

THE DECOUPLING PROBLEM: INVARIANTS, PARAMETER
VARIATIONS, COMPOSITE SYSTEMS, AND TIME-VARYING
LINEAR SYSTEMS.

by

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A B S T R A C T

In this thesis, some decoupling problems for linear multivariable dynamic systems are considered. A (minimal) realization of a system with a given set of high frequency characteristics, fixed by some quantities relevant to the decoupling problem, is obtained. A set of decoupling invariants is shown to be independent and complete, for the class of decouplable systems which preserve observability under arbitrary state feedback. A (unique) canonical form is thus defined. A necessary and sufficient condition for the output feedback decoupling problem to have a solution is given, for this class of systems.

The decoupling problem of composite linear systems is considered for cascade, feedback, and parallel connections, and conditions for the overall system to be decouplable are given for all three cases.

The effect of parameter variations on the decoupling properties of a given system is studied. Variations on both, the plant of the system and the input map, are considered, and the results include those found in previous literature.

The decoupling problem of discrete time-varying systems via state feedback is solved and the state feedback decoupling pairs are characterized. Finally, the dynamic state feedback decoupling problem of continuous time-varying linear systems is considered, and necessary and sufficient conditions for the existence of a solution are obtained. An algorithm to find a (time-varying) decoupling compensator is given.

A MARGARITA

Me duele este niño hambriento
como una grandiosa espina,
y su vivir ceniciento
revuelve mi alma de encina.

Me da su arado en el pecho,
y su vida en la garganta
y sufro viendo el barbecho
tan grande bajo su planta.

¿Quién salvará a este chiquillo
menor que un grano de avena?
¿De donde saldrá el martillo
verdugo de esta cadena?

Que salga del corazón
de los hombres jornaleros,
que antes de ser hombres son
y han sido niños yunteros.

Miguel Hernández

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C H A P T E R I

In this introductory chapter, we shall describe, in general terms, the decoupling problem. We also present a brief survey of the field, with particular emphasis on those topics directly related to the contributions of this thesis. In section 3, all the new results contained in the main body of the dissertation are summarized.

1) Introduction.

A large number of dynamical systems found in practice are multivariable in nature, i.e., they have more than one input and ^{more than} one output. One of the main difficulties encountered when trying to control such systems is the presence of interaction between inputs and outputs. That is, in general, control action on one of the inputs affects more than one output.

The available synthesis techniques for single-input single-output systems are not easily extended to the multivariable case when coupling is present. On the other hand, most of the techniques developed so far for general interactive multivariable systems are not powerful or simple enough to provide a suit-

able answer to the synthesis problem. Moreover, in some control problems of practical interest, such as certain process control problems or vehicle landing systems, it is desirable or even necessary to include non-interaction as part of the design requirements.

Thus, if one is presented with an interactive multi-variable system, and interaction is not "helpful" or is not desired at all, one may attempt to transform the given system into a non-interactive or decoupled one, by some practicable means. One would be left then, with a system decomposed into smaller subsystems each controlled independently of the others. If the resulting subsystems are one-input one-output systems, then all the well known results available for synthesizing such systems can be applied using individual control loops. Even if the subsystems are still multivariable, the synthesis problem would be reduced to a set of independent subproblems, each of smaller dimension and hopefully simpler to control.

Finding the means for transforming an interactive multivariable system into a non-interactive one, the conditions under which this is possible, and the characteristics one may achieve for the resulting non-interactive system are the basic questions which

constitute the decoupling problem.

It is clear that if noninteraction is to be achieved, this will have to be accomplished by specifying the inputs or controls of the system in a suitable manner, for these are the only variables which can be manipulated directly. Different types of decoupling problems arise, according to the different functions used to specify the controls of the system; namely, decoupling by state or output feedback or the "restricted decoupling" problem, and decoupling by dynamic compensation and state or output feedback, or the "extended decoupling" problem.

A further classification of the decoupling problem can be made, according to the nature of the resulting subsystems, i.e., if the subsystems are to be single-input single-output, then we refer to the problem as "one-to-one decoupling", whereas if the resulting subsystems are multi-input and/or multi-output, the problem is known as "block decoupling".

The relationship of decoupling to other problems such as closed-loop observability, closed-loop stability (or pole placement in the case of linear time invariant systems), data sensitivity, etc, rests on the idea that one is interested in the qualitative and quantitative characteristics associated with the response of the resulting noninteractive system.

Other control problems, such as invertibility or output-controllability are related to decoupling through the existence question.

What has been said so far, applies to arbitrary multivariable dynamical systems. In this dissertation only linear multivariable systems will be considered, and attention will be concentrated on the one-to-one decoupling problem.

2) Brief Survey.

We will try to survey the most important ideas and developments in the solution of the decoupling problem as briefly as possible. In particular, attention will be given to those sub-problems or specific questions which are related to the topics studied in this thesis. A more comprehensive and detailed state of the art may be found in [2]. By sketching only the outstanding ideas very briefly, it is hoped that it will be possible to situate in the vast field of decoupling, the contributions made by this dissertation, which we will summarize in the next section.

In the last decade, the decoupling problem has attracted the attention of many researchers and as a consequence major advancements have been made towards its solution. Most of the results so far avai-

lable on the different facets of the problem however, apply only to linear time-invariant multivariable systems. Few of these results have been extended to the linear time varying case, and even fewer to nonlinear systems.

In general terms we say that a system is decoupled if;

- (i) the variations in the j^{th} output of the system due to any variations in the i^{th} input; $i \neq j$, are identically zero, and
- (ii) the variations in the j^{th} output of the system due to any variations in the j^{th} input are not identically zero.

Intuitively, condition (i) of this definition, accounts for non-interaction, whereas condition(ii) is included simply to ensure that some degree of control exists on the outputs of the non-interactive system.

The above conditions are not the only possible definition of a decoupled system, nor are they the most rigorous. However, it suits our purposes as it is both intuitive and general.

A clearer picture of a one-to-one decoupled

system is possible if we restrict our attention to linear time-invariant systems. Consider the constant linear system $S=(A,B,C)$ described by the set of linear differential equations

$$\dot{x}(t) = A x(t) + B u(t) \quad \text{---(1.1)}$$

$$y(t) = C x(t) \quad \text{---(1.2)}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t), y(t) \in \mathbb{R}^m$ are the input and output vectors respectively, and A, B , and C are constant matrices of appropriate dimensions.

The system $S=(A,B,C)$ is said to be one-to-one decoupled, if its transfer function matrix $H(s)$, given by

$$H(s) = C(Is-A)^{-1}B \quad \text{---(1.3)}$$

is a nonsingular diagonal matrix (see Ref.[21]).

2.1 State-feedback decoupling: Existence of a solution.

B.S. Morgan [35], was the first to pose the decoupling problem in the state space description, and he gave a sufficient condition for a solution to exist for a somewhat restricted class of linear systems.

The control law proposed to achieve noninteraction

was a state feedback law, i.e. a law of the form

$$\bar{u}(t) = F x(t) + G v(t) \quad (1.4)$$

where $v(t) \in \mathbb{R}^m$ is the closed-loop input and G is an $m \times m$ nonsingular matrix.

A year later, in 1965, Z.V. Rekasius [53], extended Morgan's results, but still considered only sufficiency.

The central question of existence for the one-to-one decoupling problem via state-feedback was not solved until 1967, when P.L. Falb and W.A. Wolovich [17], gave a necessary and sufficient condition, for the existence of a solution.

This condition is given in terms of the nonsingularity of a matrix formed by certain products of the parameters of the system. More precisely, if we denote the i^{th} row of C by c_i , and define

$$d_i = \min \{ j : c_i A^j B \neq 0; j=0,1,\dots,n-1 \} \quad (1.5)$$

or $d_i = n-1$ if $c_i A^j B = 0$ for all j ,
and the $m \times m$ matrix B^* as

$$B^* = \begin{bmatrix} c_1 A^{d_1} B \\ \vdots \\ c_m A^{d_m} B \end{bmatrix} \quad (1.6)$$

then, [17] there exists a feedback pair (F,G) in eq. (1.4) such that the closed-loop system $S(F,G) = (A+BF, BG, C)$ is decoupled, if and only if B^* is non-singular.

This has been proved since then, in many different ways and by several authors. The approach used by E.G. Gilbert [21] is particularly important, because it gave structure to the decoupling problem. He gave an alternative definition for d_i ; $i=1, \dots, m$, and B^* , in the frequency domain, and as he was to use it later to build a canonical form, emphasized the invariant nature of the d_i 's under similarity ($z=Qx$; Q nonsingular) and state feedback ($u=Fx + Gv$) transformations. Thus, the d_i 's and B^* became associated with certain fixed "high frequency characteristics" of the system.

Two other papers are essential to the existence question. Both give solutions to special subcases of the block decoupling problem, which include the one-to-one decoupling problem as studied by Falb and Wolovich, and Gilbert as a subcase. We remark that the general block decoupling problem remains unsolved to-date, although the cases for which a solution has been found include most practical situations.

The first of these papers is that of W.M. Wonham

and A.S. Morse [86]. Their approach is of a geometric nature and is based on the concept of controllability subspaces. Necessary and sufficient conditions for the existence of a solution to the block decoupling problem are given for two important subcases (i.e. $\text{rank } [C]=n$, and more important, $\text{rank } [B]=m$) in terms of the controllability subspaces. Their results are mathematically beautiful and rigorous. However, due to their geometric nature, they are difficult to translate into a direct design procedure, i.e., a computer algorithm.

The second essential paper mentioned above is that of L.M. Silverman and H.J. Payne [60]. They reproved the results of Wonham and Morse, for the more general case where a direct input-output map is allowed in eq. (1.2) (i.e., $y(t)=Cx(t) + Du(t)$) and solved a third subcase (when $\text{rank } [G]=m$). Their approach is algebraic and based on a structure algorithm previously used by Silverman in [61] to find the inverse of a linear system. Their results show clearly a relationship between the decoupling and invertibility problems.

2.2 State-feedback decoupling: Characterization of the closed-loop noninteracting systems.

We now turn our attention to the type of closed -

loop decoupled systems which may be obtained. When a solution to the one-to-one decoupling problem via state-feedback exists, it is known that it is not unique. Thus, one may obtain non-interacting systems with different performance characteristics, according to the decoupling state-feedback pair chosen. It is desirable to characterize the class of decoupling laws and to determine the attainable closed-loop subsystems. As the location of the poles of a system and its response are highly related, one would like to be able to place the closed-loop poles arbitrarily while decoupling, or at least to ensure stability.

Z.V.Rekasius [53], was the first to look at the decoupling and pole assignment problem, although he gave no definite answer. Later, Falb and Wolovich [17], succeeded in characterizing the class of decoupling feedback pairs, although only implicitly, and showed that at least $m + \sum_{i=1}^m d_i$ ($\leq n$) poles of the system are assignable while decoupling.

E.G.Gilbert [21], was able to give a definite answer to both questions. He established the number of poles which can be freely assigned while decoupling (i.e., $\sum_{i=1}^m p_i$, where $p_i = d_i + r_i + 1$ *, $i=1, \dots, m$) and gave an explicit characterization of the state feedback

* $r_i = \dim Q_i - d_i - 1$, where

$$Q_i = \{ \gamma: \gamma \in \mathbb{R}^n; \gamma A^j b_k = 0 \text{ for } k=1, \dots, m, k \neq i, \text{ and } j=0, \dots, n-1 \}$$

where b_k denotes the k^{th} column of B .

decoupling laws. Further, he established in the frequency domain, the form of the achievable non-interacting closed-loop subsystems. Similar and equivalent results were obtained later by Wolovich and Falb [83], Wonham and Morse [86] geometrically, Silverman and Payne [60] and others [29],[39],[46].

The question of zero-assignment, also closely related to the response of a system, has not received as yet due attention, apart from very few results [23],[31].

Qualitative characteristics of decoupled systems have been studied. In particular I.H. Mufti [40] and P.C. Chandrasekharan [11], established conditions for a system decoupled by state feedback to be observable, in terms of the d_i 's and the p_i 's.

2.3 Sensitivity and decoupling.

A further question of significant importance is the effect that parameter variations have on the decouplability properties of a system. As, in general, the parameters of a given system are known in practice only approximately, it is possible to arrive to a wrong conclusion regarding the possibility of effecting non-interactive control. The same is true when the parameters of the system suffer variations.

Thus, it is important to be able to say under which parameter variations or errors, the decoupling properties of a given system remain invariant.

One can distinguish two basic problems. First, the "data sensitivity and decoupling problem", studied by E. Fabian and W.M. Wonham [16], which is that of finding under which variations there exists a feedback law which decouples both the nominal system and the perturbed system, and such that the closed-loop outputs of both are identical for all inputs, all initial conditions, and all times. Using a geometric approach, Fabian and Wonham solved this problem for variations in A, the plant of the system, and B, the input-state map.

The second problem, which is less restrictive, is that of characterizing the set of variations, under which a decouplable system remains decouplable. This problem was analyzed by V. Sankaran [58] who called it the "input-output invariance problem". He only considered parameter variations in B, and in C, the state-output map, but did not consider variations in A. Moreover, Sankaran's results are far from giving a complete answer.

Results on the sensitivity of decoupled systems on a different line can also be found in the

literature [28],[48],[69],[71],[77]. The approach in these papers is to include the error due to parameter variations as part of the design considerations and try to minimize it.

2.4 Output feedback decoupling.

In many practical instances, the states of the system are not available for measurement. In these cases, even if a solution to the state-feedback decoupling problem is known to exist, it may not be possible to implement it, unless an estimate of the state is obtained first. However, this is not an economical situation if it can be avoided. As the outputs of the system are always available for control purposes, an output feedback law, i.e. a law of the form

$$u(t) = K y(t) + G v(t) \quad \text{_____} (1.7)$$

is a possible alternative, to achieve non-interaction.

It should be clear that the output-feedback decoupling problem is a special case of the state feedback decoupling problem. Namely, when there exists a solution to

$$F = KC \quad \text{_____} (1.7a)$$

with F a state-feedback decoupling matrix. Thus, a system will be decouplable via output-feedback only if it is decouplable via state-feedback.

Several necessary and sufficient conditions for the existence of a solution to the output-feedback decoupling problem are available in the literature. That given by Falb and Wolovich [17], algebraically, is in implicit form and therefore it is difficult to use. However it was the first solution proposed for the existence question. J.W. Howze [24] also solved the problem, but his conditions are rather cumbersome. S.H. Wang and E.J. Davison [81], attacked the problem in the frequency domain, and proved a necessary and sufficient condition constructively. M.J. Denham[[13] and S.H. Wang [80] used the geometric approach and studied the more general block decoupling problem via output-feedback, for those subcases for which a solution via state-feedback is known. The main difference between their approaches is that Wang did not have recourse to the state feedback solution formulated by Wonham and Morse, but introduced instead the concept of (A,B,C) -controllability subspaces. A further necessary and sufficient condition was given by A.K. Majumdar and A.K. Choudhury [33], but theirs is also given in an implicit form, i.e., it requires the vector $c_i A^{d_i+1}$ to lie

in a subspace which in general, depends on the feedback matrix K . (Their results were originally formulated for linear time-varying systems).

As in the state-feedback case, the pole assignment problem and the question of characterizing the class of output-feedback decoupling pairs (K, G) , have been considered by several authors, [13], [24], [81]. As expected, the degrees of freedom when output-feedback is used to achieve noninteraction are less than for the state-feedback situation (i.e., only m poles of the system may be arbitrarily placed while decoupling via output-feedback).

2.5 Dynamic decoupling

When a given system is not decouplable by state-feedback, it may still be possible to achieve noninteraction if some dynamics is allowed in the control law.

Z.V. Rekasius [53], was the first to suggest this, and he showed that if dynamic state-feedback, i.e. feedback of the form

$$u(s) = F(s) x(s) + G(s) v(s) \quad \text{_____} (1.8)$$

where $F(s)$ and $G(s)$ are proper rational matrices of appropriate dimensions, is used instead of eq.(1.4),

then every invertible system may be decoupled.

E.G. Gilbert [21], formalized the idea, and classified the set of linear systems into systems with no-inherent coupling. (i.e., decouplable by static state-feedback of the form (1.4)), systems with weak-inherent coupling (i.e., invertible systems for which no static-decoupling feedback pair exists) and systems with strong-inherent coupling (i.e., non-invertible systems). By means of an example, he showed how a precompensator connected in cascade with the plant, can transform a system with weak-inherent coupling into one with no-inherent coupling, and thus decouplable by static state-feedback.

From then on, different algorithms for finding precompensators which yield an overall decouplable system have proliferated. For obvious economic reasons, a precompensator of minimal order is desirable. Such compensators were found by several authors, using algebraic [9],[22], [54],[62],[78] and geometric approaches, [14],[36],[80], for both, state and output feedback.

Dynamic precompensation has been used not only to achieve noninteraction when this is not possible by using state-feedback alone, but also to obtain more freedom in the closed-loop system. In [25]

precompensation is used to achieve complete pole-placement while decoupling, for those systems where this is not possible using static feedback. In [10] it is used to maintain observability while decoupling.

A detailed account of the precompensation schemes used for decoupling and a discussion of their relative merits may be found in [2].

It should also be clear that one can view the dynamic decoupling problem as equivalent to the decoupling problem of composite systems connected in cascade. Visualized in this light, the cascade connection is the only configuration of the decoupling of composite systems which has received proper attention. Very little or nothing has been published on the feedback or parallel connections.

2.6 Extensions to other classes of dynamical systems.

All the results reviewed in previous sections extend directly to linear discrete-time-invariant systems (e.g., see [30],[72]). An interesting study of a hybrid situation (i.e., discrete feedback and continuous systems) was made by J.C. Johnson [26], and the decoupling problem for sampled-data systems has also been studied [20],[66], although the results so far available are far from complete.

The extension of the results to the time-varying situation or the non-linear case, presents greater technical difficulties. Of all the different definitions of a decoupled system given for the linear time invariant case, only some apply to linear time-varying systems. (e.g. one can no longer speak of a nonsingular diagonal transfer function matrix). Moreover, those definitions which may be generalized to the time-varying situation are no longer necessarily equivalent. (see Appendix A.1). Further, in the time-varying situation concepts such as complete, total or uniform decoupling become meaningful (just as with qualitative properties like controllability or observability). That is, decoupling has to be defined as a time dependent property.

In spite of the above, some results have been generalized to the time-varying situation. In particular, some of those regarding static decoupling.

W.A. Porter [50], using certain differential operators defined recursively on the parameters of the system, defined a set of integer valued functions $d_i(t)$; $i=1, \dots, m$ and a matrix $B^*(t)$ which are the time-varying equivalent of d_i ; $i=1, \dots, m$ and B^* in

eqs. (1.5) and (1.6). However, he was only able to show that the nonsingularity of $B^*(t)$ is sufficient for the existence of a solution to the state-feedback decoupling problem. (Porter's results apply to both, continuous and discrete time-varying systems).

Later on, several authors [33],[56],[63],[67],[76], showed that $B^*(t)$ nonsingular is also necessary, and gave characterizations of the class of state-feedback decoupling pairs $(F(t)G(t))$. Yet, to do this the assumption that the d_i 's are constant was required.

The problem of pole-placement while decoupling, which is no longer meaningful, was changed into the weaker problem of closed-loop stability (e.g.[76]).

The approaches used to obtain these results, are extensions of those used by Falb and Wolovich [17] and Gilbert [21] in the time-invariant case.

For the discrete time-varying situation, the only extension of Porter's results known to the author i.e., a necessary and sufficient condition, and a characterization of the class of decoupling state-feedback pairs, is that proposed by V. Sankaran and M.D. Srinath [57]. However, we believe their results should be reviewed.

Some results on the sensitivity of decoupled

time-varying systems have been obtained by V. Sankaran [58] and S.G. Tzafestas and P.N. Paraskevoupoulos [71] but much more research is needed in this area.

The time-varying output-feedback decoupling problem has also been studied, and as we mentioned before, Majumdar and Choudhury [33] found a necessary and sufficient condition for the problem to have a solution, in an implicit form. To date, this remains the only condition known, although it is almost certain that some of those given for time-invariant systems are likely to hold with appropriate modifications for time-varying systems.

On the dynamic decoupling problem of time-varying systems almost nothing is known. All the results so far available are for the output feedback case [33],[63],[64], and even then, a complete answer has not been given.

The situation for non-linear systems presents even stronger technical difficulties. Most of the researchers who have worked in this field have restricted their attention to the subclass of non-linear systems linear in the control. Even for this case, where functions equivalent to the d_i 's can be defined, things are less tractable. The d_i 's are not only

functions of time, as in the linear time-varying case, but also functions of the state and even the input. Thus, it has been usually assumed when solving the problem, that the d_i 's are constant for all possible $(x,t) \in (R^n, T)$. This assumption is more critical here, for even when one is able to obtain a diagonal input-output map, without it, interaction may occur through the d_i 's, [51].

Answers to the existence question, for the nonlinear static state-feedback decoupling problem have been proposed in [34],[42],[59],[65],[68],[70]. The problem of decoupling via output-feedback has not been considered at all in the literature, and very few results [43],[1], exist for the dynamic decoupling problem.

The ideas of noninteractive control have also been applied to other types of systems such as distributed parameter systems [52], but with little success. The decoupling of time-delay systems has received some attention only lately [44],[74],[75].

Before closing this brief survey, we ought to point out that other types of non-interaction, such as triangular decoupling [37],[79], where interaction is allowed in one direction only, and asymptotic decoupling [85], where the target is noninteraction in

the steady-state, have also been considered in the literature. For specific problems these types of interaction may offer advantages over diagonal or block diagonal decoupling.

3 Contributions of the thesis.

The main aims of the research leading to this thesis were

- i) To study the decoupling problem of linear time-invariant systems with a view to obtaining results giving more insight into the structure of the problem.
- ii) To analyze the effect of parameter variations on the decoupling properties of the system.
- iii) To study the noninteracting control problem for composite linear systems, and
- iv) To generalize some of the results available for linear time-invariant systems to the time-varying situation (continuous and discrete).

Accordingly, the contents of the thesis are as follows:

In chapter II, a procedure to obtain a minimal order realization of a system with given d_i 's and a given matrix B^* (not necessarily non-singular) is given. This shows that $\{d_i; i=1, \dots, m\}$ and B^* are

an independent set of invariants under similarity and state-feedback ($u=Fx + v$) transformations. From the frequency domain definition of the d_i 's and B^* , it is clear that this minimal realization is a solution to the problem of realizing a system with certain high frequency characteristics fixed, but otherwise arbitrary. It is evident however, that in general the d_i 's and B^* are not a complete set of independent invariants. These properties are desirable, because any set of invariants possessing them is a solution to a universality problem, and the search for canonical forms may be considered such a problem.

In the next section of Chapter II, we show that for the subclass of linear systems satisfying

$$m + \sum_{i=1}^m d_i = n \quad \text{_____} (1.9)$$

the set of d_i 's, does constitute a complete set of independent invariants, and thus, we are able to define a (unique) canonical form, completely determined by the d_i 's. In proving this, we first show that any decouplable system satisfying eq. (1.9) is controllable, observable, and preserves observability under arbitrary state-feedback. It is this last characteristic which makes $\Omega = \{ \text{class of decouplable systems satisfying (1.9)} \}$, interesting.

In the last section of chapter II, we give a necessary and sufficient condition for the output-feedback decoupling problem to have a solution, for systems belonging to Ω . This condition may be considered a particular case of a known result [33], but the proof given is different and we believe, simpler. Moreover, for any member of Ω , the necessary and sufficient condition is explicit, whereas for the general case considered in [33], it is dependent on the feedback matrix, and thus difficult to verify. Finally, we show that for any system belonging to Ω , the independence of the observability subspaces is a sufficient condition for the existence of a solution to the output feedback decoupling problem.

In Chapter III, we consider the decoupling problem for composite systems. As regards the cascade connection, we have already said in the survey that many results are available in the literature. Here, we give some necessary conditions on the class of precompensators which yield overall decouplable systems. One such condition is that the precompensator must be input-observable with input-observability index bounded above by the integer α in Silverman's structure algorithm [61].

In the section devoted to the feedback connection of composite systems, we show that the use of dynamic state feedback or dynamic output feedback,

does not enlarge the class of decouplable systems; that is given a nondecouplable plant, there is no dynamic (state or output) feedback which, alone, yields an overall decouplable system. This clearly shows that its use in connection with precompensation, as in the more general compensation schemes found in the literature, is simply to obtain more design flexibility.

In Chapter III we also consider the parallel connection, and give a necessary and sufficient condition for the overall composite system to be decouplable by static-state-feedback. In the discussion it is shown that given two decouplable systems connected in parallel, it is not necessarily true that the composite system will be decouplable.

In Chapter IV we analyze the effect parameter variations have on the decouplability properties of a system. Variations in both A , the plant of the system and B , the input-state map are considered. A necessary and sufficient condition for the data sensitivity and decoupling problem to have a solution is given. This condition is stated in terms of a matrix related to observability, and it is intuitively clear since it requires that the variations lie in the closed-loop unobservable subspace. The approach used is algebraic in essence, and the proof is simpler than that of Fabian and Wonham [16]. The equivalence between their result and ours is shown rigorously.

The other problem considered in Chapter IV is that of total input-output invariance. Again, the effect of variations in A and B is analysed. A characterization of the parameter variation subspaces such that the given decouplable system remains decouplable is given. The main results, which extend those of V. Sankaran [58], are presented in a way which resembles those on robustness of controllers [47].

In Chapter V, the decoupling problem for linear discrete time varying systems is considered. A necessary and sufficient condition for a solution to exist via state-feedback is given.

These results review those of Sankaran and Srinath [57], which are not completely satisfactory. In proving the results the assumption that $d_i(k)$ is constant is made at an appropriate stage.

Chapter VI deals with the dynamic state-feedback problem for continuous time-varying systems. It is shown that the dynamic state-feedback decoupling problem is equivalent to the invertibility problem, just as in the time-invariant case. Again, assuming constancy of $d_i(t)$; $i=1, \dots, m$, and that the parameters of the system are continuously differentiable time functions, a necessary and sufficient

condition in terms of $A(t)$, $B(t)$ and $C(t)$ for the problem to have a solution is given. This condition is difficult to test in general, as it involves some non-fixed integers. Thus, a procedure to find the precompensator is proposed which at the same time yields a necessary and sufficient condition for the existence question. For this the concept of rank essentiality [9] is extended to the time-varying situation. The question of minimal order precompensation is posed, and the technical difficulties one would have to overcome for its solution are discussed. In the last section of Chapter VI, some of the difficulties which would arise if the assumption that $d_i(t)$; $i=1, \dots, m$, are constant is removed, are considered. The possibility of using precompensation to obtain an overall system, with constant d_i 's over the time interval of interest, which at the same time is decouplable is discussed.

Finally, Chapter VII contains the conclusions and some suggestions for further research.

CHAPTER II. Decoupling invariants.

1). Introduction.

Recent literature, [12],[18],[27],[49],[55],[82], shows a growing interest in establishing sets of invariants with respect to equivalence classes defined by different transformations, such as similarity and state feedback. The value of showing that a set of invariants is complete and independent is that a set with such properties is known to be a solution to a universality problem [32], and the search for canonical forms may be considered such a problem. To be able to obtain a (unique) canonical form is of importance since it may help understanding and simplify the solution of certain control problems.

Similarity and state feedback transformations define equivalence relations and therefore establish a partition on the set of linear systems into equivalence classes. Each of these equivalence classes or orbits is completely determined by a particular value of the elements of the complete set of independent invariants. That is to say, two linear systems belong to the same orbit, if and only if they have the same set of invariants.

When solving the one-to-one decoupling problem

via state feedback it was found, [17], [21], that certain quantities which play a key role in its solution were invariant under similarity and state feedback transformations. These quantities were the set $\{d_i\}$; $i=1, \dots, m$, which we will refer to as the set of decoupling invariants, and the matrix B^* , defined by eqs. (1.5) and (1.6) respectively.

In this chapter, we show that the set of invariants $\{d_i, B^*\}$ is independent. That is, we show that given an arbitrary set $\{d_i, B^*\}$, there exists at least one linear system $S=(A, B, C)$ which has $\{d_i, B^*\}$ as its set of invariants. (B^* is not restricted to be nonsingular, i.e., nondecouplable systems are also considered). A realization of $\{d_i, B^*\}$ is obtained, which has minimum order among all the possible realizations of the given set. Two examples are given to illustrate the realization algorithm.

If we state the equivalent definition of the d_i 's and B^* , given by E.G. Gilbert [21] in the frequency domain, i.e.

$$d_i = \min \left\{ j: \lim_{s \rightarrow \infty} s^{j+1} H_i(s) \neq 0; j=0, 1, \dots, n-1 \right\} \quad (2.1)$$

or $d_i = n-1$ if $H_i(s) = 0$.

where n is the state space dimension of $S=(A, B, C)$, and $H_i(s)$ is the i^{th} row of its transfer function matrix, and

$$B_i^* = \lim_{s \rightarrow \infty} s^{d_i+1} H_i(s) \quad (2.2)$$

where B_i^* denotes the i^{th} row of B^* , it is clear that the realization problem above, may be regarded as that of finding a minimal realization of a system having certain high frequency gain properties fixed, but arbitrary otherwise.

Now, a set of invariants is said to be complete under a class of transformations $\mathcal{T}$, if given two systems S_1 and S_2 having the same set of invariants, we can find a transformation $T \in \mathcal{T}$ such that $T: S_1 \mapsto S_2$. With this definition in mind, and as $\{d_i, B^*\}$ contains no information about the dimension of the system (other than a lower bound), it may be trivially shown that it is not a complete set of invariants under similarity and state feedback transformations. This is so, because two systems of different dimension may have identical sets of invariants $\{d_i, B^*\}$ and neither of the transformations mentioned alter the dimension of a system.

In order to find a complete set of independent invariants under these transformations, either the set of invariants $\{d_i, B^*\}$ must be enlarged, or the class of systems under study specialized. This last option is taken in section 3, where it is shown that,

for the class of linear decouplable systems for which

$$m + \sum_{i=1}^m d_i = n \quad \text{--- (2.3)}$$

holds, the set of decoupling invariants $\{d_i\}; i=1, \dots, m$ is complete and independent under similarity and state feedback transformations. Since $\{d_i\}$ is complete and independent, a unique canonical form completely determined by $\{d_i\}$ exists and is given.

During the development, it is shown that systems belonging to $\Omega = \{S: S \text{ is decouplable and satisfies (2.3)}\}$ are both controllable and observable and preserve observability under arbitrary state feedback. It is precisely this last property which makes Ω interesting and which characterizes it.

In the last section of the chapter, a necessary and sufficient condition for any system $S \in \Omega$ to be decouplable via output feedback is found. Although this condition may be considered a special case of that given in [33], our proof is simpler. Moreover, in our case it is in an explicit form, which makes it easy to use. Finally, based on this result, we show that the independence of the observability subspaces of a system $S \in \Omega$ is sufficient to ensure the existence of an output feedback decoupling pair.

2). Realization of the set of decoupling invariants.

Let us consider an m-input m-output linear time invariant dynamical system $S=(A,B,C)$, described by the set of differential equations

$$\dot{x}(t) = A x(t) + B u(t) \quad \text{--- (2.4)}$$

$$y(t) = C x(t) \quad \text{--- (2.5)}$$

where $x(t) \in \mathbb{R}^n$; $u(t), y(t) \in \mathbb{R}^m$ and A, B , and C are **constant** matrices of appropriate dimensions.

It is known [17],[21], that a solution to the state feedback decoupling problem exists, if and only if, the matrix B^* defined by eq.(2.2), or

$$B^* = \begin{bmatrix} c_1 A^{d_1} B \\ \vdots \\ c_m A^{d_m} B \end{bmatrix} = \begin{bmatrix} D_1 \\ \vdots \\ D_m \end{bmatrix} = \begin{bmatrix} D_{1,1} & \dots & D_{1,m} \\ \vdots & & \vdots \\ D_{m,1} & \dots & D_{m,m} \end{bmatrix} \quad \text{--- (2.6)}$$

is a nonsingular $m \times m$ matrix, where c_i denotes the i^{th} row of C , and d_i ; $i=1, \dots, m$, are defined by eq.(2.1) or eq.(1.5).

E.G.Gilbert [21], showed that the set $\{d_i, D_{i,j}\}$; $i, j=1, \dots, m$, is invariant under similarity (i.e., a change of basis in the state space, $z=Qx$; Q an $n \times n$ nonsingular matrix) and state feedback (i.e., $u=Fx+v$) transformations. (Notice that no change of basis in

the input space is considered in the state feedback transformation. If $u=Fx+Gv$, then $B^* \mapsto B^*G$ and is no longer invariant).

We now centre our attention on the problem of independence. That is, we show that given a set $\{d_i, D_{i,j}\}$, it is possible to find a system $S=(A,B,C)$ which has precisely this set as its set of invariants.

Let us assume a given arbitrary set $\{d_i, D_{i,j}\}$; $i,j=1,\dots,m$, and let us find a realization of it. We first see that the dimension of the input and output spaces of the realization must obviously be equal to m , i.e., equal to the given number of d_i 's. On the other hand, the set $\{d_i, D_{i,j}\}$ does not contain any information about the dimension of the state space, n , other than a lower bound on its value. In fact, by fixing d_i ; $i=1,\dots,m$, we have fixed the minimal difference between the degrees of the numerator and denominator of the elements in the rows of the transfer function of the realization (see eq.(2.1)). Thus, the possibility of finding two realizations of $\{d_i, D_{i,j}\}$ of different dimension, exists. Theorem 2.1 below, gives the smallest order any realization of $\{d_i, D_{i,j}\}$ can have.

Let

$$\text{rank}[B^*] = r \leq m \quad \text{---(2.7)}$$

then

Theorem 2.1. The smallest possible order ν , of any realization of a given set of invariants $\{d_i, D_{i,j}\}$, is given by

$$\nu = r + \sum_{i=1}^r d_{\alpha_i} \quad (2.8)$$

where $\{\alpha_i ; i=1, \dots, r\}$ is the set of integers satisfying

(i) $D_{\alpha_1}, D_{\alpha_2}, \dots, D_{\alpha_r}$, are r linearly independent rows of B^* , and

$$(ii) \sum_{i=1}^r d_{\alpha_i} \geq \sum_{i=1}^r d_{\beta_i} \quad ; \text{where } \{\beta_i ; i=1, \dots, r\}$$

is any other set of integers satisfying (i).

Proof.- Let $S=(A,B,C)$ be any realization of $\{d_i, D_{i,j}\}$.

Let n be the state space dimension of S . We claim that the smallest possible value of n is ν .

Let $\{\pi_i ; i=1, \dots, r\}$ be any set of integers satisfying (i), and define $p(\pi_i)$ as

$$p(\pi_i) = r + \sum_{i=1}^r d_{\pi_i}$$

and the $(p(\pi_i) \times n)$ matrix $U(d_{\pi_i})$ by

$$[U(d_{\pi_i})]^T = [(c_{\pi_1})^T, (c_{\pi_1} A)^T, \dots, (c_{\pi_1} A^{d_{\pi_1}})^T, \dots, (c_{\pi_r})^T, (c_{\pi_r} A)^T, \dots, (c_{\pi_r} A^{d_{\pi_r}})^T]$$

where $(\cdot)^T$ denotes the transpose of a matrix. Further, let U_i denote the i^{th} row of $U(d_{\pi_i})$. By successively postmultiplying $U(d_{\pi_i})$ by $B, AB, A^2B, \dots$, and using the fact that $D_{\pi_1}, D_{\pi_2}, \dots, D_{\pi_r}$ are a set of r linearly independent vectors, it is easy to see that

$$\sum_{i=1}^{p(\pi_i)} \lambda_i U_i = 0$$

implies $\lambda_i = 0$ for every i . i.e., the rows of $U(d_{\pi_i})$ are linearly independent and thus

$$\text{rank}[U(d_{\pi_i})] = p(\pi_i)$$

Therefore, $n \geq p(\pi_i)$, for every set $\{\pi_i\}$ satisfying (i). From (ii) and (2.7), $\gamma = \max\{p(\pi_i)\}$ and therefore $n \geq \gamma$, which proves theorem 2.1.

