

IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY
Petroleum Engineering Section
Department of Mineral Resources Engineering

A THEORETICAL STUDY OF CONING BEHAVIOUR
IN POROUS MEDIA

by

Christine Dharshini Gnanathurai

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ABSTRACT

The occurrence of water and gas coning is common in petroleum reservoir production. The modelling of coning phenomena is generally performed by finite difference numerical simulators. Such models are not only expensive but also have several uncertainties associated with their accuracy.

In this work, a theoretical investigation of the problem is made. Models are developed principally to test the accuracy of numerical coning simulators. They are also expected to act as inexpensive means of predicting coning behaviour.

Coning problems are classed mathematically as radial, free surface problems. In general, such problems cannot be solved analytically. Therefore it is at least of theoretical interest to study the analogous 2-dimensional plane flow problem, which has scope for analytical treatment. A technique based on the theory of complex variables is presented for the 2-dimensional problem. The free boundary in the physical plane is mapped onto a fixed boundary and the problem is solved in the transformed plane. Transformations from one plane to another are obtained numerically.

The radial problem is studied using a boundary integral technique. The governing differential equations are transformed onto an equivalent set of integral equations which depend solely on the boundary data. The integration range is further confined to the free surface, by formulating the problem as a basic flow problem together with a perturbation flow problem. Although the

integral equations cannot be solved exactly, their numerical solution requires boundary discretization only. Therefore, for a given computer storage and machine time, the model achieves far greater accuracy than is possible by conventional domain discretization methods.

The model developed in this work is used to determine frontal positions, critical flow-rates, critical cone heights, and breakthrough times. Comparisons, where possible, are made with existing solutions and several checks on the accuracy of the numerical scheme are carried out.

A simple model is also presented to predict the limiting water/gas-oil-ratio after breakthrough.

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*This thesis is dedicated
to
my mother*

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CHAPTER 1

INTRODUCTION

The unwanted production of water or gas by interface coning phenomena occurs frequently. Its occurrence influences the cost of oil production, the degree of reservoir depletion and the efficiency of the recovery mechanism. It is therefore of vital importance, to the petroleum engineer, to understand the mechanism underlying the coning phenomenon, and also to predict, and if possible prevent, the adverse effects of coning.

Analogous problems also arise in engineering hydrology when wells pumping fresh water begin to produce salt water due to the movement of the fresh water - salt water interface towards the pumping well. . In this instance coning has the drastic effect of polluting the fresh water.

A number of workers both in the petroleum industry and in engineering hydrology have given consideration to interface coning problems (2, 10, 34, 36). In this work we shall focus our attention on two types of coning problems. First we shall consider the case of plane-flow upconing in the 2-dimensional plane. The second problem, which we shall devote the greater part of this work to, is the one, more frequently encountered in practice and is the axi-symmetric or radial coning problem.

1.1 The plane-flow coning problem

Plane-flow or two-dimensional interface coning can occur in one of two instances.

Consider the situation shown in figure 1.1. Here L_1 and L_2 represent vertical oil wells penetrating the entire permeable stratum, and the shaded and unshaded areas represent the water and oil saturated regions respectively. In this case the permeable stratum is so thin that the flow of fluids can be considered as a two-dimensional flow in the plane of the stratum. Thus the oil saturated and water saturated regions can be considered as two-dimensional and the producing wells may be regarded as point sinks in the two-dimensional plane. Interface coning in such instances is often referred to as 'cusping' in the petroleum literature.

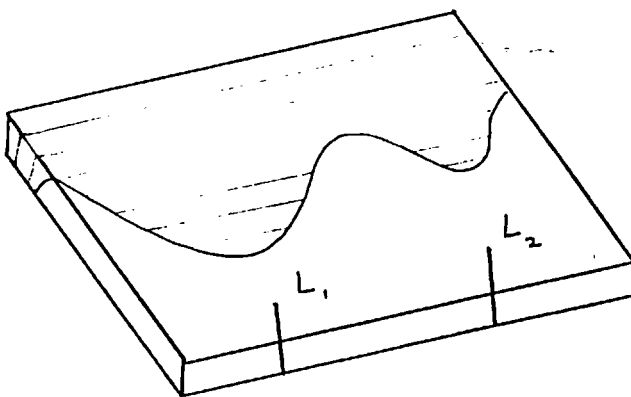


Figure 1.1 'Cusping' towards the producing wells L_1 and L_2

The second type of plane flow coning is sometimes referred to as

'wedging' in the engineering hydrology literature. It arises, in practice, in instances of horizontal well production (19). Figure 1.2a is a diagrammatic representation of the problem. Here L_3 represents a horizontal well, and the shaded region shows the water 'wedging' towards the well. In this instance, the overall flow pattern clearly represents a two-dimensional flow. It is therefore adequate, for purposes of mathematical analysis, to consider a vertical section of the flow pattern (figure 1.2b) and represent the well as a point-sink.

Although the geometry of the reservoir and the well positioning are different, in the two cases described above, the underlying mechanism which causes the interface to cone towards the well is identical. As a result of the oil production from the well, the pressure distribution established in the two fluids, will cause the interface to be drawn from its initial static equilibrium position towards the well. The movement of the interface will be slow at first but become faster as it approaches the well. Under certain conditions, a new equilibrium may be reached before the coned interface reaches the production well. Otherwise, the interface will eventually reach the producing well in a cusp-like form.

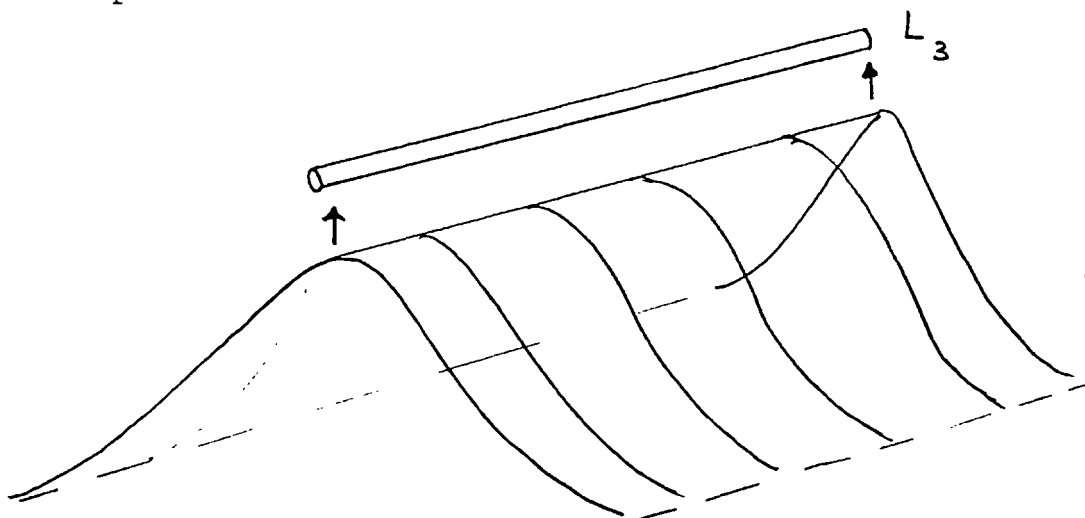


Figure 1.2a : 'Wedging' towards a horizontal well

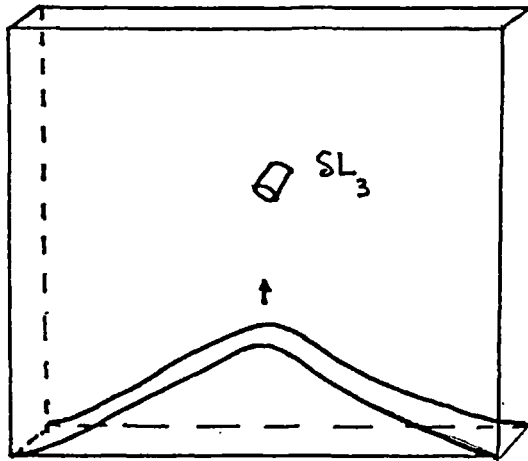


Figure 1.2b: Vertical cross-section of wedging

1.2 Radial coning problems

Radial coning occurs in petroleum reservoirs when a well partially penetrates an oil formation that is underlain by water or overlain by gas. The essential difference between plane-flow coning and radial coning is that the latter, as its name suggests, is a radial phenomenon. Thus the reservoir may be regarded as being cylindrical in shape, and in the coned surface axi-symmetric about the well (fig 1.3a). The term 'radial up-coning' is used when water cones upwards towards the producing interval, and 'radial down-coning' is used when gas cones downwards. The mechanism underlying both water and gas coning is identical, and for this reason we shall focus our attention on water coning.

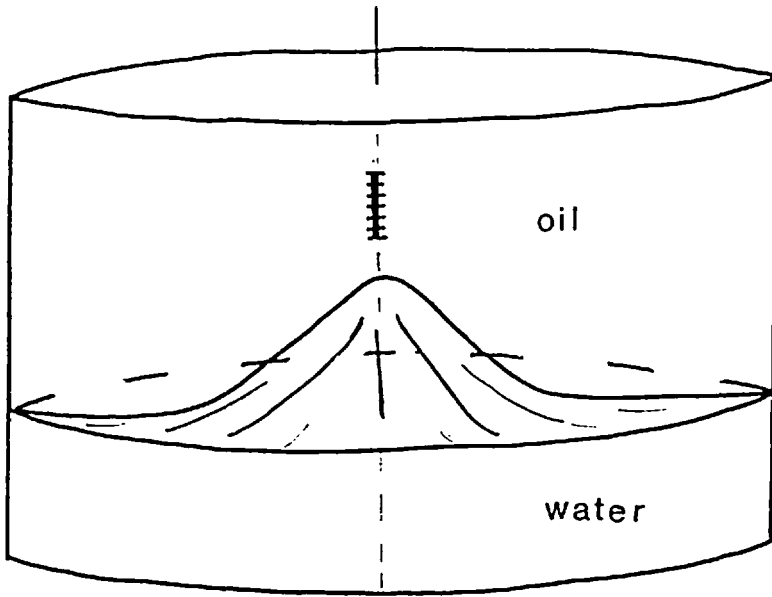


Figure 1.3A: Radial coning

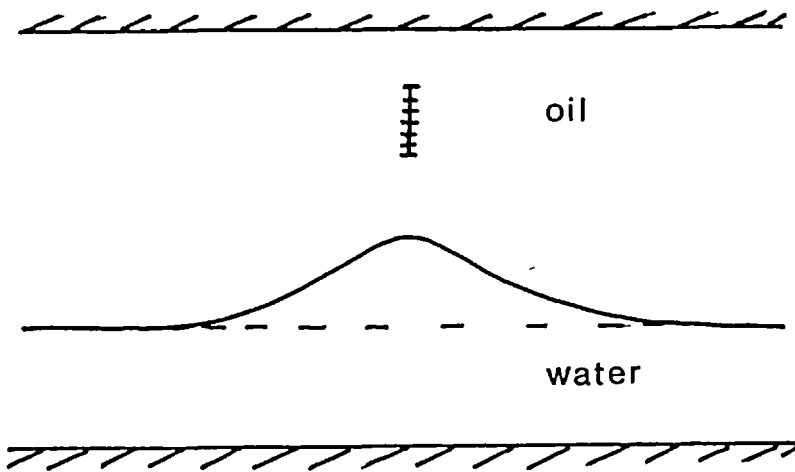


Figure 1.3B: Vertical cross-section of radial coning

Water coning is caused by pressure gradients operating around the well bore of a production well, which causes the oil-water contact to be raised from its initial horizontal level to form an upconed surface. For small well production rates, the interface reaches a new stable, upconed equilibrium position. However, for production rates exceeding a certain critical level, known as the critical flow-rate, the cone becomes unstable and water breaks through into the well. The coning mechanism is therefore a rate sensitive phenomenon.

The tendency for the water to cone is partially offset by gravitational forces, since water has a higher density than oil. The stability of the water cone is therefore determined by two forces - gravitational forces arising from the difference in the densities of the oil and water, and the potential gradients causing the flow of fluid towards the well bore. Sub-critical rate cones correspond to the situation where a balance exists between the two forces.

For super critical rates, the pressure gradients exceed the gravitational forces, and therefore after a certain time, the cone becomes unstable and breaks through into the well. The cone instability occurs at some distance between the original oil-water contact and the bottom of the well, known as the critical height. Once cone instability occurs at the critical height, the cone breaks through and water enters the well almost instantaneously. The time at which the water first enters the well is known as the water breakthrough time, and it is a factor of engineering importance.

Once the water reaches the well, the cone continues to rise

within the well completion albeit slowly. If the reservoir production mechanism is a lateral drive system i.e. if the fluids enter from the extremities of the reservoir, with the original oil water contact staying fixed at the outer boundary, the water cone will stabilize at some height within the well completion, and a limiting water oil ratio will be reached. If, however, the reservoir production mechanism is a bottom water drive i.e. if the oil water contact is continually moving upwards at the extremities of the reservoir, then the water oil ratio will continue to increase and after some time, the water will reach the well completion, and solely water will be produced from the well.

1.3 The modelling of coning phenomena

A mathematical model can be constructed for both plane-flow and radial coning, by applying conservation of mass to each phase, relating flow rates with pressure or potential by Darcy's law, and relating pressure across the interface by capillary pressure. These equations together with the boundary and initial conditions, provide a system of non-linear equations which define what is known mathematically as a free-boundary problem. In such problems the shape of part or all of the boundary is unknown and dependent on the pressure or potential distribution of the fluids, while this distribution in turn depends on the shape of the unknown boundary.

Two-dimensional steady state free-boundary problems have scope for analytical development. Several researchers in Fluid Mechanics

(26, 26, 37) have developed solutions to free boundary problems including jets, water-waves, and Stefan problems, by utilizing the powerful theory of complex variables. The techniques used in two-dimensional flow cannot, however, be extended to three-dimensional or radial problems, due to certain limitations inherent in the theory of complex variables. Thus in the case of three-dimensional or radial free boundary problems virtually no analytical tools are available for the complete solution of the problem.

Radial coning problems have in the past been studied by one of two means - a coning simulator, or a coning correlation. A coning simulator is a detailed computer program, which employs solely numerical methods. It is, therefore, known to suffer from severe discretization and truncation errors such as numerical dispersion. Simulators are also very costly in computer requirements and the data input for these simulators can be very lengthy in their preparation. Coning correlations, on the other hand, are almost invariably based on experimental scaled models and are therefore reliable only when the conditions of the laboratory experiments, are reproduced closely, if not exactly. There is also considerable disagreement between the various correlations put forward for the prediction of coning behaviour especially with respect to the prediction of breakthrough time and therefore these correlations are rather limited in their use.

1.4 Objectives of this investigation

The objectives of this work are two-fold. Coning simulators,

though widely used in industry have several uncertainties associated with their accuracy, and there is therefore a need to develop test solutions to check the accuracy of simulators. To this end, we shall not focus our attention on the detail of coning problems - such as different production schedules, varying boundary conditions, complex reservoir geometrics etc., but seek to develop as accurate a solution as possible to the basic coning problem. Many simplifying assumptions will therefore be made. Analytical methods will be used where possible, and numerical methods will be employed only where they are absolutely essential. Discretion will be exercised in the choice of numerical methods as accuracy is the prime consideration.

The other objective of this investigation is to develop an inexpensive model (or models) for the prediction of coning behaviour. We shall be interested in following the movement of the water front as it progresses from its static position to either steady state equilibrium or breakthrough position, and in determining factors of engineering importance - such as, critical production rate, breakthrough time and limiting water oil ratio. The model to be developed must, however, be inexpensive in computer storage and machine time requirements, and be simple in its input of data.

CHAPTER 2

2-DIMENSIONAL, PLANE-FLOW, CONING PROBLEMS

2.1 Introduction

Reservoir coning phenomena are generally radial, free surface problems and do not lend themselves readily to analytical solution. Numerical methods using finite difference or finite element technique are generally used to solve such problems.

These methods discretize the partial differential equations which model the physical system, and the discretized algebraic equations are solved numerically. Such methods are not only expensive in computer requirements, but also have several numerical difficulties associated with them. One such difficulty is the occurrence of numerical dispersion, which smears sharp fronts, thereby distorting the solution. Furthermore, for a numerical method to be considered 'accurate' it is necessary to perform an asymptotic analysis by refining the grid system and studying its effect on the solution. For a reservoir coning simulation, however, the dimensions of the smallest grid blocks are often of the order of a few feet near the well bore. Hence it is not feasible to refine the grid system any further to conduct an asymptotic analysis. Therefore, the accuracy of a reservoir scale coning simulation is open to question.

In view of the difficulties associated with coning simulators, and the absence of analytical solutions to the radial coning problem, it is at least of theoretical interest to study the equivalent 2-dimensional plane flow coning problem. These problems have scope for analytical development and may be used as test solutions to check the accuracy of numerical simulators. In this chapter we therefore focus our attention on 2-dimensional plane flows in porous media. Such flows may be formulated in terms of complex variables by defining a flow potential

and a stream function that satisfy the Cauchy-Riemann conditions.

The major mathematical difficulty associated with free boundary problems is the need to satisfy boundary conditions, which are invariably non-linear, on an a priori unknown surface. Transformation techniques are frequently used for this class of problem, as they can be used to map an unknown surface onto a fixed boundary. The Method of Hodographs is based on this principle and is discussed in this chapter. We examine its application to the up-coning problem in the vertical x-y plane.

A second solution technique, also based on the theory of complex variables, is developed for cases where the hodograph is multi-valued, and transformations from one plane to another become prohibitively difficult. Here, the flow domain in the physical plane is mapped onto a simple flow domain in another complex plane. The relevant transformations are obtained by fitting a series and evaluating its co-efficients numerically. For numerical purposes, a non-linear convergence formula, based on Padé' approximants, is used to accelerate the convergence of the series. The proposed model can be extended to the case of upconing towards several point sinks and to anisotropic systems.

2.2 Applications of complex variable theory to flow through porous media

2.2.1 Darcy's Law

The fundamental law governing the flow of fluids through porous

media, under laminar flow conditions, is Darcy's Law (1876).

Darcy's law is empirical, and was discovered by the French engineer, Darcy, during his work for the Dijon public water supply system.

By flowing water through a sand pack Darcy established, that for any flow rate, the velocity of the flow was directly proportional to the potential gradient

$$v = \frac{Q}{A} = K \left(\frac{d\phi}{DL} \right) \quad (K, \text{ constant}).$$

We shall now write Darcy's law in terms of a velocity potential, which will then enable us to formulate 2-dimensional porous media problems in terms of complex functions. Complex variable theory has been used widely in classical hydrodynamics (4, 28, 36) and is a particularly valuable tool in the solution of free boundary problems (15, 27, 37).

Defining a fluid potential, for the phase i , as

$$\phi_i = \left(\frac{p_i}{\rho_i} - gy \right), \quad (2.1)$$

where, y is measured positive downwards from some datum level, Darcy's law may be written as

$$\underline{u}_i = - \frac{K_i \cdot \rho_i}{\mu_i} \nabla \phi_i \quad (2.2)$$

$\underline{u}_i$ is the specific discharge vector (the discharge per unit cross sectional area) and is related to the average velocity, $\underline{U}$, by the relationship

$$\underline{U} = \frac{\underline{u}}{\sigma} ,$$

where σ is the porosity of the medium. $\underline{u}$ has the dimensions of velocity (Lt^{-1}) and authors often call it 'filter velocity' or 'discharge velocity' (31, 36). We shall refer to it simply as velocity in this chapter. Therefore the term velocity will be used to mean the specific discharge, $\underline{u}$, and not the average velocity $\underline{U}$.

For an isotropic, homogeneous porous medium with the fluid density and viscosity kept constant, Darcy's law simplifies to

$$\underline{u} = \nabla \phi_i$$
$$(u, v) = \left(\frac{\partial \phi}{\partial x^i}, \frac{\partial \phi}{\partial y^i} \right) \quad (2.3)$$

where ϕ_i is now the velocity potential,

$$\phi_i = - \frac{K_i \rho_i}{\mu_i} \left(\frac{P}{\rho_i} - g y \right) . \quad (2.4)$$

2.2.2 The complex potential

The equation of continuity for incompressible flows is, by conservation of mass,

$$\nabla \cdot \underline{u} = 0. \quad (2.5)$$

Defining a stream function ψ , such that,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad (2.6)$$

(equations (2.6) satisfy the equation of continuity (2.5)) and comparing equations (2.3) and (2.6) we note that,

$$\left. \begin{aligned} \frac{\partial \Phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \frac{\partial \Phi}{\partial y} &= -\frac{\partial \psi}{\partial x} \end{aligned} \right\} \quad (2.7)$$

Equations (2.7) are known as the Cauchy-Riemann conditions.

It is also known that a single-valued function $S = P(x,y) + i Q(x,y)$, for which $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ are continuous in some region R , is analytic in R if and only if the Cauchy-Riemann conditions are satisfied (17, 41).

Since Φ and ψ satisfy the Cauchy-Riemann conditions, we may now define a complex fluid potential, ω , that is an analytic function of z .

$$\omega = \Phi + i \psi$$

and $z = x + iy$ (the z -plane).

Then,

$$\begin{aligned} \frac{d\omega}{dz} &= \frac{\partial \Phi}{\partial x} + i \frac{\partial \psi}{\partial x} \\ &= \frac{\partial \psi}{\partial y} - i \frac{\partial \Phi}{\partial y} \\ &= u - iv. \end{aligned} \quad (2.8)$$

The curves $\Phi = \text{constant}$ are curves of equal velocity potential, and the curves $\psi = \text{constant}$ are the streamlines. Since by equation (2.7),

$$\nabla\Phi \cdot \nabla\psi = \frac{\partial\Phi}{\partial x} \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y} \frac{\partial\psi}{\partial y} = 0,$$

we deduce that these two systems of curves are orthogonal.

2.3 The Method of Hodographs

The use of the theory of functions in fluid flow has been found to be effective in a wide variety of steady, 2-dimensional flow problems. The Hodograph method is one such method based on the theory of functions and conformal mappings. It has been found to give exact solutions to a number of free surface porous media problems with simple geometries. By free surface problems, we mean problems where certain boundary conditions need to be applied on an a priori unknown surface. Pressure continuity applies across such surfaces.

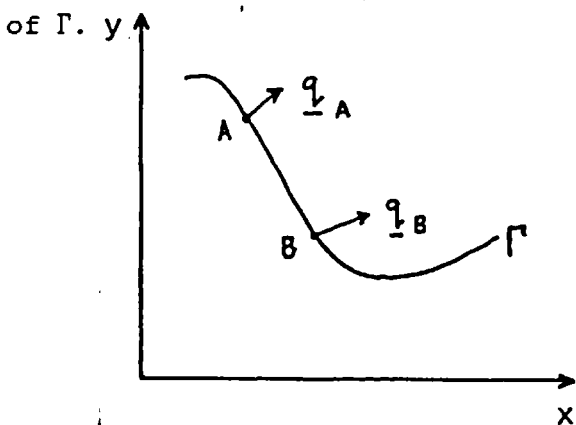
2.3.1 The Hodograph Plane

The literal meaning of the term hodograph is a 'graph of velocities'. The earliest description of its use was given by Hamel (20) and has since been used for a variety of simple problems occurring in ground water hydrology and reservoir engineering

(5, 6, 22, 33, 36).

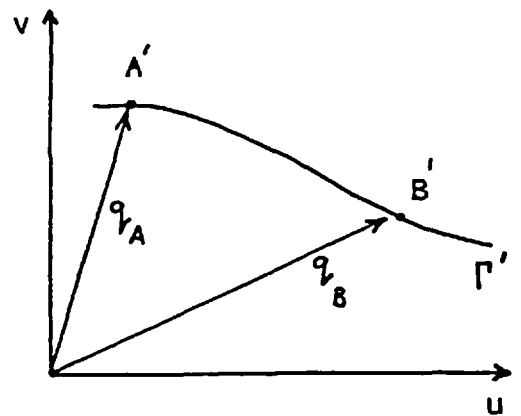
The main advantage in using a hodograph plane is that where as the geometric shape and position of a free surface or interface in the physical plane are unknown a priori, they are completely specified in the hodograph plane. In this section we shall consider its application to steady flows.

Let Γ be some curve in a 2-dimensional plane field of flow in the z -plane (See Fig 2.1a). We know that a certain velocity q , with components u and v in the x and y directions respectively, exists at every point along the curve - q_A at point A , q_B at point B etc. . We now consider another plane with a co-ordinate system (u,v) . We now draw the vectors q_A , q_B etc. . The end points (A',B') of these vectors form a curve Γ' (see Figure 2.1b) which describes the velocity at the points $(A,B,...)$ in the physical plane. Γ' is referred to as the hodograph representation of Γ .



Physical plane z - plane

Figure 2.1a



Hodograph plane q - plane

Figure 2.1b

The transformation from Γ to Γ' may be considered as a conformal mapping procedure. If in the entire flow domain, no velocity vector occurs at more than one point, the mapping from the physical

plane to the hodograph plane is in one to one correspondence.

However, if the same velocity occurs at more than one point, the mapping of the flow domain in the hodograph plane takes the form of a Riemann surface (14) - many sheets of planes.

Let us now consider two special cases of Γ and derive their corresponding boundaries in the hodograph plane.

(i) Impermeable boundaries i.e. boundaries where $\psi = \text{constant}$.

When such a boundary takes the form of a straight line, its representation in the hodograph plane is found quite simply.

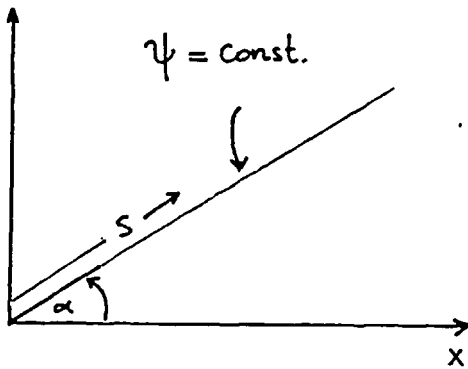


Figure 2.2a

Physical plane of impermeable boundary

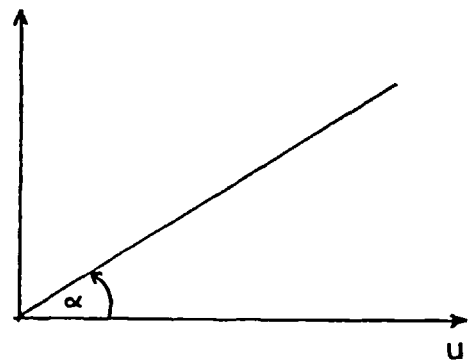


Figure 2.2b

Hodograph plane of impermeable boundary

If s is the distance measured along the boundary and α is the angle the boundary makes with the x-axis (Figure 2.1a),

$$\begin{aligned} \frac{d\psi}{ds} &= \frac{\partial\psi}{\partial x} \frac{dx}{ds} + \frac{\partial\psi}{\partial y} \frac{dy}{ds} = 0 \\ &= \frac{\partial\psi}{\partial x} \cos \alpha + \frac{\partial\psi}{\partial y} \sin \alpha \\ &= -v \cos \alpha + u \sin \alpha = 0. \end{aligned}$$

Therefore $v = u \tan \alpha$, and the hodograph is represented as a straight line passing through the origin (Figure 2.2b).

(ii) A stationary abrupt interface between a moving liquid and a stationary liquid.

The above case is a special case of a free surface. It is a case of interest to the reservoir engineer who frequently has to deal with oil water systems separated by an interface - Figure 2.3a.

Let us first examine the boundary conditions which need to be imposed on this surface - which we shall refer to as $y = \eta$.

By equation (2.1),

$$\phi_0 = \frac{-K_0 \rho_0}{\mu_0} \left(\frac{p}{\rho_0} + gy \right) \quad (2.9)$$

and
$$\phi_w = \frac{-K_w \rho_w}{\mu_w} \left(\frac{p}{\rho_w} + gy \right) . \quad (2.10)$$

Then by continuity of pressure (neglecting capillary pressure) we have by eliminating p between (2.9) and (2.10)

$$\eta = \frac{\mu_0}{K_0 (\rho_w - \rho_0) g} \phi_0 - \frac{\mu_w}{K_w (\rho_w - \rho_0) g} \phi_w .$$

The water phase is in hydrostatic equilibrium, and therefore the velocity potential ϕ_w is constant

$$\eta = \kappa \phi_0 + c \quad (2.11)$$

where

$$\kappa = \frac{\mu_0}{K_0 (\rho_w - \rho_0) g} .$$

We shall choose c to be zero, for convenience. Hence

$$\eta = \kappa \phi_0.$$

A steady state interface can be represented by an equation whose general form is :

$$F(x,y) = 0 \quad . \quad (2.12)$$

Now since the elevation of a point on the interface is given by $y = \eta(x,y)$, (2.12) may be written as

$$F(x,y) \equiv y - \eta(x,y) = 0$$

or

$$F(x,y) \equiv y - \kappa \phi_0 = 0. \quad (2.13)$$

In general for a moving interface, the material surface condition $\frac{DF}{Dt} = 0$ needs to be imposed. i.e. any particles on the interface must move in the direction and at the speed of the interface. However, here the oil water interface is stationary. Therefore,

$$\nabla \phi_0 \cdot \underline{n} = 0, \quad (2.14)$$

where $\underline{n}$ is the normal to the interface.

Hence,

$$\nabla \phi_0 = \nabla (y - \kappa \phi_0) = 0$$

which after simplification becomes,

$$u^2 + \left(v - \frac{1}{2\kappa} \right)^2 = \left(\frac{1}{2\kappa} \right)^2. \quad (2.15)$$

Equation (2.15) describes the boundary of the hodograph and is the equation of a circle radius $\frac{1}{2\kappa}$ centred at $\left(0, \frac{1}{2\kappa} \right)$.

Figure 2.3b.

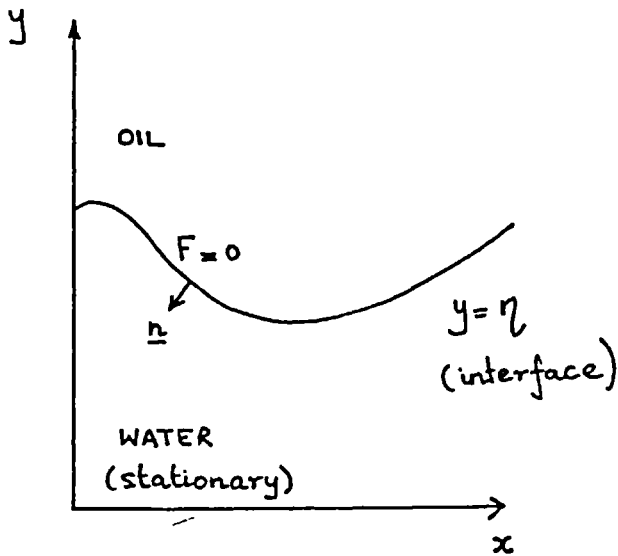


Figure 2.3a

Physical-plane of stationary abrupt interface

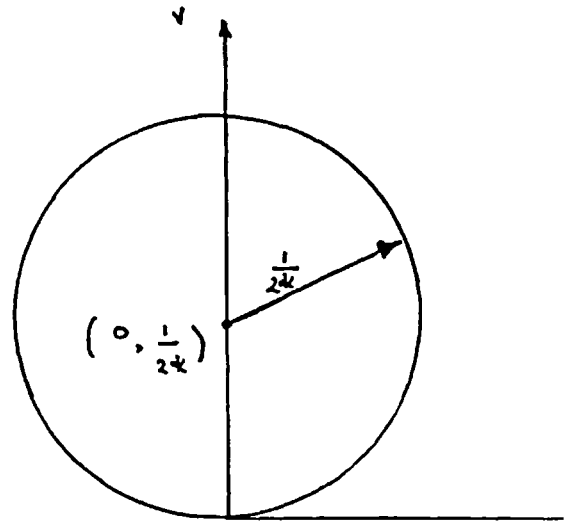


Figure 2.3b

Hodograph plane of stationary abrupt interface

2.3.2 Method of solution

In this section we shall examine the procedure for solving 2-dimensional flows by the method of hodographs. Consider the functions,

$$q = u + iv$$

$$\tilde{q} = u - iv$$

where u and v are the velocities in the x, y directions and $\tilde{q}$

is the complex conjugate of q . The $\tilde{q}$ -plane is sometimes referred to as the inverse hodograph plane.

By equation (2.8)

$$\frac{d\omega}{dz} = u - iv = \tilde{q} \quad (2.16)$$

and

$$\left| \frac{d\omega}{dz} \right| = (u^2 + v^2)^{1/2} = |q|.$$

We also have

$$\frac{dz}{d\omega} = \frac{1}{\frac{d\omega}{dz}} = \frac{1}{u-iv} = \frac{\omega}{|q|^2}.$$

From (2.16),

$$dz = \frac{1}{\tilde{q}} d\omega$$

and therefore,

$$z = \int \frac{d\omega}{\tilde{q}(\omega)} + \text{const} \quad (2.17)$$

Equation (2.17) relates ω to points z in the physical plane. Knowledge of $\tilde{q}$ as a function of ω is essential to the application of this method. This can be obtained by mapping the z -plane onto both the $\tilde{q}$ and the ω planes, and then mapping ω onto $\tilde{q}$.

The following steps need to be followed to solve the problem by the hodograph method.

- (i) Given Γ the boundary of the flow domain, map it onto the hodograph plane (q-plane).
- (ii) Transform the hodograph to the inverse hodograph plane ($\tilde{q}$). This is done simply by reflecting the contour in the $\tilde{q}$ plane on the real (u) axis.
- (iii) Map the boundary Γ onto the complex potential plane ω .
- (iv) Determine a conformal mapping (14) $\tilde{q} = \tilde{q}(\omega)$ that maps the $\tilde{q}$ -plane onto the ω -plane. This mapping is not always derived directly. One other intermediate plane, or several such planes, may be necessary. Two transformations that prove to be very effective in achieving the conformal mapping $\tilde{q} = \tilde{q}(\omega)$ are the inverse transformation and the Schwarz-Christoffel transformation.

The inverse transformation $\tilde{q}' = \frac{1}{\tilde{q}}$, maps straight lines and circles passing through the origin onto straight lines.

For instance, consider in the q-plane the case of a circle passing through the origin with radius $\frac{a}{2}$. (centre $(0, \frac{a}{2})$).

Any point on the circle may be written as

$$\tilde{q} = \frac{a}{2} (i + e^{i\theta}). \quad (2.18)$$

Then, applying the inverse transformation $\tilde{q}' = \frac{1}{\tilde{q}}$ we get,

$$\begin{aligned} \tilde{q}' &= \frac{2}{a(i + e^{i\theta})} \\ \tilde{q}' &= \frac{\cos\theta}{a(1 + \sin\theta)} - \frac{i}{a} \end{aligned} \quad (2.19)$$

Equation (2.19) represents a straight line at a distance $\frac{1}{a}$ below the real axis. Similarly for the case of lines and circles (not passing through the origin) we can verify the corresponding form of the inverse mapping.

Since the boundary of the hodograph often involves a portion of a circle (especially for free surface flows) the inverse transformation proves to be effective in reducing hodographs to polygons in an intermediate complex plane.

The Schwarz-Christoffel transformation (35) maps the interior of any polygonal boundary (including infinite polygons i.e. semi infinite strips) onto the upper half complex plane.

Consider figures (2.4a) and (2.4b).

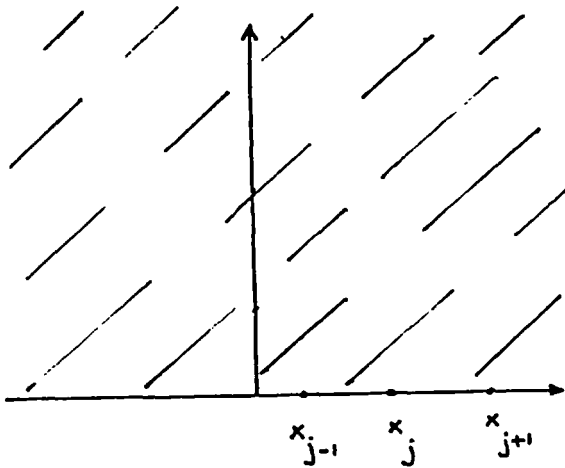


Figure 2.4a

z-plane (Schwarz-Christoffel transformation)

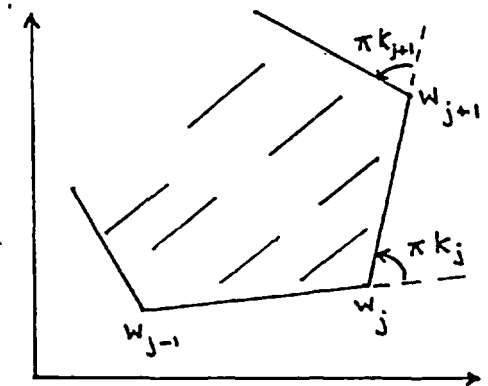


Figure 2.4b

w-plane (Schwarz-Christoffel transformation)

The conformal transformation that maps the shaded region of Figure 2.4a to the shaded region of Figure 2.4b is

$$\omega = f(z) = \alpha \int_0^z (\lambda - x_1)^{-k_1} (\lambda - x_2)^{-k_2} \dots (\lambda - x_n)^{-k_n} d\lambda + \beta$$

(Schwarz-Christoffel)

where the points $x_1, x_2, \dots, x_n$ denote points on the real axis in the z -plane such that $x_1 < x_2 < \dots < x_n$. $k_1, k_2, \dots, k_n$ are constants that when multiplied by π represent the abrupt changes of direction in the vertices of the polygon. α and β are two constants that determine enlargements, rotation and translation, but do not influence the resulting shape of the figure in the transformal plane. They may be chosen by choosing to map particular points ω_p, ω_q onto z_p, z_q say.

