

Synchronisation in Dynamically Coupled Maps

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by

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I declare that all the work presented in this thesis is my own, unless fully acknowledged and referenced accordingly.

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Abstract

The central aim of this thesis is to better understand the dynamics of a set of dynamically coupled map systems previously introduced by Ito & Kaneko in a series of papers (Phys. Rev. Lett. **88** (2002), no. 2, 028701 and Phys. Rev. E **67** (2003), no. 4, 046226). The current work extends Ito & Kaneko's studies to clarify the changes in macrodynamics induced by the differences in microdynamics between the two systems. A third system is also introduced that has a minor change to the microdynamics from nonlinear to linear output function in the externally coupled system.

The dynamics of these three dynamically-coupled maps is also compared with their simplified systems with static coupling. The previous studies of these dynamically-coupled maps showed a partitioning of the parameter space into regions of different macrodynamics. Here, an in-depth study is presented of the behaviour of the systems as they cross the boundary between one region and another. The behaviour across this boundary is shown to be much more complicated than suggested in the previous studies.

These three systems of dynamically-coupled maps all differ in the form of their microscopic couplings, yet two of the systems are shown to produce similar macrodynamics, whereas the third differs dramatically by almost any measure of the macrodynamics.

The time it takes for the systems to synchronise, both the dynamically-coupled and static-coupled systems, is investigated. It is shown that the introduction of dynamical-couplings stops the systems from synchronising quasi-instantaneously. Details of potential consequences of this in the field of neuroscience are discussed.

A brief study of the effect of driving the systems with external stimuli is presented. The different microscopic coupling forms cause different responses to the external stimuli. Some of the responses are similar to that observed by the visual cortex area of the brain.

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Chapter 1

Introduction

Networks are a natural language to describe complex systems; the constituent elements being the nodes, and the interactions between these elements being represented by the links of the network. It is hoped that through the investigation of networks, it will be possible to better understand the underlying dynamics that bring about the macroscopic behaviours displayed by large complex systems.

Before continuing any further, let us spend a little time on a brief discussion to address the question of, what exactly is a complex system? There are many working definitions in use, which one to use is as yet a matter of taste. Therefore it is important to clarify this point from the outset.

Amongst the many working definitions of complex systems there are a number of key features that appear in most. It is widely accepted that complex systems are those composed of many interacting constituent parts. It is often the case that the dynamics of the individual elements are well understood, these being known as the microdynamics. The system as a whole however, when viewed at a different level of course graining can display behaviours quite different from these microdynamics, often referred to as macrodynamics. Whilst the macrodynamics are of course a direct result of the cumulative effect of the many constituent parts and their interactions, these macrodynamics may not be obvious or anticipated simply by knowing the microdynamics. As such, they are often labelled as emergent phenomena; being an unexpected behaviour or result that is not easily predictable by simply looking at the microdynamics of one individual constituent part. This was summed up by Anderson's

influential paper “More is Different”[And72]:

The ability to reduce everything to simple fundamental laws does not imply the ability to start from those laws and reconstruct the universe. In fact, the more the elementary particle physicists tell us about the nature of fundamental laws, the less relevance they seem to have to the very real problems of the rest of science, much less to those of society.

These eloquent sentences highlight the importance of considering the relevant hierarchical level of a complex system. Different levels may be characterised by different measures and require different methods or tools of analysis. Choosing an inappropriate measure for the level in question could lead to intractable difficulties of analysis or to missing a pattern that would be obvious by a different measure.

Let us elucidate these ideas through an example system that, by anyone’s definition, is definitely complex: the brain. The brain itself comprises around 10^9 neurons, with each neuron connecting to roughly 10^4 other neurons. This gives a mind-boggling estimate of the number of connections as 10^{13} [Squ08]. In a naive attempt to model the brain, we could start by taking a highly regarded simple model of a neuron, such as the Hodgkin-Huxley integrate and fire model [HH52]. We are not interested in any of the details of this model here, only that it is generally accepted as a good model of how individual neurons behave and comprises a system of differential equations. It would clearly be an arduous task to attempt simulation of the entire brain based upon this Hodgkin-Huxley neuronal model. Even if we could undertake such a mammoth computational task, would we gain much insight or new knowledge from the endeavour? Efforts, would likely gain more knowledge, understanding and insight through analysis at a different hierarchical level of the brain: not worrying about modeling the exact details of individual neurons, but considering collections of neurons and how their connections lead to macroscopic behaviours, like oscillations, synchronisation and other empirically observed phenomena.

Another classic example of a complex system is ant colonies. The individual parts being the ants who interact with one another through pheromone secretion and touch. The colony as a whole, functions without any one ant having a global overview of what is going on, or of what needs to be done. The colony appears to run seamlessly with tasks being

done and the labour being divided, although not necessarily in an equitable fashion. How the ants self-organise in such a seemingly efficient manner is the focus of ongoing research.

What is clear about complex systems is that there are a number of key features they all possess:

Open Boundaries: They interact with their surroundings. Additionally, the location of the boundary demarking what constitutes the system may be difficult to determine and be more dictated by conventions.

Many Parts: The system is made up of many connected constituent parts. To date, many complex systems studied have been made up of identical units. This is not a restriction however, and systems with nonidentical units can and should of course still be considered as complex.

Emergent Behaviour: The system displays macroscopic dynamics, not readily predictable from the microscopic dynamics of the individual units.

Nonlinearity: Small perturbations may cause large effects. If both the units and the interactions were linear this would necessarily rule out emergent behaviour.

These are not intended to be a definition of what a complex system is, but they are useful features to recall when considering studies of complex systems.

The general methods appropriate and fruitful for studying complex systems is an open question in itself. As mentioned, complex systems are made up of many parts so it is perhaps natural to look to the tools of statistical mechanics for inspiration. Alternatively, we might focus on the nonlinearities of the individual units and/or interactions and therefore turn to tools from dynamical systems theory. However, since the connections are such an important component in themselves, we may focus on their role by studying the network of all these connections.

The study of networks for gaining insights into complex systems has been so popular, it could be said that networks have become a paradigm for the study of complex systems. Barabási has published an interesting discussion of the use of networks towards understanding complexity [Bar07]. The last decade has seen an explosion of research into networks (see for example: [AB02, DM02, New03, DM03, Eva04, BLM⁺06, dFCOJT⁺08]). The

nodes (sometimes referred to as vertices) of the network are used to represent the individual components of the complex system, with the connections (or dependencies) being mapped out by the links of the network (sometimes referred to as edges).

Much of the study into networks has focused on the topological properties of real-world networks and trying to understand how they may arise through various models of network creation. In many of the early models, the only dynamic element of the system was the addition or removal of nodes and links (see for example [WS98][AJB99][VEM08][KRR01][BE01]). These models have been of importance for understanding possible mechanisms that lead to the creation of real-world networks with the particular topological properties that have been observed empirically.

There have been many claims of finding power law degree distributions in empirical networks: such as the network of citations in scientific papers [Red98][New01a][New01b]; long-distance phone call patterns [APR99][ALH03]; the network of webpages that comprises the World Wide Web [AJB99][Ba00]; the backbone of routers comprising the internet [FFF99]; and of course the network of film actors [BA99]. All of these claims of degree distributions that follow a power law had low exponents ($1 < \gamma < 3$). Theoretical arguments as to why power laws with larger coefficients will never be seen in real networks has been given in [DM02].

A breakthrough in understanding why so many networks have come to have power law degree distributions came with the discovery that a simple “rich-get-richer” model of network growth produces power law degree distributions [BA99]. In this model, nodes are added at each timestep, and the new nodes choose which existing nodes to connect to in direct proportionality with the degree of the node. The case of nonlinear attachment was considered by Krapivsky et al. [KRL00]: they considered the case of attachment proportional to the degree, k , raised to some power: k^α . It was found that nonlinear preferential attachment, $\alpha \neq 1$ destroyed the scale-free property of the networks produced.

A limitation of these studies with relevance to complex systems research is that these network studies only go so far; they are missing any interpretation of certain aspects of real-world systems. In real systems, both the nodes and connections are dynamic; it is important to understand the interplay between these two dynamic elements; how the changing connections affect the node dynamics and how the dynamics of the nodes affects their con-

nectivity.

A natural progression from the study of the structure of real-world networks is therefore, to investigate how these structures influence the dynamics of processes occurring on them. For example, how the results of epidemiology and disease spreading may be effected when infection occurs across networks with different topologies. A brief review of basic disease spreading on networks is given in [New03]. Specific examples include the work of Brockmann et al. with their study of the spread of diseases via the network of international flights [BHG06] and Dorogovtsev et al. give a concise overview of the effect of network topology on basic disease spreading models such as the SIR model [DGM08].

A slightly more abstract branch of networks research has focused on studying dynamic units, or nodes, that have specific interconnections. By this, it is meant that the system comprises of maps, or oscillators, with fixed dependencies on one another. For example, nearest-neighbour coupling through lattices, or other network topologies etc. This started by considering interactions on simple lattices, through nearest-neighbour interactions; extending to other connectivities on lattices, to all-to-all connections and finally to connectivities dictated by general network topologies. More details of this field of research will be given in Chapter 3.

The correlated behaviour that can arise between nonlinear oscillators as a result of coupling between them is of great importance in diverse fields such as neuroscience, biology, physics, engineering and economics.

In neuroscience and biology, it is important to understand how cells and functional units, that are themselves highly nonlinear, are able to coordinate their responses at a higher organisational level. Sometimes these coordinated responses are of benefit, e.g. waves propagating across the heart enable the muscle cells to produce regular heartbeats (for details, see for example Section 4.1.6 of [PRK01], Section 7.3 of [BKO⁺02] and Section 5.1.2 of [ADGK⁺08] and references therein). In contrast, too much coordination between functional units can also be a pathology. For example, too much coordination between certain areas of the brain has been associated with the onset of epileptic seizures [EP75, MLDE00] and Parkinson's disease [Fre83, EK90]. In the case of Parkinson's disease, it is research into understanding coupled, nonlinear oscillators that led to a breakthrough in treatment, known as deep brain stimulation (see for example [HPT05], [Pop06], [FGR⁺07]

and [HPT07]).

It was however in a physical system that synchronisation was first noted: as early as the seventeenth century, Christian Huygens noticed that two pendulum clocks weakly coupled through their individual connections to the same wall always entrained with one another [Huy73]. Whilst a widely ridiculed engineering failure has been attributed to the ability of complete strangers to walk in synchrony with one another. This is due to the weak coupling introduced when there are insufficient bridge supports. This is what happened upon the opening of the millennium pedestrian bridge in London June 2000 (details of the problem and solution can be seen on the architects webpage, [taeotMB] as well as in the article by Strogatz [SAM⁺05]).

Since synchronisation between periodic oscillators is observed so widely, it is of clear importance to understand how and why it occurs. Consequently, synchronisation between periodic oscillators has been the focus of extensive study [Kur84, PRK01, WC02, BJPS09]. The Kuramoto model is probably the most influential example of a coupled oscillator system, as it allows the computation of the exact coupling strength necessary for synchronisation to occur. Additionally, many other models of coupled oscillators are found to be equivalent to the Kuramoto model under certain conditions. For a review of the Kuramoto model, see for example [ABV⁺05], [Str00] and [Kur84]. This shall be discussed in more detailed in Chapter 3.

Whilst the maths is saved until later, the reason that oscillators are able to synchronise when coupled together is that they alter their effective frequencies relative to one another. These effective frequencies can entrain, resulting in synchronisation only if the mismatch between the individual oscillator's frequencies is sufficiently small.

In the 1980s, it was first shown that it is possible for two chaotic oscillators to completely synchronise when coupled together [FY83, YF83]. This was a surprising result due to the inherent sensitivity to initial conditions in chaotic systems and their consequent unpredictability; this has led to an explosion of research in this area in order to fully understand how, why and when chaotic signals will synchronise. Since this seminal research, the phenomenon of chaotic synchronisation has been widely observed in systems of different types of oscillators and maps coupled in a wide variety of configurations. This includes both theoretical models as well as experimental systems such as coupled laser

systems [ACBS07, AK98] and coupled electronic oscillators [KP95, WBS08]. A potential application of synchronised chaos displayed by these circuits is as a means of secure communication. This was first proposed in [CO93], but many weaknesses have since been highlighted in this scheme, (see for example [AL04] and references therein).

In the theoretical models, the maps were initially coupled simply through nearest neighbour interactions on a lattice: Coupled Map Lattices (CMLs). This was then extrapolated to both global coupling: Globally Coupled Maps (GCMs), as well as many different network topologies: Coupled Map Networks (CMNs). However, there has been little work to address the case when the connections coevolve along with the oscillators or maps. Such systems are of great importance, since as already mentioned, few connections in real-world systems are fixed from the outset. As shall be discussed in Chapter 3, this particular problem has been provisionally investigated by Ito and Kaneko [IK02][IK03], through a model of coupled maps whose connections evolve according to Hebbian dynamics. It is these models, which form the basis of investigation herein this thesis. Other work by Zanette and Mikhailov has looked at the continuous time case of oscillators coupled through coevolving connections [ZM04] as well as in [ACK06] and [ZK06].

The studies presented by Ito and Kaneko [IK02][IK03] gave an incomplete partitioning of the parameter space for their system of dynamically coupled logistic maps. In their works, it was implied that the parameter space has three distinct areas characterised by their globally different behaviours. In Chapter 3, previously known results for this system are given, and in Chapter 4 it is shown how this parameter space partitioning is not as simple as suggested by Ito and Kaneko; whilst there are indeed three distinct behaviours, the transitions in parameter space between these different phases is topologically non-trivial and certainly not sharp.

In accepting the importance of understanding interactions, we must also turn our attention to the question of the form these interactions take. What constitutes a realistic form for interactions? What difference does the form make to the emergent behaviour of the whole system? It is immensely important to consider these questions carefully whenever our intention is to model a real-world system. The question as to the difference the form makes to emergent behaviour has motivated the work discussed in Chapters 4 & 5, where our quest is to understand the effects on the macroscopic dynamics that occur as a result of

changes to the microscopic coupling.

The models studied here may not apply directly in a one-to-one manner for any specific real-world system, it is nonetheless important for us to study such simplified abstract cases in order to understand the underlying conditions which lead to synchronous behaviour.

In the rest of this chapter, the importance of synchronisation to many diverse systems shall be discussed, before returning to our example of the brain as a complex system and highlighting some important aspects of the necessary neuroscience.

1.1 Synchronisation

As briefly mentioned previously, synchronisation plays an important role in a diverse array of real systems. For some it is pathological, and for others it is crucial for their effective functioning. It is therefore a phenomenon that we must understand from a fundamental level. It should be noted that whilst the formal definition of synchronisation is left until Chapter 2, for now it is sufficient to think of synchronisation as simply the phenomenon whereby two (or more) signals or states become entrained and therefore evolve in unison.

The quintessential example of synchronous behaviour is the flashing of certain species of fireflies: swarms of hundreds or even thousands of fireflies (colloquially known as glow-worms) are seen to flash in synchrony [BB68]. This is during the mating season when the male fireflies flash in order to try and attract a mate. However, the precise reason as to why the fireflies flash in synchrony is still not agreed upon. Another example known to anyone who has ever had difficulty getting up in the morning, is the difficulty of maintaining the synchrony between two variables of different period: the natural circadian rhythm of humans is not exactly 24 hours, but is distributed. Some people have a slightly shorter natural frequency and others slightly longer [Win80]. Different estimates for the mean natural circadian period range from 23 hours [SC98] to 25 hours [CDS⁺99]. Since we live in a 24 hour world we continually experience the difficulty of synchrony between mismatched frequencies! Another system where synchronisation has been found is in the menstrual cycles of women: it has been widely observed that women who spend large amounts of time together find their menstrual cycles become entrained. This has been explained through the secretion of different levels of certain hormones at different times

during their cycle; other women subconsciously detect the change in the smell of the sweat and their menstrual cycles are adjusted. Again, there is no explanation as to why this should happen [McC71].

A man-made system that relies on precise synchronisation in order to function is that of the Global Positioning System (GPS): in order for the GPS system to function, the time-pieces on each and every satellite in the system must be all synchronised with a time here on Earth [AAH97].

Empirical evidence from brain studies have also suggested that synchronisation plays an important role in cognition [REKS97], in particular, synchronisation plays an integral role in conscious awareness [RGL⁺99][MMP⁺07]. Too much neuronal synchronisation has been associated with known pathologies such as epilepsy [EP75][MLDE00] and Parkinson's disease [Fre83][EK90]. Whilst too little synchronisation has been linked to schizophrenia [LHV05]. It is clear that synchronisation plays an important role in brain dynamics. A careful balance of the correct level of synchrony must be achieved: too little or too much could be pathological, yet a high enough level is required in order to raise awareness from sub-conscious to conscious.

1.2 The brain as a complex system

When discussing brain networks, there are many levels that can be represented: from the individual neurons connected by synapses up to effective connections denoting the direct or indirect influence one collection of neurons has over another. It is hoped that we can shed light on the latter of these two hierarchical levels. Given the present state of knowledge about the brain, it would be a near impossible task to present a model intended to represent a one-to-one model of effective brain dynamics, this is consequently not the aim here. However, it is hoped that we can gain insights into possible mechanisms that are able to generate behaviours similar to those generated by the effective connections within the brain.

The human brain is made up of billions of neuronal cells coupled together by their synaptic connections. Whilst it is conceivable to understand the macrodynamics that arise from just a handful of such neurons, it has not as yet been possible to connect the microscopic interactions that occur at neuronal level to macroscopic phenomena such as con-

scious thought or even the oscillations observed empirically through brain imaging techniques like EEG (electroencephalography) or MRI (Magnetic-Resonance Imaging). Whilst it is not claimed here that we can show a step-by-step understanding of how the brain works from microscopic dynamics through to macroscopic dynamics. It will be shown that even relatively simple coupled-map systems are able to display some of the behaviours not yet understood in the brain.

The brain comprises of order 10^9 neurons with 10^{13} connections, as such it is clearly not realistic to simply model the brain in its entirety from microscopic principles. In order to make progress towards understanding its workings we must therefore undertake something akin to a renormalization or course-graining and try to model the brain at different hierarchical levels with appropriate intricacy. Through understanding the mechanisms and dynamics at play on different functional levels, from the specific chemical reactions involved in an individual synapse firing right up to identifying regions that work together on particular functions or tasks, we can hope to gain a better overall understanding of the workings of the brain.

It is a well-established notion within the neuroscience community that synchronisation between functional areas within the brain is a physiological measure of brain state [VLRM01]. By this, it is meant that our unified thought process arises as a direct consequence of coordinated activity between anatomically separated brain regions [Poc00, REKS97]. In light of this fact, it is clear that the task of understanding how synchronisation between distinct areas of a network is able to arise is of huge importance. Particularly, when such networks have similarities with the brain such as plasticity, weighted connections and chaotic node dynamics, since it is also well-established that the behaviour of single neurons are often chaotic [KAR04].

Whilst the subsystems of the nervous system are highly interconnected, they retain a high level of autonomy. It is therefore a prime example of a system that might be expected to display some behaviours similar to those seen in models comprising ensembles of interconnected units [RVSA06], particularly if the units themselves display chaotic behaviour as is the case for the models studied here.

Unlike many other networks that can be considered to be sparse networks ¹, the brain is highly connected. As a consequence, there are no isolated regions within the brain; each region is highly interconnected, with no area acting in isolation without interference from other areas. If we hope to gain insights regarding the brain from high-level models, it is therefore necessary to consider network systems that are highly dynamical in both the node-state and interconnections. Allowing for the possibility of allowing external stimuli would also be desirable. Such a dynamically coupled network is the subject of study in this thesis: in Chapter 4 and Chapter 5, we will look at the properties of this model within its parameter space and see that there are regions where synchronisation occurs and how these give way into regions where the system splits into distinct clusters. In Chapter 6, the same systems are considered, but with an external stimulus. Similarities are shown between the behaviour of this driven system and dynamics displayed by the visual cortex area of the brain.

¹the number of nodes is of the same order as the number of connections.

We shall not cease from exploration. And the end of all our exploration, will be to arrive where we started. And know the place for the first time.

- T. S. Eliot

Chapter 2

Definitions and terminology

Chapter summary and outline

In this chapter, we define a number of key concepts and elucidate upon terminology that varies greatly depending on ones field. Notation that shall be used throughout the remainder of the thesis is also introduced.

2.1 The concept of phase

Depending on whether one lives in the physical or mathematical world, the concept of phase can be very different. We shall discuss each in turn, and it should be clear from the context which is meant in subsequent chapters.

The concept of *phase* in the physical world is that of a macroscopic state of a system. A simple example would be in the discussion of the states of matter: we may talk of materials in the gas, liquid or solid *phase* of matter. It is of course from this definition that Statistical Mechanics terminology of Phase Transitions comes. Whereby, it is the specific behaviour that is found as one crosses the transition between one phase and another that is of special interest eg. from liquid water to solid ice.

In the mathematical world, however, *phase* can be best understood by considering the following simple example: take the signal generated by, $f(t) = \sin(\omega t - \phi)$. Here, we define the phase, P to be $(\omega t - \phi)$. Notice that this function, P , is monotonically increasing in time, t ; but any specific value of the signal, f , does not uniquely define the value of the

phase: in this example, f , has the same value for values of phase, P , separated by 2π .

This mathematical concept of *phase* can be expanded so that it may apply to more general, non-periodic signals. According to Pikovsky, Rosenblum and Kurths [PRK01], the important properties that must be maintained of the phase are the monotonicity and proportionality with time. In order to do this for a non-periodic signal, we may make use of Poincaré sections¹ by simply defining that the phase of a given signal has increased by 2π each time it passes through the Poincaré section. From this, we can calculate the average period of the signal as the average time between each intersection of the Poincaré section from one particular direction. Consider a scalar signal in $[0, 1]$, we could define the signal to have completed one full period each time it crosses 0.5 from below. For a periodic signal, this period will always be identical, whereas for chaotic signals this will vary, but we can take the average to produce a mean period. The inverse of this average period gives us a quasi-frequency for any signal, including those that are chaotic. This notion of quasi-frequency of a chaotic signal is elucidated using the logistic map as an example in Section 2.11.

2.2 Synchronisation

As discussed in Chapter 1, synchronisation is the term used to describe the phenomenon in which the trajectories of two or more variables coincide with one another. They become entrained, and future evolution occurs in unison.

More precisely, two signals can have various degrees of synchronisation. At the lowest level of synchronicity, there exists a functional relationship between the two signals, x^1 and x^2 (ie. $x^1 = g(x^2)$). The next level up in degree of synchronicity, is that signals become phase synchronised: the simplest example of phase synchrony is that of one-to-one synchrony. In this situation, the difference between the phases of the signals is constant in time. Whenever the first signal is at its maximum point, the second signal will always be the same proportion through its period. More complicated dynamics are possible through n - m

¹Recall that a Poincaré section is a convenient method of reducing an N -dimensional trajectory into an $(N - 1)$ -dimensional map. By simply noting the points at which the trajectory crosses a surface through the phase space. See Ott's book for a simple introduction to them on p. 9 [Ott02]

phase synchrony, where n, m are integers. For every n times the first signal passes through the Poincaré section, the second will have intersected it m times. Finally, it is possible for signals, x^1 and x^2 , to become completely synchronised, such that $|x^1 - x^2| = 0$.

In this thesis, we are interested in this final scenario of complete synchronisation. Complete synchronisation in a system characterised by the signals $x^1(n), x^2(n), \dots, x^m(n)$ is the state such that $x^i(n) = x^j(n)$, where $i, j = 1 \dots m$ and n denotes the time. We define the signals to be synchronised when,

$$\lim_{n \rightarrow \infty} \sum_{\text{all pairs } i,j} |x^i(n) - x^j(n)| = 0 \quad (2.2.1)$$

In practice, for numerical studies, we must consider systems to be synchronised if this limit is less than some threshold.

2.3 Measure

Loosely, a measure is simply a generalisation of the concept of length, area, volume etc. to abstract sets and spaces.

By use of a simple example, we may easily see why it is necessary to have a more general definition of measure than the basic notion of length, area etc. as is used in Euclidean space. Consider the Cantor set gained via the familiar algorithm of “removing the middle-third”: starting from the unit interval, $[0, 1]$, we remove the middle-third, $(\frac{1}{3}, \frac{2}{3})$; this leaves us with two disjoint sets, $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$; we repeat this same process with the two segments left, thus leaving four segments: $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$. We can continue this process of removing the middle-third an infinite number of times. Through this process, we are left with infinitely many disjoint points, a schematic of the first six iterations is shown in fig. 2.1. A remarkable property of the set obtained after applying this algorithm many times is that, if we add up the lengths of the segments removed, we find they add up to 1. Have we, then, removed everything? This is indeed known **not** to be the case. We may easily see that it is not the case by considering the end points from each iteration: $0, 1, \frac{1}{3}, \frac{2}{3}$ etc. These end points are never removed, since at each iteration, we remove an open set of points from the interval. How then is this possible: we have taken the unit interval, $[0, 1]$

and removed segments which have a combined length of 1, but are still left with an infinite number of points? We clearly need another “measure” for the set of points removed (and those remaining).

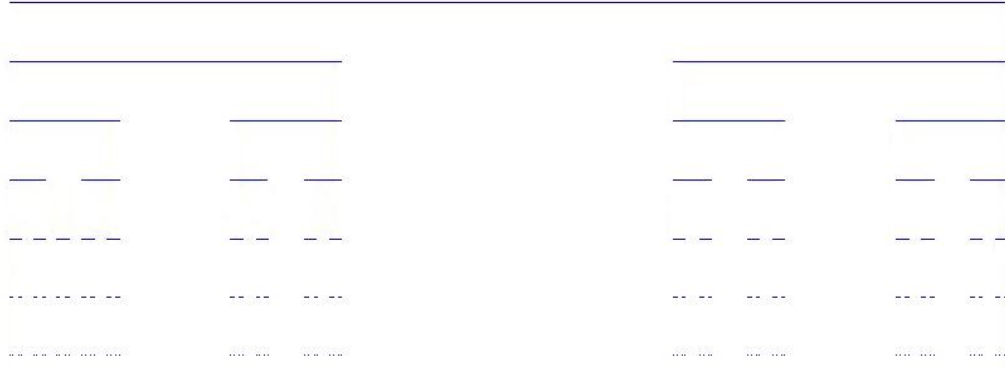


Figure 2.1: An illustration of the Cantor set generated by iteratively removing the middle third, starting from the unit interval $[0, 1]$.

A useful generalisation is to replace our notion of “length” with another measure in order to make meaningful remarks about the characteristics of this Cantor set, as well as other sets with nontrivial topology. Measures are exactly the extension we require in order to make meaningful comparisons between sets. A measure is a function that takes a set, A , and returns a positive real number: $\mu : A \longrightarrow [0, \infty)$. For a function to be a measure, it must also have the following properties:

Definition 2.3.1 *A function μ is a measure on a collection of subsets $\mathcal{B}$; $\mu : \mathcal{B} \longrightarrow \mathbb{R}_{\geq 0}$ that has the following properties:*

- (i) *Measures are always non-negative: $\mu(A) \geq 0 \forall A \in \mathcal{B}$.*
- (ii) *The measure of the union of disjoint subsets is equal to the sum of the measures of the subsets: $\mu(\bigcup_i A_i) = \sum_i (\mu(A_i))$.*
- (iii) *The measure of the empty set is always zero: $\mu(\emptyset) = 0$.*

A commonly used measure is the *Lebesgue measure*, $\mu_L(A)$ which gives the length, area or volume of a set. Another important class of measures that we will come across later

is that of *absolutely continuous invariant measures*; for a function $f : A \mapsto B$, μ is an invariant measure only if $\mu(A) = \mu(B)$. It is always possible to construct a measure that is invariant for a function f . However, it is not guaranteed that every function f will have an absolutely continuous invariant measure. For a measure, μ , to be an absolutely continuous invariant measure, μ must first be invariant by satisfying the criterion just mentioned and also be absolutely continuous with respect to for example the Lebesgue measure. So if $\mu_L(A) = 0$ and μ is an absolutely continuous invariant measure, $\mu(A) = 0$ must also be true. Whilst every function will have at least one invariant measure, proving the existence of absolutely continuous invariant measures is highly nontrivial and consequently an active field of research in itself.

2.4 Ergodicity

Once again, the physical and mathematical worlds have a difference in terminology: in this case, the same concept is given different names by the two communities. What mathematicians refer to as ergodicity is referred to as self-averaging by physicists.

The rule of thumb used to explain what it means for a system to be ergodic (or self-averaging) is the equivalence of averages taken over space and time. This means that if we take the time average of a specific component from a vector signal, this gives the same result as taking the average of the same component measured at one instant of time, but over many samples of an ensemble of the system.

This definition can be made more rigorous by considering a pair of measurable, forward-invariant sets A and B , where $A \cap B$ has measure zero. In order for the system to be ergodic, A or B , must have measure zero. In other words, one cannot decompose the phase space into distinct forward-invariant sets.

2.5 Lyapunov exponents

Lyapunov exponents are used to quantify the rate at which nearby orbits of a dynamical system diverge (or converge). Let us consider a generic dynamical system $f : [0, 1] \rightarrow [0, 1]$, with two nearby initial points x_0 and $x'_0 = x_0 + \delta_0$. We will limit ourselves to

consider the discrete time situation $x_{n+1} = f(x_n)$ since this is the situation considered within this thesis. The initial perturbation, δ_0 , after n timesteps is given by,

$$\delta_n = |x'_n - x_n| = |f^n(x'_0) - f^n(x_0)|$$

Where the notation $f^n(\cdot)$ denotes that the function f has been applied n times to its argument. The exponential separation is then quantified by the Lyapunov exponent, λ via,

$$\delta_n = \delta_0 e^{(n\lambda(x_0))}$$

Clearly, there are three cases of behaviour:

$\lambda > 0$: nearby orbits diverge,

$\lambda = 0$: perturbations to the orbit neither grow nor shrink,

$\lambda < 0$: nearby orbits converge.

By the following simple algebra, these Lyapunov exponents may be expressed as,

$$\begin{aligned} \lambda &= \lim_{n \rightarrow \infty} \lim_{\delta_0 \rightarrow 0} \frac{1}{n} \ln \left| \frac{f^n(x'_0) - f^n(x_0)}{\delta_0} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \frac{df^n(x_0)}{dx_0} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \prod_{k=0}^{n-1} f'(x_k) \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n-1} \ln |f'(x_k)|. \end{aligned} \tag{2.5.1}$$

Notice that the convergence of the limit in eqn. (2.5.1) is of crucial importance in order to ensure the Lyapunov exponent calculated is representative of the system's behaviour. This issue is guaranteed by Oseledec's multiplicative ergodic theorem, as shall be discussed next.

The concept of Lyapunov exponents may also be generalised to higher dimensional systems. In this situation, there can be perturbations in any of the D dimensions of phase

space. We therefore have D corresponding Lyapunov exponents:

$$\lambda_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \ln |J(x_i)| \quad (2.5.2)$$

where $J(x_i)$ is the Jacobian of the system.

For a simple example refer to Section 2.11 where the Lyapunov exponents for the logistic map are presented.

The Lyapunov exponent just introduced in eqn. (2.5.1) gives a measure of the average separation between nearby trajectories, averaged over the whole phase space. However, it is possible for a system to have some regions of phase space where trajectories converge, and others where they diverge. In taking the average in such a system where the divergence dominates results in a positive Lyapunov exponent. This Lyapunov exponent will not be representative of the dynamics in the regions of phase space where trajectories converge. If however, we only take the average over a small region of phase space, we can obtain a value that better reflects the local dynamics without being swamped by the global dynamics. This localised measure of the separation of trajectories has been labelled the *local Lyapunov exponent*; for studies using them, see for example [Gal99] and [AS05]. Clearly, there will be many local Lyapunov exponents for any system, since in theory there should be one for every point in phase space!

For high-dimensional systems that have an invariant submanifold, whether perturbations transverse to the submanifold grow or shrink is determined by the Lyapunov exponent in that direction; the so called transverse Lyapunov exponent, $\lambda_{\perp}$.

Oseledec's Multiplicative Ergodic Theorem

As noted before, the convergence of the limit in the Lyapunov exponents is crucial for them to be well defined. This is assured, under certain conditions, by Oseledec's famous multiplicative ergodic theorem [Ose68].

First, let us define a cocycle. Let X denote the phase space; T denote the set of all timesteps; f denotes a function from X to itself that describes the evolution from one timestep to the next; $C : X \times T \longrightarrow \mathbb{R}^{k \times k}$; $\mathbb{I}_k$ denotes the k -dimensional unit matrix; and

$k \in \mathbb{N}^*$. C is a cocycle if the following conditions hold:

1. $C(\underline{x}, 0) = \mathbb{I}_k \quad \forall \underline{x} \in X.$
2. $C(\underline{x}, m + n) = C(f^n(\underline{x}), m)C(\underline{x}, n) \quad \underline{x} \in X \text{ \& } n, m \in T.$

The theorem requires μ to be an absolutely continuous invariant measure on X , and M to be a cocycle. Under these conditions, for almost all $\underline{x}$ (under μ) and for any nonzero vector, $\underline{u}$, in $\mathbb{R}^k$, the following limits exist:

$$\lambda_{\mu, \underline{u}} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{\| (M(\underline{x}, n) \underline{u}) \|}{\| \underline{u} \|} \quad (2.5.3)$$

By comparing these with the Lyapunov exponent form of eqn. (2.5.1), we clearly see that, for Lyapunov exponents to be well defined for any function, f , our main requirement is that there exists an invariant measure, μ , on the phase space. The derivative satisfies the cocycle requirement. A clear discussion of Oseledec's multiplicative ergodic theorem and its consequences for Lyapunov exponents is given in [ER85]. The question as to whether an invariant measure exists for the logistic map, which is used as the underlying map in this thesis, is a nontrivial one and shall be discussed in more detail in section 2.11.

