

HEAT AND MASS TRANSFER
IN TURBULENT BOUNDARY LAYERS

by

Suhas Vasant Patankar

Thesis submitted for the degree of
Doctor of Philosophy
in the Faculty of Engineering
University of London.

June 1967.

Abstract

The foundation is laid of a general theory of heat and mass transfer in turbulent boundary layers. Both the mathematical and the physical aspects of the theory are dealt with.

The mathematical contribution mainly consists of an implicit, finite-difference procedure for solving the boundary-layer equations. The novel features of the procedure include : the employment of a grid which adjusts its width so as to conform to the thickness of the layer in which the dependent variables vary significantly ; and the use, near a wall, of algebraic relationships expressing the once-for-all integrations obtained by the neglect of the longitudinal convection there. Calculations are presented to demonstrate that the solution procedure is accurate, economical and widely applicable. It is pointed out that a good mathematical tool prepares the road for research into physical hypotheses.

On the physical side, van Driest's hypothesis is used to illustrate the algebraic relationships for the region near a wall ; their validity is tested by reference to experimental data including some of the present author. These relations and Prandtl's mixing-length hypothesis are used in the new solution procedure to make predictions of heat-transfer rates in many different situations ; comparison with experimental data shows satisfactory agreement and indicates directions of further development.

Details of the computer programme used and of the experiments performed are given in appendices.

Preface

This thesis represents one of the outcomes of my research activities for the last three years. It mainly describes the best of my knowledge and recommendations concerning the subject at present, out of the several matters invented, explored and pursued during this period.

It seems to me appropriate to give here a brief outline of the history that led to the present thesis. I consider myself very fortunate in having come to Imperial College (in October 1964) when the construction of a general theory of turbulent boundary layer had just started. After completing a survey [46]* of experimental temperature profiles, I recognised the need for introducing an additional parameter in the then-existing temperature-profile expression. The development of the resulting two-parameter method [49] needed the rest of my first year at Imperial College. Then began a search for integral methods with more flexible profiles and without the unpleasant matrix singularities (present in the method of Ref.[49]). After various difficulties were overcome, the new integral method using flexible polynomial profiles was ready for use just before the summer of 1966; the necessary relations for the fluxes at a wall were also formulated [48].

At this stage, I was about to start my experimental work, thinking that the theoretical construction was almost complete. However, a trip to the U S A last summer, during which I allowed myself to explore a different type of solution procedure, significantly changed the situation. For the new finite-difference method turned out to be more attractive and promising than the profile methods, and therefore, it seemed more profitable to devote time to the development of this new method. The result was that I could spend relatively short time in performing experiments.

* Square brackets indicate reference numbers.

This sequence of events has also influenced the layout of this thesis. The contribution made by the new calculation procedure appeared so significant that I have given it the pride of place in this thesis, the earlier methods being only briefly mentioned. The work (including my experiments) relating to the physical aspects has been presented as illustrative material.

It is true that, in the course of time, the calculation procedure described here is also likely to be superseded. What will be of lasting value to me is the training I received from Professor Spalding. The confident and optimistic attitude that I learnt from him enabled me to force my way towards new methods and techniques despite intermediate obstacles. Quite apart from the boundary-layer theory, there was much to be learnt from Spalding as regards technical writing. Being inspired by the principle of providing ease and facility for the reader, Spalding very carefully maintains the continuity of thought, guarding against ambiguities, errors and sudden jumps. I always enjoyed his merciless correction of the reports I wrote, although it meant considerable discredit to me.

The work reported in this thesis was carried out during the tenure of a scholarship from I.C.I. (India) Pvt. Ltd. Thanks are also due to Northern Research and Engineering Corporation, Cambridge (Massachusetts) where, during the short period spent by Professor Spalding and myself, we developed the early stages of the calculation procedure to be described in this thesis.

I wish to acknowledge gratefully the help and guidance given by many individuals. My respect and gratitude for Professor Spalding cannot be adequately expressed in words. It must, however, be mentioned here that the work presented in this thesis is throughout based upon his suggestions, ideas and contributions. I have found it very difficult to give him the due credit in many places, because either the relevant material has not been

published, or it is hard to distinguish his contribution from mine. Nevertheless, much is due to him.

My coming to Imperial College was largely due to the advice and help given by Mr T. Mukerjee, whose affection and encouragement have always been valuable to me. At the beginning of my work here, Dr.C.L.V. Jayatillaka patiently explained to me the details of the prediction methods used in those days; Dr W.B. Nicoll gave me insight into numerical methods; detailed information and training of experimental techniques was given by Dr. J.H. Whitelaw and Dr. E. Baker; several refinements of the calculation procedure resulted because Mr S.J. Peerless repeatedly pointed out to me the unsatisfactoriness of the solutions; Mr P. Dale helped in some of the computational work. Thanks are also due to Mr M. Wolfshtein, Mr M.P. Escudier and Dr. D.P. Lal for useful information and discussions. Finally, I cannot afford to forget thanking that most helpful person - Miss M.P. Steele - who, by her continual assistance in non-scientific matters, has made this scientific work possible, and who has efficiently organised the typing and reproduction of this thesis.

London.)
)
June, 1967.)

S.V. Patankar.

Contents

	Page
<u>Abstract</u>	2
<u>Preface</u>	3
<u>0. Introduction</u>	10
0.1 The problem considered	10
0.2 Present status of the turbulent-boundary-layer theory	11
0.3 Scope and outline of the present contribution	14
0.3-1 The historical development	14
0.3-2 Outline of thesis	17
<u>Part I</u>	
<u>The mathematical contribution</u>	
<u>1. Equations of the boundary layer</u>	21
1.1 The partial differential equations for axi-symmetrical flow	21
1.1-1 Restrictions	21
1.1-2 Co-ordinate system	22
1.1-3 The Conservation equations	23
1.2 Initial and boundary conditions	24
1.2-1 Initial profiles of dependent variables	24
1.2-2 The boundary conditions	24
1.3 Auxiliary relations	25
1.3-1 Thermodynamic relationships	25
1.3-2 Exchange laws	26
1.3-3 Expression for the effective viscosity	26
1.3-4 Effective Prandtl and Schmidt number	29
1.3-5 The region near a wall	29
1.4 Equations of the one-dimensional boundary layer near a wall	32
1.4-1 The Couette-flow assumption	33
1.4-2 Implications of the generalised van Driest hypothesis	34
1.4-3 Use of the adiabatic-wall enthalpy	36
1.4-4 Some analytical solutions	38
1.4-5 Explicit forms of the relationships	43
1.5 Choice of solution procedure	45

2.	<u>A finite-difference solution procedure</u>	Page 48
2.1	Choice of co-ordinate system	48
2.2	Conservation equations in $x \sim \omega$ co-ordinates	50
2.3	The entrainment rates	51
2.4	The difference equation	54
2.5	Slip values at boundaries	60
2.5-1	What are slip values	60
2.5-2	Slip-value relations for a wall boundary	62
2.5-3	Slip-value relations for a free boundary	65
2.5-4	Slip-value relations for a symmetry-line boundary	66
2.6	Treatment of special boundary conditions	68
2.6-1	The case of prescribed total flux through the wall	68
2.6-2	The case of a symmetry-line boundary	69
2.7	Solution of the difference equations	71
2.8	Miscellaneous matters	72
2.8-1	Grid-control formulae	72
2.8-2	Calculation of the normal distance	73
2.8-3	A characteristic thickness of the layer	75
2.8-4	Choice of the forward step	76
3.	<u>Capabilities of the solution procedure:</u>	
	<u>demonstration and appreciation</u>	77
3.1	Purpose of the chapter	77
3.2	Some boundary-layer calculations	77
3.2-1	Plane turbulent mixing layer	77
3.2-2	Compressible laminar boundary layer on flat plate	80
3.2-3	Compressible turbulent boundary layer on flat plate	82
3.2-4	General remarks	86
3.3	Achievements and shortcomings of the present method	87
3.4	Suggestions for future work	88

Part II

Illustrations of research into physical hypotheses

	Page
<u>4. Some wall-flux relationships</u>	91
4.1 Motivation	91
4.2 The numerical integrations	92
4.3 The algebraic formulae	95
4.3-1 Relationships for laminar flow	95
4.3-2 Drag law for turbulent flow	96
4.3-3 Heat-flux or diffusional-flux law for turbulent flow	101
4.4 Examination of validity of the relationships	103
4.5 Further developments	108
4.6 Summary of the relationships for turbulent flow	109
 <u>5. Implications of the mixing-length hypothesis</u>	 112
5.1 Purpose of the chapter	112
5.2 Preliminary choices	112
5.2-1 The hypotheses used	112
5.2-2 Choice of constants	113
5.2-3 Details of computation	114
5.3 Comparison with uniform-property data	114
5.3-1 Cases of prescribed wall temperature	116
5.3-2 Cases of prescribed heat flux	116
5.4 Comparison with non-uniform-property flat-plate data	121
5.4-1 The flat-plate drag	121
5.4-2 The flat-plate Stanton number	121
5.5 The case of a wall jet	124
5.6 Concluding remarks	126

<u>References</u>	Page
	130
<u>Nomenclature</u>	138
<u>Appendix A. A computer programme for solution of the</u>	
<u>boundary-layer equations</u>	145
A.1 Introduction	145
A.1-1 Purpose of the present appendix	145
A.1-2 Scope and limitations of the programme	145
A.2 Conventions and symbols used in the programme	147
A.3 Description of the subroutines	151
A.3-1 Subroutines of general validity	155
A.3-2 Subroutines where the physical hypotheses enter	157
A.3-3 Subroutines which specify a particular problem	159
A.4 The user's quick-reference guide	163
A.5 Listing of the computer programme	165
 <u>Appendix B. Experimental investigation</u>	 183
B.1 Purpose and outline of the experiments	183
B.2 Description of the apparatus and instrumentation	184
B.2-1 The porous plate	184
B.2-2 Measurement of velocity	185
B.2-3 Measurement of concentration	185
B.2-4 Measurement of flow rate	190
B.3 The procedure	190
B.4 Results of the experiments	191

Introduction

0.1 The problem considered

Turbulent boundary layers occur in many situations of engineering interest. In several cases, the designer needs the ability to predict the rates of heat and mass transfer. Even when the main interests lie in hydrodynamic quantities such as the pressure recovery in a diffuser or the spread of a jet, the heat- and mass-transfer processes make their presence felt through the non-uniformity of fluid properties, chemical reaction and other effects. The present thesis concerns a general theory for predicting these transfer processes.

A general theory should be able to meet the demands of different physical situations. The boundary layer may be on a wall which is smooth or rough, permeable or impermeable; the density and other fluid properties may be non-uniform because of temperature or concentration gradients; pressure gradient, kinetic heating and chemical reactions may be present; the geometries can be plane, axi-symmetrical or three-dimensional.

It should be noted that the term "boundary layer" is being used in this thesis in a wider sense than is usually done. It includes the "conventional" boundary layer on a wall in which the velocity rises monotonically with distance from the wall, as well as the flows with velocity maxima which develop immediately downstream of film-cooling slots. In addition, we shall regard certain free turbulent flows like mixing layers, jets and wakes as boundary layers. In general, we shall consider those flows as boundary layers in which the gradients of the flow properties are much larger in the cross-stream direction than those in the stream-wise direction.

The final goal of the work reported in this thesis is to provide a sound theory for prediction of various aspects of turbulent boundary layers under the above-mentioned conditions.

0.2 Present status of the turbulent-boundary-layer theory

The available prediction methods. Although research into turbulent boundary layers has been in progress for over forty years and several text books exist which describe useful theoretical and experimental findings, the available prediction procedures are unsatisfactory.

The most common method of predicting heat transfer is by way of the Reynolds analogy, sometimes modified to take into account the effect of the Prandtl number. For an isothermal flat plate, this method is indeed satisfactory. Chi and Spalding [11] found that all the available experimental data for an isothermal flat plate in air could be correlated by use of 0.58 as the ratio of the Stanton number to the drag coefficient, regardless of Reynolds number, Mach number or wall-to-mainstream temperature ratio. However, the case of an isothermal flat plate in air represents a very tiny fraction of practically interesting boundary-layer flows. The Reynolds-analogy factor changes considerably when effects of pressure gradient, mass transfer and non-uniform wall temperature are present and when the fluid has a Prandtl number different from that of air.

A more refined procedure can be seen in a method put forward by Spalding [70] for the step-wall-temperature problem. This method employs the velocity and the eddy-conductivity distributions appropriate to a turbulent Couette flow, and is, therefore, applicable only when the thermal boundary layer is much thinner than the velocity boundary layer. Exact numerical solutions based on this method have been published (Refs. [32], [24], [68]) for various Prandtl numbers and for cases where a step exists in either the wall temperature or the heat flux. The cases of prescribed non-uniform wall temperature or heat flux could be solved by use of superposition techniques. Although the method of Ref. [70] gives good agreement with experiment for flat-plate cases (as shown, for example, in Refs. [51], [69], [25]), its unsatisfactoriness for boundary layers with pressure gradient has been demonstrated in Refs. [37] and [4].

It was observed in Ref.[71] that the numerical solutions of Refs. [24] and [68] exhibited certain regularities; in particular, the effect of large Prandtl numbers could be expressed mainly through an increased resistance of the laminar sub-layer. A formula for this resistance was obtained, after examination of a large number of experimental data, by Spalding and Jayatilaka [81].

Apart from their individual restrictions, the above-mentioned methods all need a prior calculation of the drag coefficient, for which most of the available methods are, it may be remarked here, unreliable and fragmentary. (It is not intended to go into details of these methods here; the reader may look into reviews such as Ref. [85].)

Because of the difficulty in predicting the drag coefficient, some authors have developed methods of heat-transfer prediction which do not need a hydrodynamic calculation. Typical is the method of Ambrok [3], in which a unique relationship is assumed to exist between the Stanton number and the enthalpy-thickness Reynolds number, and the solution is obtained by use of the integral Steady-Flow Energy equation. This method has been successfully applied by Moretti and Kays [40] to problems of non-uniform wall temperature by the superposition technique. Kutateladze and Leont'ev [34] also recommend a method of the similar type. In spite of the success over a limited range and the relative simplicity of the Ambrok-type methods, they cannot be accepted for general use; because a complete neglect of the hydrodynamic behaviour will certainly lead to unrealistic predictions in several cases.

Only a few authors have considered the effect of non-uniform density or of mass transfer (for example, Refs. [34], [79], [6], [60]). In many cases, these theories do not fit the experimental data with reasonable accuracy and sometimes show unrealistic trends.

General remarks. It should be noted that, in the present state of knowledge, the theory of the turbulent boundary layer differs from that of the laminar one in one important respect. Unlike the laminar flow, the physics of the turbulent flow is little known. Therefore, being forced to resort to restricted empirical relationships, authors often use approximate mathematical methods. Then, depending upon the needs of the method used, the empirical relationships enter at various levels of the theoretical structure. For example, the procedure of Dvorak and Head [18] employs a velocity-profile assumption and an entrainment law for the hydrodynamic -boundary-layer prediction, followed by an exact numerical solution of the partial differential equation for temperature. In such methods, it is difficult to ascertain whether the lack of agreement with experiment is caused by the empirical input, by the mathematical approximation or by both. The attitude that our insecure knowledge of the physical matters about turbulent flows is worthy of only approximate mathematical treatment has, in fact, blocked, to a large extent, the way to obtain better physical hypotheses.

The choice of the mathematical method and of empirical relationships is, in many cases, not made with a view to achieving sufficient generality or at least providing possibilities of extension. No theory exists which satisfies the major requirements (stated in Section 0.1) of a general theory.

The need for a new and general theory is thus evident. The present work lays down the foundation of such a theory and indicates how the rest of the construction may be completed.

It must be admitted here that some of the above remarks about lack of accuracy and of generality do injustice to the methods recently developed at Imperial College. However, the mention of these methods has been purposely suppressed here so far. The reason is that since these methods represent the immediate historical and logical background of the present work,

they can be better outlined in the following section where at the same time the origins of various important aspects of the present contribution will be revealed.

0.3 Scope and outline of the present contribution

0.3-1 The historical development

An outstanding paper by Spalding [72] has been the main source of inspiration for the present work. This paper not only pointed out the need for a general theory, but also demonstrated the possibility by treating a wide variety of problems in a unified way. The novel and most striking feature was that the theory in this paper made no distinction between the "conventional" boundary layers and those with velocity maxima (wall-jet flows). This was made possible by the ingenious use of two-component profiles for velocity and temperature and of an entrainment law. Further applications of the method in Ref. [72] have been published in Refs. [42] and [43].

Later the emphasis shifted from the entrainment equation to the integral kinetic-energy equation [73]. Although the latter could be used with an empirical function for the shear-work integral (see Refs. [21], [22]), it seemed possible to feed the physical information at a more fundamental level via an effective-viscosity hypothesis. Prandtl's mixing-length hypothesis [53] was a satisfactory choice and this became ready for use after Escudier [20] recommended a generally-applicable mixing-length distribution.

At this stage, the theory began to assume a new look. It became clear that an effective-viscosity hypothesis enabled us, at least in principle, to obtain solutions of the relevant partial differential equations. The enthusiasm at that time was for the use of integral equations with profile assumptions. It was, however, recognised that since integral equations represented a mathematical tool, any number of them could be generated by use of weighting

functions [74] and that the obvious physical significance of some of them was irrelevant to their utility. Thus the unrealistic implications (shown in Ref. [43]) of the temperature profile in Ref. [72] could be corrected by use of an additional parameter in the profile and an additional integral equation [49].

This work created new problems as it solved the earlier ones. Use of simultaneous integral equations met with matrix singularity. This was overcome in Ref. [49] by minor modifications of some algebraic expressions; it was thought that the singularities would disappear as more parameters were used. Then followed a prolonged effort, largely unreported, to eliminate these singularities; several co-ordinate systems, weighting functions and profile expressions were tried. At one stage, in an attempt to effect algebraic simplification, use of piece-wise linear profiles and sub-domain integral equations gave the integral method almost an appearance of the finite-difference method. The tiresome fact, however, remained: the matrix singularities were always present.

It was found later that the main origin of this trouble was more fundamental. Integral methods involved an essential use of a thickness of the boundary layer, which being arbitrary must be specified in an arbitrary manner, especially for highly-flexible profiles. The above-mentioned attempts failed because they lacked this specification. At this stage, the entrainment law was restored to the theory, but for a different purpose: it provided the necessary specification of the boundary-layer thickness in a general and efficient way. This time the entrainment law, unlike the one in Ref. [72], was not empirical; it was derived from the equation of motion [75].

The use of this entrainment law in integral methods has been described in Ref. [76]. This paper contains the final comprehensive account of the integral methods in the present stream of research. The emphasis there is on polynomial profiles and subdomain equations. The cross-stream co-ordinate

is a form of the Howarth variable $\int \rho \, dy$ which makes the case of non-uniform density almost as easy.

In all the work mentioned so far, both the dependent and independent variables were made dimensionless, by use of, for example, the main-stream velocity or the temperature difference across the layer as the normalising parameter. It was later recognised that this practice was unnecessary and sometimes troublesome, because the normalising parameters could pass through zero. Another recognition that came at this stage concerned the presence of reverse flow. So far it was thought that the theory could eventually be extended to cases of reverse flow. Indeed, in Ref. [76] the stream function was disqualified as a satisfactory cross-stream variable, because it would exhibit turning values in the presence of reverse flow. But the reverse flow was, it became clear, altogether beyond the scope of forward-integration techniques; for then the elliptic equations could not be truncated to a parabolic form and the downstream events influenced the upstream ones.

When logarithmic velocity and temperature profiles were used, as in Refs. [72], [49], [43], it was straight forward to relate the profile parameters to the shear stress and heat flux at the wall via the "universal law of the wall". With the use of polynomial profiles, as suggested in Ref. [76], it seemed desirable to employ, very near the wall, special relationships for wall shear stress and heat flux. Preliminary recommendations for such wall-flux relationships were given in Ref. [76], and more elaborate ones in Ref. [48].

At this stage, the unsatisfactoriness of the integral methods had once again started to manifest itself. The algebraic complication had become considerable even with the use of simple subdomain equations, and the latter were believed to give an accuracy inferior to that given by other types of integral equation. Matrix inversion was necessary at every integration step and instability was observed when the number of parameters was large.

Then the finite-difference method, which is the main achievement reported in this thesis, came on the scene. It incorporates the merits of the earlier methods, eliminates their shortcomings, and possesses additional advantages. In particular, the specification of the boundary-layer thickness by an entrainment law and the use of the wall-flux relationships near a wall are the two noteworthy features of this method, which incidentally originated from the work on integral methods. Also it is more than a mere coincidence that the finite-difference equation in the present work is formed by way of a miniature integral equation.

0.3-2 Outline of thesis

The purpose of the present work is to provide a general theory which will satisfy the requirements outlined in Section 0.1. Certain simplifications are, in fact, essential to bring the analysis and computation within the reach of present capabilities. For this reason, we shall confine our attention to two-dimensional boundary layers.

A general theory of the turbulent boundary layer will consist of mathematical and physical aspects. Part I of the thesis deals with the main mathematical contribution. It is here that a widely-applicable, accurate and economical calculation method will be described. A good mathematical tool, however, does not guarantee that we shall make physically - realistic predictions; but it prepares the way. For then the implications of any physical hypothesis can be worked out without their being obscured by mathematical approximation. Subsequent comparisons of the predictions with experimental data will show whether the hypothesis used is satisfactory or needs either replacement or refinement. An economical computation method makes it possible for an extensive and speedy research programme to be undertaken to obtain satisfactory physical hypotheses. Part II of the present thesis provides some illustrations of such research concerning the physical aspects.

The theory presented in Part I deals with hydrodynamic as well as heat- and mass-transfer phenomena. For the most part, the treatment is applicable to both turbulent and laminar boundary layers. Although presented as a mathematical contribution, Part I contains considerable theoretical and experimental information about turbulent flows. This inclusion is intentional; firstly, this makes the treatment more concrete; also an insight into the physical behaviour helps us to arrive at an appropriate mathematical method; lastly, the attempt is to provide the reader with some information and references from the related literature simultaneously with the development of the mathematical structure.

Part I consists of three chapters. The partial differential equations representing the conservation laws are given in Section 1.1. Various other relations must be supplied before the problem becomes mathematically solvable. Therefore, the initial and boundary conditions (Section 1.2), the relations for exchange terms and expressions for the exchange coefficients (Section 1.3) are outlined. The region near the wall has been given special importance in the present work; because the neglect of the convection terms there allows a one-dimensional treatment and consequent saving of the computational effort. This treatment is given in Section 1.4. In Section 1.5 is discussed the problem of choosing a procedure to solve the equations in Section 1.1; the conclusion anticipates the next chapter.

Chapter 2 describes the finite-difference solution procedure in detail. In the beginning, the main novelty of the present method is described: a co-ordinate system is chosen (Section 2.1) such that the grid can be made always to conform to the thickness of the layer. The calculation of the entrainment rates which define the grid is outlined in Section 2.3. The rest of the chapter deals with various details of the finite-difference procedure including the formation of the difference equations, treatment of the boundary conditions, solution of the resulting algebraic equations etc.

Chapter 3 demonstrates the capabilities of the method by way of a few examples. The mathematical accuracy of the numerical method is here tested by comparison with some available exact solutions. A summary of the main achievements is given and suggestions for future work are included.

Part II shows the direction of future progress. Out of the two chapters there, Chapter 4 concerns the region near the wall. Here some algebraic relationships are obtained which fit the Cquette-flow analyses outlined in Section 1.4. The validity of these relations is examined with reference to some experimental data. As a contribution to this, data from experiments performed by the present author are also presented. The investigation concerns the effect of mass transfer through the wall on the diffusional transfer of a tracer gas in a radial wall jet.

The implications of a form of the Prandtl mixing-length hypothesis [53] are worked out in Chapter 5. In addition to illustrating the versatility of the calculation procedure, the computations presented in this chapter enable us to assess the satisfactoriness of the mixing-length hypothesis from the comparisons, which have been made, with experimental data. Here directions of the future work are again outlined.

A computer programme based on the finite-difference solution procedure has been included, along with its comprehensive description, in Appendix A. Appendix B contains the details of the experimental investigation made by the present author.

Part IThe mathematical contribution

Chapter 1

Equations of the boundary layer

1.1 The partial differential equations for axi-symmetrical flow

1.1-1 Restrictions

In this chapter will be presented the differential equations expressing various conservation laws as applied to steady axi-symmetrical flows. The treatment is, of course, valid for plane flows, which may be regarded as members of the axi-symmetrical family. We shall assume that the only significant shear stresses, heat fluxes and diffusional fluxes are caused by the gradients in the direction normal to the streamlines. This is the well-known part of the boundary-layer approximation. The other, less-frequently - mentioned part is that reverse flow must be absent; only then can the general elliptic equations be truncated to a parabolic form. This assumption of the absence of reverse flow is implicit in the use, which we shall later make, of the stream function or the non-dimensional stream function as an independent cross-stream variable; for, had reverse flow existed, the stream function would become non-monotonic and hence unsatisfactory as an independent variable.

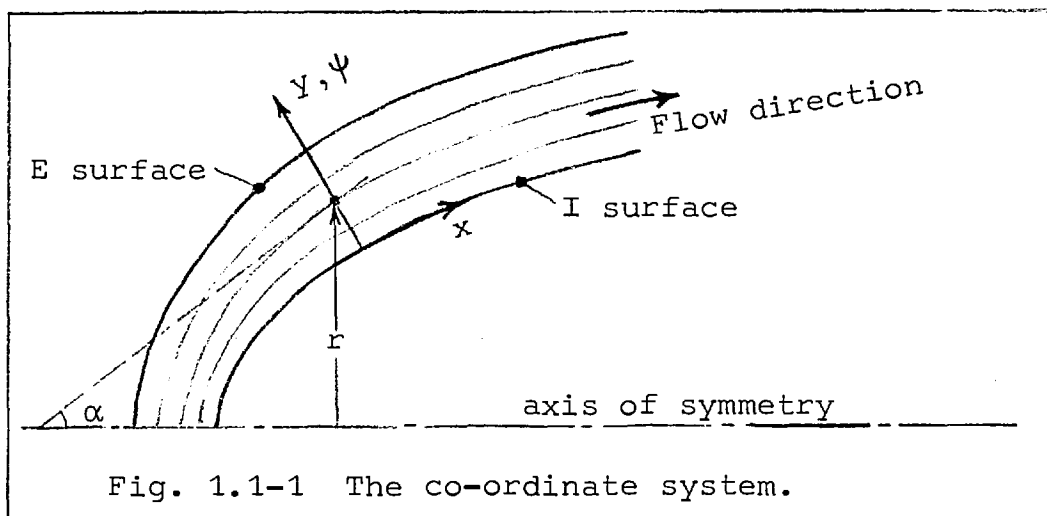
Although we shall deal with turbulent flows, only the time-averaged values of the variables will appear in the following pages. The contribution of the fluctuating components will be lumped in the form of effective shear stresses and fluxes, which in turn will be expressed through the effective exchange coefficients. The advantage of this formulation is that the equations have the same appearance as those for laminar boundary layers. Indeed, the solution procedure to be described in this thesis (Chapter 2) is applicable to laminar as well as turbulent flows, and we shall make use of this fact while testing the mathematical accuracy of the numerical method.

Any further restrictions are, in fact, not necessary. However, we shall make some more restrictions for convenience. We shall assume that the angle of inclination of the streamlines with the symmetry axis varies slowly with the downstream distance, and that swirl velocity (i.e. velocity in the plane perpendicular to the symmetry axis), body forces and thermal radiation are absent. (Should the reader be interested in the effects of strongly curved streamlines or of swirl, he is advised to see Ref. [50] for the corresponding equations; the solution procedure described in this thesis will still be useful to him after minor modifications.) The main implication of these restrictions is that then pressure is uniform in the cross-stream direction.

It may be asserted that the main solution procedure to be described later involves no further restrictions. The restrictions outlined here serve as a reminder of the limitations of the theory presented. On the other hand, the fact that they are few testifies to the generality of the theory; it gives an indication that the ambitious goal set out in Section 0.1 can actually be achieved.

1.1-2 Co-ordinate system

Fig. 1.1-1 shows the co-ordinate system which we shall use. The flow region of interest will be considered to lie between two imaginary surfaces, for which the subscripts I (internal) and E (external) will be used. The streamwise co-ordinate



x will be approximately parallel to the streamlines. The x direction makes an angle α with the axis of symmetry; α can vary with x , but only slowly. The distance y is measured outward from the $\perp$ surface and normal to the x direction. The distance from the axis of symmetry, i.e. the radius r , of any point in the flow field is given by:

$$r = r_{\perp} + y \cos \alpha \quad . \quad (1.1-1)$$

The boundary - layer equations can be expressed in a compact form by use of a stream function ψ as the cross-stream variable. This stream function is defined such that it identically satisfies the continuity equation. Thus we define:

$$x = \text{fixed:} \quad d\psi \equiv \rho u r dy \quad , \quad (1.1-2)$$

where ρ and u are respectively the fluid density and velocity in the x direction*. The $x \sim \psi$ system is known as the von Mises co-ordinate system.

1.1-3 The conservation equations

Given below are the conservation equations in $x \sim \psi$ co-ordinates.

The symbols used are defined in Nomenclature.

Conservation of momentum in x direction:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \psi} (\tau r) - \frac{1}{\rho u} \frac{dp}{dx} \quad . \quad (1.1-3)$$

Conservation of chemical species j :

$$\frac{\partial m_j}{\partial x} = - \frac{\partial}{\partial \psi} (J_j r) + \frac{R_j}{\rho u} \quad . \quad (1.1-4)$$

Conservation of stagnation enthalpy:

$$\frac{\partial \tilde{h}}{\partial x} = - \frac{\partial}{\partial \psi} \{ (J_h - u\tau) r \} \quad . \quad (1.1-5)$$

* Strictly speaking, eq. (1.1-2) does not define ψ completely. The complete definition of ψ implies that eq. (1.1-2) is valid for any two orthogonal directions x and y (in a plane passing through the axis of symmetry), u being the velocity perpendicular to the element dy .

Before proceeding to solve these equations, we shall need to know how the quantities such as τ , J_j , J_h are related to the dependent variables. These links will be provided in Section 1.3. In the mean time, we shall acquaint ourselves with different initial and boundary conditions which will be encountered in flow problems of practical interest.

1.2 Initial and boundary conditions

1.2-1 Initial profiles of dependent variables

To solve the parabolic boundary-layer equations, it is necessary to know the values of all the variables at a station upstream of the region of interest. These initial profiles can be obtained from experiments, empirical relations, approximate theories or guesswork. In turbulent boundary layers, the downstream events are usually not very sensitive to the upstream conditions. This renders the task of providing accurate initial profiles rather unimportant and hence less frightening. It is, however, desirable that the initial profiles should correspond to the real flow in certain gross features; for example, the mass, momentum or enthalpy flux implicit in the initial profiles should be approximately equal to that in the real flow under consideration.

1.2-2 The boundary conditions

Three types of boundary. As mentioned earlier, the flow region of interest will be bounded by two imaginary surfaces, I and E . Although these surfaces can, in principle, be arbitrarily located, only the following three types of boundary will suffice for all the problems of practical interest.

- (1) The boundary surface may coincide with a wall; this will be so in all the "conventional" boundary layers arising from flow over solid bodies.
- (2) The boundary surface may be adjacent to an inviscid stream. We shall refer to this type as a "free boundary". Examples of such a boundary are the outer edge of a jet or of a boundary layer on a solid body.

- (3) The boundary surface may coincide with a line of symmetry. A typical example is the centre line of a jet. The line of symmetry mentioned here is not to be confused with the axis of symmetry. For a plane jet, the centre line is a line of symmetry, whereas the axis of symmetry lies, by definition, infinitely far away.

Information along a boundary. The solution of the boundary-layer equations will need, in addition to the initial profiles, information relating to the I and E boundaries. This information is usually given in terms of the values of the dependent variables along the boundary; sometimes gradients of the variables are specified. In the particular case of the symmetry-line boundary, the gradients normal to the symmetry line are zero. In the case of a stationary-wall boundary, the velocity along the wall is always zero; but if the wall is porous, prescription of the mass-transfer rate through the wall is necessary. While describing the solution procedure, we shall show how to deal with different kinds of boundary condition. More complicated boundary conditions can be treated by obvious combinations of the procedures given.

Prescription of pressure. Because of the presence of the pressure-gradient term in the momentum-conservation equation, we need to know the variation of pressure with downstream distance. In external-flow situations, usually the velocity along a free boundary is given; this can be related to the pressure gradient via the Euler equation, i.e. (1.1-3) applied to inviscid flow. If the flow is internal, the pressure cannot be specified prior to the solution of the boundary-layer equations. Then pressure becomes an additional unknown and the solution is made possible by use of an additional relationship expressing the continuity for the whole duct.

1.3 Auxiliary relations

1.3-1 Thermodynamic relationships

The dependent variables and fluid properties are linked by many thermodynamic relationships. For example, the stagnation enthalpy is given by:

$$\tilde{h} = h + u^2/2, \quad (1.3-1)$$

where h is the specific enthalpy; here the kinetic energy associated with motion in the y direction and with the turbulent fluctuations is neglected.

Relationships connecting the fluid density with enthalpy and concentration will also be needed for our purposes. These, being well known, will not be presented here.

1.3-2 Exchange laws

Quantities like shear stress, diffusional flux and heat flux appear in the conservation equations. Had we been concerned solely with laminar boundary layers, these quantities could have been expressed through Newton's law of viscosity, Fick's law of diffusion and Fourier's law of heat conduction. For turbulent flow, it is convenient to assume that the effective shear stress, heat flux etc. also obey these laws; but the molecular exchange property is replaced by an effective exchange coefficient which is usually much larger than its laminar counterpart.

Thus, with obvious notation, we form the following exchange laws:

$$\tau = \mu_{eff} (\partial u / \partial y) \quad , \quad (1.3-2)$$

$$J_j = - (\mu_{eff} / \sigma_{j,eff}) (\partial m_j / \partial y) \quad , \quad (1.3-3)$$

$$J_h = - (\mu_{eff} / \sigma_{h,eff}) (\partial h / \partial y) \quad . \quad (1.3-4)$$

1.3-3 Expression for the effective viscosity

If the flow is laminar, μ_{eff} becomes the same as laminar viscosity and no problem arises. For the turbulent flow, however, our main weakness lies in the lack of precise knowledge regarding the effective viscosity. There are a few hypotheses available for evaluation of μ_{eff} . For example, the one by Clauser [13] makes μ_{eff} proportional to the displacement

thickness of the boundary layer. This hypothesis, although quite good for a restricted range of boundary layers, does not have sufficient generality for our purposes; for the displacement thickness may take zero and even negative values for flows with either velocity maxima or non-uniform density, and has no meaning for flows in absence of walls.

The two hypotheses which appear to possess wider applicability are those of Prandtl ([53] and [54]). The first one of these is the well-known mixing-length hypothesis. This has been many times used and is known to give reasonably good agreement between the predictions and experimental data. It is this hypothesis to which we shall mainly refer in this thesis. The second proposal of Prandtl [54] connects the effective viscosity with the kinetic energy of the fluctuating motion. The use of this hypothesis necessitates the solution of a partial differential equation for the turbulent kinetic energy. Therefore, the empirical functions and constants for this hypothesis and their implications have so far been little explored.

It is important to note here that the use of the mixing-length hypothesis in the present work is mainly illustrative. The solution procedure to be described in Chapter 2 is capable of handling any hypothesis with almost equal ease.

Prandtl's mixing-length hypothesis can be expressed as:

$$\mu_{eff} = \rho \ell^2 \left| \partial u / \partial y \right|, \quad (1.3-5)$$

where ℓ is the mixing length. For flows remote from walls, ℓ is usually taken as uniform across the layer and proportional to the thickness of the layer. The constant of proportionality lies between 0.07 and 0.1 for different types of free turbulent flow [1]. For a boundary layer on wall, the variation of ℓ in the outer part is similar to that in free turbulent flows; but near the wall, ℓ is proportional to the distance from the wall.

Such a variation of the mixing length was first used by Hudimoto [30] and is now substantiated by the experimental data collected by Escudier [20], who after examining the data recommended the following variation of ℓ :

$$\left. \begin{array}{ll} 0 < y \leq \lambda y_\ell / K & : \quad \ell = Ky \\ \lambda y_\ell / K < y & : \quad \ell = \lambda y_\ell \end{array} \right\} \quad (1.3-6)$$

where λ and K are constants, y is the distance from the wall, and y_ℓ is a characteristic thickness of the layer. Escudier's collection includes various types of boundary layer : those with zero, adverse or favourable pressure gradient, those with or without velocity maxima. The interesting fact is that all the data show roughly the same quantitative relation between the mixing length and the distance from the wall. This demonstrates the wider applicability of eq. (1.3-5) and (1.3-6) taken together. Further, Maise and McDonald [38] have shown, from experimental data in compressible boundary layers, that upto the Mach number of 5, effect of compressibility on mixing length is negligible. By use of eq. (1.3-5) and (1.3-6), successful comparisons with experimental data are presented in Refs. [49] and [10] ; these provide further, though rather indirect, justification for the applicability of these equations.

The implausibility of the mixing-length hypothesis resides in its implication that the effective viscosity vanishes at the point of zero velocity gradient. This, when combined with an assumption of finite effective Prandtl number, leads to the unrealistic conclusion that the heat flux is zero at the point of zero velocity gradient. This defect in the hypothesis is, however, likely to cause little harm except in some exceptional circumstances.

1.3-4 Effective Prandtl and Schmidt number

In Section 1.3-2 we introduced the exchange - coefficient ratios $\sigma_{j,eff}$ and $\sigma_{h,eff}$. These are respectively the effective Schmidt number and the effective Prandtl number. The available experimental data for the value of $\sigma_{h,eff}$ in turbulent pipe flows are summarised in the review made by Kestin and Richardson [33]; the data suggest that $\sigma_{h,eff}$ remains roughly uniform across the pipe and has a value around 0.8. Jayatilaka [31] recommended a value of 0.9 for $\sigma_{h,eff}$ or $\sigma_{j,eff}$ after examination of a very large number of pipe-flow data for both heat and mass transfer.

The values of $\sigma_{j,eff}$ and $\sigma_{h,eff}$ in free turbulent flows, however, seem to be lower. For axisymmetrical jets, several authors report a value of 0.7 (see Refs. [23], [29]). For plane jets, wakes and mixing layers, a value around 0.5 has been found to be appropriate ([55], [90] and [1]).

1.3-5 The region near a wall

Need for special treatment. The above-mentioned formulae for the effective viscosity and other exchange properties take into account only the turbulent contribution and neglect the laminar part. This is satisfactory for the most part, because the turbulent contribution is much greater than the laminar one. In the vicinity of a wall, however, the magnitude of the turbulent viscosity diminishes (as can be seen from eq. (1.3-5) and (1.3-6)), and becomes comparable to the laminar viscosity. The effective Schmidt and Prandtl numbers also approach their laminar values very near the wall. Thus, a more detailed hypothesis for μ_{eff} describing the respective roles of turbulent and molecular viscosities is needed for the region near a wall. Indeed, the hypothesis for this region is even more important, because usually very steep gradients of velocity and other variables exist near a wall, and because the shear stress and fluxes at the wall are of greater practical interest.

Expression for μ_{eff} near a wall. There are many proposals

in the literature for the effective viscosity near a wall. Most of them spring from an expression for a "universal law of the wall" and an assumption of uniform shear stress. These proposals have been listed in Refs. [31] and [33]. All such expressions have been designed to accord with the experimental evidence in the absence of pressure gradient, non-uniform fluid properties, and mass transfer at the wall. For the more general case, they will need replacement by a more fundamental hypothesis. The hypothesis which appears most suitable at present is that of van Driest [89]; its modification, which we shall make, almost suggests itself. We shall use this hypothesis in the following form:

$$\mu_{eff} = \mu + \rho K^2 y^2 [1 - \exp\{-y\sqrt{\tau\rho}/(\mu A_+)\}]^2 |\partial u/\partial y|, \quad (1.3-7)$$

where A_+ is a constant. The hypothesis implies that μ_{eff} is composed of the laminar viscosity μ and the turbulent viscosity $\rho K^2 y^2 |\partial u/\partial y|$, which gets damped near the wall in an exponential fashion. Van Driest was concerned with a layer in which the shear stress was everywhere equal to that at the wall, and therefore, he used the value of the wall shear stress in the exponential term; as an obvious modification, it has been replaced by the local shear stress in the fluid in eq. (1.3-7).

It is eq. (1.3-7) that will be used for the wall-near region in the present work; it also appears consistent with our use of eq. (1.3-5) and (1.3-6) for the fully - turbulent part of the boundary layer. Once again, our purpose in employing a particular hypothesis is to make the treatment more concrete; the hypothesis, however, does not limit the capabilities of the solution procedure to be described.

Expressions for $\sigma_{j,eff}$ and $\sigma_{h,eff}$. The effective Schmidt or Prandtl number can be obtained, near a wall, by use of a formula of the form:

$$\frac{\mu_{eff}}{\sigma_{eff}} = \frac{\mu}{\sigma} + \frac{\rho K^2 y^2}{\sigma_t} [1 - \exp\{-y\sqrt{\tau\rho}/(\mu A_+)\}]^2 |\partial u/\partial y|, \quad (1.3-8)$$

where σ and σ_t are respectively the laminar and turbulent Schmidt or Prandtl numbers. The value of σ_t will be the same as that of σ_{eff} for the fully-turbulent region; thus the discussion in Section 1.3-4 applies to σ_t as well.

The role of the Couette-flow analysis. So far, sufficient information has been given regarding the initial and boundary conditions and the exchange laws, so that the solution of the conservation equations of Section 1.1-3 appears to be possible at least in principle. Before starting to discuss, in Section 1.5 below, the problem of solving these equations, we foresee a difficulty associated with the wall-near region. As mentioned earlier, both the dependent variables and the effective exchange coefficients vary steeply in this region. As a consequence, whichever numerical method of solution we use, good accuracy for this region can be purchased by only spending excessive computational effort in this narrow region. One of the novelties of the calculation procedure to be described in this thesis is that it overcomes this difficulty. The device used for this will be briefly outlined now. Near a wall, the velocity u is small and therefore, the x -wise convection is locally negligible. Thus, there exists a one-dimensional boundary layer near the wall; this is also known as the "Couette flow". Then the flow problem in this region is essentially that of solving ordinary rather than partial differential equations, and what is more attractive is that such Couette-flow solutions can be carried out once-for-all, to be merely used as asymptotes or boundary conditions in the particular solutions of the partial differential equations. Of course, it would be very convenient if the results

of these Couette-flow analyses were expressed in the form of algebraic relationships; some examples of these will be presented later in Chapter 4.

The treatment of the one-dimensional region near a wall, therefore, forms an important part of our theoretical framework. This treatment will now be given in Section 1.4 below. Should the reader wish to skip this rather long section, it would suffice for him to note at this stage that the use, near the wall, of the once-for-all Couette-flow solutions takes a considerable burden off the subsequent solutions of the partial differential equations for particular problems. The analysis in Section 1.4 leads to relationships like eq. (1.4-92), (1.4-93) and (1.4-94), which we shall later use in our solution procedure.

1.4 Equations of the one-dimensional boundary layer near a wall

The purpose of this section is to provide a general treatment of the one-dimensional boundary layer near a wall. The main elements of the method, however, can be put forward without handling the completely general equations which tend to make the treatment more involved. For this reason, we shall make restriction to a chemically inert fluid with uniform properties. It is easy to see that this restriction is inessential and can be removed when so desired.

We shall deal with the distributions of velocity u , and of a general dependent variable ϕ which can stand for m_j in the absence of chemical reaction and for $\tilde{h}$ when either the kinetic heating is negligible or the effective Prandtl number equals unity everywhere. It will be shown later that almost the same treatment can be made applicable to the case of finite kinetic heating and non-unity $\sigma_{h,eff}$ by defining an adiabatic-wall enthalpy.

1.4-1 The Couette-flow assumption

When the x -wise convection is neglected, the solution of the resulting one-dimensional conservation equations can be expressed as*:

$$\tau = \tau_S + \frac{dp}{dx} y + \dot{m}_S'' u \quad , \quad (1.4-1)$$

$$\text{and} \quad J = J_S + \dot{m}_S'' (\phi_S - \phi) \quad . \quad (1.4-2)$$

Here the subscript S denotes the conditions at the wall. These relations can be put into dimensionless form by use of:

$$\tau_+ \equiv \tau / \tau_S \quad , \quad (1.4-3)$$

$$y_+ \equiv y \sqrt{\tau_S \rho} / \mu \quad , \quad (1.4-4)$$

$$u_+ \equiv u / \sqrt{\tau_S / \rho} \quad , \quad (1.4-5)$$

$$p_+ \equiv \mu (dp/dx) / (\tau_S^{\frac{3}{2}} \rho^{\frac{1}{2}}) \quad , \quad (1.4-6)$$

$$m_+ \equiv \dot{m}_S'' / \sqrt{\tau_S \rho} \quad , \quad (1.4-7)$$

$$J_+ \equiv J / J_S \quad , \quad (1.4-8)$$

$$\phi_+ \equiv (\phi_S - \phi) \sqrt{\tau_S \rho} / J_S \quad , \quad (1.4-9)$$

leading to:

$$\tau_+ = 1 + p_+ y_+ + m_+ u_+ \quad , \quad (1.4-10)$$

$$\text{and} \quad J_+ = 1 + m_+ \phi_+ \quad . \quad (1.4-11)$$

Using the exchange laws of Section 1.3-2, we can also write:

$$\tau_+ = (\mu_{\text{eff}} / \mu) / (du_+ / dy_+) \quad , \quad (1.4-12)$$

$$\text{and} \quad J_+ = (\mu_{\text{eff}} / \mu \sigma_{\text{eff}}) / (d\phi_+ / dy_+) \quad . \quad (1.4-13)$$

Here the partial derivatives have been replaced by the ordinary ones, because the problem is one-dimensional.

* These expressions are given here without proof as they are well known and easy to derive. The variation of radius r is ignored because of the thinness of the region under consideration.

1.4-2 Implications of the generalised van Driest hypothesis

The hypothesis. The above equations will yield solutions for u and ϕ only after the use of expressions for μ_{eff} and σ_{eff} . For this purpose we shall use van Driest's hypothesis as given by eq. (1.3-7), which in dimensionless form will appear as:

$$\frac{\mu_{\text{eff}}}{\mu} = 1 + K^2 y_+^2 [1 - \exp(-y_+ \sqrt{\tau_+}/A_+)]^2 (du_+/dy_+) \quad (1.4-14)$$

Here the modulus sign for (du_+/dy_+) has been dropped for obvious reasons. Similarly, eq. (1.3-8) can be rewritten as:

$$\frac{\mu_{\text{eff}}}{\mu \sigma_{\text{eff}}} = \frac{1}{\sigma} + \frac{K^2 y_+^2}{\sigma_t} [1 - \exp(-y_+ \sqrt{\tau_+}/A_+)]^2 (du_+/dy_+) \quad (1.4-15)$$

Although two constants, K and A_+ , appear in these expressions, general Couette-flow solutions can be obtained by use of only one constant, and then there exists a particular solution for every value of K . This will be made possible by way of the following definitions:

$$y_* \equiv K y_+ \quad , \quad (1.4-16)$$

$$u_* \equiv K u_+ \quad , \quad (1.4-17)$$

$$A_* \equiv K A_+ \quad . \quad (1.4-18)$$

Then we have:

$$\frac{\mu_{\text{eff}}}{\mu} = 1 + y_*^2 [1 - \exp(-y_* \sqrt{\tau_+}/A_*)]^2 (du_*/dy_*) \quad , \quad (1.4-19)$$

and

$$\frac{\mu_{\text{eff}}}{\mu \sigma_{\text{eff}}} = \frac{1}{\sigma} + \frac{y_*^2}{\sigma_t} [1 - \exp(-y_* \sqrt{\tau_+}/A_*)]^2 (du_*/dy_*) \quad . \quad (1.4-20)$$

Further, if we define:

$$p_* \equiv p_+/K \quad , \quad (1.4-21)$$

$$m_* \equiv m_+/K \quad , \quad (1.4-22)$$

$$\text{and} \quad \phi_* \equiv K \phi_+ \quad , \quad (1.4-23)$$

then the equations (1.4-10) and (1.4-11) can be rewritten as:

$$\tau_+ = 1 + p_* y_* + m_* u_* \quad , \quad (1.4-24)$$

and $J_+ = 1 + m_* \phi_* \quad . \quad (1.4-25)$

The profiles of u and ϕ . Since the equation (1.4-12) can be written as:

$$\tau_+ = \frac{\mu_{eff}}{\mu} \frac{du_*}{dy_*} \quad , \quad (1.4-26)$$

substitution from eq. (1.4-19) and (1.4-24) into eq. (1.4-26), followed by some algebraic manipulation leads to:

$$\frac{du_*}{dy_*} = \frac{2 (1 + p_* y_* + m_* u_*)}{1 + \sqrt{1 + 4 y_*^2 (1 + p_* y_* + m_* u_*) [1 - \exp(-y_* \sqrt{1 + p_* y_* + m_* u_*} / A_*)]^2}} \quad (1.4-27)$$

Similarly, for the dimensionless flux J_+ we have:

$$J_+ = \frac{\mu_{eff}}{\mu \sigma_{eff}} \frac{d\phi_*}{dy_*} \quad , \quad (1.4-28)$$

which in combination with eq. (1.4-20) and (1.4-25) will give:

$$\frac{d\phi_*}{dy_*} = \frac{(1 + m_* \phi_*) \sigma_t}{\frac{1}{\sigma/\sigma_t} + y_*^2 [1 - \exp(-y_* \sqrt{1 + p_* y_* + m_* u_*} / A_*)]^2 (du_*/dy_*)} \quad (1.4-29)$$

For a given value of A_* , and with the obvious boundary condition:

$$y_* = 0 : \quad u_* = 0, \quad \phi_* = 0 \quad , \quad (1.4-30)$$

equations (1.4-27) and (1.4-29) can be solved to yield profiles of u_* and ϕ_* in the following form:

$$u_* = u_*(y_*, p_*, m_*) \quad , \quad (1.4-31)$$

and $\phi_* = \phi_*(y_*, p_*, m_*, \sigma/\sigma_t, \sigma_t) \quad . \quad (1.4-32)$

Some such solutions, obtained numerically, will be later presented in Chapter 4. Under special circumstances analytical solutions are obtainable;

we shall obtain some of these in Section 1.4-4 below. In the mean time, we shall turn to the equation for the stagnation enthalpy and show how the adiabatic-wall enthalpy can be made use of.

1.4-3 Use of the adiabatic-wall enthalpy

Here we shall show how the above treatment for $\emptyset$ can be made useful to obtain $\tilde{h}$ even when kinetic heating is not negligible and the effective Prandtl number is non-unity.

The adiabatic-wall relations. Neglecting the x -wise convection, we obtain, from eq. (1.1-5), the relation:

$$J_h = J_{h,S} + \dot{m}''_S (\tilde{h}_S - \tilde{h}) + u\tau \quad (1.4-33)$$

For an adiabatic wall, $J_{h,S}$ is zero; thus,

$$J_{h,ad} = \dot{m}''_S (\tilde{h}_{ad,S} - \tilde{h}_{ad}) + u\tau \quad (1.4-34)$$

where the subscript "ad" is a reminder of the adiabatic-wall condition.

Now, the equations (1.3-1), (1.3-2) and (1.3-4) enable us to write eq.

(1.4-34) as:

$$\frac{\mu_{eff}}{\sigma_{h,eff}} \frac{d\tilde{h}_{ad}}{dy} + \left(1 - \frac{1}{\sigma_{h,eff}}\right) \mu_{eff} \frac{du^2/2}{dy} + \dot{m}''_S (\tilde{h}_{ad,S} - \tilde{h}_{ad}) = 0 \quad (1.4-35)$$

the dimensionless form of which is:

$$\frac{\mu_{eff}}{\mu \sigma_{h,eff}} \frac{d\tilde{h}_{*,ad}}{dy_*} = \left(1 - \frac{1}{\sigma_{h,eff}}\right) \frac{\mu_{eff}}{\mu} \frac{du_*^2/2}{dy_*} + m_* \tilde{h}_{*,ad} \quad (1.4-36)$$

$$\text{here } \tilde{h}_{*,ad} \equiv K^2 \tilde{h}_{+,ad} \quad (1.4-37)$$

$$\text{and } \tilde{h}_{+,ad} \equiv \frac{\rho(\tilde{h}_{ad,S} - \tilde{h}_{ad})}{\tau_S} \quad (1.4-38)$$

Solution of eq. (1.4-36), by use of eq. (1.4-19), (1.4-20) and (1.4-27), can be obtained in the following form:

$$\tilde{h}_{*,ad} = \tilde{h}_{*,ad} (y_*, p_*, m_*, \sigma_h/\sigma_{h,t}, \sigma_{h,t}) \quad (1.4-39)$$

The recovery factor H , defined by:

$$H \equiv 1 + \frac{(\tilde{h}_{ad,S} - \tilde{h}_{ad})}{u^2/2}, \quad (1.4-40)$$

can then be expressed as:

$$H = 1 + 2\tilde{h}_{*,ad}/u^2. \quad (1.4-41)$$

When the wall is not adiabatic. We shall now define a new quantity $\tilde{\tilde{h}}$ as:

$$\tilde{\tilde{h}} \equiv \tilde{h} + (H - 1) u^2/2, \quad (1.4-42)$$

where the recovery factor H is to be substituted from eq. (1.4-39) and (1.4-41). It can be easily seen that, for an adiabatic wall, $\tilde{\tilde{h}}$ will be the same as $\tilde{h}_{ad,S}$. If we now substitute for $\tilde{h}$ from eq. (1.4-42) into (1.4-33) and then eliminate H via eq. (1.4-40), and $(\tilde{h}_{ad,S} - \tilde{h}_{ad})$ via (1.4-34), we obtain:

$$J_h - J_{h,ad} = J_{h,S} + \dot{m}''_S (\tilde{h}_S - \tilde{\tilde{h}}), \quad (1.4-43)$$

which can be expressed as:

$$J_{+,h} = 1 + m_+ h_+, \quad (1.4-44)$$

$$\text{where} \quad J_{+,h} \equiv \frac{J_h - J_{h,ad}}{J_{h,S}}, \quad (1.4-45)$$

$$\text{and} \quad h_+ \equiv \frac{(\tilde{h}_S - \tilde{\tilde{h}}) \sqrt{\tau_S \rho}}{J_{h,S}}. \quad (1.4-46)$$

The equation (1.4-44) is identical in form to eq. (1.4-11). What remains to be shown is that $J_{+,h}$ obeys a relation similar to eq. (1.4-13).

Since:

$$J_h - J_{h,ad} = -\frac{\mu_{eff}}{\sigma_{h,eff}} \left\{ \frac{d\tilde{h}}{dy} - \left(\frac{d\tilde{h}}{dy} \right)_{ad} \right\}, \quad (1.4-47)$$

substitution from eq. (1.4-42) will yield:

$$J_h - J_{h,ad} = -\frac{\mu_{eff}}{\sigma_{h,eff}} \left\{ \frac{d\tilde{h}}{dy} - \left(\frac{d\tilde{h}}{dy} \right)_{ad} \right\}. \quad (1.4-48)$$

Now $\tilde{h}$ as defined by eq. (1.4-42) will be same as $\tilde{h}_{ad,S}$ (hence independent of y) under the adiabatic-wall condition; thus the second term in the curly brackets in eq. (1.4-48) can be put equal to zero. The resulting equation in dimensionless form will be:

$$J_{+,h} = \frac{\mu_{eff}}{\mu \sigma_{h,eff}} \frac{dh_+}{dy_+}, \quad (1.4-49)$$

which is similar to eq. (1.4-13). After noting that $J_{+,h}$, h_+ , $\sigma_{h,eff}$ and $\tilde{h}$ are respectively analogous to J_+ , ϕ_+ , σ_{eff} and ϕ , we can use all the treatment of the ϕ_+ equation for h_+ as well. Then the connection with the local value of $\tilde{h}$ can be made via eq. (1.4-42) in which H is substituted from a separate analysis as mentioned above.

1.4-4 Some analytical solutions

It has been mentioned that eq. (1.4-27) and (1.4-29) will in general be solved by numerical integration; however, some insight into these solutions can be obtained by the consideration of some simple cases.

(a) The laminar asymptotes. If we restrict our attention to very small values of Y_* , in the limit the laminar condition will be attained. Then, eq. (1.4-27) reduces to:

$$\frac{du_*}{dy_*} = 1 + p_* Y_* + m_* u_*, \quad (1.4-50)$$

which has the following implications:

$$p_* = m_* = 0 \quad : \quad u_* = y_* \quad ; \quad (1.4-51)$$

$$p_* \neq 0, m_* = 0 \quad : \quad u_* = y_* + p_* y_*^2 / 2 \quad ; \quad (1.4-52)$$

$$p_* = 0, m_* \neq 0 \quad : \quad u_* = \{\exp(m_* y_*) - 1\} / m_* \quad . \quad (1.4-53)$$

Similarly, eq. (1.4-29) becomes:

$$\frac{d\phi_*}{dy_*} = \sigma (1 + m_* \phi_*) \quad , \quad (1.4-54)$$

which has the solution:

$$\phi_* = \sigma \{\exp(m_* y_*) - 1\} / m_* \quad ; \quad (1.4-55)$$

$$\text{for } m_* \rightarrow 0, \quad \phi_* = \sigma y_* \quad . \quad (1.4-56)$$

(b) The case of unity turbulent Prandtl or Schmidt number. It

seems tiresome that in the list of arguments in eq. (1.4-32), σ_t has to be included in addition to the ratio σ / σ_t . Let us now explore the possibility of generalising a solution obtained for $\sigma_t = 1$ to any arbitrary value of σ_t . Let $\phi_{*(1)}$ stand for the value of ϕ_* at the same value of y_* and of σ / σ_t , but when σ_t equals unity. Then, from eq. (1.4-29) we obtain:

$$\frac{d\phi_*}{d\phi_{*(1)}} = \frac{(1 + m_* \phi_*) \sigma_t}{1 + m_* \phi_{*(1)}} \quad . \quad (1.4-57)$$

It must be emphasised that ϕ_* and $\phi_{*(1)}$ refer to the same value of σ / σ_t and not to the same value of σ . Integration of eq. (1.4-57) now yields:

$$\phi_* = \frac{(1 + m_* \phi_{*(1)})^{\sigma_t} - 1}{m_*} \quad , \quad (1.4-58)$$

which has the limiting form:

$$m_* \rightarrow 0, \quad \phi_* = \sigma_t \phi_{*(1)} \quad . \quad (1.4-59)$$

Thus instead of eq. (1.4-32), we can use:

$$\boxed{\phi_{*(1)} = \phi_{*(1)}(y_*, p_*, m_*, \sigma/\sigma_t)} \quad , \quad (1.4-60)$$

followed by conversion to ϕ_* via eq. (1.4-58).

(c) The fully-turbulent asymptotes. For large values of y_* , the contribution of the laminar viscosity and the damping due to the exponential term are negligible. Then eq. (1.4-27) can be expressed as:

$$\frac{du_*}{dy_*} = \frac{(1 + p_* y_* + m_* u_*)^{\frac{1}{2}}}{y_*} \quad . \quad (1.4-61)$$

$$\text{For } p_* = m_* = 0, \text{ we have: } u_* = \ln(E_* y_*) \quad , \quad (1.4-62)$$

where E_* is an integration constant.

For $p_* \neq 0$ and $m_* = 0$, we have:

$$u_* = 2(\sqrt{1 + p_* y_*} - 1) + \ln \left\{ \frac{4E_* y_*}{2 + p_* y_* + 2\sqrt{1 + p_* y_*}} \right\} \quad , \quad (1.4-63)$$

where the integration constant E_* will now depend on p_* . (This result is essentially same as the one obtained by Townsend [87] from considerations of turbulent kinetic energy.)

For $p_* = 0$ and $m_* \neq 0$, we have:

$$u_* = \ln(E_* y_*) + \frac{m_*}{4} \{\ln(E_* y_*)\}^2 \quad , \quad (1.4-64)$$

the integration constant E_* being a function of m_* . (This type of result has been obtained in Refs. [12], [39], [8], and [83].)

For the fully-turbulent region and $p_* = 0$, we can obtain,

from eq. (1.4-27) and from an equation for $\phi_{*(1)}$ (derived from eq. (1.4-29)), the relation:

$$\frac{d\phi_{*(1)}}{du_*} = \frac{1 + m_*\phi_{*(1)}}{1 + m_*u_*}, \quad (1.4-65)$$

which, on integration, gives:

$$\frac{\ln(1 + m_*\phi_{*(1)})}{m_*} = \frac{\ln(1 + m_*u_*)}{m_*} + P_*, \quad (1.4-66)$$

where the integration constant P_* will, in general, be a function of m_* and σ/σ_t . The limiting form of eq. (1.4-66), for $m_* \rightarrow 0$, will be:

$$\phi_{*(1)} = u_* + P_*. \quad (1.4-67)$$

(Spalding and Jayatillaka [81] have used a form of eq. (1.4-67), and Spalding [72] has later generalised it by way of an expression similar to eq. (1.4-66).)

(d) The case of large σ/σ_t , $p_* = m_* = 0$. It is easy to show that eq. (1.4-67) is the solution of the differential equation:

$$\frac{d\phi_{*(1)}}{du_*} = \sigma_{\text{eff}}, \quad (1.4-68)$$

which is valid when $p_* = m_* = 0$. Integrating eq. (1.4-68), we obtain:

$$\phi_{*(1)} = u_* + \int_0^{u_*} (\sigma_{\text{eff}} - 1) du_*. \quad (1.4-69)$$

For large values of y_* (and hence of u_*), the integrand in the second term on the right of eq. (1.4-69) will vanish; then the upper limit can be replaced by infinity. Thus we obtain an expression for P_* in eq. (1.4-67) as:

$$P_* = \int_0^{\infty} (\sigma_{\text{eff}} - 1) du_*. \quad (1.4-70)$$

When σ/σ_t is very large, the significant contribution to P_* comes from the region of small values of u_* . The asymptotic value of P_* , therefore, can be found by using only the first term in the series expansion of the exponential term in eq. (1.4-20) and by assuming that y_* approximately equals u_* in this region. Then,

$$P_* = \int_0^{\infty} \left\{ \frac{1 + (u_*^4/A_*^2)}{(\sigma_t/\sigma) + (u_*^4/A_*^2)} \right\} du_* \quad , \quad (1.4-71)$$

which gives the required asymptotic expression:

$$\text{large } \sigma/\sigma_t : \quad P_* = \frac{\pi}{4 \sin(\pi/4)} A_*^{\frac{1}{2}} (\sigma/\sigma_t)^{-\frac{1}{4}} (\sigma/\sigma_t - 1) \quad . \quad (1.4-72)$$

(e) The recovery factor for $m_* = 0$. If we put $m_* = 0$ in eq. (1.4-36), then integration yields:

$$\tilde{h}_{*,ad} = \int_0^{u_*^2/2} (\sigma_{h,eff} - 1) du_*^2/2 \quad . \quad (1.4-73)$$

If σ_h and $\sigma_{h,t}$ are equal, $\sigma_{h,eff}$ will be uniform throughout and then we obtain:

$$\tilde{h}_{*,ad} = (\sigma_{h,t} - 1) u_*^2/2 \quad . \quad (1.4-74)$$

It should be noted that, even when σ_h is different from $\sigma_{h,t}$, the value of $\sigma_{h,eff}$ significantly differs from $\sigma_{h,t}$ only in the region of small u_* where the value of the integral in eq. (1.4-73) will be small anyway; thus, eq. (1.4-74) will be sufficiently accurate for large values of u_* even when $\sigma_h \neq \sigma_{h,t}$.