Thus using the inverse transformation and the Schwarz-Christoffel transformation it is usually possible to evaluate $\tilde{q} = \tilde{q}(\omega)$.

(v) Using equation (2.17) we can integrate to obtain,

$$z = \int_{\omega=\omega_1}^{\omega} \frac{1}{\tilde{q}(\omega)} d\omega = x+iy.$$

Comparing real and imaginary parts, parametric equations for x and y can be obtained.

$$x = x(\Phi, \psi)$$

$$y = y(\Phi, \psi) \quad .$$

2.3.3 Application of the method of hodographs to fingering or wedging problems

'Fingering' here refers to problems in which the permeable stratum is sufficiently thin that the flow of fluids can be treated as a 2-dimensional flow in the plane of the stratum. Similarly the terms 'wedging' is used to mean the up-coning of one fluid towards another lighter fluid in the 2-dimensional vertical plane.

Kidder (22) obtained by the method of hodographs, a mathematically exact solution to the free boundary problem of the fingering of water towards a line of oil producing wells that penetrate a thin stratum. By defining a dimensionless potential, dimensionless velocity and a dimensionless co-ordinate system, the relevant flow contours in the physical, hodograph and potential planes were obtained. Then by use of the inverse transformation and the Schwarz-Christoffel transformation, parametric equations of the free boundary, for the critical case were obtained. The critical case is when the oil production Q assumes the largest value for which a stable free boundary can exist.

$$x = 1 - \frac{1}{\pi} [\beta + 2(1+a) \tan^{-1} \{ (\sin \beta) / (1+2a - \cos \beta) \}]$$

$$y = \frac{1}{\pi} \ln \{ 1 + [(1+2a)/2a^2] (1 - \cos \beta) \}$$

$$(0 \leq \beta \leq \pi)$$

where

$$a = \frac{\mu}{g \Delta \rho D Q} \quad \text{and } D = \text{distance between neighbouring wells.}$$

Later Bear (6) applied the method of hodographs to the problem of upconing towards a point sink in the vertical plane (wedging). The aquifer and the oil domain were assumed to be of infinite extent, with no impermeable boundaries present within the flow domain.

Let us extend Bear's rather idealistic case, to the case with an upper impermeable boundary present.

Figure 2.5a represents the flow domain. AB represents the steady state oil-water interface, C the point sink and $A_\infty MD$ the upper impermeable boundary. Since the problem is symmetric about the y-axis we shall consider only half the flow domain.

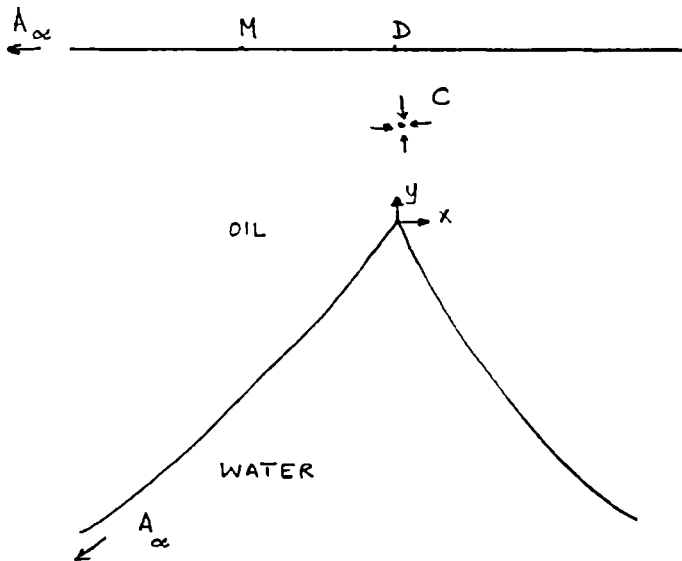


Figure 2.5a
z-physical plane

In the first instance, we shall limit our attention to the critical case i.e. when the interface is drawn up to its maximum stable position.

Figure 2.5b shows a representation of the inverse hodograph plane $\tilde{q}$ and figure 2.5c the complex potential plane ω . We note that $\tilde{q}$ is many-valued and is a Riemann surface. i.e. zero velocities appear at A and D. We next perform the inverse transformation $q' = \frac{K}{\tilde{q}}$ - as shown in figure 2.5d. Figure 2.5e shows an intermediate reference plane onto which ω may be mapped. If the shaded region of figure 2.5d too can be mapped onto the corresponding region of 2.5e then a relationship between ω and $\tilde{q}$ is obtainable and a solution possible. Because of the presence of the branch cut, AMD, it is not possible to obtain the mapping of 2.5d onto 2.5e, by the Schwarz-Christoffel transformation. It is now necessary to consider transformations that map complex planes onto Riemann surfaces. This considerably increases the complexity of the mathematical analysis.

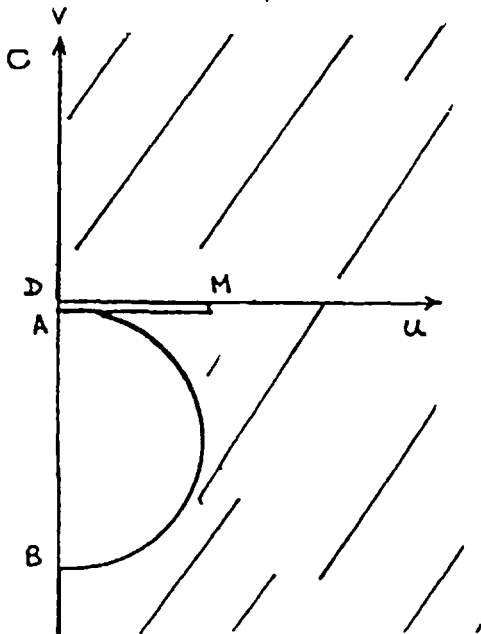


Figure 2.5b

$\tilde{q}$ - inverse hodograph plane

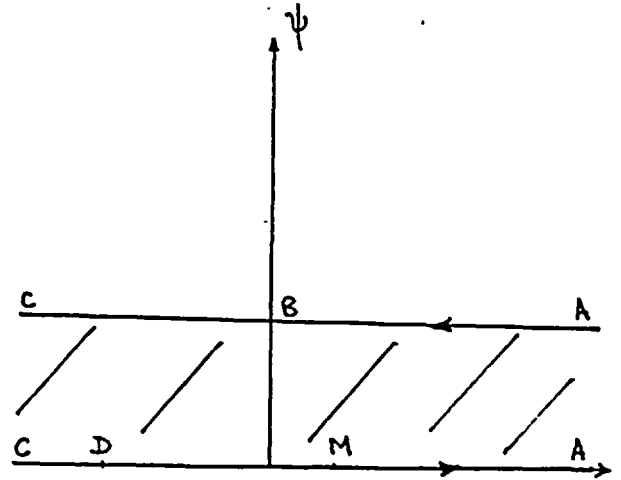


Figure 2.5c

ω - potential plane

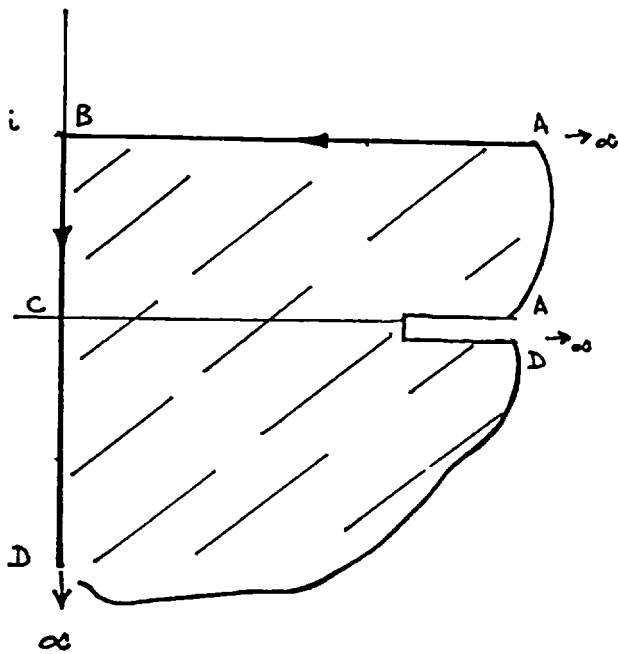


Figure 2.5d

$$\tilde{q}' = \frac{\kappa}{q} \text{ plane}$$

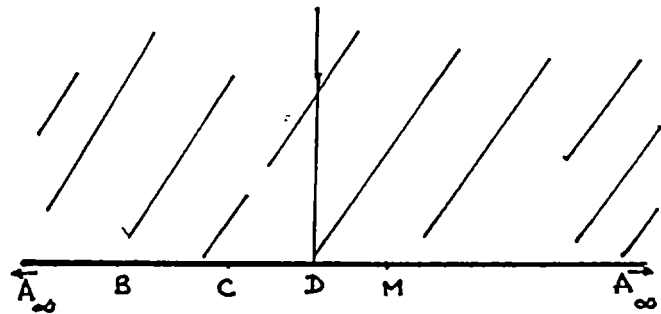


Figure 2.5e

intermediate reference
plane i.e. upper half plane

If it is indeed possible to find the transformation that maps the shaded region of 2.5d onto 2.5e surfaces, it is necessary to make a careful study of Riemann surfaces. A few problems involving many - valuedness in the hodograph plane have been solved by Strack (40) and De Josseling De Jong (16).

2.4 A new alternate (series) solution technique

In view of the complexity and restrictive nature of the method of hodographs to problems involving many -valuedness in the velocity field we have developed a new semi-analytical method similar to the perturbation series method developed recently for the computation of steady water waves and other free streamline problems (26, 37).

This method is similar to the hodograph method, in that it utilizes the powerful properties of complex variable theory. It also maps the unknown free surface onto a known boundary. However, the transformation from the free boundary to the known fixed boundary is achieved by constructing a perturbation series and evaluating its co-efficients numerically. The fixed boundary onto which we map the free boundary is chosen such that the flow problem may be solved easily in the new plane.

2.4.1 Formulation of the up-coning problem

Consider again the problem of the upconing of water towards a point sink operating in the oil formation (Fig 2.6a). The flow will be taken to be a 2-dimensional steady flow in the vertical x-y plane.

We shall assume that the flow is irrotational and incompressible. Since the problem is one of steady flow, the water will be assumed to be at rest within the cone and the free surface will be modelled as a streamline. We shall, therefore, for a given sink strength, determine the interface profile of limiting stability i.e. the profile of the highest possible stable cone that can lie static beneath the operating sink. Later, we shall see how one may determine the unique interface position, i.e. sub-critical case, given some other information about the flow.

2.4.2 The physical model

The model in the z -plane is as shown in figure 2.6a. At position D , distance d below the origin, we have a point sink of strength q . $A_\alpha C A'_\alpha$ is an upper impermeable boundary and is modelled as a streamline $\psi = \psi_1$. $A_\alpha E B F A'_\alpha$ is the oil water contact and is also modelled as a streamline $\psi = \psi_2$. The stream lines $\psi = \psi_1$ and $\psi = \psi_2$ are such that $\psi_2 - \psi_1 = \frac{q}{2h}$, is the distance between $A_\alpha C A'_\alpha$ and the initial oil water contact. The portion EBF of the streamline $\psi = \psi_2$, represents the upconed interface or free surface.

At large distances away from the sink the flow is uniform and parallel to the real axis with velocity $\pm \frac{q}{2h}$.

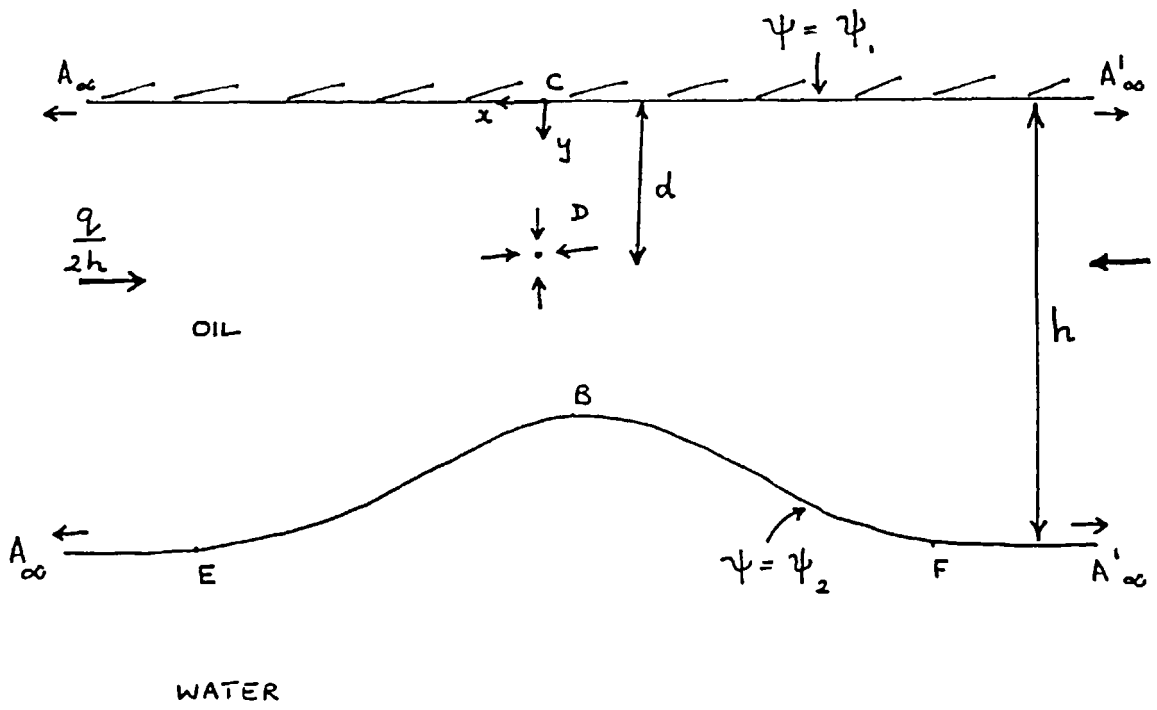


Figure 2.6a z -plane

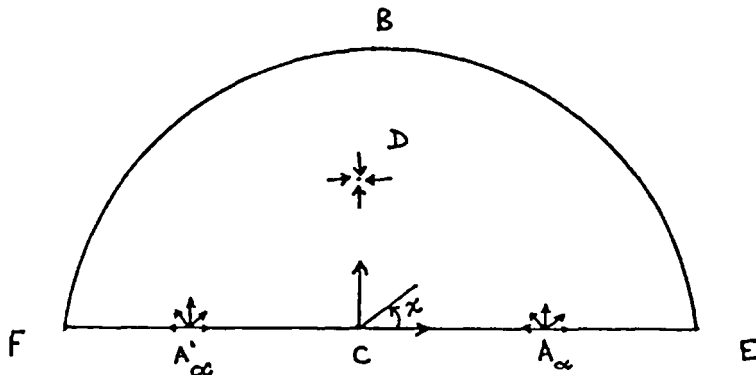


Figure 2.6b
 ζ - plane

$$\zeta = re^{i\chi}$$

$$\zeta = e^{i\chi} \quad \text{EBF}$$

2.4.3 Mathematical development

Rather than solve the problem in the physical plane, where the shape of the free streamline EBE is unknown, we map EBF onto the circumference of the unit semicircle in a new plane - the ζ -plane. Corresponding points are denoted by the same letter in the physical and the new plane.

At this stage the transformation $z = z(\zeta)$ that maps the flow domain in the z -plane onto the semicircular region of the ζ -plane is unknown. We shall, however, use our existing knowledge of the flow behaviour in the z -plane to determine the form of this transformation $z = z(\zeta)$. (Henceforth $(u - iv)$ as expressed in equation (2.8) will be referred to as the complex velocity).

In the z -plane D represents a point sink. The complex potential due to such a point sink operating in an infinite domain is

$$\omega = \frac{q}{2\pi} \ln(z-d) \quad (2.20)$$

where q is the sink strength and d its distance from the origin:

At A_α in the z -plane, we have a uniform flow with velocity $+\frac{q}{2h}$. The complex potential near A_α is, therefore,

$$\omega(z) = +\frac{q}{2\pi} z \quad (2.21)$$

Equation (2.21) can also be considered as the complex potential due to a point sink operating at the infinite point.

Transforming the flow domain from the z -plane onto the ζ -plane, we see that the flow in the ζ -plane is due to the presence

of a point sink at the point D (say $\zeta = i\ell$) and two point sources of strength $\frac{q}{2}$ at A_α and $A'_\alpha(\zeta = \pm a, \text{ say})$.

We shall now determine the complex velocity in the ζ -plane, noting that $A_\alpha C A'_\alpha$ and $A_\alpha E B F A'_\alpha$ are streamlines.

Suppose the point sources and sinks operate in an infinite domain, then the corresponding flow potential in the ζ -plane is found by summing the complex potential due to the individual sources and sinks.

$$\omega = \frac{1}{2\pi} \left(\frac{q}{2} \ln(\zeta - a) + \frac{q}{2} \ln(\zeta + a) - q \ln(\zeta - i\ell) \right). \quad (2.22)$$

Introducing the no-flow boundary $EA_\alpha CA'_\alpha, F$, the potential is modified, by placing image points, and is

$$\begin{aligned} \omega = \frac{1}{2\pi} & \left(\frac{q}{2} \ln(\zeta - a) + \frac{q}{2} \ln(\zeta + 2) - q \ln(\zeta - i\ell) \right. \\ & \left. + \frac{q}{2} \ln(\zeta - a) + \frac{q}{2} \ln(\zeta - a) - q \ln(\zeta + i\ell) \right). \end{aligned} \quad (2.23)$$

Finally introducing the semi-circular no-flow boundary EBF, the complex potential is found by the circle theorem (28) i.e. (2.23) and the addition due to the inverse points in the circular arc EBF.

$$\begin{aligned} \omega = \frac{1}{2\pi} & \left\{ q \ln(\zeta - a) + q \ln(\zeta + a) - q \ln(\zeta - i\ell) - q \ln(\zeta + i\ell) \right. \\ & \left. + q \ln\left(\frac{1}{\zeta} - a\right) + q \ln\left(\frac{1}{\zeta} + a\right) - q \ln\left(\frac{1}{\zeta} - i\ell\right) - q \ln\left(\frac{1}{\zeta} + i\ell\right) \right\} \\ & = \frac{q}{2\pi} \left(\ln(\zeta^2 - a^2) + \ln\left(\zeta^2 - \frac{1}{a^2}\right) - \ln(\zeta^2 + \ell^2) - \ln\left(\zeta^2 + \frac{1}{\ell^2}\right) + \ln\left(-\frac{a^2}{\ell^2}\right) \right) \\ & \dots \quad (2.24) \end{aligned}$$

(2.24) is the complex potential in the ζ -plane. We can now obtain the complex velocity in the z -plane.

$$u - iv = \frac{d\omega}{dz} = \frac{d\omega}{d\zeta} \frac{d\zeta}{dz}$$

$$= \frac{q}{\pi} \frac{(a^2 + \ell^2)(a^2 \ell^2 + 1)}{a^2 \ell^2} \left(\frac{\zeta(\zeta^2 - 1)(\zeta^2 + 1)}{(\zeta^2 - a^2)(\zeta^2 + \ell^2)(\zeta^2 - \frac{1}{a^2})(\zeta^2 + \frac{1}{\ell^2})} \right) \frac{d\zeta}{da} .$$

... (2.25)

From our knowledge of the flow pattern in the z -plane we can gain some insight into the form of $\left(\frac{d\zeta}{dz} \right) \cdot \left(\frac{d\zeta}{dz} \right)$ must make provision for all points of zero and infinite velocity and remove any that are not (in the z -plane).

Disregarding the effect of $\left(\frac{d\zeta}{dz} \right)$ for the moment, we note that within the domain of interest ($|\zeta| \leq 1$) the complex velocity in the z -plane is zero at $\zeta = 0$, $\zeta = \pm 1$ and $\zeta = \pm i$ and infinite at $\zeta = \pm a$, $\zeta = \pm i\ell$. However, in the z -plane $\zeta = \pm a$ corresponds to A_α , A'_α , which are not points of infinite velocity but uniform velocity. Therefore $\left(\frac{d\zeta}{dz} \right)$ must contain a factor $(\zeta + a)(\zeta - a)$ to remove this unwanted singularity.

i.e.

$$\frac{d\zeta}{dz} = (\zeta^2 - a^2) M(\zeta) \quad (2.26)$$

where $M(\zeta)$, in general, has no zeros or poles and is analytic within $|\zeta| \leq 1$. Further by Schwartz lemma (41) we can obtain

analytic continuation in the lower half plane, and represent $M(\zeta)$ by a power series.

Considering the inverse of equation (2.26), we obtain,

$$\frac{dz}{d\zeta} = \frac{M'(\zeta)}{\zeta^2 - a^2} \quad (2.27)$$

where $M'(\zeta)M(\zeta) = 1$

$M'(\zeta)$ may also be expressed as a power series.

$$M'(\zeta) = \sum_{k=0}^{\alpha} \alpha_k \zeta^k \quad (2.28)$$

For mathematical convenience we can re-write $M'(\zeta)$ as

$$M'(\zeta) = b_0 + b_1 \zeta + (\zeta^2 - a^2) \sum_{k=0}^{\alpha} a_k \zeta^k \quad (2.29)$$

The complex velocity in the z -plane is now

$$u - iv = \frac{g}{\pi} \frac{(a^2 + \ell^2)(a^2 \ell^2 + 1)}{a^2 \ell^2} \frac{\zeta(\zeta^2 - 1)(\zeta^2 + 1)}{(\zeta^2 + \ell^2)(\zeta^2 - \frac{1}{a^2})(\zeta^2 + \frac{1}{\ell^2})} \frac{1}{b_0 + b_1 \zeta + (\zeta^2 - a^2) \sum_{k=0}^{\alpha} a_k \zeta^k} \quad (2.30)$$

$b_0, b_1, a_0, a_1, \dots, a_{12} \dots$ are in general complex co-efficients.

However, on $EA_{\alpha}CA'_{\alpha}F$ (i.e. $\zeta = \text{real}$). The complex velocity is wholly real and therefore we deduce that $b_0, b_1, a_0, \dots, a_n, \dots$ are in fact real.

Also when $\zeta = \pm a$, $u - iv = \pm \frac{g}{2h}$.

Therefore, $b_1 \equiv 0$, and $b_0 = \frac{-2ah}{\pi}$.

$\left(\frac{dz}{d\zeta}\right)$ may now be written as,

$$\left(\frac{dz}{d\zeta}\right) = \frac{-2ah}{\pi} \frac{1}{(\zeta^2 - a^2)} + \sum_{k=0}^{\infty} a_k \zeta^k. \quad (2.31)$$

Integrating equation (2.31) and using the symmetry of the flow about CDB, we obtain the mapping from the z-plane to the ζ -plane.

$$z = \frac{-h}{\pi} \ln \frac{(\zeta-a)}{(\zeta+a)} + \sum_{k=0}^{\infty} a_{2k} \frac{\zeta^{(2k+1)}}{(2k+1)} + ih. \quad (2.32)$$

The constant of integration (ih) is found by substituting $\zeta = 0$ when $z = 0$.

Therefore the equation of the free streamline is
(substituting $\zeta = e^{i\chi}$)

$$\left. \begin{aligned} z &= \frac{-h}{\pi} \ln \frac{(e^{i\chi}-a)}{(e^{i\chi}+a)} + \sum_{k=0}^{\infty} \frac{a_{2k}}{(2k+1)} e^{i(2k+1)\chi} + ih \\ \text{or} \\ x &= \frac{-h}{\pi} \ln \frac{|1+a^2-2a\cos\chi|}{|1+a^2+2a\cos\chi|} + \sum_{k=0}^{\infty} \frac{a_{2k}}{(2k+1)} \cos(2k+1)\chi \\ y &= h - \frac{h}{\pi} \tan^{-1} \left(\frac{2a\sin\chi}{1-a^2} \right) + \sum_{k=0}^{\infty} \frac{a_{2k}}{(2k+1)} \sin(2k+1)\chi \\ 0 &\leq \chi \leq \pi \end{aligned} \right\} \quad (2.33)$$

The equations for the interface profile have now been derived and it remains to determine the unknown co-efficients (a, a_0, a_2, a_4) .

We require that pressure be continuous across EBF (neglecting capillary pressure). There, equation (2.11) holds and it may be written as

$$\text{Im}\{z\} = \kappa \text{Re}\{\omega\} + \text{constant} \quad \text{on } \zeta = e^{i\chi}.$$

Differentiating along the streamline $\zeta = e^{i\chi}$,

$$\text{Im}\left(\frac{dz}{d\chi}\right) = \kappa \text{Re}\left(\frac{d\omega}{d\chi}\right). \quad (2.34)$$

Substituting for $\frac{dz}{d\chi}$ and $\frac{d\omega}{d\chi}$ we obtain a Fourier cosine series of the form.

$$\sum_{k=0}^{\infty} a_{2k} \cos(2k+1)\chi = f(\chi, a, \ell) \quad (2.35)$$

where,

$$\begin{aligned} f(\chi, a, \ell) = & \frac{2q\kappa}{\pi} \sin 2\chi \left(\frac{a^2}{1+a^4-2a^2 \cos 2\chi} + \frac{\ell^2}{1+\ell^4+2\ell^2 \cos \chi} \right) \\ & + \frac{h2a(1-a^2) \cos \chi}{\pi(1+a^2)-4a^2 \cos^2 \chi}. \end{aligned} \quad (2.36)$$

We note that $f(\chi, a, \ell)$ is such that,

$$\begin{aligned} \int_0^{\pi} f(\chi) \cos j\chi \, d\chi &= 0 \quad j = \text{even} \\ &\neq 0 \quad j = \text{odd.} \end{aligned}$$

Therefore, $f(\chi, a, \ell)$ is consistent with the left hand side of equation (2.35).

The co-efficients a_{2k} are,

$$a_{2k} = \frac{2}{\pi} \int_0^{\pi} f(\chi, a, \ell) \cos(2k+1)\chi \, d\chi \quad (2.37)$$

for all $k = 0, 1, 2, \dots$

$\zeta = i\ell$ is the 'inverse' point of the point sink $z = id$ in the ζ -plane. Therefore from equation (2.32),

$$d = \frac{-h}{\pi} \tan^{-1} \left(\frac{2a\ell}{\ell^2 - a^2} \right) + \sum_{k=0}^{\infty} \frac{a_{2k} (-1)^k \ell^{(2k+1)}}{(2k+1)} + h \quad (2.38)$$

For numerical purposes we shall truncate the infinite series of equation (2.32) and consider a finite number of terms a_{2k} ($k=0, 1, \dots, N$). N will be chosen to be large enough such that the various perturbation series converge.

Then for the unknown co-efficients $(a_0, a_2, \dots, a_{2N}, a, \ell)$ $(N+3)$ in number we have so far derived $(N+2)$ equations i.e. $(N+1)$ from (2.37) and one from (2.38). It is apparent that we require additional information about the flow behaviour to derive the $(N+3)^{rd}$ equation which will then complete the system of equations for the unknowns. In addition to the value of the sink strength q , one more parameter must be given to ensure a unique solution. This requirement of an additional degree of freedom has also been encountered in the solution of coning problems by the method of hodographs (6.43).

2.4.4 Limiting case of stability

For the case where the upconed interface is drawn up to its maximum height and is about to breakthrough, the point B is no longer a stagnation point but one which has a non-zero velocity in the vertical direction. This case corresponds to the critical case for which the water is still at rest.

Hence,

$$u - iv \neq 0 \quad \text{when } \zeta = i.$$

Examining (2.25) we see that $\frac{d\zeta}{dz}$ must now also remove the factor $(\zeta - i)$ from the numerator. Therefore $M'(\zeta)$ in equation (2.27) must have a factor $(\zeta - i)$ i.e. $M'(i) = 0$

$$M'(i) = \frac{-2ah}{\pi} - (a^2 + 1) \sum_{k=0}^N a_{2k} (-1)^k = 0 \quad (2.39)$$

A complete system of non-linear algebraic equations for the unknowns of the limiting case have been derived. In section 2.4.6 we outline how the unknown co-efficients may be evaluated numerically.

2.4.5 Sub-critical case

(i) Unique interface profile for a specified external distance of influence.

Here the term 'external distance of influence' will be used to mean the x-distance beyond which the oil-water interface is assumed to be undisturbed. It will refer to points E and F in figure 2.6a. Suppose the external distance of influence is x_e , then substituting $x = x_e$ and $\chi = 0$, or 2π , we obtain,

$$x_e = \frac{-h}{\pi} \ln \frac{|1+a^2-2a|}{|1+a^2+2a|} + \sum_{k=0}^N \frac{a_{2k}}{(2k+1)} \quad (2.40)$$

Equations (2.37), (2.38) and (2.40) together with equation (2.33) completely specify the unique interface profile for a given external distance of influence.

(ii) Unique interface profile given a fixed water volume.

If a fixed water volume is specified an additional equation may be obtained. The volume beneath the cone may be found by integrating equation (2.33). By equating the volume integral to the fixed water volume an additional equation may be derived.

2.4.6 Numerical Procedure

The numerical procedure is as follows

(i) As indicated earlier truncate the infinite series and consider a finite number of co-efficients (a_0, a_{2N}). N must be chosen to be sufficiently large such that convergence of the series is obtained.

(ii) Replace the integral expressions for a_{2k} by numerical integration formula. The trapezoidal integration rule was found to be suitable.

$$a_{2\pi} = \frac{2}{\pi} \frac{\pi}{2M} \left(f'_k(x_1, a, \ell) + f'_k(x_{M+1}, a, \ell) + 2 \sum_{i=2}^M f'_k(x_i, a, \ell) \right)$$

$$k = 0, 1, \dots, N \quad (2.41)$$

where

$$f_k'(x_i, a, \ell) = f(x_i, a, \ell) \cos(2k+1)x_i$$

$$x_i = (i-1) \frac{\pi}{M}$$

$$i = 1, \dots, (M+1).$$

M is chosen to be a large number and represents the number of mesh points in the integration range. For practical purposes $M = 40$ was found to give sufficient accuracy.

(iii) An iterative solution technique such as Newton's method may be used to solve the set of non-linear algebraic equations.

(iv) The convergence of the perturbation series of (2.33) can be further accelerated by the use of Padé approximants. Given a series $S(z)$, Padé approximants (3) are a sequence of rational functions in z which frequently have better convergence properties than the original sequence of partial sums $\{S_\ell(z)\}$.

In our computations we used the Shanks non-linear transformations (also known as the Epsilon algorithm) based on Padé approximants, to accelerate the convergence of the sequences (38).

Suppose S_α is the sum of the infinite series and S_{n+1}, S_n, S_{n-1} are partial sums then an approximation to S_α is

$$S_\alpha \sim \frac{S_{n+1} S_{n-1} - S_n^2}{S_{n+1} + S_{n-1} - 2S_n}.$$

2.5 Extension to upconing towards several point sinks

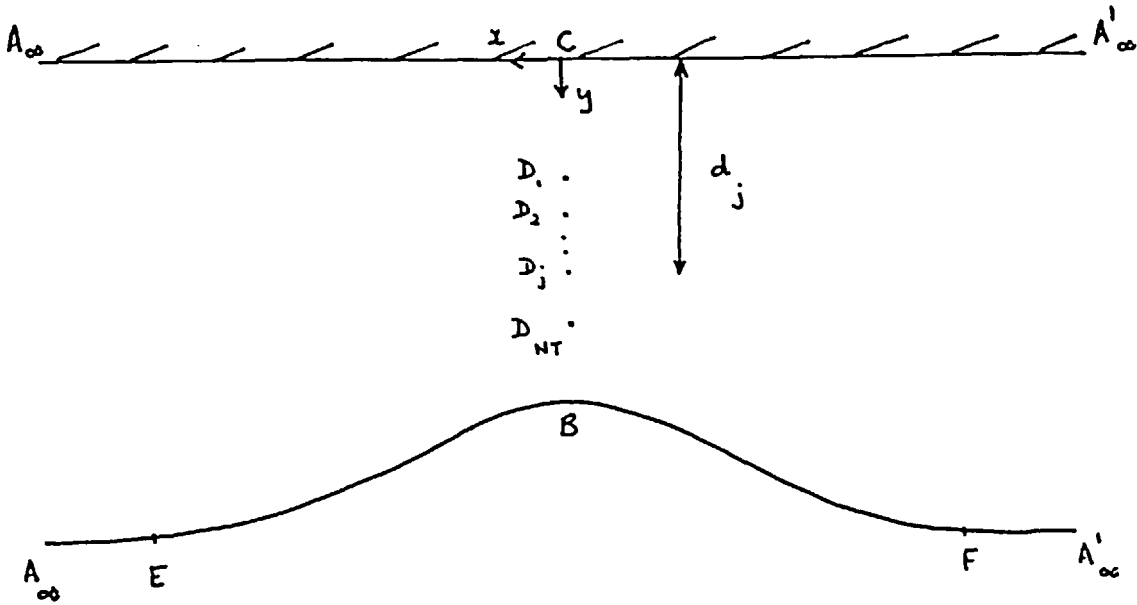


Figure 2.7a : z - plane

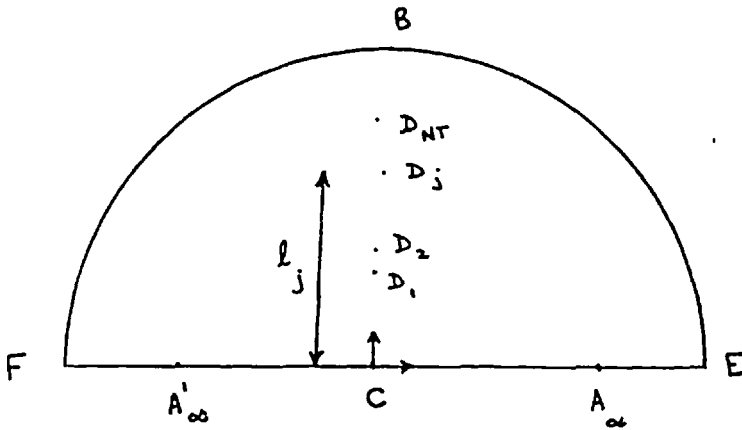


Figure 2.7b : ζ - plane

The problem under consideration is a generalized case of the one discussed in section 2.4.

We extend our analysis to include N point sinks ($D_1, \dots, D_j, \dots, D_{NT}$) positioned at distance $d_1, \dots, d_j, \dots, d_{NT}$ respectively below the origin,

in the z -plane. Each point sink will be of strength q .

The complex potential in the ζ plane is easily verified to be

$$\omega = \frac{NT \cdot q}{2\pi} \ln(\zeta^2 - a^2) \left(\zeta^2 - \frac{1}{a^2}\right) - \frac{q}{2\pi} \sum_{i=1}^{NT} \ln(\zeta^2 + \ell_i^2) \left(\zeta^2 + \frac{1}{\ell_i^2}\right) + \text{constant.}$$

The complex velocity $u - iv$, in the z -plane is now

$$u - iv = \frac{(\zeta^2 - 1)(\zeta^2 + 1)P(\zeta)}{\zeta(\zeta^2 - a^2)\left(\zeta^2 - \frac{1}{a^2}\right)} \prod_{i=1}^{NT} \frac{1}{(\zeta^2 + \ell_i^2)\left(\zeta^2 + \frac{1}{\ell_i^2}\right)} \frac{q}{\pi} \left(\frac{d\zeta}{dz}\right)$$

where

$$P(\zeta) = NT \cdot \prod_{i=1}^N (\zeta^2 + \ell_i^2) \left(\zeta^2 + \frac{1}{\ell_i^2}\right) - (\zeta^2 - a^2) \left(\zeta^2 - \frac{1}{a^2}\right) \sum_{i=1}^{NT} \prod_{\substack{j=1 \\ j \neq i}}^{NT} (\zeta^2 + \ell_j^2) \left(\zeta^2 + \frac{1}{\ell_j^2}\right)$$

$\left(\frac{d\zeta}{dz}\right)$ must remove any unwanted singularities and is therefore of the form

$$\frac{d\zeta}{dz} = \zeta^2 (\zeta^2 - a^2) M(\zeta). \quad (2.42)$$

$M(\zeta)$ may be expressed as a power series. Inverting equation (2.43)

and integrating we obtain

$$z = \frac{-h}{\pi} \ln \frac{(\zeta - a)}{(\zeta + a)} + \sum_{k=1}^{\infty} \frac{a_{2k}}{(2k-1)} \zeta^{(2k-1)} + ih.$$

NT equations may be obtained using the fact that $z = id_j$ is mapped onto $\zeta = i\ell_j$. Therefore,

$$d_j = \frac{-h}{\pi} \tan^{-1} \left(\frac{2a\ell_j}{\ell_j^2 - a^2} \right) + \sum_{k=1}^N \frac{a_{2k}}{(2k-1)} (\ell_j)^{2k-1} (-1)^{(k+1)} + h$$

$$j = 1, \dots, NT \quad . \quad (2.43)$$

The pressure continuity condition, after some manipulation, becomes

$$\sum_k a_{2k} \cos(2k-1)\pi = f(\chi, \underline{\ell}, a)$$

$$\underline{\ell} = (\ell_1, \dots, \ell_{NT})$$

where

$$f(\chi, \underline{\ell}, a) = \frac{2q\kappa}{\pi} \left(\frac{a^2 \sin 2\chi \cdot NT}{1+a^4 - 2a^2 \cos 2\chi} + \sum_{j=1}^{NT} \frac{\ell_j^2 \sin 2\chi}{1+\ell_j^4 - 2\ell_j^2 \cos 2\chi} \right)$$

$$- \frac{h}{\pi} \frac{2a(a^2-1) \cos \chi}{(1+a^2)^2 - 4a^2 \cos^2 \chi} \quad (2.44)$$

Equations (2.43) and (2.44) together with one other equation (which can be obtained as for the single point sink case) completely specify the interface profile.

