the value of $S_t(\omega)$ occurs in the $\mathcal{F}_t$ -conditional density function and as $S_t(\omega) \leq M_{t,T}^S$ this provides a lower bound for the possible values of B . With these remarks in mind, we define

$$\frac{\partial}{\partial B} F_{t,X_T}^Q(B, \omega) \triangleq \frac{\partial}{\partial y} \rho(B, X_t) \triangleq f_{t,X_T}^Q(B, \omega). \quad (3.39)$$

We call $f_{t,X_T}^Q(B, \omega)$ the $\mathcal{F}_t$ -conditional density function of X_T , and note that

$$E^Q [1_{\{X_T \leq B\}} | \mathcal{F}_t] = F_{t,X_T}^Q(B, \omega). \quad (3.40)$$

If X_T is Q -a.s. strictly positive, we will write

$$F_{t,X_T}^Q(B, \omega) = \int_0^B f_{t,X_T}^Q(b, \omega) db. \quad (3.41)$$

Remark 3.33. Since $f_{t,X_T}^Q(b, \omega)$ is the composition of a deterministic function with X_t and B and is smooth, then as a function of $\omega \in \Omega$, f_{t,X_T}^Q is $\mathcal{F}_t$ -measurable. In fact, for a Borel set $\mathcal{B} \subseteq \mathbb{R}$, $(\frac{\partial \rho}{\partial y})^{-1}(\mathcal{B})$ is a Borel set and $X_t^{-1}((\frac{\partial \rho}{\partial y})^{-1}(\mathcal{B}))$ is in $\mathcal{F}_t$.

For now we are interested in $f_{t,X_T}^Q(b, \omega)$ as a function of the real parameter, b . Moreover, $b \mapsto f_{t,X_T}^Q(b, \omega)$ is a continuous function. In view of the relation

$$E^Q [1_{\{X_T \leq B\}} | \mathcal{F}_t] = \int_{-\infty}^B f_{t,X_T}^Q(b, \omega) db, \quad (3.42)$$

$b \mapsto f_{t,X_T}^Q(b, \omega)$ is non-negative Q -a.s.. Moreover, $0 \leq E^Q [1_{\{X_T \leq B\}} | \mathcal{F}_t] \leq 1_\Omega$, so that

$$\int_{-\infty}^{\infty} f_{t,X_T}^Q(b, \omega) db = \lim_{B \rightarrow \infty} \int_{-\infty}^B f_{t,X_T}^Q(b, \omega) db = 1. \quad (3.43)$$

Hence, $f_{t,X_T}^Q(b, \omega)$ gives us a probability measure on $\mathbb{R}$, Q -a.s.. However, its null sets can be non-null Lebesgue sets. Now let us construct $E^Q[X_T | \mathcal{F}_t]$ from $E^Q [1_{\{X_T \leq B\}} | \mathcal{F}_t]$, and $B \in \mathbb{R}$, by the next theorem.

Theorem 3.34. Let $X_T \in L^1(\Omega)$ be as above. Then

$$E^Q[X_T | \mathcal{F}_t] = \int_{-\infty}^{\infty} b f_{t,X_T}^Q(b, \omega) db, \quad (3.44)$$

where $f_{t,X_T}^Q(b, \omega)$ is the $\mathcal{F}_t$ -conditional density function of X_T .

Proof. Let us define X_T^n by

$$X_T^n = \begin{cases} \sum_{i=1}^{n2^{n+1}} \left(-n + \frac{i}{2^n}\right) 1_{[-n+\frac{i-1}{2^n}, -n+\frac{i}{2^n})} (X_T) & \text{if } |X_T| \leq n, \\ n & \text{if } X_T \geq n, \\ -n & \text{if } X_T \leq -n. \end{cases}$$

Then $X_T^n \rightarrow X_T$ Q -a.s., and since $X_T \in L^1(\Omega)$, then by dominated convergence theorem we have $X_T^n \rightarrow X_T$ in $L^1(\Omega)$. So, $E^Q[X_T^n | \mathcal{F}_t] \rightarrow E^Q[X_T | \mathcal{F}_t]$ in L^1 norm also. Some subsequence will converge Q -a.s.. Moving to that subsequence (but not changing our notation), we have

$$\begin{aligned} E^Q[X_T^n | \mathcal{F}_t] &= E^Q \left[\sum_{i=1}^{n2^{n+1}} \left(-n + \frac{i}{2^n}\right) 1_{[-n+\frac{i-1}{2^n}, -n+\frac{i}{2^n})} (X_T) | \mathcal{F}_t \right] \\ &\quad + E^Q [n 1_{\{X_T \geq n\}} | \mathcal{F}_t] + E^Q [(-n) 1_{\{X_T < -n\}} | \mathcal{F}_t] \\ &= \sum_{i=1}^{n2^{n+1}} \int_{-n+\frac{i-1}{2^n}}^{-n+\frac{i}{2^n}} \left(-n + \frac{i}{2^n}\right) f_{t, X_T}^Q(b, \omega) db \\ &\quad + \int_n^\infty (n) f_{t, X_T}^Q(b, \omega) db + \int_{-\infty}^{-n} (-n) f_{t, X_T}^Q(b, \omega) db \\ &\rightarrow \int_{-\infty}^\infty b f_{t, X_T}^Q(b, \omega) db \quad \text{as } n \rightarrow \infty, Q - \text{a.s..} \end{aligned}$$

□

This result is a special case of the next theorem.

Theorem 3.35. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a Lebesgue measurable function and let X_T be as above. If g is such that $g(X_T)$ belongs to $L^1(\Omega)$ and $g(b)f_{t, X_T}^Q(b, \omega)$ is Lebesgue integrable for Q -almost everywhere ω , then

$$E^Q[g(X_T) | \mathcal{F}_t] = \int_{-\infty}^\infty g(b) f_{t, X_T}^Q(b, \omega) db, \quad (3.45)$$

where $f_{t, X_T}^Q(B, \omega)$ is the $\mathcal{F}_t$ -conditional density of X_T .

Proof. Rewrite the proof of the previous theorem with $g(X_T^n)$ replacing X_T^n . We consider integrability of $g(b)f_{t, X_T}^Q(b, \omega)$ in explicit cases later. □

One can regard $f_{t,X_T}^Q(b, \omega)$ as a family of measures indexed by ω . There are some details, of how $f_{t,X_T}^Q(b, \omega)$ behaves as the real parameters which depend upon ω , of which $X_t(\omega)$ is one, approach the boundary of their values, which may need consideration. There is a discussion of this for the case $X_T = S_T$ in Section 3.3.1. One consequence of this theorem is that it allows us to define $E^Q[T(X_T)|\mathcal{F}_t]$ when $T \in S'(\mathbb{R})$ is a distribution. Let (φ_n) be a sequence of test function with defining T , then

$$E^Q[\varphi_n(X_T)|\mathcal{F}_t] = \int_{-\infty}^{\infty} \varphi_n(b) f_{t,X_T}^Q(b, \omega) db.$$

If, for example, $f_{t,X_T}^Q(b, \omega)$ is Q -a.s. a test function, then

$$\begin{aligned} E^Q[\varphi_n(X_T)|\mathcal{F}_t] &= \int_{-\infty}^{\infty} \varphi_n(b) f_{t,X_T}^Q(b, \omega) db \\ &\rightarrow \int_{-\infty}^{\infty} T(b) f_{t,X_T}^Q(b, \omega) db \\ &\triangleq E^Q[T(X_T)|\mathcal{F}_t]. \end{aligned}$$

We shall see that this is the case with $X_T = S_T$. Even if $f_{t,X_T}^Q(b, \omega)$ is *not* a test function—as is the case when $X_T = M_{t,T}^S$ —one can still compose a distribution with $M_{t,T}^S$. See Section 3.3.2 for detail.

Example 3.36. For $\delta_B(\cdot)$ the Dirac delta function at B ,

$$E^Q[\delta_B(X_T)|\mathcal{F}_t] = \int_{-\infty}^{\infty} \delta_B(b) f_{t,X_T}^Q(b, \omega) db = f_{t,X_T}^Q(B, \omega).$$

Notice that this is a function of $\omega \in \Omega$.

3.3.1 Conditional Expectation of the Composition of a Distribution with S_T

In this section, we continue to consider the conditional expectation of a composition of distribution with a random variable. Here, let us consider the case that $X_T = S_T$.

Definition 3.37. Let f_{t,S_T}^Q denote the $\mathcal{F}_t$ -conditional density function of S_T , corresponding

to the distribution function $F_{t,S_T}^Q(s, \omega)$. Note that there is a generalization of the usual idea of a density and we can think of $f_{t,S_T}^Q(s, \omega)$ as “the $\mathcal{F}_t$ -conditional probability that $S_T = s$ ”, i.e.,

$$f_{t,S_T}^Q(s, \omega) := \frac{\partial}{\partial s} F_{t,S_T}^Q(s, \omega) \quad (3.46)$$

and is given by equation (C.15). We extend $f_{t,S_T}^Q(s, \omega)$ to all of $\mathbb{R}$ by defining it to be zero for $s < 0$.

Lemma 3.38. For Q -a.s. $\omega \in \Omega$, $f_{t,S_T}^Q(s, \omega)$, given by equation (C.15), is a test function.

Proof. We must proof that $f_{t,S_T}^Q(s, \omega)$ is infinitely differentiable at zero and rapidly decreasing as $|s| \rightarrow \infty$. That is, to show that $f_{t,S_T}^Q(s, \omega)$ and its derivatives with respect to the variable s decay faster than any polynomial as $|s| \rightarrow \infty$. Since we define $f_{t,S_T}^Q(s, \omega) = 0$ for $s < 0$, we need to make sure $f_{t,S_T}^Q(s, \omega)$ is infinitely differentiable at 0, i.e. as $s \rightarrow 0$. Let $A = 1/(\sigma\sqrt{T-t})$, $C = -A(\ln x + \mu(T-t))$ and take $D = 1/(\sqrt{2\pi}\sigma\sqrt{T-t})$. In our calculation we can with loss of generality let $D = 1$. Then we simplify $f_{t,S_T}^Q(s, \omega)$ as

$$f_{t,S_T}^Q(s, \omega) = s^{-1} e^{-\frac{1}{2}(A \ln s + C)^2}. \quad (3.47)$$

Then take the first three orders of partial derivatives with respect to s , and we get

$$\begin{aligned} \frac{\partial f_{t,S_T}^Q(s, \omega)}{\partial s} &= s^{-2} e^{-\frac{1}{2}(A \ln s + C)^2} \{-A(A \ln s + C) - 1\}, \\ \frac{\partial^2 f_{t,S_T}^Q(s, \omega)}{\partial^2 s} &= s^{-3} e^{-\frac{1}{2}(A \ln s + C)^2} \cdot \\ &\quad \{3(A \ln s + C)A + (A \ln s + C)^2 A^2 - A^2 + 2\} \\ \frac{\partial^3 f_{t,S_T}^Q(s, \omega)}{\partial^3 s} &= s^{-4} e^{-\frac{1}{2}(A \ln s + C)^2} \cdot \\ &\quad \{-A^3(A \ln s + C)^2 + (2A^2 - 6A^2)(A \ln s + C)^2 \\ &\quad + (A^3 - 11A)(A \ln s + C) + 6A^2 - 6\} \\ &\quad \vdots \end{aligned}$$

By induction, we find the partial derivatives have the general form

$$\frac{\partial^n f_{t,S_T}^Q(s, \omega)}{\partial^n s} = s^{-(n+1)} e^{-\frac{1}{2}(A \ln s + C)^2} h((A \ln s + C)), \quad (3.48)$$

where $h(\cdot)$ is a polynomial. Then for any $m \in \mathbb{N}$, assume the limit

$$\lim_{s \rightarrow \infty} s^m \frac{\partial^n f_{t,S_T}^Q(s, \omega)}{\partial^n s} = 0 \quad \text{and} \quad \lim_{s \rightarrow 0} s^m \frac{\partial^n f_{t,S_T}^Q(s, \omega)}{\partial^n s} = 0.$$

Take the $(n+1)$ -th partial derivative:

$$\frac{\partial^{n+1} f_{t,S_T}^Q(s, \omega)}{\partial^{n+1} s} = s^{-(n+2)} e^{-\frac{1}{2}(A \ln s + C)^2} g((A \ln s + C)), \quad (3.49)$$

where $g((A \ln s + C)) = A \frac{\partial h((A \ln s + C))}{\partial s} - (A(A \ln s + C) + (n+1))h((A \ln s + C))$ is still a polynomial function. Then take the limit:

$$\lim_{s \rightarrow \infty} s^m \frac{\partial^{n+1} f_{t,S_T}^Q(s, \omega)}{\partial^{n+1} s} = 0 \quad \text{and} \quad \lim_{s \rightarrow 0} s^m \frac{\partial^{n+1} f_{t,S_T}^Q(s, \omega)}{\partial^{n+1} s} = 0,$$

which complete the proof by induction. $\square$

The next theorem shows us that for $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ a Borel measurable function, integrals

$$\int_0^\infty g(s) f_{t,S_T}^Q(s, \omega) ds$$

will exist for a class of functions *larger* than $L^2(\mathbb{R}^+)$.

Theorem 3.39. *Let $L^1([0, \infty), \bar{\mathcal{B}}, \lambda)$ be Lebesgue measure space. Consider a space*

$$L^1([0, \infty), \bar{\mathcal{B}}, \mu_x),$$

where $\mu_x(E) = \int_E f_x(s) ds$, for $E \in \bar{\mathcal{B}}$ and

$$f_x(s) = \frac{1}{s\sqrt{2\pi\sigma}\sqrt{T-t}} \exp \left\{ -\frac{1}{2} \left(\frac{\ln(s/x) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \right)^2 \right\}.$$

Then we have

$$L^1([0, \infty), \overline{\mathcal{B}}, \lambda) \subset L^1([0, \infty), \overline{\mathcal{B}}, \mu_x).$$

Proof. By Lemma 3.38, we know $\lim_{s \rightarrow \infty} f_x(s) = 0$ and $\lim_{s \rightarrow 0} f_x(s) = 0$. Besides, $f_x(s)$ is differentiable and has a unique maximum at $s^* = x \exp \{(r - \frac{3}{2}\sigma^2)(T - t)\}$ such that

$$f_x(s^*) = \frac{1}{x\sqrt{2\pi\sigma}\sqrt{T-t}} \exp \{-(r - \sigma^2)(T - t)\}.$$

Hence $f_x(s) \in L^\infty([0, \infty), \overline{\mathcal{B}}, \lambda)$.

For $X \in L^1([0, \infty), \overline{\mathcal{B}}, \lambda)$, of course $\int_{\mathbb{R}^+} X d\lambda < \infty$, we want to show that $X \in L^1([0, \infty), \overline{\mathcal{B}}, \mu_x)$. Since X is $\overline{\mathcal{B}}$ -measurable we only need to show that $\int_{\mathbb{R}^+} X d\mu_x < \infty$. We already know that for a simple function, g , we have

$$\int_{\mathbb{R}^+} g(s) d\mu_x = \int_{\mathbb{R}^+} g(s) f_x(s) ds.$$

This formula extends to every element of $L^1([0, \infty), \overline{\mathcal{B}}, \mu_x)$, so X is μ_x integrable if and only if $X f_x$ is λ -integrable. It follows that $X f_x \in L^1([0, \infty), \overline{\mathcal{B}}, \lambda)$. This is because

$$|X f_x| \leq \|f_x(s)\|_\infty |X|.$$

Hence, $X \in L^1([0, \infty), \overline{\mathcal{B}}, \mu_x)$. □

Corollary 3.40. For each $x > 0$, we have

$$\bigcup_{p=1}^{\infty} L^p([0, \infty), \overline{\mathcal{B}}, \lambda) \subseteq L^1([0, \infty), \overline{\mathcal{B}}, \mu_x).$$

Proof. We have $Y \in L^p(\lambda) \Leftrightarrow |Y|^p \in L^1(\lambda)$, then $|Y|^p \in L^1(\mu_x) \Rightarrow Y \in L^p(\mu_x)$ which is the subset of $L^1(\mu_x)$. □

Remark 3.41. For each x , $L^1([0, \infty), \overline{\mathcal{B}}, \mu_x)$ contains $S(\mathbb{R})$, the test functions.

By the theorem above, we can define

$$E^Q[g(S_T)|\mathcal{F}_t] := \int_0^\infty g(s) f_{t,S_T}^Q(s, \omega) ds \quad (3.50)$$

for a large class of g 's for which $g(S_T) \in L^1(\Omega)$. Note that while the right-hand side of the above equation exists for a large class of functions, g , we must restrict these, so that $g(S_T)$ lies in L^1 , otherwise $E^Q[g(S_T)]$ may not be defined.

Example 3.42. Take the approach to the Dirac delta function that it is obtained as the limit of a sequence of “smooth spikes”, $\varphi_n(\cdot)$, converging to $\delta_B(\cdot)$ in the sense of distribution. Then we have

$$\begin{aligned} E^Q [\varphi_n(S_T) | \mathcal{F}_t] &= \int_0^\infty \varphi_n(s) f_{t,S_T}^Q(s, \omega) ds \\ &\rightarrow \int_0^\infty \delta_B(s) f_{t,S_T}^Q(s, \omega) ds = f_{t,S_T}^Q(B, \omega) \\ &\triangleq E^Q [\delta_B(S_T) | \mathcal{F}_t]. \end{aligned}$$

So we have shown that $\int_0^\infty \delta_B(s) f_{t,S_T}^Q(s, \omega) ds$ makes perfectly good sense, and we have given a complete treatment of the example in [4].

The next few lemmas compute various Malliavin derivatives under the conditional expectation. We do this directly, by approximation. We are aware of the treatment of distribution in [44], in particular Theorem 6.16, which shows that any continuous function can be approximated by C^∞ functions (infinitely differentiable function) uniformly on compact sets, and its distributional derivatives too. But we prefer to get our results by elementary methods.

Lemma 3.43. For $G = 1_{\{S_T < B\}} \in L^2(\Omega)$, we have

$$E^Q [D_t 1_{\{S_T < B\}} | \mathcal{F}_t] = -E^Q [\delta_B(S_T) D_t S_T | \mathcal{F}_t] \quad Q - a.s.. \quad (3.51)$$

Proof. Let $\Psi_{0,B,n}$ be the functions defined in equation (B.1) and denote $T_{\Psi_{0,B,n}}$ by the distributional derivative of $\Psi_{0,B,n}$. By Lemma B.1, we know

$$\lim_{n \rightarrow \infty} T_{\Psi_{0,B,n}}(\cdot) = \delta_0(\cdot) - \delta_B(\cdot)$$

in the sense of distribution. For $S_T \in \mathbb{D}_{1,2}$, we have

$$\Psi_{0,B,n}(S_T) \rightarrow 1_{[0,B]}(S_T), \quad Q - a.s..$$

By Proposition 2.29, we know $\Psi_{0,B,n}(S_T) \in \mathbb{D}_{1,2} \subset L^2(\Omega)$. Note that $\|\Psi_{0,B,n}(S_T)\|_2 \leq 1$, for all n . So $\{\Psi_{0,B,n}^2(S_T) : n \in \mathbb{N}\}$ is bounded in $\|\cdot\|_2$ and therefore a uniformly integrable set. The Vitali convergence theorem (see III. 6.15 of [19]) tells us that

$$\Psi_{0,B,n}(S_T) \rightarrow 1_{[0,B]}(S_T) \quad \text{in } L^2(\Omega).$$

Consequently, by Remark 3.26

$$E^Q[D_t \Psi_{0,B,n}(S_T) | \mathcal{F}_t] \rightarrow E^Q[D_t 1_{\{S_T < B\}} | \mathcal{F}_t]. \quad (3.52)$$

By Proposition 2.29, we have

$$E^Q[D_t \Psi_{0,B,n}(S_T) | \mathcal{F}_t] = E^Q[\Psi'_{0,B,n}(S_T) D_t S_T | \mathcal{F}_t].$$

Since $\Psi_{0,B,n}$ is differentiable, by Proposition 1.23 we know $\Psi'_{0,B,n} = T_{\Psi_{0,B,n}}$. Moreover, by Lemma 3.38, we know f_{t,S_T}^Q is a test function and we have

$$\begin{aligned} E^Q[\Psi'_{0,B,n}(S_T) D_t S_T | \mathcal{F}_t] &= \int_0^\infty \Psi'_{0,B,n}(s) \sigma s f_{t,S_T}^Q(s, \omega) ds \\ &= \int_0^\infty T_{\Psi_{0,B,n}}(s) \sigma s f_{t,S_T}^Q(s, \omega) ds \\ &\rightarrow - \int_0^\infty \delta_B(s) \sigma s f_{t,S_T}^Q(s, \omega) ds \quad Q - a.s. \\ &= -E^Q[\delta_B(S_T) D_t S_T | \mathcal{F}_t], \end{aligned} \quad (3.53)$$

in the sense of distribution. Note that we use the fact that $S_T > 0$, Q -a.s., so $\delta_0(S_T) = 0$, Q -a.s. Therefore,

$$E^Q[D_t 1_{\{S_T < B\}} | \mathcal{F}_t] = -E^Q[\delta_B(S_T) D_t S_T | \mathcal{F}_t] \quad Q - a.s..$$

□

Lemma 3.44. For $G = (S_T - K)^+ \in L^2(\Omega)$, we have

$$E^Q[D_t(S_T - K)^+|\mathcal{F}_t] = E^Q[1_{[K,\infty)}(S_T)D_t S_T|\mathcal{F}_t]. \quad (3.54)$$

Proof. Let Φ_n be given by equation (B.8) and T_{Φ_n} denotes the distributional derivative of Φ_n . By Lemma B.2, we know

$$\lim_{n \rightarrow \infty} T_{\Phi_n}(\cdot) = 1_{[K,\infty)}(\cdot)$$

in the sense of distribution. Note the sequence (Φ_n) approximate $(x - K)^+$ pointwise and for each $n \in \mathbb{N}$, $(x - K)^+ \leq \Phi_n(x) \leq x^+$. For $S_T \in L^2(\Omega)$ a non-negative stochastic variable, we have

$$\Phi_n(S_T) \rightarrow (S_T - K)^+, \quad Q - a.s.$$

and $(S_T - K)^+ \leq \Phi_n(S_T) \leq S_T$, Q-a.s. Therefore, $\Phi_n^2(S_T) \leq S_T^2$, which means that $\{\Phi_n^2(S_T) : n \in \mathbb{N}\}$ is a uniformly integrable set. By Vitali Convergence Theorem (see III. 6.15 of [19]), we have

$$\Phi_n(S_T) \rightarrow (S_T - K)^+, \quad \text{in } L^2(\Omega).$$

Consequently, by Remark 3.26

$$E^Q[D_t \Phi_n(S_T)|\mathcal{F}_t] \rightarrow E^Q[D_t(S_T - K)^+|\mathcal{F}_t]. \quad (3.55)$$

By equation (B.12) we know the partial derivative of Φ_n is bounded, then by Proposition 2.29 we have

$$E^Q[D_t \Phi_n(S_T)|\mathcal{F}_t] = E^Q[\Phi'_n(S_T)D_t S_T|\mathcal{F}_t].$$

Since Φ_n is differentiable, by Proposition 1.23 we know $T_{\Phi_n} = \Phi'_n$. Moreover, by Lemma

3.38, we know f_{t,S_T}^Q is a test function and we have

$$\begin{aligned} E^Q[\Phi_n'(S_T)D_t S_T | \mathcal{F}_t] &= \int_0^\infty T_{\Phi_n}(s) \sigma s f_{t,S_T}^Q(s, \omega) 1_{\{t \leq T\}} ds \\ &\rightarrow \int_0^\infty 1_{[K, \infty)}(s) \sigma s f_{t,S_T}^Q(s, \omega) 1_{\{t \leq T\}} ds \\ &= E^Q[1_{[K, \infty)}(S_T) D_t S_T | \mathcal{F}_t], \end{aligned} \quad (3.56)$$

in the sense of distribution. Therefore,

$$E^Q[D_t(S_T - K)^+ | \mathcal{F}_t] = E^Q[1_{[K, \infty)}(S_T) D_t S_T | \mathcal{F}_t].$$

□

Lemma 3.45. For $G = (S_T - K)^+ 1_{\{S_T \leq B\}} \in L^2(\Omega)$ with $B > K$, we have

$$\begin{aligned} &E^Q[D_t((S_T - K)^+ 1_{\{S_T \leq B\}}) | \mathcal{F}_t] \\ &= E^Q[1_{\{K < S_T \leq B\}} D_t S_T | \mathcal{F}_t] - E^Q[(S_T - K)^+ \delta_B(S_T) D_t S_T | \mathcal{F}_t] \quad Q - a.s.. \end{aligned} \quad (3.57)$$

Proof. Let Φ_n and $\Psi_{0,B,n}$ be the functions defined in equation (B.8) and equation (B.1). For $S_T \in \mathbb{D}_{1,2}$, we have

$$\begin{aligned} \Phi_n(S_T) &\rightarrow (S_T - K)^+ \quad Q - a.s., \\ \Psi_{0,B,n}(S_T) &\rightarrow 1_{[0,B]}(S_T) \quad Q - a.s.. \end{aligned}$$

Therefore,

$$\Phi_n(S_T) \Psi_{0,B,n}(S_T) \rightarrow (S_T - K)^+ 1_{[0,B]}(S_T), \quad Q - a.s..$$

Note that $0 \leq \Phi_n(S_T) \Psi_{0,B,n}(S_T) \leq S_T$, for all n . So $\{\Phi_n^2(S_T) \Psi_{0,B,n}^2(S_T) : n \in \mathbb{N}\}$ is a uniformly integrable set. Then Vitali convergence theorem (see III. 6.15 of [19]) tells us that

$$\Phi_n(S_T) \Psi_{0,B,n}(S_T) \rightarrow (S_T - K)^+ 1_{[0,B]}(S_T), \quad \text{in } L^2(\Omega). \quad (3.58)$$

Consequently, by Remark 3.26 we have

$$E^Q [D_t (\Phi_n(S_T) \Psi_{0,B,n}(S_T))] \rightarrow E^Q [D_t ((S_T - K)^+ 1_{[0,B]}(S_T)) | \mathcal{F}_t]. \quad (3.59)$$

Note that $\Phi_n(x) \Psi_{0,B,n}(x)$ is a continuously differentiable function and $|\Phi'_n(x) \Psi_{0,B,n}(x)| \leq |\Phi'_n(x)| < 1$ and $|\Phi_n(x) \Psi'_{0,B,n}(x)| < M_n$ for some constant M_n for each n . So by Proposition 2.29, we have

$$D_t (\Phi_n(S_T) \Psi_{0,B,n}(S_T)) = \Phi_n(S_T) \Psi'_{0,B,n}(S_T) D_t S_T + \Psi_{0,B,n}(S_T) \Phi'_n(S_T) D_t S_T.$$

Then taking the conditional expectation of the first term in the above equation, we have

$$\begin{aligned} & E^Q [\Psi_{0,B,n}(S_T) \Phi'_n(S_T) D_t S_T | \mathcal{F}_t] \\ &= \int_0^\infty \Psi_{0,B,n}(s) T_{\Phi_n}(s) \sigma s f_{t,S_T}^Q(s, \omega) ds \\ &\rightarrow \int_0^\infty 1_{[0,B]}(s) 1_{[K,\infty)}(s) \sigma s f_{t,S_T}^Q(s, \omega) ds \quad Q - a.s. \\ &= E^Q [1_{[K,B]}(S_T) D_t S_T | \mathcal{F}_t] \quad Q - a.s., \end{aligned} \quad (3.60)$$

in the sense of distribution and

$$\begin{aligned} & E^Q [\Phi_n(S_T) T_{\Psi_{0,B,n}}(S_T) D_t S_T | \mathcal{F}_t] \\ &= \int_0^\infty \Phi_n(s) T_{\Psi_{0,B,n}}(s) \sigma s f_{t,S_T}^Q(s, \omega) ds \\ &\rightarrow - \int_0^\infty (s - K)^+ \delta_B(s) \sigma s f_{t,S_T}^Q(s, \omega) ds \quad Q - a.s. \\ &= -E^Q [(S_T - K)^+ \delta_B(S_T) D_t S_T | \mathcal{F}_t] \quad Q - a.s. \end{aligned} \quad (3.61)$$

in the sense of distribution. Therefore,

$$\begin{aligned} & E^Q [D_t ((S_T - K)^+ 1_{\{S_T < B\}}) | \mathcal{F}_t] \\ &= E^Q [1_{\{K < S_T \leq B\}} D_t S_T | \mathcal{F}_t] - E^Q [(S_T - K)^+ \delta_B(S_T) D_t S_T | \mathcal{F}_t] \quad Q - a.s.. \end{aligned}$$

□

3.3.2 Conditional Expectation of the Composition of a Distribution with $M_{t,T}^S$

In this section, we continue to consider the conditional expectation of a composition of distribution with a random variable. Here, let us consider the case that $X_T = M_{t,T}^S$.

Definition 3.46. Let $f_{t,M_{t,T}^S}^Q$ denote the $\mathcal{F}_t$ -conditional density function of $M_{t,T}^S$, corresponding to the distribution function $F_{t,M_{t,T}^S}^Q(m, \omega)$. We can think of $f_{t,M_{t,T}^S}^Q(m, \omega)$ as the $\mathcal{F}_t$ conditional probability that $M_{t,T}^S = m$, i.e.

$$f_{t,M_{t,T}^S}^Q(m, \omega) := \frac{\partial}{\partial m} F_{t,M_{t,T}^S}^Q(m, \omega).$$

Note that for each fixed $S_t(\omega) = x > 0$, $f_{t,M_{t,T}^S}^Q(m, \omega)$ is a continuous function on $[x, \infty)$ with

$$\lim_{m \rightarrow \infty} f_{t,M_{t,T}^S}^Q(m, \omega) = 0. \quad (3.62)$$

Also notice that we cannot have $x > m$, because $x = S_t = M_{t,t}^S \leq M_{t,T}^S$. The time t -conditional distribution function $Q(M_{t,T}^S \leq m | \mathcal{F}_t)$ only makes sense for $x \leq m$, the probability that $\{M_{t,T}^S < S_t\}$ is zero. So the time t -conditional distribution function only works for certain values. The time t conditional density function only exists for the same range of values as the time t -conditional distribution function does. The time t -conditional density is actually

$$1_{[x, \infty)}(m) f_{t,M_{t,T}^S}^Q(m, \omega) := f_{t,M_{t,T}^S}^{x,Q}(m, \omega). \quad (3.63)$$

where $f_{t,M_{t,T}^S}^{x,Q}(m, \omega)$ is given by equation (C.23). Notice that as $m \rightarrow x$, we have

$$f_{t,M_{t,T}^S}^{x,Q}(m, \omega) \rightarrow \frac{2}{x\sigma\sqrt{T-t}} n\left(\frac{-\mu(T-t)}{\sigma\sqrt{T-t}}\right) - \frac{2\mu}{x\sigma^2} N\left(\frac{-\mu(T-t)}{\sigma\sqrt{T-t}}\right), \quad (3.64)$$

where $n(\cdot)$ is the standard normal density function and given by equation (C.7) and $N(\cdot)$ is the standard normal cumulative distribution function and given by equation (C.6). Note that for each $x > 0$, we have that $f_{t,M_{t,T}^S}^{x,Q}(B, \omega)$ is infinitely differentiable at $m = x$. Then we multiply it by the indicator $1_{[x, \infty)}(m)$, which puts a jump in it if one regards it to exist for $m < x$, but a perfectly logical and consisted view is that the time t -conditional density

function is given by a smooth function of μ, σ, T, t, x, m , in the region $\{(m, x) : 0 < x \leq m\}$. However, it is still true that

$$0 \leq E^Q \left[1_{\{M_{t,T}^S \leq B\}} | \mathcal{F}_t \right] \leq 1_\Omega \quad Q - a.s. \quad (3.65)$$

and

$$E^Q \left[1_{\{M_{t,T}^S \leq B\}} | \mathcal{F}_t \right] \rightarrow 1_\Omega \quad \text{as } B \rightarrow \infty, \quad Q - a.s.. \quad (3.66)$$

But $1_{\{M_{t,T}^S \leq B\}} = 1_{\{S_t \leq M_{t,T}^S \leq B\}}$, so the “left end point is random”. For $B < S_t(\omega)$, the indicator is zero at that ω . However the time t -conditional density function still gives us a probability measure, but it is zero until x . As before, we can define integrals for a wide class of functions of $M_{t,T}^S$.

Remark 3.47. Let $\Psi_{0,B,n}$ be the functions defined in equation (B.1). Although by Lemma B.1, we know $T_{\Psi_{0,B,n}}(\cdot) \rightarrow \delta_0(\cdot) - \delta_B(\cdot)$ in the sense of distribution, we cannot immediately make our conclusion such as

$$\int_0^\infty T_{\Psi_{0,B,n}}(m) f_{t,M_{t,T}^S}^{x,Q}(m, \omega) dm \rightarrow \int_0^\infty \delta_B(m) f_{t,M_{t,T}^S}^{x,Q}(m, \omega) dm \quad Q - a.s.,$$

in the sense of distribution, because $f_{t,M_{t,T}^S}^{x,Q}(m, \omega)$ is not a test function. However, for each $x \neq B$, it is still true that

$$\int_0^\infty \Psi'_{0,B,n}(m) f_{t,M_{t,T}^S}^{x,Q}(m, \omega) dm \rightarrow \int_0^\infty \delta_B(m) f_{t,M_{t,T}^S}^{x,Q}(m, \omega) dm.$$

When $x = B$, we cannot make sense of the delta function at B . But this only occurs on Q -null set $\{S_t(\omega) = B\}$. So long as $x \neq B$, we take the limit at points of continuity of the integrand. With these in mind, we have the following lemmas.