Substitution of eq. (1.4-74) into (1.4-41) yields the simple result:

$$H = \sigma_{h,t} \quad . \quad (1.4-75)$$

1.4-5 Explicit forms of the relationships

The relationships such as eq. (1.4-31) or (1.4-60) represent the main outcome of our Couette-flow analysis so far. We shall later use them to relate the values of y , u , ϕ , dp/dx , $\dot{m}_S''$ etc. to τ_S and J_S ; but since almost every dimensionless quantity used above contains τ_S , the process will necessarily be iterative. This feature can be removed by casting the relationships like (1.4-31) into convenient explicit forms.

For this purpose, we shall define new dimensionless groups and arrange them so as to exhibit the wall shear stress, heat flux etc. as functions of the known quantities. Here we shall merely introduce the symbols and show their connection with the earlier ones; in Chapter 2 we shall assume that explicit algebraic relationships connecting these new symbols are obtainable from the Couette-flow analysis of the present section. Examples of the actual formulae will be presented in Chapter 4.

We shall define:

$$R \equiv \frac{\rho u y}{\mu}, \quad (1.4-76)$$

$$F \equiv \frac{y}{\rho u^2} \frac{dp}{dx}, \quad (1.4-77)$$

$$M \equiv \frac{\dot{m}_S''}{\rho u}, \quad (1.4-78)$$

$$s \equiv \frac{\tau_S}{\rho u^2}, \quad (1.4-79)$$

$$S \equiv \frac{J_S}{\rho u (\phi_S - \phi)}, \quad (1.4-80)$$

and further,

$$R_* \equiv K^2 R, \quad (1.4-81)$$

$$F_* \equiv F/K^2, \quad (1.4-82)$$

$$M_* \equiv M/K^2, \quad (1.4-83)$$

$$s_* \equiv s/K^2, \quad (1.4-84)$$

$$S_* \equiv S/K^2. \quad (1.4-85)$$

Now if we note that:

$$R_* = u_* y_* \quad , \quad (1.4-86)$$

$$F_* = p_* y_* / u_*^2 \quad , \quad (1.4-87)$$

$$M_* = m_* / u_* \quad , \quad (1.4-88)$$

$$s_* = 1/u_*^2 \quad , \quad (1.4-89)$$

$$S_{*(1)} = 1/(u_* \phi_{*(1)}) \quad , \quad (1.4-90)$$

$$S_* = 1/(u_* \phi_*) \quad , \quad (1.4-91)$$

it is easy to see that the information contained in the profile relations such as eq. (1.4-31), (1.4-60) and (1.4-39) can be expressed in the following convenient forms:

$$s_* = s_* \{R_*, F_*, M_*\} \quad , \quad (1.4-92)$$

$$S_{*(1)} = S_{*(1)} \{R_*, F_*, M_*, \sigma/\sigma_t\} \quad , \quad (1.4-93)$$

$$H = H\{R_*, F_*, M_*, \sigma_h/\sigma_{h,t}, \sigma_{h,t}\} \quad . \quad (1.4-94)$$

Here the significance of $S_{*(1)}$ can be easily understood; indeed, eq. (1.4-58) enables us to relate S_* to $S_{*(1)}$ in the following manner:

$$S_* = \frac{M_*}{(1 + M_*/S_{*(1)})^{\sigma_t} - 1} \quad , \quad (1.4-95)$$

which has the limiting form:

$$M_* \rightarrow 0 : \quad S_* = S_{*(1)} / \sigma_t \quad . \quad (1.4-96)$$

Now we have completed the erection of the theoretical framework for the one-dimensional region close to a wall. It has been indicated how the results of the analysis for this region can be expressed by way of convenient symbols. A one-dimensional region, however, does not exist in the outer part of the boundary layer. There the partial differential equations of Section 1.1-3 will have to be solved; we shall now

turn to the question of choosing a suitable procedure for doing this.

1.5 Choice of solution procedure

Requirements of the procedure sought. Our efforts are directed towards the long-term objective of establishing a turbulent-boundary-layer theory as sound as, let us say, the theory of heat conduction. The choice of solution procedure for solving the partial differential equations of Section 1.1-3 should therefore conform to this final aim. The solution procedure should allow us to work out the true implications of an effective-viscosity hypothesis without introducing mathematical inaccuracies. It should retain its utility even when the present hypotheses are discarded and replaced by new ones. It should be computationally inexpensive, simple to use and widely applicable.

Known methods of solution. There are three main techniques for solving parabolic partial differential equations. The first of these, to be quickly dismissed as unacceptable for the present purposes, involves use of finite differences in the stream-wise direction and solution of ordinary differential equations in the cross-stream direction. This presents a succession of two-point boundary-value problems, to be solved by the iterative procedure of Hartree and Womersley [28]. This iteration is necessary at every station as the integration proceeds downstream. The result is that the required computing time is excessively large. A description of this method can be found in Refs. [65], [66], [67], where the reported computing time is larger by a factor of over 500 than that for the method of Ref. [49] or for the one to be described in this thesis.

The second method of solution involves conversion of the partial differential equations into ordinary ones by integrating them in the cross-stream direction after multiplication by a weighting function; the resulting

equations are solved in conjunction with assumptions regarding the algebraic forms of the dependent-variable profiles. The method is known as the "profile" or "integral" method. Although the earlier applications of the integral method were motivated by the obvious physical significance of some integral equations, the integral method is nowadays increasingly recognised as a mathematical tool to solve the parent partial differential equations.

Use of a few integral equations gives good value for the computing expense, provided that the profile family is judiciously chosen for a particular class of problems under consideration. This point is illustrated by Refs. [47] and [49]*. Ref. [47] demonstrates the ability to obtain, by use of only two integral equations, a solution within less than 2% of the available exact solution. Ref. [49] makes this point indirectly : it shows how, by use of physically realistic hypotheses and profiles, the integral method can give results, which agree with experimental measurements.

It is, of course, true that profiles with a few parameters cannot represent all the situations of practical interest. For example, the power-law profiles, which have been very widely used in the past, cannot take shapes appropriate for exhibiting maxima. As a result, profiles with small flexibility cannot give good accuracy over the entire range of conditions.

It is certainly possible to use many parameters in the profile and to solve the corresponding number of integral equations. This method, outlined in Ref. [76], ensures wider applicability and higher accuracy; but objections can be made which are no less serious. The algebraic complication

* Both of these papers are being submitted as subsidiary material along with this thesis.

increases the computing time. Different algebraic expressions are necessary for plane and axi-symmetrical flows, and for different types of variation of the fluid density. All the necessary integrals of the profiles may not be analytically obtainable ; thus numerical evaluation of quadratures is required. Prior to forward integration , the ordinary differential equations for the parameters are obtained via matrix inversion. This increases the computing time in proportion to the square of the number of parameters. The forward integration has to be performed by explicit methods , because the equations are usually strongly non-linear and there is no obvious way of linearising them. Then as the number of parameters is increased , the stability considerations may impose severe limitations on the size of the forward step , thus increasing the computing time further. All these considerations make the integral method less attractive for our purpose of obtaining accurate solutions of the partial differential equations with modest computing time.

Lastly, the finite-difference methods are a standard means of solving parabolic partial differential equations. These methods are known to give unlimited accuracy , but also to be very expensive in computing time. In the following pages will be introduced certain innovations that have now changed this situation. The method to be described in the next chapter retains the advantages of the finite-difference methods and incorporates devices which largely reduce the computing time. The main saving of the computational effort comes from the use of a grid that always adjusts its width so as to conform to the thickness of the layer, and from the use of the Couette-flow relationships near a wall. The implicit nature of the difference formulae allows use of large forward steps without incurring the danger of instability. Treatment of axi-symmetrical rather than plane flows , and of non-uniform density offers no special difficulty.

Chapter 2

A finite-difference solution procedure

2.1 Choice of co-ordinate system

The co-ordinate system used has a particular importance in a finite-difference procedure; because attention is given to the values of the dependent variables only at a finite number of points which are usually arranged as a grid in the chosen co-ordinate system. The most common practice is to employ a rectangular grid in $x \sim y$ co-ordinates. It is easy to see that this is disadvantageous for boundary-layer problems. Usually, boundary layers grow in thickness in the downstream direction. If the grid spacing is such that a certain minimum number of grid points lie within the thickness of the layer at the upstream end, then it will prove to be excessively fine and hence wasteful at the downstream end; similarly, a spacing designed for the downstream end will be too coarse for the upstream region and hence the accuracy there will be poor.

The above-mentioned inefficiency of $x \sim y$ grid is the main reason why finite-difference methods are regarded as very expensive. The same is true when the stream function ψ is used as a cross-stream co-ordinate instead of y . The procedures, which are sometimes used, for varying the grid spacing as the computation proceeds are inelegant, troublesome and unsuited for general application.

The ideal system will be to have a grid which always fits the boundary-layer region. There are different ways of ensuring this. Lew [36], for example, while obtaining solutions for laminar mixing of chemically-reacting streams used the dimensionless stream-wise velocity as the cross-stream variable. This had the desired effect, because no matter how much the mixing layer grew, the velocity varied across the layer between fixed limits. It should, however, be noted that velocity is not a convenient cross-stream variable for general purposes. It is necessary that the independent

cross-stream variable varies monotonically across the layer. When the velocity profile exhibits a maximum, velocity as an independent variable ceases to be satisfactory.

We shall use below the dimensionless stream function ω as the cross-stream variable. It will be seen that this combines the advantages of stream-function co-ordinates with those of restricting the boundary layer to a finite range in ω . We shall define ω as follows:

$$\omega \equiv (\psi - \psi_I) / (\psi_E - \psi_I) \quad , \quad (2.1-1)$$

where ψ_I and ψ_E are the values of the stream function at the I and E boundaries of the layer; ω is so defined that it equals zero at the I boundary and unity at E. We shall choose ψ_I and ψ_E such that all the important variations in the dependent variables take place at ω values between zero and unity.

From the definition of ψ , we have:

$$\frac{d\psi_I}{dx} = -r_I \dot{m}''_I \quad , \quad (2.1-2)$$

$$\text{and} \quad \frac{d\psi_E}{dx} = -r_E \dot{m}''_E \quad , \quad (2.1-3)$$

where $\dot{m}''_I$ and $\dot{m}''_E$ are the rates of mass transfer across the I and E surfaces. We shall use eq. (2.1-2) and (2.1-3) to determine the variation of ψ_I and ψ_E . How to obtain $\dot{m}''_I$ and $\dot{m}''_E$ is an important problem, which will be discussed in Section 2.3 below.

2.2 Conservation equations in $x \sim \omega$ co-ordinates

Now we shall transform the equations of Section 1.1-3 into $x \sim \omega$ system, at the same time substituting from the exchange laws of Section 1.3-2. Thus we have:

Conservation of momentum:

$$\frac{\partial u}{\partial x} + \frac{\{r_I \dot{m}''_I + \omega(r_E \dot{m}''_E - r_I \dot{m}''_I)\}}{(\psi_E - \psi_I)} \frac{\partial u}{\partial \omega} = \frac{\partial}{\partial \omega} \left\{ \frac{r^2 \rho u \mu_{eff}}{(\psi_E - \psi_I)^2} \frac{\partial u}{\partial \omega} \right\} - \frac{1}{\rho u} \frac{dp}{dx} . \quad (2.2-1)$$

Conservation of chemical species j :

$$\frac{\partial m_j}{\partial x} + \frac{\{r_I \dot{m}''_I + \omega(r_E \dot{m}''_E - r_I \dot{m}''_I)\}}{(\psi_E - \psi_I)} \frac{\partial m_j}{\partial \omega} = \frac{\partial}{\partial \omega} \left\{ \frac{r^2 \rho u \mu_{eff}}{(\psi_E - \psi_I)^2 \sigma_{j,eff}} \cdot \frac{\partial m_j}{\partial \omega} \right\} + \frac{R_j}{\rho u} . \quad (2.2-2)$$

Conservation of stagnation enthalpy:

$$\begin{aligned} \frac{\partial \tilde{h}}{\partial x} + \frac{\{r_I \dot{m}''_I + \omega(r_E \dot{m}''_E - r_I \dot{m}''_I)\}}{(\psi_E - \psi_I)} \frac{\partial \tilde{h}}{\partial \omega} &= \frac{\partial}{\partial \omega} \left\{ \frac{r^2 \rho u \mu_{eff}}{(\psi_E - \psi_I)^2 \sigma_{h,eff}} \frac{\partial \tilde{h}}{\partial \omega} \right\} \\ &+ \frac{\partial}{\partial \omega} \left\{ \frac{r^2 \rho u \mu_{eff}}{(\psi_E - \psi_I)^2} \left(1 - \frac{1}{\sigma_{h,eff}} \right) \frac{\partial (u^2/2)}{\partial \omega} \right\} . \end{aligned} \quad (2.2-3)$$

These equations possess the common form:

$$\frac{\partial \phi}{\partial x} + (a + b\omega) \frac{\partial \phi}{\partial \omega} = \frac{\partial}{\partial \omega} \left(c \frac{\partial \phi}{\partial \omega} \right) + d , \quad (2.2-4)$$

$$\text{where: } a \equiv r_I \dot{m}''_I / (\psi_E - \psi_I) , \quad (2.2-5)$$

$$b \equiv (r_E \dot{m}''_E - r_I \dot{m}''_I) / (\psi_E - \psi_I) , \quad (2.2-6)$$

$$\text{and } c \equiv r^2 \rho u \mu_{eff} / \{(\psi_E - \psi_I)^2 \sigma_{eff}\} . \quad (2.2-7)$$

Here ϕ stands for the dependent variable and d is the term on the right-hand side which does not contain $\partial \phi / \partial \omega$. Table 2.2-1 gives the particular expressions for d .

Table 2.2-1. Expressions for d .

Dependent variable	Expression for d
u	$-\frac{1}{\rho u} \frac{dp}{dx}$
m_j	$R_j / (\rho u)$
$\tilde{h}$	$\frac{\partial}{\partial \omega} \left\{ \frac{r^2 \rho u \mu_{eff}}{(\psi_E - \psi_I)^2} \left(1 - \frac{1}{\sigma_{h,eff}} \right) \frac{\partial (u^2/2)}{\partial \omega} \right\}$

2.3 The entrainment rates

The specification of the rates of mass transfer across the I and E surfaces indirectly determines the boundaries of the layer to be considered. Our requirement is that the region $0 \leq \omega \leq 1$ should contain all points having significant gradients of the dependent variables. When the boundary coincides with a symmetry line or a wall, the problem of specifying $\dot{m}''_I$ and $\dot{m}''_E$ is straightforward. At the symmetry line, the mass-transfer rate across it is zero; at a wall, the mass flux is either specified or deducible from given data.

When the boundary is adjacent to a free stream, then the mass-transfer rate across it, commonly known as the entrainment rate, demands more ingenuity for its specification. Two cases of this kind should now be distinguished. In the first case, the edge of the layer can be established without arbitrariness; we shall be mainly concerned with this case in the present work; for in a turbulent flow which is assumed to obey the mixing length hypothesis, the effective viscosity vanishes along a definite edge of the layer. In the second case, the boundary is truly arbitrary; because the effective viscosity does not vanish along a definite boundary line and the approach to the free-stream conditions is asymptotic. In such a case the boundary should be established solely with reference to our computational requirement of restricting the important variations in ϕ to a finite domain in ω .

A straightforward way of obtaining an expression for the entrainment at a free boundary (for which we shall use the subscript G) would be to apply the partial differential equation (2.2-1) near to that boundary.

Just outside the G boundary we have:

$$\left(\frac{\partial u}{\partial x}\right)_G = -\left(\frac{1}{\rho u} \frac{dp}{dx}\right)_G \quad . \quad (2.3-1)$$

Now if we assume the same equality to be true just inside the G boundary, we obtain (whether ω equals zero or unity at the boundary under consideration) the following expression:

$$r_G \dot{m}''_G = \lim_{\omega \rightarrow \omega_G} \left[\frac{\frac{\partial}{\partial \omega} \left\{ \frac{r^2 \rho u \mu_{eff}}{(\psi_E - \psi_I)} \frac{\partial u}{\partial \omega} \right\}}{\partial u / \partial \omega} \right], \quad (2.3-2)$$

which can also be written as:

$$r_G \dot{m}''_G = \lim_{Y \rightarrow Y_G} \left[\frac{\frac{\partial}{\partial Y} \left\{ r \mu_{eff} \frac{\partial u}{\partial Y} \right\}}{\partial u / \partial Y} \right]. \quad (2.3-3)$$

Now it is easy to judge the satisfactoriness of this equation for the specification of $\dot{m}''_G$. The limit will be finite if μ_{eff} is proportional to $\partial u / \partial Y$; then this specification will be satisfactory. Use of the mixing-length hypothesis (as expressed by eq.(1.3-5), in which such a proportionality exists), therefore, enables us to employ eq.(2.3-3) as the entrainment law. For laminar flows, however, μ_{eff} does not vanish with $\partial u / \partial Y$; then eq.(2.3-3) gives infinite entrainment rate. We shall now outline the procedure to be adopted for such cases.

Since the unsatisfactoriness of eq. (2.3-3) is caused by the vanishing of $\partial u / \partial Y$ at the G boundary, it would be profitable to apply eq. (2.2-1) somewhat away from the G boundary. Let ω_B stand for the value of ω at which eq.(2.2-1) is applied; of course, ω_B lies between zero and unity. Then we have:

$$r_I \dot{m}''_I + \omega_B (r_E \dot{m}''_E - r_I \dot{m}''_I) = \left[\frac{\frac{\partial}{\partial \omega} \left\{ \frac{r^2 \rho u \mu_{eff}}{(\psi_E - \psi_I)} \frac{\partial u}{\partial \omega} \right\}}{(\partial u / \partial \omega)} \right]_B - \frac{(\psi_E - \psi_I)}{(\partial u / \partial \omega)_B} \left(\frac{du_B}{dx} + \frac{1}{\rho_B u_B} \frac{dp}{dx} \right) \quad (2.3-4)$$

Now for computational efficiency, we shall try to locate the boundary such that for a particular value of ω_B , u_B attains a predetermined value $\tilde{u}_B$.

These values can be chosen such that, at $\omega = 1$, making u equal to the free-stream velocity will not cause significant error. For example, we can use: $\omega_B = 0.9$, $\tilde{u}_B = 0.99 u_E$. Then all the quantities on the right-hand side of eq. (2.3-4) can be evaluated in terms of the values of the variables at a particular value of x , except the term du_B/dx the value of which can be found from the current value of u_B and the value $\tilde{u}_B$

desired at the downstream station. When only one of the boundaries is adjacent to a free stream, the entrainment at that boundary is to be found from eq. (2.3-4) by substituting the value of $\dot{m}''$ for the other boundary. When both the boundaries are free, two equations like (2.3-4) should be used, their simultaneous solution yielding the values of $\dot{m}''_I$ and $\dot{m}''_E$.

The finite-difference formulae for evaluation of the entrainment rates will be presented in Section 2.8 below.

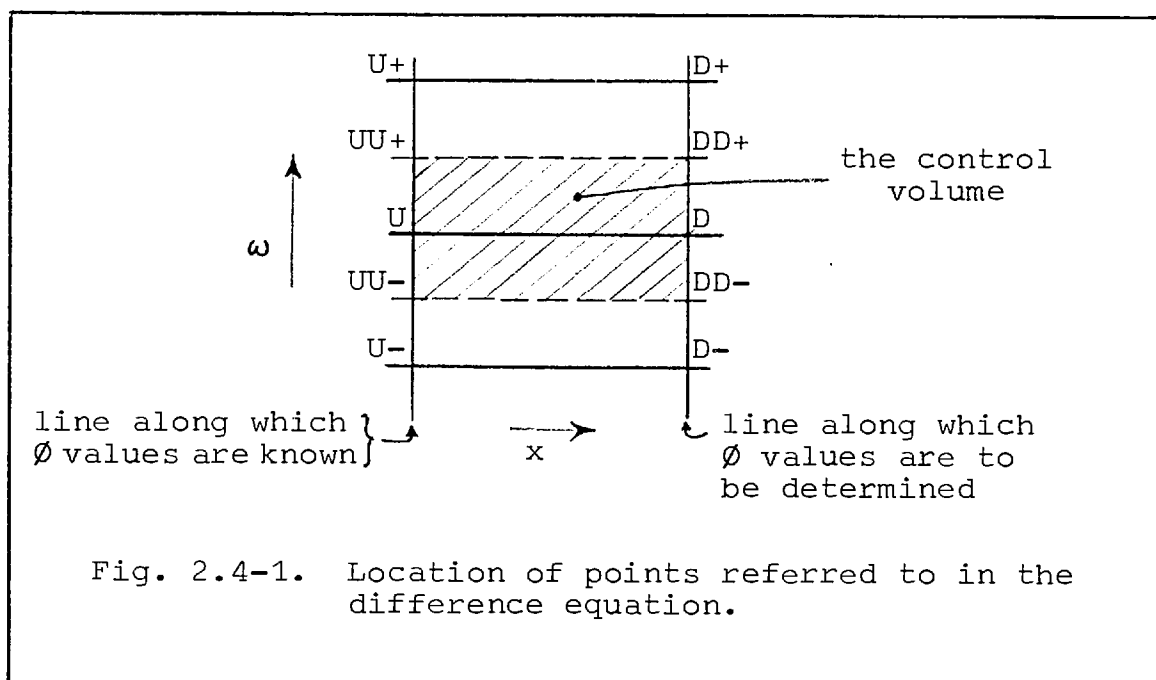
One last remark regarding the entrainment rates will now be made. The above-mentioned practices are the ones which will be used in the present work and which seem best at present. But they are not the only ones. An expression for $\dot{m}''_G$ can equally well be derived from any other conservation equation, or can be based upon a different principle. It is necessary to remember that the entrainment across a free boundary is essentially arbitrary; and all the formulations are satisfactory as long as they accord with the requirements of computational efficiency.

The formation of $x \sim \omega$ co-ordinate system and expressions for the entrainment rates complete the preparatory work for the finite-difference scheme, details of which will now be described.

2.4 The difference equation

General outline. In this section we shall express the general partial differential equation (2.2-4) in finite-difference form. The equation will be solved by step-by-step forward integration. Therefore, at every step in the integration, the values of ϕ will be known at discrete values of ω and at one value of x ; the task will be to obtain the values of ϕ at the same values of ω , but at a downstream value of x . By repetition of this basic operation, the whole field of interest can be covered.

The discrete values of ω , which are decided beforehand, define a grid; a portion of this is shown in Fig. 2.4-1. The points U and D represent the upstream and downstream nodes at a given ω , points at nearby values of ω will be called



$U+$, $U-$, $D+$, $D-$. The point midway between U and $U+$ is called $UU+$; similarly $UU-$, $DD-$, $DD+$ lie midway between the corresponding points. These four midway points form a control volume (shown shaded) which will be used in forming the difference equation.

The partial derivatives in eq.(2.2-4) with respect to ω can be evaluated in terms of the ϕ values at x_U or x_D or as a weighted mean of both. Evaluation at x_U would correspond to the explicit Binder-Schmidt method (Refs. [7], [62]). This method is unsuitable for boundary-layer work, because stability requirements for this method put an upper limit on the step in the x direction, proportional to the fluid velocity which may be sometimes very small. Use of the arithmetic mean of the values of $\partial/\partial\omega$ terms at x_U and x_D is similar to the procedure of Crank and Nicholson [17], which is free from the limitations on step length. It can, however, be shown, at least in simple cases, that use of the value only at x_D can be more accurate for large values of the forward step and is always stable. This is also more convenient. We shall, therefore, use the practice of evaluating the $\partial/\partial\omega$ terms from the ϕ values at x_D . It is desirable, again in the interests of convenience, to have the resulting difference equations linear in ϕ . Therefore, the coefficients such as a, b, c will always be evaluated from the upstream (i.e. known) values of ϕ .

The usual way to obtain a finite-difference equation from a partial differential equation is to express the terms in the latter in a Taylor-series-expansion form. There is, however, another way which will be used in the present work. We can formulate a miniature integral equation over the control volume shown in Fig. 2.4-1; this, coupled with an assumption regarding the nature of variation of ϕ between the grid points, yields the desired finite-difference equation. In other words, the finite-difference equation is obtained by expressing each term in the parent partial differential equation as an integrated average over a small control volume. The advantage of this formulation is that, unlike the conventional method, it ensures that the conservation equation will be satisfied over any part of the boundary layer. We shall assume that, in the ω direction, ϕ varies linearly with ω between the grid points. The variation in the x direction will be considered to be stepwise, the values of ϕ for the interval from x_U to x_D , except at x_U , being uniform and equal to those at x_D ;

this is in conformity with our above-mentioned decision of evaluating the $\partial/\partial\omega$ terms at x_D . A linear variation in the x direction would correspond to the Crank-Nicholson procedure.

Following these general principles, we can now proceed to express the various terms in eq.(2.2-4) in finite-difference form. The reader who is reluctant to go through the algebra that now follows can safely jump to the remarks about the pressure-gradient term at the end of Section 2.4, after noting that eq.(2.4-31) and (2.4-35) represent the outcomes of the intermediate algebra.

The convection terms. The terms on the left-hand side of eq.(2.2-4) can at first be expressed as:

$$\partial\phi/\partial x \approx \left\{ \int_{x_U}^{x_D} \int_{\omega_{DD-}}^{\omega_{DD+}} (\partial\phi/\partial x) d\omega dx \right\} / \{ (x_D - x_U) (\omega_{DD+} - \omega_{DD-}) \} , \quad (2.4-1)$$

$$(a + b\omega) (\partial\phi/\partial\omega) \approx \left\{ \int_{\omega_{DD-}}^{\omega_{DD+}} (a + b\omega) (\partial\phi/\partial\omega)_{x=x_D} d\omega \right\} / (\omega_{DD+} - \omega_{DD-}) . \quad (2.4-2)$$

Substitution of the linear $\phi \sim \omega$ profile between the grid points leads, after some algebra, to:

$$\frac{\partial\phi}{\partial x} \approx P_1 (\phi_{D+} - \phi_{U+}) + P_2 (\phi_D - \phi_U) + P_3 (\phi_{D-} - \phi_{U-}) , \quad (2.4-3)$$

$$\text{where } P_1 \equiv \frac{\omega_{D+} - \omega_D}{4(x_D - x_U)(\omega_{D+} - \omega_{D-})} , \quad (2.4-4)$$

$$P_2 \equiv \frac{3}{4(x_D - x_U)} , \quad (2.4-5)$$

$$\text{and } P_3 \equiv \frac{\omega_D - \omega_{D-}}{4(x_D - x_U)(\omega_{D+} - \omega_{D-})} ; \quad (2.4-6)$$

$$a \frac{\partial\phi}{\partial\omega} \approx Q (\phi_{D+} - \phi_{D-}) , \quad (2.4-7)$$

$$\text{where } Q \equiv \frac{a}{\omega_{D+} - \omega_{D-}} ; \quad (2.4-8)$$

and: $b\omega \frac{\partial \phi}{\partial \omega} \approx R_1 \phi_{D+} + R_2 \phi_D + R_3 \phi_{D-} \quad , \quad (2.4-9)$

where $R_1 \equiv \frac{b}{4} \left(\frac{\omega_{D+} + 3\omega_D}{\omega_{D+} - \omega_{D-}} \right) \quad , \quad (2.4-10)$

$R_2 \equiv -\frac{b}{4} \quad , \quad (2.4-11)$

and $R_3 \equiv -\frac{b}{4} \left(\frac{\omega_{D-} + 3\omega_D}{\omega_{D+} - \omega_{D-}} \right) \quad . \quad (2.4-12)$

The complete convection term can now be expressed as:

$\frac{\partial \phi}{\partial x} + (a + b\omega) \frac{\partial \phi}{\partial \omega} \approx g_1 \phi_{D+} + g_2 \phi_D + g_3 \phi_{D-} + g_4 \quad , \quad (2.4-13)$

where $g_1 \equiv P_1 + Q + R_1 \quad , \quad (2.4-14)$

$g_2 \equiv P_2 + R_2 \quad , \quad (2.4-15)$

$g_3 \equiv P_3 - Q + R_3 \quad , \quad (2.4-16)$

and $g_4 \equiv -P_1 \phi_{U+} - P_2 \phi_U - P_3 \phi_{U-} \quad . \quad (2.4-17)$

It is important to note that the g 's are expressible in terms of known quantities including the ϕ values at $x=x_U$.

The flux term. According to the already-mentioned principles, the expression for the flux term will be:

$\frac{\partial}{\partial \omega} (c \frac{\partial \phi}{\partial \omega}) \approx \frac{2}{\omega_{D+} - \omega_{D-}} \left\{ c_{UU+} \left(\frac{\phi_{D+} - \phi_D}{\omega_{D+} - \omega_D} \right) - c_{UU-} \left(\frac{\phi_D - \phi_{D-}}{\omega_D - \omega_{D-}} \right) \right\} \quad , \quad (2.4-18)$

where the subscripts $UU+$ and $UU-$ denote that c 's are evaluated at the upstream station and midway between the corresponding grid points.

If we now introduce:

$g_5 \equiv \frac{2 c_{UU+}}{(\omega_{D+} - \omega_{D-})(\omega_{D+} - \omega_D)} \quad , \quad (2.4-19)$

and $g_6 \equiv \frac{2 c_{UU-}}{(\omega_{D+} - \omega_{D-})(\omega_D - \omega_{D-})} \quad , \quad (2.4-20)$

then the flux term can be written as:

$\frac{\partial}{\partial \omega} (c \frac{\partial \phi}{\partial \omega}) \approx g_5 (\phi_{D+} - \phi_D) - g_6 (\phi_D - \phi_{D-}) \quad . \quad (2.4-21)$

The source term. Finally, the term d , here referred to as the source term, remains to be put into finite-difference form. From Table 2.2-1, it can be seen that the actual expression for d can be at times quite complicated. We shall, therefore, consider it sufficient to regard the value of d , over the control volume, as uniform and equal to that at D ; but again, since d may not be linear in ϕ , we shall obtain d_D from the following linearised formula:

$$d_D \approx d_U + \left(\frac{\partial d}{\partial \phi} \right)_U (\phi_D - \phi_U) \quad . \quad (2.4-22)$$

We shall make one exception to this treatment of the source term. In the equation for velocity u , the nature of the source term is fairly simple; moreover, the solution of this equation is the most important as it influences the solutions of all other equations. Therefore, we shall express the source term for the velocity equation by assuming it to vary linearly with ω between grid points. Thus,

$$d \approx \frac{\int_{\omega_{DD-}}^{\omega_{DD+}} (d)_{x=x_D} d\omega}{(\omega_{DD+} - \omega_{DD-})} \quad , \quad (2.4-23)$$

$$\text{and } (d)_{x=x_D} \approx (d)_{x=x_U} + \left(\frac{\partial d}{\partial u} \right)_{x=x_U} \{ (u)_{x=x_D} - (u)_{x=x_U} \} \quad . \quad (2.4-24)$$

Using the actual expression for d as $-(dp/dx)/(\rho u)$, we obtain:

$$d \approx s_1 u_{D+} + s_2 u_D + s_3 u_{D-} + s_4 \quad , \quad (2.4-25)$$

$$\text{where } s_1 \equiv \frac{P_1}{\rho_{U+} u_{U+}^2} \frac{dp}{dx} (x_D - x_U) \quad , \quad (2.4-26)$$

$$s_2 \equiv \frac{P_2}{\rho_U u_U^2} \frac{dp}{dx} (x_D - x_U) \quad , \quad (2.4-27)$$

$$s_3 \equiv \frac{P_3}{\rho_{U-} u_{U-}^2} \frac{dp}{dx} (x_D - x_U) \quad , \quad (2.4-28)$$

$$s_4 \equiv -2 \frac{dp}{dx} (x_D - x_U) \left\{ \frac{P_1}{\rho_{U+} u_{U+}} + \frac{P_2}{\rho_U u_U} + \frac{P_3}{\rho_{U-} u_{U-}} \right\} \quad (2.4-29)$$

The complete difference equation. Having explained how individual terms can be expressed in finite-difference form, we can now compile the complete difference equation as follows:

$$g_1 \phi_{D+} + g_2 \phi_D + g_3 \phi_{D-} + g_4 = g_5 (\phi_{D+} - \phi_D) - g_6 (\phi_D - \phi_{D-}) + d_U + \left(\frac{\partial d}{\partial \phi} \right)_U (\phi_D - \phi_U) \quad , \quad (2.4-30)$$

which can be reduced to the form:

$$\phi_D = A \phi_{D+} + B \phi_{D-} + C \quad , \quad (2.4-31)$$

$$\text{where } A \equiv \frac{g_5 - g_1}{g_2 + g_5 + g_6 - (\partial d / \partial \phi)_U} \quad , \quad (2.4-32)$$

$$B \equiv \frac{g_6 - g_3}{g_2 + g_5 + g_6 - (\partial d / \partial \phi)_U} \quad , \quad (2.4-33)$$

$$C \equiv \frac{d_U - (\partial d / \partial \phi)_U \phi_U - g_4}{g_2 + g_5 + g_6 - (\partial d / \partial \phi)_U} \quad . \quad (2.4-34)$$

These expressions will be valid also for the velocity equation if eq.(2.4-22) is used for the pressure-gradient term. Use of eq. (2.4-25), however, leads to the following slightly-different expressions:

$$u_D = A_u u_{D+} + B_u u_{D-} + C_u \quad , \quad (2.4-35)$$

$$\text{where } A_u \equiv \frac{g_5 - g_1 + s_1}{g_2 + g_5 + g_6 - s_2} \quad , \quad (2.4-36)$$

$$B_u \equiv \frac{g_6 - g_3 + s_3}{g_2 + g_5 + g_6 - s_2} \quad , \quad (2.4-37)$$

$$C_u \equiv \frac{s_4 - g_4}{g_2 + g_5 + g_6 - s_2} \quad . \quad (2.4-38)$$

Here the subscript "u" is used as a reminder that these expressions refer to the equation for velocity u . Equations (2.4-31) and (2.4-35) represent the final outcome of our finite-difference formulation. How to solve such equations will be explained in Section 2.7.

When the pressure gradient is not known. All the algebraic rearrangements so far have been made with one policy: bringing together the groups of symbols which are calculable in terms of known quantities and retaining the values of ϕ at $x=x_D$ as unknowns in the equation. Thus A , B , C can be known prior to obtaining ϕ values at $x = x_D$. This remark cannot, however, be made about A_u , B_u , C_u unless the pressure gradient dp/dx is known. If the flow is confined and hence the value of dp/dx is not known beforehand, the algebraic rearrangement should display dp/dx as an unknown, and then the method of solving the resulting difference equations will be somewhat different. The present author has worked out a way of doing this and has explained it in Ref. [50]. However, the details will not be given here; firstly because the illustrative material in this thesis does not need them, and secondly because the associated algebra is tedious, though not essentially difficult.

2.5 Slip values at boundaries

2.5-1 What are slip values

In forming the finite-difference equation we have assumed that the variation of ϕ between the grid points is linear in ω . This assumption is usually quite good (provided that there are sufficient grid points across the layer) everywhere except in the interval near the boundary. Near a wall, for example, a straight line in $u \sim \omega$ passing through the true u value at the wall would be a poor representation of the reality in which the variations are much steeper. On the other hand, these boundary regions are important because they determine various quantities such as the wall fluxes, entrainment rates and symmetry-line values.

For this reason, we shall use a "false" or a "slip" value of ϕ at the boundary such that the $\phi \sim \omega$ line passing through this value rather than the true one is a better representation for the interval near the boundary. Incidentally, the introduction of the slip values also makes it easier to handle those boundary conditions in which the gradients of ϕ rather than the values are specified at the boundary; this matter will be dealt with in Section 2.6 below.

The definition of a slip value, which we shall use, should be in conformity with our requirements. Fig. 2.5-1 shows the scheme of subscripts that we shall use. The grid lines divide the interval from $\omega = 0$ to $\omega = 1$ into N strips. Subscripts 1 and 2 denote respectively the true and slip values at the I boundary; subscripts 3, 4, ..., $N+1$ are used for the successive grid lines upto the E boundary; the slip and true values at the E boundary are denoted by subscripts

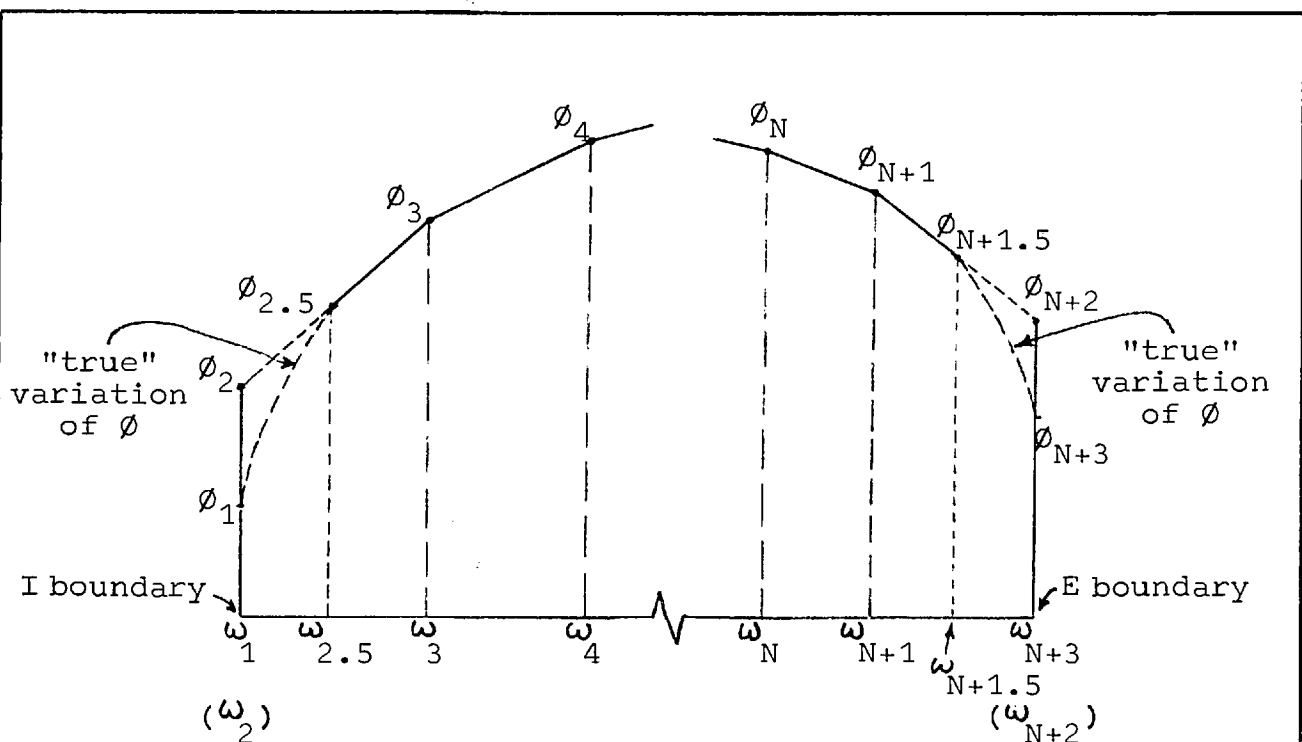


Fig. 2.5-1. The scheme of subscripts for the true and slip values of ϕ .

$N+2$ and $N+3$ respectively. Obviously, the values of ω_2 and ω_{N+2} if used at all, are identical with those of ω_1 and ω_{N+3} respectively. The subscript 2.5 refers to a line midway between the I boundary and the grid line 3 ; similarly, $N+1.5$ is used for a line midway between $N+1$ and the E boundary. The lines with subscripts 2.5 and $N+1.5$ have special significance, because they represent the boundaries up to which the control volumes used for the difference equations will extend. The function of the slip value ϕ_2 can thus be seen to be that of orienting the $\phi_2 \sim \phi_3$ line so that a better representation is obtained for the region from $\omega_{2.5}$ to ω_3 than that given by the $\phi_1 \sim \phi_3$ line. We shall, therefore, define the slip value ϕ_2 as the one which enables us to obtain, from a linear $\phi \sim \omega$ interpolation between the points 2 and 3 , the correct slope and value of ϕ at the point 2.5 . Similar remarks apply to the slip value ϕ_{N+2} .

What are the correct values of the slope and value at 2.5 depends upon the nature of the boundary and on our assumptions regarding the flow properties there. These expressions will be developed in the following sub-sections. While obtaining these expressions, we shall regard the density as uniform, and so also the radius, except near the axis of symmetry. A little reflection will show that this is unlikely to introduce significant error. In what follows, the relations are given only for the I boundary; similar relations will be valid for the E boundary. To facilitate the construction of expressions for the E boundary, we shall retain the terms ω_1 and y_1 in the following treatment, although they equal zero for the I boundary. Incidentally, y_2 in general will not equal y_1 ; the variable y will also be treated like ϕ and will have a slip value.

2.5-2 Slip-value relations for a wall boundary

Near a wall, we shall assume that the velocity profile is of the power-law type:

$$u \propto |(y - y_1)|^\beta \quad . \quad (2.5-1)$$

How to ascribe a value to β will be shown later. Since, by definition, :

$$(\omega - \omega_1) \propto \int_{y_1}^y u \, dy \quad , \quad (2.5-2)$$

we obtain:

$$u \propto |(\omega - \omega_1)|^{\beta/(1+\beta)} \quad . \quad (2.5-3)$$

This relation allows us, by matching the slope and the ordinate at 2.5 , to arrive at:

$$u_2 = \frac{1}{1 + 2\beta} u_3 \quad . \quad (2.5-4)$$

When ϕ stands for a dependent variable other than u , we shall again assume a power-law profile for ϕ , but with a different exponent. Thus,

$$(\phi - \phi_1) \propto |(y - y_1)|^\gamma \quad , \quad (2.5-5)$$

which, by use of eq.(2.5-1) and (2.5-3), can be expressed as:

$$(\phi - \phi_1) \propto |\omega - \omega_1|^{\gamma/(1+\beta)} \quad . \quad (2.5-6)$$

Use of the slip-value definition then yields:

$$\phi_2 = \phi_3 \left(\frac{1 + \beta - \gamma}{1 + \beta + \gamma} \right) + \phi_1 \left(\frac{2\gamma}{1 + \beta + \gamma} \right) \quad . \quad (2.5-7)$$

In this case, the slip value of y can be obtained, it is easy to see, by replacing ϕ by y and γ by unity. Thus:

$$y_2 = y_3 \left(\frac{\beta}{2 + \beta} \right) + y_1 \left(\frac{2}{2 + \beta} \right) \quad . \quad (2.5-8)$$

Determination of β and γ . The exponents β and γ in the profiles near a wall will be chosen such that the slopes at the point 2.5 are consistent with the values, at that point, of the shear stress and flux as obtained from the Couette-flow analysis. The power-law relationships imply:

$$\beta = \left[\frac{(y - y_1)}{u} \frac{\partial u}{\partial y} \right]_{2.5} \quad , \quad (2.5-9)$$

and
$$\gamma = \left[\frac{(Y - Y_1)}{(\phi - \phi_1)} \frac{\partial \phi}{\partial Y} \right]_{2.5} . \quad (2.5-10)$$

By use of the definitions in Section 1.4, the above relations can be expressed as *:

$$\beta = \left[\frac{(S + F + M) R}{\mu_{eff}/\mu} \right]_{2.5} , \quad (2.5-11)$$

and
$$\gamma = \left[\frac{(S + M) R}{\mu_{eff}/(\mu \sigma_{eff})} \right]_{2.5} . \quad (2.5-12)$$

So far, no particular hypothesis has been introduced here. For turbulent flow, if we substitute for μ_{eff} from the mixing-length formula near the wall in the form:

$$\mu_{eff} = \rho K^2 (Y - Y_1)^2 | \partial u / \partial Y | , \quad (2.5-13)$$

which can be expressed as:

$$\left(\frac{\mu_{eff}}{\mu} \right)_{2.5} = K^2 \beta R_{2.5} , \quad (2.5-14)$$

we obtain:

$$\beta = (S + F + M)_{2.5}^{1/2} / K , \quad (2.5-15)$$

and
$$\gamma = (S + M)_{2.5} \sigma_{eff} / (K^2 \beta) . \quad (2.5-16)$$

* The expression for γ given by eq.(2.5-12) is valid only when ϕ

satisfies the conditions given at the beginning of Section 1.4, i.e. when it stands for m_j in the absence of chemical reaction and for $\tilde{h}$ with either negligible kinetic heating or $\sigma_{h,eff}$ unity everywhere. For finite kinetic heating and non-unity $\sigma_{h,eff}$, it can be shown that:

$$\gamma = \left[\frac{(S_h + M) R}{\mu_{eff}/(\mu \sigma_{h,eff})} - \frac{u^2/2}{\tilde{h} - \tilde{h}_S} \left\{ \frac{(1 - H) S_h R}{\mu_{eff}/(\mu \sigma_{h,eff})} - 2\beta(1 - \sigma_{h,eff}) \right\} \right]_{2.5} . \quad (2.5-12a)$$

In using eq.(2.5-14), we have implicitly assumed that the point 2.5 is sufficiently far from the wall for the laminar viscosity to be unimportant there; indeed, our attempt is to confine the region of steep gradients and of the influence of laminar viscosity to the space between the wall and the point 2.5.

2.5-3 Slip-value relations for a free boundary

In the region near a free boundary, a turbulent flow obeying the mixing-length hypothesis can be shown to exhibit a velocity profile parabolic in distance [1]. This assumption looks plausible even for flows which obey other effective-viscosity laws. We therefore start by assuming:

$$(u - u_1) \propto (y - y_1)^2, \quad (2.5-17)$$

from which follows:

$$(\omega - \omega_1) \propto |u - u_1|^{\frac{1}{2}} (2u_1 + u). \quad (2.5-18)$$

Application of the slip-value definition leads, after some algebra, to:

$$u_2 = \frac{16u_1^2 - 4u_1u_3 + u_3^2}{2(u_1 + u_3) + (84u_1^2 - 12u_1u_3 + 9u_3^2)^{\frac{1}{2}}} \quad (2.5-19)$$

This expression is satisfactory in all but one respect: it is not linear. In order to solve this equation, without iteration, simultaneously with the difference equations of Section 2.4, it is necessary to express the relation between the downstream values of u_1 , u_2 and u_3 in a linear fashion. This can be done by a rearrangement of eq.(2.5-19) as follows:

$$u_2 = u_3 G + u_1 (1 - G), \quad (2.5-20)$$

$$\text{where } G \equiv \frac{u_2 + u_3 - 8u_1}{5u_2 + 5u_3 + 8u_1} \quad (2.5-21)$$

Since G is a ratio of u 's, it will vary slowly and hence its value calculated by using the upstream values of u can be conveniently used.

The profile for a dependent variable ϕ other than u will be taken as power-law type. Thus:

$$(\phi - \phi_1) \propto |y - y_1|^n \quad . \quad (2.5-22)$$

It is an implication of the mixing-length hypothesis that the exponent n is related to σ_{eff} via the relation*:

$$n = 2 \sigma_{eff} \quad . \quad (2.5-23)$$

Use of the definition of slip value and of eq.(2.5-20) yields:

$$\phi_2 = \phi_3 G_\phi + \phi_1 (1 - G_\phi) \quad , \quad (2.5-24)$$

$$\text{where } G_\phi \equiv \frac{G + (2 - n)/(2 + n)}{1 + G(2 - n)/(2 + n)} \quad . \quad (2.5-25)$$

The slip value of y can be obtained from this by putting $n = 1$.
Thus:

$$y_2 = y_3 \left(\frac{3G + 1}{3 + G} \right) + y_1 \left(\frac{2(G - 1)}{3 + G} \right) \quad . \quad (2.5-26)$$

2.5-4 Slip-value relations for a symmetry-line boundary

Near a symmetry line, if we note that in eq.(2.2-4), a is zero, ω is small and the terms $\partial\phi/\partial x$ and d cannot be very different from their values on the symmetry line, then we get the limiting form:

$$\frac{\partial}{\partial \omega} \left(c \frac{\partial \phi}{\partial \omega} \right) = \frac{d\phi_1}{dx} - d_1 \quad . \quad (2.5-27)$$

* Spalding [78] gave a very interesting proof of this. The reader can arrive at this relation first by obtaining, from the general equation (2.2-4), an expression (similar to eq.(2.3-3)) for the entrainment rate $\dot{m}''_G$, and then by requiring that the latter should be independent of σ_{eff} when the mixing-length hypothesis is used.

We shall further assume that μ_{eff} is uniform for the region near the symmetry line; this assumption is more in agreement with reality than with the mixing-length hypothesis. This departure from the latter is intentional. Just as near the free boundary we accepted the plausible implications of the mixing-length theory as generally applicable, here we accept, for general use, the consequences of the constant - μ_{eff} assumption because of its simplicity and nearness to reality. The differences between alternative practices are again going to be small.

Near a symmetry line, the cases of plane and axi-symmetrical flow must be distinguished.

For plane flow, c will be uniform for the region under consideration.

Then integration of eq.(2.5-27) will give:

$$(\phi - \phi_1) \propto (\omega - \omega_1)^2, \quad (2.5-28)$$

from which it is easy to derive the slip-value relation:

$$\phi_2 = -\frac{1}{3}\phi_3 + \frac{4}{3}\phi_1. \quad (2.5-29)$$

Evidently, the relation for velocity u will be similar; thus:

$$u_2 = -\frac{1}{3}u_3 + \frac{4}{3}u_1. \quad (2.5-30)$$

For these conditions,

$$(\omega - \omega_1) \propto (y - y_1), \quad (2.5-31)$$

and hence:

$$y_2 = y_1. \quad (2.5-32)$$

For axi-symmetrical flow, the radius r is same as the distance y ,

and therefore:

$$(\omega - \omega_1) \propto (y - y_1)^2, \quad (2.5-33)$$

$$c \propto (y - y_1)^2, \quad (2.5-34)$$

the substitution of which into eq. (2.5-27) leads, after integration, to:

$$(\phi - \phi_1) \propto (\omega - \omega_1) \quad . \quad (2.5-35)$$

Thus the $\phi \sim \omega$ profile is linear and hence the slip value of ϕ equals its true value. Therefore:

$$\phi_2 = \phi_1 \quad , \quad (2.5-36)$$

$$\text{and} \quad u_2 = u_1 \quad . \quad (2.5-37)$$

The slip value of y can be expressed as:

$$y_2 = \frac{1}{3} y_3 + \frac{2}{3} y_1 \quad . \quad (2.5-38)$$

2.6 Treatment of special boundary conditions

The difference equations of Section 2.4 will form, together with the slip-value relations of Section 2.5, a solvable set of algebraic equations, provided that the true ϕ values at the boundaries are known from the boundary conditions. Sometimes, instead of the boundary value of ϕ , the gradient of ϕ or the flux caused by it is specified. Then it becomes necessary to rewrite the slip-value relations by eliminating the unknown boundary value of ϕ via the given gradient condition. Two such cases will now be described.

2.6-1 The case of prescribed total flux through the wall

When the total flux J_{tot} through the wall is given, it can be related to the diffusional flux J_S by:

$$J_{\text{tot}} = J_S + \dot{m}''_S \phi_1 \quad , \quad (2.6-1)$$

which, in dimensionless form, will appear as:

$$\frac{J_{\text{tot}}}{\rho u (\phi_1 - \phi)} = S + M + \frac{\phi}{\phi_1 - \phi} M \quad . \quad (2.6-2)$$

Application of eq. (2.6-2) at the point 2.5 followed by elimination of ϕ_1 with the help of eq. (2.5-7) finally leads to*:

$$\phi_2 = \phi_3 \left(\frac{1 - VM}{1 + VM} \right)_{2.5} + 2 J_{\text{tot}} \left(\frac{V}{\rho u (1 + VM)} \right)_{2.5}, \quad (2.6-3)$$

where $V \equiv \frac{Y}{(1 + \beta)(S + M)}$. (2.6-4)

Use of eq. (2.5-16) which is based on the mixing-length hypothesis enables us to write:

$$V = \frac{\sigma_{\text{eff}}}{K^2 \beta (1 + \beta)} . \quad (2.6-5)$$

2.6-2 The case of a symmetry-line boundary

Here we shall derive relations for the general dependent variable ϕ ; these will be applicable even when ϕ stands for the velocity u . Once again, the cases of plane and axisymmetrical flow will be distinguished.

Plane flow. The solution of eq. (2.5-27) for the plane flow can be expressed as:

$$\phi - \phi_1 = \left(\frac{d\phi_1}{dx} - d_1 \right) \frac{\rho_1 u_1 \sigma_{\text{eff}} (y - y_1)^2}{2 \mu_{\text{eff}}} . \quad (2.6-6)$$

* Remarks similar to those made for eq. (2.5-12) apply to eq. (2.6-3). When ϕ stands for $\tilde{h}$, the corresponding equation would be:

$$\tilde{h}_2 = \tilde{h}_3 \left(\frac{1 - V_h M_h}{1 + V_h M_h} \right)_{2.5} + 2 J_{h, \text{tot}} \left(\frac{V_h}{\rho u (1 + V_h M_h)} \right)_{2.5}, \quad (2.6-3a)$$

where $V_h \equiv \frac{Y}{(1 + \beta)(S_h + M)}$, (2.6-4a)

and $M_h \equiv M + (1 - H) S_h \frac{u^2/2}{\tilde{h}}$. (2.6-4b)

We shall use the finite-difference representation:

$$\frac{d\phi_1}{dx} - d_1 \approx \frac{\phi_{1,D} - \phi_{1,U}}{x_D - x_U} - d_{1,U} - \left(\frac{\partial d_1}{\partial \phi} \right)_U (\phi_{1,D} - \phi_{1,U}) \quad , \quad (2.6-7)$$

and satisfy eq. (2.6-6) at the point 2.5, so that:

$$\phi_{2.5,D} - \phi_{1,D} = \left\{ \frac{\phi_{1,D} - \phi_{1,U}}{x_D - x_U} - d_{1,U} - \left(\frac{\partial d_1}{\partial \phi} \right)_U (\phi_{1,D} - \phi_{1,U}) \right\} \times \frac{\rho_1 u_1 \sigma_{eff}}{2 \mu_{eff}} (y_{2.5} - y_1)^2 \quad . \quad (2.6-8)$$

(Wherever neither U nor D is used as a subscript, the upstream values are to be used.) Now the task is to eliminate $\phi_{1,D}$ between eq.(2.6-8) and (2.5-29). The resulting expression can be written as:

$$\phi_{2,D} = \phi_{3,D} \left(\frac{2 - k_1 k_3}{2 + 3k_1 k_3} \right) - \frac{4 k_2 k_3}{2 + 3k_1 k_3} \quad , \quad (2.6-9)$$

$$\text{where} \quad k_1 \equiv \frac{1}{x_D - x_U} - \left(\frac{\partial d_1}{\partial \phi} \right)_U \quad , \quad (2.6-10)$$

$$k_2 \equiv - (k_1 \phi_{1,U} + d_{1,U}) \quad , \quad (2.6-11)$$

$$\text{and} \quad k_3 \equiv \frac{\rho_1 u_1 \sigma_{eff} (y_{2.5} - y_1)^2}{4 \mu_{eff}} \quad . \quad (2.6-12)$$

Axi-symmetrical flow. A very similar derivation is applicable to the case of axi-symmetrical flow. The basic equation would be:

$$\phi - \phi_1 = \left(\frac{d\phi_1}{dx} - d_1 \right) \cdot \frac{\rho_1 u_1 \sigma_{eff} (y - y_1)^2}{4 \mu_{eff}} \quad . \quad (2.6-13)$$

Substitution of $(d\phi_1/dx - d_1)$ from eq. (2.6-7) followed by elimination of $\phi_{1,D}$ with the help of eq. (2.5-36) leads to:

$$\phi_{2,D} = \phi_{3,D} \left(\frac{2}{2 + k_1 k_3} \right) - \frac{k_2 k_3}{2 + k_1 k_3} \quad , \quad (2.6-14)$$

where the k 's are defined as before.

2.7 Solution of the difference equations

Let us retrace the steps taken so far. We formulated the partial differential equations in a suitable co-ordinate system. They were expressed in the finite-difference form (in Section 2.4) which involved the satisfaction of the integral equation over a control volume extending over half an interval on either side of a grid line. Thus we obtain one difference equation for every grid line except those at the boundaries. The difference equations for the grid lines 3 and $N + 1$ will use the slip values ϕ_2 and ϕ_{N+2} and not the true values ϕ_1 and ϕ_{N+3} ; for only then the linear-profile assumption will be appropriate. The relations for the slip values have been obtained, in a linear form, in Section 2.5, and they have been made, in Section 2.6, independent of the boundary values when the latter are not given. Thus we now possess a set of linear algebraic equations which can be solved. Moreover, this set has a special characteristic: only three unknowns appear, in a particular order, in every equation. This allows us to solve these equations by simple successive - substitution formulae (to be described below) rather than by standard matrix-inversion technique. Therefore, unlike the latter, the solution procedure needs computing time proportional to the number of equations, and not to its square.

Using the subscript scheme given in Section 2.5-1, we can express the algebraic equations as:

$$\phi_i = A_i \phi_{i+1} + B_i \phi_{i-1} + C_i \quad , \quad (2.7-1)$$

where $i=2, 3, 4, 5, \dots, N+2$. The values of ϕ_1 and ϕ_{N+3} will either be known or have zero coefficient.

Equation (2.7-1) can be transformed into the simpler form:

$$\phi_i = A'_i \phi_{i+1} + B'_i \quad , \quad (2.7-2)$$

where

$$\left. \begin{aligned} A'_i &= \frac{A_i}{1 - B_i A'_{i-1}} \\ B'_i &= \frac{B_i B'_{i-1} + C_i}{1 - B_i A'_{i-1}} \\ A'_2 &= A_2 \\ B'_2 &= B_2 \phi_1 + C_2 \end{aligned} \right\} \quad (2.7-3)$$

After calculation of A' and B' for $i=2, 3, \dots, N+2$, it is a simple matter to obtain ϕ'_s from eq.(2.7-2) by successive substitution starting from ϕ_{N+3} .

2.8 Miscellaneous matters

2.8-1 Grid-control formulae

The general principles of evaluating the free-boundary entrainment rates which control the physical width of the grid have already been outlined in Section 2.3. The actual formulae for $\dot{m}''_{\Gamma}$ and $\dot{m}''_E$ will depend upon the type of effective-viscosity hypothesis used. We shall describe two cases below.

Free boundary obeying the mixing-length hypothesis. When the flow is assumed to obey the mixing-length hypothesis, the entrainment across a free boundary can be specified without arbitrariness; for then the limit in eq. (2.3-3) converges to a finite value. Substitution of eq. (1.3-5) into (2.3-3), and the assumption of zero velocity gradient at the free boundary leads, according to whether the Γ or E boundary is in question, to:

$$\dot{m}''_I = 2 \rho_I \ell_I^2 \left| \partial^2 u / \partial Y^2 \right|_I, \quad (2.8-1)$$

or

$$\dot{m}''_E = -2 \rho_E \ell_E^2 \left| \partial^2 u / \partial Y^2 \right|_E. \quad (2.8-2)$$

Further, use of a velocity profile parabolic in distance between the I boundary and the point 2.5 enables us to evaluate $\left| \partial^2 u / \partial Y^2 \right|_I$, the final result being:

$$\dot{m}''_I = 8 \rho_I \ell_I^2 \frac{|u_2 + u_3 - 2u_1|}{(Y_2 + Y_3 - 2Y_1)^2}. \quad (2.8-3)$$

The corresponding expression for $\dot{m}''_E$ is:

$$\dot{m}''_E = -8 \rho_E \ell_E^2 \frac{|u_{N+1} + u_{N+2} - 2u_{N+3}|}{(Y_{N+1} + Y_{N+2} - 2Y_{N+3})^2}. \quad (2.8-4)$$

Free boundary with non-vanishing effective viscosity. We shall here merely outline how different terms in eq. (2.3-4) can be put into finite-difference form. The important term is du_B/dx . This can be expressed as:

$$\frac{du_B}{dx} \approx \frac{\tilde{u}_B - u_B}{x_D - x_U}, \quad (2.8-5)$$

where u_B is the value of u which we want to achieve at $\omega = \omega_B$, $x = x_D$. The terms $\partial u / \partial \omega$ and $\partial / \partial \omega \{ r^2 \rho u_{eff} (\partial u / \partial \omega) / (\psi_E - \psi_I)^2 \}$ can be evaluated by use of formulae similar to eq. (2.4-7) and (2.4-18), but the upstream values of u will have to be used as the downstream values are not known.

2.8-2 Calculation of the normal distance

The finite-difference procedure enables us to obtain $\phi \sim \omega$ profiles. The physical distance to which the intervals in ω correspond, however, varies as the forward integration proceeds. It is, therefore, necessary to compute the distance Y for every grid point at every integration step.

As a preparation, we need the value of $(\psi_E - \psi_I)$ which can be obtained by a finite-difference form of eq. (2.1-2) and (2.1-3), as follows:

$$(\psi_E - \psi_I)_D = (\psi_E - \psi_I)_U + (r_I \ddot{m}_I - r_E \ddot{m}_E)_U (x_D - x_U) \quad (2.8-6)$$

Now, since by definition, we have:

$$x = \text{fixed:} \quad r \, dy = (\psi_E - \psi_I) (\rho u)^{-1} d\omega, \quad (2.8-7)$$

$$\text{and} \quad r = r_I + y \cos \alpha, \quad (2.8-8)$$

we obtain, after integration of eq. (2.8-7), :

$$y = \frac{2I}{\{r_I^2 + (r_I^2 + 2I \cos \alpha)^{\frac{1}{2}}\}}, \quad (2.8-9)$$

$$\text{where } I \equiv (\psi_E - \psi_I) \int_0^\omega (\rho u)^{-1} d\omega \quad (2.8-10)$$

The integral I can be evaluated by use of trapezoidal rule. Thus, with obvious notation, :

$$I_{i+1} - I_i = \frac{1}{2} (\psi_E - \psi_I) \{ (\rho_{i+1} u_{i+1})^{-1} + (\rho_i u_i)^{-1} \} (\omega_{i+1} - \omega_i) \quad (2.8-11)$$

This formula, however, may not be a good approximation near the boundaries. It may appear that use of slip values instead of true values at the boundary will be satisfactory; but our slip values have been designed to give the correct ordinate and slope at the midway point and not the correct value of the integral over the interval. Therefore, for the half-intervals near the boundaries we shall use "exact" expressions for the integral I consistent with our assumed profiles. Such expressions for the I boundary are given below; expressions for the E boundary can be similarly worked out.