2.6 Extension to anisotropic reservoirs

In the previous sections we confined our attention to isotropic systems. In this section we shall see that solution for 2-dimensional

flow problems in reservoirs with anisotropic, but homogeneous, permeability can be obtained from the solutions to corresponding isotropic problems by a simple transformation of the co-ordinate system.

Taking the co-ordinate axes x and y (in the physical plane) to be principle directions of permeability, the Darcy velocity may be expressed as,

$$\underline{u} = \underline{\tilde{K}} \cdot \nabla \Phi'$$

where $\underline{\tilde{K}} = (K_x, K_y)$. K_x and K_y being the constant permeabilities in the x and y directions respectively ($K_x \neq K_y$). The potential Φ' in this case is,

$$\Phi' = -\frac{\rho}{\mu} \left(\frac{P}{\rho} - gy \right),$$

and is not as defined in equation (2.1).

Substituting the expression for $\underline{u}$ in the equation of continuity $\nabla \cdot \underline{u} = 0$, we obtain

$$K_x \frac{\partial^2 \Phi}{\partial x^2} + K_y \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad . \quad (2.44)$$

Equation (2.44) can be reduced to Laplace's equation by means of the geometric transformation,

$$\bar{x} = x, \quad \bar{y} = \frac{1}{\gamma} y \quad (2.45)$$

where,

$$\gamma^2 = \frac{K_y}{K_x}$$

Introducing a stream function Ψ such that

$$\underline{u} = (u, v) = \sqrt{K_x K_y} \left(\frac{\partial \psi}{\partial x}, -\frac{\partial \psi}{\partial y} \right) \quad (2.46)$$

we see that in the $\bar{z}$ - plane $(\bar{x}, \bar{y})$, the Cauchy - Riemann conditions

$$\frac{\partial \Phi}{\partial \bar{x}} = \frac{\partial \psi}{\partial \bar{y}}, \quad \frac{\partial \Phi}{\partial \bar{y}} = -\frac{\partial \psi}{\partial \bar{x}} \quad \text{apply.}$$

Also the flow-rate between two streamlines ψ_M and ψ_N is

$$Q = \sqrt{K_x K_y} (\psi_N - \psi_M).$$

The equation of continuity (2.44) is now transformed to:

$$\frac{\partial^2 \Phi}{\partial \bar{x}^2} + \frac{\partial^2 \Phi}{\partial \bar{y}^2} = 0.$$

Hence, we have reduced the anisotropic system to an equivalent isotropic system. The transformation of co-ordinates (2.45) represents a distortion of the actual dimensions of the flow region.

The analysis now is as for the isotropic problem in Sections (2.4), (2.5), ensuring that the transformed boundary and interface conditions are used. Once the problem has been solved in the equivalent isotropic system, the interface profile in the z -plane can easily be obtained by the co-ordinate transformation (2.45).

2.7 Numerical results

The model developed in section (2.4) was tested on 2 example problems with the following data.

Both problems had the same rock and fluid properties.

$$\begin{aligned}\rho_0 &= 0.6 \text{ g/cc} & (600 \text{ kg m}^{-3}) \\ \rho_w &= 1.39 \text{ g/cc} & (1390 \text{ kg m}^{-3}) \\ \mu_0 &= .09 \text{ cp} & (9. \times 10^{-5} \text{ Pa} \cdot \text{s}) \\ K_0 &= .8007 \times 10^5 \text{ mD} & (0.79 \times 10^{-10} \text{ m}^2)\end{aligned}$$

Reservoir distances:

Problem 1

Problem 2

$$h = 6.56 \text{ ft} \quad (200 \text{ cm}) \qquad h = 1.46 \text{ ft} \quad (44.6 \text{ cm})$$

$$d = 6.07 \text{ ft} \quad (185 \text{ cm}) \qquad d = 0.61 \text{ ft} \quad (18.5 \text{ cm})$$

q - sink strength (cc/sec)

(i) Limiting interface properties

The proposed new model was used to compute limiting profiles of stability. Figures (2.8a) and (2.8b) show these profiles for a range of sink strengths. Figure 2.8a is for the data of problem 1, and 2.8b for the data of problem 2.

We observe that the height of the water cone increases monotonically as Q decreases. Theoretically, if $Q = \infty$, the height of the cone is zero, and if $Q = 0$, the height is maximum i.e. up to the point of the point sink.

If the strength is Q , the limiting case solution gives a

volume of water beneath the oil. If Q is increased above Q_1 to Q_2 , some water will flow out until its volume beneath the oil is that consistent with the solution for $Q = Q_2$. If the flow-rate increases further, the volume decreases further. However, if the flow-rate is now decreased to Q_1 again, say, the flow cannot return to the limiting case for $Q = Q_1$, because the water, once lost cannot be regained. The flow will then be given by a solution for the sub-critical case, with the volume of water equal to that which is now available, but with $Q = Q_1$.

(ii) Sub-critical case

In figure 2.9 the interface profiles for problem 1 are shown for a given external distance of influence. Unique interface profiles can also be determined given a fixed water volume. We note that the free surface profiles for the sub-critical case are much flatter than those of the limiting case (figure 1b). This is to be expected as the velocity at the peak of the interface is zero, for the sub-critical case. We see that for the sub-critical case, the height of the cone decreases as the flow rate decreases.

(iii) Comparison with the Hodograph solution

We have compared our solutions for the limiting case against that of the hodograph solution due to Bear (6). Solutions at the peak of the cone are compared in table 2.1. It should be noted that the hodograph solution is for the problem of upconing in an infinite domain - infinite oil and water and no impermeable boundaries.

Table 2.1

Flow rate	Bear's solution	Model solution	
		Problem 1	Problem 2
Q	h_i	h_i	h_i
90	6.06	5.68	7.03
70	4.71	4.57	5.48
40	2.69	2.35	3.49

(h_i - distance between the top of the cone and the point sink).

The solutions obtained, for problems 1 and 2, from the proposed model are in reasonable agreement with the hodograph solution. However, the solution to problem 1 is distinctly in better agreement. This is to be expected as the data for problem 1 better approximates the infinite reservoir assumption(i.e. no impermeable boundaries present) that is inherent in the hodograph solution.

(iv) Influence of the number of series terms on solution

In figure 2.10 two interface profiles have been computed with $N = 9$ and $n = 18$ terms in the series summation. Agreement between the two solutions is excellent, indicating that $N = 9$ provides sufficient accuracy.

(v) Influence of the number of mesh points, in integration range, on solution

In the figure (2.11) we examine the influence of the number of mesh points on the integration involved in equation (2.37). The two solutions almost coincide indicating that $M = 40$ gives a sufficiently

fine mesh for the integration range.

Figure (2.12) shows the limiting interface profiles for the problem of coning toward 2 point sinks placed at $d_1 = 20$ cm and $d_2 = 15$ cm (problem 2). Q represents the total strength of the 2 sinks.

Figure (2.13) displays limiting interface profiles for the case of anisotropic permeability (With $K_y/K_x = 1.33$) for problem 2. The limiting cone heights are lower than those for the isotropic case (fig 2.8b). This is to be expected as the permeability in the vertical direction is higher than in the horizontal direction.

2.8 Summary

2 dimensional, steady problems in porous media have been discussed. The application of the hodograph method to free surface problems has also been investigated. Although the hodograph method does provide exact solutions to some problems with relatively simple geometries for most practical cases, its approach is too complex to be considered. A new alternative, more flexible, technique is developed for the problem of up-coning in the 2-dimensional vertical plane. The unknown free surface is first mapped onto a unit semi circular boundary and the problem is then solved in the transformal plane. Transformations from one plane to another are achieved numerically. The proposed model can be extended to the case of up-coning towards several point sinks and to anisotropic systems.

The model has been tested on two example problems. Comparisons with the hodograph solution are made, and some checks on the accuracy of the numerical scheme have been carried out.

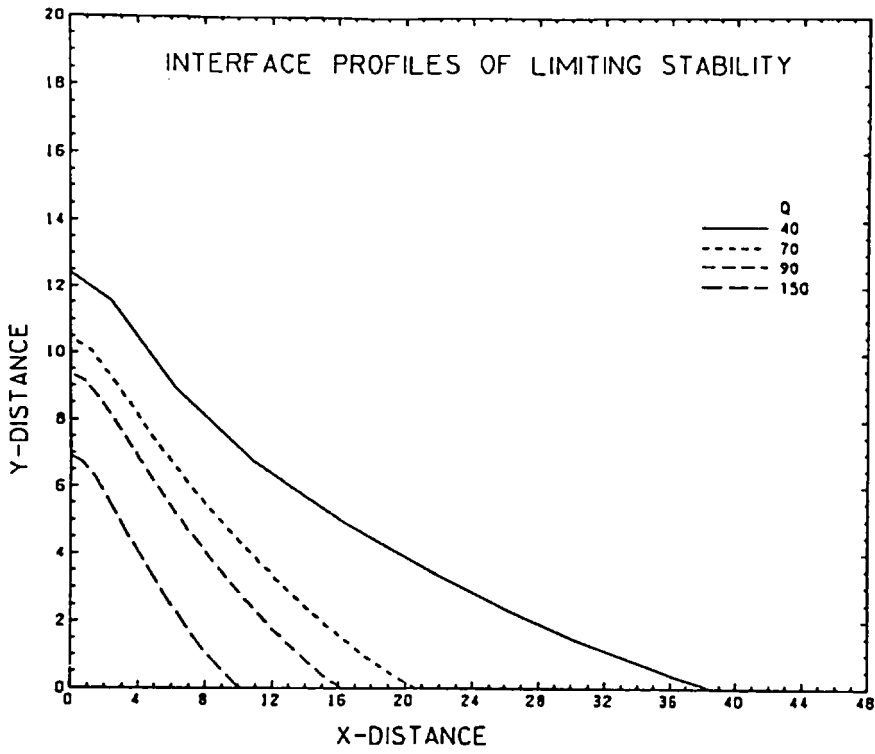


Fig 2.8a: Limiting profiles of stability (problem 1)

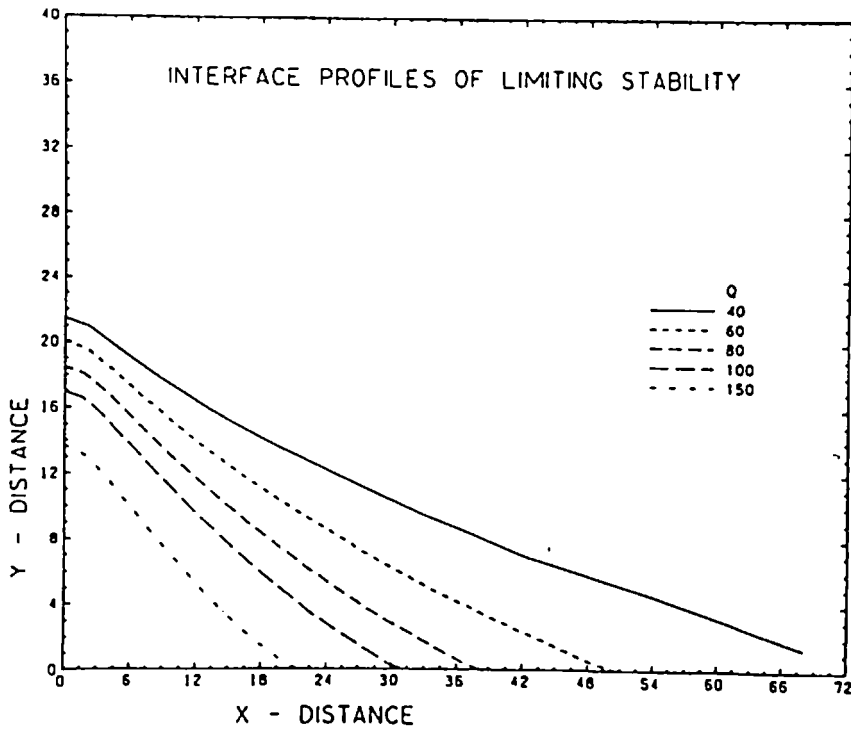


Fig 2.8b: Limiting profiles of stability (problem 2)

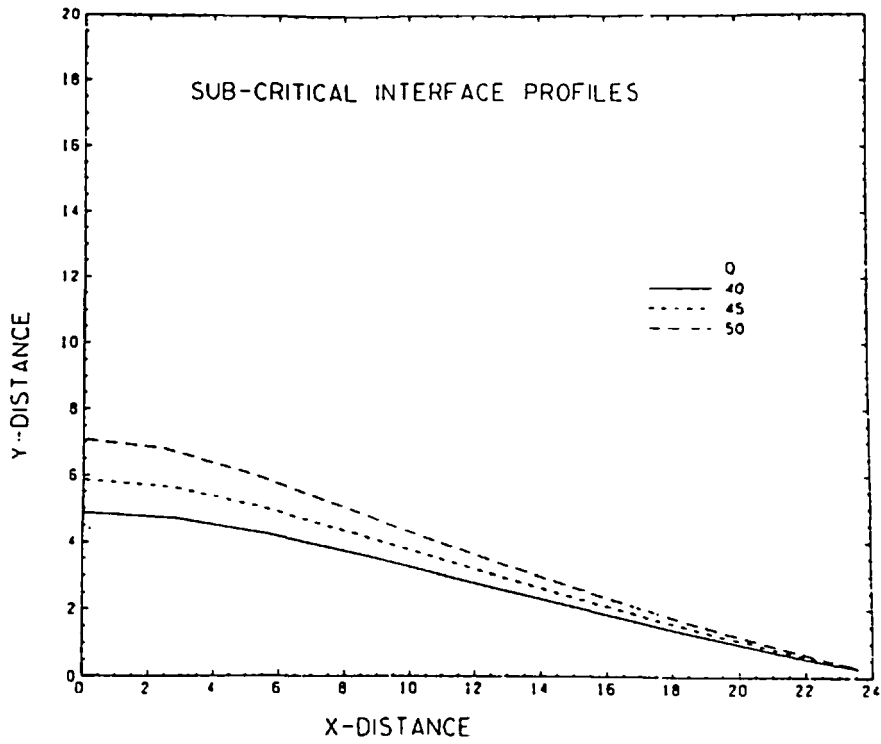


Figure 2.9: Sub-critical interface profiles (problem 1)

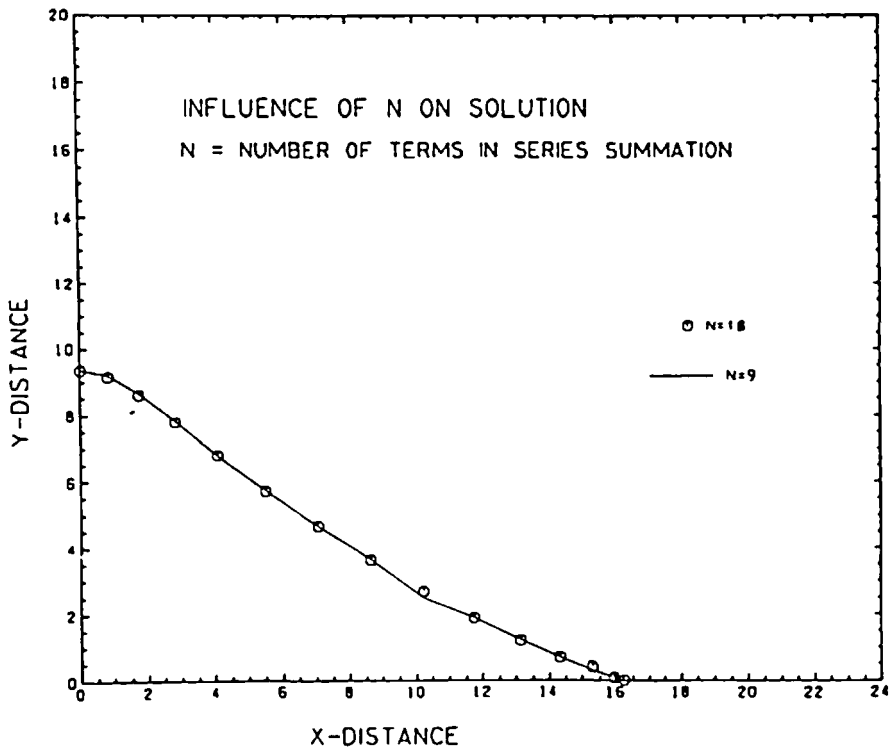


Figure 2.10: Influence of the number of series terms on the solution

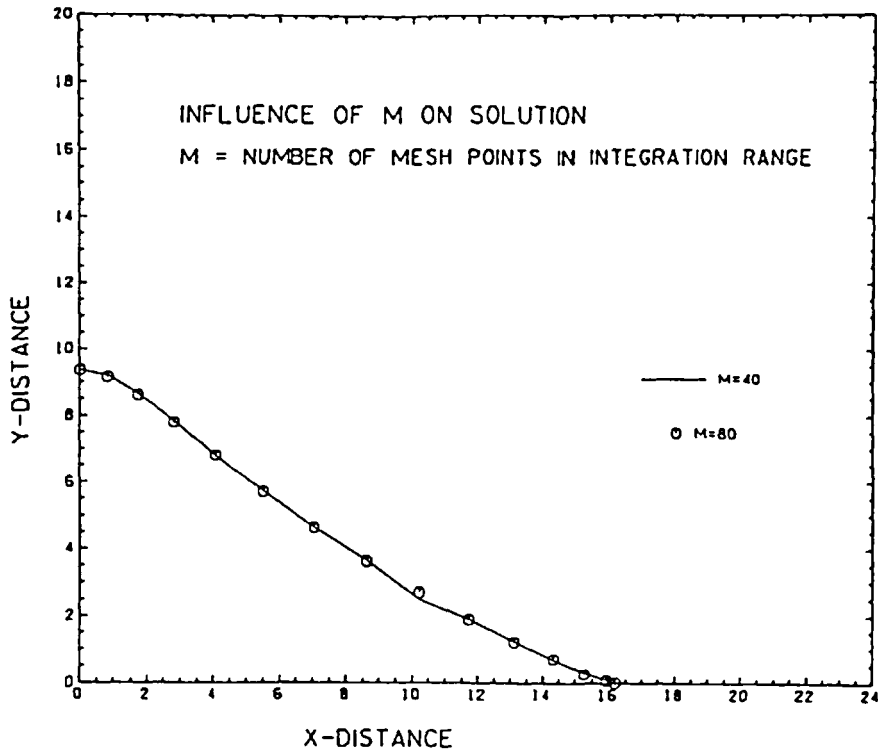


Figure 2.11: Influence of mesh points on solution

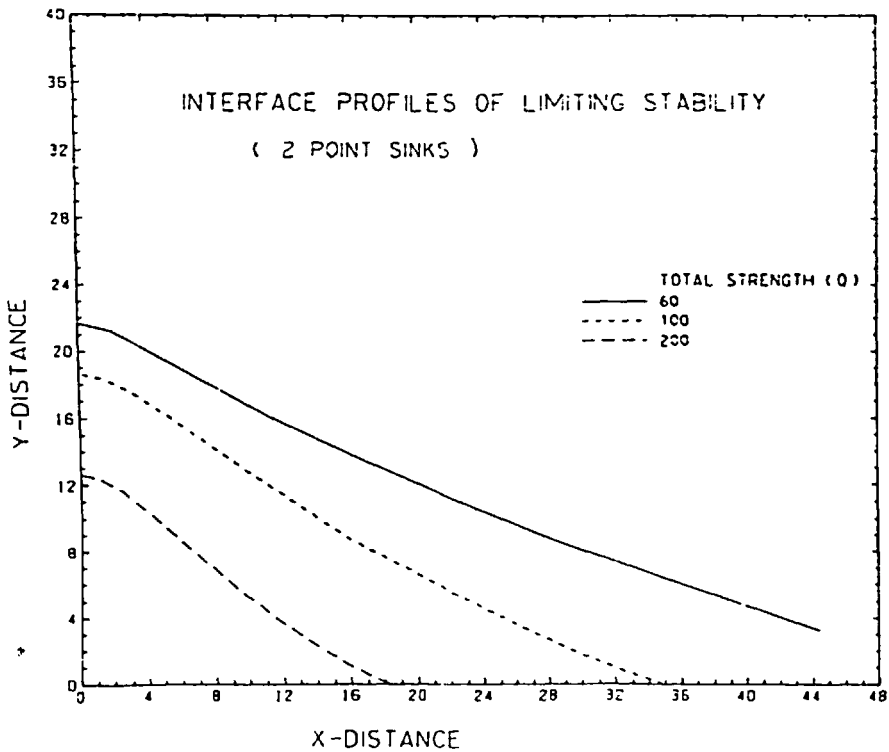


Figure 2.12: Limiting profiles of stability - 2 point sinks

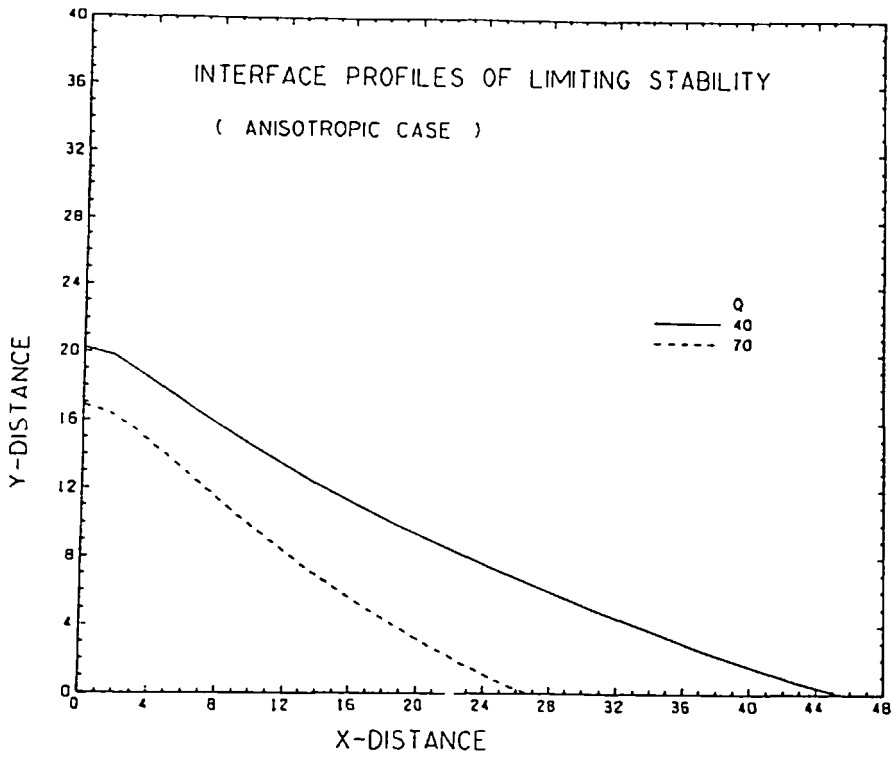


Figure 2.13: Limiting profiles of stability - anisotropic case

CHAPTER 3

THE RADIAL CONING PROBLEM - STEADY STATE

3.1 Introduction

We have already seen that the occurrence of coning in petroleum reservoirs, is detrimental to the economics of oil production. In this chapter we focus our attention on steady state radial coning phenomena, as a first step towards understanding reservoir coning behaviour.

Although radial coning problems are usually solved by sophisticated numerical simulators, the relatively high computational cost involved in such a reservoir scale simulation has led to the development of several coning correlations. We review some of the better known coning correlations, dealing with the steady state problem, and discuss their limitations. Most of these correlations adopt either a simplified analytical approach or are based on experimental work on scale models. Although they are easy to use, they do contain many simplifying assumptions and therefore their accuracy and value is open to question. In this chapter we formulate a new mathematical model, and develop a novel boundary integral equation method (BIEM) for the solution of the steady-state coning equations.

The essence of the BIEM is the transformation of the governing differential equations into an equivalent set of integral ones. This transformation is achieved by Green's 2^{nd} identity. The transformed integral equations involve boundary data only. Hence any discretization that is necessary is introduced only at the boundaries, and not over

the entire flow domain. This not only improves the accuracy of the scheme, but also minimizes computer requirements. Furthermore, in the case of coning, by formulating the problem in terms of a perturbation potential and by devising a Green's function, such that its normal derivative vanishes over part of the boundary surface, the integration range is further confined to the oil water contact only.

3.2 A review of some existing literature

In this section we shall briefly discuss some of the existing literature on the steady state coning phenomenon mainly with respect to the determination of critical production rates and interface profiles.

3.2.1 Muskat and Wycoff (1934)

The first analytical study of the coning problem was made by Muskat and Wycoff (34). Their study deals with the case, where the water lies static beneath the oil zone in an elevated or coned position. The reservoir formation is assumed to be homogeneous with the upper portion saturated with oil and the lower portion with water. The up-coned oil water interface is treated as a limiting streamline. In this instance of steady state, the permeability to water and the porosity do not enter the analysis of the problem.

By equating pressure across the oil water interface, Muskat and Wycoff found the shape of the coned surface to depend on the pressure distribution in the oil domain. However, the pressure

distribution itself depends on the shape of the cone. The simultaneous determination of the pressure distribution and the shape of the cone can be solved for only by numerical methods. Muskat and Wycoff, therefore, made the simplifying assumption that the nature of the pressure distribution is effectively the same as that for the case where there is no cone and oil is flowing in a sand between two parallel impermeable boundaries into a partially penetrating well. The pressure or potential distribution for the latter case may be found easily (12).

The equation of the upconed interface, $z = y(r)$, was thus found to be

$$\sum_n a_n \Phi_0(b_n, r, y) = A + \frac{B}{\Delta P} \left(1 - \frac{y}{h}\right)$$

$$A = \frac{4}{h} \sum_n a_n b_n \log \frac{4h}{r_e} ; \quad B = gh\Delta\rho(\Phi_e - A).$$

with $\Phi = \frac{K}{\mu}(p - p_0 - \rho_0 gz)$, h , the oil thickness, r_e , the formation boundary, and Φ_e the potential at r_e . a_n and b_n represent the density and the depth of hypothetical flux elements and may be found in Muskat (ref 33).

Once the physical constants of the system, the well penetration, and the pressure differential, ΔP , are given the equation for the upconed interface may be solved for. Muskat and Wycoff suggest a graphical technique. Plot the left hand side of the equation as a function of y and find the intersection of the curve so formed with the straight line representing the right hand side

of the equation for the interface. Depending on the slope of the lines there will be two roots, one root, or no roots (intersections). If there are two roots, both roots represent solutions that could be in hydrostatic equilibrium, but only one would be in dynamic equilibrium. The condition for dynamic equilibrium is simply that the vertical pressure gradients in the oil zone must be less than that in the water zone.

Thus for a specified pressure drop and well penetration Muskat and Wycoff were able to establish whether a cone was stable or not. They also established the existence of a critical cone height - a distance between the original interface and the bottom of the well, beyond which instability appears and water breakthrough occurs almost instantaneously.

Although the work of Muskat and Wycoff contains many simplifying assumptions (including identical oil and water mobility), it does present the fundamental physical principles underlying the behaviour of a cone. Their concept of critical cone height has since been used widely in other coning correlations.

3.2.2 Bournazel and Jeanson (1971)

Bournazel and Jeanson (7) proposed a correlation that gives the critical production rate, the breakthrough time and the water oil ratio after breakthrough. It is an extension of the earlier work of Sobocinsky and Cornelius (39). The correlations were derived mainly from experimental scale models, although a simplified analytical approach was used for post breakthrough behaviour.

Two dimensionless variables a reduced cone height, z_D , and a reduced breakthrough time, t_D , were defined as follows.

$$z_D = \frac{2\pi \Delta\rho g k_x h_0^2 (1 - \frac{h_p}{h_0})}{\mu_0 q}$$

$$t_D = \frac{\Delta\rho g K_{\frac{1}{2}} t_{BT}}{\mu_0 \sigma h_0} f(M)$$

where $f(M)$ is a function of the mobility ratio

$$f(M) = \frac{1+M^\alpha}{2} \quad \alpha = 0.7 \text{ (from experimental data).}$$

Then, by plotting z_D against t_D using experimental data, the resulting curve was found to asymptotically approach a constant value of 4.3.

$$z_D \rightarrow 4.3 \quad \text{as} \quad t_D \rightarrow \infty$$

and the expression for the critical production rate was found to be

$$q_c = \frac{1.466 \Delta\rho g K h_0^2 (1 - \frac{h_p}{h_0})}{\mu_0} .$$

where h_0 is the oil formation thickness and h_p the well penetration.

Bournazel and Jeanson's correlation are easy to use and are suitable for hand calculation. However, their expression for critical production rate is based solely on experimental scale models and

therefore its accuracy on a reservoir scale is open to question.

3.2.3 Chierici, Ciucci, and Pizzi (1964)

Chierici, Ciucci and Pizzi (13) used potentiometric models to reproduce the pressure distribution for a range of reservoir dimensions. The critical production rate was taken to be

$$q_c = 2\pi g \left(\frac{h_0^2 \Delta \rho K_{e0}}{B_0 \mu_0} \right) \psi(r_d, \epsilon, \delta_w)$$

where K_{e0} is the effective permeability to oil and B_0 the oil formation factor. The variable ψ is a function of the reservoir dimensions, the perforation interval factor and its position within the reservoir. It is this factor which is found using the potentiometric models, and graphs of ψ are given for various values of r_d , ϵ and δ_w . The variables r_d , ϵ and δ_w are geometry dependent.

3.2.4 Camy (1975)

Camy (11) made a theoretical analysis of the problem of cone evolution. By allowing time assume large values results for the steady state problem may also be easily derived.

Camy used the technique of small perturbations to linearise the non-linear equations of the free surface boundary value problem. The resulting equations were then solved analytically.

Camy's analysis can be applied to the following type of system

- (i) a homogeneous reservoir
- (ii) small oil-water transition zone and hence negligible capillary pressure

- (iii) incompressible fluids
- (iv) 'no-flow' upper and lower boundaries
- (v) fixed oil water contact at infinity

A diagrammatic representation of Camy's model is in Figure 3.1.

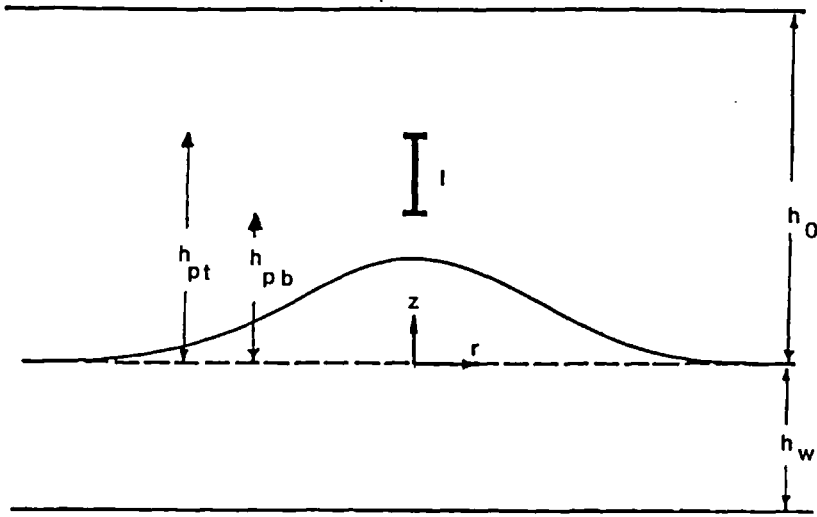


Figure 3.1: A diagrammatic representation of Camy's model

Camy defined a velocity potential for each phase $i = 0, w$ as

$$\phi_i = \frac{KK}{\mu_i} \frac{ri}{\rho_i} \left(z + \frac{p}{\rho_i} \right) \quad i = 0, w .$$

The governing equations and boundary conditions were

$$\nabla^2 \phi_0 = 0 \quad (\text{in the oil zone}) \quad (3.1)$$

$$\nabla^2 \phi_w = 0 \quad (\text{in water zone}) \quad (3.2)$$

$$\frac{\partial \phi_0}{\partial z} = 0 \quad \text{on } z = h_0 \quad (3.3)$$

$$\frac{\partial \phi_0}{\partial z} = 0 \quad \text{on } z = -h_w . \quad (3.4)$$

Let the interface position be written as $F \equiv z - \eta(r, t) = 0$

For the period up to water breakthrough, the motion of any particle on the oil water interface is exactly the same as the oil water interface itself. This is expressed mathematically as

$$\frac{DF}{Dt} = 0 \quad (3.5)$$

where $\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + V \cdot \nabla \right)$, and V is the velocity of the interface.

Equation (3.5) is known as the material surface condition.

Note that $V = - \frac{1}{\sigma} \nabla \Phi_i$ ($i = 0, w$), where σ is the effective porosity. Then after substitution of the velocity potentials Φ_i into equation (3.5), and after some simplification the equations that represent the behaviour of the oil-water contact are obtained.

$$b_0 \frac{\partial \Phi_0}{\partial t} - b_w \frac{\partial \Phi_w}{\partial t} - a_0 (\nabla \Phi_0)^2 + a_w \nabla \Phi_0 \cdot \nabla \Phi_w - \frac{\partial \Phi_0}{\partial z} = 0 \quad (3.6)$$

and,

$$b_0 \frac{\partial \Phi}{\partial t} - b_w \frac{\partial \Phi_w}{\partial t} + a_w (\nabla \Phi_w)^2 - a_0 \nabla \Phi_0 \cdot \nabla \Phi_w - \frac{\partial \Phi_w}{\partial z} = 0 \quad (3.7)$$

where

$$a_0 = \frac{\mu_0}{K K_{ro} \Delta \rho}$$

$$a_w = \frac{\mu_w}{K K_{rw} \Delta \rho}$$

$$b_0 = \sigma a_0$$

$$b_w = \sigma a_w$$

Equations (3.6) and (3.7) are non-linear and have to be satisfied on the interface position. However, the interface position itself is unknown a priori, and therefore an exact analytical solution to the equations is not possible. Camy, therefore, linearized the equations by the method of small perturbations.

The variables Φ_0 , Φ_w and h were expanded in terms of a small parameter ϵ ,

$$\Phi_0(r, z, t) = \Phi_0^{(0)}(r, z) + \epsilon \Phi_0^{(1)}(r, z, t) + \epsilon^2 \Phi_0^{(2)}(r, z, t) + \dots$$

$$\Phi_w(r, z, t) = \Phi_w^{(0)}(r, t) + \epsilon \Phi_w^{(1)}(r, z, t) + \epsilon^2 \Phi_w^{(2)}(r, z, t) + \dots$$

$$h(r, t) = h^{(0)}(r) + \epsilon h^{(1)}(r, t) + \epsilon^2 h^{(2)}(r, t) + \dots$$

$\Phi_0^{(0)}$, $\Phi_w^{(0)}$ and $h^{(0)}$ represent the non-perturbed steady state.

A first order analysis was used and the governing equations for $\Phi_0^{(1)}$ and $\Phi_w^{(1)}$ were solved. In so doing, higher order terms in the expansions were neglected.

Representing the well by a line sink, using the method of images and after some algebraic manipulation an expression for $h(r, t)$ was found.

$$h(r, t) = \frac{qa_0}{2\pi\ell} \int_0^\infty \left[\left(\frac{\sinh\lambda(h_0 - h_{PB}) - \sinh\lambda(h_0 - h_{Pt})}{\lambda \sinh\lambda h_0} \right) x \right. \\ \left. \left(1 - \exp\left(\frac{-\lambda t}{a_w \cosh\lambda_w + a_0 \cosh\lambda h_0} \right) \right) \right] J_0(\lambda, r) d\lambda \quad (3.8)$$

where $\ell = h_{PT} - h_{PB}$

and $J_0(\lambda r)$ is a Bessel function.

In the case of a well penetrating a small interval, with a thick oil and water zone, the first order solution (equation 3.8) takes a simpler form that is obtained by letting h_0 and h_w tend to infinity.

$$h(r, t) = \frac{qa_0}{\pi\ell} \int_0^\infty \left(\left(\frac{\ell}{\pi} \sin \frac{\lambda\ell}{2} \exp(-\lambda(\frac{h_{PT}+h_{PB}}{2})) \right) \times \right. \\ \left. \left(1 - \exp(-\frac{\lambda t}{a_0+a_w}) \right) J_0(\lambda r) \right) d\lambda$$

which becomes after performing the integration,

$$h(r, t) = \frac{qa_0}{2\pi\ell} \left(\ln \frac{h_{PT} + (h_{PT}^2 + r^2)^{1/2}}{h_{PB} + (h_{PB}^2 + r^2)^{1/2}} \right. \\ \left. - \ln \frac{h_{PT} + t^* + \{(h_{PT}^2 + t^{*2}) + r^2\}^{1/2}}{h_{PB} + t^* + \{(h_{PB}^2 + t^{*2}) + r^2\}^{1/2}} \right) \quad (3.9)$$

where $t^* = \frac{t}{(a_0+a_w)}$.

Camy's solution is a first order linearized solution and therefore cannot be expected to represent accurately the interface position of large cones. However, for small flow-rate cones, where non-linearities are less dominant, the solution is expected to be reasonably accurate.