2.6 Attractors and basins of attraction

Roughly speaking, an attractor is a subset of phase space to which trajectories of a typical initial condition will converge. For any one attractor in a phase space, the set of points that tend to it for large times is known as its basin of attraction. There are many precise definitions of attractors in common usage, so it is of great importance to be clear as to what exactly is meant when referring to attractors. Unless otherwise stated, attractors in this thesis are those as in definition 2.6.3, though I have introduced other definitions that are used elsewhere for completeness.

Definition 2.6.1 *A basic attractor is any closed subset A of the phase space X that satisfies the following:*

- (i) *The set of points whose trajectories asymptotically end up entirely contained in A , known as the basin of attraction $\beta(A)$, must have strictly positive D -dimensional Lebesgue measure.*
- (ii) *There must not be a smaller closed set $A' \subset A$ such that the measure of the difference between $\beta(A')$ & $\beta(A)$ is zero.*

Some other commonly used definitions of attractors include the following:

Definition 2.6.2 *A Topological Attractor is a subset, A , of the phase space X , such that if we take any point in the neighbourhood of A , it will also be attracted to A .*

Definition 2.6.3 *A weaker notion of an attractor is that of a Milnor attractor [Mil85]; A is a Milnor attractor if the basin of attraction, $\beta(A)$ has strictly positive measure.*

Clearly, a Milnor attractor is an attractor in a measure theoretic sense only and is a much weaker notion of attractor than a topological attractor. A Milnor attractor can be a union of disjoint subsets and therefore allows for the possibility of highly nontrivial topologies as discussed next. In contrast to either of the other two definitions, the basin of attraction for a Milnor attractor need not be dense anywhere.

In a chaotic system the attractor is often a set of points that is geometrically a fractal, attractors that have this fractal structure are called *strange attractors*. A famous example of a strange attractor is that of the Lorenz system; the attractor has structure similar in shape to a butterfly.

Globally riddled basins

A new type of basin of attraction called a *Riddled Basin* was introduced by Alexander et al. [AYYK92]. A riddled basin is a set of points of positive measure, but it is nowhere dense. Thus, the basin is “riddled” with a set of points that are not in the basin. Here is a concise definition of what is meant by a riddled basin:

Definition 2.6.4 *A basin, $\beta(A)$, is said to be riddled if for all $x \in \beta(A)$, and for all $\epsilon > 0$, there exists a ball, of radius ϵ , centred on x , denoted $\mathcal{B}_\epsilon(x)$. The set of points within $\mathcal{B}_\epsilon(x)$ whose limit set is not within A has non-zero measure.*

A related situation can occur for systems with two distinct attractors, A & C , with basins $\beta(A)$ & $\beta(C)$. Here, it is possible for $\beta(A)$ to be riddled with $\beta(C)$ and vice versa. When both $\beta(A)$ is riddled with $\beta(C)$, and $\beta(C)$ is riddled with $\beta(A)$, the basins are said to be *intertwined* and the following condition is satisfied for all $x \in \beta(A) \cup \beta(C)$:

$$\mu(\mathcal{B}_\epsilon(x) \cap \beta(A)) \cdot \mu(\mathcal{B}_\epsilon(x) \cap \beta(C)) > 0 \quad (2.6.1)$$

This means that an arbitrarily small ball, $\mathcal{B}_\epsilon(x)$, centered around any point $x \in \beta(A)$ contains a set of points of positive measure that belongs to $\beta(C)$. Thus any disk in $\beta(A) \cup \beta(C)$ will contain both points from the basin of attraction for A and C .

Locally riddled basins

A generalisation of the concept of globally riddled basins was introduced by Ashwin, Buesci and Stewart in [ABS94]. Their generalisation was to allow an attractor, A , to be *locally riddled*. If A is an attractor in the Milnor sense only, and for any neighbourhood of A , V , there will be a non-empty set of points that do *not* remain in V , $U(V)^c$. Notice that this is the complement of the set of points that will remain in V , $U(V)$. For A to be *locally riddled*,

$$\mu_L(\mathcal{B}_\epsilon(x) \cap U(V)^c) > 0, \quad (2.6.2)$$

for all $x \in A$. This is more general than the previous definition of a globally riddled basin since the “riddling” only occurs within a neighbourhood of the attractor, $V(A)$. In contrast, for the globally riddled basin, the “riddling” occurs about every point in the basin of attraction and not just in a neighborhood about the attractor.

A comprehensive text that includes the concepts introduced in this section has been written by Buescu [Bue97]. They are also discussed in the book by Moselkilde, Maistrenko and Postnov [MMP02].

2.7 Bifurcations

The concept of bifurcations was introduced by Poincaré in 1879 [Poi79], and has been used in numerous fields since. There are many precise types of bifurcation but all relate to a qualitative change in properties in response to a small change in a specific parameter, similar to how order parameters control phase transitions in statistical mechanics.

There are two bifurcation types not usually covered in standard bifurcation texts and often seen in coupled-map systems. They are as follows:

Definition 2.7.1 *A riddling bifurcation is said to occur when the cycle of lowest period embedded in an attractor of a chaotic system loses its transverse stability.*

The point in parameter space where this occurs is easily calculable for the types of systems considered here, as will be seen later.

Definition 2.7.2 *A blowout bifurcation is said to occur when the transverse Lyapunov exponent, $\lambda_{\perp}$, changes sign.*

When the transverse Lyapunov exponent is negative, the invariant submanifold will be an attractor since trajectories will converge to it. The case of changing from negative to positive therefore corresponds to the invariant submanifold losing stability on the average. This can lead to a number of different situations depending on whether the linear or nonlinear mechanisms dominate the global dynamics.

2.8 On-off intermittency

A type of behaviour often seen in coupled map systems has been labeled as *on-off intermittency* by Platt in [PST93]. This is an extreme form of intermittent or bursting behaviour. Bursting behaviour is such that the system spends most of the time displaying one type of dynamics, interspersed by small *bursts* away from this before returning to the original dynamics. For coupled map systems, with the coupling strength very close to the critical coupling strength where the synchronous state becomes linearly stable ($\lambda_{\perp}$), the system can often spend long-periods of time very close to the synchronous submanifold. This is interspersed by bursts away from the synchronous state, that after only a short-period of time

will be once again folded back close to the synchronous state. Such behaviour is highly intermittent and has been labeled on-off intermittency; the *on* referring to the synchronous state and the *off* referring to the bursts of asynchrony.

2.9 Stability or instability of attractors

Whilst there are many definitions of an attractor, there are possibly even more manners in which they are defined to be stable. A few types of stability often referred to with regard to coupled-map systems are:

Definition 2.9.1 *An attractor, A , is said to be Lyapunov stable if for every open neighbourhood, U , of A , there is an open neighbourhood, V , whose forward projection under the dynamics, f , is also contained in U : $f^n(V) \subset U$ for positive integers n .*

Definition 2.9.2 *An attractor, A , is said to be asymptotically stable if it is Lyapunov stable and there is also an open neighbourhood around the attractor, W , such that the trajectory of all points in the attractor are contained within the neighbourhood of the attractor: $\omega(x) \subset A \forall x \in W$.*

Clearly, asymptotic stability is a stronger type of stability than Lyapunov stability. In the same way, an attractor can lose its stability to a greater or lesser extent:

Definition 2.9.3 *An invariant set, A , is said to be a chaotic saddle if there is a neighbourhood, U , of A , such that $\beta(A) \cap U$ is greater than A but has zero Lebesgue measure.*

Definition 2.9.4 *Consider a dynamical system that contains an invariant subspace that has an attractor, A . If all points not within the invariant subspace eventually leave a neighbourhood of A , then A is said to be a normally repelling chaotic saddle.*

Discussions of these concepts can also be found in Buescu's book [Bue97], and papers by Ashwin, Buescu and Stewart [ABS94][ABS96].

2.10 Network laplacians

Any network can be defined by an Adjacency Matrix A_{ij} . For an unweighted network, this matrix will have entries 1 if $j \sim i$, which means node j is connected to node i , or 0 otherwise. For a weighted network, the adjacency matrix will be denoted as w_{ij} , with each entry taking a value denoting the strength of connection between the two nodes. Any weighted network can be normalised so that $w_{ij} \in [0, 1]$.

The laplacian is only defined for undirected networks and is defined as follows [Moh97]:

$$L(i, j) = \begin{cases} d_i & \text{if } i = j, \\ -A_{ij} & \text{if } i \neq j. \end{cases} \quad (2.10.1)$$

where d_i denotes the degree of node i ; for an unweighted network it is calculated as $\sum_i A_{ij}$ and for a weighted network as $\sum_i w_{ij}$. The spectrum of a network is the set of eigenvalues of this laplacian, L . They are always real and non-negative for undirected and unweighted networks due to the symmetry of A_{ij} . The smallest positive eigenvalue has been named the algebraic connectivity of the network [Fie73] or spectral gap of the laplacian.

There are several useful studies characterising the laplacians of various network topologies: [GKK01, JB08]. The links between the laplacian spectrum and properties of the network are further elucidated in [CDS97]. Several results related to the synchronisability of networks have been derived that give bounds on the eigenvalues of the laplacian of a network, as shall be discussed in Chapter 3.

2.11 The Logistic Map

A number of useful results and properties of the logistic map are collected together here for completeness. Some properties such as the existence of absolutely continuous invariant measures are of importance for results presented in later chapters. The logistic map is a much-studied function of mathematics:

$$f(x) = a \cdot x \cdot (1 - x), \quad (2.11.1)$$

where $a \in [0, 4]$ and $x \in [0, 1]$, thus ensuring that $f(x) \in [0, 1]$ so $f : [0, 1] \rightarrow [0, 1]$.

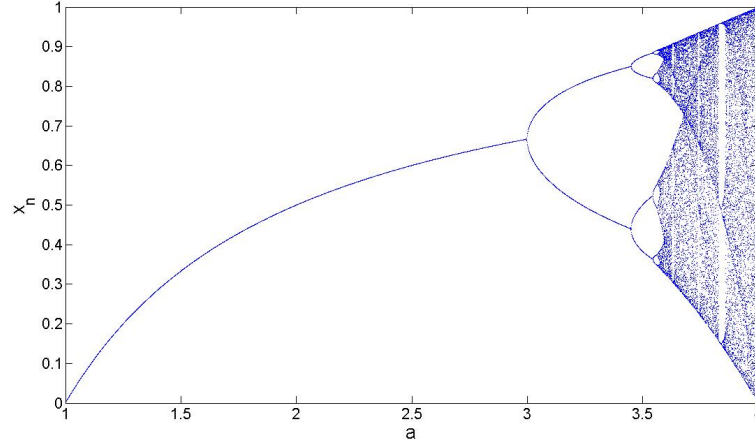


Figure 2.2: This is a bifurcation diagram of the logistic map, eqn. (2.11.1). It is a graph of the points of the orbit $f^n(x_0)$ for values of the parameter $a \in [1, 4]$ and for 100 timesteps, $n \in [500, 600]$. The x_0 are chosen uniformly at random in $[0, 1]$.

Its behaviour is highly dependent on the value of the nonlinearity parameter, a . For values of $a \leq 1$, $x = 0$ is a stable fixed point, and all initial values will end up at this point. For $1 < a < 3$, the fixed point $x = 1 - \frac{1}{a}$ is stable. A period doubling bifurcation occurs at $a = 3$ and the 2-cycle with values $x = \frac{a+1 \pm \sqrt{(a+1) \cdot (a-3)}}{2 \cdot a}$ comes into existence. For greater values of a , the system undergoes what is known as a period doubling cascade. This means that initially, 2-cycles appear, followed by cycles of period 4, 8, 16 etc. At a value of $a \simeq 3.57$ the map becomes chaotic, featuring dense orbits. This can be seen in the bifurcation diagram in fig. 2.2.

Lyapunov exponents

For this simple 1-dimensional mapping, $x_{n+1} = f(x_n)$, the Jacobian is simply given by the derivative $f'(x) = a - 2 \cdot a \cdot x$. By using eqn. (2.5.1), the Lyapunov exponent of the logistic

map is given by,

$$\begin{aligned}\lambda &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln |f'(x_k)| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln |a - 2 \cdot a \cdot x_k|\end{aligned}$$

In reality of course, we calculate the finite time Lyapunov exponent by taking n sufficiently large, since it is clearly not possible to simulate an infinite timeseries. We can see in fig. 2.3 that the Lyapunov exponent for the logistic map is highly sensitive to the nonlinearity parameter, a .

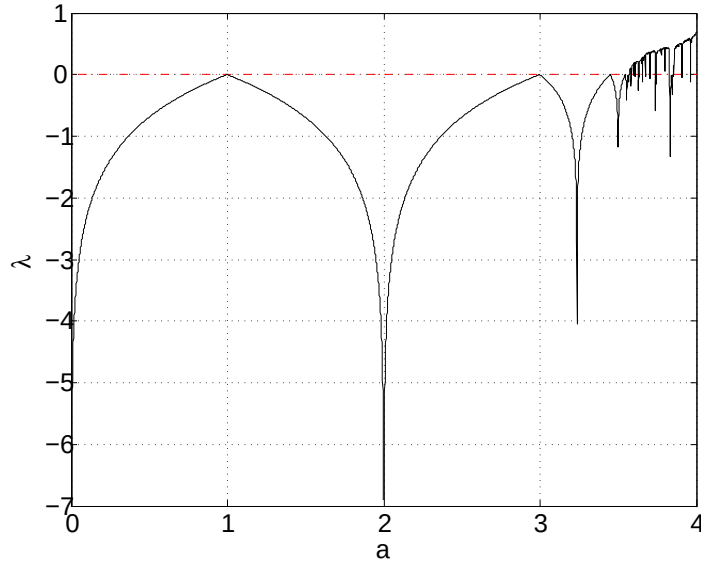


Figure 2.3: Lyapunov exponent of the logistic map. This is based on a finite time approximation, with $n = 10^4$. The first 500 iterations are discounted so as to ensure only the dynamics on any attractor is counted.

The question of whether there exists an absolutely continuous invariant measure for the logistic map is a nontrivial one. There have been many papers published discussing this very question; some prove that the set of nonlinearity parameters for which there exists an absolutely continuous invariant measure is of non-zero lebesgue measure [Jak81, LT06], whilst others prove the existence of an absolutely continuous invariant measure for specific values of the nonlinearity parameter. There are two such nonlinearity parameter values for

which it is known there exists an absolutely continuous invariant measure: $a = 4$ [UN47] and $a \approx 3.68$ [Rue77], which is the value of a such that the third iteration under f , is the unstable fixed point, $x^* = 1 - \frac{1}{a}$.

Since the Lyapunov exponent is only proven to be well-defined when an absolutely continuous invariant measure exists, it may seem surprising that there are so few specific parameter values for which an absolutely continuous invariant measure has been proven to exist. It is also useful to note with relevance to the models studied here that an absolutely continuous invariant measure of a map, f , will be an absolutely continuous invariant measure of a system of such maps when they are coupled together [Kel00].

A timeseries generated by the logisitic map with $a = 4$, often known as the Ulam-von Neumann Map, has an analytical expression for the distribution of its points [UN47]:

$$P(x) = \frac{1}{\pi \sqrt{x \cdot (1-x)}}. \quad (2.11.2)$$

In fig. 2.4 we can see that this theoretical distribution does indeed tie-in well with the empirical, normalised histogram of points for a timeseries generated by the logistic map, f , with $a = 4$.

Quasi-frequency

As we can see in fig. 2.5, the quasi-frequency of the logistic map is highly irregular for $a \gtrsim 3.57$. This is entirely expected since this is the parameter value above which the logistic map becomes chaotic. Several of the so-called windows of periodicity within the chaotic region are visible; these are ranges of nonlinear parameter where the map once again displays periodic dynamics. For example, the period-3 window can be seen as a measured quasi-frequency of 0.33 at $3.83 \lesssim a \lesssim 3.85$. This period-3 window can also be seen in the bifurcation diagram in fig. 2.2. The quasi-frequency of relevance to the work presented later is 0.28 at $a = 3.97$.

Further details of the behaviour of the logistic map, can be found in the following: section 1.3 of [BS95]; Chapter 10 of [Str94]; and [Tho81], which gives details on statistics of timeseries generated by the logistic map.

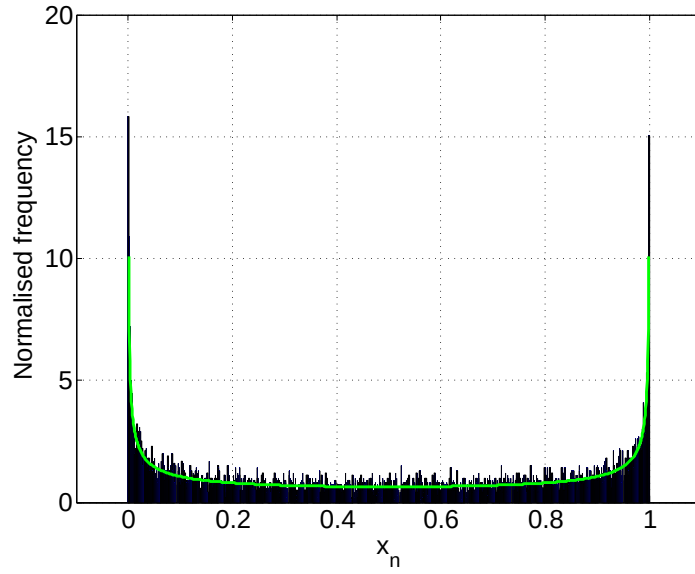


Figure 2.4: A histogram of the normalised frequency of points in a timeseries of length 10^4 for the logistic map with $a = 4$. The green line shows the theoretically expected distribution of eqn. (2.11.2).

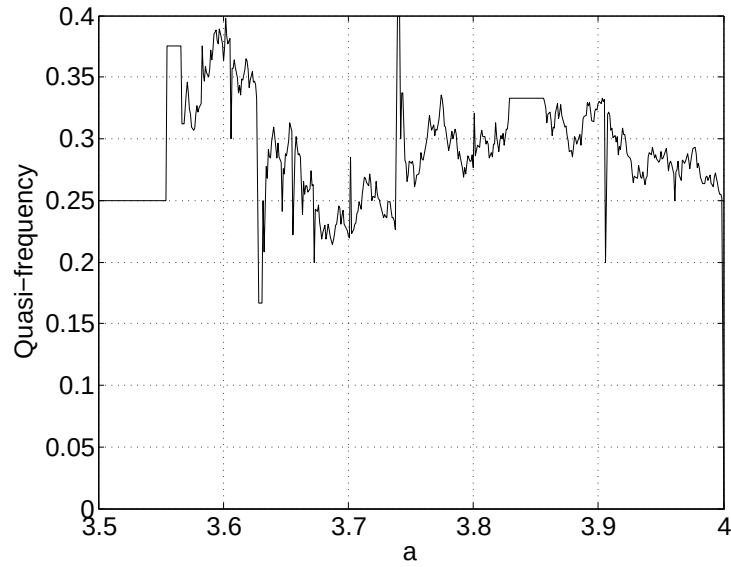


Figure 2.5: The quasi-frequency of the logistic map is shown here as a function of the nonlinearity parameter a . This has been measured using a timeseries of 10^4 steps, excluding a transient of 1000 timesteps. The Poincaré section was taken at 0.5 and only intersections from below are counted.

Chapter 3

Background of coupled systems

3.0.1 Chapter summary and outline

In this chapter, a brief review of the Kuramoto model will be given, along with a review of previous studies of coupled-map systems. The previous works will be discussed in increasing order of intricacy; starting with the simplest, static, global connectivities and moving through to complex network connectivities that are also dynamic. A review of the limited previous literature that has considered coupled-map systems in receipt of an external stimulus is also included.

3.1 The Kuramoto Model

The Kuramoto model is one of the simplest and most studied examples of a system of nonlinearly coupled oscillators:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{k}{N} \cdot \sum_j \sin(\theta_j - \theta_i), \quad (3.1.1)$$

where $k \in [0, 1]$ denotes the coupling strength between the oscillators with phases θ_i and $i = 1, \dots, N$; N denotes the total number of oscillators and ω_i is their natural frequency, chosen randomly from a specified distribution, $g(\omega)$.

For coupling strengths above a critical coupling, k_c , the oscillator phases synchronise and all oscillators continue their evolution at the same frequency. The critical coupling

strength value for which this transition to synchronous behaviour occurs was solved directly by Kuramoto [Kur84]. The details of the derivation have also been repeated in many reviews, see for example p. 279-283 of [PRK01] and the concise review of Acebrón et al. [ABV⁺05].

In addition to the model being one of the few examples where the transition to synchronisation is exactly solvable, it is of huge importance because there are lots of more complicated models of coupled oscillators that are equivalent to the Kuramoto model in certain limits.

3.2 Coupled maps

As mentioned in Chapter 1, models of coupled-maps are ideal candidates for considering systems composed of interacting dynamic components. Coupled-map systems are made up of dynamical nodes characterised by some state, referred to here as the node-state. In the coupled map systems discussed here this state is usually a scalar, for example in the case of coupled logistic maps. It is of course possible for the node-states to be vector quantities, for example coupled Hénon maps are the subject of study in [LC03] and coupled Rössler oscillators are considered in [HCP94]. The future evolution of these dynamical nodes is influenced by the previous state of other nodes they interact with as well as their own previous state. So, the future state of any one component will not only depend on itself, but on the state of all the other nodes it interacts with. In the case of *global* coupling, that will be every other node. Interactions of many other types have also been considered: from nearest-neighbours on a lattice, through nodes on a complex network, to nodes coupled via a coevolving network. Unless specified otherwise, the coupled map systems discussed here use the logistic map as their underlying map, f .

As might be anticipated, initial work with coupled maps involved studying maps placed on a lattice, the so called Coupled Map Lattices (CMLs) (for an overview, see the focus issue on CMLs: Chaos Vol 2, issue 3 (1992), [PCDG⁺96] and [Kan92]). These CML systems consist of nodes placed on an underlying lattice that is used to denote the interactions. The simplest examples have nearest-neighbour interactions. More complicated scenarios have since been considered by taking longer-range interactions on the lattice or by replacing the

lattice by a complex network. Notice that by taking the limit of long-range interactions in these coupled maps, we end up with all-to-all coupling. In such a situation, the underlying lattice or network may be negated since it no longer plays a part in the dynamics. Such systems of Globally Coupled Maps (GCMs) were introduced at the start of the 1990's by Kunihiko Kaneko [Kan90]. These systems are the precursors to those that will be studied here.

3.2.1 Globally coupled maps, with static connectivity

The first Globally Coupled Map system to be studied [Kan90] was:

$$x_{n+1}^i = f\left((1 - c) \cdot x_n^i + \frac{c}{N} \cdot \sum_j x_n^j\right). \quad (3.2.1)$$

The underlying map, f , was taken to be the logistic map: $f(x) = a \cdot x \cdot (1 - x)$. By taking the transformation $x_n^i = f(y_n^i)$ the following related system is obtained:

$$y_{n+1}^i = (1 - c) \cdot f(y_n^i) + \frac{c}{N} \cdot \sum_j f(y_n^j). \quad (3.2.2)$$

These two systems, given by eqns. (3.2.1) & (3.2.2), were found to display equivalent dynamics [Kan90]. Notice that here, all nodes are equivalent in the sense that they each have equal influence on every other node in the system. Whether this criterion is crucial for the two systems to display equivalent dynamics is an open question; one that shall be addressed in Chapters 4 and 5.

These systems, eqns. (3.2.1) and (3.2.2), demonstrate two general classes of coupled map systems: eqn. (3.2.2) describes systems where the nodes first undergo their (usually) nonlinear transformation, f , before being coupled together. These are therefore examples of *externally* coupled schemes. This is in contrast to eqn. (3.2.1), where the nodes are coupled together and only then is the nonlinear transformation, f , applied. This is therefore an example of an *internally* coupled scheme.

It was found that the systems of eqns. (3.2.1) & (3.2.2) display different behaviour in different regions of the global parameter space. These dynamics were characterised into

four different phases¹, depending on the values of a , the nonlinearity parameter of the logistic map and c , the coupling strength between nodes. The four phases of the parameter space are as follows:

Coherent: The basin of attraction for the state where all nodes are synchronised occupies (almost) all of the phase space. Therefore, almost all realisations result in a completely synchronised system.

Ordered: The system (almost) always ends up in a state comprising a few clusters.

Partially ordered: Depending on the initial conditions, the system will comprise either a few large clusters, or many small clusters. The average cluster distribution across many initial conditions is therefore bimodal.

Turbulent: No nodes end up in the same cluster; there is no synchronisation between any pair of nodes.

In these descriptions of the phases, nodes are said to belong to the same cluster if they are completely synchronised, ie. $x_n^i = x_n^j$.

Kaneko highlighted that GCMs give a mean-field type description of CMLs and are therefore useful since they reproduce the basic phases shown by the CMLs. This allows GCMs to be used in order to gain insights into, for example, the transitions between the regions of parameter space characterised by different behaviours.

Whilst the introduction of these GCM systems is of importance to understanding the roots of synchronisation phenomena, in order for these coupled map systems to be of most use for gaining insights into the mechanisms of synchronisation in complex systems, more realistic connectivities than all-to-all are needed. The recent huge research efforts into real-world network topologies gives a useful starting point for generalising the basic all-to-all coupled map systems of eqns. (3.2.1) & (3.2.2).

However, a lot of the previous studies of coupled-maps have focused on systems consisting of two or three coupled maps. Since, for such small systems, straight-forward analytical progress is possible without becoming intractable. Much effort has focused on the

¹This is of course phase in the physics sense as discussed in Section 2.1.

stability of the fully synchronised state. The commonly used measure for determining this stability analytically is the transverse Lyapunov exponent. That is, the Lyapunov exponent corresponding to the direction in phase space transverse to the synchronous submanifold (see for example the discussion in Section 13.2 of [PRK01]).

For the basic GCM of eqns. (3.2.1) and (3.2.2), the transverse Lyapunov exponent gives the following criterion for the stability of the synchronous state:

$$\lambda_{\perp} = \lambda_0 + \ln(1 - c) < 0, \quad (3.2.3)$$

where λ_0 is the Lyapunov exponent of the underlying map f .

As highlighted by Pecora and Carroll [PC98], the Lyapunov exponent cannot however describe the absolute stability of the synchronous state. As a global measure, it is insensitive to local instabilities, so whilst $\lambda_{\perp} < 0$ tells us that the synchronous state is on average attracting, it is still possible for the synchronous state to be repelling in some parts of phase space only to be re-attracted at a later point along the orbit. When such a situation arises this can lead to the behaviour of on-off intermittency or bursting [PST93]. To gain knowledge of the absolute stability for certain classes of systems, the Master Stability Function developed in [PC98] may be used. Another means to overcome the problem of the global nature of Lyapunov exponents is to use local Lyapunov exponents; instead of averaging the dynamics over the entirety of phase space, these measures only incorporate the dynamics of the localised region of phase space. Studies involving this approach include [Gal99] and [AS05].

The partitioning of parameter space for system (3.2.2) as presented by Kaneko in [Kan90] was further refined by Maistrenko, Popovych and Yanchuk in [MPY02]. Their revised phase diagram highlighted the precise parameter values at which the system undergoes first, a riddling bifurcation and then a blowout bifurcation. The former of these was shown to occur at a higher coupling strength relative to the latter and the curve denoting the blowout bifurcation was found to be fractal.

This work also highlighted how the loss of stability of the coherent state (the blowout bifurcation) can lead to a number of different behaviours of the system, depending on which other cluster states are stable. For high a -values, where the attractor of the logistic

map is one-piece chaotic ($a > 3.678$), the basin of attraction for the coherent attractor fills the entirety of phase space at high coupling strength, c . As c is lowered, the fixed point $x^* = 1 - \frac{1}{a}$ is the first periodic orbit to lose stability (the riddling bifurcation). For a slightly lower c -value, the synchronous 2-cycle embedded in the chaotic coherent state undergoes a period-doubling bifurcation and an asynchronous 4-cycle is produced.

Again, when considering a fixed a -value, if the blowout bifurcation occurs at a coupling strength higher than when the stable two-cluster state comes into existence, the blowout bifurcation results in the coherent state turning into a high-dimensional chaotic state, with no synchronisation between nodes. This will pervade until c is lowered to such a point that the 2-cluster state stabilises. However, if these asynchronous periodic cycles stabilise before the blowout bifurcation, there is a coexistence of states in this region of the a - c parameter space, since both the synchronous and asynchronous states are stable solutions.

The question of stability of the synchronous state for higher dimensional maps was considered in [Gle99]. In the work of Glendinning, the critical coupling strength, $c_{crit.} = 1 - \exp^{-\lambda(a)}$, was considered. This is the coupling strength, above which the synchronous state of eqn. (3.2.2) is stable according to the transverse Lyapunov exponent; therefore, $c_{crit.}$ also denotes the point at which the system undergoes the blowout bifurcation. In particular, it was noted that this boundary of stability is fractal in parameter regions where the underlying map is chaotic. An implication of this is that typical paths through the parameter space at a fixed value of c often show complicated sequences of synchronous and asynchronous behaviour.

Another coupled-map system that has been the focus of much study is as follows:

$$\begin{aligned} x_{n+1} &= f_a(x_n) + c \cdot (y_n - x_n), \\ y_{n+1} &= f_a(y_n) + c \cdot (x_n - y_n). \end{aligned} \tag{3.2.4}$$

Ashwin, Buescu and Stewart [ABS94] found that the loss of stability of the synchronous state does not coincide with the transverse Lyapunov exponent becoming positive. For this system, the loss in stability is much more gradual and occurs over a range of coupling strengths, c : the synchronous state of the system first loses asymptotic stability, so that the synchronous state becomes an attractor in the Milnor sense only. In this scenario, the basin

of attraction can become locally riddled. Upon further reducing the coupling strength, the synchronous state becomes Lyapunov unstable ($\lambda_{\perp} > 0$) and the system undergoes a blowout bifurcation. The synchronous state is now a chaotic saddle but a riddled basin of attraction is still admissible. Finally, for extremely low coupling strengths, the synchronous state becomes a normally repelling chaotic saddle and the system is never seen in the synchronous state.

Maistrenko et al. have also studied the system of eqn. (3.2.4) in [MMPM98]. They showed that the fixed point of the logistic map, $x^* = 1 - \frac{1}{a}$, remains transversally stable for $a > 3$ if:

$$-\frac{a-1}{2} < c < -\frac{a-3}{2}.$$

Therefore, for $a > 3$ and c -values that lie outside these bounds, the fixed point becomes a repelling node since both eigenvalues for the system are then greater than 1. The theoretical curve where this fixed point loses its stability of course denotes where the system undergoes a riddling bifurcation. The transverse stability of the period 2, 4, 6 and 8 orbits are also presented in [MMPM98]. Using this information along with the behaviour of the transverse Lyapunov exponent, Maistrenko et al. deduced some regions of the parameter space where the synchronous state is absolutely stable.

The generalisation of this system to systems of larger numbers of maps coupled in a similar manner was considered by Maistrenko, Popovych and Hasler [MPH00]:

$$x_{n+1}^i = f\left(x_n^i + \sum_{j \neq i} \varepsilon^{ij} (x_n^i - x_n^j)\right), \quad (3.2.5)$$

where $\varepsilon^{ij} \in [0, 1]$ denotes the coupling strength between node i and j . In this work of Maistrenko, Popovych and Hasler, it was proven that this system is topologically conjugate to

$$x_{n+1}^i = f(x_n^i) + \sum_{j \neq i} \varepsilon^j (f(x_n^i) - f(x_n^j)), \quad (3.2.6)$$

under the condition that $\sum_{j=1}^N \varepsilon^j \neq 1$. This work also considered the stability of the clustering states: the situation where the system comprises distinct clusters within which all nodes are synchronised. Maistrenko, Popovych and Hasler introduced the terminology

of n -cluster states to denote the system consists of n clusters within which, the nodes are completely synchronised.

3.2.2 Coupled map networks, with static connectivity

All systems considered until now have had coupling schemes that cannot be thought of as realistic; we have little hope of gaining insights into how interactions in real-world systems affect their dynamics unless we look at topologies inspired by the real-world. There are some studies that utilise coupling schemes inspired by topologies of real-world networks; such systems of coupled maps whose connectivity is given by a network are often referred to as Coupled Map Networks (CMNs). In these systems, the connections between nodes are described by the adjacency matrix of the network, A_{ij} (see for example [WC02] for a concise review of networks and CMNs). It is possible for this network to be directed and weighted, but most studies in the literature to date have used undirected and unweighted networks; such networks allow the use of the network laplacian as introduced in Section 2.10.

A common goal of studying CMNs has been to identify on which networks the dynamical nodes will be able to synchronise. A significant contribution to this issue was given by Barahona and Pecora [BP02]. Here, the ability of generic maps or oscillators to synchronise given a particular network was analysed. This is known as the synchronisability of a network. Very similar results were also presented by Jost and Joy in [JJ01], where the synchronisability of CML and GCM systems was considered.

Recall the small-world networks introduced by Watts and Strogatz [WS98]: starting with a pristine network² and with probability p , we rewire one end of each link to another node chosen at random. Through this rewiring process, shortcuts are introduced and for a range of rewiring probabilities, the resultant network has the small-world property that the average distance between any pair of nodes scales as $\ln N$, where N is the number of nodes in the network.

There have been claims that any number of these rewirings will lead to a network that

²A pristine network is one where all the nodes have exactly the same degree and it is connected e. g. a pristine network of degree two is a cycle.

is synchronisable [GH00] [WC02]. However, Barahona and Pecora showed in [BP02] that this is not sufficient to guarantee the synchronisability of the following coupled system for a network whose topology is characterised by the laplacian, G :

$$\dot{x}^i = F(x^i) + \sigma \cdot \sum_{j=1}^N G_{ij} \cdot H(x^j), \quad (3.2.7)$$

where F is the underlying map and H an output function.