For a wall boundary:

$$I_{2.5} = \frac{(\psi_E - \psi_I) (1 + \beta) (\omega_3 - \omega_1)}{\rho (u_2 + u_3)} \quad (2.8-12)$$

For a free boundary:

$$I_{2.5} = \frac{3(\psi_E - \psi_I)(\omega_3 - \omega_1)}{\rho(u_2 + u_3 + 4u_1)} \quad (2.8-13)$$

For a symmetry-line boundary:

$$I_{2.5} = \frac{(\psi_E - \psi_I)(\omega_3 - \omega_1)}{\rho_1 u_1} \quad (2.8-14)$$

It will be seen from eq.(2.8-12) that evaluation of $Y_{2.5}$ needs the value of β ; it will now be recalled that β is to be obtained from $R_{2.5}$ which contains $Y_{2.5}$. However, since β varies very slowly, iteration can be avoided by substituting the previous value of β in eq. (2.8-12).

2.8-3 A characteristic thickness of the layer.

The calculation of the normal distance Y will give us the physical thickness of the layer between the I and E boundaries. This is the thickness to which we have confined our computation so that the latter remains efficient. As long as this requirement of computational efficiency is not seriously contravened, variations in this nominal thickness due to slightly-different formulations of entrainment rates or due to different spacing of the grid lines will be quite acceptable. Now, it will be recalled that, except near a wall, the mixing length is proportional to a characteristic thickness of the layer, Y_ℓ . Certainly it will not be desirable to use the nominal arbitrary thickness of the layer as this characteristic thickness. We shall, therefore, define such a characteristic thickness as can be used for mixing-length calculation.

The characteristic thickness of the layer, Y_ℓ , is defined as the distance between two points, one near the I boundary and the other near E ; when the boundary coincides with a wall or a symmetry line, such a point lies on the boundary; when the boundary is free, the point is located such that the

velocity there differs from that of the adjacent free stream by a definite fraction of the maximum velocity difference across the layer. The location of such a point near a free boundary can be obtained by linear interpolation, between the grid points, of velocity with distance; only in the half-interval near the boundary, the velocity profile should be taken as parabolic in distance.

2.8-4 Choice of the forward step

The forward integration can be performed when the size of the forward step, $(x_D - x_U)$, has been decided. The implicit nature of our finite-difference scheme will ensure stability even for large steps; however, a reasonably small size of step will be desirable for good accuracy. The most-economical step size for a particular class of problems can be found by experience. A simple procedure is to make the step length proportional to the thickness of the layer, i.e. to put:

$$x_D - x_U = \text{const.} \times y_E, \quad (2.8-15)$$

where the constant can be around 0.5. This type of specification will be quite satisfactory for most of the turbulent boundary layers where the thickness of the layer varies approximately linearly with the longitudinal distance. For laminar boundary layer on a flat plate, a step length proportional to the square of the layer thickness would be more appropriate.

In some situations, the growth of the layer thickness is very slow, for example in a mixing layer between two streams of nearly-equal velocities; in such cases we can choose the step length so as to entrain an extra quantity of fluid, during that step, equal to a definite fraction of the quantity of fluid already existing in the layer. This rule can be expressed as:

$$x_D - x_U = \text{const.} \times (\psi_E - \psi_I)_U / (r_I \dot{m}_I'' - r_E \dot{m}_E'') , \quad (2.8-16)$$

where the value of the constant can be around 0.05.

Chapter 3

Capabilities of the solution procedure : demonstration and appreciation

3.1 Purpose of the chapter

The development of the theoretical structure including a convenient calculation procedure has now been completed. Although, in the previous chapters, explanations have been given of the choice of the solution procedure and of its various special features, the statements about the merits of the method may still appear as assertions or even speculations. The reader may like to be shown, at this stage, examples of computations performed by use of the method ; these can also establish that our solutions, obtained in a modest amount of computing time, agree satisfactorily with available exact solutions.

It is, therefore, the purpose of this chapter to present some computations and to compare them with exact solutions. (Comparison with experiment is also an important matter, which we shall present, in its appropriate place (in Part II : Chapter 5) , while dealing with the physical aspects of the problem.) We shall also summarise the achievements of the solution procedure and outline the directions of future development.

3.2 Some boundary-layer calculations

3.2-1 Plane turbulent mixing layer

The problem. The physical situation considered in our first example is the one treated analytically by Tollmien [86]; Fig. 3.2-1 shows the important features of this situation. It consists of the mixing layer between a stream of uniform velocity u_{∞} and temperature T_{∞} , and a stagnant fluid at temperature T_E . The fluid has uniform density ; the effective viscosity is given by the mixing-length formula (1.3-5) with the mixing length ℓ

proportional to the characteristic thickness of the layer, y_ℓ ; the effective Prandtl number has a uniform value of 0.5. The problem is to calculate the profiles of velocity and temperature, the spread of the mixing layer etc.

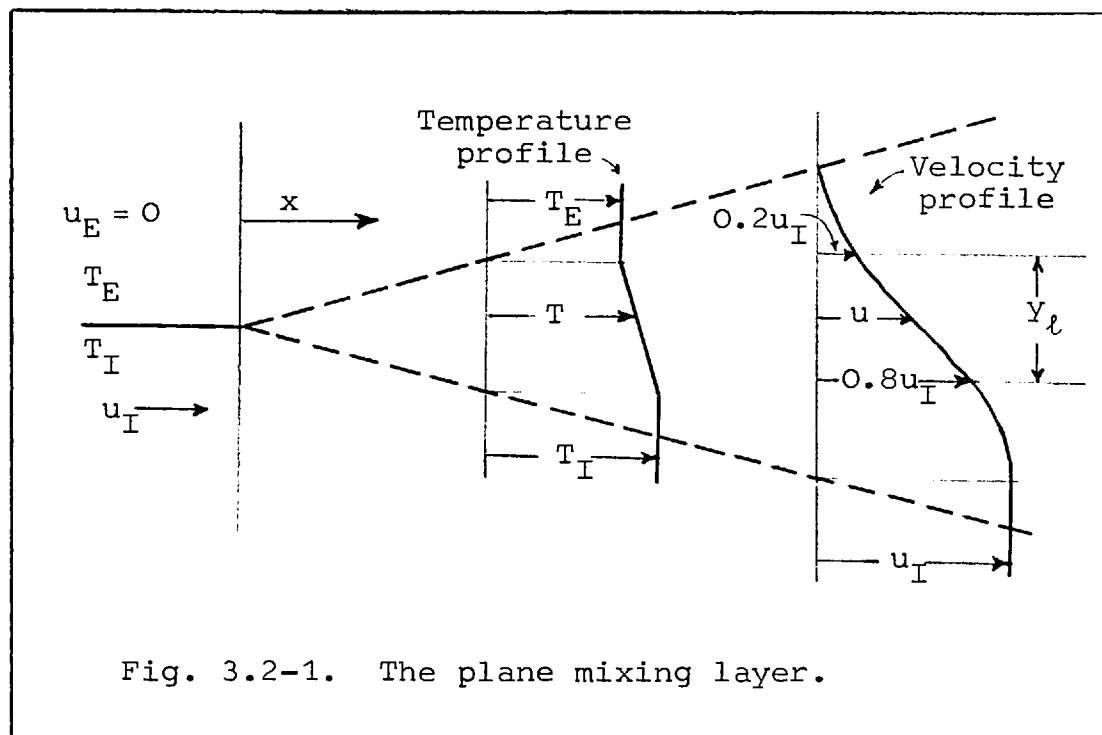


Fig. 3.2-1. The plane mixing layer.

Details of solution procedure. Here the partial differential equations to be solved are those for velocity and temperature, the kinetic heating being taken as negligible. We shall take the characteristic thickness y_ℓ as the distance between the two points where the velocity equals respectively 0.2 and 0.8 times u_I . Three sets of calculation will be presented with 11, with 16 and with 21 grid lines across the layer. The integration will be terminated when the profiles of velocity and temperature cease to change. In this equilibrium state, the layer thickness will grow linearly with the downstream distance x .

Results. Fig. 3.2-2 shows the comparison of the velocity profiles obtained by the present method with Tollmien's exact solution. The agreement is indeed satisfactory. When the effective Prandtl number is 0.5, the exact

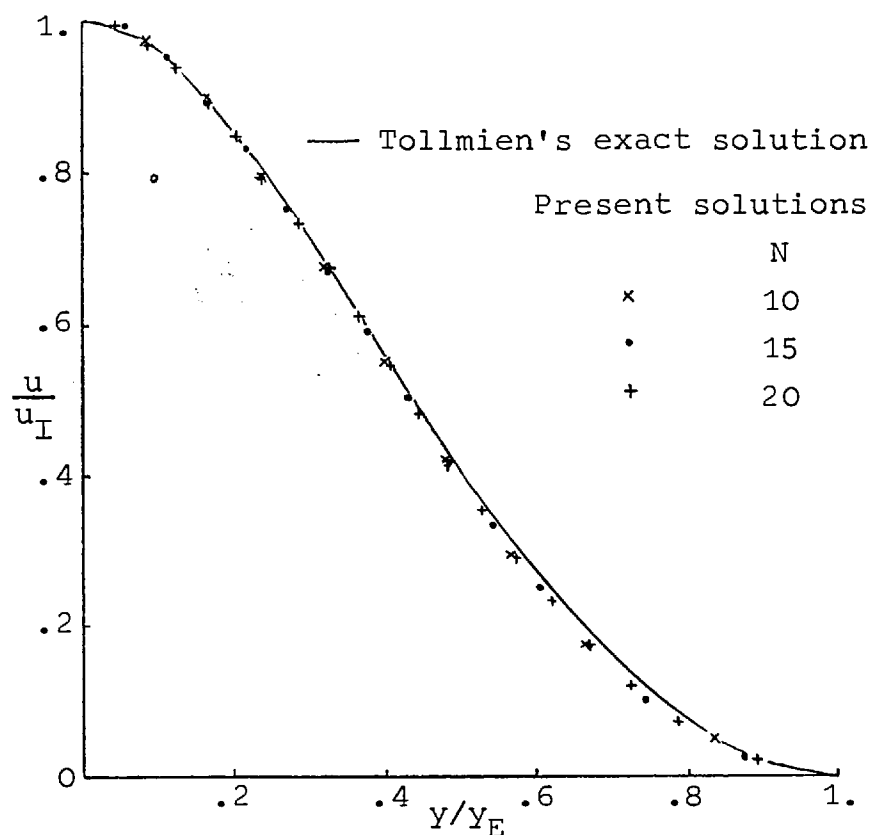


Fig. 3.2-2. Mixing-layer velocity profile.

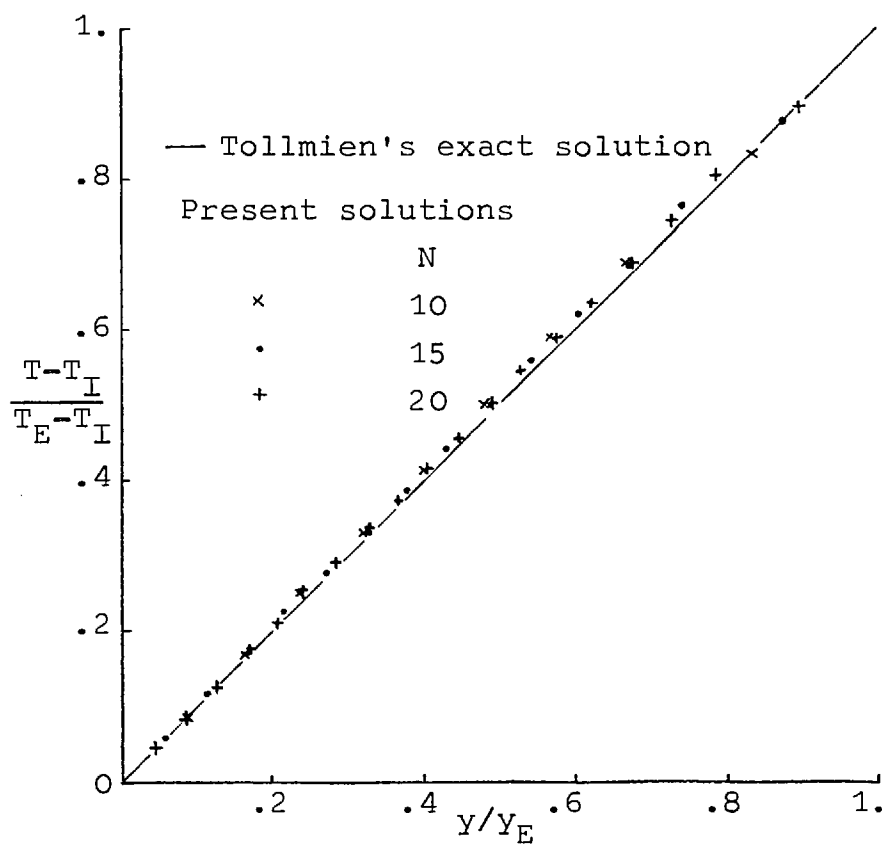


Fig. 3.2-3. Mixing-layer temperature profile.

temperature profile happens to be linear with the cross-stream distance ;
Fig. 3.2-3 shows that our solutions agree with the exact solution also in this respect. It can be deduced from Tollmien's solution that :

$$\frac{y_\ell/x}{\lambda^2} = 3.94 \quad . \quad (3.2-1)$$

The present solutions give this quantity as 4.01 , 3.96 and 3.92 for respectively 11, 16 and 21 grid lines. It can be concluded that use of 16 grid lines gives sufficient accuracy, at least for this problem.

3.2-2 Compressible laminar boundary layer on flat plate

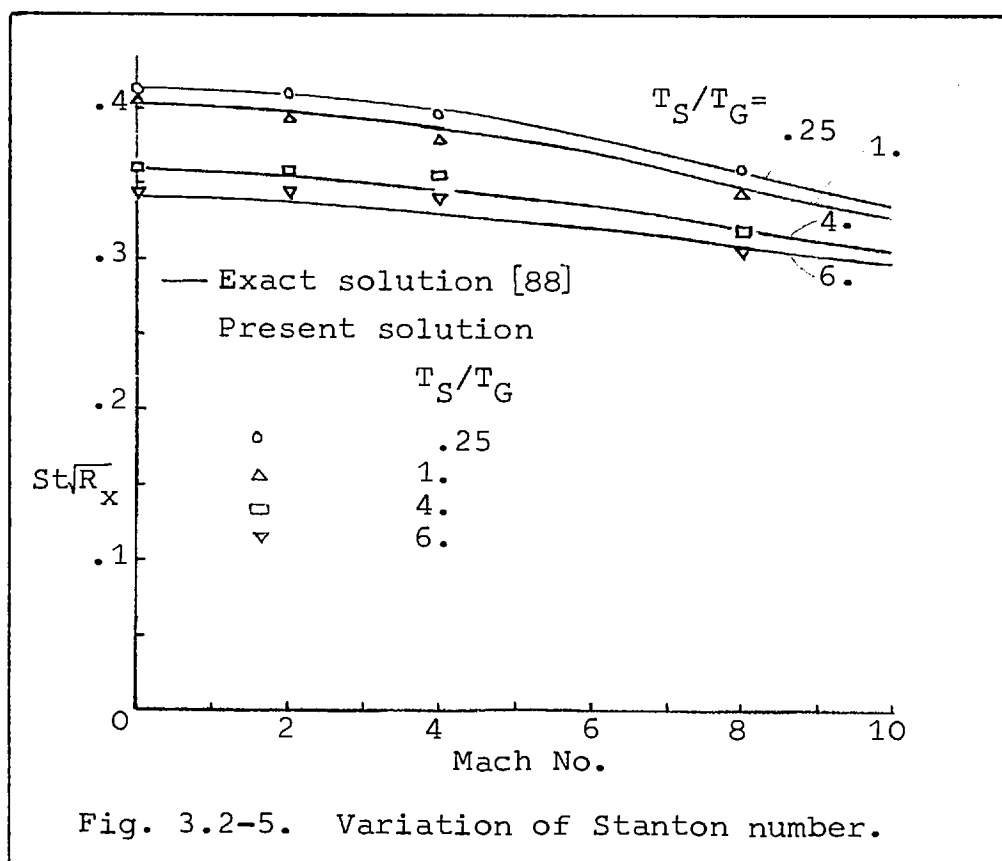
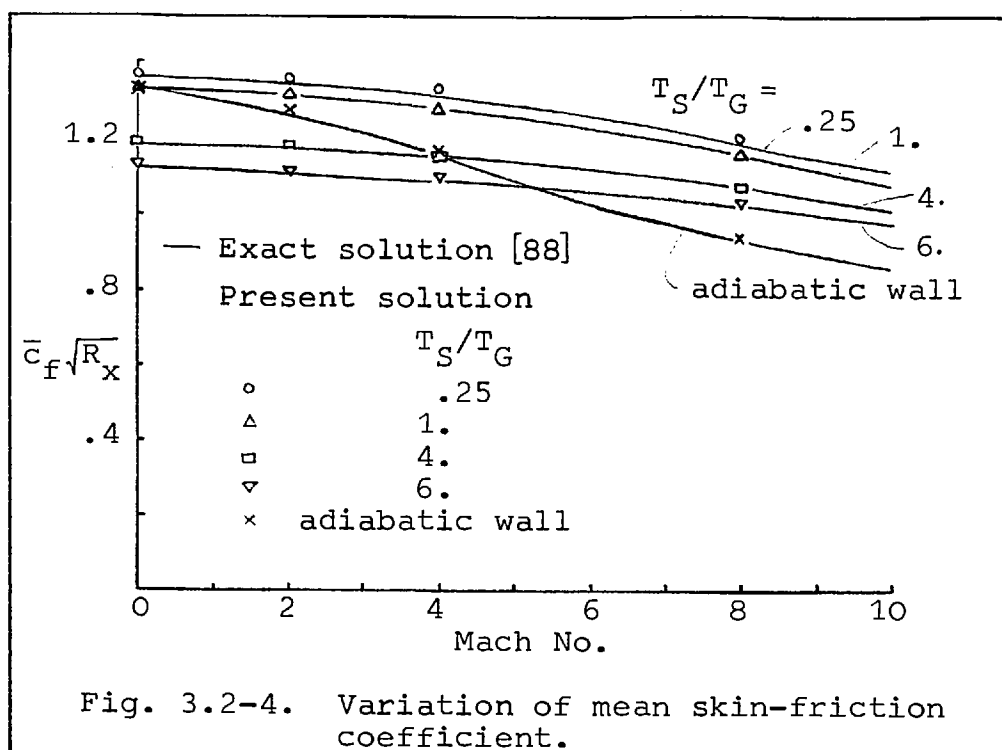
The problem. Van Driest [88] has presented exact numerical solutions for a laminar boundary layer on an impermeable isothermal flat plate ; the specification of the fluid properties includes : uniform specific heats (their ratio, γ , equal to 1.4) , uniform Prandtl number (equal to 0.75) , density inversely proportional to the absolute temperature , and viscosity variation given by the Sutherland law, namely :

$$\frac{\mu}{\mu_G} = \left(\frac{T}{T_G} \right)^{\frac{1}{2}} \frac{1.505}{1 + 0.505(T_G/T)} \quad , \quad (3.2-2)$$

where μ and T respectively stand for viscosity and absolute temperature, while the subscript G denotes the conditions in the main stream.

We shall apply the solution procedure of Chapter 2 to this problem and compare the results with those of van Driest. The task is to calculate the drag coefficient and the Stanton number for various Mach numbers and for various wall-to-mainstream temperature ratios.

Details of solution procedure. In this case the partial differential equations to be solved are (2.2-1) and (2.2-3). The number of grid lines across the layer will be taken as 16. The grid-control procedure should , of course, be that for the boundary with non-vanishing viscosity ; the boundary



will be located so that, on the grid next to the free boundary, the velocity would equal 0.999 times the free-stream velocity u_G . The wall-flux relations to be used are those obtainable from the laminar asymptotes given in Section 1.4-4 ; the actual formulae used will later be presented in Chapter 4. The integration is to be continued until the velocity and temperature profiles show no further change.

Results. Figs. 3.2-4 and 3.2-5 respectively show the variations of $\bar{C}_f \sqrt{R_x}$ and $St \sqrt{R_x}$ with Mach number , for various temperature ratios. The full lines represent the exact solutions from Ref. [88] and the points show the present solution. Here , once again, the present calculation procedure proves its ability to give accurate solutions of the equations, with a reasonably small number of grid lines.

Since the entrainment rate is arbitrary and is used merely to ensure computational efficiency , we can expect that the results would be insensitive to the precise way in which the grid is controlled. Indeed, this was found to be the case ; the results changed little when the desired value of the velocity on the grid line next to the free boundary was varied from $0.999 u_G$ to $0.95 u_G$.

3.2-3 Compressible turbulent boundary layer on flat plate

Having demonstrated the satisfactory agreement of the present-method solutions with some of the available exact solutions, we now present some computations for the turbulent flat-plate boundary layer showing the variation of the Stanton number with Reynolds number, for various Prandtl numbers, Mach numbers and wall-to-mainstream temperature ratios.

The fluid-property variation and hypotheses used. For the computations to be presented in this sub-section, the density is taken as inversely proportional to the absolute temperature ; the viscosity variation

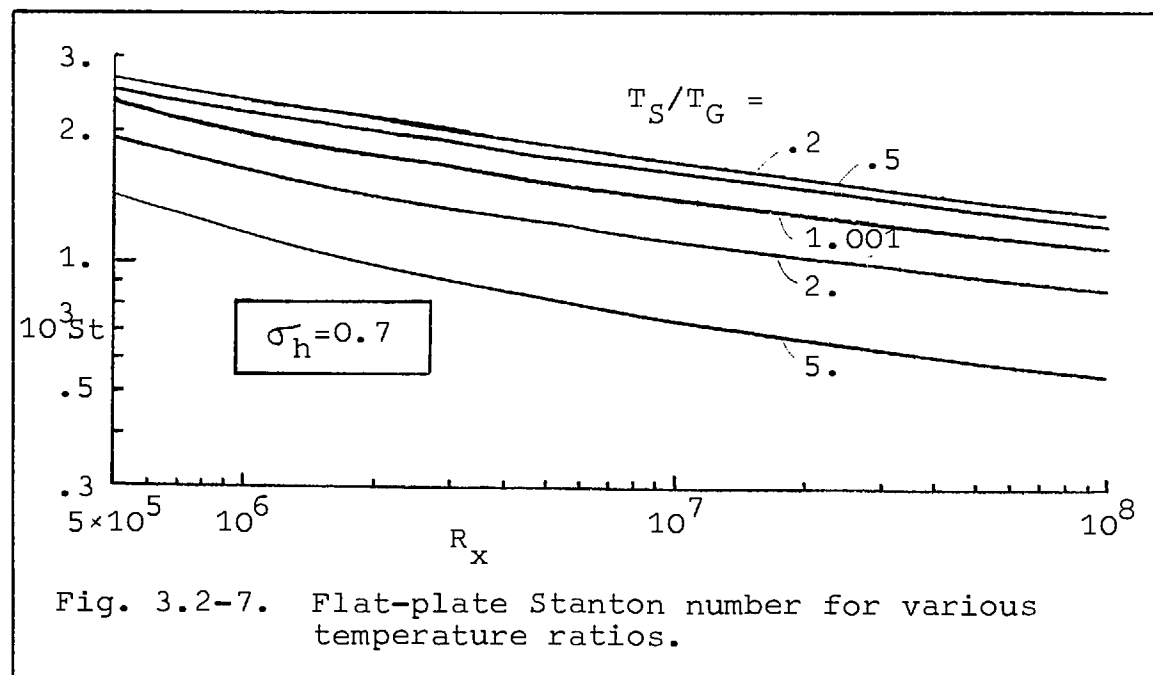
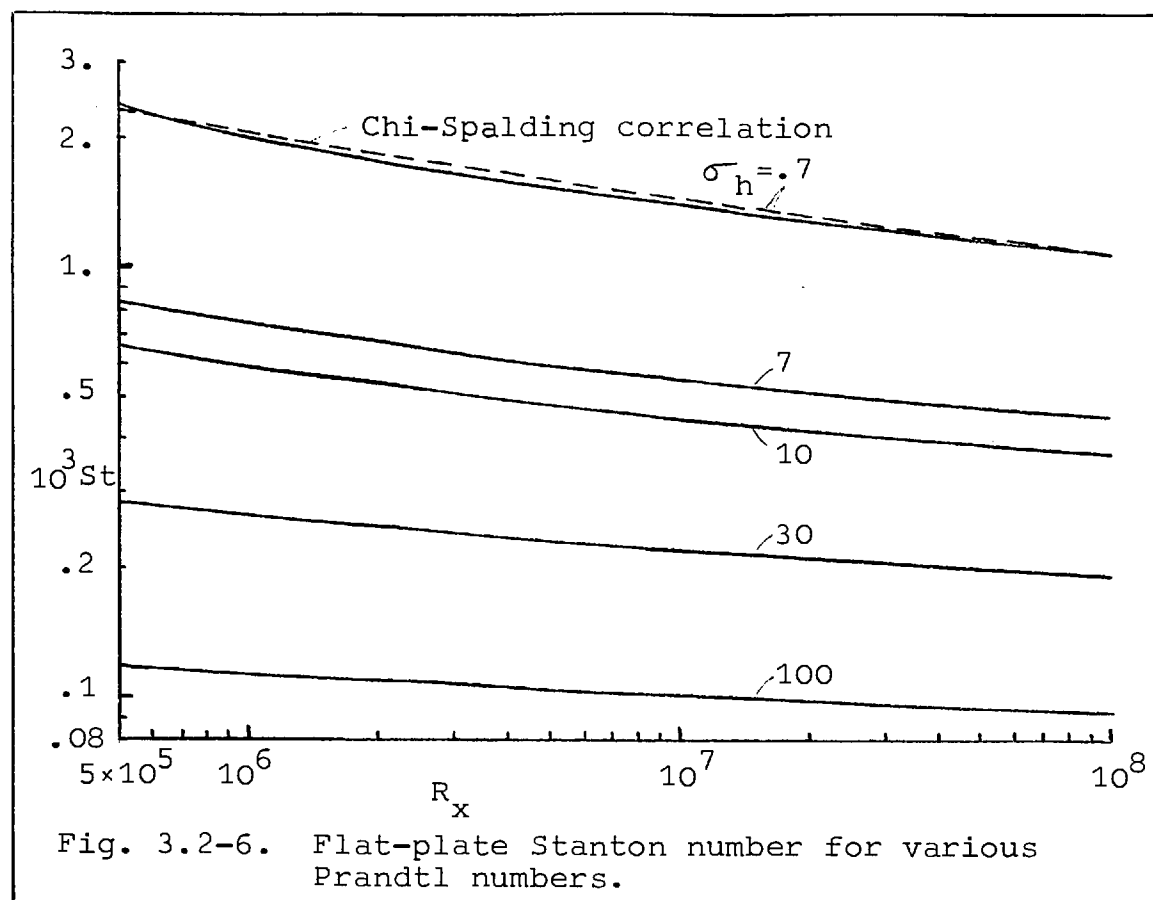
is given by :

$$\frac{\mu}{\mu_G} = \left(\frac{T}{T_G} \right)^{0.76}, \quad (3.2-3)$$

where the subscript G denotes the free-stream quantities. The effective viscosity is calculated from the mixing-length formula (1.3-5) with the mixing-length distribution according to eq. (1.3-6). The thickness y_ℓ is taken as the distance, from the wall, of the point where the velocity equals $0.99 u_G$. The values of the mixing-length constants that we shall use are : $K = 0.435$, $\lambda = 0.09$. The effective Prandtl number will be given a uniform value of 0.9 everywhere except very near the wall where the wall-flux relationships are to be used. These relationships are based on van Driest's hypothesis as described in Section 1.4 ; the actual algebraic expressions will be found in Chapter 4. We shall use the range of the length Reynolds number R_x from 5×10^5 to 10^8 , and the correlation of Spalding and Chi [80] to obtain the appropriate value of the momentum thickness at the beginning of the integration ; the initial velocity and temperature profiles can then be adjusted accordingly.

Effect of Prandtl number. Fig. 3.2-6 shows curves of the Stanton number against Reynolds number for various values of the laminar Prandtl number ; the temperature difference across the layer and the Mach number are assumed to be small. The broken curve represents the correlation of Chi and Spalding [11] for air, i.e. for $\sigma_h = 0.7$. It will be noticed that the effect of the Reynolds number is small at large Prandtl numbers.

Effect of temperature ratio. The influence of wall-to-mainstream temperature ratio on the Stanton number is shown in Fig. 3.2-7 ; here the laminar Prandtl number is 0.7 and the Mach number is small. It can be seen that a wall colder than the free stream is less effective in increasing the Stanton number than is a wall hotter than the free stream in decreasing it.



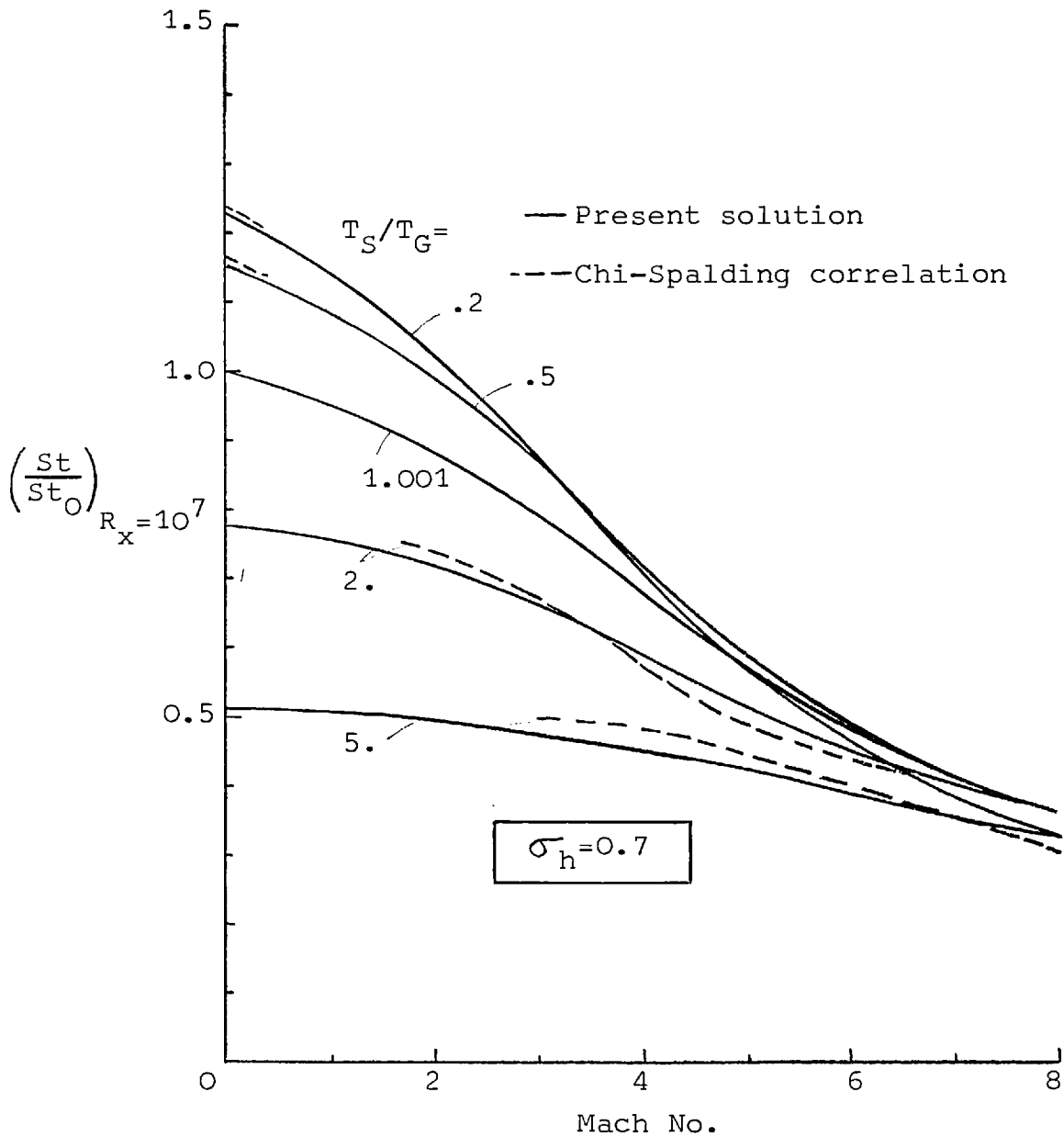


Fig. 3.2-8. Effect of Mach number and temperature ratio on flat-plate Stanton number.

Here St_0 stands for the Stanton number under uniform-property conditions.

Effect of Mach number. The ratio of the actual Stanton number to the one under uniform-property conditions for the same R_x has been plotted, in Fig. 3.2-8, against the Mach number for different temperature ratios. The broken curves show the Chi-Spalding [11] correlation; these are drawn only where the correlations are certainly likely to be valid.

We shall later present, in Chapter 5, comparisons with experimental data. Here our purpose is to demonstrate the capabilities of the calculation procedure; but the results obtained are also realistic as can be seen from the satisfactory agreement, in Figs. 3.2-6 and 3.2-8, of our solutions with the Chi-Spalding correlation. In all the computations for the Prandtl number of 0.7, the ratio of the Stanton number to the drag coefficient was very nearly equal to 0.58 — a value recommended by Chi and Spalding. The value of the recovery factor was around 0.93, whereas experimental data suggest a value of about 0.9. In this connection, it should be remembered that the value of the recovery factor mainly depends upon the value of the effective Prandtl number; it can be shown that the recovery factor is larger than the effective Prandtl number (see, for example, Ref. [82]).

3.2-4 General remarks

The computations described above were performed by use of a computer programme the details of which are given in Appendix A. Here one sees the convenience brought about by a theoretical framework which maintains a high degree of generality; because then a single computer programme can be used for a wide variety of problems. Introduction of the effects of non-uniform density or of Mach number has been very little extra trouble.

The computing time required for all the above calculations is quite modest. On an average, 1000 integrations can be performed in one minute of computing time on the IBM 7090 computer. This applies to the use of 16

grid lines across the layer and to solution of two partial differential equations, for example, those for velocity and stagnation enthalpy. The computing time required for flows near walls is slightly larger than that for free jets, wakes and mixing layers, because the wall problem involves use of wall-flux relationships which are somewhat time-consuming.

3.3 Achievements and shortcomings of the present method

The present method incorporates the advantages of the earlier methods without sharing their defects. The finite-difference technique possesses simplicity of concept ; our use of an implicit formulation ensures stability ; linearisation of the difference equations eliminates the need for iteration ; and the resulting set of algebraic equations allows solution by simple substitution technique.

The use of the non-dimensional stream function ω offers two advantages : firstly, the equations then become somewhat simpler ; and secondly, the grid can be made to conform to the thickness of the boundary layer by an appropriate variation of the boundary values of the stream function, ψ_I and ψ_E . This self-adjusting grid ensures that all the grid points are efficiently utilised as they always lie in the region where significant variations of the dependent variables are present.

Although about 16 grid lines have been shown to give good accuracy, it is always possible to test, by increasing the number of grid lines, whether we have obtained the true solution of the equations ; use of finer spacing in the cross-stream direction does not put a limit on the size of step in the stream-wise direction. Use of the once-for-all integrations for the one-dimensional region near a wall enables us to obtain accurate solutions for this region without excessive computation.

Being of the finite-difference variety, the method is widely applicable ; for, the profiles of velocity and of other variables can easily take various shapes required in different physical situations. Since the algebra involved in obtaining the difference equation is generally applicable, no additional complication is introduced by different variations of the fluid properties or by the variation of radius in the cross-stream direction.

There is one class of situations in which the present method may not prove to be as efficient as the earlier methods. When the thermal boundary layer is much thinner than the velocity boundary layer, only a small proportion of the total number of grid points will lie within the thermal layer, as our grid is designed to fit the velocity layer. The integral methods described in Refs. [47], [49] and [76] incorporate devices specially suited for the treatment of thin thermal layers. The present method would need a fine grid for accuracy in this situation. However, such situations arise only immediately downstream of sudden discontinuities in the thermal boundary condition at a wall and, perhaps, are not very important.

3.4 Suggestions for future work

Application of the present method. In this chapter, some applications of our method have been demonstrated. Some more illustrations will be found in Chapter 5. This, however, does not exhaust the types of physical situation that can be dealt with by this method. Use of the present calculation procedure in various situations is one of the tasks for the future. In this connection, it is important to note that this method (by way of the computer programme described in Appendix A) opens up many new possibilities. Thus, for predicting the wall-temperature distribution in a film-cooling situation, it is not necessary to employ, as in Ref. [44], empirical relationships between the quantities at the slot and the properties of the velocity and temperature profiles at some distance downstream ; now the mixing layer that develops from the slot can be computed until it touches the wall,

followed by a calculation for the boundary layer on the wall, from that point onwards. (This type of work has been reported in Ref. [15].) Similar mixing-layer calculation in the initial region would be useful for the prediction of the development of a jet. Application to flows with combustion, swirl, body forces etc. will in the long run be worth making.

Confined flows. Application of the present method to confined flows has been outlined in Ref. [50]. However, no computations have so far been made to test this procedure. In addition to this test, it will be necessary, in future, to develop methods and computer programmes for complex confined-flow situations in which several jets within a duct and the boundary layers on the confining walls are to be simultaneously considered.

Improvement of the physical hypotheses. The main contribution of the calculation method described in this thesis is that it prepares the road for research into physical hypotheses. Some illustrations of such research will be presented in Part II. It will be shown there how, with the help of the present method, exact implications of particular hypotheses can be worked out and how the subsequent comparison with experimental data can reveal the direction in which the hypotheses should be modified. A considerable amount of such research into the physical hypotheses remains to be done. The mathematical contribution of this thesis makes it possible that such research could be carried out conveniently, quickly and economically.

Part II

Illustrations of research into physical hypotheses

Chapter 4

Some wall - flux relationships

4.1 Motivation

One of the novelties of the calculation procedure described in Part I is the use of once-for-all integrations for the one-dimensional layer near a wall. In Section 1.4, it has been shown how these integrations can be performed by use of a hypothesis such as that of van Driest [89]. If the results of these integrations are expressed as algebraic formulae, they can be conveniently used in subsequent computations. These formulae will be called as "wall-flux relationships" because they relate the flux, at the wall, of momentum, matter or enthalpy to the values of velocity, concentration, temperature etc. in the one-dimensional layer. The reader will recall that the slip-value relations for a wall boundary (given in Section 2.5-2) involve the use of dimensionless drag and flux (s and S) which are the outcomes of the wall-flux relationships. Therefore, before we proceed to make predictions in Chapter 5, for comparison with experimental data, it is necessary to prepare these relationships.

Although wall-flux formulae have been introduced in the present work in a new and general form, their use, though indirect, can be found in earlier works; for example, the method of Ref. [49] makes use of the relations:

$$u_+ = \frac{1}{K} \ln(E y_+) \quad , \quad (4.1-1)$$

$$\text{and} \quad \phi_+ = \sigma_t (u_+ + P) \quad , \quad (4.1-2)$$

as implicit forms of wall-flux laws. (The symbols appearing in these equations have already been defined in Section 1.4, except E and P which can be regarded as integration constants.) Quite apart from their use in prediction methods, the wall-flux relationships to be presented

here can be used to obtain the shear stress and the flux at the wall from measured profiles of velocity and other variables; thus they serve the same purpose as, but have a wider range of applicability than, the "Clauser plot" [13] for determining the wall shear stress.

We shall show below how the numerical integrations for the one-dimensional layer can be put into algebraic forms. The process of arriving at these formulae is not always obvious or easy, and certainly the resulting formulae are not unique. What is presented below serves as an illustration and is more than sufficient for most of the experimental situations dealt with in Chapter 5. A summary of the formulae for turbulent flow will be found in Section 4.6.

4.2 The numerical integrations

Choice of A_* . It has been mentioned in Section 1.4 that the value of A_* must be given for the integration of eq. (1.4-27) and (1.4-29). To obtain good agreement with some experimental data, van Driest [89] used: $A_+ = 26$ and $K = 0.4$, which imply 10.4 as the value of A_* . We shall try to choose A_* so as to obtain good agreement with the following well-known experimental results for smooth walls:

- (1) the constants in the logarithmic law of the wall given by eq. (4.4-1) are: $K = 0.4$, $E = 9.025$;
- (2) the Spalding-Jayatillaka [81] formula for P in eq. (4.1-2) has the asymptote:

$$P = 9.24 (\sigma / \sigma_t)^{3/4}, \quad (4.2-1)$$

for large values of σ / σ_t .

It can be shown that, when A_* equals 11, these two conditions can be satisfied with good accuracy. With $A_* = 11$, the

numerical integration of eq. (1.4-27) will give, for zero values of p_* and m_* , the integration constant E_* introduced in eq. (1.4-62) as 22.74; this is equivalent to $E = 9.096$ for $K = 0.4$. Also, use of $A_* = 11$ in eq. (1.4-72), in conjunction with $K = 0.4$, leads to:

$$\text{large } \sigma/\sigma_t: \quad P = 9.2 (\sigma/\sigma_t)^{3/4}, \quad (4.2-2)$$

which is to be compared with eq. (4.2-1). The value of A_* will therefore be taken as 11 for the rest of the work in this chapter. The reader should note that since our choice of A_* has been made with reference to smooth-wall data, the relationships to be developed here will apply only to smooth walls. In general, A_* will depend upon the roughness height and the type of roughness; only indirect information about this dependence can be found in Ref. [31].

Profiles of u_* and $\phi_*(1)$ — Using the value of A_* , we can now obtain numerical integrations of eq. (1.4-27) for various values of p_* and m_* . Some representative results are shown in Figs. 4.2-1 and 4.2-2. It should be noted that, for negative values of p_* or m_* , real solution ceases to exist after the local shear stress vanishes according to eq. (1.4-24).

Similar integrations can be performed of eq. (1.4-29) to yield profiles of ϕ_* , or rather of $\phi_*(1)$, after putting σ_t equal to unity. Integrations for $p_* = 0$ will enable us to deduce values of the integration constant P_* used in eq. (1.4-66).

If we now recall the symbols and definitions given in Section 1.4-5, it will be apparent that the above integrations can give us the values of s_* and $S_*(1)$ as a function of R_* , F_* , M_* and σ/σ_t . What is not satisfactory at this stage is that these functional relationships are numerical and not algebraic. We shall now turn to the task of casting

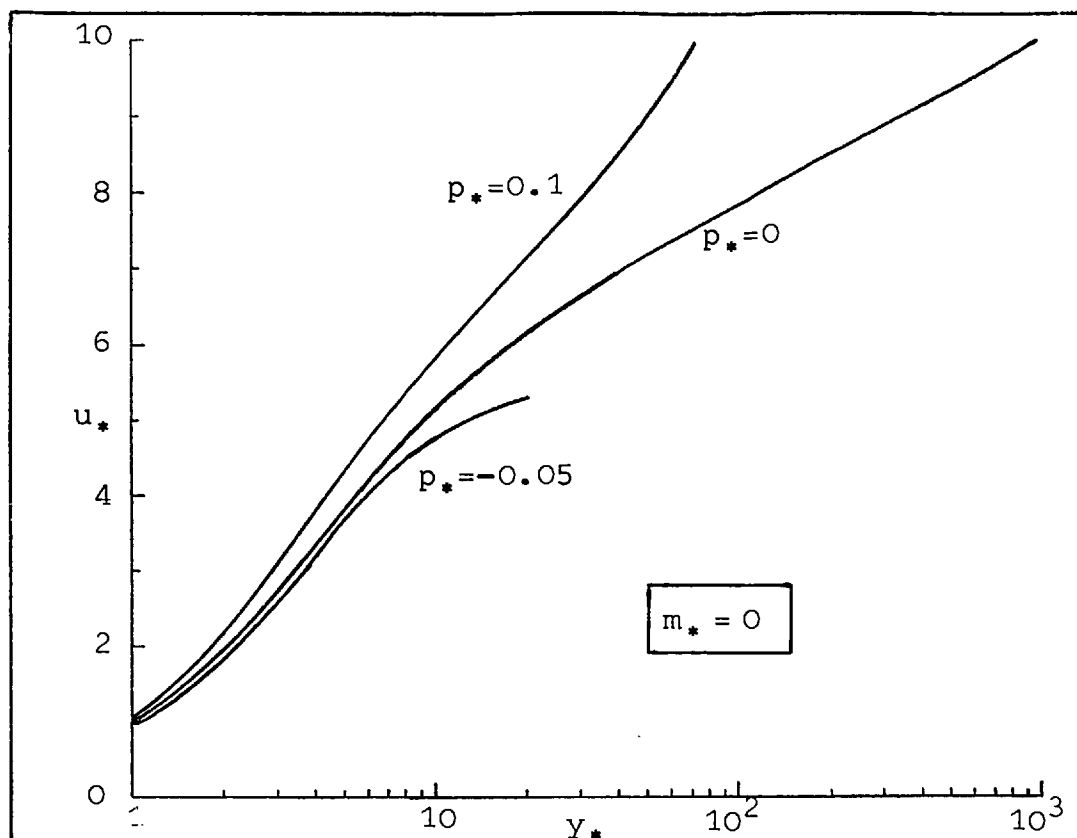


Fig. 4.2-1. Dimensionless velocity profiles for various values of p_* .

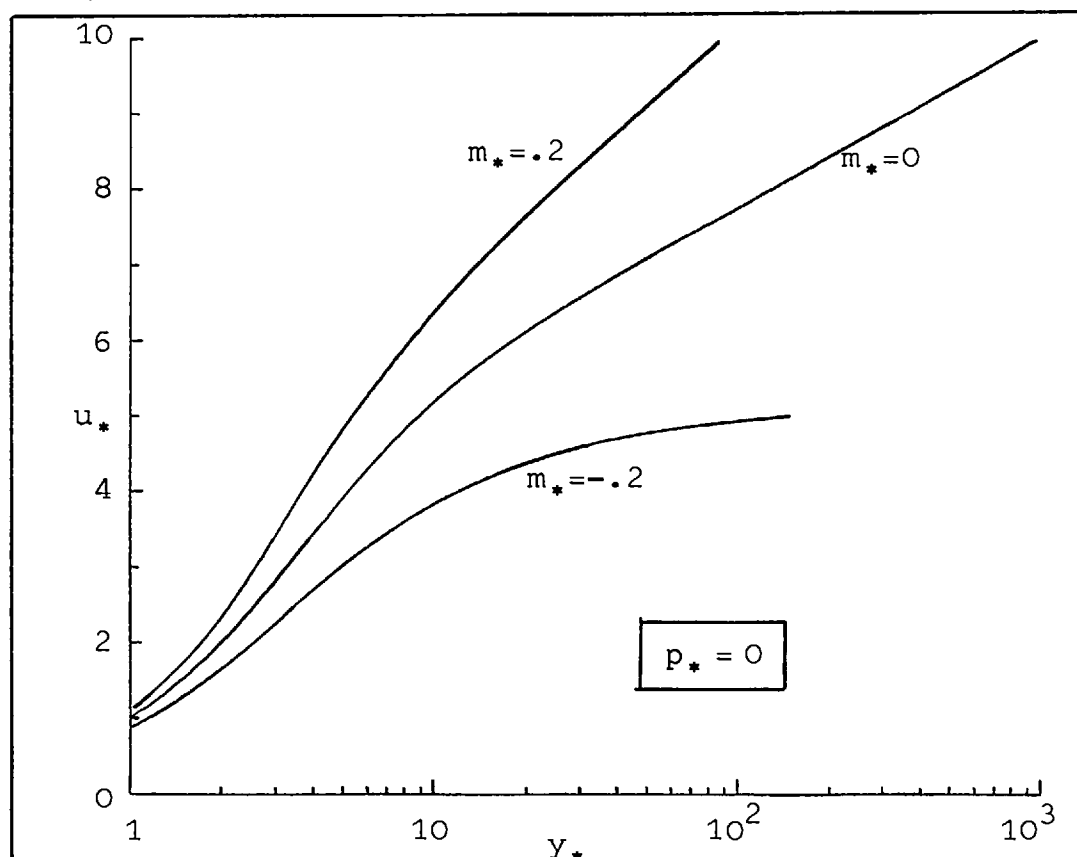


Fig. 4.2-2. Dimensionless velocity profiles for various values of m_* .

these results into algebraic formulae.

4.3 The algebraic formulae

4.3-1 Relationships for laminar flow

Although the wall-flux formulae have a special relevance to turbulent flow (because of the steep gradients in the wall-near region), we shall need them also for the laminar flow to maintain uniformity of the calculation procedure. Moreover, as the formulae for laminar conditions are easy to derive, they will quickly illustrate relationships of their kind. The subsequent formulae for turbulent flow should ideally reduce, very near the wall, to those for laminar flow, because the flow there is indeed laminar.

From the definitions of various dimensionless quantities and from the analytical solutions given in Section 1.4-4 (a), the following formulae can be constructed :

$$\text{For } M = 0 : \quad s = \frac{1}{R} - \frac{F}{2} \quad . \quad (4.3-1)$$

$$\text{For } F = 0 : \quad s = \frac{M}{\exp(MR) - 1} \quad . \quad (4.3-2)$$

$$\text{For all } F, M: \quad S = \frac{M}{\sigma \{ \exp(MR) - 1 \}} \quad , \quad (4.3-3)$$

which has the limiting form :

$$M \rightarrow 0 : \quad S = \frac{1}{\sigma R} \quad . \quad (4.3-4)$$

From eq. (1.4-41) and (1.4-73), it is easy to show that the recovery factor is given by :

$$\text{for } M = 0 : \quad H = \sigma_h \quad . \quad (4.3-5)$$

4.3-2 Drag law for turbulent flow

Flat plate with zero mass transfer. As a convenient basis for further formulae, we shall first obtain the $s_* \sim R_*$ relation for the case of zero F_* and M_* . Fig. 4.3-1 shows this relation as obtained from the numerical results.

It is possible to fit this by the expression :

$$s_{*,0} = R_*^{-1} - 0.1561 R_*^{-0.45} + 0.08723 R_*^{-0.3} + 0.03713 R_*^{-0.18}, \quad (4.3-6)$$

with a maximum discrepancy of 3% for the range $0 < R_* < 10^5$; the subscript 0 has been added to s_* as a reminder that this refers to the condition : $F_* = M_* = 0$.

Effect of pressure gradient on drag. When the pressure gradient is non-zero, eq. (1.4-63) applies at large values of R_* . In the limit when the boundary layer approaches separation (i.e. $\tau_S \rightarrow 0$), we get :

$$\text{for } p_* y_* \rightarrow \infty : \quad u_* = 2 \sqrt{p_* y_*}. \quad (4.3-7)$$

Here the integration constant has been regarded as negligible ; this can be expected for large values of $p_* y_*$ and is also substantiated by the numerical integrations*. We now obtain, from eq. (4.3-7) that :

$$F_{*, \text{sep}} \equiv \left(\frac{p_* y_*}{u_*^2} \right)_{\tau_S \rightarrow 0} = \frac{1}{4}. \quad (4.3-8)$$

* Equation (4.3-7) is the solution given by Townsend [87] and Spalding [77], it is interesting to remark, from different assumptions about the diffusion of turbulent kinetic energy. The velocity profiles reported by Straiford [84] for a separating boundary layer show agreement with the one-half-power law as given by eq. (4.3-7).

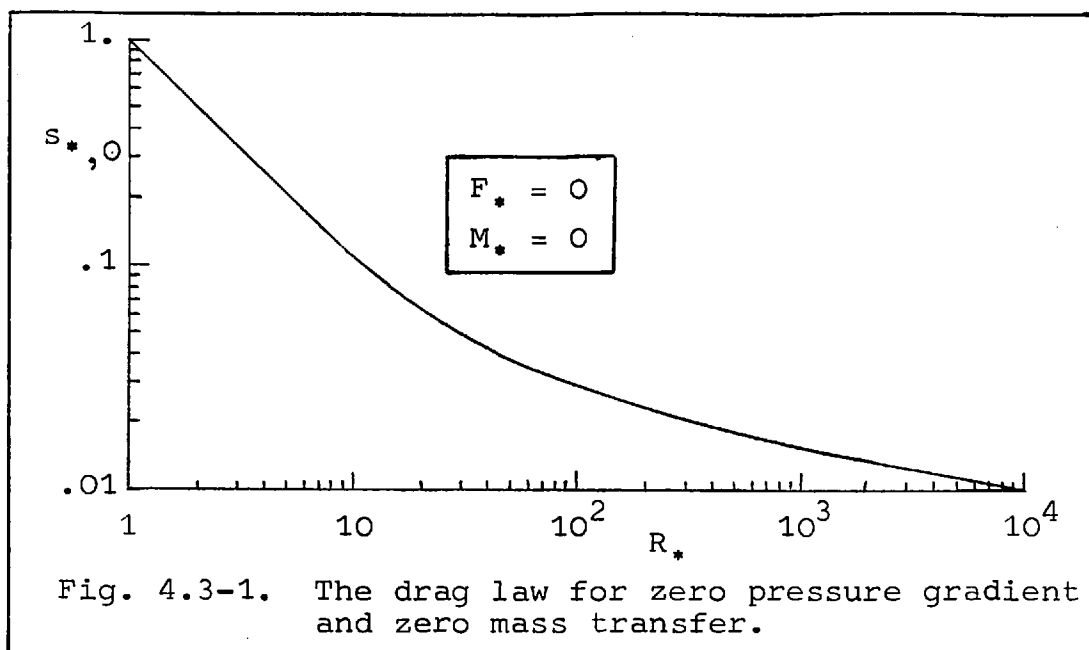


Fig. 4.3-1. The drag law for zero pressure gradient and zero mass transfer.

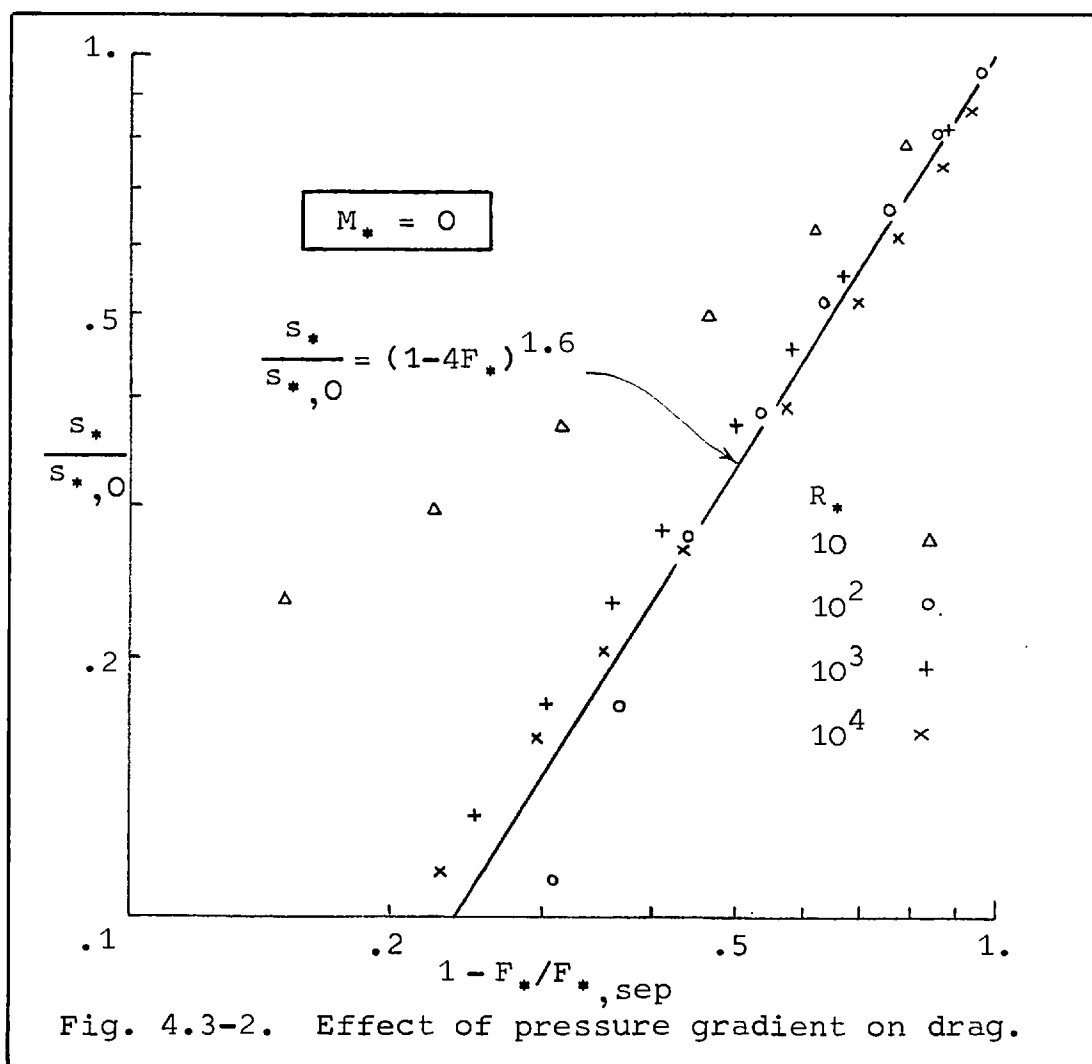


Fig. 4.3-2. Effect of pressure gradient on drag.

This indicates that, for large values of R_* , the value of F_* will not exceed $1/4$. Indeed, the value of $F_{*,sep}$ serves as a convenient scaling factor for F_* . Fig. 4.3-2 shows a plot of $(s_*/s_{*,0})$ against $(1 - F_*/F_{*,sep})$. It will be seen that the points for large values of R_* lie near to the line represented by :

$$s_*/s_{*,0} = (1 - 4 F_*)^{1.6} \quad (4.3-9)$$

By modification of this, the following formula has been devised to give good agreement also for smaller values of R_* .

$$\frac{s_*}{s_{*,0}} = \left[1 - \frac{4 F_* R_*}{(12.8^{2.5} + R_*^{2.5})^{0.4}} \right]^{1.6} \quad (4.3-10)$$

Fig. 4.3-3 shows the exact values of $(s_*/s_{*,0})$ plotted against those computed from eq. (4.3-10). The agreement may be regarded as satisfactory; although the discrepancy is larger at small values of $(s_*/s_{*,0})$, accuracy is usually of little importance in this region.

Effect of mass transfer on drag. Fig. 4.3-4 shows a plot of the exact values of $(s_*/s_{*,0})$ against M_* for various values of R_* . It can be seen that the horizontal distance between any two curves of constant R_* is approximately uniform for all values of $(s_*/s_{*,0})$. This suggests that all the curves could be made to collapse on a single curve if the abscissa were multiplied by a certain quantity depending upon the value of R_* . Then an equation for this single curve can be obtained. The following formula has been designed in this manner.

$$\frac{s_*}{s_{*,0}} = \left[1 - \frac{M_*}{7.74 R_*^{-1.17} + 0.956 R_*^{-0.25}} \right]^4 \quad (4.3-11)$$

From Fig. 4.3-5 it can be seen that this formula gives good agreement with exact values for both positive and negative values of M_* .

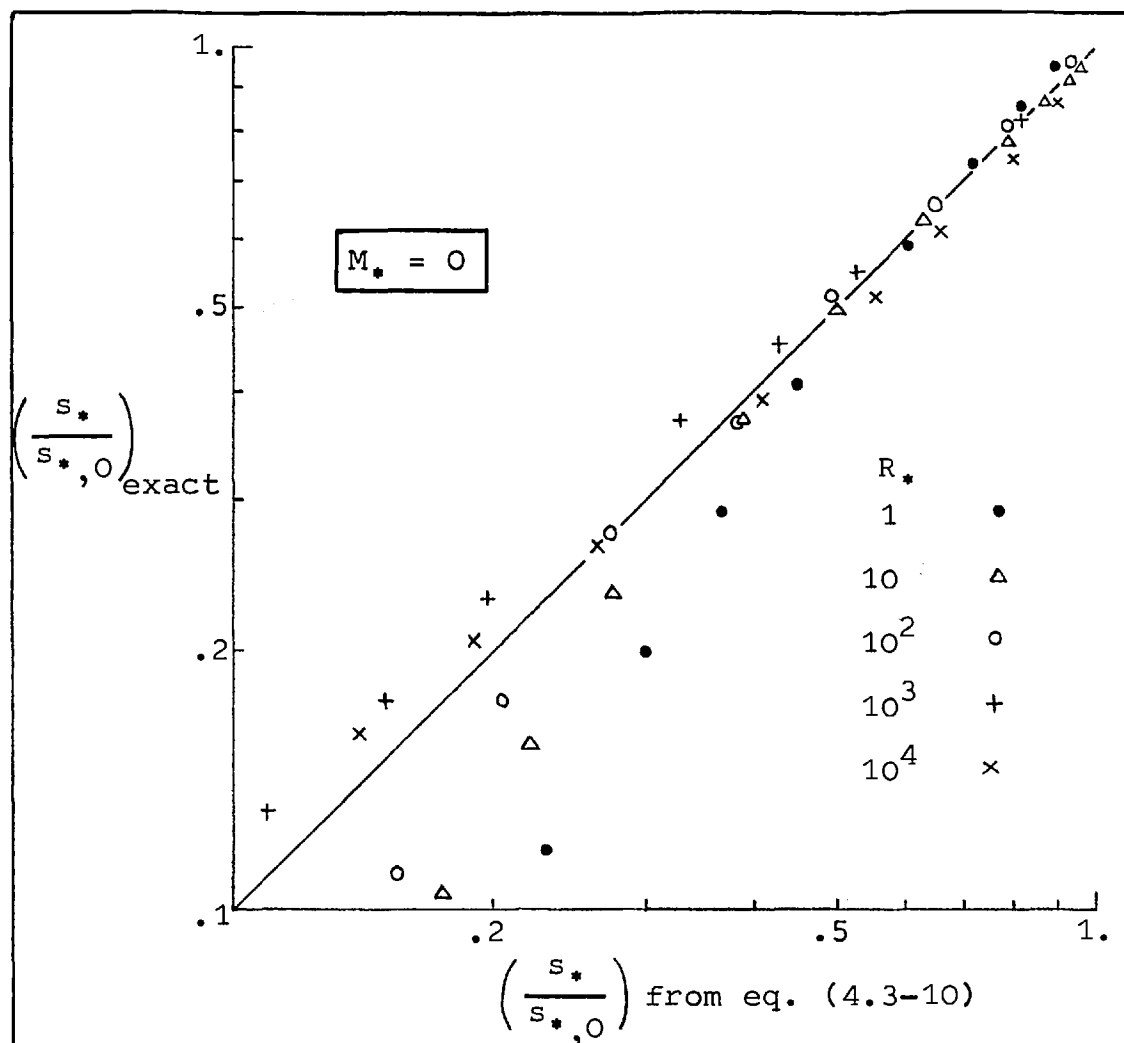


Fig. 4.3-3. Comparison of the exact drag-ratio values with those computed from eq. (4.3-10).