3.3 Boundary Integral Methods

3.3.1 Discussion of Numerical Methods

The numerical methods most widely used at present, in reservoir simulation, tackle the governing equations directly in the form in which they were derived, without any further mathematical manipulation. For instance, finite difference methods approximate the differential operators in the equations by simpler, localized algebraic ones valid at a series of nodes within the region. Thus, the precision of the numerical solutions obtained is entirely dependant on the fineness of the mesh used to define the nodal points. Consequently large systems of simultaneous algebraic equations are always generated as part of the solution technique. This is particularly evident in the case of finite-difference coning simulators, where the converging nature of the flow itself requires the construction of a very fine mesh near the well bore. Thus ensuring accuracy, in addition, would require a mesh so fine that it would be impossible to construct in practice, due to computational limitations. Even if this were possible numerical instability problems are likely to be severe due to too fine a mesh. There is therefore no measure of the precision of the numerical solution obtained from finite-difference simulators.

Finite element methods on the other hand represent the region of interest by non-infinitesimal (finite) elements which are assembled to provide an approximation to the real system. It is also a whole-body discretization scheme and therefore leads to

a very large number of finite elements. It is effective for problems where the knowledge of the variable is required throughout the domain of interest. In the case of free-boundary problems, such as the coning problem, where the flow information is required mostly on the OWC it is ineffective in that it is a whole body discretization scheme and therefore presents the same difficulties as the finite difference method.

The Boundary Integral Equation Method (BIEM) adopts a completely different approach, by attempting to integrate analytically the set of governing differential equations, before proceeding to any discretization or approximation. By integration of the differential equations one would expect to obtain a set of equations which involve values of the variables only at the extremities of the range of integration, i.e. on the boundary of the region. The boundary integral equations thus obtained are a statement of the exact solution to the problem posed, and numerical errors due to discretization arise only on the boundaries due to the inability to carry out the integrations in closed form. Numerical integration is also a far more stable and precise process than numerical differentiation.

Finally the computational requirements for the BIEM, particularly with respect to storage, are clearly very low in contrast to the requirements for the conventional domain or whole body discretization methods. This in turn enables one to achieve far greater accuracy, for a given storage and machine time, than is possible by the conventional methods.

The boundary integral equation method has recently gained popularity in the numerical solution of certain partial differential equations. Where applicable, it has been found to be far more accurate than conventional domain discretization methods such as the finite difference or finite element method.

3.3.2 The integral equation

The basis of boundary integral methods is Green's 2nd identity. Green's 2nd identity states that if two functions U and V are continuous and have continuous first and second partial derivatives in the closure of a domain D , bounded by a surface S , then:

$$\int_D (U \nabla^2 V - V \nabla^2 U) dV = \int (U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n}) ds. \quad (3.10)$$

$\underline{n}$ is the unit outward normal directed from the surface S .

If two functions U and V are chosen such that they are harmonic in the domain D , then from equation (3.10) the volume integral on the left hand side of (3.10) vanishes altogether. This property can be exploited to reduce the dimensionality of the problem by one.

We shall choose U to be some function Φ , which is analytic and harmonic in the domain D . V will be chosen to be the three-dimensional Green's function for an infinite space which satisfies the equation

$$\nabla^2 g = -4\pi \delta(x-x_i) \delta(y-y_i) \delta(z-z_i)$$

where $\delta(\cdot)$ is the Dirac delta function and $g = \frac{1}{R}$,
 $R = [(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{\frac{1}{2}}$, which is singular at
 $P_i(x_i, y_i, z_i)$. P_i is known as the base point for the Green's
function.

With the above choice of U and V if P_i is chosen
inside the domain D , equation (3.10) becomes,

$$-4\pi \Phi(x_i, y_i, z_i) = \int (\Phi \frac{\partial g}{\partial n} - g \frac{\partial \Phi}{\partial n}) ds \quad (3.11)$$

Thus the solution for Φ at any interior points can be
obtained by performing the surface integration of equation (3.11),
if Φ and $\frac{\partial \Phi}{\partial n}$ are known throughout S .

If P_i is chosen on the boundary S , then equation (3.11) becomes,

$$-\alpha_i \Phi(x_i, y_i, z_i) = \int (\Phi \frac{\partial g}{\partial n} - g \frac{\partial \Phi}{\partial n}) ds \quad (3.12)$$

where $\alpha_i = 2\pi$, if the boundary is smooth

$= 4\pi \times (\text{solid interior angle at } P_i)$, if not.

In a well posed problem, both Φ and $\frac{\partial \Phi}{\partial n}$ are not known on
the surface S - instead either Φ or $\frac{\partial \Phi}{\partial n}$ or a relationship between
them is known at all points on the boundary. The integral equation
(3.12) can, therefore, be used to solve for the 'missing data'.
Once the missing data on the boundary are solved for, equation
(3.11) may be used to obtain the solution at any interior point.

3.3.3 Solution of the boundary integral equation.

Equation (3.12) cannot, in general, be solved analytically.

It can, however, be readily solved by numerical methods.

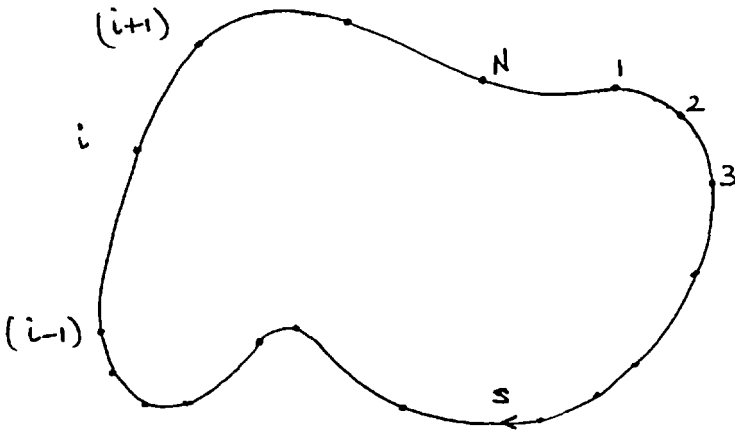


Figure 3.2 : Typical boundary contour

Since we can choose the base points, P_i , anywhere on the surface S , by selecting N points on the boundary, we can from equation (3.12) generate N integral equations relating Φ and $\frac{\partial \Phi}{\partial n}$. i.e. one equation for each choice of P_i ($i=1, N$). Since either Φ or $\frac{\partial \Phi}{\partial n}$ or a relationship between Φ and $\frac{\partial \Phi}{\partial n}$ is specified at each node, from the given boundary conditions, there are sufficient equations to solve for the unknown boundary values at the nodes. By discretizing the surface, the N integral equations can be further reduced to a system of N simultaneous equations.

3.3.4 Application to axi-symmetric problems

Many free surface problems, including the reservoir coning problem, are axi-symmetric. Axi-symmetric problems are problems where all the physical variables and the geometry of the domain are independent of the cylindrical co-ordinate θ .

The adaptation of the boundary integral equation method to axi-symmetric flow problems was first developed by Lennon et al (23).

Consider, once again, the free space Green's function, g ,

$$\begin{aligned} g &= \frac{1}{R} = \{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2\}^{-\frac{1}{2}} \\ &= \{(r\cos\theta - r_i \cos\theta_i)^2 + (r\sin\theta - r_i \sin\theta_i)^2 + (z-z_i)^2\}^{-\frac{1}{2}} \\ &\dots \end{aligned} \quad (3.13)$$

in cylindrical co-ordinates (r, θ, z) .

Although the problem is axi-symmetric g is still a function of the co-ordinate θ . Substituting, once again, a harmonic function Φ and g in equation (3.10), we obtain

$$-\alpha \Phi(r_i, z_i) = \oint_{\Gamma} \left(\Phi \int_0^{2\pi} \frac{\partial g}{\partial n} d\theta - \frac{\partial \Phi}{\partial n} \int_0^{2\pi} g d\theta \right) r d\xi. \quad (3.14)$$

In equation (3.14), the surface element dS , has been expressed as $(r d\theta d\xi)$ and Γ is the generating boundary contour of the surface S .

Defining an 'axi-symmetric Green's function' as,

$$G(r, z; r_i, z_i) = \int_0^{2\pi} g d\theta, \quad (3.15)$$

equation (3.15) becomes ,

$$-\alpha_i \Phi(r_i, z_i) = \oint_{\Gamma} \left(\Phi \frac{\partial G}{\partial n} - G \frac{\partial \Phi}{\partial n} \right) r d\xi \quad . \quad (3.16)$$

Thus, by introduction of the function G , the integral equations have been reduced to line integrals along the generating contours.

The numerical procedure for the solution of these equations is similar to that for 2-dimensional problems (9, 24, 25). The axi-symmetric Green's function G , however, must be evaluated numerically.

Equation (3.15) may be rewritten as

$$\begin{aligned} G &= \int_0^{2\pi} \frac{1}{(a-b\cos\theta)^{\frac{1}{2}}} d\theta \\ &= \frac{4}{(a+b)^{\frac{1}{2}}} K(m) \end{aligned} \quad (3.17)$$

where, $a = r_i^2 + r^2 + z_d^2$, $b = 2rr_i$ and $z_d = z - z_i$. $K(m)$ is the complete elliptic integral of the first kind,

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{(1-m\sin^2\theta)^{\frac{1}{2}}} \quad (0 \leq m \leq 1)$$

where the modulus m and complementary modulus m_1 are

$$\begin{aligned} m &= \frac{4rr_i}{(r+r_i)^2 + z_d^2} \\ m_1 &= \frac{(r-r_i)^2 + z_d^2}{(r+r_i)^2 + z_d^2} \end{aligned}$$

For numerical computation, it is convenient to express the elliptic integral $K(m)$, as a polynomial. i.e.

$$K(m) = (a_0 + a_1 m_1 + \dots + a_4 m_1^4) + (b_0 + b_1 m_1 + \dots + b_4 m_1^4) \ln\left(\frac{1}{m_1}\right) + \varepsilon(m) \quad (3.18)$$

with,

$$\begin{aligned} a_0 &= 1.38629436112 & b_0 &= 0.5 \\ a_1 &= 0.09666344259 & b_1 &= 0.12498593597 \\ a_2 &= 0.03590092383 & b_2 &= 0.06880248576 \\ a_3 &= 0.03742563713 & b_3 &= 0.03328355346 \\ a_4 &= 0.01451196212 & b_4 &= 0.00441787012 \end{aligned}$$

and $\varepsilon(m) \leq 2 \times 10^{-8}$ - the upper bound for the error of this approximation.

Higher order polynomial may be used to approximate $K(m)$ if wished (1).

The normal derivative of G , along the boundary contour is,

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r} n_r + \frac{\partial G}{\partial z} n_z$$

where

$$\frac{\partial G}{\partial z} = - \frac{-4z d}{(a+b)^{1/2} (a-b)} E(m)$$

and

$$\frac{\partial G}{\partial r} = \frac{4}{(a+b)^{3/2}} \left[-(r+r_i) E(m) \frac{(a+b)}{(a-b)} + 4r_i \left(\frac{dK}{dm} \right) \right]$$

Using the identity,

$$E(m) = m_1 \left\{ 2m \frac{dk}{dm} + K(m) \right\}$$

$\frac{\partial G}{\partial n}$ becomes,

$$\frac{\partial G}{\partial n} = \frac{2n}{r(a+b)^{\frac{1}{2}}} \left(E(m) - K(m) \right) - \frac{4 \left[n \frac{(r-r_i) + n z_d^2}{r} \right]}{(a-b)(a+b)^{\frac{1}{2}}} E(m) \quad (0 \leq m < 1)$$

... (3.19)

where $E(m)$ is the complete elliptic integral of the 2nd kind,

$$E(m) = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{\frac{1}{2}} d\theta \quad 0 \leq m \leq 1$$

$E(m)$ may also be approximated as a polynomial. i.e.

$$E(m) = [1 + c_1 m_1 + \dots + c_4 m_1^4] + [d_1 m_1 + \dots + d_4 m_1^4] \ln\left(\frac{1}{m_1}\right) + \epsilon(m)$$

with,

$$c_1 = 0.44325141463 \quad d_1 = 0.24998368310$$

$$c_2 = 0.06260601220 \quad d_2 = 0.09200180037$$

$$c_3 = 0.04757483546 \quad d_3 = 0.04069697526$$

$$c_4 = 0.01746596451 \quad d_4 = 0.00426449639$$

and $\epsilon(m) < 2 \times 10^{-8}$.

Numerical expressions for G and $\frac{\partial G}{\partial n}$ have now been found and it remains to solve the integral equation (3.16).

3.3.5 Numerical procedure

The numerical procedure is as follows:

- (i) Select a finite number of base points P_i ($i=1, \dots, N$) to represent the generating boundary contour Γ_j .
- (ii) Use the points P_i ($i=1, \dots, N$) successively as origins for the Green's function (3.17).
- (iii) Use linear (or higher order) interpolation formulae to represent Φ and $\frac{\partial \Phi}{\partial n}$ over the generating contour Γ .

For example over a small element Γ_j between a pair of nodal points P_j and P_{j+1} , Φ and $\frac{\partial \Phi}{\partial n}$ may be approximated as,

$$\Phi = [N_{1j} \quad N_{2j}] \begin{pmatrix} \Phi_j \\ \Phi_{j+1} \end{pmatrix}$$

and

$$\frac{\partial \Phi}{\partial n} = [N_{1j} \quad N_{2j}] \begin{pmatrix} \frac{\partial \Phi}{\partial n_j} \\ \frac{\partial \Phi}{\partial n_{j+1}} \end{pmatrix},$$

for $\xi_j \leq \xi \leq \xi_{j+1}$, and

$$N_{1j} = \frac{\xi_{j+1} - \xi}{\xi_{j+1} - \xi_j}; \quad N_{2j} = \frac{\xi - \xi_j}{\xi_{j+1} - \xi_j}.$$

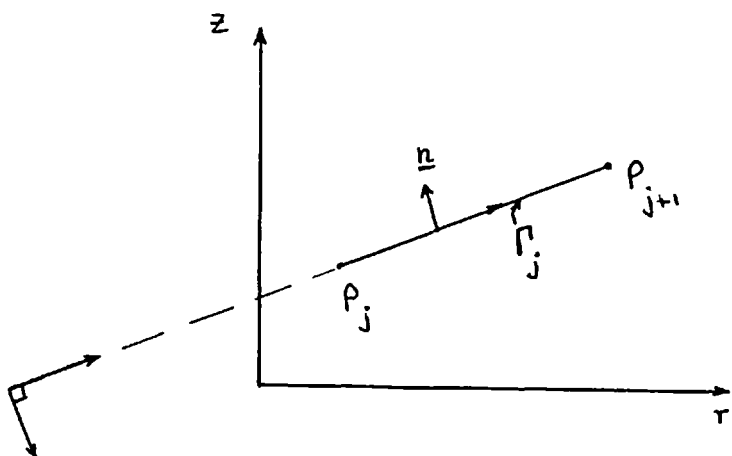


Figure 3.3 : Boundary element Γ_j

ξ represents the distance along the line segment measured from an origin. (Figure 3.3).

(iv) Assemble the expressions for ϕ and $\frac{\partial \phi}{\partial n}$ in the integral equation (3.16) to obtain.

$$\begin{aligned}
 -\alpha_i \phi_i &= \sum_{j=1}^N \left[\int_{\xi_j}^{\xi_{j+1}} \begin{pmatrix} N_{1j} \\ N_{2j} \end{pmatrix} \frac{\partial G}{\partial n} r d\xi \right]^T \begin{pmatrix} \phi_j \\ \phi_{j+1} \end{pmatrix} \\
 &- \sum_{j=1}^N \left[\int_{\xi_j}^{\xi_{j+1}} \begin{pmatrix} N_{1j} \\ N_{2j} \end{pmatrix} G r d\xi \right]^T \begin{pmatrix} \frac{\partial \phi_j}{\partial n} \\ \frac{\partial \phi_{j+1}}{\partial n} \end{pmatrix}
 \end{aligned}$$

(i = 1, ..., N)

which becomes after some simplification,

$$-\alpha_i \Phi_i = [H_{i1} \ H_{i2} \ \dots \ H_{iN}] \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} - [G_{i1} G_{i2} \dots G_{iN}] \begin{pmatrix} \frac{\partial \Phi_1}{\partial n} \\ \frac{\partial \Phi_2}{\partial n} \\ \vdots \\ \frac{\partial \Phi_N}{\partial n} \end{pmatrix} \quad (i=1, \dots, N) \quad (3.20)$$

where,

$$H_{ij} = \left[\int_{\Gamma_{j-1}} N_{ij} \frac{\partial G}{\partial n} r d\xi + \int_{\Gamma_j} N_{2j} \frac{\partial G}{\partial n} r d\xi \right] \quad (3.21)$$

and

$$G_{ij} = \left[\int_{\Gamma_{j-1}} N_{ij} G r d\xi + \int_{\Gamma_j} N_{2j} G r d\xi \right]$$

(i, j = 1, \dots, N).

Equation (3.20) may be further simplified to

$$[\tilde{H}'] [\tilde{\Phi}] - [\tilde{G}] \left(\frac{\partial \tilde{\Phi}}{\partial \tilde{n}} \right) = 0 \quad (3.22)$$

where

$$H'_{ij} = H_{ij}, \quad \forall i, i \neq j$$

$$= H_{ij} + \alpha_i, \quad i = j.$$

The integrals H'_{ij} and G_{ij} can easily be evaluated using a Gaussian quadrature formula. When $i \neq j$ and $i \neq (j+1)$, the integrals

have regular behaviour and may be evaluated by the regular Gaussian quadratic,

$$\int_{\xi_j}^{\xi_{j+1}} f(G \text{ or } \frac{\partial G}{\partial n}, r, \xi) d\xi \approx \frac{\xi_{j+1} - \xi_j}{2} \sum_{k=1}^M w_k f(G(\xi_k) \text{ or } \frac{\partial G}{\partial n}(\xi_k), r(\xi_k), \xi_k)$$

with

$$\xi_k = [(\xi_{j+1} - \xi_j) \chi_k + \xi_{j+1} - \xi_j] / 2$$

where w_k are Gaussian quadratic weights, and χ_k Gaussian quadratic points (1).

In the instance when P_i is located at either P_j or P_{j+1} , the integrals (3.20) have an integrable singularity in the expression for $k(m)$. This can be overcome by writing $k(m)$ as

$$k(m) = k^*(m) - b_0 \ln(m_1)$$

where $k^*(m) = a_0 + \sum_k [a_k m_1^k + b_k m_1^k \ln(1/m_1)]$.

$k^*(m)$ is a regular function along Γ_j . The integrals (3.20) now contain a regular function as its integrand plus an additional integrand with a logarithmic singularity. The latter integral can be evaluated accurately by the special Gaussian quadrature formula,

$$\int_0^1 f(x) \ln x \, dx = \sum_{k=1}^M w_k^S f(X_k^S)$$

w_k^S and X_k^S are special Gaussian quadrature weights and points (1).

(v) The system of algebraic equation (3.22) may now be solved using any of the well known matrix techniques. A Gaussian elimination technique is suitable as the matrices $[\tilde{H}']$ and $[\tilde{G}]$ are invariably full.

3.4 Application of the BIEM to the steady state radial coning problem

3.4.1 The steady state mathematical model

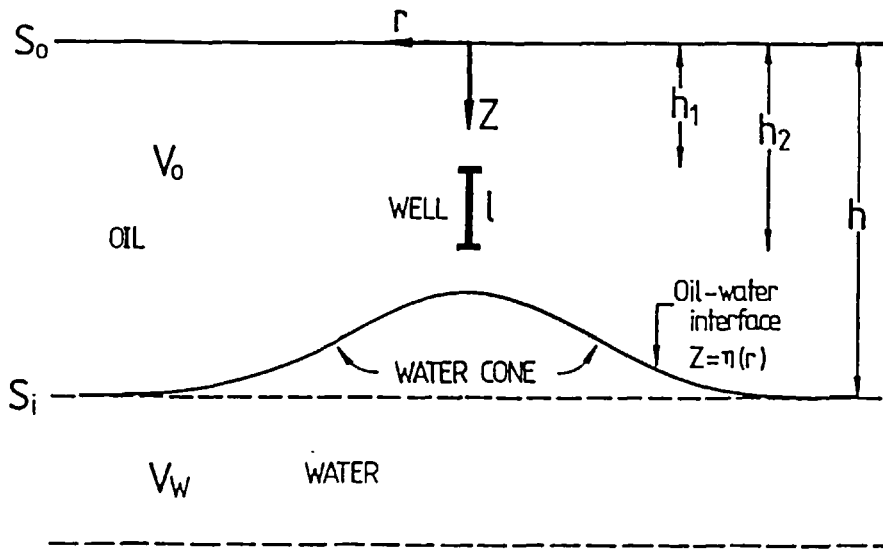


Figure 3.4: The radial coning model

A diagrammatic representation of the model is shown in figure 3.4. A cylindrical co-ordinate system (r,z) is used and the co-ordinate z is measured positive downwards. V_0 and V_w represent the oil and water domains respectively. The water is assumed to be in hydrostatic equilibrium. The two fluids are taken to be incompressible and the porous medium homogeneous.

Governing equations:

The velocity potentials are defined as,

$$\phi_i = \frac{k_i \rho_i}{\mu_i} \left(\frac{p}{\rho_i} - gz \right) \quad (i = 0, w) . \quad (3.23)$$

Then, by conservation of mass,

$$\nabla^2 \phi_0 = 0 , \text{ in } V_0 . \quad (3.24)$$

Since the water phase is at rest within the cone, for the steady state case

$$\phi_w = \text{const} = 0, \text{ say in } V_w . \quad (3.25)$$

Interface conditions :

Let the oil water interface S_i be represented by the equation $z = \eta(r)$.

Pressure is continuous across S_i (by neglecting capillary pressure)

$$p_0 = p_w \quad \text{on } S_i .$$

Therefore by equations (3.23) and (3.25)

$$\eta(r) = a_0 \phi_0 \quad \text{on } S_i \quad (3.26)$$

with $a_0 = \frac{\mu_0}{k_0 \Delta \rho g}$.

Since in the steady state the interface is not moving, but is at rest, it may be represented as a surface with no-flow across it.

$$\frac{\partial \Phi_0}{\partial n} = 0 \quad \text{on } z = \eta(r) \quad (3.27)$$

($\underline{n}$ is the outward normal).

Equation (3.27) may be written as

$$\begin{aligned} \nabla \Phi_0 \cdot \underline{n} &= \nabla \Phi_0 \cdot \nabla (z - \eta) \\ &= \frac{\partial \Phi_0}{\partial r} \frac{\partial \eta}{\partial r} - \frac{\partial \Phi_0}{\partial z} = 0 \end{aligned} \quad (3.28)$$

Boundary conditions :

The upper boundary S_0 in figure 3.4 is an impermeable barrier and has no flow across it. Therefore,

$$\frac{\partial \Phi_0}{\partial n} = 0 \quad \text{on } z = 0. \quad (3.29)$$

Well condition:

The apex of the highest stable cone (critical case) is known to remain at a distance below the bottom of the well much larger than the well radius (7,8,11). It is therefore reasonable to assume that the radius of the well has little influence on the solution.

Therefore, approximating the well by a line sink,

$$\lim_{r \rightarrow 0} r \frac{\partial \Phi}{\partial r} = -\frac{q}{2\pi\ell} \quad (3.30)$$

where q is the oil production rate.

External condition:

At very large distances away from the well, the oil water contact is undisturbed i.e. horizontal.

Therefore

$$\lim_{r \rightarrow \infty} \eta(r, t) = h. \quad (3.31)$$

Equations (3.26) and (3.31) also imply that,

$$\lim_{r \rightarrow \infty} \phi_0 = \frac{h}{a_0} = k, \text{ say.} \quad (3.32)$$

3.4.2 Dimensionless variables

For convenience in the development of the model the following dimensionless variables will be introduced.

$$(r_D, z_D) = \left(\frac{r}{L}, \frac{z}{L} \right)$$

$$\phi_D = \frac{\phi}{k'L}$$

$$(u_D, v_D) = \left(u \frac{\sigma}{k'}, v \frac{\sigma}{k'} \right) \quad u, v - \text{velocities}$$

$$q_D = \frac{q}{k' L^2}$$

where L is a typical length scale - say the thickness of the oil formation h , and σ the effective porosity.

$$k' = \frac{k_0 \rho_0}{\mu_0} g.$$

Equations (3.24) to (3.31), except the pressure continuity equation - equation (3.26), remain unaltered in the new dimensionless variables.

Equation (3.26) is modified to,

$$\eta_D = a_{0D} \phi_{0D}$$

where $a_{0D} = a_0 k'.$

In subsequent discussions, the use of subscript D to denote dimensionless variables will be discontinued. If dimensional variables are used it will be explicitly stated. Otherwise all variables will be assumed to be dimensionless.

3.4.3 The boundary integral equations:

Choosing the function ϕ to be ϕ_0 , equation (3.16) becomes

$$\alpha_i \phi_{0i} + \int_{h_1}^{h_2} \frac{q}{\ell} g(z) dz = \int_{\Gamma_0}^{\Gamma_i} \left(\phi_0 \frac{\partial G}{\partial n} - G \frac{\partial \phi_0}{\partial n} \right) r d\xi \quad (3.33)$$

where Γ_0 and Γ_i are the generating contours of the surfaces S_0 and S_i . The additional term $\int_{h_1}^{h_2} \frac{q}{\ell} g(z) dz$ appears because of the singularity at the line sink.

As outlined in subsection 3.4.4, the integral equation (3.33) can be used to generate a set of linear algebraic equations of the form,

$$[\underline{H}'] [\underline{\phi}_0] + [\underline{G}] \left[\frac{\partial \phi_0}{\partial n} \right] = [\underline{q}] \quad (3.34)$$

where $[\underline{q}]$ is an additional vector representing the effect of the line sink.

$\frac{\partial \phi_0}{\partial n}$ is specified throughout the boundary Γ_0 and Γ_i .

Therefore, rearranging equation (3.34) we obtain

$$[\underline{H}'] [\underline{\phi}_0] = [\underline{b}]$$

where all the known quantities appear through the vector $\{\underline{b}\}$. $\{\phi_0\}$ can now be solved for.

3.5 Improved BIEM solution

The additional vector $[\underline{q}]$, in equation (3.34), appears due to the effect of the line sink singularity. This term may be removed

by the introduction of a basic flow and a perturbation flow. The basic flow potential will represent the singular portion of the solution and the perturbation potential the non-singular solution. Furthermore, the basic flow will be solved exactly and it only remains to solve the perturbation potential numerically.

3.5.1 The basic flow problem

The basic flow will be taken as the flow towards the line sink ($h_1 \leq z \leq h_2$) in a semi-infinite, domain, bounded above, at $z = 0$, by a 'no-flow' boundary. Let the potential due to this flow be L .

On imposing the condition that,

$$\lim_{r \rightarrow \infty} L(r, z) = \phi_{\infty} = k,$$

and by using the method of images, the basic potential is calculated to be,

$$L = \frac{q}{4\pi\ell} \left\{ \ln \left[\left[(z-h_2)^2 + r^2 \right]^{\frac{1}{2}} + (z-h_2) \right] \ln \left[\left[(z+h_1)^2 + r^2 \right]^{\frac{1}{2}} + (z+h_1) \right] \right. \\ \left. - \ln \left[\left[(z-h_1)^2 + r^2 \right]^{\frac{1}{2}} + (z-h_1) \right] \ln \left[\left[(z+h_2)^2 + r^2 \right]^{\frac{1}{2}} + (z+h_2) \right] \right\} + k.$$

3.5.2 The perturbation flow problem

The total flow for the coning problem can now be considered as

the basic flow together with the flow disturbance created by superimposing the water cone on the basic flow. This flow disturbance will be referred to as the perturbation flow.

$$\Phi_0 = L + \tilde{\Phi}_0 \quad . \quad (3.35)$$

The governing equations and boundary conditions now become,

$$\nabla^2 \tilde{\Phi}_0 = 0 \quad (3.36)$$

$$\frac{\partial \tilde{\Phi}_0}{\partial n} = 0 \quad , \quad \text{on } z = 0 \quad (3.37)$$

(noting that $\frac{\partial L}{\partial n} = 0$, on $z = 0$)

$$\frac{\partial \tilde{\Phi}_0}{\partial n} = \frac{-\partial L}{\partial n} \quad \text{on } z = \eta(r) \quad (3.38)$$

$$\eta(r) = a_0(L + \tilde{\Phi}_0) \text{ on the interface} \quad (3.39)$$

$$(\tilde{\Phi}_0)_\alpha = 0 \quad . \quad (3.40)$$

Substituting $\Phi = \tilde{\Phi}_0$, the integral equation (3.16) now becomes

$$-\alpha_i \tilde{\Phi}_{0i} = \int_{\Gamma_0} \tilde{\Phi}_0 \frac{\partial G}{\partial n} r d\xi + \int_{\Gamma_i} \left(\tilde{\Phi}_0 \frac{\partial G}{\partial n} - G \frac{\partial \tilde{\Phi}_0}{\partial n} \right) r d\xi \quad (3.41)$$

On examining equation (3.41) it is apparent that the only variable present in the Γ_0 integrand is $\tilde{\Phi}_0$, which appears together

with $\frac{\partial G}{\partial n}$. Therefore, choosing a new Green's function g' , such

that $\frac{\partial G'}{\partial n} = 0$ along Γ_0 , the Γ_0 integrand can be made to vanish altogether.

The Green's function associated with the axi-symmetric Green's function G' is

$$g' = \left| (r\cos\theta - r_i\cos\theta_i)^2 + (r\sin\theta - r_i\sin\theta_i)^2 + (z - z_i)^2 \right|^{\frac{1}{2}} \\ + \left| (r\cos\theta - r_i\cos\theta_i)^2 + (r\sin\theta - r_i\sin\theta_i)^2 + (z + z_i)^2 \right|^{-\frac{1}{2}}$$

with $G' = \int_0^{2\pi} g' d\theta.$

The boundary integral equation now becomes,

$$-\alpha' \phi_{0i} = \int_{\Gamma} \left(\phi_0 \frac{\partial G}{\partial n} - G \frac{\partial \phi_0}{\partial n} \right) r d\xi \quad (3.42)$$

with $\alpha' = 6\pi$ for P_i on the boundary.

Therefore, by choice of Green's function, such that it satisfies the no-flow condition on Γ_0 , the integration range has been confined to Γ_i only. The perturbation potential $\tilde{\phi}_0$, in equation (3.42), may now be solved for at discrete points P_i along the interface Γ_i .

3.5.3 Numerical procedure for location of the steady state interface

The algorithm, for the location of Γ_i , is as follows :

(i) Use an initial approximation, $z^{(0)} = \eta^{(0)}(r)$, for the interface and solve for the first approximation to the oil perturbation potentials $[\tilde{\phi}_0^{(1)}]$. The initial undisturbed oil-water contact is an obvious choice for $z^{(0)} = \eta^{(0)}(r)$.

(ii) Using the pressure continuity condition, equation (3.39), adjust the free surface position, such that

$$\eta^{(n+1)} = a_0 (L + \tilde{\phi}_0^{(n)})$$

(iii) Set ϵ_d to be some prescribed tolerance.

Evaluate $\epsilon^{(n)}$,

$$\text{where } \epsilon^{(n)} = |\eta^{(n+1)} - \eta^{(n)}|.$$

If $\epsilon^{(n)} > \epsilon_d$, repeat the procedure starting at step (ii).

If $\epsilon^{(n)} \leq \epsilon_d$, the interface has been located to within the prescribed tolerance.

It must be noted that Γ_i can be an infinite boundary. The problem of modelling an infinite boundary, that is undisturbed at large distances away from the origin, can be overcome by a truncation method.

The free surface is assumed to be undisturbed for $r \geq r_n$.

$$\text{i.e. } \eta = h \quad \text{for } r \geq r_n.$$

To ensure that the truncation of the free boundary has no effect on the solution r_n was continually moved outward until no change in the solution was detected. The truncation method is

particularly effective for radial problems, such as the coning problem, as the radial domain of influence is fairly small. In our experience $r_n = 10 \times (\text{height of oil formation})$ was found to be large enough.

3.7 Numerical results and accuracy

The steady state model developed in the previous section, was tested on a typical reservoir whose rock and fluid properties are given below.

Table 3.1

Impermeable boundary	at datum ($z = 0$)
oil water contact	100 ft. below datum (30.48m below datum)
oil formation thickness	100 ft (30.48m)
well completion interval	40 to 50ft (12-19 to 15.24m)
ρ_w	1.05 g/cc ($1.05 \cdot 10^3 \text{ kg/m}^3$)
ρ_o	0.65 g/cc ($6.5 \cdot 10^2 \text{ kg/m}^3$)
k_o	100 mD ($9.869 \times 10^{-4} \text{ m}^2$)
μ_w	0.4 cp ($4.0 \times 10^{-4} \text{ Pa.s}$)
μ_o	0.25 cp ($2.5 \times 10^{-4} \text{ Pa.s}$)

All computations were performed in dimensionless variables, as defined in subsection 3.4.2. The length scale, L , was taken to be 100ft. In the graphical displays that follow, to obtain reservoir distances (in feet) multiply the dimensionless distances

by 100. The production rates, Q , shown in figures (3.5) to (3.11), are expressed in dimensional units of barrels per day.

3.6.4 Steady state interface profiles

In figure (3.5) are shown the interface profiles, for a range of flow rates Q . As the flow rate increases, the interface gradient at the cone apex becomes noticeably sharper.

The interface profile for $Q = 200$ b/d was obtained using the initial oil-water contact, $z = 100$ ft, as a first approximation, and then iterating using the scheme outlined in subsection 3.5.3. Three iterations were found to be necessary before the interface was located to within (.1)ft. For subsequent production rates ($Q = 300, 400, \dots, 458$), the interface profile of the previous lower production rate was used as an initial approximation to the oil water contact. With this scheme, the average computer time required, to compute a steady state interface on the CDC 6400 machine, was found to be about 20 c.p.u. seconds.

3.6.2 Critical production rate

Figure 3.6 shows a plot of interface gradient near the cone apex, against production rate. The curve asymptotes towards a constant value and this value is the critical production rate. Close to this asymptotic value, a small increase in flow rate produces a large increase in interface gradient, indicating instability. For the given data, the critical production rate is

estimated to be 470 b/d.

Associated with the critical production rate is the critical cone height - the reservoir height above which instability develops. The critical cone height estimate is made using the steady state interface profile for a production rate that is just below the critical. For this particular reservoir the critical height is estimated to be 32 feet.

3.6.3 Comparison with the linearized solution

In figure (3.7) we compare our solution for the interface profile against the linearized solution of Camy (ref.11). Both solutions are computed for the case when the impermeable upper boundary, S_0 , is absent. In the boundary integral model, a 12 nodal point grid was chosen between $r = 0$ and $r = r_n$. r_n was taken to be $10 \times (\text{oil formation thickness})$.

Comparison between the two solutions is good for small production rates. However, for larger production rates there is substantial disagreement. This is to be expected as the linearized solution breaks down when the cone becomes large and non-linearities become important.

3.6.4 Influence of the number of boundary points on numerical solution

Numerical solutions, using $N = 12$ and $N = 24$ boundary points, were obtained for 3 different production rates ($Q = 100, 300, 400$ b/d)

In figure (3.8) we compare the two sets of interface profiles. Agreement between the two solution is excellent, indicating that $N = 12$ boundary points is sufficiently accurate.

3.6.5 Effect of well positioning on critical production rate.

The well production interval was kept constant at 10 feet, but its position within the depth of the oil thickness was varied. Figure 3.9 shows its effect on the critical production rate. The graph is almost linear.

The critical flow-rate for the case of a well penetrating from the top of the oil formation was compared with the results of Bournazel and Jeanson (7). The solutions agree to within 6% of each other.

3.6.6 Influence of anisotropy

The model outlined in section (3.5) may be easily extended to the case of anisotropic permeability. This is achieved by a simple co-ordinate transformation,

$$x_a = x_i \quad , \quad y_i = \left(\frac{k_r}{k_z} \right) y_a$$

which reduces the equation

$$\frac{k_r}{k_z} \left(\frac{\partial^2 \Phi}{\partial r_a^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r_a} \right) + \frac{\partial^2 \Phi}{\partial z_a^2} = 0$$

to

$$\frac{\partial^2 \phi}{\partial r_i^2} + \frac{1}{r} \frac{\partial \phi}{\partial r_i} + \frac{\partial^2 \phi}{\partial z_i^2} = 0 \quad (3.43)$$

with

$$\phi = \frac{k_{zi} \rho_i}{\mu_i} \left(\frac{p}{\rho_i} - gz \right) \quad i = 0, w$$

The subscript a and i refer to anisotropic and isotropic systems. We see that equation (3.43) is the governing equation for an isotropic system. Therefore, an anisotropic problem may be solved by solving an equivalent isotropic system.

In figure (3.10) are the steady state interface profiles for a reservoir with rock and fluid properties as in table (3.1), but with $k_r = 1000$ mD and $k_z = 100$ mD. The critical production rate for this case was found to be about 500 b/d. In figure 3.11 we compare the steady state interface profiles of the isotropic and anisotropic systems. The cone heights for both cases are almost identical (anisotropic cone height slightly lower than isotropic cone height), but the shape of the interface profiles are substantially different. The cone is very much broader in the anisotropic case. This is to be expected as $k_r = 10k_z$ and therefore one would expect the water to flow more easily in the lateral direction, than for the case $k_r = k_z$ (isotropic).

3.7 Summary

By using the powerful properties of potential theory, a

boundary integral equation has been derived, which relates the perturbation velocity potential, $\tilde{\phi}_0$, on the oil water interface to its normal derivatives. The perturbation potential and its derivatives in the interior are not used, although these can easily be calculated from surface values, by a further application of Green's theorem. Hence the number of independent variables in the computation is of order N , and not N^2 , and for a given storage and machine time much greater accuracy can be obtained, than is possible by conventional domain methods.

