SOME PROPERTIES OF SERIES SUMMABLE BY ABEL METHODS.

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ABSTRACT

The thesis is mainly concerned with the possibility of obtaining non-trivial limitation theorems for (A,λ_n) summability methods. It is shown that for a large class of methods, including the (A,n^q) methods where $q<1$, the trivial limitation theorem cannot be improved. However, it is found that non-trivial limitation theorems do exist when the sequence $\{\lambda_n\}$ satisfies certain conditions. In particular, the following conditions are considered, where $\phi(x) = \log x$ or x^q and $q<1$.

(i) $\phi(\lambda_{n+1})-\phi(\lambda_n) \geq c > 0$.

(ii) $\Sigma[\phi(\lambda_n)]^{-1}$ converges.

Finally it is shown that these results can be used to obtain a number of inclusion relations between various (A,λ_n) methods.

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NOTATION

The symbols Σ and $\int$ should be read as $\sum_{1}^{\infty}$ and $\int_{-\infty}^{\infty}$. Exceptionally it will be convenient to use $\int$ to denote $\int_{0}^{\infty}$ but this will be apparent from the text.

The sequence $\{\lambda_n\}$ should always be understood to mean a strictly-increasing, divergent sequence of positive numbers starting at λ_1. Where $\{\log\lambda_n\}$ is also discussed it will be assumed that $\lambda_1 > 1$.

Other notation should be unambiguous.

CONTENTS

Introduction

Chapter 1

It is shown that for a certain class of (A, λ_n) methods no non-trivial limitation theorems can exist.

Chapter 2

A non-trivial limitation sequence is obtained under the assumption that $\lambda_{n+1}^q - \lambda_n^q \geq c > 0$ for each n and some $q<1$.

Chapter 3

This contains the high indices theorem which deals with the methods for which $\log\lambda_{n+1} - \log\lambda_n \geq c > 0$.

Chapter 4

A limitation theorem is proved under the assumption that $\Sigma \lambda_n^{-q}$ converges for some $q<1$.

Chapter 5

The condition that $\Sigma(\log\lambda_n)^{-1}$ should converge is considered. The case when $\lambda_{n+k+1}/\lambda_n \geq \phi > 1$ for some positive k is also mentioned.

Chapter 6

Some inclusion theorems are obtained.

INTRODUCTION

0.1

A series, Σa_n, will be said to be summable (A) to a sum s (that is, summable in the Abel sense to s) if $\Sigma a_n x^n$ converges to a function f(x) for $|x|<1$ and $f(x)\to s$ as $x\to 1-0$. The method derives its name from Abel's theorem on the continuity of power series, which establishes that if a series converges then it is summable (A) to the same sum. A summability method with this property is said to be regular. An example of a series which does not converge but is summable (A) is the series $\Sigma(-1)^{n+1}$. We have

$$\sum (-1)^n x^{n+1} = \frac{x}{x+1} \to \frac{1}{2} \quad \text{as} \quad x \to 1-0.$$

It follows that $\Sigma(-1)^{n+1}$ is summable (A) to $\frac{1}{2}$. A less trivial example is the series $\Sigma(-1)^{n+1}n^{-z}$ which is is summable (A) to $(1-2^{1-z})\zeta(z)$ for every complex z.

We shall mainly be concerned with a generalisation of this method. Suppose that $\{\lambda_n\}$ is a strictly increasing, divergent sequence of positive numbers. Then Σa_n will be said to be summable (A,λ_n) to sum s if $\Sigma a_n e^{-x\lambda_n}$ converges to a function f(x) for $x>0$ and $f(x)\to s$ as $x\to 0+$. The (A) and (A,n) methods are then identical. A few examples are given below.

The series Σz^n converges to $z(1-z)^{-1}$ only for $|z|<1$. However, it is summable $(A,1+n\log n)$ to this function for every z not in the interval $[1,\infty)$. This example leads to some useful results in the theory of analytic continuation.

We have already seen that the series $\Sigma(-1)^{n+1}$ is summable (A,n) to $\frac{1}{2}$. In fact, the series is summable (A,n^k) to $\frac{1}{2}$ for each positive k. It is also summable $(A,\log(n+1))$ to $\frac{1}{2}$. However, if we take $\{\lambda_n\}$ to be the sequence $1,3,4,6,7,\ldots$ we obtain that

$$\sum (-1)^{n+1} e^{-\lambda_n x} = e^{-x}(1-e^{-2x})/(1-e^{-3x}) \to \tfrac{2}{3} \quad \text{as} \quad x \to 0+$$

And so, with this choice of $\{\lambda_n\}$, $\Sigma(-1)^{n+1}$ is summable (A,λ_n) to $\frac{2}{3}$. The series is not summable at all by the $(A,\log\log(n+2))$ method. Nor is it summable (A,ϕ^n) where $\phi>1$.

Consider also the series Σn^{-1-ic} where $c\neq 0$. This is summable $(A,1+\log n)$ but is not summable (A,n).

The question of the relationships connecting some of the (A,λ_n) methods is discussed in Chapter 6.

0.2.

The (A,λ_n) methods are all regular. This is easily proved by the following argument.

Suppose that Σa_n converges to s. We lose nothing by assuming $s=0$ so let us do this. Now put $s_n = \sum_1^n a_i$. Then s_n is bounded, say by H, and given $\varepsilon>0$ there is an N such that for $n>N$, $|s_n|<\varepsilon$. Also, for $x>0$,

$$\sum a_n e^{-\lambda_n x} = \sum s_n (e^{-\lambda_n x} - e^{-\lambda_{n+1} x})$$

And

$$\sum_{N+1}^{\infty} \left| e^{-\lambda_n x} - e^{-\lambda_{n+1} x} \right| < 1 .$$

It follows that

$$\left| \sum a_n e^{-\lambda_n x} \right| < H\left(e^{-\lambda_1 x} - e^{-\lambda_N x}\right) + \varepsilon \to \varepsilon \quad \text{as} \quad x \to 0+$$

Hence

$$\sum a_n e^{-\lambda_n x} \to 0 \quad \text{as} \quad x \to 0+$$

0.3

A limitation theorem for a given summability method (P) will be said to be a theorem of the type, ' Σa_n summable (P) implies $a_n = O(u_n)$ '. The sequence $\{u_n\}$ will be said to be a limitation sequence. For (A, λ_n) summability it follows trivially from the definition that the sequence $\{e^{\varepsilon \lambda_n}\}$ is a limitation sequence for each $\varepsilon > 0$. The bulk of this thesis is concerned with obtaining non-trivial limitation sequences for certain (A, λ_n) methods. This leads to some inclusion theorems which connect various of the (A, λ_n) methods. An inclusion theorem will be said to be one of the type,' Σa_n summable (P) implies Σa_n summable (Q).'

CHAPTER 1

1.1

Suppose that Σa_n is summable (A,λ_n). Then $\Sigma a_n e^{-\varepsilon\lambda_n}$ converges for each $\varepsilon>0$, and hence $a_n = O(e^{\varepsilon\lambda_n})$ for each $\varepsilon>0$. This simple limitation theorem will be called the 'crude' limitation theorem for the (A,λ_n) method. At first sight it does not appear that anything better may be true, and in fact this chapter will show that for a large class of (A,λ_n) methods nothing better is true. However, in general the crude limitation sequences are not the best that can be obtained, and the following chapters will be concerned with obtaining others.

Suppose $\{u_n\}$ and $\{v_n\}$ are two limitation sequences for an (A,λ_n) method and that $u_n = o(v_n)$. Then we shall say that $\{u_n\}$ is stronger than $\{v_n\}$. If we only have that for some subsequence $\{n_r\}$, $u_{n_r} = o(v_{n_r})$, then we shall say that $\{u_n\}$ is stronger than $\{v_n\}$ on the subsequence $\{n_r\}$.

In this and the following two chapters we shall consider only limitation sequences which are stronger than every crude limitation sequence. It will constantly be useful to refer to the sequences $\{\mu_n^*\}$ and $\{\mu_n\}$, where

$$\mu_n^* = \frac{\lambda_n}{\lambda_{n+1}-\lambda_n} \quad , \quad \mu_n = \max\left[\mu_n^*, \mu_{n-1}^*\right] .$$

In particular, we shall show that if $\underline{\lim}\ \frac{\mu_n^*}{n} > 1$, then there is no limitation sequence stronger than every

crude limitation sequence.

1.2

We shall first prove a theorem of Austin* about the sequence $\{\mu_n\}$ defined in § 1.1.

THEOREM 1.1

Suppose $\{u_n\}$ is a limitation sequence for (A,λ_n) summability which is stronger than every crude limitation sequence. Then

$$\mu_n = O(u_n) .$$

Proof

Suppose the conclusion false. Then either

$$\lim \left[\frac{\lambda_{n+1}-\lambda_n}{\lambda_n}\right] u_n = 0 \quad \text{or} \quad \lim \left[\frac{\lambda_{n+1}-\lambda_n}{\lambda_n}\right] u_{n+1} = 0 .$$

Suppose the former. To obtain a contradiction we now construct a series, Σa_n, summable (A,λ_n) but such that it is not true that $a_n = O(u_n)$.

Let n_1 be the least integer n such that

$$\frac{\lambda_{n+1}-\lambda_n}{\lambda_n} u_n < 1 .$$

Then define n_ν so that

(i) $n_\nu > n_{\nu-1} + 1$

(ii) $\lambda_{n_\nu} > \nu$

(iii) $u_{n_\nu}\left(\frac{\lambda_{n_\nu+1}-\lambda_{n_\nu}}{\lambda_{n_\nu}}\right) < \frac{1}{\nu^3}$

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* M.C.Austin, J.L.M.S., 26 (1951), 304-307.

Now let

$$a_n = \begin{cases} \nu u_{n_\nu} & , \ n = n_\nu \\ -\nu u_{n_\nu} & , \ n = n_\nu + 1 \\ 0 & , \ \text{otherwise} \end{cases}$$

Then

$$0 \leq a_{n_\nu} e^{-x\lambda_{n_\nu}} + a_{n_\nu+1} e^{-x\lambda_{n_\nu+1}} = \nu u_{n_\nu} \left(e^{-x\lambda_{n_\nu}} - e^{-x\lambda_{n_\nu+1}} \right)$$

$$= \nu u_{n_\nu} x \int_{\lambda_{n_\nu}}^{\lambda_{n_\nu+1}} e^{-xy}\, dy \leq \nu u_{n_\nu} x (\lambda_{n_\nu+1} - \lambda_{n_\nu}) e^{-x\lambda_{n_\nu}}$$

$$= \nu u_{n_\nu} \left(\frac{\lambda_{n_\nu+1} - \lambda_{n_\nu}}{\lambda_{n_\nu}} \right) \cdot x\lambda_{n_\nu} e^{-x\lambda_{n_\nu}} \leq \frac{1}{\nu^2} \cdot \frac{1}{e} .$$

It follows that $\Sigma(a_{n_\nu} e^{-x\lambda_{n_\nu}} + a_{n_\nu+1} e^{-x\lambda_{n_\nu+1}})$ is uniformly convergent for $x>0$. And so if $f(x)$ is its sum, then $f(x) \to 0$ as $x \to 0+$.

To complete the proof we just need to show that $f(x) = \Sigma a_n e^{-\lambda_n x}$, since then Σa_n is summable (A, λ_n) to 0.

Now since $\{u_n\}$ is stronger than every crude limitation sequence it follows that for every $x>0$, $u_n = O(e^{\frac{1}{2}x\lambda_n})$. Hence

$$|a_{n_\nu+1} e^{-x\lambda_{n_\nu+1}}| \leq |a_{n_\nu} e^{-x\lambda_{n_\nu}}| = \nu u_{n_\nu} e^{-x\lambda_{n_\nu}}$$

$$= O(\nu e^{+\frac{1}{2}x\lambda_{n_\nu}} e^{-x\lambda_{n_\nu}})$$

$$= O(\nu e^{-\frac{1}{2}x\lambda_{n_\nu}}) = O(\nu e^{-\frac{1}{2}x\nu}) .$$

Hence $\Sigma a_n e^{-x\lambda_n}$ is absolutely convergent, and the result follows.

If $\lim \left[\frac{\lambda_{n+1} - \lambda_n}{\lambda_n} \right] u_{n+1} = 0$ the proof follows the same lines.

Now if $\{u_n\}$ is stronger than every crude limitation

sequence, then $\overline{\lim}\ \frac{\log u_n}{\lambda_n} = 0$. Hence it follows from the theorem that if

$$\overline{\lim}\ \frac{\log \mu_n}{\lambda_n} > 0$$

then no such limitation sequence $\{u_n\}$ can exist. A typical example when this is true is the (A, λ_n) method with $\lambda_n = \log n$.

This result allows us to prove the following lemma.

LEMMA 1.2

Suppose* $\underline{\lim}\ \frac{\mu_n^*}{n \log n} > 1$.

Then there are no limitation sequences for (A, λ_n) summability which are stronger than every crude limitation sequence.

Proof

We merely need to show that $\overline{\lim}\ \frac{\log \mu_n}{\lambda_n} > 0$.