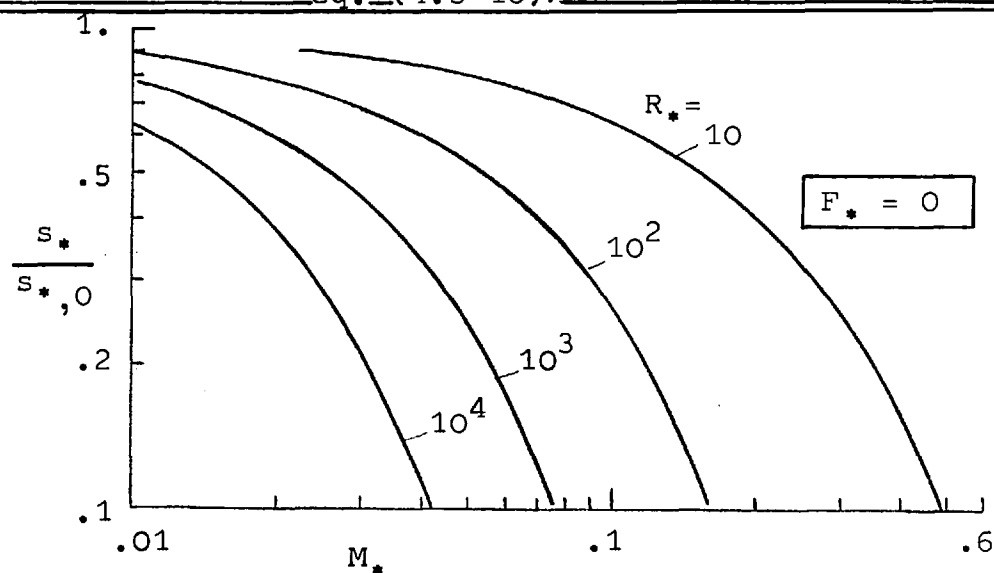


Fig. 4.3-4. Effect of mass transfer on drag.

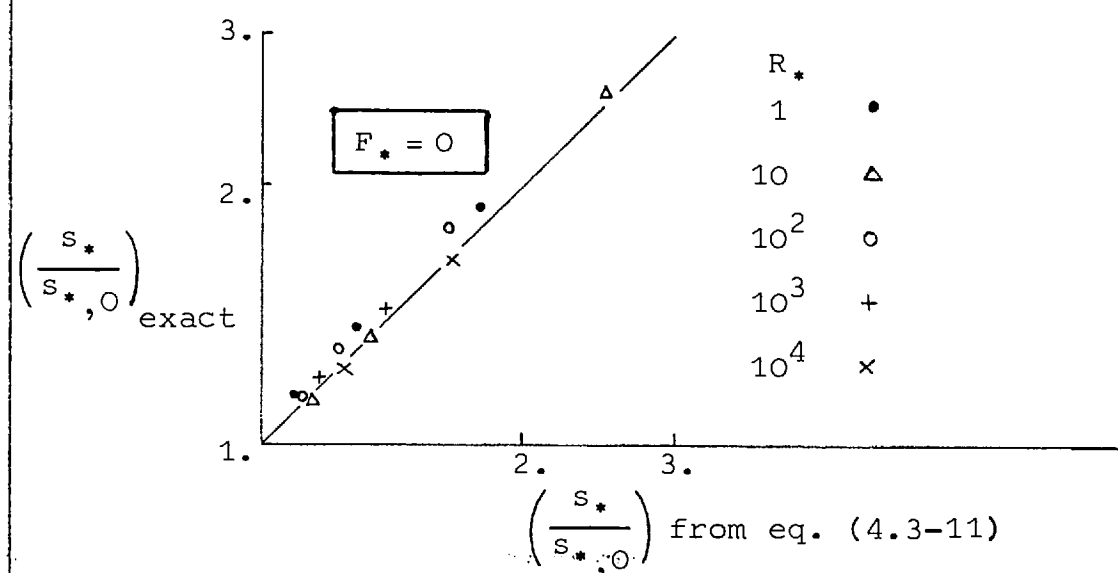
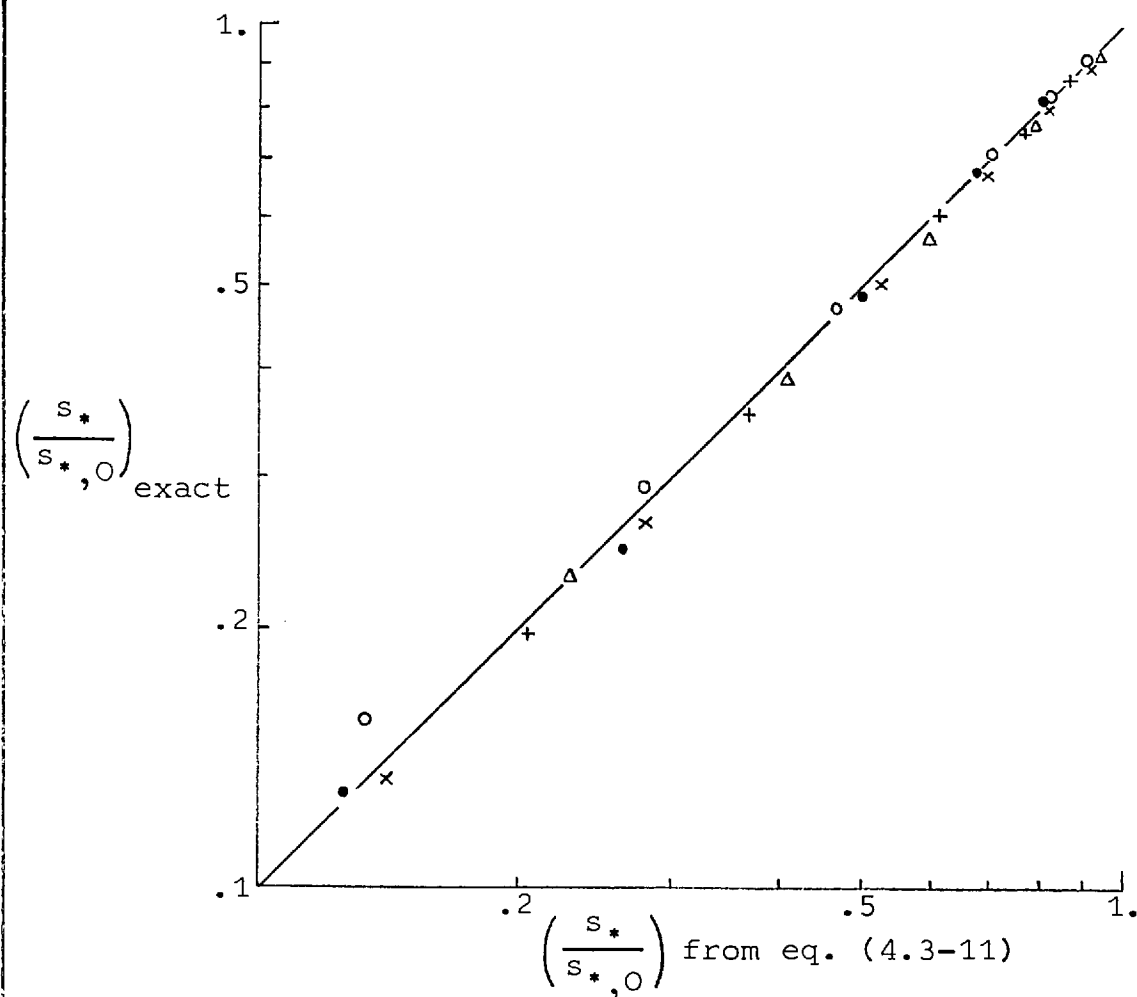


Fig. 4.3-5. Comparison of the exact drag-ratio values with those computed from eq. (4.3-11).

4.3-3 Heat-flux or diffusional-flux law for turbulent flow

Flat plate with zero mass transfer. Here we seek a formula for the dimensionless flux $S_{*(1)}$ in terms of R_* and σ/σ_t . If we use eq. (1.4-67) as our starting point, we can express it as :

$$S_{*(1),0} = \frac{S_{*,0}}{1 + P_*(s_{*,0})^{\frac{1}{2}}}, \quad (4.3-12)$$

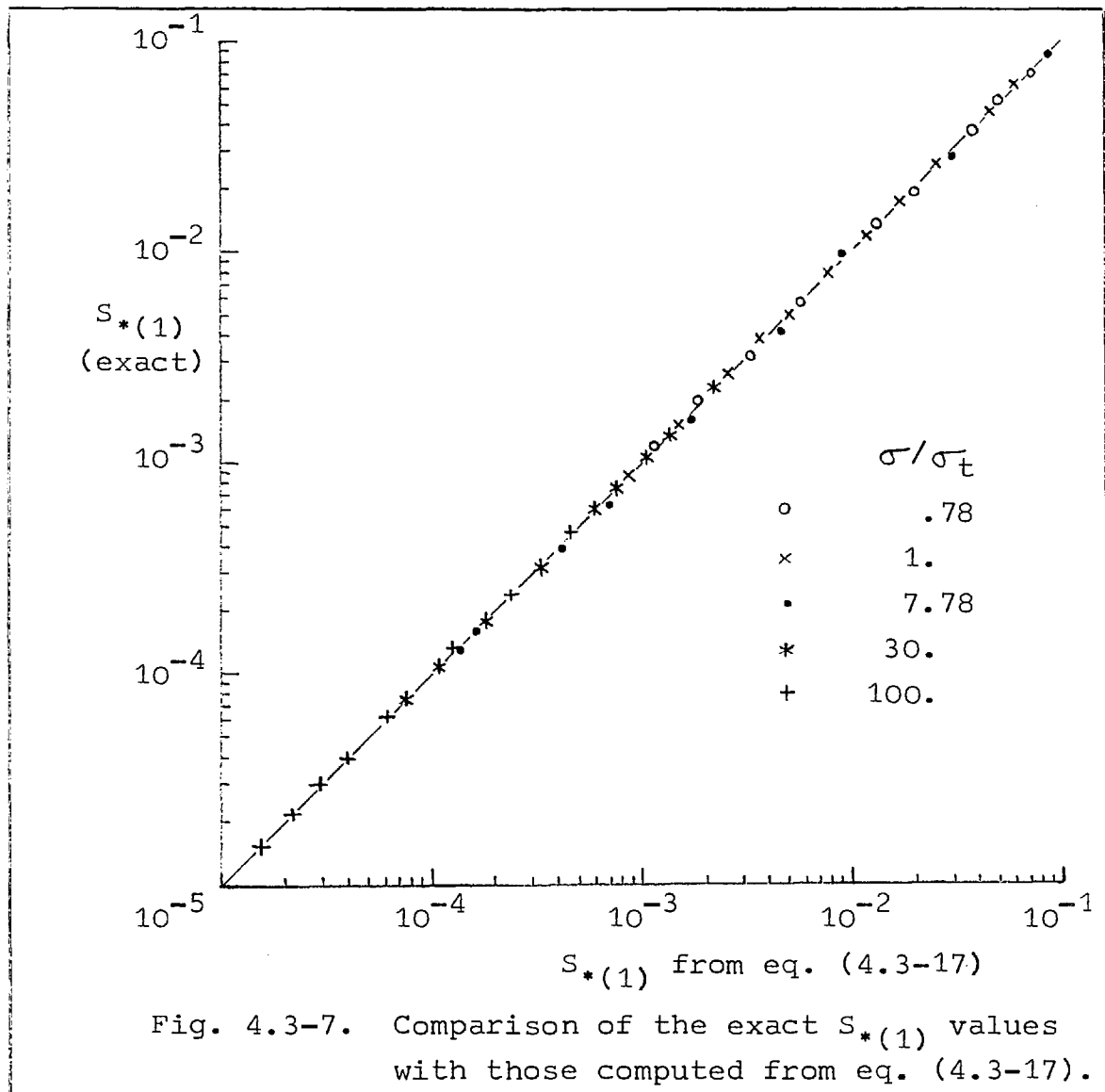
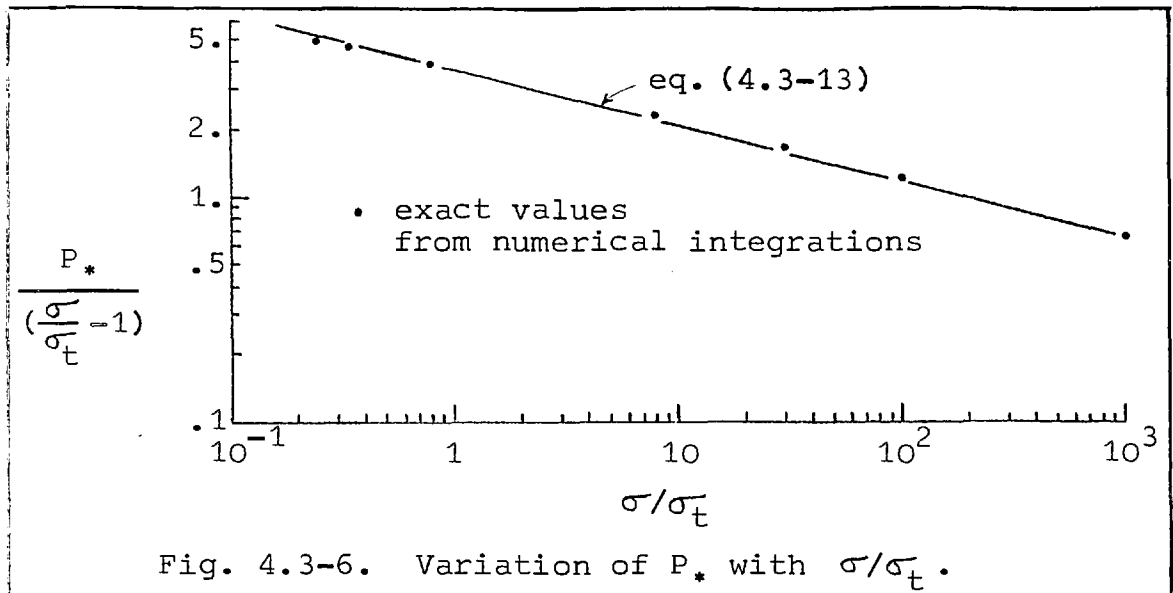
where the subscript 0 denotes the condition : $F_* = M_* = 0$, and where $s_{*,0}$ is to be substituted from eq. (4.3-6). Now only P_* remains to be expressed as a function of σ/σ_t . The analytical expression (1.4-72), derived in Section 1.4-4 (d), for large values of σ/σ_t can be written as :

$$P_* = 3.68 (\sigma/\sigma_t)^{-0.25} \{(\sigma/\sigma_t) - 1\}, \quad (4.3-13)$$

in which the value of A_* has been substituted. Fig. 4.3-6 shows $P_*/\{(\sigma/\sigma_t) - 1\}$ plotted against (σ/σ_t) ; it will be seen that the line representing eq. (4.3-13) is in agreement with exact values for not only the large values of (σ/σ_t) , but also the small ones. We shall, therefore, adopt eq. (4.3-13) as a general formula for P_* . This equation along with eq. (4.3-12) completes the flux law for zero pressure gradient and zero mass transfer*.

Effect of pressure gradient. Once again we shall consider the case of separating boundary layer. When $p_* y_*$ tends to infinity, it can be shown from eq. (1.4-29) by use of eq. (4.3-7) that :

* It is true that eq. (4.3-12) is not satisfactory for very small values of R_* . A formula which has the correct asymptote also at small values of R_* has been given in Ref. [48]; however, it is complicated and this complication seems unnecessary, because, as mentioned in Section 2.5-3, we shall try always to apply the wall-flux relationships at sufficiently large values of R_* .



$$\phi_{*(1)} = \text{const.} , \quad (4.3-14)$$

which, in conjunction with eq. (4.3-7) , gives :

$$S_{*(1)} \propto R_*^{-1/3} . \quad (4.3-15)$$

It was found from the numerical solutions that the constant of proportionality depends only on σ/σ_t ; thus we obtain , for the separating boundary layer , that :

$$S_{*(1), \text{sep}} = \frac{1.725}{(P_* + 6.8)^{1.165}} R_*^{-1/3} \quad (4.3-16)$$

Now a formula can be constructed which will fit the numerical solutions also for all the intermediate values of F_* . For example :

$$S_{*(1)} = S_{*(1),0} (1 - F') + F' S_{*(1), \text{sep}} , \quad (4.3-17)$$

$$\text{where } F' \equiv \frac{0.25 F_* R_*}{1 + 0.0625 R_*} . \quad (4.3-18)$$

The agreement of this formula with the exact numerical results can be seen from Fig. 4.3-7 where points are plotted for various values of R_* , F_* and σ/σ_t .

Effect of mass transfer. We shall not consider here the effect of mass transfer on $S_{*(1)}$ for all values of σ/σ_t . It can be seen that when σ/σ_t equals unity, $S_{*(1)}$ becomes the same as s_* provided that the pressure gradient is zero ; thus eq. (4.3-11) can be regarded as giving $S_{*(1)}$ as well.

4.4 Examination of validity of the relationships

Van Driest's hypothesis is as yet largely untried. As far as the present author is aware, this thesis describes the first large-scale use of that hypothesis. In this use, we have generalised the hypothesis and also applied it to heat transfer. It is now necessary to examine the extent to which the van Driest law (or rather the set of our algebraic relationships

expressing its implications) is in agreement with reality. This can be done in two ways : wherever detailed measurements for the region very close to the wall are available , a direct test of the hypothesis is possible ; when only certain gross features of the flow situation are known , these can be compared with the predictions made by use of our wall-flux laws. Such indirect tests will be presented in the next chapter ; here we shall look for direct verification.

(a) The hydrodynamic quantities. In the published literature, there are few sets of reliable hydrodynamic data which can lead to a direct decisive test of the hypothesis. Direct measurements of wall shear stress , and of velocity profile very close to the wall, are difficult to make. Pai [45] , while reviewing the available methods of measuring skin friction, points out the unsatisfactoriness of almost all of them. Apart from the implication of the hypothesis in giving the well-established "universal" law of the wall for impermeable flat-plate situation, and the one-half-power law for the separating boundary layer (in agreement with Stratford's [84] results) , there is little that can be said with certainty on the hydrodynamic side.

(b) Effect of Prandtl number on wall flux. For heat transfer or diffusional transfer of matter , the hypothesis can be said to predict the effect of Prandtl or Schmidt number correctly. Fig. 4.4-1 shows that our formula for P_* (eq. (4.3-13)) is in close agreement with the Spalding - Jayatillaka [81] formula , namely* :

$$P = 9.24 [(\sigma/\sigma_t)^{3/4} - 1] [1 + 0.28 \exp(-0.007 \sigma/\sigma_t)] , \quad (4.4-1)$$

* Here the connection between P_* and P is made by noting that $P_* = KP$; K has been put equal to 0.4 - a value which Spalding and Jayatillaka [81] used.

which represents a correlation of a large number of experimental data.

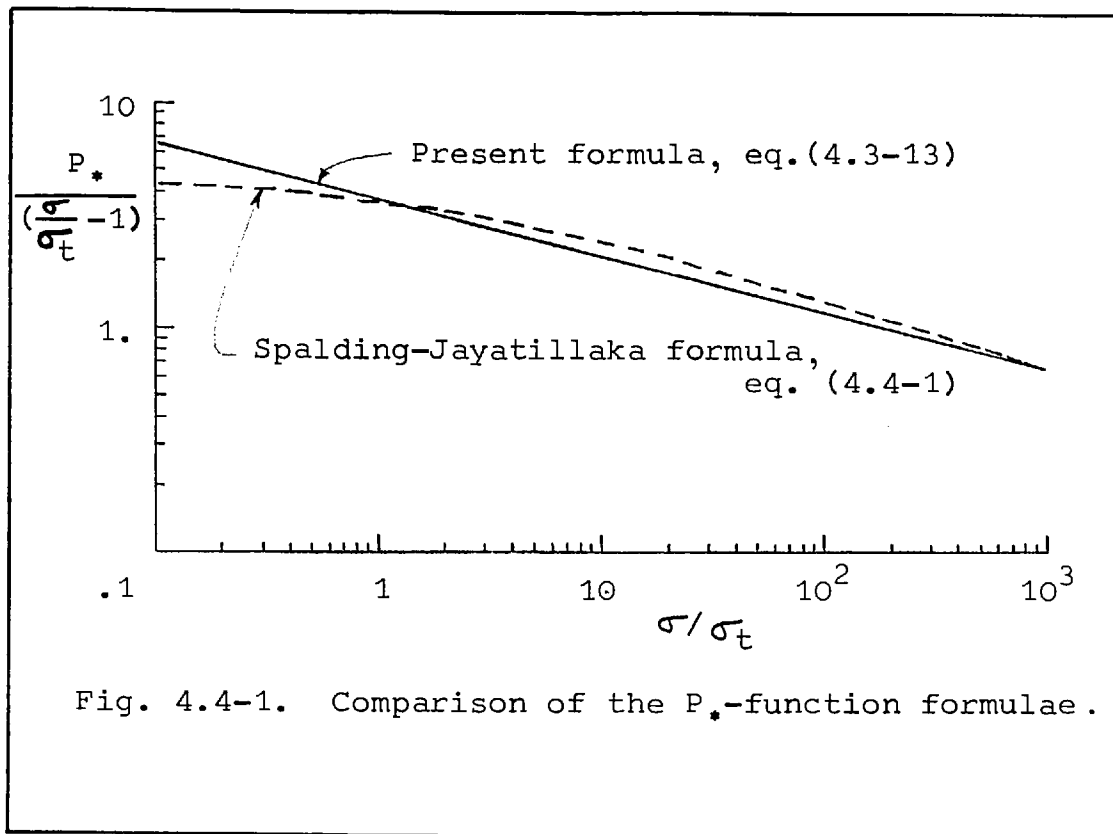


Fig. 4.4-1. Comparison of the P_* -function formulae.

(c) Effect of pressure gradient on wall flux. Although many data are available for the effect of pressure gradient on heat transfer, detailed velocity and temperature profiles have been given by few. The only available complete measurements are those of Perry et. al.[52] for heat transfer in adverse pressure gradient. Pai [45] , after comparison of these data with the present * wall -flux relationships, reports an agreement within 20% with the wall heat flux.

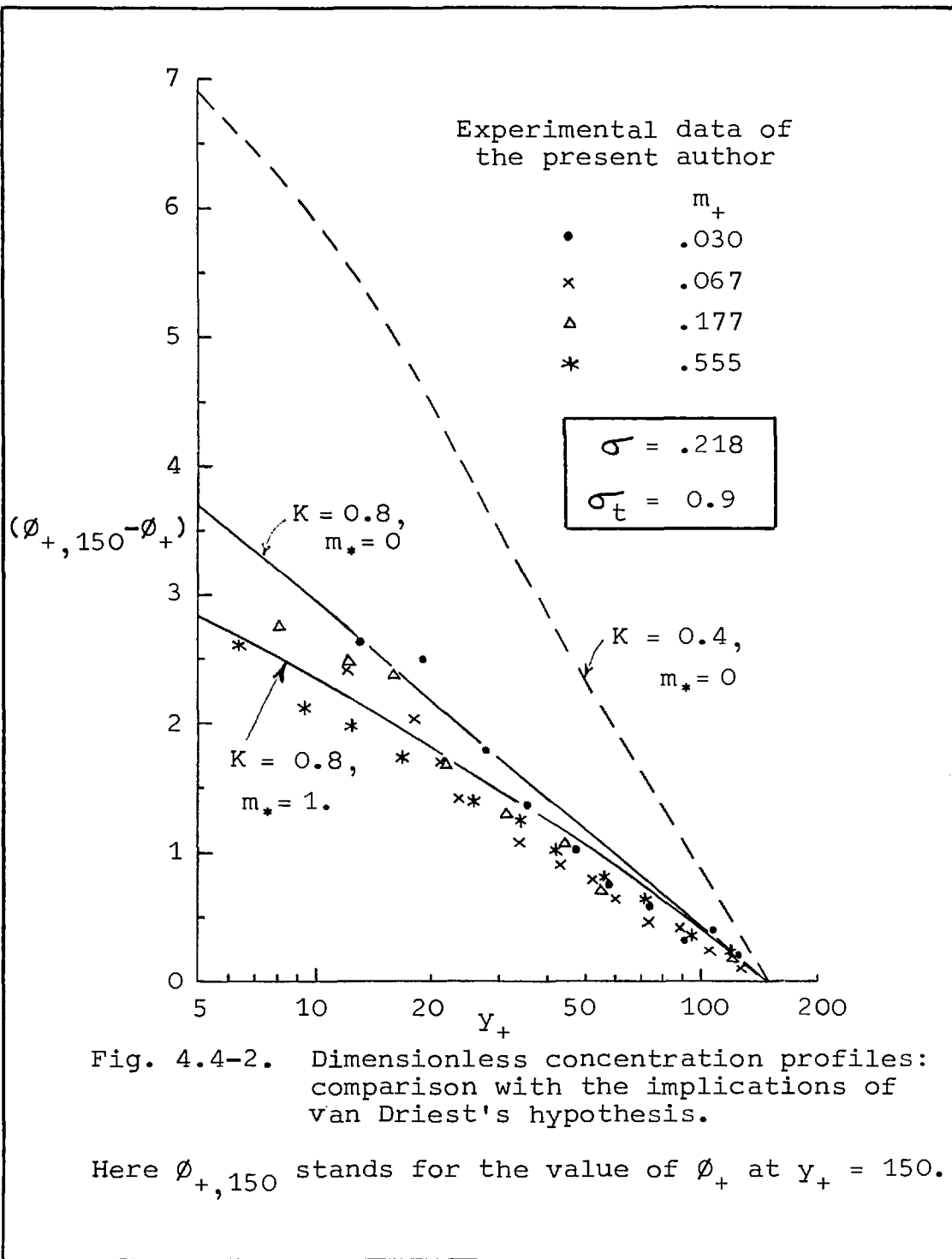
* Pai actually used the relationships given by the present author in Ref. [48] ; the present formulae differ from these in only minor details.

(d) Effect of mass transfer on wall flux. Once again, few workers have reported detailed measurements in situations involving mass transfer at the wall. Recognising the need for such data, the present author performed some experiments on a radial wall jet blown on a porous surface through which air was injected with helium gas as a tracer. The details of this experimental investigation are given in Appendix B ; here we shall present some of the results of these experiments and compare them with the implications of van Driest's hypothesis.

Details of data reduction. Fig. 4.4-2 shows dimensionless profiles of helium concentration measured for various blowing rates at a distance of 12" from the slot. The largest blowing rate used very nearly corresponds to the "blow-off" condition. The skin-friction values required to obtain ϕ_+ and y_+ were taken from Baker's [5] measurements* on the same apparatus. The profiles have been drawn up to $y_+ = 150$; this distance roughly corresponds to half the distance at which the velocity reaches a maximum, and so we can expect our Couette-flow analysis to hold in this region.

Details of theoretical curves. As mentioned earlier, integration of eq. (1.4-29) will yield $\phi_* \sim y_*$ profiles in the form (1.4-32). We shall use the value of laminar Schmidt number for helium in air as 0.218 (obtained by use of the formula in Ref. [56]), and of the turbulent Schmidt number as 0.9. The connection of ϕ_* and

* Baker applied a turbulence correction to his skin-friction values. Although his correction does not seem very convincing, we have used here the "corrected" values as Baker does not report the "raw" values except for the no-blowing case. His skin-friction values show considerable scatter ; the values used here have been obtained from a mean curve.



Y_* (as defined by eq. (1.4-23) and (1.4-16)) with ϕ_+ and Y_+ is to be made through the mixing-length constant K . If we use the conventional value of K , namely 0.4 , the resulting $\phi_+ \sim Y_+$ profile for the no-blowing case will be as shown in Fig. 4.4-2 by the broken curve . Obviously here the discrepancy between theory and experiment is very large .

It will now be recalled that experiments in rather unusual circumstances are being considered here . The conventional value of K may not be applicable to this situation . Indeed, in Chapter 5 we shall show that good agreement with wall-jet velocity profiles can be obtained if K is taken as 0.8 . In Fig. 4.4-2, therefore, theoretical curves with $K = 0.8$ have also been drawn for $m_* = 0$ and 1 ; the value $m_* = 1$ corresponds to $m_+ = 0.8$ which is larger than the highest value of m_+ in the experiments .

Remarks. The agreement with experiment is satisfactory with $K = 0.8$ and the effect of mass transfer also seems to be correctly predicted . It appears that use of a turbulent - Schmidt-number value smaller than 0.9 will improve the agreement still further . However, in view of the uncertainties in the concentration measurements (mentioned in Appendix B) and in the skin-friction values, we shall refrain from drawing firm conclusions .

On the whole, the implications of van Driest's hypothesis seem encouraging ; however, further research is needed to recommend its use with confidence .

4.5 Further developments

The present chapter, in conformity with its illustrative nature, has outlined a few main relationships for drag and heat flux . To obtain a complete set of wall-flux formulae is a major undertaking which it will take some time to accomplish . This complete set will include effects of non-

uniform density and viscosity, and the simultaneous effect of pressure gradient and mass transfer. Further stages will have to account for roughness. Formulae can be devised which give closer agreement with the exact implications of the hypothesis than the present formulae do.

We shall here make no recommendation about roughness. Simultaneous pressure gradient and mass transfer can be taken into account, for example, by a suitable combination of eq. (4.3-10) and (4.3-11); however, this needs examination. For non-uniform fluid properties, we shall use the present formulae but substitute the mean value of the density in the region considered and the value of viscosity at the wall. This is akin to the use of a reference-temperature method [19]. That this practice is adequate at present will be shown by the comparison with experimental data in Chapter 5. For the recovery factor H , we shall adopt the simple result (arrived at in Section 1.4-4 (e)) that H equals the turbulent Prandtl number.

Establishing the validity of the wall-flux relationships with reference to experimental data is an important matter. Indeed, this should be undertaken simultaneously with the preparation of better and general formulae; for blind extrapolations of any hypothesis may result into a waste of effort.

4.6 Summary of the relationships for turbulent flow

Here we shall assemble all the turbulent-flow formulae presented in this chapter. The main dimensionless quantities are defined by :

$$R \equiv \frac{\rho u y}{\mu} \quad , \quad F \equiv \frac{y}{\rho u^2} \frac{dp}{dx} \quad , \quad M \equiv \frac{\dot{m}''_S}{\rho u} \quad ,$$

$$s \equiv \frac{\tau_S}{\rho u^2} \quad , \quad S \equiv \frac{J_S}{\rho u (\phi_S - \phi)} \quad ,$$

where the fluid density ρ is the average value between the wall and the point under consideration and the viscosity μ is evaluated at the wall.

These quantities are related to the corresponding ones with

* as subscript via :

$$R_* \equiv RK^2, \quad F_* \equiv F/K^2, \quad M_* \equiv M/K^2,$$

$$s_* \equiv s/K^2, \quad S_* \equiv S/K^2.$$

$$\text{Also, } S_* = M_* / \{ (1 + M_*/S_{*(1)})^{\sigma_t} - 1 \}.$$

The drag law. For $F_* = M_* = 0$, we have :

$$s_{*,0} = R_*^{-1} - 0.1561 R_*^{-0.45} + 0.08723 R_*^{-0.3} + 0.03713 R_*^{-0.18};$$

then for finite F_* values and $M_* = 0$:

$$\frac{s_*}{s_{*,0}} = \left[1 - \frac{4 F_* R_*}{(12.8^{2.5} + R_*^{2.5})^{0.4}} \right]^{1.6};$$

and for finite M_* values and $F_* = 0$:

$$\frac{s_*}{s_{*,0}} = \left[1 - \frac{M_*}{7.74 R_*^{-1.17} + 0.956 R_*^{-0.25}} \right]^4.$$

The flux law. For $F_* = M_* = 0$, we have :

$$S_{*(1),0} = \frac{s_{*,0}}{1 + P_*(s_{*,0})^{\frac{1}{2}}},$$

where $P_* = 3.68 (\sigma / \sigma_t)^{-0.25} (\frac{\sigma}{\sigma_t} - 1)$.

For finite F_* , and $M_* = 0$:

$$S_*(1) = S_*(1)_0 (1 - F') + F' S_*(1)_{\text{sep}} ,$$

where $F' \equiv \frac{0.25 F_* R_*}{1 + 0.0625 R_*} ,$

and $S_*(1)_{\text{sep}} = \frac{1.725}{(P_* + 6.8)^{1.165}} R_*^{-1/3} .$

For finite M_* , and $F_* = 0$:

$$S_*(1) = s_* , \quad \text{if } \frac{\sigma}{\sigma_t} = 1 .$$

The recovery factor. For $M_* = 0$ we have :

$$H = \sigma_{h,t} .$$

Chapter 5

Implications of the mixing-length hypothesis

5.1 Purpose of the chapter

The success of a general theory of the turbulent boundary layer depends upon the validity, over a wide range of conditions, of the physical hypotheses used. It has been emphasised so far that the mathematical contribution of this thesis will facilitate the research into physical matters. The present chapter provides an illustration of such research.

Here we shall use a form of Prandtl's mixing-length hypothesis in the calculation procedure of Chapter 2, and compare the resulting predictions with available experimental data. In a way, this chapter is similar to Chapter 3 where the capabilities of the procedure have been demonstrated. However, the emphasis here is different. In the present chapter, the merits of the method are not intended to be shown (although they will manifest themselves) ; the validity of the hypothesis is our main concern. Therefore, discrepancies between theory and experiment, if present, will be regarded as the shortcomings of the hypothesis and not as a reflection on the calculation procedure.

We shall first choose a set of constants by reference to some well-known experimental results ; then compare the predictions with data over a wider range. The comparisons will indicate the directions in which the hypothesis can be refined.

5.2 Preliminary choices

5.2-1 The hypotheses used

The physical situations considered in this chapter will all involve

the presence of a wall. Two sets of hypotheses will, therefore, be used : one for the fully-turbulent region away from the wall, and the other for the wall-near region. Our calculation procedure will, for the most part of the layer, employ the hypotheses of the first kind ; only the slip-value relations for the wall boundary will need the wall-flux relationships which are based upon the wall-region hypotheses.

For the fully-turbulent part, we shall use the mixing-length hypothesis as given by eq. (1.3-5), with the mixing-length distribution according to eq. (1.3-6). The effective Prandtl number will be regarded as uniform across the layer.

The wall-flux relationships summarised in Section 4.6 will be used in the slip-value relations near a wall. Thus, we shall, in effect, employ van Driest's hypothesis for the wall-near region.

5.2-2 Choice of constants

The above set of hypotheses needs three constants : the two mixing-length constants K and λ , and the value of the effective Prandtl number $\sigma_{h,eff}$ for the fully-turbulent region.

The value of λ will depend upon the way in which the characteristic thickness of the layer, y_ℓ , is defined. We shall take y_ℓ as the distance, from the wall, of a point (near the free boundary) at which the velocity differs from the free-stream velocity by one per cent of the maximum velocity difference across the layer.

The mixing-length constants, K and λ , will be chosen so as to give good agreement with the following hydrodynamic results : (i) the Spalding - Chi [80] correlation for the flat-plate drag, and (ii) Cole's [16] empirical relation between the shape factor and Reynolds number on

a flat plate. Figs. 5.2-1 and 5.2-2 show the comparison of our predictions with these two correlations ; the good agreement shown in these figures has been obtained by use of : $K = 0.435$, $\lambda = 0.09$. We shall use these values below.

The value of the effective Prandtl number will be selected by reference to the flat-plate heat-transfer data of Reynolds, Kays and Kline [57] (which are in agreement with the Chi-Spalding [11] recommendation that the Reynolds-analogy factor equals 1.16). In Fig. 5.2-3 are shown our predictions and the experimental data for the Stanton number ; here we have used 0.9 as the value of the effective (or turbulent) Prandtl number. We shall adopt this value throughout the present chapter.

5.2-3 Details of computation

We shall make the predictions by use of the computer programme described in Appendix A, which is based on the procedure given in Chapter 2. The number of grid lines across the layer will be taken as 16. We shall choose the size of the forward step such that the quantity of fluid entrained during the step would be 5% of the amount of fluid already existing in the layer.

It is necessary to establish, at this stage, that the above-mentioned grid size will give sufficient accuracy. For this purpose some of the calculations have been repeated by increasing the number of grid lines and by decreasing the size of the forward step ; the results show little difference.

5.3 Comparison with uniform-property data

Most of the published experimental data belong to the situations in which the fluid density, viscosity etc. are almost uniform. We shall first present comparisons of our predictions with such data. In the physical situations considered here, we shall encounter various pressure gradients and

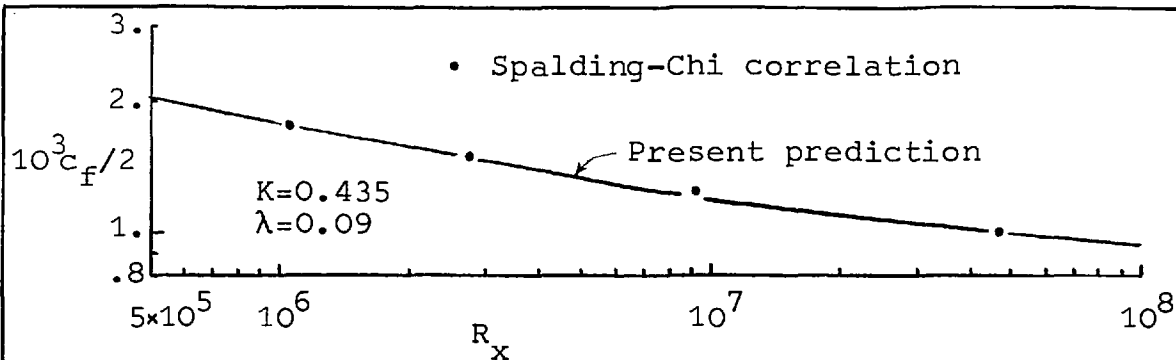


Fig. 5.2-1. Comparison with Spalding-Chi correlation for flat-plate drag.

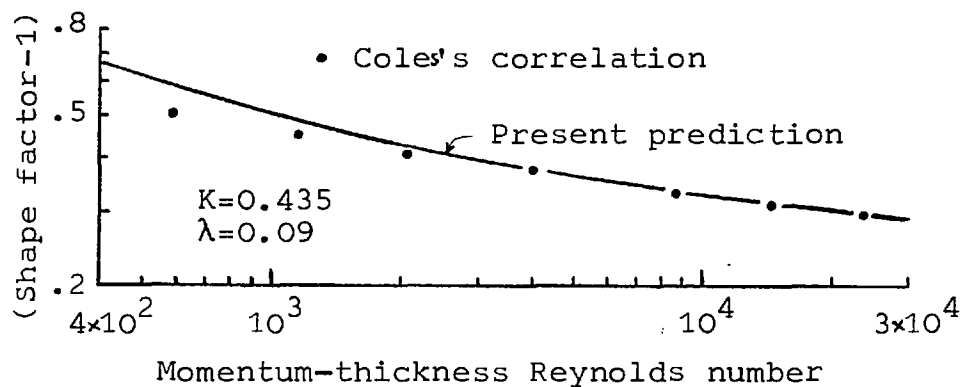


Fig. 5.2-2. Comparison with Coles's correlation for shape factor.

Here shape factor is the ratio of displacement thickness to momentum thickness.

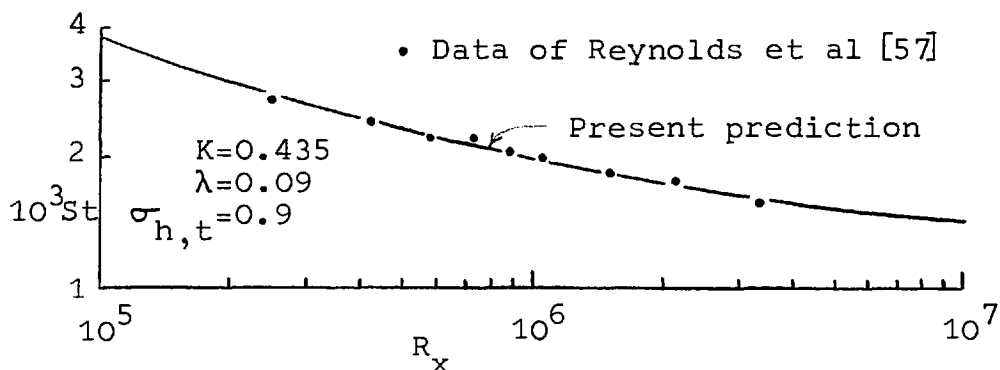


Fig. 5.2-3. Comparison with flat-plate data for heat transfer.

different types of thermal boundary condition. In every case, the variation of the Stanton number with downstream distance will be presented.

5.3-1 Cases of prescribed wall temperature

The agreement of our prediction with the flat-plate data of Reynolds, Kays and Kline [57] has already been shown in Fig. 5.2-3 ; Fig. 5.3-1 displays some more flat-plate cases in which the wall temperature is prescribed. The agreement of our predictions with the data is very satisfactory. It should be noted that Fig. 5.3-1 contains cases with a sudden step in the wall temperature ; the agreement is good even in these cases despite the remarks made at the end of Section 3.3.

Figs. 5.3-2 and 5.3-3 show the cases of respectively favourable and adverse pressure gradients. The agreement is, once again, satisfactory. However, systematic deviations can be noted; we overestimate the Stanton number in favourable pressure gradient and underestimate it in adverse pressure gradient.

In Fig. 5.3-4 are included cases in which regions of very strong favourable pressure gradient exist. In these regions, the experimental values of the Stanton number fall appreciably below the predicted values. This may be attributed to the so-called "laminarisation" phenomenon studied by Launder [35] , Moretti and Kays [40] and others.

5.3-2 Cases of prescribed heat flux

The other, most-common type of thermal boundary condition is met when the heat flux through the wall is specified. How to deal with such a problem has already been described in Section 2.6-1. Here we shall present some applications of that procedure.

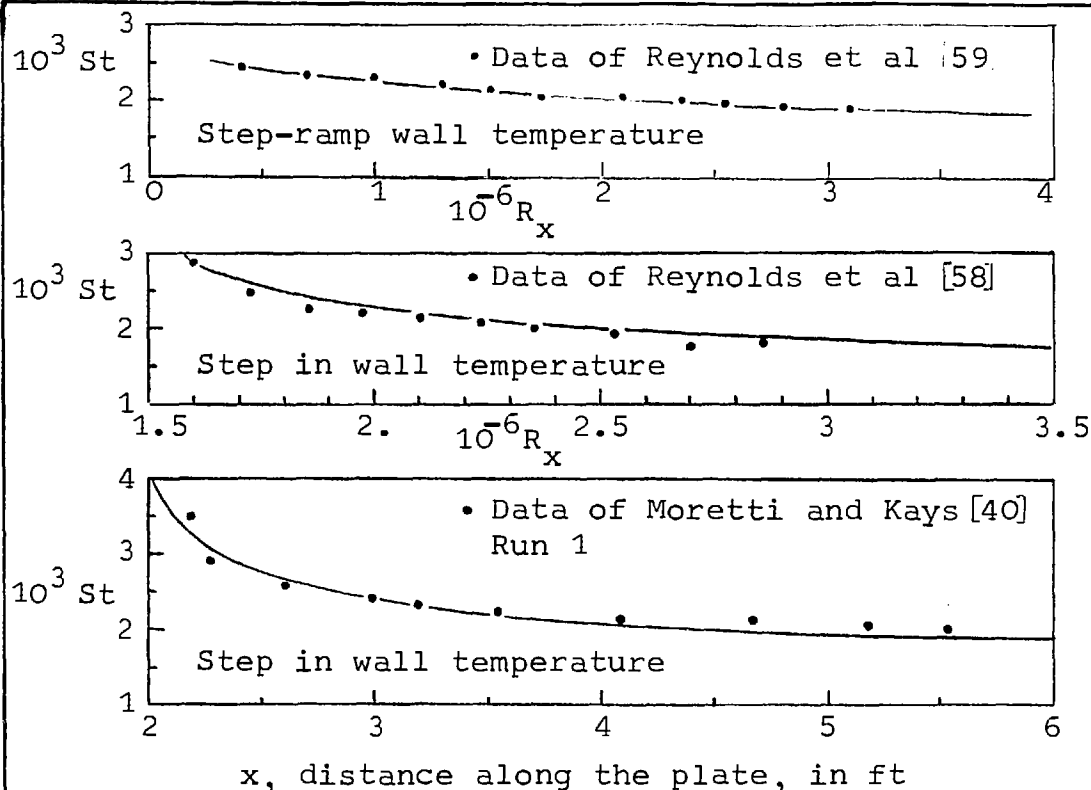


Fig. 5.3-1. Comparison with data for flat-plate given-wall-temperature cases. The full lines represent our predictions.

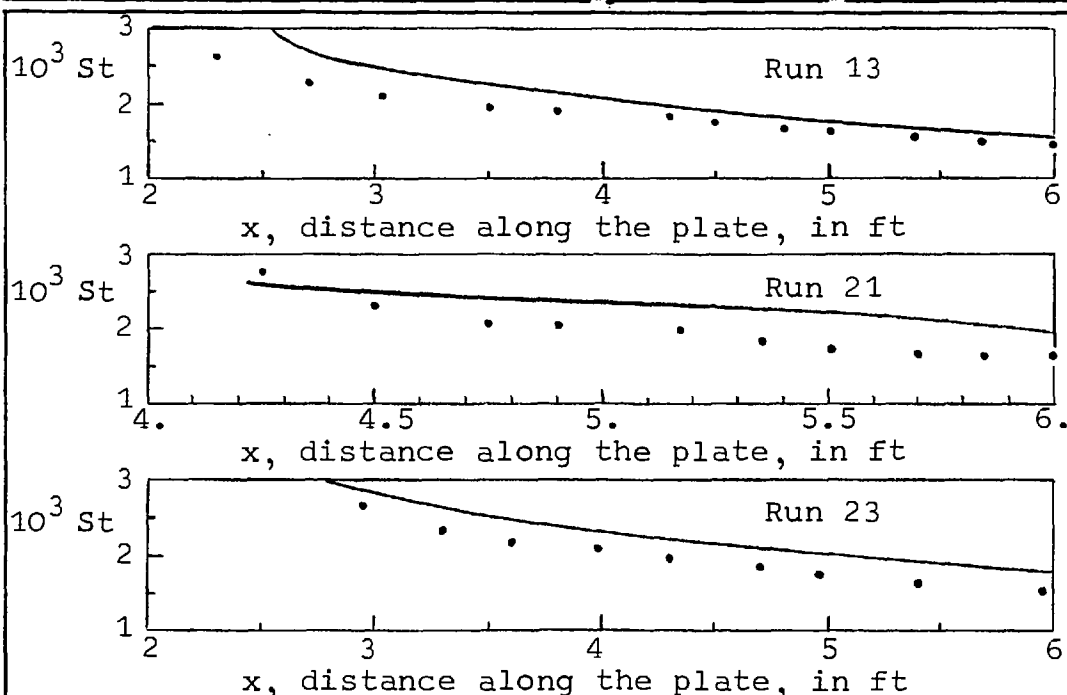


Fig. 5.3-2. Comparison with experimental data: favourable pressure gradient, given step-ramp wall temperature. Dots represent data of Moretti and Kays [40]; the full lines represent our predictions.

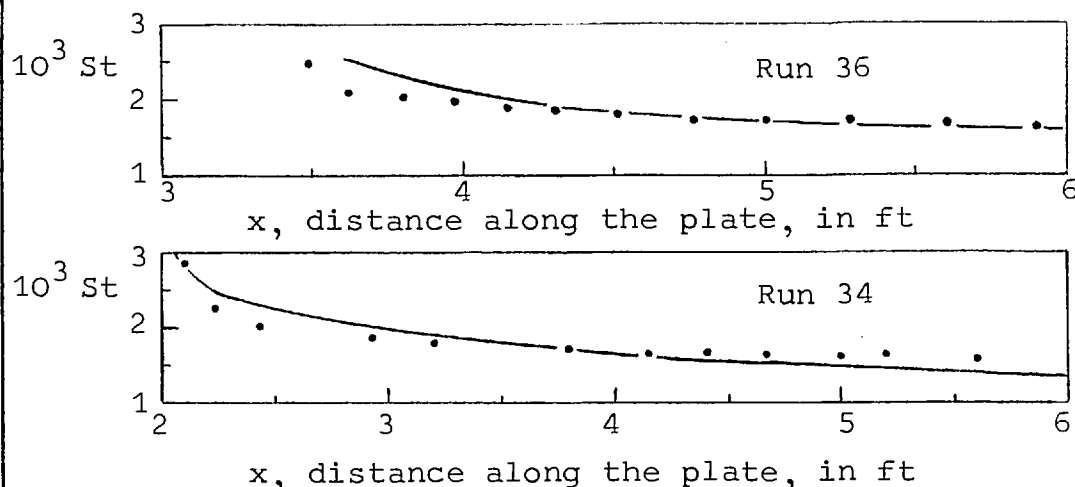


Fig. 5.3-3. Comparison with experimental data: adverse pressure gradient, step in wall temperature. Dots represent data of Moretti and Kays[40]; the full lines represent our predictions.

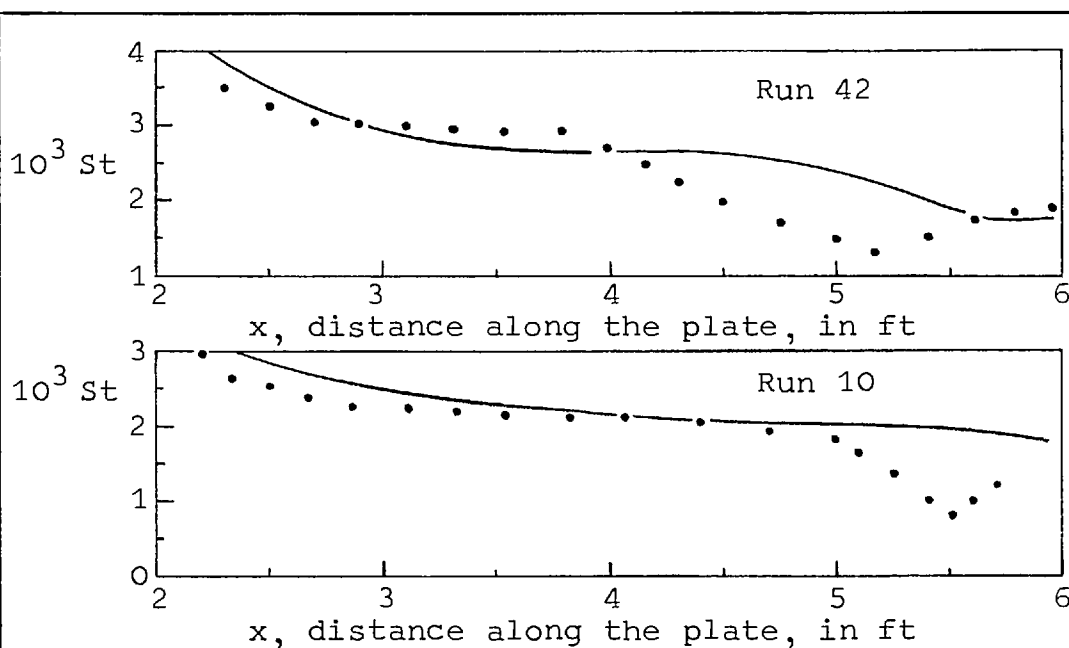


Fig. 5.3-4. Comparison with experimental data: strong favourable pressure gradient, step in wall temperature. Dots represent data of Moretti and Kays[40]; the full lines represent our predictions.

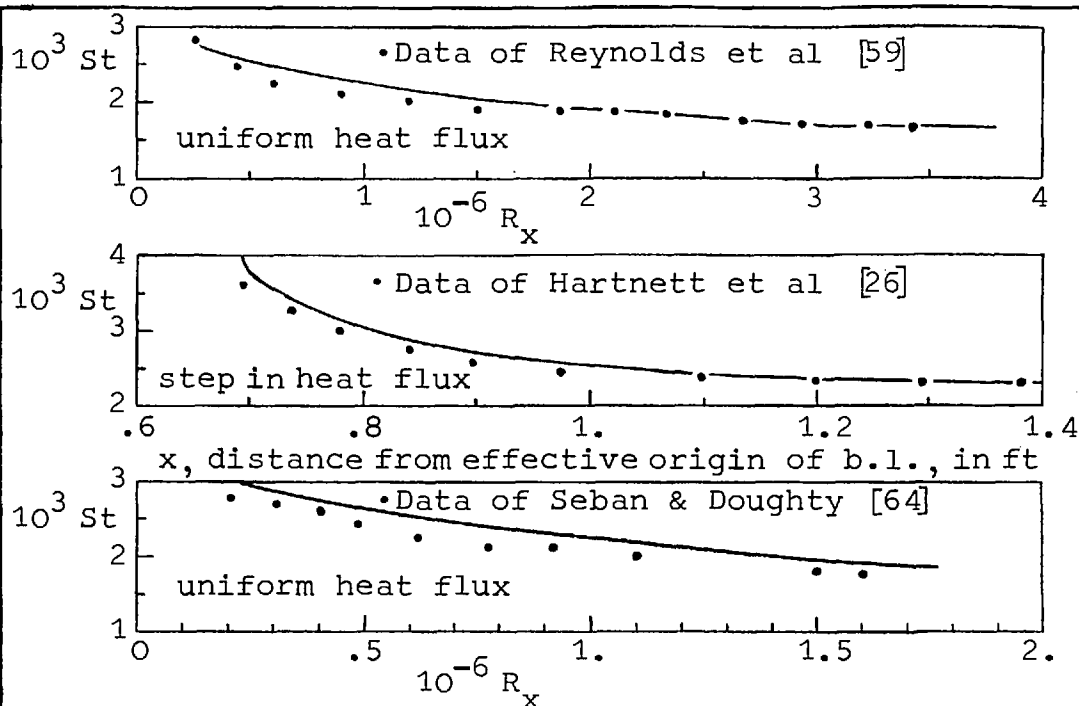


Fig. 5.3-5. Comparison with data for flat-plate given-heat-flux cases. The full lines represent our predictions.

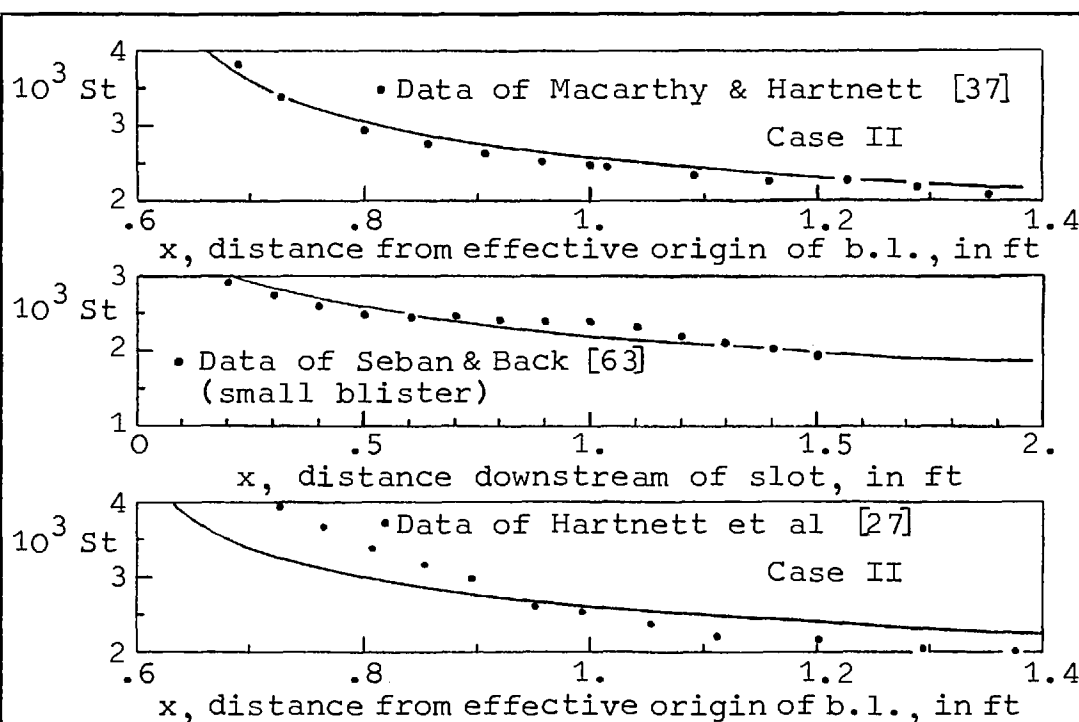


Fig. 5.3-6. Comparison with experimental data: favourable pressure gradient, step in heat flux. The full lines represent our predictions.

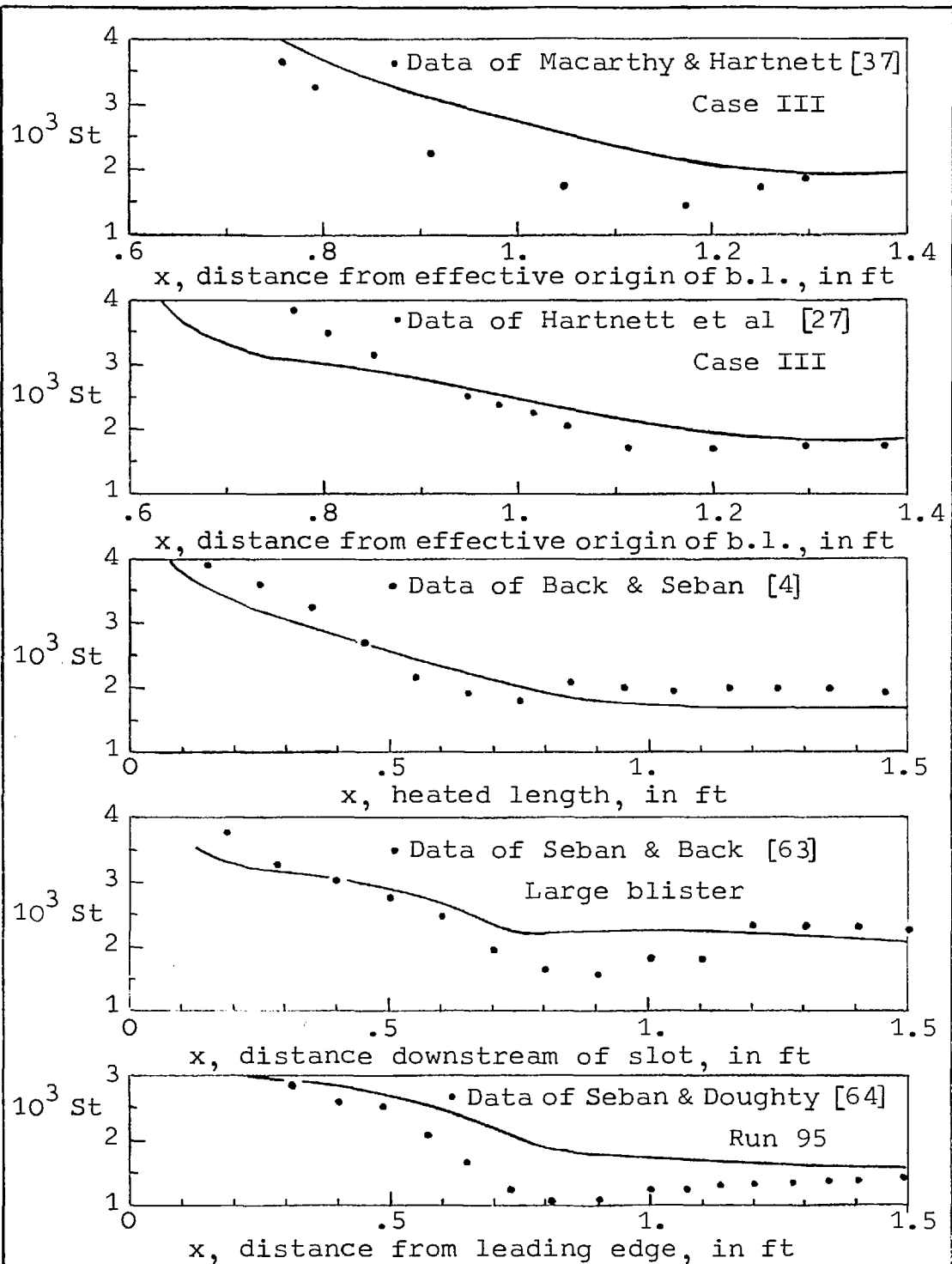


Fig. 5.3-7. Comparison with experimental data: strong favourable and adverse pressure gradients, step in heat flux. The full lines represent our predictions.

In Fig. 5.3-5 are shown the flat-plate cases ; Fig. 5.3-6 contains cases of favourable pressure gradient, while in Fig. 5.3-7 are included situations of adverse and favourable pressure gradients, sometimes strong enough to cause the so-called "laminarisation". Wherever this effect is absent, the agreement of our predictions with experimental data is satisfactory.

5.4 Comparison with non-uniform-property flat-plate data

We shall now consider the compressible turbulent boundary layer on a flat plate. Some calculations for this situation have already been presented in Section 3.2-3, where the details about the fluid-property variation are given. Here we shall compare our predictions with some of the experimental data.

5.4-1 The flat-plate drag

Fig. 5.4-1 shows the effect of Mach number and wall-to-mainstream temperature ratio on the drag coefficient. Here the ratio of the actual drag to the drag under uniform-property condition for the same value of the Reynolds number R_x has been plotted. The broken curves show the correlation of Spalding and Chi [80], which is based upon a large number of experimental data. Further, Fig. 5.4-2 displays comparison with some drag data taken from Schlichting [61]. In both of these figures, the agreement of our predictions with the experimental results is very satisfactory.

5.4-2 The flat-plate Stanton number

In Chapter 3, we have presented our Stanton-number results and compared them with the Chi-Spalding [11] correlation (Fig. 3.2-8); the agreement there is quite good. Here in Fig. 5.4-3 we compare our predictions with the experimental data of Chi and Spalding, obtained at low Mach number for various wall-to-mainstream temperature ratios. Once again, the predictions agree well with the experimental results.