The proposed model has been tested on an example problem and steady state interface profiles for a range of sub-critical flow-rates have been obtained. A simple graphical technique has been used to predict critical production rate. The accuracy of the technique has been demonstrated and computer requirements have been found to be modest.

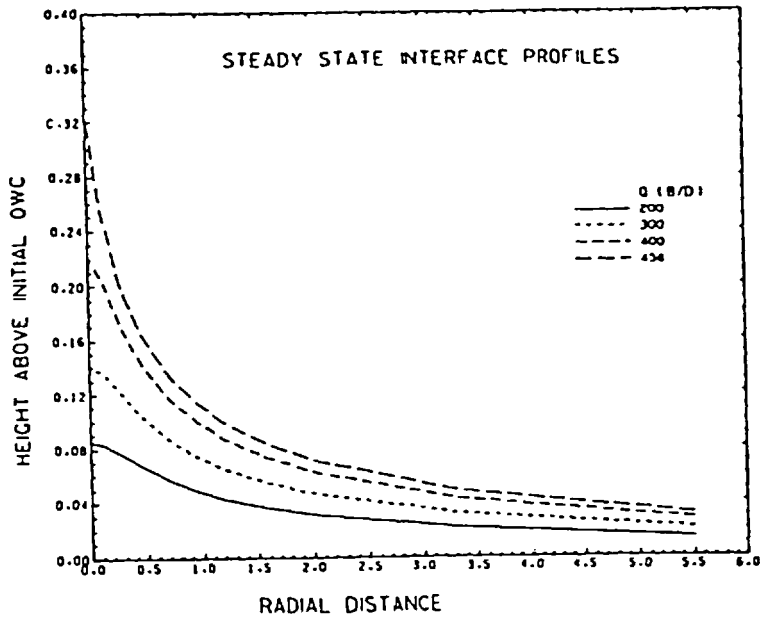


Figure 3.5: Steady-state interface profiles

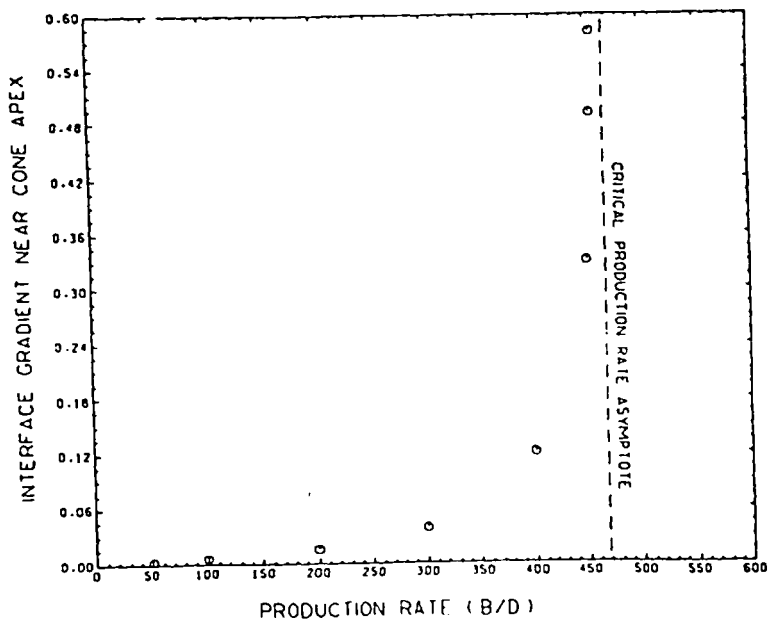


Figure 3.6: Critical production rate correlation

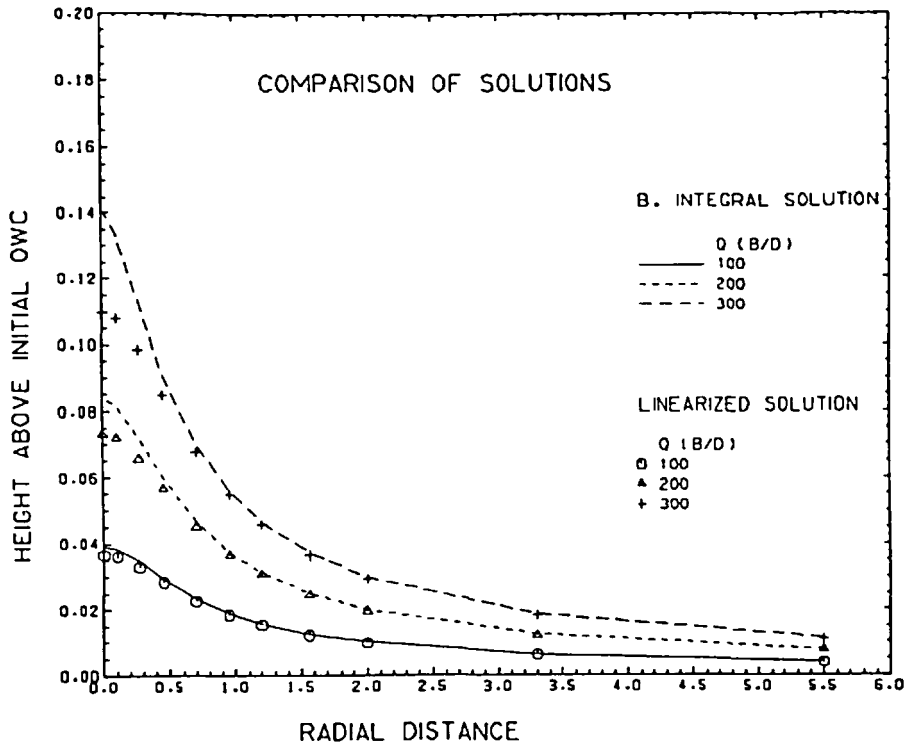


Figure 3.7: Comparison of linearized and BIEM solutions

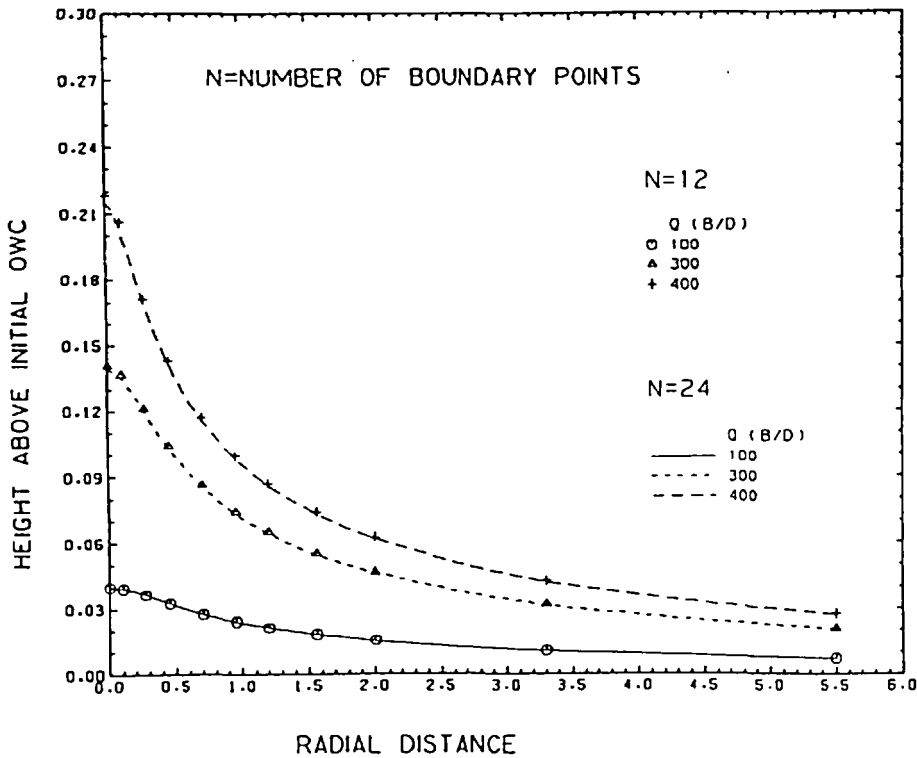


Figure 3.8: Influence of boundary points on solution

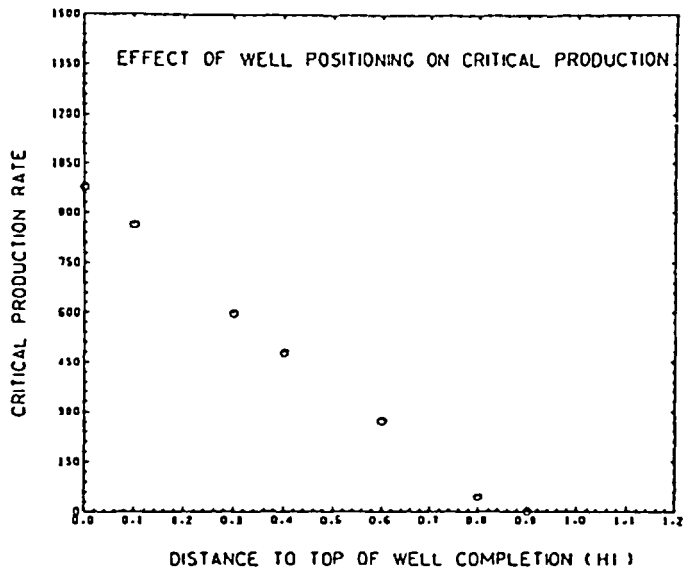


Figure 3.9: Effect of well positioning on critical production rate

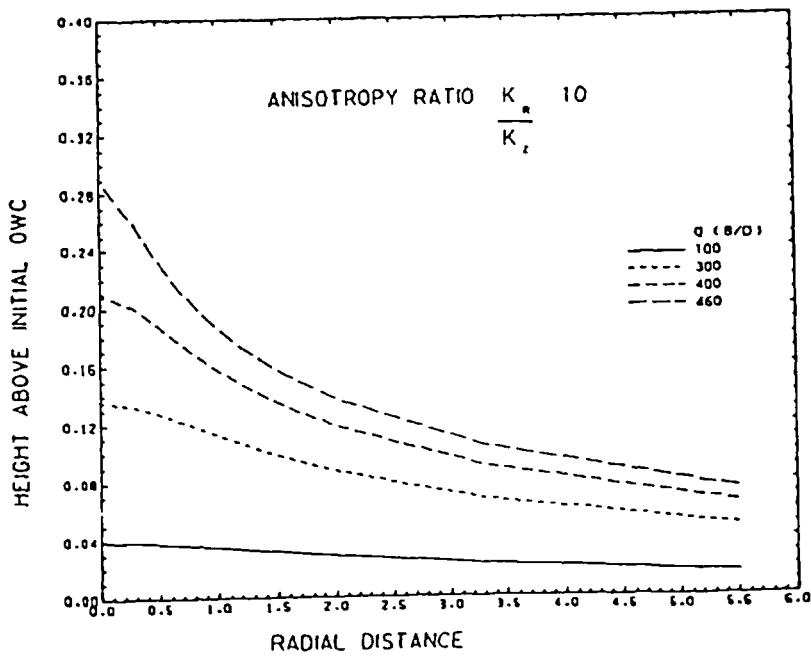


Figure 3.10: Steady state interface profiles (anisotropic case)

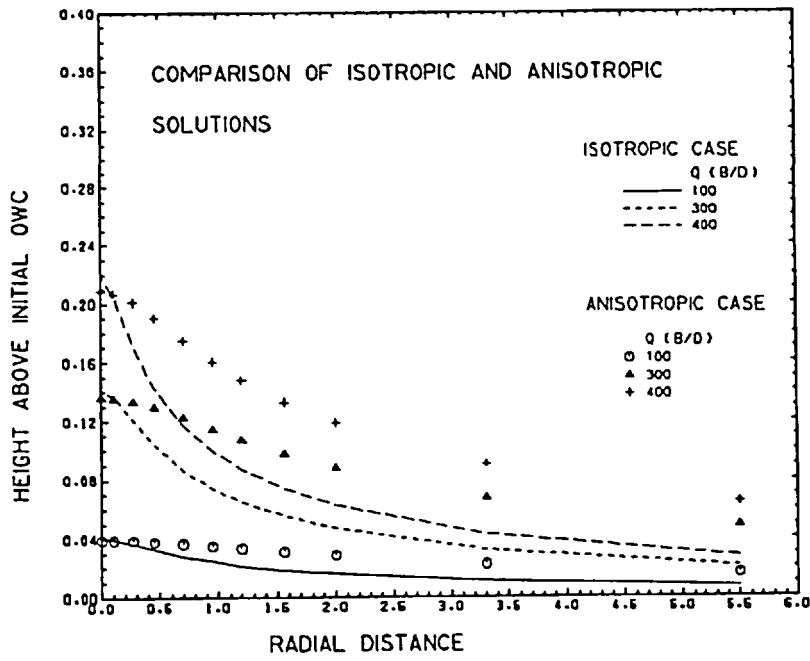


Figure 3.11: Comparison of isotropic and anisotropic solutions

CHAPTER 4

RADIAL CONING - UNSTEADY STATE

4.1 Introduction

Economic necessities often make production above the critical flow rate inevitable. In conditions of super-critical well production, after a certain time interval has elapsed, water is bound to be produced simultaneously with the oil. This situation is clearly detrimental to oil recovery and therefore prediction of water breakthrough time is a factor of immense importance to the reservoir engineer.

Few coning correlations exist to predict water breakthrough time. The two most widely used are that of Sobocinsky and Cornelius (39) and Bournazel and Jeanson (7), which are both based on experimental data. We briefly review these two correlations.

As an extension of the steady state model, of Chapter 3, we introduce time into the governing equations and boundary conditions to develop a new unsteady state coning model. The equations are valid up to the time of the cone reaching either its steady state equilibrium position or critical height. The problem is, again, subdivided into a basic flow and a perturbation flow problem. An exact solution is developed for the basic flow problem. Hence it is merely the non-singular or perturbation flow problem that requires numerical solution.

A new boundary integral type solution is developed, and a finite difference time stepping procedure is introduced to advance in time.

The conventional application of boundary integral type methods to problems, require contour integration over the entire boundary contour. We demonstrate that for the coning problem the contour integration range can be confined to the oil-water contact only. This is achieved by devising Green's functions, that satisfy certain boundary conditions on the impermeable boundaries present within the flow domain. The proposed model therefore, has the outstanding advantage that the only variables which need to be calculated and stored, at each time level, are those at the oil water contact which is precisely where the flow information is needed.

The use of a constant time step size, throughout the length of a computation, is clearly very expensive in computer time. We therefore suggest the use of a variable time stepping scheme, such that the maximum potential change over each time step is checked.

The new coning model is used, on an example problem, to trace the position of the water cone at any time up to water breakthrough for super critical production rates, and at any time up to steady state position for sub-critical rates. The influence of various parameters on coning behaviour is also investigated.

4.2 Literature survey of correlation predicting breakthrough time

4.2.1 Sobocinsky and Cornelius (39)

Sobocinsky and cornelius presented a correlation for predicting the behaviour of a water cone as it developed from the static oil-

water contact to breakthrough conditions. It is applicable to homogeneous reservoirs with incompressible fluids, and with the well penetrating from the top of the oil formation.

The correlation is based on experimental data from a laboratory sand-packed model and on results from a computer program for a 2-dimensional system (43). The computer work involved the use of a program for two-dimensional two phase, incompressible fluid flow, and used input data that described the laboratory model. The water breakthrough times from the computer predictions were found to be nearly identical with experimental runs for the two separate cases. For this reason, the computer model alone was used to obtain the necessary breakthrough time and cone build up data for the correlation.

A dimensionless cone height z and a dimensionless time t_D were defined using the scaling criteria for the immiscible displacement of oil by water (18).

$$z = \frac{.00307 \Delta \rho k_0 h_c}{\mu_0 q_0 B_0} \quad (4.1a)$$

$$t_D = \frac{.00137 \Delta \rho k g(M) t}{\mu_0 \sigma h_0 F_h} \quad (4.1b)$$

$$g(M) = (1+M^\alpha) \quad (\text{field units})$$

In equations (4.1a) and (4.1b) h_c is the height of the apex of the water cone above the average oil water contact, B_0 is the oil formation volume factor, F_h is the horizontal to vertical permeability ratio, M the water oil mobility ratio ($= \mu_0 (k_w) / \mu_w (k_0)_{wc}$)

and $\alpha = 0.5$ for $M < 1$; 0.6 for $1 < M < 10$.

Having defined the dimensionless cone height Z and dimensionless time t_D , Sobocinsky and Cornelius plotted Z against t_D , to obtain a basic cone build up curve, a breakthrough curve, and 4 departure curves. (See figure 4.1). The breakthrough curve traces the movement of the cone from static conditions to breakthrough, when it is produced at a constant uninterrupted rate. The basic build up cone traces the apex of the water cone before water breakthrough, while the departure curves describe the rise of the apex as it approaches the breakthrough curve. The correlations apply only to coning situations for which breakthrough conditions fall on figure 4.1 i.e. for Z at breakthrough, t_D is less than 7.5.

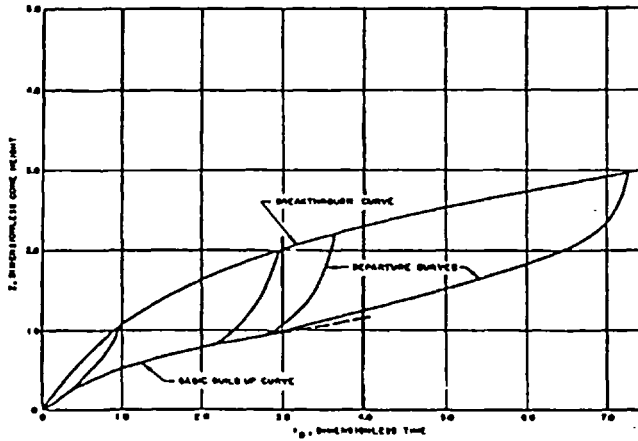


Figure 4.1: Dimensionless cone height vs dimensionless time

The details for calculating breakthrough time are as follows:

- (1) Defining the cone height at breakthrough as

$h_c = h_0 - h_p$, where h_p is the well penetration and h_0 the oil formation thickness, calculate the dimensionless cone height at breakthrough using equation (4.1).

- (2) Using figure (4.1) obtain the dimensionless time t_D , the corresponds to z at breakthrough.

- (3) Calculate t , the actual breakthrough time in days, using equation (4.1b)

Inherent in Sobocinsky and Cornelius' correlations are assumptions concerning scaling criteria and certain model parameters. Inspection of their scaling criteria reveals that all scaling groups are not fixed by arbitrarily fixing one. Furthermore, one coning model with one set of fluids and one set of boundary conditions cannot supply enough data to produce a generalized correlation. Sobocinsky and Cornelius themselves conclude that additional data from different systems are needed to develop a correlation that is more generalized than the one they present.

4.2.2 Bournazel and Jeanson (6)

Bournazel and Jeanson, as part of a general study of the entire coning phenomenon, investigated the breakthrough time correlation of Sobocinsky and Cornelius. They found the correlation to be satisfactory provided a slight modifications of the parameter $g(M) = 1 + M^\alpha$, appearing in equation (4.1b), was made. After a

wide exploration of different values of the mobility ratio ($M = 0.14$ to 7.3). Bournazel and Jeanson found $\alpha = 0.7$ to be more suitable for the evaluation of $g(M)$.

To facilitate hand calculation, Bournazel and Jeanson researched a simple analytic form of $z = f(t_D)$ and found the equation,

$$z = \frac{3t_D}{1+0.7t_D} \quad (4.2)$$

to be in good agreement with the experimental data from 2 different sized laboratory models.

The procedure for evaluating the actual breakthrough time is identical to the correlation of Sobocinsky and Cornelius, except that the dimensionless breakthrough time may now be derived directly from equation (4.2).

The breakthrough time correlation of Bournazel and Jeanson is open to the same criticism as that of Sobocinsky and Cornelius, as it is merely a modification of the work of the latter. However, the fact that Bournazel and Jeanson have confirmed the results of Sobocinsky and Cornelius, after a slight change, give the correlations more authority. The correlations are therefore useful, at least, as order of magnitude estimates of breakthrough time.

4.3 A mathematical model of the unsteady state coning phenomenon

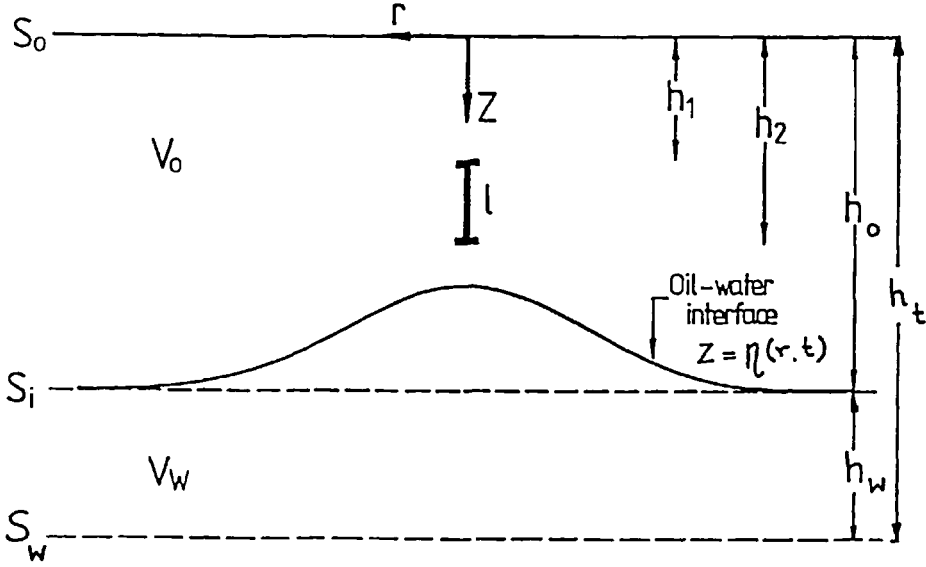


Figure 4.2: Radial coning model

A diagrammatic representation of the model is shown in figure 4.2, and is similar to the model of Section 3.4.2 except for a few changes. Cylindrical co-ordinates are used once again and V_0 and V_w represent the oil and water domains. In this case, however, the water is no longer assumed to be in hydrostatic equilibrium but is infact continually moving upwards, in response to the well production rate. The interface S_i is represented by $z = \eta(r, t)$ in the mathematical model.

Velocity potentials for the oil and water phase are defined as before

$$\Phi_i = \frac{k_i \rho_i}{\mu_i} \left(\frac{p}{\rho} - gz \right) \quad (i = 0, w) \quad (4.3)$$

4.3.1 Basic and perturbation flows

As for the steady state case, the concept of a basic flow is introduced to remove the singularity at the well. The basic flow will, once again, be taken to be the flow towards the line sink located at $h_1 \leq z \leq h_2$, in the presence of the no-flow boundary at S_0 , but with no water cone present. Hence, the basic flow potential is,

$$L = \frac{q}{4\pi\ell} \left\{ \ln \left[\left[(z-h_2)^2 + r^2 \right]^{\frac{1}{2}} + (z-h_2) \right] \left[\left[(z+h_1)^2 + r^2 \right]^{\frac{1}{2}} + (z+h_1) \right] \right. \\ \left. - \ln \left[\left[(z-h_1)^2 + r^2 \right]^{\frac{1}{2}} + (z-h_1) \right] \left[\left[(z+h_2)^2 + r^2 \right]^{\frac{1}{2}} + (z+h_2) \right] \right\} + c \\ \dots (4.4)$$

where $c = \lim_{r \rightarrow \infty} L(r, z)$.

The water zone thickness is known to have little effect on coning behaviour until after water breakthrough into the well (7.8). Therefore, no impermeable boundary has been placed, to confine the water zone for the moment. If however, the solution in the presence of an additional lower boundary is required, its effect on the solution for the basic flow potential can be included (see section 4.5).

With the definition of the basic flow potential, as in equation (4.4), the total flow can be considered as the basic flow together with the perturbation flow

$$\Phi_0 = L + \tilde{\Phi}_0 \quad (4.5)$$

$$\Phi_w = L + \tilde{\Phi}_w \quad (4.6)$$

ϕ_0 and ϕ_w are the total velocity potentials, and $\tilde{\phi}_0$ and $\tilde{\phi}_w$ are perturbation velocity potentials for the oil and water phase respectively. $\tilde{\phi}_0$ and $\tilde{\phi}_w$ are functions that are non-singular throughout the entire flow domain.

4.3.2 The governing equations and boundary conditions

Governing equations:

By conservation of mass,

$$\nabla^2 \tilde{\phi}_0 = 0 \quad (4.7)$$

$$\nabla^2 \tilde{\phi}_w = 0 \quad (4.8)$$

Interface conditions:

On $z = \eta(r, t)$

- continuity of pressure neglecting capillary pressure

$$P_0 = P_w \text{ on } z = \eta(r, t) .$$

Therefore, using equation (4.3)

$$\eta(r, t) = (a_0 + a_w)L + a_0 \tilde{\phi}_0 + a_w \tilde{\phi}_w \quad (4.9)$$

with
$$a_0 = \frac{\mu_0}{k_0 (\rho_w - \rho_0) g}$$

$$a_w = \frac{\mu_w}{k_w (\rho_w - \rho_0) g}$$

- continuity of velocity normal to the interface

$$\frac{\partial \phi_0}{\partial n} = \frac{\partial \phi_w}{\partial n} \quad .$$

Therefore,

$$\frac{\partial \tilde{\phi}_0}{\partial n} = \frac{\partial \tilde{\phi}_w}{\partial n} \quad (4.10)$$

- material surface condition. i.e. any fluids particles at the interface move in the direction and at the velocity of the interface.

$$\frac{DF}{Dt} = 0, \quad F \equiv z - \eta(r, t) = 0$$

where $\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} - \nabla \phi_i \cdot \nabla \right)$ $i = 0$ or w .

$$\text{Noting that, } \frac{\partial \phi_i}{\partial n} = \nabla \phi_i \cdot \underline{n} = \nabla \phi_i \cdot \frac{\nabla(z-\eta)}{|\nabla(z-\eta)|} \quad .$$

$$\frac{\partial \eta}{\partial t} = - \frac{\partial \phi_i}{\partial n} \left(1 + \left(\frac{\partial \eta}{\partial r} \right)^2 \right)^{\frac{1}{2}} \quad .$$

Writing $\frac{\partial \eta}{\partial r} = \tan \beta$, where β is the angle the interface makes with the horizontal,

$$\frac{\partial \eta}{\partial t} = - \frac{\partial \phi_i}{\partial n} \left(\frac{1}{\cos \beta} \right)$$

or,

$$\frac{\partial \eta}{\partial t} = - \frac{\partial}{\partial n} (L + \tilde{\phi}_i) \left(\frac{1}{\cos \beta} \right) \quad (4.11)$$

(i = 0 or w).

Boundary condition on S_0 :

S_0 is an upper impermeable boundary and therefore has no flow across it

$$\frac{\partial \phi_0}{\partial n} = \frac{\partial L}{\partial n} + \frac{\partial \tilde{\phi}_0}{\partial n} = 0$$

but $\frac{\partial L}{\partial n} = 0$ on S_0 ,

therefore,

$$\frac{\partial \tilde{\phi}_0}{\partial n} = 0 \quad \text{on } S_0. \quad (4.12)$$

Far-field conditions:

At large distances away from the well the oil water contact is undisturbed.

Therefore,

$$\left. \begin{array}{l} \tilde{\phi}_0, \tilde{\phi}_w \rightarrow 0 \\ \text{and } \eta(r, t) = h_0 \end{array} \right\} \text{as } r \rightarrow \infty. \quad (4.13)$$

Initial conditions:

$$\left. \begin{array}{l} \eta(r,0) = h_0 \\ \tilde{\phi}_0, \tilde{\phi}_w \text{ specified} \end{array} \right\} , \text{ for all } r \quad (4.14)$$

4.3.3 Dimensionless variables:

The following dimensionless variables are introduced.

$$(r_D, z_D) = \left(\frac{r}{L}, \frac{z}{L} \right)$$

$$\phi_{iD} = \frac{\phi_i}{k' L} \quad (i = 0, w)$$

$$(u_D, v_D) = \left(\frac{u\sigma}{k'}, \frac{v\sigma}{k'} \right)$$

$$q_D = \frac{q}{k' L^2}$$

$$t_D = \frac{tk'}{L\sigma}$$

where L is a typical length scale, σ the effective porosity and $k' = \frac{k_0 \phi_0 g}{\mu_0}$.

In the new dimensionless variables equations (4.7) to (4.14) remain unaltered except for equation (4.9) which now becomes,

$$\eta_D = (a_{0D} + a_{wD})L + a_{0D}\tilde{\phi}_{0D} + a_{wD}\tilde{\phi}_{wD}$$

with

$$a_{0D} = a_0 k' \text{ and } a_{wD} = a_w k'.$$

Dimensionless variables will be used, for the discussion in the rest of this chapter, and the subscript D will be dropped from now on. If dimensionl variables are used it will be explicitly stated.

4.4 A new boundary integral, unsteady state coning model

4.4.1 Integral equations

In section (3.3.3) of Chapter 3 we showed that Green's 2nd identity could be reduced to the form

$$-\alpha_i \Phi(r_i, z_i) = \oint_{\Gamma} \left(\Phi \frac{\partial G}{\partial n} - \frac{G \partial \Phi}{\partial n} \right) r d\xi \quad (4.15)$$

for axi-symmetric flows. (see equation (3.16)).

For the domain V_0 , choosing Φ to be $\tilde{\phi}_0$ and G to be G' , where

$$G' = \int_0^{2\pi} g' d\theta$$

$$\begin{aligned} \text{with } g' = & \left| (r \cos \theta - r_i \cos \theta_i)^2 + (r \sin \theta - r_i \sin \theta_i)^2 + (z - z_i)^2 \right|^{-\frac{1}{2}} \\ & + \left| (r \cos \theta - r_i \cos \theta_i)^2 + (r \sin \theta - r_i \sin \theta_i)^2 + (z + z_i)^2 \right|^{-\frac{1}{2}} \end{aligned} \quad (4.16)$$

the integral equation becomes,

$$-\alpha_{0i} \tilde{\phi}_{0i} = \int_{\Gamma_i} r \left(\tilde{\phi}_0 \frac{\partial G'}{\partial n} - G' \frac{\partial \tilde{\phi}}{\partial n} \right) d\xi \quad (4.17)$$

where Γ_i is the generating contour of S_i and $\underline{n}$ the unit normal directed away from the boundary surface of V_0 .

Similarly for the domain V_w , choosing ϕ to be $\tilde{\phi}_w$ and G to be G' , the integral equation becomes,

$$-\alpha_{wi} \tilde{\phi}_{wi} = \int_{\Gamma_i} r \left(\tilde{\phi}_w \frac{\partial G'}{\partial n'} - G' \frac{\partial \tilde{\phi}_w}{\partial n'} \right) d\xi \quad (4.18)$$

where n' is the unit normal directed away from V_w .

Noting that $\underline{n}' = -\underline{n}$ and $\frac{\partial \tilde{\phi}_w}{\partial n} = \frac{\partial \tilde{\phi}_0}{\partial n}$ on Γ_i , equation (4.18) becomes

$$\alpha_{wi} \tilde{\phi}_{wi} = \int_{\Gamma_i} r \left(\tilde{\phi}_w \frac{\partial G'}{\partial n} - G' \frac{\partial \tilde{\phi}_0}{\partial n} \right) d\xi. \quad (4.19)$$

It now remains to solve for the unknown variables, $\tilde{\phi}_0$, $\tilde{\phi}_w$ and $\frac{\partial \phi_0}{\partial n}$, using the integral equations (4.17) and (4.19) together with the material surface condition on Γ_i .

4.4.2 Time stepping equations:

The material surface condition, equation (4.11), relates the perturbation potentials on the interface to its normal derivatives, at any time t . Substituting the pressure continuity condition

(equation(4,9)) in the material surface condition we obtain,

$$\frac{\partial}{\partial t}(a_0 \tilde{\phi}_0 + a_w \tilde{\phi}_w) = - \frac{\partial}{\partial n} (L + \tilde{\phi}_i) \frac{1}{(\cos \beta)} \quad . \quad (4.20)$$

We now write equation (4.20) in finite difference form for a small time step, Δt , as follows:

$$\begin{aligned} a_0 \tilde{\phi}_0^{(k+1)} + a_w \tilde{\phi}_w^{(k+1)} &= a_0 \tilde{\phi}_0^{(k)} + a_w \tilde{\phi}_w^{(k)} \\ &- \frac{\Delta t}{\cos \beta^{(k)}} \left[\left(\frac{\partial L}{\partial n} + \frac{\partial \tilde{\phi}_0}{\partial n} \right)^{(k+1)} \theta + \left(\frac{\partial L}{\partial n} + \frac{\partial \tilde{\phi}_0}{\partial n} \right)^{(k)} (1-\theta) \right] \end{aligned} \quad \dots (4.21)$$

where the superscripts k and $(k+1)$ refer to the successive time levels $k\Delta t$ and $(k+1)\Delta t$, and θ is a time weighting factor ($0 < \theta < 1$) which is set to (0.5) for all computations.

Equation (4.21) may now be used to advance in time.

4.4.3 Numerical procedure:

The algorithm, we have developed for the solution of the equation is as follows:

(i) Select a finite number of base points P_j ($j=1, \dots, N$) to represent the initial, undisturbed oil water contact.

(ii) Represent the independent variables $\tilde{\phi}_0$, $\tilde{\phi}_w$ and $\frac{\partial \tilde{\phi}_0}{\partial n}$, on any segment, $\Gamma_j(P_j, P_{j+1})$, on the interface by linear or higher order interpolation formulae (as for steady state case).

(iii) Use the points P_j successively as origins for the Green's function and obtain $2N$ integral equations from equations (4.17) and (4.19). Substitute the interpolation formulae for the independent variables to derive $2N$ algebraic equations of the form,

$$[\underline{H}^0][\underline{\tilde{\phi}}_0] + [\underline{G}^0] \left(\frac{\partial \tilde{\phi}_0}{\partial n} \right) = 0 \quad (4.22)$$

$$[\underline{H}^w][\underline{\tilde{\phi}}_w] - [\underline{G}^w] \left(\frac{\partial \tilde{\phi}_0}{\partial n} \right) = 0 \quad . \quad (4.23)$$

The coefficients of the matrices $[\underline{H}^0]$ and $[\underline{G}^0]$ of equation (4.22) are identical to the co-efficients of the matrices $[\underline{H}']$ and $[\underline{G}]$, respectively in equation (3.22) of Chapter 3.

The co-efficients of the matrices in equation (4.23) are as follows :

$$H_{ij}^w = H_{ij}^0, \text{ if } i \neq j$$

$$H_{ij}^w = H_{ij}^0 - \alpha_{oi} - \alpha_{wi}, \text{ if } i = j$$

and,

$$G_{ij}^w = G_{ij}^0, \text{ for all } i, j.$$

(iv) The time stepping equation, equation (4.21) is written at each of the nodes, P_j , as,

$$a_0 \tilde{\phi}_j^{(k+1)} + a_w \tilde{\phi}_{w_j}^{(k+1)} + c_j^{(k)} \frac{\partial \tilde{\phi}_0^{(k+1)}}{\partial n} = d_j^{(k)} \quad (4.24)$$

$$(j = 1, \dots, N)$$

$$(k = 0, 1, 2, \dots)$$

where,

$$c_j^{(k)} = \frac{1}{(\cos \beta_j)^{(k)}} \quad (4.25)$$

$$d_j^{(k)} = - \frac{\Delta t}{(\cos \beta_j)^{(k)}} \left[\frac{\partial L}{\partial n} + (1-\theta) \frac{\partial \tilde{\phi}_0^{(k)}}{\partial n} \right] \\ + a_0 \tilde{\phi}_{0j}^{(k)} + a_w \tilde{\phi}_{wj}^{(k)} \quad (4.26)$$

The subscript (j) refers to the nodal number and the subscript (k) refers to the time level.

Conditions at time level $t^{(k)} = k\Delta t$, $k = 0, 1, 2, \dots$ ($k = 0$, refers to initial conditions) are used to evaluate the coefficients $c_k^{(k)}$ and $d_j^{(k)}$.

(v) Assemble the time stepping equations together with equations (4.22) and (4.23) to obtain a matrix equation of the form,

$$[\underline{\tilde{A}}][\underline{x}^{(k+1)}] = [\underline{b}] \quad (4.27)$$

where $[\underline{x}^{k+1}]^T = \left(\tilde{\phi}_{0_1}, \dots, \tilde{\phi}_{0_N}, \tilde{\phi}_{w_1}, \dots, \tilde{\phi}_{w_N}, \frac{\partial \tilde{\phi}_{0_1}}{\partial n}, \dots, \frac{\partial \tilde{\phi}_{0_N}}{\partial n} \right)$.

The matrix $\underline{\tilde{A}}$ is of the form,

$$\begin{pmatrix} \eta^0 & \vdots & \eta^0 & \vdots & \eta^0 \\ \hline \eta^0 & \vdots & \eta^w & \vdots & \eta^w \\ \hline a_0 & \vdots & a_w & \vdots & c_1 \\ a_0 & \vdots & a_w & \vdots & c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_0 & \vdots & a_w & \vdots & c_N \end{pmatrix}$$

and the vector $\underline{b}$ is of the form,

$$\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ d_1 \\ d_2 \\ \vdots \\ d_N \end{pmatrix}.$$

The matrix equation (4.27) is solved using any of the well known algorithms for the solution of systems of equations. A Gaussian elimination algorithm is suitable.