Now, there exists an $N > 3$ such that for $n > N$

$$n \log n < \frac{\lambda_n}{\lambda_{n+1} - \lambda_n} .$$

Hence

$$\frac{\lambda_{n+1}}{\lambda_n} < 1 + \frac{1}{n \log n} .$$

But

$$\lambda_{n+1} = \frac{\lambda_{n+1}}{\lambda_n} \cdot \frac{\lambda_n}{\lambda_{n-1}} \cdots \frac{\lambda_{N+1}}{\lambda_N} \cdot \lambda_N .$$

Thus

$$\lambda_{n+1} \leq \lambda_N \prod_N^n \left(1 + \frac{1}{j \log j}\right) = \lambda_N e^{\sum_N^n \log\left(1 + \frac{1}{j \log j}\right)} .$$

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* Note that there are no (A, λ_n) methods with $n(\log n)^\alpha = O(\mu_n^*)$ where $\alpha > 1$, since then $\lambda_n = O(1)$

Hence $\lambda_{n+1} \leq \lambda_N e^{\sum_N^n (j\log j)^{-1}}$.

But $\sum_N^n \frac{1}{j\log j} \leq \int_{N-1}^{n} \frac{dx}{x\log x} = \log\log n - \log\log(N-1)$.

Hence for some H, $\lambda_n < H\log n$, for each n.

Thus $\frac{\log \mu_n}{\lambda_n} > \frac{\log(n\log n)}{H\log n} = \frac{1}{H}\left\{1 + \frac{\log\log n}{\log n}\right\} \to \frac{1}{H}$ as $n \to \infty$.

Hence $\overline{\lim}\ \frac{\log \mu_n}{\lambda_n} > 0$.

1.3

In fact the class of (A,λ_n) methods for which there are no limitation sequences stronger than every crude limitation sequence is much wider than the previous lemma might suggest. In particular the (A,n) method falls into this class.

THEOREM 1.3

There is no limitation sequence for (A,n) summability which is stronger than every crude limitation sequence.

Proof

Suppose $\{u_n\}$ is a limitation sequence for (A,n) summability which is stronger than every crude limitation sequence.

Now $\Sigma(-1)^n n$ is summable (A,n) to $-\frac{1}{4}$ and so certainly $u_n > 1$ for every n. And, since $\{u_n\}$ is stronger than every crude limitation sequence, $\Sigma u_n x^n$ converges for every $x<1$. It follows that if we define v_n to be

nu_n then

$$\exists \lim |V_n|^{\frac{1}{n}} = 1 \quad . \qquad [1]$$

Now let $n_k = 4^k$ and define w_n so that

$$w_n = \begin{cases} V_{\frac{3}{2}n_k} & , \ n_k \leq n \leq 2n_k \\ 0 & , \ \text{otherwise} \end{cases}$$

Then for $n_k \leq n \leq 2n_k$

$$|V_{\frac{3}{2}n_k}|^{\frac{1}{2n_k}} \leq |w_n|^{\frac{1}{n}} \leq |V_{\frac{3}{2}n_k}|^{\frac{1}{n_k}} .$$

It follows by [1] that $\overline{\lim} |w_n|^{\frac{1}{n}} = 1$, and so $\Sigma w_n x^n$ converges for every $x<1$.

Now define

$$a_n = \begin{cases} \frac{1}{\binom{n_k}{\frac{n_k}{2}}} \cdot (-1)^j \binom{n_k}{j} V_{\frac{3}{2}n_k} & , \ n = n_k + j \\ & \quad (0 \leq j \leq n_k) \\ 0 & , \ \text{otherwise} \end{cases}$$

Then $a_{\frac{3}{2}n_k} = \frac{3}{2} n_k u_{\frac{3}{2}n_k}$. So $\{u_n\}$ does not dominate $\{a_n\}$. Hence if Σa_n is summable (A,n) we have a contradiction.

Now $|a_n| < w_n$ for each n. It follows that $\Sigma a_n x^n$ converges absolutely for each $x<1$, to a function, say $f(x)$, and

$$f(x) = \sum_1 a_n x^n = \sum_{k=1}^{\infty} V_{\frac{3}{2}n_k} \frac{x^{n_k}}{\binom{n_k}{\frac{n_k}{2}}} \sum_{j=0}^{n_k} (-1)^j x^j \binom{n_k}{j}$$

$$= \sum_{k=1}^{\infty} V_{\frac{3}{2}n_k} \binom{n_k}{\frac{n_k}{2}}^{-1} \{x(1-x)\}^{n_k} .$$

But $\binom{n_k}{\frac{n_k}{2}}^{-1} \{x(1-x)\}^{n_k} < \left(\frac{1}{4}\right)^{n_k}$ and $\sum_{k=1}^{\infty} V_{\frac{3}{2}n_k} \left(\frac{1}{4}\right)^{n_k}$ converges

by comparison with $\Sigma w_n (\frac{1}{4})^n$.

Hence $f(x) \to 0$ as $x \to 1-0$. It follows that Σa_n is summable (A,n) to zero and so $\{u_n\}$ cannot be a limitation sequence.

This theorem can be generalised quite easily, by using known results, to the class of (A,λ_n) methods which have $\lambda_n = f(n)$, where $f(y)$ is a strictly increasing, unbounded function for $y \geq 1$ and has a completely monotonic derivative on this range. The methods with $\lambda_n = n^q$ $(0<q\leq 1)$ and $\lambda_n = \log n$ all belong to this class.

Under these cnditions, $e^{-xf(y)}$ is a completely monotonic function of y on the range $y \geq 1$, for each $x>0$. Hence* $\{e^{-x\lambda_n}\}$ is a completely monotonic sequence for each $x>0$. It follows that** there is a function $\alpha(x,t)$ such that

$$e^{-x\lambda_n} = \int_0^1 t^n d\alpha(x,t)$$

$$\text{where } V[\alpha(x,t)]_0^1 \leq 1.$$

This in effect, transforms the problem from one dealing with the sequence $\{\lambda_n\}$ to one dealing with the sequence $\{n\}$. And this problem we have already solved.

However, this theorem is contained in the more general theorem which follows, so that further detail would be superfluous.

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* See Widder's 'The Laplace Transform', p.158.

**See Widder's 'The Laplace Transform', p.108. In the notation of the book,

$$V[\alpha(x,t)]_0^1 \leq \overline{\lim_k} \sum_0^k |\lambda_{k,m}| = \overline{\lim_k} \sum_0^k \lambda_{k,m} = \mu_0 = e^{-x\lambda_1} \leq 1.$$

1.4

We now show that Theorem 1.3 generalises to the class of (A, λ_n) methods which have $\underline{\lim} \frac{\mu_n^*}{n} > 1$.

THEOREM 1.4

Suppose $\underline{\lim} \frac{\mu_n^*}{n} > 1$.

Then there are no limitation sequences for (A, λ_n) summability which are stronger than every crude limitation sequence.

The proof of this theorem is rather long and is most conveniently dealt with by proving a series of lemmas.

LEMMA 1.5

Suppose $\underline{\lim} \frac{\mu_n^*}{n} > 1$. Then $\lambda_{n+1} - \lambda_n = O(1)$.

Proof

There exists an N such that for $n>N$, $\frac{\mu_n^*}{n} > 1$. Hence for $n>N$

$$\lambda_{n+1} - \lambda_n < \frac{\lambda_n}{n}$$

Also $$\frac{\lambda_{n+1}}{\lambda_n} < 1 + \frac{1}{n}.$$

It follows that $$\lambda_{n+1} < \lambda_N \prod_{j=N}^{n} \left(1 + \frac{1}{j}\right) = O(n).$$

And hence $$\lambda_{n+1} - \lambda_n = O(1).$$

LEMMA 1.6

Suppose $h_0, h_1 \ldots h_n$ are n+1 numbers and that

$$H^{(p)}_{q_1, q_2 \ldots q_r} = \sum_{\substack{\sum_1^r \theta_i = p \\ 0 < \theta_i \leq p}} h_{q_1}^{\theta_1} h_{q_2}^{\theta_2} \ldots h_{q_r}^{\theta_r}$$

where p and $q_1, q_2 \ldots q_r$ are non-negative integers and $q_i \neq q_j$, $0 \leq q_i \leq n$ $(i=1 \ldots r)$.

Then,

(i) Rearranging the order of the $q_1 \ldots q_r$ does not alter the value of $H^{(p)}_{q_1 q_2 \ldots q_r}$.

(ii) If $r>s>1$ then

$$H^{(p)}_{q_1 \ldots q_r} = H^{(p)}_{q_1 \ldots q_s} + H^{(p-1)}_{q_1 \ldots q_s} H^{(1)}_{q_{s+1} \ldots q_r} + H^{(p-2)}_{q_1 \ldots q_s} H^{(2)}_{q_{s+1} \ldots q_r} + \ldots + H^{(p)}_{q_{s+1} \ldots q_r} .$$

$$\text{(iii)} \quad H^{(p-1)}_{q_1 \ldots q_{r+1}} = \frac{H^{(p)}_{q_2 \ldots q_{r+1}} - H^{(p)}_{q_1 \ldots q_r}}{h_{q_{r+1}} - h_{q_1}} .$$

Proof

(i) This is obvious.

(ii) It is clear that

$$H^{(p)}_{q_1 \ldots q_r} = h^p_{q_1} + h^{p-1}_{q_1} H^{(1)}_{q_2 \ldots q_r} + \ldots + H^{(p)}_{q_2 \ldots q_r} .$$

The result then follows by induction.

(iii)

$$H^{(p)}_{q_2 \ldots q_{r+1}} - H^{(p)}_{q_1 \ldots q_r} = H^{(p)}_{q_{r+1} q_2 \ldots q_r} - H^{(p)}_{q_1 \ldots q_r}$$

$$= \left(h^p_{q_{r+1}} + h^{p-1}_{q_{r+1}} H^{(1)}_{q_2 \ldots q_r} + \ldots + H^{(p)}_{q_2 \ldots q_r} \right) - \left(h^p_{q_1} + h^{p-1}_{q_1} H^{(1)}_{q_2 \ldots q_r} + \ldots + H^{(p)}_{q_2 \ldots q_r} \right)$$

$$= \left(h^p_{q_{r+1}} - h^p_{q_1} \right) + \left(h^{p-1}_{q_{r+1}} - h^{p-1}_{q_1} \right) H^{(1)}_{q_2 \ldots q_r} + \ldots + \left(h_{q_{r+1}} - h_{q_1} \right) H^{(p-1)}_{q_2 \ldots q_r}$$

$$= \left(h_{q_{r+1}} - h_{q_1} \right) \left\{ H^{(p-1)}_{q_1 q_{r+1}} + H^{(p-2)}_{q_1 q_{r+1}} H^{(1)}_{q_2 \ldots q_r} + \ldots + H^{(p-1)}_{q_2 \ldots q_r} \right\}$$

$$= \left(h_{q_{r+1}} - h_{q_1} \right) H^{(p-1)}_{q_1 \ldots q_{r+1}} \quad \text{by (ii)} .$$

Before going any further, a few points of notation are necessary. Given a table of values, $f_i = f(x_i)$ for $i = 0,1,\ldots n$, of a function $f(x)$, the expressions $\Delta^{(r)}f_o$ and $\nabla^{(r)}f_n$ are familiar as r^{th} order forward and backward differences of the table. Thus

$$\Delta^{(0)}f_i = \nabla^{(0)}f_i = f_i$$

and

$$\Delta^{(r)}f_i = \Delta^{(r-1)}f_{i+1} - \Delta^{(r-1)}f_i$$

$$\nabla^{(r)}f_i = \nabla^{(r-1)}f_i - \nabla^{(r-1)}f_{i-1}$$

These definitions are of most use in connection with tables of a function evaluated at equal intervals of the argument. However, we shall be concerned with the function e^{-xy} evaluated at the points $y = \lambda_j$ $(j = m-n,..n)$. To deal with this case we introduce 'divided backward differences' denoted by $\underline{\nabla}^{(r)}f_i$ and defined by

$$\underline{\nabla}^{(0)}f_i = f_i$$

$$\underline{\nabla}^{(r)}f_i = \frac{\underline{\nabla}^{(r-1)}f_{i-1} - \underline{\nabla}^{(r-1)}f_i}{x_i - x_{i-r}} \qquad [1]$$

where $f_i = f(x_i)$.

We shall also define an entity $N^{(p)}_{q_1 q_2 \ldots q_r}$ as the $H^{(p)}_{q_1 q_2 \ldots q_r}$ of the previous lemma, but with h_i replaced by i for $i = 0,1,\ldots n$. [2]

From now on, whenever m and n are defined so that $m>n$, we shall take h_i to be defined by

$$h_i = \lambda_m - \lambda_{m-i} \qquad [3]$$

We are now in a position to prove the remaining lemmas.

LEMMA 1.7

If m>n, then

$$\underline{\nabla}^{(n)} e^{-x\lambda_m} = \sum_{j=0}^{m}{}' \frac{1}{\prod_{i=0}^{n}{}'(\lambda_j - \lambda_i)} e^{-x\lambda_{m-j}}$$

where the dash indicates the omission of the term with i=j.

Proof

Suppose we have a table $f_i = f(x_i)$, $(i = 0,..n)$.

Then $\underline{\nabla}^{(0)} f_r = f_r$.

