

Long time behaviour and particle approximation of a generalized Vlasov dynamic

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Abstract

In this paper, we are interested in a generalised Vlasov equation, which describes the evolution of the probability density of a particle evolving according to a generalised Vlasov dynamic. The achievement of the paper is twofold. Firstly, we obtain a quantitative rate of convergence to the stationary solution in the Wasserstein metric. Secondly, we provide a many-particle approximation for the equation and show that the approximate system satisfies the propagation of chaos property.

Keywords:

Long time behaviour, particle approximation, propagation of chaos, generalised Vlasov dynamic, Wasserstein metric.

1. Introduction

1.1. The main equation

This paper is concerned with the long time behaviour and particle approximation of solutions of the following equation

$$\partial_t \rho_t = -p \cdot \nabla_q \rho_t + [\beta q + A(q) + B * \rho_t(q) - \lambda z] \cdot \nabla_p \rho_t + \operatorname{div}_z [(\lambda p + \alpha z) \rho_t] + \Delta_z \rho_t. \quad (1)$$

The spatial domain is $\mathbf{R}^{3d}$ with coordinates (q, p, z) , with q, p and z each in $\mathbf{R}^d$. Subscripts as in ∇_q and Δ_z are used to indicate that the differential operators act only on those variables. The unknown is a time-dependent probability measure $\rho: [0, T] \rightarrow \mathcal{P}(\mathbf{R}^{3d})$; A and B are given $\mathbf{R}^d$ -to- $\mathbf{R}^d$ maps; α, β and λ are given positive constants. Throughout the paper, ρ_t denotes

both the density of a probability measure at time t and the corresponding probability measure. Finally, the convolution $B * \rho_t(q)$ is defined by

$$B * \rho_t(q) = \int_{\mathbf{R}^{3d}} B(q - q') \rho_t(q', p, z) dq' dp dz. \quad (2)$$

It is well-known that Eq. (1) is the forward Kolmogorov equation of the following stochastic differential equation (SDE)

$$\begin{cases} dQ_t = P_t dt, \\ dP_t = -\beta Q_t dt - A(Q_t) dt - B * \rho_t(Q_t) dt + \lambda Z_t dt, \\ dZ_t = -\lambda P_t dt - \alpha Z_t dt + \sqrt{2} dW_t, \end{cases} \quad (3)$$

where $(W_t)_{t \geq 0}$ is the d -dimensional standard Wiener process.

The law of the $\mathbf{R}^{3d}$ -valued process $(Q_t, P_t, Z_t)_{t \geq 0}$ evolving according to the SDE (3) is a solution of Eq. (1) at time t . Systems of the form as in (1) and (3) have been studied by many authors. When $B = 0$ and A is a gradient of a potential $A = \nabla V$, (3) is called a generalized Langevin dynamic in [12, 15]. It is also a special case of the so-called Nose-Hoover-Langevin dynamic studied in [16, 13, 2, 11, 3]. In these papers (3) is considered as a stochastic perturbation, via the external heat bath variable Z_t , of the classical deterministic Hamiltonian system for (Q_t, P_t) . The generalised Langevin and the Nose-Hoover-Langevin dynamics play an important role in molecular dynamics because they maintain canonical sampling of the original Hamiltonian system while being ergodic. When $B \neq 0$, the involution term $B * \rho_t$, which is often known as Vlasov term or mean-field term in the literature, models the self-interaction of the underlying physical system. In this case, (3) belongs to the class of weakly interacting particle systems and have been used widely in applied sciences, especially in plasma physics, see e.g., [8, 9, 14, 10] and references therein. Combining both cases, the full dynamic (3) can also be seen as stochastic perturbation (via the augmented variable Z) of Vlasov's dynamic similarly as the generalised Langevin dynamic. This motivates the name in the title of the present paper.

The purpose of this paper is twofold. First, we prove the existence and uniqueness of a stationary solution as well as convergence to the stationary solution of other solutions of (1). Second, we provide a particle approximation of solutions of (1). We address both issues using the stochastic interpretation (3) of (1) and the so-called coupling technique introduced in [18, 20] and used further in [6, 5]. We now state the assumptions and main results of the present paper.

1.2. Assumptions and main results of the paper

Throughout the paper, we make the following assumption.

Assumption 1. B is an odd map on $\mathbf{R}^{3d}$. Furthermore, A and B are Lipschitz, i.e., there are non-negative constants C_A, C_B such that for all $q_1, q_2 \in \mathbf{R}^d$

$$|A(q_1) - A(q_2)| \leq C_A |q_1 - q_2|, \quad |B(q_1) - B(q_2)| \leq C_B |q_1 - q_2|. \quad (4)$$

The assumption that B is odd is motivated by the fact that in applications, B is often a gradient of a symmetric interaction potential which depends only on the distance of two different particles in the underlying physical system. The Lipschitz conditions (4) are standard assumptions to ensure the existence and uniqueness of the SDE (3) with square-integrable initial data, see e.g., [14]. It also guarantees the well-posedness of Eq. (1): it admits a unique measure solution $\rho_t \in \mathcal{P}_2(\mathbf{R}^{3d})$ for any $t > 0$ provided that the initial data $\rho_0 \in \mathcal{P}_2(\mathbf{R}^{3d})$, where $\mathcal{P}_2(\mathbf{R}^{3d})$ denotes the subspace of probability measures on $\mathbf{R}^{3d}$ with finite second moments.

The first result of the present paper is the following theorem concerning the existence and uniqueness of a stationary solution as well as convergence towards to the stationary solution of all solutions of (1). The trend to equilibrium is measured in terms of the Wasserstein distance W_2 on $\mathcal{P}_2(\mathbf{R}^{3d})$. In particular, $W_2(\mu, \nu)$, with $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^{3d})$, can be formulated as the *infimum* over a set of all coupled random variables that distribute according to μ and ν respectively (more details follow in Definition 12). The usage of the Wasserstein distance has two advantages. Firstly, it makes use of the relationship between the SDE (3) and Eq. (1), which is the underlying idea of the coupling technique, see Section 2. Secondly, the infimum formulation of the Wasserstein distance is convenient in estimating upper bounds, which is needed to prove convergence, by choosing any admissible couple of random variables.

Theorem 1. *Under Assumption 1, for all positive α, β and λ there exists a positive constant η_0 such that, if $0 \leq C_A + C_B < \eta_0$, then there exist positive constant C and C' such that*

$$W_2(\rho_t, \bar{\rho}_t) \leq C' e^{-Ct} W_2(\rho_0, \bar{\rho}_0), \quad \text{for all } t \geq 0, \quad (5)$$

for all solutions $(\rho_t)_{t \geq 0}$ and $(\bar{\rho}_t)_{t \geq 0}$ of (1) with respectively initial data ρ_0 and $\bar{\rho}_0$ in $\mathcal{P}_2(\mathbf{R}^{3d})$. Moreover, (1) has a unique stationary solution ρ_∞ and all solutions $(\rho_t)_{t \geq 0}$ converge towards it exponentially in the Wasserstein distance

$$W_2(\rho_t, \rho_\infty) \leq C' e^{-Ct} W_2(\rho_0, \rho_\infty). \quad (6)$$

We recall that C_A and C_B are respectively the Lipschitz constants of the confinement A and interaction B forces. The condition that $0 \leq C_A + C_B < \eta_0$ in the theorem means that they are linear-like forces, and that the interaction is weak enough. The proof of this theorem is given in Section 3.1.