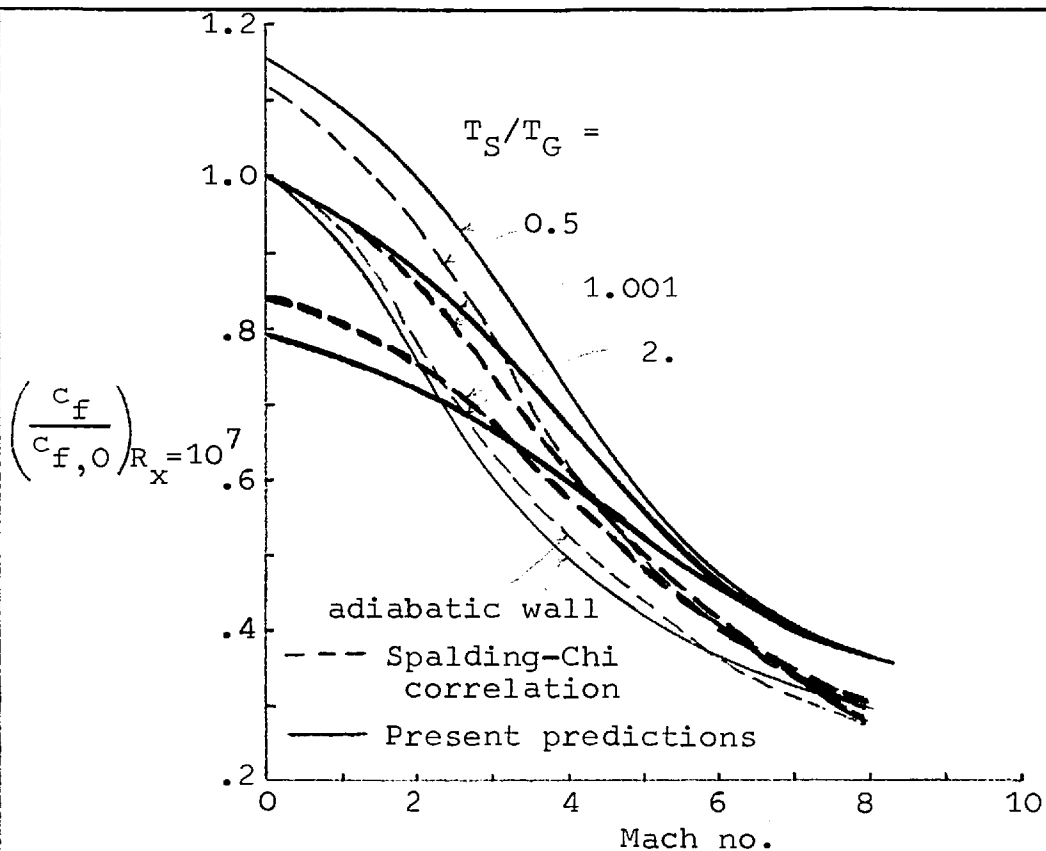


Fig. 5.4-1. The drag of a compressible turbulent boundary layer on a flat plate. Here $c_{f,0}$ stands for the drag coefficient under uniform-property conditions.

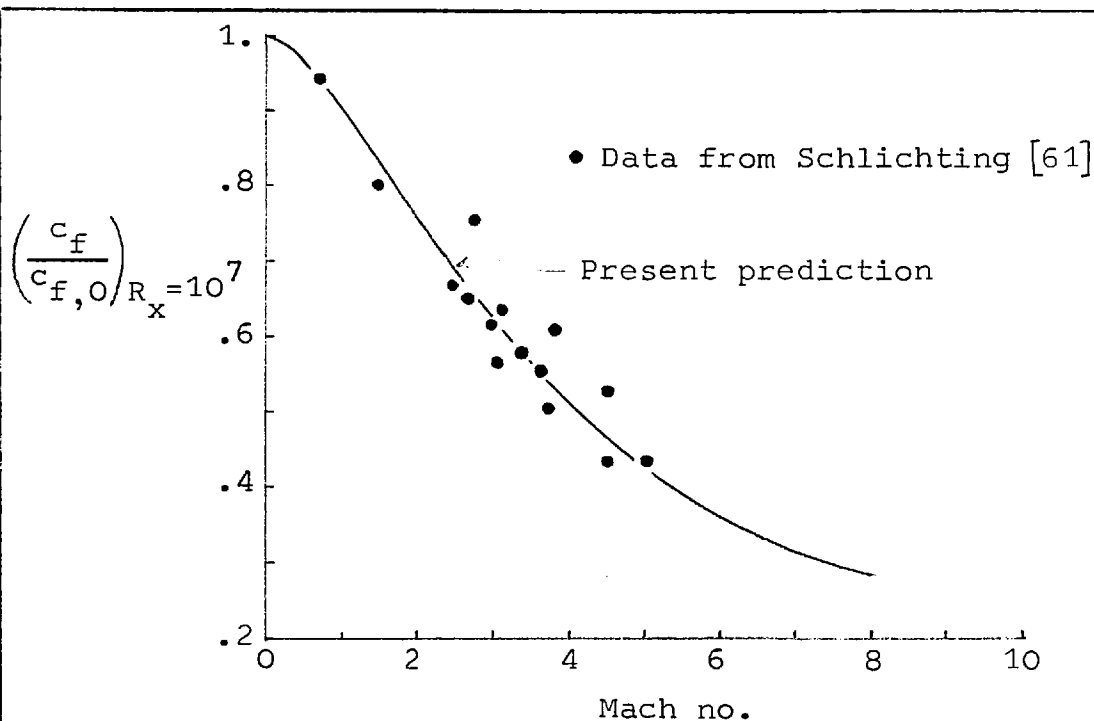
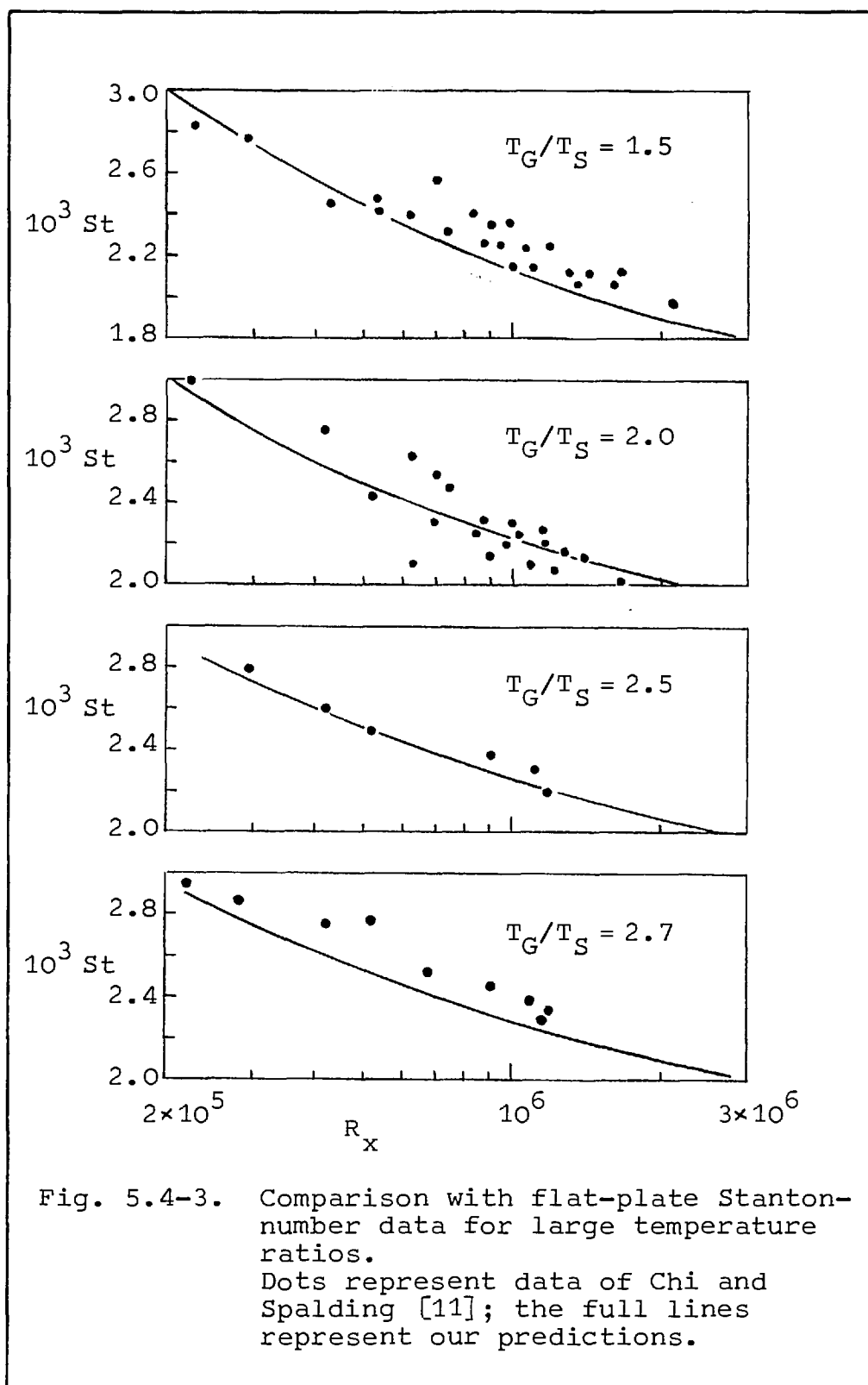


Fig. 5.4-2. Influence of Mach number on adiabatic-flat-plate drag: comparison with experimental data.

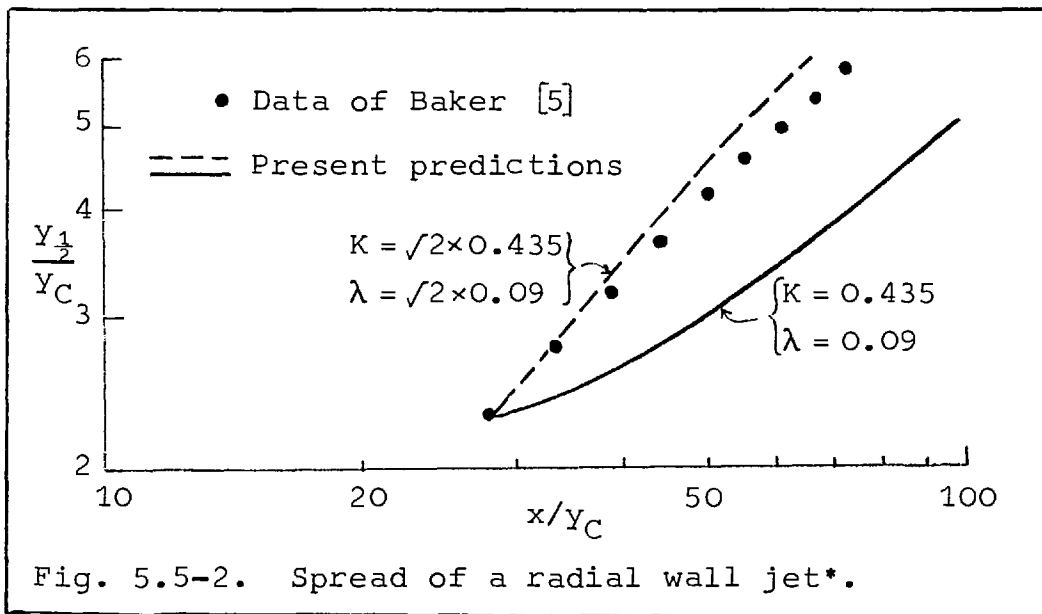
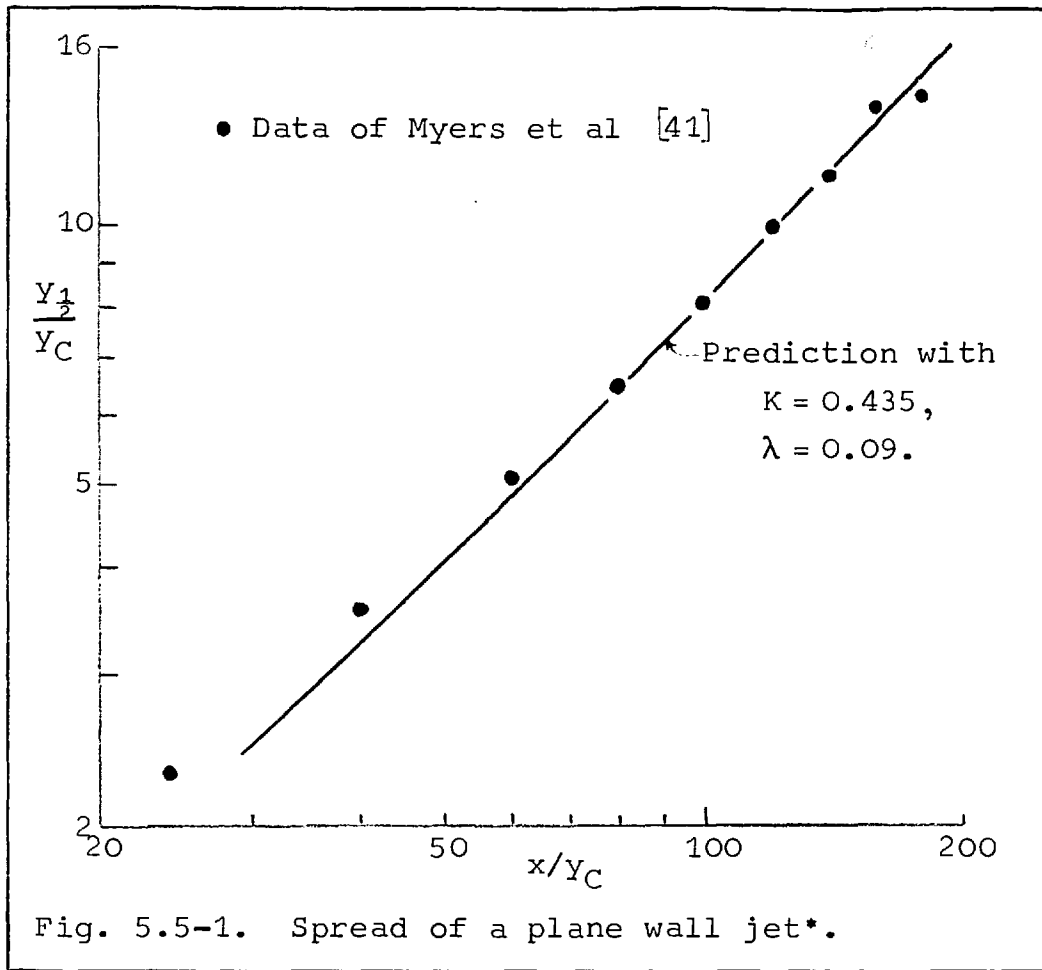


5.5 The case of a wall jet

In this chapter, we selected a set of hypotheses, adjusted the free constants so as to fit certain well-known experimental data, and then used the same set of hypotheses and constants to make predictions for many different physical situations. Although we have not exhausted all the available data of this type, and although our values of the constants may not be the best ones, there are indications that our choices (listed in Section 5.2) will be satisfactory over a wide range except in the regions of "laminarisation". We shall now examine the implications of the mixing-length hypothesis in a somewhat different type of flow : a uniform-property wall jet in stagnant surroundings.

Spread of the wall jet. Using the same set of hypotheses and constants, we can predict the spread of a plane wall jet as shown in Fig. 5.5-1 ; the agreement with the data of Myers, Schauer and Eustis [41] is very satisfactory. If we now make a similar prediction for the spread of a radial wall jet, we shall find that the agreement, as shown in Fig. 5.5-2 (the full curve), is unsatisfactory. This is not unexpected ; for experiments show similar velocity profiles and the same angle of spread in plane and radial wall jets ; a simple mass and momentum balance enables us to conclude that the effective viscosity in a radial wall jet should be twice as large as that in a plane wall jet. Consequently, we can expect a better agreement in the radial-wall-jet case if we use the values of the mixing-length constants, K and λ , which are $\sqrt{2}$ times larger than those for the plane one ; that it is indeed so can be seen from the comparison, in Fig. 5.5-2, of the broken curve with Baker's [5] experimental data.

So far, we have been concerned with only the gross features of the flow. Although usually these features are practically more important, greater insight into the suitability of the hypotheses can be obtained by the examination of details such as the shape of the velocity profile. Now we shall disclose that the values of K and λ which we have used do not produce



* Here y_C stands for the slot width and $y_{1/2}$ is the distance of the point at which the velocity equals one-half of the maximum velocity.

good agreement with the velocity profiles both in plane and radial wall jets.

Velocity profile. Fig. 5.5-3 shows the dimensionless velocity profile ; the dots represent the data of Baker [5] . On such a dimensionless plot, experimental profiles in plane and radial wall jets collapse on more-or-less a single curve. The predicted profile shape depends upon only the ratio of K to λ and not on their absolute values . For $K = 0.435$ and $\lambda = 0.09$, the predicted profile has been shown by the full line in Fig. 5.5-3 ; it will be seen that the agreement with experiment is not satisfactory.

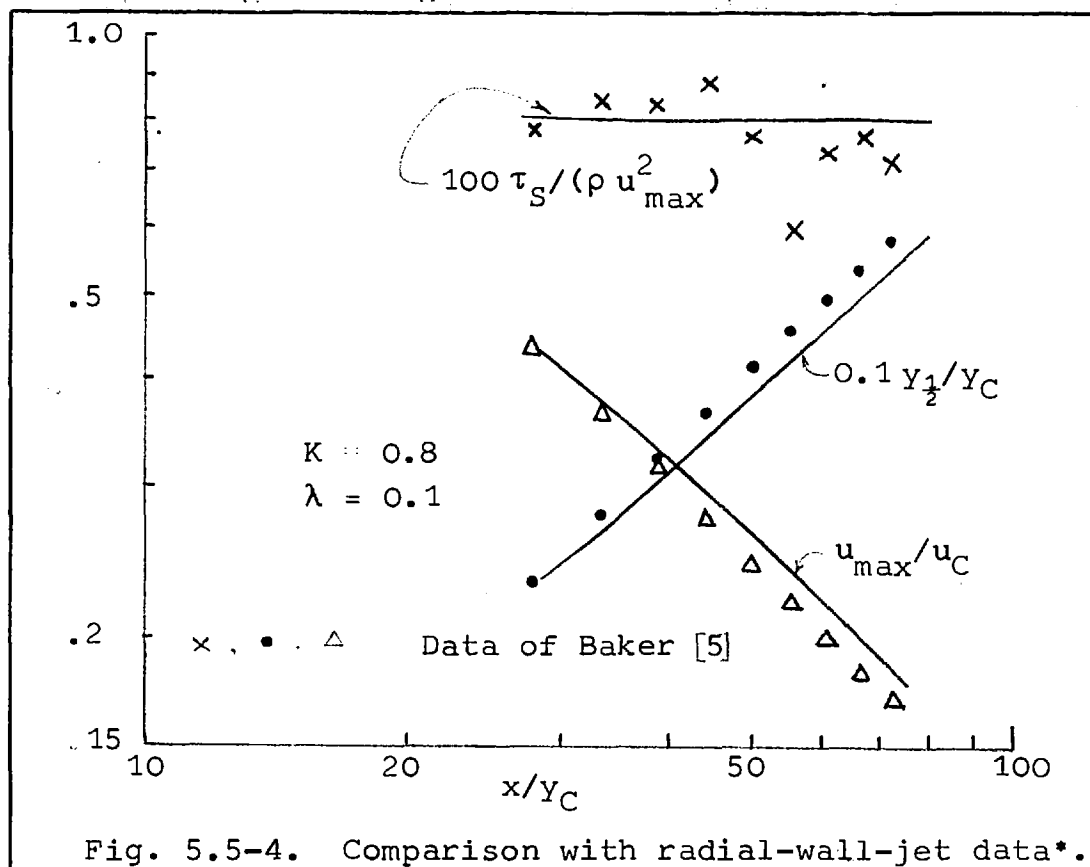
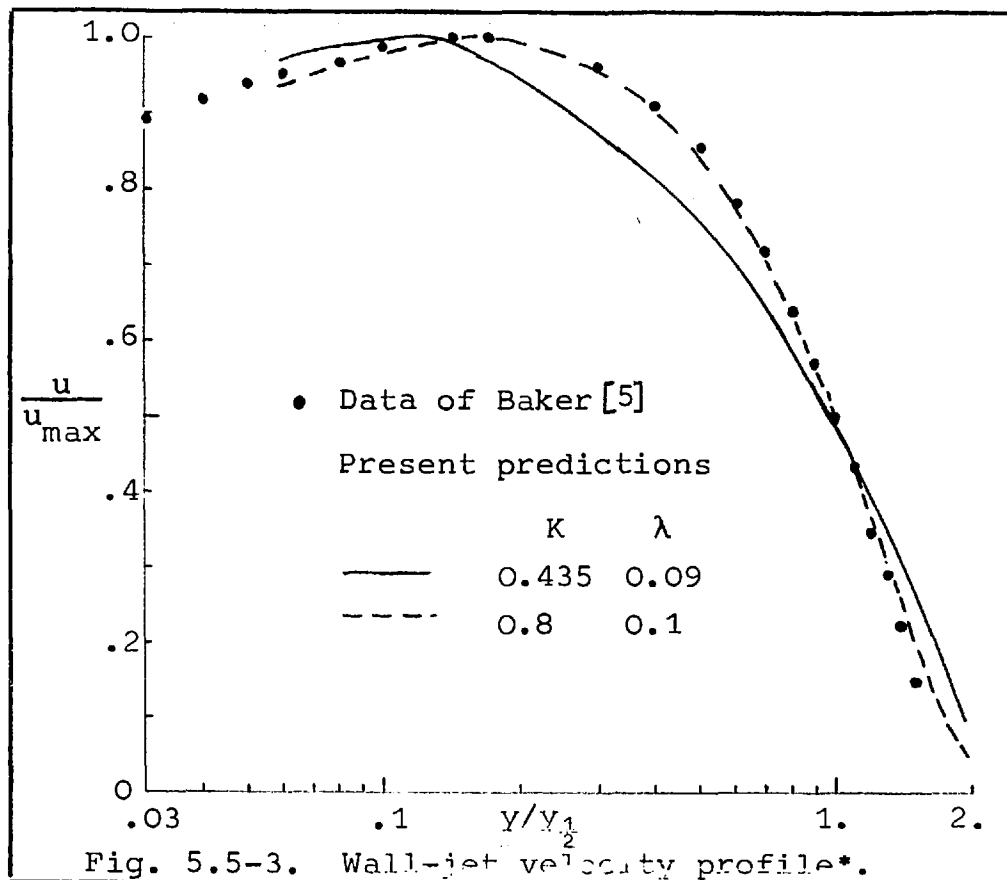
Adjustment of constants. It is possible to find a new set of values of K and λ which will give agreement, in most respects, with Baker's data on radial wall jet. Figs. 5.5-3 and 5.5-4 display the predictions obtained by use of : $K = 0.8$, $\lambda = 0.1$. In Fig. 5.5-3 the broken curve can be seen to follow the data very well. The comparison with the spread of the jet, decay of the maximum velocity, and shear stress* at the wall has been shown in Fig. 5.5-4. It can be concluded that, for a radial wall jet, $K = 0.8$ and $\lambda = 0.1$ represents a better set than $K = 0.435$ and $\lambda = 0.09$. It will now be recalled that in Section 4.4 (d) we obtained good agreement for the concentration profiles by use of $K = 0.8$.

The surprising values of the constants for the wall jets suggest the need for further research ; we shall discuss this matter in the next section.

5.6 Concluding remarks

Achievements and shortcomings of the mixing-length hypothesis. In this chapter, we have demonstrated that we can obtain, by use of the

* Baker's "uncorrected" values of shear stress have been used.



*Here $u_{\max}$ and u_C stand for the maximum velocity and the slot velocity respectively.

mixing-length hypothesis, realistic predictions of the heat -transfer rates in the presence of various pressure gradients, different kinds of thermal boundary condition, and effects of compressibility. Good agreement has been shown also for some hydrodynamic quantities such as drag, shape factor and wall-jet spread. What is important is that the same equations, the same set of hypotheses and constants (and, in fact, the same computer programme too) have been used for all these predictions, for only then we can progress towards the goal of providing a general theory. The comparisons in this chapter indicate that the mixing-length hypothesis is very promising.

A closer examination of our predictions will show that pressure gradient produces systematic deviations in the predicted heat-transfer rate. It may be that the mixing-length constants should depend on the pressure gradient—higher values for adverse and lower ones for favourable pressure gradients. However, this needs further study.

Another shortcoming of the hypothesis is its inability to predict the low Stanton-number values in the regions of the so-called "laminarisation". A more general hypothesis will be needed for these regions.

The fact that quite different values of the constants K and λ are necessary to predict correctly the various aspects of a radial wall jet suggests the need for further examination. It may be that the mixing-length model is too crude for the wall-jet flows in which there exists a strong diffusion of the turbulent kinetic energy towards the wall from the outer region.

Future work. The present chapter represents only a beginning of the search for a satisfactory physical hypothesis ; the tasks for the future are many. The immediate work would be to complete the comparisons

with experiment by using all available data. This will make it possible to arrive at the set of constants which gives agreement over the whole range. In this connection, it should be remembered that not all the data are reliable and, therefore, we need not expect perfect agreement in every case.

The next stage of research will include refinement of the hypothesis. The values of K and λ , for example, can be made functions of the pressure gradient. This will improve agreement with experiment and may even predict the "laminarisation". Similarly, these constants can depend upon the angle between the flow direction and the axis of symmetry; then we shall get a smooth change in the constants, from the plane to the radial situation. Further modifications of the hypothesis will be needed when the swirl velocity is to be accounted for. It will soon become necessary to perform new experiments.

After the mixing-length theory is fully exploited, we shall become clear about its merits and limitations. Certain situations will stand out which cannot be adequately described by the mixing-length model. At this stage, inquiry into new hypotheses will be profitable.

The present thesis equips the future worker with a convenient mathematical tool; also a way forward has been indicated through examples, suggestions and advice. It would not be before very long that we have a satisfactory general theory of the turbulent boundary layer.

References

1. ABRAMOVICH, G.N. (1963) The Theory of Turbulent Jets (English Transln.) M.I.T. Press.
2. ALPINIERI, L.J. (1963) "An experimental investigation of the turbulent mixing on non-homogeneous co-axial jets." Poly. Inst. of Brooklyn, PIBAL Rep. No. 789.
3. AMBROK, G.S. (1957) "Approximate solution of equations for the thermal boundary layer with variations in boundary layer structure". Sov. Phys. Tech. Phys., 2, 1979.
4. BACK, L.H. and SEBAN, R.A. (1965) "On constant property turbulent boundary layers with variable temperature or heat flow at the wall". J. Heat Transfer, C87, 151 - 156.
5. BAKER, E. (1967) "Influence of mass transfer on surface friction at a porous surface". Ph. D. Thesis, University of London.
6. BARTZ, D.R. (1954) "An approximate solution of compressible turbulent boundary-layer development and convective heat transfer in convergent-divergent nozzles". ASME Paper No. 54-A-153.
7. BINDER, L. (1911) Doctoral Dissertation of the Technische Hochschule, München, Wilhelm Knapp, Halle.
8. BLACK, T.J. and SARNECKI, A.J. (1958) "The turbulent boundary layer with suction or injection". ARC. Report 20, 501, F.M.2745.
9. BRITISH STANDARD CODE (1943) "Flow measurement" B.S. 1042. Published by British Standards Institution, London, 1951.
10. CARMICHAEL, A.D. and PUSTINTSEV, G.N. (1966) "Prediction of turbulent boundary layer development in conical diffusers". J. Mech. Eng. Sci., 8, No. 4, 426 - 436.
11. CHI, S.W. and SPALDING, D.B. (1966) "Influence of temperature ratio on heat transfer to a flat plate through a turbulent boundary layer in air". Proc. of 3rd Internat. Heat Transfer Conf., Chicago, 1966. (AIChE), II, 41 - 49.
12. CLARKE, J.H., MENKES, H.R. and LIBBY, R.A. (1955) "A provisional analysis of turbulent boundary layers with injection". J. Aeronaut. Sci., 22, 255 - 260.

13. CLAUSER, F.H. (1954). "Turbulent boundary layers in adverse pressure gradients". J. Aeronaut. Sci., 21, (2), 91 - 108.
14. CLAUSER, F.H. (1956) "The turbulent boundary layers". Advances in Applied Mechanics, 1 - 51.
15. COLE, E.H., SPALDING, D.B. and STOLLERY, J.L. (1967) "Film-cooling effectiveness calculated by finite-difference procedure". Imperial College, Mech.Eng.Dept. Rep.EHT/TN/3.
16. COLES, D.E. (1962) "The turbulent boundary layer in a compressible fluid". Rand Corp. Rept. R - 403 - pr.
17. CRANK, J. and NICHOLSON, P. (1947) "A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type". Proc. Cambridge Phil.Soc., 43, 50.
18. DVORAK, F.A. and HEAD, M.R. (1967) "Heat transfer in the constant property turbulent boundary layer". Int. J. Heat Mass Tr., 10, 61 - 81.
19. ECKERT, E.R.G. (1955) "Engineering relations for friction and heat transfer to surfaces in high velocity flow". J. Aeronaut. Sci., 22, 585 - 586.
20. ESCUDIER, M.P. (1965) "The distribution of the mixing length in turbulent flows near walls". Imperial College, Mech.Eng.Dept. Rep. TWF/TN/1.
21. ESCUDIER, M.P., NICOLL, W.B., SPALDING, D.B. and WHITELOW, J.H. (1966) "Decay of a velocity maximum in a turbulent boundary layer". Imperial College, Mech. Eng. Dept. Rep. TWF/TN/13.
22. ESCUDIER, M.P. and SPALDING, D.B. (1965) "A note on the turbulent uniform-property hydrodynamic boundary layer on a smooth impermeable wall ; comparisons of theory with experiment". A.R.C. C.P. 875 (1966)
23. FORSTALL, W. and SHAPIRO, A.H. (1950) "Momentum and mass transfer in coaxial gas jets". J. Appl. Mech., 17, 4, 399.
24. GARDNER, G.O. and KESTIN, J. (1963) "Calculation of the Spalding function over a range of Prandtl numbers". Int. J. Heat Mass Tr., 6, 289 - 299.

25. GOSMAN, A.D., IRIBARNE, A. and SPALDING, D.B. (1967)
"A theoretical and experimental investigation of diffusion-controlled electrolytic mass transfer between a falling liquid film and a wall".
To be published in Int. J. Heat Mass Transfer.
26. HARTNETT, J.P., BIRKEBAK, R.C. and ECKERT, E.R.G. (1961)
"Velocity distributions, temperature distributions and heat transfer for air injected through a tangential slot into a turbulent boundary layer". J. Heat Transfer, C83, 293.
27. HARTNETT, J.P., BIRKEBAK, R.C. and ECKERT, E.R.G. (1961)
"Velocity distributions, temperature distributions, effectiveness and heat transfer in cooling of a surface with a pressure gradient."
Int. Developments in Heat Transfer, Part IV, 682.
28. HARTREE, D.R. and WOMERSLEY, J.R. (1937) "A method for the numerical or mechanical solution of certain types of partial differential equations." Proc. Roy. Soc. London, Series A, 161, 353.
29. HINZE, J.O. and Van der HEGGE ZIJNEN, B.G. (1949)
Appl. Sci. Res., 1A, 435.
30. HUDIMOTO, B. (1951) "Momentum equations of the boundary layer and their application to the turbulent boundary layer."
Kyoto Univ. Memoirs of Fac. of Eng., XIII, No. 4.
31. JAYATILLAKA, C.L.V. (1966) "The influence of Prandtl number and surface roughness on the resistance of the laminar sub-layer to momentum and heat transfer." Ph.D. Thesis, University of London.
32. KESTIN, J. and PERSEN, L.N. (1962) "Application of Schmidt's method to the calculation of Spalding's function and of the skin-friction coefficient in turbulent flow." Int. J. Heat Mass Transfer, 5, 143 - 152.
33. KESTIN, J. and RICHARDSON, P.D. (1963) "Heat transfer across turbulent, incompressible boundary layers." Int. J. Heat Mass Transfer, 6, 147 - 189.
34. KUTATELADZE, S.S., and LEONT'EV, A.I. (1964) Turbulent Boundary Layers in Compressible Gases, Arnold, London.
35. LAUNDER, B.E. (1963) "The turbulent boundary layer in a strongly negative pressure gradient." Massachusetts Inst. of Tech., Gas Turbine Lab. Rep. No. 71.

36. LEW, H.G. (1966) "Nonsimilar mixing of chemically reacting streams." Proc. of 3rd Internat. Heat Transfer Conference, Chicago, (AIChE), 11, 1 - 11.
37. MACARTHY, T.F. and HARTNETT, J.P. (1964) "Turbulentnye Pogranishnye Sloi s Prodol'nym Gradientom Davleniya i Teploobmenom". Vsesoyuznoe Soveshehanie po Teplo-i Massoobmeny, Minsk.
38. MAISE, G. and McDONALD, H. (1966) "Mixing length and kinematic eddy viscosity in a compressible boundary layer." United Aircraft Research Laboratories, East Hartford, Connecticut.
39. MICKLEY, H.S. and DAVIS, R.S. (1957) "Momentum transfer for flow over a flat plate with blowing." NACA TN 4017.
40. MORETTI, P.M. and KAYS, W.M. (1964) "Heat transfer through an incompressible turbulent boundary layer with varying free-stream velocity and varying surface temperature." Stanford University, Mech. Eng. Dept. Rep. No. PG - 1.
41. MYERS, G.E., SCHAUER, J.J. and EUSTIS, R.H. (1961) "The plane turbulent wall jet. Part 1 : Jet development and friction factor." Stanford University, Mech. Eng. Dept. Tech. Rep. No. 1.
42. NICOLL, W.B. and ESCUDIER, M. P. (1966) "The entrainment function in turbulent boundary-layer and wall-jet calculations." J. Fluid Mech., 25, Pt. 2, 337-366.
43. NICOLL, W.B. and WHITELAW, J.H. (1965) "The solution of the Unified Theory equations and their application to problems involving heat transfer." Imperial College, Mech. Eng. Dept. Rep. IC/HRJ/24.
44. NICOLL, W.B. and WHITELAW, J.H. (1967) "The effectiveness of the uniform density, two-dimensional wall jet." Int. J. Heat Mass Transfer, 10, 623 - 639.
45. PAI, B.R. (1967) "Heat and mass transfer through the turbulent boundary layer - A survey." Imperial College, Mech. Eng. Dept. Rep.
46. PATANKAR, S.V. (1965) "An investigation to determine the best expression for the wake component of ϕ - profile." Imperial College, Mech. Eng. Dept., Rept. TWF/TN/2.

47. PATANKAR, S.V. (1966) "Heat transfer across a turbulent boundary layer : application of a profile method to the step-wall-temperature problem." *Int. J. Heat Mass Transfer*, 9, 829 - 834.
48. PATANKAR, S.V. (1966) "Wall-shear-stress and heat-flux laws for turbulent boundary layer with a pressure gradient : use of van Driest's eddy-viscosity hypothesis". Imperial College, Mech. Eng. Dept. Rep. TWF/TN/14.
49. PATANKAR, S.V. and SPALDING, D.B. (1966) "A calculation procedure for heat transfer by forced convection through two-dimensional uniform-property turbulent boundary layers on smooth impermeable walls." *Proc. of 3rd Internat. Heat Transfer Conference, Chicago, (1966) (A.I.Ch.E.)*, II, 50 - 63.
50. PATANKAR, S.V. and SPALDING, D.B. (1967) "A finite-difference procedure for solving the equations of the two-dimensional boundary layer." Imperial College, Mech. Eng. Dept. Rep. TWF/TN/20. To be published in *Int. J. Heat Mass Transfer*.
51. PATANKAR, S.V. and TAYLOR, R.G. (1965) "Diffusion from a line source in a turbulent boundary layer : comparison of theory and experiment." *Int. J. Heat Mass Transfer*, 8, 1172 - 1175.
52. PERRY, A.E., BELL, J.B. and JOUBERT, P.N. (1966) "Velocity and temperature profiles in adverse pressure gradient turbulent boundary layers." *J. Fluid Mech.*, 25, (2), 299 - 320.
53. PRANDTL, L. (1925) "Bericht über Untersuchungen zur ausgebildeten Turbulenz." *ZAMM*, 5, 136.
54. PRANDTL, L. (1945) "Über ein neues Formelsystem für die ausgebildete Turbulenz." *Nachrichten der Akad. Wiss., Göttingen, Mathphys.* pp. 6 - 19, Van den Loock & Ruprecht, Göttingen.
55. REICHARDT, H. (1942) "Gesetzmäßigkeiten der freien Turbulenz". *VDI - Forschungsheft*, 414, 1942; 2nd ed., 1951.
56. REID, R.C. and SHERWOOD, T.K. (1958) The Properties of Gases and Liquids, McGraw Hill, New York.
57. REYNOLDS, W.C., KAYS, W.M. and KLINE, S.J. (1958) "Heat transfer in the turbulent incompressible boundary layer. I : Constant wall temperature." *NASA Memo. No. 12 - 1 - 58W*.

58. REYNOLDS, W.C., KAYS, W.M. and KLINE, S.J. (1958) "Heat transfer in the turbulent incompressible boundary layer. II : Step wall temperature distribution." NASA Memo. No. 12-2-58W.
59. REYNOLDS, W.C., KAYS, W.M. and KLINE, S.J. (1958) "Heat transfer in the turbulent incompressible boundary layer. III : Arbitrary wall temperature and heat flux." NASA Memo. No. 12-3-58W.
60. ROTTA, J.C. (1965) "Heat transfer and temperature distribution in turbulent boundary layers at supersonic and hypersonic flow." Aerodynamische Versuchsanstalt, Report 65A 07.
61. SCHLICHTING, H. (1960) Boundary Layer Theory, 4th Ed., McGraw Hill, New York.
62. SCHMIDT, E. (1924) "Föppl's Festschrift". p. 179. Julius Springer, Berlin.
63. SEBAN, R.A. and BACK, L.H. (1961) "Effectiveness and heat transfer for a turbulent boundary layer with tangential injection and variable free-stream velocity." ASME paper No. 61-WA-156.
64. SEBAN, R.A. and DOUGHTY, D.L. (1956) "Heat transfer to turbulent boundary layers with variable free-stream velocity." Trans. ASME, 78, 217.
65. SMITH, A.M.O. and CLUTTER, D.W. (1965) "Machine calculation of compressible laminar boundary layers." AIAA Jnl., 3, No. 4, 639 - 647.
66. SMITH, A.M.O. and JAFFE, N.A. (1965) "General method for solving the nonequilibrium boundary layer equations of a dissociating gas." AIAA paper No. 65 - 129.
67. SMITH, A.M.O., JAFFE, N.A. and LIND, R.C. (1965) "Study of a general method of solution of the incompressible turbulent boundary layer equations." Bureau of Naval Weapons Fluid Mechanics and Flight Dynamics Branch, RRRE - 4.
68. SMITH, A.G. and SHAH, V.L. (1962) "The calculation of wall and fluid temperatures for the incompressible turbulent boundary layer, with arbitrary distribution of wall heat flux." Int. J. Heat Mass Tr., 5, 1179 - 1189.

69. SMITH, A.G. and SHAH, V.L. (1962) "Study of heat and mass transfer from non-isothermal surfaces." The College of Aeronautics, Cranfield, Note No. 135.
70. SPALDING, D.B. (1961) "Heat transfer to a turbulent stream from a surface with a step-wise discontinuity in wall temperature." Int. Developments in Heat Transfer, Part II, 439 - 446.
71. SPALDING, D.B. (1964) "Contribution to the theory of heat transfer across a turbulent boundary layer." Int. J. Heat Mass Tr., 7, 743 - 761.
72. SPALDING, D.B. (1964) "A unified theory of friction, heat transfer and mass transfer in the turbulent boundary layer and wall jet." A.R.C. Rep. 25, 925, FM 3459, HMT 41.
73. SPALDING, D.B. (1965) "The kinetic-energy-deficit equation of the turbulent boundary layer." AGARDograph, Pt. 1, 191 - 244.
74. SPALDING, D.B. (1964/65) Lectures on turbulent jets, wakes and boundary layers. Imperial College, Mech. Eng. Dept.
75. SPALDING, D.B. (1965) "A relation between entrainment rate, shear-stress gradient and velocity gradient." Imperial College, Mech. Eng. Dept. Rep. TWF/TN/8.
76. SPALDING, D.B. (1966) "Progress in the development of a unified theory of friction, heat transfer and mass transfer in boundary layers and jets." Imperial College, Mech.Eng. Dept. Rep. TWF/TN/10.
77. SPALDING, D.B. (1967) "Heat transfer from turbulent separated flows". J. Fluid Mech., 27, Pt. 1, 97 - 109.
78. SPALDING, D.B. (1967) Private communication.
79. SPALDING, D.B., AUSLANDER, D.M. and SUNDARAM, T.R. (1964) "The calculation of heat and mass transfer through the turbulent boundary layer on a flat plate at high Mach numbers, with and without chemical reaction." Supersonic Flow, Chemical Processes and Radiative Transfer, pp. 211 - 276 Pergamon Press.
80. SPALDING, D.B. and CHI, S.W. (1964) "The drag of a compressible turbulent boundary layer on a smooth flat plate with and without heat transfer." J. Fluid Mech., 18, Pt. 1, 117 - 143.

81. SPALDING, D.B. and JAYATILLAKA, C.L.V. (1965) "A survey of the theoretical and experimental information on the resistance of the laminar sub-layer to heat and mass transfer." Proc. 2nd. All - Union Conf. on Heat and Mass Transfer, Minsk, USSR, 1964. 2, 234 - 264, (In Russian) ; English translation, Rand. Corp., California (1966) - to be published.
82. SPENCE, D.A. (1960) "Velocity and enthalpy distributions in the compressible turbulent boundary layer on a flat plate." J. Fluid Mech., 8, Pt. 3, 368 - 387.
83. STEVENSON, T.N. (1964) "Boundary layers with suction or injection." Ph.D. Thesis, University of London.
84. STRATFORD, B.S. (1959) "An experimental flow with zero skin friction throughout its region of pressure rise." J. Fluid. Mech., 5, 17.
85. THOMPSON, B.G.J. (1964) "A critical review of existing methods of calculating the turbulent boundary layer." A.R.C. 26, 109.
86. TOLLMIE, W. (1926) "Berechnung turbulenter Ausbreitungsvorgänge." ZAMM. 6, 6, 468. Also translated as NACA TM 1085, 1945.
87. TOWNSEND, A.A. (1961) "Equilibrium layers and wall turbulence." J. Fluid Mech., 11, 97.
88. Van DRIEST, E.R. (1952) "Investigation of laminar boundary layer in compressible fluids using the Crocco method." NACA TN 2597.
89. Van DRIEST, E.R. (1956) "On turbulent flow near a wall." J. Aeronaut. Sci., 23, 1007.
90. Van der HEGGE ZIJNEN, B.G. (1958)
 - (a) "Measurements of the velocity distribution in a plane turbulent jet of air." Appl. Sci. Res. Section A, 7, 256.
 - (b) "Measurements of the distribution of heat and matter in a plane turbulent jet of air." Appl. Sci. Res. Section A, 7, 277.
 - (c) "Measurements of turbulence in a plane jet of air by the diffusion method and by the hot-wire method." Appl. Sci. Res., Section A, 7, 293.
91. WHITELAW, J.H. (1966) "An experimental investigation of the two-dimensional wall jet." Imperial College, Mech. Eng. Dept., Rep. IC/HRJ/36.

Nomenclature

Symbol	Meaning	Equation of first mention
a	a group of symbols in the convection term	(2.2-4)
A	a coefficient in the difference equation	(2.4-31)
A_u	a coefficient in the difference equation for velocity u	(2.4-35)
A_+	the damping constant in van Driest's hypothesis	(1.3-7)
A_*	the transformed damping constant	(1.4-18)
A'	transformed coefficient	(2.7-2)
b	a group of symbols in the convection term	(2.2-4)
B	a coefficient in the difference equation	(2.4-31)
B_u	a coefficient in the difference equation for velocity u	(2.4-35)
B'	transformed coefficient	(2.7-2)
c	a group of symbols in the diffusion term	(2.2-4)
c_f	drag coefficient ($\equiv 2\tau_s / (\rho_G u_G^2)$)	-
$\bar{c}_f$	mean drag coefficient	-
C	a coefficient in the difference equation	(2.4-31)
C_u	a coefficient in the difference equation for velocity u	(2.4-35)
d	the source term	(2.2-4)
E	an integration constant	(4.1-1)
E_*	an integration constant	(1.4-62)

Symbol	Meaning	Equation of first mention
F	a pressure-gradient parameter	(1.4-77)
F_*	$\equiv F/K^2$	(1.4-82)
F'	a group of symbols involving F_*	(4.3-17)
g_1, g_2, g_3, g_4	Coefficients in the difference form of the convection term	(2.4-13)
$\left. \begin{matrix} g_5 \\ g_6 \end{matrix} \right\}$	coefficients in the difference form of the diffusion term	(2.4-20)
G	a quantity in the free-boundary slip-value relation for u	(2.5-20)
$G_\emptyset$	a quantity in the free-boundary slip-value relation for $\emptyset$	(2.5-24)
h	the specific enthalpy	(1.3-1)
h_+	dimensionless enthalpy	(1.4-44)
$\tilde{h}$	the stagnation enthalpy	(1.1-5)
$\tilde{h}_{+,ad}$	dimensionless enthalpy under adiabatic conditions	(1.4-37)
$\tilde{h}_{*,ad}$	$\equiv K^2 \tilde{h}_{+,ad}$	(1.4-36)
$\tilde{\tilde{h}}$	the enthalpy that would exist at the wall had the conditions been adiabatic	(1.4-42)
H	recovery factor for the region near the wall	(1.4-40)
I	an integral $(\equiv (\psi_E - \psi_I) \int_0^\omega (\rho u)^{-1} d\omega)$	(2.8-9)
J	diffusional flux in the positive y direction	(1.4-2)
J_+	dimensionless flux	(1.4-8)
J_{tot}	total flux through the wall	(2.6-1)
k_1, k_2, k_3	groups of symbols in the symmetry-line relations	(2.6-9)

Symbol	Meaning	Equation of first mention
K	a mixing-length constant	(1.3-6)
ℓ	the mixing length	(1.3-5)
m_j	mass fraction of chemical species j	(1.1-4)
m_+	dimensionless mass-transfer rate	(1.4-7)
m_*	$\equiv m_+/K$	(1.4-22)
$\dot{m}''$	mass-transfer rate across a boundary	(1.4-1)
M	a mass-transfer parameter	(1.4-78)
M_*	$\equiv M/K^2$	(1.4-83)
M_h	transformed mass-transfer parameter	(2.6-4b)
n	exponent in the ϕ profile near a free boundary	(2.5-22)
N	number of strips across the layer	-
p	pressure	(1.1-3)
p_+	dimensionless pressure gradient	(1.4-6)
p_*	$\equiv p_+/K$	(1.4-21)
P	resistance of the laminar sublayer	(4.1-2)
P_*	an integration constant	(1.4-66)
P_1, P_2, P_3	coefficients in the difference form of the convection term	(2.4-3)
Q	coefficient in the difference form of the convection term	(2.4-7)
r	distance from the axis of symmetry	(1.1-1)
R	a Reynolds number for the region near a wall	(1.4-76)

Symbol	Meaning	Equation of first mention
R_*	$\equiv R K^2$	(1.4-81)
R_j	rate of generation of the chemical species j	(1.1-4)
R_x	a Reynolds number ($\equiv \rho_G u_G x / \mu_G$)	-
R_1, R_2, R_3	coefficients in the difference form of the convection term	(2.4-9)
s	dimensionless shear stress at the wall	(1.4-79)
s_*	$\equiv s / K^2$	(1.4-84)
s_1, s_2, s_3, s_4	coefficients in the difference form of the source term for the velocity equation	(2.4-25)
S	dimensionless diffusional flux	(1.4-80)
S_*	$\equiv S / K^2$	(1.4-85)
$S_*(1)$	value of S_* for $\sigma_t = 1$ and for the same σ / σ_t	(1.4-90)
St	the Stanton number	-
T	absolute temperature	(3.2-2)
u	velocity in the longitudinal direction	(1.1-2)
u_+	dimensionless velocity	(1.4-5)
u_*	$\equiv K u_+$	(1.4-17)
u_B	velocity desired at the downstream station for grid control	(2.8-5)
v_S	velocity at the wall in the positive Y direction	-
V	a group of symbols in the relations for the flux boundary condition	(2.6-3)

Symbol	Meaning	Equation of first mention
x	distance in the stream-wise direction	(1.1-2)
y	distance in the cross-stream direction, measured outward from the Γ boundary	(1.1-1)
y_+	dimensionless distance	(1.4-4)
y_*	$\equiv K y_+$	(1.4-16)
y_ℓ	a characteristic thickness of the layer	(1.3-6)
α	the angle made by the x direction with the axis of symmetry	(1.1-1)
β	exponent in the velocity profile near a wall	(2.5-1)
γ	exponent in the ϕ profile near a wall *	(2.5-5)
λ	a mixing-length constant	(1.3-6)
μ	laminar viscosity of the fluid	(1.3-7)
μ_{eff}	effective viscosity	(1.3-2)
ρ	density of the fluid	(1.1-2)
σ	laminar Prandtl or Schmidt number	(1.3-8)
σ_{eff}	effective Prandtl or Schmidt number	(1.3-8)
σ_t	turbulent Prandtl or Schmidt number	(1.3-8)
τ	local shear stress	(1.1-3)
τ_+	dimensionless shear stress	(1.4-3)

* At only one place in this thesis (p.80) , γ stands for the ratio of the specific heat at constant pressure to the one at constant volume. However, no confusion is likely to occur.

Symbol	Meaning	Equation of first mention
ϕ	a general dependent variable	(1.4-2)
ϕ_+	dimensionless form of ϕ	(1.4-9)
ϕ_*	$\equiv K \phi_+$	(1.4-23)
$\phi_{*(1)}$	value of ϕ_* for $\sigma_t=1$ and for the same σ/σ_t	(1.4-57)
ψ	a stream function	(1.1-2)
ω	dimensionless stream function	(2.1-1)
<u>Subscripts *</u>		
ad	adiabatic-wall condition	(1.4-34)
B	a boundary used in the grid-control formula	(2.3-4)
D	the downstream point on a portion of the grid	(2.4-1)
D+ D- DD+ DD- }	points near to and at the same value of x as point D	(2.4-3) (2.4-3) (2.4-1) (2.4-1)
E	the external boundary of the layer	(2.1-1)
G	a free boundary ; conditions in the free stream	(2.3-1)
h	pertaining to the enthalpy	(1.1-5)
I	the internal boundary of the layer	(1.1-1)
j	pertaining to chemical species j	(1.1-4)
sep	separating boundary layer	(4.3-8)
S	a wall boundary	(1.4-1)

* Additional subscripts are defined in Section 2.5-1.

Symbol	Meaning	Equation of first mention
t	fully-turbulent condition	(1.3-8)
U	the upstream point on a portion of the grid	(2.4-1)
$U+$	points near to and at the same value of x as point U	(2.4-3)
$U-$		(2.4-3)
$UU+$		(2.4-18)
$UU-$		(2.4-18)
O	condition with zero pressure gradient and zero mass transfer ; also for uniform-property condition	(4.3-6)

Appendix A

A computer programme for solution of the boundary - layer equations

A.1 Introduction

A.1-1 Purpose of the present appendix

Development of an accurate, widely-applicable and economical method of solving the boundary-layer equations has been the central theme of the present thesis. A computer programme is a necessary link between the formal description of the method in terms of symbols and the practically - useful predictions in terms of numbers. The computer programme that has been developed for making the predictions presented in this thesis is, like the solution procedure, fairly general and it is, therefore, felt that many prospective users of the method would benefit if the programme along with its comprehensive description were available to them. The purpose of the present appendix is to fulfil this need.

A.1-2 Scope and limitations of the programme

The limitations of the calculation method described in the main body of the thesis apply also to the computer programme. The programme is directly useful for flows with known pressure gradient. Modifications will be necessary for including the treatment of confined flows by the method given in Ref. [50]. The programme is written in FORTRAN IV language and can be run on IBM 7090 computer.

The programme is so designed that the partial differential equation for velocity u will always be solved; in addition, any desired number of conservation equations of the type (2.2-4) for other dependent variables can be solved. The existing DIMENSION and COMMON statements limit this number of additional equations to 2, and the number of strips N across the layer to 40; but these can easily be changed if desired.

The programme handles all the dependent variables and other auxiliary quantities in a destructive way ; i.e. the values stored at a time refer to only one value of x , and these are destroyed and replaced by the values for the next downstream station as the calculation proceeds . This reduces the necessary storage space and much of the complication .

The scope of a flexible computer programme such as the present one depends largely upon the ingenuity of the user . The programme can be used, for example, to solve the partial differential equations (not given in the present thesis, but to be found in Ref.[50]) for the velocity of swirl or the kinetic energy of fluctuating motion ; and there is no reason why, with proper care, it cannot be applied to a problem of transient heat conduction or laminar-flame propagation .

The subroutines in the present programme can be divided into three groups . Those of the first group are valid for flow problems of all types . In the second-group subroutines enters the assumption regarding the type of flow such as laminar, turbulent etc . In Section A.5, where the listing of the whole programme will be given, we shall present, for the second group, subroutines for turbulent flow using the mixing-length hypothesis and the wall-flux relationships of Chapter 4 ; preparation of corresponding subroutines for laminar flow or for turbulent flow with a different hypothesis seems to be straightforward . The last group includes subroutines which refer to a particular physical situation ; here enter the initial and boundary conditions, fluid properties and other data of the problem . Only examples of such subroutines will be given in Section A.5 ; however, a precise description will be given of what such a subroutine is supposed to do, so that the user should have no difficulty in either tailoring the sample subroutines to his own needs or writing new ones .

A.2 Conventions and symbols used in the programme

Before we go into the finer details of various subroutines, a general familiarity with the programme can be developed by the knowledge of different conventions and symbols used therein.

Subscripts for the grid lines. The numbering of the lines of constant ω closely corresponds to that used in Chapter 2. Subscripts 1 and NP3 are used for respectively the I and E boundaries, 2 and NP2 for corresponding slip values. NP1, NP2, NP3 stand for respectively $N+1$, $N+2$ and $N+3$ where N is the number of strips across the layer.

Dependent variables. The array $U(I)$ is used for velocities at grid points. The dependent variables of equations other than the velocity equation are stored in the form $F(J, I)$, where J denotes serial number of the dependent variable and I , the grid line. We shall use this serial number J to refer to various quantities related to $F(J, I)$. J can vary from 1 to NPH where NPH, the number of ϕ equations, is given by :

$$NPH = NEQ - 1, \quad (A.2-1)$$

where NEQ is the number of partial differential equations to be solved, one of which is always the velocity equation.

Specification of the type of boundary. KIN and KEX are the two numbers specifying the type of respectively the I and E boundaries. Each of these numbers can take the value 1, 2 or 3, according to whether the boundary is wall, free or symmetry line respectively. KASE is a quantity derived from KIN and KEX ; KASE equals 1 when at least one of the boundaries is a wall ; otherwise KASE is 2. INDI (J) and INDE (J) refer to the ϕ boundary condition at a wall existing at respectively the I and E surfaces ; J stands for the serial number of the ϕ equation involved. INDI (J) and INDE (J) take the value 1 or 2 according to whether the value of ϕ or the corresponding flux is specified.

Effects of radius. KRAD is the number which supplies the information regarding the inclusion or otherwise of the radius effects. When KRAD equals zero, it is an instruction to neglect the variation of radius across the layer ; otherwise the radius effects are properly accounted for. This device is mainly useful for the exact treatment of plane flows without making the radius infinitely large. It should be noted that $KRAD = 0$ suppresses the variation of radius merely across the layer ; the variation of radius in the stream direction is always taken into account.

List of FORTRAN symbols. Other symbols do not need much explanation. Given below is the list of all the important FORTRAN symbols used in the programme and their algebraic equivalents wherever possible ; where the symbol has a direct connection with a particular subroutine, the name of that subroutine is also given.

FORTRAN symbol	Meaning	Related subroutine
A (J, I)	A	COEFF, SLIP
AJI(J)	$J_{tot,E}$	) WALL, SLIP
AJE(J)	$J_{tot,I}$	
AJFS	ϕ_S or J_{tot}	FBC
AK	K	
AL	ℓ	VEFF
ALMG	λ	VEFF, ENTRN
AM	$\dot{m}''_S$	MASS
AM	$\dot{m}''_S / (\rho u)$	WF1, WF2
AME	$\dot{m}''_E$	
AMI	$\dot{m}''_I$	
AMU	μ (reference value)	VISCO

FORTTRAN Symbol	Meaning	Related subroutine
AU (I) B (J, I)	A_u B_u	) COEFF, SLIP)
BETA	β	WALL
BU (I) C (J, I)	B_u C_u	) COEFF, SLIP)
CS	d_U	SOURCE
CSALFA	$\cos \alpha$	RAD, READY
CU (I)	C_u	COEFF, SLIP
D (J, I)	$(\partial d / \partial \emptyset)_U$	COEFF
DEN	ρ (reference value)	DENSTY
DPDX	dp/dx	PRE
DS	$(\partial d / \partial \emptyset)_U$	SOURCE
DX	$(x_D - x_U)$	MAIN, COEFF
EMU	μ_{eff}	VEFF, COEFF
F	$(\gamma dp/dx) / (\rho u^2)$	WF1, WF2
F (J, I)	$\emptyset$	
FR	fraction in the definition of y_ℓ	LENGTH
G1(I), G2(I), G3(I)	g_1, g_2, g_3	COEFF
GAMA (J)	γ	WALL
IND	equals 1 or 2 according to whether $\emptyset_S$ or J_{tot} is specified	FBC
INDE (J) INDI (J)	IND for the E boundary IND for the I boundary	) WALL, SLIP)
INTG	number of integrations performed	MAIN
KASE	equals 1 or 2 according to whether or not a wall boundary exists	

FORTRAN Symbol	Meaning	Related subroutine
KEX	specifies the type of the E boundary	
KIN	specifies the type of the I boundary	
KRAD	when equals zero, suppresses the cross-stream radius effects.	READY
N	N	BEGIN
NEQ	number of partial differential equations to be solved	
NP1	$N + 1$	
NP2	$N + 2$	
NP3	$N + 3$	
NPH	number of $\emptyset$ equations	
OM (I)	ω	
P, P (J)	P_*	WF2, WALL
P1, P2, P3	P_1, P_2, P_3	COEFF
PEI	$(\psi_E - \psi_I)_U$	
PR, PR (J)	σ	WF2, WALL
PREF (J)	σ_{eff}, σ_t	COEFF, WALL
PRT	σ_t	WF2
Q	Q	COEFF
R	$\rho u y / \mu$	WF1, WF2
R (I)	r	
R1	r_I	RAD
R1, R2, R3	R_1, R_2, R_3	COEFF
RHO (I)	ρ	DENSTY

FORTRAN Symbol	Meaning	Related subroutine
S	$\tau_S / (\rho u^2)$	WF1
S	$J_S / \{\rho u (\phi_S - \phi)\}$	WF2
S1(I), S2(I), S3(I)	s_1, s_2, s_3	)
SA	a	)
SB	b	) COEFF
SC (I)	c	)
TAUE	τ_E	) WALL
TAUI	τ_I	)
U (I)	u	
UMAX	maximum value of u	) LENGTH
UMIN	minimum value of u	)
X	x	FBC, RAD
XD	x_D	)
XL	value of x at which computation is to be terminated	)
XP	value of x previous to x_U	) MAIN
XU	x_U	)
Y (I)	y	READY
YEM	y for a point near the E boundary	) LENGTH
YIP	y for a point near the I boundary	)
YL	y_L	VEFF, LENGTH

A.3 Description of the subroutines

Fig A.3-1 shows the flow diagram of the computer programme. It indicates the sequence of operations, the inter-connections of various parts of the programme and their functions in brief. This will serve as the common thread that links the descriptions of individual subroutines which now follow. Listings of all the subroutines are given in Section A.5. The FORTRAN symbols used below have been defined above.

