

Free Surface Flow Simulations Using High Order Algorithms

by

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Abstract

The aim of this work is to formulate a computational algorithm to simulate the motion of a viscous fluid with a free surface boundary undergoing oscillations of sufficient amplitude to induce nonlinearities. To achieve this aim stability complications caused by the spatial discretisation of a two dimensional fluid domain are investigated by analysing the eigenspectrum of the semi-discrete, linear, inviscid solution system. The conditions for stability are established and procedures to overcome instability are formulated and tested. Unsteady, inviscid flows with nonlinear boundary conditions are then examined.

Free surface flows of a viscous, incompressible fluid undergoing unsteady motion are computationally generated using a high order temporal and spatial discretisation scheme to solve the arbitrary Lagrangian-Eulerian Navier-Stokes and continuity equations. The discretisation concepts are presented and the governing equations for fluid flow within a moving frame of reference are formulated.

The suitability of differing free surface boundary conditions is investigated by

simulating forced and free oscillations of a contained body of fluid for increasing Reynolds number. The flow structures they promote are compared and the effect of Reynolds number is investigated, in terms of the general flow characteristics and the choice of boundary condition. The dissipative nature of the boundary conditions are compared with analytically predicted decay rates for the oscillation of a free surface under the force of gravity[37].

Various models to overcome the moving free surface contact point and no-slip boundary conditions are devised and their respective merits studied by observing the production of contact wall boundary layers and advected vorticity layers. The additional dissipation generated by the introduction of the formulated contact wall boundary conditions is assessed and compared with analytically predicted decay rates of a freely oscillating bounded fluid[35].

Preliminary three dimensional results are given for free surface flow within a container undergoing forced oscillations. Three dimensional effects are investigated for uni-directional and bi-directional horizontal motion of the containing tank for $Re = 111$.

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Chapter 1

Introduction

1.1 Overview

As in all areas of computational fluid dynamics, the need for more accurate and efficient numerical algorithms is of great importance. For example, computational wave radiation solutions rely on the reduction of numerical dissipation to give precise results. To promote industrial confidence in numerical wave tanks there should be little dispersion or attenuation of the generated waves. In this work the unstructured *hp*/Spectral element method, which ensures fast convergence rates and minimal phase errors, is used. This method incorporates multi-domain spectral methods and high-order finite element methods to enable the efficient discretisation of complex geometries. The *hp* finite element techniques of Szabo & Babuska [69] and the spectral element methods of Gottlieb & Orszag [20] have been extended to their present unstructured form by Sherwin & Karniadakis [67]. Though the concept is simple, where the solution is represented by a set of polynomials of arbitrarily chosen order within each element, the application of this technique is not easy. The advantages of the *hp*/Spectral element make the effort needed to apply it worthwhile. Convergence can be reached by increasing the number of elements or by increasing the order of the polynomial expansion basis within each cell. The convergence

properties when increasing the number of elements is identical to that for low order schemes. The main advantage is when the polynomial order is increased, providing fast convergence and minimal diffusion and dispersion errors and therefore producing highly accurate long time integration of unsteady flow problems. This method has not yet been applied to the solution of inviscid free surface flows and it is hoped that the convergence properties that it provides will result in the precise simulation of such flows.

A major application of numerical solutions of free surface flows is the evaluation of wave forces on offshore structures, where the accurate prediction of the loads placed on marine installations is of paramount importance when considering the structural parameters of the body. Traditionally the majority of the work has been based on potential theory which treats the flow as inviscid and irrotational. Though linear theory can accurately predict these forces for small wave amplitudes, for naturally occurring sea conditions fully nonlinear calculations are essential. Boundary element methods have been used extensively to computationally simulate such free surface flows [7, 17], though recent understanding of the computational efficiency of these algorithms has shown that the finite element method is a more effective numerical technique when a large number of degrees of freedom is used to discretise the fluid domain [4, 82]. The finite element method allows the fluid domain to be discretised using unstructured elements, making the introduction of complex geometries a relatively straight forward procedure. The finite element method has therefore become popular when simulating unsteady, inviscid free surface flows [4, 84].

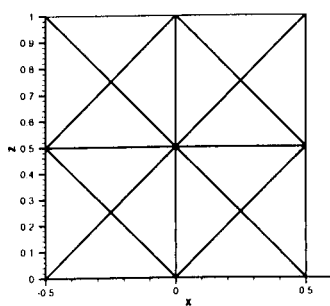
A common theme of many boundary element [42, 52] and finite element computations [29, 82] is the use of a smoothing technique to inhibit the formation of a free surface saw-tooth pattern, which is characteristic of an unstable system. Though this phenomenon has been widely observed for nonlinear flow, minimal investigation or explanation has been attempted. Longuet-Higgins & Cokelet [42] suggested it was due to the *'physical growth of short gravity waves by horizontal compression of the crests of longer waves'* and proposed that these would be realistically damped

by viscosity. They developed five and seven point smoothing formulae which take advantage of the fact that alternate nodes on the free surface form smooth curves. This technique to stabilise the numerical solution is now widely used [10, 52]. It was reported by Tsai & Yue [73] that the instabilities are different from those associated with discrete time integration, such as the wrong choice of time step or time integration scheme and are intrinsic to the spatial discretisation of the system. Moore [47] and Roberts [62] speculated that the instabilities were caused by a resonant interaction between the numerically induced discrete waves. They used linearised eigenvalue analysis techniques to prove that the removal of certain high wavenumber modes could stabilise the system.

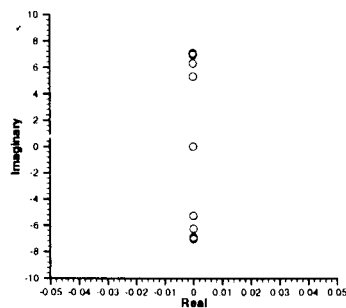
In the course of this research it became apparent that the spatial discretisation of the fluid domain was greatly influencing the stability of the computational solution [63]. This phenomena is investigated using a novel approach to generate the eigenspectrum of the semi-discrete solution system which reduces the computational problem to the prime variables on the free surface. The system evaluates a set of semi-discrete equations relating the temporal derivative of the discrete values of ϕ , the vector potential, and ζ , the surface elevation, and is of the form,

$$\begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}_t = \mathbf{L} \begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}, \quad (1.1)$$

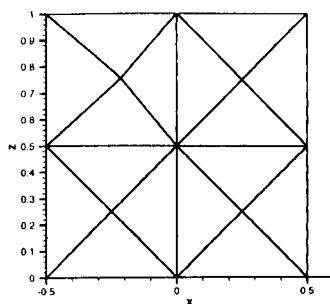
where b indicates the values on the free surface boundary, t represents differentiation in time and $\mathbf{L}$ is a linear operator. Eigenvalues of $\mathbf{L}$ which contain a positive real component indicate an unstable system. An example of the dependency of the stability on the spatial discretisation is shown in figure 1.1 where two meshes are shown with their corresponding eigenvalues. The mesh configuration in figure 1.1(a) is locally symmetric, that is all the elements on the free surface have a vertical line of symmetry. The spatial discretisation produces purely imaginary eigenvalues shown in figure 1.1(b) indicating stability, whilst the slightly asymmetric mesh 1.1(c) is unstable due to the occurrence of positive real eigenvalues.



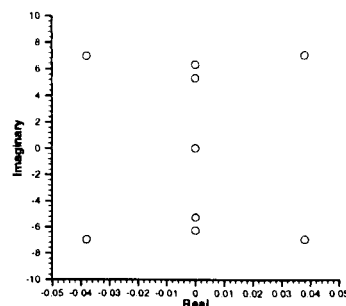
(a) Mesh a



(b)



(c) Mesh b



(d)

Figure 1.1: Comparison of eigenvalues of spatially semi-discrete system for contrasting meshes

Though potential theory facilitates highly accurate solutions for many flow scenarios, there are some areas where viscosity must be considered, such as:

- Viscous damping of wave excited oscillations of offshore structures,
- Flow past a structure at low Keulegan-Carpenter number,
- Wave generation of submerged and floating bodies,
- Flow separation around a body causing drag forces,
- Damping of sloshing waves in a container.

A problematic area of viscous free surface flow is the nature of the free surface contact point on a wall. Though the no-slip condition is a basic premise of fluid dynamics, the motion of a free surface contradicts this condition. It can be seen experimentally that the contact point of the free surface, whilst theoretically adhering to a no-slip condition, is in relative motion and various models are available to overcome this contradiction, both of an analytical nature [12] and based on empirical data [72]. The most widely used computational model is the no-shear force or friction free condition [27], which is referred to in this work as the slip condition. This boundary condition does not promote the generation of vorticity at free surface contact walls, where it is theorised that much of the dissipation of the energy of the fluid occurs [35, 45]. The nature of the free surface, in terms of its shape and velocity, is also affected by the boundary conditions placed on the contact walls [12]. Therefore models which allow the free surface contact point to be in motion whilst also producing appreciable boundary layers on free surface contact walls are necessary to allow the full investigation of the free surface flow of a contained body of fluid or the flow past a surface piercing structure. To this end contact wall boundary conditions are formulated in this work which adhere to the proviso of a moving contact point, whilst also promoting the generation of boundary layers at the contact wall.

The governing equations of viscous flow are solved by utilising the arbitrary Lagrangian-Eulerian formulation which allow the mesh points to move with a velocity different to that of the velocity of the fluid, though some constraints are placed for free surface flow. This procedure alleviates the production of highly deformed elements which would affect the stability of the computations. The first application of the ALE formulation was based on the finite difference scheme [23] and has been extensively utilised for free surface problems using the finite element method [27, 60]. Ho [26] developed an ALE approach to simulate free surface flow using a spectral element discretisation based on conforming quadrilaterals and this formulation has been extended by Warburton & Karniadakis [80] for hybrid discretisations incorporating triangular and tetrahedral elements [66, 67], where the vorticity generation

induced by a cylinder close to a free surface is investigated. This *hp*/Spectral element formulation ensures fast convergence rates on deformed elements [80] and also allows the fluid domain to be discretised using an unstructured mesh [64]. Therefore complex geometries such as surface piercing cylinders and submerged bodies can be incorporated into the computational domain. The benefits of the ALE and *hp*/Spectral element formulations have therefore been used in this work to generate solutions to unsteady free surface flows with viscosity.

In order to study the various flow effects of inviscid and viscous fluids with a free surface, the problem of a free surface wave undergoing free or forced oscillations is investigated in this work. Free oscillations occur when the free surface of a fluid has an initial prescribed deformation and is allowed to oscillate with gravity as the only external force. When the containing tank is in motion the fluid is said to be undergoing forced oscillations. The fluid motion generated by these conditions is commonly known as sloshing. Sloshing has the benefit of allowing the investigation of important problems in free surface fluid dynamics, such as the of nonlinearities, generation of boundary layers at the free surface and walls and the complex issue of the determination of the nature of the free surface contact point with a no-slip boundary, to take place in a controlled environment.

Sloshing has many areas of engineering application. The motion of any floating vessel on the open sea is affected by the motion of the sea itself. If the wave induced oscillations of the vessel are of the same resonant frequency as any fluid containment system within it, the forces and moments resulting from the fluid motion will affect the dynamic behaviour of the entire vessel and also place large structural loads on the surrounding walls. Oil cargo ships and liquid natural gas carriers often have tank dimensions such that the high resonant sloshing frequencies are in the same range as the induced ship motions [68], leading to violent sloshing motions and large impact loads.

Advancements in space flight are dependent upon the understanding of the forces and moments produced by a sloshing liquid in a microgravity environment where

viscous and free surface effects dominate the flow. Similarly sloshing effects on the dynamic stability of light aircrafts also have to be considered.

Sloshing has applications on a larger scale, for example the fluid motion within a harbour or lake can be affected by tidal oscillations and earthquake disturbances. These motions become increasingly important when a dam is used to contain a large body of water which becomes excited due to earthquake activity or landslides. Not only are the impact loads on the dam important, but also the possibility of the fluid over-spilling.

1.2 Outline of Thesis

The thesis consists of 7 chapters, with Chapter 2 containing a review of analytical and numerical solutions of free surface flows, primarily those concerned with free and forced oscillations of the fluid which are relevant to this research. Chapter 3 provides a brief introduction to *hp*/Spectral element methods and illustrates the convergence properties of the technique by solving the one-dimensional Poisson equation. Chapter 4 derives the governing equations of motion for an inviscid fluid with a free surface and investigates the stability of the linear problem by evaluating the eigenspectrum of the spatially semi-discrete solution system. Von Neumann analysis is used to identify applicable time integration schemes and test case results given for sloshing in a container. Contained in Chapter 5 is the derivation of the arbitrary Eulerian-Lagrangian Navier-Stokes and continuity equations for viscous free surface flows. Varying free surface and contact wall boundary conditions are investigated by simulating free and forced oscillations of a fluid. Preliminary results of three-dimensional free surface flow undergoing forced oscillations are presented in Chapter 6. Chapter 7 contains the conclusions of this research and applicable further work.

Chapter 2

Literature Review

This review will mainly be concerned with the solutions of free surface flows undergoing free and forced oscillations, which can be categorised into two main areas:

- Analytical solutions,
- Numerical solutions.

These will form the main sections of the review. Model test are included where they have been used for comparison with analytical and numerical solutions.

2.1 Analytical Solutions

Analytical solutions are generally based on inviscid irrotational free surface flow equations, also known as potential theory, where the velocity, pressure and accelerations are represented in terms of the potential and the viscous and rotational properties of the flow are ignored.

2.1.1 Linear Solutions

Accurate potential theory solutions to linear problems are only achievable by employing simplifications of the fluid system. The simplifications generally used are

given in Abramson [2] as:

- The body of fluid must be contained within a rigid tank,
- The fluid is inviscid,
- The fluid is incompressible,
- The flow is irrotational,
- The fluid must contain no sinks or sources,
- There must be small displacement, velocities and slopes of the free surface.

The most basic solutions to the free surface flow with linear boundary conditions are the standing and travelling wave in an unbounded fluid, where the wave is sinusoidal in shape and oscillates sinusoidally in time. When the wave is bounded the solution is modified to account for the boundary condition on the side walls. Linear solutions of this type are presented in Lamb [36] for a rectangular container (art. 190) and for a circular container in terms of their eigenfrequencies and eigenfunctions. The symmetric and asymmetric eigenfrequencies and eigenfunctions are given in (art. 258) Lamb[36] for a container of two sides which are at forty degrees to the horizontal. For a circular section, the first eigenfrequency or ‘slowest mode’ is evaluated approximately.

Another article of interest is Abramson [2], which contains a comprehensive review of the linear response of fluid in various container configurations. The natural frequencies, mode shapes, potential functions, free surface elevation, forces and moments are given for various structures undergoing harmonic oscillation. Included in this paper are the natural frequencies and functions for rectangular tanks and circular cylindrical tanks of various compartmental configurations including ring sector, circular sector and annular divisions. The fluid response due to translational and rotational excitation are derived, as is the resulting motion due to tank bending.

2.1.2 Nonlinear Solutions

Due to the various constraints placed on the linear solutions, for many flow scenarios nonlinear solutions are essential in order to correctly predict the values of the prime variables, such as free surface elevation and pressure. Abramson [2] categorises the phenomena responsible for the promotion of nonlinearities into three classes:

- tank geometry,
- coupling or instabilities of the sloshing modes,
- large amplitude excitation and response.

The geometry of the tank can produce non linear effects for small amplitude excitation and response and Abramson [2] discusses in some detail the nonlinear flow characteristics of fluid sloshing in a cylindrical compartmented tank. The nonlinear effects are exaggerated due to the decreasing area of the cross section of each sector as shown in figure 2.1.

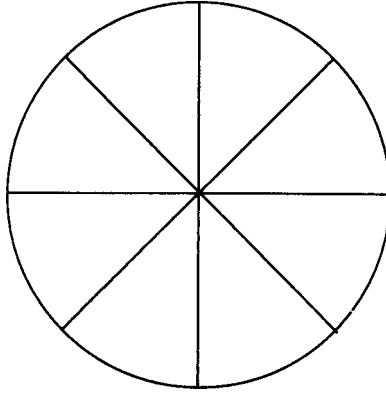


Figure 2.1: Cross sectional area of cylindrical compartmented tank

An example of the instabilities generated by the nonlinear effects of sloshing is given by Abramson [2], where an instability occurs when a vertical cylindrical tank is forced to oscillate laterally. The liquid rotates about the vertical axis while also undergoing a normal lateral sloshing motion. This scenario occurs due to the first

asymmetric sloshing mode transforming itself into a rotating mode, the rotational velocity of which is sinusoidal. This rotational motion coupled with that of the normal lateral sloshing modes leads to a complicated ‘beating’ effect.

The nonlinear effect of large amplitude excitation and response has been the subject of much analytical research. Moiseev’s general perturbation theory, which is explained in Abramson [2] has been the basis for many studies into forced and free oscillatory sloshing. The potential and free surface elevation are taken to be in the form of a power series in $\varepsilon^{\frac{1}{3}}$ where ε is a small parameter equal to the amplitude of the oscillations. This perturbed solution is then substituted into the governing equations and the motion solved by identifying terms of the same power.

Faltinsen [15] uses Moiseev’s general theory to predict the nonlinear response to a fluid in a rectangular tank undergoing forced lateral and forced roll oscillations to third order $O(\varepsilon)$. The results are evaluated for finite and infinite depth and good agreement is found with unpublished experimental studies.

The third order solution of Faltinsen [15] relies on the evaluation of a cubic equation which is dependent on Ω the frequency of the forcing motion. Depending on the value of Ω , the equation and therefore the calculated flow have one, two or three mathematical solutions. These solutions are graphically illustrated in figure 2.2 taken from Faltinsen [15]. In this figure σ_1 represents the lowest natural frequency of the system evaluated from linear theory and Ω is the forcing frequency. If Ω is less than σ_J there are three possible mathematical solutions. The solution the physical system takes is dependent on how the value of Ω is approached. When Ω is approached by starting from a lower frequency then y_1 will be the achieved amplitude and if Ω is approached from a higher frequency y_3 will be the maximum wave amplitude. y_2 will never be chosen by the physical system as it is an unstable solution, resulting in the physical solution jumping from y_4 to y_5 if the oscillations are started from a low frequency. For oscillations commencing at a higher frequency the solution will travel along the upper branch until it reaches some frequency σ_H where it will jump to y_7 due to a physical limitation on the size of the response.

These observations are shared by the unpublished experimental data. In the work

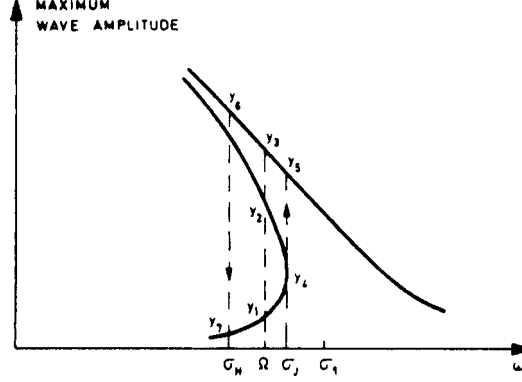


Figure 2.2: Schematic of the maximum wave amplitude for forced oscillations [15]

of Penney & Price's [56], who also use Moiseev's general formulation, the shape of a standing wave in an infinite depth fluid is evaluated to fifth order in terms of the predominant amplitude of the linear wave, ε . They found four interesting results:

- the free surface contains no nodes,
- the free surface is never completely flat,
- the nonlinearities increase the time period of the fluid motion,
- the limiting form of the standing wave, that is the shape it takes just before breaking, has a crest of 90° as oppose to that of 120° for a progressive wave.

The shape of the free surface predicted by Penny & Price [56] is in good agreement with the experimental results of Taylor [71].

The analytical results of Penny & Price [56] are also consistent with those of Tadjbakhsh & Keller [70] who primarily used the perturbation theory to predict the shape and frequency of standing waves in a fluid of finite depth up to the third order in terms of the amplitude of the surface wave motion. They also modified their results to produce an infinite depth solution which is consistent with the results of

Penny & Price [56]. Tadjbakhsh & Keller's solution [70] indicates that there exists a depth of fluid where the nonlinear effects change from producing a softening effect, that is an increase in the time period, to a hardening effect, that is a decrease in the time period. This depth is found to be approximately equal to 0.17λ where λ is the wavelength of the surface wave. This method was extended to three dimensional waves by Verma & Keller [77] with similar results.

More recently Solaas [68] has expanded the work of Faltinsen [15] to include the nonlinear response of fluid undergoing harmonically forced motion in arbitrarily shaped tanks, including V-shaped tanks of thirty and forty degrees inclination of the walls. The solution is evaluated by using a combined analytical and numerical method. A boundary element method is used to determine the eigenvalues and frequencies of the problem as well as the velocity potential and free surface elevation for each boundary value problem in the perturbation scheme.

2.1.3 Analytical solutions with damping

All the above solutions are based on potential theory which describes an irrotational, inviscid fluid. Therefore no damping mechanism is present and any transient effects excited at start up will influence the motion of the fluid for all time. In order to incorporate viscous damping into analytical solutions many authors have used energy arguments to predict viscous dissipation. The decay constant is defined in Abramson [2] as

$$\alpha = \ln \frac{(\text{Maximum amplitude of any oscillation})}{(\text{Maximum amplitude one cycle later})} , \quad (2.1)$$

where

$$a = a_0 e^{-\alpha n} , \quad (2.2)$$

, a is the amplitude of the free surface oscillations, a_0 the initial amplitude and n is the number of cycles. When considering a linear system, the total energy of a wave

at its peak amplitude is equal to the square of the wave elevation [2, 36]. Therefore

$$\alpha = \ln \left(\frac{\text{Energy of any oscillation}}{\text{Energy one cycle later}} \right)^{\frac{1}{2}}. \quad (2.3)$$

In terms of the energy lost over one cycle, ΔE and the total energy, E the decay constant can be written,

$$\alpha = \frac{\Delta E}{2E}. \quad (2.4)$$

Using these principles Lamb [36] and Landau & Lifshitz [37] used energy arguments to establish the decay constant for a progressive wave in an infinitely deep and long fluid as,

$$\alpha = \frac{4\pi\nu k^2}{\omega} \quad (2.5)$$

where ν is the kinematic viscosity, k the wavenumber and ω the wavefrequency. The above relationship was reached under the conditions

$$\lambda^2\omega \gg \nu, \quad (2.6)$$

$$a \ll \lambda, \quad (2.7)$$

where λ represents the wavelength of the free surface wave. These conditions ensure that the body of the fluid is essentially irrotational and inviscid allowing the use of potential theory to evaluate the velocities and energy. Landau & Lifshitz [37] and Lamb [36] continue by approximately evaluating the damping co-efficient of a fluid with arbitrary viscosity. For certain limiting conditions, i.e. $\nu k^2 \gg (gk)^{\frac{1}{2}}$, solutions occur where the motion of the fluid is not oscillatory. Instead the fluid slowly returns to its mean position which is analogous to critical damping.

Keulegan [35] attempted to evaluate the amount of viscous damping occurring in the boundary layers at the side walls and floor of a container holding a fluid with a freely oscillating free surface. The dissipation in the boundary layers is assumed to be produced by viscosity and velocity gradients, where the flow within the boundary layers is laminar. Following Boussinesque the particle velocities in the boundary

layers are derived and the resulting loss of energy is calculated. The amount of energy dissipated at the walls is found to be dependent on $\left(\frac{\nu}{\omega}\right)^{\frac{1}{2}}$. The loss of energy in the main body of the fluid, equation (2.5), is also taken into consideration. A series of experiments is then undertaken to calculate the damping rate of a standing wave in different sized rectangular containers. The tanks were made out of two materials; lucite and glass which are hydrophobic and hydrophilic respectively. For the smaller glass basins the energy losses were slightly larger than those predicted by the theoretical evaluation. Keulegan attributes this to surface tension effects which were neglected in the theoretical damping prediction. When the smaller lucite tanks were used the decay rate was many times that predicted theoretically. Interfacial effects were deemed responsible for the discrepancy. For the larger tanks where these surface effects may be neglected satisfactory agreement was found.

Case & Parkinson [6] make identical assumptions to those held by Keulegan [35], that is the viscous dissipation in the boundary layers is the primary source of damping and that the flow within these layers is laminar. They use a similar method to predict the dissipation of freely oscillating surface waves of small amplitude in a vertical circular cylinder. Also included in their evaluation of the decay constant is the damping due to the main body of fluid determined by Lamb [36]. The theoretical decay rates were compared with experimentally calculated values and good agreement was found. The cylinder used was first polished to diminish the effect of surface roughness on the dissipation. When this was not possible the decay rates were found to be two to three times higher than those theoretically predicted.

Miles [45] attempted to explain the discrepancies between the predicted and recorded decay rates by Keulegan [35] and Case & Parkinson [6]. He theorised that an appropriate length scale for the thickness of the boundary layer at the walls is

$$\delta = \left(\frac{2\nu}{\omega}\right)^{\frac{1}{2}}, \quad (2.8)$$

and his analysis was subject to the condition,

$$\epsilon = k\delta \ll 1. \quad (2.9)$$

Miles [45] therefore argues that the theoretically derived damping constant due to dissipation occurring in the main body of the fluid should not be included in the evaluation of the total damping constant. This is due to the dissipation of the main body of fluid being of $O(\epsilon^2)$ and the damping due to the boundary layers and surface contamination is of $O(\epsilon)$ and all higher orders are ignored in their derivation.

The decay rates predicted by Keulegan [35] and Case & Parkinson [6] for viscous dissipation at the boundary walls are reproduced by Miles [45]. He further theorises that the discrepancies in the predicted and recorded decay rates can be attributed to capillary hysteresis and surface contamination effects. The predicted decay rate due to surface contamination is derived and an empirically evaluated rate of dissipation due to capillary hysteresis is formulated.

Other authors to include damping in analytical, inviscid models for free surface flow include Ursell, Dean and Yu [75] who developed a model to predict the free surface motion of waves travelling in a channel generated by a wavemaker. They include a value for the dissipation of energy due to the boundary layer at the side walls and the floor, though not the end walls, which is again dependent on $(\frac{\nu}{\omega})^{\frac{1}{2}}$.

2.1.4 Shallow water solutions

Chester [8] includes a damping and dispersion term in his nonlinear theory of shallow water, the condition for shallow water is $\frac{d}{l} \ll 1$ and d is the depth of the undisturbed fluid and l is the length of the container. Again boundary layer effects are evaluated in a similar manner as above [35] with similar damping rates. Also predicted by this theory is the formation of bores and travelling waves at forcing frequencies near the first natural frequency of the fluid. Experimental results of fluid response to forced harmonic motions of the tank near resonance and varying depth to length ratios of $\frac{1}{48}, \frac{1}{24}, \frac{1}{12}$ and amplitudes of the forcing oscillations are given in Chester & Bones [9]. It was experimentally found that for small amplitude oscillations of the tank away from resonance, the free surface motion of the fluid can be represented

by the sinusoidal form predicted by linear theory. Near resonance the waveform could take multiple peaks within one forcing period depending on the depth to height ratio and the forcing amplitude. For $\frac{d}{l} = \frac{1}{48}$ a large step in the elevation occurs at the wall of the container followed by multiple high frequency peaks. For a resonant frequency close to the second natural frequency sub-harmonic waves were observed where the frequency of the surface oscillations was approximately a third of the forcing frequency. Response curves of maximum amplitude against forcing frequency are also provided. The theory of Chester [8] was found to be in good agreement with the experimental results of Chester & Bones [9].

Verhagen & Wijngaarden [76] also studied the formation of bores and travelling waves in shallow water undergoing forced oscillations. Their theory was based on one-dimensional gas flow where a hydraulic jump occurs. Comparison between theory and experiments were found to be in reasonable agreement for a tank with height to length ratio equal to 0.075 undergoing forced roll oscillations. No dissipation was included in their analytical theory.

Lepelletier & Raichlen [39] investigated the transient effects and state steady state motion associated with lateral forced oscillations of a shallow fluid. They evaluated the linear fluid response from the dispersive, dissipative long wave equations analytically, whilst the nonlinear response is calculated using a finite element formulation. The theoretical calculations were compared with experimental results for height to length ratio of $\frac{1}{10}$ in the vicinity of the first natural frequency of the fluid. The linear theory was found to be inadequate near resonance whereas the nonlinear calculations were found to be in good agreement in terms of wave shape and height. Transient effects were also found to be important as the largest recorded amplitude for each forcing frequency was found during the transient stage of the free surface motion.

2.2 Numerical Procedures

As the order of analytical solutions incorporating perturbation techniques increases the size of the problem obviously increases to an unmanageable state. Numerical procedures have therefore become popular in solving fluid dynamic problems including free surface flows for both inviscid and viscous fluids. The difficulties in numerically solving a viscous flow with a moving boundary are listed by Floryan & Rasmussen [18] as:

- interfacial boundary conditions are nonlinear and of a mixed type have to be evaluated accurately at the boundary,
- the field equations are nonlinear and boundary layers are possible,
- the solution domain has irregular, constantly changing geometry,
- the interface may undergo large distortions and non-analytic cusp-like interfaces are possible,
- tracking the shape of the interface, i.e. its curvature, demands high accuracy to account properly for the surface tension effects,
- presence of singularities at the contact points poses serious difficulties for accurate determination of the location of the boundary.

Many of the above problems also appear in the solution of inviscid flows. Therefore the numerical implementation of nonlinear boundary conditions on a moving interface is a challenging task. This section will be split into those numerical procedures used to solve inviscid free surface flows and those used to solve viscous free surface flows. The review will emphasise finite element procedures, but also contain other solution techniques including finite volume and boundary element techniques.

2.2.1 Inviscid flow

Boundary element methods

Boundary element techniques have been used to solve inviscid, irrotational, incompressible free surface flows for many years. According to Yeung [85] this can be attributed to the ability to reduce by one the spatial dimensions to be solved since only the boundary of the domain needs to be discretised. Furthermore, the physical quantities of primary interest are those on the boundaries such as wave height and pressure and therefore no computational work is wasted on the evaluation of interior values.

Longuet & Higgins [44] developed a mixed Eulerian and Lagrangian boundary element method to investigate progressive waves. They compared computationally derived wave shapes with analytical solutions and good agreement was found. Overturning waves were then studied by applying an instantaneous asymmetric surface pressure on a progressive, periodic two dimensional Stokes wave. They conclude that a travelling wave does not necessarily develop a sharp corner or singularity before the free surface overturns. Instead the waves can curl over and plunge towards the forward face of the wave whilst a smooth profile is retained. Also addressed in this paper are numerical stability problems which arise when simulating free surface flows.

Faltinsen [16] developed a two dimensional boundary integral scheme to facilitate the simulations of forced lateral sloshing in a container. The method can be applied to arbitrary shaped tanks on the condition that the walls are vertical at the free surface. An artificial damping term was introduced to the governing Euler equations in order to dissipate the effect on the flow motion due to the initially excited natural frequencies, as would occur in a viscous fluid. For small oscillations of the tank away from resonance the computational results compare well with linear analytical solutions and in the vicinity of resonance the free surface exhibits the nonlinear attributes of higher peaks than troughs, though at resonance the computations

break down. A draw back of the method is the surface elevation for all cases being dependent on the damping term resulting in a user defined solution.

More recently Dutta & Laha [13] have used the boundary element method to estimate the linear sloshing frequencies due to forced oscillation of a contained fluid. The oscillations are limited to small displacements, leading to the linearised free surface boundary conditions. The computed first three sloshing modes are compared to the experimental values reported by Abramson *et al.*[1]. The effect of compartmenting a cylindrical tank on the natural frequencies is given, as is the effect on decreasing the fluid depth in compartmentalised cylindrical and conical tanks. Forces and moments are produced for forced sloshing of cylindrical and conical tanks. It is claimed that the computer resources needed to produce the results is less than that needed by equivalent finite element techniques due to the eigenvalue problem being equal to the number of unknowns on the free surface.

Finite element methods

Finite element techniques have more recently become involved in the solution of inviscid, incompressible free surface flows. The finite element method allows the use of an structured grid meaning complex geometries can be incorporated into the domain.

Ikegawa [30] developed a finite element technique to solve inviscid, incompressible free surface flows. The method uses a Lagrangian formulation to investigate the sinusoidal forced motion of a fluid in a container for which qualitative results are given [30] showing some nonlinear attributes.

Washizu *et al.*[81] expand the method of Ikegawa [30] in order to study forced vertical sinusoidal motion of a fluid in a rectangular container. The resulting wave profiles are said to be similar to those experimentally generated by Taylor [71] and again illustrate the nonlinear effect of higher peaks than troughs.

The method is further extended by Nakayama and Washizu [48] to enable forced roll oscillations. The finite element results are compared with linear theory to high-

light the nonlinear effects. The pressure at the right hand wall of the tank is compared with experimental results and good agreement is found.

Wu & Eatock Taylor [82, 83] developed a two dimensional finite element method to solve free surface flows using a linear expansion basis and triangular elements. They utilise a reference frame which is stationary in the horizontal direction and is updated vertically by using an analytical expression which distributes the mesh points exponentially. This configuration is chosen since the potential decays in this manner. In [82] the advantages and disadvantages of two different finite element schemes are compared. In the first scheme, called the finite element method (FEM), the potential is expressed in terms of the shape functions and the velocity is then evaluated by using consistent finite difference techniques. This leads to a discontinuous velocity field, which is projected onto a continuous field using finite element techniques. In the second scheme, called the mixed finite element method (MFEM), both the potential and the velocity are represented by shape functions and both appear in the variational statement. This results in the velocity field being automatically continuous. A Runge-Kutta scheme of the fourth order was used to temporally integrate the nonlinear boundary conditions.

Excellent agreement is observed for converged values of the velocity potential and vertical velocity on an impulsively started wavemaker and the vertical velocity of the fluid on the free surface with an unpublished analytical solution by Peregrine. The MFEM does not seem to give any better results than the FEM, whilst needing much more memory and being CPU intensive. Good agreement is found when using the FEM between the free surface elevation at different time steps after starting the wavemaker when compared to Peregrine's analytical solution.

Using a Stoke's perturbation expansion of the velocity potential, a second order solution is derived for a standing wave freely oscillating in a tank. Good agreement is seen between the analytical and computational solution and any discrepancies are attributed to the computational code capturing higher order nonlinearities.

Wu & Eatock Taylor [83] investigate the computational resources needed to eval-

uate nonlinear free surface flows using a finite element technique and a boundary element technique. The finite element method is found to be superior in terms of memory allocation and CPU time as the number of modes increases. This is attributed to the sparseness of the finite element solution matrix due to the connections between nodes only occurring when they are physically linked. Although the matrix dimension is smaller when using the boundary element technique all nodes are linked leading to a full matrix. It is noted that when the domain includes a complicated structure or the system contains multiple bodies, bandwidth may not be easy to control for the finite element method.

The finite element technique is then expanded to three dimensions in Wu *et al.*[84]. The boundary conditions for forced lateral sloshing are derived for a reference frame moving with the velocity of the prescribed motion of the tank. Two dimensional results are very favourably compared with the linear analytical solution derived from Faltinsen [16] and with the experimental data of Okamoto & Kawahara [50]. Nonlinear effects such as higher peaks than troughs and an increase in the time period are observed, as is the formation of travelling waves and bores. The occurrence of travelling waves and bores are shown to be dependent on the height to length ratio with a small ratio promoting these features. A ratio of $\frac{1}{8}$ was found to promote the occurrence of a travelling wave and a ratio of $\frac{1}{25}$ to promote a bore.

Three dimensional flows were studied by oscillating the tank in the two horizontal directions and also vertically for a range of frequencies and amplitudes. When a tank of depth to length ratio of $\frac{1}{4}$ is oscillated in both horizontal directions at close to the natural frequencies in those directions large displacements of the free surface occur at diagonally opposing corners, whilst far smaller displacements are evident at the remaining corners. It is also shown that when the forcing frequencies are increased to close to the second natural frequency of the fluid the wave amplitudes are greatly decreased even though the amplitudes of the forcing velocity and acceleration are greater.

Wu *et al.* citeWunow continued by generating Faraday waves by vertically os-

cillating the container coupled with some initial horizontal velocity. The frequency of the vertical excitation is approximately twice that of the first natural frequency. The resulting free surface motion is at a frequency equal to that of the natural frequency of the fluid and not at the frequency of the excitation. When the fluid is continuously horizontally excited in addition to being vertically excited the free surface shape is changed though the maximum amplitude at the corners remains the same. It is also discovered that the maximum amplitude during the transient period simulated is solely dependent on the acceleration amplitude.

The ratio of height to length is then decreased to $\frac{1}{8}$. The oscillations are purely horizontal and close to the first natural frequencies of the fluid in the x and y direction. These motions produce a progressive wave travelling diagonally across the container. When the depth to length ratio is decreased to $\frac{1}{25}$ a bore travelling diagonally across the container is formed.

2.2.2 Viscous Flows

Finite Volume Methods

The most commonly used finite difference methods implement to solve incompressible viscous free surface flows are the closely related Marker and Cell (MAC) method and the Volume of Fluid (VOF) method.

The MAC method was developed by Harlow & Welch [21] and is based on the concept that the computational domain is represented by a Eulerian mesh with marker particles initially placed in the cells containing fluid. These particles are then subsequently moved with the computationally calculated local velocities. If a cell has no marker particle within it then it holds no fluid. A cell containing a marker particle adjacent to an empty cell is considered to be a surface element.

Harlow & Welch [22] evaluated large amplitude free surface motions using simplified free surface boundary conditions. The boundary conditions were evaluated by neglecting the normal viscous stresses and using the incompressibility constraint

when evaluating the velocity. The linear stress boundary conditions were implemented by Hirt & Shannon [25] and were later improved by Nichols & Hirt [49] by implementing the pressure on the surface instead of the cell centre. Examples illustrating the corrective effect of the improved conditions are provided. The improvements were found to be only significant for very low Reynolds numbers typically less than ten.

A similar technique is the fractional Volume of Fluid method where each element holds information, denoted by F , on the proportion of its volume taken up by fluid. If the element is completely full then F has a value of one and if empty has a zero value. The position of the free surface is ascertained by evaluating the derivatives of F . The method is fully described in Hirt & Nichols [24] and also included in this work are numerical test cases studying broken dam cases, undular bores and breaking bores, which are compared with experimental and analytical results.

More recently the MAC method has been extended to three dimensions by Park *et al.* [54]. The boundary conditions on the free surface are the same as those adopted by Harlow & Welch [21, 22], where surface tension and viscous stresses are neglected. The interactions of large amplitude waves, generated by prescribing a velocity profile on a side wall, with a stationary three dimensional body were also investigated. The computational results are compared with experimental and computational solutions for run up on the cylinder which has a no-slip boundary condition imposed on it.

An improved MAC method is used to investigate unsteady free surface flows by Armenio [3]. To study the convergence of the computations a contained fluid is forced to oscillate sinusoidally at a frequency far from resonance at $Re = 313$. Time periodic free surface motion is observed for large sway motion. The velocity in the vertical direction along a horizontal line indicates the formation of boundary layers at the free surface walls due to a no-slip condition being implemented. Two test cases of forced roll motion are performed for Reynolds' number equal to 11.5×10^6 and 1×10^6 . It is observed that the thickness of the boundary layers decreases significantly with increasing Reynolds' number.

Finite Element Methods

Ramaswamy *et al.*[61] developed a finite element method to solve incompressible, viscous free surface flows using a Lagrangian treatment of the computational mesh. In this approach the mesh nodes move with the fluid velocity, resulting in the reference frame moving with the material frame. It is not apparent how the contradiction of no-slip and moving contact line is handled though from the results a no-permeability or slip condition is presumed to be used. The boundary conditions on the free surface require that the normal stress is equal to the atmospheric pressure and the tangential stress vanishes. The free oscillation of a standing wave was investigated for both an inviscid and a highly viscous fluid and time histories of a point on the free surface. The viscous fluid experiences a gradual damping of the amplitude of the oscillations not observed for the inviscid fluid.

Small amplitude forced roll oscillations are then studied for the inviscid and viscous fluids. The time history of the point of maximum displacement for the inviscid fluid describes a ‘beating’ pattern, whilst this effect is only temporary for the viscous fluid and a steady state or time periodic solution is achieved. No comparisons with analytical solutions, model tests or other computational findings are provided.

Ramaswamy and Kawahara [60] presented an arbitrary Lagrangian Eulerian description of the fluid domain. Using this method the mesh nodal points can move independently of the fluid motion, resulting in the ability to handle larger deformations of the fluid surface than a purely Lagrangian representation. This technique allows the solution of the Navier-Stokes equations in two ways: 1) using a Lagrangian representation where the mesh moves with the fluid, 2) using an ALE representation where the nodes move with some arbitrary velocity. In practice the computations were performed using a Lagrangian method until the mesh became distorted, remeshing then occurred, where the mesh nodes were moved to a more advantageous configuration and finally the convective flux calculations were performed to take into consideration the motion of the mesh. The computations were

then allowed to continue using the Lagrangian representation until the mesh became distorted again. The dynamic boundary conditions were the same as those used by Ramaswamy *et al.*[61] and the kinematic condition ensures that the surface moves with the fluid.

The motion of a solitary two dimensional travelling wave in a channel is studied and compared to an analytical solution. The agreement is said to be close, though no quantitative results are given. Large amplitude sloshing is then investigated for rectangular tanks and containers with inclined side walls. The nonlinear effects of higher peaks and troughs and an increase in the period are identified for rectangular tanks.

Ramaswamy [59] extended the computational test cases to include a solitary progressive wave and multiple progressive waves travelling in a channel of variable depth. It was found that as the depth of the channel decreased the solitary wave separated into two distinct wave forms with different velocities.

Heurta and Liu [27] use a finite element technique incorporating the ALE formulation to investigate large free surface motion where the maximum amplitudes were approximately fifty percent of the depth of the fluid at rest. The usual boundary conditions are implemented on the free surface and a ‘friction-less’ boundary condition on all walls. The computations were compared with experimental results for high Reynolds’ numbers and good agreement was found between the first natural frequencies. This value was evaluated by measuring the maximum amplitude of the free surface for a range of frequencies and the frequency producing the largest displacement was presumed to be the first natural frequency. When the fluid is forced to laterally oscillate at this frequency the free surface is shown to initially increase in amplitude until a constant value is reached wherein the motion becomes time periodic.

Okamoto & Kawahara [50] used a Lagrangian finite element method to simulate sloshing in arbitrary shaped two dimensional containers including those with multiple side walls. Forced harmonic sloshing for both lateral and rotational dis-

placements were performed on a variety of container shapes including rectangular, hexagonal and octagonal geometries. Very large amplitudes are achieved such that the profile of the free surface is a double valued function of the horizontal co-ordinate. Experimental test cases were performed and compared with free surface profiles generated computationally. The numerical method is considered sufficiently accurate to calculate free surface profiles and velocity and pressure distributions.

Chapter 3

hp/Spectral Elements Formulation

3.1 One Dimensional Problem

The solution procedure of an one dimensional Helmholtz problem is used to introduce the theoretical and practical use of *hp*/Spectral elements. The one dimensional Helmholtz equation can be represented by

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \lambda u + f = 0 . \quad (3.1)$$

where L is a linear operator and f some forcing function. If suitable boundary conditions are satisfied this equation is said to be in its strong form. The function u is typically continuous, needing an infinite number of independent functions to exactly represent it. A discrete function, u^δ , is therefore used to numerically approximate u using a finite number, n , of functions. When the operator L acts on the function u^δ , a residual occurs denoted by R ,

$$L(u^\delta) = R(u^\delta) , \quad (3.2)$$

and as $n \rightarrow \infty$, $R(u^\delta) \rightarrow 0$ since u^δ approaches u . The condition, along with the boundary conditions, on this equation is that the residual is orthogonalised by a test or weight function, v^δ , such that the inner product of R with v^δ is zero,

$$\int_{\Omega} v^\delta R(u^\delta) dx = 0 , \quad (3.3)$$

or introducing new notation,

$$(v, R(u^\delta))_\Omega = 0 , \quad (3.4)$$

where $(\cdot)_\Omega$ denotes the inner product in the region Ω . The approximate solution or trial solution can be represented by

$$u^\delta = \sum_{i=1}^{ndof} \hat{u}_i N_i(x) , \quad (3.5)$$

and the corresponding discrete test function by

$$v^\delta = \sum_{i=1}^{ndof} \hat{v}_i M_i(x) , \quad (3.6)$$

where $\hat{u}_i$ and $\hat{v}_i$ are sets of expansion coefficients multiplied by an expansion bases, $N_i(x)$ and $M_i(x)$ respectively. The choice of N_i and M_i dictates the type of weighted residual method used. If $M_i(x) = N_i(x)$ then a Galerkin method is used. An alternative choice of $M_i(x)$ leads to a Petrov-Galerkin formulation. The exact choice of expansion basis is irrespective of this derivation and will be discussed later. At this stage it only necessary to know that the test functions, v , are by definition zero at the Dirichlet or essential boundary conditions.

