

**INVESTIGATIONS ON VISCOUS DAMPING
OF VIBRATING SYSTEMS.**

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ABSTRACT

Investigations have been made on the damping of various forms of vibrating systems, particular attention being paid to the effect of and the measurement of fluid viscosity. Two distinct experimental techniques have been developed for these investigations. The first method was used to study the damping of a freely vibrating strip or rod, partially or wholly immersed in a fluid medium, the damping being measured by a counting technique where the number of cycles of vibration for the amplitude to decay from one value to half this value were counted electronically. The second method was used to investigate the damping of a longitudinal travelling wave propagated along a wire, the far-end section of which was immersed in a liquid medium, and to measure either the reflection coefficient of the gas-liquid boundary or the attenuation of the wave within the loaded medium. From both types of measurement, under favourable conditions, the viscosities of the liquid medium can be determined.

By suitable choice of the vibrating system, a considerable range of fluid viscosities can be covered. For example, the viscosity of a gas was determined by using a thin-walled tube as the vibrating system and applying the counting method to

measure the damping, while for liquids of very high viscosity the propagation method was found to be more appropriate.

The frequency of the vibration used in these methods was restricted experimentally to the 1 - 100 Kc/s range. At these frequencies the penetration of the viscous wave in the fluid medium is very small (for example, for a fluid of viscosity 1 poise and density 1 gm/cc., the skin depth at 10 Kc/s is 5.6×10^{-3} cm.), and does not permit the direct investigation of the wave system.

The effect on the damping of the thickness of thin film has, however, been investigated, and the experimental results are in accordance with theoretical predictions.

C O N T E N T S

	Page No.
<u>I. INTRODUCTION</u>	1
1. Free vibrating systems	2
2. Forced vibrating systems	7
3. Travelling wave systems	7
<u>II. BASIC THEORY</u>	10
1. Infinite visco-elastic medium	13
2. Bounded visco-elastic medium	14
3. Cylindrical waves	19
<u>III. EXPERIMENTAL SYSTEM INVOLVING THE ELECTRONIC</u>	
<u>COUNTING TECHNIQUE</u>	25
1. Operation of the electronic circuit	25
2. Discussion	29
<u>IV. TORSIONALLY VIBRATING RODS</u>	31
1. Theory	34
2. Experimental systems for measuring viscosity	
(a) Beryllium copper rod	38
(b) Quartz rod	47
(c) Double clamped beryllium copper rod	54
<u>V. VARIATION OF DAMPING WITH THICKNESS OF A LIQUID</u>	
<u>FILM</u>	59
1. Liquid film with free surface	
(a) Low frequency system	59
(b) High frequency system	63
2. Liquid film bounded by solids	72

<u>VI. GAS DAMPING OF A ROD IN LONGITUDINAL</u>	
<u>OSCILLATION</u>	75
1. Theory	75
2. Experimental	84
3. Results	94
4. Discussion of results	98
<u>VII. TRAVELLING WAVE SYSTEM</u>	99
1. Theory	99
2. Experimental	106
3. Results	115
4. Discussion of results	119
<u>APPENDIX</u>	121
<u>BIBLIOGRAPHY</u>	130
<u>ACKNOWLEDGMENTS</u>	132

I. INTRODUCTION

Measurements of the external damping of a vibrating system immersed in a fluid will give information about the visco-elastic properties of the fluid. In order to measure the external damping of the system the total damping is evaluated and the internal damping, determined separately, is subtracted.

Several methods have been described for measuring the damping of a vibrating system due to a surrounding visco-elastic medium but in general the methods available can be subdivided into three classes: (1) those which use the free vibrations of a specimen and in which the amplitude time decay is measured; (2) those which employ a specimen in forced vibration; and (3) those which measure the attenuation and phase of a travelling wave.

The principal apparatus to be described, employing a free vibrating specimen, will fall into class (1). A similar method has been described by Roth & Rich (1953), and this will therefore be considered at length. Also included in this class is the method employing a low frequency torsional pendulum. An application of this method has been used by the author for the study of the damping of a liquid film.

The methods of class (2) have not been applied in this research and will consequently only be mentioned briefly.

The travelling wave method has been used by McSkimin (1952) and Carruthers (1959). Both used pulse techniques to measure the attenuation and phase of a wave propagated along a rod immersed in a fluid medium. In contrast, the author has used a continuous wave technique.

1. 1. Free Vibrating Systems.

In the method used by Roth & Rich (1953) a metal strip is excited at its fundamental longitudinal frequency while immersed in a fluid. To obtain the wave equation for the free vibrations of such a system it is convenient to use the concept of 'driving point impedance' (Z_0). This quantity is defined as the ratio of the shear stress (Y_x) to particle velocity ($\partial \xi / \partial t$) in the medium at the surface of the vibrating strip and hence can be evaluated in terms of the viscosity and elasticity of the medium and the boundary conditions imposed on it.

The wave equation thus obtained is

$$\frac{\partial^2 \xi}{\partial t^2} + 2K_1 \frac{\partial \xi}{\partial t} = c_m^2 \frac{\partial^2 \xi}{\partial x^2}$$

where C_m is the wave velocity in the strip and K_1 is the damping term, which Roth & Rich have shown is proportional to Z_0 , which is in general complex. K_1 however, should also include a term to account for internal energy loss in the vibrating strip but Roth & Rich are able to ignore this as being negligibly small. Since only one half of the strip was immersed in the liquid, the magnitude of the damping term was, for small damping, approximately one half of that for full immersion.

By contrast, in the experiments of the author the internal damping of the vibrating specimens was not negligibly small and varying lengths of the rod were immersed in the fluid medium. In consequence of the changed experimental conditions, the analytical procedure adopted in the present experiments was to calculate the energy lost by the specimen per cycle of vibration (ΔW) and also the total energy of vibration (W) and thus determine the logarithmic decrement

$$\delta = \frac{\Delta W}{2W} \quad (\text{for small values of } \delta).$$

To calculate the energy lost by the vibrating specimen to the liquid, we note that $Y_L = \frac{\partial \xi}{\partial t} Z_0 = \frac{\partial \xi}{\partial t} (R_0 + iX_0)$ where R_0 is the resistive component and X_0 a term which merely alters the effective mass of the

vibrating system and therefore the frequency of vibration.

In actual practice the frequency change is negligibly small and can be ignored.

It is thus possible to relate the damping of a vibrating system, which is due to the surrounding fluid medium to R_0 , and this can be expressed in terms of the visco-elastic properties of the fluid.

In the 'ultra-viscoson', the apparatus which is due to Roth & Rich, the strip is excited magneto-strictively and when its amplitude of vibration reaches a certain predetermined level the 'drive' is switched off and the amplitude decays to another predetermined value. On reaching this value the 'drive' is switched on and the greater the fluid damping the quicker the rate of switching 'on and off'. An analogue computer converts this rate into viscosity. It is essential for the apparatus to be calibrated against standard liquids.

In the author's apparatus the logarithmic decrement of the vibrating system is determined by counting the number of cycles of vibration (N) which occur while the amplitude decays from one value to half this value. Thus a direct determination of R_0 is obtained. By using different standard viscous liquids the theory can be verified.

The 'ultra-viscoson' uses a very thin vibrating strip and the corresponding damping due to the surrounding liquid will generally be much greater than the internal damping of the strip itself. This internal damping is constant at a given temperature and will therefore not affect the accuracy of this calibrated apparatus. There is also damping due to radiation resistance which is dependent on the dimensions of the vibrating strip and also ρc (where ρ is the density of the fluid and c the velocity of compressional waves in the medium). This can be of the same order as the viscous damping and depending as it does on the properties of the liquid will cause errors in the calibration. The author's apparatus uses a torsionally vibrating rod as the vibrating system for liquids to eliminate all possible radiation damping. For gases the radiation and viscous damping can be separated by a method involving the variation of the gas pressure and therefore in this case longitudinal vibrations were used.

A torsional pendulum has been used by many authors to measure the viscosity of a fluid. Lamb (Ref.) gives a solution for an oscillating lamina in a viscous fluid. Others, such as Hopkins & Toye (1950) and Roscoe & Bainbridge

(1958), have used a hollow cylindrical vessel containing the fluid for the precise measurement of viscosity. More recently Ashwin, Hagyard, Saunders & Young (1960) have adopted a dimensional analysis to solve the case of an oscillating cylinder immersed in a very viscous liquid.

In the research mentioned above the fluid is in general considered to extend to infinity. This is justified when the fluid container is relatively small owing to the rapid attenuation of the viscous waves. For example, Ashwin, Hagyard, Saunders & Young say that "Depth of immersion has little effect", and take as an example a particular fluid and bob which "could be used as little as 1 mm. below the liquid surface with little effect on the damping."

The author's interest in the torsional pendulum has, however, been confined to the study of the damping of a liquid film with its thickness. The general theory derived in section II.3(a)(i) is applicable to this particular case.

I. 2. Forced vibrating systems.

The methods which employ a force vibrated specimen generally measure the electrical impedance of the driving system and observe the change in this when the specimen is surrounded by a fluid. The impedance is determined at resonance of the specimen. Mason (1947) uses an ADP crystal vibrating torsionally for his system, but unfortunately this has a coupled 'ring type' vibration which generates compressional waves in the liquid and therefore suffers from the same disadvantages as the 'ultra-viscoson'. The Woodward (1953) vibrating plate viscometer has a plate which vibrates in its own plane and is immersed in the fluid. The edge of the plate, however, will generate compressional waves and is therefore not satisfactory. This apparatus, like the ultra-viscoson, had to be calibrated.

I. 3. Travelling wave systems.

In the pulse techniques used by McSkimin (1952) and Carruthers (1959), measurement was made of the attenuation and phase of the wave propagated along a rod immersed in the fluid medium. In the author's apparatus two similar

parameters were measured using a continuous wave technique. For a purely viscous medium the viscosity could be calculated from either measurement. For a visco-elastic medium, however, the two measurements should enable the evaluation of both viscosity and rigidity.

The theory needed for the calculation of the visco-elastic properties of the fluid is simplified if it can be assumed that plane waves rather than cylindrical waves are propagated into the fluid medium. McShimin (1952) assumes the waves are plane and then applies a correction to convert the plane wave characteristic impedance to the cylindrical characteristic impedance. This, however, employs a method of successive approximations and cannot be used to obtain a general solution for cylindrical waves.

In the method used by the author, a large error will result in the calculations if the problem is considered in terms of plane waves and as a general solution is necessary for the evaluation of viscosity from the measured parameters, a theory has been derived for cylindrical waves. This is, however, restricted to a viscous medium only.

The apparatus used by the author consists of a long thin nickel wire held vertically and excited into longitudinal

vibration by a transducer at the upper end. A length of the wire at the bottom is surrounded by the fluid medium. The longitudinal waves travelling down the wire are partially reflected at the boundary between the 'free' and 'loaded' sections of the wire, causing standing waves in the upper section. In the 'loaded' section of the wire the wave is exponentially attenuated. From the measurement of the standing wave ratio or the attenuation the viscosity can be calculated by means of the theory mentioned above.

II. BASIC THEORY

In order to obtain an equation for the stress in the fluid in terms of its rheological/visco-elastic properties it is necessary either to use a model to explain the mechanism of the behaviour of the fluid, or to adopt the approach of Roth & Rich. The latter method is easier and allows a general solution for the driving point impedance in terms of frequency dependent viscosity and rigidity coefficients. These coefficients can be interpreted in terms of rheological properties.

For the purpose of the present work the method followed by Roth & Rich has been adopted and in accordance with their point of view all strains in the fluid in phase with the stress are considered to result from a single rigidity coefficient function (g) and those 90° out of phase as resulting from a viscosity coefficient function (η). Because all steady state alternating strains can be resolved into these two components, this approach does not reduce the generality of the analysis.

Thus the shear stress γ_x produced in a visco-elastic fluid medium by a transverse elastic wave of angular frequency ω travelling in the y direction is related to the shear stress by the equation

$$\gamma_x = g \frac{\partial \xi}{\partial y} + \eta \frac{\partial^2 \xi}{\partial t \partial y}$$

where ξ is the particle displacement in the x direction and g and η are frequency-dependent rigidity and viscosity coefficient functions:

but

$$\frac{\partial \gamma_x}{\partial y} = \rho \frac{\partial^2 \xi}{\partial t^2}$$

where ρ is the density of the fluid medium.

The wave equation for shear waves propagated in this visco-elastic medium thus obtained is

$$\rho \frac{\partial^2 \xi}{\partial t^2} = g \frac{\partial^2 \xi}{\partial y^2} + \eta \frac{\partial^3 \xi}{\partial t \partial y^2}$$

Assuming the solution at a particular frequency to be

$$\xi = \xi_0 e^{\Gamma y} e^{i\omega t}$$

it is found that
$$\Gamma = \pm \left[\frac{i\omega\rho}{g + i\omega\eta} \right]^{1/2}$$

and
$$\xi = [Ae^{-\Gamma y} + Be^{\Gamma y}] e^{i\omega t}$$

where A and B respectively are the amplitude of wave travelling in the positive and negative y directions.

The characteristic acoustic wave impedance Z_c can now be determined. This is the impedance offered by an infinite layer of visco-elastic medium to a shear wave travelling in one direction.

$$\xi = \xi_0 e^{\Gamma y} e^{i\omega t}$$

$$\therefore \gamma_x = (g + i\omega\eta)\Gamma$$

$$\therefore Z_c = \frac{-\gamma_x}{\partial \xi / \partial t} = \frac{1}{i} [g + i\omega\eta]^{1/2}$$

If the medium is bounded at a distance l and terminated with an impedance Z_R , then the driving point impedance of the medium Z_0 is

$$Z_0 = Z_c \left[\frac{1 + Z_c/Z_R \tanh \Gamma l}{Z_c/Z_R + \tanh \Gamma l} \right]$$

Many special cases of this equation have been considered by Roth & Rich.

II. 1. Infinite visco-elastic medium.

In the case of an infinite viscous medium the plane shear wave travels away from the vibrating surface and

$$Z_o = Z_c = (i\omega\rho\eta)^{\frac{1}{2}} = \left(\frac{\omega\rho\eta}{j_2t}\right)^{\frac{1}{2}}(1 + j)$$

The distance travelled by the wave for the amplitude to decrease to $1/e$ of its original value, called the skin depth, is $\left(\frac{2\eta}{\omega\rho}\right)^{1/2}$

If the medium is visco-elastic, the impedance depends on the relative value of $\omega\eta$ and g . Writing $g/\omega\eta = n$ then

$$Z_o = \left(\frac{\omega\rho\eta}{j_2t}\right)^{\frac{1}{2}} \left[\{ (n^2 + 1)^{1/2} + n \}^{\frac{1}{2}} + i \{ (n^2 + 1)^{1/2} - n \}^{\frac{1}{2}} \right]$$

For n small

$$Z_o \approx \left(\frac{\omega\rho\eta}{j_2t}\right)^{\frac{1}{2}} \left[(1 + n/2) + i(1 - n/2) \right]$$

II. 2. Bounded visco-elastic medium.

(a) Free boundary. Here $Z_R = 0$ and hence $Z_o = Z_c$

$\tanh \Gamma l$. This expression can be expanded in terms of $\omega \rho \eta$ and g .

(i) Viscous medium, i.e. $g = 0$. Roth & Rich have obtained the following expression:-

$$Z_o = (R_o + iX_o)$$

$$= \frac{1}{2} \left(\frac{\omega \rho \eta}{2} \right)^{\frac{1}{2}} \left[\frac{\sinh 2\Theta - \sin 2\Theta}{\cos^2 \Theta + \sinh^2 \Theta} + i \frac{\sinh 2\Theta + \sin 2\Theta}{\cos^2 \Theta + \sinh^2 \Theta} \right] \quad \text{--- (1)}$$

where $\Theta = \left(\frac{\omega \rho \eta}{2} \right)^{\frac{1}{2}} l$

The variation of damping with thickness for a viscous medium is given in Fig.(1) where $R_o / \left(\frac{\omega \rho \eta}{2} \right)^{\frac{1}{2}}$ is plotted against Θ .

(ii) Visco-elastic medium. The solution for this case is more complex and Roth & Rich have only evaluated Z_o for a particular value of $\omega \eta$ and l .

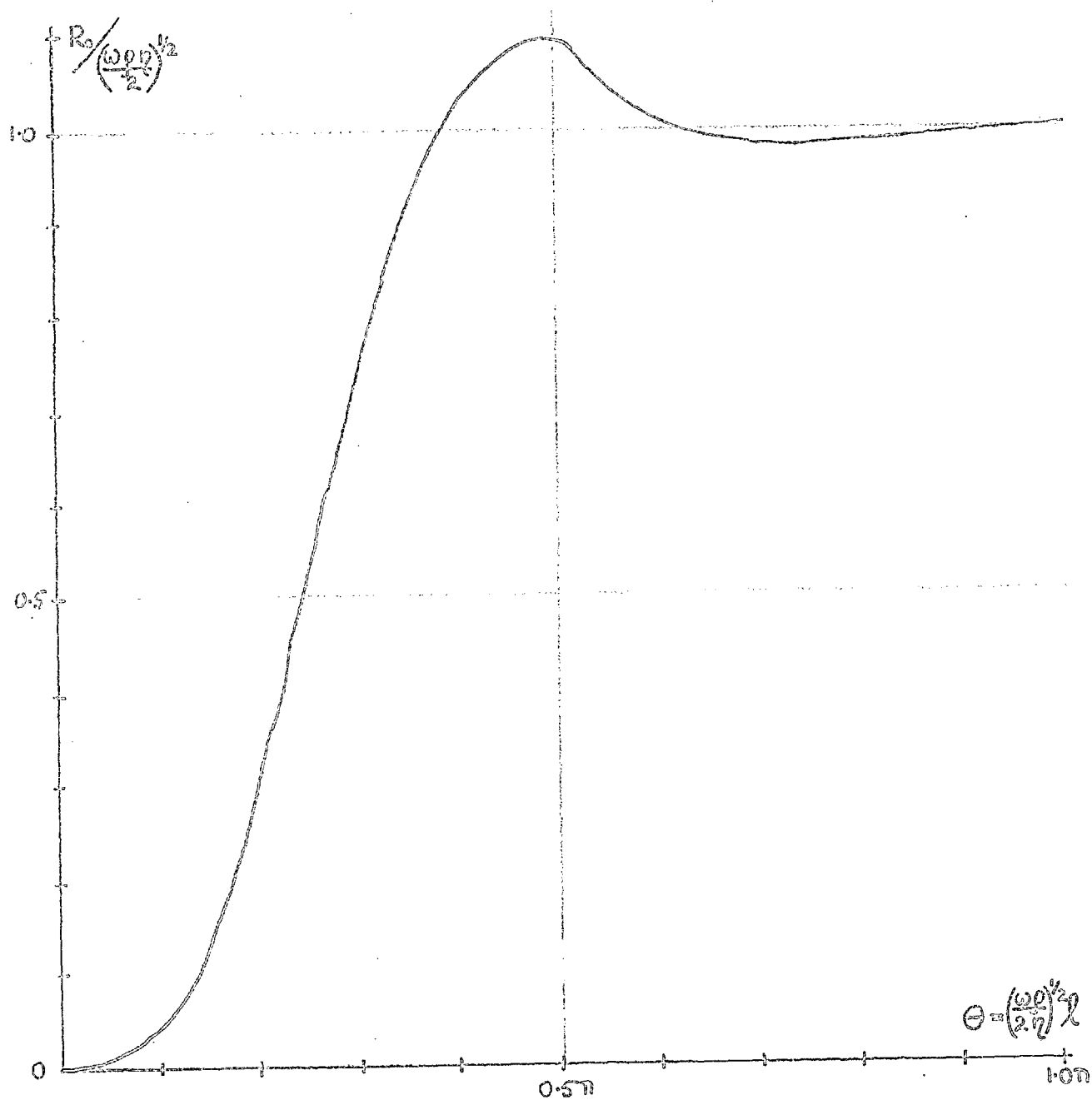


Fig.(1) Variation of Damping with thickness
of bounded Viscous Layer ($Z_R = 0$)

(b) Solid boundary: Z_c will be $\ll Z_R$ and hence
 $Z_o = Z_c / \tanh \Gamma l$.

By expanding this it can be shown

$$Z_o = \left(\frac{\omega p \eta}{2}\right)^{\frac{1}{2}} \left[(p^{-\frac{1}{2}} C - p^{\frac{1}{2}} D) + i(p^{\frac{1}{2}} C + p^{-\frac{1}{2}} D) \right]$$

where $p = (n^2 + 1)^{\frac{1}{2}} - n$

$$n = g/\omega \eta$$

$$C = \frac{\tanh X}{\cos^2 Y \tanh^2 X + \sin^2 Y}$$

$$D = \frac{-\tan Y}{\sinh^2 X + \cosh^2 X \tan^2 Y}$$

$$X = \Theta p^{\frac{1}{2}} (n^2 + 1)^{-\frac{1}{2}} ; Y = \Theta p^{-\frac{1}{2}} (n^2 + 1)^{-\frac{1}{2}}$$

(1) Viscous medium, i.e. $g = 0, \therefore n = 0; p = 1; X = Y = \Theta$

and $Z_o = \left(\frac{\omega p \eta}{2}\right)^{\frac{1}{2}} [(C - D) + i(C + D)]$

which written in full becomes

$$\begin{aligned} Z_o &= (R_o + i X_o) \\ &= \left(\frac{\omega p \eta}{2}\right)^{\frac{1}{2}} 2 \left[\frac{\cos^2 \Theta + \sinh^2 \Theta}{\sinh^2 2\Theta + \sin^2 2\Theta} \right] \\ &\quad \times \left[(\sinh 2\Theta + \sin 2\Theta) + i(\sinh 2\Theta - \sin 2\Theta) \right] \end{aligned}$$

In Fig. (2) $R_0/(\omega \rho \eta)^{1/2}$ is again plotted against Θ to show the variation of damping with thickness. The smallest value of the damping occurs when $\ell = \lambda/4$ where λ is the wavelength of the shear waves.

(ii) Visco-elastic medium. Corresponding graphs to Fig. (2) could be plotted for various values of n but the calculation is tedious. A minimum would occur at $\ell = \lambda/4$ followed by a maximum at $\ell = \lambda/2$ and other maxima and minima. An expression can be derived relating the value of the first minimum $(R_0)_{\min.}$ and the first maximum $(R_0)_{\max.}$ to the value corresponding to an infinite thickness $(R_0)_{\infty}$. The wave length of shear waves in a visco-elastic medium is given by the equation

$$\lambda = 2\pi \left(\frac{2\eta}{\omega \rho} \right)^{1/2} (n^2 + 1)^{1/4} \sec \left(\frac{\tan^{-1} 1/n}{2} \right)$$

This can be simplified to give $\lambda = 2\pi \ell / \Theta (n^2 + 1)^{1/2} p^{1/2}$

and thus $X = \frac{2\pi \ell}{\lambda} p$

$$Y = \frac{2\pi \ell}{\lambda}$$

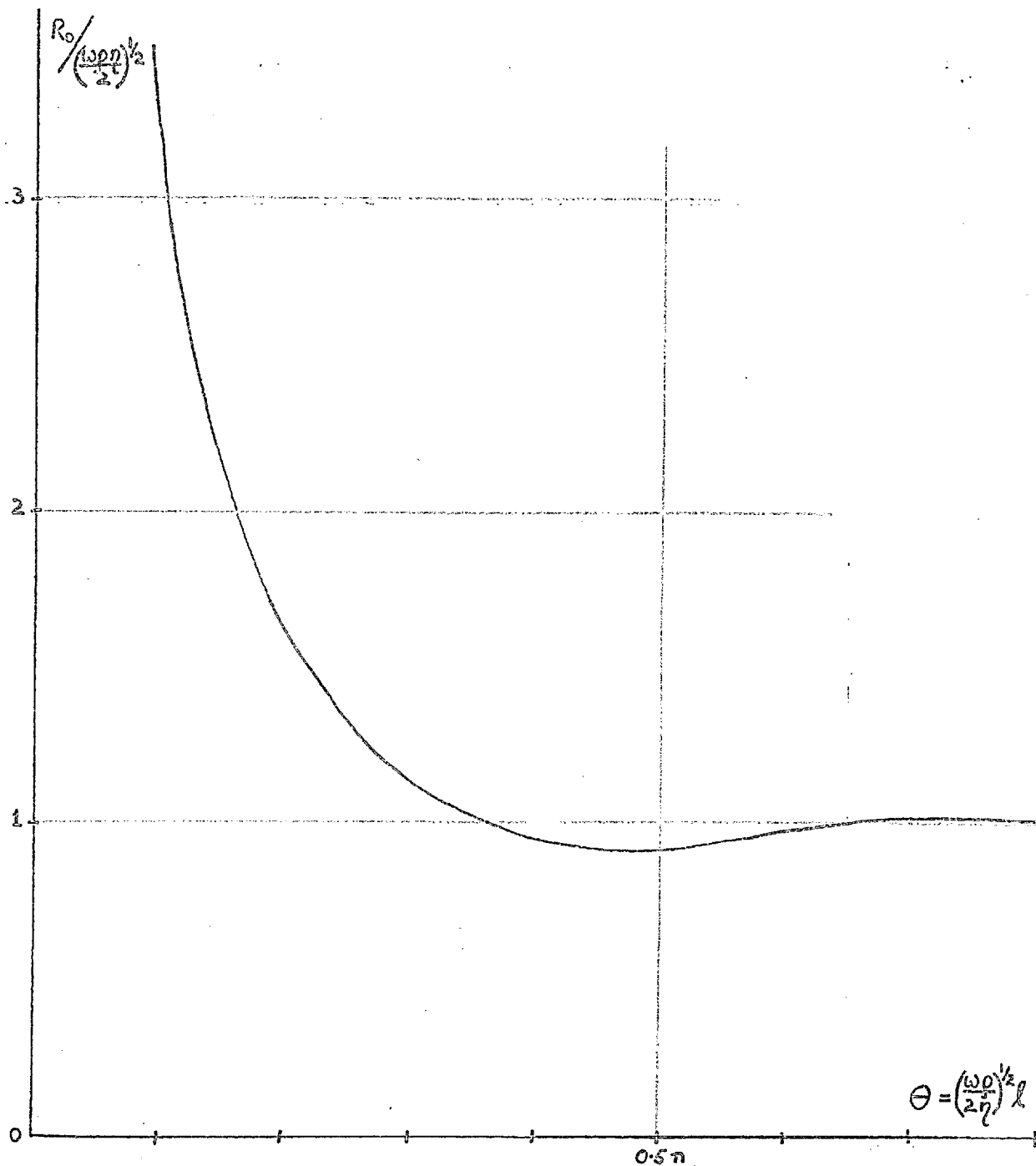


Fig.(2) Variation of Damping with thickness
of bounded Viscous Layer ($z_R \gg z_c$)

It follows that

$$\frac{(R_o)_{\min}}{(R_o)_{\infty}} = \frac{1}{\tanh \pi/2p}$$

$$\frac{(R_o)_{\max.}}{(R_o)_{\infty}} = \tanh \pi/2p$$

$$\text{where } p = (n^2 + 1)^{1/2} - n$$

$$n = g/\omega\eta$$

Curves of $\frac{(R_o)_{\max}}{(R_o)_{\infty}}$ and $\frac{(R_o)_{\min}}{(R_o)_{\infty}}$ against n are plotted in Fig.(3).

II. 3. Cylindrical waves.

In deriving the expressions for the driving point impedance it has been assumed that plane waves are propagated in the fluid medium. This cannot be considered true in the case of propagation along a wire where the dimensions of the vibrating system are small. In this case cylindrical waves are propagated into the medium and the corresponding wave equation will be

$$\rho \frac{\partial^2 \xi}{\partial t^2} = g \left(\frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial r} \right) + \eta \frac{\partial}{\partial t} \left(\frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial r} \right)$$

where r is radial.