(vi) Once $\underline{x}^{k+1}$ is solved, the pressure continuity condition is used to update the interface position by shifting each nodal point, P_j , forward in time.

i.e

$$\eta_j^{(k+1)} = (a_0 + a_w)L + a_0 \tilde{\phi}_0^{(k)} + a_w \tilde{\phi}_w^{(k)}.$$

(vii) If steady state (i.e. if $|\eta^{(k+1)} - \eta^{(k)}| < \epsilon_d$,

where ϵ_d is small) or critical height is reached the algorithm is terminated. If not, the procedure is repeated starting at step (ii).

The critical height is evaluated, for a particular set of reservoir data, from the steady state model of Chapter 3.

4.5 Modification to include water zone thickness

In this section we include the effect of water zone thickness by considering an aquifer that is bounded below by an impermeable boundary. In figure (4.2) the lower impermeable boundary is denoted by S_w and it appears at a distance h_t below the origin. The water zone is of thickness h_w .

4.5.1 Governing equations and boundary conditions

For this case we introduce two basic flow potentials, L_1 and L_2 for the oil and water phase respectively, so that,

$$\Phi_o = L_1 + \tilde{\Phi}_o$$

$$\Phi_w = L_2 + \tilde{\Phi}_w$$

where $\tilde{\Phi}_o$ and $\tilde{\Phi}_w$ are perturbation potentials.

L_1 will be chosen to be identical to the basic potential L in equation (4.4). L_2 , however, will be made to represent the potential due to a line sink in the presence of a lower impermeable boundary S_w at $z = h_t$. L_2 is readily derived by placing an image line sink below the plane $z = h_t$, and by summing the contribution due to the real and image sinks.

Hence,

$$L_1 = \frac{q}{4\pi\ell} \left(\ln |[(z-h_2)^2 + r^2]^{\frac{1}{2}} + (z-h_2) | [(z+h_1)^2 + r^2]^{\frac{1}{2}} + (z+h_1) | \right. \\ \left. - \ln |[(z-h_1)^2 + r^2]^{\frac{1}{2}} + (z-h_1) | [(z+h_2)^2 + r^2]^{\frac{1}{2}} + (z+h_2) | \right) + c_1$$

and

$$L_2 = \frac{q}{4\pi\ell} \ln |[(z-h_2)^2 + r^2]^{\frac{1}{2}} + (z+h_2) | [(z-2h_t+h_1)^2 + r^2]^{\frac{1}{2}} + (z-2h_t+h_1) | \\ - \ln |[(z-h_1)^2 + r^2]^{\frac{1}{2}} + (z-h_1) | [(z-2h_t+h_2)^2 + r^2]^{\frac{1}{2}} + (z-2h_t+h_2) | \Big) + c_2$$

where $c_1 = \lim_{r \rightarrow \infty} L_1(r, z)$

and

$$c_2 = \lim_{r \rightarrow \infty} L_2(r, z).$$

The governing equations and boundary conditions now become,

$$\nabla^2 \tilde{\phi}_0 = 0 \quad \underline{r} \in V_0 \quad (4.28)$$

$$\nabla^2 \tilde{\phi}_w = 0 \quad \underline{r} \in V_w. \quad (4.29)$$

On the interface, continuity of pressure: $\eta = (a_0 L_1 + a_w L_2) + a_0 \tilde{\phi}_0 + a_w \tilde{\phi}_w$ (4.30)

continuity of velocity: $\frac{\partial \tilde{\phi}}{\partial n} = \frac{\partial \tilde{\phi}}{\partial n} + \frac{\partial L_1}{\partial n} - \frac{\partial L_2}{\partial n}$

material surface condition: $\frac{\partial \eta}{\partial t} = - \frac{\partial}{\partial n} (L_1 + \tilde{\phi}_0) \frac{1}{\cos \beta}$ (4.31)

$$\dots \quad (4.32)$$

On the impermeable boundary S_0 , ($z=0$)

$$\frac{\partial \tilde{\phi}_0}{\partial n} = 0 \quad , \quad (4.33)$$

On the impermeable boundary S_w , ($z = h_t$)

$$\frac{\partial \phi_w}{\partial n} = 0 \quad . \quad (4.34)$$

Far field condition:

$$\left. \begin{array}{l} \tilde{\phi}_0, \tilde{\phi}_w \rightarrow 0 \\ \eta(r,t) = h \end{array} \right\} \quad \text{as } r \rightarrow \infty \quad (4.35)$$

Initial conditions ($t = 0$) :

$$\left. \begin{array}{l} \eta(r,0) = h_0 \\ \tilde{\phi}_0, \tilde{\phi}_w \text{ specified} \end{array} \right\} \quad (4.36)$$

4.5.2 Boundary integral equations

Setting G in equation (4.15), to $G' = \int_0^{2\pi} g' d\theta$, (g , as in equation (4.16)) we obtain, for the domain V_0 ,

$$- \alpha_{0i} \tilde{\phi}_{0i} = \int_{\Gamma_i} r \left(\tilde{\phi}_0 \frac{\partial G'}{\partial n} - G' \frac{\partial \phi_0}{\partial n} \right) d\xi \quad . \quad (4.37)$$

For the domain V_w , we choose G to be G^2 , such that,

$\frac{\partial G^2}{\partial n} = 0$, on $\Gamma_w - \Gamma_w$ being the generating contour of the surface S_w .

$$G^2 = \int_0^{2\pi} g^2 d\theta$$

$$g^2 = \left| (r \cos \theta - r_i \cos \theta_i)^2 + (r \sin \theta - r_i \sin \theta_i)^2 + (z - z_i)^2 \right|^{-\frac{1}{2}} \\ + \left| r (\cos \theta - r_i \cos \theta_i)^2 + (r \sin \theta - r_i \sin \theta_i)^2 + (z - 2h_t + z_i)^2 \right|^{-\frac{1}{2}}.$$

The integral equation (4.15) now becomes,

$$\alpha_{wi} \tilde{\phi}_{wi} = \int_{\Gamma_i} r \left(\tilde{\phi}_w \frac{\partial G^2}{\partial n} - G^2 \frac{\partial \tilde{\phi}_w}{\partial n} \right) d\xi.$$

Substituting equation (4.31), to eliminate $\frac{\partial \tilde{\phi}_w}{\partial n}$, the boundary integral equation for the water phase becomes

$$\alpha_{wi} \tilde{\phi}_{wi} = \int_{\Gamma_i} r \left(\tilde{\phi}_w \frac{\partial G^2}{\partial n} - G^2 \left(\frac{\partial \tilde{\phi}_o}{\partial n} - \frac{\partial L_2}{\partial n} - \frac{\partial L_1}{\partial n} \right) \right) d\xi. \quad (4.38)$$

Thus by devising two Green's functions, G' and G^2 , such that their normal derivatives vanish along S_0 and S_w respectively, we have reduced the boundary integral equation for the oil and water phase to equations of the form (4.37) and (4.38). Integration therefore does not have to be performed throughout the entire boundary contour, $\Gamma_0 + \Gamma_i + \Gamma_w$, but merely along the oil-water contact Γ_i .

Using the procedure outlined in Section 4.4.3, equations

(4.37) and (4.38) are written in matrix form as follows:

$$[\underline{\tilde{H}}^0][\underline{\tilde{\phi}}_0] + [\underline{\tilde{G}}^0]\left(\frac{\partial \underline{\tilde{\phi}}_0}{\partial n}\right) = [\underline{0}] \quad (4.39)$$

$$[\underline{\tilde{H}}^w][\underline{\tilde{\phi}}_w] + [\underline{\tilde{G}}^w]\left(\frac{\partial \underline{\tilde{\phi}}_w}{\partial n}\right) = [\underline{f}] \quad (4.40)$$

with,

$$\begin{aligned} H_{ij}^0 &= \int_{\Gamma_{j-1}} N_{ij} \frac{\partial G_j'}{\partial n} r d\xi + \int_{\Gamma_j} N_{2j} \frac{\partial G_j'}{\partial n} r d\xi, \quad (\forall i, j \ i \neq j) \\ &= \int_{\Gamma_{j-1}} N_{ij} \frac{\partial G_j'}{\partial n} r d\xi - \int_{\Gamma_j} N_{2j} \frac{\partial G_j'}{\partial n} r d\xi + \alpha_{0i}, \quad (i = j) \end{aligned}$$

$$G_{ij}^0 = \int_{\Gamma_{j-1}} N_{ij} G_j' r d\xi + \int_{\Gamma_j} N_{2j} G_j' r d\xi \quad \forall i, j$$

$$\begin{aligned} H_{ij}^w &= \int_{\Gamma_{j-1}} N_{ij} \frac{\partial G_i^2}{\partial n} r d\xi + \int_{\Gamma_j} N_{2j} \frac{\partial G_j^2}{\partial n} r d\xi \quad (\forall i, j \ i \neq j) \\ &= \int_{\Gamma_{j-1}} N_{ij} \frac{\partial G_j^2}{\partial n} r d\xi - \int_{\Gamma_j} N_{2j} \frac{\partial G_j^2}{\partial n} r d\xi - \alpha_{wi} \quad (i = j) \end{aligned}$$

$$G_{ij}^w = \int_{\Gamma_{j-1}} N_{ij} G_j^2 r d\xi + \int_{\Gamma_j} N_{2j} G_j^2 r d\xi \quad \forall i, j$$

$$f_i = \sum_{j=1}^N G_{ij}^2 \left(\frac{\partial L_{1j}}{\partial n} - \frac{\partial L_{2j}}{\partial n} \right) \quad \forall i = 1, \dots, N.$$

The time stepping equations (4.24) remain unchanged except for the modification that L in the expression for $d_j^{(k)}$ is replaced by L_1 .

Thus assembling the matrix equations (4.39) and (4.40) together with the time stepping equations, we obtain the equation,

$$[\underline{\tilde{A}}'] [\underline{x}']^{k+1} = [\underline{b}']$$

where,

$$\underline{\tilde{A}}' = \begin{pmatrix} \tilde{H}^0 & 0 & \tilde{G}^0 \\ \tilde{O} & \tilde{H}^w & \tilde{G}^w \\ a_0 & 0 & c_1 \\ a_0 & a_w & c_2 \\ \cdot & \cdot & \cdot \\ 0 & a_w & c_N \end{pmatrix}$$

$$[\underline{x}']^{k+1} = \begin{pmatrix} \tilde{\phi}_0 \\ \tilde{\phi}_w \\ \frac{\partial \tilde{\phi}_0}{\partial n} \end{pmatrix} \quad [\underline{b}'] = \begin{pmatrix} 0 \\ f \\ d \end{pmatrix}$$

The procedure for the location of the interface is identical to that outlined in Section (4.4.3), except that the equation,

$$\eta_j^{(k+1)} = a_0 L_1 + a_w L_2 + a_0 \tilde{\phi}_0^{(k)} + a_w \tilde{\phi}_w^{(k)}$$

is now used to update the interface position.

4.6 Numerical results, checks on accuracy and improvement of model

The new unsteady state coning model was tested in a reservoir with the following rock and fluid properties.

Table 4.1

Impermeable boundary	at datum (z=0)	
oil water contact	100ft. below datum (30.48m below datum)	
oil formation thickness	100 ft (30.48 m)	
well completion interval	40 to 50 ft. (12.19 to 15.24 m)	
ρ_w	1.05 g/cc (1.05×10^3 kg/m ³)	
ρ_o	0.65 g/cc (6.5×10^2 kg/m ³)	
k_w	80mD (7.89×10^{-14} m ²)	
k_o	100mD (9.869×10^{-14} m ²)	
μ_w	0.4 cp (4×10^{-4} Pz.s)	
μ_o	0.25 cp (2.5×10^{-4} Pa.s)	
σ	0.2	
Critical flow rate	} for this reservoir	470 b/d (8.6493×10^{-4} m ³ /sec)
Critical cone height		

All numerical computations were performed in dimensionless variables. In the graphical displays to follow, production rates are given in the dimensionless units of barrels per day. To obtain reservoir distances (in feet), multiply dimensionless distances by 100.

Where time is shown in dimensionless units multiply by (27.7) to obtain time in days.

Water cone development

Figure (3.3) traces the cone evolution up to the steady state position for a sub critical flow rate. Steady state is assumed to be reached when the change in the position of the interface over a large time step is less than some small value (say .1 feet). In figure (3.4) we compare the steady state position obtained from the unsteady model against the cone profile obtained directly from the steady state model. Agreement between the two solutions is excellent. Steady state for a production rate of 300 b/d is therefore reached, in practice, after about 3079 days.

Figure (3.5) traces the evolution of the cone, until it reaches the critical height, for a super-critical flow rate of 1087 b/d. We note that the interface gradient of the cone becomes sharper as we approach the critical height of 32 feet after 122.6 days. In figure (3.6) we compare the steady state interface profile of an almost critical case cone ($Q = 458$ b/d, $Q_c \sim 470$ b/d) against the interface profile of the super-critical ($Q = 1087$ b/d) cone whose apex is just below the critical height. Although the shape of the cones are considerably different at large distances away from the well, near the well there is close agreement. It is therefore reasonable to assume that a supercritical cone does indeed become unstable and breaks through at the critical height, which is determined from

the steady state analysis. The development of a super-critical cone may also be considered as succession of steady state cones.

Checks on accuracy and improvement of model

(i) Comparison with the linearized solution :

In order to assess the performance of the proposed boundary integral model, comparisons where possible were made with an analytical solution. The analytical solution is the linearized or small perturbation solution which has been outlined in subsection (3.2.4). Interface profiles have been compared for a sub-critical (figure 3.7)) and super-critical flow rate (figure (3.8)), for the case where there is no upper impermeable boundary present. The comparison is excellent for the sub-critical case. As to be expected, the linearized solution breaks down when the cone becomes large and non-linearities become important. Therefore, there is substantial disagreement at larger times for the super critical case.

(ii) Influence of the number of boundary points in the numerical solution.

The influence of boundary points on the numerical model was investigated by deriving solutions using N , $2N$, $4N$ and $8N$ boundary points for a range of flow-rates at different time levels.

In figure (3.9) we plot $-\log(\text{error})$ against the number of boundary points for a typical flow-rate at three time levels. From the graph we deduce that the relative error e_N , over the whole interface behaves approximately as $A \exp(-BN)$, where A and B are constants. The relative error, e_N , is defined as

$$e_M = \left(\sum_{i=1}^M \left[\frac{z_{8N}(x_i) - z_M(x_i)}{z_{8N}(x_i)} \right]^2 \right)^{\frac{1}{2}} \frac{1}{M}$$

(iii) Influence of time step-size on numerical solution

(a) Constant time step model

A major difficulty with all finite difference time marching processes is that the solution obtained for the first few time steps is likely to be in error by around 7 to 10%. This is partly due to the forward difference approximation of using initial data at $t = 0$ to calculate the solution at the next time step and so on, and partly due to approximating a step change of input in the physical system by a mathematical description (29). In view of this difficulty, it is necessary to use very small time steps (of the order of $\frac{1}{4}$ of a day) in field units) for the first few time levels. After that a much larger time step may be used.

In order to assess the influence of time step size on the numerical solution, we performed a series of tests on the model reservoir whose characteristics are given in table (4.1). The upper impermeable boundary, S_0 , was, however, neglected. In figure (3.10) we compare the interface profiles obtained by using a constant dimensionless time step size (Δt) of 0.1 and 0.2. The time level, $t = 3.2$ (88.6 days) is very nearly at breakthrough position. Agreement between the two sets of profiles is good indicating that the solution converges to the true numerical solutions asymptotically.

Figure (3.11) examines the percentage relative change in the numerical solution from $\Delta t = 0.1$ (the reference case) to $\Delta t = 0.2, 0.4$.

We see that the change in the solution is at worst 7.6% and therefore conclude that the constant time step size solutions is stable.

(b) Variable time step-potential control scheme

As it is the change in potential or pressure that determines how far the interface progresses we developed a scheme whereby, the criteria for the selection of the time step size was made to depend on the amount of change in potential over the previous time step. Thus at each time level, the time step size was chosen, such that the maximum potential change over the last time step was less than some prescribed value. We took this value to be 1% of the minimum initial potentials.

As for the case of constant time step size, the first few time steps were taken to be very small. After that the time step size, Δt , was calculated as follows :

$$\Delta t^{(n+1)} = \Delta t^{(n)} \times \frac{k}{\left[\Delta \phi_{\max} \right]^{(n)}}$$

with

$$k = 1\% \times \min[\phi_0^{(0)}, \phi_w^{(0)}].$$

In figure (3.12) we compare the interface profiles obtained using the variable time step-potential control scheme and the constant time step model. For this particular case of a super-critical flow-rate of 1087 b/d, the constant time step model reached the breakthrough position after 35 time steps, whilst the variable time step model reached breakthrough position in 16 steps.

In figure (3.12) we see that the agreement between the two solutions is excellent. We therefore conclude that the same degree of accuracy can be obtained with far fewer time steps using the variable time step-potential control scheme.

(iv) The effect of the explicit evaluation of β (the angle the interface makes with the horizontal).

In the time stepping equations, equation (4.24), appear the co-efficients $c_j^{(k)}$, $d_j^{(k)}$. These co-efficients contain a factor $\cos\beta_j$ (see equations (4.25), (4.26)) which is calculated explicitly at each time level.

To examine the influence of this explicit evaluation of $(\cos\beta)$, the numerical model was adapted to iterate on the interface position using three successive evaluations of $(\cos\beta)$ on the new time level. i.e. the first evaluation of the interface at time level $(k+1)$, $\eta^{(k+1)1}$ was calculated using $\cos\beta^{(k)}$. Then to calculate $\eta^{(k+1)2}$ $\cos\beta^{(k+1)1}$ was used, and to calculate $\eta^{(k+1)3}$ $\cos\beta^{(k+1)2}$ was used and so on. This procedure produced an insignificant change in solution i.e. $|\eta^{(k+1)1} - \eta^{(k+1)4}|$ was very small. We therefore conclude that for a reasonably small time step an explicit evaluation of $(\cos\beta)$ produces a very accurate solution.

4.7 The influence of various parameters on coning behaviour

(i) The effect of production rate on breakthrough time

Water breakthrough times obtained for a range of super-critical flow rates (using the data of table 4.1) are shown in figure (3.13). The asymptotic behaviour of this curve as the

flow-rate falls towards the critical ($q_c = 470$ b/d) can be seen.

(ii) The effect of well penetration on critical production rate and breakthrough time

Figure (3.14a) shows the effect of well penetration (as a percentage of the total oil formation thickness), on critical production rate. In figure (3.14b) we plot breakthrough time against well penetration for a super-critical production rate of 1200 b/d. For a well penetration greater or equal to 80%, water breakthrough occurs very quickly (in less than 20 days).

(iii) The effect of oil viscosity on critical production rate and breakthrough time.

Figures (3.15a) and (3.15b) show the effect of oil viscosity, i.e. as the water viscosity is kept constant the effect of the mobility ratio, on coning behaviour. An increase in mobility ratio is seen to adversely affect both the critical production rate and the breakthrough time.

Summary

A new mathematical model has been developed to follow coning behaviour in time. By appropriate interface with analytical solutions developed for the basic flow problem, and by suitable choice of Green's function, the numerical calculations are shown to be necessary only along the oil water contact.

A variable time step-potential control feature has also been introduced so that the maximum potential change over each time level is

checked. This additional feature minimizes the number of time steps that need to be used, without incurring any loss of accuracy in the solution.

The accuracy of the model has been demonstrated and it may be used as a suitable model for predicting water front positions, water breakthrough times and other details of engineering importance.

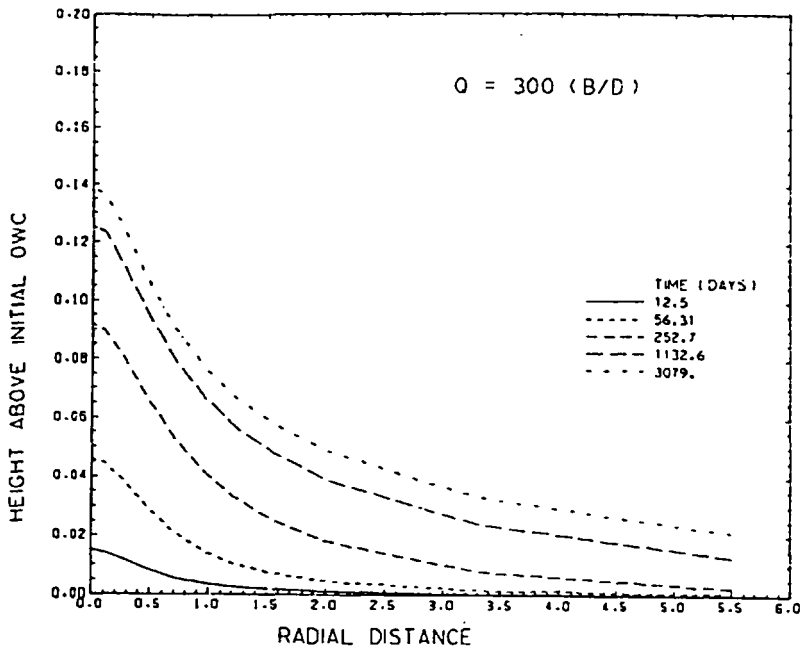


Figure 4.3: Sub-critical interface profiles

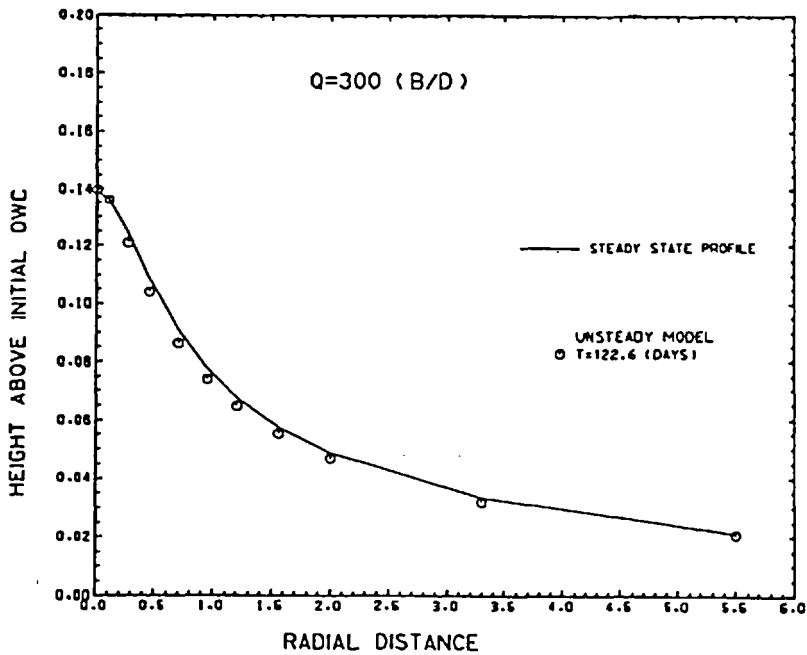


Figure 4.4: Comparison of steady state and unsteady state profiles for a sub-critical cone

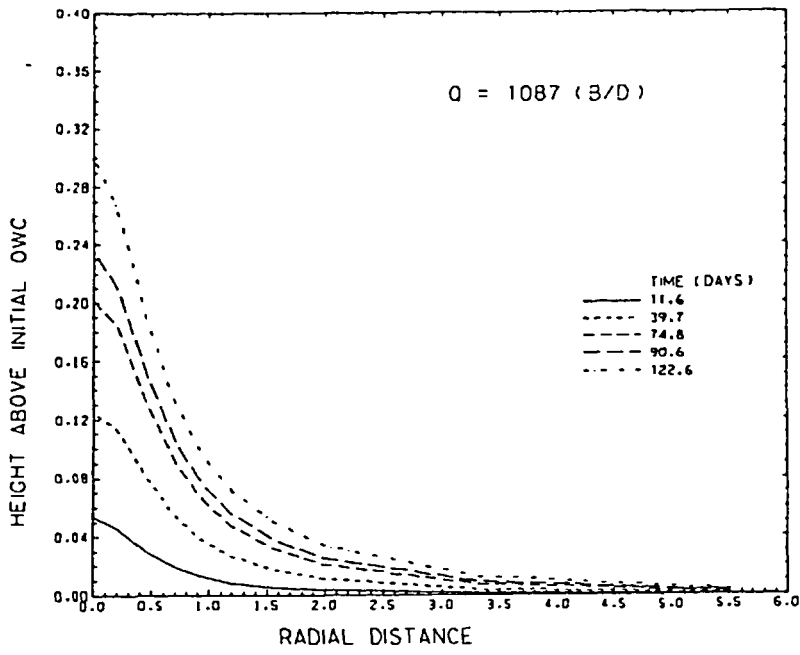


Figure 4.5: Super-critical interface profiles

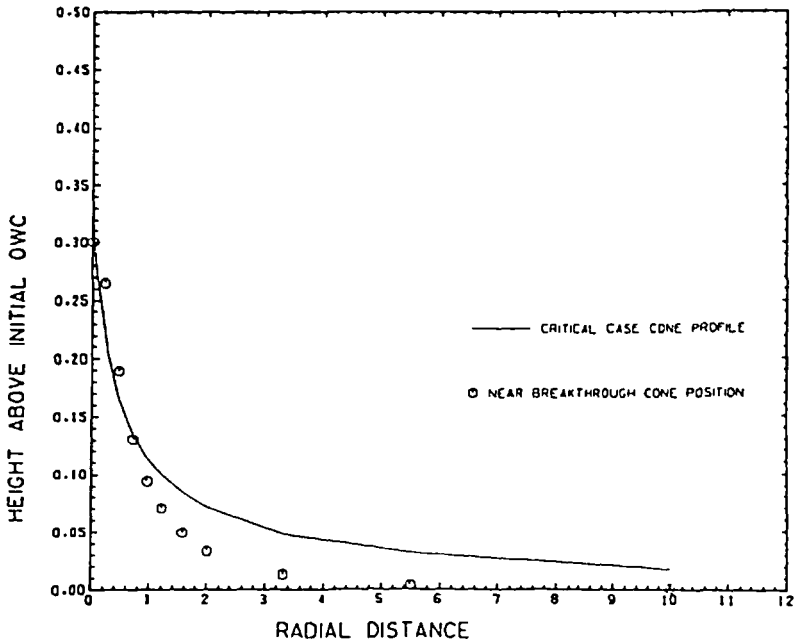


Figure 4.6: Comparison of a near-critical steady state profile and a super-critical near breakthrough profile

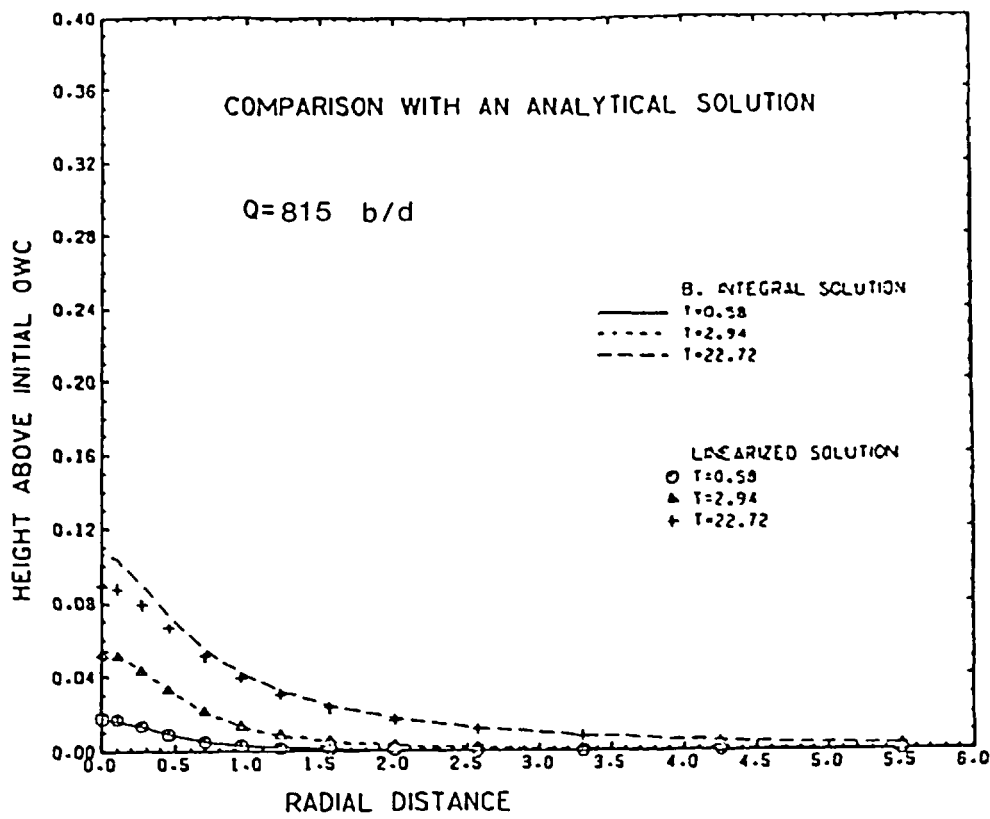


Figure 4.7: Comparison with an analytical solution (sub-critical case)

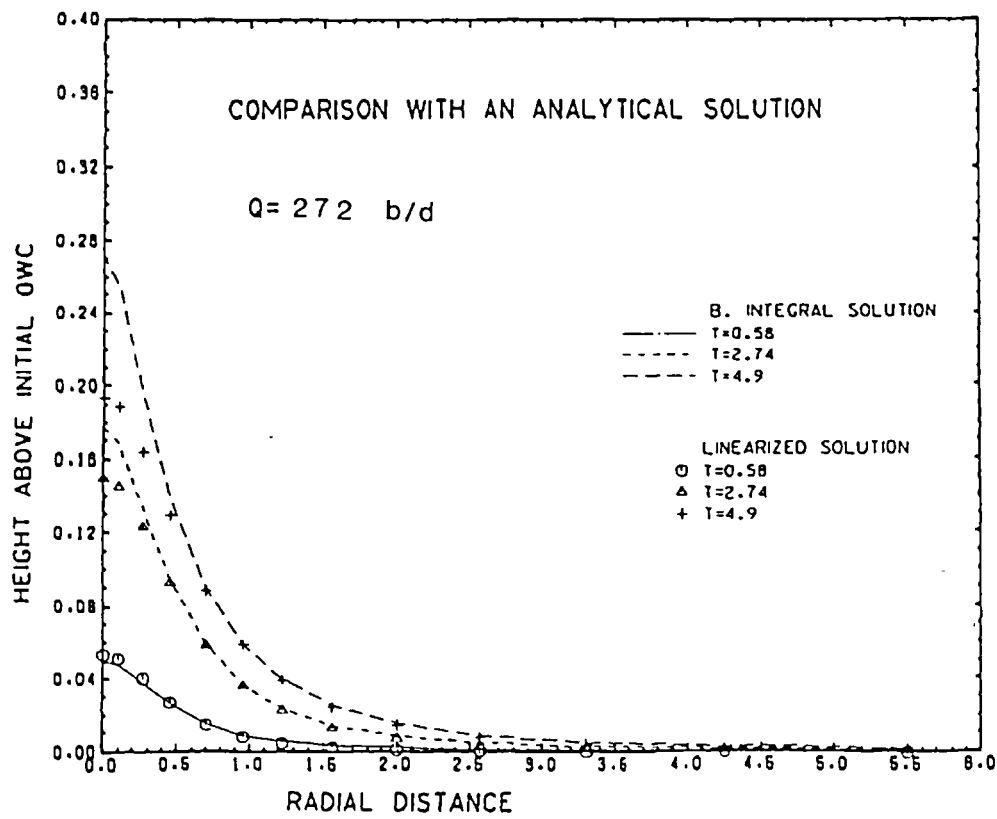


Figure 4.8: Comparison with an analytical solution (super-critical case)

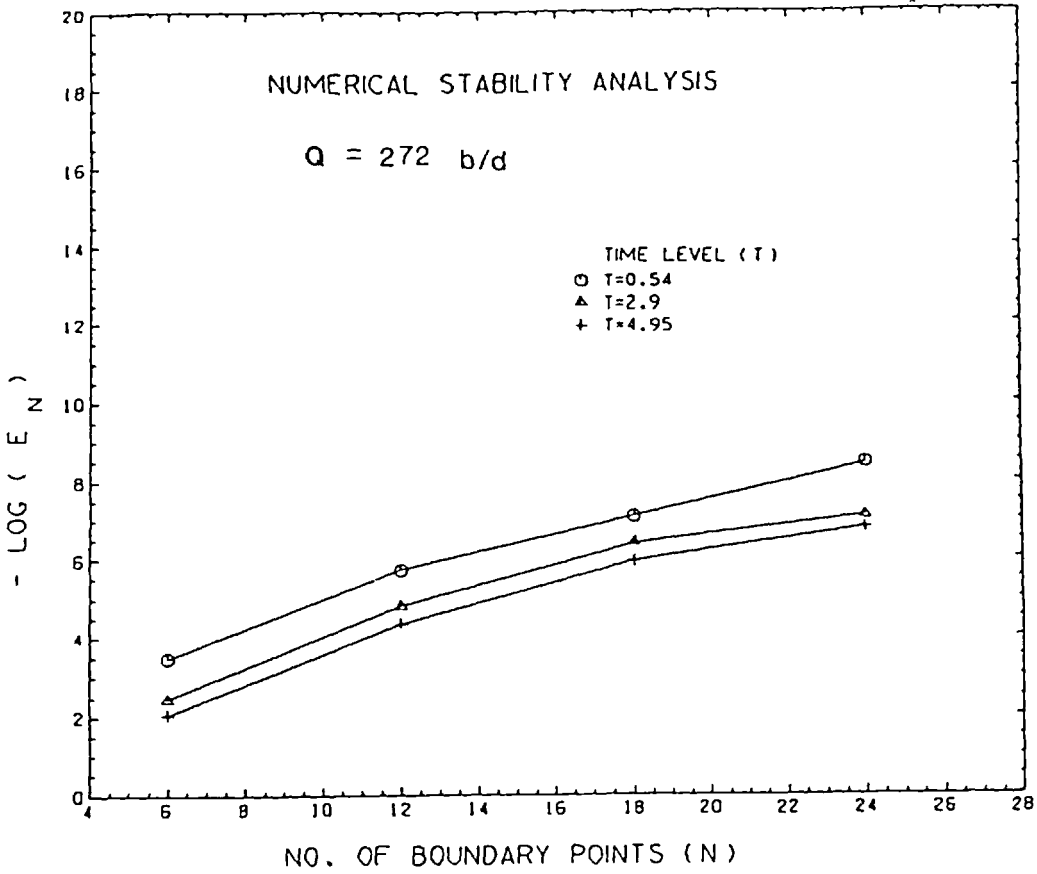


Figure 4.9: Influence of boundary points on relative error

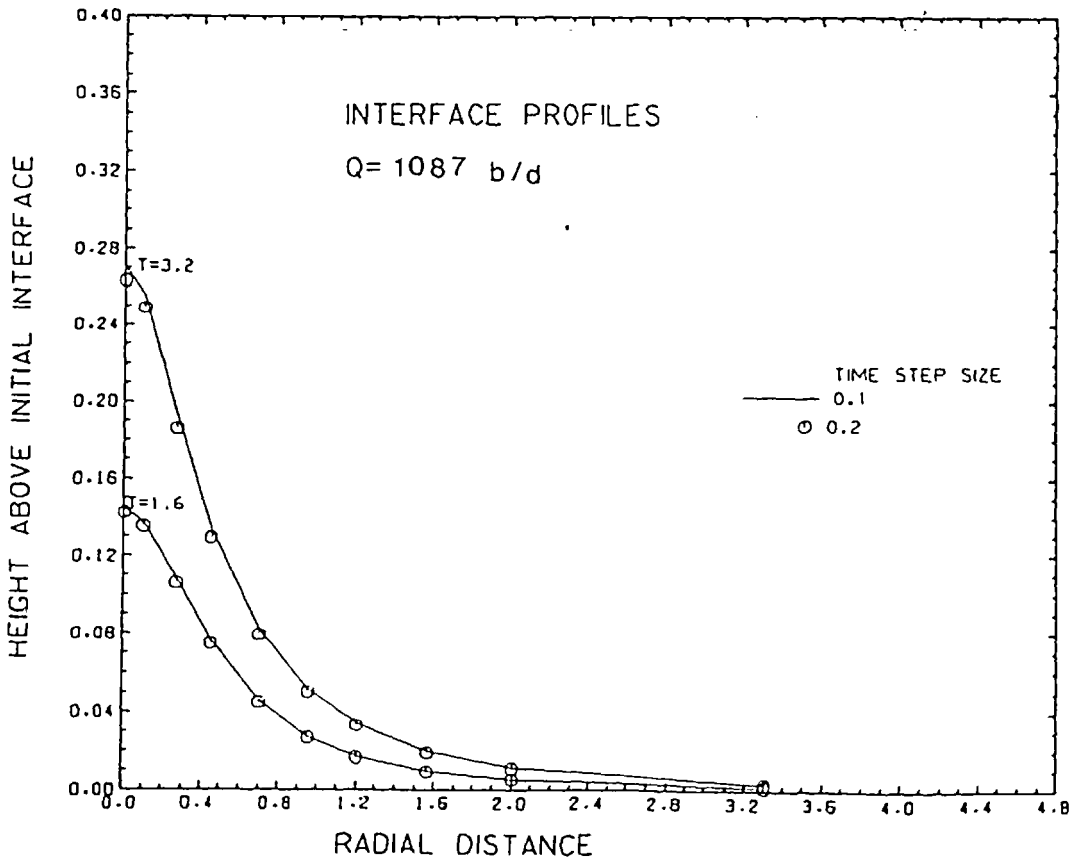


Figure 4.10: Influence of time step size on solution

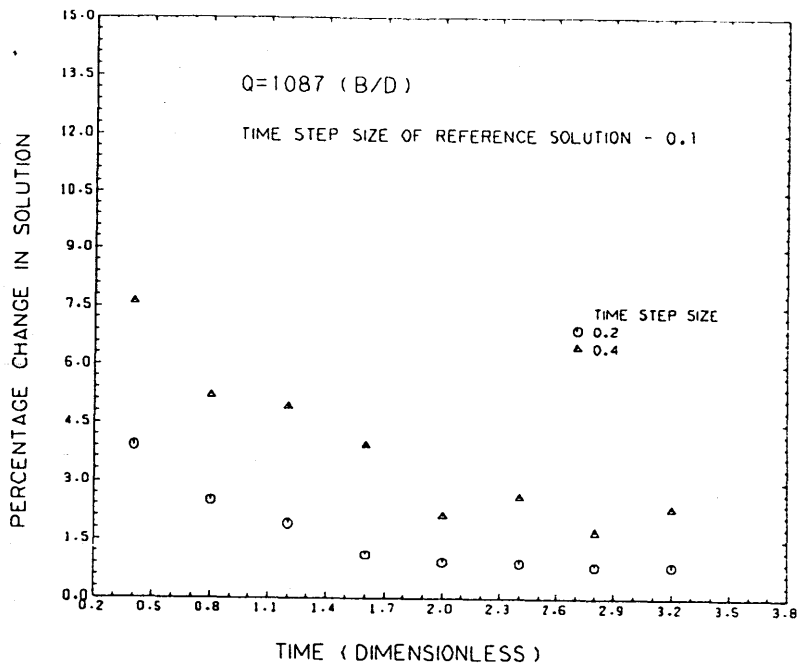


Figure 4.11: Influence of time step size on percentage change in solution

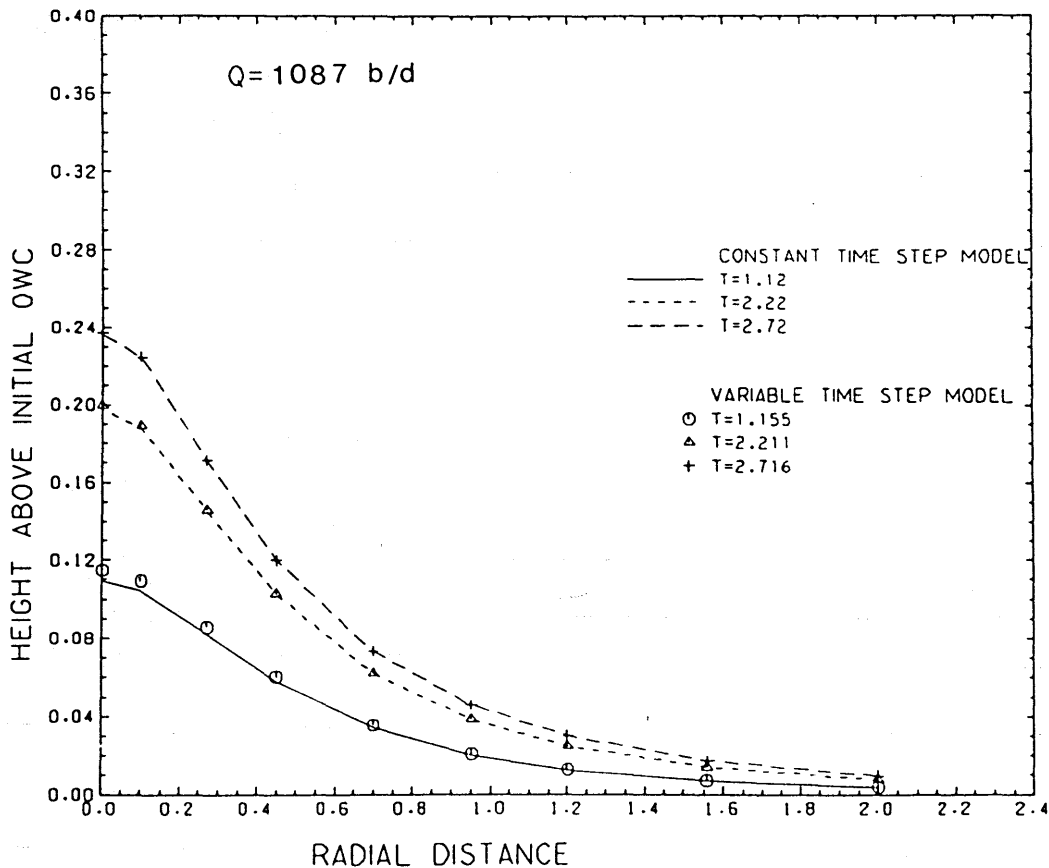


Figure 4.12: Comparison of constant time step and variable time step model solutions