If equation (3.1) is multiplied by the test function, v , and integrated over the domain Ω then,

$$(v, L(u)) = \int_{\Omega} v \left(\frac{\partial^2 u}{\partial x^2} + \lambda u + f \right) = 0 . \quad (3.7)$$

The solution domain is bounded by $\Omega = \{x|a < x < b\}$ and the boundary conditions are specified such that $u(a) = c_D$ is a Dirichlet boundary condition, and $\frac{\partial u}{\partial x}(b) = c_N$ is a Neumann boundary condition. By rearranging equation (3.7) and integrating the second derivatives by parts, equation (3.7) can be rewritten as

$$\left[v \frac{\partial u}{\partial x} \right]_a^b - \int_{\Omega} \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dx + \lambda \int_{\Omega} v u dx + \int_{\Omega} f v dx = 0 . \quad (3.8)$$

The first term of equation (3.8) can be reduced to $v(b) c_N$, as $v(a) = 0$ and $\frac{\partial u}{\partial x}(b) = c_N$. By replacing the infinite expansion of u and v in with the finite expansions u^δ and v^δ equation (3.8) is represented in its Galerkin formulation,

$$v(b) c_N - \int_{\Omega} \frac{\partial v^\delta}{\partial x} \frac{\partial u^\delta}{\partial x} dx + \lambda \int_{\Omega} v^\delta u^\delta dx + \int_{\Omega} v^\delta f dx = 0 . \quad (3.9)$$

It is this equation that is solved numerically to evaluate the value of u^δ . Before the system can be solved the Dirichlet boundary conditions must be incorporated into the equation. This is done by decomposing the trial function, u^δ , into its constituent parts,

$$u^\delta = u^{\delta H} + u^{\delta D} , \quad (3.10)$$

where $u^{\delta H}$ represents the homogeneous value of the solution which is zero on the Dirichlet boundaries, i.e. $u^{\delta H}(b) = 0$ and belongs to the same set of expansions as the test functions. $u^{\delta D}$ are the known Dirichlet boundary conditions, i.e. $u^{\delta D}(b) = c_D$. Equation (3.9) is therefore rewritten as

$$\begin{aligned} - \int_{\Omega} \frac{\partial v^\delta}{\partial x} \frac{\partial u^{\delta H}}{\partial x} dx + \lambda \int_{\Omega} v^\delta u^{\delta H} dx = -v^\delta(b) c_N - \\ \int_{\Omega} f v^\delta dx + \int_{\Omega} \frac{\partial v^\delta}{\partial x} \frac{\partial u^{\delta D}}{\partial x} dx - \lambda \int_{\Omega} v^\delta u^{\delta D} dx , \end{aligned} \quad (3.11)$$

with all the unknowns on the left hand side. The equation now contains all the information that is needed to solve the system. To achieve a solution system that can be solved numerically as a set of algebraic, dependent set of equations, the functions $u^{\delta H}$ and v^δ are substituted for their expansion coefficient and basis form.

The resulting system is therefore

$$\begin{aligned} - \int_{\Omega} \sum_{i=1}^{ndof} \hat{u}_i^{\delta H} \frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x} dx + \lambda \int_{\Omega} \sum_{i=1}^{ndof} \hat{u}_i^{\delta H} N_i \cdot N_j dx = -v^\delta(b) c_N - \\ \int_{\Omega} f \cdot N_j dx + \int_{\Omega} \frac{\partial u^{\delta D}}{\partial x} \cdot N_j dx - \lambda \int_{\Omega} u^{\delta D} \cdot N_j dx \quad \forall \quad j , \end{aligned} \quad (3.12)$$

and can be reduced to matrix form:

$$\mathbf{A} \mathbf{u}^H = \mathbf{f} , \quad (3.13)$$

where $\mathbf{A}$ and $\mathbf{f}$ are known and defined as

$$\mathbf{A}[i, j] = - \int_{\Omega} \frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x} dx + \lambda \int_{\Omega} N_i \cdot N_j dx , \quad (3.14)$$

$$\mathbf{f}[j] = -v^{\delta}(b) c_N - \int_{\Omega} f \cdot N_j dx + \int_{\Omega} \sum_{i=1}^{ndof} \frac{\partial u^{\delta D}}{\partial x} \cdot N_j dx - \lambda \int_{\Omega} \sum_{i=1}^{ndof} u^{\delta D} \cdot N_j dx , \quad (3.15)$$

and

$$\mathbf{u}^H[i] = \hat{u}_i^{\delta H} . \quad (3.16)$$

$\mathbf{A}$ is invertible and has been proved to be positive definite [28]. Therefore $\mathbf{u}^H$ can be evaluated from the following equation,

$$\mathbf{u}^H = \mathbf{A}^{-1} \mathbf{b} . \quad (3.17)$$

This system has $ndof$ local degrees of freedom, where $ndof = P \times nel$, nel is the number of elements which discretise the domain and P is the order of the highest order polynomial in the local expansion basis. If h , the size of the elements, is decreased, while P is kept fixed then this is the traditional finite element or h -type approach, whereas if h is fixed and P is increased then the spectral element or p -type technique is used. The hp /Spectral technique increases both P and nel to achieve convergence and therefore uses the advantageous components of each technique.

3.1.1 h -type Approach

When using the h -type technique the solution domain is divided into non-overlapping sub-domains or elements. In one dimension this is just a series of lines, in two dimensions typically triangles and in three dimensions tetrahedrals. If the domain is represented as Ω and the elements as Ω_e then

$$\Omega = \bigcup_{e=1}^{nel} \Omega_e . \quad (3.18)$$

When Ω is one dimensional where $\Omega = \{x|a < x < b\}$ and is discretised using $nel + 1$ points,

$$a < x_0 < x_1 < \dots < x_{e-1} < x_e < x_{e+1} < \dots < x_{nel-1} < x_{nel} = b , \quad (3.19)$$

then the e^{th} element can be defined as

$$\Omega_e = \{x|x_{e-1} < x < x_e\} . \quad (3.20)$$

For ease of computation this arbitrary local element is mapped onto a standard element,

$$\Omega_{st} = \xi|-1 < \xi < 1 , \quad (3.21)$$

using a simple linear transformation. The global co-ordinate in the e^{th} element can be defined using the standard co-ordinate thus,

$$x = X_e(\xi) = \frac{1-\xi}{2}x_{e-1} + \frac{1+\xi}{2}x_e, \quad \xi \in \Omega_{st} . \quad (3.22)$$

and as the transformation $X(\xi)$ is linear it has an inverse of form

$$\xi = X_e^{-1}(x) = 2\frac{x - x_{e-1}}{x_e - x_{e-1}} - 1, \quad x \in \Omega_e . \quad (3.23)$$

The global expansion basis can then be described in terms of the local co-ordinates

$$\Phi_p(x) = \phi_p(\xi), \quad x \in \Omega_e , \quad (3.24)$$

where $\xi = X_e^{-1}(x)$ and therefore the global solution can be defined in terms of the local expansion basis and expansion co-efficients,

$$u^\delta(x) = \sum_{i=1}^{ndof} \hat{u}_i \Phi_i(x) = \sum_{e=1}^{nel} \sum_{p=0}^P \phi_p^e(\xi) \hat{u}_p^e , \quad (3.25)$$

where P is the polynomial order of the expansion. Equation (3.25) cannot be evaluated without further restraints on the value of $\hat{u}$ at the boundary points of each global element. In order to be able to solve the system for $u^\delta(x)$ the right hand side

of the system must be bounded. This is ensured by making the expansion basis C_0 continuous. If the expansion basis used is of order one, the local expansion basis consists of two modes,

$$\phi_0 = \frac{1 - \xi}{2} , \quad (3.26)$$

$$\phi_1 = \frac{1 + \xi}{2} . \quad (3.27)$$

The constraint put on the boundary values when considering a domain split into 3 elements is,

$$\hat{u}_1^1 = \hat{u}_0^2 , \quad (3.28)$$

$$\hat{u}_1^2 = \hat{u}_0^3 . \quad (3.29)$$

The co-efficients $\hat{u}_0^2$ and $\hat{u}_0^3$ represent those related to the expansion basis ϕ_0 in the second and third element respectively and the co-efficients $\hat{u}_1^1$ and $\hat{u}_1^2$ represent those related to the expansion basis ϕ_1 in the first and second element respectively. The discrete value of u at the intersection of the elements is therefore identical and the number of unknowns decreases from 6 to 4. The global matrix can now be formed by summing the contribution from each local matrix. The sparseness of this matrix is dependent on the form of expansion basis used [64].

3.2 *p*-type Approach

The *p*-type approach uses an expansion basis to represent the required function. Typically, the higher the number of modes which make up the basis the more accurate the solution, therefore convergence is achieved by increasing the number of modes which form the expansion basis. The choice of expansion basis is not only important with respect to the accuracy of the solution, but greatly affects the sparseness of the solution matrix and therefore the computational time needed to solve the system.

3.2.1 Expansion Bases

Traditionally polynomials have been used to form higher order expansion bases and the technique used here is no different. The two main categories of expansion bases are modal and nodal expansions. Modal expansions are mainly used in this work though these are transformed into nodal expansions which are used to perform numerical differentiation due to efficiency concerns. The expansions that have been adopted in two and three dimensions for this work first proposed by Dubiner [11] in two dimensions and later expanded to three dimensions by Sherwin & Karniadakis [34, 64], and are based upon generalised family known as the Jacobi polynomials. These polynomials are the family of polynomial solutions to the Sturm-Liouville problem and for $-1 < x < 1$, have the form

$$\frac{d}{dx} \left[(1-x)^{1+\alpha} (1+x)^{1+\beta} \frac{d}{dx} u_p(x) \right] = \lambda_p (1-x)^\alpha (1+x)^\beta u_p(x) , \quad (3.30)$$

where

$$u_p = P_p^{\alpha,\beta}(x) , \quad (3.31)$$

and

$$\lambda_p = -p(\alpha + \beta + p + 1) . \quad (3.32)$$

$P_p^{\alpha,\beta}(x)$ represents a Jacobi polynomial of order p .

Jacobi polynomials have two main advantages. They have the orthogonality relationship

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_p^{\alpha,\beta}(x) P_q^{\alpha,\beta}(x) dx = C \delta_{pq} , \quad (3.33)$$

with C depending on α, β and p , which helps maintain sparse matrices and they can be produced by a three term recursive operation. The expansion bases therefore have the form

$$\phi_1(\xi) = \frac{1-\xi}{2} \quad p = 0 \quad (3.34)$$

$$\phi_p(\xi) = \frac{(1-\xi)}{2} \frac{(1+\xi)}{2} P_{p-1}^{1,1}(\xi) \quad 1 < p < P \quad (3.35)$$

$$\phi_P(\xi) = \frac{1+\xi}{2} \quad p = P \quad (3.36)$$

The first six modes can be seen in figure 3.1, it can be seen that they form a hierarchical basis starting if the two linear modes are ignored. This has the advantage of increasing the accuracy of the solution when more modes are added irrespective of the number of quadrature points.

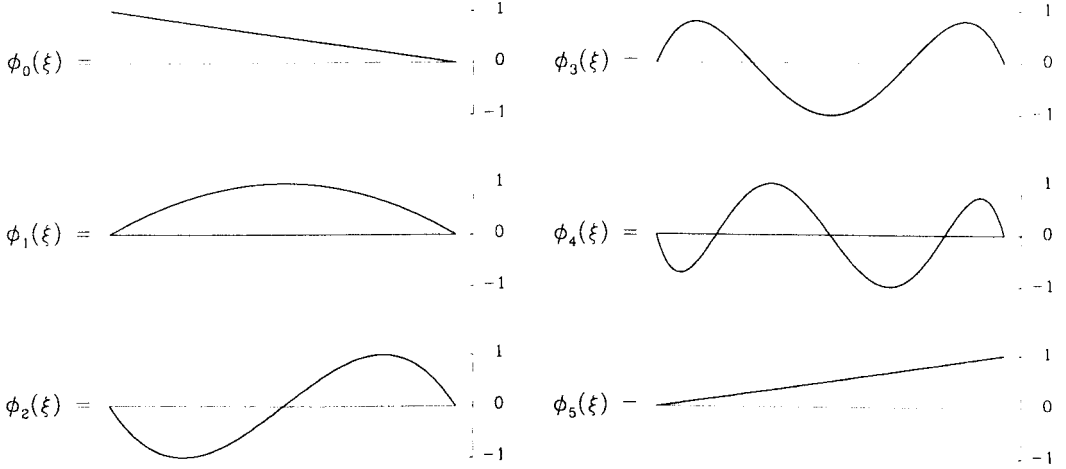


Figure 3.1: Shape of modal expansion modes for a polynomial of order 5.

In order to enable integration, values of ξ are chosen to be the Gauss-Lobatto-Legendre Quadrature abscissae with np distinct points where $-1 \leq \xi \leq 1$. This enables integration to be performed on a function by finite summation of the form

$$\int_a^b f(x) dx = \sum_0^{np-1} J w_i f(\xi_i) + R(u) , \quad (3.37)$$

where

$$R(u) = \int_a^b \varepsilon(u) d\varepsilon . \quad (3.38)$$

J is the Jacobian equal to $\frac{b-a}{2}$ and w_i are the Gauss-Lobatto-Legendre weights. The integration is exact, i.e. $R(u) = 0$, if $P < 2np - 3$. Further explanation of Gauss quadrature integration including the generation of x_i and w_i can be found in [34, 69].

The other main category of expansion modes are nodal ones. Here each expansion basis is one at a single node and zero at all others. Nodal expansions often use a

Lagrangian polynomial through the zeros of the Gauss-Labatto polynomials. This type of nodal p-type elements have become known as “spectral elements” [55]. A typical form of the Lagrangian interpolants is

$$\phi_i(\xi) = -\frac{(1-\xi^2) L'_P(\xi)}{P(P+1) L_P(\xi_p)(\xi-\xi_p)} \quad 0 \leq p \leq P \quad (3.39)$$

where there are $N + 1$ nodal points and L_P the Legendre polynomial of order P . The first six modes can be seen in figure 3.2. Each mode is of polynomial order P as

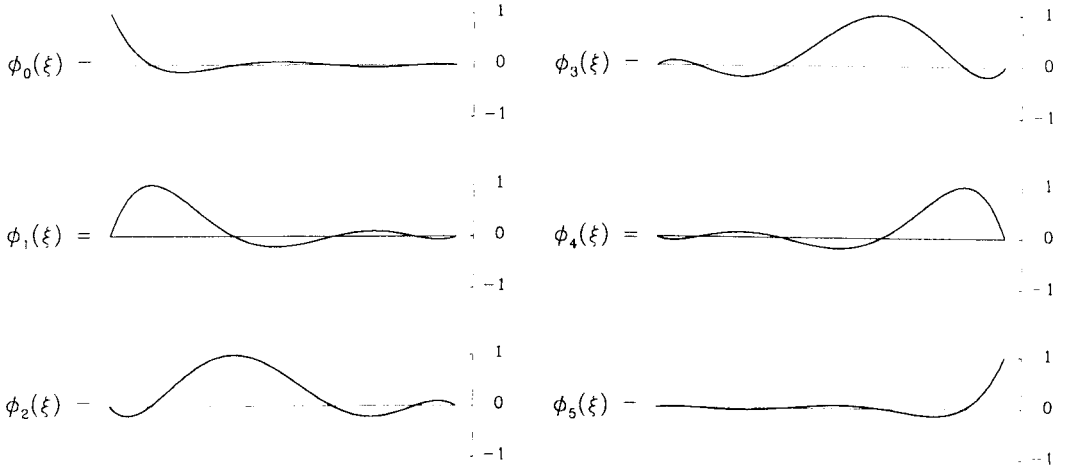


Figure 3.2: Nodal expansion modes for a polynomial order of $p = 5$.

oppose to a modal expansion basis where the modes range from order 1 through to P . Nodal expansions are therefore not hierarchical. The main use of nodal expansions in this thesis and the code is to perform differentiation and evaluation of non-linear products. If we consider a function u represented discretely as before by a set of coefficients and an expansion basis,

$$u^\delta(\xi) = \sum_{p=0}^P \hat{u}_p \phi_p(\xi) , \quad (3.40)$$

we can differentiate to obtain,

$$\frac{du^\delta(\xi)}{d\xi} = \sum_{p=0}^P \hat{u}_p \frac{d\phi_p(\xi)}{d\xi} . \quad (3.41)$$

When the expansion basis is a modal one, the differentiation is said to be performed in modal space [20, 5]. If a nodal expansion basis is used, then the differentiation is said to be physical or collocation differentiation, so called as the coefficients, $\hat{u}_p$, represent the physical or real value of u^δ at the quadrature points, ξ_i due to Kroneka Delta nature of the expansion basis. Equation (3.40) can be rewritten as

$$u^\delta(\xi) = \sum_{i=0}^{Q-1} u^\delta(\xi_i) h_i(\xi) , \quad (3.42)$$

where $Q \geq P + 1$, ξ_i are a set of quadrature points with $0 \leq i \leq Q - 1$ and h_i a Lagrangian expansion basis. By differentiating equation (3.42) the derivative of u^δ can be represented by,

$$\frac{du^\delta(\xi)}{d\xi} = \sum_{i=0}^{Q-1} u^\delta(\xi_i) \frac{dh_i(\xi)}{d\xi} . \quad (3.43)$$

At the nodal points

$$\left. \frac{du^\delta(\xi)}{d\xi} \right|_{\xi=\xi_i} = \sum_{j=0}^{Q-1} d_{ij} u^\delta(\xi_i) , \quad (3.44)$$

where

$$d_{ij} = \left. \frac{dh_j(\xi)}{d\xi} \right|_{\xi=\xi_i} . \quad (3.45)$$

3.2.2 Error Estimates

The error associated with a one dimensional Poisson or Helmholtz equation as described in section 3.1, where the domain is split into equally sized elements of length h and an expansion basis of order P is used within the elements, can be represented by,

$$\| \varepsilon \| \leq C h^{\mu-1} P^{-(k-1)} \| u \| ^k , \quad (3.46)$$

where k is the differentiability of the u , $\varepsilon = u - u^\delta$, $\mu = \min(k, P + 1)$. Therefore if u is smooth, $k \geq P + 1$, then the error algebraically proportional to h , $\log \varepsilon \propto \log h$

and exponentially proportional to P , $\log \varepsilon \propto P$. A simple method to illustrate these convergence rates is to computationally solve the one dimensional Poisson equation,

$$-\frac{\partial^2 u}{\partial x^2} = f, \quad (3.47)$$

where $f = n^2 \pi^2 \sin(n\pi x)$ and therefore $u = \sin(n\pi x)$, using the *hp*/spectral element method. The computations are performed on a one dimensional mesh with $-1 \leq x \leq 1$. As n increases the error decreases for fixed h and P , but the rate of convergence is constant. These characteristics are shown in figures 3.3 and 3.4. The error shown in these graphs is the L_2 error where

$$\varepsilon = \left(\frac{\int_A (u(x) - u^\delta(x))^2 dx}{\int_A dx} \right)^{\frac{1}{2}}, \quad (3.48)$$

where A is the area of the domain, Ω .

3.3 Extension to Higher Dimensions

One dimensional problems are obviously of limited use and to investigate more applicable problems the theory in the previous chapter must be extended to two and three dimensions. The expansion of the Galerkin formulation to higher dimensions is straight forward and will not be recounted here. The formulation of the new expansion basis is a more complicated task and only a brief explanation is given below. A more rigorous account of the intricacies of multi-dimensional spectral elements for a variety of configurations, including triangles and tetrahedrals, is given by Sherwin & Karniadakis [67, 66] and expanded by Sherwin [64]. The theory was then further developed by Warburton [78] and Warburton *et al.* [79] to include many other element geometries including prisms and cuboids which are capable of forming a hybrid mesh.

For structured domains, i.e. those that are made up of quadrilaterals in two dimensions and hexahedrons in three dimensions, the expansion basis can be constructed by a simple product of the one dimensional tensors in each of the Cartesian

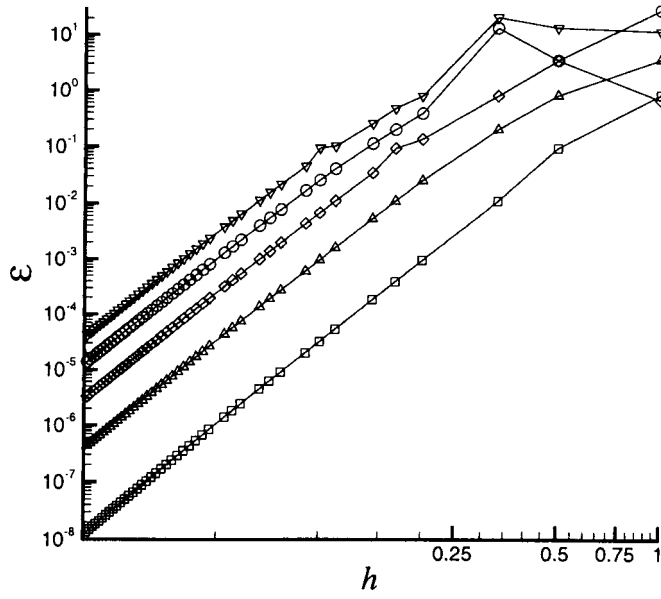


Figure 3.3: Error (ε) of u^δ against element size (h) for $-\frac{\partial^2 u}{\partial x^2} = n^2 \pi^2 \sin(n\pi x)$ where $p = 4$ $\square$ $n = 2$, $\triangle$ $n = 4$, $\diamond$ $n = 6$, $\bigcirc$ $n = 8$ and ∇ $n = 10$

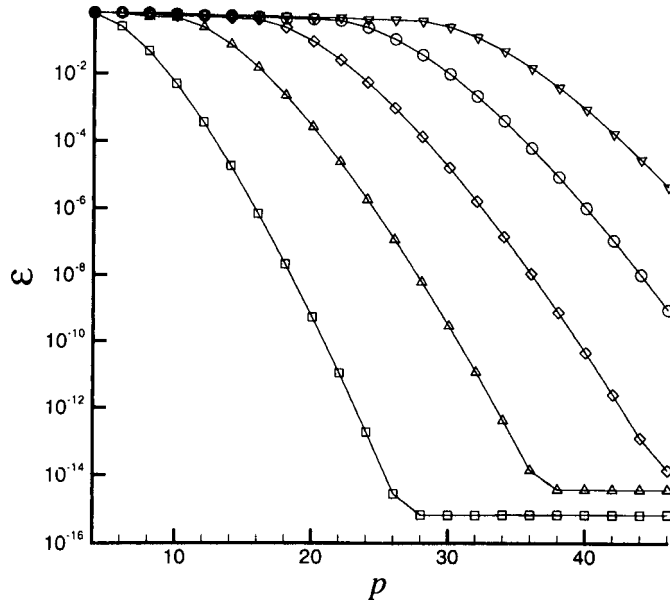


Figure 3.4: Error (ε) of u^δ against polynomial order (p) for $-\frac{\delta^2 u}{\delta x^2} = n^2 \pi^2 \sin(n\pi x)$ where $h = 2$ $\square$ $n = 2$, $\triangle$ $n = 4$, $\diamond$ $n = 6$, $\bigcirc$ $n = 8$ and ∇ $n = 10$

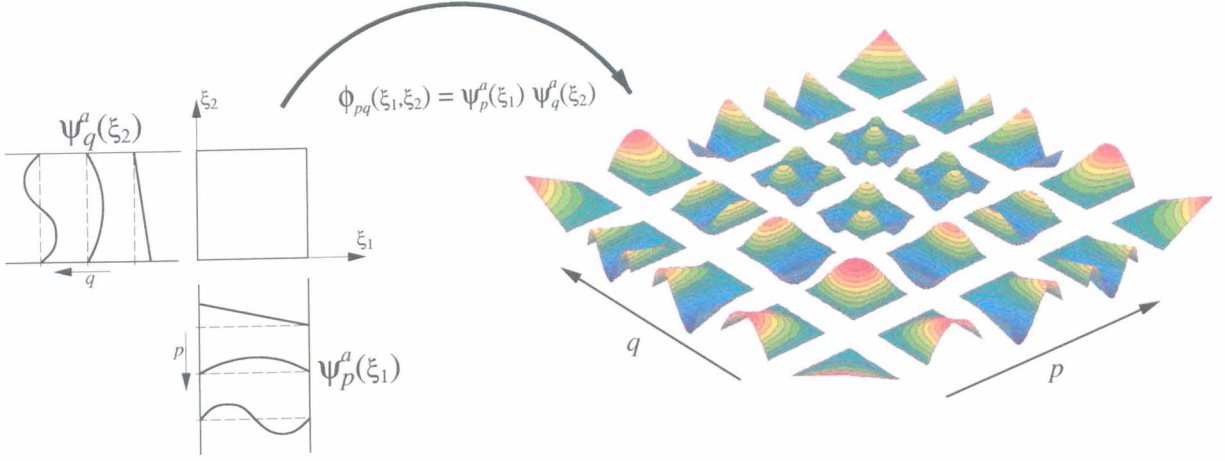


Figure 3.5: Construction of a two dimensional expansion basis from the product of two one dimensional modal expansions of order $p = 4$ defined in equation (3.50) (edge and face modes have been scaled by a factor of 4 and 16 respectively) [34].

co-ordinate systems, geometries

$$\phi_{pq}(\xi_1, \xi_2) = \psi_p(\xi_1)\psi_q(\xi_2) \quad 0 \leq p, q; p \leq P_1, q \leq P_2, \quad (3.49)$$

$$\phi_{pqr}(\xi_1, \xi_2, \xi_3) = \psi_p(\xi_1)\psi_q(\xi_2)\psi_r(\xi_3) \quad 0 \leq p, q, r; p \leq P_1, q \leq P_2, r \leq P_3 \quad (3.50)$$

where ξ_1, ξ_2, ξ_3 are the standard Cartesian co-ordinates and ψ the one dimensional modal expansion basis of order P described in equations (3.34-3.36). P_1, P_2 and P_3 are the orders of the one dimensional expansion basis and can take different values. ϕ_{pq} represents the two dimensional basis and ϕ_{pqr} the three dimensional basis. The resulting two dimensional basis can be seen in figure 3.5.

To be able to calculate fluid flow around complex and within complex domains, there is a need for unstructured elements, primarily triangles in two-dimensions and tetrahedrons in three dimensions. Until relatively recently the use of unstructured methods incorporating the *hp*/Spectral element method has been very limited. A suitable modal expansion was first proposed by Dubiner [11] in two dimensions and later extended to three dimensions by Sherwin [64, 65].

For an unstructured *hp*/Spectral element code to be practical, the efficiency of

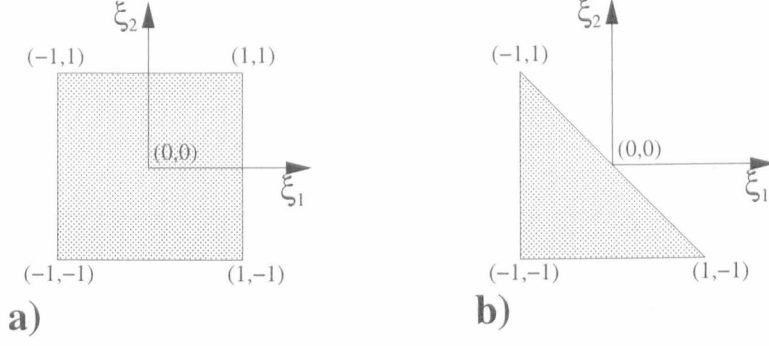


Figure 3.6: Standard regions for the (a) quadrilateral, and (b) triangular expansion in terms of the Cartesian coordinates ξ, ξ_2 . [34]

the computations has to be comparable to a structured method. A distinct advantage of the structured method is its ability to construct two and three dimension expansion basis by taking the tensor product of the one dimensional basis as in equations (3.49) and (3.50). To retain this characteristic in unstructured elements a mapping from a structured element, a quadrilateral, to a unstructured element, a triangle is developed. A quadrilateral, Q^2 , can be represented by two Cartesian coordinates (ξ_1, ξ_2) such that

$$Q^2 = \{(\xi_1, \xi_2) \mid -1 \leq \xi_1, \xi_2 \leq 1\}. \quad (3.51)$$

A triangle, T^2 , can be represented in a similar fashion by

$$T^2 = \{(\xi_1, \xi_2) \mid -1 \leq \xi_1, \xi_2; \xi_1 + \xi_2 \leq 0\}. \quad (3.52)$$

The standard regions are shown in figure 3.6. A coordinate mapping of,

$$\begin{aligned} \eta_1 &= 2 \frac{1 + \xi_1}{1 - \xi_2} - 1 \\ \eta_2 &= \xi_2 \end{aligned} \quad (3.53)$$

is adopted, resulting in η , the new local coordinate, describing the triangle thus

$$T^2 = \{(\eta_1, \eta_2) \mid -1 \leq \eta_1, \eta_2 \leq 1\}. \quad (3.54)$$

Under this transformation the standard triangle, T^2 , is converted to the standard quadrilateral, Q^2 , illustrated in figure 3.7. In the same way the one dimensional expansion modes can be transformed from a quadrilateral element to a triangular one;

$$\phi_{pq}(\xi_1, \xi_2) = \psi_p(\eta_1(\xi_1, \xi_2))\psi_q(\eta_2(\xi_2)). \quad (3.55)$$

Though C_0 continuity cannot easily be enforced using the basis above. A more general form of the expansion basis is produced by repetition of the one dimensional expansion bases. The one dimensional expansion basis tensor, $\psi_p^a(\eta_1)$ is multiplied by a two dimensional $\psi_{pq}^b(\eta_2)$, where

$$\psi_i^a(\xi) = \begin{cases} \left(\frac{1-\xi}{2}\right) & i = 0 \\ \left(\frac{1-\xi}{2}\right) \left(\frac{1+\xi}{2}\right) P_{i-1}^{1,1}(\xi) & 1 \leq i \leq I-1 \\ \left(\frac{1+\xi}{2}\right) & i = I \end{cases} \quad (3.56)$$

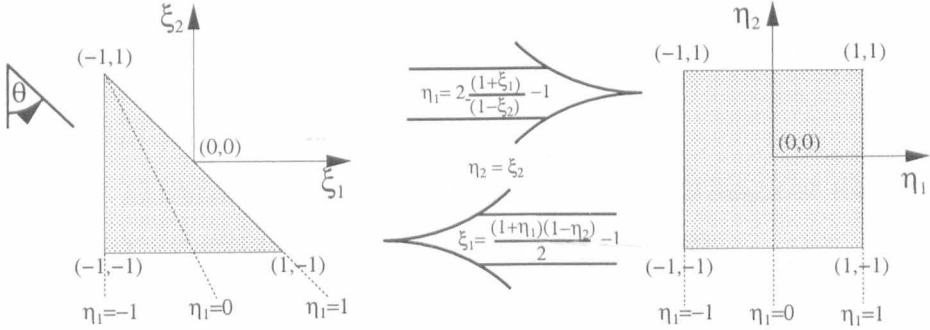


Figure 3.7: Triangle to rectangle transformation [34]

$$\psi_{ij}^b(\xi) = \begin{cases} \psi_j^a(\xi) & i = 0, \quad 0 \leq j \leq J \\ \left(\frac{1-\xi}{2}\right)^{i+1} & 1 \leq i \leq I-1, \quad j = 0 \\ \left(\frac{1-\xi}{2}\right)^{i+1} \left(\frac{1+\xi}{2}\right) P_{j-1}^{2i+1,1}(\xi) & 1 \leq i \leq I-1 \quad 1 \leq j \leq J-1 \\ \psi_j^a(\xi) & i = I, \quad 0 \leq j \leq J \end{cases} \quad (3.57)$$

The resulting expansion modes are illustrated in figure 3.8. This technique can also be extended to three dimensions to transform hexahedrons to tetrahedrons [66].

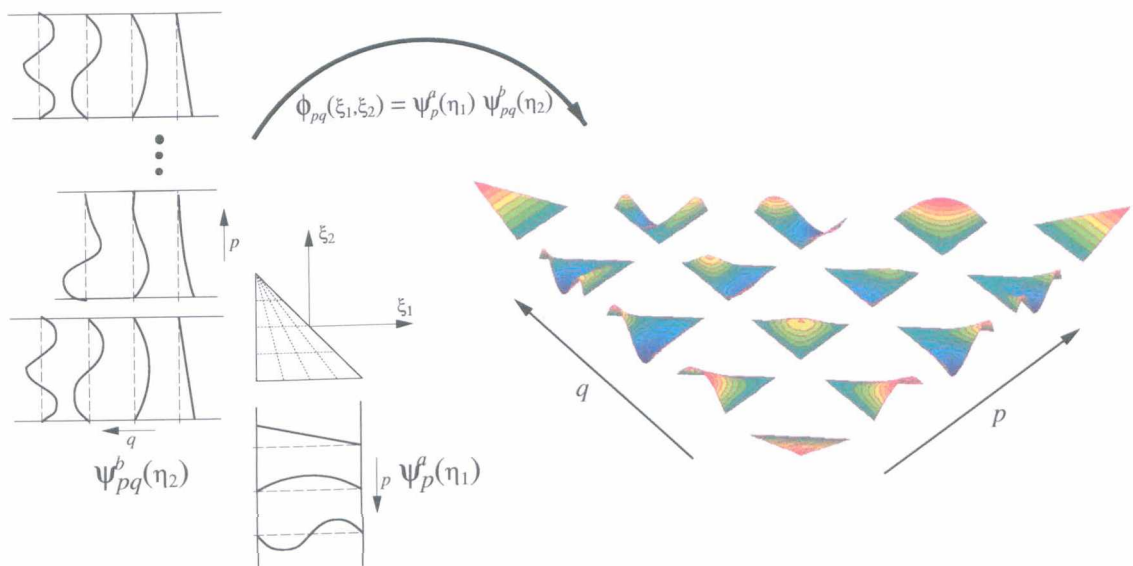


Figure 3.8: Construction of a fourth-order ($p = 4$) triangular expansion using the product of two tensors $\psi_p^a(\eta_1)$ and $\psi_{pq}^b(\eta_2)$ [34].

3.4 Summary

The theoretical and practical basis of an *hp*/Spectral computational code have been presented. The full mechanism to formulate a solution to the one dimensional Helmholtz, Poisson and Laplace problems has been illustrated and the error conver-

gence properties of the hp /Spectral element method shown. The theory was then extended to incorporate multi-dimensions.

These procedures are used to facilitate the simulation of inviscid and viscous free surface flow in the following chapters.

Chapter 4

Inviscid Free Surface Flow

4.1 Introduction

The aim of this research is to formulate a computational algorithm capable of simulating a viscous, rotational flow with a free surface. To enhance the understanding of these free surface flows and to identify the flow characteristics that these properties promote, inviscid, irrotational free surface flow is first considered. In this chapter the linear and nonlinear free surface boundary conditions are derived and the stability of the system with respect to the spatial and temporal discretisation is ascertained. In addition, results are given for free surface flows using linear and nonlinear boundary conditions.

4.2 Inviscid Free Surface Governing Equations

A fixed two dimensional Cartesian coordinate system Ox^*z^* is defined, where the undisturbed free surface is described by $z^* = 0$, and z^* points vertically upwards. The free surface height can be represented as $z^* = \zeta^*(x, t)$ and the depth of a containing tank is d and the length l as shown in figure 4.1. As the fluid is irrotational and inviscid the velocity, $\mathbf{u}^* = (u_x^*(x^*, z^*; t), u_z^*(x^*, z^*; t))$, is represented by

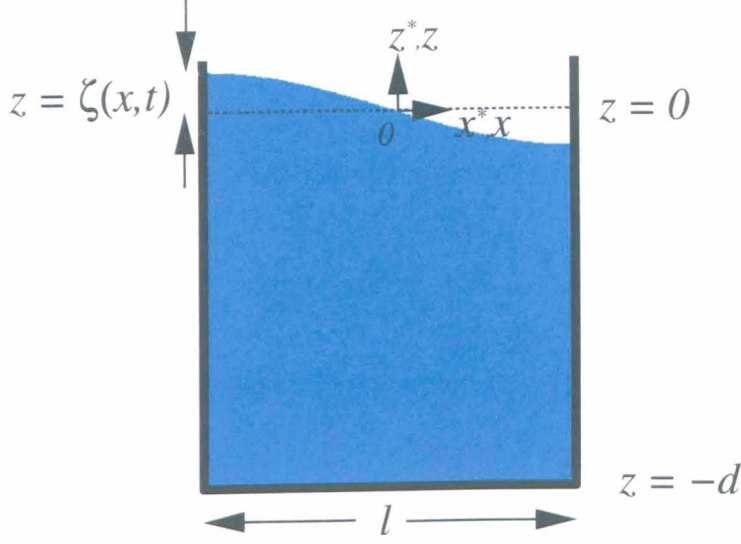


Figure 4.1: Definition of frame of reference and wall boundary conditions for contained free surface system

the gradient of a velocity potential, $\phi^*(x^*, z^*; t)$, such that,

$$\mathbf{u}^* = \nabla \phi^* , \quad (4.1)$$

where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z} \right). \quad (4.2)$$

and by considering that the fluid is incompressible, the velocity potential can be evaluated by solving the Laplacian,

$$\nabla^2 \phi^* = 0 , \quad (4.3)$$

For the solution system to be fully defined the Laplacian (4.21) must be supplemented by suitable boundary conditions.

4.2.1 Boundary Conditions

Eulerian Boundary Conditions

The following boundary conditions are well known and are not derived here. A full derivation is given in Lighthill [40]. The dynamic boundary condition which ensures that the pressure on the free surface is balanced by the atmospheric pressure is,

$$\frac{\partial \phi^*}{\partial t} = -g\zeta^* - \frac{1}{2} \nabla \phi^* \cdot \nabla \phi^* , \quad (4.4)$$

where g is force of gravity per unit mass. The kinematic boundary condition is formulated by considering that the free surface is a bounding surface with no transfer of matter across the surface. The free surface can be described by,

$$S(x^*, z^*; t) = z^*(x^*, z^*; t) - \zeta^*(x^*; t) . \quad (4.5)$$

and the condition of a bounding surface is satisfied by

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + u_x^* \frac{\partial S}{\partial x^*} = 0 . \quad (4.6)$$

Therefore a particle on the free surface remains there, such that,

$$\frac{dz^*}{dt} = \frac{\partial \zeta^*}{\partial t} + u_x^* \frac{\partial \zeta^*}{\partial x^*} , \quad (4.7)$$

and in terms of potential flow,

$$\frac{\partial \phi^*}{\partial z^*} = \frac{\partial \zeta^*}{\partial t} + \frac{\partial \phi^*}{\partial x^*} \frac{\partial \zeta^*}{\partial x^*} . \quad (4.8)$$

The boundary condition on a wall is the non-permeability condition,

$$\frac{\partial \phi^*}{\partial n} = V_w \cdot \mathbf{n} , \quad (4.9)$$

where V_w is the velocity of the wall and can be spatially and temporally dependent and $\mathbf{n}$ is its normal.

Governing Equations in a Moving Reference Frame

In the course of this work the free surface flow of a fluid contained in a moving tank is considered. To facilitate this, a moving reference frame is considered, Oxz , which is initially coincident with the fixed reference frame, Ox^*z^* , and moves with the tank velocity $\mathbf{U}$, where

$$\mathbf{U} = \frac{d\mathbf{X}_t}{dt} = (U_x(t), U_z(t)) , \quad (4.10)$$

and

$$\mathbf{X}_t = (X_t(t), Z_t(t)) , \quad (4.11)$$

where $\mathbf{X}_t$ describes the displacements of the tank. The governing equations in the moving frame of reference were evaluated by Wu *et al.*[84] and are reproduced here. In the fluid domain,

$$\nabla^2 \phi = 0 , \quad (4.12)$$

where

$$\phi^* = \phi + xU_x + zU_z , \quad (4.13)$$

and on the walls,

$$\frac{\partial \phi}{\partial n} = 0 . \quad (4.14)$$

On the free surface the kinematic boundary condition is represented as,

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial x} \frac{\partial \zeta}{\partial x} , \quad (4.15)$$

and the dynamic boundary condition as,

$$\left. \frac{\partial \phi}{\partial t} \right|_{\frac{\partial \zeta}{\partial t}} = \frac{\partial \phi}{\partial z} \frac{\partial \zeta}{\partial t} - g\zeta - \frac{1}{2} \nabla \phi \cdot \nabla \phi - x \frac{dU_x}{dt} - \zeta \frac{dU_z}{dt} , \quad (4.16)$$

where $\left. \frac{\partial \phi}{\partial t} \right|_{\frac{\partial \zeta}{\partial t}}$ denotes that the time derivative is evaluated on the free surface for fixed x . Care must be taken with the initial conditions of ϕ on the free surface such that,

$$\phi(x, \zeta, 0) = \phi^*(x^*, \zeta^*, 0) - xU_x(x, \zeta, 0) - zU_z(x, \zeta, 0) . \quad (4.17)$$

Linearised Boundary Conditions

The standard linearised version of the free surface boundary conditions (4.15,4.16) are obtained by considering small velocities ($\nabla\phi \ll 1$) and gradients of the free surface ($\frac{\partial\zeta}{\partial x} \ll 1$) and ignoring second order terms. The resulting linearised, free surface boundary conditions are therefore,

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\phi}{\partial z} \quad (4.18)$$

$$\frac{\partial\phi}{\partial t} = -g\zeta - x\frac{dU_x}{dt} - z\frac{dU_z}{dt} , \quad (4.19)$$

which are evaluated on $z = 0$.

hp/Spectral Element Formulation

To solve the system numerically the two-dimensional fluid domain is spatially discretised by a finite number of elements and the velocity potential is represented in terms of a set of modal expansions which vary with h , length of the elements, and p , the polynomial order of the function, and a corresponding set of coefficients.

The velocity potential therefore takes the form,

$$\phi \approx \phi^\delta = \sum_{i=1}^{ndof} \hat{\phi}_i N_i(x, y) , \quad (4.20)$$

where ϕ is the exact solution, ϕ^δ is the numerical approximation and $\hat{\phi}_i$ are the coefficients of the modal expansions bases $N_i(x, y)$. To solve the Laplacian

$$\nabla^2\phi = 0 , \quad (4.21)$$

ϕ is split into its constituent parts,

$$\nabla^2\phi^H + \nabla^2\phi^D = 0 , \quad (4.22)$$

where ϕ^H represents the homogeneous value of the potential within the domain with zero Dirichlet boundary conditions, and ϕ^D are the known potential Dirichlet boundary conditions on the free surface. The weak form of the solution is found by

multiplying equation (4.22) by a test function, v^δ , and integrating over the domain. Following the Galerkin approach the test function is set to the same form as the trial function, ϕ^δ , i.e.

$$v^\delta = \sum_{j=1}^{ndof} \hat{v}_j N_j(x, y) , \quad (4.23)$$

where the symbols have the same meaning as above. Equation (4.22) then becomes

$$\int_{\Omega} \nabla^2 \phi^{\delta H} \cdot v^\delta d\Omega + \int_{\Omega} \nabla^2 \phi^{\delta D} \cdot v^\delta d\Omega = 0 , \quad (4.24)$$

where Ω is the area of the domain. Utilising Green's Theorem equation (4.24) can be represented as

$$-\int_{\Omega} \nabla \phi^{\delta H} \cdot \nabla v^\delta d\Omega = \int_{\Omega} \nabla \phi^{\delta D} \cdot \nabla v^\delta d\Omega - \int_S (\nabla \phi^{\delta H} \cdot \mathbf{n}) v^\delta dS , \quad (4.25)$$

where S represents the boundary and $\mathbf{n}$ is its outward normal. By utilising equation (4.14) and substituting equations (4.20) and (4.23) into equation 4.25 the solution system can be written as,

$$-\int_{\Omega} \sum_{j=1}^{ndof} \hat{\phi}_i^H \nabla N_i \cdot \nabla N_j d\Omega = \int_{\Omega} \nabla \phi^{\delta D} \cdot \nabla N_j d\Omega \quad \forall j, \quad (4.26)$$

In matrix form this can be written as

$$\mathbf{L} \boldsymbol{\phi}^H = \mathbf{f} , \quad (4.27)$$

where $\mathbf{L}$ and $\mathbf{f}$ are known and defined as

$$\mathbf{L}[i, j] = - \int_{\Omega} \nabla N_i \cdot \nabla N_j d\Omega , \quad (4.28)$$

$$\mathbf{f}[j] = \int_{\Omega} \nabla \phi^{\delta D} \cdot \nabla N_j dA , \quad (4.29)$$

and

$$\boldsymbol{\phi}^H[i] = \hat{\phi}_i^H \quad (4.30)$$

To solve for the unknowns $\mathbf{L}$ is inverted and multiplied through, i.e.

$$\boldsymbol{\phi}^H = \mathbf{L}^{-1} \mathbf{f} . \quad (4.31)$$

To evaluate the kinematic boundary condition the velocity of the fluid necessitates the differentiation of ϕ . A problem arises due to C_0 continuous nature of the modal basis across elements and therefore the velocity and free surface will be discontinuous. To recover the continuous velocity a procedure set out in [82] can be followed, which uses the Galerkin approach to evaluate a continuous velocity field, $\mathbf{u}$, over the whole domain, i.e.

$$\int_{\Omega} \nabla \phi \cdot \mathbf{N}_i d\Omega = \int_{\Omega} \sum_{i=1}^{ndof} \hat{u}_i \mathbf{N}_i \cdot \mathbf{N}_j d\Omega = \int_{\Omega} \mathbf{u} \cdot \mathbf{N}_j d\Omega \quad \forall j . \quad (4.32)$$

An alternative approach is to use a simple up-winding scheme, which was found to provide greater numerical stability when using the nonlinear, free surface boundary conditions.