Lemma 3.48. For $G = 1_{\{M_{0,T}^S < B\}} \in L^2(\Omega)$, we have

$$\begin{aligned} E^Q \left[D_t 1_{\{M_{0,T}^S < B\}} | \mathcal{F}_t \right] &= E^Q \left[D_t \left(1_{\{M_{0,t}^S < B\}} 1_{\{M_{t,T}^S < B\}} \right) | \mathcal{F}_t \right] \quad Q - a.s. \\ &= -E^Q \left[\delta_B(M_{0,T}^S) D_t M_{0,T}^S | \mathcal{F}_t \right] \quad Q - a.s. \end{aligned} \quad (3.67)$$

$$= -1_{\{M_{0,t}^S < B\}} \sigma B f_{t,M_{t,T}^S}^{x,Q}(B, \omega) \quad Q - a.s.. \quad (3.68)$$

Proof. Let $\Psi_{0,B,n}$ be the functions defined in equation (B.1). As the procedure in Lemma 3.43, we have

$$E^Q [D_t \Psi_{0,B,n} (M_{0,T}^S) | \mathcal{F}_t] = E^Q [\Psi'_{0,B,n} (M_{0,T}^S) D_t M_{0,T}^S | \mathcal{F}_t].$$

For $t \in [0, T]$, we have

$$\begin{aligned} & E^Q [\Psi'_{0,B,n} (M_{0,T}^S) D_t M_{0,T}^S | \mathcal{F}_t] \\ &= \int_0^\infty \Psi'_{0,B,n}(m) \sigma m 1_{\{M_{0,t}^S \leq m\}} f_{t,M_{t,T}^S}^{x,Q}(m, \omega) dm \\ &\rightarrow - \int_0^\infty \delta_B(m) \sigma m 1_{\{M_{0,t}^S \leq m\}} f_{t,M_{t,T}^S}^{x,Q}(m, \omega) dm \quad Q - a.s. \\ &= -1_{\{M_{0,t}^S < B\}} \sigma B f_{t,M_{t,T}^S}^{x,Q}(B, \omega) \quad Q - a.s. \\ &= -E^Q [\delta_B (M_{0,T}^S) D_t M_{0,T}^S | \mathcal{F}_t] \quad Q - a.s.. \end{aligned}$$

Therefore,

$$E^Q [D_t 1_{\{M_{0,T}^S < B\}} | \mathcal{F}_t] = -E^Q [\delta_B (M_{0,T}^S) D_t M_{0,T}^S | \mathcal{F}_t] \quad Q - a.s..$$

□

3.3.3 Conditional Expectation of the Composition of a Distribution with S_T and $M_{t,T}^S$

In this section, we consider the extension of our formula (3.45). We suppose that we have two non-negative, adapted assets X and Y . Define

$$E^Q [1_{\{X_T \leq s, Y_T \leq m\}} | \mathcal{F}_t] = H(s, m), \quad (3.69)$$

where H is some twice continuously differentiable function of s, m , with $x = X_t(\omega) > 0$ and $y = Y_t(\omega) > 0$. The parameters σ, r, T , etc will be deprecated. The mixed partial derivative;

$$\frac{\partial H(s, m)}{\partial s \partial m} = f_{t, X_T, Y_T}^{x, y, Q}(s, m, \omega) \quad (3.70)$$

is the time t -conditional density function of (X_T, Y_T) . Let $g(s, m)$ be a simple function of the form, $\sum_{i=1}^k \alpha_i 1_{R_i}(s, m)$, where $\alpha_i \in \mathbb{R}$ and $R_i = E_i \times F_i$ is bounded rectangle in $\mathbb{R}$. Then

$$E^Q[g(X_T, Y_T)|\mathcal{F}_t] = \int_{\mathbb{R}^+ \times \mathbb{R}^+} g(s, m) f_{t, X_T, Y_T}^{x, y, Q}(s, m, \omega) ds dm. \quad (3.71)$$

This extends to all $g : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ which are integrable with respect to every $f_{t, X_T, Y_T}^{x, y, Q}$ and for which $g(X_T, Y_T)$ is an integrable random variable. As before;

$$0 \leq E^Q [1_{\{X_T \leq s, Y_T \leq m\}} | \mathcal{F}_t] \leq 1_\Omega. \quad (3.72)$$

So that $f_{t, X_T, Y_T}^{x, y, Q}(s, m, \omega) \geq 0$ is continuous for s and m in the appropriate range of values, and

$$\int_{\mathbb{R}^+ \times \mathbb{R}^+} f_{t, X_T, Y_T}^{x, y, Q}(s, m, \omega) ds dm = 1 \quad Q - a.s.. \quad (3.73)$$

That is, it gives us a probability measure on $\mathbb{R}^+ \times \mathbb{R}^+$ almost surely. So the class of functions, g , we can compose with (X_T, Y_T) is very large. As before with $f_{t, M_t, T}^{x, Q}$, the region for which the expression $f_{t, X_T, Y_T}^{x, y, Q}(s, m, \omega)$ makes sense, and therefore equation (3.71) makes sense, needs to be determined for each pair (X_T, Y_T) .

Remark 3.49. Let Φ_n and $\Psi_{0, B, n}$ be the functions defined in equation (B.8) and equation (B.1). The function $\Psi_{0, B, n}(m) \Phi_n'(s) s \rightarrow 1_{[0, B]}(m) 1_{[K, \infty)}(s) s$, almost everywhere in (m, s) and $|\Psi_{0, B, n}(m) \Phi_n'(s) s| \leq s$ which is integrable with respect to the measure given by $f_{t, M_{t, T}^S, S_T}^{x, Q}(m, s, \omega)$. By the dominated convergence theorem, we have

$$\begin{aligned} & \int_0^\infty \int_0^m \Psi_{0, B, n}(m) \Phi_n'(s) s f_{t, M_{t, T}^S, S_T}^{x, Q}(m, s, \omega) ds dm \\ \rightarrow & \int_0^\infty \int_0^m 1_{[0, B]}(m) 1_{[K, \infty)}(s) s f_{t, M_{t, T}^S, S_T}^{x, Q}(m, s, \omega) ds dm \quad Q - a.s.. \end{aligned}$$

Let $\hat{\Phi}_n$ be the functions defined in equation (B.17). Now let us consider,

$$\lim_{n \rightarrow 0} \int_0^\infty \int_0^m \hat{\Phi}_n(s) \Psi_{0, B, n}'(m) \sigma m 1_{\{M_{0, t}^S \leq m\}} f_{t, M_{t, T}^S, S_T}^{x, Q}(m, s, \omega) ds dm.$$

First of all, $\Psi_{0, B, n}'(m)$ amounts to the symmetric pulse with unit integral on $[B, B + \frac{1}{n}]$,

also $\hat{\Phi}_n(s) \rightarrow (s - K)^+$ uniformly on $[0, B + 1]$. Indeed, for all large n , $\hat{\Phi}_n(s)$ only differs from $(s - K)^+$ on $[K - \frac{1}{n}, K + \frac{1}{n}]$, for $s \in [0, B + 1]$. For large enough n , and a fixed x , $f_{t, M_{t,T}^S, S_T}^{x, Q}(m, s, \omega)$ is a continuous function on the rectangle $[B, B + \frac{1}{n}] \times [K - \frac{1}{n}, K + \frac{1}{n}]$. This argument works for all $x \neq B$, and the set of ω such that $x = B$ is Q -null set. This allows us to prove that

$$\begin{aligned} & \int_0^\infty \int_0^m \hat{\Phi}_n(s) \Psi'_{0,B,n}(m) \sigma m 1_{\{M_{0,t}^S \leq m\}} f_{t, M_{t,T}^S, S_T}^{x, Q}(m, s, \omega) ds dm \\ \rightarrow & - \int_0^\infty \int_0^m (s - K)^+ \delta_B(m) \sigma m 1_{\{M_{0,t}^S \leq m\}} f_{t, M_{t,T}^S, S_T}^{x, Q}(m, s, \omega) ds dm \quad Q - a.s. \end{aligned}$$

With this remark in mind, we have the following lemma.

Lemma 3.50. For $G = (S_T - K)^+ 1_{\{M_{0,T}^S < B\}}$, we have

$$\begin{aligned} & E^Q \left[D_t \left((S_T - K)^+ 1_{\{M_{0,T}^S < B\}} \right) | \mathcal{F}_t \right] \\ = & 1_{\{M_{0,t}^S < B\}} E^Q \left[1_{[0,B]}(M_{t,T}^S) 1_{[K,\infty)}(S_T) D_t S_T | \mathcal{F}_t \right] \\ - & 1_{\{M_{0,t}^S < B\}} E^Q \left[(S_T - K)^+ \delta_B(M_{t,T}^S) D_t M_{t,T}^S | \mathcal{F}_t \right]. \quad Q - a.s.. \end{aligned} \tag{3.74}$$

Proof. Note that $(S_T - K)^+ 1_{\{M_{0,T}^S < B\}} = (S_T - K)^+ 1_{\{M_{0,t}^S < B\}} 1_{\{M_{t,T}^S < B\}}$ Q -a.s.. Let Φ_n and $\Psi_{0,B,n}$ be the functions defined in equation (B.8) and (B.1). We modify Φ_n here by making it as a bounded function. The reason for this is that the product $\Phi_n(x) \Psi_{0,B,n}(y) \Psi_{0,B,n}(z)$ has partial derivative with respect to y or z which are unbounded. Because Φ_n is unbounded and this prevents us from using the chain rule given by Proposition 2.29. So we define $\hat{\Phi}_n$ be given by equation (B.17). Notice that $\hat{\Phi}_n(x) \rightarrow (x - K)^+$ for every x . For $S_T \in \mathbb{D}_{1,2}$ we have $\hat{\Phi}_n(S_T) \rightarrow (S_T - K)^+$, Q -a.s.. For $M_{0,t}^S, M_{t,T}^S \in \mathbb{D}_{1,2}$, we have $\Psi_{0,B,n}(M_{0,t}^S) \rightarrow 1_{[0,B]}(M_{0,t}^S)$, Q -a.s. and $\Psi_{0,B,n}(M_{t,T}^S) \rightarrow 1_{[0,B]}(M_{t,T}^S)$, Q -a.s.. Therefore,

$$\hat{\Phi}_n(S_T) \Psi_{0,B,n}(M_{0,t}^S) \Psi_{0,B,n}(M_{t,T}^S) \rightarrow (S_T - K)^+ 1_{[0,B]}(M_{0,t}^S) 1_{[0,B]}(M_{t,T}^S), \quad Q - a.s..$$

Note that $0 \leq \hat{\Phi}_n(S_T) \Psi_{0,B,n}(M_{0,t}^S) \Psi_{0,B,n}(M_{t,T}^S) \leq S_T$, for all n . Therefore, we have that $\left\{ \hat{\Phi}_n^2(S_T) \Psi_{0,B,n}^2(M_{0,t}^S) \Psi_{0,B,n}^2(M_{t,T}^S) : n \in \mathbb{N} \right\}$ is a uniformly integrable set. Then Vitali

convergence theorem (see III. 6.15 of [19]) tells us that

$$\begin{aligned} & \hat{\Phi}_n(S_T) \Psi_{0,B,n}(M_{0,t}^S) \Psi_{0,B,n}(M_{t,T}^S) \\ \rightarrow & (S_T - K)^+ 1_{[0,B]}(M_{0,t}^S) 1_{[0,B]}(M_{t,T}^S), \quad \text{in } L^2(\Omega). \end{aligned} \quad (3.75)$$

Consequently, by Remark 3.26 we have

$$\begin{aligned} & E^Q \left[D_t \left(\hat{\Phi}_n(S_T) \Psi_{0,B,n}(M_{0,t}^S) \Psi_{0,B,n}(M_{t,T}^S) \right) | \mathcal{F}_t \right] \\ \rightarrow & E^Q \left[D_t \left((S_T - K)^+ 1_{[0,B]}(M_{0,t}^S) 1_{[0,B]}(M_{t,T}^S) \right) | \mathcal{F}_t \right]. \end{aligned} \quad (3.76)$$

Let $\rho_n(x, y, z) = \hat{\Phi}_n(x) \Psi_{0,B,n}(y) \Psi_{0,B,n}(z)$, which is continuously differentiable. Then

$$\begin{aligned} \left| \frac{\partial \rho_n}{\partial x} \right| &= |\hat{\Phi}'_n(x) \Psi_{0,B,n}(y) \Psi_{0,B,n}(z)| \leq |\hat{\Phi}'_n(x)| \leq 1, \\ \left| \frac{\partial \rho_n}{\partial y} \right| &= |\hat{\Phi}_n(x) \Psi'_{0,B,n}(y) \Psi_{0,B,n}(z)| < M_n \\ \left| \frac{\partial \rho_n}{\partial z} \right| &= |\hat{\Phi}_n(x) \Psi_{0,B,n}(y) \Psi'_{0,B,n}(z)| < M_n, \end{aligned}$$

for some constant M_n for each n . So by Proposition 2.29, we have

$$\begin{aligned} D_t \left(\hat{\Phi}_n(S_T) \Psi_{0,B,n}(M_{0,t}^S) \Psi_{0,B,n}(M_{t,T}^S) \right) &= \Psi_{0,B,n}(M_{0,t}^S) \Psi_{0,B,n}(M_{t,T}^S) \hat{\Phi}'_n(S_T) D_t S_T \\ &+ \hat{\Phi}_n(S_T) \Psi_{0,B,n}(M_{0,t}^S) \Psi'_{0,B,n}(M_{t,T}^S) D_t M_{t,T}^S \\ &+ \hat{\Phi}_n(S_T) \Psi_{0,B,n}(M_{t,T}^S) \Psi'_{0,B,n}(M_{0,t}^S) D_t M_{0,t}^S. \end{aligned}$$

Then taking the conditional expectation of the first term in the above equation, and by noticing $\Psi_{0,B,n}(M_{0,t}^S)$ is $\mathcal{F}_t$ -measurable, we have

$$\begin{aligned} & E^Q \left[\Psi_{0,B,n}(M_{0,t}^S) \Psi_{0,B,n}(M_{t,T}^S) \hat{\Phi}'_n(S_T) D_t S_T | \mathcal{F}_t \right] \\ = & \Psi_{0,B,n}(M_{0,t}^S) \int_0^\infty \int_0^m \Psi_{0,B,n}(m) \hat{\Phi}'_n(s) \sigma s f_{t,M_{t,T}^S, S_T}^{x,Q}(m, s, \omega) ds dm \\ \rightarrow & 1_{\{M_{0,t}^S < B\}} \int_0^\infty \int_0^m 1_{[0,B]}(m) 1_{[K,\infty)}(s) \sigma s f_{t,M_{t,T}^S, S_T}^{x,Q}(m, s, \omega) ds dm \quad Q - a.s. \\ = & 1_{\{M_{0,t}^S < B\}} E^Q \left[1_{[0,B]}(M_{t,T}^S) 1_{[K,\infty)}(S_T) D_t S_T | \mathcal{F}_t \right], \end{aligned} \quad (3.77)$$

and for $0 \leq t \leq T$, we have

$$\begin{aligned}
& E^Q \left[\hat{\Phi}_n(S_T) \Psi_{0,B,n}(M_{0,t}^S) \Psi'_{0,B,n}(M_{t,T}^S) D_t M_{t,T}^S | \mathcal{F}_t \right] \\
&= \Psi_{0,B,n}(M_{0,t}^S) E^Q \left[\hat{\Phi}_n(S_T) \Psi'_{0,B,n}(M_{t,T}^S) D_t M_{t,T}^S | \mathcal{F}_t \right] \\
&= \Psi_{0,B,n}(M_{0,t}^S) \int_0^\infty \int_0^m \hat{\Phi}_n(s) \Psi'_{0,B,n}(m) \sigma m f_{t,M_{t,T}^S, S_T}^{x,Q}(m, s, \omega) ds dm \\
&\rightarrow -1_{\{M_{0,t}^S < B\}} \int_0^\infty \int_0^m (s - K)^+ \delta_B(m) \sigma m f_{t,M_{t,T}^S, S_T}^{x,Q}(m, s, \omega) ds dm \quad Q - a.s. \\
&= -1_{\{M_{0,t}^S < B\}} E^Q \left[(S_T - K)^+ \delta_B(M_{t,T}^S) D_t M_{t,T}^S | \mathcal{F}_t \right]. \tag{3.78}
\end{aligned}$$

Notice that $D_t M_{0,t}^S = 0$ by Remark 2.38, then the third term is zero. Hence, we have

$$\begin{aligned}
& E^Q \left[D_t \left((S_T - K)^+ 1_{\{M_{0,T}^S < B\}} \right) | \mathcal{F}_t \right] \\
&= 1_{\{M_{0,t}^S < B\}} E^Q \left[1_{[0,B]}(M_{t,T}^S) 1_{[K,\infty)}(S_T) D_t S_T | \mathcal{F}_t \right] \\
&- 1_{\{M_{0,t}^S < B\}} E^Q \left[(S_T - K)^+ \delta_B(M_{t,T}^S) D_t M_{t,T}^S | \mathcal{F}_t \right]. \quad Q - a.s..
\end{aligned}$$

□

We have offered our own proof for the calculation of the Malliavin derivative of the indicator function $1_{\{S_T > B\}}$ under the conditional expectation, the Malliavin derivative of the function $(S_T - K)^+$ under the conditional expectation and the Malliavin derivative of the product $(S_T - K)^+ 1_{\{M_{0,T}^S < B\}}$ under the conditional expectation. We need those in Chapter 5 when we apply Malliavin calculus to find the hedging formula for the exotic barrier options. Let us think about the well known European Call option and Digital barrier option as examples before we finish this section.

Example 3.51. Consider the European call option $G = (S_T - K)^+$. Since $G \in L^2(\Omega)$, it is attainable by a self-financing portfolio $h = (h^0, h^1)$. By Proposition 3.28 and Lemma 3.44, we get

$$\begin{aligned}
h_t^1 &= e^{-r(T-t)} \sigma^{-1} S_t^{-1} E^Q [D_t (S_T - K)^+ | \mathcal{F}_t] \\
&= e^{-r(T-t)} S_t^{-1} E^Q [S_T 1_{\{S_T > K\}} | \mathcal{F}_t].
\end{aligned}$$

For S given by equation (1.18), we have $S_T = S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma W_T \right\}$. Let $F = S_0 e^{rT}$ and $\xi = \exp \left\{ -\frac{1}{2} \sigma^2 T + \sigma W_T \right\}$ and defining an equivalent probability measure Q^* by $dQ^*/dQ = \xi$, we have by Girsanov theorem,

$$\begin{aligned} E^Q[S_T 1_{\{S_T > K\}} | \mathcal{F}_t] &= F E^Q[\xi 1_{\{S_T > K\}} | \mathcal{F}_t] \\ &= F E^Q[\xi | \mathcal{F}_t] E^{Q^*}[1_{\{S_T > K\}} | \mathcal{F}_t] \\ &= S_t e^{r(T-t)} E^{Q^*}[1_{\{S_T > K\}} | \mathcal{F}_t]. \end{aligned}$$

Therefore,

$$h_t^1 = E^{Q^*}[1_{\{S_T > K\}} | \mathcal{F}_t] = 1 - F_{t,S_T}^{Q^*}(K, \omega), \quad (3.79)$$

where $F_{t,S_T}^{Q^*}$ is the $\mathcal{F}_t$ -conditional cumulative distribution function of S_T under the measure Q^* and is given by equation (C.45).

Example 3.52. Consider the digital barrier option $G = 1_{\{M_{0,T}^S > B\}}$. Since $G \in L^2(\Omega)$, it is attainable by a self-financing portfolio $h = (h^0, h^1)$. By Proposition 3.28 and Lemma 3.48, we get

$$\begin{aligned} h_t^1 &= e^{-r(T-t)} \sigma^{-1} S_t^{-1} E^Q \left[D_t 1_{\{M_{0,T}^S > B\}} | \mathcal{F}_t \right] \\ &= e^{-r(T-t)} \sigma^{-1} S_t^{-1} E^Q \left[\delta_B(M_{0,T}^S) D_t M_{0,T}^S | \mathcal{F}_t \right] \end{aligned}$$

For $0 \leq t \leq T$, $M_{0,t}^S \leq M_{t,T}^S$ means that the maximum value of S in the interval $[0, T]$ happens in the interval $[t, T]$, hence

$$\begin{aligned} h_t^1 &= e^{-r(T-t)} \sigma^{-1} S_t^{-1} E^Q \left[\delta_B(M_{0,T}^S) \sigma M_{0,T}^S 1_{\{M_{0,t}^S \leq M_{t,T}^S\}} | \mathcal{F}_t \right] \\ &= e^{-r(T-t)} S_t^{-1} E^Q \left[\delta_B(M_{t,T}^S) M_{t,T}^S 1_{\{M_{0,t}^S \leq M_{t,T}^S\}} | \mathcal{F}_t \right] \\ &= e^{-r(T-t)} S_t^{-1} \int_0^\infty \delta_B(m) m 1_{\{M_{0,t}^S \leq m\}} f_{t,M_{t,T}^S}^{x,Q}(m, \omega) dm \\ &= e^{-r(T-t)} S_t^{-1} B 1_{\{M_{0,t}^S \leq B\}} f_{t,M_{t,T}^S}^{x,Q}(B, \omega), \end{aligned}$$

where $f_{t,M_{t,T}^S}^{x,Q}$ denotes the $\mathcal{F}_t$ -conditional density function of $M_{t,T}^S$ under the measure Q and is given by equation (C.23).

3.4 Discussion

The next proposition is stated in [5] and claims that when a stochastic variable F belongs to $\mathbb{D}_{1,2}$ and φ is a *piecewise Lipschitz* function, such that $\varphi(F) \in L^2(\Omega)$, we still have the chain rule for the Malliavin derivative of $D_t\varphi(F)$.

Proposition 3.53 (See Corollary 5.3 in [5]). *Suppose $F = (F_1, \dots, F_n)$ is a stochastic vector with each $F_i \in \mathbb{D}_{1,2}$, and that the law of F is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$. Let $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ be a piecewise Lipschitz function such that $\varphi(F) \in L^2(\Omega)$. Then $D_t\varphi(F) \in \mathbb{D}_{-1,2}(L^2([0, T]))$, and*

$$D_t\varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F) D_t F_i, \quad (3.80)$$

where $\frac{\partial \varphi}{\partial x_i}(F)$ is interpreted as an element of $\mathbb{D}_{-1,2}$ for all i .

As a generation of the chain rule of Proposition 2.31 (which is Proposition 1.2.4 in [35]), it would be very useful for some of our specific calculations. However, we look at the proof and find it is not transparent and difficult to follow. So we have provided explicit proofs for the cases that this result would cover. While we looked for a proof of this result we noted a remark in Chapter 6 of Nualart's book [35], where he states that, for a function $\Phi(\cdot)$, “By means of an approximation procedure these formulas still hold although the function Φ and its derivative are *not Lipschitz*. We just need Φ to be *piecewise continuous with jump discontinuities with linear growth*. In particular, we can apply these formulas to the case of a European call option $\Phi(x) = (x - K)^+$ or digital option $\Phi(x) = 1_{\{x > K\}}$ ”. But there is no mention of a chain rule for *piecewise Lipschitz* functions of elements of $\mathbb{D}_{1,2}$.

For a piecewise Lipschitz F , each “piece” will be continuous, indeed absolutely continuous and possess a derivative almost everywhere. One can envisage that by smoothing out the jump discontinuities one might be able to prove a version of Bermin's Corollary 5.3 of [5] by approximation.

In Malliavin and Thalmaier [32], there is a version of Nualart's Proposition 1.2.4 stated as Theorem 1.10, but not proved. They also examine two methods for dealing with barrier

style options in their Chapter 2, computation of Greeks and Integration by parts formulas. Based upon ‘pathwise smearing’, it has its roots in a P.D.E. view. They provide two techniques for computing the Greek. One uses martingale methods, the other a mixture of p.d.e.’s and use of Brownian motion reflected at the barrier.

Chapter 4

Malliavin Calculus in the Multi-dimensional Case

In this chapter, we extend the concepts of Malliavin calculus from the one-dimensional case to the multi-dimensional case. The reason for this is for the consideration of barrier structures which rely on more than one asset. We will give the derivation of Malliavin derivative of random variable driven by multi-dimensional Brownian motions, and find the replicating strategy of a contingent claim in which there are multiple stocks.

We let $[0, T]$ be a fixed finite time-interval and assume (W_t) is a two-dimensional Brownian motion on the complete filtered probability space $(\Omega, \mathcal{F}, Q)$, i.e., $W_t = (W_t^1, W_t^2)$, where W_t^i is a one-dimensional Brownian motion for $i = 1, 2$ and W_t^1, W_t^2 are strongly orthogonal, i.e. $W_t^1 W_t^2$ is an L^1 martingale, $\langle W_t^1, W_t^2 \rangle = 0$ and moreover W_t^1, W_t^2 are independent.

Definition 4.1. Let $C_p^\infty(\mathbb{R}^n)$ denote the set of all infinitely continuously differentiable functions $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, such that φ and all its partial derivatives have polynomial growth.

Let $f \in C_p^\infty(\mathbb{R}^{2n}) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, and $t_1, \dots, t_n$ be time points in $[0, T]$, and set

$$\begin{aligned} F(\omega) &= f(W_{t_1}, W_{t_2}, \dots, W_{t_n}) \\ &\equiv f(W_{t_1}^1, W_{t_1}^2, W_{t_2}^1, W_{t_2}^2, \dots, W_{t_n}^1, W_{t_n}^2). \end{aligned}$$

Then F is said to be a *smooth* stochastic variable and denote $\mathcal{S}$ by the class of smooth

stochastic variables.

In this section, our Ω is all continuous functions on $[0, T]$ with values in $\mathbb{R}^2$ and initial value zero: $C_0([0, T], \mathbb{R}^2)$. Brownian motion is the evaluation map as before: $\omega \in C_0([0, T], \mathbb{R}^2)$, and $\omega = (\omega^1, \omega^2)$, where ω^i is a continuous $\mathbb{R}$ -valued function on $[0, T]$ with $\omega^i(0) = 0$ for $i = 1, 2$. Note that, as in the one-dimensional case, we take a concrete realization of two-dimensional Brownian motion. We have $W_t(\omega) = (\omega^1(t), \omega^2(t)) = (W_t^1(\omega), W_t^2(\omega))$.

Definition 4.2. Define the Cameron-Martin space, $\mathcal{H}$, a subspace of $C_0([0, T], \mathbb{R}^2)$, by

$$\mathcal{H} = \left\{ \gamma : [0, T] \rightarrow \mathbb{R}^2; \gamma(t) = \int_0^t \dot{\gamma}(s) ds; |\gamma|_{\mathcal{H}}^2 = \int_0^T |\dot{\gamma}|^2 dt \right\}. \quad (4.1)$$

Remark 4.3. Note that $\dot{\gamma} : [0, T] \rightarrow \mathbb{R}^2$ is $(\mathcal{B}([0, T]), \mathcal{B}^2)$ -measurable and

$$\int_0^T |\dot{\gamma}|^2 dt = \int_0^T \left(\left(\dot{\gamma}^1 \right)^2 + \left(\dot{\gamma}^2 \right)^2 \right) dt < \infty.$$

First, consider a very special case. Take a random variable $F : \Omega \rightarrow \mathbb{R}$ of the form

$$F(\omega) = f(W_t(\omega)), \quad (4.2)$$

where the deterministic function $f \in C_p^\infty : \mathbb{R}^2 \rightarrow \mathbb{R}$, and consider $f(x) = f(x^1, x^2)$ composed with $W_t = (W_t^1, W_t^2)$.

Proposition 4.4. For $F(\omega)$ defined by equation (4.2), the directional derivative of $F(\omega)$ is given by

$$\mathbb{D}_\gamma F(\omega) = \frac{\partial f}{\partial x^1}(\omega^1(t), \omega^2(t)) \gamma^1(t) + \frac{\partial f}{\partial x^2}(\omega^1(t), \omega^2(t)) \gamma^2(t), \quad (4.3)$$

for $\gamma \in \mathcal{H}$.

Proof. We offer the following proof. By definition of directional derivative and assuming

$\gamma^1(t) \neq 0$ and $\gamma^2(t) \neq 0$, we have

$$\begin{aligned}
\mathbb{D}_\gamma F(\omega) &= \lim_{\epsilon \rightarrow 0} \frac{F(\omega + \epsilon\gamma) - F(\omega)}{\epsilon} \\
&= \lim_{\epsilon \rightarrow 0} \frac{f(\omega^1(t) + \epsilon\gamma^1(t)) - f(\omega^1(t))}{\epsilon} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (f(\omega^1(t) + \epsilon\gamma^1(t), \omega^2(t) + \epsilon\gamma^2(t)) - f(\omega^1(t), \omega^2(t))) \\
&= \gamma^1(t) \lim_{\epsilon \rightarrow 0} \frac{f(\omega^1(t) + \epsilon\gamma^1(t), \omega^2(t) + \epsilon\gamma^2(t)) - f(\omega^1(t), \omega^2(t) + \epsilon\gamma^2(t))}{\epsilon\gamma^1(t)} \\
&\quad + \gamma^2(t) \lim_{\epsilon \rightarrow 0} \frac{f(\omega^1(t), \omega^2(t) + \epsilon\gamma^2(t)) - f(\omega^1(t), \omega^2(t))}{\epsilon\gamma^2(t)} \\
&= \frac{\partial f}{\partial x^1}(\omega^1(t), \omega^2(t)) \gamma^1(t) + \frac{\partial f}{\partial x^2}(\omega^1(t), \omega^2(t)) \gamma^2(t).
\end{aligned}$$

□

It is easy to check that for each $\omega \in \Omega$ the map $\gamma \mapsto \mathbf{D}_\gamma F(\omega)$ is a continuous linear functional on the Cameron Martin Space, and by the Riesz representation theorem, there exists a unique stochastic variable $\nabla F(\omega) \in \mathcal{H}$, such that

$$\mathbf{D}_\gamma F(\omega) = \langle \nabla F(\omega), \gamma \rangle_{\mathcal{H}} = \int_0^T D_s F(\omega) \cdot \dot{\gamma}(s) ds, \quad (4.4)$$

with $D_s F(\omega)$ an $\mathbb{R}^2$ -valued stochastic process, $(D_s^1 F(\omega), D_s^2 F(\omega))$.

Proposition 4.5. *For F defined by equation (4.2), the Malliavin derivative of F is the stochastic vector process $D_s F = (D_s^1 F, D_s^2 F)$ with components*

$$\begin{aligned}
D_s^1 F(\omega) &= \frac{\partial f}{\partial x^1}(W_t(\omega)) 1_{[0,t]}(s), \\
D_s^2 F(\omega) &= \frac{\partial f}{\partial x^2}(W_t(\omega)) 1_{[0,t]}(s).
\end{aligned}$$

Proof. The map $\gamma \mapsto \mathbf{D}_\gamma F$ is a continuous linear map of the two-dimensional Cameron-Martin space into $\mathbb{R}$. Now $\gamma = (\gamma^1, \gamma^2)$ and γ^i are elements of the one-dimensional Cameron-Martin space, $i = 1, 2$. By Proposition 2.12, we know $\gamma^i \mapsto \gamma^i(s)$ is a continuous linear functional on the one-dimensional Cameron-Martin space, and from Proposition 4.4

we have

$$\begin{aligned}
 \mathbf{D}_\gamma F(\omega) &= \frac{\partial f}{\partial x^1}(W_t(\omega)) \gamma^1(t) + \frac{\partial f}{\partial x^2}(W_t(\omega)) \gamma^2(t) \\
 &= \frac{\partial f}{\partial x^1}(W_t(\omega)) \int_0^t \dot{\gamma}^1 ds + \frac{\partial f}{\partial x^2}(W_t(\omega)) \int_0^t \dot{\gamma}^2 ds \\
 &= \int_0^T \left(\frac{\partial f}{\partial x^1}(W_t(\omega)) 1_{[0,t]}(s), \frac{\partial f}{\partial x^2}(W_t(\omega)) 1_{[0,t]}(s) \right) \cdot \left(\dot{\gamma}^1, \dot{\gamma}^2 \right) ds. \quad (4.5)
 \end{aligned}$$

Comparing equation (4.4) with equation (4.5), we get

$$D_s F(\omega) = \left(\frac{\partial f}{\partial x^1}(W_t(\omega)) 1_{[0,t]}(s), \frac{\partial f}{\partial x^2}(W_t(\omega)) 1_{[0,t]}(s) \right). \quad (4.6)$$

□

Let us rerun this calculation but with a more general smooth stochastic variable, $F(\omega)$.

Definition 4.6. Let the random variable $F : \Omega \rightarrow \mathbb{R}$ be of the form

$$F(\omega) = f(W_{t_1}(\omega), \dots, W_{t_n}(\omega)), \quad (4.7)$$

where $f = f(x_1^1, x_1^2, x_2^1, x_2^2, \dots, x_n^1, x_n^2)$ and $f \in C_p^\infty : \mathbb{R}^{2n} \rightarrow \mathbb{R}$.

For F defined by equation (4.7), we calculate $\mathbf{D}_\gamma F$ by the definition of directional derivative and work it out in the explicit realization of (W_t) as the evaluation map on $C_0([0, T], \mathbb{R}^2)$. The derivative quotient is

$$\begin{aligned}
 &\frac{F(\omega + \epsilon \gamma) - F(\omega)}{\epsilon} \\
 &= \frac{f(\omega(t_1) + \epsilon \gamma(t_1), \dots, \omega(t_n) + \epsilon \gamma(t_n)) - f(\omega(t_1), \dots, \omega(t_n))}{\epsilon}.
 \end{aligned}$$

Recall how one breaks this into parts: formally,

$$\begin{aligned}
& f(\omega(t_1) + \epsilon\gamma(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
& - f(\omega(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
& + f(\omega(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
& - f(\omega(t_1), \omega(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
& \vdots \\
& + \{f(\omega(t_1), \dots, \omega(t_n) + \epsilon\gamma(t_n)) - f(\omega(t_1), \dots, \omega(t_n))\}.
\end{aligned}$$

However, we have to do this for each component of W_{t_j} . So, for example, looking at the W_{t_1} term, and writing out its components only:

$$\begin{aligned}
& f(\omega(t_1) + \epsilon\gamma(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
& - f(\omega(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
= & f(\omega^1(t_1) + \epsilon\gamma^1(t_1), \omega^2(t_1) + \epsilon\gamma^2(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
& - f(\omega^1(t_1), \omega^2(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
= & f(\omega^1(t_1) + \epsilon\gamma^1(t_1), \omega^2(t_1) + \epsilon\gamma^2(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
& - f(\omega^1(t_1), \omega^2(t_1) + \epsilon\gamma^2(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
& + f(\omega^1(t_1), \omega^2(t_1) + \epsilon\gamma^2(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)) \\
& - f(\omega^1(t_1), \omega^2(t_1), \omega(t_2) + \epsilon\gamma(t_2), \dots, \omega(t_n) + \epsilon\gamma(t_n)).
\end{aligned}$$

Dividing by ϵ and then multiplying by $\frac{\gamma^1}{\gamma^1}$ for the first difference and $\frac{\gamma^2}{\gamma^2}$ for the second difference, and taking the limit and we get

$$\frac{\partial f}{\partial x_1^1}(W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \gamma^1(t_1) + \frac{\partial f}{\partial x_1^2}(W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \gamma^2(t_2)$$

So each formal difference gives us two derivative terms, one for the first variable of the pair and one for the second. Then we get the next proposition.

