Imperial College London

Non-Ideal Compressible Fluid Formula Handbook

A Reference for Thermodynamics and Fluid Dynamics of Perfect Gases and Non-Ideal Fluids

by

Jinhong Wang

Department of Mechanical Engineering

Imperial College London

Preface

This handbook has been compiled with the aim of providing a clear, organised, and accessible collection of fundamental equations, assumptions, and modelling frameworks that underpin the study of thermodynamics and fluid dynamics for both perfect gases and non-ideal fluids.

The document systematically tidies up and clarifies many behaviours, derivations, and relationships that are frequently encountered in research and industrial practice but are often scattered across various textbooks and publications. Covering topics from basic assumptions and equations of state to compressible flows, turbulence, and turbomachinery, it serves as a concise yet comprehensive reference for students, researchers, and practising engineers alike.

Since this booklet serves as a formula handbook for quick equation retrieval and practical reference, explicit in-text citations are not always provided. The major theories and results compiled here draw primarily from the following foundational works:

- A. H. Shapiro, "The dynamics and thermodynamics of compressible fluid flow," *New York: Ronald Press*, 1953
- H. Tennekes and J. L. Lumley, *A First Course in Turbulence*. The MIT Press, 1972
- H. Schlichtiing and K. Gersten, *Boundary Layer Theory*. 1992, vol. 42, 307–311
- J. Blazek, *Computational Fluid Dynamics: Principles and Applications: Third Edition*. 2015, 1–447
- K. Masatsuka, "I do like CFD,," *Cradle*, vol. 1, 2013
- E. M. Greitzer *et al.*, *Internal Flow Concepts and Applications*. 2004
- F. M. White, *Viscous Fluid Flow 3rd Ed.* McGraw-Hill, 2006
- J. Anderson, *Modern Compressible Flow*. 1990
- W. H. Press *et al.*, *Numerical Recipes: The Art of Scientific Computing*
- A. Guardone *et al.*, "Nonideal compressible fluid dynamics of dense vapours and supercritical fluids," *Annual Review of Fluid Mechanics*, vol. AA, pp. 1–64, 2023

By making this work openly accessible, it is hoped that it will contribute to the wider scientific community, offering a reliable point of reference that can help reinforce understanding, reduce repetitive derivations, and support rigorous analysis in both teaching and advanced research.

I welcome constructive feedback and collaborative suggestions to further improve and expand this resource for the benefit of all who study or work in the fields of thermodynamics and fluid dynamics.

Jinhong Wang

July 2025

© 2025 Jinhong Wang
This document is distributed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License (CC BY-NC-SA 4.0).
It is intended to be freely available to the academic community for research and educational use.

Contents

Nomenclature **6**

1 Assumptions and Modelling **8**
1.1 Thermodynamics Assumptions 8
1.2 Types of Flows 8
1.2.1 Steady vs. Unsteady 8
1.2.2 Uniform vs. Non-uniform 8
1.2.3 Inviscid vs. Viscous 8
1.2.4 Laminar vs. Turbulent 8
1.2.5 Incompressible vs. Compressible . . 8
1.2.6 Irrotational and Rotational 8
1.3 Viscous Fluid Modelling 8
1.3.1 Newtonian Fluid 8
1.3.2 Generalised Newtonian Fluid 8
1.3.3 Other Non-Newtonian Fluid 8

2 Thermodynamics Basics **9**
2.1 Definition of Properties 9
2.2 Stagnation Properties 9
2.3 Fundamental Equations 9
2.4 Maxwell Relationships 9
2.5 Important Conservation Laws 9
2.6 Equation of State 9
2.6.1 Ideal Gas Law 9
2.6.2 Cubic Equation of State 9
2.6.3 Virial Equations of State 10
2.6.4 Multiparameter Equations of State . 10
2.7 Helmholtz Equations 10
2.8 Properties 11
2.8.1 Compressibility 11
2.8.2 Coefficient of Thermal Expansion . . 11
2.8.3 Heat Capacities 11
2.8.4 Viscosity 11
2.8.5 Fundamental Derivative 11
2.8.6 Grüneisen Parameter 11
2.9 Thermodynamic Relations 11
2.9.1 Specific Heat Relationship 11
2.9.2 Entropy Difference 12
2.9.3 Clausius-Clapeyron Relation 12
2.9.4 Joule–Thomson effect 12
2.9.5 Crocco's theorem 12

3 Fluid Dynamics Basics **13**
3.1 Reynolds transport theorem (RTT) 13
3.2 Conservation laws 13
3.3 Euler's Equation 13
3.4 Incompressible Navier-Stokes Equations . . 13
3.4.1 Conservation of mass 13
3.4.2 Conservation of momentum 13
3.5 Compressible Navier-Stokes Equations . . . 14
3.6 Entropy Transport Equation 14
3.7 Jacobian Matrices 14
3.7.1 Convective Flux Jacobian 14
3.7.2 Velocity Jacobian 15
3.8 Strain Tensors 15
3.9 Shear Stress Tensor 15
3.10 Dissipation Function 16
3.11 Vorticity 16
3.12 Circulation 16
3.13 Boundary Layer Thickness 16

4 Compressible Flow **17**
4.1 Governing Equations 17
4.1.1 Isentrope Equations 17
4.1.2 Isentropic Expansion Coefficient/Exponents 17
4.1.3 Speed of Sound 17
4.1.4 Compressibility factor 17
4.2 Generalised One-dimensional Flow 17
4.3 Perfect Gas vs. Non-Ideal Fluid Flow . . . 18
4.4 Quasi-1D Nozzle 19
4.4.1 General Analysis 19
4.4.2 Perfect vs. Non-Ideal Fluid Nozzle . 19
4.5 Shock 20
4.5.1 General Equations 20
4.5.2 Perfect vs. Non-Ideal Fluid Normal Shock Relations 20
4.5.3 Oblique Shock 20
4.6 Prandtl-Meyer Function 20

5 Turbulence **22**
5.1 Turbulence Scales 22
5.1.1 Integral Scale 22
5.1.2 Kolmogorov Microscales 22
5.1.3 Scale reations 22
5.2 Averaging Methods 22
5.2.1 Fluctuation 22
5.2.2 Reynolds (Time) Average 22
5.2.3 Favre Average 22
5.3 Turbulence Study Parameters 22
5.3.1 Skin Friction Coefficient 22
5.3.2 Correlation Coefficients 23
5.4 Vortex Identification Criteria 23
5.4.1 Q Criterion 23
5.4.2 Δ Criterion 23
5.4.3 λ_2 Criterion 23
5.5 Energy Spectrum of Turbulence 23
5.5.1 Turbulent Kinetic Energy 23
5.5.2 Dissipation of Turbulent Kinetic Energy 23
5.5.3 Energy Cascade 23
5.6 Tollmien–Schlichting Wave 23
5.7 Boussinesq (Eddy Viscosity) Approximation 23
5.8 Turbulent Stress Tensors 24
5.9 Reynolds-Averaged Incompressible Navier-Stokes Equation (RANS) 24
5.10 Favre-Averaged Compressible Navier-Stokes Equations 24
5.11 Transport Equation for Reynolds Stress . . 25
5.12 Transport Equation for Turbulent Kinetic Energy 25
5.13 Transport Equation for Vortices 25
5.14 Turbulence Modelling 25
5.14.1 Subgrid-scale (SGS) Modelling for LES 25
5.14.2 RANS Model 25
5.15 Turbulent Boundary Layer 26
5.15.1 Logarithmic Law of the Wall 26
5.15.2 Spalding's Wall Function 27
5.15.3 Wall Function Implementation in CFD 27
5.16 Boundary Layer with Compressibility . . . 27
5.16.1 Semi-local Values 27
5.16.2 Van Driest (VD) Transformation . . 27
5.16.3 Trettel Larsson (TL) Transformation 27
5.17 Walz's Equation 27

5.18 Morkovin Hypothesis . . . 27
5.19 Turbulent Prandtl Number . . . 28
5.20 Transition . . . 28
5.21 Entropy Generation . . . 28

6 Mixture **29**
6.1 Mixture Type . . . 29
6.1.1 Gibbs-Dalton's Law of Mixture . . . 29
6.1.2 Amagat's Law of Mixture . . . 29
6.2 Mixture Fractions . . . 29
6.2.1 Volume Fraction . . . 29
6.2.2 Mass Fraction . . . 29
6.2.3 Molar Fraction . . . 29
6.3 Two-phase Mixture . . . 29
6.3.1 Wet Vapour Fractions . . . 29
6.3.2 Wet Vapour Fraction Relationships . 29
6.3.3 Speed of Sound . . . 29

7 Turbomachinery **31**
7.1 Isentropic Efficiency . . . 31
7.2 Axial Compressor Analysis . . . 31
7.3 Turbomachinery Loss . . . 31
7.3.1 General Equations . . . 31
7.3.2 Loss mechanism . . . 31
7.3.3 Loss types . . . 32

8 Non-Dimensional Numbers **33**

Appendix A Mathematics **34**
A.1 Calculus Basics . . . 34
A.2 Partial Differential Equations Classification 34
A.3 Quasi-linear Partial Differential Equations Classification . . . 34
A.3.1 Method of Cramers Rule . . . 34
A.3.2 Eigenvalue Method . . . 34
A.4 Method of Characteristics . . . 34
A.5 Behaviours of Physical Governing Equations 35
A.5.1 Hyperbolic . . . 35
A.5.2 Parabolic . . . 35
A.5.3 Elliptic . . . 35
A.6 General Forms of Partial Differential Equations . . . 35
A.7 Physical Governing Partial Differential Equations . . . 35
A.7.1 Heat Equation . . . 35
A.8 Boundary Conditions . . . 35
A.9 Trigonometric formulas . . . 35
A.10 Index Notations . . . 36
A.10.1 Kronecker Delta . . . 36
A.10.2 Levi-Civita Symbol . . . 36
A.10.3 Others . . . 36
A.11 Tensor basics . . . 36
A.11.1 Tensor Construction . . . 36
A.11.2 Special Matrices . . . 37
A.11.3 Coordinate Transformation of Vectors 37
A.12 Tensor Operators . . . 37
A.12.1 Dot Operator . . . 37
A.12.2 Cross Operator . . . 37
A.12.3 Dyadic Operator . . . 37
A.12.4 Scalar Triple Product . . . 37
A.12.5 Vector Triple Product . . . 37
A.12.6 Gradient Operator . . . 38
A.12.7 Divergence Operator . . . 38
A.12.8 Curl Operator . . . 38
A.12.9 Laplacian Operator . . . 38
A.12.10 Inner Product . . . 38
A.12.11 Material derivative . . . 38
A.13 Theorems . . . 39
A.14 Linear Systems . . . 39
A.15 Eigenvalues and Eigenvectors . . . 39
A.16 Random Variables . . . 39
A.16.1 Expectation (μ) . . . 39
A.16.2 Variance (σ^2) . . . 39
A.16.3 Covariance . . . 39
A.17 Correlation Coefficient . . . 39
A.18 Probability Density . . . 40
A.18.1 Probability Density . . . 40
A.18.2 Joint Probability Density . . . 40
A.19 Fourier Transform . . . 40
A.20 Singular Value Decomposition (SVD) . . . 40
A.20.1 Full SVD . . . 40
A.20.2 Economy SVD . . . 40
A.20.3 Truncated SVD . . . 40
A.21 Proper Orthogonal Decomposition (POD) . 40

Appendix B Derivations **42**
B.1 Fundamental Equations . . . 42
B.2 Legendre Transformation . . . 42
B.2.1 Enthalpy . . . 42
B.2.2 Helmholtz Free Energy . . . 42
B.2.3 Gibbs Free Energy . . . 42
B.3 Helmholtz Equations . . . 42
B.3.1 First Helmholtz Equations . . . 42
B.3.2 Second Helmholtz Equations . . . 43
B.4 Navier-Stokes Equations . . . 43
B.4.1 Mass . . . 43
B.4.2 Momentum . . . 43
B.4.3 Energy . . . 44
B.5 1D Inviscid Navier-Stokes Equation Primitive Form . . . 45
B.5.1 Auxiliary Relations . . . 45
B.5.2 Jacobian . . . 46
B.5.3 Perfect Gas . . . 47
B.6 Speed of Sound . . . 47
B.7 Isentropic Relations . . . 48
B.7.1 Perfect Gas . . . 48
B.7.2 Non-Ideal Fluid . . . 48
B.8 Nozzle Analysis . . . 49
B.8.1 General Relations . . . 49
B.8.2 Area-Mach-Relations . . . 50
B.8.3 Mass flow rate . . . 50
B.8.4 Exit Mach number for isentropic flow 50
B.9 Normal Shock Relation . . . 51
B.9.1 General . . . 51
B.9.2 Perfect Gas . . . 51
B.9.3 Non-Ideal Fluid . . . 52
B.10 Prandtl-Meyer Function . . . 52
B.11 Loss Mechanism . . . 53
B.11.1 Boundary Layer Loss . . . 53
B.11.2 Constant Area Mixing Case . . . 54
B.11.3 Mixing Loss . . . 54
B.12 Turbulence . . . 55
B.12.1 Eddy Viscosity Models . . . 55
B.13 Mixture . . . 55
B.13.1 Mixture Properties . . . 55
B.13.2 Mixture Fraction Relations . . . 56
B.13.3 Wet Vapour Speed of Sound . . . 56
B.13.4 Relaxation Models . . . 57

Nomenclature

Romen Letters

$\overline{R}$	Universal ideal gas constant 8.3144621 [J K^{-1} mol^{-1}]
$\underline{\underline{D}}$	Deformation (strain) tensor
$\vec{F}$	Flux vector
$\vec{G}$	Body force vector [N]
$\vec{n}$	Normal vector
$\vec{W}$	State vector
A	Helmholtz free energy [J]
a, b, k	Local constants or parameters in the scheme
c	Speed of sound [m s^{-1}]
C_d	Dissipation coefficient
C_p	Constant-pressure heat capacity [J K^{-1}]
C_v	Constant-volume heat capacity [J K^{-1}]
C_{dis}	Discharge coefficient
C_{pb}	Base pressure coefficient
E	Internal Energy [J]
F	Force [N]
G	Gibbs free energy [J], Grüneisen Parameter
g	Clearance gap size [m]
H	Enthalpy [J K^{-1}]
J	Jacobian
k_t	Turbulent kinetic energy [J]
M	The Mach number, momentum [kg m s^{-1}]
m	Mass [kg]
m_s	Isentropic exponent for TP
N	Molar number [mol kg^{-1}]
n_s	Isentropic exponent for PV
P	Absolute pressure [Pa]
R	Ideal gas constant [J K^{-1} kg^{-1}]
S	Entropy [J K^{-1}]
T	Absolute temperature [K]
t	Time [s], Blade thickness [m]
U	Velocity [m s^{-1}]
V	Volume [m^3]
Z	Compressibility factor

Greek Letters

α	Thermal expansion coefficient, angle of attack [°], volume/void fraction
α_i	Absolute blade angle, subscript i refers to location in stage of a component [°]
β	Mach line angle [°], shock line angle [°]
β_i	Relative blade angle, subscript i refers to location in stage of a component [°]
χ	Mass fraction/vapour quality/dryness fraction
δ	Kronecker Delta, deviation angle [°]
ϵ	Levi-Civita symbol
η	Efficiency
Γ	Fundamental derivative
γ	Heat capacity ratio
κ	Compressibility, Von Kármán constant, isentropic expansion coefficient
Λ	Degree of reaction
μ	Dynamic viscosity [kg m^{-1} s^{-1}]
ν	Prandtle-Meyer function (angle) [°], kinematic viscosity [m^2 s^{-1}]
ω	The angular velocity of shaft [rad s^{-1}]
Φ	Mole fraction
ϕ	Flow coefficient, arbitrary property
ψ	Work coefficient
ρ	Density [kg m^{-3}]
σ	Total stress [Pa]
τ	Viscous (shear) stress [Pa]
θ	Boundary layer thickness [m], flow deflection angle [°]
$\underline{\underline{\Omega}}$	Spin tensor

Symbols

$\Im$	Impulse function [kg m s^{-2}]
∇	Nabla operator
$\otimes$	Dyadic product

Abbreviations

Bi	Biot number
Ek	Eckert number
Eu	Euler number
Fo	Fourier number
Gr	Grashof number
Nu	Nusselt number
Pr	Prandtl number

Re	Reynolds number
LCV	Low calorific value
SGS	Subgrid scale

Sub- and superscripts

$\dot{\{\ \}}$	Rate of change of the property
$\{\ \}'$	Fluctuation term in Reynolds averge
$\{\ \}''$	Fluctuation term in Favre averge
$\{\ \}^*$	Property at the throat of nozzle
$\{\ \}^*$	Semi-local values for tubulent boundary layer
$\{\ \}^+$	Reduced property
$\{\ \}^o$	Ideal-gas part of SW EOS
$\{\ \}^r$	Real-gas part of SW EOS
$\{\ \}^T$	Transpose of the tensor
$\{\ \}_0$	Stagnation property
$\{\ \}_1$	Property right before normal shock plane
$\{\ \}_2$	Property right after normal shock plane
$\{\ \}_c$	Property at critical point
$\{\ \}_e$	Property at the exit of nozzle
$\{\ \}_g$	Property of saturated vapour
$\{\ \}_i$	Property at the reference plane of nozzle
$\{\ \}_l$	Property of saturated liquid
$\{\ \}_m$	Property of mixture
$\{\ \}_p$	Property at the pressure surface
$\{\ \}_s$	Property at the suction surface
$\{\ \}_{ref}$	Reference property
$\hat{\{\ \}}$	Nondimensionlised property
$\overline{\{\ \}}$	Property divided by the molar number
$\overline{\{\ \}}$	Reynolds-averaged property
$\underline{\underline{\{\ \}}}$	Second order tensor
$\underline{\{\ \}}$	First order tensor (vector)
$\vec{\{\ \}}$	First order tensor (vector)
$\widetilde{\{\ \}}$	Favre-averaged property

Operators

$\Delta\{\ \}$	Difference, Laplace operator
$\frac{D\{\ \}}{Dt}$	Material derivative
$\nabla \cdot \{\ \}$	Divergence operator
$\nabla\{\ \}$	Gradient operator
$\nabla \times \{\ \}$	Curl operator
$\nabla^2\{\ \}$	Laplace operator
$\left(\frac{\partial\{\ \}}{\partial\{\ \}}\right)_X$	Partial differentiation holding property X as constant
$\mathrm{EOS}_{\{\ \}}$	Equation of state calculation of $\{\ \}$
$X(\{\ \})$	Property X is a function of *, equivalent to $X = f(\{\ \})$
$\{X, x\}$	X is the Extensive property and x is the corresponding specific property (divided by mass)

1 Assumptions and Modelling

1.1 Thermodynamics Assumptions

Thermally Perfect means ideal gas law $Pv = RT$ applies.

Calorically Perfect means heat capacity values C_p and C_v are constant.

	Perfect	Ideal	Non-ideal
EOS	$Pv = RT$	$Pv = RT$	$Pv = ZRT$
Z	1	1	$\neq 1$
C_p, C_v	Constant	$f(T)$	$f(P,T)$
R	$C_p - C_v$	$C_p - C_v$	$\neq C_p - C_v$

1.2 Types of Flows

1.2.1 Steady vs. Unsteady

Stready flows are without time variant terms.

1.2.2 Uniform vs. Non-uniform

Uniform flows' velocity at a given instant of time is the same at any point of the fluid and has the same magnitude in all directions, the flow depth does not change with distance as well.

Non-uniform flows' velocity changes at every instant for any point at a given time.

1.2.3 Inviscid vs. Viscous

Inviscid flow are with $\mu = 0$ and is governed by Euler's equations.

Viscous flow are with $\mu \neq 0$ and is governed by Navier-Stokes equations.

1.2.4 Laminar vs. Turbulent

Laminar Flow are with smooth movement to the fluid flow with parallel streamlines.

Turbulent flow is a type of flow in which the inertial forces of the fluid dominate and is generally seen to flow with a higher velocity at any given point and time.

1.2.5 Incompressible vs. Compressible

Incompressible can be assumed for liquid or flow with $M < 0.3$. The flow is solenoidal as $\nabla \cdot \vec{U} = 0$.

$$\begin{aligned} \rho &= Constant \\ \nabla \cdot \vec{U} &= 0 \end{aligned} \tag{1}$$

The energy equation is decoupled with pressure and velocity for incompressible flows.

Compressible flow is a more generic assumption:

$$\rho \neq Constant \tag{2}$$

1.2.6 Irrotational and Rotational

Irrotational flow $\nabla \times \vec{U} = 0$
Rotational flow $\nabla \times \vec{U} \neq 0$

1.3 Viscous Fluid Modelling

1.3.1 Newtonian Fluid

Viscous stresses arising from its flow are at every point linearly correlated to the local strain rate $\tau = \mu \frac{\partial u}{\partial y}$, and in matrix forms:

$$\underline{\underline{\tau}} = \underline{\underline{\mu}} \left(\nabla \vec{U} \right) \tag{3}$$

hence:

$$\begin{aligned} \underline{\underline{\tau}} &= \mu \left(\nabla \vec{U} + \left(\nabla \vec{U} \right)^T \right) - \frac{2}{3} \mu \left(\nabla \cdot \vec{U} \right) \underline{\underline{I}} \\ &= 2\mu \underline{\underline{D}} + \lambda \left(\nabla \cdot \vec{U} \right) \underline{\underline{I}} \end{aligned} \tag{4}$$

The total stress in fluid is

$$\underline{\underline{\sigma}} = \underline{\underline{\tau}} - P \underline{\underline{I}} \tag{5}$$

1.3.2 Generalised Newtonian Fluid

Although this type of fluid is non-Newtonian (i.e. non-linear) in nature, its constitutive equation is a generalised form of the Newtonian fluid.

$$\underline{\underline{\tau}} = \mu_{eff} \left(\dot{\gamma} \right) \dot{\gamma} \tag{6}$$

Power-law Fluid A.k.a. Ostwald–de Waele relationship:

$$\tau = K \left(\frac{\partial u}{\partial y} \right)^n \quad , \quad \mu_{eff} = K \left(\frac{\partial u}{\partial y} \right)^{n-1} \tag{7}$$

where K is flow consistency index and n is the behaviour index. For Newtonian fluids $n = 1$.

Cross Fluid

$$\mu_{eff} = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + (k\dot{\gamma})^n} \tag{8}$$

where $\dot{\gamma}$ is the shear rate, μ_0 and μ_∞ are zero- and infinite shear viscosity.

Carreau Fluid

$$\mu_{eff} = \mu_\infty + (\mu_0 - \mu_\infty) \left(1 + (\lambda\dot{\gamma})^2 \right)^{\frac{n-1}{2}} \tag{9}$$

where λ is characteristic time.

1.3.3 Other Non-Newtonian Fluid

A wide range of non-Newtonian fluid types exists: viscoelastic, timp-dependent viscosity (Rheopectic or Thixotropic), and non-Newtonian viscosity (shear thickening/thinning).

2 Thermodynamics Basics

2.1 Definition of Properties

Defining the thermodynamic properties:

$$\begin{aligned} E &= E(S,V) \\ S &= S(E,V) \\ T &= \left(\frac{\partial E}{\partial S}\right)_V \\ P &= -\left(\frac{\partial E}{\partial V}\right)_S \\ H &= E + PV \\ A &= E - TS \\ G &= H - TS \end{aligned} \tag{10}$$

2.2 Stagnation Properties

For specific total energy and stagnation enthalpy (not the same thermodynamic state):

$$\begin{aligned} e_0 &= e + \frac{u^2}{2} \\ h_0 &= h + \frac{u^2}{2} \\ &= e_0 + \frac{P}{\rho} \end{aligned} \tag{11}$$

For ideal gas:

$$\begin{aligned} e &= C_v T \\ e_0 &= C_v T_0 \\ h &= C_p T \\ h_0 &= C_p T_0 \end{aligned} \tag{12}$$

with manipulation:

$$\begin{aligned} e &= \frac{R}{\gamma-1}T = \frac{1}{\gamma-1}\frac{P}{\rho Z} \\ h &= \frac{\gamma R}{\gamma-1}T = \frac{\gamma}{\gamma-1}\frac{P}{\rho Z} \\ &= \frac{c^2}{(\gamma-1)\,Z} \\ e_0 &= \frac{1}{\gamma-1}\frac{P}{\rho Z} + \frac{u^2}{2} \\ h_0 &= \frac{\gamma}{\gamma-1}\frac{P}{\rho Z} + \frac{u^2}{2} \end{aligned} \tag{13}$$

2.3 Fundamental Equations

Derivatives of internal energy, enthalpy, Helmholtz and Gibbs free energy.

$$\begin{aligned} dE &= T\mathrm{d}S - P\mathrm{d}V \\ dH &= T\mathrm{d}S + V\mathrm{d}P \\ dA &= -S\mathrm{d}T - P\mathrm{d}V \\ dG &= -S\mathrm{d}T + V\mathrm{d}P \end{aligned} \tag{14}$$

The Gibbs equations:

$$\begin{aligned} \mathrm{d}e &= T\mathrm{d}s - P\mathrm{d}v \\ \mathrm{d}h &= T\mathrm{d}s + v\mathrm{d}P \end{aligned} \tag{15}$$

2.4 Maxwell Relationships

From the fundamental equations:

$$\begin{aligned} -\left(\frac{\partial T}{\partial V}\right)_S &= \left(\frac{\partial P}{\partial S}\right)_V \\ \left(\frac{\partial T}{\partial P}\right)_S &= \left(\frac{\partial V}{\partial S}\right)_P \\ \left(\frac{\partial S}{\partial V}\right)_T &= \left(\frac{\partial P}{\partial T}\right)_V \\ -\left(\frac{\partial S}{\partial P}\right)_T &= \left(\frac{\partial V}{\partial T}\right)_P \end{aligned} \tag{16}$$

2.5 Important Conservation Laws

First Law of Thermodynamics

$$\mathrm{d}Q - \mathrm{d}W = \mathrm{d}E \tag{17}$$

where for quasi-equilibrium process:

$$\mathrm{d}W = P\mathrm{d}V \tag{18}$$

Steady Flow Energy Equation

$$\begin{aligned} \dot{Q} - \dot{W} = \quad & \dot{m}_2\left(h + \frac{1}{2}u^2 + gz\right)_2 \\ & - \dot{m}_1\left(h + \frac{1}{2}u^2 + gz\right)_1 \end{aligned} \tag{19}$$

Bernoulli Equation Valid for incompressible flows only

$$P_1 + \frac{1}{2}\rho u_1^2 + \rho g z_1 = P_2 + \frac{1}{2}\rho u_2^2 + \rho g z_2 \tag{20}$$

2.6 Equation of State

For consistency, knowing all is important.

$$\begin{aligned} T &= T(S,V,N_1,...,N_m); \\ P &= P(S,V,N_1,...,N_m); \\ \mu &= \mu(S,V,N_1,...,N_m); \end{aligned} \tag{21}$$

General form of EOS:

$$f(P,V,T) = 0 \tag{22}$$

2.6.1 Ideal Gas Law

Thermally perfect gas:

$$PV = N\overline{R}T = RT \tag{23}$$

2.6.2 Cubic Equation of State

General form:

$$P = \frac{RT}{V-b+c} - \frac{\alpha}{V^2+\delta V+\epsilon} \tag{24}$$

where α, b, c, δ are given for different models.

Noble-Able gas

$$P\,(V-b) = NRT \tag{25}$$

Van der Waals gas [11]

$$\left(P+\frac{a}{\overline{V}^2}\right)\left(\overline{V}-b\right)=RT \tag{26}$$

where

$$\begin{aligned} a&=3P_c*\overline{V}_c^2\\ b&=\frac{\overline{V}_c}{3}\end{aligned} \tag{27}$$

which is also written as:

$$\begin{aligned} a&=\frac{27\left(RT_c\right)^2}{64P_c}\\ b&=\frac{RT_c}{8P_c}\end{aligned} \tag{28}$$

Redlich-Kwong [12]

$$P=\frac{RT}{\overline{V}-b}-\frac{a}{\sqrt{T}\,\overline{V}\left(\overline{V}+b\right)} \tag{29}$$

where:

$$\begin{aligned} a&\approx 0.45724\frac{R^2T_c^2}{P_c}\\ b&\approx 0.07780\frac{RT_c}{P_c}\end{aligned} \tag{30}$$

Peng-Robinson [13]

$$P=\frac{RT}{\overline{V}-b}-\frac{a\alpha}{\overline{V}^2+2b\overline{V}-b^2} \tag{31}$$

where

$$\begin{aligned} a&\approx 0.42748\frac{R^2T_c^{\frac{5}{2}}}{P_c}\\ b&\approx 0.08664\frac{RT_c}{P_c}\\ \alpha&=\left(1+k\left(1-T_r^{\frac{1}{2}}\right)\right)^2\\ k&\approx 0.37464+1.54226\omega-0.26992\omega^2\\ T_r&=\frac{T}{T_c}\end{aligned} \tag{32}$$

2.6.3 Virial Equations of State

Compressibility factor Z is fitted as polynomials [14]:

$$Z=\frac{PV}{RT}=A+\frac{B}{V}+\frac{C}{V^2}+\frac{D}{V^3}+... \tag{33}$$

2.6.4 Multiparameter Equations of State

Such equations are usually formulated using Helmholtz energy A:

Span-Wagner Equation of State [15]

$$\phi\left(\delta,\tau\right)=\phi^o\left(\delta,\tau\right)+\phi^r\left(\delta,\tau\right) \tag{34}$$

where

$$\begin{cases}\phi=\dfrac{A}{RT} & \text{Reduced Helmholtz energy}\\ \tau=\dfrac{T_c}{T} & \text{Reduced Temperature}\\ \delta=\dfrac{\rho}{\rho_c} & \text{Reduced Density}\end{cases} \tag{35}$$

and superscript o and r mean ideal and residual fluid behaviour, respectively.

The ideal-gas part:

$$\begin{aligned}\phi^o\left(\delta,\tau\right)=&\frac{h^o_{ref}}{RT_c}\tau-\frac{s^o_{ref}}{R}-1+\ln\frac{\delta\tau_{ref}}{\delta_{ref}\tau}\\&-\frac{\tau}{R}\int_{\tau_{ref}}^{\tau}\frac{c^o_p}{\tau^2}\mathrm{d}\tau+\frac{1}{R}\int_{\tau_{ref}}^{\tau}\frac{c^o_p}{\tau}\mathrm{d}\tau\end{aligned} \tag{36}$$

where ideal-gas heat capacity is

$$\frac{c^o_p}{R}=4+\sum_{i=1}^{2}v_i\left(\frac{u_i}{T}\right)^2\frac{e^{\frac{u_i}{T}}}{\left(e^{\frac{u_i}{T}}-1\right)^2} \tag{37}$$

and u_i and v_i are coefficients.

The real part:

$$\begin{aligned}\phi^r\left(\delta,\tau\right)=&\sum_{k=1}^{I_{pol}}N_k\tau^{t_k}\delta^{d_k}+\sum_{k=1+I+pol}^{I_{pol}+I_{exp}}N_k\tau^{t_k}\delta^{d_k}e^{-\delta^{l_k}}\\&+\sum_{k=1+I+pol+I_{exp}}^{I_{pol}+I_{exp}+I_{sph}}N_k\tau^{t_k}\delta^{d_k}e^{-\eta_k(\delta-\varepsilon_k)^2-\beta_k(\tau-\gamma_k)^2}\end{aligned} \tag{38}$$

where all parameters are empirically fitted fluid-related values.

Hence:

$$\begin{aligned}\frac{P\left(\delta,\tau\right)}{\rho RT}&=1+\delta\frac{\partial\phi^r}{\partial\delta}\\ \frac{s\left(\delta,\tau\right)}{R}&=\tau\frac{\partial\phi}{\partial\tau}-\phi\\ \frac{h\left(\delta,\tau\right)}{RT}&=1+\tau\frac{\partial\phi}{\partial\tau}+\delta\frac{\partial\phi^r}{\partial\delta}\\ \frac{e\left(\delta,\tau\right)}{RT}&=\tau\frac{\partial\phi}{\partial\tau}\\ \frac{c\left(\delta,\tau\right)^2}{RT}&=1+2\delta\frac{\partial\phi^r}{\partial\delta}+\delta^2\frac{\partial^2\alpha^r}{\partial\delta^2}\\&\quad-\frac{\left(1+\delta\dfrac{\partial\phi^r}{\partial\delta}-\delta\tau\dfrac{\partial^2\alpha^r}{\partial\tau\partial\delta}\right)^2}{\tau^2\left(\dfrac{\partial^2\alpha^o}{\partial\tau^2}+\dfrac{\partial^2\alpha^r}{\partial\tau^2}\right)}\end{aligned} \tag{39}$$

2.7 Helmholtz Equations

First Helmholtz equation

$$\begin{aligned}\left(\frac{\partial E}{\partial V}\right)_T&=\left[T\left(\frac{\partial P}{\partial T}\right)_V-P\right]\\&=T^2\left(-\frac{\partial\frac{P}{T}}{\partial T}\right)_V\end{aligned} \tag{40}$$

Second Helmholtz equation

$$\left(\frac{\partial H}{\partial P}\right)_T=\left[V-T\left(\frac{\partial V}{\partial T}\right)_P\right] \tag{41}$$

2.8 Properties

2.8.1 Compressibility

Isothermal compressibility

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \tag{42}$$

Isobaric compressibility

$$\kappa_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \tag{43}$$

Isentropic compressibility (rarely used)

$$\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S \tag{44}$$

2.8.2 Coefficient of Thermal Expansion

Change of volume due to temperature change.

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \tag{45}$$

2.8.3 Heat Capacities

Specific heat at constant volume/pressure.

$$\begin{aligned} C_v &= \left(\frac{\partial E}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \\ &= -T\left(\frac{\partial V}{\partial T}\right)_S\left(\frac{\partial P}{\partial T}\right)_V \\ C_p &= \left(\frac{\partial H}{\partial T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P \\ &= T\left(\frac{\partial P}{\partial T}\right)_S\left(\frac{\partial V}{\partial T}\right)_P \end{aligned} \tag{46}$$

For mass specific heat capacity:

$$c_p = \left(\frac{\partial e}{\partial T}\right)_P + \frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P \tag{47}$$

2.8.4 Viscosity

Dynamic viscosity μ and **kinematic** viscosity ν.

$$\nu = \frac{\mu}{\rho} \tag{48}$$

Polynomial Dependency

$$\mu = \sum_{x=0}^{N} C_n T^n \tag{49}$$

Power-law Viscosity

$$\mu = \mu_{ref}\left(\frac{T}{T_{ref}}\right)^s \tag{50}$$

Sutherland Model

$$\mu = \mu_{ref}\left(\frac{T}{T_{ref}}\right)^{\frac{3}{2}}\frac{T_{ref}+C}{T+C} \tag{51}$$

where C is Sutherland's constant that is empirically fitted.

2.8.5 Fundamental Derivative

A non-dimensional fluid derivative that is influential to the flow variables:

$$\begin{aligned} \Gamma &= 1 + \frac{\rho}{c}\left(\frac{\partial c}{\partial \rho}\right)_S \\ &= \frac{v^3}{2c^2}\left(\frac{\partial^2 P}{\partial v^2}\right)_S \\ &= \frac{c^4}{2v^3}\left(\frac{\partial^2 v}{\partial P^2}\right)_S \end{aligned} \tag{52}$$

which sometimes denoted as $\tilde{\Gamma}$. The value for perfect gas is $\frac{\gamma+1}{2}$.