4.3 Stability Analysis of Spatial Discretisation

To investigate the stability properties of different spatial discretisations three structured mesh configurations are considered. They are shown in figure 4.2 where the first is a locally symmetric quadrilateral mesh, the second a locally symmetric triangular mesh and the third a locally asymmetric triangular mesh. All three meshes are of size $d = 1$ and $l = 1$. A mesh is defined as locally symmetric if the elements on the free surface are symmetric when reflected about a vertical axis through the mid-point of the free surface side of the element. The motivation to investigate asymmetric meshes comes from the wish to undertake nonlinear flow solutions which necessitates a moving mesh. This inevitably introduces distortion to the grid, as would the introduction of complex geometries. All symmetric and asymmetric meshes used in this chapter are of the same pattern as those shown in figure 4.2 and shall be denoted by the number of elements on each side i.e. figure 4.2 a) is a 10×10 quadrilateral mesh,

b) a 10×10 symmetrical triangular mesh and c) a 10×10 asymmetrical triangular mesh. It is believed that whatever numerical effect a distorted grid has on linear flow, it will have a similar effect on nonlinear inviscid and viscous flow. Therefore it is a worthwhile area of interest.

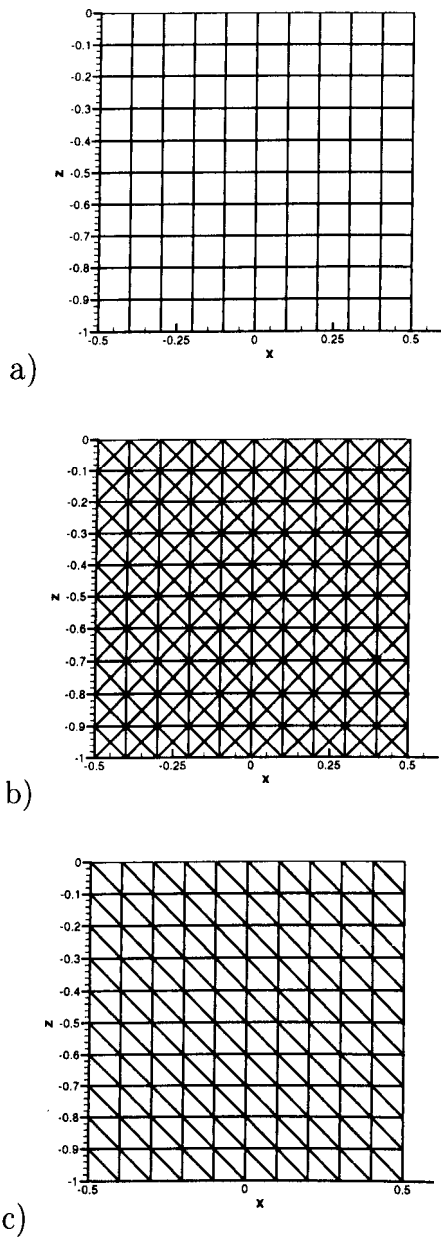


Figure 4.2: Structured meshes for stability analysis: a) quadrilateral mesh
b) symmetric triangular mesh c) asymmetric triangular mesh

In order to investigate how the computations are affected by the different spatial discretisations, the three meshes are used to solve the same free surface problem of a standing wave in a stationary tank, represented by the Laplacian of the velocity potential in equation (4.12), wall boundary conditions in equation (4.14), linear free surface boundary conditions in equations (4.18), (4.19) and the initial conditions,

$$\phi(x, z; 0) = 0 \quad (4.33)$$

$$\zeta(x; 0) = -a_0 \sin(kx) \quad (4.34)$$

where $a_0 = 0.1$ and $k = \frac{\pi}{l}$. The solution of this problem is a standing wave of constant wavelength, amplitude and time period represented by

$$\phi = \frac{a_0 g \cosh(k(z+d))}{\omega \cosh(kd)} \sin(kx) \sin(\omega t) \quad (4.35)$$

where g is gravity which unless otherwise stated is equal to 9.81 and ω is the first natural frequency of the oscillation, where $\omega^2 = gk \tanh(kd)$. Using the *hp*/Spectral element solver described in Chapter 3 with $p = 5$, a time-step, Δt of 0.01 and a second-order time integration scheme, the three meshes gave differing results for long time studies. The symmetric meshes were stable over long periods of time as shown in figure 4.3, where the time history of the point of maximum displacement on the free surface is shown for over 50 time units. The displacement, frequency and wavelength are all constant and stability is achieved. This also indicates the minimal dissipation errors inherent in the *hp*/Spectral element method. In contrast to this the asymmetric mesh became unstable after relatively few periods. The free surface profiles for the symmetric and asymmetric meshes after approximately 2 oscillations is shown in figures 4.4. The asymmetric triangular mesh profile is similar to that of the familiar saw-tooth pattern reported by Longuet-Higgins & Cokelet [42]. Decreasing the time step for any given temporal discretisation had no effect on the stability of the problem, the absolute time of the onset of instability was unaltered. To investigate the effect of the spatial discretisation a semi-discrete formulation of the problem was developed. This formulation reduced the problem to the solution values on the free surface boundary.

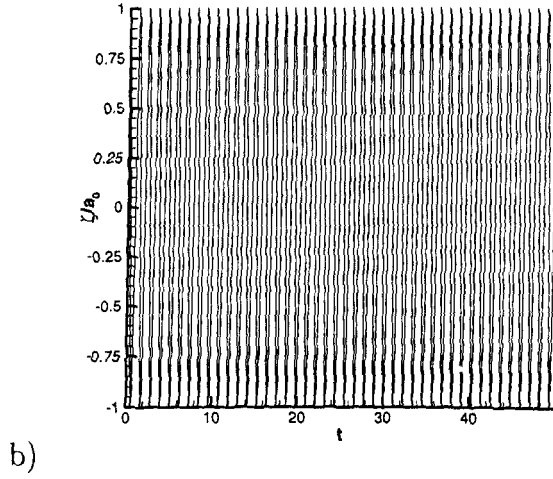
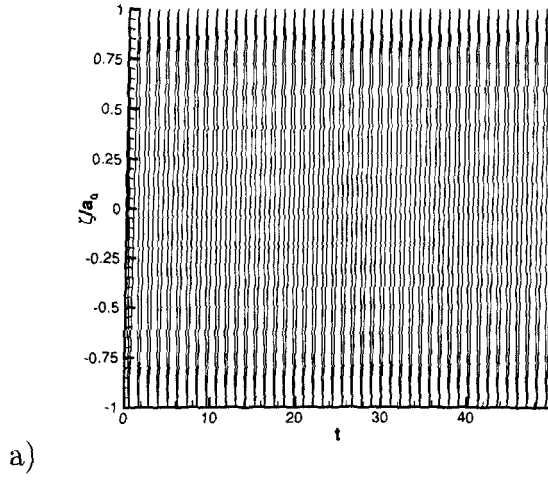


Figure 4.3: Time history of point on free surface which undergoes maximum displacement using a) the symmetric quadrilateral mesh and b) the symmetric triangular mesh

4.4 Formulation of Semi-discrete Eigenvalue and Eigenvector Analysis

In order to formulate a semi-discrete system to investigate the stability of the numerical problem the Laplacian of the governing equation (4.12) is decomposed into

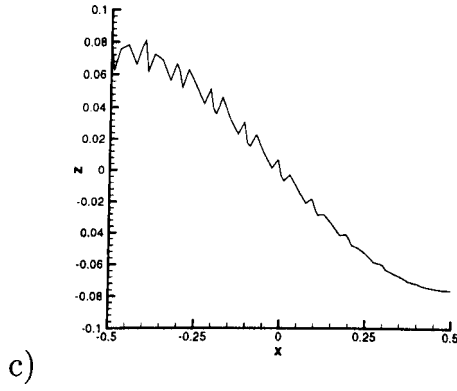
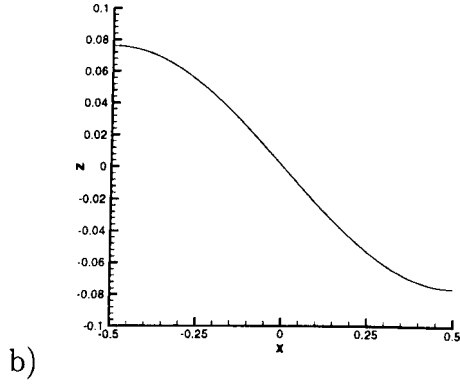
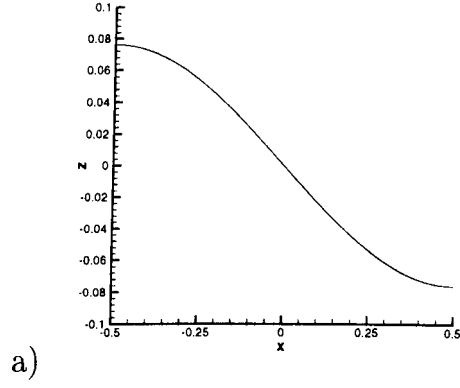


Figure 4.4: Free surface profile at $t = 2.39$ for a) quadrilateral mesh, b) symmetric triangular mesh and c) asymmetric triangular mesh

its interior and free surface boundary components,

$$\mathbf{L}\boldsymbol{\phi} = \begin{bmatrix} \mathbf{L}_{ii} & \mathbf{L}_{ib} \\ \mathbf{L}_{bi} & \mathbf{L}_{bb} \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}_i \\ \boldsymbol{\phi}_b \end{bmatrix} = \mathbf{0} , \quad (4.36)$$

where the subscript b refers to the degrees of freedom on the free surface boundary and i refers to all remaining degrees of freedom. The first row of this system can be rearranged to obtain,

$$\mathbf{L}_{ii}\phi_i + \mathbf{L}_{ib}\phi_b = \mathbf{0} \Rightarrow \phi_i = -\mathbf{L}_{ii}^{-1}\mathbf{L}_{ib}\phi_b . \quad (4.37)$$

Equation (4.37) therefore expresses the interior degrees of freedom in terms of the free surface boundary degrees of freedom. This is possible due to the elliptic nature of the Laplace equation. The differential of ϕ in the z -direction can similarly be represented by an operator $\mathbf{D}$ such that,

$$\mathbf{D}\phi = \begin{bmatrix} \mathbf{D}_{ii} & \mathbf{D}_{ib} \\ \mathbf{D}_{bi} & \mathbf{D}_{bb} \end{bmatrix} \begin{bmatrix} \phi_i \\ \phi_b \end{bmatrix} . \quad (4.38)$$

By considering the bottom line of equation (4.38), an expression for the differential on the boundary is

$$\left. \frac{\partial \phi}{\partial z} \right|_b = \mathbf{D}_{bi}\phi_i + \mathbf{D}_{bb}\phi_b , \quad (4.39)$$

where $\left. \frac{\partial \phi}{\partial z} \right|_b$ is a vector of the values of $\frac{\partial \phi}{\partial z}$ on the free surface. By substituting (4.37) into (4.39) a relationship for the differential in terms of the boundary values of the velocity potential is obtained, i.e.

$$\left. \frac{\partial \phi}{\partial z} \right|_b = -\mathbf{D}_{bi}\mathbf{L}_{ii}^{-1}\mathbf{L}_{ib}\phi_b + \mathbf{D}_{bb}\phi_b = \left[\mathbf{D}_{bb} - \mathbf{D}_{bi}\mathbf{L}_{ii}^{-1}\mathbf{L}_{ib} \right] \phi_b = \mathbf{N}_D\phi_b , \quad (4.40)$$

where $\mathbf{N}_D$ can be thought of as a discrete Dirichlet-Neumann operator. By combining (4.40) with the linear free surface boundary conditions

$$\frac{\partial \phi}{\partial t} = -g\zeta , \quad (4.41)$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} , \quad (4.42)$$

the semi-discrete solution system of the linear free surface motion is now formed,

$$\begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}_t = \begin{bmatrix} \mathbf{0} & -g\mathbf{I} \\ \mathbf{N}_D & \mathbf{0} \end{bmatrix} \begin{bmatrix} \phi_b \\ \zeta \end{bmatrix} . \quad (4.43)$$

The eigenvalues of the operating matrix dictate the stability of the scheme. Thus the eigenvalues, s_n , must be found, such that

$$\begin{bmatrix} \mathbf{0} & -g\mathbf{I} \\ \mathbf{N}_D & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_n \\ \boldsymbol{\eta}_n \end{bmatrix} = s_n \begin{bmatrix} \boldsymbol{\psi}_n \\ \boldsymbol{\eta}_n \end{bmatrix}, \quad (4.44)$$

where $\boldsymbol{\psi}_n$ and $\boldsymbol{\eta}_n$ denote the eigenvectors. This system can be rearranged to form

$$-s_n \boldsymbol{\psi}_n - g \boldsymbol{\eta}_n = \mathbf{0}, \quad (4.45)$$

$$\mathbf{N}_D \boldsymbol{\psi}_n - s_n \boldsymbol{\eta}_n = \mathbf{0}, \quad (4.46)$$

and by substituting equation (4.45) into equation (4.46) an equation for the eigenvalues of the full matrix in terms of those of the Neumann-Dirichlet operator matrix is obtained,

$$s_n^2 \mathbf{I} \boldsymbol{\eta}_n + g \mathbf{N}_D \boldsymbol{\eta}_n = \mathbf{0}. \quad (4.47)$$

If the eigenvalues and eigenvectors of $\mathbf{N}_D$ are denoted by μ_n and $\boldsymbol{\alpha}_n$ respectively such that,

$$\mathbf{N}_D \boldsymbol{\alpha}_n = \mu_n \boldsymbol{\alpha}_n, \quad (4.48)$$

where μ_n is complex, then

$$s_n^2 = -\mu_n g, \quad (4.49)$$

or

$$s_n = \pm i (\mu_n g)^{\frac{1}{2}}. \quad (4.50)$$

Therefore considering the general form of μ_n by stating its phase and magnitude,

$$\mu_n = |\mu_n| e^{i\theta}, \quad (4.51)$$

then the eigenvalues of the full matrix system are related to those of the Neumann-Dirichlet operator by the following relationship,

$$s_n = \pm i g^{\frac{1}{2}} |\mu_n|^{\frac{1}{2}} e^{i\frac{\theta}{2}} \quad (4.52)$$

$$= \pm g^{\frac{1}{2}} |\mu_n|^{\frac{1}{2}} e^{i(\frac{\theta}{2} + \frac{\pi}{2})}. \quad (4.53)$$

If $\theta = 0$ then μ_n is real, resulting in purely imaginary s_n eigenvalues, and the matrix system (4.43) will have a purely dispersive solution. If $\theta \neq 0$ then the eigenvalues, s_n , have a positive and negative real component resulting in an unstable system. For stability of this inviscid, incompressible linear free surface flow the eigenvalues of the matrix are required to be purely imaginary.

Validation of Numerical Implementation

To measure the accuracy of the numerical implementation the analytical values of the system's eigenvalues are compared with the eigenvalues of the numerical system. If the analytical representation of ϕ is used, where

$$\phi = A \cosh(k_n(z+d)) \sin(k_n x) \sin(\omega_n t) , \quad (4.54)$$

where $k_n = \frac{n\pi}{l}$, $\omega_n = (gk_n \tanh(k_n d))^{\frac{1}{2}}$ for $n = 0, 1, 2, 3, \dots$ and A is dependent on the initial conditions. then system (4.43) can be represented by

$$\begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}_t = \begin{bmatrix} 0 & -g\mathbf{I} \\ k_n \tanh(k_n d) & 0 \end{bmatrix} \begin{bmatrix} \phi_b \\ \zeta \end{bmatrix} . \quad (4.55)$$

The resulting eigenvalues this system are therefore,

$$\Lambda_n = \pm (gk_n \tanh(k_n d))^{\frac{1}{2}} i . \quad (4.56)$$

The first non-zero, analytical eigenvalue is represented by $k_1 = \pi$ and has a value of $\Lambda_1 = \pm 5.541131i$. Figure 4.5 (a) shows the plot of the error, ε , between this analytical value and that produced computationally on a 2×2 symmetric triangular mesh as the order of the expansions, p , increases and figure 4.5(b) shows the error as the size of the elements, h , decreases for similar shaped symmetric triangular meshes. Figure 4.5(a) shows that $\log(\varepsilon) \propto p$ and figure 4.5(b) that $\log(\varepsilon) \propto \log(h)$ proving standard hp convergence.

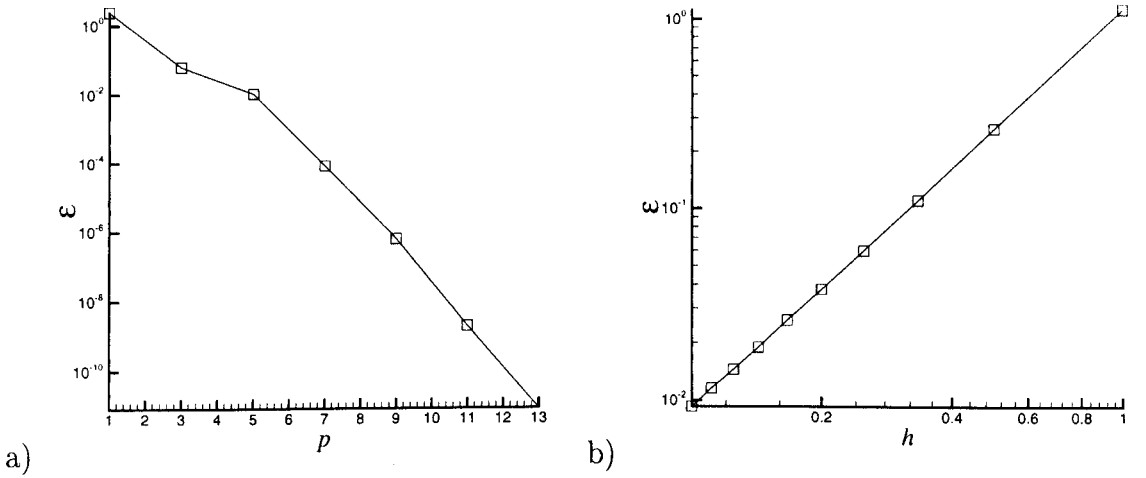


Figure 4.5: First Eigenvalue Error for a) increasing polynomial order p and b) decreasing element size, h

4.4.1 Spatial Discretisation Comparison

To illustrate the above analysis the eigenspectrums of the 10×10 meshes were considered. The numerically evaluated eigenvalues and eigenvectors corresponding to a second order expansion basis, $p = 2$, can be seen in figures 4.6 and 4.7 respectively. The eigenspectrum for the two locally symmetric meshes clearly indicates a stable solution whilst the solution for the asymmetric mesh is unstable. The first three corresponding eigenvectors are shown in figure 4.7. Note the sinusoidal form of the eigenvectors corresponding to the analytical solution. The asymmetric mesh, however, leads to a set of asymmetric eigenvectors biased in the same direction as the free surface elements of the asymmetric mesh.

An increase in the number of degrees of freedom results in the solution of higher wavenumbers, k_n . This leads to an increase in the magnitude of the highest real eigenvalue when considering the asymmetric mesh. Therefore as the degrees of freedom of the solution system increases, the more unstable the solution becomes. Figure 4.8 illustrates the increasing magnitude of the highest eigenvalue as (a) p increases

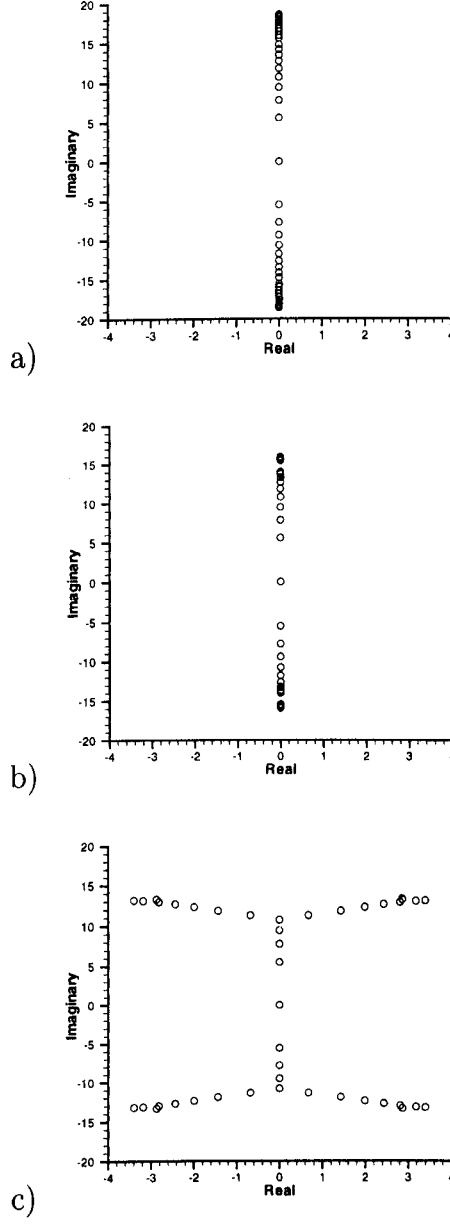


Figure 4.6: Eigenvalues indicating stability for (a) symmetric quadrilateral and (b) triangular meshes and eigenvalues indicating instability for (c) asymmetric mesh

($h = \frac{1}{3}$) and (b) as h decreases ($p = 3$). The variation of the maximum eigenvalue is described by $\log(s_{max}) \propto p$ and $\log(s_{max}) \propto \log h$, which is the same relationship

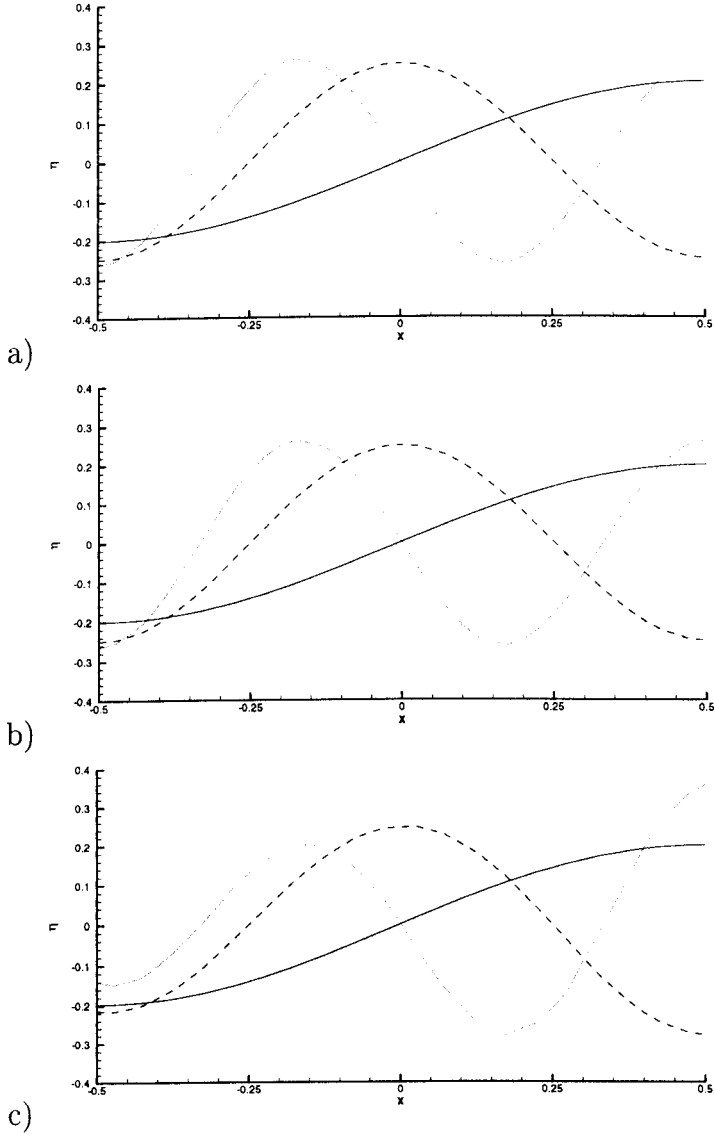


Figure 4.7: Eigenvectors for (a) symmetric quadrilateral mesh, (b) symmetric triangular mesh and (c) asymmetric triangular mesh

between the error, ε , and h and p .

4.4.2 Mechanisms for Stability

Unstructured meshes play an important part in free surface computations, particularly when a complex geometry is included in the domain. Furthermore, when it

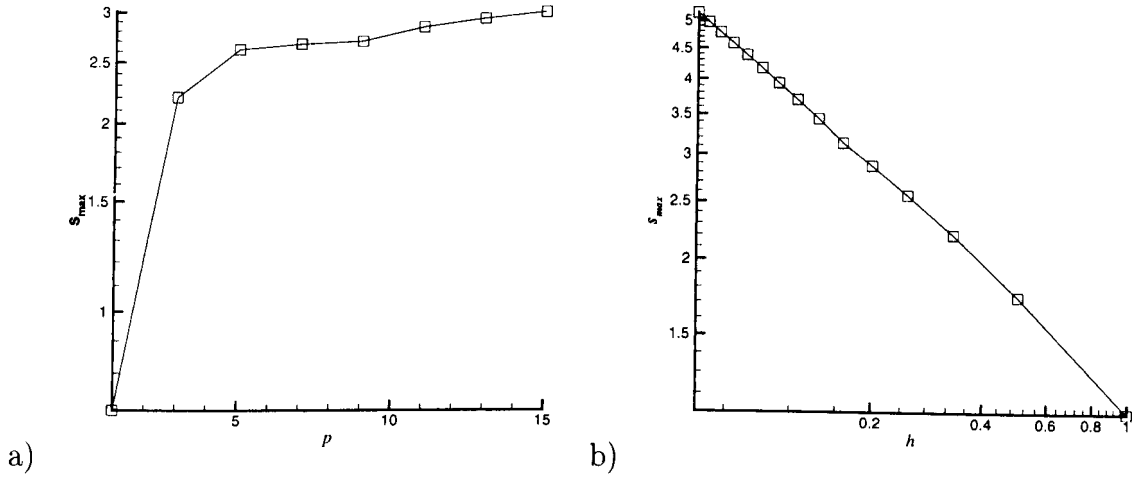


Figure 4.8: Magnitude of the maximum eigenvalue for a) increasing polynomial order p ($h = \frac{1}{3}$) and b) decreasing element size h ($p = 3$)

is necessary to move the mesh, i.e. for nonlinear flow, the integrity of the elements cannot be guaranteed. Therefore a mechanism is needed to numerically enforce the stability of the computations.

Removal of high modes

The following procedure was proposed by both Moore [47] and Roberts [62], who theorised that the incorrect phase relation of the discretised system lead to a resonance between the lower frequency physical waves and the high frequency numerically induced discrete waves causing the instability for the nonlinear problem. It is theorised by Tsai & Yue [74] that high wavenumber instabilities should arise in an inviscid model due to strong nonlinear interactions causing energy to transfer to and accumulate at the highest wavenumber modes of the discretisation. Though this theory was advanced for nonlinear motion the same process appears to occur in the linear analysis. Figure 4.9 shows two eigenspectrums for a 10×10 asymmetric triangular mesh when (a) $p = 2$ and (b) when $p = 5$. It is clearly seen that the unstable eigenvalues are those with a larger magnitude imaginary part and if equation (4.56)

is considered these unstable eigenvalues relate to higher wavenumber eigenvectors, thus substantiating the theory of high wavenumber instabilities. Furthermore if the eigenspectrums are studied closely it can be seen that the unstable eigenvectors for the $p = 2$ case become stable for $p = 5$. Therefore it is the high wavenumbers relative to the discretisation that are causing the instabilities.

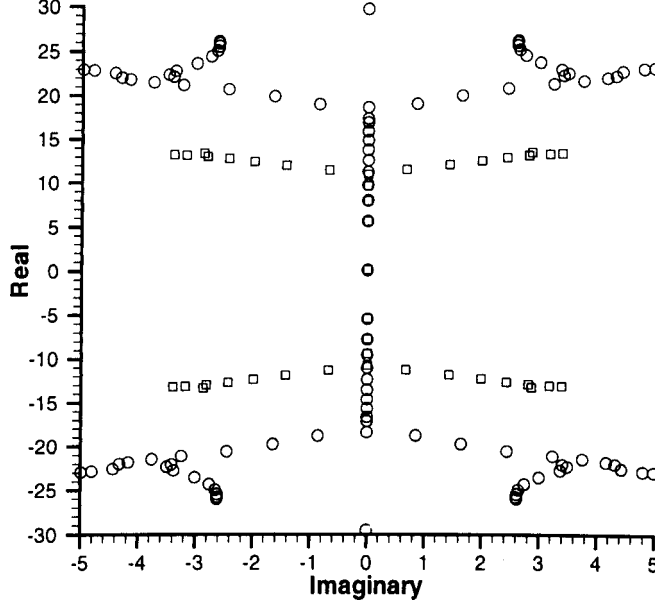


Figure 4.9: Eigenvalues for a 10×10 asymmetric triangular mesh with
 $\square$ $p = 2$ and $\circ$ $p = 5$

To remove the instability the recommendation of Moore [47] and Roberts[62] is the removal of the higher modes. Using the *hp*/Spectral technique a similar technique can be applied. The higher modes are removed by neglecting the components of the matrix $\mathbf{D}$ which relate to high frequency modal shape functions and therefore the higher frequency eigenvectors. This is done by formulating the matrix $\mathbf{D}$ of the same order as the expansion basis and then zeroing the contributions of the matrix which contain contributions from the higher order modal shape functions. Figure 4.10 (a) shows the resulting purely imaginary eigenvalues for computations using

$p = 3$ with the removal of the highest mode. Compare this to figure 4.10 (b) which shows the eigenvalues for $p = 2$ with mode no removal. The two solutions are of the same order, but by removing the highest mode the solution system has been stabilised. Unfortunately, as the original polynomial order increases more modes

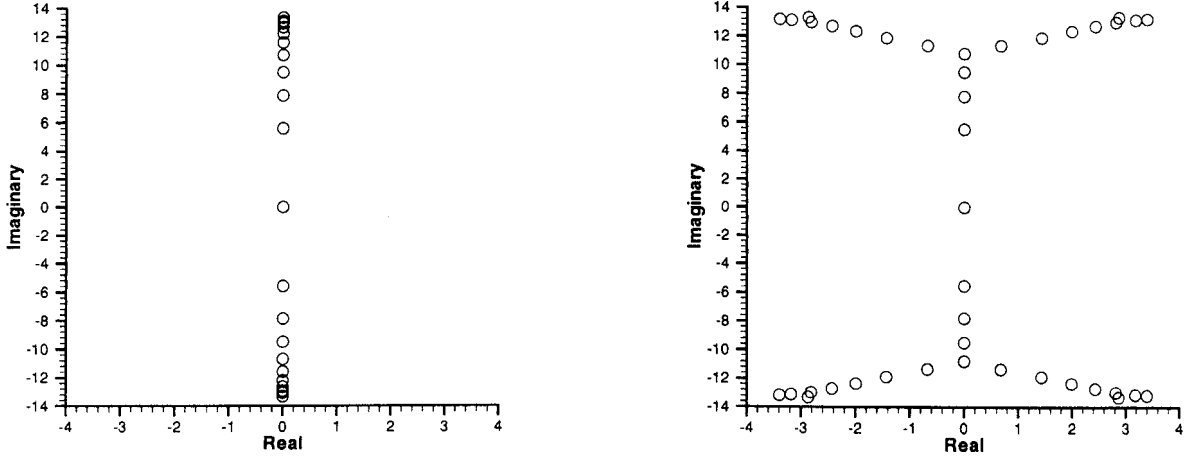


Figure 4.10: Eigenvalues for asymmetric mesh (a) with removal of highest mode and (b) with no removal of modes

have to be removed to enforce stability, as shown in table 4.1, resulting in expensive computational calculations for high accuracy solutions.

Addition of a Diffusive Term

Artificial diffusion is commonly used in compressible flow simulation to remove numerical instabilities and is attractive in finite element methods due to the ease with which it can be implemented. To this end a diffusive term is added to the kinematic boundary condition to enforce stability. The kinematic boundary condition then becomes,

$$\frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} - \mu_s \frac{\partial^2 \zeta}{\partial x^2}, \quad (4.57)$$

Number of modes	Number of modes removed with 3×3 mesh	Number of modes removed with 5×5 mesh
2	1	1
3	2	2
4	2	2
5	3	3
6	3	3
7	4	4
8	4	4
9	4	5
10	5	5
11	5	5
12	7	6
13	7	6
14	7	7
15	7	7

Table 4.1: Number of modes which require removal to stabilise the free surface solution system

where μ_s is a constant dependent upon the discretisation properties h and p . The matrix system (4.43) is modified thus,

$$\begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}_t = \begin{bmatrix} 0 & -g\mathbf{I} \\ \mathbf{N}_D & \mu_s \mathbf{D}_x^2 \end{bmatrix} \begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}, \quad (4.58)$$

where $\mathbf{D}_x$ is the equivalent operator to $\mathbf{D}$ in the x direction. The lowest value of μ_s which produces a stable solution is denoted by $\mu_{critical}$ or μ_c . This added term has the benefit of diffusing high frequency displacements of the free surface such as the saw-tooth effect, but also damps the entire system. The resulting eigenvalues of the

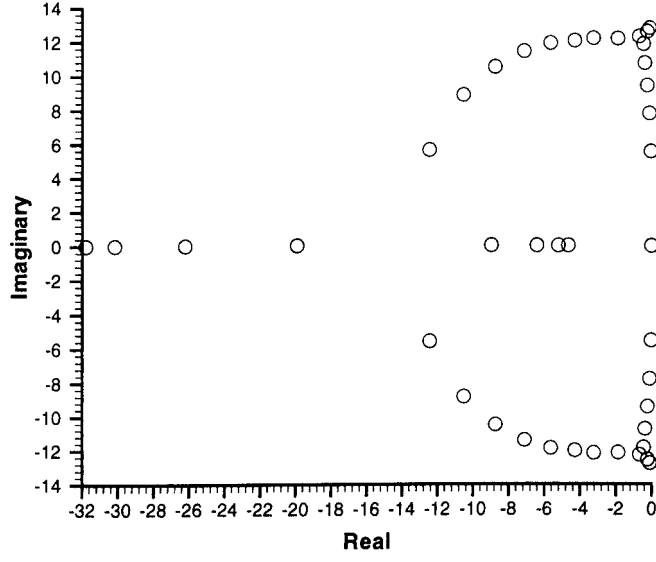


Figure 4.11: Eigenvalues for asymmetric mesh with added diffusion term

semi-discrete system with a polynomial expansion basis of order 2 and $\mu_c = 3 \times 10^{-4}$ evaluated on an asymmetric triangular mesh (figure 4.2 (c)) are shown in figure 4.11. All real parts of the eigenvalues are non-positive, indicating stability.

Before the relationship between μ_c , h , p and the asymmetry of the mesh can be assessed, the meaning of mesh asymmetry or skewness must be mathematically quantified. It is also necessary to formulate a characteristic length representing the amount of skewness of an element. Figure 4.12 shows a typical skewed element illustrating the length σ which represents the salient asymmetry of the problem. The line AB is denoted by the the vertical unit vector, $\mathbf{e}_z$, and the line CD is runs from the intercept of the centre of the element's free surface side and its interior vertex. The angle θ is the angle between these two lines. If the coordinates of the vertices are denoted by X_1 , X_2 and X_3 , where $X_j = (x_j, y_j)$ then

$$\mathbf{AB} = \mathbf{e}_z \quad (4.59)$$

$$\mathbf{CD} = [x_3 - (\frac{x_1+x_2}{2})]\mathbf{e}_x + [y_3 - (\frac{y_1+y_2}{2})]\mathbf{e}_z, \quad (4.60)$$

and

$$|\mathbf{CD}| \sin \theta = \frac{|\mathbf{AB} \times \mathbf{CD}|}{|\mathbf{AB}|} . \quad (4.61)$$

The characteristic length of the skewness of an element, σ , as

$$\sigma = |\mathbf{AB} \times \mathbf{CD}| \quad (4.62)$$

and the non-dimensionalised skewness of an element, κ , as

$$\kappa = \frac{\sigma}{h} , \quad (4.63)$$

where the size of an element, h , is represented by the length of the element's side on the free surface. Figure 4.13 shows the values of μ_c for varying values of σ ,

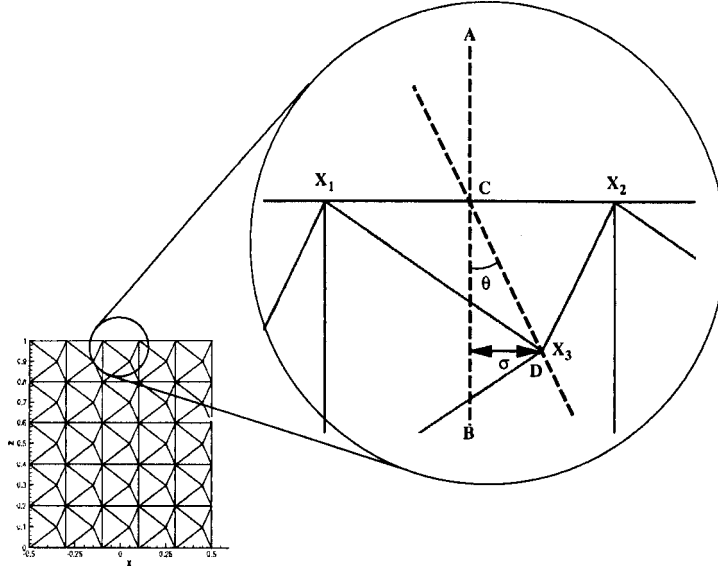


Figure 4.12: Evaluation of σ

where μ_c was evaluated to 8 decimal places, by finding the lowest possible value of μ_s which gave purely imaginary eigenvalues. When $\sigma = 0$ the mesh is symmetric and increasing σ represents an increase in the asymmetry of the mesh resulting in an increase in μ_c . It can be seen that the data collapses to a single curve purely

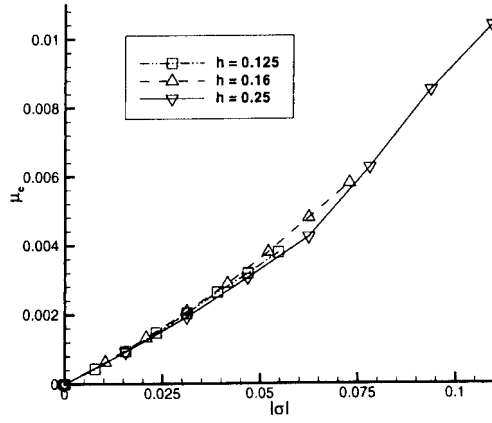


Figure 4.13: μ_c for increasing σ and varying h with $p = 2$

dependent on σ . It is theorised that the dependence of the solution's stability on the skewness of an element, with respect to the vertical, comes from the dependence of the kinematic boundary condition on the accurate solution of $\frac{\partial \phi}{\partial z}$. A largely skewed element will not accurately evaluate this derivative for at least one of its free surface vertices. To illustrate this point the values of $\frac{\partial \phi}{\partial z}$, where ϕ is described by equation (4.35), for a 1×1 assymetric triangular mesh at the points $X_1 = (-\frac{l}{2}, 0.0)$ and $X_2 = (\frac{l}{2}, 0.0)$ taken from the triangle on the free surafce, are given in table 4.2. It can be seen that the error for high polynomial order of the expansion at X_1 is an order higher to that at X_2 .

Figure 4.14 shows the value of μ_c for increasing values of p performed on a 1×1 , 2×2 and 4×4 assymetric meshes corresponding to element sizes $h = 1.0$, $h = 0.5$ and $h = 0.25$ respectively. A relationship of the form $\log(\mu_c) \propto p$ is apparent. It is therefore theorised that the relationship is of the form,

$$\mu_c = C\sigma^{\alpha_c p} , \quad (4.64)$$

where α_c and C are positive constants. To test this theory the value of μ_c was measured for increasing σ on a 5×5 mesh for differing values of p . The results are

Polynomial order	Error at X_1	Error at X_2
1	0.392346	0.238121
3	0.0759745	0.0129629
5	2.85346×10^{-04}	4.91659×10^{-04}
7	4.0874×10^{-05}	4.44574×10^{-06}
9	5.27677×10^{-07}	4.60273×10^{-08}
11	7.54204×10^{-09}	1.74898×10^{-10}

Table 4.2: Error of $\frac{\partial \phi}{\partial z}$ on an asymmetric triangular mesh for increasing polynomial order of the expansion basis

shown in figure 4.15 and support the hypothesis, as $\log(\mu_c) \propto \log(|\sigma|)$ with the steepness of the lines increasing with p . By utilising the information contained in figures 4.14 and 4.15, μ_c can be numerically bounded by taking C as $\frac{1}{10}$ and α_c as 0.2. Equation (4.64) is consistent with the requirement that $\mu_c \rightarrow 0$ upon convergence. Further more it is consistent with the hp /Spectral element approximation such that μ_c should decay as $h^{\alpha_c p}$, due to σ being proportional to h for different sizes of similar shaped elements. This procedure is only valid if the accuracy of the solution behaves identically as before i.e. $\varepsilon \propto h^p$. The numerical validation of this can be seen in figures 4.16 and 4.18 where the error of the first eigenvalue is plotted against increasing p and h respectively. In figure 4.16 mesh (a) is a 5×5 symmetric mesh whilst meshes (b) to (e) refer to those in figure 4.17. Figure 4.18 illustrates the rate of convergence, which is consistent with hp convergence i.e. $\log(\varepsilon) \propto \log(h)$. It can be seen in both figures that the rate of convergence is proportional to the skewness of the mesh, with a distorted mesh undergoing slower convergence.