The ability of a network with laplacian, G , to be synchronisable for a given map or oscillator is given by a restriction on the range of non-zero eigenvalues of G [BP02]. First, Barahona and Pecora looked at the maximum Lyapunov exponent, λ_{max} for many different particular choices of function, F and H . When they looked at how λ_{max} varies with coupling α ,³ Barahona and Pecora showed that for a large class of maps and oscillators, as α is increased from zero, the value of λ_{max} decreases and at a particular value of α , the value of λ_{max} becomes negative; the value of α at which this occurs is denoted α_1 . However, non-intuitively, as α is increased further, λ_{max} once again becomes positive for $\alpha > \alpha_2$. This is nonintuitive since it had been largely expected and observed that coupled maps synchronise only for coupling strengths above a certain threshold. The finding that there is actually a bounded range of coupling strengths where synchronisation can occur was unexpected.

The crucial piece of the jigsaw elucidated in [BP02] is that in order for a given network, with eigenvalues, θ_l of the laplacian, the following criterion must be satisfied if the system is to be synchronisable: $\sigma \cdot \theta_l \in (\alpha_1, \alpha_2)$.

It has also been shown by Atay, Biyikoğlu and Jost [AB05], that knowing the degree distribution of the underlying network is not sufficient in order to predict the synchronisability of a CMN system.

3.2.3 Systems with time-varying couplings

As highlighted earlier, the connections in real-world systems are dynamic, it therefore seems natural that we should extend these coupled map systems to incorporate this fea-

³ α is defined as a generic coupling strength; related to the coupling σ as $\alpha = \sigma \cdot \theta$, where θ are the eigenvalues of the laplacian G .

ture into the models.

An attempt to study coupled maps on time-varying networks was presented in [ACK06]. In this work, the time evolution of the network was imposed in an arbitrary manner: the connectivity of the nodes being dictated by an ensemble of networks. Each network in the ensemble has a time-period associated with it and after being used for that length of time, the connectivity is instantaneously switched to the next network in the ensemble. The synchronisability on these time-varying networks was compared to the synchronisability of the time-average of the ensemble of networks; this time-averaged network gives a static network with which to compare the synchronisability of the time-varying networks. It was found that the ability of the system with the time-varying connections to synchronise was only different to the time-averaged case if the laplacians of the different networks in the ensemble did not commute.

This work is of interest from a theoretical viewpoint since it is of importance to try and understand the circumstances necessary for synchronisation to be possible. However, since the time dependence of the connectivity is imposed in such an arbitrary fashion, it is not obvious what insight is gained into real world systems.

Similar arbitrary switching between network topologies was presented in [SB04], where it was shown that even for networks that are unconnected (comprising of several distinct clusters) at any particular timestep, dynamic processes on them are still able to globally synchronise. This is only possible if the dynamics of the network switching is sufficiently fast compared to the process taking place on the network. The results were expanded in [SBR04], where sufficient conditions for the fast switching network to synchronise were given.

Boccaletti et al. [BHC⁺06] presented a study of the appearance of synchronisation in dynamical networks without making any explicit assumptions on the time scale responsible for the variation of the coupling. In this study, it was found that the synchronisability of a network could be significantly improved by introducing dynamical connections. This work appears to draw inspiration from real-world systems, giving a less abstract view of this line of enquiry into synchronisation on dynamical networks.

A study using results from graph theory was presented by Atay and Biyikoğlu [AB05]. Their results are based on the previously discussed results of Barahona and Pecorra [BP02]

(as well as the continuous time equivalent by Li and Chen [LC03]) that the ratio of smallest to largest nonzero eigenvalue of the network laplacian must be larger than some threshold that is dependent on the coupling strength and the maximal Lyapunov exponent of the underlying map, in order for a particular network to be synchronisable for the dynamics of that particular underlying map.

Atay and Biyikoğlu[AB05] also considered whether it was possible to easily deduce the synchronisability of a network that is obtained from altering a network (adding/removing links) or by creating one network from two or more simpler networks. Their aim was to do this without recalculating the eigenvalues of the laplacian for the new network from first principles. One of their surprising findings was that when we take two networks and join them by creating links between them, the synchronisability of the whole larger network can worsen as the synchronisability of the two individual networks improves [AB05].

These works of Atay and Biyikoğlu show that the problem of understanding the interplay between the dynamics of the network and those taking place on the network is certainly nontrivial.

Some progress towards autonomous dynamical connections was made by Zanette and Mikhailov, [ZM04]. They considered a model of coupled oscillators whose connections are autonomous. The connections are determined by an internal state of the oscillators, θ_i . In this work, the coupling strength is

$$J_{ij}(t) = J \cdot \cos [\theta_j(t) - \theta_i(t)] \quad (3.2.8)$$

and the internal state θ_i evolves according to $\dot{\theta}_i = \omega_i = w_i + \frac{1}{N_j} \sum_{j=1}^N J_{ij}(t) \cdot \sin(\theta_j - \theta_i)$, with the ω_i chosen at random for each oscillator from a fixed distribution $g(\omega)$. Notice that the coupling here is undirected due to the symmetry in the coupling function and therefore when it is static, $J_{ij}(t) = J_{ij}$, this model reduces to the Kuramoto model discussed earlier.

A coupled-map system with connections that are determined by a coevolving network that is both directed and weighted was introduced in a series of papers by Ito & Kaneko [IK02][IK03]. The systems consist of N nodes with a map f associated to each, just like the previous simpler GCM and CML systems. The node-state of node i at timestep n is denoted by x_n^i . It is with coupling strength c and connection strength w_n^{ij} , that node j

is influenced by node i and therefore alters its trajectory accordingly. The mathematical descriptions of the systems studied is as follows:

Model A, internally coupled:

$$x_{n+1}^i = f \left[(1 - c) \cdot x_n^i + c \cdot \sum_{j=1}^N w_n^{ij} \cdot x_n^j \right]. \quad (3.2.9)$$

Model B, externally coupled:

$$x_{n+1}^i = (1 - c) \cdot f(x_n^i) + c \cdot \sum_{j=1}^N w_n^{ij} \cdot f(x_n^j). \quad (3.2.10)$$

Where in each system, the w_n^{ij} evolve according to

$$w_{n+1}^{ij} = \frac{[1 + \delta \cdot g(x_n^i, x_n^j)] \cdot w_n^{ij}}{\sum_{j=1}^N [1 + \delta \cdot g(x_n^i, x_n^j)] \cdot w_n^{ij}}, \quad (3.2.11)$$

and Hebbian dynamics⁴ is ensured by

$$g(x_n^i, x_n^j) = 1 - 2 \cdot |x_n^i - x_n^j|. \quad (3.2.12)$$

Both these systems are initiated in the same manner: $w_0^{ij} = \frac{1}{N-1}$ and the x_0^i are distributed uniformly in $(0, 1]$. $\delta = 0.1$ is a parameter that governs the plasticity of the network. The Hebbian dynamics used in these models seems an important step towards more realistic connectivities between nodes. It seems likely that in many real-world systems, alike “nodes” will be likely to have stronger connections between them.

Ito and Kaneko’s papers, [IK02][IK03], demonstrated that, similar to the simpler GCM systems of eqns. (3.2.1) and (3.2.2), the a - c parameter space has distinct regions characterised by qualitatively different behaviours of the system. The regions are similar to those found for the simpler all-to-all connected GCM, the main difference being that there are only three regions of distinct behaviour for these particular coupled map systems:

⁴See [Sej99] for a concise review of Hebbian dynamics.

Coherent: the whole system synchronises and comprises one connected component. Namely there is only one cluster of N nodes whose states x_n^i synchronise after some short initial transient.

Ordered: the system splits into a number of distinct clusters. The nodes within any given cluster have states x_n^i that are synchronised. The intra-cluster connection strengths are of negligible strength relative to the inter-cluster connection strengths.

Disordered: there is no synchronisation between any pair of nodes. Each node-state evolves independently.

In Ito and Kaneko's paper published in 2003, [IK03], a schematic of the subdivision of parameter space is given, similar to that shown in fig. 3.1. From this, we see that depending on the specific a -value chosen, the boundary between the phases occurs at different coupling strengths. The general trend is that for higher coupling strengths the system is coherent. As we look at lower and lower coupling strength, the system passes first to the ordered state and then to the disordered state.

From Ito and Kaneko's series of papers [IK02, IK03], it is not clear which of the two systems, eqn. (3.2.9) or eqn. (3.2.10), the results presented corresponded to. It was assumed that the two systems display identical dynamics as is the case for internally and externally coupled maps with fixed all-to-all connections [Kan06]. It therefore seems necessary to clarify whether this is indeed the case, the answer to this open question should become clear in Chapters 4 and 5.

An interesting type of system with adaptive connections between maps or oscillators was introduced by Zhou & Kurths [ZK06]:

$$\dot{x}_i = F(x_i) + \sum_{j=1}^N G_{ij} \cdot [H(x_j) - H(x_i)], \quad (3.2.13)$$

where $F(x)$ governs the local dynamics and $H(x)$ is a linear output function. The connection matrix G_{ij} is split into an unweighted adjacency matrix, A_{ij} , and a coupling strength matrix, W_{ij} . All the coupling strengths are positive and determined by the local synchronisation properties: nodes become more strongly coupled if they are not synchronised with

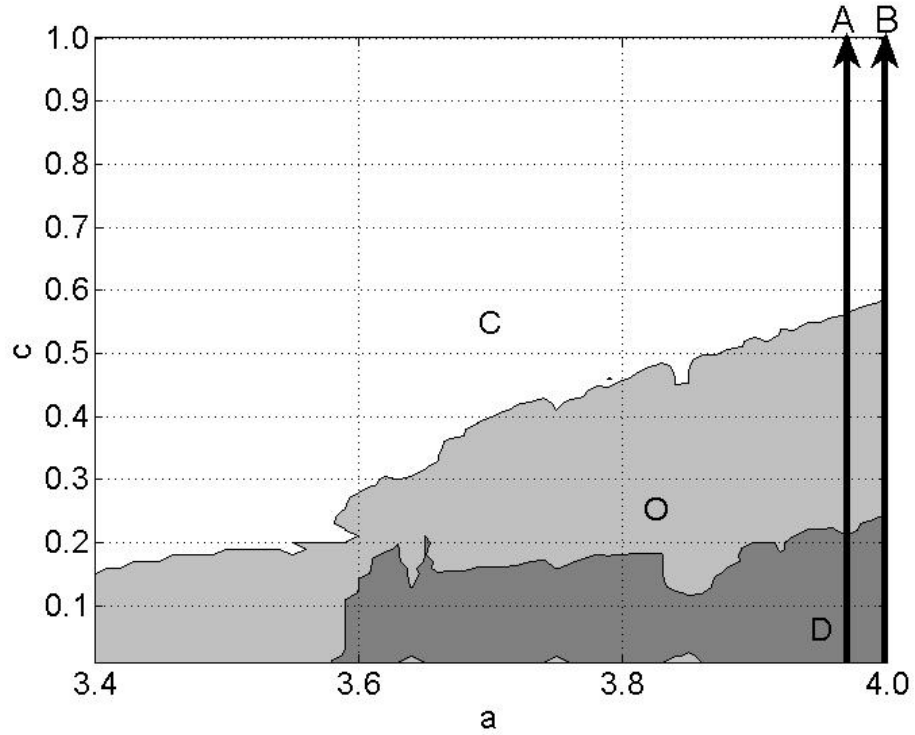


Figure 3.1: The global behaviours of the system of eqn. (3.2.9), Model A, in different areas of the parameter space: coherent (C), the whole system synchronises. Ordered (O), the system consists of several clusters within which the nodes are synchronised. Disordered (D), there is no synchronisation between any pair of nodes, so their node-states all evolve independently. This is a reproduction of that shown as fig. 1 in [IK03]. Lines A & B are for reference purposes in later chapters; they are at $a = 3.97$ and $a = 4.0$ respectively.

their immediate neighbours. Through such an adaptive scheme of strengthening connections between unlike nodes, it was found that for a given unweighted network A_{ij} , it was possible to synchronise systems on the network that were unable to synchronise when there was no adaptation, the case when $W_{ij} = 1$.

A less abstract study involving models of opinion formation across a network of evolving interactions was presented by Holme and Newman in [HN06]. Their model incorporated two aspects of earlier models of opinion formation: firstly, people are likely to change their opinion to be aligned with those they are connected to; secondly, people are more likely to be connected to others who share the same opinion. In the model presented in [HN06], both processes are present with a parameter that governs which is dominant. They found there was a phase transition between the regimes when one or other of these processes is dominant.

The study into complex networks with dynamical connections is very much still active, with recent interesting contributions including the paper of Tang et al. [TSM⁺] where properties of empirical dynamical networks are presented. This work is somewhat reminiscent of the early studies into static networks, only taking it to the next level and acknowledging the time dimension that real networks clearly possess. The empirical networks are compared to randomised versions of themselves, and it is found that the real networks possess what is termed as a temporal small world behaviour. That is to say, they have a short temporal path and high temporal clustering coefficient.

3.2.4 Effect of transmission delays

Clearly, in real-world systems, interactions are not instantaneous, models that incorporate non-instantaneous interactions are therefore of obvious interest. Despite this rather obvious and interesting extension, there are few papers where delays have been incorporated into models of coupled maps or oscillators. This becomes even more surprising since the few studies where delays have been considered have found delays dramatically change the dynamics displayed by the systems.

Several of these studies consider the effect delays have on GCMs, where the delay is distance dependent. Obviously, this requires the system to be embedded in a vector

space or placed on a lattice so as to allow easy calculation of distance between nodes [Jia00, Zan00, JKM02, MM03, MM05].

For systems with time delay τ_{ij} between nodes i and j , the concept of synchronisation must also be considered with care; a generally used notion of synchronisation is that each node *perceives* the system to be synchronised:

$$x_j(t - \tau_{ij}) = x_i(t). \quad (3.2.14)$$

Martí and Masoller [MM03] considered this system of coupled logistic maps:

$$x_{n+1}^i = (1 - c) f(x_n^i) + \frac{c}{N} \sum_{j=1}^N f(x_{n-\tau_{ij}}^j), \quad (3.2.15)$$

where τ_{ij} is the distance-dependent delay between nodes i and j . In [Jia00] a simpler version was considered, with $\tau_{ij} = k \forall i, j$. Martí and Masoller found the coherent behaviour of the system to be time-independent, with the value of node-state coinciding with the fixed point, $x^* = 1 - \frac{1}{a}$, of the logistic map. This happened even for a -values where the fixed point is unstable for the single logistic map.

Masoller and Martí have also extended these results to consider systems of CMNs [MM05]:

$$x_{n+1}^i = (1 - c) \cdot f(x_n^i) + \frac{c}{b_i} \cdot \sum_{j=1}^N A_{ij} \cdot f(x_{n-\tau_{ij}}^j), \quad (3.2.16)$$

where A_{ij} is the adjacency matrix of the underlying network and b_i is the number of neighbours of node i : $b_i = \sum_j A_{ij}$

In [MM05], Masoller and Martí considered several distributions of delays: exponential, gaussian, and constant. They found the dynamics displayed in the coherent region were highly sensitive to the type of delay imposed. Delays with exponential or gaussian distribution were found to stabilise the fixed point of the logistic map, $x^* = 1 - \frac{1}{a}$, leading to time-independent behaviour in the coherent state. In contrast, when delays were constant, $\tau_{ij} = k \forall i, j$ the coherent state was characterised by aperiodic dynamics. This was shown for $\tau_{ij} = 0$ ie. no delay, and $\tau_{ij} = 3$. It is not clear from their work whether this behaviour is typical for other values of uniform delay. However, from the results in [Ata03], this be-

haviour that the uniform delays do not stabilise the fixed point, when distributed delays do, might be anticipated.

Atay et al. have also shown [AJW04] that systems of coupled maps with transmission delays are able to completely synchronise for lower coupling strength than the same system with instantaneous couplings. They also show that for a scale-free or random connectivity network, with an odd time-delay τ , the system is able to completely synchronise for much lower coupling strengths than is typical of coupled map systems without delays. There is a small range of coupling strengths around $c \sim 0.2$, where this was observed; typical coupling strengths for the onset of the coherent state are usually around $c \sim 0.5$.

Additionally, Atay et al. highlight that when the systems are in the coherent state, only those systems with time delays will exhibit behaviour sensitive to the coupling strength. This is easy to see when we consider the effective equations of motion when the system of eqn. (3.2.16) or eqn. (3.2.15) is coherent:

$$x_{n+1} = (1 - c) \cdot f(x_n) + c \cdot f(x_{n-\tau}). \quad (3.2.17)$$

Clearly for $\tau = 0$ the evolution will be governed purely by the underlying map and is no longer affected by the coupling strength.

Atay and Karabacak [AK06] considered the following system:

$$x_{n+1}^i = f(x_n^i) + \frac{1}{b_i} \cdot \sum_{j=1}^N A_{ij} \cdot H(x_n^i, x_{n-\tau}^j), \quad (3.2.18)$$

where H is an output function with the following property, $H(x, x) = 0 \forall x \in \mathbb{R}$, and other symbols have their usual meaning. They proved that such systems are only able to stabilise the fixed-point of the map f if the time-delay τ is odd.

3.3 Transition to the coherent state

We have seen that there have been many studies showing that coupled maps (and oscillators) will synchronise for sufficiently high coupling strength, when coupled via a variety of different connectivities. There have also been a few studies attempting to characterise the

transition to the fully synchronised (coherent) state.

Ahlers and Pikovsky in [AP02] considered the transition to the fully synchronous state in a system of CMLs:

$$\begin{aligned} u_1(x, n) &= (1 - \gamma) \cdot (1 + \epsilon \Delta) \cdot f(u_1(x, n)) + \gamma \cdot (1 + \epsilon \Delta) \cdot f(u_2(x, n)), \\ u_2(x, n) &= (1 - \gamma) \cdot (1 + \epsilon \Delta) \cdot f(u_2(x, n)) + \gamma \cdot (1 + \epsilon \Delta) \cdot f(u_1(x, n)), \end{aligned} \quad (3.3.1)$$

where $u_{1,2}$ are the node-state variables, γ and ϵ are both coupling constants and the discrete laplacian is used: $\Delta v(x) = v(x - 1) - 2 \cdot v(x) + v(x + 1)$. In their paper, Ahlers and Pikovsky consider the behaviour for fixed $\epsilon = \frac{1}{3}$ as γ is varied; x denotes the discrete space of the lattice and n is the usual discrete time; f describes the local dynamics and periodic boundary conditions are implemented. They studied the behaviour of what has been termed the synchronisation error: $w(x, n) = u_1(x, n) - u_2(x, n)$, the difference between the two state-variables at each site. The average absolute synchronisation error, $\langle |w(x, n)| \rangle_{x,n}$,⁵ displayed qualitatively different behaviours when the underlying map f had a discontinuity in comparison to when the underlying map f was continuous. Additionally, the following power-law behaviour was reported:

$$\langle |w| \rangle_{x,n} \sim (\gamma_c - \gamma)^\beta, \quad (3.3.2)$$

$$\langle |w| \rangle_x \sim n^{-\delta}, \quad (3.3.3)$$

where γ_c denotes the critical value of coupling such that $\lambda_\perp = 0$. They claim that the values of β and δ found from numerical studies imply that depending on whether the underlying map has a discontinuity or not, the systems belong to a specific universality class: maps with discontinuities belong to the directed percolation (DP) universality class; those that are continuous have critical exponents similar to the bounded Kardar-Parisi-Zhang (BKPZ) universality class.

Additionally, maps that have a strong nonlinearity, such as a discontinuity like the Bernoulli map display asynchronous behaviour for a small range of coupling strengths

⁵ $\langle \dots \rangle_{x,n}$ denotes the average over space, x , and timesteps, n . The system is always evolved past an initial transient before the averaging is done.

when the synchronous state is linearly stable. In contrast, continuous maps, like the logistic map, were found to display the transition to the synchronous state at a coupling strength that coincides exactly with the critical coupling strength where the synchronous state becomes linearly stable according to $\lambda_{\perp} < 0$.

Similar findings of similarity between the transition to synchronisation and the critical exponents that characterise the DP universality class were reported in [GLPT03] and [GH06]. The work of Ginelli et al. also discussed the relation between the stability of the synchronous state and the transverse Lyapunov exponent, $\lambda_{\perp}$. In contrast to previous findings reported by Shuai, Wong and Chen in [SWC97], Ginelli et al. claim that in order for synchronisation to occur in spatially extended systems, $\lambda_{\perp} < 0$ is a necessary condition. The work of Shuai, Wong and Chen [SWC97], found that a system of globally coupled maps was in fact able to synchronise for a small range of coupling strengths where $\lambda_{\perp} > 0$. This result of synchronisation despite the positivity of the transverse Lyapunov exponent is not entirely unexpected; the Lyapunov exponent is such a coarse global measure after all.

Gade and Hu considered a system that incorporated both the local coupling of CML and the global coupling of GCM:

$$x^i(t+1) = (1 - \epsilon - \gamma) \cdot f(x^i(t)) + \frac{\epsilon}{2} [f(x^{i+1}(t)) + f(x^{i-1}(t))] + \frac{\gamma}{N} \cdot \sum_{j=1}^N f(x^j(t)), \quad (3.3.4)$$

where ϵ and γ are the local and global coupling strengths and periodic boundary conditions are employed. Gade and Hu considered several order parameters in their work and concluded that the synchronisation transition belongs to the directed percolation (DP) universality class [GH06]. They looked at the time-dependence of the following quantities at the critical coupling strength where $\lambda_{\perp} = 0$,

$$d = \frac{1}{N} \sum_{i=1}^N |x_i(t) - x_{i+1}(t)|,$$

$$\rho = \frac{1}{N} \sum_{i=1}^N |x_i(t) - \langle x(t) \rangle|,$$

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^N (x_i(t) - \langle x(t) \rangle)^2.$$

Gade and Hu found all these quantities displayed power law behaviour. It is based on the exponents of these power laws that they claimed the transition to synchronisation to be in the DP universality class.

3.4 The effect of external stimuli on coupled maps

Despite it being of clear importance to consider the effect that an external stimulus has on the dynamical behaviour of coupled-maps, little work in this direction has been undertaken. Earlier, it was highlighted how ubiquitous it is in complex systems that determining where the boundary of the system lies is more a matter of convention; they are often considered as having open boundaries. This very fact that they do not operate in isolation, but interact with other systems and the general environ around them means we must consider the effect external stimuli have on any systems we might like to consider as prototypes for complex systems.

There are a few works that have considered the effect of external stimuli: Neumann et al. [NSMF03] applied an additive quasiperiodic stimulus to the system of eqn. (3.2.4) so that it becomes:

$$\begin{aligned} x_{n+1} &= f_a(x_n) + c \cdot (y_n - x_n) + \varepsilon \cdot \cos(2\pi\theta_n), \\ y_{n+1} &= f_a(y_n) + c \cdot (x_n - y_n) + \varepsilon \cdot \cos(2\pi\theta_n), \\ \theta_{n+1} &= \theta_n + \omega \pmod{1}, \end{aligned} \tag{3.4.1}$$

where ε denotes the amplitude and ω the frequency of the external stimulus. When this system synchronises, $x_n = y_n$, the dynamics reduce to

$$\begin{aligned} x_{n+1} &= f_a(x_n) + \varepsilon \cdot \cos(2\pi\theta_n), \\ \theta_{n+1} &= \theta_n + \omega \pmod{1}. \end{aligned} \tag{3.4.2}$$

Clearly, for a stimulus of zero amplitude, this reduces to the uncoupled logistic map, so the dynamics seen in the synchronous state is governed only by the nonlinearity parameter of

the logistic map. The changes triggered by quasiperiodic stimuli to the single logistic map have also been previously investigated in [HH94] & [KFP98]. The effects of stimulating the coupled system were found to be similar to those observed to occur when the quasiperiodic stimulus was applied to the single logistic map.

The general effect of the quasiperiodic stimulus is that it reduces in size the region of parameter space where synchronisation occurs. As the amplitude of the stimulus is increased, the size of this region reduces; for sufficiently high amplitude stimuli, synchronisation is suppressed entirely, across the whole a - c parameter space.

The same quasiperiodic stimulus was applied to eqn. (3.2.4) in a multiplicative manner and considered by Shrimali et al. [SPRF05]:

$$\begin{aligned} x_{n+1} &= [1 + \varepsilon \cdot \cos(2\pi\theta_n)] \cdot f_a(x_n) + c \cdot (y_n - x_n) \\ y_{n+1} &= [1 + \varepsilon \cdot \cos(2\pi\theta_n)] \cdot f_a(y_n) + c \cdot (x_n - y_n) \\ \theta_{n+1} &= \theta_n + \omega \pmod{1} \end{aligned} \tag{3.4.3}$$

This work investigated the structure of the basins of attractors in the regions of bistability; regions of parameter space where both synchronous and asynchronous states were found to coexist. They observed a power-law increase in the volume of the basin of attraction for the synchronous state at the onset of bistability. Similar to the system without the external stimulus, ie. $\varepsilon = 0$, this system with stimulus and low coupling strength, the basin of attraction for the asynchronous state occupies the entirety of the phase space. As the coupling strength is increased, holes appear in this basin that are attracted to the synchronous state. These holes increase in number as the coupling strength is increased and occupy two main bands that merge at a critical coupling strength. As the coupling strength is increased yet further, eventually the whole phase space is occupied by the basin of attraction for the synchronous state.

3.5 Concluding remarks

Coupled-map systems are a diverse class of systems; there are a multitude of manners to couple the map, be it internally, externally, linearly, nonlinearly etc. Additionally, there are an endless choice of network topologies to choose from for the connectivity, and the possibility of weighted and/or directed connections adds yet more possibilities. Then there is the possibility of transmission delays and external stimuli. These possibilities create a seemingly endless collection of possible systems that constitute coupled-map systems, with many open questions and potential applications.

Only systems comprising two or three maps simply coupled together could be considered to be well studied. There have been some initial forays into the realm of more intricate systems comprising larger numbers of maps, but there are many open questions regarding their behaviour. The possibility for this type of system being able to give insights into some behaviours of real-world systems is also still in its infancy; the initial introduction of connection topologies inspired by real-world connectivities has given a good start towards this goal, that has been continued by considering the effect transmission delays have on the dynamics.

With increases in computing power, it should become easier to study the behaviour of intricate coupled-map systems. The lack of computing power in the past will have limited this progress, as these systems require intensive computations in order to simulate them.

The systems introduced by Ito and Kaneko with coevolving connections, eqns. (3.2.9) & (3.2.10), form the subject of investigation throughout this thesis. The question of whether these two systems, the internally coupled and externally coupled, do display identical dynamics, as was assumed by Ito and Kaneko, will be answered. In addition the nature of the transition across the boundary between the coherent and ordered regions will be explored. An initial foray into the response of these systems to external stimuli will also be considered.

The whole is more than the sum of the parts.

- Aristotle

Chapter 4

Results 1: Partitioning of the parameter space

4.0.1 Chapter summary and outline

In this chapter, many of the statistical aspects of synchronisation in coupled map networks with coevolving connections are presented. The models that form the basis of this study are either re-introduced or introduced for the first time. Their global behaviours are discussed, contrasting two methods for identifying clusters of nodes, discussing the parameter-space phase diagram for each model and analysing the onset to synchronisation. Additionally, the contrast between the models as one increases the coupling strength is highlighted. It will be shown how the analytical curves for the riddling and blowout bifurcations tie-in with the numerically observed behaviours. The transition from the synchronous, 1-cluster state is analysed; the transition for coupled-map models with internal coupling and external coupling with nonlinear output function are found to be very different to the model with external coupling and linear output function.

4.1 The Models

The models that form the basis of this work are those introduced by Ito & Kaneko in their series of papers [IK02, IK03]. These papers present studies of two particular coupled-map

systems, and a slight alteration to one of these models is made to produce Model C below. Model C is an alteration to Model B so that the coupling takes the form referred to in the literature as linear coupling as opposed to the nonlinear version used by Ito & Kaneko. This alternative form, Model C, also allows for non-instantaneous updating between nodes, but without invoking an explicit time-delay by hand. The three models that will be studied here are given as follows:

Model A, internally coupled:

$$x_{n+1}^i = f \left[(1 - c) \cdot x_n^i + c \cdot \sum_{j=1}^N w_n^{ij} \cdot x_n^j \right], \quad (4.1.1)$$

Model B, externally, nonlinearly coupled:

$$x_{n+1}^i = (1 - c) \cdot f(x_n^i) + c \cdot \sum_{j=1}^N w_n^{ij} \cdot f(x_n^j), \quad (4.1.2)$$

Model C, externally, linearly coupled:

$$x_{n+1}^i = (1 - c) \cdot f(x_n^i) + c \cdot \sum_{j=1}^N w_n^{ij} \cdot x_n^j, \quad (4.1.3)$$

where, in each system, the w_n^{ij} evolve according to,

$$w_{n+1}^{ij} = \frac{[1 + \delta \cdot g(x_n^i, x_n^j)] \cdot w_n^{ij}}{\sum_{j=1}^N [1 + \delta \cdot g(x_n^i, x_n^j)] \cdot w_n^{ij}}, \quad (4.1.4)$$

and Hebbian dynamics [Sej99] are ensured by,

$$g(x_n^i, x_n^j) = 1 - 2 \cdot |x_n^i - x_n^j|. \quad (4.1.5)$$

All these systems are initiated in the same manner: $w_0^{ij} = \frac{1}{N-1}$ for $i \neq j$ and $w_0^{ij} = 0$ for $i = j$, the x_0^i are distributed uniformly in $(0, 1]$ and δ is a constant governing the plasticity of the connections and is set to 0.1 throughout this work. N gives the number of maps

coupled together in the system and $N = 100$ unless otherwise stated.

It can easily be shown using eqn. (4.1.4) that the total connection strength of the system is conserved in time. Thus, any increase of connection strength between a specific pair of nodes from one timestep to the next will have a corresponding decrease in connection strength elsewhere in the system. Additionally, the connection strengths between nodes i and j are not necessarily symmetric, so node i may connect to node j with a different connection strength than j connects to node i : $w_{ij} \neq w_{ji}$. The function g ensuring Hebbian dynamics through eqn. (4.1.5) gives limits as $g \in [-1, 1]$.

4.2 Determination of clusters

In a previous work of Ito and Kaneko studying Models A & B [IK03], the global a - c parameter space was partitioned according to how many clusters the system splits into. However, there are several methods we may use to denote that two nodes belong to the same cluster. Two such criteria to say that nodes are connected and therefore in the same cluster are *connection-strength threshold* and *node-state similarity*, defined as follows:

Connection-strength threshold: Nodes are said to be connected if the connection strength, w_n^{ij} between two nodes is greater than some threshold value: $w_n^{ij} > \frac{1}{N-1}$. This threshold value is chosen to be the same as the initialisation value.

Node-state similarity: Nodes are said to be connected if the difference between the node-states is below some threshold: $|x_n^i - x_n^j| < \mu$. Throughout the present work, $\delta = 10^{-8}$ is used, unless otherwise specified. It should be noted that results are not significantly altered for other values of δ , both greater and smaller.

Recall, depending on the values of the parameters a and c , Models A & B were found by Ito & Kaneko to display different behaviours [IK02][IK03], as was shown schematically in fig. 3.1:

Coherent: the whole system synchronises.

Ordered: the system consists of several clusters within which the nodes are synchronised.

Disordered: there is no synchronisation between any pair of nodes.

In these definitions, the clusters referred to are clusters defined by the node-state similarity method, as was used throughout the works by Ito & Kaneko.

Since the connection strengths between nodes evolve according to Hebbian dynamics, we might expect that determining the clusters by either method described above would result in comparable classifications. This is not found to be the case for any of the three models; figs. 4.1 and 4.2 show the number of clusters produced by the different models across a range of parameter values, a and c , using these two different cluster determination methods. We can see by comparing these two respective parameter space diagrams that the number of clusters across the a - c parameter space is different for these two cluster determination methods.

The difference is greatest for Model C: with the node-state similarity cluster determination, shown in fig. 4.1(c), the system comprises one or two clusters for all a - c values except two distinct bands of parameter values, where the system splits into many clusters, of the same order as the number of nodes. However, with the connection-strength threshold cluster determination method, as shown in fig. 4.2(c), the system comprises only one cluster for almost all parameter values. There are some distinct a -values where the system comprises a few clusters but only for a narrow-band of coupling strengths. Only for extremely low c -values and high a -values does the system ever enter a state with many clusters.

When applying the connection-strength threshold method of cluster determination to Models A & B, once again in the corner of parameter space with low c -values and high a -values, the system is found to comprise only one cluster. So, whilst the network of connections is of sufficient weight to produce a connected network, the node-states themselves are not synchronised as shown by the number of clusters counted via the node-state similarity method in figs. 4.1(a) & 4.1(b).

This leads us to question what meaning we should take from the cluster numbers found through the connection-strength threshold method. Since the connection strengths obey Hebbian dynamics, the stronger connections should represent nodes whose node-states are similar. It could be suggestive that whilst the node-states do not completely synchronise, they are correlated. Or, perhaps the connection-strength threshold method finding only

one cluster in the corner of parameter space with high- a and low- c could be due to all nodes being equally uncorrelated from one another. Since connection-strengths between similar nodes are strengthened at the detriment of nodes that are not alike, if all nodes are equally dissimilar, many connection strengths may be maintained above the threshold value at which they are initialised and therefore be counted as connected by the connection-strength cluster determination method.

Whatever the reason, it clearly depends upon the application of the model one has in mind as to which of these cluster determination methods will be most appropriate. It is however quite clear that these two cluster determination methods are not equivalent and give very different information about the systems, even despite the co-dependence of the connections on the node-states and vice versa. It is also clear that the linear coupling implemented in Model C starkly changes the dynamics of the system. Whilst Models A & B appear to show similar global behaviours. This can be seen easily in figs. 4.1(d) & 4.2(d), where the difference between the number of clusters in Model A and Model B is shown across the parameter space. There are patchy regions of difference produced by both cluster determination methods; clustered along the boundary regions where, for example, the system changes from being in a 1-cluster state to a k -cluster state ($k > 1$) as c is reduced. They are also seen at parameter values where the connection strength threshold method finds the system changes from a k -cluster state ($k > 1$) to a 1-cluster state as c is reduced further.