To find a minimal system (i.e., of dimension $n=\gamma$) which realizes $\{d_i, D_{i,j}\}$, one proceeds as follows. Let the usual orthonormal basis for R be given by

$$M^T = [(e_{\alpha_1,0})^T, (e_{\alpha_1,1})^T, \dots, (e_{\alpha_1,d_{\alpha_1}})^T, \dots, (e_{\alpha_r,0})^T, \dots, (e_{\alpha_r,d_{\alpha_r}})^T]$$

i.e., M is the $\gamma \times \gamma$ identity matrix.

For every $i \notin \{\alpha_j ; j=1, \dots, r\}$ let

$$D_i = \bar{\Phi}_{i,1} D_{\alpha_1} + \bar{\Phi}_{i,2} D_{\alpha_2} + \dots + \bar{\Phi}_{i,r} D_{\alpha_r} \quad \text{---(2.9)}$$

where $\bar{\Phi}_{i,j}$ are suitable scalars and which obviously exist as $\{D_{\alpha_1}, \dots, D_{\alpha_r}\}$ span the row space of B^* .

Then, define

$$c_i = e_{\alpha_i, 0} \quad \text{for } i \in \{\alpha_1, \dots, \alpha_r\} \quad (2.10)$$

and

$$c_i = [c_{i,1}, c_{i,2}, \dots, c_{i,r}] \quad \text{for } i \notin \{\alpha_1, \dots, \alpha_r\} \quad (2.11)$$

where $c_{i,j} = [0, \dots, 0, \bar{\Phi}_{i,j}, 0, \dots, 0]$ is a $d_{\alpha_j} + 1$ -row vector

and $\bar{\Phi}_{i,j}$ is in the $(d_{\alpha_j} + 1 - d_i)^{\text{th}}$ position.

Define A by the relations

$$e_{\alpha_i, j} A = e_{\alpha_i, j+1} \quad \text{for } j=0, 1, \dots, d_{\alpha_i} - 1 \quad (2.12)$$

and

$$e_{\alpha_i, d_{\alpha_i}} A = 0 \quad (2.13)$$

Remark 2.1. Instead of eq.(2.13) above, we may use

the form

$$e_{\alpha_i, d_{\alpha_i}} A = \sum_{j=1}^i \sum_{k=0}^{\min(d_{\alpha_i}+1, d_{\alpha_j})} \gamma_{\alpha_i, j, k} e_{\alpha_i, k} + \sum_{j=i+1}^r \sum_{k=0}^{\min(d_{\alpha_i}, d_{\alpha_j})} \gamma_{\alpha_i, j, k} e_{\alpha_i, k}$$

where $\gamma_{\alpha_i, j, k}$ are arbitrary scalars. This would have been more general but would have complicated the notation, and for our purposes is not necessary.

Finally, define the $\nu \times m$ matrix B as

$$B = \begin{bmatrix} B_{\alpha_1} \\ B_{\alpha_2} \\ \vdots \\ B_{\alpha_r} \end{bmatrix} \quad \text{--- (2.14)}$$

where B_{α_i} is the $(d_i+1) \times m$ matrix given by

$$B_{\alpha_i} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ D_{\alpha_i,1} & \dots & D_{\alpha_i,m} \end{bmatrix} \quad \text{--- (2.15)}$$

A direct but tedious verification shows that $S=(A,B,C)$, with A,B and C defined as in eqs.(2.10)-(2.15) above, satisfies

$$c_i A^k B = 0 \quad \text{for } i=1, \dots, m; \quad k=0, 1, \dots, (d_i-1) \quad \text{---}$$

and

$$c_i A^{d_i} B = [D_{i,1}, D_{i,2}, \dots, D_{i,m}] \neq 0$$

i.e., $S=(A,B,C)$ has the given $\{d_i, D_{i,j}\}$ as its set of invariants. This shows

Theorem 2.2. The set of invariants $\{d_i, D_{i,j}\}$ is independent.

To illustrate the realization procedure just described, we give two examples; one where B^* is non-singular and one where it is singular.

Example 2.1. Let $\{d_i, D_{i,j}\}$ be given as

$$d_1=0 ; d_2=3 ; d_3=1$$

and

$$B^* = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} ; \det B^* \neq 0.$$

So, $\text{rank}[B^*] = 3 = r$, and the minimum possible order of $S=(A,B,C)$ which realizes $\{d_i, D_{i,j}\}$ is

$$v = m + \sum_{i=1}^m d_i = 7$$

From eqs.(2.10) to (2.15), we obtain

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

With A, B and C so defined, we verify that

$$c_1 B = [1 \ 0 \ 2] \Rightarrow d_1=0 ; D_1=[1 \ 0 \ 2]$$

$$c_2 B = [0 \ 0 \ 0]$$

$$c_2 A B = [0 \ 0 \ 0]$$

$$c_2 A^2 B = [0 \ 0 \ 0]$$

$$c_2 A^3 B = [3 \ 0 \ 0] \Rightarrow d_2=3 ; D_2=[3 \ 0 \ 0]$$

and

$$c_3 B = [0 \ 0 \ 0]$$

$$c_3 A B = [0 \ 2 \ 1] \Rightarrow d_3=1 ; D_3=[0 \ 2 \ 1]$$

which is precisely the given set of invariants. A direct verification shows that the realization is truly minimal, in that it is both controllable and observable.

Example 2.2. Let $\{d_i, D_{i,j}\}$ be

$$d_1=1 ; d_2=3 ; d_3=0$$

and

$$B^* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} ; \text{rank}[B^*] = 2$$

We now find all possible sets of two linearly

independent rows of B^* and their associated values $p(\pi_i)$, i.e.,

$$a). \{D_1, D_2\}; \pi_1=1, \pi_2=2 \text{ and } p_1(\pi_i)=1+2=3$$

$$b). \{D_1, D_3\}; \pi_1=1, \pi_2=3 \text{ and } p_2(\pi_i)=1+2=3$$

$$c). \{D_2, D_3\}; \pi_1=2, \pi_2=3 \text{ and } p_3(\pi_i)=2+3=5$$

Therefore,

$$V = \max_j p_j(\pi_i) = 6$$

and the minimal realization will be of order 6.

We see that $D_3 = D_2 - D_1$; so, $\phi_{3,1}=-1$ and $\phi_{3,2}=1$, then

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & & & & \\ 0 & 0 & & & & \\ \hline & & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 \\ & & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

With A, B and C so defined,

$$c_1 B = [0 \quad 0 \quad 0]$$

$$c_1 A B = [1 \quad 0 \quad 0] \Rightarrow d_1=1; D_1 = [1 \quad 0 \quad 0]$$

$$c_2 B = [0 \quad 0 \quad 0]$$

$$c_2 A B = [0 \quad 0 \quad 0]$$

$$c_2 A^2 B = [0 \quad 0 \quad 0]$$

$$c_2 A^3 B = [1 \quad 1 \quad 0] \Rightarrow d_2=3 ; D_2 = [1 \quad 1 \quad 0]$$

$$c_3 B = [0 \quad 1 \quad 0] \Rightarrow d_3=0 ; D_3 = [0 \quad 1 \quad 0]$$

which is the given set of invariants. Again, a direct calculation shows that $S=(A,B,C)$ so defined is both controllable and observable.

Remark 2.2. It must be noticed that for a given set of invariants to be consistent, no row of B^* should be identically zero, for otherwise, it would contradict the definition of the corresponding d_i .

Remark 2.3. If the given matrix B^* is nonsingular, then theorem 2.1 above follows directly from the results in [17]. The results contained in this section have been published by the author in [4].

3). A complete set of independent decoupling invariants.

The set $\{d_i, D_{i,j}\}$ considered in the realization problem of the previous section, is invariant under similarity and feedback transformations of the form $u(t) = F x(t) + v(t)$. However, if we consider the

more general state feedback law

$$u(t) = F x(t) + G v(t) \quad \text{--- (2.16)}$$

where $v(t) \in \mathbb{R}^m$ and G is a nonsingular $m \times m$ matrix, then B^* does not remain invariant (i.e., $B^* \mapsto B^*G$). From $\{d_i, D_{i,j}\}$ the only quantities that remain invariant under such state feedback are $\{d_i\}$. However, unless we restrict the class of systems under consideration, it is again clear that they do not constitute a complete set of independent invariants, completeness being the absent property.

We shall first give some results on the observability of decoupled systems which will allow us to characterize an important subclass of the set of linear systems, for which the d_i 's are a complete set of independent invariants under similarity and state feedback transformations.

Let c_i denote the i^{th} row of C . From the definition of $d_i; i=1, \dots, m$, (eq.(1.5)), a simple calculation shows that for any feedback pair (F, G) ,

$$c_i (A+BF)^k = c_i A^k \quad ; k=0, 1, \dots, d_i \quad ; i=1, \dots, m \quad \text{--- (2.17)}$$

and

$$c_i (A+BF)^k = c_i A^{d_i} (A+BF)^{k-d_i} \quad ; k=(d_i+1), \dots, n; \quad i=1, \dots, m \quad \text{--- (2.18)}$$

Further, let the matrix U be defined by

$$U = \begin{bmatrix} c_1 \\ c_1 A \\ \vdots \\ c_1 A^{d_1-1} \\ \hline \vdots \\ \hline c_m \\ c_m A \\ \vdots \\ c_m A^{d_m-1} \end{bmatrix} \quad (2.19)$$

then the following lemma, which may be considered a special case of theorem 2.1, is easily established.

Lemma 2.1. (Falb and Wolovich [17]). Let $S=(A,B,C)$ be a decouplable system. Then

$$\text{rank}[U] = m + \sum_{i=1}^m d_i$$

and thus

$$m + \sum_{i=1}^m d_i \leq n$$

We may now show

Theorem 2.3. Let $S=(A,B,C)$ be a decouplable system.

The closed-loop system $S(F,G)=(A+BF,BG,C)$ is observable

for all possible feedback pairs (F,G) in eq.(2.16),
if and only if

$$m + \sum_{i=1}^m d_i = n \quad \text{_____ (2.20)}$$

Proof. (If). The closed-loop system $S(F,G)$, for arbitrary (F,G) will be observable if its observability matrix

$$W = \begin{bmatrix} C \\ C(A+BF) \\ \vdots \\ C(A+BF)^{n-1} \end{bmatrix}$$

is of rank n . From eq.(2.17), it is clear that U in eq.(2.19) is a submatrix of W (for all (F,G)). Thus,

$$\text{rank}[W] \geq \text{rank}[U]$$

and from lemma 2.1 and eq.(2.20)

$$\text{rank}[W] \geq n$$

but as W is an $(nm) \times n$ matrix, equality must hold, i.e. the closed-loop system is observable for all (F,G) .

(Only if). To show that eq.(2.20) is necessary, we first assume $m + \sum_{i=1}^m d_i < n$. Next, we show that there exists a feedback pair (F,G) for which $S(F,G)$ the closed-loop system is not observable. This feed-

back pair is

$$F^* = -B^{*-1} A^* \quad (2.21)$$

and

$$G^* = B^{*-1} \quad (2.22)$$

where

$$A^* = \begin{bmatrix} c_1 A^{d_1+1} \\ c_2 A^{d_2+1} \\ \vdots \\ c_m A^{d_m+1} \end{bmatrix}$$

Remark 2.4. The feedback pair just defined, decouples the system. E.G. Gilbert [21], called it an "integrator decoupling" (I.D.) feedback, because the transfer function of the closed-loop system $S(F^*, G^*)$ is

$$H(\cdot, F^*, G^*) = \text{diag}\{s^{d_i+1}\}$$

i.e., the i^{th} output is the (d_i+1) -fold integral of the i^{th} input.

From eq.(2.18), with (F^*, G^*) so defined,

$$\begin{aligned} c_i (A + BF^*)^{d_i+1} &= c_i A^{d_i+1} + c_i A^{d_i} BF^* \\ &= c_i A^{d_i+1} - c_i A^{d_i} B B^{*-1} A^* \\ &= c_i A^{d_i+1} - e_i A^* \\ &= 0 \end{aligned}$$

where e_i is the i^{th} unit-row vector. Therefore

$$c_i(A+BF^*)^{d_i+k} = 0 \quad \text{for } k=1,2,\dots$$

and the only nonzero row vectors in W are those of U .

This implies

$$\text{rank}[W] = \text{rank}[U] = m + \sum_{i=1}^m d_i < n$$

and the closed-loop system is not observable. This proves the theorem.

Theorem 2.3 above effectively states that Ω , the class of systems defined by

$$\Omega = \{S=(A,B,C): S \text{ is decouplable and (2.20) holds}\}$$

is the subset of the set of decouplable systems, which preserve observability under all possible state feedback transformations, and this property completely characterizes Ω .

Two further properties of Ω are apparent.

Theorem 2.4. If $S \in \Omega$ then S is observable.

Proof. This is an obvious consequence of lemma 2.1 above. The converse, i.e., if S is decouplable and and observable then (2.20) holds, is not true in general, as can be seen from the following example.

Example 2.3. Let $S=(A,B,C)$ be given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, $d_1=0$, $d_2=0$, and

$$B^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so, $\det\{B^*\} = 1 \neq 0$, and the system is decouplable.

Further, with no feedback

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and $\text{rank}[W]=3$. The system is observable. Yet

$$m + \sum_{i=1}^m d_i = 2 \neq n = 3$$

so eq.(2.20) is not satisfied.

Theorem 2.5. Let $S \in \Omega$. Then S is controllable.

Proof. The proof is not as direct as that of theorem

2.4. If S is decouplable, then at least $m + \sum_{i=1}^m d_i$

poles of the system can be arbitrarily assigned [17], [21] (furthermore, whilst simultaneously decoupling the system). So, if (2.20) is satisfied, n poles of the system can be arbitrarily specified. This in turn is only possible if and only if the system is controllable [41], so theorem 2.5 follows.

As in theorem 2.4 above, the converse is not true. i.e., in general, S being decouplable and controllable does not necessarily imply that eq.(2,20) is satisfied. To illustrate this we may use example 2.3 above. The system defined there is decouplable. A simple calculation shows that its controllability matrix $\bar{W}$, is given by

$$\bar{W} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = [B, AB, A^2B]$$

which is of rank 3, and thus S is controllable. Yet eq.(2.20) does not hold.

Theorem 2.3 above was published by the author in a less general form in [3], and a similar result was established by I.H.Mufti [40]. Theorems 2.4 and 2.5 were proved by Chandrasekharan [11], independently, but using a different approach. In our view his argument is more involved (see also [3]).

Having characterized Ω , the class of decouplable systems which preserve observability under arbitrary state feedback, we are ready to show our main result of this section, namely that $\{d_i\} ; i=1, \dots, m$, is a complete set of independent invariants under similarity and state feedback transformations for the class of systems Ω .

That $\{d_i\} ; i=1, \dots, m$, is independent should be clear from section 2 of this chapter. In fact, given a set $\{d_i\} ; i=1, \dots, m$, it is easy to verify that the system $S_c = (A_c, B_c, C_c)$ where

$$A_c = \begin{bmatrix} \begin{array}{c|c} \begin{matrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{matrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} \\ \hline \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} \end{array} & \begin{matrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{matrix} \end{bmatrix} ; B_c = \begin{bmatrix} \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{matrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} \\ \hline \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} \end{bmatrix}$$

$\xleftarrow{\quad d_1+1 \quad} \quad \xleftarrow{\quad d_m+1 \quad} \quad \xleftarrow{\quad m \quad} \quad (2.23)$

$$C_c = \begin{bmatrix} \begin{array}{c|c} \begin{matrix} 1 & 0 & 0 & \dots & 0 \end{matrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} \\ \hline \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} \end{array} \quad \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} \end{bmatrix} \quad \begin{matrix} \uparrow \\ m \\ \downarrow \end{matrix} \quad (2.24)$$

has $\{d_i\}$; $i=1, \dots, m$, as its set of decoupling invariants (and belongs to Ω). Therefore, only completeness remains to be shown.

Theorem 2.6. $\{d_i\}$; $i=1, \dots, m$, is a complete set of invariants (for systems belonging to Ω), under state feedback and similarity transformations.

Proof. Let $S=(A,B,C)$ and $\hat{S}=(\hat{A},\hat{B},\hat{C})$ be two systems, $S, \hat{S} \in \Omega$, having the same set of decoupling invariants $\{d_i\}$; $i=1, \dots, m$. Let (F^*, G^*) and $(\hat{F}^*, \hat{G}^*)$ be their integrator decoupling feedback laws respectively, then both $S_{ID} = (A+BF^*, BG^*, C)$ and $\hat{S}_{ID} = (\hat{A}+\hat{B}\hat{F}^*, \hat{B}\hat{G}^*, \hat{C})$ have the same transfer function matrix. For if they have the same set of d_i 's, then

$$H_{ID}(\cdot) = \text{diag}\{s^{-(d_i+1)}\} = \hat{H}_{ID}(\cdot)$$

From theorems 2.3 to 2.5, and the fact that controllability is preserved under state feedback, S_{ID} and $\hat{S}_{ID}$ are both minimal realizations of the same transfer function matrix. So, there exists a nonsingular matrix T such that

$$\hat{A}+\hat{B}\hat{F}^* = T(A+BF^*)T^{-1} \quad \text{_____} (2.25)$$

$$\hat{B}\hat{G}^* = TBG^* \quad \text{_____} (2.26)$$

$$\hat{C} = CT^{-1} \quad \text{_____} (2.27)$$

from (2.26), calling

$$G^* (\hat{G}^{*-1}) = G$$

we obtain

$$\hat{B} = TBG \quad \text{_____} (2.28)$$

where G is nonsingular.

Substituting (2.28) in (2.25), we get

$$\hat{A} = T(A+BF^*)T^{-1} - TBG\hat{F}^*$$

or

$$\hat{A} = T\{A+B[F^* - G\hat{F}^*T]\}T^{-1}$$

and if we let

$$F = F^* - G\hat{F}^*T$$

then

$$\hat{A} = T(A+BF)T^{-1} \quad \text{_____} (2.29)$$

Eqs. (2.27)-(2.29) show that S and $\hat{S}$ belong to the same equivalence class of the partition of the set of linear systems $S \in \Omega$ introduced by the similarity and state-feedback transformations. Thus, $\{d_i\}; i=1, \dots, m$ is complete, which proves the theorem.

It is clear now, that each equivalence class is determined by a particular value of the d_i 's and therefore every element of any equivalence class is equivalent to a system given by (2.23)-(2.24). So,

we are justified in calling (2.23)-(2.24) a canonical form. That this canonical form is unique is obvious.

4). Decoupling by output feedback.

When solving the decoupling problem via output-feedback, [23],[24],[33],[80],[81], a common assumption is that the systems under consideration are controllable and observable. Here we make the stronger assumption that eq. (2.20) holds.

An output feedback law given by

$$u(t) = K y(t) + G v(t) \quad \text{_____} (2.30)$$

where G is nonsingular, is clearly a special case of the state feedback law of eq.(2.16). That is, when there exists a matrix $K \in \mathbb{R}^{m \times m}$ which is a solution to

$$\bar{F} = KC \quad \text{_____} (2.31)$$

This obviously means that a given system $S = (A,B,C)$ will be decouplable by output feedback only if it is decouplable by state feedback.

To give a condition which is both necessary and sufficient for the existence of a solution to the decoupling problem via output feedback, we need the following preliminary results.

Let Q be a similarity transformation $z=Qx$, such that $Q: S=(A,B,C) \mapsto \tilde{S}=(\tilde{A},\tilde{B},\tilde{C})=(QAQ^{-1},QB,CQ^{-1})$. Then

Lemma 2.2. $S=(A,B,C)$ is decouplable by output feedback if and only if $\tilde{S}=(\tilde{A},\tilde{B},\tilde{C})$ is decouplable by output feedback.

Proof. Let $\tilde{c}_i$ be the i^{th} row of $\tilde{C}$. Then

$$\begin{aligned}\tilde{c}_i(\tilde{A}+\tilde{B}\tilde{K}\tilde{C})^j\tilde{B} &= c_i Q^{-1}(QAQ^{-1}+QBKCQ^{-1})^j QB \\ &= c_i(A+BKC)^j B \quad ; i=1,\dots,m; j=0,1,\dots\end{aligned}$$

for any matrix $K \in \mathbb{R}^{m \times m}$. The lemma follows directly from the necessary and sufficient condition given by Falb and Wolovich [17], i.e., $S=(A,B,C)$ is decouplable by output feedback if and only if B^* in eq.(2.6) is nonsingular and there exists a matrix K such that

$$\text{rank} \begin{bmatrix} c_i(A+BKC)^{n-1}B \\ c_i(A+BKC)^{n-2}B \\ \vdots \\ c_i(A+BKC)^{d_i}B \\ 0 \end{bmatrix} = 1 \quad \text{for } i=1,\dots,m$$

Lemma 2.3. There exists a solution K to the matrix equation $F=KC$ if and only if

$$\mathcal{N}[C] \subseteq \mathcal{N}[F] \quad (2.32)$$

where $\mathcal{N}[\cdot]$ stands for "null space".

Proof. The proof of this result is well known (e.g. [14]) and is omitted here.

Lemma 2.4. (I.H.Mufti,[39]). If $S=(A,B,C)$ is decouplable by state feedback and eq.(2.20) holds, then the class of decoupling feedback pairs is given by

$$F = -B^{*-1} L(\gamma) \quad (2.33)$$

where

$$L(\gamma) = \begin{bmatrix} c_1 A^{d_1+1} + \sum_{j=0}^{d_1} \gamma_j^{(1)} c_1 A^j \\ \vdots \\ c_m A^{d_m+1} + \sum_{j=0}^{d_m} \gamma_j^{(m)} c_m A^j \end{bmatrix} \quad (2.34)$$

and $\gamma_j^{(i)}$ are arbitrary scalars, and

$$G = B^{*-1} \Delta \quad (2.35)$$

where Δ is an arbitrary diagonal nonsingular constant matrix.

Now let us consider any system $S=(A,B,C) \in \Omega$, i.e., S is decouplable by state feedback and satisfies (2.20). It is clear that the matrix $U \in \mathbb{R}^{n \times n}$ in eq.(2.19) is a nonsingular matrix. If we use U as a similarity transformation, i.e., $U: S=(A,B,C) \mapsto \tilde{S}=(\tilde{A}, \tilde{B}, \tilde{C})$, then $\tilde{C}$ has

the form

$$\tilde{C} = CU^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \dots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}$$

$\begin{array}{ccccccc} \overbrace{\hspace{1.5cm}}^{d_1+1} & \overbrace{\hspace{1.5cm}}^{d_2+1} & \overbrace{\hspace{1.5cm}}^{d_3+1} & \dots & \overbrace{\hspace{1.5cm}}^{d_m+1} \end{array}$

(2.36)

and therefore, any column vector $w \in \mathcal{N}[\tilde{C}]$ must be of the form

$$w^T = [0 \quad X \dots X \mid 0 \quad X \dots X \mid \dots \mid 0 \quad X \dots X] \quad (2.37)$$

$\begin{array}{ccccccc} \overbrace{\hspace{1.5cm}}^{d_1+1} & \overbrace{\hspace{1.5cm}}^{d_2+1} & \overbrace{\hspace{1.5cm}}^{d_3+1} & \dots & \overbrace{\hspace{1.5cm}}^{d_m+1} \end{array}$

where the X's denote elements possibly different from zero.

Theorem 2.7. Let $S=(A,B,C) \in \Omega$. Then S is decouplable by output feedback if and only if

$$c_i A^{d_i+1} \in \text{span}\{c_1, \dots, c_m, c_i A, c_i A^2, \dots, c_i A^{d_i}\}; i=1, \dots, m$$

(2.38)

Proof. (If). Assume eq.(2.38) holds. i.e., there exist scalars $\alpha_j^{(i)}, \beta_j^{(i)}$, such that

$$c_i A^{d_i+1} = \sum_{j \neq i} \alpha_j^{(i)} c_j + \sum_{j=0}^{d_i} \beta_j^{(i)} c_i A^j \quad ; i=1, \dots, m. \quad (2.39)$$

A direct calculation shows that $\tilde{c}_i \tilde{A}^{d_i+1}$ is given

by

$$\tilde{c}_i^{\tilde{A}^{d_i+1}} = c_i^{A^{d_i+1}} U^{-1} = [z_1, \dots, z_m] \quad (2.40)$$

where $z_k \in \mathbb{R}^{d_k}$; $k=1, \dots, m$ are of the form

$$z_i = [X, \dots, X] \quad ; \quad z_j = [X, 0, \dots, 0] \quad ; j \neq i \quad (2.41)$$

where again, the X's represent scalars, possibly different from zero. Let $\tilde{L}(\gamma) = L(\gamma) U^{-1}$. i.e., $\tilde{L}(\gamma)$ is the matrix $L(\gamma)$ corresponding to $\tilde{S} = (\tilde{A}, \tilde{B}, \tilde{C})$. By an appropriate choice of $\gamma_j^{(i)}$ in eq.(2.34), $\tilde{L}(\gamma)$ can be written as

$$\tilde{L}(\gamma) = \left[\begin{array}{ccc|ccc|ccc|ccc} X & 0 & \dots & 0 & X & 0 & \dots & 0 & X & 0 & \dots & 0 \\ X & 0 & \dots & 0 & X & 0 & \dots & 0 & X & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ X & 0 & \dots & 0 & X & 0 & \dots & 0 & X & 0 & \dots & 0 \end{array} \right]$$

$\underbrace{\hspace{1.5cm}}_{d_1+1} \quad \underbrace{\hspace{1.5cm}}_{d_2+1} \quad \dots \quad \underbrace{\hspace{1.5cm}}_{d_m+1}$

(2.42)

From eqs.(2.37) and (2.42), if $w \in \mathcal{N}[\tilde{C}]$ then $w \in \mathcal{N}[\tilde{L}(\gamma)]$ and thus

$$\mathcal{N}[\tilde{C}] \subseteq \mathcal{N}[\tilde{L}(\gamma)] = \mathcal{N}[F] \quad (2.43)$$

and from lemmas 2.2 and 2.3, there exists a matrix K such that the output feedback pair (K, B^{*-1}) decouples $S=(A, B, C)$.

(Only if). Assume $c_i^{A^{d_i+1}}$ depends linearly on some row of U different from those already in eq.(2.38).

Then for some $j \neq i$, z_j in eq.(2.40) has at least two elements different from zero, which implies that $\tilde{L}(\gamma)$ will have at least one non-zero element in a column other than those with X's in eq.(2.42). Thus, for some $w \in \mathcal{N}[\tilde{C}]$,

$$\tilde{L}(\gamma) w \neq 0$$

and this implies that

$$\mathcal{N}[\tilde{C}] \not\subseteq \mathcal{N}[F]$$

So, $\tilde{S}=(\tilde{A}, \tilde{B}, \tilde{C})$ cannot be decoupled by output feedback (lemma 2.3), nor can $S=(A, B, C)$ as a consequence (lemma 2.2), which proves the theorem.

To obtain $\tilde{L}(\gamma)$ as in eq.(2.42) we must assign specific values to $\gamma_j^{(i)}$; $j=1, \dots, d_i$; $i=1, \dots, m$. This means that only m parameters are left free. i.e., $\gamma_o^{(i)}$; $i=1, \dots, m$. To each set of values $\gamma_o^{(i)}$ there will correspond an output feedback decoupling matrix K . Further, it is clear that only m closed-loop poles may be arbitrarily assigned while decoupling by output feedback, each corresponding to a $\gamma_o^{(i)}$; $i=1, \dots, m$.

Let us define now

$$\mathcal{P}_i = \text{span}\{c_i, c_i A, \dots, c_i A^{d_i+1}\} ; i=1, \dots, m \quad \text{---(2.44)}$$

then,

Corollary 2.7.1. Let $S=(A,B,C) \in \Omega$. If the subspaces $\mathcal{P}_i$ are mutually independent subspaces, then there exists a decoupling output feedback control law.

Proof. In this case, $\alpha_j^{(i)}$ in eq.(2.39) are all zero, and the choice

$$\gamma_j^{(i)} + \beta_j^{(i)} = 0 \quad \text{for } j=1, \dots, d_i ; i=1, \dots, m$$

and

$$\gamma_o^{(i)} + \beta_o^{(i)} = \gamma^{(i)} \neq 0 \quad ; i=1, \dots, m$$

yields $L(\gamma) = \Gamma C$, with $\Gamma = \text{diag}\{\gamma^{(i)}\}$ which clearly implies that $\mathcal{N}[C] \subseteq \mathcal{N}[F]$.

Corollary 2.7.2. Let $S=(A,B,C) \in \Omega$. If the observability subspaces of S are independent, then there is a solution to the decoupling problem via output feedback.

Proof. The i^{th} observability subspace of S is, by definition $\text{span}\{c_i, c_i A, \dots, c_i A^{n-1}\}$, and thus the corollary follows trivially from corollary 2.7.1 above.

The results contained in this section may also be found in [5].

To end this chapter, we must point out that although those systems belonging to Ω have some nice properties (as shown in sections 3 and 4 above), they have one possible practical disadvantage. i.e., the

high sensitivity of their decoupling properties to parameter variations. This will be shown later on in chapter IV.

CAPTER III. Decoupling of composite linear systems.

1). Introduction.

In this chapter we will consider the problem of decoupling composite linear systems connected in cascade, in both state and output feedback and in parallel, via state feedback. As such, these problems have received little or no attention in the decoupling literature.

The cascade connection, and to a certain extent also the feedback connection, have so far only been studied when associated with the dynamic decoupling problem. That is, when a non-decouplable system $S = (A, B, C)$ is given and a dynamic precompensator $\hat{S} = (\hat{A}, \hat{B}, \hat{C})$ is to be found such that $\bar{S} = (\bar{A}, \bar{B}, \bar{C})$, the cascade connection of $\hat{S}$ followed by S , is decouplable by state feedback. (see fig.3.1, section 2 below). So, it is only natural to expect that the results thus obtained, concern the conditions which the given system S must satisfy in order that such a precompensator exists. Here we are concerned with the properties which S and $\hat{S}$, the precompensator, must possess in order that the composite system is decouplable.

For example, it is well known (e.g. [53]) that

a necessary and sufficient condition for the existence of a solution to the dynamic decoupling problem is that the given system is invertible. In section 2 below, we show that given two systems S and $\hat{S}$ connected in cascade, it is necessary that both are invertible if the composite system is to be state feedback decouplable. Sufficiency is lost because we no longer have the freedom to choose $\hat{S}$ appropriately.

Proceeding further, we obtain a necessary and sufficient condition for the composite system to be decouplable by state feedback. The approach used is algebraic, and is based directly on the main result of Falb and Wolovich [17], and the condition thus obtained is expressed in terms of the parameters of the subsystems S and $\hat{S}$. An analysis of this necessary and sufficient condition shows that in general, the decouplability of the composite system is not determined by the decoupling properties of either S or $\hat{S}$. By means of two examples, we show that S and $\hat{S}$ decouplable does not necessarily imply that $\bar{S}$ is decouplable, and that $\bar{S}$ can be decouplable even if both S and $\hat{S}$ are not.

However, if $\hat{S}$ is decouplable and $\{\hat{d}_i\}; i=1, \dots, m$, its decoupling invariants, are all equal, it is possible to show that the possibility of decoupling the

composite system $\bar{S}$, rests exclusively on whether S is decouplable or not. We also show that the system acting as a precompensator, that is $\hat{S}$, must be input-observable, (with input observability index bounded above), if the composite system is to be decouplable.

In section 3 we analyze the decoupling problem of feedback connected composite systems, and consider two possible cases; (a).- When the system in the feedback path is driven by the output of the feedforward system (see fig. 3.2, section 3 below) and (b).- When it is driven by the state of the feedforward system. (see fig.3.3, section 3 below). In contrast with the parallel connection, for the feedback (both state and output) connection, the decoupling properties of the composite system are completely determined by those of the system in the feedforward path. That is, $\bar{S}$, the composite system is decouplable by state feedback if and only if S , the feed forward system is. Taken in the context of the dynamic decoupling problem, this result shows that dynamic feedback alone, cannot transform a system with weak inherent coupling into one with no-inherent coupling, [2]. Therefore, the dynamic feedback which results from applying a state feedback pair (F,G) to a pre-compensated system only provides possible degrees of freedom in the design.

In the last section of the chapter, the decoupling of composite systems formed by the connection in parallel of two systems is studied. We first obtain a necessary and sufficient condition for the composite system $\bar{S}$ to be decouplable by state feedback, based, once more, directly on the results of Falb and Wolovich [17]. As in the cascade connection, it is possible to show from this condition that, S and $\hat{S}$, the systems connected in parallel being both decouplable, does not necessarily imply that $\bar{S}$, the composite system is; and that $\bar{S}$ may be decouplable even if neither of the subsystems is. At first sight this may seem contrary to expectations, but if we notice that the states of each subsystem are being feedback not in independent loops, but through the same input vector, then the result is heuristically reasonable. A set of necessary conditions for the existence of a feedback pair (F,G) which decouples the composite system are also given.

The results of this last section can be used to study the effect of parameter variations on the decoupling properties of a given system, particularly for variations in the input map B , and the output map C . However, this will not be done here because a more ad-hoc approach, which will also allow us to consider variations in A , the plant of the system,

will be presented in the next chapter.

We also point out, that, with trivial modifications in the definition of $\{d_i; i=1, \dots, m\}$, the set of decoupling invariants, the results contained in this chapter can be easily extended to the more general case where a direct input-output map is introduced in the output equation of the state-space description of the systems considered.

3.2 Cascade connection.

Let us consider two constant linear systems $S=(A,B,C)$ and $\hat{S}=(\hat{A},\hat{B},\hat{C})$ described respectively by

$$\dot{x}(t) = A x(t) + B u(t) \quad \text{_____} (3.1)$$

$$y(t) = C x(t) \quad \text{_____} (3.2)$$

and

$$\dot{\hat{x}}(t) = \hat{A} \hat{x}(t) + \hat{B} \hat{u}(t) \quad \text{_____} (3.3)$$

$$\hat{y}(t) = \hat{C} \hat{x}(t) \quad \text{_____} (3.4)$$

where $x(t) \in \mathbb{R}^n$, $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$, $u(t), y(t), \hat{u}(t)$ and $\hat{y}(t) \in \mathbb{R}^m$ and $A, B, C, \hat{A}, \hat{B},$ and $\hat{C}$ are matrices of appropriate dimensions.

Consider now the composite system $\bar{S}=(\bar{A},\bar{B},\bar{C})$ formed by the cascade connection of $\hat{S}$ followed by S .

that is, set

$$\hat{y}(t) = u(t) \quad (3.5)$$

This is illustrated in the following figure

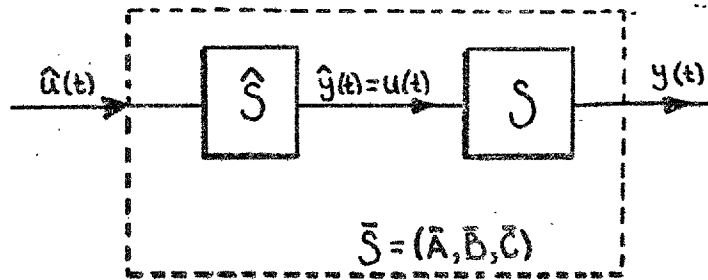


Fig. 3.1

Fig. 3.1.- $\bar{S} = (\bar{A}, \bar{B}, \bar{C})$; the cascade connection of $\hat{S}$ followed by S .

Remark 3.1

For the cascade connection to be meaningful, it is obvious that $\hat{y}(t)$ and $u(t)$ must be of the same dimension, m , say. The assumption that $\hat{u}(t)$ and $y(t)$ also have dimension m is made in order to be able to consider the one-to-one decoupling problem for $\hat{S}$ and S as well as for $\bar{S}$. In most problems, (e.g., when $\hat{S}$ is considered as a dynamic precompensator to be fixed by the designer) this assumption constitutes no serious restriction.