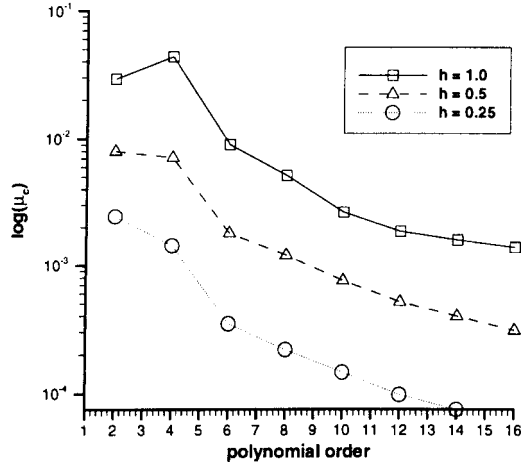


Figure 4.14: μ_c for varying order of expansion basis where h represents the size of the element edge on the free surface

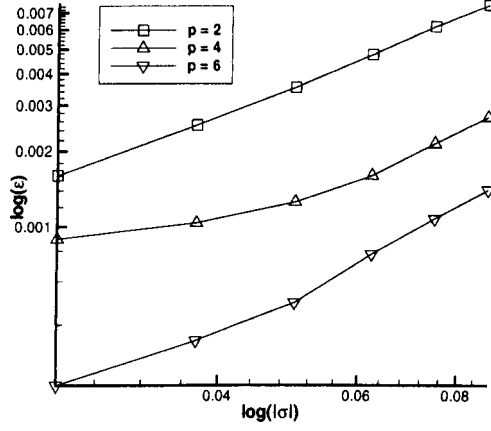


Figure 4.15: μ_c for increasing σ and varying p with $h = 0.2$

Smoothing Formulae

Another widespread method to stabilise the solution is to use a smoothing formula on the free surface or values of the potential on the free surface. The pioneers of

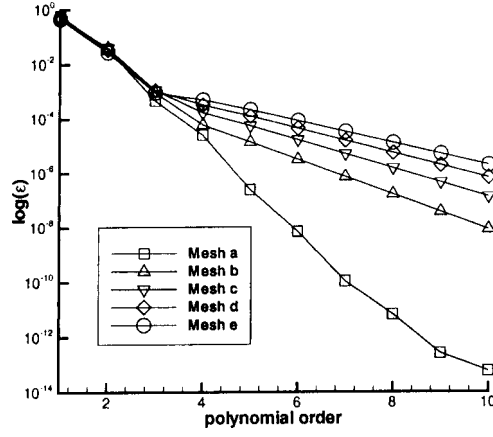
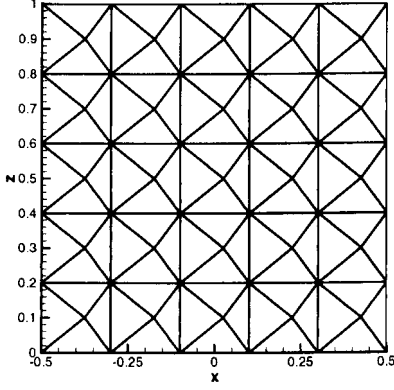


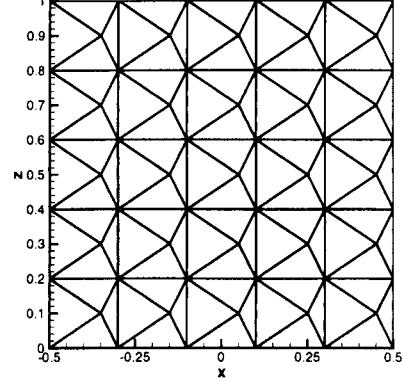
Figure 4.16: First eigenvalue error for varying order of expansion basis for $h = 0.2$

this method where Longuet-Higgins and Cokelet [44] who used a boundary element technique to simulate progressive, periodic free surface waves. They noticed that the unstable saw-tooth pattern was made up by the computed free surface points lying alternatively above and below a smooth curve. They further surmised that the phenomenon may be physical as it is similar to the growth of short gravity waves by horizontal compression of the crests of longer waves [43, 57]. Due to the lack of viscosity in the model these high-frequency waves are not damped out. A function is said to have a saw-tooth pattern if alternate points lie on a smooth curve. Therefore the function, $f(x)$, specified at a series of equidistant points, x_j , where $j = 1, 2, 3, \dots$, is represented by a smooth mean curve and an oscillating function with period $2h$ where $h = x_j - x_{j-1}$ such that,

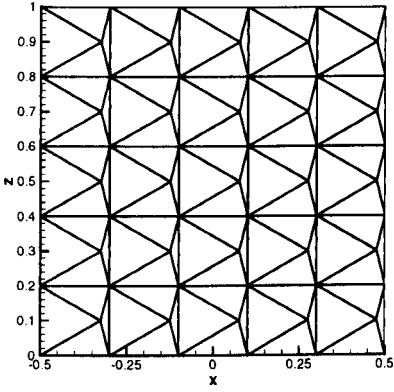
$$\begin{aligned}
 h(x) = & \left(a_0 + a_1x + a_2x^2 + \dots + a_nx^n \right) + \\
 & (-1)^j \left(b_0 + b_1x + b_2x^2 + \dots + b_nx^n \right). \quad (4.65)
 \end{aligned}$$



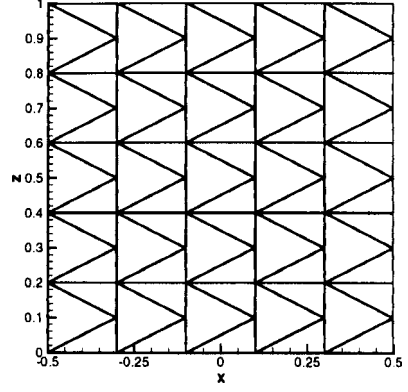
Mesh (b) : $\sigma = 0.125h$



Mesh (c) : $\sigma = 0.25h$



Mesh (d) : $\sigma = 0.375h$



Mesh (e) : $\sigma = 0.49h$

Figure 4.17: Meshes used for first eigenvalue error analysis

The coefficients can then be chosen so $h(x) = f(x)$ at $2n + 1$ points from $(j - n)$ to $(j + n)$. If the oscillating part is then ignored a smooth function, $h^*(x)$ is left,

$$h^*(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n. \quad (4.66)$$

By taking different values of n different smoothing formulas are obtained. If $n = 2$ the method produces a 5-point smoothing stencil,

$$f_j^* = \frac{1}{16} (-f(x_{j-2}) + 4f(x_{j-1}) + 10f(x_j) + 4f(x_{j+1}) - f(x_{j+2})), \quad (4.67)$$

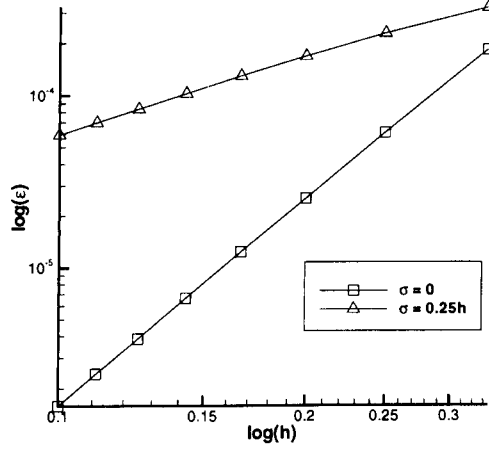


Figure 4.18: First eigenvalue error for varying order of element size for $p = 4$

and if $n=3$ a 7-point smoothing formula is produced,

$$f_j^* = \frac{1}{32} (-f(x_{j-3}) + 9f(x_{j-1}) + 16f(x_j) + 9f(x_{j+1}) - f(x_{j+3})). \quad (4.68)$$

The stencil can be applied every time-step, at every 5 time-steps or whenever the user wishes. A more expansive and applied explanation of the technique to formulate multi-point smoothing formulas is supplied by Dold [10]. To investigate the stabilising effect the smoothing formula has on the *hp*/Spectral element technique used in this work, the spatial discretisation method described above must be changed to a spatial and time discretisation scheme. The resulting analysis follows von Neumann analysis [31] where the solution system is represented in fully discretised form as

$$\Psi^{n+1} = \mathbf{A}\Psi^n; \quad (4.69)$$

where $\mathbf{A}$ is the amplification matrix and the superscripts denote the time level of the solution, ϕ , such that ϕ^n represents the solution at $t = n\Delta t$, where Δt is the value of the time-step. If the eigenvalues of $\mathbf{A}$ are denoted by Λ then stability is achieved if $\max\{|\Lambda|\} \leq 1$, i.e. the spectral radius of $\mathbf{A}$ is less than 1 [38]. Furthermore as the

system should have no damping the ideal requirement is $\Lambda(i) = 1 \quad \forall \quad i$. When the following first order time integration scheme is used to calculate the solution values on the boundary,

$$\zeta^{n+1} = \zeta^n + \Delta t \frac{\partial \phi}{\partial z}, \quad \phi^{n+1} = \phi^n - \Delta t g \zeta^{n+1}, \quad (4.70)$$

the modified spatial discretisation solution system is

$$\begin{bmatrix} \mathbf{I} & g\Delta t \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}^{n+1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \Delta t \mathbf{N}_D & \mathbf{I} \end{bmatrix} \begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}^n. \quad (4.71)$$

This system represents a first order in time accurate solution. Following von Neumann analysis the eigenvalues of $\mathbf{A}$ must be evaluated where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} & g\Delta t \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \Delta t \mathbf{N}_D & \mathbf{I} \end{bmatrix} \quad (4.72)$$

The eigenspectrum for the periodic symmetric and asymmetric 10×10 meshes can be seen in figure 4.19 for $p = 2$ and $\Delta t = 0.01$. The circles shown on the graphs have a radius of one and eigenvalues on or within the circle denote a stable solution. The minimum, λ_{min} , maximum, λ_{max} , and average, λ_{ave} values of the magnitudes of the eigenvalues are shown in table 4.3. These results are consistent with the spatial

Mesh	quadrilateral symmetric	triangular symmetric	triangular asymmetric
λ_{min}	1.00000000	1.00000000	0.96653591
λ_{max}	1.00000000	1.00000000	1.03462271
λ_{ave}	1.00000000	1.00000000	1.00023997

Table 4.3: Minimum, maximum and average eigenvalue magnitudes for 10×10 quadrilateral symmetric, triangular symmetric and triangular asymmetric meshes

discretisation stability results earlier in this chapter. In order to fully implement

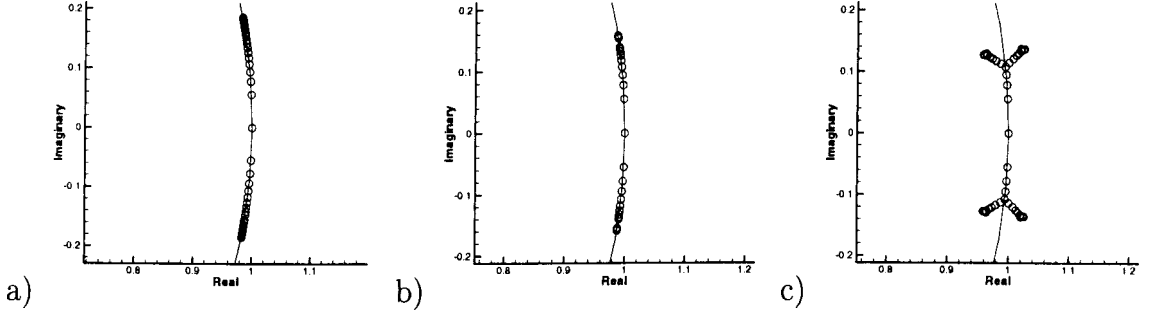


Figure 4.19: Eigenvalues for von Neumann analysis performed on (a) a 10×10 quadrilateral mesh, (b) a 10×10 triangular symmetric mesh and (c) a 10×10 triangular asymmetric mesh. The circle of radius equal to one represents the region of stability

the smoothing stencil the cases are limited to those where $p = 1$, thus ensuring the solution points are equidistant. In addition the eigenvalues are produced for periodic meshes, that is the prime variables on the right hand side of the mesh are equal to those on the left. In order to stabilise the solution system the smoothing stencil is either applied to ζ^n or ϕ^n . In the following analysis it is applied to ζ^n . The resulting von Neumann system is therefore,

$$\begin{bmatrix} \mathbf{I} & g\Delta t \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}^{n+1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \Delta t \mathbf{N}_D & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix} \begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}^n \quad (4.73)$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \Delta t \mathbf{N}_D & \mathbf{S} \end{bmatrix} \begin{bmatrix} \phi_b \\ \zeta \end{bmatrix}^n. \quad (4.74)$$

where for a 10×10 mesh configuration,

$$S = S_5 = \begin{bmatrix} \frac{5}{8} & \frac{1}{4} & -\frac{1}{16} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{4} & -\frac{1}{16} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} \\ -\frac{1}{16} & \frac{1}{4} & \frac{5}{8} & \frac{1}{4} & -\frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{16} & \frac{1}{4} & \frac{5}{8} & \frac{1}{4} & -\frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{16} & \frac{1}{4} & \frac{5}{8} & \frac{1}{4} & -\frac{1}{16} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{4} & \frac{5}{8} & \frac{1}{4} & -\frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{4} & \frac{5}{8} & \frac{1}{4} & -\frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{4} & \frac{5}{8} & \frac{1}{4} & -\frac{1}{16} \\ -\frac{1}{16} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{4} & \frac{5}{8} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{16} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{4} & \frac{5}{8} \end{bmatrix} \quad (4.75)$$

for the 5-point smoothing formula and

$$S = S_7 = \begin{bmatrix} \frac{1}{2} & \frac{9}{32} & 0 & -\frac{1}{32} & 0 & 0 & 0 & -\frac{1}{32} & 0 & \frac{9}{32} \\ \frac{9}{32} & \frac{1}{2} & \frac{9}{32} & 0 & -\frac{1}{32} & 0 & 0 & 0 & -\frac{1}{32} & 0 \\ 0 & \frac{9}{32} & \frac{1}{2} & \frac{9}{32} & 0 & -\frac{1}{32} & 0 & 0 & 0 & -\frac{1}{32} \\ -\frac{1}{32} & 0 & \frac{9}{32} & \frac{1}{2} & \frac{9}{32} & 0 & -\frac{1}{32} & 0 & 0 & 0 \\ 0 & -\frac{1}{32} & 0 & \frac{9}{32} & \frac{1}{2} & \frac{9}{32} & 0 & -\frac{1}{32} & 0 & 0 \\ 0 & 0 & -\frac{1}{32} & 0 & \frac{9}{32} & \frac{1}{2} & \frac{9}{32} & 0 & -\frac{1}{32} & 0 \\ 0 & 0 & 0 & -\frac{1}{32} & 0 & \frac{9}{32} & \frac{1}{2} & \frac{9}{32} & 0 & -\frac{1}{32} \\ -\frac{1}{32} & 0 & 0 & 0 & -\frac{1}{32} & 0 & \frac{9}{32} & \frac{1}{2} & \frac{9}{32} & 0 \\ 0 & -\frac{1}{32} & 0 & 0 & 0 & -\frac{1}{32} & 0 & \frac{9}{32} & \frac{1}{2} & \frac{9}{32} \\ \frac{9}{32} & 0 & -\frac{1}{32} & 0 & 0 & 0 & -\frac{1}{32} & 0 & \frac{9}{32} & \frac{1}{2} \end{bmatrix} \quad (4.76)$$

for the 7-point smoothing formula. The resulting eigenspectrums for the von Neumann matrix system with no smoothing, with the 5-point stencil implemented and the 7-point stencil implemented can be seen in figure 4.20 and the minimum, maximum and average values for the magnitude of the eigenvalue are shown in table 4.4. It can be seen that the 5-point stencil does not ensure stability for the system when used in conjunction with the first order time integration scheme (4.70),

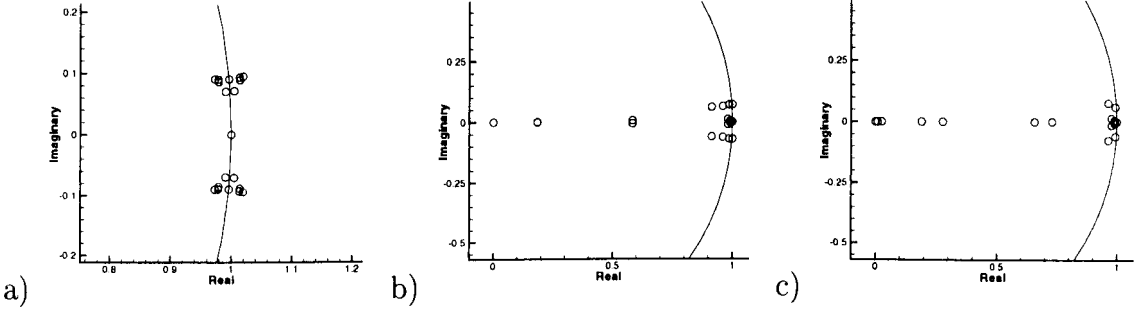


Figure 4.20: Eigenvalues for von Neumann analysis with (a) no smoothing, (b) 5-point stencil and (c) 7-point stencil

Smoothing stencil	none	5-point	7-point
Λ_{min}	0.97687508	0.00000000	0.00000000
Λ_{max}	1.02367234	1.00261863	1.00000000
Λ_{ave}	1.00012152	0.81018000	0.73631454

Table 4.4: Minimum, maximum and average eigenvalue magnitudes for 10×10 triangular asymmetric mesh using no smoothing, a 5-point stencil and a 7-point stencil

whilst the 7-point stencil does enforce stability. The average eigenvalue indicates the amount of artificial damping introduced into the system, which shows that the 7-point stencil introduces more damping to the system than the 5-point system. An example of the effect the stencils have on unstable eigenvectors can be seen in figure 4.21. The top eigenvector is an eigenvector for the solution system shown in equation (4.71) where no smoothing formula has been applied. The eigenvector has a corresponding eigenvalue of magnitude 1.0193511. The middle eigenvector is from the solution system utilising the 5-point stencil and has a corresponding eigenvalue of magnitude 0.18638098. The 7-point stencil is used to produce the bottom eigenvector which has a corresponding eigenvalue of zero magnitude. The unstable

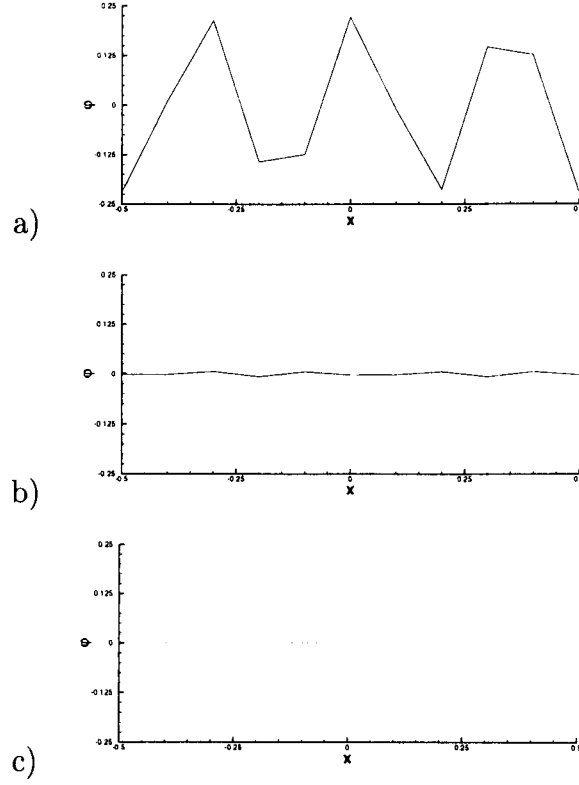


Figure 4.21: Eigenvectors of the von Neumann solution system shown in equation (4.74) where (a) $S=0$, (b) $S=S_5$ and (c) $S=S_7$

eigenvector is transformed into a stable one by the smoothing stencils at the expense of the energy within the system.

4.5 Results of Linear and Nonlinear Free Surface Flow

Though linear, inviscid, irrotational free surface flow has been extensively studied both analytically and numerically, some results are presented here in order to show the accuracy, convergence and stability properties of the *hp*/Spectral Laplace equation solver which is used for linear and nonlinear flow and also constitutes a major part of the Navier-Stokes solver. This can easily be achieved as exact solutions exist

for many flow scenarios. Linear flow is also studied in order to facilitate the comparison and validation of the more relevant nonlinear and Navier-Stokes free surface flow.

4.5.1 Validation Methods

The results for linear and nonlinear free surface flows are compared against two analytically derived solutions described in the following sections.

Linear Standing Wave

The first is the exact solution of the linear, standing wave problem, which was simulated in section 4.3, where the elevation of the potential can be expressed as,

$$\phi = \frac{a_0 g \cosh(k(z+d))}{\omega \cosh(kd)} \sin(kx) \sin(\omega t) , \quad (4.77)$$

for the initial conditions,

$$\phi(x, z; 0) = 0 , \quad (4.78)$$

$$\zeta(x; 0) = -a_0 \sin(kx) . \quad (4.79)$$

where $k = \frac{\pi}{l}$. This provides an opportunity to compare the numerical results with an exact solution, enabling the spatial and temporal convergence of the numerical algorithm to be assessed.

Amplitude Modulation Response to Forced Oscillations

One of the most important applications of the study of free surface flow is to simulate the sloshing of a fluid contained in a tank. Much analytical work has been done on this subject particularly by Faltinsen [15, 16]. Wu *et al.* [84] derived a linear solution for the motion of a fluid undergoing horizontal forced oscillations from the results of Faltinsen [16]. For a tank motion prescribed by

$$X_t = a_x \sin(\omega_x t) , \quad (4.80)$$

$$Z_t = 0 , \quad (4.81)$$

the solution is stated by Wu *et al.*[84] as,

$$\phi = a_x \sum_{n=0}^{\infty} \left[C_n \cos \omega_x t - \left\{ C_n + \frac{H_n}{\omega_x^2} \right\} \cos(\omega_n t) \right] \frac{\cosh(k_n(z+d))}{\cosh(k_n d)} \sin(k_n x) , \quad (4.82)$$

and

$$\zeta = \zeta_1 + \zeta_2, \quad (4.83)$$

where,

$$\zeta_1 = \frac{a_x}{g} \left(x \omega_x^2 + \sum_{n=0}^{\infty} C_n \omega_x \sin k_n x \right) \sin \omega_x t , \quad (4.84)$$

$$\zeta_2 = -\frac{a_x}{g} \sum_{n=0}^{\infty} \omega_n \left(C_n + \frac{H_n}{\omega_x^2} \right) \sin k_n x \sin \omega_n t , \quad (4.85)$$

and

$$H_n = \omega_x^3 \frac{4(-1)^n}{L k_n^2} , \quad (4.86)$$

$$C_n = \frac{H_n}{\omega_n^2 - \omega_x^2} . \quad (4.87)$$

The wave number and wave frequencies are evaluated by,

$$k_n = \frac{(2n+1)\pi}{l} , \quad (4.88)$$

$$\omega_n^2 = g k_n \tanh k_n d . \quad (4.89)$$

From this analytical solution it can be seen that the natural frequencies affect the amplitude of the free surface motion and the frequency of the ensuing motion is affected by the frequency of the tank motion and the natural frequencies.

4.5.2 Non-dimensionalisation

The results in the following sections are non-dimensionalised such that,

$$\begin{aligned} x &= \frac{\tilde{x}}{d}, & z &= \frac{\tilde{z}}{d}, & \zeta &= \frac{\tilde{\zeta}}{d}, \\ a_0 &= \frac{\tilde{a}_0}{d}, & l &= \frac{\tilde{l}}{d}, & \tau &= \left(\frac{g}{d}\right)^{\frac{1}{2}} \tilde{t} \end{aligned} \quad (4.90)$$

where a tilde denotes the dimensional values.

4.5.3 Results for Linear Free Surface Flow

To investigate the accuracy of the unsteady, linear, free surface code a standing wave motions were simulated. No smoothing is used for these flow conditions as a mesh consisting of quadrilaterals or triangles can be used due to the domain being of a simple geometric shape.

Standing wave

The computations were performed on a symmetric triangular mesh consisting of 200 elements shown in figure 4.22. The size of the tank is described by $l = 1$ and the amplitude of the initial wave is $a_0 = 0.1$. Figure 4.23(a) shows the $\mathbf{H}^1$ error of the computational results against the analytical results in equation (4.77). The error for increasing polynomial order of the expansion basis, p , is evaluated after 10 timesteps when $\Delta\tau = 3 \times 10^{-14}$, this effectively negates the effect of the time error. In figure 4.23(b) the $\mathbf{H}^1$ error is shown for $P = 8$ at $\tau = 0.062$ for increasing $\Delta\tau$. The $\mathbf{H}^1$ error plots demonstrate that $\log(\varepsilon) \propto P$ and $\log(\varepsilon) \approx \log(\Delta t)$ as expected since a first order time integration scheme was used to discretise the linear, free surface boundary conditions.

Forced horizontal oscillation of fluid in a Rectangular tank

The computations are performed on a 10×10 symmetric quadrilateral mesh with $l = 2$, $a_x = 0.004$, $P = 5$ and $\Delta\tau = 0.0044$. The mesh used is shown in figure 4.24 and the initial conditions are,

$$\phi(x, z; 0) = -a_x x \omega_x, \quad (4.91)$$

$$\zeta(x, 0) = 0. \quad (4.92)$$

The time history of the height of the free surface at $x = -\frac{l}{2}$ is shown in figure 4.25 for $\omega_x = 0.5\omega_0$ and for $\omega_x = 0.9\omega_0$ where ω_0 is the first natural frequency. Both the computational results and the analytical values are shown, it can be seen that

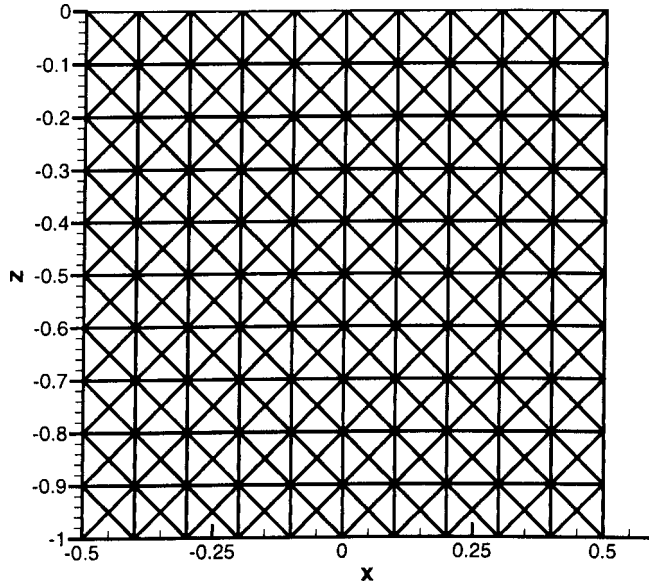
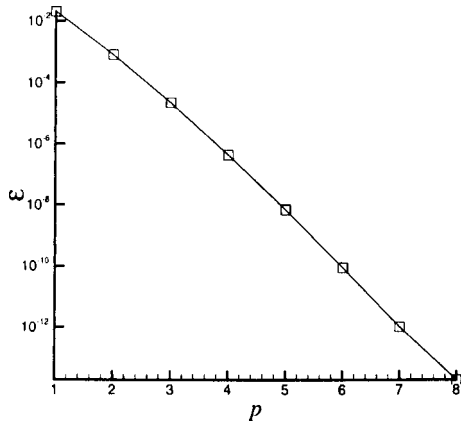


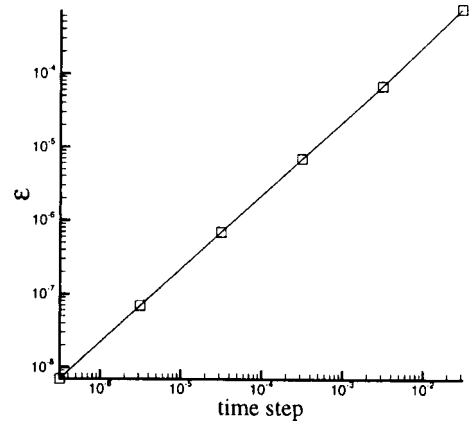
Figure 4.22: Mesh used for linear standing wave simulations

good agreement is found. Under these initial conditions all the natural frequencies of the fluid are excited at start up and, as there is no dissipation in the solution, play a major role in the motion of the fluid. The motion is primarily driven by the first natural frequency, ω_0 , and the forcing frequency, ω_x . The difference between these frequencies defines the wave envelope. The time-history of a point on the free surface for $\omega_x = 0.5\omega_0$ and $\omega_x = 0.9\omega_0$ are shown in figure 4.25.

The time history of the free surface for $\omega_x = 0.5\omega_0$ shows the elevation oscillating at a frequency which is twice that of the forcing frequency. This is due to the first natural frequency, which is double the forcing frequency, affecting the fluid motion. The ‘beating’ can clearly be seen in figure 4.25(b), where the closeness of ω_x and ω_0 produces a long wave envelope.



(a) Error convergence for increasing polynomial order



(b) Error convergence for decreasing $\Delta\tau$

Figure 4.23: H^1 error plots of computational against analytical values for a linear free surface standing wave.

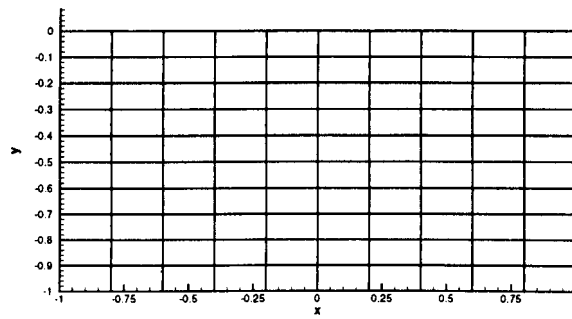
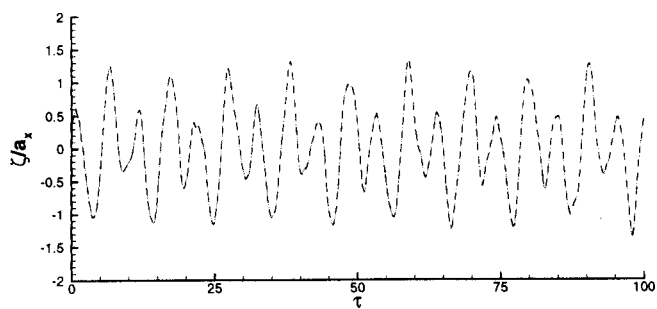
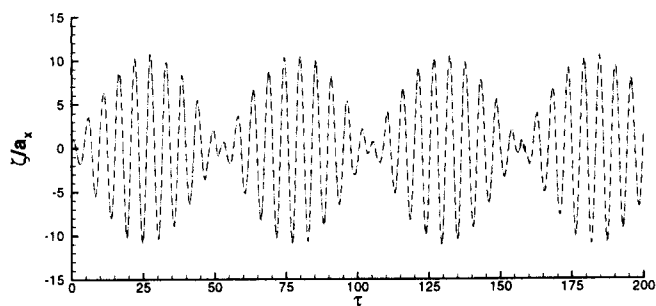


Figure 4.24: Mesh used for forced oscillations of a free surface fluid



(a) $\omega_x = 0.5\omega_0$



(b) $\omega_x = 0.9\omega_0$

Figure 4.25: Time history of particle on the free surface undergoing forced lateral oscillations: - - - - - computational results, $\cdot\cdot\cdot\cdot\cdot$ analytical results

4.5.4 Nonlinear Free Surface Flow

Nonlinear flow was numerically simulated using the free surface boundary conditions described in equations (4.15) and (4.16). The mesh must be updated at each time step so that it contains the entire fluid domain. This is ensured by solving a Laplacian equation of the form,

$$\nabla^2 z_m = 0, \quad (4.93)$$

to generate the interior mesh points, z_m . The Dirichlet boundary conditions for z_m on the free surface are evaluated from the kinematic boundary condition, whilst the Dirichlet boundary conditions for the walls are linearly evaluated according to their distance from the free surface. This motion of the mesh points ensures the accurate evaluation of $\frac{\partial \phi}{\partial z}$ on the free surface, therefore it was found that no smoothing was needed for the test cases undertaken below. Second order Crank-Nicolson and Adams-Bashforth schemes were used to propagate the solution in time.

Forced Horizontal Oscillation of Fluid in a Rectangular Tank

The initial conditions, mesh configuration and description of the motion of the box are identical to those used in the previous section. Four test cases were simulated for $\omega_x = 0.5\omega_0$, $\omega_x = 0.9\omega_0$, $\omega_x = 1.1\omega_0$ and $\omega_x = 0.999\omega_0$. Figure 4.26 illustrates the spatial convergence of the numerical solution for $\omega_x = 0.999\omega_0$. The dissipative nature of the solution for $P = 1$ can be seen, as can the rapid convergence as p increases. The solutions for $P = 3$ and $P = 5$ are almost identical. Convergence in time can be seen in figure 4.27 where no difference is apparent for all solutions.

Figure 4.28(a) shows the analytical and computational results for $\omega = 0.5\omega_0$ and very good agreement is seen. Figures 4.28(b) and 4.28(c) are close to the resonant frequency, though agreement is still good due to relatively low amplitude of the free surface elevation. Nonlinear effects are apparent in figure 4.28(d) where the forcing frequency is almost identical to the first natural frequency and the amplitude increases greatly. Initially the numerical results and the linear, analytical results

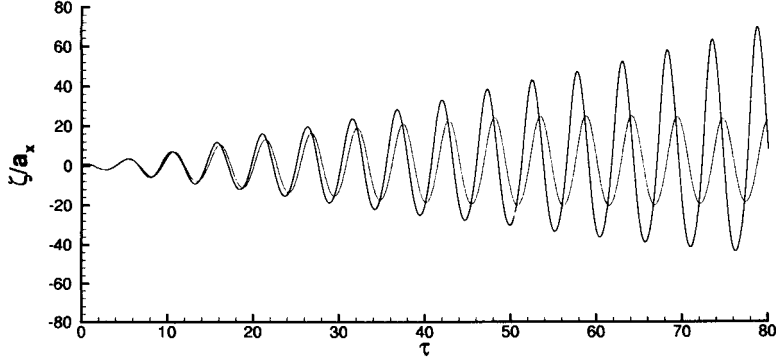


Figure 4.26: Spatial convergence of nonlinear numerical solution : ———
 $P = 1$, ——— $P = 3$, ——— $P = 5$

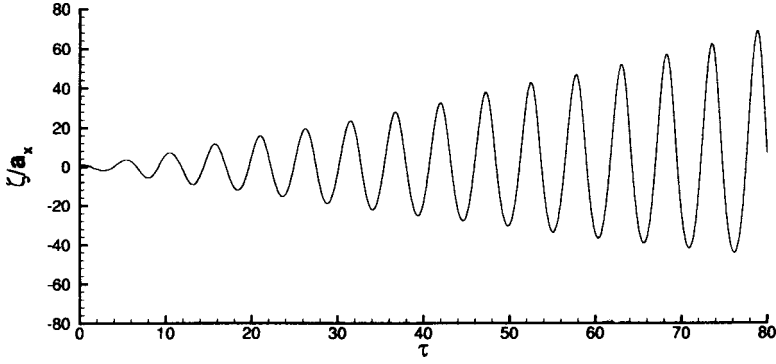
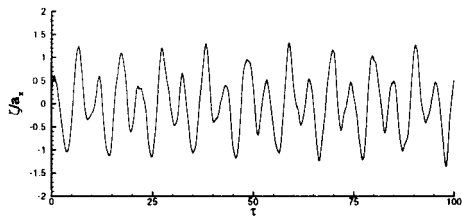
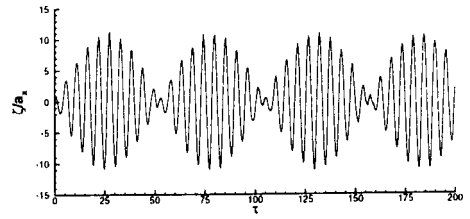


Figure 4.27: Temporal convergence of nonlinear numerical solution : ———
 $\tau = 2.21 \times 10^{-2}$, ——— $\tau = 8.86 \times 10^{-3}$, ——— $\tau = 4.429 \times 10^{-3}$

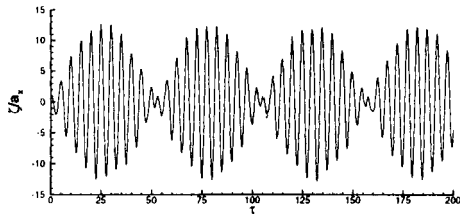
are in good agreement, though as the amplitude increases nonlinear characteristics are seen in the computational simulation. These nonlinear characteristics of an elongation of the time period coupled with higher, steeper peaks and lower, shallower troughs were all observed by Wu *et al.* [84] for fluid motion under similar conditions.



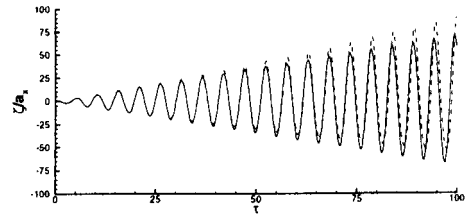
(a) $\omega_x = 0.5\omega_0$



(b) $\omega_x = 0.99\omega_0$



(c) $\omega_x = 1.1\omega_0$



(d) $\omega_x = 0.999\omega_0$

Figure 4.28: Time histories of surface elevation at $x = -\frac{l}{2}$. — analytical results; — — — — — computational results

4.6 Summary

Stability problems inherent to the spatial discretisation of the fluid domain when simulating inviscid free surface flow with linear boundary conditions have been identified and mechanisms to enforce stability formulated and tested. The discretisation of the fluid domain by quadrilaterals has been shown to provide a stable solution for linear and nonlinear flow simulations. The next chapter discusses the effect of the introduction of viscosity to the governing parameters.

Chapter 5

Navier-Stokes Free-Surface Flow

5.1 Introduction

To fully investigate the flow of a fluid contained within a tank, the effect of viscosity must be included. In this chapter the governing equations of viscous fluid flow are stated, namely the Navier-Stokes and continuity equations. To facilitate the determination of fluid flow with a moving free surface boundary, undergoing excitation due to the harmonic oscillation of the containing tank, the governing equations are derived for a reference frame which moves with the tank and with some arbitrary velocity to incorporate the oscillating free surface. The high order temporal discretisation scheme is described in order to illustrate the solution procedure of the computational algorithm.

Free surface flow undergoing free and forced oscillations are then investigated. The free surface boundary conditions are derived and three variations of these are studied in order to ascertain the flow effects they promote. The no-slip boundary condition on the side walls is overcome by the formulation of three models which allow the free surface contact point to be in motion. The effect these models have on the fluid flow are compared, particularly when considering boundary layer generation and the decay rates of free oscillations.

5.2 Governing Equations of Free Surface Flows with Viscosity

A fixed frame of reference is defined as Ox^*z^* where z^* points vertically upwards. The flow of a viscous, incompressible fluid under the influence of gravity is therefore governed by the continuity equation,

$$\nabla \cdot \mathbf{u}^* = 0 , \quad (5.1)$$

where $\mathbf{u}^*$ has components $\mathbf{u}^* = (u_x^*(x, z; t), u_z^*(x, z; t))$, and by the Navier-Stokes equations,

$$\frac{\partial \mathbf{u}^*}{\partial t} + (\mathbf{u}^* \cdot \nabla) \mathbf{u}^* = -\frac{\nabla p^*}{\rho} + \nu \nabla^2 \mathbf{u}^* + \mathbf{G} , \quad (5.2)$$

where $p^*(x, z; t)$ is the pressure, $\mathbf{G} = (0, -g)$, g is the force of gravity per unit mass, ν is the kinematic viscosity and ρ the density. The basic premises of the tank and fluid configuration are consistent with those for inviscid flow (see figure 4.1), where the fluid is considered to be contained within a rigid tank of length l and depth d , where the side walls are vertical and the floor is horizontal. The undisturbed free surface of the fluid is described by $z^* = 0$ and the deformation of the free surface is denoted by ζ^* where $z^* = \zeta^*(x; t)$. The tank undergoes rigid body oscillations at some velocity $\mathbf{U}$ where

$$\mathbf{U} = (U_x(t), U_z(t)) , \quad (5.3)$$

and

$$\mathbf{U} = \frac{d\mathbf{X}_t}{dt} , \quad (5.4)$$

$$\mathbf{X}_t = (X_t(t), Z_t(t)) . \quad (5.5)$$

$\mathbf{X}_t$ therefore describes the displacement on the tank.

A moving frame of reference, denoted by Oxz , is attached to the tank and is initially coincident with Ox^*z^* . The governing equations can be evaluated in the

moving frame of reference by considering

$$\nabla \Big|_{\mathbf{U}} = \nabla, \quad (5.6)$$

$$\frac{\partial \mathbf{u}^*}{\partial t} \Big|_{\mathbf{U}} = \frac{\partial \mathbf{u}^*}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{u}^*, \quad (5.7)$$

and therefore equations (5.1) and (5.2) can be written as,

$$\nabla \cdot \mathbf{u}^* = 0, \quad (5.8)$$

$$\frac{\partial \mathbf{u}^*}{\partial t} \Big|_{\mathbf{U}} + ((\mathbf{u}^* - \mathbf{U}) \cdot \nabla) \mathbf{u}^* = -\frac{\nabla p^*}{\rho} + \nu \nabla^2 \mathbf{u}^* + \mathbf{G}. \quad (5.9)$$

If $\mathbf{u}^*$ is split into its relative and forced velocity constituent parts, such that

$$\mathbf{u}^* = \mathbf{u} + \mathbf{U}, \quad (5.10)$$

then equations (5.8) and (5.9) become,

$$\nabla \cdot \mathbf{u} = 0, \quad (5.11)$$

$$\frac{\partial \mathbf{u}}{\partial t} \Big|_{\mathbf{U}} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u} - \frac{d\mathbf{U}}{dt} + \mathbf{G}. \quad (5.12)$$

In order for the prime variables to be evaluated on the free surface the reference frame Oxz must be allowed to move with some additional velocity $\mathbf{w}$ where,

$$\mathbf{w} = (w_x(x, z; t), w_z(x, z; t)) , \quad (5.13)$$

and the value of $\mathbf{w}$ is constrained on the free surface by the kinematic boundary condition. Using similar arguments and utilising the Reynolds transport theorem the continuity and Navier-Stokes equations can be represented as [26];

$$\nabla \cdot \mathbf{u} = 0, \quad (5.14)$$

$$\frac{\partial \mathbf{u}}{\partial t} \Big|_{\mathbf{U}+\mathbf{w}} + ((\mathbf{u} - \mathbf{w}) \cdot \nabla) \mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u} - \frac{d\mathbf{U}}{dt} + \mathbf{G}. \quad (5.15)$$

This is the standard arbitrary Lagrangian-Eulerian formulation used by many investigators to simulate free surface flow in stationary tanks [79, 60, 59, 27]. The conditions placed on the boundary of the fluid will be discussed in section 5.4. For convenience the notation to denote the velocity of the frame of reference that the time derivative is evaluated on will be relaxed in subsequent discussions.

5.3 Splitting Scheme for Navier-Stokes solver

The computational code, *NεκTαr* -ALE [80], solves the governing incompressible arbitrary Lagrangian-Eulerian (ALE) equations by spatially discretising the domain using *hp*/Spectral elements, described in chapter 3, and temporally discretising by using a high order splitting method. The splitting scheme used has been extensively documented and tested [33] and only a brief overview is given here.

The ALE governing equations (5.14) and (5.15) can be represented as

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla p}{\rho} + \nu \mathbf{L}(\mathbf{u}) + \mathbf{N}(\mathbf{u}, \mathbf{w}) + \mathbf{F}, \quad (5.16)$$

where

$$\mathbf{L}(\mathbf{u}) = \nabla^2 \mathbf{u}, \quad (5.17)$$

$$\mathbf{N}(\mathbf{u}, \mathbf{w}) = -((\mathbf{u} - \mathbf{w}) \cdot \nabla) \mathbf{u}, \quad (5.18)$$

$$\mathbf{F} = -\frac{d\mathbf{U}}{dt} + \mathbf{G}. \quad (5.19)$$

The time derivative in the Navier-Stokes equation can be represented as

$$\frac{\partial \mathbf{u}^{n+1}}{\partial t} = \frac{1}{\Delta t} \left(\gamma_0 \mathbf{u}^{n+1} - \sum_{q=0}^{Je-1} \alpha_q \mathbf{u}^{n-q} \right), \quad (5.20)$$

where the superscript index n refers to the relevant value being at time level $t_n = n\Delta t$, Je is the order of the time integration and γ_0 and α_i are the stiffly stable time integration co-efficients [33], where for consistency

$$\gamma_0 = \sum_{q=0}^{Je-1} \alpha_q. \quad (5.21)$$

Though both the stiffly stable and the Adams family integration scheme use the same technique of introducing a small amount of dissipation to stabilise the convective terms, the stiffly stable scheme is chosen due to its broader stability domain [33]. Further information on the use and properties of stiffly stable time integration schemes is given by Gear[19].

