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Phase Space Dynamics of Non-Hermitian Quantum Systems

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Declaration of Originality

I, Katherine Holmes, declare that the work in this thesis is my own. The work of others has been appropriately referenced. A full list of references is given in the bibliography.

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Abstract

In this thesis, we shall study phase space dynamics of non-Hermitian quantum systems in the semiclassical limit. I derive the (semi)classical dynamics generated by non-Hermitian Hamiltonians from quantum dynamical equations for two quasi-probability distributions, the Husimi and Wigner distributions. The methodology presented applies to generic Hamiltonians expressed as power series of n -dimensional position and momentum operators and the quasi-probability distributions are allowed to have arbitrary initial conditions. In the Husimi and Wigner representations, we uncover an important function, the norm landscape, contained within the classical dynamics, which evolves over time and phase space. The norm landscape describes the classical analogue of loss and gain, a fundamental characteristic of non-Hermitian system.

This thesis analyses the quantum and classical phase space dynamics of several examples of non-Hermitian systems. The examples include linear systems, which in some cases have exact analytic solutions in either the Husimi or Wigner representation; non-linear systems, including a dissipative and a PT-symmetric Hamiltonian, and a two-dimensional billiard model. The conclusion of the thesis summarises preliminary results for further research projects.

Acknowledgements

I started the PhD with a very Taoist mindset of "I'm not trying to get a PhD, I'm trying to have a nice day", knowing to myself that a nice day was full of learning, collaborating and, ultimately, working towards the completion of a PhD. For a large part, this PhD has been a succession of nice days and I have many wonderful people to thank for that.

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As this chapter comes to a close, I reflect that in aspiring to have nice days, I will have achieved a PhD. I am extremely grateful for this experience.

Dedication

Whilst Schrödinger's cat resides in perpetual superposition, my dear cat Holly will
live on forever in my heart.

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Chapter 1

Background

1.1 Introduction

The phase space, spanned by position and momentum (or generalised canonical variables) is the home of classical dynamics. Classically, we may describe position and momentum with a point or with joint probability distributions on phase space however on a quantum level, we cannot. Quantum descriptions on the other hand have to fulfil the uncertainty relation and we rely on quasi-probability distributions to represent a quantum state on phase space [1]. These quasi-probability distributions have enabled a long and fruitful study of quantum dynamics. In the context of classical-quantum correspondence, the quantum dynamics of quasi-probability distributions can be approximated in a *semiclassical limit* that yields the underlying classical dynamics [2, 3].

In this thesis I will analyse the semiclassical limit of quantum dynamics generated by non-Hermitian Hamiltonians. Non-Hermitian Hamiltonians can be used to effectively describe decay and damping [4, 5]. While the study of non-Hermitian Hamiltonians in some form began in the early days of quantum mechanics [6], the majority of quantum research has centred around Hermitian Hamiltonians. In the last few decades, however, the field of non-Hermitian quantum theories has experienced a boom in research [7]. This excitement goes beyond theory into experiments, with applications in wave-guides [8, 9], many-body systems [10–12], optical experiments [13, 14], and biological systems [15, 16]. While much has been learned, the quantum-classical correspondence of non-Hermitian systems still remains largely unexplored.

The research in this thesis contributes to the foundations of non-Hermitian quantum theory, through a semiclassical and classical phase space approach. I will begin the thesis with a brief summary of the relevant background of conventional (Hermitian) quantum dynamics on phase space in the semiclassical limit and a short introduction to non-Hermitian quantum theory, before outlining the structure and crediting my original publications that the thesis builds upon [17, 18].

1.2 Classical Mechanics on Phase Space

Classical physics builds upon the notion that an object or particle possesses a set of physical observables. The state of a system is described by these real-valued observables and may be represented as a point in the phase space. In this thesis we consider systems on a phase space spanned by $2n$ canonical phase space variables q_j and p_j , the real-valued components of (generalised) position q and momentum p , where $j = 1, \dots, n$. For simplicity, in what follows we discuss one-dimensional systems, $n = 1$. Generalisation to higher dimensions are straight forward.

We consider joint probability distributions $f(p, q)$ on phase space. There are three axioms a joint probability distribution needs to fulfil:

1. The distribution is normalised

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p, q) dq dp = 1. \quad (1.1)$$

2. The distribution is non-negative, $0 \leq f(p, q) \leq 1$.

3. One can obtain the marginal probability distributions by integrating over one of the variables

$$f(q) = \int_{-\infty}^{\infty} f(p, q) dp, \quad f(p) = \int_{-\infty}^{\infty} f(p, q) dq. \quad (1.2)$$

From the probability distribution $f(p, q)$ we can calculate the expectation value of an observable $A(p, q)$ as

$$\mathbb{E}(A(p, q)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(p, q) f(p, q) dp dq. \quad (1.3)$$

The covariance of two observables $A(p, q)$ and $B(p, q)$ is defined as

$$\text{cov}(A, B) = \mathbb{E}\left((A - \mathbb{E}(A))(B - \mathbb{E}(B))\right). \quad (1.4)$$

Phase space is the natural setting for Hamiltonian dynamics, where conservative systems are described by real-valued Hamiltonian functions $H(p, q)$ that correspond to a conserved total energy E . The dynamics of these systems are governed by Hamilton's equations of motion

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}. \quad (1.5)$$

The solutions to Hamilton's equations for arbitrary initial phase space points determines the phase space flow $\Phi_t(p, q) = (p(t), q(t))$, which is known as Hamiltonian flow.

The time evolution of a joint probability distribution $f(t, p, q)$ follows from Hamilton's equations of motion as

$$\frac{\partial f}{\partial t} = -\{H, f\}_{p,q}, \quad (1.6)$$

which is the Liouville equation. Here $\{, \}_{p,q}$ denotes the Poisson bracket with respect to (p, q) , which is defined as

$$\{A, B\}_{p,q} = \frac{\partial A}{\partial p} \frac{\partial B}{\partial q} - \frac{\partial A}{\partial q} \frac{\partial B}{\partial p}, \quad (1.7)$$

for two phase space functions $A(p, q)$ and $B(p, q)$. For an initial distribution $f_0(p, q)$ at time $t = 0$, the time-evolved distribution is of the form

$$f(t, p, q) = f_0(\Phi_t^{-1}(p, q)), \quad (1.8)$$

where $\Phi_t^{-1}(p, q)$ is the inverse of $\Phi_t(p, q)$, which I denote as the backwards flow. That is $f(t, p, q)$ is the initial probability distribution transported along the Hamiltonian flow. This can be seen for example by applying the method of characteristics (see Appendix B) to equation (1.6). Alternatively, one can describe a point in the real 2-dimensional phase space by a complex variable z , which, with a view on quantum mechanics, we define as

$$z = \frac{1}{\sqrt{2\hbar}}(q + ip), \quad (1.9)$$

with complex conjugate $z^* = \frac{1}{\sqrt{2\hbar}}(q - ip)$.

Hamilton's equations of motion (1.5) in terms of z take the form

$$i\hbar \frac{dz}{dt} = \frac{\partial H}{\partial z^*} \quad (1.10)$$

and the Liouville equation is given by

$$i\hbar \frac{\partial f}{\partial t} = \frac{\partial H}{\partial z} \frac{\partial f}{\partial z^*} - \frac{\partial f}{\partial z} \frac{\partial H}{\partial z^*} = -\{H, f\} \quad (1.11)$$

with

$$\{A, B\} = \frac{i}{\hbar} \frac{\partial A}{\partial z} \frac{\partial B}{\partial z^*} - \frac{i}{\hbar} \frac{\partial B}{\partial z} \frac{\partial A}{\partial z^*}. \quad (1.12)$$

1.3 Quantum States and Observables

There are many conceptional differences between classical and quantum physics. One of the most prominent difference concerns the notion of states and observables. In classical Hamiltonian mechanics the state of a system is given by the phase space variables (p, q) (or a joint probability distribution $f(p, q)$) and observables are real-valued functions on phase space. In quantum physics, a state of a system is described by a linear operator known as the density operator $\hat{\rho}$. Any density operator can be expressed as

$$\hat{\rho} = \sum_j P_j |\psi_j\rangle \langle \psi_j|, \quad (1.13)$$

where $|\psi_j\rangle$ are Hilbert space vectors and P_j are real positive numbers with $\sum_j P_j = 1$. Note that the decomposition (1.13) is in general not unique. Density operators have the following properties

1. $\text{Tr}(\hat{\rho}) = 1$.
2. Non-negativity.
3. Hermiticity, $\hat{\rho} = \hat{\rho}^\dagger$.

For the special case that the density operator can be written as $\hat{\rho} = |\psi\rangle \langle \psi|$ we say that the state is pure and can work directly with $|\psi\rangle$.

Quantum observables are represented by linear Hermitian operators. The possible outcomes for a measurement of the observable are given by the eigenvalues of the corresponding operator. If a system is in a state $\hat{\rho}$, the expectation value of an observable $\hat{A}$ is given by $\langle \hat{A} \rangle = \text{Tr}(\hat{A}\hat{\rho})$ which reduces to $\langle \psi | \hat{A} | \psi \rangle$ for a pure state.

The Hamiltonian operator $\hat{H}$ encodes the energy E of the system. The Hamiltonian

also generates the time evolution of the state; the dynamics is given by the solution of the von-Neumann equation,

$$i\hbar \frac{d\hat{\rho}}{dt} = [\hat{H}, \hat{\rho}], \quad (1.14)$$

with commutator bracket

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \quad (1.15)$$

For pure states the von-Neumann equation reduces to the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle. \quad (1.16)$$

To solve Schrödinger's equation one can first solve the time-independent Schrödinger equation

$$\hat{H} |\phi_n\rangle = E_n |\phi_n\rangle, \quad (1.17)$$

where for simplicity we consider the case of a discrete spectrum of energy eigenvalues $\{E_n\}$ and corresponding eigenstates $\{|\phi_n\rangle\}$. Due to the Hermiticity of the Hamiltonian, its eigenstates form a basis of the state space and an initial state $|\psi\rangle$ can be expanded as

$$|\psi\rangle = \sum_n c_n |\phi_n\rangle, \quad (1.18)$$

where $\sum_n |c_n|^2 = 1$, and $|c_n|^2$ is the probability associated to $|\phi_n\rangle$. Then, for this initial condition, the solution to the Schrödinger equation is given by

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t} |\phi_n\rangle. \quad (1.19)$$

1.4 Position, Momentum and Phase Space in Quantum Mechanics

The operators $\hat{q}$ and $\hat{p}$, representing position and momentum, fulfil the canonical commutation relation

$$[\hat{q}, \hat{p}] = i\hbar \hat{I}. \quad (1.20)$$

The position and momentum operators have continuous spectra, $q \in \mathbb{R}$ and $p \in \mathbb{R}$, with corresponding eigenstates $|q\rangle$ and $|p\rangle$, such that

$$\hat{q}|q\rangle = q|q\rangle, \quad \hat{p}|p\rangle = p|p\rangle, \quad (1.21)$$

which satisfy the generalised orthonormality conditions

$$\langle q|q'\rangle = \delta(q - q'), \quad \langle p|p'\rangle = \delta(p - p'). \quad (1.22)$$

The eigenstates span the Hilbert space, i.e.,

$$\int_{-\infty}^{\infty} dq |q\rangle \langle q| = \hat{I}, \quad \int_{-\infty}^{\infty} dp |p\rangle \langle p| = \hat{I}, \quad (1.23)$$

and can be used to represent states in position and momentum basis.

The position and momentum representations of a pure state $|\psi\rangle$ are given by $\psi(q) = \langle q|\psi\rangle$ and $\tilde{\psi}(p) = \langle p|\psi\rangle$, respectively. We also have

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}pq}, \quad (1.24)$$

and thus

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq \psi(q) e^{-\frac{i}{\hbar}pq}, \quad (1.25)$$

i.e. position and momentum representation are related by a Fourier transform. Furthermore, the non-commutativity of the phase space operators from (1.20) leads to Heisenberg's uncertainty principle

$$\Delta\hat{q}\Delta\hat{p} \geq \frac{\hbar}{2}, \quad (1.26)$$

where $\Delta\hat{q}$ and $\Delta\hat{p}$ denote the standard deviation of the position and momentum observables. The uncertainty prevents the construction of joint probability distributions for quantum states in the classical sense. One can, however, define quantum phase space representations, which can correspond to quantum states with minimum uncertainty, coherent states, which I will briefly introduce in what follows.

One can view the position and momentum operators as real and imaginary components of a non-Hermitian operator known as the annihilation operator

$$\hat{a} = \sqrt{\frac{\mu}{2\hbar}} \left(\hat{q} + \frac{i}{\mu} \hat{p} \right), \quad (1.27)$$

where $\mu > 0$ is a physical parameter that I generally set to $\mu = 1$ in this thesis. The adjoint to the annihilation operator is the creation operator $\hat{a}^\dagger$, defined as

$$\hat{a}^\dagger = \sqrt{\frac{\mu}{2\hbar}} \left(\hat{q} - \frac{i}{\mu} \hat{p} \right). \quad (1.28)$$

The two operators $\hat{a}$ and $\hat{a}^\dagger$ are often called the ladder operators. They fulfil the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{I}. \quad (1.29)$$

A notable quantum system in the context of ladder operators is the quantum harmonic oscillator, given by

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger\hat{a} + \frac{1}{2} \right), \quad (1.30)$$

with real valued frequency ω . The eigenvalue spectrum is given by $\{\hbar\omega(n + \frac{1}{2})\}$ and the eigenstates are denoted by $|n\rangle$ for $n \in \mathbb{N}_0$. We often call $|n\rangle$ the harmonic oscillator basis and it has special relations to the ladder operators. The actions of the ladders operators on the harmonic oscillator basis states are given by

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \text{and} \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \quad (1.31)$$

The normalised eigenstates of the annihilation operator are known as coherent states $|z\rangle$,

$$\hat{a}|z\rangle = z|z\rangle, \quad (1.32)$$

with $z \in \mathbb{C}$. One can express coherent states as a displaced harmonic oscillator ground state ($n = 0$) such that

$$|z\rangle = \hat{D}(z)|0\rangle, \quad (1.33)$$

where

$$\hat{D}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}}, \quad (1.34)$$

is the displacement operator. We can identify $z = \frac{1}{\sqrt{2\hbar}}(q + ip)$ where q and p are the expectation values of $\hat{q}$ and $\hat{p}$, respectively. Thus, there is a one-to-one correspondence between points in phase space and coherent states. In the harmonic oscillator basis, coherent states are expanded as

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle. \quad (1.35)$$

Coherent states are minimum uncertainty states that satisfy $\Delta q = \Delta p = \sqrt{\frac{\hbar}{2}}$. The set of coherent states forms an over-complete basis of the Hilbert space, i.e., they form a resolution of the identity

$$\int \frac{dz dz^*}{\pi} |z\rangle \langle z| = \hat{I}, \quad (1.36)$$

but coherent states are not mutually orthogonal. The overlap of two coherent states is given by

$$|\langle z_1 | z_2 \rangle|^2 = e^{-|z_1 - z_2|^2}. \quad (1.37)$$

1.5 Operator Ordering and Classical Symbols

In this thesis, we study Hamiltonians that can be expressed as a power series of position and momentum operators $\hat{q}$ and $\hat{p}$, that is,

$$\hat{H} = \sum_{n,m} H_{nm} (\hat{p}^n \hat{q}^m + \hat{q}^m \hat{p}^n), \quad (1.38)$$

with $H_{nm} \in \mathbb{R}$. Equivalently, in terms of $\hat{a}$ and $\hat{a}^\dagger$ we can write

$$\hat{H} = \sum_{n,m} \tilde{H}_{nm} \hat{a}^{\dagger n} \hat{a}^m, \quad (1.39)$$

with $\tilde{H}_{nm} = \tilde{H}_{mn}^*$ and $(\hat{a}^{\dagger n} \hat{a}^m)^\dagger = \hat{a}^{\dagger m} \hat{a}^n$. We assume the coefficients H_{nm} in (1.38) to be independent of $\hbar$ and thus the coefficients in (1.39) scale with $\hbar$ as $\tilde{H}_{nm} \in O(\hbar^{\frac{n+m}{2}})$.

The order of non-commuting operators within the Hamiltonian matters. In any term of the form $\hat{a}^{\dagger n} \hat{a}^m$ one can repeatedly use the commutation relation (1.29) to rearrange the order, resulting in equal expressions with a different functional form. There are two important operator ordering conventions that we use throughout this thesis: normal and symmetric ordering [19]. An operator is normal ordered if all the creation operators are on the left, and the annihilation operators are on the right, as in equation (1.39). There are $\binom{n+m}{n}$ ways to order a product of m annihilation and n creation operators (and vice versa). The average over all these yields the symmetric ordering. For example we have

$$\begin{aligned} n=1, m=1 : & \quad \frac{1}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger), \\ n=2, m=2 : & \quad \frac{1}{6} (\hat{a}^{\dagger 2} \hat{a}^2 + \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a}^2 \hat{a}^\dagger + \hat{a} \hat{a}^{\dagger 2} \hat{a} + \hat{a} \hat{a}^\dagger \hat{a} \hat{a}^\dagger + \hat{a}^2 \hat{a}^{\dagger 2}). \end{aligned}$$

We denote the symmetric order of a function $\hat{A}(\hat{a}^\dagger, \hat{a})$ by $\{\hat{A}\}$.

Given an expansion of an operator in powers of $\hat{a}$ and $\hat{a}^\dagger$ (or $\hat{p}$ and $\hat{q}$) in any ordering, one can replace the operators with the corresponding classical phase space variables (i.e. z and z^* or p and q) to obtain a classical phase space function called a "symbol"

of the operator.

The choice of ordering introduces an ambiguity in the correspondence between quantum operators and classical phase space functions. The symbol

$$A^{(S)}(z, z^*) = \sum_{n,m} \tilde{A}_{nm} z^{*n} z^m, \quad (1.40)$$

belonging to the symmetric ordering

$$\hat{A} = \sum_{n,m} \tilde{A}_{nm} \{\hat{a}^{\dagger n} \hat{a}^m\}, \quad (1.41)$$

of an operator, is often called the Wigner-Weyl symbol [20, 21]. The symbol

$$A^{(N)}(z, z^*) = \sum_{n,m} A_{nm}^{(N)} z^n z^{*m}, \quad (1.42)$$

belonging to the normal-ordered operator

$$\hat{A} = \sum_{n,m} A_{nm} \hat{a}^{\dagger n} \hat{a}^m, \quad (1.43)$$

is known as the Wick symbol [21, 22]. The Wick symbol is identical to the expectation value of the operator in coherent states

$$A^{(N)}(z, z^*) = \langle z | \hat{A} | z \rangle, \quad (1.44)$$

viewed as a function of (z, z^*) . Generally, we denote a given Hermitian Hamiltonian operator as $\hat{H}$ with the symbol $H(z, z^*)$, and it will always be stated in the text which ordering is being used.

1.6 Quasi-probability Distributions on Phase Space

In classical mechanics, we can represent the state as a joint probability distribution in phase space. However, in quantum mechanics, this is not possible because even the concept of a phase space point violates the Heisenberg uncertainty principle (1.26). One can, however, define quasi-probability distributions [1] for quantum systems, that fulfil some, but not all, of the axioms of a joint probability distribution. There is a multitude of phase space distributions that allow for the representation of quantum states, and thus the comparison with classical systems. In this thesis, I study two important types of phase space distributions, the Husimi and Wigner distributions,

that I will briefly introduce in what follows.

1.6.1 Husimi Representation

The Husimi distribution $Q(z, z^*)$ [23] is a quasi-probability distribution, that can be defined as the expectation value of the density operator in coherent states, as

$$Q(z, z^*) = \langle z | \hat{\rho} | z \rangle, \quad (1.45)$$

i.e, the Wick symbol of the density operator. For a pure state $|\psi\rangle$ we have $Q(z, z^*) = |\langle z | \psi \rangle|^2$. This distribution fulfils two of the three joint probability distribution axioms

1. The Husimi distribution is normalised

$$\int \int Q(z, z^*) \frac{dz dz^*}{\pi} = 1. \quad (1.46)$$

2. The Husimi distribution is non negative, $0 \leq Q(z, z^*) \leq 1$.

However, in general we cannot integrate over either phase space variable to retrieve the marginal probability distributions of the wave function for either position or momentum, i.e. in general for a pure state $|\psi\rangle$ we have $\int_{-\infty}^{\infty} Q(z, z^*) dp \neq |\psi(q)|^2$ and $\int_{-\infty}^{\infty} Q(z, z^*) dq \neq |\tilde{\psi}(p)|^2$.

The Husimi distribution is well suited for Hamiltonians comprised of ladder operators such as (1.39) and states represented in the harmonic oscillator basis $\{|n\rangle\}$. The Husimi distribution for a harmonic oscillator eigenstate $|n\rangle$ is given by

$$Q(z, z^*) = \frac{|z|^{2n}}{n!} e^{-|z|^2}. \quad (1.47)$$

The Husimi distribution of a general pure state $|\psi\rangle = \sum_{n \geq 0} \psi_n |n\rangle$ is given as

$$Q(z, z^*) = e^{-|z|^2} |F^*(z)|^2, \quad (1.48)$$

where

$$|F^*(z)|^2 = \sum_{n, m \geq 0} \psi_n \psi_m^* \frac{z^n z^{*m}}{\sqrt{n!} \sqrt{m!}}. \quad (1.49)$$

The Husimi distribution of a displaced harmonic oscillator eigenstate $\hat{D}(z_c)|n\rangle$, can

be expressed as

$$Q(z, z^*) = \frac{|z - z_c|^{2n}}{n!} e^{-|z - z_c|^2}. \quad (1.50)$$

Three examples of Husimi distributions for harmonic oscillator states are shown in figure 1.1 for the ground state $|0\rangle$, the first excited state $|1\rangle$ and the second excited state $|2\rangle$. All states are centred around the origin. The ground state is a Gaussian, whilst the excited states are localised around rings, further out from the centre for higher excited states.

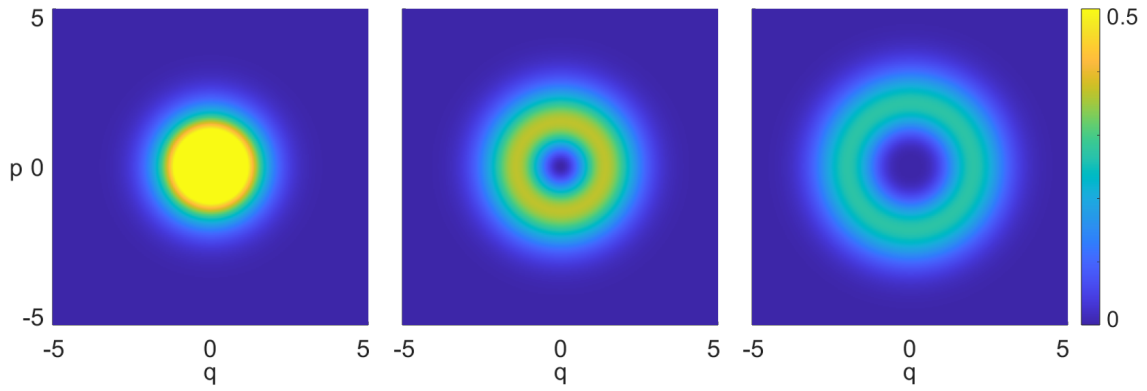


Figure 1.1: The Husimi distribution for the the ground state (left), the first excited state (middle) and the second excited state (right) of the harmonic oscillator.

The dynamical equation of the Husimi distribution follows from the definition

$$Q(z, z^*, t) = \langle z | \hat{\rho}(t) | z \rangle, \quad (1.51)$$

in which we take the expectation value in coherent states of the Von Neumann equation (1.14) to yield

$$\frac{dQ}{dt} = \frac{i}{\hbar} \langle z | \hat{\rho} \hat{H} | z \rangle - \frac{i}{\hbar} \langle z | \hat{H} \hat{\rho} | z \rangle. \quad (1.52)$$

Considering normal ordered Hamiltonians (1.42) and using the eigenvalue relations (1.32) this becomes [24]

$$\begin{aligned} \frac{dQ}{dt} &= \frac{i}{\hbar} \sum_{m,n=0} \langle z | \hat{\rho} H_{mn} (\hat{a}^\dagger)^n \hat{a}^m | z \rangle - \langle z | H_{mn} (\hat{a}^\dagger)^m \hat{a}^n \hat{\rho} | z \rangle, \\ &= \frac{i}{\hbar} \sum_{m,n=0} H_{mn} z^m \langle z | \hat{\rho} (\hat{a}^\dagger)^n | z \rangle - H_{mn} (z^*)^m \langle z | \hat{a}^n \hat{\rho} | z \rangle. \end{aligned} \quad (1.53)$$

Let us use the identities [25–27]

$$\hat{a}^\dagger|z\rangle\langle z| = |z\rangle\langle z|\left(\frac{\overleftarrow{\partial}}{\partial z} + z^*\right), \quad |z\rangle\langle z|\hat{a} = \left(\frac{\overrightarrow{\partial}}{\partial z^*} + z\right)|z\rangle\langle z|, \quad (1.54)$$

see appendix A for details. Here the superscript arrow denotes the direction of the differential. Differentials without an arrow always act to the right. The dynamical equation for the Husimi distribution thus takes the form

$$\frac{dQ}{dt} = \sum_{m,n} \frac{i}{\hbar} H_{mn} Q \left(z^* + \frac{\overleftarrow{\partial}}{\partial z} \right)^n z^m - \frac{i}{\hbar} H_{mn} z^{*m} \left(z + \frac{\partial}{\partial z^*} \right)^n Q, \quad (1.55)$$

with

$$\left(z + \frac{\partial}{\partial z^*} \right)^n Q = z^n Q + n z^{n-1} \frac{\partial Q}{\partial z^*} + \frac{n(n-1)}{2!} z^{n-2} \frac{\partial^2 Q}{\partial z^{*2}} + \dots, \quad (1.56)$$

$$Q \left(z^* + \frac{\overleftarrow{\partial}}{\partial z} \right)^n = (z^*)^n Q + n (z^*)^{n-1} \frac{\partial Q}{\partial z} + \frac{n(n-1)}{2!} (z^*)^{n-2} \frac{\partial^2 Q}{\partial z^2} + \dots \quad (1.57)$$

For a generic Hamiltonian, (1.55) is a partial differential equation of infinite order. For Hamiltonians that are polynomials of order k in $\hat{a}$ and $\hat{a}^\dagger$, i.e. of the form (1.39), (1.55) terminates at k -derivative.

1.6.2 Wigner Representation

The second quantum phase space distribution we consider is known as the Wigner distribution [28]. In terms of the phase space variables p and q , it can be defined as the Wigner-Weyl symbol of the density operator. Starting from the position representation this can be expressed as

$$W(p, q) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \langle q + x | \hat{\rho} | q - x \rangle e^{\frac{2ipx}{\hbar}} dx, \quad (1.58)$$

or equivalently, starting from the momentum representation as

$$W(p, q) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \langle p + \xi | \hat{\rho} | p - \xi \rangle e^{-\frac{2iq\xi}{\hbar}} d\xi. \quad (1.59)$$

Alternatively, the Wigner distribution may be defined by a combination of expectation values of the density operators over displaced harmonic oscillator basis states

[1, 29] as

$$W(z, z^*) = \sum_{n=0}^{\infty} (-1)^n \langle n | \hat{D}^\dagger(z) \hat{\rho} \hat{D}(z) | n \rangle, \quad (1.60)$$

where $\hat{D}(z)$ is the displacement operator defined in (1.34). The Husimi distribution on the other hand consists of the $n = 0$ terms alone.

The Wigner distribution fulfils the following two joint-probability axioms

1. The distribution is normalised

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(p, q) dq dp = 1. \quad (1.61)$$

2. We can obtain the marginal probability distributions by integrating over one of the phase space variables. I.e. for pure states $|\psi\rangle$ it holds

$$|\psi(q)|^2 = \int_{-\infty}^{\infty} W(p, q) dp, \quad |\tilde{\psi}(p)|^2 = \int_{-\infty}^{\infty} W(p, q) dq, \quad (1.62)$$

where $|\psi(q)|^2$ and $|\tilde{\psi}(p)|^2$ are the probability distributions in position and momentum representation.

Notably, the missing axiom is the lack of definite positivity; the Wigner function can be negative. This is particularly unusual in the comparison with classical joint-probability distributions, however it is commonly accredited that the negativity of the Wigner distribution is a strong indicator of the "quantumness" of the state and is a valuable characteristic. According to Hudson's theorem [30], the Wigner function of a pure state is non-negative if and only if the state is Gaussian. In relation the Husimi representation, the Husimi distribution was originally defined as a Gaussian smoothing of the Wigner function [23] as

$$Q(z, z^*) = \frac{2}{\pi} \int W(\chi, \chi^*) e^{-2|z-\chi|^2} d\chi d\chi^*, \quad (1.63)$$

where the smoothing removes the negativity.

As an example, the Wigner distribution for a displaced harmonic oscillator state $\hat{D}(z_c)|n\rangle$ is given by

$$W(p, q) = \frac{((-1)^n)}{\pi} e^{-(q-q_c)^2 + (p-p_c)^2} L_n(2((q-q_c)^2 + (p-p_c)^2)), \quad (1.64)$$

where $L_n(x)$ denotes the n th Laguerre polynomial. Figure 1.2 depicts the Wigner distributions for the ground state $|0\rangle$, the first excited state $|1\rangle$ and the second excited state $|2\rangle$ of the harmonic oscillator centered at the origin. The ground state is non-negative everywhere, however the excited states have regions of negative values of the Wigner distribution.

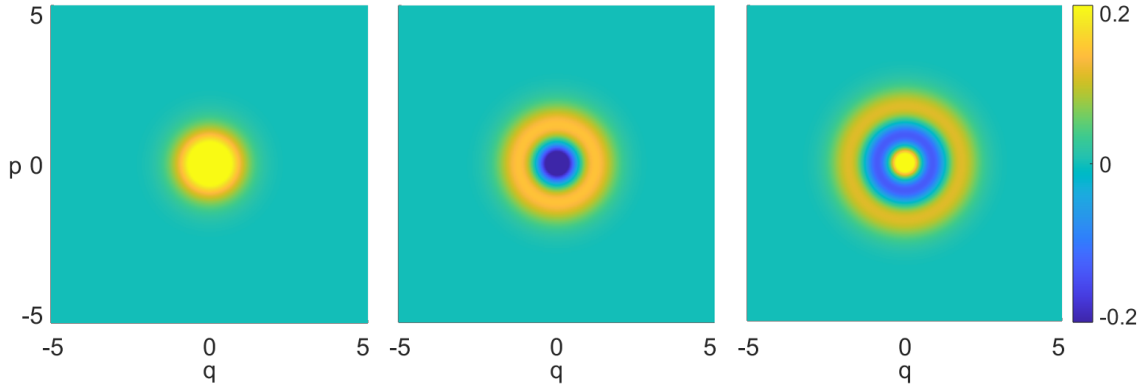


Figure 1.2: The Wigner distribution for the the ground state (left), the first excited state (middle) and the second excited state (right).

The dynamical equation for the Wigner function is given by the von Neumann equation in Wigner representation,

$$\frac{dW}{dt} = \frac{-i}{\hbar}(H * W - W * H), \quad (1.65)$$

where the Hermitian Hamiltonian $\hat{H}$ has a symmetrically ordered Weyl symbol in canonical co-ordinates $H(p, q)$, and $H * W$ denotes the Moyal product [31] defined as

$$H * W = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k H \left(\overleftarrow{\frac{\partial}{\partial p}} \overrightarrow{\frac{\partial}{\partial q}} - \overleftarrow{\frac{\partial}{\partial q}} \overrightarrow{\frac{\partial}{\partial p}} \right)^k W. \quad (1.66)$$

Note that while the dynamical equations for the Wigner and the Husimi equation agree in leading order of $\hbar$ (as we shall discuss in some detail in section 1.7.1) the higher order terms differ.

1.6.3 Example: Harmonic Oscillator

As a simple example for quantum phase space dynamics let us briefly consider a quantum harmonic oscillator, with the Hamiltonian

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \hat{I} \right), \quad (1.67)$$

where $\omega \in \mathbb{R}$. The normal ordered symbol is

$$H^{(N)}(z, z^*) = \hbar\omega \left(z^*z + \frac{1}{2} \right) = \omega \left(\frac{p^2 + q^2}{2} + \frac{1}{2} \right), \quad (1.68)$$

In rewriting the Hamiltonian (1.67) as

$$\hat{H} = \omega \left(\frac{\hat{p}^2 + \hat{q}^2}{2} \right), \quad (1.69)$$

we see that the symmetrical ordered symbol is

$$H^{(S)}(z, z^*) = \hbar\omega z^*z = \omega \left(\frac{p^2 + q^2}{2} \right). \quad (1.70)$$

The quantum and classical dynamics agree for this system for both Husimi and Wigner distributions, as the corresponding dynamical equations (1.52) and (1.65) are exact in the first differential order.

For the Husimi distribution, the dynamical equation (1.55) is given by

$$\frac{\partial Q}{\partial t} - i\omega z \frac{\partial Q}{\partial z} + i\omega z^* \frac{\partial Q}{\partial z^*} = 0. \quad (1.71)$$

The corresponding Hamiltonian flow is given by

$$\Phi_t(z, z^*) = (ze^{-i\omega t}, z^*e^{i\omega t}). \quad (1.72)$$

Thus, an initial Husimi distribution $Q_0(z, z^*)$ evolves in time as

$$Q(t, z, z^*) = Q_0(\Phi_t^{-1}(z, z^*)), \quad (1.73)$$

where $\Phi_t^{-1}(z, z^*) = (ze^{i\omega t}, z^*e^{-i\omega t})$.

Equivalently, (1.71) can be rewritten in terms of the real phase space coordinates p and q as

$$\frac{\partial Q}{\partial t} - \omega q \frac{\partial Q}{\partial p} + \omega p \frac{\partial Q}{\partial q} = 0, \quad (1.74)$$

and we have

$$\Phi_t(q, p) = (q \cos(\omega t) + p \sin(\omega t), p \cos(\omega t) - q \sin(\omega t)), \quad (1.75)$$

and

$$Q(t, p, q) = Q_0(\Phi_t^{-1}(p, q)), \quad (1.76)$$

where $\Phi_t^{-1}(q, p) = (q \cos(\omega t) - p \sin(\omega t), p \cos(\omega t) + q \sin(\omega t))$.

Despite the different appearance of the dynamical equation (1.65) for the Wigner function, this too, reduces to the same classical Liouville equation (1.71)/(1.74) for the harmonic oscillator Hamiltonian. Thus, the time-evolved Wigner function is also given by

$$W(t, p, q) = W_0(\Phi_t^{-1}(p, q)), \quad (1.77)$$

with the same classical flow.

1.7 Dynamics in the Semiclassical Limit

Let us now study the dynamics of the quantum systems in the semiclassical limit, $\hbar \rightarrow 0$, in both the Husimi and Wigner representations. I begin with the Husimi distribution, where we discuss the scaling of the different terms with $\hbar$ in some detail. The principles are the same for the Wigner distribution as well as in the evaluation of non-Hermitian systems that form the main subject of this thesis.

1.7.1 Husimi Representation

To begin, let us summarise the $\hbar$ dependence and scaling of the phase space variables and the Hamiltonian. In section 1.4 we saw that $z \in O(\hbar^{-\frac{1}{2}})$, and $z^n \in O(\hbar^{-\frac{n}{2}})$, and thus, in the semiclassical limit $\hbar \rightarrow 0$, we have $|z|^2 \rightarrow \infty$. The differential terms with respect to (z, z^*) can be shown to be of order $\frac{\partial}{\partial z} \in O(\hbar^{\frac{1}{2}})$ and $\frac{\partial^k}{\partial z^k} \in O(\hbar^{\frac{k}{2}})$. The symbol of the Hamiltonian has order $H(z, z^*) \in O(\hbar^0)$ and such does not effect the order in $\hbar$ of the terms in (1.39).

Let us consider the dynamics of the Husimi distribution in the semiclassical limit. We start with the expansion of (1.55) as

$$\begin{aligned} i\hbar \frac{\partial Q}{\partial t} &= \sum_{m,n} H_{mn} (z^*)^m \left(z^n Q + n z^{n-1} \frac{\partial Q}{\partial z^*} + \dots \right) - H_{mn} z^m \left((z^*)^n Q + n (z^*)^{n-1} \frac{\partial Q}{\partial z} + \dots \right), \\ &= (H Q - Q H) + \left(\frac{\partial H}{\partial z} \frac{\partial Q}{\partial z^*} - \frac{\partial H}{\partial z^*} \frac{\partial Q}{\partial z} \right) + \left(\frac{\partial^2 H}{\partial z^2} \frac{\partial^2 Q}{\partial z^{*2}} - \frac{\partial^2 H}{\partial z^{*2}} \frac{\partial^2 Q}{\partial z^2} \right) + \dots \end{aligned} \quad (1.78)$$

The coefficients of the higher-order differential terms of the Husimi distribution have decreasing orders in $\hbar^{-\frac{1}{2}}$. The terms containing the k-th derivate of the Husimi function themselves are of orders $O(\hbar^{\frac{k}{2}})$, or lower, which can be seen as follows.

For the Husimi distribution itself we have $Q(z, z^*) \in O(\hbar^0)$, and thus we expect

k-differential of the Husimi distribution decrease with increasing k. However, if we consider a localised Husimi distribution the order of the k-th derivative is $\frac{\partial^k Q}{\partial z^k} \in O((z - z_c)^k Q)$, which might at first glance appear to be of magnitude $O(\hbar^{\frac{-k}{2}})$. However the localised Husimi distribution itself is negligible outside the region $|z - z_c| \in O(\hbar^0)$. Thus, all that remains to be considered is the order of coefficients of the terms $\frac{\partial^k Q}{\partial z^k}$ which, as mentioned, is $O(\hbar^{\frac{k}{2}})$.

Therefore, the dynamical equation for the Husimi distribution in the semiclassical limit reduces to

$$i\hbar \frac{\partial Q}{\partial t} = \frac{\partial H}{\partial z} \frac{\partial Q}{\partial z^*} - \frac{\partial H}{\partial z^*} \frac{\partial Q}{\partial z}, \quad (1.79)$$

which we recognise as the classical Liouville equation

$$\frac{\partial Q}{\partial t} = -\{H, Q\}, \quad (1.80)$$

where

$$\{H, Q\} = \frac{i}{\hbar} \frac{\partial H}{\partial z} \frac{\partial Q}{\partial z^*} - \frac{i}{\hbar} \frac{\partial H}{\partial z^*} \frac{\partial Q}{\partial z} = \frac{\partial H}{\partial p} \frac{\partial Q}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial Q}{\partial p}. \quad (1.81)$$

The solution has the form

$$Q(t, p, q) = Q_0(\Phi_t^{-1}(p, q)), \quad (1.82)$$

where $Q_0(p, q)$ is the initial Wigner distribution and $\Phi_t(p_0, q_0) = (p, q)$ is the Hamiltonian flow.