Proposition 4.7. *For $F(\omega)$ defined by equation (4.7), the directional derivative of F is*

given by

$$\mathbf{D}_\gamma F(\omega) = \sum_{j=0}^n \frac{\partial f}{\partial x_j^1}(W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \gamma^1(t_j) + \sum_{j=0}^n \frac{\partial f}{\partial x_j^2}(W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \gamma^2(t_j). \quad (4.8)$$

Again, it is easy to see $\gamma \mapsto \mathbf{D}_\gamma F(\omega)$ is a continuous linear functional on the Cameron-Martin space, so there is $\mathbb{R}^2$ -valued $\nabla F(\omega)$ belonging to Cameron-Martin space such that

$$\mathbf{D}_\gamma F(\omega) = \langle \nabla F(\omega), \gamma \rangle_{\mathcal{H}} = \int_0^T D_t F(\omega) \cdot \dot{\gamma}(t) dt. \quad (4.9)$$

As before

$$\begin{aligned} & \mathbf{D}_\gamma F(\omega) \\ &= \sum_{j=0}^n \frac{\partial f}{\partial x_j^1}(W_{t_1}, \dots, W_{t_n}) \gamma^1(t_j) + \sum_{j=0}^n \frac{\partial f}{\partial x_j^2}(W_{t_1}, \dots, W_{t_n}) \gamma^2(t_j) \\ &= \sum_{j=0}^n \frac{\partial f}{\partial x_j^1}(W_{t_1}, \dots, W_{t_n}) \int_0^{t_j} \dot{\gamma}^1(t) dt + \sum_{j=0}^n \frac{\partial f}{\partial x_j^2}(W_{t_1}, \dots, W_{t_n}) \int_0^{t_j} \dot{\gamma}^2(t) dt \\ &= \int_0^T \left(\sum_{j=0}^n \frac{\partial f}{\partial x_j^1}(W_{t_1}, \dots, W_{t_n}) 1_{[0, t_j]}(t) \dot{\gamma}^1(t) + \sum_{j=0}^n \frac{\partial f}{\partial x_j^2}(W_{t_1}, \dots, W_{t_n}) 1_{[0, t_j]}(t) \dot{\gamma}^2(t) \right) dt \\ &= \int_0^T \left(\sum_{j=0}^n \frac{\partial f}{\partial x_j^1}(W_{t_1}, \dots, W_{t_n}) 1_{[0, t_j]}(t), \sum_{j=0}^n \frac{\partial f}{\partial x_j^2}(W_{t_1}, \dots, W_{t_n}) 1_{[0, t_j]}(t) \right) \cdot \left(\dot{\gamma}^1(t), \dot{\gamma}^2(t) \right) dt. \end{aligned}$$

Comparing the last line of the above equation with equation (4.9), we get

$$D_t F(\omega) = \left(\sum_{j=0}^n \frac{\partial f}{\partial x_j^1}(W_{t_1}, \dots, W_{t_n}) 1_{[0, t_j]}(t), \sum_{j=0}^n \frac{\partial f}{\partial x_j^2}(W_{t_1}, \dots, W_{t_n}) 1_{[0, t_j]}(t) \right).$$

So we have the next proposition:

Proposition 4.8. *Let $F : \Omega \rightarrow \mathbb{R}$ be of the form*

$$F(\omega) = f(W_{t_1}(\omega), \dots, W_{t_n}(\omega)),$$

where $f = f(x_1^1, x_1^2, x_2^1, x_2^2, \dots, x_n^1, x_n^2)$ and $f \in C_p^\infty : \mathbb{R}^{2n} \rightarrow \mathbb{R}$. The Malliavin deriva-

tive of F is the stochastic vector process $D_t F = (D_t^1 F, D_t^2 F)$ with components

$$\begin{aligned} D_t^1 F(\omega) &= \sum_{j=0}^n \frac{\partial f}{\partial x_j^1} (W_{t_1}, \dots, W_{t_n}) 1_{\{t \leq t_j\}}, \\ D_t^2 F(\omega) &= \sum_{j=0}^n \frac{\partial f}{\partial x_j^2} (W_{t_1}, \dots, W_{t_n}) 1_{\{t \leq t_j\}}. \end{aligned}$$

Now let us consider the case where the stochastic variable has the form $\varphi(F)$, with $F = (F_1, F_2)$ and $\varphi \in C_p^\infty : \mathbb{R}^2 \rightarrow \mathbb{R}$ and F_i is of the form

$$F_i = \psi_i (W_{i,t_1}, \dots, W_{i,t_{n_i}}), \quad (4.10)$$

with $\psi_i \in C_p^\infty : \mathbb{R}^{2n_i} \rightarrow \mathbb{R}$ for $i = 1, 2$. Then we have the following proposition.

Proposition 4.9. *Let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C_p^∞ function and $F = (F_1, F_2)$ be a stochastic vector and F_i is given by equation (4.10) for all $i = 1, 2$. Then,*

$$D_t \varphi(F) = (D_t^1 \varphi(F), D_t^2 \varphi(F)), \quad (4.11)$$

where each component is given by

$$D_t^1 \varphi(F) = \left(\frac{\partial \varphi}{\partial F_1}, \frac{\partial \varphi}{\partial F_2} \right) \cdot (D_t^1 F_1, D_t^1 F_2), \quad (4.12)$$

$$D_t^2 \varphi(F) = \left(\frac{\partial \varphi}{\partial F_1}, \frac{\partial \varphi}{\partial F_2} \right) \cdot (D_t^2 F_1, D_t^2 F_2). \quad (4.13)$$

Proof. Note that $\varphi(\psi_1, \psi_2) : \mathbb{R}^{2(n_1+n_2)} \rightarrow \mathbb{R}$, and

$$\begin{aligned} &\varphi(F_1, F_2) \\ &= \varphi \left(\psi_1(x_{1,1}^1, x_{1,1}^2, x_{1,2}^1, x_{1,2}^2, \dots, x_{1,n_1}^1, x_{1,n_1}^2), \psi_2(x_{2,1}^1, x_{2,1}^2, x_{2,2}^1, x_{2,2}^2, \dots, x_{2,n_2}^1, x_{2,n_2}^2) \right). \end{aligned}$$

We can see that $\varphi(\psi_1, \psi_2)$ is continuously differentiable. By using the usual chain rule, we can work out the specific form of any partial derivative. So the first component of $D_t \varphi(F)$,

is given by

$$\begin{aligned}
D_t^1 \varphi(F) &= \sum_{j=0}^{n_1} \frac{\partial \varphi}{\partial x_{1,j}^1} (\psi_1(W_{1,t_1}, \dots, W_{1,t_{n_1}}), \psi_2(W_{2,t_1}, \dots, W_{2,t_{n_2}})) D_t W_{1,t_j}^1 \\
&\quad + \sum_{j=0}^{n_2} \frac{\partial \varphi}{\partial x_{2,j}^1} (\psi_1(W_{1,t_1}, \dots, W_{1,t_{n_1}}), \psi_2(W_{2,t_1}, \dots, W_{2,t_{n_2}})) D_t W_{2,t_j}^1 \\
&= \sum_{j=0}^{n_1} \frac{\partial \varphi}{\partial \psi_1} (\psi_1(W_{1,t_1}, \dots, W_{1,t_{n_1}}), \psi_2(W_{2,t_1}, \dots, W_{2,t_{n_2}})) \frac{\partial \psi_1}{\partial x_{1,j}^1} D_t W_{1,t_j}^1 \\
&\quad + \sum_{j=0}^{n_2} \frac{\partial \varphi}{\partial \psi_2} (\psi_1(W_{1,t_1}, \dots, W_{1,t_{n_1}}), \psi_2(W_{2,t_1}, \dots, W_{2,t_{n_2}})) \frac{\partial \psi_2}{\partial x_{2,j}^1} D_t W_{2,t_j}^1 \\
&= \frac{\partial \varphi}{\partial \psi_1} \sum_{j=0}^{n_1} \frac{\partial \psi_1}{\partial x_{1,j}^1} D_t W_{1,t_j}^1 + \frac{\partial \varphi}{\partial \psi_2} \sum_{j=0}^{n_2} \frac{\partial \psi_2}{\partial x_{2,j}^1} D_t W_{2,t_j}^1 \\
&= \frac{\partial \varphi}{\partial \psi_1} D_t^1 \psi_1 + \frac{\partial \varphi}{\partial \psi_2} D_t^1 \psi_2 \\
&= \left(\frac{\partial \varphi}{\partial F_1}, \frac{\partial \varphi}{\partial F_2} \right) \cdot (D_t^1 F_1, D_t^1 F_2),
\end{aligned}$$

with a similar derivation for the second component of $D_t F(\omega)$. So we get

$$D_t \varphi(F) = (D_t^1 \varphi(F), D_t^2 \varphi(F)),$$

where each component is given by

$$\begin{aligned}
D_t^1 \varphi(F) &= \left(\frac{\partial \varphi}{\partial F_1}, \frac{\partial \varphi}{\partial F_2} \right) \cdot (D_t^1 F_1, D_t^1 F_2), \\
D_t^2 \varphi(F) &= \left(\frac{\partial \varphi}{\partial F_1}, \frac{\partial \varphi}{\partial F_2} \right) \cdot (D_t^2 F_1, D_t^2 F_2).
\end{aligned}$$

□

In Proposition 1.2.1 of [35], Nualart shows the Malliavin derivative extends beyond the smooth random variables. In particular, for the case $k = 2$ the space $\mathbb{D}_{1,2}$, which is the domain of the closure of the derivative in $L^2(\Omega)$, is the closure of the smooth random variables with respect to the norm

$$\|F\|_{1,2} = \left[E[|F|^2] + E \left[\|D_t F\|_{L^2([0,T], \mathbb{R}^2)}^2 \right] \right]^{1/2}. \quad (4.14)$$

Nualart proves the following chain rule, Proposition 1.2.3 of [35], and subsequently proves Proposition 1.2.4 of [35], for a Lipschitz function φ .

Proposition 4.10. *[See Proposition 1.2.3 in [35]] Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F_1, \dots, F_n)$ be a stochastic vector and $F_i \in \mathbb{D}_{1,2}$ for $i = 1, \dots, n$. Then $\varphi(F) \in \mathbb{D}_{1,2}$, and*

$$D_t \varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F) D_t F_i. \quad (4.15)$$

For our two-dimensional case, we have the chain rule in equation (4.15) explicitly,

$$D_t \varphi(F) = (D_t^1 \varphi(F), D_t^2 \varphi(F)), \quad (4.16)$$

with the components given by

$$D_t^1 \varphi(F) = \left(\frac{\partial \varphi}{\partial F_1}, \dots, \frac{\partial \varphi}{\partial F_n} \right) \cdot (D_t^1 F_1, \dots, D_t^1 F_n), \quad (4.17)$$

$$D_t^2 \varphi(F) = \left(\frac{\partial \varphi}{\partial F_1}, \dots, \frac{\partial \varphi}{\partial F_n} \right) \cdot (D_t^2 F_1, \dots, D_t^2 F_n). \quad (4.18)$$

4.1 Extending the Hedging portfolio by using multi-dimensional Malliavin calculus

In this section, first we consider an extended Black-Scholes model driven by multi-dimensional Brownian motion, and assume that under the risk-neutral probability measure Q there are two risky assets S^1 and S^2 and one risk-free asset R in the market. The evolution of the risk-neutralized process for S^1 and S^2 are as follows:

$$dS_t^1 = rS_t^1 dt + \sigma_1 S_t^1 dW_t^1, \quad (4.19)$$

$$dS_t^2 = rS_t^2 dt + \sigma_2 S_t^2 \left(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right). \quad (4.20)$$

where $\{W_t, t \in [0, T]\}$ is a standard two-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, Q)$, and $r, S_0^1, S_0^2, \sigma_1 \neq 0$ and $\sigma_2 \neq 0$ are all positive con-

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stants, and ρ is the correlation between W_t^1 and W_t^2 . We follow [5] to give two examples as an application of the chain rule for the two-dimensional case.

Example 4.11. For a fixed time $t \in [0, T]$, we have S_t^1 given by equation (4.19) and S_t^2 given by equation (4.20) belong to $\mathbb{D}_{1,2}$, and

$$D_s S_t^1 = (\sigma_1 S_t^1 1_{[0,t]}(s), 0), \quad (4.21)$$

$$D_s S_t^2 = (\rho \sigma_2 S_t^2 1_{[0,t]}(s), \sqrt{1 - \rho^2} \sigma_2 S_t^2 1_{[0,t]}(s)). \quad (4.22)$$

In fact, if we take

$$S_t^1 = \varphi_{S^1}(W_t^1, W_t^2) \quad \text{and} \quad S_t^2 = \varphi_{S^2}(W_t^1, W_t^2),$$

where φ_{S^1} and φ_{S^2} are given by

$$\begin{aligned} \varphi_{S^1}(x^1, x^2) &= S_0^1 \exp \left\{ \left(r - \frac{1}{2} \sigma_1^2 \right) t + \sigma_1 x^1 \right\}, \\ \varphi_{S^2}(x^1, x^2) &= S_0^2 \exp \left\{ \left(r - \frac{1}{2} \sigma_2^2 \right) t + \rho \sigma_2 x^1 + \sqrt{1 - \rho^2} \sigma_2 x^2 \right\}, \end{aligned}$$

and then following the same idea for the proof as we did in Proposition 2.32, we approximate S_t^1 and S_t^2 by Wiener polynomials. Due to the closability of the Malliavin derivative, we have the results.

Example 4.12. For a fixed time $t \in [0, T]$ and S_t^1 given by equation (4.19), we have

$$D_t M_{0,T}^{S^1} = \left(M_{0,T}^{S^1} 1_{\{M_{0,t}^S \leq M_{t,T}^S\}} \sigma, 0 \right). \quad (4.23)$$

We follow the derivation of $D_t M_{0,T}^S$ from the earlier Section 2.2.2 and use the function φ_n given by

$$\varphi_n(S_{t_1}^1, S_{t_2}^1, \dots, S_{t_n}^1) := \max_{1 \leq i \leq n} S_{t_i}^1;$$

then we can easily get the result. Notice that although S_t^1 is driven by two-dimensional Brownian motion $W_t = (W_t^1, W_t^2)$, the term W_t^2 is zero all the time for S_t^1 .

As before, we can extend the Malliavin derivative to all $L^2(\Omega)$ and the Clark-Ocone for-

mula [27] extends likewise, and we have the following proposition of the multi-dimensional Clark-Ocone formula, which we need for the following theorem.

Proposition 4.13. *For every $G \in L^2(\Omega)$, we have*

$$G = E^Q[G] + \int_0^T E^Q[(D_t G)^* | \mathcal{F}_t] dW_t, \quad (4.24)$$

where W_t is $\mathbb{R}^d$ -valued Brownian motion and $W_t = (W_t^1, \dots, W_t^d)^*$, $0 \leq t \leq T$ and $D_t G = (D_t^1 G, \dots, D_t^d G)^*$. Here $(\dots)^*$ means the transpose of the vector, and note that

$$E^Q[(D_t G)^* | \mathcal{F}_t] = (E^Q[D_t^1 G | \mathcal{F}_t], \dots, E^Q[D_t^d G | \mathcal{F}_t]). \quad (4.25)$$

Theorem 4.14. *Any contingent claim $G \in L^2(\Omega)$ can be replicated by the self-financing portfolio $h = (h^0, h^1, h^2)$ defined by*

$$h_t^0 = e^{-rt}(V_t^h - h_t^1 S_t^1 - h_t^2 S_t^2), \quad (4.26)$$

$$h_t^1 = \frac{e^{-r(T-t)}}{\sigma_1 S_t^1 \sqrt{1-\rho^2}} \left(\sqrt{1-\rho^2} E^Q[D_t^1 G | \mathcal{F}_t] - \rho E^Q[D_t^2 G | \mathcal{F}_t] \right), \quad (4.27)$$

$$h_t^2 = \frac{e^{-r(T-t)}}{\sigma_2 S_t^2 \sqrt{1-\rho^2}} E^Q[D_t^2 G | \mathcal{F}_t]. \quad (4.28)$$

Here h_t^0 denotes the number of units to be held at time t in the locally risk-free asset R_t , and h_t^1, h_t^2 denote the number of units to be held in the stocks S_t^1, S_t^2 at time t .

Proof. Consider the value of the portfolio at time t : $V_t^h = h_t^0 R_t + h_t^1 S_t^1 + h_t^2 S_t^2$ and the corresponding stochastic equation is

$$\begin{aligned} dV_t^h &= h_t^0 dR_t + h_t^1 dS_t^1 + h_t^2 dS_t^2 \\ &= h_t^0 r R_t dt + h_t^1 r S_t^1 dt + h_t^1 \sigma_1 S_t^1 dW_t^1 \\ &\quad + h_t^2 r S_t^2 dt + h_t^2 \sigma_2 S_t^2 \left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2 \right) \\ &= r V_t^h dt + h_t^1 \sigma_1 S_t^1 dW_t^1 + h_t^2 \sigma_2 S_t^2 \left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2 \right). \end{aligned}$$

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Then the discounted value process of V_t^h is given by

$$\begin{aligned} d(e^{-rt}V_t^h) &= e^{-rt}dV_t^h - rV_t^he^{-rt}dt \\ &= e^{-rt}h_t^1\sigma_1S_t^1dW_t^1 + e^{-rt}h_t^2\sigma_2S_t^2\rho dW_t^1 + e^{-rt}h_t^2\sigma_2S_t^2\sqrt{1-\rho^2}dW_t^2. \end{aligned}$$

Thus, by using Itô's formula and the Girsanov kernel for the Radon-Nikodym derivative, we have

$$\begin{aligned} V_T^h &= e^{rT}V_0^h + \int_0^T e^{r(T-t)} (h_t^1\sigma_1S_t^1 + h_t^2\sigma_2S_t^2\rho) dW_t^1 \\ &\quad + \int_0^T e^{r(T-t)} h_t^2\sigma_2S_t^2\sqrt{1-\rho^2}dW_t^2 \\ &= e^{rT}V_0^h + \int_0^T e^{r(T-t)} \left(\sigma_1h_t^1S_t^1 + \rho\sigma_2h_t^2S_t^2, \sqrt{1-\rho^2}\sigma_2h_t^2S_t^2 \right) d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}. \end{aligned}$$

We can identify the coefficients in the Clark-Ocone formula 4.24. We see

$$E^Q[D_t^1G|\mathcal{F}_t] = e^{r(T-t)} (\sigma_1h_t^1S_t^1 + \rho\sigma_2h_t^2S_t^2), \quad (4.29)$$

$$E^Q[D_t^2G|\mathcal{F}_t] = e^{r(T-t)} \sqrt{1-\rho^2}\sigma_2h_t^2S_t^2. \quad (4.30)$$

Solving the above two equations, we have that the number of units to be held at time t in the stock S^1 and S^2 are given by

$$\begin{aligned} h_t^1 &= \frac{e^{-r(T-t)}}{\sigma_1S_t^1\sqrt{1-\rho^2}} \left(\sqrt{1-\rho^2}E^Q[D_t^1G|\mathcal{F}_t] - \rho E^Q[D_t^2G|\mathcal{F}_t] \right), \\ h_t^2 &= \frac{e^{-r(T-t)}}{\sigma_2S_t^2\sqrt{1-\rho^2}} E^Q[D_t^2G|\mathcal{F}_t]. \end{aligned}$$

and consequently the number of units to be held at time t in the locally risk-free asset R_t is given by

$$h_t^0 = e^{-rt}(V_t^h - h_t^1S_t^1 - h_t^2S_t^2).$$

Note that the initial amount of money required to replicate the contingent claim G by the self-financing portfolio h is given by $V_0^h = e^{-rT}E^Q[G]$. $\square$

Let us end this section with the following lemma which we need in the next chapter.

Lemma 4.15. *For the random variable $G = (S_T^2 - K)^+ 1_{\{M_{0,T}^{S^1} < B\}}$, we have*

$$E^Q [D_t G | \mathcal{F}_t] = (E^Q [D_t^1 G | \mathcal{F}_t], E^Q [D_t^2 G | \mathcal{F}_t])$$

with

$$\begin{aligned} E^Q [D_t^1 G | \mathcal{F}_t] &= E^Q \left[1_{\{S_T^2 > K\}} 1_{\{M_{0,T}^{S^1} < B\}} D_t^1 S_T^2 | \mathcal{F}_t \right] \\ &\quad - E^Q \left[(S_T^2 - K)^+ \delta_B \left(M_{0,T}^{S^1} \right) D_t^1 M_{0,T}^{S^1} | \mathcal{F}_t \right] \quad Q - a.s. \end{aligned} \quad (4.31)$$

$$E^Q [D_t^2 G | \mathcal{F}_t] = E^Q \left[1_{\{S_T^2 > K\}} 1_{\{M_{0,T}^{S^1} < B\}} D_t^2 S_T^2 | \mathcal{F}_t \right] \quad Q - a.s. \quad (4.32)$$

Proof. Let $\hat{\Phi}_n$ and $\Psi_{0,B,n}$ be the functions defined in equation (B.17) and equation (B.1). We give the proof as we did in the proof of Lemma 3.50. For $S_T^2 \in \mathbb{D}_{1,2}$, $\hat{\Phi}_n(S_T^2) \rightarrow (S_T^2 - K)^+$, Q -a.s. and for $M_{0,T}^{S^1} \in \mathbb{D}_{1,2}$, $\Psi_{0,B,n}(M_{0,T}^{S^1}) \rightarrow 1_{[0,B]}(M_{0,T}^{S^1})$, Q -a.s.. Therefore,

$$\hat{\Phi}_n(S_T^2) \Psi_{0,B,n}(M_{0,T}^{S^1}) \rightarrow (S_T^2 - K)^+ 1_{[0,B]}(M_{0,T}^{S^1}), \quad Q - a.s..$$

Note that $0 \leq \hat{\Phi}_n(S_T^2) \Psi_{0,B,n}(M_{0,T}^{S^1}) \leq S_T^2$, for all n . So $\{\hat{\Phi}_n^2(S_T^2) \Psi_{0,B,n}^2(M_{0,T}^{S^1}) : n \in \mathbb{N}\}$ is a uniformly integrable set. Then Vitali convergence theorem (see III. 6.15 of [19]) tells us that

$$\hat{\Phi}_n(S_T^2) \Psi_{0,B,n}(M_{0,T}^{S^1}) \rightarrow (S_T^2 - K)^+ 1_{[0,B]}(M_{0,T}^{S^1}), \quad \text{in } L^2(\Omega). \quad (4.33)$$

Now let $\rho_n(x, y) = \hat{\Phi}_n(x) \Psi_{0,B,n}(y)$ with composition of S_T^2 and $M_{0,T}^{S^1}$. Now we use the fact that $G \rightarrow E^Q[D_t G | \mathcal{F}_t]$ is continuous on $L^2(\Omega)$. There is a proof in chapter 5 of [46]. However, it is more elementary to simply rewrite the proof of Proposition 3.24 for the case of two-dimensional Brownian motion and the Malliavin derivative. We have

$$E^Q [D_t \rho_n(S_T^2, M_{0,T}^{S^1}) | \mathcal{F}_t] \rightarrow E^Q \left[D_t \left((S_T^2 - K)^+ 1_{[0,B]}(M_{0,T}^{S^1}) \right) | \mathcal{F}_t \right], \quad (4.34)$$

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with

$$E^Q \left[D_t^1 \rho_n \left(S_T^2, M_{0,T}^{S^1} \right) | \mathcal{F}_t \right] \rightarrow E^Q \left[D_t^1 (S_T^2 - K)^+ 1_{\{M_{0,T}^{S^1} < B\}} | \mathcal{F}_t \right], \quad (4.35)$$

$$E^Q \left[D_t^2 \rho_n \left(S_T^2, M_{0,T}^{S^1} \right) | \mathcal{F}_t \right] \rightarrow E^Q \left[D_t^2 (S_T^2 - K)^+ 1_{\{M_{0,T}^{S^1} < B\}} | \mathcal{F}_t \right]. \quad (4.36)$$

By Proposition 4.10, we have

$$D_t \rho_n \left(S_T^2, M_{0,T}^{S^1} \right) = \left(D_t^1 \rho_n \left(S_T^2, M_{0,T}^{S^1} \right), D_t^2 \rho_n \left(S_T^2, M_{0,T}^{S^1} \right) \right)$$

with

$$\begin{aligned} D_t^1 \rho_n(S_T^2, M_{0,T}^{S^1}) &= \hat{\Phi}'_n(S_T^2) \Psi_{0,B,n}(M_{0,T}^{S^1}) D_t^1 S_T^2 + \hat{\Phi}_n(S_T^2) \Psi'_{0,B,n}(M_{0,T}^{S^1}) D_t^1 M_{0,T}^{S^1}, \\ D_t^2 \rho_n(S_T^2, M_{0,T}^{S^1}) &= \hat{\Phi}'_n(S_T^2) \Psi_{0,B,n}(M_{0,T}^{S^1}) D_t^2 S_T^2 + \hat{\Phi}_n(S_T^2) \Psi'_{0,B,n}(M_{0,T}^{S^1}) D_t^2 M_{0,T}^{S^1}. \end{aligned}$$

By taking the conditional expectation, we have

$$E^Q[D_t \rho_n(S_T^2, M_{0,T}^{S^1}) | \mathcal{F}_t] = \left(E^Q[D_t^1 \rho_n(S_T^2, M_{0,T}^{S^1}) | \mathcal{F}_t], E^Q[D_t^2 \rho_n(S_T^2, M_{0,T}^{S^1}) | \mathcal{F}_t] \right) \quad (4.37)$$

Now compute the first component of the above vector and we get,

$$\begin{aligned} &E^Q \left[D_t^1 \rho_n(S_T^2, M_{0,T}^{S^1}) | \mathcal{F}_t \right] \\ &= E^Q \left[\hat{\Phi}'_n(S_T^2) \Psi_{0,B,n}(M_{0,T}^{S^1}) D_t^1 S_T^2 | \mathcal{F}_t \right] + E^Q \left[\hat{\Phi}_n(S_T^2) \Psi'_{0,B,n}(M_{0,T}^{S^1}) D_t^1 M_{0,T}^{S^1} | \mathcal{F}_t \right]. \end{aligned}$$

Moreover, by Example 4.11 we have

$$\begin{aligned} &E^Q \left[\hat{\Phi}'_n(S_T^2) \Psi_{0,B,n}(M_{0,T}^{S^1}) D_t^1 S_T^2 | \mathcal{F}_t \right] \\ &= E^Q \left[\hat{\Phi}'_n(S_T^2) \Psi_{0,B,n}(M_{0,T}^{S^1}) \rho \sigma_2 S_T^2 | \mathcal{F}_t \right] \\ &= \int_0^\infty \int_0^\infty \hat{\Phi}'_n(s) \Psi_{0,B,n}(m) \rho \sigma_2 s f_{t, M_{0,T}^{S^1}, S_T^2}^{x,Q}(m, s, \omega) dm ds \\ &\rightarrow \int_0^\infty \int_0^\infty 1_{[K, \infty)}(s) 1_{[0, B]}(m) \rho \sigma_2 s f_{t, M_{0,T}^{S^1}, S_T^2}^{x,Q}(m, s, \omega) dm ds \quad Q - a.s. \\ &= E^Q \left[1_{\{S_T^2 > K\}} 1_{\{M_{0,T}^{S^1} < B\}} \rho \sigma_2 S_T^2 | \mathcal{F}_t \right] \quad Q - a.s. \end{aligned} \quad (4.38)$$

and by Example 4.12 we have

$$\begin{aligned}
& E^Q \left[\hat{\Phi}_n(S_T^2) \Psi'_{0,B,n}(M_{0,T}^{S^1}) D_t^1 M_{0,T}^{S^1} | \mathcal{F}_t \right] \\
&= E^Q \left[\hat{\Phi}_n(S_T^2) \Psi'_{0,B,n}(M_{0,T}^{S^1}) M_{0,T}^{S^1} 1_{\{M_{0,t}^{S^1} \leq M_{t,T}^{S^1}\}} \sigma_1 | \mathcal{F}_t \right] \\
&= \int_0^\infty \int_0^\infty \hat{\Phi}_n(s) \Psi'_{0,B,n}(m) m 1_{\{M_{0,t}^{S^1} \leq m\}} \sigma_1 f_{t,M_{t,T}^{S^1}, S_T^2}^{x,Q}(m, s, \omega) dm ds \\
&\rightarrow - \int_0^\infty \int_0^\infty (s - K)^+ \delta_B(m) m 1_{\{M_{0,t}^{S^1} \leq m\}} \sigma_1 f_{t,M_{t,T}^{S^1}, S_T^2}^{x,Q}(m, s, \omega) dm ds \quad Q - a.s. \\
&= -E^Q \left[(S_T^2 - K)^+ \delta_B(M_{0,T}^{S^1}) M_{0,T}^{S^1} 1_{\{M_{0,t}^{S^1} \leq M_{t,T}^{S^1}\}} \sigma_1 | \mathcal{F}_t \right]. \tag{4.39}
\end{aligned}$$

Therefore, the first component of the vector equation (4.37) converges Q -a.s. to

$$\begin{aligned}
E^Q \left[D_t^1 \rho_n(S_T^2, M_{0,T}^{S^1}) | \mathcal{F}_t \right] &\rightarrow E^Q \left[1_{\{S_T^2 > K\}} 1_{\{M_{0,T}^{S^1} < B\}} D_t^1 S_T^2 | \mathcal{F}_t \right] \quad Q - a.s. \\
&\quad - E^Q \left[(S_T^2 - K)^+ \delta_B(M_{0,T}^{S^1}) D_t^1 M_{0,T}^{S^1} | \mathcal{F}_t \right]. \tag{4.40}
\end{aligned}$$

Now compute the second component of the vector equation (4.37) and we get

$$\begin{aligned}
& E^Q \left[D_t^2 \rho_n(S_T^2, M_{0,T}^{S^1}) | \mathcal{F}_t \right] \\
&= E^Q \left[\Phi'_n(S_T^2) \Psi_{0,B,n}(M_{0,T}^{S^1}) \sqrt{1 - \rho^2} \sigma_2 S_T^2 | \mathcal{F}_t \right] \\
&= \int_0^\infty \int_0^\infty \Phi'_n(s) \Psi_{0,B,n}(m) \sqrt{1 - \rho^2} \sigma_2 s f_{t,M_{0,T}^{S^1}, S_T^2}^{x,Q}(m, s, \omega) dm ds \\
&\rightarrow \int_0^\infty \int_0^\infty 1_{[K, \infty)}(s) 1_{[0, B]}(m) \sqrt{1 - \rho^2} \sigma_2 s f_{t,M_{0,T}^{S^1}, S_T^2}^{x,Q}(m, s, \omega) dm ds \\
&= E^Q \left[1_{\{S_T^2 > K\}} 1_{\{M_{0,T}^{S^1} < B\}} \sqrt{1 - \rho^2} \sigma_2 S_T^2 | \mathcal{F}_t \right] \quad Q - a.s.. \tag{4.41}
\end{aligned}$$

We complete the proof by comparing equation (4.35) with equation (4.40) and equation (4.36) with equation (4.41). $\square$

Chapter 5

Hedging Exotic Barrier Options

After the extension in previous chapters, in this chapter we use the extended Malliavin calculus approach to find the self-financing replicating portfolios that generate the square-integrable payoff functions of barrier options. Barrier options are a family of options with the common property that their terminal payoffs are functions of the maximum or the minimum value of the underlying security. That means, the terminal payoff of barrier options depends on the whole trajectory of the underlying security, so barrier options are typical path-dependent contingent claims.

If we consider barrier options as alternatives to ordinary options, we find that barrier options are always cheaper than a similar option without barrier. Hence, barrier options are normally attractive alternatives to ordinary options for an investor. There have been a large number of papers published in this area, among them [4], [5] and [12], which develop and price new types of exotic barrier options. We can find from these papers that the most important feature of all kinds of barrier options is, of course, that the payoff functions of the options are discontinuous in the sense that the owner of a barrier option receives at maturity either the amount zero or the payoff of a standard option, depending on the history of the underlying.

Based on the Black and Scholes framework, see [9], we will in this thesis show how to hedge the Digital Barrier Option with a random time, Protected Barrier Option and Rainbow Barrier Option studied in [12], by using Malliavin Calculus.

5.1 Digital Barrier Option with a random time

A digital barrier option is an option that pays a set amount if the maximum or minimum value of the underlying asset is above or below the predetermined barrier level, or nothing at all.

Definition 5.1. Let τ denote the first time that the process S hits the barrier level B :

$$\tau(\omega) = \inf \{t : S_t(\omega) \geq B\}, \quad \omega \in \Omega. \quad (5.1)$$

Remark 5.2. From the definition of τ , we have $S_\tau = B$. If, for $\omega \in \Omega$, S never reaches the barrier level, our definition gives $\tau = \infty$.

However, by the term digital barrier option with a random time, we mean the maturity is a fixed time T , but the payoff depends on the random time τ and the payment time is a random time which is the minimum of the time τ and the maturity T . Write $\min\{\tau, T\} \equiv \tau \wedge T$ and see $E[(\tau \wedge T)^2] \leq T^2 < \infty$.

In this section, we first derive the price of the digital barrier option with a random time. Then we find the self-financing portfolio that generates the digital barrier option with a random time by using Malliavin calculus. Finally we compare the results we get between the traditional Δ -hedging approach and the Malliavin approach.

Definition 5.3 (Digital Barrier Option with a Random Time). Consider a claim that pays one at the first time that S hits the barrier level B before the fixed maturity time T , i.e. pays one at time τ , zero otherwise. Formally, the payoff is defined by $1_{\{\tau \leq T\}}$ at time τ or, equivalently to $e^{r(T-\tau)}1_{\{\tau \leq T\}}$ at time T . So under the risk-neutral measure Q and at time $0 \leq t \leq \tau$, the value of this claim is given by

$$\text{DBRT}(t) = e^{rt} E^Q [e^{-r\tau} 1_{\{\tau \leq T\}} | \mathcal{F}_t]. \quad (5.2)$$

Remark 5.4. Notice that when $\tau = T$, then it becomes the ordinary digital barrier option.

5.1.1 Δ -Hedging for the Digital Barrier Option with a Random Time

Now let us first consider at time zero, i.e., $t = 0$, the price of the digital barrier option with a random time.

Proposition 5.5. *By using Girsanov's theorem and changing measure, we can show*

$$\begin{aligned} E^Q [e^{-r\tau} 1_{\{\tau \leq T\}}] &= e^{\frac{\mu - \tilde{\mu}}{\sigma^2} b} E^{\tilde{Q}} [1_{\{\tau \leq T\}}] \\ &= e^{-b} E^{\tilde{Q}} [1_{\{\tau \leq T\}}], \end{aligned} \quad (5.3)$$

where $E^{\tilde{Q}}$ denotes the expectation under the probability measure $\tilde{Q}$, $\tilde{\mu}$ is given by equation (C.37), μ by equation (C.2) and b by equation (C.4). Note also that $(\mu - \tilde{\mu})/\sigma^2 = -1$.