If the right hand side of the Navier-Stokes equation (5.16) is denoted by $\mathbf{f}$, then using the relationships

$$\frac{1}{\Delta t} \sum_{q=0}^{Je-1} \alpha_q (\mathbf{u}^{n+1} - \mathbf{u}^{n-q}) = \frac{1}{\Delta t} \sum_{q=0}^{Je-1} \alpha_q \int_{(n-q)\Delta t}^{(n+1)\Delta t} \frac{\partial \mathbf{u}}{\partial t} dt, \quad (5.22)$$

$$= \frac{1}{\Delta t} \sum_{q=0}^{Je-1} \alpha_q \int_{(n-q)\Delta t}^{(n+1)\Delta t} \mathbf{f} dt, \quad (5.23)$$

$$(5.24)$$

the equations (5.16) can be written in its temporally discrete form,

$$\begin{aligned} \frac{1}{\Delta t} \left(\gamma_0 \mathbf{u}^{n+1} - \sum_{q=0}^{Je-1} \alpha_q \mathbf{u}^{n-q} \right) = & -\frac{1}{\rho} \nabla p^{n+1} + \mathbf{N}(\mathbf{u}^{n+1}, \mathbf{w}^{n+1}) \\ & + \nu \mathbf{L}(\mathbf{u}^{n+1}) + \mathbf{F}^{n+1}. \end{aligned} \quad (5.25)$$

and further represented in terms of the explicit and implicit terms;

$$\begin{aligned} \frac{1}{\Delta t} \left(\gamma_0 \mathbf{u}^{n+1} - \sum_{q=0}^{Je-1} \alpha_q \mathbf{u}^{n-q} \right) = & -\frac{1}{\rho} \nabla p^{n+1} + \sum_{q=0}^{Je-1} \beta_q \mathbf{N}(\mathbf{u}^{n-q}, \mathbf{w}^{n-q}) \\ & + \nu \mathbf{L}(\mathbf{u}^{n+1}) + \mathbf{F}^{n+1}. \end{aligned} \quad (5.26)$$

The pressure term can be expressed as a scalar field,

$$p^{n+1} = \bar{p}^{n+1}, \quad (5.27)$$

with $\bar{p}$ ensuring that $\mathbf{u}^{n+1}$ is incompressible. The time integration co-efficients β_q can be evaluated by utilising Taylor expansions to extrapolate an arbitrary function at time level $(n+1)\Delta t$, f^{n+1} , in terms of f^{n-q} for $0 \leq q < Je$. The values of the time integration co-efficients, γ_0 , α_q and β_q are given in table 5.1 up to the third order.

The temporally discretised Navier-Stokes equation (5.26) is then split into three

Co-efficient	1 st Order	2 nd Order	3 rd Order
γ_0	1	$\frac{3}{2}$	$\frac{11}{6}$
α_0	1	2	3
α_1	0	$-\frac{1}{2}$	$-\frac{3}{2}$
α_2	0	0	$-\frac{1}{3}$
β_0	1	2	3
β_1	0	-1	-3
β_2	0	0	1

Table 5.1: Time integration co-efficients for a high-order splitting scheme to solve the Navier-Stokes equations

sub-steps,

$$\hat{\mathbf{u}} - \sum_{q=0}^{Je-1} \alpha_q \mathbf{u}^{n-q} = \Delta t \left(\sum_{q=0}^{Je-1} \beta_q N(\mathbf{u}^{n-q}, \mathbf{w}^{n-q}) + \mathbf{F}^{n+1} \right), \quad (5.28)$$

$$\frac{\hat{\mathbf{u}} - \hat{\mathbf{u}}}{\Delta t} = -\frac{1}{\rho} \nabla \bar{p}^{n+1}, \quad (5.29)$$

$$\frac{\gamma_0 \mathbf{u}^{n+1} - \hat{\mathbf{u}}}{\Delta t} = \nu \mathbf{L}(\mathbf{u}^{n+1}), \quad (5.30)$$

$$(5.31)$$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}$ are intermediate values of $\mathbf{u}$. $\hat{\mathbf{u}}$ is assumed to be incompressible such that,

$$\nabla \cdot \hat{\mathbf{u}} = 0 \quad (5.32)$$

and by taking the divergence of (5.29) to form a Poisson equation, $\bar{p}$ is evaluated such that,

$$\nabla^2 \bar{p}^{n+1} = \rho \nabla \cdot \left(\frac{\hat{\mathbf{u}}}{\Delta t} \right), \quad (5.33)$$

coupled with suitable boundary conditions. The final step to determine $\mathbf{u}^{n+1}$ is the evaluation of equation (5.30) from the solution of the Helmholtz equation,

$$\gamma_0 \mathbf{u}^{n+1} - \Delta t \nu \mathbf{L}(\mathbf{u}^{n+1}) = \hat{\mathbf{u}} \quad (5.34)$$

A further step is needed after $\mathbf{u}^{n+1}$ has been evaluated in order to establish the value of $\mathbf{w}^{n+1}$, where $\mathbf{w}$ is the velocity of the mesh points $\mathbf{X}_w = \mathbf{X}_w(x, y, z; t)$ and

$$\frac{d\mathbf{X}_w}{dt} = \mathbf{w} . \quad (5.35)$$

The mesh velocity is arbitrary everywhere except on the free surface and can either be specified explicitly or evaluated from a Laplacian equation

$$\nabla^2 \mathbf{w} = 0 , \quad (5.36)$$

as is done here following Ho [26]. More recent work [41] has advocated using a variable coefficient within the Laplacian matrix to enhance smoothing and prevent sudden distortions. The free surface boundary condition on $\mathbf{w}$ is evaluated using the kinematic boundary condition and will be derived in section 5.4. The next step then begins by evaluating the mesh point locations using the stiffly stable time integration co-efficients used previously (see Table 5.1);

$$\gamma_0 \mathbf{X}_w^{n+1} = \sum_{q=0}^{Je-1} \alpha_q \mathbf{X}_w^{n-q} + \Delta t \sum_{q=0}^{Je-1} \beta_q \mathbf{w}^{n-q} . \quad (5.37)$$

5.4 Boundary Conditions

In the following sections the pressure and velocity boundary conditions are formulated for wall boundaries and the free surface. The wall boundaries are decomposed into walls in contact with the free surface, contact walls, and those fully submerged and therefore not in contact with the free surface. The boundary demarkations are shown in figure 5.1.

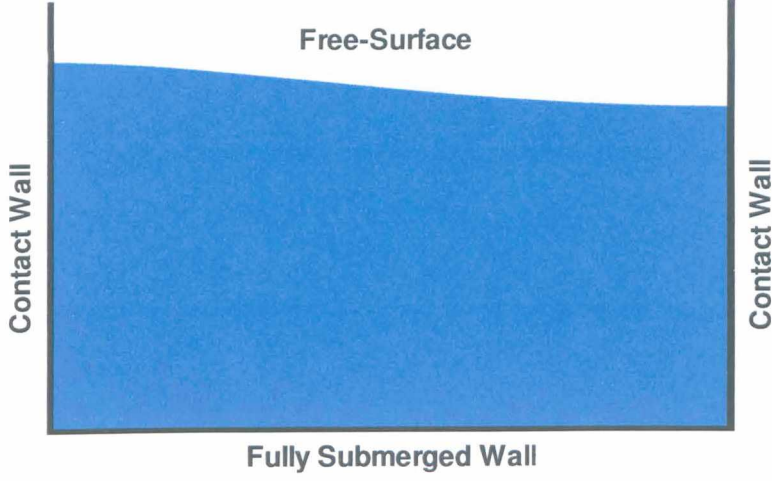


Figure 5.1: Composition of wall boundaries

5.4.1 Wall Boundary Conditions

Pressure boundary condition

The pressure boundary condition on the walls is found by taking the contributions of the temporally discretised Navier-Stokes equation (5.16) in the normal direction,

$$\frac{1}{rho} \frac{\partial \bar{p}}{\partial n} = \mathbf{n} \cdot \left(-\frac{\partial \mathbf{u}^{n+1}}{\partial t} + \sum_{q=0}^{Je-1} \beta_q \mathbf{N}(\mathbf{u}^{n-q}, \mathbf{w}^{n-q}) + \nu \mathbf{L}(\mathbf{u}^{n+1}) + \mathbf{F}^{n+1} \right), \quad (5.38)$$

where $\frac{\partial \mathbf{u}}{\partial t}$ normal to the wall is zero and $\mathbf{F}^{n+1}$ is generally known at time level $(n+1)\Delta t$. The linear operator contribution to (5.38) is not known at $(n+1)\Delta t$ and must be explicitly evaluated such that,

$$\begin{aligned} \frac{1}{rho} \frac{\partial p}{\partial n} = \mathbf{n} \cdot & \left(\sum_{q=0}^{Je-1} \beta_q \mathbf{N}(\mathbf{u}^{n-q}, \mathbf{w}^{n-q}) \right. \\ & \left. + \nu \sum_{q=0}^{Je-1} \beta_q \mathbf{L}(\mathbf{u}^{n-q}) + \mathbf{F}^{n+1} \right). \end{aligned} \quad (5.39)$$

This form of the pressure boundary condition was found to be unstable [51] and using the following vector calculus identity

$$\nabla^2 \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \mathbf{u}) , \quad (5.40)$$

the boundary condition was modified to enforce divergence of $\mathbf{u}$ such that [33],

$$\begin{aligned} \frac{1}{rho} \frac{\partial \bar{p}}{\partial n} = & \mathbf{n} \cdot \left(-\frac{\partial \mathbf{u}^{n+1}}{\partial t} + \sum_{q=0}^{Je-1} \beta_q \mathbf{N}(\mathbf{u}^{n-q}, \mathbf{w}^{n-q}) \right. \\ & \left. + \nu \sum_{q=0}^{Je-1} \beta_q (-\nabla \times (\nabla \times \mathbf{u})^{n-q}) + \mathbf{F}^{n+1} \right) . \end{aligned} \quad (5.41)$$

Velocity Boundary Condition on Fully Submerged Walls

The velocity boundary condition on walls not in contact with the free surface, generally the floor wall, is the no-slip condition and is denoted in the fixed frame of reference Ox^*z^* as,

$$u_x^* = U_x , \quad (5.42)$$

$$u_z^* = U_z . \quad (5.43)$$

Therefore in the moving frame of reference Oxz ,

$$u_x = u_z = 0 . \quad (5.44)$$

Velocity Boundary Condition on Free Surface Contact Walls

The velocity boundary condition specified on walls in contact with the free surface is not well posed due to the contradiction of the moving free surface and a no-slip condition on the bounding wall. Three alternative free surface contact wall boundary conditions are proposed below. They are then tested in the following sections for various flow scenarios and their effect on the fluid flow examined.

‘Slip’ wall boundary condition

The contact wall boundary problem has been overcome by many authors [27, 79] by adopting a slip boundary condition on the free surface contact walls such that the non-permeability condition is maintained, whilst the tangential velocity is allowed to take a non-zero value by implementing a Neumann boundary condition at the wall. The resulting boundary conditions for the vertical side walls are,

$$u_x = 0 , \quad (5.45)$$

$$\frac{\partial u_z}{\partial x} = 0 , \quad (5.46)$$

These conditions relate to a no-shear force premise at the walls.

‘Semi-noslip’ wall boundary conditions

Another alternative is to place no-slip boundary conditions on some fraction of the wall and slip conditions on the remainder, with the slip conditions occupying the upper portion of the wall. An example is shown in figure 5.2 where the red elements indicate a no-slip boundary condition on the wall and the blue elements a slip boundary. The designated fraction of slip elements against no-slip elements is arbitrary and therefore user defined. In all the test cases investigated in this work the slip length, l_s , is arbitrarily chosen as $\frac{l_s}{d} = \frac{1}{32}$. The arbitrary Lagrangian-Eulerian formulation enables the preservation of the slip length by allowing the no-slip element boundaries to move along the boundaries.

‘Semi-slip’ wall boundary condition

The semi-slip boundary condition is formulated by following the ideas of Dussan [12], who analytically studied the lowering of an infinitely long, inclined flat plate into a fluid. At the contact point, that is the point at which the free surface meets the plate, the fluid is allowed to slip and the no-slip boundary condition is approached with increasing distance from the contact point. The resulting fluid boundary conditions

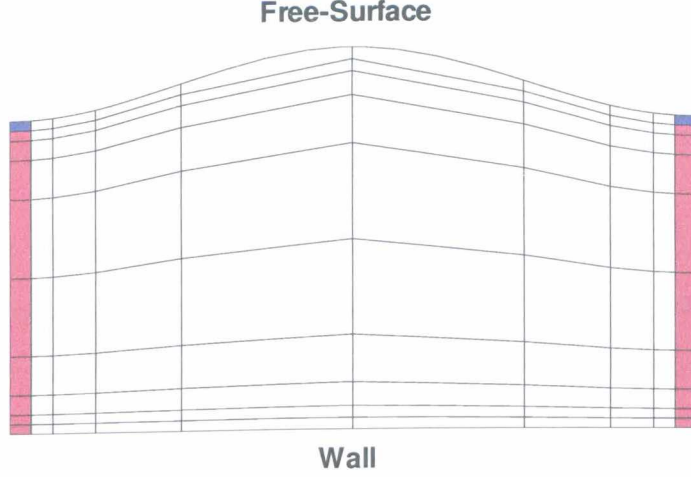


Figure 5.2: Placement of slip/no-slip boundary conditions for a ‘no-slip’ wall boundary condition

on the flat plate are,

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{V}_f \cdot \mathbf{n} \quad (5.47)$$

$$\mathbf{u} \cdot \mathbf{s} = U_n(r) \mathbf{V}_f \cdot \mathbf{s} \quad (5.48)$$

where $\mathbf{V}_f$ is the velocity of the flat plate and $\mathbf{n}$ and $\mathbf{s}$ the unit normal and tangent to the plate respectively, r is the distance along the flat-plate from the contact point and

$$U_n(r) = \frac{r^n}{1 + r^n} \quad (5.49)$$

where $0 < n < \infty$ and therefore $\lim_{r \rightarrow \infty} U_n = 1$, resulting in the no-slip condition being reached at infinity.

In a similar manner a semi-slip condition is computationally implemented on the contact walls, denoted by S' with Ω' representing the remainder of the boundary and subdomain, by regulating the velocity of the fluid depending on the distance from the contact point. This is achieved by modifying the values of the intermediate

velocity, $\hat{u}_z$, at the vertical contact walls according to the arbitrary expression,

$$\hat{u}_z^\dagger = H(\hat{u}_z) = \hat{u}_z \frac{\sinh(k_{ss}(z+d))}{\sinh(k_{ss}(\zeta_w+d))} \quad \text{on } S', \quad (5.50)$$

$$= \hat{u}_z \quad \text{on } \Omega \quad (5.51)$$

where ζ_w is the displacement of the contact point on the wall and k_{ss} is defined by the initial conditions for free oscillations and is equal to the first asymmetric wavenumber, $\frac{\pi}{l}$, for forced oscillations. $\hat{u}_z^\dagger$ is therefore not only dependent on the distance from the contact point, but also on the nature of the flow. The three semi-discrete sub-steps (5.28-5.30) used to propagate the values of $\mathbf{u}$ and $\bar{p}$ are therefore modified to,

$$\hat{\mathbf{u}} - \sum_{q=0}^{Je-1} \alpha_q \mathbf{u}^{n-q} = \Delta t \left(\sum_{q=0}^{Je-1} \beta N(\mathbf{u}^{n-q}, \mathbf{w}^{n-q}) + \mathbf{F}^{n+1} \right), \quad (5.52)$$

$$\hat{u}_x^\dagger = \hat{u}_x \quad (5.53)$$

$$\hat{u}_z^\dagger = H(\hat{u}_z) \quad (5.54)$$

$$\frac{\hat{\mathbf{u}} - \hat{\mathbf{u}}^\dagger}{\Delta t} = -\frac{1}{\rho} \nabla \bar{p}^{n+1}, \quad (5.55)$$

$$\frac{\gamma_0 \mathbf{u}^{n+1} - \hat{\mathbf{u}}}{\Delta t} = \nu \mathbf{L}(\mathbf{u}^{n+1}), \quad (5.56)$$

$$(5.57)$$

and the pressure is evaluated by

$$\nabla^2 \bar{p}^{n+1} = \rho \nabla \cdot \left(\frac{\hat{\mathbf{u}}^\dagger}{\Delta t} \right). \quad (5.58)$$

The contact wall boundary conditions placed on the evaluation of $\mathbf{u}$ by equation (5.56) are the slip boundary conditions (5.45-5.46).

5.4.2 Free Surface Boundary Conditions

It will be seen that the pressure and velocity boundary conditions for the fluid on the free surface are purely spatially dependent and therefore can be formulated directly for $\mathbf{u}$, the velocity relative to the moving reference frame. The pressure

and velocity boundary conditions on the free surface are both formulated from the dynamic constraint of continuity of normal momentum flux across the free surface, whilst assuming negligible momentum on the air side and neglecting surface tension. The component of the stress tensor in the outward normal direction is therefore,

$$\sigma_{ij}n_j = 0 , \quad (5.59)$$

by substituting in the expression for the stress tensor the free surface dynamic condition becomes

$$-pn_i + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j = 0 , \quad (5.60)$$

for zero atmospheric pressure.

Free Surface Pressure Boundary Condition

The normal component of the dynamic free surface boundary condition (5.60) gives a Dirichlet boundary condition for the pressure,

$$-pn_i n_i + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j n_i = 0. \quad (5.61)$$

This condition enforces that the externally applied stresses are balanced by the internal stresses. In two dimensions where $\mathbf{n} = (n_x, n_z)$ and $\mathbf{u} = (u_x, u_z)$ and neglecting surface tension

$$-p + 2\mu \left(\frac{\partial u_x}{\partial x} n_x^2 + \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) n_x n_z + \frac{\partial u_z}{\partial z} n_z^2 \right) = 0 . \quad (5.62)$$

By dividing through by n_z^2 the condition is represented in terms of the velocity and free surface gradients,

$$-p + \frac{2\mu}{1 + \left(\frac{\partial \zeta}{\partial x} \right)^2} \left(\frac{\partial u_x}{\partial x} \left(\frac{\partial \zeta}{\partial x} \right)^2 - \left(\frac{\partial u_x}{\partial z} \frac{\partial u_z}{\partial x} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial u_z}{\partial z} \right) = 0 . \quad (5.63)$$

where $\frac{\partial \zeta}{\partial x} = -\frac{n_x}{n_z}$. The linearised condition is found by presuming all gradients are small and neglecting any products of these terms to give,

$$-p + 2\mu \frac{\partial u_z}{\partial z} = 0 . \quad (5.64)$$

Free Surface Velocity Boundary Condition

The tangential component of equation (5.60) gives a Neumann boundary condition for the free surface. The tangential contribution to the dynamic free surface boundary condition is

$$\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j s_i = 0 , \quad (5.65)$$

where $\mathbf{s}$ is the unit tangent vector. By substituting $\mathbf{s} = (n_z, -n_x)$ the condition becomes,

$$2 \frac{\partial u_x}{\partial x} n_x n_z + \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) n_z^2 - \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) n_x^2 - 2 \frac{\partial u_z}{\partial z} n_z n_x = 0 . \quad (5.66)$$

The necessary Neumann boundary conditions for the velocity are not immediately obvious, though by rearranging equation (5.66) two Neumann boundary conditions for the velocity can be formulated as,

$$\frac{\partial u_x}{\partial n} = -\frac{\partial u_x}{\partial x} n_x - \frac{\partial u_x}{\partial z} \frac{\partial \zeta}{\partial x} n_x - \frac{\partial u_z}{\partial x} \left(\frac{\partial \zeta}{\partial x} n_x + n_z \right) + 2 \frac{\partial u_z}{\partial z} n_x , \quad (5.67)$$

$$\frac{\partial u_z}{\partial n} = 2 \frac{\partial u_x}{\partial x} n_z + \frac{\partial u_x}{\partial z} \left(\frac{n_z}{n_x} n_z - n_x \right) + \frac{\partial u_z}{\partial x} \frac{n_z}{n_x} n_z - \frac{\partial u_z}{\partial z} n_z . \quad (5.68)$$

The expression for $\frac{\partial u_x}{\partial n}$ above is suitable for implementation in the free surface code, whereas the expression for $\frac{\partial u_z}{\partial n}$ is not, due to the susceptibility of the term $\frac{n_z}{n_x}$ to take very large values in many free surface scenarios, for example at start up where n_x is invariably very small or zero at some position. Equation (5.68) can be re-arranged and combined with the continuity equation to form a more suitable expression for $\frac{\partial u_z}{\partial n}$ where,

$$\begin{aligned} \frac{\partial u_z}{\partial n} = & \frac{\partial u_x}{\partial x} \left(2 \frac{\partial \zeta}{\partial x} \frac{1}{n_z} - n_z \right) + \frac{\partial u_x}{\partial z} n_x \left(\left(\frac{\partial \zeta}{\partial x} \right)^2 - 1 \right) \\ & + \frac{\partial u_z}{\partial x} \left(\frac{\partial \zeta}{\partial x} \right)^2 n_x - 2 \frac{\partial u_z}{\partial z} \frac{\partial \zeta}{\partial x} n_x . \end{aligned} \quad (5.69)$$

To linearise the velocity boundary condition, equation (5.66) is divided through by n_z^2 and the condition is represented in terms of the velocity and free surface gradients,

$$2 \frac{\partial u_x}{\partial x} \frac{\partial \zeta}{\partial x} + \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \left(\left(\frac{\partial \zeta}{\partial x} \right)^2 - 1 \right) - 2 \frac{\partial u_z}{\partial z} \frac{\partial \zeta}{\partial x} = 0 . \quad (5.70)$$

Using the same assumptions as before to linearise, the boundary condition becomes,

$$\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0 . \quad (5.71)$$

As the assumptions used to linearise the condition implied that $n_x \approx 0$ the linearised Neumann condition for u is

$$\frac{\partial u_x}{\partial n} = -\frac{\partial u_z}{\partial x} n_z . \quad (5.72)$$

The continuity condition

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0 , \quad (5.73)$$

is used in conjunction with linearised free surface condition (5.71) to formulate the boundary condition on u_z where,

$$\frac{\partial u_z}{\partial n} = -\frac{\partial u_x}{\partial z} n_x - \frac{\partial u_x}{\partial x} n_z . \quad (5.74)$$

Though as above the value of n_x can be taken as zero to be consistent with the linearisation principles. The velocity $\mathbf{w}$ is chosen such that $w_x = 0$. This is done to limit the distortion of the elements and therefore promote a stable solution. The kinematic boundary condition is therefore represented as,

$$u_z = w_z + u_x \frac{\partial \zeta}{\partial x} , \quad (5.75)$$

which places a constraint on w_z to be used as a Dirichlet boundary condition when evaluating the velocity of $\mathbf{w}$ in the interior of the fluid using equation (5.36). The choice of the Dirichlet boundary conditions for w_z on the side and floor walls is arbitrary and they are either allowed to move with the fluid $w_z = u_z$ or linearly evaluated from the velocity of the free surface at the wall. For example at the wall $x = -\frac{l}{2}$,

$$w_z \left(-\frac{l}{2}, z; t \right) = \frac{z + d}{\zeta + d} w_z \left(-\frac{l}{2}, \zeta; t \right) . \quad (5.76)$$

In general the first approach is used for the slip and semi-slip boundary conditions and a modified version of the second, to account for the preservation of the slip length, is used for the semi-noslip boundary condition.

5.5 Validation Methods

In this section the analytical solutions of Landau & Lifshitz [37] and Keulegan [35] to predict the damping of gravity or freely oscillating waves are formulated to facilitate the validation of the computational model. A linear solution of the long wave equations is also described by following the work of Lepelletier & Raichlen [39] in order to validate the computational results for forced oscillations of a contained fluid.

5.5.1 Damping of Free Surface Waves

The damping of free surface waves has been extensively analytically and experimentally covered [6, 35, 45]. The free surface is prescribed by an initial displacement which is then ‘released’ and allowed to freely oscillate under the influence of gravity. The decay rate is then found by evaluating the decrease in wave height for subsequent oscillations and a governing equation for the decay of the wave height is formulated such that $a = a_0 e^{-\alpha n}$ where a_0 is the initial wave displacement of a point on the free surface, α is the non-dimensional decay constant and α the dimensional decay constant and a is the value of the amplitude after n oscillations where $n = \frac{t}{T}$ and T is the time period of the flow. The decay constant can also be represented dimensionally by $\tilde{\alpha}$ where $a = a_0 e^{-\tilde{\alpha} t}$ and a is the value of the amplitude after time t .

The dissipation of energy from the wave is attributed to four main features by Miles [45]; a) the ‘boundary layer’ at the free surface and the volume of the fluid, b) the boundary layers at the fixed boundaries, c) capillary hysteresis and d) surface contamination effects. If the decay is presumed to be linear [6, 35, 45] the decay constant is therefore the sum of its contributing parts,

$$\alpha = \alpha_s + \alpha_w + \alpha_l + \alpha_c, \quad (5.77)$$

where α_s is the free surface ‘boundary layer’ and volume contribution, α_w the wall

boundary layer contribution, α_l the capillary hysteresis contribution [45, 46] and α_c the surface contamination contribution [36, 45]. In this work only the contributions from the surface and the wall will be considered.

Free Surface ‘Boundary Layer’ Damping and Volume Damping

α_s was evaluated by both Landau and Lifshitz [37] for unbounded standing waves and Lamb [36] for unbounded progressive waves using differing energy arguments as $\frac{4\pi\nu k^2}{\omega}$ with the restrictions of

$$\lambda^2\omega \gg \nu, \quad a \ll \lambda, \quad (5.78)$$

where λ is the wavelength, $k = \frac{2\pi}{\lambda}$ and $\omega^2 = gk$. Both methods represent the flow of an unbounded fluid which is irrotational away from the free surface and are therefore unreliable for arbitrary viscosity or when the depth of the fluid is small compared to the wavelength, due to damping effects from the no-slip condition on the floor wall.

To evaluate the decay constant for an unbounded fluid with arbitrary viscosity the body of the fluid has to be considered rotational. Landau & Lifshitz [37] sought a solution for arbitrary viscosity by considering the continuity equation and linearising the Navier-Stokes equations in two dimensions for $\mathbf{u} = (u_x, u_z)$;

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0, \quad (5.79)$$

$$\frac{\partial u_x}{\partial t} = \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (5.80)$$

$$\frac{\partial u_z}{\partial t} = \nu \left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial z} - g. \quad (5.81)$$

Landau & Lifshitz [37] theorised that $\mathbf{u}$ must be dependent on t and x such that $\mathbf{u} = f(z) e^{i\chi t + ikx}$ and $f(z) \rightarrow 0$ as $z \rightarrow -\infty$ where χ is a complex term describing the frequency and damping of $\mathbf{u}$. They therefore found

$$u_x = e^{-i\chi t + ikx} (Ae^{kz} + Be^{mz}), \quad (5.82)$$

$$u_z = e^{-i\chi t + ikx} \left(-iAe^{kz} - \frac{ik}{m} Be^{mz} \right), \quad (5.83)$$

$$p = \frac{e^{-i\chi t + ikx} \chi A e^{kz}}{k} - gz, \quad (5.84)$$

where $m = (k^2 - i\frac{\chi}{\nu})^{\frac{1}{2}}$. The linearised, free surface boundary conditions, shown in equations (5.64) and (5.74), are;

$$-p + 2\nu \frac{\partial u_z}{\partial z} = 0, \quad (5.85)$$

$$\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = 0, \quad (5.86)$$

and both boundary conditions are further simplified by evaluating on $z = 0$. The relationships for u_x , u_z and p given in equations (5.82-5.84) are substituted into equations (5.85) and (5.86) and by eliminating A and B from the resulting equations the following dispersion relation combining χ and k must be satisfied;

$$\left(2 - \frac{i\chi}{\nu k^2} \right)^2 + \frac{g}{\nu^2 k^3} = 4 \left(1 - \frac{i\chi}{\nu k^2} \right)^{\frac{1}{2}}. \quad (5.87)$$

The solution of (5.87) gives a complex value of χ in terms of k , g and ν . The real part of χ represents the frequency of the oscillations and the imaginary part gives the damping constant. The solutions that have a physical meaning are those with a negative imaginary part, indicating a damped solution. Landau & Lifshitz [37] present two solutions based on limiting cases. The first is when $\nu k \ll (gk)^{\frac{1}{2}}$ which gives a solution of $\chi = \pm (gk)^{\frac{1}{2}} - 2\nu k^2 i$ this is the same result as found by Landau & Lifshitz [37] and Lamb [36] using energy arguments. For the opposing limit $\nu k \gg (gk)^{\frac{1}{2}}$ two aperiodic solutions are found with $\chi = -\frac{g}{2\nu k} i$ and another of the order νk^2 ($\approx -0.91\nu k^2$ (Lamb [36])), which is considered too large. To solve for completely arbitrary cases the dispersion relation in equation (5.87) must be solved numerically to evaluate the value of χ for any choice of ν . This is done by substituting $\chi = \omega + \tilde{\alpha}_s i$ into equation (5.87) and solving the two resulting simultaneous equations by the Newton-Raphson [58] method for non-linear systems of equations. Table 5.2 contains values of $\tilde{\alpha}_s$ using the limiting approximations compared with the appropriate numerically approximated decay constant for $k = \frac{2\pi}{\lambda}$ where $\lambda = 1$. It can be seen that the limiting approximations are adequate for the extreme cases they represent, though for values of the kinematic viscosity between

these limits there is a distinct loss of accuracy. Therefore the numerically evaluated decay constants are used in the following sections.

ν	10	1	0.1	0.01	0.001	0.0001
α_{sl}	0.0780655	0.780655	7.89568	0.789568	0.0789568	0.00789568
α_{sn}	0.0780771	0.792631	3.34692	0.661195	0.0749878	0.00777046

Table 5.2: Comparison between the value of the decay constant evaluated by the limiting approximations, $\tilde{\alpha}_{sl}$ and the numerically evaluated decay constant, $\tilde{\alpha}_{sn}$ for sloshing when $k = 2\pi$

Wall Boundary Damping

The formulation of α_w has been analytically investigated by many authors including Case & Parkinson [6] for closed three dimensional circular basins and Keulegan [35] for closed three dimensional rectangular basins and Miles [45] for both. Here the method of Keulegan [35] is followed to produce an evaluation of α_w in a two dimensional tank, where,

$$\alpha_w = \frac{\Delta E}{2E}, \quad (5.88)$$

and ΔE is the energy loss during one oscillation due to the wall boundary layers and E is the total energy of the wave. The main body of the fluid, that is the fluid which is away from the boundaries, is considered irrotational, thus providing the limiting value of the fluid velocity at the outer edge of the boundary layers. The flow within these boundary layers is considered to be laminar. According to second order theory the velocity potential of an inviscid, incompressible, irrotational sloshing wave is,

$$\begin{aligned} \phi = & A_1 \cosh(k(z+d)) \cos\left(k\left(x + \frac{l}{2}\right)\right) \cos(\omega t) \\ & + A_2 \cosh(2k(z+d)) \cos\left(2k\left(x + \frac{l}{2}\right)\right) \sin(2\omega t) \end{aligned} \quad (5.89)$$

where,

$$A_1 = -\frac{a_0 g}{\omega} \frac{1}{\cosh(kd)} , \quad (5.90)$$

$$A_2 = \frac{3}{16} A_1 \frac{a_0 k}{\sinh^3(kd)} , \quad (5.91)$$

and $k = \frac{\pi}{l}$ and $\omega^2 = gk \tanh(kd)$. The linearised particle velocities in the irrotational core of the fluid domain are therefore,

$$u_{x0} = -\frac{a_0 g k \cosh(k(z+d))}{\omega \cosh(kd)} \sin\left(k\left(x + \frac{l}{2}\right)\right) \cos(\omega t) \quad (5.92)$$

$$u_{z0} = \frac{a_0 g k \sinh(k(z+d))}{\omega \cosh(kd)} \cos\left(k\left(x + \frac{l}{2}\right)\right) \cos(\omega t) . \quad (5.93)$$

By denoting $z' = z + d$ and $x' = x + \frac{l}{2}$ and following Keulegan [35] the fluid velocities, u and w near the vertical wall at $x' = 0$ are,

$$\begin{aligned} u_x &= 0 , \\ u_z - u_{z0} &= -\frac{a_0 g k \sinh(kz')}{\omega \cosh(kd)} e^{-\beta x'} \cos(\omega t - \beta x') , \end{aligned} \quad (5.94)$$

and at the floor wall $z' = 0$,

$$\begin{aligned} u_x - u_{x0} &= \frac{a_0 g k \sin(kx')}{\omega \cosh(kd)} e^{-\beta z'} \cos(\omega t - \beta z') , \\ u_z &= 0 , \end{aligned} \quad (5.95)$$

where $\beta = \left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}$.

These velocities represent transverse waves travelling from the no-slip boundaries into the body of the fluid with amplitude dependent on the distance from the boundaries. The value of β is the rate at which the velocities in the no-slip boundary layers, $\mathbf{u}$, increases to the velocity of the body of the fluid, $\mathbf{u}_0$. The inverse of this ratio is taken as an appropriate length scale, δ , for this problem. The value of δ therefore represents the distance over which the velocity within the boundary layers increases by a factor of e . Miles [45] theorised that the proceeding method to evaluate the energy dissipation of a sloshing fluid due to viscosity has the limitation,

$$\epsilon = k\delta \ll 1 . \quad (5.96)$$

The rate of viscous dissipation within a fluid of area Ω completely bounded by no-slip walls is given by Lamb [36] as,

$$\frac{dE}{dt} = \mu \int_{\Omega} (\nabla \times \mathbf{u})^2 d\Omega, \quad (5.97)$$

where μ is the dynamic viscosity of the fluid. Over one oscillation the viscous dissipation due to the no-slip boundary at the wall $x' = 0$ is given by Keulegan [35] as,

$$\Delta E_x = \mu \int_0^d \int_0^\infty \int_0^{\frac{2\pi}{\omega}} (\nabla \times \mathbf{u})^2 dt dx' dz', \quad (5.98)$$

which reduces to

$$\Delta E_x = \mu \int_0^d \int_0^\infty \int_0^{\frac{2\pi}{\omega}} \left(\frac{\partial u_z}{\partial x'} \right)^2 dt dx' dz', \quad (5.99)$$

when it is considered that $u_x = 0$ on $x' = 0$. When equation (5.94) is substituted into equation (5.99) and the integration is completed the dissipation of energy due to viscosity at the wall $x' = 0$ is represented by,

$$\Delta E_x = \frac{g^2 a_0^2 \pi}{\omega^2} \frac{\pi}{2} \left(\frac{\mu \rho}{2\omega} \right)^{\frac{1}{2}} \frac{k}{\cosh^2(kd)} \left[\frac{1}{2} \sinh(2kd) - kd \right]. \quad (5.100)$$

The energy dissipation at the floor wall, $z' = 0$, is evaluated by following the same procedure and is represented by,

$$\Delta E_z = \frac{g^2 a_0^2 \pi^2}{\omega^2} \frac{\pi}{2} \left(\frac{\mu \rho}{2\omega} \right)^{\frac{1}{2}} \frac{k}{\cosh^2(kd)}. \quad (5.101)$$

Therefore the total energy dissipated due to the no-slip boundaries is,

$$\Delta E = 2\Delta E_x + \Delta E_z. \quad (5.102)$$

The energy of the wave, E , can be evaluated using the potential description of the wave by,

$$\frac{E}{2} = \frac{1}{2} \rho g \int_0^l \zeta_{max}^2 dx', \quad (5.103)$$

where ζ_{max} is the maximum free surface elevation evaluated using equation (5.89). The right hand side of equation (5.103) denotes the potential energy of the wave and the total energy is double the potential energy. By neglecting terms containing $a_0^2 k^2$ and higher powers, the energy is

$$E = \frac{1}{4} \frac{\pi \rho g a_0^2}{k} . \quad (5.104)$$

The decay constant is therefore

$$\alpha_w = \frac{\Delta E}{2E} \quad (5.105)$$

$$= k \left(\frac{2\nu}{\omega} \right)^{\frac{1}{2}} \frac{1}{\sinh(kd) \cosh(kd)} \left[\frac{\sinh(2kd)}{2} - kd + \frac{\pi}{2} \right] \quad (5.106)$$

α_w is therefore proportional to ϵ , whilst it can be shown that α_s is proportional to ϵ^2 , resulting in α_w being the primary source of dissipation when $\epsilon \ll 1$.

To be consistent with the evaluation of the decay constant due to the surface boundary layers evaluated by Landau & Lifshitz [37], α_w is divided by the time period of the oscillations such that,

$$\tilde{\alpha}_w = \frac{\alpha_w \omega}{2\pi} . \quad (5.107)$$

5.5.2 Analytical Derivation of Long Wave Forced Sloshing

To further test the validity of the Navier-Stokes free surface code, an analytical, linear solution for a fluid with a free surface undergoing forced oscillations is compared with computationally generated results.

The two dimensional system studied is similar to that described in figure 4.1. The fluid is contained within a rigid rectangular basin with length l and mean water depth d . A stationary co-ordinate system is denoted by $0x^*z^*$ and a moving co-ordinate system attached to the basin is denoted by $0xz$. The basin undergoes translational motion in the x^* direction prescribed by velocity $U_x(t)$. The analytical solution was evaluated by Lepelletier and Raichlen [39] using the nonlinear, dispersive, dissipative

long wave equations, $\frac{d}{l} \ll 1$ and their method is followed here. They are written in dimensional form as

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (d + \zeta) \bar{u}}{\partial x} = 0 , \quad (5.108)$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + g \frac{\partial \zeta}{\partial x} - \frac{1}{3} d^2 \frac{\partial^3 \bar{u}}{\partial x^2 \partial t} + \frac{1}{d} \left(\frac{\nu \omega_x}{2} \right)^{\frac{1}{2}} \bar{u} + \frac{\partial U_x}{\partial t} = 0 , \quad (5.109)$$

where $\zeta(x, t)$ denotes the wave elevation; $\bar{u}(x, t)$ is the velocity averaged in the z direction in the $0xz$ frame of reference; ν is the kinematic viscosity and ω_x is the frequency of the horizontal oscillations of the tank. The boundary and initial conditions are prescribed as;

$$\zeta(x, 0) = 0 , \quad (5.110)$$

$$\bar{u}(x, 0) = 0 , \quad (5.111)$$

$$\bar{u}\left(-\frac{l}{2}, t\right) = 0 , \quad (5.112)$$

$$\bar{u}\left(\frac{l}{2}, t\right) = 0 . \quad (5.113)$$

The dimensional form of the governing equations (5.108) and (5.109) can be written in a linearised non-dimensional form by assuming small velocities and free surface gradients. The resulting equations are;

$$\frac{\partial \zeta}{\partial \tau_l} + \frac{\partial \bar{u}}{\partial x} = 0 , \quad (5.114)$$

$$\frac{\partial \bar{u}}{\partial \tau_l} + \frac{\partial \zeta}{\partial x} - \frac{1}{3} \beta_l \frac{\partial^3 \bar{u}}{\partial^2 x \partial \tau_l} + f^0(\delta_l \tau_l) + \gamma_s \bar{u} = 0 , \quad (5.115)$$

where the dimensional variables, denoted by a tilde, have been non-dimensionalised as follows:

$$\tilde{\zeta} = H \zeta , \quad (5.116)$$

$$\tilde{x} = lx , \quad (5.117)$$

$$\tilde{t} = \frac{l}{(gd)^{\frac{1}{2}}} \tau_l , \quad (5.118)$$

$$\tilde{\bar{u}} = \frac{H}{l} (gd)^{\frac{1}{2}} \bar{u} , \quad (5.119)$$

$$\frac{d\tilde{U}}{d\tilde{t}} = A_c f^0(\omega_x \tilde{t}) , \quad (5.120)$$

where $H = \frac{A_c l}{g}$ and f^0 is generally a sinusoidal function. These non-dimensional terms are used in this section to be consistent with those used by Lepelletier & Raichlen [39]. All non-dimensional terms proceeding this section are described in section 5.6. The non-dimensional parameters in equation (5.115) are defined as;

$$\beta_l = \frac{d^2}{l^2} , \quad (5.121)$$

$$\gamma_s = \left(\frac{\nu \omega_x}{2} \right)^{\frac{1}{2}} \frac{l}{d (gd)^{\frac{1}{2}}} , \quad (5.122)$$

$$\delta_l = \frac{\omega_x l}{(gd)^{\frac{1}{2}}} . \quad (5.123)$$

where γ_s represents the dissipative nature of the motion.

The non-dimensional governing equations (5.115) are solved analytically by Lepelletier & Raichlen [39] and an expression for $\zeta(x, \tau)$ is found,

$$\begin{aligned} \zeta(x, \tau) = & -\text{Re} \left\{ \frac{1}{\kappa} \frac{\sin(\kappa x)}{\cos\left(\frac{\kappa}{2}\right)} e^{i\delta_l \tau_l} \right\} \\ & - 2 \sum_{n=0}^{\infty} (-1)^n \sin(a_n x) \text{Re} \left\{ f_n e^{s_n \tau_l} \right\} , \end{aligned} \quad (5.124)$$

where

$$a_n = (2n + 1) \pi , \quad (5.125)$$

$$\kappa = \frac{\delta_l}{\left(1 - \frac{1}{3}\beta_l \delta_l^2\right)^{\frac{1}{2}}} \left(1 - i \frac{\gamma_s}{2\delta_l}\right) , \quad (5.126)$$

$$\delta_l < \left(\frac{3}{\beta_l}\right)^{\frac{1}{2}} \quad (5.127)$$

$$f_n = 4 \frac{s_n}{(s_n^2 + \delta_l^2)} \frac{1 + \frac{1}{3}\beta_l s_n^2}{(2s_n + \gamma_s (1 - \frac{1}{3}\beta_l s_n^2))} , \quad (5.128)$$

$$s_n = \frac{-\gamma_s - 2ia_n (1 + \frac{1}{3}\beta_l a_n^2)}{2 (1 + \frac{1}{3}\beta_l a_n^2)} . \quad (5.129)$$

In addition to the inequality in equation (5.127), other limitations are $\beta_l \ll 1$ and $\gamma_s \ll 1$, which ensure the validity of the governing long wave equations (5.109).

The linear solution consists of two contributing parts. The second represents the contribution to the motion of all the natural modes initially excited. These

transients are damped by the viscous term, eventually leaving only the steady state solution, represented by the first part of equation (5.124) which is dependent on the frequency of the driving motion.

The dimensional natural frequencies, σ_n , of the long wave system were found by Lepelletier & Raitchlen [39] by neglecting dissipation as,

$$\frac{\sigma_n l}{(gd)^{\frac{1}{2}}} = (2n + 1) \pi \left[1 - \frac{1}{6} (2n + 1)^2 \pi^2 \left(\frac{d}{l} \right)^2 \right]. \quad (5.130)$$

5.6 Non-dimensionalisation

The Reynolds number is defined by,

$$Re = \frac{(gd)^{\frac{1}{2}} d}{\nu}, \quad (5.131)$$

for all the test cases in the following sections and the governing parameters of the fluid motion are represented as non-dimensional values such that,

$$x = \frac{\tilde{x}}{d}, \quad z = \frac{\tilde{z}}{d}, \quad \zeta = \frac{\tilde{\zeta}}{d}, \quad l = \frac{\tilde{l}}{d}, \quad a = \frac{\tilde{a}}{d}, \quad (5.132)$$

$$k = \tilde{k}d, \quad \omega = \left(\frac{d}{g} \right)^{\frac{1}{2}} \tilde{\omega}, \quad u_x = (gd)^{-\frac{1}{2}} \tilde{u}_x, \quad u_z = (gd)^{-\frac{1}{2}} \tilde{u}_z, \quad (5.133)$$

$$p = (\rho gd)^{-1} \tilde{p}, \quad t = \tau = \left(\frac{g}{d} \right)^{\frac{1}{2}} \tilde{t}, \quad (5.134)$$

where a tilde denotes the dimensional quantities.