2.8.6 Grüneisen Parameter

Describes the effect that changing the volume of a crystal lattice has on its vibrational properties, and, as a consequence, the effect that changing temperature has on the size or dynamics of the crystal lattice.

$$G = v\left(\frac{\partial P}{\partial E}\right)_V \tag{53}$$

and for non-ideal fluid $U = C_v T$

$$\begin{aligned} G &= \frac{1}{\rho C_v}\left(\frac{\partial P}{\partial T}\right)_v \\ &= \frac{v}{C_v}\left(\frac{\partial S}{\partial V}\right)_T \\ &= \frac{v}{C_v}\left(\frac{\partial S}{\partial P}\right)_T\left(\frac{\partial P}{\partial V}\right)_T \\ &= -\frac{v}{C_v}\left(\frac{\partial V}{\partial T}\right)_P\left(\frac{\partial P}{\partial V}\right)_T \\ &= \frac{v}{C_v}\frac{\kappa_P}{\kappa_T} \end{aligned} \tag{54}$$

2.9 Thermodynamic Relations

2.9.1 Specific Heat Relationship

Specific heat ratio

$$\gamma = \frac{C_p}{C_v} = \frac{\left(\frac{\partial P}{\partial V}\right)_S}{\left(\frac{\partial P}{\partial V}\right)_T} \tag{55}$$

Difference

$$\begin{aligned} C_p - C_v &= T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_P \\ &= \left(\left(\frac{\partial E}{\partial V}\right)_T + P\right)\left(\frac{\partial V}{\partial T}\right)_P \end{aligned} \tag{56}$$

and in terms of α and κ:

$$C_p - C_v = \frac{T\alpha^2 V}{\kappa_T} \tag{57}$$

and another handy relation for perfect gas:

$$\begin{aligned} R &= C_p - C_v \\ C_v &= \frac{R}{\gamma - 1} \\ C_p &= \frac{\gamma R}{\gamma - 1} \end{aligned} \tag{58}$$

2.9.2 Entropy Difference

By definition:

$$\mathrm{d}S = \left(\frac{\mathrm{d}Q}{T}\right)_{rev} \tag{59}$$

hence:

$$\begin{aligned} \mathrm{d}s &= \left(\frac{\mathrm{d}e + \mathrm{d}w}{T}\right)_{rev} \\ &= \left(\frac{\mathrm{d}e + P\mathrm{d}v}{T}\right)_{rev} \end{aligned} \tag{60}$$

For non-ideal fluids:

$$\frac{\mathrm{d}s}{R} = \frac{\mathrm{d}e}{RT} - Z\frac{\mathrm{d}\rho}{\rho} \tag{61}$$

For thermally perfect gas, entropy change for a process:

$$s_2 - s_1 = \int_1^2 \frac{c_v}{T}\mathrm{d}T + \int_1^2 \frac{R}{v}\mathrm{d}v \tag{62}$$

For perfect gas:

$$\begin{aligned} s_2 - s_1 &= c_v \ln\frac{T_2}{T_1} + R\ln\frac{v_2}{v_1} \\ &= c_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1} \\ &= c_v \ln\frac{P_2}{P_1} + c_p\ln\frac{v_2}{v_1} \end{aligned} \tag{63}$$

2.9.3 Clausius-Clapeyron Relation

The Clausius–Clapeyron relation characterises behaviour of a closed system during a phase change at constant temperature and pressure. L is latent heat.

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \frac{L}{T\Delta v} = \frac{\Delta s}{\Delta v} \tag{64}$$

2.9.4 Joule–Thomson effect

Joule–Thomson effect (also known as the Joule–Kelvin effect or Kelvin–Joule effect) describes the temperature change of a non-ideal fluid or liquid (as differentiated from an ideal gas) when it is forced through a valve or porous plug while keeping it insulated so that no heat is exchanged with the environment.

$$\mu_{JT} = \left(\frac{\partial T}{\partial P}\right)_H = \frac{V}{C_p}\left(\alpha T - 1\right) \tag{65}$$

2.9.5 Crocco's theorem

Crocco's theorem is an aerodynamic theorem relating the flow velocity, vorticity, and stagnation pressure (or entropy) of a potential flow.

$$\frac{D\vec{U}}{Dt} = T\nabla s - \nabla h \tag{66}$$

Other forms:

Stagnation pressure

$$\vec{U}\times\vec{\omega} = v\nabla P_0 \tag{67}$$

Entropy

$$T\frac{\mathrm{d}s}{\mathrm{d}n} = \frac{\mathrm{d}h_0}{\mathrm{d}n} + u\omega \tag{68}$$

Momentum

$$\frac{\partial\vec{U}}{\partial t} + \nabla\left(\frac{u^2}{2} + h\right) = \vec{U}\times\vec{\omega} + T\nabla s + \vec{G} \tag{69}$$

3 Fluid Dynamics Basics

3.1 Reynolds transport theorem (RTT)

Rate of change in CV is equal to the generation plus net flux.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\phi \ \mathrm{d}V = \int_V \frac{\partial \rho\phi}{\partial t} \ \mathrm{d}V + \int_A \rho\phi\vec{U}\cdot\vec{n} \ \mathrm{d}A \tag{70}$$

where ϕ is property per unit mass (specific property). Using divergence law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\phi\right) = \frac{\partial}{\partial t}\left(\rho\phi\right) + \nabla\cdot\left(\rho\phi\vec{U}\right) \tag{71}$$

3.2 Conservation laws

Classifications:

1. **Integral Form:** finite control volume that allows discontinuities (more fundamental)
   (a) **Conservation Form:** Element fixed in space ($\frac{\partial \rho u}{\partial x}$)
      i. **Strong Form:** No flow variables outside of partial derivatives (Eq. (73))
      ii. **Weak Form:** Derivatives of x, y, z, and t explicitly appearing on RHS (Eq. (85) for momentum and energy)
   (b) **Non-conservation Form:** Element moving with the flow ($\rho\frac{\partial \rho}{\partial x} + u\frac{\partial u}{\partial x}$)
2. **Partial differential form:** infinitesimally small volume that assumes flow properties are differentiable and continuous.
   (a) **Conservation Form**
      i. **Strong Form**
      ii. **Weak Form**
   (b) **Non-conservation Form**

3.3 Euler's Equation

Governing equation for inviscid flow:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\vec{U}\right) = 0 \\ \dfrac{\partial}{\partial t}\left(\rho\vec{U}\right) + \nabla\cdot\left(\rho\vec{U}\otimes\vec{U}\right) = -\nabla P + \rho\vec{F} \\ \dfrac{\partial}{\partial t}\left(\rho\left(e+\dfrac{1}{2}|U|^2\right)\right) + \nabla\cdot\left(\rho\vec{U}\left(h+\dfrac{1}{2}|U|^2\right)\right) = \\ \qquad \nabla\cdot\left(k\nabla T\right) + \vec{U}\cdot\vec{F} + Q_{gen} \end{cases} \tag{72}$$

In a more compact (strong conservative) form:

$$\frac{\partial \vec{W}}{\partial t} + \frac{\partial \vec{F}}{\partial x} = \vec{C} \tag{73}$$

where $\vec{W}$ is the vector of conserved quantities (primitive variables or solution vector), $\vec{F}$ is vector of fluxes and $\vec{C}$ is the source term vector.

$$\vec{W} = \begin{pmatrix} \rho \\ \rho\vec{U} \\ \rho e_0 \end{pmatrix} \tag{74}$$

$$\vec{F} = \begin{pmatrix} \rho\left(\vec{U}\cdot\vec{n}\right) \\ \rho\vec{U}\left(\vec{U}\cdot\vec{n}\right) + P\vec{n} \\ \rho\left(\vec{U}\cdot\vec{n}\right)h_0 \end{pmatrix} \tag{75}$$

$$\vec{C} = \begin{pmatrix} 0 \\ \vec{F} \\ Q + \vec{F}\cdot\vec{U} \end{pmatrix} \tag{76}$$

3.4 Incompressible Navier-Stokes Equations

3.4.1 Conservation of mass

The balance of mass is 0:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \int_V \frac{\partial \rho}{\partial t} \ \mathrm{d}V + \int_A \rho\left(\vec{U}\cdot\vec{n}\right) \ \mathrm{d}A = 0 \tag{77}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\vec{U}\right) = 0 \tag{78}$$

$$\frac{\partial \rho}{\partial t} + \vec{U}\cdot\left(\nabla\rho\right) + \rho\nabla\cdot\vec{U} = 0 \tag{79}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\vec{U} = 0 \tag{80}$$

3.4.2 Conservation of momentum

The balance of momentum is the net force.

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = \int_V \frac{\partial \rho\vec{U}}{\partial t} \ \mathrm{d}V + \int_A \rho\vec{U}\left(\vec{U}\cdot\vec{n}\right) \ \mathrm{d}A = \sum\vec{F} \tag{81}$$

Derived from momentum balance of RTT:

$$\frac{\partial \vec{U}}{\partial t} + \left(\vec{U}\cdot\nabla\right)\vec{U} = -\frac{1}{\rho}\nabla P + \frac{1}{\rho}\nabla\cdot\underline{\underline{\tau}} + \vec{G} \tag{82}$$

where $\left(\vec{U}\cdot\nabla\right)$ is the advection operator.

$$\begin{aligned} \left(\vec{U}\cdot\nabla\right)\vec{U} &= \left(\nabla\vec{U}\right)\vec{U} \\ &= \vec{U}\cdot\left(\nabla\otimes\vec{U}\right) \\ &= \frac{1}{2}\nabla\left(\vec{U}\cdot\vec{U}\right) - \vec{U}\times\left(\nabla\times\vec{U}\right) \end{aligned} \tag{83}$$

Simplified Navier-Stokes Equation From the Cauchy equation and assumptions of

1. Newtoinian fluid: $\underline{\underline{\tau}} = 2\mu\underline{\underline{D}} + \lambda\nabla\cdot\vec{U}\underline{\underline{I}}$;
2. Incompressible flow: $\nabla\cdot\vec{U} = 0$;
3. Constant μ,

Navier Stokes equation can be derived:

$$\frac{\partial \vec{U}}{\partial t} + \left(\nabla\vec{U}\right)\vec{U} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2\vec{U} + \vec{G} \tag{84}$$

3.5 Compressible Navier-Stokes Equations

Navier-Stokes equations are a set of conservation equations for mass, momentum, and energy (in conservation form here):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{U}\right) = 0 \\ \dfrac{\partial}{\partial t}\left(\rho \vec{U}\right) + \nabla \cdot \left(\rho \vec{U} \otimes \vec{U}\right) = -\nabla P + \nabla \cdot \underline{\underline{\tau}} + \rho \vec{F} \\ \dfrac{\partial}{\partial t}\left(\rho\left(e + \dfrac{1}{2}|U|^2\right)\right) + \nabla \cdot \left(\rho \vec{U}\left(e + \dfrac{1}{2}|U|^2\right)\right) = \\ \qquad \nabla \cdot (k \nabla T) + \nabla \cdot \left(-P\vec{U} + \underline{\underline{\tau}} \cdot \vec{U}\right) + \vec{U} \cdot \vec{F} + Q_{gen} \end{cases} \tag{85}$$

For weaker forms LHS is taken as ρ times the material derivative of ϕ ($\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{U}$, $\rho \frac{D\vec{U}}{Dt}$, and $\rho \frac{De_0}{Dt}$)

In conservative forms:

$$\frac{\partial \vec{W}}{\partial t} + \frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{G}}{\partial y} + \frac{\partial \vec{H}}{\partial z} = \vec{C} \tag{86}$$

$$\vec{W} = \begin{pmatrix} \rho & \rho u_1 & \rho u_2 & \rho u_3 & \rho e_0 \end{pmatrix}^T \tag{87}$$

$$\vec{F} = \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + P - \tau_{xx} \\ \rho u_1 u_2 - \tau_{yx} \\ \rho u_1 u_3 - \tau_{zx} \\ \rho u_1 h_0 - u_1 \tau_{xx} - u_2 \tau_{xy} - u_3 \tau_{xz} \end{pmatrix} \tag{88}$$

$$\vec{G} = \begin{pmatrix} \rho u_2 \\ \rho u_2 u_1 - \tau_{xy} \\ \rho u_2^2 + P - \tau_{yy} \\ \rho u_2 u_3 - \tau_{zy} \\ \rho u_2 h_0 - u_1 \tau_{yx} - u_2 \tau_{yy} - u_3 \tau_{yz} \end{pmatrix} \tag{89}$$

$$\vec{H} = \begin{pmatrix} \rho u_3 \\ \rho u_3 u_1 - \tau_{xz} \\ \rho u_3 u_2 - \tau_{yz} \\ \rho u_3^2 + P - \tau_{zz} \\ \rho u_3 h_0 - u_1 \tau_{zx} - u_2 \tau_{zy} - u_3 \tau_{zz} \end{pmatrix} \tag{90}$$

$$\vec{C} = \begin{pmatrix} 0 \\ \vec{F} \\ Q + \vec{F} \cdot \vec{U} \end{pmatrix} \tag{91}$$

3.6 Entropy Transport Equation

Transport equations are derived using energy balance and Gibbs Equation.

$$\rho \frac{Ds}{Dt} = \frac{\Phi}{T} + \frac{k}{T}\left(\frac{\partial T}{\partial x_i}\right)^2 + \frac{\partial}{\partial x_i}\left(\frac{k}{T}\frac{\partial T}{\partial x_i}\right) \tag{92}$$

or in other forms

$$\rho \frac{Ds}{Dt} + \nabla \cdot \left(\frac{\vec{q}}{T}\right) = \frac{\Phi}{T} + \frac{k}{T}\left(\frac{\partial T}{\partial x_i}\right)^2 \tag{93}$$

In integral terms

$$\begin{aligned} \int_V \rho \frac{Ds}{Dt} \mathrm{d}V &= \int_V \frac{\Phi}{T} \mathrm{d}V + \int_V \frac{k}{T}\left(\frac{\partial T}{\partial x_i}\right)^2 \mathrm{d}V \\ &\quad + \int_V \frac{\partial}{\partial x_i}\left(\frac{k}{T}\frac{\partial T}{\partial x_i}\right) \mathrm{d}V \\ &= \int_V \frac{\Phi}{T} \mathrm{d}V + \int_V \frac{k}{T}\left(\frac{\partial T}{\partial x_i}\right)^2 \mathrm{d}V \\ &\quad + \int_A \left(\frac{k}{T}\frac{\partial T}{\partial x_i}\right) \cdot \hat{n} \, \mathrm{d}A \end{aligned} \tag{94}$$

3.7 Jacobian Matrices

3.7.1 Convective Flux Jacobian

Using 1D inviscid NS equations form of Eq. (73) without source term:

$$\vec{W} = \begin{pmatrix} \rho \\ \rho u \\ \rho e_0 \end{pmatrix} \quad , \quad \vec{F} = \begin{pmatrix} \rho u \\ \rho u^2 + P \\ \rho u h_0 \end{pmatrix} \tag{95}$$

Convective Flux Jacobian is the partial derivative of flux vector $\vec{F}$ w.r.t. flow state vector $\vec{W}$:

$$\underline{\underline{J}} = \frac{\partial \vec{F}}{\partial \vec{W}} = \begin{pmatrix} \frac{\partial F_1}{\partial W_1} & \frac{\partial F_2}{\partial W_1} & \frac{\partial F_3}{\partial W_1} \\ \frac{\partial F_1}{\partial W_2} & \frac{\partial F_2}{\partial W_2} & \frac{\partial F_3}{\partial W_2} \\ \frac{\partial F_1}{\partial W_3} & \frac{\partial F_2}{\partial W_3} & \frac{\partial F_3}{\partial W_3} \end{pmatrix} \tag{96}$$

which can be written as :

$$\underline{\underline{J}} = \underline{\underline{R \Lambda L}} \tag{97}$$

In general cases:

$$\underline{\underline{J}} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial P}{\partial W_1} - u^2 & \frac{\partial P}{\partial W_2} + 2u & \frac{\partial P}{\partial W_3} \\ u\left(\frac{\partial P}{\partial W_1} - h_0\right) & h_0 + \frac{\partial P}{\partial W_2} u & \left(1 + \frac{\partial P}{\partial W_3}\right) u \end{pmatrix} \tag{98}$$

or equivalently

$$\underline{\underline{J}} = \begin{pmatrix} 0 & 1 & 0 \\ c^2 - u^2 - \frac{h_0 - u^2}{\rho}\frac{\partial P}{\partial e} & -\frac{u}{\rho}\frac{\partial P}{\partial e} + 2u & \frac{1}{\rho}\frac{\partial P}{\partial e} \\ u\left(c^2 - h_0\right) - \frac{u}{\rho}\frac{\partial P}{\partial e}\left(h_0 - u^2\right) & h_0 - \frac{u^2}{\rho}\frac{\partial P}{\partial e} & u + \frac{u}{\rho}\frac{\partial P}{\partial e} \end{pmatrix} \tag{99}$$

For a non-ideal fluid with (ρ, e) flash ($P = P(\rho, e)$):

$$\begin{aligned} \frac{\partial P}{\partial W_1} &= \frac{\partial P}{\partial \rho} + \frac{1}{\rho}\left(\frac{1}{2}u^2 - e\right)\frac{\partial P}{\partial e} \\ \frac{\partial P}{\partial W_2} &= -\frac{u}{\rho}\frac{\partial P}{\partial e} \\ \frac{\partial P}{\partial W_3} &= \frac{1}{\rho}\frac{\partial P}{\partial e} \end{aligned} \tag{100}$$

An important relationship is

$$\frac{\partial P}{\partial W_2} = -u\frac{\partial P}{\partial W_3} \tag{101}$$

The speed of sound is defined as

$$\begin{aligned} c^2 &= \frac{\partial P}{\partial W_1} + \left(h_0 - u^2\right)\frac{\partial P}{\partial W_3} \\ &= \frac{\partial P}{\partial \rho} + \frac{P}{\rho^2}\frac{\partial P}{\partial e} \end{aligned} \tag{102}$$

and the eigenvalues are found to be

$$\begin{aligned} \lambda_1 &= u - c \\ \lambda_2 &= u \\ \lambda_3 &= u + c \end{aligned} \tag{103}$$

$$\underline{\underline{\Lambda}} = \begin{pmatrix} u-c & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u+c \end{pmatrix} \tag{104}$$

and the corresponding eigenvectors are

$$\vec{r_1} = \begin{pmatrix} 1 \\ u-c \\ h_0 - uc \end{pmatrix} \ , \ \vec{r_2} = \begin{pmatrix} 1 \\ u \\ h_0 - \dfrac{c^2\rho}{\dfrac{\partial P}{\partial e}} \end{pmatrix} \ , \ \vec{r_3} = \begin{pmatrix} 1 \\ u+c \\ h_0 + uc \end{pmatrix} \tag{105}$$

hence

$$\underline{\underline{R}} = \begin{pmatrix} 1 & 1 & 1 \\ u-c & u & u+c \\ h_0 - uc & h_0 - \dfrac{c^2\rho}{\dfrac{\partial P}{\partial e}} & h_0 + uc \end{pmatrix} \tag{106}$$

and

$$\underline{\underline{L}} = \underline{\underline{R}}^{-1} \tag{107}$$

For ideal gases:

$$\begin{aligned} \frac{\partial P}{\partial W_1} &= \frac{\gamma-1}{2}\frac{W_2^2}{W_1^2} = \frac{\gamma-1}{2}u^2 \\ \frac{\partial P}{\partial W_2} &= -\left(\gamma-1\right)\frac{W_2}{W_1} = -\left(\gamma-1\right)u \\ \frac{\partial P}{\partial W_3} &= \gamma - 1 \end{aligned} \tag{108}$$

hence for an ideal gas the convective flux Jacobian matrix is:

$$\underline{\underline{J}} = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{\gamma-3}{2}u^2 & \left(3-\gamma\right)u & \gamma-1 \\ \dfrac{\gamma-1}{2}u^3 - uh_0 & h_0 - \left(\gamma-1\right)u^2 & \gamma u \end{pmatrix} \tag{109}$$

The eigenvalues are the same and the eigenvectors are:

$$\vec{r_1} = \begin{pmatrix} 1 \\ u-c \\ h_0 - uc \end{pmatrix} \ , \ \vec{r_2} = \begin{pmatrix} 1 \\ u \\ \dfrac{u^2}{2} \end{pmatrix} \ , \ \vec{r_3} = \begin{pmatrix} 1 \\ u+c \\ h_0 + uc \end{pmatrix} \tag{110}$$

Therefore:

$$\underline{\underline{R}} = \begin{pmatrix} 1 & 1 & 1 \\ u-c & u & u+c \\ h_0 - uc & \dfrac{u^2}{2} & h_0 + uc \end{pmatrix} \tag{111}$$

$$\underline{\underline{L}} = \begin{pmatrix} \dfrac{1}{2}\left(\dfrac{\gamma-1}{2c^2}u^2 + \dfrac{u}{c}\right) & -\dfrac{1}{2}\left(\dfrac{\gamma-1}{c^2}u + \dfrac{1}{c}\right) & \dfrac{\gamma-1}{2c^2} \\ 1 - \dfrac{\gamma-1}{2c^2}u^2 & \dfrac{\gamma-1}{c^2}u & -\dfrac{\gamma-1}{c^2} \\ \dfrac{1}{2}\left(\dfrac{\gamma-1}{2c^2}u^2 - \dfrac{u}{c}\right) & -\dfrac{1}{2}\left(\dfrac{\gamma-1}{c^2}u - \dfrac{1}{c}\right) & \dfrac{\gamma-1}{2c^2} \end{pmatrix} \tag{112}$$

For non-ideal fluids: The simplification for perfect gas is not applicable and the original form of Jacobian needs to be maintained.

3.7.2 Velocity Jacobian

Velocity Jacobian is the gradient of the velocity vector.

$$\underline{\underline{J}} = \underline{\underline{grad}}\ \vec{U} = \nabla\vec{U} = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{pmatrix} = u_{i,j} \tag{113}$$

with Helmholtz's decomposition into irrotational and solenoidal part:

$$\underline{\underline{J}} = \underline{\underline{D}} + \underline{\underline{\Omega}} \tag{114}$$

where

$$\begin{aligned} \underline{\underline{D}} &= \frac{1}{2}\left(\underline{\underline{J}} + \underline{\underline{J}}^T\right) \\ \underline{\underline{\Omega}} &= \frac{1}{2}\left(\underline{\underline{J}} - \underline{\underline{J}}^T\right) \end{aligned} \tag{115}$$

3.8 Strain Tensors

Strain or deformation tensor is the symmetric part of velocity Jacobian:

$$\underline{\underline{D}} = \frac{1}{2}\left(\nabla\vec{U} + \left(\nabla\vec{U}\right)^T\right) \qquad D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{116}$$

$D^\star$ is the deviatoric part of the strain tensor.

$$\underline{\underline{D^\star}} = \underline{\underline{D}} - \frac{1}{3}\,\mathrm{Tr}\left(\underline{\underline{D}}\right) = \underline{\underline{D}} - \frac{1}{3}\left(\nabla\cdot\vec{U}\right)\underline{\underline{I}} \tag{117}$$

$$D^\star_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{118}$$

3.9 Shear Stress Tensor

Shear (viscous) stress tensor is the total stress tensor without hydrostatic pressure:

$$\underline{\underline{\sigma}} = \underline{\underline{\tau}} - \underline{\underline{P}} \tag{119}$$

Viscous stress tensor is related to strain tensor by viscosity:

$$\begin{aligned}\tau &= \mu\left(\nabla\vec{U} + \left(\nabla\vec{U}\right)^T\right) - \frac{2}{3}\mu\left(\nabla\cdot\vec{U}\right)\underline{\underline{I}} \\ &= 2\mu\underline{\underline{D}} - \frac{2}{3}\mu\left(\nabla\cdot\vec{U}\right)\underline{\underline{I}} = 2\mu\underline{\underline{D^\star}}\end{aligned} \tag{120}$$

or in index notation

$$\tau_{ij} = \mu\left(\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) \tag{121}$$

3.10 Dissipation Function

The Dissipation function is defined as:

$$\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \mu\frac{\partial u_i}{\partial x_j}\left(\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) \tag{122}$$

In explicit terms:

$$\begin{aligned}\Phi = \mu\Bigg(&2\left(\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 + \left(\frac{\partial u_3}{\partial z}\right)^2\right) \\ &+ \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right)^2 \\ &+ \left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right)^2 \\ &- \frac{2}{3}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right)^2\Bigg)\end{aligned} \tag{123}$$

3.11 Vorticity

Pseudovector field that describes the local spinning motion of a continuum.

$$\vec{\omega} = \nabla\times\vec{U} \tag{124}$$

and its relationship with spin tensor:

$$\omega_i = -\epsilon_{ijk}\Omega_{jk} \quad \text{or} \quad \Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k \tag{125}$$

3.12 Circulation

Circulation is the integral of the velocity around a closed contour.

$$\Gamma = \int_\ell \vec{U}\cdot\mathrm{d}\ell = \int_A \nabla\times\vec{U}\cdot\hat{n}\,\mathrm{d}A = \int_A \vec{\omega}\cdot\hat{n}\,\mathrm{d}A \tag{126}$$

where countercloscckwise is positive.

3.13 Boundary Layer Thickness

Displacement thickness:

$$\delta_1(x) = \int_0^Y \left(1 - \frac{\rho(x,y)\,u(x,y)}{\rho_\infty u_\infty}\right)\mathrm{d}y \tag{127}$$

Momentum thickness:

$$\theta(x) = \int_0^Y \frac{\rho(x,y)\,U(x,y)}{\rho_\infty U_\infty}\left(1 - \frac{u(x,y)}{u_\infty}\right)\mathrm{d}y \tag{128}$$

Energy thickness:

$$\delta^*(x) = \int_0^Y \frac{\rho(x,y)\,U(x,y)}{\rho_\infty U_\infty}\left(1 - \frac{u(x,y)^2}{u_\infty^2}\right)\mathrm{d}y \tag{129}$$

4 Compressible Flow

4.1 Governing Equations

4.1.1 Isentrope Equations

Polytropic isentrope

$$\begin{aligned} PV^{n_s} &= Constant \\ TP^{-m_s} &= Constant \end{aligned} \tag{130}$$

This is a common assumption to make but have problems in some thermo ranges such as near the critical point, where the exponents varies significantly. Corrections of this form are proposed in [16].

Callendar isentrope [17]

$$P\left(V-b\right)^{n} = A \tag{131}$$

Rayleigh adiabat [18]

$$\left(P+B\right)V^{n} = A \tag{132}$$

Walker isentrope [19]

$$PV^{m}e^{\lambda T} = A \tag{133}$$

4.1.2 Isentropic Expansion Coefficient/Exponents

$$\kappa = \frac{c^2\rho}{P} = n_s \tag{134}$$

4.1.3 Speed of Sound

Sound speed of the flow:

$$\begin{aligned} c = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} = \sqrt{-v^2\left(\frac{\partial P}{\partial v}\right)_S} = \sqrt{\frac{\kappa P}{\rho}} \\ = \frac{\frac{P}{\rho^2} - \left(\frac{\partial e}{\partial \rho}\right)_P}{\left(\frac{\partial e}{\partial P}\right)_\rho} \end{aligned} \tag{135}$$

and the Mach number is defined as:

$$M = \frac{U}{c} \tag{136}$$

4.1.4 Compressibility factor

Ratio of real volume and ideal volume.

$$Z = f(T,P) = \frac{PV}{RT} \tag{137}$$

4.2 Generalised One-dimensional Flow

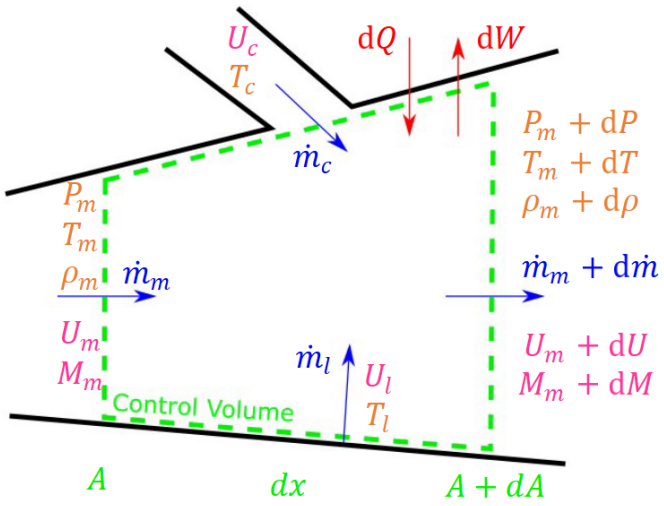


Figure 1: Generalised 1D flow control volume analysis [1]

Mass conservation:

$$\mathrm{d}\dot{m} = \dot{m}_g + \dot{m}_l \tag{138}$$

Energy conservation:

$$\begin{aligned} \dot{m}\left(\mathrm{d}Q - \mathrm{d}W_x\right) = \mathrm{d}h + \mathrm{d}\frac{U^2}{2} \\ + \left(h_{gT} - h_g + \frac{U^2 - U_g^2}{2}\right)\frac{\mathrm{d}\dot{m}_g}{\dot{m}} \\ + \left(h_{lT} - h_l + \frac{U^2 - U_l^2}{2}\right)\frac{\mathrm{d}\dot{m}_l}{\dot{m}} \end{aligned} \tag{139}$$

Momentum conservation:

$$\begin{aligned} PA + P\mathrm{d}A - \left(P + \mathrm{d}P\right)\left(A + \mathrm{d}A\right) - \tau_w \mathrm{d}A_w - \mathrm{d}F_x \\ = \left(\dot{m} + \dot{m}_g + \dot{m}_l\right)\left(U + \mathrm{d}U\right) - U_{gx}\dot{m}_g - U_{lx}\dot{m}_l - U\dot{m} \end{aligned} \tag{140}$$

Since

$$\begin{aligned} \tau_w &= \frac{f\rho U^2}{2} \\ \frac{A_w}{A} &= \frac{4\mathrm{d}x}{D} \end{aligned} \tag{141}$$

the equation can be re-written as:

$$\begin{aligned} \frac{\mathrm{d}P}{P} + \kappa M^2\frac{\mathrm{d}U}{U} + \frac{\kappa M^2}{2}\left(4f\frac{\mathrm{d}x}{D} + \frac{\mathrm{d}F_x}{\frac{1}{2}\kappa PAM^2}\right) \\ + \kappa M^2\left(1-y\right)\frac{\mathrm{d}\dot{m}}{\dot{m}} = 0 \end{aligned} \tag{142}$$

where

$$\begin{aligned} y_g = \frac{U_{gx}}{U},\ y_l = \frac{U_{lx}}{U} \\ \left(1-y\right)\frac{\mathrm{d}\dot{m}}{\dot{m}} = \left(1-y_g\right)\frac{\mathrm{d}\dot{m}_g}{\dot{m}} + \left(1-y_l\right)\frac{\mathrm{d}\dot{m}_l}{\dot{m}} \\ \frac{\mathrm{d}\dot{m}}{\dot{m}} = \frac{\mathrm{d}\dot{m}_g}{\dot{m}} + \frac{\mathrm{d}\dot{m}_l}{\dot{m}} \\ y\frac{\mathrm{d}\dot{m}}{\dot{m}} = y_g\frac{\mathrm{d}\dot{m}_g}{\dot{m}} + y_l\frac{\mathrm{d}\dot{m}_l}{\dot{m}} \end{aligned} \tag{143}$$

Entropy Generation

$$\mathrm{d}\bar{s} = \mathrm{d}s + \left(s - s_g\right)\frac{\mathrm{d}\dot{m}_g}{\dot{m}} + \left(s - s_l\right)\frac{\mathrm{d}\dot{m}_l}{\dot{m}} \tag{144}$$

4.3 Perfect Gas vs. Non-Ideal Fluid Flow

	Perfect Gas	Non-Ideal Fluid (Polytropic Assumption)
Gas Law	$Pv = RT$	$Pv = ZRT$
Compressibility	$\kappa_T = \frac{1}{P}$ $\kappa_P = \frac{1}{T}$	$\kappa_T = -\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T = \frac{1}{P} - \frac{1}{Z}\left(\frac{\partial Z}{\partial P}\right)_T$ $\kappa_P = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P = \frac{1}{T} + \frac{1}{Z}\left(\frac{\partial Z}{\partial T}\right)_P$
Heat Capacity Difference	$C_p - C_v = \frac{R}{M} = Constant$	$C_p - C_v = T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P$ $= \frac{R}{Z}\left(Z + T\left(\frac{\partial Z}{\partial T}\right)_P\right)\left(Z + T\left(\frac{\partial Z}{\partial T}\right)_V\right)$
Heat Capacity Ratio	$\gamma = \frac{C_p}{C_v} = Constant$	$\gamma = \left(\frac{\partial P}{\partial V}\right)_S \Big/ \left(\frac{\partial P}{\partial V}\right)_T$
Isentropic Indicies (polytropic assumption)	$n_s = \gamma$ $m_s = \frac{\gamma - 1}{\gamma}$	$n_s = -\frac{v}{P}\left(\frac{\partial P}{\partial v}\right)_S = -\frac{v\gamma}{P}\left(\frac{\partial P}{\partial v}\right)_T$ $= \frac{\rho}{P}\left(\frac{\partial P}{\partial \rho}\right)_S = \frac{\gamma}{\kappa_T P}$ $m_s = \frac{P}{T}\left(\frac{\partial T}{\partial P}\right)_S = \frac{\gamma - 1}{\gamma}\frac{\kappa_T}{\kappa_P}\frac{P}{T}$
Fundamental Derivative	$\Gamma = \frac{\gamma + 1}{2}$	$\Gamma = \frac{1}{2}\left(n_s + 1 - \frac{v}{n_s}\left(\frac{\partial n_s}{\partial v}\right)_S\right)$
Non-isentropic Exponent	$\Psi = \gamma$	$\Psi = \rho\left(\frac{\partial c^2}{\partial P}\right)_\rho = n_s + P\left(\frac{\partial n_s}{\partial P}\right)_\rho$
Speed of Sound	$c = \sqrt{\gamma RT} = \sqrt{\frac{\gamma P}{\rho}}$ $= \sqrt{(\gamma - 1)\left(h_0 - \frac{U^2}{2}\right)}$	$c = \sqrt{n_s P v} = \sqrt{\frac{n_s P}{\rho}}$ $= \sqrt{n_s ZRT}$
Isentropic Relations	$\frac{T_0}{T} = \left(1 + \frac{\gamma - 1}{2}M^2\right)$ $\frac{P_0}{P} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma - 1}}$ $\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma - 1}}$	$\frac{T_0}{T} = \left(1 + \frac{n_s - 1}{2}M^2\right)^{\frac{m_s n_s}{n_s - 1}}$ $\frac{P_0}{P} = \left(1 + \frac{n_s - 1}{2}M^2\right)^{\frac{n_s}{n_s - 1}}$ $\frac{\rho_0}{\rho} = \left(1 + \frac{n_s - 1}{2}M^2\right)^{\frac{1}{n_s - 1}}$
Compressibility Relation	$Z_0 = Z = 1$	$\frac{Z_0}{Z} = \left(1 + \frac{n_s - 1}{2}M^2\right)^{1 - \frac{m_s n_s}{n_s - 1}}$
Entropy Difference	$s_2 - s_1 = c_v \ln\frac{T_2}{T_1} + R\ln\frac{v_2}{v_1}$ $= c_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1}$ $= c_p \ln\frac{v_2}{v_1} + c_v \ln\frac{P2}{P1}$	No explicit expression
Grüneisen Parameter	$G = \gamma - 1$	$G = v\left(\frac{\partial P}{\partial E}\right)_V = n_s m_s$

4.4 Quasi-1D Nozzle

4.4.1 General Analysis

Conservation laws for mass, momentum and energy, respectively, where U is velocity here instead of energy.

$$\begin{aligned} \rho U A &= Constant \\ P + \rho U^2 &= Constant \\ h + \frac{1}{2}U^2 &= Constant \end{aligned} \tag{145}$$

and in differential form:

$$\begin{aligned} \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}U}{U} + \frac{\mathrm{d}A}{A} &= 0 \\ \mathrm{d}P + \rho U \mathrm{d}U &= 0 \\ \mathrm{d}h + U\mathrm{d}U &= 0 \end{aligned} \tag{146}$$

Other important relations

$$\frac{\mathrm{d}A}{A} = \left(M^2 - 1\right)\frac{\mathrm{d}U}{U} \tag{147}$$

$$\frac{\mathrm{d}U}{U} = \left(\frac{1}{1 + (\Gamma - 1)\,M^2}\right)\frac{\mathrm{d}M}{M} \tag{148}$$

$$\frac{\mathrm{d}A}{A} = \left(\frac{M^2 - 1}{1 + (\Gamma - 1)\,M^2}\right)\frac{\mathrm{d}M}{M} \tag{149}$$

$$\left(\frac{\mathrm{d}M}{\mathrm{d}x}\right)^2 = \left(\frac{\Gamma}{2A}\frac{\mathrm{d}^2 A}{\mathrm{d}x^2}\right)_{M=1} \tag{150}$$

Flow behaviours

$$\begin{cases} M > 1 & \begin{cases} \Gamma > 1 & \mathrm{d}M \propto \mathrm{d}A \\ \Gamma < 1 & \mathrm{d}M \propto -\mathrm{d}A \end{cases} \\ M = 1 & \mathrm{d}A = 0 \\ M < 1 & \begin{cases} \Gamma > 1 - \frac{1}{M^2} & \mathrm{d}M \propto -\mathrm{d}A \\ \Gamma < 1 - \frac{1}{M^2} & \mathrm{d}M \propto \mathrm{d}A \end{cases} \end{cases} \tag{151}$$

4.4.2 Perfect vs. Non-Ideal Fluid Nozzle

	Perfect Gas	Non-Ideal Fluid (Polytropic Assumption)
Area-Mach Number Relation Arbitrary Reference	$\frac{A}{A_i} = \frac{M_i}{M}\left(1 + \frac{\gamma-1}{2}M_i^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}\left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}$	$\frac{A}{A_i} = \frac{M_i}{M}\sqrt{\frac{n_{s_i}}{n_s}}\left(1 + \frac{n_{s_i}-1}{2}M_i^2\right)^{-\frac{n_{s_i}+1}{2(n_{s_i}-1)}}\left(1 + \frac{n_s-1}{2}M^2\right)^{\frac{n_s+1}{2(n_s-1)}}$
Area-Mach Number Relation Critical Reference	$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$	$\frac{A}{A^*} = \frac{1}{M}\sqrt{\frac{n_s^*}{n_s}}\left(\frac{2}{n_s^*+1}\right)^{\frac{n_s^*+1}{2(n_s^*-1)}}\left(1 + \frac{n_s-1}{2}M^2\right)^{\frac{n_s+1}{2(n_s-1)}}$
Mass Flow Rate	$\dot{m} = \frac{\gamma P_0 A M}{\sqrt{C_p T_0\,(\gamma-1)}}\left(1 + \frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$	$\dot{m} = AM\sqrt{n_s \rho_0 P_0}\left(1 + \frac{n_s-1}{2}M^2\right)^{-\frac{n_s+1}{2(n_s-1)}}$
Critical Mass Flow Rate	$\dot{m}^* = A^*\sqrt{\gamma\rho_0 P_0}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$	$\dot{m}^* = A^*\sqrt{n_s^*\rho_0 P_0}\left(\frac{n_s^*+1}{2}\right)^{-\frac{n_s^*+1}{2(n_s^*-1)}}$
Non-dimensional Mass Flow Rate	$\overline{m} = \frac{\dot{m}\sqrt{C_p T_0}}{P_0 A} = \frac{\gamma}{\sqrt{\gamma-1}}M\left(1 + \frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$	No explicit expression
Exit Mach Number	$M_e^2 = \sqrt{\frac{1}{(\gamma-1)^2} + \frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left(\frac{P_{01}A_1^*}{P_e A_e}\right)^2} - \frac{1}{\gamma-1}$	No explicit expression

4.5 Shock

4.5.1 General Equations

Mach Line Angle (Shock Angle)

$$\beta = \sin^{-1}\frac{1}{M} \tag{152}$$

Shock Strength

$$\frac{P_2 - P_1}{P_1} \tag{153}$$

Rayleigh line

$$m^2 = \frac{P_2 - P_1}{v_1 - v_2} = \frac{P_2 - P_1}{\frac{1}{\rho_1} - \frac{1}{\rho_2}} \tag{154}$$

Rankine-Hugoniot Equation

$$h_2 - h_1 = \frac{1}{2}(P_2 - P_1)(v_1 + v_2) \tag{155}$$

4.5.2 Perfect vs. Non-Ideal Fluid Normal Shock Relations

	Perfect Gas	Non-Ideal Fluid
Static Pressure	$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)$	$\frac{P_2}{P_1} = \frac{1+\kappa_1 M_1^2}{1+\kappa_2 M_2^2}$
Stagnation Pressure	$\frac{P_{02}}{P_{01}} = \left(\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\right)^{\frac{\gamma}{\gamma-1}} \left(\frac{\gamma+1}{2\gamma M_1^2-(\gamma-1)}\right)^{\frac{1}{\gamma-1}}$	$\frac{P_{02}}{P_{01}} = \left(\frac{1+\kappa_1 M_1^2}{1+\kappa_2 M_2^2}\right)\frac{P_{02}}{P_2}\frac{P_1}{P_{01}}$
Static Temperature	$\frac{T_2}{T_1} = \frac{\left(2\gamma M_1^2-(\gamma-1)\right)\left((\gamma-1)M_1^2+2\right)}{(\gamma+1)^2 M_1^2}$	$\frac{T_2}{T_1} = \frac{Z_1\kappa_2 M_2^2\left(1+\kappa_1 M_1^2\right)^2}{Z_2\kappa_1 M_1^2\left(1+\kappa_2 M_2^2\right)^2}$
Total Enthalpy	$\frac{H_{02}}{H_{01}} = 1$	$\frac{H_{02}}{H_{01}} = 1$
Stagnation Temperature	$\frac{T_{02}}{T_{01}} = 1$	$\frac{T_{02}}{T_{01}} \neq 1$
Density	$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}$	$\frac{\rho_2}{\rho_1} = \frac{\kappa_1 M_1^2\left(1+\kappa_2 M_2^2\right)}{\kappa_2 M_2^2\left(1+\kappa_1 M_1^2\right)}$
Mach Number	$M_2^2 = \frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}$	No explicit expression
Speed	$\frac{U_2}{U_1} = \frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}$	No explicit expression
Entropy Generation	$\Delta S = -R\ln\frac{P_{02}}{P_{01}} \approx R\frac{2\gamma}{3(\gamma+1)^2}\left(M_1^2-1\right)^3$	No explicit expression

4.5.3 Oblique Shock

Equations were derived based on assumption that tangential component of velocity does not change across shock.