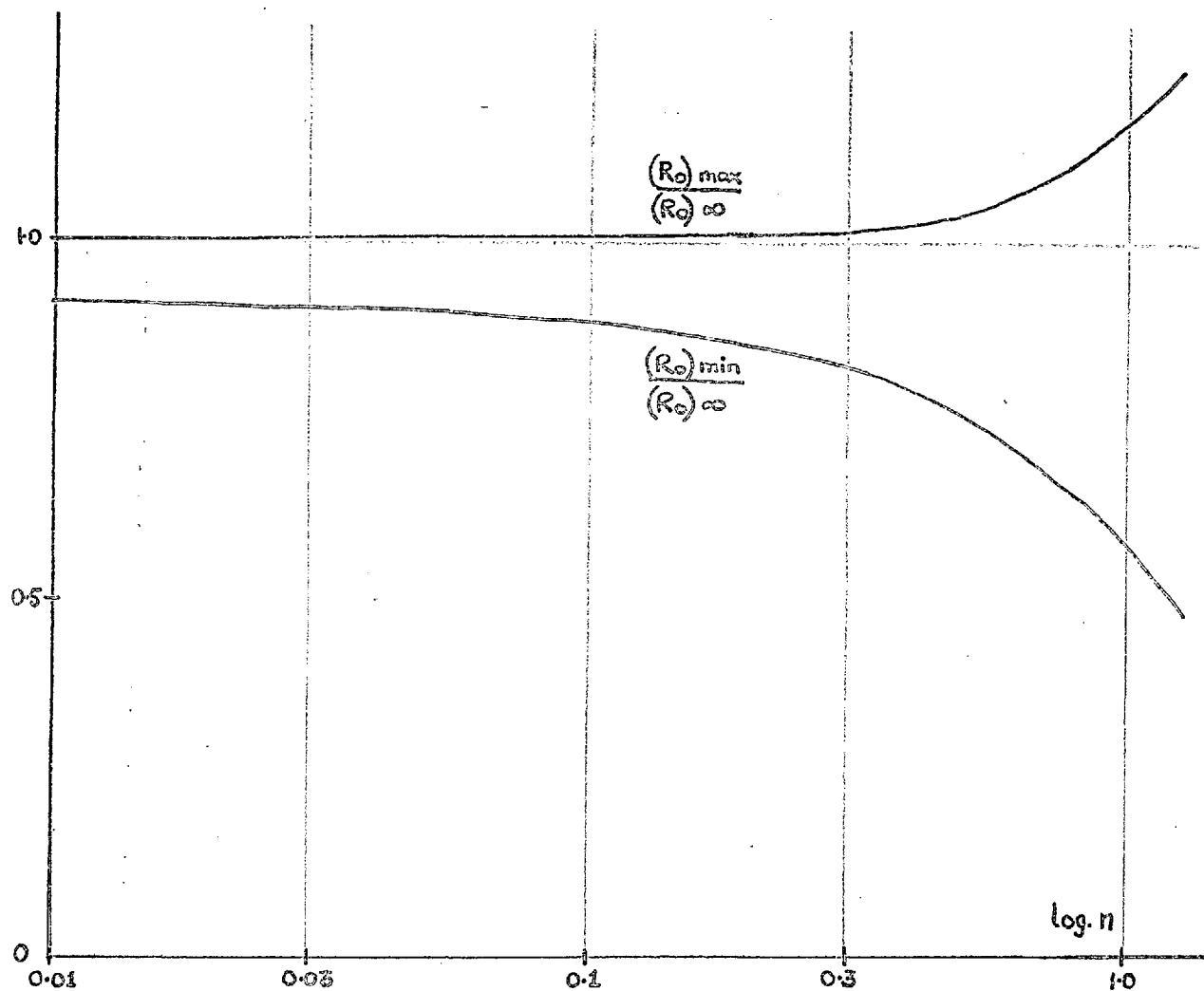


Fig.(3) Relative Damping Curves for Visco elastic Medium

Confining our solution to the case where $g = 0$ and assuming it to be of the form $\xi = U(r) e^{-i\omega t}$ the following equation in $U(r)$ is obtained:-

$$\frac{\partial^2 U(r)}{\partial r^2} + \frac{\partial U(r)}{r \partial r} + i \left(\frac{\omega p}{\eta} \right) U(r) = 0$$

Writing $k^2 = i \left(\frac{\omega p}{\eta} \right)$, i.e. $k = \pm \left(1 + i \left(\frac{\omega p}{2\eta} \right) \right)^{1/2}$

then $U(r) = A H_0^{(1)}(kr) + B H_0^{(2)}(kr)$

where $H_0^{(1)}$ and $H_0^{(2)}$ are Hankel functions of the first and second kind.

As $r \rightarrow \infty$, the boundary condition for an outgoing wave will be $U(r) \rightarrow 0$

and also $H_0^{(1)}(kr) \sim \left(\frac{2}{\pi kr} \right)^{1/2} e^{i(kr - \pi/4)}$

$$H_0^{(2)}(kr) \sim \left(\frac{2}{\pi kr} \right)^{1/2} e^{-i(kr - \pi/4)}$$

Selecting the +ve root for k , only $H_0^{(1)}$ will fulfil the boundary condition, therefore $B = 0$. For the -ve root, $H_0^{(2)}$ is applicable and $A = 0$. The solutions obtained from these two roots are, however, identical.

Using the +ve root and substituting for k in the limiting value of $H_0^{(1)}(kr)$ as $r \rightarrow \infty$

$$H_0^{(1)}(kr) \sim \left(\frac{2}{\pi(1+i)\left(\frac{\omega\rho}{2\eta}\right)^{1/2}r} \right)^{1/2} e^{-\left(\frac{\omega\rho}{2\eta}\right)^{1/2}r + i\left\{\left(\frac{\omega\rho}{2\eta}\right)^{1/2}r - \frac{\pi}{4}\right\}}$$

which is the correct form for an outgoing wave.

To calculate the characteristic acoustic impedance Z_0

$$\begin{aligned} Z_0 &= \frac{\eta \frac{\partial^2 \xi}{\partial t \partial r}}{\partial \xi / \partial t} \\ &= \frac{-i\omega\eta k H_1^{(1)}(kr) e^{-i\omega t}}{-i\omega H_0^{(1)}(kr) e^{-i\omega t}} \end{aligned}$$

To find the value for Z_0 at the surface of a cylindrical wire of radius a , let $m = a \left(\frac{\omega\rho}{\eta}\right)^{1/2}$ which is the ratio of the radius of the wire to the skin depth of the outgoing wave.

Thus

$$Z_0 = \frac{\omega\rho a}{m} \sqrt{i} \frac{H_1^{(1)}(\sqrt{i}m)}{H_0^{(1)}(\sqrt{i}m)}$$

To test this equation we can find the value of Z_0 as $a \rightarrow \infty$

which should give the plane wave value of $Z_0 = \left(\frac{\omega \mu \eta}{2}\right)^{1/2} (1 + i)$

$$\text{As } r \rightarrow \infty \quad H_1^{(0)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr - 3\pi/4)}$$

$$H_0^{(0)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr - \pi/4)}$$

$$\text{Thus} \quad \frac{H_1^{(0)}(kr)}{H_0^{(0)}(kr)} \rightarrow e^{i\pi/2} = -i$$

$$\begin{aligned} \text{and } Z_0 &\rightarrow -i(1+i)\left(\frac{\omega \mu \eta}{2}\right)^{1/2} \\ &= (1-i)\left(\frac{\omega \mu \eta}{2}\right)^{1/2} \end{aligned}$$

The difference in the sign of the imaginary part is simply due to considering an $e^{-i\omega t}$ time variation of ξ in the cylindrical wave instead of $e^{+i\omega t}$ which was used in the plane wave derivation.

The values of the Hankel functions $H_0^{(1)}(\sqrt{1}m)$ and $H_1^{(1)}(\sqrt{1}m)$

are tabulated in Jahnke & Emde. (1945)

$$\text{Let } H_0^{(1)}(\sqrt{1}m) = U_0 + iV_0$$

$$H_1^{(1)}(\sqrt{1}m) = \frac{U_1 + iV_1}{\sqrt{i}}$$

$$\text{Then } Z_0 = \frac{\omega \rho a}{m} \frac{(U_0 U_1 + V_0 V_1) + i(U_0 V_1 - U_1 V_0)}{U_0^2 + V_0^2}$$

III. EXPERIMENTAL SYSTEM INVOLVING THE ELECTRONIC COUNTING TECHNIQUE.

A modified version of the apparatus described by Pattison (1954) has been used to measure the damping of the various vibratory systems. This apparatus uses an electronic counting technique to measure the number of oscillations which a freely vibrating specimen executes when the amplitude decays from any one value to a half of this value. The modifications were designed to improve the accuracy of the method and make it work continuously.

III. 1. Operation of the electronic circuit.

The vibrating specimen is excited by closing switch S_1 in Fig.(4). The amplified signal from the pick-up coil is then fed to the 'driving-coil' and provided the loop gain is greater than one and there is a phase shift of approximately $\pi/2$ in the amplifier circuit, the specimen will start to vibrate at a harmonic frequency.

On opening switch S_1 , and thus disconnecting the driving force, the vibrations of the specimen will freely decay. The output signal from the pre-amplifier (b) in Fig.(5) will

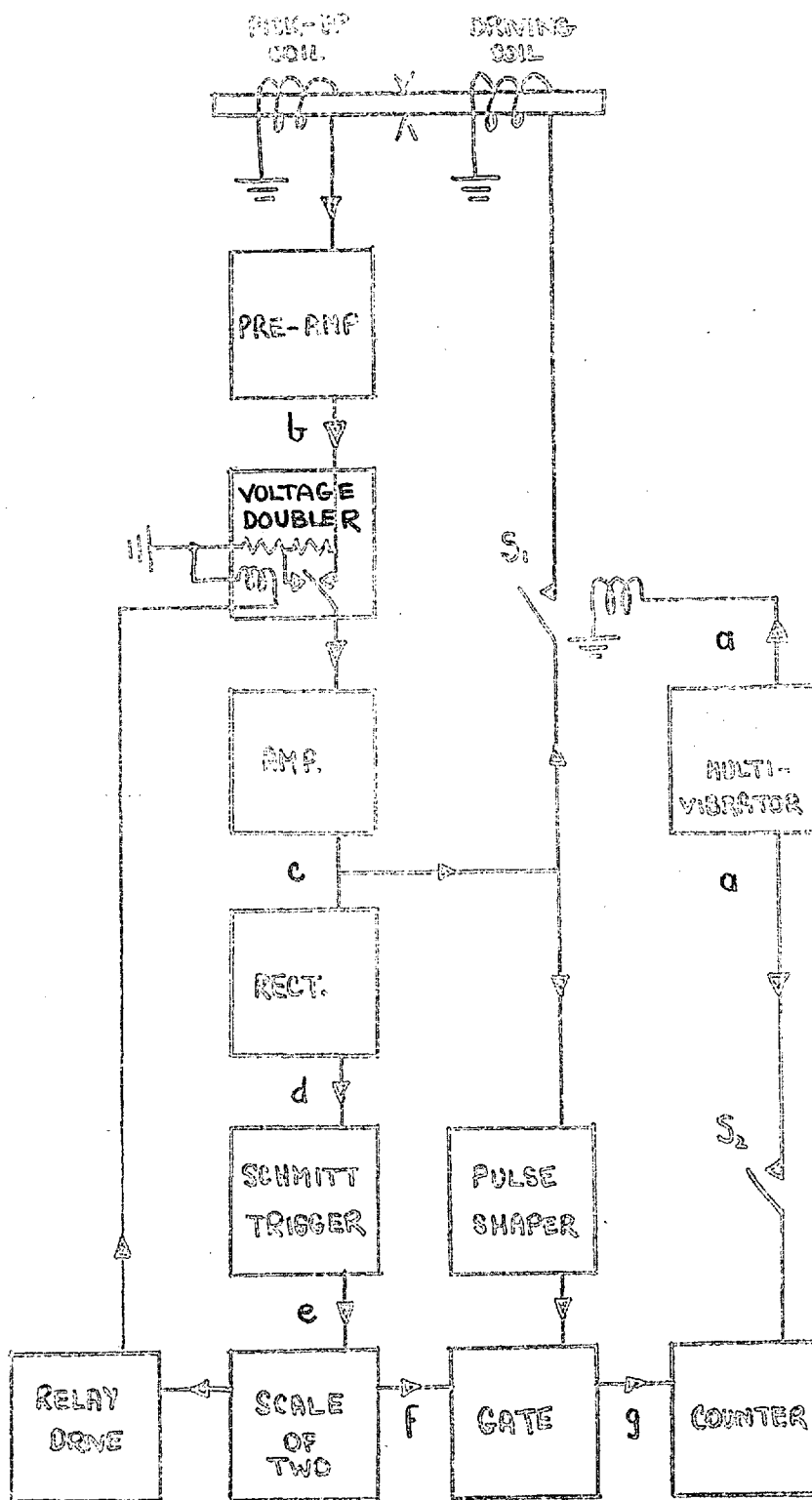


Fig.(4) Schematic Circuit Diagram

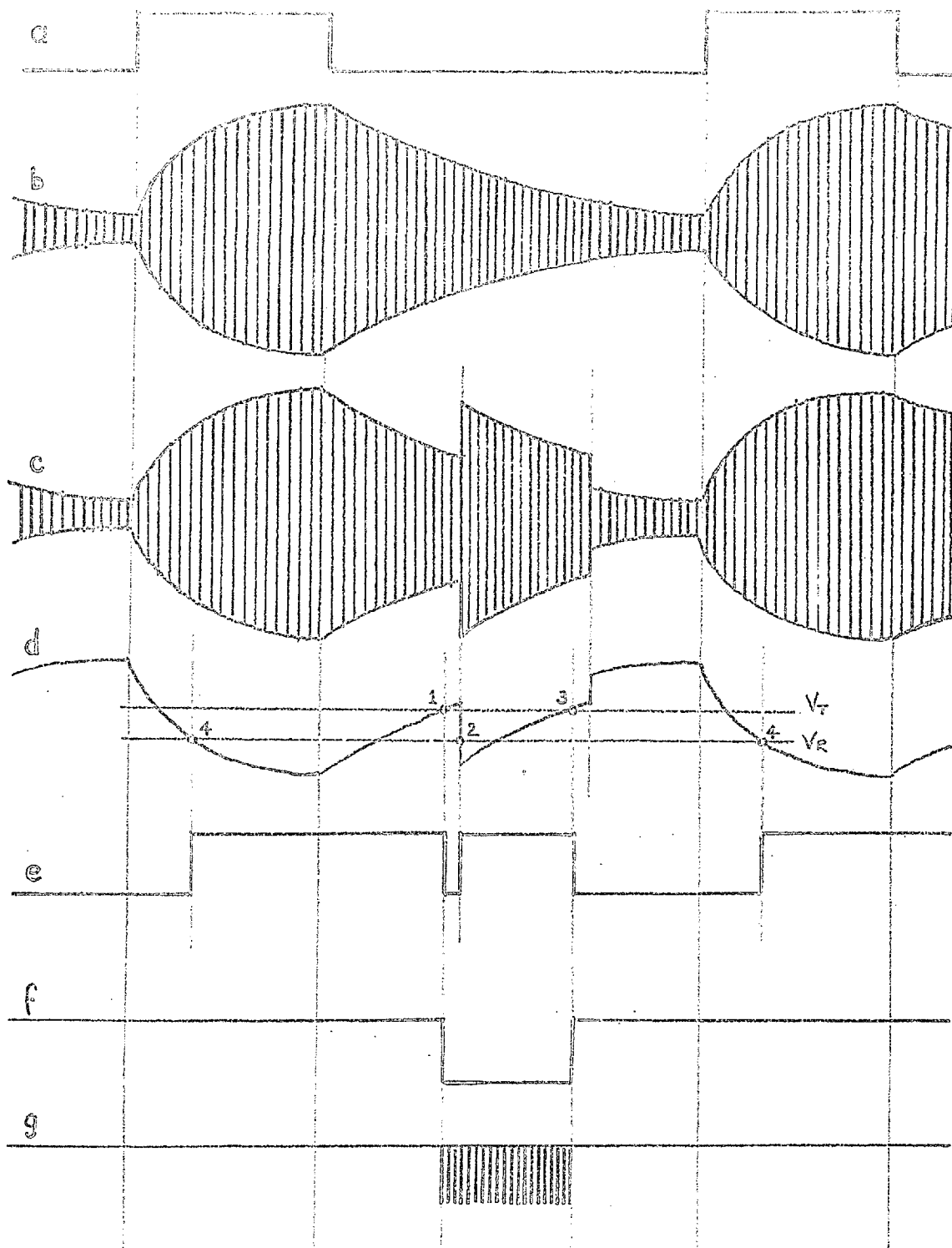


Fig.(5) Circuit Waveforms at indicated Points in Fig.(4)

then be an exponentially decreasing voltage. The 'voltage doubling circuit' sends exactly half this signal to the amplifier. The negative half of this signal is then smoothed (d) and fed to the 'Schmitt Trigger' circuit. When the voltage rises to the value of V_T (see Fig.5 d₍₁₎) the trigger 'fires' to produce a negative going pulse which operates the 'Scale of Two'. The output from the 'Scale of Two' opens the gate to allow pulses at the frequency of the vibrating rod from the 'Pulse Shaper' to be fed to the counter and also operates the relay drive which allows the full signal to be fed to the amplifier. Thus the amplitude of the signal is doubled. The voltage fed to the 'Schmitt Trigger' decreases and resets it as the voltage passes a value of V_R (d₍₂₎). The positive-going output pulse does not affect the subsequent parts of the circuit. When the decaying voltage fed to the Schmitt trigger again reaches V_T the trigger 'fires' for a second time, causing the 'Gate' to close and the relay to reset.

The counter will record the number of oscillations N that the specimen has performed during the interval of time that the amplitude of the freely decaying specimen has decreased from one value to half this value.

It can be shown that the log. decrement $\delta = \frac{\log 2}{N}$

The switch S_2 is operated by a multi-vibrator circuit which 'cycles' the apparatus. This circuit also resets the counter at the end of each operation and so permits continuous working. The time for the apparatus to complete one cycle of operation is approximately 2 sec. and so every two seconds a new determination of damping can be made. If the counter is not reset at the end of each cycle, an accumulative number appears. If resetting is done after every ten counts the 'mean value of ten counts' will be obtained as the counter reading every 20 sec.

III. 2. Discussion.

The apparatus is inherently inaccurate to one cycle of vibration of the specimen, as the counter will only count integer numbers and also because the Schmitt trigger tends to operate at a particular phase of a vibration. Thus in order to obtain a high accuracy in measuring damping it is necessary to have as large an 'N' as possible. If the 'voltage doubler' altered the ratio by a larger value the count could be increased, but this would tend to reduce

the accuracy of determining the voltage levels by the
'Schmitt trigger'. It is thus desirable to employ
high mechanical 'Q' materials for the vibrating systems.

IV. TORSIONALLY VIBRATING RODS

The methods available for exciting or detecting torsional oscillations of a rod generally require some kind of attachment to the rod. A simple method is to place a magnet across the end of the rod as shown in Fig.(6). If an alternating current is supplied to the electro-magnet, the alternating field produced causes the rod to vibrate. Conversely, this system can be used to detect the vibrations. Another method is to wind an electrically conducting coil along the surface of the rod, also shown in Fig.(6). If a radial magnetic field is applied and an alternating current passed through the coil, the rod will vibrate torsionally. A similar coil wound along the rod but orthogonal to the first will detect the vibrations. This system has been used in preference to the first because 'drive' and 'detection' can take place in the same part of the rod. In a metal specimen the coils are wound in grooves in the surface of the rod, thus being made flush with the surface, and are held in position with araldite. For a quartz rod the coils are painted on the surface with colloidal silver. The position of excitation and clamping will determine

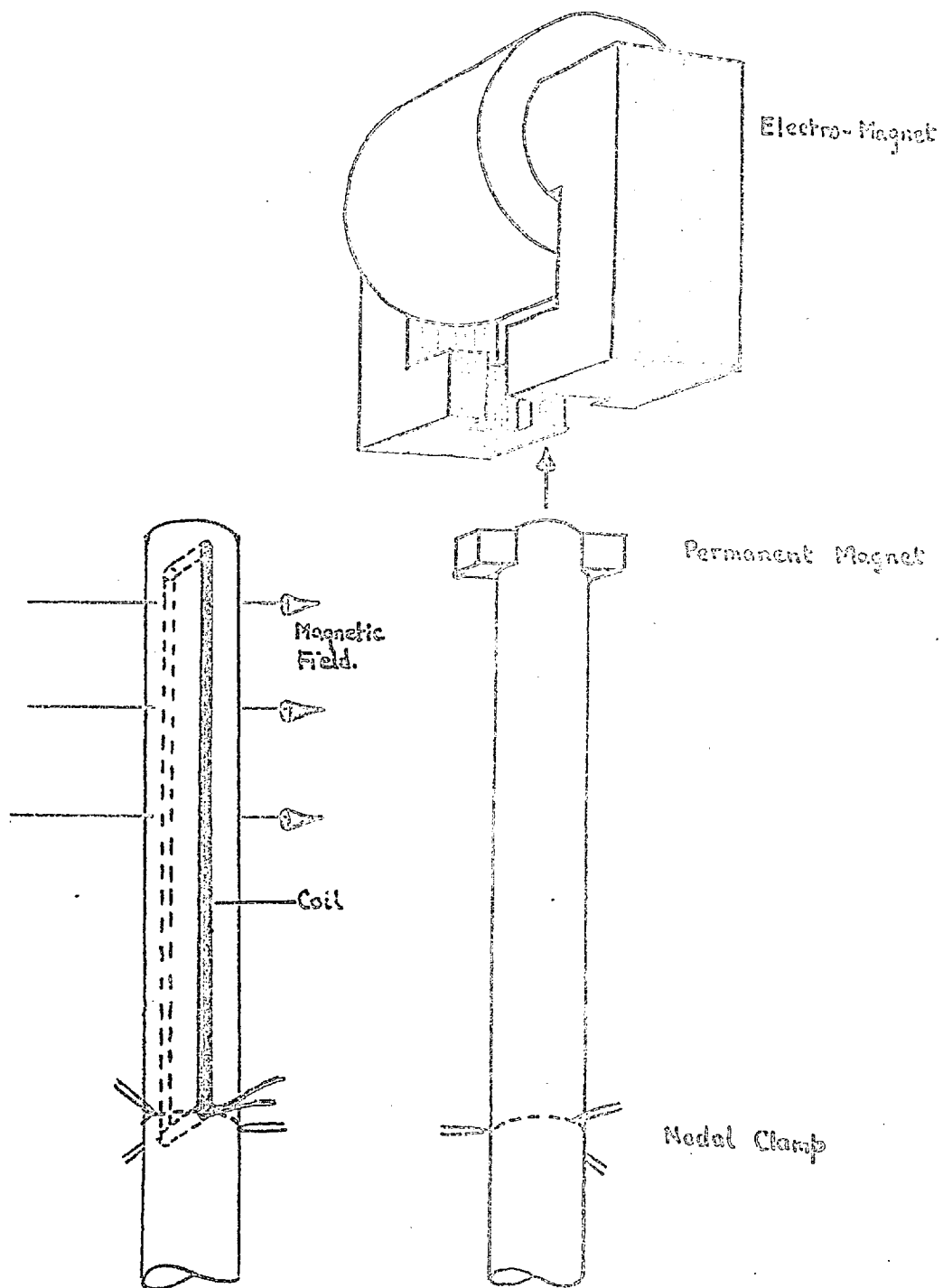


Fig.(6). Torsionally vibrating systems.

at which harmonic frequency the rod will vibrate. In the simplest arrangement the rod is clamped centrally and the excitation and detection coils situated over one half of the rod so that the rod will vibrate at its fundamental frequency. By reversing the direction of the radial magnetic field near the end of the rod it is possible to excite it at a third harmonic. In another arrangement the rod is clamped at two nodal positions and 'excited' and 'detected' at the centre. The rod will now vibrate at its second harmonic frequency.

One disadvantage of this type of driving system for metal rods is that eddy current damping will be introduced due to the oscillation taking place in a radial magnetic field. Non-metal would not have this disadvantage and a quartz rod was therefore selected for one vibrating system.

Several types of clamping arrangements are possible. For a single nodal clamp adjustable screws can be used but for a metal rod the best system is a metal disc 'heat shrunk' on the rod. If this is thin, little additional damping results. When clamping at two nodal positions, needle points can be used but for a robust arrangement two metal rings are employed.

IV. 1. Theory.

The damping of the viscous liquid can be calculated by first evaluating the energy lost per cycle of vibration (ΔW); which is proportional to the real part of the driving point impedance (R_0).

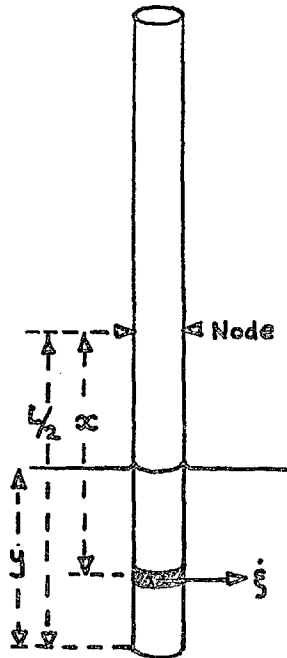
The shear stress $\gamma_x = \dot{\xi} R_0$ where $\dot{\xi}$ is the particle velocity

$$\text{and } \dot{\xi} = \dot{\xi}_0 \sin \omega t$$

The work done on an elemental area dA in time dt will be

$$\gamma_x dA \dot{\xi} dt$$

$$\therefore \Delta W = \int_0^L \int_0^{2\pi/\omega} R_0 dA \dot{\xi}_0^2 \sin^2 \omega t dt \quad (2)$$



But $\dot{\xi}_0$ varies from zero at the nodes to a maximum value ($\dot{\xi}_{0 \max}$) at the anti-nodes. For the rod considered with one node in the centre and length L , $\dot{\xi}_0 = \dot{\xi}_{0 \max} \sin \frac{\pi x}{L}$

$$\text{and for a rod of radius } R \quad dA = 2\pi R dx$$

If the rod is immersed to a depth y as shown in diagram

$$\begin{aligned}\Delta W &= R_o (\dot{\xi}_o \max)^2 2\pi R \int_{L-y}^{L/2} \sin^2 \frac{x\pi}{L} dx \int_0^{2\pi/\omega} \sin^2 \omega t dt \\ &= R_o (\dot{\xi}_o \max)^2 \frac{\pi^2 R}{\omega} \left[y + \frac{L}{2\pi} \sin \frac{2\pi y}{L} \right] \quad \text{--- (3)}\end{aligned}$$

In addition, some energy is lost from the end of the rod.