Now suppose

$$\underline{\nabla}^{(s)} f_r = \sum_{j=r-s}^{r}{}' \frac{f_j}{\prod_{i=r-s}^{r}{}'(x_i - x_j)} .$$

Then, by [1],

$$\underline{\nabla}^{(s+1)} f_r = \frac{\underline{\nabla}^{(s)} f_{r-1} - \underline{\nabla}^{(s)} f_r}{x_r - x_{r-s-1}}$$

$$= \frac{1}{x_r - x_{r-s-1}} \left\{ \sum_{j=r-s-1}^{r-1}{}' \frac{f_j}{\prod_{r-s-1}^{r-1}{}'(x_i - x_j)} - \sum_{j=r-s}^{r}{}' \frac{f_j}{\prod_{r-s}^{r}{}'(x_i - x_j)} \right\}$$

$$= \frac{1}{x_r - x_{r-s-1}} \left\{ \frac{f_{r-s-1}}{\prod_{r-s-1}^{r-1}(x_i - x_{r-s-1})} + \sum_{j=r-s}^{r-1}{}' f_j \left(\frac{1}{\prod_{r-s-1}^{r-1}{}'(x_i - x_j)} - \frac{1}{\prod_{r-s}^{r}{}'(x_i - x_j)} \right) - \frac{f_r}{\prod_{r-s}^{r}(x_i - x_j)} \right\}$$

$$= \sum_{j=r-s-1}^{r}{}' \frac{f_j}{\prod_{r-s-1}^{r}{}'(x_i - x_j)}$$

because

$$\frac{1}{\prod_{r-s-1}^{r-1}{}'} - \frac{1}{\prod_{r-s}^{r}{}'} = \frac{x_r - x_{r-s-1}}{\prod_{r-s-1}^{r}{}'(x_i - x_j)}$$

It follows that

$$\underline{\nabla}^{(n)} f_m = \sum_{i=m-n}^{m} \frac{1}{\prod_{t=m-n}^{m}{}' (x_i - x_j)} f_j$$

Hence, taking $f_{m-j} = e^{-x\lambda_{m-j}}$, $(j = 0,1,\ldots n)$, we obtain that

$$\underline{\nabla}^{(n)} e^{-x\lambda_m} = \sum_{j=0}^{n} \frac{1}{\prod_{i=0}^{n}{}' (h_j - h_i)} e^{-x\lambda_{m-j}}$$

since, by [3],

$$\lambda_{m-i} - \lambda_{m-j} = h_j - h_i$$

LEMMA 1.8

$$\underline{\nabla}^{(n)} e^{-x\lambda_m} = \sum_{j=n}^{\infty} e^{-x\lambda_m} \frac{x^j}{j!} H_{1\ldots n}^{(j-n)}$$

Proof

Since h_i is defined as $\lambda_m - \lambda_{m-i}$, $(i = 0,1,\ldots n)$, it follows that

$$\underline{\nabla}^{(0)} e^{-x\lambda_{m-k}} = e^{-x\lambda_m} e^{xh_k} = \sum_{j=0}^{\infty} e^{-x\lambda_m} \frac{x^j}{j!} h_k^j = \sum_{j=0}^{\infty} e^{-x\lambda_m} \frac{x^j}{j!} H_k^{(j)}$$

Now suppose that

$$\underline{\nabla}^{(s)} e^{-x\lambda_{m-k}} = \sum_{j=s}^{\infty} e^{-x\lambda_m} \frac{x^j}{j!} H_{k,k+1,\ldots k+s}^{(j-s)}$$

Then, by [1],

$$\underline{\nabla}^{(s+1)} e^{-x\lambda_{m-k}} = \frac{\underline{\nabla}^{(s)} e^{-x\lambda_{m-k-1}} - \underline{\nabla}^{(s)} e^{-x\lambda_{m-k}}}{\lambda_{m-k} - \lambda_{m-k-s-1}}$$

$$= \frac{\underline{\nabla}^{(s)} e^{-x\lambda_{m-k-1}} - \underline{\nabla}^{(s)} e^{-x\lambda_{m-k}}}{h_{k+s+1} - h_k}$$

Hence

$$\underline{\nabla}^{(s+1)} e^{-x\lambda_{m-k}} = \sum_{j=s}^{\infty} e^{-x\lambda_m} \frac{x^j}{j!} \left\{ \frac{H^{(j-s)}_{k+1,\ldots k+s+1} - H^{(j-s)}_{k,\ldots k+s}}{h_{k+s+1} - h_k} \right\}$$

$$= \sum_{j=s+1}^{\infty} e^{-x\lambda_m} \frac{x^j}{j!} H^{(j-s-1)}_{k,k+1,\ldots k+s+1} .$$

by Lemma 1.6(111).

It follows that

$$\underline{\nabla}^{(n)} e^{-x\lambda_m} = \sum_{j=n}^{\infty} e^{-x\lambda_m} \frac{x^j}{j!} H^{(j-n)}_{0,1\ldots n} .$$

But $h_0 = 0$ and hence by Lemma 1.6(11)

$$H^{(j-n)}_{0,1\ldots n} = H^{(j-n)}_{1\ldots n} .$$

Thus

$$\underline{\nabla}^{(n)} e^{-x\lambda_m} = \sum_{j=n}^{\infty} e^{-x\lambda_m} \frac{x^j}{j!} H^{(j-n)}_{1,2\ldots n} .$$

LEMMA 1.9

$$\sum_{j=n}^{\infty} e^{-2nx} \frac{x^j}{j!} N^{(j-n)}_{1,2\ldots n} = \frac{(+1)^n e^{-nx} (1-e^{-x})^n}{n!} .$$

Proof

$$\sum_{j=n}^{\infty} e^{-2nx} \frac{x^j}{j!} N^{(j-n)}_{1\ldots n} = \underline{\nabla}^{(n)} e^{-2nx} = \frac{(-1)^n \nabla^{(n)} e^{-2nx}}{n!} = \frac{(-1)^n \Delta^{(n)} e^{-nx}}{n!}$$

$$= \frac{(+1)^n e^{-nx}}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} e^{-jx} = \frac{(+1)^n e^{-nx} (1-e^{-x})^n}{n!} .$$

Proof of Theorem 1.4

By Lemma 1.5, $\lambda_{n+1} - \lambda_n = O(1)$. But summability (A, λ_n) is equivalent to summability $(A, k\lambda_n)$ where $k>0$. We may therefore assume that

$$\lambda_{n+1} - \lambda_n < 1 \ . \qquad [4]$$

It follows that $\lambda_n < n$ for each n.

We may also assume that $\lim \frac{\log n}{\lambda_n} = 0$. Since otherwise $\lim \frac{\log n}{\lambda_n} > 0$. But this, together with the assumption of the theorem that $\lim \frac{\mu_n^*}{n} > 1$ means that $\lim \frac{\log \mu_n}{\lambda_n} > 0$. And we have shown by Theorem 1.1 that this condition implies the conclusion of Theorem 1.4.

The condition $\lim \frac{\log n}{\lambda_n} = 0$, allows us to say that the abcissae of convergence and absolute convergence of any Dirichlet series $\Sigma a_n e^{-x\lambda_n}$ are the same. However, the condition is more useful in another way.

Suppose $\{u_n\}$ is a limitation sequence for (A, λ_n) summability which is stronger than every crude limitation sequence. Then if $v_n = \lambda_n u_n$, certainly $v_n > 1$ for each n, and $\Sigma v_n e^{-x\lambda_n}$ converges for every $x>0$.

Now let $n_k = 4^k$, and let m_k be the smallest integer such that $\lambda_{m_k} \geq 2n_k$. Then certainly $\lambda_{m_k - m_k} > \lambda_{m_{k-1}}$. For convenience let us write $l_k = m_k - n_k$ and suppose that ν_k is defined so that $l_k \leq \nu_k \leq m_k$. Now define

$$w_n = \begin{cases} v_{\nu_k} \ , & l_k \leq n \leq m_k, \ (k=0,1\ldots) \\ 0 \ , & \text{otherwise} . \end{cases}$$

It is clear that Σw_n does not converge.

Suppose

$$v_n' = \begin{cases} v_{\nu_k} & , n = \nu_k \quad (k=0,1\ldots) \\ 0 & , \text{otherwise} \end{cases}$$

Then the abcissa of convergence of $\Sigma v_n' e^{-x\lambda_n}$ is zero, but $\Sigma v_n'$ does not converge.

We now use the following result*: if $\Sigma a_n e^{-x\lambda_n}$ is a Dirichlet series such that Σa_n diverges and

$$\overline{\lim_{N\to\infty}} \frac{\log|\sum_{n=1}^{N} a_n|}{\lambda_N} = 0$$

then the abcissa of convergence of $\Sigma a_n e^{-x\lambda_n}$ is zero.

We see that

$$\overline{\lim_{N\to\infty}} \frac{\log|\sum_{1}^{N} v_n'|}{\lambda_N} = 0.$$

But $\sum_{1}^{N} w_n \leq N \sum_{1}^{N} v_n'$. It follows that

$$\overline{\lim_{N\to\infty}} \frac{\log|\sum_{1}^{N} w_n|}{\lambda_N} \leq \overline{\lim_{N\to\infty}} \frac{\log N}{\lambda_N} + \overline{\lim_{N\to\infty}} \frac{\log|\sum_{1}^{N} v_n'|}{\lambda_N} = 0.$$

Hence $\Sigma w_n e^{-x\lambda_n}$ converges for each $x>0$.

In the preceding lemmas, 1.6-1.9, take n and m to be n_k and m_k respectively, where these are defined as above.

Then, by [3], $h_i = \lambda_{m_k} - \lambda_{m_k - i}$ $(i = 0,1\ldots n)$. But $\lambda_{m+1} - \lambda_m < 1$. It follows that for each i

$$h_i < i \,. \qquad [5]$$

---

* See for example Widder's 'The Laplace Transform', p.43.

Now, by Lemma 1.7, we have that

$$\underset{\sim}{\nabla}^{(n_k)} e^{-x\lambda_{n_k}} = \sum_{j=0}^{n_k} C_{k,j}\, e^{-x\lambda_{n_k-j}} \qquad [6]$$

$$\text{where} \quad C_{k,j} = \Big[\prod_{i=0}^{n_k}{}' (h_j - h_i)\Big]^{-1}.$$

Now define $C_k = |C_{k,K}| = \max_{0\le j\le n_k} |C_{k,j}|$. [7]

Fix the ν_k used in the definition of w_n so that $\nu_k = m_k - K$. Then certainly $1_k \le \nu_k \le m_k$.

Now define

$$a_n = \begin{cases} \dfrac{C_{k,j}}{C_k} v_{\nu_k}, & n = m_k - j \quad (0 \le j \le n_k) \\ 0, & \text{otherwise} \end{cases}$$

Then $$|a_{\nu_k}| = \left|\frac{C_{k,K}}{C_k} v_{\nu_k}\right| = v_{\nu_k} = \lambda_{\nu_k} u_{\nu_k}$$

Hence if Σa_n is summable (A,λ_n) it follows that $\{u_n\}$ cannot be a limitation sequence for (A,λ_n) summability. This will be a contradiction.

Now, by [7], $$\left|\frac{C_{k,j}}{C_k}\right| \le 1, \quad (j = 0, 1 \ldots n_k)$$

Hence $\Sigma a_n e^{-x\lambda_n}$ is absolutely convergent for every $x>0$ by comparison with $\Sigma w_n e^{-x\lambda_n}$. Now suppose $\Sigma a_n e^{-x\lambda_n}$ has the sum $f(x)$. Then

$$f(x) = \sum a_n e^{-\lambda_n x} = \sum_{k=1}^{\infty} \frac{v_{\nu_k}}{C_k} \sum_{j=0}^{n_k} C_{k,j}\, e^{-x\lambda_{n_k-j}}$$

Hence, by [6],

$$f(x) = \sum_{k=1}^{\infty} \frac{v_{\nu_k}}{C_k}\, \underset{\sim}{\nabla}^{(n_k)} e^{-x\lambda_{n_k}} \qquad [8]$$

But by Lemma 1.8

$$\underline{\nabla}^{(n_k)} e^{-x\lambda_{m_k}} = \sum_{j=n_k}^{\infty} e^{-x\lambda_{m_k}} \frac{x^j}{j!} H_{1\ldots n_k}^{(j-n_k)}$$

And by Lemma 1.9

$$\frac{1}{n_k!}\left\{e^{-x}(1-e^{-x})\right\}^{n_k} = \sum_{j=n_k}^{\infty} e^{-2n_k x} \frac{x^j}{j!} N_{1\ldots n_k}^{(j-n_k)}$$

Also for each $i = 0, 1, \ldots n_k$, $h_i < i$, by [5]. It follows that

$$H_{1\ldots n_k}^{(j-n_k)} < N_{1\ldots n_k}^{(j-n_k)}$$

Hence

$$e^{x\lambda_{m_k}} \underline{\nabla}^{(n_k)} e^{-x\lambda_{m_k}} \leq \frac{1}{n_k!} e^{xn_k}(1-e^{-x})^{n_k}$$

and so

$$\underline{\nabla}^{(n_k)} e^{-x\lambda_{m_k}} \leq \frac{1}{n_k!}\left\{e^{-x}(1-e^{-x})\right\}^{n_k}$$

since $\lambda_{m_k} \geq 2n_k$.

Therefore from [8] we obtain that

$$|f(x)| \leq \sum_{k=1}^{\infty} v_{\nu_k} \frac{1}{n_k! C_k}\left\{e^{-x}(1-e^{-x})\right\}^{n_k}$$

But

$$C_k^{-1} = \left|\prod_{i=0}^{n_k}{}'(h_k - h_i)\right| < n_k!$$

because $|h_k - h_i| < |K-i|$.

Hence

$$|f(x)| \leq \sum_{k=1}^{\infty} v_{\nu_k}\left\{e^{-x}(1-e^{-x})\right\}^{n_k} \qquad [9]$$

But since $e^{-x}(1-e^{-x}) \leq \frac{1}{4}$ for $x \geq 0$ and also $n_k \geq \frac{1}{2}\lambda_{m_k} - 1$, it follows that the series, [9], is uniformly convergent. ($\Sigma 4 v_{\nu_k}(\frac{1}{2})^{\lambda_{m_k}}$ converges by comparison with $\Sigma w_m e^{-x\lambda_m}$).

Hence $f(x) \to 0$ as $x \to 0+$.