Exit Mach Number in Flow Direction

$$M_2 = \frac{1}{\sin(\beta-\theta)}\sqrt{\frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}} \tag{156}$$

Flow Property Relations Substitute $M_1 = M_1\sin\beta$ where β is the shock angle ($\beta = \frac{\pi}{2}$for normal shock).

θ-β-M Relation

$$\tan\theta = 2\cot\beta\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos 2\beta)+2} \tag{157}$$

where θ is the deflection angle of flow.

4.6 Prandtl-Meyer Function

ν is Prandtl-Meyer Function, which determines the angle through which a sonic flow must turn to reach a particular Mach number. By convection $\nu(1) = 0$.

$$\mathrm{d}\nu = \sqrt{M^2-1}\,\frac{\mathrm{d}U}{U} \tag{158}$$

$$\mathrm{d}\nu = \frac{\sqrt{M^2-1}}{1+(\Gamma-1)M^2}\frac{\mathrm{d}M}{M} \tag{159}$$

$$\theta = \nu(M_2) - \nu(M_1) \tag{160}$$

The differential form of mach number M and the flow deflection angle θ relation.

$$\frac{\mathrm{d}M}{M} = \left(1+\frac{\gamma-1}{2}M^2\right)\frac{-\mathrm{d}\theta}{\sqrt{M^2-1}} \tag{161}$$

Integrating for explicit form (Prandtl-Meyer Function):

$$\nu(M) = \int_{\nu_1}^{\nu_2}\mathrm{d}\nu = \int_{\theta_1}^{\theta_2}-\mathrm{d}\theta = \int_{M_1}^{M_2}\frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{\mathrm{d}M}{M} \tag{162}$$

For a specific problem with inlet Mach number $M_1 = 1$ and for exit Mach number M, $\theta_1 = 0$, $\theta_2 = \theta$:

$$\begin{aligned}\nu(M) = &\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} \\ &-\tan^{-1}\sqrt{M^2-1}\end{aligned} \tag{163}$$

When $M\to\infty$, $\tan^{-1}\infty \to \frac{\pi}{2}$:

$$\nu_{max} = \theta_{max} = \frac{\pi}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}}-1\right) \approx 130.5^\circ \tag{164}$$

For isentropic expansion:

$$\nu(M_2) = \nu(M_1) + \theta \tag{165}$$

For isentropic compression:

$$\nu(M_2) = \nu(M_1) - \theta \tag{166}$$

5 Turbulence

5.1 Turbulence Scales

5.1.1 Integral Scale

The correlation distance of a process in terms of space or time.

Length

$$\ell = \int_0^\infty R_{xx}\left(\vec{x}\right) \mathrm{d}\vec{x} \tag{167}$$

Time

$$T = \int_0^\infty R_{xx}\left(t\right) \mathrm{d}t \tag{168}$$

For isotropic homogeneous turbulence:

$$\ell = \frac{\int_0^\infty k^{-1} E\left(k\right) \mathrm{d}k}{\int_0^\infty E\left(k\right) \mathrm{d}k} \tag{169}$$

where $E\left(k\right)$ is the energy spectrum and k here is wave number.

5.1.2 Kolmogorov Microscales

Smallest scales in turbulent flow: viscosity dominates the flow and turbulent kinetic energy is dissipated into thermal energy.

Length

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} \tag{170}$$

Time

$$\tau_\eta = \sqrt{\frac{\nu}{\varepsilon}} \tag{171}$$

Velocity

$$u_\eta = \left(\nu\varepsilon\right)^{\frac{1}{4}} \tag{172}$$

where ε is the average rate of dissipation of turbulence kinetic energy per unit mass, and ν is the kinematic viscosity of the fluid.

The Reynolds number formed with these quantities is unity:

$$\frac{\eta u_\eta}{\nu} = 1 \tag{173}$$

5.1.3 Scale reations

L is characteristic length and ℓ is the interal length. U is free stream velocity and u is the characteristic velocity fluctuation.

$$\begin{aligned}
\frac{\ell}{L} &\sim \left(\frac{\nu}{UL}\right)^{-\frac{1}{2}} = Re^{-\frac{1}{2}} \\
\frac{\eta}{\ell} &\sim \left(\frac{u\ell}{\nu}\right)^{-\frac{3}{4}} = Re^{-\frac{3}{4}} \\
\frac{\tau_\eta u}{\ell} &\sim \frac{\tau_\eta}{t} = \left(\frac{u\ell}{\nu}\right)^{-\frac{1}{2}} = Re^{-\frac{1}{2}} \\
\frac{u_\eta}{u} &\sim \left(\frac{u\ell}{\nu}\right)^{-\frac{1}{4}} = Re^{-\frac{1}{4}}
\end{aligned} \tag{174}$$

5.2 Averaging Methods

5.2.1 Fluctuation

The fluctuations for two different averaging methods:

$$\begin{aligned}
f &= \overline{f} + f' \\
f &= \widetilde{f} + f''
\end{aligned} \tag{175}$$

5.2.2 Reynolds (Time) Average

Turbulence is seen as a statistical phenomenon.

$$\overline{f} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} f \mathrm{d}t \tag{176}$$

Properties of Reynolds averaging:

$$\begin{aligned}
\overline{\overline{f}} &= \overline{f} \\
\overline{f'} &= 0 \\
\overline{\overline{f}g'} &= 0 \\
\overline{\overline{f}g} &= \overline{f}\,\overline{g} \\
\overline{f+g} &= \overline{f} + \overline{g} \\
\overline{\frac{\partial f}{\partial x}} &= \frac{\partial \overline{f}}{\partial x}
\end{aligned} \tag{177}$$

5.2.3 Favre Average

With density variation considerations:

$$\widetilde{f} = \frac{\overline{\rho f}}{\overline{\rho}} \tag{178}$$

Properties of Favre averaging:

$$\begin{aligned}
f'' &= f - \overline{f} - \frac{\overline{\rho' f'}}{\overline{\rho}} \\
\widetilde{f''} &= -\frac{\overline{\rho' f'}}{\overline{\rho}} \\
\overline{f''} &\neq 0 \\
\overline{\rho f''} &= 0 \\
\overline{\rho f} &= \overline{\rho}\widetilde{f} \\
\overline{f} &= \widetilde{f} + \overline{f''} \\
\overline{fg} &= \widetilde{f}\widetilde{g} + \overline{f''g} + \widetilde{f}\overline{g''}
\end{aligned} \tag{179}$$

For incompressible flows only:

$$\begin{aligned}
\overline{f} &= \widetilde{f} \\
f'' &= f - \overline{f} = f' \\
\overline{f''} &= 0
\end{aligned} \tag{180}$$

5.3 Turbulence Study Parameters

5.3.1 Skin Friction Coefficient

Local

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_e^2} \tag{181}$$

Total

$$C_F = \frac{F_{drag}}{\frac{1}{2}\rho U_e^2 A} \tag{182}$$

5.3.2 Correlation Coefficients

The general form of correlation:

$$R_{AB} = \frac{\overline{A'(\vec{x},t)\,B'(\vec{x}+\Delta\vec{x},t+\Delta t)}}{\sqrt{\overline{A'(\vec{x},t)^2}}\sqrt{\overline{B'(\vec{x}+\Delta\vec{x},t+\Delta t)^2}}} \tag{183}$$

Space correlation: $\Delta\vec{x} \neq 0$, $\Delta t = 0$ and A, B are $\vec{U}$

Autocorrelation: $\Delta\vec{x} = 0$, $\Delta t \neq 0$ and A, B are $\vec{U}$

Parameter Correlations

$$R_{AB} = \frac{\overline{A'B'}}{\sqrt{\overline{A'^2}}\,\sqrt{\overline{B'^2}}} \tag{184}$$

5.4 Vortex Identification Criteria

5.4.1 Q Criterion

Based on the second invariant of the velocity gradient tensor:

$$Q = \frac{1}{2}\left(||\underline{\underline{\Omega}}||^2 - ||\underline{\underline{D}}||^2\right) > 0 \tag{185}$$

where $\underline{\underline{\Omega}}$ and $\underline{\underline{D}}$ are the vorticity and deformation tensors. In index notations:

$$Q = \frac{1}{2}\left(\Omega_{ij}\Omega_{ij} - D_{ij}D_{ij}\right) \tag{186}$$

5.4.2 Δ Criterion

Based on second (Q) and third ($R = \det\left(\nabla\vec{U}\right)$) invariants of velocity gradient tensor.

$$\Delta = \left(\frac{1}{2}R\right)^2 + \left(\frac{1}{3}Q\right)^3 > 0 \tag{187}$$

5.4.3 λ_2 Criterion

Calculate the second largest eigenvalues of $\underline{\underline{D}}^2 + \underline{\underline{\Omega}}^2$.

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \tag{188}$$

5.5 Energy Spectrum of Turbulence

Energy spectrum is the Fourier transform of the autocorrelation.

5.5.1 Turbulent Kinetic Energy

Turbulent kinetic energy is usually a measure of the energy associated with the turbulent motion of a fluid.

$$k_t = \frac{1}{2}\overline{\vec{U'}\cdot\vec{U'}} = \frac{1}{2}\left(\overline{u_i'u_i'}\right) = \int_0^\infty E(k)\,\mathrm{d}k \tag{189}$$

or for Favre averaged turbulent kinetic energy:

$$k_t = \frac{1}{2}\frac{\overline{\rho u_i'' u_i''}}{\overline{\rho}} \tag{190}$$

5.5.2 Dissipation of Turbulent Kinetic Energy

Diffusion goes as the Laplacian of velocity, the dissipation rate may be written in terms of the energy spectrum.

$$\varepsilon = 2\nu\int_0^\infty k^2 E(k)\,\mathrm{d}k \tag{191}$$

and is to the order of $\varepsilon \sim \frac{u^3}{\ell}$

5.5.3 Energy Cascade

An energy cascade involves the transfer of energy from large scales of motion to the small scales (called a direct energy cascade) or a transfer of energy from the small scales to the large scales (called an inverse energy cascade).

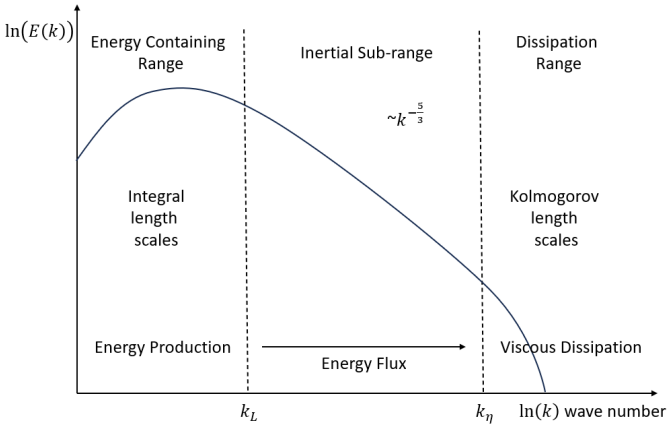


Figure 2: Turbulence energy cascade

For Energy in inertial subrange, Kolmogorov -5/3 law:

$$E(k) = C\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}} \tag{192}$$

5.6 Tollmien–Schlichting Wave

A streamwise unstable wave which arises in a bounded shear flow.

5.7 Boussinesq (Eddy Viscosity) Approximation

This assumption ignores small-scale vortices and includes their effects on large-scale motion with an effective viscosity called eddy viscosity (μ_t, which can be both positive or negative). The overall effective viscosity μ_{eff} of the flow

$$\begin{aligned} \mu_{eff} &= \mu + \mu_t \\ \frac{\mu_{eff}}{Pr} &= \frac{\mu}{Pr} + \frac{\mu_t}{Pr_t}, \quad \lambda_2 < 0 \end{aligned} \tag{193}$$

Boussinesq's hypothesis for turbulence modelling assumes that the **Reynolds stress tensor** is aligned with the **strain tensor** $\underline{\underline{D}}$ of the mean flow.

Reynolds Averaging:

$$\begin{aligned} -\overline{\rho}\overline{\vec{U'}\otimes\vec{U'}} &= \mu_t\left(2\underline{\underline{\overline{D}}} - \frac{2}{3}\left(\nabla\cdot\overline{\vec{U}}\right)\underline{\underline{I}}\right) - \frac{2}{3}\overline{\rho}k_t\underline{\underline{I}} \\ &= 2\mu_t\underline{\underline{\overline{D^\star}}} - \frac{2}{3}\overline{\rho}k_t\underline{\underline{I}} \end{aligned} \tag{194}$$

where μ_t is eddy viscosity and k_t is the turbulent kinetic energy. Or in index notations:

$$\begin{aligned}-\overline{\rho}\overline{u_i'u_j'} &= \mu_t\left(\frac{\partial\overline{u}_i}{\partial x_j}+\frac{\partial\overline{u}_j}{\partial x_i}-\frac{2}{3}\frac{\partial\overline{u}_k}{\partial x_k}\delta_{ij}\right)-\frac{2}{3}\overline{\rho}k_t\delta_{ij}\\ &= 2\mu_t\overline{D}_{ij}-\frac{2\mu_t}{3}\frac{\partial\overline{u}_k}{\partial x_k}\delta_{ij}-\frac{2}{3}\overline{\rho}k_t\delta_{ij}\\ &= 2\mu_t\overline{D^\star}_{ij}-\frac{2}{3}\overline{\rho}k_t\delta_{ij}\end{aligned} \tag{195}$$

Favre Averaging:

$$-\overline{\rho}\widetilde{\vec{U}''\otimes\vec{U}''}=\mu_t\left(2\underline{\underline{\widetilde{D}}}-\frac{2}{3}\left(\nabla\cdot\widetilde{\vec{U}}\right)\underline{\underline{I}}\right)-\frac{2}{3}\overline{\rho}k_t\underline{\underline{I}} \tag{196}$$

where

$$-\overline{\rho}\widetilde{\vec{U}''\otimes\vec{U}''}=-\overline{\rho\vec{U}''\otimes\vec{U}''} \tag{197}$$

and in index notations:

$$\begin{aligned}-\overline{\rho u_i''u_j''} &= \mu_t\left(\frac{\partial\widetilde{u}_i}{\partial x}_j+\frac{\partial\widetilde{u}_j}{\partial x_i}-\frac{2}{3}\frac{\partial\widetilde{u}_k}{\partial x_k}\delta_{ij}\right)-\frac{2}{3}\overline{\rho}k\delta_{ij}\\ &= 2\mu_t\widetilde{D}_{ij}-\frac{2\mu_t}{3}\frac{\partial\widetilde{u}_k}{\partial x_k}\delta_{ij}-\frac{2}{3}\overline{\rho}k_t\delta_{ij}\\ &= 2\mu_t\widetilde{D^\star}_{ij}-\frac{2}{3}\overline{\rho}k_t\delta_{ij}\end{aligned} \tag{198}$$

5.8 Turbulent Stress Tensors

Total stress tensor:

$$\underline{\underline{\tau}}^{tot}=\underline{\underline{\tau}}^{lam}+\underline{\underline{\tau}}^{turb}=\underline{\underline{\overline{\tau}}}+\underline{\underline{\tau}}^{turb}=\widetilde{\tau}+\overline{\tau''}+\underline{\underline{\tau}}^{turb} \tag{199}$$

Reynolds Averaging

$$\begin{aligned}\underline{\underline{\tau}}^{turb} &= -\overline{\rho}\overline{\vec{U}'\otimes\vec{U}'}\\ \tau_{ij}^{turb} &= -\overline{\rho}\overline{u_i'u_j'}\end{aligned} \tag{200}$$

Laminar stress as $\overline{\tau''}=\overline{\tau'}=0$:

$$\overline{\tau}=\mu\left(\nabla\overline{\vec{U}}+\left(\nabla\overline{\vec{U}}\right)^T\right)-\frac{2}{3}\mu\left(\nabla\cdot\overline{\vec{U}}\right)\underline{\underline{I}} \tag{201}$$

Favre Averaging

$$\begin{aligned}\underline{\underline{\tau}}^{turb} &= -\overline{\rho}\widetilde{\vec{U}'\otimes\vec{U}'}\\ \tau_{ij}^{turb} &= -\overline{\rho}\widetilde{u_i''u_j''}=-\overline{\rho u_i''u_j''}\end{aligned} \tag{202}$$

There will be non-zero $\overline{\tau''}$ but is normally approximated to zero

$$\begin{aligned}\overline{\tau} &= \widetilde{\tau}+\overline{\tau''}\\ &= \mu\left(\nabla\widetilde{\vec{U}}+\left(\nabla\widetilde{\vec{U}}\right)^T\right)-\frac{2}{3}\mu\left(\nabla\cdot\widetilde{\vec{U}}\right)\underline{\underline{I}}\\ &\quad+\left(\nabla\overline{\vec{U}''}+\left(\nabla\overline{\vec{U}''}\right)^T\right)-\frac{2}{3}\mu\left(\nabla\cdot\overline{\vec{U}''}\right)\underline{\underline{I}}\end{aligned} \tag{203}$$

but the latter two terms are normally ignored due to small fluctuations.

5.9 Reynolds-Averaged Incompressible Navier-Stokes Equation (RANS)

Incompressible NS equations are usually Reynolds averaged.

$$\begin{aligned}\nabla\cdot\overline{\vec{U}} &= 0\\ \frac{\partial\rho\overline{\vec{U}}}{\partial t}+\nabla\cdot\left(\rho\overline{\vec{U}}\otimes\overline{\vec{U}}\right) &= -\nabla\overline{P}+\nabla\cdot\underline{\underline{\tau}}^{tot}\end{aligned} \tag{204}$$

where

$$\underline{\underline{\tau}}^{tot}=\overline{\tau}-\overline{\rho\overline{\vec{U}'\otimes\vec{U}'}}=2\left(\mu+\mu_t\right)\overline{D^\star}_{ij}-\frac{2}{3}\overline{\rho}k_t\underline{\underline{I}} \tag{205}$$

5.10 Favre-Averaged Compressible Navier-Stokes Equations

Due to the density variation, compressible NS equation must be Favre-averaged:

$$\begin{aligned}\frac{\partial\overline{\rho}}{\partial t}+\nabla\cdot\left(\overline{\rho}\widetilde{\vec{U}}\right) &= 0\\ \frac{\partial}{\partial t}\left(\overline{\rho}\widetilde{\vec{U}}\right)+\nabla\cdot\left(\overline{\rho}\widetilde{\vec{U}}\otimes\widetilde{\vec{U}}\right) &= -\nabla\overline{P}+\nabla\cdot\underline{\underline{\tau}}^{tot}\\ \frac{\partial}{\partial t}\left(\overline{\rho}\widetilde{E}\right)+\nabla\cdot\left(\overline{\rho}\widetilde{\vec{U}}\widetilde{H}\right) &= \nabla\cdot\left(\underline{\underline{\tau}}^{tot}\widetilde{\vec{U}}\right)-\nabla\cdot\vec{q^{tot}}+\nabla\cdot\vec{K}\end{aligned} \tag{206}$$

where (with approximations in $\overline{\tau}$)

$$\begin{aligned}\overline{P} &= \mathrm{EOS}\left(\overline{\rho},\widetilde{T}\right)\\ \widetilde{E} &= \widetilde{e}+\frac{\widetilde{\vec{U}}\cdot\widetilde{\vec{U}}}{2}+k\\ \widetilde{H} &= \widetilde{h}+\frac{\widetilde{\vec{U}}\cdot\widetilde{\vec{U}}}{2}+k\\ k_t &= \frac{\overline{\rho\vec{U}''\cdot\vec{U}''}}{2\overline{\rho}}\\ \underline{\underline{\tau}}^{tot} &= \overline{\underline{\underline{\tau}}}-\overline{\rho\vec{U}''\otimes\vec{U}''}\\ &= 2\left(\mu+\mu_t\right)\underline{\underline{\widetilde{D^\star}}}-\frac{2}{3}\overline{\rho}k_t\underline{\underline{I}}\\ \vec{q^{tot}} &= -\frac{\mu}{Pr}\nabla h+\overline{\rho\vec{U}''h''}\\ \vec{K} &= \overline{\underline{\underline{\tau}}\vec{U}''}-\overline{\rho\vec{U}''\frac{\vec{U}''\cdot\vec{U}''}{2}}\end{aligned} \tag{207}$$

Or in index notations:

$$\begin{aligned}\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial\overline{\rho}\widetilde{u}_j}{\partial x_j} &= 0\\ \frac{\partial\overline{\rho}\widetilde{u}_i}{\partial t}+\frac{\partial\overline{\rho}\widetilde{u}_i\widetilde{u}_j}{\partial x_j} &= -\frac{\partial\overline{P}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\overline{\tau_{ji}}-\overline{\rho}\widetilde{u_i''u_j''}\right)\\ \frac{\partial\overline{\rho}\widetilde{E}}{\partial t}+\frac{\partial\overline{\rho}\widetilde{u}_j\widetilde{H}}{\partial x_j} &= -\frac{\partial\overline{\rho}\widetilde{h''u_j''}}{\partial x_j}-\frac{\partial\overline{\rho}\widetilde{u}_i\widetilde{u_i''u_j''}}{\partial x_j}\\ &\quad-\frac{\partial}{\partial x_j}\left(\frac{\overline{\rho}\widetilde{u_i''u_i''u_j''}}{2}\right)+\frac{\partial\overline{u_i\tau_{ij}}}{\partial x_j}-\frac{\partial\overline{q}}{\partial x_j}\end{aligned} \tag{208}$$

For Boussinesq approximations:

$$
\begin{aligned}
-\overline{\rho\vec{U}''\otimes\vec{U}''} &= \mu_t\left(\nabla\widetilde{\vec{U}}+\left(\nabla\widetilde{\vec{U}}\right)^T\right)\\
&\quad-\frac{2}{3}\mu_t\left(\nabla\cdot\widetilde{\vec{U}}\right)\underline{\underline{I}}-\frac{2}{3}\overline{\rho}k_t\underline{\underline{I}}\\
-\overline{\rho\vec{U}''h''} &= -\frac{\mu_t}{Pr_t}\nabla\widetilde{h}
\end{aligned}
\tag{209}
$$

5.11 Transport Equation for Reynolds Stress

Starting from the Favre averaged momentum conservation equation Eq. (208) and using Eq. (202) for Reynolds stress calculation.

$$
\begin{aligned}
\frac{\partial\tau_{kl}^{turb}}{\partial t}+\widetilde{u}_j\frac{\partial\tau_{kl}^{turb}}{\partial x_j} &= -\left(\tau_{kj}^{turb}\frac{\partial\widetilde{u}_l}{\partial x_j}+\tau_{lj}^{turb}\frac{\partial\widetilde{u}_k}{\partial x_j}+\tau_{kl}^{turb}\frac{\partial\widetilde{u}_j}{\partial x_j}\right.\\
&\quad\left.-\overline{u_l''}\frac{\partial\overline{P}}{\partial x_k}-\overline{u_k''}\frac{\partial\overline{P}}{\partial x_l}\right)\\
&\quad+\left(\overline{P'\frac{\partial u_k''}{\partial x_l}}+\overline{P'\frac{\partial u_l''}{\partial x_k}}\right)\\
&\quad-\left(\frac{\partial}{\partial x_l}\overline{(P'u_k'')}+\frac{\partial}{\partial x_k}\overline{(P'u_l'')}\right.\\
&\quad\left.+\frac{\partial}{\partial x_j}\overline{(\rho u_k''u_l''u_j'')}\right)\\
&\quad+\left(\overline{u_l''\frac{\partial\tau_{kj}'}{\partial x_j}+u_k''\frac{\partial\tau_{lj}'}{\partial x_j}}\right.\\
&\quad\left.+\overline{u_l''}\frac{\partial\tau_{kj}}{\partial x_j}+\overline{u_k''}\frac{\partial\tau_{lj}}{\partial x_j}\right)
\end{aligned}
\tag{210}
$$

5.12 Transport Equation for Turbulent Kinetic Energy

Reynolds Averaging

$$
\begin{aligned}
\frac{\partial\overline{\rho}k_t}{\partial t}+\frac{\partial\overline{\rho u}_jk_t}{\partial x_j} &= -\overline{\rho}\overline{u_i'u_j'}\frac{\partial\overline{u}_i}{\partial x_j}-\frac{\partial}{\partial x_j}\left(\overline{u_j'\left(\frac{1}{2}\rho u_i'u_i'\right)}\right)\\
&\quad-\overline{u_k'\frac{\partial P}{\partial x_k}}+\frac{\partial}{\partial x_j}\left(\overline{u_i'\tau_{ij}}\right)-\overline{\tau_{ij}\frac{\partial u_i'}{\partial x_j}}
\end{aligned}
\tag{211}
$$

Favre Averaging

$$
\begin{aligned}
\frac{\partial\overline{\rho}k_t}{\partial t}+\frac{\partial\overline{\rho}\widetilde{u}_jk_t}{\partial x_j} &= -\overline{\rho}\widetilde{u_i''u_j''}\frac{\partial\widetilde{u}_i}{\partial x_j}-\frac{\partial}{\partial x_j}\left(\overline{u_j''\left(\frac{1}{2}\rho u_i''u_i''\right)}\right)\\
&\quad-\overline{u_k''\frac{\partial P}{\partial x_k}}+\frac{\partial}{\partial x_j}\left(\overline{u_i''\tau_{ij}}\right)-\overline{\tau_{ij}\frac{\partial u_i''}{\partial x_j}}
\end{aligned}
\tag{212}
$$

5.13 Transport Equation for Vortices

For votices (defined as the curl of velocity field):

$$
\frac{\partial}{\partial t}\left(\frac{\omega}{\rho}\right)+u_j\frac{\partial}{\partial x_j}\left(\frac{\omega}{\rho}\right)=\frac{\omega_j}{\rho}\frac{\partial\vec{U}}{\partial x_j}+\frac{1}{\rho^3}\nabla\rho\times\nabla P+\frac{1}{\rho}\mathrm{rot}\left(\vec{\Sigma}\right)
\tag{213}
$$

where

$$
\nabla\rho\times\nabla P=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial\rho}{\partial x_1} & \dfrac{\partial\rho}{\partial x_2} & \dfrac{\partial\rho}{\partial x_3}\\ \dfrac{\partial P}{\partial x_1} & \dfrac{\partial P}{\partial x_2} & \dfrac{\partial P}{\partial x_3}\end{vmatrix}
\tag{214}
$$

and

$$
\mathrm{rot}\left(\vec{\Sigma}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3}\\ \dfrac{1}{\rho}\dfrac{\partial\tau_{1j}}{\partial x_j} & \dfrac{1}{\rho}\dfrac{\partial\tau_{2j}}{\partial x_j} & \dfrac{1}{\rho}\dfrac{\partial\tau_{3j}}{\partial x_j}\end{vmatrix}
\tag{215}
$$

5.14 Turbulence Modelling

5.14.1 Subgrid-scale (SGS) Modelling for LES

Subgrid stress tensor $\underline{\underline{\tau_{SGS}}}=\overline{u_iu_j}-\overline{u}_i\overline{u}_j$.

Smagorinsky Model

$$
\nu_t=C\Delta^2\sqrt{2\overline{D}_{ij}\overline{D}_{ij}}
\tag{216}
$$

5.14.2 RANS Model

Mixing Length Model

$$
\nu_t=\frac{\mu_t}{\overline{\rho}}=\left|\frac{\partial\overline{U}}{\partial y}\right|l_m^2
\tag{217}
$$

Spalart-Allmaras Model A viscosity-like variable $\check{\nu}$ is introduced for this model:

$$
\nu_t=\check{\nu}f_{\nu1},\qquad f_{\nu1}=\frac{\chi^3}{\chi^3+C_{\nu1}^3},\qquad \chi=\frac{\check{\nu}}{\nu}
\tag{218}
$$

One additional transport equation:

$$
\begin{aligned}
\frac{\partial\check{\nu}}{\partial t}+\nabla\cdot\left(\check{\nu}\vec{U}\right) &= C_{b1}\left(1-f_{t2}\right)\check{D}\check{\nu}\\
&\quad+\frac{1}{\sigma}\left(\nabla\cdot\left(\left(\nu+\check{\nu}\right)\nabla\check{\nu}\right)+C_{b2}\left|\nabla\check{\nu}\right|^2\right)\\
&\quad-\left(C_{w1}f_w-\frac{C_{b1}}{\kappa^2}f_{t2}\right)\left(\frac{\check{\nu}}{d}\right)^2+f_{t1}\Delta U^2
\end{aligned}
\tag{219}
$$

where

$$
\begin{aligned}
\check{D} &= \sqrt{2\Omega_{ij}\Omega_{ij}}+\frac{\check{\nu}}{\kappa^2d^2}f_{\nu2}\\
f_{\nu2} &= 1-\frac{\chi}{1+\chi f_{\nu1}}\\
f_w &= g\left(\frac{1+C_{w3}^6}{g^6+C_{w3}^6}\right)^{\frac{1}{6}}\\
g &= r+C_{w2}(r^6-r)\\
r &= \frac{\check{\nu}}{\check{D}\kappa^2d^2}
\end{aligned}
\tag{220}
$$

and

$$
\begin{aligned}
f_{t1} &= C_{t1}g_te^{-C_{t2}\frac{\omega_t^2}{\Delta U^2}\left(d^2+g_t^2d_t^2\right)}\\
f_{t2} &= C_{t3}e^{-C_{t4}\chi^2}
\end{aligned}
\tag{221}
$$

where d is the distance to the cloest surface and the constants are:

$$\begin{aligned}\sigma &= \frac{2}{3}\\ C_{b1} &= 0.1355\\ C_{b2} &= 0.622\\ \kappa &= 0.41\\ C_{w1} &= \frac{C_{b1}}{\kappa^2} + \frac{1+C_{b2}}{\sigma}\\ C_{w2} &= 0.3\\ C_{w3} &= 2\\ C_{\nu 1} &= 7.1\\ C_{t1} &= 1\\ C_{t2} &= 2\\ C_{t3} &= 1.1\\ C_{t4} &= 2\end{aligned} \tag{222}$$

A modification is made:

$$\begin{aligned}C_{t3} &= 1.2\\ C_{t4} &= 0.5\end{aligned} \tag{223}$$

K-Epsilon Model Two additional transport equations:

Turbulent kinetic energy k_t

$$\frac{\partial \overline{\rho} k_t}{\partial t} + \frac{\partial \overline{\rho}\widetilde{u}_j k_t}{\partial x_j} = -\overline{\rho}\widetilde{u_i'' u_j''}\frac{\partial \widetilde{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k_t}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\overline{\mu}\frac{\partial k_t}{\partial x_j}\right) - \overline{\rho}\varepsilon \tag{224}$$

Rate of dissipation of turbulent kinetic energy ε

$$\frac{\partial \overline{\rho}\varepsilon}{\partial t} + \frac{\partial \overline{\rho}\widetilde{u}_j \varepsilon}{\partial x_j} = -C_{1\varepsilon}\frac{\varepsilon}{k_t}\overline{\rho}\widetilde{u_i'' u_j''}\frac{\partial \widetilde{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\overline{\mu}\frac{\partial \varepsilon}{\partial x_j}\right) - C_{2\varepsilon}\overline{\rho}\frac{\varepsilon^2}{k_t} \tag{225}$$

where Eddy viscosity μ_t is:

$$\mu_t = C_\mu \overline{\rho}\frac{k_t^2}{\varepsilon} \tag{226}$$

with $C_{1\varepsilon} = 1.44, C_{2\varepsilon} = 1.92, C_\mu = 0.09, \sigma_k = 1.0, \sigma_\varepsilon = 1.3$.