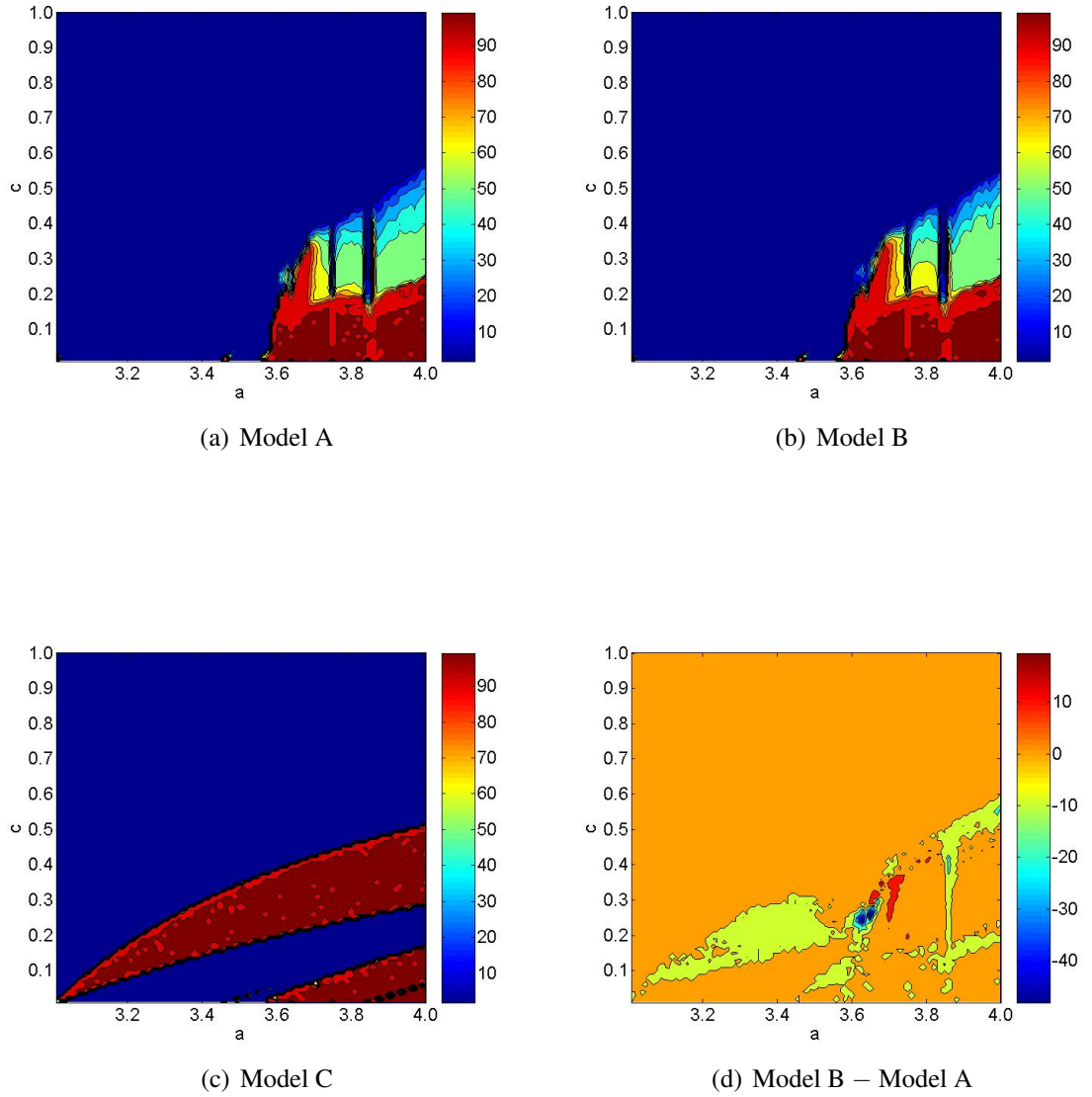


Figure 4.1: The number of clusters as determined by the node-state similarity across the parameter space for the specific Models as indicated. The average of 100 realisations is shown with different initial conditions and the simulations are run to a maximum timestep of 10^4 .

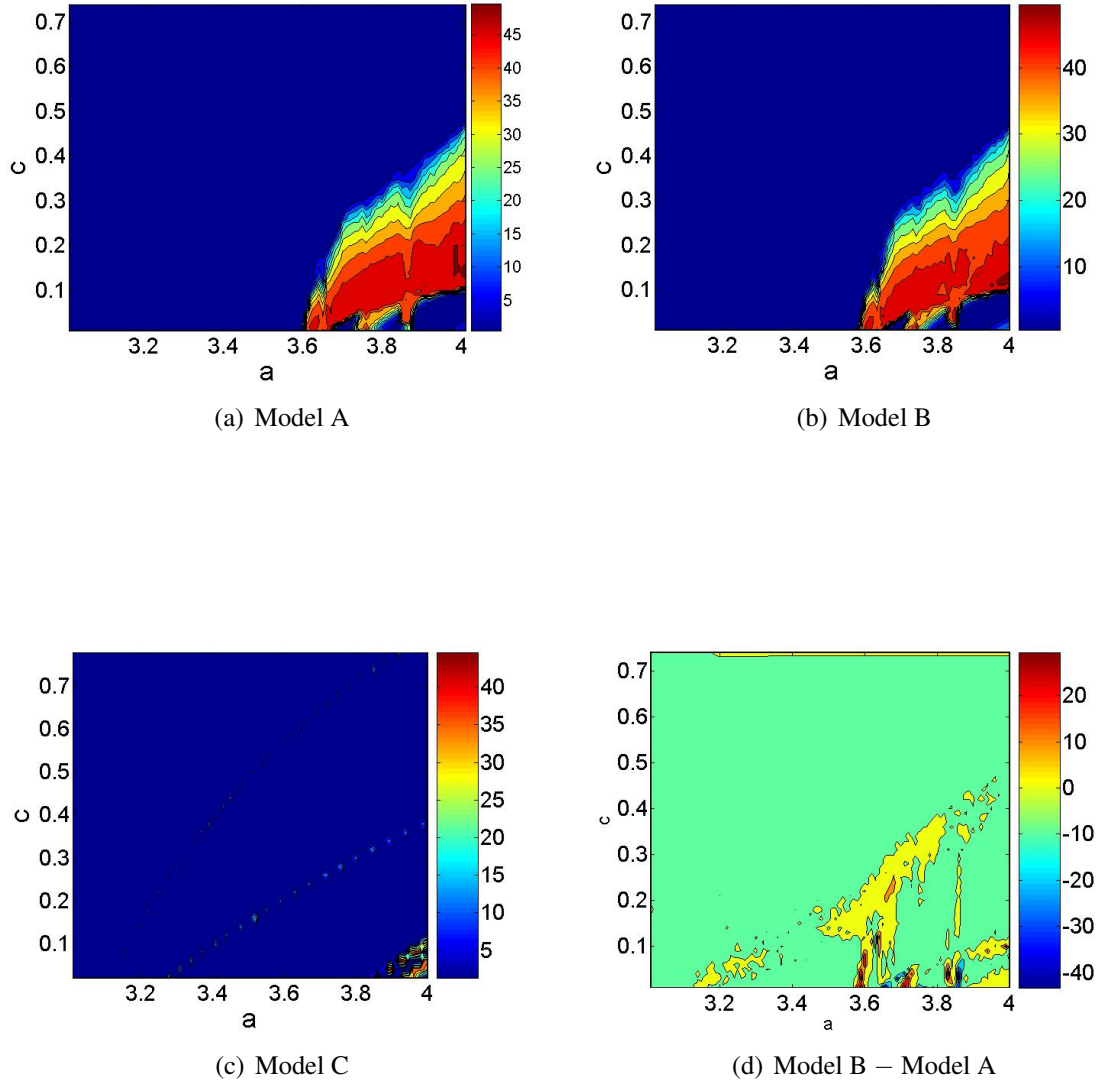


Figure 4.2: The number of clusters as determined by the connection-strength threshold method, across the parameter space for Models as indicated. The average of 100 realisations is shown with different initial conditions, the simulations are run to a maximum timestep of 10^4 .

4.3 Blowout bifurcation

For sufficiently high coupling strength, c , all three models display completely synchronous behaviour, $x_n^i = x_n^j \forall i, j$, as characterises the coherent state. Stability of this state can be calculated via the Lyapunov exponent corresponding to the direction transverse to this synchronous submanifold. The point at which the transverse Lyapunov exponent becomes positive is where the system undergoes a blowout bifurcation. These Lyapunov exponents can be calculated for all three models if we assume that the connection strengths are homogeneous and static:

$$w_n^{ij} = \begin{cases} \frac{1}{N-1} & \text{for } i \neq j, \\ 0 & \text{for } i = j. \end{cases}$$

This is a reasonable assumption to make since the connection strengths do become static when nodes synchronise and numerical checks find them to be well described by this homogeneous distribution, as has been reported previously [IK03]. In making this assumption, the three systems simplify to the following:

Fixed A, simplified internally coupled:

$$x_{n+1}^i = f \left[(1 - c) \cdot x_n^i + \frac{c}{N-1} \cdot \sum_{j \neq i} x_n^j \right]. \quad (4.3.1)$$

Fixed B, simplified externally, nonlinearly coupled:

$$x_{n+1}^i = (1 - c) \cdot f(x_n^i) + \frac{c}{N-1} \cdot \sum_{j \neq i} f(x_n^j). \quad (4.3.2)$$

Fixed C, simplified externally, linearly coupled:

$$x_{n+1}^i = (1 - c) \cdot f(x_n^i) + \frac{c}{N-1} \cdot \sum_{j \neq i} x_n^j. \quad (4.3.3)$$

We now calculate the Lyapunov exponent via

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |J|.$$

For Model A, we use the derivative,

$$\begin{aligned} \frac{\partial x_{n+1}^i}{\partial x_n^i} &= \frac{\partial}{\partial x_n^i} \left[f(x_n^i \cdot (1 - \frac{c \cdot N}{N-1}) + \frac{c}{N-1} \cdot \sum_{j=1}^n x_n^j) \right] \\ &= \left[f'(x_n^i) - \frac{2 \cdot a \cdot c}{N-1} \cdot \sum_{j=1}^N x_n^j + \frac{2 \cdot a \cdot c \cdot N}{N-1} \cdot x_n^i \right] \cdot \left[1 - \frac{c \cdot N}{N-1} \right]. \end{aligned}$$

Recall, we are considering the stability of the synchronous state where $x_n^j = x_n^i$. Thus, we obtain the Lyapunov exponent as,

$$\begin{aligned} \lambda_i &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T \ln \left| \left[f'(x_n^i) - \frac{2 \cdot a \cdot c}{N-1} \cdot \sum_{j=1}^N x_n^j + \frac{2 \cdot a \cdot c \cdot N}{N-1} \cdot x_n^i \right] \cdot \left[1 - \frac{c \cdot N}{N-1} \right] \right| \\ &= \ln \left| 1 - \frac{c \cdot N}{N-1} \right| + \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T \ln \left| \left[f'(x_n^i) - \frac{2 \cdot a \cdot c \cdot N}{N-1} \cdot x_n^i + \frac{2 \cdot a \cdot c \cdot N}{N-1} \cdot x_n^i \right] \right| \\ &= \ln \left| 1 - \frac{c \cdot N}{N-1} \right| + \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T \ln |f'(x_n^i)|. \end{aligned}$$

There also exist $(N-1)$ degenerate Lyapunov exponents associated with the derivatives,

$$\frac{\partial x_{n+1}^i}{\partial x_n^j} = \frac{c}{N-1} \cdot \left[a - \frac{2 \cdot a \cdot c}{N-1} \cdot x_n^j - a \cdot x_n^i \left(1 - \frac{c \cdot N}{N-1} \right) \right].$$

With values,

$$\begin{aligned} \lambda_j &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T \ln \left| \frac{c}{N-1} \cdot \left[a - \frac{2 \cdot a \cdot c}{N-1} \cdot x_n^j - a \cdot x_n^i \left(1 - \frac{c \cdot N}{N-1} \right) \right] \right| \\ &= \ln \left| \frac{c}{N-1} \right| + \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T \ln \left| \left[a - \frac{2 \cdot a \cdot c}{N-1} \cdot x_n^j - a \cdot x_n^i \left(1 - \frac{c \cdot N}{N-1} \right) \right] \right|. \end{aligned}$$

The transverse Lyapunov exponent, $\lambda_{\perp}$, for Model A is the Lyapunov exponent, λ_i just calculated. Through similar simple algebra for Models B & C, we obtain the transverse Lyapunov exponents for these models respectively. The transverse Lyapunov exponents for the three models are as follows,

Models A & B:

$$\lambda_{\perp} = \ln \left| 1 - \frac{c \cdot N}{N-1} \right| + \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T \ln \left| f'(x_n^i) \right|. \quad (4.3.4)$$

Model C:

$$\lambda_{\perp} = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T \left[\ln \left| f'(x_n^i) \right| + \ln \left| (1-c) - \frac{c}{(N-1) \cdot f'(x_n^i)} \right| \right]. \quad (4.3.5)$$

The point when $\lambda_{\perp}$ becomes positive denotes where the system undergoes a blowout bifurcation. We can see where in parameter space this occurs in fig. 4.3; the line separating the white area from the grey areas is the line of parameter values along which the blowout bifurcation occurs.

It may appear from eqns. (4.3.4) & (4.3.5) that whilst Models A & B have the same criterion for the loss of stability of the synchronous state (the blowout bifurcation), Model C has a different criterion. Fig. 4.3(a) shows the values of this analytical expression for the transverse Lyapunov exponent for Models A & B across the a - c parameter space and fig. 4.3(b) shows the values of the expression for Model C. By comparing these two figures, we see that whilst analytically the expressions are very different, this does not affect the numerical values across parameter space. In fact, the two criteria have a maximum difference of $\sim 3\%$ across the whole parameter space, but for most a - c values, the difference is $< 0.1\%$.

We will now look at the behaviour of the system in simulations so as to observe what happens as we move through parameter space and across the blowout bifurcation curve. By looking at the behaviour at fixed a and varying c we can see what happens to the behaviour as we cross this blowout bifurcation curve; we might expect to see the system change from displaying coherent behaviour when $\lambda_{\perp} < 0$ to perhaps a two-cluster state when $\lambda_{\perp} > 0$, as has been implied by Ito and Kaneko's work [IK03].

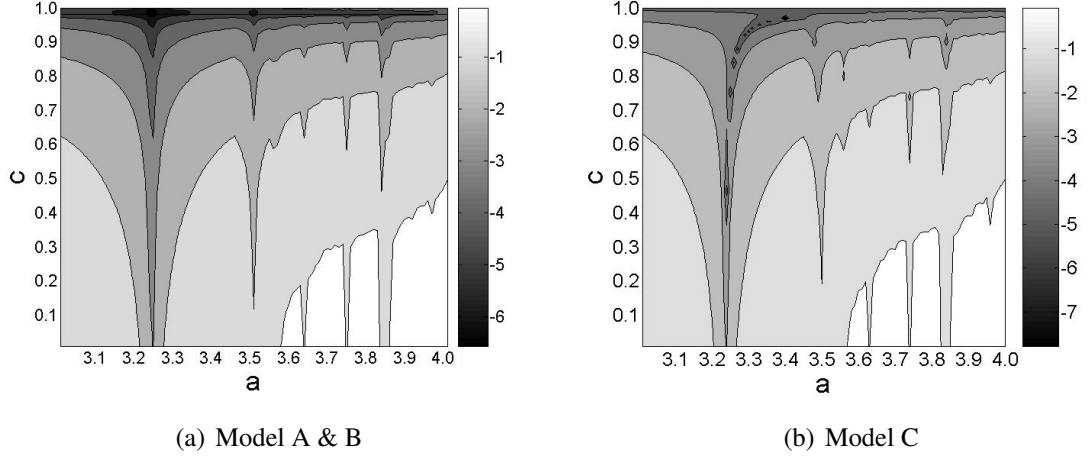


Figure 4.3: Parameter space with regions shown where the transverse Lyapunov exponents are positive (white area) and negative (all other regions). These areas of $\lambda_{\perp} < 0$, are where the synchronous state is theoretically stable. These calculations of $\lambda_{\perp}$ are done for a timeseries of 10^4 ; this was found to be sufficiently long for the limits of eqns 4.3.4 and 4.3.5 to converge, that is, except for $a = 4.0$, where the analytic value of the Lyapunov exponent for the logistic map is used, $\ln(2)$. As can be seen, the numerical values of $\lambda_{\perp}$ for the three different models are very similar across the parameter space. They are in fact less than 0.1% different for most parameter values. At high c -values for $3.2 \lesssim a \lesssim 3.4$, this difference rises to as much as $\sim 3\%$, but we are not especially interested in this region.

In order to analyse this change in behaviour, we must choose a basic measure of whether the system is synchronised. One such measure is the variance of the node-state: $\sigma_n^2 = \sum_i (x_n^i - \bar{x}_n)^2$ where $\bar{x}_n$ is the average of all node-states at the n th timestep. Obviously, if all nodes are synchronised, $\sigma_n^2 \rightarrow 0$ as $n \rightarrow \infty$. In order to gain a representative measure of the system, rather than at a single timestep, we take the average of σ_n^2 : $\sigma^2 = \frac{1}{s-r} \sum_{n=r}^s \sigma_n^2$. We will take this average over a suitably large number of timesteps and choose our start time, r , so as to exclude any initial transient¹. We will see later how the time evolution of σ_n^2 compares between the different models and coupling strengths, c . In fig 4.4 we can see the behaviour of σ^2 for the three models and different coupling strengths, c . The average has been taken over 1000 timesteps: $9000 \leq n < 10,000$. It shows that the transition from

¹Much care has been taken when choosing the time after which it can be deemed that the system has reached its final state and the transient can reliably be considered over. By running many simulations for up to 10^7 timesteps, it has been concluded that after ~ 5000 timesteps that system is sufficiently converged to its final state, so a transient in excess of this will be needed when studying the final state of any system.

coherent state (at high c) to ordered state (at lower c) occurs in a very different manner for Model C as compared to Models A & B. What is the same for Models A, B, and C, is that the coupling strength corresponding to loss of stability according to the analytical criterion $\lambda_{\perp} = 0$ does not coincide with what is observed in simulations; all three systems are found numerically to synchronise when the synchronous state is linearly unstable, $\lambda_{\perp} > 0$. However, when we observe numerically what happens for the simplified systems of type A and B with fixed all-to-all couplings of eqns. (4.3.1) and (4.3.2) we see that whilst the loss of stability of the synchronous state according to $\lambda_{\perp}$ does not coincide with the numerical point in parameter space where the system stops reaching the coherent state, characterised by $\sigma^2 = 0$, the systems do not synchronise until the coupling strength is such that $\lambda_{\perp} < 0$.

By comparing the full Models with coevolving connections to the simplified systems with static connections in fig. 4.4, we see that the coevolving connections for Models A & B result in a higher coupling strength to be needed in order for the coherent state to be reached. Whereas for Model C, the two versions both stop reaching the coherent state at the same coupling strength. For Models A & B, there is a range of coupling strengths for which the synchronous state is linearly stable ($\lambda_{\perp} < 0$) yet the system does not completely synchronise. Since this does not occur for the similarly coupled systems with fixed all-to-all coupling, it must be a result of the coevolution of the connections, in contrast to previous claims that it only occurs in systems of coupled, discontinuous maps [GLPT03][GH06][AP02].

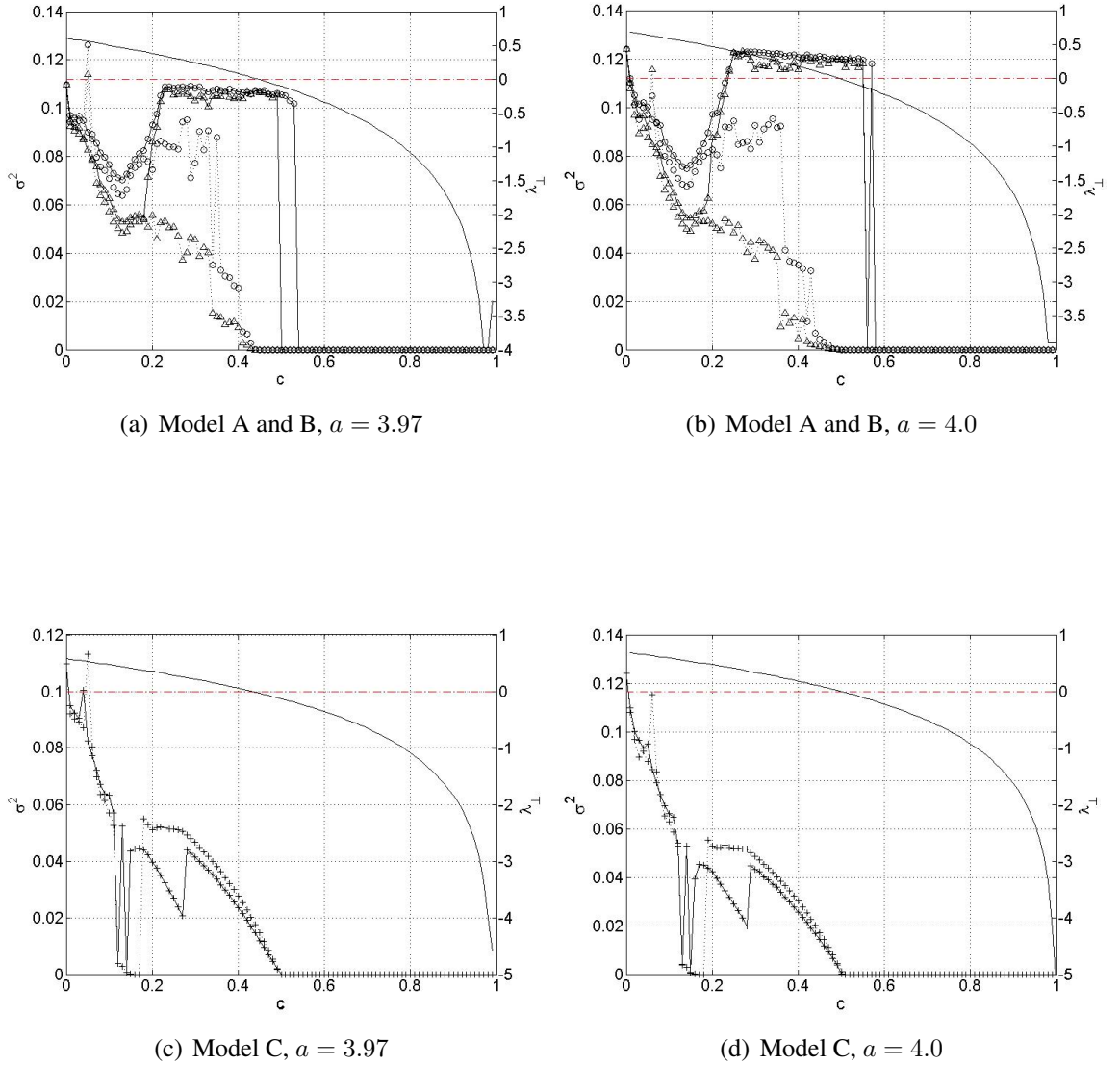


Figure 4.4: The node-state variance, σ^2 averaged over 1000 timesteps, n , is plotted versus the coupling strength, c . Also plotted versus the coupling strength, c , the transverse Lyapunov exponent values as the solid line without any marker. The other solid lines denote those models with coevolving connections, whereas dashed lines denote the simplified systems with all-to-all fixed connections, ie. $w_{ij}^i = \text{constant} \forall i, j$. Models of type A (internally coupled): $\circ$, type B (external with nonlinear coupling): $\triangle$ and type C (external with linear coupling): $+$.

4.4 Riddling bifurcation

As defined in Chapter 2, a system undergoes a riddling bifurcation when the first periodic orbit embedded in the chaotic attractor loses stability. Since the fixed point of the logistic map, $x^* = 1 - \frac{1}{a}$, is also a fixed point of all the coupled systems, we may look at the stability of this fixed point in order to see at which parameter values this riddling bifurcation takes place for Models A, B & C. Consider a perturbation away from the fully-synchronised system state that is on the fixed point for Model A:

$$\begin{aligned} x_{n+1}^i &= x^* + \delta_{n+1}^i \\ &= f \left[(1 - c) \cdot (x_n^i + \delta_n^i) + c \cdot \sum_j w_n^{ij} (x_n^j + \delta_n^j) \right] \\ &= x^* + (1 - c) \cdot (2 - a) \cdot \delta_n^i + c \cdot (2 - a) \cdot \sum_j w_n^{ij} \delta_n^j. \end{aligned}$$

We assume a perturbation in the i th direction only and re-arranging this expression, we obtain the following criterion for the stability of the fixed point:

$$\frac{1 - a}{2 - a} < c < \frac{3 - a}{2 - a}. \quad (4.4.1)$$

For this perturbation analysis to linear order, similar simple algebra also leads to exactly the same criterion for Models B & C. Thus, all three models undergo a riddling bifurcation at the same point in parameter space, this is shown in fig. 4.5. When we consider the second order terms in the perturbation analysis, the evolution of δ_{n+1}^i for Model A is different to the evolution followed by δ_{n+1} for Models B and C. In the next section, we will see how these analytical bounds on the stability of the fixed point to linear order tie-in with the behaviours displayed by the models.

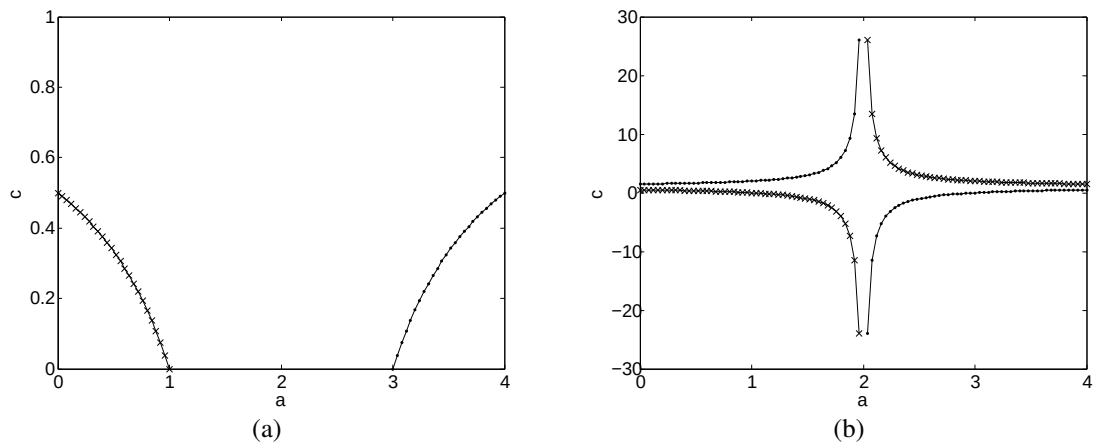


Figure 4.5: The maximum ($\times$) and minimum ($\cdot$) bounds on coupling strength, c , for which the fixed point is stable according to eqn. (4.4.1) are shown as a function of nonlinearity parameter, a . In (a) we show the bounds applicable in these models where $c \in [0, 1]$, whereas (b) shows the full bounds of eqn. (4.4.1).

4.5 Bifurcations across parameter space

Fig. 4.6 shows where in the parameter space the riddling and blowout bifurcations occur relative to one another. In addition, these theoretical curves are superimposed on the previous plots of the number of clusters². From these, we can see that the change from coherent to ordered state for Models A & B roughly coincides with the blowout bifurcation. Whereas for Model C, it coincides with the riddling bifurcation, since the riddling bifurcation occurs at higher c -values, the blowout bifurcation causes no change to Model C's dynamics. In contrast, for Models A & B, as shall be seen later, the coherent state is characterised by aperiodic dynamics, so the destabilisation of the fixed point has no visible effect on the system dynamics. These differences between Models A & B, and C will be explained further in Chapter 5.

4.6 Transition from ordered to coherent region

In the previous section, we saw that our analytically derived expressions for the loss of stability of the synchronous state do not tie-in exactly with where we observe numerically that these systems stop displaying this coherent behaviour, this is seen clearly in Table. 4.1 by comparing the coupling strength values that correspond to the riddling & blowout bifurcations, with the coupling strength value where the system is found to synchronise numerically. This leads us to ask the question as to where exactly do the boundaries between these regions of parameter space occur? Let us consider the transition between the ordered state and the coherent state that occurs for Models A & B just above the blowout bifurcation. By inspection of figs. 4.1 and 4.2 we see that for $a = 3.97$ (the value for which these systems are previously studied) this transition occurs at $c \approx 0.55$. This is higher than the theoretical location that the transition would take place if it were governed purely by the linear stability of the synchronous state. Both Models A & B do not necessarily reach the coherent state, despite it being linearly stable.

Generally, when we compare the behaviour of Models A & B at the same parameter values close to this ordered-coherent boundary, Model B has a higher fraction of initial

²As measured using the node-state similarity method.

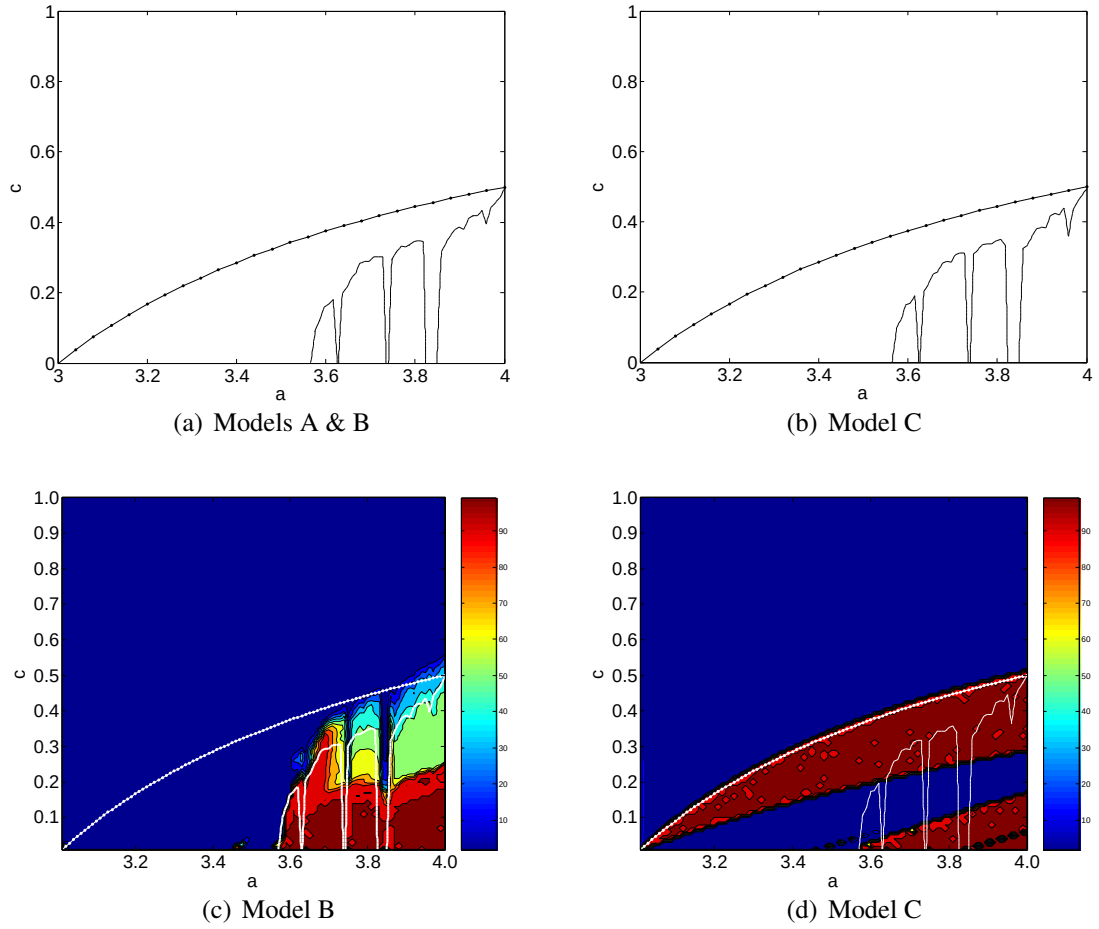


Figure 4.6: The riddling bifurcation (solid line with $\cdot$) and blowout bifurcation (solid line) are shown together along with the global behaviour from simulations, across the parameter space. The top row shows only the analytical bifurcation lines in parameter space given by eqns. (4.4.1) and (4.3.4). The bottom row shows these same analytical lines relative to the numerically calculated number of clusters observed according to the node-state similarity criterion, as previously shown in fig. 4.1.

conditions attracted to the coherent state. Figs. 4.7 and 4.8 show some example node-state evolutions for a number of different initial conditions. By comparing figs. 4.7(b) and 4.7(d) we can see that of five random initial conditions with $c = 0.50$ and $a = 3.97$, none result in the coherent state for Model A, whereas two lead to the coherent state for Model B. From comparing the early timesteps of these realisations, figs. 4.7(a) and 4.7(c), there is no obvious visual distinction to suggest this outcome. In fact, from fig. 4.7(a), we might expect

System Type	$Min(c) _{\lambda_{\perp} < 0}$	$c _{x^* \text{ stable}}$	$Min(c) _{\sigma^2 \simeq 0}$
Model A	0.45	0.49	0.54
Model B	0.45	0.49	0.50
Model C	0.44	0.49	0.50
Fixed A	0.45	0.49	0.45
Fixed B	0.45	0.49	0.44
Fixed C	0.44	0.49	0.50

Table 4.1: For $a = 3.97$, the values shown in this table for the different models are as follows: (1) $Min(c)|_{\lambda_{\perp} < 0}$: the lowest value of c for which $\lambda_{\perp} < 0$; (2) $c|_{x^* \text{ stable}}$: the value of c where the fixed point gains/loses stability; (3) $Min(c)|_{\sigma^2 \simeq 0}$: the lowest value of c for which $\sigma^2 \simeq 0$. These values have been recorded from data with a resolution in c of 0.01. Fixed A refers to the simplified version of Model A, with fixed connections, $w_n^{ij} = \text{constant}$. Fixed B & C are similarly defined.

two of these realisations of Model A to end up in the coherent state. Fig. 4.8 shows a similar trend for a higher coupling strength of $c = 0.55$. At this slightly higher coupling strength, for both models, the attractor for the coherent state clearly occupies a larger volume of the phase space as seen by a larger fraction of this small set of initial conditions resulting in the coherent state. Notice that for this a -value, for both these values of coupling strength, the synchronous state is linearly stable; recall the values of the blowout bifurcation given in table 4.1.