It is evident that the vector $\bar{x}^T(t) = [\hat{x}(t)^T, x(t)^T]^T$, where $(\cdot)^T$ denotes "transpose", qualifies as the state vector of the composite system $\bar{S}$. Thus, using eq. (3.5) and substituting (3.4) in (3.1), the following state space description of $\bar{S} = (\bar{A}, \bar{B}, \bar{C})$ is obtained;

$$\dot{\bar{x}} = \bar{A} \bar{x}(t) + \bar{B} \bar{u}(t) = \begin{bmatrix} \hat{A} & 0 \\ \hat{B}\hat{C} & A \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} \hat{u}(t) \quad (3.6)$$

$$y(t) = \bar{C} \bar{x}(t) = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ x(t) \end{bmatrix} \quad (3.7)$$

In what follows we shall investigate the state feedback decoupling properties of $\bar{S} = (\bar{A}, \bar{B}, \bar{C})$ and will try to relate them to the parameters of $S = (A, B, C)$ and $\hat{S} = (\hat{A}, \hat{B}, \hat{C})$. The first obvious result is

Theorem 3.1. Let $\bar{S}$ be the cascade connection of $\hat{S}$ followed by S . A necessary condition for $\bar{S}$ to be decouplable by state feedback is that both S and $\hat{S}$ are invertible.

Proof. Let $H(s)$, $\hat{H}(s)$ and $\bar{H}(s)$ denote the transfer function matrices of S , $\hat{S}$ and $\bar{S}$ respectively. Then

$$\bar{H}(s) = H(s) \hat{H}(s) \quad (3.8)$$

and if either $H(s)$ or $\hat{H}(s)$ are singular matrices, then

$\det(\bar{H}(s)) = 0$, which implies that $\bar{S}$ has strong inherent coupling [21], and no control law can effect decoupling.

Define now $\{d_i, B^*\}$, $\{\hat{d}_i, \hat{B}^*\}$, and $\{\bar{d}_i, \bar{B}^*\}$ as in eqs.(1.5) and (1.6) in terms of the parameters of S , $\hat{S}$ and $\bar{S}$, respectively. Then, from [17],[21], the systems S , $\hat{S}$ and $\bar{S}$ will be decouplable by state feedback if and only if the matrices B^* , $\hat{B}^*$ and $\bar{B}^*$ are nonsingular.

From the form of $\bar{A}$ in eq.(3.6), a direct calculation shows that

$$\bar{A}^k = \begin{bmatrix} \hat{A}^k & 0 \\ \sum_{j=0}^{k-1} A^j B \hat{C} \hat{A}^{k-1-j} & \hat{A}^k \end{bmatrix} \quad ; k=1, 2, \dots$$

and thus,

$$\bar{c}_i \bar{A}^k \bar{B} = \sum_{j=0}^{k-1} [c_i A^j B] [\hat{C} \hat{A}^{k-1-j} \hat{B}] \quad ; k=1, 2, \dots \quad ; i=1, \dots, m \quad \text{---(3.9)}$$

From the definition of d_i ; $i=1, \dots, m$, we have

$$\bar{c}_i \bar{A}^k \bar{B} = 0 \quad \text{for } k \leq d_i \quad ; i=1, \dots, m \quad \text{---(3.10)}$$

and

Lemma 3.1. Let $\bar{S}$ be the cascade connection of $\hat{S}$ followed

by S. Then, $\{\bar{d}_i ; i=1, \dots, m\}$, the set of decoupling invariants of $\bar{S}$, satisfies

$$\bar{d}_i \geq d_i + 1 + \hat{d}^* ; i=1, \dots, m \quad \text{--- (3.11)}$$

where $\hat{d}^* = \min_i \{\hat{d}_i\}$.

Proof. That $\bar{d}_i \geq d_i + 1$ is clear from (3.10) above.

With this in mind, from eq.(3.9) we obtain

$$\begin{aligned} \bar{c}_i \bar{A}^{d_i+1+k} \bar{B} &= [c_i A^{d_i+k} B] [\hat{C}\hat{B}] + [c_i A^{d_i+k-1} B] [\hat{C}\hat{A}\hat{B}] + \dots \\ &\dots + [c_i A^{d_i} B] [\hat{C}\hat{A}^k \hat{B}] ; k=0, 1, \dots, \end{aligned} \quad \text{--- (3.12)}$$

for $i=1, \dots, m$, and as by definition of $\hat{d}_i$ and $\hat{d}^*$, $\hat{C}\hat{A}^k \hat{B} = 0$ for $k < \hat{d}^*$, then

$$\bar{c}_i \bar{A}^{d_i+1+k} \bar{B} = 0 \quad \text{for } k=0, 1, \dots, \hat{d}^*-1 ; i=1, \dots, m$$

i.e., $\bar{d}_i \geq d_i + 1 + \hat{d}^*$, and lemma 3.1 follows.

Notice that

$$\bar{c}_i \bar{A}^{d_i+1+\hat{d}^*} \bar{B} = [c_i A^{d_i} B] [\hat{C}\hat{A}^{\hat{d}^*} \hat{B}] ; i=1, \dots, m$$

thus equality in (3.11) holds if and only if

$$[c_i A^{d_i} B]^T \notin \mathcal{N} [\hat{C}\hat{A}^{\hat{d}^*} \hat{B}]^T$$

In fact, define

$$L_i^k = [c_i A^{d_i+k} B, c_i A^{d_i+k-1} B, \dots, c_i A^{d_i} B] ; k=0, 1, \dots$$

and

$$\hat{W}_k = \begin{bmatrix} \hat{C}\hat{B} \\ \hat{C}\hat{A}\hat{B} \\ \vdots \\ \hat{C}\hat{A}^k\hat{B} \end{bmatrix} \quad k=0,1,\dots$$

and let

$$\beta_i = \min\{k: [L_i^k] [\hat{W}_k] \neq 0; k=0,1,\dots\}, \quad i=1,\dots,m$$

then

$$\bar{d}_i = d_i + 1 + \beta_i \quad i=1,\dots,m \quad (3.13)$$

and obviously $\beta_i \geq \hat{d}^*$ for all $i=1,\dots,m$, which proves the lemma.

Let now $\beta^* = \max_i \{\beta_i\}$, and define the $m(\beta^*+1)$ -row vector $\mathcal{L}_i(\beta_i)$; $i=1,\dots,m$, as

$$\mathcal{L}_i(\beta_i) = [L_i^{\beta_i}, 0_{\eta_i}]^T \quad ; i=1,\dots,m$$

where 0_{η_i} is an $m(\beta^*-\beta_i)$ -zero row vector, and

$$\mathcal{L}(\beta_i) = \begin{bmatrix} \mathcal{L}_1(\beta_i) \\ \vdots \\ \mathcal{L}_m(\beta_i) \end{bmatrix}$$

then,

Theorem 3.2. Let $\bar{S}$ be the cascade connection of $\hat{S}$ followed by S . Then, $\bar{S}$ will be decouplable by state feedback if and only if

$$\text{rank}\{[\mathcal{L}(\beta_i)] [\hat{W}_{\beta*}]\} = m \quad \text{--- (3.14)}$$

Proof. From eq.(3.13) and the definition of $\bar{B}^*$, $\hat{W}_{\beta*}$ and $\mathcal{L}(\beta_i)$, it follows that

$$\bar{B}^* = [\mathcal{L}(\beta_i)] [\hat{W}_{\beta*}] \quad \text{--- (3.15)}$$

and thus, theorem 3.2 follows.

Let us analyze eq.(3.15) in more detail. It is obvious that $\text{rank}[\mathcal{L}(\beta_i)] = m$ and $\text{rank}[\hat{W}_{\beta*}] = m$, are both necessary conditions for $\bar{S}$ to be decouplable. In general however, it is not possible to relate these conditions to the decoupling properties of S and $\hat{S}$. (Although from the form of the matrix $\mathcal{L}(\beta_i)$ it is easy to see that the first of these conditions will be satisfied whenever S is decouplable). In fact, the following two propositions show that the nonsingularity of $\bar{B}^*$ is not related in any way to the nonsingularity of the matrices B^* and $\hat{B}^*$.

Proposition 3.1. Let $\bar{S}$ be the cascade connection of $\hat{S}$ followed by S . If S and $\hat{S}$ are both decouplable, it does not necessarily follow that the composite system is decouplable.

Proof. To prove this, consider the following example, where S and $\hat{S}$ are both decouplable but $\bar{S}$ is not. Let $\hat{S}=(\hat{A},\hat{B},\hat{C})$ be given by

$$\hat{A} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & \bigcirc & \\ -1 & -3 & -3 & & & \\ \hline & & & 0 & 1 & 0 \\ & \bigcirc & & 0 & 0 & 1 \\ & & & -1 & -3 & -3 \end{array} \right] \quad \hat{B} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \\ 1 & -3 \\ -3 & 6 \end{array} \right]$$

$$\hat{C} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & \bigcirc & \\ \hline & \bigcirc & & 1 & 0 & 0 \end{array} \right]$$

and let $S=(A,B,C)$ be given by

$$A = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & \bigcirc & \\ 0 & 0 & 1 & & & \\ -1 & -3 & -3 & & & \\ \hline & & & 0 & 1 & 0 \\ & \bigcirc & & 0 & 0 & 1 \\ & & & -1 & -3 & -3 \end{array} \right] \quad B = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ -3 & 1 \\ \hline 0 & 0 \\ 0 & 1 \\ 1 & -3 \end{array} \right]$$

$$C = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & \bigcirc & \\ \hline & \bigcirc & & 1 & 0 & 0 \end{array} \right]$$

A direct calculation shows that

$$\begin{aligned} \hat{c}_1 \hat{B} &= [0, 0] \\ \hat{c}_1 \hat{A} \hat{B} &= [0, 0] \\ \hat{c}_1 \hat{A}^2 \hat{B} &= [1, 0] \Rightarrow \hat{d}_1 = 2 \\ \hat{c}_2 \hat{B} &= [0, 1] \Rightarrow \hat{d}_2 = 0 \end{aligned} \left\{ \hat{B}^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \det[B^*] = 1 \neq 0 \right.$$

i.e., the system $\hat{S}=(\hat{A}, \hat{B}, \hat{C})$ is decouplable. Further

$$\begin{aligned} c_1 B &= [0, 0] \\ c_1 AB &= [1, 0] \Rightarrow d_1 = 1 \\ c_2 B &= [0, 0] \\ c_2 AB &= [0, 1] \Rightarrow d_2 = 1 \end{aligned} \left\{ B^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \det[B^*] = 1 \neq 0 \right.$$

i.e., the system $S=(A, B, C)$ is also decouplable.

Now, the composite system $\bar{S}=(\bar{A}, \bar{B}, \bar{C})$ is given from eqs.(3.6) and (3.7), by

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -3 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & -3 \\ -3 & 6 \\ 0 \\ 0 \end{bmatrix}$$

and

$$C = \left[\begin{array}{cc|ccc} \circ & \circ & 1 & 0 & 0 & \circ \\ \hline & & \circ & 1 & 0 & 0 \end{array} \right]$$

Thus,

$$\left. \begin{array}{l} \bar{c}_1 \bar{B} = [0,0] \\ \bar{c}_1 \bar{A} \bar{B} = [0,0] \\ \bar{c}_1 \bar{A}^2 \bar{B} = [0,0] \\ \bar{c}_1 \bar{A}^3 \bar{B} = [0,1] \Rightarrow \bar{d}_1 = 3 \\ \bar{c}_2 \bar{B} = [0,0] \\ \bar{c}_2 \bar{A} \bar{B} = [0,0] \\ \bar{c}_2 \bar{A}^2 \bar{B} = [0,1] \Rightarrow \bar{d}_2 = 2 \end{array} \right\} \bar{B}^* = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

and $\det[\bar{B}^*] = 0$, which implies that the composite system $\bar{S}$ is not decouplable.

Proposition 3.2. Let $\bar{S}$ be the cascade connection of $\hat{S}$ followed by S . If S and $\hat{S}$ are both non-decouplable it is not necessarily true that $\bar{S}$ is non-decouplable.

Proof. We shall use an example again to show this.

Let $S=(A,B,C)$ and $\hat{S}=(\hat{A},\hat{B},\hat{C})$ be given as below

$$A = \left[\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & & & & & & \\ 0 & 0 & 1 & & 0 & & & 0 & \\ 0 & 0 & 0 & & & & & & \\ \hline & 0 & & 0 & 1 & 0 & 0 & & \\ & 0 & & 0 & 0 & 1 & 0 & & 0 \\ & 0 & & 0 & 0 & 0 & 1 & & \\ & 0 & & 0 & 0 & 0 & 0 & & \\ \hline & 0 & & & & & & 0 & 1 & 0 \\ & & & & & & & 0 & 0 & 1 \\ & & & & & & & 0 & 0 & 0 \end{array} \right] ; B = \left[\begin{array}{ccc} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{array} \right]$$

$$C = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & & 0 & & & 0 & \\ & 0 & & 1 & 0 & 0 & 0 & & 0 \\ & 0 & & & 0 & & & 1 & 0 & 0 \end{array} \right]$$

and

$$\hat{A} = \left[\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & & & & & & \\ 0 & 0 & 1 & & 0 & & & 0 & \\ 0 & 0 & 0 & & & & & & \\ \hline & 0 & & 0 & 1 & & & 0 & \\ & & & 0 & 0 & & & & \\ \hline & 0 & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 & 0 \end{array} \right] ; \hat{B} = \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{array} \right]$$

$$\hat{C} = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & & 0 & & & 0 & \\ & 0 & & 1 & 0 & & & 0 & \\ & 0 & & & 0 & & & 1 & 0 & 0 \end{array} \right]$$

Then,

$$\left. \begin{aligned} \hat{c}_1 \hat{B} &= [1 \ 0 \ -1] \Rightarrow \hat{d}_1=0 \\ \hat{c}_2 \hat{B} &= [0 \ 0 \ 0] \\ \hat{c}_2 \hat{A} \hat{B} &= [1 \ 1 \ 0] \Rightarrow \hat{d}_2=1 \\ \hat{c}_3 \hat{B} &= [0 \ 0 \ 0] \\ \hat{c}_3 \hat{A} \hat{B} &= [0 \ 0 \ 0] \\ \hat{c}_3 \hat{A}^2 \hat{B} &= [0 \ 1 \ 1] \Rightarrow \hat{d}_3=2 \end{aligned} \right\} \quad \hat{B}^* = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and $\det[\hat{B}^*] = 0$, thus $\hat{S}=(\hat{A}, \hat{B}, \hat{C})$ is not decouplable.

Similarly,

$$\left. \begin{aligned} c_1 B &= [0 \ 0 \ -1] \Rightarrow d_1=0 \\ c_2 B &= [0 \ 0 \ 1] \Rightarrow d_2=0 \\ c_3 B &= [0 \ 0 \ -1] \Rightarrow d_3=0 \end{aligned} \right\} \quad B^* = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

and $\det[B^*] = 0$, which shows that $S=(A, B, C)$ is not decouplable either.

However, a direct calculation shows that

$$L_1^0 \hat{W}_0 = L_1^1 \hat{W}_1 = L_1^2 \hat{W}_2 = 0 \quad \text{and} \quad L_1^3 \hat{W}_3 = [0 \ 0 \ -1]$$

$$L_2^0 \hat{W}_0 = L_2^1 \hat{W}_1 = L_2^2 \hat{W}_2 = 0 \quad \text{and} \quad L_2^3 \hat{W}_3 = [1 \ 0 \ 0]$$

$$L_3^0 \hat{W}_0 = L_3^1 \hat{W}_1 = L_3^2 \hat{W}_2 = 0 \quad \text{and} \quad L_3^3 \hat{W}_3 = [1 \ 1 \ -1]$$

and therefore $\beta_1 = \beta_2 = \beta_3 = 3$, and

$$\begin{aligned} \bar{B}^* &= [\mathcal{L}(\beta_i)] [\hat{W}_3] = \\ &= \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \end{aligned}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

which is a nonsingular matrix ($\det \bar{B}^* = -1$), and thus, $\bar{S}=(\bar{A},\bar{B},\bar{C})$ is decouplable by state feedback.

Although in general, as propositions 3.1 and 3.2 above show, it is not possible to relate the decoupling properties of S and $\hat{S}$ with those of $\bar{S}$, we may still consider special cases where this is possible. In fact, these are cases where from the form of $\mathcal{L}(\beta_i)$ and $\hat{W}_{\beta^*}$, and the fact that they must be full rank matrices if $\bar{S}$ is to be decouplable, one can obtain some information about the decoupling properties of S and $\hat{S}$. For example,

Corollary 3.2.1. Let $\bar{S}$ be the cascade connection of $\hat{S}$ followed by S . Let $\hat{S}$ be decouplable and $\hat{d}_1 = \hat{d}_2 = \dots = \hat{d}_m = \hat{d}^*$. Then, the composite system $\bar{S}$ will be decouplable by state feedback if and only if S is decouplable by state feedback.

Proof. If $\hat{d}_1 = \hat{d}_2 = \dots = \hat{d}_m = \hat{d}^*$, then

$$\hat{B}^* = \hat{C} \hat{A}^{\hat{d}^*} \hat{B}$$

and if $\hat{S}$ is decouplable, $\hat{B}^*$ is nonsingular. This necessarily implies $\beta_i = \hat{d}^*$ for every $i=1, \dots, m$, and from eq.(3.15)

$$\bar{B}^* = \begin{bmatrix} c_1 A^{d_1} B \\ \vdots \\ c_m A^{d_m} B \end{bmatrix} \quad [\hat{C} \hat{A}^{\hat{d}^*} \hat{B}] = B^* \hat{B}^*$$

and as $\hat{B}^*$ is nonsingular then $\text{rank}[\bar{B}^*] = m$ if and only if $\text{rank}[B^*] = m$ and corollary 3.21 follows.

Corollary 3.2.2. Let $\bar{S}$ be the cascade connection of $\hat{S}$ followed by S . $\bar{S}$ will be decouplable only if $\hat{S}$ is input observable with input observability index smaller than or equal to β^* .

Proof. The corollary follows directly from eq.(3.15) and the definition of input observability, i.e., $\hat{S}$ is input observable if and only if $\text{rank}[\hat{W}_{n-1}] = m$.

The condition imposed upon $\hat{S}$ in corollary 3.2.1, i.e., $\hat{d}_1 = \hat{d}_2 = \dots = \hat{d}_m$, is restrictive and we do not claim it will be encountered frequently in practice. The main interest of the result is theoretical. However, in the context of the dynamic decoupling problem, corollary 3.2.1 is of help in that it states that the first nonzero matrix in the unknown sequence $\hat{C}\hat{B}, \hat{C}\hat{A}\hat{B}, \dots, \hat{C}\hat{A}^k\hat{B}, \dots$, must be chosen to be singular if the overall system is to be decouplable.

A very nice feature of eq.(3.15) is that $\bar{B}^*$ is expressed in terms of two matrices each exclusively determined by one of the subsystems S and $\hat{S}$. This was used to develop a procedure to obtain a minimal order precompensator for the dynamic decoupling problem. Although this procedure was obtained in an independent manner, it was found to be almost identical with that proposed by M.C.Cremer [9] and for that reason will be omitted here.

3.3. Feedback connection.

We shall now consider the decoupling problem for the feedback connection. Therefore, assume we are given two linear systems $S=(A,B,C)$ and $\hat{S}=(\hat{A},\hat{B},\hat{C})$ as described by eqs.(3.1)-(3.4), and let them be connected as in fig.3.2 below.

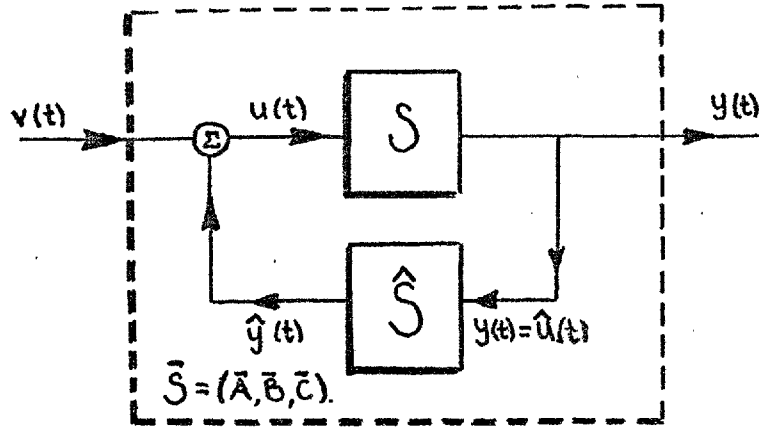


Fig.3.2. $\bar{S}=(\bar{A},\bar{B},\bar{C})$; the output feedback connection of S and $\hat{S}$.

The resulting system $\bar{S}$, will be called in this case, the output feedback composite system to make clear that $y(t)$, the output of the system in the feed-forward path drives the system $\hat{S}$ in the feedback path. i.e.,

$$y(t) = \hat{u}(t) \quad \text{--- (3.16)}$$

To complete the output feedback connection we set

$$u(t) = \hat{y}(t) + v(t) \quad \text{--- (3.17)}$$

where $v(t) \in \mathbb{R}^m$ is the closed-loop input.

As in the case of the cascade connection, it is clear that $\bar{x}^T = [\hat{x}(t)^T, x(t)^T]^T$ qualifies as the state vector for $\bar{S}=(\bar{A},\bar{B},\bar{C})$. Thus, from eqs.(3.1)-(3.4) and (3.16),(3.17), the output feedback composite system $\bar{S}$ is described by the set of linear differen-

tial equations

$$\dot{\bar{x}}(t) = \begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{x}(t) \end{bmatrix} = \bar{A}\bar{x}(t) + \bar{B}v(t) = \begin{bmatrix} \hat{A} & \hat{B}C \\ \hat{B}\hat{C} & A \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v(t) \quad (3.18)$$

and

$$y(t) = \bar{C}\bar{x}(t) = [0 \quad c] \begin{bmatrix} \hat{x}(t) \\ x(t) \end{bmatrix} \quad (3.19)$$

Let $\bar{A}^k$ be partitioned as

$$\bar{A}^k = \begin{bmatrix} \bar{A}_{1,1}^k & \bar{A}_{1,2}^k \\ \bar{A}_{2,1}^k & \bar{A}_{2,2}^k \end{bmatrix} ; k=1,2,\dots$$

where $\bar{A}_{1,1}^k$ is an $\hat{n} \times \hat{n}$ matrix and $\bar{A}_{2,2}^k$ an $n \times n$ matrix, and $\bar{A}_{1,2}^k$ and $\bar{A}_{2,1}^k$ are of appropriate dimensions.

From the form of $\bar{C}$ and $\bar{B}$ in eqs.(3.18) and (3.19),

$$\bar{c}_i \bar{A}^k \bar{B} = c_i \bar{A}_{2,2}^k B ; k=1,2,\dots \quad (3.20)$$

and we only need to find $\bar{A}_{2,2}^k$ in terms of the parameters of the subsystems S and $\hat{S}$. After a direct and simple, but tedious manipulation, we obtain

$$\bar{A}_{2,2} = A$$

$$\bar{A}_{2,2}^2 = A^2 + B[\hat{C}\hat{B}]C$$

$$\bar{A}_{2,2}^3 = A^3 + [AB, B] \begin{bmatrix} \hat{C}\hat{B} & 0 \\ \hat{C}\hat{A}\hat{B} & \hat{C}\hat{B} \end{bmatrix} \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$\bar{A}_{2,2}^4 = A^4 + [A^2B, AB, B] \begin{bmatrix} \hat{C}\hat{B} & 0 & 0 \\ \hat{C}\hat{A}\hat{B} & \hat{C}\hat{B} & 0 \\ \hat{C}\hat{A}^2\hat{B} & \hat{C}\hat{A}\hat{B} & \hat{C}\hat{B} \end{bmatrix} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} + B[\hat{C}\hat{B}][CB][\hat{C}\hat{B}]C$$

$$\bar{A}_{2,2}^5 = A^5 + [A^3B, A^2B, AB, B] \begin{bmatrix} \hat{C}\hat{B} & 0 & 0 & 0 \\ \hat{C}\hat{A}\hat{B} & \hat{C}\hat{B} & 0 & 0 \\ \hat{C}\hat{A}^2\hat{B} & \hat{C}\hat{A}\hat{B} & \hat{C}\hat{B} & 0 \\ \hat{C}\hat{A}^3\hat{B} & \hat{C}\hat{A}^2\hat{B} & \hat{C}\hat{A}\hat{B} & \hat{C}\hat{B} \end{bmatrix} \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} +$$

$$+ [AB, B] \begin{bmatrix} \hat{C}\hat{B} & 0 \\ \hat{C}\hat{A}\hat{B} & \hat{C}\hat{B} \end{bmatrix} \begin{bmatrix} CB & 0 \\ CAB & CB \end{bmatrix} \begin{bmatrix} \hat{C}\hat{B} & 0 \\ \hat{C}\hat{A}\hat{B} & \hat{C}\hat{B} \end{bmatrix} \begin{bmatrix} C \\ CA \end{bmatrix}$$

and in general

$$\bar{A}_{2,2}^k = A^k + \left\{ \text{Terms whose first factor is of the form } A^j B; \text{ for some } j \leq k-2 \right\}.$$

$$k=1, 2, \dots \quad \text{---(3.21)}$$

Thus if we define $\{d_i, B^*\}$ and $\{\bar{d}_i, \bar{B}^*\}$ as before, then substituting (3.21) in (3.20) we obtain

$$\bar{c}_i \bar{A}^k \bar{B} = 0 \quad \text{for } k=0, 1, \dots, d_i-1; i=1, \dots, m$$

and

$$\bar{c}_i \bar{A}^{d_i} \bar{B} = c_i A^{d_i} B \neq 0 \quad ; i=1, \dots, m$$

which imply that $\bar{d}_i = d_i$; $i=1, \dots, m$, and

$$\bar{B}^* = B^* \quad \text{--- (3.22)}$$

Therefore

Theorem 3.3. Let $\bar{S}$ be the output feedback connection of S and $\hat{S}$ as shown in fig.3.2 above. Then $\bar{S}$ is decouplable by state feedback if and only if S , the system in the feedforward path is decouplable.

Proof. The theorem follows directly from eq.(3.22).

A second feedback connection of two given systems is possible if $x(t)$, the state vector of S (instead of $y(t)$ the output vector) is used to drive $\hat{S}$. This will be called the state feedback connection to distinguish it from the previous case, and is represented in fig.3.3 below.

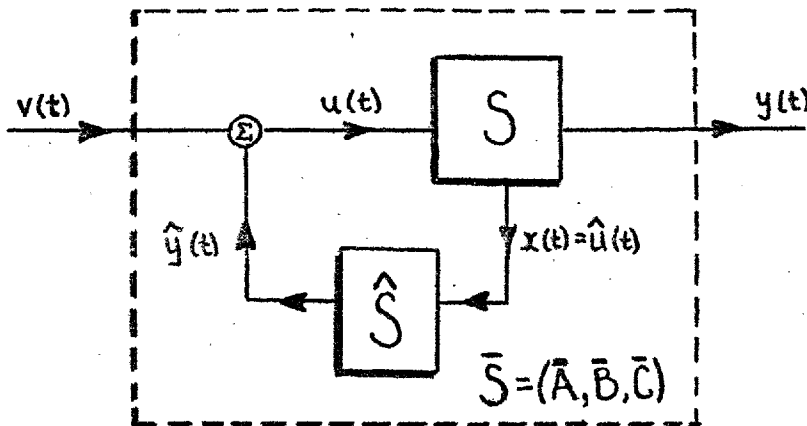


Fig.3.3. $\bar{S}=(A, B, C)$; the state feedback connection of S and $\hat{S}$.

To obtain $\bar{S}=(\bar{A},\bar{B},\bar{C})$, we set

$$\hat{u}(t) = x(t) \quad \text{_____} (3.23)$$

and

$$u(t) = \hat{y}(t) + v(t) \quad \text{_____} (3.24)$$

Remark 3.2. Notice that for this connection to be feasible, we require that $\hat{u}(t)$ is of dimension equal to n , and as $\hat{y}(t)$ must still be of dimension m , we will not be able to consider the one-to-one decoupling problem for $\hat{S}$. To obtain a feasible connection, maintaining the dimension of $\hat{u}(t)$ equal to m , one could considered using a linear map $K \in \mathbb{R}^{m \times n}$ between $x(t)$ and $\hat{u}(t)$, i.e.,

$$\hat{u}(t) = K x(t)$$

to connect the two systems S and $\hat{S}$ in the feedback path. However, this is not necessary. As will be shown below, the decouplability of the composite system is dominated by S and not $\hat{S}$, and thus, there is no need to consider the one-to-one decoupling problem for $\hat{S}$.

Using $\bar{x}(t)^T = [\hat{x}(t)^T, x(t)^T]$ again as the state vector of the composite system, we obtain

$$\dot{\bar{x}}(t) = \begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{x}(t) \end{bmatrix} = \bar{A}\bar{x}(t) + \bar{B}v(t) = \begin{bmatrix} \hat{A} & \hat{B} \\ B\hat{C} & A \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v(t) \quad \text{_____} (3.25)$$

and

$$y(t) = \bar{C}\bar{x}(t) = [0 \quad C] \begin{bmatrix} \hat{x}(t) \\ x(t) \end{bmatrix} \quad (3.26)$$

and with $\bar{A}^k$ partitioned as before,

$$\bar{c}_i \bar{A}^k \bar{B} = c_i \bar{A}_{2,2}^k B$$

where again, a straightforward but tedious algebraic manipulation shows that

$$\bar{A}_{2,2}^k = A^k + \left\{ \text{Terms whose first factor is } A^j B \text{ for} \right. \\ \left. \text{some } j \leq k-2 \right\} \quad ; k=1,2,\dots \quad (3.27)$$

and therefore

$$\bar{c}_i \bar{A}^k \bar{B} = 0 \quad \text{for } k=0,1,\dots,d_i-1 ; i=1,\dots,m$$

and

$$\bar{c}_i \bar{A}^{d_i} \bar{B} = c_i A^{d_i} B \neq 0 \quad ; i=1,\dots,m$$

which imply that $\bar{d}_i = d_i ; i=1,\dots,m$, and $\bar{B}^* = B^*$, and the following theorem follows.

Theorem 3.4. Let $\bar{S}$ be the state feedback connection of S and $\hat{S}$ (see fig.3.3 above). Then $\bar{S}$ is decouplable by state feedback if and only if S is decouplable by state feedback.

Theorems 3.3 and 3.4 show that in general, in contrast with the results of section 2 above, in the feedback connection the system in the feedforward

path completely determines the decouplability of the overall composite system.

It is a simple matter to verify that if the (state or output) feedback composite system $\bar{S}$ is decouplable, then a decoupling pair is

$$F = -[\hat{C}, B^{*-1}A^*]$$

and

$$G = B^{*-1}\Delta$$

where

$$A^* = \begin{bmatrix} c_1 A & d_{1+1} \\ \vdots & \\ c_m A & d_{m+1} \end{bmatrix}$$

and Δ is an arbitrary nonsingular diagonal matrix.

As we pointed out in the introduction of this chapter, theorems 3.3 and 3.4, if interpreted suitably, show clearly that dynamic feedback alone is not sufficient to solve the dynamic decoupling problem. It is the use of dynamic compensation in the feedforward path which transforms a nondecouplable system into a decouplable one.

3.4. Parallel connection.

In this last section of chapter III, we study the state feedback decoupling problem for systems connected in parallel. Let $S=(A,B,C)$ and $\hat{S}=(\hat{A},\hat{B},\hat{C})$ be described by eqs.(3.1)-(3.4) and let $\bar{S}=(\bar{A},\bar{B},\bar{C})$ be the composite system driven by $\bar{u}(t)=u(t)=\hat{u}(t)$, and whose output $\bar{y}(t)$ is

$$\bar{y}(t) = y(t) + \hat{y}(t) \quad (3.28)$$

i.e.,

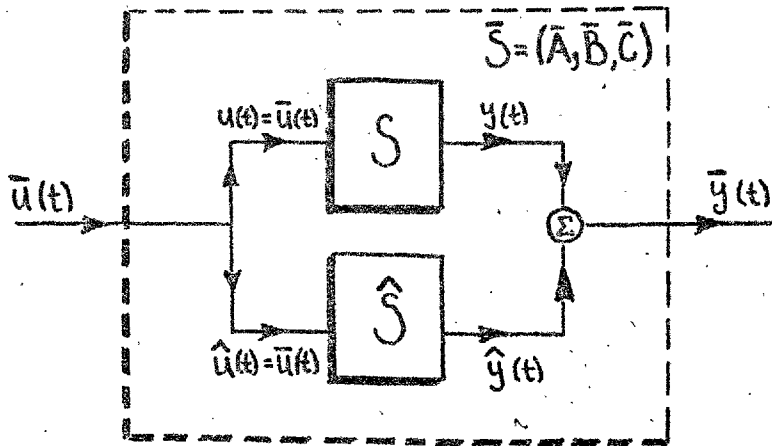


Fig.3.4. $\bar{S}=(\bar{A},\bar{B},\bar{C})$; the parallel connection of S and $\hat{S}$.

As in the previous sections, $\bar{x}(t)^T = [\hat{x}(t)^T, x(t)^T]$ qualifies as the state vector of $\bar{S}$. Substituting (3.28) in eqs.(3.1)-(3.4), and with $\bar{u}(t)=u(t)=\hat{u}(t)$, we obtain the following state space description of S

$$\dot{\bar{x}}(t) = \begin{bmatrix} \hat{x}(t) \\ x(t) \end{bmatrix} = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) = \begin{bmatrix} \hat{A} & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} \hat{B} \\ B \end{bmatrix} \bar{u}(t) \quad (3.29)$$

and

$$\bar{y}(t) = \bar{C} \bar{x}(t) = [\hat{C} \quad c] \begin{bmatrix} \hat{x}(t) \\ x(t) \end{bmatrix} \quad (3.30)$$

and from the form of $\bar{A}$ in eq.(3.29),

$$\bar{A}^k = \begin{bmatrix} \hat{A}^k & 0 \\ 0 & A^k \end{bmatrix}; k=1,2,\dots$$

Therefore

$$\bar{c}_i \bar{A}^k \bar{B} = \hat{c}_i \hat{A}^k \hat{B} + c_i A^k B \quad ; k=1,2,\dots \quad ; i=1,\dots,m \quad (3.31)$$

If $\{d_i, B^*\}$, $\{\hat{d}_i, \hat{B}^*\}$ and $\{\bar{d}_i, \bar{B}^*\}$ are defined as usual, we have

Lemma 3.2. Let $\bar{S}$ be the parallel connection of S and $\hat{S}$. Then

$$\bar{d}_i \geq \min\{d_i, \hat{d}_i\} \quad ; i=1,\dots,m \quad (3.32)$$

Further, the inequality in (3.32) holds if and only if

$$(i) \quad d_i = \hat{d}_i \quad \text{and}$$

$$(ii) \quad c_i A^{d_i} B = -\hat{c}_i \hat{A}^{\hat{d}_i} \hat{B}$$

Proof. From eq.(3.31) and the definition of d_i and $\hat{d}_i$,

$$\bar{c}_i \bar{A}^k \bar{B} = 0 \quad \text{for } k=0,1,\dots,\min\{d_i, \hat{d}_i\}-1; i=1,\dots,m.$$

and (3.32) follows. Now, if $d_i \neq \hat{d}_i$, and we let

$r_i = \min \{d_i, \hat{d}_i\}$, then from (3.31)

$$\bar{c}_i \bar{A}^{r_i} \bar{B} = \begin{cases} c_i A^{d_i} B \neq 0 & \text{if } r_i = d_i, \text{ and} \\ \hat{c}_i \hat{A}^{\hat{d}_i} \hat{B} \neq 0 & \text{if } r_i = \hat{d}_i \end{cases}$$

which implies that equality holds in (3.32). Thus, assume $d_i = \hat{d}_i$. From eq. (3.31), $\bar{c}_i \bar{A}^{d_i} \bar{B}$ will be zero, if and only if

$$c_i A^{d_i} B = -\hat{c}_i \hat{A}^{d_i} \hat{B}$$

and the second part of the lemma follows.