1.7.2 Wigner Representation

For the semiclassical dynamics of the Wigner distribution derived from (1.65), we begin by expanding the Moyal product (1.66)

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k H \left(\overleftarrow{\frac{\partial}{\partial p}} \overrightarrow{\frac{\partial}{\partial q}} - \overleftarrow{\frac{\partial}{\partial q}} \overrightarrow{\frac{\partial}{\partial p}} \right)^k W &= HW + \frac{i\hbar}{2} \frac{\partial H}{\partial q} \frac{\partial W}{\partial p} - \frac{i\hbar}{2} \frac{\partial H}{\partial p} \frac{\partial W}{\partial q} \\ &\quad - \frac{\hbar^2}{8} \frac{\partial^2 H}{\partial p^2} \frac{\partial^2 W}{\partial q^2} + \frac{\hbar^2}{4} \frac{\partial^2 H}{\partial q \partial p} \frac{\partial^2 W}{\partial p \partial q} - \frac{\hbar^2}{8} \frac{\partial^2 H}{\partial q^2} \frac{\partial^2 W}{\partial p^2} + \dots, \end{aligned} \quad (1.83)$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k W \left(\overleftarrow{\frac{\partial}{\partial p}} \overrightarrow{\frac{\partial}{\partial q}} - \overleftarrow{\frac{\partial}{\partial q}} \overrightarrow{\frac{\partial}{\partial p}} \right)^k H &= WH + \frac{i\hbar}{2} \frac{\partial W}{\partial q} \frac{\partial H}{\partial p} - \frac{i\hbar}{2} \frac{\partial W}{\partial p} \frac{\partial H}{\partial q} \\ &\quad - \frac{\hbar^2}{8} \frac{\partial^2 W}{\partial p^2} \frac{\partial^2 H}{\partial q^2} + \frac{\hbar^2}{4} \frac{\partial^2 W}{\partial q \partial p} \frac{\partial^2 H}{\partial p \partial q} + \frac{\hbar^2}{8} \frac{\partial^2 W}{\partial q^2} \frac{\partial^2 H}{\partial p^2} + \dots \end{aligned} \quad (1.84)$$

Then we see that all the terms in (1.65) with even values of k cancel each other out, and we find

$$\frac{\partial W}{\partial t} = \frac{\partial H}{\partial q} \frac{\partial W}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial W}{\partial q} - \frac{\hbar^2}{24} H \left(\frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} \right)^3 W + O(\hbar^4). \quad (1.85)$$

In the semiclassical limit, $\hbar \rightarrow 0$, the leading order terms again yield the Liouville equation (1.6)

$$\frac{\partial W}{\partial t} = \frac{\partial H}{\partial q} \frac{\partial W}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial W}{\partial q}. \quad (1.86)$$

Thus we have

$$W(t, p, q) = W_0(\Phi_t^{-1}(p, q)), \quad (1.87)$$

where $W_0(p, q)$ is the initial Wigner distribution and $\Phi_t(p_0, q_0) = (p, q)$ is the Hamiltonian flow.

1.8 Open Quantum Systems

So far I have discussed conservative systems, where both energy and overall probability are constant throughout the time-evolution. In practice no system is ever fully isolated from its environment, leading to decay, damping and dissipation (or the opposite effects of growth and amplification). In quantum dynamical systems, we can model this effect with either the Schrödinger/von Neumann dynamics for non-Hermitian Hamiltonians [32–35] or with Lindblad dynamics [36–38]. While this thesis is mainly concerned with non-Hermitian quantum systems and their classical counterparts, these two models of open quantum dynamics are closely related, and I shall conclude this background chapter with a brief introduction to both.

1.8.1 Non-Hermitian Hamiltonians

In recent years, quantum systems with non-Hermitian Hamiltonians have attracted significant attention and have led to a diverse and active field of research. Non-Hermitian Hamiltonians therefore appear in waveguide [39–43], dynamical systems [10, 33], scattering problems [44] and even biological system [15, 16]. In particular, the field of Parity-Time, "PT", symmetric quantum theories [7, 45, 46], i.e. the study of non-Hermitian Hamiltonians that can correspond to real valued eigenvalues, is very active. There have been a great number of papers centred around the exploration of topological effects and exceptional points [4, 8, 47–49]. In this section

we will briefly summarise some of the properties of non-Hermitian systems, as well as introduce the subset of non-Hermitian PT-symmetric Hamiltonians.

Let us denote a general non-Hermitian Hamiltonian by $\hat{K}$, decomposed as

$$\hat{K} = \hat{H} - i\hat{\Gamma}, \quad (1.88)$$

with two Hermitian operators $\hat{H} = \hat{H}^\dagger$ and $\hat{\Gamma} = \hat{\Gamma}^\dagger$. Here I use the convention of a minus sign in front of the imaginary part, such that a positive $\hat{\Gamma}$ corresponds to a decay/damping.

Non-Hermitian quantum theories typically consider pure state dynamics, generated by the Schrödinger equation with a non-Hermitian Hamiltonian $\hat{K}$. One can also consider a non-Hermitian extension of the von Neumann equation given by [50]

$$i\hbar \frac{d\hat{\rho}}{dt} = \hat{K}\hat{\rho} - \hat{\rho}\hat{K}^\dagger, \quad (1.89)$$

which reduces to the Schrödinger equation in the case of pure states.

There are two crucial differences between non-Hermitian and Hermitian Hamiltonians. The first is that the eigenstates of the system $\{|\phi_n\rangle\}$ do not have to be mutually orthogonal. The second is that the energy spectrum $\{E_n\}$ may be complex-valued such that $E_n = \epsilon_n - i\gamma_n$ for real-valued ϵ_n and γ_n . Even in cases where all eigenvalues are real, not all expectation values can be real, and in particular the classical symbol of the Hamiltonian (for any ordering) is complex valued,

$$K(z, z^*) = H(z, z^*) - i\Gamma(z, z^*), \quad (1.90)$$

with $H, \Gamma \in \mathbb{R}$. While non-Hermitian Hamiltonians can have purely real eigenvalues or orthogonal eigenvectors, if both the eigenvalues are real and the eigenvectors are orthogonal, the Hamiltonian must be Hermitian [33].

The non-Hermiticity has profound effects on the dynamics of the system. Consider an initial eigenstate $|\psi(0)\rangle = |\phi_n\rangle$, then the time-evolved state is given by

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\frac{\hat{K}t}{\hbar}} |\phi_n\rangle, \\ &= e^{-i(\epsilon_n - i\gamma_n)\frac{t}{\hbar}} |\phi_n\rangle. \end{aligned} \quad (1.91)$$

For $\gamma_n \neq 0$ the norm $w = \langle \psi | \psi \rangle$ evolves as [51, 52]

$$\begin{aligned} w_n(t) &= e^{-2\gamma_n \frac{t}{\hbar}} \langle \phi_n | \phi_n \rangle, \\ &= e^{-2\gamma_n \frac{t}{\hbar}} w_n(0), \end{aligned} \quad (1.92)$$

thus a complex eigenenergy where $\gamma_n > 0$ describes an exponential decay of the overall probability in the corresponding eigenstate. This exponential decay has been the original motivation of Gamow to consider eigenstates with complex energies in the context of nuclear decay [6].

Away from what is known as exceptional points (with which we will not concern ourselves in the present thesis) the eigenstates $|\phi_n\rangle$ of a Hamiltonian still form a basis of the Hilbert space and an arbitrary initial state can be expanded as $|\psi(0)\rangle = \sum_n c_n |\phi_n\rangle$. The time-evolution is then given by

$$|\psi(t)\rangle = \sum_n c_n e^{-i\epsilon_n \frac{t}{\hbar}} e^{-\gamma_n \frac{t}{\hbar}} |\phi_n\rangle. \quad (1.93)$$

Thus, the contributions of eigenstates with complex energies decay over time, which leads to a dampening in the dynamics of expectation values for observables. Even if all the eigenvalues are real, however, the overall norm of a generic initial state $|\psi(0)\rangle$ is not constant in time. We have, for $\gamma_n = 0$ for all n ,

$$\langle \psi(t) | \psi(t) \rangle = \sum_{n,m} c_n^* c_m e^{-i(w_m - w_n) \frac{t}{\hbar}} \langle \phi_n | \phi_m \rangle, \quad (1.94)$$

which oscillates due to the non-orthogonality of the eigenstates [39]. As a simple example of a non-Hermitian Hamiltonian, let us consider a harmonic oscillator with a complex frequency $\omega - i\gamma$,

$$\hat{K} = \hbar(\omega - i\gamma) \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \hat{I} \right), \quad (1.95)$$

with $\omega \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+$. This non-Hermitian system has energy eigenvalues $\{E_n\} = \hbar(\omega - i\gamma)(n + \frac{1}{2})$ and the eigenstates are just the (orthogonal) harmonic oscillator eigenstates $\{|n\rangle\}$. An initial state $|\psi(0)\rangle = \sum_n c_n |n\rangle$ evolves in time as

$$|\psi(t)\rangle = e^{-i\frac{\omega}{2}t} e^{-\frac{\gamma}{2}t} \sum_n c_n e^{-i\omega nt} e^{-\gamma nt} |n\rangle. \quad (1.96)$$

This is not periodic but as long as $c_0 \neq 0$, asymptotically $|\psi(t)\rangle$ approaches the ground state $|0\rangle$, which has the slowest decay rate. Generally, the most stable

state of non-Hermitian systems may not be the ground state, however for this non-Hermitian harmonic oscillator the ground state is the most stable state. Thus, the expectation values of position and momentum of the time-evolved state will also spiral into the origin. For an initial coherent state $|z_c\rangle$, centred at z_c , the time evolved state is

$$|\psi(t)\rangle = e^{-i\frac{\omega}{2}t} e^{-\frac{\gamma}{2}t} e^{-\frac{|z_c|^2}{2}} \sum_n \frac{(z_c e^{i\omega t} e^{-\gamma t})^n}{\sqrt{n!}} |n\rangle. \quad (1.97)$$

Defining

$$z_c(t) = z_c e^{i\omega t} e^{-\gamma t}, \quad (1.98)$$

this can be written as

$$|\psi(t)\rangle = e^{-i\frac{\omega}{2}t} e^{-\frac{\gamma}{2}t} e^{-\frac{|z_c|^2}{2}(1-e^{-2\gamma t})} |z_c(t)\rangle. \quad (1.99)$$

Thus, the state will stay a coherent state, whilst it spirals into the origin, displaying the qualitative behaviour of a damped oscillator.

For more general Hamiltonians, an initial coherent state will not stay coherent over time. Approximating the time dependent state with a coherent state, however, yields equations of motion for the expectation values of position and momentum, which are given by Hamilton's equations (1.2) in the Hermitian case. For non-Hermitian Hamiltonians that are analytic functions of $\hat{a}$ and $\hat{a}^\dagger$ (or $\hat{q}$ and $\hat{p}$), it has been shown in [52] that a coherent state approximation [53] yields the dynamical equations

$$i\hbar \frac{dz}{dt} = \frac{\partial K}{\partial z^*} \quad (1.100)$$

i.e.

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} - \frac{\partial \Gamma}{\partial p}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p} - \frac{\partial \Gamma}{\partial q}, \quad (1.101)$$

for the expectation values of position and momentum. Here the anti-Hermitian part yields a gradient flow that can be interpreted as a damping term for positive Γ . In addition, the global (total) norm $w_G(t) = \langle \psi | \psi \rangle$ of the state changes in time [51] according to

$$\frac{dw_G}{dt} = -\frac{2}{\hbar} \langle \hat{\Gamma} \rangle w_G. \quad (1.102)$$

One may consider the global norm as the total probability of the system, which for non-Hermitian systems has gain/loss profile generated by the expectation value of

the imaginary part

$$\langle \hat{\Gamma} \rangle = \frac{\langle \psi | \hat{\Gamma} | \psi \rangle}{\langle \psi | \psi \rangle}. \quad (1.103)$$

The decaying global norm may be interpreted as a survival probability, describing whether a population of particles initially within the system remains within the system or escapes into the environment [54, 55]. Another interpretation is found within the field of optical systems, where the fluctuating global norm is related to the amplification of light [56, 57]. Historically, we can trace this notion back to Gamow in 1928, who described nuclear decay in terms of complex-valued systems [6].

In this thesis, I will occasionally mention the evolution of the global norm for a given system in which there are some interesting behaviours. However, it will not be my principal focus as I will exclusively depict the renormalised Husimi and Wigner distributions in figures, which are not affected by the global norm.

Non-Hermitian Hamiltonians with purely real eigenvalues are often referred to as quasi-Hermitian. This is related to the weaker requirement of pseudo-Hermiticity [34], which can, but doesn't have to, lead to purely real eigenvalues. A Hamiltonian is pseudo-Hermitian if there exists a linear operator $\hat{\rho}$ such that $\hat{H}^\dagger = \hat{\rho} \hat{H} \hat{\rho}^{-1}$. A special type of these are PT-symmetric Hamiltonians [7, 45], that are invariant under simultaneous parity and time-reversal. These Hamiltonians satisfy the commutation relation $[\hat{P}\hat{T}, \hat{K}] = 0$ where $\hat{P}$ is a parity operator with $\hat{P}^2 = \hat{I}$ with $[\hat{P}, \hat{T}] = 0$ and $\hat{T}$ is a time-reversal operator that maps i to $-i$. PT-symmetric Hamiltonians with purely real spectra are said to be in the unbroken phase of PT-symmetry. Otherwise, the Hamiltonian is said to have broken PT-symmetry. The point of transition between unbroken and broken PT-symmetry is an exceptional point.

A simple example of a PT-symmetric and quasi-Hermitian Hamiltonian is a harmonic oscillator with a linear imaginary part [58]

$$\hat{K}_{PT} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \hat{I} \right) - i\hbar^{\frac{1}{2}} \delta (\hat{a} + \hat{a}^\dagger) \quad (1.104)$$

where coefficient $\delta \in \mathbb{R}$. This Hamiltonian is PT-symmetric with respect to the PT operator

$$\hat{P} : \quad \hat{q} \rightarrow -\hat{q}, \quad \hat{p} \rightarrow -\hat{p}, \quad \hat{T} : \quad i \rightarrow -i, \quad \hat{p} \rightarrow -\hat{p}. \quad (1.105)$$

The system has unbroken PT-symmetry and the eigenenergies are real-valued,

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) + \frac{\hbar\delta^2}{\omega^2}. \quad (1.106)$$

This can be seen explicitly by observing that $\hat{K}_{PT}$ is related to a simple harmonic oscillator, $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$, by a similarity transformation and an energy shift as

$$\hat{K}_{PT} = e^{\frac{\delta}{\hbar\omega}\hat{p}} \hat{H} e^{\frac{-\delta}{\hbar\omega}\hat{p}} + \frac{\hbar\delta^2}{\omega^2}. \quad (1.107)$$

The eigenvectors are thus given by $|\phi_n\rangle = e^{\frac{\delta}{\hbar\omega}\hat{p}}|n\rangle$, and are not orthogonal for $\delta \neq 0$. The dynamics of expectation values and the norm are periodic.

In many ways PT-symmetric quantum systems can be viewed as living at the boundary between closed and open quantum systems, and their special properties have made them very popular, leading to a vast amount of literature and applications [4, 7, 39–41, 59, 60]. In this thesis I will discuss examples of PT-symmetric, and more general non-Hermitian Hamiltonians.

1.8.2 Lindblad Master Equation

A seemingly quite different approach to open quantum systems starts from considering a system coupled to an external environment or "bath". Under a Born-Markov approximation the dynamics of systems coupled to a bath can be described by the Lindblad (Gorini-Kossakowshi-Sudarshan-Lindblad) Master equation for the density operator [36–38]. The Lindblad equation finds applications in many fields including quantum optics [61, 62], thermodynamics [63, 64] and quantum decoherence [65–67]. It describes the evolution of the density operator $\hat{\rho}$ in a system described by a Hamiltonian $\hat{H}$, coupled to an environment, effectively modelled by the Lindblad operators $\hat{L}_k$,

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}[\hat{H}, \hat{\rho}] + \sum_k \hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2} \hat{L}_k^\dagger \hat{L}_k \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}_k^\dagger \hat{L}_k. \quad (1.108)$$

The operators, $\hat{L}_k$ are also known as jump operators [68]. The dynamics generated by this equation are non-unitary but the time-evolved density operator maintains the properties of unit trace, non-negativity and Hermiticity. In general it does not preserve the purity of a state. The Lindblad dynamics is closely related to non-Hermitian dynamics in various ways.

First, the Lindblad dynamics can be viewed as an average over non-Hermitian von Neumann dynamics with the effective non-Hermitian Hamiltonian $\hat{K} = \hat{H} - \frac{i}{2} \sum_k \hat{L}_k^\dagger \hat{L}_k$, interrupted by "quantum jumps" represented by the terms $\hat{L}_k \hat{\rho} \hat{L}_k^\dagger$ in the Lindblad equation. In particular in the case of a "non-click" dynamics, when only realisations are considered in which no jump has occurred, Lindblad dynamics can be used to implement non-Hermitian Hamiltonians $\hat{K}$ [60, 69, 70]. Note that in this formulation the $\hat{\Gamma}$ part of the Hamiltonian is necessarily positive.

The second connection between Lindblad and non-Hermitian Schrödinger dynamics is on a more structural level. This can be seen when we write the Lindblad equation (1.108) as the differential equation

$$\hbar \frac{\partial \hat{\rho}}{\partial t} = \hat{\mathcal{K}} \hat{\rho}, \quad (1.109)$$

where $\hat{\mathcal{K}}$ is a linear super-operator [71]. One can then interpret (1.109) as a Schrödinger equation where $\hat{\rho}$ is viewed as a pure state vector (in a Hilbert space that is a tensor product of the original Hilbert space with itself) and $-i\hat{\mathcal{K}}$ as a general non-Hermitian Hamiltonian. In this thesis we will mostly view Lindblad dynamics as a motivation for non-Hermitian systems and will not consider the Lindblad equation as such.

In summary, this introductory chapter primarily serves as a literature review, focusing on two key aspects: firstly, the phase space dynamics of Hermitian classical and quantum systems, and secondly, the nature of non-Hermitian Hamiltonians.

I began this chapter by covering the foundations of classical and quantum mechanics, with a focus on the phase space dynamics of systems. Special attention was given to the quasi-phase space Husimi and Wigner distributions, which are used to depict a quantum state on phase space. The dynamics of a system in the Husimi and Wigner representations can be described by either an exact dynamical equation derived from the quantum evolution of the state or by approximating this equation in the semiclassical limit, which returns classical dynamical equations. A complete analysis of a system on phase space may include comparing the quantum and classical dynamics of the time-evolved state represented by both the Husimi and Wigner distributions.

This chapter also introduced non-Hermitian Hamiltonians and highlighted some of the crucial differences between them and their Hermitian counterparts. Specifically, the energy and the global norm of the state are generally not conserved quantities, except in special cases where the Hamiltonian is PT-symmetric and in an unbroken

PT phase. I also provided physical interpretations of the decaying global norm, and an example of a non-Hermitian harmonic oscillator experiencing dampening effects. This chapter concluded with a brief section on the Lindblad description of open quantum systems and their relation to non-Hermitian Hamiltonians.

While there has been intense research interest in non-Hermitian quantum systems over the past decades, their behaviour in semiclassical limit remains poorly understood. This incomplete understanding stands in the way of the development of semiclassical methods to approximate and describe non-Hermitian systems. This motivates the research in this thesis, which predominately focuses on the semiclassical description of non-Hermitian systems on phase space. In the following chapters I will present an original study.

1.9 Thesis Outline

This thesis presents original work based on two publications, [17] published in Physical Review Letters and [18] has been accepted and pending publication in Physical Review E, with further work in preparation for publication.

We begin with chapter 2, the backbone of this thesis. In this chapter I write a semiclassical dynamical equation of non-Hermitian Hamiltonians for the Husimi distribution (published in [17]) and the Wigner distribution. I provide an explicit closed form solution for the time-evolved distributions, comprised of a propagated initial distribution along a phase space flow and an additional phase space function that I call a norm landscape. I will show that the norm landscape is the dominating factor in the damping effects that the distributions experience on phase space. The method presented here works for non-Hermitian Hamiltonians that are generic functions of $\hat{a}$ and $\hat{a}^\dagger$, (or equivalently $\hat{p}$ and $\hat{q}$) and for arbitrary initial states. Chapter 2 presents two linear systems with analytic solutions to illustrate the effects of the phase space flow and the norm landscape for both Husimi and Wigner distributions. The chapter ends with an example of a free particle with a piecewise constant imaginary potential, where the semiclassical treatment is not exact.

In chapter 3 I present further examples of the Husimi (published in [17]) and Wigner distribution dynamics of two non-linear systems, a PT-symmetric Hamiltonian and a dissipative non-Hermitian Mexican Hat (Higgs) potential. Each example is accompanied by a discussion of the Hermitian counterparts of the non-Hermitian Hamiltonians. I compare the quantum and classical dynamics and demonstrate the influence of the norm landscape in the quantum dynamics. We observe in both examples that

the quantum and classical non-Hermitian systems have a closer correspondence than the Hermitian counterparts.

In chapter 4 I introduce a higher dimensional dynamical system, a billiard model with additional loss (published in [18]). The chapter is predominately focused on the classical description. In this section I provide an analysis and depiction of loss experienced by the billiard dynamics on the Poincaré phase space, represented by intensity landscapes. This chapter contains examples of dynamics for three billiard geometries: a circular, elliptic and oval billiard.

I will conclude with chapter 5 which presents a topic for further research. Following on from chapter 4, I consider the quantum counterpart described by a non-Hermitian Hamiltonian and present preliminary results on a semiclassical analysis, implementing the method from Chapter 2. Lastly, I discuss the application of the semiclassical method to the higher-dimensional Glauber R-representation.

I have included three appendices, providing further information on the method of characteristics, D-algebra identities and numerical implementation of semiclassical method, respectively.

Chapter 2

Phase Space Dynamics of Non-Hermitian Systems in the Semiclassical Limit

As we have seen in Chapter 1, the dynamics of the Husimi and Wigner distributions for a Hermitian system in the semiclassical limit are equivalent, both reducing to the Liouville equation (1.6). Now, I will apply the same methodology to non-Hermitian systems and derive a closed-form expression for the time-evolved Husimi and Wigner distributions. The resulting closed-form expression will show that both the Husimi and Wigner distributions contain a function, which I denote as a *norm landscape*, that varies over time and phase space and is the principal source of dampening in a non-Hermitian system. Furthermore, while Hamilton's equations of motion still generates the flow Wigner distribution for non-Hermitian systems, we will see that a different set of equations generates the flow for the Husimi distribution. The difference in flow between the two distributions also causes a difference in their respective norm landscapes. Thus, we must use both distributions in our analysis to obtain a full picture of the dampening experienced by non-Hermitian systems and the effect on the system dynamics. This section will include examinations of non-Hermitian systems with linear imaginary terms, namely a non-Hermitian harmonic oscillator, a non-Hermitian free particle and a free particle with an absorbing region bounded on position space.

In this Chapter I consider non-Hermitian Hamiltonians of the form

$$\hat{K} = \hat{H} - i\hat{\Gamma} = \sum_{n,m} K_{nm} \hat{a}^{\dagger n} \hat{a}^m, \quad (2.1)$$

where $K_{nm} = H_{nm} - i\Gamma_{nm}$, with $H_{nm}, \Gamma_{nm} \in \mathbb{C}$ and $H_{mn} = H_{nm}^*$ and $\Gamma_{mn} = \Gamma_{nm}^*$. I consider the semiclassical limit of the dynamics for both the Husimi and Wigner distributions and reveal an additional modulation of the time-evolved distributions, not present in the Hermitian counterpart. The results for the Husimi distribution have been published in [17]. The results for the Wigner distribution were obtained afterwards, and are currently being prepared for publication.

2.1 Semiclassical Dynamics for the Husimi Representation

Let us write the dynamical equation for the Husimi distribution by starting from the von Neumann equation (1.8.1) for non-Hermitian Hamiltonians and take the expectation value in coherent states such that

$$\frac{\partial Q}{\partial t} = \frac{i}{\hbar} \langle z | \hat{\rho} \hat{K}^\dagger | z \rangle - \frac{i}{\hbar} \langle z | \hat{K} \hat{\rho} | z \rangle. \quad (2.2)$$

Inserting the family of Hamiltonians (2.1) and a similar treatment to (1.55), using the identities (1.32) and (1.54), we obtain

$$\begin{aligned} \frac{\partial Q}{\partial t} &= \sum_{m,n} \frac{i}{\hbar} K_{mn} Q \left(z^* + \frac{\overleftarrow{\partial}}{\partial z} \right)^n z^m - \frac{i}{\hbar} K_{mn}^* z^{*m} \left(z + \frac{\partial}{\partial z^*} \right)^n Q, \\ &= \frac{i}{\hbar} \left(K^* Q - K Q + \frac{\partial K^*}{\partial z^*} \frac{\partial Q}{\partial z} - \frac{\partial K}{\partial z} \frac{\partial Q}{\partial z^*} + \frac{\partial^2 K^*}{\partial z^{*2}} \frac{\partial^2 Q}{\partial z^2} - \frac{\partial^2 K}{\partial z^2} \frac{\partial^2 Q}{\partial z^{*2}} + \dots \right), \end{aligned} \quad (2.3)$$

where $K = K^{(N)}(z, z^*)$ is the (normal-ordered) Wick symbol of $\hat{K}$. As with the Hermitian case (1.55), solving the full differential equation for the system is equivalent to solving the von Neumann equation (1.8.1). In the semiclassical limit of $\hbar \rightarrow 0$, (2.3) truncates to the first-order partial differential equation

$$\frac{\partial Q_{sc}}{\partial t} = -\frac{2}{\hbar} \Gamma Q_{sc} + \frac{i}{\hbar} \frac{\partial K^*}{\partial z^*} \frac{\partial Q_{sc}}{\partial z} - \frac{i}{\hbar} \frac{\partial K}{\partial z} \frac{\partial Q_{sc}}{\partial z^*}, \quad (2.4)$$

where Q_{sc} denotes the Husimi distribution in the semiclassical limit. Here I have used the fact that $\frac{\partial^n Q}{\partial z^n}$ and $\frac{\partial^n Q}{\partial z^{*n}}$ are of lower order than $O(\hbar^{-\frac{n}{2}})$ and disappear in the limiting case, as discussed in section 1.7.1. Note that while equation (2.4) already implicitly appeared in [72] it was not further discussed there. The analysis and interpretation of equation (2.4), as well as for the Wigner distribution which we shall derive shortly, is a main contribution of this thesis.

In the majority of this thesis I work in the semiclassical limit with rescaled units such that $\hbar = 1$, except when explicitly stated. Equation (2.4) can be solved by the method of characteristics yielding

$$Q_{sc}(t, z, z^*) = Q_0(\Phi_t^{-1}(z, z^*))w(t, z, z^*), \quad (2.5)$$

where $Q_0(z, z^*)$ is the initial Husimi distribution, $\Phi_t^{-1}(z, z^*)$ is the backwards flow, and the function $w(t, z, z^*)$ is generated by

$$\dot{w}(t, z, z^*) = -2\Gamma(\Phi_t^{-1}(z, z^*))w(t, z, z^*), \quad (2.6)$$

with initial condition $w(0, z, z^*) = 1$. From here we will drop the subscription notation of Q_{sc} as the Husimi distribution Q will always be a semiclassical approximation unless stated otherwise.

The phase space flow Φ_t , along which the Husimi distribution is transported and is equivalent to the phase space itself propagated along the backwards flow $\Phi_t^{-1}(z, z^*)$, does not follow Hamilton's equations of motion, but instead is influenced by the non-Hermitian part of the Hamiltonian. Specifically, Φ_t is generated by the equations of motion

$$\frac{dz}{dt} = -i\frac{\partial K^*}{\partial z^*}, \quad \frac{dz^*}{dt} = i\frac{\partial K}{\partial z}, \quad (2.7)$$

which in the real canonical coordinates (p, q) are expressed as

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} + \frac{\partial \Gamma}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p} + \frac{\partial \Gamma}{\partial p}. \quad (2.8)$$

The time-evolved Husimi distribution (2.5) comprises two parts: an initial Husimi distribution propagated along the flow $\Phi_t(p, q)$, and another phase space function, denoted by $w(t, p, q)$ which I call the *norm landscape*. Interestingly, considering that for the Hamiltonian $\hat{K} = \hat{H} - i\hat{\Gamma}$ a positive Γ corresponds to decay, the flow in equations (2.7) and (2.8) a positive Γ does not describe dampening but instead an amplification. This is in contrast to the phase space flow (1.8.1) arising from the coherent state approximation and is generated by K^* rather than K .

Here, the part of the flow generated by Γ is a gradient flow moving towards the maximum of Γ , i.e., towards the areas of maximal loss. However the flow is not the principle driver of either loss or gain the system, as evident by quantum states evolving under a non-Hermitian Hamiltonian will tend towards a stable state that

experiences the least amount of loss. The dampening effect experienced within the quantum dynamics is entirely captured by the norm landscape found within the underlying classical evolution of the Husimi distribution. The norm landscape reflects the time-dependent norm of a state, in dependence on the position and momentum [51, 59, 73]. From (2.6) we recognise that the norm landscape, generated by the propagated non-Hermitian part of the Hamiltonian locally changes exponentially which dominates the dynamics of the Husimi distribution. In the case of a Hermitian Hamiltonian, $\Gamma = 0$, the norm landscape is the identity function over phase space for all time.

It is illuminating to study the dynamics of the Husimi distribution in terms of these two separate components that is unique to non-Hermitian systems. I will discuss a number of examples in the remainder of this thesis. Interestingly, the dynamics for the Wigner function also consists of two similar contributions, however, the underlying phase space flow is different from that of the Husimi flow, as we shall show in what follows.

2.2 Semiclassical Dynamics for the Wigner Representation

To obtain the semiclassical dynamics for the Wigner distribution for non-Hermitian Hamiltonians, I start with the dynamical equation [35] expressed as

$$\frac{dW}{dt} = \frac{i}{\hbar}(W * H - H * W) - \frac{1}{\hbar}(\Gamma * W + W * \Gamma), \quad (2.9)$$

where $K = K^{(S)}(p, q)$, $H = H^{(S)}(p, q)$ and $\Gamma = \Gamma^{(S)}(p, q)$ are the (symmetrically-ordered) Weyl symbols of $\hat{K} = \hat{H} - i\hat{\Gamma}$. After expanding the Moyal products (1.66) to extract the higher differential terms we see that the dynamical equation have non-vanishing contributions from the Hermitian part, H , in odd powers of the semiclassical parameter and from the non-Hermitian part, Γ , in even powers. Explicitly we have

$$\begin{aligned} \frac{\partial W}{\partial t} = & -\frac{2}{\hbar}\Gamma W + \frac{\partial H}{\partial q}\frac{\partial W}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial W}{\partial q} \\ & + \frac{\hbar}{2}\left(\frac{\partial^2 \Gamma}{\partial p^2}\frac{\partial^2 W}{\partial q^2} - 2\frac{\partial^2 \Gamma}{\partial p \partial q}\frac{\partial^2 W}{\partial q \partial p} + \frac{\partial^2 \Gamma}{\partial q^2}\frac{\partial^2 W}{\partial p^2}\right) + \dots \end{aligned}$$

In semiclassical limit $\hbar \rightarrow 0$ the dynamical equation truncates to

$$\frac{\partial W_{sc}}{\partial t} = -\frac{2}{\hbar}\Gamma W_{sc} + \frac{\partial H}{\partial q} \frac{\partial W_{sc}}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial W_{sc}}{\partial q}, \quad (2.10)$$

where W_{sc} denotes the Husimi distribution in the semiclassical limit. This equation is similar to the semiclassical equation (1.2) for Hermitian systems in that the flow $\Phi_t(p, q)$ is generated by Hamilton's equations of motion

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}. \quad (2.11)$$

Interestingly, the trajectories for the system do not experience modulation from the non-Hermitian part of the Hamiltonian. This contrasts with the flow (2.8) in the Husimi representation. There is however, one key similarity that (2.10) shares with the Husimi distribution counterpart (2.4): the extra term $-\frac{2}{\hbar}\Gamma W$. This term once again manifests as a norm landscape $w(t, p, q)$ in the general solution

$$W_{sc}(t, p, q) = W_0(\Phi_t^{-1}(p, q))w(t, p, q), \quad (2.12)$$

where $W_0(p, q)$ is the initial Wigner distribution and $\Phi_t^{-1}(p, q)$ is the backwards flow corresponding to (2.11). The norm landscape is given as the solution of

$$\dot{w}(t, p, q) = -2\Gamma(\Phi_t^{-1}(p, q))w(t, p, q), \quad (2.13)$$

with initial condition $w(0, p, q) = 1$. As with the Husimi dynamics in the semiclassical limit, I will drop the subscription notation of W_{sc} as the Wigner distribution W will always be a semiclassical approximation unless stated

The dynamics for the norm landscape in the Wigner representation takes a similar form to that for the Husimi distribution. However, in general the flow Φ_t differs between the two distributions and thus the resulting norm landscapes will be different.

2.3 Numeric and Analytic Implementation

I have calculated numerically the solutions (2.5) and (2.12) for a number of examples in this thesis. The MATLAB code I constructed and used to generate the classical dynamics of the Husimi distribution is provided in appendix C. The code is made up of three functions: *Classical Husimi distribution* which generates the dynamics, *deq* where we specify the Hamiltonian symbol and the dynamical equations to solve, and *rho0*, the initial Husimi distribution. Appendix C presents the code with an

example system of a non-Hermitian harmonic oscillator with initial displaced second excited state $|2\rangle$. Whilst not included in print, the code has also been adapted to higher dimensional systems, and to generate the classical dynamics for the Wigner distribution.

The algorithm works as follows: I define a grid on a phase space with discretised position, q , and momentum, p , co-ordinates. We then use the *ODE45* solver to solve the time-evolved trajectories of the backwards flow Φ_t^{-1} , and corresponding value of the norm landscape $w(t, p, q)$, for every initial phase space point up to a specified time. The code uses parallel computing through the inclusion of *parfor* on the nested for-loop which span all points on the phase space grid.

To obtain the time-evolved Husimi distribution I insert these trajectories into the initial distribution ρ_0 and then multiply this with the norm landscape. Before generating the plots, the Husimi distribution is normalised by division of the global norm of the state.

Depending on the grid spacing and the final time, this code takes between seconds and minutes to compute, as was the run time for the examples in this thesis on my 8-core PC.

To further explore the resulting semiclassical dynamics I will consider two analytically solvable examples. First I consider a non-Hermitian harmonic oscillator for the Husimi and Wigner distributions. For this system the semiclassical dynamics is exact for the Husimi distribution, but not for the Wigner distribution. Then I turn to the example of a free particle with a complex pre-factor. For the Hermitian version the semiclassics is exact in the Wigner distribution but not in the Husimi distribution. In the non-Hermitian case, both distributions are approximations. The non-Hermitian harmonic oscillator and free particle offer ideal settings to compare the fundamental difference between the two representations and to analyse the contributions of the transported initial distribution and the norm landscapes.

2.4 Non-Hermitian Harmonic Oscillator

My first example is a complex harmonic oscillator [52, 74], which I previously introduced in section 1.8.1, with the Hamiltonian

$$\hat{K} = \hbar(\omega - i\gamma) \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \hat{I} \right), \quad (2.14)$$

where $\omega \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+$. As discussed in section 1.8.1, this can be interpreted as a type of damped oscillator.

2.4.1 Husimi Representation

The non-Hermitian harmonic oscillator has the Wick symbol

$$K^{(N)}(z, z^*) = \hbar(\omega - i\gamma) \left(z^* z + \frac{1}{2} \right) \quad (2.15)$$

The full quantum dynamical equation for the Husimi distribution (2.4) is simply given by

$$\frac{\partial Q}{\partial t} + (\gamma - i\omega)z \frac{\partial Q}{\partial z} + (\gamma + i\omega)z^* \frac{\partial Q}{\partial z^*} = -2\gamma \left(z^* z + \frac{1}{2} \right) Q. \quad (2.16)$$

The flow is found as

$$\Phi_t(z, z^*) = \left(z e^{(\gamma - i\omega)t}, z^* e^{(\gamma + i\omega)t} \right), \quad (2.17)$$

with backwards

$$\Phi_t^{-1}(z, z^*) = \left(z e^{-(\gamma - i\omega)t}, z^* e^{-(\gamma + i\omega)t} \right). \quad (2.18)$$

In canonical coordinates, the flow is generated by

$$\frac{dq}{dt} = \omega p + \gamma q, \quad (2.19)$$

$$\frac{dp}{dt} = -\omega q + \gamma p, \quad (2.20)$$

which can be combined to the single differential equation

$$\frac{d^2 q}{dt^2} - \gamma \frac{dq}{dt} + (\omega^2 + \gamma^2)q = 0, \quad (2.21)$$

which leads to the flow as expressed in p and q

$$\Phi_t(p, q) = (q e^{\gamma t} \cos(\omega t) - p e^{\gamma t} \sin(\omega t), p e^{\gamma t} \cos(\omega t) + q e^{\gamma t} \sin(\omega t)). \quad (2.22)$$

That is, a harmonic oscillator with negative damping, i.e. amplification.

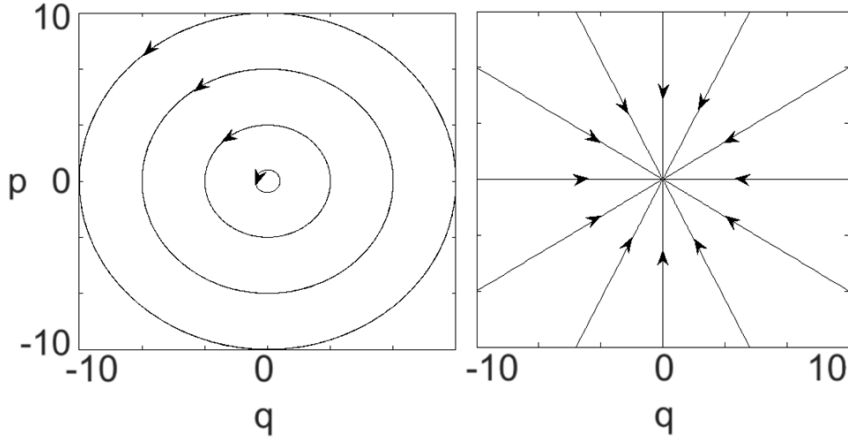


Figure 2.1: The backwards flow generated by the real part H (left) and imaginary part Γ (right) of the complex harmonic oscillator (2.14).

The trajectories along which the Husimi distribution is transported are outwards spirals, moving away from the origin. The Hermitian part of the Hamiltonian causes the clockwise rotation of trajectories around the origin. The other contribution comes from the non-Hermitian part, providing the negative damping, resulting in the factor of $e^{2\gamma t}$ in the trajectories. This turns an otherwise elliptic fixed point at the origin into an unstable spiral fixed point. As γ is positive, the trajectories spiral outwards on the forward flow. The two separate contributions of the backward flow are represented as gradient plots in figure 2.1.

While the contribution from the transported initial Husimi distribution seems to thus correspond to an amplified, rather than a damped, oscillator, the influence of the norm landscape overcompensates this effect and leads to an overall dampening. The time-evolution of the norm landscape is governed by equation (2.6) with initial condition $w(0, z, z^*) = 1$. Inserting $\Gamma^{(N)} = \gamma(z^*z + \frac{1}{2})$ and the flow (2.18) into equation (2.6) yields

$$\dot{w}(t, z, z^*) = -2\gamma(z^*ze^{-2\gamma t} + \frac{1}{2})w(t, z, z^*), \quad (2.23)$$

which can be directly integrated to yield

$$w(t, z, z^*) = e^{-z^*z(1-e^{-2\gamma t})-\gamma t}. \quad (2.24)$$

Note that the global norm of the non-Hermitian harmonic oscillator is $w_G = e^{-\gamma t}$, would result in the norm landscape to approach zero, $w(t, z, z^*) \rightarrow 0$, over the

whole phase space as $t \rightarrow \infty$. Generally we renormalise the Husimi and Wigner distributions, which cancels out the global norm, to ensure viability in our figures. The logarithm of the norm landscape for intermediate time steps are shown in Figure 2.4.1.

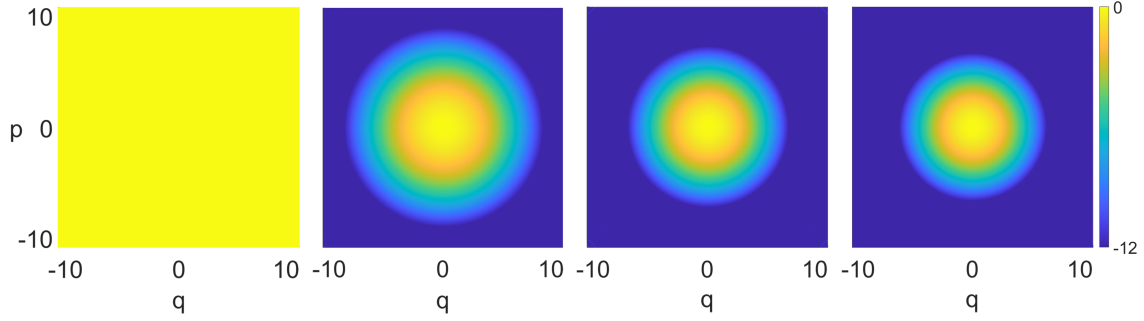


Figure 2.2: The norm landscapes corresponding to the Husimi distribution on a logarithmic scale for the non-Hermitian harmonic oscillator (2.14) with $\omega = 1$ and $\gamma = 0.15$, at times (from left to right) $t = 0$, $t = \frac{2\pi}{3}$, $t = \frac{4\pi}{3}$ and $t = 2\pi$.