Equation (2) is again used but in this case $dA = 2\pi r dr$

and $\dot{\xi}_o = \frac{r}{R} (\dot{\xi}_o \max)$

$$\begin{aligned}\therefore \Delta W_{\text{end}} &= R_o (\dot{\xi}_o \max)^2 \frac{2\pi}{R^2} \int_0^R r^3 dr \int_0^{2\pi/\omega} \sin^2 \omega t dt \\ &= R_o (\dot{\xi}_o \max)^2 \frac{\pi^2 R^2}{2\omega}\end{aligned}$$

$$\text{Thus } \Delta W_{\text{Total}} = R_o (\dot{\xi}_o \max)^2 \frac{\pi^2 R}{\omega} \left[\frac{R}{2} + y + \frac{L}{2\pi} \sin \frac{2\pi y}{L} \right]$$

The total energy stored in the rod (W) is given by

$$W = \frac{\pi R^2 L \rho_m (\dot{\xi}_o \max)^2}{8}$$

where ρ_m is the
density of the rod material

Thus

$$\delta = \frac{\Delta W}{2W} = \frac{4\pi R_0}{\rho_m L R \omega} \left[\frac{R}{2} + y + \frac{L}{2\pi} \sin \frac{2\pi y}{L} \right] \quad \text{--- (4)}$$

The variation of δ with depth of immersion y will depend on

$$f(y) = \left[y + \frac{L}{2\pi} \sin \frac{2\pi y}{L} \right]$$

In Fig.(7) this function has been plotted for $L = 30.45$ cm. showing that for values of y near $L/2$ the value of the function is nearly constant, which indicates that it is not necessary to locate the level of the liquid surface to a high degree of accuracy if the level is close to the nodal position, i.e. for $y = .45 L$ the function $f(y)$

has a value only 0.2% different from the value for $y = L/2$

$$\text{In equation (4) when } y = L/2 \quad \delta = \frac{2\pi R_0}{\rho_m L R \omega} [R + L] \quad \text{--- (5)}$$

$$\text{when } y = 0 \quad \delta = \frac{2\pi R_0}{\rho_m L \omega}$$

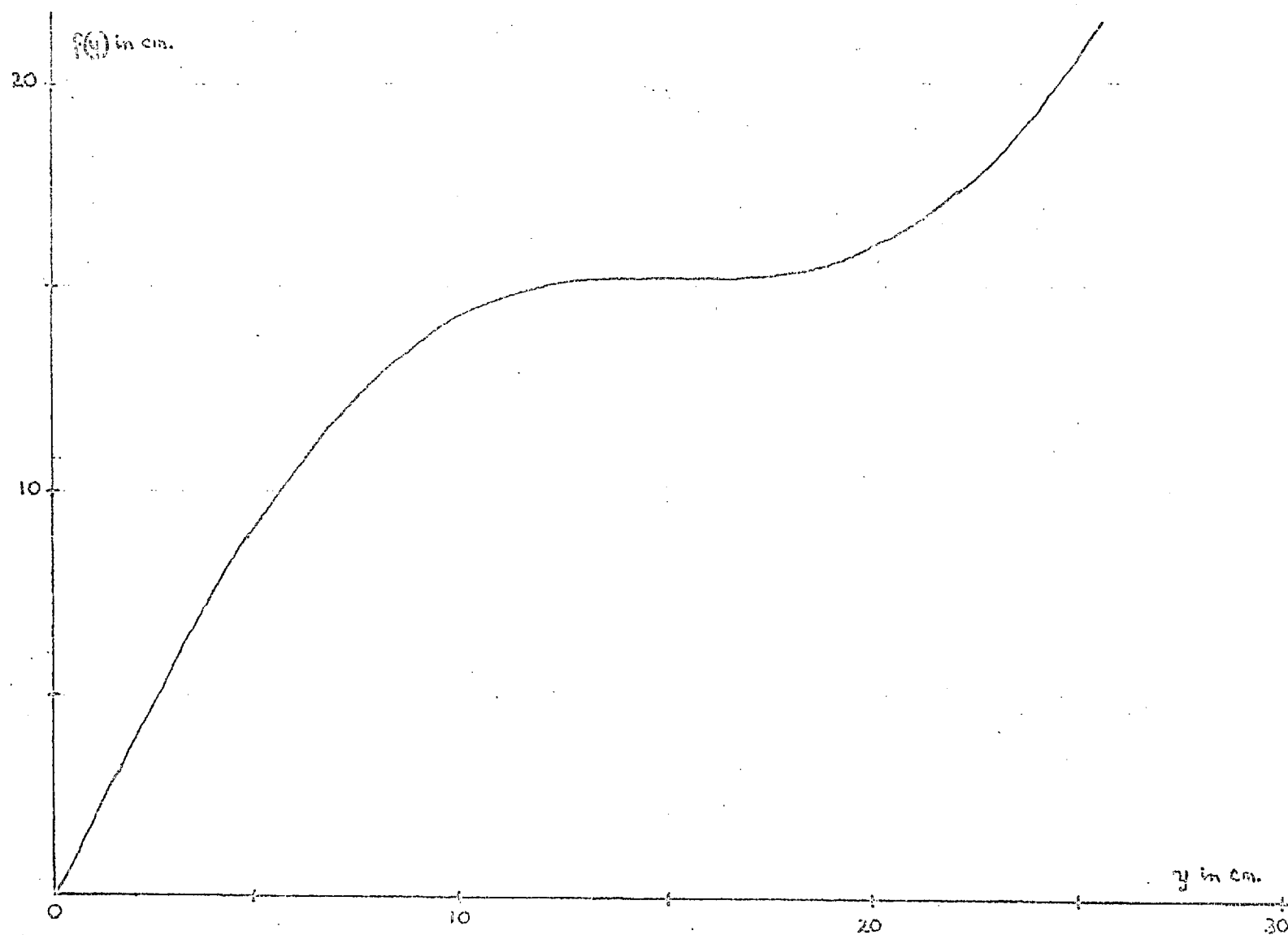


Fig.(7). Variation of damping with depth of immersion.

As R, L, ρ_m and ω are all constants for the apparatus, a measurement of δ will give a value for R_0 which for an infinitely extended viscous liquid is equal to $(\omega \rho \eta / 2)^{1/2}$ and thus η can be determined.

IV. 2. Experimental systems for measuring viscosity.

In applying the foregoing theory to practice, particular care is needed in the choice of materials (see page 30), and two low damping materials were employed, a metal beryllium copper and a non-metal quartz. Each had its particular advantages and disadvantages, as will be evident from the description below.

(a) Beryllium copper rod.

The experimental system is shown in Fig.(8), and with the field direction the same in each section of the magnet arrangement, the rod is excited at its fundamental frequency. By reversing the upper group of magnets (M_1 and M_2) with respect to the lower group (M_3 and M_4), the rod can be excited at its third harmonic.

The rod is clamped at the centre by three screws and

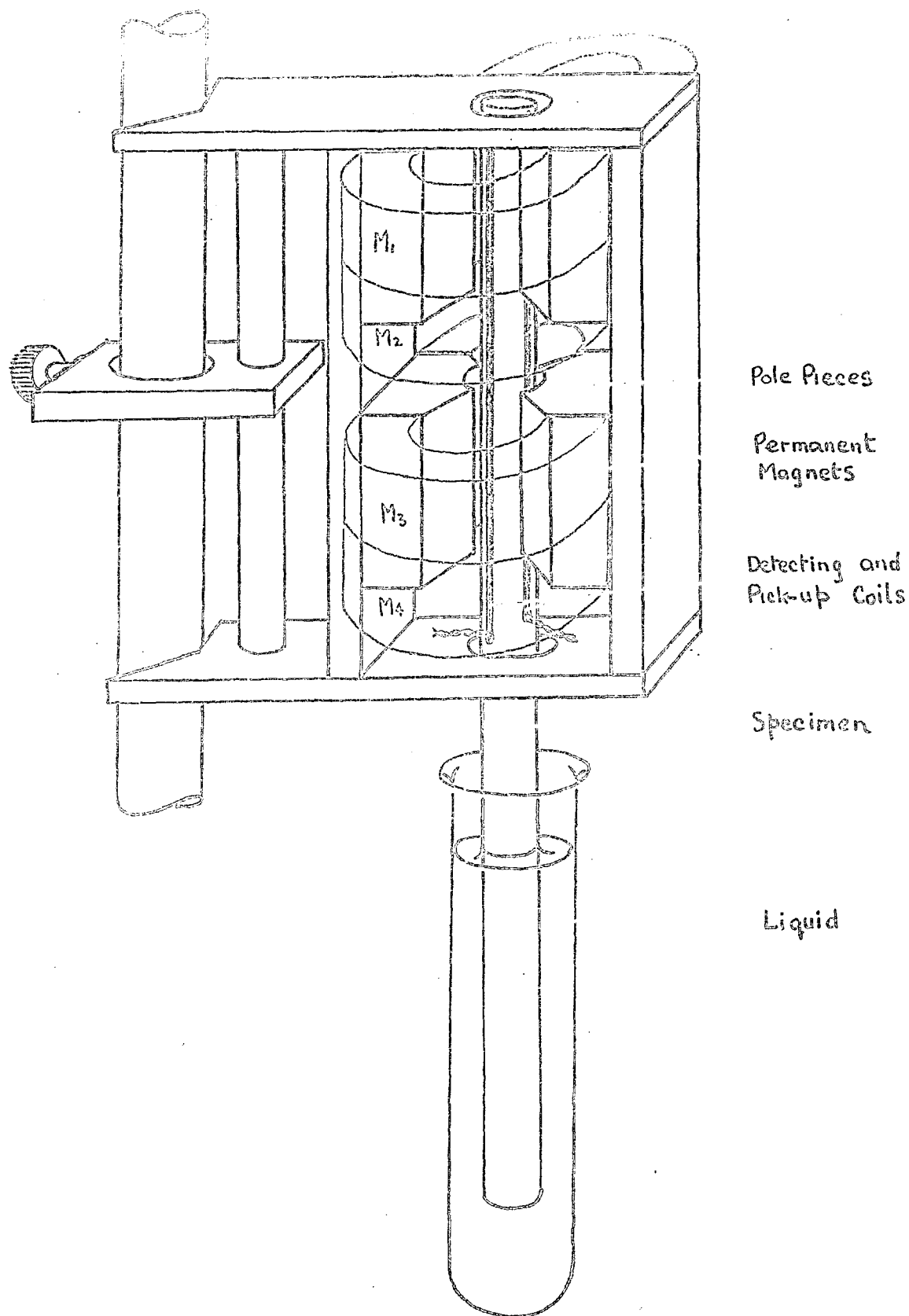


Fig.(8). Single Clamp Beryllium Copper Rod.

its damping is almost independent of slight movements of the clamping position. The strength of the magnetic field, however, influences the eddy current damping, which in theory is proportional to the square of the magnetic field strength. Measurements of damping and the magnetic field are plotted in Fig.(9). The field should therefore be as low as possible consistent with obtaining a reasonable output signal from the detecting coil.

In order to confirm that equation (4) is applicable to this system, preliminary experiments were made with the rod immersed to various depths (y) in water. If the total damping of the system is measured and plotted against $f(y) = [y + L/2\pi \sin 2\pi y/L]$ then the resultant graph should be linear as was experimentally verified, Fig.(10). The viscosity of water may be calculated from the slope and gave a value which was approximately 4% low. Although only a trial experiment, it is of interest to look for an explanation of this error. Firstly, in measuring the level of the liquid to the base of the meniscus, the damping due to the water meniscus was ignored. This extra damping would depend on the position of the meniscus along the rod and, being greatest

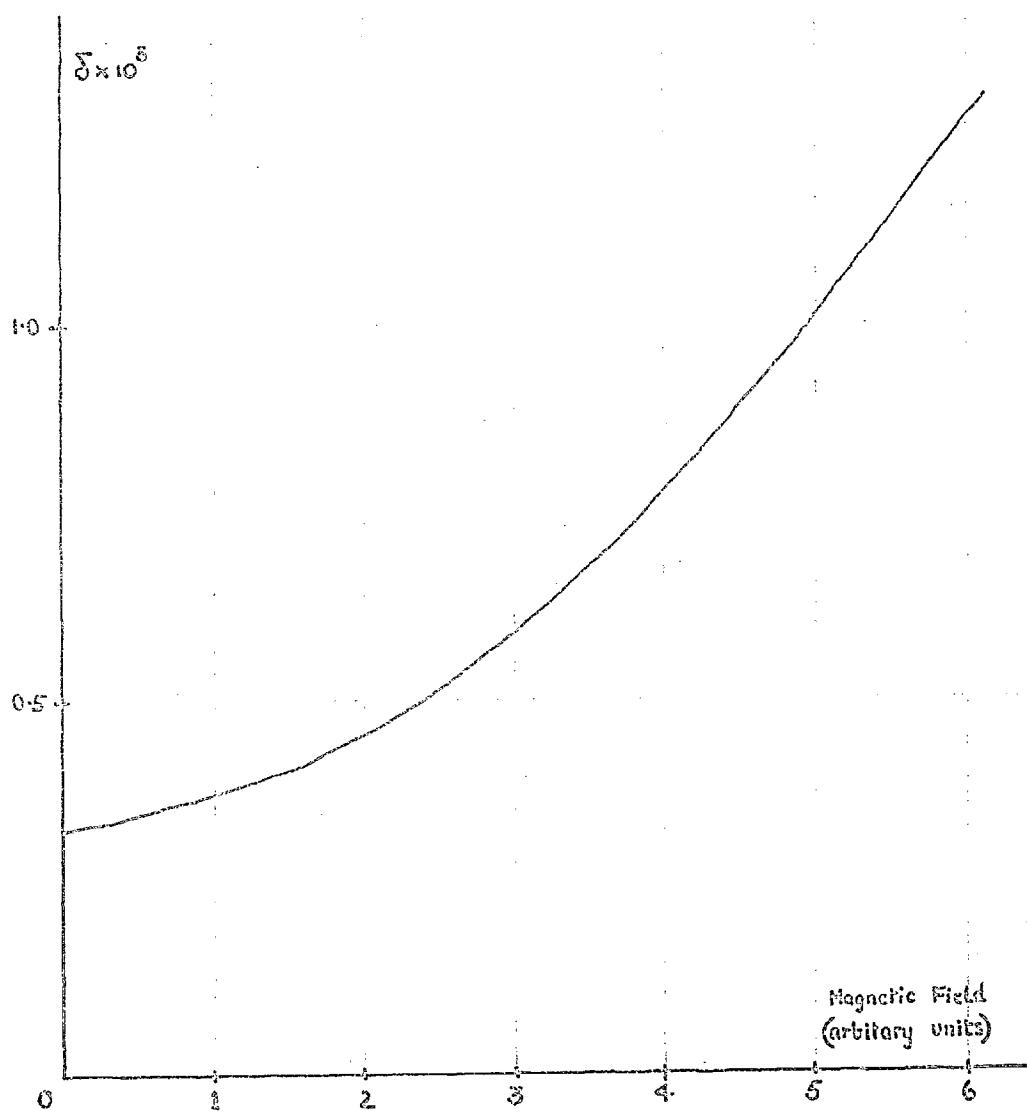


Fig.(9). Variation of damping with magnetic field.

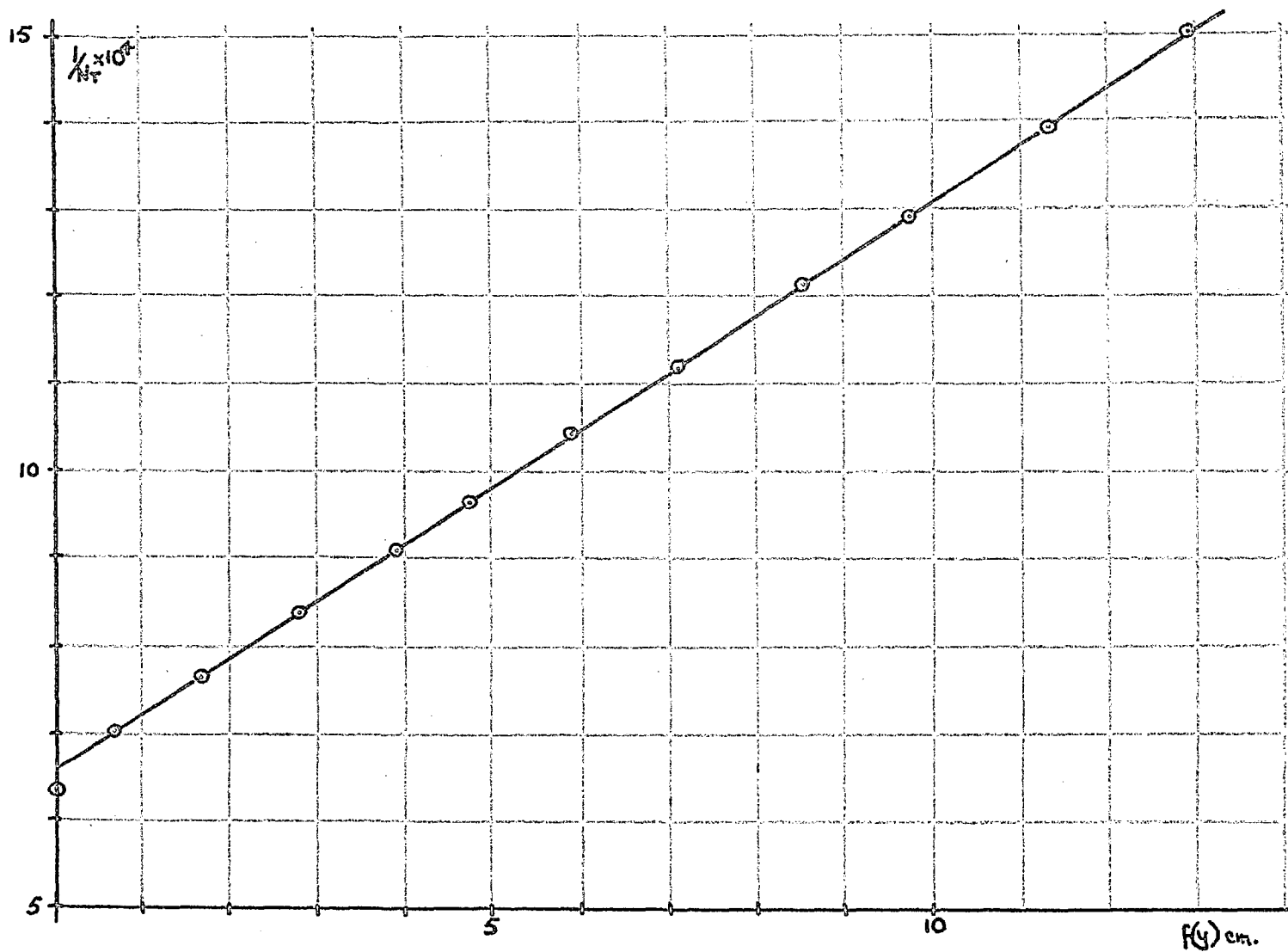


Fig.(10). Variation of damping with $f(y)$.

at an end, would tend to decrease the slope of the resultant graph and introduce an error of perhaps 2% in the value of viscosity. Secondly, no temperature control was used and during the 30 minutes or so duration of the experiment the internal as well as the external damping of the system could change to give rise to the order of the discrepancy observed.

To obtain a more reliable value for viscosity, the following procedure was adopted. The internal damping of the rod was measured and the rod was then immersed in water to within 1 cm. of the centre. The new value of damping was measured and the temperature of the water observed.

To calculate the resultant viscosity, equation (5) was used which includes the damping due to the end of the rod.

For an infinite viscous medium $R_0 = \left(\frac{\omega \rho \eta}{2}\right)^{1/2}$ and thus $\rho \eta = K_2 (\delta)^2$

where K_2 is a constant depending on the constants of the vibrating system. For the rod described, $K_2 = 1.89 \times 10^4$ poise gm/cc. and values of the viscosity of water were obtained which agree with the accepted values within a maximum error of 2% (see Fig.(10a)). Temperature measurements were, however, not very precise and as a 0.5°C temperature change

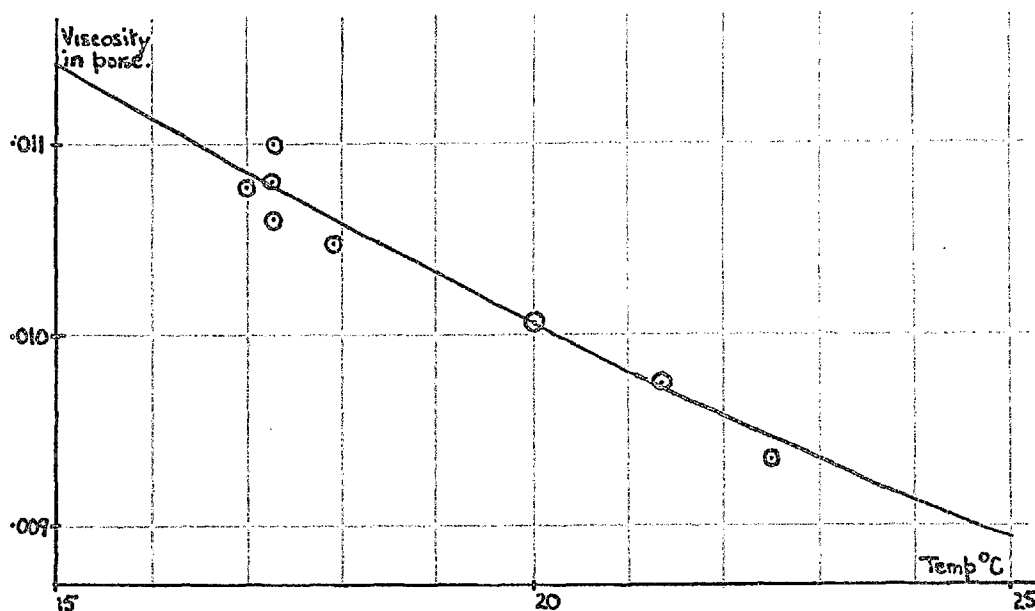


Fig.(10a). Experimental values of the viscosity of water at different temperatures. The continuous line is plotted from values taken from Kaye and Laby.

produces approximately a 1% variation of viscosity, the error due to the apparatus might well be only 1%.

With the rod adjusted to vibrate at its third harmonic, the rod was immersed to a depth of $y = L/6$ and the corresponding damping measured. Again the resultant viscosity was within 2% of the expected value.

The viscosity of water-alcohol mixtures was also investigated. This information was required for the experiment described in Section V. 1 (b), and although values for viscosity are given in Landolt & Börnstein (1933), these have not been measured by a vibrational technique. It was therefore decided to confirm these values by using the rod vibrating at its third harmonic frequency (11.2 Kc/s.). Results obtained are plotted as dotted circles in Fig.(10b). The continuous line is a plot of the data obtained from Landolt & Börnstein.

In order to investigate the possibility of using the experimental system as a means of measuring a small quantity of liquid, say less than 0.1 cc., a set of observations were made in which a water drop covered one plane end face of the rod. The choice of water was less favourable to the accuracy of the absolute determination of η than if a more viscous liquid had

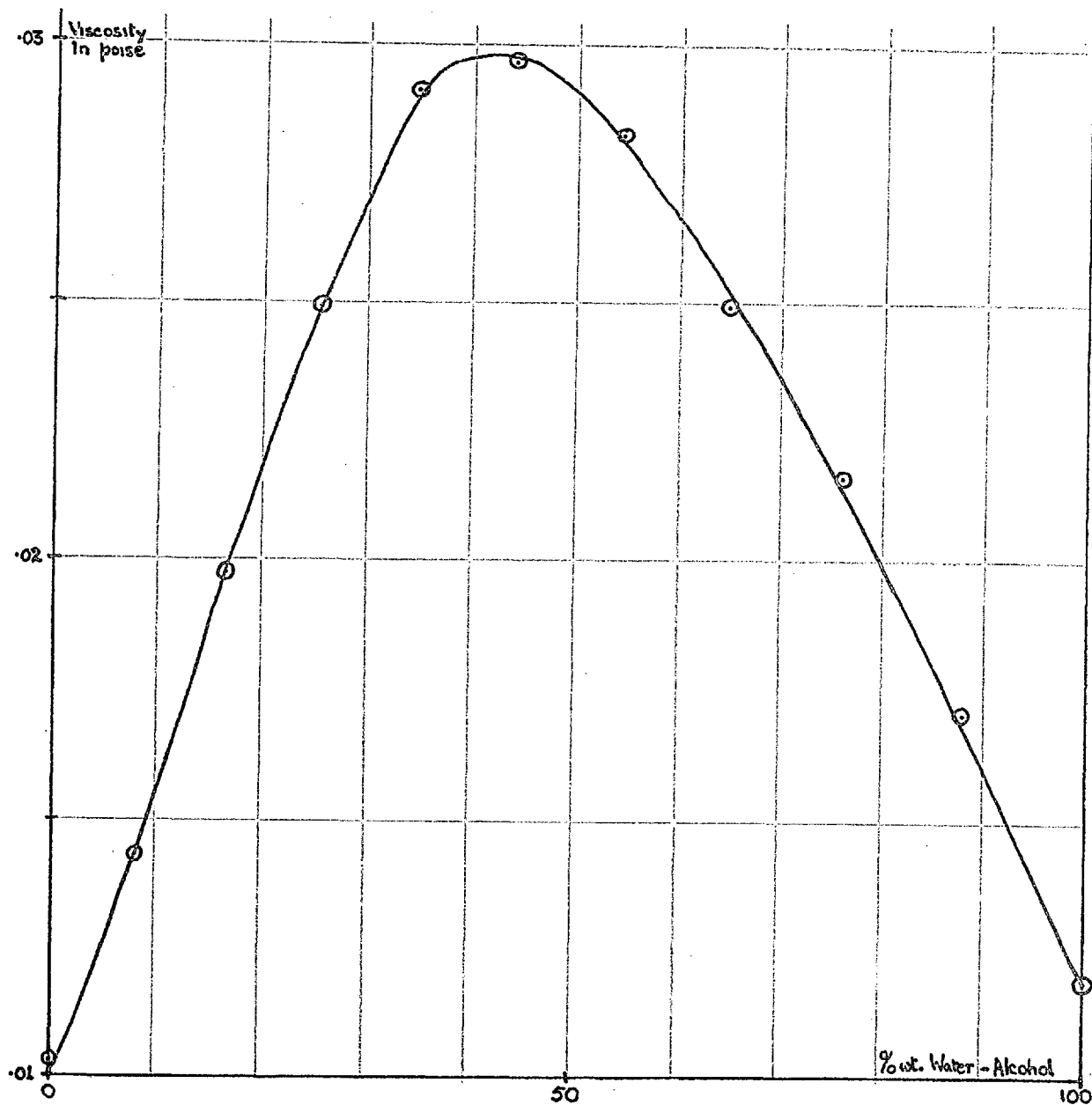


Fig.(10b). Experimental values of the viscosity of water-alcohol mixtures at approx. 20°C. The continuous line is plotted from values taken from Landolt and Bärnstein.

been chosen. Successive experiments were performed with and without water and the readings obtained are tabulated in Table (1). A graph showing the steady drift of these is shown in Fig.(11). The calculation of viscosity from these readings gave a value of $.93 \times 10^{-2}$ poise at 19°C , which is approximately 10% low. This discrepancy could arise from a change of 0.3 cycles in the observed difference count of 5.3 cycles. This error being only a fraction of one cycle, instead of the probable error of one which can arise for each individual count, indicates that if means of several counts are used for the calculation, the error in N is less than half a cycle.

Further experiments performed with a standard oil instead of water gave results agreeing with the accepted value to within 3% and this error could be accounted for by an uncertainty of the knowledge of the temperature to better than 1°C .

IV. 2. (b) Quartz rod.

In order to attempt to make more precise measurements by end damping experiments it was decided to use a quartz rod. The principal advantages of this compared with the beryllium copper rod are that the internal damping is very small and that

Table 1. Experiment with water on the end of rod.