We now describe the setup and state the second main result of the present paper. Let $(W^i)_{i \geq 1}$ be independent standard Brownian motions on $\mathbf{R}^d$, and $(Q_0^i, P_0^i, Z_0^i)_{i \geq 1}$ be independent random vectors on $\mathbf{R}^{3d}$ with law $\rho_0 \in \mathcal{P}(\mathbf{R}^{3d})$ and independent of $(W^i)_{i \geq 1}$. Let $(Q_t^{(N)}, P_t^{(N)}, Z_t^{(N)})_{t \geq 0} = (Q_t^{1,N}, \dots, Q_t^{N,N}, P_t^{1,N}, \dots, P_t^{N,N}, Z_t^{1,N}, \dots, Z_t^{N,N})_{t \geq 0}$ be the solution of the following SDE in $(\mathbf{R}^{3d})^N$:

$$\begin{cases} dQ_t^{i,N} = P_t^{i,N} dt, \\ dP_t^{i,N} = -\beta Q_t^{i,N} dt - A(Q_t^{i,N}) dt - \frac{1}{N} \sum_{j=1}^N B(Q_t^{i,N} - Q_t^{j,N}) dt + \lambda Z_t^{i,N} dt, \\ dZ_t^{i,N} = -\lambda P_t^{i,N} dt - \alpha Z_t^{i,N} dt + \sqrt{2} dW_t^i, \\ (Q_0^{i,N}, P_0^{i,N}, Z_0^{i,N}) = (Q_0^i, P_0^i, Z_0^i). \end{cases} \quad (7)$$

This is an interacting particle system since there is an interacting term $\frac{1}{N} \sum_{j=1}^N B(Q_t^{i,N} - Q_t^{j,N})$ in the SDE for each i . However, the interaction is of weak type in the sense that the interacting term for each i depends only on the empirical measure μ_t^N associated to the whole system

$$\frac{1}{N} \sum_{j=1}^N B(Q_t^{i,N} - Q_t^{j,N}) = \int_{\mathbf{R}^{3d}} B(Q_t^{i,N} - q) \mu_t^N(dq dp dz),$$

where

$$\mu_t^N(dq dp dz) := \frac{1}{N} \sum_{i=1}^N \delta_{(Q_t^{i,N}, P_t^{i,N}, Z_t^{i,N})}(dq dp dz). \quad (8)$$

On the other hand, we consider the following non-interacting particle

system, which is a many-particle version of the SDE (3)

$$\begin{cases} d\tilde{Q}_t^i = \tilde{P}_t^i dt, \\ d\tilde{P}_t^i = -\beta \tilde{Q}_t^i dt - A(\tilde{Q}_t^i) dt - B * \mu_t(\tilde{Q}_t^i) dt + \lambda \tilde{Z}_t^i dt, \\ d\tilde{Z}_t^i = -\lambda \tilde{P}_t^i dt - \alpha \tilde{Z}_t^i dt + \sqrt{2} dW_t^i, \\ (\tilde{Q}_0^i, \tilde{P}_0^i, \tilde{Z}_0^i) = (Q_0^i, P_0^i, Z_0^i), \end{cases} \quad (9)$$

where μ_t is the distribution of $(\tilde{Q}_t^i, \tilde{P}_t^i, \tilde{Z}_t^i)$. Since the initial data and the driving Brownian motions are independent $(Q_t^i, P_t^i, Z_t^i)_{t \geq 0}$ with $i \geq 1$ are independent. Furthermore, they are identically distributed and their common law evolves according to (1), so μ_t is the solution of (1) at time t with initial data ρ_0 . In comparison with the system in (7), the interacting term is replaced by $B * \mu_t(\tilde{Q}_t^i)$, i.e., integrating of $B(Q_t^{i,N} - \cdot)$ with respect to μ_t instead of the empirical measure μ_t^N .

In the second main result of the present paper, we prove that as N becomes larger and larger, the weakly interacting processes $(Q_t^{i,N}, P_t^{i,N}, Z_t^{i,N})_{t \geq 0}$ in (7) behaves more and more like the independent processes $(\tilde{Q}_t^i, \tilde{P}_t^i, \tilde{Z}_t^i)_{t \geq 0}$ in (9). In other words, the particle system (7) satisfies the property of propagation of chaos as defined in [17], which we recall in Section 2.

Theorem 2. *Let (Q_0^i, P_0^i, Z_0^i) , $i = 1, \dots, N$ be N independent $\mathbf{R}^{3d}$ -valued random variables with law $\rho_0 \in \mathcal{P}_2(\mathbf{R}^{3d})$. Let $(Q_t^{i,N}, P_t^{i,N}, Z_t^{i,N})_{t \geq 0, 1 \leq i \leq N}$ and $(\tilde{Q}_t^i, \tilde{P}_t^i, \tilde{Z}_t^i)_{t \geq 0}$ respectively be the solution of (7) and (9) with initial datum (Q_0^i, P_0^i, Z_0^i) , $i = 1, \dots, N$. Under Assumption 1, for all positive α, β, λ there exists a positive constant η_0 such that, if $0 \leq C_A + C_B < \eta_0$, then there exists a positive constant C , independent of N , such that for all $i = 1, \dots, N$*

$$\sup_{t \geq 0} \mathbb{E} \left(\left| Q_t^{i,N} - \tilde{Q}_t^i \right|^2 + \left| P_t^{i,N} - \tilde{P}_t^i \right|^2 + \left| Z_t^{i,N} - \tilde{Z}_t^i \right|^2 \right) \leq \frac{C}{N}. \quad (10)$$

As a consequence, the particle system (7) satisfies the property of propagation of chaos. In addition, the law $\rho_t^{(1,N)}$ of any $(Q_t^{i,N}, P_t^{i,N}, Z_t^{i,N})$ converges to the law μ_t of $(\tilde{Q}_t^i, \tilde{P}_t^i, \tilde{Z}_t^i)$ uniformly in t and in the Wasserstein metric

$$\sup_{t \geq 0} W_2(\rho_t^{(1,N)}, \mu_t)^2 \leq \frac{C}{N}. \quad (11)$$

We stress that this theorem not only proves the convergence but also provides a quantitative rate of convergence. This theorem is proved in Section 3.2.

1.3. Organization of the paper

The rest of the paper is organized as follows. In Section 2, we recall some preliminary knowledge. We present the proof of Theorem 1 in Section 3.1 and that of Theorem 2 in Section 3.2.

2. Preliminaries

2.1. Wasserstein metric

We recall that $\mathcal{P}_2(\mathbf{R}^{3d})$ denotes the space of probability measures μ on $\mathbf{R}^{3d}$ with finite second moment, i.e.,

$$\int_{\mathbf{R}^{3d}} [|q|^2 + |p|^2 + |z|^2] \mu(dqdpdz) < \infty.$$

The Wasserstein distance W_2 between two probability measures $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^{3d})$ is defined via

$$W_2(\mu, \nu)^2 = \inf_{\Pi} \left\{ \mathbb{E}(|Q - \tilde{Q}|^2 + |P - \tilde{P}|^2 + |Z - \tilde{Z}|^2) : \right. \\ \left. \text{Law}(Q, P, Z) = \mu, \text{Law}(\tilde{Q}, \tilde{P}, \tilde{Z}) = \nu \right\}, \quad (12)$$

where the infimum is taken over all joint distributions Π with marginals μ and ν respectively [21, Definition 6.1]. We note that convergence in Wasserstein metric is equivalent to narrow convergence (i.e., tested against continuous and bounded functions) plus convergence of the second moments. The Wasserstein metric plays an important role in many fields of mathematics such as optimal transport and dissipative evolution equations. We refer to [19, 21] and [1] for expositions on the topic.

More generally, given a positive definite quadratic form $\mathcal{Q}$ on $\mathbf{R}^{3d}$, one can define a $\mathcal{Q}$ -Wasserstein distance $W_{2\mathcal{Q}}(\mu, \nu)$ between $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^{3d})$ via

$$W_{2\mathcal{Q}}(\mu, \nu)^2 = \inf_{\Pi} \left\{ \mathbb{E}\mathcal{Q}(Q - \tilde{Q}, P - \tilde{P}, Z - \tilde{Z}) : \right. \\ \left. \text{Law}(Q, P, Z) = \mu, \text{Law}(\tilde{Q}, \tilde{P}, \tilde{Z}) = \nu \right\}, \quad (13)$$

where the infimum is taken over all joint distributions Π with marginals μ and ν respectively. Moreover, since $\sqrt{\mathcal{Q}}$ is equivalent to the Euclidean norm $\sqrt{|q|^2 + |p|^2 + |z|^2}$, there exist positive constants C and C' such that

$$CW_2(\mu, \nu) \leq W_{2\mathcal{Q}}(\mu, \nu) \leq C'W_2(\mu, \nu), \quad (14)$$

for $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^{3d})$.

The underlying idea of the coupling technique introduced in [18, 20] and used further in [6, 5] is based on (14). First, we find a positive definite quadratic form $\mathcal{Q}$ on $\mathbf{R}^{3d}$ such that (3) is dissipative with respect to $\mathcal{Q}$. Then we apply (14) to obtain estimations in terms of the Wasserstein metric. We note that the Wasserstein distance can be defined via a variety of different ways, but for the purpose of this paper, the definition above will be the most useful because of two reasons as already mentioned in the introduction: it is facilitated with the coupling technique and is convenient in obtaining upper bound estimates.