The time-evolved Husimi distribution for an initial Husimi distribution $Q_0(z, z^*)$ is thus given by

$$Q(z, z^*, t) = Q_0(ze^{-(\gamma-i\omega)t}, z^*e^{-(\gamma+i\omega)t})e^{-z^*z(1-e^{-2\gamma t})-\gamma t}. \quad (2.25)$$

For an initial n -th excited state displaced to z_c , where the initial Husimi distribution is given by (1.6.1), the time-evolved Husimi distribution is found as

$$Q(z, z^*, t) = e^{-|z|^2(1-e^{-2\gamma t})-(2n+1)\gamma t} \frac{|z - z_c e^{(-i\omega+\gamma)t}|^{2n}}{n!} e^{|z - z_c e^{(-i\omega+\gamma)t}|^2}. \quad (2.26)$$

An explicit calculation confirms that an initial coherent state remains coherent over time and moves on a trajectory spiralling into the origin as we have already deduced in Section 1.8.1 in the harmonic oscillator basis. A general state on the other hand moves along a different spiral (that depends on the particular state) into the origin, and deforms asymptotically into a coherent state. Figure 2.3 depicts the dynamics of the Husimi distribution for the Hermitian and non-Hermitian harmonic oscillators for an initial second excited state $|2\rangle$, displaced to $z_c = \frac{1}{\sqrt{2}}(4 + 2i)$. The white lines depict the expectation values of the evolved state over time up to the time of the plot. This example illustrates that whilst the underlying transport does not, on its own, tend towards the origin, the Husimi distribution does, because the norm landscape localises around the origin. Furthermore, the Husimi distribution of the second excited state does not retain its initial ring structure, and instead deforms

according to the dynamics of flow and the norm landscape. The Husimi distribution over large time will decay into a Gaussian localised at the origin, as expected from the quantum spectrum.

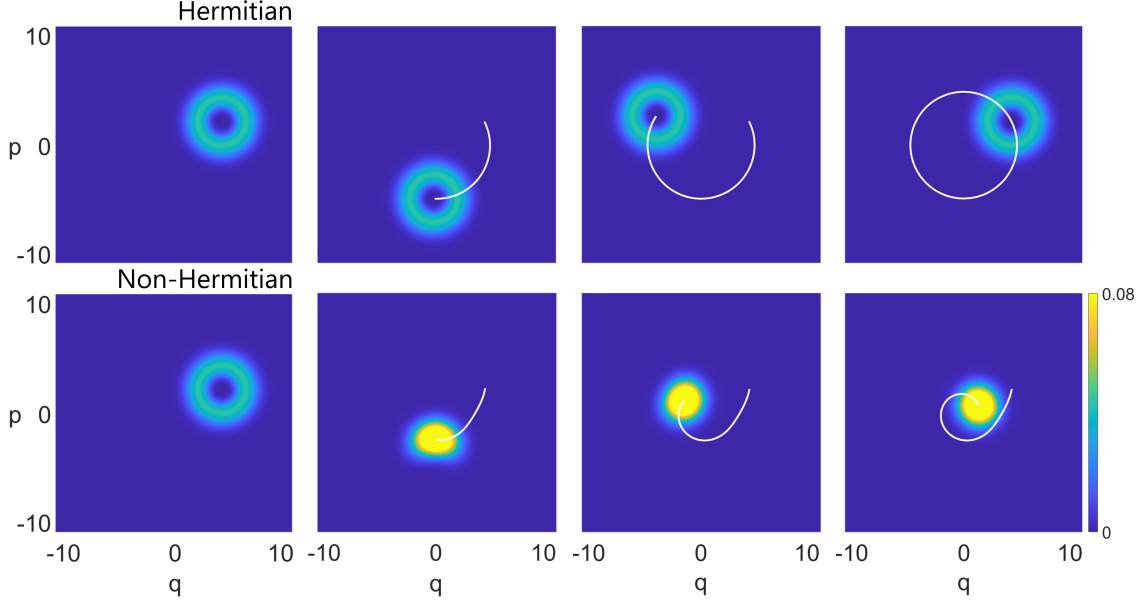


Figure 2.3: The dynamics of the Husimi distributions for the a Hermitian harmonic oscillator with $\omega = 1$ (top row) and a complex harmonic oscillator with $\omega = 1$ and $\gamma = 0.15$ (bottom row) at times (from left to right) $t = 0, t = \frac{2\pi}{3}, t = \frac{4\pi}{3}$ and $t = 2\pi$ for an initial second excited state, $\hat{D}(z_c)|2\rangle$, localised around $z_c = \frac{1}{\sqrt{2}}(4 + 2i)$. The white lines depict the trajectories of the expectation values up to the specified times.

2.4.2 Wigner Representation

For the semiclassical dynamics of the Wigner distribution I consider the Weyl symbol of the complex harmonic oscillator (2.14) as a function of (p, q) , given by

$$K^{(s)}(p, q) = \frac{(\omega - i\gamma)}{2}(q^2 + p^2), \quad (2.27)$$

where $\omega \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+$. Unlike for the Husimi distribution, the classical dynamics for the Wigner distribution in the case of $\gamma \neq 0$ is an approximation. The full quantum dynamical equation for $W(t, p, q)$ is generated by the second order differential equation

$$\frac{dW}{dt} + \omega p \frac{dW}{dq} - \omega q \frac{dW}{dp} + \hbar \gamma \frac{d^2 W}{dq^2} + \hbar \gamma \frac{d^2 W}{dp^2} = -\gamma(q^2 + p^2)W. \quad (2.28)$$

For the Hermitian harmonic oscillator, $\gamma = 0$, this again reduces to the first order differential equation (1.74). In the top two rows of Figure 2.4 I show an example of

the quantum dynamics of the Hermitian and non-Hermitian systems for the same initial state and parameters as in the Husimi discussion in Figure 2.3. As with the Husimi distribution, for the Wigner distribution for both the quantum and classical dynamics, we see the distribution spiral into the origin and tend towards the ground state. As expected, the spiralling motion of the exact quantum Wigner distribution matches that of the Husimi distribution, as evident from the plotted lines of the expectation value. The bottom row of Figure 2.4 displays the classical dynamics of the Wigner distribution which, in contrast to the Husimi dynamics, is an approximation for $\gamma \neq 0$.

In the semiclassical limit, (2.28) truncates to the classical equation

$$\frac{dW}{dt} + \omega p \frac{dW}{dq} - \omega q \frac{dW}{dp} = -\gamma(q^2 + p^2)W. \quad (2.29)$$

The solution of the classical dynamics are of the form (2.12) where the flow is the same as that of the Hermitian harmonic oscillator generated by (2.11). Despite the equivalent flow between the Hermitian and non-Hermitian case, the classical dynamics of the Wigner distribution under the non-Hermitian Hamiltonian is a spiral, the shape of which depends on the initial state. This can be seen from the white lines plotted in Figure 2.4 depicting the expectation of the state.

The norm landscape of the system is generated by the equation

$$\dot{w}(t, p, q) = -\gamma(q^2 + p^2)w(t, p, q), \quad (2.30)$$

for initial condition $w(0, p, q) = 1$ and has the solution

$$w(t, p, q) = e^{-\gamma(q^2 + p^2)t}. \quad (2.31)$$

In contrast to the Husimi norm landscape, which asymptotically approaches a Gaussian distribution, the Wigner norm landscape will continue to shrink asymptotically towards the origin. The logarithms of this norm landscape at the times corresponding to Figure 2.4 are depicted in Figure 2.5.

The classically time-evolved Wigner distribution is given by

$$W(t, p, q) = W_0(q \cos(\omega t) - p \sin(\omega t), p \cos(\omega t) + q \sin(\omega t))e^{-\gamma(q^2 + p^2)t}. \quad (2.32)$$

While the classical dynamics cannot increase the negativity of the Wigner distri-

bution, it preserves some negativity for finite times. In fact, for the non-Hermitian case, we see in Figure 2.4 the classical dynamics preserves more negativity for longer than the quantum dynamics.

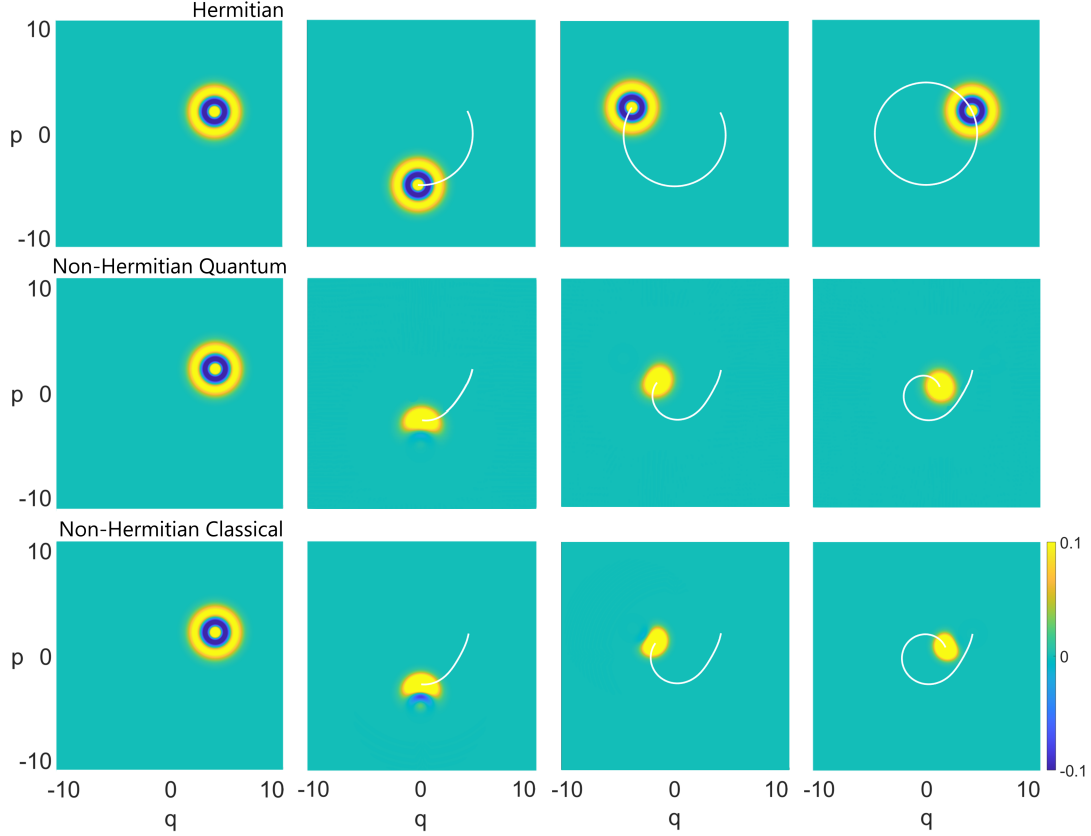


Figure 2.4: The time-evolved Wigner distribution following: the quantum dynamics of the Hermitian (top); non-Hermitian (middle) harmonic oscillator with dampening parameter $\gamma = 0.15$; the classical dynamics of the non-Hermitian harmonic oscillator (bottom). Snapshots of the Wigner distributions are shown at times (from left to right) $t = 0$, $t = \frac{2\pi}{3}$, $t = \frac{4\pi}{3}$ and $t = 2\pi$, for the second excited state initially displaced to $(p, q) = (2, 4)$.

Finally, let us compare the semiclassical dynamics of the Wigner distribution to that of the Husimi distribution in section 2.4.1 and in Figure 2.3. In both cases, the Hermitian harmonic oscillator rigidly rotates around the origin and for the non-Hermitian system, the respective distributions spiral into the origin and decay from excited states to Gaussians. In the non-Hermitian case the dynamics of the distribution differ slightly due to the different flows and the effect of the norm landscapes. The forwards flow for the Husimi distribution is a scaling of the Wigner distribution such that

$$\Phi_t^{Husimi}(q, p) = e^{\gamma t} \Phi_t^{Wigner}(q, p). \quad (2.33)$$

The difference in the resulting norm landscapes partially compensates this difference

in the flow, leading to similar dynamics.

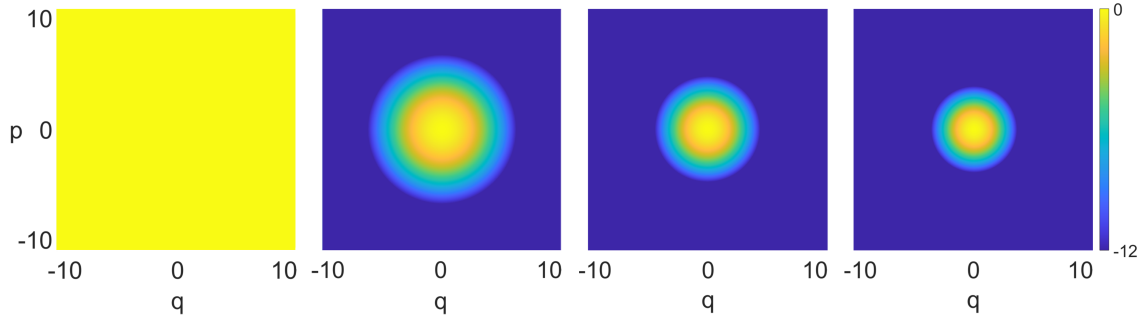


Figure 2.5: The norm landscapes corresponding to the Wigner distribution on a logarithmic scale for the non-Hermitian harmonic oscillator (2.14) with $\omega = 1$ and $\gamma = 0.15$, at times (from left to right) $t = 0$, $t = \frac{2\pi}{3}$, $t = \frac{4\pi}{3}$ and $t = 2\pi$.

2.5 Non-Hermitian Free Particle

My second linear non-Hermitian system is the complex free particle

$$\hat{K} = \hbar \left(\frac{1}{2} - i\gamma \right) \frac{(\hat{a} - \hat{a}^\dagger)^2}{2} = \left(\frac{1}{2} - i\gamma \right) \hat{p}^2, \quad (2.34)$$

where $\gamma \in \mathbb{R}^+$. The additional non-Hermitian part can be viewed as a friction, that dampens the kinetic energy. Unlike the non-Hermitian Harmonic oscillator, the classical dynamics for the Husimi distribution is an approximation for the Hermitian and non-Hermitian free particle. Meanwhile, the classical dynamics for the Wigner distribution is exact for only the Hermitian case and it is an approximation for the non-Hermitian system.

2.5.1 Husimi Representation

The non-Hermitian free particle has the Wick symbol

$$K^{(N)}(z, z^*) = \hbar \left(\frac{1}{4} - i\frac{\gamma}{2} \right) (1 - (z - z^*)^2). \quad (2.35)$$

The full quantum dynamics are given by the equation

$$\begin{aligned} \frac{\partial Q}{\partial t} + \left(\gamma + \frac{i}{2} \right) (z^* - z) \left(\frac{\partial Q}{\partial z^*} - \frac{\partial^2 Q}{\partial z^{*2}} \right) + \left(\gamma - \frac{i}{2} \right) (z - z^*) \left(\frac{\partial Q}{\partial z} - \frac{\partial^2 Q}{\partial z^2} \right) \\ = -\gamma (1 - (z - z^*)^2). \end{aligned} \quad (2.36)$$

The dynamical equation in the semiclassical limit for the Husimi distribution is

$$\frac{\partial Q}{\partial t} + \left(\gamma + \frac{i}{2}\right)(z^* - z)\frac{\partial Q}{\partial z^*} + \left(\gamma - \frac{i}{2}\right)(z - z^*)\frac{\partial Q}{\partial z} = -\gamma(1 - (z - z^*)^2). \quad (2.37)$$

This system is better suited for an analysis in the real phase space variables p and q , in which the flow is given by the solution of equations of motion

$$\frac{dq}{dt} = p, \quad \frac{dp}{dt} = 2\gamma p, \quad (2.38)$$

yielding

$$\Phi_t(p, q) = \left(pe^{2\gamma t}, q + \frac{p}{2\gamma}(e^{2\gamma t} - 1)\right). \quad (2.39)$$

The dynamical equation for the norm landscape is given by

$$\frac{dw}{dt} = -2\gamma e^{4\gamma t} w, \quad (2.40)$$

with solution

$$w(t, p, q) = e^{-\frac{p^2}{2}(e^{4\gamma t} - 1)}, \quad (2.41)$$

for an initial norm landscape $w(0, p, q) = 1$. Thus, the Husimi distribution at time $t > 0$ is given by

$$Q(t, p, q) = Q_0 \left(pe^{-2\gamma t}, q + \frac{p}{2\gamma}(e^{-2\gamma t} - 1)\right) e^{-\frac{p^2}{2}(e^{4\gamma t} - 1)}. \quad (2.42)$$

For the Hermitian system where $\gamma = 0$, we see that trajectories simply have constant momentum. We can see this in the classical dynamics of Figure 2.6 the upper row of which shows the time evolution of the Husimi distribution of an initial displaced first excited harmonic oscillator state (propagated with the classical Liouville equation). The bottom row of the same figure depicts the corresponding full quantum dynamics, which shows a similar evolution, but as the distribution becomes stretched in the classical dynamics, the quantum contributions prevent the distribution from stretching too thin.

For the non-Hermitian system, the flow incorporates a negative friction, generated by Γ , that leads to an acceleration. Just as in the complex harmonic oscillator, this effect is over compensated by the norm landscape, resulting in an overall dissipation.

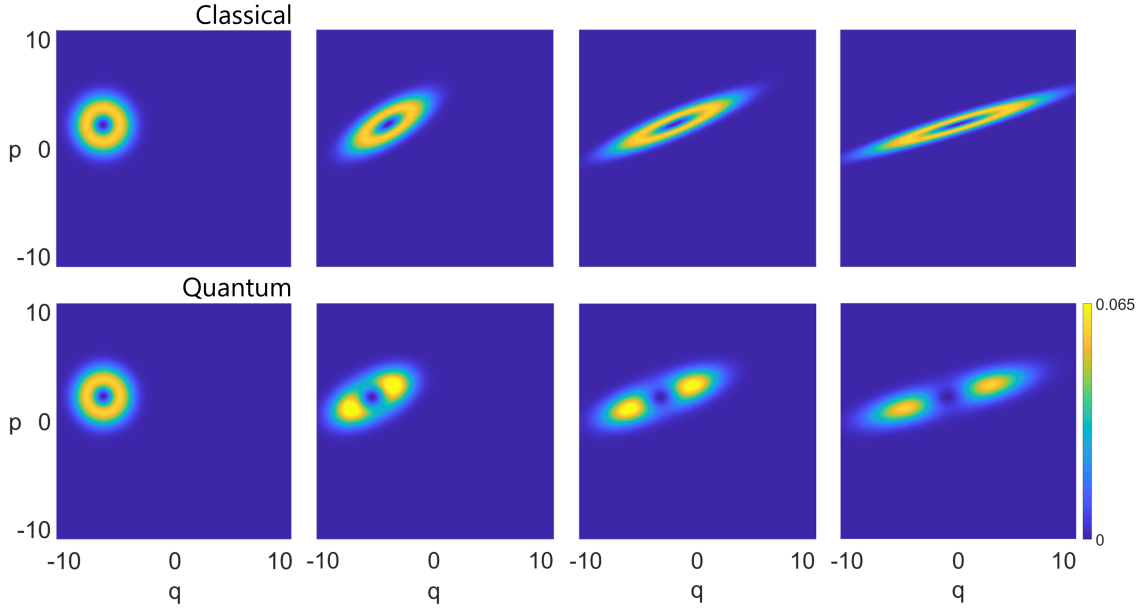


Figure 2.6: The classical (top row) and quantum (bottom row) dynamics of the Husimi distributions for the Hermitian free particle at times (from left to right) $t = 0$, $t = 1$, $t = 2$ and $t = 3$, for the first excited state $|1\rangle$ initially displaced to $z_0 = \frac{1}{\sqrt{2}}(-6 + 2i)$.

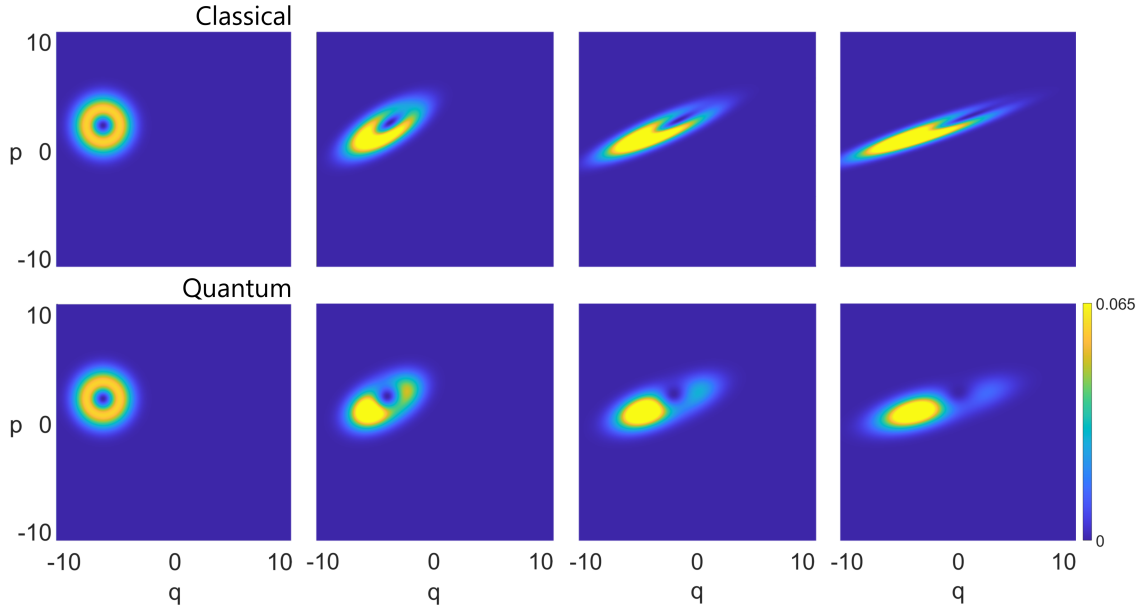


Figure 2.7: The classical (top row) and quantum (bottom row) dynamics of the Husimi distributions for the non-Hermitian free particle with dampening parameter $\gamma = 0.05$ at times (from left to right) $t = 0$, $t = 1$, $t = 2$ and $t = 3$, for the first excited state $|1\rangle$ initially displaced to $z_0 = \frac{1}{\sqrt{2}}(-6 + 2i)$.

Figure 2.7 shows the quantum and classical dynamics for a non-Hermitian free particle with a small friction parameter $\gamma = 0.05$, which we can compare with Figure 2.6.

We see that whilst for a Hermitian free particle, the classical dynamics preserved a (stretched) ring structure of the excited state, the state for the non-Hermitian free particle in both the classical and quantum distributions localises on the left side of the distribution, i.e. the dispersion is reduced compared to the Hermitian case.

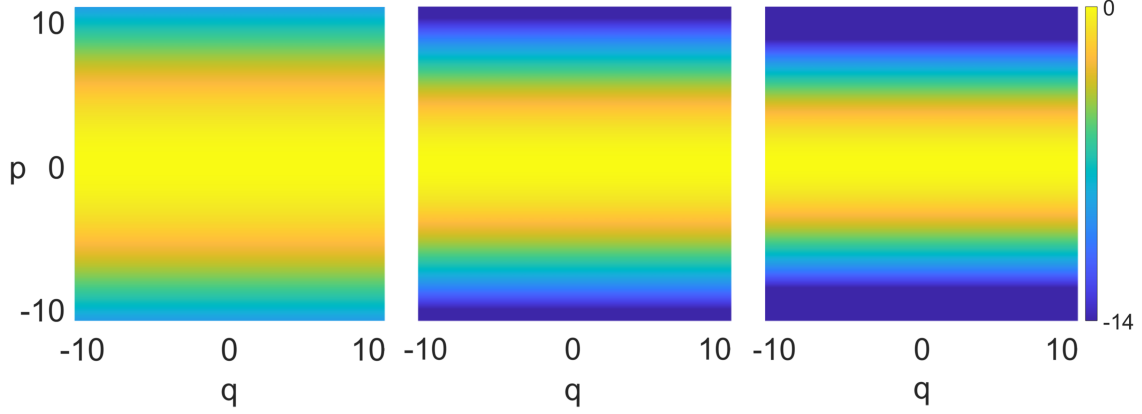


Figure 2.8: The norm landscapes corresponding to the Husimi distribution on a logarithmic scale for the non-Hermitian free particle with dampening parameter $\gamma = 0.05$ at times (from left to right) $t = 1$, $t = 2$ and $t = 3$.

Over time, the Husimi distribution under quantum dynamics develops only one strongly localised region, in contrast to the two regions observed in the quantum dynamics of the Hermitian case. For the non-Hermitian free particle, the Husimi distribution in both dynamics has the highest intensity values near zero momentum. This reflects the behaviour of the norm landscape plotted in Figure 2.5.1 which localises on phase space along $p = 0$. The asymptotic state is the momentum eigenstate belonging to $p = 0$.

2.5.2 Wigner Representation

Here the Weyl symbol in (p, q) of the non-Hermitian free particle (2.34), given by

$$K^{(S)}(p, q) = \left(\frac{1}{2} - i\gamma \right) p^2. \quad (2.43)$$

The Hermitian free particle has the same quantum and classical dynamics for the Wigner distribution generated by

$$\frac{\partial W}{\partial t} + p \frac{\partial W}{\partial q} = 0, \quad (2.44)$$

with the solution

$$W(t, p, q) = W_0(p, q - pt), \quad (2.45)$$

for an initial Wigner distribution $W_0(p, q)$.

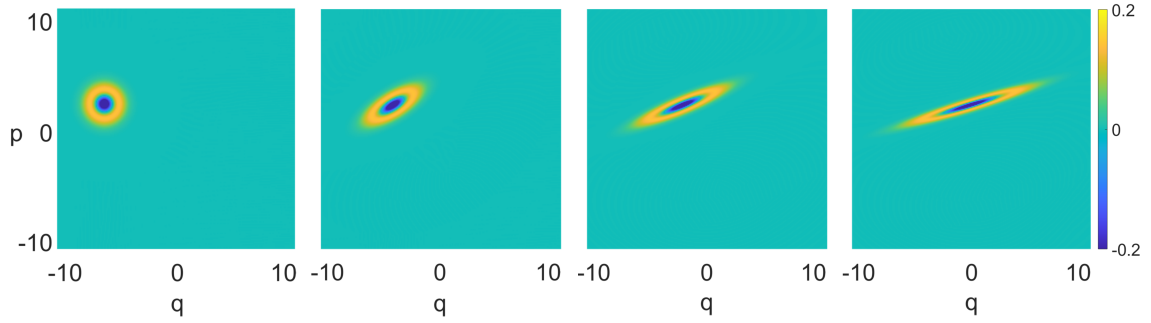


Figure 2.9: The classical (top row) and quantum (bottom row) dynamics of the Wigner distributions for the Hermitian free particle at times (from left to right) $t = 0$, $t = 1$, $t = 2$ and $t = 3$, for the first excited state $|1\rangle$ initially displaced to $z_0 = \frac{1}{\sqrt{2}}(-6 + 2i)$.

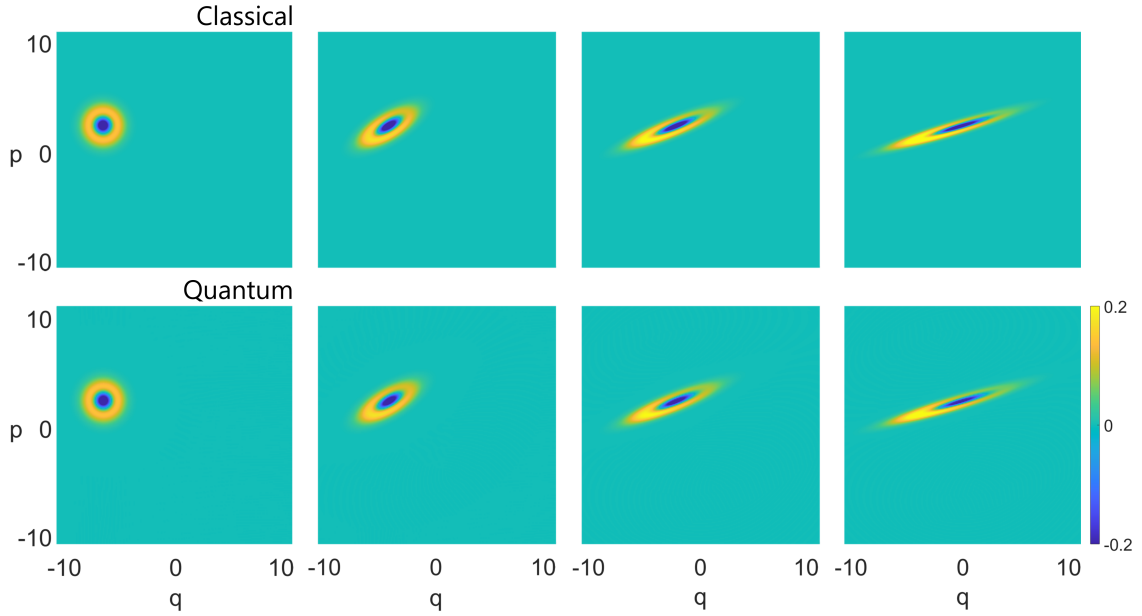


Figure 2.10: The classical (top row) and quantum (bottom row) dynamics of the Husimi distributions for the non-Hermitian free particle with dampening parameter $\gamma = 0.05$ at times (from left to right) $t = 0$, $t = 1$, $t = 2$ and $t = 3$, for the first excited state $|1\rangle$ initially displaced to $z_0 = \frac{1}{\sqrt{2}}(-6 + 2i)$.

The time-evolved Wigner distribution is shown in Figure 2.9. The distribution maintain regions of positive and negative values whilst being stretched across the phase-space in the positive q -direction. The dynamics for the Wigner distribution is very similar to the classical dynamics of the Husimi distribution in Figure 2.6.

For the non-Hermitian free particle, the quantum and classical dynamics are no longer the same as the full quantum dynamics is given by

$$\frac{\partial W}{\partial t} + p \frac{\partial W}{\partial q} - 2\hbar\gamma \frac{\partial^2 W}{\partial q^2} = -2\gamma p^2 W, \quad (2.46)$$

which truncates to

$$\frac{\partial W}{\partial t} + p \frac{\partial W}{\partial q} = -2\gamma p^2 W, \quad (2.47)$$

in the semiclassical limit. The flow is the same for the Hermitian free particle, as both systems are governed by Hamilton's equations of motion, as discussed in section 2.2. The equation of motion for the norm landscape is given by

$$\dot{w} = -2\gamma p^2 w, \quad (2.48)$$

which is trivially integrated to

$$w(t, p, q) = e^{-2\gamma p^2 t}. \quad (2.49)$$

The analytic solution for the classical Wigner evolution is thus given by

$$W(t, p, q) = W_0(p, q - pt) e^{-2\gamma p^2 t}. \quad (2.50)$$

The quantum and classical dynamics of the Wigner distribution for the non-Hermitian free particle are shown in Figure 2.10. The corresponding snapshots of the logarithm of the classical norm landscape are depicted in Figure 2.11.

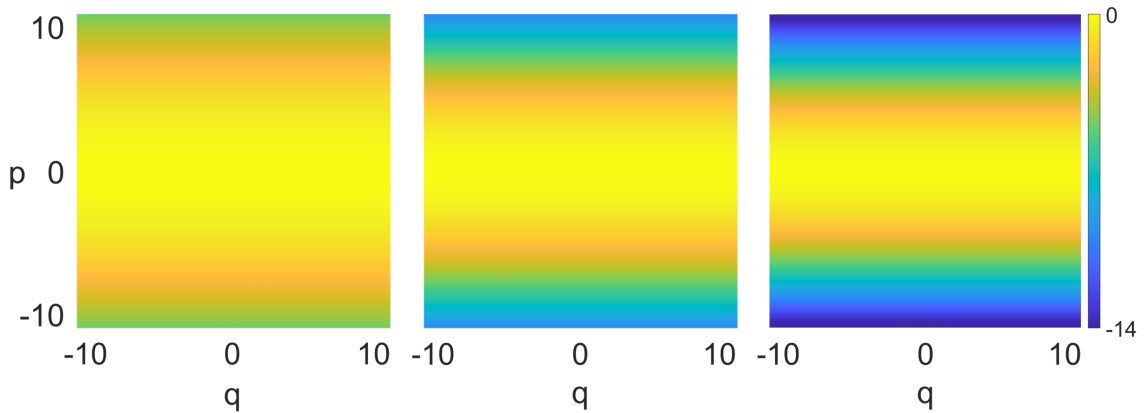


Figure 2.11: The norm landscapes corresponding to the Wigner distribution on a logarithmic scale for the non-Hermitian free particle with dampening parameter $\gamma = 0.05$ at times (from left to right) $t = 1$, $t = 2$ and $t = 3$.

The quantum and classical dynamics are very similar for the small friction parameter $\gamma = 0.05$, due to the resulting small effect of the second order terms in (2.46). For the Husimi distribution on the other hand the neglected second derivative already appears in the Hermitian case, leading to larger discrepancies between the quantum and semiclassical evolution. The quantum and classical dynamics for the Wigner dynamics stretch further across the phase-space and do not localise as dramatically as the corresponding the Husimi distribution. Just as in the Husimi description, the Wigner function localises around the region $p = 0$ for longer times. That arises from the corresponding norm landscape, as shown in Figure 2.11, localises around $p = 0$, where large absolute values of the momentum experience dampening.

I will now move to the discussion an example of a Hermitian free particle with a piecewise constant imaginary potential, where the classical description is not exact.

2.6 Free Particle with an Absorbing Region

To conclude this chapter, let us study the dynamics of a free particle intersecting a bounded absorbing region. That is, a Hamiltonian of the form

$$\hat{K} = \frac{\hat{p}^2}{2} - i\hat{\Gamma}(\hat{q}), \quad (2.51)$$

where $\hat{\Gamma}$ is a quantisation of the classical indicator function on the absorbing interval, which without loss of generality we choose as $[-r, r]$ with $r \in \mathbb{R}^+$. I can write

$$\Gamma(q) = \gamma (\Theta(q + r) - \Theta(q - r)), \quad (2.52)$$

where $\Theta(q)$ is the Heaviside function

$$\Theta(q) = \begin{cases} 0 & \text{for } q < 0, \\ 1 & \text{for } q \geq 0. \end{cases} \quad (2.53)$$

Of course, $\Gamma(q)$ is not an analytic function, and thus we would have to regularise it before applying the semiclassical formalism. Just as in the case of a real valued Hermitian hard wall potential, we expect this to lead to additional quantum effects on top of the classical Liouvillian dynamics. It is, nevertheless, instructive, to consider the resulting classical dynamics, (2.5) and (2.12), for a formally piecewise constant absorbing potential. The trajectories for the classical Wigner dynamics are not influenced by the imaginary part of the Hamiltonian in any case, and the whole

effect of the absorption is described by the norm landscape alone. This is also the case for the Husimi dynamics when the imaginary potential is constant, as we will briefly demonstrate. For $|q| > r$, the contribution from the imaginary potential is zero and the flow (2.8) is simply that generated by the Hermitian free particle. For $|q| < r$ then the flow is given by the solution of the equations

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} + \frac{\partial \Gamma}{\partial p} = 0, \quad (2.54)$$

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} + \frac{\partial \Gamma}{\partial q} = \frac{\partial}{\partial p}\left(\frac{p^2}{2}\right) + \frac{\partial}{\partial q}(-\gamma) = p, \quad (2.55)$$

which are also the dynamical equations for the Hermitian free particles. Thus, the phase space flow is not affected by the piecewise constant imaginary potential on either side of the transitions at $|q| = r$ where the flow is

$$\Phi_t(p, q) = (p, q + pt), \quad (2.56)$$

for both the Husimi and the Wigner. We will now study a smooth function in which we can apply our semiclassical method to evaluate the Wigner dynamics and norm landscapes for all values of q .

Instead of the piecewise constant absorption function, it is convenient for numerical simulations to consider a regularised smooth Hamiltonian function [75]. Specifically, I consider the Weyl symbol

$$K^{(s)}(p, q) = \frac{p^2}{2} - i\gamma \left(1 - \tanh \left(\left(\frac{q}{r} \right)^{2s} \right) \right), \quad (2.57)$$

where $r \in \mathbb{R}^+$ is the radius of the absorbing region and $s \in \mathbb{R}^+$ determines the steepness of the imaginary potential. In the limit $s \rightarrow \infty$, this approaches the piecewise constant function (2.51). The smooth imaginary parts of the potential, $\Gamma^{(s)}(q) = \gamma(1 - \tanh((\frac{q}{r})^{2s}))$ for $r = 1$ and $s = 10$ are displayed in Figure 2.12 for loss strength values $\gamma = 1$ and $\gamma = 20$. In what follows I consider only the Wigner representation, to illustrate the behaviour of the system. The classical and quantum dynamics for the Wigner distribution are depicted for small loss strength $\gamma = 1$ for loss in the interval $[-1, 1]$ at different times in Figures 2.13. The initial wave packet is centred at $(p, q) = (5, -7)$. Classical and quantum descriptions yield similar free particle motion, in that the distributions move through the imaginary potential relatively undisturbed. We observe slight interference patterns emerging in the quantum dynamics at the boundaries of the absorbing region.

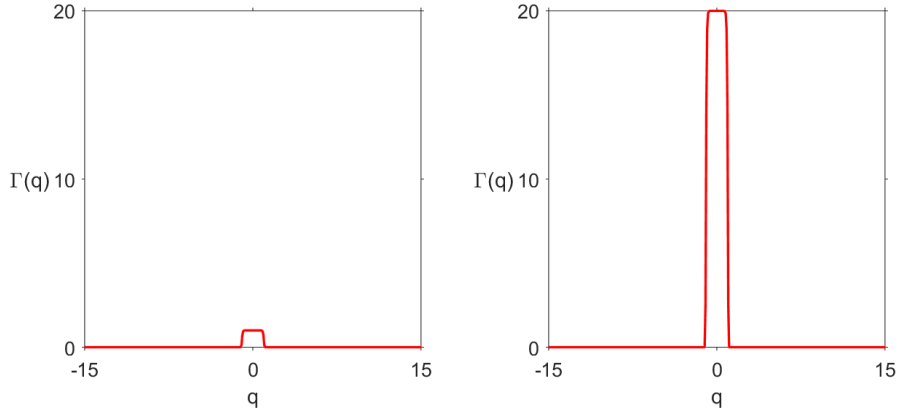


Figure 2.12: The imaginary part of the smooth billiard potential (2.57) are plotted for parameters $r = 1, s = 10$ with loss strength $\gamma = 1$ (left) and $\gamma = 20$ (right) .