	N_I	N_T	N_I	N_T	N_I	N_T	N_I	N_T	N_I
Temp. of water °C		18.7		18.8		18.9		19.1	
Means of ten counts	582.6	576.8	581.7	576.7	581.7	576.3	581.4	576.3	581.4
	2.2	6.8	2.0	6.5	2.0	6.3	1.8	6.5	1.5
	2.2	6.8	1.8	6.7	1.9	6.2	1.8	6.1	1.5
	2.5	6.7	1.8	6.5	2.0	6.2	1.7	6.3	1.2
	2.0	6.6	2.0	6.5	1.9	6.2	1.9	6.6	1.4
	1.9	6.7	2.0	6.6	1.8	6.3	1.7	6.4	1.6
	2.0	6.5	1.8	6.7	1.8	6.4	1.6	6.4	1.4
	1.9	6.5	1.7	6.3	1.8	6.3	1.6	6.3	1.2
	2.1	6.4	1.8	6.6	1.9	6.6	1.7	6.2	1.3
Mean of one hundred	582.15	576.64	581.84	576.57	581.86	576.33	581.68	576.35	581.36

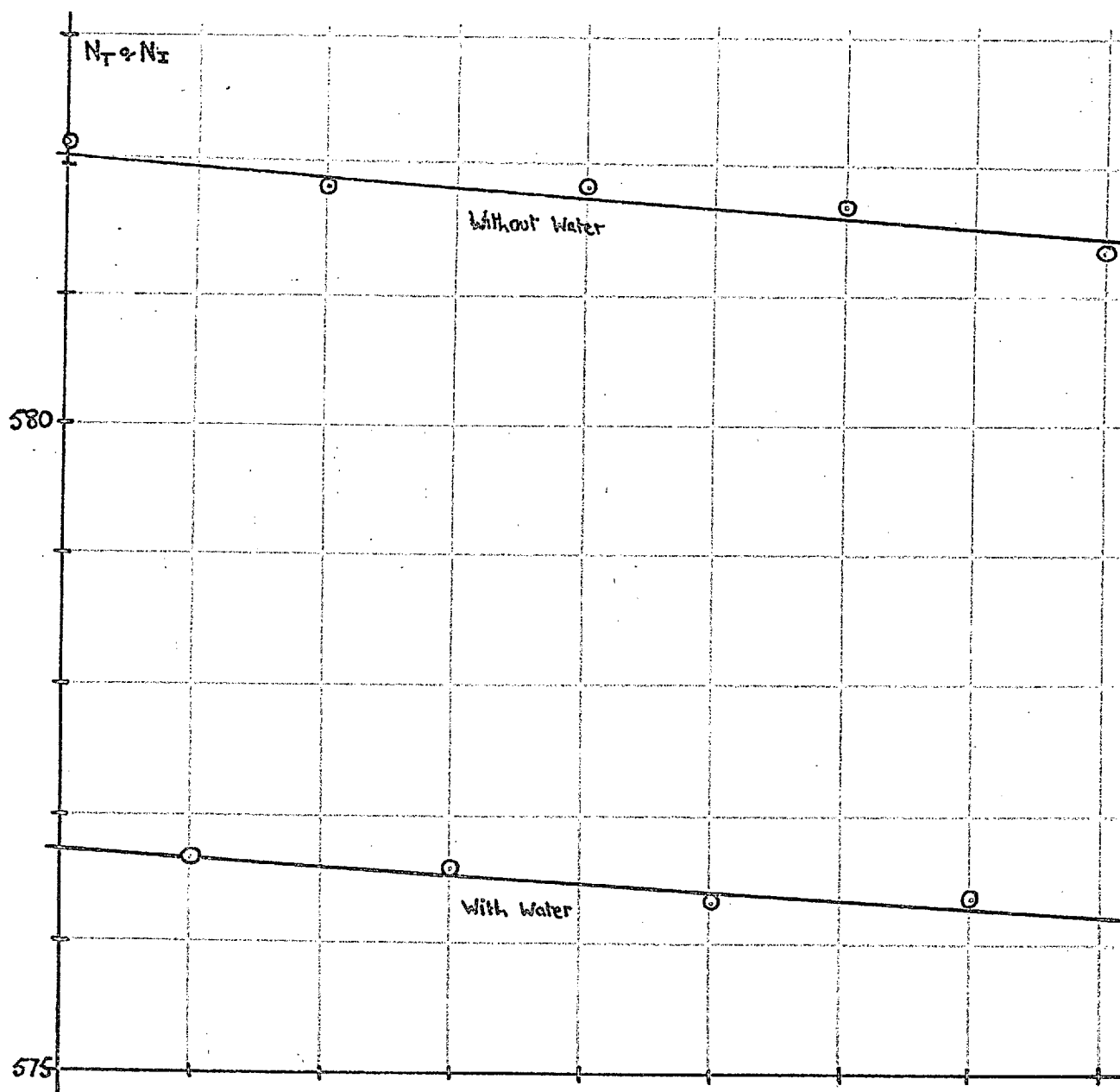


Fig.(11). Damping measurements with and without water on the end of rod.

because the density of quartz is approximately one-third that of beryllium copper, the end damping will be correspondingly increased. A further advantage is that the internal damping of quartz shows little variation with temperature.

The driving and 'pick-up' coils were painted on the surface of the quartz rod with colloidal silver, the rod having previously been cleaned and heated in an oven to red heat. The rod was then reheated to bake the silver on to the surface. The conducting coils had a resistance of approx. 20 ohms each and as this was much too high, a copper film was electro-deposited on the silver. This reduced the coil resistance to approx. 0.5 ohms. The clamping system for the rod was designed to give connections to the ends of the coils and, also, to act as cross over connections between two parts of each coil. To facilitate this, two nodal clamps were used, each having four needle point contacts with the rod. The arrangement of rod and magnetic field is shown in Fig.(12). On excitation, the rod will vibrate torsionally at its second harmonic frequency. The fluid medium can be applied to either of the free ends of the rod. The vibrating system is placed inside a water jacket to maintain it at a given temperature.

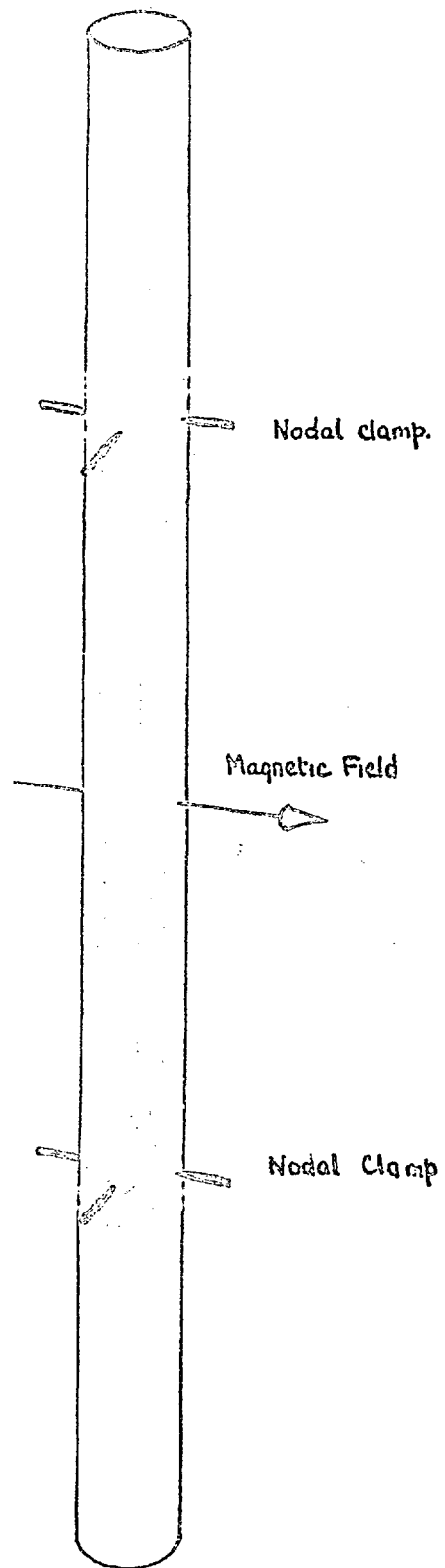


Fig.(12). Quartz Rod.

Measurements of the total damping of the rod with the lower portion immersed in an oil sample were made over a period of several days. No temperature control was employed but the temperature of the enclosure was recorded. The results obtained are plotted in Fig.(13). These results indicate the repeatability of readings obtained with this system and show how ($\rho\eta$) for the oil varies with temperature. The calculated viscosity x density at 20°C was found to be 01.77 poise gm/cc. and the rate of change with temperature 2.1% per °C.

It was found to be difficult to maintain the rod in vibration owing to poor contact being made between the needle supports and the coils. Eight connections had to be made simultaneously and each adjustment of the rod tended to remove some of the plating from the coil ends. Soldering of these connections was attempted but even the use of low melting point solder resulted in the coil-ends breaking away from the rod.

Although the quartz rod has obvious advantages, it was decided to abandon this system, due to the difficulty mentioned above.

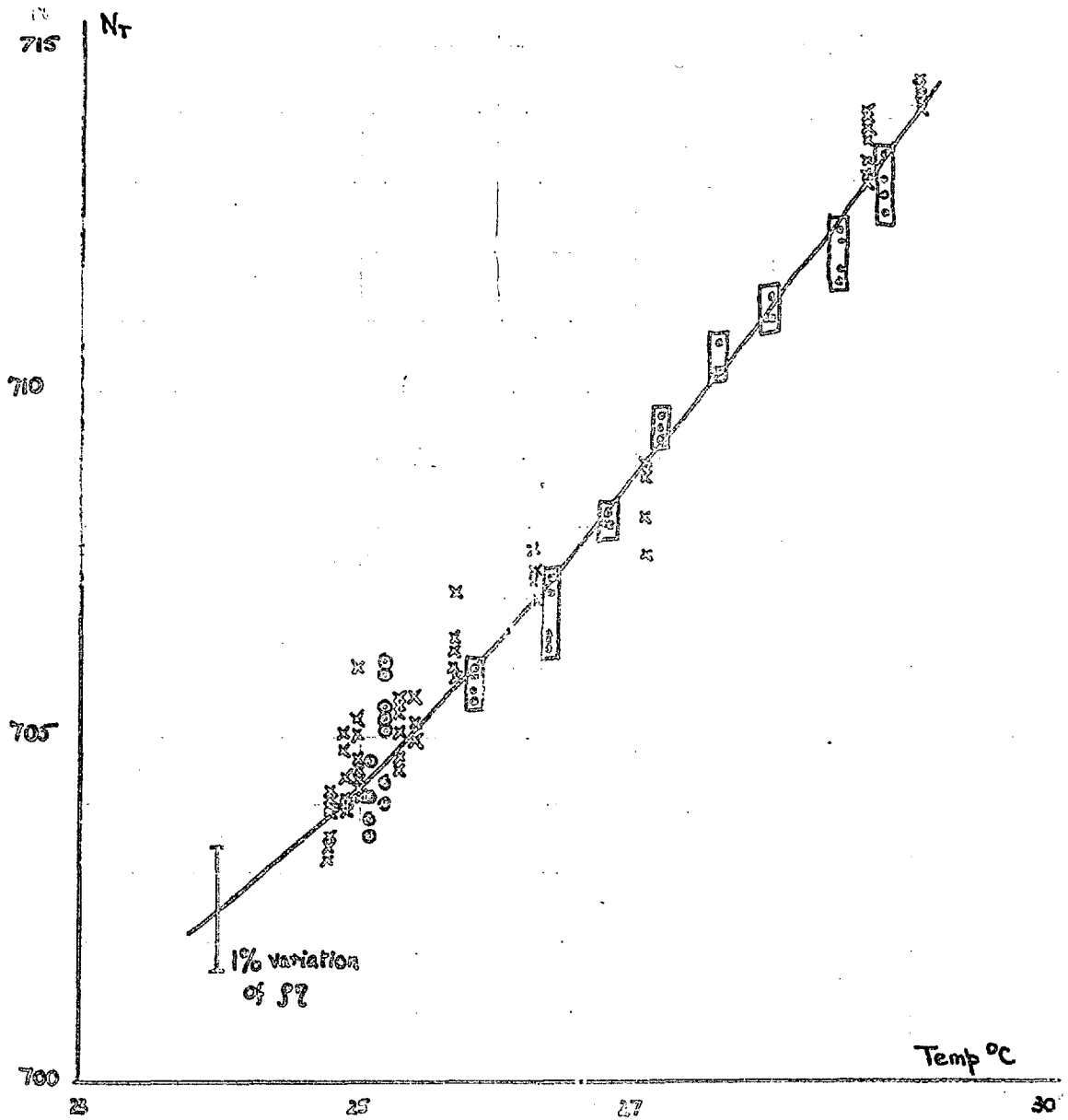


Fig.(13). Damping of quartz rod by oil sample with temperature.

IV. 2. (c) Double clamped Beryllium-Copper Rod.

In order to be able to measure as large a range of viscosity as possible with one vibrating system, a beryllium-copper rod was selected with two nodal clamps.

For such a system it can be shown (see Equation (5) page 36)

$$\rho\eta = (\delta_\eta)^2 (\rho_m R)^2 \frac{4f}{\pi} \left(\frac{L}{L+2R} \right) \frac{1}{S^2}$$

where δ_η is the logarithmic decrement due to the fluid medium surrounding either or both ends of the rod, S being 1 or 2 depending on whether one or both ends are immersed.

$\rho\eta$ is thus determined by the square of δ_η and the selected values of rod parameters. The range of $\rho\eta$ is therefore dependent on the range of the logarithmic decrement which can be measured by the electronic apparatus, which for a theoretical accuracy of 0.5% is $0 - 3.0 \times 10^{-3}$. But $\delta_\eta = \delta_T - \delta_I$ where δ_I is the logarithmic decrement due to the internal damping of the rod and δ_T the decrement obtained when the rod is immersed in the fluid medium. It is obviously desirable to make δ_I as small as possible and for the rod selected this was found to be 0.5×10^{-3} . The maximum value

of δ_η is thus 2.5×10^{-3} . The minimum value, determined to the same accuracy, will be approx. equal to the internal damping. The range of δ_η is therefore $0.5 - 2.5 \times 10^{-3}$. The corresponding range of η will have a variation of a factor of 25. As it is a simple matter to make S in the equation 1 or 2, the total possible variation with one specimen is 100. For the rod used in this experiment with $R = 3/16$ " and $L = 60$ cm. the range is $0.005 - 0.5$ poise gm/cc.

The beryllium copper-rod is supported at two nodal points by metal disc heat shrunk on to it. These discs are machined to a thickness of 20 thousandths of an inch and clamped in a suitable container as illustrated in Fig.(14). The fluid whose viscosity is to be measured can be pumped around either end of the rod. The whole apparatus is contained in a combined water-bath and water jacket system.

As an initial experiment to determine the constancy of the system, the same oil sample used with the quartz rod was circulated around one end of the rod. Readings were taken over a period of several days and the results obtained are shown in Fig.(15). The variation of damping with temperature is not the same as that obtained for the quartz rod. This is because the internal damping of the rod changes with temperature. In a subsidiary experiment the internal damping was found to

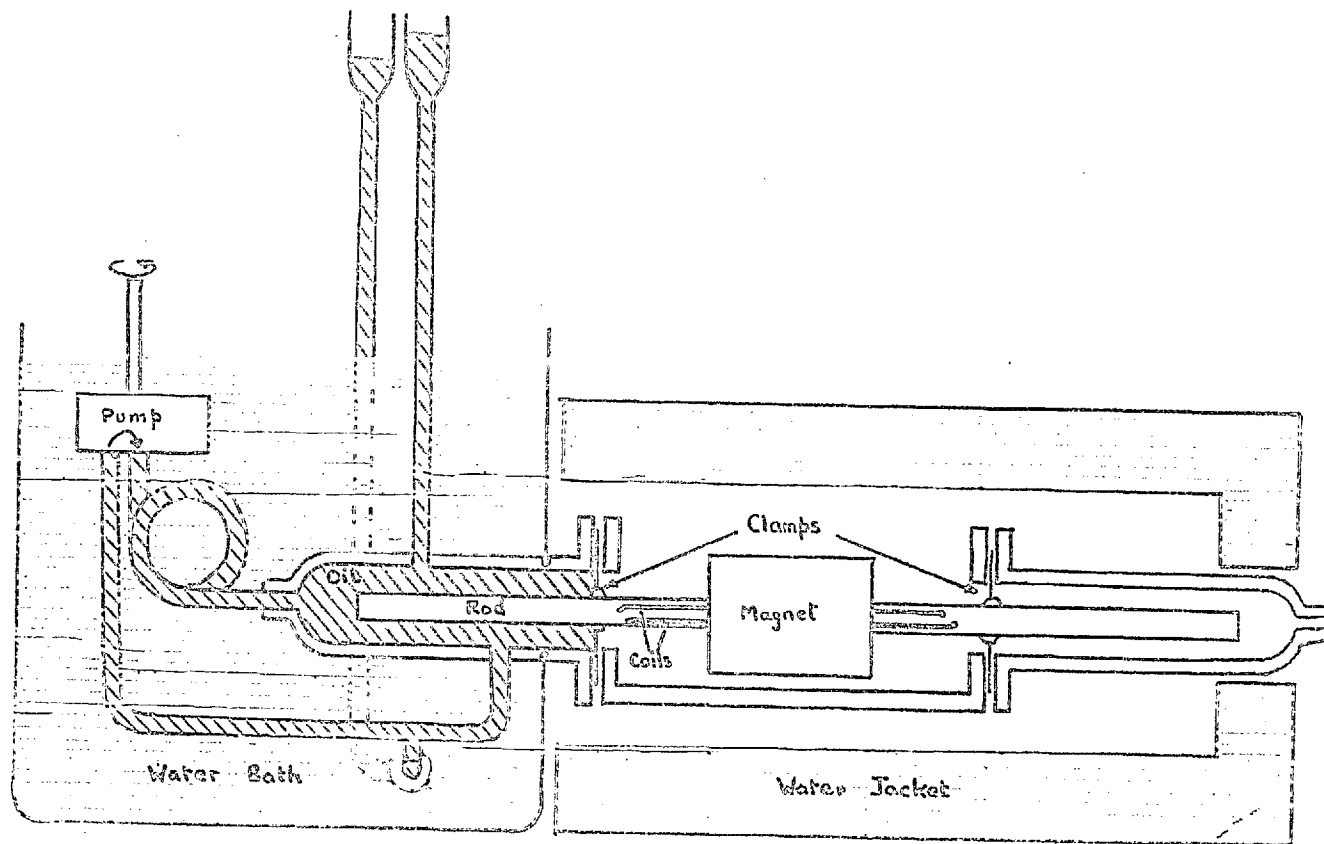


Fig.(14). Double clamped Beryllium Copper Rod.

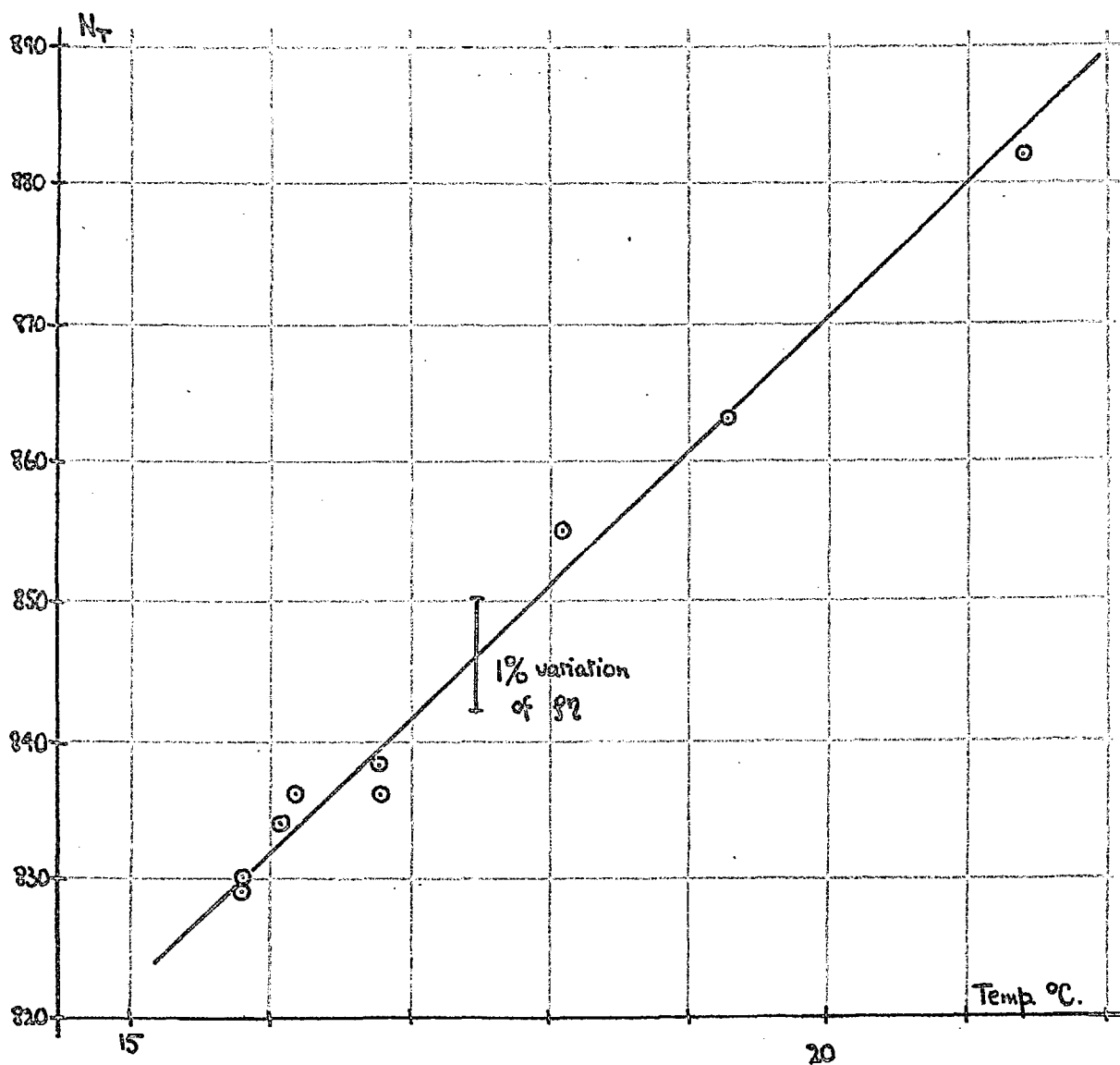


Fig.(15). Damping of double clamped beryllium copper rod by oil sample with temperature.

increase with temperature. Making allowance for this effect, the calculated temperature change of viscous damping agreed fairly well with the value obtained with the quartz where it is assumed that the internal damping is temperature independent. The calculated viscosity x density of the oil was found to be .0166 poise gm/cc. at 24°C.

Using a standard silicone oil (20 centistoke at 25°C) to surround one end of the rod, the corresponding viscosity was calculated to be 21.8 centistoke at 18.8°C. Using information available regarding the temperature variation of this oil, the value at 18.8°C was found to be 21.5 c.s.

V. VARIATION OF DAMPING WITH THICKNESS OF A LIQUID FILM

In the section on the basic theory of visco-elastic damping, the effect of thickness of a bounded fluid layer has been considered. The experiments to be described have been designed to confirm the equations derived in this section.

V. 1. Liquid film with free surface.

(a) Low frequency system

The thickness of a water film with a free surface necessary to cause maximum damping (see Fig.1) will depend on the frequency of the vibrating specimen. At $f = 2000$ cps, $\ell = 2 \times 10^{-3}$ cm., while for $f = 0.2$ cps, $\ell = 2 \times 10^{-1}$ cm. In order to measure the damping of a water film as a function of the thickness, it was decided to use a low frequency so that the thickness could be determined to a reasonable accuracy. A heavy cylindrical disc attached to a steel torsion wire was therefore selected as the vibrating system. The disc is immersed in a dish of water to the level of its upper surface. By adding known volumes of water to the dish, a film, of thickness which can be determined, will cover the upper surface of the disc. The damping of this system can be determined by measuring the amplitude of successive vibrations. The amplitude is

observed by using a beam of light reflected from a mirror attached to the torsion wire, the image of a crosswire being focussed on a scale.

The apparatus used for this experiment is shown in Fig.(16). To excite the disc into torsional vibrations the torsion head is rotated and returned to its normal position. The amplitudes of successive vibrations are recorded as deflections of the light beam. In order that these should be linearly related to the torsional amplitude, a circular scale should have been used. As this was not available, a straight scale was employed and was placed so that when the deflection was half the maximum value recorded, the light beam was normal to the scale. This arrangement can be shown to give good linearity for the larger deflections but would need some correction for small deflections. The small deflections could not, however, be measured with great accuracy and were therefore ignored. In order to check the linearity, a graph of log. deflection versus number of swing was plotted and is shown in Fig.(17).

In order to calculate the logarithmic decrement for the disc, fifty successive swings were recorded and the logarithm of each was found. The mean values of the logarithms of the first and second twenty-five swings were subtracted, and from this value the decrement was determined. Good accuracy could be expected from this method.

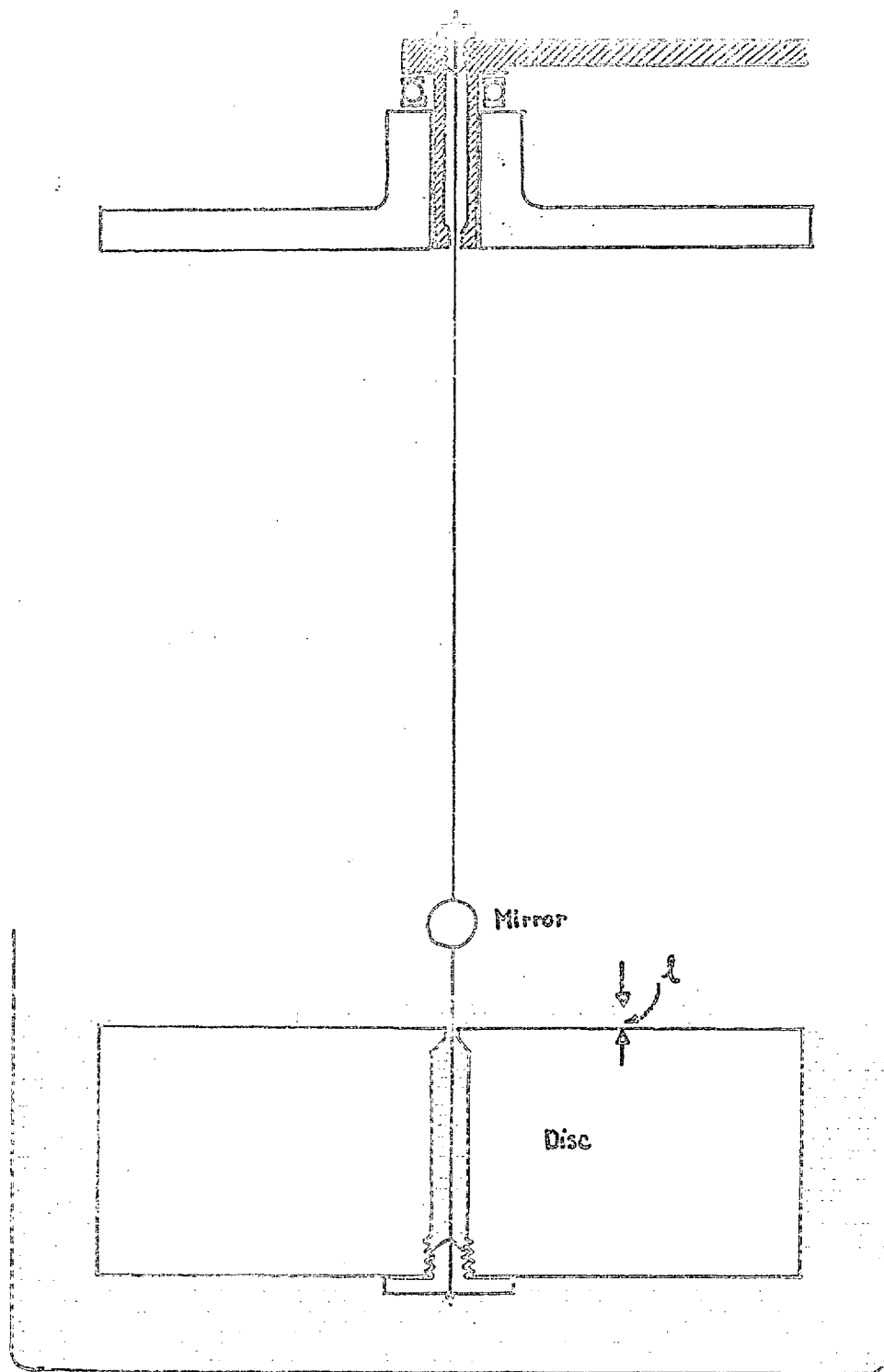


Fig.(16). Torsional Disc.