Define now the following sets of integers.

$$J = \{i: d_i < \hat{d}_i; i=1, \dots, m\}$$

$$\hat{J} = \{i: d_i > \hat{d}_i; i=1, \dots, m\}$$

and $\tilde{J} = \{i: d_i = \hat{d}_i; i=1, \dots, m\}$

and let

$$\tilde{B}_i^* = c_i A^{d_i + \eta_i} B + \hat{c}_i \hat{A}^{d_i + \eta_i} \hat{B} \quad \forall i \in \tilde{J}$$

where

$$\eta_i = \min \left\{ j: c_i A^{d_i + j} B \neq -\hat{c}_i \hat{A}^{d_i + j} \hat{B} \right\}; i \in \tilde{J}$$

Then,

Theorem 3.5. Let $\bar{S}$ be the parallel connection of S and $\hat{S}$. Then $\bar{S}$ will be decouplable by state-feedback

if and only if

(i).- $\{B_i^*; i \in J\} = \{R_1\}$; $\{\hat{B}_i^*; i \in \hat{J}\} = \{R_2\}$ and $\{\tilde{B}_i^*; i \in \tilde{J}\} = \{R_3\}$ are sets of linearly independent vectors and

(ii).- $Q_j \cap \sum_{i \neq j} Q_i = 0$ $i, j=1,2,3$.

where $Q_i = \text{span}\{R_i\}$

Proof. The basis of our result is again the main theorem of [5], i.e. $\bar{S}$ is decouplable if and only if $\bar{B}^*$ is nonsingular. From eq. (3.31) and the definition of $J, \hat{J}$ and $\tilde{J}$, it is clear that $\bar{B}_i^* = B_i^* \forall i \in J$; $\bar{B}_i^* = \hat{B}_i^* \forall i \in \hat{J}$ and $\bar{B}_i^* = \tilde{B}_i^* \forall i \in \tilde{J}$. Thus

$$\bar{B}^* = P \begin{bmatrix} \{B_i^*; i \in J\} \\ \{\hat{B}_i^*; i \in \hat{J}\} \\ \{\tilde{B}_i^*; i \in \tilde{J}\} \end{bmatrix} = PR$$

where P is a permutation matrix, and

$$\text{rank } \{\bar{B}^*\} = \text{rank } \{R\}$$

Now, if (i) is not satisfied, $\text{rank } \{R\} < m$, and $\bar{S}$ will not be decouplable. So, assume (i) holds. Then,

$$\text{span} \{ \{R_1\}, \{R_2\}, \{R_3\} \} = \mathbb{R}^m$$

if and only if (ii) holds.

i.e., $\text{rank}[\bar{B}^*] = m$ if and only if (ii) is satisfied and theorem 3.5 follows.

Corollary 3.5.1. Let $\bar{S}$ be the parallel connection of S and $\hat{S}$. Then $\bar{S}$ will be decouplable by state feedback only if

(i) $\text{rank}[B] \geq p$, where p is the number of elements in J .

(ii) $\text{rank}[\hat{B}] \geq q$, where q is the number of elements in $\hat{J}$.

(iii) $\text{rank} \begin{bmatrix} \hat{B} \\ B \end{bmatrix} = m$

(iv) $\text{rank}[\hat{C}, C] = m$

Proof. Conditions (i) and (ii) are obvious from theorem 3.5 (condition (i)). Conditions (iii) and (iv) follow from the fact that the transfer function matrix of the overall system $\bar{S}$ must be nonsingular if $\bar{S}$ is to be decouplable, which implies $\text{rank}[\bar{C}] = \text{rank}[\bar{B}] = m$. [45].

From theorem 3.5, it is evident that the relationship between the decoupling properties of $\bar{S}$ and those of S and $\hat{S}$ are not straightforward in general. In this sense, the situation is somewhat similar to that of the cascade connection, i.e.,

Proposition 3.3. Let $\bar{S}$ be the parallel connection of S and $\hat{S}$. If S and $\hat{S}$ are both decouplable by state feedback, it is not necessarily true that $\bar{S}$ is decouplable by state feedback.

Proof. Let $S=(A,B,C)$ be such that $d_1=1$, $d_2=2$, and

$$B^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\hat{S}=(\hat{A},\hat{B},\hat{C})$ be such that $\hat{d}_1=2$, $\hat{d}_2=1$, and

$$\hat{B}^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

That such systems can be found is clear, because $\{d_i, B^*\}$ is an independent set of invariants (section 2, chapter I). S and $\hat{S}$ are both decouplable, but $\bar{S}$, their parallel connection is not because a simple calculation shows that $\bar{d}_1=1$, $\bar{d}_2=1$ and

$$\bar{B}^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

i.e., $\det[\bar{B}^*]=0$.

Proposition 3.4. Let $\bar{S}$ be the parallel connection of S and $\hat{S}$. If neither S nor $\hat{S}$ are decouplable it does not necessarily follow that $\bar{S}$ is not decouplable.

Proof. Consider $S=(A,B,C)$ and $\hat{S}=(\hat{A},\hat{B},\hat{C})$ such that

$$d_1=0, d_2=2, \text{ and } B^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

and

$$\hat{d}_1=1, \hat{d}_2=2 \text{ and } \hat{B}^* = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Neither S nor $\hat{S}$ are decouplable. Yet, $\bar{S}=(\bar{A}, \bar{B}, \bar{C})$ will have the following set of invariants $\{\bar{d}_i, \bar{B}^*\}$

$$\bar{d}_1=0, \bar{d}_2=2 \text{ and } \bar{B}^* = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and as $\det[\bar{B}^*] = 1 \neq 0$, then $\bar{S}$ is decouplable.

Although the above propositions show that the situation is similar to that encountered in the cascade connection, for the parallel connection there are two obvious cases when the decouplability of $\bar{S}$ is "dominated" by one of the subsystems. These are;

(a). When $d_i < \hat{d}_i \forall i=1, \dots, m$. In this case $\bar{B}^*=B^*$ and $\bar{S}$ is decouplable if and only if S is, and

(b). When $d_i > \hat{d}_i \forall i=1, \dots, m$, where $\bar{B}^*=\hat{B}^*$ and $\bar{S}$ is decouplable if and only if $\hat{S}$ is decouplable.

As we mentioned in section 1, the results in this chapter can be easily extended to systems having a more general output equation, i.e.,

$$y(t) = Cx(t) + Du(t)$$

To do so, one only needs to modify the definition

of d_i ; $i=1, \dots, m$ (and consequently of $\hat{d}_i$ and $\bar{d}_i$), as

$$d_i = \begin{cases} 0 & \text{if } D_i \neq 0 \\ \min\{j: c_i A^{j-1} B \neq 0 ; j=1, 2, \dots, n-1\}, & \text{if } D_i = 0 \end{cases}$$

for $i=1, \dots, m$, where D_i denotes the i^{th} row of D . (Not to be confused with D_i of chapter II (eq.(2.6)), the i^{th} row of B^*).

CHAPTER IV. Decoupling and parameter variations.

1). Introduction.

In spite of the great amount of research which has been done on the noninteracting control problem, very little is known about the effect parameter variations have on the decoupling properties of a given system. Yet, this is of great practical importance, because, in general, the parameters of most systems suffer small variations due, for example, to varying environmental conditions, ageing, etc. Moreover, in practice, the parameters of a given system are known only approximately (except for some parameters which may be considered "natural" zeros or ones, and one may reach a wrong conclusion regarding the possibility of effecting non-interacting control. It is then very important to be able to say under which parameter variations or estimation errors, the decoupling properties of a system remain invariant.

Let us consider a given linear time-invariant system $S = (A, B, C)$, which we will call the "nominal" system, described by the set of differential equations

$$\dot{x}(t) = A x(t) + B u(t) \quad (4.1)$$

$$y(t) = C x(t) \quad (4.2)$$

where $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m$ and A, B , and C are constant matrices of appropriate dimensions.

If the parameters of the nominal system vary, i.e., if $A \mapsto A + \delta A$, $B \mapsto B + \delta B$, and $C \mapsto C + \delta C$, then $S \mapsto S_\delta = (A + \delta A, B + \delta B, C + \delta C)$, where S_δ the "perturbed" system is described by

$$\dot{\hat{x}}(t) = (A + \delta A) \hat{x}(t) + (B + \delta B) u(t) \quad (4.3)$$

$$\hat{y}(t) = (C + \delta C) \hat{x}(t) \quad (4.4)$$

Do the nominal and the perturbed systems have the same decoupling properties? In what follows, we will consider two different problems:

- (i) The decoupling and data sensitivity problem [16], or total-input-output invariance problem [58], which can be stated as: Find the class of admissible parameter variations such that there exists a state-feedback pair (F, G) , (i.e. a control law as given by eq. (1.4)), such that
 - (a).- (F, G) decouples $S = (A, B, C)$ and
 - (b).- The outputs of $S(F, G) = (A + BF, BG, C)$ and $S_\delta(F, G) = [(A + \delta A) + (B + \delta B)F, (B + \delta B)G, C + \delta C]$ are equal for

all closed-loop inputs $v(t) \in \mathbb{R}^m$, and all initial conditions $x_0 = x(0) = \hat{x}(0)$, for all times $t \geq 0$.

(ii). The input-output invariance problem [58]. That is, under what parameter variations δA , δB and δC does $S_\delta = ((A + \delta A), (B + \delta B), (C + \delta C))$ remain decouplable.

Of these two problems, the decoupling and data sensitivity problem is perhaps of greater practical importance, because if there exists a decoupling feedback pair satisfying (i).(a) and (i).(b), it means that the closed-loop system is insensitive to parameter variations and retains noninteraction, which are highly desirable properties. However, the input-output invariance problem is also of importance when decoupling is to be included as part of the design requirements.

The results presented below for the decoupling and data sensitivity problem are equivalent to those of [16], but the approach used here is simpler and more direct. Further, in practice, it is easier to calculate the subspace characterizing the class of perturbations for which the problem has a solution using theorem 4.1 below, than it is using the beautiful characterization given in [16], and our result may be readily interpreted intuitively.

The results for the input-output invariance problem extend those in [58]. Moreover, we also consider possible parameter variations in A, the plant of the system. Some of the results are given in a way that resemble those of [47] on the robustness of controllers and it is in this sense that we may consider the input-output invariance problem as that of finding conditions for the existence of "robust decoupling" configurations.

2). Decoupling and data sensitivity.

In this section, we will assume $\delta C = 0$, i.e., we will only consider parameter variations in A, the plant of the system and B, the input map. This might seem restrictive, but in a large number of practical situations the outputs of the system are only a subset of the states. Thus, the parameters of C, which are then only ones and zeros, appear "naturally", and one does not expect variations. Furthermore, if $\delta C \neq 0$, then clearly condition (i).(b) above in the statement of the problem, cannot hold, and the decoupling and data sensitivity problem has no solution. (Notice that if $\delta C \neq 0$, then $y(0) = C x(0)$ and $\hat{y}(0) = (C + \delta C) \hat{x}(0)$ will not be identical for all possible initial conditions $x(0) = \hat{x}(0) \in \mathbb{R}^n$).

Let us define $\{d_i; i=1, \dots, m\}$, the set of decoupling invariants of $S=(A,B,C)$ as usual, (see eq. (1.5)), and, with the i^{th} row of C denoted c_i , define

$$W_i(o) = \begin{bmatrix} c_i \\ \vdots \\ c_i A^{d_i-1} \\ c_i A^{d_i} \\ \vdots \\ c_i A^{d_i} \end{bmatrix} \quad (4.5)$$

and

$$W(o) = \begin{bmatrix} W_1(o) \\ \vdots \\ W_m(o) \end{bmatrix} \quad (4.6)$$

Also let B^* be defined as in eq. (1.6), and

$$A^* = \begin{bmatrix} & & d_1+1 \\ c_1 A & & \\ \vdots & & \\ & d_m+1 \\ c_m A & & \end{bmatrix} \quad (4.7)$$

and let $\hat{d}_i; i=1, \dots, m; \hat{B}^*$ and $\hat{A}^*$ be the corresponding matrices for the perturbed system. $S_\delta = (A+\delta A, B+\delta B, C)$.

By taking successive derivatives of $y_i(t)$, the i^{th} component of $y(t)$ in eq. (4.2), it is straightforward to obtain, for the closed loop system $S(F,G)$

$$y_i^{(j)}(t) = c_i A^j x(t) ; j=0,1,\dots,d_i ; i=1,\dots,m \quad (4.8)$$

and

$$y_i^{(d_i+1)}(t) = (c_i A^{d_i+1} + c_i A^{d_i} B F) x(t) + c_i A^{d_i} B G v(t); i=1, \dots, m \quad (4.9)$$

where $y_i^{(k)}(t)$, denotes the k^{th} derivative of $y_i(t)$.

Similarly, for $\hat{y}_i(t)$, the i^{th} component of $\hat{y}(t)$ in eq. (4.4), we obtain, for the closed loop system $S_0(F, G)$

$$\hat{y}_i^{(j)}(t) = c_i (A + \delta A)^j \hat{x}(t); j=0, 1, \dots, \hat{d}_i; i=1, \dots, m \quad (4.10)$$

and

$$\begin{aligned} \hat{y}_i^{(\hat{d}_i+1)}(t) &= [c_i (A + \delta A)^{\hat{d}_i+1} + c_i (A + \delta A)^{\hat{d}_i} (B + \delta B) F] \hat{x}(t) \\ &+ c_i (A + \delta A)^{\hat{d}_i} (B + \delta B) G v(t); i=1, \dots, m \quad (4.11) \end{aligned}$$

Now if we denote the image of a map by $\text{Im}(\cdot)$ and the null space by $\mathcal{N}(\cdot)$, we have

Theorem 4.1. Let $S=(A, B, C)$ be given by eqs. (4.1), (4.2).

The decoupling and data sensitivity problem has a solution if and only if

- (1). B^* is nonsingular, and
- (2). $\text{Im}(\delta A) + \text{Im}(\delta B) \subset \mathcal{N}[W^{(0)}]$

Proof. Condition (1) above, simply ensures that there exists a feedback pair (F, G) which decouples $S=(A, B, C)$, [17], [21]. Thus, only (i). (b) in the statement of the

problem (section 1) remains to be satisfied.

(If). Assume that $\text{Im}(\delta A) + \text{Im}(\delta B) \subset \mathcal{N}[W^{(0)}]$.
then, it is clear that

$$c_i A^j = c_i (A + \delta A)^j \quad ; j=0,1,\dots,d_i+1 \quad ; i=1,\dots,m \quad (4.12)$$

In fact, it is very easy to show eq.(4.12) holds
if and only if $\text{Im}(\delta A) \subset \mathcal{N}[W^{(0)}]$. This means that

$$c_i (A + \delta A)^j (B + \delta B) = c_i A^j (B + \delta B) = c_i A^j B + c_i A^j \delta B$$

$$j=0,1,\dots,d_i \quad ; i=1,\dots,m \quad (4.13)$$

and as $\text{Im}(\delta B) \subset \mathcal{N}[W^{(0)}]$, then

$$c_i (A + \delta A)^j (B + \delta B) = c_i A^j B \quad ; j=0,1,\dots,d_i \quad ; i=1,\dots,m$$

which implies

$$\hat{d}_i = d_i \quad ; i=1,\dots,m \quad (4.14)$$

and

$$\hat{B}^* = B^* \quad (4.15)$$

and from eqs.(4.12),(4.14) and (4.7),

$$\hat{A}^* = A^* \quad (4.16)$$

Define now

$$F = -B^{*-1} A^*$$

and

$$G = B^{*-1} \Delta$$

with $\Delta = \text{diag}\{\lambda_i\}$, nonsingular. With (F,G) so defined
the feedback law

$$u(t) = Fx(t) + Gv(t)$$

integrator decouples $S=(A,B,C)$, (see remark 2.4), and thus for the closed-loop system $S(F,G)$,

$$y_i^{(d_i+1)}(t) = \lambda_i v_i(t) \quad ; i=1, \dots, m \quad (4.17)$$

From eqs.(4.14)-(4.16), the same feedback pair, i.e.

$$u(t) = F\hat{x}(t) + Gv(t)$$

is also an integrator decoupling feedback law for the perturbed system $S_\delta=(A+\delta A, B+\delta B, C)$, and the output of the closed-loop system $S_\delta(F,G)$ is given by

$$\hat{y}_i^{(d_i+1)}(t) = \lambda_i v_i(t) \quad ; i=1, \dots, m \quad (4.18)$$

Hence, from eqs.(4.17) and (4.18)

$$y_i^{(d_i+1)}(t) = \hat{y}_i^{(d_i+1)}(t) \quad ; i=1, \dots, m, \quad \forall t \geq 0 \quad (4.19)$$

for every $x_0 = \hat{x}_0 = x(0)$ and all $v(t) \in \mathbb{R}^m$. Furthermore, differentiating (4.19)

$$y_i^{(d_i+k)}(t) = \hat{y}_i^{(d_i+k)}(t) \quad ; i=1, \dots, m \quad ; k=1, 2, \dots; \quad \forall t \geq 0 \quad (4.20)$$

for all initial conditions and closed-loop inputs.

Define now $y_i^{(j)}(t) = z_{i,j}(t)$ and $\hat{y}_i^{(j)}(t) = \hat{z}_{i,j}(t)$; $i=1, \dots, m$; $j=0, 1, \dots, d_i+1$. Then,

$$\dot{z}_i(t) = \begin{bmatrix} \dot{z}_{i,0}(t) \\ \dot{z}_{i,1}(t) \\ \vdots \\ \dot{z}_{i,d_i}(t) \\ \dot{z}_{i,d_i+1}(t) \end{bmatrix} = Mz_i(t) + Nv_i(t) =$$

$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} z_{i,0}(t) \\ z_{i,1}(t) \\ \vdots \\ z_{i,d_i}(t) \\ z_{i,d_i+1}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda_i \end{bmatrix} v_i(t) \quad (4.21)$$

and

$$\dot{\hat{z}}_i(t) = \begin{bmatrix} \dot{\hat{z}}_{i,0}(t) \\ \dot{\hat{z}}_{i,1}(t) \\ \vdots \\ \dot{\hat{z}}_{i,d_i}(t) \\ \dot{\hat{z}}_{i,d_i+1}(t) \end{bmatrix} = M\hat{z}_i(t) + Nv_i(t) \quad (4.22)$$

with M and N as in eq.(4.21) above. Comparing these two sets of differential equations it is evident that $z_i(t) = \hat{z}_i(t)$; $i=1, \dots, m$, for all $t \geq 0$ and all $v_i(t)$; $i=1, \dots, m$, if and only if their initial conditions are identical, i.e., if and only if $z_i(0) = \hat{z}_i(0)$, or

equivalently,

$$y_i^{(j)}(0) = \hat{y}_i^{(j)}(0) \quad ; i=1, \dots, m \quad ; j=0, 1, \dots, d_i+1 \quad (4.23)$$

With $t=0$ in eq.(4.19), it follows that (4.23) is satisfied for $i=1, \dots, m$, and $j=d_i+1$. From eqs.(4.8), (4.10) and (4.12), and $x(0) = \hat{x}(0)$, eq.(4.23) follows for $j=0, 1, \dots, d_i$; $i=1, \dots, m$. Therefore, the closed-loop outputs of the systems $S(F, G)$ and $S_{\hat{G}}(F, G)$ are identical for all $t \geq 0$, all initial conditions x_0 , and all inputs $v(t)$, and sufficiency follows.

(Only if). To prove the necessity of condition (2) of theorem 4.1, we first notice that in order to satisfy (i).(b) in the statement of the problem, it is necessary that

$$\hat{d}_i = d_i \quad ; i=1, \dots, m \quad (4.24)$$

If this is not so and, without loss of generality say, $d_i > \hat{d}_i$ for some i , then, assuming zero initial conditions we obtain from eq.(4.8)

$$y_i^{(\hat{d}_i+1)}(0) = 0 \quad (4.25)$$

whereas from eq.(4.11)

$$\hat{y}_i^{(\hat{d}_i+1)}(0) = c_i(A+\delta A)^{\hat{d}_i}(B+\delta B)G v(0)$$

and as by definition of $\hat{d}_i$, $c_i(A+\delta A)^{\hat{d}_i}(B+\delta B) \neq 0$, and

G is nonsingular, then

$$\hat{y}_i^{(\hat{d}_i+1)}(0) \neq 0 \text{ for some } v(0) \quad (4.26)$$

and eqs.(4.25) and (4.26) violate (i).(b). Thus, (4.24) must hold. Comparing now eqs.(4.8) and (4.10), it is also obvious that for (i).(b) to be satisfied for all initial conditions $x_0 = \hat{x}_0 = x(0)$, it is necessary that

$$c_i A^j = c_i (A + \delta A)^j ; j=0,1,\dots,d_i ; i=1,\dots,m \quad (4.27)$$

Furthermore, from eqs.(4.9) and (4.11), assuming $x_0 = \hat{x}_0 = 0$, it is also required that

$$\hat{B}^* = B^* \quad (4.28)$$

and again from eqs.(4.9) and (4.11), but now assuming $v(0)=0$,

$$\hat{A}^* = A^* \quad (4.29)$$

Clearly, the necessary conditions (4.27) and (4.29) hold if and only if $\text{Im}(\delta A) \subset \mathcal{N}[W^{(0)}]$, and the necessary conditions expressed by (4.24) and (4.28), imply $\text{Im}(\delta B) \subset \mathcal{N}[W^{(0)}]$. Thus, to satisfy (i).(b) it is necessary that

$$\text{Im}(\delta B) + \text{Im}(\delta A) \subset \mathcal{N}[W^{(0)}]$$

and theorem 4.1 follows.

For the class of systems Ω , defined in chapter II, (i.e., $\Omega = \{S: S \text{ is decouplable and } m + \sum_{i=1}^m d_i = n\}$), the following corollary holds

Corollary 4.1.1. The decoupling and data sensitivity problem has no solution for the class of decouplable linear systems which preserve observability under arbitrary state feedback.

Proof. As we have shown in chapter II, section 3, Ω , the class of decouplable linear systems which preserve observability under arbitrary state feedback, is characterized by the equality $m + \sum_{i=1}^m d_i = n$. Further, the rows of $W^{(0)}$ are linearly independent vectors [17], and therefore $\mathcal{N}[W^{(0)}] = 0$ and any non-trivial variation (i.e., $\neq 0$) in A or B violates condition (2) of theorem 4.1.

This means that although the systems belonging to Ω have the very desirable property of preserving observability, they are extremely sensitive to any parameter variations.

The class of systems less sensitive to parameter variations is, on the other hand, characterized by $d_i = 0; i=1, \dots, m$. For these systems the variations for which a solution to the decoupling and data sensitivity problem exists, must lie in the $(n-m)$ -

dimensional subspace $\mathcal{N}[C]$.

Let us now establish the equivalence between theorem 4.1 above and theorem 7 of [16], obtained by Fabian and Wonham using a geometric approach, i.e., the decoupling and data sensitivity problem has a solution if and only if

- (i) $S=(A,B,C)$ is decouplable, and
- (ii) $\text{Im}(\delta A) + \text{Im}(\delta B) \subset \gamma^{(0)}$, where

$$\gamma^{(0)} = \sup \{ \gamma / \gamma \subset A^{-1}(\gamma + \mathcal{B}) \cap \mathcal{N}[C] \} \quad (4.30)$$

where $A^{-1}(\gamma) = \{ z : Az \in \gamma \}$, $\mathcal{B} = \text{Im}(B)$ and $\sup\{.\}$ denotes the maximal subspace with respect to the partial ordering introduced by inclusion.

All we need to establish the equivalence is to show that $\gamma^{(0)} = \mathcal{N}[W^{(0)}]$. For this, we require the following preliminary results. Define

$$W_i^{(-j)} = \begin{bmatrix} c_i \\ \vdots \\ d_i^{(-j)} \\ c_i A \end{bmatrix} ; i=1, \dots, m \quad (4.31)$$

where $W_i^{(-j)} = 0$ whenever $d_i < j$, and let

$$W^{(-j)} = \begin{bmatrix} W_1^{(-j)} \\ \vdots \\ W_m^{(-j)} \end{bmatrix} \quad (4.32)$$

then,

Lemma 4.1. Let $\mathcal{R}$ and $\mathcal{P}$ be subspaces of $\mathbb{R}^n$. Let $\mathcal{R} \subset \mathcal{P}$. Then $\mathcal{R} = \mathcal{P}$ if and only if $\dim \mathcal{R} = \dim \mathcal{P}$.

The proof of this is obvious and well known, and is omitted here.

Lemma 4.2. Let $S=(A,B,C)$ be decouplable. Then,

- (i) $\mathcal{B} \subseteq \mathcal{N}[W^{(-1)}]$
- (ii) Equality holds in (i) if and only if $m + \sum_{i=1}^m d_i = n$
- (iii) $\mathcal{B} \cap \mathcal{N}[W^{(0)}] = \{0\}$.

Proof. To show (i) assume $x \in \mathcal{B}$, then there exists $y \in \mathbb{R}^m$ such that $By=x$. Thus $W^{(-1)}x = W^{(-1)}By = 0$, i.e., $x \in \mathcal{N}[W^{(-1)}]$, and $\mathcal{B} \subset \mathcal{N}[W^{(-1)}]$. To show (ii), notice that $S=(A,B,C)$ decouplable means $\text{rank}[B] = m$, [45], i.e., $\dim \mathcal{B} = m$. Also, the rows of $W^{(-1)}$ are linearly independent if S is decouplable, i.e.,

$\dim \mathcal{N}[W^{(-1)}] = n - \sum_{i=1}^m d_i$, and as $m + \sum_{i=1}^m d_i \leq n$ ([17], lemma), then, $\dim \mathcal{N}[W^{(-1)}] \geq m$. i.e., $\dim \mathcal{N}[W^{(-1)}] \geq \dim \mathcal{B}$, with equality holding if and only if $m + \sum_{i=1}^m d_i = n$, (ii) follows from lemma 4.1.

To prove (iii) let $x \neq 0 \in \mathcal{B}$. Then, $By=x$ for some $y \neq 0 \in \mathbb{R}^m$ and

$$W^{(0)}x = W^{(0)}By = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_1^* \\ \vdots \\ 0 \\ \vdots \\ 0 \\ B_m^* \end{bmatrix} y$$

which is different from zero because $S=(A,B,C)$ is decouplable and B^* is therefore a nonsingular matrix. Thus $x \in \mathcal{N}[W^{(0)}]$ and (iii) follows.

Lemma 4.3. Let $S=(A,B,C)$ be decouplable. Then,

$$\mathcal{B} \oplus \mathcal{N}[W^{(0)}] = \mathcal{N}[W^{(-1)}] \quad (4.33)$$

Proof. Let $z \in \mathcal{B} \oplus \mathcal{N}[W^{(0)}]$. Then there exist (unique) vectors $x \in \mathcal{B}$ and $y \in \mathcal{N}[W^{(0)}]$ such that $z = x + y$.

So,

$$W^{(-1)}z = W^{(-1)}x + W^{(-1)}y$$

but as $y \in \mathcal{N}[W^{(0)}]$, then clearly $y \in \mathcal{N}[W^{(-1)}]$ and as $x \in \mathcal{B}$, from lemma 4.2, (i), $x \in \mathcal{N}[W^{(-1)}]$. Then, $W^{(-1)}z=0$ i.e., $z \in \mathcal{N}[W^{(-1)}]$ and

$$\mathcal{B} \oplus \mathcal{N}[W^{(0)}] \subset \mathcal{N}[W^{(-1)}] \quad (4.34)$$

If S is decouplable, then $\dim \mathcal{B} = m$, and

$$\dim \mathcal{N}[W^{(0)}] = n - (m + \sum_{i=1}^m d_i) ; \dim \mathcal{N}[W^{(-1)}] = n - \sum_{i=1}^m d_i$$

As it is well known that $\dim(\mathcal{X} + \mathcal{Y}) = \dim \mathcal{X} + \dim \mathcal{Y}$ for any two disjoint subspaces $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^n$, then

$$\dim\{\mathcal{B} \oplus \mathcal{N}[W^{(0)}]\} = m+n-(m+\sum_{i=1}^m d_i) = \dim \mathcal{N}[W^{(-1)}]$$

and from eq.(4.34) and lemma 4.1, lemma 4.3 follows.

We can now establish

Lemma 4.4. Let $S=(A,B,C)$ be decouplable. Then

$$\mathcal{N}[W^{(0)}] = A^{-1}(\mathcal{N}[W^{(0)}] + \mathcal{B}) \cap \mathcal{N}[C] \quad (4.35)$$

Proof. Let $z \in \mathcal{N}[W^{(0)}]$. Then

$$Cz = 0 \quad (4.36)$$

and

$$W^{(-1)}Az = 0$$

Thus, $Az \in \mathcal{N}[W^{(-1)}]$, and from lemma 4.3, $Az \in \mathcal{B} + \mathcal{N}[W^{(0)}]$.

This together with eq.(4.36) implies that

$$z \in A^{-1}(\mathcal{B} + \mathcal{N}[W^{(0)}]) \cap \mathcal{N}[C]. \text{ i.e.,}$$

$$\mathcal{N}[W^{(0)}] \subset A^{-1}(\mathcal{N}[W^{(0)}] + \mathcal{B}) \cap \mathcal{N}[C] \quad (4.37)$$

Now, let $z \in A^{-1}(\mathcal{N}[W^{(0)}] + \mathcal{B}) \cap \mathcal{N}[C]$. Then $Cz=0$

and $Az \in \mathcal{B} + \mathcal{N}[W^{(0)}]$, which from lemma 4.3 implies

$Az \in \mathcal{N}[W^{(-1)}]$. This together with (4.36) imply

$z \in \mathcal{N}[W^{(0)}]$, i.e.,

$$A^{-1}(\mathcal{N}[W^{(0)}] + \mathcal{B}) \cap \mathcal{N}[C] \subset \mathcal{N}[W^{(0)}] \quad (4.38)$$

and the lemma follows from eqs.(4.37) and (4.38).

The above lemma 4.4, shows that $\mathcal{N}[W^{(0)}]$ is a member of the class of subspaces satisfying $\mathcal{V} \subset A^{-1}(\mathcal{V} + \mathcal{B}) \cap \mathcal{N}[C]$. Lemma 4.5 below shows that it is the maximal subspace belonging to this class and thus, that $\mathcal{V}^{(0)} = \mathcal{N}[W^{(0)}]$.

Let $\mathcal{P}_j$ be any subspace independent of $\mathcal{N}[W^{(-j+1)}]$ such that $\mathcal{P}_j \subset \mathcal{N}[W^{(-j)}]$, and define $\mathcal{V}_{(-j)} = \mathcal{N}[W^{(-j+1)}] + \mathcal{P}_j$. Then

Lemma 4.5. Let $S=(A,B,C)$ be decouplable. Then

$$A^{-1}(\mathcal{V}_{(-j)} + \mathcal{B}) \cap \mathcal{N}[C] \subset \mathcal{V}_{(-j)} \quad \text{for } j=1,2,\dots \quad (4.39)$$

Proof. From lemma 4.2, (i), $\mathcal{B} \subset \mathcal{N}[W^{(-1)}]$. Then, obviously, $\mathcal{B} \subset \mathcal{N}[W^{(-j)}]$, $j=1,2,\dots$, so that

$$(\mathcal{V}_{(-j)} + \mathcal{B}) = (\mathcal{N}[W^{(-j+1)}] + \mathcal{P}_j + \mathcal{B}) \subseteq \mathcal{N}[W^{(-j)}]$$

This implies that

$$A^{-1}(\mathcal{V}_{(-j)} + \mathcal{B}) \subseteq A^{-1}(\mathcal{N}[W^{(-j)}])$$

and

$$A^{-1}(\mathcal{V}_{(-j)} + \mathcal{B}) \cap \mathcal{N}[C] \subseteq A^{-1}(\mathcal{N}[W^{(-j)}]) \cap \mathcal{N}[C]$$

But clearly,

$$A^{-1}(\mathcal{N}[W^{(-j)}]) \cap \mathcal{N}[C] \subseteq \mathcal{N}[W^{(-j+1)}]$$

thus,

$$A^{-1}(\mathcal{V}_{(-j)} + \mathcal{B}) \cap \mathcal{N}[C] \subseteq \mathcal{N}[W^{(-j+1)}]$$

and as by definition $\mathcal{N}[W^{(-j+1)}] \subset \mathcal{V}_{(-j)}$, then

$$A^{-1}(\mathcal{V}_{(-j)} + \mathcal{B}) \cap \mathcal{N}[C] \subset \mathcal{V}_{(-j)}$$

and lemma 4.5 follows.

As $\mathcal{N}[W^{(0)}] \subset \mathcal{V}_{(-j)}$, and $\mathcal{V}_{(-j)}$ is not a member of the class of subspaces $\mathcal{V} \subset A^{-1}(\mathcal{V} + \mathcal{B}) \cap \mathcal{N}[C]$, for any $\mathcal{P}_j$ and $j=1,2,\dots$, but $\mathcal{N}[W^{(0)}]$ does belong to this class, then $\mathcal{N}[W^{(0)}] = \mathcal{V}^{(0)}$.

From the intuitive point of view, the meaning of $\text{Im}(\delta A) + \text{Im}(\delta B) \subset \mathcal{N}[W^{(0)}]$ is more clear than condition (ii) above eq.(4.30). Roughly speaking, it requires the parameter variations to be unobservable from the output of the closed-loop decoupled system. This is not so clear if one uses $\mathcal{V}^{(0)}$

3. Input-output invariance problem.

In this section we shall be concerned with finding under which parameter variations a given decouplable system will remain decouplable.

Let us first consider variations in the input map only, i.e., $S=(A,B,C) \mapsto S_{\delta B}=(A,B+\delta B,C)$. Then, we can prove

Theorem 4.2. Let S be decouplable. If $\text{Im}(\delta B) \subset \mathcal{B}$, i.e., $\delta B = BY$ for some (uniquely determined) $Y \in \mathbb{R}^{m \times m}$, then the system $S_{\delta B}=(A,B+\delta B,C)$ is decouplable if and only if $(I + Y)$ is a nonsingular matrix.

Proof. (Only if). It is well known [45] and easy to establish that a necessary condition for $S_{\delta B}$ to be decouplable is $\text{rank}(B+\delta B) = m$. Thus, if $\text{Im}(\delta B) \subset \mathcal{B}$ this condition can be expressed as $\text{rank}[B(I+Y)] = m$, which applying Sylvester's inequality, holds if and only if $(I+Y)$ is nonsingular. (Recall that S is decouplable and therefore $\text{rank}[B] = m$).

(If). $\text{Im}(\delta B) \subset \mathcal{B}$ implies, from lemma 4.2, (i), that $\text{Im}(\delta B) \subset \mathcal{N}[W^{(-1)}]$, and this means that $\hat{d}_i \geq d_i$, $\forall i$. Assume $(I+Y)$ is nonsingular. Then,

$$c_i A^{d_i} (B+\delta B) = c_i A^{d_i} B(I+Y) \quad ; i=1, \dots, m.$$

which is non-zero by definition of d_i (i.e., $c_i A^{d_i} B \neq 0$).

Thus $\hat{d}_i = d_i$, and

$$\hat{B}^* = A^{**}(B + \delta B) = A^{**}B(I + Y) = B^*(I + Y)$$

where A^{**} is defined by $A^{**}A = A^*$. Therefore $\hat{B}^*$ is nonsingular and theorem 4.2 follows.

As stated, theorem 4.2 is of little practical importance because it requires the knowledge of Y , and thus of δB . However, based on it, we can prove a further result which shows the robustness of the decouplability of a system for the class of perturbations defined by $\text{Im}(\delta B) \subset \mathcal{B}$, i.e.,

Theorem 4.3. Let S be a decouplable system. Then, for every δB such that $\text{Im}(\delta B) \subset \mathcal{B}$, there exists an $\alpha > 0$ such that $S_{\delta B} = (A, B + \epsilon(\delta B), C)$ is decouplable for every ϵ such that $|\epsilon| < \alpha$.

Proof. From theorem 4.2 above, $S_{\delta B} = (A, B + \epsilon(\delta B), C)$, ($\text{Im}(\delta B) \subset \mathcal{B}$), will be decouplable if and only if $(I + \epsilon Y)$ is nonsingular. For $\epsilon = 0$, $(I + \epsilon Y)$ is obviously nonsingular. Let $\epsilon \neq 0$ and denote $s = 1/\epsilon$. Then, $(I + \epsilon Y) = \epsilon(Is + Y)$, and if we let λ_i ; $i=1, \dots, m$, be the eigenvalues of $(-Y)$ and λ_i^* ; $i=1, \dots, r$ those which are real, it is obvious that $\det(Is + Y) \neq 0$ for every s such that $|s| > |\lambda_{\max}^*| = \max\{|\lambda_i^*|\}$. Therefore,

$\det(I+\varepsilon Y) \neq 0$ for every ε such that $|\varepsilon| < \frac{1}{|\lambda_{\max}^*|}$,

and this together with theorem 4.2 imply theorem 4.3.

This result is in a useful form because it simply ensures that whenever $\text{Im}(\delta B) \subset \mathcal{B}$, then the system will remain decouplable provided that the perturbations are sufficiently small.