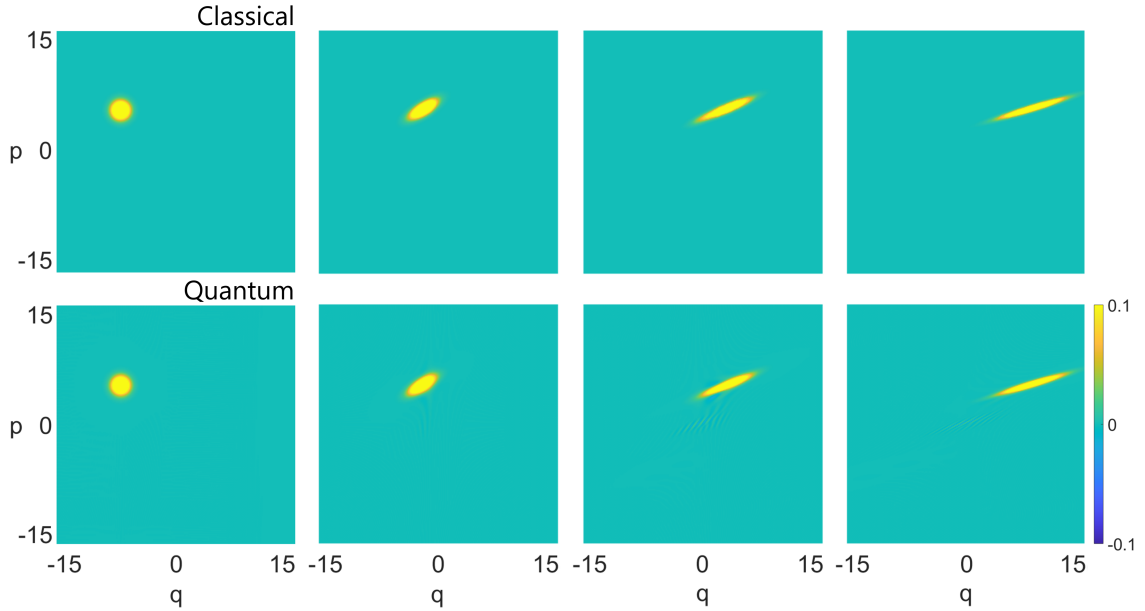


Figure 2.13: The classical (top row) and quantum (bottom row) dynamics of the Wigner distributions for the free particle with an absorbing region with parameters $r = 1, s = 10$ and $\gamma = 1$ at times (from left to right) $t = 0, t = 1, t = 2$ and $t = 3$, for the ground state $|0\rangle$ initially displaced to $z_0 = \frac{1}{\sqrt{2}}(-7 + 5i)$.

Figure 2.14 shows the norm landscape corresponding to the set up in Figure 2.13 with loss strengths $\gamma = 1$. The norm landscape, for any time can be separated into the regions $\log(w(t)) < 0$ and $\log(w(t)) = 0$, with boundaries around $\min(-1 + pt, -1)$ and $\max(1 + pt, 1)$, which correlate to trajectories on phase space that have and have not intersect the absorbing region. Overtime, the valley in the norm landscape spreads over more of the phase spaces. The norm landscape shows a flat minimum

around the trapezium shaped region bounded by $(p > 0, -1 < q < -1 + pt)$ and $(p < 0, 1 < q < 1 - pt)$ corresponding to trajectories that remain in the absorbing region for all time. Beyond the bounded region and within $\log(w(t)) < 0$, we see a gradient on the norm landscape corresponding to trajectories that pass through the absorbing region at different rates and experience partial absorption. The norm landscape for $\gamma = 20$ has the same structure as Figure 2.14 up to a scaling.

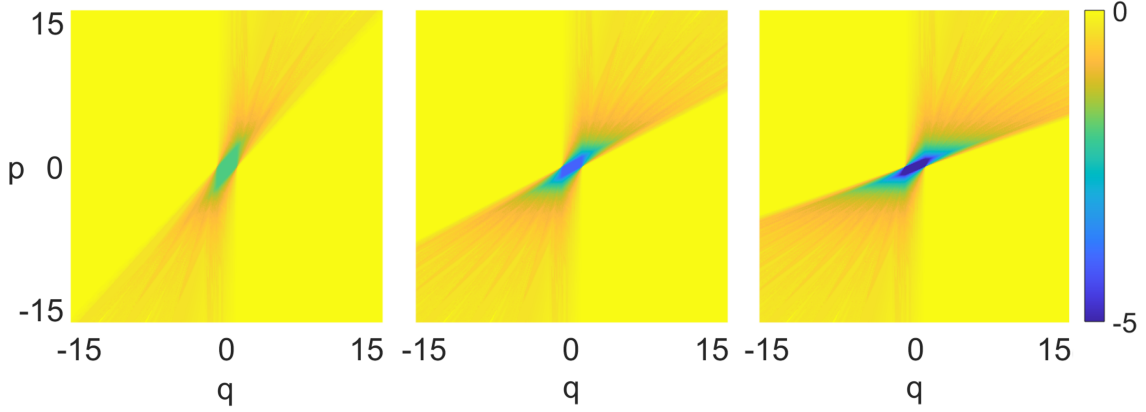


Figure 2.14: Snapshots of the logarithm of the norm landscape for the free particle with an absorbing region with parameters $r = 1, s = 10$ and $\gamma = 1$ at times (from left to right) $t = 1, t = 2$ and $t = 3$

The classical and quantum dynamics are plotted in Figure 2.15 for $\gamma = 20$. For large gamma for the quantum dynamics we see an interesting phenomenon, where the imaginary potential acts similar to a Hermitian "hard wall" potential. That is, the distribution is mostly reflected off the absorbing boundary. We see this at $t = 1$ of Figure 2.15 where the unabsorbed distribution hits an imaginary vertical wall. Then at $t = 2$ we see the distribution reflect, such that the momentum for the majority of the Wigner distribution has flipped sign. Finally, at $t = 3$ the Wigner distribution moves away from the absorbing region. This is a pure quantum effect as evident from the classical dynamics. The Wigner distribution under classical dynamics is almost fully absorbed upon intersecting the imaginary potential at $t = 2$, and we see the remains of the renormalised distribution at the boundary of the absorbing region.

Figure 2.16 shows the norm of the time-evolved distribution for the quantum (solid line) and classical (dashed line) dynamics for $\gamma = 1$ (left) and $\gamma = 20$ (right). For small values of the absorption, where the quantum scattering does not occur, the classics and quantum norms have strong agreement. For the high loss strength, we see that while the norm for the classics almost reaches zero (total absorption), the

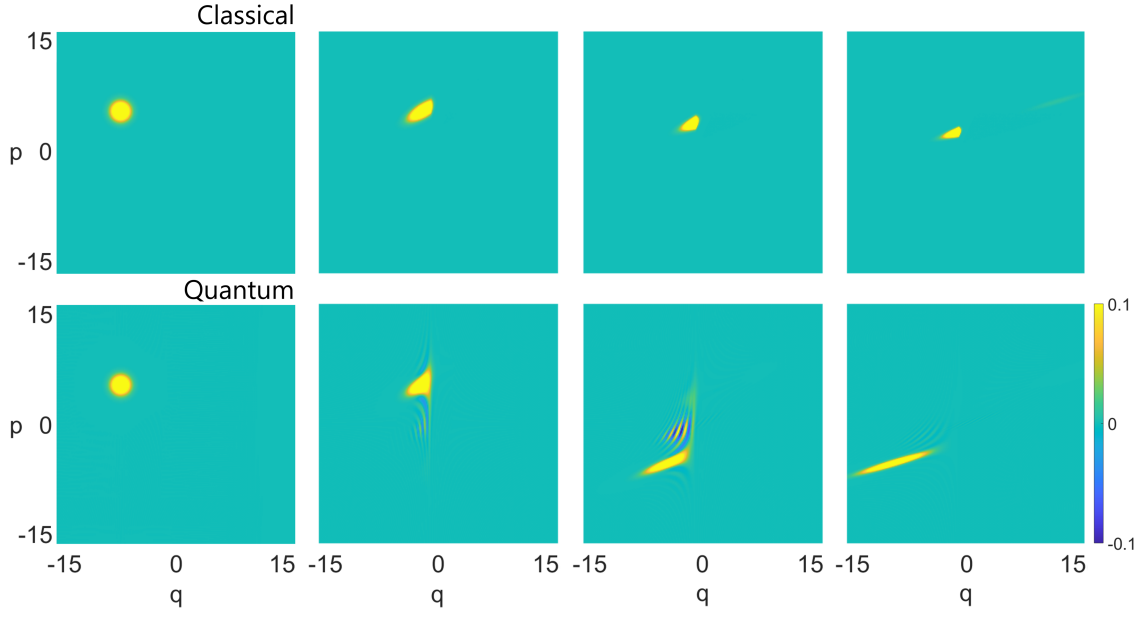


Figure 2.15: The classical (top row) and quantum (bottom row) dynamics of the Wigner distributions for the free particle with an absorbing region with parameters $r = 1$, $s = 10$ and $\gamma = 20$ at times (from left to right) $t = 0$, $t = 1$, $t = 2$ and $t = 3$, for the ground state $|0\rangle$ initially displaced to $z_0 = \frac{1}{\sqrt{2}}(-7 + 5i)$.

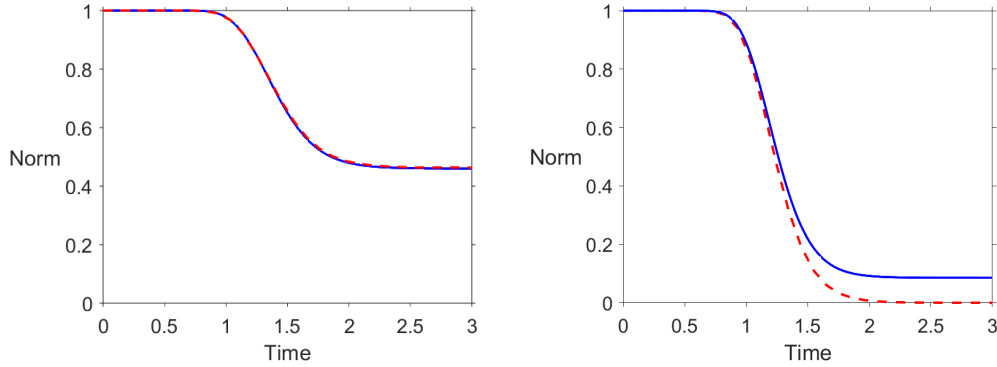


Figure 2.16: Plots of the time-dependent global norm of the wavepacket over time. The figure compares the norm of the state evolved by the classical (dashed red line) and quantum dynamics (solid blue line) for two potentials: $\gamma = 1$ (left), and $\gamma = 20$ (right).

quantum dynamics only sees partial absorption since part of the Wigner distribution has been reflected from the absorbing region and continues to evolve in the opposite direction without loss. In section 5.1 I will come back to a similar setup of a smooth two-dimensional quantum billiard, with an internal region of constant absorption.

In this chapter, I derived the dynamics of the Husimi and Wigner distributions

generated by non-Hermitian Hamiltonians in the semiclassical limit, which saw the emergence of a norm landscape and highlighted the stark differences to their Hermitian counterparts. I presented the time-evolved Husimi and Wigner distributions for two examples: a non-Hermitian harmonic oscillator and a non-Hermitian free particle. The non-Hermitian harmonic oscillator exhibits the same quantum and classical dynamics for the Husimi distribution as the non-Hermitian free particle does for the Wigner distribution. The fact that the Husimi and Wigner distributions in each example do not yield the same dynamics warrants further studies and raises the question of which distribution is most suitable for depicting phase space dynamics in non-Hermitian systems.

The chapter concluded with an evaluation of a free particle Hamiltonian with a constant imaginary potential with an equivalent flow between the Husimi and Wigner distributions. This piecewise Hamiltonian was regularised such that we can apply our semiclassical method and numerically generate the system dynamics. The norm landscape was presented alongside every system to pinpoint the regions in phase space most affected by dampening. Within this section, I also discussed how I numerically simulated the dynamics of the Husimi and Wigner distributions, as well as their corresponding norm landscapes. In this section, I referred to [appendix C](#) for a copy of my MATLAB code. In the next chapter we will consider systems without analytic solutions that display non-trivial dynamical behaviours and norm landscapes.

Chapter 3

Non-Linear Non-Hermitian System

In this section I present two examples of non-Hermitian quantum systems with underlying non-linear classical flow which we refer to as non-linear non-Hermitian systems. The first example is a PT-symmetric Kerr Hamiltonian and the second is a non-Hermitian variation of the Mexican Hat (Higgs) potential. I will begin each example with a brief discussion of the spectra and eigenstates of the quantum Hamiltonian (Hermitian and non-Hermitian versions). I also discuss the classical flow associated to this, using the Wick symbol of the Hamiltonian but I will also note the differences for Weyl symbol. I will then analyse the quantum and classical dynamics of the Husimi and Wigner distributions for Hermitian and non-Hermitian versions of both examples.

3.1 PT-Symmetric Kerr Hamiltonian

Let us begin with the example of a PT-symmetric variation of a Kerr Hamiltonian with a linear imaginary potential, given by

$$\hat{K} = \hbar \hat{a}^\dagger \hat{a} + \hbar^2 \beta \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} - i\sqrt{\hbar} \frac{\epsilon}{\sqrt{2}} (\hat{a}^\dagger + \hat{a}), \quad (3.1)$$

where β and $\epsilon \in \mathbb{R}^+$, which reduces to a Hermitian Kerr Hamiltonian [24] for $\epsilon = 0$.

The PT-symmetric Hamiltonian has a balanced gain-loss profile in the q direction, from the term $\hat{q} = \frac{\sqrt{\hbar}}{\sqrt{2}} (\hat{a}^\dagger + \hat{a})$. As discussed in section 1.8.1, the linear version of this system with $\beta = 0$, has unbroken PT-symmetry, and thus purely real eigenvalues. We observe numerically that the PT-symmetric Hamiltonian (3.1) is also in the unbroken PT-symmetry phase for the parameters $\hbar = 1$, $\beta = 0.25$ and $\epsilon = 1$, with

only real eigenvalues. The first 50 eigenvalues are shown in the left plot of Figure 3.1. The Husimi distribution of the ground state, the eigenstate that corresponds to the minimum eigenvalue, is shown in right plot of Figure 3.1. The ground state is localised around a point $(p, q) = (p_0, 0)$ where $p_0 > 0$. The high intensity region of the eigenstate is in the upper plane $p > 0$ which also tails off into the lower plane $p < 0$. This corresponds to the classical trajectories.

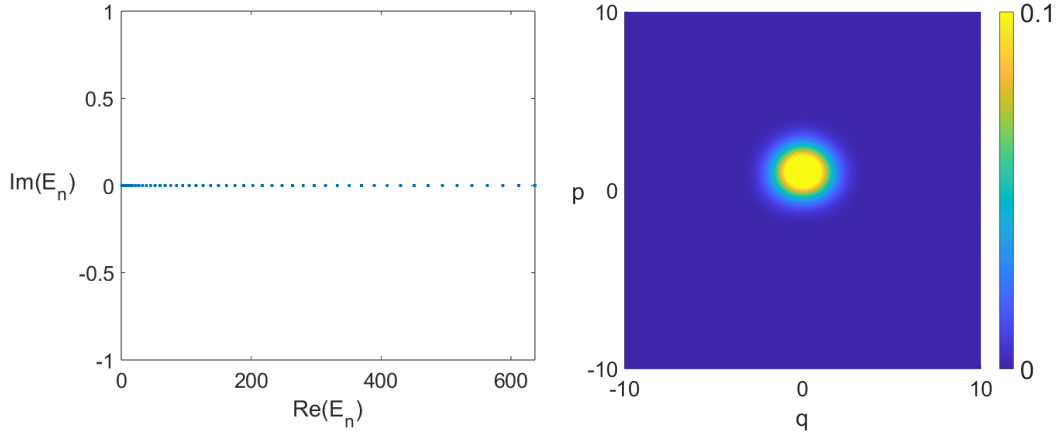


Figure 3.1: The eigenvalue spectrum on the complex plane for the PT-symmetric Hamiltonian with $\beta = 0.25$, $\epsilon = 1$ (left) and the Husimi distribution of the ground state (right).

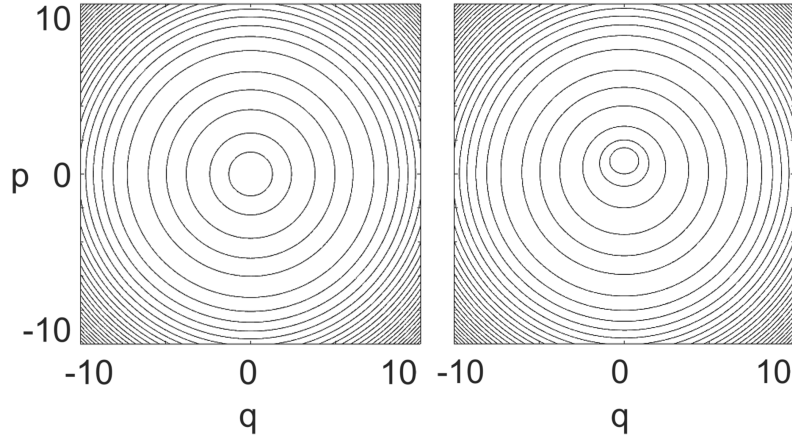


Figure 3.2: The phase space trajectories for the Hermitian Kerr Hamiltonians (left) and the non-Hermitian PT-symmetric Hamiltonian (right) with $\beta = 0.25$, $\epsilon = 1$.

The Wick symbol of the PT-symmetric Hamiltonian in position and momentum variables is given by

$$K^{(N)}(p, q) = \frac{p^2 + q^2}{2} + \frac{\beta}{4}(p^2 + q^2)^2 - i\epsilon q. \quad (3.2)$$

The corresponding phase space flow in coherent state approximation (1.8.1) is equivalent to the Hamiltonian flow of the real-valued Hamiltonian

$$\tilde{H}^{(N)}(p, q) = \frac{p^2 + q^2}{2} + \frac{\beta}{4}(p^2 + q^2)^2 - \epsilon p, \quad (3.3)$$

From $\tilde{H}^{(N)}(p, q)$ we can deduce that there is a single elliptic fixed point located at $q = 0$ and $p_0 > 0$, where p_0 is the single real root of the polynomial $p_0^3 + \frac{1}{\beta}p_0 + \frac{\epsilon}{\beta} = 0$. This can be seen in Figure 3.2 which show the contour plots for the Kerr and PT-symmetric Hamiltonians, where the centres lie at $(0, 0)$ and $(p_0, 0)$ respectively. Note that in contrast to the linear case $\beta = 0$, in general each trajectory has a different frequency increasing with the increasing distance to the origin. For the Weyl symbol, the Kerr and PT-symmetric Hamiltonians have an elliptic fixed point at the origin.

From Figure 3.1 we see that ground state of the PT-symmetric system indeed sits at this fixed point. For the Hermitian Kerr Hamiltonians the ground state is just the ground state of the harmonic oscillator, and sits at the origin.

Lastly, note that the norm of a time-evolved eigenstate is constant for the PT-symmetric system. However, as we will see, the classical of norm landscapes fluctuate between positive and negative values over the phase space. Over time the norm landscapes develop an intricate structure which has profound effects on the Husimi and Wigner dynamics.

3.1.1 The Husimi Representation

Let us begin with the Husimi dynamics for the Hermitian case. The normal ordered Wick symbol of the Hermitian Kerr Hamiltonian, from equation (3.1) where $\epsilon = 0$, is given by

$$H^{(N)}(z, z^*) = \hbar z^* z + \hbar^2 \beta z^{*2} z^2, \quad (3.4)$$

where $\beta \in \mathbb{R}^+$.

The quantum dynamics of the Husimi distribution are governed by the second-order partial differential equation

$$\frac{\partial Q}{\partial t} = iz(1 + 2\hbar\beta z^* z) \frac{\partial Q}{\partial z} - iz^*(1 + 2\hbar\beta z z^*) \frac{\partial Q}{\partial z^*} + i\hbar\beta z^2 \frac{\partial^2 Q}{\partial z^2} - i\hbar\beta z^{*2} \frac{\partial^2 Q}{\partial z^{*2}}, \quad (3.5)$$

which was solved explicitly in [24].

The classical dynamics are given by the leading order terms in (3.5), in which the higher derivatives of the Husimi distribution vanish, which yields the Liouville equation

$$\frac{\partial Q}{\partial t} = iz(1 + 2\hbar\beta z^*z)\frac{\partial Q}{\partial z} - iz(1 + 2\hbar\beta z^*z)\frac{\partial Q}{\partial z^*}, \quad (3.6)$$

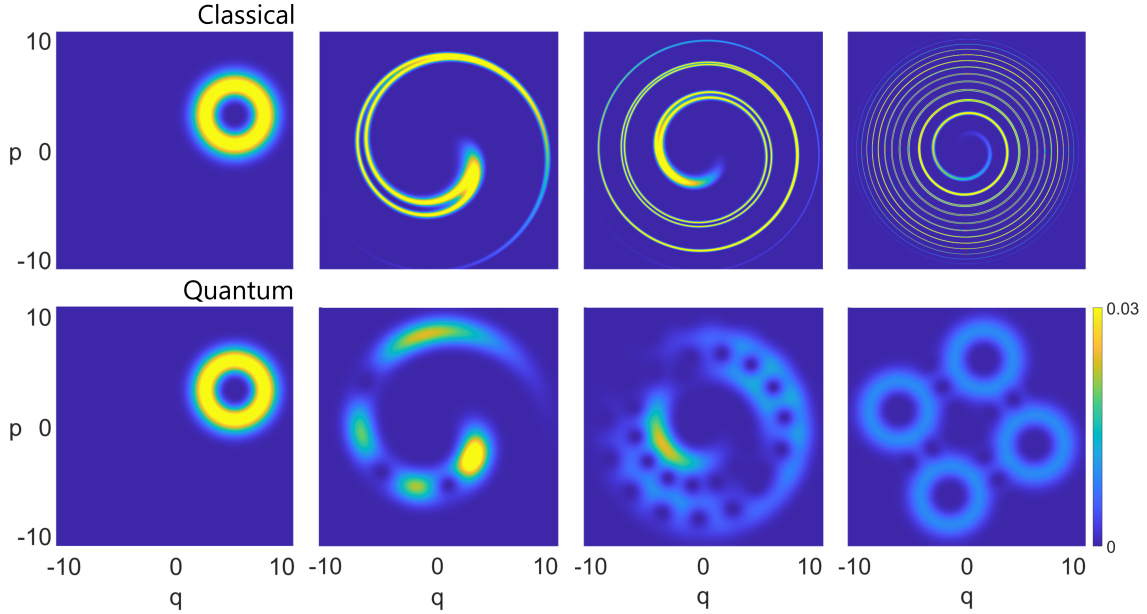


Figure 3.3: The classical (top row) and quantum (bottom row) dynamics of the Husimi distributions for the Kerr Anharmonic Oscillator for parameter $\beta = 0.25$ at times (from left to right) $t = 0$, $t = \frac{\pi}{10}$, $t = \frac{\pi}{4}$ and $t = \pi$, for the third excited state initially displaced to $z_c = \frac{1}{\sqrt{2}}(5 + 3i)$.

Figure 3.3 illustrates the time-evolved Husimi distribution for the Kerr Anharmonic Oscillator for parameter $\beta = 0.25$ for an initial state $\hat{D}(z_c)|3\rangle$ evolving under classical and quantum dynamics. This has already been investigated in [24] for an initial coherent state. For short and intermediate times, a similar structure can be seen between the classical and quantum dynamics as the Husimi distributions both curl around the origin in a clockwise direction. The Husimi distribution in both cases follows the phase space flow depicted in Figure 3.2 which due to the different frequencies of the trajectories, leads to a whorling motion, as the parts of the distribution further from the fixed point move faster [24]. However the two dynamics are distinct even at short times as “holes” emerge in the quantum dynamics due to quantum interference effects. For long time, a fine-scale whorl develops in the classical dynamics of the Husimi distribution, while the quantum dynamics goes through partial revivals of the initial distribution.

I will now consider the PT-symmetric system (3.1), with classical Wick symbol

$$K^{(N)}(z, z^*) = \hbar z^* z + \hbar^2 \beta z^{*2} z^2 - i\sqrt{\hbar} \frac{\epsilon}{\sqrt{2}}(z^* + z), \quad (3.7)$$

where β and $\epsilon \in \mathbb{R}^+$.

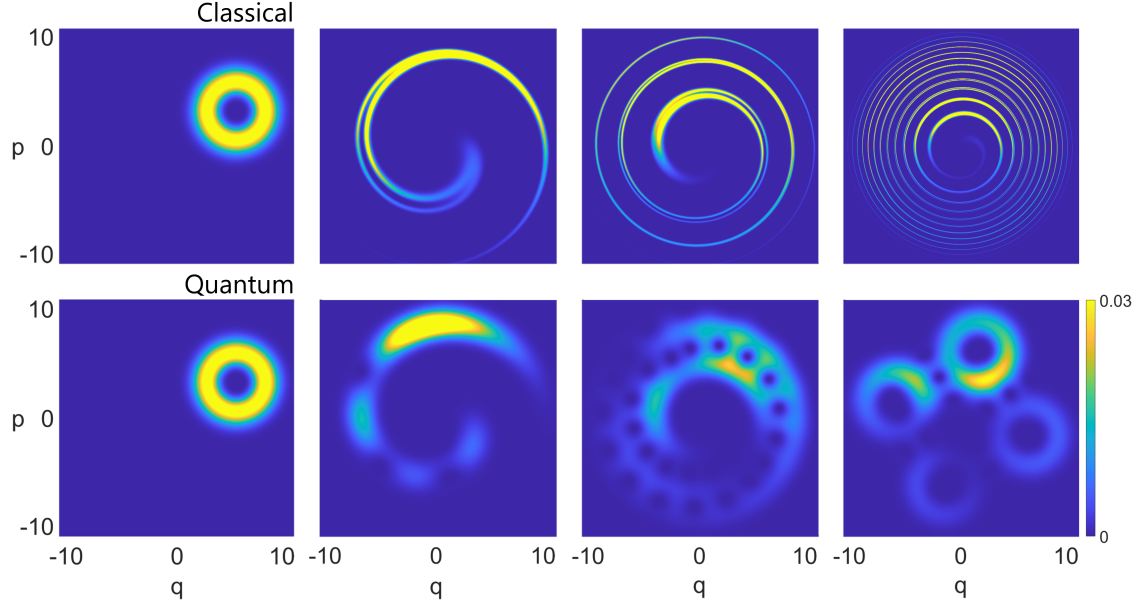


Figure 3.4: The classical (top row) and quantum (bottom row) dynamics of the Husimi distributions for the non-Hermitian PT-symmetric anharmonic oscillator with $\beta = 0.25$ and $\epsilon = 1$ at times (from left to right) $t = 0$, $t = \frac{\pi}{10}$, $t = \frac{\pi}{4}$ and $t = \pi$, for the third excited state initially displaced to $z_c = \frac{1}{\sqrt{2}}(5 + 3i)$.

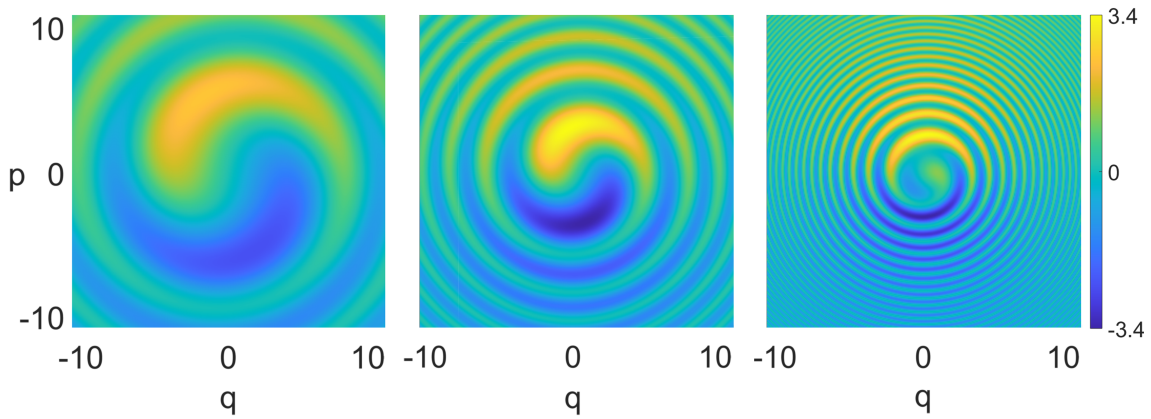


Figure 3.5: The norm landscapes corresponding to the Husimi distribution on a logarithmic scale for the non-Hermitian PT-symmetric anharmonic oscillator with $\beta = 0.25$ and $\epsilon = 1$ at times (from left to right) $t = 0$, $t = \frac{\pi}{10}$, $t = \frac{\pi}{4}$ and $t = \pi$.

Figure 3.4 shows the time-evolved Husimi distribution of the PT-symmetric Hamiltonian with $\beta = 0.25$ and $\epsilon = 1$ following classical and quantum dynamics. The Husimi distributions follows the classical flow which leads to the whorling motion already observed in the Hermitian case. Again, this motion is especially prominent in the classical dynamics but it can be also be seen in the quantum dynamics for short times. We see this for example at $t = \frac{\pi}{10}$ in Figures 3.3 and 3.4. However, as we reach $t = \frac{\pi}{4}$ the quantum interference is a prominent feature of the distribution, and only the classical dynamics sees a fine whorl. By the final time plotted, $t = \pi$, there is little correlation between the structure of the classical and quantum Husimi distributions. However, one can still see a similarity between the classical and quantum plots for all snapshots in that there exists a high intensity region for the Husimi distribution around the upper half of the phase-space.

The norm landscape corresponding to the Husimi distribution in Figure 3.4 is plotted in Figure 3.5. The gain-loss profile of the norm landscape over the phase space allows for the overall norm to be conserved depending on the initial distribution. In general the overall norm will oscillate in time. The value of the norm $w(t, p, q)$ at a given point p and q returns to its initial value over a full rotation, because the evolution spends equal time in gain and loss regions. For illustration, consider an initial point on the right side of phase space, as it travels clockwise through the $q > 0$ plane, the intensity of the distribution will decay to a minimum. Then, as it travels through the region of phase space where $q < 0$, the intensity grows reaching its maximum intensity at $q = 0$. After one period the norm of the time-evolved point will be the same as the initial one. However, each trajectory has their own frequency, and each starting point along a trajectory corresponds to a different evolution of the norm, which gives rise to the characteristic whorl structure in figure 3.5. While the quantum Husimi dynamics deviates from the whorl structure of the classical evolution, the region of intensity for the time-evolved Husimi distribution corresponds to the classical norm landscape.

3.1.2 The Wigner Representation

Now let us consider the Wigner dynamics for which we need the Weyl symbol of the Kerr Hamiltonian (3.1). In position and momentum variables, this is given as

$$K^{(S)}(p, q) = \frac{1 - 2\hbar\beta}{2}(q^2 + p^2) + \frac{\beta}{4}(q^2 + p^2)^2 - i\epsilon q - \frac{\hbar}{2} + \frac{\hbar^2\beta}{4}, \quad (3.8)$$

where β and $\epsilon \in \mathbb{R}^+$ and I choose to set $\hbar = 1$.

I show an example of the quantum and classical dynamics of the Wigner distribution for the Hermitian Kerr Hamiltonian, $\epsilon = 0$, with an initial coherent state in Figure 3.6. This highlights an interesting property of non-linear systems: the negativity of the Wigner distribution may emerge in the quantum evolution even for a non-negative initial state [76]. I demonstrate this in the quantum dynamics depicted in the bottom row of Figure 3.6. For time $t > 0$ we see regions over the phase space that oscillate between positive and negative values. At the final time $t = \pi$ we see four localised non-negative regions aside regions of oscillating positive and negative values, this is a four-cat state. This feature of the quantum dynamics for the Wigner distribution is not present in the classical dynamics, where the Wigner distribution remains non-negative for all time [76].

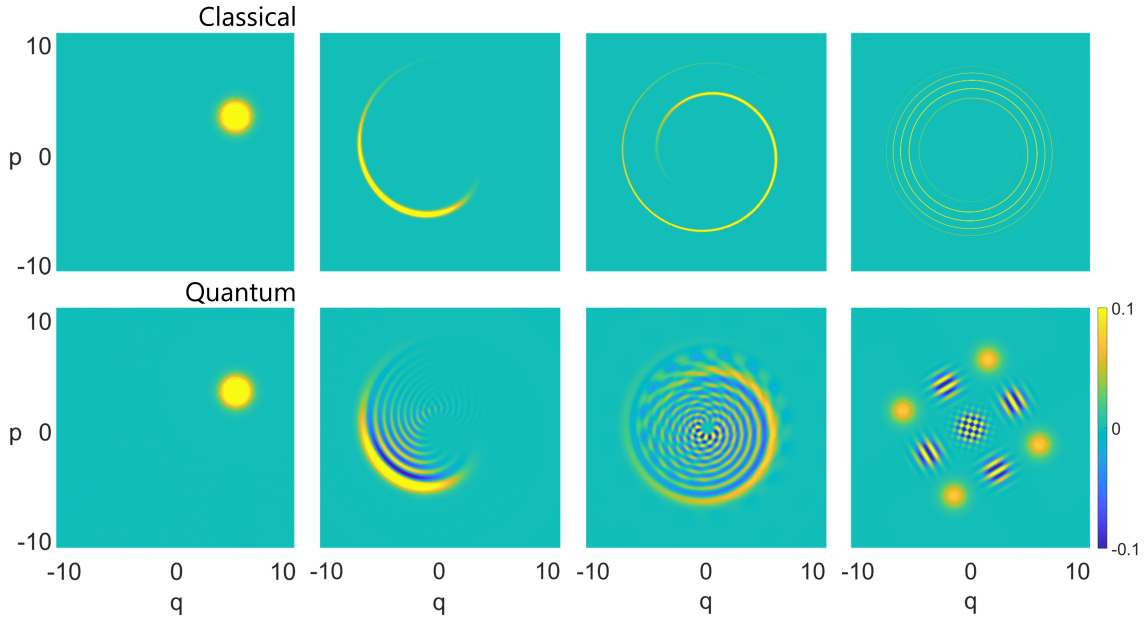


Figure 3.6: The classical (top row) and quantum (bottom row) dynamics of the Wigner distributions for the Kerr Anharmonic Oscillator for parameter $\beta = 0.25$ at times (from left to right) $t = 0$, $t = \frac{\pi}{10}$, $t = \frac{\pi}{4}$ and $t = \pi$, for the ground state initially displaced to $z_c = \frac{1}{\sqrt{2}}(5 + 3i)$.

The classical and quantum dynamics of the Kerr Hamiltonian for an initial third excited state is shown in Figure 3.7. The classical dynamics show a whorling structure around the elliptic fixed point at the origin. Furthermore, the concentric positive and negative regions of the initial distribution are preserved in the time-evolved distribution, even as it whorls. This is especially noticeable at time $t = \frac{\pi}{10}$.

Between the classical and quantum dynamics, there is some similarity for small time, as we see at $t = \frac{\pi}{10}$ where both Wigner distributions have curled around the origin. However, we do not see the concentric positive and negative regions in the quantum

dynamics which appear in the classical dynamics. Instead there are localised positive and negative regions of the distribution that make up a whorl structure.

Furthermore, in the quantum dynamics we see interference patterns that ripple around the origin. At time $t = \pi$ we see faint images of the initial distribution copied across the phase space in four locations, and between them are interference patterns. Beyond the times shown, the dynamics for the quantum system will repeat in reverse and at period $t = 2\pi$, we will see the resurgence of the initial distribution. All the while, the classical system will continue to stretch and whorl the initial distribution into an increasingly fine structure.

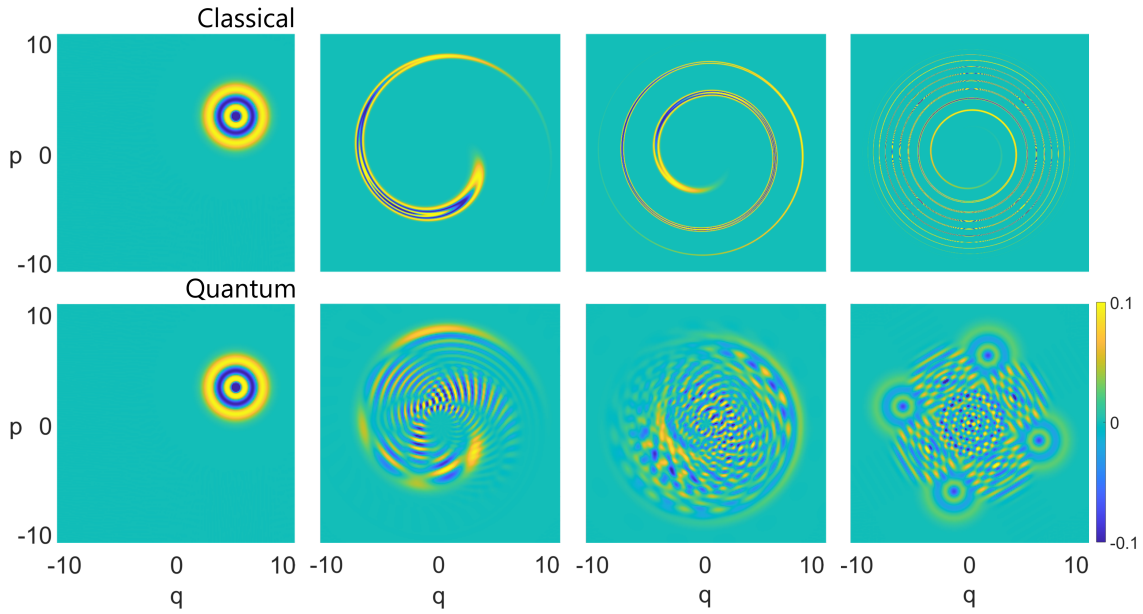


Figure 3.7: The classical (top row) and quantum (bottom row) dynamics of the Wigner distributions for the Kerr Anharmonic Oscillator for parameter $\beta = 0.25$ at times (from left to right) $t = 0$, $t = \frac{\pi}{10}$, $t = \frac{\pi}{4}$ and $t = \pi$, for the third excited state initially displaced to $z_c = \frac{1}{\sqrt{2}}(5 + 3i)$.

The phase space flow guiding the Wigner distribution for the PT-symmetric system is the same as that of the Hermitian Kerr oscillator, with trajectories depicted in the left plot of Figure 3.2. The quantum and classical dynamics of the time-evolved Wigner distribution for the non-Hermitian PT-symmetric case with $\epsilon = 1$ have been plotted in Figure 3.8.

The addition of the imaginary term in (3.8) term has a similar effect to the Wigner distribution as we saw for the Husimi distribution in Figure 3.4. The time-evolved Wigner distribution for both dynamics experiences damping for $q > 0$ and gain for $q < 0$, leading to high intensity values of the Wigner distribution in the top half

of phase space and low values on the bottom half. The motion of the distribution is unaffected by the non-Hermiticity and thus these modulations of intensities are a feature of the underlying norm landscape, which are depicted in Figure 3.9. The norm landscape develops a similar structure as Figure 3.5, with interlocking spirals with regions of high and low intensity values for $p > 0$ and $p < 0$ respectively. However, there is a greater contrast between these high and low intensity regions.

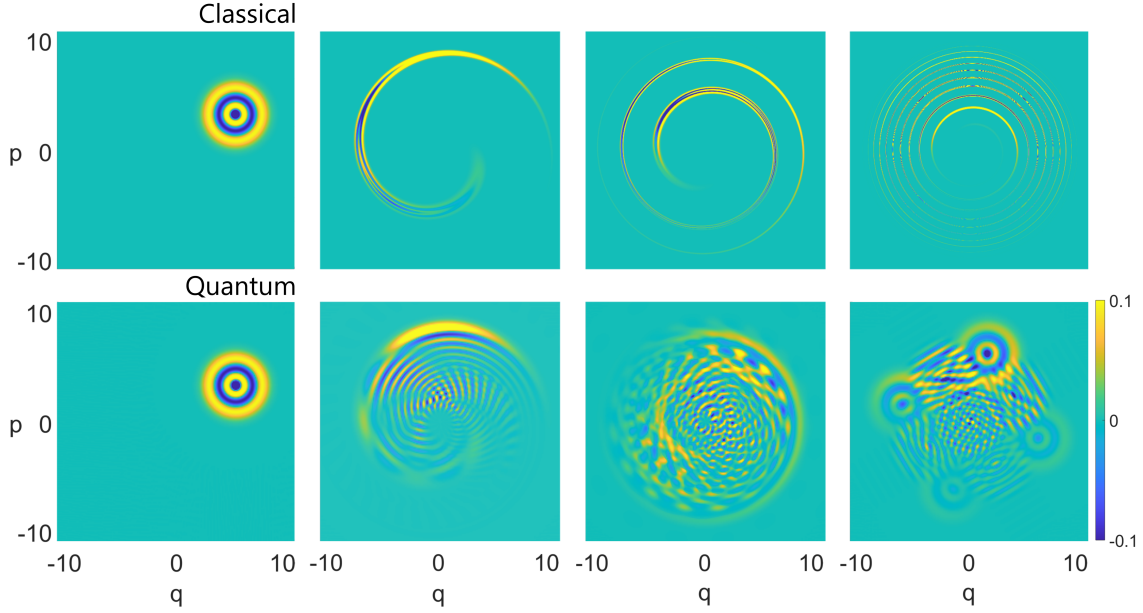


Figure 3.8: The classical (top row) and quantum (bottom row) dynamics of the Wigner distributions for the non-Hermitian PT-symmetric anharmonic oscillator with $\beta = 0.25$ and $\epsilon = 1$ at times (from left to right) $t = 0$, $t = \frac{\pi}{10}$, $t = \frac{\pi}{4}$ and $t = \pi$, for the third excited state initially displaced to $z_c = \frac{1}{\sqrt{2}}(5 + 3i)$.