Proof. Note that the term $e^{-r\tau}$ is a function of a random time, so we can't remove it directly from the term $E^Q [e^{-r\tau} 1_{\{\tau \leq T\}}]$. This term is dealt with by a change of measure. Here we follow some similar steps in [12], but in more detail. Note we can always change the drift of the arithmetic Brownian motion X in equation (C.1) from μ to $\tilde{\mu}$ and find a change of measure, Q and $\tilde{Q}$, so that under $\tilde{Q}$, X_t follows the process $X_t = \tilde{\mu}t + \sigma\tilde{W}_t$, $t \in [0, T]$, with $\tilde{W}_t$ a $\tilde{Q}$ Brownian motion. We rewrite X_t as $X_t = \tilde{\mu}t + \sigma(W_t + \frac{\mu - \tilde{\mu}}{\sigma}t)$, and let

$$\tilde{W}_t = W_t + \frac{\mu - \tilde{\mu}}{\sigma}t. \quad (5.4)$$

By Girsanov's theorem, the kernel is given by

$$Z_t = \exp \left\{ -\frac{\mu - \tilde{\mu}}{\sigma} W_t - \left(\frac{\mu - \tilde{\mu}}{\sigma} \right)^2 \frac{t}{2} \right\}.$$

Observe that $X_\tau = b$, i.e., $\mu\tau + \sigma W_\tau = b$, a fixed number, and also observe that

$$\begin{aligned} \exp \left\{ \frac{\mu - \tilde{\mu}}{\sigma} W_t \right\} &= \exp \left\{ \frac{\mu - \tilde{\mu}}{\sigma^2} \sigma W_t + \frac{\mu - \tilde{\mu}}{\sigma^2} \mu t - \frac{\mu - \tilde{\mu}}{\sigma^2} \mu t \right\} \\ &= \exp \left\{ \left(\frac{\mu - \tilde{\mu}}{\sigma^2} \right) (\sigma W_t + \mu t) - \left(\frac{\mu - \tilde{\mu}}{\sigma^2} \right) \mu t \right\}. \end{aligned}$$

So at time τ , we have

$$\exp \left\{ \frac{\mu - \tilde{\mu}}{\sigma} W_\tau \right\} = \exp \left\{ \frac{\mu - \tilde{\mu}}{\sigma^2} b - \frac{\mu - \tilde{\mu}}{\sigma^2} \mu \tau \right\}.$$

Multiply by 1 inside E^Q , where

$$\begin{aligned} 1 &= \exp \left\{ -\frac{\mu - \tilde{\mu}}{\sigma^2} \sigma W_\tau + \frac{\mu - \tilde{\mu}}{\sigma^2} \sigma W_\tau \right\} \exp \left\{ \frac{\mu - \tilde{\mu}}{\sigma^2} \mu \tau - \frac{\mu - \tilde{\mu}}{\sigma^2} \mu \tau \right\} \\ &\quad \exp \left\{ -\frac{(\mu - \tilde{\mu})^2}{\sigma^2} \frac{\tau}{2} + \frac{(\mu - \tilde{\mu})^2}{\sigma^2} \frac{\tau}{2} \right\} \\ &= Z_\tau \exp \left\{ \frac{\mu - \tilde{\mu}}{\sigma^2} b - \frac{\mu - \tilde{\mu}}{\sigma^2} \mu \tau + \frac{(\mu - \tilde{\mu})^2}{\sigma^2} \frac{\tau}{2} \right\}, \end{aligned}$$

which we must combine with $e^{-r\tau}$. If we collect the exponents involving τ and determine $\tilde{\mu}$ by equation

$$\frac{1}{2} \left(\frac{\mu - \tilde{\mu}}{\sigma} \right)^2 - \mu \frac{\mu - \tilde{\mu}}{\sigma^2} - r = 0,$$

then

$$\tilde{\mu} \equiv \sqrt{\mu^2 + 2r\sigma^2}.$$

Then we have,

$$\begin{aligned} E^Q [e^{-r\tau} 1_{\{\tau \leq T\}}] &= E^Q \left[Z_\tau \exp \left\{ \frac{\mu - \tilde{\mu}}{\sigma^2} X_\tau - \frac{\mu - \tilde{\mu}}{\sigma^2} \mu \tau + \frac{(\mu - \tilde{\mu})^2}{\sigma^2} \frac{\tau}{2} - r\tau \right\} 1_{\{\tau \leq T\}} \right] \\ &= E^Q \left[Z_\tau \exp \left\{ \frac{\mu - \tilde{\mu}}{\sigma^2} b \right\} 1_{\{\tau \leq T\}} \right] \\ &= e^{-b} E^Q [Z_\tau 1_{\{\tau \leq T\}}]. \end{aligned}$$

Now Z_t is a Q -martingale and $E^Q[Y Z_T] = E^Q[Y]$, but if Y is $\mathcal{F}_\tau$ measurable and bounded, then

$$\begin{aligned} E^Q[Y Z_T] &= E^Q [E^Q [Y Z_T | \mathcal{F}_\tau]] \\ &= E^Q [Y E^Q [Z_T | \mathcal{F}_\tau]] \\ &= E^Q [Y Z_\tau], \end{aligned}$$

and therefore

$$E^Q[Y Z_\tau] = E^{\tilde{Q}}[Y] \quad (5.5)$$

Since $1_{\{\tau \leq T\}}$ is $\mathcal{F}_\tau$ measurable, the claim value can be represented as

$$E^Q [e^{-r\tau} 1_{\{\tau \leq T\}}] = e^{-b} E^{\tilde{Q}} [1_{\{\tau \leq T\}}],$$

where $E^{\tilde{Q}}$ denotes the expectation under the probability measure $\tilde{Q}$. Notice that since $Z_{\tau \wedge t}$ is a martingale, [28], and by the Martingale Convergence theorem, we have

$$E[Z_\tau] = \lim_{t \rightarrow \infty} E[Z_{\tau \wedge t}].$$

Moreover, for any t , $E[Z_{\tau \wedge t}]$ is constant, hence $E[Z_{\tau \wedge t}] = E[Z_{\tau \wedge 0}] = E[Z_0] = 1$. Hence

$$E[Z_\tau] = 1.$$

So Z_τ gives us a change of measure which agrees with $\tilde{Q}$ on the $\mathcal{F}_\tau$ -measurable random variables. $\square$

Proposition 5.6. *At time zero, $t = 0$, the price of digital barrier option with a random time is given by*

$$\text{DBRT}(0) = \frac{S_0}{B} \left(1 - F_{M_{0,T}^S}^{\tilde{Q}}(B) \right), \quad (5.6)$$

where $F_{M_{0,T}^S}^{\tilde{Q}}(B)$ is given by equation (C.38).

Proof. For a fixed time T , the event that S following the process equation (1.18) hits the barrier level B after T is equal to the event that $\{M_{0,T}^S < B\}$. Also $\tilde{Q}(\tau > T) = \tilde{Q}(M_{0,T}^S \leq B)$. Then finally by Proposition 5.5, under the measure $\tilde{Q}$ we have the time-zero price of digital barrier option with random time as

$$\begin{aligned} \text{DBRT}(0) &= E^Q [e^{-r\tau} 1_{\{\tau \leq T\}}] \\ &= e^{\frac{\mu - \tilde{\mu}}{\sigma^2} b} E^{\tilde{Q}} [1_{\{\tau \leq T\}}] \\ &= e^{\frac{\mu - \tilde{\mu}}{\sigma^2} b} \left(1 - \tilde{Q}(M_{0,T}^S \leq B) \right) \\ &= \frac{S_0}{B} \left(1 - F_{M_{0,T}^S}^{\tilde{Q}}(B) \right), \end{aligned}$$

where $F_{M_{0,T}^S}^{\tilde{Q}}(B)$ is given by equation (C.38). $\square$

According to the Δ -hedging approach, we then differentiate the equation (5.6) with respect to the variable S_0 and get

$$h_0^1 = \frac{1}{B} \left(1 - F_{M_{0,T}^S}^{\tilde{Q}}(B) \right) - \frac{S_0}{B} \frac{\partial}{\partial S_0} F_{M_{0,T}^S}^{\tilde{Q}}(B).$$

By equation (C.40), we have

$$\begin{aligned} h^1(0) &= \frac{1}{B} \left(1 - F_{M_{0,T}^S}^{\tilde{Q}}(B) \right) + \frac{\partial}{\partial B} F_{M_{0,T}^S}^{\tilde{Q}}(B) \\ &= \frac{1}{B} \left(1 - F_{M_{0,T}^S}^{\tilde{Q}}(B) \right) + f_{M_{0,T}^S}^{\tilde{Q}}(B), \end{aligned} \quad (5.7)$$

where $f_{M_{0,T}^S}^{\tilde{Q}}(B)$ is given by equation (C.39).

Now let us consider the price of the digital barrier option with a random time, at time $t > 0$. Under the risk-neutral measure Q , the time t price is given by

$$\begin{aligned} \text{DBRT}(t) &= e^{rt} E^Q \left[e^{-r\tau} 1_{\{\tau \leq T\}} | \mathcal{F}_t \right] \\ &= e^{rt} E^Q \left[e^{-r\tau} 1_{\{\tau \leq t\}} 1_{\{\tau \leq T\}} | \mathcal{F}_t \right] + e^{rt} E^Q \left[e^{-r\tau} 1_{\{\tau > t\}} 1_{\{\tau \leq T\}} | \mathcal{F}_t \right] \\ &= e^{rt} E^Q \left[e^{-r\tau} 1_{\{\tau \leq t\}} | \mathcal{F}_t \right] + e^{rt} E^Q \left[e^{-r\tau} 1_{\{t < \tau \leq T\}} | \mathcal{F}_t \right]. \end{aligned}$$

Since $e^{-r\tau}$ is an $\mathcal{F}_\tau$ -measurable random variable, and any $\mathcal{F}_\tau$ -measurable random variable Y is such that $Y 1_{\{\tau \leq t\}}$ is $\mathcal{F}_t$ -measurable, then $e^{-r\tau} 1_{\{\tau \leq t\}}$ is $\mathcal{F}_t$ -measurable, so

$$e^{rt} E^Q \left[e^{-r\tau} 1_{\{\tau \leq t\}} | \mathcal{F}_t \right] = e^{r(t-\tau)} 1_{\{\tau \leq t\}}. \quad (5.8)$$

We consider only then the second term where the option is still alive. So we have the next proposition:

Proposition 5.7. For $0 < t < \tau$, we have the following equation:

$$e^{rt} E^Q \left[e^{-r\tau} 1_{\{t < \tau \leq T\}} | \mathcal{F}_t \right] = 1_{\{M_{0,t}^S < B\}} \frac{S_t}{B} \left(1 - F_{t, M_{t,T}^S}^{x, \tilde{Q}}(B, \omega) \right), \quad (5.9)$$

where $F_{t, M_{t,T}^S}^{x, \tilde{Q}}(B, \omega)$ is given by equation (C.41).

Proof. Observe, $Z_\tau = S_\tau e^{-r\tau}/S_0 = B e^{-r\tau}/S_0$, so look at

$$\begin{aligned} e^{rt} E^Q [e^{-r\tau} 1_{\{t < \tau \leq T\}} | \mathcal{F}_t] &= e^{rt} \frac{S_0}{B} E^Q [Z_\tau 1_{\{t < \tau \leq T\}} | \mathcal{F}_t] \\ &= e^{rt} \frac{S_0}{B} E^Q [Z_\tau | \mathcal{F}_t] E^{\tilde{Q}} [1_{\{t < \tau \leq T\}} | \mathcal{F}_t] \\ &= e^{rt} \frac{S_0}{B} Z_{\tau \wedge t} E^{\tilde{Q}} [1_{\{t < \tau \leq T\}} | \mathcal{F}_t], \end{aligned}$$

using the Bayes law in the last line of the equation. Notice that

$$Z_t = \exp \left\{ \sigma W_t - \frac{1}{2} \sigma^2 t \right\} = \exp \left\{ X_t - \left(\mu + \frac{1}{2} \sigma^2 \right) t \right\} = \frac{S_t}{S_0} e^{-rt},$$

and Z_t is a Q -martingale. So

$$\begin{aligned} e^{rt} E^Q [e^{-r\tau} 1_{\{t < \tau \leq T\}} | \mathcal{F}_t] &= e^{rt} \frac{S_0}{B} Z_t E^{\tilde{Q}} [1_{\{t < \tau \leq T\}} | \mathcal{F}_t] \\ &= \frac{S_t}{B} E^{\tilde{Q}} [1_{\{M_{0,t}^S < B\}} 1_{\{M_{t,T}^S > B\}} | \mathcal{F}_t] \\ &= \frac{S_t}{B} 1_{\{M_{0,t}^S < B\}} E^{\tilde{Q}} [1_{\{M_{t,T}^S > B\}} | \mathcal{F}_t] \\ &= 1_{\{M_{0,t}^S < B\}} \frac{S_t}{B} \left(1 - F_{t, M_{t,T}^S}^{x, \tilde{Q}}(B, \omega) \right). \end{aligned}$$

Notice $1_{\{M_{0,t}^S < B\}}$ is $\mathcal{F}_t$ -measurable, and $F_{t, M_{t,T}^S}^{x, \tilde{Q}}(B, \omega)$ is given by equation (C.41). $\square$

Proposition 5.8. For $0 < t < \tau$, the time t price of the digital barrier option with a random time is given by

$$\text{DBRT}(t) = 1_{\{M_{0,t}^S < B\}} \frac{S_t}{B} \left(1 - F_{t, M_{t,T}^S}^{x, \tilde{Q}}(B, \omega) \right), \quad (5.10)$$

where $F_{t, M_{t,T}^S}^{x, \tilde{Q}}(B, \omega)$ is given by equation (C.41).

Remark 5.9. We not only cover the derivation of the time-zero price in [12], but also we obtain any time- t price of the security.

According to the Δ -hedging approach, we differentiate the equation (5.10) with respect

to the variable S_t and get formally

$$h_t^1 = 1_{\{M_{0,t}^S < B\}} \left(\frac{1}{B} \left(1 - F_{t,M_{t,T}^S}^{x,\tilde{Q}}(B, \omega) \right) - \frac{S_t}{B} \frac{\partial}{\partial S_t} F_{t,M_{t,T}^S}^{x,\tilde{Q}}(B, \omega) \right).$$

Using equation (C.43), we have

$$\begin{aligned} h_t^1 &= 1_{\{M_{0,t}^S < B\}} \left(\frac{1}{B} \left(1 - F_{t,M_{t,T}^S}^{x,\tilde{Q}}(B, \omega) \right) + \frac{\partial}{\partial B} F_{t,M_{t,T}^S}^{x,\tilde{Q}}(B, \omega) \right) \\ &= 1_{\{M_{0,t}^S < B\}} \left(\frac{1}{B} \left(1 - F_{t,M_{t,T}^S}^{x,\tilde{Q}}(B, \omega) \right) + f_{t,M_{t,T}^S}^{x,\tilde{Q}}(B, \omega) \right), \end{aligned} \quad (5.11)$$

where $f_{t,M_{t,T}^S}^{x,\tilde{Q}}(B, \omega)$ is given by equation (C.42).

5.1.2 Hedging Digital Barrier Option with a Random Time by using Malliavin Calculus

In this section, we formally derive the hedging portfolio of digital barrier option with random time by using the Malliavin-calculus approach. Note that the random time τ is a key feature of this option, and we then approximate τ defined in equation (5.1) on a dyadic partition of $[0, T]$. We denote the partitions by $t_i^n = \frac{i}{2^n}T$, where $i = 0, 1, \dots, 2^n$, $n \in \mathbb{N}$ and $t_0^n = 0$ and $t_{2^n}^n = T$. We define a sequence $\{\tau_n\}$ as follows:

$$\tau_n = \sum_{i=1}^{2^n} t_i^n 1_{(t_{i-1}^n, t_i^n]}(\tau) + \infty 1_{\{\tau > T\}}. \quad (5.12)$$

Remark 5.10. The approximation sequence τ_n defined in equation (5.12), has the following properties:

- $\{\tau_n \leq 0\} = \{\tau = 0\} \in \mathcal{F}_0$.
- On $0 < \tau \leq T$, $\tau_n \rightarrow \tau$ in $L^2(\Omega)$.
- From the way in which τ_n is constructed, we have $\tau_n \geq \tau_{n+1}$, for $\omega \in \{0 \leq \tau \leq T\}$.
- For $0 < t \leq T$, we have $\{\tau_n \leq t\} = \bigcup_{t_i^n \leq t} \{t_{i-1}^n < \tau \leq t_i^n\}$, and each term $\{t_{i-1}^n < \tau \leq t_i^n\}$ is $\mathcal{F}_{t_i^n}^n$ -measurable, hence $\{\tau_n \leq t\} \in \mathcal{F}_t$.

Hence τ_n is a stopping time.

What we need is that $\tau_n 1_{(0,T]}(\tau) \rightarrow \tau 1_{(0,T]}(\tau)$ in $L^2(\Omega)$ and also recall that the payoff of Digital barrier option with a random time at time t is given by

$$e^{rt-r\tau} 1_{(0,T]}(\tau). \quad (5.13)$$

Hence, in order to find the hedging portfolio h_t^1 by equation (3.34), we need to approximate $\tau 1_{(0,T]}$ by a decreasing sequence of random times $\tau_n 1_{(0,T]}(\tau)$ and calculate the Malliavin derivative of $e^{-r\tau_n} 1_{(0,T]}(\tau)$ under the conditional expectation, $E^Q [D_t (e^{-r\tau_n} 1_{(0,T]}(\tau)) | \mathcal{F}_t]$. Since the sets $\{t_{i-1}^n < \tau \leq t_i^n\}$ are disjoint with each other, we can write $e^{-r\tau_n} 1_{(0,T]}(\tau)$ as follows:

$$e^{-r\tau_n} 1_{(0,T]}(\tau) = \sum_{i=1}^{2^n} e^{-rt_i^n} 1_{(t_{i-1}^n, t_i^n]}(\tau). \quad (5.14)$$

Assume $t \in (t_j^n, t_{j+1}^n]$ for some j , and split the sum above into three parts:

$$\begin{aligned} & \sum_{i=1}^{2^n} e^{-rt_i^n} 1_{\{t_{i-1}^n < \tau \leq t_i^n\}} \\ &= \sum_{i=1}^j e^{-rt_i^n} 1_{\{t_{i-1}^n < \tau \leq t_i^n\}} + e^{-rt_{j+1}^n} 1_{\{t_j^n < \tau \leq t_{j+1}^n\}} + \sum_{i=j+2}^{2^n} e^{-rt_i^n} 1_{\{t_{i-1}^n < \tau \leq t_i^n\}}. \end{aligned} \quad (5.15)$$

Notice that each indicator function of the first sum term in the above equation is $\mathcal{F}_{t_j^n}$ -measurable and time point t_j^n is strictly less than time t , and by Proposition 2.28, we have

$$\sum_{i=1}^j e^{-rt_i^n} D_t 1_{\{t_{i-1}^n < \tau \leq t_i^n\}} = 0. \quad (5.16)$$

We split the indicator $1_{\{t_j^n < \tau \leq t_{j+1}^n\}}$ into two parts, one with $t < \tau$, i.e.,

$$1_{\{t_j^n < \tau \leq t_{j+1}^n\}} = 1_{\{t_j^n < \tau \leq t\}} + 1_{\{t < \tau \leq t_{j+1}^n\}}. \quad (5.17)$$

Notice also $1_{\{t_j^n < \tau \leq t\}} = 1_{\{M_{0,t_j^n}^S < B\}} 1_{\{M_{t_j^n,t}^S \geq B\}} = 1_{\{M_{0,t_j^n}^S < B\}} - 1_{\{M_{0,t}^S < B\}}$. Taking the

conditional expectation and by Lemma 3.48, we have

$$\begin{aligned}
& E^Q \left[D_t 1_{\{t_j^n < \tau \leq t\}} | \mathcal{F}_t \right] \\
&= E^Q \left[D_t 1_{\{M_{0,t_j^n}^S < B\}} | \mathcal{F}_t \right] - E^Q \left[D_t 1_{\{M_{0,t}^S < B\}} | \mathcal{F}_t \right] \\
&= -E^Q \left[\delta_B(M_{0,t_j^n}^S) D_t M_{0,t_j^n}^S | \mathcal{F}_t \right] + E^Q \left[\delta_B(M_{0,t}^S) D_t M_{0,t}^S | \mathcal{F}_t \right] \\
&= 0,
\end{aligned}$$

due to Proposition 2.28 and Remark 2.38. Hence,

$$\begin{aligned}
& E^Q \left[D_t \left(e^{-r\tau_n} 1_{(0,T]}(\tau) \right) | \mathcal{F}_t \right] \\
&= e^{-rt_{j+1}^n} E^Q \left[D_t 1_{\{t < \tau \leq t_{j+1}^n\}} | \mathcal{F}_t \right] + \sum_{i=j+2}^{2^n} e^{-rt_i^n} E^Q \left[D_t 1_{\{t_{i-1}^n < \tau \leq t_i^n\}} | \mathcal{F}_t \right]. \quad (5.18)
\end{aligned}$$

In fact, from the above equation, we can see that without loss of generality we can modify our partition and start our approximation from time point t on the interval $(t, T]$, such that $t_i^n = t + \frac{i}{2^n}(T - t)$, where $i = 0, 1, \dots, 2^n$ and $n \in N$ and $t_0^n = t$ and $t_{2^n}^n = T$.

Theorem 5.11. For $0 \leq t \leq \tau \leq T$, and a fixed n , the conditional expectation of the Malliavin derivative of the indicator function $1_{\{t_{i-1}^n < \tau \leq t_i^n\}}$ is given by

$$E^Q \left[D_t 1_{\{t_{i-1}^n < \tau \leq t_i^n\}} | \mathcal{F}_t \right] = 1_{\{M_{0,t}^S < B\}} \sigma B \left\{ f_{t, M_{t, t_{i-1}^n}^S}^{x, Q}(B, \omega) - f_{t, M_{t, t_i^n}^S}^{x, Q}(B, \omega) \right\} \quad (5.19)$$

Proof. Note that although we start our partition from t , we have to remember that, for $t \leq \tau$, we have implicit information that $M_{0,t}^S < B$, so we have

$$\begin{aligned}
1_{\{t_{i-1}^n < \tau \leq t_i^n\}} &= 1_{\{M_{0,t}^S < B\}} 1_{\{M_{t, t_{i-1}^n}^S < B\}} 1_{\{M_{t_{i-1}^n, t_i^n}^S \geq B\}} \\
&= 1_{\{M_{0,t}^S < B\}} 1_{\{M_{t, t_{i-1}^n}^S < B\}} \left(1 - 1_{\{M_{t_{i-1}^n, t_i^n}^S < B\}} \right) \\
&= 1_{\{M_{0,t}^S < B\}} 1_{\{M_{t, t_{i-1}^n}^S < B\}} - 1_{\{M_{0,t}^S < B\}} 1_{\{M_{t, t_i^n}^S < B\}}.
\end{aligned}$$

Taking the conditional expectation of the Malliavin derivative of the indicator $1_{\{t_{i-1}^n < \tau \leq t_i^n\}}$,

and according to Lemma 3.48 we have

$$\begin{aligned}
& E^Q \left[D_t 1_{\{t_{i-1}^n < \tau \leq t_i^n\}} | \mathcal{F}_t \right] \\
&= E^Q \left[D_t \left(1_{\{M_{0,t}^S < B\}} 1_{\{M_{t,t_{i-1}^n}^S < B\}} \right) | \mathcal{F}_t \right] - E^Q \left[D_t \left(1_{\{M_{0,t}^S < B\}} 1_{\{M_{t,t_i^n}^S < B\}} \right) | \mathcal{F}_t \right] \\
&= 1_{\{M_{0,t}^S < B\}} \sigma B \left\{ f_{t,M_{t,t_i^n}^S}^{x,Q}(B, \omega) - f_{t,M_{t,t_{i-1}^n}^S}^{x,Q}(B, \omega) \right\}.
\end{aligned}$$

□

Moreover, we know $e^{-r\tau} 1_{(0,T]}(\tau) \in L^2(\Omega)$, and $e^{-r\tau_n} 1_{(0,T]}(\tau) \rightarrow e^{-r\tau} 1_{(0,T]}(\tau)$ in $L^2(\Omega)$. So by Remark 3.26, we have

$$E^Q \left[D_t \left(e^{-r\tau_n} 1_{(0,T]}(\tau) \right) | \mathcal{F}_t \right] \rightarrow E^Q \left[D_t \left(e^{-r\tau} 1_{(0,T]}(\tau) \right) | \mathcal{F}_t \right] \quad \text{in } L^2([0, T] \times \Omega).$$

Hence, take the conditional expectation on equation (5.14), we have

$$\begin{aligned}
& E^Q \left[D_t \left(e^{-r\tau_n} 1_{\{\tau \leq T\}} \right) | \mathcal{F}_t \right] \\
&= \sum_{i=1}^{2^n} e^{-rt_i^n} E^Q \left[D_t 1_{\{t_{i-1}^n < \tau \leq t_i^n\}} | \mathcal{F}_t \right] \\
&= 1_{\{M_{0,t}^S < B\}} \sigma B \sum_{i=1}^{2^n} e^{-rt_i^n} \left\{ f_{t,M_{t,t_i^n}^S}^{x,Q}(B, \omega) - f_{t,M_{t,t_{i-1}^n}^S}^{x,Q}(B, \omega) \right\} \quad (5.20) \\
&\rightarrow 1_{\{M_{0,t}^S < B\}} \sigma B \int_t^T e^{-rl} d \left(f_{t,M_{t,l}^S}^{x,Q}(B, \omega) \right) \quad \text{in } L^2([0, T] \times \Omega).
\end{aligned}$$

At this point, the integral is just a formal expression for a limit we know to exist. So we write

$$E^Q \left[D_t \left(e^{-r\tau} 1_{\{\tau \leq T\}} \right) | \mathcal{F}_t \right] = 1_{\{M_{0,t}^S < B\}} \sigma B \int_t^T e^{-rl} d \left(f_{t,M_{t,l}^S}^{x,Q}(B, \omega) \right). \quad (5.21)$$

where $f_{t,M_{t,l}^S}^{x,Q}(B, \omega)$ is given by equation (C.23).

Now let us find the replicating portfolio h_t^1 by equation (3.34), and consider the case

where $t \geq 0$. We have

$$\begin{aligned} h_t^1 &= e^{rt} \frac{1}{\sigma S_t} E^Q [D_t (e^{-r\tau} 1_{\{\tau \leq T\}}) | \mathcal{F}_t] \\ &= 1_{\{M_{0,t}^S < B\}} \frac{B}{S_t} e^{rt} \int_t^T e^{-rl} \frac{\partial}{\partial l} f_{t, M_{t,l}^S}^{x,Q} (B, \omega) dl, \end{aligned} \quad (5.22)$$

where $f_{t, M_{t,l}^S}^{x,Q} (B, \omega)$ is given in equation (C.23).

Before we finish this section, let us consider a special case of the digital barrier option with a random time, that is the normal digital barrier option. Therefore, the payoff at time maturity T is $1_{\{\tau \leq T\}}$. In the following example, let us calculate the hedging strategy for the digital barrier option by the Malliavin approach and the Δ -hedging approach.

Example 5.12. *The price of the Digital barrier option, at time t , under the risk-neutral measure Q is given by*

$$\begin{aligned} \text{DB}(t) &= e^{-r(T-t)} E^Q [1_{\{\tau \leq T\}} | \mathcal{F}_t] \\ &= e^{-r(T-t)} E^Q [1_{\{t < \tau \leq T\}} | \mathcal{F}_t] + e^{-r(T-t)} E^Q [1_{\{\tau \leq t\}} | \mathcal{F}_t] \end{aligned} \quad (5.23)$$

For $\tau \leq t$, it means the underlying stock S has already hit the barrier level. Therefore for the hedging, there is no number of units to be held in stock but one holds money amount $e^{-r(T-t)}$.

Now we consider the case $t < \tau$,

$$\begin{aligned} \text{DB}(t) &= e^{-r(T-t)} E^Q [1_{\{t < \tau \leq T\}} | \mathcal{F}_t] \\ &= e^{-r(T-t)} 1_{\{M_{0,t}^S < B\}} E^Q [1_{\{M_{t,T}^S > B\}} | \mathcal{F}_t] \\ &= e^{-r(T-t)} 1_{\{M_{0,t}^S < B\}} Q (M_{t,T}^S > B | \mathcal{F}_t) \\ &= e^{-r(T-t)} 1_{\{M_{0,t}^S < B\}} \left(1 - F_{t, M_{t,T}^S}^{x,Q} (B, \omega) \right), \end{aligned} \quad (5.24)$$

where $F_{t, M_{t,T}^S}^{x,Q} (B, \omega)$ is given by equation (C.22). By using the Δ -hedging approach, we have the number of units to be held in stock S at time t is given by

$$h_t^1 = -e^{-r(T-t)} 1_{\{M_{0,t}^S < B\}} \frac{\partial}{\partial S_t} F_{t, M_{t,T}^S}^{x,Q} (B, \omega), \quad (5.25)$$

and due to equation (C.25), we have

$$\begin{aligned} h_t^1 &= e^{-r(T-t)} 1_{\{M_{0,t}^S < B\}} \frac{B}{S_t} \frac{\partial}{\partial B} F_{t,M_{t,T}^S}^{x,Q}(B, \omega) \\ &= e^{-r(T-t)} 1_{\{M_{0,t}^S < B\}} \frac{B}{S_t} f_{t,M_{t,T}^S}^{x,Q}(B, \omega), \end{aligned} \quad (5.26)$$

where $f_{t,M_{t,T}^S}^{x,Q}(B, \omega)$ is given in equation (C.23).

Now consider the Malliavin approach and find the replicating portfolio, h_t^1 by equation (3.34) for $t \geq 0$. We have

$$\begin{aligned} h_t^1 &= e^{-r(T-t)} \frac{1}{\sigma S_t} E^Q [D_t (1_{\{\tau \leq T\}}) | \mathcal{F}_t] \\ &= 1_{\{M_{0,t}^S < B\}} \frac{B}{S_t} e^{-r(T-t)} \int_t^T \frac{\partial}{\partial l} f_{t,M_{t,l}^S}^{x,Q}(B, \omega) dl \\ &= 1_{\{M_{0,t}^S < B\}} \frac{B}{S_t} e^{-r(T-t)} f_{t,M_{t,T}^S}^{x,Q}(B, \omega), \end{aligned} \quad (5.27)$$

where $f_{t,M_{t,T}^S}^{x,Q}(B, \omega)$ is given in equation (C.23) and notice that $f_{t,M_{t,t}^S}^{x,Q}(B, \omega) = 0$.

From equation (5.26) and (5.27), we see the Malliavin approach and the Δ -Hedging approach match each other and give the same results for the digital barrier option.

5.1.3 Numerical Implementation

In this section, we implement our calculation for the Digital barrier option with a random time numerically. In the next two tables, we list different combinations of model parameters and see that the differences between the results from the Malliavin Calculus approach and the Δ -Hedging approach are very small – in fact, they are almost zero. We can conclude that the formula we derived in Chapter 5 and Appendix are correct and that they have been correctly implemented.

For $t = 0$, we numerically implement the equation (5.7) and the equation (5.22) with $t = 0$:

$$h_0^1 = \frac{B}{S_0} \int_0^T e^{-rl} \frac{d}{dl} f_{M_{0,l}^S}^Q(B) dl, \quad (5.28)$$

where $f_{M_{0,l}^S}^Q(B)$ is given in equation (C.21). It is complicated to calculate equation (5.28)

in explicit form, although we know the density function $f_{M_{0,t}^S}^Q(B)$ explicitly. However, we estimate the integral in equation (5.28) by numerical calculation and split interval $[0, T]$ into 2^n partitions with $n = 13$. We implement it by using the standard central-difference method [40] to perform the numerical integration. VBA codes are provided in Appendix.

Figure 5.1: $t = 0$ Comparison of the Malliavin calculus results and the Δ -Hedging results

the payoff is One at time tau								delta-hedging	Malliavin App n=13	
S0	Barrier	t	T	vol	r	mu_tilde	mu	h(0)	h(0)	dif
39	50	0	1	30%	10%	0.145	0.055	4.80369%	4.80366%	0.00003%
40	50	0	1	30%	10%	0.145	0.055	4.93395%	4.93392%	0.00003%
41	50	0	1	30%	10%	0.145	0.055	5.03144%	5.03141%	0.00003%
40	49	0	1	30%	10%	0.145	0.055	5.11833%	5.11830%	0.00003%
40	50	0	1	30%	10%	0.145	0.055	4.93395%	4.93392%	0.00003%
40	51	0	1	30%	10%	0.145	0.055	4.73978%	4.73975%	0.00003%
40	50	0	0.9	30%	10%	0.145	0.055	5.08628%	5.08625%	0.00003%
40	50	0	1	30%	10%	0.145	0.055	4.93395%	4.93392%	0.00003%
40	50	0	1.1	30%	10%	0.145	0.055	4.78732%	4.78729%	0.00003%
40	50	0	1	25%	10%	0.1313	0.0688	5.40235%	5.40232%	0.00003%
40	50	0	1	30%	10%	0.145	0.055	4.93395%	4.93392%	0.00003%
40	50	0	1	35%	10%	0.1613	0.0388	4.54890%	4.54887%	0.00003%
40	50	0	1	30%	5%	0.095	0.005	5.01052%	5.01051%	0.00002%
40	50	0	1	30%	10%	0.145	0.055	4.93395%	4.93392%	0.00003%
40	50	0	1	30%	15%	0.195	0.105	4.78893%	4.78888%	0.00004%

For $t > 0$, we numerically implement equation (5.11) and equation (5.22). Again we estimate the integral in equation (5.22) by numerical calculation and split interval $[0, T]$ into 2^n partitions with $n = 13$. We implement it by using the standard central-difference method [40] to perform the numerical integration. VBA codes are provided in Appendix.