2.2. Propagation of chaos

We now recall the definition of propagation of chaos of a particle system introduced in [17]. Let $X_t^{i,N} := (Q_t^{i,N}, P_t^{i,N}, Z_t^{i,N})$, and denote $dx^{i,N} = dq^{i,N} dp^{i,N} dz^{i,N}$ for $i = 1, \dots, N$. Let $\rho_t^{(N)}(dx^{1,N}, \dots, dx^{N,N})$ be the joint law of $(X^{1,N}, \dots, X^{N,N})$ where for each $i = 1, \dots, N$, $X_t^{i,N}$ satisfies (7). Let $k \geq 1$ be a fixed integer, and let $(i_1, \dots, i_k)$ be any k -uple of $[1, N]$. The $(i_1, \dots, i_k)$ -marginal $\rho_t^{(i_1, \dots, i_k)}$ of $\rho_t^{(N)}$ is defined via

$$\rho_t^{(i_1, \dots, i_k)}(A_1 \times \dots \times A_k) := \rho_t^{(N)}(\mathbf{R} \times \dots \times A_1 \times \dots \times A_k \times \dots \times \mathbf{R}),$$

for all Borel sets $A_1, \dots, A_k \subset \mathcal{B}(\mathbf{R})$ and in the right hand side of the above expression A_j is at position i_j for $1 \leq j \leq k$. We recall that μ_t is the law of $(\tilde{Q}_t^1, \tilde{P}_t^1, \tilde{Z}_t^1)$, which solves Eq. (1). The particle system (7) is said to satisfy the property of propagation of chaos with respect to μ_t if $\rho^{(i_1, \dots, i_k)}$ converges narrowly to $(\mu_t)^{\otimes k}$ for all k -uple $(i_1, \dots, i_k)$. According to [17, Proposition 2.2], this is equivalent to the statement that the empirical measure μ_t^N converges narrowly to μ_t . We refer to [17] for more information on the topic.

3. Proofs of the main theorems

The proofs of Theorem 1 and Theorem 2 are presented respectively in Section 3.1 and Section 3.2. The proofs follow the procedure of [6, 5] (see also [18, 20]) which use the stochastic interpretation (3) of (1) and the coupling technique explained in the previous section.

3.1. *Stationary solution and long time behaviour of the main equation: proof of Theorem 1*

The main ingredient of the proof of Theorem 1 is the following proposition, which is the content of the coupling technique.

Proposition 1. Under the assumption of Theorem 1, there exists a positive constant C and a positive definite quadratic form $\mathcal{Q}$ on $\mathbf{R}^{3d}$ such that

$$W_{2\mathcal{Q}}(\rho_t, \bar{\rho}_t) \leq e^{-Ct} W_{2\mathcal{Q}}(\rho_0, \bar{\rho}_0), \quad \text{for all } t \geq 0. \quad (15)$$

Proof. Let (Q_t, V_t, Z_t) and $(\bar{Q}_t, \bar{P}_t, \bar{Z}_t)$ be two $\mathbf{R}^{3d}$ -valued stochastic processes evolving according to (3) with the same Brownian motion $(W_t)_{t \geq 0}$ in $\mathbf{R}^d$. Let $(q_t, p_t, z_t) := (Q_t - \bar{Q}_t, P_t - \bar{P}_t, Z_t - \bar{Z}_t)$ be the difference between them. Then (q_t, p_t, z_t) evolves according to

$$\begin{aligned} dq_t &= p_t dt, \\ dp_t &= -\beta q_t dt - (A(Q_t) - A(\bar{Q}_t)) dt - (B * \rho_t(Q_t) - B * \bar{\rho}_t(\bar{Q}_t)) dt + \lambda z_t dt, \\ dz_t &= -\lambda p_t dt - \alpha z_t dt. \end{aligned} \quad (16)$$

Let $a_1, \dots, a_5$ be positive constants, which will be specified later on. Define

$$\mathcal{Q}(q, p, z) = a_1 |q|^2 + a_2 |p|^2 + a_3 |z|^2 + 2a_4 q \cdot p + 2a_5 q \cdot z + 2p \cdot z. \quad (17)$$

Assume that (q_t, p_t, z_t) evolves according to (16). Then we have

$$\begin{aligned} & \frac{d}{dt} \mathcal{Q}(q_t, p_t, z_t) \\ &= \frac{d}{dt} (a_1 |q_t|^2 + a_2 |p_t|^2 + a_3 |z_t|^2 + 2a_4 q_t \cdot p_t + 2a_5 q_t \cdot z_t + 2p_t \cdot z_t) \\ &= 2a_1 q_t \cdot p_t + 2a_2 p_t \cdot [-\beta q_t - (A(Q_t) - A(\bar{Q}_t)) - (B * \rho_t(Q_t) - B * \bar{\rho}_t(\bar{Q}_t)) + \lambda z_t] \\ & \quad + 2a_3 z_t \cdot [-\lambda p_t - \alpha z_t] + 2a_4 |p_t|^2 \\ & \quad + 2a_4 q_t \cdot [-\beta q_t - (A(Q_t) - A(\bar{Q}_t)) - (B * \rho_t(Q_t) - B * \bar{\rho}_t(\bar{Q}_t)) + \lambda z_t] \\ & \quad + 2a_5 p_t \cdot z_t + 2a_5 q_t \cdot [-\lambda p_t - \alpha z_t] + 2p_t \cdot [-\lambda p_t - \alpha z_t] \\ & \quad + 2z_t \cdot [-\beta q_t - (A(Q_t) - A(\bar{Q}_t)) - (B * \rho_t(Q_t) - B * \bar{\rho}_t(\bar{Q}_t)) + \lambda z_t] \\ &= (2a_1 - 2\lambda a_5 - 2a_2 \beta) q_t \cdot p_t + (2\lambda a_4 - 2\alpha a_5 - 2\beta) q_t \cdot z_t \\ & \quad + (2\lambda a_2 - 2\lambda a_3 + 2a_5 - 2\alpha) p_t \cdot z_t - 2\beta a_4 |q_t|^2 - (2\lambda - 2a_4) |p_t|^2 \\ & \quad - (2\alpha a_3 - 2\lambda) |z_t|^2 - (2a_2 p_t + 2a_4 q_t + 2z_t)(A(Q_t) - A(\bar{Q}_t)) \\ & \quad - (2a_2 p_t + 2a_4 q_t + 2z_t)(B * \rho_t(Q_t) - B * \bar{\rho}_t(\bar{Q}_t)). \end{aligned} \quad (18)$$

Now we estimate the last two terms on the right-hand side of (18) using the assumptions on A and B . The first term is bounded from above by

$$\begin{aligned}
& - (2a_2p_t + 2a_4q_t + 2z_t)(A(Q_t) - A(\bar{Q}_t)) \\
& \leq 2a_2C_A|q_t||p_t| + 2a_4C_A|q_t|^2 + 2C_A|q_t||z_t| \\
& \leq a_2C_A(|q_t|^2 + |p_t|^2) + 2a_4C_A|q_t|^2 + C_A(|q_t|^2 + |z_t|^2) \\
& = (a_2 + 2a_4 + 1)C_A|q_t|^2 + a_2C_A|p_t|^2 + C_A|z_t|^2.
\end{aligned} \tag{19}$$

The second term is a bit more intricate. Let Π_t be the joint law of $(Q_t, P_t, Z_t; \bar{Q}_t, \bar{P}_t, \bar{Z}_t)$ on $\mathbf{R}^{3d} \times \mathbf{R}^{3d}$. Then its marginals on $\mathbf{R}^{3d}$ are distributions ρ_t and $\bar{\rho}_t$ of (Q_t, P_t, Z_t) and $(\bar{Q}_t, \bar{P}_t, \bar{Z}_t)$ respectively. We have

$$\begin{aligned}
B * \rho_t(Q_t) - B * \bar{\rho}_t(\bar{Q}_t) &= \int_{\mathbf{R}^{3d}} B(Q_t - q) d\rho_t(q, p, z) - \int_{\mathbf{R}^{3d}} B(\bar{Q}_t - \bar{q}) d\bar{\rho}_t(\bar{q}, \bar{p}, \bar{z}) \\
&= \int_{\mathbf{R}^{6d}} (B(Q_t - q) - B(\bar{Q}_t - \bar{q})) d\Pi_t(q, p, z; \bar{q}, \bar{p}, \bar{z}).
\end{aligned}$$