It follows that Σa_n is summable (A,λ_n) to zero, which furnishes the required contradiction.

1.5

We have proved Theorem 1.4 for the class of (A,λ_n) methods which have $\underline{\lim} \frac{\mu_n^*}{n} > 1$. In Chapter 2, we shall show that limitation sequences stronger than every crude limitation sequence do in fact exist for (A,λ_n) methods with $\overline{\lim} \frac{\mu_n}{n} < 1$. For (A,λ_n) methods falling between these two extremes the position is not yet clear. Certainly the result of Theorem 1.4 holds for a wider class of methods than that for which it was proved. (For instance it holds for summability (A,n) as was shown in Theorem 1.3). However, for such cases as $(A,n\log n)$ summability, when, as with (A,n) summability, $\lim \frac{\mu_n}{n} = 1$, nothing can be said either way as yet. It seems probable though, that if any limitation sequences of the type we are discussing do exist for these intermediate cases, then they will not represent any great improvement on the crude limitation sequences. For example*, $\Sigma(-1)^n x^n$ is summable $(A,n\log n)$ to $(1+x)^{-1}$, for each $x>-1$. It follows that if $\{u_n\}$ is a limitation sequence for $(A,n\log n)$ summability, then $e^{\alpha n} = O(u_n)$, for each $\alpha>0$.

Exactly where the line falls which divides those methods for which no overall improvement on the crude limitation theorems is possible from those where this is possible, is a question of some interest. Perhaps a more delicate condition on the behaviour of the sequence $\{\mu_n^*\}$ than that considered is called for.

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* See, for example, Hardy's 'Divergent Series', p.78.

As might be expected we can use the method of Theorem 1.3 to obtain a result for the Abel summability of integrals. We conclude Chapter 1 by saying something about this.

Suppose $\int a(t)e^{-xt}dt$ converges to a function $f(x)$ for every $x>0$ and that $f(x)\to s$ as $x\to 0+$. Then let us say that $\int a(t)dt$ is summable [A] to s. With this definition it is not difficult to show that the summability [A] of $\int a(t)dt$ implies that for each positive ε and δ

$$\int_x^{x+\delta} a(t)\,dt = O(e^{\varepsilon x}) \quad \text{as} \quad x\to\infty .$$

The analogue of Theorem 1.3 is then the following result. There is no function, $u(x)$, with $u(x) = O(e^{\varepsilon x})$ for each $\varepsilon>0$ and such that the summability [A] of $\int a(t)dt$ implies that for each ε and δ

$$\int_x^{x+\delta} a(t)\,dt = O(u(x)) \quad \text{as} \quad x\to\infty$$

A modification of the proof of Theorem 1.3 will prove this as well.

CHAPTER 2

2.1

In Chapter 1, it was shown that if $\underline{\lim} \frac{\mu_n}{n} > 1$, then there are no limitation sequences for (A, λ_n) summability stronger than every crude limitation sequence. In this chapter, it is shown that such limitation sequences do exist when $\overline{\lim} \frac{\mu_n}{n} < 1$. A typical example is the (A, λ_n) method with $\lambda_n = n^p$, where $p>1$.

We begin with the following lemma.

LEMMA 2.1

Suppose that $\overline{\lim} \frac{\mu_n}{n} < s < 1$. Then for some $c>0$ and for every n

$$\lambda_{n+1}^s - \lambda_n^s \geq c.$$

Proof

There is an N such that for $n \geq N$

$$\frac{\lambda_{n+1}}{\lambda_n} > 1 + \frac{1}{sn}$$

Hence
$$\lambda_{n+1} > \lambda_N \prod_{j=N}^{n} \left(1 + \frac{1}{sj}\right)$$

It follows that there is an $\alpha>0$ such that

$$\lambda_{n+1} > \alpha \prod_{j=1}^{n} \left(1 + \frac{1}{sj}\right)$$

But $$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{j=1}^{\infty} \left(1+\frac{z}{j}\right) e^{-\frac{z}{j}}$$

where γ is Euler's Constant.

Hence, for large n,

$$\prod_{j=1}^{n} \left(1+\frac{1}{sj}\right) > \frac{1}{2} \frac{s e^{-\gamma/s}}{\Gamma(\frac{1}{s})} \prod_{j=1}^{n} e^{-\frac{1}{sj}}$$

and also

$$\sum_{j=1}^{n} \frac{1}{j} > \frac{1}{2}\gamma + \log n$$

It follows that there is a $\beta>0$ such that $\lambda_n^s > \beta n$. Also, for large n,

$$\left(\frac{\lambda_{n+1}}{\lambda_n}\right)^s - 1 > \left(1+\frac{1}{sn}\right)^s - 1$$

$$= \frac{1}{n} - \frac{s(1-s)}{2!}\left(\frac{1}{ns}\right)^2 + \cdots > \frac{1}{2n}.$$

Hence, for large n, we have that

$$\lambda_{n+1}^s - \lambda_n^s > \frac{1}{2n} \lambda_n^s > \frac{1}{2}\beta.$$

The result then follows.

In the following it will be convenient to introduce a new notation. If $p>1$ is defined we shall write

$$\Lambda_n^p = \lambda_n .$$

2.2

We now prove Theorem 2.2 which is fundemental to this chapter. It is stated here in a rather different

form to that in which it was first proved by Miss Cartwright* but the proof is essentially the same.

THEOREM 2.2

Let $p>1$, $\beta<\frac{\pi}{2}(1-\frac{1}{p})$ and $\Lambda_n^p = \lambda_n$.

Suppose that $\Sigma a_n e^{-\lambda_n w}$ is absolutely convergent for $w>0$ to a function $f(w)$ which is bounded for $w>0$.

Then $\Sigma a_n e^{-\Lambda_n z}$ is uniformly summable (A, λ_n) to $F(z)$ in the region $|\arg z|\leq\beta$, where,

$$F(z) = \frac{p}{2\pi i}\int_0^\infty f(u^p)\frac{du}{u}\int_{K-i\infty}^{K+i\infty}\left(\frac{z}{u}\right)^{-s}\frac{\Gamma(s)}{\Gamma(\frac{s}{p})}\,ds$$

for each $K>0$.

To prove Theorem 2.2 we need two lemmas.

LEMMA 2.3

Let $\phi(w,s) = \Sigma a_n \Lambda_n^{-s} e^{-w\lambda_n}$. Then for $R(s)>0$ and $w>0$,

$$\phi(w^p,s) = \frac{p}{\Gamma(\frac{s}{p})}\int_0^\infty u^{s-1} f(u^p+w^p)\,du$$

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* M.L.Cartwright, P.L.M.S., 31 (1930), 81-96.

Miss Cartwright states the theorem for the case $\Lambda_n = n$, but, as she notes, the extension to the above is immediate. Also her statement does not mention the uniformity of the summability but, as Kuttner pointed out, her proof is sufficient to prove this as well.

Proof

$$\Lambda_n^{-s}\Gamma\left(\frac{s}{\rho}\right) = \int_0^\infty y^{\frac{s}{\rho}-1} e^{-y\Lambda_n^\rho}\,dy = \rho\int_0^\infty u^{s-1}e^{-u^\rho\lambda_n}\,du .$$

Multiply both sides by $a_n e^{-w^\rho\lambda_n}$ and sum. Then

$$\Gamma\left(\frac{s}{\rho}\right)\sum a_n\Lambda_n^{-s}e^{-w^\rho\lambda_n} = \rho\int_0^\infty u^{s-1}f(u^\rho+w^\rho)\,du ,$$

term by term integration being justified by the absolute convergence of $\Sigma a_n e^{-w^\rho\lambda_n}$ for $w>0$.

LEMMA 2.4

Suppose $f(w,z) = \Sigma a_n e^{-\Lambda_n z}e^{-\lambda_n w}$. Then, for $R(z)>0$, $|\arg z|<\beta$ and $w>0$,

$$f(w^\rho,z) = \frac{\rho}{2\pi i}\int_0^\infty f(u^\rho+w^\rho)\frac{du}{u}\int_{K-i\infty}^{K+i\infty}\left(\frac{z}{u}\right)^{-s}\frac{\Gamma(s)}{\Gamma\left(\frac{s}{\rho}\right)}\,ds ,$$

where $K>0$.

Proof

For $K>0$ and $|\arg z|<\frac{\pi}{2}$,

$$e^{-\Lambda_n z} = \frac{1}{2\pi i}\int_{K-i\infty}^{K+i\infty}\Lambda_n^{-s}z^{-s}\Gamma(s)\,ds .$$

Multiply both sides by $a_n e^{-w^\rho\lambda_n}$ and sum. Then, for $K>0$, $w>0$ and $|\arg z|<\frac{\pi}{2}$,

$$f(w^\rho,z) = \frac{1}{2\pi i}\int_{K-i\infty}^{K+i\infty}z^{-s}\Gamma(s)\,\phi(w^\rho,s)\,ds ,$$

term by term integration being justified by the absolute convergence of $\Sigma a_n\Lambda_n^{-s}e^{-w^\rho\lambda_n}$ for $w>0$.

Substituting the result of Lemma 2.3

$$f(w^\rho,z) = \frac{\rho}{2\pi i}\int_{K-i\infty}^{K+i\infty} z^{-s}\frac{\Gamma(s)}{\Gamma(\frac{s}{\rho})}\int_0^\infty u^{s-1} f(u^\rho+w^\rho)\,du\,ds \qquad [1]$$

for $K>0$, $w>0$ and $|\arg z|<\frac{\pi}{2}$.

Now, for $|\arg z|\leq\beta$, the double integral

$$\int_{K-i\infty}^{K+i\infty}\int_0^\infty \left| \frac{z^{-s}\Gamma(s)u^{s-1}f(u^\rho+w^\rho)}{\Gamma(\frac{s}{\rho})}\right| |ds|\,du$$

converges by comparison with

$$\int_{-\infty}^{\infty}\int_0^\infty e^{(|\theta|-\frac{\pi}{2}+\frac{\pi}{2\rho}+\epsilon)|t|}\left(\frac{u}{r}\right)^K e^{-\lambda_1 u^\rho}|dt|\,\frac{du}{u}$$

where $K>0$, $w>0$, $z = re^{i\theta}$, $s = K+it$ and ε is any positive number.

For $|\arg z|\leq\beta$, we may therefore invert the order of integration of [1] and obtain the result of the lemma.

Proof of Theorem 2.2

For $u>0$, $r>0$, $K>0$ and $|\theta|\leq\beta$, write

$$I\left(\frac{z}{u},K\right) = \frac{1}{2\pi i}\int_{K-i\infty}^{K+i\infty}\left(\frac{z}{u}\right)^{-s}\frac{\Gamma(s)}{\Gamma(\frac{s}{\rho})}\,ds = O\left\{\left(\frac{u}{r}\right)^K\right\}.$$

It follows that

$$\int_0^1 \left| I\left(\frac{z}{u},K\right)\right|\frac{du}{u}$$

converges. Now suppose $w_0>0$. Then $f(u^\rho+w^\rho) = O(1)$ as $u\to 0+$, uniformly for $0\leq w\leq w_0$. Also $f(u^\rho+w^\rho) = O(e^{-\lambda_1 u^\rho})$ as $u\to\infty$, uniformly for $0\leq w\leq w_0$. Hence

$$\int_0^\infty f(u^\rho + w^\rho)\, I\left(\frac{z}{u}, K\right) \frac{du}{u}$$

is uniformly convergent for $0 \le w \le w_0$. Thus for $K>0$, $r>0$ and $|\theta| \le \beta$

$$\exists\ F(z) = \lim_{w \to 0+} f(w^\rho, z) = p\int_0^\infty f(u^\rho) I\left(\frac{z}{u}, K\right) \frac{du}{u}$$

We now show that $F(z)$ is a uniform limit for $|\arg z| \le \beta$. Consider

$$|F(z) - f(w^\rho, z)| < p\int_0^\infty |I\left(\frac{z}{u}, K\right)|\, |f(u^\rho) - f(u^\rho + w^\rho)| \frac{du}{u}$$

$$= p\int_0^\infty |I\left(\frac{e^{i\theta}}{v}, K\right)|\, |f(r^\rho v^\rho) - f(r^\rho v^\rho + w^\rho)| \frac{dv}{v}.$$

But there exists an w_0 such that for $0 \le w \le w_0$ and every positive r and v, $|f(r^D v^D) - f(r^D v^D + w^D)| < \varepsilon$. And anticipating the result of Theorem 2.5, which does not use the uniformity of the limit in its proof, it is clear that

$$\int_0^\infty |I\left(\frac{e^{i\theta}}{v}, K\right)| \frac{dv}{v}$$

converges and is bounded for $|\theta| \le \beta$. It follows that $F(z)$ is a uniform limit for $|\arg z| \le \beta$.

Theorem 2.5 now follows quite naturally. This is again due to Miss Cartwright[9].

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[9] ibid.

THEOREM 2.5

Let $p>1$, $\beta<\frac{\pi}{2}(1-\frac{1}{p})$ and $\Lambda_n^p=\lambda_n$.

Suppose Σa_n is summable (A,λ_n) to α, and further that $\Sigma a_n e^{-\lambda_n w}$ is absolutely convergent for $R(w)>0$.

Then if $F(z)$ is the sum of the series $\Sigma a_n e^{-\Lambda_n z}$ in the (A,λ_n) sense, $F(z)\to\alpha$ as $z\to 0$, uniformly in the angle $|\arg z|\leq\beta$.