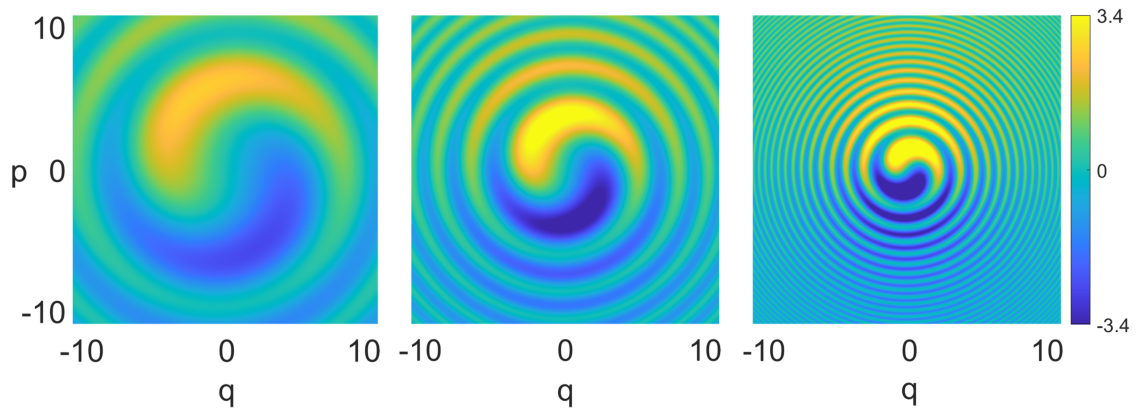


Figure 3.9: The norm landscapes corresponding to the Wigner distribution on a logarithmic scale for the non-Hermitian PT-symmetric anharmonic oscillator with $\beta = 0.25$ and $\epsilon = 1$ at times (from left to right) $t = \frac{\pi}{10}$, $t = \frac{\pi}{4}$ and $t = \pi$.

3.2 Tilted Mexican Hat Potential

In this section I study a non-Hermitian tilted version of a Mexican Hat (Higgs) Hamiltonian [52, 77], given by

$$\hat{K} = -\hbar\hat{a}^\dagger\hat{a} - i\hbar\gamma\hat{a}^\dagger\hat{a} + \hbar^2\beta\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \sqrt{\hbar}\frac{\delta}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}), \quad (3.9)$$

where β, δ and $\gamma \in \mathbb{R}^+$. The terms scaled by real-valued coefficient δ (i.e. the δ -term) is proportional to the position operator $\hat{q}$ and thus, linearly increases the potential for $q > 0$ and decreases it for $q < 0$. The imaginary part of the Hamiltonian describes a dampening of strength proportional to γ that, on its own, would dampen the system into the harmonic oscillator ground state.

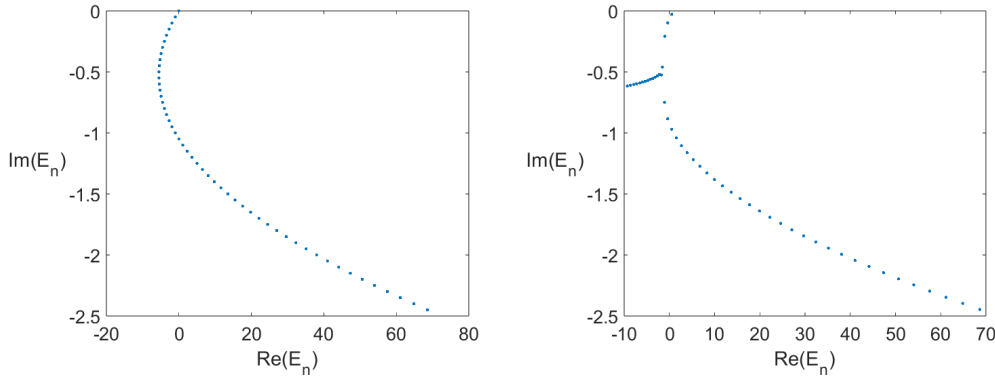


Figure 3.10: The eigenvalue spectrum plotted on a complex axis for the non-Hermitian Mexican Hat potential with $\gamma = 0.05$, $\beta = 0.05$ and $\delta = 0$ (left) and $\delta = 1$ (right).

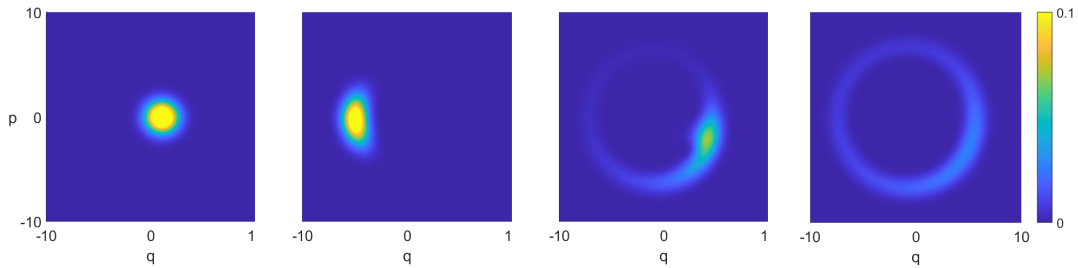


Figure 3.11: Husimi distribution of eigenstates with eigenvalues $E_n = 0.51 - 0.28i$ (left), $E_n = -10.03 - 0.62i$ (middle left), $E_n = -2.23 - 0.52i$ (middle right) and $E_n = -1.79 - 0.52i$ (right).

The spectrum for the non-Hermitian system is complex valued. In Figure 3.10 the 50 eigenvalues with the lowest real parts have been plotted for $\hbar = 1$, $\beta = 0.05$, $\gamma = 0.05$, $\delta = 0$ (left) and $\delta = 1$ (right). For $\delta = 0$, the Hamiltonian is diagonal in

the harmonic oscillator basis, with eigenvalues $E_n = \hbar^2 \beta n^2 - \hbar(1 + i\gamma)n$. Thus, the eigenvalues lie on a curve in the complex plane, which is given by

$$\text{Re}(E) = \frac{\beta}{\gamma^2} \text{Im}(E)^2 + \frac{1}{\gamma} \text{Im}(E), \quad (3.10)$$

with $\text{Im}(E) < 0$.

For the case that $\delta = 1$, we see that eigenvalues are organised around three branches in the complex plane: One set of eigenvalues with $\text{Re}(E_n) < -2$, another for $-0.5 < \text{Im}(E_n) < 0$, and the last where $\text{Re}(E_n) > -2$ and $\text{Im}(E_n) < -0.5$. The first two branches have a finite number of eigenvalues, whilst the last branch holds infinitely many eigenvalues. In Figure 3.11 we plot the Husimi distribution of four selected eigenstates. The left plot of Figure 3.11 shows the most stable state, i.e., the eigenstate that corresponds to the eigenvalue with the maximum imaginary part, which belongs to the top branch. The middle left plot shows the "ground state", i.e., the eigenstate with the minimum value of the real part. The middle right plot shows an eigenstate corresponding to the point at which the three branches meet, and the right plot shows an eigenstate corresponding to an eigenvalue of the bottom branch, specifically, the eigenvalue with the largest real part. The most stable state resembles a coherent state, which is localised around a point $p = 0, q > 0$ in phase space. The "ground state" on the other hand is located around a point $p = 0, q < 0$, and has a slightly bent and squeezed shape. The state at the intersection of the branches has a high peak and a long ridge encircling the points at which the ground and most stable states are localised. The "higher excited" state shows a ring structure akin to a higher excited harmonic oscillator state, which can be understood by the fact that the β -term dominates for large values of the (real part of the) energy. We do, however, observe a non-uniform distribution of intensity along the ring structure, which is due to the δ -term in the Hamiltonian. The structure of the eigenstates is closely related to the classical phase space trajectories.

The Wick symbol of the Hamiltonian (3.9) is given by

$$K^{(N)}(z, z^*) = -\hbar(1 + i\gamma)z^*z + \hbar^2\beta z^{*2}z^2 + \sqrt{\hbar}\frac{\delta}{\sqrt{2}}(z^* + z). \quad (3.11)$$

In the coherent state approximation, the trajectories for the Hamiltonian are generated by

$$\frac{dz}{dt} = -\gamma z + i \left(z - 2\hbar\beta z^{*2}z - \frac{\delta}{\sqrt{\hbar}\sqrt{2}} \right). \quad (3.12)$$

The Hermitian Hamiltonian has three fixed points. For $\hbar = 1, \beta = 0.05, \gamma = 0$ and $\delta = 1$ these fixed points are a global minimum located at $(p_0, q_0) = (0, -4.907)$, a local maximum at $(p_1, q_1) = (0, 1.060)$ and a local minimum/saddle at $(p_2, q_2) = (0, 3.847)$. The fixed points for the non-Hermitian Mexican Hat, where $\gamma = 0.05$, lie in similar regions with $(p_0, q_0) = (1.198, -4.746)$, $(p_1, q_1) = (0.056, 1.056)$ and $(p_2, q_2) = (0.746, 3.790)$. The trajectories for all versions of the Mexican Hat Hamiltonian are depicted in Figure 3.12.

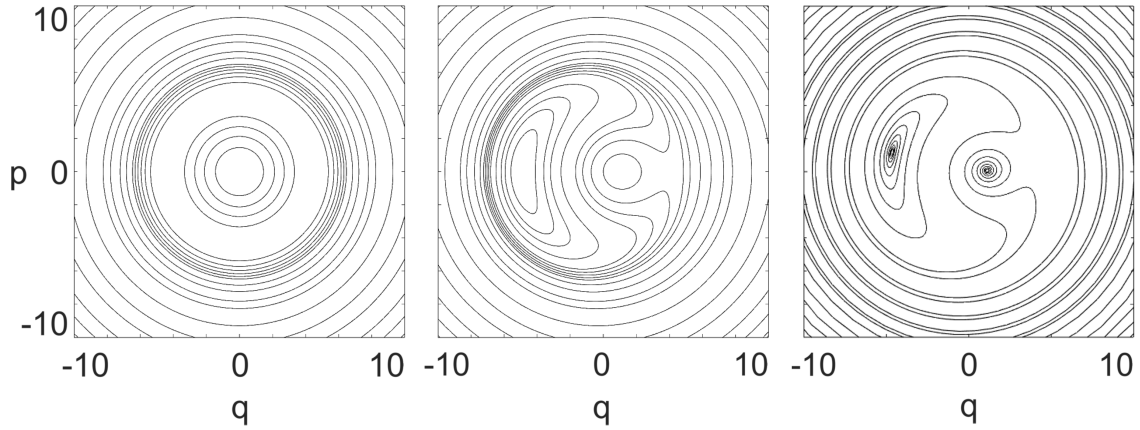


Figure 3.12: The phase space trajectories for the Mexican Hat potential (left), the tilted Mexican Hat potential (middle) and the non-Hermitian Mexican Hat potential (right) for parameters $\hbar = 1, \beta = 0.05, \delta = 1$ and $\gamma = 0.05$.

The left plot of Figure 3.12 depicts the Mexican Hat with $\gamma = \delta = 0$ where the maximum at the origin is surrounded by elliptic trajectories and a radial band for the minima. The middle plot shows the titled Mexican Hat $\gamma = 0, \delta = 1$, with elliptic trajectories around the global minimum and the local maximum separated by the saddle point orbit. The right plot shows the trajectories of the non-Hermitian Mexican Hat $\gamma = 0.05, \delta = 1$, where, in the coherent state approximation, trajectories spiral into two stable spiral fixed points. We observe that the Husimi distributions of the eigenstates depicted in Figure 3.11 are organised according to the fixed point structure seen in the middle plot of Figure 3.12. Note that the most stable state corresponds to the local maximum of the energy in the Hermitian case.

From Hamilton's equations of motion for the Weyl symbol for the Hermitian and non-Hermitian systems, the trajectories are similar to the middle plot of Figure 3.12 but in both cases for our chosen parameters the fixed points are $(p_0, q_0) = (0, -5.091)$, $(p_1, q_1) = (0, 0.948)$, which are both elliptic, and $(p_2, q_2) = (0, 4.144)$.

3.2.1 The Husimi Representation

I first consider the Hermitian Mexican Hat potential (3.11) with $\gamma = \delta = 0$. We show snapshots of the quantum and classical dynamics of the Husimi distribution for an initial state $\hat{D}(z_c)|2\rangle$ in Figure 3.13. The classical dynamics of the Mexican Hat potential has a stretching and whorling [24] behaviour, that we have already seen in the previous example. The classical Husimi distribution maintains a ring structure for the displaced excited state $\hat{D}(z_c)|2\rangle$ even though the ring is stretched along a clockwise phase space flow. By $t = 8$ we observe the typical fine whorl structure. For short time the correspondence between the classical and quantum dynamics is clear. Until $t = 2$ the quantum dynamics sees the same clockwise motion that attempts to stretch the distribution around the fixed point. However, even for short time $t = 0.5$ the quantum dynamics does not develop fine structures and quickly loses the ring structure of the initial distribution. By time $t = 8$ the quantum dynamics are drastically different to the classical dynamics, as we see the Husimi distribution spread out and encircle the origin, whilst developing many holes from self interference. Note that for times $t = n\pi$ where $n \in \mathbb{Z}^+$ we would observe (partial) revivals, as discussed in the previous example.

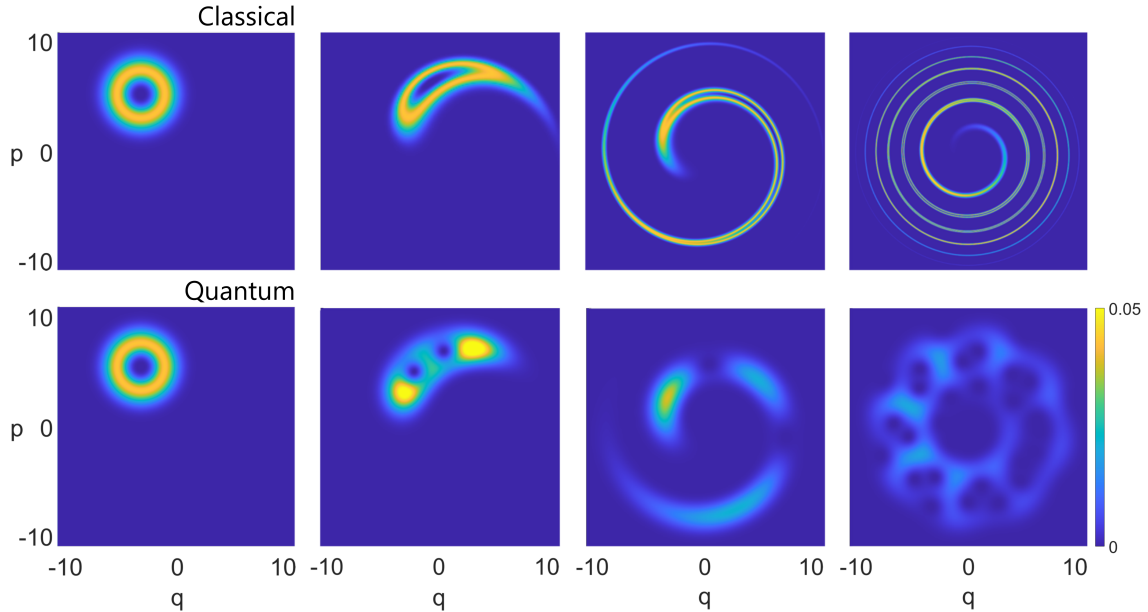


Figure 3.13: The classical (top row) and quantum (bottom row) dynamics of the Husimi distributions for the Hermitian Mexican Hat potential for parameters $\hbar = 1, \beta = 0.05$ and $\delta = 0$ at times (from left to right) $t = 0, t = 0.5, t = 2$ and $t = 8$, for the second excited state initially displaced to $z_c = \frac{1}{\sqrt{2}}(-3 + 5i)$.

Now consider the tilted Mexican Hat with a non-Hermitian potential (3.11). The time-evolved Husimi distribution is of the form (2.5) with trajectories generated by

$$\frac{dz}{dt} = \gamma z + i \left(z - 2\hbar\beta z^{*2}z - \frac{\delta}{\sqrt{\hbar}\sqrt{2}} \right). \quad (3.13)$$

In contrast to the coherent state approximation, the trajectories spiral along which the Husimi distribution is transported outwards away from the local and global minima of the system.

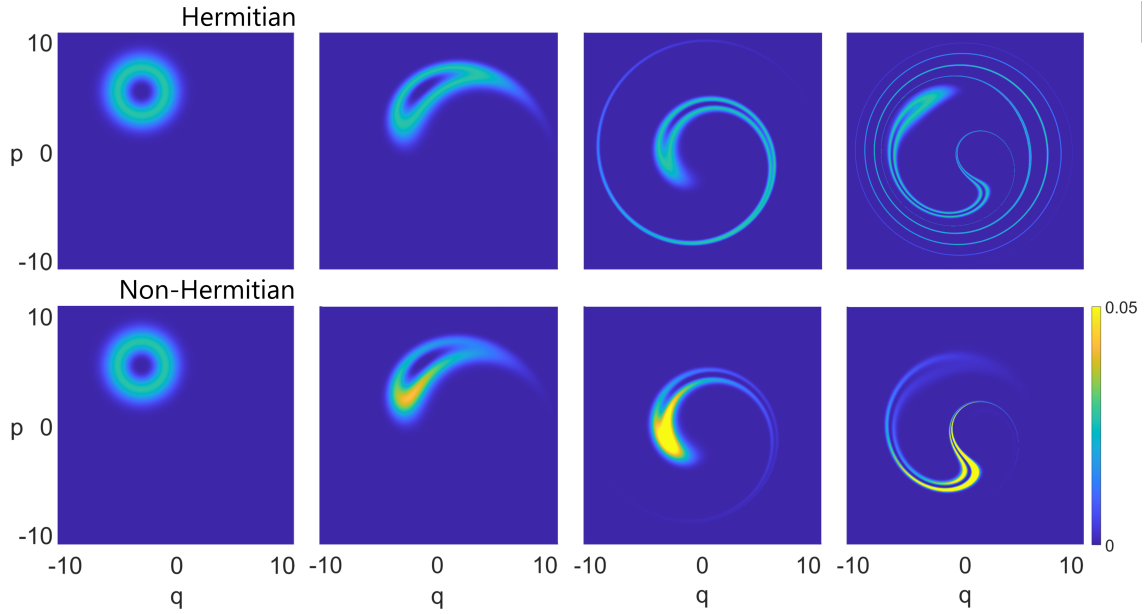


Figure 3.14: Husimi distributions following classical dynamics are shown for the Hermitian (top row) and non-Hermitian (bottom row) Mexican Hat anharmonic oscillator, for $\hbar = 1, \beta = 0.05$ and $\delta = 1$ and $\gamma = 0$ (top) and $\gamma = 0.05$ (bottom) at times (left to right) $t = 0, t = 0.5, t = 2$ and $t = 8$, for an initially displaced second excited state localised at $z_c = \frac{1}{\sqrt{2}}(-3 + 5i)$.

The classical dynamics of the Husimi distribution for a Hermitian and a non-Hermitian tilted Mexican Hat potentials are shown in Figure 3.14. Up to $t = 2$, we see clearly that in both cases the Husimi distribution rotates and whorls, similar to the non-tilted Mexican Hat in Figure 3.13. In contrast to Figure 3.13, the tilted Hamiltonians curl around the fixed points of the system, in agreement with the middle and right plots of Figure 3.12. The Hermitian system has a uniformity in intensity values, whilst the non-Hermitian system from $t = 0.5$ has higher intensity values on localised regions. The regions in which the classical dynamics localise correspond to the larger values in the norm landscape, which is plotted in Figure 3.15.

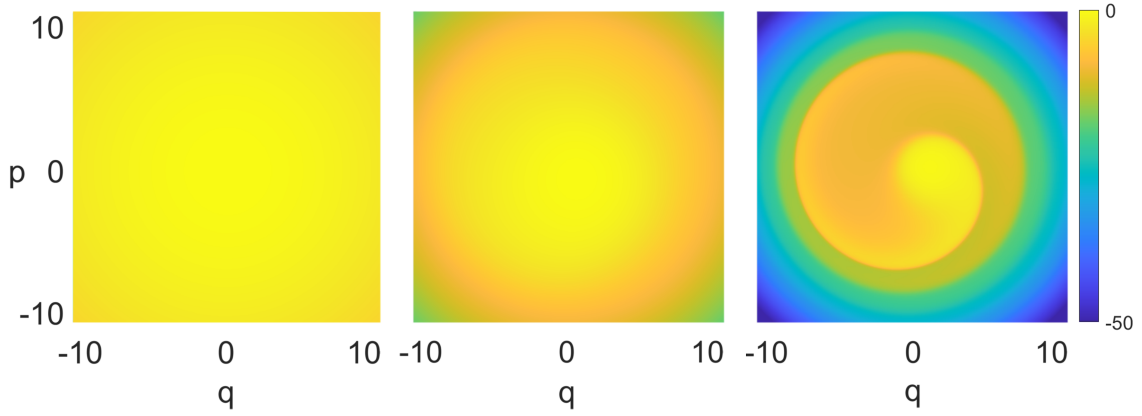


Figure 3.15: The norm landscapes corresponding to the Husimi distribution on a logarithmic scale for the Mexican Hat anharmonic oscillator at times (from left to right) $t = 0.5$, $t = 2$ and $t = 8$.

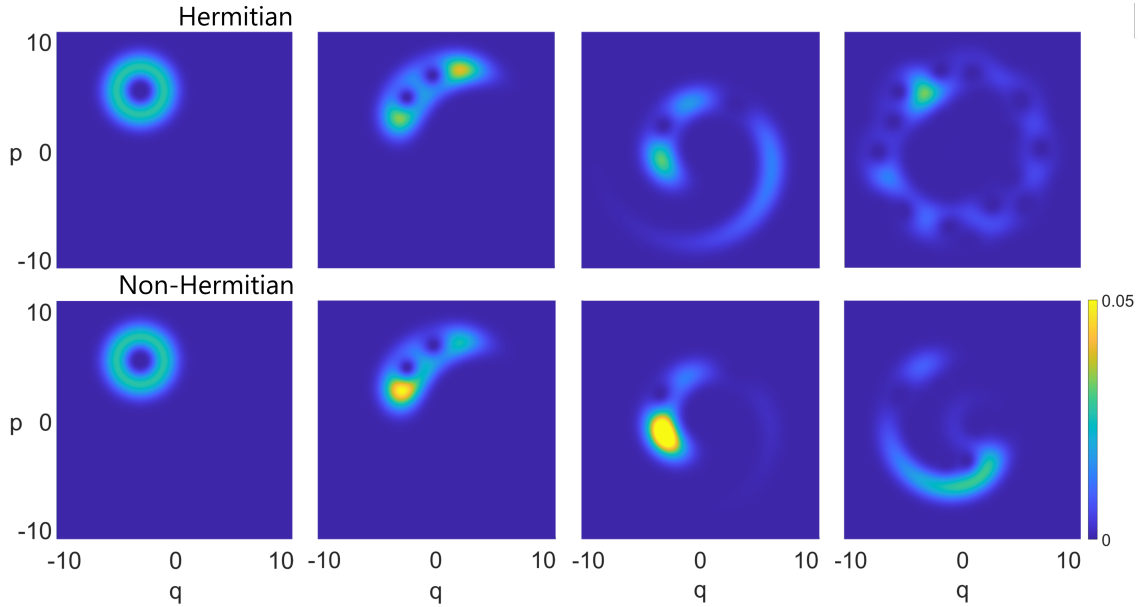


Figure 3.16: The Husimi distributions following the quantum dynamics are shown for the Hermitian (top row) and non-Hermitian (bottom row) Mexican Hat anharmonic oscillator. Both systems have $\hbar = 1$, $\beta = 0.05$ and $\delta = 1$, the Hermitian system has $\gamma = 0$, and the non-Hermitian system has $\gamma = 0.05$ at times (left to right) $t = 0$, $t = 0.5$, $t = 2$ and $t = 8$, for an initially displaced second excited state localised at $z_c = \frac{1}{\sqrt{2}}(-3 + 5i)$.

The norm landscape localises around the region of the stable fixed point that belongs to the most stable quantum state (even though this belongs to a local maximum of the energy in the Hermitian case). Thus, the parts of the Husimi distribution that follow trajectories that spiral into that fixed point decrease the least in intensity. Over longer times then plotted here, the surviving portion of the Husimi distribution

is localised on this fixed point resembling the most stable state depicted in the left plot of Figure 3.11.

Lastly, I present the quantum dynamics of the Husimi distribution in Figure 3.16 for the Hermitian and non-Hermitian Mexican Hat. As we saw in the classical dynamics, the quantum dynamics between the Hermitian and non-Hermitian systems develop in very similar manners for short time, up until $t = 2$. We also observe the uniformity of intensity along the Husimi distribution in the Hermitian case, whilst the non-Hermitian case sees the localised regions of intensity, which we identify as resulting from the norm landscape in the classical case. At time $t = 8$, the Husimi distribution for non-Hermitian case still share some similarities with the corresponding classical case. Both the classical and quantum dynamics sees the Husimi distribution spread along similar areas of phase space. Furthermore, the quantum dynamics does not result in the Husimi distribution developing many holes and aligns more with the classical dynamics which can never develop any holes. These similarities are not present in the quantum dynamics of the Hermitian system at $t = 8$. The quantum and classical correspondence is stronger for longer times in the non-Hermitian system than for the Hermitian counterpart.

3.2.2 The Wigner Representation

For the Wigner dynamics I consider the Weyl symbol of the non-Hermitian tilted Mexican Hat,

$$K^{(S)}(p, q) = -\frac{1 + 2\hbar\beta + i\gamma}{2}(p^2 + q^2) + \frac{\beta}{4}(p^2 + q^2)^2 + \delta q + \frac{\hbar(1 + i\gamma)}{2} + \frac{\hbar^2\beta}{4} \quad (3.14)$$

where $\beta, \delta \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^+$.

To begin, I consider the Mexican Hat Hamiltonian with $\gamma = \delta = 0$, with the classical and quantum dynamics of the Wigner distribution shown in Figure 3.17 for the same initial state considered in the Husimi representation. The classical dynamics are very similar to the classical dynamics for the Husimi distribution; as the structure of the initial distribution $\hat{D}(z_c)|2\rangle$ is maintained whilst the distribution is stretched and whorls over time. For the Wigner distribution of the displaced second excited state, the inner negative region is preserved such that, even at later time $t = 8$ we see an inner negative band within the fine structure. For the quantum dynamics, up to $t = 2$ there remain some high intensity regions that share similar structures to the classical dynamics. However, outside of these regions the Wigner distribution develops ripples of alternating positive and negative values. This is a uniquely

quantum behaviour, as we saw in the section 3.1.2. For large time, at $t = 8$, a lace-like structure develops, further distinguishing the distribution under quantum dynamics from the classical counterpart.

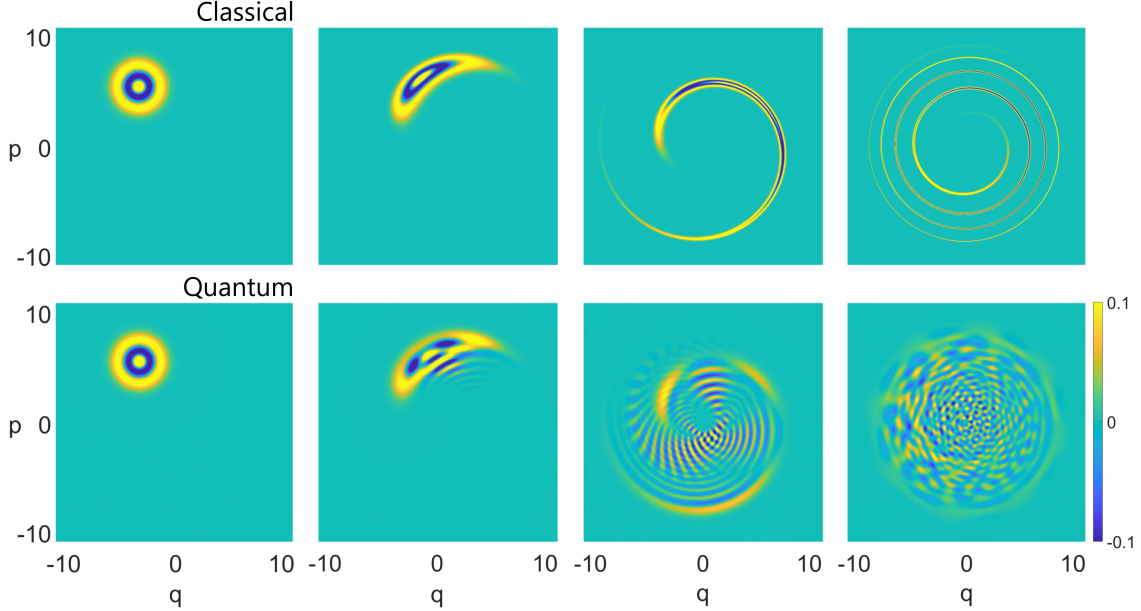


Figure 3.17: The classical (top row) and quantum (bottom row) dynamics of the Wigner distributions for the Hermitian Mexican Hat potential for parameters $\hbar = 1, \beta = 0.05$ and $\delta = 0$ at times (from left to right) $t = 0, t = 0.5, t = 2$ and $t = 8$, for the second excited state initially displaced to $z_c = \frac{1}{\sqrt{2}}(-3 + 5i)$.

The time-evolved Wigner distribution follows the phase space flow generated by Hamilton's equations of motion

$$\frac{dq}{dt} = -(1 + \hbar\beta)p + \beta p(p^2 + q^2), \quad (3.15)$$

$$\frac{dp}{dt} = (1 + \hbar\beta)q - \beta q(q^2 + p^2) - \delta. \quad (3.16)$$

For both the Hermitian and non-Hermitian systems, the trajectories follow similar orbits to those in the middle plot of Figure 3.12.

The classical dynamics of the Hermitian and non-Hermitian tilted Mexican Hat are plotted in Figure 3.18. Again, we see that the negativity of the initial distribution is preserved over time and is present at fine-scale at $t = 8$. At this time, we can see the tail of both distributions begin to wrap around the region of the local maximum of the Hamiltonian. For the non-Hermitian case, the Wigner distribution is highly peaked with the tails that appeared in the Hermitian case suppressed by the damping. This corresponds with the area where the norm landscape localises.

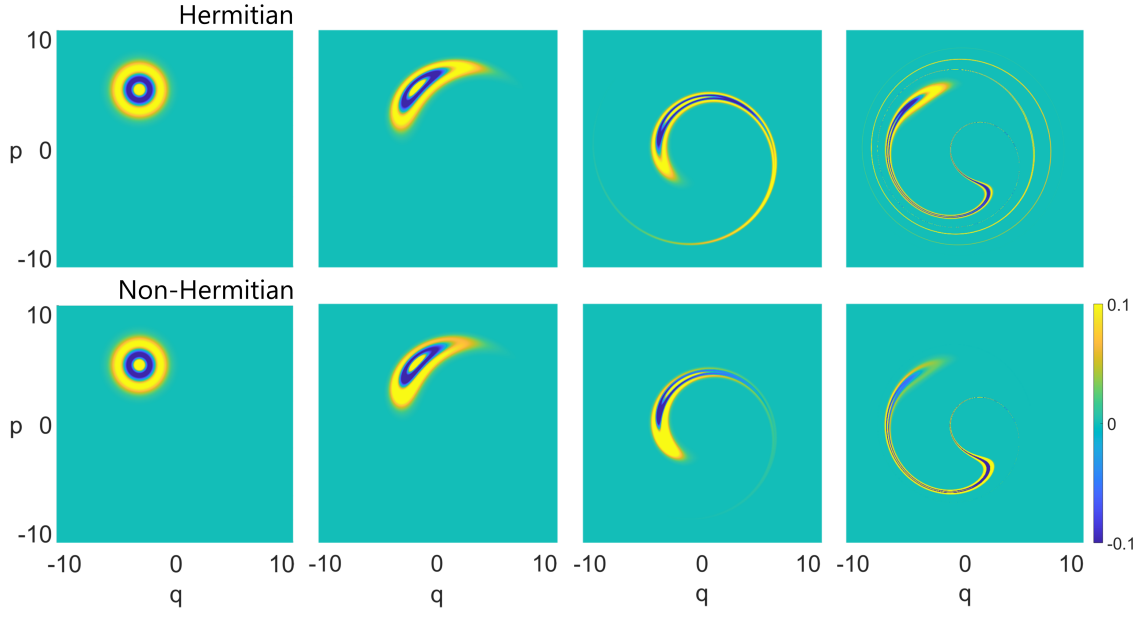


Figure 3.18: The Wigner distributions following the quantum dynamics are shown for the Hermitian (top row) and non-Hermitian (bottom row) Mexican Hat anharmonic oscillator. Both systems have $\hbar = 1, \beta = 0.05$ and $\delta = 1$, the Hermitian system has $\gamma = 0$, and the non-Hermitian system has $\gamma = 0.05$ at times (left to right) $t = 0, t = 0.5, t = 2$ and $t = 8$, for an initially displaced second excited state localised at $z_c = \frac{1}{\sqrt{2}}(-3 + 5i)$.

The norm landscape is shown in Figure 3.19, and is similar to the norm landscape for the Husimi distribution, however the regions of highest and lowest damping are more defined at $t = 8$ than in 3.15.

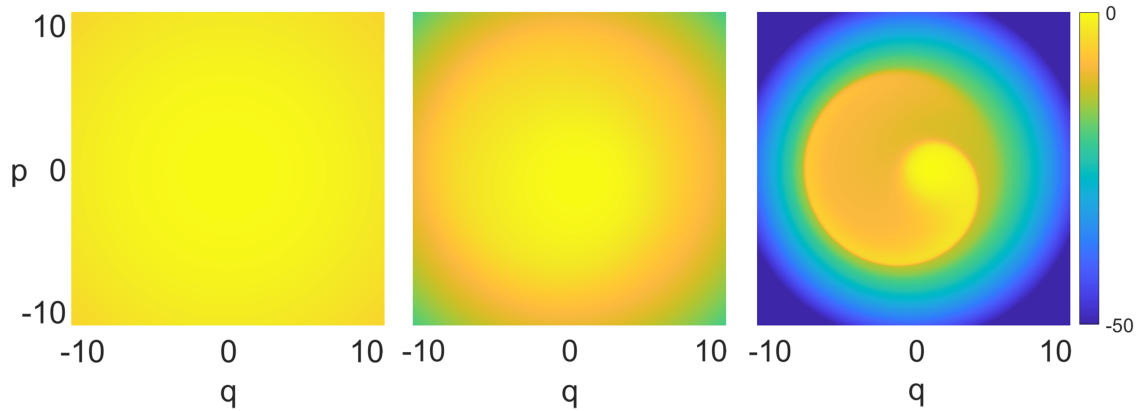


Figure 3.19: The norm landscapes corresponding to the Wigner distribution on a logarithmic scale for the Mexican Hat anharmonic oscillator at times (from left to right) $t = 0.5, t = 2$ and $t = 8$.

The quantum dynamics of the Hermitian and non-Hermitian tilted Mexican Hat are plotted in Figure 3.20. For short times, (e.g. $t = 0.5$), the time-evolved

Wigner distribution is similar for the Hermitian and non-Hermitian systems. As time increases, the Hermitian system sees the Wigner distribution stretch and rotate around the phase-space, interfering with itself and creating ripples. Meanwhile, the non-Hermitian system localises around similar regions of phase-space to where the Wigner distribution for the classical dynamics localises. This is particularly noticeable at $t = 2$. At $t = 8$, the quantum dynamics for the Hermitian system see the distribution spread across the phase space and develop a similar lace-pattern to that observed in Figure 3.17. In contrast, whilst the Wigner distribution for the non-Hermitian system has some interference patterns, it is largely localised on the region that the corresponding norm landscape has high values. Again, as for the Husimi distribution, there is a stronger quantum-classical correspondence for the non-Hermitian system than for the Hermitian system.

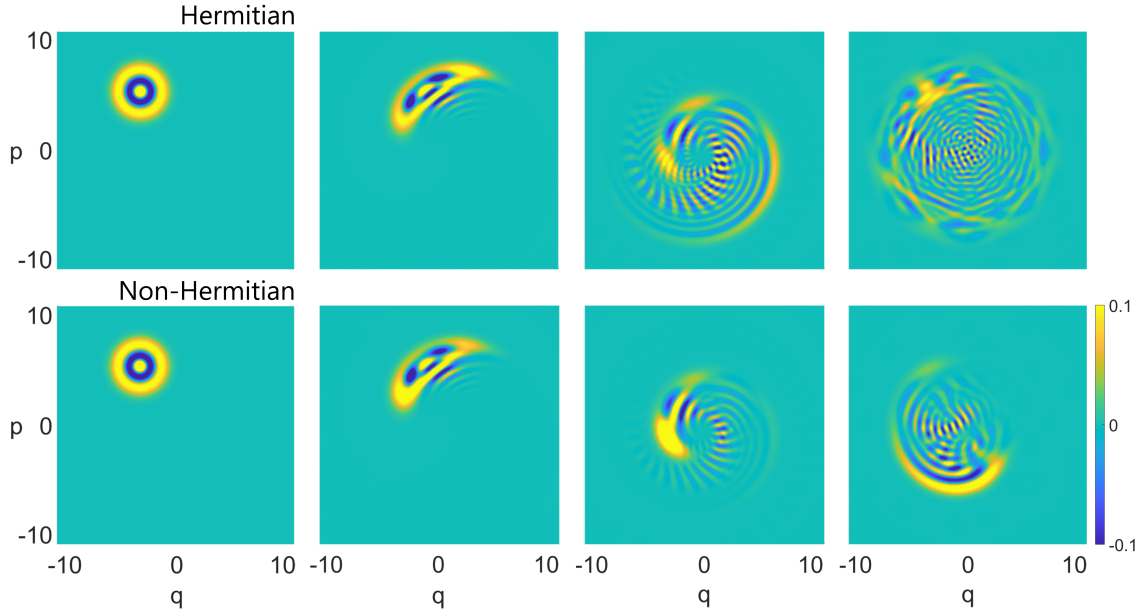


Figure 3.20: The Husimi distributions following the quantum dynamics are shown for the Hermitian (top row) and non-Hermitian (bottom row) Mexican Hat anharmonic oscillator. Both systems have $\hbar = 1, \beta = 0.05$ and $\delta = 1$, the Hermitian system has $\gamma = 0$, and the non-Hermitian system has $\gamma = 0.05$ at times (left to right) $t = 0$, $t = 0.5$, $t = 2$ and $t = 8$, for an initially displaced second excited state localised at $z_c = \frac{1}{\sqrt{2}}(-3 + 5i)$.