5.2 Protected Barrier Option

In this section, we deal with the so-called “partial” or “protected” barrier option, referring to [5] and [12]. The “partial” barrier option means there is a time period, say $[0, t^*]$ and $t^* \leq T$, for which the predetermined barrier level disappears. We find the partial barrier options described in [5] is slightly different from the ones in [12]. One way it differs is that the protected barrier option in [12] has a rebate amount which is paid if S is greater than

Figure 5.2: $t = 0.5$ Comparison of the Malliavin calculus results and the Δ -Hedging results

the payoff is One at time tau								delta-hedging	Malliavin App	n=13
St	Barrier	t1	t2	vol	r	mu_tilde	mu	h(t)	h(t)	diff
39	50	0.5	1	30%	10%	0.145	0.055	5.09495%	5.09494%	0.00002%
40	50	0.5	1	30%	10%	0.145	0.055	5.57826%	5.57825%	0.00002%
41	50	0.5	1	30%	10%	0.145	0.055	6.01080%	6.01078%	0.00002%
40	49	0.5	1	30%	10%	0.145	0.055	6.05683%	6.05681%	0.00002%
40	50	0.5	1	30%	10%	0.145	0.055	5.57826%	5.57825%	0.00002%
40	51	0.5	1	30%	10%	0.145	0.055	5.10083%	5.10081%	0.00002%
40	50	0.5	0.9	30%	10%	0.145	0.055	5.51142%	5.51140%	0.00001%
40	50	0.5	1	30%	10%	0.145	0.055	5.57826%	5.57825%	0.00002%
40	50	0.5	1.1	30%	10%	0.145	0.055	5.51205%	5.51203%	0.00002%
40	50	0.5	1	25%	10%	0.1313	0.0688	5.60594%	5.60592%	0.00002%
40	50	0.5	1	30%	10%	0.145	0.055	5.57826%	5.57825%	0.00002%
40	50	0.5	1	35%	10%	0.1613	0.0388	5.38206%	5.38205%	0.00002%
40	50	0.5	1	30%	5%	0.095	0.005	5.38757%	5.38756%	0.00001%
40	50	0.5	1	30%	10%	0.145	0.055	5.57826%	5.57825%	0.00002%
40	50	0.5	1	30%	15%	0.195	0.105	5.72058%	5.72056%	0.00003%
40	50	0.4	1	30%	10%	0.145	0.055	5.51205%	5.51203%	0.00002%
40	50	0.5	1	30%	10%	0.145	0.055	5.57826%	5.57825%	0.00002%
40	50	0.6	1	30%	10%	0.145	0.055	5.51142%	5.51140%	0.00001%

the barrier at time t^* and is also paid at time τ if the time τ occurs before time T . This alone makes it different. However, the term τ appearing in the formula in [5] is the length of the monitoring period, i.e. protected period; it is not a random time, but a fixed time.

According to [12], the Protected Barrier Option is an up-and-out call option with a protection period for a fixed period of time, $[0, t^*]$, at the start of the option's life during which the barrier disappears. At the end of this fixed period, the call is knocked out and a rebate C_1 is paid if the underlying stock price at time t^* satisfies $S_{t^*} \geq B > K$, where B is the given barrier level and K is the strike level. Otherwise, the up-and-out call remains alive until the first time after the protection period ends and the underlying stock price hits the barrier, or until expiration, whichever comes first. That is, if after the protection period has elapsed, the underlying stock price hits the barrier prior to expiration, the up-and-out call is knocked out and a constant rebate C_2 is paid at time τ . We denote by τ^p the first time after t^* that S hits B conditional on S_{t^*} is below the barrier level at time t^* , i.e.,

$$\tau^p(\omega) = \inf\{s : t^* < s < T, M_{t^*,s}^S \geq B, S_{t^*} < B\}. \quad (5.29)$$

So on $\{S_{t^*} \geq B\}$, $\tau^p = \infty$. If on the other hand, the stock price has not hit the barrier by the expiration date T , the up-and-out call becomes a standard call, and consequently is exercised if the underlying stock price S_T , finishes above the strike K , and expires worthless otherwise. So the payoff of a Protected Barrier Option at maturity T is

$$G = e^{r(T-t^*)} C_1 1_{\{S_{t^*} \geq B\}} + e^{r(T-\tau^p)} C_2 1_{\{\tau^p < T, S_{t^*} < B\}} + (S_T - K)^+ 1_{\{\tau^p \geq T, S_{t^*} < B\}}. \quad (5.30)$$

The initial price of this option has already been derived in the paper [12]. In order to find the replicating portfolio for the protected barrier option, we need to calculate units of the asset S at time t . Now let us break G into three parts, we let

$$A_1 : = C_1 e^{r(T-t^*)} 1_{\{S_{t^*} \geq B\}}, \quad (5.31)$$

$$A_2 : = C_2 e^{r(T-\tau^p)} 1_{\{\tau^p < T, S_{t^*} < B\}}, \quad (5.32)$$

and since the set $\{\omega : \tau^p(\omega) \geq T, S_{t^*} < B\}$ is equivalent to the set $\{\omega : M_{t^*, T}^S(\omega) < B\}$,

$$A_3 := (S_T - K)^+ 1_{\{\tau^p \geq T, S_{t^*} < B\}} = (S_T - K)^+ 1_{\{M_{t^*, T}^S < B\}}. \quad (5.33)$$

Then we give the calculation of the Malliavin derivative of A_1 , A_2 and A_3 under the conditional expectation in the next three subsections.

5.2.1 Malliavin derivative of A_1 and $E^Q [D_t A_1 | \mathcal{F}_t]$

In this section, let us compute the Malliavin derivative of $A_1 = C_1 e^{r(T-t^*)} 1_{\{S_{t^*} > B\}}$ under the conditional expectation. According to Lemma 3.43, we have

$$\begin{aligned} E^Q [D_t A_1 | \mathcal{F}_t] &= C_1 e^{r(T-t^*)} E^Q [D_t 1_{\{S_{t^*} > B\}} | \mathcal{F}_t] \\ &= C_1 e^{r(T-t^*)} E^Q [\delta_B(S_{t^*}) D_t S_{t^*} | \mathcal{F}_t] \\ &= 1_{[0, t^*]}(t) C_1 e^{r(T-t^*)} \int_0^\infty \delta_B(s) \sigma s f_{t, S_{t^*}}^Q(s, \omega) ds \\ &= 1_{[0, t^*]}(t) C_1 e^{r(T-t^*)} \sigma B f_{t, S_{t^*}}^Q(B, \omega), \end{aligned} \quad (5.34)$$

where $f_{t,S_{t^*}}^Q(B, \omega)$ denotes the $\mathcal{F}_t$ -conditional density function of S_{t^*} under the risk-neutral measure Q and is given by equation (C.15).

5.2.2 Malliavin Derivative of A_2 and $E^Q [D_t A_2 | \mathcal{F}_t]$

In this section, let us compute the Malliavin derivative of $A_2 = C_2 e^{r(T-\tau^p)} 1_{\{\tau^p < T, S_{t^*} < B\}}$ under the conditional expectation. Notice that by the definition for τ^p , if $\tau^p < \infty$, then $S_{t^*} < B$. Therefore, the set $\{\tau^p < T, S_{t^*} < B\}$ is equivalent to the set $\{t^* < \tau^p < T\}$. In this section, we will adopt the same idea that we used in the previous section in the calculation of the Malliavin derivative of $e^{-r\tau}$ under the conditional expectation occurring in the Digital barrier option with a random time by approximation.

For $t < t^*$, we approximate $\tau^p 1_{(t^*, T)}(\tau)$ on a dyadic partition of $[t^*, T]$ and denote the partitions by $t_0^n = t^*$ and $t_i^n = t^* + \frac{i}{2^n}(T - t^*)$, where $i = 1, \dots, 2^n$ and $n \in \mathbb{N}$ and $t_{2^n}^n = T$. Define a sequence $\{\tau_n^p\}$ as follows:

$$\tau_n^p = \sum_{i=1}^{2^n} t_i^n 1_{\{t_{i-1}^n < \tau^p \leq t_i^n\}} + \infty 1_{\{\tau^p > T\}}. \quad (5.35)$$

Note that $e^{-r\tau_n^p} 1_{\{t^* < \tau^p < T\}} = e^{-r\tau_n^p} 1_{\{t^* < \tau^p \leq T\}}$, Q -a.s. and in $L^2(\Omega)$. For a fixed n ,

$$\begin{aligned} 1_{\{t_{i-1}^n < \tau^p \leq t_i^n\}} &= 1_{\left\{M_{t^*, t_{i-1}^n}^S < B\right\}} 1_{\left\{M_{t_{i-1}^n, t_i^n}^S \geq B\right\}} \\ &= 1_{\left\{M_{t^*, t_{i-1}^n}^S < B\right\}} - 1_{\left\{M_{t^*, t_i^n}^S < B\right\}}. \end{aligned}$$

By Lemma 3.48, we have

$$\begin{aligned} E^Q \left[D_t 1_{\left\{M_{t^*, t_{i-1}^n}^S < B\right\}} | \mathcal{F}_t \right] &= -E^Q \left[\delta_B \left(M_{t^*, t_{i-1}^n}^S \right) D_t M_{t^*, t_{i-1}^n}^S | \mathcal{F}_t \right] \\ &= -E^Q \left[\delta_B \left(M_{t^*, t_{i-1}^n}^S \right) \sigma M_{t^*, t_{i-1}^n}^S 1_{[0, \alpha(t^*, t_{i-1}^n)]}(t) | \mathcal{F}_t \right] \\ E^Q \left[D_t 1_{\left\{M_{t^*, t_i^n}^S < B\right\}} | \mathcal{F}_t \right] &= -E^Q \left[\delta_B \left(M_{t^*, t_i^n}^S \right) D_t M_{t^*, t_i^n}^S | \mathcal{F}_t \right] \\ &= -E^Q \left[\delta_B \left(M_{t^*, t_i^n}^S \right) \sigma M_{t^*, t_i^n}^S 1_{[0, \alpha(t^*, t_i^n)]}(t) | \mathcal{F}_t \right]. \end{aligned}$$

Since $t < t^*$, then $1_{[0, \alpha(t^*, t_{i-1}^n)]}(t) = 1$ and $1_{[0, \alpha(t^*, t_i^n)]}(t) = 1$. Therefore, we get

$$\begin{aligned}
& E^Q [D_t (e^{-r\tau_n} 1_{\{t^* < \tau^p < T\}} 1_{\{S_{t^*} < B\}}) | \mathcal{F}_t] \\
&= \sum_{i=1}^{2^n} e^{-rt_i^n} E^Q [D_t 1_{\{t_{i-1}^n < \tau^p \leq t_i^n\}} | \mathcal{F}_t] \\
&= \sum_{i=1}^{2^n} e^{-rt_i^n} \sigma B \left(f_{t, M_{t^*, t_i}^S}^Q(B, \omega) - f_{t, M_{t^*, t_{i-1}}^S}^Q(B, \omega) \right) \\
&\rightarrow \sigma B \int_{t^*}^T e^{-rl} \frac{d}{dl} f_{t, M_{t^*, l}^S}^Q(B, \omega) dl \quad \text{as } n \rightarrow \infty \text{ in } L^2([0, T] \times \Omega). \quad (5.36)
\end{aligned}$$

Therefore, for $t < t^*$, the Malliavin derivative $D_t A_2$ under the conditional expectation is

$$E^Q [D_t A_2 | \mathcal{F}_t] = C_2 e^{rT} \sigma B \int_{t^*}^T e^{-rl} \frac{\partial}{\partial l} f_{t, M_{t^*, l}^S}^Q(B, \omega) dl, \quad (5.37)$$

where $f_{t, M_{t^*, l}^S}^Q(B, \omega)$ denotes the $\mathcal{F}_t$ -conditional density function of $M_{t^*, l}^S$ under the risk-neutral measure Q and is given by equation (C.34).

For $t^* < t < \tau^p$, notice that the event $\{\omega : t^* < \tau^p \leq T \text{ and } S_{t^*} < B\}$ is equivalent to the event $\{\omega : M_{t^*, t}^S < B \text{ and } t < \tau^p \leq T\}$. Therefore, we adopt the same idea as in the previous section to approximate τ^p on a dyadic partition of $[t, T]$ and denote the partitions by $t_0^n = t$ and $t_i^n = t + \frac{i}{2^n}(T - t)$, where $i = 1, \dots, 2^n$ and $n \in \mathbb{N}$ and $t_{2^n}^n = T$. Define a sequence $\{\tau_n^p\}$ as follows:

$$\tau_n^p = \sum_{i=1}^{2^n} t_i^n 1_{\{t_{i-1}^n < \tau^p \leq t_i^n\}} + \infty 1_{\{\tau^p > T\}}. \quad (5.38)$$

By Theorem 5.11, we have

$$\begin{aligned}
& E^Q [D_t (e^{-r\tau_n} 1_{\{t^* < \tau \leq T\}} 1_{\{S_{t^*} < B\}}) | \mathcal{F}_t] \\
&= \sum_{i=1}^{2^n} e^{-rt_i^n} E^Q [D_t (1_{\{M_{t^*, t}^S < B\}} 1_{\{t_{i-1}^n < \tau^p \leq t_i^n\}}) | \mathcal{F}_t] \\
&= 1_{\{M_{t^*, t}^S < B\}} \sigma B \sum_{i=1}^{2^n} e^{-rt_i^n} \left(f_{t, M_{t^*, t_i^n}^S}^{x, Q}(B, \omega) - f_{t, M_{t^*, t_{i-1}^n}^S}^{x, Q}(B, \omega) \right) \\
&\rightarrow 1_{\{M_{t^*, t}^S < B\}} \sigma B \int_t^T e^{-rl} \frac{d}{dl} f_{t, M_{t^*, l}^S}^{x, Q}(B, \omega) dl \quad \text{as } n \rightarrow \infty \text{ in } L^2([0, T] \times \Omega).
\end{aligned}$$

Therefore, for $t^* < t < \tau < T$, the Malliavin derivative $D_t A_2$ under the conditional expectation is

$$E^Q[D_t A_2 | \mathcal{F}_t] = 1_{\{M_{t^*,t}^S < B\}} C_2 e^{rT} \sigma B \int_t^T e^{-rl} \frac{d}{dl} f_{t,M_{t,l}^S}^{x,Q}(B, \omega) dl, \quad (5.39)$$

where $f_{t,M_{t,l}^S}^{x,Q}(B, \omega)$ denotes the $\mathcal{F}_t$ -conditional density function of $M_{t,l}^S$ under the risk-neutral measure Q and is given by equation (C.23).

Remark 5.13. Notice that if we set $t^* = 0$, i.e., there is no protected period, and set $C_2 = 1$, then we find equation (5.39) is exactly the same as equation (5.22).

5.2.3 Malliavin Derivative of A_3 and $E^Q[D_t A_3 | \mathcal{F}_t]$

In this section, let us compute the Malliavin derivative of $A_3 = (S_T - K)^+ 1_{\{M_{t^*,T}^S < B\}}$ under the conditional expectation. By Lemma 3.50 we have:

$$\begin{aligned} & E^Q[D_t A_3 | \mathcal{F}_t] \\ &= E^Q \left[1_{\{M_{t^*,T}^S < B\}} 1_{\{S_T > K\}} D_t S_T | \mathcal{F}_t \right] - E^Q \left[(S_T - K)^+ \delta_B(M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right] \\ &= \sigma E^Q \left[S_T 1_{\{M_{t^*,T}^S < B\}} 1_{\{S_T > K\}} | \mathcal{F}_t \right] + K E^Q \left[1_{\{S_T > K\}} \delta_B(M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right] \\ &\quad - E^Q \left[S_T 1_{\{S_T > K\}} \delta_B(M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right]. \end{aligned}$$

Since S follows the process in equation (1.18), we have $S_T = S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma W_T \right\}$. Let $F = S_0 e^{rT}$ and $\xi = \exp \left\{ -\frac{1}{2} \sigma^2 T + \sigma W_T \right\}$ and define an equivalent probability mea-

sure Q^* by $dQ^*/dQ = \xi$. By Girsanov's theorem, we have

$$\begin{aligned}
& E^Q [D_t A_3 | \mathcal{F}_t] \\
&= \sigma F E^Q \left[\xi 1_{\{M_{t^*,T}^S < B\}} 1_{\{S_T > K\}} | \mathcal{F}_t \right] + K E^Q \left[1_{\{S_T > K\}} \delta_B (M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right] \\
&\quad - F E^Q \left[\xi 1_{\{S_T > K\}} \delta_B (M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right] \\
&= \sigma F E^Q [\xi | \mathcal{F}_t] E^{Q^*} \left[1_{\{M_{t^*,T}^S < B\}} 1_{\{S_T > K\}} | \mathcal{F}_t \right] + K E^Q \left[1_{\{S_T > K\}} \delta_B (M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right] \\
&\quad - F E^Q [\xi | \mathcal{F}_t] E^{Q^*} \left[1_{\{S_T > K\}} \delta_B (M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right] \\
&= \sigma e^{r(T-t)} S_t E^{Q^*} \left[1_{\{M_{t^*,T}^S < B\}} 1_{\{S_T > K\}} | \mathcal{F}_t \right] \tag{5.40}
\end{aligned}$$

$$+ K E^Q \left[1_{\{S_T > K\}} \delta_B (M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right] \tag{5.41}$$

$$- e^{r(T-t)} S_t E^{Q^*} \left[1_{\{S_T > K\}} \delta_B (M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right] \tag{5.42}$$

For $t < t^*$, we compute the conditional expectation in equation (5.40):

$$E^{Q^*} \left[1_{\{M_{t^*,T}^S < B\}} 1_{\{S_T > K\}} | \mathcal{F}_t \right] = F_{t, M_{t^*,T}^S, S_T}^{Q^*}(B, B, \omega) - F_{t, M_{t^*,T}^S, S_T}^{Q^*}(B, K, \omega) \tag{5.43}$$

where $F_{t, M_{t^*,T}^S, S_T}^{Q^*}(\cdot, \cdot, \omega)$ denotes $\mathcal{F}_t$ -conditional joint distribution function of $M_{t^*,T}^S$ and S_T under the measure Q^* and is given by equation (C.47). Now compute the conditional expectation in equation (5.41):

$$\begin{aligned}
& E^Q \left[1_{\{S_T > K\}} \delta_B (M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right] \\
&= E^Q \left[1_{\{S_T > K\}} \delta_B (M_{t^*,T}^S) \sigma M_{t^*,T}^S | \mathcal{F}_t \right] \\
&= \int_0^\infty \int_K^m \delta_B(m) \sigma m f_{t, M_{t^*,T}^S, S_T}^Q(m, s, \omega) ds dm \\
&= \int_0^\infty \delta_B(m) \sigma m \left(\frac{\partial F_{t, M_{t^*,T}^S, S_T}^Q}{\partial m}(m, m, \omega) - \frac{\partial F_{t, M_{t^*,T}^S, S_T}^Q}{\partial m}(m, K, \omega) \right) dm \\
&= \sigma B \left(\frac{\partial F_{t, M_{t^*,T}^S, S_T}^Q}{\partial m}(m, m, \omega)|_{m=B} - \frac{\partial F_{t, M_{t^*,T}^S, S_T}^Q}{\partial m}(m, K, \omega)|_{m=B} \right), \tag{5.44}
\end{aligned}$$

where $F_{t, M_{t^*,T}^S, S_T}^Q(\cdot, \cdot, \omega)$ denotes $\mathcal{F}_t$ -conditional joint cumulative distribution function of $M_{t^*,T}^S$ and S_T under the measure Q and is given by equation (C.36). Now compute the

conditional expectation under the measure Q^* in equation (5.42):

$$\begin{aligned} & E^{Q^*} [1_{\{S_T > K\}} \delta_B (M_{t^*, T}^S) D_t M_{t^*, T}^S | \mathcal{F}_t] \\ &= \sigma B \left(\frac{\partial F_{t, M_{t^*, T}^S, S_T}^{Q^*}}{\partial m} (m, m, \omega) |_{m=B} - \frac{\partial F_{t, M_{t^*, T}^S, S_T}^{Q^*}}{\partial m} (m, K, \omega) |_{m=B} \right), \end{aligned} \quad (5.45)$$

where $F_{t, M_{t^*, T}^S, S_T}^{Q^*}(\cdot, \cdot, \omega)$ is given by equation (C.47). Hence, by putting equations (5.43), (5.44) and (5.45) into equation (5.40), (5.41) and (5.42) we have, for $t < t^*$,

$$\begin{aligned} & E^Q [D_t A_3 | \mathcal{F}_t] \\ &= \sigma e^{r(T-t)} S_t \left(F_{t, M_{t^*, T}^S, S_T}^{Q^*}(B, B, \omega) - F_{t, M_{t^*, T}^S, S_T}^{Q^*}(B, K, \omega) \right) \\ &+ K \sigma B \left(\frac{\partial F_{t, M_{t^*, T}^S, S_T}^Q}{\partial m} (m, m, \omega) |_{m=B} - \frac{\partial F_{t, M_{t^*, T}^S, S_T}^Q}{\partial m} (m, K, \omega) |_{m=B} \right) \\ &- e^{r(T-t)} S_t \sigma B \left(\frac{\partial F_{t, M_{t^*, T}^S, S_T}^{Q^*}}{\partial m} (m, m, \omega) |_{m=B} - \frac{\partial F_{t, M_{t^*, T}^S, S_T}^{Q^*}}{\partial m} (m, K, \omega) |_{m=B} \right), \end{aligned} \quad (5.46)$$

where $F_{t, M_{t^*, T}^S, S_T}^{Q^*}$ is given by equation (C.47) and $F_{t, M_{t^*, T}^S, S_T}^Q$ by equation (C.36).

For $t^* < t \leq T$, we compute the conditional expectation under in equation (5.40).

Noticing that $1_{\{M_{t^*, t}^S < B\}}$ is $\mathcal{F}_t$ -measurable, we have

$$\begin{aligned} & E^{Q^*} [1_{\{M_{t^*, T}^S < B\}} 1_{\{S_T > K\}} | \mathcal{F}_t] \\ &= 1_{\{M_{t^*, t}^S < B\}} E^{Q^*} [1_{\{M_{t^*, T}^S < B\}} 1_{\{S_T > K\}} | \mathcal{F}_t] \\ &= 1_{\{M_{t^*, t}^S < B\}} \left\{ F_{t, M_{t^*, T}^S, S_T}^{x, Q^*}(B, B, \omega) - F_{t, M_{t^*, T}^S, S_T}^{x, Q^*}(B, K, \omega) \right\}, \end{aligned} \quad (5.47)$$

where $F_{t, M_{t^*, T}^S, S_T}^{x, Q^*}$ denotes $\mathcal{F}_t$ -conditional joint cumulative distribution function of $M_{t, T}^S$ and S_T under the measure Q^* and is given by equation (C.48). Now compute the conditional expectation in equation (5.41). Since under the condition $\{\omega : M_{t^*, t}^S \leq M_{t, T}^S\}$, S takes its

maximum value in the interval $[t, T]$, we get

$$\begin{aligned}
& E^Q \left[1_{\{S_T > K\}} \delta_B (M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t \right] \\
&= E^Q \left[1_{\{S_T > K\}} \delta_B (M_{t,T}^S) \sigma M_{t,T}^S 1_{\{M_{t^*,t}^S \leq M_{t,T}^S\}} | \mathcal{F}_t \right] \\
&= \int_{M_{t^*,t}^S}^{\infty} \int_K^m \delta_B(m) \sigma m f_{t,M_{t,T}^S,S_T}^{x,Q}(m, s, \omega) ds dm \\
&= \int_{M_{t^*,t}^S}^{\infty} \delta_B(m) \sigma m \left(\frac{\partial F_{t,M_{t,T}^S,S_T}^{x,Q}}{\partial m}(m, m, \omega) - \frac{\partial F_{t,M_{t,T}^S,S_T}^{x,Q}}{\partial m}(m, K, \omega) \right) dm \\
&= 1_{\{M_{t^*,t}^S < B\}} \sigma B \left(\frac{\partial F_{t,M_{t,T}^S,S_T}^{x,Q}}{\partial m}(m, m, \omega) |_{m=B} - \frac{\partial F_{t,M_{t,T}^S,S_T}^{x,Q}}{\partial m}(m, K, \omega) |_{m=B} \right) \quad (5.48)
\end{aligned}$$

where $F_{t,M_{t,T}^S,S_T}^{x,Q}$ denotes $\mathcal{F}_t$ -conditional joint cumulative distribution function of $M_{t,T}^S$ and S_T under the measure Q and is given by equation (C.19). Now we compute the conditional expectation in equation (5.42):

$$\begin{aligned}
& E^{Q^*} \left[1_{\{S_T > K\}} \delta_B (M_{t^*,T}^S) D_t M_{t^*,T}^S | \mathcal{F}_t | \mathcal{F}_t \right] \\
&= 1_{\{M_{t^*,t}^S < B\}} \sigma B \left(\frac{\partial F_{t,M_{t,T}^S,S_T}^{x,Q^*}}{\partial m}(m, m, \omega) |_{m=B} - \frac{\partial F_{t,M_{t,T}^S,S_T}^{x,Q^*}}{\partial m}(m, K, \omega) |_{m=B} \right) \quad (5.49)
\end{aligned}$$

where $F_{t,M_{t,T}^S,S_T}^{x,Q^*}$ denotes $\mathcal{F}_t$ -conditional joint cumulative distribution function of $M_{t,T}^S$ and S_T under the measure Q^* and is given by equation (C.48). Hence, by putting equations (5.47), (5.48) and (5.49) into equations (5.40), (5.41) and (5.42), we get, for $t^* < t < T$,

$$\begin{aligned}
& E^Q [D_t A_3 | \mathcal{F}_t] \quad (5.50) \\
&= \sigma e^{r(T-t)} S_t 1_{\{M_{t^*,t}^S < B\}} \left(F_{t,M_{t,T}^S,S_T}^{x,Q^*}(B, B, \omega) - F_{t,M_{t,T}^S,S_T}^{x,Q^*}(B, K, \omega) \right) \\
&\quad + \sigma K B 1_{\{M_{t^*,t}^S < B\}} \left(\frac{\partial F_{t,M_{t,T}^S,S_T}^{x,Q}}{\partial m}(m, m, \omega) |_{m=B} - \frac{\partial F_{t,M_{t,T}^S,S_T}^{x,Q}}{\partial m}(m, K, \omega) |_{m=B} \right) \\
&\quad - \sigma e^{r(T-t)} S_t B 1_{\{M_{t^*,t}^S < B\}} \left(\frac{\partial F_{t,M_{t,T}^S,S_T}^{x,Q^*}}{\partial m}(m, m, \omega) |_{m=B} - \frac{\partial F_{t,M_{t,T}^S,S_T}^{x,Q^*}}{\partial m}(m, K, \omega) |_{m=B} \right),
\end{aligned}$$

where $F_{t,M_{t^*,T},S_T}^{x,Q^*}$ is given by equation (C.48) and $F_{t,M_{t^*,T},S_T}^{x,Q}$ by equation (C.19).

For hedging purpose by using the Malliavin approach and putting equations (5.34), (5.37), (5.46) together, we have the number of units to be held in the stock S according to equation (3.34), for $t \leq t^*$:

$$\begin{aligned} h_t^1 = & \frac{e^{r(t-t^*)}BC_1}{S_t} f_{t,S_{t^*}}^Q(B, \omega) + \frac{e^{rt}BC_2}{S_t} \int_{t^*}^T e^{-rl} \frac{\partial}{\partial l} f_{t,M_{t^*,l}^S}^Q(B, \omega) dl \\ & + F_{t,M_{t^*,T},S_T}^{Q^*}(B, B, \omega) - F_{t,M_{t^*,T},S_T}^{Q^*}(B, K, \omega) \\ & + \frac{KB}{e^{r(T-t)}S_t} \left(\frac{\partial F_{t,M_{t^*,T},S_T}^Q}{\partial m}(m, m, \omega)|_{m=B} - \frac{\partial F_{t,M_{t^*,T},S_T}^Q}{\partial m}(m, K, \omega)|_{m=B} \right) \\ & - B \left(\frac{\partial F_{t,M_{t^*,T},S_T}^{Q^*}}{\partial m}(m, m, \omega)|_{m=B} - \frac{\partial F_{t,M_{t^*,T},S_T}^{Q^*}}{\partial m}(m, K, \omega)|_{m=B} \right), \quad (5.51) \end{aligned}$$

where $f_{t,S_{t^*}}^Q$ is given by equation (C.15), $f_{t,M_{t^*,l}^S}^Q$ by equation (C.34), $F_{t,M_{t^*,T},S_T}^{Q^*}$ by equation (C.47) and $F_{t,M_{t^*,T},S_T}^Q$ by equation (C.36).

For $t^* < t < \tau$, by putting equations (5.39), (5.50) together, we have

$$\begin{aligned} h_t^1 = & 1_{\{M_{t^*,t}^S < B\}} e^{rt} \frac{C_2 B}{S_t} \int_t^T e^{-rl} \frac{\partial}{\partial l} f_{t,M_{t,l}^S}^{x,Q}(B, \omega) dl \\ & + 1_{\{M_{t^*,t}^S < B\}} \left(F_{t,M_{t^*,T},S_T}^{x,Q^*}(B, B, \omega) - F_{t,M_{t^*,T},S_T}^{x,Q^*}(B, K, \omega) \right) \\ & + 1_{\{M_{t^*,t}^S < B\}} \frac{KB}{e^{r(T-t)}S_t} \left(\frac{\partial F_{t,M_{t^*,T},S_T}^{x,Q}}{\partial m}(m, m, \omega)|_{m=B} - \frac{\partial F_{t,M_{t^*,T},S_T}^{x,Q}}{\partial m}(m, K, \omega)|_{m=B} \right) \\ & + 1_{\{M_{t^*,t}^S < B\}} B \left(\frac{\partial F_{t,M_{t^*,T},S_T}^{x,Q^*}}{\partial m}(m, K, \omega)|_{m=B} - \frac{\partial F_{t,M_{t^*,T},S_T}^{x,Q^*}}{\partial m}(m, m, \omega)|_{m=B} \right), \quad (5.52) \end{aligned}$$

where $f_{t,M_{t,l}^S}^{x,Q}$ is given by equation (C.23), $F_{t,M_{t^*,T},S_T}^{x,Q^*}$ by equation (C.48) and $F_{t,M_{t^*,T},S_T}^{x,Q}$ by equation (C.19).

5.2.4 Numerical Implementation

In this section, we implement our calculation for the Protected barrier option numerically. Comparing the results, we see the Malliavin hedging results are the same as those using the Δ -hedging approach.

In our numerical calculation, we assume $t = 0$, set the rebate $C_1 = C_2$. and implement equation (5.51) and equation (D.5) numerically.

Again, we estimate the integral in equation (5.51) by splitting the interval $[t^*, T]$ into 2^n partitions with $n = 12$. and implement it by using the standard central difference method [40] to perform the numerical integration. VBA codes are provided in Appendix. See Figure 5.3.

Figure 5.3: Comparison of the Malliavin Calculus results and the Δ -Hedging results $t = 0$

ProtectedBarrier													delta-hedging	Malliavin	n=12
S0	K	B	Rb	t1	t2	vol	r	Divi	mu	mu_tilde	mu_star	Price(0)	h(0)	h(0)	dif
39	40	50	5	0.5	1	30%	10%	0%	6%	14.5%	14.5%	2.4467	0.1810	0.1810	0.0000
40	40	50	5	0.5	1	30%	10%	0%	6%	14.5%	14.5%	2.6254	0.1762	0.1762	0.0000
41	40	50	5	0.5	1	30%	10%	0%	6%	14.5%	14.5%	2.7988	0.1703	0.1703	0.0000
40	39	50	5	0.5	1	30%	10%	0%	6%	14.5%	14.5%	2.7579	0.1716	0.1716	0.0000
40	40	50	5	0.5	1	30%	10%	0%	6%	14.5%	14.5%	2.6254	0.1762	0.1762	0.0000
40	41	50	5	0.5	1	30%	10%	0%	6%	14.5%	14.5%	2.5157	0.1799	0.1799	0.0000
40	40	49	5	0.5	1	30%	10%	0%	6%	14.5%	14.5%	2.6727	0.1788	0.1788	0.0000
40	40	51	5	0.5	1	30%	10%	0%	6%	14.5%	14.5%	2.6005	0.1735	0.1735	0.0000
40	40	50	4	0.5	1	30%	10%	0%	6%	14.5%	14.5%	2.1823	0.1385	0.1385	0.0000
40	40	50	6	0.5	1	30%	10%	0%	6%	14.5%	14.5%	3.0685	0.2140	0.2140	0.0000
40	40	50	5	0.4	1	30%	10%	0%	6%	14.5%	14.5%	2.6944	0.1852	0.1852	0.0000
40	40	50	5	0.6	1	30%	10%	0%	6%	14.5%	14.5%	2.5532	0.1686	0.1686	0.0000
40	40	50	5	0.5	0.9	30%	10%	0%	6%	14.5%	14.5%	2.5459	0.1814	0.1814	0.0000
40	40	50	5	0.5	1.1	30%	10%	0%	6%	14.5%	14.5%	2.6986	0.1716	0.1716	0.0000
40	40	50	5	0.5	1	25%	10%	0%	7%	13.1%	13.1%	2.6435	0.2044	0.2044	0.0000
40	40	50	5	0.5	1	35%	10%	0%	4%	16.1%	16.1%	2.6247	0.1548	0.1548	0.0000
40	40	50	5	0.5	1	30%	5%	0%	1%	9.5%	9.5%	2.4023	0.1831	0.1831	0.0000
40	40	50	5	0.5	1	30%	15%	0%	11%	19.5%	19.5%	2.8280	0.1660	0.1660	0.0000

5.3 Rainbow Barrier Option

In this section, as an application, we apply the multi-dimensional Malliavin calculus we developed in Chapter 4 to find the hedging portfolio of Rainbow Barrier Options [12]. Rainbow Barrier Options signify that there are two underlying securities. One of them

determines the value of the potential payoff, while the other determines whether the owner at maturity will receive the potential payoff or not. More precisely, it is an extension to a European up-and-out call. Like a European up-and-out call, this option is knocked out if the underlying stock price S^1 hit a predetermined barrier $B > S_0^1$ before the option expires at time T . In contrast to a European up-and-out call, if S^1 is below the barrier level all the time prior to T , then the payoff at T is $[S_T^2 - K]^+$; here S_T^2 is the terminal price of asset S^2 . So the payoff at time T of Rainbow barrier option is

$$G = [S_T^2 - K]^+ 1_{\{M_{0,T}^{S^1} < B\}}.$$

Assume S_t^1 and S_t^2 are in the standard Black-Scholes model and given by equation (4.19) and equation (4.20). The Black-Scholes value of Rainbow Barrier Option at time zero is given by [12]. In this section, we compute the Malliavin derivative of G under the conditional expectation, which are given by equation (4.26), (4.27) and (4.28), such that we find the replicating portfolio of Rainbow barrier option via Malliavin calculus.