Fig. A.3-1. Flow diagram of the computer programme

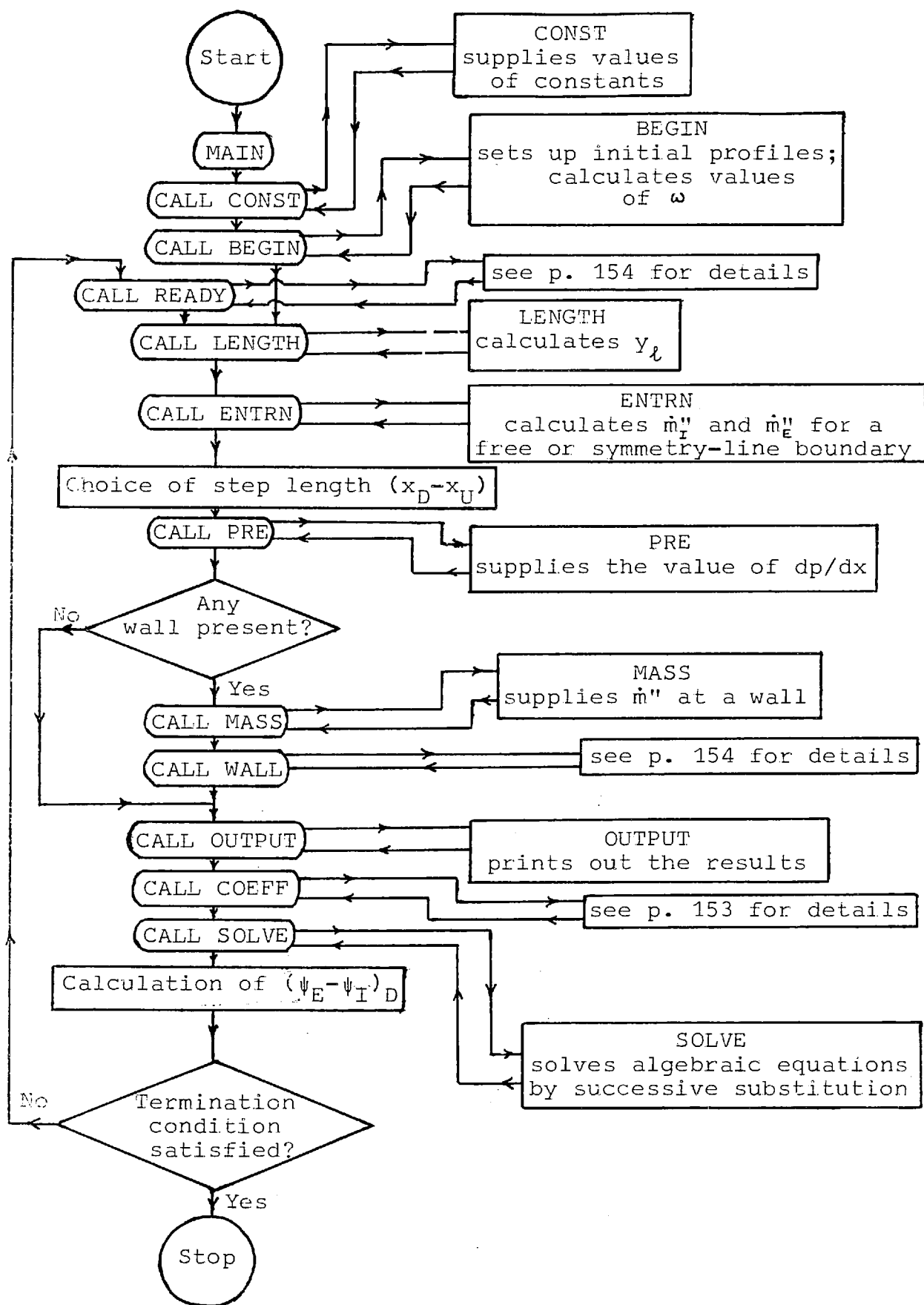


Fig. A.3-1. (continued)

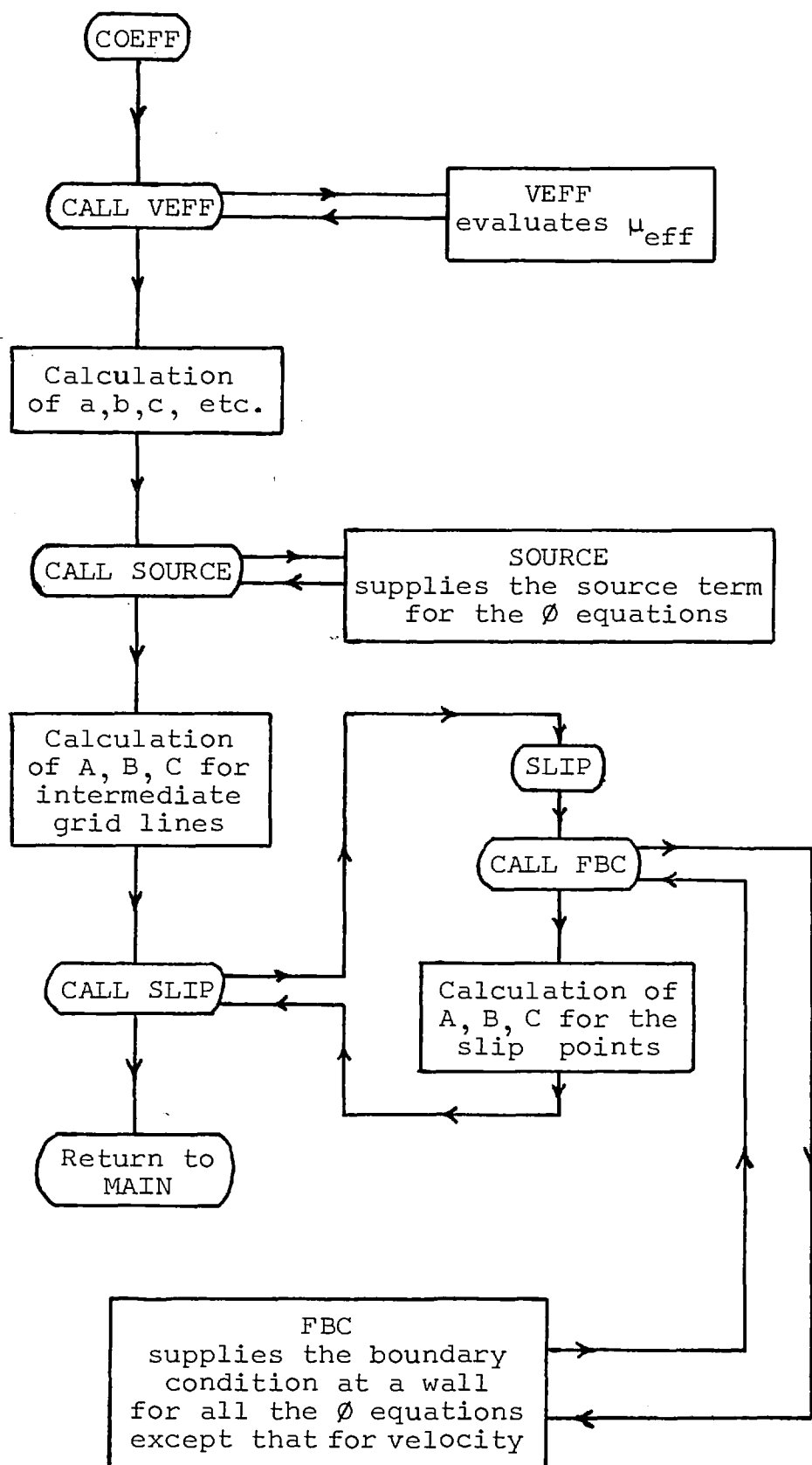
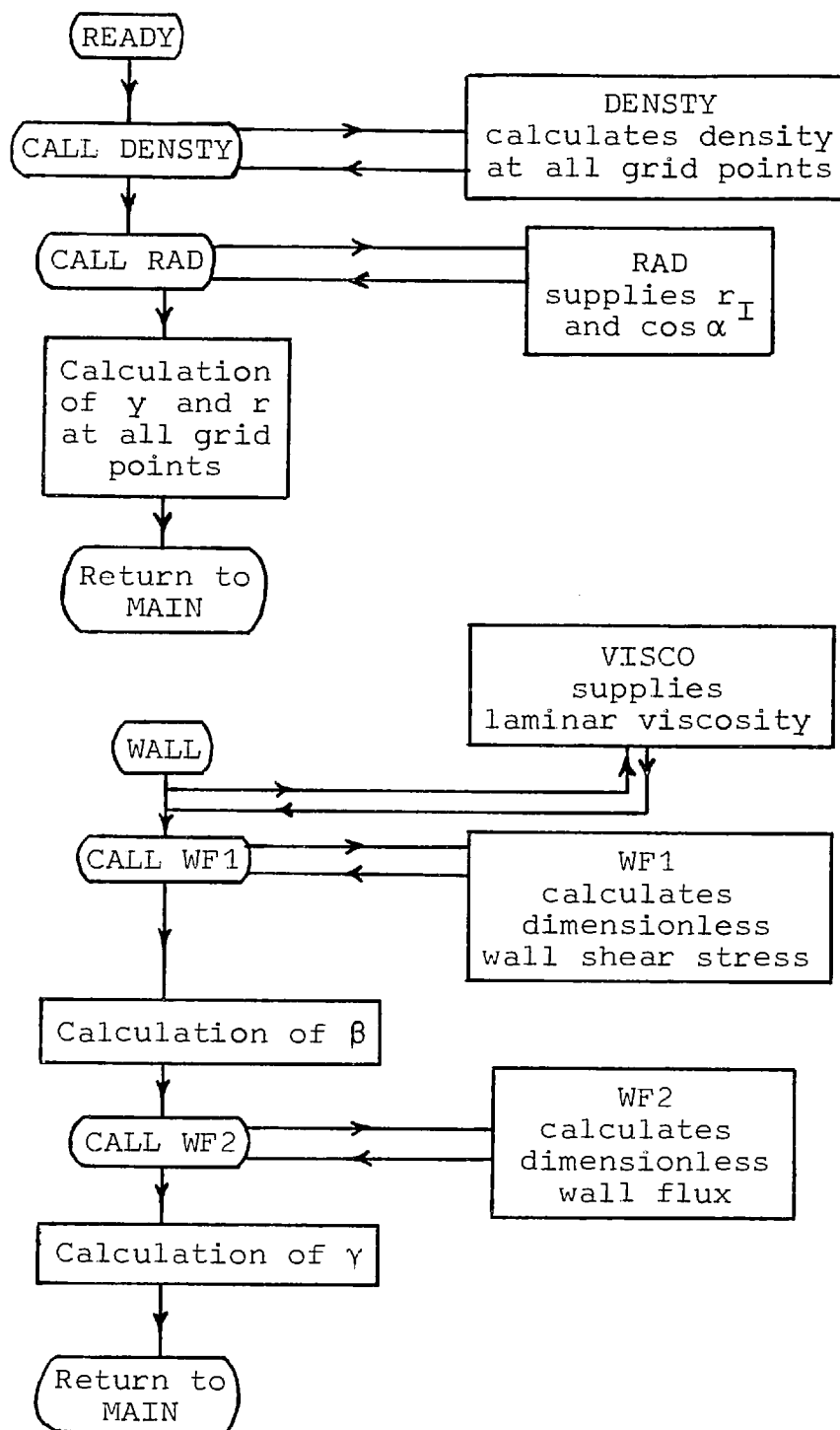


Fig. A.3-1. (concluded)



A.3-1 Subroutines of general validity

MAIN

Strictly speaking, the main programme cannot be called a "subroutine" according to FORTRAN terminology ; however, this point is of little significance for the present purposes. The main programme starts the computation and controls the sequence of operations. The choice of forward step is made here. The listing given in Section A.5 contains the use of eq. (2.8-16) ; the user may like to change this. For free boundaries the values of free-stream velocity are set here by use of the Euler equation. When the special boundary conditions of Section 2.6 exist, the true boundary values of the corresponding dependent variables are set here after the integration by use of the slip-value relations of Section 2.5. Use of array SC(I) is temporary. New value of $(\psi_E - \psi_I)$ is here obtained by use of eq. (2.8-6). The listing contains a stopping condition which terminates the integration when x exceeds a predetermined value XL ; the user may alter this condition to suit his requirements.

COEFF

The subroutine COEFF can be described as the heart of this programme. Its purpose is to obtain the coefficients A_u, B_u, C_u mentioned in eq. (2.4-35) and A, B, C in eq. (2.4-31) for all the $\emptyset$ equations and for all the intermediate grid points. It starts off with the calculation of g 's for the velocity equation followed by the c 's of the convection term. In the end various terms are assembled according to eq. (2.4-36) to (2.4-38) and (2.4-32) to (2.4-34). During this assembling process, various registers have been used as temporary storage space. The sets of coefficients are completed by obtaining the remaining ones (i.e. those with subscripts 2 and $N + 2$) from a separate subroutine SLIP to which we shall now turn.

SLIP

The relations given in Section 2.5 connecting the slip values to the neighbouring true values have been expressed in the form of the corresponding coefficients in the subroutine SLIP. These coefficients are here computed both for the I and E boundaries, the actual expressions depending upon the nature of the boundary. For a symmetry-line boundary the cases of plane and axi-symmetrical flow are distinguished. For a wall boundary, the boundary condition is obtained from subroutine FBC ; then appropriate expressions for slip coefficients are used according to whether the flux or value of ϕ at the wall is specified.

SOLVE

The subroutine SOLVE performs the mathematical operation (described in Section 2.7) of solving simultaneous equations of the type (2.7-1). The calling statement is :

CALL SOLVE (A, B, C, F, NP3) , where A, B, C are the arrays for the given coefficients, F is the array for the values of ϕ , and NP3 is the size of the array F which includes the known values at the boundaries.

READY

After every integration, we obtain the values of u and other ϕ 's for known values of ω . In the subroutine READY is undertaken the calculation of the corresponding normal distance y and radius r for every grid point ; this makes the stage ready for the performance of the next integration. READY needs the values of density ρ at all grid points and the specification of r_I and α ; these are obtained from subroutines DENSITY and RAD. The formulae used in READY are given in Section 2.8-2.

A.3-2 Subroutines where the physical hypotheses enter

VEFF

The subroutine VEFF is intended to supply the value of μ_{eff} according to the hypothesis used. The calling statement is :

CALL VEFF (I, I + 1, EMU) ,

which indicates that the required μ_{eff} value, EMU, is to be evaluated midway between the grid lines I, I + 1.

The listing in Section A.5 uses the mixing-length hypothesis in the form given by eq. (1.3-5). The characteristic length y_l , as defined in Section 2.8-3, is obtained from the subroutine LENGTH.

LENGTH

The task of evaluating the characteristic thickness of the layer, in accordance with the definition in Section 2.8-3, is performed in the subroutine LENGTH. At first the maximum and minimum values of velocity are found ; then the search proceeds near the two boundaries. For laminar-flow problems this subroutine can be dispensed with. It should, however, be noted that although primarily intended for obtaining a length scale for evaluation of the mixing length, this subroutine can have general utility ; it could be used, for example, to obtain the value of $y_{\frac{1}{2}}$ in a free jet or wall jet, where $y_{\frac{1}{2}}$ is the distance of the point at which the velocity equals one-half of the maximum velocity.

ENTRN

The evaluation of the entrainment rates, $\dot{m}''_I$ and $\dot{m}''_E$, at free or symmetry-line boundaries is done in the subroutine ENTRN. At the symmetry line, the corresponding $\dot{m}''$ is put equal to zero. The listing in Section A.5 contains the use of eq. (2.8-3) and (2.8-4) for the free boundaries. For flows which are laminar or use other μ_{eff} hypotheses, different expressions will be appropriate. Even for turbulent

flows with mixing-length hypothesis, the user is free to include here arbitrary specifications for the entrainment rates provided that sufficient care is taken.

WALL

The purpose of the subroutine WALL is to evaluate the exponents β and γ for the region near a wall boundary. Obviously therefore, this subroutine is not called into action if no wall boundary exists. As a by-product of this subroutine values are obtained of the wall shear stress and various fluxes at the wall. The subroutine derives its main information from two other subroutines, WF1 and WF2, which incorporate the wall-flux relationships.

The listing in Section A.5 uses eq. (2.5-15) and (2.5-16) for β and γ .

WF1

The wall-flux relationship concerning the momentum transfer is contained in the subroutine WF1. The calling statement is :

CALL WF1 (R, F, AM, S) ,

where R, F and AM are given quantities, while S is the output of the subroutine.

The listing in Section A.5 is based on the relationships described in Chapter 4.

WF2

Similar to WF1 is the subroutine WF2 which is relevant when ϕ stands for a variable other than u . The calling statement is :

CALL WF2 (R, F, AM, PR, PRT, P, S) ,

where S is the output of the subroutine and all other arguments are the

given quantities.

Once again, the relationships in Chapter 4 are the basis of the listing of WF2 in Section A.5.

A.3-3 Subroutines which specify a particular problem

SOURCE

When the partial differential equation is written in the form (2.2-4), the actual expression for d is the main feature that distinguishes one ϕ equation from the other. In our treatment of the velocity equation, a definite expression (as given in Table 2.2-1) has been used for the source term d . To supply the source terms for all other ϕ equations is the purpose of the subroutine SOURCE. The calling statement is :

CALL SOURCE (J, I, CS, DS) ,

where J is the serial number of the ϕ equation, I refers to the grid line at which the source term is required, and CS and DS form the output of this subroutine. Normally, the source term will be required at only the intermediate grid lines ; i.e. I can take values from 3 to $N + 1$. However, if a symmetry-line boundary exists, the subroutine SOURCE should be able to supply the source term also at that boundary.

In Section A.5 , two examples of this subroutine are presented. One of them puts the source term equal to zero; this is appropriate when ϕ stands for m_j and R_j is zero. The other example gives the source term when ϕ stands for $\tilde{h}$.

CONST

The values of different constants including some fluid properties, mixing-length constants etc. are to be given by the user in the subroutine CONST. There are various ways of feeding in these numbers. It is good

practice to give all the necessary constants in this subroutine alone and use the corresponding symbols in other subroutines. Those constants which are functions of others can also be computed in this subroutine ; for example,

P_* can be evaluated from the values of σ and σ_t by use of eq. (4.3-13). The sample subroutine in Section A.5 illustrates this.

DENSTY

The purpose of the DENSTY subroutine is to evaluate the density at all the grid points as a function of the dependent variables. Actual expressions will vary according to the type of the problem and the nature of the fluid. The symbol DEN, the value of which may be given in the subroutine CONST, can be conveniently used as a reference value of density. The example given in Section A.5 makes the density inversely proportional to the absolute temperature ; there $F(I, I)$ is supposed to stand for the absolute temperature and DEN for the density at the E boundary.

VISCO

The function subprogramme VISCO (I) is employed to obtain the laminar viscosity at any grid point I. In turbulent-flow problems, the use of this function is mainly near a wall. For laminar flows, more extensive use can be made. The example in Section A.5 gives the laminar viscosity as a function of the absolute temperature $F(I, I)$, when AMU represents the viscosity at the E boundary.

RAD

The subroutine RAD supplies the geometrical information regarding the problem. The task of this subroutine is to give the values of r_I and $\cos\alpha$ for a given value of x . The calling statement is :

CALL RAD (X, R1, CSALFA) .

For plane flows, the user is advised to set R1 to a constant value such as 1. It is not necessary to make it very large, and it must not be put equal to

zero in this case. The value of $R1$ for plane flows has no physical significance ; it is a dummy value.

Three examples of this subroutine will be given in Section A.5. The first one is applicable to plane flows ; the second one is for a radial wall jet. The third example can be used for an axisymmetric mixing layer subsequently turning into a jet ; here the radius r_I is calculated by taking into account the entrainment into the mixing layer.

PRE

The specification of the pressure gradient is through the subroutine PRE. The calling statement is :

```
CALL PRE ( XU, XD, DPDX )      ,
```

where the value of pressure gradient $DPDX$ is the output, while XU and XD are the two given values of x between which the pressure-gradient value is required.

There are a number of ways in which the pressure gradient can be calculated from the data of the problem. The example in Section A.5 shows one such way ; there given values of velocity at a free boundary are used in conjunction with the Euler equation to obtain the pressure gradient.

MASS

When a wall exists at a boundary, the mass-transfer rate through the wall should be supplied ; this is done by the subroutine MASS. The calling statement is :

```
CALL MASS ( XU, XD, AM )      ,
```

where XU and XD are two given values of x ; the subroutine is supposed to supply the mass-transfer rate AM between XU and XD . The example in Section A.5 is a simple version of this subroutine, useful for an impermeable wall.

FBC

At a wall boundary, the velocity u (relative to the wall) is always zero ; however, we need to know the boundary conditions for other dependent variables. These are to be supplied by the subroutine FBC. (The name is derived from " ϕ boundary conditions".) The calling statement is :

CALL FBC (X, J, IND, AJFS) .

Here X and J are the given quantities which stand for the distance x and the serial number of the ϕ equation under consideration. IND and AJFS are the outputs ; IND will be 1 or 2 according to whether the value of ϕ or the corresponding flux is specified, and the register AJFS will then contain that specified value. Two examples are given in Section A.5 ; the first one specifies $F(1,1)$ as a linear function of x , while the other gives a uniform flux at the wall .

When no wall boundary exists, the subroutines MASS and FBC are not called.

BEGIN

The initial profiles and other auxiliary quantities are to be specified in the subroutine BEGIN. Although the user can arrange to specify the initial profiles by any method of his choice, a large portion of this subroutine which concerns setting up of slip values and calculation of ω'_s should be left unaltered.

In this subroutine, it is necessary to specify the values of KRAD, NEQ, KIN, KEX, N ; the initial value of x ; the initial $u \sim y$ profiles and the corresponding profiles of other dependent variables. In the profile specification, the slip values namely those at nodes 2 and $N + 2$ need not be given, as they will be calculated in this subroutine. The user should note that this subroutine is a convenient place for reading in all the data for the problem including the boundary conditions, property

values etc. ; these quantities can be transferred to their respective subroutines via COMMON statements.

In the example given in Section A.5, all the necessary values are read in from data cards. This is convenient when experimental profiles are available. Alternatively, one can use algebraic expressions to generate the initial profiles. The values of β and γ are each put equal to 0.143 which is a well-known value for turbulent flows near walls. This should be changed when laminar flows are to be dealt with. The slip-value relations used in the example are from Section 2.5, and the calculation of $(\psi_E - \psi_I)$ and ω is in accordance with eq. (1.1-2) and (2.1-1).

OUTPUT

The instructions for printing out the results are to be contained in the OUTPUT subroutine. The rest of the programme makes no demand from this subroutine, except that of no interference, and therefore, the user has complete freedom to make arrangements here to print out any information of his choice. The integer variable INTG which counts the number of integrations can be profitably used to obtain the print-out after a desired number of integrations. In the example in Section A.5, the value of x and the corresponding velocity and temperature profiles are arranged to be printed out after every 5 integrations.

A.4 The user's quick-reference guide

The details of the computer programme given so far and the listings in Section A.5 complete the information necessary for the user to proceed on his own. However, after acquiring a general understanding of the programme, he may not like to go through this entire information every time he wants to prepare the programme for a particular problem. In such a case, he should make sure that he has taken the following ten steps. These steps express the same information in a concise and readily-usable form.

- (1) Keep the subroutines COEFF, SLIP, SOLVE and READY unaltered.
- (2) Make sure that in MAIN the specification of step length ($x_D - x_U$) and the condition for terminating the computation are suitable for your purposes.
- (3) Decide about the effective-viscosity formula. If the mixing-length hypothesis in the present form is what you want, keep the subroutines VEFF, LENGTH, ENTRN, WALL, WF1 and WF2 unaltered. For any other hypothesis or for laminar flow, suitably modify these subroutines.
- (4) Decide the values of KRAD, NEQ, KIN, KEX, N; supply these in BEGIN along with the initial value of x and initial profiles. See whether the nature of the boundaries (i.e. KIN or KEX value) is likely to change in your flow problem; if so, arrange for the value of KIN or KEX to be correctly altered at a convenient place in the programme. Change the values of β and γ in BEGIN if the flow is not turbulent. Try, as far as possible, to confine all the READ statements to this subroutine.
- (5) Prepare the SOURCE subroutine appropriate to the ϕ equations you need. If a wall boundary exists, arrange FBC to give the desired ϕ boundary conditions *, and MASS to give the mass-transfer rate through the wall.
- (6) Arrange for the appropriate calculation of fluid properties in DENSTY and VISCO.
- (7) Prepare the PRE subroutine to give the value of the pressure gradient for your flow problem, and RAD to give the geometrical information regarding the I boundary.
- (8) Supply the necessary constants in CONST.
- (9) Arrange for the desired print out in OUTPUT.
- (10) Prepare the data cards by noting the sequence of the READ statements in the whole of the programme.

* Normally, the boundary-condition treatment in SLIP will be adequate; but modifications will be needed for treating complex boundary conditions.

A.5 Listing of the computer programme

In this section, we shall present a complete listing of the computer programme. Here the subroutines will appear in the following order (which is same as the one used in Section A.3) :

page

Subroutines of general validity

MAIN	...	...	...	166
COEFF	...	...	...	167
SLIP	...	...	...	169
SOLVE	...	...	...	171
READY	...	...	...	171

Subroutines based on the mixing-length hypothesis

VEFF	...	...	...	172
LENGTH	...	...	...	173
ENTRN	...	...	...	174
WALL	...	...	...	174
WF1	...	...	...	175
WF2	...	...	...	176

Subroutines which specify a particular problem

SOURCE	...	...	...	176
CONST	...	...	...	177
DENSTY	...	...	...	177
VISCO	...	...	...	177
RAD	...	...	...	178
PRE	...	...	...	178
MASS	...	...	...	179
FBC	...	...	...	179
BEGIN	...	...	...	179
OUTPUT	...	...	...	182

MAIN

```

COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
1/B/BETA,GAMA(2),TAUI,TAUE,AJI(2),AJE(2),INDI(2),INDE(2)
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/C/SC(43),AU(43),BU(43),CU(43),A(2,43),B(2,43),C(2,43)
COMMON /L/AK,ALMG
16 CONTINUE
  INTG=0
  XL=20.
  CALL CONST
  CALL BEGIN
  AMI=0.
  AME=0.
  GO TO 25
15 CALL READY
25 CONTINUE
  INTG=INTG+1
  CALL LENGTH
  CALL ENTRN
C CHOICE OF FORWARD STEP
  FRA=.05
  DX=FRA *PEI/(R(1)*AMI-R(NP3)*AME)
  IF(DX.GT..5*Y(NP3))DX=.5*Y(NP3)
  XD=XU+DX
77 CONTINUE
  CALL PRE(XU,XD,DPDX)
  IF(KASE.EQ.2) GO TO 26
  IF(KIN.EQ.1)CALL MASS(XU,XD,AMI)
  IF(KEX.EQ.1)CALL MASS(XU,XD,AME)
  CALL WALL
26 CALL OUTPUT
  CALL COEFF
C SETTING UP VELOCITIES AT A FREE BOUNDARY
  IF(KEX.EQ.2)U(NP3)=SQRT(U(NP3)*U(NP3)-2.*(XD-XU)*DPDX/RHO(NP3))
  IF(KIN.EQ.2)U(1)=SQRT(U(1)*U(1)-2.*(XD-XU)*DPDX/RHO(1))
  CALL SOLVE(AU,BU,CU,U,NP3)
C SETTING UP VELOCITIES AT A SYMMETRY LINE
  IF(KIN.NE.3) GO TO 71
  U(1)=U(2)
  IF(KRAD.EQ.0)U(1)=.75*U(2)+.25*U(3)
71 IF(KEX.EQ.3)U(NP3)=.75*U(NP2)+.25*U(NP1)
72 CONTINUE
  IF(NEQ.EQ.1) GO TO 30
  DO 45 J=1,NPH
  DO 46 I=2,NP2
    AU(I)=A(J,I)
    BU(I)=B(J,I)
  46 CU(I)=C(J,I)
  DO 47 I=1,NP3
  47 SC(I)=F(J,I)
  CALL SOLVE(AU,BU,CU,SC,NP3)

```

```

DO 48 I=1,NP3
48 F(J,I)=SC(I)
  IF(KASE.EQ.2) GO TO 81
  SETTING UP WALL VALUES OF F
    IF(KIN.EQ.1.AND.INDI(J).EQ.2) F(J,1)=((1.+BETA+GAMA(J))*F(J,2)-
1(1.+BETA-GAMA(J))*F(J,3))*5/GAMA(J)
    IF(KEX.EQ.1.AND.INDE(J).EQ.2) F(J,NP3)=((1.+BETA+GAMA(J))*F(J,NP2)-
1(1.+BETA-GAMA(J))*F(J,NP1))*5/GAMA(J)
  SETTING UP SYMMETRY-LINE VALUES OF F
81 IF(KIN.NE.3) GO TO 82
  F(J,1)=F(J,2)
  IF(KRAD.EQ.0) F(J,1)=.75*F(J,2)+.25*F(J,3)
82 IF(KEX.EQ.3) F(J,NP3)=.75*F(J,NP2)+.25*F(J,NP1)
45 CONTINUE
30 XP=XU
  XU=XD
  PEI=PEI+DX*(R(1)*AMI-R(NP3)*AME)
  THE TERMINATION CONDITION
  IF(XU.LT.XL) GO TO 15
  GO TO 16
END

```

COEFF

```

SUBROUTINE COEFF
COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
1/B/BETA,GAMA(2),TAUI,TAUE,AJI(2),AJE(2),INDI(2),INDE(2)
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/C/SC(43),AU(43),BU(43),CU(43),A(2,43),B(2,43),C(2,43)
COMMON /L/AK,ALMG
DIMENSION G1(43),G2(43),G3(43),D(2,43),S1(43),S2(43),S3(43)
: CALCULATION OF SMALL C 'S
DO 98 I=2,NP1
  RA=.5*(R(I+1)+R(I))
  RH=.5*(RHO(I+1)+RHO(I))
  UM=.5*(U(I+1)+U(I))
  CALL VEFF(I,I+1,EMU)
98 SC(I)=RA*RA*RH*UM*EMU/(PEI*PEI)
  THE CONVECTION TERM
  SA=R(1)*AMI/PEI
  SB=(R(NP3)*AME-R(1)*AMI)/PEI
  DX=XD-XU
DO 71 I=3,NP1
  OMD=OM(I+1)-OM(I-1)
  P2=.25/DX
  P3=P2/OMD
  P1=(OM(I+1)-OM(I))*P3
  P3=(OM(I)-OM(I-1))*P3
  P2=3.*P2

```

```

Q=SA/OMD
R2=-SB*.25
R3=R2/OMD
R1=-(OM(I+1)+3.*OM(I))*R3
R3=(OM(I-1)+3.*OM(I))*R3
G1(I)=P1+Q+R1
G2(I)=P2+R2
G3(I)=P3-Q+R3
CU(I)=-P1*U(I+1)-P2*U(I)-P3*U(I-1)

```

C THE DIFFUSION TERM

```

AU(I)=2./OMD
BU(I)=SC(I-1)*AU(I)/(OM(I)-OM(I-1))
AU(I)=SC(I)*AU(I)/(OM(I+1)-OM(I))
IF(NEQ.EQ.1) GO TO 33
DO 34 J=1,NPH
C(J,I)=-P1*F(J,I+1)-P2*F(J,I)-P3*F(J,I-1)
CALL SOURCE(J,I,CS,D(J,I))
C(J,I)=-C(J,I)+CS-F(J,I)*D(J,I)
A(J,I)=AU(I)/PREF(J)
B(J,I)=BU(I)/PREF(J)

```

34 CONTINUE

C SOURCE TERM FOR VELOCITY EQUATION

```

33 S1(I)=DPDX*DX
S2(I)=P2*S1(I)/(RHO(I)*U(I))
S3(I)=P3*S1(I)/(RHO(I-1)*U(I-1))
S1(I)=P1*S1(I)/(RHO(I+1)*U(I+1))
CU(I)=-CU(I)-2.*(S1(I)+S2(I)+S3(I))
S1(I)=S1(I)/U(I+1)
S2(I)=S2(I)/U(I)
S3(I)=S3(I)/U(I-1)

```

71 CONTINUE

C COEFFICIENTS IN THE FINAL FORM

```

DO 91 I=3,NP1
RL=1./(G2(I)+AU(I)+BU(I)-S2(I))
AU(I)=(AU(I)+S1(I)-G1(I))*RL
BU(I)=(BU(I)+S3(I)-G3(I))*RL
91 CU(I)=CU(I)*RL
IF(NEQ.EQ.1) GO TO 76
DO 92 J=1,NPH
DO 92 I=3,NP1
RL=1./(G2(I)+A(J,I)+B(J,I)-D(J,I))
A(J,I)=(A(J,I)-G1(I))*RL
B(J,I)=(B(J,I)-G3(I))*RL
92 C(J,I)=C(J,I)*RL
76 CALL SLIP
RETURN
END

```

SLIP

```

SUBROUTINE SLIP
COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/B/BETA,GAMA(2),TAUI,TAUE,AJI(2),AJE(2),INDI(2),INDE(2)
COMMON /L/AK,ALMG
1/C/SC(43),AU(43),BU(43),CU(43),A(2,43),B(2,43),C(2,43)
SLIP COEFFICIENTS NEAR THE I BOUNDARY FOR VELOCITY EQUATION
CU(2)=0.
CU(NP2)=0.
GO TO (71,72,73),KIN
71 BU(2)=0.
AU(2)=1./(1.+2.*BETA)
GO TO 74
72 SQ=84.*U(1)*U(1)-12.*U(1)*U(3)+9.*U(3)*U(3)
BU(2)=8.*(2.*U(1)+U(3))/(2.*U(1)+7.*U(3)+SQRT(SQ))
AU(2)=1.-BU(2)
GO TO 74
73 BU(2)=0.
CALL VEFF(2,3,EMU)
AK1=1./DX-DPDX/(RHO(1)*U(1)*U(1))
AK2=-U(1)*AK1+DPDX/(RHO(1)*U(1))
AJ=RHO(1)*U(1)*.25*(Y(2)+Y(3))*2/EMU
IF(KRAD.EQ.0) GO TO 75
AU(2)=2./(2.+AJ*AK1)
CU(2)=-.5*AJ*AK2*AU(2)
GO TO 74
75 CU(2)=1./(2.+3.*AJ*AK1)
AU(2)=CU(2)*(2.-AJ*AK1)
CU(2)=-CU(2)*4.*AJ*AK2
SLIP COEFFICIENTS NEAR THE E BOUNDARY FOR VELOCITY EQUATION
74 GO TO (81,82,83),KEX
81 AU(NP2)=0.
BU(NP2)=1./(1.+2.*BETA)
GO TO 84
82 SQ=84.*U(NP3)*U(NP3)-12.*U(NP3)*U(NP1)+9.*U(NP1)*U(NP1)
AU(NP2)=8.*(2.*U(NP3)+U(NP1))/(2.*U(NP3)+7.*U(NP1)+SQRT(SQ))
BU(NP2)=1.-AU(NP2)
GO TO 84
83 AU(NP2)=0.
CALL VEFF(NP1,NP2,EMU)
BK1=1./DX-DPDX/(RHO(NP3)*U(NP3)*U(NP3))
BK2=-U(NP3)*BK1+DPDX/(RHO(NP3)*U(NP3))
BJ=RHO(NP3)*U(NP3)*.25*(2.*Y(NP3)-Y(NP1)-Y(NP2))*2/EMU
CU(NP2)=1./(2.+3.*BJ*BK1)
BU(NP2)=CU(NP2)*(2.-BJ*BK1)
CU(NP2)=-CU(NP2)*4.*BJ*BK2
84 IF(NEQ.EQ.1)RETURN
SLIP COEFFICIENTS NEAR THE I BOUNDARY FOR OTHER EQUATIONS
DO 54 J=1,NPH

```

```

C(J,2)=0.
C(J,NP2)=0.
GO TO (41,42,43),KIN
41 CALL FBC(XD,J,INDI(J),QI)
IF(INDI(J).EQ.1) GO TO 61
AJI(J)=QI
A(J,2)=1.
B(J,2)=0.
C(J,2)=8.*(1.+2.*BETA)*PREF(J)*AJI(J)/(AK*AK*BETA*(1.+BETA)*(1.+
1BETA)*(3.*RHO(2)+RHO(3))*U(3))
GO TO 44
61 F(J,1)=QI
A(J,2)=(1.+BETA-GAMA(J))/(1.+BETA+GAMA(J))
B(J,2)=1.-A(J,2)
GO TO 44
42 A(J,2)=(U(2)+U(3)-8.*U(1))/(5.*(U(2)+U(3))+8.*U(1))
GF=(1.-PREF(J))/(1.+PREF(J))
A(J,2)=(A(J,2)+GF)/(1.+A(J,2)*GF)
B(J,2)=1.-A(J,2)
GO TO 44
43 B(J,2)=0.
CALL SOURCE(J,1,CS,DS)
AK1=1./DX-DS
AK2=-AK1*F(J,1)-CS
AJF=AJ*PREF(J)
IF(KRAD.EQ.0) GO TO 45
A(J,2)=2./(2.+AJF*AK1)
C(J,2)=-.5*AJF*AK2*A(J,2)
GO TO 44
45 C(J,2)=1./(2.+3.*AJF*AK1)
A(J,2)=C(J,2)*(2.-AJF*AK1)
C(J,2)=-C(J,2)*4.*AJF*AK2
C SLIP COEFFICIENTS NEAR THE E BOUNDARY FOR OTHER EQUATIONS
44 GO TO (51,52,53),KEX
51 CALL FBC(XD,J,INDE(J),QE)
IF(INDE(J).EQ.1) GO TO 31
AJE(J)=QE
B(J,NP2)=1.
A(J,NP2)=0.
C(J,NP2)=-8.*(1.+2.*BETA)*PREF(J)*AJE(J)/(AK*AK*BETA*(1.+BETA)*
1(1.+BETA)*(RHO(NP1)+3.*RHO(NP2))*U(NP1))
GO TO 54
31 F(J,NP3)=QE
B(J,NP2)=(1.+BETA-GAMA(J))/(1.+BETA+GAMA(J))
A(J,NP2)=1.-B(J,NP2)
GO TO 54
52 B(J,NP2)=(U(NP2)+U(NP1)-8.*U(NP3))/(5.*(U(NP2)+U(NP1))+8.*U(NP3))
GF=(1.-PREF(J))/(1.+PREF(J))
B(J,NP2)=(B(J,NP2)+GF)/(1.+B(J,NP2)*GF)
A(J,NP2)=1.-B(J,NP2)
GO TO 54
53 A(J,NP2)=0.
CALL SOURCE(J,NP3,CS,DS)
BK1=1./DX-DS

```

```

BK2=-BK1*F(J,NP3)-CS
BJF=BJ*PREF(J)
C(J,NP2)=1./(2.+3.*BJF*BK1)
B(J,NP2)=C(J,NP2)*(2.-BJF*BK1)
C(J,NP2)=-C(J,NP2)*4.*BJF*BK2

```

```

54 CONTINUE
RETURN
END

```

SOLVE

```

SUBROUTINE SOLVE(A,B,C,F,NP3)
THIS SOLVES EQUATIONS OF THE FORM
F(I) = A(I)*F(I+1) + B(I)*F(I-1) + C(I)
FOR I=2,NP2
DIMENSION A(NP3),B(NP3),C(NP3),F(NP3)
NP2=NP3-1
B(2) = B(2)*F(1) + C(2)
DO 48 I=3,NP2
T = 1./(1.-B(I)*A(I-1))
A(I) = A(I)*T
48 B(I) = (B(I)*B(I-1) + C(I))*T
DO 50 I=2,NP2
J=NP2-I+2
50 F(J)=A(J)*F(J+1)+B(J)
RETURN
END

```

READY

```

SUBROUTINE READY
COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
IXL,DX,INTG,CSALFA
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
1/B/BETA,GAMA(2),TAUI,TAUE,AJI(2),AJE(2),INDI(2),INDE(2)
CALL DENSTY
CALL RAD(XU,R(1),CSALFA)
Y NEAR THE I BOUNDARY
GO TO (71,72,73),KIN
71 Y(2)=(1.+BETA)*OM(3)*4./((3.*RHO(2)+RHO(3))*(U(2)+U(3)))
GO TO 74
72 Y(2)=12.*OM(3)/((3.*RHO(2)+RHO(3))*(U(2)+U(3)+4.*U(1)))
GO TO 74
73 Y(2)=.5*OM(3)/(RHO(1)*U(1))
74 Y(3)=Y(2)+.25*OM(3)*(1./(RHO(3)*U(3))+2./(RHO(3)*U(3)+RHO(2)*U(2)+
1)

```

```

C   Y 'S FOR INTERMEDIATE GRID POINTS
      DO 50 I=4,NP1
50  Y(I)=Y(I-1)+.5*(OM(I)-OM(I-1))*(1./(RHO(I)*U(I))+1./(RHO(I-1))*
      1U(I-1)))
C   Y NEAR THE E BOUNDARY
      Y(NP2)=Y(NP1)+.25*(OM(NP2)-OM(NP1))*(1./(RHO(NP1)*U(NP1))+2./
      1(RHO(NP1)*U(NP1)+RHO(NP2)*U(NP2)))
      GO TO (81,82,83),KEX
81  Y(NP3)=Y(NP2)+(1.+BETA)*(OM(NP2)-OM(NP1))*4./((RHO(NP1)+3.*RHO(NP2
      1))*U(NP1)+U(NP2)))
      GO TO 84
82  Y(NP3)=Y(NP2)+12.*(OM(NP2)-OM(NP1))/((RHO(NP1)+3.*RHO(NP2))*(U(NP2
      1)+U(NP1)+4.*U(NP3)))
      GO TO 84
83  Y(NP3)=Y(NP2)+.5*(OM(NP2)-OM(NP1))/(RHO(NP3)*U(NP3))
84  IF(CSALFA.EQ.0..OR.KRAD.EQ.0) GO TO 51
      DO 52 I=2,NP3
52  Y(I)=2.*Y(I)*PEI/(R(1)+SQRT(R(1)*R(1)+2.*Y(I)*PEI*CSALFA))
      GO TO 56
51  DO 54 I=2,NP3
54  Y(I)=PEI*Y(I)/R(1)
56  Y(2)=2.*Y(2)-Y(3)
      Y(NP2)=2.*Y(NP2)-Y(NP1)
C   CALCULATION OF RADII
      DO 57 I=2,NP3
      IF(KRAD.EQ.0)R(I)=R(1)
      IF(KRAD.NE.0)R(I)=R(1)+Y(I)*CSALFA
57  CONTINUE
      RETURN
      END

```

VEFF

```

SUBROUTINE VEFF(I,IP1,EMU)
COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
COMMON /L/AK,ALAQ
1/L1/YL,UMAX,UMIN,FR,YIP,YEM

```

C THIS SUBROUTINE USES THE MIXING-LENGTH HYPOTHESIS

```

      AL=ALMG*YL
      IF(KASE.EQ.2) GO TO 66
      IF(KIN.EQ.1)YM=(Y(I)+Y(IP1))*0.5
      IF(KEX.EQ.1)YM= Y(NP3)-.5*(Y(I)+Y(IP1))
      IF(YM.LT.AL/AK)AL=AK*YM
66  EMU=.5*(RHO(I)+RHO(IP1))*AL*AL*ABS((U(I)-U(IP1))/(Y(I)-Y(IP1)))
      RETURN
      END

```

LENGTH

SUBROUTINE LENGTH

COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
 1XL,DX,INTG,CSALFA
 1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
 1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
 1/L1/YL,UMAX,UMIN,FR,YIP,YEM

C SEARCH FOR THE MAXIMUM AND MINIMUM VELOCITIES

40 UMAX=U(1)
 UMIN=U(1)
 DO 41 J=3,NP3
 IF(J.EQ.NP2)GO TO 41
 IF(U(J).GT.UMAX)UMAX=U(J)
 IF(U(J).LT.UMIN)UMIN=U(J)
 41 CONTINUE

DIF=ABS(UMAX-UMIN)*FR

C SEARCH NEAR THE I BOUNDARY

IF(KIN.NE.2) GO TO 43
 U21=ABS(.5*(U(2)+U(3))-U(1))
 IF(U21.LT.DIF) GO TO 47
 YIP=SQRT(DIF/U21)*(Y(2)+Y(3))*0.5
 GO TO 44
 47 J=2
 48 J=J+1
 UJ1=U(J)-U(1)
 IF(ABS(UJ1).GE.DIF) GO TO 49
 GO TO 48
 49 A1=1.
 IF(UJ1.LT.0.)A1=-1.
 YIP=Y(J-1)+(Y(J)-Y(J-1))*(U(1)+A1*DIF-U(J-1))/(U(J)-U(J-1))
 GO TO 44

43 YIP=0.

C SEARCH NEAR THE E BOUNDARY

44 IF(KEX.NE.2) GO TO 45
 U21=ABS(.5*(U(NP1)+U(NP2))-U(NP3))
 IF(U21.LT.DIF) GO TO 50
 YEM=SQRT(DIF/U21)*(.5*(Y(NP1)+Y(NP2))-Y(NP3))+Y(NP3)
 GO TO 46
 50 J=NP2
 51 J=J-1
 UJ1=U(J)-U(NP3)
 IF(ABS(UJ1).GE.DIF) GO TO 52
 GO TO 51
 52 A1=1.
 IF(UJ1.LT.0.)A1=-1.
 YEM=Y(J+1)+(Y(J)-Y(J+1))*(U(NP3)+A1*DIF-U(J+1))/(U(J)-U(J+1))
 GO TO 46
 45 YEM=Y(NP3)
 46 YL=YEM-YIP
 RETURN

END

ENTRN

SUBROUTINE ENTRN

```

COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
COMMON /L/AK,ALMG
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
1/L1/YL,UMAX,UMIN,FR,YIP,YEM

```

C THIS SUBROUTINE USES THE MIXING-LENGTH HYPOTHESIS

GO TO (71,72,73),KIN

71 GO TO 74

72 AMI=8.*RHO(1)*((ALMG*YL)/(Y(2)+Y(3)))*2*ABS(U(2)+U(3)-2.*U(1))

GO TO 74

73 AMI=0.

74 GO TO (81,82,83),KEX

81 RETURN

82 AME=-8.*RHO(NP3)*((ALMG*YL)/(Y(NP1)+Y(NP2)-2.*Y(NP3)))*2*ABS(

1U(NP1)+U(NP2)-2.*U(NP3))

RETURN

83 AME=0.

RETURN

END

WALL

SUBROUTINE WALL

```

COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
1/B/BETA,GAMA(2),TAUI,TAUE,AJI(2),AJE(2),INDI(2),INDE(2)
COMMON /L/AK,ALMG

```

C CALCULATION OF BETA FOR THE E BOUNDARY

IF(KEX.NE.1) GO TO 15

YI=Y(NP3)-.5*(Y(NP1)+Y(NP2))

UI=.5*(U(NP2)+U(NP1))

RH=.25*(3.*RHO(NP2)+RHO(NP1))

RE=RH*UI*YI/VISCO(NP3)

FP=DPDX*YI/(RH*UI*UI)

AM=AME/(RH*UI)

CALL WF1(RE,FP,AM,S)

BETA=SQRT(ABS(S+FP+AM))/AK

TAUE=S*RH*UI*UI

```

      IF(NEQ.EQ.1) GO TO 36
CALCULATION OF GAMA 'S FOR THE E BOUNDARY
      DO 35 J=1,NPH
      CALL WF2(RE,FP,AM,PR(J),PREF(J),P(J),SF)
      GAMA(J)=(SF+AM)*PREF(J)/(AK*AK*BETA)
      IF(INDI(J).EQ.1)AJE(J)=SF*RH*UI*(F(J,NP2)+F(J,NP1)-2.*F(J,NP3))*0.5
35 CONTINUE
36 IF(KIN.NE.1)RETURN
CALCULATION OF BETA FOR THE I BOUNDARY
15 YI=.5*(Y(2)+Y(3))
   UI=.5*(U(2)+U(3))
   RH=.25*(3.*RHO(2)+RHO(3))
   RE=RH*UI*YI/VISCO( 1 )
   FP=DPDX*YI/(RH*UI*UI)
   AM=AMI/(RH*UI)
   CALL WF1(RE,FP,AM,S)
   BETA=SQRT(ABS(S+FP+AM))/AK
   TAU1=S*RH*UI*UI
   IF(NEQ.EQ.1) RETURN
CALCULATION OF GAMA 'S FOR THE I BOUNDARY
      DO 38 J=1,NPH
      CALL WF2(RE,FP,AM,PR(J),PREF(J),P(J),SF)
      GAMA(J)=(SF+AM)*PREF(J)/(AK*AK*BETA)
      IF(INDI(J).EQ.1)AJI(J)=SF*RH*UI*(2.*F(J,1)-F(J,2)-F(J,3))*0.5
38 CONTINUE
      RETURN
      END

```

WF1

```

SUBROUTINE WF1(R,F,AM,S)
COMMON /L/ AK,ALMG
1/WL/STO,AKS,RT,FT,AMT
AKS=AK*AK
RT=R*AKS
ST=1./RT-.1561*RT**(-.45)+.08723*RT**(-.3)+.03713*RT**(-.18)
STO=ST
IF(F.EQ.0.) GO TO 15
FT=F/AKS
FM=1.-4.*FT*RT/(585.+RT**2.5)**.4
IF(FM.LT.0.)FM=0.
ST=ST*FM**1.6
GO TO 16
15 IF(AM.EQ.0.) GO TO 16
AMT=AM/AKS
AMM=1.-AMT/(7.74*RT**(-1.17)+.956*RT**(-.25))
ST=ST*AMM**4
16 S=ST*AKS
      RETURN
      END

```

WF2

```

SUBROUTINE WF2(R,F,AM, PR ,PRT,P,S)
COMMON /L/AK,ALMG
1/WL/ST0,AKS,RT,FT,AMT
ST1=ST0/(1.+P*SQRT(ST0))
IF(F.EQ.0.) GO TO 15
SSEP=1.725*RT**(-.3333)*(P+6.8)**(-1.165)
FD=.25*FT*RT/(1.+0.0625*RT)
ST1=ST1*(1.-FD)+FD*SSEP
15 ST=ST1/PRT
S=ST*AKS
RETURN
END

```

SOURCE (EXAMPLE 1)

C FOR CONSERVATION OF CHEMICALLY-INERT SUBSTANCE

```

SUBROUTINE SOURCE(J,I,CS,DS)
CS=0.
DS=0.
RETURN
END

```

SOURCE (EXAMPLE 2)

C FOR CONSERVATION OF STAGNATION ENTHALPY
C CAUTION- USE CONSISTENT UNITS

```

SUBROUTINE SOURCE(J,I,CS,DS)
COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/C/SC(43),AU(43),BU(43),CU(43),A(2,43),B(2,43),C(2,43)
CS=SC(I)*(U(I+1)*U(I+1)-U(I)*U(I))/(OM(I+1)-OM(I))
CS=CS-SC(I-1)*(U(I)*U(I)-U(I-1)*U(I-1))/(OM(I)-OM(I-1))
CS=(1.-1./PREF(J))*CS/(OM(I+1)-OM(I-1))
DS=0.
RETURN
END

```

CONST

```

SUBROUTINE CONST
COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
COMMON /L/AK,ALMG
1/L1/YL,UMAX,UMIN,FR,YIP,YEM
AK=.435
ALMG=.09
FR=.01
PREF(1)=.9
PR(1)=.7
PRRAT=PR(1)/PREF(1)
P(1)=3.68*(PRRAT-1.)*PRRAT**(-.25)
DEN=.075
AMU=.000012
RETURN
END

```

DENSTY

```

SUBROUTINE DENSTY
COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
DO 45 I=1,NP3
45 RHO(I)=DEN*F(1,NP3)/F(1,I)
RETURN
END

```

VISCO

```

FUNCTION VISCO(1)
COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
VISCO=AMU*(F(1,1)/F(1,NP3))**.76
RETURN
END

```

RAD (EXAMPLE 1)

C APPLICABLE TO PLANE FLOWS

```

SUBROUTINE RAD(X,R1,CSALFA)
CSALFA=1.
R1=1.
RETURN
END

```

RAD (EXAMPLE 2)

C APPLICABLE TO A RADIAL WALL JET

```

SUBROUTINE RAD(X,R1,CSALFA)
CSALFA=0.
R1=X.
RETURN
END

```

RAD (EXAMPLE 3)

C APPLICABLE TO AXISYMMETRIC MIXING LAYER AND JET

```

SUBROUTINE RAD(X,R1,CSALFA)
COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
IXL,DX,INTG,CSALFA
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
CSALFA=1.
IF(KIN.EQ.3) GO TO 17
IF(X.EQ.0.) GO TO 15
R1=R(1)*(R(1)-2.*AMI*(X-XP)/(RHO(1)*U(1)))
IF(R1.LT.0.)R1=0.
R1=SQRT(R1)
RETURN
15 R1=50.
RETURN
17 R1=0.
RETURN
END

```

PRE

```

SUBROUTINE PRE(XU,XD,DPDX)
COMMON /PR/UGU,UGD
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD

```

C HERE UGU AND UGD STAND FOR FREE-STREAM VELOCITIES AT XU AND XD
 $DPDX = (UGU + UGD) * (UGU - UGD) * .5 * RHO(NP3) / (XD - XU)$
 RETURN
 END

MASS

C APPLICABLE TO AN IMPERMEABLE-WALL SITUATION

SUBROUTINE MASS(XU,XD,AM)
 AM=0.
 RETURN
 END

FBC (EXAMPLE 1)

C PRESCRIBES A STEP-RAMP WALL TEMPERATURE

SUBROUTINE FBC(X,J,IND,AJFS)
 IND=1
 $AJFS = 100. + 5. * X$
 RETURN
 END

FBC (EXAMPLE 2)

C PRESCRIBES UNIFORM FLUX

SUBROUTINE FBC(X,J,IND,AJFS)
 IND=2
 AJFS=50.
 RETURN
 END

BEGIN

SUBROUTINE BEGIN
 COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
 1XL,DX,INTG,CSALFA
 1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
 1/B/BETA,GAMA(2),TAUI,TAUE,AJI(2),AJE(2),INDI(2),INDE(2)

```

1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
C PROBLEM SPECIFICATION
  READ(5,42)KRAD,NEQ,KEX,KIN,N
42 FORMAT(4I1,I2)
  KASE=2
  IF(KIN.EQ.1.OR.KEX.EQ.1)KASE=1
  XU=0.
  NPH=NEQ-1
  NP1=N+1
  NP2=N+2
  NP3=N+3
C INITIAL VELOCITY PROFILE
  READ(5,43)Y(1),U(1),(Y(I),U(I),I=3,NP1),Y(NP3),U(NP3)
43 FORMAT(6F10.0)
C CALCULATION OF SLIP VELOCITIES AND DISTANCES
  BETA=.143
  GO TO (71,72,73),KIN
71 U(2)=U(3)/(1.+2.*BETA)
  Y(2)=Y(3)*BETA/(2.+BETA)
  GO TO 74
72 U11=U(1)*U(1)
  U13=U(1)*U(3)
  U33=U(3)*U(3)
  SQ=84.*U11-12.*U13+9.*U33
  U(2)=(16.*U11-4.*U13+U33)/(2.*(U(1)+U(3))+SQRT(SQ))
  Y(2)=Y(3)*(U(2)+U(3)-2.*U(1))*5/(U(2)+U(3)+U(1))
  GO TO 74
73 IF(KRAD.NE.0) GO TO 89
  U(2)=(4.*U(1)-U(3))/3.
  Y(2)=0.
  GO TO 74
89 U(2)=U(1)
  Y(2)=Y(3)/3.
74 GO TO (75,76,77),KEX
75 U(NP2)=U(NP1)/(1.+2.*BETA)
  Y(NP2)=Y(NP3)-(Y(NP3)-Y(NP1))*BETA/(2.+BETA)
  GO TO 78
76 U11=U(NP1)*U(NP1)
  U13=U(NP1)*U(NP3)
  U33=U(NP3)*U(NP3)
  SQ=84.*U33-12.*U13+9.*U11
  U(NP2)=(16.*U33-4.*U13+U11)/(2.*(U(NP1)+U(NP3))+SQRT(SQ))
  Y(NP2)=Y(NP3)-(Y(NP3)-Y(NP1))*(U(NP2)+U(NP1)-2.*U(NP3))*5/
1(U(NP2)+U(NP1)+U(NP3))
  GO TO 78
77 U(NP2)=(4.*U(NP3)-U(NP1))/3.
  Y(NP2)=Y(NP3)
78 CONTINUE
  IF(NEQ.EQ.1) GO TO 45
  DO 88 J=1,NPH
C INITIAL PROFILES OF OTHER DEPENDENT VARIABLES
  READ(5,43)F(J,1),(F(J,I),I=3,NP1),F(J,NP3)
C CALCULATION OF CORRESPONDING SLIP VALUES
  GAMMA(J)=.143

```

```

      GO TO (81,82,83),KIN
81  F(J,2)=F(J,1)+(F(J,3)-F(J,1))*(1.+BETA-GAMA(J))/(1.+BETA+GAMA(J))
      GO TO 84
82  G=(U(2)+U(3)-8.*U(1))/(5.*(U(2)+U(3))+8.*U(1))
      GF=(1.-PREF(J))/(1.+PREF(J))
      GF=(G+GF)/(1.+G*GF)
      F(J,2)=F(J,3)*GF+(1.-GF)*F(J,1)
      GO TO 84
83  F(J,2)=F(J,1)
      IF(KRAD.EQ.0)F(J,2)=(4.*F(J,1)-F(J,3))/3.
84  GO TO (85,86,87),KEX
85  F(J,NP2)=F(J,NP3)+(F(J,NP1)-F(J,NP3))*(1.+BETA-GAMA(J))/(1.+BETA-
1GAMA(J))
      GO TO 88
86  G=(U(NP2)+U(NP1)-8.*U(NP3))/(5.*(U(NP2)+U(NP1))+8.*U(NP3))
      GF=(1.-PREF(J))/(1.+PREF(J))
      GF=(G+GF)/(1.+G*GF)
      F(J,NP2)=F(J,NP1)*GF+(1.-GF)*F(J,NP3)
      GO TO 88
87          F(J,NP2)=(4.*F(J,NP3)-F(J,NP1))/3.
88  CONTINUE
45  CONTINUE
      CALL DENSTY
C  CALCULATION OF RADII
      CALL RAD(XU,R(1),CSALFA)
      IF(CSALFA.EQ.0..OR.KRAD.EQ.0) GO TO 27
      DO 28 I=2,NP3
28  R(I)=R(1)+Y(I)*CSALFA
      GO TO 29
27  DO 30 I=2,NP3
30  R(I)=R(1)
29  CONTINUE
C  CALCULATION OF OMEGA VALUES
      OM(1)=0.
      OM(2)=0.
      DO 49 I=3,NP2
49  OM(I)=OM(I-1)+.5*(RHO(I)*U(I)*R(I)+RHO(I-1)*U(I-1)*R(I-1))*
1(Y(I)-Y(I-1))
      PEI=OM(NP2)
      DO 59 I=3,NP1
59  OM(I)=OM(I)/PEI
      OM(NP2)=1.
      OM(NP3)=1.
      IF(NEQ.EQ.1)RETURN
      DO 69 J=1,NPH
      IF(KEX.EQ.1)INDE(J)=1
      IF(KIN.EQ.1)INDI(J)=1
69  CONTINUE
      RETURN
      END

```

OUTPUT

```

SUBROUTINE OUTPUT
COMMON /GEN/PEI,AMI,AME,DPDX,PREF(2),PR(2),P(2),DEN,AMU,XU,XD,XP,
1XL,DX,INTG,CSALFA
1/V/U(43),F(2,43),R(43),RHO(43),OM(43),Y(43)
1/C/SC(43),AU(43),BU(43),CU(43),A(2,43),B(2,43),C(2,43)
COMMON /L/AK,ALMG
1/L1/YL,UMAX,UMIN,FR,YIP,YEM
1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD
1/B/BETA,GAMA(2),TAUI,TAUE,AJI(2),AJE(2),INDI(2),INDE(2)
IF(INTG.NE.1) GO TO 15
WRITE(6,49)(OM(I),I=1,NP3)
49 FORMAT(24H1THE VALUES OF OMEGA ARE/(11F10.4))
15 CONTINUE
IF(FLOAT(INTG-1)/5..NE.FLOAT((INTG-1)/5))RETURN
WRITE(6,51)XU
WRITE(6,51)(Y(I),I=1,NP3),(U(I),I=1,NP3),(F(1,I),I=1,NP3)
51 FORMAT(1P12E10.2)
RETURN
END

```

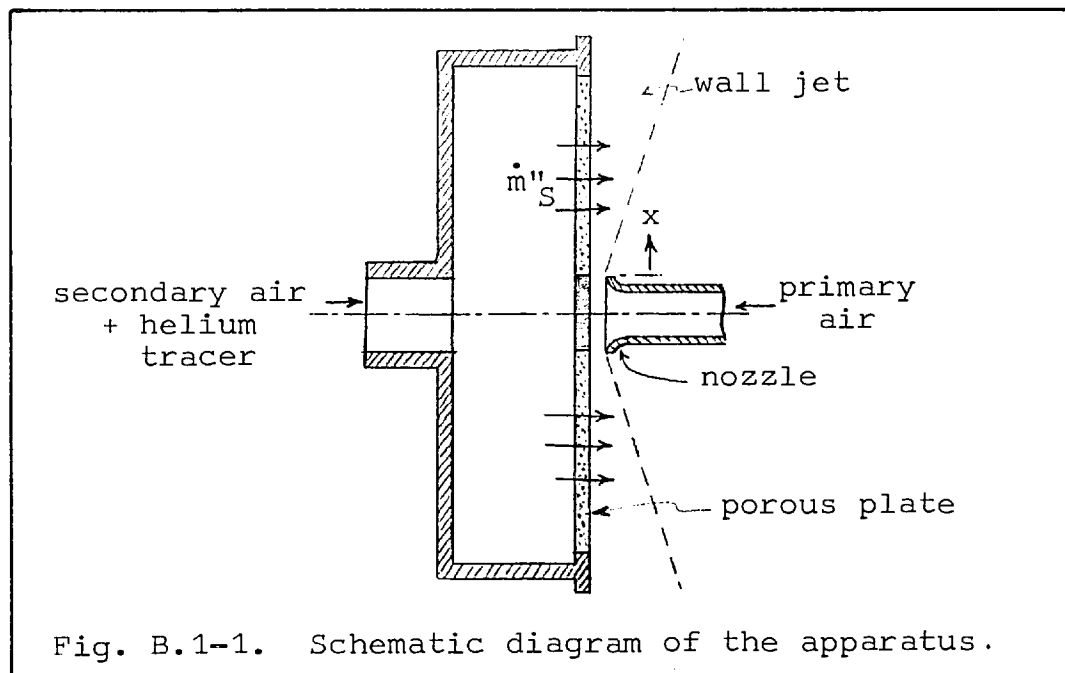
Appendix B

Experimental Investigation

B.1 Purpose and outline of the experiments

The need for experimental data using which one can test the validity of the wall-flux relationships has been made clear in Chapter 4. The main purpose of the experimental work to be described here is to fulfil this need to some extent. Experimental data are here obtained for the diffusional flux at the wall and for the velocity and concentration profiles in the presence of finite mass transfer through the wall.

The experiments were performed on a radial wall jet. The system is schematically shown in Fig. B.1-1 ; it is the same as the one used by Baker [5] , and for this reason, frequent reference will here be made to Baker's thesis where additional details can be found. The use of a radial-wall-jet system makes it easy to ensure that the flow is truly two-dimensional and that pressure gradients are absent. The radial wall jet is here formed on a porous plate through which mass may be transferred into the wall jet at



a finite rate. In the present work, helium was used as a tracer gas in the secondary air passing through the porous plate. The porous plate used was hydrodynamically smooth and fluid properties were almost uniform.

B.2 Description of the apparatus and instrumentation

B.2-1 The porous plate

For the present study, it was necessary to use a plate having uniform permeability, and sufficient strength to resist the necessary pressure drop without bulging. It was desirable that the plate should be smooth, so that the effects of mass transfer would not be obscured by those of roughness. Baker [5] has given detailed information on different available types of porous material. The material used in the present experiments was Porosint which is made by sintering bronze spheres. Grade B plate, having 0.003 inches as the nominal sphere diameter, was chosen. Baker checked the spatial uniformity of the mass-transfer rate through Porosint plates by use of a bubble meter and found it satisfactory.

The plate (of 24" x 28" size) used in the experiments was made up of four Porosint plates of 12" x 14" size (which was the largest available size) soldered to one another at the edges. The plate area under the nozzle was made impermeable by filling in with Araldite so that the primary air would not pass through the plate at the impingement point. Under one of the Porosint plates on which the measurements were conducted, a separate mass-transfer chamber was attached and the flow rate to it was separately measured; this chamber was enclosed in a bigger chamber which provided air to the other three plates. This arrangement made it possible to obtain an accurate measurement of the mass-transfer rate through the test section irrespective of the small differences in the individual permeability of the four Porosint plates. Helium was added as a tracer gas to the smaller mass-transfer chamber only. This reduced the required quantity of helium

by a factor of four ; the departures from two-dimensionality resulting from this were believed to be small.

B.2-2 Measurement of velocity

In the present investigation, the measurement of velocity profiles was, in fact, not necessary; because results of Baker [5] who made extensive velocity measurements on the same apparatus could have been used. Nevertheless, it was thought desirable to make these measurements, in case the test conditions such as the slot height and slot velocity should be slightly different. It was not possible to make velocity measurements very close to the wall ; because, as reported by Baker, the impact-probe viscous effects, effects of turbulent fluctuations and the danger of the probe hole getting clogged by dirt put a limit on the smallest size of the flattened probe to be used : for velocities greater than 15 ft/s, the smallest allowable height of flattened probe was 0.023 inches. A rake consisting of five pitot tubes was used for the velocity measurement. The probe nearest to the wall was a flattened one of 0.023 inches height, the next one of 0.03 inches diameter, and the three outer probes were of 0.05 inches diameter each. The spacing between the probes was approximately 0.2 inches. The rake could be mounted on a traversing mechanism with a micrometer to measure the position of the rake. The exact position when the probe touched the wall could be determined by the standard electrical-contact method. Dynamic pressures were measured on an inclined bank of manometers in inches of paraffin, except for the slot velocity profile for which a vertical U-tube manometer was used.

B.2-3 Measurement of concentration

Concentration measurements required in the present study were of two kinds : measurement of helium concentration in the mass-transfer chamber, and concentration-profile measurement in the wall jet. Each was accomplished by obtaining a sample of the helium-air mixture from the

appropriate location and analysing it with the aid of a gas chromatograph. The method of sampling proved to be the important part of the whole procedure ; we shall turn to the details of this a little later.