Finally, the Reynolds stresses can be calculated as in Section 5.7.

K-Omega Model Two additional transport equations are needed:

$$\frac{\partial \overline{\rho} k_t}{\partial t} + \frac{\partial \overline{\rho}\widetilde{u}_j k_t}{\partial x_j} = \overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} - B^*\overline{\rho}\omega k_t + \frac{\partial}{\partial x_j}\left(\left(\overline{\mu} + \sigma_k \frac{\partial k_t}{\partial x_j}\right)\right) \tag{227}$$

and

$$\frac{\partial \overline{\rho}\omega}{\partial t} + \frac{\partial \overline{\rho}\widetilde{u}_j \omega}{\partial x_j} = \frac{A\omega}{k_t}\overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} - B\overline{\rho}\omega^2 + \frac{\partial}{\partial x_j}\left((\overline{\mu} + \sigma_\omega)\frac{\partial k_t}{\partial x_j}\right) + \frac{\overline{\rho}\sigma_d}{\omega}\frac{\partial k_t}{\partial x_j}\frac{\partial \omega}{\partial x_j} \tag{228}$$

$$\mu_t = \frac{\overline{\rho} k_t}{\omega} \tag{229}$$

where ω is the specific eddy dissipation.

5.15 Turbulent Boundary Layer

5.15.1 Logarithmic Law of the Wall

u^+ and y^+ are non-dimensional velocities and wall coordinates, respectively. u_τ is friction velocity or shear velocity.

$$\begin{aligned}u_\tau &= \sqrt{\frac{\overline{\tau}_w}{\overline{\rho}_w}}\\ u^+ &= \frac{\overline{u}}{u_\tau}\\ y^+ &= \frac{\overline{\rho}_w u_\tau y}{\overline{\mu}_w}\end{aligned} \tag{230}$$

The Logarithmic law of the wall states that (κ is Von Kármán constant and C^+ is another constant, experimentally fitted to be 0.41 and 5.0 respectively for air, E is 9.8):

$$u^+ = \frac{1}{\kappa}\ln y^+ + C^+ \tag{231}$$

or

$$u^+ = \frac{1}{\kappa}\ln\left(E y^+\right) \tag{232}$$

or in dimensional form:

$$u = \frac{u_\tau}{\kappa}\ln\frac{y}{y_0} \tag{233}$$

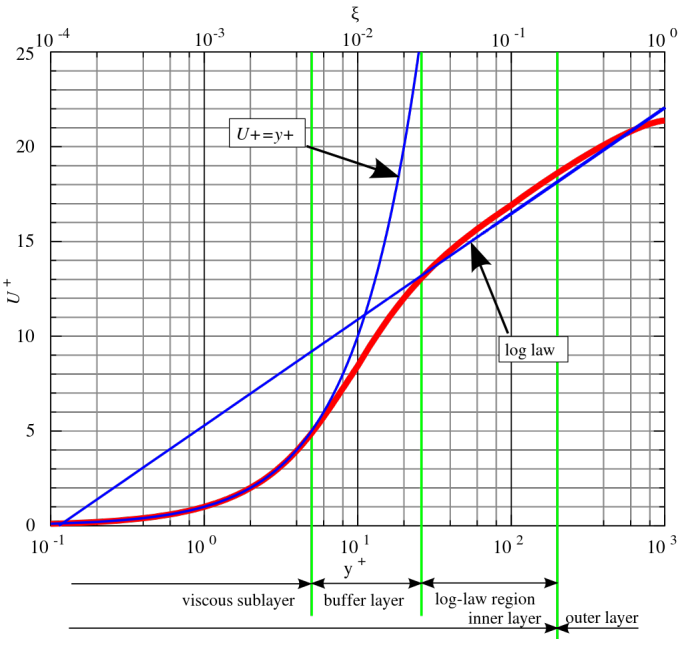


Figure 3: Turbulence log law of the wall. Source: `https://en.wikipedia.org/wiki/Law_of_the_wall`

As shown in Fig. 3:

$$\begin{cases} y^+ < 5 & y^+ = u^+ & \text{Viscous Sub-layer}\\ 5 < y^+ < 30 & & \text{Buffer Layer}\\ y^+ > 30 & u^+ = \frac{1}{\kappa}\ln y^+ + C^+ & \end{cases} \tag{234}$$

5.15.2 Spalding's Wall Function

An empirically fitted function to logarithmic law. A smooth function for $y^+ < 300$ [20].

$$y^+ = u^+ + \frac{1}{E}\left(e^{\kappa u^+} - 1 - \frac{\kappa u^+}{1!} - \frac{(\kappa u^+)^2}{2!} - \frac{(\kappa u^+)^3}{3!} - \frac{(\kappa u^+)^4}{4!}\right) \tag{235}$$

5.15.3 Wall Function Implementation in CFD

Criterion : By equating the viscous sublayer equation to log law region ($\kappa = 0.41$, $C^+ = 5.5$):

$$y^+ = \frac{1}{\kappa}\ln y^+ + C^+ \tag{236}$$

the critical y^+ is 11.45.

$$\begin{cases} y^+ < 11.45 \quad \tau_w = \mu\dfrac{\partial u}{\partial y} = \mu\dfrac{u_p}{y_p} \\ y^+ > 11.45 \quad \tau_w = \mu\dfrac{\partial u}{\partial y} = \dfrac{u_\tau u_p}{\frac{1}{\kappa}\ln y^+ + C^+} \end{cases} \tag{237}$$

where subscript p indicates the value at the first element away from the wall.

The second equation comes from

$$\tau_w = \rho u_\tau^2 = \rho u_\tau \frac{u}{u^+} = \frac{\rho u_\tau u_p}{\frac{1}{\kappa}\ln y^+ + C^+} \tag{238}$$

5.16 Boundary Layer with Compressibility

5.16.1 Semi-local Values

For compressible flow with density variations, semi-local values are used:

$$\begin{aligned} u_\tau^* &= \sqrt{\frac{\overline{\tau}_w}{\overline{\rho}}} \\ y^* &= \frac{\overline{\rho} u_\tau^* y}{\overline{\mu}} \end{aligned} \tag{239}$$

Reynolds Numer:

$$\begin{aligned} Re_G^* &= \frac{\rho_B u_B l}{\overline{\mu}} \\ Re_\tau^* &= \frac{l}{\delta_v^*} = \frac{\overline{\rho} u_\tau^* l}{\overline{\mu}} \end{aligned} \tag{240}$$

Mach Number:

$$\begin{aligned} \overline{Ma} &= \frac{\overline{u}}{c} \\ Ma_G^* &= \frac{u_B}{\overline{c}} \end{aligned} \tag{241}$$

5.16.2 Van Driest (VD) Transformation

To account for the effect of eddy diffusivity in the viscous sublayer, the Van Driest hypothesis states that the eddy diffusivity decreases as one nears the wall, instead of two regions where it is "on" or "off. It considers the density variation in the boundary layer for compressible flows. It is defined as:

$$u_{VD} = \int_0^{\widetilde{u}} \sqrt{\frac{\overline{\rho}}{\overline{\rho}_w}}\, \mathrm{d}\widetilde{u} \tag{242}$$

where u_{VD} is the average velocity in the boundary layer [21].

5.16.3 Trettel Larsson (TL) Transformation

Taking both density and viscosity variation into consideration [22]:

$$u_{TL} = \int_0^{\widetilde{u}} \sqrt{\frac{\overline{\rho}}{\overline{\rho}_w}}\left(1 + \frac{1}{2\overline{\rho}}\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}y}y - \frac{1}{\overline{\mu}}\frac{\mathrm{d}\overline{\mu}}{\mathrm{d}y}y\right)\mathrm{d}\widetilde{u} \tag{243}$$

5.17 Walz's Equation

A relation between mean velocity and mean temperature in the turbulent boundary layer:

$$\frac{\widetilde{T}}{\widetilde{T}_\delta} = 1 + \frac{\widetilde{T}_r - \widetilde{T}_w}{\widetilde{T}_\delta}\left(\frac{\widetilde{u}}{\widetilde{u}_\delta} - 1\right) + r\frac{\gamma - 1}{2}\overline{M}_\delta^2\left(1 - \left(\frac{\widetilde{u}}{\widetilde{u}_\delta}\right)^2\right) \tag{244}$$

where subscript δ refers to boundary layer edge, $\widetilde{T}_r$ is the recovery temperature and r is the recovery factor, where

$$\frac{T_r}{T_\delta} = 1 + r\frac{\gamma - 1}{2}M_e^2 \tag{245}$$

For non-ideal fluids:

$$\frac{\widetilde{h}}{\widetilde{h}_\delta} = 1 + \frac{\widetilde{h}_r - \widetilde{h}_w}{\widetilde{h}_\delta}\left(\frac{\widetilde{u}}{\widetilde{u}_\delta} - 1\right) + r\frac{\widetilde{u}_\delta^2}{2\widetilde{h}_\delta}\left(1 - \left(\frac{\widetilde{u}}{\widetilde{u}_\delta}\right)^2\right) \tag{246}$$

where

$$\widetilde{h}_r = \widetilde{h}_\delta + r\frac{\widetilde{u}_\delta^2}{2} \tag{247}$$

is the recovery enthalpy.

5.18 Morkovin Hypothesis

The turbulent Mach number is defined as

$$M_t = \frac{\sqrt{\overline{u'^2}}}{\overline{c}} \tag{248}$$

where

$$\sqrt{\overline{u'^2}} = \sqrt{\overline{u_1'^2 + u_2'^2 + u_3'^2}} \tag{249}$$

When M_t is small, the fluctuation in pressure, density and temperature are small compared to the averaged values. Methods like Favre average and van Drient transformation can be used to match compressible and incompressible turbulence statistical characters.

5.19 Turbulent Prandtl Number

Pr_t is the ratio of turbulent heat transfer and kinetic energy exchange.

$$Pr_t = \frac{\overline{u_1' u_2'} \dfrac{\partial \overline{T}}{\partial y}}{\overline{u_2' T'} \dfrac{\partial \overline{u}}{\partial y}} = \frac{\mu_t c_p}{K_t} \tag{250}$$

where turbulent viscosity is

$$\mu_t = \frac{\overline{\rho} \overline{u_1' u_2'}}{\dfrac{\partial \overline{u_1}}{\partial y}} \tag{251}$$

and turbulent diffusivity is

$$K_t = \frac{-c_p \overline{\rho} \overline{u_2' T'}}{\dfrac{\partial \overline{T}}{\partial y}} \tag{252}$$

For Favre averaged turbulent Prandtl number:

$$Pr_t = \frac{\widetilde{u_1'' u_2''} \dfrac{\partial \widetilde{h}}{\partial y}}{\widetilde{u_2'' h''} \dfrac{\partial \widetilde{u}}{\partial y}} \tag{253}$$

5.20 Transition

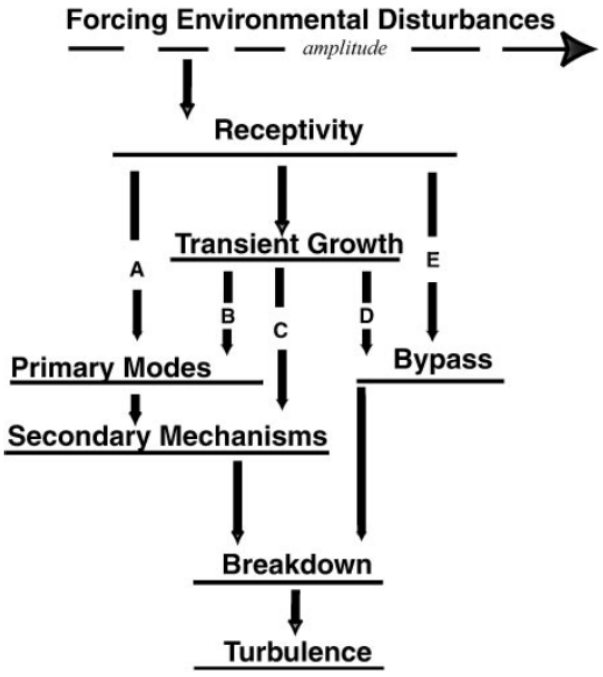


Figure 4: The path from receptivity to laminar-turbulent transition [23]

5.21 Entropy Generation

Volumetric entropy generation is

$$S_{gen} = \frac{\Phi}{T} \tag{254}$$

The terms can be split into mean and fluctuating terms [24]: Velocity terms:

$$\begin{aligned}
S_{gen-\overline{U}} = \frac{\mu}{\overline{T}} \Bigg(& 2 \left(\left(\frac{\partial \overline{u}_1}{\partial x} \right)^2 + \left(\frac{\partial \overline{u}_2}{\partial y} \right)^2 + \left(\frac{\partial \overline{u}_3}{\partial z} \right)^2 \right) \\
& + \left(\frac{\partial \overline{u}_1}{\partial y} + \frac{\partial \overline{u}_2}{\partial x} \right)^2 + \left(\frac{\partial \overline{u}_1}{\partial z} + \frac{\partial \overline{u}_3}{\partial x} \right)^2 \\
& + \left(\frac{\partial \overline{u}_2}{\partial z} + \frac{\partial \overline{u}_3}{\partial y} \right)^2 \\
& - \frac{2}{3} \left(\frac{\partial \overline{u}_1}{\partial x} + \frac{\partial \overline{u}_2}{\partial y} + \frac{\partial \overline{u}_3}{\partial z} \right)^2 \Bigg) \\
S_{gen-U'} = \frac{\mu}{\overline{T}} \Bigg(& 2 \left(\overline{\left(\frac{\partial u_1'}{\partial x} \right)^2} + \overline{\left(\frac{\partial u_2'}{\partial y} \right)^2} + \overline{\left(\frac{\partial u_3'}{\partial z} \right)^2} \right) \\
& + \overline{\left(\frac{\partial u_1'}{\partial y} + \frac{\partial u_2'}{\partial x} \right)^2} + \overline{\left(\frac{\partial u_1'}{\partial z} + \frac{\partial u_3'}{\partial x} \right)^2} \\
& + \overline{\left(\frac{\partial u_2'}{\partial z} + \frac{\partial u_3'}{\partial y} \right)^2} \\
& - \frac{2}{3} \overline{\left(\frac{\partial u_1'}{\partial x} + \frac{\partial u_2'}{\partial y} + \frac{\partial u_3'}{\partial z} \right)^2} \Bigg)
\end{aligned} \tag{255}$$

Temperature terms:

$$\begin{aligned}
S_{gen-\overline{T}} &= \frac{k}{\overline{T}^2} \left(\left(\frac{\partial \overline{T}}{\partial x} \right)^2 + \left(\frac{\partial \overline{T}}{\partial y} \right)^2 + \left(\frac{\partial \overline{T}}{\partial z} \right)^2 \right) \\
S_{gen-T'} &= \frac{k}{\overline{T}^2} \left(\overline{\left(\frac{\partial T'}{\partial x} \right)^2} + \overline{\left(\frac{\partial T'}{\partial y} \right)^2} + \overline{\left(\frac{\partial T'}{\partial z} \right)^2} \right)
\end{aligned} \tag{256}$$

6 Mixture

6.1 Mixture Type

6.1.1 Gibbs-Dalton's Law of Mixture

For a mixture of non-reacting gases. Also known as the law of partial pressure.

$$\frac{P_i}{P} = \frac{n_i}{n} \tag{257}$$

and for other properties:

$$\begin{aligned} V_m &= V_i \\ T_m &= T_i \\ P_m &= \sum_i P_i \\ m_m &= \sum_i m_i \\ \rho_m &= \sum_i \alpha_i \rho_i \end{aligned} \tag{258}$$

6.1.2 Amagat's Law of Mixture

More suitable for two-phase mixtures. Also known as law of partial volumes.

$$\frac{V_i}{V} = \frac{n_i}{n} \tag{259}$$

and for other properties:

$$\begin{aligned} P_m &= P_i \\ T_m &= T_i \\ V_m &= \sum_i V_i \\ m_m &= \sum_i m_i \\ \rho_m &= \sum_i \alpha_i \rho_i \end{aligned} \tag{260}$$

6.2 Mixture Fractions

6.2.1 Volume Fraction

A.K.A. void fraction. Dependent properties includes density.

$$\alpha_i = \frac{V_i}{V_m} \tag{261}$$

6.2.2 Mass Fraction

A.K.A vapour quality. Dependent properties includes most extensive properties: entropy s, enthalpy h, internal energy u, specific volume v etc.

$$\chi_i = \frac{m_i}{m_m} \tag{262}$$

6.2.3 Molar Fraction

The fraction of molecule number of different species.

$$\Phi_i = \frac{n_i}{n_m} \tag{263}$$

6.3 Two-phase Mixture

6.3.1 Wet Vapour Fractions

Volume Fraction

$$\begin{aligned} \alpha &= \frac{V_g}{V_m} = \frac{V_g}{V_l + V_g} \\ 1 - \alpha &= \frac{V_l}{V_m} = \frac{V_l}{V_l + V_g} \end{aligned} \tag{264}$$

Mass Fraction

$$\begin{aligned} \chi &= \frac{m_g}{m_m} = \frac{m_g}{m_l + m_g} \\ &= \frac{\alpha \rho_g}{\alpha \rho_g + (1 - \alpha) \rho_l} \\ 1 - \chi &= \frac{m_l}{m_m} = \frac{m_l}{m_l + m_g} \end{aligned} \tag{265}$$

6.3.2 Wet Vapour Fraction Relationships

Mixture Density

$$\begin{aligned} \rho_m &= \alpha \rho_g + (1 - \alpha) \rho_l \\ &= \frac{1}{\dfrac{\chi}{\rho_g} + \dfrac{1 - \chi}{\rho_l}} \end{aligned} \tag{266}$$

Mixture Extensive Properties Entropy s, enthalpy h, internal energy u, specific volume v:

$$\phi_m = \chi \phi_g + (1 - \chi) \phi_l \tag{267}$$

$\alpha - \chi$ Relation

$$\begin{aligned} \alpha &= \frac{\rho_m}{\rho_g} \chi \\ \chi &= \frac{\rho_g}{\rho_m} \alpha \end{aligned} \tag{268}$$

and

$$\alpha = \frac{1}{1 + S \dfrac{1 - \chi}{\chi} \dfrac{\rho_g}{\rho_l}} \tag{269}$$

where S is the velocity fraction of vapour and liquid stream $\frac{u_g}{u_l}$.

$$\begin{aligned} \rho_m &= \frac{\rho_l \rho_g}{\chi \rho_l + (1 - \chi) \rho_g} \\ \alpha &= \frac{\chi \rho_l}{\chi \rho_l + (1 - \chi) \rho_g} \\ 1 - \alpha &= \frac{(1 - \chi) \rho_g}{\chi \rho_l + (1 - \chi) \rho_g} \end{aligned} \tag{270}$$

6.3.3 Speed of Sound

By definition:

$$c_m = \sqrt{\left(\frac{\partial P}{\partial \rho_m} \right)_S} \tag{271}$$

Thermodynamic Equilibrium

$$
\begin{aligned}
c_m = & \left(\frac{\alpha \rho_m}{\rho_g} \left(\frac{\partial \rho_g}{\partial P} \right)_S + \frac{(1-\alpha)\,\rho_m}{\rho_l} \left(\frac{\partial \rho_l}{\partial P} \right)_S + \right. \\
& \left. \frac{\rho_m^2 \left(\rho_g - \rho_l\right)}{\rho_l \rho_g} \left(\frac{\partial \chi}{\partial P} \right)_S \right)^{-\frac{1}{2}} \\
= & \left(\frac{\alpha \rho_m}{\rho_g c_g^2} + \frac{(1-\alpha)\,\rho_m}{\rho_l c_l^2} + \frac{\rho_m^2 \left(\rho_g - \rho_l\right)}{\rho_l \rho_g} \left(\frac{\partial \chi}{\partial P} \right)_S \right)^{-\frac{1}{2}}
\end{aligned}
\tag{272}
$$

Relaxation Models The equilibrium models with different assumptions: Mechanical (pressure $P = P_i$), thermal (temperature $T = T_i$), and chemical (chemical potential $\mu = \mu_i$):

$$
c_m = \rho_m \left(Z_P + Z_{PT} + Z_{PT\mu} \right) \tag{273}
$$

where:

$$
\begin{aligned}
Z_P &= \frac{\alpha}{\rho_g c_g^2} + \frac{1-\alpha}{\rho_l c_l^2} \\
Z_{PT} &= \frac{1}{T} \frac{C_{P,g} C_{P,l} \left(\zeta_l - \zeta_g\right)^2}{C_{P,g} + C_{P,l}} \\
&= T \frac{C_{P,g} C_{P,l}}{C_{P,g} + C_{P,l}} \left(\frac{G_l}{\rho_l c_l^2} - \frac{G_g}{\rho_g c_g^2} \right)^2 \\
Z_{PT\mu} &= T \left(C_{P,g} + C_{P,l} \right) \\
&\quad \left(\frac{\rho_l - \rho_g}{\rho_g \rho_l \left(h_l - h_g\right)} + \frac{\zeta_g C_{P,g} + \zeta_l C_{P,l}}{T \left(C_{P,g} + C_{P,l} \right)} \right)^2
\end{aligned}
\tag{274}
$$

and

$$
\begin{aligned}
\zeta &= \left(\frac{\partial T}{\partial P} \right)_S = \frac{GT}{\rho c^2} \\
G &= \frac{1}{\rho} \left(\frac{\partial P}{\partial E} \right)_V \\
c_p &= \left(\frac{\partial H}{\partial T} \right)_P = T \left(\frac{\partial S}{\partial T} \right)_P \\
C_P &= \rho \alpha c_p
\end{aligned}
\tag{275}
$$

Frozen Homogeneous Expression

$$
\begin{aligned}
c_{\text{bubble}} &= \left(\frac{\alpha \rho_m}{\rho_g c_g^2} + \frac{(1-\alpha)\,\rho_m}{\rho_l c_l^2} \right)^{-\frac{1}{2}} \\
c_{\text{liquid}} &= \left(\alpha \rho_l + (1-\alpha)\,\rho_g \right)^{\frac{1}{2}} \\
&\quad \left(\frac{\alpha \rho_l}{c_g^2} + \frac{(1-\alpha)\,\rho_g}{c_l^2} \right)^{-\frac{1}{2}}
\end{aligned}
\tag{276}
$$

7 Turbomachinery

7.1 Isentropic Efficiency

Since $\Delta S > 0$:

$$\begin{aligned} \eta_{comp} &= \frac{h_{2s} - h_1}{h_2 - h_1} \\ \eta_{turb} &= \frac{h_1 - h_2}{h_1 - h_{2s}} \end{aligned} \tag{277}$$

7.2 Axial Compressor Analysis

Euler Turbomachinery Equation

$$\begin{aligned} \dot{W} &= \dot{m}U\left(C_{\theta_2} - C_{\theta_3}\right) \\ &= \dot{m}UC_Z\left(\tan\alpha_2 - \tan\alpha_3\right) \\ &= \dot{m}UC_Z\left(\tan\beta_2 - \tan\beta_3\right) \end{aligned} \tag{278}$$

Degree of Reaction

$$\begin{aligned} \Lambda = \frac{\Delta h_{rotor}}{\Delta h_{stage}} &= \frac{h_2 - h_3}{h_{01} - h_{03}} = \frac{h_2 - h_3}{h_{02} - h_{03}} \\ &= \frac{\frac{1}{2}C_Z^2\left(\tan^2\beta_3 - \tan^2\beta_2\right)}{UC_Z\left(\tan\beta_2 - \tan\beta_3\right)} \\ &= -\frac{C_Z}{2U}\left(\tan\beta_2 + \tan\beta_3\right) \end{aligned} \tag{279}$$

Special cases:

Λ	Flow Angle	Note
0	$\beta_3 = -\beta_2$	Symmetric rotor
0.5	$\alpha_3 = -\beta_2$ $\beta_3 = -\alpha_2$	Mirrored velocity triangle

Stage Loading and Flow Coefficient

$$\begin{aligned} \psi &= \frac{\dot{W}}{\dot{m}U^2} = \frac{-\Delta h}{U^2} \\ \phi &= \frac{C_Z}{U} \end{aligned} \tag{280}$$

and

$$\psi = \phi\left(\tan\beta_2 - \tan\beta_3\right) \tag{281}$$

7.3 Turbomachinery Loss

7.3.1 General Equations

Stagnation Pressure Loss Coefficient

$$\begin{aligned} Y_{comp} &= \frac{P_{01} - P_{02}}{P_{01} - P_1} \\ Y_{turb} &= \frac{P_{01} - P_{02}}{P_{02} - P_2} \end{aligned} \tag{282}$$

Energy Loss Coefficient

$$\begin{aligned} \zeta_{comp} &= \frac{h_2 - h_{2s}}{h_{01} - h_1} = \frac{h_2 - h_{2s}}{\frac{1}{2}U_2^2} \\ \zeta_{turb} &= \frac{h_2 - h_{2s}}{h_{02} - h_2} = \frac{h_2 - h_{2s}}{\frac{1}{2}U_1^2} \end{aligned} \tag{283}$$

Entropy Loss Coefficient

$$\begin{aligned} \zeta_{comp} &= \frac{T_2\Delta S}{h_{01} - h_1} \\ \zeta_{turb} &= \frac{T_2\Delta S}{h_{02} - h_2} \end{aligned} \tag{284}$$

Polytropic Efficiency

$$\eta_{poly} = \frac{v\mathrm{d}P}{\mathrm{d}h} \tag{285}$$

For perfect gas:

$$\eta_{poly} = \frac{\gamma - 1}{\gamma}\frac{\ln\frac{P_2}{P_1}}{\ln\frac{T_2}{T_1}} \tag{286}$$

and for non-ideal fluid:

$$\eta_{poly} = \frac{\sum(\Delta s)_h}{(\Delta s)_p} = \frac{\sum(\Delta s)_h}{\Delta s + \sum(\Delta s)_h} \tag{287}$$

Entropy Rise Coefficient

$$\zeta = \frac{\overline{T}_0\left(s_e - s_i\right)}{0.5\bar{u}_i^2} \tag{288}$$

7.3.2 Loss mechanism

Boundary Layer Loss Definition:

$$\begin{aligned} \dot{S}_a &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\theta \left(\rho U_x\left(s - s_\theta\right)\right)\ \mathrm{d}y \\ &= \int_0^\theta \frac{1}{T}\tau_{yx}\ \mathrm{d}U_x \end{aligned} \tag{289}$$

Approximation model:

$$\dot{S}_a = \int_0^x \frac{\rho U_\theta^3 C_d}{T_\theta}\mathrm{d}x \tag{290}$$

where dissipation coefficient is:

$$C_d = \frac{T_\theta \dot{S}_a}{\rho_\theta U_\theta^3} \tag{291}$$

Mixing Loss The non-dimensional form of the impulse function of general mixing of two streams:

$$\widetilde{F}_m = \frac{I_m}{\dot{m}_m\sqrt{C_pT_{0m}}} = \frac{\sqrt{\gamma - 1}\left(1 + \gamma M_m^2\right)}{\gamma M_m\sqrt{\left(1 + \frac{\gamma - 1}{2}M_m^2\right)}} \tag{292}$$

and the impulse function $\Im$ is defined as:

$$\Im = PA + \dot{m}_1U_1 + \dot{m}_2U_2 \tag{293}$$

$$\begin{aligned} \Delta s = C_p\frac{\dot{m}_c}{\dot{m}_m}\Bigg(&\left(1 + \frac{\gamma - 1}{2}M_m^2\right)\frac{T_{0c} - T_{0m}}{T_{0m}} \\ &+ \left(\gamma - 1\right)M_m^2\left(1 - \frac{U_c\cos\alpha}{U_m}\right)\Bigg) \end{aligned} \tag{294}$$

For main flow m is mixed with flow c, assuming $T_{0m} = T_{0c}$; $\dot{m}_c \ll \dot{m}_m$ and incompressible flow:

$$T\dot{m}_m \Delta s = T\dot{S} = \dot{m}_c \left(U_m^2 - U_m U_c \cos\alpha\right) \tag{295}$$

For modelling purposes, an approximation below assumes all the relative kinetic energy of the small flow $\{\}_c$ is lost.

$$\dot{S} = \frac{1}{2T_m}\dot{m}_c \left(\left(U_m - U_c\cos\alpha\right)^2 + \left(U_c \sin\alpha\right)^2\right) \tag{296}$$

Shock Loss For adiabatic flow:

$$\begin{aligned}\Delta s &= R\ \ln\left(\frac{P_{02}}{P_{01}}\right) \\ &= c_v\left(\ln\left(\frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}\right)\right. \\ &\quad \left. -\gamma\ \ln\left(\frac{(\gamma+1)\,M_1^2}{(\gamma-1)\,M_1^2+2}\right)\right) \\ &= c_v \frac{2\gamma\,(\gamma-1)}{3\,(\gamma+1)^2}\left(M_1^2-1\right)^3 \\ &\quad + \mathcal{O}\left(M_1^2-1\right)^4\end{aligned} \tag{297}$$

For oblique shock approximation:

$$\Delta s = c_v \frac{2\gamma\,(\gamma-1)}{3\,(\gamma+1)^2}\left(M_1^2\sin^2\theta - 1\right)^3 \tag{298}$$

Other handy relations:

$$\frac{\Delta P}{P_1} = \frac{2\gamma}{\gamma+1}\left(M_1^2-1\right) \tag{299}$$

$$\Delta s \approx R\frac{\gamma+1}{12\gamma^2}\left(\frac{\Delta P}{P_1}\right)^3 \tag{300}$$

7.3.3 Loss types

1. Profile loss
   (a) Attached boundary loss

$$\dot{S} = \sum C_s \int_0^1 \frac{C_d \rho U_\theta^3}{T}\mathrm{d}\left(\frac{x}{C_s}\right) \tag{301}$$

   (b) Separated boundary loss
   (c) Trailing edge loss

$$\frac{T\Delta S}{0.5U^2} = -\frac{C_{pb}t}{w} + \frac{2\theta}{w} + \left(\frac{\delta+t}{w}\right)^2 \tag{302}$$

   which involves drag, mixing and blockage loss, and base pressure coefficient with reference at blade exit condition:

$$C_{pb} = \frac{P_b - P_{ref}}{0.5\rho V_{ref}^2} \tag{303}$$

2. Tip leakage loss

$$T_m\dot{S} = C_{dis} g \int_0^1 U_s^2\left(1 - \frac{U_p}{U_s}\right)\sqrt{U_s^2 - U_p^2}\frac{\mathrm{d}z}{C} \tag{304}$$

   where discharge coefficient is around 0.8.
3. Endwall loss
4. Shock loss
5. Others
   (a) Unsteadiness
   (b) Windage
   (c) Leaving kinetic energy

8 Non-Dimensional Numbers

Name	Expression	Interpretation	Notes
Reynolds	$Re = \frac{\rho U L}{\mu}$	$\frac{\text{inertial force}}{\text{viscous force}}$	All fluid problems
Euler	$Eu = \frac{P}{\rho U^2}$	$\frac{\text{pressure force}}{\text{inertial force}}$	Pressure and pressure difference are of interest
Eckert	$Ek = \frac{U^2}{\Delta H}$	$\frac{\text{kinetic energy}}{\text{enthalpy}}$	Dissipation is of interest
Biot	$Bi = \frac{hL}{k_s}$	$\frac{\text{conductive resistance}}{\text{convective resistance}}$	Heat transfer
Fourier	$Fo = \frac{\alpha t}{L^2}$	$\frac{\text{rate of conduction}}{\text{rate of heat storage}}$	Heat transfer
Nusselt	$Nu = \frac{hL}{k_f}$	$\frac{\text{convection}}{\text{pure conduction}}$	Heat transfer
Prandtl	$Pr = \frac{\nu}{\alpha} = \frac{\mu C_p}{k}$	$\frac{\text{momentum diffusion}}{\text{heat diffusion}}$	Heat transfer
Schmidt	$Sc = \frac{\nu}{\rho D}$	$\frac{\text{momentum diffusion}}{\text{mass diffusion}}$	Heat transfer
Turbulent Prandtl	$Pr_t = \frac{\epsilon_M}{\epsilon_H} = \frac{\mu C_p}{k}$	$\frac{\text{eddy viscosity}}{\text{eddy heat diffusivity}}$	Heat transfer
Turbulent Schmidt	$Sc_t = \frac{\epsilon_M}{\epsilon_D}$	$\frac{\text{eddy viscosity}}{\text{eddy mass diffusivity}}$	Turbulence
Grashof	$Gr = \frac{g\kappa_P \left(T_s - T_\infty\right) L^3}{\nu^2}$	$\frac{\text{buoyancy}}{\text{viscocity}}$	Heat transfer
C_f	$\frac{\tau}{\frac{1}{2}\rho U_\infty^2}$	drag coefficient	Fluids
C_D/L	$\frac{F}{\frac{1}{2}\rho U_\infty^2 S}$	drag(F=D) / lift(F=L) coefficient	Aircraft Design
$\widehat{\dot{m}_a}$	$\frac{\dot{m}_a\sqrt{C_p T_{02}}}{P_{02}A}$	air mass flow rate	Aircraft Design
$\widehat{\dot{m}_f}$	$\frac{\dot{m}_f LCV}{P_{02}A\sqrt{C_p T_{02}}}$	fuel mass flow rate	Aircraft Design
$\widehat{\omega}$	$\frac{\omega D}{\sqrt{\gamma R T_{02}}}$	rotational speed	Aircraft Design
$\widehat{F_{trust}}$	$\frac{\dot{m}_a U + P_a A}{A P_{02}}$	thrust	Aircraft Design
ψ	$\frac{\dot{W}}{\dot{m}U^2} = \frac{-\Delta h}{U^2}$	stage loading coefficient	Turbomachinery
ϕ	$\frac{C_Z}{U}$	flow coefficient	Turbomachinery
Λ	$\frac{\Delta h_{rotor}}{\Delta h_{stage}}$	degree of reaction	Turbomachinery

A Mathematics

A.1 Calculus Basics

Chain rule

$$\frac{\partial X}{\partial Z} = \frac{\partial X}{\partial Y}\frac{\partial Y}{\partial Z} \tag{305}$$

Cyclic rule

$$\left(\frac{\partial X}{\partial Y}\right)_Z \left(\frac{\partial Y}{\partial Z}\right)_X \left(\frac{\partial Z}{\partial X}\right)_Y = -1 \tag{306}$$

Higher order derivatives

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \tag{307}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}y^2} = -\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \div \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 \tag{308}$$

A.2 Partial Differential Equations Classification

For a general form of second-order PDE:

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_x + e\phi_y + f = 0 \tag{309}$$

the classification is based on the shape of characteristic equation:

$$\begin{aligned} b^2 - 4ac < 0, &\quad \text{Elliptic} \\ b^2 - 4ac = 0, &\quad \text{Parabolic} \\ b^2 - 4ac > 0, &\quad \text{Hyperbolic} \end{aligned} \tag{310}$$

From a numerical point of view, hyperbolic and parabolic PDEs are initial value problems (time evolution) while elliptic PDEs are boundary value problem (static solution).

First order PDEs are all hyperbolic.

A.3 Quasi-linear Partial Differential Equations Classification

A general form of quasi-linear PDE with 2 spacial dimension:

$$\begin{aligned} a_1\frac{\partial u}{\partial x} + b_1\frac{\partial u}{\partial y} + c_1\frac{\partial v}{\partial x} + d_1\frac{\partial v}{\partial y} = f_1 \\ a_2\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + c_2\frac{\partial v}{\partial x} + d_2\frac{\partial v}{\partial y} = f_2 \end{aligned} \tag{311}$$

where u, v are functions of (x, y), a, b, c, d, f can be functions of (x, y, u, v).