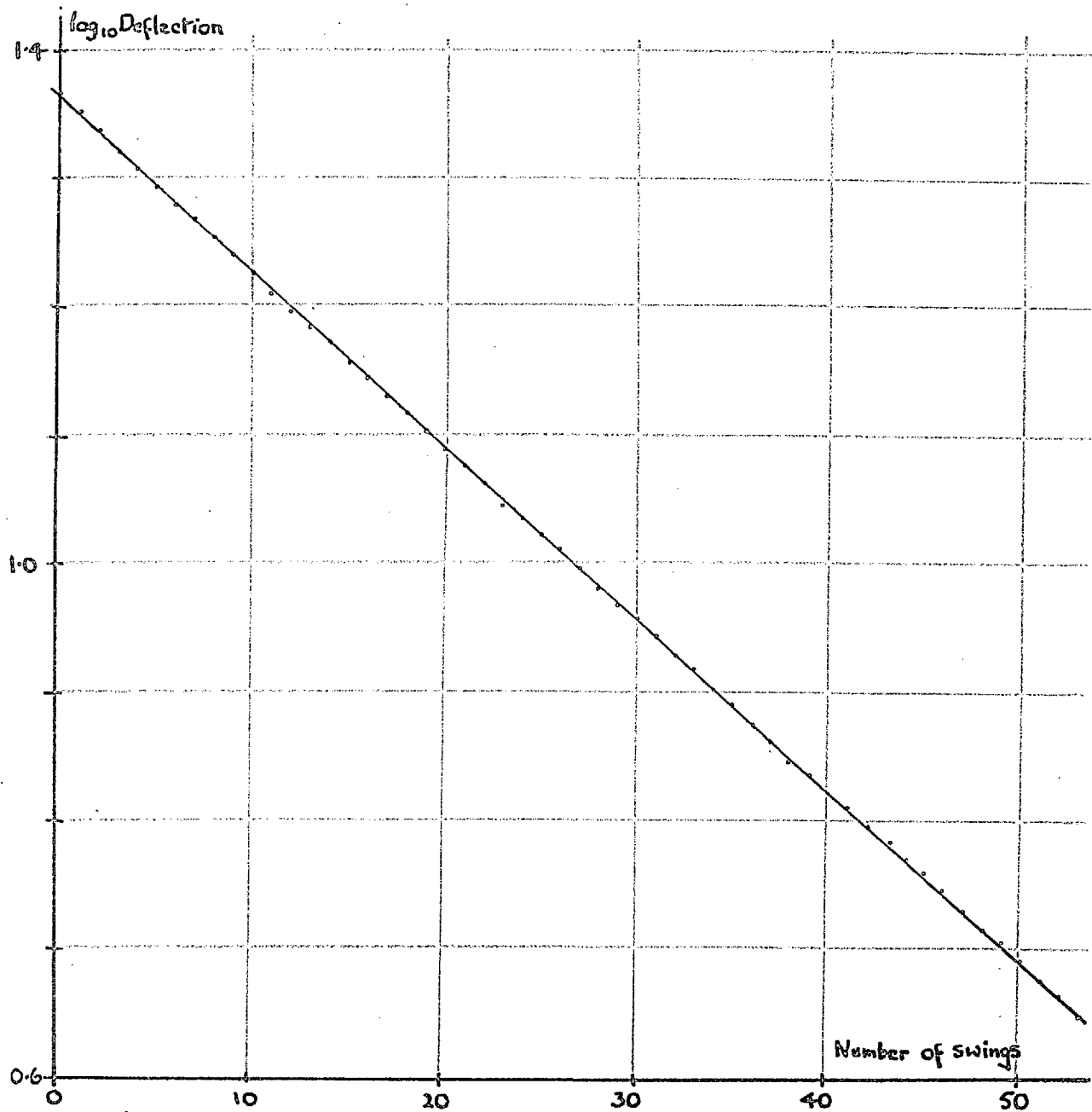


Fig.(17). Logarithmic deflection of torsional disc against number of swings.

An experiment was performed by removing known volumes of water from the dish, the level of which was already covering the upper surface of the disc. The corresponding logarithmic decrements were measured. The results obtained are plotted in Fig.(18). Using the known values of viscosity and density of water and the parameters of the disc, the theoretical curve obtained from equation (1) has also been plotted and, as can be seen, good agreement results.

In this experiment it is possible to observe the amplitude and phase of particles in the viscous wave simply by suspending particles in the water. At a film thickness which produces maximum damping, the vibrations of particles on the surface of the film can be clearly seen to be $\pi/2$ out of phase with the disc itself, and the amplitude of vibration to have decreased to approximately one-fifth of the amplitude at the surface.

V. 1. (b) High frequency system.

If the previous experiment is performed at higher frequencies, the method of controlling the thickness of the liquid film will be more difficult. One possible method is to control the rate of evaporation of the film

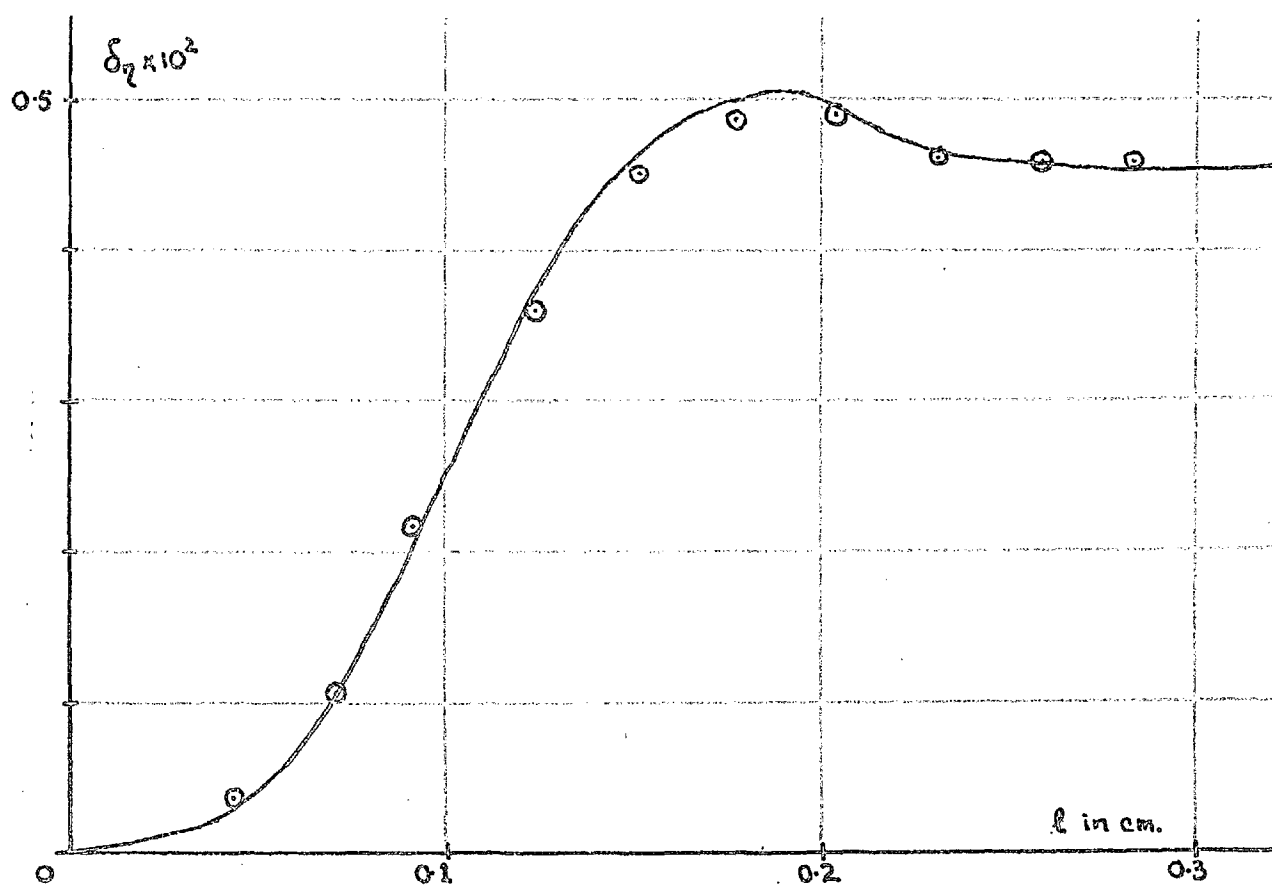
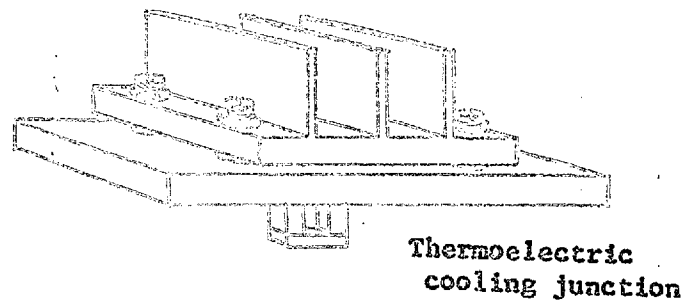


Fig.(18). Logarithmic decrement of torsional disc against depth of water layer.

of liquid; the thickness will therefore decrease at a constant rate. Although there are several disadvantages, such as that the temperature of the film will fall and that the constitution of the liquid might change as the volume decreased, this method was chosen and the apparatus illustrated in Fig.(19) was constructed.

The vibrating system consisted of a steel strip 12" long and $\frac{3}{4}$ " wide supported at two nodal positions by three needles as shown in the diagram. The strip can be excited magnetostrictively into longitudinal vibrations at its second harmonic frequency by the driving coil situated near one node. The pick-up coil is placed near the other node and detects the amplitude of the vibrations. A ridge is placed across the strip at each node to retain the liquid sample poured on to the centre of the strip. Accurate levelling is necessary to ensure a liquid film of uniform thickness, and this can be done by adjusting two screws which move the needle supports. The method of adjustment is to place on the strip a small dish, containing mercury, and to reflect a beam of light from this to form an image on a suitable distant surface. The image position is marked and the



Thermoelectric
cooling junction

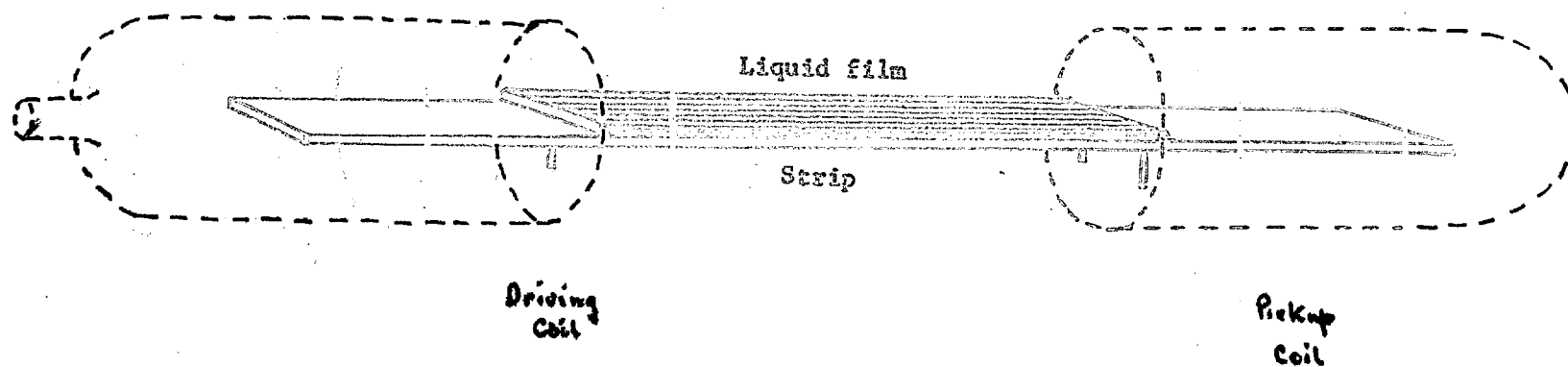
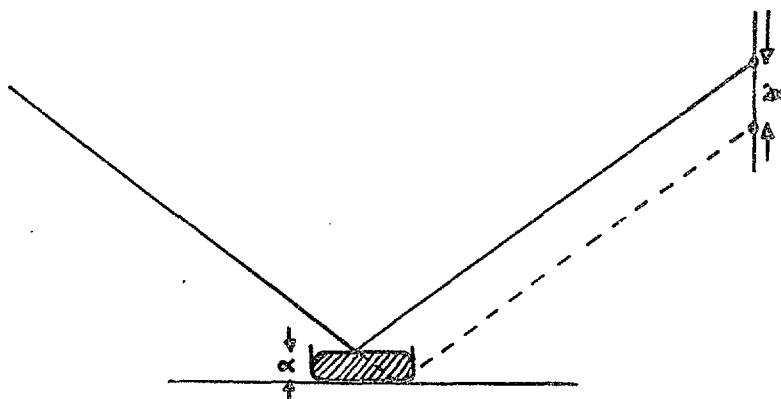


Fig.(19). Apparatus for the controlled evaporation of a liquid film on a strip by a thermoelectric cooling junction.

mercury dish removed. The light reflected from the metal strip will form another image, the position of which is moved by means of the two adjusting screws until the image lies immediately below the position of the first image at a distance of twice the height of the mercury surface above the strip.



In order to control the rate of evaporation of the liquid, the strip is placed in a vacuum enclosure. In this enclosure, immediately above the strip, is a thermoelectric cooling junction (Goldsmid, 1955) capable of cooling at the rate of 0.5 watt at a temperature of approximately 20°C below the ambient temperature. The liquid vapour condenses on the cold surface at a rate controlled by the current ^{*} flowing through the junction.

* The current is maintained constant and thus the thickness of the film is linearly related to the time of evaporation.

In the experiment described, 0.5 cc of liquid were condensed in approximately 20 minutes.

Although several liquids were selected for this experiment, ethyl alcohol was the only liquid which would maintain itself as a thin film during the evaporation. The results obtained for two alcohol samples are plotted in Fig.(20). These do not agree with the expected theoretical curve. The discrepancy can be attributed to two factors, both of which produce an increase of viscosity of the liquid film and thus cause the observed rise.

One factor is due to the cooling of the liquid film. As the strip is thermally isolated from the surroundings, any heat loss due to evaporation will cause a fall in temperature of the strip and liquid film. This has been estimated to be approximately 20°C for the complete evaporation of 0.5 cc of alcohol. The second factor is due to a change in percentage water concentration in the liquid film. For boiling at reduced pressures the vapour phase of alcohol-water mixture is predominantly alcohol (International Critical Tables, 1933). Thus if 1.5% water is present initially in the liquid, as evaporation takes place this will increase inversely in proportion to the

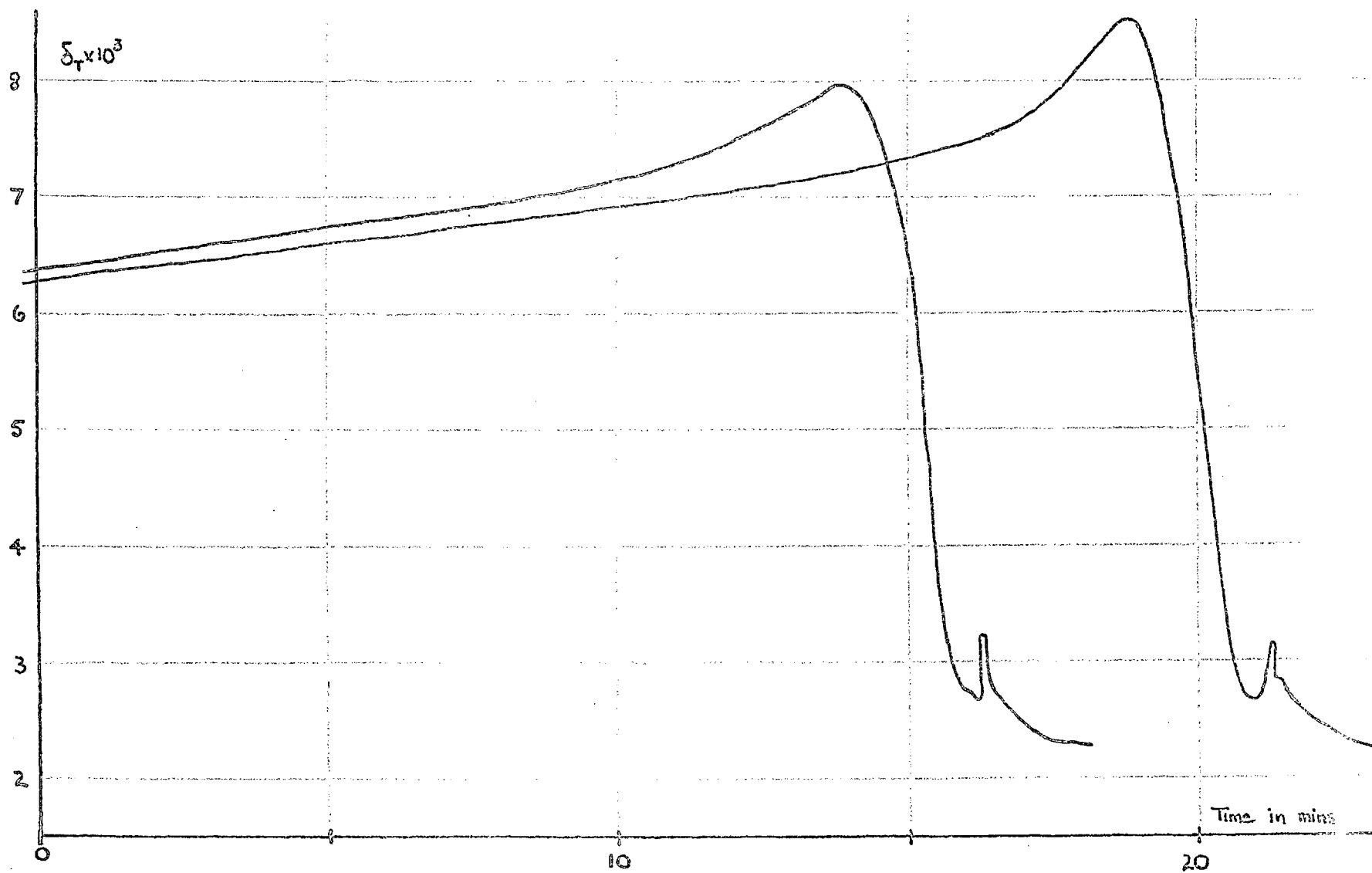


Fig.(20). Damping of vibrating strip against time of evaporation of liquid film.

volume of liquid remaining. This will cause a fairly sharp rise of viscosity of the film as it gets very thin.

If the theoretical curve for damping $\tau_0 \left(= R_0 / \left(\frac{\omega \eta}{2} \right)^{1/2} \right)$ shown in red in Fig.(21) is multiplied by the square root of the most likely viscosity values (also shown in red in this Figure), the black curve of damping so obtained is very similar to the practical results shown here as dotted circles. The increase at the end of the curves in Fig.(20) are interesting because these occur when the expected water-alcohol concentration reaches approximately 60%. The liquid would then be expected to break up into drops, and in fact this was observed. The increase in damping is due to sudden increase in thickness which more than compensates for the decrease which would be expected due to change in area.

This experiment does not directly confirm equation (1), but when due allowance is made for changing viscosity, the experimental results are in good agreement with the theory.

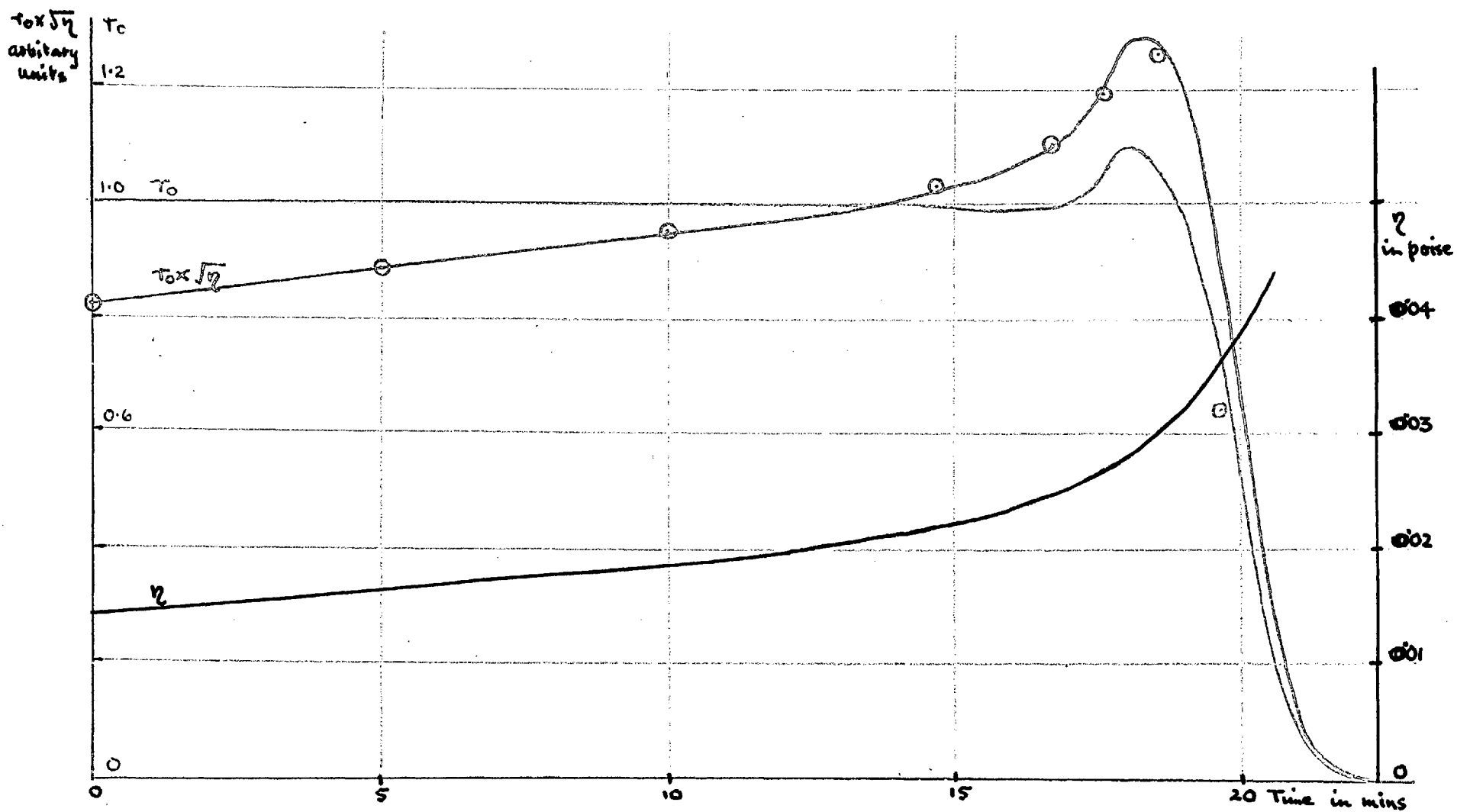


Fig.(21). Theoretical damping curve (black) for a liquid film of changing viscosity and thickness. Practical results shown as dotted circles.

V. 2. Liquid film bounded by solids.

By interposing a liquid film between the end of the beryllium-copper rod described earlier and the micrometer head illustrated in Fig.(22), it is possible to measure the damping of a liquid film, bounded by solid surfaces, as a function of thickness.

To perform the experiment, the internal damping of the rod is first determined. The micrometer head is levelled and raised to just make contact with the rod. The zero reading of the micrometer is read and the micrometer lowered. Liquid can now be forced into the gap by the screw piston arrangement. Damping measurements were taken for several thicknesses and the results plotted in Fig.(23). The circles and crosses correspond to two separate experiments. The corresponding theoretical curve obtained from the known viscosity and density of the liquid is also drawn. It can be seen that very good agreement has been obtained.

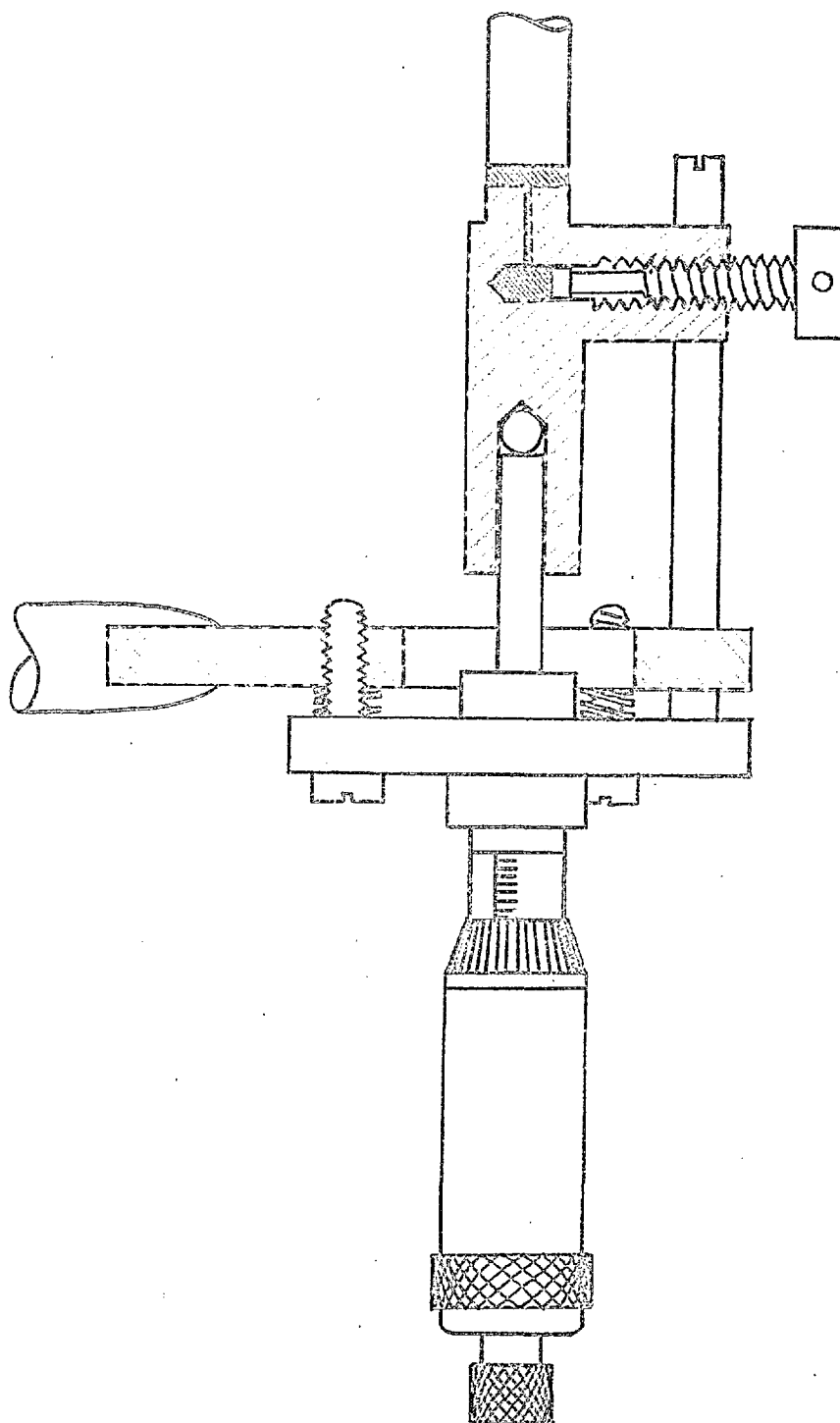


Fig.(22). Micrometer head for adjusting the thickness of a liquid film bounded by a solid surface.