Hence

$$\begin{aligned}
& -2\mathbb{E} [q_t \cdot (B * \rho_t(Q_t) - B * \bar{\rho}_t(\bar{Q}_t))] \\
&= -2 \int_{\mathbf{R}^{12d}} (Q - \bar{Q}) \cdot (B(Q - q) - B(\bar{Q} - \bar{q})) d\Pi_t(Q, P, Z; \bar{Q}, \bar{P}, \bar{Z}) d\Pi_t(q, p, z; \bar{q}, \bar{p}, \bar{z}) \\
&\stackrel{(*)}{=} - \int_{\mathbf{R}^{12d}} ((Q - q) - (\bar{Q} - \bar{q})) \cdot (B(Q - q) - B(\bar{Q} - \bar{q})) d\Pi_t(Q, P, Z; \bar{Q}, \bar{P}, \bar{Z}) d\Pi_t(q, p, z; \bar{q}, \bar{p}, \bar{z}) \\
&\leq C_B \int_{\mathbf{R}^{12d}} |(Q - q) - (\bar{Q} - \bar{q})|^2 d\Pi_t(Q, P, Z; \bar{Q}, \bar{P}, \bar{Z}) d\Pi_t(q, p, z; \bar{q}, \bar{p}, \bar{z}) \\
&= 2C_B \left[\int_{\mathbf{R}^{6d}} |Q - \bar{Q}|^2 d\Pi_t(Q, P, Z; \bar{Q}, \bar{P}, \bar{Z}) - \left(\int_{\mathbf{R}^{6d}} (Q - \bar{Q}) d\Pi_t(Q, P, Z; \bar{Q}, \bar{P}, \bar{Z}) \right)^2 \right] \\
&\leq 2C_B \mathbb{E}|q_t|^2,
\end{aligned}$$

where we have used the fact that B is odd to obtain the equality $(*)$. So, we have just proved that

$$-\mathbb{E} [q_t \cdot (B * \rho_t(Q_t) - B * \bar{\rho}_t(\bar{Q}_t))] \leq C_B \mathbb{E}|q_t|^2. \tag{20}$$

Similarly, we have

$$\begin{aligned}
& -2\mathbb{E} [p_t \cdot (B * \rho_t(Q_t) - B * \bar{\rho}_t(\bar{Q}_t))] \\
& = -2 \int_{\mathbf{R}^{12d}} (P - \bar{P}) \cdot (B(Q - q) - B(\bar{Q} - \bar{q})) d\Pi_t(q, p, z; \bar{q}, \bar{p}, \bar{z}) d\Pi_t(Q, P, Z; \bar{Q}, \bar{P}, \bar{Z}) \\
& = - \int_{\mathbf{R}^{12d}} ((P - p) - (\bar{P} - \bar{p})) \cdot (B(Q - q) - B(\bar{Q} - \bar{q})) d\Pi_t(q, p, z; \bar{q}, \bar{p}, \bar{z}) d\Pi_t(Q, P, Z; \bar{Q}, \bar{P}, \bar{Z}) \\
& \leq C_B \int_{\mathbf{R}^{12d}} |(P - p) - (\bar{P} - \bar{p})| \cdot |(Q - q) - (\bar{Q} - \bar{q})| d\Pi_t(q, p, z; \bar{q}, \bar{p}, \bar{z}) d\Pi_t(Q, P, Z; \bar{Q}, \bar{P}, \bar{Z}) \\
& \leq \frac{C_B}{2} \int_{\mathbf{R}^{12d}} [| (P - p) - (\bar{P} - \bar{p}) |^2 + | (Q - q) - (\bar{Q} - \bar{q}) |^2] d\Pi_t(q, p, z; \bar{q}, \bar{p}, \bar{z}) d\Pi_t(Q, P, Z; \bar{Q}, \bar{P}, \bar{Z}) \\
& = \frac{C_B}{2} \int_{\mathbf{R}^{12d}} [| (P - \bar{P}) - (p - \bar{p}) |^2 + | (Q - \bar{Q}) - (q - \bar{q}) |^2] d\Pi_t(q, p, z; \bar{q}, \bar{p}, \bar{z}) d\Pi_t(Q, P, Z; \bar{Q}, \bar{P}, \bar{Z}) \\
& \leq C_B \mathbb{E}(|q_t|^2 + |p_t|^2),
\end{aligned}$$

i.e.,

$$-2\mathbb{E} [p_t \cdot (B * \rho_t(Q_t) - B * f[\bar{\rho}_t](\bar{Q}_t))] \leq C_B \mathbb{E}(|q_t|^2 + |p_t|^2). \quad (21)$$

In the same way, we also obtain

$$-2\mathbb{E} [z_t \cdot (B * \rho_t(Q_t) - B * \bar{\rho}_t(\bar{Q}_t))] \leq C_B \mathbb{E}(|q_t|^2 + |z_t|^2). \quad (22)$$

Substituting estimates from (19) to (22) into (18), we get

$$\begin{aligned}
\frac{d}{dt} \mathcal{Q}(q_t, p_t, z_t) & \leq (2a_1 - 2\lambda a_5 - 2a_2\beta) q_t \cdot p_t + [2\lambda a_4 - 2\alpha a_5 - 2\beta] q_t \cdot z_t \\
& \quad + [2\lambda a_2 - 2\lambda a_3 + 2a_5 - 2\alpha] p_t \cdot z_t - [2\beta a_4 - (a_2 + 2a_4 + 1)(C_A + C_B)] |q_t|^2 \\
& \quad - [2\lambda - 2a_4 - a_2(C_A + C_B)] |p_t|^2 - [2\alpha a_3 - 2\lambda - (C_A + C_B)] |z_t|^2.
\end{aligned} \quad (23)$$

Set $\eta := C_A + C_B$. We choose $a_1, \dots, a_5$ such that

$$\begin{cases} 2a_1 - 2\lambda a_5 - 2a_2\beta = 0, \\ 2\lambda a_4 - 2\alpha a_5 - 2\beta = 0, \\ 2\lambda a_2 - 2\lambda a_3 + 2a_5 - 2\alpha = 0, \\ 2\beta a_4 - (a_2 + 2a_4 + 1)\eta > 0, \\ 2\lambda - 2a_4 - a_2\eta > 0, \\ 2\alpha a_3 - 2\lambda - \eta > 0, \end{cases} \quad (24)$$

and that $\mathcal{Q}(q, p, z)$ is a positive definite quadratic form on $\mathbf{R}^{3d}$. Firstly, we choose $a_4 = \frac{\lambda}{2}$ and express a_1, a_2, a_5 in terms of a_3 . Then we show that there exists a_3 and $\eta_0 > 0$ such that all the requirements are fulfilled for $0 < \eta < \eta_0$. Conditions in (24) are equivalent to

$$\begin{cases} a_5 = \frac{\lambda a_4 - \beta}{\alpha} = \frac{\lambda^2/2 - \beta}{\alpha}, \\ a_2 = a_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda}, \\ a_1 = \lambda a_5 + \beta a_2 = \lambda a_5 + \beta \left(a_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda} \right), \\ 2\beta a_4 - (a_2 + 2a_4 + 1)\eta = \lambda\beta - \left(a_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda} + \lambda + 1 \right)\eta > 0, \\ 2\lambda - 2a_4 - a_2\eta = \lambda - \left(a_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda} \right)\eta > 0, \\ 2\alpha a_3 - 2\lambda - \eta > 0. \end{cases} \quad (25)$$

Let $a_3 = 2 + \tilde{a}_3$, where $\tilde{a}_3$ will be chosen later on. Then we have

$$\begin{aligned} \mathcal{Q}(q, p, z) &= \left(a_5 \left(\lambda - \frac{\beta}{\lambda} \right) + \beta \left(a_3 + \frac{\alpha}{\lambda} \right) \right) |q|^2 + \left(a_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda} \right) |p|^2 + a_3 |z|^2 + \lambda q \cdot p \\ &\quad - 2a_5 q \cdot z + 2p \cdot z \\ &= \left(a_5 \left(\lambda - \frac{\beta}{\lambda} \right) + \beta (\tilde{a}_3 + \frac{\alpha}{\lambda} + 2) \right) |q|^2 + \left(\tilde{a}_3 + \frac{\alpha}{\lambda} + 2 - \frac{a_5}{\lambda} \right) |p|^2 + (2 + \tilde{a}_3) |z|^2 \\ &\quad + \lambda q \cdot p - 2a_5 q \cdot z + 2p \cdot z \\ &= \left(a_5 \left(\lambda - \frac{\beta}{\lambda} \right) + \beta (\tilde{a}_3 + \frac{\alpha}{\lambda} + 2) - a_5^2 - \frac{\lambda^2}{4} \right) |q|^2 + \left(\tilde{a}_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda} \right) |p|^2 + \tilde{a}_3 |z|^2 \\ &\quad + \left(\frac{\lambda}{2} q + p \right)^2 + (a_5 q + z)^2 + (p + z)^2. \end{aligned}$$