A characteristic line is a curve in the domain of the PDE along which the solution of the PDE is constant (PDE becomes ODE). For characteristic lines the derivatives of u and v are inderterminant. Two methods can be applied:

A.3.1 Method of Cramers Rule

For total derivatives of u, v:

$$\begin{aligned} \mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y \\ \mathrm{d}v = \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y \end{aligned} \tag{312}$$

system can be rewritten in a linear form $\underline{\underline{A}}\ \underline{x} = \underline{b}$:

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ \mathrm{d}x & \mathrm{d}y & 0 & 0 \\ 0 & 0 & \mathrm{d}x & \mathrm{d}y \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} \tag{313}$$

For $\underline{x}$ to be indeterminant, $|\underline{\underline{A}}| = 0$.

$$\begin{aligned} (a_1c_2 - a_2c_1)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\ -(a_1d_2 - a_2d_1 + b_1c_2 - b_2c_1)\frac{\mathrm{d}y}{\mathrm{d}x} \\ +(b_1d_2 - b_2d_1) = 0 \end{aligned} \tag{314}$$

which is second order ODE in form of $ax^2 + bx + c = 0$. Hence similarly:

$$\begin{aligned} b^2 - 4ac < 0, &\quad \text{Elliptic} \\ &\quad \text{(imaginary characteristic lines)} \\ b^2 - 4ac = 0, &\quad \text{Parabolic} \\ b^2 - 4ac > 0, &\quad \text{Hyperbolic} \\ &\quad \text{(two real characteristic lines)} \end{aligned} \tag{315}$$

A.3.2 Eigenvalue Method

For $\underline{w}$:

$$\underline{w} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{316}$$

the quasi linear system can be re-written:

$$\begin{aligned} \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}\frac{\partial \underline{w}}{\partial x} + \begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix}\frac{\partial \underline{w}}{\partial y} = 0 \\ \underline{\underline{K}}\frac{\partial \underline{w}}{\partial x} + \underline{\underline{M}}\frac{\partial \underline{w}}{\partial y} = 0 \\ \frac{\partial \underline{w}}{\partial x} + \underline{\underline{K}}^{-1}\underline{\underline{M}}\frac{\partial \underline{w}}{\partial y} = 0 \\ \frac{\partial \underline{w}}{\partial x} + \underline{\underline{N}}\frac{\partial \underline{w}}{\partial y} = 0 \end{aligned} \tag{317}$$

The eigenvalues of $\underline{\underline{N}}$ determine the classification. If all eigenvalues are real, system is hyperbolic. If all eigenvalues are complex, the equation is elliptic. If the eigenvalues are a mix of both real and complex number, the equations exhibits a mixed hyperbolic-elliptic nature.

A.4 Method of Characteristics

Two real characteristic lines are needed for method of characteristics. Followsing are the steps:

1. Identify the characteristic curves: These are the curves along which the solution is constant. They are determined by the initial conditions of the problem.
2. Transform the partial differential equation: Using the chain rule, the partial differential equation can be transformed into a system of ordinary differential equations along the characteristic curves.

3. Solve the system of ordinary differential equations: This can typically be done using standard techniques for solving first-order differential equations, such as separation of variables or integrating factors.

4. Express the solution in terms of the original variables: Once the system of ordinary differential equations has been solved, the solution can be expressed in terms of the original variables of the problem.

A.5 Behaviours of Physical Governing Equations

A.5.1 Hyperbolic

Two real characteristic curves exist through a given point. Following flows are governed by hyperbolic PDEs:

- Steady inviscid supersonic flow
- Unsteady inviscid flow (w.r.t. time, hence always marching in time)

A.5.2 Parabolic

- Steady boundary layer flows
- "Parabolized" viscous flows
- Unsteady thermal conduction
- Unsteady incompressible Couette flows.

A.5.3 Elliptic

The disturbance is left everywhere in the domain.

- Steady subsonic inviscid flow
- Incompressible inviscid flow

A.6 General Forms of Partial Differential Equations

Advection' Equation

$$\frac{\partial \phi}{\partial t} + a\frac{\partial \phi}{\partial x} = 0 \tag{318}$$

Diffusion Equation

$$\frac{\partial \phi}{\partial t} = \nu\frac{\partial^2 \phi}{\partial x^2} \tag{319}$$

Parabolic.

Burgers' Equation

$$\frac{\partial \phi}{\partial t} + \phi\frac{\partial \phi}{\partial x} = \nu\frac{\partial^2 \phi}{\partial x^2} \tag{320}$$

Hyperbolic or parabolic if ν is large.

Inviscid Burgers' Equation

$$\frac{\partial \phi}{\partial t} + \phi\frac{\partial \phi}{\partial x} = 0 \tag{321}$$

Hyperbolic.

Laplace Equation

$$\nabla^2\phi = \Delta\phi = 0 \tag{322}$$

Elliptic.

Poisson Equation

$$\nabla^2\phi = f(x) \tag{323}$$

Elliptic.

Diffusion Equation

$$\nabla^2\phi - k\frac{\partial \phi}{\partial t} = f(x) \tag{324}$$

Wave Function

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{325}$$

Hyperbolic.

Biharmonic Equation

$$\Delta^2\phi = 0 \tag{326}$$

A.7 Physical Governing Partial Differential Equations

A.7.1 Heat Equation

From governing equation:

$$\nabla \cdot k\nabla T + \dot{q}_{gen} = \rho c_p\frac{\partial T}{\partial t} \tag{327}$$

with constant k:

$$\Delta T + \frac{\dot{q}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{328}$$

without heat generation:

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} \tag{329}$$

The PDE is parabolic and hence an initial value problem.

A.8 Boundary Conditions

Dirichlet

$$\phi = C \tag{330}$$

Neumann

$$\frac{\partial \phi}{\partial x} = C \tag{331}$$

Robin

$$a\phi + b\frac{\partial \phi}{\partial x} = C \tag{332}$$

A.9 Trigonometric formulas

Identities

$$\begin{aligned} \sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \sin\theta \end{aligned} \tag{333}$$

Relations

$$\begin{aligned}\sin^2\theta+\cos^2\theta&=1\\ \sec^2\theta-\tan^2\theta&=1\\ \csc^2\theta-\cot^2\theta&=1\end{aligned}\tag{334}$$

Complementary Angles

$$\begin{aligned}\sin(90^\circ-\theta)&=\cos\theta\\ \cos(90^\circ-\theta)&=\sin\theta\\ \tan(90^\circ-\theta)&=\cot\theta\\ \cot(90^\circ-\theta)&=\tan\theta\\ \sec(90^\circ-\theta)&=\csc\theta\\ \csc(90^\circ-\theta)&=\sec\theta\end{aligned}\tag{335}$$

Sine Relations

$$\begin{aligned}\sin(\alpha+\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \sin(\alpha-\beta)&=\sin\alpha\cos\beta-\cos\alpha\sin\beta\end{aligned}\tag{336}$$

Cosine Relations

$$\begin{aligned}\cos(\alpha+\beta)&=\cos\alpha\cos\beta-\sin\alpha\sin\beta\\ \cos(\alpha-\beta)&=\cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}\tag{337}$$

A.10 Index Notations

A.10.1 Kronecker Delta

Identity matrix. Note that $\delta_{kk}=3$ (implying summation).

$$\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}\tag{338}$$

A.10.2 Levi-Civita Symbol

Even cycle mean 123, 231, 312 and odd cycle is 321, 213, 132.

$$\epsilon_{ijk}=\begin{cases}1 & \text{if } ijk \text{ is even permutation}\\ -1 & \text{if } ijk \text{ is odd permutation}\\ 0 & \text{otherwise}\end{cases}\tag{339}$$

hence

$$\begin{aligned}\epsilon_{ijk}&=\epsilon_{jki}=\epsilon_{kij}\\ \epsilon_{ijk}&=-\epsilon_{jik}\end{aligned}\tag{340}$$

and $\delta-\epsilon$ identity:

$$\epsilon_{ijk}\epsilon_{ipq}=\delta_{jp}\delta_{kq}-\delta_{kp}\delta_{jq}\tag{341}$$

A.10.3 Others

Coordinate vector

$$\vec{x}=\underline{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=x_1\vec{e}_1+x_2\vec{e}_2+x_3\vec{e}_3\tag{342}$$

Index notation for partial differentiation (e.g. for 3D)

$$f_{,i}=\frac{\partial f}{\partial x_i}=\frac{\partial f}{\partial x_1}\vec{e}_1+\frac{\partial f}{\partial x_2}\vec{e}_2+\frac{\partial f}{\partial x_3}\vec{e}_3\tag{343}$$

hence

$$x_{i,j}=\delta_{ij}\tag{344}$$

Determinate of a tensor

$$\mathrm{Det}\left(\underline{\underline{A}}\right)=\frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}A_{pi}A_{qj}A_{rk}=\epsilon_{ijk}A_{i1}A_{j2}A_{k3}\tag{345}$$

Trace

$$\mathrm{Tr}\left(\underline{\underline{A}}\right)=A_{ii}\tag{346}$$

Inverse of a tensor

$$A^{-1}_{ij}=\frac{1}{2\ \mathrm{Det}\left(\underline{\underline{A}}\right)}\epsilon_{ipq}\epsilon_{jrs}A_{rp}A_{sq}\tag{347}$$

Modulus of a vector

$$|\underline{u}|=\sqrt{\underline{u}\cdot\underline{u}}=\sqrt{u_iu_i}\tag{348}$$

Unit normal vector

$$\underline{n}=\frac{\nabla\phi}{|\nabla\phi|}\quad n_i=\frac{\phi_{,i}}{\sqrt{\phi_{,j}\phi_{,j}}}\tag{349}$$

Nabla operator ∇

$$\nabla_i=\frac{\partial}{\partial x_i}\tag{350}$$

A.11 Tensor basics

A.11.1 Tensor Construction

Vector and tensor

$$\vec{u}=\underline{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=u_1\vec{e}_1+u_2\vec{e}_2+u_3\vec{e}_3\tag{351}$$

$$\underline{\underline{A}}=\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix}\tag{352}$$

Vector and matrix bases

$$\begin{pmatrix}\vec{e}_1\\\vec{e}_2\\\vec{e}_3\end{pmatrix}\quad\begin{pmatrix}\vec{e}_1\vec{e}_1&\vec{e}_1\vec{e}_2&\vec{e}_1\vec{e}_3\\\vec{e}_2\vec{e}_1&\vec{e}_2\vec{e}_2&\vec{e}_2\vec{e}_3\\\vec{e}_3\vec{e}_1&\vec{e}_3\vec{e}_2&\vec{e}_3\vec{e}_3\end{pmatrix}\tag{353}$$

Unit vector $\vec{e}$

$$\begin{aligned}\vec{e}_i\cdot\vec{e}_j&=\delta_{ij}\\ \vec{e}_i\times\vec{e}_j&=\epsilon_{ijk}\vec{e}_k\end{aligned}\tag{354}$$

Cylindrical Coordinates

$$\vec{e}=\begin{pmatrix}\vec{e}_r\\\vec{e}_\theta\\\vec{e}_z\end{pmatrix}\qquad \mathrm{d}\vec{e}=\begin{pmatrix}\mathrm{d}r\\r\mathrm{d}\theta\\\mathrm{d}z\end{pmatrix}\tag{355}$$

where:

$$\begin{aligned}\frac{\partial\vec{e}_r}{\partial\theta}&=\vec{e}_\theta\\ \frac{\partial\vec{e}_\theta}{\partial\theta}&=-\vec{e}_r\end{aligned}\tag{356}$$

Spherical Coordinates

$$\vec{e} = \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\phi \end{pmatrix} \tag{357}$$

where:

$$\begin{aligned} \frac{\partial \vec{e}_r}{\partial \theta} &= \sin\phi\ \vec{e}_\theta \\ \frac{\partial \vec{e}_\theta}{\partial \theta} &= -\cos\phi\ \vec{e}_\phi - \sin\phi\ \vec{e}_r \\ \frac{\partial \vec{e}_\phi}{\partial \theta} &= \cos\phi\ \vec{e}_\theta \\ \frac{\partial \vec{e}_r}{\partial \phi} &= \vec{e}_\phi \\ \frac{\partial \vec{e}_\phi}{\partial \phi} &= -\vec{e}_r \end{aligned} \tag{358}$$

A.11.2 Special Matrices

Identity Matrix

$$\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{359}$$

Unitary Matrix

$$\underline{\underline{A}}^{-1} = \underline{\underline{A}}^{T} \tag{360}$$

and hence

$$\underline{\underline{I}} = \underline{\underline{A}}\ \underline{\underline{A}}^{T} = \underline{\underline{A}}^{T}\underline{\underline{A}} \tag{361}$$

Hermitian matrix A.k.a. self-adjoint matrix.

$$\underline{\underline{A}} = \overline{\underline{\underline{A}}^{T}} = \underline{\underline{A}}^{H} \tag{362}$$

where $\underline{\underline{A}}^{H}$ is the conjugate transpose of the matrix $\underline{\underline{A}}$

Orthogonal Vectors

$$\begin{aligned} \underline{n}\cdot\underline{n} = \underline{m}\cdot\underline{m} = \underline{l}\cdot\underline{l} = 1 \\ \underline{n}\cdot\underline{m} = \underline{m}\cdot\underline{l} = \underline{n}\cdot\underline{l} = 0 \end{aligned} \tag{363}$$

and

$$\underline{n}\otimes\underline{n} + \underline{m}\otimes\underline{m} + \underline{l}\otimes\underline{l} = \underline{\underline{I}} \tag{364}$$

A.11.3 Coordinate Transformation of Vectors

Cylindrical to Cartesian:

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} \tag{365}$$

Cartesian to cylindrical:

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \tag{366}$$

Spherical to Cartesian:

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \cos\theta\sin\phi & -\sin\theta & \cos\theta\cos\phi \\ \sin\theta\sin\phi & \cos\theta & \sin\theta\cos\phi \\ \cos\phi & 0 & -\sin\phi \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \\ u_\phi \end{pmatrix} \tag{367}$$

Cartesiam to Spherical:

$$\begin{pmatrix} u_r \\ u_\theta \\ u_\phi \end{pmatrix} = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta & \cos\theta & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \tag{368}$$

A.12 Tensor Operators

A.12.1 Dot Operator

Vector dot product

$$(\underline{a}\cdot\underline{b}) = a_i v_i \tag{369}$$

Vector dot with tensor

$$\begin{aligned} \left(\underline{\underline{A}}\cdot\underline{a}\right)_i &= A_{ij}a_j \\ \left(\underline{a}\cdot\underline{\underline{A}}\right)_i &= a_j A_{ji} \end{aligned} \tag{370}$$

Tensor dot with tensor

$$\begin{aligned} \left(\underline{\underline{A}}\cdot\underline{\underline{B}}\right)_{ij} &= A_{ik}B_{kj} \\ \left(\underline{\underline{A}}:\underline{\underline{B}}\right) &= A_{ij}B_{ij} \\ \left(\underline{\underline{A}}\cdot\cdot\underline{\underline{B}}\right) &= A_{ij}B_{ji} \end{aligned} \tag{371}$$

A.12.2 Cross Operator

Cross vectors

$$(\underline{a}\times\underline{b})_i = \epsilon_{ijk}a_j b_k \tag{372}$$

and

$$\underline{a}\times\underline{b} = -\underline{b}\times\underline{a} \tag{373}$$

Vector cross with tensor

$$\begin{aligned} \left(\underline{a}\times\underline{\underline{A}}\right)_{ij} &= \epsilon_{ipq}a_p A_{qj} \\ \left(\underline{\underline{A}}\times\underline{a}\right)_{ij} &= \epsilon_{jpq}a_q A_{ip} \end{aligned} \tag{374}$$

Tensor cross with tensor

$$\left(\underline{\underline{A}}\times\underline{\underline{B}}\right)_i = \epsilon_{ijk}A_{jl}B_{kl} \tag{375}$$

A.12.3 Dyadic Operator

Dyadic tensor builds a second-rank tensor out of two vectors:

$$(\underline{a}\otimes\underline{b})_{ij} = a_i b_j \tag{376}$$

for some identities

$$\begin{aligned} (\underline{a}\otimes\underline{b})\,\underline{c} &= \underline{a}\,(\underline{b}\cdot\underline{c}) \\ \underline{a}\otimes\underline{b} &\neq \underline{b}\otimes\underline{a} \end{aligned} \tag{377}$$

A.12.4 Scalar Triple Product

Calculates 1/6 of the volume composed of three vectors:

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = \underline{b}\cdot(\underline{c}\times\underline{a}) = \underline{c}\cdot(\underline{a}\times\underline{b}) \tag{378}$$

A.12.5 Vector Triple Product

Lagrange's formula or vector product expansion

$$\underline{a}\times(\underline{b}\times\underline{c}) = \underline{b}\,(\underline{c}\cdot\underline{a}) - \underline{c}\,(\underline{a}\cdot\underline{b}) \tag{379}$$

A.12.6 Gradient Operator

Gradient reduces the order of tensor.

$$\nabla f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z \tag{380}$$

and in index notations:

$$\begin{aligned} \left(\nabla f\right)_i &= f_{,i} \\ \left(\nabla \underline{u}\right)_{ij} &= u_{i,j} \\ \left(\underline{u}\nabla\right)_{ij} &= u_{j,i} \end{aligned} \tag{381}$$

Important identities:

$$\begin{aligned} \nabla\left(\phi+\psi\right) &= \nabla\phi + \nabla\psi \\ \nabla\left(\phi\psi\right) &= \psi\nabla\phi + \phi\nabla\psi \\ \nabla\left(\phi\underline{u}\right) &= \nabla\phi\otimes\underline{u} + \phi\nabla\underline{u} \\ \nabla\left(\underline{u}\cdot\underline{v}\right) &= \left(\underline{u}\cdot\nabla\right)\underline{v} + \left(\underline{v}\cdot\nabla\right)\underline{u} \\ &\quad + \underline{u}\times\left(\nabla\times\underline{v}\right) \\ &\quad + \underline{v}\times\left(\nabla\times\underline{u}\right) \end{aligned} \tag{382}$$

In cylindrical coordinates:

$$\nabla\underline{u} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta}{r} & \dfrac{\partial u_r}{\partial z} \\ \dfrac{\partial u_\theta}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r} & \dfrac{\partial u_\theta}{\partial z} \\ \dfrac{\partial u_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta} & \dfrac{\partial u_z}{\partial z} \end{pmatrix} \tag{383}$$

In spherical coordinates:

$$\nabla\underline{u} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{r\sin\phi}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta}{r} & \dfrac{1}{r}\dfrac{\partial u_r}{\partial\phi} - \dfrac{u_\phi}{r} \\ \dfrac{\partial u_\theta}{\partial r} & \dfrac{1}{r\sin\phi}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r} + \dfrac{u_\phi}{r\tan\phi} & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\phi} \\ \dfrac{\partial u_z}{\partial r} & \dfrac{1}{r\sin\phi}\dfrac{\partial u_\phi}{\partial\theta} - \dfrac{u_\theta}{r\tan\theta} & \dfrac{1}{r}\dfrac{\partial u_z}{\partial\phi} + \dfrac{u_r}{r} \end{pmatrix} \tag{384}$$

A.12.7 Divergence Operator

"Dot product" of nabla and the vector or tensor:

$$\left(\nabla\cdot\underline{u}\right) = u_{i,i} \tag{385}$$

For operation with tensor:

$$\begin{aligned} \left(\nabla\cdot\underline{\underline{A}}\right)_i &= A_{ji,j} \\ \left(\underline{\underline{A}}\cdot\nabla\right)_i &= A_{ij,j} \end{aligned} \tag{386}$$

Important identities:

$$\begin{aligned} \nabla\cdot\left(\underline{u}+\underline{v}\right) &= \nabla\cdot\underline{u} + \nabla\cdot\underline{v} \\ \nabla\cdot\left(\phi\underline{u}\right) &= \phi\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\phi \\ \nabla\cdot\left(\underline{u}\times\underline{v}\right) &= \left(\nabla\times\underline{u}\right)\cdot\underline{v} - \left(\nabla\times\underline{v}\right)\cdot\underline{u} \end{aligned} \tag{387}$$

A.12.8 Curl Operator

"Cross product" of nabla and the vector or tensor:

$$\left(\nabla\times\underline{u}\right)_i = \epsilon_{ijk}u_{k,j} \tag{388}$$

For tensors:

$$\left(\nabla\times\underline{\underline{A}}\right)_{ij} = \epsilon_{ipq}A_{qj,p} \tag{389}$$

Important identities:

$$\begin{aligned} \nabla\times\left(\underline{u}+\underline{v}\right) &= \nabla\times\underline{u} + \nabla\times\underline{v} \\ \nabla\times\left(\phi\underline{u}\right) &= \phi\left(\nabla\times\underline{u}\right) - \left(\underline{u}\times\nabla\right)\phi \\ &= \phi\left(\nabla\times\underline{u}\right) + \left(\nabla\phi\right)\times\underline{u} \\ \nabla\times\left(\phi\nabla\psi\right) &= \nabla\phi\times\nabla\psi \\ \nabla\times\left(\underline{u}\times\underline{v}\right) &= \underline{u}\left(\nabla\cdot\underline{v}\right) - \underline{v}\left(\nabla\cdot\underline{u}\right) \\ &\quad + \left(\underline{v}\cdot\nabla\right)\underline{u} - \left(\underline{u}\cdot\nabla\right)\underline{v} \end{aligned} \tag{390}$$

In cylindrical coordinates:

$$\nabla\times\underline{u} = \frac{1}{r}\begin{pmatrix} \dfrac{\partial u_z}{\partial\theta} - \dfrac{\partial r u_\theta}{\partial z} \\ r\left(\dfrac{\partial u_r}{\partial z} - \dfrac{\partial u_z}{\partial r}\right) \\ \dfrac{\partial r u_\theta}{\partial r} - \dfrac{\partial u_r}{\partial\theta} \end{pmatrix} \cdot \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_z \end{pmatrix} \tag{391}$$

In spherical coordinates:

$$\nabla\times\underline{u} = \begin{pmatrix} \dfrac{1}{r\sin\phi}\left(\dfrac{\partial u_\theta\sin\phi}{\partial\phi} - \dfrac{\partial u_\phi}{\partial\theta}\right) \\ \dfrac{1}{r}\left(\dfrac{\partial r u_\phi}{\partial r} - \dfrac{\partial u_r}{\partial\phi}\right) \\ \dfrac{1}{r}\left(\dfrac{1}{\sin\phi}\dfrac{\partial u_r}{\partial\theta} - \dfrac{\partial r u_\theta}{\partial r}\right) \end{pmatrix} \cdot \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\phi \end{pmatrix} \tag{392}$$

A.12.9 Laplacian Operator

Laplace operator is the divergence $(\nabla\cdot)$ of the gradient (∇f).

$$\Delta f = \nabla^2 f = \nabla\cdot\left(\nabla f\right) \tag{393}$$

and for index notations

$$\begin{aligned} \left(\nabla^2\underline{u}\right)_i &= u_{i,jj} \\ \left(\nabla^2\underline{\underline{A}}\right)_{ij} &= A_{ij,kk} \end{aligned} \tag{394}$$

A.12.10 Inner Product

Inner product space (or, rarely, a Hausdorff pre-Hilbert space[1][2]) is a real vector space or a complex vector space with an operation called an inner product.

$$\langle y_1, y_2\rangle = \int y_1(x)y_2(x)\ \mathrm{d}x \tag{395}$$

A.12.11 Material derivative

Change rate of ϕ in an element in the velocity field.

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \left(\vec{U}\cdot\nabla\right)\phi \tag{396}$$

where $\left(\vec{U}\cdot\nabla\right)$ is an operator defined as:

$$\left(\vec{U}\cdot\nabla\right) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{397}$$

with index notation:

$$\left[\left(\vec{U}\cdot\nabla\right)\phi\right] = \phi_{,i}U_i \tag{398}$$

A special case when $\phi = \vec{U}$:

$$\left(\vec{U}\cdot\nabla\right)\vec{U} = \left(\nabla\vec{U}\right)\vec{U} \tag{399}$$

with index notation:

$$\left[\left(\vec{U}\cdot\nabla\right)\vec{U}\right]_i = U_{i,j}U_j \tag{400}$$

A.13 Theorems

Divergence law for vectors:

$$\begin{aligned}\int_R \nabla\cdot\underline{u}\ \mathrm{d}V &= \int_{\delta R}\underline{u}\cdot\underline{n}\ \mathrm{d}A\\ \int_R u_{i,i}\ \mathrm{d}V &= \int_{\delta R} u_i n_i\ \mathrm{d}A\end{aligned} \tag{401}$$

and for tensors:

$$\begin{aligned}\int_R \nabla\cdot\underline{\underline{U}}\ \mathrm{d}V &= \int_{\delta R}\underline{n}\cdot\underline{\underline{U}}\ \mathrm{d}A\\ \int_R U_{ji,j}\ \mathrm{d}V &= \int_{\delta R} U_{ji}n_j\ \mathrm{d}A\\ \int_R \underline{\underline{U}}\cdot\nabla\ \mathrm{d}V &= \int_{\delta R}\underline{\underline{U}}\cdot\underline{n}\ \mathrm{d}A\\ \int_R U_{ij,j}\ \mathrm{d}V &= \int_{\delta R} U_{ij}n_j\ \mathrm{d}A\end{aligned} \tag{402}$$

Stoke's Theorem

$$\int_S \nabla\times\underline{u}\cdot\underline{n}\ \mathrm{d}S = \int_{\partial S}\underline{u}\ \mathrm{d}l \tag{403}$$

A.14 Linear Systems

Matrix form of a linear system is:

$$\underline{\underline{A}}\ \underline{x} = \underline{b} \tag{404}$$

and the solution of $\underline{x}$ is:

$$\underline{x} = \underline{\underline{A}}^{-1}\underline{b} \tag{405}$$

From Cramer's rule:

$$x_i = \frac{\mathrm{Det}(\underline{\underline{A_i}})}{\mathrm{Det}(\underline{\underline{A}})} = \frac{|\underline{\underline{A_i}}|}{|\underline{\underline{A}}|} \tag{406}$$

where $\underline{\underline{A_i}}$ is matrix $\underline{\underline{A}}$ with the i^{th} column substituted by $\underline{b}$.

A.15 Eigenvalues and Eigenvectors

for $n\times n$ tensor $\underline{\underline{A}}$:

$$\begin{aligned}\left(\underline{\underline{A}}-\lambda\underline{\underline{I}}\right)\underline{r} &= 0\\ \underline{l}^T\left(\underline{\underline{A}}-\lambda\underline{\underline{I}}\right) &= 0\end{aligned} \tag{407}$$

it has n Eigenvalues $\lambda_1,\lambda_2,...,\lambda_n$, each associated with eigenvectors $\underline{r}_1,\underline{r}_2,...,\underline{r}_n$, and left eigenvectors $\underline{l}_1,\underline{l}_2,...,\underline{l}_n$.

$$\begin{aligned}\underline{\underline{\Lambda}} &= \underline{\underline{R}}^{-1}\underline{\underline{A}}\ \underline{\underline{R}}\\ \underline{\underline{\Lambda}} &= \underline{\underline{L}}\ \underline{\underline{A}}\ \underline{\underline{R}}\\ \underline{\underline{A}} &= \underline{\underline{R}}\ \underline{\underline{\Lambda}}\ \underline{\underline{R}}^{-1}\\ \underline{\underline{A}} &= \underline{\underline{R}}\ \underline{\underline{\Lambda}}\ \underline{\underline{L}}\end{aligned} \tag{408}$$

where right Eigenvector matrix $\underline{\underline{R}}$ is given by:

$$\underline{\underline{R}} = \begin{pmatrix}\underline{r}_1 & \underline{r}_2 & \cdots & \underline{r}_n\end{pmatrix} \tag{409}$$

left Eigenvector matrix $\underline{\underline{L}} = \underline{\underline{R}}^{-1}$:

$$\underline{\underline{L}} = \begin{pmatrix}\underline{l}_1^T\\ \underline{l}_2^T\\ \vdots\\ \underline{l}_n^T\end{pmatrix} \tag{410}$$

$$\Lambda = \begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & \lambda_n\end{pmatrix} \tag{411}$$

For symmetric matrix $\underline{\underline{A}}$, Eigenvector matrix is orthorgonal $\underline{\underline{L}} = \underline{\underline{R}}^T$:

A.16 Random Variables

A.16.1 Expectation (μ)

Discrete

$$\mathbb{E}[X] = \sum_{x\in\mathcal{X}} xP(X=x) \tag{412}$$

Continuous

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)\ \mathrm{d}x \tag{413}$$

A.16.2 Variance (σ^2)

Discrete

$$\mathbb{V}[X] = \mathbb{E}[(X-\mu)^2] = \sum_{x\in\mathcal{X}}(x-\mu)^2P(X=x) \tag{414}$$

Continuous

$$\mathbb{V}[X] = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\ \mathrm{d}x = \int_{-\infty}^{\infty} x^2 f(x)\ \mathrm{d}x - \mu^2 \tag{415}$$

A.16.3 Covariance

Measures the linear dependence between X_1 and X_2.

$$\begin{aligned}\mathrm{Cov}[X_1,X_2] &= \mathbb{E}\left[(X_1-\mu_1)(X_2-\mu_2)\right]\\ &= \mathbb{E}[X_1X_2] - \mu_1\mu_2\end{aligned} \tag{416}$$

A.17 Correlation Coefficient

Non-dimensionalized covariance

$$R_{X,Y} = \mathrm{Corr}[X_1,Y_2] = \frac{\mathrm{Cov}[X_1,X_2]}{\sqrt{\mathbb{V}[X_1]\mathbb{V}[X_2]}} \tag{417}$$

A.18 Probability Density

A.18.1 Probability Density

$$B\left(U\right)\Delta U=\lim_{T\to\infty}\frac{1}{T}\sum\left(\Delta t\right) \tag{418}$$

where $B\left(U\right)$ is a probablity density.

$$B\left(U\right)\geq 0,\int_{-\infty}^{\infty}B\left(U\right)\mathrm{d}U=1 \tag{419}$$

First Moment

$$U_{first}=\int_{-\infty}^{\infty}UB\left(U\right)\mathrm{d}U \tag{420}$$

Second Moment (Variance)

$$\sigma^2=U_{second}=\int_{-\infty}^{\infty}U^2B\left(U\right)\mathrm{d}U \tag{421}$$

Third Moment (Skewness)

$$S=\frac{U_{third}}{\sigma^3}=\frac{1}{\sigma^3}\int_{-\infty}^{\infty}U^3B\left(U\right)\mathrm{d}U \tag{422}$$

Fourth Moment (Kurtosis / Flatness factor)

$$K=\frac{U_{fourth}}{\sigma^4}=\frac{1}{\sigma^4}\int_{-\infty}^{\infty}U^4B\left(U\right)\mathrm{d}U \tag{423}$$

A.18.2 Joint Probability Density

$B\left(U,V\right)$ is joint probablity density.

$$B\left(U,V\right)\geq 0,\int\int_{-\infty}^{\infty}B\left(U,V\right)\mathrm{d}U\mathrm{d}V=1 \tag{424}$$

A.19 Fourier Transform

Forward:

$$F\left(\xi\right)=\int_{-\infty}^{\infty}f(x)e^{-i2\pi\xi x}\mathrm{d}x \tag{425}$$

Inverse:

$$f\left(x\right)=\int_{-\infty}^{\infty}F\left(\xi\right)e^{i2\pi\xi x}\mathrm{d}\xi \tag{426}$$

A.20 Singular Value Decomposition (SVD)

A.20.1 Full SVD

For any matrix $\underline{\underline{Y}}$ with dimension $n\times m$:

$$\underline{\underline{Y}}=\underline{\underline{U}}\,\underline{\underline{\Sigma}}\,\underline{\underline{V}}^{-1} \tag{427}$$

where $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are unitary matrices: ($\underline{\underline{U}}\underline{\underline{U}}^T=\underline{\underline{I}}$) with dimension $n\times n$ and $m\times m$, respectively. $\underline{\underline{\Sigma}}$ is a diagonal matrix of dimension $n\times m$ with decreasing and non-negative diagonal entries.

A.20.2 Economy SVD

By trimming $\underline{\underline{U}}$ and $\underline{\underline{\Sigma}}$ in full SVD (if $m>n$, and vice versa):

$$\underline{\underline{Y}}=\underline{\underline{\hat{U}}}\,\underline{\underline{\hat{\Sigma}}}\,\underline{\underline{V}}^{-1} \tag{428}$$

where $\underline{\underline{\hat{U}}}$ and $\underline{\underline{\hat{\Sigma}}}$ have dimensions $n\times m$ and $m\times m$. respectively.

A.20.3 Truncated SVD

By removing more columns for a "good enough" rank-r approximation of $\underline{\underline{Y}}$:

$$\underline{\underline{Y}}\approx\underline{\underline{Y_r}}=\underline{\underline{U_r}}\,\underline{\underline{\Sigma_r}}\,\underline{\underline{V_r}}^{-1}=\sum_{j=1}^{r}\sigma_j\vec{u}_j\vec{v}_j^{-1} \tag{429}$$

where $\underline{\underline{U_r}}$, $\underline{\underline{\Sigma_r}}$, and $\underline{\underline{V_r}}$ have dimensions $n\times r$, $r\times r$, and $r\times m$. respectively. $\vec{u}_j$ and $\vec{v}_j^{-1}$ are j^{th} columns of $\underline{\underline{U_r}}$ and $\underline{\underline{V_r}}$ The required optimisation is the minimisation of

$$\left|\left|\underline{\underline{Y}}-\underline{\underline{Y_r}}\right|\right|_F=\sqrt{\sum_{j=1}^{n}\sum_{k=1}^{m}\left|Y_{jk}-\left(Y_r\right)_{jk}\right|^2} \tag{430}$$

A.21 Proper Orthogonal Decomposition (POD)

To modal decompose a vector $\vec{u}\left(\vec{x},\vec{t}\right)$ field into a set of deterministic spatial functions:

$$\vec{u}\left(\vec{x},\vec{t}\right)=\sum_{k=1}^{\infty}a_k(\vec{t})\phi_k(\vec{x}) \tag{431}$$

Using the method of snapshot with n spacial elements and p times samples:

$$\begin{aligned}\underline{\underline{Y}}&=\begin{pmatrix}y\left(x_1,t_1\right)&\cdots&y\left(x_1,t_p\right)\\\vdots&\ddots&\vdots\\y\left(x_n,t_1\right)&\cdots&y\left(x_n,t_p\right)\end{pmatrix}\\&=\begin{pmatrix}\vec{y}\left(\vec{x},t_1\right)&\cdots&y\left(\vec{x},t_1\right)\end{pmatrix}\end{aligned} \tag{432}$$

For multiple properties, the snapshots are vertically stacked.

$$\underline{\underline{Y}}=\begin{pmatrix}y_1\left(x_1,t_1\right)&\cdots&y_1\left(x_1,t_p\right)\\\vdots&\ddots&\vdots\\y_1\left(x_n,t_1\right)&\cdots&y_1\left(x_n,t_p\right)\\y_2\left(x_1,t_1\right)&\cdots&y_2\left(x_1,t_p\right)\\\vdots&\ddots&\vdots\\y_2\left(x_n,t_1\right)&\cdots&y_2\left(x_n,t_p\right)\end{pmatrix} \tag{433}$$

Inner product:

$$\begin{aligned}\langle\vec{y}_j,\vec{y}_k\rangle_x&=\int_{\Omega_x}\vec{y}_1(x,t_j)\vec{y}_2(x,t_k)\ \mathrm{d}x=\overline{\vec{y}_j^T}W_x\vec{y}_k\\\langle\vec{z}_j,\vec{z}_k\rangle_t&=\int_{\Omega_t}\vec{z}_1(x_j,t)\vec{z}_2(x_k,t)\ \mathrm{d}t=\overline{\vec{z}_j^T}W_t\vec{z}_k\end{aligned} \tag{434}$$

For the eigenvalue problem of temporal correlation matrix $\underline{\underline{Y}}\,\underline{\underline{Y}}^T$:

$$\underline{\underline{Y}}\,\underline{\underline{Y}}^T\vec{\varphi}=\lambda\vec{\varphi} \tag{435}$$

The eigenvector are simply the principal axes of the cloud of data y in vector space.

From the SVD in the previous section:

$$\underline{\underline{Y}} = \underline{\underline{U}}\ \underline{\underline{\Sigma}}\ \underline{\underline{V}}^T = \sum_{j=1}^{r} \sigma_j \vec{\varphi}_j \vec{v}_j^T \tag{436}$$

The original field can hence be reconstructed using r POD modes.

$$\underline{\underline{Y}}_j \approx \underline{\underline{Y}}_{r,j} = \sum_{i=1}^{d} a_i(j) \vec{v}_i \tag{437}$$

where $a_i(j) = (\vec{v}_i, \vec{X}_j)$ is the time coefficient of POD basis $\vec{v}_i$

B Derivations

B.1 Fundamental Equations

Internal energy E is a function of extensive properties S, V. Same for S.

$$\begin{aligned} E &= E(S,V) \\ S &= S(E,V) \end{aligned} \tag{438}$$

hence,

$$\begin{aligned} \mathrm{d}E &= \left(\frac{\partial E}{\partial S}\right)_V \mathrm{d}S + \left(\frac{\partial E}{\partial V}\right)_S \mathrm{d}V \\ \mathrm{d}S &= \left(\frac{\partial S}{\partial E}\right)_V \mathrm{d}E + \left(\frac{\partial S}{\partial V}\right)_E \mathrm{d}V \end{aligned} \tag{439}$$

By definition:

$$T = \left(\frac{\partial E}{\partial S}\right)_V \; ; \; P = -\left(\frac{\partial E}{\partial V}\right)_S \tag{440}$$

hence:

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V \tag{441}$$

rearrange:

$$\mathrm{d}S = \frac{1}{T}\mathrm{d}E + \frac{P}{T}\mathrm{d}V \tag{442}$$

hence:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_V \; ; \; \frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_E \tag{443}$$

B.2 Legendre Transformation

The Legendre transformation is a mathematical process that changes one function into another using its derivative. By defining

$$Y = Y(X); \; \lambda = \frac{dY}{dX}\left(\lambda\left(X_i\right) = \frac{\partial Y}{\partial X_i}\text{for multiple}X\right) \tag{444}$$

and

$$\psi\left(\lambda\right) = Y - \lambda X \tag{445}$$

series of thermodynamic properties can be defined with $Y = U$ and $X = N_i$.