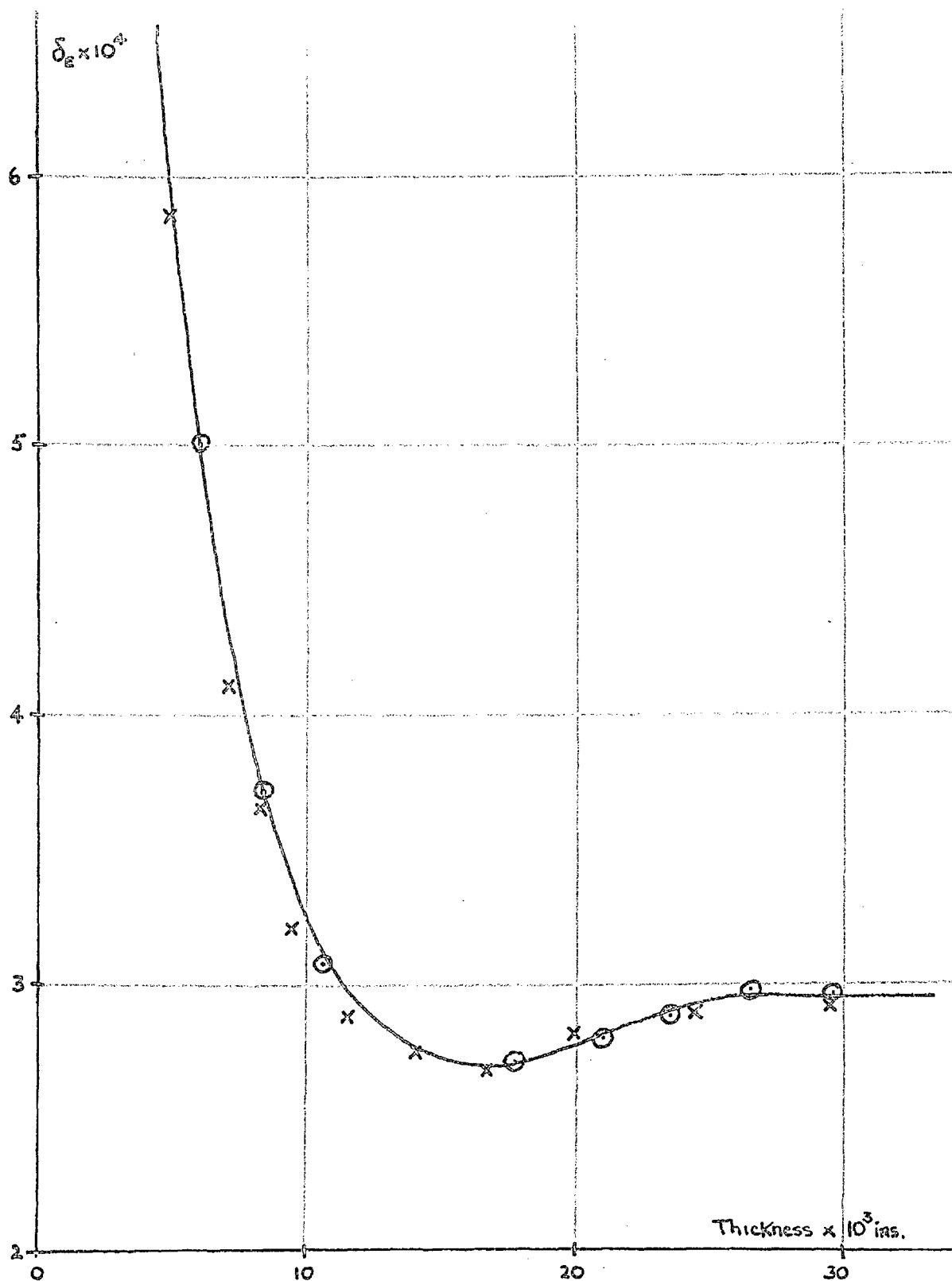


Fig.(23). Damping of a liquid film bounded by a solid with thickness.

VI. GAS DAMPING OF A ROD IN LONGITUDINAL OSCILLATION

In a number of methods used for investigating internal damping in solids, the specimen is in the form of a cylindrical rod supported at a node of vibration so that it is able to vibrate under free-free conditions. Under such conditions, external sources of damping are present in addition to that under study. These are: (a) viscous damping along the curved surface; (b) acoustic radiation from the vibrating surfaces; and (c) damping loss in the specimen support. A theoretical evaluation of these damping sources may be made without difficulty.

VI. 1. Theory.

The viscous damping along the curved surface of the rod can be evaluated using the method adopted for the torsionally vibrating rod. In this case the energy lost per cycle (ΔW_η) will be the same as equation (3) where $y = L$ and thus $\Delta W_\eta = R_0 (\dot{\xi}_{\max})^2 \frac{\pi^2 PL}{\omega}$

If the total energy of the rod is known, the corresponding logarithmic decrement δ_η can be determined.

The calculation of the damping effects due to the acoustic radiation from a vibrating rod is simplified by considering the end faces of the rod and the curved surface as two separate sources of radiation, and this condition may be reproduced by suitable baffles surrounding the ends of the rod.

The real component R_1 of the radiation impedance of a rigid circular piston of radius R , vibrating in a plane infinite baffle, has been obtained by Morse (1948) in the form

$$R_1 = \pi R^2 \rho c \left[1 - \frac{J_1(2kR)}{kR} \right]$$

where c is the velocity of sound in the medium of density ρ , $k = \omega/c$, and J_1 is the Bessel function of the first kind of order one.

Therefore, considering the circular ends of a vibrating cylindrical specimen as piston radiators, the energy dissipated in one cycle of vibration by these pistons is

$$\Delta W_{R_1} = R_1 (\dot{\xi}_0 \max)^2 \frac{2\pi}{\omega}$$

Hence, the decrement due to radiation from the ends of the rod δ_{R_1} can be found.

The acoustic energy radiated by the curved surface of a vibrating rod may be calculated approximately by treating it as short cylinders expanding and contracting normally and using Morse's equation for the energy radiated per unit length. The cylinder sections vibrate in phase so that the total radiated energy from the curved surface of the rod in each cycle is given by

$$\Delta W_{R_2} = \pi^3 \rho R^2 \int_{-1/2}^{1/2} u^2 dx$$

where u is the peak radial velocity which is related to the peak longitudinal velocity ($\dot{\xi}_0 \text{ max.}$) by the ratio $2\mu R/L$ where μ is the Poisson's ratio.

Thus

$$\Delta W_{R_2} = 2\pi^3 \rho R^4 \mu^2 (\dot{\xi}_0 \text{ max.})^2 / L$$

Hence, the decrement due to radiation of energy from the curved surface of the rod δ_{R_2} can be evaluated.

The remaining source of external damping is provided by the support for the vibrating rod. Various designs may be employed but a common one consists of a single wire attached at the node of longitudinal free-free vibration. Since the wire is fixed at a node, the only velocity amplitude acting on it is u , the peak radial velocity at the surface of the rod. The wire is considered to be small enough in cross-section to neglect variations in radial velocity across it and to ignore longitudinal velocities.

If it is assumed that all the energy taken from the specimen by the support is dissipated in the wire alone, and not reflected by the fixture from which the wire is hung, then the elastic energy ΔW_s absorbed in one cycle of vibration is given by

$$\Delta W_s = 2\pi(u^2 \rho_s c_s / \omega) \sigma$$

where ρ_s is the density of the suspension wire material,

c_s is the velocity of sound in the wire, and σ is the cross-sectional area of the wire.

The corresponding decrement introduced by the specimen support wire δ_s can thus be determined. The simplest way of reducing this decrement to as small a value as possible is to reduce the cross-sectional area of the suspending wire. This may be done in such a way that a more positive support is obtained by using three rigid needles with the points contacting the rod surface symmetrically in the plane of the node of longitudinal vibrations, instead of a single wire.

When the damping in a bar vibrating under free-free conditions is measured by an instrument which does not introduce further damping effects, the measured decrement is the sum of the decrements corresponding to internal damping in the bar δ_I and the external damping.

$$\delta_T = \delta_I + (\delta_\gamma + \delta_{R_1} + \delta_{R_2} + \delta_s) \text{ ————— } (5)$$

The magnitude of the external damping represented by the last four terms on the right-hand side of equation (5) is a function of certain characteristics of the specimen bar and surrounding medium. For a given size and material of

the specimen and for a given medium, the external damping is a function of the density of the medium.

The viscous damping δ_η is a function of $\sqrt{\rho}$ where ρ is the density of the medium, since the viscosity of the medium is unchanged by alteration of the pressure. Since the velocity of sound in the medium is also unchanged by change in pressure, provided that the mean free path of the molecules does not become comparable in magnitude to the wavelength of the sound wave, the radiation damping $(\delta_{R_1} + \delta_{R_2})$ is directly proportional to density. Hence equation (5) may be written

$$\delta_T = \delta_I + \delta_S + A\sqrt{\rho} + B\rho \text{ ————— (6)}$$

where A and B are constants for this system.

For specimens of a reasonable size, say 9 in. long and $\frac{1}{4}$ in. diameter, the values for decrements given in Table (2) are obtained for free-free oscillation in air at 0°C and 760 mm. of mercury pressure. The frequency of each specimen has been determined from known values of Young's modulus.

Table 2Typical values of damping

Material	Frequency (c/s)	$\delta_z \times 10^5$	$\delta_\theta \times 10^5$	$\delta_N \times 10^5$	$\delta_R \times 10^5$	$\delta_e \times 10^5$	$\delta_e/\delta_T \times 100\%$
Nickel	10420	73	0.30	0.72	0.011	0.50	2.1
Aluminium	11170	5	0.95	2.52	0.042	1.86	51.8

Considerable divergence exists between published values for internal damping due in many cases to the vibrational stresses applied being sufficient in magnitude to cause irreversible internal changes. The value tabulated for the internal damping of nickel obtained by Ide (1931) for annealed nickel at zero magnetic field and small vibrational stress is the smallest value known to the author for this metal. The internal damping value for aluminium was obtained by Hanstock & Murray (1946) using small vibrational stresses. The decrement due to suspension damping in Table (2) is for three taut steel wires of 0.5 mm. diameter fixed symmetrically in the plane of the node of vibration.

Table (2) shows that the maximum influence of external damping is exerted by the piston radiation and that support damping is the next in order of importance. The effect of external damping on an aluminium specimen is greater than that on a nickel specimen of the same size, owing to the relatively smaller total energy of vibration W at the same peak axial velocity of the free ends of the rods, as well as the lower internal damping.

The support damping may be easily reduced by replacing the 0.5 mm. diameter wires by steel needles, each having a much smaller contact area with the specimen. Both radiation damping decrements are linear functions of the density of the surrounding atmosphere and thus may be reduced by a factor of 1/100 by vibrating the specimen in an enclosure at 7.6 mm. of mercury. This would also reduce the viscous damping by a factor of 1/10. The improved suspension and reduced pressure enclosure reduce the ratio of external to total measured damping to approximately 0.1% for the nickel and 3% for the aluminium specimen.

Where values of the internal damping of a material are required, it is necessary to reduce the influence of the external sources of damping as much as possible, but if it is intended to measure external damping as a check on the equations given here for the logarithmic decrements of external damping, then the opposite process is necessary.

VI. 2. Experimental.

In order to measure the external damping on a vibrating rod due to the surrounding gaseous medium, it is necessary to reduce the internal damping and the external damping introduced by the specimen suspension as much as possible. Further, since the damping due to surrounding gas will be relatively small compared with the total damping effect from other sources, it is necessary to employ a very accurate measuring technique.

Thus, needle supports were used and Monel metal was chosen for the rod material since this metal was found by Ide (1931) to have an internal friction decrement of 3×10^{-4} ratio in zero magnetic field, which was small compared with his results for other magnetic materials.

A further increase in the surface damping relative to internal damping can be obtained by using a tube with plugged ends instead of a solid rod. This has the effect of reducing W , the total energy of vibration on the specimen, and hence all the decrements for external damping (including suspension) are increased. There is a limit to the enhancement of the surface damping effects

by reducing the value of W . The air enclosed in the tube will exert a damping effect on the vibrations, which is considerably smaller than that of the same volume of metal. However, W is evaluated from the vibrational energy contained in the solid tube so that the tube walls must not be so thin that the energy in the enclosed air is comparable in magnitude to that in the tube alone.

A magnetic material was chosen for the specimen rod because of the ease with which vibrations can be produced by magnetostrictive effects using a coaxial drive coil.

The electronic apparatus previously described gave a constant range of three cycles in any set of consecutive readings, whatever the value of N , with a standard deviation of 0.87. Hence, for this experiment in which $N > 1500$ cycles, the expected error in N evaluated from ten consecutive readings was less than $\pm 0.67 \times \frac{100 \cdot 0.87}{1500 \sqrt{10-1}} \%$ that is, less than $\pm 0.015\%$

A Monel metal tube 9 in. long and $\frac{1}{8}$ in. overall diameter was used in the experiment. The tube wall thickness was 0.0055 in. and the ends of the tube were sealed by Perspex disks 0.05 in. thick. The tube was supported at its centre by three needles symmetrically fixed and mounted in a glass enclosure as illustrated in Fig.(24).

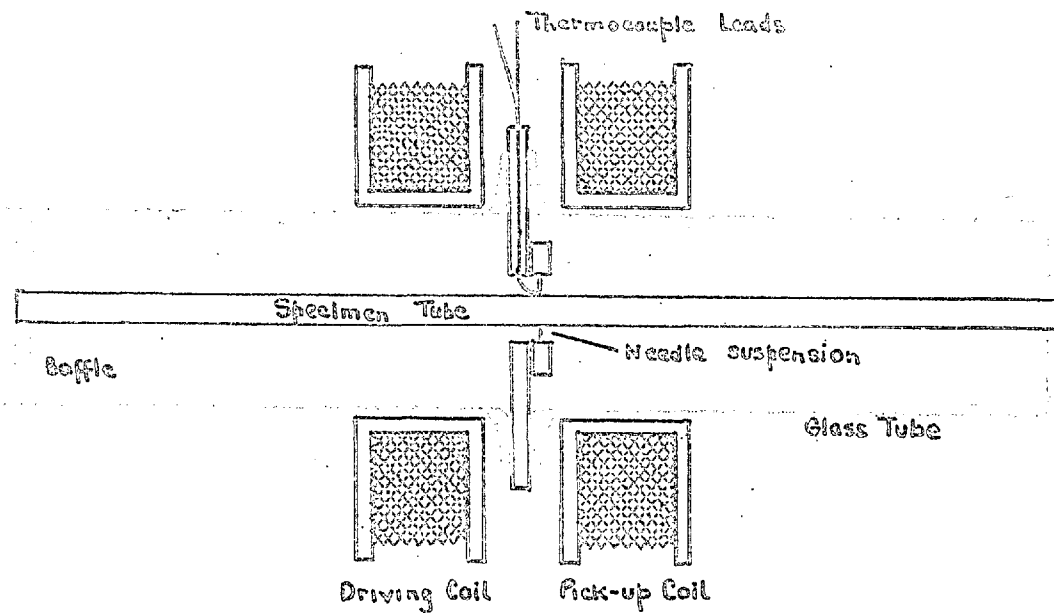


Fig.(24). Apparatus used to measure damping of monel metal tube by gases.

The tube ends were located in the central holes of two brass disks of $\frac{1}{8}$ in. thickness fixed to the glass enclosure. These disks acted as baffles to separate the piston radiation region from the cylindrical radiation region. A small copper constantan thermocouple junction was fixed to the tube surface near to the node of fundamental longitudinal vibration, the leads being carried out through holes in the Perspex support which were subsequently sealed. A further thermocouple attached to the glass enclosure measured the ambient temperature at which the experiments were performed.

The specimen support ring was symmetrically perforated so that the two flanged tubes forming a vacuum-tight enclosure could be evacuated by a pump connected to one of the inlet ports at either end. The enclosure was pumped out and filled from a pressure cylinder with the gas under investigation. Liquid oxygen traps were used to ensure dry gases and the pressures inside the enclosure were measured by a mercury manometer and a McLeod low-pressure gauge.

The enclosure walls were lined with glass wool to reduce acoustic coupling between the vibrating specimen and the enclosure. The usual system of feed back loop to excite the specimen was not found practical in this case due to the large direct coupling between the driving and pick up coils. Instead, a decade oscillator was used to drive the specimen. The electronic measuring instrument connected to the pick-up coil was brought into action by a switch in the driver coil circuit cutting off the oscillator output. The decaying signal output from the pick-up coil initiated a count of the number of cycles for a half amplitude decay which always started from the same signal amplitude. Thus, provided that the magnetostrictive eddy-current effects producing the signals always occur under the same conditions, measurements will always be taken for the same amplitude range of mechanical vibration of the given specimen. In this way, the effects of amplitude-dependent damping are minimized.

The specimen tube was placed between the poles of an electromagnet which provided a polarizing magnetic

field corresponding to maximum magnetic damping in the specimen. The maximum damping, rather than the minimum, was used in order to work on a position of stability in the damping/magnetic induction curve for the specimen material.

In a subsidiary experiment, the temperature of the tube and the external temperature were recorded for a normal run of this apparatus. The results obtained are plotted in Fig.(25). It can be seen quite clearly how the temperature of the tube falls at each pressure decrease and then rises to a value which tends to follow the external temperature. The time to reach this value is of the order of 3 minutes. A corresponding graph of the damping of this specimen is also given, Fig.(26), showing how the count N rises to a steady value, after each pressure change. The procedure adopted to obtain the results which follow was to change the pressure slowly and then allow three minutes before the damping measurements were taken. As the magnetic properties of Monel metal are strongly influenced by temperature change at room temperature, it was necessary to carry out experiments when the ambient temperature was not changing.

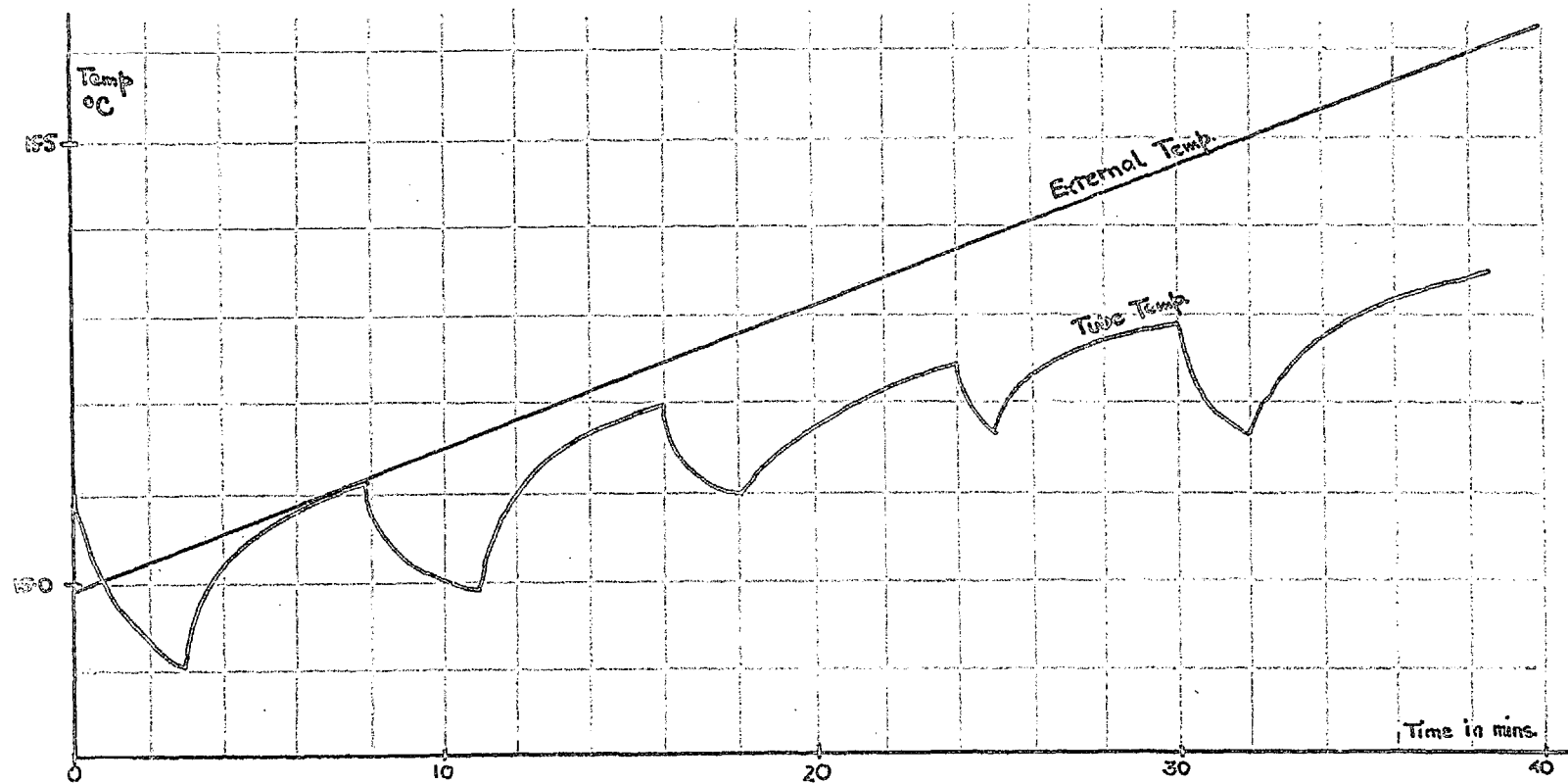


Fig. (25). Variation of tube temperature during normal run of apparatus.

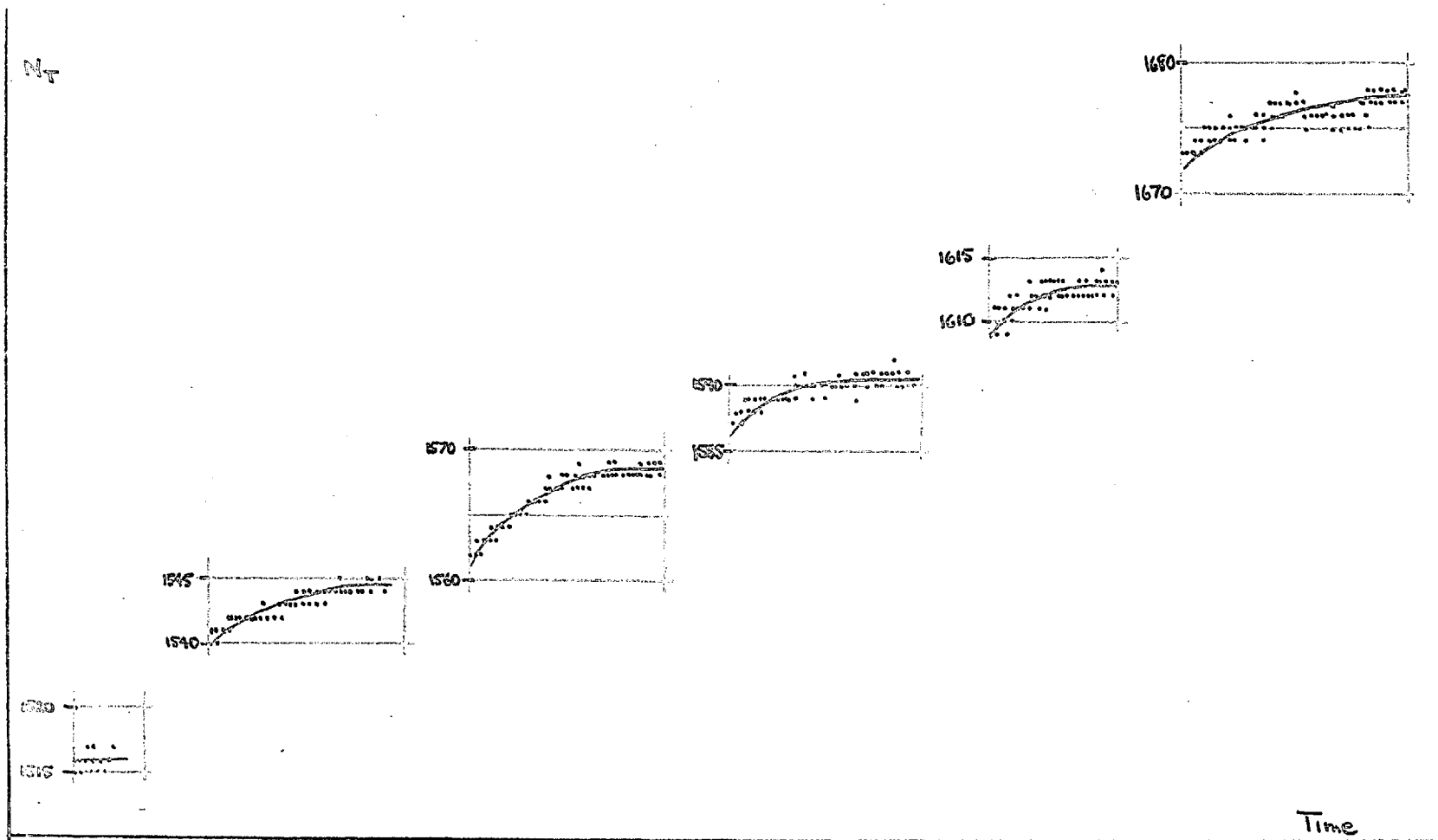


Fig.(26). Variation of damping during normal run of apparatus.

For an open tube of external diameter $2R$, length L , wall thickness t , vibrating with a peak longitudinal velocity $(\dot{\xi}_0 \text{ max.})$

$$W = \pi(2R - t)t L \rho_m (\dot{\xi}_0 \text{ max.})^2$$

If this tube is plugged at each end by thin plane plugs of mass M each, the resulting addition to W is $M(\dot{\xi}_0 \text{ max.})^2$. To ^{minimise} ~~avoid~~ errors in the evaluation of $2W/(\dot{\xi}_0 \text{ max.})^2$ due to the difficulty in measuring t accurately, $2W/(\dot{\xi}_0 \text{ max.})^2$ may be obtained by subtracting one-half of the mass of the tube, measured before plugging, from the total mass of the tube and plugs after fixing.

For the tube used in the experiments described $2W/(\dot{\xi}_0 \text{ max.})^2 = 1.348 \text{ g.}$, $f = 9777 \text{ c/s}$, $L = 22.82 \text{ cm.}$, $R = 0.159 \text{ cm.}$, and $\mu = 0.326$.

Hence, in equation (6)

$$A = 0.0758 \sqrt{\eta} \text{ c.g.s.}$$

$$B = \left\{ 6.007 \left[1 - \frac{J_1(kR)}{kR} \right] c + 135.8 \right\} \times 10^{-6} \text{ cgs. — (7)}$$

If ρ_0 is the density of the gas surrounding the specimen at room temperature and pressure p_0 , then at any subsequent pressure p

$$\delta_T = \delta_I + \delta_S + A\sqrt{\rho_0/p_0}\sqrt{p} + B(\rho_0/p_0)p$$

A simpler relation between δ_T and p may be obtained.

If δ_T' is the measured decrement at a given pressure p_1 , then for any other pressure p

$$\delta_T - \delta_T' = A\sqrt{\rho_0/p_0}[\sqrt{p} - \sqrt{p_1}] + B\rho_0/p_0[p - p_1]$$

or

$$\frac{\delta_T - \delta_T'}{\sqrt{p} - \sqrt{p_1}} = A\sqrt{\rho_0/p_0} + B\rho_0/p_0[\sqrt{p} + \sqrt{p_1}] \quad \text{--- (8)}$$

This equation gives a linear relationship between a function of δ_T and p and a function of p

$$f(\delta_T, p) = A\sqrt{\rho_0/p_0} + B\rho_0/p_0 \phi(p) \quad \text{--- (9)}$$

VI. 3. Results

Six different pressures were used for the four gases studied and at each pressure ten half-decay readings were taken. The mean values of N and δ_T are given in Table (3).

Taking p_1 as 134 mm. of mercury and also $p_1 \ll 1 \text{ mm.}$ in equation (8), nine values of the functions $f(\delta_T p)$ and $\phi(p)$ were obtained in equation (9) for each gas. These data are plotted in Fig.(27) and the slopes and intercepts evaluated, giving the following constants (Table 4).

The theoretical values of B obtained by the substitution of known data in equation (7) are consistently high compared with the experimental determinations given in Table (4), and also appear to vary in an unsystematic fashion.

Substituting for A and inserting the values given for p_0 by Kaye & Laby (1948) at a pressure p_0 of 760 mm. of mercury, values for absolute viscosity η are derived. These data are given in Table (5) together with the comparative values obtained by interpolation from the smoothed figures published by Johnson & McCloskey (1940), using the oscillating disk method.