Now we choose $\tilde{a}_3$ such that

$$\begin{cases} f_1(\tilde{a}_3) := a_5 \left(\lambda - \frac{\beta}{\lambda} \right) + \beta (\tilde{a}_3 + \frac{\alpha}{\lambda} + 2) - a_5^2 - \frac{\lambda^2}{4} > 0, \\ f_2(\tilde{a}_3) := \tilde{a}_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda} > 0, \\ f_3(\tilde{a}_3) := \tilde{a}_3 > 0, \\ f_4(\tilde{a}_3) := \tilde{a}_3 + \frac{\alpha}{\lambda} + \lambda + 3 - \frac{a_5}{\lambda} > 0, \\ f_5(\tilde{a}_3) := 2\alpha(2 + \tilde{a}_3) - 2\lambda > 0. \end{cases} \quad (26)$$

Since all f_i , with $i = 1, \dots, 5$, are linear functions of $\tilde{a}_3$ with positive slopes, it is obvious that there exists $\tilde{a}_3$ that is large enough so that all of them are positive. We choose such a $\tilde{a}_3$ and define

$$\eta_0 := \min \left\{ \frac{\lambda\beta}{f_4(\tilde{a}_3)}, \frac{\lambda}{2 + f_2(\tilde{a}_3)}, f_5(\tilde{a}_3) \right\} > 0. \quad (27)$$

It then follows from (26) and (27) that (25) are fulfilled for every $0 < \eta < \eta_0$ and $\mathcal{Q}$ is a positive definite quadratic on $\mathbf{R}^{3d}$.

From (23) it follows that there exists a positive constant C , depending only on $\alpha, \beta, \lambda, C_A$ and C_B such that

$$\frac{d}{dt} \mathbb{E} \mathcal{Q}(q_t, p_t, z_t) \leq -C \mathbb{E} [|q_t|^2 + |p_t|^2 + |z_t|^2], \quad \text{for all } t \geq 0. \quad (28)$$

Since in a finite dimensional space, all norms are equivalent to the Euclidean norm, the right hand side of (28) is bounded above by $-C \mathbb{E} \mathcal{Q}(q_t, p_t, z_t)$ for some positive constant C , i.e.,

$$\frac{d}{dt} \mathbb{E} \mathcal{Q}(q_t, p_t, z_t) \leq -C \mathbb{E} \mathcal{Q}(q_t, p_t, z_t), \quad \text{for all } t \geq 0. \quad (29)$$

By Gronwall's inequality, we obtain that

$$\mathbb{E} \mathcal{Q}(q_t, p_t, z_t) \leq e^{-Ct} \mathbb{E} \mathcal{Q}(q_0, p_0, z_0), \quad \text{for all } t \geq 0.$$

We can re-write the above inequality using definition of (q_t, p_t, z_t) as follows

$$\mathbb{E} \mathcal{Q}((Q_t, P_t, Z_t) - (\bar{Q}_t, \bar{P}_t, \bar{Z}_t)) \leq e^{-Ct} \mathbb{E} \mathcal{Q}((Q_0, P_0, Z_0) - (\bar{Q}_0, \bar{P}_0, \bar{Z}_0)),$$

for all $t \geq 0$. We then proceed as follows

$$\begin{aligned} W_{2\mathcal{Q}}(\rho_t, \bar{\rho}_t)^2 &\leq \inf_{(Q_0, P_0, Z_0), (\bar{Q}_0, \bar{P}_0, \bar{Z}_0)} \mathbb{E} \mathcal{Q}((Q_t, P_t, Z_t) - (\bar{Q}_t, \bar{P}_t, \bar{Z}_t)) \\ &\quad e^{-Ct} \inf_{(Q_0, P_0, Z_0), (\bar{Q}_0, \bar{P}_0, \bar{Z}_0)} \mathbb{E} \mathcal{Q}((Q_0, P_0, Z_0) - (\bar{Q}_0, \bar{P}_0, \bar{Z}_0)) \\ &= e^{-Ct} W_{2\mathcal{Q}}(\rho_0, \bar{\rho}_0)^2, \end{aligned}$$

which is equivalent to (15).

This completes the proof of the proposition. $\square$

The following lemma will be used later to prove the existence and uniqueness of the stationary measure.

Lemma 3. [7, Lemma 7.3] *Let $(\mathcal{M}, \text{dist})$ be a complete metric space and $S(t)$ be a continuous semigroup on $(\mathcal{M}, \text{dist})$. Assume that there exists $0 < L(t) < 1$ such that*

$$\text{dist}(S(t)(x), S(t)(y)) \leq L(t) \text{dist}(x, y),$$

for all $t > 0$, and x, y in $\mathcal{M}$. Then there exists a unique stationary point $x_\infty \in \mathcal{M}$, i.e., $S(t)(x_\infty) = x_\infty$ for all $t \geq 0$.

We are now ready to present the proof of Theorem 1.

Proof of Theorem 1. Let $\mathcal{Q}(q, p, z)$ be the positive definite quadratic form on $\mathbf{R}^{3d}$ obtained in Proposition 1. Since $\mathcal{Q}$ is equivalent to the canonical form $|q|^2 + |p|^2 + |z|^2$, there exists positive constants C' and C'' such that

$$W_2(\rho_t, \bar{\rho}_t) \leq C'' W_{2\mathcal{Q}}(\rho_t, \bar{\rho}_t) \leq C'' e^{-C't} W_{2\mathcal{Q}}(\rho_0, \bar{\rho}_0) \leq C' e^{-C't} W_2(\rho_0, \bar{\rho}_0),$$

for all $t \geq 0$ and all solutions $(\rho_t)_{t \geq 0}$ and $(\bar{\rho}_t)_{t \geq 0}$ of (1) by Proposition 1. This proves the first assertion (5) of Theorem 1. Since $\sqrt{\mathcal{Q}}$ is a norm on $\mathbf{R}^{3d}$, which is equivalent to the Euclidean norm $\sqrt{|q|^2 + |p|^2 + |z|^2}$, according to [4], the space $(\mathcal{P}_2(\mathbf{R}^{3d}), W_{2\mathcal{Q}})$ is a complete metric space. The existence of a unique stationary measure ρ_∞ is then followed from the contraction property of Proposition 1 and Lemma 3. Moreover, taking $(\bar{\rho}_t)_{t \geq 0} \equiv \rho_\infty$ in (5) we obtain (6). $\square$

3.2. Particle approximation of the main equation: proof of Theorem 2

We start with proving that the second moments of all solutions of (1) are finite provided that the second moment of the initial data is finite.

Lemma 4. *Under Assumption 1, for all positive α, β and λ there exists a positive constant η_0 such that, if $0 \leq C_A + C_B < \eta_0$, then $\sup_{t \geq 0} \int_{\mathbf{R}^{3d}} (|q|^2 + |p|^2 + |z|^2) d\rho_t(q, p, z)$ is finite for all solutions $(\rho_t)_{t \geq 0}$ of (1) with initial data $\rho_0 \in \mathcal{P}_2(\mathbf{R}^{3d})$.*

Proof. The assertion of this lemma can be obtained using Theorem 1. Indeed, let ρ_∞ be the unique stationary measures of (1) obtained in Theorem 1. For all solutions $(\rho_t)_{t \geq 0}$ of (1) with initial data $\rho_0 \in \mathcal{P}_2(\mathbf{R}^{3d})$, from Theorem 1, we have for all $t \geq 0$

$$\begin{aligned} \int_{\mathbf{R}^{3d}} (|q|^2 + |p|^2 + |z|^2) d\rho_t(q, p, z) &= W_2(\rho_t, \delta_0)^2 \\ &\leq 2 (W_2(\rho_t, \rho_\infty)^2 + W_2(\delta_0, \rho_\infty)^2) \\ &\stackrel{(6)}{\leq} 2 (C' e^{-C't} W_2(\rho_0, \rho_\infty)^2 + W_2(\delta_0, \rho_\infty)^2) \\ &\leq 2 (C' W_2(\rho_0, \rho_\infty)^2 + W_2(\delta_0, \rho_\infty)^2). \end{aligned}$$

$\square$

Remark 1. It is worth pointing out that the finiteness of the second moments, and hence the property of propagation of chaos of the system (7), is independent of the long time behaviour of the solutions. Lemma 4 can be proved directly without using Theorem 1 but using again the coupling technique. Since it is similar to the proof of Theorem 1, we only sketch the main steps here. This proof also demonstrates further the usefulness of the coupling technique.

A sketch of proof of Lemma 4 using coupling technique. Let ρ_t be a solution of (1) with initial data ρ_0 in $\mathcal{P}_2(\mathbf{R}^{3d})$.