A corollary of theoretical importance is easily derived, i.e.,

Corollary 4.3.1. Let $S=(A,B,C)$ be decouplable and $\text{Im}(\delta B) \subset \mathcal{B}$. i.e., $BY = \delta B$ for some (unique) $Y \in \mathbb{R}^{m \times m}$. Then, if Y is skew-symmetric, $S_{\delta B}=(A,B+\delta B,C)$ is decouplable.

Proof. If Y is skew-symmetric, then all the eigenvalues of $(-Y)$ are imaginary, except possibly one, which is zero. Thus, from theorem 4.3, $S_{\delta B}=(A,B+\varepsilon(\delta B),C)$ will be decouplable for all ε ; $|\varepsilon| < \infty$, and corollary 4.3.1. follows.

A result similar to theorem 4.3 is still possible if we enlarge the class of admissible variations δB , thus

Theorem 4.4. Let $S=(A,B,C)$ be decouplable. Then, for every δB such that $\text{Im}(\delta B) \subset \mathcal{N}[W^{(-1)}]$, there exists $\alpha > 0$ such that $S_{\delta B}=(A,B+\varepsilon(\delta B),C)$ is decouplable for every $|\varepsilon| < \alpha$.

Proof. From lemma 4.3, $\mathcal{N}[W^{(-1)}] = \mathcal{B} \oplus \mathcal{N}[W^{(0)}]$. Thus, $\text{Im}(\delta B) \subset \mathcal{N}[W^{(-1)}]$ implies that there exist unique matrices N and Y ; $W^{(0)}N = 0$, such that

$$\delta B = N + BY \quad (4.40)$$

Now, consider $A^{**}[B+\varepsilon(\delta B)]$. From eq.(4.40),

$$A^{**}[B+\varepsilon(\delta B)] = A^{**}B(I+\varepsilon Y) + \varepsilon A^{**}N$$

but as $W^{(0)}N = 0$, then $A^{**}N=0$ and

$$A^{**}[B+\varepsilon(\delta B)] = A^{**}B(I+\varepsilon Y) = B^*(I+\varepsilon Y)$$

and following a similar argument to that of theorem 4.3, $S_{\delta B}=(A,B+\varepsilon(\delta B),C)$ will be decouplable for all $|\varepsilon| < \frac{1}{\lambda_{\max}^*}$ and theorem 4.4 follows.

Let us now consider parameter variations in A , the plant of the system. i.e., $S=(A,B,C) \mapsto S_{\delta A}=(A+\delta A,B,C)$. Then,

Theorem 4.5. Let $S=(A,B,C)$ be decouplable. If $\text{Im}(\delta A) \subset \mathcal{N}[W^{(-1)}]$, then $S_{\delta A}=(A+\delta A,B,C)$ is also decouplable.

Proof. If $\text{Im}(\delta A) \subset \mathcal{N}[W^{(-1)}]$, then

$$c_i(A+\delta A)^j = c_i A^j \quad ; j=0,1,\dots,d_i \quad ; i=1,\dots,m$$

and therefore

$$c_i(A+\delta A)^j B = c_i A^j B \quad ; j=0,1,\dots,d_i \quad ; i=1,\dots,m$$

which implies that $\hat{d}_i = d_i \quad ; i=1,\dots,m$, and

$$\hat{B}^* = B^*$$

Thus, $\hat{B}^*$ is nonsingular because $S=(A,B,C)$ is decouplable, and theorem 4.5 follows.

The results given above for the input-output invariance problem are far from being a complete answer to it. In fact, the cases we have considered are those where the d_i 's remain invariant, but the problem may still have a solution even if the d_i 's vary.

This can be easily seen from the results of section 4, chapter III, on the decoupling of systems connected in parallel. If we consider only variations on B (or C), the perturbed system $S_{\delta B}=(A,B+\delta B,C)$, may be considered to be the parallel connection of $S=(A,B,C)$ and $\tilde{S}=(A,\delta B,C)$, and it is clear that $\tilde{S}$ may "dominate" the decouplability of $S_{\delta B}$. This is illustrated by the following example. Let $S=(A,B,C)$,

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then, $d_1=1$, $d_2=1$, and

$$B^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So, the nominal system S is decouplable. Consider now, for example, a matrix δB of the form

$$\delta B = \begin{bmatrix} 1 & X \\ X & X \\ 0 & 1 \\ X & X \end{bmatrix}$$

where the X 's represent elements possibly different from zero. Then, for $\tilde{S}$ a direct calculation shows that $\tilde{d}_1=0$, $\tilde{d}_2=0$, and $\tilde{S}$ "dominates" the decoupling of $S_{\delta B}$ (theorem 3.5). i.e.,

$$\hat{B}^* = \tilde{B}^* = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}$$

and although the d_i 's have varied, $S_{\delta B}$ is still decouplable.

Cases like this where the response of the closed-

loop system is completely determined by the variations are however, not desirable.

Finally, notice that although the perturbed system S_δ will remain decouplable if the variations satisfy the conditions given in theorems 4.4 and 4.5, the feedback pair (F,G) which decouples S , does not necessarily decouple S_δ .

Little emphasis has been placed in the case where $\delta C \neq 0$ in this section, because as we mentioned before C will in general, consist of "natural" elements (ones and zeros). However, in contrast with the data sensitivity and decoupling problem of section 2, it is no longer true that the input-output invariance problem will have no solution if $\delta C \neq 0$.

The results contained in this chapter can also be found in [6].

CHAPTER V. Decoupling of discrete time-varying systems.

1). Introduction.

In the previous three chapters we have analyzed some problems associated with the noninteracting control of linear continuous time invariant systems. In this chapter we will consider nonstationary discrete time systems and will solve the basic existence question for one-to-one decoupling.

In the brief survey given in chapter I, we mentioned that most of the results available for the decoupling problem were obtained for continuous time invariant systems, although they also hold for discrete time invariant systems with trivial notational modifications. Some of the results have been extended to the continuous time varying situation by several authors, [33], [50], [56], [63], [67], [76], and the basic existence question has been solved. The only attempts to extend these results to linear discrete time-varying systems are those of W.A.Porter [50] and V.Sankaran and M.D.Srinath [57]. Of these, Porter's paper gives only a sufficient condition for the existence of a decoupling state feedback law. Sankaran and Srinath extended Porter's results, and found a condition for

the existence of a solution to the decoupling problem which is both necessary and sufficient. Further, they gave a characterization of a class of decoupling feedback pairs. However, their paper contains some undesirable concepts arising partly from their definition of a decoupled system.

Remark 5.1. They require the variations in $y_i(k+1)$, the i^{th} output of the system at time $k+1$, due to variations in $u_j(k)$; $j \neq i$, the j^{th} input of the system at time k , to be identically zero. This ensures noninteraction in one time step, but interaction may still occur through the input at previous times, i.e., the variations of $y_i(k+1)$ due to variations in $u_j(h)$, $j \neq i$, $h=k-1, k-2, \dots$, may be different from zero.

Below, we reconsider the problem and under the assumption that the decoupling invariants are constant over the time interval of interest, we give a necessary and sufficient condition for the existence of a solution to the decoupling problem via state feedback.

2). A necessary and sufficient condition.

Consider the linear discrete time varying system $S=(A(k),B(k),C(k))$, described by the set of linear difference equations

$$x(k+1) = A(k)x(k) + B(k)u(k) \quad (5.1)$$

and

$$y(k) = C(k)x(k) \quad (5.2)$$

where $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^m$, and $x(k) \in \mathbb{R}^n$, are the input, output and state vectors respectively, and $A(k)$, $B(k)$ and $C(k)$ are time dependent matrices of appropriate dimensions defined over the time interval of interest $[0, k_f]$. Consider the state feedback law given by

$$u(k) = F(k)x(k) + G(k)v(k) \quad (5.3)$$

where $v(k) \in \mathbb{R}^m$ is the external or closed-loop input vector, $G(k)$ is a nonsingular matrix for every $k \in [0, k_f]$, and $F(k)$ is an appropriate matrix.

If we apply this feedback law to the system of eqs.(5.1),(5.2), the closed-loop system

$S(F,G)=(A^c(k),B^c(k),C^c(k))$ is described by

$$x(k+1) = A^c(k)x(k) + B^c(k)v(k) \quad \text{_____} (5.4)$$

and

$$y^c(k) = C^c(k)x(k) \quad \text{_____} (5.5)$$

where

$$A^c(k) = A(k) + B(k)F(k)$$

$$B^c(k) = B(k)G(k)$$

and

$$C^c(k) = C(k)$$

In general terms we shall say that the system S in eqs.(5.1), (5.2) is decouplable by state feedback if there exists a pair of matrices $(F(k),G(k))$, such that each component of the output vector of the closed-loop system of eqs.(5.4),(5.5) is independently controlled by its correspondent component in the external input vector.

Although this intuitive definition is helpful in understanding the problem, in what follows we shall need a rigorous formalization of it. Let

$y_i(t)$ and $u_i(k)$ be the i^{th} components of $y(k)$ and $u(k)$ respectively, and let $c_i(k)$ denote the i^{th} row of $C(k)$ and $b_j(k)$ denote the j^{th} column of $B(k)$, and define the following set of operators

$$D_{i,0}(k) = c_i(k) \quad (5.6)$$

and

$$D_{i,r}(k) = D_{i,r-1}(k) A(k-r) \quad ; r=1,2,\dots,k \quad (5.7)$$

for $i=1,\dots,m$ and every $k \in [0, k_f]$. Similarly, for the parameters of the closed-loop system, define

$$D_{i,0}^c(k) = c_i^c(k) \quad (5.8)$$

and

$$D_{i,r}^c(k) = D_{i,r-1}^c(k) A^c(k-r) \quad ; r=1,2,\dots,k \quad (5.9)$$

for $i=1,\dots,m$ and $k \in [0, k_f]$, where $c_i^c(k)$ denotes the i^{th} row of $C^c(k)$.

Definition 5.1. The system $S=(A(k),B(k),C(k))$ is

said to be decouplable by state feedback from time

$k_0 \in [0, k_f]$, if there exists a sequence of feedback pairs $\{F(k), G(k)\}$, such that $S(F, G)$ is decoupled for some

$k > k_0$.

Definition 5.2. A system $S=(A(k),B(k),C(k))$ is said to be decoupled at time k , if

(i). $D_{i,r}(k)b_j(k-1-r) = 0$; $r=0,1,\dots,k-1$; $i \neq j$,
for $i,j=1,\dots,m$, and

(ii). $D_{i,r}(k)b_i(k-1-r) \neq 0$, for at least one
 $r \in \{0,1,\dots,k-1\}$, and $i=1,\dots,m$.

We justify this definition as follows. After simple algebraic manipulations and repeated substitution of eq.(5.1) in eq.(5.2), it is easy to see that

$$y_i(k) = D_{i,k}(k)x(0) + \sum_{r=0}^{k-1} D_{i,r}(k)B(k-1-r)u(k-1-r) \quad (5.10)$$

and thus, neglecting the effect of initial conditions, condition (i) above simply requires that the variations of $y_i(k)$, due to variations on $u_j(k-1-r)$; $j \neq i$; $r \in \{0,1,\dots,k-1\}$, be zero. Condition (ii) simply states that the variations of the i^{th} output due to any variations on the i^{th} input $u_i(k-1-r)$; $r \in \{0,1,\dots,k-1\}$ must not be identically zero. This second condition can be expressed more neatly as

$$\text{rank}[D_{i,0}(k)b_i(k-1), D_{i,1}(k)b_i(k-2), \dots, D_{i,k-1}(k)b_i(0)] = 1 \quad (5.11)$$

Let us now define the scalar valued functions of k ; $d_i(k)$; $i=1, \dots, m$, as follows

$$d_i(k) = \min \{ j ; D_{i,j}(k)B(k-1-j) \neq 0 ; j=0,1,\dots,k-1 \} \quad (5.12)$$

Remark 5.2. Notice that if for some $i \in \{1, \dots, m\}$, $D_{i,j}(k)B(k-1-j)=0$ for every $j \in \{0,1,\dots,k-1\}$, then $y_i(k)$ will be independent of the value of all the inputs and thus, no control may be exercised upon it. If this happens the system will not be decouplable, because $\{d_i(k) ; i=1, \dots, m\}$ is invariant under state feedback transformations (see theorem 5.1 below), and therefore $D_{i,j}(k)B(k-1-j)=0$ for every $j \in \{0,1,\dots,k-1\}$ implies $D_{i,j}^c(k)B(k-1-j)=0$ for every $j \in \{0,1,\dots,k-1\}$, which clearly contradicts eq.(5.11) (or condition (ii) in def.5.1). If this happens for every $k \in [0, k_f]$, then the decoupling problem is trivially non-solvable. Therefore, we shall assume that for every i , $i \in \{1, \dots, m\}$, there exists some $k \in [0, k_f]$ such that $D_{i,j}(k)B(k-1-j) \neq 0$ for some $j \in \{0,1,\dots,k-1\}$. The continuous time invariant analogue of the above discussion would be when $c_i A^j B = 0$ for $j=0,1,\dots,n-1$, and d_i in eq.(1.5) is not defined. In this case, the i^{th} row of the transfer function matrix would be identically zero, and the decoupling problem is trivially non-solvable.

Theorem 5.1.- The set of functions $d_i(k); i=1, \dots, m$ defined by eq.(5.12), is invariant under arbitrary state feedback transformations of the form (5.3).

To prove this theorem, we need the following

Lemma 5.1 $D_{i,j}^c(k) = D_{i,j}(k)$ for $j=0, 1, \dots, d_i(k); i=1, \dots, m$ for every $k \in [0, k_f]$.

Proof. We proceed by induction. For $j=0$, equality holds trivially, i.e., $D_{i,0}^c(k) = C_i^c(k) = C_i(k) = D_{i,0}(k)$. Assume then that $D_{i,j}^c(k) = D_{i,j}(k)$ holds for some $j < d_i(k)$.

From eq.(5.9)

$$\begin{aligned} D_{i,j+1}^c(k) &= D_{i,j}^c(k) [A(k-1-j) + B(k-1-j)F(k-1-j)] \\ &= D_{i,j}(k) A(k-1-j) + D_{i,j}(k) B(k-1-j)F(k-1-j) \end{aligned}$$

and as $j < d_i(k)$, then

$$D_{i,j}(k) B(k-1-j) = 0$$

so,

$$D_{i,j+1}^c(k) = D_{i,j}(k) A(k-1-j); i=1, \dots, m \quad \text{and}$$

from eq.(5.7)

$$D_{i,j+1}^c(k) = D_{i,j+1}(k); i=1, \dots, m$$

which proves Lemma 5.1.

Proof of Theorem 5.1.- From the lemma above, eq.

(5.12), and the fact that $G(k)$ is a nonsingular matrix for all $k \in [0, k_f]$, it follows that

$$D_{i,j}^c(k) \cdot B^c(k-1-j) = D_{i,j}(k) \cdot B(k-1-j) \cdot G(k-1-j) = 0$$

for every $j \leq d_i(k)$, and

$$D_{i,d_i}^c(k) \cdot B^c(k-1-d_i) = D_{i,d_i}(k) \cdot B(k-1-d_i) \cdot G(k-1-d_i) \neq 0$$

for $i=1, \dots, m$. So, with $d_i^c(k)$ given by the familiar definition

$$d_i^c = \min \{ j : D_{i,j}^c(k) \cdot B^c(k-1-j) \neq 0 ; j=0, 1, \dots, k-1 \} \quad (5.13)$$

$i=1, \dots, m$, it is clear that

$$d_i^c(k) = d_i(k) \quad ; i=1, \dots, m$$

and the theorem follows.

To proceed we will now make the assumption that $d_i(k) ; i=1, \dots, m$, are independent of k , i.e., we will assume that $d_i(k) = \text{constant}$ for every $k \in [0, k_f] ; i=1, \dots, m$. Mathematically this is expressed by

(i). If $D_{i,j}(k) \cdot B(k-1-j) = 0$ for some $k \in [0, k_f]$,

then $D_{i,j}(k) \cdot B(k-1-j) = 0$ for every $k \in [0, k_f] ; (k \geq j+1)$

and

(ii). If $D_{i,j}(k) \cdot B(k-1-j) \neq 0$ for some $k \in [0, k_f]$,

then $D_{i,j}(k) B(k-l-j) \neq 0$ for every $k \in [0, k_f]; (k \geq j+1)$.

An equivalent assumption is usually made in the continuous time varying case [30],[33] so that the problem becomes tractable. In the discrete time varying situation, ([57] in particular), although $d_i(k)$ are defined as functions of time, the constancy assumption is made implicitly in the development. Otherwise the proposed input-output representations contained in these papers cannot be obtained.

Remark 5.3 A discussion of the difficulties arising from the removal of this assumption, will be postponed until later. It will be included in Chapter VI, where we analyze the dynamic decoupling problem for continuous time varying systems.

In our case, the assumption will allow us to give an equivalent but simpler definition for $d_i(k)$, than that of eq.(5.12), i.e.

$$d_i = d_i(k) = \min\{j: D_{i,j}(j+1) B(0) \neq 0; j=0,1,\dots\} \quad (5.14)$$

and a number of facts become apparent. Firstly, if we go back to Remark 5.2, it is clear that the assumption made in it, (i.e., there exists some $k \in [0, k_f]$

such that $D_{i,j}(k) B(k-l-j) = 0$ for some $j \in \{0,1,\dots,m\}$),

simply implies that the d_i 's are required to be finite, and to have values within $[0, k_f]$.

It is also evident now, that for any $i=1, \dots, m$, condition (ii) in definition 5.2, cannot be satisfied for $k < d_i$ and therefore, if $d^* = \max_i \{d_i\}$, the system will not be decouplable for $k < d^*$. This is so because, although non-interaction (i.e., condition (i), definition 5.2) may be achieved at any time, at least d^* time intervals must elapse before we can exercise some control upon all of the outputs.

We are now ready to state and prove the main results in this section. Let $S=(A(k), B(k), C(k))$ be any system described by eqs.(5.1)-(5.2), and let $\{d_i\}$ be its set of decoupling invariants, with $d_i = \hat{d}_1, \forall i \in J_1$; $d_i = \hat{d}_2, \forall i \in J_2$; ... ; $d_i = \hat{d}_q, \forall i \in J_q$, where $J_i \subset \{1, \dots, m\}$; $i=1, \dots, q$; $J_i \cap J_j = \emptyset \quad \forall i \neq j$, and $\bigcup_{i=1}^q J_i = \{1, \dots, m\}$.

Then

Theorem 5.2. The system $S=(A(k), B(k), C(k))$ is decouplable from time $k_0 \in [0, k_f]$ if and only if, for $j=1, \dots, q$, the sets of vectors

$$\{D_{i, d_i}(k) B(k-1-d_i) ; i \in J_j\}$$

are linearly independent for some $k \geq k_0 + d^* + 1, (k \in [0, k_f])$.

Proof. (If). By repeated substitution of eq.(5.5)

in eq.(5.4), and after some direct algebraic manipulations, it is easy to obtain

$$y_i^c(k) = D_{i,k}^c(k)x(k_0) + \sum_{r=0}^{k-k_0-1} D_{i,r}^c(k)B^c(k-1-r)v(k-1-r) \quad (5.15)$$

which from the definition of d_i^c and theorem 5.1, can be rewritten as

$$y_i^c(k) = D_{i,k}^c(k)x(k_0) + \sum_{r=d_i}^{k-k_0-1} D_{i,r}^c(k)B^c(k-1-r)v(k-1-r) \quad (5.16)$$

for $i=1, \dots, m$.

Let us consider $D_{i,r}^c(k)$; $d_i < r \leq k-k_0-1$. Applying eq.(5.9) repeatedly we obtain

$$\begin{aligned} D_{i,r}^c(k) &= D_{i,r-1}^c(k) A^c(k-r) \\ &= D_{i,r-2}^c(k) A^c(k-r+1)A^c(k-r) \\ &= \dots = D_{i,d_i}^c(k)A^c(k-d_i-1)A^c(k-d_i-2)\dots A^c(k-r) \\ &= D_{i,d_i}^c(k)[A(k-d_i-1)+B(k-d_i-1)F(k-d_i-1)] A^c(k-d_i-1) \\ &\quad \dots A^c(k-r) \end{aligned} \quad (5.17)$$

for $i=1, \dots, m$.

Define $B^*(k-\hat{d}_p-1) \in \mathbb{R}^{m \times m}$; $p=1, \dots, q$, as a matrix whose i^{th} row, $i \in J_p$, is given by $D_{i,d_i}^c(k) B(k-d_i-1)$, i.e.,

$$\begin{aligned} B_i^*(k-\hat{d}_p-1) &= [D_{i,d_i}(k) B(k-d_i-1)] \\ &= [D_{i,\hat{d}_p}(k) B(k-\hat{d}_p-1)] ; \forall i \in J_p \quad \text{--- (5.18)} \end{aligned}$$

If for $p=1, \dots, q$, the sets of vectors $\{D_{i,d_i}(k)B(k-d_i-1) ; i \in J_p\}$ are linearly independent, then it is clear that we can define $B_i^*(k-\hat{d}_p-1) ; i \in \{1, \dots, m\}-J_p$, so that $B^*(k-\hat{d}_p-1)$ is a nonsingular matrix. That is, if the condition of theorem 5.2 is satisfied, then there exist vectors $z_{i,p}(k-\hat{d}_p-1) ; i \in \{1, \dots, m\}-J_p$, (not unique) such that if

$$B_i^*(k-\hat{d}_p-1) = z_{i,p}(k-\hat{d}_p-1) ; \forall i \in \{1, \dots, m\}-J_p \quad \text{--- (5.19)}$$

$B^*(k-\hat{d}_p-1)$ is nonsingular.

Let $A^*(k-\hat{d}_p-1)$ be an $m \times n$ matrix whose i^{th} row; $i \in J_p$ is given by $D_{i,d_i+1}(k)$, i.e.,

$$A_i^*(k-\hat{d}_p-1) = D_{i,d_i+1}(k) = D_{i,\hat{d}_p+1}(k) ; \forall i \in J_p \quad \text{--- (5.20)}$$

and arbitrary otherwise. If we define now $F(k-\hat{d}_p-1)$, $G(k-\hat{d}_p-1) ; p=1, \dots, q$, by

$$F(k-\hat{d}_p-1) = -B^{*-1}(k-\hat{d}_p-1) A^*(k-\hat{d}_p-1) \quad \text{--- (5.21)}$$

and

$$G(k-\hat{d}_p-1) = B^{*-1}(k-\hat{d}_p-1) \Delta \quad \text{--- (5.22)}$$

where $\Delta = \text{diag}\{\lambda_i\}$ is nonsingular, then, substituting

in eq.(5.17) we obtain

$$\begin{aligned} D_{i,r}^c(k) &= [D_{i,d_i+1}(k) - D_{i,d_i+1}(k)] A^c(k-d_i-2) \dots A^c(k-r) \\ &= 0 \quad \text{for } d_i < r \leq k-k_0-1, \quad i=1, \dots, m \end{aligned} \quad (5.23)$$

Thus, from eq.(5.16),

$$\begin{aligned} y_i^c(k) &= D_{i,d_i}^c(k) B^c(k-d_i-1) v(k-d_i-1) \\ &= D_{i,d_i}(k) B(k-d_i-1) G(k-d_i-1) v(k-d_i-1) ; i=1, \dots, m \end{aligned}$$

and with $G(k-d_i-1)$ as defined by eq.(5.22), we obtain

$$y_i^c(k) = \lambda_i v_i(k-d_i-1) ; i=1, \dots, m \quad (5.24)$$

and the closed-loop system is clearly decoupled at time k . (To verify that conditions (i)-(ii) of def. 5.2 are satisfied by $S(F,G)$ is straight forward).

(Only if). To prove necessity, assume S is decouplable from $k_0 \in [0, k_f]$. Then, there exists a sequence of feedback pairs $\{F(k), G(k)\}$ such that conditions (i) and (ii) of definition 5.2 are satisfied for some $k > k_0$, i.e.,

$$D_{i,r}^c(k) B^c(k-l-r) = \alpha_{i,r} e_i ; i=1, \dots, m$$

for some k and all $r \leq k-l$, where $\alpha_{i,r}$ are arbitrary scalars and e_i denotes the i^{th} unit-row vector.

By definition of d_i^c and theorem 5.1, it is clear that

$$D_{i,d_i}^c(k) B^c(k-d_i-1) = \alpha_{i,d_i} e_i \neq 0 \quad ; i=1, \dots, m$$

so, $\alpha_{i,d_i} \neq 0$. This can be expressed as

$$D_{i,d_i}(k) B(k-d_i-1) G(k-d_i-1) = \alpha_{i,d_i} e_i \quad ; i=1, \dots, m \quad \underline{\hspace{1cm}} (5.24)$$

Thus, for every $i \in J_p$; $p=1, \dots, q$,

$$\begin{aligned} [D_{i,\hat{d}_p}(k) B(k-\hat{d}_p-1)] ; i \in J_p] G(k-\hat{d}_p-1) &= \\ &= \text{diag}\{\alpha_{i,d_i}\} [\{e_i\} ; i \in J_p] \end{aligned}$$

and as $\alpha_{i,d_i} \neq 0$, $\forall i \in \{1, \dots, m\}$, and $G(k-\hat{d}_p-1)$ is non-singular, then

$$\text{rank}[\{D_{i,\hat{d}_p}(k) B(k-\hat{d}_p-1)\} ; i \in J_p] = \text{rank}[\{e_i\} ; i \in J_p]$$

for every $p=1, \dots, q$. i.e., the vectors

$D_{i,d_i}(k) B(k-d_i-1) ; i \in J_p$ must be linearly independent

for all $p=1, \dots, q$, and necessity follows.

Corollary 5.2.1. Let $S=(A(k), B(k), C(k))$ be given.

If $d_i \neq d_j$; $i \neq j$, $i, j=1, \dots, m$, then S is decouplable from all $k_0 \in [0, k_f - d^* - 1]$.

Proof. If $d_i \neq d_j$, $\forall i \neq j$, $i, j=1, \dots, m$, then J_p ; $p=1, \dots, q$, contain only one element each (i.e., $q=m$), and as by

definition of d_i , and the assumption that $\{d_i\}$ is independent of k ,

$$D_{i,d_i}(k) B(k-d_i-1) \neq 0 \quad \forall k$$

Thus, the condition of theorem 5.2 is satisfied for all $k \in [0, k_f - d^* - 1]$.

Theorem 5.2 and corollary 5.2.1, are very interesting results. Indeed, whereas in the continuous time situation a system can be decoupled if and only if a matrix $B^*(t)$ is nonsingular, for discrete time-varying systems this is not the case (except if $d_i \neq d_j$, $\forall i, j=1, \dots, m$, which is a very particular case).

Theorem 5.2, is less demanding. A partial explanation of this is that for continuous time systems, control action is instantaneous, at least theoretically, whereas in the discrete time situation there is a delay in the response to a control, and this delay is not necessarily the same in general, for all the outputs.

Notice also that $B^*(k-\hat{d}_p-1)$ and $A^*(k-\hat{d}_p-1)$; $p=1, \dots, q$, defined by eqs.(5.18)-(5.20), depend on the parameters of the system at future times (i.e., on $D_{i,d_i}(k)$ and $D_{i,d_i+1}(k)$ respectively). This, however, should not be a matter for concern. It is the discrete-time equivalent to the dependence of $\{F(t), G(t)\}$ on the derivatives of the parameters of the system in the continuous case (e.g. [33]).

In theorem 5.2, we have given conditions for a system S to be decouplable from $k_0 \in [0, k_f]$. That is, conditions for the existence of a sequence of feedback pairs such that the closed-loop system $S(F, G)$ is decoupled at some time $k \geq k_0 + d^* + 1$. This is the discrete-time equivalent of "point-wise" decoupling (see appendix A.1, def.A.1) in the continuous time situation. A further problem of interest would be to obtain conditions for the existence of a sequence of feedback pairs such that the closed-loop system $S(F, G)$ is decoupled over a time interval $[k_1, k_2]$; $k_1 \geq k_0 + d^* + 1$.

Without loss of generality, assume that $\hat{d}_i > \hat{d}_{i+1}$; $i=1, \dots, q-1$. Define the time intervals

$$T_i = [(k_1 - \hat{d}_i - 1), (k_2 - \hat{d}_i - 1)] ; i=1, \dots, q \quad (5.25)$$

$$T_0 = T_{q+1} = 0 \quad (5.26)$$

and let

$$\bar{K}_{l,j} = \left\{ \bigcap_{p=l}^j T_p \right\} - \left\{ \bigcup_{s=0}^{l-1} T_s \right\} - T_{j+1} \quad ; l=1, \dots, q ; j=l, \dots, q. \quad (5.27)$$

Then,

Theorem 5.3. There exists a sequence of state feedback pairs $\{F(k), G(k)\}$ such that the closed-loop system is decoupled for every $k \in [k_1, k_2]$, for some

$k_1 \geq k_0 + d^* + 1$, if and only if for every $k \in \bar{K}_{l,j}$, the vectors

$$\{D_{i,\hat{d}_l}(k - \hat{d}_l - \hat{d}_j) B(k - \hat{d}_j - 1)\}; i \in J_l \quad ; l=1, \dots, q,$$

$j=1, \dots, q$, are linearly independent.

Proof. The necessary and sufficient condition of theorem 5.3 above is a direct consequence of the condition given in theorem 5.2 for decoupling at some point $k \geq k_0 + d^* + 1$, noticing that for theorem 5.3, it must be satisfied for every $k \in [k_1, k_2]$; (for some $k_1 \geq k_0 + d^* + 1$). To make this more evident, consider Fig.5.1 below.

To decouple the system at k_1 , appropriate control action would have to be taken at the points marked by $\bullet$ in the figure, exactly as was done in the proof of theorem 5.2. That is, for example, $B^*(k_1 - 1 - \hat{d}_2)$ would have its i^{th} row, for $i \in J_2$, defined by

$$B_i^*(k_1 - 1 - \hat{d}_2) = D_{i,\hat{d}_2}(k_1) B(k_1 - 1 - \hat{d}_2) \quad \forall i \in J_2 \quad \text{--- (5.29)}$$

and the remaining rows, $z_{i,2}(k_1 - 1 - \hat{d}_2)$, would be chosen so that $B^*(k_1 - 1 - \hat{d}_2)$ is nonsingular.

Now, to decouple the i^{th} output of the system for any $i \in J_1$ at time $k_1 + \hat{d}_1 - \hat{d}_2$, an appropriate

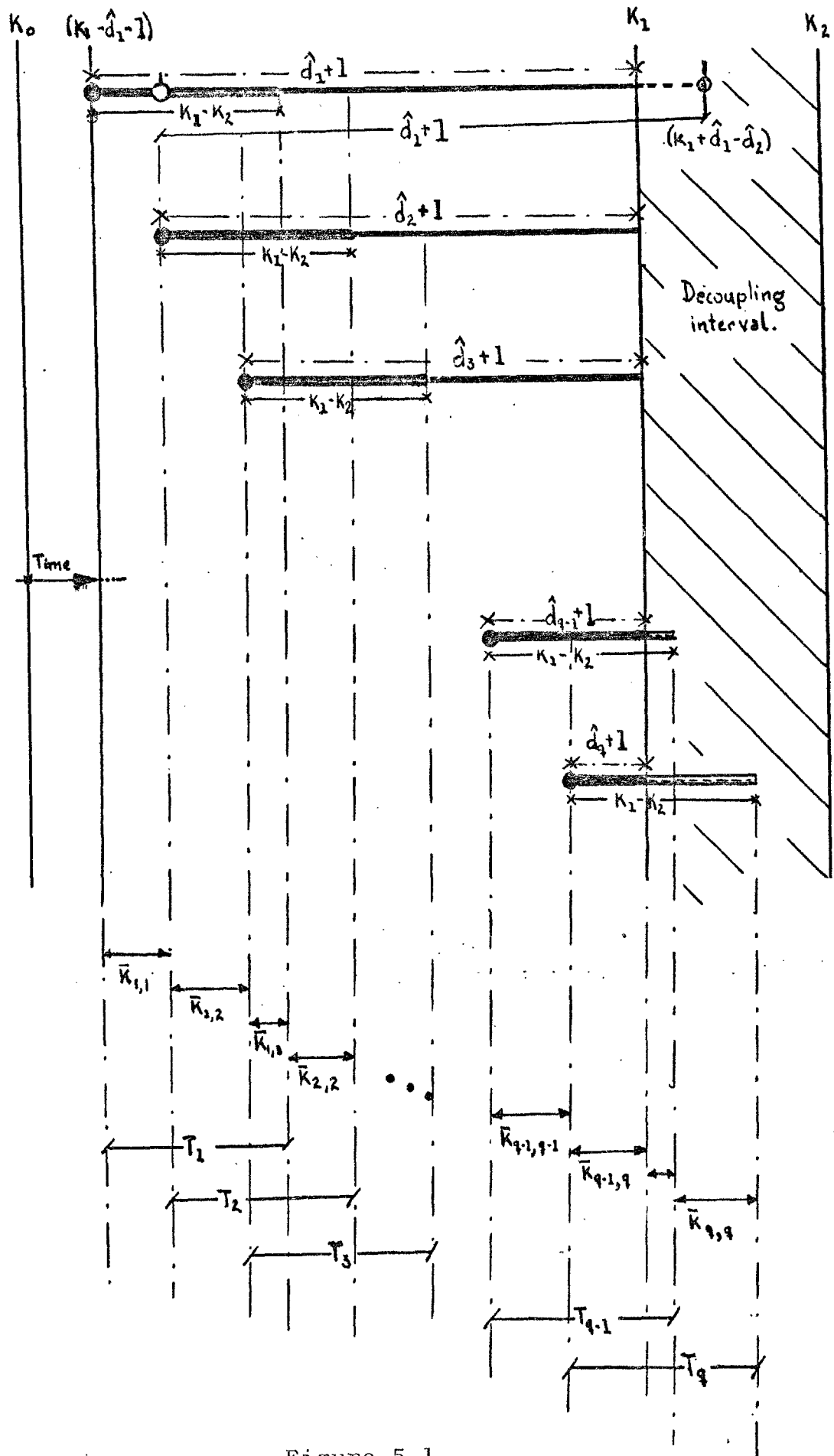


Figure 5.1.

control must be applied at the point marked $\circ$ in fig.5.1. i.e., at $k = k_1 - 1 - \hat{d}_2$. This, from theorem 5.2, would imply fixing the i^{th} row of $B^*(k_1 - 1 - \hat{d}_2)$; $\forall i \in J_1$, as

$$B_i^*(k_1 - 1 - \hat{d}_2) = D_{i, \hat{d}_1}(k_1 + \hat{d}_1 - \hat{d}_2) B(k_1 - 1 - \hat{d}_2) ; \forall i \in J_1 \quad (5.30)$$

Therefore, to be able to decouple at both, k_1 and $k_1 + \hat{d}_1 - \hat{d}_2$, eqs.(5.29) and (5.30) must be satisfied simultaneously. i.e., some of the vectors $z_{i,2}(k_1 - 1 - \hat{d}_2)$ are no longer arbitrary, but have been fixed by eq.(5.30). As we still require $B^*(k_1 - 1 - \hat{d}_2)$ to be nonsingular to achieve non-interaction, not only must $\{B_i^*(k_1 - 1 - \hat{d}_2) ; i \in J_1\}$ and $\{B_i^*(k_1 - 1 - \hat{d}_2) ; i \in J_2\}$ be sets of linearly independent vectors, but also

$$\{B_i^*(k_1 - 1 - \hat{d}_2) ; i \in J_1 \cup J_2\}$$

which is a stronger condition.

If the time interval $[k_1, k_2]$ is big enough, it is clear that over some subinterval, no $z_{i,p}(\cdot)$'s will be arbitrary, because the decoupling conditions will fix them all. Over this subinterval, the condition for decoupling given in theorem 5.3, can be seen to correspond with the nonsingularity of the matrix $B^*(k)$, and a direct analog with the continuous time situation can be established.