In this chapter, I generated and discussed the Husimi and Wigner distribution dynamics of two non-Hermitian systems with non-linear classical flows: a PT-symmetric Kerr Hamiltonian and a non-Hermitian tilted Mexican Hat potential. For each system, the quantum and classical dynamics of the Husimi and Wigner distributions were numerically generated at different times, along with the dynam-

ics of the Hermitian counterparts for comparison. We found that in both examples, the correspondence between the classical and quantum dynamical evolution of the system was stronger in the non-Hermitian case than in the Hermitian case. This correspondence is primarily the result of the norm landscape, which is a dominant component of the classically evolved distributions in the semiclassical limit that underlies the quantum evolution. Throughout this chapter, we have consistently observed that the Husimi and Wigner distributions exhibit higher intensity values in regions corresponding to the peaks of the norm landscape.

For the PT-symmetric system, the norm landscape does not alter the phase space dynamics of the non-Hermitian system, as it exhibits the same structure as its Hermitian counterpart. However, the norm landscape does create balanced regions of amplification and damping that affect the intensity of the Husimi and Wigner distributions. Meanwhile, for the non-Hermitian tilted Mexican Hat potential, both distributions tend towards the region around the fixed point corresponding to the most stable quantum state in which the norm landscape localises. This is seen in both the classical and quantum dynamics of the non-Hermitian system. This chapter showcases applications of the results from chapter 2 and demonstrates the potential to be a high-impact tool in the analysis of non-trivial non-Hermitian dynamics. A natural extension of this work is to explore the correspondence of the quantum and classical dynamics of these non-linear systems for decreasing values of $\hbar$ as this parameter approaches zero in the semiclassical limit. In the next chapter we will examine a higher dimensional non-Hermitian system, a billiard model with internal loss.

Chapter 4

Billiard Models with Internal Loss

4.1 Introduction

So far I have considered only one-dimensional examples. For practical applications it is important to move beyond this, and in particular to consider systems with chaotic or mixed dynamics. A standard toy model for dynamical systems and quantum-classical correspondence are two-dimensional mathematical billiards. The classical billiard model [78] describes the motion of a free point particle within a two-dimensional spatial region enclosed by a hard wall boundary. The trajectories of the classical particle consist of frictionless free motion along straight lines, between elastic collisions with the boundary. The resulting dynamics may be regular, chaotic or mixed, depending on the shape of the billiard boundary [79]. In this chapter we consider three shapes, a circular and an elliptic billiard with regular dynamics, and an oval shape with mixed dynamics.

The quantum counterpart is the quantum billiard model [80, 81], described by the Schrödinger equation for a free particle, within the interior of the billiard boundary with suitable boundary conditions. Viewing the boundary as an infinite high potential wall leads to the typical Dirichlet boundary condition where the wave function vanishes on the boundary. The quantum billiard is a toy model of quantum optical systems in QED cavities [82–86] and quantum dots [87–90]. Quantum billiards with classically chaotic counterparts are standard systems in the field of "quantum chaos" [91–93].

In recent years, open billiard models with partial or total losses at the boundary have been studied intensely, particularly in the context of optical applications [94–97]. In [98] a rectangular quantum billiard with piecewise constant imaginary parts in a stripe-pattern has been investigated. Regions of total absorption in the interior of the billiard have been introduced in [99], and the case that the billiard has an internal amplifier has been studied in [100].

In this chapter I consider a billiard with an interior region of partial loss, modelled by a constant imaginary potential within the loss region. I focus on investigate the dynamics of the corresponding classical system. We begin with a short introduction to classical two-dimensional Birkhoff billiards and the corresponding billiard maps on a two-dimensional Poincaré phase space. Then I will introduce the non-Hermitian generalisation considered here, and introduce the concept of an intensity landscape on the Poincaré phase space, which we analyse in some detail for a circular, an elliptic, and an oval billiard with a central circular region of finite absorption.

I follow this chapter with an outlook chapter 5 which includes preliminary results of applying the semiclassical approach developed in the thesis to investigate the dynamical quantum-classical correspondence of the non-Hermitian model. The results on the classical billiards have been submitted for publication [18].

4.2 Classical Billiard Dynamics

A classical billiard system is fully specified by its boundary curve. In what follows, we consider only Birkhoff billiards which have convex boundaries [101]. The dynamics of a billiard can be regular, chaotic or mixed [102–104]. Integrable billiards [105], such as elliptic billiards, lead to regular motion whilst chaotic motion is characterised by the exponential divergence between two trajectories with neighbouring initial conditions. Mixed motion features trajectories that are either regular or chaotic depending on their initial conditions. Often, smooth convex billiards produce mixed dynamics. The boundary curve of a Birkhoff billiard can be expressed in polar coordinates as the radius $r(\phi)$ in dependence on the polar angle $\phi \in (0, 2\pi)$.

The trajectory of a free particle can be described entirely by the points at which the particle hits the boundary and the corresponding angles of reflection. An example trajectory for an elliptic billiard is show in figure 4.1. The Poincaré-Birkhoff coordinates [106], $S \in (0, 1)$, the arc length of a point of impact along the billiard boundary, and $p \in (-1, 1)$, the tangential velocity given by the cosine of the angle of reflection $p(\alpha) = \cos(\alpha)$, span a reduced phase space, fully encoding the trajectories.

The dynamics are given by an iterative map T between the Poincaré coordinates (S_n, p_n) of a collision, i.e. the n -th iteration, to those at the next, (S_{n+1}, p_{n+1}) , i.e. the $n + 1$ -th iteration. The dynamics on phase space are displayed as Poincaré diagrams that plot the set of (S_n, p_n) for multiple trajectories.

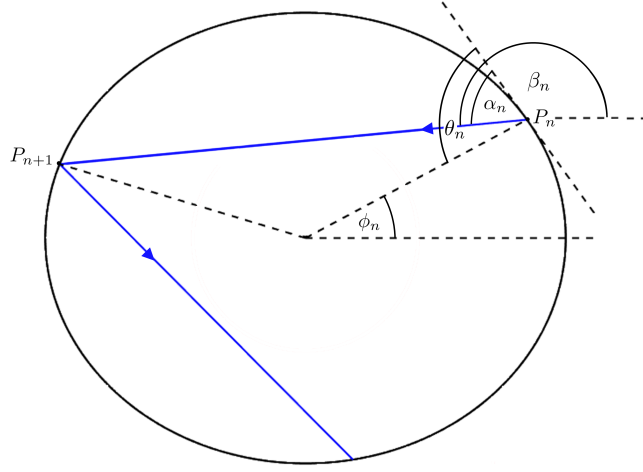


Figure 4.1: Illustration of two segments of a trajectory with relevant angles.

I will now summarise how the map T can be deduced from the billiard boundary curve, closely following reference [107], and using the circular billiard as a simple example. The normalised arc length along the billiard boundary up to the point with polar angle ϕ , measured from the reference point $\phi = 0$, is given by

$$S(\phi) = \frac{1}{L} \int_0^\phi d\phi' \sqrt{r^2(\phi') + \left(\frac{dr(\phi')}{d\phi'} \right)^2}, \quad (4.1)$$

where L is the circumference of the billiard boundary,

$$L = \int_0^{2\pi} d\phi \sqrt{r^2(\phi) + \left(\frac{dr(\phi)}{d\phi} \right)^2}. \quad (4.2)$$

Given the Poincaré coordinates of the n -th boundary collision $(S(\phi_n), p(\alpha_n))$, to find the coordinates of the $(n + 1)$ th collision we first need to determine the angles θ_n and β_n (see Figure 4.1). The angle θ_n is given by

$$\theta_n = \text{mod}(\arctan \left(-r(\phi_n) \left(\frac{dr(\phi_n)}{d\phi_n} \right)^{-1} \right), \pi), \quad (4.3)$$

and the angle β_n is given by

$$\beta_n = \text{mod}(\pi + \phi_n + \alpha_n - \theta_n, 2\pi). \quad (4.4)$$

We can now construct the line $l(\phi)$ of the trajectory from the point P_n

$$l(\phi) = r(\phi_n) \frac{\sin(\beta_n - \phi_n)}{\sin(\beta_n - \phi)}, \quad (4.5)$$

that will intersect the boundary of the billiard at P_{n+1} . Thus, the roots of the equation

$$l(\phi) - r(\phi) = 0, \quad (4.6)$$

are the polar angles associated to the two collisions, ϕ_n and ϕ_{n+1} . In the common case that there is no analytic solution for (4.6), a numerical solver can be utilised. In this thesis, this has been implemented on *MATLAB* using *vpasolve*.

From ϕ_n , ϕ_{n+1} , θ_n and θ_{n+1} we can calculate the reflection angle α_{n+1} as

$$\alpha_{n+1} = \text{mod}(\phi_{n+1} - \phi_n + \theta_n - \theta_{n+1} - \alpha_n, \pi), \quad (4.7)$$

from which we then deduce $S_{n+1} = S(\phi_{n+1})$ and $p_{n+1} = \cos(\alpha_{n+1})$.

As a short illustrative example, let us consider a billiard with circular boundary. The radius of the circle billiard is constant, given by

$$r_{\text{circle}}(\phi) = R, \quad (4.8)$$

where $R \in \mathbb{R}^+$, and the arc length is given by $S = \frac{\phi}{2\pi}$. Consider the initial polar co-ordinate and angle of reflection (ϕ_0, α_0) . From (4.3) and (4.4) we find $\theta_0 = \phi_0 - \frac{\pi}{2}$ and $\beta_n = \frac{\pi}{2} + \alpha_0$. The line $l(\phi)$ connecting P_0 and P_1 is given by

$$l(\phi) = \frac{R \sin(\frac{\pi}{2} + \alpha_0 - \phi_0)}{\sin(\frac{\pi}{2} + \alpha_0 - \phi)}. \quad (4.9)$$

Then by solving (4.6) for ϕ_1 we find $\phi_1 = \text{mod}(\phi_0 + 2\alpha_0, 2\pi)$, thus from (4.7) we see that the reflection angle is constant $\alpha_1 = \alpha_0$.

By induction it can then be shown that

$$\theta_n = \theta_0 + 2n\alpha_0 \quad (4.10)$$

$$\beta_n = \frac{\pi}{2} + \alpha_0(1 + 2n), \quad (4.11)$$

and thus the billiard map is given by

$$T^{(n)}(S, p) : (S(\phi_0), p_0) \rightarrow (S(\text{mod}(\phi_0 + 2n\alpha_0, 2\pi)), p_0). \quad (4.12)$$

The angle of reflection is constant for every iteration, independent of the initial Birkhoff co-ordinate ϕ_0 . An example of a trajectory in a circular billiard is depicted in the left plot of Figure 4.2. The Poincaré diagram, depicted in the right plot of the same figure, consists of horizontal lines of uniformly distributed points, which reflect the dynamical dependence on the reflection angle only.

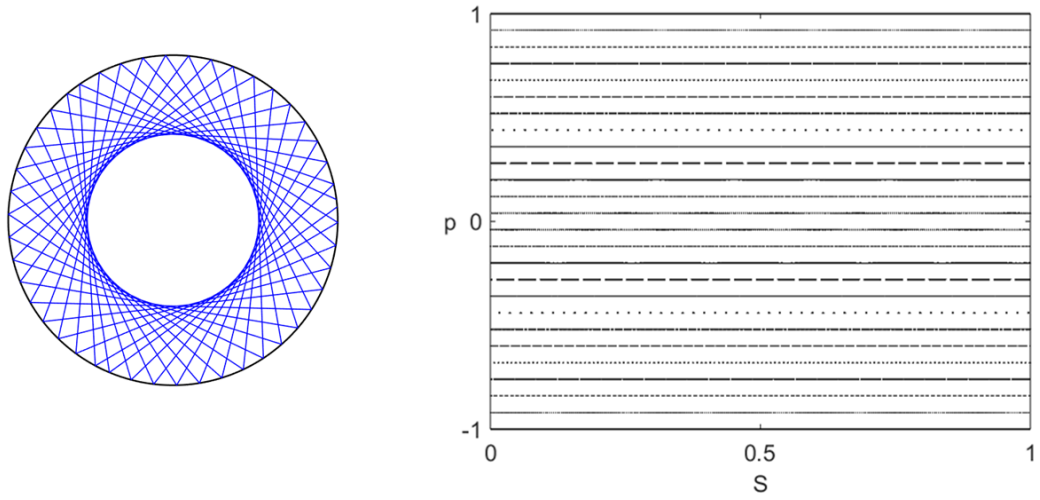


Figure 4.2: Circular billiard (left) with an example regular trajectory (blue), depicted with the corresponding Poincaré diagram (right)

The complexity for analytic solutions quickly increases when moving away from the circle billiard. An analytic solution for an elliptic billiard with boundary

$$r_{\text{ellipse}}(\phi) = \frac{b}{\sqrt{1 - \epsilon^2 \cos \phi^2}}, \quad (4.13)$$

where $b \in \mathbb{R}^+$ and $0 \leq \epsilon < 1$, is found as [102, 103, 108]

$$\phi_{n+1} = \text{mod}\left(-\phi_n + 2 \arctan\left(\frac{b(a \tan(\phi_n) + b \tan(\alpha_n))}{a(b - a \tan(\phi_n) \tan(\alpha_n))}\right), \pi\right), \quad (4.14)$$

$$\alpha_{n+1} = \text{mod}\left(-\alpha_n + \arctan\left(\frac{b}{a \tan(\phi_n)}\right) - \arctan\left(\frac{b}{a \tan(\phi_{n+1})}\right), \pi\right), \quad (4.15)$$

where $a = \sqrt{b^2(1 - \epsilon^2)^{-1}}$.

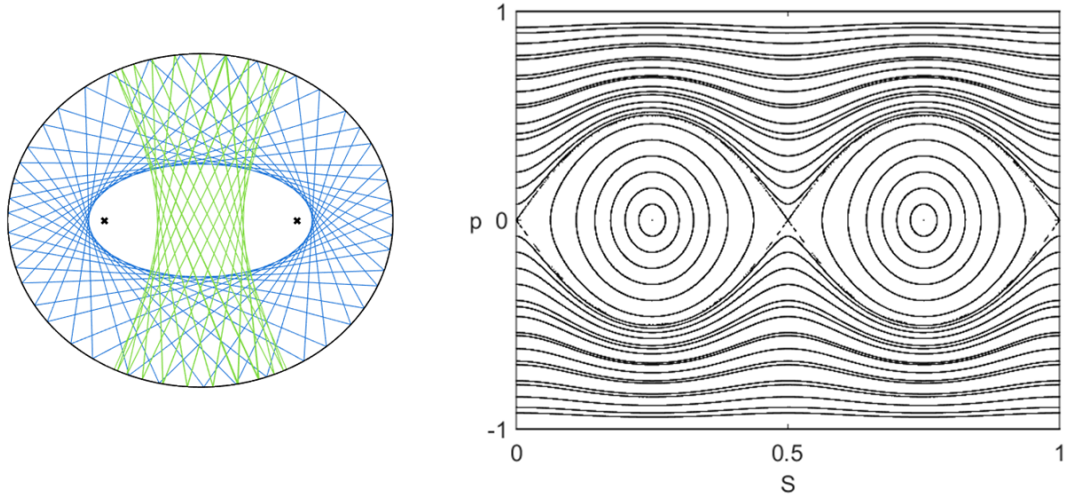


Figure 4.3: Elliptic billiard (left) with two example trajectories: one loop (blue) and one box (green), depicted with the corresponding Poincaré diagram (right)

The left plot of Figure 4.3 shows an example of an elliptic billiard with $b = 1$ and $\epsilon = 0.5$, the corresponding Poincaré diagram is depicted on the right. There are two types of trajectories within the ellipse, box trajectories and loop trajectories [109]. In Figure 4.3 an example of an box trajectory is depicted in green and an example of a loop trajectory in blue. Box trajectories pass through the vertical centre of the billiard between the two focal points $F_{\pm}$, marked by black crosses in the plot. These trajectories have points of impact that always alternate between the upper half and the lower half of the billiard. An extreme case is the bouncing ball trajectory that iterates between just two points on opposite sites. Meanwhile, loop trajectories never cross between the two focal points and instead skid along to boundary of the billiard. Loop trajectories confined to large values of $|p|$ are known as whispering gallery trajectories.

As a result, the Poincaré diagram for the ellipse, depicted in the right plot in Figure 4.3, has two distinct regions. The first region surrounds the elliptic two cycle at $(S_{FP,1}, p_{FP,1}) = (0.25, 0)$ and $(S_{FP,2}, p_{FP,2}) = (0.75, 0)$ (the bouncing ball cycle) and corresponds to box trajectories which alternate between the left side of phase space ($0 < S \leq 0.5$) and the right side ($0.5 < S < 1$). The bouncing ball trajectory traverses vertically through the centre of the billiard. The second region consists of the loop trajectories. The regions belonging to loop and box trajectories are divided on Poincaré diagram by the separatrix, the line connecting the hyperbolic two cycle $(S_{FP,3}, p_{FP,3}) = (0, 0)$ and $(S_{FP,4}, p_{FP,4}) = (0.5, 0)$. The heteroclinic trajectories at

some point pass through a focal point, which leads on to a horizontal (unstable) bouncing ball motion travelling between the two hyperbolic fixed points.

Lastly, I introduce the oval billiard [110] with boundary

$$r_{\text{oval}}(\phi) = (1 + c) + c \cos(2\phi), \quad (4.16)$$

where $c \in \mathbb{R}^+$. The oval billiard does not have an explicit map like the previous integrable billiards, and thus we use only numerical methods to arrive at the dynamics. The left plot of Figure 4.4 shows an example of an oval billiard with $a = 1$ and $c = 0.25$. The corresponding Poincaré diagram is depicted on the right. In the left plot two example trajectories are shown, an regular trajectory in green and a chaotic trajectory in blue.

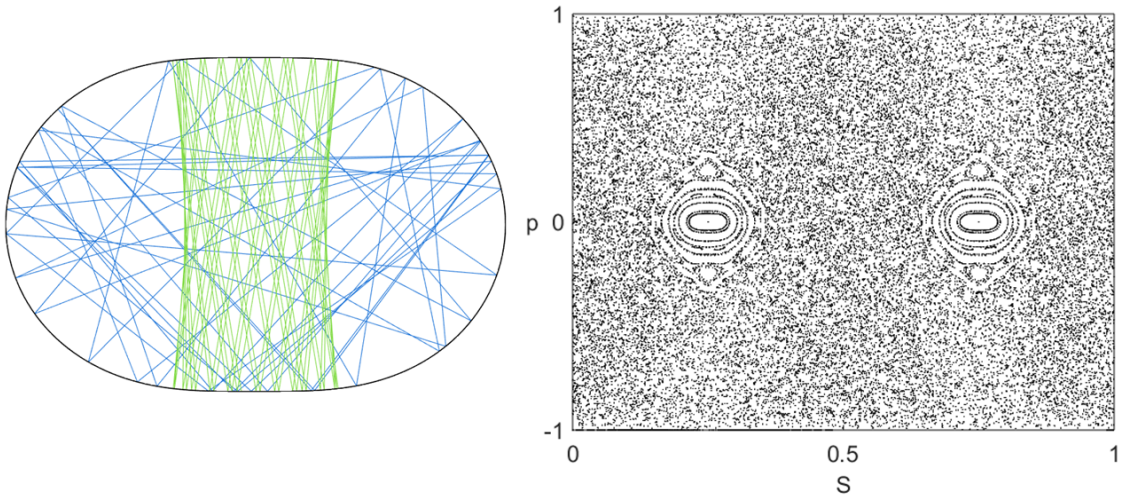


Figure 4.4: Oval billiard (left) with two example trajectories: one chaotic (blue) and one regular (green), depicted with the corresponding Poincaré diagram (right).

The Poincaré diagram for the oval billiard share a similarity with the Poincaré diagram for the elliptic billiard, in that there exists regular orbits structured around the elliptical bouncing ball two cycle at $(S_{FP,1}, p_{FP,1}) = (0.25, 0)$ and $(S_{FP,2}, p_{FP,2}) = (0.75, 0)$. However, outside of those regions, a chaotic sea spans the phase space. The chaotic sea, despite seeming unstructured on phase space, harbours some structured motions, for example, the stable and unstable manifolds around hyperbolic fixed points, and sticky trajectories [111]. Sticky trajectories are particularly interesting as the trajectories belong to the chaotic sea, outside of regular islands, but for a finite (and often large) number of iterations are alternating between localised regions of phase space. These regions are displayed in red and blue on the Poincaré section in

figure 4.5, alongside the first 40 iterations of an example trajectory corresponding to the blue regions. Here we see that the sticky trajectory follows a bow tie shape, until it breaks free from this motion and returns to the chaotic sea. Note that the sets of trajectories corresponding to the blue and red regions in figure 4.5 have similar structures but travel in the opposite direction.

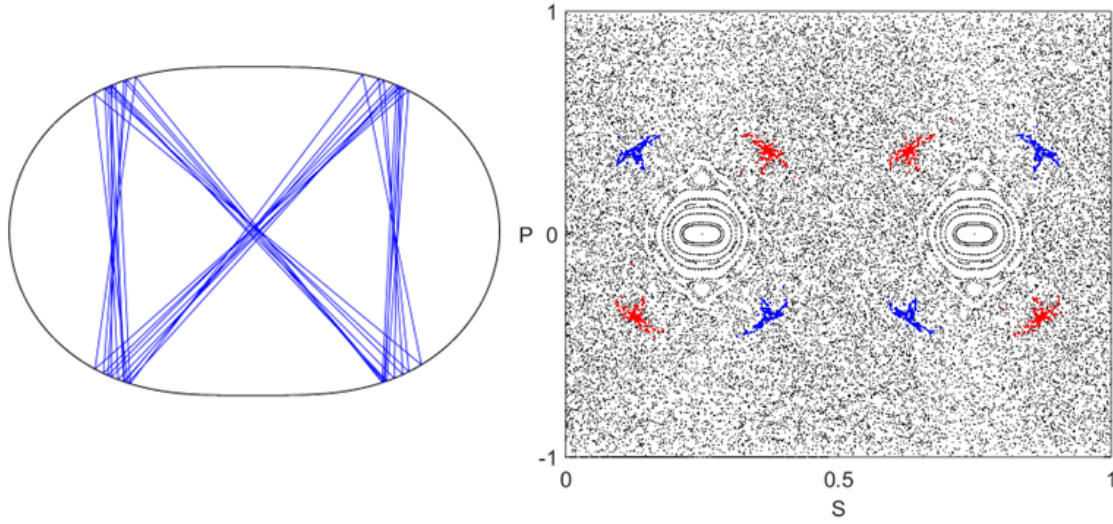


Figure 4.5: Sticky trajectories in an oval billiard. The left hand plot is an example of a sticky trajectory for the first 40 iterations. The right hand plot shows the Poincaré section of the oval billiard with two sets of phase-space points corresponding to sticky trajectories highlighted in blue and red, respectively. The phase-space points of the example trajectory (left) belong to the blue regions on the Poincaré section.

4.3 A Classical Billiard with an Internal Absorbing Region

In this thesis, I introduce a model of a billiard with an absorbing region (AR) with radius r_{AR} located at the centre of the billiard, which does not change the trajectories but does reduce an intensity value, where I assume an exponential decay of the intensity within the absorbing region.

The dynamical quantum system is a free particle within the respective billiard region with an additional imaginary potential

$$\hat{\Gamma} = \gamma \mathbf{1}_{AR}(\hat{x}, \hat{y}), \quad (4.17)$$

where $\mathbf{1}_{AR}$ denotes the indicator function of the absorbing area in coordinate space.

As discussed in section 2.6 for a one-dimensional free particle with an absorbing region, this Hamiltonian is strictly speaking outside the allowed set of operators for our semiclassical theory, as it is not an analytic function of $\hat{x}$ and $\hat{y}$. Nevertheless, I shall consider the dynamics of the classical flow resulting from the Husimi and Wigner dynamics (2.4) and (2.10) as we expect a similar quantum-classical correspondence as in Hermitian quantum billiards, that are also not analytic, leading to distinct quantum behaviours on top of the classical features.

The classical dynamics, in the spirit of chapter 2, takes place in the four-dimensional phase space. Again, similar to the case of the one-dimensional free particle with a constant absorbing region, the phase space trajectories are not influenced by the imaginary part of the Hamiltonian, and thus the Wigner and the Husimi dynamics yield the same classical counterpart. This consists of the Hamiltonian billiard dynamics augmented with the norm landscape. The full trajectories of the Hamiltonian billiard can be described on the reduced two-dimensional Poincaré phase space.

In what follows, I shall consider the norm landscape on this same reduced phase space, associating to every initial condition (S, p) the resulting value of the norm after n iterations, $w(n, S, p)$ with an initial $w(0, S, p) = 1$. I will refer to $w_n = w(n, S, p)$ as an intensity landscape.

The decrease of the trajectory's intensity is proportional to the length of the trajectory within the absorbing region. This is analogous to time spent within the region, and thus, the intensity maps are conceptually similar to what is known as *true time* maps [94, 97] relating to the full length of trajectories between points of impact on the billiard boundary.

Furthermore, the intensity landscapes typically change from iteration to iteration, and can evolve highly non-trivial structures. At each iteration they are divided into two regions - the set of initial conditions corresponding to trajectories that do not intersect the absorbing region up to the n -th iteration, forming a plateau where $w_n = 1$, and the complement of this set, which is affected by the absorption and on which w_n has a valley structure with $w_n < 1$. The plateau region is the intersection of the sets that do not enter the absorbing region in the n -th iteration. Let us denote the set of initial points corresponding to trajectories that intersect the absorbing region in the n -th iteration by $\mathcal{S}_n$. These sets are related to each other by the recursion

$$\mathcal{S}_{n+1} = T^{-1}\mathcal{S}_n = T^{-n}\mathcal{S}_1, \quad (4.18)$$

where T^{-1} is the inverse of the billiard map. Thus, the sets $\mathcal{S}_n$ can be found from

the set $\mathcal{S}_1$ (that intersects the absorbing region in the first iteration) and the billiard map of the absorption-free system.

The boundary of the set $\mathcal{S}_1$ has been considered in [112] where a billiard with internal circular scatterer has been studied. For the case of a hole located at the centre of the billiard with radius r_{AR} , if

$$\sin(\phi_0 - \alpha_0 - \theta_0) = \frac{r_{AR}}{r(\phi_0)} \quad (4.19)$$

then the trajectory is tangent to the hole (scatterer). Thus, we can define the boundary of the set $\mathcal{S}_1$ on phase space by two curves $p^{(+)}$ and $p^{(-)}$ given by

$$p^{(\pm)} = \cos \left(\phi_0 \pm \arcsin\left(\frac{r_{AR}}{r(\phi_0)}\right) - \zeta(\phi_0), \text{mod } \pi \right), \quad (4.20)$$

where

$$\zeta(\phi) = \arctan \left(\frac{\left(\frac{d}{d\phi} r(\phi) \right) \sin(\phi) + r(\phi) \cos(\phi)}{\left(\frac{d}{d\phi} r(\phi) \right) \cos(\phi) - r(\phi) \sin(\phi)} \right). \quad (4.21)$$

In the limit of $\gamma \rightarrow \infty$ the absorbing region acts as a hole, leading to a situation similar to the one studied in [99]. In this case the intensity jumps from one to zero as soon as the absorbing region is first encountered and the propagation terminates. For systems with holes, escape time diagrams are often used to visualise the dependence of the escape time on the initial conditions, specifically, colour-coding each point of the coordinate space to show at which iteration the given trajectory intersects the hole and terminates. The escape time diagrams comprise the sets $\mathcal{S}_1, \mathcal{S}_2 \setminus \mathcal{S}_1, \mathcal{S}_3 \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$, etc, each associated to a different colour.

For finite values of γ , in each iteration in which a trajectory intersects the absorbing region, it experiences a loss of intensity, such that

$$w_{n+1} = e^{-\gamma \Delta_n} w_n, \quad (4.22)$$

with $w_0 = 1$, and where Δ_n denotes the length of the trajectory segment within the absorbing region in this iteration. Iterating from a point of impact $P_n = (S_n, p_n)$, the particle will pass through the absorbing region if $p^{(-)} < p_n < p^{(+)}$ following equation (4.20). I denote the two points of intersection with the absorbing region where the particle enters and exits the region by $I_n^{(1)}$ and $I_n^{(2)}$. The length of the trajectory spent within the absorbing region, Δ_n , is the distance between $I_n^{(1)}$ and $I_n^{(2)}$. The value of Δ_n is zero if there is no intersection, and is positive otherwise.

In case of an intersection, the length Δ_n can be expressed analytically as

$$\Delta_n = r_{\text{AR}} \sqrt{2(1 - \cos(\kappa_n))}. \quad (4.23)$$

where κ_n is the angle between the two lines that connect the centre of the billiard to each of the intersection points $I_n^{(1)}$ and $I_n^{(2)}$ respectively. Figure 4.6 illustrates the relevant angles and points. Given that we know the points P_n and P_{n+1} , and the radius of the absorbing region, through careful trigonometric analysis we can deduce κ_n and thus Δ_n .

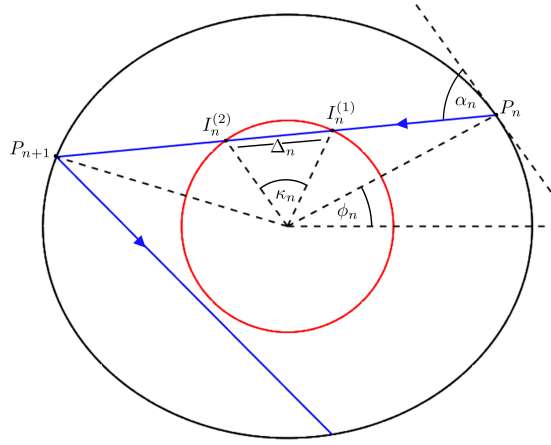


Figure 4.6: Illustration of a trajectory that interacts the absorbing region.

Let us now analyse the resulting intensity landscapes for our three example billiards.

4.3.1 Circular Billiard

I begin with a short discussion of a circular billiard with constant radius $r(\phi) = R$, for which the intensity landscape has a simple analytic form. Intersections with the absorbing region occur when $|p| \leq \frac{R}{r_{\text{AR}}}$. The length Δ_n for each intersection is independent of the iteration number and simply given by

$$\Delta_n = 2\sqrt{r_{\text{AR}}^2 - R^2 \cos^2(\alpha_0)}. \quad (4.24)$$

Thus, the resulting intensity landscape is given by

$$w_n(p) = \begin{cases} 1 & \text{for } |p| > \frac{R}{r_{\text{AR}}}, \\ e^{-2n\gamma\sqrt{r_{\text{AR}}^2 - R^2 \cos^2(\alpha_0)}} & \text{otherwise.} \end{cases} \quad (4.25)$$

In figure 4.7 I depict the intensity landscape (right) compared to the Poincaré diagram with boundary curves given by (4.20) (left) and the set $\mathcal{S}_1$ depicted as black

points on the phase space (middle) for the case that the patch radius is $r_{AR} = 0.5$. As expected, the boundary curves are the straight lines $|p| = r_{AR}$ for all $S(\phi)$. The intensity landscapes has a simple cosine-structure on top of the set $\mathcal{S}_1$ which is also independent of the arc length $S(\phi)$. The trajectories with the largest absorption are the two-cycle trajectories, where $p = 0$.

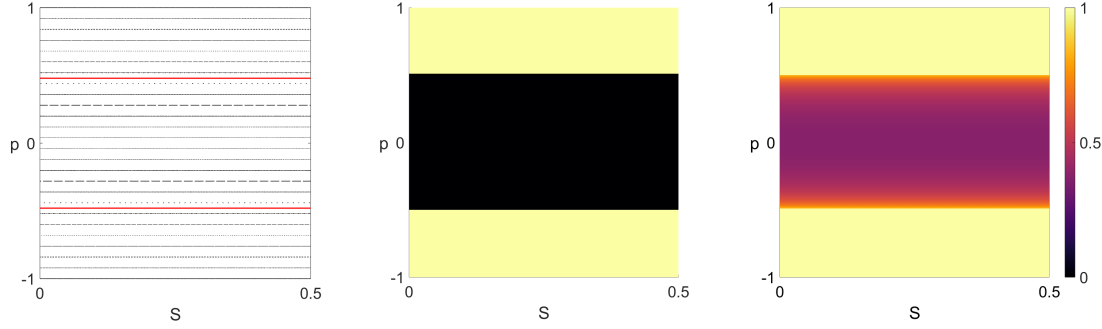


Figure 4.7: Poincaré diagram with boundary curves given by (4.20) (left), the set $\mathcal{S}_1$ depicted as black points (middle) and the intensity landscape (right) of the circular billiard with $R = 1$, $r_{AR} = 0.5$, and absorption strength $\gamma = 1$.

4.3.2 Elliptic Billiard

Next I look at a second regular billiard, the ellipse, again with a circular absorbing region. The boundary of the billiard is described by

$$r_{\text{ellipse}}(\phi) = \frac{b}{\sqrt{1 - \epsilon^2 \cos^2 \phi}}, \quad (4.26)$$

where $0 \leq \epsilon < 1$, and $b \in \mathbb{R}^+$. In what follows we use parameters $b = 1$ and $\epsilon = 0.5$.

While still integrable, the elliptic billiard presents richer dynamics than the circular billiard, owing to the coexistence of elliptic and hyperbolic fixed points, with corresponding box and loop trajectories, separated by a heteroclinic orbit (the separatrix). This section aims to evaluate how both trajectory types contribute to the intensity landscape for absorbing regions of radii $r_{AR} = 0.1$ and $r_{AR} = 0.7$.

To gain some intuition we begin by considering the problem in configuration space. In the left plots of Figure 4.8 I depict the elliptic billiard and the absorbing region (red circle) of radius $r_{AR} = 0.1$ (top figure) and $r_{AR} = 0.7$ (bottom figure) together with examples of box (green) and loop (blue) trajectories. The initial conditions of these trajectories are highlighted in Poincaré sections with corresponding boundary curves on the right column of Figure 4.8. Here we see that, for both sizes of absorbing region, contributions will come from both box and loop trajectories to various

degrees. This is because the elliptic and hyperbolic fixed points are contained within the set $\mathcal{S}_1$. For the case that r_{AR} is small, very few loop trajectories are captured and the trajectories that do not intersect the absorbing region are whispering-gallery type modes with sufficiently large initial values of $|p|$. For a larger r_{AR} , more loop trajectories intersect with the patch. In any case for a non-vanishing r_{AR} , box trajectories, given enough iterations will travel through the centre of the billiard and intersect with the absorbing region.

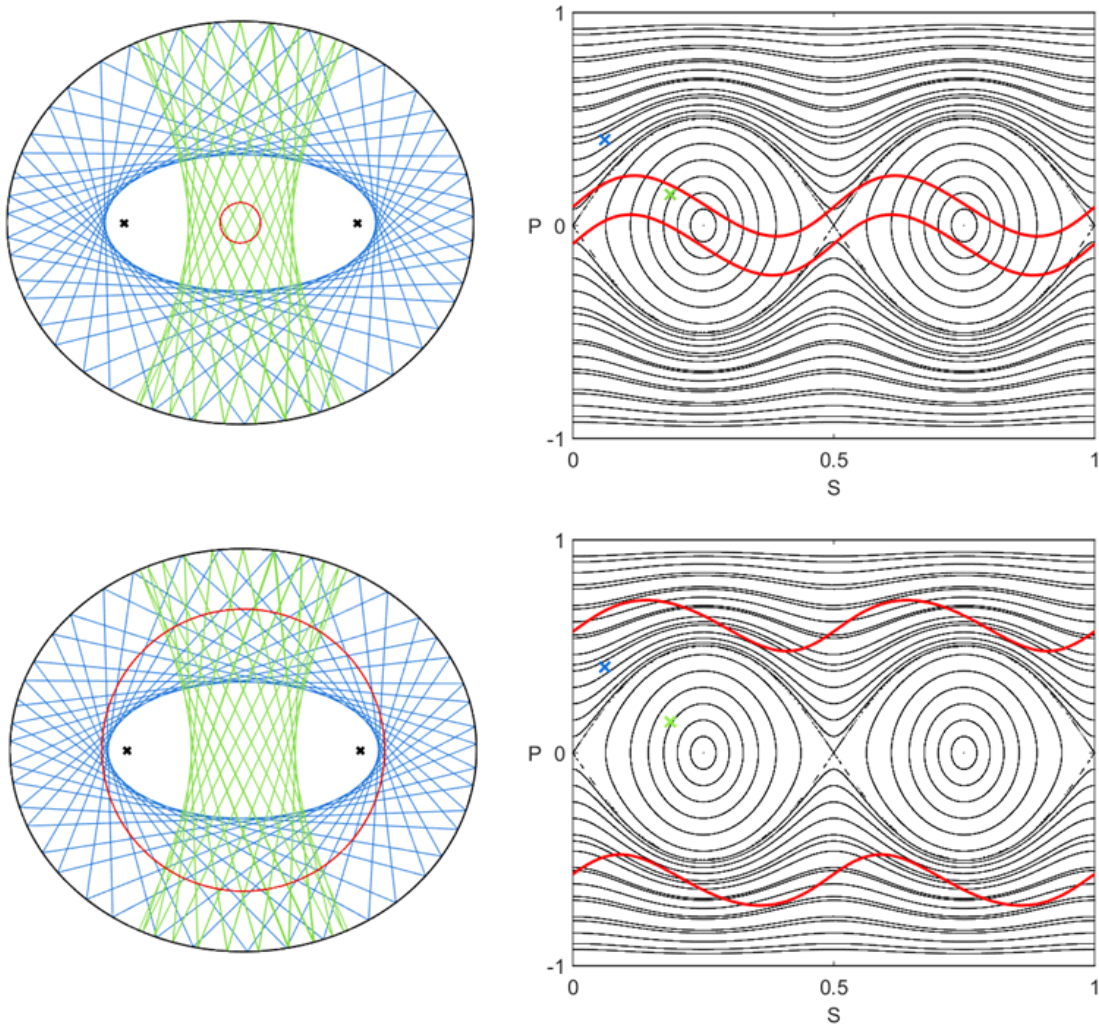


Figure 4.8: Elliptic billiard with $b = 1$ and $\epsilon = 0.5$ and absorbing regions of radius $r_{AR} = 0.1$ (top) and $r_{AR} = 0.7$ (bottom). Left plots: Billiard boundary in coordinate space with boundary of the absorbing region (red circle), focal points (black crosses), and two example trajectories: one loop (blue) and one box (green). Right plots: Corresponding Poincaré sections spanning half of the phase space $s \in (0, 0.5)$, with initial conditions for trajectories on top (green and blue crosses) and boundaries of the set $\mathcal{S}_1$ (red lines).

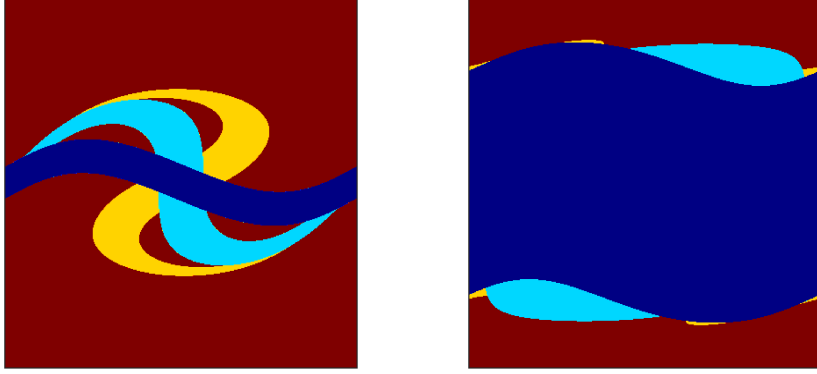


Figure 4.9: Escape time diagrams and loss distributions of the elliptic billiard for $r_{AR} = 0.1$ (left) and $r_{AR} = 0.7$ (right). The escape time plots depict intersection with the absorbing region after $n = 1$ (dark blue), $n = 2$ (light blue), $n = 3$ (orange) iterations and no intersection up to $n = 3$ (red).