Analysis of the sample. A Shandon KG-2 gas chromatograph was used which contained a two-metre long column packed with a Linde Molecular Sieve to separate the permanent gases, which passed through a thermal-conductivity cell. The cell was connected to an automatic recorder which traced lines of different peak heights corresponding to the proportions of different components of the mixture. Nitrogen was used as a carrier gas ; therefore the only two significant peaks were of helium and oxygen. The column temperature was maintained at 60°C so that the analysis of 1 ml of sample was complete in less than three minutes. Samples were injected into the chromatograph by use of a 1 ml gas-tight syringe. Had the volume of mixture injected by the syringe remained precisely the same, the peak height of helium would have served as a measure of its concentration. However, as a precaution against slight variations in the volume of mixture injected, the ratio of helium peak height to oxygen peak height was used as a measure of the helium concentration. To regard this ratio as proportional to the mass fraction of helium implies a linear calibration curve for the gas chromatograph. It was indeed linear, as can be seen from the curve given in Ref. [91] ; departures from linearity exist for ratios of helium to oxygen peak heights below 0.3. In the present experiments, the concentrations in the region of interest were always kept in the linear region.

Sampling method. There seemed to be no particular difficulty in obtaining a true sample from the mass-transfer chamber. That the concentration was uniform all over the chamber was confirmed by analysing samples drawn from different points in the chamber. The effect of the rate at which the sample was sucked was understandably negligible.

The measurement of concentration profiles was fraught with more difficulty. The crucial point was the measurement of concentration at the wall, because only this would enable us to obtain the diffusional component of the total flux at the surface. Use of sampling tubes inserted in the porous plate and made flush with the surface was considered to be undesirable ; the reason, in addition to the constructional difficulty, was that the concentration thus measured would be strongly influenced by the presence of the sampling tube itself, and by the velocity caused by sucking a sample in a direction opposite to that of the mass transfer through the wall.

The alternative, therefore, was the measurement of concentrations very near the wall, followed by some form of extrapolation to obtain the value at the wall. For this purpose, the same pitot probes as used for the velocity measurement could, in principle, be used. The samples could then be sucked into gas bottles by means of a vacuum pump, in the same way as Nicoll and Whitelaw [44] did for measurement of the impermeable wall concentrations. The suction rate, however, should be so low as not to disturb the flow in the small-velocity region near the wall. This requirement would become increasingly more stringent with the use of smaller probes for measurements sufficiently near the wall ; for then the probe would not only be in a region of very small velocities, but also induce a larger velocity for the same volumetric suction rate.

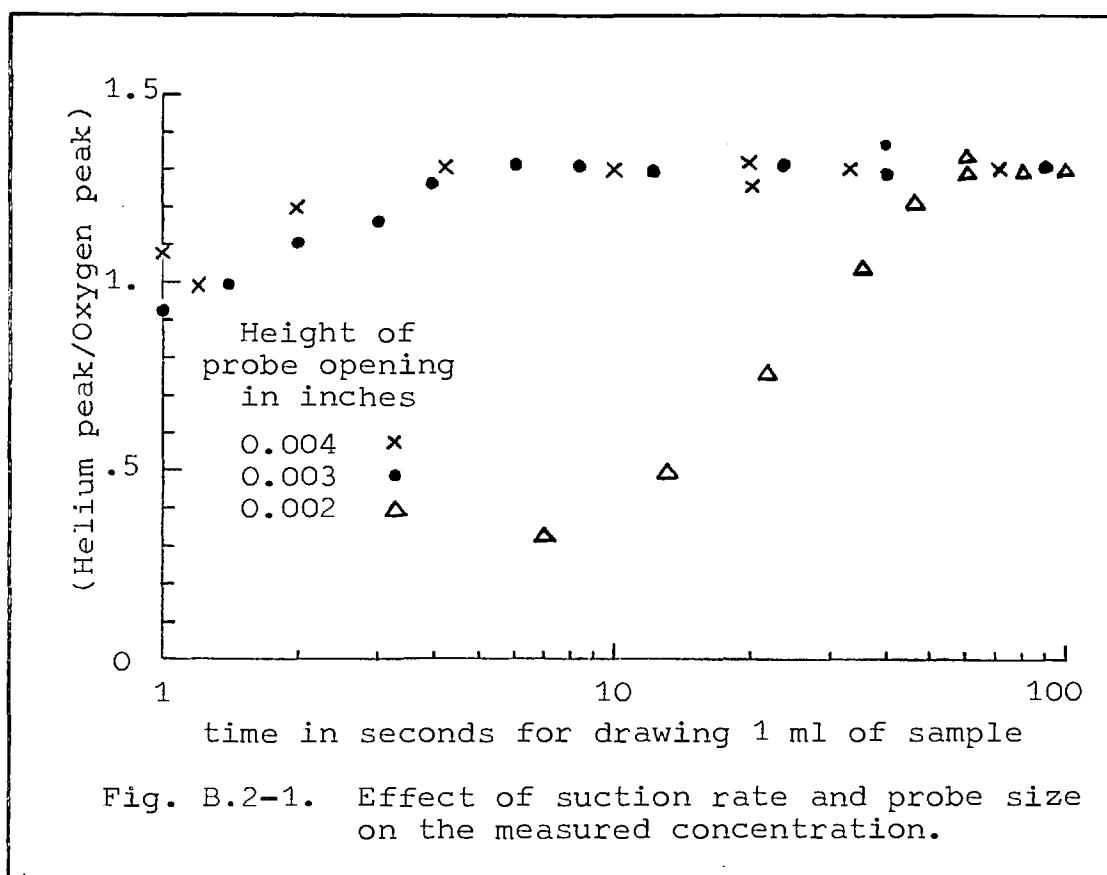
A simple calculation showed that the time required for flushing the connecting pipe line and the sampling bottle and for the subsequent collection of the sample would be impracticably long if the suction rate had to be kept small enough ; also then it would be difficult to maintain the flow rate of helium precisely constant over this long period of time.

It was, therefore, decided to obtain only 1 ml of sample directly from the probe and to dispense with the sampling bottles and connecting pipe

lines. The probe then consisted of a short stainless-steel tube fitted with a thick rubber diaphragm at one end, the other end being placed in the wall jet, facing the flow. This probe could be mounted on the same traversing mechanism as for the velocity - measuring probe. A sample could be drawn with the aid of the gas-tight syringe by piercing the rubber diaphragm with the needle of the syringe. The flushing problem was reduced to that of flushing the short length of probe the volume of which was much smaller than that of the syringe. Necessary supports were provided for the probe and for the hands of the experimenter so that the probe would not be displaced during the drawing of a sample.

Effect of suction rate. Although the arrangement described above made it seem likely that the allowable suction rate would also be a practicable one, it was necessary to establish the actual magnitude of this safe suction rate. The suction rate could be varied by changing the speed at which the plunger in the syringe was drawn. A certain amount of practice was necessary to achieve the ability to draw the plunger at a uniform speed ; but this skill was not very difficult to acquire. The effect of suction rate was studied by drawing samples at various plunger speeds and analysing them on the chromatograph. The time taken by the plunger to travel the length of the syringe was the measure of the plunger speed. Fig. B.2-1 shows the effect of suction rate on the concentration measured, for three probes of different sizes. The ordinate is the ratio of the peak heights of helium to oxygen obtained on the chromatograph, while the time for filling up the syringe is the abscissa. It can be seen that at first the measured value of the helium concentration increases with the decrease of suction rate, and then remains constant for all small suction rates. The effect is, understandably, more pronounced for the smallest probe. What is interesting and useful is the fact that, for the two larger probes, the suction rate will have no influence as long as the syringe - filling time is kept more than five seconds. Incidentally, five to ten seconds happens to be the most convenient time

interval in which the plunger can be comfortably drawn. Fig. B.2-1 shows results of a single typical test ; similar tests were carried out at different stations along the plate and for different rates of mass transfer. All these substantiated the above-mentioned conclusion. The reproducibility of the results was also satisfactory (to within less than 2 per cent).



Effect of probe size. To study the effect of probe size, three geometrically similar flattened probes were made. The heights of the rectangular opening were respectively 0.004, 0.003 and 0.002 inches. Concentration measurements were taken under identical conditions with each probe touching the wall. The results were as shown in Fig. B.2-1. The interesting thing to note this time is that the asymptotic values of concentration for all the three probes are very close to one another. Indeed,

the differences between them are smaller than the uncertainty introduced by the chromatograph. Similar behaviour was observed in tests carried out at different stations and for different mass-transfer rates. Therefore, it was concluded that the concentration value obtained by use of any one of the probes touching the wall could be taken as the value of the wall concentration. The probe with 0.003 inches as the height of opening was used in all the subsequent measurements (both at the wall and in the boundary layer) ; the minimum allowable plunger-drawing time for the smallest probe was too large.

B.2-4 Measurement of flow rate

The flow rate of the total supply of secondary air and that of the separate supply to the mass-transfer chamber under the test plate were measured with orifices made to British Standard Specifications [9] . It was not necessary to measure independently the flow rate of helium ; because the concentration of helium in the mass-transfer chamber could give, in conjunction with the measured value of the flow rate of air, sufficient information to arrive at the helium flow rate. However, a rotameter was used to obtain a measure of the helium flow rate so that the latter could be adjusted to the same value every time a sample was drawn.

B.3 The procedure

In the beginning, the plate was aligned so that the flow was truly axi-symmetrical. The measurements began with the slot velocity profile followed by the velocity profile at 7 inches from the slot. Immediately after the measurement of velocity profile there, the concentration profile was measured. The helium gas was not supplied continuously. After the helium supply was started, its flow rate was adjusted, the probe was flushed by drawing a sample and throwing it away, and a fresh sample was carefully drawn immediately, after which the helium supply was shut off. The sample was injected into the chromatograph, the probe position was

changed and only then the helium supply was restarted and adjusted. The concentration of the mixture in the mass-transfer chamber under the test plate was measured four times during the measurement of one concentration profile. The flow rate of helium was chosen such that the helium concentration at the wall would be of the order of one per cent by volume. Concentrations in the region more than 0.5 inches away from the wall were not measured ; because all the interesting information which was sought resided wholly in the region very near the wall ; moreover, the chromatograph was neither linear nor accurate for such small concentrations as would be found away from the wall.

Velocity and concentration profiles were measured at six stations : 7, 8, 9, 10, 11 and 12 inches from the slot for a particular mass-transfer rate through the wall. (The distance of the slot from the axis of symmetry was 3 inches.) Then the mass-transfer rate was adjusted to a new value and the whole procedure was repeated. Four different injection velocities at the wall were used : 0.1, 0.2, 0.4 and 0.8 ft/s.

B.4 Results of the experiments

The measurements of velocity and concentration profiles are given in Table B.4-1 and Table B.4-2 respectively. The internal consistency of these measurements can be checked by calculating the total streamwise flux of helium via integration of the profiles ; this should then agree with the amount of helium supplied through the wall. When such checks were made, the helium flux obtained from the integration of the profiles turned out to be about 50 to 100 per cent larger than the helium flow rate through the wall.

It is true that such a mass balance cannot be made with certainty in the present case ; because small spacial variations in the porosity of the plate, departures from two-dimensionality, and non-homogeneity of

Table B.4-1. Velocity profiles

(a) at 7 inches from the slot

$v_S = 0.1$ ft/s		$v_S = 0.2$ ft/s		$v_S = 0.4$ ft/s		$v_S = 0.8$ ft/s	
y inches	u ft/s	y inches	u ft/s	y inches	u ft/s	y inches	u ft/s
0.0138	71.097	0.0138	71.471	0.0138	60.709	0.0138	47.963
0.0174	76.072	0.0172	76.865	0.0175	65.615	0.0178	50.779
0.0225	81.812	0.0212	81.149	0.0222	71.102	0.0212	55.535
0.0265	84.123	0.0273	84.493	0.0270	74.243	0.0269	60.335
0.0324	86.372	0.0339	87.358	0.0367	78.514	0.0337	64.584
0.0364	87.575	0.0391	89.102	0.0467	82.366	0.0397	68.202
0.0410	88.367	0.0485	91.150	0.0621	85.665	0.0499	72.164
0.0466	89.738	0.0583	92.988	0.0826	88.290	0.0596	74.914
0.0512	90.895	0.0697	94.303	0.1067	90.480	0.0750	79.179
0.0717	92.981	0.0849	95.438	0.1362	91.555	0.0895	82.002
0.0915	93.914	0.1139	96.400	0.1620	90.839	0.1258	85.917
0.1018	94.469	0.1447	96.241	0.1788	89.756	0.1454	87.375
0.1314	93.542	0.1677	95.760	0.1872	89.392	0.1557	88.239
0.1465	93.355	0.1788	94.139	0.1920	88.659	0.1862	86.795
0.1624	92.981	0.1822	93.812	0.2117	88.290	0.1987	86.503
0.1918	91.087	0.2041	92.491	0.2271	87.175	0.2149	86.503
0.2064	89.932	0.2233	91.150	0.2476	86.045	0.2400	85.622
0.2367	85.968	0.2347	90.472	0.2717	84.515	0.2693	84.433
0.2565	84.330	0.2499	89.102	0.3012	82.167	0.2908	83.226
0.2668	83.290	0.2789	87.181	0.3270	79.750	0.3104	81.693
0.2800	82.237	0.2943	85.578	0.3588	77.678	0.3307	80.760
0.2934	80.740	0.3588	80.578	0.3625	76.834	0.3588	79.179
0.3412	76.983	0.3789	78.253	0.3720	75.980	0.3628	78.859
0.3714	73.508	0.3935	76.865	0.3817	74.681	0.3719	77.566
0.3962	70.358	0.4147	74.217	0.3917	73.803	0.4046	74.576
0.4167	68.347	0.4447	71.039	0.4071	72.013	0.4200	72.513
0.4365	66.012	0.4589	69.727	0.4276	70.178	0.4345	71.460
0.4468	64.678	0.4743	67.485	0.4517	67.574	0.4708	67.456
0.4764	61.643	0.5127	63.733	0.4812	64.106	0.4904	65.169
0.5212	56.939	0.5288	60.767	0.5070	61.244	0.5288	61.167
0.5514	53.466	0.5541	57.648	0.5517	55.664	0.5328	60.335
0.5767	47.602	0.5733	54.915	0.5617	53.878	0.5487	58.202
0.5966	46.114	0.5847	53.207	0.5771	52.030	0.5746	55.078
0.6168	44.576	0.6147	50.841	0.5976	50.114	0.6045	51.765
0.6300	43.388	0.6289	49.305	0.6217	47.782	0.6408	47.699
0.6464	41.330	0.6443	47.070	0.6512	43.867	0.6807	43.252
0.6774	37.806	0.6597	45.066	0.6770	41.577	0.7188	38.946
0.6912	36.397	0.6827	42.246	0.7188	35.665	0.7447	35.553
0.7068	34.428	0.7188	38.030	0.7517	31.281	0.7800	30.994
0.7366	32.874	0.7323	36.796	0.7671	29.675	0.8093	27.539
0.7460	27.694	0.7441	34.195	0.7876	27.389	0.8308	25.637
0.7667	25.736	0.7747	30.383	0.8117	24.894	0.8504	22.485
0.8200	21.288	0.8047	27.176	0.8412	22.119	0.8976	17.417
0.8515	17.713	0.8497	22.189	0.8670	18.941	0.9256	12.316
0.8812	16.700	0.8727	19.216	0.8810	14.831	0.9516	8.709
0.8968	0.	0.9571	0.	0.9670	0.	0.9961	0.

Table B.4-1. (continued)

(b) at 8 inches from the slot

$v_S = 0.1$ ft/s		$v_S = 0.2$ ft/s		$v_S = 0.4$ ft/s		$v_S = 0.8$ ft/s	
y inches	u ft/s	y inches	u ft/s	y inches	u ft/s	y inches	u ft/s
0.0138	60.232	0.0138	61.606	0.0138	49.435	0.0138	35.442
0.0166	61.052	0.0169	65.657	0.0187	54.843	0.0178	38.007
0.0202	66.879	0.0211	68.793	0.0243	59.369	0.0220	41.748
0.0245	70.673	0.0271	72.118	0.0289	61.699	0.0273	45.180
0.0315	73.937	0.0325	74.200	0.0371	65.036	0.0375	50.123
0.0376	75.927	0.0375	75.607	0.0443	67.169	0.0474	53.458
0.0425	76.903	0.0471	77.746	0.0542	69.575	0.0586	56.873
0.0527	78.502	0.0574	79.534	0.0700	72.712	0.0719	59.833
0.0617	79.602	0.0674	80.993	0.0906	75.409	0.0881	62.904
0.0723	80.841	0.0825	82.142	0.1156	76.952	0.1027	64.627
0.0819	81.606	0.0969	82.993	0.1406	77.862	0.1180	66.543
0.1013	82.212	0.1121	83.556	0.1656	78.464	0.1318	67.944
0.1157	82.514	0.1273	84.254	0.1950	78.464	0.1474	69.089
0.1306	82.665	0.1421	84.254	0.2192	77.256	0.1643	70.440
0.1475	82.514	0.1620	84.254	0.2350	77.104	0.1834	71.105
0.1619	82.363	0.1856	83.836	0.2556	76.338	0.1923	69.543
0.1769	82.061	0.1921	82.710	0.2806	75.409	0.2236	70.217
0.1917	81.606	0.1975	82.569	0.3056	73.835	0.2531	70.440
0.2063	80.994	0.2121	81.856	0.3306	72.874	0.2677	69.993
0.2373	78.184	0.2224	81.713	0.3600	70.913	0.2968	69.993
0.2469	77.866	0.2475	80.703	0.3821	70.247	0.3124	69.316
0.2663	76.579	0.2619	79.828	0.3893	69.912	0.3484	68.174
0.2807	75.599	0.2923	78.346	0.3992	69.236	0.3723	67.944
0.2956	74.773	0.3071	77.444	0.4150	68.210	0.3924	66.778
0.3125	73.262	0.3270	76.837	0.4356	66.466	0.4169	65.592
0.3269	72.752	0.3506	74.985	0.4606	64.674	0.4477	63.153
0.3419	71.546	0.4449	68.452	0.4856	62.078	0.4768	61.643
0.3567	70.497	0.4571	67.764	0.5106	60.545	0.5093	59.570
0.3713	69.788	0.4723	66.012	0.5400	58.170	0.5328	57.423
0.4067	66.506	0.4871	64.939	0.5692	55.270	0.5423	56.873
0.4269	64.995	0.5070	63.664	0.5850	53.979	0.5624	54.907
0.4463	63.837	0.5306	61.796	0.6056	51.757	0.6031	52.270
0.4756	61.457	0.6249	53.899	0.6306	49.908	0.6330	49.174
0.4925	59.610	0.6423	51.225	0.6556	47.496	0.6468	47.880
0.5219	57.272	0.6571	49.361	0.6806	44.954	0.6793	45.526
0.5665	53.224	0.6770	48.884	0.7100	42.260	0.7188	43.408
0.5873	50.591	0.7006	46.173	0.7339	41.132	0.7270	41.936
0.5969	48.841	0.7949	39.322	0.7592	38.169	0.7524	40.020
0.6456	45.414	0.8019	36.221	0.7750	36.598	0.7769	37.172
0.6625	42.881	0.8171	34.904	0.7956	34.277	0.8077	34.996
0.7213	38.613	0.8323	33.534	0.8206	33.584	0.8230	33.623
0.7475	35.248	0.8471	32.828	0.8456	30.658	0.8524	31.700
0.7773	32.688	0.8670	32.107	0.8706	27.422	0.8693	29.653
0.8356	28.199	0.8906	29.042	0.9000	25.651	0.9584	22.416
0.8525	25.902	0.9849	18.111	0.9635	16.792	1.0562	12.531
0.9265	22.293	1.0999	0.	1.0637	6.855	1.1518	0.

Table B.4-1. (continued)

(c) at 9 inches from the slot

$v_s = 0.1$ ft/s		$v_s = 0.2$ ft/s		$v_s = 0.4$ ft/s		$v_s = 0.8$ ft/s	
y inches	u ft/s	y inches	u ft/s	y inches	u ft/s	y inches	u ft/s
0.0138	53.976	0.0138	51.243	0.0138	39.582	0.0138	27.449
0.0155	55.281	0.0176	55.057	0.0175	42.691	0.0173	29.648
0.0175	57.938	0.0234	59.604	0.0243	48.643	0.0205	31.695
0.0204	60.085	0.0288	62.141	0.0301	51.360	0.0264	35.436
0.0229	61.645	0.0334	63.372	0.0358	53.644	0.0350	38.207
0.0280	64.039	0.0438	66.057	0.0477	56.546	0.0441	41.553
0.0332	65.264	0.0535	67.788	0.0582	58.764	0.0541	44.295
0.0381	66.466	0.0640	69.337	0.0680	60.375	0.0649	46.878
0.0424	66.940	0.0745	70.579	0.0833	62.967	0.0801	49.484
0.0485	68.113	0.0858	71.530	0.0990	64.472	0.0945	51.657
0.0568	69.380	0.1082	72.734	0.1188	65.457	0.1098	53.742
0.0670	70.287	0.1239	73.263	0.1438	67.382	0.1243	55.183
0.0782	70.848	0.1573	74.050	0.1694	67.619	0.1391	56.864
0.0882	71.515	0.1946	73.919	0.1938	68.325	0.1593	58.362
0.1038	72.395	0.2088	73.263	0.2179	68.325	0.1799	59.692
0.1177	72.722	0.2185	72.999	0.2330	67.382	0.1823	58.496
0.1335	72.722	0.2395	72.602	0.2640	67.619	0.1914	58.898
0.1480	72.940	0.2732	71.665	0.2838	66.906	0.2000	59.031
0.1636	72.940	0.3036	70.988	0.3344	65.212	0.2091	59.823
0.1785	72.504	0.3414	69.476	0.3588	64.966	0.2191	60.215
0.2074	72.286	0.3626	69.476	0.3829	63.974	0.2299	60.605
0.2229	71.846	0.3784	68.496	0.4032	63.473	0.2451	61.121
0.2320	70.287	0.3985	67.503	0.4283	62.712	0.2748	61.632
0.2432	69.948	0.4090	66.503	0.4440	61.684	0.3041	61.632
0.2827	68.692	0.4308	65.470	0.4638	60.375	0.3243	61.887
0.3130	67.294	0.4532	64.279	0.4888	59.306	0.3449	61.377
0.3286	66.346	0.4689	63.676	0.5144	57.388	0.3588	61.887
0.3435	65.867	0.5023	61.517	0.5388	56.404	0.3800	61.377
0.4018	63.168	0.5214	60.248	0.5629	54.823	0.3891	61.121
0.4332	61.000	0.5396	59.116	0.5830	53.345	0.4395	60.085
0.4488	60.085	0.5438	58.294	0.6140	51.516	0.4693	58.764
0.4627	59.289	0.5685	56.953	0.6338	50.100	0.5043	57.413
0.4930	57.112	0.5895	55.754	0.6508	48.313	0.5323	56.029
0.5235	55.424	0.6232	53.094	0.6844	46.629	0.5591	54.323
0.5679	52.338	0.6389	51.991	0.7088	44.882	0.5951	52.858
0.5820	50.333	0.6723	50.100	0.7408	43.064	0.6393	50.114
0.6032	48.897	0.6914	48.534	0.7632	41.553	0.6743	47.872
0.6485	46.063	0.7284	46.502	0.7883	39.582	0.7314	45.346
0.6935	43.042	0.7690	43.499	0.8040	38.766	0.7699	43.037
0.7224	40.968	0.7908	41.686	0.8238	37.721	0.7851	41.929
0.7535	38.374	0.8132	39.790	0.8488	35.313	0.8148	39.816
0.7932	35.591	0.8623	36.763	0.8744	33.928	0.8643	36.311
0.8227	33.292	0.8996	34.036	0.8988	31.987	0.9054	34.081
0.8530	31.332	0.9591	30.443	0.9229	30.451	0.9783	29.114
0.8835	29.778	1.0538	25.621	0.9733	26.822	1.0865	21.700
0.9279	26.395	1.1736	10.763	1.0732	19.588	1.1684	16.809

Table B.4-1. (continued)

(d) at 10 inches from the slot

$v_s = 0.1$ ft/s		$v_s = 0.2$ ft/s		$v_s = 0.4$ ft/s		$v_s = 0.8$ ft/s	
y inches	u ft/s	y inches	u ft/s	y inches	u ft/s	y inches	u ft/s
0.0138	46.158	0.0138	42.690	0.0138	33.848	0.0138	21.525
0.0179	50.499	0.0172	44.862	0.0170	35.679	0.0187	22.962
0.0236	54.496	0.0206	48.101	0.0230	40.287	0.0254	26.813
0.0322	57.099	0.0245	50.825	0.0296	43.332	0.0315	29.099
0.0415	59.045	0.0297	53.112	0.0375	46.176	0.0354	30.177
0.0513	60.530	0.0383	55.590	0.0488	48.528	0.0460	33.202
0.0671	61.979	0.0510	58.235	0.0636	51.084	0.0556	35.527
0.0821	63.268	0.0604	59.580	0.0798	53.220	0.0662	37.282
0.1068	64.280	0.0702	60.635	0.0992	55.560	0.0817	40.170
0.1319	64.780	0.0849	61.672	0.1240	57.392	0.0975	42.490
0.1563	64.905	0.0999	62.817	0.1486	59.166	0.1243	45.045
0.1819	65.029	0.1149	63.570	0.1993	60.233	0.1519	47.798
0.2071	64.905	0.1299	64.190	0.2498	61.280	0.1978	50.717
0.2315	64.280	0.1453	64.804	0.2642	60.233	0.2004	49.763
0.2471	63.268	0.1602	65.170	0.3136	60.233	0.2110	50.083
0.2718	62.498	0.1836	65.170	0.3643	59.301	0.2206	50.717
0.2969	61.979	0.2202	65.654	0.3746	59.435	0.2312	51.032
0.3213	61.061	0.2352	64.436	0.3825	59.435	0.2467	51.962
0.3469	60.128	0.2649	63.942	0.3938	59.166	0.2625	52.725
0.3721	59.181	0.2949	63.694	0.4086	58.626	0.2893	52.877
0.3965	58.496	0.3486	62.691	0.4248	58.081	0.3169	53.775
0.4121	57.662	0.3852	61.414	0.4442	57.530	0.3628	54.072
0.4271	56.816	0.3960	61.026	0.4690	56.974	0.3804	54.366
0.4518	55.668	0.4054	60.766	0.4936	56.130	0.3910	54.072
0.4769	54.791	0.4299	59.713	0.5443	54.403	0.4112	53.775
0.5013	53.298	0.4599	58.642	0.5525	53.518	0.4267	54.072
0.5269	52.073	0.4749	58.099	0.5638	52.920	0.4693	52.877
0.5521	50.818	0.4903	57.413	0.5786	52.315	0.4969	52.877
0.5765	50.017	0.5052	56.579	0.5948	51.549	0.5428	51.344
0.5821	48.544	0.5286	55.874	0.6142	50.457	0.5504	51.032
0.5971	46.852	0.5652	54.292	0.6390	49.823	0.5610	50.875
0.6218	45.807	0.5852	52.511	0.6636	48.199	0.5812	50.083
0.6469	45.096	0.6149	50.668	0.7143	45.830	0.5967	49.603
0.6713	43.641	0.6299	49.880	0.7280	45.130	0.6125	48.954
0.6969	42.136	0.6603	48.430	0.7346	45.130	0.6669	46.785
0.7221	40.575	0.6986	46.425	0.7425	44.420	0.7237	45.398
0.7465	39.363	0.7352	44.329	0.7538	44.060	0.7365	44.689
0.7563	38.743	0.7433	44.329	0.7686	43.332	0.7510	43.968
0.7721	37.257	0.7752	42.130	0.7848	42.591	0.7712	42.677
0.7871	36.380	0.7899	41.371	0.8042	41.837	0.8025	41.346
0.8118	35.024	0.8049	40.205	0.8290	40.680	0.8293	40.566
0.8369	34.090	0.8349	38.390	0.8536	38.675	0.8569	38.547
0.8613	32.390	0.8886	36.048	0.9043	35.679	0.9028	36.415
0.8869	31.119	0.9252	33.306	0.9548	32.894	0.9531	33.680
0.9121	30.064	1.1063	24.540	1.0455	27.637	1.0522	29.099
0.9365	28.408	1.1800	22.519	1.1430	21.108	1.1479	24.962

Table B.4-1. (continued)

(e) at 11 inches from the slot

$v_s = 0.1$ ft/s		$v_s = 0.2$ ft/s		$v_s = 0.4$ ft/s		$v_s = 0.8$ ft/s	
y inches	u ft/s	y inches	u ft/s	y inches	u ft/s	y inches	u ft/s
0.0138	41.384	0.0138	-37.766	0.0138	29.303	0.0138	16.961
0.0214	47.150	0.0174	40.986	0.0166	30.369	0.0310	23.987
0.0302	50.359	0.0215	44.329	0.0208	32.883	0.0367	25.284
0.0408	52.441	0.0249	46.082	0.0256	35.889	0.0455	27.115
0.0525	54.292	0.0285	47.102	0.0283	37.408	0.0577	29.378
0.0617	55.046	0.0345	48.756	0.0356	39.071	0.0718	31.479
0.0711	55.936	0.0445	50.980	0.0444	41.823	0.0922	34.391
0.0822	56.812	0.0545	52.360	0.0537	43.133	0.1177	37.503
0.0942	57.532	0.0639	53.409	0.0696	45.466	0.1429	39.576
0.1057	57.960	0.0747	54.438	0.0851	47.183	0.1672	40.965
0.1159	58.102	0.0896	55.304	0.1044	48.839	0.1934	42.872
0.1272	58.525	0.1042	56.579	0.1303	51.686	0.2105	42.685
0.1381	58.666	0.1356	57.688	0.1538	51.993	0.2227	43.794
0.1483	58.806	0.1641	58.371	0.1796	52.601	0.2368	44.157
0.1810	59.086	0.1793	58.777	0.1816	54.530	0.2572	44.875
0.1952	58.385	0.1940	59.046	0.1933	52.902	0.2827	45.931
0.2058	58.243	0.2195	58.099	0.2094	51.993	0.3079	46.793
0.2267	57.960	0.2397	58.235	0.2187	52.601	0.3322	47.640
0.2472	57.960	0.2692	58.099	0.2346	53.500	0.3584	47.974
0.2809	57.389	0.2845	57.963	0.2501	53.796	0.4051	48.306
0.3031	57.101	0.3147	57.826	0.2694	54.384	0.4168	48.471
0.3224	56.667	0.3443	57.413	0.3188	54.384	0.4372	47.974
0.3460	55.936	0.3590	57.136	0.3446	54.091	0.4627	48.306
0.3588	56.229	0.3624	57.688	0.3733	54.676	0.4879	48.471
0.3752	55.641	0.3699	57.136	0.3894	54.530	0.5122	48.306
0.4161	54.444	0.3795	56.858	0.4146	54.091	0.5384	47.807
0.4392	53.220	0.3995	56.439	0.4301	53.500	0.5851	47.303
0.4609	52.598	0.4492	55.304	0.4753	52.902	0.6072	46.622
0.4831	51.650	0.4806	54.146	0.4988	52.145	0.6327	45.931
0.5140	50.359	0.5091	53.261	0.5246	51.686	0.6579	45.230
0.5364	49.368	0.5243	52.812	0.5358	51.377	0.6822	44.875
0.5558	48.357	0.5897	50.039	0.5687	50.282	0.7084	44.338
0.5767	47.671	0.6192	49.080	0.6194	48.512	0.7551	43.058
0.5972	46.623	0.6506	47.438	0.6453	47.518	0.7627	42.685
0.6207	45.551	0.7224	45.214	0.6688	46.844	0.7768	41.929
0.6422	44.453	0.7495	43.243	0.6946	45.815	0.7972	41.929
0.6633	43.328	0.7689	42.317	0.7406	44.405	0.8227	40.573
0.6840	42.754	0.7946	41.371	0.7587	43.864	0.8479	39.978
0.7188	41.384	0.8092	40.597	0.7746	42.948	0.8722	39.170
0.7352	40.580	0.8245	39.809	0.8094	41.441	0.8984	38.137
0.7761	38.498	0.8547	38.596	0.8353	40.273	0.9451	36.202
0.7992	37.192	0.8691	37.766	0.8588	39.677	0.9949	34.391
0.8209	36.296	0.8843	39.409	0.8846	38.248	1.0489	31.983
0.8431	35.377	0.8990	36.048	0.9338	36.329	1.0975	29.917
0.8624	34.193	1.0202	31.346	1.0096	32.883	1.1906	26.519
0.8860	33.463	1.2005	20.299	1.1544	25.843		

Table B.4-1. (concluded)

(f) at 12 inches from the slot

$v_s = 0.1$ ft/s		$v_s = 0.2$ ft/s		$v_s = 0.4$ ft/s		$v_s = 0.8$ ft/s	
y inches	u ft/s	y inches	u ft/s	y inches	u ft/s	y inches	u ft/s
0.0138	36.178	0.0138	34.233	0.0138	24.532	0.0138	14.934
0.0168	37.448	0.0169	35.816	0.0173	26.398	0.0176	15.965
0.0194	39.867	0.0212	39.596	0.0212	29.244	0.0234	16.934
0.0234	42.697	0.0258	41.927	0.0256	30.826	0.0355	20.352
0.0274	44.309	0.0310	43.229	0.0295	32.331	0.0561	24.604
0.0319	45.523	0.0351	44.314	0.0362	33.768	0.0755	27.071
0.0373	46.706	0.0408	45.374	0.0432	35.372	0.0955	28.782
0.0422	47.533	0.0500	46.918	0.0544	36.906	0.1157	30.917
0.0482	48.506	0.0600	47.920	0.0642	38.584	0.1299	32.179
0.0524	48.986	0.0709	49.063	0.0754	39.796	0.1513	33.868
0.0621	49.931	0.0816	49.704	0.0939	41.739	0.1753	35.251
0.0720	50.551	0.0956	50.963	0.1149	42.862	0.1788	34.796
0.0818	51.468	0.1101	51.734	0.1372	43.956	0.1826	34.796
0.0928	51.769	0.1248	52.039	0.1626	45.024	0.1884	35.251
0.1157	53.107	0.1400	52.644	0.1877	45.895	0.2005	35.700
0.1315	53.253	0.1550	53.391	0.2132	47.088	0.2211	37.014
0.1464	53.690	0.1713	53.539	0.2292	47.423	0.2405	38.284
0.1625	53.835	0.1854	53.834	0.2589	47.921	0.2605	38.698
0.1927	53.835	0.2751	52.191	0.2799	47.756	0.2807	39.512
0.2414	53.835	0.2898	52.644	0.3022	47.756	0.2949	39.914
0.2578	53.107	0.3050	52.795	0.3276	47.756	0.3163	40.704
0.2807	53.107	0.3200	52.795	0.3527	47.589	0.3403	41.093
0.2965	52.812	0.4050	52.795	0.3782	48.086	0.3684	42.052
0.3114	52.516	0.4266	52.343	0.4092	48.740	0.3805	42.241
0.3275	52.368	0.4551	52.039	0.4389	48.414	0.4011	42.429
0.3577	51.468	0.4850	51.273	0.4599	47.756	0.4205	42.988
0.4064	50.858	0.5163	50.495	0.4822	47.423	0.4405	42.988
0.4268	50.858	0.5362	49.864	0.5582	46.239	0.4607	43.357
0.4607	49.931	0.5408	49.545	0.5792	45.723	0.4749	43.173
0.4914	48.506	0.5501	49.224	0.5904	45.375	0.4963	43.357
0.5377	47.359	0.5650	48.739	0.6299	44.315	0.5203	43.357
0.5968	45.007	0.5859	48.249	0.6522	43.956	0.5505	42.988
0.6078	43.956	0.5966	47.920	0.6776	42.862	0.5711	42.988
0.6465	42.515	0.6251	47.087	0.7282	42.116	0.5905	42.988
0.6775	41.775	0.6398	46.579	0.7345	42.677	0.6307	42.429
0.7077	40.832	0.6700	45.894	0.7594	41.739	0.6449	42.052
0.7564	38.676	0.7004	44.847	0.7692	40.584	0.7188	41.671
0.7671	38.676	0.7188	44.493	0.7989	40.584	0.7405	41.237
0.7868	37.862	0.7308	44.135	0.8422	38.584	0.8005	39.713
0.8207	36.178	0.7407	43.229	0.8676	37.544	0.8563	38.698
0.8365	35.745	0.7759	42.676	0.9182	36.037	0.9055	37.442
0.8514	34.862	0.8151	41.165	0.9444	35.147	0.9566	36.143
0.8977	33.495	0.8763	38.788	0.9939	34.002	1.0085	34.796
0.9464	31.331	0.8904	37.962	1.0819	30.826	1.0556	33.394
1.0381	27.912	1.0005	34.464	1.1778	28.698	1.1061	31.931
1.1829	23.017	1.1008	30.567			1.2034	28.223

(a) at 7 inches from the slot

(b) at 8 inches from the slot

[illegible]

Table B.4-2. (continued)

(c) at 9 inches from the slot

$v_S = 0.1 \text{ ft/s}$		$v_S = 0.2 \text{ ft/s}$		$v_S = 0.4 \text{ ft/s}$		$v_S = 0.8 \text{ ft/s}$	
$\phi_S/(\phi_T - \phi_S) = .105$		$\phi_S/(\phi_T - \phi_S) = .239$		$\phi_S/(\phi_T - \phi_S) = .577$		$\phi_S/(\phi_T - \phi_S) = 1.5$	
Y in.	$(\phi_S - \phi)/\phi_S$	Y in.	$(\phi_S - \phi)/\phi_S$	Y in.	$(\phi_S - \phi)/\phi_S$	Y in.	$(\phi_S - \phi)/\phi_S$
0.0082	0.0242	0.0082	0.1565	0.0072	0.0150	0.0077	0.1082
0.0126	0.0212	0.0113	0.2540	0.0102	0.1708	0.0114	0.2277
0.0181	0.2016	0.0143	0.2978	0.0145	0.2662	0.0153	0.3053
0.0237	0.3658	0.0194	0.3826	0.0192	0.3265	0.0239	0.3447
0.0287	0.4274	0.0248	0.4119	0.0247	0.3333	0.0291	0.3465
0.0341	0.4750	0.0299	0.4558	0.0296	0.3768	0.0342	0.3890
0.0387	0.5051	0.0392	0.5041	0.0396	0.4249	0.0437	0.4379
0.0431	0.5271	0.0493	0.5457	0.0543	0.4823	0.0541	0.4556
0.0539	0.5305	0.0589	0.4866	0.0695	0.5175	0.0642	0.4950
0.0637	0.6041	0.0690	0.5919	0.0904	0.5591	0.0841	0.5545
0.0739	0.6284	0.0848	0.6325	0.1161	0.6041	0.1096	0.5782
0.0835	0.6306	0.1003	0.6626	0.1680	0.6566	0.1344	0.6058
0.1031	0.7045	0.1203	0.7043	0.2201	0.7085	0.1843	0.6485
0.1290	0.7264	0.1469	0.7219	0.2992	0.7412	0.2871	0.7401
0.1532	0.7730	0.2056	0.7760	0.3969	0.8019	0.3868	0.7862
0.2045	0.8335	0.3019	0.8245	0.4934	0.8129	0.4833	0.8266
0.3044	0.8505	0.4485	0.8713				
0.5026	0.9067						

(d) at 10 inches from the slot

$v_S = 0.1 \text{ ft/s}$		$v_S = 0.2 \text{ ft/s}$		$v_S = 0.4 \text{ ft/s}$		$v_S = 0.8 \text{ ft/s}$	
$\phi_S/(\phi_T - \phi_S) = .121$		$\phi_S/(\phi_T - \phi_S) = .277$		$\phi_S/(\phi_T - \phi_S) = .730$		$\phi_S/(\phi_T - \phi_S) = 1.68$	
Y in.	$(\phi_S - \phi)/\phi_S$	Y in.	$(\phi_S - \phi)/\phi_S$	Y in.	$(\phi_S - \phi)/\phi_S$	Y in.	$(\phi_S - \phi)/\phi_S$
0.0068	0.0011	0.0071	0.1085	0.0087	0.2137	0.0072	0.0640
0.0109	0.0109	0.0095	0.1077	0.0139	0.2889	0.0109	0.1685
0.0153	0.0465	0.0136	0.3336	0.0182	0.3481	0.0145	0.1762
0.0191	0.2421	0.0205	0.3902	0.0274	0.4370	0.0205	0.2667
0.0240	0.3512	0.0260	0.4485	0.0378	0.4530	0.0307	0.3367
0.0295	0.3916	0.0326	0.4893	0.0519	0.5178	0.0404	0.3810
0.0389	0.4746	0.0390	0.5299	0.0804	0.5891	0.0558	0.4238
0.0486	0.5346	0.0487	0.5732	0.1124	0.6335	0.0729	0.4691
0.0591	0.5802	0.0594	0.5882	0.1579	0.6981	0.0905	0.5048
0.0742	0.6050	0.0699	0.6106	0.2117	0.7417	0.1180	0.5544
0.0887	0.6529	0.0825	0.6395	0.2611	0.7703	0.1691	0.6161
0.1037	0.6700	0.0986	0.6789	0.3302	0.8055	0.2408	0.6729
0.1188	0.6905	0.1180	0.6994	0.4514	0.8407	0.3418	0.7476
0.1336	0.7095	0.1444	0.7331			0.4058	0.7725
0.1546	0.7410	0.2635	0.8079			0.4651	0.7876
0.1742	0.7501	0.4614	0.8772				
0.1992	0.7719						
0.2486	0.8113						
0.2994	0.8349						
0.3747	0.8518						

(e) at 11 inches from the slot

$v_S = 0.1 \text{ ft/s}$		$v_S = 0.2 \text{ ft/s}$		$v_S = 0.4 \text{ ft/s}$		$v_S = 0.8 \text{ ft/s}$	
$\phi_S/(\phi_T - \phi_S) = .151$		$\phi_S/(\phi_T - \phi_S) = .312$		$\phi_S/(\phi_T - \phi_S) = .805$		$\phi_S/(\phi_T - \phi_S) = 2.33$	
y in.	$(\phi_S - \phi)/\phi_S$	y in.	$(\phi_S - \phi)/\phi_S$	y in.	$(\phi_S - \phi)/\phi_S$	y in.	$(\phi_S - \phi)/\phi_S$
0.0081	0.0307	0.0079	0.2331	0.0079	0.0863	0.0073	0.0932
0.0133	0.2328	0.0097	0.2151	0.0118	0.1673	0.0114	0.1940
0.0182	0.3631	0.0154	0.3029	0.0172	0.2851	0.0157	0.2569
0.0247	0.4427	0.0221	0.3692	0.0226	0.3714	0.0200	0.3451
0.0303	0.4863	0.0290	0.4453	0.0274	0.3653	0.0244	0.3336
0.0383	0.5106	0.0399	0.5027	0.0353	0.4122	0.0347	0.3703
0.0493	0.5832	0.0496	0.5362	0.0424	0.4449	0.0450	0.4249
0.0595	0.6022	0.0610	0.5807	0.0532	0.4866	0.0593	0.4710
0.0686	0.6322	0.0757	0.6235	0.0685	0.5419	0.0853	0.5192
0.0785	0.6624	0.0910	0.6416	0.0834	0.5655	0.1104	0.5634
0.0887	0.6653	0.1155	0.6945	0.1041	0.6038	0.1609	0.6204
0.1041	0.6956	0.1411	0.7265	0.1342	0.6281	0.2348	0.6851
0.1189	0.7114	0.1896	0.7528	0.1843	0.6804	0.3109	0.7088
0.1390	0.7373	0.2896	0.8238	0.2589	0.7348	0.3911	0.7565
0.1641	0.7865	0.3929	0.8522	0.3314	0.7779	0.4767	0.7943
0.1897	0.7742	0.4800	0.8569	0.4226	0.8027		
0.2390	0.7954						
0.3401	0.8486						
0.4393	0.8762						

(f) at 12 inches from the slot

$v_S = 0.1 \text{ ft/s}$		$v_S = 0.2 \text{ ft/s}$		$v_S = 0.4 \text{ ft/s}$		$v_S = 0.8 \text{ ft/s}$	
$\phi_S/(\phi_T - \phi_S) = .156$		$\phi_S/(\phi_T - \phi_S) = .341$		$\phi_S/(\phi_T - \phi_S) = .953$		$\phi_S/(\phi_T - \phi_S) = 2.71$	
y in.	$(\phi_S - \phi)/\phi_S$	y in.	$(\phi_S - \phi)/\phi_S$	y in.	$(\phi_S - \phi)/\phi_S$	y in.	$(\phi_S - \phi)/\phi_S$
0.0077	0.1648	0.0078	0.1741	0.0069	0.1429	0.0085	0.1022
0.0112	0.1883	0.0117	0.2491	0.0103	0.1933	0.0125	0.2008
0.0160	0.3301	0.0153	0.3726	0.0135	0.2118	0.0165	0.2295
0.0206	0.4141	0.0219	0.4369	0.0186	0.3390	0.0225	0.2787
0.0277	0.4787	0.0276	0.4710	0.0266	0.4117	0.0287	0.2884
0.0333	0.5307	0.0337	0.4972	0.0373	0.4506	0.0340	0.3503
0.0437	0.5607	0.0391	0.5240	0.0466	0.5242	0.0452	0.3799
0.0535	0.6137	0.0470	0.5597	0.0619	0.5403	0.0560	0.4262
0.0640	0.6044	0.0571	0.5690	0.0783	0.5919	0.0745	0.4721
0.0730	0.6404	0.0669	0.6027	0.1026	0.6187	0.0954	0.5064
0.0836	0.6664	0.0824	0.6313	0.1277	0.6547	0.1253	0.5656
0.0928	0.6975	0.0961	0.6449	0.1536	0.6763	0.1605	0.5902
0.1024	0.6976	0.1223	0.6935	0.2031	0.7266	0.1946	0.6346
0.1180	0.7276	0.1477	0.7227	0.2577	0.7324	0.2452	0.6752
0.1379	0.7353	0.1775	0.7451	0.3132	0.7671	0.3172	0.7036
0.1628	0.7866	0.2273	0.7754	0.3661	0.7865	0.4212	0.7564
0.1881	0.7921	0.3040	0.8016	0.4559	0.8245	0.5121	0.7869
0.2384	0.7987	0.4058	0.8330				
0.2879	0.8148	0.4998	0.8545				

the helium-air mixture can all contribute to discrepancies. However, the mass balance gives an indication that the concentration values may be in error. Since the measurement of both the concentration profile and the helium flux through the wall depends upon obtaining and analysing a sample, there is no way of knowing whether the profile values of concentration are too small or too large.

One cause of inaccuracy in the concentration measurement may be the effect of the internal geometry of the probe. Alpinieri [2] has reported that the actual geometry of the probe could significantly alter the value of the concentration measured. It may be that a phenomenon known as "preferential sampling" is responsible for this: the lighter molecules of helium get drawn into the probe in preference to the heavier ones of air, and this process is facilitated more by certain probe geometries than others. Investigation of this probe-geometry effect requires further study.

Reprinted from

J. Heat Mass Transfer. Vol. 9, pp. 829-834. Pergamon Press 1966. Printed in Great Britain

**HEAT TRANSFER ACROSS A TURBULENT BOUNDARY LAYER:
APPLICATION OF A PROFILE METHOD TO THE
STEP-WALL-TEMPERATURE PROBLEM**

S. V. PATANKAR

Mechanical Engineering Department, Imperial College of Science and Technology, London

(Received 7 March 1966 and in revised form 21 April 1966)



PERGAMON PRESS
OXFORD NEW YORK LONDON PARIS

HEAT TRANSFER ACROSS A TURBULENT BOUNDARY LAYER: APPLICATION OF A PROFILE METHOD TO THE STEP-WALL-TEMPERATURE PROBLEM

S. V. PATANKAR

Mechanical Engineering Department, Imperial College of Science and Technology, London

(Received 7 March 1966 and in revised form 21 April 1966)

NOMENCLATURE*

a ,	a constant;
b ,	exponent in the temperature profile;
c_f ,	local drag coefficient;
E ,	a constant;
K ,	a constant;
N_{Pr} ,	laminar Prandtl number;
$N_{Pr,t}$,	turbulent Prandtl number;
N_{St} ,	Stanton number;
S_T ,	non-dimensional heat-transfer coefficient (12);
T ,	temperature;
u^+ ,	non-dimensional velocity along the wall;
x^+ ,	non-dimensional distance along the wall;
X ,	non-dimensional distance (5);
y^+ ,	non-dimensional distance normal to the wall;
δ ,	non-dimensional thickness of the temperature profile;
θ ,	non-dimensional temperature (10);
ξ ,	non-dimensional distance (4);
ϕ ,	turbulent contribution to the non-dimensional total viscosity.

Subscripts

G ,	main stream;
S ,	wall.

1. INTRODUCTION

FOLLOWING a method put forward by Spalding [1], Gardner and Kestin [2] obtained exact numerical solutions of the partial differential equation for heat transfer through uniform-property "universal" turbulent boundary layer, for various Prandtl numbers. In a later paper [3], Spalding generalized the solutions in reference [2] for non-unity turbulent Prandtl number and also gave exact analytical solutions for the case of $N_{Pr} = N_{Pr,t}$ with a power-law velocity profile.

* Nomenclature is as in reference [3] where further details can be found. The numbers in parentheses denote the defining equations.

It is the purpose of the present communication to present a profile method for solving the same problem and to examine the extent to which the resulting approximate solutions agree with the exact solutions. The special feature of this method is that two integral equations are used in conjunction with a power-law temperature profile, the two free parameters in the profile being the exponent and the "thickness" of the profile.

The step-wall-temperature problem of a turbulent boundary layer is taken here to illustrate the method; however, the method can also be applied to many problems of the boundary-layer type in fluid mechanics, heat conduction and diffusion.

2. DESCRIPTION OF THE METHOD

Here all the equations will be given in the generalized form according to reference [3].

2.1 The partial differential equation

The temperature T in a uniform-property universal turbulent boundary layer has been shown in reference [3] to be governed by:

$$\frac{\partial T}{\partial(x^+/N_{Pr,t})} = \frac{1}{u^+(1+\phi)} \frac{\partial}{\partial u^+} \times \left\{ \frac{N_{Pr,t}/N_{Pr} + \phi}{1+\phi} \frac{\partial T}{\partial u^+} \right\}. \quad (1)$$

Here ϕ is to be obtained from the universal velocity profile by the relation:

$$1 + \phi = \frac{dy^+}{du^+}. \quad (2)$$

Equation (1) represents the conservation of enthalpy in the boundary layer for uniform values of density and specific heat (i.e. small velocities and temperature differences). The two fundamental assumptions made in obtaining this equation are: that there is a universal relationship between u^+ and y^+ ; and that the shear stress is uniform across the boundary layer. These assumptions are known to hold quite well, at least in the part of the boundary layer nearer to the wall and,

therefore, equation (1) can be expected to give correct predictions of heat transfer whenever the thermal boundary layer is appreciably thinner than the velocity boundary layer.

Equation (1) can be rewritten in a more convenient form, in terms of a new independent variable ξ , as:

$$\frac{\partial T}{\partial X} = \frac{1}{u^+(N_{Pr,t}/N_{Pr} + \phi)} \frac{\partial^2 T}{\partial \xi^2}; \quad (3)$$

where

$$\xi \equiv \int_0^{u^+} \frac{1 + \phi}{(N_{Pr,t}/N_{Pr} + \phi)} du^+, \quad (4)$$

and

$$X \equiv x^+/N_{Pr,t}. \quad (5)$$

Here we consider the step-wall-temperature problem; so the boundary conditions are:

$$\left. \begin{aligned} X = 0, \quad \xi \geq 0 \\ \text{all } X, \quad \xi \rightarrow \infty \end{aligned} \right\} \quad T = T_G; \quad (6)$$

$$X > 0, \quad \xi = 0: \quad T = T_s. \quad (7)$$

2.2 Integral equations

From the partial differential equation (3), we form two integral equations by integration with respect to ξ from 0 to ∞ , after multiplication respectively by unity and T , as weighting functions. The resulting integral equations are:

$$\frac{d}{dX} \int_0^\infty u^+(N_{Pr,t}/N_{Pr} + \phi) \theta d\xi = -\left(\frac{\partial \theta}{\partial \xi}\right)_s; \quad (8)$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dX} \int_0^\infty u^+(N_{Pr,t}/N_{Pr} + \phi) \theta^2 d\xi &= -\left(\frac{\partial \theta}{\partial \xi}\right)_s \\ &\quad - \int_0^\infty \left(\frac{\partial \theta}{\partial \xi}\right)^2 d\xi; \quad (9) \end{aligned}$$

where θ is the dimensionless temperature defined by:

$$\theta \equiv \frac{T - T_G}{T_s - T_G}. \quad (10)$$

In order to solve these integral equations we need an assumption for the temperature profile.

2.3 The temperature profile

We shall use the following two-parameter temperature profile.

$$\left. \begin{aligned} \xi \leq \delta: \quad \theta &= (1 - \xi/\delta)^b, \\ \xi > \delta: \quad \theta &= 0. \end{aligned} \right\} \quad (11)$$

Here the two parameters are: b , the exponent; and δ which

has the significance of the "thickness" of the temperature profile.

The dimensionless heat-transfer coefficient, S_T , defined by:

$$S_T \equiv \frac{N_{St} N_{Pr}}{(c_f/2)^{1/2}}, \quad (12)$$

can be shown to be equal to $-(N_{Pr}/N_{Pr,t})(\partial \theta / \partial \xi)_s$. Therefore, in terms of the profile parameters, we obtain:

$$S_T = \left(\frac{N_{Pr}}{N_{Pr,t}} \right) \cdot \frac{b}{\delta}. \quad (13)$$

3. SOLUTION OF THE INTEGRAL EQUATIONS

Here we apply the method outlined above to the following three cases:

- (i) laminar velocity profile: $y^+ = u^+$; $N_{Pr} = N_{Pr,t}$;
- (ii) seventh-power-law velocity profile: $y^+ = au^{+7}$; $N_{Pr} = N_{Pr,t}$; and
- (iii) velocity profile given by the universal law of the wall due to Spalding [4]; $N_{Pr}/N_{Pr,t} = 0.71, 1, 7, 30, 100, 1000$.

Exact analytical solutions for the first two cases are given in reference [3]; for the third case, exact numerical solutions are presented in reference [2].

It can be easily seen that, when $N_{Pr} = N_{Pr,t}$, the new independent variable ξ becomes the same as u^+ .

3.1 Laminar velocity profile: $y^+ = u^+$; $N_{Pr} = N_{Pr,t}$

Here

$$1 + \phi = \frac{dy^+}{du^+} = 1. \quad (14)$$

Substituting the expression for the temperature profile and eliminating δ from the two integral equations, namely (8) and (9), we get:

$$6b^2 - 7b - 2 = 0. \quad (15)$$

This equation has the positive root $b = 1.404$; this is the relevant one in the present problem. Integration of equation (8) then yields:

$$\delta = \{1.5b(b+1)(b+2)\}^{1/3} X^{1/3}; \quad (16)$$

and

$$S_T = b/\delta = b^{2/3} \{1.5(b+1)(b+2)\}^{-1/3} X^{-1/3}. \quad (17)$$

Substituting the value of b , we get, as the final result:

$$S_T = 0.543 X^{-1/3}. \quad (18)$$

The corresponding exact solution given in reference [3] is

$$S_T = 0.53835 X^{-1/3}. \quad (19)$$

The agreement is very satisfactory.

3.2 *Seventh-power-law velocity profile*: $y^+ = au^{+7}$; $N_{Pr} = N_{Pr,t}$
Here

$$1 + \phi = \frac{dy^+}{du^+} = 7au^{+6}. \quad (20)$$

Eliminating δ from the two integral equations, we get:

$$\frac{(2b+1)(2b+2)(2b+3)\dots(2b+8)}{(b+1)(b+2)(b+3)\dots(b+8)} = \frac{2b-1}{2(b-1)}. \quad (21)$$

The relevant root of this equation is $b = 1.11$. Now the integration of (8) yields:

$$\delta = \left\{ \frac{9b(b+1)(b+2)(b+3)\dots(b+8)}{56a(7!)} \right\}^{1/9} X^{1/9}; \quad (22)$$

and

$$S_T = b/\delta =$$

$$b^{8/9} \left\{ \frac{56a(7!)}{9(b+1)(b+2)(b+3)\dots(b+8)} \right\}^{1/9} X^{-1/9}. \quad (23)$$

Substituting the value of b as obtained above, and taking $a = 2.412 \times 10^{-7}$ as used in reference [3], we get, as the final solution:

$$S_T = 0.1504X^{-1/9}. \quad (24)$$

The exact solution for this case, as given in reference [3], is

$$S_T = 0.1479X^{-1/9}. \quad (25)$$

Again, the coefficients agree within 2 per cent.

3.3 *Universal law of the wall, various Prandtl numbers*

For this case, we use the following law of the wall due to Spalding [4]:

$$y^+ = u^+ + \frac{1}{E} \left\{ e^{Ku^+} - 1 - Ku^+ - \left(\frac{Ku^+}{2!} \right)^2 - \left(\frac{Ku^+}{3!} \right)^3 - \left(\frac{Ku^+}{4!} \right)^4 \right\}, \quad (26)$$

with $K = 0.4$, and $E = 9.025$.

This gives:

$$1 + \phi = \frac{dy^+}{du^+} = 1 + \frac{K}{E} \left\{ e^{Ku^+} - 1 - Ku^+ - \frac{(Ku^+)^2}{2!} - \frac{(Ku^+)^3}{3!} \right\} \quad (27)$$

In this case, it can be seen that the exponent b does not have a constant value, but changes with X . Therefore, the two integral equations should be treated as two simultaneous ordinary differential equations with b and δ as the dependent variables. Then these can be solved by the standard techniques of numerical forward integration.

Unlike the first two cases, here it is necessary to have u^+ as a function of ξ and $N_{Pr}/N_{Pr,t}$. In the present work, this function was obtained by evaluating the quadrature in equation (4) numerically.

The solutions were obtained from $X = 1$ to $X = 10^6$ for $N_{Pr}/N_{Pr,t} = 0.71, 1, 7, 30, 100$ and 1000 . For the starting values at $X = 1$, the results of the first case (described in Section 3.1 above) were used. Thus the values of b and δ at the start of the integration were given by:

$$b = 1.404, \quad (28)$$

and

$$\delta = \{1.5b(b+1)(b+2)\}^{1/3} X^{1/3} (N_{Pr}/N_{Pr,t})^{1/3} \quad (29)$$

[This is a modified form of equation (16); the modification accounts for the fact that here $N_{Pr} \neq N_{Pr,t}$.] Since the law of the wall given by equation (26) has $y^+ = u^+$ as the asymptote for small u^+ , the results of the first case form a good starting point at low values of X .

The difference between the present results and the exact solutions is everywhere less than 2 per cent*; this can be seen from Table 1 where the results of the present computations are given with the exact values from reference [2]. Thus, by use of two integral equations it has been possible to attain good accuracy with much less computing time than was needed for the exact solutions.

4. CONCLUSIONS

(i) An approximate method for the solution of the partial differential equation for heat transfer in a uniform-property universal turbulent boundary layer has been described. The method is of profile type; two integral equations are used to obtain the two free parameters in the temperature profile.

(ii) Approximate solutions obtained by use of this method have been compared with available exact solutions. The agreement is very good.

ACKNOWLEDGEMENT

The author's thanks are due to Professor D. B. Spalding for suggesting the method and for his encouragement.

* In the case of $N_{Pr}/N_{Pr,t} = 30$, the present results differ from the exact values by about 4.5 per cent for large values of X . This is surprising in view of the good agreement for all the other values of $N_{Pr}/N_{Pr,t}$. The reason may be that the exact values from reference [2] are somewhat in error in this case. This conclusion is further supported by the comparison of Table 1 and Table 2 in reference [3]; for large values of X , the value of $(\xi - u^+)$ from Table 1 should be equal to $[(c_f/2)^{1/2}/N_{St} - 1/S_{T,1}]$ from Table 2. It can be seen that this condition is not satisfied with sufficient accuracy for only $N_{Pr}/N_{Pr,t} = 30$.