5.7 Small Amplitude Freely Oscillating Waves

In this section the theoretically predicted damping rates for a freely sloshing fluid are compared with those generated by the numerical algorithm. To eliminate the necessity to consider the influence of the boundary condition and the contact point of the free surface on the fixed vertical walls, a periodic domain is utilised such that the values of the prime variables on the left hand side of the domain are identical to those on the right hand side. This has the benefit of allowing the decay rate

due to the free surface boundary layer to be isolated and evaluated, though there is a minimal contribution to the decay from the floor wall. No surface tension or contamination effects are simulated in the computations. The decay constant is therefore taken as $\alpha = \alpha_s + \alpha_w$ where α_s is the contribution from the surface which can be evaluated from the results following Landau & Lifshitz [37] formulated in section 5.5.1 and α_w is the contribution from the floor wall boundary layer and is evaluated following Keulegan [35] in section 5.5.1.

As the fluid domain is a simple geometric shape for the following test cases, quadrilateral meshes were used and therefore the solution required no artificial smoothing.

5.7.1 Validation of Damping of Gravity Waves

The fluid is initially at rest within a periodic domain of size $l = 2$, with an initial free surface displacement described by,

$$\zeta(x; 0) = -a_0 \cos(kx), \quad (5.135)$$

where $k = \frac{2\pi}{l}$ and $a_0 = 0.005$ thus ensuring a linear wave. The mesh used contains 100 quadrilateral elements and is shown in figure 5.3 at its initial displacement. The elements are smaller at the surface and floor to capture the boundary layer adequately, whilst relatively large elements occur in the body of the mesh due to the small velocity gradient present in this area. The pressure and velocity boundary conditions on the free surface are the fully non-linear conditions specified in equations (5.63), (5.68) and (5.69). The spatial and temporal convergence of a suitable test case is investigated to ensure a convergent solution is achievable. The Reynolds number of the test case is 110.736 and the mesh configuration is shown in figure 5.3. Figure 5.4(a) shows the time history of the free surface at $x = \frac{l}{2}$ for decreasing values of $\Delta\tau$. It can be seen from the graph that convergence in time is satisfied as the results are nearly identical when using $\tau = 4.429 \times 10^{-3}$ and $\tau = 2.215 \times 10^{-3}$. Figure 5.4(b) shows the time history of the free surface for increasing values of

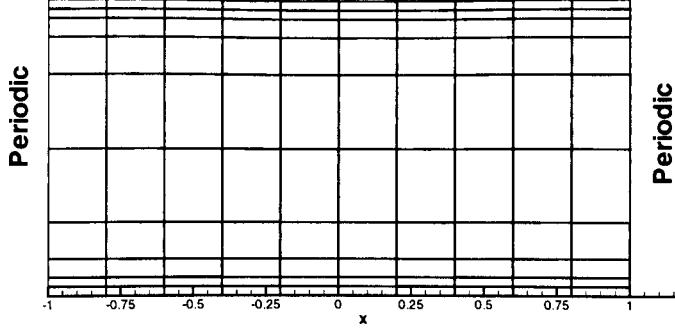


Figure 5.3: Mesh used for small amplitude sloshing

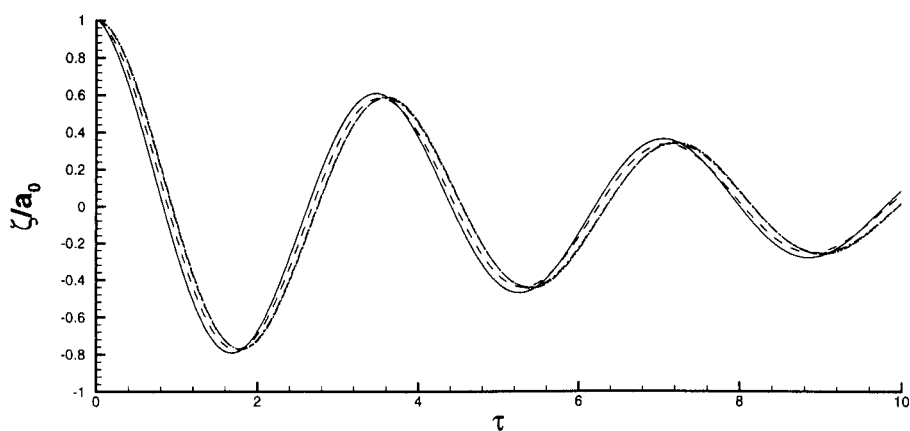
p , rapid convergence in p is achieved. For values of $P = 5$, $P = 7$ and $P = 9$ the results are indistinguishable.

A series of cases for differing Reynolds number is now investigated to ensure the computational accuracy of the Navier-Stokes free surface solver. Table 5.3 contains information on the governing parameters for each case from A to F. The periodic domain is identical to that considered above, with $\Delta\tau = 4.429 \times 10^{-3}$ and $P = 5$.

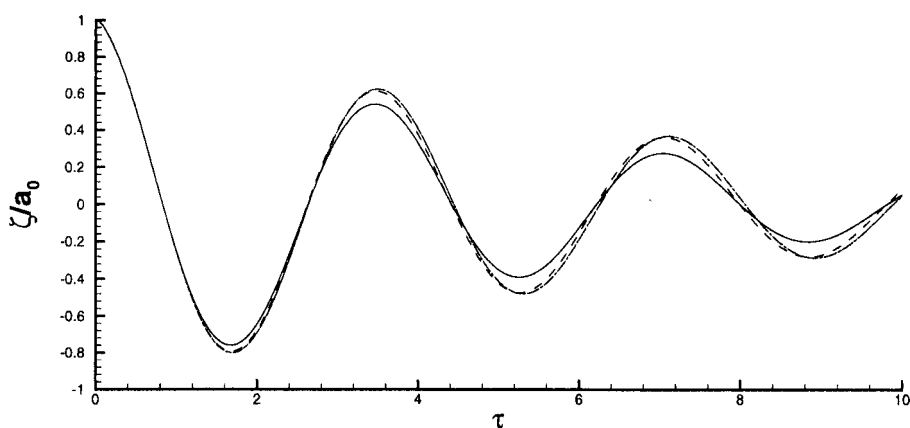
The resulting fluid motion can be seen in figures 5.5 and 5.6 where $Re = 110.736$ and 1107.36 respectively where the contours represent vorticity, Θ , where,

$$\Theta = \nabla \times \mathbf{u} , \quad (5.136)$$

The motion of the free surface produces alternating vorticity layers within the fluid. These layers are formed at and propagate away from the free surface and the floor wall. These propagating vorticity layers are clearly shown in figure 5.5 and figure 5.6, where the minimum value of vorticity is represented by the colour blue and the maximum by red. This representation is used for all contour plots in this work. It is seen that the depth of penetration of the vorticity layers is dependent on the Reynolds number, the vorticity layers present in the fluid when $Re = 110.736$ spread further into the body of the fluid than those in the fluid when $Re = 1107.36$.



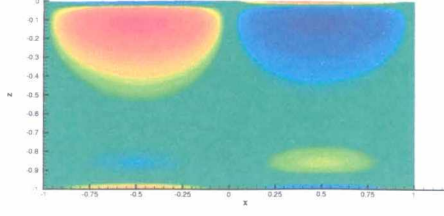
(a) Temporal convergence for decreasing time-step: — $\Delta\tau = 4.429 \times 10^{-2}$, --- $\Delta\tau = 2.215 \times 10^{-2}$, - · - · - $\Delta\tau = 4.429 \times 10^{-3}$, · · · · · $\Delta\tau = 2.215 \times 10^{-3}$, - · - · - · $\Delta\tau = 4.429 \times 10^{-4}$.



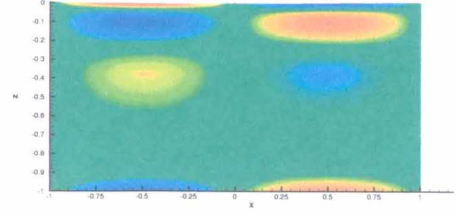
(b) Spatial convergence for increasing polynomial order : — $p = 1$, --- $p = 3$, - · - · - $p = 5$, · · · · · $p = 7$, - · - · - · $p = 9$.

Figure 5.4: Time history of free surface at $x = -\frac{l}{2}$ showing 5.4(a) temporal convergence and 5.4(b) spatial convergence

Figure 5.7 contains the time histories of the elevation of the free surface at $x = -\frac{l}{2}$. Also included on the graph is the total predicted decay rate α_p where

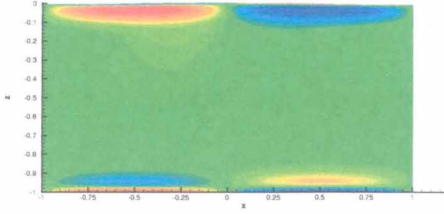


(a) $\tau = 1.86$

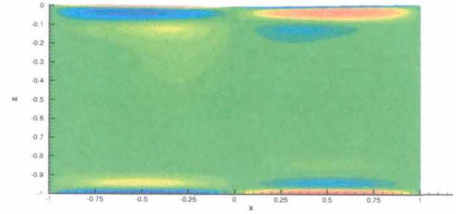


(b) $\tau = 3.72$

Figure 5.5: Vorticity contours of fluid flow when sloshing for initial amplitude $a_0 = 0.005$ and $Re = 110.736$. $\Theta_{min} = -0.015$ $\Theta_{max} = 0.015$



(a) $\tau = 1.86$

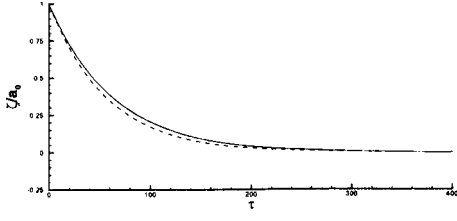


(b) $\tau = 3.72$

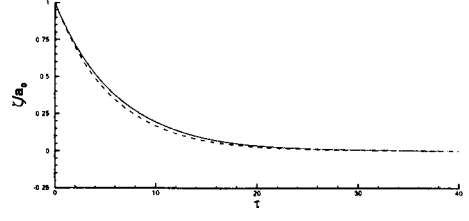
Figure 5.6: Vorticity contours of fluid flow when sloshing for initial amplitude $a_0 = 0.0025$ and viscosity $Re = 1107.36$. $\Theta_{min} = -0.022$ $\Theta_{max} = 0.022$

$\alpha_p = \alpha_s + \alpha_w$ for oscillatory flow and $\alpha_p = \alpha_s$ for the highly viscous aperiodic flow. α_w is not included in the evaluation of the predicted decay rate for the aperiodic case as it is assumed in the theoretical evaluation of α_w that the flow is oscillating at its natural frequency. The predicted decay rate, α_p , is included in table 5.3, this can be compared to the computational decay rate, α_c which is also shown.

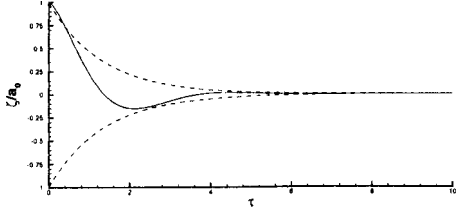
The results shown in figure 5.7 are in good qualitative agreement, though the computational decay rates are approximately ten percent different to the predicted decay rates, except for cases D and F. These discrepancies are attributed to the



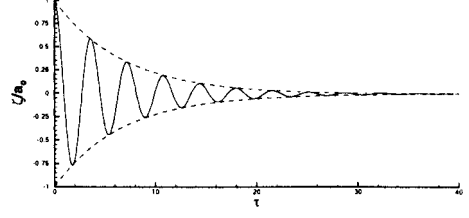
(a) $Re = 0.110736$



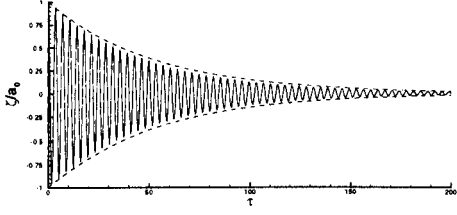
(b) $Re = 1.10736$



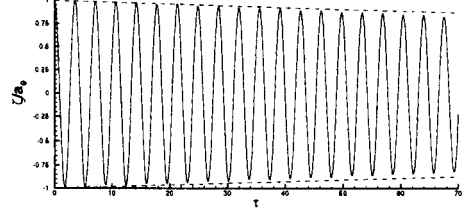
(c) $Re = 11.0736$



(d) $Re = 110.736$



(e) $Re = 1107.36$



(f) $Re = 11073.6$

Figure 5.7: Time history of free surface displacement at $x = -\frac{l}{2}$ for increasing Reynolds number with predicted decay rate as dashed line.

relative shallowness of the domain. The theory to predict the decay rate due to the free surface ‘boundary layer’ is based on the velocity of the fluid exponentially decaying to zero as z approaches negative infinity. As the depth of the fluid in the test cases A to F is only half of the wavelength, the velocity of the fluid will not decrease exponentially to zero. Instead a boundary layer will form at the floor

Test Case	Re	α_p	α_c	$\frac{\alpha_c}{\alpha_p}$
A	0.110736	1.76268×10^{-2}	1.59111×10^{-2}	0.902665
B	1.10736	1.78945×10^{-1}	1.64371×10^{-1}	0.918551
C	11.0736	7.55609×10^{-1}	8.25554×10^{-1}	1.09257
D	110.736	1.50323×10^{-1}	1.53886×10^{-1}	1.02370
E	1107.36	1.90531×10^{-2}	2.07846×10^{-2}	1.09088
F	11073.6	1.91568×10^{-3}	2.76411×10^{-3}	1.44289

Table 5.3: Test cases A to F for sloshing with $k = \frac{2\pi}{l}$, $a_0 = 0.005$, $l = 2$ giving a comparison between the predicted decay constant, α_p and the computed decay constant, α_c

wall and although the damping due to the this boundary layer has been taken into account, the effect of finite depth on the strength of the free surface boundary layer and the effect of the interaction of the boundary layers have not been taken into account. The theory should be accurate when the distance from the surface to the floor wall is large compared to the wavelength of the free surface displacement as specified by Landau & Lifshitz [37]. To test this assumption the cases A to F are performed on a domain of size $l = 0.2$, ensuring little contribution to the dissipation from the floor wall and no interaction between the vorticity layers generated by the surface and the floor wall. The mesh is made up of 200 elements in a similar pattern to the previous mesh in figure 5.3.

When compared to the decay rates generated using the shallow domain, $l = 2$, the difference between the predicted decay rate and the computational decay rate has been reduced for all cases except case D. The large difference between the predicted and computed decay rates for case F can be attributed to the diffusion inherent in the stiffly stable time integration scheme used to propagate the Navier-Stokes equations. It is more apparent for the lower viscosities due to the smaller decay

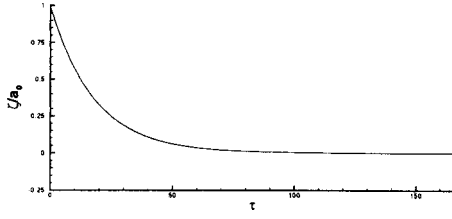
Test Case	Re	α_p	α_c	$\frac{\alpha_c}{\alpha_p}$
A	3.50178	5.57409×10^{-2}	5.59406×10^{-2}	1.00358
B	35.0178	5.65876×10^{-1}	5.62732×10^{-1}	0.994444
C	350.178	2.38943	2.41594	1.01109
D	3501.78	4.72041×10^{-1}	4.54660×10^{-1}	0.96318
E	35017.8	5.92026×10^{-2}	6.15687×10^{-2}	1.03997
F	350178	5.72630×10^{-3}	7.67816×10^{-3}	1.34086

Table 5.4: Test cases A to F for sloshing with $k = \frac{2\pi}{l}$, $a_0 = 0.0005$ and $l = 0.2$ giving a comparison between the predicted decay constant, α_p and the computed decay constant, α_c

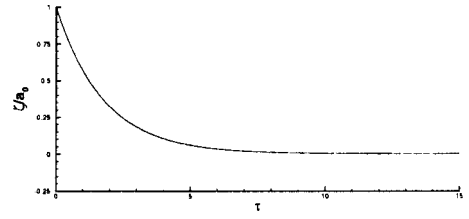
rates present.

Modification of the Time Period Due to Viscosity

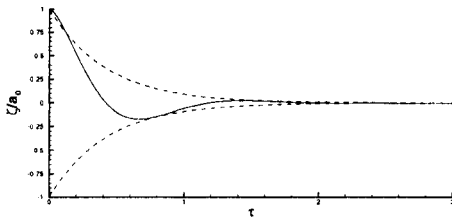
The introduction of viscosity has had an effect on the time period, T , of the oscillatory motion of the free surface. For the shallower tank $l = 2$ the effect is negligible for the higher Reynolds numbers, $Re = 1107.36$ and 11073.6 , though it is apparent when $Re = 110.736$. Table 5.5 contains the value of the time period for various Reynolds numbers and the time period of an inviscid, irrotational fluid undergoing the equivalent linear motion which is denoted by T_0 . The natural frequency of the fluid has therefore been affected by the introduction of viscosity. It will be seen that this effect is increased when the fluid is bounded by side walls.



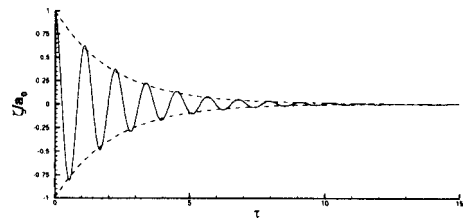
(a) $Re = 3.50178$



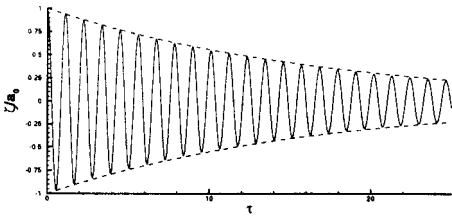
(b) $Re = 35.0178$



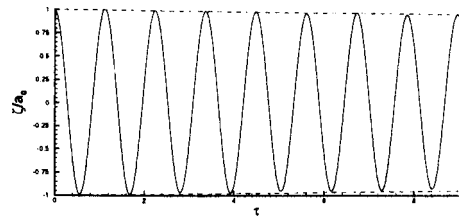
(c) $Re = 350.178$



(d) $Re = 3501.78$



(e) $Re = 35017.8$



(f) $Re = 350178$

Figure 5.8: Time history of free surface at $x = -\frac{l}{2}$ for $l = 0.2$ with increasing Reynolds number with predicted decay rate as dashed line.

Re	110.736	1107.36	11073.6	inviscid
T	3.58950	3.55208	3.55200	3.55153
$\frac{T}{T_0}$	1.01069	1.00015	1.00013	1.00000

Table 5.5: Comparison between the value of the linear, inviscid time period and the time period of a fluid with a free surface and bounded by a floor wall

5.8 Large Amplitude Sloshing for Alternative Implementations of the Free Surface Boundary Conditions

As many different free surface boundary conditions are used by various authors when evaluating free surface flows, it is intended here to compare their respective effect on the fluid motion for a number of scenarios, the first of which is standing wave sloshing in a stationary tank of size $l = 2$. In order to investigate these effects, large amplitude sloshing is studied for a number of test cases at increasing Reynolds number.

The boundary conditions investigated are:

(a) the zero boundary conditions used by Harlow & Welch [21, 22] and Park et al [54] are:

$$p = 0 ; \quad (5.137)$$

$$\frac{\partial u_x}{\partial n} = 0 , \quad (5.138)$$

$$\frac{\partial u_z}{\partial n} = 0 ; \quad (5.139)$$

(b) the linear boundary conditions used by Hirt & Shannon [25] and Nichols & Hirt

[49]:

$$p = \frac{2}{Re} \left(\frac{\partial u_z}{\partial z} \right), \quad (5.140)$$

$$\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0; \quad (5.141)$$

(c) the nonlinear boundary conditions used by Ramaswamy [59] and Ramaswamy & Kawahara [60] are:

$$-p + \frac{2}{Re \left(1 + \left(\frac{\partial \zeta}{\partial x} \right)^2 \right)} \left(\frac{\partial u_x}{\partial x} \left(\frac{\partial \zeta}{\partial x} \right)^2 - \left(\frac{\partial u_x}{\partial z} \frac{\partial u_z}{\partial x} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial u_z}{\partial z} \right) = 0, \quad (5.142)$$

$$2 \frac{\partial u_x}{\partial x} \frac{\partial \zeta}{\partial x} + \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \left(\left(\frac{\partial \zeta}{\partial x} \right)^2 - 1 \right) - 2 \frac{\partial u_z}{\partial z} \frac{\partial \zeta}{\partial x} = 0. \quad (5.143)$$

The mesh for the test cases is made up of 100 quadrilateral elements and is shown

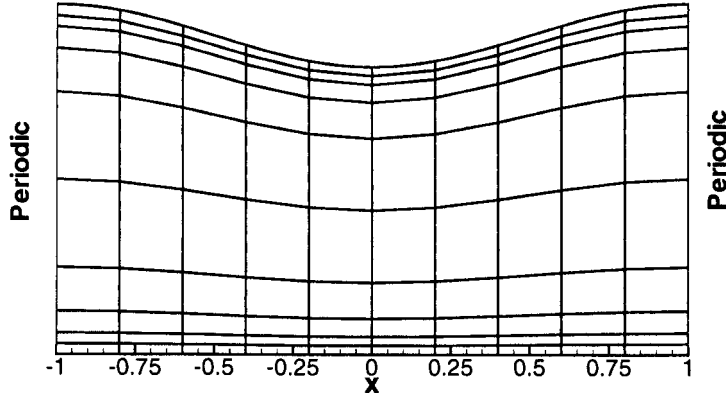


Figure 5.9: Mesh used for large amplitude sloshing

in its initial state in figure 5.9, where the free surface is prescribed by,

$$\zeta(x; 0) = -a_0 \cos(kx), \quad (5.144)$$

where $k = \frac{2\pi}{l}$, $a_0 = 0.1$ and $l = 2$. Figure 5.10 contains 20 snapshots of the free surface displacement for time steps of $\tau = 0.354$ from $0 \leq \tau \leq 7.08$ at Reynolds number

110.736. The height of the box is $\frac{3d}{2}$. Non-linear effects of shallow troughs and steep peaks can be seen, as can the more dissipative nature of the motion when using the linear and non-linear boundary conditions. The primary observation, however, is the similarity of the free surface profiles, particularly between the non-linear and linear boundary conditions, though there is no significant difference even when using the zero boundary condition. This implies that the free surface nonlinear effects are primarily generated by the evaluation of the governing equations in a moving domain. Figures 5.11, 5.12 and 5.13 contain snapshots of the fluid flow when using the zero boundary conditions, linear boundary conditions and nonlinear boundary conditions respectively for $Re = 110.736$. The contours represent the vorticity of the fluid and each set of snapshots represents approximately one oscillation of the fluid.

The flow fields are more complicated than those for small amplitude sloshing and there is interaction between the vorticity layers produced by the floor wall and those produced at the surface. For all boundary condition types the vorticity layers at the free surface are at their strongest when the displacement is closest to zero, this corresponds to the free surface attaining its maximum velocity. These vorticity layers are then advected from the surface as the displacement of the free surface reaches its maximum and are replaced by another layer of opposing sign. It is the change in sign of the velocity which leads to an opposite sign vorticity layer. It can therefore be seen that vorticity layers exist within the fluid domain, not merely at the boundaries.

The difference in the vorticity layer strengths can clearly be seen between the zero boundary conditions and the linear and non-linear boundary conditions, where the zero boundary conditions produces a weaker boundary layer at the free surface due to the derivative of the velocities in the normal directions being set to zero. The strength of the linear and non-linear free surface boundary layers are comparable.

An interesting feature of the influence of the vorticity layers on the fluid motion can be seen in figure 5.14, where the contours again indicate vorticity and the

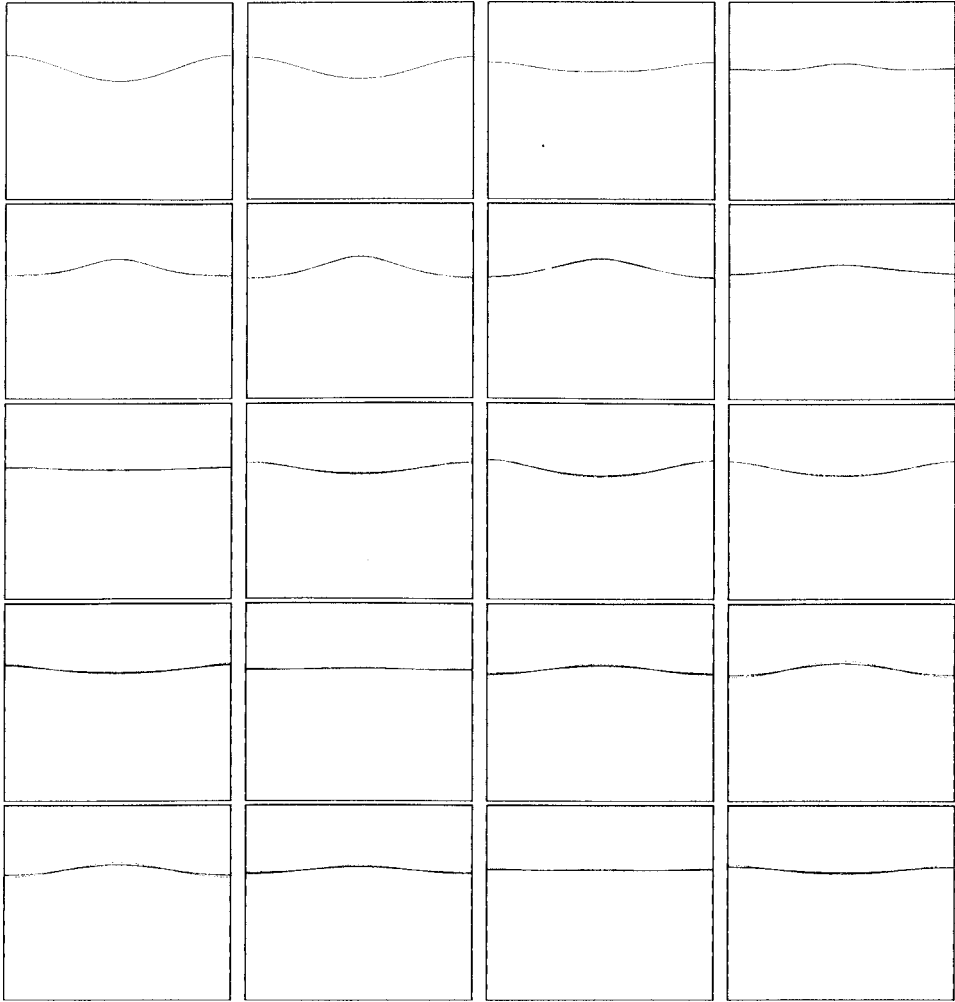
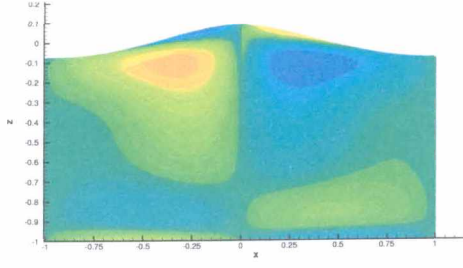
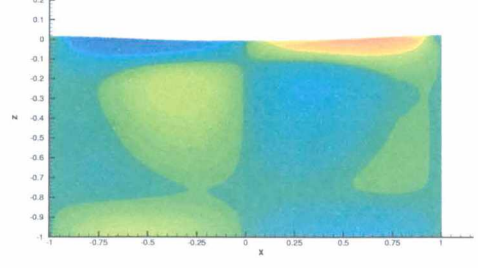


Figure 5.10: Snapshots of free surface displacement for standing wave motion using — Non-linear boundary conditions, — Linear boundary conditions, — Zero boundary conditions

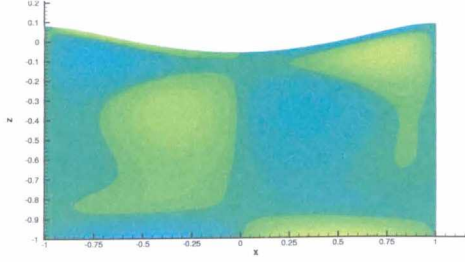
direction of the fluid motion is represented by arrows. The opposing vorticity layers in the two halves of the fluid have set up a circulation within the domain which is counter-productive to the free surface sloshing motion, resulting in the fluid below the free surface maximum having positive vertical velocity, whilst the free surface itself is descending. A saddle point is therefore formed, increasing the horizontal flow of the sloshing liquid.



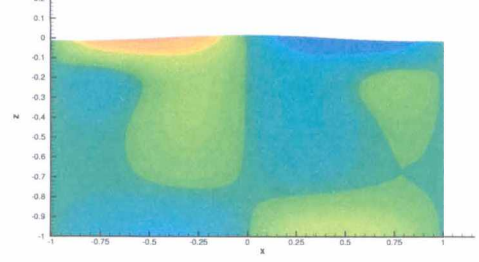
(a) $\tau = 1.86$



(b) $\tau = 2.79$



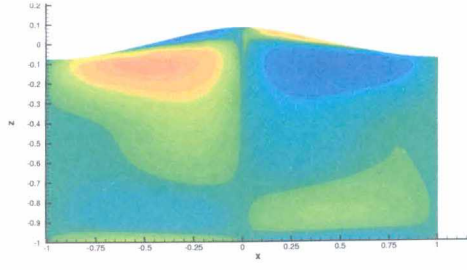
(c) $\tau = 3.72$



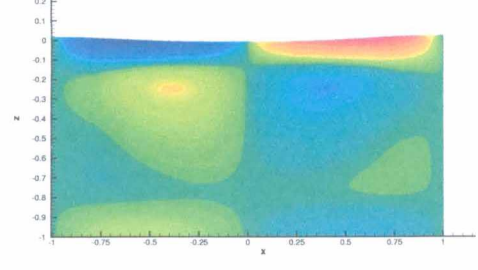
(d) $\tau = 4.65$

Figure 5.11: Vorticity contours of fluid flow when sloshing using the zero boundary conditions. $\Theta_{min} = -0.685$ $\Theta_{max} = 0.685$

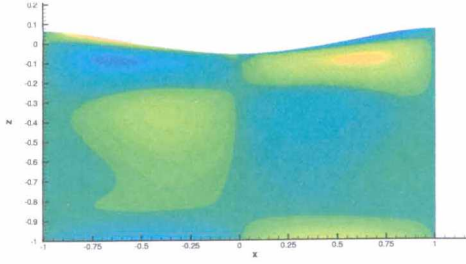
These factors contribute to differences between the damping rates of the motion depending on the boundary conditions applied. Figure 5.15 contains the time histories of the free surface elevation at $x = -\frac{l}{2}$ for three representative cases of increasing Reynolds number. For low Reynolds numbers there is little difference between the motion for the linear and non-linear boundary conditions due to the similarity in shear layer strength. The zero boundary conditions produce a less dissipative solution due to the weaker surface vorticity layers. At $Re = 11073.6$ there is highly non-linear motion with no substantial difference between the boundary conditions used, particularly between the non-linear and linear conditions. At the lowest Reynolds number the linear boundary condition produces an erroneous solution. The reasons behind this were unfortunately unable to be ascertained.



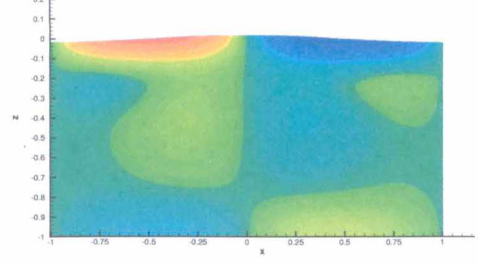
(a) $\tau = 1.86$



(b) $\tau = 2.79$



(c) $\tau = 3.72$

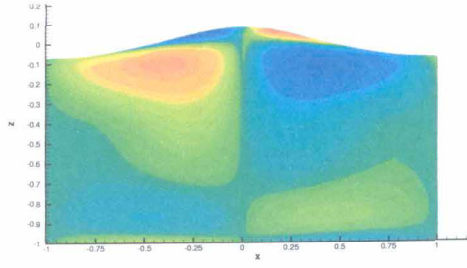


(d) $\tau = 4.65$

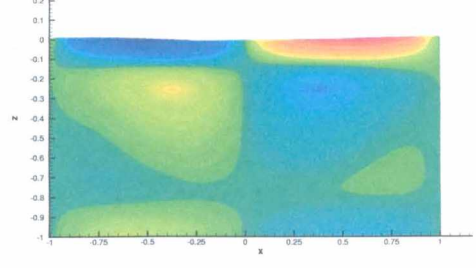
Figure 5.12: Vorticity contours of fluid flow when sloshing using the linear boundary conditions. $\Theta_{min} = -0.685$ $\Theta_{max} = 0.685$

5.8.1 Sloshing of a Fully Bounded Fluid

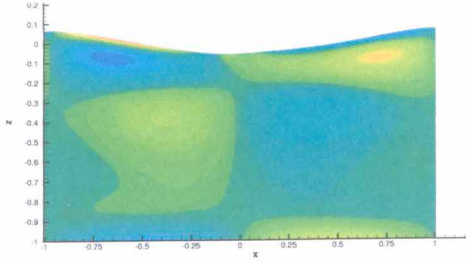
In this section the theoretically predicted and computational decay rates of a fully bounded, free surface standing wave in a stationary tank are compared. The periodic boundary conditions on the side walls used in the last section are replaced with wall boundary conditions, though as these walls are in contact with the free surface the normal no-slip condition cannot be applied. Instead the three modified boundary conditions specified in section 5.4 are implemented. These are the semi-noslip condition, the semi-slip condition and the slip condition. Initially small amplitude standing waves are studied in order to investigate the applicability of the three modified wall boundary conditions by comparing the various computational



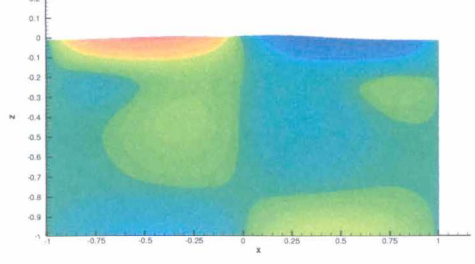
(a) $\tau = 1.86$



(b) $\tau = 2.79$



(c) $\tau = 3.72$



(d) $\tau = 4.65$

Figure 5.13: Vorticity contours of fluid flow when sloshing using the non-linear boundary conditions. $\Theta_{min} = -0.685$ $\Theta_{max} = 0.685$

decay rates with the theoretical decay rates predicted by Keulegan [35] and Landau & Lifshitz [37]. The flow fields are then studied, in particular the effect the differing wall boundary conditions have on the boundary layers at the contact walls.

Large amplitude sloshing is then simulated to achieve a nonlinear solution in order to ascertain any resulting differences in the fluid motion from the linear simulations.

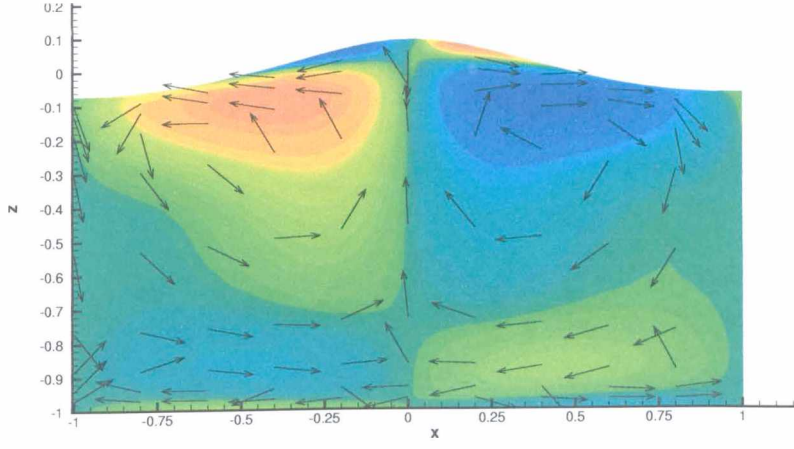


Figure 5.14: Sloshing fluid shortly after maximum displacement has been achieved where a saddle point is formed directly under the point of maximum deflection

Convergence

To ensure convergence a suitable test case was studied. The fluid is initially at rest with a free surface deformation of,

$$\zeta(x; 0) = -a_0 \sin(kx) , \quad (5.145)$$

where $a_0 = 0.01$, $k = \frac{\pi}{l}$, $l = 2$ and $Re = 1107.36$. The values of the free surface elevation at $x = -\frac{l}{2}$ at $\tau = 5.2$ using the differing contact wall boundary conditions can be seen in table 5.6 for increasing p ($\Delta\tau = 4.429 \times 10^{-3}$) and in table 5.7 for decreasing $\Delta\tau$ ($P = 5$). The values were measured at $\tau = 5.2$ as this relates to the point in time when the free surface has undergone approximately one period. It can be seen from the data that convergence is attained for all boundary conditions.

Small Amplitude Freely Oscillating Waves

Four test cases for differing values of Reynolds number were performed, each using the three modified contact wall boundary conditions. The initial conditions used to

Boundary condition	Semi-noslip	semi-slip	slip
$P = 3$	0.979381	0.956971	0.987631
$P = 5$	0.942267	0.965218	0.987849
$P = 7$	0.940375	0.971783	0.988895

Table 5.6: Values of $\frac{\zeta}{a_0}$ at $x = -\frac{l}{2}$ for increasing p

Boundary condition	Semi-noslip	semi-slip	slip
$\Delta\tau = 8.858 \times 10^{-3}$	0.976300	0.980723	1.00493
$\Delta\tau = 4.429 \times 10^{-3}$	0.962538	0.958285	0.987096
$\Delta\tau = 2.215 \times 10^{-3}$	0.955729	0.945225	0.978707

Table 5.7: Values of $\frac{\zeta}{a_0}$ at $x = -\frac{l}{2}$ for decreasing $\Delta\tau$

simulate the fluid motion are

$$u_x(x, z; 0) = 0, \quad (5.146)$$

$$u_z(x, z; 0) = 0, \quad (5.147)$$

$$\begin{aligned} \zeta(x; 0) = & -a_0 \sin(kx) - a_0 \left(\frac{1}{4}a_0 k\right) N_1 \cos(2kx)) \\ & -a_0 \left(\frac{1}{4}a_0 k\right) N_2 \cos(2kx)) , \end{aligned} \quad (5.148)$$

where

$$N_1 = \frac{\cosh(2k)}{\sinh(2k)}, \quad (5.149)$$

$$N_2 = \frac{\cosh^2 k (\cosh(2k) + 2)}{\sinh^2(k) \sinh(2k)} \quad (5.150)$$

The initial value of ζ is the second order approximation to the solution of the non-linear inviscid, irrotational governing equations for free surface flow presented by Keulegan [35] at a point in time corresponding to the maximum amplitude of the

wave. In order to produce a low amplitude wave the governing parameters are $k = \frac{\pi}{l}$, $a_0 = 0.01$ and $l = 2$.

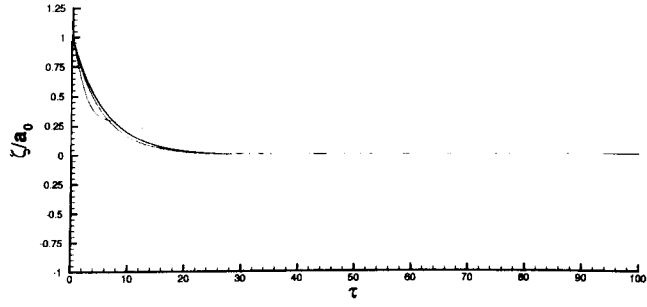
The motion is simulated for $Re = 11.0736, 110.736, 1107.36$ and 11073.6 , using the non-linear boundary conditions for pressure and velocity on the free surface and taking $\Delta\tau = 4.4294 \times 10^{-3}$ and $P = 5$. The computational mesh used is shown in figure 5.16 in its initial state. The mesh has small elements at all boundaries in order to fully resolve the boundary layers that are produced by the various boundary conditions specified. The time histories of the elevation of the free surface at $x = -\frac{l}{2}$ for the various Reynolds numbers and contact wall boundary conditions are shown in figure 5.17. Also included on the time-histories is the theoretically predicted decay rate, α_p , where $\alpha_p = \alpha_w + \alpha_s$. α_s is derived from the dispersion relation evaluated by Landau & Lifshitz in section 5.5.1 and α_w is evaluated for the contact and floor walls from Keulegan's equation (5.106) for the viscous dissipation at no-slip boundaries. Though the predicted decay rate is shown it will not always be accurate for the Reynolds numbers used in simulating the free surface motion due to the assumption of

$$\epsilon = k \left(\frac{2\nu}{\omega} \right)^{\frac{1}{2}} \ll 1, \quad (5.151)$$

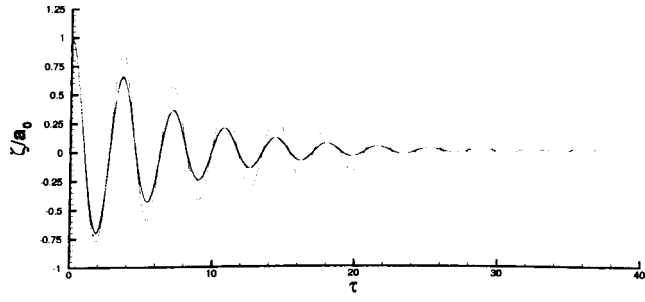
when evaluating the viscous dissipation due to the no-slip boundaries. Table 5.8 contains the values of ϵ for corresponding Reynolds numbers where it can be seen that accurate comparisons can only be made for $Re = 11073.6$.

Re	11.0736	110.736	1107.36	11073.6
ϵ	0.609326	0.192686	0.0609326	0.0192686

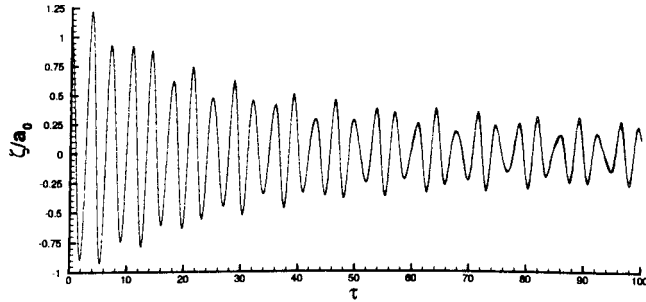
Table 5.8: Comparison between the value of ϵ and the Reynolds number Re for sloshing when $k = \frac{\pi}{l}$



(a) $Re = 1.10736$



(b) $Re = 110.736$



(c) $Re = 11073.6$

Figure 5.15: Time history of free surface at $x = -\frac{l}{2}$ for — Non-linear boundary conditions, — Linear boundary conditions, — Zero boundary conditions

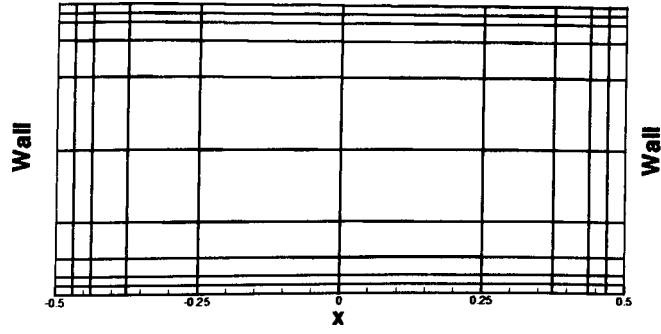
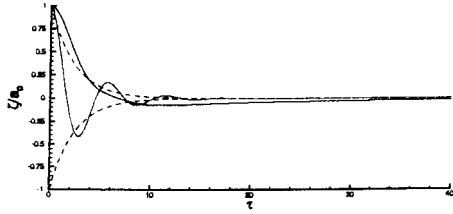
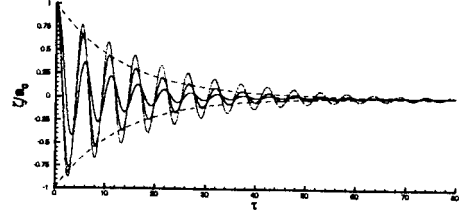


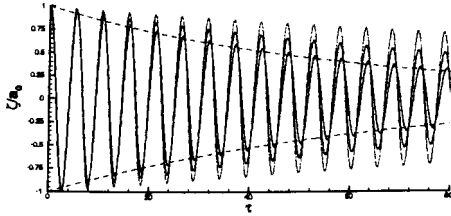
Figure 5.16: Mesh used for small amplitude fully bounded sloshing



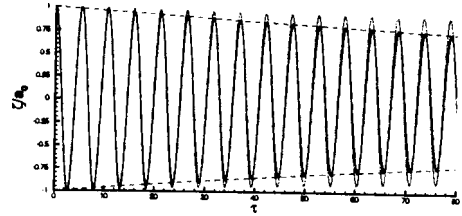
(a) $Re = 11.0736$



(b) $Re = 110.736$



(c) $Re = 1107.36$



(d) $Re = 11073.6$

Figure 5.17: Time history of free surface at $x = -\frac{l}{2}$ with increasing Reynolds number using — Semi-noslip boundary conditions, — Semi-slip boundary conditions, — Slip boundary conditions, — — — — — Predicted decay.