Step 1) We define

$$\mathcal{Q}(q, p, z) = a_1|q|^2 + a_2|p|^2 + a_3|z|^2 + 2a_4q \cdot p + 2a_5q \cdot z + 2p \cdot z, \quad (30)$$

where $a_1, \dots, a_5$ be positive constants, which will be specified later on.

Step 2) We compute $\frac{d}{dt} \int_{\mathbf{R}^{3d}} \mathcal{Q}(q, p, z) \rho_t(dqdpdz)$. Similarly as in the proof of Theorem 1, there exist positive constants $a_1, \dots, a_5$ such that $\mathcal{Q}(q, p, z)$ is a positive definite quadratic form on $\mathbf{R}^{3d}$ and that

$$\begin{aligned} \frac{d}{dt} \int \mathcal{Q}(q, p, z) \rho_t(dqdpdz) &\leq C_1 - C_2 \int_{\mathbf{R}^{3d}} (|q|^2 + |p|^2 + |z|^2) \rho_t(dqdpdz) \\ &\leq C_1 - C_3 \int_{\mathbf{R}^{3d}} \mathcal{Q}(q, p, z) \rho_t(dqdpdz), \end{aligned}$$

for some positive constants C_1, C_2 and C_3 .

Step 3) Applying the Gronwall's lemma, we obtain that

$$\sup_{t \geq 0} \int_{\mathbf{R}^{3d}} \mathcal{Q}(q, p, z) \rho_t(dqdpdz) < \infty,$$

provided that initially $\int_{\mathbf{R}^{3d}} \mathcal{Q}(q, p, z) \rho_0(dqdpdz) < \infty$. This is equivalent to the statement that

$$\sup_{t \geq 0} \int_{\mathbf{R}^{3d}} (|q|^2 + |p|^2 + |z|^2) \rho_t(dqdpdz) < \infty,$$

if initially ρ_0 has finite second moment. This completes the proof of the Lemma.

□

We are now in the position to prove Theorem 2.

Proof of Theorem 2. We recall that for each $1 \leq i \leq N$, the law μ_t of $(\tilde{Q}_t^i, \tilde{P}_t^i, \tilde{Z}_t^i)$ is the solution to Eq. (1) at time t with the initial datum ρ_0 and that the two processes $(\tilde{Q}_t^i, \tilde{P}_t^i, \tilde{Z}_t^i)_{t \geq 0}$ and $(Q_t^{i,N}, P_t^{i,N}, Z_t^{i,N})_{t \geq 0}$ are driven by the same Brownian motions and initial datum. Define $(q_t^i, p_t^i, z_t^i) := (Q_t^{i,N}, P_t^{i,N}, Z_t^{i,N}) - (\tilde{Q}_t^i, \tilde{P}_t^i, \tilde{Z}_t^i)$, then (q_t^i, p_t^i, z_t^i) satisfies the following SDE

$$\begin{cases} dq_t^i = p_t^i dt, \\ dp_t^i = -\beta q_t^i dt - (A(Q_t^{i,N}) - A(\tilde{Q}_t^i)) dt - \frac{1}{N} \sum_{j=1}^N (B(Q_t^{i,N} - Q_t^{j,N}) - B * \mu_t(\tilde{Q}_t^i)) dt + \lambda z_t^i dt, \\ dz_t^i = -\lambda p_t^i dt - \lambda \alpha z_t^i dt, \\ (q_0^i, p_0^i, z_0^i) = (0, 0, 0). \end{cases} \quad (31)$$

The underlying idea of remaining of the proof will be similar as that of Theorem 1 and Lemma 4. We will find a positive definite quadratic form $\mathcal{D}$ on $\mathbf{R}^{3d}$ such that the SDE (31) is dissipative with respect to $\mathcal{D}$. Let $a_1, \dots, a_5$ be positive constants to be chosen later on. We define

$$\mathcal{D}(q, p, z) := a_1 |q|^2 + a_2 |p|^2 + a_3 |z|^2 + 2a_4 q \cdot p + 2a_5 q \cdot z + 2p \cdot z. \quad (32)$$

Let (q_t^i, p_t^i, z_t^i) be a solution to the SDE (31). We have

$$\begin{aligned}
& \frac{d}{dt} \mathcal{D}(q_t^i, p_t^i, z_t^i) \\
&= \frac{d}{dt} (a_1 |q_t^i|^2 + a_2 |p_t^i|^2 + a_3 |z_t^i|^2 + 2a_4 q_t^i \cdot p_t^i + 2a_5 q_t^i \cdot z_t^i + 2p_t^i \cdot z_t^i) \\
&= 2a_1 q_t^i \cdot p_t^i + 2a_4 |p_t^i|^2 + 2a_5 p_t^i \cdot z_t^i \\
&\quad + (2a_3 z_t^i + 2a_5 q_t^i + 2p_t^i) \cdot (-\lambda p_t^i - \alpha z_t^i) \\
&\quad + (2a_2 p_t^i + 2a_4 q_t^i + 2z_t^i) \cdot \left(-\beta q_t^i - (A(Q_t^{i,N}) - A(\tilde{Q}_t^i)) - \frac{1}{N} \sum_{j=1}^N (B(Q_t^{i,N} - Q_t^{j,N}) \right. \\
&\quad \left. - B * \mu_t(\tilde{Q}_t^i)) + \lambda z_t^i \right) \\
&= (2a_1 - 2\lambda a_5 - 2\beta a_2) q_t^i \cdot p_t^i + (2a_5 - 2\lambda a_3 - 2\alpha + 2\lambda a_2) p_t^i \cdot z_t^i + (2\lambda a_4 - 2\alpha a_5 - 2\beta) q_t^i \cdot z_t^i \\
&\quad - 2\beta a_4 |q_t^i|^2 - (2\lambda - 2a_4) |p_t^i|^2 - (2\alpha a_3 - 2\lambda) |z_t^i|^2 \\
&\quad - (2a_2 p_t^i + 2a_4 q_t^i + 2z_t^i) \cdot (A(Q_t^{i,N}) - A(\tilde{Q}_t^i)) \\
&\quad - \frac{1}{N} \sum_{j=1}^N (2a_2 p_t^i + 2a_4 q_t^i + 2z_t^i) \cdot (B(Q_t^{i,N} - Q_t^{j,N}) - B * \mu_t(\tilde{Q}_t^i)). \quad (33)
\end{aligned}$$

Similarly as in (19) we have

$$\begin{aligned}
& -(2a_2 p_t^i + 2a_4 q_t^i + 2z_t^i) (A(Q_t^{i,N}) - A(\tilde{Q}_t^i)) \leq 2a_2 C_A |q_t^i| |p_t^i| + 2a_4 C_A |q_t^i|^2 + 2C_A |q_t^i| |z_t^i| \\
& \leq a_2 C_A (|q_t^i|^2 + |p_t^i|^2) + 2a_4 C_A |q_t^i|^2 + C_A (|q_t^i|^2 + |z_t^i|^2) \\
& = (a_2 + 2a_4 + 1) C_A |q_t^i|^2 + a_2 C_A |p_t^i|^2 + C_A |z_t^i|^2. \quad (34)
\end{aligned}$$

Substituting (34) into (33), we obtain

$$\begin{aligned}
& \frac{d}{dt} \mathcal{D}(q_t^i, p_t^i, z_t^i) \\
& \leq (2a_1 - 2\lambda a_5 - 2\beta a_2) q_t^i \cdot p_t^i + (2a_5 - 2\lambda a_3 - 2\alpha + 2\lambda a_2) p_t^i \cdot z_t^i + (2\lambda a_4 - 2\alpha a_5 - 2\beta) q_t^i \cdot z_t^i \\
& \quad - (2\beta a_4 - (a_2 + 2a_4 + 1) C_A) |q_t^i|^2 - (2\lambda - 2a_4 - 2a_2 C_A) |p_t^i|^2 - (2\alpha a_3 - 2\lambda - C_A) |z_t^i|^2 \\
& \quad - \frac{1}{N} \sum_{j=1}^N (2a_2 p_t^i + 2a_4 q_t^i + 2z_t^i) \cdot (B(Q_t^{i,N} - Q_t^{j,N}) - B * \mu_t(\tilde{Q}_t^i)). \quad (35)
\end{aligned}$$

We first choose $a_1, \dots, a_5$ such that the cross terms in the right-hand side of

(35) vanish, i.e.,

$$\begin{cases} 2a_1 - 2\lambda a_5 - 2a_2\beta = 0, \\ 2\lambda a_2 - 2\lambda a_3 + 2a_5 - 2\alpha = 0, \\ 2\lambda a_4 - 2\alpha a_5 - 2\beta = 0. \end{cases} \quad (36)$$