Proof

Write $u=rv$, and otherwise use the notation of Theorem 2.2. Then, for $K>0$, $r>0$ and $|\theta|\leq\beta$,

$$F(re^{i\theta}) = p\int_0^\infty f(r^p v^p)\, I\left(\frac{e^{i\theta}}{v},K\right)\frac{dv}{v} \qquad [1]$$

Consider the integral

$$\frac{1}{2\pi i}\int_C \left(\frac{e^{i\theta}}{v}\right)^{-s}\frac{\Gamma(s)}{\Gamma(\frac{s}{p})}\,ds$$

where C is the rectangle with vertices K_1-iT, K_1+iT, K_2+iT and K_2-iT, where $-1<K_1<0<K_2$. The integrand is regular for $R(s)>-1$, and provided $|\theta|<\beta$, the contribution from the horizontal sides vanishes as T tends to infinity. It follows that

$$\int_0^\infty \left|I\left(\frac{e^{i\theta}}{v},K\right)\right|\frac{dv}{v} = \int_0^1 \left|I\left(\frac{e^{i\theta}}{v},K_2\right)\right|\frac{dv}{v} + \int_1^\infty \left|I\left(\frac{e^{i\theta}}{v},K_1\right)\right|\frac{dv}{v}$$

and so is uniformly convergent for $|\theta|\leq\beta$.

Also $f(r^p v^p)\to\alpha$ as $r\to 0+$, and $f(r^p v^p)$ is bounded for positive r and v. We can therefore proceed to the limit under the integral sign of [1].

Hence

$$\lim_{r\to 0+} F(re^{i\theta}) = p\alpha \int_0^\infty I\left(\frac{e^{i\theta}}{v}, k\right) \frac{dv}{v}$$

$$= \frac{p\alpha}{2\pi i}\left\{ \int_0^1 \frac{dv}{v} \int_{K_2-i\infty}^{K_2+i\infty} v^s e^{-i\theta s} \frac{\Gamma(s)}{\Gamma\left(\frac{s}{p}\right)} ds + \int_1^\infty \frac{dv}{v} \int_{K_1-i\infty}^{K_1+i\infty} v^s e^{-i\theta s} \frac{\Gamma(s)}{\Gamma\left(\frac{s}{p}\right)} ds \right\}$$

uniformly for $|\theta| \leq \beta$

Both double integrals converge absolutely for $|\theta| \leq \beta$ and so we can invert the orders of integration.

If $R(s) = K_2 > 0$ then $\int_0^1 v^{s-1} dv = \frac{1}{s}$.

If $R(s) = K_1 < 0$ then $\int_1^\infty v^{s-1} dv = -\frac{1}{s}$.

Hence

$$\lim_{r\to 0+} F(re^{i\theta}) = \frac{p\alpha}{2\pi i}\left\{ \int_{K_1-i\infty}^{K_1+i\infty} - \int_{K_2-i\infty}^{K_2+i\infty} \right\} \frac{e^{-i\theta s}}{s} \frac{\Gamma(s)}{\Gamma\left(\frac{s}{p}\right)} ds$$

Consider this integral around the rectangle C used previously. The horizontal contributions vanish as T tends to infinity, and we are left to compute the residue at the simple pole at the origin. But

$$\lim_{s\to 0} e^{-i\theta s} \frac{\Gamma(s)}{\Gamma\left(\frac{s}{p}\right)} = \lim_{s\to 0} \frac{\sin \frac{\pi s}{p}}{\sin \pi s} = \frac{1}{p} .$$

Thus

$$\lim_{r\to 0+} F(re^{i\theta}) = \alpha$$

uniformly for $|\theta| \leq \beta$.

2.3

In the following we shall only be concerned with sequences $\{\lambda_n\}$ such that $\overline{\lim_{n\to\infty}} \frac{\log n}{\lambda_n} = 0$, in which case

the abcissae of convergence and absolute convergence of the series $\Sigma a_n e^{-\lambda_n w}$ are the same. By Theorems 2.2 and 2.5, the single assumption that Σa_n is summable (A, λ_n) is then sufficient, as we shall see, to obtain that the function $f(w,z) = \Sigma a_n e^{-\lambda_n w} e^{-\Lambda_n z}$ is uniformly bounded for $w>0$ in the angle $|\arg z| \leq \beta < \frac{\pi}{2}(1-\frac{1}{p})$.

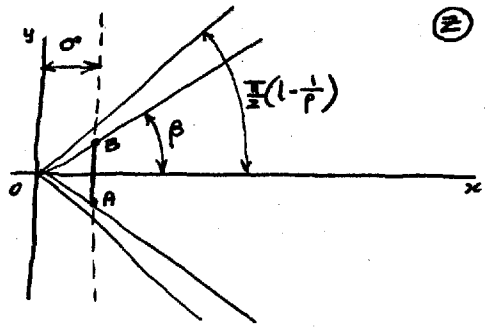
Now, if $\Sigma A_n e^{-\Lambda_n it}$ converges absolutely to $g(t)$, then $g(t)$ is uniformly almost periodic and

$$|A_n| \leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|\,dt .$$

Hence, writing $f(w, \sigma'+it) = g(t)$, we obtain that

$$|a_n e^{-\lambda_n w} e^{-\Lambda_n \sigma'}| \leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(w,\sigma'+it)|\,dt .$$

However, for each fixed $\sigma'>0$, we have information about $f(w,\sigma'+it)$ only for a finite range of t. This range is in fact the line segment AB in the diagram below.



The following theorem of Ingham* resolves this problem for the case when $\Lambda_n - \Lambda_{n-1} \geq \gamma > 0$.

---

* A.E.Ingham, P.C.P.S., 46 (1950), 535-537.

THEOREM 2.6

Suppose there is a $\gamma > 0$ such that for each n, $\Lambda_n - \Lambda_{n-1} \geq \gamma$, and that g(t) is defined by

$$g(t) = \sum_{n=N}^{N'} A_n e^{-i\Lambda_n t},$$

then

$$|A_m| \leq \frac{\gamma}{\pi} \int_{-\frac{\pi}{\gamma}}^{\frac{\pi}{\gamma}} |g(t)|\, dt$$

a similar result holding for the case of infinite series which converge uniformly on $[-\frac{\pi}{\gamma}, \frac{\pi}{\gamma}]$.

Proof

Consider any particular Λ_m, say A_q. Write

$$g(t)e^{\Lambda_q t i} = \sum_{N}^{N'} A_n e^{-(\Lambda_n - \Lambda_q)ti} = \sum_{-M+1}^{M-1} b_m e^{-\nu_m ti}$$

where M is a sufficiently large integer and $\nu_0 = 0$, $b_0 = A_q$ while $\gamma m \leq \nu_m < \gamma(m+1)$ and $-\gamma m \geq \nu_{-m} > -\gamma(m+1)$, for m = 1,2,..,M-1.

This arrangement is possible because $(-\gamma, \gamma)$ cannot contain any $\Lambda_n - \Lambda_q$ except 0, while no interval, $[\gamma m, \gamma(m+1))$, or $(-\gamma(m+1), -\gamma m]$, can contain more than one of the numbers $\Lambda_n - \Lambda_q$. If a $\Lambda_n - \Lambda_q$ is contained in the interval $[\gamma m, \gamma(m+1))$ take ν_m to be this. Otherwise take ν_m to be γm. Similarly for the interval $(-\gamma(m+1), -\gamma m]$. For $\gamma M > \max(\Lambda_{N'} - \Lambda_q, \Lambda_q - \Lambda_N)$ all the terms of g(t) will be accounted for.

Let

$$P(u) = \prod_{-M+1}^{M+1}{}' (u - \nu_m) \quad , \quad Q(u) = \prod_{-M}^{M} (u - \gamma_m)$$

where $\prod'$ indicates the omission of the term, $(u - \nu_0)$.

Resolving into partial fractions,

$$\frac{P(u)}{Q(u)} = \sum_{-M}^{M} \frac{K_m}{u - \gamma m}$$

where

$$K_m = \frac{P(\gamma m)}{Q'(\gamma m)} \quad \text{and} \quad \sum_{-M}^{M} K_m = \lim_{u \to \infty} \frac{uP(u)}{Q(u)} = 0 .$$

Now define

$$k(t) = \sum_{-M}^{M} (-1)^m K_m e^{\gamma m t i} .$$

Then

$$\begin{aligned} K(u) &= \int_{-\pi/\gamma}^{\pi/\gamma} k(t)\, e^{-ut i}\, dt \\ &= \sum_{-M}^{M} (-1)^m K_m \int_{-\pi/\gamma}^{\pi/\gamma} e^{(\gamma m - u) t i}\, dt \\ &= \sum_{-M}^{M} (-1)^m K_m \frac{2 \sin (\gamma m - u)\frac{\pi}{\gamma}}{\gamma m - u} = 2 \sin \frac{u\pi}{\gamma} \frac{P(u)}{Q(u)} . \end{aligned}$$

Now

$$\int_{-\frac{\pi}{\gamma}}^{\frac{\pi}{\gamma}} k(t)\, g(t)\, e^{\lambda_2 t i}\, dt = \sum_{-M+1}^{M-1} b_m K(\gamma_m) .$$

But $K(\gamma_m) = 0$ for $m \neq 0$ because $P(\gamma_m) = 0$ and $2\sin \frac{u\pi}{\gamma}[Q(u)]^{-1}$ is an integral function of u. Also

$$K(\gamma_0) = K(0) = \frac{2\pi}{\gamma} \frac{P(0)}{Q'(0)} = \frac{2\pi}{\gamma} K_0 \neq 0 .$$

Hence

$$\left| b_0 \frac{2\pi}{\gamma} K_0 \right| \leq \max_{-\frac{\pi}{\gamma} \leq t \leq \frac{\pi}{\gamma}} |k(t)| \int_{-\frac{\pi}{\gamma}}^{\frac{\pi}{\gamma}} |g(t)|\, dt$$

Now $Q'(\gamma m)$ takes the signs + and - alternatively as m takes in succession the values M, M-1, ..., -M.

The same is true of $-P(\gamma m)$ if we count $m-\gamma_m$ and $-m-\gamma_m$, where $m \geq 1$, as $-$ and $+$ respectively (even if they are zero), except that $-P(\gamma m)$ keeps the same sign for $m = 1, 0$ and -1. This determines the signs of the K_m. Thus

$$K_0 = \frac{P(0)}{Q(0)} > 0 \quad , \quad K_m = \frac{P(\gamma m)}{Q'(\gamma m)} \leq 0 \quad (m \neq 0).$$

It follows that

$$\max_{-\frac{\pi}{\gamma} \leq t \leq \frac{\pi}{\gamma}} |k(t)| \leq \sum_{-M}^{M} |K_m| = K_0 - \sum_{-M}^{M}{}' K_m = 2K_0 .$$

Therefore

$$\left| b_0 \frac{2\pi}{\gamma} K_0 \right| = \left| A_q \frac{q\pi}{\gamma} K_0 \right| \leq 2K_0 \int_{-\pi/\gamma}^{\pi/\gamma} |g(t)|\,dt$$

Hence

$$|A_q| \leq \frac{\gamma}{\pi} \int_{-\pi/\gamma}^{\pi/\gamma} |g(t)|\,dt$$

The result for uniformly convergent series follows immediately.

2.4

We are now in a position to prove the limitation theorem we have been leading up to. This is a generalisation of a theorem of Kuttner*, who proved it for the case $\lambda_n = n^p$, where $p > 1$.

---

* B. Kuttner, J.L.M.S., 37 (1962), 123-125.

THEOREM 2.7

Let $p>1$ and $\Lambda_n^p = \lambda_n$. Suppose that $\Lambda_n - \Lambda_{n-1} \geq \gamma > 0$, for some $\gamma > 0$ and every n. Then if $\sum a_n$ is summable (A, λ_n)

$$a_n = O(e^{K\Lambda_n})$$

for each $K > \frac{\pi}{\gamma} \tan \frac{\pi}{2p}$.

Proof

Since $\Lambda_n - \Lambda_{n-1} \geq \gamma > 0$, the abcissae of convergence and absolute convergence of the series $\sum a_n e^{-\lambda_n w}$ are the same. We can therefore use Theorems 2.2 and 2.5.

Now

$$f(w,z) = \sum a_n e^{-\lambda_n w} e^{-\Lambda_n z},$$

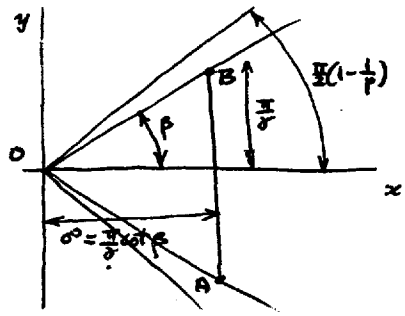
and, by Theorem 2.2, $f(w,z) \to F(z)$ as $w \to 0+$, uniformly for $|\arg z| \leq \beta < \frac{\pi}{2}(1-\frac{1}{p})$. Furthermore, by Theorem 2.5, $F(z) \to \alpha$ as $z \to 0$, uniformly for $|\arg z| \leq \beta$, where α is the (Λ, λ_n) sum of $\sum a_n$. It follows that $F(z)$ is bounded, say by H, for $|\arg z| \leq \beta$.