For the lowest Reynolds number, $Re = 11.0736$, there is little difference between the motion of the free surface when either the slip condition or the semi-slip condition is used on the contact walls. The motion corresponding to these two boundary conditions is periodic in nature as opposed to the motion due to the semi-noslip boundary condition when a predominantly aperiodic motion is seen due to the increased damping.

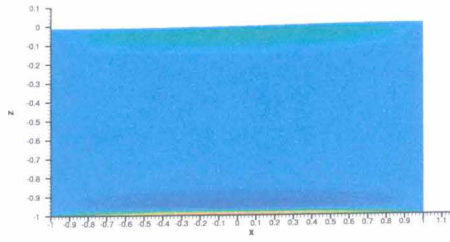
As the Reynolds number is increased the contrasting nature of the motion when the varying boundary conditions are applied decreases, i.e. the time histories of the elevation of the free surface converge, particularly when using the semi-slip and semi-noslip conditions. Figures 5.18, 5.19 and 5.20 contain 4 flow field snapshots

Test Case	Re	α_p	Slip $\left(\frac{\alpha_c}{\alpha_p}\right)$	Semi-slip $\left(\frac{\alpha_c}{\alpha_p}\right)$	Semi-noslip $\left(\frac{\alpha_c}{\alpha_p}\right)$
A	11.0736	4.17449×10^{-1}	1.02542	1.02831	
B	110.736	7.71333×10^{-2}	0.670313	0.924685	1.25550
C	1107.36	1.59633×10^{-2}	0.279553	0.561853	0.933965
D	11073.6	4.12230×10^{-3}	0.708628	1.08771	1.01159

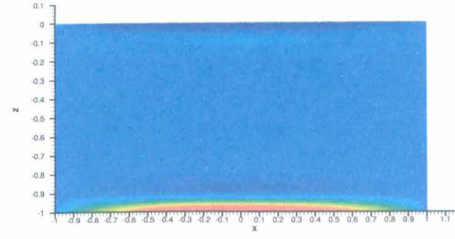
Table 5.9: Test cases A to D for sloshing of a bounded fluid with $k = \frac{\pi}{l}$, $a_0 = 0.01$, $l = 2$ providing a comparison between the predicted decay constant, α_p , and the computed decay constant, α_c , for the slip boundary condition, the semi-slip boundary condition and the semi-noslip boundary condition

representing approximately one oscillation of the fluid motion for the slip, semi-slip and noslip wall contact boundary conditions respectively when $Re = 1107.36$.

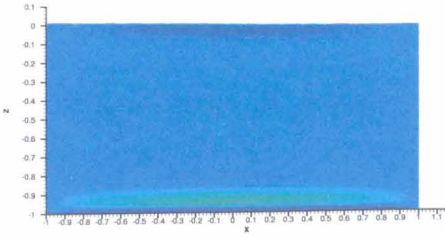
The absence of a boundary layer when using the slip boundary condition is evident from figure 5.18. The slip condition provides no mechanism for producing boundary layers due to the condition of no shear force on the boundary. Miles [45]



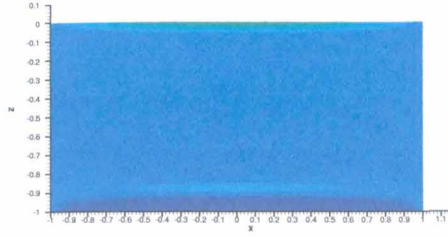
(a) $\tau = 2.66$



(b) $\tau = 3.99$



(c) $\tau = 5.32$

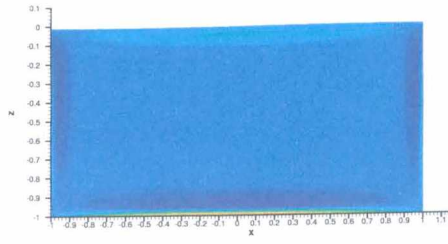


(d) $\tau = 6.64$

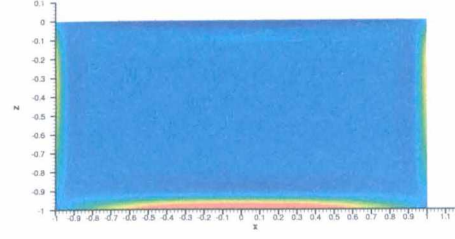
Figure 5.18: Vorticity contours of fluid flow when sloshing using the slip contact wall boundary conditions for $Re = 1107.36$. $\Theta_{min} = -0.14$ $\Theta_{max} = 0.33$

and Keulegan [35] believed that most of the dissipation is in the area of the contact walls, therefore the slip boundary condition is not fully capturing this damping and therefore producing a solution which is not dissipative enough. This can be seen in figure 5.17 where the slip boundary condition produces the least dissipative solution.

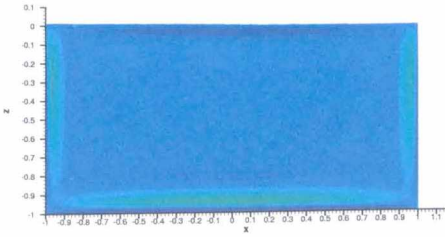
The boundary layer produced by the semi-slip boundary condition is evident in figure 5.19. The generation of a boundary layer is due to the constraint placed on the intermediate velocity $\hat{u}_z$. The semi-noslip condition produces the strongest boundary layer, where the largest value of vorticity is at the boundary between the no-slip element and the slip element, due to the velocity gradient $\frac{\partial u_z}{\partial x}$ being particularly high in this area. This rapid jump in velocity could have a destabilising



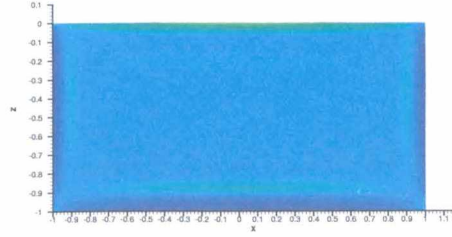
(a) $\tau = 2.66$



(b) $\tau = 3.99$



(c) $\tau = 5.32$

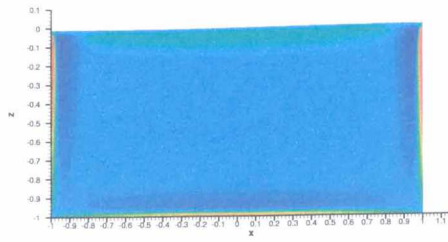


(d) $\tau = 6.64$

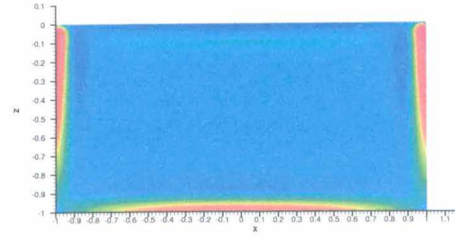
Figure 5.19: Vorticity contours of fluid flow when sloshing using the semi-slip contact wall boundary conditions for $Re = 1107.36$. $\Theta_{min} = -0.14$ $\Theta_{max} = 0.33$

effect on the computations for flows with higher values of ω and a_0 , i.e. faster flowing simulations.

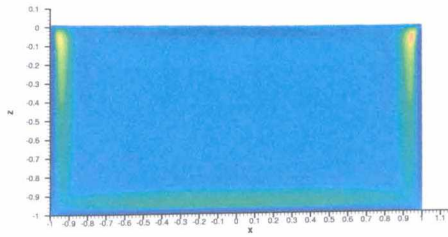
Contained in figure 5.21 are free surface profiles for varying kinematic viscosity and wall contact boundary conditions. The aspect ratio of the axis is exaggerated in order to highlight the ‘lagging’ of the free surface contact point. The profiles are shown for $-\frac{l}{2} \leq x \leq \frac{l}{2}$ and $-0.02 \leq z \leq 0.02$. This ‘lagging’ is apparent for the semi-noslip boundary conditions at low Reynolds numbers, $Re = 11.0736$, 110.736 , and as the Reynolds number is increased the amount of ‘lagging’ decreases and the profiles converge. The ‘lagging’ is presumed to be proportional to the size of the boundary layer and therefore the findings here are consistent with those of



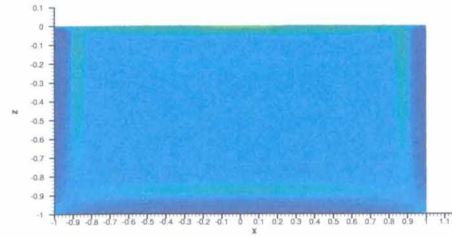
(a) $\tau = 2.66$



(b) $\tau = 3.99$



(c) $\tau = 5.32$



(d) $\tau = 6.64$

Figure 5.20: Vorticity contours of fluid flow when sloshing using the semi-noslip contact wall boundary conditions for $Re = 1107.36$. $\Theta_{min} = -0.14$
 $\Theta_{max} = 0.33$

Armenio [3] who found that the size of the boundary layer decreased for increasing Reynolds number. It can also be seen that although the semi-slip condition produces a significant boundary layer at the contact walls, there is little difference between the shape of the free surface profiles when the slip or semi-slip contact wall boundary condition is applied. Though it is implied by the time-histories in figure 5.17 that the difference would increase in time.

Modification of Time Period Due to Contact Wall Boundary Conditions

In section 5.7.1 it was found that the time period of the oscillatory motion of a free surface was affected by the presence of viscosity when the fluid is bounded by a

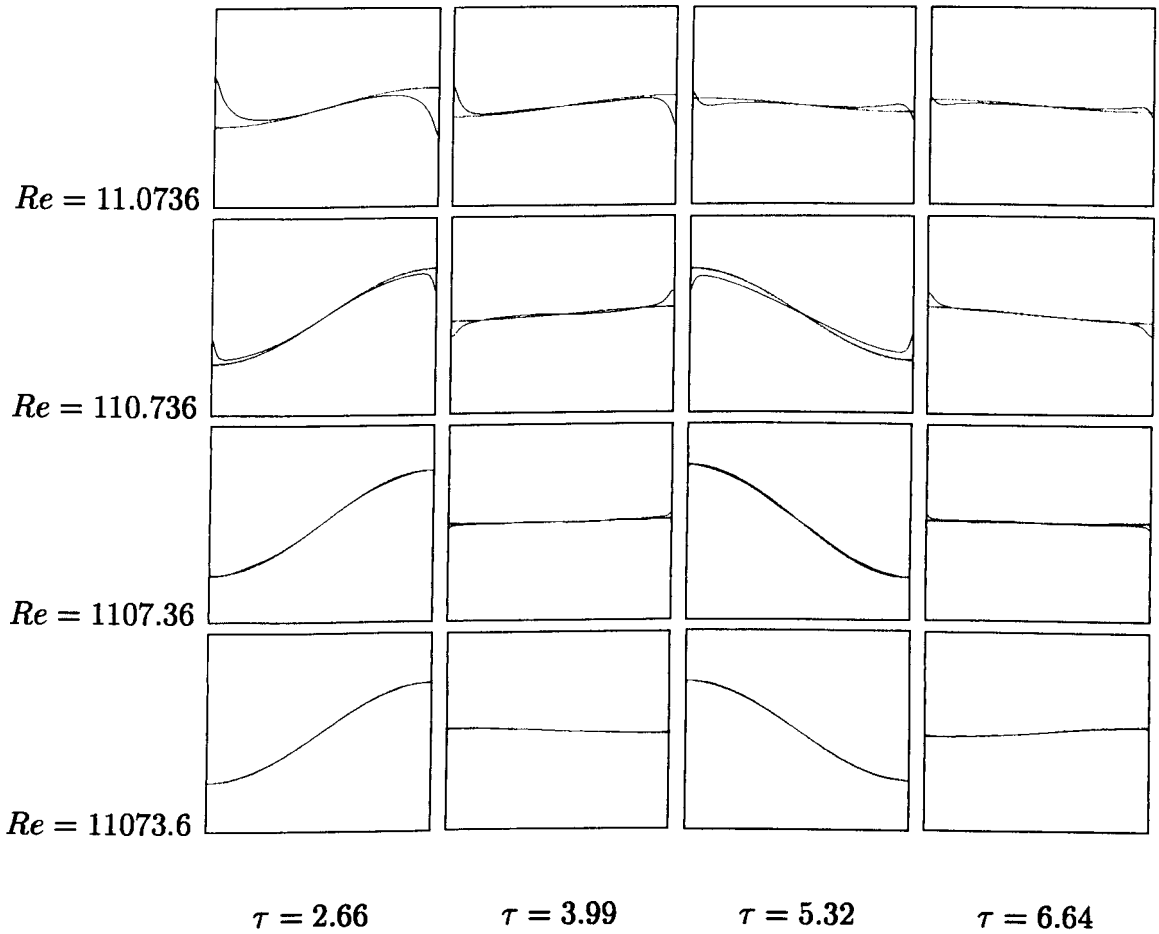


Figure 5.21: Snapshots of free surface displacement for standing wave motion using — Semi-noslip boundary conditions, - - - Semi-slip boundary conditions, . . . Slip boundary conditions

floor wall, where a decrease in the Reynolds number led to a slight increase in the time period. This effect is increased when the free surface fluid is fully contained by a floor wall and side contact walls. Table 5.10 contains values of the time period of the motion for varying Reynolds numbers and contact wall boundary conditions. The values of the time period are represented as a fraction of the linear, inviscid, irrotational time period evaluated using $\omega^2 = k \tanh k$. It can be seen that the magnitude of the Reynolds number has a substantial effect on the time period of the motion. The introduction of contact walls has an effect on the time period of

the motion depending on the type of boundary condition implemented, where the semi-noslip condition produces the largest effect, due to the ‘lagging’ of the free surface contact points seen in figure 5.24.

Boundary Condition	Type	$Re = 110.736$	$Re = 1107.36$	$Re = 11073.6$
Semi-noslip	T	5.43023	5.28070	5.27941
	$\frac{T}{T_0}$	1.03733	1.00876	1.00852
Semi-slip	T	5.33217	5.25625	5.25434
	$\frac{T}{T_0}$	1.01860	1.00401	1.00373
Slip	T	5.28069	5.24876	5.23946
	$\frac{T}{T_0}$	1.00876	1.00267	1.00089

Table 5.10: Comparison between the value of the linear, inviscid time period and the time period of a viscous, fully bounded fluid

Large Amplitude Sloshing

To fully appreciate the differences between the flow fields generated by the various contact wall boundary conditions, large amplitude standing waves are simulated. The non-linear free surface boundary condition is used for all calculations in this section. The initial conditions are the same as those for low amplitude sloshing described in the section above, except $a_0 = 0.1$. Therefore the initial conditions are,

$$u_x(x, z; 0) = 0, \quad (5.152)$$

$$u_z(x, z; 0) = 0, \quad (5.153)$$

$$\begin{aligned} \zeta(x; 0) = & -a_0 \sin(kx) - a_0 \left(\frac{1}{4}a_0 k\right) N_1 \cos(2kx) \\ & -a_0 \left(\frac{1}{4}a_0 k\right) N_2 \cos(2kx). \end{aligned} \quad (5.154)$$

where

$$N_1 = \frac{\cosh(2k)}{\sinh(2k)}, \quad (5.155)$$

$$N_2 = \frac{\cosh^2 k (\cosh(2k) + 2)}{\sinh^2(k) \sinh(2k)} \quad (5.156)$$

for $k = \frac{\pi}{l}$ and $l = 2$.

The semi-noslip contact wall boundary condition was found to cause an unstable solution for these larger amplitude cases. The extremely slow velocity of the contact point and the high values of the vorticity at the boundary of the slip/no-slip element produced a solution which was prone to overlapping of elements near the free surface. Figure 5.22 shows a snapshot of the fluid just prior to the solution ‘blowing-up’. The vorticity field and the boundaries of the elements are shown and the interaction of the free surface and the high vorticity region can be seen. All attempts at increasing the stability of the motion by producing a finer mesh and smaller timesteps failed and therefore the semi-noslip condition was not used to simulate any large amplitude standing waves. A possible course of action would have been to increase the size of the slip length, l_s , this was decided against as it would have led to user defined solutions for varying flow scenarios, resulting in the inability to accurately compare the fluid flow.

The time-histories for the elevation of the free surface at $x = -\frac{l}{2}$ are shown in figure 5.23 using the slip and semi-slip wall boundary conditions for increasing Reynolds number. The time-histories indicate nonlinear effects in the fluid motion. These nonlinearities are heavily damped in the low Reynolds number cases, $Re = 11.0736$ and $Re = 110.736$, and the solution appears linear even at large amplitudes. This is due to two reasons; a) the nonlinearities are at a higher frequency than the linear contributions and are therefore more heavily damped and b) the nonlinearities are proportional to the amplitude of the motion and as this decreases so the effect of the nonlinearities decrease. The gradual decline of the nonlinear effects can be seen in the elevation time-history for $Re = 1107.36$, where the decline is understandably faster when the semi-slip boundary condition is used as opposed to the slip boundary

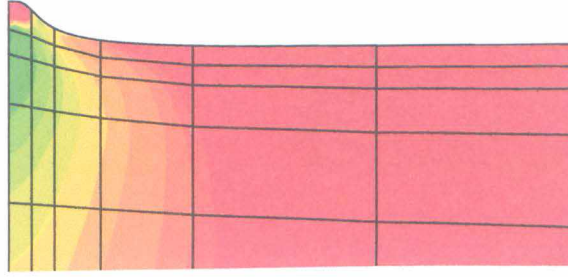
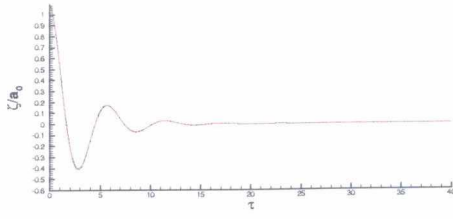


Figure 5.22: Fluid undergoing standing wave sloshing shortly before ‘blow-up’. Vorticity contours and the element boundaries are shown.

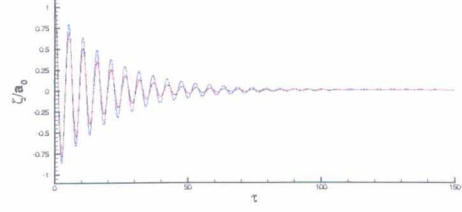
condition. The nonlinearities are more evident in the high Reynolds number case of $Re = 11073.6$ and continue to affect the solution after a high number of oscillations.

The free surface profiles contained in figure 5.24 illustrate the linearity of the low Reynolds number solution, $Re = 11.0736$. The profile appears sinusoidal in shape and has the same elevation at the peaks and troughs. As the Reynolds number is increased the profiles appear more nonlinear in nature; the peaks are higher than the troughs and the centre point is not stationary. The profiles are shown for $-\frac{l}{2} \leq x \leq \frac{l}{2}$ and $0.15 \leq z \leq 0.15$.

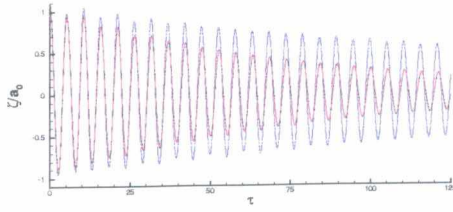
Figures 5.25 and 5.26 contain four flow field snapshots representing approximately one oscillation of the fluid motion for the various wall contact boundary conditions when $Re = 1107.36$. The vorticity layers generated by all the boundaries contain a higher level of vorticity than those generated by small amplitude sloshing due to the increased velocity of the fluid at the walls. The motion when using the slip boundary conditions produces shear layers at the free surface and floor wall which are generated and gradually diffuse as they travel towards the centre of the fluid, much the same as for periodic standing wave sloshing. When the semi-slip contact wall boundary condition is applied vorticity layers of the same sign interact



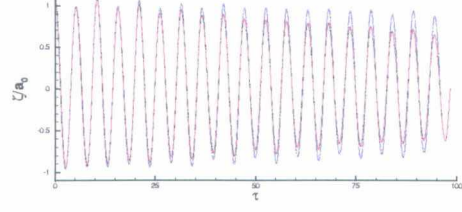
(a) $Re = 11.0736$



(b) $Re = 110.736$



(c) $Re = 1107.36$



(d) $Re = 11073.6$

Figure 5.23: Time history of free surface at $x = -\frac{l}{2}$ with increasing Reynolds number using — Semi-slip boundary conditions, — Slip boundary conditions.

and form a rectangular layer within the body of the fluid, this can be seen in figure 5.26 at $\tau = 3.99$ and $\tau = 6.64$.

5.8.2 Forced Oscillation Sloshing

In this section viscous free surface flow due to translational excitation is investigated. The bounding tank of the fluid is harmonically oscillated as a rigid body. The ensuing fluid motion for small amplitude oscillations is compared to the linear solution based on the viscous shallow water equations in section 5.5.2.

The amplitude of the motion is then increased in order to identify changes in the fluid flow patterns due to the implementation of the zero, linear and nonlin-

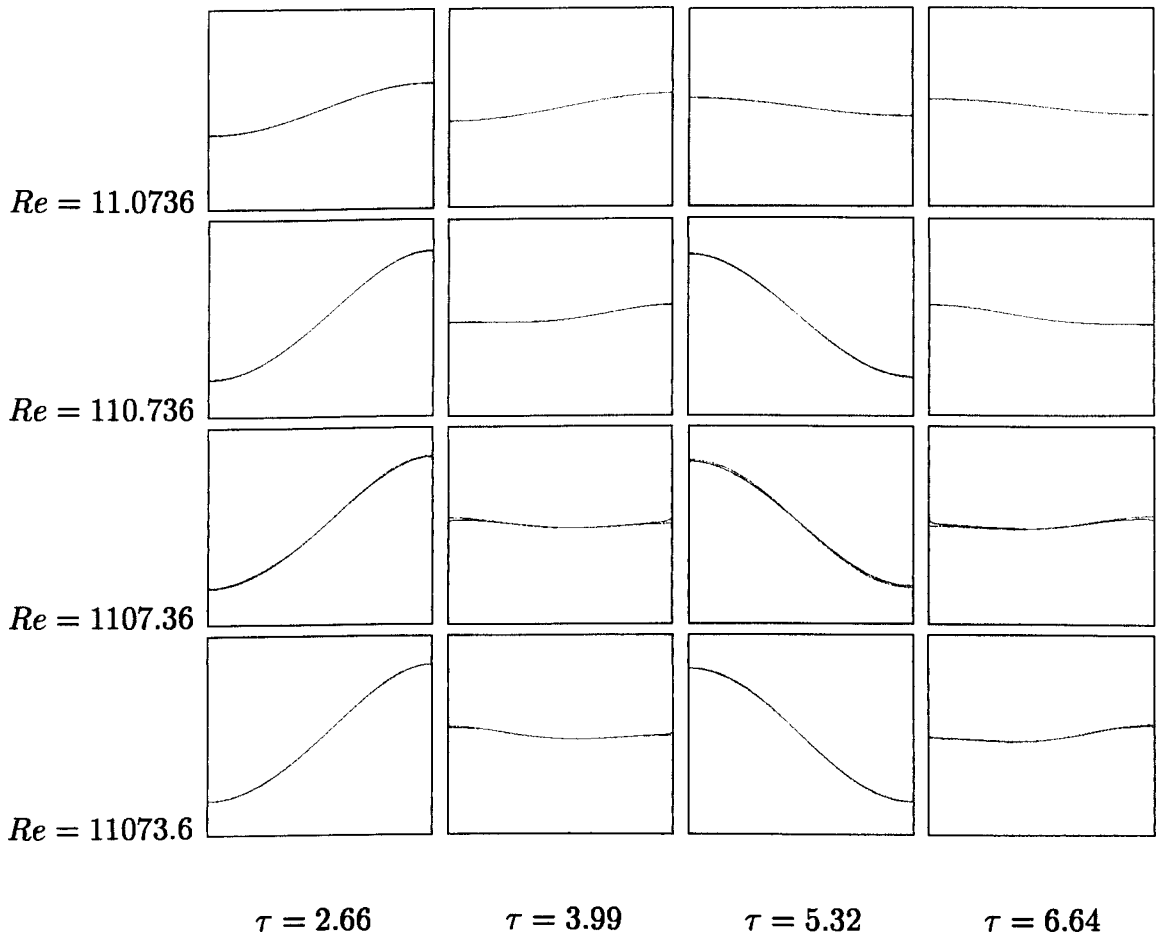
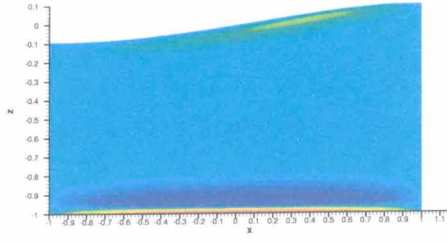


Figure 5.24: Snapshots of free surface displacement for large amplitude standing wave motion using — Semi-slip boundary conditions, - - Slip boundary conditions

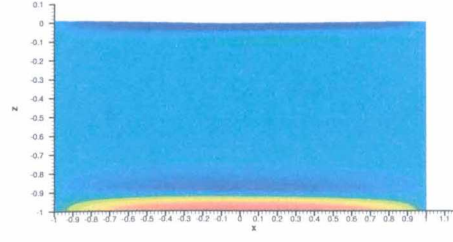
ear boundary conditions on the free surface and the slip, semi-slip and semi-noslip boundary conditions on the contact walls.

Computational and Analytical Comparison of Long Wave Forced Sloshing

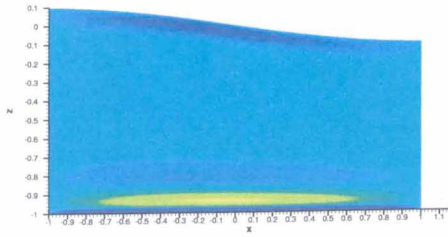
To ensure the validity of the analytical, linear solution of Lepelletier & Raitchlen [39] the following parameters were used to compare the computational and analytical



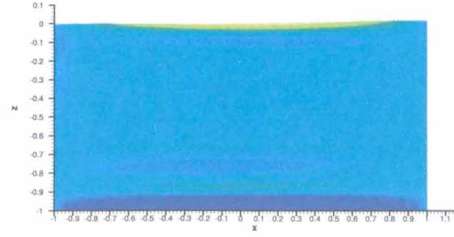
(a) $\tau = 2.66$



(b) $\tau = 3.99$



(c) $\tau = 5.32$



(d) $\tau = 6.64$

Figure 5.25: Vorticity contours of fluid flow when large amplitude standing wave sloshing using the slip contact wall boundary conditions. $\Theta_{min} = -0.57$ $\Theta_{max} = 1.9$

results:

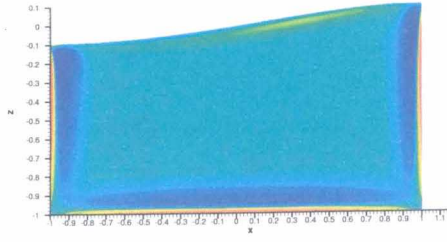
$$l = 10, \quad (5.157)$$

$$U_x = a_0 \cos(\omega_x \tau), \quad (5.158)$$

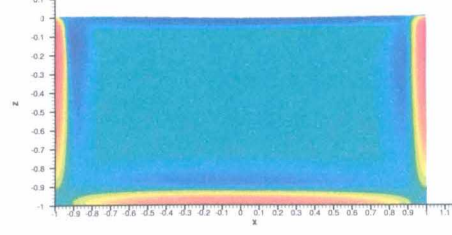
$$U_z = 0, \quad (5.159)$$

$$a_0 = 0.0025 \quad (5.160)$$

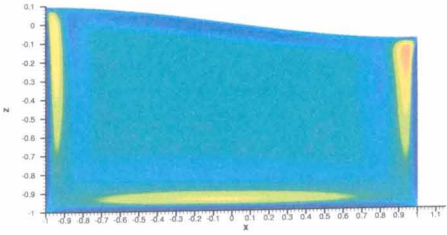
$$Re = 9904.54. \quad (5.161)$$



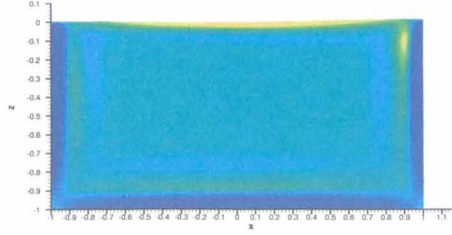
(a) $\tau = 2.66$



(b) $\tau = 3.99$



(c) $\tau = 5.32$



(d) $\tau = 6.64$

Figure 5.26: Vorticity contours of fluid flow when large amplitude standing wave sloshing using the semi-slip contact wall boundary conditions. $\Theta_{min} = -0.44$ $\Theta_{max} = 1.9$

and ω_x takes the values $\omega_x = 0.5\sigma_0$ and $\omega_x = 0.99\sigma_0$ where the first natural frequency, σ_0 is evaluated from equation (5.130). The initial conditions are therefore

$$u_x(x, z; 0) = u_z(x, z, 0) = 0, \quad (5.162)$$

$$\zeta(x; 0) = 0 \quad (5.163)$$

The solution was evaluated using an expansion basis of order 5, with $\tau = 9.9045 \times 10^{-5}$ and the computational mesh used comprised of 200 quadrilateral elements and is shown in figure 5.27.

Slip boundary conditions are implemented on the side walls as the dissipation term, γ_s , used by Lepelletier & Raitchlen [39] only accounts for viscous dissipation

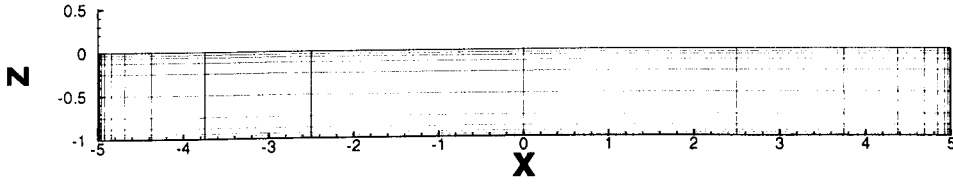


Figure 5.27: Mesh used for shallow water forced oscillation sloshing

from the floor wall. The computational and analytical time histories of the free surface elevation at $x = -\frac{l}{2}$ for $\omega_x = 0.5\sigma_0$, $\omega_x = 0.99\sigma_0$ are shown in figure 5.28.

Good agreement is seen between the analytical and computational solutions for $\omega_x = 0.5\sigma_0$ and initially for $\omega_x = 0.99\sigma_0$. As the amplitude of the free surface motion increases the computational results capture the nonlinear effects neglected by the analytical solution. These effects include those seen for inviscid flow such as an increase in amplitude at the peaks and troughs and an elongation of the time period of the oscillation.

Effect of Differing Boundary Conditions

To investigate the effect of the various boundary conditions on the free surface and the contact walls on the flow field, the bounded fluid is subjected to rigid body oscillations. The motion of the tank are described by,

$$X_t = a_x \sin(\omega_x \tau) , \quad (5.164)$$

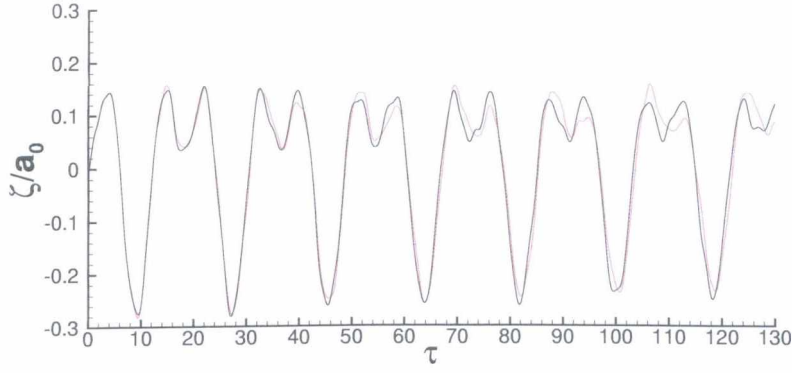
$$Z_t = a_z \sin(\omega_z \tau) . \quad (5.165)$$

and the initial conditions of the fluid are,

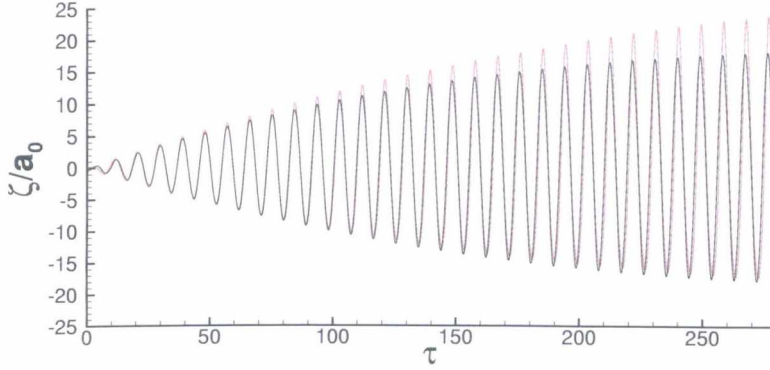
$$u_x(x, z; 0) = -a_x \omega_x , \quad (5.166)$$

$$u_z(x, z; 0) = -a_z \omega_z , \quad (5.167)$$

$$\zeta(x; 0) = 0 . \quad (5.168)$$



(a) $\omega_x = 0.5\sigma_0$



(b) $\omega_x = 0.99\sigma_0$

Figure 5.28: Time history of free surface elevation at $x = -\frac{l}{2}$: ——— analytical solution, ——— computational solution

Only horizontal motions of the tank are considered in this work i.e. $a_z = 0$.

A representative case is considered to investigate the effect of the various boundary conditions. This case is described by the parameters $a_x = 0.004$, $l = 2$ and $\omega_x = 0.938\omega_0$ where ω_0 is the linear, inviscid first natural frequency, such that $\omega_n^2 = k_n \tanh k_n$ and $k_n = \frac{(2n+1)\pi}{l}$ for $n = 0, 1, 2, \dots$. The mesh used for the simulations consisted of 140 quadrilateral elements and can be seen in figure 5.29. A

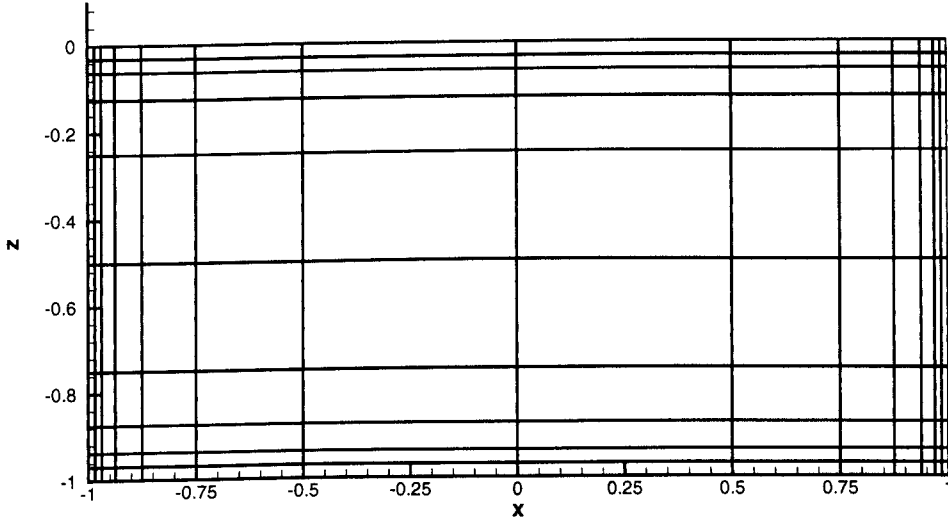


Figure 5.29: Mesh used for forced oscillation sloshing

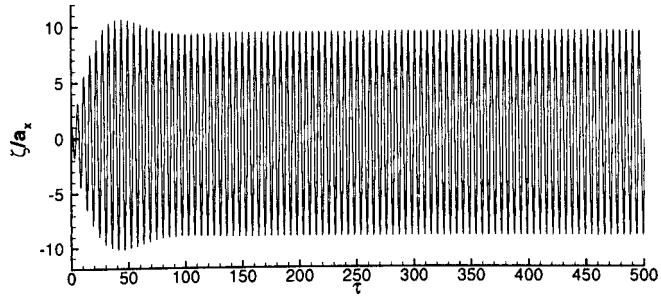
fifth order expansion basis and a time step, $\Delta\tau$, of 4.429×10^{-3} were used in the calculations.

The time-history of the displacement of the free surface at $x = -\frac{l}{2}$ for $Re = 110.736$ can be seen in figure 5.30 for differing free surface boundary conditions coupled with the semi-slip contact wall boundary condition and figure 5.31 for the slip and semi-slip wall boundary conditions coupled with the non-linear free surface boundary condition. The semi-noslip boundary condition was implemented and as in section 5.8.1, became unstable very quickly due to the large velocity gradients at the slip/no-slip boundary and slow moving contact point.

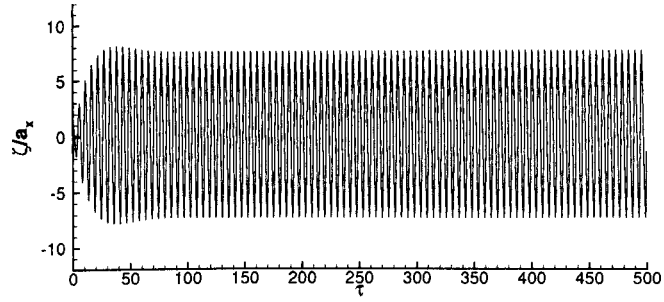
Initially all the natural frequencies are excited, as is the case for inviscid free surface forced sloshing. An amplitude modulated response is again evident with the motion mainly dependent on the driving frequency ω_x and the first natural frequency of the viscous fluid, ω_v , where the first natural frequency is close but not equal to ω_0 and is dependent on the Reynolds number, see sections 5.7.1 and 5.8.1. The

modulation wave envelope is defined by $2\Delta\omega \approx |\omega_v - \omega_x|$.

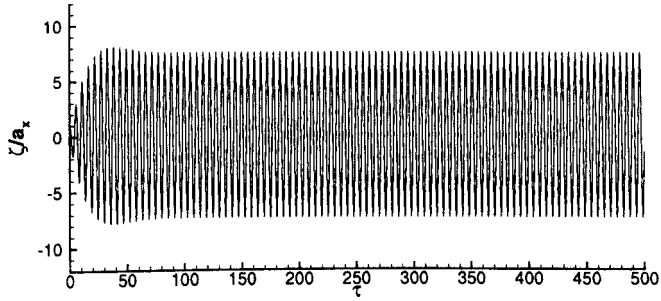
In contrast to the inviscid case, the motion due to the natural frequency is dissipated by the viscosity and therefore the segment in which the motion is affected by the natural frequencies is referred to as the transient stage of the motion. After the transient stage the motion reaches a time periodic state where the amplitude is constant. The rate at which the transients decay is dependent on the energy loss of the system due to viscosity, which has been shown to be dependent on the boundary conditions used on the free surface and the contact walls in the previous sections. This dependency on the boundary conditions can be seen in figure 5.30 where the transient stage is shortest for the non-linear and linear free surface boundary conditions. Furthermore it can be seen in figure 5.31 that the transient stage is shorter and the time periodic amplitude reduced when the semi-slip boundary condition is used when compared to the free surface elevation when using the slip boundary condition.



(a) Zero boundary conditions

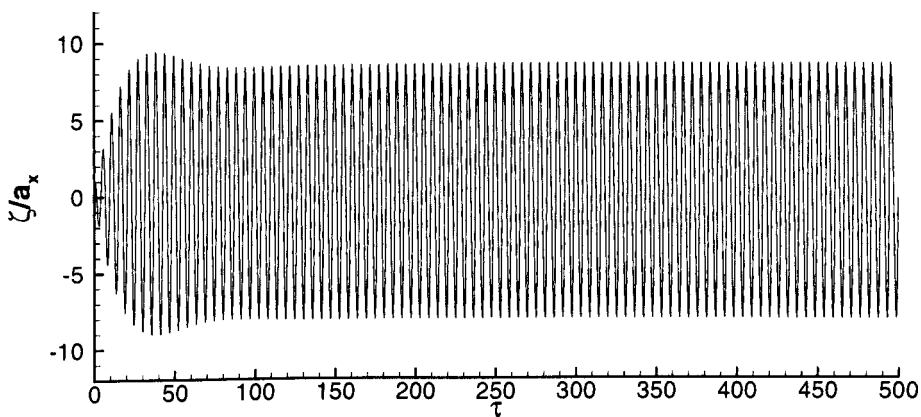


(b) Linear free surface boundary conditions

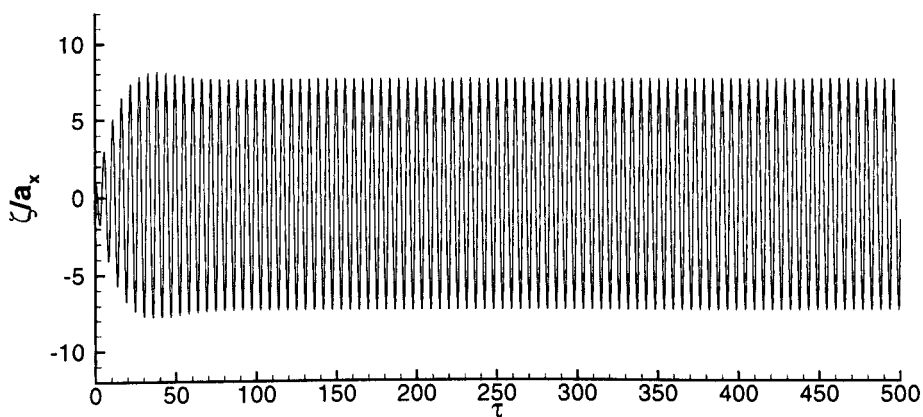


(c) Nonlinear free surface boundary conditions

Figure 5.30: Time history of free surface at $x = -\frac{l}{2}$ where $a_x = 0.004$, $\omega_x = 0.938\omega_0$ and $Re = 110.736$ for differing free surface boundary conditions



(a) Slip contact wall boundary conditions



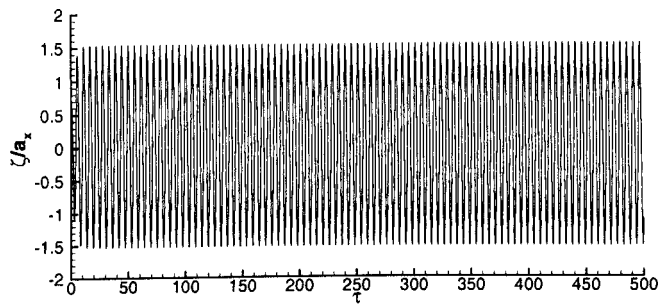
(b) Semi-slip contact wall boundary conditions

Figure 5.31: Time history of free surface at $x = -\frac{l}{2}$ where $a_x = 0.004$, $\omega_x = 0.938\omega_0$ and $Re = 110.736$ for differing contact wall boundary conditions

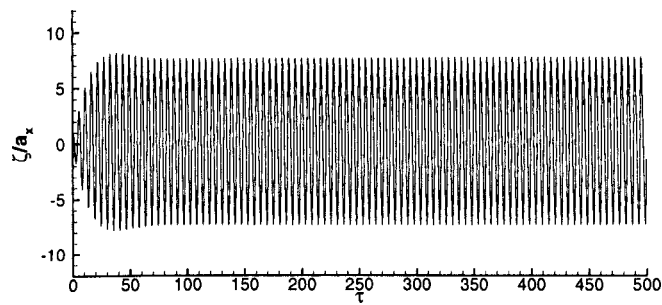
Effect of Increasing Reynolds Number

To investigate the effect of an increase in the Reynolds number on the motion of the fluid, three test cases were carried out for progressively increasing Reynolds number. The mesh configuration, motion of the tank and the initial conditions are identical to those used to investigate boundary condition effects. The time histories of the free surface displacement at $x = \frac{l}{2}$ for increasing Reynolds number are illustrated in figure 5.32. The dependency of the steady state amplitude, transient amplitude and the length of the transient stage on the Reynolds number can clearly be seen. The natural frequencies of the solution which were excited at start up are very quickly damped when $Re = 11.0736$, resulting in a steady state solution being attained before a ‘beating’ effect is initialised. The high amount of damping inherent in the solution for $Re = 11.0736$, leads to low wave amplitudes and results in a predominantly linear solution in terms of the respective heights of the peaks and troughs. For $Re = 110.736$ the solution undergoes one ‘beat’ and then reaches its steady state solution, whilst for $Re = 11073.6$ the transient stage is elongated. The duration of the transient stage is dependent on the damping of the system and therefore dependent on the Reynolds number of the flow.