Table 3 Damping measurements in various gases

	Mean half-decay N cycles				Measured damping $\delta_T \times 10^7$			
	Nitrogen at 17.9°C	Oxygen at 17.5°C	Air at 16.6°C	Carbon dioxide at 18.1°C	Nitrogen at 17.9°C	Oxygen at 17.5°C	Air at 16.6°C	Carbon dioxid at 18.1°C
Pressure p (mmHg)								
1	1702.1	1684.7	1687.3	1759.2	4072.3	4114.4	4180.0	3940.1
134	1641.6	1616.9	1625.5	1684.3	4222.4	4286.9	4264.1	4115.4
230	1622.6	1595.8	1606.3	1660.8	4271.9	4343.6	4315.2	4173.6
348	1603.9	1575.4	1587.6	1637.7	4321.7	4399.8	4366.0	4298.1
504	1583.8	1553.3	1567.3	1612.7	4376.5	4462.4	4422.6	4298.1
760	1556.1	1523.7	1540.0	1578.6	4454.4	4549.1	4501.0	4390.9

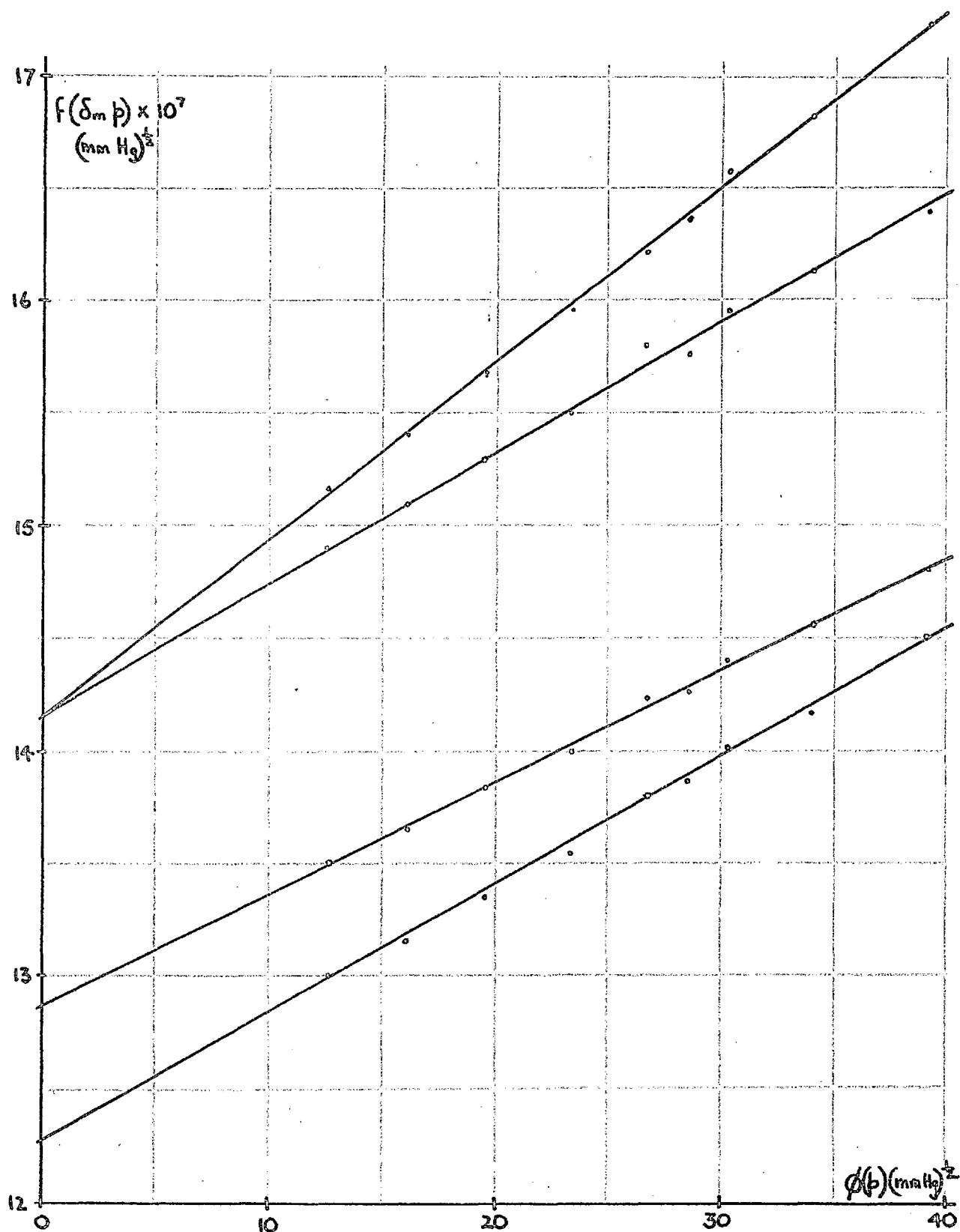


Fig.(27). Relation between damping and pressure.

Table 4Values of damping constants

	Experimental determinations		Theoretical value
	$A/(P/P_0) \times 10^7$ (mmHg) ^{-1/2}	$B(P/P_0) \times 10^6$ (mmHg) ⁻¹	$B(P/P_0) \times 10^6$ (mmHg) ⁻¹
Nitrogen	12.27	0.554	1.25
Oxygen	14.15	0.581	1.55
Air	12.87	0.494	1.34
Carbon dioxide	14.15	0.787	3.57

Table 5The absolute viscosity of various gases

	Absolute viscosity $\eta \times 10^6$ poise			
	Nitrogen at 18° C	Oxygen at 18° C	Air at 17° C	Carbon dioxide at 18° C
Present work	170	198	178	143
Johnston & McCloskey	176	202	180	145

VI. 4. Discussion of results.

The viscosity values obtained are in good agreement with published values and it is concluded that with certain experimental refinements, such as using a vibrator material which is less influenced by temperature change and placing the gas-filled enclosure in a temperature-controlled bath, a precision method for the measurement of gas viscosities could be developed, using the principles described.

A theoretical treatment of the radiation damping would need to be far more complex than that attempted here. For the simple apparatus described, the assumptions, of infinite baffles with pistons radiating into free space uncoupled to other parts of the vibrating specimen or the rest of the apparatus, are obviously inapplicable. However, the plotted curves demonstrate that the radiation damping is proportional to gas density and does not interfere with viscous damping. It is possible that the radiation damping effect exerted by various structures could be investigated using apparatus based on that described.

VII. TRAVELLING WAVE SYSTEM

If a longitudinal travelling wave is propagated along a wire, a length of which is immersed in a fluid medium, reflection will occur at the fluid boundary and the wave will be heavily attenuated in the loaded part (henceforth referred to as the 'loaded' section of the wire).

Standing waves will be formed in the free part of the wire and by measuring the standing wave ratio the acoustic input impedance of the loading fluid can be obtained. Similar information can be obtained by measuring the attenuation of the wave in the loaded wire.

VII. 1. Theory.

To derive the wave equation for a longitudinal wave travelling along a wire immersed in a fluid medium, we consider the resultant force acting on a small element of the wire. If X_x is the stress in the x direction at one end of the element, and $X_{x+\delta x}$ is the stress at the other, then the stress on the element

$$X_{x+\delta x} - X_x = \frac{\partial X_x}{\partial x} \delta x$$

The other force acting on the element is that due to the fluid medium. If Y_x is the stress exerted on

the element by the fluid medium, the resultant force on the element is

$$\pi a^2 \frac{\partial X_x}{\partial x} \delta x - 2\pi a \delta x Y_x$$

where a is radius of wire.

But

$$X_x = E \frac{\partial \xi}{\partial x} \quad \text{and} \quad Y_x = Z_o \frac{\partial \xi}{\partial t}$$

where ξ is the particle displacement and E is a modulus of elasticity and $Z_o = (R_o + iX_o)$ is the complex input impedance of the fluid medium.

This resultant force causes the element to accelerate, and we can write down the equation of motion of the element δx

$$\pi a^2 E \frac{\partial^2 \xi}{\partial x^2} \delta x - 2\pi a \delta x Z_o \frac{\partial \xi}{\partial t} = \pi a^2 \delta x \rho_m \frac{\partial^2 \xi}{\partial t^2}$$

This equation ignores any energy loss in the wire itself;

or

$$\frac{\partial^2 \xi}{\partial t^2} + \omega(h + ih') \frac{\partial \xi}{\partial t} = c^2 \frac{\partial^2 \xi}{\partial x^2} \quad \text{—————} \quad (10)$$

where $h = \frac{2R_o}{\omega a \rho_m}$; $h' = \frac{2X_o}{\omega a \rho_m}$; $c^2 = \frac{E}{\rho_m}$; ρ_m = density of wire

Assuming the particle displacement is of the form

$$\xi = \xi_0 e^{i\omega t} e^{-(\alpha + i\beta)x}$$

and substituting in the wave equation to evaluate α and β , the attenuation and phase constants, we obtain

$$-\omega^2 + i\omega^2(h + ih') = c^2(\alpha + i\beta)^2$$

or

$$\{i\omega^2 h - \omega^2(1 + h')\}^{1/2} = c(\alpha + i\beta)$$

Expanding the square root term into real and imaginary parts we find

$$\alpha = \frac{1}{\sqrt{2}} \frac{\omega}{c} \left\{ [(1 + h')^2 + h^2]^{1/2} - (1 + h') \right\}^{1/2}$$

$$\beta = \frac{1}{\sqrt{2}} \frac{\omega}{c} \left\{ [(1 + h')^2 + h^2]^{1/2} + (1 + h') \right\}^{1/2}$$

The attenuation constant α is proportional to the measured attenuation in the wire, being = $\frac{20\alpha}{\log_e 10} \text{ db cm}^{-1}$

It can be seen that when no fluid medium is present, i.e. $h = h' = 0$, the attenuation is zero, as is to be expected if no energy is lost in the wire.

When the fluid medium is absent, the phase constant $\beta = \omega/c$ and so the velocity $v = c$, the undamped longitudinal wave velocity. For a finite value of input impedance Z_0 , $\beta > \omega/c$ and thus $v < c$

In order to calculate the reflection coefficient of the boundary between the loaded and unloaded wire, we proceed as follows. The stress at any point in the wire is

$S_x = \int \rho_m \frac{\partial^2 \xi}{\partial t^2} dx$ which is evaluated by using the solution of equation (10). After substituting for ξ we obtain

$$\begin{aligned} S_x &= \int -\rho_m \omega^2 \xi_0 e^{i\omega t} e^{-(\alpha+i\beta)x} dx \\ &= \frac{-\rho_m \omega^2}{-(\alpha+i\beta)} \xi_0 e^{i\omega t} e^{-(\alpha+i\beta)x} + \text{const} \end{aligned}$$

Hence the stress amplitude = $\frac{\rho_m \omega^2}{(\alpha+i\beta)} \xi_0$

The volume displacement amplitude is equal to the product of the particle velocity amplitude and the cross sectional area of the wire,

$$\text{i.e. volume displacement amplitude} = i\omega\pi a^2\xi_0$$

Hence the characteristic impedance of the wire

$$Z_2 \left(\frac{\text{stress amplitude}}{\text{volume displacement}} \right) = \frac{-p_m\omega}{(\beta - i\alpha)\pi a^2}$$

In the special case of an unloaded wire $\alpha = 0$ and $\beta = \omega/c$ and the impedance becomes $Z_1 = \frac{-p_m\omega}{\pi a^2}$

The modulus of the complex reflection coefficient P as defined by $\frac{Z_2 - Z_1}{Z_2 + Z_1}$ is given by

$$|P| = \frac{\{[(\omega/c)^2 - \beta^2 - \alpha^2]^2 + 4\alpha^2(\omega/c)^2\}^{1/2}}{(\omega/c + \beta)^2 + \alpha^2}$$

$$\text{and the phase angle } \phi = \tan^{-1} \frac{2\alpha\omega/c}{(\omega/c)^2 - \beta^2 - \alpha^2}$$

The standing wave ratio (db) produced by this reflection = $20 \log_{10} \frac{1+|P|}{1-|P|}$

Thus, two measurable quantities, the attenuation per wavelength and the standing wave ratio, can be expressed in terms of α and β , which in turn depend on h and h' , which are proportional to the real and imaginary parts of the input impedance of the fluid medium (see page 24).

$$h = \frac{2\rho}{m\rho_m} \frac{U_0 U_1 + V_0 V_1}{U_0^2 + V_0^2}$$

$$h' = \frac{2\rho}{m\rho_m} \frac{U_0 V_1 - U_1 V_0}{U_0^2 + V_0^2}$$

In order to evaluate h and h' as a function of m , a value of ρ/ρ_m , the ratio of the density of the fluid medium to that of the wire material, must be assumed. A suitable value for most liquids using a nickel wire would be $1/8$, and thus this value was used.

The attenuation per wavelength and the standing wave ratio have been plotted in Fig.(28) as functions of m which is proportional to $(\omega/\eta)^{1/2}$

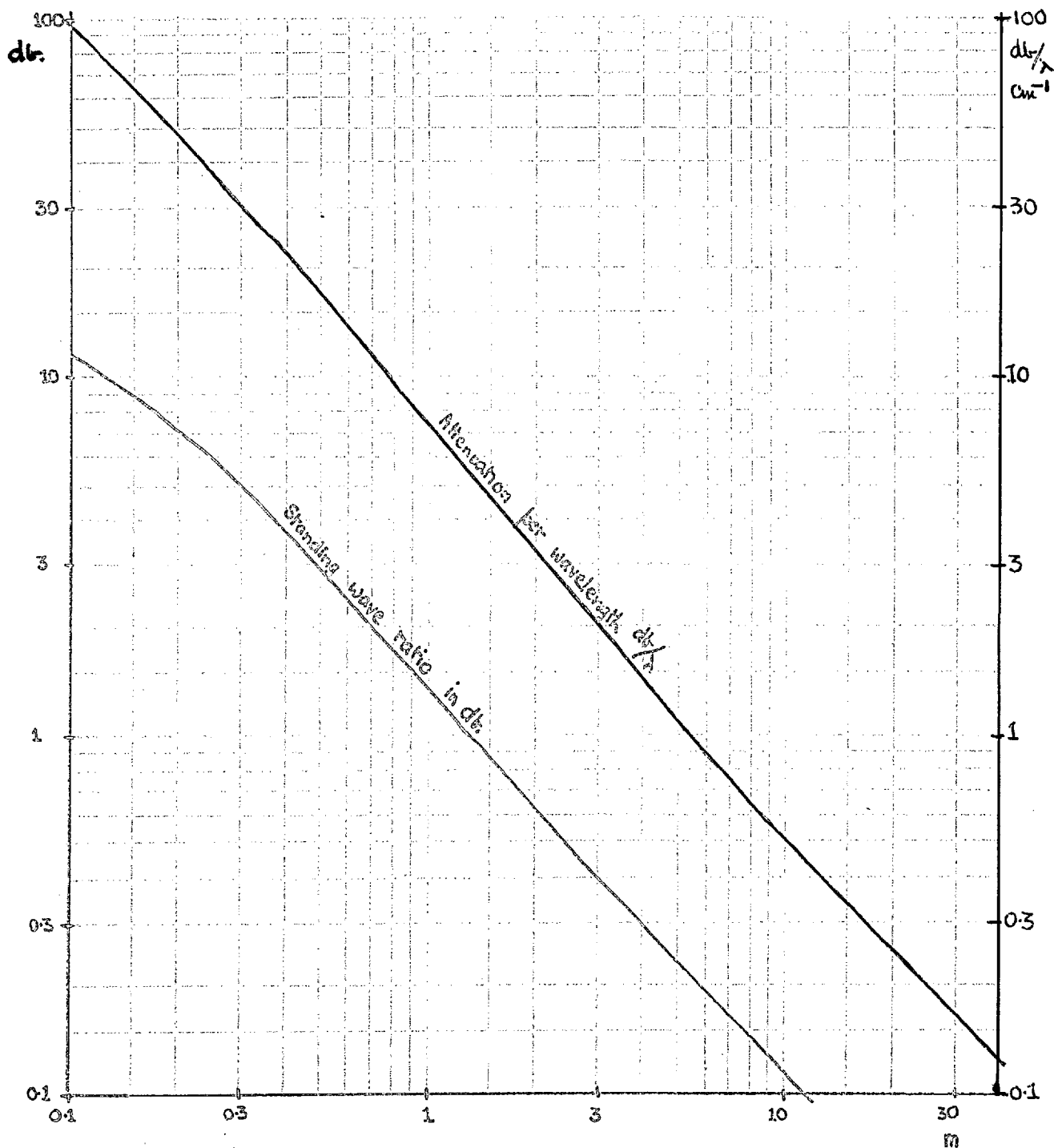


Fig. (28). Attenuation per wave length and standing wave ratio with m for a wave propagated along a wire partially immersed in a viscous medium.

VII. 2. Experimental.

The upper end of a wire is attached to a transducer to excite in it longitudinal vibrations while a mass is attached to the lower end to keep the wire straight and in a vertical position [see Fig.(29)]. The wire passes co-axially through a glass tube which contains the fluid under investigation and also allows the detecting coil to travel freely along the loaded and unloaded lengths of the wire. Magnetostriction is used as the method of detecting the vibrations in the wire, and because of this nickel wire is chosen. A polarizing field coil is wound on the same bobbin as the detecting coil, thus ensuring that the polarizing field at the position in the wire being measured is always the same, as the coil moves along. The current in the field coil is adjusted to give a maximum signal in the detecting coil. The output from the detecting coil is fed to a Bruel & Kjer level recorder (fitted with a logarithmic potentiometer). The 'paper drive' of this is coupled by a flexible drive to a 'winch' which moves the detecting coil. A trace as illustrated in Fig.(30a) can therefore be obtained of signal level against distance along the wire and from this the standing wave ratio and attenuation can easily be read off.

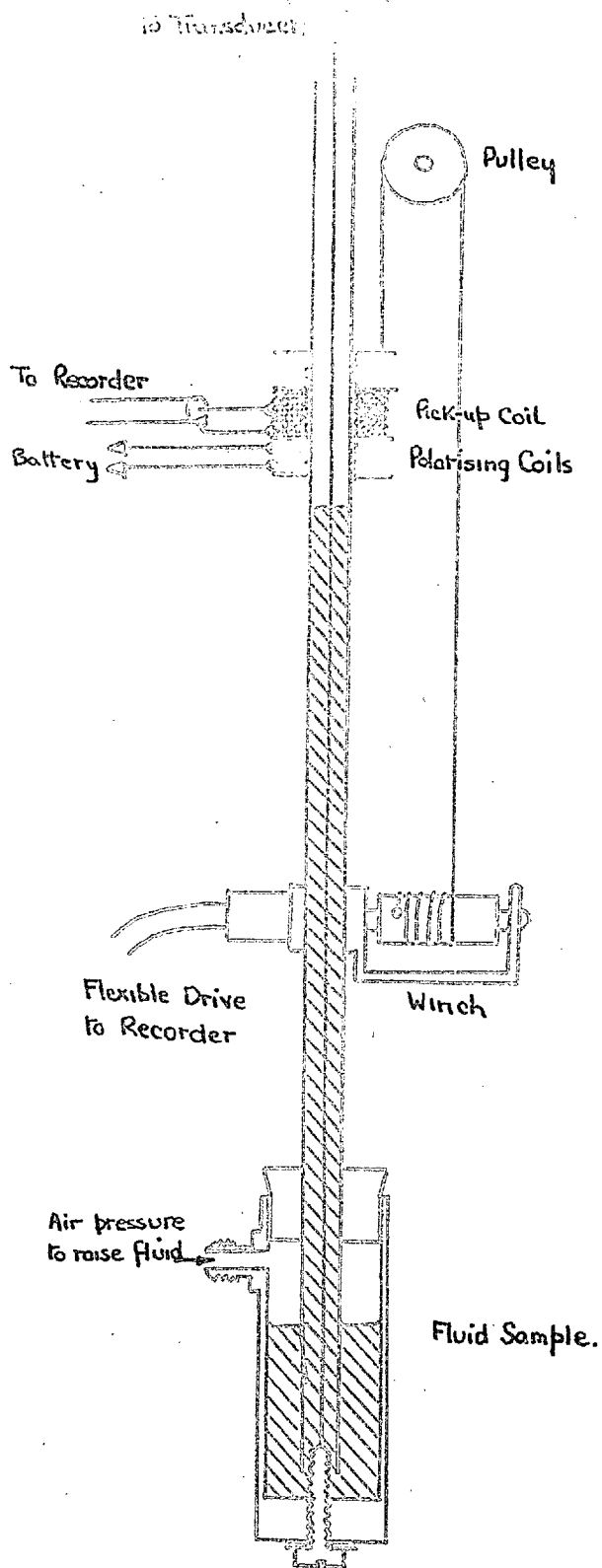
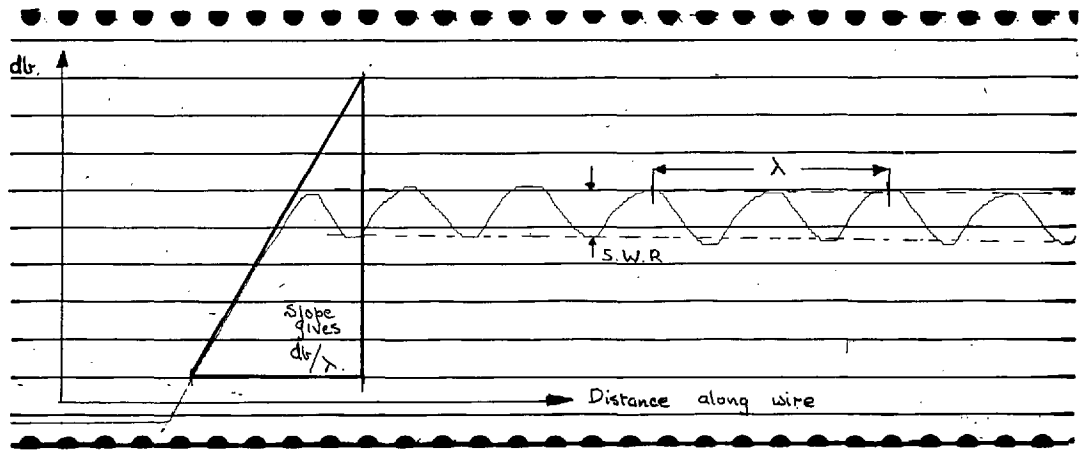


Fig. (29). Travelling Wave System.

(a)



(b)

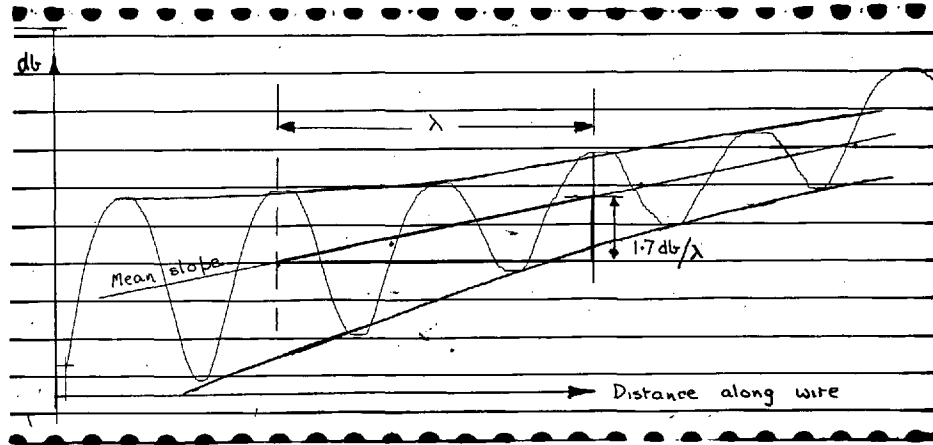


Fig (30) Level Recorder Trace

In order for the signal strength to be proportional to the vibration amplitude at all points along the wire, the wire must be carefully demagnetised and given annealing treatment before use. This is done by passing an A.C. heating current through the wire and raising its temperature to red heat. The wire should now be slowly cooled but it was found that if this was done a periodic non-uniformity of signal output arose. The same results were obtained when a D.C. heating current was used. Uniformity of signal output was obtained by suddenly switching off the heating current.

Owing to the relatively large size of the detecting coil this will not apparently give a signal proportional to the amplitude at a point in the wire but rather a mean value of the amplitude along a certain length. When the coil is used to measure the standing wave ratio, for example, it might be expected that the peaks would be flattened and an incorrect ratio obtained. The signal from the coil will be proportional to the integrated product of a coil 'function' and the wave displacement in the wire. This coil function using the reciprocity

principle will depend on the distribution of magnetic field along the axis of the coil if it were carrying a current. The approximate form of this function is that of a single turn of radius r given by

$$\frac{r^2}{\{r^2 + (x - d)^2\}^{3/2}}$$

where x is the distance along the axis of the coil from any reference zero and d the value of x at the centre of the coil. To justify this approximation, the distribution of field was investigated with a 'search coil' and it was found possible to fit this curve to the plotted function as shown in Fig.(31a). The value of r so obtained agreed (approximately) with the estimated mean value of the radius of the coil.

Firstly, let us consider the coil response in the standing wave section of the wire. Here the wave equation can be expressed as $A \sin(\omega t - \beta x) + B \sin(\omega t + \beta x)$ or $(A + B) \sin \omega t \cos \beta x + (A - B) \cos \omega t \sin \beta x$

where A and B are the incident and reflected wave amplitudes.

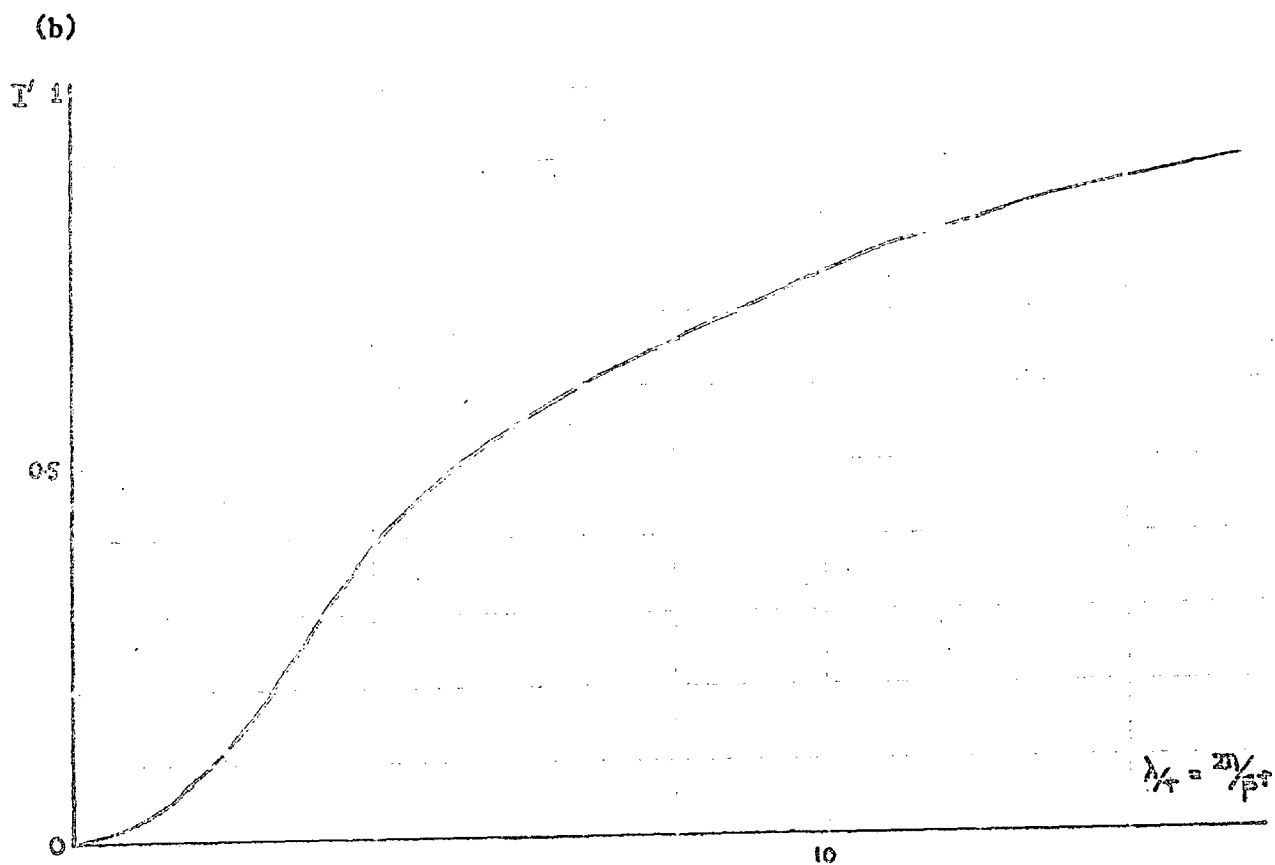
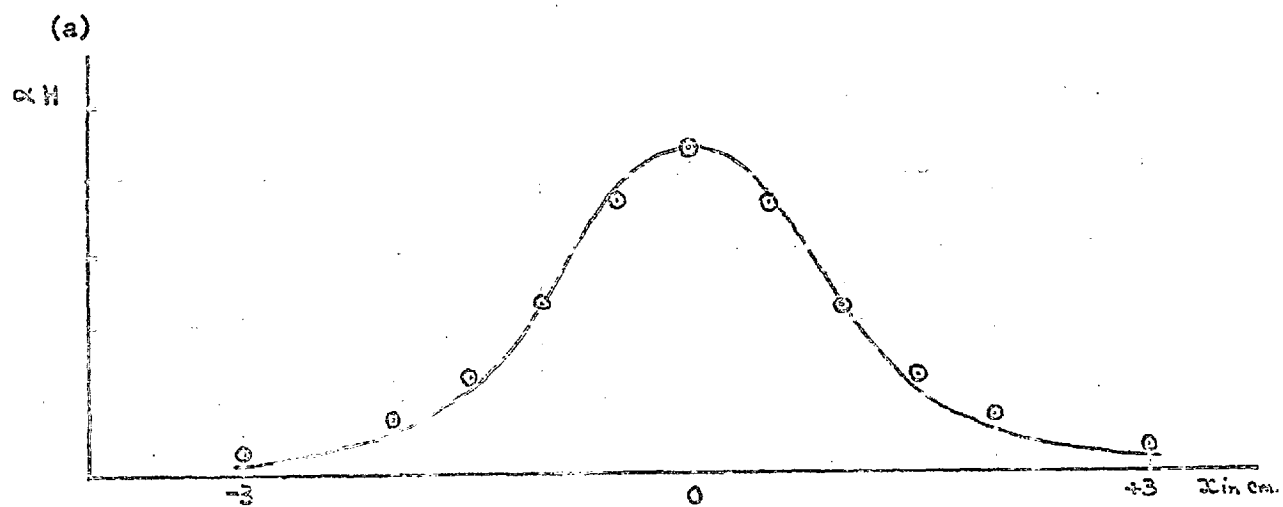


Fig.(31). (a) Field distribution along axis of coil.