If decoupling over a time interval $[k_1, k_2]$ is possible, and again assuming this interval is big enough, after a period of time it may be possible to enlarge the class of decoupling feedback pairs, by defining

$$F(k) = -B^{*-1}(k) [A^*(k) - M(k)]$$

where $M(k)$ is a matrix whose i^{th} row, $M_i(k)$, will be of the form

$$M_i(k) = D_{i, \hat{1}_p}(k + d_p) ; \forall i \in J_p$$

This characterization of the class of decoupling pairs is the discrete time-varying equivalent of that given by Falb and Wolovich [17] for continuous time-invariant systems.

CHAPTER VI. Dynamic decoupling of linear time-
varying systems.

1). Introduction.

In this chapter we will study the extended or dynamic decoupling problem via state feedback for time varying systems. That is, given a nondecouplable time varying plant, we will find conditions for the existence of a precompensator (also time varying), such that the overall system is decouplable by state feedback. This problem has been extensively studied for time invariant systems and solutions have been found using different approaches, [9], [22], [36], [54], [62], [78], [80]. These results have not yet been extended to the time varying situation, except some regarding the dynamic decoupling problem via output feedback. What follows is an attempt to fill this gap.

The first technical difficulty arising in the study of noninteracting control for time varying systems is that of suitably defining what is meant by a decoupled system. Whereas for stationary linear systems all the existing definitions are equivalent

(see [2]), in the time varying situation this is no longer true. A discussion of the definitions of a decoupled system is given in appendix A.1.. Throughout this chapter we will adopt the definition given by Majumdar and Choudhury [33] (see definitions A.1.1-A.1.3 in appendix A.1) for uniform decoupling. i.e., consider the linear time varying system

$\Sigma(t) = (A(t), B(t), C(t))$, described by

$$\dot{x}(t) = A(t) x(t) + B(t) u(t) \quad \text{_____} (6.1)$$

$$y(t) = C(t) x(t) \quad \text{_____} (6.2)$$

where $u(t), y(t) \in \mathbb{R}^m$; $x(t) \in \mathbb{R}^n$, and $A(t), B(t), C(t)$ are matrices of appropriate dimensions, whose elements are continuous functions of time with sufficient continuous derivatives defined over the time interval of interest $T = [t_0, t_1]$. Then, $\Sigma(t)$ is said to be uniformly decoupled if

(i). The variations of $y_i(t)$, (the i^{th} element of $y(t)$) due to any variations on $u_j(t)$; $j \neq i$ ($u_j(t)$ denotes the j^{th} element of $u(t)$), are identically zero for all $t \in T$, and

(ii). The variations in $y_i(t)$ due to any variations in $u_i(t)$; $i=1, \dots, m$, are not zero for all $t \in T$.

Define the following set of differential operators

$$(L_A^{(0)} c_i)(t) = c_i(t) \quad \text{--- (6.3)}$$

and

$$(L_A^{(k)} c_i)(t) = \frac{d}{dt} [(L_A^{(k-1)} c_i)(t)] + (L_A^{(k-1)} c_i)(t) A(t) \quad \text{--- (6.4)}$$

where $c_i(t)$ is the i^{th} row of $C(t)$. Let

$$d_i(t) = \min \left\{ j: (L_A^{(j)} c_i)(t) B(t) \neq 0; j=0,1,\dots \right\} \quad i=1,\dots,m. \quad \text{--- (6.5)}$$

for each $t \in T$, and

$$B^*(t) = \begin{bmatrix} (d_1(t)) \\ (L_A^{(d_1(t))} c_1)(t) B(t) \\ \vdots \\ (d_m(t)) \\ (L_A^{(d_m(t))} c_m)(t) B(t) \end{bmatrix}; \forall t \in T. \quad \text{--- (6.6)}$$

Then, it is well known that

Theorem 6.1. [33],[76]. There exists a state feed-back law given by

$$u(t) = F(t) x(t) + G(t) v(t) \quad \text{--- (6.7)}$$

where $v(t) \in \mathbb{R}^m$, such that the closed-loop system is uniformly decoupled if and only if the $m \times m$ matrix $B^*(t)$ is nonsingular for every $t \in T$.

To be able to establish this result, [33], [76], it is required first, that $d_i(t); i=1, \dots, m$, are finite for every $t \in T$, and second, that they remain constant over the time interval of interest.

Remark 6.1. If the d_i 's are not constant, it is usually said in the literature that by the continuity and differentiability assumptions made on the parameters of the system, if $d_i(t) = k$ for some $t \in T$, then $d_i(t) = k$ for some subinterval of T containing t . Following this argument, the analysis may then be carried out in these subintervals where the d_i 's are constant. However, two points must be taken into consideration; first, these subintervals are not necessarily all of positive measure, nor closed, and secondly, even if they are of positive measure, some practical difficulties arise near the boundaries of the subintervals, (i.e., unbounded feedback gains). We will discuss the effect of removing the constancy assumption for $d_i(t)$ later, in section 4.

Let us assume now that the plant of a given system is not uniformly decouplable by state feedback, i.e., $B^*(t)$ is singular for at least one $t \in T$. Is it still possible to decouple the system uniformly by any other means? In the time invariant case the

answer to this question is well known [9],[21],[22], [36],[53],[54],[62],[78]; the use of dynamic feedback ($u=F(s)x+G(s)v$) enlarges the class of decouplable systems. Indeed, a time invariant system is decouplable by dynamic precompensation and state feedback if and only if it is invertible (and there are invertible systems for which B^* is singular).

A similar result can be obtained for regular [61] time varying systems.

Remark 6.2. A system $\Sigma(t)$ described by eqs.(6.1),(6.2) is said to be regular if the matrices $[C(t)B(t)]$ and $[\bar{D}_{(k)}(t)^T, (\tilde{C}_{(k)}(t)B(t))^T]$ have constant rank on $[t_0, \infty)$ for $k=1, \dots, n-1$, where $\bar{D}_{(k)}(t)$ and $[\tilde{C}_{(k)}(t)B(t)]$ are defined by Silverman's structure algorithm. i.e., Let $[C(t)B(t)]$ be of constant rank q_1 ($< n$) on $[t_0, \infty)$. Then, there exists a nonsingular differentiable matrix S_1 such that

$$S_1 [C(t)B(t)] = \begin{bmatrix} \bar{D}_{(1)}(t) \\ 0 \end{bmatrix}$$

where $\bar{D}_{(1)}(t)$ is formed by the first q_1 linearly independent rows (on $[t_0, \infty)$) of $[C(t)B(t)]$. Let

$$S_1 [(L_A^{(1)} C)(t)] \triangleq \begin{bmatrix} \bar{C}_{(1)}(t) \\ \tilde{C}_{(1)}(t) \end{bmatrix}$$

where $\bar{C}_{(1)}(t)$ has q_1 rows and $\tilde{C}_{(1)}(t)$ has $m-q_1$ rows.

Now, define the $m \times m$ matrix $D_{(1)}(t)$ as

$$D_{(1)}(t) = \begin{bmatrix} \bar{D}_{(1)}(t) \\ [\tilde{C}_{(1)}(t)B(t)] \end{bmatrix}$$

and let it be of rank $q_2 (\geq q_1)$, constant on $[t_0, \infty)$.

Then there exists S_2 , nonsingular and differentiable, such that

$$S_2 \cdot D_{(1)}(t) = \begin{bmatrix} \bar{D}_{(2)}(t) \\ 0 \end{bmatrix}$$

where $\bar{D}_{(2)}(t)$ is formed by the first q_2 linearly independent rows (on $[t_0, \infty)$) of $D_{(1)}(t)$. Now define

$$S_2 \begin{bmatrix} \bar{C}_{(1)}(t) \\ (L_A^{(1)} \tilde{C}_{(1)})(t) \end{bmatrix} = \begin{bmatrix} \bar{C}_{(2)}(t) \\ \tilde{C}_{(2)}(t) \end{bmatrix}$$

where $\bar{C}_{(2)}(t)$ has q_2 rows and $\tilde{C}_{(2)}(t)$, $m-q_2$ rows.

In general, if $D_{(k)}(t)$ is defined by

$$D_{(k)}(t) \triangleq \begin{bmatrix} \bar{D}_{(k)}(t) \\ \tilde{C}_{(k)}(t)B(t) \end{bmatrix}$$

and $\text{rank}[D_{(k)}(t)] = q_{k+1}$, constant on $[t_0, \infty)$, there exists a matrix S_{k+1} , nonsingular and differentiable,

such that

$$S_{k+1} \cdot D_{(k)}(t) = \begin{bmatrix} \bar{D}_{(k+1)}(t) \\ 0 \end{bmatrix}$$

where $\bar{D}_{(k+1)}(t)$ consists of the first q_{k+1} linearly independent rows (on $[t_0, \infty)$) of $D_{(k)}(t)$. The matrix $\tilde{C}_{(k+1)}(t)$ would be defined by

$$S_{k+1} \begin{bmatrix} \bar{C}_{(k)}(t) \\ (L_A^{(1)} \tilde{C}_{(k)})(t) \end{bmatrix} = \begin{bmatrix} \bar{C}_{(k+1)}(t) \\ \tilde{C}_{(k+1)}(t) \end{bmatrix}$$

where $\bar{C}_{(k+1)}(t)$ has q_{k+1} rows and $\tilde{C}_{(k+1)}(t)$ has $m - q_{k+1}$ rows.

Theorem 6.2. A regular time-varying system is uniformly decouplable by dynamic precompensation and state feedback if and only if it is uniformly invertible. (i.e., invertible for all $t \in T$).

Proof. (If). If the system is uniformly invertible, then it is obvious that the inverse system considered as a precompensator will yield a uniformly decoupled composite system.

(Only if). Assume that a given system $\Sigma(t)$ is uniformly decouplable by dynamic feedback, i.e., there exists a precompensator $\hat{\Sigma}(t)$, such that the cascade connection of $\hat{\Sigma}(t)$ and $\Sigma(t)$ is decouplable by state feedback. Thus, if we define $B_c^*(t)$ as in eq.(6.6) for the parameters of the composite system, $B_c^*(t)$ will be nonsingular for every $t \in T$, (by theorem 6.1), and clearly, the invertibility condition given in [61] is satisfied by the composite system $\Sigma_c(t)$ at all times $t \in T$. i.e., there exists an integer α such that $\text{rank}[D_{(\alpha)}(t)] = m$ for every $t \in [t_0, t_1]$, where α is the minimum integer k such that $\text{rank}[D_{(k)}] = m$, with $D_{(k)}(t)$ as given in remark 6.2 above. Therefore, $\Sigma_c^{-1}(t)$, the inverse system of $\Sigma_c(t)$ exists and is representable by a set of differential equations of the form (6.1),(6.2). Now, if we connect in cascade $\Sigma_c^{-1}(t)$ and $\hat{\Sigma}(t)$, (see fig.6.1 below), the resulting system has $y(t)$ as its input and $u(t)$ as its output, and is obviously realizable and causal, being representable by a set of differential equations of the form (6.1),(6.2). Thus, the cascade connection of $\Sigma_c^{-1}(t)$ and $\hat{\Sigma}(t)$ is an inverse for $\Sigma(t)$. i.e., $\Sigma^{-1}(t)$ exists and uniform decouplability implies uniform invertibility and theorem 6.2 follows.

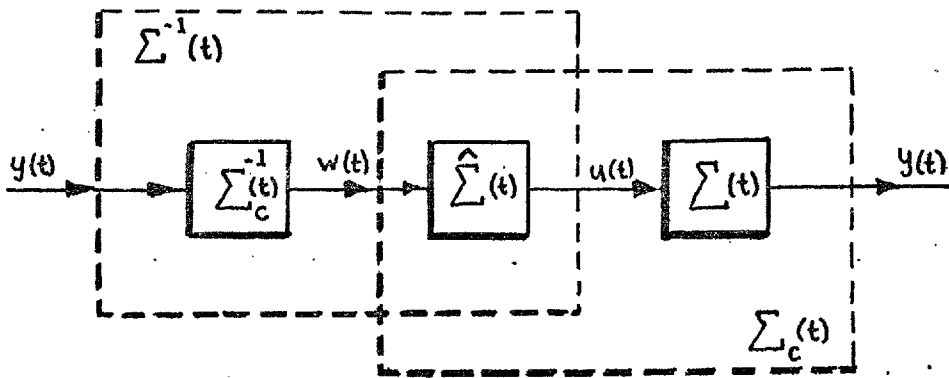


Fig. 6.1.

2). Necessary and sufficient condition.

Knowing that the use of dynamic elements enlarges the class of uniformly decouplable systems, in this section we will give a necessary and sufficient condition for the dynamic state feedback decoupling problem to have a solution, in terms of the parameters of the given plant.

So, consider the precompensator $\hat{\Sigma}(t) = (\hat{A}(t), \hat{B}(t), \hat{C}(t), \hat{D}(t))$, described by

$$\dot{\hat{x}}(t) = \hat{A}(t) \hat{x}(t) + \hat{B}(t) w(t) \quad (6.8)$$

and

$$u(t) = \hat{C}(t) \hat{x}(t) + \hat{D}(t) w(t) \quad (6.9)$$

where $w(t) \in \mathbb{R}^m$ is the overall input, $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$, and $\hat{A}(t), \hat{B}(t), \hat{C}(t)$ and $\hat{D}(t)$ are matrices of appropriate dimensions with continuous differentiable elements.

The dynamic state feedback decoupling problem can be stated as finding $\hat{A}(t), \hat{B}(t), \hat{C}(t)$ and $\hat{D}(t)$ such that the cascade connection of $\hat{\Sigma}(t)$ followed by $\Sigma(t)$, i.e., the composite system $\Sigma_c(t) = (A_c(t), B_c(t), C_c(t))$ is decouplable.

$\Sigma_c(t)$ may be represented by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t)\hat{C}(t) \\ 0 & \hat{A}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B(t)\hat{D}(t) \\ \hat{B}(t) \end{bmatrix} w(t) \quad (6.10)$$

and

$$y(t) = [C(t) \quad 0] \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \quad (6.11)$$

This is the time-varying version of the cascade connection discussed in chapter III, eqs.(3.6),(3.7). The order in which $x(t)$ and $\hat{x}(t)$ appear has been changed here for convenience, and a direct input-output map $\hat{D}(t)$ has been introduced in the precompensator.

Let us define the differential operators

$$(\mathcal{L}_A^{(0)} B)(t) = B(t) \quad (6.12)$$

and

$$(\mathcal{L}_A^{(k)} B)(t) = \frac{d}{dt} [(\mathcal{L}_A^{(k-1)} B)(t)] - A(t) (\mathcal{L}_A^{(k-1)} B)(t) \quad (6.13)$$

for $k=1,2,\dots$. If we define differential operators similar to those in eqs.(6.3),(6.4) in terms of the parameters of $\sum_c(t)$, the composite system, then after some tedious algebraic manipulation we obtain

$$\begin{aligned} (L_{A_c}^{(j)} C_{c_i})(t) &= [(L_A^{(j)} c_i)(t), 0] ; j=0,1,\dots,d_i(t) \\ & ; i=1,\dots,m \end{aligned} \quad (6.14)$$

and

$$\begin{aligned} (L_{A_c}^{(d_i(t)+k)} C_{c_i})(t) &= [(L_A^{(d_i(t)+k)} c_i)(t), M_i^{(d_i(t)+k)}(t)] \\ & k=1,2,\dots ; i=1,\dots,m \end{aligned} \quad (6.15)$$

for all $t \in T$, where $C_{c_i}(t)$ denotes the i^{th} row of $C_c(t)$ and $M_i^{(d_i(t)+k)}$; $k=1,2,\dots ; i=1,\dots,m$, is given by

$$M_i^{(d_i(t)+k)}(t) = \sum_{j=0}^{k-1} \binom{k}{j} (L_A^{(d_i(t)+j)} c_i)(t) \sum_{r=0}^{k-1-j} \binom{k-1-j}{r} (L_A^{(r)} B)(t) (L_A^{(k-j-r-1)} \hat{c})(t) \quad (6.16)$$

Theorem 6.3. Let $(d_c)_i(t) \triangleq \min \{j ; (L_{A_c}^{(j)} C_{c_i})(t) B_c(t) \neq 0 ; j=0,1,\dots\}$ for every $t \in T$. Then $(d_c)_i(t) \geq d_i(t) ; i=1,\dots,m$, for all $t \in T$.

Proof. From eq.(6.14)

$$(L_{A_c}^{(j)} C_{c_i})(t) B_c(t) = [(L_A^{(j)} c_i)(t) B(t)] \hat{D}(t) ; j=0,1,\dots,d_i(t)$$

and from the definition of $d_i(t)$,

$$(L_{A_c}^{(j)} c_{c_i})(t) B_c(t) = 0 \quad ; j=0,1,\dots,d_i(t)-1; i=1,\dots,m$$

for all $t \in T$. Thus, $(d_c)_i(t) \geq d_i(t)$ and theorem 6.3 follows.

Notice that this result is true for time varying $(d_c)_i$ and d_i . In the remainder of the chapter we will assume that $d_i(t)$; $i=1,\dots,m$, are constant, and later, a similar assumption will be imposed on $(d_c)_i$; $i=1,\dots,m$.

Let us rewrite $M_i^{d_i+k}(t)$; $k=1,2,\dots$, in eq.(6.16) as follows

$$M_i^{d_i+k}(t) = \sum_{j=0}^{k-1} \binom{k}{j} (L_A^{(d_i+j)} c_i)(t) \left[\binom{k-1-j}{0} (L_A^{(0)} B)(t), \dots, \binom{k-1-j}{k-1-j} (L_A^{(k-1-j)} B)(t) \right] \begin{bmatrix} (L_{\hat{A}}^{(k-1-j)} \hat{C})(t) \\ \vdots \\ (L_{\hat{A}}^{(0)} \hat{C})(t) \end{bmatrix}$$

or more explicitly

$$\begin{aligned}
 M_i^{d_i+k}(t) = & \binom{k}{0} (L_A^{(d_i)} c_i)(t) \left[\binom{k-1}{0} (\mathcal{L}_A^{(0)} B)(t), \dots, \binom{k-1}{k-1} (\mathcal{L}_A^{(k-1)} B)(t) \right] \begin{bmatrix} (L_{\hat{A}}^{(k-1)} \hat{C})(t) \\ \vdots \\ (L_{\hat{A}}^{(0)} \hat{C})(t) \end{bmatrix} \\
 & + \binom{k}{1} (L_A^{(d_i+1)} c_i)(t) \left[\binom{k-2}{0} (\mathcal{L}_A^{(0)} B)(t), \dots, \binom{k-2}{k-2} (\mathcal{L}_A^{(k-2)} B)(t) \right] \begin{bmatrix} (L_{\hat{A}}^{(k-2)} \hat{C})(t) \\ \vdots \\ (L_{\hat{A}}^{(0)} \hat{C})(t) \end{bmatrix} \\
 & + \dots + \binom{k}{k-1} (L_A^{(d_i+k-1)} c_i)(t) \left[\binom{0}{0} (\mathcal{L}_A^{(0)} B)(t) \right] \left[(L_{\hat{A}}^{(0)} \hat{C})(t) \right]
 \end{aligned}
 \tag{6.17}$$

Thus, if we define

$$Q_{i,0}^{(d_i+k)}(t) = [(L_A^{(d_i+k)} c_i)(t)] [(\mathcal{L}_A^{(0)} B)(t)] ; k=0,1,\dots$$

$$i=1,\dots,m \tag{6.18}$$

and

$$Q_{i,j}^{(d_i+k)}(t) = \sum_{p=0}^{k-j} \binom{k}{p} \binom{k-1-p}{k-j-p} (L_A^{(d_i+p)} c_i)(t) \times (\mathcal{L}_A^{(k-j-p)} B)(t)$$

$$\text{for } k=0,1,\dots; j=1,\dots,k; i=1,\dots,m \tag{6.19}$$

then, from eq.(6.15) and rearranging terms in (6.17),
we obtain

$$\begin{aligned}
 & (d_i + k) \\
 & (L_{A_c} \quad C_{c_i})(t) B_c(t) = \\
 & = [Q_{i,0}^{(d_i+k)}(t), Q_{i,1}^{(d_i+k)}(t), \dots, Q_{i,k}^{(d_i+k)}(t)] \begin{bmatrix} \hat{D}(t) \\ (L_{\hat{A}}^{(0)} \hat{C})(t) \hat{B}(t) \\ \vdots \\ (L_{\hat{A}}^{(k-1)} \hat{C})(t) \hat{B}(t) \end{bmatrix}
 \end{aligned}$$

$$\text{for } k=0,1,\dots; i=1,\dots,m, \text{ for all } t \in T \quad (6.20)$$

Define $k_i(t); i=1,\dots,m$, as

$$\begin{aligned}
 k_i(t) &= \min \left\{ k: (L_{A_c} \quad C_{c_i})^{(d_i+k)}(t) B_c(t) \neq 0; k=0,1,\dots \right\} \\
 & i=1,\dots,m; t \in T \quad (6.21)
 \end{aligned}$$

i.e.,

$$(d_c)_i(t) = d_i + k_i(t); i=1,\dots,m; t \in T \quad (6.22)$$

We shall assume from now on that $k_i(t)$, and thus $(d_c)_i(t); i=1,\dots,m$, are independent of t over the interval of interest $[t_0, t_1]$. If they are not, by continuity and differentiability assumptions, they will be constant over some subinterval of T and the analysis is carried out over these subintervals. (see remark 6,1).

If we let $\{z_i(t); i=1,\dots,m\}$ be any set of

continuous and differentiable linearly independent vectors over T , $(z_i(t) \in \mathbb{R}^m, i=1, \dots, m)$ then the dynamic decoupling problem can be reformulated as finding matrices $\hat{A}(t), \hat{B}(t), \hat{C}(t)$, and $\hat{D}(t)$, such that

$$(i). \quad (L_{A_c}^{(d_i+j)} C_{c_i})(t) B_c(t) = 0 \quad ; j=0, 1, \dots, k_i-1; \\ i=1, \dots, m. \quad (6.23)$$

for all $t \in T$, where $k_i, i=1, \dots, m$, are some positive integers, and

$$(ii). \quad (L_{A_c}^{(d_i+k_i)} C_{c_i})(t) B_c(t) = z_i(t); \quad i=1, \dots, m, \\ \text{for all } t \in T. \quad (6.24)$$

From eq.(6.20) these two conditions can be expressed as

$$\begin{bmatrix} Q_{i,0}^{(d_i)}(t) & , & 0 & \dots & 0 & , & 0 \\ Q_{i,0}^{(d_i+1)}(t) & , & Q_{i,1}^{(d_i+1)}(t) & \dots & 0 & , & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ Q_{i,0}^{(d_i+k_i-1)}(t) & , & Q_{i,1}^{(d_i+k_i-1)}(t) & \dots & Q_{i,k_i-1}^{(d_i+k_i-1)}(t) & , & 0 \\ Q_{i,0}^{(d_i+k_i)}(t) & , & Q_{i,1}^{(d_i+k_i)}(t) & \dots & Q_{i,k_i-1}^{(d_i+k_i)}(t) & , & Q_{i,k_i}^{(d_i+k_i)}(t) \end{bmatrix} \begin{bmatrix} \hat{D}(t) \\ (L_{\hat{A}}^{(0)} \hat{C})(t) \hat{B}(t) \\ \vdots \\ (L_{\hat{A}}^{(k_i-2)} \hat{C})(t) \hat{B}(t) \\ (L_{\hat{A}}^{(k_i-1)} \hat{C})(t) \hat{B}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z_i(t) \end{bmatrix} \quad (6.25)$$

for $i=1, \dots, m$, and every $t \in T$, for some $k_i \geq 0; i=1, \dots, m$.

Thus, if we let $k^* = \max\{k_i\}$ and define $\bar{Q}_i(t)$ to be the $k_i \times [m(k_i-1)]$ matrix given by

$$\bar{Q}_i(t) = \begin{bmatrix} \begin{matrix} (d_i) \\ Q_{i,0}(t) \end{matrix} & 0 & \dots & 0 & 0 \\ \begin{matrix} (d_i+1) \\ Q_{i,0}(t) \end{matrix} & \begin{matrix} (d_i+1) \\ Q_{i,1}(t) \end{matrix} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \begin{matrix} (d_i+k_i-1) \\ Q_{i,0}(t) \end{matrix} & \begin{matrix} (d_i+k_i-1) \\ Q_{i,1}(t) \end{matrix} & \dots & \begin{matrix} (d_i+k_i-1) \\ Q_{i,k_i-1}(t) \end{matrix} & 0 \end{bmatrix} \quad (6.26)$$

and $\hat{Q}_i(t)$ to be the $k_i \times [m(k^*-1)]$ matrix given by

$$\hat{Q}_i(t) = [\bar{Q}_i(t) \mid 0] \quad (6.27)$$

and let $\Gamma(t)$ be the $m \times [m(k^*-1)]$ matrix given by

$$\Gamma(t) = \begin{bmatrix} \begin{matrix} (d_1+k_1) \\ Q_{1,0}(t) \end{matrix}, \dots, \begin{matrix} (d_1+k_1) \\ Q_{1,k_m}(t) \end{matrix}, \dots, \begin{matrix} (d_1+k_1) \\ Q_{1,k_1}(t) \end{matrix}, \dots, 0, \dots, 0 \\ \begin{matrix} (d_2+k_2) \\ Q_{2,0}(t) \end{matrix}, \dots, \begin{matrix} (d_2+k_2) \\ Q_{2,k_m}(t) \end{matrix}, \dots, \begin{matrix} (d_2+k_2) \\ Q_{2,k_1}(t) \end{matrix}, \dots, \begin{matrix} (d_2+k_2) \\ Q_{2,k_2}(t) \end{matrix}, \dots, 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \begin{matrix} (d_m+k_m) \\ Q_{m,0}(t) \end{matrix}, \dots, \begin{matrix} (d_m+k_m) \\ Q_{m,k_m}(t) \end{matrix}, \dots, 0, \dots, 0, \dots, 0 \end{bmatrix} \quad (6.28)$$

Then, eqs.(6.25) can be written as

$$\begin{bmatrix} \hat{Q}_1(t) \\ \hat{Q}_2(t) \\ \vdots \\ \hat{Q}_m(t) \\ \hat{T}(t) \end{bmatrix} \begin{bmatrix} \hat{D}(t) \\ (L_{\hat{A}}^{(0)} \hat{C})(t) \hat{B}(t) \\ \vdots \\ (L_{\hat{A}}^{(k^*-2)} \hat{C})(t) \hat{B}(t) \\ (L_{\hat{A}}^{(k^*-1)} \hat{C})(t) \hat{B}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ Z(t) \end{bmatrix} \quad (6.29)$$

where $Z(t)$ is the nonsingular matrix (for all $t \in T$) whose i^{th} row is $z_i(t)$.

If we could find matrices $\hat{A}(t), \hat{B}(t), \hat{C}(t)$ and $\hat{D}(t)$ such that for some integers $k_i; i=1, \dots, m$, eq.(6.29) was satisfied, then $B_c^*(t) = Z(t)$ for every $t \in T$, and from theorem 6.1, the composite system would be uniformly decouplable by state feedback.

Notice that if eq.(6.29) is written in a concise form as

$$Y(t) X(t) = W(t) \quad (6.30)$$

all the unknown parameters of the problem, i.e., $\hat{A}(t), \hat{B}(t), \hat{C}(t)$ and $\hat{D}(t)$, are in $X(t)$, which is a very desirable feature, allowing us to prove

Theorem 6.4. There exists a precompensator $\hat{\Sigma}(t)$ such that the overall system $\hat{\Sigma}_c(t)$ of eqs.(6.10)-(6.11) is uniformly decouplable by state feedback if and only if for some set of positive integers k_i ; $i=1, \dots, m$,

$$\text{rank} \begin{bmatrix} \hat{Q}_1(t) \\ \hat{Q}_2(t) \\ \vdots \\ \hat{Q}_m(t) \\ \Gamma(t) \end{bmatrix} = \text{rank} \begin{bmatrix} \hat{Q}_1(t) & 0 \\ \hat{Q}_2(t) & 0 \\ \vdots & \vdots \\ \hat{Q}_m(t) & 0 \\ \Gamma(t) & Z(t) \end{bmatrix} \quad \forall t \in T. \quad (6.31)$$

where $Z(t)$ is an arbitrary nonsingular matrix with continuous and differentiable elements over $[t_0, t_1]$.

Proof. It is a well known fact that a matrix equation of the form $Y X = W$ will have a solution X , if and only if $\text{rank}[Y] = \text{rank}[Y, W]$. If Y, X and W are time dependent, then obviously, this must be satisfied for all $t \in T$. Thus eq.(6.31) is a necessary and sufficient condition for the existence of a solution to eq.(6.29). If a solution does exist, then it is a simple matter to show that there exists at least one representation $\hat{A}(t), \hat{B}(t), \hat{C}(t), \hat{D}(t)$ which reproduces the solution. This will be shown in the proof of

theorem 6.6 (section 3 of this chapter). Although we will come back to this later, we point out that the problem of finding a minimal order representation is a very difficult one.

Remark 6.3. Notice that if $[\hat{Q}_1(t)^T, \dots, \hat{Q}_m(t)^T, T(t)^T]^T$, ($(\cdot)^T$ denoting "transpose") is a matrix of constant rank over T , then by the implicit function theorem, the solution to eq.(6.29), when one exists, will be continuous and differentiable. If this matrix is not of constant rank, then one may partition T into open subintervals over which it is of constant rank, and carry out the analysis over these subintervals. However, it is not possible to ensure continuity at the times where the rank changes and thus, the solution of the dynamic decoupling problem would probably require a different precompensator for each subinterval (i.e., a precompensator with discontinuous parameters).

Remark 6.4. Theorem 6.4 above, extends theorem 3.2 to the time varying case, although it has been stated in a different fashion, better suited to the dynamic decoupling problem.

Although theorem 6.4 has been stated for the uniform decoupling problem, it is evident that with trivial modifications it will hold for other types

of decoupling, i.e., total, complete, etc. (see appendix A.1). For example, if eq.(6.31) holds for every open subinterval of T , then $\Sigma_c(t)$ will be totally decouplable.

From eq.(6.29) it is also easy to establish

Corollary 6.4.1. For the overall system $\Sigma_c(t)$ to be uniformly decouplable by state feedback, the pre-compensator $\hat{\Sigma}(t)$ must be uniformly input observable with input observability index smaller or equal to k^* . Further, its direct input-output map $\hat{D}(t)$ must be singular for all $t \in T_\phi$, where T_ϕ denotes the set of points where $\Sigma(t)$ is not decouplable.

Proof. That $\hat{\Sigma}(t)$ must be uniformly input observable with input observability index smaller or equal to k^* follows from eq.(6.29) because as $Z(t)$ is nonsingular for all $t \in T$, then applying Sylvester's inequality it must be true that

$$\text{rank} \begin{bmatrix} \hat{D}(t) \\ (L_A^{(0)} \hat{C})(t) \hat{B}(t) \\ \vdots \\ (L_A^{(k^*-1)} \hat{C})(t) \hat{B}(t) \end{bmatrix} = m \quad \forall t \in T.$$

Now, if $\hat{D}(t)$ is nonsingular for any $t \in T_\phi$, then

$$(L_A^{(d_i)} c_i)(t) \hat{B}(t) \hat{D}(t) \neq 0 \quad ; i=1, \dots, m$$

and

$$B_c^*(t) = B^*(t) \hat{D}(t)$$

which is singular for every $t \in T_0$, and thus, $\Sigma_c(t)$ is not decouplable uniformly.

Corollary 6.4.1 above is the time varying version of corollary 3.2.2.

3). Procedure to obtain the precompensator.

Theorem 6.4 above, gives a necessary and sufficient condition for the existence of a solution to the dynamic decoupling problem in terms of a set of integers k_i ; $i=1, \dots, m$, which have not been fixed, i.e., it is sufficient that eq.(6.31) holds for some combination of values k_i ; $i=1, \dots, m$. Thus, in practice, theorem 6.4 is of little help because the process of finding a priori a set of values k_i ; $i=1, \dots, m$ for which eq.(6.31) is satisfied (if there is one), would be, in general, excessively long and tedious.

Below we give a procedure to determine whether or not a given system is dynamically decouplable, which at the same time yields the smallest possible value for k_i ; $i=1, \dots, m$, when a solution to the problem exists.

Before describing the algorithm and our main result, we shall give a definition and state a few

results which will be needed later. The proofs are omitted here to avoid breaking the exposition unnecessarily, but are given in appendix A.2.

Definition 6.1. Let $A(t)$ be any $p \times q$ matrix ($p < q$), whose i^{th} row is denoted by $a_i(t)$. Then $a_i(t)$ is said to be a rank-essential (r.e.)-row of $A(t)$ at $t \in [t_0, t_1]$, if $a_i(t)$ cannot be expressed as a linear combination of $a_j(t); j=1, \dots, p; j \neq i$, at $t \in T$. If $a_i(t)$ is a r.e.-row of $A(t)$ for all $t \in T$, then $a_i(t)$ is said to be a uniformly rank essential (u.r.e.)-row of $A(t)$.

This definition extends the concept of rank-essentiality [9] to the time-varying situation.

Consider an arbitrary matrix $X(t)$ of dimension $p \times q$ ($p < q$) and let $x_i(t)$ denote its i^{th} row. Define $I = \{1, 2, \dots, p\}$ and $I_0 = \{i: i \in I \text{ and } x_i(t) \text{ is a u.r.e.-row of } X(t)\}$, where I_0 has r elements ($r \leq p$). Let $J_0 \subseteq I_0$ be an arbitrary subset of I_0 with r_0 elements ($r_0 \leq r$).

Lemma 6.1. For any arbitrary set of indices $J_0 \subseteq I_0$, $\text{rank}[X(t)] = r_0 + \text{rank}[\{x_i(t)\}; i \in I - J_0]$ for every $t \in T$.

Proof. See appendix A.2.

Theorem 6.5. Let $X(t)$ be an arbitrary $p \times q$ matrix ($p \leq q$). Let $X(t)$ be partitioned as

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

where $X_2(t)$ is an $r \times q$ arbitrary matrix. Let $Z(t)$ be any $r \times r$ nonsingular matrix over $[t_0, t_1]$. Then,

$$\text{rank} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \text{rank} \begin{bmatrix} X_1(t) & 0 \\ X_2(t) & Z(t) \end{bmatrix} \quad \forall t \in T \quad (6.32)$$

if and only if all the rows of $X_2(t)$ are u.r.e.-rows of $X(t)$.

Proof. See appendix A.2.

With these results in mind, let us assume we are given a system whose $B^*(t)$ matrix is singular for every $t \in T_\phi \subseteq T$ (T_ϕ as previously defined). From eq. (6.18) we can rewrite $B^*(t)$ as

$$B^*(t) = \begin{bmatrix} \begin{matrix} (d_1) \\ (L_A^{c_1})(t)B(t) \end{matrix} \\ \vdots \\ \begin{matrix} (d_m) \\ (L_A^{c_m})(t)B(t) \end{matrix} \end{bmatrix} = \begin{bmatrix} \begin{matrix} (d_1) \\ Q_{1,o}(t) \end{matrix} \\ \vdots \\ \begin{matrix} (d_m) \\ Q_{m,o}(t) \end{matrix} \end{bmatrix} \quad \forall t \in T$$

Let us fix $k_i=0$; $i=1, \dots, m$, $\forall t \in [t_0, t_1]$ in eq.(6.22).
Then $\Gamma(t)$ in eq.(6.28) becomes $B^*(t)$, and as $\hat{Q}_i(t)$;
 $i=1, \dots, m$, (eq.(6.27)) are non-existent in this case,
eq.(6.29) can be written as

$$\Gamma(t) \hat{D}(t) = [B^*(t)] \hat{D}(t) = Z(t)$$

and there exists a solution to the uniform dynamic
decoupling problem if and only if (theorem 6.4)

$$\text{rank}[B^*(t)] = \text{rank}[B^*(t), Z(t)] \quad \forall t \in T$$

Clearly this does not hold, because $B^*(t)$ is
singular over T_ϕ and on the other hand $Z(t)$ is non-
singular over $T(\supset T_\phi)$.

Thus, let

$$J_o = \{i: \{Q_{i,o}^{(d_i)}(t)\} \text{ is a u.r.e.-row of } B^*(t)\} \quad \text{---(6.33)}$$

and let r_o denote the number of elements in J_o .