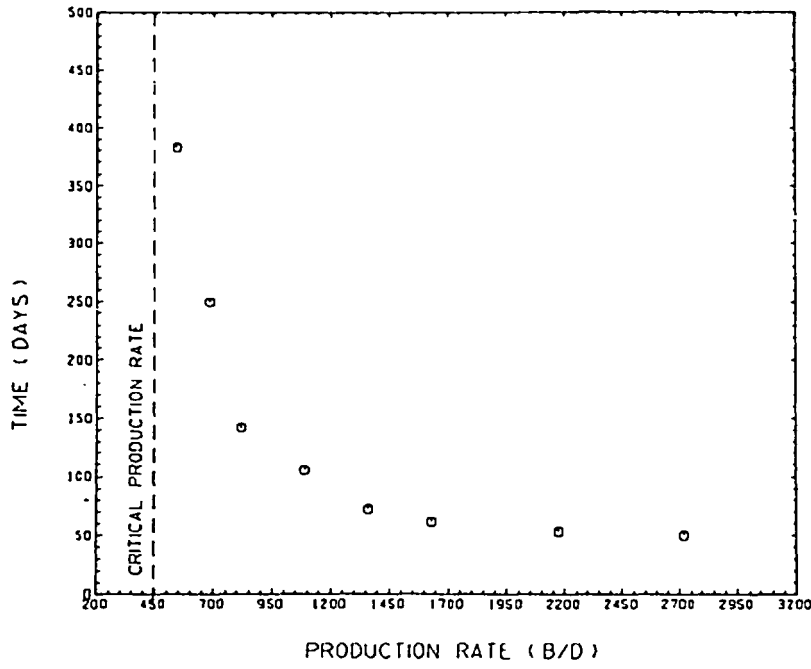


Figure 4.13: Influence of production rate on water breakthrough time

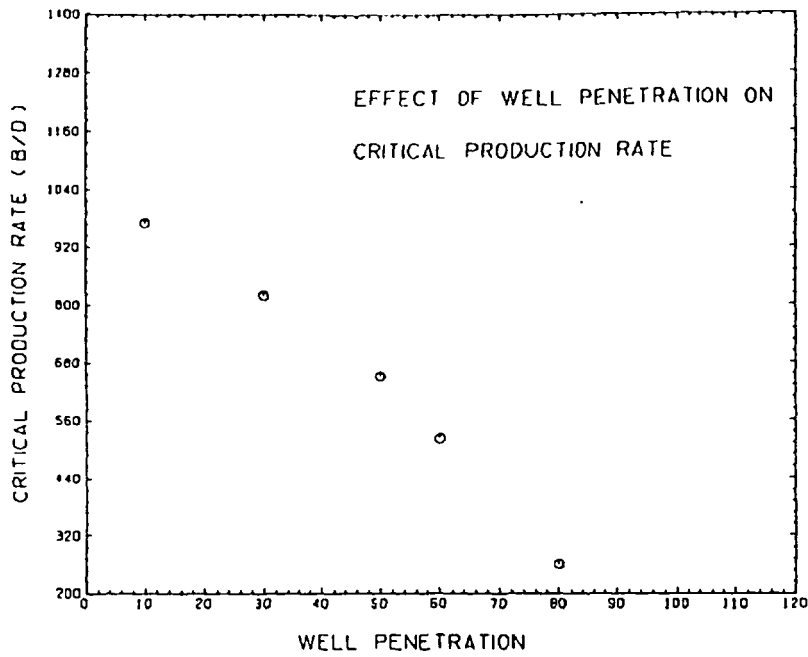


Figure 4.14a: Effect of well penetration on critical production rate

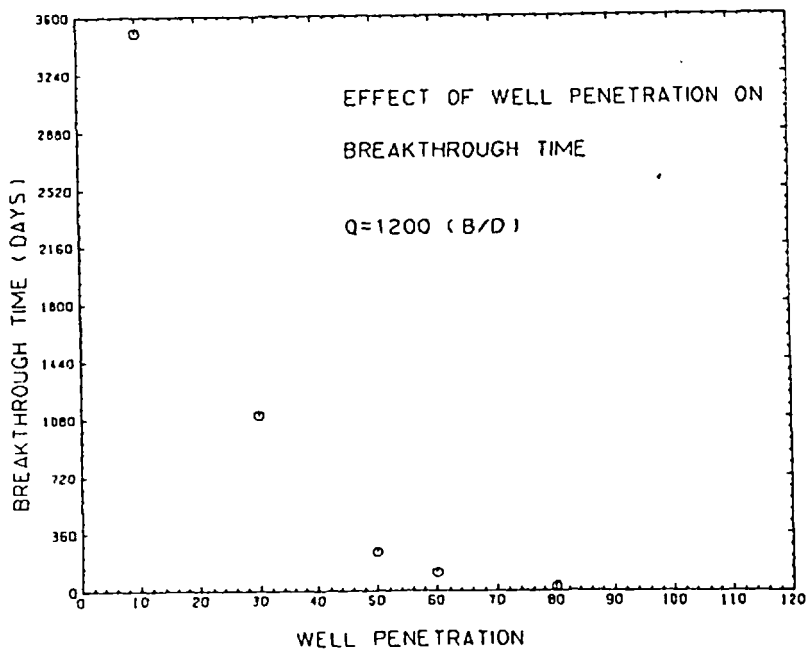


Figure 4.14b: Effect of well penetration on breakthrough time

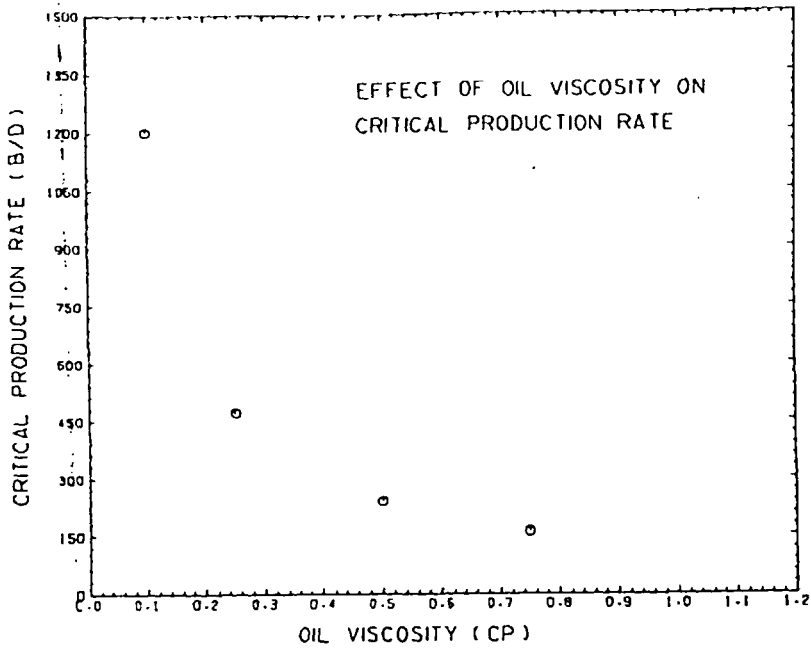


Figure 4.15a: Effect of oil viscosity on critical production rate

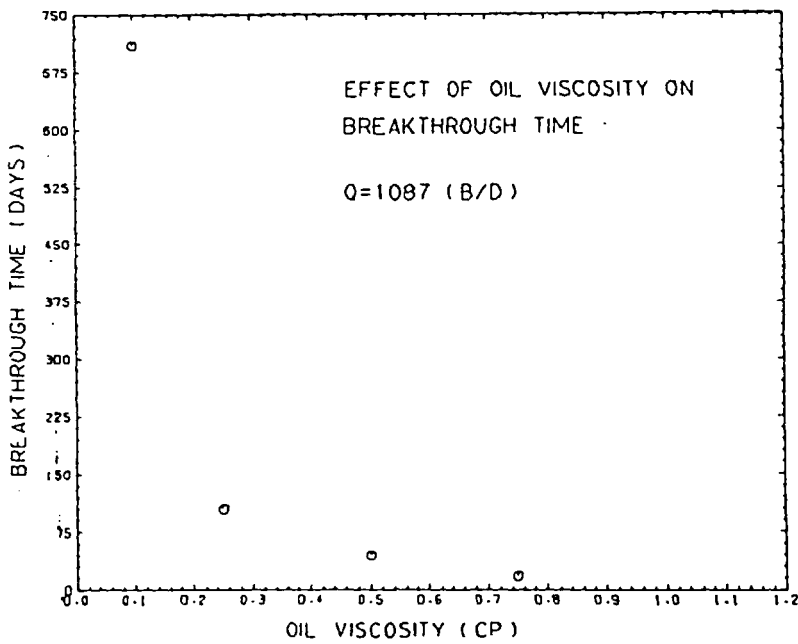


Figure 4.15b: Effect of oil viscosity on breakthrough time

CHAPTER 5

THE ACCURACY OF A FINITE DIFFERENCE SIMULATOR AND THE CONING CORRELATIONS AS COMPARED WITH THE BOUNDARY INTEGRAL MODEL

5.1 Introduction

The new boundary integral coning model developed in this work has been used to study water coning behaviour, up to the instant of water breakthrough. An example using rock and fluid properties from a typical petroleum reservoir is given to demonstrate the model, and both the steady state and unsteady state modes are used. The three most widely used coning correlations, those of Bournazel and Jeanson, Chierici et al, and Sobocinsky and Cornelius are then used, with the identical reservoir rock and fluid properties, to estimate critical production rate and, where possible, water breakthrough time. A water breakthrough time estimate has also been obtained from a fully implicit finite difference coning simulator and is presented. Numerical results obtained from the two sources - the coning correlations and the finite difference simulator are then compared with the boundary integral solution.

5.2 Reservoir rock and fluid properties

The example reservoir is horizontal and laterally fed, with an upper impermeable boundary. The initial characteristics are given in table (5.1), and were obtained from the data used for the

fast coning evaluation method of Bournazel and Jeanson (7).

external feeding radius	r_e	656.17 ft (200 m)
well radius	r_w	.328 ft (.1m)
oil zone thickness	h_o	131.23 ft (40m)
water zone thickness	h_w	65.62 ft (20 m)
total thickness	h_t	196.85 ft (60 m)
penetration depth	h_p	32.81 ft (10 m)
horizontal permeability to oil	k_r	150 mD ($1.48 \times 10^{-13} \text{ m}^2$)
relative permeability ratio	(k_w/k_o)	0.6
anisotropy ratio	(k_r/k_z)	1.0
effective porosity	σ	0.1
viscosity of oil	μ_o	4.0 cP ($4. \times 10^{-3} \text{ Pa.s}$)
viscosity of water	μ_w	0.8 cP ($8. \times 10^{-4} \text{ Pa.s}$)
density of oil	ρ_o	0.75 g/cm ³ (750 kg/m ³)
density of water	ρ_w	1.05 g/cm ³ (1050 kg/m ³)
flow-rate	q	314.7 b/d ($5.8 \times 10^{-4} \text{ m}^3/\text{s}$)

Table 5.1: Rock and fluid properties, for a typical petroleum reservoir

5.3 The boundary integral coning model solution

The steady state model, described in Chapter 3, was used to

compute the interface profiles which are shown in figure (5.1). The critical production rate was then obtained from the graphical display, figure (5.2). For the reservoir with rock and fluid properties as in table 5.1, the critical production rate is estimated to be 104 b/d, and the critical cone height 71 ft.

The unsteady state model, described in Chapter 4, was then used to compute the water breakthrough time, at a production rate of 314.7 b/d. The water cone reached the critical height after about 138 days. Interface profiles at various times up to breakthrough are shown in figure (5.3). Figure (5.4) traces the movement of the cone apex in time.

5.4 Comparison of Solutions

5.4.1 The coning correlations

(i) Bournazen and Jeanson

The critical flow-rate was obtained from the formula, described in subsection (3.2.2)

i.e.

$$q_c = \frac{1.466 \Delta \rho \text{ gKx } h_0^2 \left(1 - \frac{h_p}{h_0}\right)}{\mu_0} \quad (\text{metric units}).$$

Hence for the given data the critical production rate is calculated to be 194 cm³/sec or 105 b/d. Now using equation (4.1a), the

dimensionless cone height , z_D , is calculated to be 1.4 . Hence from equation (4.3) , $t_D = 0.72$, and

$$t_{BT} = \frac{\mu_0 \sigma h_0}{\Delta \rho g K_z g(M) t_D} \approx 192 \text{ days.}$$

Therefore, at a super-critical production rate of 314.7 b/d the fast coning correlation of Bournazel and Jeanson, predicts a water breakthrough time of 192 days.

(ii) Chierici et al

Chierici et al present a formula for the critical flow rate only (see section 3.2.3). In metric units it is written as

$$q_c = 5.256 \times 10^{-3} h_0^2 \Delta \rho \frac{k_0}{\mu_0} \psi(r_{De}, \epsilon, \delta)$$

where

$$r_{De} = \frac{r_e}{h_0} = 5.$$

$$\delta_w = \frac{L}{h} = 0.75$$

(with L = distance between the owc and the well)

$$\epsilon = \frac{b}{h} = 0.25$$

(with b = completion interval).

Using potentiometric models, Chierici et al determined

$\psi(r_{De}, \varepsilon, \delta)$ within the following range of its parameters.

$$\begin{array}{rcccl} 5. & \leq & r_{De} & \leq & 80. \\ 0. & \leq & \varepsilon & \leq & 0.75 \\ 0.01 & \leq & \delta & \leq & 0.9 \end{array}$$

Values of ψ are presented in the form of curves for a range of values of r_{De} . The graph for $r_{De} = 5$ is shown in figure (5.5), and we note that for $\delta_w = 0.75$, and $\varepsilon = 0.25$, ψ assumes a value of 0.29. Hence q_c is calculated to be 208 cc/sec or 113 b/d.

(iii) Sobocinsky and Cornelius

The dimensionless cone height z_D is as for the Bournazel and Jeanson correlation (see Section 4.2.1), and for this example reservoir assumes a value of 1.4.

The graph of dimensionless cone height against dimensionless time (figure 4.1 i.e. $t_D \approx 1.4$, for $z_D = 1.4$) is similar to the shape of the breakthrough curve obtained from the boundary integral unsteady state model (figure 5,4).

5.4.2 The numerical simulator

In collaboration with Societe Nationale Elf Aquitaine (Production), a number of computer runs on their fully implicit, finite difference coning simulator were made. The data shown in table (5.1) were used. No attempt was made to obtain an estimate for the critical flow-rate, as this would have required a trial and error process and would therefore have been very costly in computer time. However, a water breakthrough time estimate was determined for a production rate of 314.7 b/d.

Water was estimated to enter the well at between 100 and 120 days.

The interpretation of the results from the numerical simulator, and its comparison with the results of the boundary integral model presented some difficulties. These difficulties arose essentially from the fact that the simulation with the finite difference simulator could not reproduce exactly the conditions set by the initial data of table (5.1). For instance, the numerical simulator generated a water oil transition zone of sizeable thickness, with linear relative permeability curves and even when capillary pressure was neglected. Furthermore, the boundary condition imposed on the external radius; a fixed oil-water contact, could not be exactly met in the simulation. The evolution of the spatial distribution of saturation showed that the oil water contact had moved up a few metres at breakthrough. The numerical simulator also used water permeabilities which were different in the aquifer zone and in the water swept zone (cone). This difference, however, is not expected to seriously alter coning behaviour until after water breakthrough.

5.5 Discussion

Table (5.2) summarizes the results, obtained for the example problem, from the three different sources. The correlation predictions for the critical flow rate are in excellent agreement with the boundary integral model estimate. In the case of water breakthrough time prediction, however, although all the solutions are of the same order of magnitude, there is a considerable variation

in the results. Sobocinsky and Cornelius' prediction for the breakthrough time is excessive at 401 days and has a relative error of 191% when compared with the boundary integral solution. Bournazel and Jeanson's fast water evaluation correlation, and the finite difference simulator are in reasonable agreement with the boundary integral solution with relative errors of 38% and 13-23% respectively. The finite difference model predicts early breakthrough at between 100 and 120 days. However, this is to be expected as numerical dispersion is a severe problem encountered in simulations. Numerical dispersion has the effect of smearing sharp fronts and thereby predicting early breakthrough times.

The study of this example reservoir, and the comparison of the available methods against the boundary integral method, has demonstrated that the prediction of the critical flow-rate is performed effectively by the coning correlation. However, the unsteady state phenomenon, with particular reference to breakthrough times, is not well predicted by the correlations. The numerical simulator solution, on the other hand, is in better agreement with the boundary integral solution - although it is costly to use.

Model/Correlation	q_c (b/d)	t_{BT} (days)	relative error for q_c	relative error for t_{BT}
Boundary Integral Model (This work)	104	138		
Bournazel and Jeanson	105.43	192	1.4%	38%
Chierici et al	113.2	-	8%	
Finite difference simulator	-	100 - 120	-	13.28%

Table 5.2: Comparison of the correlations and the finite difference simulator against the boundary integral model solutions

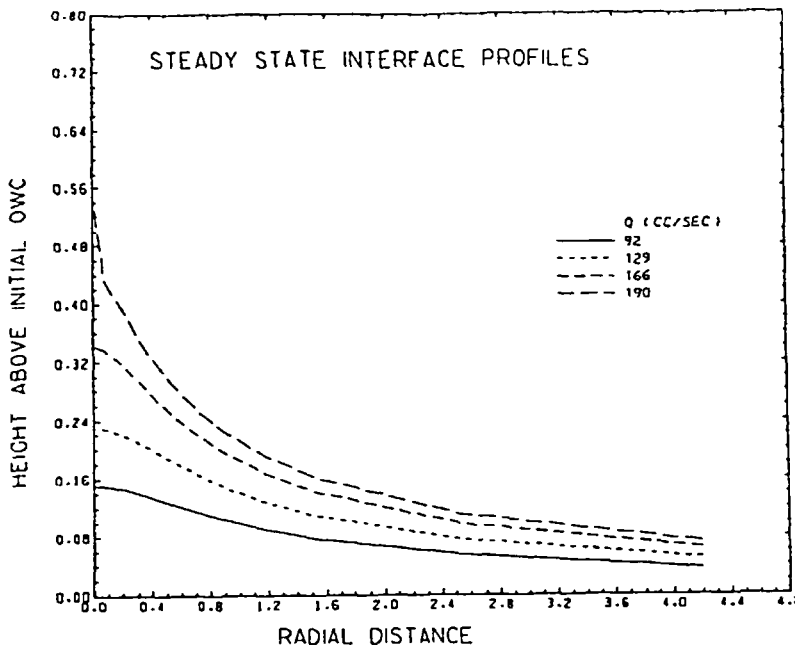


Figure 5.1: Steady state interface profiles for example problem

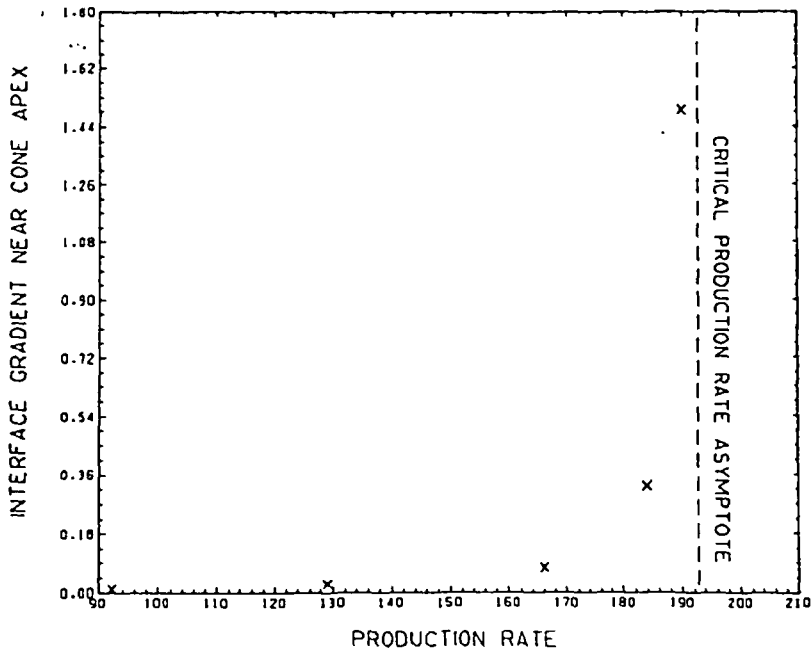


Figure 5.2: Critical production rate correlation

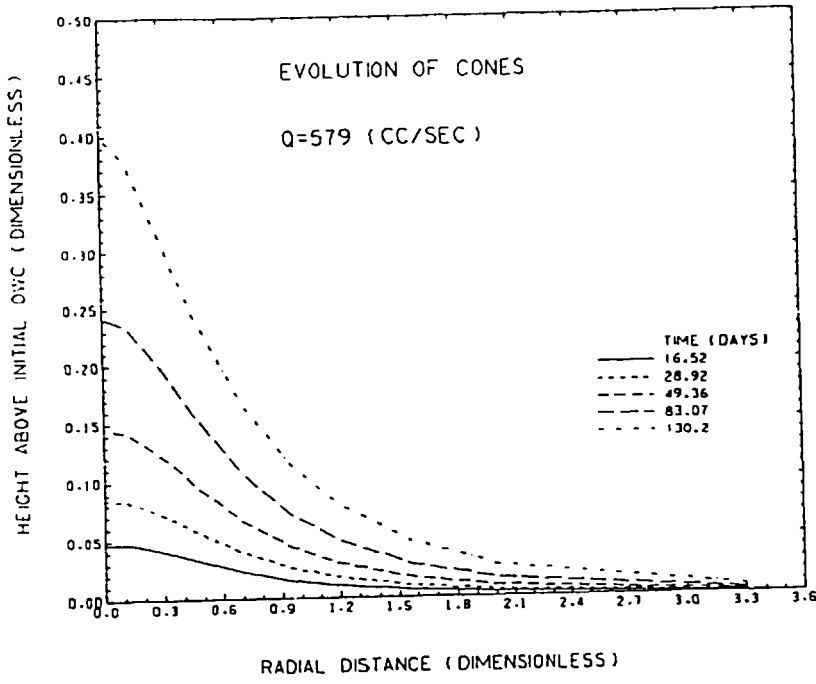


Figure 5.3: Evolution of cones for example problem

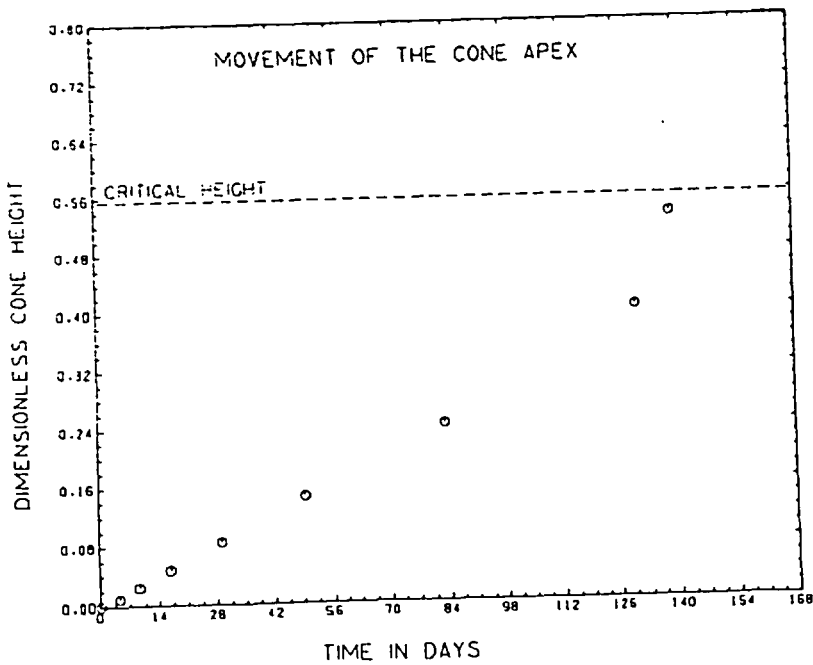


Figure 5.4: Movement of the cone apex in time

CHAPTER 6

POST-BREAKTHROUGH CONING BEHAVIOUR

6.1 Introduction

We have so far restricted our study of coning behaviour to the cone evolution process up to the time when water breaks through into the well. However, for well performance forecasts, the prediction of the evolution of the water cut, (or water oil ratio) subsequent to water breakthrough is an important part of the analysis. As production continues the water oil ratio increases until it reaches its limit. The rate of increase and the limiting water oil ratio are dependant on the conditions at the reservoir boundaries.

In the case of bottom drive reservoirs the oil-water contact is continually rising at the extremities of the reservoir. In such situations it is reasonable to expect the evolution of the water oil ratio to be unbounded and to tend to infinity as time increases (21,30). In a lateral drive system, however, because the interface does not move at an infinite radius (or at the external radius), the water oil ratio vs production time curve will be monotonically increasing but will be bounded above. This upper bound is often referred to as the limiting water oil ratio, and will be reached after a finite time following breakthrough.

In this chapter we focus our attention mainly on ways of evaluating the limiting water oil for a lateral drive system. We

discuss some of the mathematical difficulties encountered in the study of post-breakthrough behaviour, and show why certain approximations have to be made to make a solution to the problem possible. The work of Bournazel and Jeanson (7) on postbreakthrough behaviour, is discussed in some detail, and is later extended to incorporate a more realistic situation. The new model enables the input of two separate values for the water permeability - the water permeability at 100% water saturation for the aquifer zone, and the water permeability at residual oil saturation for the water swept zone (cone). Bournazel and Jeanson's experimental correlation to predict the evolution of the water oil ratio is also discussed.

Our model is used to study the post breakthrough behaviour of an example problem, and comparison with the correlation results of Bournazel and Jeanson are made. The influence of various parameters on postbreakthrough coning behaviour is also investigated.

6.2 Description of the problem and boundary conditions

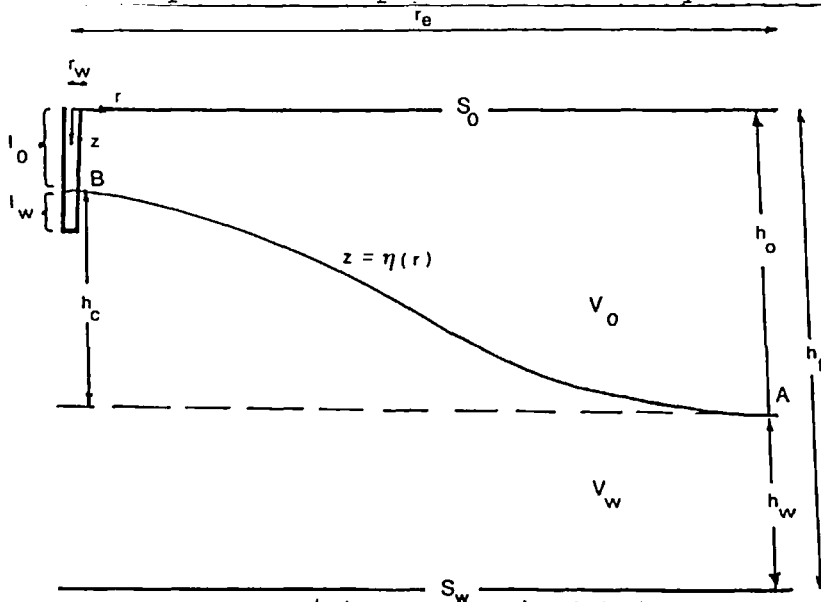


Figure 6.1: post-breakthrough stabilized cone model

The limiting water oil ratio in a lateral drive system is required. Let us, therefore, consider the steady state behaviour of a horizontal cylindrical reservoir (figure 6.1) whose oil water contact is undisturbed at the external radius r_e . The well will be assumed to partially penetrate the oil formation from the top.

We shall assume that the water cone has reached the producing interval of the well and is stabilized at a distance h_0 above the original oil water contact. Let the total production rate be q where $q = q_0 + q_w$; q_0 is the oil production rate and q_w the water production rate. Oil is produced from the upper section, length ℓ_0 , of the production interval and water from the lower section, length ℓ_w . Also present in the reservoir are upper and lower impermeable boundaries S_0 and S_w , located at $z = 0$ and $z = h_t$ respectively.

The governing equations:

By conservation of mass,

$$\nabla^2 \phi_0 = 0, \quad \text{for } \underline{r} \in V_0 \quad (6.1)$$

$$\nabla^2 \phi_w = 0, \quad \text{for } \underline{r} \in V_w. \quad (6.2)$$

Boundary conditions:

On the impermeable boundaries S_0 and S_w ,

$$\frac{\partial \phi_0}{\partial n} = 0 \quad (z = 0) \quad (6.3)$$

and

$$\frac{\partial \phi_w}{\partial n} = 0 \quad (z = h_t). \quad (6.4)$$

On the oil water interface, which we shall write as $z = \eta(r)$, we require

- continuity of pressure

$$p_0 = p_w \quad (6.5)$$

- continuity of normal velocity

$$\frac{\partial \phi_0}{\partial n} = \frac{\partial \phi_w}{\partial n} \quad (6.6)$$

At the external radius $r = r_e$,

$$\eta(r_e) = h_0 \quad (6.7)$$

and

$$\left. \begin{aligned} \phi_0(r_e) &= \phi_{0_{re}} \\ \phi_w(r_e) &= \phi_{w_{re}} \end{aligned} \right\} \quad (6.8)$$

where $\phi_{0_{re}}$ and $\phi_{w_{re}}$ are constants.

Well boundary conditions:

The production well cannot now be represented as a line sink of negligible radius, as the local flow behaviour near the well is also of interest.

The total well production rate is q , therefore,

$$2\pi r_w \left(\frac{\partial \phi_0}{\partial r} \right)_{rw} = \frac{q_0}{\ell_0} \quad (6.9)$$

$$2\pi r_w \left(\frac{\partial \phi_w}{\partial r} \right)_{rw} = \frac{q_w}{\ell_w} \quad (6.10)$$

with $q = q_o + q_w$ and $\ell = \ell_o + \ell_w$.

On the well surface $r = r_w$, we also require that

$$\left. \begin{aligned} \phi_o &= \phi_{o_{rw}} \\ \phi_w &= \phi_{w_{rw}} \end{aligned} \right\} \quad (6.11)$$

where $\phi_{o_{rw}}$ and $\phi_{w_{rw}}$ are constants. The evaluation of these constants, for a given reservoir problem, poses some difficulty. In the case of single phase flow, Muskat postulates that one may assume the flux density over the well to be uniform and take as the well potential, $\phi_{w_{rw}}$, an "effective average", which Muskat found to be at three quarters the distance from the top of the stratum to the bottom of the well (ref. Muskat). In the case of two-phase flow, however, where two fluids are being produced from different sections of the production interval, the choice of the constants $\phi_{o_{rw}}$ and $\phi_{w_{rw}}$ in equation (6.11) is unclear. No published literature exists on the determination of the constants $\phi_{o_{rw}}$ and $\phi_{w_{rw}}$ in this situation, and a thorough investigation of the well boundary condition is necessary before these constants can be found to obtain a solution to the equations (6.1) to (6.11).

As the mathematical difficulty is due to the boundary conditions arising from the two-phase flow at the well, it will be necessary to approximate the water oil contact AB as a limiting streamline, and construct an equivalent single phase system to reproduce the two-phase systems shown in figure (6.1). First let us introduce

the concept of flow resistance factors, and examine the equations that determine the limiting water oil ratio.

6.3 Flow resistance factors:

For single phase flow, we define a fluid potential head, ϕ , as follows

$$\phi = p + \rho g y, \quad (6.12)$$

so that

$$\underline{u} = - \frac{k}{\mu} \nabla \phi. \quad (6.13)$$

We now introduce a flow resistance factor, γ , (sometimes referred to as the "shape parameter") such that,

$$\gamma = [\phi(r_e) - \phi(r_w)] \frac{kh}{\mu q} \quad (6.14)$$

where q represents the production rate, h the formation thickness, and $\phi(r_e)$ and $\phi(r_w)$ the potentials heads at the external and well radii respectively.

For the case of a well completely penetrating the formation, it is known (31) that :

$$\gamma = \frac{h}{2\pi} \ln \left(\frac{r_e}{r_w} \right).$$

For the case of a well partially penetrating the formation,

Muskat (31) obtained the following approximation to γ .

$$\gamma = \frac{1}{2\pi} \left[\frac{1}{2\bar{h}} (\beta - \alpha) - \ln \left(\frac{4h}{re} \right) \right] \quad (6.15)$$

where

$$\alpha = \ln \frac{\Gamma(1+0.875\bar{h}) \cdot \Gamma(1+0.125\bar{h})}{\Gamma(1-0.975\bar{h}) \cdot \Gamma(1-0.125\bar{h})}$$

and

$$\beta = \ln \left[\frac{0.875\bar{h} + \left(\frac{r_w^2}{4h^2} + (0.875\bar{h})^2 \right)^{\frac{1}{2}}}{-0.125\bar{h} + \left(\frac{r_w^2}{4h^2} + (0.125\bar{h})^2 \right)^{\frac{1}{2}}} \right]$$

where $\bar{h} = \frac{h}{h_c}$ is the dimensionless penetration.