Now let us find the hedging portfolio h_t^1 according to equation (4.27). The number of units that we should hold in S^1 is given by

$$h_t^1 = \frac{e^{-r(T-t)}}{\sigma_1 S_t^1} \left(E^Q [D_t^1 G | \mathcal{F}_t] - \frac{\rho}{\sqrt{1-\rho^2}} E^Q [D_t^2 G | \mathcal{F}_t] \right).$$

By Lemma 4.15, we know $E^Q [D_t^1 G | \mathcal{F}_t]$ is given by equation (4.31) and $E^Q [D_t^2 G | \mathcal{F}_t]$ by equation (4.31). Therefore,

$$\begin{aligned} h_t^1 &= -\frac{e^{-r(T-t)}}{\sigma_1 S_t^1} E^Q \left[(S_T^2 - K)^+ \delta_B \left(M_{0,T}^{S^1} \right) D_t^1 M_{0,T}^{S^1} | \mathcal{F}_t \right] \\ &= \frac{K e^{-r(T-t)}}{S_t^1} E^Q \left[1_{\{S_T^2 > K\}} \delta_B \left(M_{0,T}^{S^1} \right) M_{0,T}^{S^1} 1_{\{M_{0,t}^{S^1} \leq M_{t,T}^{S^1}\}} | \mathcal{F}_t \right] \\ &\quad - \frac{e^{-r(T-t)}}{S_t^1} E^Q \left[S_T^2 1_{\{S_T^2 > K\}} \delta_B \left(M_{0,T}^{S^1} \right) M_{0,T}^{S^1} 1_{\{M_{0,t}^{S^1} \leq M_{t,T}^{S^1}\}} | \mathcal{F}_t \right]. \end{aligned} \quad (5.53)$$

Now let us calculate the first expectation term of equation (5.53):

$$\begin{aligned}
& E^Q \left[1_{\{S_T^2 > K\}} \delta_B \left(M_{0,T}^{S^1} \right) M_{0,T}^{S^1} 1_{\{M_{0,t}^{S^1} \leq M_{t,T}^{S^1}\}} | \mathcal{F}_t \right] \\
&= \int_K^\infty \int_{M_{0,t}^{S^1}}^\infty \delta_B(m) m f_{t, M_{t,T}^{S^1}, S_T^2}^{x,Q}(m, s, \omega) dm ds \\
&= \int_K^\infty B f_{t, M_{t,T}^{S^1}, S_T^2}^{x,Q}(B, s, \omega) 1_{\{M_{0,t}^{S^1} < B\}} ds \\
&= 1_{\{M_{0,t}^{S^1} < B\}} B \left(\frac{\partial F_{t, M_{t,T}^{S^1}, S_T^2}^{x,Q}}{\partial m}(m, \infty, \omega) |_{m=B} - \frac{\partial F_{t, M_{t,T}^{S^1}, S_T^2}^{x,Q}}{\partial m}(m, K, \omega) |_{m=B} \right) \quad (5.54)
\end{aligned}$$

where $F_{t, M_{t,T}^{S^1}, S_T^2}^{x,Q}$ denotes the $\mathcal{F}_t$ -conditional joint cumulative distribution function of $M_{t,T}^{S^1}$ and S_T^2 under the measure Q and is given by equation (C.61). Now since S_t^2 follows the process in equation (4.20), we have $S_T^2 = S_0^2 \exp \{ (r - \frac{1}{2}\sigma_2^2)T + \rho\sigma_2 W_T^1 + \sqrt{1 - \rho^2}\sigma_2 W_T^2 \}$. Let

$$\begin{aligned}
\xi &= \exp \left\{ -\frac{1}{2}\sigma_2^2 T + \rho\sigma_2 W_T^1 + \sqrt{1 - \rho^2}\sigma_2 W_T^2 \right\} \\
&= \exp \left\{ -\frac{1}{2}\rho^2\sigma_2^2 T + \rho\sigma_2 W_T^1 - \frac{1}{2}(1 - \rho^2)\sigma_2^2 T + \sqrt{1 - \rho^2}\sigma_2 W_T^2 \right\},
\end{aligned}$$

and define an equivalent probability measure $\hat{Q}$ by $d\hat{Q}/dQ = \xi$. By Girsanov's theorem, we have $\hat{W}_t^1 = W_t^1 - \rho\sigma_2 t$ and $\hat{W}_t^2 = W_t^2 - \sqrt{1 - \rho^2}\sigma_2 t$ are standard Brownian motions on the probability space $(\Omega, \mathcal{F}, \hat{Q})$. Note that under the measure $\hat{Q}$

$$\begin{aligned}
\frac{dS_t^1}{S_t^1} &= r dt + \sigma_1 d(\hat{W}_t^1 + \rho\sigma_2 t) = (r + \rho\sigma_1\sigma_2)dt + \sigma_1 d\hat{W}_t^1, \\
\frac{dS_t^2}{S_t^2} &= r dt + \sigma_2 \left(\rho d(\hat{W}_t^1 + \rho\sigma_2 t) + \sqrt{1 - \rho^2} d(\hat{W}_t^2 + \sqrt{1 - \rho^2}\sigma_2 t) \right) \\
&= (r + \sigma_2^2)dt + \sigma_2 \left(\rho d\hat{W}_t^1 + \sqrt{1 - \rho^2} d\hat{W}_t^2 \right).
\end{aligned}$$

Thus the second expectation in equation (5.53) becomes

$$\begin{aligned}
& E^Q \left[S_T^2 1_{\{S_T^2 > K\}} \delta_B \left(M_{0,T}^{S^1} \right) M_{0,T}^{S^1} 1_{\{M_{0,t}^{S^1} \leq M_{t,T}^{S^1}\}} | \mathcal{F}_t \right] \\
&= S_0^2 e^{rT} E^Q \left[\xi 1_{\{S_T^2 > K\}} \delta_B \left(M_{0,T}^{S^1} \right) M_{0,T}^{S^1} 1_{\{M_{0,t}^{S^1} \leq M_{t,T}^{S^1}\}} | \mathcal{F}_t \right] \\
&= S_0^2 e^{rT} E^Q [\xi | \mathcal{F}_t] E^{\hat{Q}} \left[1_{\{S_T^2 > K\}} \delta_B \left(M_{0,T}^{S^1} \right) M_{0,T}^{S^1} 1_{\{M_{0,t}^{S^1} \leq M_{t,T}^{S^1}\}} | \mathcal{F}_t \right] \\
&= S_t^2 e^{r(T-t)} E^{\hat{Q}} \left[1_{\{S_T^2 > K\}} \delta_B \left(M_{0,T}^{S^1} \right) M_{0,T}^{S^1} 1_{\{M_{0,t}^{S^1} \leq M_{t,T}^{S^1}\}} | \mathcal{F}_t \right] \\
&= 1_{\{M_{0,t}^{S^1} < B\}} S_t^2 e^{r(T-t)} B \left(\frac{\partial F_{t,M_{t,T}^{S^1},S_T^2}^{x,\hat{Q}}}{\partial m} (m, \infty, \omega) |_{m=B} - \frac{\partial F_{t,M_{t,T}^{S^1},S_T^2}^{x,\hat{Q}}}{\partial m} (m, K, \omega) |_{m=B} \right), \tag{5.55}
\end{aligned}$$

where $F_{t,M_{t,T}^{S^1},S_T^2}^{x,\hat{Q}}$ denotes the $\mathcal{F}_t$ -conditional joint cumulative distribution function of $M_{t,T}^{S^1}$ and S_T^2 under the measure $\hat{Q}$ and is given by equation (C.62). Finally, put equations (5.54) and (5.55) into equation (5.53); we get

$$\begin{aligned}
h_t^1 &= \frac{KB}{e^{r(T-t)} S_t^1} 1_{\{M_{0,t}^{S^1} < B\}} \left(\frac{\partial F_{t,M_{t,T}^{S^1},S_T^2}^{x,Q}}{\partial m} (m, \infty, \omega) |_{m=B} - \frac{\partial F_{t,M_{t,T}^{S^1},S_T^2}^{x,Q}}{\partial m} (m, K, \omega) |_{m=B} \right) \\
&\quad - \frac{BS_t^2}{S_t^1} 1_{\{M_{0,t}^{S^1} < B\}} \left(\frac{\partial F_{t,M_{t,T}^{S^1},S_T^2}^{x,\hat{Q}}}{\partial m} (m, \infty, \omega) |_{m=B} - \frac{\partial F_{t,M_{t,T}^{S^1},S_T^2}^{x,\hat{Q}}}{\partial m} (m, K, \omega) |_{m=B} \right), \tag{5.56}
\end{aligned}$$

where $F_{t,M_{t,T}^{S^1},S_T^2}^{x,Q}$ is given by equation (C.61) and $F_{t,M_{t,T}^{S^1},S_T^2}^{x,\hat{Q}}$ is given by equation (C.62).

Now let us find the hedging portfolio h_t^2 according to equation (4.28). By Lemma 4.15, $E^Q [D_t^2 G | \mathcal{F}_t]$ is given by equation (4.32). Therefore,

$$\begin{aligned}
h_t^2 &= \frac{e^{-r(T-t)}}{\sigma_2 S_t^2 \sqrt{1-\rho^2}} E^Q \left[1_{\{S_T^2 > K\}} 1_{\{M_{0,T}^{S^1} < B\}} \sqrt{1-\rho^2} \sigma_2 S_T^2 | \mathcal{F}_t \right] \\
&= \frac{e^{-r(T-t)}}{S_t^2} E^Q \left[1_{\{S_T^2 > K\}} 1_{\{M_{0,T}^{S^1} < B\}} S_T^2 | \mathcal{F}_t \right].
\end{aligned}$$

By changing measure and under an equivalent probability measure $\hat{Q}$, we have

$$\begin{aligned}
 h_t^2 &= \frac{e^{-r(T-t)}}{S_t^2} S_t^2 e^{r(T-t)} E^{\hat{Q}} \left[1_{\{S_T^2 > K\}} 1_{\{M_{0,T}^{S^1} < B\}} | \mathcal{F}_t \right] \\
 &= 1_{\{M_{0,t}^{S^1} < B\}} E^{\hat{Q}} \left[1_{\{S_T^2 > K\}} 1_{\{M_{t,T}^{S^1} < B\}} | \mathcal{F}_t \right] \\
 &= 1_{\{M_{0,t}^{S^1} < B\}} \left\{ F_{t, M_{t,T}^{S^1}, S_T^2}^{x, \hat{Q}}(B, \infty, \omega) - F_{t, M_{t,T}^{S^1}, S_T^2}^{x, \hat{Q}}(B, K, \omega) \right\}, \quad (5.57)
 \end{aligned}$$

where $F_{t, M_{t,T}^{S^1}, S_T^2}^{x, \hat{Q}}$ denotes the $\mathcal{F}_t$ -conditional joint cumulative distribution function of $M_{t,T}^{S^1}$ and S_T^2 under the measure $\hat{Q}$ and is given by equation (C.62).

5.3.1 Numerical Implementation

In this section, we implement our calculation for the Rainbow barrier option numerically. Comparing the results, we see the Malliavin hedging results are the same as the ones by using the Δ -hedging approach.

We assume $t = 0$ and implement our calculation in equation (5.56), equation (5.57), equation (D.8) and equation (D.9) numerically in VBA environment. See Figure 5.4.

Figure 5.4: Comparison of the Malliavin Calculus results and the Δ -Hedging results $t = 0$

RainbowBarrier									Formula		Malliavin			
S1_0	S2_0	Strike	Barrier	T	vol1	vol2	rho	r	w.r.t S1	w.r.t S2	h1_0	h2_0	diff_S10	diff_S20
39	40	40	50	1	0.3	0.3	0	0.1	-0.3265	0.3629	-0.3265	0.3629	0.0000	0.0000
40	40	40	50	1	0.3	0.3	0	0.1	-0.3328	0.3291	-0.3328	0.3291	0.0000	0.0000
41	40	40	50	1	0.3	0.3	0	0.1	-0.3367	0.2948	-0.3367	0.2948	0.0000	0.0000
40	39	40	50	1	0.3	0.3	0	0.1	-0.2995	0.3144	-0.2995	0.3144	0.0000	0.0000
40	40	40	50	1	0.3	0.3	0	0.1	-0.3328	0.3291	-0.3328	0.3291	0.0000	0.0000
40	41	40	50	1	0.3	0.3	0	0.1	-0.3676	0.3428	-0.3676	0.3428	0.0000	0.0000
40	40	39	50	1	0.3	0.3	0	0.1	-0.3593	0.3432	-0.3593	0.3432	0.0000	0.0000
40	40	41	50	1	0.3	0.3	0	0.1	-0.3078	0.3148	-0.3078	0.3148	0.0000	0.0000
40	40	40	49	1	0.3	0.3	0	0.1	-0.3430	0.3011	-0.3430	0.3011	0.0000	0.0000
40	40	40	51	1	0.3	0.3	0	0.1	-0.3216	0.3556	-0.3216	0.3556	0.0000	0.0000
40	40	40	50	0.9	0.3	0.3	0	0.1	-0.3221	0.3431	-0.3221	0.3431	0.0000	0.0000
40	40	40	50	1.1	0.3	0.3	0	0.1	-0.3418	0.3165	-0.3418	0.3165	0.0000	0.0000
40	40	40	50	1	0.25	0.3	0	0.1	-0.3683	0.3655	-0.3683	0.3655	0.0000	0.0000
40	40	40	50	1	0.35	0.3	0	0.1	-0.3044	0.3011	-0.3044	0.3011	0.0000	0.0000
40	40	40	50	1	0.3	0.25	0	0.1	-0.2979	0.3361	-0.2979	0.3361	0.0000	0.0000
40	40	40	50	1	0.3	0.35	0	0.1	-0.3684	0.3252	-0.3684	0.3252	0.0000	0.0000
40	40	40	50	1	0.3	0.3	-0.5	0.1	-0.3313	0.4279	-0.3313	0.4279	0.0000	0.0000
40	40	40	50	1	0.3	0.3	0.5	0.1	-0.2738	0.2301	-0.2738	0.2301	0.0000	0.0000
40	40	40	50	1	0.3	0.3	0	0.05	-0.2869	0.3354	-0.2869	0.3354	0.0000	0.0000
40	40	40	50	1	0.3	0.3	0	0.15	-0.3731	0.3136	-0.3731	0.3136	0.0000	0.0000

Chapter 6

Conclusion and Future Research

The thesis demonstrates that Malliavin calculus can be useful for pricing and hedging of different securities with a discontinuous payoff function, say Barrier style options. We have shown in this thesis how to use the extended Malliavin calculus to generate the the hedging portfolio formula for these contingent claims in $L^2(\Omega)$. We have compared the results we get from the Malliavin approach with those from the Δ -hedging approach.

We have developed a lot of useful propositions for the Malliavin derivative via the relationship between the directional derivative and the Malliavin derivative, and given a more detailed treatment in the thesis.

We then make a mathematical contribution by providing an elementary calculus for the composition of a generalized function with a stochastic variable in the presence of a conditional expectation, i.e.,

$$E[g(X)|\mathcal{F}_t] = \int_0^\infty g(b)f_X(b, \omega)db, \quad (6.1)$$

We have been unable to find this result in the literature. It may prove useful subsequently.

We find the explicit hedging formula for the two-dimensional securities by using the extended Malliavin calculus. The calculations we mentioned in the thesis can be extended to higher dimensions to many possible applications, say, a basket of equities.

The computation of the Malliavin derivative of the stopping time discussed in the thesis suggests two things.

- That the Malliavin derivative of other random times may be computed so long as the level sets: $\tau \leq t$ can be identified in terms of other sets whose Malliavin derivative are known.
- That more general payoffs, involving functions of the time τ , say $g(\tau)$, could be priced and hedged for suitable functions g .

Again, we have been unable to find an explicit computation of the Malliavin derivative of a random time in the literature.

We restricted our assets to the Black-Scholes model in order to find the explicit formula for the replicating portfolio of the exotic barrier options and make a explicit formula comparing with a classical delta hedging formula. However, we also notice that Malliavin calculus can be extended to assets which do not follow the Black-Scholes model. For example, in [1], a Heston stochastic volatility model is considered,

$$dS_t = \alpha S_t dt + \sqrt{v_t} S_t dW_t^1, \quad (6.2)$$

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dW_t^2, \quad (6.3)$$

where W^1 and W^2 are two correlated Brownian motions. The explicit expression for the Malliavin derivative of v_t is given by [1]. Moreover, we also believe that our expression in equation (6.1) can be extended to a model where the volatility σ becomes a function, $\sigma(S_t)$, of the current stock value S_t . Bermin suggests as much in equation 7.1 of [5]. It would benefit from considering stochastic volatility model, but from the theoretical point of view, in order to find the explicit formula, the Black-Scholes model is enough for our purpose. We leave it as future research work.

There are several areas for the application of the Malliavin calculus, such as the calculation of “Greeks”, hedging portfolio selection. For the application to sensitivity analysis and computation of Greeks in Finance, there are a large amount of published papers in this area, see [6], [7], [20], [21],[22] and [29]. These papers address far more general payoff functions than we consider here, but the approach is to simulate the payoff rather than seek an analytic form. The method is to use integration by parts formula to obtain the “Malliavin Weight” rather than employ the Clark-Ocone formula directly.

The so-called Greeks represent the pricing sensitivities of financial derivatives with respect to small changes of model parameters, say

$$Greek := \frac{E^x[f(V_T)] - E^{x+\delta}[f(V_T)]}{\delta}, \quad (6.4)$$

where x represents the model parameter and $f(V_T)$ denotes the discounted payoff function. Often, the greeks cannot be represented in explicit formula and need numerical implementation. Practically, given a model parameter x , by using Monte Carlo simulation one generates millions of scenarios and takes the average of the discounted payoffs. Then one changes the model parameter from x to $x + \delta$ and regenerates the scenarios and take the average of the discounted payoffs. However, as far as *integration by parts formula* is concerned, one can transform

$$\frac{E^{x+\delta}[f(V_T)] - E^x[f(V_T)]}{\delta} = E^x[f(V_T)M.W.], \quad (6.5)$$

where $M.W.$ represents a random variable named Malliavin weight. One needs the Malliavin derivative to calculate the Malliavin weight. One advantage of this transform is that for a discontinuous payoff, one avoids the difficult of the differentiability of the discontinuous function, but one needs to specify the model, i.e. the diffusion process.

Greeks calculation is one area application of the Malliavin calculus and normally it requires numerical calculation and simulations in implicit form. However, our thesis has another direction of the application of the Malliavin calculus, i.e. the extension of the *Clark-Ocone formula*, as a generalization of the Itô representation formula, from which we can find the hedging portfolio formula explicitly.

We also considered a problem where the numerical value of the volatility changes at time τ , the first time a security hits a barrier level. Consider that a risky asset, S , has dynamics $S_t = S_0 \exp \{ \sigma_1 W_t + (r - \sigma_1^2/2) t \}$ unless at time, τ , it hits a barrier level B for the first time. Thereafter, the volatility becomes σ_2 . The dynamic process of S_t is,

$$\begin{aligned} S_t = & S_0 \exp \{ \sigma_1 W_t + (r - \sigma_1^2/2) t \} 1_{[0, \tau \wedge T)}(t) \\ & + S_\tau \exp \{ \sigma_2 (W_t - W_\tau) + (r - \sigma_2^2/2) (t - \tau) \} 1_{(\tau \wedge T, T]}(t). \end{aligned}$$

We imagine that when S hits the level B , the market changes its view of S and this results in a change of volatility. We consider an option on S with payoff $(S_T - K)^+$. This payoff has two components:

$$\begin{aligned}
 & (S_T - K)^+ 1_{\{\tau \leq T\}} + (S_T - K)^+ 1_{\{\tau > T\}} \\
 = & \left(S_\tau e^{\sigma_2(W_T - W_\tau) + (r - \sigma_2^2/2)(T - \tau)} - K \right)^+ 1_{\{\tau \leq T\}} + \left(S_0 e^{\sigma_1 W_T + (r - \sigma_1^2/2)T} - K \right)^+ 1_{\{\tau > T\}} \\
 = & (S_T^2 - K)^+ 1_{\{\tau \leq T\}} + (S_T^1 - K)^+ 1_{\{\tau > T\}} \\
 = & (S_T^2 - K)^+ 1_{\{M_{0,T}^{S^1} \geq B\}} + (S_T^1 - K)^+ 1_{\{M_{0,T}^{S^1} < B\}} .
 \end{aligned}$$

The possible extensions of this problem is, to a finite sequence of times, τ_n , which are the first times of hitting barriers B_n , after time τ_{n-1} where S hit barrier B_{n-1} . However, we do not have time to continue this problem here, so we leave it as a possible future work.

Appendix A

Wiener polynomial

Lemma A.1. For $f(s), g(s) \in L^2([0, T])$, we have

- a) For $f_n(s) \in L^2([0, T])$ such that $\|f - f_n\|_2 \rightarrow 0$, we have

$$\int_0^T f_n(s) dW_s \rightarrow \int_0^T f(s) dW_s \quad \text{in } \|\cdot\|_p, \quad \forall p \in [1, \infty). \quad (\text{A.1})$$

- b) Every Wiener polynomial is in $L^2(\Omega)$.
- c) Every Wiener polynomial is in $L^{2r}(\Omega)$, for $r = 1, 2, \dots$

Proof. a) We write $X = \int_0^T f(s) dW_s$ and $X_n = \int_0^T f_n(s) dW_s$. By Itô isometry property, we know that

$$X - X_n \sim N(0, \|f - f_n\|_2^2). \quad (\text{A.2})$$

Since $\|f - f_n\|_2 \rightarrow 0$, we have

$$X - X_n \rightarrow 0 \quad \text{in } L^2(\Omega). \quad (\text{A.3})$$

For $1 \leq p \leq 2$, we have $L^2(\Omega) \subseteq L^p(\Omega) \subseteq L^1(\Omega)$ and $\|\cdot\|_1 \leq \|\cdot\|_p \leq \|\cdot\|_2$. Then

$$\|X - X_n\|_p \rightarrow 0, \quad (\text{A.4})$$

for each $p \in [1, 2]$. So $X_n \rightarrow X$ in $\|\cdot\|_p$ for every $p \in [1, 2]$. Now for $p \in (2, \infty)$, we write

$\sigma_n^2 = \|f - f_n\|_2^2$ and have,

$$\begin{aligned} E[|X - X_n|^p] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_n^2}} |x|^p e^{-\frac{1}{2}\left(\frac{x}{\sigma_n}\right)^2} dx \\ &= \sigma_n^p \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left|\frac{x}{\sigma_n}\right|^p e^{-\frac{1}{2}\left(\frac{x}{\sigma_n}\right)^2} d\left(\frac{x}{\sigma_n}\right) \\ &= \sigma_n^p \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} t^p e^{-\frac{1}{2}t^2} dt \\ &< \infty. \end{aligned}$$

This result shows several things;

- Replace $X - X_n$ with X or X_n , and we see that $X, X_n \in L^p(\Omega)$ for $1 \leq p < \infty$.
- Let $n \rightarrow \infty$ and observe $\sigma_n^p \rightarrow 0$, so that $X_n \rightarrow X$ in $\|\cdot\|_p$ for every $p \in [1, \infty)$.

b) Let $l \in \mathbb{N}$ and consider X^l . By a), $X \in L^p(\Omega)$ for $1 \leq p < \infty$. Of course, $X \in L^{2l}(\Omega)$. So

$$X^l \in L^2(\Omega). \quad (\text{A.5})$$

For $g_n(s) \in L^2([0, T])$ such that $\|g - g_n\|_2 \rightarrow 0$, we write $Y = \int_0^T g(s) dW_s$ and $Y_n = \int_0^T g_n(s) dW_s$. Let $k \in \mathbb{N}$. We know that X^l and Y^k are both in $L^2(\Omega)$ as are X^{2l} and Y^{2k} , therefore

$$E[X^{2l}Y^{2k}] \leq \|X^{2l}\|_2 \|Y^{2k}\|_2 < \infty.$$

So

$$X^l Y^k \in L^2(\Omega). \quad (\text{A.6})$$

Now let $Z_i = \int_0^T h_i(s) dW_s$, for $i = 1, \dots, m$. Suppose that we know that all expressions of the form

$$A \equiv Z_1^{l_1} Z_2^{l_2} \dots Z_{m-1}^{l_{m-1}} \in L^2(\Omega) \quad (\text{A.7})$$

for $l_1, \dots, l_{m-1} \in \mathbb{N}$. Then write $Z_1^{l_1} Z_2^{l_2} \dots Z_{m-1}^{l_{m-1}} Z_m^{l_m} = AZ_m^{l_m}$. Note that $A^2 \in L^2(\Omega)$ and $Z_m^{2l_m} \in L^2(\Omega)$, then

$$E[(AZ_m^{l_m})^2] = E[A^2 Z_m^{2l_m}] \leq \|A^2\|_2 \|Z_m^{2l_m}\|_2 < \infty.$$

So

$$Z_1^{l_1} Z_2^{l_2} \cdots Z_{m-1}^{l_{m-1}} Z_m^{l_m} \in L^2(\Omega). \quad (\text{A.8})$$

The principle of mathematical induction tells us that all finite products of finite powers of *deterministic* stochastic integrals are elements of $L^2(\Omega)$. So every Wiener Polynomial is in $L^2(\Omega)$.

c) Since $X^l \in L^2(\Omega)$, for $l \geq 1$, then $E[X^{2rl}] < \infty$, for $r = 1, 2, \dots$. That is,

$$X^l \in L^{2r}(\Omega). \quad (\text{A.9})$$

So $Y^k \in L^{2r}(\Omega)$ too and X^{2rl}, Y^{2rk} are in $L^2(\Omega)$. Therefore,

$$E[X^{2rl} Y^{2rk}] \leq \|X^{2rl}\|_2 \|Y^{2rk}\|_2 < \infty.$$

So $X^l Y^k \in L^{2r}(\Omega)$. Re-run the induction proof above, we have

$$E[A^{2r} Z_m^{2rl_m}] \leq \|A^{2r}\|_2 \|Z_m^{2rl_m}\|_2 < \infty. \quad (\text{A.10})$$

So all Wiener Polynomials are in $L^{2r}(\Omega)$, for $r = 1, 2, \dots$ □

Lemma A.2. Every Wiener Polynomial in the variables $\int_0^T f(s) dW_s, \dots, \int_0^T h(s) dW_s$ is the limit in $L^2(\Omega)$ of the sequence of Wiener Polynomials in the variables $\int_0^T f_n(s) dW_s, \dots, \int_0^T h_n(s) dW_s$, where $f_n(s), \dots, h_n(s) \in L^2([0, T])$ such that $\|f - f_n\|_2 \rightarrow 0, \dots, \|h - h_n\|_2 \rightarrow 0$.

Proof. We write $X = \int_0^T f(s) dW_s$ and $X_n = \int_0^T f_n(s) dW_s$. Consider $X^l, l \in \mathbb{N}$. By Lemma A.1, we know $X_n \rightarrow X$ in $L^2(\Omega)$ and therefore Q -a.s. So

$$X_n^l \rightarrow X^l, \quad Q - a.s.$$

Now

$$E[X_n^{2l}] = \|f_n\|_2^{2l} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} t^{2l} e^{-\frac{t^2}{2}} dt,$$

and $(\|f_n\|_2)$ is a convergent and therefore bounded sequence. This shows that

$$\sup_n \|X_n^l\|_2 < \infty.$$

Now $L^2(\Omega)$ is a reflexive space and a bounded subset is uniformly integrable. This means that (X_n^l) satisfy the conditions of Vitali's Convergence Theorem, (see III. 6.15 of [19]) and we conclude that

$$X_n^l \rightarrow X^l, \quad \text{in } L^2(\Omega). \quad (\text{A.11})$$

We write $Y = \int_0^T g(s) dW_s$, $Y_n = \int_0^T g_n(s) dW_s$, $Z = \int_0^T h(s) dW_s$ and $Z_n = \int_0^T h_n(s) dW_s$. Consider $X_n^l Y_n^m \cdots Z_n^r$ with $X_n \rightarrow X$, $Y_n \rightarrow Y$, $\dots$, $Z_n \rightarrow Z$ in $L^2(\Omega)$. It follows that they all converge Q -a.s. as well and so

$$X_n Y_n \cdots Z_n \rightarrow XY \cdots Z, \quad Q - a.s.$$

Note that each of $X_n, Y_n, \dots, Z_n$, lie in every $L^p(\Omega)$, $p \in [1, \infty)$, as do their powers, $X_n^l, Y_n^m, \dots, Z_n^r$. Suppose that there are b factors in this product of powers, by Hölder's inequality

$$\|X_n^l Y_n^m \cdots Z_n^r\|_2 \leq \|X_n^l\|_{2b} \|Y_n^m\|_{2b} \cdots \|Z_n^r\|_{2b}.$$

Now

$$\|X_n^l\|_{2b}^{2b} = E[X_n^{2bl}] = \|f_n\|_2^{2bl} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^{2bl} e^{-\frac{x^2}{2}} dx, \quad (\text{A.12})$$

so that $\|X_n^l\|_{2b} = \|f_n\|_2^l m_{2bl}^{1/2b}$, where m_{2bl} is the $2bl$ -th absolute moment of the standard normal random variable. Noting that $(\|f_n\|_2)$ is a convergent sequence, we have that

$$\sup_n \|X_n^l\|_{2b} < \infty.$$

Similarly, $\sup_n \|Y_n^k\|_{2b} < \infty, \dots, \sup_n \|Z_n^r\|_{2b} < \infty$. So

$$\sup_n \|X_n^l Y_n^k \cdots Z_n^r\|_2 < \infty.$$

Therefore, the set $\{X_n^l Y_n^k \cdots Z_n^r : n \in \mathbb{N}\}$ is a uniformly integrable set in $L^2(\Omega)$. Once

again the Vitali Convergence theorem tells us that

$$X_n^l Y_n^k \cdots Z_n^r \rightarrow X^l Y^k \cdots Z^r, \quad \text{in } L^2(\Omega), \quad (\text{A.13})$$

with which we complete our proof. □

Appendix B

Distributions

Lemma B.1. For a fixed constant $B > 0$, let $\Psi_{0,B,n}$ be given by, see Figure B.1,

$$\Psi_{0,B,n}(x) = \begin{cases} 0 & x \leq -\frac{1}{n} \text{ or } x > B + \frac{1}{n}, \\ \int_{-\frac{1}{n}}^x \lambda_{-\frac{1}{n},0}(t) dt & -\frac{1}{n} < x \leq 0, \\ 1 & 0 < x \leq B, \\ 1 - \int_{B-x}^0 \lambda_{-\frac{1}{n},0}(t) dt & B < x \leq B + \frac{1}{n}, \end{cases} \quad (\text{B.1})$$

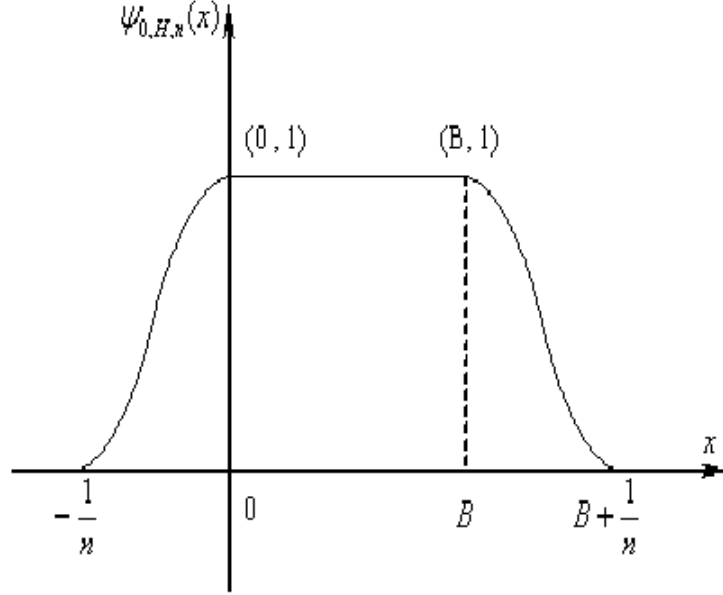
where $\lambda_{-\frac{1}{n},0}$ is given by equation (1.6) and a sequence of positive test function such that $\lambda_{-\frac{1}{n},0}$ is non-zero in $(-\frac{1}{n}, 0)$, zero outside of this interval and such that $\int_{-\frac{1}{n}}^0 \lambda_{-\frac{1}{n},0}(x) dx = 1$. Denote $T_{\Psi_{0,B,n}}$ by the distributional derivative of $\Psi_{0,B,n}$. Then

$$\lim_{n \rightarrow \infty} T_{\Psi_{0,B,n}}(\cdot) = \delta_0(\cdot) - \delta_B(\cdot) \quad (\text{B.2})$$

in the sense of distribution.