Also let $I=\{1, \dots, m\}$ and define

$$H_{(1)}(t) = \begin{bmatrix} \{Q_{i,o}^{(d_i)}(t), 0\} & ; i \in I-J_o \\ \hline \{Q_{i,o}^{(d_i)}(t), 0\} & ; i \in J_o \\ \{Q_{i,o}^{(d_i+1)}(t), Q_{i,l}^{(d_i+1)}(t)\} & ; i \in I-J_o \end{bmatrix} \quad \forall t \in T \quad \text{---(6.34)}$$

Now, with $k_i=0$ for $i \in J_o$, and $k_i=1$ for $i \in I-J_o$,

we must consider finding a solution for

$$[H_{(1)}(t)] \begin{bmatrix} \hat{D}(t) \\ (L_{\hat{A}}^{(0)} \hat{C})(t) \hat{B}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ Z(t) \end{bmatrix} \quad \forall t \in T \quad (6.35)$$

From theorem 6.5 above, there exists a solution to eq.(6.35) if and only if the last m rows of $H_{(1)}(t)$ are u.r.e.-rows. If this is the case, then the parameters of the precompensator would be found first solving eq.(6.35) and then finding $\hat{A}(t), \hat{B}(t), \hat{C}(t)$ and $\hat{D}(t)$ to match this solution. (See the proof of theorem 6.6 below).

If the last m rows of $H_{(1)}(t)$ are not u.r.e.-rows, then define

$$J_{1,0} = \{i: \{Q_{i,0}^{(d_i)}(t), 0\} \text{ is a u.r.e.-row of } H_{(1)}(t); i \in I\}$$

and

$$J_{1,1} = \{i: \{Q_{i,0}^{(d_i+1)}(t), Q_{i,1}^{(d_i+1)}(t)\} \text{ is a u.r.e.-row of } H_{(1)}(t); i \in I - J_0\}$$

and let

$$H_{(2)}(t) = \begin{bmatrix} \{Q_{i,o}^{(d_i)}(t), 0, 0\} & ; i \in I-J_{1,o} \\ \{Q_{i,o}^{(d_i+1)}(t), Q_{i,1}^{(d_i+1)}(t), 0\} & ; i \in I-J_{1,o}-J_{1,1} \\ \hline \{Q_{i,o}^{(d_i)}(t), 0, 0\} & ; i \in J_{1,o} \\ \{Q_{i,o}^{(d_i+1)}(t), Q_{i,1}^{(d_i+1)}(t), 0\} & ; i \in J_{1,1} \\ \{Q_{i,o}^{(d_i+2)}(t), Q_{i,1}^{(d_i+2)}(t), Q_{i,2}^{(d_i+2)}(t)\} & ; i \in I-J_{1,o}-J_{1,1} \end{bmatrix} \quad \begin{array}{c} \downarrow \\ m \end{array} \quad (6.36)$$

and see if a solution exists for

$$[H_{(2)}(t)] \begin{bmatrix} \hat{D}(t) \\ (L_{\hat{A}}^{(0)} \hat{C})(t) \hat{B}(t) \\ (L_{\hat{A}}^{(1)} \hat{C})(t) \hat{B}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Z(t) \end{bmatrix} \quad (6.37)$$

Again, from theorem 6.3, a solution will exist if and only if the last m -rows of $H_{(2)}(t)$ are u.r.e.-rows. If a solution exists, it would yield $(d_c)_i = d_i$ for all $i \in J_{1,o}$; $(d_c)_i = d_i + 1$ for all $i \in J_{1,1}$; and $(d_c)_i = d_i + 2$ for all $i \in I-J_{1,o}-J_{1,1}$ and $B_c^*(t) = Z(t)$, nonsingular over T . If the last m rows of $H_{(2)}(t)$ are not u.r.e.-rows we would have to define $H_{(3)}(t)$ in a similar way as we did with $H_{(2)}(t)$ and go a step further. In general, if the last m rows of $H_{(q-1)}(t)$ are not u.r.e.-rows, then we define in the q^{th} step of the algorithm the matrix $H_{(q)}(t)$ as

$$H_q(t) = \begin{bmatrix} \{ Q_{i,0}^{(d_i)}(t), 0, \dots, 0, 0 \}; i \in I - J_{q-1,0} \\ \{ Q_{i,0}^{(d_i+1)}(t), Q_{i,1}^{(d_i+1)}(t), \dots, 0, 0 \}; i \in I - J_0 - J_{q-1,1} \\ \vdots \\ \{ Q_{i,0}^{(d_i+q-1)}(t), Q_{i,1}^{(d_i+q-1)}(t), \dots, Q_{i,q-1}^{(d_i+q-1)}(t), 0 \}; i \in I - J_{q-2} - J_{q-1,q-1} \\ \hline \{ Q_{i,0}^{(d_i)}(t), 0, \dots, 0, 0 \}; i \in J_{q-1,0} \\ \{ Q_{i,0}^{(d_i+1)}(t), Q_{i,1}^{(d_i+1)}(t), \dots, 0, 0 \}; i \in J_{q-1,1} \\ \vdots \\ \{ Q_{i,0}^{(d_i+q-1)}(t), Q_{i,1}^{(d_i+q-1)}(t), \dots, Q_{i,q-1}^{(d_i+q-1)}(t), 0 \}; i \in J_{q-1,q-1} \\ \{ Q_{i,0}^{(d_i+q)}(t), Q_{i,1}^{(d_i+q)}(t), \dots, Q_{i,q-1}^{(d_i+q)}(t), Q_{i,q}^{(d_i+q)}(t) \}; i \in I - J_{q-1} \end{bmatrix}$$

where

$$J_{k,l} = \{j: j \in I - J_{l-1} \text{ ; and } \{Q_{j,0}^{(d_j+l)}(t), Q_{j,1}^{(d_j+l)}(t), \dots, Q_{j,l}^{(d_j+l)}(t), Q_{(k-l)}\}$$

is a u.r.e.-row of $H_{(k)}(t)$

for $l=0,1,\dots,k$, and $k=0,1,\dots$, where $0_{(k-l)}$ is a $(k-l)m$ -zero row vector,

$$J_{-1} = \{\emptyset\}$$

and

$$J_k = \bigcup_{j=0}^k J_{k,j}$$

With $H_{(q)}(t)$ so defined, we would seek a solution for

$$[H_{(q)}(t)] \begin{bmatrix} \hat{D}(t) \\ (L_{\hat{A}}^{(0)} \hat{C})(t) \hat{B}(t) \\ \vdots \\ (L_{\hat{A}}^{(q-1)} \hat{C})(t) \hat{B}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ z(t) \end{bmatrix} \quad (6.39)$$

which from theorem 6.5 exists if and only if the last m rows of $H_{(q)}(t)$ are uniform rank essential, and thus,

Theorem 6.6. There exists a precompensator $\hat{\Sigma}(t)$ such that the composite system $\Sigma_c(t)$ is uniformly decouplable by state feedback, if and only if there exists an integer, say α , such that the last m rows of $H_{(\alpha)}(t)$ are u.r.e.-rows.

Proof. From the discussion above and theorem 6.5, it is obvious that the existence of an integer α for which eq.(6.39) has a solution is necessary. To show sufficiency it remains to be shown that, when there exists an α such that eq.(6.39) has a solution, we can actually find matrices $\hat{A}(t), \hat{B}(t), \hat{C}(t)$ and $\hat{D}(t)$, which will reproduce it. So, let the solution of eq.(6.39) ($q=\alpha$) be denoted by $X(t)$ and let it be partitioned as

$$X(t) = \begin{bmatrix} X_0(t) \\ X_1(t) \\ \vdots \\ X_\alpha(t) \end{bmatrix}$$

where $X_i(t)$; $i=0,1,\dots,\alpha$, are $m \times m$ matrices. Define

$$\hat{D}(t) = X_0(t) \quad \text{--- (6.40)}$$

and

$$\hat{A}(t) = \begin{bmatrix} \bar{A}_1 & 0 & \dots & 0 \\ 0 & \bar{A}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{A}_m \end{bmatrix}; \bar{A}_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{--- (6.41)}$$

where $\bar{A}_i \in \mathbb{R}^{(\alpha-1) \times (\alpha-1)}$,

$$\hat{C}(t) = [\bar{c}_1, \bar{c}_2, \dots, \bar{c}_m]; \bar{c}_i = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad \begin{matrix} i^{\text{th}} \text{ row} \\ \text{--- (6.42)} \end{matrix}$$

where $\bar{c}_i \in \mathbb{R}^{m \times (\alpha-1)}$, and finally

$$\hat{B}(t) = \begin{bmatrix} \bar{B}_1(t) \\ \bar{B}_2(t) \\ \vdots \\ \bar{B}_m(t) \end{bmatrix}; \bar{B}_i(t) = \begin{bmatrix} \{X_1(t)\}_i \\ \{X_2(t)\}_i \\ \vdots \\ \{X_{\alpha-1}(t)\}_i \end{bmatrix}; \bar{B}_i \in \mathbb{R}^{(\alpha-1) \times m} \quad (6.43)$$

where $\{X_j(t)\}_i$ denotes the i^{th} row of $X_j(t)$.

With $\hat{A}(t)$ and $\hat{C}(t)$ so defined, it is evident that

$$(L_{\hat{A}}^{(j)} \hat{C})(t) = \hat{C} \hat{A}^j; j=0,1,\dots$$

and a direct and simple verification shows that

$$(L_{\hat{A}}^{(j)} \hat{C})(t) \hat{B}(t) = \hat{C} \hat{A}^j \hat{B}(t) = X_{j+1}(t)$$

for $j=0,1,\dots,\alpha-1$, and every $t \in T$. Thus $\hat{A}(t), \hat{B}(t), \hat{C}(t)$ and $\hat{D}(t)$ as defined by eqs.(6.40)-(6.43), reproduce the solution of eq.(6.39). Therefore they are a solution to the uniform decoupling problem, and theorem 6.6 follows.

Remark 6.5. When solving eq.(6.39), one only has to consider the last m rows of $H_{(\alpha)}(t)$ and those which form a basis for the span of the remaining rows of $H_{(\alpha)}(t)$. The rest of the rows of $H_{(\alpha)}(t)$ provide no additional information and may be disregarded.

When particularized to the time invariant situation,

the above results correspond with those of M.Cremer [9]. However, for time invariant systems it is possible to show in a direct manner that whenever the given system is invertible, there exists an integer α for which the algorithm terminates. Further, α is the same integer which terminates the structure algorithm of Silverman [60],[61]. In the time varying case this is still an open problem, but from theorem 6.2, it must be possible to show directly that if $\sum (t)$ is invertible, there exists an α for which the last m rows of $H_{(\alpha)}(t)$ are u.r.e.-rows.

The order of the realization of the precompensator obtained in the proof of theorem 6.6 (eqs.(6.40)-(6.43)), is unnecessarily large. It may be possible to find a precompensator of smaller order which reproduces the solution of eq.(6.39). Notice however, that to do so, time varying $\hat{A}(t)$ and $\hat{C}(t)$ matrices must be considered, in order to have the number of free parameters which would be required. Thus, one would have to solve a set of (αm) nonlinear differential equations $((L_{\hat{A}}^{(j)} \hat{C})(t) \hat{B}(t) = X_{j+1}(t) ; j=0,1,\dots,\alpha-1)$ which is certainly a problem of greater magnitude than our original problem.

Furthermore, from the construction of the algorithm leading to theorem 6.6, it is clear that when a solution exists it has the smallest possible values of $(d_c)_i ; i=1,\dots,m$. In the time invariant case this

implies that the precompensator is of minimal order [9]. For time varying systems the situation is more complicated, for even if we assume that a precompensator of order $(\alpha-1)$ was found, it would be minimal only if we impose the restriction that precompensators must be of constant order over the time interval of interest. (In practice, this restriction is reasonable). This is due to the fact that in each step of the algorithm a search is made only for u.r.e.-rows. Notice however, that at all times $t \in T$, the number of r.e.-rows of $H_{(k)}(t)$; $k=1,2,\dots$, is greater than or equal to the number of u.r.e.-rows. Hence, if we apply the algorithm locally (i.e., searching for r.e.-rows at each $t \in T$) and denote by $\alpha(t)$ the smallest integer such that the last m rows of $H_{(\alpha(t))}(\cdot)$ evaluated at t are r.e.-rows, obviously $\alpha(t) \leq \alpha$, for every $t \in T$. Therefore, provided one could ensure continuity, it may be possible to obtain a precompensator of variable order $\alpha(t) \leq \alpha \forall t \in T$, which is a solution to the dynamic decoupling problem.

4). Discussion of the difficulties arising from the removal of the assumption $d_i(t) = \text{Constant}$.

As we have pointed out in section 1 of this chapter, to obtain theorem 6.1 it is assumed that

$d_i(t) = \text{Constant}$ for $i=1, \dots, m$. We also mentioned (remark 6.1) that if this is not the case, having assumed continuity and differentiability of the parameters of $\Sigma(t)$, then $d_i(t)=k$ for $t \in T$, implies $d_i(t)=k$ for some interval containing t (k being an arbitrary integer). It is customary [33],[56], in this situation to split the time interval of interest T into subintervals over which the d_i 's are constant and to carry on the analysis over these subintervals. However, at the discontinuity points of $d_i(t)$ some technical and practical problems arise. At these points, the feedback gains may be discontinuous, or worse, unbounded. To illustrate this, consider the following time varying system $\Sigma(t)=(A(t),B(t),C(t))$ where

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} ; B(t) = \begin{bmatrix} f(t) & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} ; C(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

defined over the time interval of interest $T=[0,2]$, and where

$$f(t) = \begin{cases} 0 & \forall t \in [0,1] \\ t^3 - 3t^2 + 3t - 1 & \forall t \in (1,2]. \end{cases}$$

Notice that for decoupling purposes $A(t), B(t)$ and $C(t)$ satisfy the continuity and differentiability

assumptions required. Simple calculations show that

$$d_1(t) = \begin{cases} 1 & \forall t \in [0, 1] \\ 0 & \forall t \in (1, 2] \end{cases}$$

and

$$d_2(t) = 0 \quad \forall t \in [0, 2]$$

Further, $B^*(t)$ is given by

$$B^*(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \forall t \in [0, 1] \\ \begin{bmatrix} f(t) & 0 \\ 0 & 1 \end{bmatrix} & \forall t \in (1, 2] \end{cases}$$

which is nonsingular for all $t \in [0, 2]$, and thus, the system $\Sigma(t)$ is uniformly decouplable. However, $B_{1,1}^*(t)$, the $(1,1)^{th}$ -element of $B^*(t)$ is discontinuous, i.e.,

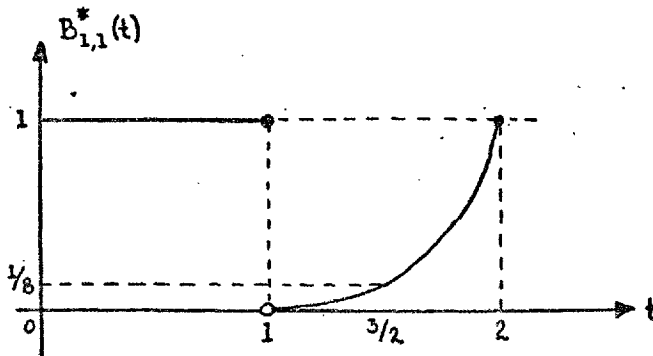


Fig.6.2

The integrator decoupling feedback pair, $(F^*(t), G^*(t))$ defined by

$$F^*(t) = -B^*(t)^{-1} A^*(t)$$

$$G^*(t) = B^*(t)^{-1} \Delta(t)$$

where $\Delta(t) = \text{diag} \{ \lambda_i(t) \}$ for all $t \in T$ is nonsingular,
and

$$A^*(t) = \begin{bmatrix} (d_1+1) & & \\ (L_A & c_1)(t) & \\ & \vdots & \\ & (d_m+1) & \\ (L_A & c_m)(t) & \end{bmatrix}$$

is given for our example by

$$F^*(t) = \begin{cases} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} & \forall t \in [0,1] \\ \begin{bmatrix} 0 & -1/f(t) & 0 \\ 0 & -1 & -1 \end{bmatrix} & \forall t \in (1,2] \end{cases}$$

and

$$G^*(t) = \begin{cases} \begin{bmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{bmatrix} & \forall t \in [0,1] \\ \begin{bmatrix} \frac{\lambda_1(t)}{f(t)} & 0 \\ 0 & \lambda_2(t) \end{bmatrix} & \forall t \in (1,2] \end{cases}$$

Fig.6.3 below depicts elements (1,1) and (1,2) of $F^*(t)$ as functions of time.

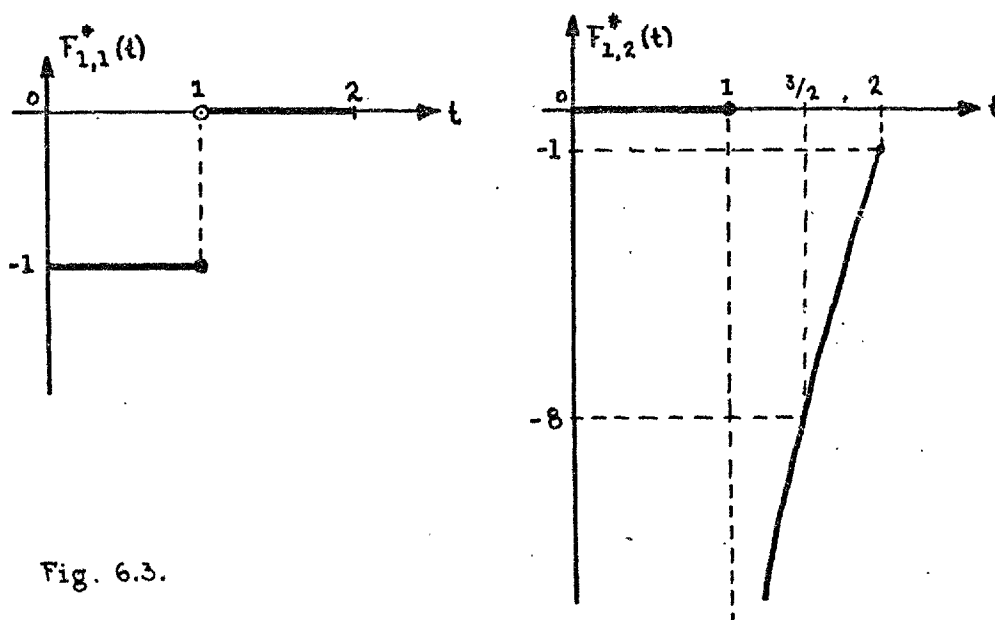


Fig. 6.3.

Clearly $F_{1,2}^*(t)$ is not bounded for $t=1^{(+)}$, and thus, although $(F^*(t), G^*(t))$ decouples $\Sigma(t)$ uniformly, it is unacceptable in practice. In this example, $d_i(t)$; $i=1,2$, are non-increasing functions, but it should be clear that a similar thing would happen if $d_i(t)$; $i=1, \dots, m$ were non-decreasing integer valued functions.

In general, if the parameters of $\Sigma(t)$ satisfy the continuity and differentiability assumptions, then if $d_i(t)$ changes its value at say $\hat{t} \in T$, either

- (i). The decoupling feedback matrices have some unbounded feedback gains as $t \rightarrow \hat{t}$ from the left if $d_i(\hat{t}^{(-)}) < d_i(\hat{t}^{(+)})$ (i.e., if $d_i(t)$ increases its value at $t=\hat{t}$), or

(ii). The decoupling feedback matrices have some unbounded feedback gains as $t \rightarrow \hat{t}$ from the right, if $d_i(\hat{t}^{(-)}) > d_i(\hat{t}^{(+)})$ (i.e., if $d_i(t)$ decreases its value at $t=\hat{t}$).

Thus, if the parameters of the system are continuous and differentiable, the feedback gains in the decoupling pair $(F(t), G(t))$ will be bounded only if the d_i 's are constant over the time interval of interest.

CHAPTER VII. Conclusions and suggestions for further
research.

In Chapters II to VI of this dissertation we have analyzed several decoupling problems extending the results already published in the literature by looking at the structure of decoupled systems in more detail, by considering problems which up to now had scarcely been studied, and by generalizing known results to the (discrete and continuous) time varying situation. However, it is clear that many interesting possibilities remain open and many problems are yet to be solved, both within the areas studied here and for the general decoupling problem.

In Chapter II it was shown that for a subclass of the set of linear systems, the set of decoupling invariants $\{d_i; i=1, \dots, m\}$ was complete and independent under similarity and state feedback transformations. A (unique) canonical representation is therefore possible, which is completely determined by the values of the d_i 's. For the systems under consideration, the set of decoupling invariants coincides with both, the observability and control-

$$q_i = \min\{j:c_1, \dots, c_m, c_i A, \dots, c_m A, \dots, c_i A^j, \dots, c_i A^j,$$

are a set of linearly dependent vectors}\}.

To establish more definite relationships between $\{d_i\}$, $\{n_i\}$ and $\{q_i\}$, is not easy and further research is needed. This could lead to establishing an (A,B,C)-canonical form based on the set of decoupling invariants suitably augmented, for general linear systems. It is our opinion that if a canonical representation with A,B and C of a specific form is to be found, one should look at input-output invariants. This makes the d_i 's a suitable starting point, although not the only one. It would also be helpful to study the partition into equivalence classes introduced by other transformations such as output feedback or even dynamic state and output feedback, and their relation to the decoupling.

ling problem.

In Chapter II, we were also able to obtain an explicit necessary and sufficient condition for the existence of a solution to the output feedback decoupling problem for the set of linear systems which preserve observability under arbitrary state feedback. If generalized to arbitrary linear systems this condition would become implicit. i.e., it would depend on the feedback matrix K in eq.(2.30), [33]. This would render it difficult to apply in practice. We feel that for the general case a simple explicit algebraic condition is still needed and may be found.

Posing the problem in the frequency domain may be helpful. For instance, it is not difficult to show that if $H(s)$ denotes the transfer function matrix of a given system $S=(A,B,C)$, then S will be decouplable by output feedback ($u(s) = K y(s) + G v(s)$) if and only if there exist matrices K and G . (G nonsingular) such that

$$K + G Q(s) = H(s)^{-1}$$

where $Q(s)$ is a diagonal nonsingular non-proper rational matrix. The possibility of carrying this result further to obtain the desired condition should be investigated. (Notice that K is the critical unknown, as G may be taken as $G = B^*{}^{-1}\Delta$).

A second approach which also looks promising is to manipulate algebraically the implicit necessary and sufficient condition given by Falb and Wolowich [17], (see the proof of Lemma 2.2), to obtain an equivalent but explicit condition, in the form of a matrix equation required to be solvable. With this approach it is likely that a nice general characterization of the class of (output and state) feedback decoupling pairs may be also obtained.

In Chapter III, we studied the problem of decoupling composite linear systems. Of the three possible connections considered, it was seen that in general, only in the feedback case the decouplability of the overall composite system was "dominated" by one of the subsystems; namely the subsystem in the feedforward path (Theorems 3.3 and 3.4). As we pointed out, considered as part of the dynamic decoupling problem, this means that dynamic feedback alone is not enough to obtain an overall decouplable system when the given plant is non-decouplable. That is, B^* of eq.(1.5) remains invariant under dynamic feedback. However, when precompensation is used to solve the dynamic state-feedback problem, the closed loop system for the most common schemes [9],[36],[62], may be diagrammatically represented as

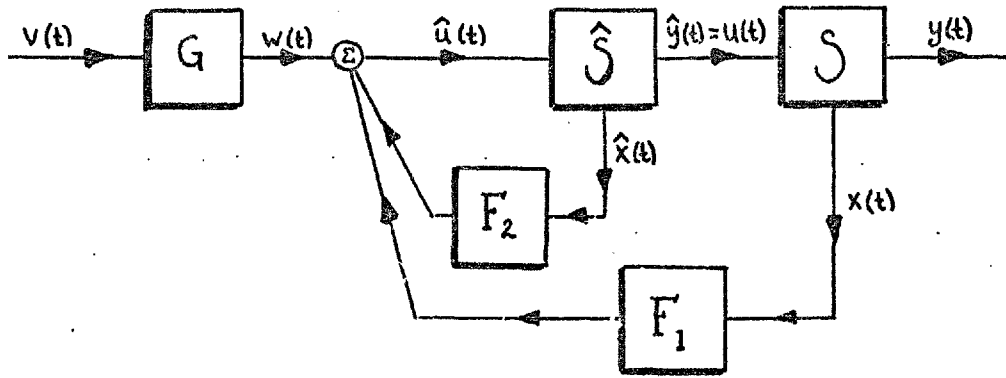


Fig. 7.1

Or equivalently, as

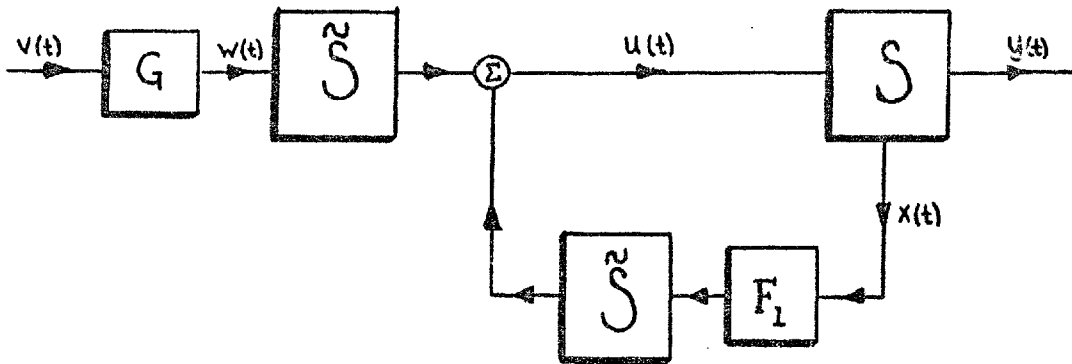


Fig. 7.2

Where the closed-loop system $\hat{S}(F_2, I)$ (i.e., system $\hat{S}$ plus the local feedback $u = F_2 \hat{x} + w$) has been denoted by $\tilde{S}$.

The dynamic compensation in the feedforward path is the crucial transformation used to obtain an overall decouplable system. The dynamic feedback in fig.

7.2, is used to obtain extra degrees of freedom in the design, and possibly to decrease the order of the required precompensator. This last fact however, has not been explicitly shown in the literature. Further, Theorem 3.4, and the fact that under a slightly more general precompensation scheme than that of Fig.7.1, [22], the minimal precompensator needed to decouple is still of the same order, indicate that this may not be the case. However, if the minimal order precompensator for decoupling was indeed of dimension strictly smaller than that of the minimal order inverse system, this would be so only due to the presence of dynamics in the feedback path. Hence the use of a more general compensation scheme, where the feedforward and feedback paths were not constrained to have the same dynamics, as in Fig.7.2, may further reduce the overall state space extension needed to solve the dynamic decoupling problem.

Propositions 3.1-3.4, have been proved by the author in the frequency domain, using a different approach based on

Proposition 7.1. Let $H(s) = \frac{N(s)}{\Delta(s)}$; $\Delta(s) = \det(Is - A)$,

be the transfer function matrix of a given system

$S=(A,B,C)$. Then, S is decouplable by state feedback if and only if $N(s)$ is a row-proper matrix.

Proof. The proposition follows directly from the Leverrier expansion of $H(s)$ (eg.[21]), and the definitions of row-properness[84], $d_i; i=1, \dots, m$ and B^* .

Now, given two proper rational matrices

$$H_1(s) = N_1(s)/\Delta_1(s) \quad \text{and} \quad H_2(s) = N_2(s)/\Delta_2(s), \quad \text{where}$$

$N_1(s)$ and $N_2(s)$ are row-proper polynomial matrices, it is relatively simple to show that $[N_1(s)*N_2(s)]$ and $[N_1(s)+N_2(s)]$ are not necessarily row-proper and thus, propositions 3.1 and 3.3 follow. Propositions 3.2 and 3.4 can be proved using a similar argument.

To prove the remaining results of chapter III using this approach is not as simple. Further research would be needed. Although it is not completely clear how the resulting proofs would improve those given here, it is possible that working in the frequency domain, a link between non-interaction and concepts such as diagonal dominance, for example, could be established.

The next problem considered in the thesis was that of parameter variations. Although the results given in chapter IV completely solve the data sensi-

tivity and decoupling problem, more research is needed to solve the input-output invariance problem. From lemma 4.3, it is clear that $\mathbb{R}^n$ can be decomposed as

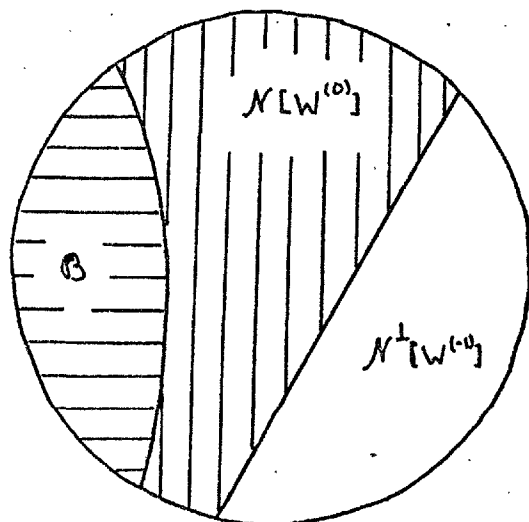


Fig.7.3. Decomposition of $\mathbb{R}^n$.

where $\mathcal{N}^\perp[W^{(-1)}]$ denotes the orthogonal complement of $\mathcal{N}[W^{(-1)}]$. In section 3, chapter III, we only considered those cases where δA and δB lie in the shaded areas of fig.7.2, (i.e., in $\mathcal{N}[W^{(-1)}]$).

However, we showed with an example that solutions to the input-output invariance problem also exist in $\mathcal{N}^\perp[W^{(-1)}]$, even though these may be unsatisfactory from the point of view of the controller. A more detailed and explicit partition of $\mathbb{R}^n$ in terms of $\mathcal{N}[W_i^{(-j)}]$; $i=1, \dots, m$; $j=0, 1, \dots$, is possible and may be useful to extend the results.

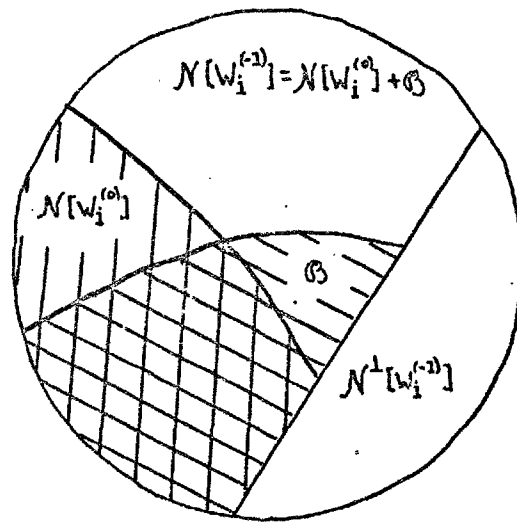


Fig.7.4. Decomposition of $\mathbb{R}^n$ in terms of $\mathcal{N}[W_i^{(-j)}]$, $j=0,1$.

Notice that $\dim\{\mathcal{N}[W_i^{(0)}] \cap B\} = m-1$; $\forall i=1, \dots, m$, and that $\mathcal{N}[W_i^{(0)}] + B = \mathcal{N}[W_i^{(-1)}]$; $\forall i=1, \dots, m$, but the sum is no longer a direct sum.

Recalling that decoupling is a generic property for m -input m -output systems [15],[16], it is almost certain that very general results will be obtainable (although not all will hold for the block-decoupling problem which is non-generic).

Further problems which should also be investigated are the effect of parameter variations in F , the state feedback matrix, and the data sensitivity and decoupling, and input-output invariance problems when output feedback is used to decouple. It is clear that the results of chapter IV will be necessary

conditions for the existence of a solution to these problems in the output feedback situation, but they almost certainly will not be sufficient.

The study of the effect that parameter variations have on the decoupling problem can be used to solve a further problem of interest. When decoupling is not possible by state feedback, one is forced to use extra dynamic elements if noninteraction is to be achieved. However, in some problems, such as some process control problems, "steady state" or "asymptotic" decoupling may be sufficient. One possible approach to this type of problems was investigated by W.A. Wolovich, [85], who defined and solved the "static" decoupling problem. A less rigid definition of an asymptotic decoupling problem can be stated as:

Find a feedback pair (F, G) such that

(i). The variations $\Delta y_{i,j}(t)$ in $y_i(t)$, the i^{th} output of the closed-loop system, due to variations in $v_j(t)$, the j^{th} closed-loop input, $j \neq i$, tend to zero as $t \rightarrow \infty$ and $|\Delta y_{i,j}(t)| < \epsilon$ for every $t \geq \tilde{t}$, where ϵ is an arbitrary small scalar ($\epsilon > 0$), and $\tilde{t} \in [0, \infty)$, and

(ii). The variations in $y_i(t)$ due to variations in $v_i(t)$ are not identically zero for all $t \in [\tilde{t}, \infty)$.

One possible approach to solve this asymptotic

decoupling problem may be as follows: Assume a given system $S=(A,B,C)$ is nondecouplable by state feedback. Due to the genericity of decoupling, there exists a system $\hat{S}=(\hat{A},\hat{B},\hat{C})$ arbitrarily close to S , which is decouplable. Thus, we may find at least one feedback pair $(\hat{F},\hat{G})$ such that $\hat{S}(\hat{F},\hat{G})=(\hat{A}+\hat{B}\hat{F},\hat{B}\hat{G},\hat{C})$ is decoupled. Now assume $(\hat{F},\hat{G})$ is applied to the original system S . Is the closed-loop system $S(\hat{F},\hat{G})=(A+B\hat{F},B\hat{G},C)$ decoupled in any sense? We believe that the answer may be in the affirmative if $\hat{S}$ is adequately chosen. That is, the system $S(\hat{F},\hat{G})$ may be considered asymptotically decoupled in the sense of the definition given above. Our belief is based on some simple examples, and we will illustrate the proposed approach by one of them.

Consider a given system $S=(A,B,C)$ where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Its transfer function matrix $H(s)$ is given by

$$H(s) = C(Is-A)^{-1}B = \begin{bmatrix} \frac{s+1}{s^2} & \frac{1}{s} \\ \frac{1}{s^2} & \frac{1}{s^2} \end{bmatrix}$$

and a simple calculation shows that $d_1=0$, $d_2=1$,
and

$$B^* = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

so, $\det(B^*) = 0$, and the system is not decouplable.