The general trend that the basin of attraction for the coherent state increases in volume as we increase the coupling strength c is shown in fig. 4.9 for Models A & B. This figure shows the percentage of realisations that result in the coherent 1-cluster state. Clearly, for larger system sizes, a higher coupling strength is required to achieve a specific fraction of realisations to end in the coherent state. Although the data shown in fig. 4.9 is not conclusive, it is suggestive that this trend does not continue to infinite system size, since the size of the coherent basin of attraction for systems of 2000 nodes is comparable with systems of 1000 nodes. What is certainly clear is that the transition from the coherent attractor's basin being of negligible volume at lower c -values to occupying the entirety of phase space at higher c -values occurs in a continuous manner; there is no c -value at which the volume suddenly jumps in size, for any of the system sizes that it has been numerically

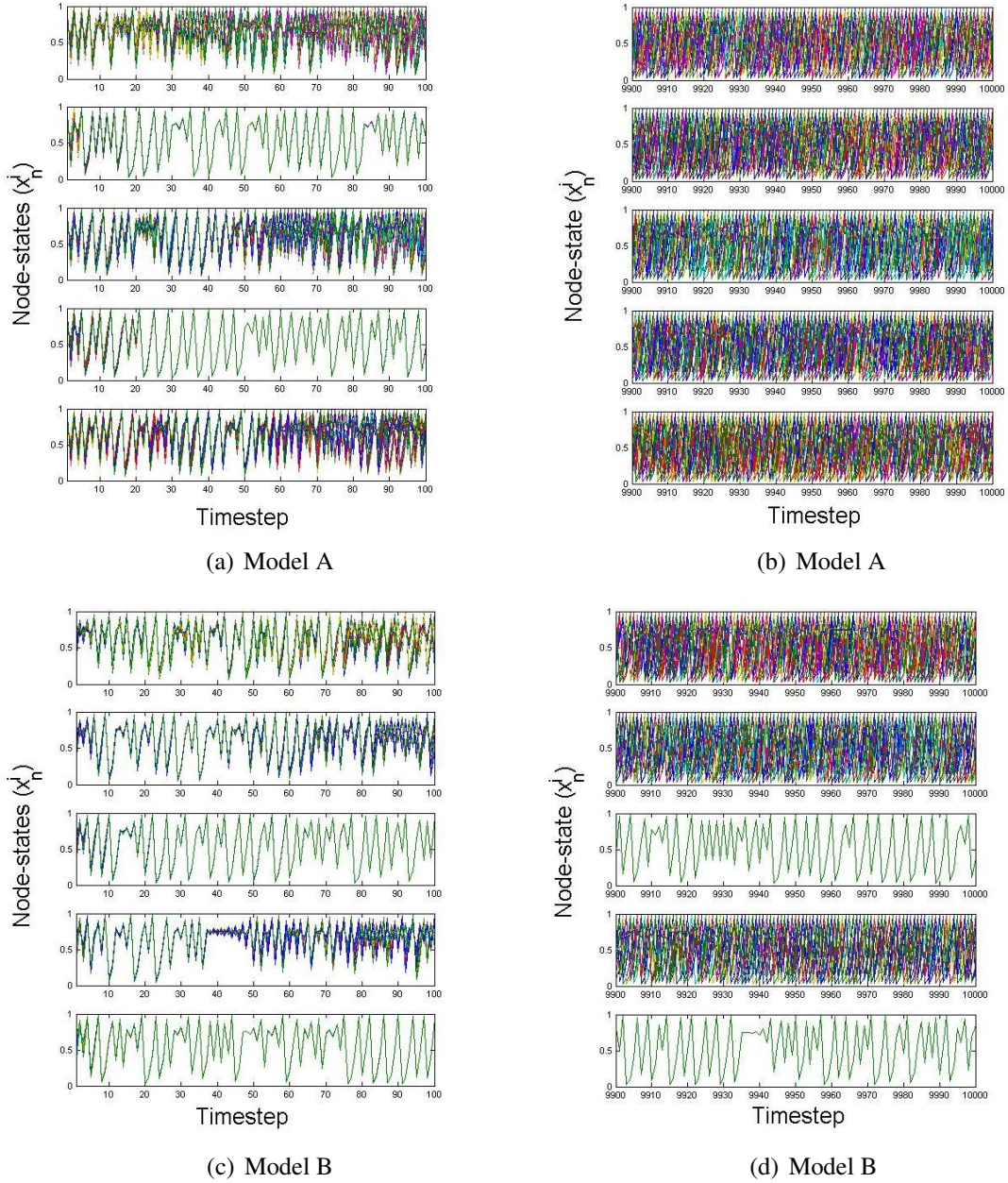
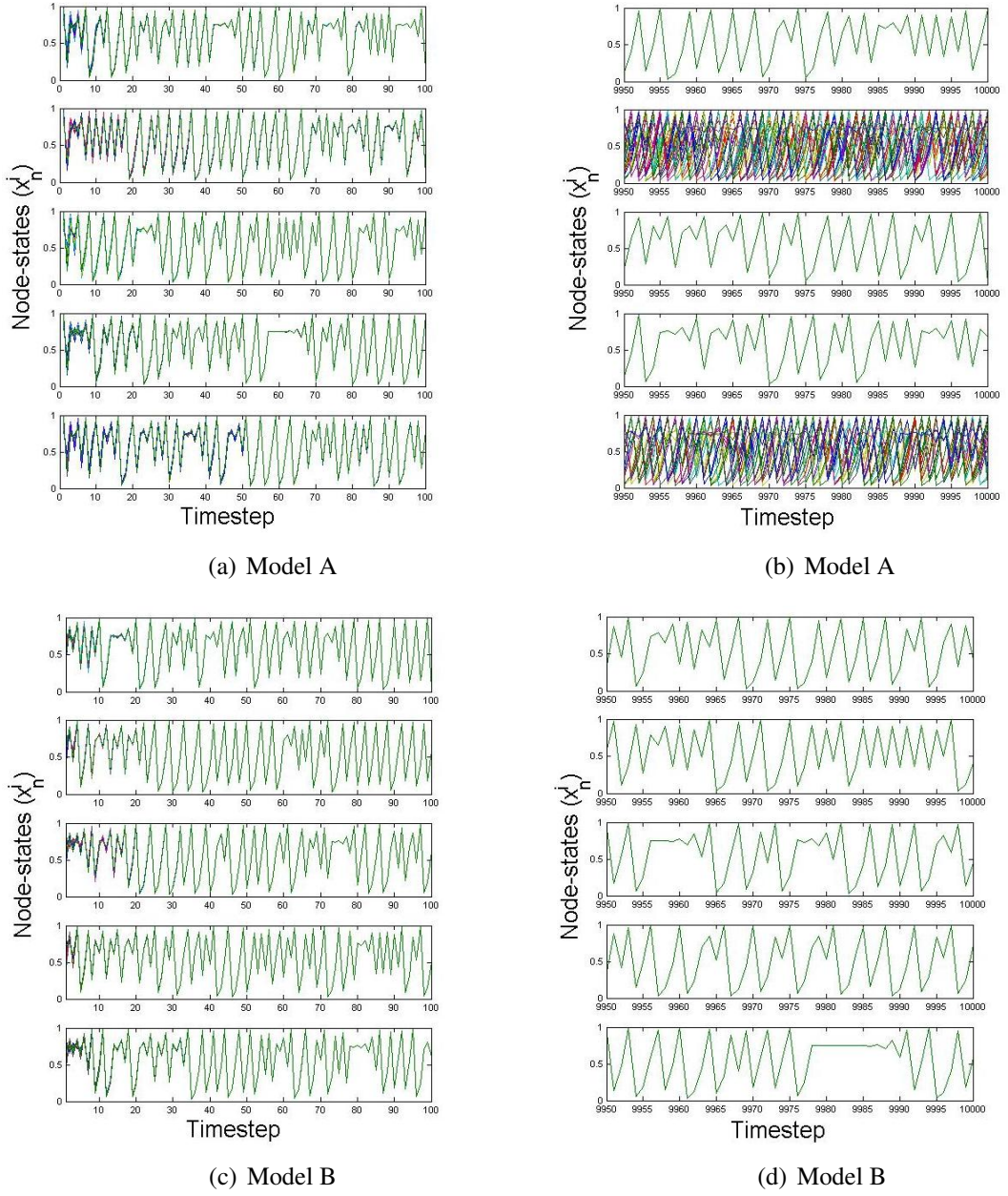


Figure 4.7: Node-state evolution for a system with $N = 100$, $a = 3.97$ and $c = 0.50$. Each graph has the resultant 100 node-states plotted on the same axes. Consequently, when the node-states completely synchronise, they lie directly on top of one another, giving the illusion that only one node-state is plotted. The left-hand column shows the first 100 timesteps and the right-hand column shows the last 100 timesteps of a simulation of 10^4 timesteps. Within the figures, are 10 random initial conditions, some of which are attracted to the coherent state, whereas others are not.

Figure 4.8: The same as Figure 4.7, only for $c = 0.55$.

feasible to simulate³.

In contrast, for Model C, the basin of attraction for the coherent state switches suddenly from not existing to occupying all of phase space. Fig. 4.10 shows the fraction of initial conditions that reach the coherent state versus coupling strength, c , for Model C. For a system size of 100 nodes and $c = 0.51$, no realisations become coherent, whereas for $c = 0.52$, all the realisations become coherent.

Fig. 4.11 shows how the node-state variance per node varies with coupling strength, c , for $a = 3.97$. The node-state variance per node, $\frac{\sigma^2}{N}$ is the average of σ_n^2 over 1000 timesteps, $9000 \leq n < 10,000$ and normalised by the number of nodes in the system for ease of comparison across a range of system sizes. It is clear to see that once again, Model C has very different characteristics to Models A & B. As already seen, for Models A & B, larger systems require stronger coupling strengths in order to achieve complete synchronisation between all nodes. In contrast, for Model C, the value of variance per node at fixed coupling strength is in fact less for larger system sizes. Since this is a measure of how synchronised the system is, unexpectedly, Model C synchronises more easily when the system is larger. Additionally, Model C shows a certain level of synchronisation is achieved for a range of much lower coupling strengths, around $c \sim 0.25$. This will be discussed in more detail in Chapter 5.

Model A has an interesting feature developing at $c \sim 0.5$ in fig. 4.11(a). As the system size is increased, the average variance per node has an increasing reduction. At this value of coupling strength, the model becomes $x_{n+1}^i = f[\frac{1}{2}x_n^i + \frac{1}{2}\sum w_n^{ij}x_n^j]$. That is, every node is equally influenced by its own previous state as the weighted average of the other nodes. It would be interesting to investigate if this trend towards synchronisation continues for even larger system sizes. Unfortunately, computational limitations have restricted this.

The sharpness of the transition to the coherent state cannot be seen in the behaviour of $\langle \sigma^2 \rangle$ as c is varied at fixed a , where $\langle \dots \rangle$ denotes the average over an ensemble of initial conditions, as was shown in fig. 4.11. Whilst σ^2 can be viewed as giving a measure

³It has not been possible to simulate any system larger than 2000 nodes due to computational limitations; for example, the numerics to generate the data for either Model A or B with 2000 nodes as shown in 4.9 took 2 months of processing using Imperial College's High Performance Computing Cluster (HPC). The actual number of processors in use at any one moment did fluctuate, but it was of order 100 at most times checked.

of the distance from the synchronous submanifold, if the node-states do not occupy the entirety of phase space, taking an average over an ensemble of initial conditions can lead to a low value for $\langle \sigma^2 \rangle$. This similarly low value for $\langle \sigma^2 \rangle$ can also be obtained if the sum of the ensemble of initial conditions converge to the coherent state ($\sigma^2 = 0$) and some have not, but the node-states occupied a larger region of phase space. The first of these two scenarios is what leads to the seemingly gradual transition for Model C seen in fig. 4.11(c). Whereas for Models A & B the gradual transition is due to the latter of these two mechanisms. This will be seen later, in Chapter 5.

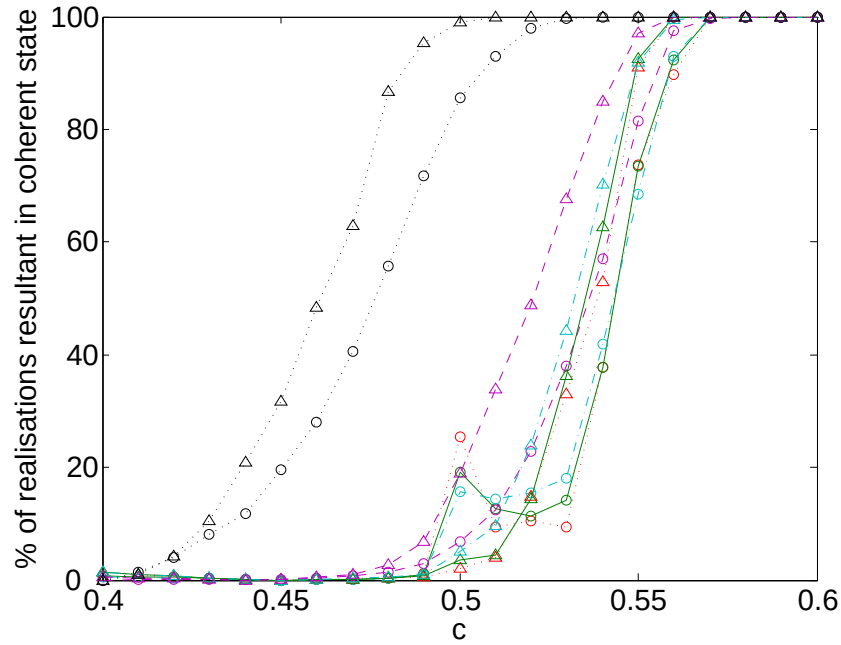


Figure 4.9: The percentage of 1000 random initial conditions that result in a system comprising one connected cluster (as measured using the connection-strength threshold cluster determination) after 10^4 timesteps and for $a = 3.97$. This graph shows the same data for Models A ($\circ$) and B ($\triangle$) as well as for different system sizes: 10 (black), 100 (purple), 500 (cyan), 1000 (green), 2000 (red).

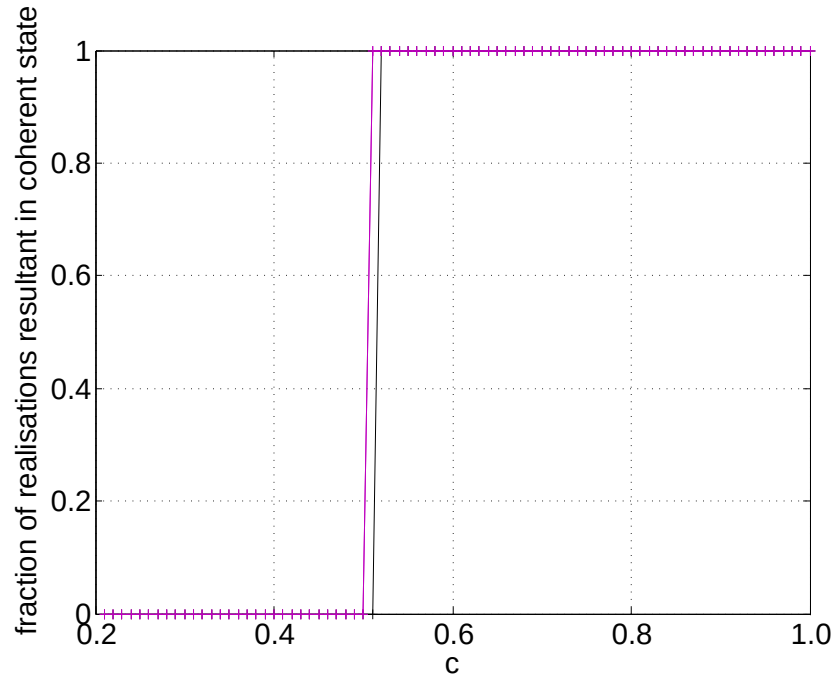


Figure 4.10: For Model C, the fraction of 500 random initial conditions that result in the coherent state is plotted as a function of the coupling strength, c . The simulations are run for 10^4 timesteps and for $a = 3.97$. This graph shows data different system sizes: 50 (black), 100 (purple), 500 (cyan).

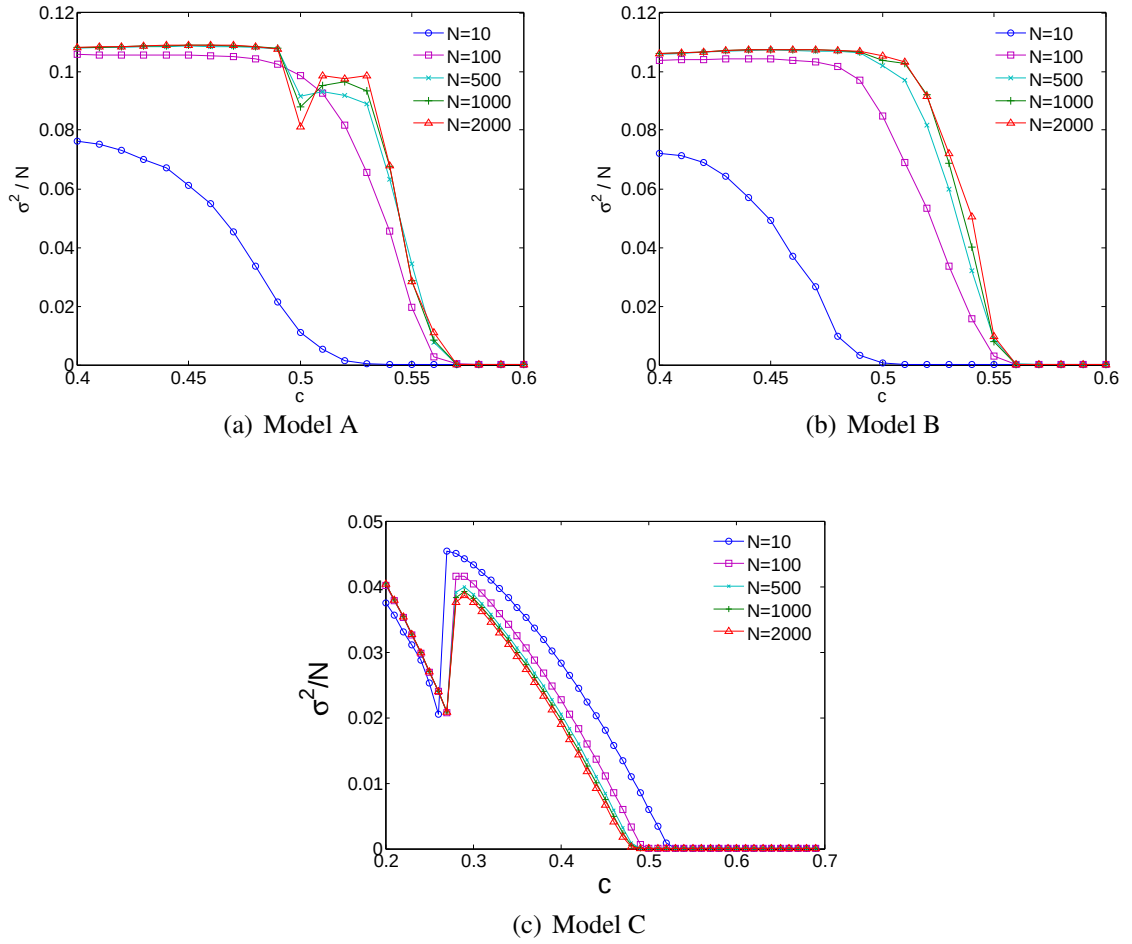


Figure 4.11: The node-state variance, σ_n^2 , is plotted against coupling strength, c , for the three different models and for different system sizes, N . In each case, σ^2 is scaled by N for ease of comparison. The data plotted is the average over 1000 random initialisations as measured at 10^4 timesteps and for $a = 3.97$.

4.7 Concluding remarks

We have seen how sensitive the macrodynamics of coupled-map systems are to changes in the microdynamics. Whilst the assumption by Ito and Kaneko that Models A & B are equivalent is reasonably accurate at the global level, the two Models do have distinct behaviours for some intermediate values of the coupling strength. This appears to become ever more apparent as the system size is increased, as suggested by the increasing size of the “dip” in $\frac{\sigma^2}{N}$ for larger system sizes shown in fig. 4.11(a).

In contrast to the similarities between Models A & B, Model C displays very different behaviours by almost any measure we choose to look at: the regions of parameter where Model C synchronises is different by both cluster determination methods; the effect of the coevolving connections is minor in comparison to the large difference it makes for Models A & B; the riddling bifurcation governs the loss of the synchronous state rather than the blowout bifurcation that governs Models A & B; the transition out of the coherent state is sharp for C in comparison to the gradual crossover for A and B.

Chapter 5

Results 2: The coherent state and its basin of attraction

5.0.1 Chapter summary and outline

This chapter discusses some topological aspects of synchronisation in the three models introduced in Chapter 4. Different dynamical behaviour displayed by Model C in the coherent state is demonstrated and explained; the approach to the synchronous state is investigated and the structure of the basin of attraction for the Models is also presented; we compare the distributions of where in phase space the node-states spend their time with the theoretical distribution for the single logistic map discussed in Chapter 2. Once again, there is a dichotomy between what is found for Model C in comparison to Models A & B. Some behaviours of the reduced models with fixed all-to-all connections are also presented for comparison with the analytical explanations where the assumption of simpler connectivity has been made.

5.1 Coherent state dynamics

When in the coherent state, Model C displays completely different dynamics to Models A & B. This can clearly be seen in fig. 5.1 which are bifurcation diagrams for the three models. It can also be seen by comparing the examples of typical node-state time-

series shown for Models A & B in fig. 5.2, with that shown for Model C in fig. 5.3. The dynamics displayed by Models A & B appears aperiodic, yet Model C displays steady-state behaviour. The value of the node-state, x_n^i , in the time-independent state of Model C coincides with the value of the fixed-point of the logistic map, $x^* = 1 - \frac{1}{a}$. As discussed in Section 2.11, for a single logistic map this fixed point is only stable for $1 < a < 3$. However, it appears to be possible to “stabilise” this fixed point for a -values where the underlying map is chaotic through the particular coupling mechanism of Model C. It is not however a result of the coevolving connections; fig. 5.4 shows the bifurcation diagrams for the simplified versions of each model so that the nodes are connected by fixed all-to-all connections. These simpler systems are consequently referred to as Fixed A, Fixed B and Fixed C. Fixed C displays the same steady-state behaviour when all nodes are synchronised, the same as Model C. Fixed A and B display similar aperiodic dynamics to Models A & B when all nodes are completely synchronised.

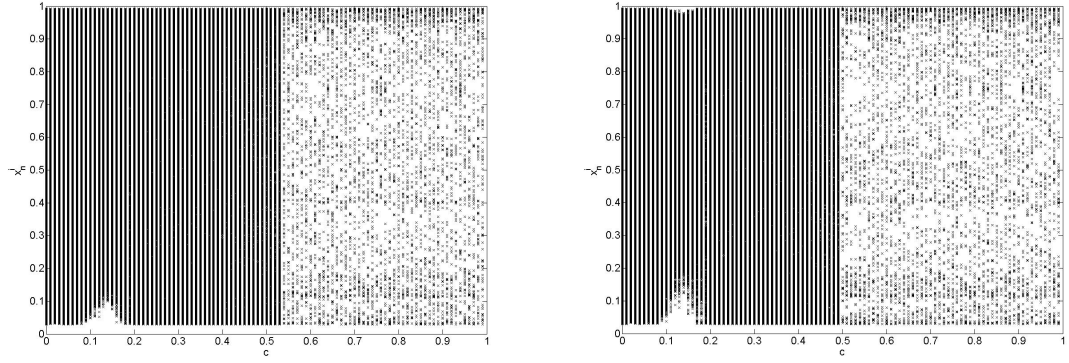
The criterion for stability of the fixed point $x^* = 1 - \frac{1}{a}$ in Model C was given by eqn. (4.4.1):

$$\frac{1-a}{2-a} < c < \frac{3-a}{2-a}.$$

According to this criterion, when $a = 3.97$, the fixed point is stable for $c \gtrsim 0.49$. However, the criterion of eqn. (4.4.1) should equally apply for Models A & B, but node-states evolving to the fixed point is not seen in these models. Nor does either Model A or B remain on the fixed point for long times, even if all node-states are initialised at the value $1 - \frac{1}{a}$.

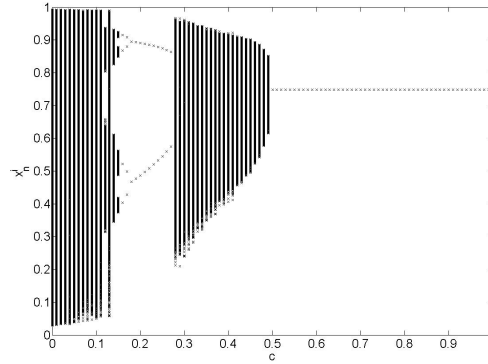
This difference of behaviours between the models can be understood by noticing from fig. 4.6 that for all a -values, the fixed point destabilises at a higher coupling strength than the synchronised state. That is, as we lower the coupling strength at a specific a -value, the system undergoes a riddling bifurcation first and then a blowout bifurcation. Therefore, for all parameter values where the fixed point is stable, the synchronous state is also stable.

Now, consider the equations governing the node-state evolution when the nodes are synchronised in each of the three models:



(a) Model A

(b) Model B



(c) Model C

Figure 5.1: These are bifurcation diagrams for the systems as labeled. They are graphs of the node-states, x_n^i for all nodes and values of $9000 \leq n < 9100$ for values of $c \in [0, 1)$ with a resolution of 0.01. Other parameters take the following values: $a = 3.97$, $N = 100$. When the node-states synchronise, this will result in a much lower density of points since for each timestep, n , there will only be one $\times$ visible in contrast to 100 if the node-states are not synchronised.

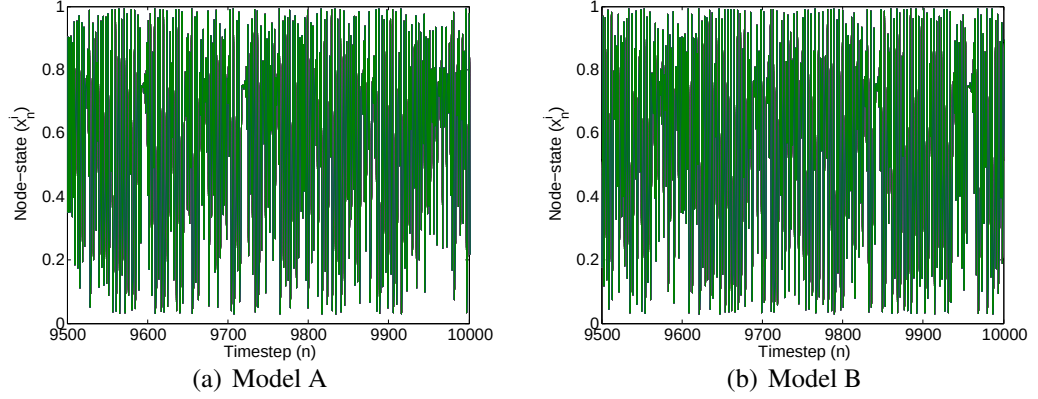


Figure 5.2: Typical evolution of the node-states (x_n^i) for Models A & B as labeled, with $c = 0.6$, $a = 3.97$ and $N = 100$. The time-evolution of each node-state, x_n^i , is represented by a line on the graph; thus, giving N lines. However, when the nodes are synchronised these all lie on top of one another giving the impression that only one node-state is plotted.

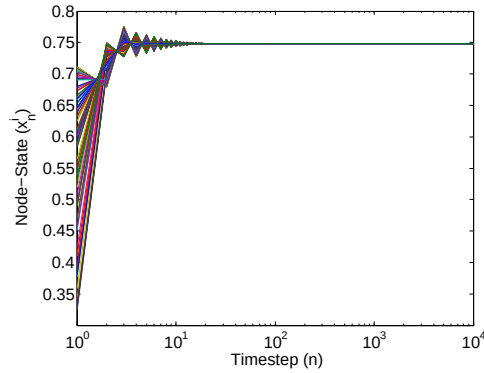


Figure 5.3: Same as Figure 5.2 only for Model C. All parameters are kept the same: $c = 0.6$, $a = 3.97$, $N = 100$. Notice that the x-axis limits are different here to fig. 5.2, this is done purely for clarity.

Model A reduces to,

$$x_{n+1}^i = f\left((1-c) \cdot x_n^i + c \cdot \sum_{j=1}^N w_n^{ij} \cdot x_n^i\right) \quad (5.1.1)$$

$$= f(x_n^i). \quad (5.1.2)$$

Model B reduces to,

$$x_{n+1}^i = (1-c) \cdot f(x_n^i) + c \cdot \sum_{j=1}^N w_n^{ij} \cdot f(x_n^i) \quad (5.1.3)$$

$$= f(x_n^i). \quad (5.1.4)$$

Model C reduces to,

$$x_{n+1}^i = (1-c) \cdot f(x_n^i) + c \cdot \sum_{j=1}^N w_n^{ij} \cdot x_n^i \quad (5.1.5)$$

$$= f(x_n^i) + c \cdot (x_n^i - f(x_n^i)). \quad (5.1.6)$$

In all of these simplifications, the property that $\sum_j w_n^{ij} = 1$ has been used.

The dynamics of the node-states when in the coherent state, for both Models A & B are simply governed by the underlying map, f ; the logistic map in our case. The nodes effectively become decoupled since the coupling strength, c , plays no part in the node-state evolution when all nodes are synchronised. In contrast, for Model C, when the system completely synchronises, the coupling still plays an active part in the node-state evolution through the extra term $c \cdot (x_n^i - f(x_n^i))$, unless of course $x_n^i = f(x_n^i)$. This situation obviously corresponds to the fixed point of the underlying map, f . In this case, f is our logistic map with fixed point $x^* = 1 - \frac{1}{a}$. Therefore, when Model C is on the fixed point, it also effectively decouples. However, once the fixed point loses stability, we should expect the dynamics to once again become dependant on the coupling strength, c . This is exactly what is found to occur in simulations.

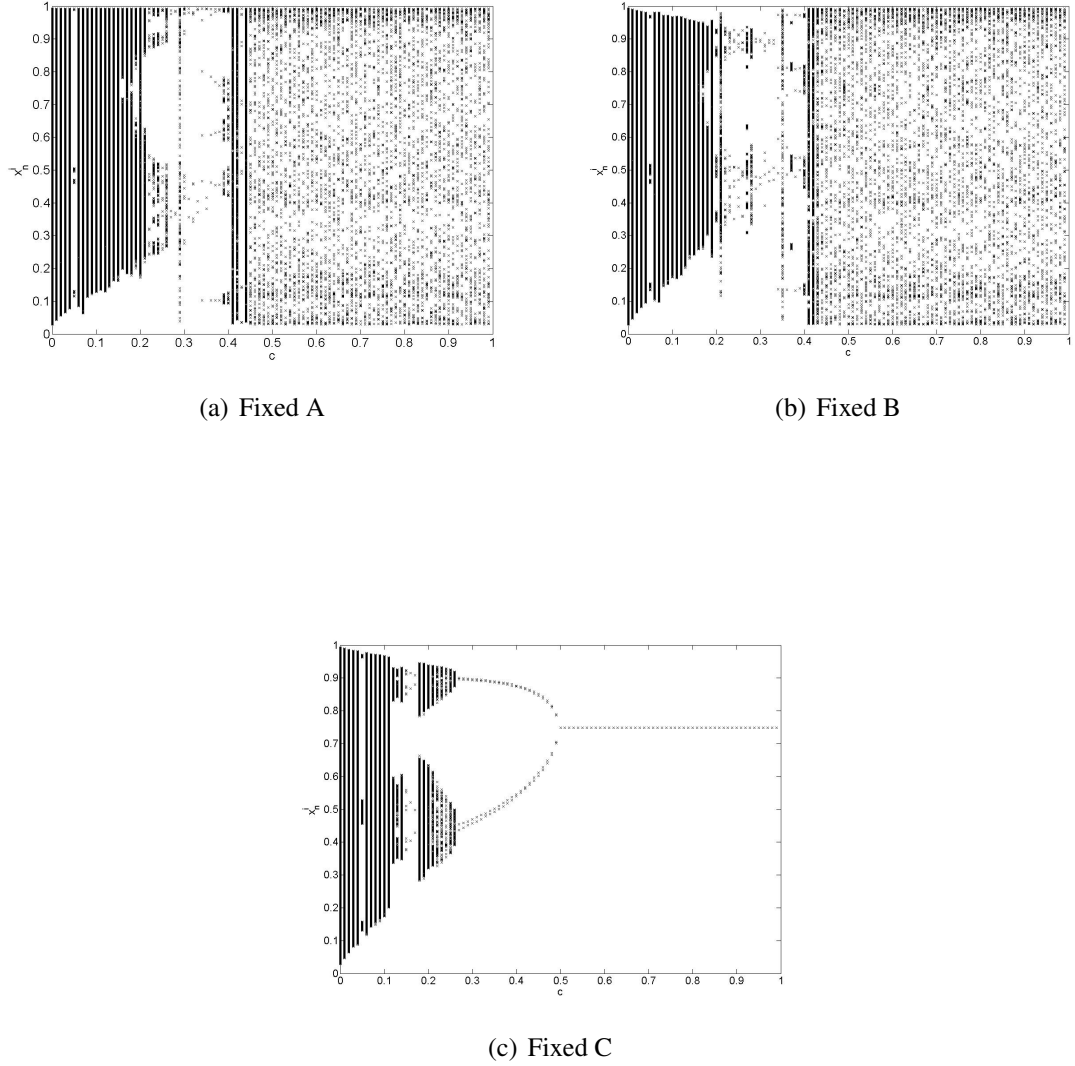


Figure 5.4: These are bifurcation diagrams for the systems as labeled; Fixed A for example is the simplified version of Model A, such that the connections are not dynamic and are from all nodes to all other nodes (all-to-all). The graphs are of the node-states, x_n^i for all nodes and values of $9000 \leq n < 9100$ for values of $c \in [0, 1)$ with a resolution of 0.01. Other parameters take the following values: $a = 3.97$, $N = 100$. When the node-states synchronise, this will result in a much lower density of points since for each timestep, n , there will only be one $\times$ visible in contrast to 100 if the node-states are not synchronised.