Proof. Note that for each n , $\Psi_{0,B,n}$ is a test function. Also it converges to the indicator function $1_{[0,B]}(x)$ pointwise on $\mathbb{R}$, i.e.,

$$\Psi_{0,B,n}(x) \rightarrow 1_{[0,B]}(x), \quad \text{as } n \rightarrow \infty. \quad (\text{B.3})$$

Figure B.1: A mesa function $\Psi_{0,B,n}$


Now for a test function $f \in S(\mathbb{R})$, we consider

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \Psi_{0,B,n}(x) f'(x) dx \\
 &= \int_{-\frac{1}{n}}^0 \Psi_{0,B,n}(x) f'(x) dx + \int_0^B \Psi_{0,B,n}(x) f'(x) dx + \int_B^{B+\frac{1}{n}} \Psi_{0,B,n}(x) f'(x) dx \\
 &= \int_{-\frac{1}{n}}^0 \Psi_{0,B,n}(x) f'(x) dx + f(B) - f(0) + \int_B^{B+\frac{1}{n}} \Psi_{0,B,n}(x) f'(x) dx . \quad (\text{B.4})
 \end{aligned}$$

Consider the first integral in the above equation, and we have

$$\begin{aligned}
 \int_{-\frac{1}{n}}^0 \Psi_{0,B,n}(x) f'(x) dx &= [\Psi_{0,B,n}(x) f(x)]_{-\frac{1}{n}}^0 - \int_{-\frac{1}{n}}^0 \lambda_{-\frac{1}{n},0}(x) f(x) dx \\
 &= f(0) - \int_{-\frac{1}{n}}^0 \lambda_{-\frac{1}{n},0}(x) f(x) dx . \quad (\text{B.5})
 \end{aligned}$$

Similarly,

$$\begin{aligned} \int_B^{B+\frac{1}{n}} \Psi_{0,B,n}(x) f'(x) dx &= [\Psi_{0,B,n}(x) f(x)]_B^{B+\frac{1}{n}} - \int_B^{B+\frac{1}{n}} \lambda_{-\frac{1}{n},0}(B-x) f(x) dx \\ &= -f(B) + \int_{-\frac{1}{n}}^0 \lambda_{-\frac{1}{n},0}(x) f(B-x) dx. \end{aligned} \quad (\text{B.6})$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \Psi_{0,B,n}(x) f'(x) dx &\rightarrow f(B) - f(0) \\ &= \int_{-\infty}^{\infty} (\delta_B(x) - \delta_0(x)) f(x) dx. \end{aligned}$$

By Proposition 1.23, we know

$$\int_{-\infty}^{\infty} \Psi_{0,B,n}(x) f'(x) dx = - \int_{-\infty}^{\infty} T_{\Psi_{0,B,n}(x)} f(x) dx.$$

Hence,

$$\lim_{n \rightarrow \infty} T_{\Psi_{0,B,n}}(\cdot) = \delta_0(\cdot) - \delta_B(\cdot) \quad (\text{B.7})$$

in the sense of distribution. $\square$

Lemma B.2. For a fixed $K > 0$, let $\Phi_n(x)$ be given by, see Figure B.2,

$$\Phi_n(x) = \begin{cases} 0 & \text{if } x \in (-\infty, K - \frac{1}{n}), \\ h_n(x) & \text{if } x \in [K - \frac{1}{n}, K + \frac{1}{n}], \\ x - K & \text{if } x \in [K + \frac{1}{n}, \infty), \end{cases} \quad (\text{B.8})$$

where

$$h_n(x) = \int_{K-\frac{1}{n}}^x g_n(r) dr, \quad x \in \left[K - \frac{1}{n}, K + \frac{1}{n} \right]. \quad (\text{B.9})$$

and

$$g_n(t) = \int_{K-\frac{1}{n}}^t \lambda_{K-\frac{1}{n}, K+\frac{1}{n}}(s) ds \quad (\text{B.10})$$

with $g_n(K + \frac{1}{n}) = 1$ and $g_n(K - \frac{1}{n}) = 0$, see Figure B.3. Let T_{Φ_n} denote the distributional

Figure B.2: The function $\Phi_n(x)$

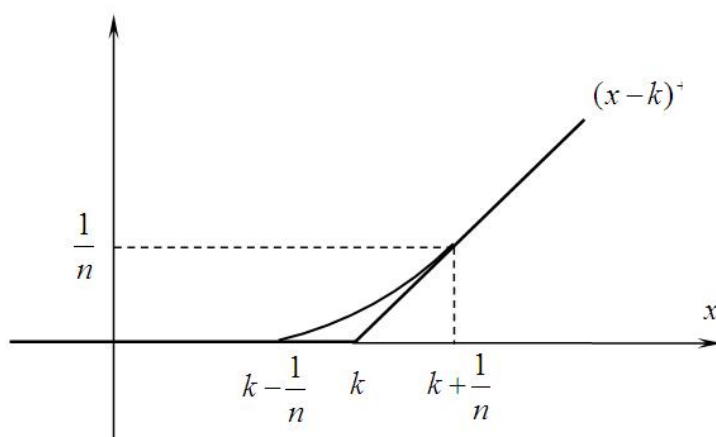
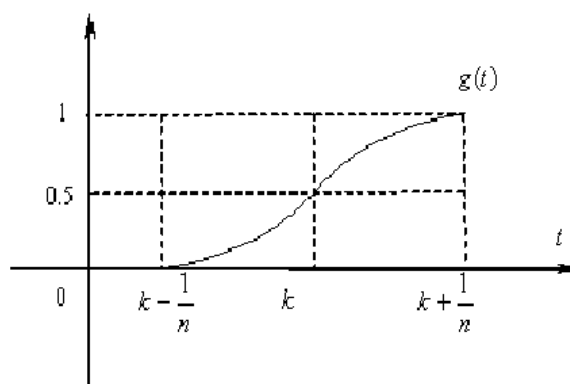


Figure B.3: The function $g(t)$



derivative of $\Phi_n(x)$. Then,

$$\lim_{n \rightarrow \infty} T_{\Phi_n}(\cdot) = 1_{[K, \infty)}(\cdot) \quad (\text{B.11})$$

in the sense of distribution.

Proof. Observe that $\Phi_n(x)$ is differentiable. For example, the first derivative is

$$\Phi'_n(x) = \begin{cases} 0 & \text{if } x \in (-\infty, K - \frac{1}{n}), \\ g_n(x) & \text{if } x \in [K - \frac{1}{n}, K + \frac{1}{n}], \\ 1 & \text{if } x \in [K + \frac{1}{n}, \infty), \end{cases} \quad (\text{B.12})$$

and its second derivative is

$$\Phi''_n(x) = \begin{cases} 0 & \text{if } x \in (-\infty, K - \frac{1}{n}), \\ \lambda_{K-\frac{1}{n}, K+\frac{1}{n}}(x) & \text{if } x \in [K - \frac{1}{n}, K + \frac{1}{n}], \\ 0 & \text{if } x \in [K + \frac{1}{n}, \infty). \end{cases}$$

The pulse $\lambda_{K-\frac{1}{n}, K+\frac{1}{n}}$ is a test function, so the function Φ_n is infinitely differentiable. Moreover Φ_n is a decreasing function as n gets larger, i.e. $\Phi_1(x) \geq \Phi_2(x) \geq \dots \Phi_n(x) \geq \dots$, and if $K > 0$, there is $N \in \mathbb{N}$ such that $K - \frac{1}{N} > 0$, so the sequence $(\Phi_n), n \in \mathbb{N}$, will approximate $(x - K)^+$ pointwise, i.e.,

$$\Phi_n(x) \rightarrow (x - K)^+, \quad \text{as } n \rightarrow \infty. \quad (\text{B.13})$$

Now for a test function $f(x) \in S(\mathbb{R})$, we consider

$$\int_{-\infty}^{\infty} \Phi_n(x) f'(x) dx = \int_{K-\frac{1}{n}}^{K+\frac{1}{n}} h_n(x) f'(x) dx + \int_{K+\frac{1}{n}}^{\infty} (x - K) f'(x) dx.$$

Consider the first integral of the above equation, we have

$$\begin{aligned} \int_{K-\frac{1}{n}}^{K+\frac{1}{n}} h_n(x) f'(x) dx &= [h_n(x) f(x)]_{K-\frac{1}{n}}^{K+\frac{1}{n}} - \int_{K-\frac{1}{n}}^{K+\frac{1}{n}} h'_n(x) f(x) dx \\ &= \frac{1}{n} f(K + \frac{1}{n}) - \int_{K-\frac{1}{n}}^{K+\frac{1}{n}} g_n(x) f(x) dx. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{K+\frac{1}{n}}^{\infty} (x-K)f'(x)dx &= [(x-K)f(x)]_{K+\frac{1}{n}}^{\infty} - \int_{K+\frac{1}{n}}^{\infty} f(x)dx \\ &= \frac{1}{n}f(K-\frac{1}{n}) - \int_{K+\frac{1}{n}}^{\infty} f(x)dx. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \Phi_n(x)f'(x)dx = - \int_K^{\infty} f(x)dx = - \int_{-\infty}^{\infty} 1_{\{x \geq K\}}f(x)dx. \quad (\text{B.14})$$

Moreover, by Proposition 1.23 we have

$$\int_{-\infty}^{\infty} \Phi_n(x)f'(x)dx = - \int_{-\infty}^{\infty} T_{\Phi_n}(x)f(x)dx. \quad (\text{B.15})$$

Therefore,

$$\lim_{n \rightarrow \infty} T_{\Phi_n}(\cdot) = 1_{[K, \infty)}(\cdot) \quad (\text{B.16})$$

in the sense of distribution. $\square$

Remark B.3. In some cases, we need to modify Φ_n here by making it a bounded function. So we define $\hat{\Phi}_n$ be equal to, see Figure B.4,

$$\hat{\Phi}_n(x) = \begin{cases} \Phi_n & \text{if } x \leq [K] + 1 + n, \\ [K] + 1 + n - K + m(x - ([K] + 1 + n)) & \text{if } x \in ([K] + 1 + n, [K] + 2 + n), \\ [K] + 2 + n - K & \text{if } x > [K] + 2 + n, \end{cases} \quad (\text{B.17})$$

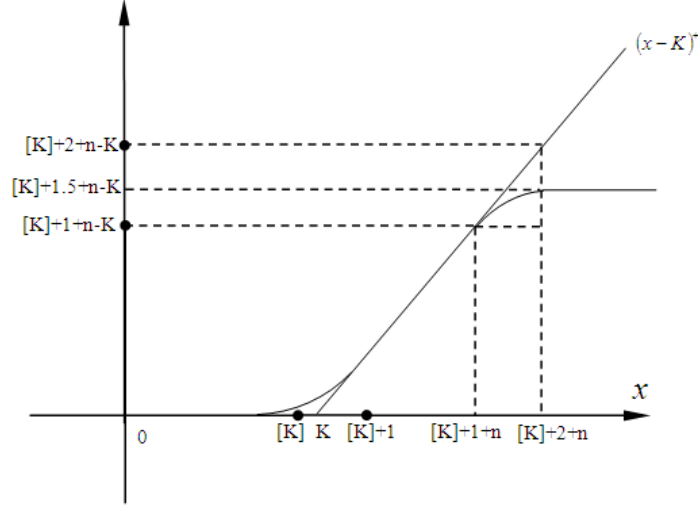
where $[K]$ denotes the integer part of K and $m(x)$ is given by,

$$m(x) = \frac{1}{2} - h(1-x) \quad x \in [0, 1], \quad (\text{B.18})$$

and

$$h(x) = \int_0^x g(r)dr, \quad x \in [0, 1], \quad (\text{B.19})$$

$$g(r) = \int_0^r \lambda_{0,1}(s)ds, \quad r \in [0, 1], \quad (\text{B.20})$$

Figure B.4: The function $\hat{\Phi}_n(x)$ 

with $g(0) = 0$, $g(1) = 1$, $h(0) = 0$ and $h(1) = \frac{1}{2}$. While, $h'(x) = g(x)$, so that $h'(0) = 0$ and $h'(1) = 1$. Here $\lambda_{0,1}$ is symmetric pulse on $[0, 1]$ with the integral $\int_0^1 \lambda_{0,1}(s) ds = 1$ and given by equation (1.6).

Appendix C

Derivation of cumulative distribution functions

In this section, we derive some cumulative distribution functions under different measures.

Definition C.1. Let B be the predetermined barrier level and K be the strike level. Let S_t be the geometric Brownian motion defined by (1.18). If $X_t \equiv \ln(S_t/S_0)$, then X_t is an arithmetic Brownian motion:

$$dX_t = \mu dt + \sigma dW_t \quad t \in [0, T], \quad (\text{C.1})$$

where

$$\mu \equiv r - \frac{\sigma^2}{2} \quad (\text{C.2})$$

is the drift of the process. Integrating equation (C.1) over time yields

$$X_t = \mu t + \sigma W_t \quad t \in [0, T]. \quad (\text{C.3})$$

Remark C.2. The derivation of arithmetic Brownian motion from geometric Brownian motion is given by Ito's formula. Notice $X_0 = 0$. Furthermore, if $S_t = B$, then define

$$b \equiv \ln \frac{B}{S_0}, \quad (\text{C.4})$$

while if $S_t = K$, then define

$$k \equiv \ln \frac{K}{S_0}. \quad (\text{C.5})$$

Notice also that X_t is normally distributed with mean μt and variance $\sigma^2 t$. Define the stochastic variable

$$M_{t_1, t_2}^X = \sup_{t \in [t_1, t_2]} X_t$$

for $0 \leq t_1 \leq t_2 \leq T$.

Let us denote $N(x)$, the cumulative distribution function of a standard normal stochastic variable, which is given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}u^2} du, \quad (\text{C.6})$$

and denote $n(x)$ by the standard normal density function which is given by

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad (\text{C.7})$$

and denote $N_2(x, y, \rho)$, the cumulative distribution function of a standard bivariate normal stochastic variable, which is given by

$$N_2(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^y \int_{-\infty}^x e^{-\frac{1}{2} \frac{u^2 - 2uv\rho + v^2}{1-\rho^2}} dudv. \quad (\text{C.8})$$

Denote by $n_z(\mu, \Omega)$ the bivariate normal density function with mean vector μ and covariance matrix Ω :

$$n_z(\mu, \Omega) = \frac{1}{2\pi} |\Omega|^{-1/2} \exp \left\{ -\frac{1}{2} (z - \mu)' \Omega^{-1} (z - \mu) \right\}.$$

We present some different cumulative distribution functions under different measures that we need in this thesis. All the calculations of the cumulative distribution function are based on the reflection principle, see, for example, [28]. Notice that the following cumulative distribution functions are the functions of some variables, say B , K , S_0 , T and

so on. In order to find the corresponding density functions, we need to have the formula to show how to differentiate the bivariate normal cumulative distribution function with respect to some variable. Denote $F(K, B)$ by

$$F(K, B) = N_2(\varphi_1(K, B), \varphi_2(K, B); \rho),$$

where $\varphi_1(K, B)$ and $\varphi_2(K, B)$ are some continuously differentiable functions with respect to K and B . Then we have

$$\begin{aligned} \frac{\partial}{\partial K} F(K, B) &= n(\varphi_1(K, B)) N\left(\frac{\varphi_2(K, B) - \rho\varphi_1(K, B)}{\sqrt{1 - \rho^2}}\right) \frac{\partial}{\partial K} \varphi_1(K, B) \\ &\quad + n(\varphi_2(K, B)) N\left(\frac{\varphi_1(K, B) - \rho\varphi_2(K, B)}{\sqrt{1 - \rho^2}}\right) \frac{\partial}{\partial K} \varphi_2(K, B), \end{aligned} \quad (\text{C.9})$$

where $N(\cdot)$ is given by equation (C.6) and $n(\cdot)$ is given by equation (C.7).

We present a proposition about a relationship between standard normal distribution function and bivariate normal cumulative distribution function. We need this proposition for our calculations in the thesis.

Proposition C.3. *We can show*

$$N(y) - N_2(x, y; \rho) = N_2(-x, y; -\rho), \quad (\text{C.10})$$

where $N(\cdot)$ is given by equation (C.6) and $N_2(\cdot, \cdot; \rho)$ is given by equation (C.8)

Proof. Notice that

$$N_2(-x, y; -\rho) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^y \int_{-\infty}^{-x} e^{-\frac{1}{2} \frac{u^2 + 2uv\rho + v^2}{1 - \rho^2}} du dv.$$

Let us change variable and set $u' = -u$ and $v' = v$ and

$$\left| \frac{\partial(u, v)}{\partial(u', v')} \right| = \left| \det \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 1.$$

Therefore, we have

$$\begin{aligned}
 N_2(-x, y; -\rho) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^y \int_x^{\infty} e^{-\frac{1}{2} \frac{u'^2 - 2u'v' + v'^2}{1-\rho^2}} \left| \frac{\partial(u, v)}{\partial(u', v')} \right| du' dv' \\
 &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^y \int_x^{\infty} e^{-\frac{1}{2} \frac{u^2 - 2uv + v^2}{1-\rho^2}} dudv.
 \end{aligned} \tag{C.11}$$

Then put equation (C.8) and (C.11) together, we have

$$\begin{aligned}
 &N_2(x, y; \rho) + N_2(-x, y; -\rho) \\
 &= \frac{1}{2\pi\sqrt{1-\rho^2}} \left(\int_{-\infty}^y \int_x^{\infty} e^{-\frac{1}{2} \frac{u^2 - 2uv + v^2}{1-\rho^2}} dudv + \int_{-\infty}^y \int_x^{\infty} e^{-\frac{1}{2} \frac{u^2 - 2uv + v^2}{1-\rho^2}} dudv \right) \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{1}{2}v^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2} \frac{(u-v\rho)^2}{1-\rho^2}} dudv \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{1}{2}v^2} dv.
 \end{aligned} \tag{C.12}$$

□

C.1 Under the measure Q

Proposition C.4. *The cumulative distribution function of S_T under the risk-neutral measure Q is given by*

$$F_{S_T}^Q(B) = N\left(\frac{\ln \frac{B}{S_0} - \mu T}{\sigma\sqrt{T}}\right), \tag{C.13}$$

where μ is given by the equation (C.2).

Proof.

$$Q(S_T \leq B) = Q(X_T \leq b) = Q\left(W_T \leq \frac{b - \mu T}{\sigma}\right).$$

Notice that W_T is normally distributed with mean 0 and variance T , so the cumulative distribution function of S_T is given by:

$$Q(S_T < B) = N\left(\frac{\ln(B/S_0) - \mu T}{\sigma\sqrt{T}}\right).$$

□

Proposition C.5. For a fixed time $t \in [0, T]$, the $\mathcal{F}_t$ -conditional cumulative distribution function of S_T under the risk-neutral measure Q is given by

$$F_{t,S_T}^Q(B, \omega) = N \left(\frac{\ln \frac{B}{S_t} - \mu(T-t)}{\sigma\sqrt{T-t}} \right), \quad (\text{C.14})$$

and the corresponding $\mathcal{F}_t$ -conditional density function of S_T under the risk-neutral measure Q is given by

$$f_{t,S_T}^Q(B, \omega) = \frac{1}{B\sigma\sqrt{T-t}} n \left(\frac{\ln \frac{B}{S_t} - \mu(T-t)}{\sigma\sqrt{T-t}} \right), \quad (\text{C.15})$$

where μ is given by the equation (C.2) and $n(\cdot)$ by equation (C.7).

Proposition C.6. For $K < B$, the joint cumulative distribution function of $M_{0,T}^S$ and S_T under the risk-neutral measure Q is given by

$$F_{M_{0,T}^S, S_T}^Q(B, K) = N \left(\frac{\ln \frac{K}{S_0} - \mu T}{\sigma\sqrt{T}} \right) - \left(\frac{B}{S_0} \right)^{\frac{2\mu}{\sigma^2}} N \left(\frac{\ln \frac{K}{S_0} - 2 \ln \frac{B}{S_0} - \mu T}{\sigma\sqrt{T}} \right), \quad (\text{C.16})$$

where μ is given by the equation (C.2).

Proof. Initially, we calculate the distribution function for a standard Brownian motion W^0 defined on a probability space $(\Omega, \mathcal{F}, Q^0)$, and by the reflection principle,

$$\begin{aligned} Q^0 \left(M_{0,T}^{W^0} \geq b, W_T^0 \leq k \right) &= Q^0 \left(M_{0,T}^{W^0} \geq b, W_T^0 \geq 2b - k \right) \\ &= Q^0 \left(W_T^0 \geq 2b - k \right) \\ &= Q^0 \left(W_T^0 \leq k - 2b \right) \\ &= N \left(\frac{k - 2b}{\sqrt{T}} \right). \end{aligned} \quad (\text{C.17})$$

Differentiating the above equation with respect to k implies that

$$Q^0 \left(M_{0,T}^{W^0} \geq b, W_T^0 \in dk \right) = \frac{1}{\sqrt{2\pi}\sqrt{T}} e^{-\frac{1}{2} \left(\frac{k-2b}{\sqrt{T}} \right)^2} dk. \quad (\text{C.18})$$

Define a probability measure Q , equivalent to Q^0 , by its Radon-Nikodym derivative $dQ/dQ^0 =$

$\exp \left\{ \frac{\mu}{\sigma} W_T^0 - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 T \right\}$. By Girsanov's theorem, $W_t = W_t^0 - \mu t / \sigma$, $t \in [0, T]$ is Q -Brownian motion. So under the measure Q , $\sigma W_t^0 = \mu t + \sigma W_t = X_t$, $t \in [0, T]$, and we get

$$\begin{aligned} Q(M_{0,T}^X \geq b, X_T < k) &= E^Q \left[1_{\{M_{0,T}^X \geq b, X_T < k\}} \right] \\ &= E^{Q^0} \left[\frac{dQ}{dQ^0} 1_{\{M_{0,T}^{W^0} \geq b, \sigma W_T^0 < k\}} \right] \\ &= E^{Q^0} \left[e^{\frac{\mu}{\sigma} W_T^0 - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 T} 1_{\{M_{0,T}^{W^0} \geq \frac{b}{\sigma}, W_T^0 < \frac{k}{\sigma}\}} \right] \\ &= e^{-\frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 T} \int_{-\infty}^{\frac{k}{\sigma}} e^{\frac{\mu}{\sigma} z} Q^0 \left(M_{0,T}^{W^0} \geq \frac{b}{\sigma}, W_T^0 \in dz \right) \end{aligned}$$

and

$$\begin{aligned} Q(M_{0,T}^X \geq b, X_T < k) &= e^{-\frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 T} \int_{-\infty}^{\frac{k}{\sigma}} e^{\frac{\mu}{\sigma} z} \frac{1}{\sqrt{2\pi T}} e^{-\frac{(z - \frac{2b}{\sigma})^2}{2T}} dz \\ &= e^{-\frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 T} \int_{-\infty}^{\frac{k}{\sigma}} e^{\frac{2b\mu}{\sigma^2} + \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 T} \frac{1}{\sqrt{2\pi T}} e^{-\frac{(z - \frac{2b}{\sigma} - \frac{\mu T}{\sigma})^2}{2T}} dz \\ &= e^{\frac{2b\mu}{\sigma^2}} N \left(\frac{k - 2b - \mu T}{\sigma \sqrt{T}} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} Q(M_{0,T}^X \leq b, X_T < k) &= Q(X_T < k) - Q(M_{0,T}^X \geq b, X_T < k) \\ &= N \left(\frac{k - \mu T}{\sigma \sqrt{T}} \right) - e^{\frac{2b\mu}{\sigma^2}} N \left(\frac{k - 2b - \mu T}{\sigma \sqrt{T}} \right). \end{aligned}$$

□

Proposition C.7. For $0 < t \leq T$, $K < B$ and $0 < x = S_t(\omega) < B$, the $\mathcal{F}_t$ -conditional cumulative distribution function of $M_{t,T}^S$ and S_T under the risk-neutral measure Q is given by

$$F_{t, M_{t,T}^S, S_T}^{x, Q}(B, K, \omega) = N \left(\frac{\ln \frac{K}{S_t} - \mu(T-t)}{\sigma \sqrt{T-t}} \right) - \left(\frac{B}{S_t} \right)^{\frac{2\mu}{\sigma^2}} N \left(\frac{\ln \frac{K}{S_t} - 2 \ln \frac{B}{S_t} - \mu(T-t)}{\sigma \sqrt{T-t}} \right), \quad (\text{C.19})$$

where μ is given by the equation (C.2).

By letting $K \rightarrow B$ in the equation (C.16) we have the follow the result.

Proposition C.8. *The cumulative distribution function of $M_{0,T}^S$ under the risk-neutral measure Q is given by*

$$F_{M_{0,T}^S}^Q(B) = N\left(\frac{\ln \frac{B}{S_0} - \mu T}{\sigma \sqrt{T}}\right) - \left(\frac{B}{S_0}\right)^{\frac{2\mu}{\sigma^2}} N\left(\frac{-\ln \frac{B}{S_0} - \mu T}{\sigma \sqrt{T}}\right), \quad (\text{C.20})$$

and the corresponding density function of $M_{0,T}^S$ under the risk-neutral measure Q is given by

$$\begin{aligned} f_{M_{0,T}^S}^Q(B) &= \frac{1}{B\sigma\sqrt{T}} n\left(\frac{\ln \frac{B}{S_0} - \mu T}{\sigma \sqrt{T}}\right) + \frac{1}{B\sigma\sqrt{T}} \left(\frac{B}{S_0}\right)^{\frac{2\mu}{\sigma^2}} n\left(\frac{-\ln \frac{B}{S_0} - \mu T}{\sigma \sqrt{T}}\right) \\ &\quad - \frac{2\mu}{B\sigma^2} \left(\frac{B}{S_0}\right)^{\frac{2\mu}{\sigma^2}} N\left(\frac{-\ln \frac{B}{S_0} - \mu T}{\sigma \sqrt{T}}\right), \end{aligned} \quad (\text{C.21})$$

where μ is given by the equation (C.2).

Proposition C.9. *For a fixed time $t \in [0, T]$ and $0 < x = S_t(\omega) < B$, the $\mathcal{F}_t$ -conditional cumulative distribution function of $M_{t,T}^S$ under the risk-neutral measure Q is given by*

$$F_{t,M_{t,T}^S}^{x,Q}(B, \omega) = N\left(\frac{\ln \frac{B}{S_t} - \mu(T-t)}{\sigma \sqrt{T-t}}\right) - \left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}} N\left(\frac{-\ln \frac{B}{S_t} - \mu(T-t)}{\sigma \sqrt{T-t}}\right), \quad (\text{C.22})$$

and the corresponding $\mathcal{F}_t$ -conditional density function of $M_{t,T}^S$ under the risk-neutral measure Q is given by

$$\begin{aligned} &f_{t,M_{t,T}^S}^{x,Q}(B, \omega) \quad (\text{C.23}) \\ &= \frac{1}{B\sigma\sqrt{T-t}} \Phi\left(\frac{\ln \frac{B}{S_t} - \mu(T-t)}{\sigma \sqrt{T-t}}\right) + \frac{1}{B\sigma\sqrt{T-t}} \left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}} \Phi\left(\frac{-\ln \frac{B}{S_t} - \mu(T-t)}{\sigma \sqrt{T-t}}\right) \\ &\quad - \frac{2\mu}{B\sigma^2} \left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}} N\left(\frac{-\ln \frac{B}{S_t} - \mu(T-t)}{\sigma \sqrt{T-t}}\right), \end{aligned}$$

where μ is given by the equation (C.2).

Remark C.10. We find

$$\begin{aligned} & \frac{\partial}{\partial S_t} F_{t, M_{t,T}^S}^{x,Q}(B, \omega) \\ = & -\frac{1}{S_t \sigma \sqrt{T-t}} \Phi\left(\frac{\ln \frac{B}{S_t} - \mu(T-t)}{\sigma \sqrt{T-t}}\right) - \frac{1}{S_t \sigma \sqrt{T-t}} \left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}} \Phi\left(\frac{-\ln \frac{B}{S_t} - \mu(T-t)}{\sigma \sqrt{T-t}}\right) \\ & + \frac{2\mu}{S_t \sigma^2} \left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}} N\left(\frac{-\ln \frac{B}{S_t} - \mu(T-t)}{\sigma \sqrt{T-t}}\right). \end{aligned} \quad (\text{C.24})$$

From equation (C.23) and (C.24), we have the relation

$$\frac{\partial}{\partial S_t} F_{t, M_{t,T}^S}^{x,Q}(B, \omega) = -\frac{B}{S_t} \frac{\partial}{\partial B} F_{t, M_{t,T}^S}^{x,Q}(B, \omega). \quad (\text{C.25})$$

Proposition C.11. If $Q(M_{0,t_1}^X \geq b, X_{t_1} < k_1, M_{t_1,t_2}^X \leq b, X_{t_2} < k_2)$ is the probability that X hits the level b before t_1 , and is below the level k_1 at time t_1 and below the level b between t_1 and t_2 and is less than level k_2 at time t_2 , then

$$\begin{aligned} & Q(M_{0,t_1}^X \geq b, X_{t_1} < k_1, M_{t_1,t_2}^X \leq b, X_{t_2} < k_2) \\ = & e^{\frac{2b\mu}{\sigma^2}} N_2\left(\frac{k_1 - 2b - \mu t_1}{\sigma \sqrt{t_1}}, \frac{k_2 - 2b - \mu t_2}{\sigma \sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}}\right) - N_2\left(\frac{k_1 - 2b + \mu t_1}{\sigma \sqrt{t_1}}, \frac{k_2 - \mu t_2}{\sigma \sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right), \end{aligned} \quad (\text{C.26})$$

where μ is given by equation (C.2), b by equation (C.4) and k by equation (C.5).

Proof. Initially, we calculate the distribution function for a standard Brownian motion W^0 defined on a probability space $(\Omega, \mathcal{F}, Q^0)$,

$$\begin{aligned} & Q^0(M_{0,t_1}^{W^0} \geq b, W_{t_1}^0 < k_1, M_{t_1,t_2}^{W^0} \leq b, W_{t_2}^0 < k_2) \\ = & Q^0(M_{0,t_1}^{W^0} \geq b, W_{t_1}^0 < k_1, W_{t_2}^0 < k_2) - Q^0(M_{0,t_1}^{W^0} \geq b, W_{t_1}^0 < k_1, M_{t_1,t_2}^{W^0} > b, W_{t_2}^0 < k_2). \end{aligned} \quad (\text{C.27})$$

By the reflection principle,

$$\begin{aligned} Q^0(M_{0,t_1}^{W^0} \geq b, W_{t_1}^0 < k_1, W_{t_2}^0 < k_2) &= Q^0(M_{0,t_1}^{W^0} \geq b, W_{t_1}^0 > 2b - k_1, W_{t_2}^0 > 2b - k_2) \\ &= Q^0(W_{t_1}^0 > 2b - k_1, W_{t_2}^0 > 2b - k_2) \\ &= Q^0(-W_{t_1}^0 < k_1 - 2b, -W_{t_2}^0 < k_2 - 2b) \\ &= N_2\left(\frac{k_1 - 2b}{\sqrt{t_1}}, \frac{k_2 - 2b}{\sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}}\right). \end{aligned} \quad (\text{C.28})$$

By the reflection principle again,

$$\begin{aligned}
& Q^0 \left(M_{0,t_1}^{W^0} \geq b, W_{t_1}^0 < k_1, M_{t_1,t_2}^{W^0} > b, W_{t_2}^0 < k_2 \right) \\
&= Q^0 \left(M_{0,t_1}^{W^0} \geq b, W_{t_1}^0 > 2b - k_1, m_{t_1,t_2}^{W^0} < b, W_{t_2}^0 < k_2 \right) \\
&= Q^0 \left(W_{t_1}^0 > 2b - k_1, W_{t_2}^0 < k_2 \right) \\
&= Q^0 \left(-W_{t_1}^0 < k_1 - 2b, W_{t_2}^0 < k_2 \right) \\
&= N_2 \left(\frac{k_1 - 2b}{\sqrt{t_1}}, \frac{k_2}{\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}} \right).
\end{aligned} \tag{C.29}$$

Substituting (C.28) and (C.29) into (C.27), we get

$$\begin{aligned}
& Q^0 \left(M_{0,t_1}^{W^0} \geq b, W_{t_1}^0 < k_1, M_{t_1,t_2}^{W^0} \leq b, W_{t_2}^0 < k_2 \right) \\
&= N_2 \left(\frac{k_1 - 2b}{\sqrt{t_1}}, \frac{k_2 - 2b}{\sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}} \right) - N_2 \left(\frac{k_1 - 2b}{\sqrt{t_1}}, \frac{k_2}{\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}} \right).
\end{aligned}$$

Differentiating the above equation with respect to k_1 and k_2 yields

$$\begin{aligned}
& Q^0 \left(M_{0,t_1}^{W^0} \geq b, W_{t_1}^0 \in dk_1, M_{t_1,t_2}^{W^0} \leq b, W_{t_2}^0 \in dk_2 \right) \\
&= \left[n_{z_1 z_2} \left(\begin{pmatrix} 2b \\ 2b \end{pmatrix}, \begin{pmatrix} t_1 & t_1 \\ t_1 & t_2 \end{pmatrix} \right) - n_{z_1 z_2} \left(\begin{pmatrix} 2b \\ 0 \end{pmatrix}, \begin{pmatrix} t_1 & -t_1 \\ -t_1 & t_2 \end{pmatrix} \right) \right] dk_1 dk_2.
\end{aligned}$$

Now define a probability measure Q , equivalent to Q^0 , by its Radon-Nikodym derivative $dQ/dQ^0 = \exp \left\{ \frac{\mu}{\sigma} W_{t_2}^0 - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 t_2 \right\}$. By Girsanov's theorem, $W_t = W_t^0 - \mu t / \sigma$, $t \in [0, t_2]$

is Q -Brownian motion. So under the measure Q , $\sigma W_t^0 = \mu t + \sigma W_t = X_t$, $t \in [0, t_2]$:

$$\begin{aligned}
 & Q(M_{0,t_1}^X \geq b, X_{t_1} < k_1, M_{t_1,t_2}^X \leq b, X_{t_2} < k_2) \\
 &= E^Q \left[1_{\{M_{0,t_1}^X \geq b, X_{t_1} < k_1, M_{t_1,t_2}^X \leq b, X_{t_2} < k_2\}} \right] \\
 &= E^{Q^0} \left[\frac{dQ}{dQ^0} 1_{\{M_{0,t_1}^{\sigma W^0} \geq b, \sigma W_{t_1}^0 < k_1, M_{t_1,t_2}^{\sigma W^0} \leq b, \sigma W_{t_2}^0 < k_2\}} \right] \\
 &= E^{Q^0} \left[e^{\frac{\mu}{\sigma} W_{t_2}^0 - \frac{1}{2}(\frac{\mu}{\sigma})^2 t_2} 1_{\{M_{0,t_1}^{W^0} \geq \frac{b}{\sigma}, W_{t_1}^0 < \frac{k_1}{\sigma}, M_{t_1,t_2}^{W^0} \leq \frac{b}{\sigma}, W_{t_2}^0 < \frac{k_2}{\sigma}\}} \right] \\
 &= \int_{-\infty}^{\frac{k_2}{\sigma}} \int_{-\infty}^{\frac{k_1}{\sigma}} e^{\frac{\mu}{\sigma} z_2 - \frac{1}{2}(\frac{\mu}{\sigma})^2 t_2} \left[n_{z_1 z_2} \left(\begin{pmatrix} \frac{2b}{\sigma} \\ \frac{2b}{\sigma} \end{pmatrix}, \begin{pmatrix} t_1 & t_1 \\ t_1 & t_2 \end{pmatrix} \right) \right. \\
 &\quad \left. - n_{z_1 z_2} \left(\begin{pmatrix} \frac{2b}{\sigma} \\ 0 \end{pmatrix}, \begin{pmatrix} t_1 & -t_1 \\ -t_1 & t_2 \end{pmatrix} \right) \right] dz_1 dz_2 \\
 &= \int_{-\infty}^{\frac{k_2}{\sigma}} \int_{-\infty}^{\frac{k_1}{\sigma}} e^{\frac{2b\mu}{\sigma^2}} n_{z_1 z_2} \left(\begin{pmatrix} \frac{2b+\mu t_1}{\sigma} \\ \frac{2b+\mu t_2}{\sigma} \end{pmatrix}, \begin{pmatrix} t_1 & t_1 \\ t_1 & t_2 \end{pmatrix} \right) \\
 &\quad - n_{z_1 z_2} \left(\begin{pmatrix} \frac{2b-\mu t_1}{\sigma} \\ \frac{\mu t_2}{\sigma} \end{pmatrix}, \begin{pmatrix} t_1 & -t_1 \\ -t_1 & t_2 \end{pmatrix} \right) dz_1 dz_2 \\
 &= e^{\frac{2b\mu}{\sigma^2}} N_2 \left(\frac{k_1 - 2b - \mu t_1}{\sigma \sqrt{t_1}}, \frac{k_2 - 2b - \mu t_2}{\sigma \sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}} \right) - N_2 \left(\frac{k_1 - 2b + \mu t_1}{\sigma \sqrt{t_1}}, \frac{k_2 - \mu t_2}{\sigma \sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}} \right).
 \end{aligned}$$

□

Proposition C.12. $Q(X_{t_1} < b, X_{t_2} \leq k, \tau > t_2)$ is the probability that X hits the barrier after t_2 , and is below its barrier at time t_1 , and is below k at time t_2 and

$$\begin{aligned}
 & Q(X_{t_1} < b, X_{t_2} \leq k, \tau > t_2) \\
 &= N_2 \left(\frac{b - \mu t_1}{\sigma \sqrt{t_1}}, \frac{k - \mu t_2}{\sigma \sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}} \right) - e^{\frac{2b\mu}{\sigma^2}} N_2 \left(\frac{b + \mu t_1}{\sigma \sqrt{t_1}}, \frac{k - 2b - \mu t_2}{\sigma \sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}} \right),
 \end{aligned} \tag{C.30}$$

where μ is given by equation (C.2), b by equation (C.4) and k by equation (C.5).