With this choice, (35) reads

$$\begin{aligned} & \frac{d}{dt} \mathcal{D}(q_t^i, p_t^i, z_t^i) \\ & \leq -(2\beta a_4 - (a_2 + 2a_4 + 1)C_A) |q_t^i|^2 - (2\lambda - 2a_4 - 2a_2 C_A) |p_t^i|^2 - (2\alpha a_3 - 2\lambda - C_A) |z_t^i|^2 \\ & \quad - \frac{1}{N} \sum_{j=1}^N (2a_2 p_t^i + 2a_4 q_t^i + 2z_t^i) \cdot (B(Q_t^{i,N} - Q_t^{j,N}) - B * \mu_t(\tilde{Q}_t^i)). \end{aligned} \quad (37)$$

Note that this estimate holds for every $i = 1, \dots, N$. By averaging (37) over i we have

$$\begin{aligned} & \frac{d}{dt} \mathcal{D}(q_t^1, p_t^1, z_t^1) \\ & \leq -(2\beta a_4 - (a_2 + 2a_4 + 1)C_A) |q_t^1|^2 - (2\lambda - 2a_4 - 2a_2 C_A) |p_t^1|^2 - (2\alpha a_3 - 2\lambda - C_A) |z_t^1|^2 \\ & \quad - \frac{2}{N^2} \sum_{i,j=1}^N (a_2 p_t^i + a_4 q_t^i + z_t^i) \cdot (B(Q_t^{i,N} - Q_t^{j,N}) - B * \mu_t(\tilde{Q}_t^i)). \end{aligned} \quad (38)$$

Next we estimate the last term in (38). Using the following identity

$$B(Q_t^{i,N} - Q_t^{j,N}) - B * \mu_t(\tilde{Q}_t^i) = B(Q_t^{i,N} - Q_t^{j,N}) - B(\tilde{Q}_t^i - \tilde{Q}_t^j) + B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i),$$

we now decompose the last term in (38) into six terms and estimate them as follows.

1) By symmetry and the Lipschitz property of B , we have

$$\begin{aligned}
& - \sum_{i,j=1}^N \mathbb{E}[q_t^i \cdot (B(Q_t^{i,N} - Q_t^{j,N}) - B(\tilde{Q}_t^i - \tilde{Q}_t^j))] \\
& = - \sum_{i,j=1}^N \mathbb{E}[(Q_t^{i,N} - \tilde{Q}_t^i) \cdot (B(Q_t^{i,N} - Q_t^{j,N}) - B(\tilde{Q}_t^i - \tilde{Q}_t^j))] \\
& = - \frac{1}{2} \sum_{i,j=1}^N \mathbb{E}[((Q_t^{i,N} - Q_t^{j,N}) - (\tilde{Q}_t^i - \tilde{Q}_t^j)) \cdot (B(Q_t^{i,N} - Q_t^{j,N}) - B(\tilde{Q}_t^i - \tilde{Q}_t^j))] \\
& \leq \frac{C_B}{2} \sum_{i,j=1}^N \mathbb{E} \left| (Q_t^{i,N} - Q_t^{j,N}) - (\tilde{Q}_t^i - \tilde{Q}_t^j) \right|^2 \\
& = \frac{C_B}{2} \sum_{i,j=1}^N \mathbb{E} \left| (Q_t^{i,N} - \tilde{Q}_t^i) - (Q_t^{j,N} - \tilde{Q}_t^j) \right|^2 \\
& = C_B \sum_{i,j=1}^N \mathbb{E} |q_t^i|^2 - C_B \mathbb{E} \left| \sum_{i=1}^N q_t^i \right|^2 \\
& \leq C_B \sum_{i,j=1}^N \mathbb{E} |q_t^i|^2 = C_B N^2 \mathbb{E} |q_t^1|^2.
\end{aligned}$$

2) By assumption on B and Young's inequality, we have

$$\begin{aligned}
& - \sum_{i,j=1}^N \mathbb{E}[p_t^i \cdot (B(Q_t^{i,N} - Q_t^{j,N}) - B(\tilde{Q}_t^i - \tilde{Q}_t^j))] \\
& = -\frac{1}{2} \sum_{i,j=1}^N \mathbb{E}[(p_t^i - p_t^j) \cdot (B(Q_t^{i,N} - Q_t^{j,N}) - B(\tilde{Q}_t^i - \tilde{Q}_t^j))] \\
& \leq \frac{C_B}{2} \sum_{i,j=1}^N \mathbb{E}[|p_t^i - p_t^j| |(Q_t^{i,N} - Q_t^{j,N}) - (\tilde{Q}_t^i - \tilde{Q}_t^j)|] \\
& \leq \frac{C_B}{2} \sum_{i,j=1}^N \mathbb{E}\left[\frac{1}{2}|p_t^i - p_t^j|^2 + \frac{1}{2}|(Q_t^{i,N} - Q_t^{j,N}) - (\tilde{Q}_t^i - \tilde{Q}_t^j)|^2\right] \\
& = \frac{C_B}{2} \left(N^2 \mathbb{E}|p_t^1|^2 - \left| \sum_{j=1}^N \mathbb{E}p_t^j \right|^2 + N^2 \mathbb{E}|q_t^1|^2 - \left| \sum_{j=1}^N \mathbb{E}q_t^j \right|^2 \right) \\
& \leq \frac{C_B}{2} N^2 \mathbb{E}[|q_t^1|^2 + |p_t^1|^2].
\end{aligned}$$

3) Similarly we obtain

$$- \sum_{i,j=1}^N \mathbb{E}[z_t^i \cdot (B(Q_t^{i,N} - Q_t^{j,N}) - B(\tilde{Q}_t^i - \tilde{Q}_t^j))] \leq \frac{C_B}{2} N^2 \mathbb{E}[|q_t^1|^2 + |z_t^1|^2].$$

4) We continue with the term

$$\begin{aligned}
& - 2a_4 \mathbb{E} \left[q_t^i \cdot \sum_{j=1}^N (B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i)) \right] \\
& \leq N\beta a_4 \mathbb{E}|q_t^i|^2 + \frac{a_4}{N\beta} \mathbb{E} \left| \sum_{j=1}^N (B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i)) \right|^2 \\
& = N\beta a_4 \mathbb{E}|q_t^i|^2 + \frac{a_4}{N\beta} \sum_{j=1}^N \mathbb{E} \left| B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i) \right|^2 \\
& \quad + \frac{a_4}{N\beta} \sum_{j \neq k} \mathbb{E} \left[(B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i)) \cdot (B(\tilde{Q}_t^i - \tilde{Q}_t^k) - B * \mu_t(\tilde{Q}_t^i)) \right].
\end{aligned} \tag{39}$$

We need to analyse further the last two terms in (39). Because B is odd, we have $B(0) = 0$. Hence for any $y \in \mathbf{R}^d$ it holds that $|B(y)| = |B(y) - B(0)| \leq C_B |y|$. It implies that

$$\begin{aligned}
& \mathbb{E} \left| B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i) \right|^2 \\
& \leq 2 \left[\mathbb{E} |B(\tilde{Q}_t^i - \tilde{Q}_t^j)|^2 + \mathbb{E} |B * \mu_t(\tilde{Q}_t^i)|^2 \right] \\
& \leq 2C_B^2 \left[\mathbb{E} |\tilde{Q}_t^i - \tilde{Q}_t^j|^2 + \int_{\mathbf{R}^{6d}} |q - q'|^2 \mu_t(q, p, z) \mu_t(q', p', z') dq dp dz dq' dp' dz' \right] \\
& \leq 8C_B^2 \int_{\mathbf{R}^{3d}} |q|^2 \mu_t(q, p, z) dq dp dz \\
& \leq M,
\end{aligned} \tag{40}$$

where in the last inequality above we have used Lemma 4 to obtain $M = 16C_B^2 (C' W_2(\rho_0, \rho_\infty)^2 + W_2(\delta_0, \rho_\infty)^2)$. Certainly M depends only on the initial second moment and the coefficients of the equation and but not on t or N .

The last term in (39) vanishes since for all $j \neq k$, we have

$$\begin{aligned}
& \mathbb{E} \left[(B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i)) \cdot (B(\tilde{Q}_t^i - \tilde{Q}_t^k) - B * \mu_t(\tilde{Q}_t^i)) \right] \\
& = \mathbb{E}_{\tilde{Q}_t^i} \left[(\mathbb{E}_{\tilde{Q}_t^j} [B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i)]) \cdot (\mathbb{E}_{\tilde{Q}_t^k} [B(\tilde{Q}_t^i - \tilde{Q}_t^k) - B * \mu_t(\tilde{Q}_t^i)]) \right] \\
& = \mathbb{E}_{\tilde{Q}_t^i} [0] = 0.
\end{aligned} \tag{41}$$

The equality above holds true because $\tilde{Q}_t^j$ and $\tilde{Q}_t^k$ are independent and have the same law, which is the first marginal of μ_t .