The corresponding escape diagrams highlighting initial conditions that encounter the absorbing region in three or less iterations, are depicted in Figure 4.9. Going forward, I often only depict half of the phase space in Figures. The behaviour on the remaining half is clear due to the symmetry of our example systems. The escape time diagrams are composed of the sets $\mathcal{S}_n$ that are depicted as follows: the dark blue regions correspond to escape at the $n = 1$ iteration, light blue indicates escape at $n = 2$, yellow escape at $n = 3$ and dark red indicates no escape up to $n = 3$. Unlike the circular billiard, trajectories that intersect the absorbing region in one iteration, i.e. in the set $\mathcal{S}_n$, may not intersect on the next, i.e. in set $\mathcal{S}_{n+1}$. An example of such a trajectory is shown in Figure 4.6. In the case that $r_{AR} = 0.1$ we see over the short time, the inner region of phase space becomes partially filled by the union of sets $\mathcal{S}_n$. This is in contrast to the case with $r_{AR} = 0.7$, where $\mathcal{S}_1$ contains a large bulk of the phase space, and successive contributions are minimal. Let us consider the sets $\mathcal{S}_n$ individually.

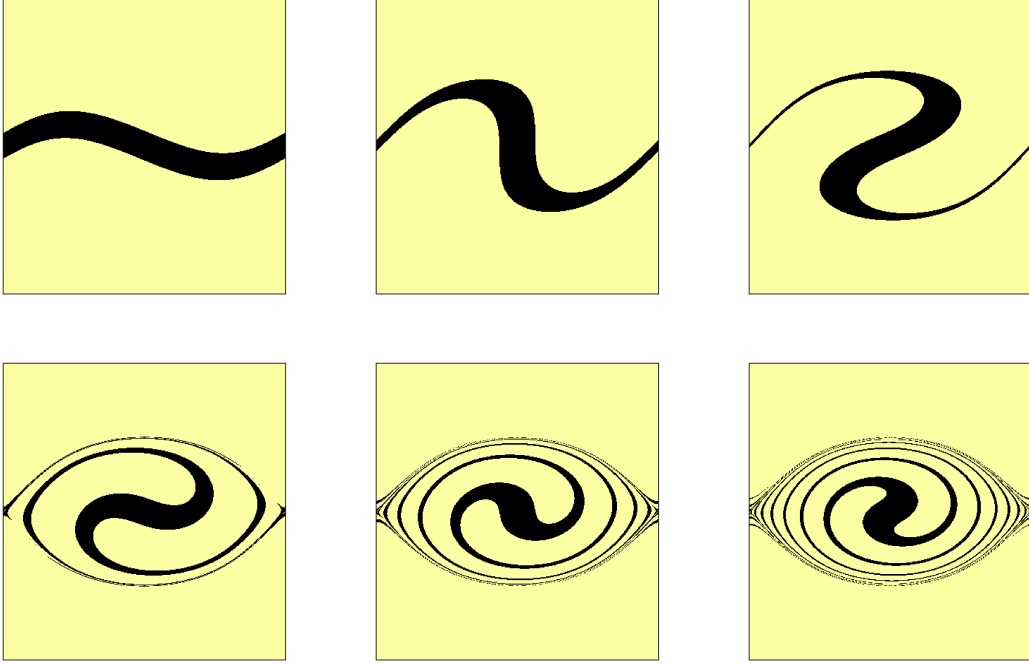


Figure 4.10: The sets $\mathcal{S}_n$ for the ellipse with an the absorbing region $r_{AR} = 0.1$. From top left to bottom right the iterations are $n = 1, 2, 3, 10, 20$, and 30 .

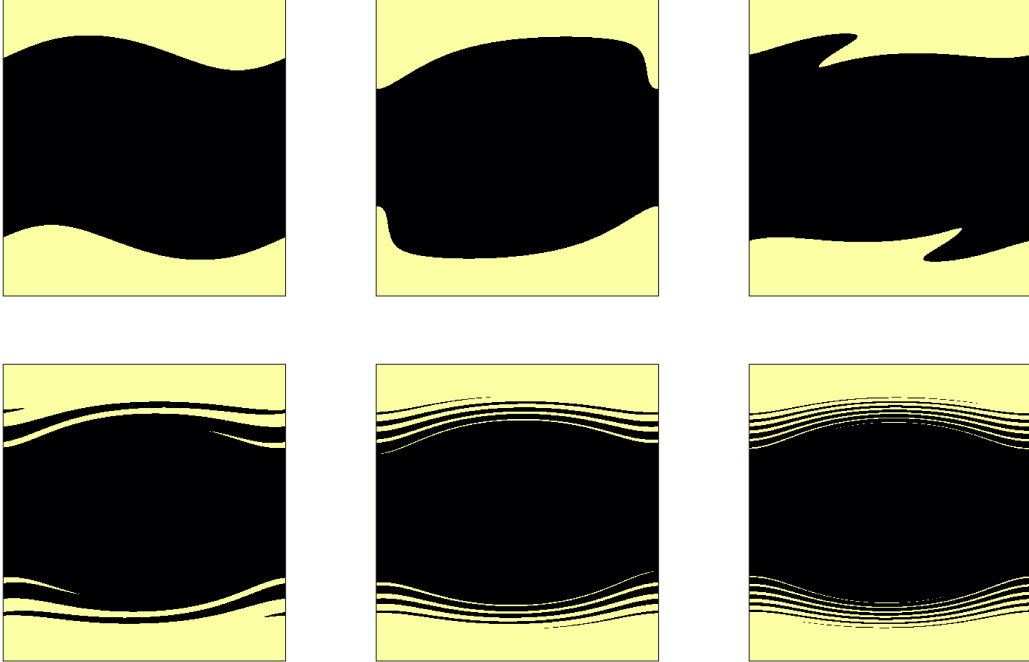


Figure 4.11: The sets $\mathcal{S}_n$ for the ellipse with an the absorbing region $r_{AR} = 0.7$. From top left to bottom right the iterations are $n = 1, 2, 3, 10, 20$, and 30 .

The sets $\mathcal{S}_n$ for $r_{AR} = 0.1$ and $r_{AR} = 0.7$ are shown in Figures 4.10 and 4.11 for $n = 1, 2, 3, 10, 20$ and 30 . In Figure 4.10 where $r_{AR} = 0.1$ we observe that at each iteration there is a set of initial conditions near $(S, p) = (0.25, 0)$ (which corresponds to the bouncing ball trajectories) that always interacts with the absorbing region. Outside of this set, we see the development of filament structures around the central region. Specifically, the sets $\mathcal{S}_n$ are related to $\mathcal{S}_1$ by successive applications of the inverse flow T^{-1} . The filaments seen are the transported set $\mathcal{S}_1$ after different numbers of backwards iterations. In the case of $r_{AR} = 0.7$, depicted in Figure 4.11, the set $\mathcal{S}_1$ contains the separatrix, i.e. it contains the entire set of phase space points belonging to box trajectories, that is, every box trajectory intersects the absorbing region in every iteration. The corresponding region on phase space is thus invariant. On the outskirts of the distribution is another filament structure that is comprised of the loop trajectories that intersect the absorbing region some of the time.

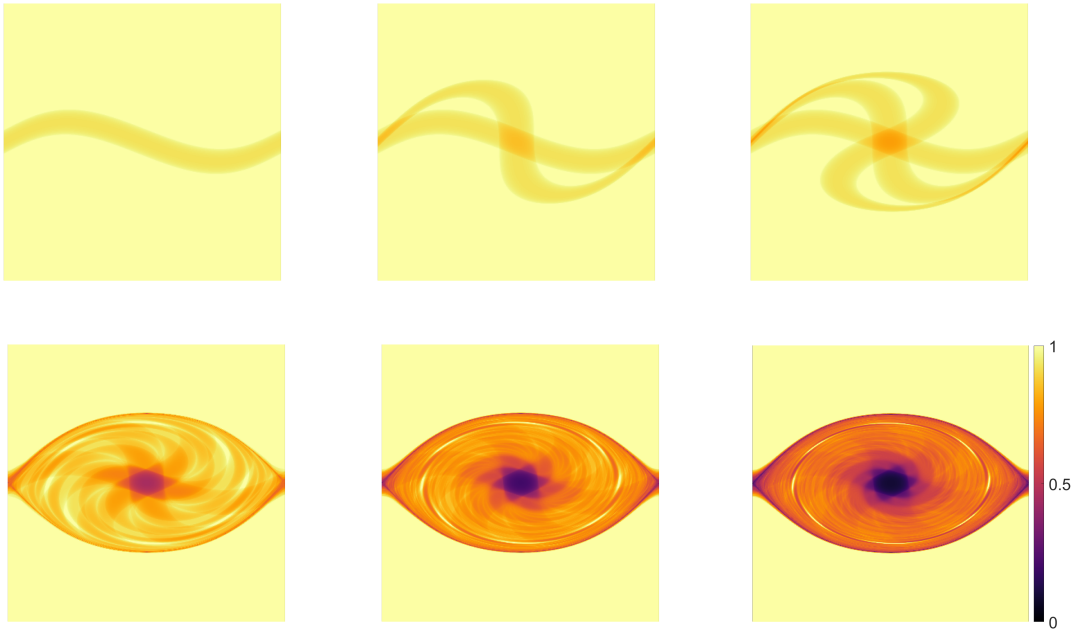


Figure 4.12: Intensity landscape for the elliptic billiard with absorbing region with radius $r_{AR} = 0.1$ and absorption rate $\gamma = 0.4$ for different numbers of iterations. From top left to bottom right: $n = 1, 2, 3, 10, 20$, and 30 .

In Figures 4.12 and 4.13, I show the intensity landscapes w_n for the ellipse with absorbing region $r_{AR} = 0.1$ with loss strength $\gamma = 0.4$ and absorbing region $r_{AR} = 0.7$ with loss strength $\gamma = 0.2$ for iterations $n = 1, 2, 3, 10, 20$ and 30 . In the latter case we choose a smaller γ than in the former case to be able to compare the intensities on the same false colour scale. Of course, the area on which the intensity is not equal to one, corresponds to the union of the sets $\mathcal{S}_n$. The intensity landscape after the first

iteration is equal to one on the complement of the set $\mathcal{S}_1$, and shows modulations in value on the set $\mathcal{S}_1$, due to the different length of trajectory segments passing through the absorbing region for different initial conditions. The logarithm of the intensity landscape for successive interactions is then simply the sum of that for the first iteration and its backwards images under the billiard map. Initial conditions that intersect the absorbing region in more iterations, i.e. the intersections of the sets $\mathcal{S}_n$ lead to a stronger loss of intensity, as is clearly visible, for example, in the star-shaped structure in the vicinity of the fixed point $(S, p) = (0.25, 0)$ for $r_{AR} = 0.1$. As a result of this, for larger numbers of iterations, we observe a pronounced valley structure along the separatrix and the two-cycles of the Poincaré dynamics, related to trajectories that have long segments in the absorbing region on all or most iterations.

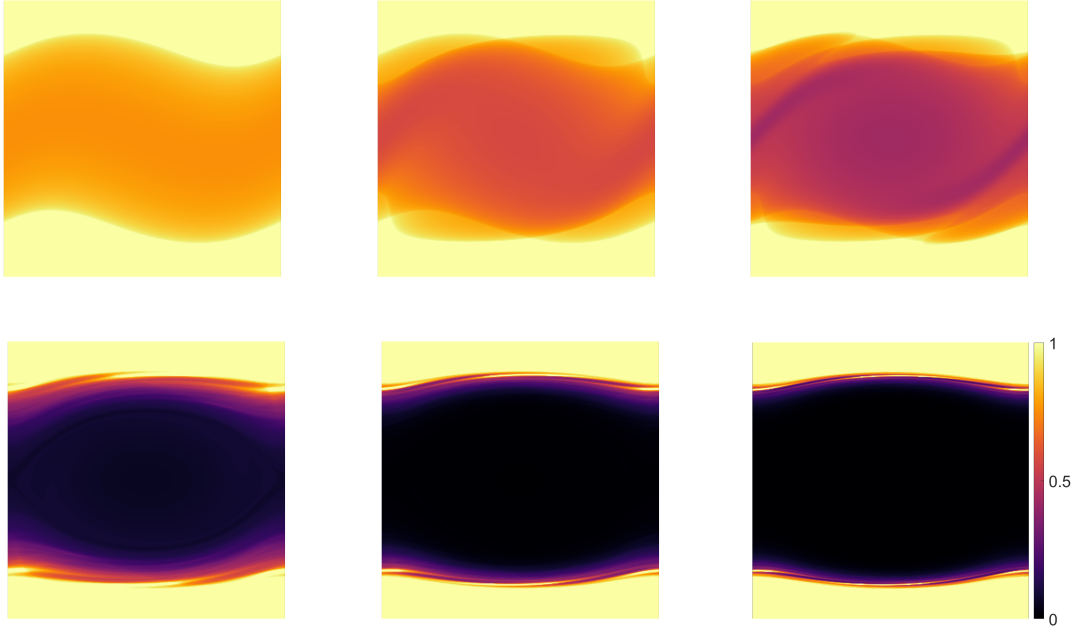


Figure 4.13: Intensity landscape for the elliptic billiard with absorbing region with radius $r_{AR} = 0.7$ and absorption rate $\gamma = 0.2$ for different numbers of iterations. From top left to bottom right: $n = 1, 2, 3, 10, 20$, and 30 .

In contrast to the delicate structure we see in $r_{AR} = 0.1$, the structure from the box trajectories for $r_{AR} = 0.7$ are simple and become almost uniform. The phase space for box trajectories is densely covered with $w_n < 1$ values for all iteration. However, as can be seen in $\mathcal{S}_2, \mathcal{S}_3$ and $\mathcal{S}_{10}$, there is a slight increase of absorption around $(S, p) = (0.25, 0)$ that tapers as we get away from this point. As with the circle billiard, this is due to the bouncing ball trajectories experiencing the maximum amount of absorption on every iteration. However, over time all box, as well as

loop, trajectories near the region correlating to the separatrix, accumulate so much absorption that the intensity landscape develops an approximately uniform area with values of $w_n \approx 0$ as we see in $\mathcal{S}_{30}$. Despite this, there remains a fine and detailed structure at the boundary between $w_n = 1$ and $w_n < 1$.

4.3.3 Oval Billiard

I conclude the examples with an oval billiard with the boundary

$$r(\phi) = (1 + c) + c \cos(2\phi), \quad (4.27)$$

where $c \in \mathbb{R}^+$ and in what follows we choose $c = 0.25$. Here we consider the two radii $r_{\text{AR}} = 0.1$ and $r_{\text{AR}} = 0.5$ of the absorbing region at the centre of the oval. Both absorbing regions intersect with regular orbits neighbouring the fixed points $(S, p) = (0.25, 0)$ and $(S, p) = (0.75, 0)$, and chaotic trajectories. This can be seen by inspection of the intersection boundary lines in the right plots of Figure 4.14. An example of regular (green) and chaotic (blue) trajectories that intersect the absorbing region are shown in the left plots of Figure 4.14 for $r_{\text{AR}} = 0.1$ (top) and $r_{\text{AR}} = 0.5$ (bottom). The initial Birkhoff co-ordinates of these trajectories are marked (green and blue crosses) on the Poincaré diagrams (right), that also depict the corresponding boundary lines of the set $\mathcal{S}_1$ for each absorbing region.

Figure 4.15 shows the escape time maps for the oval billiards up to iteration $n = 3$, for the same colour-coding as Figure 4.9. The plot for $r_{\text{AR}} = 0.1$ highlights the evolution of the initial set $\mathcal{S}_1$ into successive sets that have a filament structure, the union of which begins to sparsely cover the phase space. Meanwhile for $r_{\text{AR}} = 0.5$, the union of $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ span large areas of the phase space and we see a filament structure develop only at the edges.

For the oval billiard with radii $r_{\text{AR}} = 0.1$ and $r_{\text{AR}} = 0.5$ the sets $\mathcal{S}_n$ are depicted in Figures 4.16 and 4.17 for $n = 1, 2, 3, 10, 20$, and 30. From the first few iterations, we clearly observe a filament structures arising from the repeated backwards iteration of the set $\mathcal{S}_1$, organised around the dynamical invariants embedded in the chaotic sea of the absorption-free dynamics. In contrast to the circular and elliptic billiards, the oval billiard has only small regular regions on the Poincaré section that do not span the entire S-axis of the phase space. This is evident for later sets $\mathcal{S}_n$ that show a clear division between the regular island and the chaotic sea, where the filament structure outside the regular region slowly makes way for the natural measure of the chaotic sea, which eventually becomes uniform.

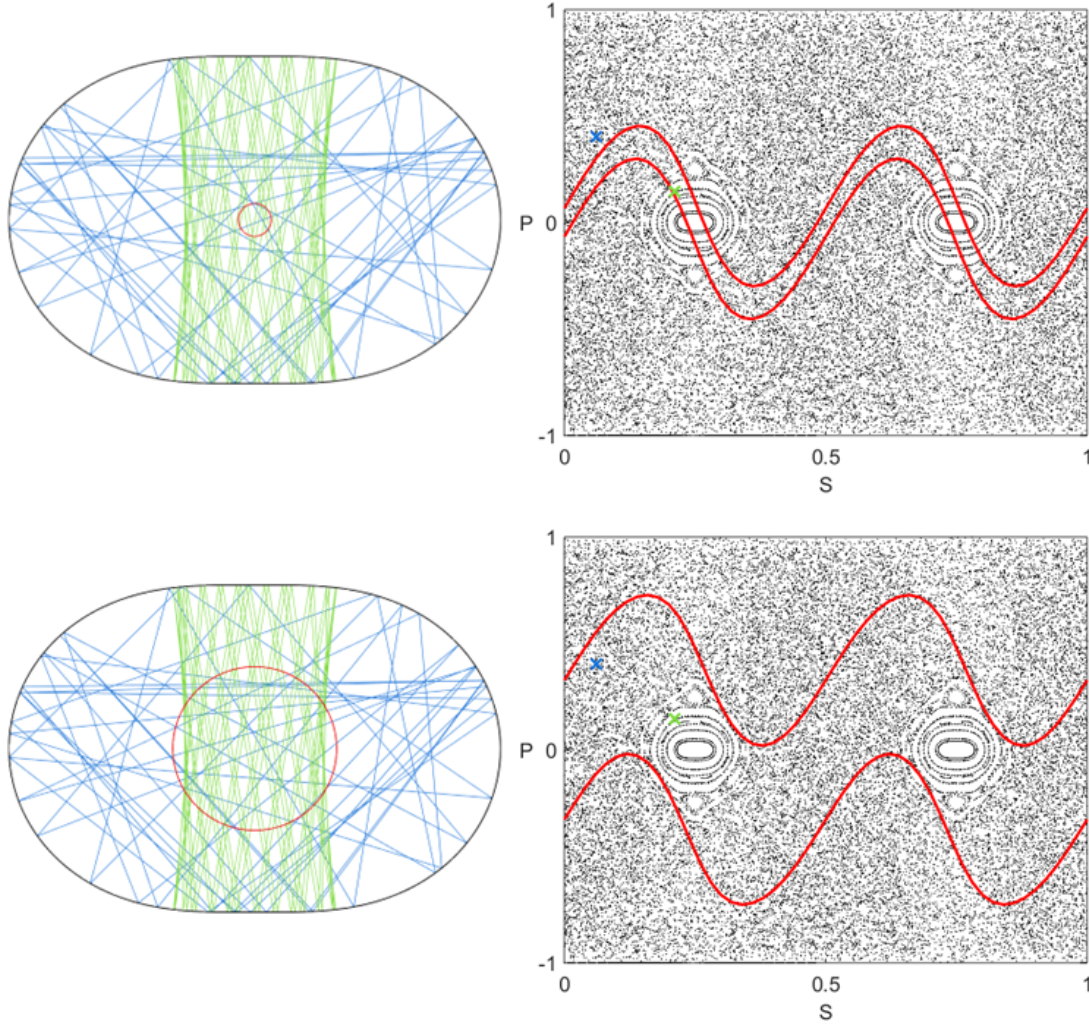


Figure 4.14: Oval billiard with $c = 0.25$ and absorbing regions of radius $r_{AR} = 0.1$ (top) and $r_{AR} = 0.5$ (bottom). Left plots: Billiard boundary-in coordinate space with boundary of the absorbing region (red circle), and two example trajectories: one regular trajectory (blue) and a chaotic (green). Right plots: Corresponding Poincaré sections spanning half of the phase space $S \in (0, 0.5)$, with initial conditions for trajectories on top (green and blue crosses) and boundaries of the set $\mathcal{S}_1$ (red lines).

For the oval with an absorbing region of radius $r_{AR} = 0.1$, we observe the typical spiral structures arising from the box trajectories. For the larger absorbing region size of radius $r_{AR} = 0.5$, most regular trajectories intersect on each iteration, and thus we only see a whorling structure near the boundary of regular trajectories and chaotic trajectories. Within the chaotic sea, in both models sticky trajectories contribute on each iteration. The regions of sticky trajectories is densely covered compared to the chaotic sea, which is especially noticeable for the case $r_{AR} = 0.5$ at $n = 20$ and $n = 30$, as the density is comparable to the covering of regular trajectories.

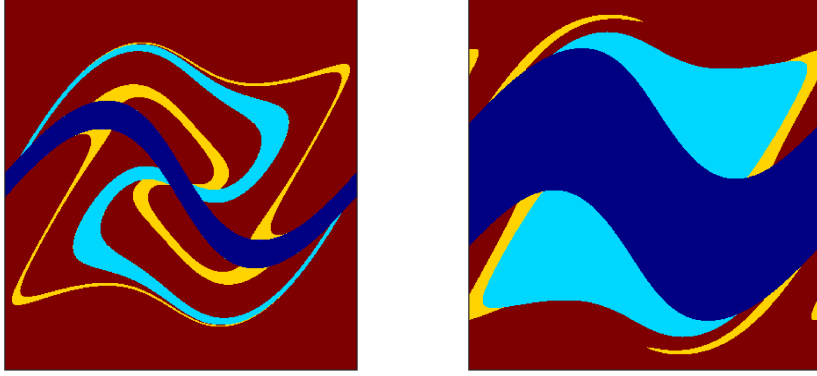


Figure 4.15: Escape time diagrams and loss distributions of the oval billiard for $r_{AR} = 0.1$ (left) and $r_{AR} = 0.5$ (right). The escape time plots depict intersection with the absorbing region after $n = 1$ (dark blue), $n = 2$ (light blue), $n = 3$ (orange) iterations and no intersection up to $n = 3$ (red).

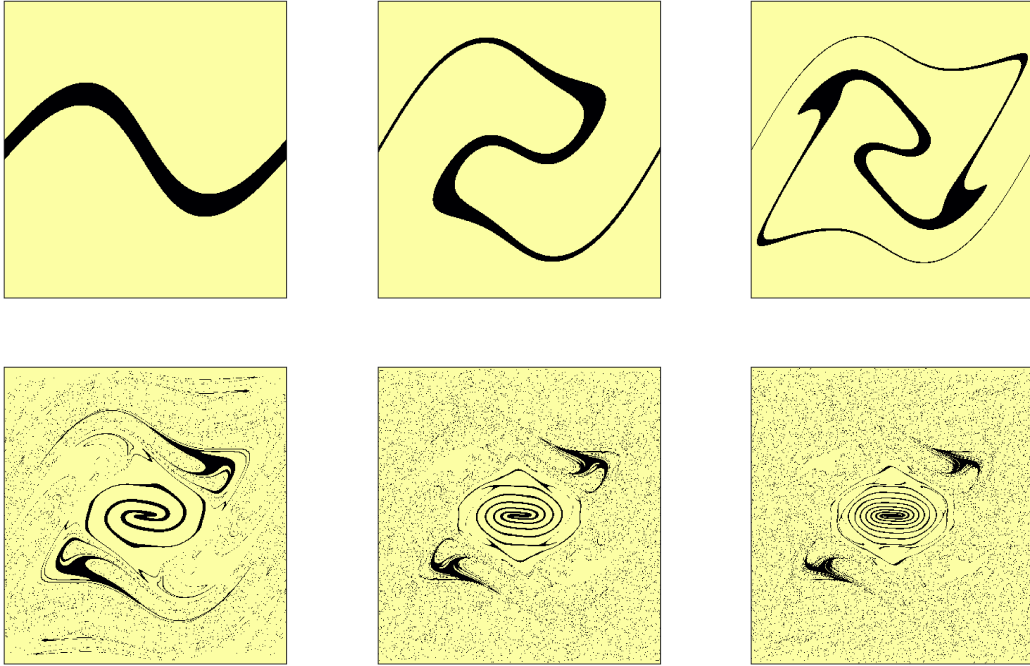


Figure 4.16: The sets S_n for the oval billiard with an the absorbing region $r_{AR} = 0.1$. From top left to bottom right the iterations are $n = 1, 2, 3, 10, 20$, and 30 .

The intensity landscapes w_n are shown in Figures 4.18 and 4.19 for the oval with absorbing region radius $r_{AR} = 0.1$ with loss strength $\gamma = 0.4$ and absorbing region radius $r_{AR} = 0.5$ with loss strength $\gamma = 0.2$ for iterations $n = 1, 2, 3, 10, 20$ and 30 . The union of the sets S_n provides the skeleton of the intensity landscapes, which become highly non-trivial for finite values of the loss parameter γ .

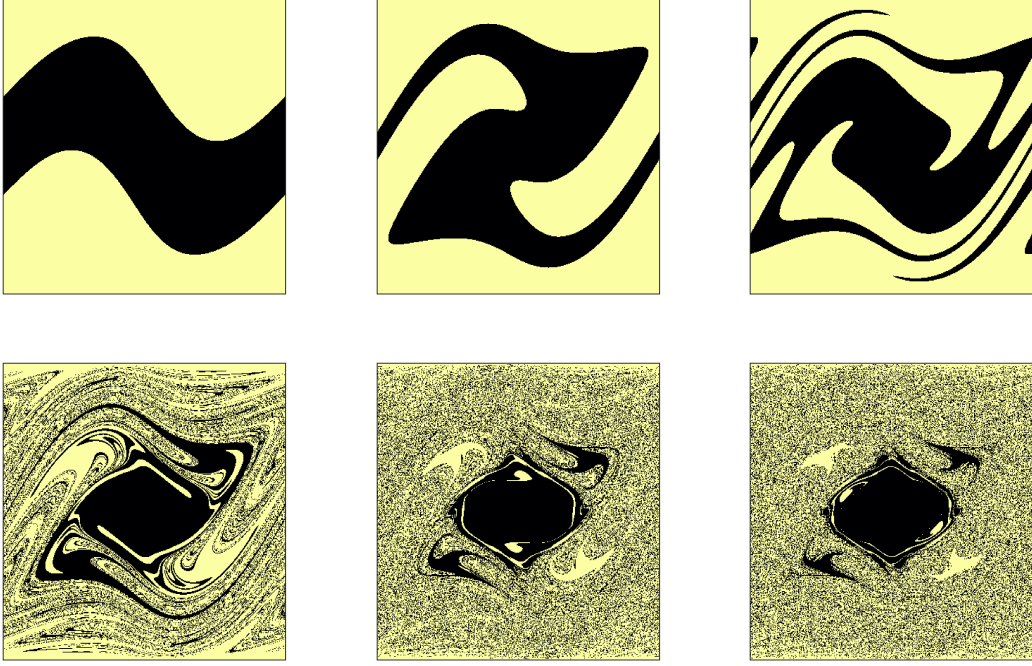


Figure 4.17: The sets $\mathcal{S}_n$ for the oval billiard with an the absorbing region $r_{AR} = 0.5$. From top left to bottom right the iterations are $n = 1, 2, 3, 10, 20$, and 30 .

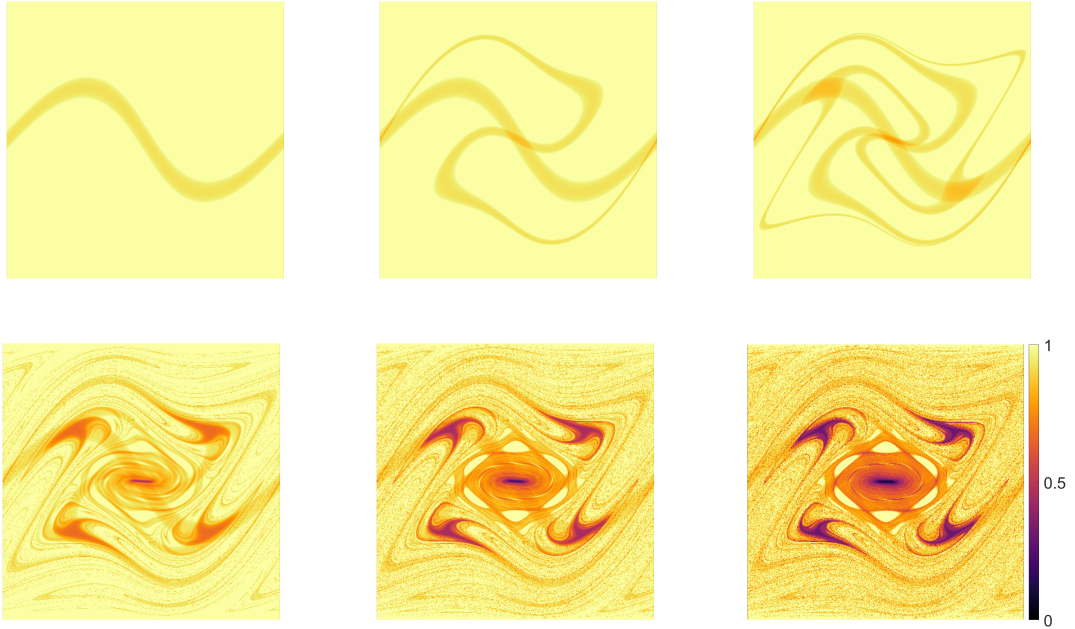


Figure 4.18: Intensity landscape for the oval billiard with absorbing region with radius $r_{AR} = 0.1$ and absorption rate $\gamma = 0.4$ for different numbers of iterations. From top left to bottom right: $n = 1, 2, 3, 10, 20$, and 30 .

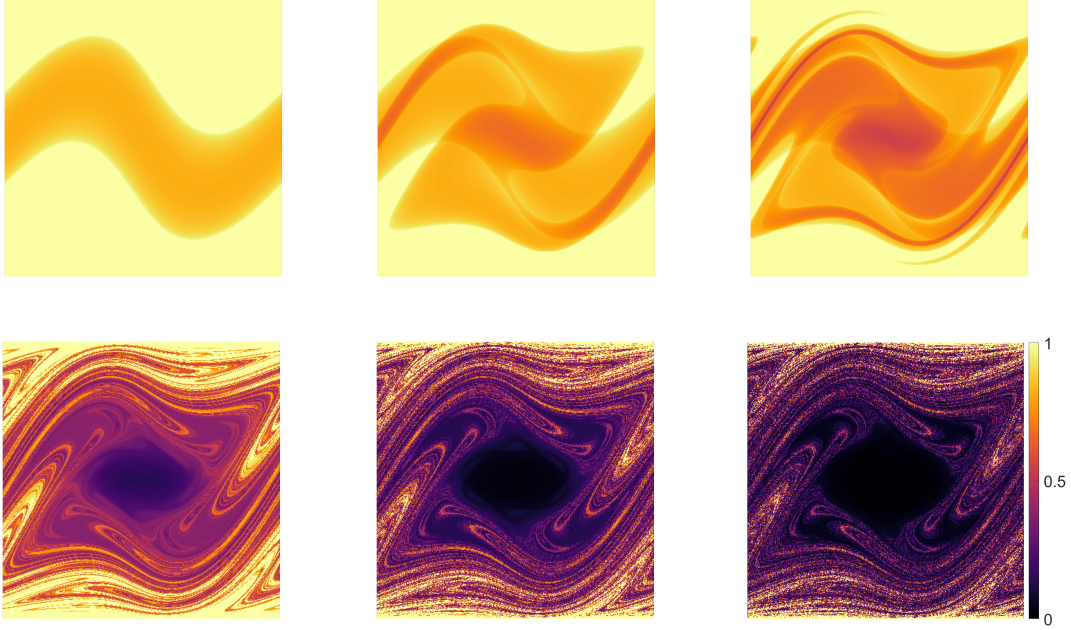


Figure 4.19: Intensity landscape for the oval billiard with absorbing region with radius $r_{AR} = 0.5$ and absorption rate $\gamma = 0.2$ for different numbers of iterations. From top left to bottom right: $n = 1, 2, 3, 10, 20$, and 30 .

For the case that $r_{AR} = 0.1$, the separatrix around the elliptic islands is well defined and we observe a similar structure around the fixed point $(S, p) = (0.25, 0)$ as in Figure 4.12 for the elliptic billiard. The only sizeable regions of phase space that do not intersect the small absorbing region are the six islands that surround the regular region at the fixed point. This is in contrast to the oval billiard with an absorbing region of size $r_{AR} = 0.5$ where all regions associated with the regular islands accumulate the greatest absorption. In both cases, amongst the chaotic trajectories the vast majority will eventually intersect the absorbing region through their natural ergodic coverage of the chaotic sea. The set of initial points that belong to trajectories that do not intersect the absorbing region up to a given number of iterations develops a fractal structure with large number of iterations, as can be seen for example for $n = 30$ in Figures 4.18 and 4.19.

Lastly, note the appearance of both sets of sticky trajectories in the later iterations. These regions have trajectories being "stuck" to them for a significant number of iterations, and as a result, intersect the absorbing region more often. The oscillation between the pairs of sticky trajectories are seen in the successive sets $\mathcal{S}_n$, as can be seen with the example sets $\mathcal{S}_{49}$ and $\mathcal{S}_{50}$ depicted in in Figure 4.20. The intensity landscape then sees both pairs at once due to accumulate loss from previous itera-

tions. These regions experience large amounts of absorption, that rivals the amount experienced by regular orbits, which can be seen at the intensity landscapes at late iterations in Figure 4.18 and Figure 4.20.

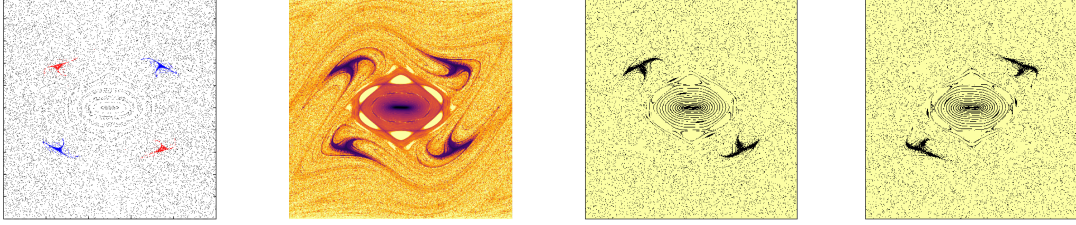


Figure 4.20: Pairs of sticky trajectories are plotted on the left half of the Poincaré diagram (left), coloured in blue and red. For an oval billiard with absorbing region with absorption rate $\gamma = 0.4$ and radius $r_{AR} = 0.1$, the intensity landscape is shown at $n = 50$ (middle left) with plots for $\mathcal{S}_{49}$ (middle right) and $\mathcal{S}_{50}$ (right).

In this chapter, I discussed the application of the semiclassical method from previous chapters to a two-dimensional dynamical system. I introduced a classical billiard model with an internal spatial region of loss, in which the analogue of a 4-dimensional norm landscape for the system, the intensity landscape, was constructed for the Poincaré-Birkhoff reduced phase space. Accompanying intensity landscapes were plots, also on the Poincaré-Birkhoff phase space, showing the set of initial coordinates of trajectories that intersect the internal absorbing region on the n -th iteration. We observed that these set diagrams were related from one iteration to the next via a backwards map, and cumulatively, the set diagrams formed the skeleton of the intensity landscapes.

I provided three examples of billiards, a circle, an ellipse and an oval, to examine the iterative phase space dynamics and observe the non-trivial structure that emerges in the intensity landscape, often illuminating phase space structures that would otherwise be shrouded. Naturally, the next step would be to investigate the phase space dynamics and intensity landscape of these three billiards, along with other billiard models, where an internal loss region is displaced from the centre and thus breaks the symmetry of the billiard. In the next chapter, I conclude this thesis with a discussion on further research topics where some progress has already been made, starting with a potential quantum counterpart to this open billiard model.

Chapter 5

Further Research

In this final chapter I propose two areas of study that have the potential for further research.

5.1 Non-Hermitian Quantum Billiard

In this section I present an outlook discussion on a non-Hermitian quantum billiard model, carrying on the research in the previous Chapter 4. Already in the one-dimensional model in section 2.6 we have observed that there are pronounced quantum effects on top of the semiclassical Wigner dynamics. Specifically, for small loss strength γ , the quantum and classical dynamics guides the distribution through the absorbing region whilst the global norm decreases upon intersection. However, for significantly large loss strength, a pure quantum effect occurs where the distribution reflects off the imaginary potential and global norm decreases at a lesser rate than in the case of small loss strength. Following on from our analysis of a two-dimensional classical billiard with internal loss, the next step would be the investigation of the resulting semiclassical Wigner dynamics in comparison to the full quantum dynamics. The results in this section are preliminary.

As discussed in section 2.6, to apply the semiclassical method from chapter 2.2, it is useful to consider a smooth Hamiltonian to approximate the piecewise defined Hamiltonian, made up of a Hermitian and non-Hermitian regions. In particular, I consider the non-Hermitian Hamiltonian

$$\hat{K} = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} + \left(\frac{\hat{q}_1^2 + \hat{q}_2^2}{r_B^2} \right)^{S_B} - i\gamma \left(1 - \tanh \left(\left(\frac{\hat{q}_1^2 + \hat{q}_2^2}{r_{\text{NH}}^2} \right)^{S_{\text{NH}}} \right) \right), \quad (5.1)$$

where $\{\hat{q}_1, \hat{q}_2\}$ and $\{\hat{p}_1, \hat{p}_2\}$ are the two-dimensional position and momentum operators respectively. The radii of the billiard boundary and non-Hermitian region are $r_B \in \mathbb{R}$ and $r_{NH} \in \mathbb{R}$ where $r_{NH} < r_B$. The steepness of the potential at the billiard boundary and at the border of the non-Hermitian region is regulated by $S_B \in \mathbb{R}$ and $S_{NH} \in \mathbb{R}$. For sufficiently large S_B and S_{NH} , the Hamiltonian is an approximation of the piecewise defined Hamiltonian. In these examples I choose the parameter values $r_B = 10, r_{NH} = 1, S_B = S_{NH} = 10$ with loss strength $\gamma = 0$ and $\gamma = 10$.

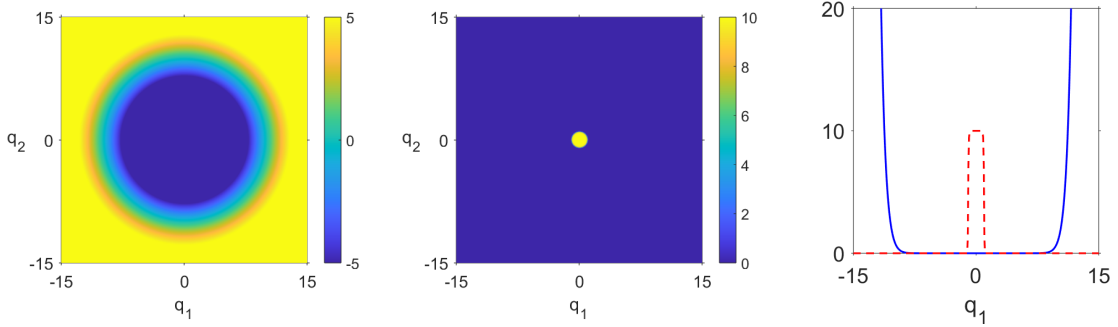


Figure 5.1: Two-dimensional plots of real (left on logarithmic scale) and imaginary (middle) parts of the smooth billiard potential (5.1). One-dimensional plot of the real (blue solid line) and imaginary (red dashed line) at $q_2 = 0$. The potential has been plotted for parameters $r_B = 10, r_{NH} = 1, S_B = S_{NH} = 10, \gamma = 10$.

Figure 5.1 shows the real (left on logarithmic scale) and imaginary (middle) parts of the potential as two-dimensional (q_1, q_2) false colour plots. A one-dimensional plot of the real (blue) and imaginary (red) potentials at $q_2 = 0$ is shown in the right plot of Figure 5.1.