Equation (6.15) shows that the flow resistance factors are solely geometry dependant.

6.4 The limiting water oil ratio equations

Consider again the two phase system shown in figure 6.1.

Equating the pressure drop, Δp , between A and B, for both fluids, we obtain from equations (6.12) and (6.14)

$$\Delta p = \gamma_0 \frac{\mu_0 q_0}{k_0 h_0} = \gamma_w \frac{\mu_w q_w}{k_w h_w} + \Delta p \, gh_c \quad (6.16)$$

where h_0 and h_w are the oil thickness and water thickness respectively, and h_c is the height of the cone. γ_0 and γ_w are flow resistance factors and represent the resistance to the oil flow in the oil zone, and the

resistance to the water flow in the water zone. They are geometry dependant factors and their evaluation is dependant on the shape of the stabilized cone, which itself is unknown and is a part of the problem.

Denoting the limiting water oil ratio as w^* and assuming that the streamlines are orthogonal to the well surface ($r=r_w$), we obtain

$$w^* = \frac{q_w}{q_0} = M \frac{\ell_w}{\ell_0} \quad (6.17)$$

where M is the mobility ratio of the system.

From the geometry of the system,

$$\ell_w = h_c + \ell - h_0 \quad (6.18)$$

and

$$\ell_0 = h_0 - h_c. \quad (6.19)$$

Therefore,

$$h_c = h_0 - \frac{M\ell}{M+w^*}. \quad (6.20)$$

Equations (6.16) to (6.20) define the limiting water oil ratio provided that the values of γ_0 and γ_w are known. However, the evaluation of γ_0 and γ_w are themselves dependant on the shape of the cone and hence on w^* . To solve the problem it is therefore necessary to introduce an approximation to γ_0 and γ_w .

6.5 The method of Bournazel and Jeanson

Bournazel and Jeanson (7) assume that the oil water contact AB is identical to a streamline in a single phase equivalent model which preserves the actual values of h_0 , ℓ_0 and thus h_c in figure 6.1. Thus, equivalent single phase oil and water systems were constructed and approximations, γ'_0 for γ_0 , and γ'_w for γ_w , were obtained by equation (6.15).

9.5.1 Equivalent single phase oil system

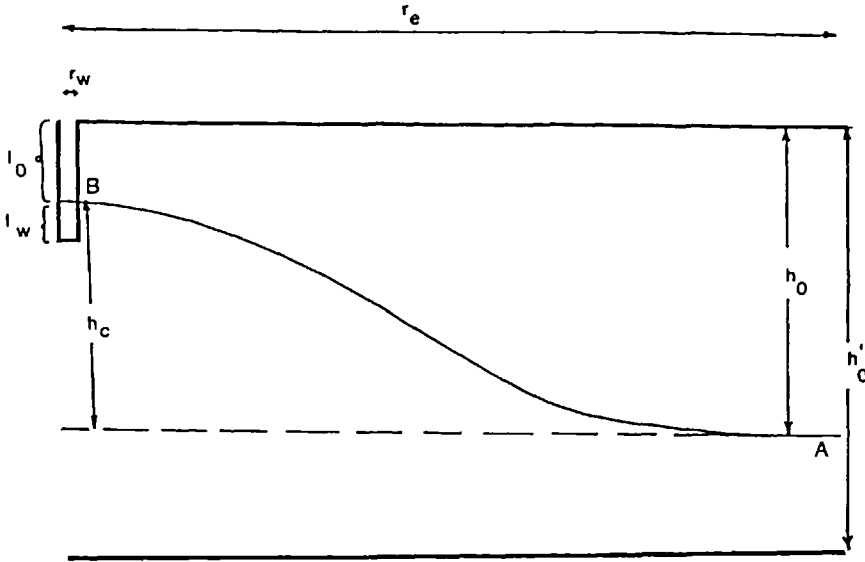


Figure 6.2: Equivalent single phase oil system

The single phase oil system, shown in figure 6.2 preserves h_c , ℓ_0 , ℓ_w and h_0 from the original system. The total thickness will now be assumed to be h'_0 .

The flow density is assumed to be constant at both the well surface ($r = r_w$) and at the external surface ($r = r_e$). Therefore,

h'_0 is such that,

$$w = M \frac{\ell_w}{\ell_0} = M \frac{(h'_0 - h_0)}{h_0} ,$$

Hence, the total thickness h'_0 is $\frac{(w^* + M)h_0}{M}$, and the dimensionless penetration is $\bar{h}'_0 = \frac{M\ell}{(w^* + M)h_0}$. γ'_0 may now be evaluated using equation

(6.15).

6.5.2 Equivalent single phase water system

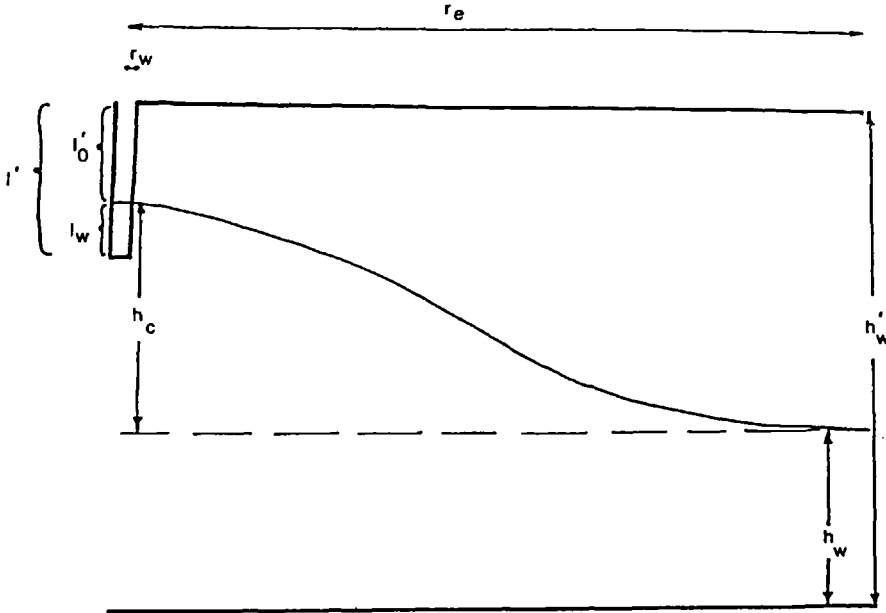


Figure 6.3: Equivalent single phase water system

The equivalent single phase system, shown in figure (6.3), preserves ℓ_w , h_w and h_c from the original system, but the total thickness h'_w , and the penetration ℓ' are altered.

Now,

$$\frac{\ell_w}{\ell'} = \frac{h_w}{h'_w} .$$

Also from figure (6.3) we observe that

$$h_c = (h'_w - h_w) - (\ell'_w - \ell_w).$$

Therefore, $h'_w = h' + h_w + \ell' - \ell_w$, which becomes after some simplification,

$$h'_w = \frac{(w+M)h_w(h_w+h_0-\ell)}{(w^*+M)h_w-w^*h_p}$$

and the dimensionless penetration $\bar{h}'_w$ given by,

$$\begin{aligned}\bar{h}'_w &= \frac{\ell'}{h'_w} \\ &= \frac{w^*\ell}{(M+w)h_w}\end{aligned}$$

The approximation γ'_w to the water flow resistance factor may now be evaluated by substituting $h'_w, \bar{h}'_w, r_w$ and r_e in equation (6.15).

$$\text{Since } q_0 = \frac{q}{1+w^*} \text{ and } q_w = q \left(\frac{w^*}{1+w^*} \right), \text{ substituting } \gamma'_0 \text{ and } \gamma'_w$$

for γ_0 and γ_w , we obtain from equation (6.16) the following implicit equation for w^* .

$$\gamma'_w \frac{w^*}{M} \frac{h_0}{h_w} + \frac{k_0 h_0^2 \Delta \rho g}{\mu_0 q} \left(1 - \frac{M\ell}{(M+w)h_0} \right) (1+w^*) - \gamma_0 = 0. \quad \dots (6.21)$$

Equation (6.21) is easily solved by adopting an interval halving algorithm. Once a solution for the limiting water oil ratio, w , is obtained the cone height may be evaluated from equation (6.20).

6.6 An improved model to predict the limiting water oil ratio

Once the water cone reaches the well completion, the effective water permeabilities (k_w) are expected to differ substantially in the aquifer zone and in the water swept zone (cone). The model of Bournazel and Jeanson, unfortunately allows the input of only one value for the permeability to water. As an improvement, we extend their model to allow the input of two different values for k_w .

Consider the system shown in figure (6.4).

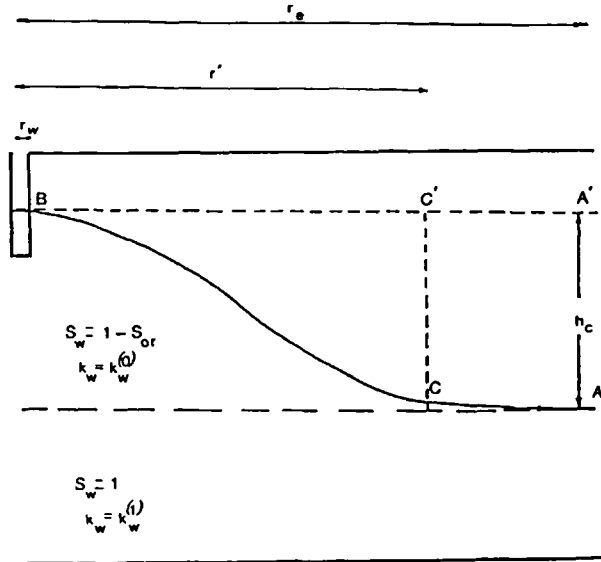


Figure 6.4 : The improved model diagram

As in figure (5.1), r_w and r_e represent the well and external radii.

Introducing a new reservoir radius r' such that;

$$\text{for } r < r'; \quad k_w = k_w^{(0)}$$

(water permeability at residual oil saturation)

$$r > r'; \quad k_w = k_w^{(1)} .$$

(water permeability at 100% water saturation)

Let us now consider the pressure drop, Δp , between the points A and B in figure (5.4).

$$\begin{aligned}\Delta p &= p_A - p_B \\ &= p_A - p_{A'} + (p_{A'} - p_{C'}) + (p_{C'} - p_B) . \\ &= \rho g h_c + (\phi_{r_e} - \phi_{r'}) + (\phi_{r'} - \phi_{rw}) .\end{aligned}$$

Let

$$\begin{aligned}\gamma_w^{(1)} &= (\phi_{r_e} - \phi_{r'}) \frac{k_w^{(1)} h}{\mu q} \\ \gamma_w^{(2)} &= (\phi_{r'} - \phi_{rw}) \frac{k_w^{(1)} h}{\mu q}\end{aligned}$$

and

$$\gamma_w^{(3)} = (\phi_{r'} - \phi_{rw}) \frac{k_w^{(0)} h}{\mu q} .$$

Then, for the water phase the pressure drop between points A and B is

$$\Delta p = \rho_w g h_c + \frac{\rho_w q_w}{k_w^{(1)} h_w} \left(\gamma_w^{(1)} - \gamma_w^{(2)} \right) + \frac{\mu_w q_w}{k_w^{(0)} h_w} \gamma_w^{(3)} . \quad (6.22)$$

Similarly for the oil phase,

$$\Delta p = p_A - p_B = \rho_o g h_c + \frac{\gamma_o \mu_o q_o}{k_o h_o} \quad (6.23)$$

with $\gamma_o = (\phi_{re} - \phi_{rw}) \frac{k_o h_o}{\mu_o q_o} .$

Hence equating the pressure drop across AB for both phases we obtain

$$\rho_0 g h_c + \frac{\gamma_0 \mu_0 q_0}{k_0 h_0} = \rho_w g h_c + \frac{\mu_w q_w}{k_w^{(1)} h_w} (\gamma^{(1)} - \gamma^{(2)}) + \frac{\mu_w q_w}{k_w^{(0)} h_w} \gamma^{(3)},$$

which becomes

$$\begin{aligned} & (\gamma_w^{(1)} - \gamma_w^{(2)}) \frac{w^* h_0}{M^{(1)} h_w} + \gamma_w^{(3)} \frac{w^* h_0}{M^{(0)} h_w} + \frac{k_0 h_0^2}{\mu_0 q} \Delta \rho g \left(1 - \frac{M^{(0)} \ell}{h^0 (M^{(0)} + w^*)} \right) \\ & - \gamma_0 = 0 \end{aligned} \quad (6.24)$$

with

$$M^{(0)} = \frac{k_w^{(0)}}{k_0} \frac{\mu_0}{\mu_w}$$

and

$$M^{(1)} = \frac{k_w^{(1)}}{k_0} \frac{\mu_0}{\mu_w}.$$

The water flow resistance factors $\gamma_w^{(1)}$, $\gamma_w^{(2)}$ and $\gamma_w^{(3)}$ are calculated as outlined in subsection (6.5.2), but with

$$h_w' = \frac{(w^* + M^{(0)}) h_w (h_w + h_0 - \ell)}{(w^* + M^{(0)}) h_w - w^* h_p}$$

and

$$\bar{h}_w' = \frac{w^* \ell}{(M^{(0)} + w) h_w}.$$

Hence to evaluate $\gamma_w^{(1)}$ we substitute h_w' , $\bar{h}_w'$, r_w and r_e in equation (6.15), and to calculate $\gamma_w^{(2)}$, and $\gamma_w^{(3)}$ we substitute h_w' , $\bar{h}_w'$, r_w and r' .

The choice of the new radius r' may appear to pose some difficulty. It can however, be estimated from the steady state interface profile of the critical case cone - the steady state critical case cone profile being the nearest approximation to the shape of the post-breakthrough cone profile.

6.7 Evolution of water oil ratio

Any theoretical attempt to study the evolution of the water oil ratio, w , will require the solution of the unsteady state equations governing post breakthrough coning behaviour. However, it is not possible to solve these equations unless a set of well defined boundary condition are specified at the well surface. Thus due to the same difficulties encountered with the steady state case, no solution could be developed. However, Bournazel and Jeanson have obtained an experimental correlation to predict the evolution of the WOR. We now review their experimental correlation.

Using their estimates of breakthrough time, t_{BT} , (Section 4.2) and limiting water oil ratio (Section 6.5), Bournazel and Jeanson obtain a correlation by plotting the function F , against a dimensionless time $\bar{t}_D$ (figure 6.5).

$$F = \frac{w(\text{experimental})}{w^* (\text{calculated})} \quad (6.25)$$

and

$$\bar{t}_D = \frac{k_z q t}{k_x \sigma h_0^3} \frac{(1+M)^{0.7}}{4 \pi (1-h_p/h_0)} \quad (6.26)$$

The w values in equation (6.25) were obtained from experimental data using scale models. The graph of the function F against $\bar{t}_D$ is shown in figure (6.5). After calculation of w^* and $\bar{t}_D$ it is possible to calculate the water oil ratio w , corresponding to a given time by using the formula.

$$w = \left(\frac{w^*}{w} \right)_{\bar{t}_D} \times w^*$$

$$= F_{\bar{t}_D} \times w^* \quad (6.27)$$

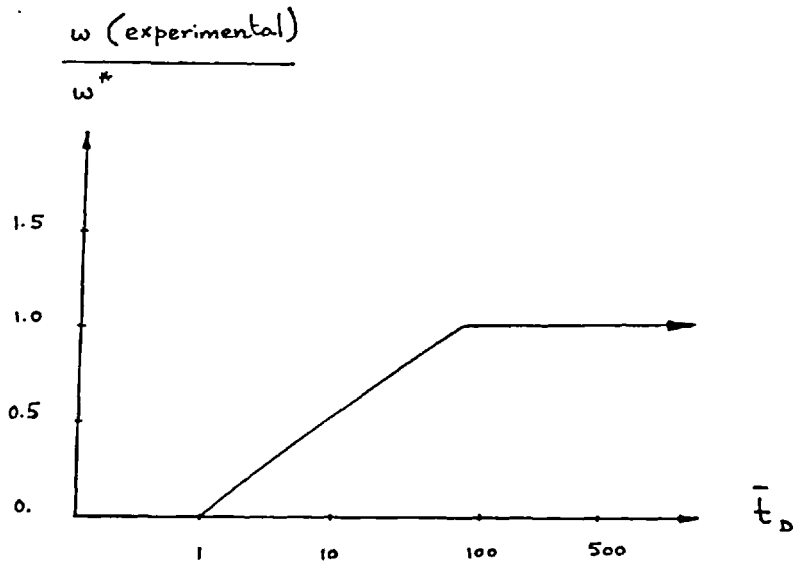


Figure 6.5: Bournazel and Jeanson's correlation of WOR evolution vs reduced time

6.8 Numerical results

The improved post-breakthrough coning model, developed in Section (6.6), was tested on an example reservoir. The rock and fluid characteristics of the example reservoir are given in table (6.1).

external radius (r_e)	1000 ft (304.8m)
well radius (r_w)	0.4 ft (.122 m)
oil zone thickness (h_o)	100 ft (30.48 m)
water zone thickness (h_w)	90 ft (27.432 m)
penetration depth (ℓ)	30 ft (9.144 m)
effective permeability to oil (k_o)	100 mD ($9.869 \times 10^{-14} \text{ m}^2$)
effective permeability to water ($k_w^{(0)}$) (in the water swept zone)	80mD ($7.895 \times 10^{-14} \text{ m}^2$)
effective permeability to water ($k_w^{(1)}$) (in the aquifer zone)	110 mD ($1.0856 \times 10^{-13} \text{ m}^2$)
viscosity of oil (μ_o)	0.25 cp ($2.5 \times 10^{-4} \text{ Pa.s}$)
viscosity of water (μ_w)	0.4 cp ($4.0 \times 10^{-4} \text{ Pa.s}$)
density of oil (ρ_o)	0.65 g/cc ($6.5 \times 10^2 \text{ kg/m}^3$)
density of water (ρ_w)	1.05 g/cc ($1.05 \times 10^3 \text{ kg/m}^3$)

Table 6.1 : Reservoir rock and fluid properties

(1) Influence of flow-rate

Figure (6.6) shows the influence of flow-rate on the limiting

water oil ratio. For a production rate of 500 (b/d), which is just above the critical flow rate, the limiting water oil ratio is small (.063); but as the flow rate is increased further, the limiting water oil ratio increases sharply.

In figure (6.7) we compare the curve obtained from our model (figure (6.6)), against the curves obtained from Bournazel and Jeanson's model with a single value for the effective permeability to water of 80 mD, and 110 mD. Agreement between our solution, and the solution obtained with $k_w = 80$ mD (throughout the reservoir) is fairly close, for production rates less than about 2000 b/d. One would expect this to be the case, as the permeability to water in the water swept part of the reservoir is likely to have more effect on the stabilized shape of the cone, than the permeability to water in the aquifer zone. However, for large super critical production rates our limiting water oil solution lies somewhere between the two limiting values obtained with the extreme values of the effective permeability to water (i.e. $k_w = 80$ and 110mD). Our results, therefore, indicate that the model of Bournazel and Jeanson (with k_w set to the permeability to water at residual oil saturation) is adequate for small super critical production rates, but fails for larger rates.

(ii) Influence of water thickness

Figure (6.8) shows the influence of water zone thickness on limiting water oil ratio. Its effect is seen to be significant for small thicknesses (for the range shown in the graph). When the water

zone thickness increases, the coning phenomenon tends towards bottom water drive.

(iii) Influence of well radius

In figure (6.9) we see the influence of well radius on limiting water oil ratio. It has little effect on the limiting WOR for small well radii. However, if the well radius is very large (greater than 2 feet) it can influence the water cut.

(iv) Influence of well penetration

The graph of well penetration against limiting water oil ratio is shown in figure (6.10). The graph shows that the minimum limiting water oil ratio is not necessarily obtained with very small well penetrations. For this particular reservoir the optimum penetration is 0.4, at which the limiting water oil ratio assumes a minimum value of 0.26. For well penetrations greater than the optimum value of 0.45, the water oil ratio steadily increases. In theory, the water oil ratio has a maximum value which is evaluated by allowing the well to penetrate the entire formation (both oil and water zones). Then assuming radial flow near the well the maximum limiting water oil ratio $w_{\max}^*$ is evaluated as follows:

$$w_{\max}^* = \frac{\frac{k_w}{\mu_w} h_w}{\frac{k_o}{\mu_o} h_o} = M^{(1)} \frac{h_w}{h_o} = 0.61875.$$

As for the boundary integral models, outlined in Chapters 3 and 4, our post-breakthrough coning model may be extended to include anisotropic reservoirs. As before, this is achieved by performing a co-ordinate transformation and reducing the anisotropic system to an equivalent isotropic one.

(v) Water oil ratio evolution

Bournazel and Jeanson's experimental correlation was used to study the water oil ratio evolution. From figure (6.5) we deduce that the limiting water oil ratio is reached when the dimensionless time $\bar{t}_D$, assume a value of approximately 63. Hence, for this reservoir the limiting water oil ratio was reached after about 3431 days.

6.9 Summary

A post-breakthrough coning model to predict the limiting water oil ratio has been developed, by extending the model by Bournazel and Jeanson. By introducing a new reservoir radius, r' , beyond which the oil water contact is assumed to be almost flat (η less than some prescribed value), two different, values for the effective permeability to water are used. The new model is, therefore, expected to better represent the real reservoir behaviour as it allows for the difference in the permeability to water in the aquifer and in the water swept zone (cone).

The model has been used on an example problem and the influence of various physical parameters on post-breakthrough behaviour has been investigated. The model is inexpensive and simple to use.

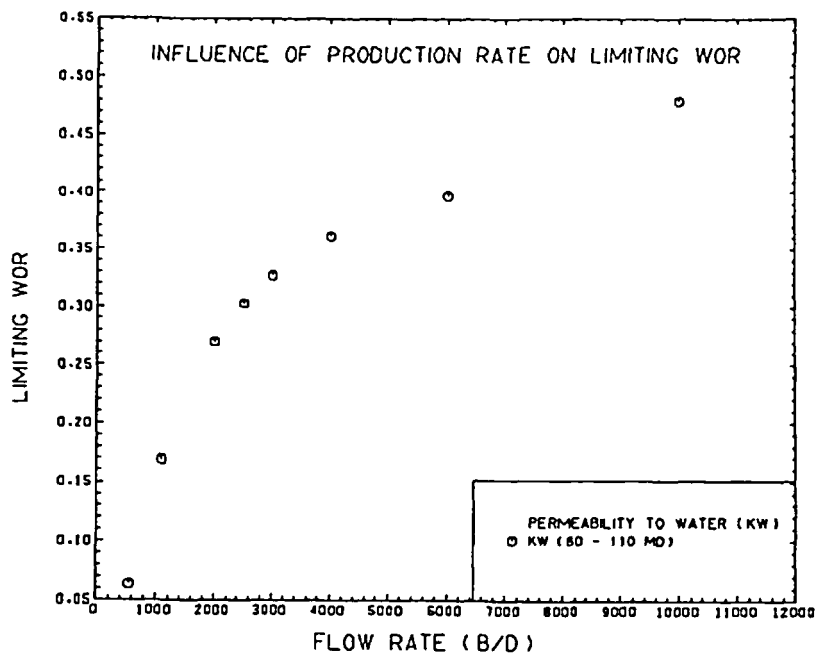


Figure 6.6: Influence of production rate on limiting WOR

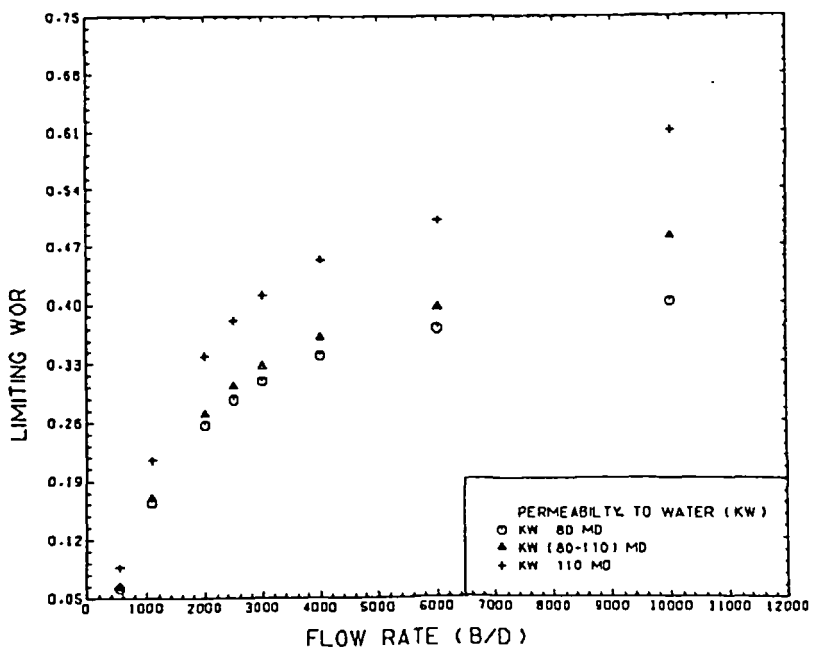


Figure 6.7: Comparison of curves showing the influence of production rate on WOR

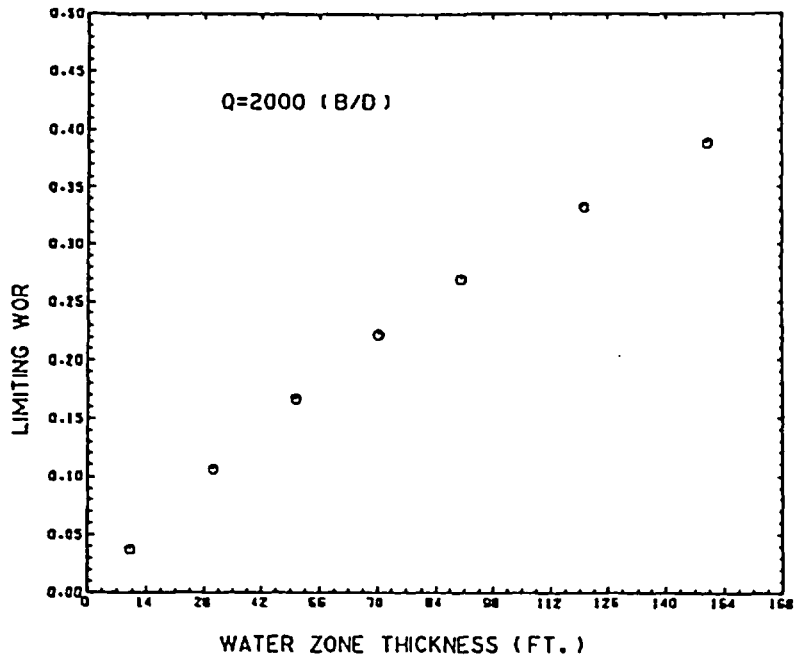


Figure 6.8: Influence of water zone thickness on limiting WOR

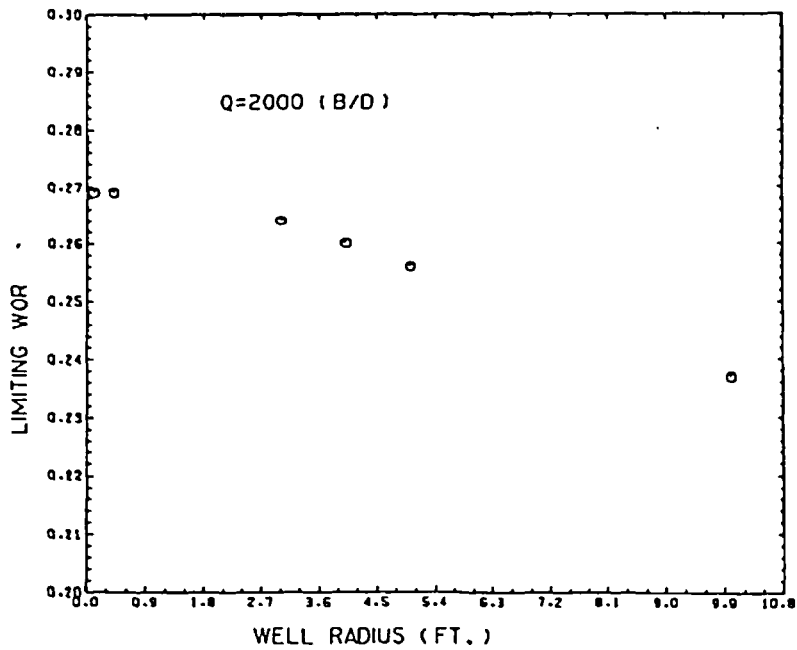


Figure 6.9: Influence of well radius on limiting WOR.

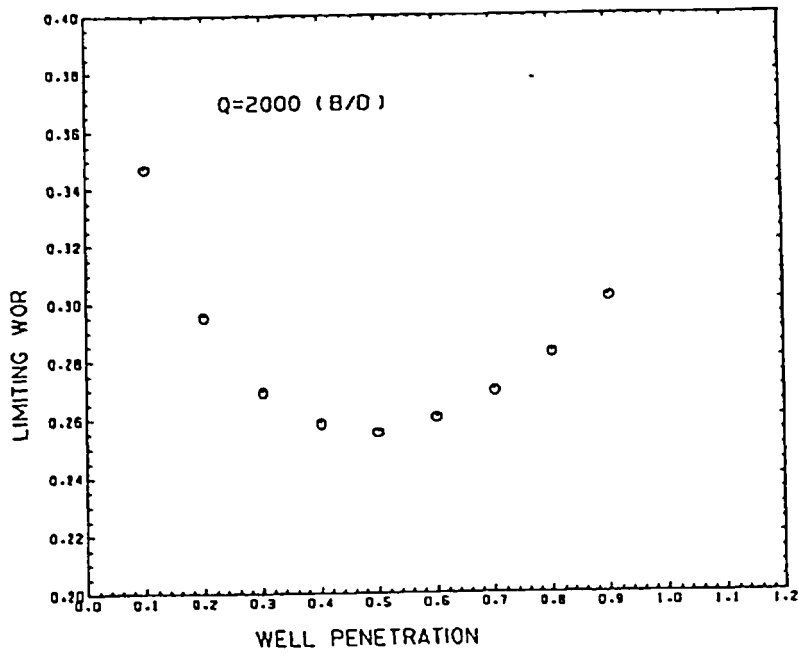


Figure 6.10: Influence of well penetration on limiting WOR

CHAPTER 7

DISCUSSION, SUGGESTIONS FOR FUTURE WORK AND CONCLUSIONS

7.1 Discussion

The methods of analysis presented in this thesis should be of direct application to reservoir coning problems. The direct semi-analytical methods, using properly defined boundary conditions, can give solutions which may be used as markers for numerical finite difference models.

In Chapter 2, a semi-analytical model was presented to analyse the problem of two dimensional, plane flow, up-coning. Although this study was performed here for purely theoretical considerations, it should find practical application in instances of near horizontal well production, where the problem of wedging (up-coning in the two-dimensional vertical plane) presents itself. A few fields with near horizontal well production are currently in existence.

The major part of this work was devoted, however, to the study of the radial coning problem. In Chapters 3 and 4, a mathematical model was developed to study coning behaviour up to the instant of cone instability and breakthrough. Perhaps the greatest asset of the proposed model is its accuracy. The mathematical analysis is performed by considering the singular and non-singular parts of the problem separately. The singular problem can be solved exactly and therefore it is necessary to solve only the non-singular problem numerically.

The boundary integral method can be utilized to this end. This technique, unlike conventional domain discretization methods (such as the finite difference or finite element method), does not discretize the governing equations describing the physical system, but transforms them into an equivalent set of integral equations, that are dependent on the boundary data only. Thus any approximations were introduced only to facilitate the numerical solution of the integral equations. By further mathematical manipulation the numerical computation can be reduced to the solution of the integral equations along the oil - water contact only. The proposal method, therefore, has the outstanding advantage of calculating and storing variables defined only at the oil water contact, which is precisely where the flow information is needed. Hence, for a given computer storage and machine time, the model achieves far greater accuracy than is possible by conventional domain discretization methods. The model therefore meets its objective of acting as an effective test for finite-difference coning simulators.

In Chapter 6, a steady-state, post-breakthrough model was developed by extending the work of Bournazel and Jeanson. The new model reflects, better, actual reservoir behaviour, as it allows for the change in the effective permeability to water, from the water swept zone to the aquifer zone. The technique is also simple to apply.

7.2 Extension of the boundary integral model to incorporate post-breakthrough behaviour

The boundary integral models, developed in Chapters 3 and 4,

have been shown to be effective in studying coning behaviour up to the instant of cone breakthrough. Their extension to post-breakthrough behaviour has not been possible, due not to a breakdown of the boundary integral method of solution but due to the complexities of the flow behaviour near the well, causing an incomplete definition of the boundary conditions. If a means of determining the constant value for the potential at the well surface can be established, by a careful investigation of the physical and mathematical systems, the equations and boundary conditions may possibly be completely specified (see Chapter 6). The basic potential could then be set to be the potential due to the flow created by the presence of two distinct line sinks, producing oil and water with strengths $\left(\frac{q}{1+w} \right)$ and $\left(\frac{qw}{1+w} \right)$ respectively. Thus if an expression for the basic potential is obtained, the perturbation oil and water potentials may be solved for using the boundary integral method of solution.

7.3 Application to layered reservoirs

In Chapters 3 and 4 of this thesis, we developed a new mathematical model based on the assumption that the reservoir medium was homogeneous (anisotropy allowed). In many practical situations the reservoir concerned contains a number of zones of porous media each having different but homogeneous properties. Thus the reservoir must be considered as a layered, but piecewise homogeneous porous media.

In this section we briefly discuss how the model proposed in Chapters 3 and 4 might possibly be extended to incorporate the

solution to coning problems in layered reservoirs, comprising any number of homogeneous zones.

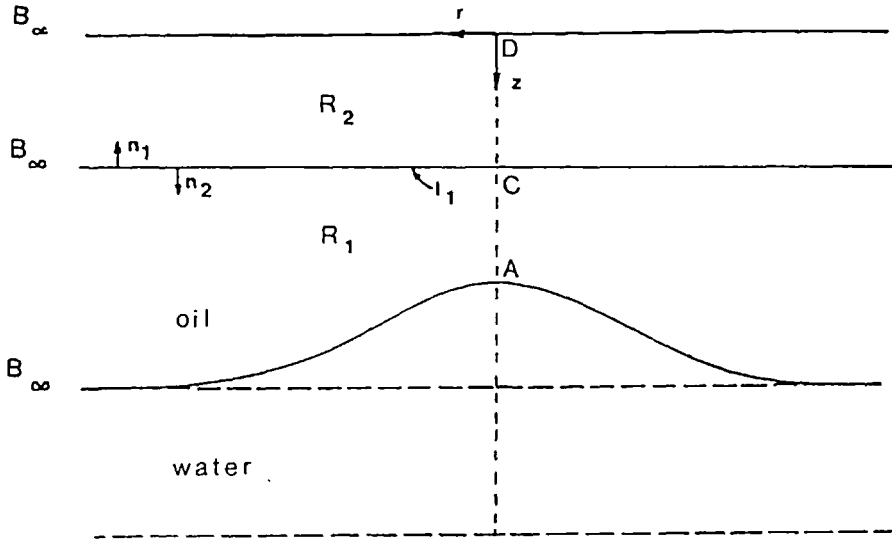


figure 7.1
2-layered reservoir system

Consider the two layered system shown in figure 7.1.

Let R_1 and R_2 denote two different zones with permeability k_1 and k_2 respectively, and let the contour CB_α represent the common interface between the two zones. On this common interface we will need to impose certain compatibility conditions. For instance the fluid head, $\phi = \frac{p}{\rho} + gz$, must be continuous. We shall also require that there be continuity of flow and hence continuity of velocity.

Thus,

$$\phi_2 = \left(\frac{k_2}{k_1} \right) \phi_1 \quad (7.10)$$

and,

$$\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} \quad (7.11)$$

where

$$\Phi_1 = \frac{k_1 \rho_1}{\mu_1} \left(\frac{p}{\rho_1} + gz \right) \quad \text{and} \quad \Phi = \frac{k_2 \rho_2}{\mu_2} \left(\frac{p}{\rho_2} + gz \right) .$$

Thus by formulating the boundary integral equations for regions R_1 and R_2 and by incorporating the compatibility conditions on the common interface CB_α a matrix system can be derived. Whereas the matrices which arise in the basic homogeneous equations are full, for layered reservoirs the equations are likely to lead to block-banded matrix systems, with one block for each zone and overlaps between blocks where any zones have a common interface.

Although the extension of the homogeneous case to the layered reservoir system is straightforward in its algorithmic approach, the computational effort required, for its implementation, is likely to increase with the increase in complexity of the reservoir system.

7.4 Concluding remarks

The boundary integral method of solution has been shown to be an accurate inexpensive and effective solution technique for the analysis of coning behaviour up to the instant of breakthrough. Perhaps, the most important extension to the work in this thesis would be in the application of the boundary integral equation method to the analysis of the post-breakthrough coning equations. A thorough investigation and understanding of the physics of flow near the well bore, in situations of simultaneous oil and water production would be necessary if this problem is to be solved.

Finally, as well as extending the understanding of the solution to coning problems, it is hoped that the methods developed in this thesis will aid in the analysis of other free-boundary type problems occurring in reservoir engineering.

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