Substituting (40) and (41) into (39), we get

$$-2a_4 \mathbb{E} \left[q_t^i \cdot \sum_{j=1}^N (B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i)) \right] \leq N\beta a_4 \mathbb{E} |q_t^i|^2 + \frac{a_4}{\beta} M,$$

from which, by summing over i , we obtain

$$\begin{aligned}
-2a_4 \sum_{i,j=1}^N \mathbb{E} \left[q_t^i \cdot (B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i)) \right] & \leq N^2 \beta a_4 \mathbb{E} |q_t^i|^2 + \frac{a_4}{\beta} MN \\
& = N^2 \beta a_4 \mathbb{E} |q_t^1|^2 + \frac{a_4}{\beta} MN.
\end{aligned} \tag{42}$$

5) Similarly we obtain the following estimate

$$-2a_2 \sum_{i,j=1}^N \mathbb{E} \left[p_t^i \cdot (B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i)) \right] \leq N^2 \lambda \mathbb{E} |p_t^1|^2 + \frac{a_2^2}{\lambda} MN.$$

6) Finally, for the last term we also get

$$-2 \sum_{i,j=1}^N \mathbb{E} \left[z_t^i \cdot (B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i)) \right] \leq N^2 \alpha a_3 \mathbb{E} |p_t^1|^2 + \frac{1}{\alpha a_3} MN.$$

Set $\eta := C_A + C_B$. Bringing all terms together, it follows from (38) that

$$\begin{aligned} \frac{d}{dt} \mathcal{D}(q_t^1, p_t^1, z_t^1) &\leq -(\beta a_4 - (a_2 + 2a_4 + 1)\eta) |q_t^1|^2 - (\lambda - 2a_4 - 2a_2\eta) |p_t^1|^2 \\ &\quad - (\alpha a_3 - 2\lambda - \eta) |z_t^1|^2 + \left(\frac{a_4}{\beta} + \frac{a_2^2}{\lambda} + \frac{1}{\alpha a_3} \right) \frac{M}{N}. \end{aligned} \quad (43)$$

Note the difference between the right hand side of the above inequality and that of (23) in Section 3.1. The difference is due to the extra terms involving $B(\tilde{Q}_t^i - \tilde{Q}_t^j) - B * \mu_t(\tilde{Q}_t^i)$ as shown in the above computations. We now choose $a_4 = \frac{\lambda}{4}$. From (36) we can express a_1, a_2 and a_5 in terms of a_3 and choose a_3 such that

$$\begin{cases} a_5 = \frac{\lambda a_4 - \beta}{\alpha} = \frac{\lambda^2/4 - \beta}{\alpha}, \\ a_2 = a_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda}, \\ a_1 = \lambda a_5 + \beta a_2 = \lambda a_5 + \beta \left(a_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda} \right), \\ \beta a_4 - (a_2 + 2a_4 + 1)\eta = \frac{\lambda}{4}\beta - \left(a_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda} + \lambda + 1 \right) \eta > 0, \\ \lambda - 2a_4 - a_2\eta = \frac{\lambda}{2} - \left(a_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda} \right) \eta > 0, \\ \alpha a_3 - 2\lambda - \eta > 0, \end{cases} \quad (44)$$

and that $\mathcal{D}(q, p, z)$ is a positive definite quadratic form on $\mathbf{R}^{3d}$, where

$$\mathcal{D}(q, p, z) = \left(a_5 \left(\lambda - \frac{\beta}{\lambda} \right) + \beta \left(a_3 + \frac{\alpha}{\lambda} \right) \right) |q|^2 + \left(a_3 + \frac{\alpha}{\lambda} - \frac{a_5}{\lambda} \right) |p|^2 + a_3 |z|^2 + \frac{\lambda}{2} q \cdot p - 2a_5 q \cdot z + 2p \cdot z.$$

Similarly as in the proof of Theorem 1 and from (43), we obtain the existence of a positive constant η_0 depending only on $\alpha, \beta, \lambda, C_A$ and C_B such that for

all $0 < C_A + C_B$, there exist $a_1, \dots, a_5$ such that $\mathcal{D}(q, p, z)$ is a positive definite quadratic form on $\mathbf{R}^{3d}$ and such that

$$\frac{d}{dt} \mathbb{E} \mathcal{D}(q_t^1, p_t^1, z_t^1) \leq -C_1 \mathbb{E}[|q_t^1|^2 + |p_t^1|^2 + |z_t^1|^2] + \frac{C_2}{N}, \quad (45)$$

for all $t \geq 0$ and for positive constants C_1 and C_2 , which depend only on the parameters of the equation and the initial second moment, but not on N . Since $\mathcal{D}$ is a positive quadratic on $\mathbf{R}^{3d}$, the right-hand side of (45) in turn is bounded by $-C_3 \mathbb{E} \mathcal{D}(q_t^1, p_t^1, z_t^1) + \frac{C_2}{N}$, for some positive constant C_3 , so that

$$\frac{d}{dt} \mathbb{E} \mathcal{D}(q_t^1, p_t^1, z_t^1) \leq -C_3 \mathbb{E} \mathcal{D}(q_t^1, p_t^1, z_t^1) + \frac{C_2}{N}. \quad (46)$$

From this inequality, we deduce that

$$\mathbb{E} \mathcal{D}(q_t^1, p_t^1, z_t^1) \leq \frac{C_4}{N}, \quad \text{for all } t \geq 0,$$

for some positive constant C_4 . Using the fact that $\mathcal{D}$ is a positive definite quadratic on $\mathbf{R}^{3d}$ again, we obtain the following bound

$$\mathbb{E}[|Q_t^{1,N} - \tilde{Q}_t^i|^2 + |P_t^{1,N} - \tilde{P}_t^i|^2 + |Z_t^{1,N} - \tilde{Z}_t^i|^2] = \mathbb{E}[|q_t^1|^2 + |p_t^1|^2 + |z_t^1|^2] \leq \frac{C}{N},$$

for all $t \geq 0$ and for some positive constant C depending only on the parameters of the equation and the initial second moment, but not on N . This completes the proof of (10).

The assertion that the particle system (7) satisfies the property of propagation of chaos is a direct consequence of (10). In deed, let $(i_1, \dots, i_k)$ be an arbitrary k -uple of $[1, N]$ and φ be a 1-Lipschitz function on $(\mathbf{R}^{3d})^k$. Then

$$\begin{aligned} & \left| \int_{(\mathbf{R}^{3d})^k} \varphi \rho_t^{(i_1, \dots, i_k)} - \int_{(\mathbf{R}^{3d})^k} \varphi (\mu_t)^{\otimes k} \right| = \left| \mathbb{E} \varphi(X_t^{i_1, N}, \dots, X_t^{i_k, N}) - \mathbb{E} \varphi(\tilde{X}_t^{i_1, N}, \dots, \tilde{X}_t^{i_k, N}) \right| \\ & \leq \left| \mathbb{E} \varphi(X_t^{i_1, N}, \dots, X_t^{i_k, N}) - \mathbb{E} \varphi(\tilde{X}_t^{i_1, N}, X_t^{i_2, N}, \dots, X_t^{i_k, N}) \right| + \dots \\ & \quad + \left| \mathbb{E} \varphi(\tilde{X}_t^{i_1, N}, \dots, \tilde{X}_t^{i_{k-1}, N}, X_t^{i_k, N}) \right| - \mathbb{E} \left| \varphi(\tilde{X}_t^{i_1, N}, \dots, \tilde{X}_t^{i_k, N}) \right| \\ & \leq \sum_{i=1}^k \mathbb{E} |X_t^{i, N} - \tilde{X}_t^{i, N}| \leq \frac{C k}{\sqrt{N}}, \end{aligned}$$

where we have used the Lipschitz property of φ and (10). This estimate implies that $\rho_t^{i_1, \dots, i_k}$ converges to $(\mu_t)^{\otimes k}$ in the 1-Wasserstein distance, which is tested against 1-Lipschitz functions. Since the 1-Wasserstein distance is equivalent to narrow convergence plus convergence of the first moments, this concludes that $\rho_t^{i_1, \dots, i_k}$ converges to $(\mu_t)^{\otimes k}$ narrowly.

Finally, the estimate (11) is followed straightforwardly thanks to (10) and definition of the Wasserstein distance (12). We complete the proof of Theorem 2. $\square$

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