Table 1. Comparison of values of S_T from present computations with exact values from reference [2]

X	$N_{Pr}/N_{Pr,t} = 0.71$		$N_{Pr}/N_{Pr,t} = 1.0$		$N_{Pr}/N_{Pr,t} = 7.0$		$N_{Pr}/N_{Pr,t} = 30.0$		$N_{Pr}/N_{Pr,t} = 100.0$		$N_{Pr}/N_{Pr,t} = 1000.0$	
	S_T present method	S_T exact	S_T present method	S_T exact	S_T present method	S_T exact	S_T present method	S_T exact	S_T present method	S_T exact	S_T present method	S_T exact
1×10	0.2250	0.2228	0.2522	0.2498	0.4822	0.4779	0.7831	0.7763	1.1698	1.1598	2.5197	2.4984
2×10	0.1789	0.1771	0.2005	0.1987	0.3830	0.3793	0.6218	0.6158	0.9286	0.9203	1.9999	1.9807
3×10	0.1567	0.1554	0.1755	0.1742	0.3350	0.3321	0.5435	0.5387	0.8115	0.8047	1.7473	1.7317
4×10	0.1428	0.1418	0.1599	0.1589	0.3047	0.3023	0.4942	0.4901	0.7376	0.7317	1.5878	1.5742
5×10	0.1330	0.1322	0.1489	0.1481	0.2833	0.2813	0.4591	0.4555	0.6851	0.6798	1.4743	1.4620
6×10	0.1256	0.1249	0.1406	0.1399	0.2670	0.2653	0.4324	0.4292	0.6450	0.6402	1.3877	1.3764
7×10	0.1198	0.1191	0.1340	0.1335	0.2540	0.2525	0.4111	0.4082	0.6131	0.6086	1.3185	1.3080
8×10	0.1150	0.1144	0.1287	0.1282	0.2434	0.2421	0.3936	0.3910	0.5868	0.5826	1.2641	1.2515
9×10	0.1111	0.1105	0.1242	0.1238	0.2344	0.2334	0.3788	0.3765	0.5645	0.5606	1.2132	1.2038
1×10^2	0.1077	0.1072	0.1204	0.1200	0.2268	0.2259	0.3661	0.3640	0.5454	0.5418	1.1717	1.1628
2×10^2	0.0891	0.0886	0.0997	0.0993	0.1845	0.1844	0.2943	0.2934	0.4363	0.4343	0.9332	0.9260
3×10^2	0.0806	0.0801	0.0906	0.0901	0.1662	0.1665	0.2613	0.2614	0.3849	0.3841	0.8185	0.8135
4×10^2	0.0756	0.0750	0.0851	0.0847	0.1562	0.1564	0.2418	0.2427	0.3536	0.3538	0.7472	0.7435
5×10^2	0.0721	0.0715	0.0814	0.0809	0.1499	0.1499	0.2292	0.2303	0.3323	0.3332	0.6971	0.6946
6×10^2	0.0694	0.0689	0.0786	0.0782	0.1457	0.1455	0.2206	0.2217	0.3168	0.3183	0.6596	0.6581
7×10^2	0.0674	0.0668	0.0765	0.0760	0.1427	0.1423	0.2144	0.2154	0.3053	0.3071	0.6302	0.6295
8×10^2	0.0657	0.0651	0.0747	0.0742	0.1404	0.1398	0.2099	0.2107	0.2964	0.2984	0.6064	0.6064
9×10^2	0.0643	0.0637	0.0732	0.0728	0.1386	0.1379	0.2066	0.2071	0.2895	0.2915	0.5867	0.5874
1×10^3	0.0631	0.0625	0.0720	0.0715	0.1371	0.1363	0.2041	0.2042	0.2840	0.2859	0.5702	0.5714
2×10^3	0.0562	0.0556	0.0649	0.0644	0.1301	0.1288	0.1945	0.1922	0.2634	0.2628	0.4879	0.4910
3×10^3	0.0528	0.0522	0.0615	0.0610	0.1271	0.1258	0.1918	0.1887	0.2593	0.2575	0.4620	0.4638
4×10^3	0.0506	0.0501	0.0592	0.0588	0.1253	0.1240	0.1904	0.1873	0.2577	0.2557	0.4521	0.4527
5×10^3	0.0491	0.0485	0.0577	0.0573	0.1240	0.1227	0.1895	0.1866	0.2569	0.2550	0.4479	0.4478
6×10^3	0.0479	0.0473	0.0564	0.0560	0.1230	0.1217	0.1888	0.1862	0.2564	0.2545	0.4459	0.4455
7×10^3	0.0469	0.0464	0.0554	0.0550	0.1222	0.1209	0.1883	0.1859	0.2560	0.2542	0.4448	0.4444
8×10^3	0.0461	0.0456	0.0546	0.0542	0.1216	0.1202	0.1879	0.1856	0.2557	0.2540	0.4441	0.4438
9×10^3	0.0454	0.0449	0.0539	0.0535	0.1210	0.1196	0.1875	0.1853	0.2554	0.2537	0.4438	0.4434

1×10^4	0-0448	0-0443	0-0532	0-0529	0-1205	0-1191	0-1872	0-1851	0-2552	0-2535	0-4435	0-4431
2×10^4	0-0411	0-0406	0-0494	0-0491	0-1173	0-1152	0-1852	0-1828	0-2539	0-2521	0-4426	0-4420
3×10^4	0-0393	0-0388	0-0474	0-0471	0-1156	0-1129	0-1841	0-1812	0-2532	0-2513	0-4422	0-4418
4×10^4	0-0380	0-0376	0-0461	0-0458	0-1144	0-1113	0-1834	0-1800	0-2527	0-2509	0-4420	0-4418
5×10^4	0-0371	0-0367	0-0451	0-0448	0-1134	0-1102	0-1828	0-1792	0-2524	0-2506	0-4418	0-4418
6×10^4	0-0364	0-0359	0-0444	0-0441	0-1127	0-1093	0-1824	0-1786	0-2521	0-2504	0-4417	0-4417
7×10^4	0-0358	0-0354	0-0437	0-0435	0-1121	0-1087	0-1820	0-1781	0-2519	0-2502	0-4416	0-4417
8×10^4	0-0353	0-0349	0-0432	0-0429	0-1116	0-1082	0-1816	0-1778	0-2517	0-2501	0-4416	0-4417
9×10^4	0-0349	0-0344	0-0427	0-0425	0-1112	0-1078	0-1814	0-1776	0-2515	0-2500	0-4415	0-4417
1×10^5	0-0345	0-0341	0-0423	0-0421	0-1108	0-1075	0-1811	0-1774	0-2514	0-2499	0-4415	0-4416
2×10^5	0-0322	0-0317	0-0398	0-0394	0-1082	0-1060	0-1795	0-1694	0-2504	0-2490	0-4412	0-4414
3×10^5	0-0310	0-0305	0-0384	0-0381	0-1067	0-1052	0-1785	0-1692	0-2499	0-2484	0-4410	0-4412
4×10^5	0-0301	0-0297	0-0375	0-0372	0-1057	0-1044	0-1778	0-1691	0-2495	0-2479	0-4409	0-4411
5×10^5	0-0295	0-0291	0-0369	0-0365	0-1049	0-1037	0-1773	0-1689	0-2491	0-2475	0-4408	0-4410
6×10^5	0-0291	0-0286	0-0363	0-0360	0-1043	0-1031	0-1769	0-1688	0-2489	0-2472	0-4407	0-4409
7×10^5	0-0287	0-0282	0-0359	0-0355	0-1037	0-1025	0-1765	0-1687	0-2487	0-2469	0-4406	0-4408
8×10^5	0-0283	0-0278	0-0355	0-0352	0-1033	0-1019	0-1762	0-1685	0-2485	0-2467	0-4406	0-4407
9×10^5	0-0280	0-0276	0-0352	0-0348	0-1029	0-1013	0-1759	0-1684	0-2483	0-2466	0-4405	0-4407
1×10^6	0-0278	0-0273	0-0349	0-0346	0-1025	0-1008	0-1757	0-1683	0-2482	0-2464	0-4405	0-4407

REFERENCES

1. D. B. SPALDING, Heat transfer to a turbulent stream from a surface with a step-wise discontinuity in wall temperature, *International Developments in Heat Transfer*, Part II, pp. 439-446. ASME, New York (1963).
2. G. O. GARDNER and J. KESTIN, Calculation of the Spalding function over a range of Prandtl numbers, *Int. J. Heat Mass Transfer*, **6**, 289-299 (1963).
3. D. B. SPALDING, Contribution to the theory of heat transfer across a turbulent boundary layer, *Int. J. Heat Mass Transfer* **7**, 743-761 (1964).
4. D. B. SPALDING, A single formula for the law of the wall, *J. Appl. Mech.* **83**, 455-458 (1961).

A CALCULATION PROCEDURE FOR HEAT TRANSFER BY FORCED
CONVECTION THROUGH TWO-DIMENSIONAL UNIFORM-PROPERTY
TURBULENT BOUNDARY LAYERS ON SMOOTH IMPERMEABLE WALLS

S.V. Patankar and D.B. Spalding

Imperial College, London, S.W.7.

ABSTRACT

A new calculation procedure has been described for heat transfer in uniform-property turbulent boundary layers. The method consists of the solution of the integral momentum and kinetic-energy equations to yield the properties of the hydrodynamic boundary layer, followed by the solution of two similar integral equations for the thermal boundary layer. Two-component velocity and temperature profiles are used; and the shear stress and the heat flux in the boundary layer are connected with the profile parameters by use of a mixing-length hypothesis. The predictions of the theory are compared with available exact solutions, and with a number of experimental data; the agreement is satisfactory when "laminarisation" is absent.

ZUSAMMENFASSUNG

Es wird ein neues Verfahren zur Berechnung des Wärmeüberganges in turbulenten Grenzschichten mit konstanten Stoffbeiwerten beschrieben. Nach dem Verfahren ergeben sich die Größen der Strömungsgrenzschicht aus der Lösung von Impuls- und Energiesatz. Dies wird gefolgt von der Lösung zweier ähnlicher integraler Gleichungen für die thermische Grenzschicht. Geschwindigkeits- und Temperaturprofile, die sich aus zwei Anteilen zusammensetzen, werden benutzt. Schubspannung und Wärmefluss in der Grenzschicht sind mit den Profilparametern durch eine Mischungsweghypothese verbunden. Die Voraussagen der Theorie werden mit vorhandenen exakten Lösungen und mit experimentellen Daten verglichen. Die Übereinstimmung ist befriedigend, wenn keine "Laminarisierung" auftritt.

АБСТРАКТ

Представлен новый метод для расчёта теплопереноса в турбулентном пограничном слое с неизменяемыми физическими свойствами. Метод состоит из решения интегральных уравнений количества движений и кинетической энергии для гидродинамического пограничного слоя с последующим решением двух подобных уравнений для теплового пограничного слоя.

Для решения использовались двух-компонентные профили скорости и температуры, а также касательные напряжения и тепловой поток в пограничном слое, связанные с указанными профилями через гипотезу о "пути перемешивания".

Результаты расчётов, проведенных на базе предлагаемого метода, сравнены с существующими точными решениями и целым рядом экспериментальных данных, и показывают удовлетворительное согласование в том случае, когда "ламинаризация" отсутствует.

1. INTRODUCTION

1.1. The problem considered

Turbulent boundary layers occur so frequently in engineering situations, that a general theory is needed for predicting the transfer processes which they provoke. To cover the situations of main interest, this theory should apply: to flows with uniform or non-uniform density; to walls

which are either smooth or rough, and either impervious to matter or engaging in mass transfer; and to geometries which are plane, axis-symmetrical or three-dimensional in character. Moreover, the theory should permit the prediction of the transfer rates of individual components of a fluid mixture, as well as that of heat; in general, some of the components may react chemically, perhaps with the evolution of heat.

Square brackets indicate reference numbers.

I.Mech.E.

The present paper concerns an important sub-class of the general problem, characterised by: uniform density of the fluid; smooth, impermeable walls; plane geometry; and heat transfer in a fluid of uniform chemical composition (air). This

sub-class is chosen because most of the published experimental data belong to it, and because the theoretical developments are easiest. We shall present a new calculation procedure for this sub-class of processes and examine, by reference to previously published experimental data, the extent to which it gives correct predictions of the heat-transfer rates.

1.2.—Present status of theoretical knowledge

Calculation procedures for the present problem, simple though it is compared with the general one, are not highly developed. They can be divided into two groups:— those for which the heat-transfer prediction is based upon a prior prediction of the relevant properties of the hydrodynamic boundary layer; and those which avoid explicit reference to the hydrodynamic problem.

Among the first group are methods of the "analogy" type, in which the local Stanton number is taken as equal to one half of the local drag coefficient, multiplied by a function of the Prandtl number (and perhaps of the momentum-thickness Reynolds number also). Also in this group are more elaborate methods, such as that presented by one of the present authors at the 1961 Heat Transfer Conference [1]; in this, the main postulate was that the velocity and eddy-conductivity distribution at any station along the wall are the same as would prevail in a turbulent Couette flow having the same shear stress at the wall; problems involving the prescription of a non-uniform temperature or heat flux at the wall were to be solved by combination of the standard solutions presented in [1] (and in subsequent papers containing further results [2], [3], [4], [5]) coupled with the superposition techniques proposed by several authors, for example Tribus and Klein [6].

Typical, and perhaps the earliest, of the methods of the second group is the procedure recommended by Ambrok [7]; in this, a unique relationship is postulated to exist between the local Stanton number and the Reynolds number based upon the local enthalpy-flux thickness of the turbulent thermal boundary layer; this relationship can be deduced, for example, from experimental data for some simple situation, such as that of the isothermal flat plate. The Stanton-number relationship is substituted in the integral form of the Steady-Flow Energy equation, an ordinary differential equation which can then be solved in the form of a quadrature. This method can be applied to problems of non-uniform wall temperature by the superposition technique just mentioned, as has

been demonstrated by Moretti and Kays [8]; a corresponding technique for prescribed non-uniform wall heat-flux could be developed.

None of the methods so far is entirely satisfactory. A fundamental obstacle to the success of procedures of the first kind is that the methods which are available for the calculation of the hydrodynamic turbulent boundary layer, a prerequisite of the heat-transfer prediction, are themselves unreliable and fragmentary; the consequent uncertainty about the value of the drag coefficient is reflected in an equal or greater uncertainty concerning the value of the Stanton number. Even when the drag coefficient can be reliably predicted, the "analogy" formulae are known to give incorrect values of the Stanton number when the temperature or heat-flux distribution of the wall is strongly non-uniform, or when a pressure gradient exists along the main stream: the reason is that, even for fluids with Prandtl numbers near unity (i.e. gases), there is little similarity between the velocity profile and the temperature profile under these conditions. Although the method of [1] has a wider validity than have the analogy methods, it gives accurate results only when the thermal boundary layer is not thicker than that region of the velocity boundary layer, near the wall, in which the velocity profile approximates to the so-called universal "law of the wall"; this condition is not always satisfied, as has been demonstrated by comparisons of the predictions of the method with experimental data by several authors ([9], [10], [11], [12]).

Methods of the second group have, it is true, given satisfactory results in many cases; however, these methods are open to the serious objection that, in reality, the velocity distribution must influence the relation between the Stanton number and the enthalpy-flux-thickness Reynolds number; so a neglect of this influence is bound to lead to error in some circumstances. For example, the Ambrok method will assuredly fail if applied to the flow which develops when a turbulent two-dimensional jet of fluid is blown along a plane smooth wall and the bulk of the fluid in the vicinity is at rest (this is the turbulent-wall-jet situation); for, the main-stream velocity, which must be inserted in the Reynolds-number and Stanton-number expressions, is zero in this case.

It can therefore reasonably be concluded that a reliable and generally applicable method of heat-transfer prediction is wanting, even for turbulent boundary layers of uniform density.

1.3. Outline of the present work

As part of a wider programme of research into turbulent boundary layers, in progress at Imperial College, the present authors are developing calculation procedures for the problems described above; we here describe the main elements of the method (in section 2) and compare its predictions with earlier theoretical and experimental results (in section 3). While developing the theory, we have aimed above all at width of applicability, and have required that the same set of equations should be employed whether the flow is a "conventional" boundary layer or a wall-jet, whether the stream velocity is uniform or varying, and for all types of thermal boundary condition at the wall. Further, although there are restrictions to uniform density and to smooth impermeable walls in the present paper, the framework of the theory has been designed so as eventually to comprehend the general conditions enumerated in section 1.1.

The calculation procedure to be presented consists of two parts: the first provides a prediction of the salient features of the hydrodynamic boundary layer; the second yields the corresponding, and more interesting, properties of the thermal layer; the two calculations proceed simultaneously. For each part there are two ordinary first-order nonlinear differential equations to be solved numerically. Those for the hydrodynamic layer are the integral momentum equation and the integral kinetic-energy equation. For the thermal boundary layer, the two equations solved are derived from the partial differential form of the Steady-Flow Energy equation by integration with respect to the distance normal to the wall, after multiplication by a weighting function; the weighting function for the first equation is unity, so that the resulting equation is the integral Steady-Flow Energy equation; the second equation, which has no name, results from the use of temperature as the weighting function.

The differential equations contain expressions involving the variation of the shear stress and the heat flux across the turbulent boundary layer; these have been evaluated, in terms of the quantities specifying the velocity and temperature boundary layers, by the use of the Prandtl [13] mixing-length formula connecting the shear stress with the local velocity gradient. However, the mixing length has not been supposed proportional to distance from the wall except in the inner fifth (approximately) of the layer; farther from the wall, the mixing length has been

assumed proportional to the boundary layer thickness itself. This assumption is akin to one first used by Hudimoto [14]; it has been shown to agree fairly well with the available experimental data for shear-stress distributions in a survey report by one of the authors' colleagues [15].

In the turbulent part of the boundary layer (i.e. outside the laminar sub-layer), the effective Prandtl number has been taken as constant at 0.9. This is one of several assumptions which are embodied in the theory, the validity of which there has as yet been no opportunity to verify; certainly no attempt has been made so far to select the set of empirical constants which gives the best overall fit to the experimental data. It can therefore be presumed that the agreement between theory and experiment which will be demonstrated is not the best that can be achieved by a theory of the present type; this report is no more than an interim one.

2. DESCRIPTION OF THE THEORY

2.1. The equations governing the development of the hydrodynamic boundary layer

Differential equations. The derivations of the integral momentum and integral kinetic-energy equations can be found in textbooks, for example [16]. We have used these equations in the following forms:

Momentum:

$$\frac{d}{dx} \left\{ R_G (I_1 - I_2) \right\} + R_G (1 - I_2) \frac{d(\ln u_G)}{dx} = \frac{c_f}{2}, \quad (2.1)$$

Kinetic energy:

$$\frac{d}{dx} \left\{ R_G (I_1 - I_3) \right\} + 2R_G (I_1 - I_3) \frac{d(\ln u_G)}{dx} = 2\bar{s}. \quad (2.2)$$

The symbols used are defined systematically in Nomenclature, section 8 below; here we merely draw attention to the significance of the main terms. The quantity

$R_G(I_1 - I_2)$ is the momentum-thickness Reynolds number, and u_G stands for the velocity of the main stream; therefore, the two terms on the left of eq. 2.1 represent respectively the rate of growth of momentum thickness and the contribution of the longitudinal pressure gradient; the term on the right is one half of the local drag coefficient. In the second equation, the quantity $R_G(I_1 - I_3)$ is the kinetic-energy-thickness Reynolds number; the whole of the left-hand side of eq. 2.2 therefore represents the rate of increase

of the kinetic-energy deficit, while the quantity $\bar{s}$ on the right-hand side stands for the dissipation integral which will be discussed shortly.

Auxiliary relations based on a velocity-profile assumption. In order to link together the quantities appearing in the above equations, we have postulated that the velocity profiles all belong to a particular two-parameter family, described by a formula of the "wall-plus-wake" type, popularised by Coles [17]; in place of the Coles wake function, we have used the simpler linear law proposed earlier by Rotta [18] and Ross and Robertson [19]; the reasons are that the algebra is then simpler, and that we shall not in any case accept the full implications of the profile assumption (See section 2.3, below). The formula for the velocity profile is thus:

$$z = z_E \left\{ 1 + \frac{\ln \xi}{\ell'} \right\} + (1 - z_E) \xi \quad (2.3)$$

Here z is the non-dimensional velocity, ξ is the non-dimensional distance from the wall (defined so that ξ equals unity at the outer edge of the boundary layer), while z_E and ℓ' are the two parameters of the profile.

We pause to note that, in order that eq. 2.3 should reduce to the well-known "law of the wall" when ξ is small, the parameter ℓ' must be defined by:

$$\ell' = \ln \left\{ E R_G (c_f/2)^{1/2} \right\} \quad (2.4)$$

where E is a constant which we have taken as 6.5, and R_G stands for the Reynolds number based upon the whole boundary-layer thickness (it is an implication of the mixing-length hypothesis, and also a great convenience, that the turbulent boundary layer has a finite thickness, unlike the laminar boundary layer). The parameter z_E , the difference of which from unity measures the magnitude of the "wake" component of the boundary-layer profile, is for the same reason connected with the local drag coefficient and with the parameter ℓ' by the equation:

$$c_f/2 = (\kappa z_E/\ell')^2 \quad (2.5)$$

where κ is a constant which we have taken as 0.4. The value of z_E varies between zero and about unity for the majority of boundary layers not resulting from wall jets and in the absence of reverse flow; if reverse flow is present, z_E is negative (we do not discuss this case here); and immediately downstream of wall jets, z_E is greater than unity.

Taken together, equations 2.4 and 2.5 comprise a drag law, connecting the local value of the drag coefficient with the local value of the boundary-layer thickness; they thus serve the same purpose as the well-known formula of Ludwig and Tillman [20], but are valid over a much wider range.

Auxiliary relation based upon the mixing-length profile. Since the quantities I_1 , I_2 and I_3 are expressible in terms of the profile parameters z_E and ℓ' by means of the relations given in the Appendix, all that is required to complete the system of equations for the hydrodynamic problem is a relationship between the dissipation integral $\bar{s}$ and the profile parameters. This is provided by the Prandtl mixing-length formula as follows:

$$\text{Since,} \quad \bar{s} = \frac{\int_0^{y_G} \tau \frac{du}{dy} dy}{\rho u_G^3} \quad (2.6)$$

$$\text{and,} \quad \tau = \ell^2 \rho \left| \frac{du}{dy} \right| \left(\frac{du}{dy} \right) \quad (2.7)$$

$$\text{we have:} \quad \bar{s} = \frac{\int_0^{y_G} \ell^2 \left| \frac{du}{dy} \right| \left(\frac{du}{dy} \right)^2 dy}{u_G^3} \quad (2.8)$$

Here u is the velocity, y the distance from the wall, and ℓ the Prandtl mixing length; the subscript G denotes conditions at the outer boundary of the layer.

With u related to y by means of the velocity-profile assumption, only an ℓ - y relationship is needed to enable $\bar{s}$ to be evaluated. This relationship we introduce in the form:

$$\left. \begin{array}{ll} 0 \leq y \leq \lambda_G y_G / \kappa & : \ell = \kappa y \\ \lambda_G y_G / \kappa < y \leq 1 & : \ell = \lambda_G y_G \end{array} \right\} \quad (2.9)$$

where:

$$\kappa = 0.4, \quad \lambda_G = 0.075 \quad (2.10)$$

The mixing-length distribution so prescribed has been selected, as mentioned already, to fit the experimental data collected by Escudier [15]. It is however a provisional and rather crude formula; it is to be expected that research will throw further light upon the quantitative relationship between the mixing-length and the geometrical and other factors which control the turbulence structure of the boundary layer.

Although the above formulae allow an explicit formula to be derived for the dissipation integral $\bar{S}$, we have preferred to evaluate the quadrature in equation 2.8 numerically whenever $\bar{S}$ is required.

2.2. The equations governing the development of the thermal boundary layer

Differential equations. The integral form of the Steady-Flow Energy equation, for the low-speed flow of a fluid of uniform specific heat, may be found in textbooks, for example [21]. We use the equation in the form:

$$\frac{1}{(T_S - T_G)} \frac{d}{dx} \left\{ (T_S - T_G) R_G I_{\theta,1} \right\} = S_S \quad (2.11)$$

Once again, we refer the reader to section 8 for a systematic set of definitions of the symbols, but here note that $(T_S - T_G)$ is the temperature difference across the boundary layer and that $I_{\theta,1}$ is an integral expression formed from the temperature profile and from the velocity profile; so the left-hand side represents, in dimensionless form, the rate of increase of the enthalpy flux in the boundary layer. The quantity on the right-hand side is the Stanton number, S_S .

The second differential equation, which we have found to be necessary for the satisfactory prediction of the development of the thermal boundary layer, is formed by analogy with the integral kinetic-energy equation; the derivation is given in detail in the Appendix. The equation is:

$$\frac{1}{(T_S - T_G)^2} \frac{d}{dx} \left\{ (T_S - T_G)^2 R_G I_{\theta,2} \right\} = 2S_S + 2\bar{S} \quad (2.12)$$

Here it will be noted that the temperature-difference bracket is squared, the result of multiplying the partial differential equation by $(T - T_G)$ before integration. The quantity $I_{\theta,2}$ is another integral expression involving the temperature and velocity profiles; and the new expression on the right-hand side, $\bar{S}$, is an analogue of the dissipation integral, which will be discussed below.

Auxiliary relations based upon the temperature-profile assumption. In order to link together the various quantities in the above equations, we have postulated that the temperature profiles belong to a particular two-parameter family, described by the formula:

$$\left. \begin{aligned} 0 < \xi \leq \xi_D \leq 1 : 1 - \theta &= (1 - \theta_D) \left\{ 1 + \frac{\ln(\xi/\xi_D)}{\mathcal{L}'_D} \right\} \\ &+ \theta_D (\xi/\xi_D) , \\ \xi_D < \xi : \theta &= 0 \end{aligned} \right\} \quad (2.13)$$

Here θ is the non-dimensional form of the temperature, defined so that θ is unity at the wall and zero in the main stream; ξ_D is the non-dimensional form of the thickness of the thermal boundary layer, which is assumed never to exceed the thickness of the velocity boundary layer. There are thus three parameters in the temperature-profile expression, θ_D , ξ_D and $\mathcal{L}'_D$; but the last two are linked together, as we shall now show.

In order that the temperature profile should take the well-known logarithmic form in the turbulent region just outside the laminar sub-layer, it is necessary that the quantity $\mathcal{L}'_D$ should be defined by:

$$\mathcal{L}'_D \equiv \mathcal{L} + \ln \xi_D + \kappa P \quad (2.14)$$

Here $\mathcal{L}'$ is one of the parameters of the velocity profile, described above, ξ_D has also just been discussed, and the quantity P is a function of the laminar and turbulent Prandtl numbers which takes account of the resistance of the laminar sub-layer to heat transfer [5]. We have adopted for the P -function the recommendation of [22], which is founded upon the examination of a very large number of experimental data; it is:

$$\begin{aligned} P &= 9.24 \left\{ (\sigma/\sigma_{tt})^{\frac{3}{4}} - 1 \right\} \left\{ 1 + 0.28 \exp\left(\frac{-0.007\sigma}{\sigma_{tt}}\right) \right\} \\ \sigma_{tt} &= 0.9 \end{aligned} \quad (2.15)$$

Here σ is the laminar Prandtl number and σ_{tt} that of the turbulent fluid.

A further consequence of the requirement that the temperature profile should reduce to its wall-law form at small ξ , is that the Stanton number should be related to θ_D and other quantities by the equation:

$$S_S = \frac{(1 - \theta_D) \kappa (c_F/2)^{\frac{1}{2}}}{\sigma_{tt} \mathcal{L}'_D} \quad (2.16)$$

This relation can be regarded as a form of local heat-transfer law; it shows that the Stanton number is particularly strongly influenced by the profile-parameter θ_D ; indeed, when θ_D is unity, the Stanton number is zero, no matter how thin is the boundary layer.

Together with the expressions given in the Appendix, linking the I_0 integrals to profile parameters, the above equations form a complete mathematical system, except that no relation has yet been provided by means of which $\bar{S}$ can be calculated. We now turn to this matter.

The expression for the heat-flux integral. The quantity $\bar{S}$, it is shown in the Appendix, must be defined by:

$$\bar{S} \equiv \frac{-\int_0^{y_G} q'' \frac{\partial T}{\partial y} dy}{\text{cpu}_G (T_S - T_G)^2}, \quad (2.17)$$

where q'' is the heat flux at the point in question, directed normally towards the wall, and T is the local temperature. The analogy with the dissipation integral, defined by eq. 2.6, should be quite clear.

Now the heat flux is connected with the local shear stress and the gradients of temperature and velocity by the relationship (a modified form of Reynolds Analogy, in reality no more than a definition of the turbulent Prandtl number σ_{tt}):

$$q'' = \frac{c}{\sigma_{tt}} \frac{\tau}{(\partial u / \partial y)} \left(\frac{\partial T}{\partial y} \right); \quad (2.18)$$

and of course the shear stress τ is connected with the velocity gradient by means of the mixing-length formula, eq. (2.7). It is therefore possible, with the aid of the above relations and of those in the Appendix, to evaluate the quadrature in eq. 2.17, and so to obtain $\bar{S}$, in terms of the parameters of the velocity and temperature profiles. Although an explicit formula can be derived, we have preferred to integrate numerically.

2.3 The use of the equations in the prediction of the development of the hydrodynamic and thermal boundary layers.

Specification of the problem. In all the cases which will be discussed below, the main-stream velocity u_0 is specified as a function of the longitudinal distance along the wall, x ; this suffices, together with knowledge of the values of the density and viscosity of the fluid, to determine the hydrodynamic part of the calculation completely.

The thermal specification comes in one or other of two forms: in the first, the wall temperature T_S is specified as a function of x ; in the second it is the heat flux through the wall that is given. Since T_G is a constant in low-speed pro-

blems, in the first case we start from a prescription of $(T_S - T_G)$, and in the second from a prescription of the product $(T_S - T_G) S_S$.

In addition, there are usually facts about the upstream end of the boundary layers to be incorporated into the statement of the mathematical problem. For example, the turbulent layer is often preceded by a laminar layer, which terminates at the point of transition. The location of this position, together with the values of the relevant properties of the boundary layers at the transition, are inputs to the turbulent-boundary-layer theory, to be obtained from experimental data or from different branches of science.

Integration of the equations. In order to predict the downstream properties of the boundary layer, the above problem specification being complete, all that is necessary is the solution of the four simultaneous ordinary differential equations 2.1, 2.2, 2.11 and 2.12. This is a relatively straightforward matter on a digital computer; we shall therefore refer here only to non-routine aspects of the matter.

As the dependent variables of the integration, we have chosen the quantities: z_E , R_G , θ_D and ξ_D ; other choices could however have been made. The solution of the four differential equations, which manifestly do not have these quantities as the operands of the differentials, therefore proceeds via the inversion of a matrix, in which such quantities as $\frac{\partial I_1}{\partial z_E}$ and $\frac{\partial I_{0,1}}{\partial \xi_D}$ appear as elements. The usual conditions regarding the non-vanishing of certain determinants have to be complied with.

When ξ_D attains the value of unity, i. e. when the thermal boundary layer has spread so as to be co-extensive with the velocity boundary layer, it has been our practice to hold ξ_D constant at that value, and at the same time to omit one differential equation, namely that in which appears the square of the temperature. Other practices are possible here, for example the retention of the fourth equation, and the adoption of an additional degree of freedom in the temperature profile; these possibilities have not yet been systematically explored.

Modifications of the auxiliary functions. It has already been mentioned that we have not accepted all the implications of the profile assumptions; there are two reasons. The first is that, when the implications of the velocity profile are expressed as a relationship between the

shape factor of the boundary layer, the drag coefficient and the local Reynolds number, systematic deviations are observed from the available experimental data; they are caused, of course, by the fact that the linear wake component of the profile formula is but a crude approximation to reality. (Incidentally, the Coles[17] wake function does not lead to a better agreement with experiment, although it gives deviations of opposite sign.) Rather than seek a more satisfactory wake profile, which would undoubtedly have led us into greater algebraic labour, we have simply adjusted some of the constants in the algebraic expressions derived for the linear wake profile.

The second modification is associated with the fact that the determinants appearing in the solution procedure do vanish in exceptional circumstances; the reason is that our profiles are such that specification of, say, the momentum thickness and kinetic-energy thickness on rare occasions fails to lead uniquely to a pair of profile parameters, z_E and ℓ' . Once again, we have simply made minor adjustments to the constants in the expressions so as to re-introduce single-valuedness in the hydrodynamic equations. In the case of the thermal equations, we avoid entering the double-valued region by an adjustment of the S function. These are devices of no physical significance; we expect them to become unnecessary when more realistic and flexible profile formulae are adopted; in the meantime we regard them as neither more nor less arbitrary than the original specification of the profile families, which indeed have made the devices necessary.

Details of the various modifications will be found in the Appendix.

3. COMPARISON WITH EARLIER RESULTS

3.1 Comparisons with the results of more exact theories.

Reynolds Analogy. We begin this section with a result which may be verified simply by examination of the equations. We expect that, if the turbulent and laminar Prandtl numbers are unity, and if the mainstream velocity u_0 and the surface temperature T_s are independent of position along the wall, the well-known equality will hold between the Stanton number and one half of the drag coefficient. This is indeed, it may be shown, a consequence of our equations, which have the additional implications:

$$1 - \theta_D = z_E, \quad (3.1)$$

$$\zeta_D = 1, \quad (3.2)$$

$$\text{and} \quad \ell_D = \ell \quad (3.3)$$

There is thus complete similarity between the velocity and temperature profiles. The present theory therefore satisfactorily passes this not very exacting test.

When the laminar Prandtl number is very large. Before proceeding to integrations of the differential equations, it is worth noting that in one further respect the theory is bound to produce the right result: when the laminar Prandtl number is very large, the Stanton-number expression 2.16 takes the asymptotic form, deducible from eqs. 2.14, and 2.15:

$$S_S = 0.1111(1 - \theta_D)(c_F/2)^{\frac{1}{2}} \sigma^{-\frac{3}{4}} \quad (3.4)$$

Now a merely qualitative consideration of the differential equation expressing the Steady-Flow Energy equation will show that, when the heat flux through the wall has the low value associated with very high Prandtl number, the measure of the longitudinal enthalpy flux in the boundary layer, θ_D , will be small. The bracket $(1 - \theta_D)$ will therefore be close to unity.

Thus we find that our equations imply, in agreement with experiment, that the heat flux at the wall is proportional to the square root of the local shear stress and to the $-\frac{3}{4}$ power of the Prandtl number; moreover, comparison of the value of the proportionality factor with that deduced by Deissler [23] from experimental data, namely 0.1117, shows that the absolute value of the heat flux will be satisfactory.

The exact computations of Gardner and Kestin [3]. In section 1.2, there was mentioned a stream of work ([1] to [5]), involving exact solutions of the partial differential equation of the thermal boundary layer. Even though the conditions for the validity of that theory are rarely satisfied experimentally, its results provide convenient standards against which we can measure the purely mathematical satisfactoriness of the present procedure; thus, comparison will show what error is introduced by our solution of two ordinary differential equations, generated from the assumption of a profile formula, in place of the full partial differential equation.

To make the comparison, it is necessary to adopt the same constants as were chosen by Gardner and Kestin ($E = 9.025$, $\kappa = 0.4$, $\sigma_{tt} = 1.0$, $\sigma = 0.71, 1.0, 7.0, 30, 100$), to set z_E equal to unity, and to make R_0 very large; further, the P -function implicit in the equations of Gardner and Kestin must be used in place of eq. 2.15. Then, with θ_D and ζ_D both initially small, the two thermal differential equations are

integrated numerically; the wall temperature is taken as uniform to conform with the Gardner-Kestin conditions.

The results obtained in this way are plotted in Fig. 1, in the form of values of $St_G / (c_p/2)^{1/2}$ versus non-dimensional distance along the wall (full lines). Also plotted are a few points from the tables computed by Gardner and Kestin. It is evident that there is very close agreement between the two sets of calculations. We conclude that, at least when the wall temperature is uniform, little error results from the use of the approximate method of solving the partial differential equation.

3.2 Comparison with experimental data for prescribed wall temperature.

The calculation procedure outlined in section 2.3 has been applied to a number of the situations with prescribed wall temperature which have been studied experimentally by various workers ([24], [25], [26], [27] and [8]). Figs. 2 to 6 display our predictions of the Stanton number, and also the experimental data. These include cases of zero, favourable and unfavourable pressure gradients, of wall jets in stagnant surroundings, and of step- or ramp-like discontinuities in the wall temperature. In Figs. 2 to 5, we also show the prediction of the Ambrok method as modified by Moretti and Kays [8].

The agreement of our predictions with experiment is fairly satisfactory in most of the cases, though not as good as that of the modified Ambrok method where the latter can be used.

In the cases shown in Fig. 5, experimental values of the Stanton number fall appreciably below the predicted values where the flow is strongly accelerated. This may be attributed to the phenomenon of "laminarisation" observed by Launder [28], Moretti and Kays [8], and others. These authors define a parameter K by:

$$K \equiv \frac{d(\ln u_G)}{dR_x}, \quad (3.5)$$

and suggest that, when K exceeds about 2×10^{-6} , the turbulent boundary layer changes into a laminar one. The regions of "laminarisation" in Fig. 5 are marked as $K > 2 \times 10^{-6}$.

3.3 Comparison with experimental data for prescribed heat flux at the wall.

As already pointed out, the thermal specification of the problem can be in terms of either the wall temperature or the heat flux through the wall. In Figs.

7 to 10 we present the comparison of our theory with experimental data ([26], [29], [30], [31], [32], [33], [34], [9], and [10]), for prescribed heat flux through the wall. Here again are included flows with uniform or variable free-stream velocity, with or without injection through a slot (including the wall jet in stagnant surroundings), and with uniform, step-wise or non-uniform heat flux through the wall. The regions in Fig. 9 where "laminarisation" is suspected are indicated; the agreement is seriously unsatisfactory only in these cases.

The case of an adiabatic wall is also one of prescribed heat flux. In Fig. 11 we present the comparison of our predictions with experimental data of Poreh and Cermak [35] for diffusion from a line source into a turbulent boundary layer on an impermeable wall (although, strictly speaking, the measurements concerned mass transfer rather than heat transfer, the difference is unimportant for present purposes). The agreement is once again satisfactory.

4. DISCUSSION

4.1 Achievements and shortcomings of the present theory.

The present theory provides a simple calculation procedure for heat transfer in turbulent boundary layers and has a wide range of applicability. The comparisons with the exact solutions of Gardner and Kestin suggest that no serious error is occasioned by our use of a profile-method solution of the temperature equation. The comparisons with experimental data are fairly good, except for strongly accelerated flows; but the agreement can probably be improved by a better selection of empirical constants such as σ_{tt} .

It is true that other theories exist which give better agreement with experiment, over a limited range of conditions; for example, the predictions of the modified Ambrok method are better than our own in the cases studied by Moretti and Kays [8]. However, the Ambrok method has not been successfully applied to the case of a wall jet in stagnant surroundings, or to the case of an adiabatic wall, etc.; nor do its predictions agree, even in form, with the exact solutions of Gardner and Kestin [3]. All other theories known to the authors are similarly limited in applicability. The present theory takes more detailed account of the hydrodynamic as well as of the thermal effects; and because of the use of two-component profiles, it has a wider range of applicability.

It is evident that the present theory cannot yet predict or account for "laminarisation".

arisation" which occurs in strongly accelerated flows. Indeed, our theory incorporates no laws or hypotheses about either the generation or the decay of turbulence.

4.2 Directions of future development.

As pointed out above, the predictions of the present method can presumably be improved by use of the "best-fit" constants, and perhaps of more realistic profiles.

Without any radical change in the calculation procedure, the theory can be extended so as to take into account the effects of mass transfer, wall roughness and kinetic heating. When the variation of density is to be considered, the four differential equations should be solved simultaneously rather than successively, the density being found from the laws of thermodynamics governing the state of the fluid.

5. CONCLUSIONS

1) A calculation procedure has been described for heat transfer in the uniform-property turbulent boundary layer. It is based on a mixing-length hypothesis of the Prandtl type and on two-component velocity and temperature profiles. It has a wider range of applicability than the earlier theories.

2) The predictions of the theory have been compared with earlier exact solutions and experimental data; the results are favourable in most cases.

6. ACKNOWLEDGEMENTS

One of us (S.V.P.) wishes to thank I.C.I. (India) for the tenure of a scholarship during the period of the work reported.

7. REFERENCES

1. D.B.Spalding, "Heat transfer to a turbulent stream from a surface with a step-wise discontinuity in wall temperature." International Developments in Heat Transfer (Proceedings of Conference organized by ASME at Boulder, Colorado, 1961): Part II, pp. 439-446.
2. J.Kestin and L.N.Persen, "Application of Schmidt's method to the calculation of Spalding's function and of the skin-friction coefficient in turbulent flow." Int.J. Heat Mass Transfer, 5, 143-152 (1962).
3. G.O.Gardner and J.Kestin, "Calculation of the Spalding function over a range of Prandtl numbers." Int. J. Heat Mass Transfer, 6, 289-299 (1963).
4. A.G.Smith and V.L.Shah, "The calculation of wall and fluid temperatures for the incompressible turbulent boundary layer, with arbitrary distribution of wall heat flux." Int.J.Heat Mass Transfer, 5, 1179-1189 (1962).
5. D.B.Spalding, "Contribution to the theory of heat transfer across a turbulent boundary layer." Int.J. Heat Mass Transfer, 7, 743-761 (1964).
6. M.Tribus and J.Klein, "Forced convection from non-isothermal surfaces", in Heat Transfer Symposium, Univ. of Michigan, p.211 (1953).
7. G.S.Ambrok, "Approximate solution of equations for the thermal boundary layer with variations in boundary-layer structure." Soviet Phys.Tech. Phys., 2, p.1979 (1957).
8. P.M.Noretti and W.M.Kays, "Heat transfer through an incompressible turbulent boundary layer with varying free-stream velocity and varying surface temperature", Rep. No.PG-1, Dept. of Mech. Eng., Stanford Univ. (1964).
9. L.H.Back and R.A.Seban, "On constant property turbulent boundary layers with variable temperature or heat flow at the wall," J. Heat Transfer, C 87, 151-156 (1965).
10. T.F.MacCarthy and J.P. Hartnett, "Turbulentnye Pogranishnye Sloi s Prodol'nyy Gradientom Davleniya i Teploobmenom" Vsesoyuznoe Soveshchaniye po Teplo-i Massoobmeny, 5-9 May 1964, Minsk.
11. A.G.Smith and V.L.Shah, "Study of heat and mass transfer from non-isothermal surfaces", The College of Aeronautics, Cranfield, Note No.135 (1962).
12. S.V.Patankar and R.G.Taylor, "Diffusion from a line source in a turbulent boundary layer: comparison of theory and experiment," Int.J.Heat Mass Transfer, 8, 1172-1175 (1965).
13. L.Prandtl, "Bericht über Untersuchungen zur ausgebildeten Turbulenz", Zeitschrift f. angew. Math. und Mech., Band 5, Heft 2, pp. 136-139 (April, 1925).
14. B.Hudimoto, "Momentum equations of the boundary layer and their application to the turbulent boundary layer", Memoirs of the Faculty of Engineering, Kyoto University, Vol. XIII, No.4, Oct. 1951 (Kyoto, Japan).
15. M.P.Escudier, "The distribution of the mixing length in turbulent flows near walls," Mech. Eng. Dept., Imperial College, TWF/TN/1 (1965).
16. H.Schlichting, "Boundary Layer Theory", 4th Ed., McGraw Hill, New York, 1960.
17. D.Coles, "The law of the wake in the turbulent boundary layer", J.Fluid Mech., 1, 191-226 (1956).

18. J. Rotta, "Über die Theorie der turbulenten Grenzschichten". Max Planck Inst. für Strömungsforschung. Mitt. No.1. (1950), transl. as "On the theory of the turbulent boundary layer", NACA TM 1344, (1953).
19. D. Ross and J.N. Robertson, "A superposition analysis of the turbulent boundary layer in an adverse pressure gradient", J. Appl. Mech., 18, 95-100 (1951).
20. H. Ludwig and W. Tillmann, "Untersuchungen über die Wandschubspannung in turbulenten Reibungsschichten", Ing. Archiv. 17, 288-299 (1949) transl. as "Investigations of the wall shearing stress in turbulent boundary layers". NACA TM 1285 (1950), also ARC 14800.
21. S.S. Kutateladze, "Fundamentals of Heat Transfer", Arnold, London (1963).
22. D.B. Spalding and C.L.V. Jayatilaka, "A survey of theoretical and experimental information on the resistance of the laminar sub-layer to heat and mass transfer." Proceedings of the Heat and Mass Transfer Conference, Minsk, (May 1964).
23. R.G. Deissler, "Analysis of turbulent heat transfer, mass transfer, and friction in smooth tubes at high Prandtl and Schmidt numbers", NACA Rep. 1210 (1955).
24. W.C. Reynolds, W.M. Kays, and S.J. Kline, "Heat transfer in the turbulent incompressible boundary layer. I - Constant wall temperature." NASA MEMO 12-1-58W, (1958).
25. W.C. Reynolds, W.M. Kays, and S.J. Kline, "Heat transfer in the turbulent incompressible boundary layer. II - Step wall temperature distribution." NASA MEMO 12-2-58W (1958).
26. W.C. Reynolds, W.M. Kays, and S.J. Kline, "Heat transfer in the turbulent incompressible boundary layer. III - Arbitrary wall temperature and heat flux." NASA MEMO 12-3-58W (1958).
27. G.E. Myers, J.J. Schauer, and R.H. Eustis, "The plane turbulent wall jet. Part II - Heat transfer." Stanford Univ., Dept. Mech. Eng., Tech. Rep. No.2, Dec. 1961.
28. B.E. Launder, "Laminarization of the turbulent boundary layer by acceleration", Rep. No. 77, Gas Turbine Lab., M.I.T. (1964).
29. J.P. Hartnett, R.C. Birkebak, and E.R.G. Eckert, "Velocity distributions, temperature distributions and heat transfer for air injected through a tangential slot into a turbulent boundary layer", J. Heat Transfer, C 83, 293 (1961).
30. J.P. Hartnett, R.C. Birkebak, and E.R.G. Eckert, "Velocity distributions, temperature distributions, effectiveness and heat transfer in cooling of a surface with a pressure gradient," International Developments in Heat Transfer, Part IV, 682 (1961).
31. R.A. Seban, "Heat transfer and effectiveness for a turbulent boundary layer with tangential fluid injection", ASME paper No. 59-A-177 (1959).
32. R.A. Seban and D.L. Doughty, "Heat transfer to turbulent boundary layers with variable free-stream velocity", Trans. ASME, 78, 217 (1956).
33. R.A. Seban and L.H. Back, "Effectiveness and heat transfer for a turbulent boundary layer with tangential injection and variable free-stream velocity," ASME paper No. 61-WA-156 (1961).
34. J. Mathieu, "Contribution a l'etude aerothermique d'un jet plan evoluant en presence d'une paroi," Publications Scientifiques et Techniques du Ministere de l'Air. No. 374 (1961).
35. M. Poreh and J.E. Cermak, "Study of diffusion from a line source in a turbulent boundary layer", Int. J. Heat Mass Transfer, 7, 1083-1095 (1964).

8. NOMENCLATURE

- b**, a constant in the I_3 -expression
c, specific heat at constant pressure
c_f, local drag coefficient ($\equiv \tau_s / (\frac{1}{2} \rho u_G^2)$)
E, a constant
I₁ $\equiv \int_0^1 z \, d\xi$
I₂ $\equiv \int_0^1 z^2 \, d\xi$
I₃ $\equiv \int_0^1 z^3 \, d\xi$
I_{0,1} $\equiv \int_0^1 \theta z \, d\xi$
I_{0,2} $\equiv \int_0^1 \theta^2 z \, d\xi$
K, acceleration parameter ($\equiv \frac{d(\ln u_G)}{dR_x}$)
ℓ, mixing length
ℓ', a parameter in the velocity profile
ℓ'_D, a parameter in the temperature profile
P, resistance of the laminar sub-layer
 $\left(\equiv \int_0^\infty \frac{\sigma_t - \sigma_{tt}}{\sigma_{tt}} du^+ \right)$
q", heat flux in the boundary layer directed normally towards the wall
R_G, Reynolds number based on the boundary-layer thickness ($\equiv \frac{\rho u_G y_G}{\mu}$)
R_x, Reynolds number based on the distance along the wall ($\equiv \int_0^x \frac{\rho u_G}{\mu} dx$)
S_S, Stanton number ($\equiv q_s'' / \{c p u_G (T_G - T_s)\}$)

- $\bar{s}$, dissipation integral $\left(\equiv \frac{\int_0^{y_G} \tau \frac{\partial u}{\partial y} dy}{\rho u_G^3} \right)$
- $\bar{S}$, heat-flux integral $\left(\equiv \frac{-\int_0^{y_G} \dot{q}'' \frac{\partial T}{\partial y} dy}{c \rho u_G (T_S - T_G)^2} \right)$
- $\bar{S}^*$, heat-flux integral obtained from linear mixing-length, velocity and temperature profiles.
- T , temperature
- u , velocity along the wall
- x , distance along the wall
- y , distance from and normal to the wall
- z , dimensionless velocity ($\equiv u/u_G$)
- z_E , a parameter in the velocity profile
- θ , dimensionless temperature ($\equiv (T - T_G)/(T_S - T_G)$)
- θ_D , a parameter in the temperature profile
- κ , a constant
- λ_G , a constant
- μ , dynamic viscosity
- ξ , dimensionless distance normal to the wall ($\equiv y/y_G$)
- ξ_D , a parameter in the temperature profile
- ρ , density
- σ , (laminar) Prandtl number
- σ_{tt} , turbulent Prandtl number
- σ_t , total Prandtl number
- τ , shear stress
- ϕ , a stream function ($\equiv \int_0^y \rho u dy$)

Subscripts

- G , outer edge of the hydrodynamic boundary layer
- S , wall.

APPENDIX

Derivation of the T^2 equation

The partial differential equation representing the conservation of enthalpy in von Mises form, is as follows:

$$\frac{\partial T}{\partial x} = \frac{1}{c} \frac{\partial \dot{q}''}{\partial \phi} \quad (A.1)$$

Multiplying this by $(T - T_G)$ and then integrating w.r.t. ϕ from 0 to ϕ_G , we obtain:

$$\frac{d}{dx} \int_0^{\phi_G} \frac{(T - T_G)^2}{2} d\phi = \frac{1}{c} \left[\dot{q}_S'' (T_G - T_S) - \int_0^{\phi_G} \dot{q}'' \frac{\partial T}{\partial \phi} d\phi \right]. \quad (A.2)$$

Re-writing in terms of y rather than ϕ yields:

$$\frac{d}{dx} \int_0^{y_G} \frac{(T - T_G)^2}{2} \rho u dy = \frac{1}{c} \left[\dot{q}_S'' (T_G - T_S) - \int_0^{y_G} \dot{q}'' \frac{\partial T}{\partial y} dy \right] \quad (A.3)$$

Using the dimensionless forms defined in section 8, we finally deduce:

$$\frac{1}{(T_S - T_G)^2} \frac{d}{dx} \left\{ (T_S - T_G)^2 R_G I_{\theta,2} \right\} = 2S_S + 2\bar{S}. \quad (A.4)$$

Auxiliary relations and various modifications

In the case of the auxiliary functions appearing in the hydrodynamic equations, we make a change in the I_2 -expression only, leaving I_1 and I_3 as derived from the velocity-profile formula (eq.2.3). Thus we have:

$$I_1 = 0.5 + (0.5 - 1/\ell') z_E \quad (A.5)$$

$$I_2 = \frac{1}{3} + \left(\frac{1}{3} - \frac{0.5}{\ell'} \right) z_E + \left(\frac{1}{3} - \frac{1.5}{\ell'} + \frac{2}{\ell'^2} \right) z_E^2, \quad (A.6)$$

and

$$I_3 = 0.25 + (0.25 - \frac{b}{\ell'}) z_E + (0.25 - \frac{7-b}{\ell'} + \frac{0.75}{\ell'^2}) z_E^2 + (0.25 - \frac{11}{6\ell'} + \frac{5.25}{\ell'^2} - \frac{6}{\ell'^3}) z_E^3, \quad (A.7)$$

$$\text{We take: } \left. \begin{array}{l} b = 0.255 \text{ for } 0 < z_E \leq 1, \\ \text{and } b = -0.25 \text{ for } z_E > 1. \end{array} \right\} \quad (A.8)$$

The value of b would be 0.333, if we accepted the full implications of the velocity profile.

In the thermal equations, we use the $I_{\theta,1}$, and $I_{\theta,2}$ expressions without modification, as follows:

$$I_{\theta,1} = \frac{z_E \xi_D}{\ell'} \left\{ \frac{(1 - \theta_D)(\ell' + \ln \xi_D - 2)}{\ell'_D} + \frac{\theta_D}{2} (\ell' + \ln \xi_D - 1.5) \right\} + (1 - z_E) \xi_D^2 \left\{ \frac{1 - \theta_D}{4\ell'_D} + \frac{\theta_D}{6} \right\}, \quad (A.9)$$

$$I_{\theta,2} = \frac{2z_E \xi_D}{\ell'} \left[\frac{(1-\theta_D)^2}{\ell'^2} (\ell' + \ln \xi_D - 3) + \frac{(1-\theta_D) \theta_D}{\ell'_D} \right. \\ \left. \left\{ \frac{3}{4} (\ell' + \ln \xi_D) - \frac{7}{4} \right\} + \frac{\theta_D^2}{6} \left\{ \ell' + \ln \xi_D - \frac{11}{6} \right\} \right] \\ + (1-z_E) \xi_D^2 \left[\frac{(1-\theta_D)^2}{4 \ell'^2} + \frac{5}{18} \frac{(1-\theta_D) \theta_D}{\ell'_D} + \frac{\theta_D^2}{12} \right]. \quad (A.10)$$

Our temperature profile near the wall is linear with velocity; however, more exact analysis shows that a power-law relation with power equal to 1.11 is better. This has the effect of multiplying $\bar{S}$ by 0.91. Since this applies strictly when $\theta_D = 0$, we use the relation

$$\bar{S} = \{\theta_D + 0.91(1-\theta_D)\} \bar{S}^* \quad (A.11)$$

where $\bar{S}^*$ is obtained from the linear mixing-length, velocity and temperature profiles.

While applying our theory to the data of [27], we have avoided entering the double-valued region for the thermal equations by keeping θ_D constant at zero and then using only one thermal equation, namely eq. 2.11, to obtain ξ_D .

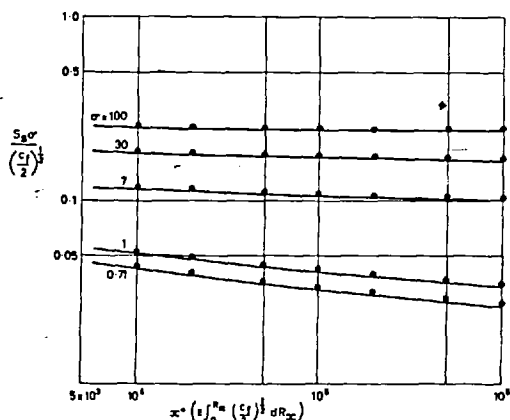


Fig.1 Comparison with exact solution, for the step-wall-temperature problem when the velocity boundary layer is much thicker than the thermal one.

Dots represent the exact solutions of Gardner and Kestin [3]; the full lines represent our solutions. The parameter is the laminar Prandtl number.

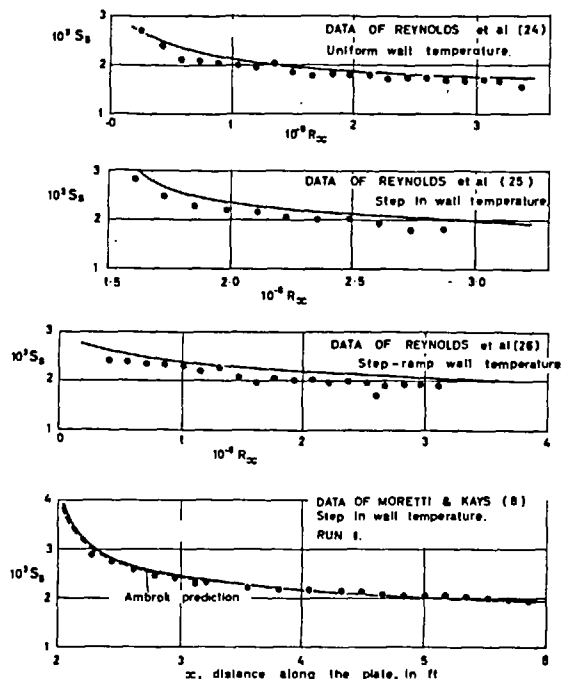


Fig.2 Comparison with experimental data for prescribed wall temperature: the flat-plate cases.

Dots represent experimental data; the full lines represent the predictions of the present theory.

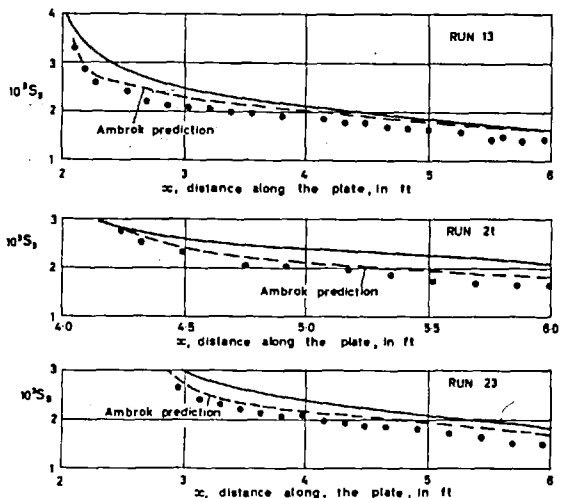


Fig.3 Comparison with experimental data: step-ramp wall temperature, favourable pressure gradient.

Dots represent experimental data of Moretti and Kays [8]; the full lines represent the predictions of the present theory.

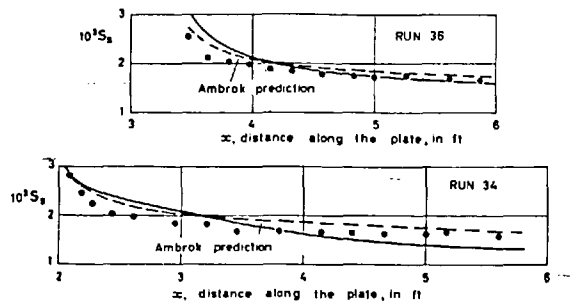


Fig. 4 Comparison with experimental data: step in wall temperature, adverse pressure gradient. Dots represent experimental data of Moretti and Kays [8]; the full lines represent the predictions of the present theory.

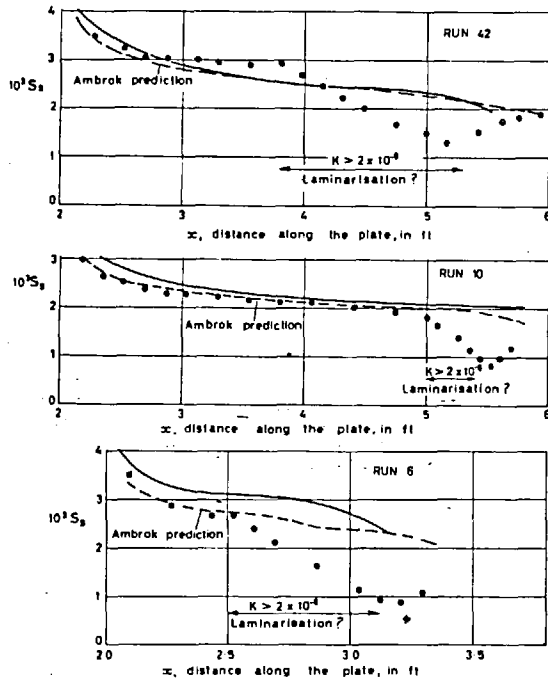


Fig. 5 Comparison with experimental data: step in wall temperature, strong favourable pressure gradient. Dots represent experimental data of Moretti and Kays [8]; the full lines represent the predictions of the present theory.

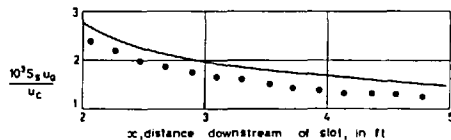


Fig. 6 Comparison with experimental data: a wall jet in stagnant surroundings with a step in wall temperature. Dots represent the experimental data of Myers, Schauer and Eustis [27], Run 23; the full line represents the predictions of the present theory. u_c stands for the velocity of the air emerging from the slot.

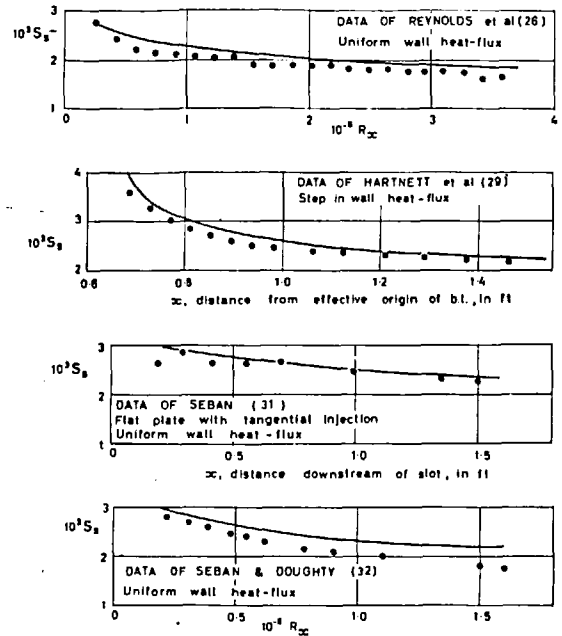


Fig. 7 Comparison with experimental data for prescribed heat flux at the wall: the flat-plate cases. Dots represent experimental data; the full lines represent the predictions of the present theory.

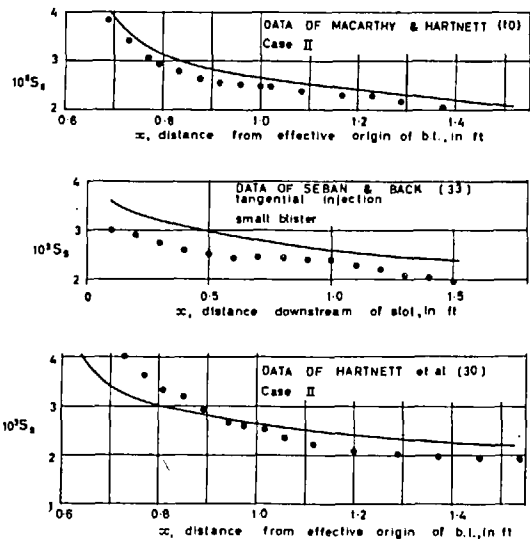


Fig. 8 Comparison with experimental data: step in wall heat-flux, favourable pressure gradient. Dots represent experimental data; the full lines represent the predictions of the present theory.

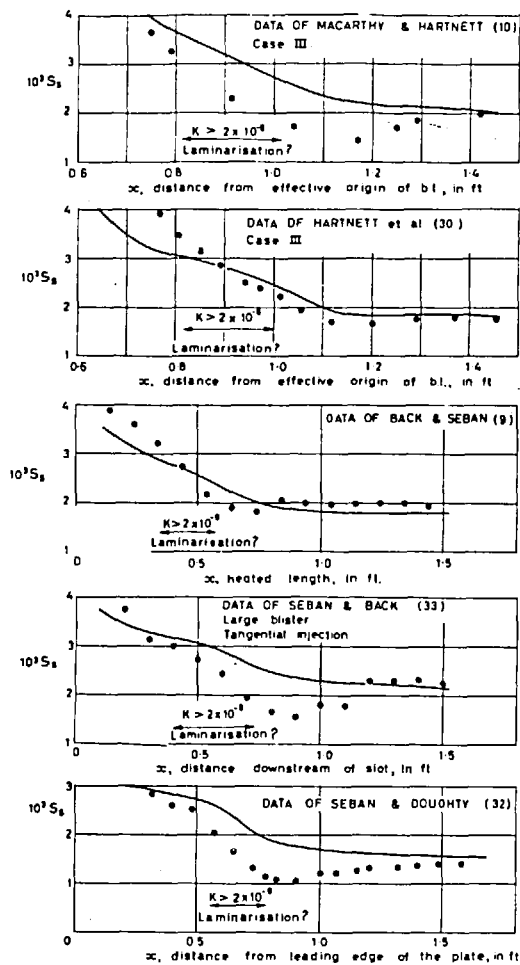


Fig.9 Comparison with experimental data: step in wall heat-flux, strong favourable and adverse pressure gradients.

Dots represent experimental data; the full lines represent the predictions of the present theory.

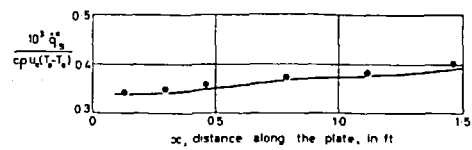


Fig.10 Comparison with experimental data: a wall jet in stagnant surroundings, non-uniform heat transfer to the wall.

Dots represent experimental data of Mathieu [34]; the full line represents the prediction of the present theory. u_c stands for the velocity of the air emerging from the slot.

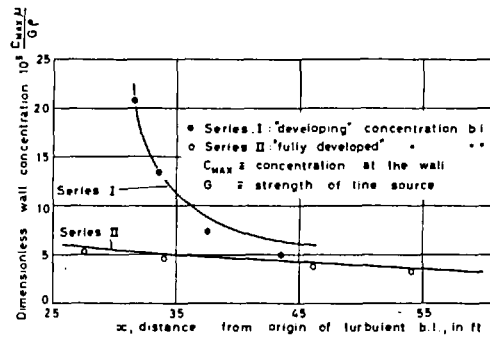


Fig.11 Comparison with experimental data for diffusion from a line source of ammonia gas into a flat-plate turbulent boundary layer.

Dots represent experimental data of Poreh and Cermak [35]; the full lines represent the predictions of the present theory.