5.2 Synchronisation transients

When we choose a sufficiently high coupling strength that the system may completely synchronise, the time required to reach this state of synchronicity is variable; being significantly shorter for higher coupling strength values close to one. We can measure this by running simulations that are initialised in the usual manner and evolved until σ_n^2 reduces below some threshold. Once this happens, each node-state is perturbed by a different random number, η . This process is continued until 10^4 timesteps have been completed and the average length of time taken to re-synchronise is obtained. For all three models, this time to re-synchronise is shorter for higher coupling strengths. However, for these systems of maps coupled via coevolving connections, the duration of this transient leading up to the synchronised state never vanishes. This is in contrast to the simpler globally coupled map system with fixed all-to-all connections previously given in eqns. (4.3.1), (4.3.2) & (4.3.3), but repeated here for clarity:

$$\textbf{Fixed A: } x_{n+1}^i = f\left[(1-c) \cdot x_n^i + \frac{c}{N-1} \cdot \sum_{j \neq i} x_n^j\right].$$

$$\textbf{Fixed B: } x_{n+1}^i = (1-c) \cdot f(x_n^i) + \frac{c}{N-1} \cdot \sum_{j \neq i} f(x_n^j).$$

$$\textbf{Fixed C: } x_{n+1}^i = (1-c) \cdot f(x_n^i) + \frac{c}{N-1} \cdot \sum_{j \neq i} x_n^j.$$

For these simpler globally coupled map systems there are some extremely high coupling strengths, $c \approx 0.99$, where the systems of type A & B are found to synchronise quasi-instantaneously as shown in figs. 5.5(a) & 5.5(b). By quasi-instantaneous, it is meant that due to the discrete nature of the system, it is not possible to record a shorter time difference than one timestep. This ability to synchronise quasi-instantaneously is not however entirely unexpected for extremely high coupling strengths in the systems with all-to-all connections, since the evolution of the node-states are dominated by the coupling term that incorporates the influence of all the other nodes. In the fixed all-to-all coupled systems, this term is identical for every node by definition.

In contrast, for the systems with connections given by a coevolving network, the interaction term does not give a simple average, but is a weighted average of the node-states

and these weighting factors are different for each individual node. Therefore, it could be expected that there would not be the same quasi-instantaneous synchronisation. Indeed, this is what is shown in fig. 5.5, the systems with coevolving connections never manage to synchronise faster than ~ 6 timesteps; even for extremely high coupling values of c close to one.

For Model C, as shown in figs. 5.5(c) & 5.5(d), for most coupling strengths, the systems with coevolving connections take exactly the same length of time to synchronise as the systems with fixed all-to-all connections. There is only a small range of coupling strengths, $0.15 \lesssim c \lesssim 0.19$, for which Fixed C synchronises when Model C does not. It is not clear why the coupling mechanism of type C is almost unaffected by the introduction of coevolving connections, whereas with couplings of type A or B, the introduction of coevolving connections does affect the time required to re-synchronise.

5.3 Approach to synchrony

Fig. 5.6 shows the evolution of σ_n^2 for all three models during the early part of typical numerical simulations and for various coupling strengths, c . Recall, $\sigma_n^2 = \sum_i (x_n^i - \bar{x}_n)^2$, so is a measure of distance between the system state and the synchronous submanifold. For the lower coupling strengths shown in fig. 5.6(a) ($c = 0.44$) and fig. 5.6(b) ($c = 0.45$), neither Model A, B nor C synchronises. This is obvious from the behaviour of σ_n^2 , since σ_n^2 is always non-zero. Notice that the timeseries of σ_n^2 has much larger amplitude fluctuations for Models A & B than for Model C. Additionally, the asymptotic values taken by σ_n^2 are often lower for Model C; for example for $c = 0.44$, σ_n^2 increases gradually for Model C until around 2000 timesteps and then fluctuates in value between ~ 1.2 and ~ 1.7 , whereas for Models A & B, σ_n^2 increases until around 400 timesteps and then fluctuates between ~ 6 and ~ 15 . This can be understood by noticing in the bifurcation diagrams, fig. 5.1, that in the asynchronous state of Model C, the node-states are confined to a smaller region of phase space than the node-states governed by Model A or B. This leads to a lower value of σ_n^2 , but does not in this case reflect a greater degree of synchronisation.

Model B shows an interesting behaviour in fig. 5.6(c) for a coupling strength, $c = 0.48$: σ_n^2 appears to plummet exponentially during the first ~ 55 timesteps and then rises

almost as quickly to a level similar to the asymptotic value of σ_n^2 for an asynchronous system. This is an example of when a system almost synchronises, but spontaneously loses synchronicity, in this case at $n \sim 55$; σ_n^2 rises for $n \gtrsim 55$, until it reaches a level similar to the unsynchronised Model A that is plotted on the same axes. At this coupling strength, the synchronous state is linearly stable, but in this case the non-linearities have clearly dominated and led to the system tending to the asynchronous state.

The lower fluctuations in the σ_n^2 timeseries of Model C is particularly apparent in fig. 5.6(d) which shows an example timeseries of σ_n^2 for $c = 0.54$. At this value of coupling strength, the majority of realisations of Model A & B and all realisations of Model C result in the coherent state. As shown in fig. 5.6(d), σ_n^2 decays almost monotonically for Model C and the decay is very well described by an exponential; see figs. 5.7 for a fit by an exponential. For $c \gtrsim 0.50$, Model C produces an exponential decay in σ_n^2 towards the synchronous state characterised by $\sigma_n^2 = 0$.¹ The exponential decay is characterised as $\sigma_n^2 \sim \beta e^{-\alpha n}$. The values of the parameters of the exponential decay, α & β , for different coupling strengths, c , are shown in figs. 5.8. This is data obtained from exponential best fit lines to the σ_n^2 timeseries of Model C for different coupling strengths. This clearly demonstrates how much quicker Model C synchronises as the coupling strength is increased. Models A & B synchronise similarly quicker for higher coupling strengths, as can be seen in figs. 5.6. Model C was chosen for the the exponential fitting to σ^2 timeseries since it has much lower amplitude fluctuations.

For coupling strengths such that the basin of attraction of the coherent state occupies the entirety of phase space for Models A & B, and therefore for Model C too, the decay of σ_n^2 to zero is approximately exponential for all three models. Figs. 5.6(e) and 5.6(f) show typical examples of the decay of σ_n^2 towards zero; whilst Model C displays the clearest exponential decay, the decay of σ_n^2 for Models A & B is noisier but still well approximated by an exponential.

In studies of other coupled-map systems, similar measures of distance from the synchronous state have been found to decay as a power law [AP02, GH06, GLPT03]. For none of the three models studied here has it been observed that σ_n^2 decays with a power

¹or $O(10^{-30})$ as seen here, which is dictated by numerical precision.

law form. In the previous studies where it was claimed that power law decay was found in similar measures of the distance from the synchronous state, this led to the conclusion that dependent upon whether the underlying map was continuous or not, systems belonged to different universality classes. Those with continuous (or weakly nonlinear) underlying maps belonged to the Directed Percolation (DP) universality class, whilst discontinuities (or strong nonlinearities) in the underlying map led to the coupled-map systems belonging to the Bounded Kardar-Parisi-Zhang (BKPZ) universality class. These previous claims of power-law decays were found to occur close to the critical coupling strength where $\lambda_{\perp} = 0$.

The investigations here have found no evidence to suggest any kind of critical behaviour in any of these systems; it is therefore not possible to suggest that any of these three Models, A, B or C, belong to any universality class. This could be related to the finding that for Models A & B in particular, the synchronous state does not lose absolute stability at a specific coupling strength; similar to other previous works (e. g. [ABS94]), there are a range of coupling strengths over which the coherent state loses its stability.

5.4 Structure of the basin of attraction for coherent state

We will now look a little more closely as to how the basin of attraction for the coherent state changes topologically from not existing to occupying the entirety of phase space as we increase the coupling strength. For low coupling strengths where $\lambda_{\perp} > 0$, the basin of attraction is non-existent since the coherent state is not an attractor in any sense; even points initialised on the synchronous submanifold do not stay there. For higher coupling strengths such that $\lambda_{\perp} < 0$, the synchronous state is linearly stable. The coupling strength has to be increased much higher than the critical coupling where $\lambda_{\perp} = 0$, before the basin of attraction occupies all of phase space. We saw this earlier by looking at the fraction of realisations that result in the synchronous state as a function of coupling strength, c . We saw that this is a continuous transition for Models A & B; at no coupling strength was there a sudden increase in the volume of the basin of attraction. How the basin changes topologically, as we shall see, gives an interesting insight into the reasoning why this leads to unpredictability as to whether a particular realisation will result in the coherent state or not, even if $\lambda_{\perp} < 0$.

The basin of attraction for the coherent state for Models A & B displays rich structure, as can be seen in figs. 5.9 and 5.10; these graphical representations of the basin of attraction are produced by fixing the initial conditions of all but two nodes to be a particular set of random numbers. Then, running simulations using this same set of random initial conditions whilst sequentially changing the other two initial conditions. By doing this procedure, it is possible to gain insight into the structure of the basin of attraction for the systems. Results from such simulations are shown in figs. 5.9 and 5.10, clearly the structure of the basin of attraction is topologically nontrivial for Models A & B. In contrast, for Model C the basin of attraction for the coherent state is found to either occupy all or none of phase space, depending on the parameter values a and c . The transition between these two situations occurs discontinuously at a specific coupling strength. This was shown earlier in fig. 4.10 where we saw that the fractions of realisations that result in the coherent state changes from zero to all realisations at a specific coupling strength, c .

The appearance of highly intricate structure to the basin of attraction for Models A & B does not occur at a coupling strength that coincides with $\lambda_{\perp}$ becoming negative. The condition $\lambda_{\perp} < 0$ is a necessary but not sufficient condition. As the coupling strength is increased, the volume of phase space occupied by the basin increases, this does not seem to reduce the topological intricacy of the structure in the basins. Eventually, the number of initial points that do not reach the coherent state becomes very small, see for example the basin of attraction for Model B at $c = 0.55$ as shown in fig. 5.10(c). Here, the basin of attraction for the coherent state occupies almost the entire slice of phase space. The few points not in the basin are surrounded by points that are in the basin and still show remnants of the complicated topological structure that was clearly visible at the lower coupling strength, $c = 0.53$, shown in fig. 5.9(c).

This complicated topology is entirely absent for Model C and all of the simplified systems, Fixed A, B or C. It therefore requires both the coevolving connections, and is highly sensitive to the particular form of coupling chosen.

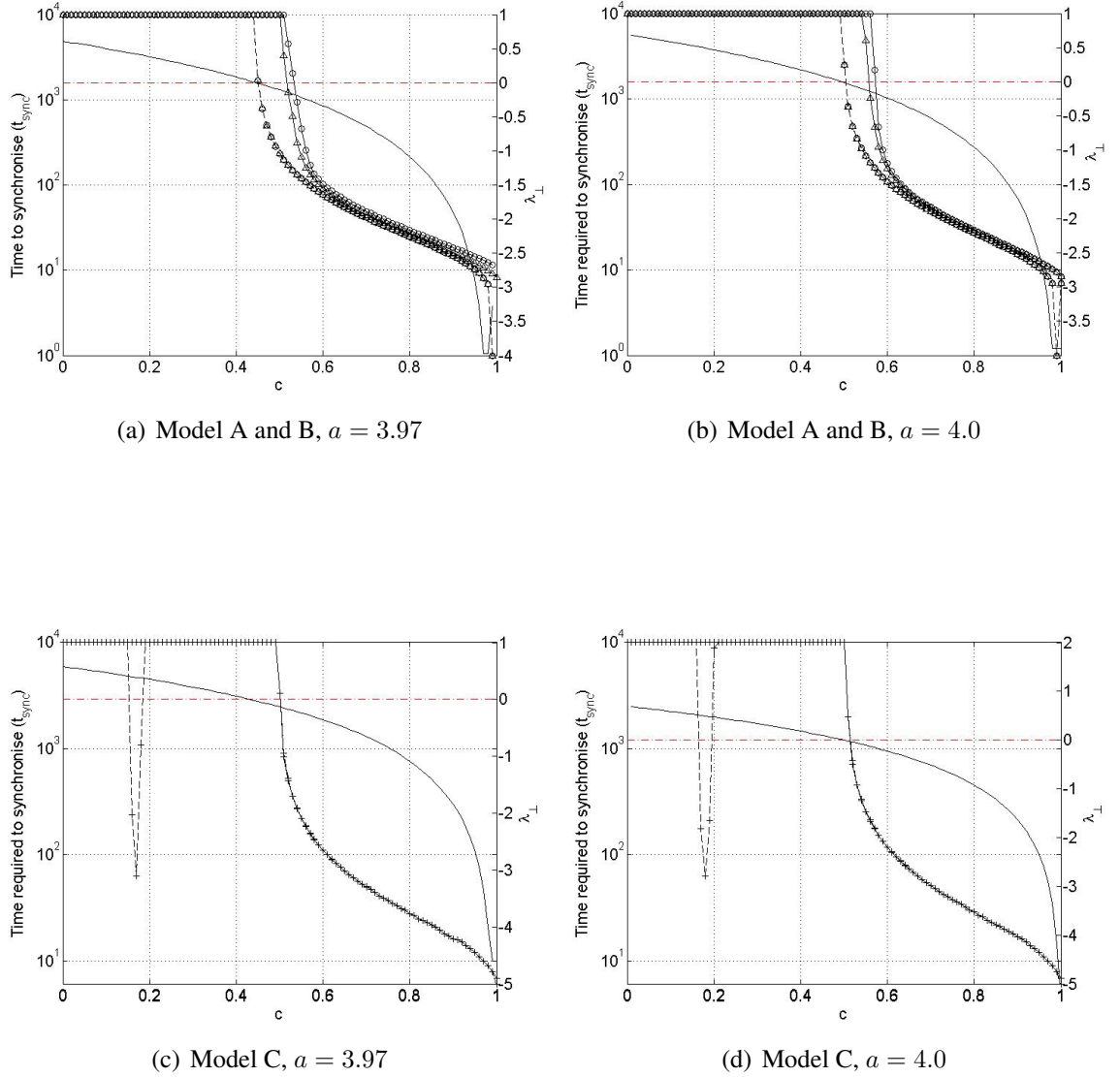


Figure 5.5: The time required to re-synchronise after perturbation from the synchronous state is plotted against the left-hand “y-axis”. The right-hand “y-axis” denotes the transverse Lyapunov exponent values, this is the solid line without any marker. The other solid lines denote those models with plastic connections, whereas dashed lines denote GCM with all-to-all fixed connections. Models of type A (internally coupled): $\circ$, type B (externally coupled): $\triangle$ and type C (externally coupled with feedback): $+$. The systems are perturbed once $\sigma_n^2 < 10^{-15}$; each node-state is perturbed by a random number $\eta < 10^{-2}$.

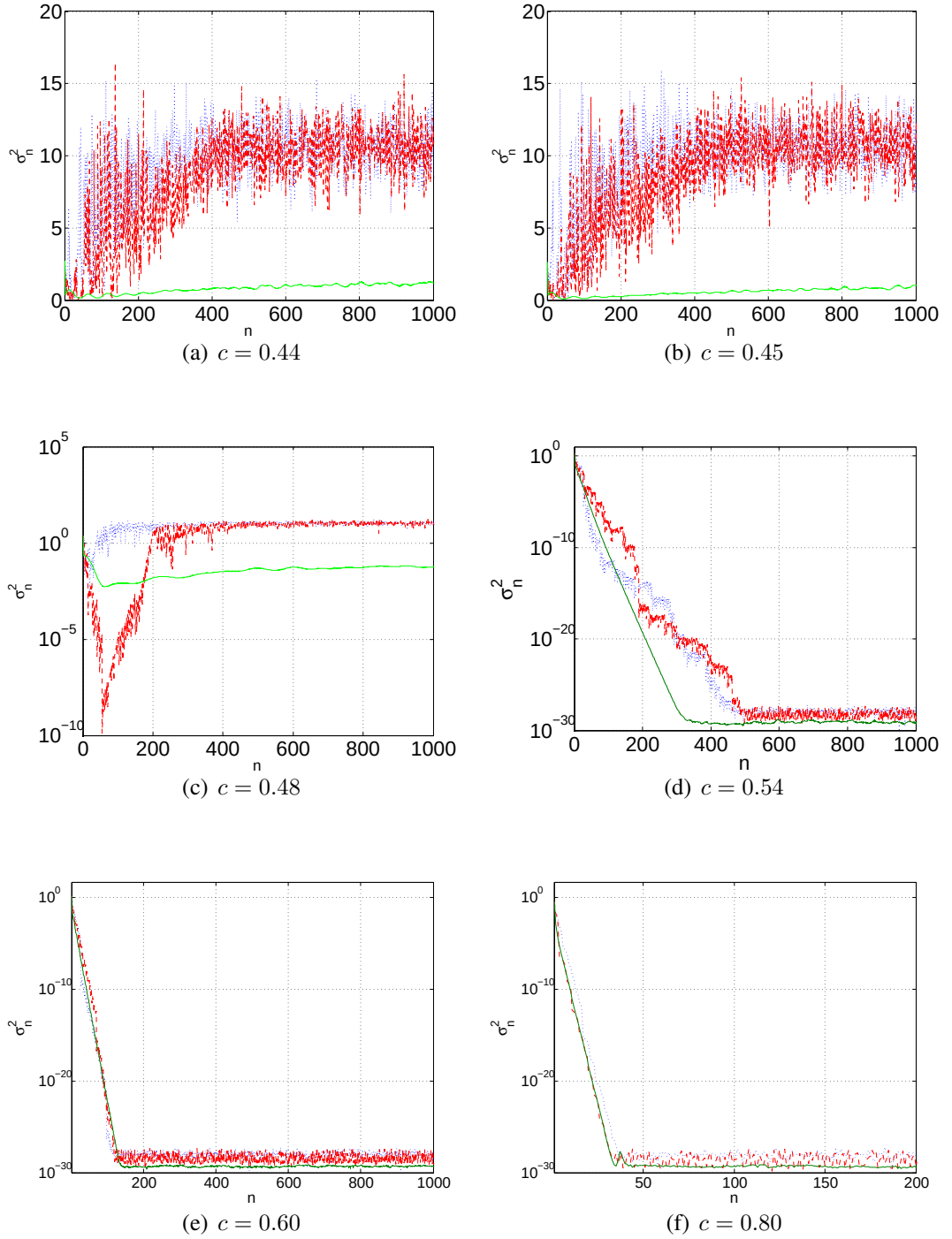


Figure 5.6: Graphs of the evolution of σ_n^2 from timestep one for Models A (blue), B (red) & C (green), for $a = 3.97$. (a)-(f) are at different coupling strengths, c , moving through the parameter space following arrow A of fig. 3.1. The simulations are run for 10^5 timesteps. Note, the data shown is not the average of an ensemble of realisations; it is the result of one realisation starting at a random point in phase space.

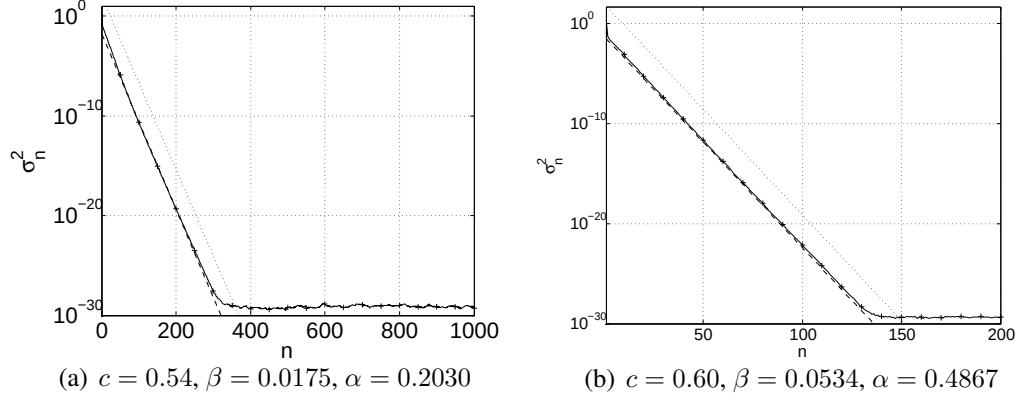


Figure 5.7: Node-state variance, σ_n^2 , for Model C, with $a = 3.97$. The solid line with + is the measured σ_n^2 , the dashed line is the exponential fit, $\beta \cdot e^{-\alpha n}$ and the dotted line is the exponential fit offset vertically for clarity.

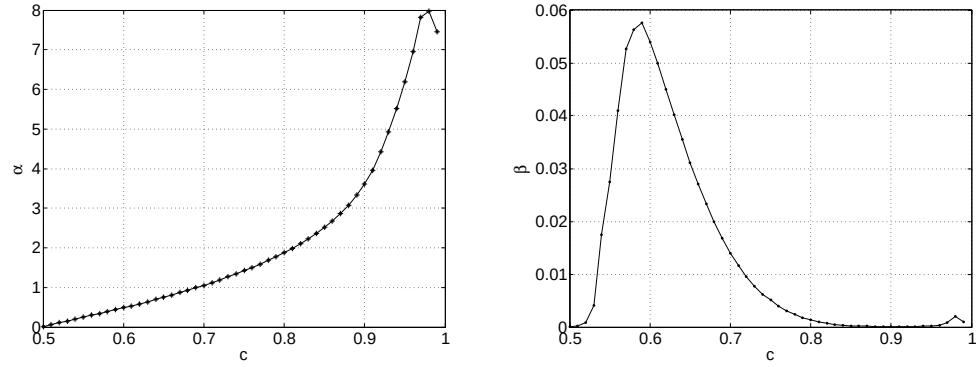


Figure 5.8: The parameters, α and β of the exponential fits, $\beta \cdot e^{-\alpha n}$, to σ_n^2 for Model C for $c \geq 0.50$.

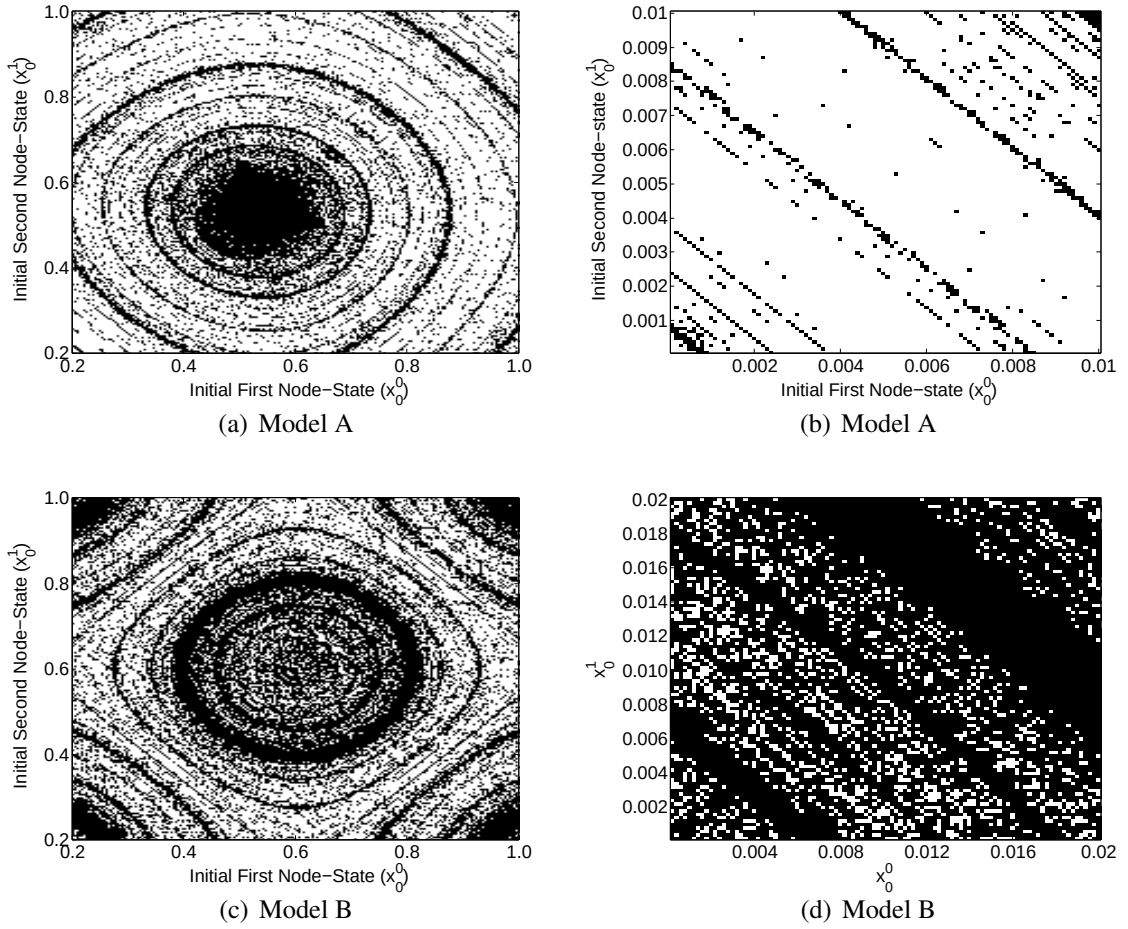


Figure 5.9: The structure of the basin of attraction for the coherent state of Models A & B, with $a = 3.97$, $c = 0.53$, and for a system size $N = 100$. The initial conditions that have reached the coherent state ($\sigma^2 < 10^{-25}$) after 10^4 timesteps are coloured black. The random initial conditions for 98 node-states are fixed and the initial node-states for the two remaining nodes (x_0^0 and x_0^1) are changed sequentially so as to sample the basin through a slice in phase space. A full slice of phase space is shown in (a) at a resolution of 0.005; a zoom of $0.0 < x_0^0, x_0^1 < 0.01$ with a resolution of 0.001 is shown in (b); (c) and (d) show the same as (a) and (b) respectively, only for Model B.

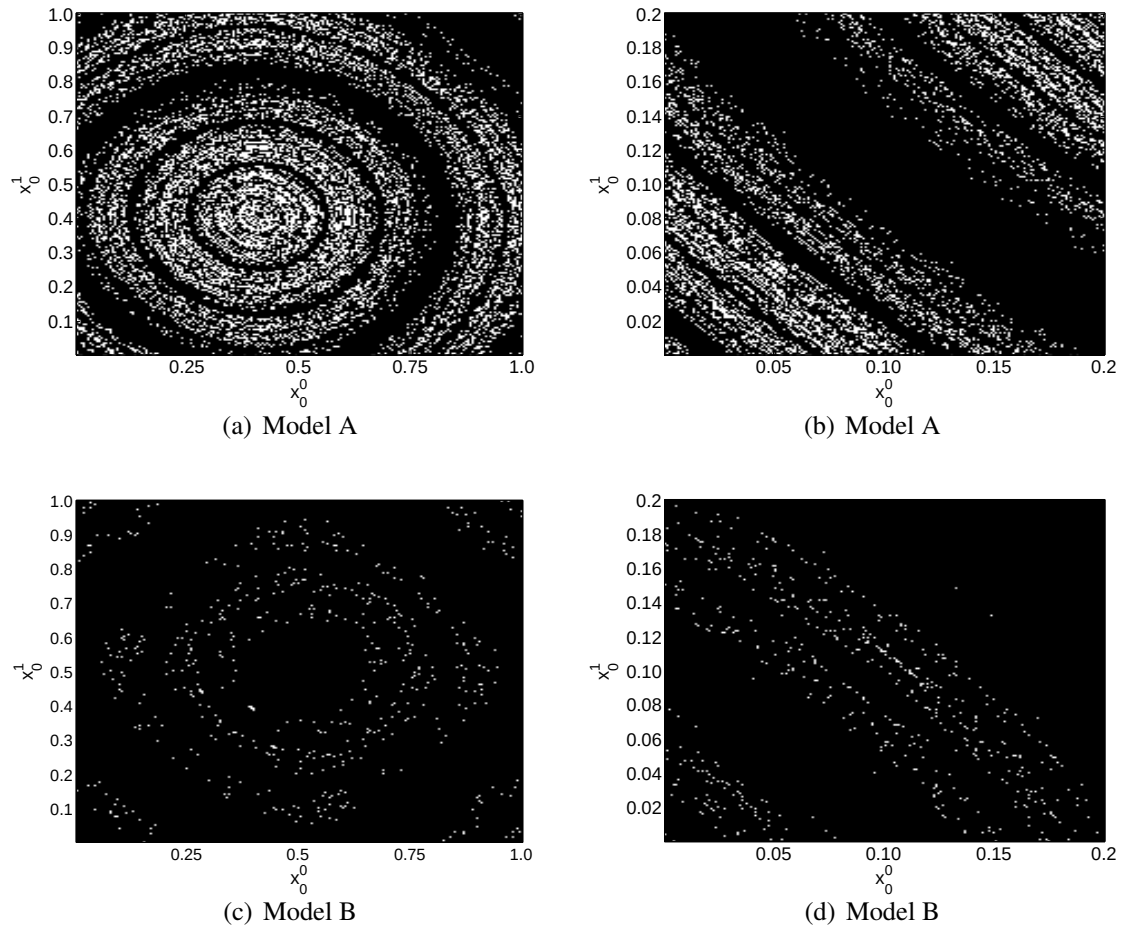


Figure 5.10: Same as fig. 5.9 only for $c = 0.55$.

5.5 Distribution in phase space

We saw in Chapter 2 that a timeseries generated by the logisitic map with $a = 4$ has points distributed according to eqn. (2.11.2), repeated here for clarity:

$$P(x) = \frac{1}{\pi \sqrt{x \cdot (1-x)}}.$$

It will be interesting to see whether the coupled systems with $a = 4$ exhibit this behaviour in the distribution of points in phase space. For Models A & B in the coherent state, it should be expected that the node-states, x_n^i , should have a similar distribution in $[0, 1]$ since their effective dynamics are governed by the logisitic map only; as was discussed in Section 5.1. As shown in figs. 5.11(e) & 5.11(f), when these systems are in the coherent state, the distribution of points of the node-state orbits follow closely the same distribution as the single logistic map that was shown in fig. 2.4. Rather unexpectedly, Model A node-states are well approximated by this distribution for all coupling strengths, see for example figs. 5.11(a) and 5.11(c).

Model B node-states are approximated by this distribution for many coupling strengths, however notice in fig. 5.11(b) how only some of the node-states follow this distribution.

In contrast, Model C node-states do not follow this distribution for any coupling strengths, some examples of their distributions can be seen in figs. 5.12. Fig. 5.12(a) shows a bimodal distribution of node-states. From the bifurcation diagram, fig. 5.1(c), this behaviour is not apparent as it would appear the node-states take all values within a sub-region of their permitted range. However, fig. 5.12(a) is suggestive that the 2-cycle has perhaps lost stability and the system now follows a quasi 2-cycle. At the coupling strength represented in fig. 5.12(b), the fixed-point is linearly stable and it is clear to see that the node-states remain in a small area around this fixed-point, $1 - \frac{1}{a}$. The corresponding node-state time-series shows this very well in fig. 5.12(d), where bursts away from the fixed-point can be seen to be folded back in towards the fixed-point, only to burst away again. This cycle of bursting dynamics appears to continue indefinitely. For completeness the distribution for a high coupling strength is also included for Model C, fig. 5.12(c). This shows, again, how the node-states spend all their time on the fixed-point, $x^* = 1 - \frac{1}{a}$.