Write $z = \sigma + it$, and take $g(t) = f(w, \sigma+it)$ and $A_n = a_n e^{-\lambda_n w} e^{-\Lambda_n \sigma}$ in Theorem 2.6. Then

$$|a_n e^{-\lambda_n w} e^{-\Lambda_n \sigma}| \leq \frac{\gamma}{\pi} \int_{-\pi/\gamma}^{\pi/\gamma} |f(w, \sigma+it)| dt . \qquad [1]$$

Now fix $\sigma = \frac{\pi}{\gamma} \cot\beta$, where $\beta < \frac{\pi}{2}(1-\frac{1}{p})$. Then throughout the range of integration of [1]

$$|\arg(\sigma+it)| \leq \beta .$$



(2)

But $f(w,z) \to F(z)$ as $w \to 0+$, uniformly on the line segment, AB. It follows that we may let $w \to 0+$ under the integral sign of [1]. We then obtain

$$|a_n e^{-\lambda_n \sigma}| \leq \frac{\gamma}{\pi} \int_{-\frac{\pi}{\gamma}}^{\frac{\pi}{\gamma}} |F(\sigma+it)|\,dt \leq \varrho H.$$

Finally, it is clear that if K exceeds $\frac{\pi}{\gamma} \tan \frac{\pi}{2\rho}$, then there is a β such that $\beta < \frac{\pi}{2}(1-\frac{1}{\rho})$ and also $\sigma = \frac{\pi}{\gamma} \cot\beta = K$. This completes the proof of the theorem.

COROLLARY 2.8

Suppose $\overline{\lim} \frac{\mu_n}{n} < s < 1$, and that Σa_n is summable (A,λ_n). Then there is an A such that

$$a_n = O(e^{A\lambda_n^s}).$$

Proof

This follows immediately from Lemma 2.1.

Theorem 2.7 is a best possible result, in the sense that no smaller lower bound for K will suffice. An example of Kuttner's, derived from the following result of E.G.Phillips* will prove this.

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* E.G.Phillips, P.L.M.S., 40 (1936), 436-448.

LEMMA 2.9

Suppose $\Sigma a_n e^{-\lambda_n s}$ has a region of absolute convergence and that it defines a function $f(s)$ with singularities s_n for $n = 1, 2, \ldots$. Suppose $p>1$, and that $\lambda_n^p = \lambda_n$. Then the series is summable (A, λ_n) in a region defined by the inequalities

$$|\arg(s-s_n)| < \tfrac{\pi}{2}(1-\tfrac{1}{p}).$$

Now take $\lambda_n = n$, so that $\lambda_n - \lambda_{n-1} = 1$. By the lemma, the series Σz^n is summable (A, n^p) for all z satisfying

$$r < e^{|\theta| \tan \frac{\pi}{2p}}, \quad |\theta| < \pi$$

where $z = re^{i\theta}$.

Taking $\theta = \pm\pi$ it is clear that Σz^n is summable (A, n^p) for values of r arbitrarily near e^A, where $A = \pi \tan \frac{\pi}{2p}$.

It follows that Theorem 2.7 is a best possible result in the sense described.

CHAPTER 3

3.1

This chapter is concerned with (A,λ_n) methods for which $\mu_n = O(1)$. In this case, $\{\lambda_n\}$ must satisfy the 'high indices' condition. This means that for some $\phi>1$ and for every n

$$\lambda_{n+1} > \phi\lambda_n .$$

Theorem 3.1, the high indices theorem, was first conjectured by Littlewood* and later proved by Hardy and Littlewood** by their method of repeated differentiation. The proof given here, however, is a simpler one due to Ingham***.

THEOREM 3.1

Suppose that Σa_n is summable (A,λ_n) and that $\mu_n = O(1)$. Then

$$a_n = O(1) .$$

Proof

For some $\phi>1$ and for every n

$$\lambda_{n+1} > \phi\lambda_n . \qquad [1]$$

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* J.E.Littlewood, P.L.M.S.,(2), 9 (1910), 434-48.
** G.H.Hardy and J.E.Littlewood, ibid., 25 (1926), 219-36.
*** A.E.Ingham, O.Q.J., 8 (1937), 1-7.

Write $f(x) = \Sigma a_n e^{-\lambda_n x}$. Then for some H,

$$|f(x)| \leq H. \qquad [2]$$

Now choose $\alpha>0$. Then $a_n e^{-\lambda_n \alpha} \to 0$ as $n \to \infty$. It follows that we can take m to be the smallest index such that

$$|a_m e^{-\lambda_m \alpha}| = \max_n |a_n e^{-\lambda_n \alpha}| \qquad [3]$$

Consider the function $p(x) = 4(2^{-x}-2^{-2x})$. It is clear that p(x) increases from zero at x=0 to a maximum of 1 at x=1. It then decreases to zero as $x\to\infty$. Finally, it is O(x) for small x, and $O(\frac{1}{x})$ for large x. Now, define

$$P(x) = [p(x)]^R.$$

Then $P(x) = [4(2^{-x}-2^{-2x})]^R = \sum_{r=0}^{R} c_r e^{-\nu_r x}$, say.

And $$\sum_{r=0}^{R} |c_r| = [4(1+1)]^R = 8^R.$$

Now define

$$F_\alpha(x) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n \alpha} P(\lambda_n x) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n \alpha} \sum_{r=0}^{R} c_r e^{-\nu_r \lambda_n x}$$

$$= \sum_{r=0}^{R} c_r \sum_{n=0}^{\infty} a_n e^{-\lambda_n(\alpha+\nu_r x)} = \sum_{r=0}^{R} c_r f(\alpha+\nu_r x).$$

Thus, by [2],

$$|F_\alpha(x)| \leq \sum_{r=0}^{R} |c_r| H = H 8^R.$$

So $F_\alpha(x)$ has a bound independent of both x and α.

Now

$$H8^R \geq |F_\alpha(\tfrac{1}{\lambda_m})| = \Big| \sum_{n=0}^{\infty} a_n e^{-\lambda_n \alpha} P(\tfrac{\lambda_n}{\lambda_m}) \Big|$$

$$= \Big| a_m e^{-\lambda_m \alpha} + \sum_{n \neq m}' a_n e^{-\lambda_n \alpha} P(\tfrac{\lambda_n}{\lambda_m}) \Big|$$

$$\geq |a_m e^{-\lambda_m \alpha}| \Big\{ 1 - \sum_{n \neq m}' P(\tfrac{\lambda_n}{\lambda_m}) \Big\} . \qquad [4]$$

But it follows from [1], that for r>0

$$\frac{\lambda_{m+r}}{\lambda_m} > \phi^r \quad \text{and} \quad \frac{\lambda_{m-r}}{\lambda_m} < \phi^{-r} .$$

Hence

$$\sum_{n \neq m}' P(\tfrac{\lambda_n}{\lambda_m}) = \sum_{n<m} P(\tfrac{\lambda_n}{\lambda_m}) + \sum_{n>m}' P(\tfrac{\lambda_n}{\lambda_m})$$

$$\leq \sum_{r=1}^{\infty} P(\phi^{-r}) + \sum_{r=1}^{\infty}{}' P(\phi^r)$$

$$\leq [p(\phi^{-1})]^{R-1} \sum_{r=1}^{\infty} p(\phi^{-r}) + [p(\phi)]^{R-1} \sum_{r=1}^{\infty}{}' p(\phi^r)$$

$$= [p(\phi^{-1})]^{R-1} \sum_{r=1}^{\infty} O(\phi^{-r}) + [p(\phi)]^{R-1} \sum_{r=1}^{\infty}{}' O(\phi^{-r}) .$$

Thus, for large enough R

$$\sum_{n \neq m}' P(\tfrac{\lambda_n}{\lambda_m}) < \tfrac{1}{2} .$$

It follows from [4] that

$$|a_m e^{-\lambda_m \alpha}| \leq 2H8^R .$$

Hence, by [3], for every n

$$|a_n e^{-\lambda_n \alpha}| \leq 2H8^R .$$

And thus, allowing α to tend to zero, we obtain that

$$a_n = O(1).$$

We have then that $\{1\}$ is a limitation sequence for (A,λ_n) summability if $\mu_n = O(1)$. Also, by Theorem 1.1, we know that if $\{u_n\}$ is a limitation sequence stronger than every crude limitation sequence, then $\mu_n = O(u_n)$. It follows that if $\{1\}$ is a limitation sequence, then $\mu_n = O(1)$. Thus $\{1\}$ is a limitation sequence for (A,λ_n) summability if, and only if, $\mu_n = O(1)$.

CHAPTER 4

4.1

Up to now we have been looking for limitation sequences which are stronger than every crude limitation sequence. To obtain theorems of this type we found it necessary to insist that the sequence $\{\lambda_n\}$ should increase not only rapidly but also steadily. We ensured this by imposing conditions on the sequence $\{\mu_n\}$ defined by

$$\mu_n = \max\left[\frac{\lambda_n}{\lambda_{n+1}-\lambda_n}, \frac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\right].$$

However, where the sequence $\{\lambda_n\}$ behaves irregularly it may be that, while no overall improvement on the crude theorem is possible, there are yet limitation sequences which are stronger than the crude ones on some subsequence. In obtaining this sort of limitation sequence we can escape with considerably weaker conditions. In this chapter, for example, we shall be largely concerned with (A, λ_n) methods for which $\Sigma \lambda_n^{-q}$ converges for some $q<1$.

4.2

We shall again use much of the notation of Chapter 2. In particular, if $p>1$ is defined we shall write $\Lambda_n^r = \lambda_n$.

THEOREM 4.1

Suppose Σa_n is summable (A, λ_n) and that, for some $p>1$, $\Sigma \Lambda_n^{-1}$ converges. Then for each $\varepsilon>0$

$$a_n = O(e^{\epsilon\Lambda_n}(1+\mu_n)^{\epsilon\Lambda_n}).$$

We first prove two lemmas.

LEMMA 4.2

$$\max\left[\frac{\Lambda_{n+1}}{\Lambda_{n+1}-\Lambda_n}, \frac{\Lambda_n}{\Lambda_n-\Lambda_{n-1}}\right] = O(1+\mu_n).$$

Proof

This follows from the fact that for $0<x<1$ $1-x > \frac{1}{p}(1-x^p)$.

Before proving Lemma 4.3 a few definitions are necessary.

We shall denote the Fourier transform of a given function by a capital letter. Thus given f(t) we define F(x) by

$$F(x) = \int e^{ixt} f(t)\,dt .$$

The convolution, $\int f(t)g(x-t)dt$, of two functions will be denoted by $f(x)*g(x)$, and the convolution of n functions, $f_1(x),\ldots,f_n(x)$, by

$$\prod_{j=1}^{n} f_j(x) . \qquad [1]$$

The Fourier transform of [1] will then be

$$\prod_{j=1}^{n} F_j(x) .$$

We shall also use the fact that if $f\in L$ and is continuous and also $F\in L$ then

$$f(t) = \frac{1}{2\pi}\int e^{-ixt} F(x)\,dx .$$

Finally let us define $\delta(t)$ and $\Delta(x)$ by

$$\delta(t) = \begin{cases} \frac{1}{2\pi} & , |t|<\pi \\ 0 & , |t|\geq\pi \end{cases}$$

and $$\Delta(x) = \int e^{ixt}\delta(t)\,dt = \frac{\sin\pi x}{\pi x} , \quad (=1 \text{ for } x=0)$$

Thus

$$\Delta(\pm 1) = 0. \qquad [2]$$

LEMMA 4.3

Suppose $\Sigma \Lambda_n^{-1}$ converges. Let $A>0$ be given and suppose that ε is any positive number. Then for each large m there are functions $h_m(t)$ and $H_m(x)$ with the following properties.

(i) $H_m(x) = \int e^{ixt} h_m(t)\,dt .$

(ii) $\int |h_m(t)|\,dt < 1 .$

(iii) $h_m(t) = 0 \quad \text{for } |t| \geq A\varepsilon$

(iv) $H_m(\Lambda_m - \Lambda_n) = 0 \quad \text{for } m \neq n$

(v) $|H_m(0)| > e^{-\frac{1}{2}\varepsilon\Lambda_m} e^{-\varepsilon\Lambda_m \log(1+\mu_m)}$

Proof

Define

$$f_m'(t) = \prod_{j=1}^{m-1} \left\{ e^{i\Lambda_j t} \Lambda_m \delta(\Lambda_m t) \right\} .$$

Then

$$J_m(x) = \prod_{j=1}^{m-1} \Delta\left(\frac{x+\Lambda_j}{\Lambda_m}\right).$$

Now, $\delta(\Lambda_m t)$ is zero for $|t| \geq \pi\Lambda_m^{-1}$. It follows that $j_m(t)$ is zero for $|t| \geq \pi m \Lambda_m^{-1}$. Also $\int\delta(t)dt = 1$ and hence

$$\int |j_m(t)|\,dt \leq \left\{\int\delta(t)dt\right\}^{m-1} = 1.$$

And by [2], for n<m,

$$J_m(\Lambda_m - \Lambda_n) = \prod_{j=1}^{m-1} \Delta\left(\frac{\Lambda_m - \Lambda_n + \Lambda_j}{\Lambda_m}\right) = 0.$$

Also define

$$k_{mN}(t) = \prod_{j=m+1}^{m+N} \left\{e^{-i\Lambda_m t}\Lambda_j \delta(\Lambda_j t)\right\}$$

Then

$$K_{mN}(x) = \prod_{j=m+1}^{m+N} \Delta\left(\frac{x-\Lambda_m}{\Lambda_j}\right)$$

Now put

$$K_m(x) = \lim_{N\to\infty} K_{mN}(x) = \prod_{j=m+1}^{\infty} \Delta\left(\frac{x-\Lambda_m}{\Lambda_j}\right).$$

That the infinite product converges for each x follows from the convergence of $\Sigma\Lambda_j^{-1}$. Now, for $N\geq 2$, $k_{mN}(t)\in L$ since it is then a continuous function zero outside a finite interval. Also $K_{mN}(x)\in L$ for $N\geq 2$. It follows that

$$k_{mN}(t) = \frac{1}{2\pi}\int e^{-ixt} K_{mN}(x)\,dx.$$

Then by Lebesgue's Convergence Theorem there is a k (t) such that

$$k_m(t) = \lim_{N\to\infty} k_{mN}(t) ,$$

and

$$K_m(x) = \prod_{m+1}^{\infty} \Delta\left(\frac{x-\Lambda_m}{\Lambda_j}\right) = \int e^{ixt} k_m(t)\,dt$$

Now, $\delta(\Lambda_j t)$ is zero for $|t| \geq \pi\Lambda_j^{-1}$. It follows that $k_m(t)$ is zero for $|t| \geq \pi \sum_{m+1}^{\infty} \Lambda_j^{-1}$. Also, for each N,

$$\int |k_{mN}(t)|\,dt \leq \left\{\int \delta(t)\,dt\right\}^N = 1$$

Hence

$$\int |k_m(t)|\,dt \leq 1$$

And by [2], for n>m,

$$K_m(\Lambda_m - \Lambda_n) = \prod_{m+1}^{\infty} \Delta\left(\frac{\Lambda_m - \Lambda_n - \Lambda_m}{\Lambda_j}\right)$$

We can now define $h_m(t) = j_m(t) * k_m(t)$, so that $H_m(x) = J_m(x).K_m(x)$. Clearly $h_m(t)$ and $H_m(x)$ satisfy (i), (ii) and (iv), and for large enough m they satisfy (iii) since $h_m(t)$ is zero for

$$|t| \geq \frac{\pi m}{\Lambda_m} + \sum_{m+1}^{\infty} \frac{\pi}{\Lambda_j}$$

and this expression tends to zero with m.