B.2.1 Enthalpy

Setting

$$X = V \tag{446}$$

and λ is hence

$$\lambda = \left(\frac{\partial E}{\partial V}\right)_S = -P \tag{447}$$

The corresponding Legendre potential is given by

$$\psi(P) = E - \lambda V = E + PV = H \tag{448}$$

which is defined as enthalpy (H).

$$\begin{aligned} \mathrm{d}H &= \mathrm{d}E + \mathrm{d}\left(PV\right) \\ &= T\mathrm{d}S - P\mathrm{d}V + P\mathrm{d}V + V\mathrm{d}P \\ &= T\mathrm{d}S + V\mathrm{d}P \end{aligned} \tag{449}$$

B.2.2 Helmholtz Free Energy

Setting

$$X = S \tag{450}$$

and λ is hence

$$\lambda = \left(\frac{\partial E}{\partial S}\right)_V = T \tag{451}$$

The corresponding Legendre potential is given by

$$\psi(T) = E - \lambda S = E - TS = A \tag{452}$$

which is defined as Helmholtz free energy (A).

$$\begin{aligned} \mathrm{d}A &= \mathrm{d}E - \mathrm{d}\left(TS\right) \\ &= T\mathrm{d}S - P\mathrm{d}V - T\mathrm{d}S - S\mathrm{d}T \\ &= -S\mathrm{d}T - P\mathrm{d}V \end{aligned} \tag{453}$$

B.2.3 Gibbs Free Energy

Setting

$$X_{1,2} = S, V \tag{454}$$

and λ is hence

$$\begin{aligned} \lambda_1 &= \left(\frac{\partial E}{\partial S}\right)_V = T \\ \lambda_2 &= \left(\frac{\partial E}{\partial V}\right)_S = -P \end{aligned} \tag{455}$$

The corresponding Legendre potential is given by

$$\psi(T,P) = E - \lambda_1 S - \lambda_2 V = E - TS + PV = H - TS = G \tag{456}$$

which is defined as Gibbs free energy (G).

$$\begin{aligned} \mathrm{d}G &= \mathrm{d}H - \mathrm{d}\left(TS\right) \\ &= T\mathrm{d}S + V\mathrm{d}P - T\mathrm{d}S - S\mathrm{d}T \\ &= -S\mathrm{d}T + V\mathrm{d}P \end{aligned} \tag{457}$$

B.3 Helmholtz Equations

B.3.1 First Helmholtz Equations

From fundamental equations:

$$T\mathrm{d}S = \mathrm{d}E + P\mathrm{d}V \tag{458}$$

$$E = E(T,V) \tag{459}$$

$$\mathrm{d}E = \left(\frac{\partial E}{\partial T}\right)_V \mathrm{d}T + \left(\frac{\partial E}{\partial V}\right)_T \mathrm{d}V \tag{460}$$

hence

$$\begin{aligned} \mathrm{d}S &= \frac{1}{T}\left(\left(\frac{\partial E}{\partial T}\right)_V \mathrm{d}T + \left(\frac{\partial E}{\partial V}\right)_T \mathrm{d}V\right) + \frac{P}{T}\mathrm{d}V \\ &= \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_V \mathrm{d}T + \frac{1}{T}\left(\left(\frac{\partial E}{\partial V}\right)_T + P\right)\mathrm{d}V \end{aligned} \tag{461}$$

Therefore

$$\begin{aligned} \left(\frac{\partial S}{\partial T}\right)_V &= \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_V \\ &= \frac{C_v}{T} \\ \left(\frac{\partial S}{\partial V}\right)_T &= \frac{1}{T}\left(\left(\frac{\partial E}{\partial V}\right)_T + P\right) \end{aligned} \tag{462}$$

rearrange last equation:

$$\left(\frac{\partial E}{\partial V}\right)_T = \left(T\left(\frac{\partial S}{\partial V}\right)_T - P\right) \tag{463}$$

from Maxwell relationship

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V \tag{464}$$

Substituting, we get the first Helmholtz equations:

$$\left(\frac{\partial E}{\partial V}\right)_T = \left(T\left(\frac{\partial P}{\partial T}\right)_V - P\right) \tag{465}$$

B.3.2 Second Helmholtz Equations

Similarly but for enthalpy:

$$\mathrm{d}H = T\mathrm{d}S + V\mathrm{d}P \tag{466}$$

$$H = H(T, P) \tag{467}$$

$$\mathrm{d}H = \left(\frac{\partial H}{\partial T}\right)_P \mathrm{d}T + \left(\frac{\partial E}{\partial P}\right)_T \mathrm{d}P \tag{468}$$

hence

$$\begin{aligned}\mathrm{d}S &= \frac{1}{T}\left(\left(\frac{\partial H}{\partial T}\right)_P \mathrm{d}T + \left(\frac{\partial H}{\partial P}\right)_T \mathrm{d}P\right) - \frac{V}{T}\mathrm{d}P \\ &= \frac{1}{T}\left(\frac{\partial H}{\partial T}\right)_P \mathrm{d}T + \frac{1}{T}\left(\left(\frac{\partial H}{\partial P}\right)_T - V\right)\mathrm{d}P\end{aligned} \tag{469}$$

Therefore

$$\begin{aligned}\left(\frac{\partial S}{\partial T}\right)_P &= \frac{1}{T}\left(\frac{\partial H}{\partial T}\right)_P \\ &= \frac{C_p}{T} \\ \left(\frac{\partial S}{\partial P}\right)_T &= \frac{1}{T}\left(\left(\frac{\partial H}{\partial P}\right)_T - V\right)\end{aligned} \tag{470}$$

rearrange last equation:

$$\left(\frac{\partial H}{\partial P}\right)_T = \left(T\left(\frac{\partial S}{\partial P}\right)_T + V\right) \tag{471}$$

from Maxwell relationship

$$-\left(\frac{\partial S}{\partial P}\right)_T = \left(\frac{\partial V}{\partial T}\right)_P \tag{472}$$

Substituting we get second Helmholtz equations:

$$\left(\frac{\partial H}{\partial P}\right)_T = \left(V - T\left(\frac{\partial V}{\partial T}\right)_P\right) \tag{473}$$

B.4 Navier-Stokes Equations

Starting from transport theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\phi \ \mathrm{d}V = \int_V \frac{\partial \rho\phi}{\partial t} \ \mathrm{d}V + \int_A \rho\phi\left(\vec{U}\cdot\vec{n}\right) \ \mathrm{d}A \tag{474}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\phi\right) = \frac{\partial}{\partial t}\left(\rho\phi\right) + \nabla\cdot\left(\rho\phi\vec{U}\right) \tag{475}$$

B.4.1 Mass

$\phi = 1$ for mass and $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \int_V \frac{\partial \rho}{\partial t} \ \mathrm{d}V + \int_A \rho\left(\vec{U}\cdot\vec{n}\right) \ \mathrm{d}A = 0 \tag{476}$$

and hence:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\vec{U}\right) = 0 \tag{477}$$

B.4.2 Momentum

$\phi = \vec{U}$ for momentum $\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = \sum \vec{F}$:

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = \int_V \frac{\partial \rho\vec{U}}{\partial t} \ \mathrm{d}V + \int_A \rho\vec{U}\left(\vec{U}\cdot\vec{n}\right) \ \mathrm{d}A = \sum \vec{F} \tag{478}$$

by divergence law:

$$\int_V \frac{\partial}{\partial t}\left(\rho\vec{U}\right) + \nabla\cdot\left(\rho\vec{U}\otimes\vec{U}\right) \ \mathrm{d}V = \sum \vec{F} \tag{479}$$

For left hand side of the equation (with volume integral neglected):

$$\begin{aligned}LHS &= \frac{\partial}{\partial t}\left(\rho\vec{U}\right) + \nabla\cdot\left(\rho\vec{U}\otimes\vec{U}\right) \\ &= \frac{\partial}{\partial t}\left(\rho\vec{U}\right) + \vec{U}\left(\nabla\cdot\left(\rho\vec{U}\right)\right) + \left(\nabla\vec{U}\right)\left(\rho\vec{U}\right)\end{aligned} \tag{480}$$

as in index notation they are equivalent:

$$\begin{aligned}\left(\nabla\cdot\left(\rho\vec{U}\otimes\vec{U}\right)\right)_i &= \left(\rho\vec{U}\otimes\vec{U}\right)_{ji,j} \\ &= \left(\rho U_j U_i\right)_{,j} \\ &= \rho U_j U_{i,j} + \rho U_i U_{j,j} \\ \left(\vec{U}\left(\nabla\cdot\left(\rho\vec{U}\right)\right)\right)_i &= \left(\nabla\cdot\left(\rho\vec{U}\right)\right)U_i \\ &= \left(\rho\vec{U}\right)_{j,j} U_i \\ &= \rho U_i U_{j,j} \\ \left(\left(\nabla\vec{U}\right)\left(\rho\vec{U}\right)\right)_i &= \left(\nabla\vec{U}\right)_{ij}\left(\rho\vec{U}\right)_j \\ &= \rho U_{j,i} U_j\end{aligned} \tag{481}$$

Therefore,

$$\begin{aligned}LHS &= \rho\frac{\partial\vec{U}}{\partial t} + \vec{U}\frac{\partial\rho}{\partial t}\left(\rho\vec{U}\right) + \vec{U}\nabla\cdot\left(\rho\vec{U}\right) + \left(\nabla\vec{U}\right)\left(\rho\vec{U}\right) \\ &= \vec{U}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{U}\right)\right) + \rho\frac{\partial\vec{U}}{\partial t} + \left(\nabla\vec{U}\right)\left(\rho\vec{U}\right) \\ &= \rho\left(\frac{\partial\vec{U}}{\partial t} + \left(\nabla\vec{U}\right)\vec{U}\right)\end{aligned} \tag{482}$$

For right hand side of the equation (with integral) with three terms describing body force, frictional forces, and pressure force respectively:

$$\begin{aligned}\int_V \mathrm{RHS} \ \mathrm{d}V &= \int_V \rho\vec{G} \ \mathrm{d}V + \int_A \underline{\underline{\tau}}\vec{n} \ \mathrm{d}A - \int_A \underline{\underline{P}}\vec{n} \ \mathrm{d}A \\ &= \int_V \rho\vec{G} \ \mathrm{d}V + \int_V \nabla\cdot\underline{\underline{\tau}} \ \mathrm{d}V - \int_V \nabla\cdot\underline{\underline{P}} \ \mathrm{d}V\end{aligned} \tag{483}$$

Since

$$\underline{\underline{\sigma}} = \underline{\underline{\tau}} - \underline{\underline{P}}$$
$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{xx} \end{pmatrix} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{xx} \end{pmatrix} - \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix} \tag{484}$$

and

$$\left(\nabla \cdot \underline{\underline{P}}\right)_i = P_{ji,j} = P_{,i} \tag{485}$$

RHS can be simplified into:

$$\begin{aligned} \int_V \text{RHS } \mathrm{d}V &= \int_V \rho \vec{G} \, \mathrm{d}V + \int_V \nabla \cdot \underline{\underline{\tau}} \, \mathrm{d}V - \int_V \nabla P \, \mathrm{d}V \\ &= \int_V -\nabla P + \nabla \cdot \underline{\underline{\tau}} + \rho \vec{G} \, \mathrm{d}V \end{aligned} \tag{486}$$

Equating LHS = RHS:

$$\rho \left(\frac{\partial \vec{U}}{\partial t} + \left(\nabla \vec{U}\right) \vec{U} \right) = -\nabla P + \nabla \cdot \underline{\underline{\tau}} + \rho \vec{F} \tag{487}$$

From definition of materials derivative:

$$\frac{D\vec{U}}{Dt} = \frac{\partial \vec{U}}{\partial t} + \left(\vec{U} \cdot \nabla\right) \vec{U} \tag{488}$$

For index notation proof:

$$\begin{aligned} \left(\vec{U} \cdot \nabla\right) \vec{U} &= u \frac{\partial \vec{U}}{\partial x} + v \frac{\partial \vec{U}}{\partial y} + w \frac{\partial \vec{U}}{\partial z} \\ \left(\left(\vec{U} \cdot \nabla\right) \vec{U}\right)_i &= U_{i,j} U_j \\ \left(\left(\nabla \vec{U}\right) \vec{U}\right)_i &= \left(\nabla \vec{U}\right)_{ij} U_j \\ &= U_{i,j} U_j \end{aligned} \tag{489}$$

Hence the Euler's equation for momentum balance can be written as:

$$\frac{D\vec{U}}{Dt} = -\frac{1}{\rho} \nabla P + \frac{1}{\rho} \nabla \cdot \underline{\underline{\tau}} + \vec{F} \tag{490}$$

B.4.3 Energy

For energy ϕ is total internal energy:

$$\begin{aligned} \frac{\mathrm{d}\vec{E_0}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \left(e + \frac{1}{2} |U|^2 \right) \mathrm{d}U \\ &= \int_V \frac{\partial}{\partial t} \left(\rho \left(e + \frac{1}{2} |U|^2 \right) \right) \mathrm{d}U + \\ &\quad \int_A \rho \left(e + \frac{1}{2} |U|^2 \right) \left(\vec{U} \cdot \vec{n} \right) \mathrm{d}A \end{aligned} \tag{491}$$

For LHS:

$$\begin{aligned} LHS &= \frac{\partial}{\partial t} \left(\rho \left(e + \frac{1}{2} |U|^2 \right) \right) + \nabla \cdot \left(\rho \vec{U} \left(e + \frac{1}{2} |U|^2 \right) \right) \\ &= \frac{\partial}{\partial t} \left(\rho E_0 \right) + \nabla \cdot \left(\rho \vec{U} E_0 \right) \\ &= E_0 \frac{\partial \rho}{\partial t} + \rho \frac{\partial E_0}{\partial t} + E_0 \nabla \cdot \left(\rho \vec{U} \right) + \left(\nabla E_0 \right) \cdot \left(\rho \vec{U} \right) \\ &= \rho \frac{\partial E_0}{\partial t} + \left(\nabla E_0 \right) \cdot \left(\rho \vec{U} \right) \\ &= \rho \frac{D E_0}{D t} \end{aligned} \tag{492}$$

From energy conservation (first law of thermodynamics):

$$\frac{\mathrm{d}\vec{E_0}}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{\mathrm{d}W}{\mathrm{d}t} \tag{493}$$

For heat transfer rate:

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= -\int_A \vec{q} \cdot \vec{n} \, \mathrm{d}A + \int_V Q_{gen} \, \mathrm{d}V \\ &= \int_V -\nabla \cdot \vec{q} + Q_{gen} \, \mathrm{d}V \end{aligned} \tag{494}$$

The calculation of work is:

$$\begin{aligned} \vec{F_s} &= \int_A \vec{T} \, \mathrm{d}A \\ \vec{W_s} &= \int_A \vec{T} \cdot \vec{U} \, \mathrm{d}A \end{aligned} \tag{495}$$

with

$$\vec{T} = \vec{n}^T \underline{\underline{\sigma}} \tag{496}$$

and $\underline{\underline{\sigma}} = \underline{\underline{\tau}} - \underline{\underline{P}} = \underline{\underline{\tau}} - P\underline{\underline{\mathcal{I}}}$ for rate of work done:

$$\begin{aligned} -\frac{\mathrm{d}W}{\mathrm{d}t} &= \int_A \underline{\underline{\sigma}} \cdot \vec{U} \vec{n} \, \mathrm{d}A + \int_V \vec{F} \cdot \vec{U} \, \mathrm{d}V \\ &= \int_A \left(\underline{\underline{\tau}} \cdot \vec{U} - P\vec{U} \right) \vec{n} \, \mathrm{d}A + \int_V \vec{F} \cdot \vec{U} \, \mathrm{d}V \\ &= \int_V \nabla \cdot \left(\underline{\underline{\tau}} \cdot \vec{U} - P\vec{U} \right) + \vec{F} \cdot \vec{U} \, \mathrm{d}V \end{aligned} \tag{497}$$

Thus:

$$\begin{aligned} \frac{\mathrm{d}\vec{E_0}}{\mathrm{d}t} &= \frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{\mathrm{d}W}{\mathrm{d}t} \\ &= \int_V -\nabla \cdot \vec{q} + \nabla \cdot \left(\underline{\underline{\tau}} \cdot \vec{U} - P\vec{U} \right) \\ &\quad + \vec{F} \cdot \vec{U} + Q_{gen} \, \mathrm{d}V \\ &= \int_V \nabla \cdot (k \nabla T) + \nabla \cdot \left(\underline{\underline{\tau}} \cdot \vec{U} - P\vec{U} \right) \\ &\quad + \vec{F} \cdot \vec{U} + Q_{gen} \, \mathrm{d}V \end{aligned} \tag{498}$$

The five terms in the equation above corresponds to heat flux via conduction, shear work, pressure work, body force work, and heat generation term in the domain.

Therefore, Euler's equation for energy balance is given by LHS=RHS:

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\rho \left(e + \frac{1}{2} |U|^2 \right) \right) + \nabla \cdot \left(\rho \vec{U} \left(e + \frac{1}{2} |U|^2 \right) \right) = \\ &\nabla \cdot (k \nabla T) + \nabla \cdot \left(-P\vec{U} + \underline{\underline{\tau}} \cdot \vec{U} \right) + \vec{U} \cdot \vec{F} + Q_{gen} \end{aligned} \tag{499}$$

B.5 1D Inviscid Navier-Stokes Equation Primitive Form

B.5.1 Auxiliary Relations

Speed of Sound Rewrite function of Pressure as:

$$P = P\left(W_1, W_2, W_3\right) \tag{500}$$

so

$$\mathrm{d}P = \frac{\partial P}{\partial W_1}\mathrm{d}W_1 + \frac{\partial P}{\partial W_2}\mathrm{d}W_2 + \frac{\partial P}{\partial W_3}\mathrm{d}W_3 \tag{501}$$

Speed of sound can be re-written as:

$$\begin{aligned} c^2 &= \left(\frac{\partial P}{\partial \rho}\right)_S \\ &= \left(\frac{\partial P}{\partial W_1}\right)_S \\ &= \frac{\partial P}{\partial W_1} + \frac{\partial P}{\partial W_2}\left(\frac{\partial W_2}{\partial W_1}\right)_S + \frac{\partial P}{\partial W_3}\left(\frac{\partial W_3}{\partial W_1}\right)_S \end{aligned} \tag{502}$$

Since

$$\frac{\partial P}{\partial W_2} = -u\frac{\partial P}{\partial W_3} \tag{503}$$

so

$$c^2 = \frac{\partial P}{\partial W_1} + \frac{\partial P}{\partial W_3}\left(\left(\frac{\partial W_3}{\partial W_1}\right)_S - u\left(\frac{\partial W_2}{\partial W_1}\right)_S\right) \tag{504}$$

Along an isentrope:

$$\begin{aligned} \mathrm{d}s = \mathrm{d}e + \mathrm{d}w &= 0 \\ -P\mathrm{d}v &= -\mathrm{d}e \\ \frac{P}{\rho^2}\mathrm{d}\rho &= \mathrm{d}e \\ \left(\frac{\partial e}{\partial \rho}\right)_S &= \frac{P}{\rho^2} \end{aligned} \tag{505}$$

$$h_0 = e + \frac{P}{\rho} + \frac{1}{2}u^2 \tag{506}$$

In addition, from the definition:

$$\begin{aligned} \left(\frac{\partial W_2}{\partial W_1}\right)_S &= \left(\frac{\partial \rho u}{\partial \rho}\right)_S \\ &= u + \rho\left(\frac{\partial u}{\partial \rho}\right)_S \end{aligned} \tag{507}$$

$$\begin{aligned} \left(\frac{\partial W_3}{\partial W_1}\right)_S &= \left(\frac{\partial \rho e_0}{\partial \rho}\right)_S \\ &= \left(\frac{\partial \rho e}{\partial \rho}\right)_S + \frac{1}{2}\left(\frac{\partial \rho u^2}{\partial \rho}\right)_S \\ &= e + \rho\left(\frac{\partial e}{\partial \rho}\right)_S + \frac{1}{2}u^2 + \rho u\left(\frac{\partial u}{\partial \rho}\right)_S \\ &= h_0 + \rho u\left(\frac{\partial u}{\partial \rho}\right)_S \end{aligned} \tag{508}$$

Therefore, continue from Eq. (504),

$$\begin{aligned} c^2 &= \frac{\partial P}{\partial W_1} + \frac{\partial P}{\partial W_3}\left(h_0 + \rho u\left(\frac{\partial u}{\partial \rho}\right)_S - u^2 - \rho u\left(\frac{\partial u}{\partial \rho}\right)_S\right) \\ &= \frac{\partial P}{\partial W_1} + \left(h_0 - u^2\right)\frac{\partial P}{\partial W_3} \end{aligned} \tag{509}$$

Alternatively:

$$\begin{aligned} P &= P\left(\rho, e\right) \\ &= \left(\frac{\partial P}{\partial \rho}\right)_e \mathrm{d}\rho + \left(\frac{\partial P}{\partial e}\right)_\rho \mathrm{d}e \end{aligned} \tag{510}$$

Thus

$$\begin{aligned} c^2 = \left(\frac{\partial P}{\partial \rho}\right)_S &= \left(\frac{\partial P}{\partial \rho}\right)_e + \left(\frac{\partial P}{\partial e}\right)_\rho \left(\frac{\partial e}{\partial \rho}\right)_S \\ &= \left(\frac{\partial P}{\partial \rho}\right)_e + \frac{P}{\rho^2}\left(\frac{\partial P}{\partial e}\right)_\rho \end{aligned} \tag{511}$$

Relation 1

$$\begin{aligned} \frac{\partial P}{\partial e} &= \frac{\partial P}{\partial \rho e_0}\frac{\partial \rho e_0}{\partial e} \\ &= \frac{\partial P}{\partial W_3}\frac{\partial}{\partial e}\left(\rho e + \frac{1}{2}\rho u^2\right) \\ &= \rho\frac{\partial P}{\partial W_3} \end{aligned} \tag{512}$$

Relation 2

$$\begin{aligned} \frac{\partial P}{\partial \rho} &= \frac{\partial P}{\partial \rho u}\frac{\partial \rho u}{\partial e} \\ &= u\frac{\partial P}{\partial W_2} \end{aligned} \tag{513}$$

Relation 3 Using cyclic rules:

$$\begin{aligned} \frac{\partial P}{\partial \rho u} &= -\frac{\partial P}{\partial \rho e_0}\frac{\partial \rho e_0}{\partial u}\frac{\partial u}{\partial \rho u} \\ &= -\frac{\partial P}{\partial W_3}\frac{\partial}{\partial u}\left(\rho e + \frac{1}{2}\rho u^2\right)\frac{1}{\rho} \\ \frac{\partial P}{\partial W_2} &= -u\frac{\partial P}{\partial W_3} \end{aligned} \tag{514}$$

Relation 4 From Eqs. (509) and (511):

$$\begin{aligned} \frac{\partial P}{\partial W_1} &= c^2 - \frac{1}{\rho}\frac{\partial \rho}{\partial e}\left(h_0 - u^2\right) \\ &= \frac{\partial P}{\partial \rho} + \frac{P}{\rho^2}\frac{\partial P}{\partial e} - \frac{1}{\rho}\frac{\partial \rho}{\partial e}\left(h - \frac{1}{2}u^2\right) \\ &= \frac{\partial P}{\partial \rho} + \frac{1}{\rho}\frac{\partial P}{\partial e}\left(\frac{P}{\rho} - h + \frac{1}{2}u^2\right) \\ &= \frac{\partial P}{\partial \rho} - \frac{e_0}{\rho}\frac{\partial P}{\partial e} + \frac{u^2}{\rho}\frac{\partial P}{\partial e} \end{aligned} \tag{515}$$

Since Eqs. (509) and (511) is equivalent:

$$\frac{\partial P}{\partial W_1} = \frac{\partial P}{\partial \rho} - \left(e - \frac{1}{2}u^2\right)\frac{\partial P}{\partial W_3} \tag{516}$$

B.5.2 Jacobian

From the definition for 1D inviscid NS equation:

$$\vec{W} = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u \\ \rho e_0 \end{pmatrix} \quad , \quad \vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \rho u \\ \rho u^2 + P \\ \rho u h_0 \end{pmatrix} \tag{517}$$

Alternative forms of F_2, F_3 can be taken:

$$\begin{aligned} F_2 &= \rho u^2 + P = \frac{\rho^2 u^2}{\rho} + P \\ &= \frac{W_2^2}{W_1} + P \\ F_3 &= \rho u h_0 = \rho u \left(e_0 + \frac{P}{\rho} \right) = \rho u e_0 + uP \\ &= \frac{W_2 W_3}{W_1} + \frac{W_2}{W_1} P \end{aligned} \tag{518}$$

Therefore the terms in Jacobian can be derived:

$$\frac{\partial F_1}{\partial W_1} = \frac{\partial W_2}{\partial W_1} = 0 \tag{519}$$

$$\frac{\partial F_1}{\partial W_2} = \frac{\partial W_2}{\partial W_2} = 1 \tag{520}$$

$$\frac{\partial F_1}{\partial W_3} = \frac{\partial W_2}{\partial W_3} = 0 \tag{521}$$

$$\begin{aligned} \frac{\partial F_2}{\partial W_1} &= -\frac{W_2^2}{W_1^2} + \frac{\partial P}{\partial W_1} \\ &= \frac{\partial P}{\partial W_1} - u^2 \end{aligned} \tag{522}$$

$$\begin{aligned} \frac{\partial F_2}{\partial W_2} &= 2\,\frac{W_2}{W_1} + \frac{\partial P}{\partial W_2} \\ &= \frac{\partial P}{\partial W_2} + 2u \end{aligned} \tag{523}$$

$$\frac{\partial F_2}{\partial W_3} = \frac{\partial P}{\partial W_3} \tag{524}$$

$$\begin{aligned} \frac{\partial F_3}{\partial W_1} &= -\frac{W_2 W_3}{W_1^2} + \frac{W_2}{W_1}\frac{\partial P}{\partial W_1} - \frac{W_2 P}{W_1^2} \\ &= -u e_0 + u\frac{\partial P}{\partial W_1} - \frac{uP}{\rho} \\ &= u\left(\frac{\partial P}{\partial W_1} - \left(e_0 + \frac{P}{\rho} \right) \right) \\ &= u\left(\frac{\partial P}{\partial W_1} - h_0 \right) \end{aligned} \tag{525}$$

$$\begin{aligned} \frac{\partial F_3}{\partial W_2} &= \frac{W_3}{W_1} + \frac{P}{W_1} + \frac{W_2}{W_1}\frac{\partial P}{\partial W_2} \\ &= e_0 + \frac{P}{\rho} + u\frac{\partial P}{\partial W_2} \\ &= h_0 + \frac{\partial P}{\partial W_2} u \end{aligned} \tag{526}$$

$$\begin{aligned} \frac{\partial F_3}{\partial W_3} &= \frac{W_2}{W_1} + \frac{W_2}{W_1}\frac{\partial P}{\partial W_3} \\ &= \left(1 + \frac{\partial P}{\partial W_3} \right) u \end{aligned} \tag{527}$$

Therefore the Jacobian can be retrieved as:

$$\underline{\underline{J}} = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{\partial P}{\partial W_1} - u^2 & \dfrac{\partial P}{\partial W_2} + 2u & \dfrac{\partial P}{\partial W_3} \\ u\left(\dfrac{\partial P}{\partial W_1} - h_0 \right) & h_0 + \dfrac{\partial P}{\partial W_2} u & \left(1 + \dfrac{\partial P}{\partial W_3} \right) u \end{pmatrix} \tag{528}$$

Using auxiliary relations (Eqs. (516), (513), and (512)) derived in the previous section:

$$\begin{aligned} \frac{\partial F_2}{\partial W_1} &= \frac{\partial P}{\partial W_1} - u^2 \\ &= \frac{\partial P}{\partial \rho} - \left(e - \frac{1}{2}u^2 \right) \frac{\partial P}{\partial W_3} - u^2 \\ &= \frac{\partial P}{\partial \rho} - \left(e - \frac{1}{2}u^2 \right) \frac{1}{\rho}\frac{\partial P}{\partial e} - u^2 \\ &= \frac{\partial P}{\partial \rho} + \frac{P}{\rho^2}\frac{\partial P}{\partial e} - \frac{P}{\rho^2}\frac{\partial P}{\partial e} - \left(e - \frac{1}{2}u^2 \right) \frac{1}{\rho}\frac{\partial P}{\partial e} - u^2 \\ &= c^2 - u^2 - \frac{1}{\rho}\left(\frac{P}{\rho} + e - \frac{1}{2}u^2 \right) \frac{\partial P}{\partial e} \\ &= c^2 - u^2 - \frac{h_0 - u^2}{\rho}\frac{\partial P}{\partial e} \end{aligned} \tag{529}$$

Using Eqs. (513) and (512):

$$\begin{aligned} \frac{\partial F_2}{\partial W_2} &= \frac{\partial P}{\partial W_2} + 2u \\ &= -u\frac{\partial P}{\partial W_3} + 2u \\ &= -\frac{u}{\rho}\frac{\partial P}{\partial e} + 2u \end{aligned} \tag{530}$$

Using Eq. (512):

$$\begin{aligned} \frac{\partial F_2}{\partial W_3} &= \frac{\partial P}{\partial W_3} \\ &= \frac{1}{\rho}\frac{\partial P}{\partial e} \end{aligned} \tag{531}$$

Using Eq. (516):

$$\begin{aligned} \frac{\partial F_3}{\partial W_1} &= u\left(\frac{\partial P}{\partial W_1} - h_0 \right) \\ &= u\left(\frac{\partial P}{\partial \rho} - \frac{e_0}{\rho}\frac{\partial P}{\partial e} + \frac{u^2}{\rho}\frac{\partial P}{\partial e} - e - \frac{P}{\rho} - \frac{1}{2}u^2 \right) \\ &= u\left(\frac{\partial P}{\partial \rho} - \frac{e}{\rho}\frac{\partial P}{\partial e} - \frac{u^2}{2\rho}\frac{\partial P}{\partial e} + \frac{u^2}{\rho}\frac{\partial P}{\partial e} - e - \frac{P}{\rho} - \frac{1}{2}u^2 \right) \\ &= u\left(\frac{\partial P}{\partial \rho} - \left(\frac{e}{\rho} + \frac{u^2}{2\rho} - \frac{u^2}{\rho} \right) \frac{\partial P}{\partial e} - h_0 \right) \\ &= u\left(\frac{\partial P}{\partial \rho} + \frac{P}{\rho^2}\frac{\partial P}{\partial e} - \frac{P}{\rho^2}\frac{\partial P}{\partial e} - \left(\frac{e}{\rho} - \frac{u^2}{2\rho} \right) \frac{\partial P}{\partial e} - h_0 \right) \\ &= u\left(c^2 - \left(\frac{P}{\rho^2} + \frac{e}{\rho} - \frac{u^2}{2\rho} \right) \frac{\partial P}{\partial e} - h_0 \right) \\ &= u\left(c^2 - h_0 \right) - \frac{u}{\rho}\frac{\partial P}{\partial e}\left(\frac{P}{\rho} + e - \frac{u^2}{2} \right) \\ &= u\left(c^2 - h_0 \right) - \frac{u}{\rho}\frac{\partial P}{\partial e}\left(h_0 - u^2 \right) \end{aligned} \tag{532}$$

Using Eq. (513):

$$\begin{aligned}\frac{\partial F_3}{\partial W_2} &= h_0 + \frac{\partial P}{\partial W_2} u \\ &= h_0 - u^2 \frac{\partial P}{\partial W_3} \\ &= h_0 - \frac{u^2}{\rho}\frac{\partial P}{\partial e}\end{aligned} \tag{533}$$

Using Eq. (512):

$$\begin{aligned}\frac{\partial F_3}{\partial W_3} &= \left(1 + \frac{\partial P}{\partial W_3}\right) u \\ &= u + \frac{u}{\rho}\frac{\partial P}{\partial e}\end{aligned} \tag{534}$$

Hence, the Jacobian can also be written as

$$\underline{\underline{J}} = \begin{pmatrix} 0 & 1 & 0 \\ c^2 - u^2 - \dfrac{h_0 - u^2}{\rho}\dfrac{\partial P}{\partial e} & -\dfrac{u}{\rho}\dfrac{\partial P}{\partial e} + 2u & \dfrac{1}{\rho}\dfrac{\partial P}{\partial e} \\ u\left(c^2 - h_0\right) & & \\ -\dfrac{u}{\rho}\dfrac{\partial P}{\partial e}\left(h_0 - u^2\right) & h_0 - \dfrac{u^2}{\rho}\dfrac{\partial P}{\partial e} & u + \dfrac{u}{\rho}\dfrac{\partial P}{\partial e} \end{pmatrix} \tag{535}$$

B.5.3 Perfect Gas

From ideal gas law:

$$\begin{aligned}P &= \rho R T \\ &= \rho \frac{C_v}{\gamma - 1} T \\ &= (\gamma - 1)\rho e \\ &= (\gamma - 1)\rho\left(e_0 - \frac{1}{2}u^2\right) \\ &= (\gamma - 1)\left(\rho e_0 - \frac{1}{2}\rho u^2\right) \\ &= (\gamma - 1) W_3 - \frac{\gamma - 1}{2}\frac{W_2^2}{W_1}\end{aligned} \tag{536}$$

Therefore:

$$\begin{aligned}\frac{\partial P}{\partial W_1} &= \frac{\gamma - 1}{2}\frac{W_2^2}{W_1^2} = \frac{\gamma - 1}{2}u^2 \\ \frac{\partial P}{\partial W_2} &= -(\gamma - 1)\frac{W_2}{W_1} = -(\gamma - 1)u \\ \frac{\partial P}{\partial W_3} &= \gamma - 1\end{aligned} \tag{537}$$

hence for an ideal gas the convective flux Jacobian matrix is:

$$\underline{\underline{J}} = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{\gamma - 3}{2}u^2 & (3 - \gamma)u & \gamma - 1 \\ \dfrac{\gamma - 1}{2}u^3 - u h_0 & h_0 - (\gamma - 1)u^2 & \gamma u \end{pmatrix} \tag{538}$$

For eigenvector $\underline{r_2}$:

$$\vec{r}_2 = \begin{pmatrix} 1 \\ u \\ h_0 - \dfrac{c^2\rho}{\dfrac{\partial P}{\partial e}} \end{pmatrix} \tag{539}$$

For perfect gases:

$$\begin{aligned}\frac{\partial P}{\partial e} &= \frac{\partial P}{\partial T}\frac{\partial T}{\partial e} \\ &= \frac{\rho R}{C_v}\end{aligned} \tag{540}$$

Therefore:

$$\begin{aligned}h_0 - \frac{c^2\rho}{\dfrac{\partial P}{\partial e}} &= h_0 - \frac{c^2 \rho C_v}{\rho R} \\ &= h_0 - \frac{\gamma R T \rho C_v}{\rho R} \\ &= h_0 - C_p T \\ &= \frac{1}{2}u^2\end{aligned} \tag{541}$$

B.6 Speed of Sound

$$c = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \tag{542}$$

$$\begin{aligned}e &= e(P, \rho) \\ \mathrm{d}e &= \left(\frac{\partial e}{\partial \rho}\right)_P \mathrm{d}\rho + \left(\frac{\partial e}{\partial P}\right)_\rho \mathrm{d}P\end{aligned} \tag{543}$$

Since:

$$\mathrm{d}q - \mathrm{d}w = \mathrm{d}e \tag{544}$$

$$T\mathrm{d}s - P\mathrm{d}v = \mathrm{d}e \tag{545}$$

Therefore,

$$\begin{aligned}T\mathrm{d}s - P\mathrm{d}v &= \left(\frac{\partial e}{\partial \rho}\right)_P \mathrm{d}\rho + \left(\frac{\partial e}{\partial P}\right)_\rho \mathrm{d}P \\ T\mathrm{d}s + \frac{P}{\rho^2}\mathrm{d}\rho &= \left(\frac{\partial e}{\partial \rho}\right)_P \mathrm{d}\rho + \left(\frac{\partial e}{\partial P}\right)_\rho \mathrm{d}P\end{aligned} \tag{546}$$

for isentropic process $\mathrm{d}s = 0$

$$\begin{aligned}\frac{P}{\rho^2}\mathrm{d}\rho &= \left(\frac{\partial e}{\partial \rho}\right)_P \mathrm{d}\rho + \left(\frac{\partial e}{\partial P}\right)_\rho \mathrm{d}P \\ \left(\frac{\partial e}{\partial P}\right)_\rho \mathrm{d}P &= \left(\frac{P}{\rho^2} - \left(\frac{\partial e}{\partial \rho}\right)_P\right)\mathrm{d}\rho\end{aligned} \tag{547}$$

hence

$$\left(\frac{\partial P}{\partial \rho}\right)_S = \frac{\dfrac{P}{\rho^2} - \left(\dfrac{\partial e}{\partial \rho}\right)_P}{\left(\dfrac{\partial e}{\partial P}\right)_\rho} \tag{548}$$

B.7 Isentropic Relations

B.7.1 Perfect Gas

From entropy difference equations for ideal gas:

$$\begin{aligned} s_2 - s_1 &= c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1} \\ &= c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1} \\ &= c_p \ln \frac{v_2}{v_1} + c_v \ln \frac{P2}{P1} \end{aligned} \tag{549}$$

for constant entropy:

$$\begin{aligned} \gamma \ln \frac{v_2}{v_1} &= -\ln \frac{P_2}{P_1} \\ P_1 v_1^\gamma &= P_2 v_2^\gamma \end{aligned} \tag{550}$$

By definition of stagnation enthalpy:

$$\begin{aligned} h_0 &= h + \frac{1}{2}U^2 \\ C_p T_0 &= C_p T + \frac{1}{2}U^2 \\ \frac{T_0}{T} &= 1 + \frac{U^2}{2C_p T_1} \\ C_p &= \frac{\gamma R}{\gamma - 1} \\ \frac{T_0}{T} &= 1 + \frac{U^2}{2\frac{\gamma R}{\gamma-1} T_1} \\ &= 1 + \frac{\gamma - 1}{2} M^2 \end{aligned} \tag{551}$$

The rest can be easily derived using isentropic exponents.