If we now consider the system $\hat{S}=(\hat{A},\hat{B},\hat{C})$, where
 $\hat{A} = A$; $\hat{B} = B$ and

$$\hat{C} = C + \epsilon = \begin{bmatrix} (1+\epsilon) & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where ϵ is an arbitrarily small scalar, then

$$\hat{c}_1 \hat{B} = \hat{c}_1 B = [(1+\epsilon), 1] \neq 0 \Rightarrow \hat{d}_1 = 0 = d_1$$

$$\hat{c}_2 \hat{B} = \hat{c}_2 B = [0, 0] = 0$$

$$\hat{c}_2 \hat{A} \hat{B} = \hat{c}_2 A B = [1, 1] \neq 0 \Rightarrow \hat{d}_2 = 1 = d_2$$

and

$$\hat{B}^* = \begin{bmatrix} (1+\epsilon) & 1 \\ 1 & 1 \end{bmatrix}$$

Hence, $\det(\hat{B}^*) = \epsilon \neq 0$ and the system $\hat{S}$, which is
obviously arbitrarily close to S , is decouplable.
The integrator decoupling feedback pair for $\hat{S}$ is
given by

$$\hat{F} = -\hat{B}^*-1\hat{A}^* = \begin{bmatrix} -\frac{1}{\epsilon} & 0 & 0 & 0 \\ \frac{1}{\epsilon} & 0 & 0 & 0 \end{bmatrix}$$

and

$$\hat{G} = \hat{B}^*-1 = \begin{bmatrix} \frac{1}{\epsilon} & -\frac{1}{\epsilon} \\ -\frac{1}{\epsilon} & \frac{\epsilon+1}{\epsilon} \end{bmatrix}$$

The closed-loop system $\hat{S}(\hat{F}, \hat{G}) = (\hat{A}_c, \hat{B}_c, \hat{C}_c)$, where

$$\hat{A}_c = \hat{A} + \hat{B}\hat{F} = \begin{bmatrix} -\frac{1}{\epsilon} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{\epsilon} & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\hat{B}_c = \hat{B}\hat{G} = \begin{bmatrix} \frac{1}{\epsilon} & -\frac{1}{\epsilon} \\ 0 & 0 \\ -\frac{1}{\epsilon} & \frac{\epsilon+1}{\epsilon} \\ 0 & 0 \end{bmatrix} \quad \hat{C}_c = \hat{C} = \begin{bmatrix} (1+\epsilon) & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has as its transfer function matrix $H(\cdot, F, G)$, where

$$H(\cdot, F, G) = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s^2} \end{bmatrix}$$

If we now apply $(\hat{F}, \hat{G})$ to the original system

$S=(A,B,C)$, a direct calculation shows that the closed-loop system $S(\hat{F},\hat{G})=(A+B\hat{F},B\hat{G},C)$ has as its transfer function matrix

$$H(\cdot,\hat{F},\hat{G}) = \begin{bmatrix} \frac{1}{s(\epsilon s+1)} & \frac{\epsilon}{(\epsilon s+1)} \\ 0 & \frac{1}{s^2} \end{bmatrix}$$

which is not one-to-one decoupled. However, some interesting remarks may be made:

(A). Notice that as $\epsilon \rightarrow 0$, $H(\cdot,\hat{F},\hat{G}) \rightarrow \hat{H}(\cdot,\hat{F},\hat{G})$ and at the limit, $H(\cdot,\hat{F},\hat{G}) = \hat{H}(\cdot,\hat{F},\hat{G})$, which is the transfer function matrix of a one-to-one decoupled system. Thus, at $\epsilon=0$, $(\hat{F},\hat{G})$ decouples $S=(A,B,C)$. (Notice that $S \rightarrow \hat{S}$ as $\epsilon \rightarrow 0$). This seems to be a contradiction, because S was said to be nondecouplable, but in fact it is not because $\epsilon=0$ implies that some elements in $\hat{F}$ and all the elements in $\hat{G}$ are infinite. What it means is that although every invertible system may be decoupled by state feedback, the feedback gains required will be finite if and only if B^* is nonsingular. When B^* is singular, a suitable precompensator of large enough order has the effect of making the otherwise infinite required feedback gains, finite.

(B). Although $H(\cdot, \hat{F}, \hat{G})$ is obviously not one-to-one decoupled, we notice that the smaller ϵ is, the further to the left of the stable complex plane the poles of the off-diagonal elements go. Thus, provided that the closed-loop inputs are of the same order of magnitude, the steady state response of the closed-loop system $S(F, G)$ will be determined by the diagonal elements of $H(\cdot, \hat{F}, \hat{G})$. In this sense, the system is "asymptotically" decoupled. The sooner the closed-loop system is required to become "asymptotically" decoupled, (i.e., the smaller $\bar{t}$ in (i) of the definition of the problem is), the smaller ϵ will have to be and the larger the feedback gains required.

In the example above, we chose to modify C for simplicity and did it in such a manner as to maintain the values of the d_i 's. It is obvious that to make the above synthesis procedure useful, a lot of work is still needed. A general algorithm to construct $\hat{S}$ must be found and a rigorous treatment of the problem should be made. The results given in chapter IV, particularly those on the input-output invariance problem and their possible extensions, should provide a good basis to do it.

A further problem of importance which is related to (A) above, is that of decoupling with bounded

state feedback. That is, if for practical reasons or constraints, the feedback gains are bounded in some way, B^* nonsingular may not be sufficient for decoupling. However, from (A) above, and as the inverse of a system used as a precompensator one-to-one decouples the system with $F = 0$, $G = I$, it is possible to say that the decoupling problem with bounded feedback will always have a solution for invertible systems (although precompensation may be needed).

Finally, the number of results available for the decoupling of linear time invariant systems which have been extended to the time-varying situation is still comparatively small. The technical difficulties which arise in the non-stationary case are large. The results in chapters V and VI represent only a step in narrowing the existing gap in the literature. Their extension to more general situations requires more research. Of prime importance is to consider the case where $d_i(t)$; $i=1, \dots, m$, are not constant integer valued functions over the time interval of interest. One possible approach is to design a precompensator such that the overall system has a constant set of decoupling invariants $\{d_i^c(t)\}$ over the time interval of interest. Such a precompensator may not exist for all systems. Further, when it does, the

overall system is likely to be non-decouplable at those times where the d_i 's of the original system were discontinuous. However, provided that the overall system thus obtained remains uniformly invertible, a second precompensator could be found using the procedure of section 3, chapter VI, to obtain an overall uniformly decouplable system. To illustrate that at least in some cases, a system with time-varying d_i 's may be compensated so that the overall system has constant d_i 's, consider the following example.

Remark 7.1. Notice that the example below is very similar to that in section 4 of chapter VI (their only difference being that the first and third rows of $A(t)$ have been interchanged). However, for the former no precompensator which yields $d_i(t)$ constant over $[0,2]$ for the overall system can be found.

Let $\Sigma(t)=[A(t),B(t),C(t)]$ be given by

$$A(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B(t) = \begin{bmatrix} f(t) & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$f(t) = \begin{cases} 0 & \forall t \in [0, 1) \\ t^3 - 3t^2 + 3t - 1 & \forall t \in [1, 2] \end{cases}$$

Then,

$$(L_A^{(0)} c_1)(t)B(t) = c_1(t)B(t) \begin{cases} = 0 & \forall t \in [0, 1] \\ \neq 0 & \forall t \in (1, 2] \end{cases}$$

$$(L_A^{(1)} c_1)(t)B(t) = c_1(t)A(t)B(t) = [1, 1] \quad \forall t \in [0, 2]$$

and

$$(L_A^{(0)} c_2)(t)B(t) = c_2(t)B(t) = [0, 1] \quad \forall t \in [0, 2]$$

Hence,

$$d_1(t) = \begin{cases} 1 & \forall t \in [0, 1] \\ 0 & \forall t \in (1, 2] \end{cases}$$

and

$$d_2(t) = 0 \quad \forall t \in [0, 2]$$

and

$$B^*(t) = \begin{cases} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \forall t \in [0, 1] \\ \begin{bmatrix} f(t) & 0 \\ 0 & 1 \end{bmatrix} & \forall t \in (1, 2] \end{cases}$$

Although $\det[B^*(t)] \neq 0 \quad \forall t \in [0, 2]$, and the system is therefore uniformly decouplable, discontinuous unbounded gains will occur in the feedback pair $(F(t), G(t))$. This makes decoupling impossible in practice.

Let $\hat{\Sigma}(t) = [\hat{A}(t), \hat{B}(t), \hat{C}(t), \hat{D}(t)]$ be a precompensator for $\Sigma(t)$, defined by

$$\hat{A}(t) = [0] ; \hat{B}(t) = [1, 0] \quad \hat{C}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\hat{D}(t) = \begin{bmatrix} g(t) & 0 \\ 2 & 1 \end{bmatrix}$$

where

$$g(t) = \begin{cases} t^3 - 3t^2 + 3t - 1 & \forall t \in [0, 1) \\ 0 & \forall t \in [1, 2] \end{cases}$$

From eqs. (6.10), (6.11) and noticing that $f(t)g(t) = 0 \quad \forall t \in [0, 2]$, the state space description of the composite system $\Sigma_c(t) = [A_c(t), B_c(t), C_c(t)]$ will be given by

$$A_c(t) = \left[\begin{array}{ccc|c} 0 & 1 & 1 & f(t) \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right] \quad B_c(t) = \left[\begin{array}{cc} 0 & 0 \\ g(t) & 0 \\ 2 & 1 \\ \hline 1 & 0 \end{array} \right]$$

$$C_c(t) = \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

and a simple calculation shows that

$$(L_{A_c}^{(0)} C_c)(t) B_c(t) = C_c(t) B_c(t) = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

and

$$(L_{A_c}^{(1)} C_c)(t) B_c(t) = C_c(t) A_c(t) B_c(t) = \begin{bmatrix} 2+f(t)+g(t) & 1 \\ 2 & 1 \end{bmatrix}$$

Hence

$$(d_c)_1(t) = 1 \quad \forall t \in [0, 2]$$

and

$$(d_c)_2(t) = 0 \quad \forall t \in [0, 2]$$

i.e., the composite system $\Sigma_c(t)$ has constant d_i 's over the time interval of interest. Notice however that

$$B_c^*(t) = \begin{bmatrix} [2+f(t)+g(t)] & 1 \\ 2 & 1 \end{bmatrix}$$

is nonsingular everywhere except at $t=1$. i.e., the time where $d_1(t)$ was discontinuous. Thus to uniformly decouple $\Sigma(t)$, a second precompensator is needed and may be found using the results of section 3, chapter VI.

Knowing when a precompensator giving constant d_i 's exists and a systematic procedure to find it when it does, are still open research topics. Probably a set of equations similar to (6.29) may be formulated to solve the problem.

The areas mentioned above are only some of those directly related to the main results contained in this thesis, but among the decoupling problems not considered in the dissertation, there are many unanswered questions. The general block-decoupling problem has not been completely solved even for linear time-invariant systems. The results so far available in the literature for decoupled non-linear systems are few and then, most of them concern non-linear systems which are linear in the control. To remark that much research is still needed to solve these and other decoupling problems seems trivial.

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APPENDIX A.1. Discussion of definitions of decoupled
time varying systems.

In this appendix we shall discuss the different definitions of decoupling which have been published in the literature for linear time varying systems. In contrast with the time invariant case, for non-stationary systems decoupling is a time dependent property. A wide variety of different types of decoupling may be defined according to the instants of time or time intervals over which a given system is non-interacting. Below we list the different available definitions and comment on their relative strength.

Let $S=(A(t),B(t),C(t))$ be an m -input, m -output linear time varying system described by the set of eqs.(6.1),(6.2) and let $T=[t_0, t_1]$ be the time interval of interest.

Definition A1.1.- (Majumdar and Choudhury,[33]).

The system S is said to be completely decoupled if

(i) The variation of $y_i(t)$ (the i^{th} element of the output vector) due to any variation of $u_j(t)$; $j \neq i$ (the j^{th} element of the input vector), is identically zero for all $t \in T$, and

(ii) The variation of $y_i(t)$ due to any variation in $u_i(t)$ is not zero for at least one $t \in T$; $i=1, \dots, m$, and all $u_i(t) \in U_i$ (the set of admissible controls).

Definition Al.2.- (Majumdar and Choudhury [33]).

If S is completely decoupled on every open subinterval of $[t_0, t_1]$, S is said to be totally decoupled.

Definition Al.3.- (Majumdar and Choudhury [33]).

If S is completely decoupled for all $t \in T$, then S is said to be uniformly decoupled.

It is evident that uniform decoupling implies total decoupling. In the time invariant case, the three definitions are indistinguishable because if S is decoupled at any time $t \in T$, then it is decoupled for all $t \in T$. The word "complete" used in definition Al.1, is a little unfortunate because it does not convey an appropriate picture of what the definition entails. A better name would be "point-wise" decoupling, but this is a rather philosophical question of not great importance.

Definition Al.4.- (Tokumaru and Iwai [67]). Let X^0 be the set of all possible values of $x_0 = x(0)$. S is said to be $[X^0, T]$ -decoupled if

(i) The value of $y_i(t)$ is independent of $u_j(t)$; $j \neq i$. i.e., $y_i(t)$ is a function of x_0 , t and $u_i(t)$ only, for any $x_0 \in X^0$ and every $t \in T$, and

(ii) $y_i(t)$ is not only a function of x_0 , t , but is also influenced by $u_i(t)$; $i=1, \dots, m$.

This definition which may also be considered of a variational nature, is less rigorous and precise than definitions A1.1 to A1.3, because there is no time specified for condition (ii). It is clear that definitions A1.1 to A1.3 imply $[X^0, T]$ -decoupling, and definition A1.4 implies at least complete decoupling.

Definition A1.5.- (Tokumaru and Iwai [67]). If S is $[X^0, T]$ -decoupled and $X^0 = \mathbb{R}^n$ and $T = [0, \infty)$, then S is said to be decoupled at large. (The notation "decoupled at large" is ours. Tokumaru and Iwai called this type of decoupling "complete", but in view of def. A1.1, this name would be confusing).

Let now $\bar{\Phi}(t, \zeta)$ be the transition matrix of S and define the impulse response matrix $H(t, \zeta)$ of S by

$$H(t, \zeta) = C(t) \bar{\Phi}(t, \zeta) B(\zeta)$$

then, the input-output behavior of S is described by

$$y(t) = \int_{t_0}^t H(t, \zeta) u(\zeta) d\zeta$$

Definition A1.6.- (Rugh [56]). S is said to be decoupled on $[t_0, t_1]$ if $H(t, \zeta)$ is diagonal.

In the time invariant case, the nonsingularity of $H(t, \zeta)$ is also an integral part of this definition (see [38]). Rugh remarks that for time-varying sys-

tems, $H(t, \zeta)$ nonsingular is not a necessary requirement. Although this is essentially correct, one must be careful. Other requirements should be introduced in the definition, such as $H_i(t, \zeta)$ not identically zero for all $t \in T; \zeta \in [t_0, t]$, where $H_i(t, \zeta)$ denotes the i^{th} row of $H(t, \zeta)$. Otherwise, a system whose impulse response matrix is $H(t, \zeta) \equiv 0$ for all $t \in T; \zeta \in [t_0, t]$ would satisfy the definition A1.6 of a decoupled system, and such a system is trivially seen not to be decoupled ($u_i(t)$ does not affect $y_i(t)$ at any time $t \in T$).

We remark that $H(t, \zeta)$ nonsingular is nevertheless not required for complete, total or uniform decoupling of defs. A1.1-A1.3. Indeed, all that is required for complete decoupling is that for every $i=1, \dots, m$, there exists a time $\hat{t}_i$ and a $\zeta_i \leq \hat{t}_i$ such that $h_i(\hat{t}_i, \zeta_i)$, the i^{th} diagonal element of $H(t, \zeta)$ evaluated at $(\hat{t}_i, \zeta_i)$, is different from zero. These times $\hat{t}_i; i=1, \dots, m$, do not necessarily have to be equal. The following example illustrates a completely decoupled system for which $H(t, \zeta)$ is singular for all $t \in T$ and all $\zeta \in [t_0, t]$.

Let S be a two-input two-output time varying linear system, such that $H(t, \zeta)$ is diagonal, i.e., $H(t, \zeta) = \text{diag} \{h_1(t, \zeta)\}$ and $h_1(t, \zeta)$ and $h_2(t, \zeta)$ are as shown in fig.A1.1.

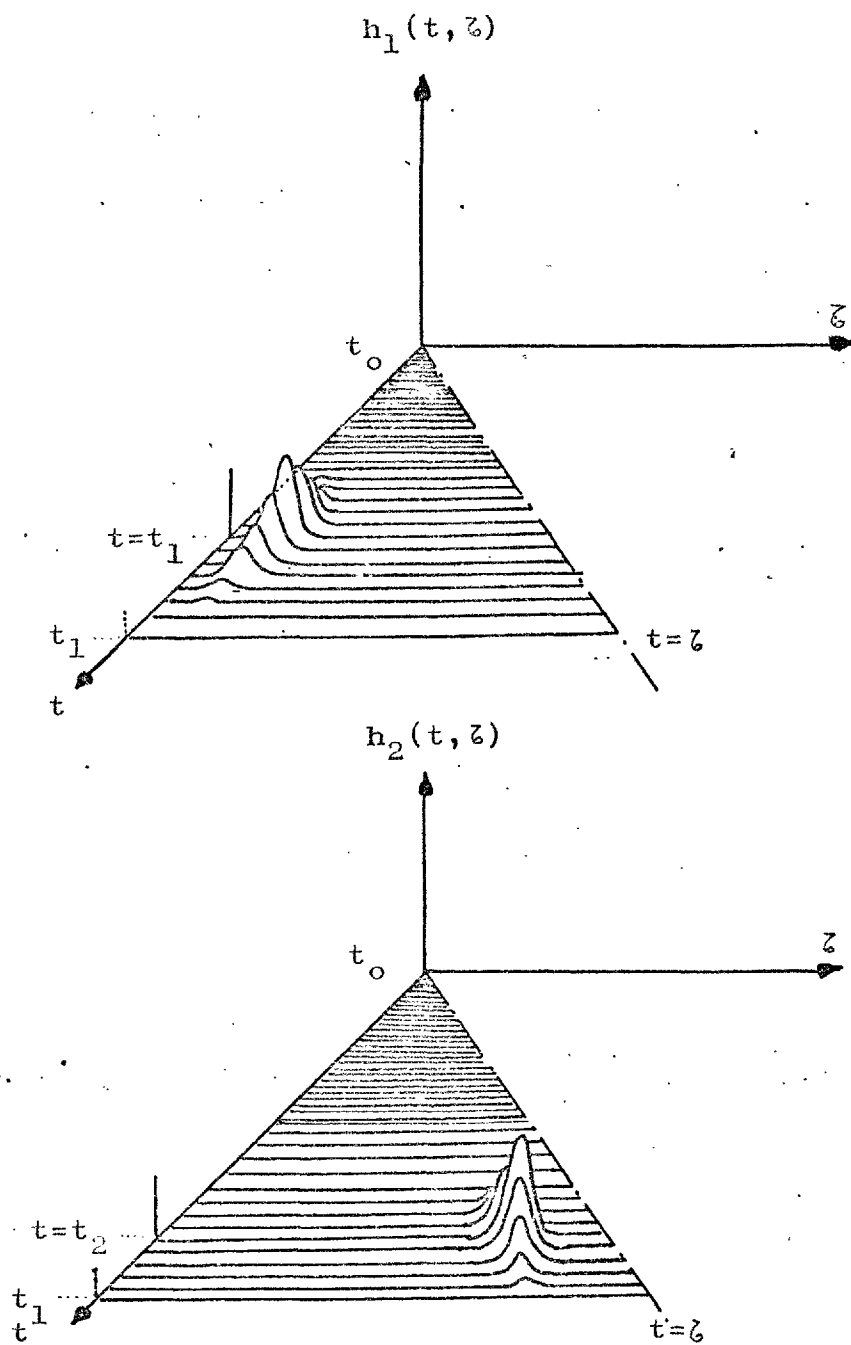


Fig. A1.1 .- Example of a completely decoupled system with singular diagonal impulse response matrix at all times.

Clearly, $H(t, \zeta)$ is singular for all $t \in T$ and all $\zeta \in [t_0, t]$. However, at $t = \hat{t}_1$, the variation of $y_1(t)$ due to a variation in $u_1(t)$ is not zero, and similarly, at $t = \hat{t}_2$, the variation of $y_2(t)$ due to any variation in $u_2(t)$ is not zero. Thus, the system satisfies definition A1.1 of a completely decoupled system. (Condition (i) is trivially satisfied as $H(t, \zeta)$ was assumed to be a diagonal matrix).

If in definition A1.6 we follow the time invariant equivalent and require $H(t, \zeta)$ to be not only diagonal, but also nonsingular for all $t \in T$; $\zeta \in [t_0, t]$, then, definition A1.6 implies definition A1.3, but is not implied by it. In fact, the assumption $H(t, \zeta)$ diagonal and nonsingular for all $t \in T$ and $\zeta \in [t_0, t]$, is the strongest possible assumption and implies that $h_i(t, \zeta)$; $i=1, \dots, m$, has the same sign for all $t \in T$; $\zeta \in [t_0, t]$. We shall illustrate this by means of a second example. Again, let S be a two-input, two-output uniformly decoupled system, with $H(t, \zeta)$ diagonal and $h_1(t, \zeta)$ and $h_2(t, \zeta)$ as shown in fig. A1.2 below.

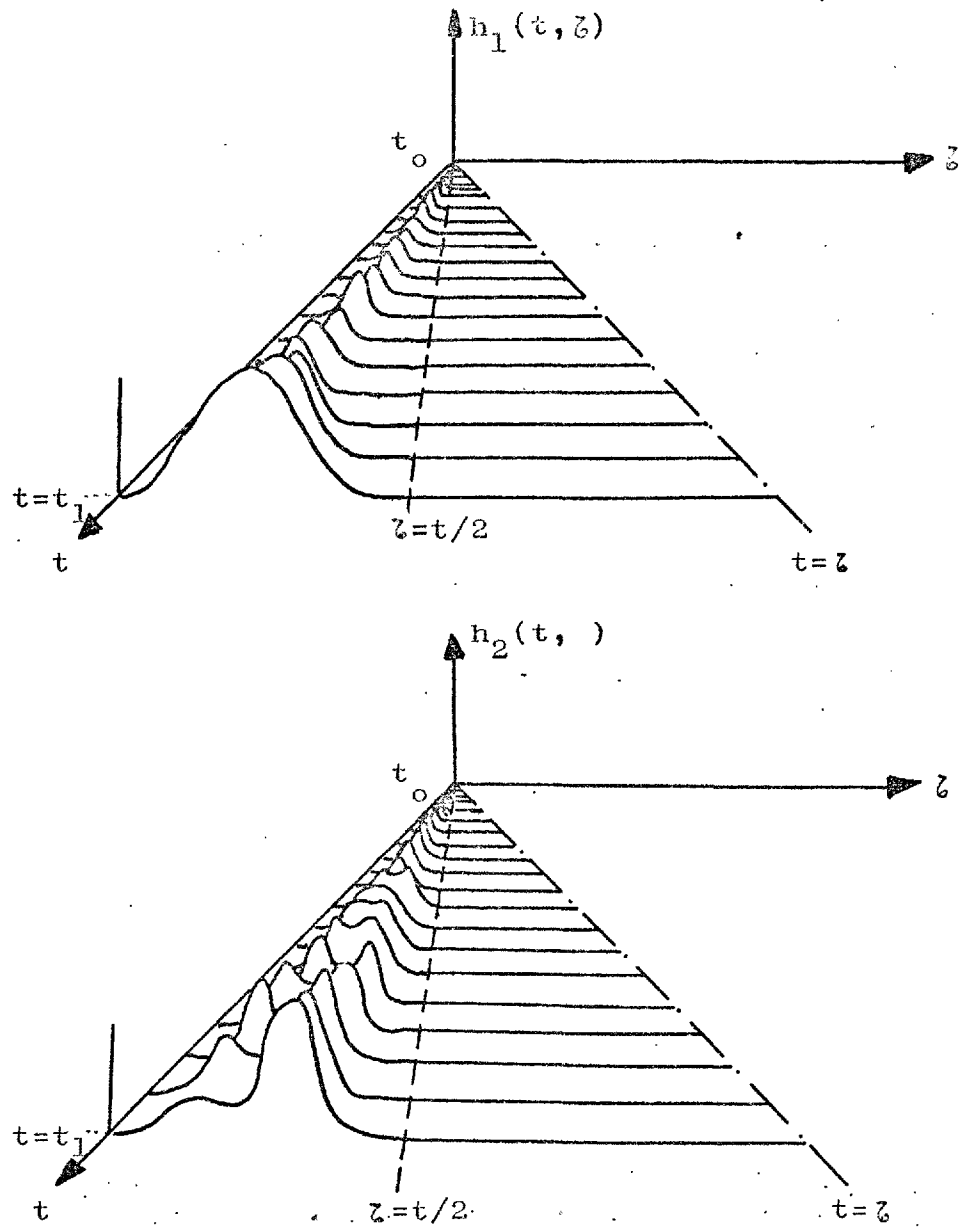


Fig. A1.2.- Example of a uniformly decoupled system with $H(t, z)$ diagonal but not a nonsingular matrix for all $t \in T$ and all $z \in [t_0, t]$.

System S so defined obviously satisfies definition A1.3 and is thus uniformly decoupled. However, for every $t \in [t_0, t_1]$ and every $\zeta \in [t/2, t] \subset [t_0, t]$; $H(t, \zeta) = 0$, and definition A1.6 with the additional requirement $H(t, \zeta)$ nonsingular for all $t \in T$ and all $\zeta \in [t_0, t]$ is not satisfied by S .

Our discussion above shows that some condition must be imposed on the diagonal elements of $H(t, \zeta)$, in order to specify any type of decoupling exactly. For example,

Definition A1.7.- (Viswanadham and Anbalagan [76]).

S is said to be uniformly decoupled almost everywhere, if

(i) $h_{i,j}(t, \zeta) = 0$ for every $t \in T$ and all $\zeta \in [t_0, t]$,

and

(ii) $h_{i,i}(t, \zeta) \neq 0$ for almost all $t \geq \zeta$ and all $\zeta \in [t_0, t]$.

Here the notation "almost everywhere" is ours, to distinguish between this definition and def. A1.3.

In our discussion above we showed that if, for any system, condition (ii) of def. A1.7 holds for every $t \in [t_0, t_1]$ and all $\zeta \in [t_0, t]$, i.e., $H(t, \zeta)$ is nonsingular for all $t \in T$ and $\zeta \in [t_0, t]$, then the system is uniformly decoupled (def. A1.3). Definition A1.7, as stated, only implies that S is completely decoupled

for almost all $t \in T$. (Thus our notational addition in def. A1.7). To see this consider once more a system S with two-dimensional input and output spaces, and $H(t, z)$ diagonal. Let $h_1(t, z)$ be as shown below, and $h_2(t, z) \neq 0$ for all $t \in T$ and all $z \in [t_0, t]$.

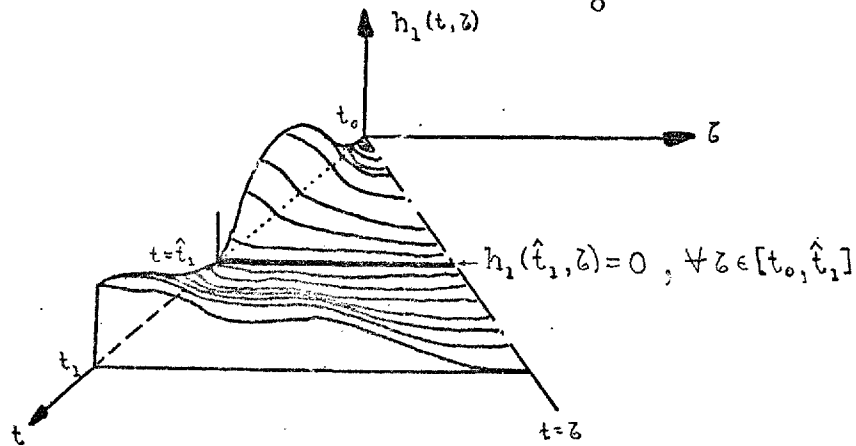


Fig. A1.3.- Example of a uniformly decoupled almost everywhere system.

That is, $h_1(t, z)$ satisfies

$$h_1(t, z) \begin{cases} \neq 0 & \text{for all } t \neq \hat{t}_1 \text{ and all } z \in [t_0, t] \\ = 0 & \text{for } t = \hat{t}_1 ; \text{ for all } z. \end{cases}$$

Thus, the system is decoupled according to def.

A1.7. However, at $t = \hat{t}_1$

$$y_1(\hat{t}_1) = \int_{t_0}^{\hat{t}_1} h_1(\hat{t}_1, z) u_1(z) dz = \int_{t_0}^{\hat{t}_1} 0 \cdot u_1(z) dz = 0$$

and $y_1(t)$ is insensitive to variations in $u_1(t)$ at $t = \hat{t}_1$, which contradicts def. A1.3 for uniform decou-

pling.

E. Freund [19], gave a definition of a decoupled system which may be considered the time varying extension of that given by Falb and Wolovich [17] for stationary systems. However, Freund's definition does not distinguish between complete, total, and uniform decoupling, and although it is otherwise rigorous, it lacks the intuitive clarity of say defs. A1.1-A1.3. We shall not state Freund's definition because this would require the introduction of long notational explanations and would add little new to our discussion.

APPENDIX A.2

Proof of Lemma 6.1.-

To prove the lemma we must show that $\text{span}\{x_i(t); i \in J_0\}$ and $\text{span}\{x_i(t); i \in I-J_0\}$ are independent subspaces for every $t \in T$. So, let $r(t) \in \text{span}\{x_i(t); i \in I-J_0\}$ for all $t \in T$. i.e.,

$$r(t) = \sum_{j \in I-J_0} \alpha_j(t) x_j(t) \quad \forall t \in T$$

for some scalar functions $\alpha_j(t)$.

If $r(t) \notin \text{span}\{x_i(t); i \in J_0\}$ for at least one $t \in T$, say for $t = \tilde{t}$, then

$$r(\tilde{t}) = \sum_{j \in J_0} \beta_j x_j(\tilde{t})$$

for some scalars β_j , and thus

$$\sum_{j \in J_0} \beta_j x_j(\tilde{t}) = \sum_{j \in I-J_0} \alpha_j(\tilde{t}) x_j(\tilde{t})$$

If $\beta_k \neq 0$ for some $k \in J_0$, then $x_k(\tilde{t})$ is expressible as a linear combination of the remaining rows of $X(\tilde{t})$, which contradicts the fact that $x_k(t); k \in J_0$ are u.r.e.-rows of $X(t)$. Thus $\beta_j = 0, \forall j \in J_0$, and $r(\tilde{t}) = 0$, which implies that $r(t) \neq 0$ is not expressible as a linear combination of $x_j(t); j \in J_0$, at any $t \in T$. Therefore

$$\text{span}\{x_i(t); i \in I - J_0\} \cap \text{span}\{x_i(t); i \in J_0\} = \{0\} ; \forall t \in T$$

and the lemma follows.

Proof of Theorem 6.5.-

(If).- Assume all the rows of $X_2(t)$ are u.r.e.-rows of $X(t)$. Then, from lemma 6.1,

$$\text{rank}[X(t)] = r + \text{rank}[X_1(t)] \quad \forall t \in T.$$

As $Z(t)$ is nonsingular over T , by definition, then

$$\text{rank} \begin{bmatrix} X_1(t) & 0 \\ X_2(t) & Z(t) \end{bmatrix} = \text{rank}[X_1(t), 0] + \text{rank}[X_2(t), Z(t)]$$

for all $t \in T$, because $\text{span}\{X_1(t), 0\}$ and $\text{span}\{X_2(t), Z(t)\}$ are obviously independent for every $t \in T$. Thus,

$$\text{rank} \begin{bmatrix} X_1(t) & 0 \\ X_2(t) & Z(t) \end{bmatrix} = r + \text{rank}[X_1(t), 0] = \text{rank}[X(t)]$$

for every $t \in T$ and sufficiency follows.

(Only if).- Assume that eq.(6.32) holds, but not all the rows of $X_2(t)$ are u.r.e.-rows of $X(t)$. Without loss of generality, write $X_2(t)$ as

$$X_2(t) = \begin{bmatrix} X_{2,1}(t) \\ X_{2,2}(t) \end{bmatrix}$$

where $X_{2,2}(t)$ contains all those rows of $X_2(t)$ which are u.r.e.-rows of $X(t)$, and is a $(\gamma \cdot q)$ matrix. From lemma 6.1,

$$\text{rank}[X(t)] = \gamma + \text{rank} \begin{bmatrix} X_1(t) \\ X_{2,1}(t) \end{bmatrix} \quad \forall t \in T.$$

and as

$$\text{rank} \begin{bmatrix} X_1(t) \\ X_{2,1}(t) \end{bmatrix} \leq \text{rank}[X_1(t)] + \text{rank}[X_{2,1}(t)] \quad \forall t \in T.$$

then,

$$\text{rank}[X(t)] \leq \gamma + \text{rank}[X_1(t)] + \text{rank}[X_{2,1}(t)] \quad \forall t \in T. \quad \text{---(A2.1)}$$

with equality holding if and only if $\text{span}\{X_1(t)\}$ and $\text{span}\{X_{2,1}(t)\}$ are independent for every $t \in T$.

On the other hand, and again from lemma 6.1, it is clear that

$$\text{rank} \begin{bmatrix} X_1(t) & 0 \\ X_2(t) & Z(t) \end{bmatrix} = \gamma + \text{rank} \begin{bmatrix} X_1(t) & 0 \\ X_{2,1}(t) & Z_1(t) \end{bmatrix}; \quad t \in T. \quad \text{---(A2.2)}$$

where $Z_1(t)$ is the $(r-\gamma) \times q$ matrix formed by the first $(r-\gamma)$ rows of $Z(t)$, and is obviously of full rank for every $t \in T$. Thus

$$\text{rank} \begin{bmatrix} X_1(t) & 0 \\ X_2(t) & Z(t) \end{bmatrix} = \gamma + \text{rank}[X_1(t)] + \text{rank}[X_{2,1}(t), Z_1(t)] \quad \text{for every } t \in T. \quad \text{---(A2.3)}$$

It is also evident that

$$\text{rank}[X_{2,1}(t)] \leq \text{rank}[X_{2,1}(t), Z_1(t)] = r - \gamma ; \forall t \in T \quad \text{---(A2.4)}$$

Then,

(i).- If equality holds in (A2.4) for every $t \in T$, substituting in eq.(A2.3) and comparing with eq.(A2.1), we obtain

$$\text{rank}[X(t)] \leq \text{rank} \begin{bmatrix} X_1(t) & 0 \\ X_2(t) & Z(t) \end{bmatrix} ; \quad \forall t \in T \quad \text{---(A2.5)}$$

with equality holding if and only if $\text{span}\{X_1(t)\}$ and $\text{span}\{X_{2,1}(t)\}$ are independent for every $t \in T$. However, we show now that these subspaces cannot be independent for all $t \in T$. Let $\{X_{2,1}(t)\}_i$, $\{X_{2,2}(t)\}_i$ and $\{X_1(t)\}_i$ be the i^{th} rows of $X_{2,1}(t)$, $X_{2,2}(t)$ and $X_1(t)$ respectively. By assumption, no row of $X_{2,1}(t)$ is a u.r.e.-row of $X(t)$. So, $\{X_{2,1}(t)\}_i ; i=1, \dots, r-$ are expressible as linear combinations of the remaining rows of $X(t)$, for at least one $t \in T$. i.e.,

$$\begin{aligned} \{X_{2,1}(t)\}_i &= \sum_{j=1}^{q-r} \alpha_{i,j} \{X_1(t)\}_j + \sum_{j=1}^{\gamma} \beta_{i,j} \{X_{2,2}(t)\}_j + \\ &\quad + \sum_{\substack{j=1 \\ j \neq i}}^{r-\gamma} \delta_{i,j} \{X_{2,1}(t)\}_j \end{aligned}$$

for at least one $t \in T$, and some scalars $\alpha_{i,j}$, $\beta_{i,j}$, $\delta_{i,j}$.

If $\beta_{i,j} \neq 0$ for some j , then $\{X_{2,2}(t)\}_j$ would be expressible as a linear combination of the remaining rows of $X(t)$, which is a contradiction. (Recall that $X_{2,2}(t)$ contains only u.r.e.-rows of $X(t)$). Thus $\beta_{i,j} = 0$ for every j , and therefore,

$$\{X_{2,1}(t)\}_i = \sum_{j=1}^{q-r} \alpha_{i,j} \{X_1(t)\}_j + \sum_{\substack{j=1 \\ j \neq i}}^{r-\gamma} \delta_{i,j} \{X_{2,1}(t)\}_j$$

for at least one $t \in T$. As we have assumed equality holds in eq.(A2.4) then $\text{rank}[X_{2,1}(t)] = r - \gamma$, for all $t \in T$, which implies that at least one $\alpha_{i,j}$ must be different from zero, and therefore $\text{span}\{X_{2,1}(t)\}$ and $\text{span}\{X_1(t)\}$ are not independent for every $t \in T$ and thus inequality holds in (A2.5) for at least one $t \in T$, which contradicts the assumption that equality was satisfied for all $t \in T$.

(ii).- Contradiction in (i) above, rests on the assumption that equality holds in (A2.4) for every $t \in T$. So, assume now

$$\text{rank}[X_{2,1}(t)] < \text{rank}[X_{2,1}(t), Z_1(t)] \quad \text{_____ (A2.6)}$$

for at least one $t \in T$. From eq.(A2.1) and substituting (A2.6) in (A2.3)

$$\text{rank}[X(t)] \leq \gamma + \text{rank}[X_1(t)] + \text{rank}[X_{2,1}(t)] <$$

$$< \text{rank} \begin{bmatrix} X_1(t) & 0 \\ X_2(t) & Z(t) \end{bmatrix}$$

for at least one $t \in T$, which again contradicts eq.(6.32).

Thus, if eq.(6.32) holds, all the rows of $X_2(t)$ must be u.r.e.-rows of $X(t)$. This proves necessity and therefore Theorem 6.3.