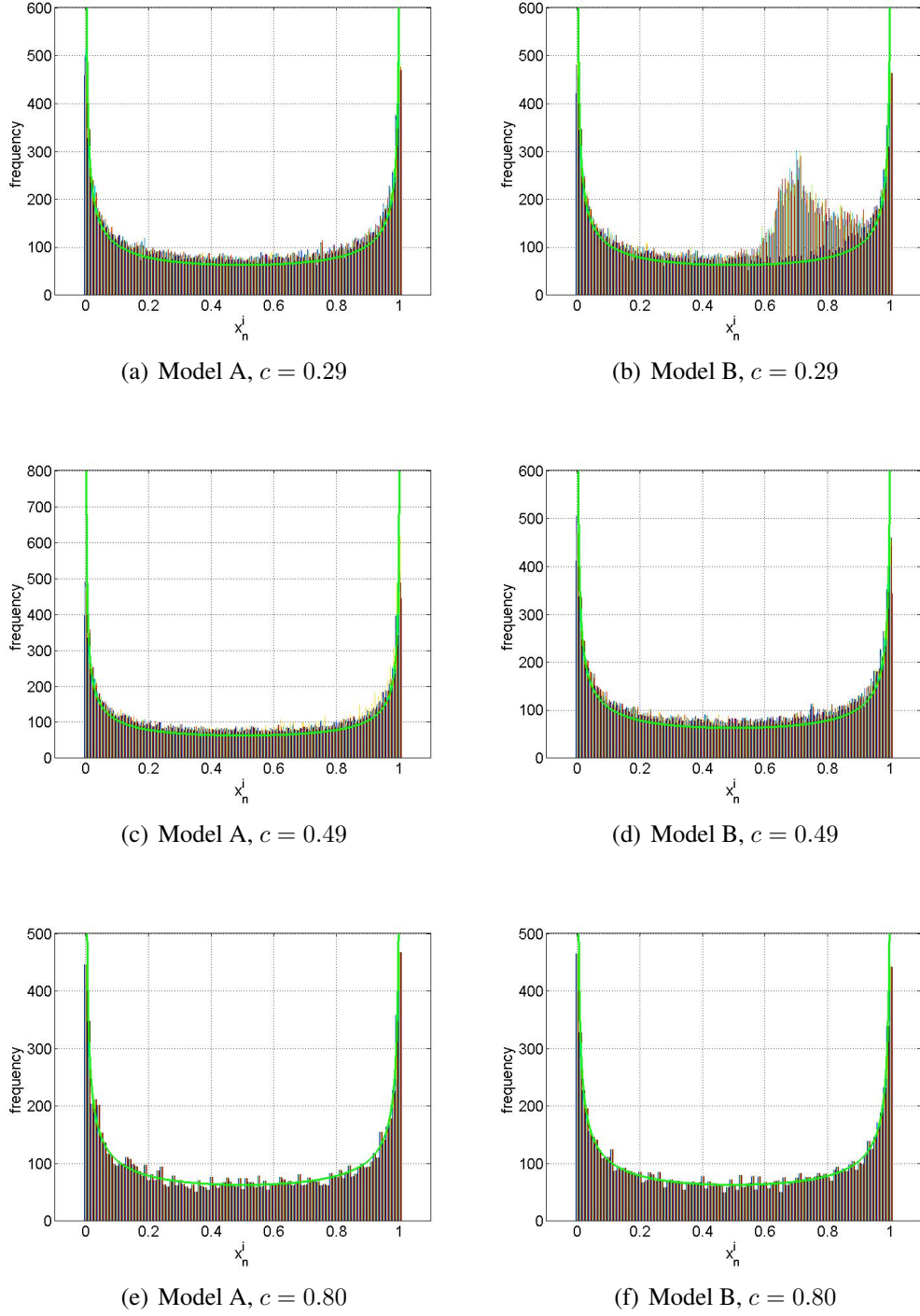


Figure 5.11: The frequency of points x_n^i for Models A & B at $a = 4$ and coupling strengths, c , as labeled. These are generated from a timeseries of length 10^4 and binning in 1001 bins equally distributed in $[0, 1]$.

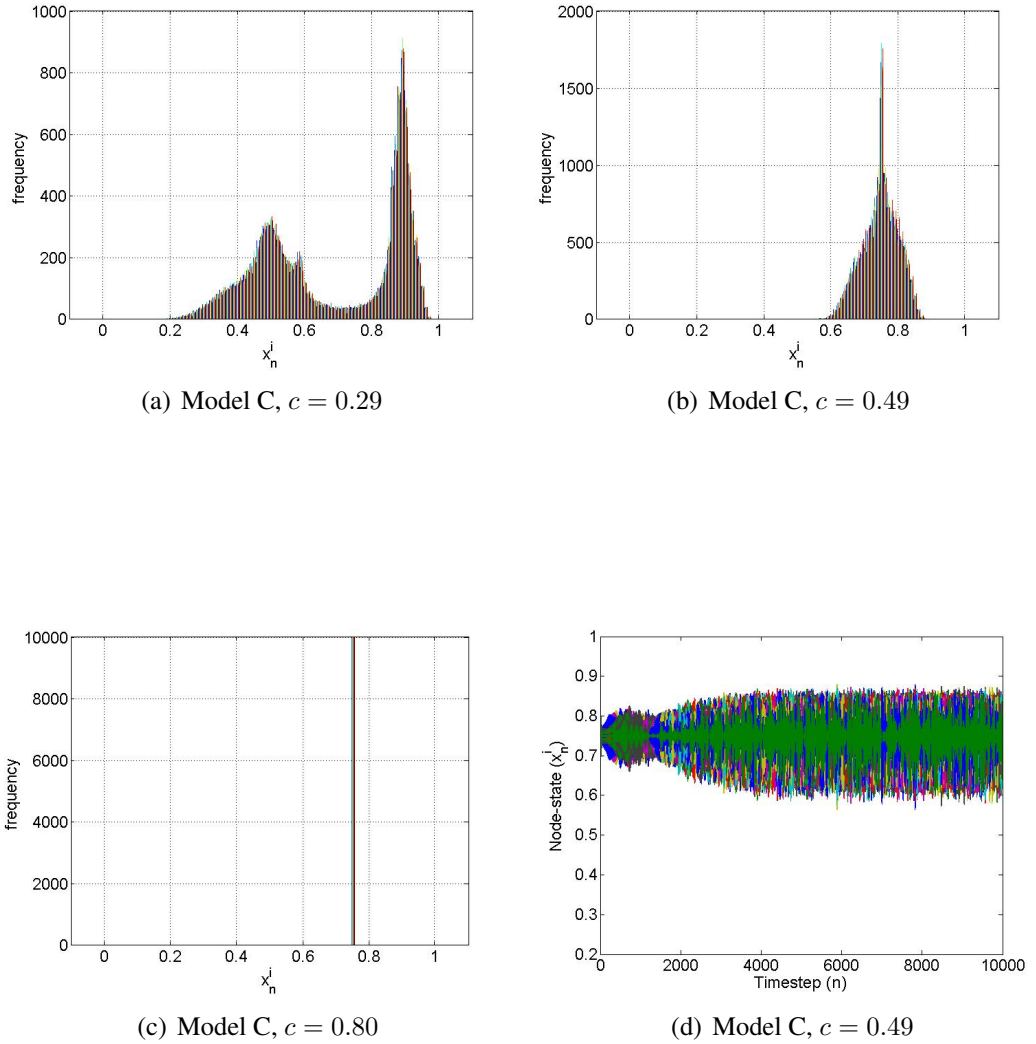


Figure 5.12: The frequency of points x_n^i for Model C at $a = 4$ and coupling strengths, c , as labeled. These are generated from a timeseries of length 10^4 and binning in 1001 bins equally distributed in $[0, 1]$. Additionally, (d) shows the full timeseries for $c = 0.49$ and corresponds to the histogram shown as (b).

5.6 Concluding remarks

At the microscopic level, the difference between Model B and Model C is that B is coupled externally with a nonlinear output function, whereas C has a linear output function. This change to the microscopic dynamics has a dramatic effect on the macroscopic dynamics that the systems display. We saw in the previous chapter that changing between internal and external coupling does not change the macroscopic dynamics greatly, yet switching between the linear and nonlinear output function does. This highlights the care that must be taken when designing models to closely model any real-world system.

The dynamics of the three Models in the coherent state were shown to be different. Models A & B produce aperiodic dynamics, whereas Model C dynamics appear to stabilise the fixed-point, $1 - \frac{1}{a}$; in the coherent state the node-states in Model C and Fixed C become time-independent and coincide with the fixed-point. This is due to the differences in the effective dynamics governing the node-states in the coherent state.

The question of how long the systems take to synchronise is an interesting one that has received little attention to date. Here it has been shown that the introduction of coevolving connections to the models makes different impacts dependant on the microscopic coupling. Models A & B have similar durations of their synchronisation transient to Fixed A & B, except at extremely high coupling strengths, where the systems with static all-to-all connections are able to synchronise quasi-instantaneously. However, Model C has synchronisation transients of the exact duration as Fixed C for all but a small range of low coupling strengths. This is important to note with relevance to understanding how synchronisation builds-up in real systems and again highlights how important it is to use a sensible microscopic structure for any model.

The approach to the synchronous state during this synchronisation transient occurs exponentially quickly. Unlike previous studies of other coupled-map systems, it was never found to be well fitted as a power law. The amplitude of oscillations about exponential decay were much larger for Models A & B when compared to Model C.

Another dichotomy between Models A & B and Model C is found in the basin of attraction of the coherent state: for Model C, as soon as the basin comes into existence it occupies the entirety of phase space. There is no coexistence of both the synchronous and

asynchronous state for Model C. This is in stark contrast to the basins for Models A & B, which show intricate topological structures in their coherent state basins of attraction across a range of coupling strengths, c . For all these c -values with intricate basins, there is a coexistence of synchronous and asynchronous states; depending on the initial conditions of the node-states as to whether the system will reach the coherent state or not.

The distribution of node-states throughout their orbits was compared to the theoretical distribution for the single logistic map at $a = 4$. This was found to be just as expected. Models A & B whose effective dynamics in the coherent state means the node-state is governed by the single logistic map, are indeed found to have node-state distributions well approximated by the analytical distribution. Model C however, never has node-state distributions that are well approximated by the analytical expression. Unexpectedly, the distribution of node-states when the systems are not coherent were also well approximated by the analytical expression for Models A & B.

When taken together, the results of this chapter and the previous one show us that the precise microscopic coupling mechanism chosen for a coupled map system can have a huge impact on the macroscopic dynamics displayed by the system. We have seen that for Models A & B the loss in stability of the synchronous state roughly coincides with the blowout bifurcation. Whereas for Model C, the loss in stability of the synchronous state is governed by the riddling bifurcation. Another major difference between the systems is seen when the time-series of the node-states in the synchronous state is compared. Only Model C enters into a steady-state, with the node-state value coinciding with the fixed point of the logistic map $1 - \frac{1}{a}$; Models A & B display non-periodic behaviour in the synchronous state.

Models A & B could be considered to display more complicated behaviour across the parameter space since their loss in stability of the coherent state roughly coincides with the blowout bifurcation, but not all initial conditions loose their stability at the same parameter values. For certain parameter values, the basin of attraction for the coherent state is topologically intricate, as was shown in figs (5.9) & s(5.10).

Chapter 6

Results 3: The effect of an external stimulus

6.0.1 Chapter summary and outline

In this chapter, we will be looking at what affect driving the systems by an external stimulus has on the dynamics and properties discussed in the previous chapters. This work is largely motivated by empirical observations of the visual cortex and how it responds to periodic stimuli.

6.1 What form of stimulus?

The not yet published results that have inspired the work that will be discussed in this chapter were carried out by the research group of Dr. Luis Diaz-Santana at City University, London. They designed an intricate experiment to try and analyse how much pre-processing of optical stimuli takes place before the optical signals are fed from the retina to the visual cortex area of the brain via the optic nerve. The basic experimental setup involves a subject watching a screen flashing periodically between blue and red. Whilst this is happening, simultaneous measurements of the retina position and visual cortex response, using EEG, are taken. As might be expected, the optical path length between the screen and the retina varied periodically as the eye focus changed in response to the red and blue inputs. How-

ever, when the response produced by the visual cortex was analysed they found something unexpected and as yet unexplained. Recall that the EEG timeseries are measurements of the changing electrical field produced by the collective electrical signals of the neurons close to where the particular electrode is attached to the subject's scalp. Diaz-Santana's group analysed the corresponding power spectra. They found the power spectra obtained had a broad frequency range, with no particular frequency being dominant and that there was no response at exactly the frequency of the stimulus. However, sometimes there were higher harmonics of the frequency of the stimulus visible in the power spectra.

Inspired by this finding, it was thought to be an interesting question as to what response the coupled-map systems studied here might have to stimuli. The following three types of stimuli have been considered:

Stimuli type X, sinusoidal:

$$x_n^{drive} = \sin^2(\omega n). \quad (6.1.1)$$

Stimuli type Y, mismatched logistic map:

$$x_{n+1}^{drive} = a_{stimulus} \cdot x_n^{drive} \cdot (1 - x_n^{drive}). \quad (6.1.2)$$

Stimuli type Z, constant:

$$x_{n+1}^{drive} = \kappa. \quad (6.1.3)$$

Where κ , $a_{stimulus}$ and ω are all constants. The sinusoidal stimulus, stimulus type X, was chosen so as to mimic the periodic stimulus used in Diaz-Santana's visual cortex experiments. A logistic map, stimulus type Y, is a natural choice to consider for brain interactions as this form of function has been correlated with the form of neural outputs [KAR04]. Finally, the constant stimulus, stimulus type Z, has been considered as a null stimulus for comparison.

In all the results presented here, the external stimulus is applied to a randomly chosen subset of the nodes. Nodes $i = 1 \dots N_{stimuli}$ receive the external stimulus and have their node-state evolutions altered as follows:

Model A nodes receiving stimuli:

$$x_{n+1}^i = f \left[(1 - c) \cdot x_n^i + c \cdot \sum_{j=1}^{N_{Tot}} w_n^{ij} \cdot x_n^j + \frac{c}{N_{Tot.} - 1} \cdot x_n^{drive} \right]. \quad (6.1.4)$$

Model B nodes receiving stimuli:

$$x_{n+1}^i = (1 - c) \cdot f(x_n^i) + c \cdot \sum_{j=1}^{N_{Tot}} w_n^{ij} \cdot f(x_n^j) + \frac{c}{N_{Tot.} - 1} \cdot f(x_n^{drive}). \quad (6.1.5)$$

Model C nodes receiving stimuli:

$$x_{n+1}^i = (1 - c) \cdot f(x_n^i) + \frac{c}{N - 1} \cdot \sum_{j=1}^{N_{Tot}} w_n^{ij} \cdot x_n^j + \frac{c}{N_{Tot.} - 1} \cdot x_n^{drive}. \quad (6.1.6)$$

Nodes $i = N_{Stimulus} \dots N_{Tot}$ do not receive the stimulus so continue to evolve according to the original equations, repeated here for clarity:

Model A nodes without stimuli:

$$x_{n+1}^i = f \left[(1 - c) \cdot x_n^i + c \cdot \sum_{j=1}^{N_{Tot}} w_n^{ij} \cdot x_n^j \right]. \quad (6.1.7)$$

Model B nodes without stimuli:

$$x_{n+1}^i = (1 - c) \cdot f(x_n^i) + c \cdot \sum_{j=1}^{N_{Tot}} w_n^{ij} \cdot f(x_n^j) + \frac{c}{N_{Tot.} - 1} \cdot f(x_n^{drive}). \quad (6.1.8)$$

Model C nodes without stimuli:

$$x_{n+1}^i = (1 - c) \cdot f(x_n^i) + \frac{c}{N - 1} \cdot \sum_{j=1}^{N_{Tot}} w_n^{ij} \cdot x_n^j + \frac{c}{N_{Tot.} - 1} \cdot x_n^{drive}. \quad (6.1.9)$$

Where N_{Tot} is the sum of the number of main nodes, N , and the number of stimuli, $N_{Stimuli}$: $N_{Tot} = N + N_{Stimuli}$. The evolution of connections between the main nodes of

the system (all, N , non-stimuli nodes) is unchanged and still given by eqn. (4.1.4). The connection strength between the stimulus and the node receiving the stimulus is fixed with value $\frac{1}{(N_{Tot}-1)}$. Thus, the periodic stimuli are not influenced by what the other nodes are doing but do have influence on those other nodes, either directly if they are connected or indirectly through their effect on those nodes that do receive the stimulus if there is no direct connection.

6.2 Effect of stimuli on time to synchronise

First let us consider how the stimuli affect how quickly the systems re-synchronise. Similar to the analysis of synchronisation transients for systems without stimulus, these stimulated systems are evolved until σ_n^2 is less than some threshold, the synchronous threshold, δ .¹ Once this occurs, every node-state, x_n^i , is perturbed by a random number, η , whilst of course ensuring that $x_n^i \in [0, 1]$ is not violated. This is continued for 10^4 timesteps and the time taken to synchronise is then calculated as the average of all the synchronisation times. In fig. 6.1 it is shown for both Model A and Fixed A how this time to synchronise changes with coupling strength, c . Recall that, Fixed A is the system with similar internal coupling form to Model A but fixed all-to-all connections, i.e. $w_n^{ij} = \text{constant } \forall i \neq j$ and 0 for $i = j$.

In fig. 6.1(a), we can see that when there are low numbers of external stimuli with respect to the system size, changing between the three types of stimuli makes no change to the time required to synchronise for Model A or Fixed A. However, this time taken to synchronise for the systems with any of the external stimuli applied is longer than the time to synchronise without an external stimulus for almost all coupling strengths, as shown in fig. 6.1(c). This figure shows that for the system with fixed all-to-all coupling, Fixed A, the system with stimuli takes longer to synchronise than that with no stimuli, for all coupling strengths. For Model A, with its coevolving connections, for $c \sim 0.5$, the stimuli actually reduce the average time to synchronise. At this coupling strength value, recall that the basin of attraction for the coherent state occupies only a small volume of phase space; only about

¹Note that σ_n^2 is calculated similarly to before: $\sigma_n^2 = \sum_{i=1}^{N_{Tot}} (x_n^i - \bar{x}_n)^2$, but it does not incorporate the external stimuli explicitly. It does include those nodes in receipt of the external stimulus, so indirectly their influence on the system is included.

10% of initial conditions reach the coherent state here. The stimuli appear to actually aid the synchronisation, but this is only for a very narrow range of coupling strengths.

As discussed previously, the shortest synchronisation transients occur for the system with fixed all-to-all couplings. Since when the external stimulus is applied to these systems it generally causes a longer duration of transient before synchrony is achieved, the Fixed A system in receipt of any stimulus no longer manages to achieve synchrony quasi-instantaneously for $a = 3.97$. However, figs. 6.1(b) & 6.1(d) show that for $a = 4$, the Fixed A system is still able to achieve quasi-instantaneous synchronisation for $c \gtrsim 0.96$. This Fixed A system with $a = 4$ receiving type X or Z stimulus is also able to achieve synchrony for coupling strengths where $\lambda_{\perp} > 0$. That is, when the synchronous state of the unstimulated system is not even linearly stable. This ability of coupled-map systems being able to synchronise when the synchronous state is not linearly stable has been previously observed for other coupled-map systems and reported in for example [SWC97].

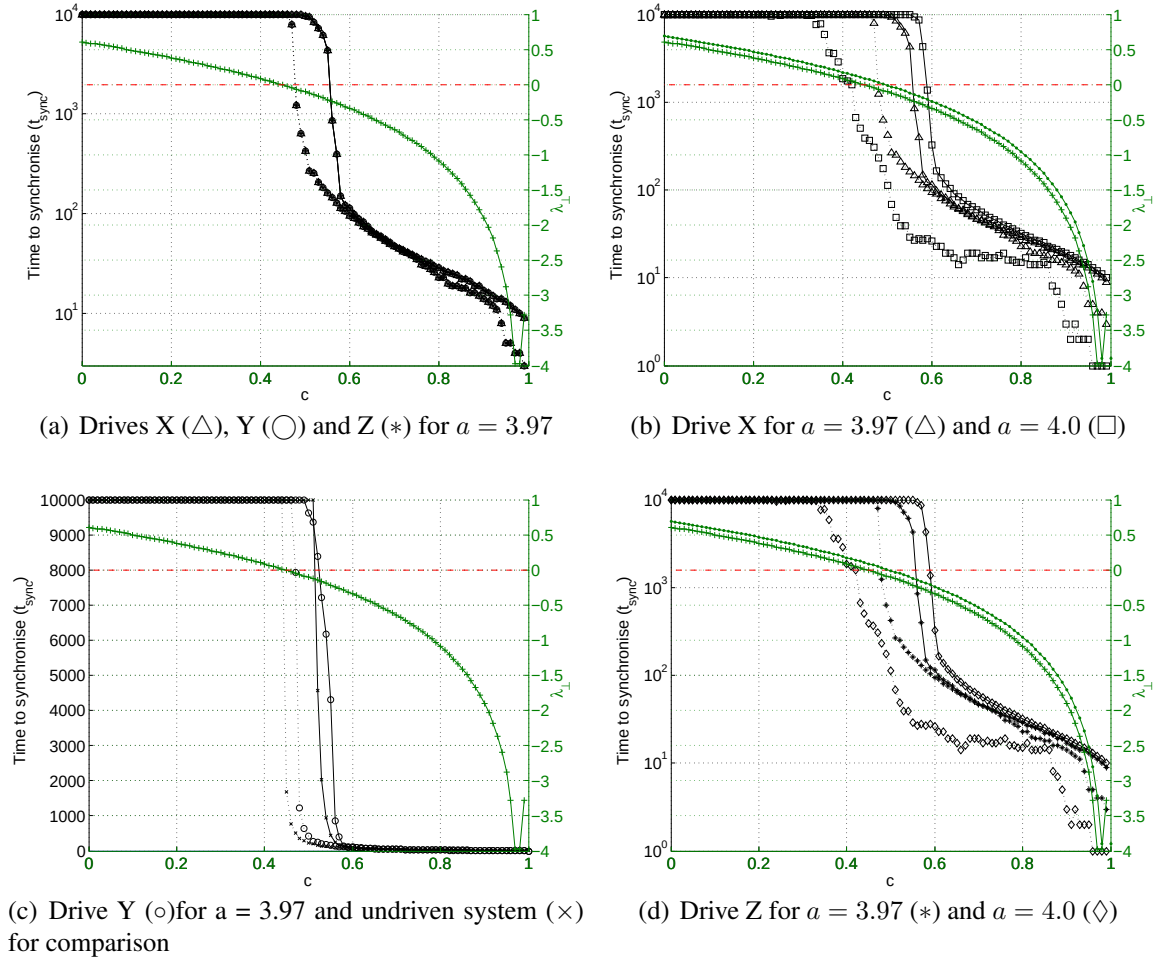


Figure 6.1: The time to synchronise for driven systems of Model A (solid lines) and the simpler Fixed A (dashed lines). The synchronous threshold, $\delta = 10^{-25}$; each system consists of 100 nodes, 5 of which receive the external stimulus. The stimuli are as follows: (1) drive type X, with $\omega = 0.28$ and $a = 3.97$ ($\triangle$); (2) drive type X, with $\omega = 0.28$ and $a = 4.00$ ($\square$); (3) drive type Y, with $a_{stimulus} = 4.0$ and $a = 3.97$ ($\circ$); (4) drive type Z of amplitude 0.1 with $a = 3.97$ ($*$) and $a = 4.0$ ($\diamond$). All data is shown as the average over 100 random initial conditions. A value of 10^4 for the time-required-to-synchronise is indicative that no synchronisation was recorded during the simulations. The transverse Lyapunov exponent is shown by the solid lines in green for $a = 3.97$ (+) and $a = 4.0$ (.).

6.3 Frequency response of systems

Before analysing what response is generated by the systems when stimulated, we will first take a look at the power spectra of the undriven systems, shown in fig. 6.2. These are obtained by taking the N node-state timeseries and generating N power spectra as the square of the absolute value of the Fourier transforms for each timeseries individually ([Jen98] and [KG95] include discussions of and general introduction to power spectra). To generate the average power spectrum for the system, the simple average of these N individual power spectra is used. It is these average power spectra that are used to consider the frequency response of the systems.

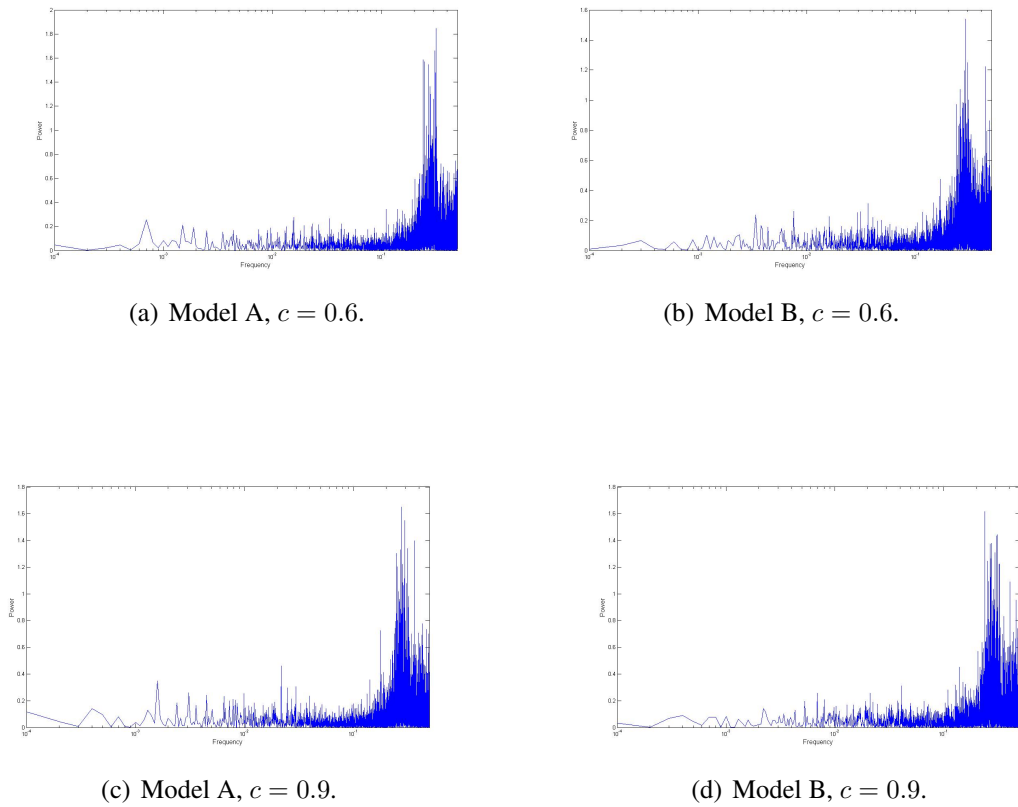


Figure 6.2: The average power spectra from the node-state timeseries of the systems as labelled. Other parameters are, $N = 100$, $a = 3.97$, and the length of the timeseries used was 10^4 timesteps.

For both Models A and B without any stimuli, the main feature of the power spectrum is a broad peak at a frequency $f \sim 0.28$. Recall that the quasi-frequency as measured for the single logistic map for the same a -value of 3.97 is 0.28. When the systems receive a small number of type X stimuli, as shown in figs. 6.3, 6.4, 6.5 and 6.6, Models A and B give different responses. For both systems, the power spectra retain largely the same form. However, Model A power spectra gain a sharp peak at exactly the frequency of the stimulus, as can be seen in figs. 6.3 & 6.4. Model B shows similar sharp peaks, but at the second or higher harmonics of the stimulus frequency; which harmonics are visible depends on the coupling strength parameter, c , as shown in figs. 6.5 & 6.6.

These basic findings are robust to changes in the stimulus frequency, ω , within the range $[0.01, 10]$. Additionally, the finding that Model A displays the extra peak in power at the stimulus frequency, whereas Model B displays extra peaks in power at higher harmonics remains unchanged. An increase to the number of stimuli simply increases the amplitude of power at the stimulus frequency or its harmonics.

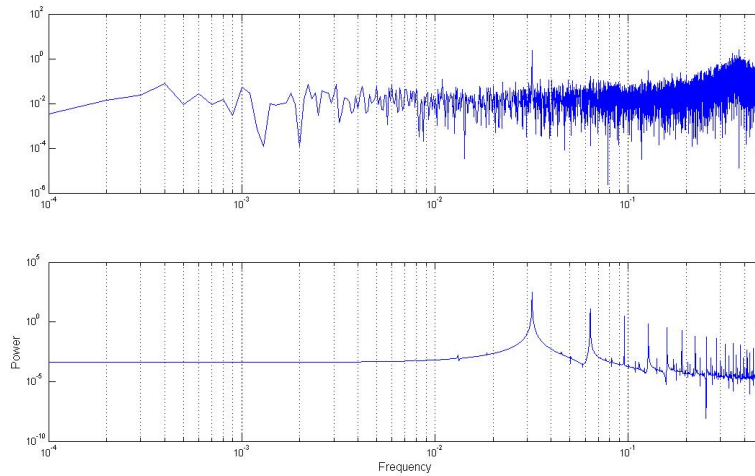


Figure 6.3: Average power spectrum of the node-state timeseries of Model A. 5% of the nodes receive stimuli of Type X with $\omega = 0.1$. Other parameters are: $c = 0.9$, $a = 3.97$ and $N = 100$ MaxTime = 10^4 . Top: main nodes only. Bottom: Stimuli only.

We have seen that the dynamics of the systems are altered when driven by a sinusoidal signal, such that there is a higher proportion of the power in the frequencies either at the driving frequency, ω , or its higher harmonics. When the systems are driven by a logistic

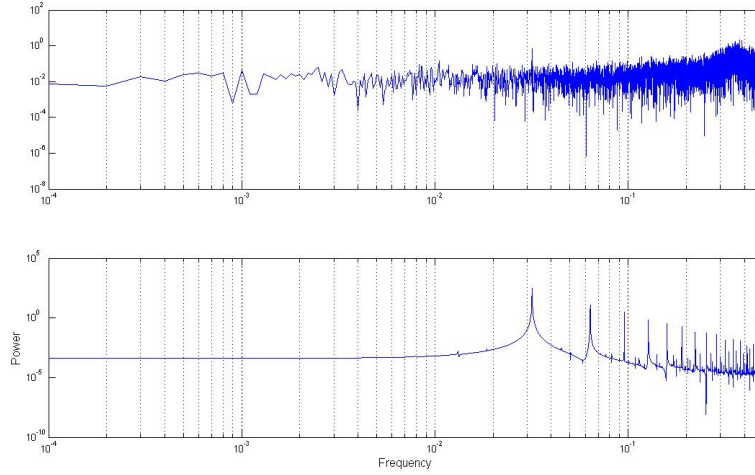


Figure 6.4: Average power spectrum of the node-state timeseries of Model A. 5% of the nodes receive stimuli of Type X with $\omega = 0.1$. Other system parameters are $c = 0.6$, $a = 3.97$, $\text{MaxTime} = 10^4$. Top: main nodes only. Bottom: stimuli only.

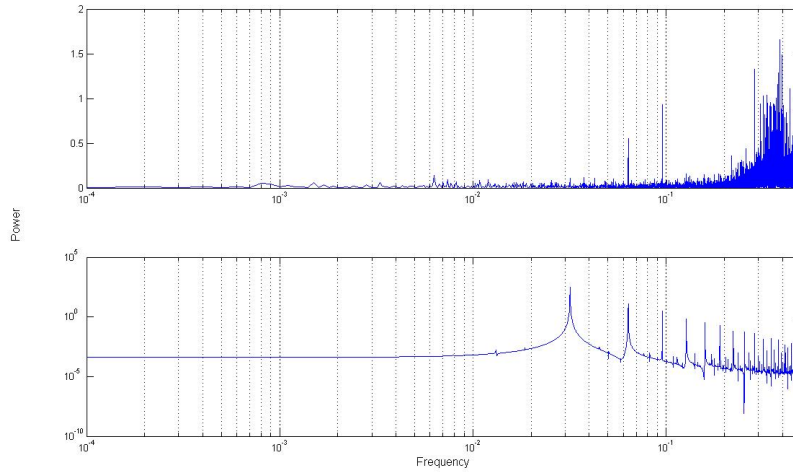


Figure 6.5: Average power spectra of the node-state timeseries of Model B. 5% of the nodes receive stimuli of Type X with $\omega = 0.1$. Other parameters are $c = 0.9$, $a = 3.97$, $\text{MaxTime} = 10^4$. Top: main nodes only. Bottom: Stimuli only.

map with mismatched nonlinearity parameter such that the stimuli are chaotic signals, the power spectra are qualitatively unchanged. This should perhaps be expected since the chaotic signal is not dominated by any particular frequency, having a broad spectrum itself. When we drive the system with a constant signal of amplitude κ , should we expect the system to have a shift of its power to lower frequencies?