B.7.2 Non-Ideal Fluid

Heat capacities

$$\begin{aligned} c_v &= \left(\frac{\partial e}{\partial T}\right)_v = -T \left(\frac{\partial v}{\partial T}\right)_s \left(\frac{\partial P}{\partial T}\right)_v \\ c_p &= \left(\frac{\partial h}{\partial T}\right)_P = T \left(\frac{\partial P}{\partial T}\right)_s \left(\frac{\partial v}{\partial T}\right)_P \end{aligned} \tag{552}$$

$$\begin{aligned} \gamma = \frac{c_p}{c_v} &= \frac{\left(\frac{\partial P}{\partial v}\right)_s}{\left(\frac{\partial P}{\partial v}\right)_T} \\ c_p - c_v &= T \left(\frac{\partial P}{\partial T}\right)_v \left(\frac{\partial v}{\partial T}\right)_P \end{aligned} \tag{553}$$

Isentropic exponents

$$\begin{aligned} P v^{n_s} &= Const \\ T P^{-m_s} &= Const \end{aligned} \tag{554}$$

Hence in derivative form:

$$\begin{aligned} 0 &= v^{n_s} \mathrm{d}P + n_s v^{n_s - 1} P \mathrm{d}v \\ 0 &= \mathrm{d}P + \frac{n_s P}{v} \mathrm{d}v \\ n_s &= -\frac{v}{P} \left(\frac{\partial P}{\partial v}\right)_S \\ n_s &= -\frac{v\gamma}{P} \left(\frac{\partial P}{\partial v}\right)_T \\ &= \frac{\gamma}{P \kappa_T} \end{aligned} \tag{555}$$

and for m_s:

$$\begin{aligned} 0 &= P^{-m_s} \mathrm{d}T - m_s T P^{-m_s - 1} \mathrm{d}P \\ 0 &= \frac{\mathrm{d}T}{T} - m_s \frac{\mathrm{d}P}{P} \\ m_s &= \frac{P}{T} \left(\frac{\partial T}{\partial P}\right)_S \end{aligned} \tag{556}$$

Then since:

$$\begin{aligned} \frac{C_p - C_v}{C_p} &= \frac{T \left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial v}{\partial T}\right)_P}{T \left(\frac{\partial P}{\partial T}\right)_S \left(\frac{\partial v}{\partial T}\right)_P} \\ &= \left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial P}\right)_S \\ &= \frac{\gamma - 1}{\gamma} \\ \frac{\kappa_T}{\kappa_P} &= \frac{\left(\frac{\partial V}{\partial P}\right)_T}{-\left(\frac{\partial V}{\partial T}\right)_P} \\ &= \frac{-\left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial P}{\partial T}\right)_V}{\left(\frac{\partial P}{\partial T}\right)_V} \\ &= \left(\frac{\partial T}{\partial P}\right)_V \end{aligned} \tag{557}$$

so

$$\left(\frac{\partial T}{\partial P}\right)_S = \frac{\gamma - 1}{\gamma} \frac{\kappa_T}{\kappa_P} \tag{558}$$

and

$$\begin{aligned} m_s &= \frac{P}{T} \left(\frac{\partial T}{\partial P}\right)_S \\ &= \frac{\gamma - 1}{\gamma} \frac{\kappa_T}{\kappa_P} \frac{P}{T} \end{aligned} \tag{559}$$

Static-Stagnation relation

$$\begin{aligned} \mathrm{d}h &= v \mathrm{d}P \\ \int_h^{h_0} \mathrm{d}h &= \int_P^{P_0} v \mathrm{d}P \end{aligned} \tag{560}$$

$$\begin{aligned} LHS &= h_0 - h \\ &= \frac{1}{2}U^2 \\ &= \frac{M^2}{2} n_s P v \end{aligned} \tag{561}$$

$$
\begin{aligned}
RHS &= \int_P^{P_0} v_0 \left(\frac{P_0}{P}\right)^{\frac{1}{n_s}} \mathrm{d}P \\
&= v_0 P_0^{\frac{1}{n_s}} \left(\frac{1}{-\frac{1}{n_s}+1} P^{-\frac{1}{n_s}+1} \right)_P^{P_0} \\
&= \frac{n_s}{n_s-1} v_0 P_0 \left(1 - \left(\frac{P}{P_0}\right)^{\frac{n_s-1}{n_s}}\right)
\end{aligned} \tag{562}
$$

by setting LHS = RHS:

$$
\begin{aligned}
\frac{M^2}{2} n_s P v &= \frac{n_s}{n_s-1} v_0 P_0 \left(1 - \left(\frac{P}{P_0}\right)^{\frac{n_s-1}{n_s}}\right) \\
(n_s-1)\frac{M^2}{2} &= \frac{v_0 P_0}{vP} \left(1 - \left(\frac{P}{P_0}\right)^{\frac{n_s-1}{n_s}}\right) \\
&= \left(\frac{P}{P_0}\right)^{\frac{1-n_s}{n_s}} \left(1 - \left(\frac{P}{P_0}\right)^{\frac{n_s-1}{n_s}}\right) \\
&= \left(\frac{P}{P_0}\right)^{\frac{1-n_s}{n_s}} - 1
\end{aligned} \tag{563}
$$

$$
\frac{P_0}{P} = \left(1 + \frac{n_s-1}{2} M^2\right)^{\frac{n_s}{n_s-1}} \tag{564}
$$

B.8 Nozzle Analysis

B.8.1 General Relations

Conservation laws: From mass conservation:

$$
\begin{aligned}
\dot{m} &= \rho U A = Constant \\
\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}U}{U} + \frac{\mathrm{d}A}{A} &= 0 \\
\frac{\mathrm{d}A}{A} &= -\frac{\mathrm{d}\rho}{\rho} - \frac{\mathrm{d}U}{U}
\end{aligned} \tag{565}
$$

Momentum conservation:

$$
\begin{aligned}
P + \rho U^2 &= Constant \\
\mathrm{d}P + 2\rho U \mathrm{d}U + U^2 \mathrm{d}\rho &= 0 \\
\frac{\mathrm{d}P}{\rho} + 2U\mathrm{d}U + U^2 \frac{\mathrm{d}\rho}{\rho} &= 0
\end{aligned} \tag{566}
$$

Then from mass conservation $\frac{\mathrm{d}\rho}{\rho} = -\frac{\mathrm{d}U}{U}$

$$
\begin{aligned}
\frac{\mathrm{d}P}{\rho} + 2U\mathrm{d}U - U\mathrm{d}U &= 0 \\
\frac{\mathrm{d}P}{\rho} + U\mathrm{d}U &= 0
\end{aligned} \tag{567}
$$

Relations for nozzle By rearrange we have $\frac{\mathrm{d}U}{U} = -\frac{\mathrm{d}P}{\rho U^2}$:

$$
\begin{aligned}
\frac{\mathrm{d}A}{A} &= -\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}P}{\rho U^2} \\
&= \frac{\mathrm{d}P}{\rho}\left(\frac{1}{U^2} - \frac{\mathrm{d}\rho}{\mathrm{d}P}\right) \\
&= \frac{\mathrm{d}P}{\rho U^2}\left(1 - M^2\right)
\end{aligned} \tag{568}
$$

and

$$
\frac{\mathrm{d}A}{A} = \left(M^2 - 1\right)\frac{\mathrm{d}U}{U} \tag{569}
$$

Since

$$
\begin{aligned}
\Gamma &= 1 + \frac{\rho}{c}\left(\frac{\partial c}{\partial \rho}\right)_S \\
&= 1 + \frac{\rho}{c}\left(\frac{\partial c}{\partial P}\right)_S \left(\frac{\partial P}{\partial \rho}\right)_S
\end{aligned} \tag{570}
$$

and

$$
c^2 = \left(\frac{\partial P}{\partial \rho}\right)_S \tag{571}
$$

By substitution and rearrangement:

$$
\begin{aligned}
\Gamma - 1 &= \frac{c}{v}\left(\frac{\partial c}{\partial P}\right)_S \\
v\mathrm{d}P &= \frac{c\mathrm{d}c}{\Gamma - 1}
\end{aligned} \tag{572}
$$

Therefore,

$$
U\mathrm{d}U = -\frac{c\mathrm{d}c}{\Gamma - 1} \tag{573}
$$

By definition:

$$
\begin{aligned}
U &= cM \\
\mathrm{d}U &= c\mathrm{d}M + M\mathrm{d}c \\
\frac{\mathrm{d}U}{U} &= \frac{\mathrm{d}M}{M} + \frac{\mathrm{d}c}{c} \\
\mathrm{d}c &= \left(\frac{\mathrm{d}U}{U} - \frac{\mathrm{d}M}{M}\right) c
\end{aligned} \tag{574}
$$

Combining the equations:

$$
\begin{aligned}
U\mathrm{d}U &= -\frac{c}{\Gamma - 1}\left(\frac{\mathrm{d}U}{U} - \frac{\mathrm{d}M}{M}\right) c \\
\frac{c^2}{\Gamma - 1}\frac{\mathrm{d}M}{M} &= \left(U + \frac{c^2}{(\Gamma-1)U}\right)\mathrm{d}U \\
\frac{\mathrm{d}M}{M} &= \left(\frac{U(\Gamma-1)}{c^2} + \frac{1}{U}\right)\mathrm{d}U \\
\frac{\mathrm{d}M}{M} &= \left(\frac{U^2(\Gamma-1) + c^2}{Uc^2}\right)\mathrm{d}U \\
\frac{\mathrm{d}U}{U} &= \left(\frac{1}{1 + (\Gamma-1)M^2}\right)\frac{\mathrm{d}M}{M}
\end{aligned} \tag{575}
$$

From

$$
\frac{\mathrm{d}A}{A} = \left(\frac{M^2 - 1}{1 + (\Gamma-1)M^2}\right)\frac{\mathrm{d}M}{M} \tag{576}
$$

$$
\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}x} = \left(\frac{1 + (\Gamma-1)M^2}{M^2 - 1}\right)\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} \tag{577}
$$

where x is along the nozzle axis. When $M = 1$, $\frac{dA}{dx} = 1$ and hence right hand side of the equation above is in form of $\frac{0}{0}$. Using L'Hopital's rule:

$$\begin{aligned}\frac{\mathrm{d}M}{\mathrm{d}x} &= \frac{M+(\Gamma-1)M^3\frac{\mathrm{d}A}{\mathrm{d}x}}{(M^2-1)A} \\ &= \frac{\left(\frac{\mathrm{d}M}{\mathrm{d}x}+3M^2(\Gamma-1)\frac{\mathrm{d}M}{\mathrm{d}x}\right)\frac{\mathrm{d}A}{\mathrm{d}x}}{2MA\frac{\mathrm{d}M}{\mathrm{d}x}-\frac{\mathrm{d}A}{\mathrm{d}x}} \\ &+ \frac{\left(M+(\Gamma-1)M^3\right)\frac{\mathrm{d}^2A}{\mathrm{d}x^2}}{2MA\frac{\mathrm{d}M}{\mathrm{d}x}-\frac{\mathrm{d}A}{\mathrm{d}x}}\end{aligned} \tag{578}$$

when $M=1$, $\frac{\mathrm{d}A}{\mathrm{d}x}=0$:

$$\begin{aligned}\frac{\mathrm{d}M}{\mathrm{d}x} &= \frac{(1+\Gamma-1)\frac{\mathrm{d}^2A}{\mathrm{d}x^2}}{2A\frac{\mathrm{d}M}{\mathrm{d}x}} \\ \left(\frac{\mathrm{d}M}{\mathrm{d}x}\right)^2 &= \left(\frac{\Gamma}{2A}\frac{\mathrm{d}^2A}{\mathrm{d}x^2}\right)_{M=1}\end{aligned} \tag{579}$$

B.8.2 Area-Mach-Relations

For AMR (Area-Mach-Relation) equation:

$$\rho U A = \rho^* U^* A^* \tag{580}$$

hence

$$\begin{aligned}\frac{A}{A^*} &= \frac{\rho^*U^*}{\rho U} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{U^*}{c^*}\frac{c^*}{c}\frac{c}{U} \\ &= \left(1+\frac{\gamma-1}{2}\right)^{\frac{1}{\gamma-1}}\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{1}{\gamma-1}} \\ &\quad \frac{M^*}{M}\sqrt{\frac{\gamma RT^*}{\gamma RT}} \\ &= \left(1+\frac{\gamma-1}{2}\right)^{\frac{1}{\gamma-1}}\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{1}{\gamma-1}} \\ &\quad \frac{1}{M}\sqrt{\frac{T^*}{T_0}\frac{T_0}{T}} \\ &= \frac{1}{M}\left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}}\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{1}{\gamma-1}} \\ &\quad \sqrt{\frac{1}{M}\left(\frac{\gamma+1}{2}\right)^{-1}\left(1+\frac{\gamma-1}{2}M^2\right)}\end{aligned} \tag{581}$$

hence

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{582}$$

The derivation follows the same principle for the arbitrary location and for non-ideal fluid.

B.8.3 Mass flow rate

The mass flow rate can be derived:

$$\begin{aligned}\dot{m} &= \rho U A = \frac{P}{RT}AU = \frac{PAMc}{RT} = PAM\sqrt{\frac{\gamma}{RT}} \\ &= P_0AM\sqrt{\frac{\gamma}{RT_0}}\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma}{\gamma-1}+\frac{1}{2}}\end{aligned} \tag{583}$$

$$\begin{aligned}\dot{m} &= P_0AM\sqrt{\frac{\gamma}{(C_p-C_v)T_0}}\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \\ &= P_0AM\sqrt{\frac{\gamma}{\left(C_p\left(1-\frac{1}{\gamma}\right)\right)T_0}}\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}\end{aligned} \tag{584}$$

For non-dimensional form:

$$\overline{m} = \frac{\dot{m}\sqrt{C_pT_0}}{P_0A} = \frac{\gamma}{\sqrt{\gamma-1}}M\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \tag{585}$$

B.8.4 Exit Mach number for isentropic flow

When calculating exit mach number: since $\dot{m} = \frac{P_0A^*}{\sqrt{T_0}}f(\gamma,R)$ and $\dot{m}$ and T_0 are constant through shock:

$$\begin{aligned}P_0A^* &= Constant \\ P_{01}A_1^* &= P_{02}A_2^* \\ P_{01}A_1^* &= P_{0e}A_e^*\end{aligned} \tag{586}$$

hence

$$\frac{P_eA_e}{P_{0e}A_e^*} = \frac{P_eA_e}{P_{01}A_1^*} \tag{587}$$

where

$$\begin{aligned}\frac{P_e}{P_{01}} &= \left(1+\frac{\gamma-1}{2}M_e^2\right)^{-\frac{\gamma}{\gamma-1}} \\ \frac{A_e}{A_1^*} &= \frac{1}{M_e}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_e^2\right)\right)^{\frac{\gamma+1}{2(\gamma-1)}}\end{aligned} \tag{588}$$

Therefore:

$$\begin{aligned}\frac{P_eA_e}{P_{0e}A_e^*} &= \frac{1}{M_e}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\left(1+\frac{\gamma-1}{2}M_e^2\right)^{-\frac{1}{2}} \\ \left(\frac{P_eA_e}{P_{0e}A_e^*}\right)^2 &= \frac{1}{M_e^2}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left(1+\frac{\gamma-1}{2}M_e^2\right)^{-1} \\ \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}} &= \left(\frac{P_eA_e}{P_{0e}A_e^*}\right)^2M_e^2\left(1+\frac{\gamma-1}{2}M_e^2\right)\end{aligned} \tag{589}$$

Using root finding for the quadratic equation above:

$$\begin{aligned}M_e^2 = &\sqrt{\frac{1}{(\gamma-1)^2}+\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left(\frac{P_{0e}A_e^*}{P_eA_e}\right)^2} \\ &-\frac{1}{\gamma-1}\end{aligned} \tag{590}$$

since

$$P_{0e}A_e^* = P_{01}A_1^* \tag{591}$$

$$\begin{aligned}M_e^2 = &\sqrt{\frac{1}{(\gamma-1)^2}+\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left(\frac{P_{01}A_1^*}{P_eA_e}\right)^2} \\ &-\frac{1}{\gamma-1}\end{aligned} \tag{592}$$

The M_e can be calculated as $\frac{P_{01}}{P_e}$ is the known pressure ratio and $\frac{A_1^*}{A_e}$ is geometrical information. Hence P_{02} can be calculated using total-static relations. The ratio of $\frac{P_{02}}{P_{01}}$ cross normal shock can be known. After that, M_1 right before shock can be calculated, and hence the normal shock location can be determined.

For non-ideal fluid flows under polytropic assumption:

$$\begin{aligned}\dot{m} = \rho U A &= \frac{\rho}{\rho_0}\cdot\rho_0\cdot\frac{U}{c}\cdot c\cdot A = \rho_0 A M\sqrt{\frac{n_s P}{\rho}}\frac{\rho}{\rho_0} \\ &= \rho_0 A M\sqrt{n_s}\sqrt{\frac{P}{P_0}\cdot\frac{\rho_0}{\rho}\cdot\frac{P_0}{\rho_0}\frac{\rho}{\rho_0}} \\ &= A M\sqrt{n_s P_0\rho_0}\sqrt{\frac{P}{P_0}\frac{\rho}{\rho_0}} \\ &= A M\sqrt{n_s P_0\rho_0}\left(1+\frac{n_s-1}{2}M^2\right)^{-\frac{n_s+1}{2(n_s-1)}}\end{aligned} \tag{593}$$

hence for non-dimensional mass flow rate:

$$\frac{\dot{m}}{A\sqrt{P_0\rho_0}} = M\sqrt{n_s}\left(1+\frac{n_s-1}{2}M^2\right)^{-\frac{n_s+1}{2(n_s-1)}} \tag{594}$$

B.9 Normal Shock Relation

B.9.1 General

Conservation equations for mass, momentum and energy:

$$\begin{aligned}\rho_1 U_1 &= \rho_2 U_2 = m \\ P_1+\rho_1 U_1^2 &= P_2+\rho_2 U_2^2 \\ h_1+\frac{1}{2}U_1^2 &= h_2+\frac{1}{2}U_2^2\end{aligned} \tag{595}$$

From mass continuity:

$$\rho_1^2 U_1^2 = \rho_2^2 U_2^2 = m^2 \tag{596}$$

substituting into momentum equation and rearrange:

$$\begin{aligned}P_1+\frac{m^2}{\rho_1} &= P_2+\frac{m^2}{\rho_2} \\ m^2 &= \frac{P_2-P_1}{\frac{1}{\rho_1}-\frac{1}{\rho_2}}\end{aligned} \tag{597}$$

hence

$$\begin{aligned}U_1^2 &= \frac{P_2-P_1}{\rho_1^2(v_1-v_2)} \\ U_2^2 &= \frac{P_2-P_1}{\rho_2^2(v_1-v_2)}\end{aligned} \tag{598}$$

Rankine-Hugoniot equation:

$$\begin{aligned}h_2-h_1 &= \frac{1}{2}\left(U_1^2-U_2^2\right) \\ &= \frac{1}{2}\frac{P_2-P_1}{(v_1-v_2)}\left(\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2}\right) \\ &= \frac{1}{2}\frac{P_2-P_1}{(v_1-v_2)}\left(v_1^2-v_2^2\right) \\ &= \frac{1}{2}(P_2-P_1)(v_1+v_2)\end{aligned} \tag{599}$$

B.9.2 Perfect Gas

Mach number derivation From energy equations:

$$\begin{aligned}c_pT_1+\frac{1}{2}U_1^2 &= c_pT_2+\frac{1}{2}U_2^2 \\ \frac{\gamma RT_1}{\gamma-1}+\frac{1}{2}U_1^2 &= \frac{\gamma RT_2}{\gamma-1}+\frac{1}{2}U_2^2 \\ \frac{c_1^2}{\gamma-1}+\frac{1}{2}U_1^2 &= \frac{c_2^2}{\gamma-1}+\frac{1}{2}U_2^2\end{aligned} \tag{600}$$

A reference state for critical point:

$$\frac{c^2}{\gamma-1}+\frac{1}{2}U^2 = \frac{c^{*2}}{\gamma-1}+\frac{1}{2}c^{*2} = \frac{c^{*2}(\gamma+1)}{2(\gamma-1)} \tag{601}$$

hence:

$$\begin{aligned}c_1^2 &= \frac{\gamma+1}{2}c^{*2}-\frac{\gamma-1}{2}U_1^2 \\ c_2^2 &= \frac{\gamma+1}{2}c^{*2}-\frac{\gamma-1}{2}U_2^2\end{aligned} \tag{602}$$

Using momentum equation divided by mass conservation equation:

$$\begin{aligned}\frac{P_1}{\rho_1U_1}+U_1 &= \frac{P_2}{\rho_2U_2}+U2 \\ \frac{c_1^2}{\gamma U_1}+U_1 &= \frac{c_2^2}{\gamma U_2}+U_2\end{aligned} \tag{603}$$

Substituting with Eq. 602 and simplifying we get Prantl relation:

$$c^{*2} = U_1U_2 \tag{604}$$

From Eq. 601

$$\begin{aligned}\frac{c^{*2}(\gamma+1)}{2(\gamma-1)} &= \frac{c^2}{\gamma-1}+\frac{1}{2}U^2 \\ (\gamma+1)\frac{c^{*2}}{U^2} &= 2\frac{c^2}{U^2}+(\gamma-1) \\ \frac{c^{*2}}{U^2} &= \frac{2+(\gamma-1)M^2}{(\gamma+1)M^2}\end{aligned} \tag{605}$$

based on Eq. 604:

$$\begin{aligned}\frac{c^*}{U_1} &= \frac{U_2}{c^*} \\ \sqrt{\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}} &= \sqrt{\frac{2+(\gamma-1)M_2^2}{(\gamma+1)M_2^2}}\end{aligned} \tag{606}$$

By rearrangement:

$$M_2^2 = \frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)} \tag{607}$$

Speed ratio

$$\frac{U_2}{U_1} = \frac{U_2U_1}{U_2^2} = \frac{a^{*2}}{U_1^2} = \frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2} \tag{608}$$

Density

$$\frac{\rho_2}{\rho_1} = \frac{U_1}{U_2} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2} \tag{609}$$

Pressure ratio

$$\begin{aligned} P_1 + \rho_1 U_1^2 &= P_2 + \rho_2 U_2^2 \\ P_2 - P_1 &= \rho_1 U_1^2 - \rho_2 U_2^2 \\ \frac{P_2}{P_1} - 1 &= \frac{1}{P_1}\left(\rho_1 U_1^2 - \rho_2 U_2^2\right) \\ &= \frac{1}{P_1}\left(\rho_1 U_1^2 - \rho_1 U_1 U_2\right) \\ &= \frac{\rho_1 U_1^2}{P_1}\left(1 - \frac{U_2}{U_1}\right) \end{aligned} \tag{610}$$

hence:

$$\begin{aligned} \frac{P_2}{P_1} &= 1 + \frac{\gamma U_1^2}{c_1^2}\left(1 - \frac{U_2}{U_1}\right) \\ &= 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right) \\ &= \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1} \end{aligned} \tag{611}$$

Temperature

$$\begin{aligned} \frac{T2}{T1} &= \frac{P_2}{\rho_2}\frac{\rho_1}{P_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} \\ &= \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1} \times \frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2} \\ &= \frac{\left(2\gamma M_1^2 - (\gamma - 1)\right)\left((\gamma - 1) M_1^2 + 2\right)}{(\gamma + 1)^2 M_1^2} \end{aligned} \tag{612}$$

Stagnation pressure ratio

$$\begin{aligned} \frac{P_{02}}{P_{01}} &= \frac{P_2 \dfrac{P_{02}}{P_2}}{P_1 \dfrac{P_{01}}{P_1}} \\ &= \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}\frac{P_{02}}{P_2}\frac{P_1}{P_{01}} \\ &= \left(\frac{(\gamma + 1) M_1^2}{(\gamma - 1) M_1^2 + 2}\right)^{\frac{\gamma}{\gamma - 1}} \\ &\quad \left(\frac{\gamma + 1}{2\gamma M_1^2 - (\gamma - 1)}\right)^{\frac{1}{\gamma - 1}} \end{aligned} \tag{613}$$

B.9.3 Non-Ideal Fluid

Static pressure Similar to ideal gas, starting from Eq. 595.

$$\begin{aligned} P_1\left(1 + \frac{\rho_1}{P_1} U_1^2\right) &= P_2\left(1 + \frac{\rho_2}{P_2} U_2^2\right) \\ P_1\left(1 + \frac{\rho_1}{P_1} M_1^2 n_{s_1}\frac{P_1}{\rho_1}\right) &= P_2\left(1 + \frac{\rho_2}{P_2} M_2^2 n_{s_2}\frac{P_2}{\rho_2}\right) \end{aligned} \tag{614}$$

hence:

$$\frac{P_2}{P_1} = \frac{1 + n_{s_1} M_1^2}{1 + n_{s_2} M_2^2} \tag{615}$$

Density From mass conservation:

$$\begin{aligned} \frac{U_1}{U_2} = \frac{\rho_2}{\rho_1} &= \frac{\frac{U_1}{c_1} c_1}{\frac{U_2}{c_2} c_2} = \frac{M_1}{M_2}\sqrt{\frac{n_{s_1} P_1}{\rho_1}}\sqrt{\frac{\rho_2}{n_{s_2} P_2}} \\ \sqrt{\frac{\rho_2}{\rho_1}} &= \frac{M_1}{M_2}\sqrt{\frac{n_{s_1}}{n_{s_2}}}\sqrt{\frac{P_1}{P_2}} \\ \frac{\rho_2}{\rho_1} &= \frac{n_{s_1} M_1^2 P_1}{n_{s_2} M_2^2 P_2} \end{aligned} \tag{616}$$

hence:

$$\frac{\rho_2}{\rho_1} = \frac{n_{s_1} M_1^2\left(1 + n_{s_2} M_2^2\right)}{n_{s_2} M_2^2\left(1 + n_{s_1} M_1^2\right)} \tag{617}$$

Temperature

$$\frac{T_2}{T_1} = \frac{Z_1 P_2 \rho_1}{Z_2 P_1 \rho_2} \tag{618}$$

$$\frac{T_2}{T_1} = \frac{Z_1 n_{s_2} M_2^2\left(1 + n_{s_1} M_1^2\right)^2}{Z_2 n_{s_1} M_1^2\left(1 + n_{s_2} M_2^2\right)^2} \tag{619}$$

B.10 Prandtl-Meyer Function

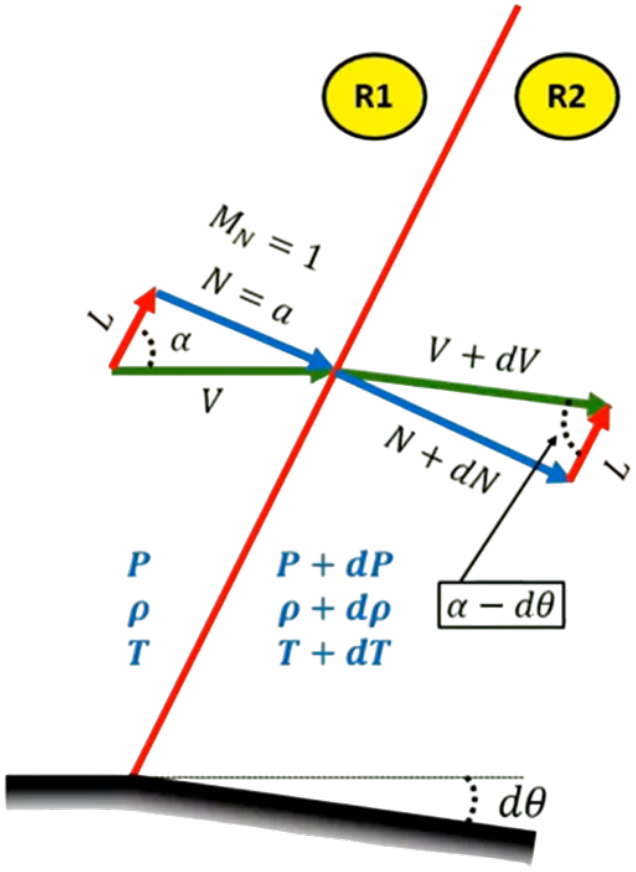


Figure 5: Prandtl-Meyer expansion wave, θ is ν

From illustration in Fig. 5, L is constant:

$$\begin{aligned} V\cos\alpha &= (V + \mathrm{d}V)\cos(\alpha - \mathrm{d}\theta) \\ &= (V + \mathrm{d}V)(\cos\alpha\cos\mathrm{d}\theta + \sin\alpha\sin\mathrm{d}\theta) \end{aligned} \tag{620}$$

Using small angle approximation:

$$\begin{aligned}(V+\mathrm{d}V)(\cos\alpha+\mathrm{d}\theta\sin\alpha)&=V\cos\alpha\\ \frac{V+\mathrm{d}V}{V}(1+\mathrm{d}\theta\tan\alpha)&=1\\ \left(1+\frac{\mathrm{d}V}{V}\right)(1+\mathrm{d}\theta\tan\alpha)&=1\end{aligned}\tag{621}$$

hence:

$$\frac{\mathrm{d}V}{V}=-\mathrm{d}\theta\tan\alpha=\frac{-\mathrm{d}\theta}{\sqrt{M^2-1}}\tag{622}$$

Since

$$\frac{\mathrm{d}U}{U}=\left(\frac{1}{1+(\Gamma-1)M^2}\right)\frac{\mathrm{d}M}{M}\tag{623}$$

substituting:

$$\mathrm{d}\theta=\frac{\sqrt{M^2-1}}{1+(\Gamma-1)M^2}\frac{\mathrm{d}M}{M}\tag{624}$$

B.11 Loss Mechanism

B.11.1 Boundary Layer Loss

From second law of thermodynamics:

$$\begin{aligned}T\frac{\mathrm{d}s}{\mathrm{d}x}&=\frac{\mathrm{d}h}{\mathrm{d}x}-\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x}\\ &=\frac{\mathrm{d}h_0}{\mathrm{d}x}-U_x\frac{\mathrm{d}U_x}{\mathrm{d}x}-\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x}\end{aligned}\tag{625}$$

from momentum balance

$$F_x-\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x}=U_x\frac{\mathrm{d}U_x}{\mathrm{d}x}\tag{626}$$

Therefore:

$$T\frac{\mathrm{d}s}{\mathrm{d}x}=\frac{\mathrm{d}h_0}{\mathrm{d}x}-F_x\tag{627}$$

From Euler's equation for energy balance:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho\left(e+\frac{1}{2}|U|^2\right)\right)+\nabla\cdot\left(\rho\vec{U}\left(e+\frac{1}{2}|U|^2\right)\right)=\\ \nabla\cdot(k\nabla T)+\nabla\cdot\left(-P\vec{U}+\underline{\underline{\tau}}\cdot\vec{U}\right)+\vec{U}\cdot\vec{F}+Q_{gen}\end{aligned}\tag{628}$$

and its weaker form

$$\begin{aligned}\rho\frac{De_0}{Dt}=-\nabla\vec{q}-\nabla\cdot\left(P\vec{U}\right)+\nabla\cdot\frac{1}{\rho}\left(\underline{\underline{\tau}}\cdot\vec{U}\right)\\ +\vec{U}\cdot\vec{F}+Q_{gen}\end{aligned}\tag{629}$$

Assuming no heat generation:

$$\begin{aligned}\frac{De_0}{Dt}=-\frac{1}{\rho}\nabla\vec{q}-\frac{1}{\rho}P\nabla\cdot\vec{U}-\frac{1}{\rho}\vec{U}\cdot\nabla P\\ +\nabla\cdot\left(\underline{\underline{\tau}}\cdot\vec{U}\right)+\vec{U}\cdot\vec{F}\end{aligned}\tag{630}$$

The material derivative of $\frac{1}{\rho}$ can be shown to equal to:

$$\begin{aligned}\frac{D}{Dt}\left(\frac{1}{\rho}\right)&=\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)+U_x\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\\ &\quad+U_y\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)+U_z\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right)\\ &=-\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial t}+U_x\frac{\partial\rho}{\partial x}+U_y\frac{\partial\rho}{\partial y}+U_z\frac{\partial\rho}{\partial z}\right)\\ &=-\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial t}+\vec{U}\cdot\nabla\rho\right)\end{aligned}\tag{631}$$

From continuity equation:

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\vec{U}\right)&=0\\ \frac{\partial\rho}{\partial t}+\vec{U}\cdot(\nabla\rho)+\rho\nabla\cdot\vec{U}&=0\\ \frac{\partial\rho}{\partial t}+\vec{U}\cdot(\nabla\rho)&=-\rho\nabla\cdot\vec{U}\end{aligned}\tag{632}$$

Therefore:

$$\begin{aligned}\frac{D}{Dt}\left(\frac{1}{\rho}\right)&=\frac{1}{\rho}\left(\nabla\cdot\vec{U}\right)\\ P\frac{D}{Dt}\left(\frac{1}{\rho}\right)&=\frac{1}{\rho}P\left(\nabla\cdot\vec{U}\right)\end{aligned}\tag{633}$$

Hence for 2D problem with assumptions, the Euler's equation reduced to:

$$\begin{aligned}\frac{D}{Dt}\left(e+\frac{1}{2}U^2\right)=-P\frac{D}{Dt}\left(\frac{1}{\rho}\right)+U_x\left(F_x-\frac{1}{\rho}\frac{\partial P}{\partial x}\right)\\ +\frac{1}{\rho}\left(\tau_{yx}\frac{\partial U_x}{\partial y}+\tau_{yz}\frac{\partial U_z}{\partial y}\right)-\frac{1}{\rho}\frac{\partial q}{\partial y}\end{aligned}\tag{634}$$

with some algebra assuming steady flow:

$$\begin{aligned}-P\frac{D}{Dt}\left(\frac{1}{\rho}\right)-U_x\frac{1}{\rho}\frac{\partial P}{\partial x}&=-U_x\left(P\frac{\partial\left(\frac{1}{\rho}\right)}{\partial x}+\frac{1}{\rho}\frac{\partial P}{\partial x}\right)\\ &=-U_x\left(\frac{\partial\frac{P}{\rho}}{\partial x}\right)\\ &=-\frac{D}{Dt}\left(\frac{P}{\rho}\right)\\ e+\frac{1}{2}U^2+\frac{P}{\rho}&=e_0+\frac{P}{\rho}\\ &=h_0\end{aligned}\tag{635}$$

Thus:

$$\begin{aligned}\frac{Dh_0}{Dt}&=F_xU_x+\frac{1}{\rho}\left(\tau_{yx}\frac{\partial U_x}{\partial y}+\tau_{yz}\frac{\partial U_z}{\partial y}\right)-\frac{1}{\rho}\frac{\partial q}{\partial y}\\ U_x\frac{\mathrm{d}h_0}{\mathrm{d}x}&=F_xU_x+\frac{1}{\rho}\left(\tau_{yx}\frac{\partial U_x}{\partial y}+\tau_{yz}\frac{\partial U_z}{\partial y}\right)-\frac{1}{\rho}\frac{\partial q}{\partial y}\\ &=U_x\left(T\frac{\mathrm{d}s}{\mathrm{d}x}+F_x\right)\\ U_xT\frac{\mathrm{d}s}{\mathrm{d}x}&=\frac{1}{\rho}\left(\tau_{yx}\frac{\partial U_x}{\partial y}+\tau_{yz}\frac{\partial U_z}{\partial y}\right)-\frac{1}{\rho}\frac{\partial q}{\partial y}\end{aligned}\tag{636}$$

given mass flow rate $\rho U_x\Delta A=\dot{m}=constant$, and times on top of equation above:

$$\begin{aligned}T\frac{\partial}{\partial x}(\rho U_xs\Delta A)&=\Delta A\left(\tau_{yx}\frac{\partial U_x}{\partial y}+\tau_{yz}\frac{\partial U_z}{\partial z}-\frac{\partial q}{\partial y}\right)\\ \int_0^\delta\frac{\partial}{\partial x}(\rho U_xs)\ \mathrm{d}y&=\int_0^\delta\frac{1}{T}\left(\tau_{yx}\mathrm{d}U_x+\tau_{yz}\mathrm{d}U_z-\mathrm{d}q\right)\end{aligned}\tag{637}$$

For adiabatic process with $\mathrm{d}q=0$, and now skew condition $\tau_{yz}=0$

$$\begin{aligned}\dot{S}_a&=\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\theta\left(\rho U_x\left(s-s_\theta\right)\right)\mathrm{d}y\\ &=\int_0^\theta\frac{1}{T}\tau_{yx}\mathrm{d}U_x\end{aligned}\tag{638}$$

The physical interpretation of the equation is the rate of change of entropy flux per unit length is equal to the rate of entropy generation per unit surface area by viscous effects in the boundary layer.