From the quantum and semiclassical description (2.12), we can numerically simulate a four-dimensional time-evolved Wigner distribution and norm landscape. To visualise the evolved state we can then integrate the Wigner distribution over the momentum variables (p_1, p_2) to obtain the time-evolved probability distribution in position space $|\psi(t, q_1, q_2)|^2$. Figures 5.2 and 5.3 show the classical and quantum dynamics of the Hermitian ($\gamma = 0$) and non-Hermitian ($\gamma = 10$) billiards, respectively. For both systems the initial conditions are $(q_1, q_2) = (-3, -3)$ and $(p_1, p_2) = (6, 6)$ for short time $t = 0, 0.5$ and 1.25 .

As demonstrated in Figure 5.2, the Hermitian potential generates a simple free-particle dynamics before it hits the billiard boundary. In this case, the classical and quantum dynamics are almost identical, as in the absence of a confining potential the quantum and semiclassical Wigner dynamics for a free particle agree exactly. These free particle dynamics are seen in part in the non-Hermitian billiard, where

the particle moves at the same rate, but the interaction with the imaginary potential has a dramatic effect which we observe in Figure 5.3.

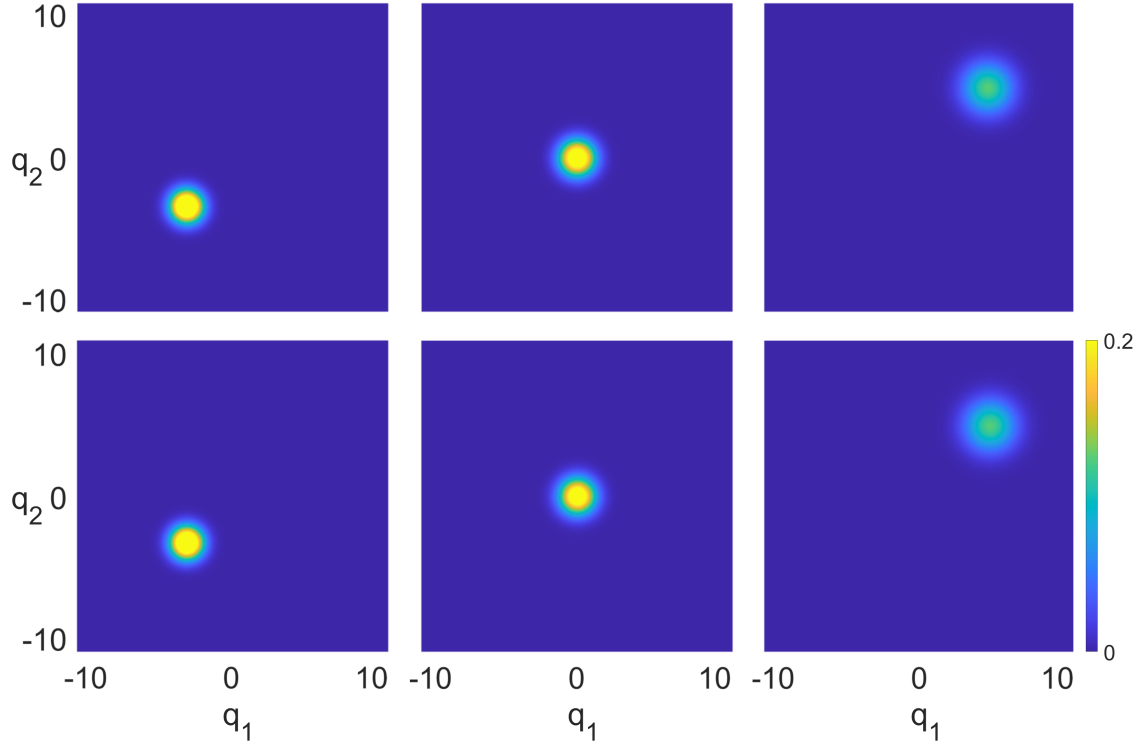


Figure 5.2: The classical (top row) and quantum (bottom row) dynamics of the probability distribution of an initial Gaussian wavepacket evolving the Hermitian ($\gamma = 0$) smooth billiard potential (5.1). The Gaussian wavepacket has initial conditions $(q_1, q_2) = (-3, -3)$ and $(p_1, p_2) = (6, 6)$, and the time shown are (from left to right): $t = 0$, $t = 0.5$ and $t = 1.25$.

Consider the classical dynamics depicted in the top row of Figure 5.3. At $t = 0.5$ the wavepacket has intersected the imaginary potential and we see that part of the distribution has experienced such strong absorption that it has partially disappeared. The parts of the wavepacket that bypass the absorbing region survive and become the new areas with the highest probability. This can be seen at $t = 1.25$ after the intersection, as the surviving renormalised wavepacket has split into two localised regions. In the quantum dynamics the wavepacket intersects the imaginary region at time $t = 0.5$ and part of it begins to be absorbed. The value of γ is relatively small, such that the absorbing region largely acts as a loss area similar to the classical case, however we do see some faint ripples in the wavepacket, reminiscent of the diffraction of a wavepacket on a Hermitian "hard wall" potential. We observe that the classical dynamics at the same time does not have these ripples, indicating this is a pure quantum effect. At $t = 1.25$ we see that wavepacket under the quantum dynamics

develops three localised regions, which cover a greater region of the position space than in the classical dynamics. We expect that for very small γ the absorbing region causes only loss, but for larger γ the absorbing region will act similar to a Hermitian hard wall, reminiscent to what we observed in the section 2.6.

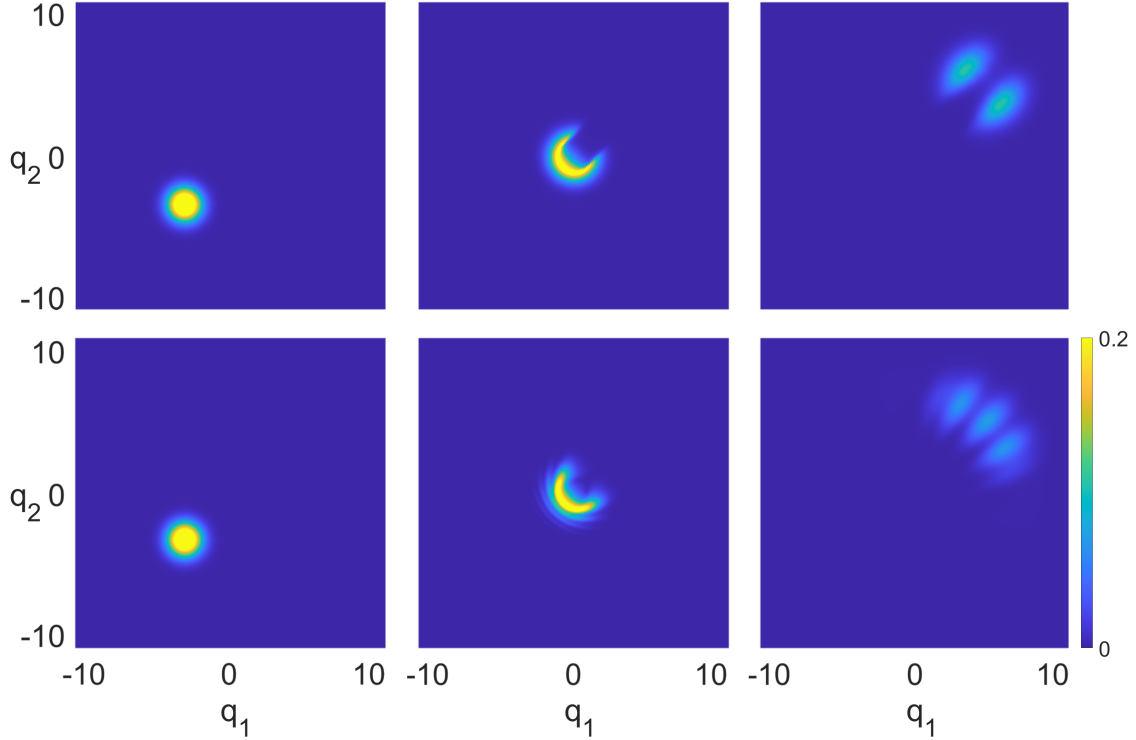


Figure 5.3: The classical (top row) and quantum (bottom row) dynamics of the probability distribution of an initial Gaussian wavepacket evolving the non-Hermitian ($\gamma = 10$) smooth billiard potential (5.1). The Gaussian wavepacket has initial conditions $(q_1, q_2) = (-3, -3)$ and $(p_1, p_2) = (6, 6)$, and the time shown are (from left to right): $t = 0$, $t = 0.5$ and $t = 1.25$.

Finally, note that the global norm in the classical and quantum dynamics agree quite well for the chosen value of $\gamma = 10$ as we see in the left plot of Figure 5.4. In Figure 5.4 I show the global norm plotted against time for the quantum dynamics (solid blue line) and the classical dynamics (dashed red line) for two values $\gamma = 10$ (left plot) and $\gamma = 100$ (right plot). There are subtle differences between the classical and quantum norm values for the case $\gamma = 10$, which maybe the result of the approximation of dynamics or potential numerical error in the simulations. In the case $\gamma = 100$ we see that the global norm for the classical dynamics decreases further than in the case $\gamma = 10$. In contrast, the global norm from the quantum dynamics is increased compared to the $\gamma = 10$ case for the larger γ , illustrating that the wavepacket is being partially reflected off the absorbing region. Also note

that the norm does not drop to zero for large γ even in the classical case, because parts of the wavepacket can bypass the absorbing region on the sides.

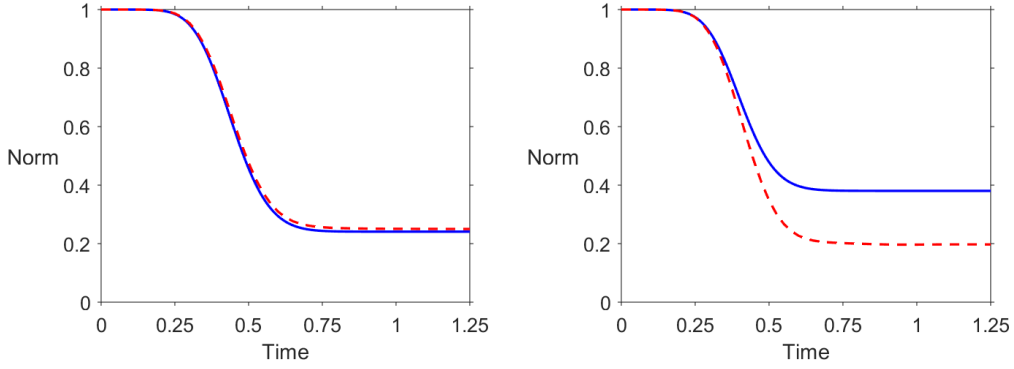


Figure 5.4: Plots of the time-dependent global norm of the wavepacket over time. The figure compares the norm of the state evolved by the classical (dashed red line) and quantum dynamics (solid blue line) for two potentials: $\gamma = 10$ (left), and $\gamma = 100$ (right).

The model proposed here would be an excellent case study for semiclassical analysis of Hermitian and non-Hermitian dynamical systems evolved on a higher-dimensional phase space. With the proposed approach, one could further unravel the foundations of quantum phenomena caused by non-Hermitian Hamiltonians through evaluating the classical dynamics of the system.

For research to progress in this direction, the first hurdle to overcome is computational cost and numerical restrictions. Specifically, the semiclassical method relies on solving the differential equation 2.10 across a four-dimensional phase space. The number of operations for these simulations are, of course, orders of magnitude larger in comparison to the simulations seen in chapters 2 and 3. That is, before considering the complication of numerically solving for Hamiltonians with steep potentials. Nevertheless, I have shown that even with these limitations, results are indeed possible and there is a great potential, with further research and optimisation, to study the classical-quantum correspondence for non-Hermitian billiard model with internal loss. Furthermore, I hope that with presenting this model and generating the system dynamics, I have provided a proof of concept in applying the semiclassical method showcased in this thesis to higher dimensional non-Hermitian systems with non-trivial Hamiltonians.

5.2 Glauber R Representation

Here I provide an outlook discussion on the application of the semiclassical method studied throughout this thesis to the Glauber R-representation [1]. The Glauber R-representation can be viewed as generalisation of the Husimi distribution on a four-dimensional phase-space. The R-representation of a state $\hat{\rho}$, as defined in [1], is

$$R(z_1, z_2) = e^{\frac{1}{2}(z_1^* z_1 + z_2^* z_2)} \langle z_1 | \psi \rangle \langle \psi | z_2 \rangle = e^{\frac{1}{2}(z_1^* z_1 + z_2^* z_2)} \tilde{R}(z_1, z_2), \quad (5.2)$$

viewed as a function of z_1 and z_2 and where I denoted $\tilde{R}(z_1, z_2) = \langle z_1 | \psi \rangle \langle \psi | z_2 \rangle$. An advantage over the Husimi representation (that only utilises the diagonal elements of $\hat{\rho}$ in the coherent state basis) is that we can retrieve the marginal probabilities in two-dimensional position and momentum as

$$R(t, q_1, q_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle q_1 | z_1 \rangle R(t, z_1, z_2) \langle z_2 | q_2 \rangle dz_1 dz_1^* dz_2 dz_2^*, \quad (5.3)$$

$$R(t, p_1, p_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle p_1 | z_1 \rangle R(t, z_1, z_2) \langle z_2 | p_2 \rangle dz_1 dz_1^* dz_2 dz_2^*. \quad (5.4)$$

In other words, $\langle q_1 | \hat{\rho} | q_2 \rangle = R(t, q_1, q_2)$ and $\langle p_1 | \hat{\rho} | p_2 \rangle = R(t, p_1, p_2)$ with $|\psi(q)|^2 = \langle q | \hat{\rho} | q \rangle$ and $|\tilde{\psi}(p)|^2 = \langle p | \hat{\rho} | p \rangle$ for pure states. The dynamics of the R-distribution for a Hermitian Hamiltonian H may be written as an extension of (1.55) with the identities (for further details see appendix A)

$$\hat{a}^\dagger |z_2\rangle \langle z_1| = (\hat{a}^\dagger |z_2\rangle) \langle z_1| = |z_2\rangle \langle z_1| \left(\overleftarrow{\frac{\partial}{\partial z_2}} + \frac{z_2^*}{2} \right), \quad (5.5)$$

$$|z_2\rangle \langle z_1| \hat{a} = |z_2\rangle (\langle z_1| \hat{a}) = \left(\overrightarrow{\frac{\partial}{\partial z_1^*}} + \frac{z_1}{2} \right) |z_2\rangle \langle z_1|. \quad (5.6)$$

The dynamical equation for the R-distribution is then found as

$$\frac{dR}{dt} = \sum_{n,m=0} iH_{n,m} R(z_1^*, z_2) \left(\overleftarrow{\frac{\partial}{\partial z_2}} + \frac{z_2^*}{2} \right)^m z_2^n - iH_{n,m} z_1^{*n} \left(\overrightarrow{\frac{\partial}{\partial z_1^*}} + \frac{z_1}{2} \right)^m R(z_1^*, z_2), \quad (5.7)$$

which is equivalent to

$$\frac{d\tilde{R}}{dt} = \sum_{n,m=0} iH_{n,m} \tilde{R}(z_1^*, z_2) \left(\overleftarrow{\frac{\partial}{\partial z_2}} + z_2^* \right)^m z_2^n - iH_{n,m} z_1^{*n} \left(\overrightarrow{\frac{\partial}{\partial z_1^*}} + z_1 \right)^m \tilde{R}(z_1^*, z_2), \quad (5.8)$$

because

$$\frac{\partial \tilde{R}}{\partial z_1} = \left(\frac{1}{2} z_1 \tilde{R} + \frac{\partial \tilde{R}}{\partial z_1} \right) e^{\frac{1}{2}(z_1^* z_1 + z_2^* z_2)}, \quad (5.9)$$

$$\frac{\partial \tilde{R}}{\partial z_2} = \left(\frac{1}{2} z_2 \tilde{R} + \frac{\partial \tilde{R}}{\partial z_2} \right) e^{\frac{1}{2}(z_1^* z_1 + z_2^* z_2)}. \quad (5.10)$$

In working with $\tilde{R}(z_1, z_2)$ we can see that for the case $z_1 = z_2$ we get back the Husimi distribution $\tilde{R}(z, z^*) = Q(z, z^*)$.

In the semiclassical limit $|z_1|^2 \rightarrow \infty$ and $|z_2|^2 \rightarrow \infty$ the dynamical equation is

$$\frac{d\tilde{R}}{dt} - 2i \frac{\partial H_2}{\partial z_2^*} \frac{\partial \tilde{R}}{\partial z_2} + 2i \frac{\partial H_1}{\partial z_1} \frac{\partial \tilde{R}}{\partial z_1^*} = i(H_2 - H_1) \tilde{R}. \quad (5.11)$$

Whilst generally, I treat (z, z^*) as independent variables that typically share the equivalent complex conjugated dynamical equations, here there remains only one dynamical equation to describe either variable z_i or z_i^* in each pair of variables. It is also reasonable to allow this characteristic equation to generate the dynamics of the missing variable by taking the complex conjugate.

$$\frac{dz_1}{dt} = -2i \frac{\partial H_1}{\partial z_1^*}, \quad (5.12)$$

$$\frac{dz_2}{dt} = -2i \frac{\partial H_2}{\partial z_2^*}. \quad (5.13)$$

I denote the solution to the four phase-space variable differential equations as the flow $(z_1(t), z_2(t)) = \Phi_t(z_1, z_2)$. The time-evolved the R-representation can then be written as

$$R(t, z_1, z_2) = e^{\frac{1}{2}(z_1^* z_1 + z_2^* z_2)} \tilde{R}_0(\Phi_t^{-1}(z_1, z_2)) G(z_1, z_2), \quad (5.14)$$

where $\tilde{R}_0(z_1, z_2) = \tilde{R}(z_1, z_2) = \langle z_1 | \psi(0) \rangle \langle \psi(0) | z_2 \rangle$ and $\Phi_t^{-1}(z_1, z_2)$ is the backwards Hamiltonian flow. Here we see a factor $G(z_1, z_2)$ that emerges from the higher dimensions of the distribution and is generated by

$$\frac{dG}{dt} = i(H_2(\Phi_t^{-1}(z_2, z_2^*)) - H_1(\Phi_t^{-1}(z_1, z_1^*))) G. \quad (5.15)$$

In the case $z_1 = z_2$ we get back the Liouville equation for the Husimi distribution (1.80).

The extension to non-Hermitian Hamiltonians (2.1) is straight forward and

the dynamical equation for $\tilde{R}(z_1, z_2)$ is

$$\frac{d\tilde{R}}{dt} = \sum_{n,m=0} iK_{n,m}^* \tilde{R}(z_1^*, z_2) \left(\overleftarrow{\frac{\partial}{\partial z_2}} + z_2^* \right)^m z_2^n - iK_{n,m} z_1^{*n} \left(\overrightarrow{\frac{\partial}{\partial z_1^*}} + z_1 \right)^m \tilde{R}(z_1^*, z_2). \quad (5.16)$$

In the the semiclassical limit we have the dynamical equation

$$\frac{d\tilde{R}}{dt} - 2i \frac{\partial K_2^*}{\partial z_2^*} \frac{\partial \tilde{R}}{\partial z_2} + 2i \frac{\partial K_1}{\partial z_1} \frac{\partial \tilde{R}}{\partial z_1^*} = i(K_2^* - K_1) \tilde{R}, \quad (5.17)$$

with generalised solution

$$R(t, z_1, z_2) = e^{\frac{1}{2}(z_1^* z_1 + z_2^* z_2)} \tilde{R}_0(\Phi^{-1}(z_1, z_2)) w(t, z_1, z_2), \quad (5.18)$$

where $\tilde{R}_0(z_1, z_2) = \langle z_1 | \psi(0) \rangle \langle \psi(0) | z_2 \rangle$ is an initial distribution. The backwards flow $\Phi^{-1}(z_1, z_2)$ is the inverse of the forwards flow generated by

$$\frac{dz_1}{dt} = -2i \frac{\partial K_1^*}{\partial z_1^*}, \quad (5.19)$$

$$\frac{dz_2}{dt} = -2i \frac{\partial K_2^*}{\partial z_2^*}. \quad (5.20)$$

Lastly, the four-dimensional norm landscape $w(t, z_1, z_2)$ is generated by the equation

$$\frac{dw}{dt} = i(K_2^*(\Phi^{-1}(z_2, z_2^*)) - K_1(\Phi^{-1}(z_1, z_1^*)))w, \quad (5.21)$$

with initial condition $w(0, z_1, z_2) = 1$.

As we observed with Hermitian systems, the dynamics for the R-distribution for non-Hermitian systems can then be reduced in the case $z_1 = z_2$ to get the dynamics of the Husimi distribution. From the Bargmann distribution for non-Hermitian systems, one may calculate the probability distribution for the wave function. This is a similar approach to chapter 5.1 where I used the Wigner distribution to evolve the system and study the dynamics of the two-dimensional wavepacket. However, using the R-distribution has the same limitations in evolving a high-dimensional phase space is costly. Further work on this involves either optimising the numerical simulations of higher dimensional phase spaces to mitigate the computing power needed. Furthermore, one could use the Stellar representation [113, 114] to describe the dynamics of the zeroes of the Husimi distribution, as potential means to get further information of the time evolved wave packet.

In this chapter, I proposed two avenues for further research based on the results presented in previous chapters. Namely, the application of the semiclassical phase space method to a 2-dimensional non-Hermitian quantum billiard with an internal spacial absorbing region and the derivation of the time-evolved Bargmann distribution in the semiclassical limit. Preliminary results on both topics have been shared, along with a summary of unresolved challenges, particularly the computational limitations of numerical simulations that need to be addressed for research progress. I shared my optimism that pursuing these research projects further will yield impactful results and further advance our understanding of the phase space dynamics of non-Hermitian systems.

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Appendix A

D-algebra Proofs

In this appendix, proofs are provided for the main D-algebra identities (5.5) and (1.54).

Identity (5.5)

$$\hat{a}^\dagger|z\rangle = |z\rangle \left(\overleftarrow{\frac{\partial}{\partial z}} + \frac{z^*}{2} \right). \quad (\text{A.1})$$

Proof:

The left hand side can be written as

$$\begin{aligned} \hat{a}^\dagger|z\rangle &= \hat{a}^\dagger \sum_{n=0} e^{-\frac{|z|^2}{2}} \frac{z^n}{\sqrt{n!}} |n\rangle \\ &= e^{-\frac{|z|^2}{2}} \sum_{n=0} \frac{z^n}{\sqrt{n!}} \sqrt{n+1} |n+1\rangle. \end{aligned} \quad (\text{A.2})$$

The first term on the right hand side, can be explicitly calculated as

$$\begin{aligned} |z\rangle \overleftarrow{\frac{\partial}{\partial z}} &= \frac{\partial}{\partial z} |z\rangle = \frac{\partial}{\partial z} \sum_{n=0} e^{-\frac{|z|^2}{2}} \frac{z^n}{\sqrt{n!}} |n\rangle, \\ &= \sum_{n=0} -\frac{z^*}{2} e^{-\frac{|z|^2}{2}} \frac{z^n}{\sqrt{n!}} |n\rangle + \sum_{n=0} e^{-\frac{|z|^2}{2}} n \frac{z^{n-1}}{\sqrt{n!}} |n\rangle, \\ &= -\frac{z^*}{2} |z\rangle + \sum_{n=1} e^{-\frac{|z|^2}{2}} \sqrt{n} \frac{z^{n-1}}{\sqrt{(n-1)!}} |n\rangle, \\ &= -\frac{z^*}{2} |z\rangle + \sum_{n'=0} e^{-\frac{|z|^2}{2}} \sqrt{n'+1} \frac{z^{n'}}{\sqrt{n'!}} |n'+1\rangle, \end{aligned} \quad (\text{A.3})$$

Using (A.2) we find

$$|z\rangle \overleftarrow{\frac{\partial}{\partial z}} = -\frac{z^*}{2} |z\rangle + \hat{a}^\dagger |z\rangle. \quad (\text{A.4})$$

Therefore

$$\hat{a}^\dagger|z\rangle = |z\rangle \left(\overleftarrow{\frac{\partial}{\partial z}} + \frac{z^*}{2} \right). \quad \square \quad (\text{A.5})$$

With the above identity, we simply take the adjoint to show that

$$\langle z|\hat{a} = \left(\overrightarrow{\frac{\partial}{\partial z^*}} + \frac{z}{2} \right) \langle z|. \quad \square \quad (\text{A.6})$$

Identity (1.54)

$$\hat{a}^\dagger|z\rangle\langle z| = |z\rangle\langle z| \left(\overleftarrow{\frac{\partial}{\partial z}} + z^* \right). \quad (\text{A.7})$$

Proof:

The left hand side, is expanded as

$$\begin{aligned} \hat{a}^\dagger|z\rangle\langle z| &= \hat{a}^\dagger \sum_{n,m=0} e^{-\frac{|z|^2}{2}} e^{-\frac{|z|^2}{2}} \frac{z^n}{\sqrt{n!}} \frac{z^{*m}}{\sqrt{m!}} |n\rangle\langle m|, \\ &= \sum_{n,m=0} e^{-|z|^2} \frac{z^n}{\sqrt{n!}} \frac{z^{*m}}{\sqrt{m!}} \sqrt{n+1} |n+1\rangle\langle m|. \end{aligned} \quad (\text{A.8})$$

The right hand side is explicitly found as

$$\begin{aligned} |z\rangle\langle z| \overleftarrow{\frac{\partial}{\partial z}} &= \frac{\partial}{\partial z} |z\rangle\langle z| = -z^*|z\rangle\langle z| + \sum_{n,m=0} e^{-|z|^2} n \frac{z^{n-1}}{\sqrt{n!}} \frac{z^{*m}}{\sqrt{m!}} |n\rangle\langle m|, \\ &= -z^*|z\rangle\langle z| + \sum_{n=1,m=0} e^{-|z|^2} \sqrt{n} \frac{z^{n-1}}{\sqrt{(n-1)!}} \frac{z^{*m}}{\sqrt{m!}} |n\rangle\langle m|, \\ &= -z^*|z\rangle\langle z| + \sum_{n',m=0} e^{-|z|^2} \sqrt{n'+1} \frac{z^{n'}}{\sqrt{n'!}} \frac{z^{*m}}{\sqrt{m!}} |n'+1\rangle\langle m|, \end{aligned} \quad (\text{A.9})$$

and thus

$$|z\rangle\langle z| \overleftarrow{\frac{\partial}{\partial z}} = -z^*|z\rangle\langle z| + \hat{a}^\dagger|z\rangle\langle z|. \quad (\text{A.10})$$

Therefore,

$$\hat{a}^\dagger|z\rangle\langle z| = |z\rangle\langle z| \left(\overleftarrow{\frac{\partial}{\partial z}} + z^* \right). \quad \square \quad (\text{A.11})$$

With the above identity, we simply take the adjoint to show that adjoint of the above

$$|z\rangle\langle z|\hat{a} = \left(\overrightarrow{\frac{\partial}{\partial z^*}} + z\right) |z\rangle\langle z|. \quad \square \quad (\text{A.12})$$

Lastly we will consider an application of the D-algebra identities to ladder operators on either side of the density operator. First we note that the expectation of an operator $\hat{A}$ in coherent states may be written as

$$\langle z|\hat{A}|z\rangle = \sum_i \langle z|i\rangle\langle i|\hat{A}|z\rangle = \sum_i \langle i|\hat{A}|z\rangle\langle z|i\rangle = \text{Tr}(\hat{A}|z\rangle\langle z|), \quad (\text{A.13})$$

which is equivalent to

$$\langle z|\hat{A}|z\rangle = \text{Tr}(|z\rangle\langle z|\hat{A}), \quad (\text{A.14})$$

because the trace is cyclic. More generally, we have

$$\text{Tr}(\hat{A}\hat{B}|z\rangle\langle z|) = \text{Tr}(\hat{B}|z\rangle\langle z|\hat{A}) = \text{Tr}(|z\rangle\langle z|\hat{A}\hat{B}). \quad (\text{A.15})$$

We can thus show that

$$\begin{aligned} \langle z|\hat{a}\hat{\rho}\hat{a}^\dagger|z\rangle &= \text{Tr}(|z\rangle\langle z|\hat{a}\hat{\rho}\hat{a}^\dagger), \\ &= \left(z + \overrightarrow{\frac{\partial}{\partial z^*}}\right) \text{Tr}(\hat{\rho}\hat{a}^\dagger|z\rangle\langle z|), \\ &= \left(z + \overrightarrow{\frac{\partial}{\partial z^*}}\right) Q\left(z^* + \overleftarrow{\frac{\partial}{\partial z}}\right), \\ &= zQ\left(z^* + \overleftarrow{\frac{\partial}{\partial z}}\right) + \overrightarrow{\frac{\partial}{\partial z^*}}Q\left(z^* + \overleftarrow{\frac{\partial}{\partial z}}\right), \\ &= |z|^2Q + Q + z\frac{\partial Q}{\partial z} + z^*\frac{\partial Q}{\partial z^*} + Q + \frac{\partial^2 Q}{\partial z^*\partial z}, \\ &= (|z|^2 + 2)Q + z\frac{\partial Q}{\partial z} + z^*\frac{\partial Q}{\partial z^*} + \frac{\partial^2 Q}{\partial z^*\partial z}. \end{aligned} \quad (\text{A.16})$$

Then it follows that

$$\langle z|\hat{a}^n\hat{\rho}\hat{a}^{\dagger m}|z\rangle = \left(z + \overrightarrow{\frac{\partial}{\partial z^*}}\right)^n Q\left(z^* + \overleftarrow{\frac{\partial}{\partial z}}\right)^m. \quad (\text{A.17})$$

Appendix B

Method of Characteristics

The method of characteristics [115] is used to solve first-order partial differential equations that are quasi-linear and hyperbolic. These equations have the form

$$\frac{\partial A}{\partial t} + \sum_j^n \vec{f}_j(t, \vec{x}, A) \frac{\partial A}{\partial x_j} - g(t, \vec{x}, A) = 0, \quad (\text{B.1})$$

with a vector $\vec{x} = (x_1, \dots, x_n)$ and functions $\vec{f}(t, \vec{x}, A) = (f_1(t, \vec{x}, A), \dots, f_n(t, \vec{x}, A))$, $g(t, \vec{x}, A)$. We are solving for the function $A(t, \vec{x})$. We use the method of characteristics in several instances during Chapters 1, 2 and 5 to find the time-evolved Husimi, Wigner and Glauber R distributions from semiclassical first-order dynamical equations. In this appendix we will walk through the steps of the method of characteristics by working through the derivation of the the Husimi distribution dynamics for non-Hermitian systems (2.4)

$$\frac{dQ}{dt} - i \frac{\partial K^*}{\partial z^*} \frac{\partial Q}{\partial z} + i \frac{\partial K}{\partial z} \frac{\partial Q}{\partial z^*} = -2\Gamma Q. \quad (\text{B.2})$$

To start, we relate the coefficients of the terms in the partial differential equation to a system of ordinary differential equations (ODEs) that is parametrised by a variable s . The geometric interpretation of this parametrisation can be found in Chapter 6 of [115]. In this instance, the equations we need to solve are

$$\frac{dt}{ds} = 1, \quad (\text{B.3})$$

$$\frac{dz}{ds} = -i \frac{\partial K^*}{\partial z^*}, \quad (\text{B.4})$$

$$\frac{dz^*}{ds} = i \frac{\partial K}{\partial z}, \quad (\text{B.5})$$

$$\frac{dQ}{ds} = -2\Gamma Q. \quad (\text{B.6})$$

With respect to the parameter s , we consider the following initial conditions,

$$t(s=0)=0, \quad z(s=0)=\xi, \quad z^*(s=0)=\xi^*, \quad Q(\xi, \xi^*, s=0)=Q_0(\xi, \xi^*). \quad (\text{B.7})$$

where we introduce a complex pair of parameters (ξ, ξ^*) .

The next step is to solve the system of equations one at a time, with respect to the parameters (s, ξ, ξ^*) . The key is to address all ODEs that contain variables independent of other equations first. For instance, in this thesis, the ODEs represented in equation (B.3) are always independent of the variables $(z, z^*, Q(z, z^*))$. As a result, the solution is simply $t=s$, meaning that the evolution of the ODEs with respect to s is equivalent to their evolution with respect to time.

Then to address the ODEs that have right-hand side that consists of functions that depend on multiple variables we need to substitute in the time-evolved variables (which may or may not be known) before proceeding to solve them. Here we are attempting to solve for a function along the characteristic curves described by time-evolved variables.

In the case of our example: the ODEs (B.4) and (B.5) for the phase-space variables (z, z^*) have the solution

$$(z, z^*) = \Phi_t(\xi, \xi^*), \quad (\text{B.8})$$

where Φ_t is a phase-space flow that transports the initial (parametrised) conditions along trajectories (known as characteristics). We often call $\Phi_t(\xi, \xi^*)$ the forwards flow, where the inverse of this function

$$(\xi, \xi^*) = \Phi_t^{-1}(z, z^*), \quad (\text{B.9})$$

is the backwards flow. Typically, we write the flow without the parametrisation, describing the trajectories as time-dependent variables

$$(z(t), z^*(t)) = \Phi_t(z, z^*), \quad (\text{B.10})$$

where $(z, z^*) = (z(0), z(0))$.

The next step is to substitute the parametrised time-evolved variables (B.8) into (B.6) such that we are solving

$$\frac{dQ}{dt} = -2\Gamma(\Phi_t(\xi, \xi^*))Q, \quad (\text{B.11})$$

In all instances within this thesis, we can use separation of variables to write a generalised solution

$$Q(t, \xi, \xi^*) = Q_0(\xi, \xi^*) e^{-2 \int_0^t \Gamma(\Phi_t(\xi, \xi^*)) dt}. \quad (\text{B.12})$$

The final step to write the time-evolved Husimi distribution is to convert the parameters ξ and ξ^* back to the phase-space variables, z and z^* . This can be easily accomplished by substituting in the backward flow (B.9).

$$Q(t, z, z^*) = Q_0(\Phi_t^{-1}(z, z^*)) e^{-2\Gamma_{\text{int}}(t, z, z^*)}, \quad (\text{B.13})$$

where

$$\Gamma_{\text{int}}(t, z, z^*) = \int_0^t \Gamma(\Phi_t(\xi, \xi^*)) dt \Big|_{(\xi, \xi^*) = \Phi^{-1}(z, z^*)}. \quad (\text{B.14})$$

The time-evolved Husimi distribution can also be expressed as

$$Q(t, z, z^*) = Q_0(\Phi_t^{-1}(z, z^*)) w(t, z, z^*), \quad (\text{B.15})$$

with norm landscape $w(z, z^*, t)$, described by

$$\dot{w}(t, z, z^*) = -2\Gamma(\Phi_t^{-1}(z, z^*)) w(t, z, z^*), \quad (\text{B.16})$$

for initial condition $w(z, z^*, 0) = 1$.

Appendix C

Numerical Implementation

In this section, I provide my MATLAB code that numerically generates the classic dynamics of the renormalised Husimi distribution. The algorithm is described in section 2.3. This code can be adapted to generate the classical dynamics of the renormalised Wigner distribution with the following minor changes: 1) Use a Weyl symbol of the Hamiltonian in function *deg*. 2) Use the backwards flow for Hamilton's equations of motion in function *deg*. 3) Define the initial function *rho0* to be a Wigner function.

My code is set up to generate the classical dynamics of the non-Hermitian harmonic oscillator from Figure 2.3 from section 2.4.1.

```
function Classical_Husimi_distribution

clock = tic
%% Set up
% Time
plot_time=[0;2*pi/3;4*pi/3;2*pi];
time_steps = 201;
time=linspace(0,max(plot_time),time_steps);
% Specify Phase Space Grid
gridsize = 10;
mesh_points = 601;
q=linspace(-gridsize,gridsize,mesh_points);
p=linspace(-gridsize,gridsize,mesh_points);
[Q,P]=meshgrid(q,p);
dq = abs(q(2) - q(1));
```

```

dp = abs(p(2) - p(1));
Z=sqrt(0.5)*(Q+1i*P);
n0=1; %Initial condition for norm
%Arrays
rho=zeros(length(p),length(q),length(time));
rho0_t=zeros(length(p),length(q),length(time));
norm_land=zeros(length(p),length(q),length(time));

%% Run ODE with ODE45 and function deq below.
%Use initial distribution function rho0 below.
%% Uses Parallel Computing Toolbox.
for ind_q=1:length(q)
    q_ic=q(ind_q); %initial condition for this ODE run
    for ind_p=1:length(p)
        p_ic=p(ind_p); %initial condition for this ODE run
        [~,znf]=ode45(@(t,z) deq(t,z),time,[q_ic,p_ic,n0]);
        %output evolved phase space points at a time as array
        zf=sqrt(0.5)*(znf(:,1)+1i*znf(:,2));
        norm_land(ind_p,ind_q,:)=znf(:,3); %The solution to the dw/dt ODE
        rho(ind_p,ind_q,:)=rho0(zf).*znf(:,3); %Final solution of the Husimi
    end
end

%% Plotting. Set up for Husimi Function 'rho'.
%Change surf plot to 'norm_land' for norm landscape.
for ind_t=1:length(time)
    norm(ind_t)=sum(sum(rho(:,:,ind_t)))*dq*dp;
    rho=rho./norm(ind_t);
    plot_2 = abs(time(ind_t) - plot_time(2));
    plot_3 = abs(time(ind_t) - plot_time(3));
    time_length = max(plot_time)/(2*time_steps);

    if time(ind_t) == 0 || plot_2 <= time_length || plot_3 <= time_length ...
        ... || time(ind_t) == plot_time(4)
    f= figure(ind_t);
    surf(q,p,rho(:,:,ind_t),'LineStyle','none')
    f.Position = [600 00 800 800];

```

```
clim([0 0.08])
c = colorbar;
c.FontSize = 30;
c.Ticks = ([0 0.08]);
view(0,90)
axis square
set(gcf,"Color",[1 1 1]);
fontsize(gca, 30,'points')
title(['t=',num2str(time(ind_t))])
xlabel('q')
ylabel('p','Rotation',0)
xticks([-10 0 10])
yticks([-10 0 10])

%% Save figures
St=num2str(ind_t);
figname = sprintf('Husimi_Non_Herm_Harmonic_Oscillator_time_%s.fig',St);
savefig(figname)
pngname = sprintf('Husimi_Non_Herm_Harmonic_Oscillator_time_%s.png',St);
saveas(gcf,pngname)
end
end
toc(clock)

function dz=deq(t,z)
q=z(1);
p=z(2);
norm=z(3);

%Specify Hamiltonian in p and q.
%Example: non-Hermitian Harmonic Oscillator (Wick symbol)
omega = 1.0;
gamma = 0.15;
%K = 0.5*(omega-1i*gamma)*(p^2+q^2 + 0.5)
%H = 0.5*omega*(p^2+q^2 + 0.5)
G = 0.5*gamma*(p^2+q^2 + 0.5); %Gamma
```

```
%Write differentials dH/dq, dH/dp, dG/dq, dG/dp
```

```
Hq = omega*q;
```

```
Hp = omega*p;
```

```
Gq = gamma*q;
```

```
Gp = gamma*p;
```

```
%% Solving the backwards flow, for the solution of the norm landscape
```

```
% For Wigner distribution set Gq = Gp = 0/
```

```
dz=[-Hp-Gq;
```

```
    Hq-Gp;
```

```
    -2*G*norm];
```

```
return
```

```
function r0=rho0(z) %Define the initial Husimi (or Wigner) distribution.
```

```
%Example: Displace n=2 harmonic oscillator eigenstate.
```

```
q0=4;
```

```
p0=2;
```

```
n=2; % Second excited state
```

```
z0=sqrt(0.5)*(q0+1i*p0);
```

```
r0=abs(z-z0).^ (2*n) ./factorial(n) .*exp(-abs(z-z0).^2); %Initial Husimi
```

```
return
```