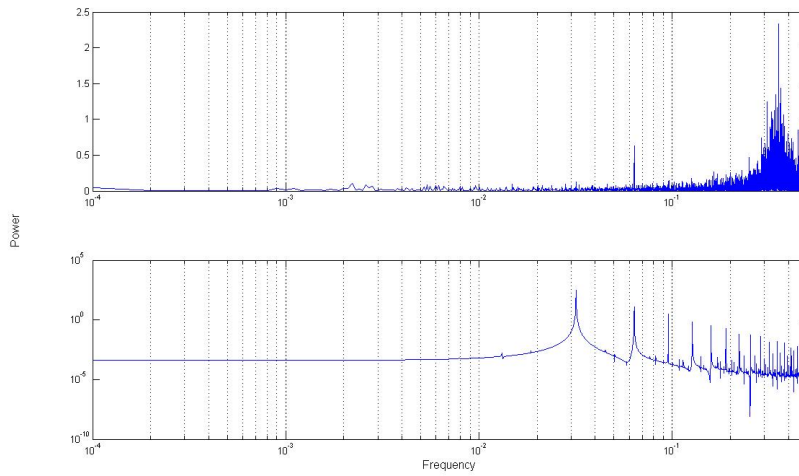
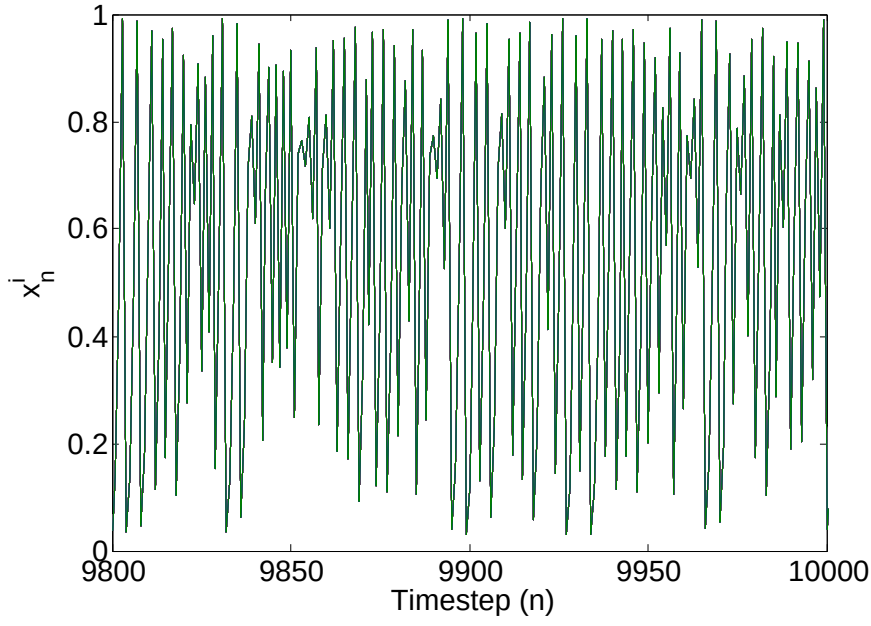


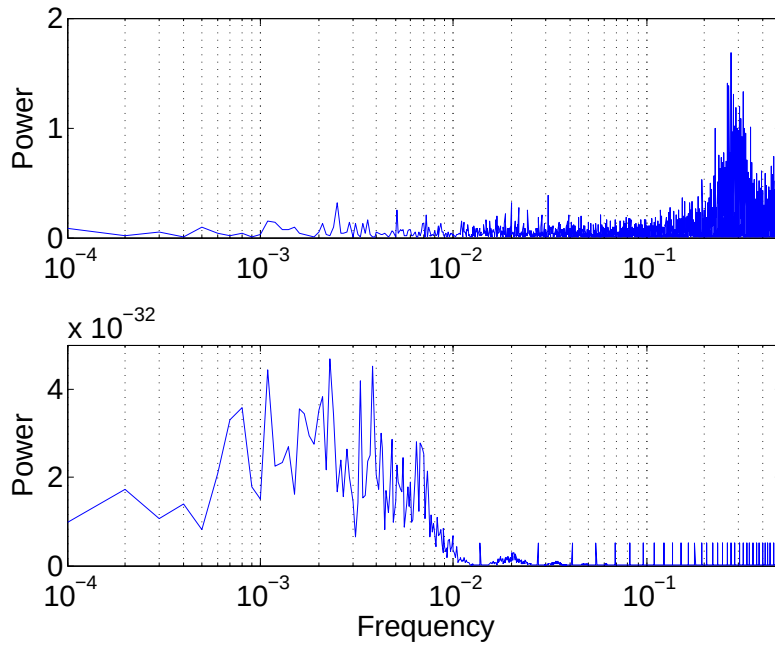
Figure 6.6: Average power spectra of the node-state timeseries of Model B. 5% of the nodes receive stimuli of Type X stimuli with $\omega = 0.1$. Other system parameters are $c = 0.6$, $a = 3.97$ and $\text{MaxTime} = 10^4$. Top: main nodes only. Bottom: stimuli only.

As can be seen by comparing the power spectra of Model A without stimuli in fig. 6.2 with those from systems in receipt of stimuli type Z shown in fig. 6.7 and 6.8, the system is qualitatively unaffected. Unlike when Model A received a periodic stimulus, no response is visible in the power spectra in response to a constant stimulus.

When we look at the simplified system with static all-to-all coupling as in eqn. (4.3.1), the results just discussed are robust. However, for high coupling strengths, when the system receives the constant drive, it undergoes a rapid oscillation death. This is different to that observed for Model C in the absence of stimuli discussed in Chapter 5, in so much as the node-states all evolve to zero: $x_n^i = 0$ for all n beyond the initial transient, rather than to

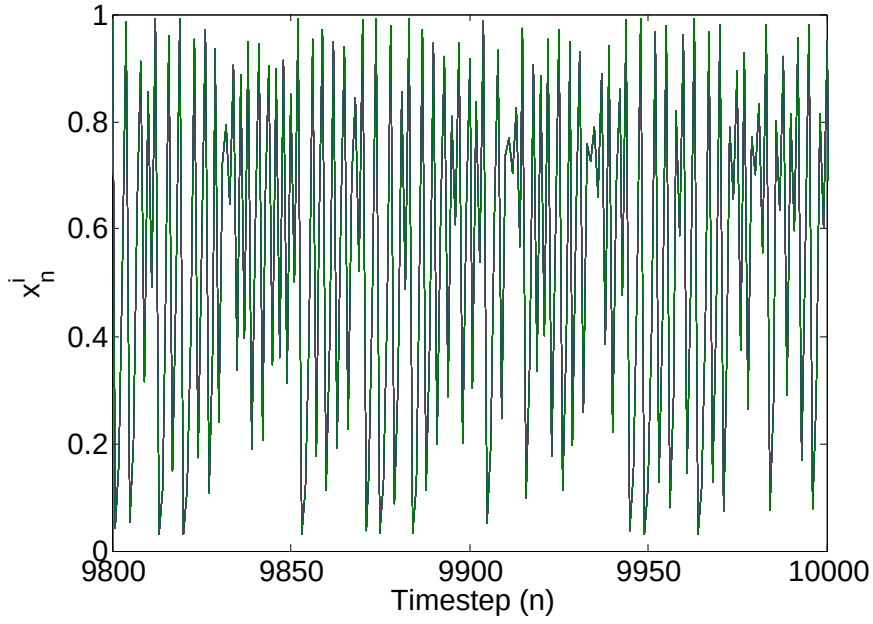


(a) Node-state timeseries

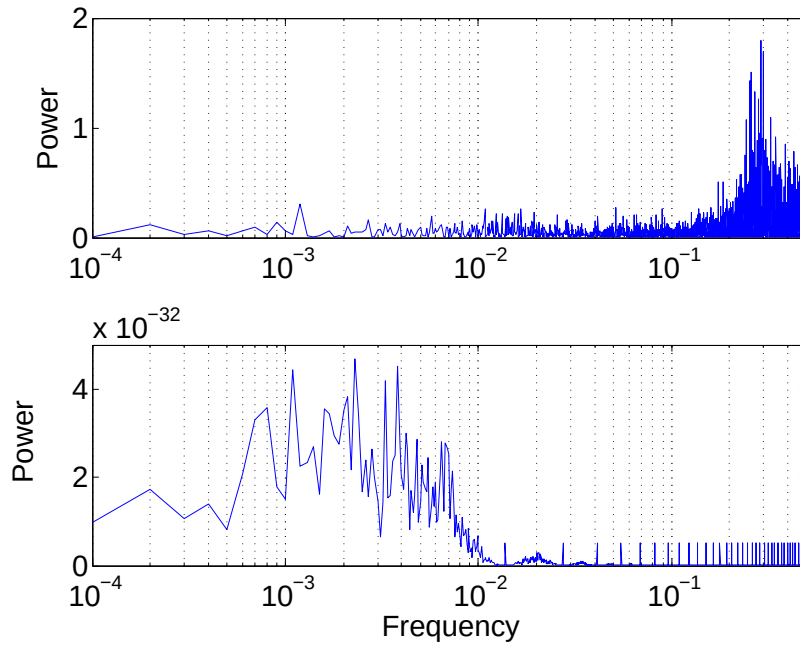


(b) Power Spectra, Top: main nodes only. Bottom: stimuli only.

Figure 6.7: Model A with drive Type Z with $\kappa = 0.1$. Other parameters are $c = 0.60$, $a = 3.97$, $\text{MaxTime} = 10^4$.



(a) Node-state timeseries



(b) Power Spectra, Top: main nodes only. Bottom: stimuli only.

Figure 6.8: Model A with drive Type Z with $\kappa = 0.1$. Other parameters are $c = 0.90$, $a = 3.97$, $\text{MaxTime} = 10^4$.

the fixed point as was the case for Model C. This oscillation death can be clearly seen in fig. 6.9, along with the associated power spectra, for completeness.

6.4 The effect of coupling mechanism to stimuli

All the results presented in this chapter are the result of simulations where the stimuli are coupled to a random subset of nodes with static connection weight. It is of course possible to allow the connectivity between the stimuli and the main nodes of the system to evolve, just like the connections between two main nodes. This minor change so that there is a Hebbian connection between stimuli and main nodes has been briefly investigated. This altered the frequency response so that the system exhibited a peak in the power spectrum at twice the driving frequency, (as well as higher harmonics), but the first harmonic was always absent for both Models A & B. It also prevented the systems from synchronising, even for high values of coupling strength, so the time to synchronise could not be calculated.

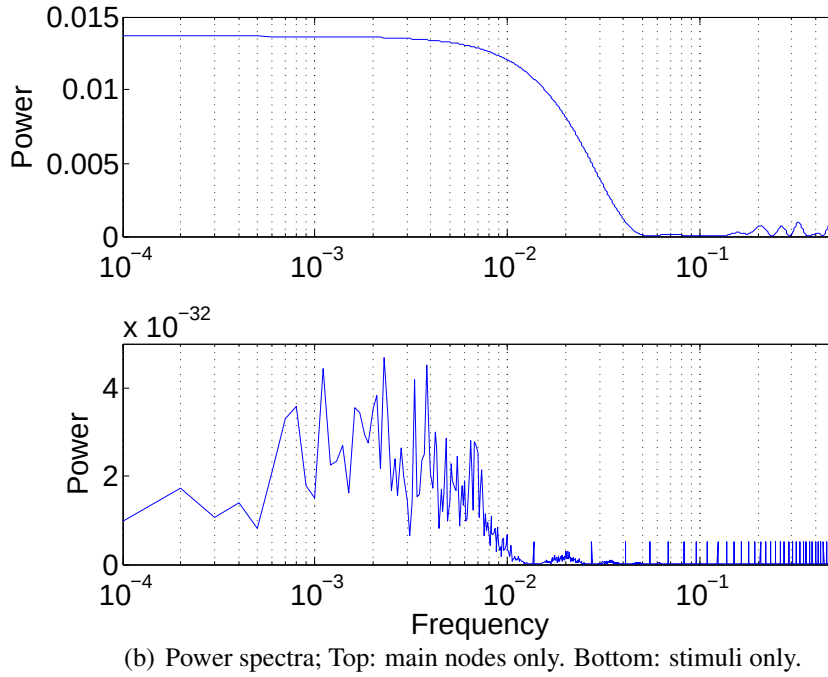
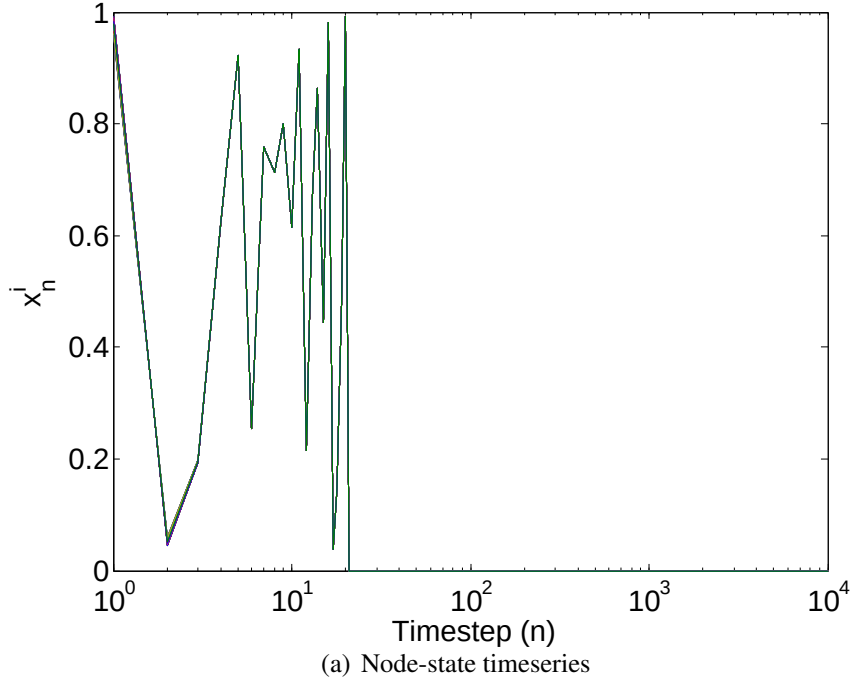


Figure 6.9: Node-state evolution (a) and power spectra (b) for the Fixed A system (Model A simplified to all-to-all coupling), with a drive Type Z of $\kappa = 0.1$, $c = 0.9$, $a = 3.97$, $\text{MaxTime} = 10^4$.

6.5 Concluding remarks

The results in this chapter give a small taster for the type of affect an external stimulus can make to the coupled-map systems studied in this thesis. There are of course infinitely many ways we could implement the addition of an external stimulus and there are equally many different types of stimuli we could use. It is not suggested that this brief study is an exhaustive analysis of the gigantic area as to the effect of external stimuli on coupled-maps. However, even from these small initial investigations we can see that an external stimulus can have unexpected affects on these systems and it could consequently be an exciting area to explore further.

The effects of the stimuli shown here have similarities to the features observed empirically in brain studies; both in the preliminary results of the group of Diaz-Santana and in other areas of brain function.

Diaz-Santana's preliminary studies found that the visual cortex response to a periodic stimulus is at frequencies corresponding to the second or higher harmonics of the stimulus frequency. This is the type of response shown by Model B. An explanation as to why the externally, nonlinearly coupled system should display this type of behaviour when the internally coupled system does not is still an open question. With more time, it would have been interesting to investigate which other forms of coupled-maps would display this behaviour too; whether it is a generic feature of the external, nonlinear coupling scheme.

The addition of external stimuli to the coupled-map systems considered here, leaves the ability of the systems with fixed connectivities to synchronise quasi-instantaneously unchanged. For the Models with coevolving connections, there is always a finite duration of the synchronisation transients. This finding strengthens the case for their being a latency between when the brain receives a stimulus and when we become aware of it.

Synchronisation in the brain has been correlated with awareness of sensory stimuli by Melloni et al. [MMP⁺07]. Since neural connectivity is changing on many timescales, so could certainly not be considered static [Squ08], we should expect there to be a delay between the stimulus and being aware of the stimulus; this latency being the required length of time for the requisite level of synchrony to build-up.

Chapter 7

Discussion and concluding remarks

7.1 Discussion of main results

Complex systems pose many unique challenges in order to make progress to understand them. The use of networks as a tool to aid the understanding of the structure of their interactions has been phenomenally successful to date. Coupled-map systems provide a simple avenue to extend this progress to incorporate dynamical evolution of the constituent elements of the system. They are a broad class of systems with many possibilities for developing different models to use for gaining insights into emergent phenomena observed in complex systems.

This thesis has presented an in-depth study of three particular systems of coupled-maps with coevolving connections. These systems are examples of autonomous toy models for studying synchronisation and other phenomena observed in complex systems. The introduction of this particular form of coevolving connections leads to a host of interesting and unexpected changes to the behaviours displayed by the systems when compared to the corresponding coupled-map systems with fixed, all-to-all connections. This use of Hebbian dynamics as a means of ensuring dynamical connections is far more appealing than some of the more ad hoc methods employed by others to introduce time evolution to the connections between maps. As a consequence, they are useful models to use in order to try and understand the mechanisms of synchronisation which is observed in so many real-world systems that comprise many parts connected by dynamic connections. Some of the

behaviours shown by these models has been shown to give insights into some interesting observations from the field of neuroscience, for example. This in turn may also suggest interesting new testable hypotheses in this field.

Models A & B were previously introduced by Ito and Kaneko, their preliminary investigations [IK02][IK03] have been extended here. It has been shown that Ito and Kaneko's assumption that Models A and B display the same behaviour is largely true. However, the behaviour of these two systems was shown in Chapters 4 & 5 to be much more complicated than was implied by Ito and Kaneko's simplistic description of the partitioning of parameter space.

The intricate structures found in the coherent state basin of attraction are reminiscent of those found by Ashwin, Buescu and Stewart for the system of eqn. (3.2.4). For this system, Ashwin, Buescu and Stewart found the stability of the synchronous state is lost gradually, across a range of coupling strengths; just as is found here for Models A & B. The transition from the basin of the coherent state occupying the entirety of phase space at high c , to it occupying none of phase space at low c occurs continuously for Models A & B, as was shown in fig. 4.9. Across this range of coupling strengths where the synchronous state remains linearly stable, yet the basin of attraction for this state does not fill the entirety of phase space, it contains holes where the system remains in the asynchronous state. This is an example of what is sometimes known as bistability: the system has two states that are both stable and in this case, it is the choice of initial conditions that determines which state the systems enter. This can also be referred to as the coexistence of these two states; for these particular parameter values, the asymptotic state of the system can either be synchronous or asynchronous, so these two states coexist together.

Conversely, for Model C, this transition occurs discontinuously, as was shown in fig. 4.10. It is interesting that there is such a difference in the characteristics of how the synchronous state loses stability between these models. Previous studies have also found that some coupled-map systems lose synchrony abruptly, whilst for others it is a gradual process across a range of coupling strengths. There is, as yet, no means to know in advance which type of transition any particular coupled-map system will follow.

Despite this dichotomy in the average nature of the transition, when the behaviour of any particular simulation that results in the synchronous state is considered, the approach

to the synchronous state for all three models occurs in the same manner: exponentially quickly, as measured using the node state variance as a distance measure between the system state and the synchronous submanifold. In simulations, this distance from the synchronous submanifold reduces exponentially until the numerical precision is exhausted.

The total duration of these synchronisation transients; the time it takes for the system to reach the synchronous submanifold, was shown in Chapter 5 to reduce as the coupling strength is increased. That is, the modulus of the parameter governing the exponential decay increases with increasing coupling strength, although not in a simple manner, as was shown in fig. 5.8. However, even for extremely high coupling strength, maps coupled via coevolving connections never synchronise instantaneously¹. In contrast, for the simpler systems with fixed all-to-all connectivity between maps, for extremely high coupling strength, the systems are found to synchronise quasi-instantaneously.

This inability for the systems with coevolving connections to synchronise quasi-instantaneously has potential relevance to the question of the timing of conscious awareness. There has been a continuing discussion in the neuroscience community for several decades, regarding the duration of the latency between when a stimulus is applied directly to the cortex and when the subject becomes aware of the stimulus. This is inherently difficult to measure, but it is now largely accepted that this latency does exist; the precise duration of this latency is not confirmed. Estimates range from tens [Poc02] to hundreds [Lib64] of milliseconds. It should however be noted that whatever the precise duration for any particular stimulus, in general, it is variable; dependant on a number of factors including the nature of the stimulus, whether it is direct cortical stimulation, visual etc. The intensity of the stimulus also has an impact on the duration, longer latencies being associated with lower intensity stimuli [Lib64, Gom02].

The results here that any system with dynamic connections always requires a finite time in order to achieve synchrony, strengthens the case for the existence of this latency before conscious awareness occurs. It has been established that a certain level of synchrony is required between disparate brain regions in order for conscious awareness to occur [MMP⁺07]. Therefore, since changes in neuronal connections are observed across

¹or quasi-instantaneously since instantaneous is not possible due to the discrete nature of time in these systems.

many timescales and these systems with dynamic connections never have an insignificant duration to their synchronisation transients, there will always be a delay between stimulus and conscious awareness of the stimulus.

The behaviours of these models suggest, that it is necessary to have a comparatively higher strength of connection in order to achieve the same duration of delay before synchronisation if the connections coevolve along with the nodes as compared with having fixed connectivity. When looking to real neuronal connections, it is expected that they should not be static in time. Indeed, changes are observed in connectivity across many timescales. We have here highlighted the difference to timings of synchrony that such time-varying connections can make. It is also interesting to note the potential extrapolations and insights possible from a greater understanding of the interplay between the levels of synchronisation in the brains of epilepsy sufferers and the potential underlying causes of this disease. Could it for example be possible that sufferers of epilepsy have less time-dependence in their neuronal connectivity? Does this allow them to have faster reaction times as a consequence? Future work is required into the existence of similar latencies before synchrony in more neurologically accurate models, as well as other simple empirical observations in order to answer these questions with certainty. It still remains, however, that coupled systems require a critical level of interaction, above which, the time it takes for them to synchronise decreases as the coupling strength is further increased.

Another aim of this thesis was to look in more detail at what changes are observed in the macroscopic dynamics as a result of changes to the microscopic coupling mechanism in coupled-map systems. This has led us on a fruitful journey of discovery and we have seen just how sensitive the behaviour of coupled-map systems can be to seemingly small changes to the microscopic coupling mechanism. Conversely, there are other changes that leave the macroscopic behaviour largely unchanged.

The difference in microscopic coupling mechanism between Model A & Model B does not lead to large differences in macroscopic behaviour; there is only a small shift as to where in parameter space the transitions between phases occur. For example, the transition from the ordered to coherent region occurs at a slightly lower coupling strength for Model B compared to Model A. However, the change in microscopic coupling from Model B to Model C, from nonlinear to linear output function, causes a huge change in the macroscopic

dynamics. This is most visible in the time evolution of nodes-states when the system is in the coherent state. For Model B (and Model A) the node-states display aperiodic dynamics, yet for Model C the node-states enter a time-independent state that coincides with the fixed point of the logistic map. It is therefore of utmost importance to choose carefully the microscopic dynamics of systems if we plan to use them as one-to-one models of real-world systems or phenomena.

The stabilising affect to fixed points of the underlying map has been previously observed in coupled-map systems [Jia00, MM03, MM05, MPM05, ABJ06], but these earlier examples required the introduction of explicit time-delays. Here we have seen that explicit time-delays are not necessary and that the linear output function of Model C is also able to stabilise the fixed point of the logistic map. This stabilising ability is not reliant upon the coevolving connections as the same stabilising affect is seen in the Fixed C system. This finding of the stabilising affect of the Model C and Fixed C coupling is similar to the stabilising affect that feedbacks can have. The addition of feedbacks is a commonly used tool to control oscillations in engineering situations, as they are a well established cause of oscillator death.²

Incorporating Hebbian dynamics into the evolution of the connections between nodes means that nodes with stronger connections are more alike relative to those with weaker connections. We might therefore expect the two cluster determination methods considered to give broadly similar partitionings of the parameter space. As we saw in Chapter 4, the partitionings obtained by the node-state similarity and connection-strength threshold methods are in fact different for the different Models; the biggest difference between these methods was observed for Model C. The two resultant partitionings of parameter space for Models A & B were both well approximated by the schematic partitioning previously published by Ito & Kaneko [IK03]. However, the impression given by this work that the transition from the ordered to coherent state is sharp and occurs along the line in parameter space where $\lambda_{\perp} = 0$, has been found not to be the case. As already discussed, the transition occurs in a much more complicated fashion with the existence of seemingly riddled basins for a range of intermediate coupling strengths. This gradual transition is not seen

²The system enters a steady-state with the absence of any oscillation; the oscillations have been “killed-off”. See [Ata03] for a discussion of this phenomenon.

for Model C, nor for any of the systems with fixed, all-to-all connectivities.

We have shown in Section 4.3 that Models A and B with coevolving connections do not synchronise for a certain range of coupling strengths even when the synchronous state is linearly stable, $\lambda_{\perp} < 0$. This is similar behaviour to that previously reported by Ahlers & Pikovsky [AP02], where they found for systems of CMLs, $\lambda_{\perp} < 0$ was not a sufficient condition for the systems to synchronise if the underlying map was discontinuous. They suggested that this finding was due to the nonlinearities of the underlying map acting as effective perturbations. This could also be the cause for $\lambda_{\perp} < 0$ not to be a sufficient condition for synchrony in Models A & B.

Since the analytical transverse Lyapunov exponent cannot explain the stability of the synchronous state, it would be interesting to be able to compare the analytically derived transverse Lyapunov exponent with one calculated empirically from a reconstruction of the dynamics using the generated timeseries. Since the systems considered here have typically been of 100 nodes, this obviously results in a 100 dimensional phase space. As explained in Appendix A, one of the crucial steps in calculating the Lyapunov exponent from a timeseries requires a frequent reorthonormalisation process which is computationally challenging in such a high-dimensional system since we need to have a suitably high density of points across the phase space. This has therefore not been possible with the time and computational power available.

In summary, it has been demonstrated that the macroscopic behaviour of coupled-map systems is not predictable simply from the microscopic dynamics. They are examples of complex systems; comprising many parts, having nonlinearities with boundaries that can be extended if they interact with external stimuli and we have observed that the collective dynamics is not readily predictable from the behaviour of the individual maps. This final characteristic constitutes the emergent behaviour that is a hall-mark of complex systems. It has also been shown that some of the specific behaviours displayed can give insights into open problems in the field of neuroscience; which interestingly was the area that inspired these models. There is of course much work still remaining with regard to these models and other autonomous coupled-map systems that will no doubt lead to many insights into very diverse complex systems in the future.

7.2 Open questions and future work

Future work could include creating a parallel code to implement the Gram-Schmidt orthonormalisation in order to calculate the Lyapunov exponents numerically by the algorithm outlined in Appendix A. By utilising parallel processing, it may be possible to scan through many trajectories at the same time in order to find a suitable nearest-neighbour without it taking a prohibitively long time or requiring the storage of a prohibitively long timeseries. This would be interesting so as to allow the comparison of the measured empirical Lyapunov exponent for the systems with coevolving connections with the analytical ones used here. Additionally, this would allow a greater study of the impact an external stimulus has on the dynamics.

There are infinitely many other characteristics that can be incorporated into these systems to better represent the real-world; for example introducing time-delays, noise, and/or inhomogeneities in the nodes, so that they are not all identical. It has not been possible to consider any of these changes due to time and computational limitations.

Another attribute that has not been altered is that of the plasticity of the coevolving connections. This is governed by the parameter, δ , that has remained fixed with value 0.1 throughout these studies, but could be interesting to vary in order to better understand the changes introduced to the macrodynamics by the introduction of coevolving connections.

The study presented here as to the effect of external stimuli is preliminary. This is a vast area, that could indeed be endless due to the wide array of types of stimuli as well as the manner in which the stimuli can be implemented.

It would be interesting to know whether sufferers of diseases related to synchrony in the brain have empirically observable characteristics that might be implied from the results presented in this thesis. For example, epileptic seizures are associated with increased synchrony in the brain. We have seen that systems with coevolving connections synchronise slower and at higher coupling strength than those with static connections. Is it a reduced plasticity in the brains of epilepsy sufferers that leads to the increases in synchrony associated with seizures? Would they have measurably faster reaction times to sensory stimuli as a result?

Conversely, schizophrenia is associated with low levels of synchrony; will the opposite

be found here of sufferers having slower reaction times to sensory stimuli? These are interesting questions that could help shed further light onto potential cause, treatments and cures for these conditions. They also appear to be easily answerable through simple experiments measuring reaction times.

Appendix A

Calculating Lyapunov exponents from timeseries

The purpose of this appendix is not to give a prescriptive description of how to calculate Lyapunov exponents directly from an experimental timeseries. For this, you are referred to the literature, see for example [PCFS80, WSSV85, Aba96, Kan04, Bj94]. Instead, it is intended to give a brief overview of the method so as to allow an appreciation of the difficulties in implementing the algorithm for such high dimensional systems as are investigated here.

The methods available to us for calculating Lyapunov exponents from timeseries are certainly not simple and cannot be treated as a black box into which we can put the timeseries and expect a Lyapunov exponent out. There are many points in the process where human judgement and experience is required. The process detailed here is as proposed by A. Wolf and co-workers in [WSSV85] and comprises several stages:

- 1 Find an appropriate time delay through analysis of the mutual information measure.
- 2 Choose an embedding dimension.
- 3 Calculate Lyapunov exponent from the trajectory in the reconstructed phase space, using Gram-Schmidt reorthonormalisation.

Each of which has its own particular pitfalls to be avoided as we will now discuss.

A.1 Choosing a time delay

From the original timeseries $x(n)$, we generate new vectors $y(n)$ in order to capture the structure of the underlying dynamics via a geometric unfolding of the original trajectory into a reconstructed phase space. These points that make up the new trajectory are given by time delayed coordinates of the original timeseries

$$y(n) = [x(n), x(n+T), x(n+2T) \dots x(n+T(d-1))] , \quad (\text{A.1.1})$$

where T denotes the duration of the chosen time-delay, and d the dimension of the reconstructed phase space, whose value will be discussed in the next section.

An indication of a good value to choose as the time-delay, T can be gained by consideration of the Average Mutual Information between $x(n)$ and $x(n+T)$ (see [Aba96] and references therein):

$$I(T) = \sum_{x(n), x(n+T)} P(x(n), x(n+T)) \cdot \ln \left[\frac{P(x(n), x(n+T))}{P(x(n)) \cdot P(x(n+T))} \right] . \quad (\text{A.1.2})$$

This average mutual information tells us quantitatively how much one can learn about $x(n)$ from the measurement $x(n+T)$, similar to a correlation function, only the mutual information contains information about the nonlinear correlations in addition to the linear correlations contained in a simple correlation function [Li90].

We calculate $I(T)$ as a function of the time delay T and choose the value of T that coincides with the first minimum of $I(T)$. It is of course advisable to check the stationarity of the finally calculated Lyapunov exponent to small changes in T about the optimum choice.

A.2 Choosing an embedding dimension

The embedding dimension is the dimensionality of space into which we will unfold the dynamics so as to (hopefully) reveal the structure of the attractor. The dimension of this space necessary will vary between timeseries, depending on the dimension of the under-

lying attractor. The embedding dimension must be larger than the dimensionality of the attractor. In fact, for an attractor that is M -dimensional, a sufficient embedding dimension of $2M$ can be used [Tak81].

Unfortunately, without prior knowledge of the dimensionality of the attractor there is no easy way to know or work out the necessary embedding dimension. It is also not wise to simply choose one so large that it will almost certainly satisfy the criterion of being more than twice the dimensionality of the attractor. This is because the computational difficulty of calculating the Lyapunov exponent increases along with the embedding dimension, as shall be discussed in the next section. It is therefore desirable to choose the smallest possible embedding dimension that still satisfies the criterion of being greater than $2M$, where M is the dimensionality of the attractor, in order to guarantee a successful embedding.

Best practice seems to suggest it sensible to check the robustness of the calculated Lyapunov exponents to small changes in the embedding dimension, just like for the time delay.

A.3 Lyapunov exponent from reconstructed phase space

Lyapunov exponents are quantitative measures of how fast two initially nearby points separate under a particular dynamics. In order to calculate the Lyapunov exponent of a time-series generated by a particular dynamics, we need to follow the dynamics of small perturbations away from a point, $\underline{x}_{t_0}$, in every direction of the phase space. In principal, we therefore find a point located in the timeseries that is sufficiently close to $\underline{x}_{t_0}$. We then "evolve" these two points separated by $L(t_0)$ in order to find how this perturbation evolves.

Within such a process, we run into some numerical difficulties: firstly, if λ_1 , the largest Lyapunov exponent, is greater (or less) than zero, the perturbation will grow (or shrink) exponentially. In this situation, even for moderate numbers of timesteps, the perturbation, $L(t_k)$ will become computationally incalculable. It will become too large ($\lambda_1 > 0$) or zero ($\lambda_1 < 0$) due to the inherently finite nature of computational numerical accuracy. A second difficulty arises due to the alignment of the perturbations along the direction corresponding to the largest Lyapunov exponent. This component of the perturbation vector grows faster than any other, and again after even a moderate numbers of timesteps, the perturbation

vector will be aligned along the direction corresponding to the largest Lyapunov exponent. A means to resolve these problems is through a series of reorthonormalisations carried out sufficiently often such that the two problems just described do not occur. The growth rate of the perturbation is independent of the length of the perturbation vectors, thus we may renormalise the vectors and not be affected by any problems associated with the numerical accuracy. At the same time as renormalising, by also reorthogonalising the vectors we may avoid the second problem of alignment. We reorthogonalise such that the first component of the perturbation vector aligns with the direction of greatest growth; the second component aligns to the direction of second greatest growth perpendicular to the first and so on for all components.

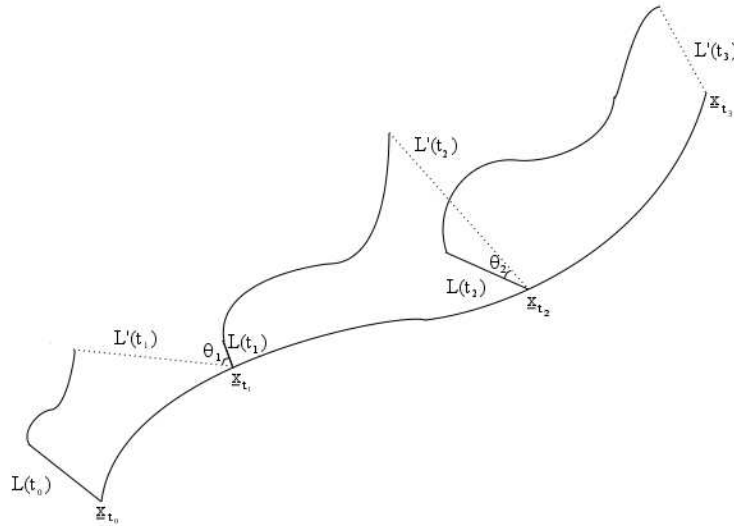


Figure A.1: A schematic representation of the fiducial trajectory and its relation to the points chosen through the reorthonormalisation procedure.

Through this process of reorthonormalisation, we have a new perturbation vector, $L(t_1)$, relative to the fiducial trajectory which is now at point x_{t_1} . This process is shown schematically in fig. A.1. The number of actual timesteps of the original timeseries between carrying out this reorthonormalisation is highly dependent on the dynamics that produced the original timeseries.

In the limit when we have an infinite timeseries, free of noise, we can calculate the maximum Lyapunov exponent by definition from,

$$\lambda_1 = \frac{1}{t_M - t_0} \sum_{k=1}^M \ln \frac{L'(t_k)}{L(t_{k-1})}, \quad (\text{A.3.1})$$

where M is the total number of times the reorthonormalisation has been done; $L(t_{k-1})$ is the initial perturbation at time t_k ; and $L'(t_{k-1})$ is the evolved perturbation at time t_k , as shown schematically in fig. A.1.

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