(b)
$$I_1 = \int_{-\infty}^{\infty} \frac{r^2}{(r^2 + x^2)^{3/2}} \cos \beta x \, dx \quad \text{with } \lambda/r$$

The coil output (I_s) when its centre is situated at a position d will be given by

$$I_s \propto \int_{-\infty}^{\infty} [(A+B) \sin \omega t \cos \beta x + (A-B) \cos \omega t \sin \beta x] \\ \times \frac{r^2}{[r^2 + (x-d)^2]}^{3/2} dx$$

Let $X = x - d \quad \therefore dX = dx$

By changing the variable to $X = x - d$ we situate the coil at $X = 0$ and effectively move the standing wave pattern along the wire.

Therefore

$$I_s \propto \int_{-\infty}^{\infty} [(A+B) \sin \omega t \cos(\beta X + \beta d) \\ + (A-B) \cos \omega t \sin(\beta X + \beta d)] f(X) dX$$

$$\therefore I_s \propto [(A+B) \sin \omega t \cos \beta d + (A-B) \cos \omega t \sin \beta d] I_1 \\ - [(A-B) \sin \omega t \sin \beta d - (A+B) \cos \omega t \cos \beta d] I_2.$$

where
$$f(X) = \frac{r^2}{(r^2 + X^2)^{3/2}}$$

$$I_1 = \int_{-\infty}^{\infty} f(X) \cos pX dX$$

$$I_2 = \int_{-\infty}^{\infty} f(X) \sin pX dX$$

But I_2 is an odd function of X and therefore equals zero.

Thus

$$I_2 \propto [(A+B) \sin \omega t \cos pd + (A-B) \cos \omega t \sin pd] I_1$$

and as I_1 is independent of d and only depends on r , the coil output has exactly the same form as the original standing wave. It is seen from the above equation that the strength of the output signal depends on the value of I_1 .

$$I_1 = \int_{-\infty}^{\infty} \frac{r^2}{(r^2 + X^2)^{3/2}} \cos pX dX$$

To evaluate this we let $X = r \tan u$ and on substitution

$$I_1 = \int_{-\pi/2}^{\pi/2} \cos(\beta r \tan u) \cos u \, du$$

This has been evaluated approximately by a graphical method and is plotted in Fig.(31b) as a function of $\lambda/r = (2\pi/\beta r)$

It is seen from this that even for a coil where $r = \lambda/4$ still nearly half the maximum signal is obtained. This consideration sets an upper limit to the frequency (f) employed with this apparatus. $f = c/\lambda$ and as $r \approx 1$ cm. the maximum wavelength will be ≈ 4 cm. and thus $f \approx 125$ Kc/s.

The coil response (I_L) in the loaded part of the wire is given by

$$I_L \propto \int_{-\infty}^{\infty} A e^{-\alpha x} \sin(\omega t - \beta x) \frac{r^2}{\{r^2 + (x-d)^2\}^{3/2}} dx$$

As before let $X = (x-d)$

$$\text{Therefore } I_L \propto A e^{-\alpha d} \cos(\omega t - \beta d) I_3 + A e^{-\alpha d} \sin(\omega t - \beta d) I_4$$

where

$$I_3 = \int_{-\infty}^{\infty} f(X) e^{-\alpha X} \sin \beta X dX$$

$$I_4 = \int_{-\infty}^{\infty} f(X) e^{-\alpha X} \cos \beta X dX$$

Both of these integrals are independent of d the position of the coil and thus the coil output has the same exponential component as the wave in the wire.

VII. 3. Results.

For the experimental test of the theory, several viscous fluids were selected covering a range of viscosities of approximately 5 to 1000 poise as determined by unidirectional flow methods. Very viscous fluids often have a much lower viscosity when measured by an alternating stress method - see, for example, Barlow & Lamb's (1959) work on the Silicone Fluids MS 200 series. It was decided to use the Silicone Fluids to see if an agreement between theoretical and experimental results could be obtained using Barlow's values for viscosity.

The author could not find any published information of this kind regarding the other fluids chosen. However, one could expect the damping to be lower than that predicted by using the known undirectional viscosity and relevant formulae. It therefore seemed better to present the results as determination of viscosity at certain frequencies, although the apparatus is not claimed to be a viscometer.

The frequency range covered by the apparatus was limited by the Transducer selected. This would only operate in the range 13.6 - 27.2 Kc/s but it was felt that this frequency variation was sufficient to test the frequency dependence shown in the theory.

The experimental results obtained for the different fluids are tabulated in Table (6). For each fluid the attenuation per wavelength was measured at three frequencies, from the level recorder traces. It was only possible to evaluate the standing wave ratio for fluids which had sufficient attenuation per wavelength to reduce the wave, which is reflected from the end, to a negligible amplitude, at the boundary between the 'loaded' and 'unloaded' sections of the wire. In general this requires a viscosity of

TABLE 6

1. Material 2. Temp. °C 3. Undirectional Viscosity	Frequency kc/s	Attenuation db/λ	m	(ω/m²) x 10 ⁻⁴	Mean (ω/m²) Viscosity	Measured S.W.R.	Calculated S.W.R.
Venetian Turp.	13.7	48	.2	215	219	6.5	7.5
21°C	19.6	39.7	.24	214	1120 poise	5.5	6.5
1300 poise (17.3°C)	27.4	34.3	.275	229		4.5	5.5
950 Oil	15.0	2.0	3	1.05	1.03		
22°C	20.0	1.6	3.6	0.97	5.0 poise		
4.1 poise (21°C)	27.2	1.4	4	1.06			
Fluxed Bitumen	13.6	20.6	.42	48.5	50	5	3.6
19°C	20.0	17.3	.49	52.5		3.5	3
1080 poise (19°C)	27.2	13.8	.59	49	250 poise	3	2.5
Shell Linea 81	13.6	5.45	1.25	5.45	5.40	1.5	1.1
19°C	20.0	4.45	1.5	5.55		1	.85
	27.2	3.55	1.8	5.30	27.4 stokes	1	.7
Vaseline	13.7	3.6	.26	127		5	6
22°C	20.5	19.7	.44	66.5		3	3.5
	27.4	15	.55	57		2	2.7
MS.200 12,500 cs.	15	2.5	2.5	1.51	1.43		
20.5°C	27.4	1.6	3.5	1.35	725 cs		
12,500 cs (25°C)							
30,000 cs	15	3	2	2.37	2.04		
19°C	20	2.4	2.6	1.85			
30,000 cs (25°C)	27.4	2	3	1.91	1050 cs.		
100,000 cs	15	3	2	2.37	2.25		
21.5°C	20	2.3	2.6	1.86			
100,000 cs (25°C)	27.4	2.4	2.6	2.52	1140 cs		

something greater than 20 poise.

For the lower viscosity fluids the amplitude of vibration within the loaded section of the wire will show an attenuated standing wave pattern due to the reflection from the end. In order to determine the attenuation per wave length, the 'mean slope' (Brüel & Kjaer, 1958) is used. A typical trace is shown in Fig.(30b).

The value of the standing wave ratio cannot be determined with very great accuracy. The principal reason for this is that the output signal from the wire when subjected to a constant vibrational amplitude can vary by approximately 1 db and as the standing wave ratio is normally of the order of 2 db, a large error is present in its determination. Another possible cause of error is the lack of perfection of the boundary between the 'loaded' and 'unloaded' section of the wire. If the boundary is not well defined (e.g. if the level of fluid fell leaving a thin layer on the unloaded section of the wire) then the standing wave ratio would be reduced.

In order to evaluate the viscosity from the measurements, the values of m are read off from Fig.(28) and (ω/m^2) determined. For a given fluid, the values of (ω/m^2) should be constant at the different frequencies. The mean value of

(ω/m^2) from the attenuation measurements has been used to calculate the viscosity.

VII. 4. Discussion of results.

In general it can be seen that the values of (ω/m^2) from the attenuation measurements at different frequencies are constant, indicating that the viscosity is relatively frequency independent in the limited range employed. Although Venetian Turpentine and the 950 oil both give viscosities agreeing fairly well with their 'undirectional' values, the other fluids give values which are considerably lower. The most marked examples are the MS 200 Silicone Fluids of which the 100,000 cs sample gave a value of only 1100 cs. The values of viscosity, however, agree quite well with those obtained by Barlow & Lamb (1959) at the same frequency. A closer study of these results indicates a decrease of viscosity with frequency (apart from one spurious result) which is also to be expected. In the case of the Fluxed Bitumen and Shell Limea 81, the calculated viscosities are approximately one quarter of the undirectional values. Vaseline shows a considerable frequency variation, but these

results can be explained qualitatively by assuming the vaseline to be visco-elastic. For example, it has been calculated that if the ratio of the attenuation per wavelength at one frequency (ω_1) and at twice the frequency was 1.3, when the fluid shows no rigidity, then the ratio would be 1.65 when $g = \omega_1 \eta$. Thus the effect of rigidity is to increase the frequency dependence of attenuation, as is observed in this experiment.

The theoretical values of the standing wave ratio have been calculated from the value of the measured viscosity and it can be seen that these are in fairly good agreement with the experimental results.

The application of the theory for the determination of damping of longitudinal vibrations in a wire, as for example in a delay line, shows that if a fluid is selected which gives a high attenuation per cm, then there is the problem to be surmounted of reducing the large standing wave ratio. In theory this can be reduced by interposing a quarter wavelength of an intermediate viscous fluid but of course this will only act effectively at one particular frequency. An alternative solution could be to produce a gradual increase of viscosity at the boundary. In practice this could be achieved by employing a temperature gradient in the fluid.

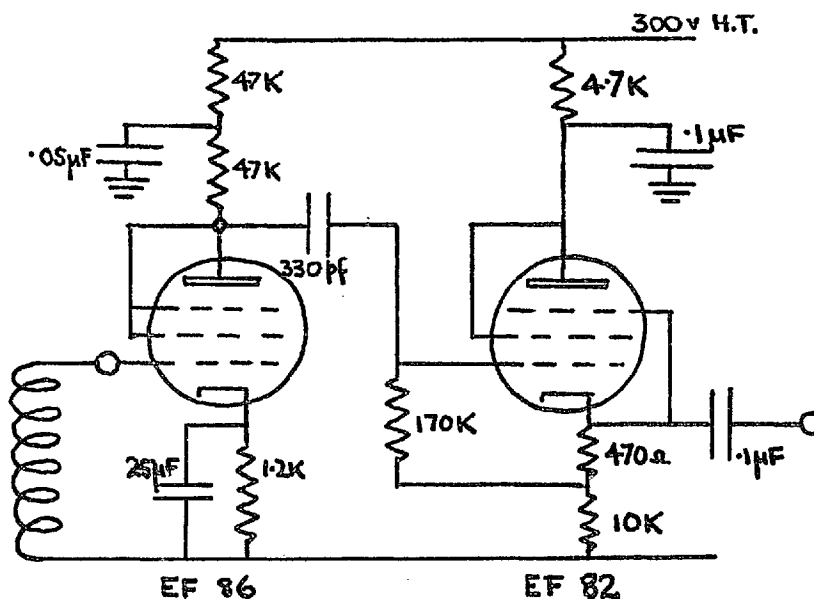
APPENDIX: Electronic Circuitry

1. Modifications of apparatus as described by Pattison (1954).

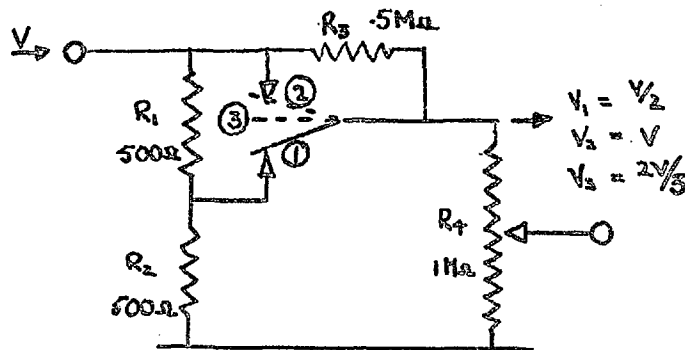
- (a) Instead of using a separate oscillator to excite the vibrating system, the output of the amplifier was fed back to the exciting coil.
- (b) Instead of using a separate amplifier to drive the counter, the output of the amplifier was again used to feed a 'pulse shaper'.
- (c) The Thyatron which operated the voltage doubling relay was replaced by a hard valve which would reset at the end of 'the count'.
- (d) A slow speed multivibrator was used to operate the 'drive' switch and to reset the counter in order to make the apparatus function automatically.

2. Description of circuits.

(a) Pre-amplifier



The pre-amplifier has been designed to have good linearity and stability for input signals of the order of 50 mV. and a low output impedance. The first stage is an EF 86 valve triode connected, selected for its low noise factor. This valve is RC coupled to an EF 82 valve which is connected as a Cathode Follower. This heavy current valve has been selected to produce signals of the order of a few volts across the 1 K resistance offered by the Voltage Doubling circuit, without distortion. The H.T. supply from a stabilised power pack is heavily decoupled.

(b) Voltage Doubling Circuit

This circuit consists of two balanced high stability non-inductively wire wound resistances of $500\ \Omega$ each connected to a high speed relay as shown in the diagram. As a continuous output is required to drive the counter, even while the relay operates, a $.5\ M\Omega$ resistance bypasses the upper relay contact. This will slightly alter the ratio of output voltage in the two relay positions.

When the relay is in position ①

$$V_{\textcircled{1}} = V \frac{R_2 R_4 / R_2 + R_4}{R_2 R_4 / R_2 + R_4 + R_1 R_3 / R_1 + R_3}$$

where $V_{\textcircled{1}}$ is the output voltage and V is the input voltage.

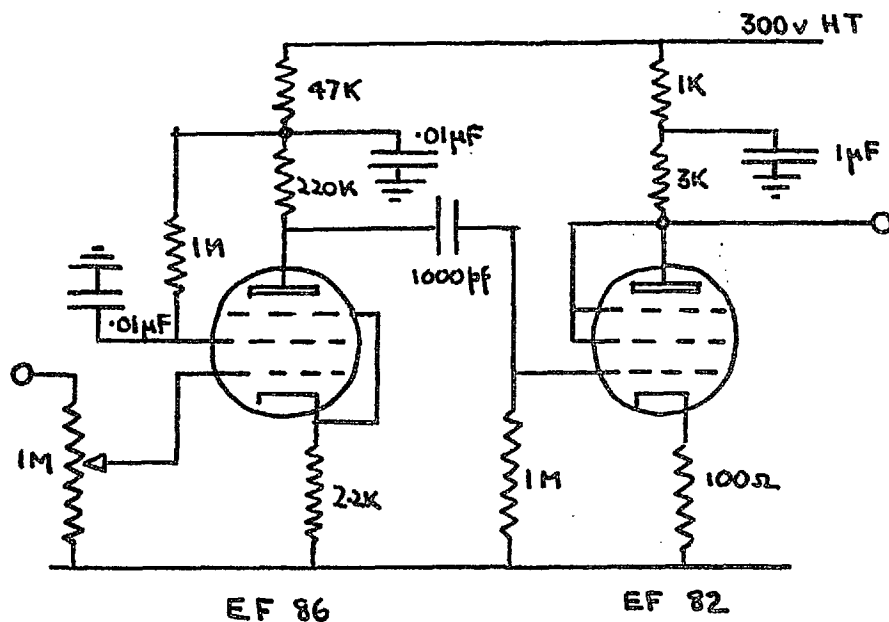
$$\text{But } R_1 = R_2 = 500\ \Omega \text{ and } R_3 \approx 1000 R_1 \\ R_4 \approx 2000 R_1$$

$$\text{Therefore } V_{\textcircled{1}} = V \frac{2000/2001}{2000/2001 + 1000/1001} \approx \frac{V}{2}$$

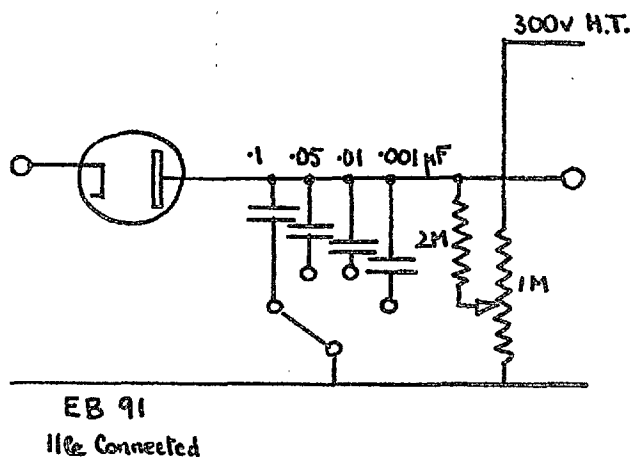
In position ②, $V_{\textcircled{2}} = V$.

When the relay is moving between the contacts, the output voltage $V_{\textcircled{3}}$ is easily seen to be $2V/3$.

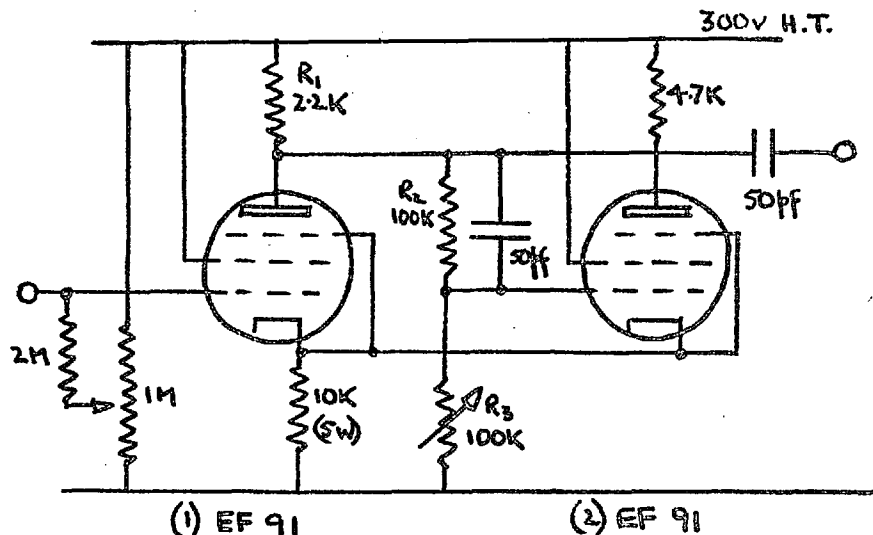
This intermediate choice of voltage for $V_{\textcircled{3}}$ is desirable. Should 'contact bounce' result as the relay moves from position ① to ② to double the gain, the signal does not fall to a value of less than $V/2$, otherwise the Schmitt Trigger will operate a second time.

(c) Amplifier

The amplifier consists of an EF 86 Pentode as the input stage and an EF 82 Triode connected output stage, which gives enough power to drive the specimens and offers a sufficiently low output impedance to feed the rectifier circuit. The H.T. supply is heavily decoupled.

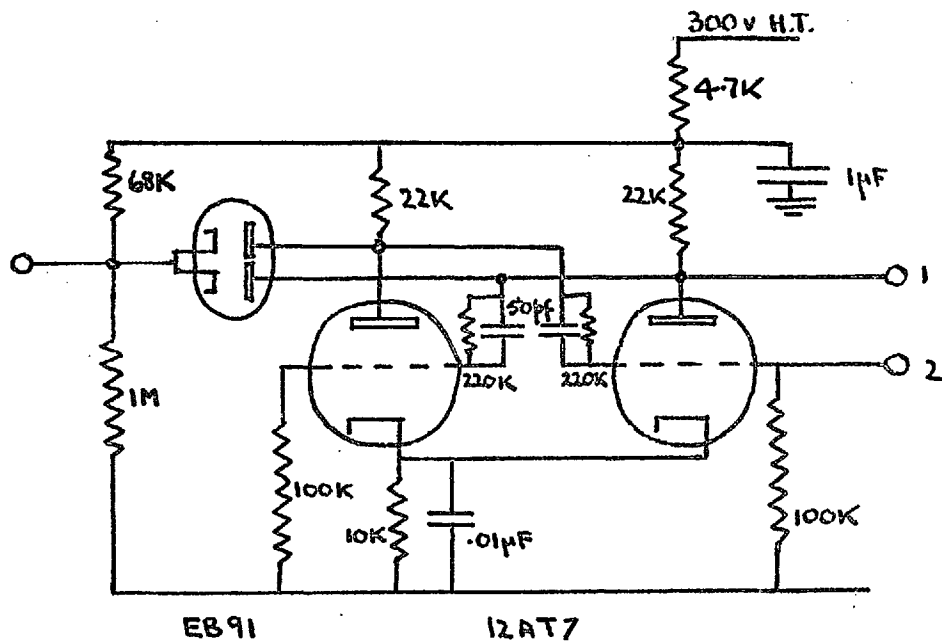
(d) Rectifier

The design considerations for this circuit have been adequately dealt with by Pattison (1954). The potentiometer (1 M) is adjusted to the mean potential of the anode of the output stage of the amplifier. Thus only the negative half of the decaying signal will be smoothed.

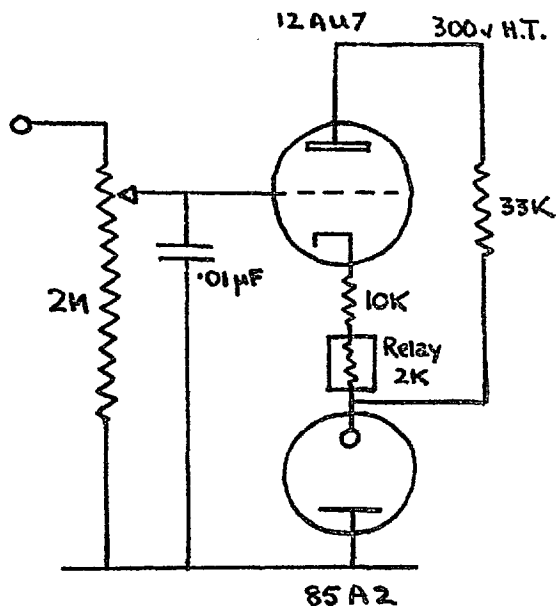
(e) Schmitt Trigger

This circuit acts as a very sensitive voltage level detector. The voltage on grid (1) is initially below cut off and thus valve (2) conducts. The common cathode potential sets itself just above the potential on Grid (2) as determined by the resistance chain of R_1 , R_2 , R_3 . R_3 can be varied to adjust the cathode potential. As the voltage on grid (1) slowly rises, a level of V_T which is just less than the mean cathode potential, is reached where valve (1) just starts to conduct. This causes the potential on grid (2) to be depressed and correspondingly the cathode which produces extra conduction of valve (1). This regenerative action causes valve (1) to conduct and valve (2) to be cut off. A negative pulse appears at the output terminal. With the potential of grid (2) at a lower level than before, the circuit does not trigger back until the input voltage approaches this value. The regenerative nature of the feed back loops again causes a sudden transition when this reset voltage V_R is reached.

The triggering voltage V_T can be adjusted by varying the 100 K Ω Potentiometer, R_3 .

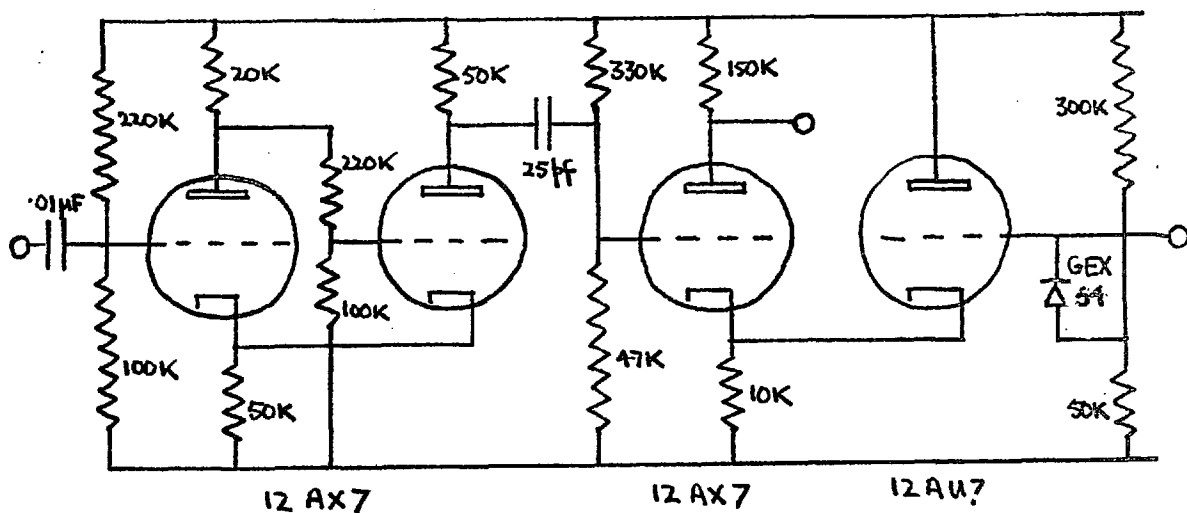
(f) Scale of Two

The function of the 'Scale of Two' is to convert the two triggering pulses from the Schmitt Trigger into a square wave 'gating pulse' produced at output (2) and a wave form to operate the relay drive produced at output (1).

(g) Relay Drive

The voltage appearing at the input terminals of this circuit depend on the state of the 'Scale of Two' circuit. This is arranged to provide a positively going square wave during the counting period. This voltage rises from 150 volts to 280 volts. With the input at 150 volts the 2 M pot. in this relay drive circuit is adjusted so that the potential appearing on the grid is about 70 volts and the valve is therefore below cut off, as the cathode rests at 85 volts due to the voltage stabilizer tube. When the input voltage suddenly rises to 280 volts the grid rises to about 130 V and the valve will conduct passing enough current to operate the relay R . In order to prevent 'relay bounce' which might be produced by the sudden passage of current, the change in grid potential is 'rounded off' by a condenser placed in parallel with the lower half of the 2 M pot.

(h) Pulse Shaper and Gating Circuit



The Pulse Shaper is a Schmitt Trigger circuit with a small 'hysteresis' and which gives a square wave output at the frequency of the sinusoidal input. This square wave is differentiated by the small coupling capacity and is fed to the grid of a triode in the gating circuit. This valve shares a cathode resistor with a second triode which is normally arranged to conduct heavily and therefore raise the common cathode potential to beyond cut off point of the first valve. Thus no signal appears at the output of the first valve.

The negative square wave output from the 'Scale of Two' is fed to the grid of the second valve. The potential on this grid cannot fall below the value determined by the potential chain to which it is connected by the crystal diode. The cathode potential falls correspondingly to this value which is adjusted to allow the positive pulses from the pulse shaper to pass the first valve. The output from this valve will be negative pulses suitable for driving a counter.

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