B.11.2 Constant Area Mixing Case

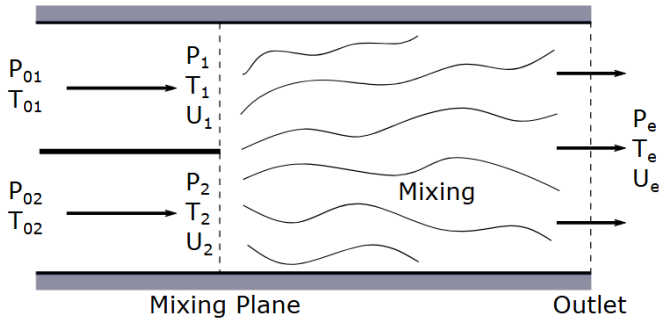


Figure 6: Mixing of two streams in a constant area duct

Assuming common and uniform static pressure P_c at the mixing interface:

$$\begin{aligned}\frac{P_{01}}{P_c} &= \left(1+\frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}} \\ \frac{P_{02}}{P_c} &= \left(1+\frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{\gamma-1}}\end{aligned} \tag{639}$$

The total momentum flux at the mixing plane C is

$$I_c = P_c\left(A_1+A_2\right)+\dot{m}_1u_1+\dot{m}_2u_2 \tag{640}$$

Neglecting wall friction in CM section, the exit momentum flux must be equal to I_c.

Assuming perfect gas and adiabatic flow:

$$T_{0m} = \frac{\dot{m}_1T_{01}+\dot{m}_2T_{02}}{\dot{m}_1+\dot{m}_2} \tag{641}$$

Therefore the non-dimensional form of impulse function can be evaluated:

$$\begin{aligned}F_m &= \frac{I_m}{\left(\dot{m}_1+\dot{m}_2\right)\sqrt{C_pT_{0m}}} \\ &= \frac{P_mA_m+\dot{m}_mu_m}{\dot{m}_m\sqrt{C_pT_m\left(1+\frac{\gamma-1}{2}M_m^2\right)}} \\ &= \frac{P_mA_m+\rho_mu_m^2A_m}{\rho_mu_mA_m\sqrt{C_pT_m\left(1+\frac{\gamma-1}{2}M_m^2\right)}} \\ &= \frac{P_m}{\rho_mM_m\sqrt{\gamma RT_m}\sqrt{C_pT_m\left(1+\frac{\gamma-1}{2}M_m^2\right)}} \\ &\quad+\frac{M_m\sqrt{\gamma RT_m}}{\sqrt{C_pT_m\left(1+\frac{\gamma-1}{2}M_m^2\right)}} \\ &= \frac{P_m}{\rho_mT_mM_m\sqrt{\gamma RC_p\left(1+\frac{\gamma-1}{2}M_m^2\right)}} \\ &\quad+\frac{M_m\sqrt{\gamma-1}}{\sqrt{\left(1+\frac{\gamma-1}{2}M_m^2\right)}} \\ &= \frac{\sqrt{\gamma-1}}{\gamma M_m\sqrt{\left(1+\frac{\gamma-1}{2}M_m^2\right)}}+\frac{M_m\sqrt{\gamma-1}}{\sqrt{\left(1+\frac{\gamma-1}{2}M_m^2\right)}}\end{aligned} \tag{642}$$

hence:

$$F_m = \frac{\sqrt{\gamma-1}\left(1+\gamma M_m^2\right)}{\gamma M_m\sqrt{\left(1+\frac{\gamma-1}{2}M_m^2\right)}} \tag{643}$$

B.11.3 Mixing Loss

From momentum conservation of generalised 1D flow:

$$\begin{aligned}\frac{\mathrm{d}P}{P}+\kappa M^2\frac{\mathrm{d}U}{U}+\frac{\kappa M^2}{2}\left(4f\frac{\mathrm{d}x}{D}+\frac{\mathrm{d}F_x}{\frac{1}{2}\kappa PAM^2}\right) \\ +\kappa M^2\left(1-y\right)\frac{\mathrm{d}\dot{m}}{\dot{m}}=0\end{aligned} \tag{644}$$

with no external forces and inviscid calculation:

$$\frac{\mathrm{d}P}{P}+\kappa M^2\frac{\mathrm{d}U}{U} = \frac{\mathrm{d}\dot{m}}{\dot{m}}\left(\kappa M^2\left(\frac{U_{x,jet}}{U}-1\right)\right) \tag{645}$$

The mass conservation equation

$$\frac{\mathrm{d}\rho}{\rho}+\frac{\mathrm{d}U_x}{U_x} = \frac{\mathrm{d}\dot{m}}{\dot{m}} \tag{646}$$

The energy conservation equation:

$$\frac{\mathrm{d}T_0}{T_0} = \frac{\mathrm{d}\dot{m}}{\dot{m}}\left(\frac{T_{0,jet}}{T_0}-1\right) \tag{647}$$

From perfect gas assumptions:

$$\frac{\mathrm{d}s}{C_p} = \frac{\mathrm{d}T}{T}-\frac{\gamma-1}{\gamma}\frac{\mathrm{d}P}{P} \tag{648}$$

The total temperature:

$$\begin{aligned}
T_0 &= T + \frac{1}{2C_p}U^2 \\
\mathrm{d}T_0 &= \mathrm{d}T + \frac{U}{C_p}\mathrm{d}U \\
\frac{\mathrm{d}T_0}{T_0} &= \frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{\mathrm{d}T}{T} + \frac{U^2}{C_pT_0}\frac{\mathrm{d}U}{U} \\
\frac{\mathrm{d}T_0}{T_0} &= \frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{\mathrm{d}T}{T} + \frac{M^2\gamma RT}{C_p\left(1+\frac{\gamma-1}{2}M^2\right)T}\frac{\mathrm{d}U}{U} \\
\frac{\mathrm{d}T_0}{T_0} &= \frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{\mathrm{d}T}{T} + \frac{(\gamma-1)M^2}{1+\frac{\gamma-1}{2}M^2}\frac{\mathrm{d}U}{U}
\end{aligned} \tag{649}$$

Therefore:

$$\begin{aligned}
\frac{\mathrm{d}T}{T} &= \left(1+\frac{\gamma-1}{2}M^2\right)\frac{\mathrm{d}T_0}{T_0} - (\gamma-1)M^2\frac{\mathrm{d}U}{U} \\
\frac{\mathrm{d}P}{P} &= \frac{\mathrm{d}\dot{m}}{\dot{m}}\left(\gamma M^2\left(\frac{U_{x,jet}}{U}-1\right)\right) - \gamma M^2\frac{\mathrm{d}U}{U}
\end{aligned} \tag{650}$$

Hence from the entropy equation:

$$\begin{aligned}
\frac{\mathrm{d}s}{C_p} &= \left(1+\frac{\gamma-1}{2}M_m^2\right)\frac{\mathrm{d}T_0}{T_0} - (\gamma-1)M_m^2\frac{\mathrm{d}U}{U} \\
&\quad - \left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\mathrm{d}\dot{m}}{\dot{m}}\left(\gamma M_m^2\left(\frac{U_{x,jet}}{U}-1\right)\right)\right. \\
&\quad \left. - \gamma M_m^2\frac{\mathrm{d}U}{U}\right) \\
&= \frac{\mathrm{d}\dot{m}}{\dot{m}}\left(\left(1+\frac{\gamma-1}{2}M_m^2\right)\frac{T_{0c}-T_{0m}}{T_{0m}}\right. \\
&\quad \left. + (\gamma-1)M_m^2\left(1-\frac{U_c\cos\alpha}{U_m}\right)\right)
\end{aligned} \tag{651}$$

with $T_{0c} = T_{0m}$ assumption:

$$\begin{aligned}
\frac{\mathrm{d}s}{C_p} &= \frac{\mathrm{d}\dot{m}}{\dot{m}}\left((\gamma-1)M_m^2\left(1-\frac{U_c\cos\alpha}{U_m}\right)\right) \\
\frac{\mathrm{d}s}{C_p} &= \frac{\mathrm{d}\dot{m}}{\dot{m}}\frac{(\gamma-1)U_m^2}{\gamma RT}\left(1-\frac{U_c\cos\alpha}{U_m}\right) \\
T\dot{m}_m\Delta s = T\dot{S} &= \dot{m}_c\left(U_m^2 - U_mU_c\cos\alpha\right)
\end{aligned} \tag{652}$$

For stagnation pressure variation:

$$\begin{aligned}
\frac{\mathrm{d}P_0}{P_0} &= \frac{\gamma}{\gamma-1}\left(\frac{\mathrm{d}T_0}{T_0} - \frac{\mathrm{d}s}{C_p}\right) \\
&= \frac{\mathrm{d}\dot{m}}{\dot{m}}\left(-\frac{\gamma M^2}{2}\left(\frac{T_{0,jet}}{T_0}-1\right) - \gamma M^2\left(1-\frac{u_{x,jec}}{u_x}\right)\right)
\end{aligned} \tag{653}$$

B.12 Turbulence

B.12.1 Eddy Viscosity Models

From 2D assumption:

$$-\overline{\rho}\overline{u_1'u_2'} = \mu_t\frac{\partial\overline{u}_1}{\partial x_2} \tag{654}$$

Since $\overline{u_1'u_2'} = \overline{u_2'u_1'}$, hence the shear stresses can be defined as

$$-\overline{\rho}\overline{u_1'u_2'} = \mu_t\left(\frac{\partial\overline{u}_1}{\partial x_2} + \frac{\partial\overline{u}_2}{\partial x_1}\right) \tag{655}$$

or in index notation:

$$-\overline{\rho}\overline{u_i'u_j'} = \mu_t\left(\frac{\partial\overline{u}_i}{\partial x_j} + \frac{\partial\overline{u}_j}{\partial x_i}\right) \tag{656}$$

The normal turbulent stresses can be written

$$-\overline{\rho}\overline{u_1'u_1'} = 2\mu_t\frac{\partial\overline{u}_1}{\partial x_1} \tag{657}$$

However, there is an internal inconsistency in this assumption for the normal turbulent stresses as their sum does not match the turbulent kinetic energy definition:

$$-\overline{\rho}\overline{u_i'u_i'} = 2\mu_t\frac{\partial\overline{u}_i}{\partial x_i} \neq -2\rho k_t \tag{658}$$

From turbulent kinetic energy:

$$k_t = \frac{1}{2}\left(\overline{u_i'u_i'}\right) = \frac{1}{2}\left(\overline{u_1'u_1'} + \overline{u_2'u_2'} + \overline{u_3'u_3'}\right) \tag{659}$$

The sum of normal turbulent stresses must hence be

$$-\overline{\rho}\overline{u_1'u_1'} - \overline{\rho}\overline{u_2'u_2'} - \overline{\rho}\overline{u_3'u_3'} = -2\rho k_t \tag{660}$$

Therefore, to correct the normal stress terms:

$$\begin{aligned}
-\overline{\rho}\overline{u_1'u_1'} &= 2\mu_t\left(\frac{\partial\overline{u}_1}{\partial x_1} - \frac{1}{3}\left(\frac{\partial\overline{u}_1}{\partial x_1} + \frac{\partial\overline{u}_2}{\partial x_2} + \frac{\partial\overline{u}_3}{\partial x_3}\right)\right) - \frac{2}{3}\overline{\rho}k_t \\
-\overline{\rho}\overline{u_2'u_2'} &= 2\mu_t\left(\frac{\partial\overline{u}_2}{\partial x_2} - \frac{1}{3}\left(\frac{\partial\overline{u}_1}{\partial x_1} + \frac{\partial\overline{u}_2}{\partial x_2} + \frac{\partial\overline{u}_3}{\partial x_3}\right)\right) - \frac{2}{3}\overline{\rho}k_t \\
-\overline{\rho}\overline{u_3'u_3'} &= 2\mu_t\left(\frac{\partial\overline{u}_3}{\partial x_3} - \frac{1}{3}\left(\frac{\partial\overline{u}_1}{\partial x_1} + \frac{\partial\overline{u}_2}{\partial x_2} + \frac{\partial\overline{u}_3}{\partial x_3}\right)\right) - \frac{2}{3}\overline{\rho}k_t
\end{aligned} \tag{661}$$

so that Eq. (660) can be guarenteed.

For consistency, the turbulence stress tensor can be written as

$$-\overline{\rho}\overline{u_i'u_j'} = \mu_t\left(\frac{\partial\overline{u}_i}{\partial x_j} + \frac{\partial\overline{u}_j}{\partial x_i} - \frac{2}{3}\frac{\partial\overline{u}_k}{\partial x_k}\delta_{ij}\right) - \frac{2}{3}\overline{\rho}k_t\delta_{ij} \tag{662}$$

where δ_{ij} is only non-zero for normal stress terms.

B.13 Mixture

B.13.1 Mixture Properties

From total mass equation:

$$\begin{aligned}
m_m &= m_l + m_g \\
V_m\rho_m &= V_l\rho_l + V_g\rho_g \\
\rho_m &= \frac{V_g}{V_m}\rho_g + \frac{V_l}{V_m}\rho_l \\
&= \alpha\rho_g + (1-\alpha)\rho_l
\end{aligned} \tag{663}$$

Derived from Amagat's law:

$$\begin{aligned}
\rho_m &= \alpha\rho_g + (1-\alpha)\rho_l \\
&= \chi\frac{\rho_m}{\rho_g}\rho_g + \left(1-\chi\frac{\rho_m}{\rho_g}\right)\rho_l \\
\frac{(1-\chi)\rho_m}{\rho_l} &= 1 - \chi\frac{\rho_m}{\rho_g} \\
1 &= \left(\frac{1-\chi}{\rho_l} + \frac{\chi}{\rho_g}\right)\rho_m \\
\rho_m &= \frac{\rho_l\rho_g}{\chi\rho_g + (1-\chi)\rho_g}
\end{aligned} \tag{664}$$

According to $\alpha - \chi$ relation:

$$\begin{aligned}
\chi &= \alpha \frac{\rho_g}{\rho_m} \\
\rho_m &= \frac{\alpha}{\chi} \rho_g \\
\frac{\alpha}{\chi} \rho_g &= \frac{\rho_l \rho_g}{\chi \rho_g + (1-\chi)\rho_g} \\
\alpha &= \frac{\chi \rho_l}{\chi \rho_g + (1-\chi)\rho_g} \\
1 - \alpha &= \frac{(1-\chi)\rho_g}{\chi \rho_g + (1-\chi)\rho_g}
\end{aligned} \tag{665}$$

B.13.2 Mixture Fraction Relations

$\alpha - \chi$ relation:

$$\begin{aligned}
\chi = \frac{m_g}{m_g + m_l} &= \frac{\rho_g V_g}{\rho_m V_m} \\
&= \alpha \frac{\rho_g}{\rho_m}
\end{aligned} \tag{666}$$

$$\begin{aligned}
1 - \alpha &= \frac{V_l}{V_m} \\
&= \frac{\frac{m_l}{\rho_l}}{\frac{m_g + m_l}{\rho_m}} \\
&= \frac{m_l}{m_g + m_l} \frac{\rho_m}{\rho_l} \\
&= (1-\chi) \frac{\rho_m}{\rho_l} \\
&= (1-\chi) \frac{\alpha \rho_g + (1-\alpha)\rho_l}{\rho_l}
\end{aligned} \tag{667}$$

therefore,

$$\begin{aligned}
1 - \alpha &= (1-\chi)(1-\alpha) + (1-\chi)\alpha \frac{\rho_g}{\rho_l} \\
(1-\alpha))(1 - 1 + \chi) &= (1-\chi)\alpha \frac{\rho_g}{\rho_l} \\
\frac{1-\alpha}{\alpha} &= \frac{1-\chi}{\chi} \frac{\rho_g}{\rho_l} \\
\frac{1}{\alpha} - 1 &= \frac{1-\chi}{\chi} \frac{\rho_g}{\rho_l} \\
\alpha &= \frac{1}{1 + \frac{1-\chi}{\chi} \frac{\rho_g}{\rho_l}}
\end{aligned} \tag{668}$$

B.13.3 Wet Vapour Speed of Sound

$$\begin{aligned}
c_m^2 &= \left(\frac{\partial P}{\partial \rho_m} \right)_S \\
c_m^{-2} &= \left(\frac{\partial \rho_m}{\partial P} \right)_S \\
&= \frac{\partial}{\partial P} \left(\alpha \rho_g + (1-\alpha)\rho_l \right) \\
&= \alpha \frac{\partial \rho_g}{\partial P} + \rho_g \frac{\partial \alpha}{\partial P} + (1-\alpha) \frac{\partial \rho_l}{\partial P} - \rho_l \frac{\partial \alpha}{\partial P} \\
&= \alpha \frac{\partial \rho_g}{\partial P} + (1-\alpha) \frac{\partial \rho_l}{\partial P} + (\rho_g - \rho_l) \frac{\partial \alpha}{\partial P}
\end{aligned} \tag{669}$$

From Eq. 665:

$$\begin{aligned}
\frac{\partial \alpha}{\partial P} &= \frac{1}{\chi \rho_l + (1-\chi)\rho_g} \left(\chi \frac{\partial \rho_l}{\partial P} + \rho_l \frac{\partial \chi}{\partial P} \right) \\
&\quad - \frac{\chi \rho_l \left(\chi \frac{\partial \rho_l}{\partial P} + \rho_l \frac{\partial \chi}{\partial P} + (1-\chi) \frac{\partial \rho_g}{\partial P} - \rho_g \frac{\partial \chi}{\partial P} \right)}{\left(\chi \rho_l + (1-\chi)\rho_g \right)^2} \\
&= \Bigg(\left(\chi \rho_l + (1-\chi)\rho_g \right) \left(\chi \frac{\partial \rho_l}{\partial P} + \rho_l \frac{\partial \chi}{\partial P} \right) - \chi^2 \rho_l \frac{\partial \rho_l}{\partial P} \\
&\quad - \chi \rho_l^2 \frac{\partial \chi}{\partial P} - \chi (1-\chi) \rho_l \frac{\partial \rho_g}{\partial P} + \chi \rho_l \rho_g \frac{\partial \chi}{\partial P} \Bigg) \\
&\quad \frac{1}{\left(\chi \rho_l + (1-\chi)\rho_g \right)^2} \\
&= \Bigg(\chi^2 \rho_l \frac{\partial \rho_l}{\partial P} + \chi (1-\chi) \rho_g \frac{\partial \rho_l}{\partial P} + \chi \rho_l^2 \frac{\partial \chi}{\partial P} \\
&\quad + (1-\chi) \rho_l \rho_g \frac{\partial \chi}{\partial P} - \chi^2 \rho_l \frac{\partial \rho_l}{\partial P} - \chi (1-\chi) \rho_l \frac{\partial \rho_g}{\partial P} \\
&\quad \chi \rho_l (\rho_l - \rho_g) \frac{\partial \chi}{\partial P} \Bigg) \frac{1}{\left(\chi \rho_l + (1-\chi)\rho_g \right)^2} \\
&= \frac{\chi (1-\chi) \left(\rho_g \frac{\partial \rho_l}{\partial P} - \rho_l \frac{\partial \rho_g}{\partial P} \right) + \rho_l \rho_g \frac{\partial \chi}{\partial P}}{\left(\chi \rho_l + (1-\chi)\rho_g \right)^2}
\end{aligned} \tag{670}$$

From last equations in Eq. 665:

$$\begin{aligned}
\frac{\partial \alpha}{\partial P} &= \frac{(1-\alpha)\chi \frac{\partial \rho_l}{\partial P}}{\chi \rho_l + (1-\chi)\rho_g} - \frac{\alpha (1-\chi) \frac{\partial \rho_g}{\partial P}}{\chi \rho_l + (1-\chi)\rho_g} + \frac{\rho_m^2 \frac{\partial \chi}{\partial P}}{\rho_l \rho_g} \\
&= \frac{\alpha (1-\alpha)}{\rho_l} \frac{\partial \rho_l}{\partial P} - \frac{\alpha (1-\alpha)}{\rho_g} \frac{\partial \rho_g}{\partial P} + \frac{\rho_m^2}{\rho_l \rho_g} \frac{\partial \chi}{\partial P}
\end{aligned} \tag{671}$$

Therefore

$$\begin{aligned}
c^{-2} &= \alpha \frac{\partial \rho_g}{\partial P} + (1-\alpha) \frac{\partial \rho_l}{\partial P} + \frac{\rho_g - \rho_l}{\left(\chi \rho_l + (1-\chi)\rho_g \right)^2} \\
&\quad \left(\chi (1-\chi) \left(\rho_g \frac{\partial \rho_l}{\partial P} - \rho_l \frac{\partial \rho_g}{\partial P} \right) + \rho_l \rho_g \frac{\partial \chi}{\partial P} \right) \\
&= \alpha \frac{\partial \rho_g}{\partial P} + (1-\alpha) \frac{\partial \rho_l}{\partial P} + (\rho_g - \rho_l) \\
&\quad \left(\frac{\alpha (1-\alpha)}{\rho_l} \frac{\partial \rho_l}{\partial P} - \frac{\alpha (1-\alpha)}{\rho_g} \frac{\partial \rho_g}{\partial P} + \frac{\rho_m^2}{\rho_l \rho_g} \frac{\partial \chi}{\partial P} \right)
\end{aligned} \tag{672}$$

Grouping terms for $\frac{\partial \chi}{\partial P}$ coefficient:

$$\text{Coeff.}\chi = \frac{\rho_m^2 (\rho_g - \rho_l)}{\rho_l \rho_g} \tag{673}$$

Grouping terms for $\frac{\partial \rho_g}{\partial P}$ coefficient:

$$\begin{aligned}
\text{Coeff.}\rho_P &= \alpha - \frac{(\rho_g - \rho_l)\,\chi\,(1-\chi)\,\rho_l}{\left(\chi\rho_l + (1-\chi)\,\rho_g\right)^2} \\
&= \alpha - \frac{\alpha\,(1-\chi)\,(\rho_g - \rho_l)}{\chi\rho_l + (1-\chi)\,\rho_g} \\
&= \alpha\left(1 - \frac{(1-\chi)\,(\rho_g - \rho_l)}{\chi\rho_l + (1-\chi)\,\rho_g}\right) \\
&= \alpha\frac{\chi\rho_l + (1-\chi)\,\rho_g - (1-\chi)\,(\rho_g - \rho_l)}{\chi\rho_l + (1-\chi)\,\rho_g} \\
&= \frac{\alpha\rho_l}{\chi\rho_l + (1-\chi)\,\rho_g}
\end{aligned} \tag{674}$$

from Eq. 664:

$$\text{Coeff.}\rho_P = \frac{\alpha\rho_m}{\rho_g} \tag{675}$$

Grouping terms for $\frac{\partial \rho_l}{\partial P}$ coefficient:

$$\begin{aligned}
\text{Coeff.}\rho_l &= (1-\alpha) + \frac{\chi\,(1-\chi)\,\rho_g\,(\rho_g - \rho_l)}{\left(\chi\rho_l + (1-\chi)\,\rho_g\right)^2} \\
&= (1-\alpha) + \frac{(1-\alpha)\,\chi\,(\rho_g - \rho_l)}{\chi\rho_l + (1-\chi)\,\rho_g} \\
&= (1-\alpha)\,\frac{\chi\rho_l + (1-\chi)\,\rho_g + \chi\rho_g - \chi\rho_l}{\chi\rho_l + (1-\chi)\,\rho_g} \\
&\quad (1-\alpha)\,\frac{\rho_g}{\chi\rho_l + (1-\chi)\,\rho_g}
\end{aligned} \tag{676}$$

Grouping terms for $\frac{\partial \rho_l}{\partial P}$ coefficient:

$$\text{Coeff.}\rho_l = \frac{(1-\alpha)\,\rho_m}{\rho_l} \tag{677}$$

Adding all terms:

$$c_m^{-2} = \frac{\alpha\rho_m}{\rho_g}\frac{\partial \rho_g}{\partial P} + \frac{(1-\alpha)\,\rho_m}{\rho_l}\frac{\partial \rho_l}{\partial P} + \frac{\rho_m^2\,(\rho_g - \rho_l)}{\rho_l\rho_g}\frac{\partial \chi}{\partial P} \tag{678}$$

hence

$$\begin{aligned}
c_m &= \left(\frac{\alpha\rho_m}{\rho_g}\left(\frac{\partial \rho_g}{\partial P}\right)_S + \frac{(1-\alpha)\,\rho_m}{\rho_l}\left(\frac{\partial \rho_l}{\partial P}\right)_S + \right. \\
&\quad \left. \frac{\rho_m^2\,(\rho_g - \rho_l)}{\rho_l\rho_g}\left(\frac{\partial \chi}{\partial P}\right)_S\right)^{-\frac{1}{2}} \\
&= \left(\frac{\alpha\rho_m}{\rho_g c_g^2} + \frac{(1-\alpha)\,\rho_m}{\rho_l c_l^2} + \frac{\rho_m^2\,(\rho_g - \rho_l)}{\rho_l\rho_g}\left(\frac{\partial \chi}{\partial P}\right)_S\right)^{-\frac{1}{2}}
\end{aligned} \tag{679}$$

B.13.4 Relaxation Models

Using Maxwell relations:

$$\begin{aligned}
\zeta &= \left(\frac{\partial T}{\partial P}\right)_S \\
&= \left(\frac{\partial T}{\partial \rho}\right)_S \left(\frac{\partial \rho}{\partial P}\right)_S \\
&= \frac{1}{c^2}\left(\frac{\partial T}{\partial \rho}\right)_S \\
&= -\frac{1}{\rho^2 c^2}\left(\frac{\partial T}{\partial v}\right)_S \\
&= \frac{1}{\rho^2 c^2}\left(\frac{\partial P}{\partial S}\right)_v \\
&= \frac{1}{\rho^2 c^2}\left(\frac{\partial P}{\partial T}\right)_v \left(\frac{\partial T}{\partial S}\right)_v
\end{aligned} \tag{680}$$

Since

$$\begin{aligned}
G &= \frac{1}{\rho}\left(\frac{\partial P}{\partial E}\right)_v \\
&= \frac{1}{\rho}\left(\frac{\partial P}{\partial T}\right)_v \left(\frac{\partial T}{\partial E}\right)_v \\
&= \frac{T}{\rho}\left(\frac{\partial P}{\partial T}\right)_v \left(\frac{\partial T}{\partial S}\right)_v
\end{aligned} \tag{681}$$

$$\begin{aligned}
\zeta &= \frac{1}{\rho^2 c^2}\left(\frac{\partial P}{\partial T}\right)_v \left(\frac{\partial T}{\partial S}\right)_v \\
&= \frac{GT}{\rho c^2}
\end{aligned} \tag{682}$$

References

[1] A. H. Shapiro, "The dynamics and thermodynamics of compressible fluid flow," *New York: Ronald Press*, 1953.

[2] H. Tennekes and J. L. Lumley, *A First Course in Turbulence*. The MIT Press, 1972.

[3] H. Schlichtiing and K. Gersten, *Boundary Layer Theory*. 1992, vol. 42, 307–311.

[4] J. Blazek, *Computational Fluid Dynamics: Principles and Applications: Third Edition*. 2015, 1–447.

[5] K. Masatsuka, "I do like CFD," *Cradle*, vol. 1, 2013.

[6] E. M. Greitzer, C. S. Tan, and M. B. Graf, *Internal Flow Concepts and Applications*. 2004.

[7] F. M. White, *Viscous Fluid Flow 3rd Ed.* McGraw-Hill, 2006.

[8] J. Anderson, *Modern Compressible Flow*. 1990.

[9] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing*.

[10] A. Guardone, P. Colonna, M. Pini, and A. Spinelli, "Nonideal compressible fluid dynamics of dense vapours and supercritical fluids," *Annual Review of Fluid Mechanics*, vol. AA, pp. 1–64, 2023.

[11] J. D. van der Waals, "An equation of state for gases and liquids," *Nobel lectures in Physics*, vol. 1, pp. 254–265, 1910.

[12] O. Redlich and J. N. Kwong, "On the thermodynamics of solutions. v. an equation of state. fugacities of gaseous solutions.," *Chemical reviews*, vol. 44, no. 1, pp. 233–244, 1949.

[13] D.-y. Peng and D. B. Robinson, "A new two-constant equation of state a new two-constant equation of state," *Ind. Eng. Chem. Fundamen.*, vol. 15, no. 1, pp. 59–64, 1976.

[14] H. K. Onnes, "Expression of the equation of state of gases and liquids by means of series."," in *Through Measurement to Knowledge*, Springer, 1991, pp. 146–163.

[15] R. Span and W. Wagner, "A new equation of state for carbon dioxide covering the fluid region from the triple-point temperature to 1100 k at pressures up to 800 mpa," *Journal of Physical and Chemical Reference Data*, vol. 25, no. 6, pp. 1509–1596, 1996.

[16] J. Wang, T. Cao, and R. Martinez-Botas, "A path-specific isentropic exponent for non-ideal compressible fluids," *Physics of Fluids*, vol. 36, no. 10, 2024.

[17] H. L. Callendar, "On the thermodynamical properties of gases and vapours as deduced from a modified form of the joule-thomson equation, with special reference to the properties of steam," *Proceedings of the Royal Society of London*, vol. 67, no. 435-441, pp. 266–286, 1901.

[18] J. W. Strutt, "Aerial plane waves of finite amplitude," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 84, no. 570, pp. 247–284, 1910.

[19] W. J. Walker, "Cxxxiv. specific heat variations in relation to the dynamic action of gases and their equations of state," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 50, no. 300, pp. 1244–1260, 1925.

[20] D. B. Spalding, "A single formula for the "law of the wall"," *Journal of Applied Mechanics, Transactions ASME*, vol. 28, no. 3, pp. 455–458, 1960.

[21] E. R. Van Driest, "Turbulent boundary layer in compressible fluids," *Journal of the Aeronautical Sciences*, vol. 18, no. 3, pp. 145–160, 1951.

[22] A. Trettel and J. Larsson, "Mean velocity scaling for compressible wall turbulence with heat transfer," *Physics of Fluids*, vol. 28, no. 2, 2016.

[23] M. V. Morkovin, "Transition in open flow systems-a reassessment," *Bull. Am. Phys. Soc.*, vol. 39, p. 1882, 1994.

[24] H. Herwig and F. Kock, "Direct and indirect methods of calculating entropy generation rates in turbulent convective heat transfer problems," *Heat and Mass Transfer/Waerme- und Stoffuebertragung*, vol. 43, no. 3, pp. 207–215, 2007.

[25] J. Anderson, *Computational Fluid Dynamics the Basics With Applications*. 1995.

[26] ANSYS Inc, "ANSYS CFX-Solver Theory Guide 2020r1," January 2020.

[27] S. Arabi, J. Y. Trépanier, and R. Camarero, "A simple extension of Roe's scheme for real gases," *Journal of Computational Physics*, vol. 329, pp. 16–28, 2017.

[28] N. D. Baltadjiev, "An Investigation of Real Gas Effects in Supercritical CO2 Compressors," Massachusetts Institute of Technology, 2012.

[29] T. Chen, "A Study of the Internal Flow of Real Gases used in Organic Rankine Cycle (ORC) Turbines," 2022.

[30] P. Cinnella, "Roe-type schemes for dense gas flow computations," *Computers and Fluids*, vol. 35, no. 10, pp. 1264–1281, 2006.

[31] J. D. Denton, "The 1993 IGTI Scholar Lecture: Loss Mechanisms in Turbomachines," *Journal of Turbomachinery*, vol. 115, no. 4, pp. 621–656, Oct. 1, 1993.

[32] G. Gori, O. L. Maitre, and P. Congedo, "A Review of some recent advancements in Non-Ideal Compressible Fluid Dynamics," *UQOP 2020 - International Conference on Uncertainty Quantification & Optimisation*, 2020.

[33] M. A. Grolmes and H. K. Fauske, "Propagation characteristics of compression and rarefaction pressure pulses in one-component vapor-liquid mixtures," *Nuclear Engineering and Design*, vol. 11, no. 1, pp. 137–142, 1970.

[34] K.-H. Grote, B. Bender, and D. Göhlich, *DUBBEL: Taschenbuch Für Den Maschinenbau*.

[35] D. Japikse, "Turbomachinery performance modeling," *SAE Technical Papers*, 2009.

[36] J. C. Leung and M. Epstein, "Non-ideal isentropic gas flow through converging-diverging nozzles," *Journal of Fluids Engineering, Transactions of the ASME*, vol. 113, no. 2, p. 311, 1991.

[37] H. Lund, "A hierarchy of relaxation models for two-phase flow," *SIAM Journal on Applied Mathematics*, vol. 72, no. 6, pp. 1713–1741, Nov. 13, 2012.

[38] P. Nederstigt, "Real gas thermodynamics and the isentropic behavior of substances," Delft University of Technology, 2017.

[39] P. L. Roe, "Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes," *Journal of Computational Physics*, vol. 43, pp. 357–372, 1981.

[40] P. A. Thompson, *Compressible fluid dynamics*.

[41] S. B. Pope, *Turbulent Flows*. Nov. 1, 2000.

[42] D. A. Sullivan, "Historical Review of Real-Fluid Isentropic Flow Models.," *American Society of Mechanical Engineers (Paper)*, vol. 103, 79 -WA/FM-1 1979.

[43] A. B. Wood and R. B. Lindsay, "A Textbook of Sound," *Physics Today*, vol. 9, no. 11, pp. 37–37, Nov. 1, 1956.

[44] J. Wang, T. Cao, and R. Martinez-Botas, "Supercritical carbon dioxide shock behavior near the critical point," *Journal of Engineering for Gas Turbines and Power*, vol. 146, no. 1, 2024.

[45] J. Wang, T. Cao, and R. Martinez-Botas, "Supercritical carbon dioxide mixing loss characteristics near the critical point," *Journal of Engineering for Gas Turbines and Power*, vol. 147, no. 5, 2025.

[46] J. Wang, B. Yang, R. Martinez-Botas, and T. Cao, "Direct numerical simulation of the supersonic turbulent boundary layer of supercritical carbon dioxide," *Physics of Fluids*, vol. 37, no. 2, 2025.