It remains to establish (v).

Choose some $\phi>1$. Let the number of Λ_n such that $\Lambda_m \leq \Lambda_n < \phi\Lambda_m$ be $\#_m+1$. Now suppose it is false that

$$\#_m = o(\Lambda_m)$$

Then for each $\varepsilon>0$ there is a subsequence $\{\Lambda_{m_r}\}$ such that

(i) $\#_{m_r} > \epsilon \Lambda_{m_r}$

(ii) $\Lambda_{m_r + \#_{m_r}} < \Lambda_{m_{r+1}}$.

Then

$$\sum_{n=1}^{\infty} \frac{1}{\Lambda_n} \geq \sum_{r=1}^{\infty} \left(\frac{1}{\Lambda_{m_r}} + \cdots\cdots + \frac{1}{\Lambda_{m_r + \#_{m_r}}} \right)$$

$$> \sum_{r=1}^{\infty} \frac{\#_{m_r}}{\Lambda_{m_r + \#_{m_r}}} > \frac{\epsilon}{\phi} \sum_{r=1}^{\infty} \frac{\Lambda_{m_r}}{\Lambda_{m_r}} = \frac{\epsilon}{\phi} \sum_{r=1}^{\infty} 1 \ .$$

It follows that $\#_m = o(\Lambda_m)$.

Now, for $0<x<1$, $\Delta(x) > 1-x$. Hence

$$|H_m(0)| = \left\{ \prod_{j=1}^{m-1} \Delta\left(\frac{\Lambda_j}{\Lambda_m}\right) \right\} \left\{ \prod_{j=m+1}^{\infty} \Delta\left(\frac{\Lambda_m}{\Lambda_j}\right) \right\} > \prod_{j=1}^{m-1} \left(1 - \frac{\Lambda_j}{\Lambda_m}\right) \prod_{j=m+1}^{\infty} \left(1 - \frac{\Lambda_m}{\Lambda_j}\right) .$$

But

$$\prod_{j=1}^{m-1} \left(1 - \frac{\Lambda_j}{\Lambda_m}\right) > \left(\frac{\Lambda_m - \Lambda_{m-1}}{\Lambda_m}\right)^m .$$

And $m = o(\Lambda_m)$. Also we have Lemma 4.2. It follows that for large m

$$\prod_{j=1}^{m-1} \left(1 - \frac{\Lambda_j}{\Lambda_m}\right) > \left(1+\mu_m\right)^{-\frac{\epsilon}{2}\Lambda_m} \cdot e^{-\frac{\epsilon}{6}\Lambda_m} . \qquad [3]$$

Also

$$\prod_{\Lambda_m < \Lambda_j < \phi\Lambda_m} \left(1 - \frac{\Lambda_m}{\Lambda_j}\right) > \left(\frac{\Lambda_{m+1} - \Lambda_m}{\Lambda_{m+1}}\right)^{\#_m} .$$

Thus, for large m,

$$\prod_{\Lambda_m < \Lambda_j < \phi\Lambda_m} \left(1 - \frac{\Lambda_m}{\Lambda_j}\right) > \left(1+\mu_m\right)^{-\frac{\epsilon}{2}\Lambda_m} \cdot e^{-\frac{\epsilon}{6}\Lambda_m} . \qquad [4]$$

Finally

$$\prod_{\lambda_j \geq \phi\lambda_m} \left(1 - \frac{\lambda_m}{\lambda_j}\right) = \exp\left\{ \sum_{\lambda_j \geq \phi\lambda_m} \log\left(1 - \frac{\lambda_m}{\lambda_j}\right)\right\}.$$

But

$$\log\left(1 - \frac{\lambda_m}{\lambda_j}\right) \geq -\frac{\lambda_m}{\lambda_j}\left\{1 + \frac{1}{\phi} + \frac{1}{\phi^2} + \ldots\right\} = -\frac{\phi}{\phi - 1}\frac{\lambda_m}{\lambda_j}.$$

Hence

$$\prod_{\lambda_j \geq \phi\lambda_m} \left(1 - \frac{\lambda_m}{\lambda_j}\right) \geq \exp\left\{-\lambda_m \frac{\phi}{\phi - 1} \sum_{m}^{\infty} \frac{1}{\lambda_j}\right\}.$$

It follows that for large enough m

$$\prod_{\lambda_j \geq \phi\lambda_m} \left(1 - \frac{\lambda_m}{\lambda_j}\right) \geq e^{-\frac{\varepsilon}{\delta}\lambda_m} \qquad [5]$$

Combining [3], [4] and [5] we obtain (v). This completes the proof of the lemma.

Proof of Theorem 4.1

First fix an $\varepsilon > 0$. Now since $\Sigma \lambda_n^{-1}$ converges, the abcissae of convergence and absolute convergence of the series $\Sigma a_n e^{-\lambda_n w}$ are the same. It follows that we may use again the results of Miss Cartwright's used in Chapter 2. Thus if we fix $\alpha < \frac{\pi}{2}\left(1 - \frac{1}{p}\right)$ and write

$$f(w,z) = \sum a_n e^{-\lambda_n w} e^{-\lambda_n z}$$

then f(w,z) is uniformly bounded, say by H, for $w > 0$ in the angle $|\arg z| \leq \alpha$.

Let us take the A of Lemma 4.3 to be $\frac{1}{2}\tan\alpha$. Then if $|t| < A\varepsilon$

$$|f(\omega, \pm\varepsilon + it)| \leq H. \qquad [1]$$

But

$$\int h_m(t) e^{i\lambda_m t} f(w, \tfrac{1}{2}\epsilon + it)\,dt = \sum a_n e^{-\lambda_n w} e^{-\frac{1}{2}\epsilon\lambda_m} H_m(\lambda_n - \lambda_m)$$

$$= a_m e^{-\lambda_m w} e^{-\frac{1}{2}\epsilon\lambda_m} H_m(0) .$$

And $h_m(t)$ is zero for $|t|>A\epsilon$ by Lemma 4.3(iii). Hence

$$|a_m e^{-\lambda_m w} e^{-\frac{1}{2}\epsilon\lambda_m}| \leq |H_m(0)|^{-1} H \int_{-A\epsilon}^{A\epsilon} |h_m(t)|\,dt \leq H.|H_m(0)|^{-1}.$$

Allowing w to tend to zero, we obtain

$$|a_m e^{-\frac{1}{2}\epsilon\lambda_m}| \leq H.|H_m(0)|^{-1} \leq H e^{-\frac{1}{2}\epsilon\lambda_m} e^{\epsilon\lambda_m \log(1+\mu_m)}$$

Hence

$$a_m = O(e^{\epsilon\lambda_m} e^{\epsilon\lambda_m \log(1+\mu_m)}) .$$

4.3

It is possible to combine the above method with that of Chapter 2. We can then obtain a limitation theorem for a somewhat wider class of (A,λ_n) methods.

Let us say, just for this chapter, that a sequence $\{\lambda_n\}$ is sparse if, for some $p>1$, the sequence $\{\lambda_n\}$ has the following property. All of the points of $\{\lambda_n\}$ belong to one of two disjoint subsequences $\{\lambda_n^{(1)}\}$ and $\{\lambda_n^{(2)}\}$ where

(i) $\lambda_{n+1}^{(1)} - \lambda_n^{(2)} \geq \gamma$, for some $\gamma>0$ and every n,

(ii) $\sum \frac{1}{\lambda_n^{(2)}}$ converges.

In the following theorem p and γ are as defined above.

THEOREM 4.4

Suppose that Σa_n is summable (A,λ_n) where $\{\lambda_n\}$ is sparse. Then for each $\varepsilon>0$ and each $k>\frac{\pi}{\gamma}\tan\frac{\pi}{2\rho}$

$$a_n = O\left(e^{k\Lambda_n}(1+\mu_n)^{\varepsilon\Lambda_n}\right).$$

Proof

Choose a large m. Clearly we can choose sequences $\{\Lambda_n^{(1)}\}$ and $\{\Lambda_n^{(2)}\}$ satisfying conditions (i) and (ii) and also such that $\Lambda_m \varepsilon \{\Lambda_n^{(1)}\}$.

After Theorem 2.6, we may take functions $k_M(t)$ and $K_M(x)$ such that

(i) $K_M(x) = \int e^{ixt} k_M(t)\,dt$.

(ii) $|k_M(t)| \leq \frac{\gamma}{\pi} K_M(0) \neq 0$.

(iii) $k_M(t) = 0$ for $|t| \geq \pi/\gamma$.

(iv) $K_M(\Lambda_m^{(1)} - \Lambda_n^{(1)}) = 0$ for $n \neq m$, $n \leq M$.

And similarly there are functions $h_m(t)$ and $H_m(x)$ which satisfy Lemma 4.3 for the subsequence $\{\Lambda_n^{(2)}\}$.

Now define

$$p_{mM}(t) = \frac{1}{K_M(0)}\left\{k_M(t) * h_m(t)\right\}.$$

Then

$$P_{mM}(x) = \frac{K_M(x)\,H_m(x)}{K_M(0)}$$

Note that for each x

$$|P_{mM}(x)| \leq \frac{\gamma}{\pi}\int |h_m(t)|\,dt \leq \frac{\gamma}{\pi}.$$

Now, for M≥m,

$$\int p_{mM}(t)\, e^{i\lambda_m t} f(w, k+rt)\, dt$$

$$= a_m e^{-\lambda_m w} e^{-\lambda_m k} P_{mM}(0) + \sum_{M+1}^{\infty} a_n e^{-\lambda_n w} e^{-\lambda_n k} P_{nM}(\lambda_m - \lambda_n).$$

But the left hand side has a bound independent of M and w, for appropriate k, while the second term on the right hand side tends to zero with M. The result then follows much as in Theorem 4.1.

CHAPTER 5

5.1

In Chapter 2 it was shown that non-trivial limitation theorems could be obtained for (A, λ_n) summability under the assumption that for some $q<1$

$$\lambda_{n+1}^q - \lambda_n^q \geq \gamma > 0 .$$

We went on to show in Chapter 3 that a considerably better result could be obtained if

$$\log \lambda_{n+1} - \log \lambda_n \geq \gamma > 0 .$$

Our first concern in this chapter will be to show that in the same way better results than those of Chapter 4 can be obtained by replacing the conditions there on $\{\lambda_n^q\}$, where $q<1$, by the same conditions on the sequence $\{\log \lambda_n\}$. We then go on to make some remarks about the case when for some $k>0$ and $\phi>1$

$$\frac{\lambda_{n+k+1}}{\lambda_n} > \phi > 1 .$$

5.2

Throughout this chapter it will be convenient to write $\Lambda_n = \log \lambda_n$. We then have

THEOREM 5.1

Suppose Σa_n is summable (A, λ_n) where $\Sigma \Lambda_n^{-1}$ converges. Then for each $\varepsilon>0$

$$a_n = O\left(\Lambda_n^{\varepsilon \Lambda_n} (1+\mu_n)^{\varepsilon \Lambda_n} \right) .$$

First we need two lemmas.

LEMMA 5.2

$$\max\left[\frac{\Lambda_{n+1}}{\Lambda_{n+1}-\Lambda_n}, \frac{\Lambda_n}{\Lambda_n-\Lambda_{n-1}}\right] = O(\Lambda_n(1+\mu_n))$$

Proof

Choose $\phi>1$. Now suppose $\Lambda_{n+1}<\phi\Lambda_n$. We have that $\log\frac{1}{x} > 1-x$ for $0<x<1$ and hence

$$\Lambda_{n+1}-\Lambda_n > 1-\frac{\lambda_n}{\lambda_{n+1}}.$$

Therefore

$$\frac{\Lambda_{n+1}}{\Lambda_{n+1}-\Lambda_n} < \phi\Lambda_n\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}.$$

And if $\Lambda_{n+1}\geq\phi\Lambda_n$ then

$$\frac{\Lambda_{n+1}}{\Lambda_{n+1}-\Lambda_n} \leq \frac{\phi}{\phi-1}.$$

The result now follows.

Lemma 5.3 is a simple extension of a result of Hardy's*.