Proof. We can decompose $Q(X_{t_1} < b, X_{t_2} \leq k, \tau > t_2)$ as follows:

$$\begin{aligned}
 & Q(X_{t_1} < b, X_{t_2} \leq k, \tau > t_2) \\
 &= Q(M_{0,t_1}^X < b, X_{t_1} < b, X_{t_2} \leq k, \tau > t_2) + Q(M_{0,t_1}^X \geq b, X_{t_1} < b, X_{t_2} \leq k, \tau > t_2) \\
 &= Q(M_{0,t_2}^X < b, X_{t_2} \leq k) + Q(M_{0,t_1}^X \geq b, X_{t_1} < b, X_{t_2} \leq k, \tau > t_2).
 \end{aligned}$$

From equation (C.16), we know that

$$Q(M_{0,t_2}^X \leq b, X_{t_2} \leq k) = N\left(\frac{k - \mu t_2}{\sigma\sqrt{t_2}}\right) - e^{\frac{2\mu}{\sigma^2}b} N\left(\frac{k - 2b - \mu t_2}{\sigma\sqrt{t_2}}\right). \quad (\text{C.31})$$

Set $k_1 = b$ and $k_2 = k$; from (C.26), we know that

$$\begin{aligned} & Q(M_{0,t_1}^X \geq b, X_{t_1} < b, X_{t_2} < k, \tau > t_2) \\ &= Q(M_{0,t_1}^X \geq b, X_{t_1} < b, M_{t_1,t_2}^X < b, X_{t_2} < k) \\ &= e^{\frac{2b\mu}{\sigma^2}} N_2\left(\frac{-b - \mu t_1}{\sigma\sqrt{t_1}}, \frac{k - 2b - \mu t_2}{\sigma\sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}}\right) - N_2\left(\frac{-b + \mu t_1}{\sigma\sqrt{t_1}}, \frac{k - \mu t_2}{\sigma\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right). \end{aligned} \quad (\text{C.32})$$

Putting (C.31) and (C.32) together, we have

$$\begin{aligned} & Q(X_{t_1} < b, X_{t_2} \leq k, \tau > t_2) \\ &= N_2\left(\frac{b - \mu t_1}{\sigma\sqrt{t_1}}, \frac{k - \mu t_2}{\sigma\sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}}\right) - e^{\frac{2b\mu}{\sigma^2}} N_2\left(\frac{b + \mu t_1}{\sigma\sqrt{t_1}}, \frac{k - 2b - \mu t_2}{\sigma\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right). \end{aligned}$$

□

Remark C.13. In fact, by letting $k \rightarrow b$ in equation (C.30), we have for $0 < t_1 < t_2$, the cumulative distribution function of M_{t_1,t_2}^S under the risk-neutral measure Q is given by

$$F_{M_{t_1,t_2}^S}^Q(B) = N_2\left(\frac{b - \mu t_1}{\sigma\sqrt{t_1}}, \frac{b - \mu t_2}{\sigma\sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}}\right) - e^{\frac{2b\mu}{\sigma^2}} N_2\left(\frac{b + \mu t_1}{\sigma\sqrt{t_1}}, \frac{-b - \mu t_2}{\sigma\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right),$$

where μ is given by equation (C.2) and b by equation (C.4).

Proposition C.14. For $0 < t < t_1 < t_2$, the $\mathcal{F}_t$ -conditional cumulative distribution function of M_{t_1,t_2}^S under the risk-neutral measure Q is given by

$$\begin{aligned} F_{t,M_{t_1,t_2}^S}^Q(B, \omega) &= N_2\left(\frac{\ln \frac{B}{S_t} - \mu(t_1 - t)}{\sigma\sqrt{t_1 - t}}, \frac{\ln \frac{B}{S_t} - \mu(t_2 - t)}{\sigma\sqrt{t_2 - t}}; \sqrt{\frac{t_1 - t}{t_2 - t}}\right) \\ &\quad - \left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}} N_2\left(\frac{\ln \frac{B}{S_t} + \mu(t_1 - t)}{\sigma\sqrt{t_1 - t}}, \frac{-\ln \frac{B}{S_t} - \mu(t_2 - t)}{\sigma\sqrt{t_2 - t}}; -\sqrt{\frac{t_1 - t}{t_2 - t}}\right), \end{aligned} \quad (\text{C.33})$$

and the corresponding $\mathcal{F}_t$ -conditional density function is given by

$$f_{t,M_{t_1,t_2}^S}^Q(B, \omega) = \frac{\partial}{\partial B} F_{t,M_{t_1,t_2}^S}^Q(B, \omega). \quad (\text{C.34})$$

Remark C.15. The set $\{M_{t_1, t_2}^X < b, X_{t_2} < k\}$ is equivalent to the set $\{X_{t_1} < b, X_{t_2} < k, \tau > t_2\}$. In fact, the equation (C.30) gives us the joint cumulative distribution function of M_{t_1, t_2}^S and S_{t_2} and for $K < B$,

$$\begin{aligned} & F_{M_{t_1, t_2}^S, S_{t_2}}^Q(B, K) \\ &= N_2\left(\frac{b - \mu t_1}{\sigma \sqrt{t_1}}, \frac{k - \mu t_2}{\sigma \sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}}\right) - e^{\frac{2b\mu}{\sigma^2}} N_2\left(\frac{b + \mu t_1}{\sigma \sqrt{t_1}}, \frac{k - 2b - \mu t_2}{\sigma \sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right), \end{aligned} \quad (\text{C.35})$$

where μ is given by the equation (C.2), b by equation (C.4) and k by equation (C.5).

Proposition C.16. For $0 \leq t < t_1 \leq t_2$ and $K < B$, the $\mathcal{F}_t$ -conditional cumulative distribution function of M_{t_1, t_2}^S and S_{t_2} under the risk-neutral measure Q is given by

$$\begin{aligned} & F_{t, M_{t_1, t_2}^S, S_{t_2}}^Q(B, K, \omega) \\ &= N_2\left(\frac{\ln \frac{B}{S_t} - \mu(t_1 - t)}{\sigma \sqrt{t_1 - t}}, \frac{\ln \frac{K}{S_t} - \mu(t_2 - t)}{\sigma \sqrt{t_2 - t}}; \sqrt{\frac{t_1 - t}{t_2 - t}}\right) \\ & \quad - \left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}} N_2\left(\frac{\ln \frac{B}{S_t} + \mu(t_1 - t)}{\sigma \sqrt{t_1 - t}}, \frac{\ln \frac{K}{S_t} - 2 \ln \frac{B}{S_t} - \mu(t_2 - t)}{\sigma \sqrt{t_2 - t}}; -\sqrt{\frac{t_1 - t}{t_2 - t}}\right), \end{aligned} \quad (\text{C.36})$$

where μ is given by the equation (C.2).

C.2 Under the measure $\tilde{Q}$

The measure $\tilde{Q}$ is equivalent to Q . Under the probability measure $\tilde{Q}$, X_t follows an arithmetic Brownian motion

$$dX_t = \tilde{\mu}dt + \sigma d\tilde{W}_t$$

with drift

$$\tilde{\mu} \equiv r + \frac{\sigma^2}{2}, \quad (\text{C.37})$$

and

$$\tilde{W}_t = W_t - \sigma t$$

is $\tilde{Q}$ Brownian motion. We present some conditional cumulative distributions under the measure $\tilde{Q}$.

Proposition C.17. *The cumulative distribution function of $M_{0,T}^S$ under the measure $\tilde{Q}$ is given by*

$$F_{M_{0,T}^S}^{\tilde{Q}}(B) = N\left(\frac{\ln \frac{B}{S_0} - \tilde{\mu}T}{\sigma\sqrt{T}}\right) - \left(\frac{B}{S_0}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} N\left(\frac{-\ln \frac{B}{S_0} - \tilde{\mu}T}{\sigma\sqrt{T}}\right), \quad (\text{C.38})$$

and the corresponding $\mathcal{F}_t$ -conditional density function of $M_{0,T}^S$ under the risk-neutral measure $\tilde{Q}$ is given by

$$\begin{aligned} f_{M_{0,T}^S}^{\tilde{Q}}(B) &= \frac{1}{B\sigma\sqrt{T}} n\left(\frac{\ln \frac{B}{S_0} - \tilde{\mu}T}{\sigma\sqrt{T}}\right) + \frac{1}{B\sigma\sqrt{T}} \left(\frac{B}{S_0}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} n\left(\frac{-\ln \frac{B}{S_0} - \tilde{\mu}T}{\sigma\sqrt{T}}\right) \\ &\quad - \frac{2\tilde{\mu}}{B\sigma^2} \left(\frac{B}{S_0}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} N\left(\frac{-\ln \frac{B}{S_0} - \tilde{\mu}T}{\sigma\sqrt{T}}\right), \end{aligned} \quad (\text{C.39})$$

where $\tilde{\mu}$ is given by equation (C.37).

Remark C.18. *We have the following relationship:*

$$\frac{\partial}{\partial S_0} F_{M_{0,T}^S}^{\tilde{Q}}(B) = -\frac{B}{S_0} \frac{\partial}{\partial B} F_{M_{0,T}^S}^{\tilde{Q}}(B). \quad (\text{C.40})$$

Proposition C.19. *For a fixed time $t \in [0, T]$ and $0 < x = S_t(\omega) < B$, the $\mathcal{F}_t$ -conditional cumulative distribution function of $M_{t,T}^S$ under the measure $\tilde{Q}$ is given by*

$$F_{t,M_{t,T}^S}^{x,\tilde{Q}}(B, \omega) = N\left(\frac{\ln \frac{B}{S_t} - \tilde{\mu}(T-t)}{\sigma\sqrt{T-t}}\right) - \left(\frac{B}{S_t}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} N\left(\frac{-\ln \frac{B}{S_t} - \tilde{\mu}(T-t)}{\sigma\sqrt{T-t}}\right), \quad (\text{C.41})$$

and the corresponding $\mathcal{F}_t$ -conditional density function of $M_{0,T}^S$ under the risk-neutral measure $\tilde{Q}$ is given by

$$\begin{aligned} &f_{t,M_{t,T}^S}^{x,\tilde{Q}}(B, \omega) \\ &= \frac{1}{B\sigma\sqrt{T-t}} n\left(\frac{\ln \frac{B}{S_t} - \tilde{\mu}(T-t)}{\sigma\sqrt{T-t}}\right) + \frac{1}{B\sigma\sqrt{T-t}} \left(\frac{B}{S_t}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} n\left(\frac{-\ln \frac{B}{S_t} - \tilde{\mu}(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad - \frac{2\tilde{\mu}}{B\sigma^2} \left(\frac{B}{S_t}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} N\left(\frac{-\ln \frac{B}{S_t} - \tilde{\mu}(T-t)}{\sigma\sqrt{T-t}}\right), \end{aligned} \quad (\text{C.42})$$

where $\tilde{\mu}$ is given by equation (C.37).

Remark C.20. We have the following relationship,

$$\frac{\partial}{\partial S_t} F_{t, M_{t,T}^S}^{x, \tilde{Q}}(B, \omega) = -\frac{B}{S_t} \frac{\partial}{\partial B} F_{t, M_{t,T}^S}^{x, \tilde{Q}}(B, \omega). \quad (\text{C.43})$$

C.3 Under the measure Q^*

Now let us introduce a measure Q^* , which is equivalent to Q . Under the probability measure Q^* , X_t follows an arithmetic Brownian motion

$$dX_t = \mu^* dt + \sigma dW_t^*$$

with drift

$$\mu^* \equiv r + \frac{\sigma^2}{2}, \quad (\text{C.44})$$

and

$$W_t^* = W_t - \sigma t$$

is Q^* Brownian motion. We present some conditional cumulative distributions under the measure Q^* .

Proposition C.21. For a fixed time $t \in [0, T]$, the $\mathcal{F}_t$ -conditional cumulative distribution function of S_T under the measure Q^* is given by

$$F_{t, S_T}^{Q^*}(K, \omega) = N\left(\frac{\ln \frac{K}{S_t} - \mu^*(T-t)}{\sigma \sqrt{T-t}}\right), \quad (\text{C.45})$$

and the corresponding $\mathcal{F}_t$ -conditional density function of S_T under the measure Q^* is given by

$$f_{t, S_T}^{Q^*}(K, \omega) = \frac{1}{K \sigma \sqrt{T-t}} n\left(\frac{\ln \frac{K}{S_t} - \mu^*(T-t)}{\sigma \sqrt{T-t}}\right), \quad (\text{C.46})$$

where μ^* is given by the equation (C.44).

Proposition C.22. For $0 \leq t < t_1 \leq t_2$ and $K < B$, the $\mathcal{F}_t$ -conditional cumulative

distribution function of M_{t_1, t_2}^S and S_{t_2} under the measure Q^* is given by

$$\begin{aligned} & F_{t, M_{t_1, t_2}^S, S_{t_2}}^{Q^*}(B, K, \omega) \\ &= N_2 \left(\frac{\ln \frac{B}{S_t} - \mu^*(t_1 - t)}{\sigma \sqrt{t_1 - t}}, \frac{\ln \frac{K}{S_t} - \mu^*(t_2 - t)}{\sigma \sqrt{t_2 - t}}; \sqrt{\frac{t_1 - t}{t_2 - t}} \right) \\ & \quad - \left(\frac{B}{S_t} \right)^{\frac{2\mu^*}{\sigma^2}} N_2 \left(\frac{\ln \frac{B}{S_t} + \mu^*(t_1 - t)}{\sigma \sqrt{t_1 - t}}, \frac{\ln \frac{K}{S_t} - 2 \ln \frac{B}{S_t} - \mu^*(t_2 - t)}{\sigma \sqrt{t_2 - t}}; -\sqrt{\frac{t_1 - t}{t_2 - t}} \right), \end{aligned} \quad (C.47)$$

where μ^* is given by the equation (C.44).

Proposition C.23. For $0 < t_1 < t \leq t_2$, $K < B$ and $0 < x = S_t(\omega) < B$, the $\mathcal{F}_t$ -conditional cumulative distribution function of M_{t, t_2}^S and S_{t_2} under the risk-neutral measure Q^* is given by

$$\begin{aligned} & F_{t, M_{t, t_2}^S, S_{t_2}}^{x, Q^*}(B, K, \omega) \\ &= N \left(\frac{\ln \frac{K}{S_t} - \mu^*(t_2 - t)}{\sigma \sqrt{t_2 - t}} \right) - \left(\frac{B}{S_t} \right)^{\frac{2\mu^*}{\sigma^2}} N \left(\frac{\ln \frac{K}{S_t} - 2 \ln \frac{B}{S_t} - \mu^*(t_2 - t)}{\sigma \sqrt{t_2 - t}} \right), \end{aligned} \quad (C.48)$$

where μ^* is given by the equation (C.44).

C.4 Under the measure $\hat{Q}$

Since S_t^1 follows geometric Brownian motion equation (4.19) and S_t^2 follows geometric Brownian motion equation (4.20), and let

$$\begin{aligned} \xi &= \exp \left\{ -\frac{1}{2} \sigma_2^2 T + \rho \sigma_2 W_T^1 + \sqrt{1 - \rho^2} \sigma_2 W_T^2 \right\} \\ &= \exp \left\{ -\frac{1}{2} \rho^2 \sigma_2^2 T + \rho \sigma_2 W_T^1 - \frac{1}{2} (1 - \rho^2) \sigma_2^2 T + \sqrt{1 - \rho^2} \sigma_2 W_T^2 \right\} \end{aligned}$$

and define an equivalent probability measure $\hat{Q}$ by $d\hat{Q}/dQ = \xi$ and by the Girsanov theorem, we have $\hat{W}_t^1 = W_t^1 - \rho \sigma_2 t$ and $\hat{W}_t^2 = W_t^2 - \sqrt{1 - \rho^2} \sigma_2 t$ are standard Brownian

motions on the probability space $(\Omega, \mathcal{F}, \hat{Q})$. Note that under the measure $\hat{Q}$

$$\begin{aligned}\frac{dS_t^1}{S_t^1} &= (r + \rho\sigma_1\sigma_2)dt + \sigma_1 d\hat{W}_t^1, \\ \frac{dS_t^2}{S_t^2} &= (r + \sigma_2^2)dt + \sigma_2 \left(\rho d\hat{W}_t^1 + \sqrt{1 - \rho^2} d\hat{W}_t^2 \right),\end{aligned}$$

or equivalently

$$S_t^1 = S_0^1 \exp \left\{ (r + \rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_1^2)t + \sigma_1 \hat{W}_t^1 \right\}, \quad (\text{C.49})$$

$$S_t^2 = S_0^2 \exp \left\{ (r + \frac{1}{2}\sigma_2^2)t + \sigma_2 \left(\rho \hat{W}_t^1 + \sqrt{1 - \rho^2} \hat{W}_t^2 \right) \right\}. \quad (\text{C.50})$$

By changing variable, let

$$X_t^1 = \ln \frac{S_t^1}{S_0^1} \quad \text{and} \quad X_t^2 = \ln \frac{S_t^2}{S_0^2} \quad t \in [0, T], \quad (\text{C.51})$$

By Itô's Lemma, under the measure Q we have

$$dX_t^1 = \mu_1 dt + \sigma_1 dW_t^1, \quad (\text{C.52})$$

$$dX_t^2 = \mu_2 dt + \sigma_2 (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2), \quad (\text{C.53})$$

where

$$\mu_1 = r - \frac{\sigma_1^2}{2}, \quad (\text{C.54})$$

$$\mu_2 = r - \frac{\sigma_2^2}{2}. \quad (\text{C.55})$$

Under the measure $\hat{Q}$, we have

$$dX_t^1 = \hat{\mu}_1 dt + \sigma_1 d\hat{W}_t^1, \quad (\text{C.56})$$

$$dX_t^2 = \hat{\mu}_2 dt + \sigma_2 (\rho d\hat{W}_t^1 + \sqrt{1 - \rho^2} d\hat{W}_t^2), \quad (\text{C.57})$$

where

$$\hat{\mu}_1 = r - \frac{\sigma_1^2}{2} + \rho\sigma_1\sigma_2, \quad (\text{C.58})$$

$$\hat{\mu}_2 = r + \frac{\sigma_2^2}{2}. \quad (\text{C.59})$$

From [12], we know $Q\left(M_{0,T}^{S^1} > B, S_T^2 > K\right)$ is given by

$$\begin{aligned} & Q\left(M_{0,T}^{S^1} > B, S_T^2 > K\right) \\ &= N_2\left(\frac{\ln \frac{S_0^1}{B} + \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{\ln \frac{S_0^2}{K} + \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) \\ &+ \left(\frac{B}{S_0^1}\right)^{\frac{2\mu_1}{\sigma_1^2}} N_2\left(\frac{\ln \frac{S_0^1}{B} - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{\ln \frac{S_0^2}{K} - 2\rho \frac{\sigma_2}{\sigma_1} \ln \frac{S_0^1}{B} + \mu_2 T}{\sigma_2 \sqrt{T}}; -\rho\right). \end{aligned} \quad (\text{C.60})$$

Notice that

$$\begin{aligned} & Q\left(M_{0,T}^{S^1} < B, S_T^2 < K\right) \\ &= Q\left(M_{0,T}^{S^1} < B\right) - Q\left(S_T^2 > K\right) + Q\left(M_{0,T}^{S^1} > B, S_T^2 > K\right) \\ &= Q\left(M_{0,T}^{S^1} < B\right) - 1 + Q\left(S_T^2 < K\right) + Q\left(M_{0,T}^{S^1} > B, S_T^2 > K\right). \end{aligned}$$

Putting equation (C.20), equation (C.13) and equation (C.60) together, we have

$$\begin{aligned} & Q\left(M_{0,T}^{S^1} < B, S_T^2 < K\right) \\ &= Q\left(M_{0,T}^{S^1} < B\right) - Q\left(S_T^2 > K\right) + Q\left(M_{0,T}^{S^1} > B, S_T^2 > K\right) \\ &= N_2\left(\frac{-\ln \frac{S_0^1}{B} - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{-\ln \frac{S_0^2}{K} - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) \\ &- \left(\frac{B}{S_0^1}\right)^{\frac{2\mu_1}{\sigma_1^2}} N_2\left(\frac{\ln \frac{S_0^1}{B} - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{-\ln \frac{S_0^2}{K} + 2\rho \frac{\sigma_2}{\sigma_1} \ln \frac{S_0^1}{B} - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right). \end{aligned}$$

Proposition C.24. For $0 \leq t \leq T$ and $0 < x = S_t^1(\omega) < B$, the $\mathcal{F}_t$ -conditional cumulative

distribution function of $M_{t,T}^{S_1^1}$ and S_T^2 under the measure Q is given by

$$\begin{aligned}
 & F_{t, M_{t,T}^{S_1^1}, S_T^2}^{x, Q}(B, K, \omega) \\
 = & N_2 \left(\frac{\ln \frac{B}{S_t^1} - \mu_1(T-t)}{\sigma_1 \sqrt{T-t}}, \frac{\ln \frac{K}{S_t^2} - \mu_2(T-t)}{\sigma_2 \sqrt{T-t}}; \rho \right) \\
 & - \left(\frac{B}{S_t^1} \right)^{\frac{2\mu_1}{\sigma_1^2}} N_2 \left(\frac{-\ln \frac{B}{S_t^1} - \mu_1(T-t)}{\sigma_1 \sqrt{T-t}}, \frac{\ln \frac{K}{S_t^2} - 2\rho \frac{\sigma_2}{\sigma_1} \ln \frac{B}{S_t^1} - \mu_2(T-t)}{\sigma_2 \sqrt{T-t}}; \rho \right),
 \end{aligned} \tag{C.61}$$

where μ_1 is given by equation (C.54) and μ_2 is given by equation (C.55).

Proposition C.25. For $0 \leq t \leq T$ and $0 < x = S_t^1(\omega) < B$, the $\mathcal{F}_t$ -conditional cumulative distribution function of $M_{t,T}^{S_1^1}$ and S_T^2 under the measure $\hat{Q}$ is given by

$$\begin{aligned}
 & F_{t, M_{t,T}^{S_1^1}, S_T^2}^{x, \hat{Q}}(B, K, \omega) \\
 = & N_2 \left(\frac{\ln \frac{B}{S_t^1} - \hat{\mu}_1(T-t)}{\sigma_1 \sqrt{T-t}}, \frac{\ln \frac{K}{S_t^2} - \hat{\mu}_2(T-t)}{\sigma_2 \sqrt{T-t}}; \rho \right) \\
 & - \left(\frac{B}{S_t^1} \right)^{\frac{2\hat{\mu}_1}{\sigma_1^2}} N_2 \left(\frac{-\ln \frac{B}{S_t^1} - \hat{\mu}_1(T-t)}{\sigma_1 \sqrt{T-t}}, \frac{\ln \frac{K}{S_t^2} - 2\rho \frac{\sigma_2}{\sigma_1} \ln \frac{B}{S_t^1} - \hat{\mu}_2(T-t)}{\sigma_2 \sqrt{T-t}}; \rho \right),
 \end{aligned} \tag{C.62}$$

where $\hat{\mu}_1$ is given by equation (C.58) and $\hat{\mu}_2$ is given by equation (C.59).

Appendix D

Δ -Hedging for Protected Barrier option and Rainbow Barrier option

The price of the Protected barrier option and Rainbow barrier option at time 0 are derived by [12]. In this section we just present the results. For the derivation of the formula in detail, we refer to [12].

Proposition D.1. *The price of the Protected Barrier option at time zero, $PBO(0)$, is given by*

$$\begin{aligned} PBO(0) = & e^{-rt^*} RN \left(\frac{\ln \frac{S_0}{B} + \mu t^*}{\sigma \sqrt{t^*}} \right) + \frac{R}{B} S_0 (G(T) - G(t^*) + H(t^*, T)) \\ & + S_0 (Func(B, B, S_0, \mu^*, t^*, T, \sigma) - Func(B, K, S_0, \mu^*, t^*, T, \sigma)) \\ & - e^{-rT} K (Func(B, B, S_0, \mu, t^*, T, \sigma) - Func(B, K, S_0, \mu, t^*, T, \sigma)), \quad (D.1) \end{aligned}$$

where

$$G(t) = N \left(\frac{\ln \frac{S_0}{B} + \tilde{\mu} t}{\sigma \sqrt{t}} \right) + \left(\frac{S_0}{B} \right)^{-\frac{2\tilde{\mu}}{\sigma^2}} N \left(\frac{\ln \frac{S_0}{B} - \tilde{\mu} t}{\sigma \sqrt{t}} \right), \quad (D.2)$$

$$\begin{aligned} H(t^*, T) = & N_2 \left(\frac{\ln \frac{S_0}{B} + \tilde{\mu} t^*}{\sigma \sqrt{t^*}}, \frac{-\ln \frac{S_0}{B} - \tilde{\mu} T}{\sigma \sqrt{T}}; -\sqrt{\frac{t^*}{T}} \right) \\ & + \left(\frac{S_0}{B} \right)^{-\frac{2\tilde{\mu}}{\sigma^2}} N_2 \left(\frac{\ln \frac{S_0}{B} - \tilde{\mu} t^*}{\sigma \sqrt{t^*}}, \frac{-\ln \frac{S_0}{B} + \tilde{\mu} T}{\sigma \sqrt{T}}; -\sqrt{\frac{t^*}{T}} \right) \quad (D.3) \end{aligned}$$

and

$$\begin{aligned} & \text{Func}(B, K, S_0, \mu, t^*, T, \sigma) \\ = & N_2 \left(\frac{-\ln \frac{S_0}{B} - \mu t^*}{\sigma \sqrt{t^*}}, \frac{-\ln \frac{S_0}{K} - \mu T}{\sigma \sqrt{T}}; \sqrt{\frac{t^*}{T}} \right) \\ & - \left(\frac{S_0}{B} \right)^{-\frac{2\mu}{\sigma^2}} N_2 \left(\frac{-\ln \frac{S_0}{B} + \mu t^*}{\sigma \sqrt{t^*}}, \frac{-\ln \frac{S_0}{K} + 2 \ln \frac{S_0}{B} - \mu T}{\sigma \sqrt{T}}; -\sqrt{\frac{t^*}{T}} \right). \end{aligned} \quad (\text{D.4})$$

where μ^* is given by equation (C.44), μ by equation (C.2) and $\tilde{\mu}$ given by equation (C.37).

Remark D.2. From Proposition C.3, we see the formula (17) in Appendix A3 in [12] is incorrect, which leads to the time zero price of the Protected barrier option formula in [12] being incorrect. What we present here is the corrected formula.

To find the the number of units to be held in stock S at time zero, we take the partial differentiation equation (D.1) with respect to S_0 and get

$$\begin{aligned} & \frac{\partial}{\partial S_0} \text{PBO}(0) \\ = & e^{-rt^*} R \frac{1}{\sigma \sqrt{t^*} S_0} \Phi \left(\frac{\ln \frac{S_0}{B} + \mu t^*}{\sigma \sqrt{t^*}} \right) + \frac{R}{B} (G(T) - G(t^*) + H(t^*, T)) \\ & + (\text{Func}(B, B, S_0, \mu^*, t^*, T, \sigma) - \text{Func}(B, K, S_0, \mu^*, t^*, T, \sigma)) \\ & + \frac{R}{B} S_0 \left(\frac{\partial}{\partial S_0} G(T) - \frac{\partial}{\partial S_0} G(t^*) + \frac{\partial}{\partial S_0} H(t^*, T) \right) \\ & + S_0 \left(\frac{\partial}{\partial S_0} \text{Func}(B, B, S_0, \mu^*, t^*, T, \sigma) - \frac{\partial}{\partial S_0} \text{Func}(B, K, S_0, \mu^*, t^*, T, \sigma) \right) \\ & - e^{-rT} K \left(\frac{\partial}{\partial S_0} \text{Func}(B, B, S_0, \mu, t^*, T, \sigma) - \frac{\partial}{\partial S_0} \text{Func}(B, K, S_0, \mu, t^*, T, \sigma) \right). \end{aligned} \quad (\text{D.5})$$

Proposition D.3. The price of Rainbow Barrier option at time zero, $RBO(0)$, is given by

$$\begin{aligned} & RBO(0) \\ = & S_0^2 P(B, K, S_0^1, S_0^2, \hat{\mu}_1, \hat{\mu}_2, \sigma_1, \sigma_2, T) - \frac{K}{e^{rT}} P(B, K, S_0^1, S_0^2, \mu_1, \mu_2, \sigma_1, \sigma_2, T), \end{aligned} \quad (\text{D.6})$$

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where

$$\begin{aligned} & P(B, K, S_0^1, S_0^2, \mu_1, \mu_2, \sigma_1, \sigma_2, T) \\ = & N_2 \left(\frac{-\ln \frac{S_0^1}{B} - \hat{\mu}_1 T}{\sigma_1 \sqrt{T}}, \frac{\ln \frac{S_0^2}{K} + \hat{\mu}_2 T}{\sigma_2 \sqrt{T}}; -\rho \right) \\ & - \left(\frac{S_0^1}{B} \right)^{-\frac{2\hat{\mu}_1}{\sigma_1^2}} N_2 \left(\frac{\ln \frac{S_0^1}{B} - \hat{\mu}_1 T}{\sigma_1 \sqrt{T}}, \frac{\ln \frac{S_0^2}{K} - 2\rho \frac{\sigma_2}{\sigma_1} \ln \frac{S_0^1}{B} + \hat{\mu}_2 T}{\sigma_2 \sqrt{T}}; -\rho \right), \end{aligned} \quad (\text{D.7})$$

and μ_1 is given by equation (C.54), μ_2 by equation (C.55), $\hat{\mu}_1$ by equation (C.58) and $\hat{\mu}_1$ by equation (C.59).

To find the the number of units to be held in stock S^1 at time zero, we take the partial differentiation equation (D.6) with respect to S_0^1 , and get

$$\begin{aligned} & \frac{\partial}{\partial S_0^1} \text{RBO}(0) \\ = & S_0^2 \frac{\partial}{\partial S_0^1} P(B, K, S_0^1, S_0^2, \hat{\mu}_1, \hat{\mu}_2, \sigma_1, \sigma_2, T) - \frac{K}{e^{rT}} \frac{\partial}{\partial S_0^2} P(B, K, S_0^1, S_0^2, \mu_1, \mu_2, \sigma_1, \sigma_2, T). \end{aligned} \quad (\text{D.8})$$

To find the the number of units to be held in stock S^2 at time zero, we take the partial differentiation equation (D.6) with respect to S_0^2 , and get

$$\begin{aligned} & \frac{\partial}{\partial S_0^2} \text{RBO}(0) \\ = & P(B, K, S_0^1, S_0^2, \hat{\mu}_1, \hat{\mu}_2, \sigma_1, \sigma_2, T) + S_0^2 \frac{\partial}{\partial S_0^2} P(B, K, S_0^1, S_0^2, \hat{\mu}_1, \hat{\mu}_2, \sigma_1, \sigma_2, T) \\ & - \frac{K}{e^{rT}} \frac{\partial}{\partial S_0^2} P(B, K, S_0^1, S_0^2, \mu_1, \mu_2, \sigma_1, \sigma_2, T). \end{aligned} \quad (\text{D.9})$$

Appendix E

VBA implementation code

Figure E.1: Integration implemented in VBA code for the Digital barrier option with a random time

```
Public Function calculate_integration_tT(t As Double, bT As Double, _
    St As Double, B As Double, r As Double, mu As Double, vol As Double)

Dim N As Integer
N = Application.Power(2, 13)

ReDim ti(N) As Double
For i = 1 To N
    ti(i) = t + i / N * (bT - t)
Next i

Dim temp As Double
Dim sum As Double
sum = Exp(-r * ti(1)) * time_t_conditional_CDF_MaxSt1_wrt_B(t, ti(1), St, B, mu, vol)

For i = 1 To N - 1
    temp = time_t_conditional_CDF_MaxSt1_wrt_B(t, ti(i + 1), St, B, mu, vol) _
        - time_t_conditional_CDF_MaxSt1_wrt_B(t, ti(i), St, B, mu, vol)
    sum = sum + Exp(-r * ti(i + 1)) * temp
Next i

calculate_integration_tT = sum

End Function
```

Figure E.2: Integration implemented in VBA code for the Protected barrier option

```
Public Function t_conditional_integration_tlt2(t1 As Double, t2 As Double, time_t As Double, _
    St As Double, B As Double, r As Double, mu As Double, vol As Double)

Dim N As Integer
N = Application.Power(2, 12)

ReDim ti(N) As Double
For i = 1 To N
    ti(i) = t1 + i / N * (t2 - t1)
Next i

Dim temp As Double
Dim sum As Double
sum = Exp(-r * t1) * (time_t_conditional_CDF_MaxStlt2_wrt_B(St, B, mu, vol, t1, ti(1), time_t) _
    - phi((Log(B / St) - mu * (t1 - time_t)) / (vol * Sqr(t1 - time_t))) / (B * vol * Sqr(t1 - time_t)))
For i = 1 To N - 1
    temp = time_t_conditional_CDF_MaxStlt2_wrt_B(St, B, mu, vol, t1, ti(i + 1), time_t) _
        - time_t_conditional_CDF_MaxStlt2_wrt_B(St, B, mu, vol, t1, ti(i), time_t)
    sum = sum + Exp(-r * ti(i + 1)) * temp
Next i

t_conditional_integration_tlt2 = sum

End Function
```

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