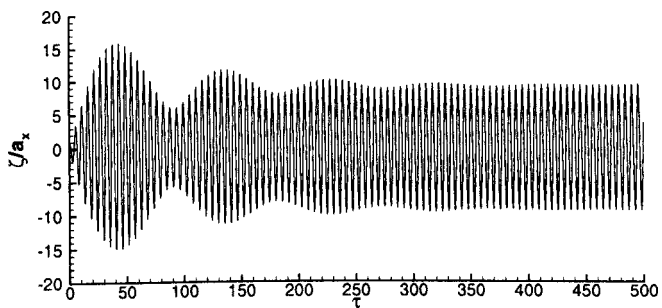
Figure 5.33 contains snapshots of the free surface at time steps of $\tau = 88.16$ beginning at $\tau = 0$. The time instant of the snapshots are highlighted by crosses on the displacement time history plot also shown in figure 5.33. The dependency of the oscillation time period on the Reynolds number can be seen from both the snapshots and the time history, as can the difference in the amplitudes of the cases. In the $Re = 11.0736$ case the transient stage is non-existent, the natural frequencies excited at start-up are almost immediately diffused.



(a) $Re = 11.0736$



(b) $Re = 110.736$



(c) $Re = 110.736$

Figure 5.32: Time history of free surface at $x = -\frac{l}{2}$ where $a_x = 0.004$, $\omega_x = 0.938\omega_0$ for differing Reynolds number

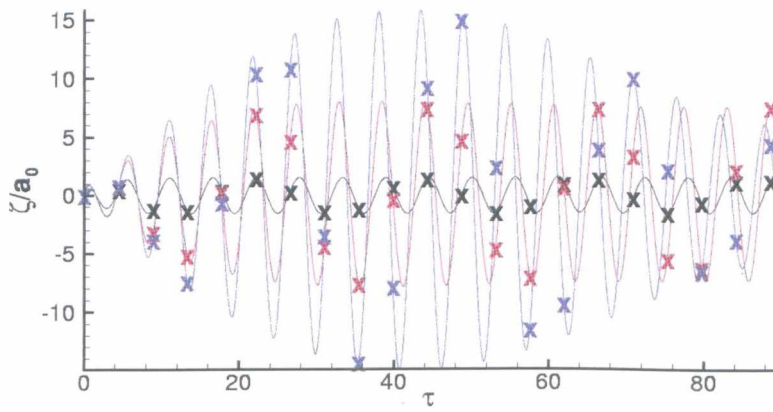
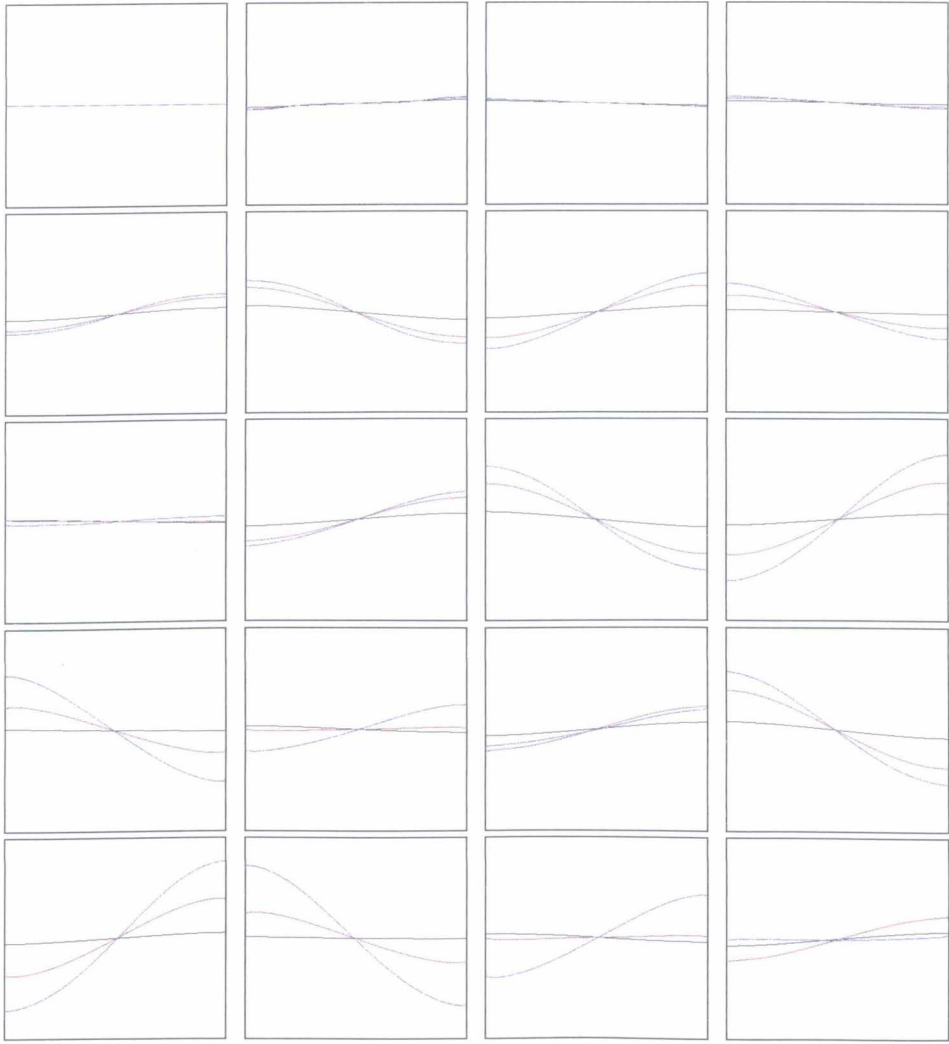


Figure 5.33: Snapshots of free surface displacement for fluid undergoing forced translational motion for — $Re = 11.0736$, — $Re = 110.736$, — $Re = 1107.36$ from $\tau = 0$ to $\tau = 84.16$

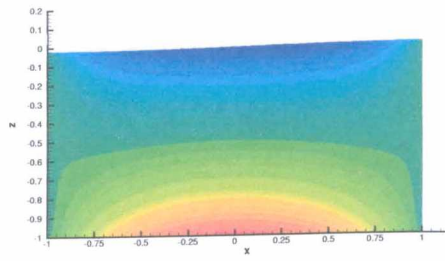
Boundary Layer Generation

Another test case is considered where the motion of the tank is now governed by $a_0 = 0.02$ and $\omega_x = 0.9$. The resulting flow field over approximately one period from $\tau = 23.92$ to $\tau = 27.91$ for $Re = 11.0736$, $Re = 110.736$ and $Re = 1107.36$ when using the non-linear free surface boundary condition and the semi-slip contact wall condition are shown in figures 5.34, 5.35 and 5.36 respectively. The same effect as seen for standing wave sloshing is produced where vorticity layers are generated at the boundaries and travel into the body of the fluid. Large differences in the size of the vorticity layers for different Reynolds number are apparent. For the $Re = 11.0736$ case the vorticity layers generated by the free surface and the floor wall almost completely fill the fluid, whereas at the lower Reynolds numbers the boundary layers are smaller in size. Though the semi-slip condition produces appreciable boundary layers for $Re = 110.736$ and $Re = 1107.36$, it fails to do so for the lowest Reynolds number case.

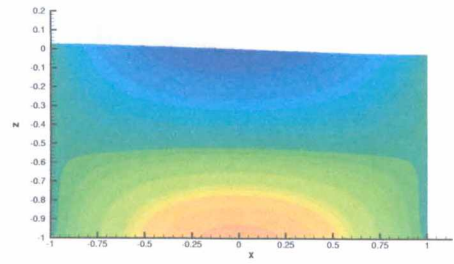
5.9 Summary

Free surface flows with viscosity have been numerically simulated and stable solutions have been achieved by using quadrilaterals to discretise the fluid domain. Good agreement has been seen between analytically derived and computationally produced decay rates of a free surface standing wave. The effect of differing free surface and contact wall boundary conditions on the flow field and free surface profiles has been investigated.

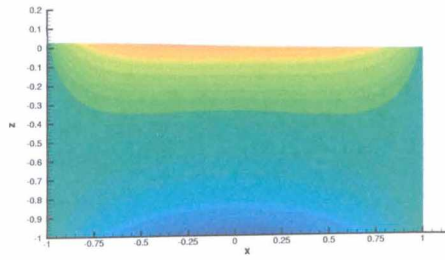
In the following chapter the governing equations for free surface flows with viscosity are extended to three dimensions.



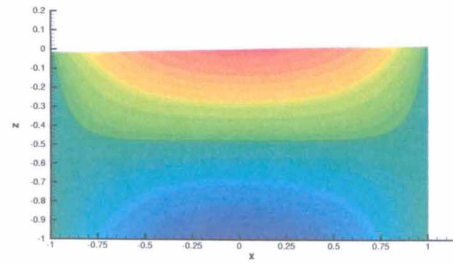
(a) $\tau = 23.92$



(b) $\tau = 25.25$



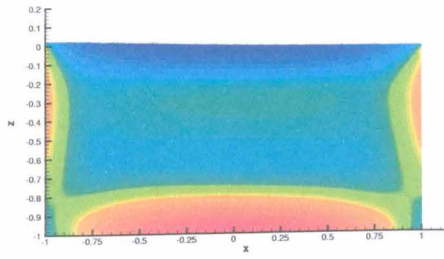
(c) $\tau = 26.58$



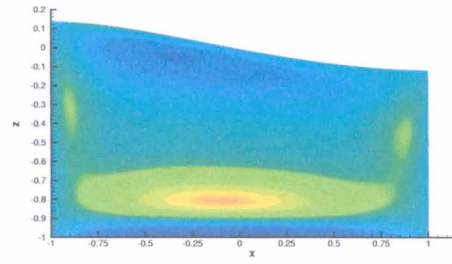
(d) $\tau = 28.34$

Figure 5.34: Fluid flow represented by vorticity contours for $Re = 11.0736$:

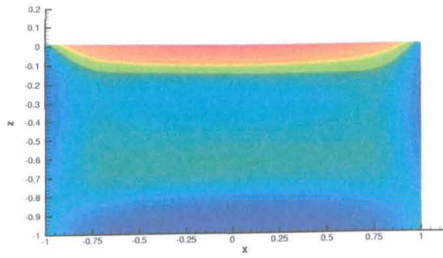
$$\Theta_{min} = -0.073 \quad \Theta_{max} = 0.086$$



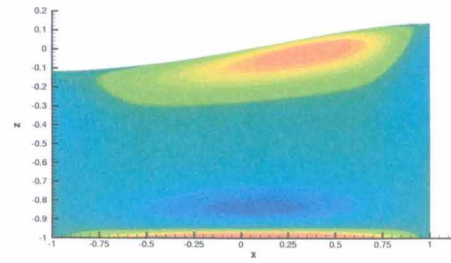
(a) $\tau = 23.92$



(b) $\tau = 25.25$



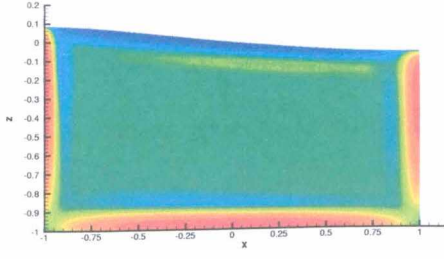
(c) $\tau = 26.58$



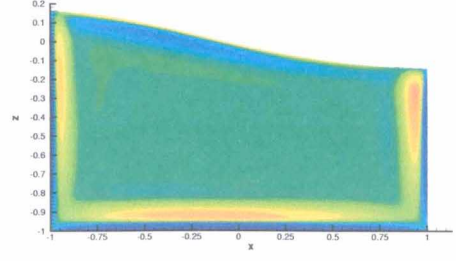
(d) $\tau = 28.34$

Figure 5.35: Fluid flow represented by vorticity contours for $Re = 110.736$:

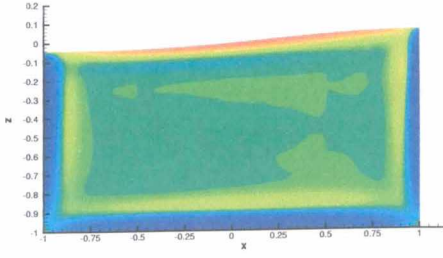
$$\Theta_{min} = -0.356 \quad \Theta_{max} = 0.481$$



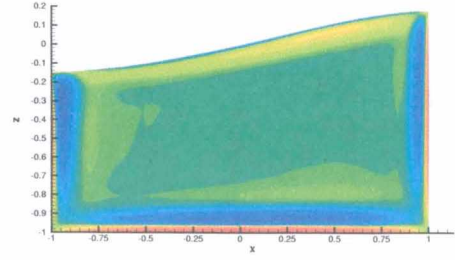
(a) $\tau = 23.92$



(b) $\tau = 25.25$



(c) $\tau = 26.58$



(d) $\tau = 28.34$

Figure 5.36: Fluid flow represented by vorticity contours for $Re = 1107.36$:

$$\Theta_{min} = -2.00 \quad \Theta_{max} = 2.00$$

Chapter 6

Three Dimensional Viscous Free-Surface Flow

6.1 Introduction

In this chapter the two dimensional analysis is extended to include a third dimension with a particular focus on forced sloshing in a tank. The computations were performed using a parallel version of the three dimensional ***NekTar*** code. A full explanation of the implementation is given by Evangelinos *et al.*[14].

6.1.1 3D Governing Equations

The basic dimensions of the tank are consistent with those used in two dimensional flow where the depth, length and breadth are denoted by d , l and b respectively. The non-dimensional arbitrary Lagrangian-Eulerian Navier-Stokes and continuity equations in three dimensions for fluid moving in a tank with velocity $\mathbf{U} = (U_x(\tau), U_y(\tau), U_z(\tau))$ are,

$$\frac{\partial \mathbf{u}}{\partial \tau} + ((\mathbf{u} - \mathbf{w}) \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} - \frac{d\mathbf{U}}{d\tau} + \mathbf{G}, \quad (6.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (6.2)$$

where $\mathbf{u}$ is the velocity of the fluid and has components,

$$\mathbf{u} = (u_x(x, y, z; \tau), u_y(x, y, z; \tau), u_z(x, y, z; \tau)), \quad (6.3)$$

$\mathbf{w}$ is the arbitrary velocity of the frame of reference and has components,

$$\mathbf{w} = (w_x(x, y, z; \tau), w_y(x, y, z; \tau), w_z(x, y, z; \tau)), \quad (6.4)$$

$p = p(x, y, z; t)$ is the pressure and $\mathbf{G} = (0, 0, -1)$. The free surface boundary conditions evaluated on $z(x, y, z; \tau) = \zeta(x, y, z; \tau)$ are the zero boundary conditions such that;

$$p = 0, \quad (6.5)$$

$$\frac{\partial u_x}{\partial n} = 0, \quad (6.6)$$

$$\frac{\partial u_y}{\partial n} = 0, \quad (6.7)$$

$$\frac{\partial u_z}{\partial n} = 0, \quad (6.8)$$

where $\mathbf{n}$ denotes the normal to the boundary. On the side walls $x = \frac{l}{2}$, $x = -\frac{l}{2}$, $y = \frac{b}{2}$, $y = -\frac{b}{2}$, a slip boundary condition is implemented,

$$u_x = 0, \quad (6.9)$$

$$u_y = 0, \quad (6.10)$$

$$\frac{\partial u_z}{\partial n} = 0, \quad (6.11)$$

and on the floor, $z = -1$, the no-slip condition is applied,

$$u_x = 0, \quad (6.12)$$

$$u_y = 0, \quad (6.13)$$

$$u_z = 0. \quad (6.14)$$

By limiting the motion of the arbitrary frame of reference to displacements in the z direction the kinematic boundary condition is,

$$u_z = w_z + u_x \frac{\partial \zeta}{\partial x} + u_y \frac{\partial \zeta}{\partial y}. \quad (6.15)$$

and the motion of the tank can be described by,

$$X_t = a_x \sin \omega_x \tau, \quad (6.16)$$

$$Y_t = a_y \sin \omega_y \tau, \quad (6.17)$$

$$Z_t = a_z \sin \omega_z \tau, \quad (6.18)$$

where,

$$\mathbf{X}_t = \mathbf{X}_t(X_t, Y_t, Z_t), \quad (6.19)$$

$$\mathbf{U} = \frac{d\mathbf{X}_t}{d\tau}. \quad (6.20)$$

The natural frequencies w_{mn} can be evaluated from the expression,

$$\omega_{mn}^2 = (k_m^2 + k_n^2)^{\frac{1}{2}} \tanh(k_m^2 + k_n^2)^{\frac{1}{2}} \quad (6.21)$$

where

$$k_m = \frac{m\pi}{l} \quad (6.22)$$

$$k_n = \frac{n\pi}{b} \quad (6.23)$$

for $m, n = 0, 1, 2, \dots$. The first natural frequency in the x and y directions are therefore $\omega_{10} = \frac{\pi}{l} \tanh \frac{\pi}{l}$ and $\omega_{01} = \frac{\pi}{b} \tanh \frac{\pi}{b}$ respectively.

6.1.2 Results for Forced Oscillations

The computational solutions were generated by evaluating the governing equations of motion for a fluid contained within a tank of size $l = 2$ and $b = 2$. The computational mesh is shown in figure 6.1 and contains 4000 prismic elements. The polynomial order of the expansion basis used was 4 and the time step $\Delta\tau = 4.429 \times 10^{-3}$. The Reynolds number for all the test cases considered is 110.736.

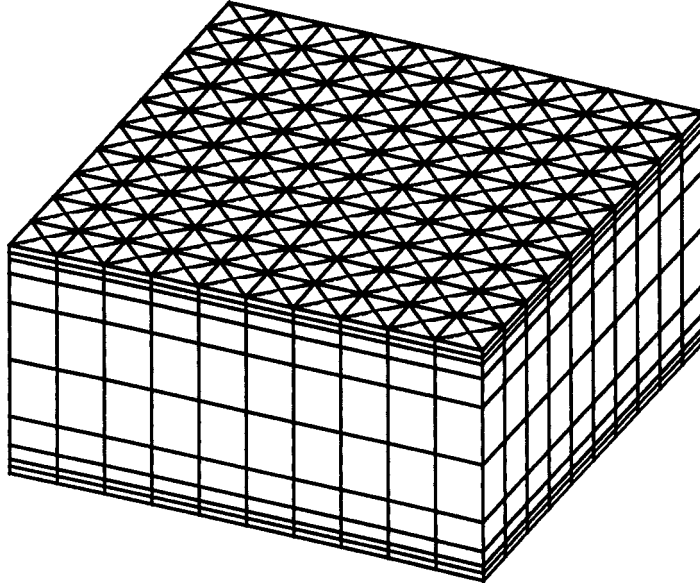


Figure 6.1: Computational mesh used for three dimensional viscous flow

Motion in One Horizontal Direction

The ensuing free surface motion due to uni-directional oscillations of the containing tank is simulated. The motion of the tank is described by,

$$X_t = a_x \sin \omega_x t, \quad (6.24)$$

$$Y_t = Z_t = 0 \quad (6.25)$$

where $a_x = 0.004$ and $\omega_x = 0.938\omega_{10}$ with initial conditions

$$\zeta(x, y; 0) = 0, \quad (6.26)$$

$$u_x(x, y, z; 0) = -a_x \omega_x, \quad (6.27)$$

$$u_y(x, y, z; 0) = 0, \quad (6.28)$$

$$u_z(x, y, z; 0) = 0. \quad (6.29)$$

The time histories of the displacement at $(\frac{l}{2}, \frac{b}{2})$ and $(\frac{l}{2}, 0)$ are shown in figure 6.2, where it is apparent that there exists a phase shift between the elevations at these points. The free surface displacement at $(\frac{l}{2}, 0)$ leads the displacement at $(\frac{l}{2}, \frac{b}{2})$.

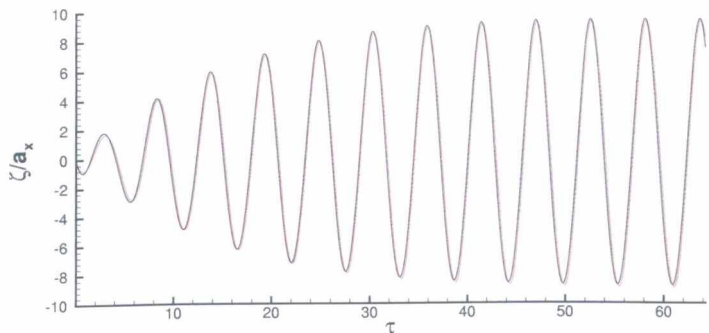
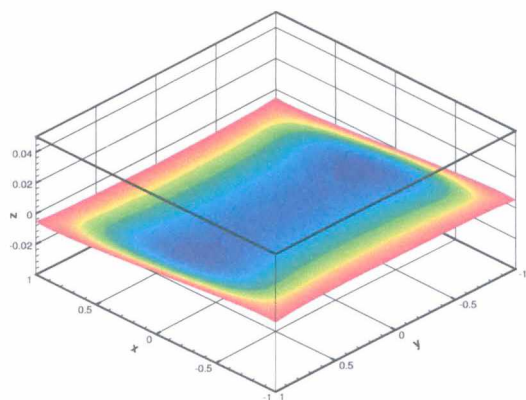
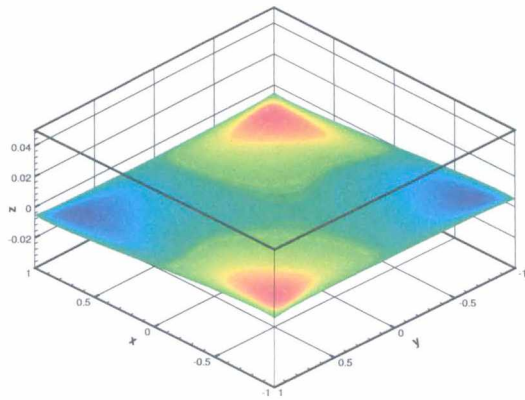


Figure 6.2: Displacement of free surface undergoing uni-directional forced oscillations at the points — ($\frac{l}{2}, \frac{b}{2}$), — ($\frac{l}{2}, 0$)

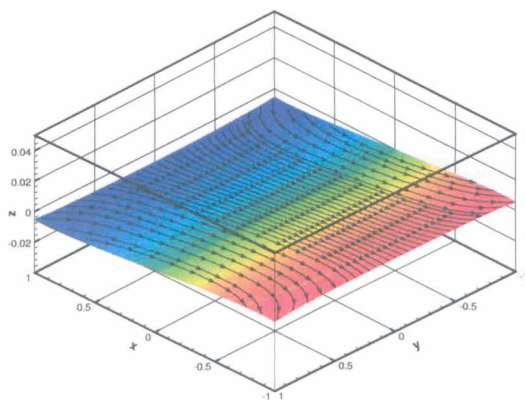
These features are due to the drag on the fluid being greater at the side walls than along the centreline plane $y = 0$. This causes a velocity gradient across the fluid in the y direction which causes an influx of fluid at the corners of the tank, raising the amplitude at these points when the free surface at the centre of the wall has begun to descend, after reaching its maximum elevation. Evidence of this is seen in figure 6.3(c), where instantaneous streamlines are shown, and in figures 6.4(b) and 6.4(c).



(a) u_x velocity contours: $\min = -4.12 \times 10^{-2}$ $\max = 0$

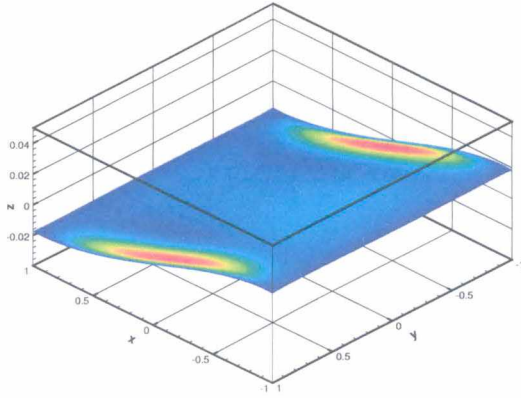


(b) u_y velocity contours: $\min = -5.39 \times 10^{-3}$ $\max = 5.38 \times 10^{-3}$

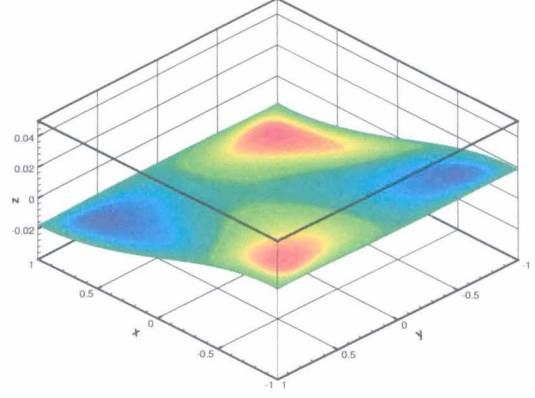


(c) u_z velocity contours: $\min = -3.73 \times 10^{-2}$ $\max = 3.90 \times 10^{-2}$

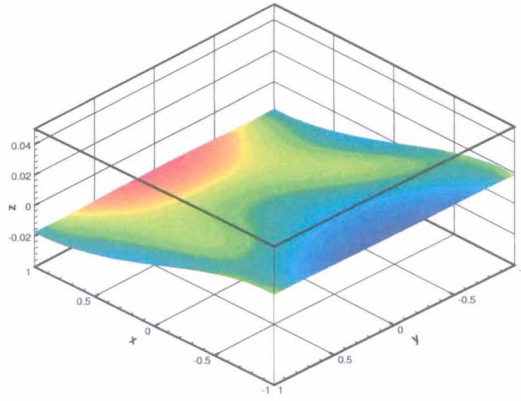
Figure 6.3: Free surface profiles showing velocity contours of fluid flow undergoing forced oscillations at $\tau = 37.65$



(a) u_x velocity contours: $\min = -1.58 \times 10^{-4}$ $\max = 1.38 \times 10^{-2}$



(b) u_y velocity contours: $\min = -3.86 \times 10^{-3}$ $\max = 3.80 \times 10^{-3}$



(c) u_z velocity contours: $\min = -4.79 \times 10^{-3}$ $\max = 4.79 \times 10^{-3}$

Figure 6.4: Free surface profiles showing velocity contours of fluid flow undergoing forced oscillations at $\tau = 44.29$

Motion in Two Horizontal Directions

The motion of the tank is now bi-directional, it is moving in the x and y directions where the oscillation of the tank is described by

$$X_t = a_x \sin \omega_x t, \quad (6.30)$$

$$Y_t = a_y \sin \omega_y t, \quad (6.31)$$

$$Z_t = 0, \quad (6.32)$$

where $a_x = a_y = 0.004$ and $\omega_x = 0.938\omega_{10}$ and $\omega_y = 0.938\omega_{01}$ with initial conditions

$$\zeta(x, y; 0) = 0, \quad (6.33)$$

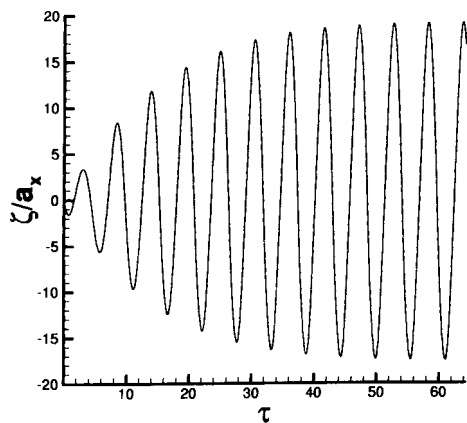
$$u_x(x, y, z; 0) = -a_x \omega_x, \quad (6.34)$$

$$u_y(x, y, z; 0) = -a_y \omega_y, \quad (6.35)$$

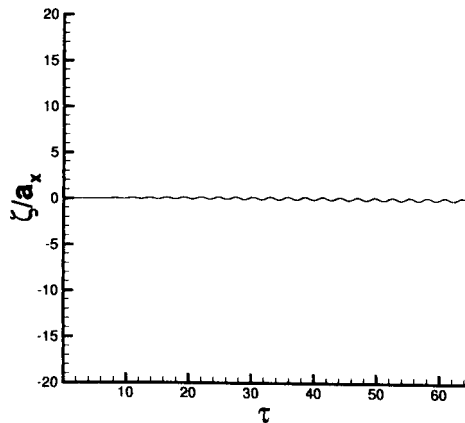
$$u_z(x, y, z; 0) = 0. \quad (6.36)$$

The time histories of the elevation of the free surface at the four corners of the tank are shown in figure 6.5, where it can be seen that the elevation at the two corners $(\frac{l}{2}, \frac{b}{2})$ and $(\frac{l}{2}, -\frac{b}{2})$ is substantially larger than that at the remaining corners. Furthermore the displacements at $(\frac{l}{2}, \frac{b}{2})$ and $(\frac{l}{2}, -\frac{b}{2})$ are π out of phase, whilst the elevation at $(-\frac{l}{2}, \frac{b}{2})$ and $(\frac{l}{2}, -\frac{b}{2})$ are in phase and oscillating at twice the frequency of the displacement at $(\frac{l}{2}, \frac{b}{2})$ and $(\frac{l}{2}, -\frac{b}{2})$. All these features are consistent with the results of Wu *et al.*[84] for free surface flow of an inviscid fluid undergoing similar forced oscillations.

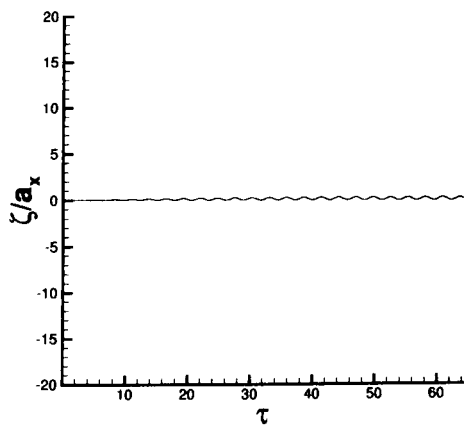
The free surface profiles at $\tau = 37.65$ and $\tau = 44.29$ are illustrated in figures 6.6 and 6.7, where figures depicting u_x and u_y velocities demonstrate the rotational symmetry of the solution. At $\tau = 44.29$ the motion of the fluid at the free surface is predominantly diagonal from the corner $(\frac{l}{2}, \frac{b}{2})$ to $(-\frac{l}{2}, -\frac{b}{2})$, illustrated by the instantaneous streamlines in figure 6.6(c). Two instantaneous streamline nodes occur at $\tau = 44.29$, shown in figure 6.7(c), where it is also apparent that the free surface



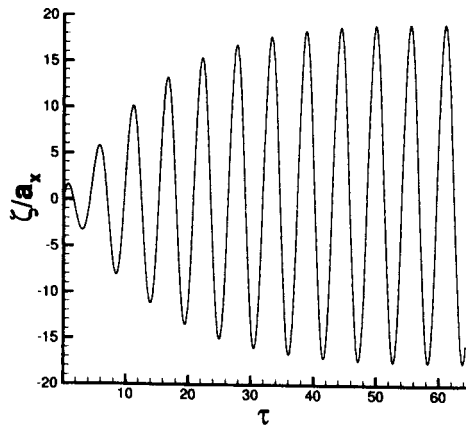
(a) $(\frac{l}{2}, \frac{b}{2})$



(b) $(\frac{l}{2}, -\frac{b}{2})$

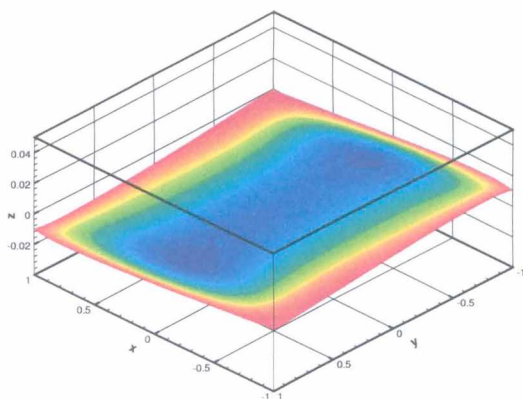


(c) $(-\frac{l}{2}, \frac{b}{2})$

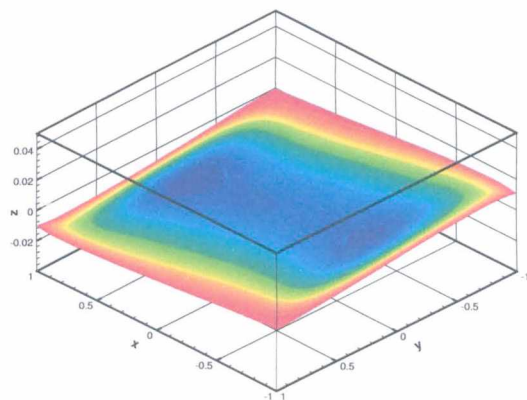


(d) $(-\frac{l}{2}, -\frac{b}{2})$

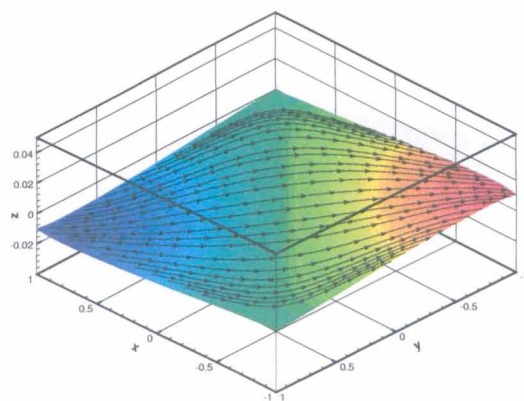
Figure 6.5: Free surface elevation at the four corners of the tank



(a) u_x velocity contours: $\min = -4.19 \times 10^{-2}$ $\max = 2.26 \times 10^{-4}$

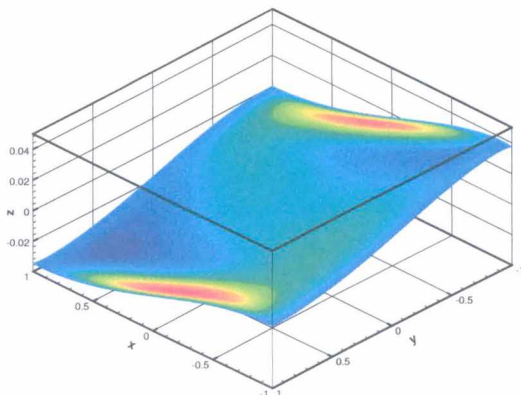


(b) u_y velocity contours: $\min = -4.19 \times 10^{-2}$ $\max = 2.30 \times 10^{-4}$

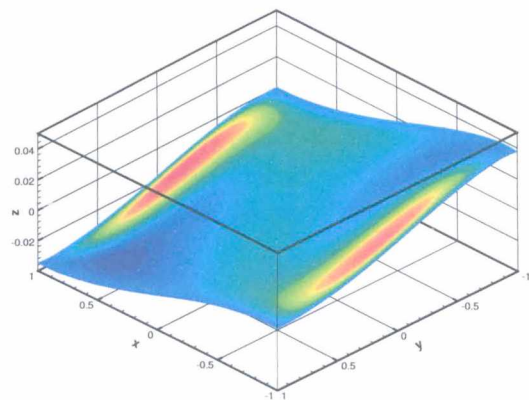


(c) u_z velocity contours: $\min = -7.47 \times 10^{-2}$ $\max = 7.91 \times 10^{-2}$

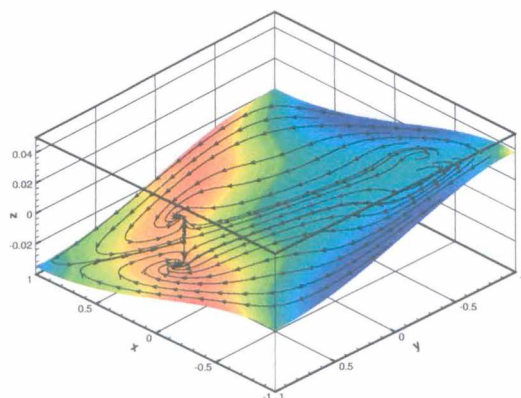
Figure 6.6: Free surface profiles showing velocity contours of fluid flow undergoing bi-directional forced oscillations at $\tau = 37.65$



(a) u_x velocity contours: $\min = -3.05 \times 10^{-3}$ $\max = 1.42 \times 10^{-2}$



(b) u_y velocity contours: $\min = -3.04 \times 10^{-3}$ $\max = 1.42 \times 10^{-2}$



(c) u_z velocity contours: $\min = -5.97 \times 10^{-3}$ $\max = 3.73 \times 10^{-3}$

Figure 6.7: Free surface profiles showing velocity contours of fluid flow undergoing bi-directional forced oscillations at $\tau = 44.29$

is ascending at corner $(-\frac{l}{2}, -\frac{b}{2})$ whilst descending along the adjoining walls $x = -\frac{l}{2}$ and $y = -\frac{b}{2}$, thus producing a phase shift between the free surface elevation at the point $(-\frac{l}{2}, -\frac{b}{2})$ and $(-\frac{l}{2}, 0)$ illustrated in figure 6.8.

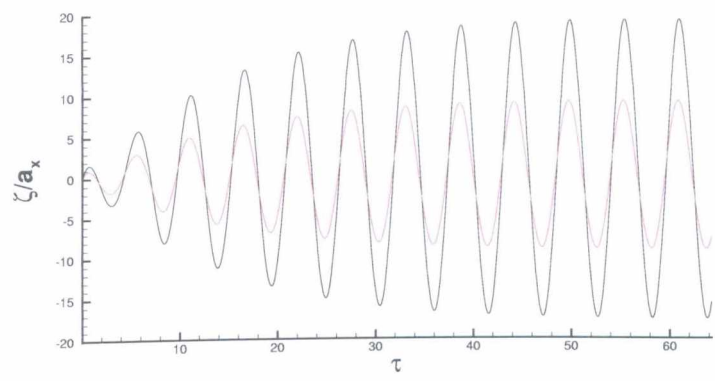


Figure 6.8: Displacement of free surface undergoing bi-directional forced oscillations at the points — $(-\frac{l}{2}, -\frac{B}{2})$, — $(-\frac{l}{2}, 0)$

6.2 Summary

Three dimensional effects of the fluid flow resulting from forced oscillations of a fluid with a free surface contained in a tank have been investigated in this chapter, for both uni-directional and bi-directional motion of the tanks. The boundary conditions used both on the free surface and the contact walls. To fully appreciate the effect of the extension to three dimensions the nonlinear free surface boundary condition and the semi-slip and semi-noslip contact wall boundary condition must be implemented.

Chapter 7

Conclusions and Further Work

7.1 Conclusions

The development and analysis of an *hp*/Spectral element technique has successfully been implemented for free surface flows with and without viscosity have been simulated with some success. The robustness of the scheme is dependent upon the correct choice of computational mesh configuration to reduce instabilities of the numerical solution. The formulation of the semi-discrete solution system for inviscid, irrotational free surface flow has enabled the determination of the criteria for stability in terms of the spatial discretisation of the fluid domain. It was found that a locally symmetric mesh produced eigenvalues indicating a stable solution for linear free surface boundary conditions. Locally symmetric meshes were therefore used in all areas of this research enabling stable simulations of nonlinear inviscid and viscous flows which need no artificial smoothing techniques, thus increasing the accuracy of and confidence in the results.

Where structured, symmetric meshes cannot be used, such as when discretising a complex geometry, procedures to stabilise the numerical solutions have been developed and analysed. The stabilisation procedures applicable to *hp*/Spectral elements are:

- the introduction of a diffusion term in the kinematic boundary condition,
- removal of high wavenumber modes.

The simulation of a standing wave using the linear free surface boundary conditions was found to contain minimal dissipation. The amplitude after long time integration showed no discernible decrease when compared to the initial conditions, thus indicating that the fast spatial convergence rates of the *hp*/Spectral element method and second order time integration ensured minimal temporal dissipation errors.

The numerical solutions of inviscid free surface flow using linear and nonlinear free surface boundary conditions were investigated and found to be in agreement with linear analytical solutions [16, 68, 84] for harmonic oscillations of the containing tank. Well known nonlinear effects of higher peaks than troughs and an elongation of the time period were observed for large amplitude nonlinear free surface motion.

When viscous flows were simulated the computational solutions of a free surface undergoing low amplitude oscillations due to the force of gravity were found to be in good agreement with the linear, analytical results of Landau & Lifshitz [37] for all Reynolds numbers tested ($Re \approx 0.1$ to $Re \approx 350000$). When the depth to length ratio of the containing tank was increased to limit the interaction of the generated vorticity layers, the difference between the numerical and analytical results was decreased, which is consistent with the assumptions of the analytical procedure.

The free surface was found to produce alternating vorticity layers which were advected into the body of the fluid. The sign and strength of the vorticity layers is dependent on the motion of the free surface and the Reynolds number. The generation of these vorticity layers was affected by the free surface boundary condition. The linear and nonlinear conditions produced stronger layers than the zero boundary condition and therefore had a more dissipative effect on the energy of the system, illustrated by the increased decay of the amplitude of the oscillations.

The introduction of side walls necessitated the implementation of various contact

wall boundary conditions, which were:

- the slip condition, where a no shear force premise is applied;
- the semi-slip condition, where the intermediate value of u_z is constrained;
- the semi-noslip condition, where a slip length, l_s is applied near the free surface.

These models led to large differences between the corresponding motion of a free surface undergoing free oscillations for low Reynolds number cases. Though as the Reynolds number increased the solutions converged in particular the semi-slip and semi-noslip condition, both of which gave an accurate prediction of the decay rate for $Re = 11073.6$ when compared to a linear, analytical solution [35, 37].

The semi-noslip condition produced appreciable boundary layers at the side walls for all Reynolds numbers investigated where the size of the boundary layer was inversely proportional to the Reynolds number, which is consistent with the results of Armenio [3]. The vorticity produced at the contact wall boundaries was advected away from the walls in a similar manner to the vorticity layers produced at the free surface. Due to stability problems the semi-noslip condition was not applied for large amplitude oscillations.

The side wall boundary layers promoted by the semi-slip condition for forced oscillations were found to be insignificant at low Reynolds numbers, $Re = 11.0763$, 110.763 , when compared to those generated by the floor and free surface. This is clearly physically inconsistent and a distinct disadvantage of this method. However, its advantage over the slip condition, which dictates a no shear force at the boundaries, is apparent when it is considered that the primary process of dissipation is due to boundary layer production on the bounding walls [35, 45].

The preliminary three-dimensional results using the zero free surface boundary conditions and slip side wall conditions were in qualitative agreement with the results for similar inviscid free surface flow produced by Wu *et al.*[84].

7.2 Further Work

The work presented here represents an initial attempt in the application of the *hp*/Spectral element method to simulate free surface flows. A major area of further study is the extension of the numerical algorithm to incorporate complex geometries into the domain, which would generate increased industrial applications. Separated flow past a surface piercing cylinder or vorticity generation of a body close to the free surface would be obvious areas of interest. Both of these scenarios would lead to more complicated free surface disturbances than were investigated in this work and also necessitate the use of unstructured grids. This could lead to numerical instabilities not seen when simulating viscous flow in this thesis and smoothing techniques may have to be incorporated. It is not certain that the formulated methods to promote stability for inviscid flow will be of use in these areas, though the addition of a diffusive, smoothing term to the kinematic boundary condition can easily be applied when using the ALE formulation. Other methods of numerical dissipation, such as spectral viscosity [32, 53] may be of use in this area.

The contact wall boundary conditions in this work were formulated to overcome a computational difficulty. They are not meant as an accurate physical representation of the nature of the fluid motion, but are an engineering solution to promote presumed characteristics of the flow. Much work needs to be done before a reliable and accurate model can be confidently, numerically implemented for all flow scenarios.

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