LEMMA 5.3

Let $\alpha<\frac{\pi}{2}$. Suppose that Σa_n is summable (A,λ_n) to s and further that $\Sigma a_n e^{-\lambda_n w}$ is absolutely convergent for $w>0$.

Then $\Sigma a_n\lambda_n^{-z}$ is uniformly summable (A,λ_n) to a function $F(z)$ in the angle $|\arg z|\leq\alpha$, and $F(z)\to s$ as $z\to 0$ uniformly in this angle.

---

* G.H.Hardy, M.M., 39(1910), 136-139.

Proof of Lemma 5.3

Let $f(t) = \Sigma a_n e^{-\lambda_n t}$. Then $f(t) \to s$ as $t \to 0+$. Now put

$$f(w,z) = \sum a_n e^{-\lambda_n w} \lambda_n^{-z}$$

$$= \sum a_n e^{-\lambda_n w} \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-\lambda_n t} dt$$

$$= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} f(w+t)\, dt \quad .$$

The integral converges uniformly for $w>0$ and hence there exists

$$F(z) = \lim_{w \to 0+} f(w,z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} f(t)\, dt \quad .$$

We must now show that $F(z)$ is a uniform limit in the angle $|\arg z| \leq \alpha$. Choose an $\varepsilon>0$. Then there is an $w_0>0$ such that for $0 \leq w \leq w_0$ and for $t>0$

$$|f(t) - f(w+t)| < \varepsilon e^{-\lambda_1 t}$$

Hence if $R(z) = \sigma$ and $|\arg z| \leq \alpha$

$$|F(z) - f(w,z)| < \left|\frac{1}{\Gamma(z)}\right| \int_0^\infty t^{\sigma-1} |f(t) - f(w+t)|\, dt$$

$$< \varepsilon \lambda_1^{-\sigma} \left| \frac{\Gamma(\sigma)}{\Gamma(z)} \right|$$

It follows that $F(z)$ is a uniform limit for $|\arg z| \leq \alpha$.

We lose nothing by assuming that $s = 0$ so let us do this. Then, given $\varepsilon>0$, there is a $\delta>0$ such that for $0<t \leq \delta$, $|f(t)|<\varepsilon$, while there is an H such that for $t>\delta$, $|f(t)|<He^{-\lambda_1 t}$.

Thus

$$|F(z)| = \left| \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} f(t)\, dt \right|$$

$$\leq \left| \frac{\epsilon}{\Gamma(z)} \right| \int_0^\delta t^{\sigma-1} dt + \left| \frac{H}{\delta^{1-\sigma}\Gamma(z)} \right| \int_\delta^\infty e^{-\lambda_1 t} dt$$

$$\leq \left| \frac{\epsilon \delta^\sigma}{z\Gamma(z)} \cdot \frac{z}{\sigma} \right| + \left| \frac{H e^{-\lambda_1 \delta}}{\lambda_1 \delta^{1-\sigma}} \cdot \frac{1}{\Gamma(z)} \right|$$

Hence $F(z) \to 0$ as $z \to 0$ uniformly in the angle $|\arg z| \leq \alpha$. This completes the proof.

Proof of Theorem 5.1

After Theorem 4.1 this presents no difficulty. In fact if we substitute Lemma 5.2 for Lemma 4.2 and Lemma 5.3 for Miss Cartwright's theorems then the proof of that theorem stands almost word for word.

It is worth noting that Theorem 3.1 cannot be obtained from Theorem 2.7 by the above technique (that is writing $\log \lambda_n$ for λ_n^2 and then using the same proof). By this method we only obtain that under the high indices condition $\{\lambda_n^\beta\}$ is a limitation sequence for each $\beta > 0$. This is a considerably weaker reult than the high indices theorem. In view of this, it seems probable that Theorem 5.1 is by no means a best possible result.

5.3

We now need to say something about the Riesz summability methods. Write

$$A^{(k)}(u) = \sum_{\lambda_n \leq u}{}' (u - \lambda_n)^k a_n .$$

Then Σa_n is said to be summable (R,λ_n,k) to s if

$$n^{-k}A^{(k)}(n) \to s \quad \text{as} \quad n \to \infty$$

Summability (R,λ_n,k) implies summability (A,λ_n) if and only if

$$\lim \frac{\log \mu_n^*}{\lambda_n} = 0$$

This is a result of Kuttner*.

The sequence $\{(1+\mu_n)^k\}$ is a limitation sequence for (R,λ_n,k) summability. This is established, for positive, integral k, by the following argument**.

Suppose that Σa_n is summable (R,λ_n,k) to s. Then

$$\sum_{\lambda_j \le w} a_j (w-\lambda_j)^k = s w^k + o(w^k)$$

Write $w = \lambda_n + jh$ for $j = 0,1,\ldots,k$, where $kh = \lambda_{n+1} - \lambda_n$. Now take the k^{th} difference of the k+1 equations so obtained. Since the k^{th} difference of $(w-\lambda_j)^k$ and that of w^k are the same and equal to a constant multiple of h^k, say ph^k, we obtain that

$$ph^k \sum_{j=1}^{n} a_j = s p h^k + o(\lambda_{n+1}^k)$$

Hence

$$\sum_{j=1}^{n} a_j = s + o\left(\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}\right)^k$$

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* B.Kuttner, P.C.P.S., 57 (1961), 61-75.

** See Hardy and Riesz's 'Theory of Dirichlet Series', p.36.

The following is a theorem stated by Korenblyum*.

THEOREM 5.4

Suppose that for some positive integer k,

$$\frac{\lambda_{n+k+1}}{\lambda_n} \geq \phi > 1 .$$

Then summability (A, λ_n) implies summability (R, λ_n, k).

The following result is an immediate corollary.

THEOREM 5.5

Suppose that for some positive integer k

$$\frac{\lambda_{n+k+1}}{\lambda_n} \geq \phi > 1 .$$

Then $\{(1+\mu_n)^k\}$ is a limitation sequence for (A, λ_n) summability.

Note that when k=1, Theorem 1.1 shows that the above result is the best possible.

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* B.I.Korenblyum, Doklady Akad. Nauk. S.S.S.R. (N.S.) 81 (1951), 725-727. (I have not yet seen the proof).

CHAPTER 6

6.1

In this chapter we consider some of the inclusion theorems which can be obtained from the limitation theorems we have obtained so far. First of all we establish the following well-known result.

THEOREM 6.1

Suppose that for each n

$$\frac{\lambda_{n+1}}{\lambda_n} \geq \phi > 1 .$$

Then Σa_n is summable (A, λ_n) if and only if it is convergent.

We need a lemma which is itself a well-known result. The lemma holds if the 'o' is replaced by 'O' but the proof is much deeper.

LEMMA 6.2

Suppose that Σa_n is summable (A, λ_n) to s and that

$$a_n = o\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$$

Then Σa_n converges to s.

Proof

Write

$$g(y) = \sum_{\lambda_n \leq y} a_n \quad , \quad f(y) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n y}$$

Then

$$j(\tfrac{1}{y}) - J(y) = \sum_{\lambda_n \le \frac{1}{y}} a_n (1 - e^{-\lambda_n y}) - \sum_{\lambda_n > \frac{1}{y}} a_n e^{-\lambda_n y}$$

Choose an $\varepsilon > 0$. Then there is an N such that for every $n > N$

$$|a_n| < \varepsilon \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \right)$$

Also there is an H such that for $n \le N$

$$|a_n| < H$$

Suppose $y < \frac{1}{N}$. Then, since $0 \le 1 - e^{-\lambda_n y} \le \lambda_n y$ for $y > 0$, we have that

$$\sum_{\lambda_n \le \frac{1}{y}} a_n (1 - e^{-\lambda_n y}) \le \sum_{n=1}^{N} H (1 - e^{-\lambda_n y}) + \varepsilon \sum_{N+1}^{\lambda_n \le \frac{1}{y}} (\lambda_n - \lambda_{n-1}) y \left\{ \frac{1 - e^{-\lambda_n y}}{\lambda_n y} \right\}$$

$$\le H N \lambda_N y + \varepsilon y \frac{1}{y} \to \varepsilon \quad \text{as} \quad y \to 0+ .$$

Also

$$\sum_{\lambda_n > \frac{1}{y}} a_n e^{-\lambda_n y} < \varepsilon \sum_{\lambda_n > \frac{1}{y}} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} e^{-\lambda_n y} < \varepsilon y \sum_{\lambda_n > \frac{1}{y}} (\lambda_n - \lambda_{n-1}) e^{-\lambda_n y}$$

$$< \varepsilon y \sum_{\lambda_n > \frac{1}{y}} \int_{\lambda_{n-1}}^{\lambda_n} e^{-uy} \, du < \varepsilon y \int_0^{\infty} e^{-uy} \, du = \varepsilon$$

It follows that

$$\lim_{y \to 0+} j(\tfrac{1}{y}) = \lim_{y \to 0+} J(y) = s .$$

Proof of Theorem 6.1

Let $f(t) = \Sigma a_n e^{-\lambda_n t}$. Then $f(t) \to s$ as $t \to 0+$. Also if $\beta > 0$ then $f(t)$ is uniformly continuous for $t \ge \beta$. It follows that if $\varepsilon > 0$ there is a $\delta > 0$ such that for every $t > 0$

$$|f(t) - f(t+\delta)| < \varepsilon .$$

In the proof of Theorem 3.1 it was shown that if $|f(t)|<H$ for $t>0$ then it follows that $|a_n|<CH$, where C is a constant which depends only on ϕ. If, therefore, in Theorem 3.1 instead of f(t) we use

$$f(t) - f(t+\delta) = \sum a_n (1-e^{-\lambda_n \delta}) e^{-\lambda_n t}$$

we will obtain that

$$a_n (1-e^{-\lambda_n \delta}) < C\epsilon$$

It follows that $a_n = o(1)$. The theorem then follows from the lemma.

6.2

By $\phi(x)$ we shall always mean a function with the following properties. For some positive A, K and δ,

(i) For $x \geq A$, $\phi(x)$ has derivatives of all orders.

(ii) For $x>0$ and $0<t<\delta$

$$\sum_{k=0}^{\infty} \left| \frac{d^k}{dx^k} e^{-t\phi(x+A)} \right| \frac{x^k}{k!} < K.$$

Also, where $\{\lambda_n\}$ is defined, by $\{\Lambda_n\}$ we shall always mean a sequence of the type $\{\phi(\lambda_n)\}$.

With these definitions, Kuttner* has proved the following theorem

THEOREM 6.3

If Σa_n is summable (A,λ_n) to s, then $\Sigma a_n e^{-\Lambda_n z}$ is summable (A,λ_n) to a function F(z), for $z>0$, and $F(z) \to s$ as $z \to 0+$.

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* B.Kuttner, P.C.P.S., 45, (1949), 186-193.

We have already seen, in Lemma 5.3 and in Theorems 2.2 and 2.5, that this holds for $\phi(z) = \log z$ and for $\phi(z) = z^q$, where $0<q<1$. Some examples of other functions for which it is true are $\phi(z) = z^q+\log z$, $z+\log z$, $z^q-\log z$, $z/\log z$, where $0<q<1$. An example for which it is not true is the function $z-\log z$.

Clearly, if we have a limitation theorem of the type, ' Σa_n summable (A,λ_n) implies $a_n = O(e^{\varepsilon\Lambda_n})$ for each $\varepsilon>0$ ', then it will follow that $\Sigma a_n e^{-\Lambda_n z}$ converges to $F(z)$ and we shall then obtain from Theorem 6.3 that the (A,λ_n) method implies the (A,Λ_n) method.

In this section we shall give examples of some of the inclusion theorems that can be obtained this way. For the case $\phi(z) = z^q$ we shall use Miss Cartwright's theorems rather than Theorem 6.3 since these give a slightly stronger result. In this connection it is useful to extend the definition of (A,λ_n) summability. We shall say that Σa_n is summable (A,λ_n,α) to s if $\Sigma a_n e^{-\lambda_n z}$ converges for $R(z)>0$ to a function $f(z)$ and $f(z)\to s$ as $z\to 0$ uniformly in the angle $|\arg z|\leq\alpha$. We then obtain

THEOREM 6.4

Suppose that for some $q_0<1$,

$$\lambda_{n+1}^{q_0} - \lambda_n^{q_0} \geq \gamma > 0$$

Then summability (A,λ_n) implies summability (A,λ_n^q,α), where $q_0<q<1$ and $\alpha = \frac{\pi}{2}(1-q)$.

Proof

It follows from Theorem 2.7 that $a_n = O(e^{\varepsilon\lambda_n^q})$ for each $\varepsilon>0$. The result then follows from Theorems 2.2 and 2.5.

THEOREM 6.5

Suppose that $\Sigma \lambda_n^{-q}$ converges for some $q<1$ and also that $\mu_n = O(\lambda_n^{\beta})$ for some $\beta>0$.

Then summability (A,λ_n) implies summability $(A, \lambda_n^{q}\log\lambda_n)$.

Proof

It follows from Theorem 4.1 that $a_n = O(e^{\varepsilon\lambda_n^{q}\log\lambda_n})$ for each $\varepsilon>0$. And Theorem 6.3 holds for $\phi(z) = z^{q}\log z$, where $0<q<1$. The result follows.

THEOREM 6.6

Suppose that $\Sigma(\log\lambda_n)^{-1}$ converges and also that $\mu_n = O(\lambda_n^{\beta})$ for some $\beta>0$.

Then summability (A, λ_n) implies summability $(A, (\log\lambda_n)^2)$.

Proof

Theorem 6.3 holds for $\phi(z) = (\log z)^2$. We then obtain the above result from Theorem 5.1.